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Stabilité stochastique, attracteur aléatoire et bifurcations homocline et heterocline

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Description des travaux

Cette thèse est consacrée à l'étude des comportements asymptotiques des équations différentielles. Elle est composée de deux parties. La première partie concerne la stabilité stochastique de certains systèmes d'équations différentielles stochastiques et l'existence d'attracteur aléatoire de l'équation stochastique de Ginzburg-Landau. La seconde partie porte sur la bifurcation homocline et heterocline.

1. Stabilité stochastique et attracteur aléatoire

1.1. Stabilité de système SIRS avec des perturbations aléatoires. Nous considérons le modèle SIRS stochastique admettant la perte d'immunité :

$$\begin{aligned} \mathrm{d}S(t) &= (-\beta S(t)I(t) - \mu S(t) + \gamma R(t) + \mu)\,\mathrm{d}t - \sigma S(t)I(t)\,\mathrm{d}w_t, \\ \mathrm{d}I(t) &= (\beta S(t)I(t) - (\lambda+\mu)I(t))\,\mathrm{d}t + \sigma S(t)I(t)\,\mathrm{d}w_t, \\ \mathrm{d}R(t) &= (\lambda I(t) - (\mu+\gamma)R(t))\mathrm{d}t, \end{aligned} \tag{1}$$

et le modèle SIRS avec retard :

$$\begin{aligned} \mathrm{d}S(t) &= (-\beta S(t)\int_0^h f(s)I(t-s)\,\mathrm{d}s - \mu S(t) + \gamma R(t) + \mu)\,\mathrm{d}t \\ &\quad -\sigma S(t)\int_0^h f(s)I(t-s)\,\mathrm{d}s\,\mathrm{d}w_t, \\ \mathrm{d}I(t) &= (\beta S(t)\int_0^h f(s)I(t-s)\,\mathrm{d}s - (\lambda+\mu)I(t))\,\mathrm{d}t \\ &\quad +\sigma S(t)\int_0^h f(s)I(t-s)\,\mathrm{d}s\,\mathrm{d}w_t, \\ \mathrm{d}R(t) &= (\lambda I(t) - (\mu+\gamma)R(t))\,\mathrm{d}t, \end{aligned} \tag{2}$$

où σ est une constante, qui représente la perturbation stochastique environnemental sur le taux β de transmission d'épidémie et w_t est un processus de Wiener réel défini sur un espace complet de probabilité $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, P)$.

Nous étudions la stabilité stochastique des systèmes SIRS ci-dessus. Nous obtenons des conditions sous lesquelles le point d'équilibre trivial est stable. Ces conditions améliorent celles données dans [**89**] pour $\gamma = 0$.

Les deux résultats principaux sont les suivants.

Théorème 1.1. *Supposons que* $0 < \beta < \lambda + \mu - \frac{\sigma^2}{2}$. *Alors l'équilibre sans épidémie* $E_0 = (1, 0, 0)$ *du système* (1) *est stochastiquement asymptotiquement stable.*

Théorème 1.2. *Si les conditions* $\max\{\lambda - \gamma,\ \beta + \gamma\} < 2\mu,\ \beta \leq \lambda + \mu - \frac{\sigma^2}{2}$ *sont vérifiées, alors l'équilibre sans épidémie* $E_0 = (1, 0, 0)$ *du système* (2) *est stochastiquement stable.*

1.2. Attracteurs aléatoire de l'équation stochastique de Ginzburg-Landau sur des domaines non bornés. Nous étudions l'équation stochastique de Ginzburg-Landau définie sur $\mathbb{R}^n$:

$$\mathrm{d}u = (\lambda + i\mu)\Delta u\mathrm{d}t - (\kappa + i\beta)|u|^2 u\mathrm{d}t - \gamma u\mathrm{d}t + \sum_{j=1}^{m} \varphi_j \mathrm{d}\omega_j(t), \tag{3}$$

avec la condition initiale

$$u(x,0) = u_0(x),\ x \in \mathbb{R}^n, \tag{4}$$

où $\lambda,\ \mu,\ \kappa,\ \beta,\ \gamma$ sont des coefficients réels, avec $\lambda > 0,\ \kappa > 0,\ \gamma > 0$; $\varphi_j \in H^2(\mathbb{R}^n) \cap W^{2,4}(\mathbb{R}^n),\ j = 1, \ldots, m$ sont indépendants de t et définis sur $\mathbb{R}^n$; $\{\omega_j\}_{j=1}^m$ sont des Wiener processus réels indépendants sur un epsace complet de probabilité $(\Omega, \mathcal{F}, P)$. Des équations différentielles stochastiques de ce type apparaissent dans beaucoup de systèmes physiques quand des forces aléatoires sont prises en comptes. Notre but est d'étudier le comportement asymptotique en temps du système dynamique généré par ce type d'équation. Plus précisement nous étudions l'existence de $\mathcal{D}$-attracteur aléatoire dans $\mathbb{L}^2(\mathbb{R}^n)$.

On obtient le résultat suivant.

Théorème 1.3. *Si $\sqrt{3}\kappa \geq |\beta|$, alors le système dynamique aléatoire ϕ de l'équation stochastique de Ginzburg-Landau avec des bruits additifs possède un unique $\mathcal{D}$-attracteur aléatoire dans $\mathbb{L}^2(\mathbb{R}^n)$.*

2. Bifurcation d'orbite homocline et d'orbite heterocline

Nous considérons le système

$$\dot{z} = f(z) + g(z, \mu), \tag{5}$$

et le système non-perturbé

$$\dot{z} = f(z), \tag{6}$$

où $z \in \mathbb{R}^{m+n+2},\ m \geq 0, n \geq 0, m+n > 0,\ \mu \in \mathbb{R}^l,\ l \geq 2,\ 0 \leq ||\mu|| \ll 1,\ g(z, 0) = 0$. On note par $||\cdot||$ la norme dans $\mathbb{R}^l$.

Nous étudirons plusieurs situations suivant des hypothèses sur le système non perturbé (6).

2.1. Bifurcation d'orbite homocline avec inclination-flip. Nous étudions d'abord la bifurcation d'orbite homocline non résonante en dimension 3 avec *inclination-flip*. C'est-à-dire que nous considérons la situation suivante : $m = 0, n = 1, l = 2, f(0) = 0$. Nous supposons que

- (H_1) Le système (6) possède une singularité hyperbolique à l'origine et la matrice $Df(0)$ a trois valeurs propres réelles simples : $-\alpha, -\beta, 1$ vérifiant $\alpha > \beta > 0$.
- (H_2) Le système (6) a une orbite homocline $\Gamma = \{z = r(t),\ t \in R\}$. Soit $e^{\pm} = \lim_{t \to \mp\infty} \frac{\dot{r}(t)}{|\dot{r}(t)|}$. Alors $e^+ \in T_0 W^u,\ e^- \in T_0 W^s$ sont les vecteurs propres unitaires correspondant aux valeurs propres 1 et $-\beta$.
- (H_3) Notons e_s^- le vecteur propre unitaire correspondant à $-\alpha$. Alors
$$\mathrm{Span}(T_{r(t)} W^u, T_{r(t)} W^s, e_s^-) = \mathbb{R}^3, \quad \text{pour } t \ll -1.$$

Le système variationnel linéaire de (6) et son système adjoint sont

$$\dot{z} = Df(r(t))z, \tag{7}$$

et

$$\dot{z} = -(Df(r(t)))^* z. \tag{8}$$

Notons $r(t) = (r^x(t), r^y(t), r^v(t))$. Soit δ suffisamment petit tel que

$$\{(x, y, v) : \ |x|, |y|, |v| < 2\delta\} \subset U.$$

Soit $T > 0$ suffisamment grand tel que $r(-T) = (\delta, 0, 0)$, $r(T) = (0, \delta, \delta_v)$, où $|\delta_v| = O(\delta^2)$.

Alors le système (7) admet une matrice fondamentale de solutions $Z(t)$ avec

$$z(-T) = \begin{pmatrix} 0 & \omega_{21} & \omega_{31} \\ 0 & 0 & \omega_{32} \\ 1 & 0 & \omega_{33} \end{pmatrix}, \qquad z(T) = \begin{pmatrix} \omega_{11} & 0 & 0 \\ \omega_{12} & 1 & 0 \\ \omega_{13} & \omega_{23} & 1 \end{pmatrix},$$

où $|\omega_{23}| \ll 1, \omega_{21} < 0, \omega_{11} \neq 0, \omega_{32} \neq 0$. Le système (8) a aussi une matrice fondamentale de solutions $\Phi(t) = (Z^{-1}(t))^*$. On note $\Phi(t) = (\phi_i^1(t), \phi_i^2(t), \phi_i^3(t))$ et

$$M_j = \int_{-T}^{T} (\phi_j(t))^* g_\mu(r(t), 0)\, \mathrm{d}t, \quad j = 1, 3.$$

Premièrement, nous étudions la bifurcation homocline d'inclination-flip non forte, c'est-à-dire que $\omega_{33} \neq 0$. Nous obtenons que le résultat de la bifurcation est unique tel que ou bien l'orbite homocline persiste ou bien une orbite périodique unique soit crée pour le système perturbé (5).

Théorème 2.1. *Sous les hypothèses* $(H_1), (H_2), (H_3)$*, si* $M_1 \neq 0$ *et* $\omega_{33} \neq 0$*, alors le système* (5) *a au plus une orbite périodique dans un petit voisinage de* Γ*. Elle existe si et seulement si* $\mu \in \{\omega_{11} M_1 \mu > 0\}$, $0 < ||\mu|| \ll 1$ *quand* $\alpha > \beta > 1$; $\mu \in \{\omega_{32}\omega_{33} M_1 \mu > 0\}$, $0 < ||\mu|| \ll 1$ *pour* $1 > \alpha > \beta > 0$ *ou* $\alpha > 1 > \beta > 0$.

Deuxièmement, nous considérons la bifurcation homocline d'inclination-flip forte, c'est-à-dire que $\omega_{33} = 0$. Le résultat de la bifurcation est aussi unique. Plus précisément on obtient le théorème suivant.

Théorème 2.2. *Supposons que les hypothèses* $(H_1), (H_2), (H_3)$ *soient vérifiées. Si* $\omega_{33} = 0$*, alors nous avons les résultats suivant.*

(1) *Si* $1 > \alpha > \beta > 0$ *et* $\delta_v \neq 0$*, alors le système* (5) *a une unique orbite 1-périodique si et seulement si* $\mu \in \{\delta_v M_1 \mu < 0\}$, $0 < ||\mu|| \ll 1$; *et il existe une surface de bifurcation de codimension 1 :* $H_1 = \{\mu : M_1\mu + h.o.t. = 0\}$ *avec le vecteur normal* M_1 *en* $\mu = 0$ *tel que* Γ *persiste pour* $\mu \in H_1$.
(2) *Si* $\alpha > 1 > \beta > 0$ *ou* $\alpha > \beta > 1$*, le système* (5) *a une unique orbite 1-périodique si et seulement si* $\mu \in \{\omega_{11} M_1 \mu > 0\}$, $0 < ||\mu|| \ll 1$; *et* H_1 *est aussi une surface de bifurcation de codimension 1 tel que* Γ *persiste pour* $\mu \in H_1$.

Et en fin, on considère le dernier cas dégénéré, c'est-à-dire que $\omega_{33} = 0, \delta_v = 0$. On obtient le théorème suivant.

Théorème 2.3. *Supposons que les hypothèses* $(H_1), (H_2), (H_3)$ *soient vérifiées. On suppose de plus que* $1 > \alpha > \beta > 0$, $\omega_{33} = 0$, $\delta_v = 0$ *et rang*$(M_1, M_3) = 2$*. Alors il existe une surface de bifurcation 1-homocline* H^1*, une surface de bifurcation d'orbite 2-pli périodique* SN^1*, une surface de bifurcation* 2^n*-périodique* P^{2^n}*, et une surface de bifurcation* 2^n*-homocline* H^{2^n} *pour tout* $n \in \mathbb{N}$*, qui admettent le même vecteur normal* M_1 *en* $\mu = 0$*, tels que le système* (5) *possède*

une orbite 1-homocline si et seulement si $\mu \in H^1$ *et* $||\mu|| \ll 1$*;*

une orbite 2-pli périodique si et seulement si $\mu \in SN^1$*;*

une orbite 2^{n-1}*-périodique ayant changé de stabilité et une nouvelle orbite* 2^n*-périodique en même temps si et seulement si* $\mu \in P^{2^n}$*;*

une orbite 2^n*-homocline si et seulement si* $\mu \in H^{2^n}$.

De plus il existe une surface de bifurcation Δ_1 *(qui est une branche de* H^1*) avec codimension 1 et vecteur normal* M_1 *tel que le système* (5) *possède une orbite 1-homocline et une orbite 1-périodique pour* $\mu \in \Delta_1$ *et* $||\mu|| \ll 1$.

Le diagramme de bifurcation suivant permet de bien illustrer nos résultats.

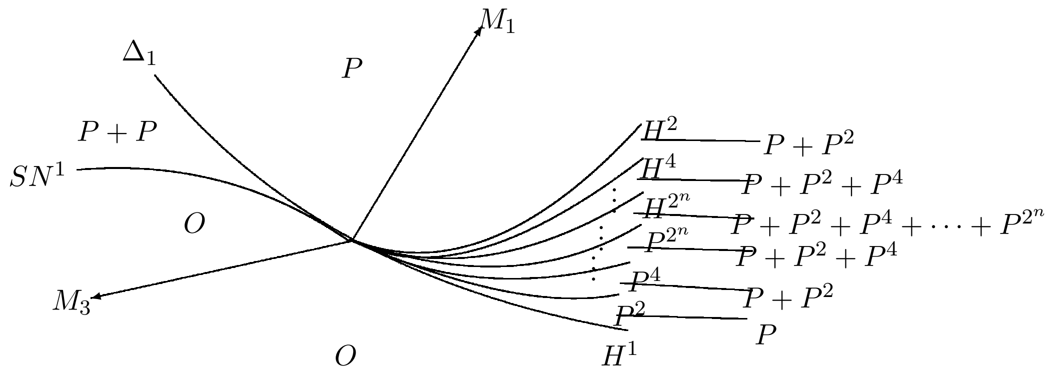


Figure 1. Diagramme de bifurcation dans le cas :
$1 > \alpha > \beta > 0,\ \alpha+\beta > 1,\ \delta_v = 0,\ \omega_{11} > 0,\ \omega_{33} = 0.$

2.2. Bifurcation d'orbites double homoclines tordus avec codimension 2. Nous étudions la bifurcation d'orbites double homoclines tordus avec codimension 2. Le système que nous considérons est dans le cas où $m \geq 0, n \geq 0, m+n > 0,\ l \geq 2,\ f(0) = 0$, et est de classe C^r. La différence avec la bifurcation d'orbite homocline d'inclination-flip du cas précédent est que la dégénérescence du champ de vecteur (6) vient exclusivement de la double homoclinicité.

Les hypothèses générales sont les suivantes.

(H_1') Le système (6) a une singularité hyperbolique à l'origine et la matrice de linéarisation à l'origine $Df(0)$ a des valeurs propres simples : $\lambda_1,\ \lambda_{2i}(i = 1, 2, \cdots, m),\ -\rho_1,\ -\rho_{2j}$ $(j = 1, 2, \cdots, n)$ vérifiant

$$-\mathrm{Re}\rho_{2j} < -\rho_1 < 0 < \lambda_1 < \mathrm{Re}\lambda_{2i}.$$

On suppose qu'il n'y a pas de résonance forte entre $-\rho_1$ et λ_1. On peut toujours suppser que $\rho_1 > \lambda_1$ sans perte de généralité.

(H_2') Le système (6) a deux orbites homoclines $\Gamma = \Gamma_1 \cup \Gamma_2$,

$$\Gamma_i = \{z = r_i(t) :\ t \in \mathbb{R},\ r_i(\pm\infty) = 0\}$$

et $\dim(T_{r_i(t)}W^s \cap T_{r_i(t)}W^u) = 1$, pour $i = 1, 2$, où W^s et W^u désignent la variété stable et instable respectivement, et T_AW est l'espace tangent de W en A.

(H_3') Soit $e_i^{\pm} = \lim_{t\to\mp\infty} \frac{\dot r_i(t)}{|\dot r_i(t)|}$, alors $e_i^+ \in T_0W^u,\ e_i^- \in \mathrm{T_0W}^s$ sont des vecteurs propres unitaires associés à λ_1 et $-\rho_1$, respectivement. De plus, $e_1^+ = -e_2^+,\ e_1^- = -e_2^-$.

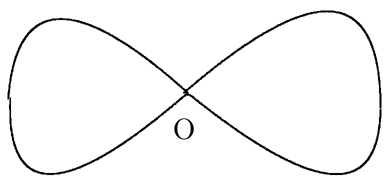
(H_4') $\mathrm{Span}\{T_{r_i(t)}W^u,\ T_{r_i(t)}W^s,\ e_i^+\} = \mathbb{R}^{m+n+2}$ quand $t \gg 1$,
$\mathrm{Span}\{T_{r_i(t)}W^u,\ T_{r_i(t)}W^s,\ e_i^-\} = \mathbb{R}^{m+n+2}$ quand $t \ll -1$.

Sous ces hypotheses nous étudions la bifurcation d'orbites double homoclines.

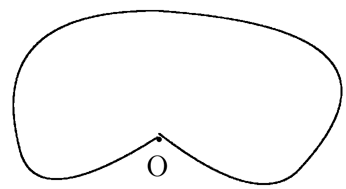
Dans le cas où il y a une seule orbite tordu, on obtient pour le système perturbé l'existence et unicité d'orbites : 1-1 double homoclines, 2-1 double homoclines, 2-1 homoclines à droite, 1-1 homoclines grand, 2-1 homoclines grand et 2-1 periodique grand.

Pour le cas où les double orbites sont tordu, nous obtenons l'existence et l'unicité d'orbites : 1-1 double homoclines, 1-2 double homoclines, 2-1 double homoclines, 2-2 homoclines double, 2-1 homocline grand, 1-2 homocline grand, 2-2 homocline grand, 2-2 homocline à droite, 2-2 orbite homocline grand, 2-2 homocline à gauche, et 2-2 periodique grand. (voir Figure 2.)

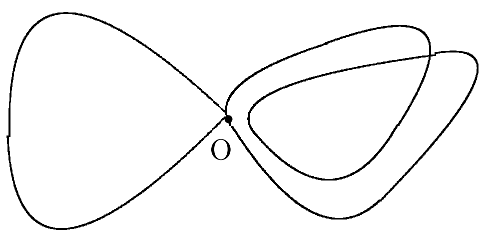
De plus, les surfaces de bifurcation et les domaines d'existence sont obtenus. Les ensembles de bifurcations sont présentés dans le plan engendré par les deux premiers vercteurs de Melnikov.



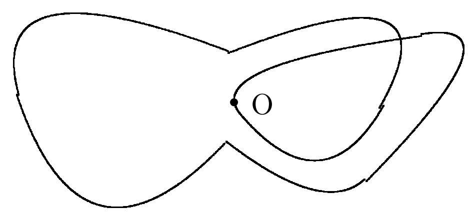
1-1 double homocline



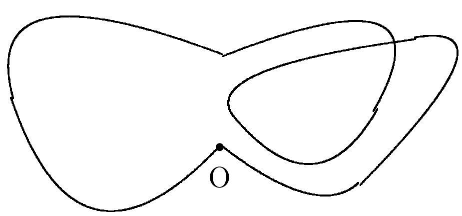
1-1 homocline grand



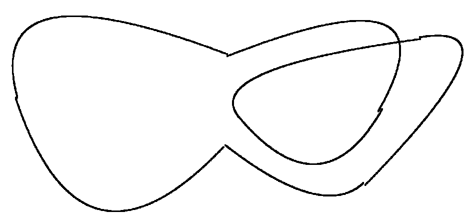
2-1 double homoclines



2-1 homocline à droite



2-1 homocline grand



2-1 periodique grand

Figure 2. Diagramme de bifurcation

2.3. Bifurcation de cycle heterodimensionnel avec orbite-flip. On se place mantenant dans le cas où $z \in \mathbb{R}^4$, $\mu \in \mathbb{R}^l$, $l \geq 2$, et les singularités du système non perturbé (6) sont p_1, p_2, *i.e.* $f(p_i) = 0$, $g(p_i, \mu) = 0$, $i = 1, 2$. On étudie la bifurcation de cycle heterodimensionnel avec orbite-flip.

Nous supposons que les conditions suivantes soient vérifiées.

(H_1'') Le système (6) a deux singularités $p_i,\ i=1,2$ et les matrices de linéarisation $Df(p_1)$ a des valeurs propres simples : $\lambda_1^1,\ \lambda_1^2,\ \lambda_1^3,\ -\rho_1^1$ vérifiant $-\rho_1^1<0<\lambda_1^1<\lambda_1^3<\lambda_1^2$; la matrice $Df(p_2)$ a des valeurs propres simples : $\lambda_2^1,\ \lambda_2^2,\ -\rho_2^1,\ -\rho_2^2$ vérifiant $-\rho_2^2<-\rho_2^1<0<\lambda_2^1<\lambda_2^2$.

(H_2'') Il existe un cycle heterocline $\Gamma=\Gamma_1\cup\Gamma_2$ reliant p_1 et p_2. Ici,
$\Gamma_i=\{z=r_i(t),\ t\in\mathbb{R}\}$ pour $i=1,2$,
$r_1(-\infty)=r_2(+\infty)=p_1,\ r_1(+\infty)=r_2(-\infty)=p_2$;
et $\dim\left(T_{r_1(t)}W_1^u\cap T_{r_1(t)}W_2^s\right)=1$.

(H_3'') Soit $e_1^\pm=\lim_{t\to\mp\infty}\frac{\dot r_1(t)}{|\dot r_1(t)|}$, alors $e_1^+\in T_{p_1}W_1^u,\ e_1^-\in T_{p_2}W_2^s$ sont des vecteurs propres associés à λ_1^1 et $-\rho_2^1$, respectivement.
Soit $e_2^{u+}=\lim_{t\to-\infty}\frac{\dot r_2(t)}{|\dot r_2(t)|},\ e_2^-=\lim_{t\to+\infty}\frac{\dot r_2(t)}{|\dot r_2(t)|}$, alors $e_2^{u+}\in T_{p_2}W_2^{uu},\ e_2^-\in T_{p_1}W_1^s$ sont des vecteurs propres associés à λ_2^2 et $-\rho_1^1$, respectivement, où W_2^{uu} est la variété stable forte de p_2.

(H_4'') $\lim\limits_{t\to+\infty}T_{r_2(t)}W_2^u=\mathrm{Span}\{e_2^-,e_1^{u+}\}$, où e_1^{u+} est le vecteur propre unitaire associé à λ_1^2. (voir Figure 3.)

(H_5'') Les systèmes sont de classe C^r et $Df(p_i),\ i=1,2$ vérifient la condition forte de Sternberg d'ordre Q avec $Q\geq 2$ et $r\geq 3Q$. En plus le nombre de Q-lissage K de $Df(p_i)$ est ≥ 4.

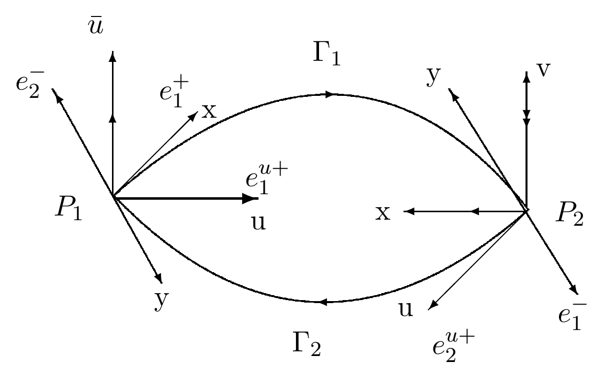


Figure 3. Cycle heterodimensionnel $\Gamma=\Gamma_1\cap\Gamma_2$

Sous les hypothèses ci-dessus, nous obtenons l'existence et l'unicité, ou non-existence d'orbite homocline, orbite heterocline, et orbite périodique. Nous donnons aussi les conditions de co-existence d'orbite homocline et d'orbite périodique. La co-existence d'orbite périodique ou d'orbite homocline avec le cycle heterodimensionnel persistant est impossible. Nous établissons aussi la surface de bifurcation d'orbite double ou triple périodique. Basant sur l'analyse de bifurcation, nous localisons les domaines d'existence.

Les travaux présentés dans cette thèse a fait l'objet de la publication de deux articles [**53, 54**] et deux préprints [**52, 55**] soumis pour publication.

Introduction

The study on dynamical systems can be dated back to the end of the 19th century. As early as in 1881, H. Poincaré started the qualitative theory of ordinary differential equations, whose topics include the stability problem, existence of periodic orbit and returning map. All these topics together with his research techniques are the initiation of dynamical system. Since 1912, G.D. Birkhoff expanded the research work on dynamical systems, taking the problem of three bodies as a background, and obtained the ergodic theorem. Many years later, the Kolmogorov-Arnold-Moser theorem is established in the fields of celestial mechanics or in Hamiltonian system with the background of the solar system stability.

During the past two decades, there has been an essential change on the study of dynamic system. This is due to the structural stability. The concept of structural stability for ordinary system was firstly proposed by A.A. Andronov and L.S. Pontrjagin in 1937 on some planar differential equations. However, it did not attract people's attention until 20 years later when M. Peck Soto gave the density theorem on two-dimensional structural stable system. S. Smale together with many other mathematicians has made great contributions to differential dynamical systems. For instance, the compact invariant subset with hyperbolic structure is still a hot topic up to now. Since the density theorem does not hold in high dimension systems, the bifurcation problem in high dimension receives more and more attention.

Let us recall some basic facts of dynamical systems. One can refer to [**87**] for more details.

We will consider dynamical systems whose state is described by an element $u = u(t)$ of a metric space H. In most cases, and particularly for dynamical systems associated to partial or ordinary differential equations, the parameter t (the time or the timelike variable) varies continuously in $\mathbb{R}$ or in some intervals of $\mathbb{R}$. In some cases, t will take only discrete values, $t \in \mathbb{Z}$ or some subset of $\mathbb{Z}$. And the space H will be a Hilbert or a Banach space.

The evolution of the dynamical system is described by a family of operators $S(t),\ t \geq 0$, which map H into itself and enjoy the semigroup properties:

- $S(t+s) = S(t)S(s), \qquad \forall s, t \geq 0,$
- $S(0) = I \qquad$ (Identity in H).

Usually, the semigroup $S(t)$ will be determined by the solution of an ordinary or a partial differential equation.

For $u_0 \in H$, the orbit or trajectory starting at u_0 is the set $\bigcup_{t\geq 0} S(t)u_0$. Similarly, when it exists, an orbit or trajectory ending at u_0 is a set of points $\bigcup_{t\leq 0} S(t)u_0$.

For $u_0 \in H$ or $B \subset H$, we define the $\omega-$limit set of u_0 (or B) as

$$\Lambda(u_0) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)u_0} \quad \text{or} \quad \Lambda(B) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B},$$

where the closures are taken in H.

A fixed point, or a stationary point, or an equilibrium point is a point $u_0 \in H$ such that

$$S(t)u_0 = u_0, \quad \text{for all} \;\; t \geq 0.$$

The orbit and the ω-limit sets of such a point are of course equal to $\{u_0\}$.

If u_0 is a stationary point of the semigroup, we define the stable and the unstable manifold of u_0 as follows.

The stable manifold $W^s(u_0)$ of u_0 is the (possibly empty) set of points u_* which belong to a complete orbit $\{u(t),\ t \in \mathbb{R}\},\ u_* = u(t_0)$ such that

$$u(t) = S(t - t_0)u_* \to u_0 \quad \text{as} \quad t \to +\infty.$$

The unstable manifold $W^u(u_0)$ of u_0 is the (possibly empty) set of points $u_* \in H$ which belong to a complete orbit $\{u(t),\ t \in \mathbb{R}\}$ such that

$$u(t) \to u_0 \quad \text{as} \quad t \to -\infty.$$

A stationary point u_0 is stable if $W^u(u_0) = \emptyset$.

We say that a set $X \subset H$ is positively invariant for the semigroup $S(t)$ if

$$S(t)X \subset X, \quad \text{for all} \quad t \geq 0.$$

It is said to be negatively invariant if

$$S(t)X \supset X, \quad \text{for all} \quad t \geq 0.$$

When the set is both positively and negatively invariant, we call it an invariant set or a functional invariant set.

Definition 1. *A set $X \subset H$ is a functional invariant set for the semigroup $S(t)$ if*

$$S(t)X = X, \quad \textit{for all} \quad t \geq 0.$$

Definition 2. *An attractor is a set $\mathfrak{A} \subset H$ that enjoys the following properties:*

(1) *$\mathfrak{A}$ is nonempty and compact.*
(2) *$\mathfrak{A}$ is an invariant set ($S(t)\mathfrak{A} = \mathfrak{A},\ t \geq 0$).*
(3) *$\mathfrak{A}$ possesses an open neighborhood $\mathfrak{u}$ such that, for every u_0 in $\mathfrak{u}$, $S(t)u_0$ converges to $\mathfrak{A}$ as $t \to \infty$:*

$$dist(S(t)u_0, \mathfrak{A}) \to 0 \qquad as\ t \to \infty.$$

The distance in (3) is understood to be the distance of a point to a set

$$dist(x, \mathfrak{A}) = \inf_{y \in \mathfrak{A}} d(x, y),$$

$d(x, y)$ denoting the distance of x to y in H.

In order to establish the existence of attractors, a useful concept is the so-called absorbing set.

Definition 3. *Let $\mathfrak{B}$ be a subset of H and $\mathfrak{u}$ an open set containing $\mathfrak{B}$, we say that $\mathfrak{B}$ is absorbing in $\mathfrak{u}$ if the orbit of any bounded set of $\mathfrak{u}$ enters into $\mathfrak{B}$ after a certain time:*

$$\begin{cases} \forall \mathfrak{B}_0 \subset \mathfrak{u},\ \mathfrak{B}_0 \ \textit{bounded} \\ \exists t_1(\mathfrak{B}_0) \ \textit{such that}\ S(t)\mathfrak{B}_0 \subset \mathfrak{B}, \qquad \forall t \geq t_1(\mathfrak{B}_0). \end{cases}$$

We also say that $\mathfrak{B}$ absorbs bounded sets of $\mathfrak{u}$.

Then, the main idea for the existence of an attractor is firstly to write a given physical nonlinear system in the form of an abstract PDEs, and secondly to establish that the corresponding semigroup $S(t)$ for this equation is continuous and meanwhile it has a bounded absorbing set. And then by employing the compact property of the semigroup, we obtain the existence of a compact attractor, which can be stated by the following theorem:

THEOREM 1. *We assume that H is a metric space and that the operator $S(t)$ is continuous, which satisfy the semigroup property. We also assume that there exists an open set $\mathfrak{u}$ and a bounded set $\mathfrak{B}$ of $\mathfrak{u}$ such that $\mathfrak{B}$ is absorbing in $\mathfrak{u}$. Then the ω-limit set of $\mathfrak{B}$, $A = \Lambda(B)$ is a compact attractor, which attracts bounded sets of $\mathfrak{u}$. It is the maximal bounded attractor in $\mathfrak{u}$ (for the inclusion relation).*

Note that if u_0 is a stationary point, the stable manifold $W^s(u_0)$ and unstable manifold $W^u(u_0)$ defined above, if not empty, are the union of trajectories defined for all time, thus they are invariant sets. Of particular interest for the understanding of the dynamics (see, for instance, J. Guckenheimer and P. Holmes [**30**]) are heteroclinic orbits which go from the unstable manifold of a stationary point u_* to the stable manifold of another stationary point $u_{**} \neq u_*$; when $u_{**} = u_*$ such a curve is called a homoclinic orbit. The points belonging to a heteroclinic (or a homoclinic) orbit are called heteroclinic (or homoclinic) points.

Homoclinic bifurcation and heteroclinic bifurcation have a great deal of significance in many complicated subject as physics, chemistry and physiology. They exist in the shock wave solution in the aerodynamics, the traveling-wave solutions of reaction-diffusion-convection systems and the viscous profiles for all magnetohydrodynamic shock waves. Homoclinic orbit and heteroclinic orbit are one of the main sources for complex dynamics while the corresponding bifurcation phenomena is the main source for the unstability of nonlinear dynamical systems. Among all bifurcation problems, they are the most difficult and the most complex.

Since the 80's of the last century, bifurcation problems of homoclinic and heteroclinic loops of planar systems have been investigated by many authors [**22, 23, 20, 24, 25, 37, 56, 66, 67, 74, 77, 79, 84, 94, 103**].

As well known that the bifurcation problem of a degenerate dynamical system on $\mathbb{R}^3$ may be very complicated. An earlier example is the case where there exists a homoclinic orbit to a hyperbolic equilibrium point. In [**84**], Shil'nikov has studied the codimension 1 homoclinic bifurcation problem with two complex conjugated eigenvalues. He has pointed out that if the eigenvalues α and β verify Re $\alpha =$ Re $\beta < 1$, then the dynamical behavior in a small neighborhood of the homoclinic orbit is chaotic. In [**85**], however, he has showed that, under generic hypothesis, the homoclinic bifurcation problem is relatively simple. Precisely, the vector field X_γ has neither homoclinic orbit nor periodic orbit in a small neighborhood of the primary homoclinic orbit Γ for $\gamma > 0$. While, for $\gamma < 0$, only one hyperbolic periodic orbit bifurcates from the homoclinic orbit.

Accordingly, more degenerate cases should be considered for more complicated dynamics. In [**96**], Yanagida has studied the inclination-flip homoclinic orbit together with two other codimension 2 homoclinic bifurcations, which are the cases of the resonant bifurcation and the orbit-flip bifurcation. Since then, many works have been devoted to this subject, see [**17, 21, 27, 35, 36, 43, 49, 67, 101, 99, 102, 104**]. Among these works, [**21**] gives the persistence condition for inclination-flip homoclinic orbits in terms of Melnikov integrals. Whereas, [**43**] studies the homoclinic doubling for an inclination-flip homoclinic orbit under the assumption $\lambda^u < -\lambda^s < 2\lambda^u$. [**17**] presents a scenario suggesting that a perturbation of an inclination-flip homoclinic orbit would lead to the occurrence of Smale horseshoes, and [**36**] proves the existence of Smale horseshoe under the condition $2\lambda^u < \min\{-\lambda^s, \lambda^{uu}\}$ by using the invariant foliation to reduce the study of the return map into the analysis of one-dimensional multivalued map. In [**67**], the author shows the existence of a strange attractor in the unfolding of an inclination-flip homoclinic orbit by comparing the Poincaré return map with the Hénon family.

Recently, the bifurcation problems of homoclinic and heteroclinic loops in high dimensional systems have been comprehensively studied as well [**13, 39, 40, 86, 88, 104, 105**].

Among all these works, not many concern the bifurcation of double homoclinic loops. However, nowadays, there is an increasing interest for the subject, for example, see [**32, 33, 38, 65, 78, 75, 90, 91, 106, 98**]. In [**91**], the authors establish the classification for the set of nonwandering points, homoclinic orbits and limit cycles, respectively. And in [**90**], the author describes the topological equivalence class of $X_\mu|_{\Omega_\mu}$ for a C^3-dynamical system X_μ in general position, where Ω_μ is a set of trajectories in a neighborhood of the double homoclinic loop. In [**65, 75**], the authors show the existence of a Lorenz attractors in the unfolding of a double homoclinic loop with a resonance condition on eigenvalues. While in [**78**], the author proves that perturbations of the initial stable double homoclinic loop can lead to creation of a Lorenz attractor. In [**38**], with the same configuration below, the existence of an invariant set (shift type) in the variant center manifold (an intersection of a center stable manifold and a center unstable manifold) is obtained for conservative and reversible vector fields.

We say that a cycle is equidimensional if all the equilibrium points in the cycle have the same index (dimension of the stable manifold) and heterodimensional if otherwise. Heterodimensional cycles have been first considered by Newhouse and Palis [**68, 69**]. In the past decades, great progresses have been achieved on the bifurcation of the homoclinic and heteroclinic cycles (see [**13, 25, 39, 40, 96, 104, 100**] and the references therein). Hyperbolic systems include many nice systems such as structurally stable systems, Axiom A systems, etc. However, contrary to the common expectation, hyperbolic systems are found not dense in Diff(M). Thereafter, the typical bifurcation phenomena in the robustly non-hyperbolic world becomes quite challenging. The famous C^1 density conjecture of Palis [**71, 72**] asserts that diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle are C^1 dense in the complement of the C^1 closure of hyperbolic systems. Some generic bifurcation through heterodimensional cycles can also be used to provide new examples of persistent transitive diffeomorphisms and persistent partically hyperbolics transitive attractors, see [**13, 18, 57, 83**]. To some extent, the study of the heterodimensional cycles is of great significance and importance.

However, as far as we know, the studies on the bifurcation problem of the heterodimensional cycles are just at the threshhold. Rademacher [**73**] analyzed homoclinic orbits near heterodimensional cycles between an equilibrium and a periodic orbit in three or higher dimensions and established conditions for the existence and uniqueness of countably infinite families of curve segments of 1-homoclinic orbits which accumulate at codimension 1 or 2 heteroclinic cycles. For other references about heterodimensional cycles, see [**9, 8, 11, 18, 19, 48**].

In [**104, 105**], the authors have devoted efforts to seeking for one simple but highly effective method to study homoclinic and heteroclinic bifurcation problems. They have established such a unique method consisting of two kinds of normal forms. The first one is, in a small neighborhood of a saddle point, by strengthening simultaneously the stable manifold, unstable manifold, as well as the strong stable manifold and strong unstable manifold, to obtain a simple normal form locally. The second one is, by selecting carefully some tangent vector bundles along the loops and some others complement to them to originally establish a moving frame globally. Since the coordinate vectors in the moving frame not only mirror the geometric invariance and the various kinds of flip properties of the corresponding stable and unstable manifolds, but also inherit the contracting or expanding instinct of these manifolds. The second normal form and hence the corresponding Poincaré mapping established in this way have very simple forms, and the key parameters in the corresponding bifurcation equations have an explicit and definite geometric and dynamical meanings.

By using this method, together with the transversality theory and the invariant manifold theory, they have completely solved several kinds of homoclinic bifurcation and heteroclinic bifurcation problems, including the complete bifurcation analysis of codimension 3 planar homoclinic loops and some codimension 2 or 3 homoclinic and heteroclinic loop bifurcation in high dimensional spaces with various kinds of degeneracy conditions.

Part 2 of this thesis is devoted to the bifurcation of homoclinic orbit and heteroclinic orbit, in which we study the non-resonant 3D homoclinic bifurcation with inclination-flip, codimension 2 bifurcation of twisted doule homoclinic loops and heterodimensional cycle bifurcation with orbit-flip by using the local active coordinates approach.

As well known, by considering deterministic equations, many important informations are lost in the investigation. So, it is important and necessary to introduce random spatio-temporal forcing in our studies.

Stochastic differential equations (SDEs) arise in mathematical models of physical systems which possess inherent noise and uncertainty. Such models have been used with great success in a variety of application areas, including biology, epidemiology, mechanics, etc. Up to now, SDEs is an important branch of the stochastic analysis, which has a great deal of significance in the filtration theorem, control theorem and potential theorem. Meanwhile, the generalization of the SDEs to the infinite dimensional space leads to stochastic partial differential equations (SPDEs).

In many applications, one usually assumes that the system under consideration is governed by a principle of causality, that is, the future state of the system is independent of the past states and is determined solely by the present. However, under closer scrutiny, it becomes apparent that the principle of causality is often only a first approximation to the true situation and that a more realistic model would include some of the past states of the system [**31**]. Then, stochastic delay differential equations (SDDEs) give us a mathematical formulation for such systems (see e.g. [**44, 45, 58, 59, 60, 61, 62**]).

In the 1980s, mathematicians Elworthy, Baxendale, Bismut, Ikeda, Watanabe, Kunita, etc. discovered that the solutions of the SDEs not only define a stochastic process but also present a stochastic diffeomorphism flow, which connected together the SDEs and stochastic dynamical systems (RDS). Meanwhile, thanks to this connection, we can look at some classic results in the SDEs in the viewpoint of RDS. L. Arnold and his work team have made new contributions on RDS (one can refer to his book [**1**] and the references there in).

When there exists noise, it is impossible to exhibit the compact invariant set stated above. Noises can make the system leave every bounded deterministic set with probability 1. Therefore, we need to give new definitions of random invariant set and random attractor for RDS so as to obtain such compact invariant set, which are not fixed, but depending on chance, and they move with time in stationary manner.

All definitions of random attractor $\mathcal{A}(\omega)$ known to the author agree in that they require that $\mathcal{A}(\omega)$ is a random compact set which is invariant under the random dynamical system (precise definitions are given in Chapter II). The definitions disagree however with respect to the class of sets which are attracted as well as the precise meaning of "attracted". Out of the three definitions we give below, the notion of a forward attractor is closest to that of an attractor for a deterministic dynamical system. It is however believed to be the least appropriate one for random dynamical systems. The concept of a pullback attractor (also called strong attractor or just attractor) has been proposed independently in [**16, 81**]. Weak attractors are recently introduced by G. Ochs. In [**70**], he highlights differences between weak and pullback attractors e.g. concerning invariance properties under random transformations.

In part 1, we study the stochastic stability of SIRS population model with random perturbations and the existence of random attractor for the stochastic Ginzburg-Landau equations on unbounded domains.

Part 1

Stochastic stability and random attractor

CHAPTER I

Stability of SIRS system with random perturbations

This chapter is devoted to the study of stabilities of an epidemiological model, which is the stochastic SIRS model with or without time delay. We shall give sufficient conditions for their stabilities.

1. Basic facts on stochastic stability

In this section, we shall investigate various types of stability for n-dimensional stochastic system:

$$\mathrm{d}X(t) = f(t, X(t))\mathrm{d}t + g(t, X(t))\mathrm{d}W_t, \tag{1.1}$$

where $f(t,x)$ is a function in $\mathbb{R}^n$ defined in $[t_0, +\infty) \times \mathbb{R}^n$, and $g(t,x)$ is a $n \times m$ matrix, f, g are locally Lipschitz functions in x and W_t is an m-dimensional Wiener process. We assume that $x = 0$ is a trivial solution of the system (1.1), *i.e.* $f(t,0) = 0$, $g(t,0) = 0$ for all $t \geq t_0$.

Firstly, let us introduce a few necessary notations and definitions, for which we can refer to [**34, 46, 60**]. Let $\mathcal{K}$ denote the family of all continuous nondecreasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(0) = 0$ and $\mu(r) > 0$ if $r > 0$. And for $h > 0$, let $S_h = \{x \in \mathbb{R}^n : \ |x| < h\}$.

Definition 1.1. *A continuous function $V(t,x)$ defined on $[t_0, +\infty) \times S_h$ is said to be positive-definite (in the sense of Lyapunov) if $V(t,0) \equiv 0$ and, for some $\mu \in \mathcal{K}$,*

$$V(t,x) \geq \mu(|x|) \qquad \textit{for all } (t,x) \in [t_0, +\infty) \times S_h.$$

A function V is said to be negative-definite if $-V$ is positive-definite.

A continuous nonnegative function $V(t,x)$ is said to be decrescent (i.e. to have an arbitrarily small upper bound) if for some $\mu \in \mathcal{K}$,

$$V(t,x) \leq \mu(|x|) \qquad \textit{for all } (t,x) \in [t_0, +\infty) \times S_h.$$

A function $V(t,x)$ defined on $[t_0, +\infty) \times R^n$ is said to be radially unbounded if

$$\lim_{|x| \to \infty} \inf_{t \geq t_0} V(t,x) = \infty.$$

Denote by $C^{1,2}(\mathbb{R}_+ \times S_h; \mathbb{R}_+)$ the family of all nonnegative functions $V(t,x)$ defined on $R_+ \times S_h$ such that they are continuously once differentiable in t and twice in x. Define the differential operator L associated with equation (1.1) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i(t,x) \cdot \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \left[g^T(t,x) \cdot g(t,x)\right]_{ij} \frac{\partial^2}{\partial x_i x_j}.$$

The action of L on a function $V(t,x) \in C^{1,2}(\mathbb{R}_+ \times S_h; \mathbb{R}_+)$ is

$$L(V)(t,x) = \frac{\partial V}{\partial t} + f^T \cdot \frac{\partial V}{\partial x} + \frac{1}{2} Tr \left[g^T \cdot \frac{\partial^2 V}{\partial x^2} \cdot g\right]. \tag{1.2}$$

Definition 1.2. (*i*) *The trivial solution of Equation* (1.1) *is said to be stochastically stable or stable in probability if for every pair of* $\epsilon \in (0,1)$ *and* $r > 0$*, there exists a* $\delta = \delta(\epsilon, r, t_0) > 0$ *such that*

$$\mathbf{P}\{|x(t;t_0,x_0)| < r \text{ for all } t \geq t_0\} \geq 1 - \epsilon$$

whenever $|x_0| < \delta$*. Otherwise, it is said to stochastically unstable.*

(*ii*) *The trivial solution of Equation* (1.1) *is said to be stochastically asymptotically stable if it is stochastically stable and moreover, for every* $\epsilon \in (0,1)$*, there exists a* $\delta = \delta(\epsilon, t_0) > 0$ *such that*

$$\mathbf{P}\{\lim_{t\to\infty} x(t;t_0,x_0) = 0\} \geq 1 - \epsilon$$

whenever $|x_0| < \delta$*.*

Now we present the following theorems which give conditions for the stability of the trivial solution of the stochastic system in terms of Lyapunov function (see [**60**]).

Theorem 1.1. (*i*) *If there exists a positive-definite function* $V(t,x) \in C^{1,2}([t_0,+\infty) \times S_h; \mathbb{R}_+)$ *such that* $L(V)(t,x) \leq 0$ *for all* $(t,x) \in [t_0,\infty) \times S_h$*, then the trivial solution of system* (1.1) *is stochastically stable.*
(*ii*) *If there exists a positive-definite decrescent function* $V(t,x) \in C^{1,2}([t_0,+\infty) \times S_h; \mathbb{R}_+)$ *such that* $L(V)(t,x)$ *is negative-definite, then the trivial solution of system* (1.1) *is stochastically asymptotically stable.*

It is well known that many problems concerning the stability of the equilibrium states of a nonlinear stochastic system can be reduced to problems concerning stability of solutions of the linear associated system (see [**34**]). Let $X(t) = 0$ be the trivial solution of system (1.1). The linear form of system (1.1) is defined as follows:

$$\mathrm{d}X(t) = F(t)X(t)\mathrm{d}t + G(t)X(t)\mathrm{d}W_t. \tag{1.3}$$

Theorem 1.2. *If the linear system* (1.3) *with constant coefficients (*$F(t)$*=*F*,* $G(t)$*=*G*) is stochastically asymptotically stable, and the coefficients of the system* (1.1) *and the coefficients of system* (1.3) *satisfy an inequality:*

$$|f(t,x) - F\cdot x| + |g(t,x) - G\cdot x| < \delta|x| \tag{1.4}$$

in a sufficiently small neighborhood of the point $x = 0$ *and with a sufficiently small constant* δ*, then the trivial solution* $X(t) = 0$ *of system (1.1) is stochastically asymptotically stable.*

Now we introduce some notations for the study of stochastic functional differential equations. Denote with $\mathcal{H}$ the space of $\mathfrak{F}_0$-adapted random variables φ, with $\varphi(s) \in \mathbb{R}^n$ for $s \leq 0$, and

$$\|\varphi\| = \sup_{s\leq 0}|\varphi(s)|, \qquad \|\varphi\|_1^2 = \sup_{s\leq 0}\mathbb{E}(|\varphi(s)|^2)$$

($\mathbb{E}$ denotes the mathematical expectation). Let $V : [0,\infty[\times\mathcal{H} \to \mathbb{R}$ be a functional defined for $t \geq 0$ and $\varphi \in \mathcal{H}$. Reduce the arbitrary functional $V(t,\varphi),\ t \geq 0,\ \varphi \in \mathcal{H}$ to the form

$$V(t,\varphi) = V(t,\varphi(0),\varphi(s)),\ s < 0$$

and define the function

$$V_\varphi(t,x) = V(t,\varphi) = V(t,x_t) = V(t,x,x(t+s)),\ s < 0,\ \varphi = x_t,\ x = \varphi(0) = x(t).$$

Let $\mathcal{D}$ be the class of functionals $V(t,\varphi)$ for which the functions $V_\varphi(t,x)$ has continuous partial derivatives with respect to x of order two, and bounded derivative for all $t \geq 0$.

Consider the following n-dimensional stochastic functional differential equations (SFDE):

$$\mathrm{d}X(t) = f(t,X_t)\mathrm{d}t + g(t,X_t)\mathrm{d}W_t, \tag{1.5}$$

where

$$X_t(s) = X(t+s) \text{ for every } s \leq 0,$$

with initial condition

$$X_0 = \varphi \in \mathcal{H}.$$

For all $V \in \mathcal{D}$, the differential operator L is given by

$$L(V) = \frac{\partial V_\varphi}{\partial t} + f^T \cdot \frac{\partial V_\varphi}{\partial x} + \frac{1}{2} Tr \left[g^T \cdot \frac{\partial^2 V_\varphi}{\partial x^2} \cdot g \right]. \tag{1.6}$$

DEFINITION 1.3. (i) *The trivial solution of SFDE* (1.5) *is said to be mean square stable if for each* $\epsilon > 0$, *there exists* $\delta(\epsilon) > 0$ *such that for any initial process* $\varphi(\theta)$, *the inequalities*

$$\sup_{\theta \leq 0} \mathbb{E}|\varphi(\theta)|^2 < \delta(\epsilon) \tag{1.7}$$

imply that $\mathbb{E}|x(t,\varphi)|^2 < \epsilon$ *for* $t \geq 0$.
(ii) *The trivial solution of SFDE* (1.5) *is said to be asymptotically mean square stable if it is mean square stable and for all functions satisfying (1.7) we have*

$$\lim_{t \to \infty} \mathbb{E}|x(t,\varphi)|^2 = 0. \tag{1.8}$$

(iii) *The trivial solution of SFDE* (1.5) *is said to be stochastically stable if for each* $\varepsilon_1 > 0$ *and* $\epsilon_2 > 0$, *there exists a* $\delta > 0$ *such that*

$$\mathbf{P}\{\sup_{t \geq 0} |x(t,\varphi)| \leq \epsilon_1\} \geq 1 - \epsilon_2$$

provided that $\mathbf{P}\{||\varphi|| \leq \delta\} = 1$.

The following theorem gives conditions for stability of equilibrium states of a SFDE (see [**46**]).

THEOREM 1.3. (i) *Suppose that there exists a functional* $V(t,\varphi) \in \mathcal{D}$ *such that*

$$c_1 \mathbb{E}(|x(t)|^2) \leq \mathbb{E}(V(t,x_t)) \leq c_2 \|x_t\|_1^2 \tag{1.9}$$

and

$$\mathbb{E}(LV(t,x_t)) \leq -c_3 \mathbb{E}(|x(t)|^2) \tag{1.10}$$

with $c_i > 0,\ i = 1,2,3$, *where* x_t *is the solution of system* (1.5) *verifying the initial condition* $x_0 = \varphi$. *Then the trivial solution of system* (1.5) *is asymptotically mean square stable.*
(ii) *Suppose that there exists a functional* $V(t,\varphi) \in \mathcal{D}$ *such that*

$$c_1 |\varphi(0)|^2 \leq V(t,\varphi) \leq c_2 \|\varphi\|^2,$$

and

$$LV(t,x_t) \leq 0$$

with $c_i > 0,\ i = 1,2$, *where* x_t *is the solution of system* (1.5) *verifying the initial condition* $x_0 = \varphi$, *for all functions* $\varphi \in \mathcal{H}$ *such that* $\mathbf{P}\{||\varphi|| \leq \delta\} = 1$ *where* $\delta > 0$ *is sufficiently small. Then the trivial solution of system* (1.5) *is stochastically stable.*

Proof. (i) The proof of part (i) can be found in [**45**].

(ii) As $c_1|\varphi(0)|^2 \leq V(t,\varphi) \leq c_2||\varphi||^2$, we have $V(t,x_t) \geq c_1|x(t)|^2$ and $V(t,0) \equiv 0$. For each $\epsilon_1 > 0$ and $\epsilon_2 \in (0,1)$, by the continuity of $V(t,\varphi)$ and the fact $V(0,0) = 0$, there exists $\delta > 0$ sufficiently small, such that

$$\frac{1}{\epsilon_2} \sup_{||\varphi|| \leq \delta} V(0,\varphi) \leq c_1 \cdot \epsilon_1^2. \tag{1.11}$$

For $\forall\ \varphi \in \mathcal{H}$ such that $\mathbf{P}\{||\varphi|| \le \delta\} = 1$, denote by $x(t) = x(t; 0, \varphi)$ and let τ be the first exit time of $x(t)$ from S_{ϵ_1}, that is

$$\tau = \inf\{t \ge 0 :\ |x(t)| > \epsilon_1\}.$$

By Ito's formula, for any $t \ge 0$,

$$V(t\wedge\tau, x_{t\wedge\tau}) = V(0,\varphi) + \int_0^{t\wedge\tau} LV(s, x_s)\,\mathrm{d}s + \int_0^{t\wedge\tau} g^T(s, x_s)\cdot\frac{\partial V_\varphi}{\partial x}\,\mathrm{d}W(s).$$

Taking the expectation on both sides and making use of the condition $LV(t, x_t) \le 0$, we obtain that

$$\mathbb{E}V(t\wedge\tau, x_{t\wedge\tau}) \le V(0,\varphi). \tag{1.12}$$

Note that $|x(t\wedge\tau)| = x(\tau) = \epsilon_1,$ if $\tau < t$. Therefore,

$$\begin{aligned}\mathbb{E}V(t\wedge\tau, x_{t\wedge\tau}) &\ge \mathbb{E}[\mathbf{I}_{\{\tau<t\}}V(\tau, x_\tau)]\\ &\ge \mathbf{P}\{\tau<t\}\cdot c_1\cdot\epsilon_1^2.\end{aligned}$$

Together with (1.11) and (1.12), one has

$$\mathbf{P}\{\tau < t\} \le \epsilon_2.$$

Letting $t \to \infty$, we get $\mathbf{P}\{\tau < \infty\} \le \epsilon_2$, that is,

$$\mathbf{P}\{\sup_{t\ge 0}|x(t)| \le \epsilon_1\} \ge 1 - \epsilon_2.$$

The proof is completed. □

2. Introduction to stochastic SIRS models

The dynamic behaviors of the SIRS models have been intensively investigated by many authors. In the 1920s, a Kermack-Mackendrick epidemic SIRS model [**42**] was proposed, in which the total population is assumed to be constant and there are infectives $I(t)$, which can pass on the disease to susceptibles $S(t)$, and the remaining members $R(t)$ which have been infected and have become unable to transmit the disease to others. Since then, many people have devoted to the study of the SIRS disease model (acquired immunity is permanent or acquired immunity is temporary) with different variations in its incidence rate, at which susceptibles become infectives, see [**50, 51, 95**].

The deterministic SIRS model existing loss of immunity is the following

$$\begin{aligned}S'(t) &= -\beta S(t)I(t) - \mu S(t) + \gamma R(t) + \mu,\\ I'(t) &= \beta S(t)I(t) - (\lambda+\mu)I(t),\\ R'(t) &= \lambda I(t) - (\mu+\gamma)R(t),\end{aligned} \tag{2.13}$$

and the deterministic SIRS model with distributed time delay is

$$\begin{aligned}S'(t) &= -\beta S(t)\int_0^h f(s)I(t-s)\,\mathrm{d}s - \mu S(t) + \gamma R(t) + \mu,\\ I'(t) &= \beta S(t)\int_0^h f(s)I(t-s)\,\mathrm{d}s - (\lambda+\mu)I(t),\\ R'(t) &= \lambda I(t) - (\mu+\gamma)R(t),\end{aligned} \tag{2.14}$$

where μ represents the birth and death rate. Moreover, all the newborns are susceptible; the constant λ represents the recovery rate of infected people and β is the transmission rate, γ is the per capita rate of loss of immunity. Of course, $\mu,\ \lambda,\ \beta \in \mathbb{R}_+^*$, $\gamma \in \mathbb{R}_+$, and $f(s)$ is a non-negative function, which is square integrable on [0,h] and satisfy $\int_0^h f(s)\,\mathrm{d}s = 1$.

Here, the non-negative constant h is the time delay, the term $\beta S(t)\int_0^h f(s)I(t-s)\,\mathrm{d}s$ can be considered as the force of infection at time t.

It is easy to see that system (2.13) always has a disease-free equilibrium (*i.e.* boundary equilibrium) $E_0 = (1, 0, 0)$.

In the case of $\gamma = 0$, the system is reduced to the SIR model (see [**50**]). The stabilities of the various forms of SIR model are studied by several authors (see [**7, 47, 95**] for example), for both the disease-free equilibrium and the endemic equilibrium E^* (*i.e.* interior equilibrium, in the domain $S > 0, I > 0, R > 0$). The results show that if $\beta < \lambda+\mu$, the disease will disappear and all population will become susceptible and the disease-free equilibrium $E_0(1, 0, 0)$ of Equation (2.13) is globally asymptotically stable. If $\beta > \lambda + \mu$, the disease always remains endemic and the endemic equilibrium $E^* = \left(\frac{\lambda+\mu}{\beta}, \frac{\mu}{\beta}\left(\frac{\beta}{\lambda+\mu}-1\right), \frac{\lambda}{\beta}\left(\frac{\beta}{\lambda+\mu}-1\right)\right)$ of Equation (2.13) is globally asymptotically stable.

In this paper, we consider the stochastic perturbation of deterministic system by introducing noises in Eqs. (2.13) and (2.14). That is the system

$$\begin{aligned}
\mathrm{d}S(t) &= (-\beta S(t)I(t) - \mu S(t) + \gamma R(t) + \mu)\,\mathrm{d}t - \sigma S(t)I(t)\,\mathrm{d}w_t, \\
\mathrm{d}I(t) &= (\beta S(t)I(t) - (\lambda+\mu)I(t))\,\mathrm{d}t + \sigma S(t)I(t)\,\mathrm{d}w_t, \\
\mathrm{d}R(t) &= (\lambda I(t) - (\mu+\gamma)R(t))\mathrm{d}t,
\end{aligned} \tag{2.15}$$

and the system with distributed delay.

$$\begin{aligned}
\mathrm{d}S(t) &= (-\beta S(t)\int_0^h f(s)I(t-s)\,\mathrm{d}s - \mu S(t) + \gamma R(t) + \mu)\,\mathrm{d}t \\
&\quad -\sigma S(t)\int_0^h f(s)I(t-s)\,\mathrm{d}s\,\mathrm{d}w_t, \\
\mathrm{d}I(t) &= (\beta S(t)\int_0^h f(s)I(t-s)\,\mathrm{d}s - (\lambda+\mu)I(t))\,\mathrm{d}t \\
&\quad +\sigma S(t)\int_0^h f(s)I(t-s)\,\mathrm{d}s\,\mathrm{d}w_t, \\
\mathrm{d}R(t) &= (\lambda I(t) - (\mu+\gamma)R(t))\,\mathrm{d}t,
\end{aligned} \tag{2.16}$$

where σ is a constant, which represents the environmental stochastic perturbation on the transmission rate β of the disease and w_t is real Wiener processes defined on a stochastic basis $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, P)$.

The case where $\gamma = 0$ is studied in [**89**]. As is shown there the noise can induce non-trivial effects in physical and biological systems. The presence of a noise source in fact can modify the behavior of corresponding deterministic evolution of the system. It is proved that the disease-free equilibrium is stable in probability under the condition $0 < \beta < \min\{\lambda + \mu - \frac{\sigma^2}{2}, 2\mu\}$. The stability of the SIR model concerning the endemic equilibrium is studied in [**12**] in a more general context, *i.e.* with general incidence function. In [**6**], the stability of the endemic equilibrium is studied for SIR system with distributed delay and with linear perturbation in noise.

We shall study the stochastic stability of the SIRS model concerning the population having loss of immunity compared with [**89**]. We obtain conditions under which the stochastic SIRS system with or without delay are stochastically asymptotically stable. Our condition for the system without delay (2.15) is $0 < \beta < \lambda + \mu - \frac{\sigma^2}{2}$ which improves that given in [**89**] for $\gamma = 0$.

3. Stability of the stochastic SIRS model

In this section, we consider the stochastic SIRS model (2.15). Under the transformation

$$u_1 = S - 1,\ u_2 = I, u_3 = R, \tag{3.17}$$

system (2.15) has the following form

$$\begin{aligned} \mathrm{d}u_1(t) &= (-\beta(u_1(t)+1)u_2(t) - \mu u_1(t) + \gamma u_3(t))\,\mathrm{d}t - \sigma(u_1(t)+1)u_2(t)\,\mathrm{d}w_t, \\ \mathrm{d}u_2(t) &= (\beta(u_1(t)+1) - \lambda - \mu)u_2(t)\,\mathrm{d}t + \sigma(u_1(t)+1)u_2(t)\,\mathrm{d}w_t, \\ \mathrm{d}u_3(t) &= (\lambda u_2(t) - (\mu+\gamma)u_3(t))\,\mathrm{d}t. \end{aligned} \tag{3.18}$$

The corresponding linearized system

$$\begin{aligned} \mathrm{d}u_1(t) &= (-\beta u_2(t) - \mu u_1(t) + \gamma u_3(t))\,\mathrm{d}t - \sigma u_2(t)\,\mathrm{d}w_t, \\ \mathrm{d}u_2(t) &= ((\beta - \lambda - \mu)u_2(t))\,\mathrm{d}t + \sigma u_2(t)\,\mathrm{d}w_t, \\ \mathrm{d}u_3(t) &= (\lambda u_2(t) - (\mu+\gamma)u_3(t))\,\mathrm{d}t. \end{aligned} \tag{3.19}$$

LEMMA 3.1. *Suppose that condition*

$$0 < \beta < \lambda + \mu - \frac{\sigma^2}{2} \tag{3.20}$$

holds. Then the trivial solution of Equation (3.19) *is stochastically asymptotically stable.*

Proof. Denote $u = (u_1, u_2, u_3)$ and consider the Lyapunov function

$$V(u) = u_1^2 + Q_2 u_2^2 + Q_3 u_3^2,$$

where

$$Q_2 = \frac{\beta^2 + E + 2\mu\sigma^2}{2\mu\,(-2\beta - \sigma^2 + 2\lambda + 2\mu)},$$

and E, Q_3 are to be determined.

Let L be the operator defined in (1.2) associated with system (3.19). One then has

$$-L(V) = 2\mu u_1^2 + 2\beta u_1 u_2 - 2\gamma u_1 u_3 + \frac{(\beta^2+E)}{2\mu} u_2^2 - 2\lambda\, Q_3 u_3 u_2 + 2\, Q_3\,(\mu+\gamma)\, u_3^2.$$

The matrix of the above quadratic form is

$$A = \begin{bmatrix} 2\,\mu & \beta & -\gamma \\ \beta & \frac{\beta^2+E}{2\mu} & -Q_3\lambda \\ -\gamma & -Q_3\lambda & 2\,Q_3\,(\mu+\gamma) \end{bmatrix}.$$

We want to choose the positive constants E and Q_3 such that the quadratic form $-L(V)$ (or the matrix A) is positive definite. We distinguish three cases.

Case 1. If $2\beta\mu + 2\beta\gamma - \gamma\lambda > 0$, then we choose

$$E = \frac{\gamma\,\lambda\,(2\,\beta\,\mu + 2\,\beta\,\gamma - \gamma\,\lambda)}{8\,(\mu+\gamma)^2} \ \text{ and } \ Q_3 = \frac{\gamma\,\left(-\gamma\,\lambda + 10\,\beta\,\mu + 10\,\beta\,\gamma\right)}{16\mu\,\lambda\,(\mu+\gamma)}.$$

We then have

$$A = \begin{bmatrix} 2\,\mu & \beta & -\gamma \\ \beta & \frac{(2\,\beta\,\mu+\gamma\,\lambda+2\,\beta\,\gamma)(4\,\beta\,\mu+4\,\beta\,\gamma-\gamma\,\lambda)}{16\mu\,(\mu+\gamma)^2} & \frac{\gamma\,(\gamma\,\lambda-10\,\beta\,\mu-10\,\beta\,\gamma)}{16\mu\,(\mu+\gamma)} \\ -\gamma & \frac{\gamma\,(\gamma\,\lambda-10\,\beta\,\mu-10\,\beta\,\gamma)}{16\mu\,(\mu+\gamma)} & \frac{\gamma\,(10\,\beta\,\mu+10\,\beta\,\gamma-\gamma\,\lambda)}{8\mu\,\lambda} \end{bmatrix}.$$

The principal minors of A are 2μ, $\frac{\gamma\lambda(2\beta\mu+2\beta\gamma-\gamma\lambda)}{8(\mu+\gamma)^2}$, and

$$\det(A) = \frac{9\gamma^2\left(2\beta\mu+2\beta\gamma-\gamma\lambda\right)^2}{128\mu\left(\mu+\gamma\right)^2}.$$

They are all positive. Hence A is positive definite.

Case 2. If $2\beta\mu+2\beta\gamma-\gamma\lambda<0$ then we take

$$E = \frac{3\lambda\gamma\left(\gamma\lambda-2\beta\mu-2\beta\gamma\right)}{2\left(\mu+\gamma\right)^2} \text{ and } Q_3 = \frac{\gamma\left(3\gamma\lambda-4\beta\gamma-4\beta\mu\right)}{4\mu\lambda\left(\mu+\gamma\right)}.$$

We obtain in this case the principal minors of A to be 2μ, $\frac{3\gamma\lambda(\gamma\lambda-2\beta\mu-2\beta\gamma)}{2(\mu+\gamma)^2}$ and

$$\det(A) = \frac{3\gamma^2\left(\gamma\lambda-2\beta\mu-2\beta\gamma\right)^2}{8\mu\left(\mu+\gamma\right)^2}.$$

Therefore the matrix A is positive definite.

Case 3. If $2\beta\mu+2\beta\gamma-\gamma\lambda=0$, then we take

$$E=\beta \text{ and } Q_3 = \frac{\gamma^2\left(1+2\beta\right)}{8\beta\mu\left(\mu+\gamma\right)}.$$

We have in this case

$$A = \begin{bmatrix} 2\mu & \beta & -\gamma \\ \beta & \frac{(\beta+1)\beta}{2\mu} & -\frac{\gamma(1+2\beta)}{4\mu} \\ -\gamma & -\frac{\gamma(1+2\beta)}{4\mu} & \frac{\gamma^2(1+2\beta)}{4\beta\mu} \end{bmatrix}.$$

Then the principal minors are 2μ, β and $\frac{\gamma^2}{8\mu}$. Therefore A is positive definite.

Finally the conclusion follows by Theorem 1.1. □

We can now give the result concerning the stability of the stochastic SIRS model.

THEOREM 3.1. *Assume that hypotheses* (3.20) *hold. Then, the disease-free equilibrium* $E_0=(1,0,0)$ *of* (2.15) *is stochastically asymptotically stable.*

Proof. Due to Lemma 3.1 and Theorem 1.2, it suffices to verify condition (1.4) in order to complete our proof.

The left-hand side of Equation (1.4) equals

$$\sqrt{(-\beta u_1u_2)^2+(\beta u_1u_2)^2}+\sqrt{(-\sigma u_1u_2)^2+(\sigma u_1u_2)^2} = \sqrt{2\beta^2u_1^2u_2^2}+\sqrt{2\sigma^2u_1^2u_2^2}$$

and it is less than $\delta|u|$ in the small neighborhood $(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$ with $\delta=\sqrt{2}\epsilon(\beta+\sigma)$. □

Since the SIR model is a special case of the above system with $\gamma=0$, we have the following consequence which improves the condition of stability as stated in [**89**].

COROLLARY 3.1. *If condition* (3.20) *is satisfied, then the stochastic SIR model (i.e. system* (2.15) *with* $\gamma=0$*) is stochastically asymptotically stable.*

4. Stochastic SIRS model with distributed time delay

In this section, we study the stability of the stochastic SIRS model with distributed time delay (2.16).

Substituting $u_1 = S - 1,\ u_2 = I,\ u_3 = R$ into system (2.16), we obtain the system

$$
\begin{aligned}
\mathrm{d}u_1(t) &= (-\beta(u_1(t)+1)\int_0^h f(s)u_2(t-s)\,\mathrm{d}s - \mu u_1(t) + \gamma u_3(t))\,\mathrm{d}t \\
&\quad -\sigma(u_1(t)+1)\int_0^h f(s)u_2(t-s)\,\mathrm{d}s\,\mathrm{d}w_t, \\
\mathrm{d}u_2(t) &= (\beta(u_1(t)+1)\int_0^h f(s)u_2(t-s)\,\mathrm{d}s - (\lambda+\mu)u_2(t))\,\mathrm{d}t \\
&\quad +\sigma(u_1(t)+1)\int_0^h f(s)u_2(t-s)\,\mathrm{d}s\,\mathrm{d}w_t, \\
\mathrm{d}u_3(t) &= (\lambda u_2(t) - (\mu+\gamma)u_3(t))\,\mathrm{d}t.
\end{aligned} \tag{4.21}
$$

We first consider the corresponding linearized system which is

$$
\begin{aligned}
&\mathrm{d}u_1(t) = (-\beta\int_0^h f(s)u_2(t-s)\,\mathrm{d}s - \mu u_1(t) + \gamma u_3(t))\,\mathrm{d}t - \sigma\int_0^h f(s)u_2(t-s)\,\mathrm{d}s\,\mathrm{d}w_t, \\
&\mathrm{d}u_2(t) = (\beta\int_0^h f(s)u_2(t-s)\,\mathrm{d}s - (\lambda+\mu)u_2(t))\,\mathrm{d}t + \sigma\int_0^h f(s)u_2(t-s)\,\mathrm{d}s\,\mathrm{d}w_t, \\
&\mathrm{d}u_3(t) = (\lambda u_2(t) - (\mu+\gamma)u_3(t))\,\mathrm{d}t.
\end{aligned} \tag{4.22}
$$

Lemma 4.1. *Suppose that condition*

$$
\max\{\lambda-\gamma,\ \beta+\gamma\} < 2\mu,\ \beta \le \lambda + \mu - \frac{\sigma^2}{2} \tag{4.23}
$$

holds, then the trivial solution of Equation (4.22) *is asymptotically mean square stable.*

Proof. Denote $C_2 = \dfrac{4\lambda + 2\mu + \sigma^2}{2(2\lambda + 2\mu - 2\beta - \sigma^2)}$. Consider a Lyapunov functional

$$
V(t,\varphi) = V_1(\varphi) + V_2(t,\varphi),
$$

where

$$
\begin{aligned}
&V_1(\varphi) = \varphi_1(0)^2 + C_2\varphi_2(0)^2 + \varphi_3(0)^2, \\
&V_2(t,\varphi) = (\beta+\sigma^2)\,(C_2+1)\int_0^h f(s)\int_{-s}^0 \varphi^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}
$$

One then has

$$
\begin{aligned}
&V_1(u_t) = u_1^2(t) + C_2u_2^2(t) + u_3^2(t), \\
&V_2(t,u_t) = (\beta+\sigma^2)\,(C_2+1)\int_0^h f(s)\int_{t-s}^t u_2^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}
$$

Let L be the differential operator defined by equation (1.6) associated to system (4.22). Then we have

$$
\begin{aligned}
L(V_1) &= -2\mu u_1^2 - 2\beta u_1\int_0^h f(s)u_2(t-s)\,\mathrm{d}s + 2\gamma u_1u_3 - 2C_2(\lambda+\mu)u_2^2 \\
&\quad +2\beta C_2u_2\int_0^h f(s)u_2(t-s)\,\mathrm{d}s + 2\lambda u_2u_3 - 2(\mu+\gamma)u_3^2 \\
&\quad +\sigma^2\,(1+C_2)\left(\int_0^h f(s)u_2(t-s)\,\mathrm{d}s\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq -2\mu u_1^2 + \beta u_1^2 + \beta \int_0^h f(s) u_2^2(t-s)\,\mathrm{d}s + \gamma u_1^2 + \gamma u_3^2 - 2(\lambda+\mu) C_2 u_2^2 \\
&\quad + \beta C_2 u_2^2 + \beta C_2 \int_0^h f(s) u_2^2(t-s)\,\mathrm{d}s + \lambda u_2^2 + \lambda u_3^2 - 2(\mu+\gamma) u_3^2 \\
&\quad + \sigma^2 (1 + C_2) \int_0^h f(s) u_2^2(t-s)\,\mathrm{d}s \\
&= -(2\mu - \beta - \gamma) u_1^2 + (\beta + \sigma^2)(1 + C_2) \int_0^h f(s) u_2^2(t-s)\,\mathrm{d}s \\
&\quad + (C_2(\beta - 2\lambda - 2\mu) + \lambda) u_2^2 - (2\mu - \lambda + \gamma) u_3^2,
\end{aligned}
$$

and

$$
L(V_2) = (\beta + \sigma^2)(C_2 + 1) \left(u_2^2(t) - \int_0^h f(s) u_2^2(t-s)\,\mathrm{d}s \right).
$$

Accordingly, we deduce that

$$
\begin{aligned}
L(V) &= L(V_1 + V_2) \\
&\leq -(2\mu - \beta - \gamma) u_1^2 + (\beta + \sigma^2)(1 + C_2) u_2^2 \\
&\quad + ((\beta - 2\lambda - 2\mu) C_2 + \lambda) u_2^2 - (2\mu - \lambda + \gamma) u_3^2 \\
&= -(2\mu - \beta - \gamma) u_1^2 - (\lambda + \mu - \frac{\sigma^2}{2} - \beta) u_2^2 - (2\mu - \lambda + \gamma) u_3^2.
\end{aligned}
$$

Thanks to condition (4.23), we get

$$
LV \leq -c|u|^2,
$$

where $c = \min\{2\mu - \beta - \gamma, \lambda + \mu - \frac{\sigma^2}{2} - \beta, 2\mu - \lambda + \gamma\}$.

Now it is clear that

$$
c_1 |u(t)|^2 \leq V(t, u_t) \leq c_2 \|u_t\|^2,
$$

with $c_1 = \min\{1, C_2\}$ and $c_2 = \max\{1, C_2, (\beta + \sigma^2)(C_2 + 1)\}$.

Hence the conclusion follows by applying (i) of Theorem 1.3, □

Now, we give the following main result of our stochastic SIRS model with distributed time delay.

THEOREM 4.1. *If condition* (4.23) *holds, then the disease-free equilibrium* $E_0 = (1, 0, 0)$ *of Equation* (2.16) *is stochastically stable.*

Proof. Let C_2 and V_1 be defined as in Lemma (4.1). Since $\beta - 2\mu + \gamma < 0$ and $\beta + \frac{\sigma^2}{2} - \lambda - \mu < 0$, one can find $\delta > 0$ such that

$$
\begin{aligned}
&\beta - 2\mu + \gamma + \sigma^2 \delta (C_2 + 1) + \beta \delta |C_2 - 2| + \sigma^2 \delta^2 (C_2 + 1) < 0, \\
&\beta + \tfrac{\sigma^2}{2} - \lambda - \mu + \sigma^2 \delta + \delta(\beta + \sigma^2) C_2 < 0.
\end{aligned}
\tag{4.24}
$$

Consider the Lyapunov functional $V(t, \varphi) = V_1(\varphi) + V_2(t, \varphi)$, where

$$
V_2(t, \varphi) = (\beta + \sigma^2 + \sigma^2 \delta)(C_2 + 1) \int_0^h f(s) \int_{-s}^0 \varphi_2(\tau)^2 \,\mathrm{d}\tau\,\mathrm{d}s.
$$

One then has

$$
V_2(t, u_t) = (\beta + \sigma^2 + \sigma^2 \delta)(C_2 + 1) \int_0^h f(s) \int_{t-s}^t u_2^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.
$$

Let L be the differential operator associated to system (2.16). Then

$$
L(V_1) = -2\beta(u_1 + 1) u_1 \int_0^h f(s) u_2(t-s)\,\mathrm{d}s - 2\mu u_1^2 + 2\gamma u_1 u_3
$$

$$
\begin{aligned}
&+2C_2\beta(u_1+1)u_2\int_0^h f(s)u_2(t-s)\,\mathrm{d}s-2(\lambda+\mu)C_2u_2^2\\
&+2\lambda u_2u_3-2(\mu+\gamma)u_3^2+\sigma^2\left(1+C_2\right)(u_1+1)^2\left(\int_0^h f(s)u_2(t-s)\,\mathrm{d}s\right)^2\\
=\;&-2\beta u_1^2\int_0^h f(s)u_2(t-s)\,\mathrm{d}s-2\beta u_1\int_0^h f(s)u_2(t-s)\,\mathrm{d}s-2\mu u_1^2+2\gamma u_1u_3\\
&+2\beta C_2u_1u_2\int_0^h f(s)u_2(t-s)\,\mathrm{d}s+2\beta C_2u_2\int_0^h f(s)u_2(t-s)\,\mathrm{d}s-2(\lambda+\mu)C_2u_2^2\\
&+2\lambda u_2u_3-2(\mu+\gamma)u_3^2+\sigma^2\left(1+C_2\right)u_1^2\left(\int_0^h f(s)u_2(t-s)\,\mathrm{d}s\right)^2\\
&+2\sigma^2\left(1+C_2\right)u_1\left(\int_0^h f(s)u_2(t-s)\,\mathrm{d}s\right)^2+\sigma^2\left(1+C_2\right)\left(\int_0^h f(s)u_2(t-s)\,\mathrm{d}s\right)^2\\
\le\;&-2\beta u_1^2\int_0^h f(s)u_2(t-s)\,\mathrm{d}s+\beta u_1^2+\beta\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s-2\mu u_1^2+\gamma u_1^2+\gamma u_3^2\\
&+\beta C_2u_1^2\int_0^h f(s)u_2(t-s)\,\mathrm{d}s+\beta C_2u_2^2\int_0^h f(s)u_2(t-s)\,\mathrm{d}s+\beta C_2u_2^2\\
&+\beta C_2\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s-2(\lambda+\mu)C_2u_2^2+\lambda u_2^2+\lambda u_3^2-2(\mu+\gamma)u_3^2\\
&+\sigma^2\left(1+C_2\right)u_1^2\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s+2\sigma^2\left(1+C_2\right)u_1\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s\\
&+\sigma^2\left(1+C_2\right)\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s\\
=\;&\left(\beta-2\mu+\gamma\right)u_1^2+\left((\beta-2\lambda-2\mu)C_2+\lambda\right)u_2^2+\beta\left(C_2-2\right)u_1^2\int_0^h f(s)u_2(t-s)\,\mathrm{d}s\\
&+\beta C_2u_2^2\int_0^h f(s)u_2(t-s)\,\mathrm{d}s+(\lambda-\gamma-2\mu)u_3^2\\
&+(\beta+\sigma^2)\left(C_2+1\right)\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s+\sigma^2\left(C_2+1\right)u_1^2\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s\\
&+2\sigma^2\left(C_2+1\right)u_1\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s,
\end{aligned}
$$

and

$$
L(V_2)=(\beta+\sigma^2+\sigma^2\delta)\left(C_2+1\right)\left(u_2^2(t)-\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s\right).
$$

Using

$$
2u_1\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s\le\delta u_1^2+\frac{1}{\delta}\left(\int_0^h f(s)u_2^2(t-s)\mathrm{d}s\right)^2,
$$

we have, thereafter,

$$
\begin{aligned}
L(V)\;=\;&L(V_1+V_2)\\
\le\;&\left(\beta-2\mu+\gamma+\sigma^2\delta\left(1+C_2\right)\right)u_1^2+\left(\beta-\lambda-\mu+\frac{\sigma^2}{2}+\sigma^2\delta(C_2+1)\right)u_2^2\\
&+\beta\left(C_2-2\right)u_1^2\int_0^h f(s)u_2(t-s)\,\mathrm{d}s+\beta C_2u_2^2\int_0^h f(s)u_2(t-s)\,\mathrm{d}s+(\lambda-\gamma-2\mu)u_3^2\\
&-\sigma^2\delta\left(C_2+1\right)\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s+\sigma^2\left(C_2+1\right)u_1^2\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s
\end{aligned}
$$

$$+\frac{\sigma^2}{\delta}\left(C_2+1\right)\left(\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s\right)^2, \tag{4.25}$$

Consider the class of process

$$\Phi=\{\varphi\in\mathcal{H}|P\{\sup_{-h\le s\le 0}|\varphi(s)|<\delta\}=1.$$

Note that for $u_t\in\Phi$,

(i) $\left|\int_0^h f(s)u_2(t-s)\,\mathrm{d}s\right|\le\delta.$

(ii) $\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s\le\delta^2.$

(iii) $\left(\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s\right)^2\le\delta^2\int_0^h f(s)u_2^2(t-s)\,\mathrm{d}s.$

Therefore, from (4.25) and (4.24), we have

$$\begin{aligned}LV &\le \Big(\beta-2\mu+\gamma+\sigma^2\delta\left(C_2+1\right)+\beta\delta|C_2-2|+\sigma^2\delta^2\left(C_2+1\right)\Big)u_1^2\\ &\quad+\left(\beta+\frac{\sigma^2}{2}-\lambda-\mu+\sigma^2\delta\left(C_2+1\right)+\beta\delta C_2\right)u_2^2+(\lambda-\gamma-2\mu)u_3^2\\ &\le 0.\end{aligned}$$

The theorem is proved using (ii) of Theorem 1.3. □

5. Numerical simulation and conclusion

We now present some computer simulation of our SIRS model using matlab. The results agree well with the above theoretical analysis. Theorem 3.1 is well verified by the following numerical simulation in Figure I.1, which shows the stability of the disease-free equilibrium E_0 under condition (3.20).

For comparison to [**89**], the computer simulations suggest also that E_0 is globally asymptotically stable also under condition

$$\lambda+\mu-\frac{\sigma^2}{2}<\beta<\lambda+\mu+\frac{\sigma^2}{2},$$

(see Figure I.2). While, if $\beta>\lambda+\mu+\frac{\sigma^2}{2}$, the disease-free equilibrium E_0 is unstable and the solution of Equation (2.15) fluctuates around its endemic equilibrium $E^*=\left(\frac{\lambda+\mu}{\beta},\frac{(\mu+\gamma)(\beta-\mu-\lambda)}{\beta(\lambda+\mu+\gamma)},\frac{\lambda(\beta-\mu-\lambda)}{\beta(\lambda+\mu+\gamma)}\right)$, (see Figure I.3). We conjecture that the loss of immunity (i.e. $\gamma\neq 0$) does not modify the stochastic stability threshold $C=\lambda+\mu+\frac{\sigma^2}{2}$ for β, depending on σ, under which E_0 is asymptotically stable (Figure I.2) and over which E_0 is unstable (Figure I.3).

Mathematically, $\sigma^2/2$ can be regarded as the intensity of the environmental stochastic perturbation on the transmission rate of the disease. We see that, for $\sigma=0$, *i.e*, there is no environmental stochastic perturbation for the transmission rate, $\beta<\beta_0\triangleq\lambda+\mu$ guarantees the disappearance of the disease, which agrees well with the classical results. Taken the environment noise into account, the introduction of the noise in the deterministic SIRS model leads the deterministic stability threshold β_0 of the disease-free equilibrium to $\hat{\beta}_0\triangleq\lambda+\mu-\frac{\sigma^2}{2}$, under which the disease-free equilibrium is stochastically stable such that the disease cannot establish itself and it will disappear finally leaving all the population susceptible. However,

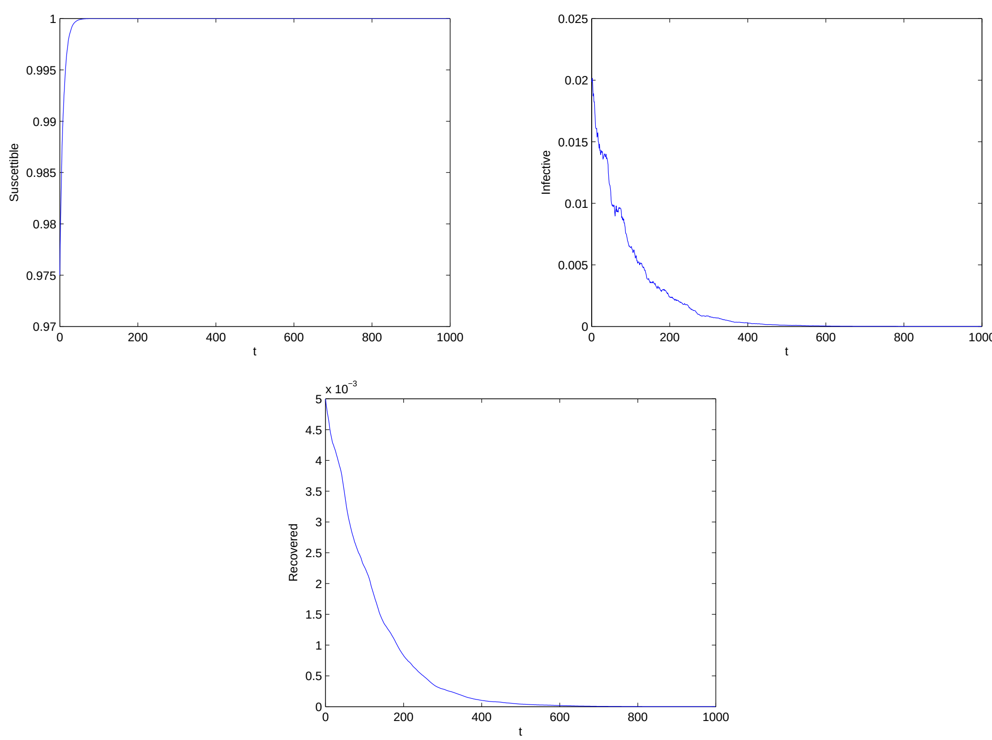


FIGURE I.1. Stochastic trajectories of SIRS model for initial condition $x_0 = 0.975,\ y_0 = 0.02,\ z_0 = 0.005$ and $\mu = 0.2,\ \sigma = 0.2,\ \lambda = 0.1,\ \gamma = 0.3,\ \beta = 0.2$ $(\lambda + \mu - \frac{\sigma^2}{2} = 0.28)$.

for the SIRS model with distributed time delay, we have much more restrictive conditions on the loss rate of immunity γ, which must be bounded in $(\lambda - 2\mu, 2\mu - \beta)$. Correspondingly, we must require that the recovery rate of infected people be two times bigger than the death rate while the transmission rate of the disease be less than two times of birth rate. In spite of this, these conditions are still realizable with higher recovery rate of the infected people but lower transmission rate of the disease.

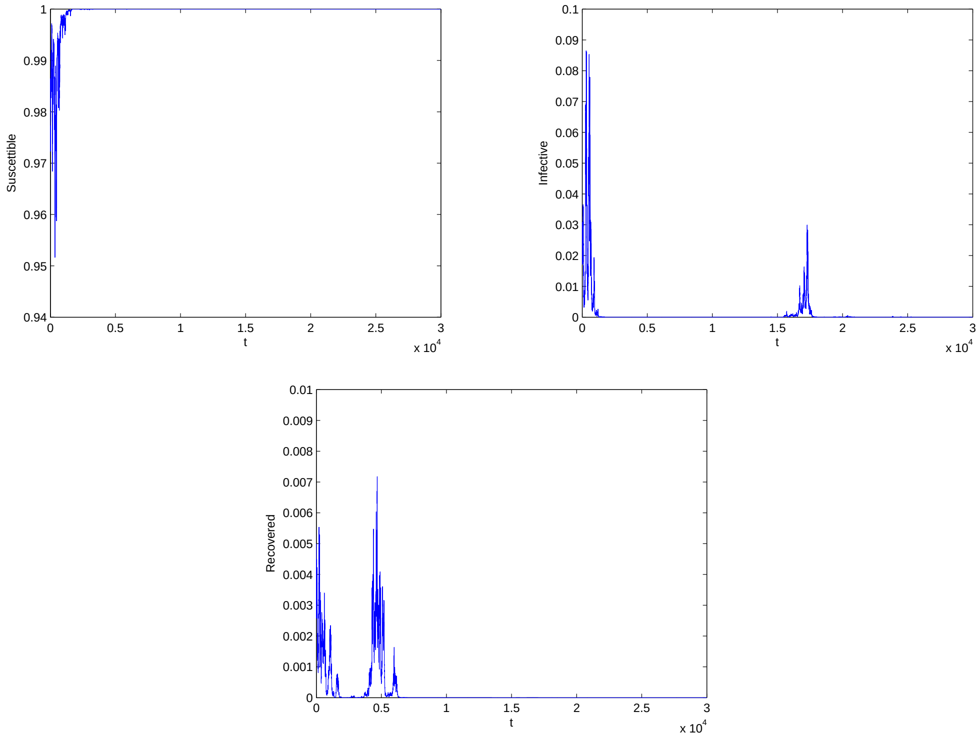


FIGURE I.2. Trajectories for initial condition $x_0 = 0.975,\ y_0 = 0.02,\ z_0 = 0.005$ and $\mu = 0.2,\ \sigma = 0.2,\ \lambda = 0.1,\ \gamma = 0.3,\ \beta = 0.31$ $(\lambda + \mu + \frac{\sigma^2}{2} = 0.32)$.

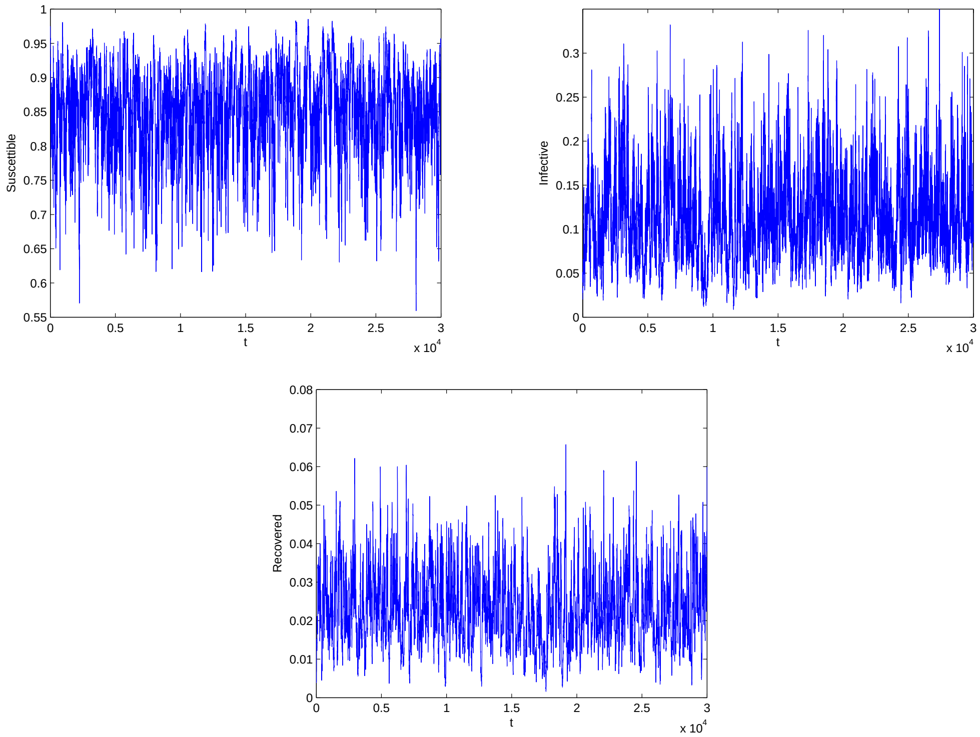


FIGURE I.3. Trajectories for initial condition $x_0 = 0.975,\ y_0 = 0.02,\ z_0 = 0.005$ and $\mu = 0.2,\ \sigma = 0.2,\ \lambda = 0.1,\ \gamma = 0.3,\ \beta = 0.36$.

CHAPTER II

Random attractor for stochastic Ginzburg-Landau equation on unbounded domains

The aim of this chapter is to prove the existence and uniqueness of a $\mathcal{D}$-random attractor for the stochastic Ginzburg-Landau equation on unbounded domain.

1. Introduction to stochastic attraction

We first give a brief introduction and some basic definitions on stochastic attraction. For more details of the definitions, one can refer to [**1, 4, 14, 15, 16, 26**].

All definitions of a random attractor $\mathcal{A}(\omega)$ known to the author agree in that they require that $\mathcal{A}(\omega)$ be a random compact set which is invariant under the random dynamical system (below we will give precise definitions). The definitions disagree however with respect to the class of sets which are attracted as well as the precise meaning of "attracted". Out of the definitions we give below the notion of a forward attractor is closest to that of an attractor for a deterministic dynamical system. It is however believed to be the least appropriate for random dynamical systems. The concept of a pullback attractor (also called strong attractor or just attractor) was proposed independently in [**16, 81**]. Weak attractors were recently introduced by G. Ochs. In [**70**], he highlights differences between weak and pullback attractors e.g. concerning invariance properties under random transformations. It is not our aim to point out such different properties but rather to list out these concepts. For comparison of various concepts of random attractors, on can refer to [**80**].

We denote by $(\Omega, \mathcal{F}, P)$ a complete probability space and X a Polish space.

DEFINITION 1.1. *Let $\{\theta_t : \ \Omega \to \Omega, \ t \in \mathbb{R}\}$ be a family of measure preserving transformation such that $(t, \omega) \to \theta_t\omega$ is measurable, $\theta_0 = id$, and $\theta_{s+t} = \theta_t \circ \theta_s$, for all $s, t \in \mathbb{R}$, then the flow θ_t together with the corresponding probability space $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system.*

DEFINITION 1.2. *A continuous random dynamical system (*RDS*) on X over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a mapping*

$$\phi : \mathbb{R}^+ \times \Omega \times X \to X, \qquad (t, \omega, x) \to \phi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable such that for P-a.e. $\omega \in \Omega$

(i) *$\phi(0, \omega, \cdot)$ is the identity on X.*
(ii) *$\phi(t + s, \omega, \cdot) = \phi(t, \theta_s\omega, \cdot) \circ \phi(s, \omega, \cdot)$ for all $t, s \in \mathbb{R}^+$.*
(iii) *$\phi(t, \omega, \cdot) : X \to X$ is continuous for all $t \in \mathbb{R}^+$.*

Hereafter, we always assume that ϕ is a continuous RDS on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$.

DEFINITION 1.3. *(Random attrator 1.) Suppose that ϕ is a RDS such that there exists a random compact set $\omega \to \mathcal{A}(\omega)$, which satisfies the following conditions:*

(i) *$\mathcal{A}(\omega)$ is invariant, that is,*

$$\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega) \quad \text{for all} \quad t \geq 0.$$

(ii) *$\mathcal{A}$ attracts every bounded deterministic set $K \subset X$,*

$$\lim_{t\to\infty} d(\phi(t,\theta_{-t}\omega)K, \mathcal{A}(\omega)) = 0.$$

Then $\mathcal{A}$ is said to be a universally or globally attracting set for ϕ.

For the existence of a random attractor defined in Definition 1.3, one has the following result:

THEOREM 1.1. ([**16**]) *Suppose that φ is an RDS on a Polish space X, and suppose that there exists a compact set $\omega \to B(\omega)$, absorbing every bounded deterministic set $K \subset X$. Then the set*

$$\mathcal{A}(\omega) = \overline{\bigcup_{K\subset X} \Lambda_K(\omega)}$$

is a global attractor for φ. Furthermore, $\mathcal{A}(\omega)$ is measurable with respect to $\mathcal{F}$ if T is discrete, and it is measurable with respect to the completion of $\mathcal{F}$(with respect to P) if T is continuous.

Random attractor defined in Definition 1.3 attracts all bounded deterministic sets. Furthermore, one introduces the collection of random subsets, depending on chance ω. And the random attractor will just attract sets in this collection.

A collection $\mathcal{D}$ of random subsets is called inclusion closed if whenever $\{E(\omega)\}_{\omega\in\Omega}$ is an arbitrary random set, and $\{F(\omega)\}_{\omega\in\Omega}$ is in $\mathcal{D}$ with $E(\omega) \subset F(\omega)$ for all $\omega \in \Omega$, then $\{E(\omega)\}_{\omega\in\Omega}$ must belong to $\mathcal{D}$. Here, $\mathcal{D}$ is called the basin of attraction. In practical applications, elements in $\mathcal{D}$ are usually tempered.

DEFINITION 1.4. *A random variable $R: \Omega \to (0,\infty)$ is called tempered with respect to the dynamical system θ if for the associated stationary stochastic process $t \to R(\theta_t\cdot)$ the invariant set for which*

$$\lim_{t\to\pm\infty} \frac{1}{t} \log R(\theta_t\omega) = 0$$

($t \to -\infty$ applies only to two-sided time) has full P-measure.

DEFINITION 1.5. *A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for P-a.e. $\omega \in \Omega$,*

$$\lim_{t\to\infty} e^{-\varepsilon t} d(B(\theta_{-t}\omega)) = 0 \quad \textit{for all } \ \varepsilon > 0,$$

where $d(B) = \sup_{x\in B} ||x||_X$.

DEFINITION 1.6. *Let $\mathcal{D}$ be a collection of random subsets of X and $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called a random absorbing set for ϕ in $\mathcal{D}$ if for every $B \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that*

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \quad \textit{for all } \ t \geq t_B(\omega).$$

DEFINITION 1.7. *Let $\mathcal{D}$ be a collection of random subsets of X. Then ϕ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for P-a.e. $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.*

We can now give a second definition of random attractor.

DEFINITION 1.8. *(Random attractor 2.) Let $\mathcal{D}$ be a collection of random subsets of X. Then a random set $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ of X is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for ϕ if the following conditions are satisfied, for P-a.e. $\omega \in \Omega$,*

(i) *$\mathcal{A}(\omega)$ is compact, and $\omega \to d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$.*

(ii) *$\mathcal{A}(\omega)$ is invariant, that is,*

$$\phi(t,\omega,\mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega) \quad \textit{for all} \;\; t \geq 0.$$

(iii) *$\mathcal{A}(\omega)$ attracts every set in $\mathcal{D}$, that is, for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,*

$$\lim_{t\to\infty} d(\phi(t,\theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where d is the Hausdorff semi-metric given by $d(Y,Z) = \sup_{y\in Y}\inf_{z\in Z}||y-z||_X$ for any $Y \subseteq X$ and $Z \subseteq X$.

The following result give a criterion for the existence of $\mathcal{D}$-random attractor.

THEOREM 1.2. ([**4, 26**]) *Let $\mathcal{D}$ be an inclusion-closed collection of random subsets of X and ϕ a continuous* RDS *on X over $(\Omega,\mathcal{F},P,(\theta_t)_{t\in R})$. Suppose that $\{K(\omega)\}_{\omega\in\Omega}$ is a closed random absorbing set for ϕ in $\mathcal{D}$ and ϕ is $\mathcal{D}$-pullback asymptotically compact in X. Then ϕ has a unique $\mathcal{D}$-random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ which is given by*

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}\phi(t,\theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

Random attractors given in Definition 1.3 and 1.8 are called pullback attractors. In practise, for fixed $\omega \in \Omega$, we consider the Ω-limit set at time t = 0 of the trajectories starting in bounded sets at time $t = \infty$. Nevertheless, the pullback absorbing property cannot guarantee that it has such a property for any forward time for almost $\omega \in \Omega$ or for all ω. Since, θ_t preserves the probability measure P, we can obtain weaker absorbtion. That is, trajectories starting from any bound random set K are forward attracted to $\mathcal{A}(\omega)$ in probability.

We now define the forward absorbing set.

DEFINITION 1.9. *Suppose $K = \{K(\omega)\}_{\omega\in\Omega}$ is bounded and $t \to \sup_{y\in K(\theta_t\omega)}||y||_H$ is tempered with respect to the dynamical system θ. And there exists a random closed set $B = \{B(\omega)\}_{\omega\in\Omega}$ and $t_0(K,\omega)$ such that*

$$\begin{aligned} \varphi(t,\omega,K(\omega)) &\subset B(\theta_t\omega), \qquad & t \geq t_0(K,\omega),\\ \varphi(t,\theta_{-t}\omega,K(\theta_{-t}\omega)) &\subset B(\omega), \qquad & t\geq t_0(K,\omega),\end{aligned}$$

and B is forward invariant, i.e., $\varphi(t,\omega,B(\omega)) = B(\theta_t\omega)$, for $t \geq 0$, $\omega\in\Omega$. Then B is called a forward absorbing set for φ.

DEFINITION 1.10. *(Random attractor 3.) Suppose that φ is an RDS such that there exists a random compact set $\omega \to \mathcal{A}(\omega)$, which satisfies the following conditions:*

(i) *$\mathcal{A}(\omega)$ is invariant, that is,*

$$\varphi(t,\omega,\mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega) \quad \textit{for all} \;\; t\geq 0, \; \omega\in\Omega.$$

(ii) *for every bounded random set $K \subset X$,*

$$\lim_{t\to\infty} d(\varphi(t,\omega)K(\omega), \mathcal{A}(\theta_t\omega)) = 0, \;\; \textit{in probability}.$$

Then $\mathcal{A}$ is said to be a weak attractor for φ.

DEFINITION 1.11. *(Random attractor 4.) Suppose φ is an RDS such that there exists a random compact set $\omega \to \mathcal{A}(\omega)$, which satisfies the following conditions:*

(i) *$\mathcal{A}(\omega)$ is invariant, that is,*

$$\varphi(t,\omega,\mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega) \quad \textit{for all} \;\; t\geq 0, \; \omega\in\Omega.$$

(ii) *for every bounded random set* $K \subset X$,

$$\lim_{t\to\infty} d(\varphi(t,\omega)K(\omega), \mathcal{A}(\theta_t\omega)) = 0, \ a.s.$$

Then $\mathcal{A}$ *is said to be a forward attractor for* φ.

In general each forward attractor and each pullback attractor is a weak attractor. Conversely, it is not true, see [**70**] for more details.

In the sequel of this chapter, we shall adopt the definition of $\mathcal{D}$-random attractor (Definition 1.8). Our aim is to use Theorem 1.2 to prove the existence and uniqueness of a $\mathcal{D}$-random attractor for the stochastic Ginzburg-Landau equation with additive noise on the entire space $\mathbb{R}^n$.

2. Problem to be considered

We shall study the following stochastic Ginzburg-Landau equation with additive noise defined in the entire space $\mathbb{R}^n$:

$$\mathrm{d}u = (\lambda + i\mu)\Delta u \mathrm{d}t - (\kappa + i\beta)|u|^2 u \mathrm{d}t - \gamma u \mathrm{d}t + \sum_{j=1}^{m} \varphi_j \mathrm{d}\omega_j(t), \tag{2.1}$$

with the initial condition

$$u(x,0) = u_0(x), \ x \in \mathbb{R}^n, \tag{2.2}$$

where λ, μ, κ, β, γ are real coefficients, with $\lambda > 0$, $\kappa > 0$, $\gamma > 0$, $\varphi_j \in H^2(\mathbb{R}^n) \cap W^{2,4}(\mathbb{R}^n)$, $j = 1, \dots, m$ being time independent defined on $\mathbb{R}^n$, and $\{\omega_j\}_{j=1}^m$ being independent two-sided real-valued Wiener processes on a complete probability space $(\Omega, \mathcal{F}, P)$. Stochastic differential equations of this type arise from many physical systems when random spatio-temporal forcing is taken into account. Our aim is to study the long time behavior of the stochastic Ginzburg-Landau equation of this type.

The existence of random attractors for the Ginzburg-Landau equation perturbed by additive white noise and multiplicative white noise on bounded domains has been investigated respectively in [**92, 97**].

Due to the difficulty that Sobolev embeddings are no longer compact and the compactness of solutions cannot be obtained using standard method, the unboundedness of the domain is a great challenging for proving the existence of an attractor. In the case of deterministic equations, this difficulty has been overcome by the energy equation approach, introduced in [**2, 3**], and then used by others to prove the asymptotic compactness of deterministic equations in unbounded domains, for example, [**10, 28, 29, 41, 63, 64, 76, 93**]. In this chapter, we prove the existence of a random attractor for the stochastic Ginzburg-Landau Equation (2.1), defined on the unbounded domain $\mathbb{R}^n$ by employing the method of tail estimates, which was firstly established in [**5**] to the case stochastic dissipative PDEs.

For the mathematical setting we introduce complex Sobolev spaces. In general, we denote by $\mathbb{X}$,, $\mathbb{Y}, \dots$, the complexified space of a function space $X, Y, \dots$. For example, $\mathbb{L}^2(\mathbb{R}^n)$ is the complexified space of $L^2(\mathbb{R}^n)$. Denote by $(\cdot,\cdot)$ and $||\cdot||_{L^2}$ the scalar product and the norm in either $L^2(\mathbb{R}^n)$ or $\mathbb{L}^2(\mathbb{R}^n)$. So, if $u \in \mathbb{L}^2(\mathbb{R}^n)$, then $u = \{u_1, u_2\}$, $u_j \in L^2(\mathbb{R}^n)$, $j = 1, 2$, and

$$||u||_{L^2} = \{||u_1||_{L^2}^2 + ||u_2||_{L^2}^2\}^{\frac{1}{2}}.$$

If $u = u_1 + iu_2$, $v = v_1 + iv_2$ are in $\mathbb{L}^2(\mathbb{R}^n)$,

$$(u,v) = \{(u_1,v_1) + (u_2,v_2)\} + i\{(u_2,v_1) - (u_1,v_2)\}.$$

The constant $c > 0$ may change their values form line to line or even in the same line.

We first obtain the continuous RDS ϕ associated with the stochastic Ginzburg-Landau Equation (2.1), with the help of Ornstein-Uhlenbeck process. Then we concentrate to get the uniform estimate on the far-field values of the solution as $t \to \infty$ and thus to further establish the asymptotic compactness of the solution operator ϕ. These lead to our main result as following:

THEOREM 2.1. *The random dynamical system ϕ of stochastic Ginzburg-Landau equation with additive noise has a unique $\mathcal{D}$-random attractor in $\mathbb{L}^2(\mathbb{R}^n)$ provided that $\sqrt{3}\kappa \geq |\beta|$.*

3. RDS associated with the stochastic Ginzburg-Landau equation on $\mathbb{R}^n$

Denote by $z(t) = z(\theta_t\omega) = \sum_{j=1}^{m} \varphi_j z_j(\theta_t\omega_j)$, where

$$z_j(t) = z_j(\theta_t\omega) = \int_{-\infty}^{t} e^{\gamma s} \mathrm{d}\omega_j(s),\ t \in \mathbb{R},$$

satisfies the one-dimensional Ornstein-Uhlenbeck equation

$$\mathrm{d}\, z_j = -\gamma z_j \mathrm{d}\, t + \mathrm{d}\, \omega_j(t).$$

Since the random variable $|z_j(\omega_j)|$ is tempered and $|z_j(\theta_t\omega_j)|$ is P-a.e. continuous, there exists a tempered function $r(\omega) > 0$ such that

$$\sum_{j=1}^{m} \left(|z_j(\omega_j)|^2 + |z_j(\omega_j)|^4\right) \leq r(\omega), \tag{3.3}$$

where $r(\omega)$ satisfies, for P-a.e. $\omega \in \Omega$,

$$r(\theta_t\omega) \leq e^{\frac{\gamma}{2}|t|} r(\omega),\ t \in \mathbb{R}, \tag{3.4}$$

thanks to the Proposition 4.3.3 in [**1**]. From Equation (3.3)-(3.4), we get for P-a.e. $\omega \in \Omega$,

$$\sum_{j=1}^{m} \left(|z_j(\theta_t\omega_j)|^2 + |z_j(\theta_t\omega_j)|^4\right) \leq e^{\gamma|t|} r(\omega),\ t \in \mathbb{R}. \tag{3.5}$$

Introduce the transformation

$$v(t) = u(t) - z(\theta_t\omega),$$

where u is the solution of Equation (2.1)-(2.2), then v should satisfy

$$\frac{\partial v}{\partial t} = (\lambda + i\mu)\triangle v - (\kappa + i\beta)|v + z|^2(v + z) - \gamma v + (\lambda + i\mu)\triangle z. \tag{3.6}$$

Similar as the procedure in [**5**], we get that Equation (3.6) has a unique solution $v(t, \omega, v_0)$ with $v(0, \omega, v_0) = v_0$, which is continuous respect to v_0 in $\mathbb{L}^2(\mathbb{R}^n)$. Let

$$u(t, \omega, u_0) = v(t, \omega, u_0 - z(\omega)) + z(\theta_t\omega),$$

then u is the solution of Equations (2.1)-(2.2). Define $\phi :\ \mathbb{R}^+ \times \Omega \times \mathbb{L}^2(\mathbb{R}^n) \to \mathbb{L}^2(\mathbb{R}^n)$ by

$$\phi(t, \omega, u_0) = u(t, \omega, u_0) = v(t, \omega, u_0 - z(\omega)) + z(\theta_t\omega), \tag{3.7}$$

for all $(t, \omega, u_0) \in \mathbb{R}^+ \times \Omega \times \mathbb{L}^2(\mathbb{R}^n)$. Then, we can claim that ϕ is a continuous random dynamical system associated with the stochastic Ginzburg-Landau equation on $\mathbb{R}^n$.

4. Existence of random attractor

In the sequel, we always assume that $\mathcal{D}$ is the collection of all tempered subsets of $\mathbb{L}^2(\mathbb{R}^n)$ with respect to $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in R})$. And then, we devote to prove that ϕ has a random absorbing set in $\mathcal{D}$, and it is also $\mathcal{D}$-pullback asymptotically compact.

PROPOSITION 4.1. *There exists $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ such that $\{K(\omega)\}_{\omega\in\Omega}$ ia a random absorbing set for ϕ in $\mathcal{D}$. Precisely, for any $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there is $t_B(\omega) > 0$ such that*

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \qquad \text{for all } t \geq t_B(\omega).$$

Proof. Multiplying Equation (3.6) by $\bar{v}$, integrating over $\mathbb{R}^n$, and taking the real part, we get

$$\frac{1}{2}\frac{d}{dt}||v||^2 = Re(\lambda+i\mu)(\triangle v, \bar{v}) - Re(\kappa+i\beta)(|v+z|^2(v+z), \bar{v}) - \gamma||v||^2 + Re(\lambda+i\mu)(\triangle z(\theta_t\omega), \bar{v}). \tag{4.8}$$

Here

$$Re(\lambda + i\mu)(\triangle v, \bar{v}) = -\lambda||\nabla v||^2, \tag{4.9}$$

$$\begin{aligned} &-Re(\kappa + i\beta)(|v + z|^2(v + z), \bar{v}) \\ &= -Re(\kappa + i\beta)(|v + z|^2(v + z), \overline{v + z}) + Re(\kappa + i\beta)(|v + z|^2(v + z), \bar{z}) \\ &= -\kappa||u||_4^4 + \int_{\mathbb{R}^n} |\kappa + i\beta| \cdot |u|^3|z|\mathrm{d}x \\ &\leq -\kappa||u||_4^4 + \frac{1}{2}\kappa||u||_4^4 + \frac{27(\kappa^2 + \beta^2)^2}{32\kappa^3}||z||_4^4 \\ &= -\frac{1}{2}\kappa||u||_4^4 + \frac{27(\kappa^2 + \beta^2)^2}{32\kappa^3}||z||_4^4, \end{aligned} \tag{4.10}$$

$$\begin{aligned} Re(\lambda + i\mu)(\triangle z(\theta_t\omega), \bar{v}) &\leq \int_{\mathbb{R}^n} |\lambda + i\mu| \cdot |\nabla z(\theta_t\omega)||\nabla v|\mathrm{d}x \\ &\leq \frac{\lambda}{2}||\nabla v||^2 + \frac{\lambda^2 + \mu^2}{2\lambda}||\nabla z||^2 \end{aligned} \tag{4.11}$$

From (4.8)-(4.11), one obtains

$$\frac{d}{dt}||v||^2 + \lambda||\nabla v||^2 + 2\gamma||v||^2 + \kappa||u||_4^4 \leq \frac{27(\kappa^2 + \beta^2)^2}{16\kappa^3}||z||_4^4 + \frac{\lambda^2 + \mu^2}{\lambda}||\nabla z||^2. \tag{4.12}$$

We can see that the right-hand side of Equation (4.12) can be bounded by

$$c \cdot \sum_{j=1}^{m} \left(|z_j(\theta_t\omega_j)|^2 + |z_j(\theta_t\omega_j)|^4\right) \triangleq h(\theta_t\omega), \tag{4.13}$$

since $z(\theta_t\omega) = \sum_{j=1}^{m} \varphi_j z_j(\theta_t\omega_j)$, where $\varphi_j \in H^2(\mathbb{R}^n) \cap W^{2,4}(\mathbb{R}^n)$.

Hence, for $\forall t \geq 0$,

$$\frac{d}{dt}||v||^2 + 2\gamma||v||^2 \leq h(\theta_t\omega), \tag{4.14}$$

which leads to

$$||v(t, \omega, v_0(\omega))||^2 \leq e^{-2\gamma t}||v_0(\omega)||^2 + \int_0^t e^{2\gamma(s-t)}h(\theta_s\omega)\mathrm{d}s, \quad \text{for all } \ t \geq 0. \tag{4.15}$$

according to Gronwall's inequality.

By replacing ω by $\theta_{-t}\omega$, we derive from (3.5) and (4.15) that, for all $t \geq 0$,

$$\begin{aligned} ||v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2 &\leq e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2 + \int_0^t e^{2\gamma(s-t)}h(\theta_{s-t}\omega)\mathrm{d}s \\ &= e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2 + \int_{-t}^0 e^{2\gamma\tau}h(\theta_\tau\omega)\mathrm{d}\tau, \\ &\leq e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2 + \int_{-t}^0 e^{2\gamma\tau}e^{-\gamma\tau}r(\omega)\mathrm{d}\tau, \\ &\leq e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2 + \frac{1}{\gamma}r(\omega). \end{aligned} \tag{4.16}$$

Replacing ω by $\theta_{-t}\omega$ in (3.7), one has

$$\phi(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega)) = v(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega)-z(\theta_{-t}\omega)) + z(\omega).$$

Thereafter,

$$\begin{aligned} &||\phi(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))||^2 \\ &= ||v(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega)-z(\theta_{-t}\omega)) + z(\omega)||^2 \\ &\leq 2||v(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega)-z(\theta_{-t}\omega))||^2 + 2||z(\omega)||^2 \\ &\leq 2e^{-2\gamma t}||u_0(\theta_{-t}\omega)-z(\theta_{-t}\omega)||^2 + \frac{2}{\gamma}r(\omega) + 2||z(\omega)||^2 \\ &\leq 4e^{-2\gamma t}\left(||u_0(\theta_{-t}\omega)||^2 + ||z(\theta_{-t}\omega)||^2\right) + \frac{2}{\gamma}r(\omega) + 2||z(\omega)||^2. \end{aligned} \tag{4.17}$$

Recall that both the random variable $||z(\omega)||^2$ and the random bounded set $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ are tempered. Then, for any $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, there exists $t_B(\omega) > 0$ such that for all $t > t_B(\omega)$,

$$\begin{aligned} 4e^{-2\gamma t}\left(||u_0(\theta_{-t}\omega)||^2 + ||z(\theta_{-t}\omega)||^2\right) &= 4\left[\left(e^{-\gamma t}||u_0(\theta_{-t}\omega)||\right)^2 + \left(e^{-\gamma t}||z(\theta_{-t}\omega)||\right)^2\right] \\ &\leq \frac{2}{\gamma}r(\omega). \end{aligned}$$

So far, for all $t > t_B(\omega)$,

$$||\phi(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))||^2 \leq \frac{4}{\gamma}r(\omega) + 2||z(\omega)||^2. \tag{4.18}$$

Select

$$K(\omega) = \{u \in \mathbb{L}^2(\mathbb{R}^n):\ ||u||^2 \leq \frac{4}{\gamma}r(\omega) + 2||z(\omega)||^2\},$$

then $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is a random absorbing set for ϕ in $\mathcal{D}$. The proof is completed. □

LEMMA 4.1. *Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$, then for any $T_1 \geq 0$ and P-a.e. $\omega \in \Omega$, it holds true for the solution $u(t,\omega,u_0(\omega))$ of Equations (2.1)-(2.2) and $v(t,\omega,v_0(\omega))$ of Equation (3.6) with $v_0(\omega) = u_0(\omega) - z(\omega)$, $t \geq T_1$, such that*

$$\int_{T_1}^t e^{2\gamma(s-t)}||u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))||_4^4\mathrm{d}s \leq e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2 + \frac{2c}{\gamma}\cdot r(\omega), \tag{4.19}$$

$$\int_{T_1}^t e^{2\gamma(s-t)}||\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2\mathrm{d}s \leq e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2 + \frac{2c}{\gamma}\cdot r(\omega). \tag{4.20}$$

Proof. Fix $T_1 \geq 0$, replace t by T_1 and then replace ω by $\theta_{-t}\omega$ in Equation (4.15), we obtain

$$||v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2 \leq e^{-2\gamma T_1}||v_0(\theta_{-t}\omega)||^2 + \int_0^{T_1} e^{2\gamma(s-T_1)}h(\theta_{s-t}\omega)\mathrm{d}s. \tag{4.21}$$

With Equations (3.5) and (4.13) in mind, multipling $e^{2\gamma(T_1-t)}$ at both side of the above equation, one can easily get

$$\begin{aligned}
& e^{2\gamma(T_1-t)}||v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2 \\
\leq\ & e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2+\int_0^{T_1}e^{2\gamma(s-t)}h(\theta_{s-t}\omega)\mathrm{d}s \\
\leq\ & e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2+\int_{-t}^{T_1-t}e^{2\gamma\tau}h(\theta_\tau\omega)\mathrm{d}\tau \\
\leq\ & e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2+c\cdot r(\omega)\int_{-t}^{T_1-t}e^{\gamma\tau}\mathrm{d}\tau \\
\leq\ & e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2+\frac{c}{\gamma}\cdot r(\omega)e^{\gamma(T_1-t)}.
\end{aligned}\tag{4.22}$$

From (4.12) − (4.13), one has

$$\frac{d}{dt}||v||^2+\lambda||\nabla v||^2+2\gamma||v||^2+\kappa||u||_4^4\leq h(\theta_t\omega).\tag{4.23}$$

Multiplying Equation (4.23) by $e^{2\gamma(s-t)}$ and then integrating from T_1 to t, we then obtain

$$\begin{aligned}
&||v(t,\omega,v_0(\omega))||^2+\lambda\cdot\textstyle\int_{T_1}^t e^{2\gamma(s-t)}||\nabla v(s,\omega,v_0(\omega))||^2\mathrm{d}s+\kappa\cdot\int_{T_1}^t e^{2\gamma(s-t)}||u(s,\omega,u_0(\omega))||_4^4\mathrm{d}s \\
&\leq e^{2\gamma(T_1-t)}||v(T_1,\omega,v_0(\omega))||^2+\textstyle\int_{T_1}^t e^{2\gamma(s-t)}h(\theta_s\omega)\mathrm{d}s.
\end{aligned}\tag{4.24}$$

Keeping the last two terms on the right-hand side of Equation (4.24), and replacing ω by $\theta_{-t}\omega$, we have

$$\begin{aligned}
&\lambda\cdot\int_{T_1}^t e^{2\gamma(s-t)}||\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2\mathrm{d}s+\kappa\cdot\int_{T_1}^t e^{2\gamma(s-t)}||u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))||_4^4\mathrm{d}s \\
&\leq e^{2\gamma(T_1-t)}||v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2+\int_{T_1}^t e^{2\gamma(s-t)}h(\theta_{s-t}\omega)\mathrm{d}s \\
&\leq e^{2\gamma(T_1-t)}||v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2+\int_{T_1-t}^0 e^{2\gamma\tau}h(\theta_\tau\omega)\mathrm{d}\tau.
\end{aligned}\tag{4.25}$$

Since the second term on the right-hand side can be bounded by

$$c\cdot r(\omega)\int_{T_1-t}^0 e^{\gamma\tau}\mathrm{d}\tau\leq\frac{c}{\gamma}\cdot r(\omega),\tag{4.26}$$

due to (3.5) and (4.13). Together with Equation (4.22), it follows

$$\begin{aligned}
&\lambda\cdot\textstyle\int_{T_1}^t e^{2\gamma(s-t)}||\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2\mathrm{d}s+\kappa\cdot\int_{T_1}^t e^{2\gamma(s-t)}||u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))||_4^4\mathrm{d}s \\
&\leq e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2+\tfrac{c}{\gamma}\cdot r(\omega)e^{\gamma(T_1-t)}+\tfrac{c}{\gamma}\cdot r(\omega) \\
&\leq e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2+\tfrac{2c}{\gamma}\cdot r(\omega)\qquad\text{for all } t\geq T_1.
\end{aligned}\tag{4.27}$$

The proof is completed. □

COROLLARY 4.1. *Let* $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ *and* $u_0(\omega)\in B(\omega)$, *then for* P*-a.e.* $\omega\in\Omega$, *there exists* $t_B(\omega)>0$ *such that the solution* $u(t,\omega,u_0(\omega))$ *of Equations (2.1)-(2.2) and* $v(t,\omega,v_0(\omega))$ *of Equation (3.6) with* $v_0(\omega)=u_0(\omega)-z(\omega)$, *satisfy the following uniform estimates, for all* $t\geq t_B(\omega)$:

$$\int_t^{t+1}||u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))||_4^4\mathrm{d}s\leq\frac{4c}{\gamma}\cdot e^{2\gamma}\cdot r(\omega)$$

$$\int_t^{t+1}||\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))||^2\mathrm{d}s\leq\frac{4c}{\gamma}\cdot e^{2\gamma}\cdot r(\omega).$$

Proof. Replacing t by $(t+1)$ and then T_1 by t in (4.19), we deduce

$$\begin{aligned} & e^{-2\gamma}\int_t^{t+1}||u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))||_4^4\mathrm{d}s \\ \leq & \int_t^{t+1} e^{2\gamma(s-t-1)}||u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))||_4^4\mathrm{d}s \\ \leq & e^{-2\gamma(t+1)}||v_0(\theta_{-t-1}\omega)||^2+\frac{2c}{\gamma}\cdot r(\omega) \\ \leq & 2e^{-2\gamma(t+1)}\left(||u_0(\theta_{-t-1}\omega)||^2+||z(\theta_{-t-1}\omega)||^2\right)+\frac{2c}{\gamma}\cdot r(\omega). \end{aligned} \tag{4.28}$$

Since both random variables $u_0(\omega)\in B(\omega)$ and $z(\omega)$ are tempered, there exists $t_B(\omega)>0$, such that for all $t\geq t_B(\omega)$,

$$2e^{-2\gamma(t+1)}\left(||u_0(\theta_{-t-1}\omega)||^2+||z(\theta_{-t-1}\omega)||^2\right)\leq\frac{2c}{\gamma}\cdot r(\omega).$$

Together with (4.28), one claims that, for all $t\geq t_B(\omega)$,

$$\int_t^{t+1}||u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))||_4^4\mathrm{d}s\leq\frac{4c}{\gamma}\cdot e^{2\gamma}\cdot r(\omega).$$

Using the same procedure as above, we can also verify that, for all $t\geq t_B(\omega)$,

$$\int_t^{t+1}||\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))||^2\mathrm{d}s\leq\frac{4c}{\gamma}\cdot e^{2\gamma}\cdot r(\omega).$$

The proof is thus completed. □

Corollary 4.2. *Let $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ and $u_0(\omega)\in B(\omega)$, then for P-a.e. $\omega\in\Omega$, there exists $t_B(\omega)>0$ such that the solution $u(t,\omega,u_0(\omega))$ of Equations (2.1)-(2.2) satisfies:*

$$\int_t^{t+1}||\nabla u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))||^2\mathrm{d}s\leq\frac{4c}{\gamma}\cdot e^{2\gamma}\cdot r(\omega),\qquad \text{for all } t\geq t_B(\omega).$$

Proof. Let $t_B(\omega)>0$ be as in Corollary 4.1, and take $t\geq t_B(\omega)$ and $s\in(t,t+1)$. Note that by Equation (3.7),

$$\begin{aligned} & ||\nabla u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))||^2 \\ = & ||\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}))+\nabla z(\theta_{s-t-1}\omega)||^2 \\ \leq & 2||\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}))||^2+2||\nabla z(\theta_{s-t-1}\omega)||^2. \end{aligned} \tag{4.29}$$

Owing to (3.5), one has

$$2||\nabla z(\theta_{s-t-1}\omega)||^2\leq c\cdot\sum_{j=1}^m|z_j(\theta_{s-t-1}\omega)|^2\leq ce^{\gamma(t+1-s)}r(\omega)\leq ce^{\gamma}r(\omega) \tag{4.30}$$

Together with Corollary 4.1, we derive

$$\begin{aligned} & \int_t^{t+1}||\nabla u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))||^2\mathrm{d}s \\ \leq & 2\int_t^{t+1}||\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))||^2\mathrm{d}s+2\int_t^{t+1}||\nabla z(\theta_{s-t-1}\omega)||^2\mathrm{d}s \\ \leq & \frac{4c}{\gamma}\cdot e^{2\gamma}\cdot r(\omega)+ce^{\gamma}r(\omega) \\ \leq & \frac{4c}{\gamma}\cdot e^{2\gamma}\cdot r(\omega), \end{aligned}$$

by integrating (4.29) with respect to s over $(t,t+1)$. The proof is completed. □

LEMMA 4.2. *Suppose $\sqrt{3}\kappa \geq |\beta|$, and let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$. Then for P-a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that for all $t \geq t_B(\omega)$,*

$$||\nabla u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))||^2 \leq \frac{4c}{\gamma} \cdot e^{2\gamma} \cdot r(\omega).$$

Proof. Multiplying Equation (3.6) by $\triangle\bar{v}$, integrating over $\mathbb{R}^n$, and then taking the real part, we get

$$\begin{aligned} &\frac{1}{2}\cdot\frac{d}{dt}||\nabla v||^2 + \lambda||\triangle v||^2 + \gamma||\nabla v||^2 \\ &= Re\left((\kappa+i\beta)(|v+z|^2(v+z), \triangle\bar{v})\right) - Re\left((\lambda+i\mu)(\triangle z(\theta_t\omega), \triangle\bar{v})\right). \end{aligned} \tag{4.31}$$

Since

$$\left(|v+z|^2(v+z), \triangle\bar{v}\right) = \left(|u|^2u, \triangle\bar{u}\right) - \left(|u|^2u, \triangle\bar{z}(\theta_t\omega)\right),$$

while

$$\left(|u|^2u, \triangle\bar{u}\right) = -\int_{\mathbb{R}^n}\left(|u|^2|\nabla u|^2 + u\nabla\bar{u}\nabla|u|^2\right)\mathrm{d}x,$$

we have

$$\begin{aligned} &Re\left((\kappa+i\beta)(|u|^2u, \triangle\bar{u})\right) \\ &= -\kappa\int_{\mathbb{R}^n}|u|^2|\nabla u|^2\mathrm{d}x - \kappa\int_{\mathbb{R}^n}Re\left(u\nabla\bar{u}\nabla|u|^2\right)\mathrm{d}x + \beta\int_{\mathbb{R}^n}Im\left(u\nabla\bar{u}\nabla|u|^2\right)\mathrm{d}x \\ &= -\kappa\int_{\mathbb{R}^n}|u|^2|\nabla u|^2\mathrm{d}x - \frac{\kappa}{2}\int_{\mathbb{R}^n}\left(\nabla|u|^2\right)^2\mathrm{d}x - \frac{\beta}{2}\int_{\mathbb{R}^n}i\left(u\nabla\bar{u} - \bar{u}\nabla u\right)\nabla|u|^2\mathrm{d}x \\ &= -\frac{1}{4}\int_{\mathbb{R}^n}\left(3\kappa(\nabla|u|^2)^2 + 2\beta i(u\nabla\bar{u} - \bar{u}\nabla u)\nabla|u|^2 + \kappa|u\nabla\bar{u} - \bar{u}\nabla u|^2\right)\mathrm{d}x \\ &\leq 0, \end{aligned} \tag{4.32}$$

provided that $\sqrt{3}\kappa \geq |\beta|$.

Therefore, for the first term at the right-hand side of Equation (4.31), we have

$$\begin{aligned} &Re\left((\kappa+i\beta)(|v+z|^2(v+z), \triangle\bar{v})\right) \\ &= Re\left((\kappa+i\beta)(|u|^2u, \triangle\bar{u})\right) - Re\left((\kappa+i\beta)(|u|^2u, \triangle\bar{z}(\theta_t\omega))\right) \\ &\leq -Re\left((\kappa+i\beta)(|u|^2u, \triangle\bar{z}(\theta_t\omega))\right) \\ &\leq |\kappa+i\beta|\cdot\int_{\mathbb{R}^n}|u|^3\cdot|\triangle z(\theta_t\omega)|\mathrm{d}x \\ &\leq \frac{3}{4}||u||_4^4 + \frac{1}{4}(\kappa^2+\beta^2)^2\cdot||\triangle z(\theta_t\omega)||_4^4. \end{aligned} \tag{4.33}$$

On the other hand, the second term at the right-hand side of Equation (4.31) can be bounded by

$$|\lambda+i\mu|\cdot\int_{\mathbb{R}^n}|\triangle z(\theta_t\omega)|\cdot|\triangle v|\mathrm{d}x \leq \lambda||\triangle v||^2 + \frac{\lambda^2+\mu^2}{4\lambda}||\triangle z(\theta_t\omega)||^2. \tag{4.34}$$

By (4.31), (4.33)-(4.34), we can see that

$$\frac{d}{dt}||\nabla v||^2 + 2\gamma||\nabla v||^2 \leq \frac{3}{2}||u||_4^4 + \frac{1}{2}(\kappa^2+\beta^2)^2\cdot||\triangle z(\theta_t\omega)||_4^4 + \frac{\lambda^2+\mu^2}{2\lambda}||\triangle z(\theta_t\omega)||^2. \tag{4.35}$$

That is,

$$\frac{d}{dt}||\nabla v||^2 \leq \frac{3}{2}||u||_4^4 + g(\theta_t\omega), \tag{4.36}$$

where

$$g(\theta_t\omega) \triangleq \frac{1}{2}(\kappa^2+\beta^2)^2\cdot||\triangle z(\theta_t\omega)||_4^4 + \frac{\lambda^2+\mu^2}{2\lambda}||\triangle z(\theta_t\omega)||^2. \tag{4.37}$$

Since $z(\theta_t\omega) = \sum_{j=1}^{m} \varphi_j z_j(\theta_t\omega_j)$, where $\varphi_j \in H^2(\mathbb{R}^n) \cap W^{2,4}(\mathbb{R}^n)$, there exists a constant $c > 0$ such that

$$g(\theta_t\omega) \leq c \cdot \sum_{j=1}^{m} \left(|z_j(\theta_t\omega_j)|^2 + |z_j(\theta_t\omega_j)|^4\right) \leq c \cdot e^{\gamma|t|} r(\omega), \text{ for all } t \in \mathbb{R}. \tag{4.38}$$

Let $t > t_B(\omega)$, $s \in (t, t+1)$, where $t_B(\omega)$ is the positive time taken in Corollary 4.1. Integrate Equation (4.36) from s to $t+1$ shows

$$\begin{aligned} &||\nabla v(t+1, \omega, v_0(\omega))||^2 \\ &\leq ||\nabla v(s, \omega, v_0(\omega))||^2 + \frac{3}{2}\int_s^{t+1} ||u(\tau, \omega, u_0(\omega))||_4^4 \mathrm{d}\tau + \int_s^{t+1} g(\theta_\tau\omega)\mathrm{d}\tau. \end{aligned}$$

Integrating the above equation with respect to s over $(t, t+1)$ leads to

$$\begin{aligned} &||\nabla v(t+1, \omega, v_0(\omega))||^2 \\ &\leq \int_t^{t+1} ||\nabla v(s, \omega, v_0(\omega))||^2 \mathrm{d}s + \frac{3}{2}\int_t^{t+1} ||u(\tau, \omega, u_0(\omega))||_4^4 \mathrm{d}\tau + \int_t^{t+1} g(\theta_\tau\omega)\mathrm{d}\tau. \end{aligned}$$

Replacing ω by $\theta_{-t-1}\omega$, we derive

$$\begin{aligned} &||\nabla v(t+1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))||^2 \\ &\leq \int_t^{t+1} ||\nabla v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))||^2 \mathrm{d}s + \frac{3}{2}\int_t^{t+1} ||u(\tau, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))||_4^4 \mathrm{d}\tau \\ &\quad + \int_t^{t+1} g(\theta_{\tau-t-1}\omega)\mathrm{d}\tau. \end{aligned} \tag{4.39}$$

Thanks to Corollary 4.1, it follows from (4.38) and (4.39) that, for all $t > t_B(\omega)$,

$$\begin{aligned} &||\nabla v(t+1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))||^2 \\ &\leq \frac{4c}{\gamma} \cdot e^{2\gamma} \cdot r(\omega) + \frac{3}{2} \cdot \frac{4c}{\gamma} \cdot e^{2\gamma} \cdot r(\omega) + \int_{-1}^{0} g(\theta_\tau\omega)\mathrm{d}\tau \\ &\leq \frac{4c}{\gamma} \cdot e^{2\gamma} \cdot r(\omega) + c \cdot r(\omega)\int_{-1}^{0} e^{-\gamma\tau}\mathrm{d}\tau \\ &\leq \frac{4c}{\gamma} \cdot e^{2\gamma} \cdot r(\omega). \end{aligned} \tag{4.40}$$

Then together with (3.3), we obtain that, for all $t > t_B(\omega)$,

$$\begin{aligned} ||\nabla u(t+1, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))||^2 &= ||\nabla v(t+1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) + \nabla z(\omega)||^2 \\ &\leq 2||\nabla v(t+1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))||^2 + 2||\nabla z(\omega)||^2 \\ &\leq \frac{4c}{\gamma} \cdot e^{2\gamma} \cdot r(\omega). \end{aligned}$$

The proof is completed. □

LEMMA 4.3. *Suppose $\sqrt{3}\kappa \geq |\beta|$, and let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$, then for every $\epsilon > 0$ and P-a.e. $\omega \in \Omega$, there exists $T^* = T(B, \omega, \epsilon) > 0$ and $R^* = R^*(\omega, \epsilon)$ such that the solution $v(t, \omega, v_0(\omega))$ of Equation (3.6) with $v_0(\omega) = u_0(\omega) - z(\omega)$ satisfies for all $t \geq T^*$,*

$$\int_{|x|\geq R^*} |v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))(x)|^2 \mathrm{d}x \leq \epsilon.$$

Proof. Let ρ be a smooth function defined on $\mathbb{R}^+$ such that $0 \le \rho(s) \le 1$ for all $s \in \mathbb{R}^+$, and

$$\rho(s) = \begin{cases} 0 & \text{for } 0 \le s \le 1, \\ 1 & \text{for } s \ge 2. \end{cases}$$

Then there exists a constant $c > 0$ such that $|\rho'(s)| \le c$, for all $s \in \mathbb{R}^+$. Multiplying Equation (3.6) by $\rho(\frac{|x|^2}{l^2})\bar{v}$, integrating it over $\mathbb{R}^n$, and then taking the real part, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|v|^2 \mathrm{d}x \\ &= Re\left((\lambda + i\mu)\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})\triangle v\bar{v}\mathrm{d}x\right) - Re\left((\kappa + i\beta)\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|v+z|^2(v+z)\bar{v}\mathrm{d}x\right) \\ &\quad -\gamma\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|v|^2\mathrm{d}x + Re\left((\lambda+i\mu)\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})\triangle z\bar{v}\mathrm{d}x\right). \end{aligned} \tag{4.41}$$

We now concentrate to estimate the terms in (4.41). Firstly,

$$\begin{aligned} \int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})\triangle v\bar{v}\mathrm{d}x &= -\int_{\mathbb{R}^n} |\nabla v|^2 \rho(\frac{|x^2|}{l^2})\mathrm{d}x - \int_{\mathbb{R}^n} \bar{v}\rho\prime(\frac{|x^2|}{l^2})\frac{2x}{l^2}\nabla v\mathrm{d}x \\ &= -\int_{\mathbb{R}^n} |\nabla v|^2 \rho(\frac{|x^2|}{l^2})\mathrm{d}x - \int_{l\le|x|\le\sqrt{2}l} \bar{v}\rho\prime(\frac{|x^2|}{l^2})\frac{2x}{l^2}\nabla v\mathrm{d}x. \end{aligned}$$

Since

$$\begin{aligned} |\int_{l\le|x|\le\sqrt{2}l} \bar{v}\rho'(\frac{|x^2|}{l^2})\frac{2x}{l^2}\nabla v\mathrm{d}x| &\le \frac{2\sqrt{2}}{l}\int_{l\le|x|\le\sqrt{2}l} |v|\cdot|\rho'(\frac{|x^2|}{l^2})|\cdot|\nabla v|\mathrm{d}x \\ &\le \frac{c}{l}\int_{\mathbb{R}^n} |v|\cdot|\nabla v|\mathrm{d}x \le \frac{c}{l}\left(||v||^2 + ||\nabla v||^2\right), \end{aligned}$$

we find that

$$Re\left((\lambda+i\mu)\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})\triangle v\bar{v}\mathrm{d}x\right) \le -\lambda\cdot\int_{\mathbb{R}^n} |\nabla v|^2\rho(\frac{|x^2|}{l^2})\mathrm{d}x + \frac{\lambda\cdot c}{l}\left(||v||^2 + ||\nabla v||^2\right). \tag{4.42}$$

Secondly,

$$\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|v+z|^2(v+z)\bar{v}\mathrm{d}x = \int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|u|^4\mathrm{d}x - \int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|u|^2\cdot u\cdot\bar{z}(\theta_t\omega)\mathrm{d}x.$$

Due to

$$\begin{aligned} |\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|u|^2\cdot u\cdot\bar{z}(\theta_t\omega)\mathrm{d}x| &\le \int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|u|^3\cdot|z(\theta_t\omega)|\mathrm{d}x \\ &\le \frac{\kappa}{2|\kappa+i\beta|}\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|u|^4\mathrm{d}x + \frac{c}{|\kappa+i\beta|}\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|z(\theta_t\omega)|^4\mathrm{d}x, \end{aligned}$$

we have

$$\begin{aligned} &-Re\left((\kappa+i\beta)\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|v+z|^2(v+z)\bar{v}\mathrm{d}x\right) \\ &= -\kappa\cdot\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|u|^4\mathrm{d}x + Re\left((\kappa+i\beta)\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|u|^2\cdot u\cdot\bar{z}(\theta_t\omega)\mathrm{d}x\right) \\ &\le -\kappa\cdot\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|u|^4\mathrm{d}x + \frac{\kappa}{2}\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|u|^4\mathrm{d}x + c\cdot\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|z(\theta_t\omega)|^4\mathrm{d}x \\ &\le -\frac{\kappa}{2}\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|u|^4\mathrm{d}x + c\cdot\int_{\mathbb{R}^n} \rho(\frac{|x^2|}{l^2})|z(\theta_t\omega)|^4\mathrm{d}x. \end{aligned} \tag{4.43}$$

Thirdly,

$$Re\left((\lambda+i\mu)\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})\triangle z\bar{v}\mathrm{d}x\right)\leq\frac{\gamma}{2}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|v|^2\mathrm{d}x+\frac{\lambda^2+\mu^2}{2\gamma}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|\triangle z|^2\mathrm{d}x. \tag{4.44}$$

Finally, from (4.41)-(4.44),

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|v|^2\mathrm{d}x+\frac{\gamma}{2}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|v|^2\mathrm{d}x+\frac{\kappa}{2}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|u|^4\mathrm{d}x+\lambda\cdot\int_{\mathbb{R}^n}|\nabla v|^2\rho(\frac{|x^2|}{l^2})\mathrm{d}x\\ &\leq\ \frac{\lambda\cdot c}{l}\left(||v||^2+||\nabla v||^2\right)+c\cdot\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|z(\theta_t\omega)|^4\mathrm{d}x+\frac{\lambda^2+\mu^2}{2\gamma}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|\triangle z|^2\mathrm{d}x,\end{aligned}$$

which implies

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|v|^2\mathrm{d}x+\gamma\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|v|^2\mathrm{d}x\\ &\leq\ \frac{\lambda\cdot c}{l}\left(||v||^2+||\nabla v||^2\right)+c\cdot\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|z(\theta_t\omega)|^4\mathrm{d}x\\ &\quad+\frac{\lambda^2+\mu^2}{\gamma}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|\triangle z|^2\mathrm{d}x.\end{aligned} \tag{4.45}$$

Proposition 4.1 together with Lemma 4.2 shows that, there is $T_1=t_B(\omega)$ such that for all $t\geq T_1$,

$$||v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2_{H^1(\mathbb{R}^n)}\leq\frac{4c}{\gamma}\cdot e^{2\gamma}\cdot r(\omega). \tag{4.46}$$

Now, multiplying (4.45) with $e^{2\gamma(s-t)}$, and then integrating it over (T_1,t) respect to s so that, for all $t\geq T_1$,

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|v(t,\omega,v_0(\omega))|^2\mathrm{d}x\\ &\leq\ e^{2\gamma(T_1-t)}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|v(T_1,\omega,v_0(\omega))|^2\mathrm{d}x\\ &\quad+\frac{\lambda\cdot c}{l}\int_{T_1}^t e^{2\gamma(s-t)}\left(||v(s,\omega,v_0(\omega))||^2+||\nabla v(s,\omega,v_0(\omega))||^2\right)\mathrm{d}s\\ &\quad+c\cdot\int_{T_1}^t e^{2\gamma(s-t)}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|z(\theta_s\omega)|^4\mathrm{d}x\mathrm{d}s\\ &\quad+\frac{\lambda^2+\mu^2}{\gamma}\int_{T_1}^t e^{2\gamma(s-t)}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|\triangle z(\theta_s\omega)|^2\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{4.47}$$

Replacing ω by $\theta_{-t}\omega$ in Equation (4.47), we obtain that, for all $t\geq T_1$,

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2\mathrm{d}x\\ &\leq e^{2\gamma(T_1-t)}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2\mathrm{d}x\\ &\quad+\frac{\lambda\cdot c}{l}\int_{T_1}^t e^{2\gamma(s-t)}||v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2\mathrm{d}s+\frac{\lambda\cdot c}{l}\int_{T_1}^t e^{2\gamma(s-t)}||\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2\mathrm{d}s\\ &\quad+c\cdot\int_{T_1}^t e^{2\gamma(s-t)}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|z(\theta_{s-t}\omega)|^4\mathrm{d}x\mathrm{d}s\\ &\quad+\frac{\lambda^2+\mu^2}{\gamma}\int_{T_1}^t e^{2\gamma(s-t)}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|\triangle z(\theta_{s-t}\omega)|^2\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{4.48}$$

We now estimate the terms in (4.48) as following.

Firstly, from (4.15), one deduces

$$||v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2 \leq e^{-2\gamma T_1}||v_0(\theta_{-t}\omega)||^2 + \int_0^{T_1} e^{2\gamma(\tau-T_1)}h(\theta_{s-t}\omega)\mathrm{d}s. \tag{4.49}$$

Thus,

$$\begin{aligned} & e^{2\gamma(T_1-t)}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2\mathrm{d}x \\ \leq\ & e^{2\gamma(T_1-t)}\left(e^{-2\gamma T_1}||v_0(\theta_{-t}\omega)||^2+\int_0^{T_1}e^{2\gamma(\tau-T_1)}h(\theta_{\tau-t}\omega)\mathrm{d}\tau\right) \\ =\ & e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2+\int_{-t}^{T_1-t}e^{2\gamma s}h(\theta_s\omega)\mathrm{d}s \\ \leq\ & e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2+\frac{2}{\gamma}c\cdot r(\omega)e^{\gamma(T_1-t)}, \end{aligned} \tag{4.50}$$

due to Equations (3.5) and (4.13). Thus, for any given $\epsilon > 0$, there is $T_2(B,\omega,\epsilon) > T_1$ such that for all $t \geq T_2$,

$$e^{2\gamma(T_1-t)}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2\mathrm{d}x \leq \epsilon. \tag{4.51}$$

Replacing T_1 by s in Equation (4.49), then we find that the second term at the right-hand side of (4.48) satisfies

$$\begin{aligned} & \frac{\lambda\cdot c}{l}\int_{T_1}^t e^{2\gamma(s-t)}||v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2\mathrm{d}s \\ \leq\ & \frac{\lambda\cdot c}{l}\int_{T_1}^t e^{-2\gamma t}||v_0(\theta_{-t}\omega)||^2\mathrm{d}s+\frac{\lambda\cdot c}{l}\int_{T_1}^t\int_0^s e^{2\gamma(\tau-t)}h(\theta_{\tau-t}\omega)\mathrm{d}\tau\mathrm{d}s \\ \leq\ & \frac{\lambda\cdot c}{l}e^{-2\gamma t}(t-T_1)||v_0(\theta_{-t}\omega)||^2+\frac{\lambda\cdot c}{l}\int_{T_1}^t\int_{-t}^{s-t}e^{2\gamma\tau}h(\theta_\tau\omega)\mathrm{d}\tau\mathrm{d}s \\ \leq\ & \frac{\lambda\cdot c}{l}e^{-2\gamma t}(t-T_1)||v_0(\theta_{-t}\omega)||^2+\frac{\lambda\cdot c}{\gamma^2 l}r(\omega), \end{aligned} \tag{4.52}$$

which implies that there exists $T_3(B,\omega,\epsilon) > T_1$ and $R_1(\omega,\epsilon) > 0$ such that for all $t \geq T_3$ and $l \geq R_1$,

$$\frac{\lambda\cdot c}{l}\int_{T_1}^t e^{2\gamma(s-t)}||v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2\mathrm{d}s \leq \epsilon. \tag{4.53}$$

From Lemma 4.1, we know that there is $T_4(B,\omega) > T_1$ such that for all $t \geq T_4$, the third term at the right-hand side of (4.48) satisfies

$$\frac{\lambda\cdot c}{l}\int_{T_1}^t e^{2\gamma(s-t)}||\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2\mathrm{d}s \leq \frac{2\lambda\cdot c}{l\gamma}r(\omega).$$

Therefore, there is $R_2(\omega,\epsilon) > 0$ such that for all $t \geq T_4$ and $l \geq R_2$ such that

$$\frac{\lambda\cdot c}{l}\int_{T_1}^t e^{2\gamma(s-t)}||\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2\mathrm{d}s \leq \epsilon. \tag{4.54}$$

Finally, note that the last two terms in (4.48) can be bounded by

$$c\cdot\int_{T_1}^t e^{2\gamma(s-t)}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})\left(|\triangle z(\theta_{s-t}\omega)|^2+|z(\theta_{s-t}\omega)|^4\right)\mathrm{d}x\mathrm{d}s \tag{4.55}$$

and $z(\theta_t\omega) = \sum_{j=1}^{m} \varphi_j z_j(\theta_t\omega_j)$, where $\varphi_j \in H^2(\mathbb{R}^n) \cap W^{2,4}(\mathbb{R}^n)$. We can find $R_3(\omega,\epsilon) > 0$ such that for all $l \geq R_3$ and $j = 1, 2, \dots, m$,

$$\int_{|x|\geq l} \left(|\varphi_j(x)|^2 + |\varphi_j(x)|^4 + |\triangle\varphi_j(x)|^2\right) \mathrm{d}x \leq \min\{\frac{\gamma\epsilon}{m^4 c r(\omega)}, \frac{\epsilon}{2m^2 r(\omega)}\}. \tag{4.56}$$

Accordingly, we have the following estimates for the last two terms in (4.48),

$$\begin{aligned}
& c\cdot\int_{T_1}^{t} e^{2\gamma(s-t)}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|z(\theta_{s-t}\omega)|^4\mathrm{d}x\mathrm{d}s + \frac{\lambda^2+\mu^2}{\gamma}\int_{T_1}^{t} e^{2\gamma(s-t)}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|\triangle z(\theta_{s-t}\omega)|^2\mathrm{d}x\mathrm{d}s \\
\leq\ & c\cdot\int_{T_1}^{t} e^{2\gamma(s-t)}\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})\left(|\triangle z(\theta_{s-t}\omega)|^2 + |z(\theta_{s-t}\omega)|^4\right)\mathrm{d}x\mathrm{d}s \\
\leq\ & c\cdot\int_{T_1}^{t} e^{2\gamma(s-t)}\int_{|x|\geq l}\left(|\triangle z(\theta_{s-t}\omega)|^2 + |z(\theta_{s-t}\omega)|^4\right)\mathrm{d}x\mathrm{d}s \\
\leq\ & cm^4\cdot\int_{T_1}^{t} e^{2\gamma(s-t)}\sum_{j=1}^{m}\int_{|x|\geq l}\left(|\triangle\varphi_j|^2|z_j(\theta_{s-t}\omega_j)|^2 + |\varphi_j|^4|z_j(\theta_{s-t}\omega_j)|^4\right)\mathrm{d}x\mathrm{d}s \\
\leq\ & \frac{\gamma\epsilon}{r(\omega)}\int_{T_1}^{t} e^{2\gamma(s-t)}\sum_{j=1}^{m}\left(|z_j(\theta_{s-t}\omega_j)|^2 + |z_j(\theta_{s-t}\omega_j)|^4\right)\mathrm{d}s \\
\leq\ & \frac{\gamma\epsilon}{r(\omega)}\int_{T_1}^{t} e^{2\gamma(s-t)}h(\theta_{s-t}\omega)\mathrm{d}s \leq \frac{\gamma\epsilon}{r(\omega)}\int_{T_1-t}^{0} e^{2\gamma\tau}h(\theta_\tau\omega)\mathrm{d}\tau \\
\leq\ & \frac{\gamma\epsilon}{r(\omega)}\int_{T_1-t}^{0} e^{\gamma\tau}\mathrm{d}\tau \leq \epsilon.
\end{aligned} \tag{4.57}$$

Let $T^* = T(B,\omega,\epsilon) = \max\{T_1, T_2, T_3, T_4\}$ and $R^* = R(\omega,\epsilon) = \max\{R_1, R_2, R_3\}$. Then from (4.48), (4.51), (4.53), (4.54) and (4.57) we know that for all $t \geq T^*$ and $l \geq R^*$,

$$\int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2\mathrm{d}x \leq 4\epsilon.$$

That is, for any $t \geq T^*$ and $l \geq R^*$,

$$\int_{|x|\geq\sqrt{2}l}|v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2\mathrm{d}x \leq \int_{\mathbb{R}^n}\rho(\frac{|x^2|}{l^2})|v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2\mathrm{d}x \leq 4\epsilon.$$

The proof is completed. □

LEMMA 4.4. *Suppose $\sqrt{3}\kappa \geq |\beta|$, and let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$, then for every $\epsilon > 0$ and P-a.e. $\omega \in \Omega$, there exist $T^* = T(B,\omega,\epsilon) > 0$ and $R^* = R^*(\omega,\epsilon)$ such that the solution $u(t,\omega,u_0(\omega))$ of Equations (2.1)-(2.2) satisfies, for all $t \geq T^*$,*

$$\int_{|x|\geq R^*}|u(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega))(x)|^2\mathrm{d}x \leq \epsilon.$$

Proof. Let T^* and R^* be the constants in Lemma 4.3. Then due to (3.3) and (4.56), we know that, for all $t \geq T^*$ and $l \geq R^*$,

$$\begin{aligned}
\int_{|x|\geq R^*}|z(\omega)|^2\mathrm{d}x &= \int_{|x|\geq R^*}|\sum_{j=1}^{m}\varphi_j z_j(\omega_j)|^2\mathrm{d}x \\
&\leq m^2\int_{|x|\geq R^*}\sum_{j=1}^{m}|\varphi_j|^2|z_j(\omega_j)|^2\mathrm{d}x \leq \frac{\epsilon}{2r(\omega)}\sum_{j=1}^{m}|z_j(\omega_j)|^2 \leq \frac{\epsilon}{2}.
\end{aligned} \tag{4.58}$$

Thus, together with Lemma 4.3, we derive, for all $t \geq T^*$ and $l \geq R^*$,

$$\int_{|x|\geq R^*} |u(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))(x)|^2 \mathrm{d}x = \int_{|x|\geq R^*} |v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))(x)+z(\omega)|^2 \mathrm{d}x$$
$$\leq \ 2\int_{|x|\geq R^*} |v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))(x)|^2 + \int_{|x|\geq R^*} |z(\omega)|^2 \mathrm{d}x \leq 3\epsilon.$$

The proof is completed. □

Now we are ready to give the $\mathcal{D}$-pullback asymptotic compactness of ϕ, based on the former uniform estimates referring to the tails of solutions.

Proposition 4.2. *Suppose that $\sqrt{3}\kappa \geq |\beta|$, then the random dynamical system ϕ is $\mathcal{D}$-pullback asymptotically compact in $\mathbb{L}^2(\mathbb{R}^n)$. That is to say, for P-a.e. $\omega \in \Omega$, the sequence $\{\phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))\}_{n=1}^{\infty}$ has a convergent subsequence in $\mathbb{L}^2(\mathbb{R}^n)$ for $t_n \to \infty$, $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_{0,n}(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$.*

Proof. Let $t_n \to \infty$, $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_{0,n}(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$. By Proposition 4.1, we know that for P-a.e. $\omega \in \Omega$,

$$\{\phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))\}_{n=1}^{\infty} \qquad \text{is bounded in } \mathbb{L}^2(\mathbb{R}^n).$$

Hence, there is a $\xi \in \mathbb{L}^2(\mathbb{R}^n)$ such that, up to a subsequence,

$$\phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega)) \to \xi \qquad \text{weakly in } \mathbb{L}^2(\mathbb{R}^n). \tag{4.59}$$

It only remains to prove the weak convergence of (4.59) is indeed strong convergence. Let $\epsilon > 0$ be small enough. Since $\xi \in \mathbb{L}^2(\mathbb{R}^n)$, there exists $R_1 = R_(\epsilon) > 0$, such that

$$\int_{|x|\geq R_1} |\xi(x)|^2 \mathrm{d}x \leq \epsilon. \tag{4.60}$$

From Lemma 4.4, there is $T_1(B,\omega,\epsilon)$ and $R_2(\omega,\epsilon) > R_1(\epsilon) > 0$, for P-a.e. $\omega \in \Omega$, such that for all $t \geq T_1$,

$$\int_{|x|\geq R_2} |\phi(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))|^2 \mathrm{d}x \leq \epsilon. \tag{4.61}$$

Since $t_n \to \infty$, let $N_1 = N_1(B,\omega,\epsilon)$ be large enough such that $t_n \geq T_1$ for every $n \geq N_1$. Hence, it follows from (4.61) that for all $n \geq N_1$,

$$\int_{|x|\geq R_2} |\phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))|^2 \mathrm{d}x \leq \epsilon. \tag{4.62}$$

On the other hand, from Proposition 4.1 and Lemma 4.2, there is $T_2 = T_2(B,\omega)$ such that for all $t \geq T_2$,

$$||\phi(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))||^2_{H^1(\mathbb{R}^n)} \leq \frac{4c}{\gamma} \cdot e^{2\gamma} \cdot r(\omega). \tag{4.63}$$

Let $N_2 = N_2(B,\omega) > N_1$ such that $t_n \geq T_2$ for $n \geq N_2$. Thus, from (4.63), we know that, for all $n \geq N_2$,

$$||\phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))||^2_{H^1(\mathbb{R}^n)} \leq \frac{4c}{\gamma} \cdot e^{2\gamma} \cdot r(\omega). \tag{4.64}$$

Denote by Q_{R_2} for the set $\{x \in \mathbb{R}^n : \ |x| \leq R_2\}$. Due to the compactness of embedding $H^1(Q_{R_2}) \hookrightarrow \mathbb{L}^2(Q_{R_2})$, we deduce from (4.64) that, up to a subsequence,

$$\phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega)) \to \xi \qquad \text{strongly in } \mathbb{L}^2(Q_{R_2}),$$

which tells us that for the given $\epsilon > 0$, there exists $N_3 = N_3(B,\omega,\epsilon) > N_2$ such that for all $n \geq N_3$,

$$||\phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega)) - \xi||^2_{\mathbb{L}^2(Q_{R_2})} \leq \epsilon. \tag{4.65}$$

By (4.60), (4.62) and (4.65), we conclude that for all $n \geq N_3$,

$$\begin{aligned} &||\phi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)) - \xi||^2_{\mathbb{L}^2(\mathbb{R}^n)} \\ \leq & \int_{|x|\geq R_2} |\phi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)) - \xi|^2 \mathrm{d}x + \int_{|x|\leq R_2} |\phi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)) - \xi|^2 \mathrm{d}x \\ \leq & 5\epsilon. \end{aligned}$$

Therefore, up to a subsequence,

$$\phi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)) \to \xi \quad \text{strongly in} \mathbb{L}^2(\mathbb{R}^n).$$

□

Up to now, we have proved that ϕ has a closed random absorbing set $\{K(\omega)\}_{\omega\in\Omega}$ in $\mathcal{D}$ by Proposition 4.1, and is $\mathcal{D}$-pull back asymptotically compact in $\mathbb{L}^2(\mathbb{R}^n)$, which is present in Proposition 4.2. So, the existence of unique $\mathcal{D}$-random attractor for ϕ stated in Theorem 2.1 immediately follows from Theorem 1.2.

Part 2

Bifurcation of homoclinic and heteroclinic orbit

CHAPTER III

Non-resonant $3D$ homoclinic bifurcation with inclination-flip

In this chapter we deal with the bifurcation problems of a 3-dimensional smooth system having a homoclinic orbit to a hyperbolic equilibrium point with "inclination-flip".

1. Hypotheses and Preliminaries

In this chapter, we consider homoclinic bifurcation with inclination-flip in dimension 3. That is, we consider the following smooth system

$$\dot{z} = f(z) + g(z, \mu), \tag{1.1}$$

and its unperturbed system

$$\dot{z} = f(z), \tag{1.2}$$

where $z \in \mathbb{R}^3,\ \mu \in \mathbb{R}^2,\ 0 \leq |\mu| \ll 1,\ f(0) = 0,\ g(z, 0) = 0.$

First of all, we assume that:

(H_1) System (1.2) has a hyperbolic equilibrium O and the relevant linearization matrix $Df(0)$ has simple real eigenvalues: $-\alpha, -\beta, 1$ satisfying $\alpha > \beta > 0$.

As the implicit function theorem gives us that the hyperbolic fixed point persists through out the unfolding, so we will always assume without loss of generality, that it is the origin, i.e., $g(0, \mu) = 0$. Moreover we assume that the eigenvalues of $Df(0)$ avoid a finite number of resonances so that system(1.1) is uniformly C^2 linearizable. Thereafter, up to a C^2 diffeomorphism, there exists U, a small neighborhood of 0 in $\mathbb{R}^3$ and V, a neighborhood of 0 in $\mathbb{R}^2$, such that for all $z \in U$ and all $\mu \in V$, system (1.1) has the following C^2 normal form:

$$\dot{x} = x, \qquad \dot{y} = -\beta(\mu)y, \qquad \dot{v} = -\alpha(\mu)v. \tag{1.3}$$

Besides, we make the following assumptions:

(H_2) System (1.2) has a homoclinic loop $\Gamma = \{z = r(t),\ t \in R\}$. Let $e^{\pm} = \lim\limits_{t \to \mp\infty} \frac{\dot{r}(t)}{|\dot{r}(t)|}$. Then $e^+ \in T_0W^u,\ e^- \in T_0W^s$ are unit eigenvectors corresponding to 1 and $-\beta$.

(H_3) Denote by e_s^- the unit eigenvector corresponding to $-\alpha$, then

$$\mathrm{Span}(T_{r(t)}W^u, T_{r(t)}W^s, e_s^-) = \mathbb{R}^3, \quad \text{for } t \ll -1.$$

With the above assumptions, the homoclinic orbit Γ is of codimension 2.

Remark 1.1. *a)* (H_3) *is equivalent to* $T_{r(t)}W^s \to e^+ \bigoplus e^-, \quad$ *when* $t \to -\infty$.

b) For the existing loop Γ, (H_2) *is generic, which guarantees that* Γ *has no orbit flip. While* (H_3) *is not generic, which indicates that* W^s *takes place inclination flip when* $t \to -\infty$ *(see Figure III.1(1)).*

2. Bifurcation equations

Now we consider the linear variational system of (1.2) and its adjoint system

$$\dot{z} = Df(r(t))z, \tag{2.4}$$

$$\dot{z} = -(Df(r(t)))^* z. \tag{2.5}$$

Denote $r(t) = (r^x(t), r^y(t), r^v(t))$ and take $T > 0$ large enough such that $r(-T) = (\delta, 0, 0)$ and $r(T) = (0, \delta, \delta_v)$, where $|\delta_v| = O(\delta^2)$ and δ is small enough so that $\{(x, y, v) : \ |x|, |y|, |v| < 2\delta\} \subset U$.

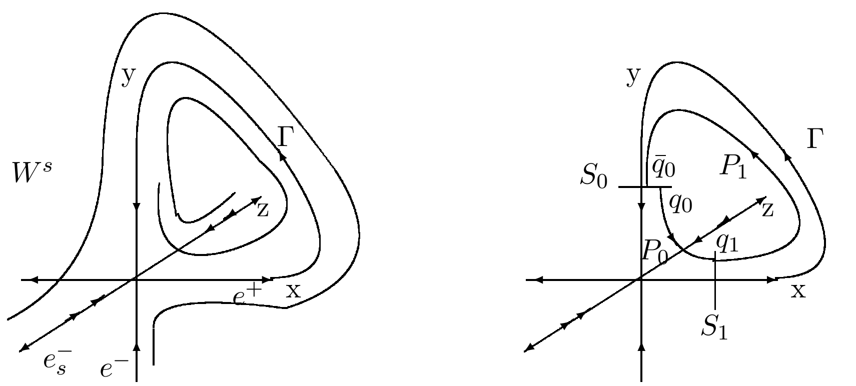
LEMMA 2.1. *There exists a fundamental solution matrix* $Z(t) = (z^1(t), z^2(t), z^3(t))$ *for system* (2.4) *with*

$$z^1(t) \in (T_{r_i(t)}W^u)^c \cap (T_{r_i(t)}W^s)^c,$$
$$z^2(t) = -\dot{r}(t)/|\dot{r}^y(T)| \in T_{r(t)}W^u \cap T_{r(t)}W^s,$$
$$z^3(t) \in T_{r(t)}W^{ss},$$

satisfying

$$z(-T) = \begin{pmatrix} 0 & \omega_{21} & \omega_{31} \\ 0 & 0 & \omega_{32} \\ 1 & 0 & \omega_{33} \end{pmatrix}, \qquad z(T) = \begin{pmatrix} \omega_{11} & 0 & 0 \\ \omega_{12} & 1 & 0 \\ \omega_{13} & \omega_{23} & 1 \end{pmatrix},$$

where $|\ \omega_{23}\ | \ll 1, \omega_{21} < 0, \omega_{11} \neq 0, \omega_{32} \neq 0.$



(1) Inclination-flip (2) Poincaré return map

FIGURE III.1

As well known from the matrix theory, system (2.5) has a fundamental solution matrix $\Phi(t) = (Z^{-1}(t))^*$. We denote $\Phi(t) = (\phi_i^1(t), \phi_i^2(t), \phi_i^3(t))$. Introduce the local active coordinates near the orbits Γ as $(z_1(t),\ z_2(t),\ z_3(t))$ with the components $N = (n_1, 0, n_3)$. Set

$$z = S(t) = r(t) + z(t)N^* = r(t) + z_1(t)n_1 + z_3(t)n_3. \tag{2.6}$$

With this notation, we can choose the cross sections

$$S_0 = \{z = S(T) :|\ x\ |, |\ y\ |, |\ v\ | < 2\delta\} \subset U,$$

$$S_1 = \{z = S(-T) :|\ x\ |, |\ y\ |, |\ v\ | < 2\delta\} \subset U.$$

Under the transformation (2.6), system (1.1) has the following form

$$\dot{n}_j = (\phi_j(t))^* g_\mu(r(t), 0)\mu + h.o.t., \qquad j = 1, 3, \tag{2.7}$$

which is C^2 and produces the map $P_1 :\ S_1 \to S_0$. Integrating both sides from $-T$ to T, we have

$$n_j(T) = n_j(-T) + M_j\mu + h.o.t., \qquad j = 1, 3, \tag{2.8}$$

where $N(T) = (n_1(T), 0, n_3(T))$, $N(-T) = (n_1(-T), 0, n_3(-T))$, and

$$M_j = \int_{-T}^{T} (\phi_j(t))^* g_\mu(r(t), 0)\,\mathrm{d}t, \quad j = 1,\ 3$$

are Melnikov vectors.

LEMMA 2.2. $M_1 = \int_{-T}^{T} (\phi_1(t))^* g_\mu(r(t), 0)\,\mathrm{d}t = \int_{-\infty}^{+\infty} (\phi_1(t))^* g_\mu(r(t), 0)\,\mathrm{d}t.$

Define $P_0 :\ S_0 \to S_1,\ q_0 \to q_1$ induced by the flow of (1.3) in the neighbourhood U of $z = 0$. Set the flying time from q_0 to q_1 as τ and the Silnikov time $s = e^{-\tau}$ (see Figure III.1(2)). Then we have

$$\begin{aligned} P_0 :\quad & q_0(x_0, y_0, v_0) \to q_1(x_1, y_1, v_1), \\ & x_0 = sx_1, \qquad y_1 = s^\beta y_0, \qquad v_1 = s^\alpha v_0, \end{aligned}$$

and $x_1 \approx \delta,\ y_0 \approx \delta$;

$$\begin{array}{ll} n_1^0 = (\omega_{11})^{-1} x_0, & n_3^0 = v_0 - \delta_v - (\omega_{11})^{-1}\omega_{13} x_0, \\ n_1^1 = v_1 - (\omega_{32})^{-1}\omega_{33} y_1, & n_3^1 = (\omega_{32})^{-1} y_1. \end{array}$$

From the above, we give the following Poincaré maps:

$$\begin{array}{l} F_1 = P_1 \circ P_0 :\ S_0 \longrightarrow S_0, \\ \bar{n}_1^0 = v_0 s^\alpha - (\omega_{32})^{-1}\omega_{33}\delta s^\beta + M_1\mu + h.o.t., \\ \bar{n}_3^0 = (\omega_{32})^{-1}\delta s^\beta + M_3\mu + h.o.t. \end{array}$$

Now, the successor function is given by $G(s, v_0) = (G_1, G_3) = (F_1(q_0) - q_0)$ as follows:

$$\begin{array}{lcl} G_1 & = & -(\omega_{11})^{-1}\delta s + v_0 s^\alpha - (\omega_{32})^{-1}\omega_{33}\delta s^\beta + M_1\mu + h.o.t., \\ G_3 & = & -v_0 + \delta_v + (\omega_{11})^{-1}\omega_{13}\delta s + (\omega_{32})^{-1}\delta s^\beta + M_3\mu + h.o.t. \end{array}$$

By solving v_0 from $G_3 = 0$ and substituting it into $G_1 = 0$, we obtain the bifurcation equation

$$-(\omega_{11})^{-1}\delta s + \delta_v s^\alpha + (\omega_{32})^{-1}\delta s^{\alpha+\beta} + M_3\mu s^\alpha - (\omega_{32})^{-1}\omega_{33}\delta s^\beta + M_1\mu + h.o.t. = 0. \tag{2.9}$$

3. Bifurcation analyses and bifurcation diagram

DEFINITION 3.1. *The strong foliation $W^{ss}_{r(t)}$ is called strong inclination flip if $t \to -\infty$, the stable manifold $W^s_{r(t)}$ is inclination flip and the strong stable component $\omega_{33} = 0$ as $t = T,\ T \gg 1$.*

We will distinguish the following cases.

Case (1): $1 > \alpha > \beta > 0$. The bifurcation equation (2.9) is reduced to the following

$$\text{if } \omega_{33} = 0, \quad \text{for } \delta_v \neq 0, \quad \delta_v s^\alpha + M_1\mu + h.o.t. = 0, \tag{3.10}$$

$$\text{for } \delta_v = 0, \quad -(\omega_{11})^{-1}\delta s + (\omega_{32})^{-1}\delta s^{\alpha+\beta} + M_3\mu s^\alpha + M_1\mu + h.o.t. = 0, \tag{3.11}$$

$$\text{if } \omega_{33} \neq 0, \quad -(\omega_{32})^{-1}\omega_{33}\delta s^\beta + M_1\mu + h.o.t. = 0. \tag{3.12}$$

Case (2): $\alpha > 1 > \beta > 0$. We obtain the following bifurcation equations

$$\text{if } \omega_{33} = 0, \quad -(\omega_{11})^{-1}\delta s + M_1\mu + h.o.t. = 0; \tag{3.13}$$

$$\text{if } \omega_{33} \neq 0, \quad -(\omega_{32})^{-1}\omega_{33}\delta s^\beta + M_1\mu + h.o.t. = 0. \tag{3.14}$$

Case (3): $\alpha > \beta > 1$. We have the following bifurcation equation

$$-(\omega_{11})^{-1}\delta s + M_1\mu + h.o.t. = 0. \tag{3.15}$$

REMARK 3.1. *Notice that, the local weak stable manifold is not unique. In fact, one can fill up a wedge area in $W^s \cap U$ with these manifolds (curves). Obviously, $\delta_v = 0$ means that in the coordinate system corresponding to the normal form (1.3), the local weak stable manifold $\Gamma \cap U$ is exactly a segment of the y-axis.*

Firstly, we assume $\omega_{33} \neq 0$, that is $W^{ss}_{r(t)}$ is not strong inclination flip. It follows from (3.12), (3.14) and (3.15) that:

THEOREM 3.1. *If $M_1 \neq 0$, $\omega_{33} \neq 0$, system (1.1) has at most one unique periodic orbit in the small neighborhood of Γ. And it does exist if and only if $\mu \in \{\omega_{11}M_1\mu > 0\}$, $0 <\mid \mu \mid \ll 1$, for $\alpha > \beta > 1$; $\mu \in \{\omega_{32}\omega_{33}M_1\mu > 0\}$, $0 <\mid \mu \mid \ll 1$, for $1 > \alpha > \beta > 0$ and $\alpha > 1 > \beta > 0$.*

THEOREM 3.2. *If $M_1 \neq 0$, $\omega_{33} \neq 0$, there exists codimension 1 bifurcation surface H_1 : $M_1\mu + h.o.t. = 0$ with normal vector M_1 at $\mu = 0$ such that Γ persists as $\mu \in H_1$.*

Secondly, we consider the case of strong inclination flip, that is, $\omega_{33} = 0$. Owing to (3.10), (3.13) and (3.15), we state the following result.

THEOREM 3.3. *Assume $\omega_{33} = 0$, then*
(1) if $1 > \alpha > \beta > 0$ and $\delta_v \neq 0$, system (1.1) has a unique 1-periodic orbit if and only if $\mu \in \{\delta_v M_1\mu < 0\}$, $0 <\mid \mu \mid \ll 1$, and there exists codimension 1 bifurcation surface $H_1 :\ M_1\mu + h.o.t. = 0$ with normal vector M_1 at $\mu = 0$ such that Γ persists for $\mu \in H_1$.
(2) if $\alpha > 1 > \beta > 0$ or $\alpha > \beta > 1$, system (1.1) has a unique 1-periodic orbit if and only if $\mu \in \{\omega_{11}M_1\mu > 0\}$, $0 <\mid \mu \mid \ll 1$, and there exists codimension 1 bifurcation surface $H_1 :\ M_1\mu + h.o.t. = 0$ with normal vector M_1 at $\mu = 0$ such that Γ persists for $\mu \in H_1$.

It then remains the case concerning the bifurcation equation (3.11) which we deal in the sequel. We state the results for this case in the following.

THEOREM 3.4. *Suppose that $1 > \alpha > \beta > 0$, $\omega_{33} = 0$, $\delta_v = 0$ and $rank(M_1, M_3) = 2$, then there exist a 1-homoclinic bifurcation surface H^1, a 2-fold periodic orbit bifurcation surface SN^1, a period-doubling bifurcation surface P^{2^n} of 2^{n-1} periodic orbit and a 2^n-homoclinic bifurcation surface H^{2^n} for $\forall\ n \in \mathbb{N}$, which share the same normal vector M_1 at $\mu = 0$, such that system (1.1) has*

a 1-homoclinic orbit if and only if $\mu \in H^1$ and $\mid \mu \mid \ll 1$;

a 2-fold periodic orbit if and only if $\mu \in SN^1$;

a 2^{n-1}-periodic orbit changing its stablility and a 2^n-periodic orbit arising at the same time if and only if $\mu \in P^{2^n}$;

a 2^n-homoclinic orbit if and only if $\mu \in H^{2^n}$.

Furthermore there exists a bifurcation surface Δ_1 (which is a branch of H^1) with codimension 1 and normal vector M_1 such that system (1.1) has a 1-homoclinic orbit as well as a 1-periodic orbit for $\mu \in \Delta_1$ and $\mid \mu \mid \ll 1$.

The rest of the paper is devoted to the proof of the above theorem which follows from several propositions.

Denote the left side of (3.11) by $F(s, \mu) = L(s, \mu) - N(s, \mu)$, where

$$\begin{aligned} L(s,\mu) &= M_3\mu s^\alpha + M_1\mu + h.o.t., \quad L(0,\mu) = F(0,\mu), \\ N(s,\mu) &= (\omega_{11})^{-1}\delta s - (\omega_{32})^{-1}\delta s^{\alpha+\beta} + h.o.t., \quad N(0,\mu) = 0, \end{aligned}$$

and

$$D_+^+ = \{\mu :\ M_1\mu > 0,\ M_3\mu > 0\}, \qquad D_-^+ = \{\mu :\ M_1\mu > 0,\ M_3\mu < 0\},$$
$$D_+^- = \{\mu :\ M_1\mu < 0,\ M_3\mu > 0\}, \qquad D_-^- = \{\mu :\ M_1\mu < 0,\ M_3\mu < 0\}.$$

It is evident that the four areas are not empty when $\text{rank}(M_1, M_3) = 2$.

PROPOSITION 3.1. *Suppose* $1 > \alpha > \beta > 0,\ \omega_{33} = 0,\ \delta_v = 0$ *and* $rank(M_1, M_3) = 2$, *then* $F(s,\mu)$ *has a unique positive zero point* $\bar{s}$ *sufficiently small such that system (1.1) has a unique periodic orbit. More precisely,*

(1) *when* $\alpha + \beta > 1$, $F(s,\mu)$ *has a unique sufficiently small positive zero point* $\bar{s}$ *for* $\mu \in D_-^+ \cup D_+^+$ *if* $\omega_{11} > 0$, *and* $\mu \in D_-^- \cup D_+^-$ *if* $\omega_{11} < 0$.
(2) *when* $0 < \alpha + \beta < 1$, $F(s,\mu)$ *has a unique sufficiently small positive zero point* $\bar{s}$ *for* $\mu \in D_-^+ \cup D_+^+$ *if* $\omega_{32} < 0$, *and* $\mu \in D_-^- \cup D_+^-$ *if* $\omega_{32} > 0$.
(3) *when* $\alpha + \beta = 1$, $F(s,\mu)$ *has a unique sufficiently small positive zero point* $\bar{s}$ *for* $\mu \in D_-^+ \cup D_+^+$ *if* $(\omega_{32})^{-1} > (\omega_{11})^{-1}$, *and* $\mu \in D_-^- \cup D_+^-$ *if* $(\omega_{32})^{-1} < (\omega_{11})^{-1}$.

Proof. When $\alpha + \beta > 1$,

$$F(s,\mu) = -(\omega_{11})^{-1}\delta s + M_3\mu s^\alpha + M_1\mu + h.o.t.$$

Let $s^\alpha = t$, then in case $\omega_{11} > 0$. If $\mu \in D_-^+$,

$$L(0,\mu) = M_1\mu + h.o.t. > 0, \qquad L'(t,\mu) = M_3\mu + h.o.t. < 0,$$
$$N'(t,\mu) = (\alpha\omega_{11})^{-1}\delta t^{\frac{1-\alpha}{\alpha}} + h.o.t. > 0.$$

So the line $W = L(t,\mu)$ and the curve $W = N(t,\mu)$ intersect at a unique sufficiently small positive point $\bar{t} < (\delta^{-1}\omega_{11}M_1\mu)^\alpha$ and F has a unique sufficiently small positive zero $\bar{s} = (\bar{t})^{1/\alpha}$.

If $\mu \in D_+^+$, then

$$L(0,\mu) = M_1\mu + h.o.t. > 0, \quad L'(t,\mu) = M_3\mu + h.o.t. > 0,$$
$$N'(t,\mu) = (\alpha\omega_{11})^{-1}\delta t^{\frac{1-\alpha}{\alpha}} + h.o.t. > 0,$$
$$N''(t,\mu) = (1-\alpha)(\alpha^2\omega_{11})^{-1}\delta t^{\frac{1-2\alpha}{\alpha}} + h.o.t. > 0.$$

Take $\bar{t} = [\delta^{-1}\omega_{11}(2M_3\mu + M_1\mu)]^\alpha$, then

$$N(\bar{t},\mu) - L(\bar{t},\mu) = 2M_3\mu + M_1\mu - M_3\mu\bar{t} - M_1\mu > M_3\mu > 0.$$

Therefore, based on the fact that $N(\cdot,\mu)$ is a monotone increasing convex function, we see that the line $W = L(t,\mu)$ and the curve $W = N(t,\mu)$ intersect uniquely at $t^* \in (0,\bar{t})$, that is, F has a unique sufficiently small positive zero point $\bar{s} \in (0, \delta^{-1}\omega_{11}(2M_3\mu + M_1\mu))$.

The proof for the rest cases can be given similarly. □

PROPOSITION 3.2. *Suppose that* $1 > \alpha > \beta > 0,\ \omega_{33} = 0,\ \delta_v = 0$ *and* $rank(M_1, M_3) = 2$, *then*
(1) *for* $\alpha + \beta > 1$, *there exists a bifurcation surface* Δ_1 *with codimension 1 and normal vector* M_1 *at* $\mu = 0$ *such that system (1.1) has an 1-homoclinic orbit as well as an 1-periodic orbit for* $\mu \in \Delta_1$ *and* $|\ \mu\ | \ll 1$.
(2) *for* $0 < \alpha + \beta < 1$, *there exists a bifurcation surface* Δ_2 *with codimension 1 and normal vector* M_1 *at* $\mu = 0$ *such that system (1.1) has an 1-homoclinic orbit as well as an 1-periodic orbit as* $\mu \in \Delta_2$ *and* $|\ \mu\ | \ll 1$.
(3) *for* $\alpha + \beta = 1$, *there exists a bifurcation surface* Δ_3 *with codimension 1 and normal vector* M_1 *at* $\mu = 0$ *such that system (1.1) has an 1-homoclinic orbit as well as an 1-periodic orbit as* $\mu \in \Delta_3$ *and* $|\ \mu\ | \ll 1$.

Proof. When $\alpha + \beta > 1$, $\mu \in \Delta_1 \triangleq \{\mu :\ F(0,\mu) = M_1\mu + h.o.t. = 0,\ \omega_{11}M_3\mu > 0\}$, we have

$$F(s,\mu) = s^\alpha[-(\omega_{11})^{-1}\delta s^{1-\alpha} + M_3\mu + h.o.t.].$$

Consequently, there are two zero points $s_1 = 0,\ s_2 = (\omega_{11}\delta^{-1}M_3\mu)^{\frac{1}{1-\alpha}} + h.o.t.$

When $0 < \alpha+\beta < 1$, $\mu \in \Delta_2 \triangleq \{\mu:\ F(0,\mu) = M_1\mu + h.o.t. = 0,\ \omega_{32}M_3\mu < 0\}$, one has

$$F(s,\mu) = s^{\alpha}[(\omega_{32})^{-1}\delta s^{\beta} + M_3\mu + h.o.t.] = 0$$

which admits $s_1 = 0,\ s_2 = (-\omega_{32}\delta^{-1}M_3\mu)^{\frac{1}{\beta}} + h.o.t.$ as its solutions.

When $\alpha+\beta = 1$, $\mu \in \Delta_3 \triangleq \{\mu:\ F(0,\mu) = M_1\mu + h.o.t. = 0,\ \omega_{11}\omega_{32}(\omega_{32}-\omega_{11})M_3\mu > 0\}$, we obtain

$$F(s,\mu) = s^{\alpha}[((\omega_{32})^{-1} - (\omega_{11})^{-1})\delta s^{1-\alpha} + M_3\mu + h.o.t.].$$

Thereafter, it has two zero points $s_1 = 0,\ s_2 = (\omega_{11}\omega_{32}(\omega_{32}-\omega_{11})^{-1}\delta^{-1}M_3\mu)^{\frac{1}{1-\alpha}} + h.o.t.$ □

PROPOSITION 3.3. *Suppose that $1 > \alpha > \beta > 0$, $\omega_{33} = 0$, $\delta_v = 0$ and $rank(M_1, M_3) = 2$, then $F(s,\mu)$ has a unique 2-fold positive zero point $\bar{s}$ such that system (1.1) has a unique 2-fold periodic orbit. Precisely speaking,*

(1) when $\alpha+\beta > 1$, $F(s,\mu)$ has a unique 2-fold positive zero point $\bar{s} = (\delta^{-1}\alpha\omega_{11}M_3\mu)^{\frac{1}{1-\alpha}} + h.o.t.$ for μ satisying $\omega_{11}M_3\mu > 0$ and the 2-fold periodic orbit bifurcation surface is SN^1 :

$$(\omega_{11})^{-1}\delta(\delta^{-1}\alpha\omega_{11}M_3\mu)^{\frac{1}{1-\alpha}} = M_3\mu(\delta^{-1}\alpha\omega_{11}M_3\mu)^{\frac{\alpha}{1-\alpha}} + M_1\mu + h.o.t.$$

with normal vector M_1 at $\mu = 0$.

(2) when $0 < \alpha+\beta < 1$, $F(s,\mu)$ has a unique 2-fold positive zero point $\bar{s} = [-(\delta(\alpha+\beta))^{-1}\alpha\omega_{32}M_3\mu]^{\frac{1}{\beta}} + h.o.t.$ for μ satisying $\omega_{32}M_3\mu < 0$ and the 2-fold periodic orbit bifurcation surface SN^1 :

$$-(\omega_{32})^{-1}\delta[-(\delta(\alpha+\beta))^{-1}\alpha\omega_{32}M_3\mu]^{1+\frac{\alpha}{\beta}} = M_3\mu[-(\delta(\alpha+\beta))^{-1}\alpha\omega_{32}M_3\mu]^{\frac{\alpha}{\beta}} + M_1\mu + h.o.t.$$

has normal vector M_1 at $\mu = 0$.

(3) when $\alpha+\beta = 1$, $F(s,\mu)$ has a unique 2-fold positive zero point

$$\bar{s} = [(\delta(\omega_{32}-\omega_{11}))^{-1}\alpha\omega_{11}\omega_{32}M_3\mu]^{\frac{1}{1-\alpha}}$$

for μ satisying $((\omega_{11})^{-1} - (\omega_{32})^{-1})M_3\mu > 0$, and the corresponding 2-fold periodic orbit bifurcation surface SN^1 :

$$\begin{aligned}((\omega_{11})^{-1} - (\omega_{32})^{-1})\delta[(\delta(\omega_{32}-\omega_{11}))^{-1}\alpha\omega_{11}\omega_{32}M_3\mu]^{\frac{1}{1-\alpha}} \\ = M_3\mu[(\delta(\omega_{32}-\omega_{11}))^{-1}\alpha\omega_{11}\omega_{32}M_3\mu]^{\frac{\alpha}{1-\alpha}} + M_1\mu + h.o.t.\end{aligned}$$

has normal vector M_1 at $\mu = 0$.

Proof. The 2-fold zero point $\bar{t}$ should satisfy

$$L(t,\mu) = N(t,\mu), \qquad L'(t,\mu) = N'(t,\mu). \tag{3.16}$$

The second equation turns out to be

$$(\alpha\omega_{11})^{-1}\delta t^{\frac{1-\alpha}{\alpha}} - (\alpha+\beta)(\alpha\omega_{32})^{-1}\delta t^{\frac{\beta}{\alpha}} + h.o.t. = M_3\mu. \tag{3.17}$$

When $\alpha+\beta > 1$, we have $\bar{t} = (\delta^{-1}\alpha\omega_{11}M_3\mu)^{\frac{\alpha}{1-\alpha}} + h.o.t.$ for $\omega_{11}M_3\mu > 0$ due to (3.17). Then from the first equation of (3.16), we get the corresponding 2-fold periodic orbit bifurcation surface SN^1 :

$$(\omega_{11})^{-1}\delta(\delta^{-1}\alpha\omega_{11}M_3\mu)^{\frac{1}{1-\alpha}} = M_3\mu(\delta^{-1}\alpha\omega_{11}M_3\mu)^{\frac{\alpha}{1-\alpha}} + M_1\mu + h.o.t.$$

with normal vector M_1 at $\mu = 0$.

The other two cases can be proofed similarly. □

Now we try to study the bifurcation of 2-homoclinic orbit and the period-doubling bifurcation for the case of $1 > \alpha > \beta > 0$, $\omega_{33} = 0$, $\delta_v = 0$. Like before, let t_1 and t_2 be the flying time from q_0 to q_1 and from q_2 to q_3, respectively, $s_i = e^{-t_i},\ i = 1, 2$. Then the second successor function can be expressed by

$$G^2(s_1, s_2, v_0, v_2) = (G_1^1, G_3^1, G_1^2, G_3^2) = (F_1(q_0) - q_2, F_1(q_2) - q_0)$$

with:

$$\begin{aligned}
G_1^1 &= -(\omega_{11})^{-1}\delta s_2 + v_0 s_1^\alpha - (\omega_{32})^{-1}\omega_{33}\delta s_1^\beta + M_1\mu + h.o.t.,\\
G_3^1 &= -v_2 + \delta_v + (\omega_{11})^{-1}\omega_{13}\delta s_2 + (\omega_{32})^{-1}\delta s_1^\beta + M_3\mu + h.o.t.\\
G_1^2 &= -(\omega_{11})^{-1}\delta s_1 + v_2 s_2^\alpha - (\omega_{32})^{-1}\omega_{33}\delta s_2^\beta + M_1\mu + h.o.t.,\\
G_3^2 &= -v_0 + \delta_v + (\omega_{11})^{-1}\omega_{13}\delta s_1 + (\omega_{32})^{-1}\delta s_2^\beta + M_3\mu + h.o.t.
\end{aligned}$$

Solving (v_0, v_2) from $(G_3^1, G_3^2) = 0$ and substituting it into $G_1^1 = 0$ and $G_1^2 = 0$, we then obtain the bifurcation equations

$$\begin{aligned}
&-(\omega_{11})^{-1}\delta s_2 + (\omega_{11})^{-1}\omega_{13}\delta s_1^{1+\alpha} + (\omega_{32})^{-1}\delta s_1^\alpha s_2^\beta + M_3\mu s_1^\alpha + M_1\mu\\
&\qquad\qquad + h.o.t. = 0, \qquad (3.18)\\
&-(\omega_{11})^{-1}\delta s_1 + (\omega_{11})^{-1}\omega_{13}\delta s_2^{1+\alpha} + (\omega_{32})^{-1}\delta s_1^\beta s_2^\alpha + M_3\mu s_2^\alpha + M_1\mu\\
&\qquad\qquad + h.o.t. = 0. \qquad (3.19)
\end{aligned}$$

It is easy to see that system (1.1) has a 2-homoclinic orbit near Γ if and only if the above equation has $s_1 = 0, s_2 > 0$ as its solution by the symmetry of G^2.

If $s_1 = 0, s_2 > 0$ is the solution of the bifurcation equation, then $s_2 = \delta^{-1}\omega_{11}M_1\mu + h.o.t.$ for $\omega_{11}M_1\mu > 0$, and the 2-homoclinic bifurcation surface

$$H^2:\ \omega_{11}M_3\mu = -\delta(\delta^{-1}\omega_{11}M_1\mu)^{1-\alpha} + h.o.t.$$

has codimension 1 with normal vector M_1 at $\mu = 0$.

Thus we have:

PROPOSITION 3.4. *Suppose* $1 > \alpha > \beta > 0,\ \omega_{33} = 0,\ \delta_v = 0$ *and* $rank(M_1, M_3) = 2$, *then there exists a unique 1-homoclinic bifurcation surface* $H^1:\ M_1\mu + h.o.t. = 0$ *with codimension 1 and normal vector* M_1 *at* $\mu = 0$, *which coincides with* Δ_1 *in the region defined by* $\{\mu:\ \omega_{11}M_3\mu > 0\}$. *For* $\mu \in H^1$ *and* $|\ \mu\ | \ll 1$, *system (1.1) has a unique 1-homoclinic orbit.*

There exists a unique bifurcation surface $H^2:\ \omega_{11}M_3\mu = -\delta(\delta^{-1}\omega_{11}M_1\mu)^{1-\alpha} + h.o.t.$ *which is well defined in the region* $\{\mu:\ \omega_{11}M_1\mu > 0,\ \omega_{11}M_3\mu < 0\}$, *such that system (1.1) has a unique 2-homoclinic orbit for* $\mu \in H^2$.

From Proposition 3.4, we know that H^1 and H^2 have the same normal vector M_1 at $\mu = 0$, and $M_3\mu = O(|\ M_1\mu\ |^{1-\alpha})$ for $\mu \in H^2$. So, there is a tongue area bounded by H^1 and H^2. And in the tongue area, there must be another bifurcation surface P^2 where a period-doubling bifurcation arises.

Similarly as in Section 2, we define

$$\begin{aligned}
P_0^j:\quad & q_{2j-2}(x_{2j-2}, y_{2j-2}, v_{2j-2}) \to q_{2j-1}(x_{2j-1}, y_{2j-1}, v_{2j-1}),\\
& x_{2j-2} = s_j x_{2j-1}, \quad y_{2j-1} = s_j^\beta y_{2j-2}, \quad v_{2j-1} = s_j^\alpha v_{2j-2},
\end{aligned}$$

and $x_{2j-1} \approx \delta,\ y_{2j-2} \approx \delta,\ j = 1, 2, \cdots$.

$$\begin{aligned}
n_1^{2j-2} &= (\omega_{11})^{-1}x_{2j-2}, & n_3^{2j-2} &= v_{2j-2} - \delta_v - (\omega_{11})^{-1}\omega_{13}x_{2j-2},\\
n_1^{2j-1} &= v_{2j-1} - (\omega_{32})^{-1}\omega_{33}y_{2j-1}, & n_3^{2j-1} &= (\omega_{32})^{-1}y_{2j-1}.
\end{aligned}$$

From the above, we give the n-th Poincaré return maps:

$$\begin{aligned}&F_1^j = P_1 \circ P_0^j : \ S_0 \to S_0, \quad q_{2j-2} \longmapsto \bar{q}_{2j-2},\\ &\bar{n}_1^{2j-2} = v_{2j-1} - (\omega_{32})^{-1}\omega_{33} y_{2j-1} + M_1\mu + h.o.t.,\\ &\bar{n}_3^{2j-2} = (\omega_{32})^{-1} y_{2j-1} + M_3\mu + h.o.t.\end{aligned}$$

Consequently, the associated n-th successor function is given by

$$\begin{aligned}G^n(s_1, \cdots, s_n, v_0, \cdots, v_{2n-2}) &= (G_1^1, G_3^1, G_1^2, G_3^2, G_1^3, G_3^3, G_1^4, G_3^4)\\ &= (F_1^1(q_0) - q_2, F_1^2(q_2) - q_4, \cdots, F_1^n(q_{2n-2}) - q_0).\end{aligned}$$

Now, we study the 4-homoclinic bifurcation surface H^4 with the condition $1 > \alpha > \beta > 0,\ \omega_{33} = 0,\ \delta_v = 0$.

Solve (v_0, v_2, v_4, v_6) from $(G_3^1, G_3^2, G_3^3, G_3^4) = 0$ and substitute it into $(G_1^1, G_1^2, G_1^3, G_1^4)$, then we get the bifurcation equation:

$$-(\omega_{11})^{-1}\delta s_2 + (\omega_{11})^{-1}\omega_{13}\delta s_1^{1+\alpha} + (\omega_{32})^{-1}\delta s_1^{\alpha} s_4^{\beta} + M_3\mu s_1^{\alpha} + M_1\mu + h.o.t. = 0, \tag{3.20}$$

$$-(\omega_{11})^{-1}\delta s_3 + (\omega_{11})^{-1}\omega_{13}\delta s_2^{1+\alpha} + (\omega_{32})^{-1}\delta s_1^{\beta} s_2^{\alpha} + M_3\mu s_2^{\alpha} + M_1\mu + h.o.t. = 0, \tag{3.21}$$

$$-(\omega_{11})^{-1}\delta s_4 + (\omega_{11})^{-1}\omega_{13}\delta s_3^{1+\alpha} + (\omega_{32})^{-1}\delta s_2^{\beta} s_3^{\alpha} + M_3\mu s_3^{\alpha} + M_1\mu + h.o.t. = 0, \tag{3.22}$$

$$-(\omega_{11})^{-1}\delta s_1 + (\omega_{11})^{-1}\omega_{13}\delta s_4^{1+\alpha} + (\omega_{32})^{-1}\delta s_3^{\beta} s_4^{\alpha} + M_3\mu s_4^{\alpha} + M_1\mu + h.o.t. = 0. \tag{3.23}$$

So, we just need to consider the above equations that admit $s_1 = 0, s_2 > 0, s_3 > 0, s_4 > 0$ as solutions. Correspondingly, from (3.20), we have $s_2 = \omega_{11}\delta^{-1}M_1\mu + h.o.t.$ for $\omega_{11}M_1\mu > 0$. Then (3.21) yields

$$s_3 = \omega_{11}\delta^{-1}[M_3\mu(\omega_{11}\delta^{-1}M_1\mu)^{\alpha} + M_1\mu] + h.o.t. = \omega_{11}\delta^{-1}M_1\mu + h.o.t.$$

for $\mu \in \{\mu :\ \omega_{11}\delta^{-1}M_1\mu > 0\} \cap \{\mu :\ M_3\mu = o(\mid M_1\mu \mid^{1-\alpha})\}$.

If $\alpha + \beta > 1$, we obtain $s_4 = \omega_{11}\delta^{-1}M_1\mu + h.o.t.$ for $\mu \in \{\mu :\ \omega_{11}M_1\mu > 0\} \cap \{\mu :\ M_3\mu = o(\mid M_1\mu \mid^{1-\alpha})\}$. Then, (3.23) gives the 4-homoclinic bifurcation surface H^4 :

$$M_3\mu(\omega_{11}\delta^{-1}M_1\mu)^{\alpha} + M_1\mu + h.o.t. = 0. \tag{3.24}$$

Here, we have that H^4 is defined on $\{\mu :\ M_3\mu = O(\mid M_1\mu \mid^{1-\alpha})\}$.

If $0 < \alpha + \beta < 1$, we get $s_4 = \omega_{11}(\omega_{32})^{-1}(\omega_{11}\delta^{-1}M_1\mu)^{\alpha+\beta} + h.o.t.$ for $\mu \in \{\mu :\ \omega_{11}M_1\mu > 0\} \cap \{\mu :\ M_3\mu = o(\mid M_1\mu \mid^{1-\alpha})\}$. Consequently, from (3.23) the 4-homoclinic bifurcation surface H^4 should be:

$$\begin{aligned}&(\omega_{32})^{-1}\delta(\omega_{11}\delta^{-1}M_1\mu)^{\beta}[\omega_{11}(\omega_{32})^{-1}(\omega_{11}\delta^{-1}M_1\mu)^{\alpha+\beta}]^{\alpha}\\ &\quad + M_3\mu[\omega_{11}(\omega_{32})^{-1}(\omega_{11}\delta^{-1}M_1\mu)^{\alpha+\beta}]^{\alpha} + h.o.t. = 0.\end{aligned} \tag{3.25}$$

In this case, we have that H^4 is defined on $\{\mu :\ M_3\mu = O(\mid M_1\mu \mid^{\beta})\}$.

If $\alpha + \beta = 1$, then $s_4 = \omega_{11}\delta^{-1}[1 + (\omega_{32})^{-1}]M_1\mu + h.o.t.$ for $\mu \in \{\mu :\ \omega_{11}M_1\mu > 0\} \cap \{\mu :\ M_3\mu = o(\mid M_1\mu \mid^{1-\alpha})\}$. Then, we get the 4-homoclinic bifurcation surface H^4:

$$M_3\mu\{\omega_{11}\delta^{-1}[1 + (\omega_{32})^{-1}]M_1\mu\}^{\alpha} + M_1\mu + (\omega_{32})^{-1}\omega_{11}[1 + (\omega_{32})^{-1}]^{\alpha}M_1\mu + h.o.t. = 0. \tag{3.26}$$

Owing to the bifurcation surface equation, we have that H^4 is defined on $\{\mu :\ M_3\mu = O(\mid M_1\mu \mid^{1-\alpha})\}$ as well.

Summing up, we get the 4-homoclinic bifurcation surface H^4 : (3.24), (3.25) or (3.26) on the parameter surface $\{\mu : \ M_3\mu = o(|\ M_1\mu\ |^{1-\alpha})\}$. Repeat the above procedure, we can also get the 2^n-homoclinic bifurcation surface H^{2^n} and the period-doubling bifurcation surface P^{2^n} for arbitrary $n \in \mathbb{N}$.

To well illustrate our main theorem, we give the following bifurcation diagrams under the assumption $1 > \alpha > \beta > 0,\ \omega_{33} = 0,\ \delta_v = 0$ and $rank(M_1, M_3) = 2$, where O represents that there is no periodic orbits, while P (resp. P^k) represents that there exists a 1-periodic (resp. k-periodic) orbit in the corresponding region.

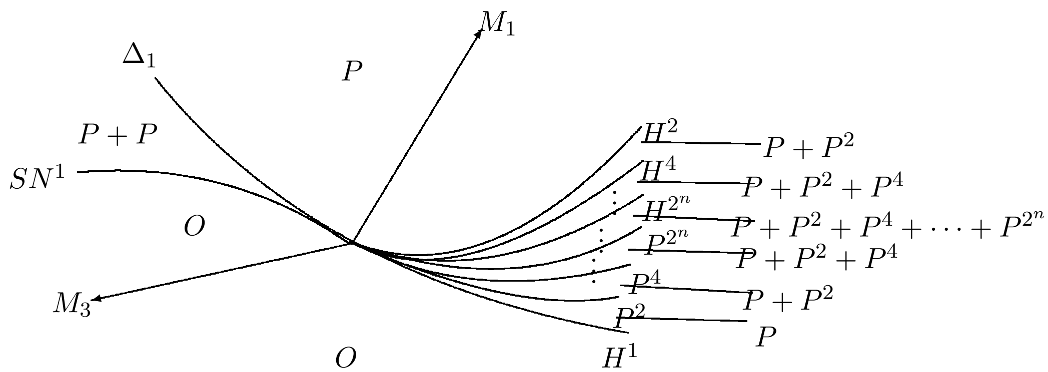


FIGURE III.2. Bifurcation diagram in case : $1 > \alpha > \beta > 0,\ \alpha + \beta > 1,\ \delta_v = 0,\ \omega_{11} > 0,\ \omega_{33} = 0$.

CHAPTER IV

Codimension 2 bifurcation of twisted double homoclinic loops

We study the bifurcation of twisted double homoclinic loops. We obtain bifurcation results in both one twisted and two twisted loops.

1. Hypotheses and Preliminaries

For fixed r, we consider the following C^r system

$$\dot{z} = f(z) + g(z,\mu), \tag{1.1}$$

where $z \in \mathbb{R}^{m+n+2}$, $m \geq 0, n \geq 0, m+n > 0$, $\mu \in \mathbb{R}^l$, $l \geq 2$, $0 \leq ||\mu|| \ll 1$, $f(0) = 0$, $g(z,0) = 0$, here $||\cdot||$ denotes the scalar product. Differently with the $3D$ homoclinic bifurcation with inclination flip in the former chapter, the degeneracy of the unperturbed vector field

$$\dot{z} = f(z), \tag{1.2}$$

comes from exclusively form the double homoclinicity, and various bifurcation manifolds and the corresponding existence regions are concretely given.

First of all, we assume that:

(H_1) System (1.2) has a hyperbolic equilibrium at the origin and the relevant linearization matrix $Df(0)$ has simple eigenvalues: λ_1, $\lambda_{2i} (i = 1, 2, \cdots, m)$, $-\rho_1$, $-\rho_{2j} (j = 1, 2, \cdots, n)$ satisfying

$$-\mathrm{Re}\rho_{2j} < -\rho_1 < 0 < \lambda_1 < \mathrm{Re}\lambda_{2i}.$$

With no strong resonance between $-\rho_1$ and λ_1 being allowed, we can always assume that $\rho_1 > \lambda_1$ without loss of generality.

Thanks to the Implicit Function Theorem, since the equilibrium of the unperturbed system (located at the origin for $\mu = 0$) is hyperbolic, this equilibrium persists and admits a continuation for small values of $||\mu||$. Up to a translation, one can assume that the equilibrium is always located at the origin.

Moreover we assume that $Df(0)$ satisfies the Sternberg condition of order Q with $Q = K([\frac{\lambda_{2m}}{\lambda_1}] + [\frac{\rho_{2n}}{\rho_1}]) + 2$, where K is the Q-smoothness of $Df(0)$, and $r \geq 3Q$, so that system (1.1) is uniformly C^K linearizable according to [**82**]. Hence, up to a C^K diffeomorphism, there exists a small neighborhood U of 0 in $\mathbb{R}^{m+n+2}$, such that for all $\mu \in \mathbb{R}^l$, $0 \leq ||\mu|| \ll 1$ and for all $(x, y, u, v) \in U$, system (1.1) has the following C^{K-1} $(K \geq 4)$ normal form:

$$\dot{x} = \lambda_1(\mu)x, \qquad \dot{y} = -\rho_1(\mu)y, \qquad \dot{u} = \lambda_2(\mu)u, \qquad \dot{v} = -\rho_2(\mu)v. \tag{1.3}$$

Here, $\lambda_2(\mu)$ is an $m \times m$ diagonal matrix with $\lambda_{21}, \lambda_{22}, \cdots, \lambda_{2m}$ as its diagonal elements and $\rho_2(\mu)$ is an $n \times n$ diagonal matrix with $\rho_{21}, \rho_{22}, \cdots, \rho_{2n}$ as its diagonal elements.

Besides, we make the following assumptions:

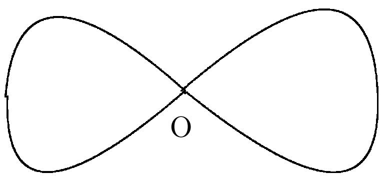
(H_2) System (1.2) has a double homoclinic loops $\Gamma = \Gamma_1 \cup \Gamma_2$,

$$\Gamma_i = \{z = r_i(t) : t \in \mathbb{R}, r_i(\pm\infty) = 0\}$$

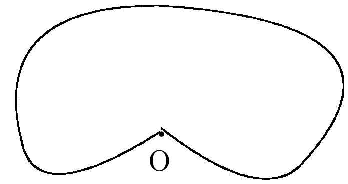
and $\dim(\mathrm{T}_{r_i(t)}\mathrm{W}^s \cap \mathrm{T}_{r_i(t)}\mathrm{W}^u) = 1,\ i = 1, 2$, where W^s and W^u designate the stable and unstable manifold respectively and $\mathrm{T}_\mathrm{A}\mathrm{W}$ is the tangent space of W at A.

(H_3) Let $e_i^{\pm} = \lim_{t\to\mp\infty} \frac{\dot r_i(t)}{|\dot r_i(t)|}$, then $e_i^+ \in \mathrm{T}_0\mathrm{W}^u,\ e_i^- \in \mathrm{T}_0\mathrm{W}^s$ are unit eigenvectors corresponding to λ_1 and $-\rho_1$, respectively. Moreover, $e_1^+ = -e_2^+,\ e_1^- = -e_2^-$.

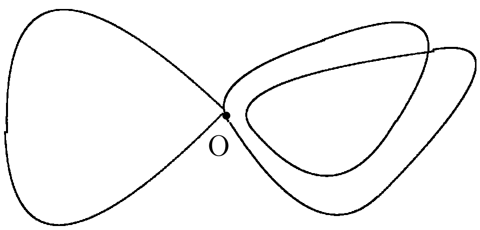
(H_4) $\mathrm{Span}\{\mathrm{T}_{r_i(t)}\mathrm{W}^u,\ \mathrm{T}_{r_i(t)}\mathrm{W}^s,\ e_i^+\} = \mathbb{R}^{m+n+2}$ as $t \gg 1$,
$\mathrm{Span}\{\mathrm{T}_{r_i(t)}\mathrm{W}^u,\ \mathrm{T}_{r_i(t)}\mathrm{W}^s,\ e_i^-\} = \mathbb{R}^{m+n+2}$ as $t \ll -1$. (see Figure IV.2)



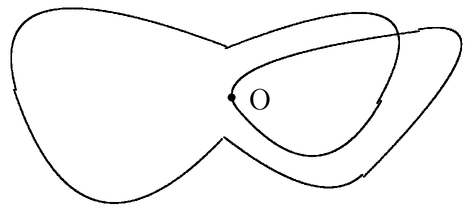
1-1 doule homoclinic orbits



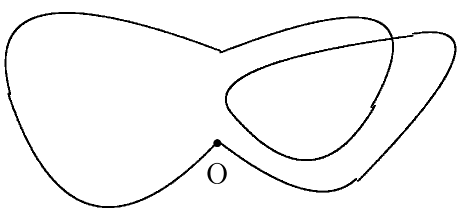
1-1 large homoclinic orbit



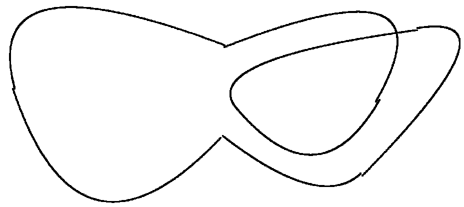
2-1 double homoclinic orbits



2-1 right homoclinic orbit



2-1 large homoclinic orbit



2-1 large periodic orbit

Figure IV.1

Remark 1.1. *For the existing loop Γ, (H_3) is generic, which guarantees that Γ has no orbit flip. While (H_4) says that both homoclinic orbit are not of inclination-flip. If both H_3 and H_4 hold, the orbit is called non-critically twisted.*

With the above assumptions, the double homoclinic loops, say Γ_1, Γ_2, are of codimension 2. Besides, a non-degenerate homoclinic orbit Γ is called non-twisted homoclinic orbit if the unstable manifold W^u has an even number of half twists along the homoclinic orbit. It is called a twisted homoclinic orbit if W^u has an odd number of half twists along Γ, see [**18**] for more details. We shall study the problems of p-q double homoclinic loops, p-q left (or right) homoclinic loop, p-q large homoclinic loop and p-q large period orbit bifurcated from the twisted double homoclinic loops in arbitrarily high-dimensional system. Here, "left" or "right" means the corresponding orbit circulates in the small neighborhood of the original double homoclinic loops whereas it just takes infinite time in the neighborhood of one orbit of the double homoclinic loops, either Γ_1 or Γ_2. And "large" means that the corresponding orbit moves around in the small neighborhood of the original double homoclinic loops and it takes infinite time in the neighborhood of each homoclinic orbit. In addition, "p-q" refers to the rounding number in each orbit's neighborhood. Precisely speaking, p-q loop will round Γ_2 p cycles, while it has winding number q in a small neighborhood of Γ_1. (see Figure IV.1.)

As in the meaning of the first approximation, the tangent vector bundles, situated in the tangent space bundles confined on the homoclinic loops, which is the intersection of the stable manifold and the unstable manifold, inherit and exhibit sufficiently the properties (such as the geometry, the invariance, the contractibility, the expansiveness, etc.) of the system near the loop. Our aim is then to select carefully some tangent vector bundles along the loops and some others complement to them to form a moving frame so as to obtain the simplest form. Let us consider the linear variational system of (1.2)

$$\dot{z} = Df(r_i(t))z, \tag{1.4}$$

and its adjoint system

$$\dot{z} = -(Df(r_i(t)))^* z. \tag{1.5}$$

Denote $r_i(t) = (r_i^x(t), r_i^y(t), r_i^u(t), r_i^v(t))$ and take $T_i^0,\ T_i^1$ large enough such that

$$r_i(-T_i^1) = ((-1)^i\delta, 0, \delta_i^u, 0), \quad r_i(T_i^0) = (0, (-1)^i\delta, 0, \delta_i^v),$$

where $||\delta_i^u||, ||\delta_i^v|| = O(\delta^\alpha),\ i = 1,2,\ \alpha = \min\limits_{j,k}\{Re\rho_{2j}/\rho_1, Re\lambda_{2k}/\lambda_1\} > 1$, and δ is small enough so that

$$\{(x,y,u,v) : |x|, |y|, ||u||, ||v|| < 4\delta\} \subset U.$$

We state the following lemma which can be found in [**40, 88**].

Lemma 1.1. *There exists a fundamental solution matrix* $Z_i(t) = (z_i^1(t), z_i^2(t), z_i^3(t), z_i^4(t))$ *for system* (1.4) *with*

$$\begin{aligned}
&z_i^1(t) \in (T_{r_i(t)}W^u)^c \cap (T_{r_i(t)}W^s)^c,\\
&z_i^2(t) = -\dot{r}_i(t)/|\dot{r}_i^y(T_i^0)| \in T_{r_i(t)}W^u \cap T_{r_i(t)}W^s,\\
&z_i^3(t) \in T_{r_i(t)}W^{uu},\\
&z_i^4(t) \in T_{r_i(t)}W^{ss}
\end{aligned}$$

satisfying

$$Z_i(-T_i^1) = \begin{pmatrix} \omega_i^{11} & \omega_i^{21} & 0 & \omega_i^{41} \\ \omega_i^{12} & 0 & 0 & \omega_i^{42} \\ \omega_i^{13} & \omega_i^{23} & I_{m\times m} & \omega_i^{43} \\ 0 & 0 & 0 & \omega_i^{44} \end{pmatrix}, \quad Z_i(T_i^0) = \begin{pmatrix} 1 & 0 & \omega_i^{31} & 0 \\ 0 & (-1)^i & \omega_i^{32} & 0 \\ 0 & 0 & \omega_i^{33} & 0 \\ \bar{\omega}_i^{14} & \omega_i^{24} & \omega_i^{34} & I_{n\times n} \end{pmatrix},$$

where, as $T_i^j \gg 1(i = 1,2,\ j = 0,1)$, $\omega_i^{12}\det\omega_i^{33}\det\omega_i^{44} \neq 0$, $(-1)^i\omega_i^{21} < 0$, $||\omega_i^{24}|| \ll 1$, $||\bar{\omega}_i^{14}|| \ll 1$, $|(\omega_i^{12})^{-1}\omega_i^{11}| \ll 1$, $||(\omega_i^{12})^{-1}\omega_i^{13}|| \ll 1$; $||(\omega_i^{21})^{-1}\omega_i^{23}|| \ll 1$, $||(\det\omega_i^{44})^{-1}\omega_i^{4k}|| \ll 1$, *for* $k \neq 4$; $||(\det\omega_i^{33})^{-1}\omega_i^{3k}|| \ll 1$, *for* $k \neq 3$.

REMARK 1.2. *In the above lemma, W^{uu} stands for the strong unstable manifold while W^{ss} stands for the strong stable manifold.*

As is well known from the matrix theory, system (1.5) has a fundamental solution matrix $\Phi_i(t) = (Z_i^{-1}(t))^* = (\phi_i^1(t), \phi_i^2(t), \phi_i^3(t), \phi_i^4(t))$. Introduce the local active coordinates $N_i = (n_i^1, 0, n_i^3, n_i^4)$, then we parametrized a point $z = (x, y, u, v)$ near the orbits Γ_i in the section $S_i(t)$ by the coordinates (n_i^1, n_i^3, n_i^4). And the section $S_i(t)$ can be written as

$$S_i(t) = \{z = r_i(t) + Z_i(t)N_i^* = r_i(t) + z_i^1(t)n_i^1 + z_i^3(t)n_i^3 + z_i^4(t)n_i^4\}. \tag{1.6}$$

Choose the cross sections, for $i = 1, 2$,

$$S_i^0 = \{z = S_i(T_i^0) : |x|, |y|, |u|, |v| < 2\delta\} \subset U,$$
$$S_i^1 = \{z = S_i(-T_i^1) : |x|, |y|, |u|, |v| < 2\delta\} \subset U.$$

With the above notation, system (1.1) has the following form

$$\dot{n}_i^j = (\phi_i^j(t))^* g_\mu(r_i(t), 0)\mu + o(||\mu||), \qquad i = 1, 2;\ j = 1, 3, 4, \tag{1.7}$$

which is C^{K-2} and produces the transition maps $P_i^1 :\ S_i^1 \to S_i^0,\ i = 1, 2$. Here, g_μ is the derivative of g with respect to μ. Integrating both sides of (1.7) from $-T_i^1$ to T_i^0, we have

$$n_i^j(T_i^0) = n_i^j(-T_i^1) + M_i^j\mu + o(||\mu||), \qquad i = 1, 2;\ j = 1, 3, 4,$$

where $N_i(T_i^0) = (n_i^1(T_i^0),\ 0,\ n_i^3(T_i^0),\ n_i^4(T_i^0))$, $N_i(-T_i^1) = (n_i^1(-T_i^1),\ 0,\ n_i^3(-T_i^1),\ n_i^4(-T_i^1))$, and $M_i^j = \int_{-T_i^1}^{T_i^0} (\phi_i^j(t))^* g_\mu(r_i(t), 0)\,\mathrm{d}t, i = 1, 2; j = 1, 3, 4$ are Melnikov vectors (see for example [**39, 40, 88, 104, 105**]).

REMARK 1.3. *The Melnikov vectors in the case $j = 1$ are given by*

$$M_i^1 = \int_{-T_i^1}^{T_i^0} (\phi_i^1(t))^* g_\mu(r_i(t), 0)\,\mathrm{d}t = \int_{-\infty}^{+\infty} (\phi_i^1(t))^* g_\mu(r_i(t), 0)\,\mathrm{d}t \quad \textit{for } i = 1, 2.$$

2. Bifurcation equations with single twisted orbit

We now study the case of a single twisted orbit which means that the following hypothesis is satisfied.

(H_5) The orbit Γ_1 is nontwisted and Γ_2 is twisted, that is, $\omega_1^{12} > 0$ and $\omega_2^{12} < 0$.

Consider the map $P_1^0 :\ S_1^0 \to S_2^1,\ q_1^0 \longmapsto q_2^1$, $\overline{P}_2^0 :\ S_2^0 \to S_2^1,\ \bar{q}_2^0 \longmapsto \bar{q}_2^1$ and $P_2^0 :\ S_2^0 \to S_1^1,\ q_2^0 \longmapsto q_1^1$ induced by the flow of (1.3) in the neighbourhood U of $z = 0$. Set the flying time from q_1^0 to q_2^1 as τ_1, $\bar{q}_2^0$ to $\bar{q}_2^1$ as τ_2, q_2^0 to q_1^1 as τ_3 and the Shilnikov time $s_k = e^{-\lambda_1\tau_k},\ k = 1, 2, 3$ (see Figure IV.2). Then we have

$$\begin{aligned}
P_1^0 :\quad & q_1^0(x_1^0, y_1^0, u_1^0, v_1^0) \to q_2^1(x_2^1, y_2^1, u_2^1, v_2^1),\\
& x_1^0 = s_1x_2^1, \quad y_2^1 = s_1^{\rho_1/\lambda_1}y_1^0, \quad u_1^0 = s_1^{\lambda_2/\lambda_1}u_2^1, \quad v_2^1 = s_1^{\rho_2/\lambda_1}v_1^0,\\[1ex]
\overline{P}_2^0 :\quad & \bar{q}_2^0(\bar{x}_2^0, \bar{y}_2^0, \bar{u}_2^0, \bar{v}_2^0) \to \bar{q}_2^1(\bar{x}_2^1, \bar{y}_2^1, \bar{u}_2^1, \bar{v}_2^1),\\
& \bar{x}_2^0 = s_2\bar{x}_2^1, \quad \bar{y}_2^1 = s_2^{\rho_1/\lambda_1}\bar{y}_2^0, \quad \bar{u}_2^0 = s_2^{\lambda_2/\lambda_1}\bar{u}_2^1, \quad \bar{v}_2^1 = s_2^{\rho_2/\lambda_1}\bar{v}_2^0,\\[1ex]
P_2^0 :\quad & q_2^0(x_2^0, y_2^0, u_2^0, v_2^0) \to q_1^1(x_1^1, y_1^1, u_1^1, v_1^1),\\
& x_2^0 = s_3x_1^1, \quad y_1^1 = s_3^{\rho_1/\lambda_1}y_2^0, \quad u_2^0 = s_3^{\lambda_2/\lambda_1}u_1^1, \quad v_1^1 = s_3^{\rho_2/\lambda_1}v_2^0,
\end{aligned}$$

where for $k = 1, 2, 3$,

$$s_k^{\lambda_2/\lambda_1} = \mathrm{diag}(s_k^{\lambda_{21}/\lambda_1}, s_k^{\lambda_{22}/\lambda_1}, \cdots, s_k^{\lambda_{2m}/\lambda_1}),\ s_k^{\rho_2/\lambda_1} = \mathrm{diag}(s_k^{\rho_{21}/\lambda_1}, s_k^{\rho_{22}/\lambda_1}, \cdots, s_k^{\rho_{2n}/\lambda_1})$$

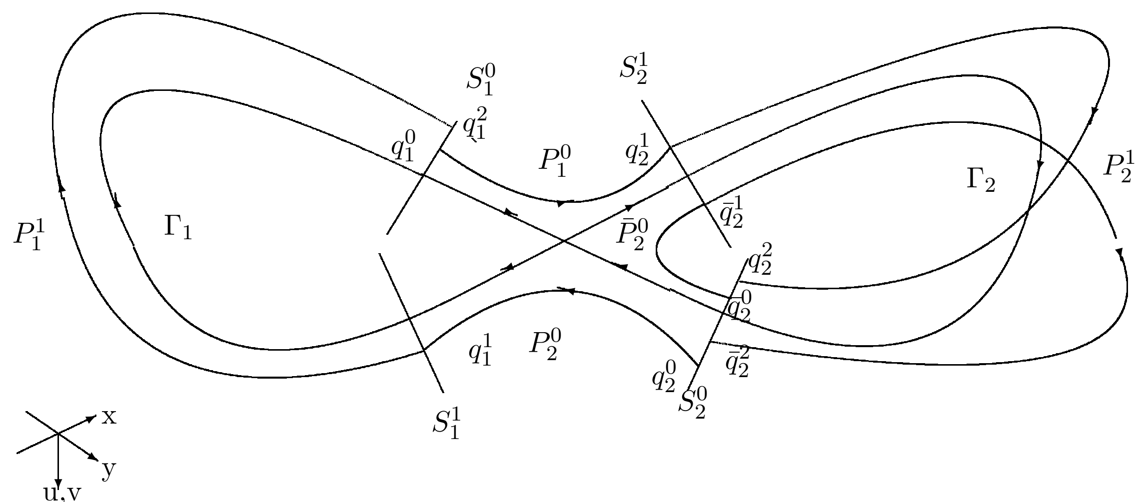


FIGURE IV.2. Poincaré map with single twisted orbit

are diagonal matrices of order m and n respectively.

To be more precise, let

$$S_{i+}^1 = \{q \in S_i^1 |\ y_q > 0\}, \quad S_{i-}^1 = \{q \in S_i^1 |\ y_q < 0\},$$
$$S_{i+}^0 = \{q \in S_i^0 |\ x_q > 0\}, \quad S_{i-}^0 = \{q \in S_i^0 |\ x_q < 0\}, \quad i = 1, 2.$$

Then,

$$P_1^0 :\ S_{1+}^0 \to S_{2-}^1, \qquad \overline{P}_1^0 :\ S_{1-}^0 \to S_{1-}^1,$$
$$P_2^0 :\ S_{2-}^0 \to S_{1+}^1, \qquad \overline{P}_2^0 :\ S_{2+}^0 \to S_{2+}^1.$$

Equipped with these formulae, we calculate the relations between

$$q_i^{2j}(x_i^{2j}, y_i^{2j}, u_i^{2j}, v_i^{2j}), \quad q_i^{2j+1}(x_i^{2j+1}, y_i^{2j+1}, u_i^{2j+1}, v_i^{2j+1}), \quad P_i^0(q_i^{2j}) = q_{i+1}^{2j+1}$$

and their new coordinates $N_i^{2j}(n_i^{2j,1}, 0, n_i^{2j,3}, n_i^{2j,4})$, $N_i^{2j+1}(n_i^{2j+1,1}, 0,\ n_i^{2j+1,3}, n_i^{2j+1,4})$ for $i = 1, 2$, where $q_3^1 = q_1^1$, and similar relations for $\bar{q}_2^{2j}$ and $\bar{q}_2^{2j+1} = \overline{P}_2^0(\bar{q}_2^{2j})$. Using (1.6) and according to the expressions of $Z_i(-T_i^1)$ and $Z_i(T_i^0)$, we obtain

$$\bar{n}_2^{2j,1} = \bar{x}_2^{2j} - \omega_2^{31}(\omega_2^{33})^{-1}\bar{u}_2^{2j}, \qquad \bar{n}_2^{2j,3} = (\omega_2^{33})^{-1}\bar{u}_2^{2j},$$
$$\bar{n}_2^{2j,4} = \bar{v}_2^{2j} - \delta_2^v - \bar{\omega}_2^{14}\bar{x}_2^{2j} + (\bar{\omega}_2^{14}\omega_2^{31} - \omega_2^{34})(\omega_2^{33})^{-1}\bar{u}_2^{2j},$$

$$\bar{n}_2^{2j+1,1} = (\omega_2^{12})^{-1}\bar{y}_i^{2j+1} - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1}\bar{v}_2^{2j+1},$$
$$\bar{n}_2^{2j+1,3} = \bar{u}_2^{2j+1} - \delta_2^u - \omega_2^{13}(\omega_2^{12})^{-1}\bar{y}_i^{2j+1} + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1}\bar{v}_2^{2j+1},$$
$$\bar{n}_2^{2j+1,4} = (\omega_2^{44})^{-1}\bar{v}_2^{2j+1},$$

$$n_i^{2j,1} = x_i^{2j} - \omega_i^{31}(\omega_i^{33})^{-1}u_i^{2j}, \qquad n_i^{2j,3} = (\omega_i^{33})^{-1}u_i^{2j},$$
$$n_i^{2j,4} = v_i^{2j} - \delta_i^v - \bar{\omega}_i^{14}x_i^{2j} + (\bar{\omega}_i^{14}\omega_i^{31} - \omega_i^{34})(\omega_i^{33})^{-1}u_i^{2j},$$

$$n_i^{2j+1,1} = (\omega_i^{12})^{-1}y_i^{2j+1} - (\omega_i^{12})^{-1}\omega_i^{42}(\omega_i^{44})^{-1}v_i^{2j+1},$$
$$n_i^{2j+1,3} = u_i^{2j+1} - \delta_i^u - \omega_i^{13}(\omega_i^{12})^{-1}y_i^{2j+1} + [\omega_i^{13}(\omega_i^{12})^{-1}\omega_i^{42} - \omega_i^{43}](\omega_i^{44})^{-1}v_i^{2j+1},$$
$$n_i^{2j+1,4} = (\omega_i^{44})^{-1}v_i^{2j+1},$$

and

$$x_i^{2j+1} \approx (-1)^i \delta,\ \bar{x}_2^{2j+1} \approx \delta, \qquad y_i^{2j} \approx (-1)^i \delta,\ \bar{y}_2^{2j} \approx \delta,\ i = 1, 2.$$

From the above, we obtain the following Poincaré maps:

$$\begin{aligned}
&F_1 = P_2^1 \circ P_1^0 :\ S_1^0 \to S_2^0,\\
&n_2^{2,1} = -(\omega_2^{12})^{-1} s_1^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1} s_1^{\rho_2/\lambda_1} v_1^0 + M_2^1\mu + o(||\mu||),\\
&n_2^{2,3} = u_2^1 - \delta_2^u + \omega_2^{13}(\omega_2^{12})^{-1} s_1^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1} s_1^{\rho_2/\lambda_1} v_1^0\\
&\qquad + M_2^3\mu + o(||\mu||),\\
&n_2^{2,4} = (\omega_2^{44})^{-1} s_1^{\rho_2/\lambda_1} v_1^0 + M_2^4\mu + o(||\mu||),
\end{aligned}$$

$$\begin{aligned}
&\overline{F}_2 = P_2^1 \circ \overline{P}_2^0 :\ S_2^0 \to S_2^0,\\
&\bar{n}_2^{2,1} = (\omega_2^{12})^{-1} s_2^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1} s_2^{\rho_2/\lambda_1} \bar{v}_2^0 + M_2^1\mu + o(||\mu||),\\
&\bar{n}_2^{2,3} = \bar{u}_2^1 - \delta_2^u - \omega_2^{13}(\omega_2^{12})^{-1} s_2^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1} s_2^{\rho_2/\lambda_1} \bar{v}_2^0\\
&\qquad + M_2^3\mu + o(||\mu||),\\
&\bar{n}_2^{2,4} = (\omega_2^{44})^{-1} s_2^{\rho_2/\lambda_1} \bar{v}_2^0 + M_2^4\mu + o(||\mu||),
\end{aligned}$$

$$\begin{aligned}
&F_3 = P_1^1 \circ P_2^0 :\ S_2^0 \to S_1^0,\\
&n_1^{2,1} = (\omega_1^{12})^{-1} s_3^{\rho_1/\lambda_1}\delta - (\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1} s_3^{\rho_2/\lambda_1} v_2^0 + M_1^1\mu + o(||\mu||),\\
&n_1^{2,3} = u_1^1 - \delta_1^u - \omega_1^{13}(\omega_1^{12})^{-1} s_3^{\rho_1/\lambda_1}\delta + [\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42} - \omega_1^{43}](\omega_1^{44})^{-1} s_3^{\rho_2/\lambda_1} v_2^0\\
&\qquad + M_1^3\mu + o(||\mu||),\\
&n_1^{2,4} = (\omega_1^{44})^{-1} s_3^{\rho_2/\lambda_1} v_2^0 + M_1^4\mu + o(||\mu||)
\end{aligned}$$

Now, the successor function

$$\begin{aligned}
G(s_1, s_2, s_3, u_1^1, \bar{u}_2^1, u_2^1, v_1^0, \bar{v}_2^0, v_2^0) &= (G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4, G_3^1, G_3^3, G_3^4)\\
&= (F_1(q_1^0) - \bar{q}_2^0, \overline{F}_2(\bar{q}_2^0) - q_2^0, F_3(q_2^0) - q_1^0)
\end{aligned}$$

is given by the following:

$$\begin{aligned}
G_1^1 &= -(\omega_2^{12})^{-1} s_1^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1} s_1^{\rho_2/\lambda_1} v_1^0 - s_2\delta\\
&\quad + \omega_2^{31}(\omega_2^{33})^{-1} s_2^{\lambda_2/\lambda_1}\bar{u}_2^1 + M_2^1\mu + o(||\mu||),\\
G_1^3 &= u_2^1 - \delta_2^u + \omega_2^{13}(\omega_2^{12})^{-1} s_1^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1} s_1^{\rho_2/\lambda_1} v_1^0\\
&\quad - (\omega_2^{33})^{-1} s_2^{\lambda_2/\lambda_1}\bar{u}_2^1 + M_2^3\mu + o(||\mu||),\\
G_1^4 &= -\bar{v}_2^0 + \delta_2^v + (\omega_2^{44})^{-1} s_1^{\rho_2/\lambda_1} v_1^0 + \bar{\omega}_2^{14}\delta s_2 - [\bar{\omega}_2^{14}\omega_2^{31} - \omega_2^{34}](\omega_2^{33})^{-1} s_2^{\lambda_2/\lambda_1}\bar{u}_2^1\\
&\quad + M_2^4\mu + o(||\mu||),\\
G_2^1 &= (\omega_2^{12})^{-1} s_2^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1} s_2^{\rho_2/\lambda_1}\bar{v}_2^0 + s_3\delta\\
&\quad + \omega_2^{31}(\omega_2^{33})^{-1} s_3^{\lambda_2/\lambda_1} u_1^1 + M_2^1\mu + o(||\mu||),\\
G_2^3 &= \bar{u}_2^1 - \delta_2^u - \omega_2^{13}(\omega_2^{12})^{-1} s_2^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1} s_2^{\rho_2/\lambda_1}\bar{v}_2^0\\
&\quad - (\omega_2^{33})^{-1} s_3^{\lambda_2/\lambda_1} u_1^1 + M_2^3\mu + o(||\mu||),\\
G_2^4 &= -v_2^0 + \delta_2^v + (\omega_2^{44})^{-1} s_2^{\rho_2/\lambda_1}\bar{v}_2^0 - \bar{\omega}_2^{14}\delta s_3 - [\bar{\omega}_2^{14}\omega_2^{31} - \omega_2^{34}](\omega_2^{33})^{-1} s_3^{\lambda_2/\lambda_1} u_1^1\\
&\quad + M_2^4\mu + o(||\mu||),
\end{aligned}$$

$$\begin{aligned}
G_3^1 &= (\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1}\delta-(\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0-s_1\delta\\
&\quad+\omega_1^{31}(\omega_1^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1+M_1^1\mu+o(||\mu||),\\
G_3^3 &= u_1^1-\delta_1^u-\omega_1^{13}(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1}\delta+[\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42}-\omega_1^{43}](\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0\\
&\quad-(\omega_1^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1+M_1^3\mu+o(||\mu||),\\
G_3^4 &= -v_1^0+\delta_1^v+(\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0+\bar\omega_1^{14}\delta s_1-[\bar\omega_1^{14}\omega_1^{31}-\omega_1^{34}](\omega_1^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1\\
&\quad+M_1^4\mu+o(||\mu||)
\end{aligned}$$

Therefore, there is a correspondence between the solution $Q=(s_1,s_2,s_3,u_1^1,\bar u_2^1,u_2^1,v_1^0,\bar v_2^0,v_2^0)$ of

$$(G_1^1,G_1^3,G_1^4,G_2^1,G_2^3,G_2^4,G_3^1,G_3^3,G_3^4)=0$$

with $s_1\ge 0, s_2\ge 0, s_3\ge 0$, and the existence of 1-1 double homoclinic loops, 2-1 double homoclinic loops, 2-1 right homoclinic loop, 1-1 large homoclinic loop, 2-1 large homoclinic loop and 2-1 large period orbit of system (1.1).

Solving $(u_2^1,\bar v_2^0,\bar u_2^1,v_2^0,u_1^1,v_1^0)$ from $(G_1^3,G_1^4,G_2^3,G_2^4,G_3^3,G_3^4)=0$ and substituting it into the equations $(G_1^1,G_2^1,G_3^1)=0$, we obtain the following bifurcation equations

$$\begin{aligned}
&-(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1}-s_2+\delta^{-1}M_2^1\mu+h.o.t.=0,\\
&(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1}+s_3+\delta^{-1}M_2^1\mu+h.o.t.=0,\\
&(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1}-s_1+\delta^{-1}M_1^1\mu+h.o.t.=0.
\end{aligned}\tag{2.8}$$

3. Bifurcation results with single twisted orbit

In this section, we study the existence, uniqueness and non-coexistence problem of *p-q* double homoclinic loops, *p-q* right homoclinic loop, *p-q* left homoclinic loop together with *p-q* large homoclinic loop and *p-q* large period orbit for a nontwisted orbit Γ_1 and a twisted Γ_2. Similarly, we can consider the corresponding problem for twisted Γ_1 and nontwisted Γ_2.

Firstly, we have the following result concerning the uniqueness and the non-coexistence.

THEOREM 3.1. *Assume that hypotheses* $(H_1)-(H_5)$ *hold. Then, for* $||\mu||$ *sufficiently small, system* (1.1) *has at most one 1-1 double homoclinic loop, or one 2-1 double homoclinic loop, or one 2-1 right homoclinic loop, or one 1-1 large homoclinic loop, or one 2-1 large homoclinic loop or one 2-1 large period orbit in the small neighbourhood of* Γ. *Moreover these orbits do not coexist.*

Proof. Let $Q=(s_1,s_2,s_3,u_1^1,\bar u_2^1,u_2^1,v_1^0,\bar v_2^0,v_2^0)$ then

$$W=\frac{\partial(G_3^1,G_1^1,G_2^1,G_3^3,G_1^3,G_2^3,G_3^4,G_1^4,G_2^4)}{\partial Q}\Big|_{Q=0,\ \mu=0}=\begin{pmatrix}W_{11}&0&0\\0&I_{3m}&0\\W_{13}&0&I_{3n}\end{pmatrix}$$

where I_k denotes the identity matrix of order k,

$$W_{11}=\mathrm{diag}(-\delta,-\delta,\delta),\quad W_{13}=\mathrm{diag}(\omega_1^{\bar 14}\delta,\omega_2^{\bar 14}\delta,-\omega_2^{\bar 14}\delta)$$

are diagonal matrices. Notice that $\det W=-\delta^3\ne 0$. According to the implict function theorem, in the neighbourhood of $(Q,\mu)=(0,0)$, there exists a unique solution $s_i=s_i(\mu)$, $u_i^1=u_i^1(\mu)$, $v_i^0=v_i^0(\mu)$, $\bar u_2^1=\bar u_2^1(\mu)$, $\bar v_2^0=\bar v_2^0(\mu)$ satisfying $s_i(0)=0$, $u_i^1(0)=0$, $v_i^0(0)=0$, $\bar u_2^1(0)=0$, $\bar v_2^0(0)=0$, for $i=1,2$.

Then, depending on the solutions s_i, one may have the following possibilities which have relations with the bifurcation problem.

If $s_1=s_2=0$, then necessarily $s_3=0$. By the uniqueness, we see that the double homoclinic loop is persistent and it is impossible to appear two different homoclinic loops near Γ_2 forming bellows configuration.

If $s_2 = s_3 = 0$ and $s_1 > 0$, then Γ_2 is persistent, and meanwhile system (1.1) has a unique 1-1 large homoclinic loop.

If $s_1 = s_3 = 0$ and $s_2 > 0$, then system (1.1) has a unique 2-1 double homoclinic loop.

If $s_1 = 0,\ s_2 > 0,\ s_3 > 0$ or $s_3 = 0,\ s_1 > 0,\ s_2 > 0$, then system (1.1) has a unique 2-1 large homoclinic loop.

If $s_2 = 0,\ s_1 > 0$ and $s_3 > 0$, then system (1.1) has a unique 2-1 right homoclinic loop.

If $s_1 > 0,\ s_2 > 0,\ s_3 > 0$, system (1.1) has a unique 2-1 large period orbit.

Clearly, the uniqueness guarantees that all these kinds of orbits do not coexist. □

REMARK 3.1. *If there exists any $p-q$ large homoclinic (or periodic) orbit for large p and q, then from (H_5), $p = 2q$ must be satisfied. However, due to the uniqueness of solution, $2q - q$ $(q > 1)$ large homoclinic orbit is impossible to exist, and all the* 2q-q $(q > 1)$ *large periodic orbits are in fact the* 2-1 *large periodic orbit.*

REMARK 3.2. *If $s_1 = s_2 = s_3 = 0$ is the solution of equation (2.8), then $G_1^j = G_2^j$, for $j = 1, 3, 4$, thus the first two equations of (2.8) are the same.*

In the sequel, we study the different bifurcation manifolds and their existence regions for the single twisted orbit case.

By substituting $s_1 = s_2 = s_3 = 0$ into the first two equations we obtain

$$M_2^1\mu + h.o.t. = 0.$$

If $M_2^1 \neq 0$, then this equation defines a manifold L_2 of codimension 1 with a normal vector M_2^1 at $\mu = 0$. One concludes that the first two equations of (2.8) have solution $s_1 = s_2 = s_3 = 0$ when $\mu \in L_2$ and $||\mu|| \ll 1$, which means that Γ_2 is persistent.

Similarly, there is a codimension 1 manifold L_1 defined by $M_1^1\mu + h.o.t. = 0$ with normal vector M_1^1 at $\mu = 0$ such that the third equation of (2.8) has solution $s_1 = s_2 = s_3 = 0$ as $\mu \in L_1$ and $||\mu|| \ll 1$. Therefore Γ_1 is persistent. Suppose $\text{rank}(M_1^1, M_2^1) = 2$, then $L_{12} = L_1 \cap L_2$ is a codimension 2 manifold with normal plane $\text{Span}\{M_1^1, M_2^1\}$ such that the double homoclinic orbit $\Gamma = \Gamma_1 \cup \Gamma_2$ is persistent for $\mu \in L_{12}$.

Substituting $s_2 = s_3 = 0$ into equations (2.8), we obtain

$$\begin{aligned} &-(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\ &\delta^{-1}M_2^1\mu + h.o.t. = 0,\\ &-s_1 + \delta^{-1}M_1^1\mu + h.o.t. = 0. \end{aligned}$$

Therefore we get $s_1 = \delta^{-1}M_1^1\mu + h.o.t.$ If $M_1^1\mu > 0$ then we have $s_1 > 0$. Substituting it into the first two equations, we obtain the codimension 2 bifurcation set H_{23}^1 such that a 1-1 large homoclinic loop bifurcates and Γ_2 persists. We have

$$\begin{aligned} H_{23}^1 :\quad &-(\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\ &\delta^{-1}M_2^1\mu + h.o.t. = 0, \end{aligned}$$

which is well defined at least in the region $\{\mu :\ M_1^1\mu > 0,\ M_2^1\mu < 0,\ M_1^1\mu = o(|\ M_2^1\mu\ |^{\lambda_1/\rho_1})\}$.

Similarly, if equation (2.8) has $s_1 = s_3 = 0,\ s_2 > 0$ as its solution, we need to have

$$\begin{aligned} &-s_2 + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\ &(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\ &\delta^{-1}M_1^1\mu + h.o.t. = 0. \end{aligned}$$

As the first equation induces $s_2 = \delta^{-1}M_2^1\mu + h.o.t.$, so we can get the bifurcation manifold for a 2-1 double homoclinic loop :

$$H_{13}^2 :\qquad \delta^{-1}M_2^1\mu + h.o.t. = 0, \qquad \delta^{-1}M_1^1\mu + h.o.t. = 0.$$

Accordingly, for $rank\{M_1^1, M_2^1\} = 2,\ \dim\mu = \ell > 2,$ and $0 < M_2^1\mu << 1$, we have $H_{13}^2 \cap \{\mu|\ s_2(\mu) > 0\} \neq \emptyset$, so there do exist 2-1 double homoclinic orbits with these conditions. If not, there exist no 2-1 double homoclinic orbits.

PROPOSITION 3.1. *There exist no p-q large homoclinic loop for any $p \geq 2,\ q \geq 1$.*

Proof. If equation (2.8) has a solution with $s_1 = 0,\ s_2 > 0, s_3 > 0.$ one has

$$\begin{aligned}&-s_2 + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\&(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1} + s_3 + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\&(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1} + \delta^{-1}M_1^1\mu + h.o.t. = 0,\end{aligned}$$

which implies that

$$s_2 = \delta^{-1}M_2^1\mu + h.o.t., \quad s_3 = -\delta^{-1}M_2^1\mu + h.o.t.$$

Hence, it is impossible to have 2-1 large homoclinic loop bifurcation for system (1.1). By using Remark (3.1), our proof is completed. □

If $s_2 = 0,\ s_1 > 0, s_3 > 0$ is the solution of (2.8), we obtain

$$\begin{aligned}&-(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\&s_3 + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\&(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1} - s_1 + \delta^{-1}M_1^1\mu + h.o.t. = 0.\end{aligned}$$

Thus,

$$s_1 = (\omega_2^{12}\delta^{-1}M_2^1\mu)^{\lambda_1/\rho_1} + h.o.t., \qquad s_3 = -\delta^{-1}M_2^1\mu + h.o.t..$$

So the codimension 1 bifurcation manifold for 2-1 right homoclinic loop is

$$H_2^{13} = \{\mu \mid -(\omega_2^{12}\delta^{-1}M_2^1\mu)^{\lambda_1/\rho_1} + \delta^{-1}M_1^1\mu + h.o.t. = 0\}$$

which is well defined at least in the region $\{\mu :\ M_1^1\mu > 0,\ M_2^1\mu < 0\}$ and has normal vector M_2^1 at $\mu = 0$.

Let μ be situated in the neighborhood of H_2^{13}, differentiating equation (2.8), take values at H_2^{13}, and denoting by $s_{i\mu}$ the gradient of $s_i(\mu)$ with respect to μ, we get

$$\begin{aligned}&-(\omega_2^{12})^{-1}\rho_1(\omega_2^{12}\delta^{-1}M_2^1\mu)^{(\rho_1-\lambda_1)/\rho_1}s_{1\mu} - \lambda_1 s_{2\mu} + \lambda_1\delta^{-1}M_2^1 + h.o.t. = 0,\\&s_{3\mu} + \delta^{-1}M_2^1 + h.o.t. = 0,\\&(\omega_1^{12})^{-1}\rho_1(-\delta^{-1}M_2^1\mu)^{(\rho_1-\lambda_1)/\lambda_1}s_{3\mu} - \lambda_1 s_{1\mu} + \lambda_1\delta^{-1}M_1^1 + h.o.t. = 0.\end{aligned}$$

Accordingly, we have $s_{2\mu} = \delta^{-1}M_2^1 + O(|\ \omega_2^{12}\delta^{-1}M_2^1\mu\ |^{(\rho_1-\lambda_1)/\rho_1})$. Therefore, $s_2 = s_2(\mu)$ increases along the direction M_2^1 in a small neighborhood of H_2^{13}.

Suppose $s_3 = 0,\ s_1 > 0, s_2 > 0$ is the solution of (2.8), then one has

$$\begin{aligned}&-(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1} - s_2 + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\&(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\&-s_1 + \delta^{-1}M_1^1\mu + h.o.t. = 0.\end{aligned}$$

Hereafter,

$$s_1 = \delta^{-1}M_1^1\mu + h.o.t., \qquad s_2 = (-\omega_2^{12}\delta^{-1}M_2^1\mu)^{\lambda_1/\rho_1} + h.o.t.,$$

and the codimension one 2-1 large homoclinic loop bifurcation manifold is

$$H_3^{12} = \{\mu \mid -(\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1} - (-\omega_2^{12}\delta^{-1}M_2^1\mu)^{\lambda_1/\rho_1} + h.o.t. = 0\}$$

with normal vector M_2^1(*resp.* M_1^1) at $\mu = 0$ as $M_2^1 \neq 0$(*resp.* $M_2^1 = 0$), which is well defined at least in the region $\{\mu \mid M_1^1\mu > 0, M_2^1\mu > 0\}$.

When $\mu \in H_3^{12}$, based on (2.8) we get

$$\begin{aligned}
&-(\omega_2^{12})^{-1}\rho_1(\delta^{-1}M_1^1\mu)^{(\rho_1-\lambda_1)/\lambda_1}s_{1\mu} - \lambda_1 s_{2\mu} + \lambda_1\delta^{-1}M_2^1 + h.o.t. = 0,\\
&(\omega_2^{12})^{-1}\rho_1[(-\omega_2^{12}\delta^{-1}M_2^1\mu)]^{(\rho_1-\lambda_1)/\rho_1}s_{2\mu} + \lambda_1 s_{3\mu} + \lambda_1\delta^{-1}M_2^1 + h.o.t. = 0,\\
&-s_{1\mu} + \delta^{-1}M_1^1 + h.o.t. = 0.
\end{aligned}$$

Then we have $s_{3\mu} = -\delta^{-1}M_2^1 + O(\mid \omega_2^{12}\delta^{-1}M_2^1\mu \mid^{(\rho_1-\lambda_1)/\rho_1})$ such that $s_3 = s_3(\mu)$ increases along the direction of the gradient $-M_2^1$ in a small neighborhood of H_3^{12}.

Now, we study the 2-1 large period orbit bifurcation and its existence regions.

Due to $(2.8)_1 - (2.8)_2$, we get

$$s_2 + s_3 = -(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1} + h.o.t.,$$

so $s_3 = o(s_1)$. Because of this, owing to $(2.8)_3$, we have

$$s_1 = \delta^{-1}M_1^1\mu + h.o.t.,$$

meanwhile $(2.8)_1$, $(2.8)_2$ produce

$$\begin{aligned}
s_2 &= \delta^{-1}M_2^1\mu - (\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1} + h.o.t.,\\
s_3 &= -\delta^{-1}M_2^1\mu - (\omega_2^{12})^{-1}(\delta^{-1}M_2^1\mu - (\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1})^{\rho_1/\lambda_1} + h.o.t.
\end{aligned}$$

From the former lines we deduce that for μ sitting on the set H^{123} defined by

$$\{\mu \mid M_1^1\mu > 0 \ \text{ and } \ (3.9) \text{ is verified}\}$$

2-1 large period orbit persists, where

$$(\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1} < \delta^{-1}\ M_2^1\mu < -(\omega_2^{12})^{-1}(\delta^{-1}M_2^1\mu - (\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1})^{\rho_1/\lambda_1}, \tag{3.9}$$

and it is nonempty when rank$\{M_1^1,\ M_2^1\} = 2$.

With the above analysis, we state the following result:

THEOREM 3.2. *Suppose that $(H_1) - (H_5)$ are fulfilled, then we have the following.*
(1) *If $M_1^1 \neq 0$, there exists a unique manifold L_1 with codimension 1 and normal vector M_1^1 at $\mu = 0$, such that system* (1.1) *has a homoclinic loop near Γ_1 if and only if $\mu \in L_1$ and $||\mu|| \ll 1$.*

If $M_2^1 \neq 0$, there exists a unique manifold L_2 with codimension 1 and normal vector M_2^1 at $\mu = 0$, such that system (1.1) *has a homoclinic loop near Γ_2.*

If rank$(M_1^1, M_2^1) = 2$, then $L_{12} = L_1 \cap L_2$ is a codimension 2 manifold and $0 \in L_{12}$ such that system (1.1) *has an 1-1 double homoclinic loop near Γ as $\mu \in L_{12}$ and $||\mu|| \ll 1$, namely, Γ is persistent.*
(2) *In the region defined by $\{\mu :\ M_1^1\mu > 0, M_2^1\mu < 0,\ M_1^1\mu = o(\mid M_2^1\mu \mid^{\lambda_1/\rho_1})\}$, there exists a unique bifurcation set H_{23}^1 which is tangent to L_2 such that system (1.1) has one 1-1 large homoclinic loop and Γ_2 persists as $\mu \in H_{23}^1$.*

In the region defined by $\{\mu :\ 0 < M_2^1\mu \ll 1\}$, there do exist a unique codimension 2 bifurcation manifold H_{13}^2 which is tangent to $L_1 \cup L_2$ at $\mu = 0$ with the normal plane span$\{M_1^1, M_2^1\}$ when rank$\{M_1^1, M_2^1\} = 2$, and for $\mu \in H_{13}^2$, system (1.1) *has a unique 2-1 double homoclinic loop near Γ.*

In the region defined by $\{\mu :\ M_1^1\mu > 0, M_2^1\mu < 0\}$, there exists a unique codimension 1 bifurcation set H_2^{13} with normal vector M_2^1 (resp. M_1^1) at $\mu = 0$ as $M_2^1 \neq 0$ (resp. $M_2^1 = 0, M_1^1 \neq 0$) such that for $\mu \in H_2^{13}$, system (1.1) *has a unique 2-1 right homoclinic loop near Γ.*

In the region defined by $\{\mu : M_1^1\mu > 0, M_2^1\mu > 0\}$, *there exists a unique 2-1 large homoclinic loop bifurcation manifold* H_3^{12} *of codimension 1 with normal vector* M_2^1 *(resp.* M_1^1*) at* $\mu = 0$ *as* $M_2^1 \neq 0$ *(resp.* $M_2^1 = 0, M_1^1 \neq 0$*) such that for* $\mu \in H_3^{12}$, *system* (1.1) *has a unique 2-1 large homoclinic loop near* Γ.

(1.3) *When* μ *belongs to the region*

$$H^{123} = \{\mu \mid M_1^1\mu > 0 \ \ \textit{and} \ \ (3.9) \ \textit{is verified}\}$$

which is bounded by H_2^{13} *and* H_3^{12}, *system (1.1) has a unique 2-1 large period orbit, and when* μ *is situated in the region* $\{\mu \mid M_1^1\mu \leq 0\} \bigcup \{\mu \mid M_2^1\mu \leq (\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1}\} \bigcup \{\mu \mid M_2^1\mu \geq -(\omega_2^{12})^{-1}(\delta^{-1}M_2^1\mu - (\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1})^{\rho_1/\lambda_1}\}$, *system* (1.1) *has no large period orbit. (see Figure IV.3)*

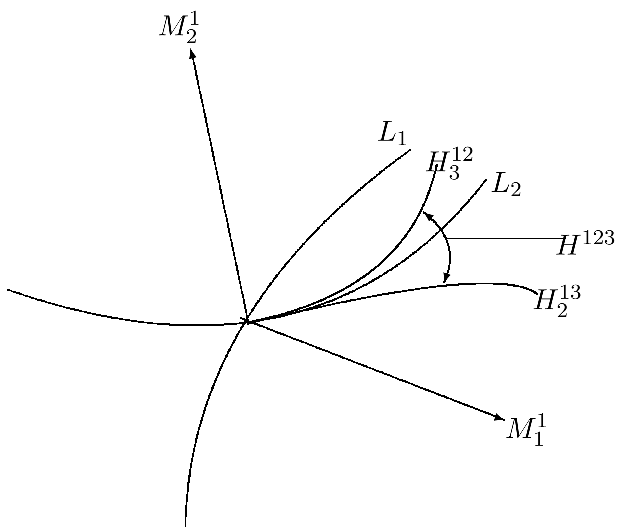


FIGURE IV.3. Bifurcation diagram in single twisted case as $rank(M_1^1, M_2^1) = 2$

4. Bifurcation with double twisted orbits

We now study the bifurcation problem of double twisted orbits, which means that the following hypothesis is verified.

(H_6) Suppose that both Γ_1 and Γ_2 are twisted, that is, $\omega_1^{12} < 0$ and $\omega_2^{12} < 0$.

Let $P_1^0, \overline{P}_2^0, P_2^0$ be the same as in 2 and let $\overline{P}_1^0 : \ S_1^0 \to S_1^1$ (see Figure IV.4) be given by

$$\overline{P}_1^0 : \ \bar{q}_1^0(\bar{x}_1^0, \bar{y}_1^0, \bar{u}_1^0, \bar{v}_1^0) \to \bar{q}_1^1(\bar{x}_1^1, \bar{y}_1^1, \bar{u}_1^1, \bar{v}_1^1),$$
$$\bar{x}_1^0 = s_4\bar{x}_1^1, \quad \bar{y}_1^1 = s_4^{\rho_1/\lambda_1}\bar{y}_1^0, \quad \bar{u}_1^0 = s_4^{\lambda_2/\lambda_1}\bar{u}_1^1, \quad \bar{v}_1^1 = s_4^{\rho_2/\lambda_1}\bar{v}_1^0,$$

where $s_4 = e^{-\lambda_1\tau_4}$ and τ_4 is the flying time from $\bar{q}_1^0$ to $\bar{q}_1^1$, $\bar{x}_1^1 \approx -\delta$, $\bar{y}_1^0 \approx -\delta$. Like above, we have

$$\bar{n}_1^{2j,1} = \bar{x}_1^{2j} - \omega_1^{31}(\omega_1^{33})^{-1}\bar{u}_1^{2j},$$
$$\bar{n}_1^{2j,3} = (\omega_1^{33})^{-1}\bar{u}_1^{2j},$$
$$\bar{n}_1^{2j,4} = \bar{v}_1^{2j} - \delta_1^v - \bar{\omega}_1^{14}\bar{x}_1^{2j} + (\bar{\omega}_1^{14}\omega_1^{31} - \omega_1^{34})(\omega_1^{33})^{-1}\bar{u}_1^{2j},$$

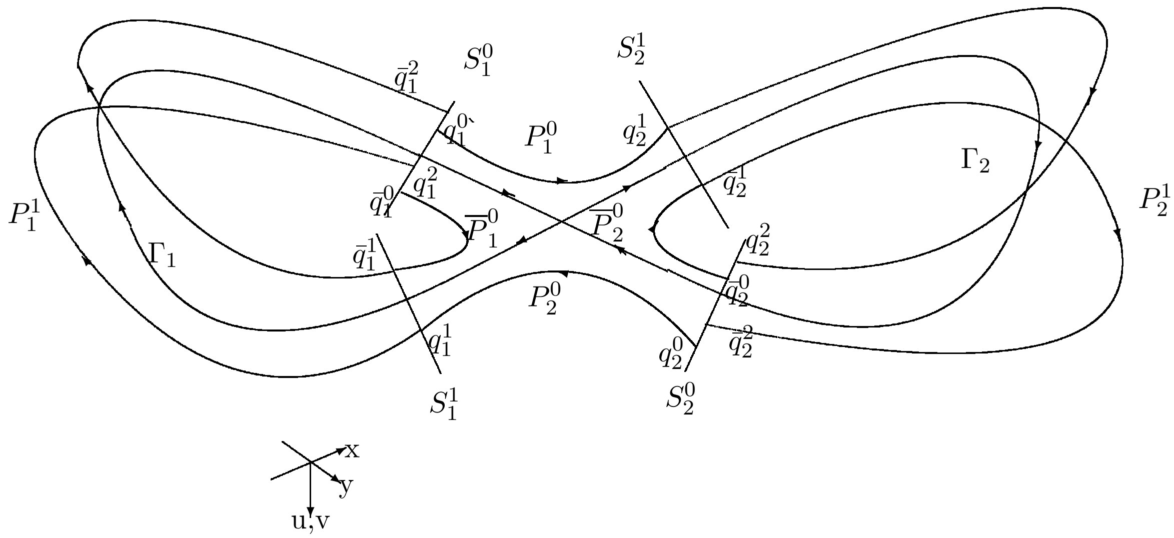


FIGURE IV.4. Poincaré map with double twisted orbits

$$\begin{aligned}
\bar{n}_1^{2j+1,1} &= (\omega_1^{12})^{-1}\bar{y}_1^{2j+1} - (\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1}\bar{v}_1^{2j+1},\\
\bar{n}_1^{2j+1,3} &= \bar{u}_1^{2j+1} - \delta_1^u - \omega_1^{13}(\omega_1^{12})^{-1}\bar{y}_1^{2j+1} + [\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42} - \omega_1^{43}](\omega_1^{44})^{-1}\bar{v}_1^{2j+1},\\
\bar{n}_1^{2j+1,4} &= (\omega_1^{44})^{-1}\bar{v}_1^{2j+1},
\end{aligned}$$

and

$$\begin{aligned}
&F_4 = P_1^1 \circ \overline{P}_1^0 :\ S_1^0 \to S_1^0,\\
&\bar{n}_1^{2,1} = -(\omega_1^{12})^{-1}s_4^{\rho_1/\lambda_1}\delta - (\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1}s_4^{\rho_2/\lambda_1}\bar{v}_1^0 + M_1^1\mu + o(||\mu||),\\
&\bar{n}_1^{2,3} = \bar{u}_1^1 - \delta_1^u + \omega_1^{13}(\omega_1^{12})^{-1}s_4^{\rho_1/\lambda_1}\delta + [\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42} - \omega_1^{43}](\omega_1^{44})^{-1}s_4^{\rho_2/\lambda_1}\bar{v}_1^0\\
&\qquad\quad + M_1^3\mu + o(||\mu||),\\
&\bar{n}_1^{2,4} = (\omega_1^{44})^{-1}s_4^{\rho_2/\lambda_1}\bar{v}_1^0 + M_1^4\mu + o(||\mu||).
\end{aligned}$$

Up to now, the successor function is given by

$$\begin{aligned}
&G(s_1, s_2, s_3, s_4, u_1^1, \bar{u}_2^1, u_2^1, \bar{u}_1^1, v_1^0, \bar{v}_2^0, v_2^0, \bar{v}_1^0)\\
&\quad = (G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4, G_3^1, G_3^3, G_3^4, G_4^1, G_4^3, G_4^4)\\
&\quad = (F_1(q_1^0) - \bar{q}_2^0, F_2(\bar{q}_2^0) - q_2^0, F_3(q_2^0) - \bar{q}_1^0, F_4(\bar{q}_1^0) - q_1^0)
\end{aligned}$$

where

$$\begin{aligned}
G_1^1 &= -(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 - s_2\delta \\
&\quad +\omega_2^{31}(\omega_2^{33})^{-1}s_2^{\lambda_2/\lambda_1}\bar{u}_2^1 + M_2^1\mu + o(||\mu||), \\
G_1^3 &= u_2^1 - \delta_2^u + \omega_2^{13}(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 \\
&\quad -(\omega_2^{33})^{-1}s_2^{\lambda_2/\lambda_1}\bar{u}_2^1 + M_2^3\mu + o(||\mu||), \\
G_1^4 &= -\bar{v}_2^0 + \delta_2^v + (\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 + \bar{\omega}_2^{14}\delta s_2 - [\bar{\omega}_1^{14}\omega_2^{31} - \omega_2^{34}](\omega_2^{33})^{-1}s_2^{\lambda_2/\lambda_1}\bar{u}_2^1 \\
&\quad +M_2^4\mu + o(||\mu||), \\
G_2^1 &= (\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1}s_2^{\rho_2/\lambda_1}\bar{v}_2^0 + s_3\delta \\
&\quad +\omega_2^{31}(\omega_2^{33})^{-1}s_3^{\lambda_2/\lambda_1}u_1^1 + M_2^1\mu + o(||\mu||), \\
G_2^3 &= \bar{u}_2^1 - \delta_2^u - \omega_2^{13}(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1}s_2^{\rho_2/\lambda_1}\bar{v}_2^0 \\
&\quad -(\omega_2^{33})^{-1}s_3^{\lambda_2/\lambda_1}u_1^1 + M_2^3\mu + o(||\mu||), \\
G_2^4 &= -v_2^0 + \delta_2^v + (\omega_2^{44})^{-1}s_2^{\rho_2/\lambda_1}v_2^-\bar{\omega}_2^{14}\delta s_3 - [\bar{\omega}_1^{14}\omega_2^{31} - \omega_2^{34}](\omega_2^{33})^{-1}s_3^{\lambda_2/\lambda_1}u_1^1 \\
&\quad +M_2^4\mu + o(||\mu||),
\end{aligned}$$

$$\begin{aligned}
G_3^1 &= (\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1}\delta - (\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 + s_4\delta \\
&\quad +\omega_1^{31}(\omega_1^{33})^{-1}s_4^{\lambda_2/\lambda_1}\bar{u}_1^1 + M_1^1\mu + o(||\mu||), \\
G_3^3 &= u_1^1 - \delta_1^u + \omega_1^{13}(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1}\delta + [\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42} - \omega_1^{43}](\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 \\
&\quad -(\omega_1^{33})^{-1}s_4^{\lambda_2/\lambda_1}\bar{u}_1^1 + M_1^3\mu + o(||\mu||), \\
G_3^4 &= -\bar{v}_1^0 + \delta_1^v + (\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 - \bar{\omega}_1^{14}\delta s_4 - [\bar{\omega}_1^{14}\omega_1^{31} - \omega_1^{34}](\omega_1^{33})^{-1}s_4^{\lambda_2/\lambda_1}\bar{u}_1^1 \\
&\quad +M_1^4\mu + o(||\mu||),
\end{aligned}$$

$$\begin{aligned}
G_4^1 &= -(\omega_1^{12})^{-1}s_4^{\rho_1/\lambda_1}\delta - (\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1}s_4^{\rho_2/\lambda_1}\bar{v}_1^0 - s_1\delta \\
&\quad +\omega_1^{31}(\omega_1 x^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1 + M_1^1\mu + o(||\mu||), \\
G_4^3 &= \bar{u}_1^1 - \delta_1^u + \omega_1^{13}(\omega_1^{12})^{-1}s_4^{\rho_1/\lambda_1}\delta + [\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42} - \omega_1^{43}](\omega_1^{44})^{-1}s_4^{\rho_2/\lambda_1}\bar{v}_1^0 \\
&\quad -(\omega_1^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1 + M_1^3\mu + o(||\mu||), \\
G_4^4 &= -v_1^0 + \delta_1^v + (\omega_1^{44})^{-1}s_4^{\rho_2/\lambda_1}\bar{v}_1^0 + \bar{\omega}_2^{14}\delta s_1 - [\bar{\omega}_1^{14}\omega_1^{31} - \omega_1^{34}](\omega_1^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1 \\
&\quad +M_1^4\mu + o(||\mu||).
\end{aligned}$$

Thereafter, there is a correspondence between the solution $Q = (s_1, s_2, s_3, s_4, u_1^1, \bar{u}_2^1, u_2^1, \bar{u}_1^1, v_1^0, \bar{v}_2^0, v_2^0, \bar{v}_1^0)$ of

$$(G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4, G_3^1, G_3^3, G_3^4, G_4^1, G_4^3, G_4^4) = 0$$

with $s_1 \geq 0, s_2 \geq 0, s_3 \geq 0, s_4 \geq 0$, and the existence of 1-1 double homoclinic loop, 1-2 double homoclinic loop, 2-1 double homoclinic loop, 2-2 double homoclinic loop, 2-1 large homoclinic loop, 1-2 large homoclinic loop, 2-2 large homoclinic loop, 2-2 right homoclinic loop, 2-2 large homoclinic loop, 2-2 left homoclinic loop and 2-2 large period orbit of system (1.1).

From equation $(G_1^3, G_1^4, G_2^3, G_2^4, G_3^3, G_3^4, G_4^3, G_4^4) = 0$, we can solve $(u_2^1, \bar{v}_2^0, \bar{u}_2^1, v_2^0, u_1^1, \bar{v}_1^0, \bar{u}_1^1, v_1^0)$ as in the former section. Substituting it into $(G_1^1, G_2^1, G_3^1, G_4^1) = 0$, we obtain the bifurcation equations

$$\begin{aligned}
&-(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1} - s_2 + \delta^{-1}M_2^1\mu + h.o.t. = 0, \\
&(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1} + s_3 + \delta^{-1}M_2^1\mu + h.o.t. = 0, \\
&(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1} + s_4 + \delta^{-1}M_1^1\mu + h.o.t. = 0, \\
&-(\omega_1^{12})^{-1}s_4^{\rho_1/\lambda_1} - s_1 + \delta^{-1}M_1^1\mu + h.o.t. = 0.
\end{aligned} \tag{4.10}$$

As a first step, let us consider the 2-2 bifurcations results with double twisted orbits. We shall study the existence, uniqueness and non-coexistence problem of the p-q double homoclinic loops, p-q large homoclinic loop, p-q left (right) homoclinic loop, p-q large period orbit for the double twisted homoclinic orbits Γ .

First, let us give the following result concerning the uniqueness and the non-coexistence.

THEOREM 4.1. *Assume that* $(H_1)-(H_4)$ *and* (H_6) *hold. Then, for* $||\mu||$ *sufficient small, system* (1.1) *has at most one 1-1 double homoclinic loops, one 1-2 double homoclinic loops, one 2-1 double homoclinic loops, one 2-2 double homoclinic loops, one 2-1 large homoclinic loop, one 1-2 large homoclinic loop, one 2-2 large homoclinic loop, one 2-2 right homoclinic loop, one one 2-2 left homoclinic loop or one 2-2 large period orbit in the small neighborhood of* Γ. *Moreover these orbits do not coexist.*

Proof. Let $Q=(s_1,s_2,s_3,s_4,u_1^1,\bar{u}_2^1,u_2^1,\bar{u}_1^1,v_1^0,\bar{v}_2^0,v_2^0,\bar{v}_1^0)$ and

$$W=\frac{\partial(G_4^1,G_1^1,G_2^1,G_3^1,G_4^3,G_1^3,G_2^3,G_3^3,G_4^4,G_1^4,G_2^4,G_3^4)}{\partial(s_1,s_2,s_3,s_4,u_1^1,\bar{u}_2^1,u_2^1,\bar{u}_1^1,v_1^0,\bar{v}_2^0,v_2^0,\bar{v}_1^0)}\,|_{Q=0,\ \mu=0},$$

then $\det W=\delta^4\neq 0$. Due to the implicit function theorem, in the neighbourhood of $(Q,\mu)=(0,0)$, there exists a unique solution $s_i=s_i(\mu),\ u_i^1=u_i^1(\mu),\ v_i^0=v_i^0(\mu),\ \ \bar{u}_i^1=\bar{u}_i^1(\mu),\ \bar{v}_i^0=\bar{v}_i^0(\mu)$ satisfying $s_i(0)=0,\ u_i^1(0)=0,\ v_i^0(0)=0,\ \bar{u}_i^1(0)=0,\ \bar{v}_i^0(0)=0,\ i=1,2$.

It indicates that, if $s_1=s_2=s_3=s_4=0$, system (1.1) has a unique 1-1 double homoclinic loops, that is to say, the double homoclinic loop Γ persists.

If $s_1=s_2=s_3=0,\ s_4>0$, then there exists a unique 1-2 double homoclinic loops, *i.e.* Γ_1 becomes a 2-homoclinic orbit and Γ_2 persists.

If $s_1=s_3=s_4=0,\ s_2>0$, then there exists a unique 2-1 double homoclinic loops, namely, Γ_2 becomes a 2-homoclinic orbit and Γ_1 persists.

If $s_1=s_3=0,\ s_2>0,s_4>0$, system (1.1) has a unique 2-2 double homoclinic loop.

If $s_1=s_4=0,\ s_2>0,s_3>0$, then Γ_1 is persistent, and meanwhile system (1.1) has a unique 2-1 large homoclinic loop.

If $s_2=s_3=0,\ s_1>0,s_4>0$, then Γ_2 is persistent, and meanwhile system (1.1) has a unique 1-2 large homoclinic loop.

If $s_1=0,\ s_2>0,s_3>0,s_4>0$, there exists a unique 2-2 large homoclinic loop.

If $s_2=0,\ s_1>0,s_3>0,s_4>0$, system (1.1) has a unique 2-2 right homoclinic loop.

If $s_3=0,\ s_1>0,s_2>0,s_4>0$, there exists a unique 2-2 large homoclinic loop.

If $s_4=0,\ s_1>0,s_2>0,s_3>0$, system (1.1) has a unique 2-2 left homoclinic loop.

If $s_1>0,\ s_2>0,\ s_3>0,s_4>0$, system (1.1) has a unique one 2-2 large period orbit.

Clearly, the uniqueness guarantees that all these kinds of orbits do not coexist. And all other cases are impossible based on the definition of the Poincaré map. □

We now study the bifurcation problem for the double twisted orbits case. It can be remarked that if $s_1=s_2=s_3=0$ ($s_1=s_3=s_4=0$) is the solution of equation (4.10), then $G_1^j=G_2^j$ ($G_3^j=G_4^j$) for $j=1,3,4$, thus the first (or last) two equations of (4.10) are the same one.

By the same reason as in 3, if $s_1=s_2=s_3=s_4=0$ is the solution of the first (or second) equation of (4.10), then we have $M_2^1\mu+h.o.t.=0$. In the case of $M_2^1\neq 0$, there exists a codimension 1 manifold L_2 with a normal vector M_2^1 at $\mu=0$, such that the first two equations of (4.10) have solution $s_1=s_2=s_3=s_4=0$ as $\mu\in L_2$ and $||\mu||\ll 1$, that is, Γ_2 is persistent. Similarly, there is a codimension 1 manifold L_1 defined by $M_1^1\mu+h.o.t.=0$ with normal vector M_1^1 at $\mu=0$ when $M_1^1\neq 0$ such that the third and the fourth equations of (4.10) have solution $s_1=s_2=s_3=s_4=0$ as $\mu\in L_1$ and $||\mu||\ll 1$, which indicates that Γ_1 is persistent. Suppose $rank(M_1^1,M_2^1)=2$, then $L_{12}=L_1\cap L_2$ is a codimension 2 manifold with normal plane $span\{M_1^1,M_2^1\}$ such that (4.10) has solution $s_1=s_2=s_3=s_4=0$ as $\mu\in L_{12}$ and $||\mu||\ll 1$, namely, the double homoclinic orbit $\Gamma=\Gamma_1\cup\Gamma_2$ is persistent.

Suppose $s_1 = s_2 = s_3 = 0,\ s_4 > 0$ is the solution of (4.10). We have $s_4 = -\delta^{-1}M_1^1\mu + h.o.t.$ for $M_1^1\mu < 0$. Substituting it into the last equation, we obtain the codimension 2 bifurcation set

$$H_{123}^4 : \quad M_2^1\mu + h.o.t. = 0, \qquad M_1^1\mu + h.o.t. = 0,$$

which is well defined at least in the region $\{\mu : \ M_1^1\mu < 0\}$ with normal plane $span\{M_1^1, M_2^1\}$ at $\mu = 0$ when $rank\{M_1^1, M_2^1\} = 2$ such that a unique 1-2 double homoclinic loop bifurcates from Γ for $\mu \in H_{123}^4$. That is, Γ_2 persists, while Γ_1 becomes a 2-homoclinic orbit.

Similarly, we get the bifurcation set

$$H_{134}^2 : \quad M_2^1\mu + h.o.t. = 0, \qquad M_1^1\mu + h.o.t. = 0,$$

such that (4.10) has solution $s_1 = s_3 = s_4 = 0,\ s_2 > 0$ as $\mu \in H_{134}^2$, that is, system (1.1) has a 2-1 double homoclinic loop near Γ. Clearly, H_{134}^2 which is well defined at least in the region $\{\mu : \ M_2^1\mu > 0\}$ when $rank\{M_1^1, M_2^1\} = 2$, has codimension 2 and normal plane $span\{M_1^1, M_2^1\}$ at $\mu = 0$.

If (4.10) has $s_1 = s_2 = s_4 = 0,\ s_3 > 0$ as its solution, then $s_3 = -\delta^{-1}M_2^1\mu + h.o.t.$. Hence, the bifurcation set

$$\begin{aligned} H_{124}^3 : \quad & M_2^1\mu + h.o.t. = 0, \qquad M_1^1\mu + h.o.t. = 0, \\ & (\omega_1^{12})^{-1}(-\delta^{-1}M_2^1\mu)^{\rho_1/\lambda_1} + \delta^{-1}M_1^1\mu + h.o.t. = 0, \end{aligned}$$

where Γ persists and an 1-1 large homoclinic orbit bifurcates near Γ, is well defined at least in the region $\{\mu : \ M_1^1\mu > 0,\ M_2^1\mu < 0\}$. When $rank\{M_1^1, M_2^1\} = 2$, it has codimension no less than 2.

Similarly, another bifurcation set

$$\begin{aligned} H_{234}^1 : \quad & M_2^1\mu + h.o.t. = 0, \qquad M_1^1\mu + h.o.t. = 0, \\ & -(\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\mu + h.o.t. = 0, \end{aligned}$$

such that Γ persists and an 1-1 large homoclinic orbit bifurcates near Γ for $\mu \in H_{2,3,4}^1$ is well defined at least in the region $\{\mu : \ M_1^1\mu > 0,\ M_2^1\mu < 0\}$. It has codimension no less than 2 as $\mathrm{rank}\{M_1^1, M_2^1\} = 2$.

Suppose $s_1 = s_3 = 0,\ s_2 > 0, s_4 > 0$ is the solution of (4.10). Consequently, we have $s_2 = \delta^{-1}M_2^1\mu + h.o.t.,\ s_4 = -\delta^{-1}M_1^1\mu + h.o.t.$. Substituting it into the second and fourth equation, the 2-2 double homoclinic loop bifurcation set $H_{13}^{24} : M_1^1\mu + h.o.t. = 0,\ M_2^1\mu + h.o.t. = 0$ is obtained, which is well defined at least in the region $\{\mu : \ M_1^1\mu < 0,\ M_2^1\mu > 0\}$ as $rank\{M_1^1, M_2^1\} = 2$. It is of codimension 2 and has normal plane $span\{M_1^1, M_2^1\}$ at $\mu = 0$.

When $\mu \in H_{13}^{24}$, system (1.1) has a unique 2-2 double homoclinic loops near Γ.

Using the same reasoning, we can obtain the bifurcation set

$$\begin{aligned} H_{24}^{13} : \quad & -(\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\mu + h.o.t. = 0, \\ & (\omega_1^{12})^{-1}(-\delta^{-1}M_2^1\mu)^{\rho_1/\lambda_1} + \delta^{-1}M_1^1\mu + h.o.t. = 0, \end{aligned}$$

which is situated in the region $\{\mu : \ M_1^1\mu > 0,\ M_2^1\mu < 0\}$ such that (4.10) has a solution $s_2 = s_4 = 0,\ s_1 > 0,\ s_3 > 0$ as $\mu \in H_{24}^{13}$ and the corresponding system (1.1) has two 1-1 large homoclinic orbits near Γ.

Ulteriorly, as the similar analysis tells us that it is impossible for (4.10) to have a solution (s_1, s_2, s_3, s_4) with $s_i = 0$ and for $j \neq i,\ s_j > 0$ or $s_i > 0$ for $i, j = 1, 2, 3, 4$. So there exists no 2-2 large period orbit.

Thanks to the above analysis, we have

THEOREM 4.2. *Suppose that $(H_1)-(H_4)$, (H_6) are valid, then*
(1) *If $M_i^1 \neq 0$, there exists a unique manifold L_i with codimension 1 and normal vector M_i^1 at $\mu = 0$, such that system (1.1) has a homoclinic loop near Γ_i if and only if $\mu \in L_i$ and $||\mu|| \ll 1$, $i = 1, 2$.*

If $rank(M_1^1, M_2^1) = 2$, then $L_{12} = L_1 \cap L_2$ is a codimension 2 manifold and $0 \in L_{12}$ such that system (1.1) has an 1-1 double homoclinic loop near Γ as $\mu \in L_{12}$ and $||\mu|| \ll 1$, $i = 1, 2$ namely, Γ is persistent.
(2) *In the region defined by $\{\mu : M_1^1\mu < 0\}$, there exists a unique codimension 2 bifurcation set H_{123}^4 such that system (1.1) has one 1-2 double homoclinic loop and Γ_2 persists.*

In the region defined by $\{\mu : M_2^1\mu > 0\}$, there exists a unique codimension 2 bifurcation set H_{134}^2 such that system (1.1) has one 2-1 double homoclinic loop and Γ_1 persists.

In the region defined by $\{\mu : M_1^1\mu < 0, M_2^1\mu > 0\}$, there exists a unique 2-2 double homoclinic loop bifurcation set H_{13}^{24} of codimension 2. For $\mu \in H_{13}^{24}$, system (1.1) has a unique 2-2 double homoclinic loop near Γ.

In the region defined by $\{\mu : M_1^1\mu > 0, M_2^1\mu < 0\}$, there exists a codimension 2 bifurcation set H_{24}^{13} such that system (1.1) has two 1-1 large homoclinic orbits near Γ for $\mu \in H_{24}^{13}$.

In the region defined by $\{\mu : M_1^1\mu > 0, M_2^1\mu < 0\}$, there exist two bifurcation sets H_{124}^3 and H_{234}^1 with codimension no less than 2, where Γ persists and an additional 1-1 large homoclinic orbit bifurcates near Γ for $\mu \in H_{124}^3 \cup H_{234}^1$ and $||\mu|| \ll 1$.
(3) *There exists no 2-2 large period orbit, 2-2 large homoclinic loop, 2-2 left homoclinic loop and 2-2 right homoclinic loop near Γ.*

As the second step, we shall consider the 1-1 bifurcations results with double twisted orbits. In this sequel, we give a further study of the 1-1 large homoclinic orbit and 1-1 large period orbit bifurcation for the case of double twisted orbits.

Consider the following Poincaré maps:

$$F_1 = P_2^1 \circ P_1^0 : S_1^0 \to S_2^0, \; F_3 = P_1^1 \circ P_2^0 : S_2^0 \to S_1^0$$

and the successor function

$$\begin{aligned} G(s_1, s_3, u_1^1, u_2^1, v_1^0, v_2^0) &= (G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4) \\ &= (F_1(q_1^0) - q_2^0, F_3(q_2^0) - q_1^0). \end{aligned}$$

Using the same procedure as the above, we have:

$$\begin{aligned}
G_1^1 = \;& -(\omega_2^{12})^{-1} s_1^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1} s_1^{\rho_2/\lambda_1} v_1^0 + s_3\delta \\
& +\omega_2^{31}(\omega_2^{33})^{-1} s_3^{\lambda_2/\lambda_1} u_1^1 + M_2^1\mu + o(||\mu||), \\
G_1^3 = \;& u_2^1 - \delta_2^u + \omega_2^{13}(\omega_2^{12})^{-1} s_1^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1} s_1^{\rho_2/\lambda_1} v_1^0 \\
& -(\omega_2^{33})^{-1} s_3^{\lambda_2/\lambda_1} u_1^1 + M_2^3\mu + o(||\mu||), \\
G_1^4 = \;& -v_2^0 + \delta_2^v + (\omega_2^{44})^{-1} s_1^{\rho_2/\lambda_1} v_1^0 - \bar\omega_2^{14}\delta s_3 - [\bar\omega_2^{14}\omega_2^{31} - \omega_2^{34}](\omega_2^{33})^{-1} s_3^{\lambda_2/\lambda_1} u_1^1 \\
& +M_2^4\mu + o(||\mu||), \\
G_2^1 = \;& (\omega_1^{12})^{-1} s_3^{\rho_1/\lambda_1}\delta - (\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1} s_3^{\rho_2/\lambda_1} v_2^0 - s_1\delta \\
& +\omega_1^{31}(\omega_1^{33})^{-1} s_1^{\lambda_2/\lambda_1} u_2^1 + M_1^1\mu + o(||\mu||), \\
G_2^3 = \;& u_1^1 - \delta_1^u - \omega_1^{13}(\omega_1^{12})^{-1} s_3^{\rho_1/\lambda_1}\delta + [\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42} - \omega_1^{43}](\omega_1^{44})^{-1} s_3^{\rho_2/\lambda_1} v_2^0 \\
& -(\omega_1^{33})^{-1} s_1^{\lambda_2/\lambda_1} u_2^1 + M_1^3\mu + o(||\mu||), \\
G_2^4 = \;& -v_1^0 + \delta_1^v + (\omega_1^{44})^{-1} s_3^{\rho_2/\lambda_1} v_2^0 + \bar\omega_1^{14}\delta s_1 - [\bar\omega_1^{14}\omega_1^{31} - \omega_1^{34}](\omega_1^{33})^{-1} s_1^{\lambda_2/\lambda_1} u_2^1 \\
& +M_1^4\mu + o(||\mu||)
\end{aligned}$$

Therefore, there is a correspondence between the solutions $Q = (s_1, s_3, u_1^1, u_2^1, v_1^0, v_2^0)$ of

$$(G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4) = 0$$

with $s_1 \geq 0, s_3 \geq 0$, and the existence of 1-1 large homoclinic loops, 1-1 large period orbit of system (1.1).

Solve $(u_2^1, v_2^0, u_1^1, v_1^0)$ from $(G_1^3, G_1^4, G_2^3, G_2^4) = 0$ and then substitute it into $(G_1^1, G_2^1) = 0$, we obtain the bifurcation equation

$$\begin{aligned} &-(\omega_2^{12})^{-1} s_1^{\rho_1/\lambda_1} + s_3 + \delta^{-1} M_2^1 \mu + h.o.t. = 0, \\ &(\omega_1^{12})^{-1} s_3^{\rho_1/\lambda_1} - s_1 + \delta^{-1} M_1^1 \mu + h.o.t. = 0. \end{aligned} \tag{4.11}$$

Similarly as in the former sections, we state the following results.

THEOREM 4.3. *Assume that* $(H_1) - (H_4)$, (H_6) *hold. Then, for* $||\mu||$ *sufficient small, system* (1.1) *has at most one 1-1 large homoclinic loop or one 1-1 large period orbit in the small neighbourhood of* Γ. *Moreover these orbits do not coexist.*

Proof. Let $Q = (s_1, s_3, u_1^1, u_2^1, v_1^0, v_2^0)$ and $W = \frac{\partial(G_2^1, G_1^1, G_2^3, G_1^3, G_2^4, G_1^4)}{\partial(s_1, s_3, u_1^1, u_2^1, v_1^0, v_2^0)}\Big|_{Q=0,\ \mu=0}$. Then $\det W = -\delta^2 \neq 0$, According to the implicit function theorem, in the neighborhood of $(Q, \mu) = (0, 0)$, there exists a unique solution $s_i = s_i(\mu),\ u_i^1 = u_i^1(\mu),\ v_i^0 = v_i^0(\mu),$ satisfying $s_i(0) = 0,\ u_i^1(0) = 0,\ v_i^0(0) = 0,\ i = 1, 2$. Then if $s_1 = s_3 = 0$, by the uniqueness, we can see that the double homoclinic loop is persistent; if $s_1 = 0,\ s_3 > 0$ or $s_3 = 0,\ s_1 > 0$, then system (1.1) has a unique 1-1 large homoclinic loop; if $s_1 > 0,\ s_3 > 0$, system (1.1) has a unique one 1-1 large period orbit.

Clearly, the uniqueness guarantees that all these kinds of orbits do not coexist. □

If (4.11) has $s_1 = s_3 = 0$ as its solution, then $M_i^1 \mu + h.o.t. = 0,\ i = 1, 2$. In the case of $M_2^1 \neq 0$, there exists a codimension 1 manifold L_2 with normal vector M_2^1 at $\mu = 0$ such that the first equation of (4.11) has solution $s_1 = s_3 = 0$ as $\mu \in L_2$ and $||\mu|| \ll 1$, that is, Γ_2 persists. Similarly, there is a codimension 1 manifold L_1 defined by $M_1^1 \mu + h.o.t. = 0$ with normal vector M_1^1 at $\mu = 0$ such that the second equation of (4.11) has solution $s_1 = s_3 = 0$ as $\mu \in L_1$ and $||\mu|| \ll 1$, that is, Γ_1 persists. Suppose $rank(M_1^1, M_2^1) = 2$, then $L_{12} = L_1 \cap L_2$ is a codimension 2 manifold with normal plane $span\{M_1^1, M_2^1\}$ such that the double homoclinic orbit $\Gamma = \Gamma_1 \cup \Gamma_2$ is persistent.

If (4.11) has solution $s_1 = 0,\ s_3 > 0$, then $s_3 = -\delta^{-1} M_2^1 \mu + h.o.t.$ for $M_2^1 \mu < 0$. Substituting it into the second equation, we obtain the bifurcation set

$$H_1^3:\ (\omega_1^{12})^{-1}(-\delta^{-1} M_2^1 \mu)^{\rho_1/\lambda_1} + \delta^{-1} M_1^1 \mu + h.o.t. = 0,$$

which is well defined in the region $\{\mu:\ M_1^1 \mu > 0,\ M_2^1 \mu < 0\}$, such that system (1.1) has a unique 1-1 large homoclinic orbit for $\mu \in H_1^3$ and $||\mu|| \ll 1$.

When $\mu \in H_1^3$, from (4.11) we have

$$\begin{aligned} &s_{3\mu} + \delta^{-1} M_2^1 + h.o.t. = 0, \\ &(\omega_1^{12})^{-1} \rho_1 (-\delta^{-1} M_2^1 \mu)^{(\rho_1 - \lambda_1)/\lambda_1} s_{3\mu} - \lambda_1 s_{1\mu} + \lambda_1 \delta^{-1} M_1^1 + h.o.t. = 0. \end{aligned}$$

As $s_{1\mu} = \delta^{-1} M_1^1 + O(|\ M_2^1 \mu\ |^{(\rho_1 - \lambda_1)/\lambda_1})$, so s_1 increases along the direction of M_1^1 for $\mu \in H_1^3$ and $||\mu|| \ll 1$.

Similarly, we get the bifurcation set

$$H_3^1:\ -(\omega_2^{12})^{-1}(\delta^{-1} M_1^1 \mu)^{\rho_1/\lambda_1} + \delta^{-1} M_2^1 \mu + h.o.t. = 0,$$

which is well defined in the region $\{\mu:\ M_1^1 \mu > 0,\ M_2^1 \mu < 0\}$, such that (4.11) has solution $s_1 > 0,\ s_3 = 0$ as $\mu \in H_3^1$, that is, system (1.1) has a unique 1-1 large homoclinic orbit near Γ for $\mu \in H_3^1$ and $||\mu|| \ll 1$. And s_3 increases along the direction $-M_2^1$.

Thus we have proved the following statement.

THEOREM 4.4. *Assume that* $(H_1)-(H_4)$, (H_6) *hold. Then*
(1) *If* $M_i^1 \neq 0$, *then there exists codimension 1 manifold* L_i *with normal vector* M_i^1 *at* $\mu = 0$ *such that* Γ_i *persists for* $\mu \in L_i$ *and* $||\mu|| \ll 1$, $i = 1, 2$.

If $rank(M_1^1, M_2^1) = 2$, *then* $L_{12} = L_1 \cap L_2$ *is a codimension 2 manifold with normal plane* $span\{M_1^1, M_2^1\}$ *such that the double homoclinic orbit* $\Gamma = \Gamma_1 \cup \Gamma_2$ *is persistent as* $\mu \in L_{12}$ *and* $||\mu|| \ll 1$.
(2) *In the region defined by* $\{\mu : \ M_1^1\mu > 0, \ M_2^1\mu < 0\}$, *there exists a unique codimension 1 bifurcation set* H_1^3 *(resp.* H_3^1*) such that system* (1.1) *has a unique 1-1 large homoclinic orbit for* $\mu \in H_1^3$ *(resp.* H_3^1*) and* $||\mu|| \ll 1$.
(3) *There is a sector* R *bounded by* H_1^3 *and* H_3^1 *such that system* (1.1) *has a unique 1-1 large period orbit for* $\mu \in R$ *and* $||\mu|| \ll 1$. *(See Figure IV.5)*

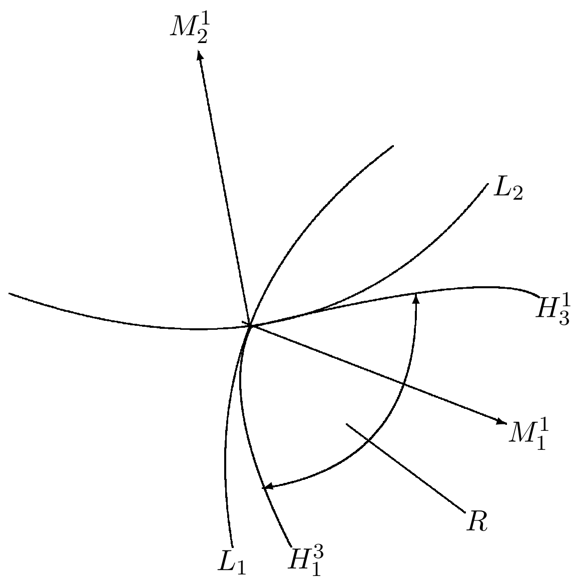


FIGURE IV.5. 1-1 bifurcation diagram in double twisted case as $rank(M_1^1, M_2^1) = 2$

CHAPTER V

Heterodimensional cycle bifurcation with orbit-flip

We consider the bifurcation problems of two heteroclinic loops to two hyperbolic equilibriums.

1. Hypotheses and preliminaries

Consider the following C^r system

$$\dot{z} = f(z) + g(z, \mu), \tag{1.1}$$

and its unperturbed system

$$\dot{z} = f(z), \tag{1.2}$$

where $z \in \mathbb{R}^4,\ \mu \in \mathbb{R}^l,\ l \geq 2,\ 0 \leq |\mu| \ll 1,\ f(p_i) = 0,\ g(p_i, \mu) = 0,\ i = 1, 2,\ g(z, 0) = 0.$

First of all, we assume that:

(H_1) System (1.2) has two hyperbolic equilibrium $p_i,\ i = 1, 2$ and the relevant linearization matrix $Df(p_1)$ has simple eigenvalues: $\lambda_1^1,\ \lambda_1^2,\ \lambda_1^3,\ -\rho_1^1$ satisfying $-\rho_1^1 < 0 < \lambda_1^1 < \lambda_1^3 < \lambda_1^2$; $Df(p_2)$ has simple eigenvalues: $\lambda_2^1,\ \lambda_2^2,\ -\rho_2^1,\ -\rho_2^2$ satisfying $-\rho_2^2 < -\rho_2^1 < 0 < \lambda_2^1 < \lambda_2^2$.

(H_2) There is a heteroclinic cycle $\Gamma = \Gamma_1 \cup \Gamma_2$ connecting p_1 and p_2. Here, $\Gamma_i = \{z = r_i(t),\ t \in R\},\ i = 1, 2,\ r_1(-\infty) = r_2(+\infty) = p_1,\ r_1(+\infty) = r_2(-\infty) = p_2$. And

$$\dim(T_{r_1(t)} W_1^u \cap T_{r_1(t)} W_2^s) = 1.$$

It is evident that Γ is a heterodimensional cycle under the assumptions (H_1) and (H_2). Besides, we make the following assumptions:

(H_3) Let $e_1^{\pm} = \lim_{t \to \mp\infty} \frac{\dot{r}_1(t)}{|\dot{r}_1(t)|}$, then $e_1^+ \in T_{p_1} W_1^u,\ e_1^- \in T_{p_2} W_2^s$ be unit eigenvectors corresponding to λ_1^1 and $-\rho_2^1$, respectively.

Let $e_2^{u+} = \lim_{t \to -\infty} \frac{\dot{r}_2(t)}{|\dot{r}_2(t)|},\ e_2^- = \lim_{t \to +\infty} \frac{\dot{r}_2(t)}{|\dot{r}_2(t)|}$, then $e_2^{u+} \in T_{p_2} W_2^{uu},\ e_2^- \in T_{p_1} W_1^s$ be unit eigenvectors corresponding to λ_2^2 and $-\rho_1^1$, respectively, where W_2^{uu} is the strong unstable manifold of p_2.

(H_4) $\lim\limits_{t \to +\infty} T_{r_2(t)} W_2^u = span\{e_2^-, e_1^{u+}\}$, where e_1^{u+} is the unit eigenvector corresponding to λ_1^2 (see Figure V.1(a)).

Remark 1.1. *For the existing* Γ, (H_3) *is generic for* Γ_1 *and not generic for* Γ_2*, which means that* Γ_2 *takes orbit-flip when* $t \to -\infty$.

Remark 1.2. *For the existing* Γ, (H_4) *is generic, which means that unstable manifold* W_2^u *satisfies the strong inclination property.*

We will always assume that:

(H_5) $r \geq 3Q$ and $Df(p_i),\ i = 1, 2$ satisfies the strong Sternberg condition of order Q and K is the Q-smoothness of $Df(p_i)$, where $Q \geq 2,\ K \geq 4$.

Under assumption (H_5), system (1.1) is uniformly C^K linearizable according to [**82**]. Hence, up to a C^K diffeomorphism, there exits U_i, a small neighborhood of p_i in $\mathbb{R}^4$ such that $p_i = (0,0,0,0)^*,\ i = 1,2$ and for all $\mu \in \mathbb{R}^l,\ 0 \le |\mu| \ll 1$ and $\forall z = (x, y, u, \bar{u}) \in U_1$, system (1.1) has the following linearization:

$$\dot{x} = \lambda_1^1(\mu)x, \qquad \dot{y} = -\rho_1^1(\mu)y, \qquad \dot{u} = \lambda_1^2(\mu)u, \qquad \dot{\bar{u}} = \lambda_1^3(\mu)\bar{u}. \tag{1.3}$$

For all $\mu \in \mathbb{R}^l,\ 0 \le |\mu| \ll 1$ and $\forall z = (x, y, u, v) \in U_2$, system (1.1) has the following linearization:

$$\dot{x} = \lambda_2^1(\mu)x, \qquad \dot{y} = -\rho_2^1(\mu)y, \qquad \dot{u} = \lambda_2^2(\mu)u, \qquad \dot{v} = -\rho_2^2(\mu)v. \tag{1.4}$$

Here, $\rho_1^1(0) = \rho_1^1,\ \lambda_1^j(0) = \lambda_1^j,\ j = 1,2,3,\ \rho_2^k(0) = \rho_2^k,\ \lambda_2^k(0) = \lambda_2^k,\ k = 1,2.$

Obviously, p_1 has one dimensional stable manifold W_1^s and three dimensional unstable manifold W_1^u. While, p_2 has two dimensional stable manifold W_2^s and two dimensional unstable manifold W_2^u.

In the new coordinate systems corresponding to (1.3) and (1.4), the local stable manifold $W^s_{1\,loc} = W_1^s \cap U_1$ is a segment of the $y-$axis, the local strong unstable manifold of p_2 is a segment of the $u-$axis, but the local weak unstable manifold of p_1 is not unique, and has the expression $\{z = (x, y, u, \bar{u}) :\ y = 0,\ u = O(|x|^{\lambda_1^2/\lambda_1^1}),\ \bar{u} = O(|x|^{\lambda_1^3/\lambda_1^1})\}$, the local weak stable manifold of p_2 is also not unique with the expression $\{z = (x, y, u, v) :\ x = u = 0,\ v = O(|y|^{\rho_2^2/\rho_2^1})\}$.

2. Bifurcation equations

Denote $r_i(t) = (r_i^x(t), r_i^y(t), r_i^u(t), r_i^{\bar{u}}(t)),\ i = 1,2$, in the small neighborhood U_1 and $r_i(t) = (r_i^x(t), r_i^y(t), r_i^u(t), r_i^v(t)),\ i = 1,2$, in the small neighborhood U_2, respectively. Take $T_i,\ i = 1,2$ large enough such that $r_1(-T_1) = (\delta, 0, \delta_u, \delta_{\bar{u}}),\ r_1(T_1) = (0, \delta, 0, \delta_v),\ r_2(-T_2) = (0, 0, \delta, 0),\ r_2(T_2) = (0, \delta, 0, 0)$, where $|\delta_u|,\ |\delta_{\bar{u}}|,\ |\delta_v| \ll \delta$ and $\delta > 0$ is small enough so that $\{(x, y, u, \bar{u}) : |x|, |y|, |u|, |\bar{u}| < 2\delta\} \subset U_1$ and $\{(x, y, u, v) : |x|, |y|, |u|, |v| < 2\delta\} \subset U_2$.

Now we consider the linear variational system of (1.2) and its adjoint system

$$\dot{z} = Df(r_i(t))z, \tag*{$(4)_i$}$$

$$\dot{z} = -(Df(r_i(t)))^* z. \tag*{$(5)_i$}$$

Lemma 2.1. *Assume $(H_1) - (H_4)$ hold,then*

(*i*) *there exists a fundamental solution matrix $Z_1(t) = (z_1^1(t), z_1^2(t), z_1^3(t), z_1^4(t))$ for system $(4)_1$ with*

$$\begin{aligned} &z_1^1(t) = \dot{r}_1(t)/|\dot{r}_1^x(-T_1)| \in T_{r_1(t)}W_1^u \cap T_{r_1(t)}W_2^s, \\ &z_1^2(t),\ z_1^3(t) \in T_{r_1(t)}W_1^u \cap (T_{r_1(t)}W_2^s)^c, \\ &z_1^4(t) \in T_{r_1(t)}W_2^s \cap (T_{r_1(t)}W_1^u)^c \end{aligned}$$

satisfying

$$Z_1(-T_1) = \begin{pmatrix} 1 & 0 & 0 & \omega_1^{41} \\ 0 & 0 & 0 & \omega_1^{42} \\ \omega_1^{13} & 1 & 0 & \omega_1^{43} \\ \omega_1^{14} & 0 & 1 & \omega_1^{44} \end{pmatrix}, \qquad Z_1(T_1) = \begin{pmatrix} 0 & \omega_1^{21} & \omega_1^{31} & 0 \\ \omega_1^{12} & \omega_1^{22} & \omega_1^{32} & 0 \\ 0 & \omega_1^{23} & \omega_1^{33} & 0 \\ \bar{\omega}_1^{14} & \omega_1^{24} & \omega_1^{34} & 1 \end{pmatrix},$$

where $\omega_1^{12} < 0,\ \omega_1^{42} \neq 0,\ \omega_1 = \begin{vmatrix} \omega_1^{21} & \omega_1^{31} \\ \omega_1^{23} & \omega_1^{33} \end{vmatrix} \neq 0,\ |\omega_1^{1i}| \ll 1,\ i = 3,4,\ |(\omega_1^{12})^{-1}\bar{\omega}_1^{14}| \ll 1,\ |\omega_1^{4j}(\omega_1^{42})^{-1}| \ll 1,\ j \neq 2,\ |\omega_1^{ij}\omega_1^{-1}| \ll 1,\ i = 2,3,\ j = 2,4.$

(ii) there exists a fundamental solution matrix $Z_2(t) = (z_2^1(t), z_2^2(t), z_2^3(t), z_2^4(t))$ *for system* $(4)_2$ *with*

$$\begin{aligned} &z_2^1(t),\ z_2^2(t) \in (T_{r_2(t)}W_2^u)^c, \\ &z_2^3(t) = \dot{r}_2(t)/|\dot{r}_2^u(-T_2)| \in T_{r_2(t)}W_2^{uu} \cap T_{r_2(t)}W_1^s, \\ &z_2^4(t) \in T_{r_2(t)}W_2^u \end{aligned}$$

satisfying

$$Z_2(-T_2) = \begin{pmatrix} \bar{\omega}_2^{11} & \bar{\omega}_2^{21} & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \bar{\omega}_2^{13} & \bar{\omega}_2^{23} & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad Z_2(T_2) = \begin{pmatrix} \omega_2^{11} & \omega_2^{21} & 0 & \omega_2^{41} \\ 0 & 0 & \omega_2^{32} & \omega_2^{42} \\ 0 & 0 & 0 & \omega_2^{43} \\ \omega_2^{14} & \omega_2^{24} & 0 & \omega_2^{44} \end{pmatrix},$$

where $\omega_2^{32} < 0,\ \omega_2^{43} \neq 0,\ |\bar{\omega}_2^{1i}| \ll 1,\ |\bar{\omega}_2^{2i}| \ll 1,\ i = 1,3,\ |\omega_2^{4j}(\omega_2^{43})^{-1}| \ll 1,\ j = 1,2,4,\ \omega_2 = \begin{vmatrix} \omega_2^{11} & \omega_2^{21} \\ \omega_2^{14} & \omega_2^{24} \end{vmatrix} \neq 0.$

Proof. (i) According to (H_2) and the definition of $z_1^1(t)$, we see that the first columns of $Z_1(-T_1)$ and $Z_1(T_1)$ are correct, and we have $\omega_1^{12} < 0$. Due to the stronger contractivity of the $u,\ \bar{u}$ (respectively v) components compared with the x (respectively y) component in $\bar{U}_1$ as $t \to -\infty$ (respectively $\bar{U}_2$ as $t \to +\infty$), we derive $|\omega_1^{13}|,\ |\omega_1^{14}| \ll 1,\ |\bar{\omega}_1^{14}(\omega_1^{12})^{-1}| \ll 1$. Let $z_1^2(t),\ z_1^3(t)$ and $z_1^4(t)$ be the solutions of $(4)_1$ with initial values $z_1^2(-T_1) = (0,0,1,0)^*$, $z_1^3(-T_1) = (0,0,0,1)^*$ and $z_1^4(T_1) = (0,0,0,1)^*$, then $\omega_1 = \begin{vmatrix} \omega_1^{21} & \omega_1^{31} \\ \omega_1^{23} & \omega_1^{33} \end{vmatrix} \neq 0$ and $\omega_1^{42} \neq 0$ follow from the transversality condition given by (H_2). The remaining inequalities can be verified easily.

(ii) Owing to (H_3), it is nature to choose $z_2^3(t) \in T_{r_2(t)}W_2^{uu}$ satisfying $z_2^3(-T_2) = (0,0,1,0)^*$. As in the small neighborhood $U_1,\ T_{p_1}W_1^s = \mathrm{span}\{(0,1,0,0)^*\}$, then we should have $z_2^3(T_2) = (0,\omega_2^{32},0,0)^*$ with $\omega_2^{32} < 0$. Consequently, $z_2^4(-T_2)$ and $z_2^4(T_2)$ can be easily given. Thanks to the strong inclination property (H_4), we have $\omega_2^{43} \neq 0$. Since $e_1^- \in (T_{p_2}W_2^u)^c$, we can choose $\hat{z}_2^2(t) \in (T_{r_2(t)}W_2^u)^c$ as the solution of system $(4)_2$ satisfying $\hat{z}_2^2(-T_2) = -e_1^- = (0,1,0,0)^*$ and $\hat{z}_2^2(T_2) = (\hat{\omega}_2^{21}, \hat{\omega}_2^{22}, \hat{\omega}_2^{23}, \hat{\omega}_2^{24})^*$. Due to the property of the solution to the linear differential equation, $z_2^2(t) = \hat{z}_2^2(t) - (\omega_2^{43})^{-1}\hat{\omega}_2^{23}z_2^4(t) - (\omega_2^{32})^{-1}[\hat{\omega}_2^{22} - (\omega_2^{43})^{-1}\tilde{\omega}_2^{23}\omega_2^{42}]z_2^3(t)$ is also one solution in $(T_{r_2(t)}W_2^u)^c$. Thus, $z_2^2(-T_2)$ and $z_2^2(T_2)$ are obtained. In the same way, the solution $z_2^1(t)$ can be chosen such that $z_2^1(-T_2)$ and $z_2^1(T_2)$ have the assigned values. Based on the fact that $detZ_2(-T_2) \neq 0$ and the Liouville formula, we have $\omega_2 = \begin{vmatrix} \omega_2^{11} & \omega_2^{21} \\ \omega_2^{14} & \omega_2^{24} \end{vmatrix} \neq 0.$ □

As well known from the matrix theory, system $(5)_i$ has a fundamental solution matrix $\Phi_i(t) = (Z_i^{-1}(t))^* = (\phi_i^1(t), \phi_i^2(t), \phi_i^3(t), \phi_i^4(t))$. For those points z very close to the orbit $\Gamma_i,\ i = 1,2$, introduce the following local moving frame coordinates:

$$z(t) = S_i(t) \triangleq r_i(t) + Z_i(t)N_i^*(t) \tag{2.6}$$

with $N_1(t) = (0, n_1^2(t), n_1^3(t), n_1^4(t))$, $N_2(t) = (n_2^1, n_2^2(t), 0, n_2^4(t))$. And choose the cross sections:

$$\begin{aligned}
S_1^0 &= \{z = S_1(-T_1):\ |x|, |y|, |u|, |\bar{u}| \le 2\delta\} \subset U_1,\\
S_2^0 &= \{z = S_2(-T_2):\ |x|, |y|, |u|, |v| \le 2\delta\} \subset U_2,\\
S_1^1 &= \{z = S_1(T_1):\ |x|, |y|, |u|, |v| \le 2\delta\} \subset U_2,\\
S_2^1 &= \{z = S_2(T_2):\ |x|, |y|, |u|, |\bar{u}| \le 2\delta\} \subset U_1.
\end{aligned}$$

Then, under transformation (2.6), system (1.1) has the following form:

$$\dot{N}_i^*(t) = \phi_i^*(t) g_\mu(r_i(t), 0)\mu + h.o.t.$$

Integrating both sides from $-T_i$ to T_i, we obtain

$$N_i^*(T_i) = N_i^*(-T_i) + \int_{-T_i}^{T_i} \phi_i^*(t) g_\mu(r_i(t), 0)\mu \,\mathrm{d}t + h.o.t.,\ i = 1, 2,$$

which produce the map $F_i^1:\ S_i^0 \to S_i^1,\ i = 1, 2,\ S_2^1 = S_0^1$. Precisely,

$$\begin{aligned}
F_1^1:\ S_1^0 \to S_1^1,\ (0, n_1^{0,2}, n_1^{0,3}, n_1^{0,4}) &\mapsto (0, \bar{n}_1^{1,2}, \bar{n}_1^{1,3}, \bar{n}_1^{1,4})\\
F_2^1:\ S_2^0 \to S_2^1,\ (n_2^{0,1}, n_2^{0,2}, 0, n_2^{0,4}) &\mapsto (\bar{n}_2^{1,1}, \bar{n}_2^{1,2}, 0, \bar{n}_2^{1,4})
\end{aligned}$$

can be expressed by:

$$\bar{n}_1^{1,j} = n_1^{0,j} + M_1^j \mu + h.o.t.,\ j = 2, 3, 4, \tag{2.7}$$

$$\bar{n}_2^{1,k} = n_2^{0,k} + M_2^k \mu + h.o.t.,\ k = 1, 2, 4, \tag{2.8}$$

where $M_1^j = \int_{-T_1}^{T_1} \phi_1^{j*}(t) g_\mu(r_1(t), 0)\,\mathrm{d}t,\ j = 2, 3, 4$ and $M_2^k = \int_{-T_2}^{T_2} \phi_2^{k*}(t) g_\mu(r_2(t), 0)\,\mathrm{d}t,\ k = 1, 2, 4$.

LEMMA 2.2.

$$M_1^j = \int_{-T_1}^{T_1} \phi_1^{j*}(t) g_\mu(r_1(t), 0)\,\mathrm{d}t = \int_{-T_1}^{+\infty} \phi_1^{j*}(t) g_\mu(r_1(t), 0)\,\mathrm{d}t,\ j = 2, 3,$$

$$M_1^4 = \int_{-T_1}^{T_1} \phi_1^{4*}(t) g_\mu(r_1(t), 0)\,\mathrm{d}t = \int_{-\infty}^{+T_1} \phi_1^{4*}(t) g_\mu(r_1(t), 0)\,\mathrm{d}t. \tag{2.9}$$

$$M_2^k = \int_{-T_2}^{T_2} \phi_2^{k*}(t) g_\mu(r_2(t), 0)\,\mathrm{d}t = \int_{-\infty}^{+\infty} \phi_2^{k*}(t) g_\mu(r_2(t), 0)\,\mathrm{d}t,\ k = 1, 2, 4. \tag{2.10}$$

Proof. To prove (2.9), it is sufficient to verify that $\phi_1^{j*}(t) g_\mu(r_1(t), 0) = 0$ for $t \ge T_1,\ j = 2, 3$, and $\phi_1^{4*}(t) g_\mu(r_1(t), 0) = 0$ for $t < -T_1$. As $r_1(T_1) = (0, \delta, 0, \delta_v)$, then $r_1(t) = (0, r_1^y(t), 0, r_1^v(t))$ for $t > T_1$ with $r_1^y(t) = O(\delta e^{-\rho_2^1(t-T_1)}) < \delta,\ r_1^v(t) = O(\delta_v e^{-\rho_2^2(t-T_1)}) < \delta_v$. Similarly, we have $r_1(t) = (r_1^x(t), 0, r_1^u(t), r_1^{\bar{u}}(t))$ with $r_1^x(t) = O(\delta e^{\lambda_1^1(t+T_1)}) < \delta,\ r_1^u(t) = O(\delta_u e^{\lambda_1^2(t+T_1)}) < \delta_u,\ r_1^{\bar{u}}(t) = O(\delta_{\bar{u}} e^{\lambda_1^3(t+T_1)}) < \delta_{\bar{u}}$ for $t < -T_1$, which is due to $r_1(-T_1) = (\delta, 0, \delta_u, \delta_{\bar{u}})$. According to the normal forms (1.3) and (1.4), we have

$$\begin{aligned}
g_\mu(r_1(t), 0) &= (0, O(\delta), 0, O(\delta_v)) \text{ for } t > T_1,\\
g_\mu(r_1(t), 0) &= (O(\delta), 0, O(\delta_u), O(\delta_{\bar{u}})) \text{ for } t < -T_1.
\end{aligned}$$

Since $\Phi_1^*(t) Z_1(t) = I$, we have $\phi_1^{j*}(t) z_1^i(t) = 0,\ j = 2, 3,\ i = 1, 4$. Denote by $\phi_1^{j*}(t) = (\phi_1^{j1}(t), \phi_1^{j2}(t), \phi_1^{j3}(t), \phi_1^{j4}(t))$, then $z_1^1(T_1) = (0, \omega_1^{12}, 0, \bar{\omega}_1^{14})^*,\ z_1^4(T_1) = (0, 0, 0, 1)^*$ implies that $\phi_1^{j2}(T_1) = \phi_1^{j4}(T_1) = 0,\ j = 2, 3$. Thereafter, we have $\phi_1^{j2}(t) = \phi_1^{j4}(t) = 0$ for $t > T_1,\ j = 2, 3$, since $Df(r_1(t))$ and its adjoint matrix are both diagonal. Likewise, we can also obtain $\phi_1^{4i}(-T_1)=0,\ i=1,3,4$. Consequently, $\phi_1^{4i}(t)=0,$ for $t<-T_1,\ i=1,3,4$. Thus, conclusion (2.9) is verified.

Set $\phi_2^{k*}(t)=(\phi_2^{k1}(t),\phi_2^{k2}(t),\phi_2^{k3}(t),\phi_2^{k4}(t)),\ k=1,2,4$, then using the same procedure as above, we can deduce that $\phi_2^{k2}(t)=0$ for $t>T_2$ and $\phi_2^{k3}(t)=0$ for $t<-T_2,\ k=1,2,4$. Further more, one can easily obtain that

$$g_\mu(r_2(t),0)=(0,O(\delta),0,0),\ \text{for } t>T_2; \qquad g_\mu(r_2(t),0)=(0,0,O(\delta),0),\ \text{for } t<-T_2.$$

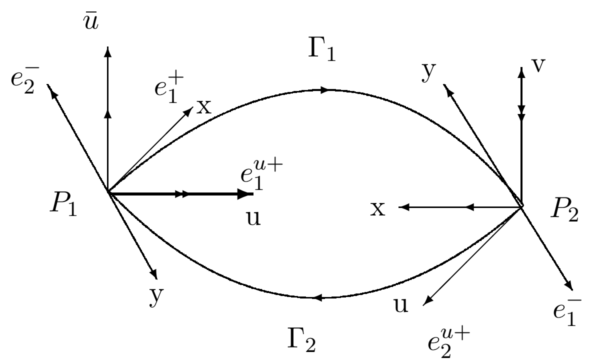
At this rate, (2.10) can be verified. □

Remark 2.1. *Under the hypothesis* $\lambda_1^3>3\lambda_1^1$ *and* $\rho_2^2>3\rho_2^1$, $\delta_u=\delta_{\bar u}=\delta_v=0$ *could be assumed. This is because under this condition, we could straighten* $\Gamma_1\cap U_1$ *and* $\Gamma_1\cap U_2$ *to be a segment of the* $x-$*axis and a segment of the* $y-$*axis, respectively. So, in the following we just consider the more general case with* $\delta_u,\ \delta_{\bar u},\ \delta_v\neq 0$.

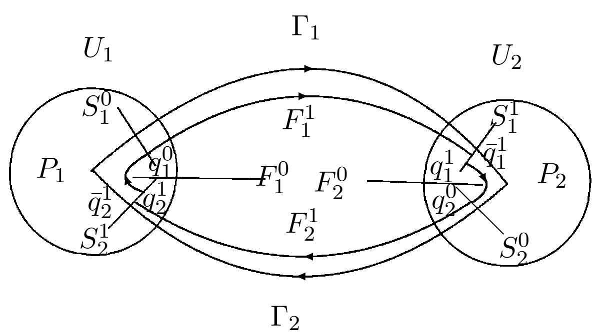
Define $F_1^0:\ S_2^1\to S_1^0,\qquad q_2^1(x_2^1,y_2^1,u_2^1,\bar u_2^1)\mapsto q_1^0(x_1^0,y_1^0,u_1^0,\bar u_1^0),$
$F_2^0:\ S_1^1\to S_2^0,\qquad q_1^1(x_1^1,y_1^1,u_1^1,v_1^1)\mapsto q_2^0(x_2^0,y_2^0,u_2^0,v_2^0),$
induced by the flow of (1.3) in the small neighborhood U_1 of $z=0$ and by the flow of (1.4) in the small neighborhood U_2 of $z=0$, respectively. Set the flying time from q_{i+1}^1 to q_i^0 as $\tau_i,\ i=1,2,\ q_3^1=q_1^1$, and the corresponding Silnikov times as $s_1=e^{-\lambda_1^1(\mu)\tau_1},\ s_2=e^{-\rho_2^1(\mu)\tau_2}$ (see Figure V.1(b)). Consequently, we have

$$F_1^0:\ S_2^1\to S_1^0$$
$$x_2^1=s_1x_1^0,\qquad y_1^0=s_1^{\rho_1^1/\lambda_1^1}y_2^1,\qquad u_2^1=s_1^{\lambda_1^2/\lambda_1^1}u_1^0,\qquad \bar u_2^1=s_1^{\lambda_1^3/\lambda_1^1}\bar u_1^0, \tag{2.11}$$
$$F_2^0:\ S_1^1\to S_2^0$$
$$x_1^1=s_2^{\lambda_2^1/\rho_2^1}x_2^0,\qquad y_2^0=s_2y_1^1,\qquad u_1^1=s_2^{\lambda_2^2/\rho_2^1}u_2^0,\qquad v_2^0=s_2^{\rho_2^2/\rho_2^1}v_1^1, \tag{2.12}$$

where $x_1^0\approx\delta,\ y_1^1\approx\delta,\ u_2^0\approx\delta,\ y_2^1\approx\delta$. And coordinates $(s_1,u_1^0,\bar u_1^0),\ (s_2,x_2^0,v_1^1)$ are called the Shilnikov coordinates.



(a) Heterodimensional cycle $\Gamma=\Gamma_1\cap\Gamma_2$



(b) Poincare map

Figure V.1

Firstly, based on the transformation (2.6) and Lemma 2.1, we give the relationship between the old coordinates:

$$q_1^0(x_1^0,y_1^0,u_1^0,\bar u_1^0),\ q_1^1(x_1^1,y_1^1,u_1^1,v_1^1),\ q_2^0(x_2^0,y_2^0,u_2^0,v_2^0),\ q_2^1(x_2^1,y_2^1,u_2^1,\bar u_2^1),$$

and the new coordinates:

$$q_1^0(0,n_1^{0,2},n_1^{0,3},n_1^{0,4}),\ q_1^1(0,n_1^{1,2},n_1^{1,3},n_1^{1,4}),\ q_2^0(n_2^{0,1},n_2^{0,2},0,n_2^{0,4}),\ q_2^1(n_2^{1,1},n_2^{1,2},0,n_2^{1,4}).$$

Precisely,

$$\left\{\begin{array}{ll} n_1^{0,2} &= u_1^0 - \delta_u - \omega_1^{43}(\omega_1^{42})^{-1} y_1^0, \\ n_1^{0,3} &= \bar{u}_1^0 - \delta_{\bar{u}} - \omega_1^{44}(\omega_1^{42})^{-1} y_1^0, \\ n_1^{0,4} &= (\omega_1^{42})^{-1} y_1^0, \end{array}\right. \tag{2.13}$$

$$\left\{\begin{array}{ll} n_1^{1,2} &= \omega_1^{-1}(\omega_1^{33} x_1^1 - \omega_1^{31} u_1^1), \\ n_1^{1,3} &= \omega_1^{-1}(\omega_1^{21} u_1^1 - \omega_1^{23} x_1^1), \\ n_1^{1,4} &= v_1^1 - \delta_v - \omega_1^{-1}[(\omega_1^{24}\omega_1^{33} - \omega_1^{23}\omega_1^{34}) x_1^1 + (\omega_1^{21}\omega_1^{34} - \omega_1^{24}\omega_1^{31}) u_1^1]. \end{array}\right. \tag{2.14}$$

$$n_2^{0,1} = v_2^0, \qquad n_2^{0,2} = y_2^0, \qquad n_2^{0,4} = x_2^0 - \bar{\omega}_2^{21} y_2^0 - \bar{\omega}_2^{11} v_2^0, \tag{2.15}$$

$$\left\{\begin{array}{ll} n_2^{1,1} &= \omega_2^{-1}\{\omega_2^{24}[x_2^1 - \omega_2^{41}(\omega_2^{43})^{-1} u_2^1] - \omega_2^{21}[\bar{u}_2^1 - \omega_2^{44}(\omega_2^{43})^{-1} u_2^1]\} \\ &= \omega_2^{-1}[\omega_2^{24} x_2^1 - \omega_2^{21}\bar{u}_2^1 - (\omega_2^{24}\omega_2^{41} - \omega_2^{21}\omega_2^{44})(\omega_2^{43})^{-1} u_2^1], \\ n_2^{1,2} &= \omega_2^{-1}\{\omega_2^{11}[\bar{u}_2^1 - \omega_2^{44}(\omega_2^{43})^{-1} u_2^1] - \omega_2^{14}[x_2^1 - \omega_2^{41}(\omega_2^{43})^{-1} u_2^1]\} \\ &= \omega_2^{-1}[\omega_2^{11}\bar{u}_2^1 - \omega_2^{14} x_2^1 - (\omega_2^{11}\omega_2^{44} - \omega_2^{14}\omega_2^{41})(\omega_2^{43})^{-1} u_2^1], \\ n_2^{1,4} &= (\omega_2^{43})^{-1} u_2^1. \end{array}\right. \tag{2.16}$$

From (2.7) (2.11) and (2.13), we can define $F_1 = F_1^1 \circ F_1^0 : \ S_2^1 \to S_1^1$ as:

$$\left\{\begin{array}{ll} \bar{n}_1^{1,2} &= u_1^0 - \delta_u - \omega_1^{43}(\omega_1^{42})^{-1}\delta s_1^{\rho_1^1/\lambda_1^1} + M_1^2\mu + h.o.t., \\ \bar{n}_1^{1,3} &= \bar{u}_1^0 - \delta_{\bar{u}} - \omega_1^{44}(\omega_1^{42})^{-1}\delta s_1^{\rho_1^1/\lambda_1^1} + M_1^3\mu + h.o.t., \\ \bar{n}_1^{1,4} &= (\omega_1^{42})^{-1}\delta s_1^{\rho_1^1/\lambda_1^1} + M_1^4\mu + h.o.t.. \end{array}\right. \tag{2.17}$$

From (2.8) (2.12) and (2.15), we can define $F_2 = F_2^1 \circ F_2^0 : \ S_1^1 \to S_2^1$ as:

$$\left\{\begin{array}{ll} \bar{n}_2^{1,1} &= s_2^{\rho_2^2/\rho_2^1} v_1^1 + M_2^1\mu + h.o.t., \\ \bar{n}_2^{1,2} &= \delta s_2 + M_2^2\mu + h.o.t., \\ \bar{n}_2^{1,4} &= x_2^0 - \bar{\omega}_2^{21}\delta s_2 - \bar{\omega}_2^{11} s_2^{\rho_2^2/\rho_2^1} v_1^1 + M_2^4\mu + h.o.t.. \end{array}\right. \tag{2.18}$$

Now, the successor function is given by

$$G(s_1, s_2, u_1^0, \bar{u}_1^0, x_2^0, v_1^1) = (G_1^2, G_1^3, G_1^4, G_2^1, G_2^2, G_2^4) = (F_1(q_2^1) - q_1^1, F_2(q_1^1) - q_2^1),$$

which can be expressed by:

$$\begin{aligned} G_1^2 &= u_1^0 - \delta_u - \omega_1^{43}(\omega_1^{42})^{-1}\delta s_1^{\rho_1^1/\lambda_1^1} - \omega_1^{-1}\omega_1^{33} s_2^{\lambda_2^1/\rho_2^1} x_2^0 + \omega^{-1}\omega_1^{31}\delta s_2^{\lambda_2^2/\rho_2^1} + M_1^2\mu + h.o.t., \\ G_1^3 &= \bar{u}_1^0 - \delta_{\bar{u}} - \omega_1^{44}(\omega_1^{42})^{-1}\delta s_1^{\rho_1^1/\lambda_1^1} - \omega_1^{-1}\omega_1^{21}\delta s_2^{\lambda_2^2/\rho_2^1} + \omega^{-1}\omega_1^{23} s_2^{\lambda_2^1/\rho_2^1} x_2^0 + M_1^3\mu + h.o.t., \\ G_1^4 &= (\omega_1^{42})^{-1}\delta s_1^{\rho_1^1/\lambda_1^1} - v_1^1 + \delta_v \\ &\quad + \omega_1^{-1}[(\omega_1^{24}\omega_1^{33} - \omega_1^{23}\omega_1^{34}) s_2^{\lambda_2^1/\rho_2^1} x_2^0 + (\omega_1^{21}\omega_1^{34} - \omega_1^{24}\omega_1^{31})\delta s_2^{\lambda_2^2/\rho_2^1}] + M_1^4\mu + h.o.t., \\ G_2^1 &= s_2^{\rho_2^2/\rho_2^1} v_1^1 - \omega_2^{-1}[\omega_2^{24}\delta s_1 - \omega_2^{21} s_1^{\lambda_1^3/\lambda_1^1}\bar{u}_1^0 - (\omega_2^{24}\omega_2^{41} - \omega_2^{21}\omega_2^{44})(\omega_2^{43})^{-1} s_1^{\lambda_1^2/\lambda_1^1} u_1^0] \\ &\quad + M_2^1\mu + h.o.t., \\ G_2^2 &= \delta s_2 - \omega_2^{-1}[\omega_2^{11} s_1^{\lambda_1^3/\lambda_1^1}\bar{u}_1^0 - \omega_2^{14}\delta s_1 - (\omega_2^{11}\omega_2^{44} - \omega_2^{14}\omega_2^{41})(\omega_2^{43})^{-1} s_1^{\lambda_1^2/\lambda_1^1} u_1^0] \\ &\quad + M_2^2\mu + h.o.t., \\ G_2^4 &= x_2^0 - \bar{\omega}_2^{11} s_2^{\rho_2^2/\rho_2^1} v_1^1 - \bar{\omega}_2^{21}\delta s_2 - (\omega_2^{43})^{-1} s_1^{\lambda_1^2/\lambda_1^1} u_1^0 + M_2^4\mu + h.o.t.. \end{aligned}$$

Define

$$\tilde{\omega}_1 \triangleq \omega_2^{-1}(\omega_2^{24}\omega_2^{41} - \omega_2^{21}\omega_2^{44})(\omega_2^{43})^{-1}, \qquad \tilde{\omega}_2 \triangleq \omega_2^{-1}(\omega_2^{11}\omega_2^{44} - \omega_2^{14}\omega_2^{41})(\omega_2^{43})^{-1}.$$

Solving $(u_1^0, \bar{u}_1^0, x_2^0, v_1^1)$ from $(G_1^2, G_1^3, G_1^4, G_2^4) = 0$, and then substituting it into $(G_2^1, G_2^2) = 0$ by using $\omega_2 \neq 0$, we obtain the bifurcation equations, which have the following three different expressions:

Case (1):

$$\delta_v s_2^{\rho_2^2/\rho_2^1} - \omega_2^{-1}\omega_2^{24}\delta s_1 + M_2^1\mu + h.o.t. = 0 \tag{2.19}$$

$$\delta s_2 + \omega_2^{-1}\omega_2^{14}\delta s_1 + M_2^2\mu + h.o.t. = 0 \tag{2.20}$$

for $\omega_2^{14} \neq 0,\ \omega_2^{24} \neq 0$.

Case (2):

$$\delta_v s_2^{\rho_2^2/\rho_2^1} - \omega_2^{-1}\omega_2^{24}\delta s_1 + M_2^1\mu + h.o.t. = 0 \tag{2.21}$$

$$\delta s_2 - \omega_2^{-1}\omega_2^{11}\delta_{\bar{u}} s_1^{\lambda_1^3/\lambda_1^1} + M_2^2\mu + h.o.t. = 0 \tag{2.22}$$

for $\omega_2^{14} = 0,\ \omega_2^{24} \neq 0$.

Case (3):

$$\delta_v s_2^{\rho_2^2/\rho_2^1} + \omega_2^{-1}\omega_2^{21}\delta_{\bar{u}} s_1^{\lambda_1^3/\lambda_1^1} + M_2^1\mu + h.o.t. = 0 \tag{2.23}$$

$$\delta s_2 + \omega_2^{-1}\omega_2^{14}\delta s_1 + M_2^2\mu + h.o.t. = 0 \tag{2.24}$$

for $\omega_2^{14} \neq 0,\ \omega_2^{24} = 0$.

3. Bifurcation results

In this section, we study the existence, uniqueness and non-coexistence of the heterodimensional cycle, homoclinic loop and periodic orbit for the heterodimensional cycle bifurcation with orbit-flip in the non-transversal orbit Γ_2.

Firstly, we have the following result concerned with the uniqueness and the non-coexistence.

THEOREM 3.1. *Suppose that $(H_1) - (H_5)$ hold and $\rho_1^1 > \lambda_1^1,\ \lambda_2^1 > \rho_2^1$. Then, for $|\mu|$ sufficient small, system (1.1) has at most one heterodimensional cycle, one homoclinic loop and one periodic orbit in the small neighbourhood of Γ and all these orbits cannot coexist if $\omega_2^{24} \neq 0$.*

Proof. Notice that

$$W = \frac{\partial(G_1^2, G_1^3, G_2^4, G_1^4, G_2^1, G_2^2)}{\partial Q}\Big|_{Q=0,\ \mu=0} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\bar{\omega}_2^{21}\delta & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -\omega_2^{-1}\omega_2^{24}\delta & 0 & 0 & 0 & 0 & 0 \\ \omega_2^{-1}\omega_2^{14}\delta & \delta & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $Q = (s_1, s_2, u_1^0, \bar{u}_1^0, x_2^0, v_1^1)$ and $\det W = \delta^2\omega_2^{-1}\omega_2^{24} \neq 0$, if $\omega_2^{24} \neq 0$. Consequently, owing to the implict function theorem, in the neighbourhood of $(Q, \mu) = (0, 0)$, there exists a unique solution $s_i = s_i(\mu),\ u_1^0 = u_1^0(\mu),\ \bar{u}_1^0 = \bar{u}_1^0(\mu),\ x_2^0 = x_2^0(\mu),\ v_1^1 = v_1^1(\mu)$ satisfying $s_i(0) = 0,\ u_1^0(0) = 0,\ \bar{u}_1^0(0) = 0,\ x_2^0(0) = 0,\ v_1^1(0) = 0,\ i = 1, 2$. Then if $s_1 = s_2 = 0$, by the uniqueness, we can see that the heterodimensional cycle is persistent; if $s_1 = 0,\ s_2 > 0$, then system (1.1) has a unique loop homoclinic to p_1; if $s_2 = 0,\ s_1 > 0$, system (1.1) has a unique loop homoclinic to p_2; if $s_1 > 0,\ s_2 > 0$, system (1.1) has a unique periodic orbit. Clearly, the implicit function theorem guarantees that all these kinds of orbits cannot coexist. □

REMARK 3.1. *If the conditions of Theorem 3.1 are valid, then we can show that there is no n-periodic or n-homoclinic orbit bifurcated from* Γ *for arbitrary* $n \geq 2$.

If $s_1 = s_2 = 0$ is the solution of the bifurcation equations (2.19) and (2.20), we have $M_2^1\mu + h.o.t. = 0,\ M_2^2\mu + h.o.t. = 0$. Suppose $rank(M_2^1, M_2^2) = 2$, then

$$L_{12} = \{\mu:\ M_2^1\mu + h.o.t. = 0,\ M_2^2\mu + h.o.t. = 0\}$$

is a codimension 2 surface with normal plane $span\{M_1^1, M_2^1\}$ at $\mu = 0$ such that system (1.1) has a unique heterodimensional loop near Γ for $\mu \in L_{12}$ and $0 < |\mu| \ll 1$.

If $s_1 = 0,\ s_2 > 0$ satisfy (2.19) and (2.20), then we calculate $s_2 = -\delta^{-1}M_2^2\mu + h.o.t. > 0$ from (2.20) for $\{\mu:\ M_2^2\mu < 0\}$. Substituting it into (2.19), we get the codimension 1 homoclinic bifurcation surface

$$H_1:\ W_1(\mu) \triangleq \delta_v(-\delta^{-1}M_2^2\mu)^{\rho_2^2/\rho_2^1} + M_2^1\mu + h.o.t. = 0$$

with normal vector M_2^1 at $\mu = 0$ such that a unique homoclinic loop $\Gamma_1^2 = \{\tilde{r}(t):\ t \in R,\ \lim_{t\to\pm\infty}\tilde{r}(t) = p_1\}$ bifurcates in the small neighborhood of Γ for $\mu \in H_1$ and $0 < |\mu| \ll 1$.

Suppose $s_2 = 0,\ s_1 > 0$ is the solution of the bifurcation equations (2.19) and (2.20). Consequently, we have $s_1 = \omega_2(\omega_2^{24})^{-1}\delta^{-1}M_2^1\mu + h.o.t. > 0$ for $\omega_2\omega_2^{24}M_2^1\mu > 0$. And the homoclinic bifurcation surface

$$H_2:\ W_2(\mu) \triangleq \omega_2^{14}(\omega_2^{24})^{-1}M_2^1\mu + M_2^2\mu + h.o.t. = 0$$

has codimension 1 with normal vector $\vec{n}_2^1 \triangleq \omega_2^{14}(\omega_2^{24})^{-1}M_2^1 + M_2^2$ at $\mu = 0$ if $\vec{n}_2^1 \neq 0$ such that system (1.1) has a unique homoclinic loop $\Gamma_2^1 = \{\bar{r}(t):\ t \in R,\ \lim_{t\to\pm\infty}\bar{r}(t) = p_2\}$ in the small neighborhood of Γ for $\mu \in H_2$ and $0 < |\mu| \ll 1$.

With the above analysis, we state the following result:

THEOREM 3.2. *Suppose that* $(H_1) - (H_5)$ *hold and* $\omega_2^{14} \neq 0,\ \omega_2^{24} \neq 0$ *hold, then*
(i) *if* $rank(M_2^1, M_2^2) = 2$, *then* L_{12} *is a codimension 2 bifurcation surface with normal plane* $span\{M_2^1, M_2^2\}$ *at* $\mu = 0$ *and* $0 \in L_{12}$ *such that system (1.1) has a unique heterodimensional loop* $\Gamma^\mu = \Gamma_1^\mu \cup \Gamma_2^\mu$ *near* Γ *as* $\mu \in L_{12}$ *and* $|\mu| \ll 1$, *namely,* Γ *is persistent. Furthermore, the persistent heteroclinic orbit* Γ_2^μ *has no orbit-flip if* $x_2^0 = -M_2^4\mu + h.o.t. \neq 0$.
(ii) *in the region defined by* $\{\mu:\ M_2^2\mu < 0\}$, *there exists a unique codimension 1 bifurcation surface* H_1 *with normal vector* M_2^1 *at* $\mu = 0$ *such that a unique loop* Γ_1^2 *homoclinic to* p_1, *bifurcates from* Γ *for* $\mu \in H_1$ *and* $0 < |\mu| \ll 1$.
In the region defined by $\{\mu:\ \omega_2\omega_2^{24}M_2^1\mu > 0\}$, *there exists a unique codimension 1 homoclinic bifurcation surface* H_2 *with normal vector* $\vec{n}_2^1$ *(if it is not zero) at* $\mu = 0$ *such that system (1.1) has a unique loop* Γ_2^1 *homoclinic to* p_2 *in the small neighborhood of* Γ *for* $\mu \in H_2$ *and* $0 < |\mu| \ll 1$. *Moreover,* Γ_2^1 *has no orbit-flip if* $x_2^0 = -M_2^4\mu + h.o.t. \neq 0$.

THEOREM 3.3. *Assume that* $(H_1) - (H_5)$ *hold and* $\omega_2^{14} \neq 0,\ \omega_2^{24} \neq 0,\ rank(M_2^1, M_2^2) = 2$ *are fulfilled, then in some small neighborhood* U_μ *of* $\mu = 0$, *there is a region* R_{12} *bounded by* H_1 *and* H_2 *such that system (1.1) has a unique periodic orbit near* Γ *as* $\mu \in R_{12}$ *and it has no periodic orbit near* Γ *as* $\mu \in U_\mu - clR_{12}$. *Corresponding to the four different combinations of the signs of* $\omega_2\omega_2^{14}$ *and* $\omega_2\omega_2^{24}$, *the region* R_{12} *and its boundaries* H_1 *and* H_2 *have four different kinds of relative position, which are shown in Figure V.2.*

Proof. Because of the similarity, we only need to consider the case $\omega_2\omega_2^{14} < 0$, $\omega_2\omega_2^{24} < 0$. Near the bifurcation surface H_1, we have $M_2^2\mu < 0$. Thus, for $0 \leq s_1 \ll 1,\ s_2 = -\omega_2^{-1}\omega_2^{14}s_1 - \delta^{-1}M_2^2\mu + h.o.t. > 0$. Substituting it into (2.19), we get

$$F(s_1, \mu) \triangleq \delta_v(-\omega_2^{-1}\omega_2^{14}s_1 - \delta^{-1}M_2^2\mu)^{\rho_2^2/\rho_2^1} - \omega_2^{-1}\omega_2^{24}\delta s_1 + M_2^1\mu + h.o.t. = 0.$$

Note that,

$$F(0,\mu) = W_1(\mu), \qquad F'_{s_1}(s_1,\mu) = -\delta\omega_2^{-1}\omega_2^{24} + h.o.t..$$

So, if $\omega_2\omega_2^{24}W_1(\mu) > 0$, $F(s_1,\mu)$ has a unique sufficiently small positive zero point $s_1 = s_1(\mu) > 0$, while, it has no small positive zero point if $\omega_2\omega_2^{24}W_1(\mu) < 0$.

Since $\omega_2\omega_2^{24} < 0$, and $W_1(\mu)$ has gradient direction M_2^1 at $\mu = 0$, we see, as μ leaves H_1 slightly along the direction $-M_2^1$, bifurcation equations (2.19) and (2.20) have a small positive solution pair $s_1(\mu) > 0$, $s_2(\mu) > 0$.

On the other hand, consider the neighborhood of H_2 by differentiating (2.19) and (2.20) with respect to μ and taking values at H_2, then we derive

$$s_{1\mu} = \delta^{-1}\omega_2(\omega_2^{24})^{-1}M_2^1 + h.o.t.,\ s_{2\mu} = -\delta^{-1}[M_2^2 + \omega_2^{14}(\omega_2^{24})^{-1}M_2^1)] + h.o.t..$$

It follows that equations (2.19) and (2.20) have a small positive solution pair $s_1(\mu) > 0$, $s_2(\mu) > 0$ as μ leaves H_2 along the direction $-[M_2^2 + \omega_2^{14}(\omega_2^{24})^{-1}M_2^1)]$ for $|\mu| \ll 1$, where $\omega_2^{14}\omega_2^{24} > 0$.

Now, combined with the uniqueness of the solution guaranteed by Theorems 3.1 and that the set $\{s_1(\mu) = 0,\ s_2(\mu) \geq 0 \text{ or } s_1(\mu) \geq 0,\ s_2(\mu) = 0\}$ consists of exactly $H_1 \cup H_2$ which divides the small neighborhood U_μ into two connected regions, the above analysis leads to the existence of the region R_{12} and locates its position. Correspondingly, the bifurcation diagram is exhibited in Figure V.2(a).

The proof is complete. □

With similar analysis to the above, we have the following statements.

THEOREM 3.4. *Suppose that* $(H_1) - (H_5)$ *hold and* $\omega_2^{14} = 0$, $\omega_2^{24} \neq 0$ *are fulfilled, then*
(i) if $rank(M_2^1, M_2^2) = 2$*, then* L_{12} *is a codimension 2 bifurcation surface with normal plane span*$\{M_1^1, M_2^1\}$ *at* $\mu = 0$ *and* $0 \in L_{12}$ *such that system (1.1) has a unique heterodimensional loop* $\Gamma^\mu = \Gamma_1^\mu \cup \Gamma_2^\mu$ *near* Γ *as* $\mu \in L_{12}$ *and* $|\mu| \ll 1$*, namely,* Γ *is persistent. Furthermore,* Γ_2^μ *has no orbit-flip if* $x_2^0 = -M_2^4\mu + h.o.t. \neq 0$.
(ii) in the region defined by $\{\mu :\ M_2^2\mu < 0\}$*, there exists a unique codimension 1 bifurcation surface* H_1 *with normal vector* M_2^1 *at* $\mu = 0$ *such that a unique loop* Γ_1^2 *homoclinic to* p_1 *bifurcates from* Γ *for* $\mu \in H_1$ *and* $0 < |\mu| \ll 1$.
In the region defined by $\{\mu :\ \omega_2\omega_2^{24}M_2^1\mu > 0\}$*, there exists a unique codimension 1 homoclinic bifurcation surface* $\hat{H}_2$ *:*

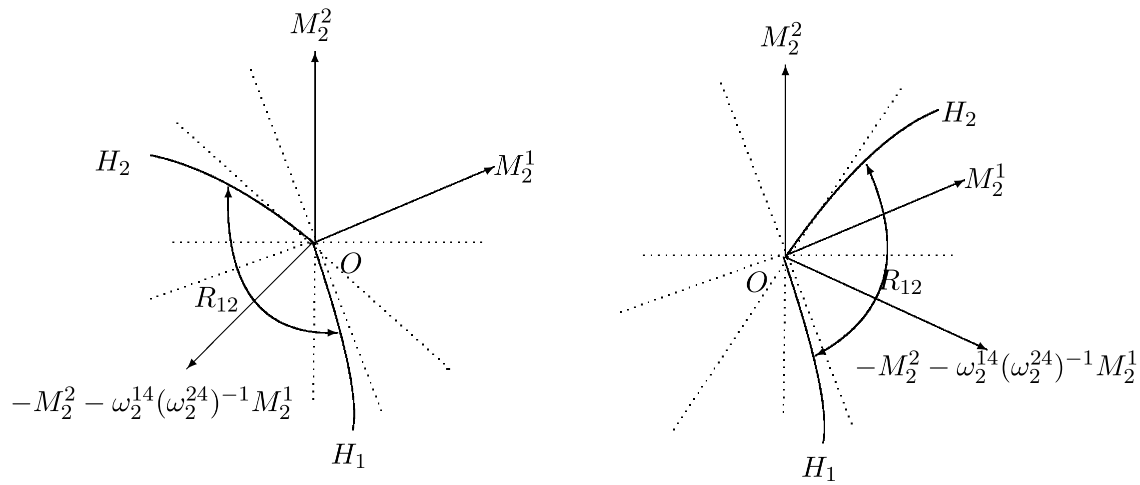
$$\hat{W}_2(\mu) \triangleq -\omega_2^{-1}\omega_2^{11}\delta_{\hat{u}}[\omega_2(\omega_2^{24}\delta)^{-1}M_2^1\mu]^{\lambda_1^3/\lambda_1^1} + M_2^2\mu + h.o.t. = 0$$

with normal vector M_2^2 *at* $\mu = 0$ *such that system (1.1) has a unique loop* Γ_2^1 *homoclinic to* p_2 *in the small neighborhood of* Γ *for* $\mu \in \hat{H}_2$ *and* $0 < |\mu| \ll 1$*. Moreover,* Γ_2^1 *has no orbit-flip if* $x_2^0 = -M_2^4\mu + h.o.t. \neq 0$.

THEOREM 3.5. *Assume that* $(H_1) - (H_5)$ *hold and* $\omega_2^{14} = 0$, $\omega_2^{24} \neq 0$ *are fulfilled, then in some small neighborhood* $\hat{U}_\mu$ *of* $\mu = 0$*, there is a region* $\hat{R}_{12}$ *bounded by* H_1 *and* $\hat{H}_2$*, such that system (1.1) has a unique periodic orbit near* Γ *as* $\mu \in \hat{R}_{12}$*, and no periodic orbit near* Γ *as* $\mu \in \hat{U}_\mu - cl\hat{R}_{12}$*. Depending on the sign of* $\omega_2\omega_2^{24}$*, the region* $\hat{R}_{12}$ *and its boundaries* H_1 *and* $\hat{H}_2$ *have 2 different kinds of relative positions, which are well illustrated in Figure V.3.*

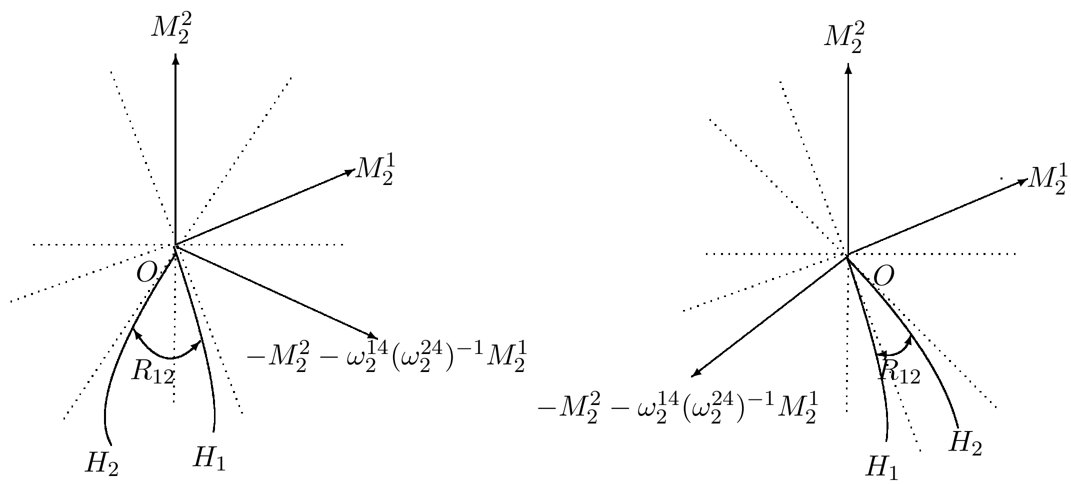
Proof. Owing to the similarity, we only consider the case $\omega_2\omega_2^{24} > 0$. Near the surface H_1, we have $M_2^2\mu < 0$. Thus, due to (2.22), for $0 \leq s_1 \ll 1$, $s_2 = \omega_2^{-1}\omega_2^{11}\delta^{-1}\delta_{\bar{u}}s_1^{\lambda_1^3/\lambda_1^1} - \delta^{-1}M_2^2\mu + h.o.t. > 0$. Substituting it into (2.21), we get

$$\hat{F}(s_1,\mu) \triangleq \delta_v(\omega_2^{-1}\omega_2^{11}\delta^{-1}\delta_{\bar{u}}s_1^{\lambda_1^3/\lambda_1^1} - \delta^{-1}M_2^2\mu)^{\rho_2^2/\rho_2^1} - \omega_2^{-1}\omega_2^{24}\delta s_1 + M_2^1\mu + h.o.t. = 0.$$



(a) When $\omega_2\omega_2^{14} < 0,\ \omega_2\omega_2^{24} < 0$.

(b) When $\omega_2\omega_2^{14} < 0,\ \omega_2\omega_2^{24} > 0$.



(c) When $\omega_2\omega_2^{14} > 0,\ \omega_2\omega_2^{24} < 0$.

(d) When $\omega_2\omega_2^{14} > 0,\ \omega_2\omega_2^{24} > 0$.

FIGURE V.2. Location of R_{12}

Note that,

$$\hat{F}(0,\mu) = W_1(\mu), \qquad \hat{F}'_{s_1}(s_1,\mu) = -\omega_2^{-1}\omega_2^{24}\delta + h.o.t..$$

So, if $\omega_2\omega_2^{24}W_1(\mu) > 0$, $\hat{F}(s_1,\mu)$ has a unique sufficiently small positive zero point $s_1 = s_1(\mu)$, while, it has no sufficient small positive zero point if $\omega_2\omega_2^{24}W_1(\mu) < 0$.

Since $\omega_2\omega_2^{24} > 0$ and $W_1(\mu)$ has gradient direction M_2^1 at $\mu = 0$, we see, as μ leaves H_1 slightly along the direction M_2^1, bifurcation equation (2.21) and (2.22) have a small positive solution pair $s_1(\mu) > 0,\ s_2(\mu) > 0$.

On the other hand, consider the neighborhood of $\hat{H}_2$. Differentiating (2.21) and (2.22) with respect to μ and taking values at $\hat{H}_2$, we obtain $s_{1\mu} = \omega_2(\omega_2^{24})^{-1}\delta^{-1}M_2^1 + h.o.t.$, $s_{2\mu} =$

$-\delta^{-1}M_2^2 + h.o.t.$. It follows that equations (2.21) and (2.22) have a small positive solution pair $s_1(\mu) > 0,\ s_2(\mu) > 0$ as μ leaves $\hat{H}_2$ along the direction $-M_2^2$ for $|\mu| \ll 1$.

Now, thanks to the uniqueness of the solution guaranteed by Theorem 3.1, the above analysis leads to the existence of the Region $\hat{R}_{12}$, which is bounded by H_1 and $\hat{H}_2$. □

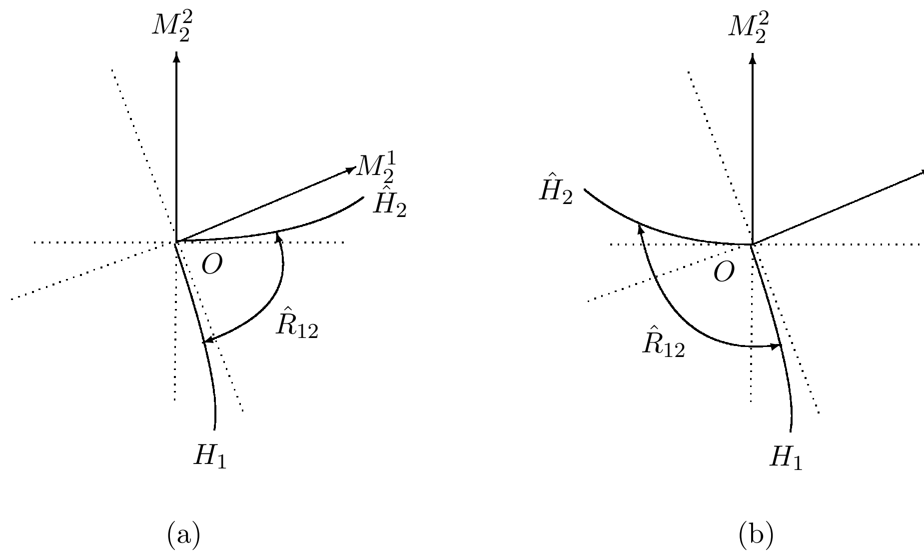


FIGURE V.3. (a) $\omega_2\omega_2^{24} > 0$ and (b) $\omega_2\omega_2^{24} < 0$.

Now, we turn to consider case (3), that is, $\omega_2^{14} \neq 0,\ \omega_2^{24} = 0$. In this case, it is possible for the coexistence of the homoclinic loop and the periodic orbit, and for the existence of the multiple periodic orbit bifurcation.

Similar to the above 2 cases, we firstly have:

THEOREM 3.6. *Suppose that* $(H_1) - (H_5)$ *and* $\omega_2^{14} \neq 0,\ \omega_2^{24} = 0$ *hold, then*
(i) *if* $rank(M_2^1, M_2^2) = 2$, *then* $L_{12} = \{\mu :\ M_2^1\mu + h.o.t. = 0,\ M_2^2\mu + h.o.t. = 0\}$ *is a codimension 2 surface with normal plane* $span\{M_2^1, M_2^2\}$ *at* $\mu = 0$ *such that system (1) has a unique heterodimensional cycle* $\Gamma^\mu = \Gamma_1^\mu \cup \Gamma_2^\mu$ *near* Γ *as* $\mu \in L_{12}$ *and* $|\mu| \ll 1$, *namely,* Γ *is persistent. Furthermore,* Γ_2^μ *has no orbit-flip if* $x_2^0 = -M_2^4\mu + h.o.t. \neq 0$.
(ii) *there is a codimension 1 homoclinic bifurcation surface*

$$H_1 :\ W_1(\mu) \triangleq \delta_v(-\delta^{-1}M_2^2\mu)^{\rho_2^2/\rho_2^1} + M_2^1\mu + h.o.t. = 0,$$

which is well defined in the region $\{\mu :\ \delta_v M_2^1\mu < 0,\ M_2^2\mu < 0\}$, *with normal vector* M_2^1 *at* $\mu = 0$ *such that a unique homoclinic loop* $\Gamma_1^2 = \{\tilde{r}(t) :\ t \in R,\ \lim_{t\to\pm\infty}\tilde{r}(t) = p_1\}$ *bifurcates from* Γ *for* $\mu \in H_1$ *and* $0 < |\mu| \ll 1$.
(iii) *there is a codimension 1 homoclinic bifurcation surface*

$$\bar{H}_2 :\ \bar{W}_2(\mu) \triangleq \omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}[-\omega_2(\omega_2^{14})^{-1}\delta^{-1}M_2^2\mu]^{\lambda_1^3/\lambda_1^1} + M_2^1\mu + h.o.t. = 0,$$

which is well defined in the region $\{\mu :\ \omega_2\omega_2^{14}M_2^2\mu < 0,\ \omega_2\omega_2^{21}\delta_{\bar{u}}M_2^1\mu < 0\}$, *with normal vector* M_2^1 *at* $\mu = 0$ *such that system (1.1) has a unique homoclinic loop* $\Gamma_2^1 = \{\bar{r}(t) :\ t \in R,\ \lim_{t\to\pm\infty}\bar{r}(t) = p_2\}$ *in the small neighborhood of* Γ *for* $\mu \in \bar{H}_2$ *and* $0 < |\mu| \ll 1$.

THEOREM 3.7. *Suppose that* $(H_1) - (H_5)$ *and* $\omega_2^{14} \neq 0,\ \omega_2^{24} = 0$, *then the periodic orbit or the homoclinic loop cannot coexist with the persistent heterodimensional cycle for system (1.1) with* $0 < |\mu| \ll 1$.

Proof. For $\mu \in L_{12}$, (2.23) and (2.24) become

$$\delta_v s_2^{\rho_2^2/\rho_2^1} + \omega_2^{-1}\omega_2^{21}\delta_{\bar{u}} s_1^{\lambda_1^3/\lambda_1^1} + o(s_1^{\lambda_1^3/\lambda_1^1}) + o(s_2^{\rho_2^2/\rho_2^1}) = 0, \tag{3.25}$$

$$s_2 = -\omega_2^{-1}\omega_2^{14} s_1 + o(s_1). \tag{3.26}$$

From Theorem 3.6, we know that system (1.1) has a unique heterodimensional cycle Γ^μ for $\mu \in L_{12}$ and $|\mu| \ll 1$. And if $\omega_2\omega_2^{14} > 0$, system (1.1) cannot have any periodic orbit to coexisting with the persistent heterodimensional cycle owing to (3.26). So, we only need to consider $\omega_2\omega_2^{14} < 0$.

By substituting (3.26) into (3.25), we get

$$\delta_v(-\omega_2^{-1}\omega_2^{14}s_1)^{\rho_2^2/\rho_2^1} + \omega_2^{-1}\omega_2^{21}\delta_{\bar{u}} s_1^{\lambda_1^3/\lambda_1^1} + h.o.t. = 0.$$

As in this situation, $0 < s_1 \ll 1$ always guarantees $0 < s_2 \ll 1$, it is sufficient to find the sufficient small positive solution s_1 for the above equation. Thereafter, $\delta_v\omega_2\omega_2^{21}\delta_{\bar{u}} < 0$ is necessary. And when $\rho_2^2/\rho_2^1 > \lambda_1^3/\lambda_1^1$, we have

$$0 < s_1 = [-\frac{\omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}}{\delta_v(-\omega_2^{-1}\omega_2^{14})^{\rho_2^2/\rho_2^1}}]^{\rho_2^1\lambda_1^1/\rho_2^2\lambda_1^1 - \rho_2^1\lambda_1^3} \text{ which cannot be as small as possible;}$$

when $\rho_2^2/\rho_2^1 < \lambda_1^3/\lambda_1^1$, we have

$$0 < s_1 = [\frac{\delta_v\omega_2(-\omega_2^{-1}\omega_2^{14})^{\rho_2^2/\rho_2^1}}{\omega_2^{21}\delta_{\bar{u}}}]^{\lambda_1^1\rho_2^1/\lambda_1^3\rho_2^1 - \lambda_1^1\rho_2^2} \text{ which cannot be sufficiently small either.}$$

So, the periodic orbit cannot coexist with the persistent heterodimensional cycle. It is obvious to see that the homoclinic loop cannot coexist with the persistent heterodimensional cycle thanks to (3.26).

The proof is completed.□

Secondly, the question is whether system (1.1) can have periodic orbit for $\mu \in H_1$ or $\bar{H}_2$, that is, the coexistence of the homoclinic loop and the periodic orbit.

Set

$$C(\mu) \triangleq -(\omega_2^{14})^{-1}\omega_2^{21}\delta_{\bar{u}}\frac{\lambda_1^3}{\lambda_1^1}[-\omega_2(\omega_2^{14}\delta)^{-1}M_2^2\mu]^{(\lambda_1^3-\lambda_1^1)/\lambda_1^1},$$

$$D(\mu) \triangleq \omega_2[(\omega_2^{14})^{-1}]^2\omega_2^{21}\delta_{\bar{u}}\frac{\lambda_1^3(\lambda_1^3-\lambda_1^1)}{(\lambda_1^1)^2}[-\omega_2(\omega_2^{14}\delta)^{-1}M_2^2\mu]^{(\lambda_1^3-2\lambda_1^1)/\lambda_1^1},$$

$$E(\mu) \triangleq -\delta_v\omega_2^{-1}\omega_2^{14}\frac{\rho_2^2}{\rho_2^1}(-\delta^{-1}M_2^2\mu)^{(\rho_2^2-\rho_2^1)/\rho_2^1},$$

$$F(\mu) \triangleq \delta_v(\omega_2^{-1}\omega_2^{14})^2\frac{\rho_2^2(\rho_2^2-\rho_2^1)}{(\rho_2^1)^2}(-\delta^{-1}M_2^2\mu)^{(\rho_2^2-2\rho_2^1)/\rho_2^1}.$$

From (2.24), we have $s_1 = \omega_2(\omega_2^{14}\delta)^{-1}(-\delta s_2 - M_2^2\mu) + h.o.t.$. By substituting it into (2.23),

$$\delta_v s_2^{\rho_2^2/\rho_2^1} + \omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}[\omega_2(\omega_2^{14}\delta)^{-1}(-\delta s_2 - M_2^2\mu)]^{\lambda_1^3/\lambda_1^1} + M_2^1\mu + h.o.t. = 0. \tag{3.27}$$

The above equation can be reformulated as

$$N(s_2,\mu) \triangleq \delta_v s_2^{\rho_2^2/\rho_2^1} + C(\mu)s_2 + \frac{1}{2}D(\mu)s_2^2 + \bar{W}_2(\mu) + O(s_2^3) = 0 \text{ for } s_2 = o(|M_2^2\mu|).$$

As for $\mu \in \bar{H}_2$, $\bar{W}_2(\mu) = 0$. So,

if $\rho_2^2/\rho_2^1 \le 2$, $\delta_v s_2^{\rho_2^2/\rho_2^1} + C(\mu)s_2 + h.o.t. = 0$. Thus, $s_2 = 0$ or $s_2 = [-(\delta_v)^{-1}C(\mu)]^{\rho_2^1/(\rho_2^2-\rho_2^1)}$, where $s_2 = o(|M_2^2\mu|)$ if and only if $\lambda_1^3/\lambda_1^1 > \rho_2^2/\rho_2^1$.

If $\rho_2^2/\rho_2^1 > 2$, then $s_2 = 0$ or $s_2 = [-(\delta_v)^{-1}C(\mu)]^{\rho_2^1/(\rho_2^2-\rho_2^1)}$, where $s_2 = o(|M_2^2\mu|)$ and $\frac{1}{2}D(\mu)s_2^2 = o(\delta_v s_2^{\rho_2^2/\rho_2^1})$ if and only if $\lambda_1^3/\lambda_1^1 > \rho_2^2/\rho_2^1$.

Since for $\mu \in \bar{H}_2,\ s_2 = 0,\ s_1 = -\omega_2(\omega_2^{14}\delta)^{-1}M_2^2\mu + h.o.t. > 0$. Accordingly, for $s_2 = [-(\delta_v)^{-1}C(\mu)]^{\rho_2^1/(\rho_2^2-\rho_2^1)} = o(|M_2^2\mu|),\ s_1 = \omega_2(\omega_2^{14}\delta)^{-1}(-\delta s_2 - M_2^2\mu) + h.o.t. > 0$. That is, there is a unique periodic orbit coexists with the homoclinic loop which is homoclinic to p_2 if and only if $\lambda_1^3/\lambda_1^1 > \rho_2^2/\rho_2^1$.

On the other hand, we get $s_2 = -\omega_2^{-1}\omega_2^{14}s_1 - \delta^{-1}M_2^2\mu + h.o.t.$ from (2.24). Substitute it into (2.23), we have

$$\delta_v(-\omega_2^{-1}\omega_2^{14}s_1 - \delta^{-1}M_2^2\mu)^{\rho_2^2/\rho_2^1} + \omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}s_1^{\lambda_1^3/\lambda_1^1} + M_2^1\mu + h.o.t. = 0. \tag{3.28}$$

It is equivalent to

$$M(s_1,\mu) \triangleq \omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}s_1^{\lambda_1^3/\lambda_1^1} + E(\mu)s_1 + \frac{1}{2}F(\mu)s_1^2 + W_1(\mu) + O(s_1^3) = 0 \text{ for } s_1 = o(|M_2^2\mu|).$$

As for $\mu \in H_1,\ W_1(\mu) = 0$. Consequently,

if $\lambda_1^3/\lambda_1^1 \le 2,\ \omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}s_1^{\lambda_1^3/\lambda_1^1} + E(\mu)s_1 + h.o.t. = 0$, we have $s_1 = 0$ or $s_1 = [-\omega_2(\omega_2^{21})^{-1}(\delta_{\bar{u}})^{-1}E(\mu)]^{\lambda_1^1/(\lambda_1^3-\lambda_1^1)}$, where $s_1 = o(|M_2^2\mu|)$ if and only if $\rho_2^2/\rho_2^1 > \lambda_1^3/\lambda_1^1$.

If $\lambda_1^3/\lambda_1^1 > 2$, then $s_1 = 0$ or $s_1 = [\omega_2(\omega_2^{21})^{-1}(\delta_{\bar{u}})^{-1}E(\mu)]^{\lambda_1^1/(\lambda_1^3-\lambda_1^1)}$, where $s_1 = o(|M_2^2\mu|)$ and $\frac{1}{2}F(\mu)s_1^2 = o(\omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}s_1^{\lambda_1^3/\lambda_1^1})$ if and only if $\rho_2^2/\rho_2^1 > \lambda_1^3/\lambda_1^1$.

As for $\mu \in H_1,\ s_1 = 0,\ s_2 = -\delta^{-1}M_2^2\mu + h.o.t. > 0$. Accordingly, for $s_1 = [\omega_2(\omega_2^{21})^{-1}(\delta_{\bar{u}})^{-1}E(\mu)]^{\lambda_1^1/(\lambda_1^3-\lambda_1^1)} = o(|M_2^2\mu|),\ s_2 = -\omega_2^{-1}\omega_2^{14}s_1 - \delta^{-1}M_2^2\mu + h.o.t. > 0$. That is, there is a unique periodic orbit coexists with the homoclinic loop which is homoclinic to p_1 if and only if $\rho_2^2/\rho_2^1 > \lambda_1^3/\lambda_1^1$.

Summing up these analysis, we can state the following result.

THEOREM 3.8. *Suppose that* $(H_1) - (H_5)$ *and* $\rho_1^1 > \lambda_1^1,\ \lambda_2^1 > \rho_2^1,\ \omega_2^{14} \ne 0,\ \omega_2^{24} = 0$, *then there is a unique periodic orbit coexisting with the homoclinic loop for system (1.1) with* $0 < |\mu| \ll 1$. *Precisely speaking,*
(i) for $0 < |\mu| \ll 1$, *there is a unique periodic orbit coexists with the homoclinic loop which is homoclinic to* p_2 *in the region* $\{\mu:\ \omega_2\omega_2^{14}M_2^2\mu < 0,\ \omega_2\omega_2^{21}\delta_{\bar{u}}M_2^1\mu < 0\}$ *if and only if* $\lambda_1^3/\lambda_1^1 > \rho_2^2/\rho_2^1$.
(ii) for $0 < |\mu| \ll 1$, *there is a unique periodic orbit coexists with the homoclinic loop which is homoclinic to* p_1 *in the region* $\{\mu:\ \delta_v M_2^1\mu < 0,\ M_2^2\mu < 0\}$ *if and only if* $\rho_2^2/\rho_2^1 > \lambda_1^3/\lambda_1^1$.

Now, we try to study the existence of the periodic orbit in the small neighborhood of H_1 and $\bar{H}_2$.

As μ leaves $\bar{H}_2$ slightly, we have $\omega_2\omega_2^{14}M_2^2\mu < 0$. Thereafter, for $0 \le s_2 \ll |M_2^2\mu|$, owing to (2.24), $s_1 = -\omega_2(\omega_2^{14})^{-1}s_2 - \omega_2(\omega_2^{14})^{-1}\delta^{-1}M_2^2\mu + h.o.t. > 0$. For $\mu \in \bar{H}_2$, from (2.23) and (2.24), we have

$$\begin{aligned}&\lambda_1^3/\lambda_1^1 \cdot \omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}s_1^{(\lambda_1^3-\lambda_1^1)/\lambda_1^1} \cdot s_{1\mu} + M_2^1 + h.o.t. = 0,\\ &\delta s_{2\mu} + \omega_2^{-1}\omega_2^{14}\delta s_{1\mu} + M_2^2 + h.o.t. = 0.\end{aligned}$$

So, we obtain $s_{2\mu} = \frac{\lambda_1^1\omega_2^{14}M_2^1}{\lambda_1^3\omega_2^{21}\delta_{\bar{u}}[-\omega_2(\omega_2^{14})^{-1}\delta^{-1}M_2^2\mu]^{(\lambda_1^3-\lambda_1^1)/\lambda_1^1}} - \delta^{-1}M_2^2 + +h.o.t.$. Accordingly, s_2 increases along the direction M_2^1 (resp. $-M_2^1$) as $\omega_2^{14}\omega_2^{21}\delta_{\bar{u}} > 0$ (resp. $\omega_2^{14}\omega_2^{21}\delta_{\bar{u}} < 0$).

Similarly, as μ leaves H_1 slightly, we have $M_2^2\mu < 0$. Therefore, for $0 \le s_1 \ll |M_2^2\mu|$, due to (24), $s_2 = -\omega_2^{-1}\omega_2^{14}s_1 - \delta^{-1}M_2^2\mu + h.o.t. > 0$. For $\mu \in H_1$, from (2.23) and (2.24), we get

$$\rho_2^2/\rho_2^1 \cdot \delta_v s_2^{(\rho_2^2-\rho_2^1)/\rho_2^1} s_{2\mu} + M_2^1 + h.o.t. = 0,$$
$$\delta s_{2\mu} + \omega_2^{-1}\omega_2^{14}\delta s_{1\mu} + M_2^2 + h.o.t. = 0.$$

Thus, $s_{1\mu} = \frac{\rho_2^1\omega_2(\omega_2^{14})^{-1}M_2^1}{\rho_2^2\delta_v(-\delta^{-1}M_2^2\mu)^{(\rho_2^2-\rho_2^1)/\rho_2^1}} - \omega_2(\omega_2^{14})^{-1}\delta^{-1}M_2^2 + h.o.t.$. That is, s_1 increases along the direction M_2^1 (resp. $-M_2^1$) as $\delta_v\omega_2\omega_2^{14} > 0$ (resp. $\delta_v\omega_2\omega_2^{14} < 0$).

Combined with the above theorem, we have

THEOREM 3.9. *Suppose that* $(H_1) - (H_5)$ *and* $\rho_1^1 > \lambda_1^1,\ \lambda_2^1 > \rho_2^1,\ \omega_2^{14} \ne 0,\ \omega_2^{24} = 0$, *then*
(1) *for* $0 < |\mu| \ll 1$, *system* (1) *have two and only two periodic orbits as* μ *is in the small one-sided neighborhood of* $\bar{H}_2$ *pointing to* M_2^1 *(resp.* $-M_2^1$*) and a unique periodic orbit in the small other-sided neighborhood of* $\bar{H}_2$ *as* $\delta_{\bar{u}}\omega_2^{14}\omega_2^{21} > 0$ *(resp.* $\delta_{\bar{u}}\omega_2^{14}\omega_2^{21} < 0$*) if and only if* $\lambda_1^3/\lambda_1^1 > \rho_2^2/\rho_2^1$.
(2) *for* $0 < |\mu| \ll 1$, *system* (1) *have two and only two periodic orbits as* μ *is in the small one-sided neighborhood of* H_1 *pointing to* M_2^1 *(resp.* $-M_2^1$*) and a unique periodic orbit in the small other-sided neighborhood of* H_1 *as* $\delta_v\omega_2\omega_2^{14} > 0$ *(resp.* $\delta_v\omega_2\omega_2^{14} < 0$*) if and only if* $\rho_2^2/\rho_2^1 > \lambda_1^3/\lambda_1^1$.
(3) *for* $0 < |\mu| \ll 1$, *there is a unique periodic orbit for system* (1) *as* μ *is in the small one-sided neighborhood of* $\bar{H}_2$ *pointing to* M_2^1 *(resp.* $-M_2^1$) *as* $\delta_{\bar{u}}\omega_2^{14}\omega_2^{21} > 0$ *(resp.* $\delta_{\bar{u}}\omega_2^{14}\omega_2^{21} < 0$) *if* $\lambda_1^3/\lambda_1^1 \le \rho_2^2/\rho_2^1$.
(4) *for* $0 < |\mu| \ll 1$, *there is a unique periodic orbit for system* (1) *as* μ *is in the small one-sided neighborhood of* H_1 *pointing to* M_2^1 *(resp.* $-M_2^1$) *as* $\delta_v\omega_2\omega_2^{14} > 0$ *(resp.* $\delta_v\omega_2\omega_2^{14} < 0$) *if* $\rho_2^2/\rho_2^1 \le \lambda_1^3/\lambda_1^1$.

In the following, we firstly try to consider the double periodic orbit bifurcation for the case (3), which is corresponding to the double positive zero point bifurcation for Eq.(2.23) and Eq.(2.24). Then, it is sufficient to consider the double positive zero point for $M(s_1,\mu)$ or $N(s_2,\mu)$. Here, we consider the double positive zero point $\bar{s}_1$ for $M(s_1,\mu)$ with $s_1 = o(|M_2^2\mu|)$.

Denote by

$$\begin{aligned} P(s_1,\mu) &= \frac{1}{2}F(\mu)s_1^2 + E(\mu)s_1 + W_1(\mu) + h.o.t., \\ Q(s_1,\mu) &= -\omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}s_1^{\lambda_1^3/\lambda_1^1} + h.o.t.. \end{aligned}$$

Then, $\bar{s}_1$ should satisfy

$$P(s_1,\mu) = Q(s_1,\mu), \tag{3.29}$$
$$P'(s_1,\mu) = Q'(s_1,\mu), \tag{3.30}$$
$$P''(s_1,\mu) \ne Q''(s_1,\mu).$$

(3.30) is equivalent to

$$-\omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}\lambda_1^3 s_1^{(\lambda_1^3-\lambda_1^1)/\lambda_1^1} + h.o.t. = \lambda_1^1F(\mu)s_1 + \lambda_1^1E(\mu) + h.o.t..$$

We denote it as $N_1(s_1,\mu) = N_2(s_1,\mu)$. Then,

(*i*). for $\omega_2\omega_2^{14} < 0,\ \omega_2^{14}\omega_2^{21}\delta_{\bar{u}}\delta_v < 0$, we have $E(\mu)\cdot F(\mu) > 0,\ \omega_2\omega_2^{21}\delta_{\bar{u}}\cdot E(\mu) > 0$, then it is easy to see that Eq.(3.30) has no positive solution. So, system (1) has no double periodic orbit for $\omega_2\omega_2^{14} < 0,\ \omega_2^{14}\omega_2^{21}\delta_{\bar{u}}\delta_v < 0$.

(*ii*). For $\omega_2\omega_2^{14} > 0,\ \omega_2^{14}\omega_2^{21}\delta_{\bar{u}}\delta_v > 0$, we have $E(\mu)\cdot F(\mu) < 0,\ \omega_2\omega_2^{21}\delta_{\bar{u}}\cdot E(\mu) < 0$. Based on the values of $N_i(0,\mu),\ i = 1,2$, and their monotonicity, we obtain Eq.(3.30) has at most one positive solution $\bar{s}_1 = o(|M_2^2\mu|)$. So, system (1) has at most one double periodic orbit for $\omega_2\omega_2^{14} > 0,\ \omega_2^{14}\omega_2^{21}\delta_{\bar{u}}\delta_v > 0$.

Set $s_1^{(\lambda_1^3-\lambda_1^1)/\lambda_1^1} = h$, then Eq.(3.30) becomes

$$-\omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}\lambda_1^3 \cdot h - \lambda_1^1 E(\mu) + h.o.t. = \lambda_1^1 F(\mu)\cdot h^{\lambda_1^1/(\lambda_1^3-\lambda_1^1)} + h.o.t.. \tag{3.31}$$

We denote by

$$N_1(h) = -\omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}\lambda_1^3 \cdot h - \lambda_1^1 E(\mu) + h.o.t.,$$
$$N_2(h) = \lambda_1^1 F(\mu)\cdot h^{\lambda_1^1/(\lambda_1^3-\lambda_1^1)} + h.o.t..$$

Then, for every positive intersection point of curves $W = N_1(h)$ and $W = N_2(h)$ with $h = o(|M_2^2\mu|^{(\lambda_1^3-\lambda_1^1)/\lambda_1^1})$, $M(s_1,\mu)$ has a corresponding double zero point bifurcation surface SN_1^h.

By solving $N_1'(h) = N_2'(h)$, we get a unique solution

$$\bar{h}^{(\lambda_1^3-2\lambda_1^1)/(\lambda_1^3-\lambda_1^1)} = -\frac{(\lambda_1^1)^2\cdot\omega_2 F(\mu)}{\lambda_1^3(\lambda_1^3-\lambda_1^1)\omega_2^{21}\delta_{\bar{u}}} + h.o.t.,$$

where $\bar{h} = o(|M_2^2\mu|^{(\lambda_1^3-\lambda_1^1)/\lambda_1^1})$ for $\rho_2^2/\rho_2^1 > \lambda_1^3/\lambda_1^1$. Substitute it into Eq.(3.31), we have

$$\begin{aligned}
&-\omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}\lambda_1^3 \triangleq -\omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}\bar{\lambda}_1^3 \\
=\ & \lambda_1^1 F(\mu)\cdot\left[-\frac{(\lambda_1^1)^2\cdot\omega_2 F(\mu)}{\lambda_1^3(\lambda_1^3-\lambda_1^1)\omega_2^{21}\delta_{\bar{u}}}\right]^{-1} + \lambda_1^1 E(\mu)\cdot\left[-\frac{(\lambda_1^1)^2\cdot\omega_2 F(\mu)}{\lambda_1^3(\lambda_1^3-\lambda_1^1)\omega_2^{21}\delta_{\bar{u}}}\right]^{-(\lambda_1^3-\lambda_1^1)/(\lambda_1^3-2\lambda_1^1)} \\
=\ & \lambda_1^1 E(\mu)\cdot\left[-\frac{(\lambda_1^1)^2\cdot\omega_2 F(\mu)}{\lambda_1^3(\lambda_1^3-\lambda_1^1)\omega_2^{21}\delta_{\bar{u}}}\right]^{-(\lambda_1^3-\lambda_1^1)/(\lambda_1^3-2\lambda_1^1)} + O(1).
\end{aligned}$$

Easily we can see from the above equation that,

(iii). for $\omega_2\omega_2^{14} > 0,\ \omega_2^{14}\omega_2^{21}\delta_{\bar{u}}\delta_v < 0$, we have $\omega_2\omega_2^{21}\delta_{\bar{u}}E(\mu) > 0$, so it is impossible for $\lambda_1^3(\mu) > 0$. Consequently, there is no tangent point for $W = N_1(h)$ and $W = N_2(h)$ at this rate. Based on the figures of the two curves, there will be at most one positive intersection point $\bar{s}_1$ satisfying $\bar{s}_1 > s_0$, where $s_0 = O(|M_2^2\mu|) > 0$ is the zero point of $N_2(s_1,\mu)$. Then, it is impossible for $\bar{s}_1 = o(|M_2^2\mu|)$. So, system (1) has no double periodic orbit for $\omega_2\omega_2^{14} > 0,\ \omega_2^{14}\omega_2^{21}\delta_{\bar{u}}\delta_v < 0$.

(iv). for $\omega_2\omega_2^{14} < 0,\ \omega_2^{14}\omega_2^{21}\delta_{\bar{u}}\delta_v > 0$, as $\omega_2\omega_2^{21}\delta_{\bar{u}}E(\mu) < 0$, then $\lambda_1^3(\mu)|_{SN_1^{\bar{h}}} > 0$. Correspondingly, the curves $W = N_1(h)$ and $W = N_2(h)$ are tangent at the unique positive point $\bar{h}$ as $\lambda_1^3(\mu) = \bar{\lambda}_1^3(\mu)$. Notice that

$$N_1(0) = -\lambda_1^1 E(\mu) + h.o.t., \qquad N_1'(h) = -\omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}\lambda_1^3 + h.o.t.,$$
$$N_2(0) \approx 0, \qquad N_2'(h) = \frac{(\lambda_1^1)^2}{\lambda_1^3-\lambda_1^1}F(\mu)h^{(2\lambda_1^1-\lambda_1^3)/\lambda_1^1} + h.o.t..$$

We claim, for $\omega_2\omega_2^{21}\delta_{\bar{u}} < 0$ (resp. $\omega_2\omega_2^{21}\delta_{\bar{u}} > 0$), the curves $W = N_1(h)$ and $W = N_2(h)$ have two positive intersection points as $\lambda_1^3(\mu) > \bar{\lambda}_1^3(\mu)$ (resp. $\lambda_1^3(\mu) < \bar{\lambda}_1^3(\mu)$) and no positive intersection points as $\lambda_1^3(\mu) < \bar{\lambda}_1^3(\mu)$ (resp. $\lambda_1^3(\mu) > \bar{\lambda}_1^3(\mu)$).

In conclusion, we have

THEOREM 3.10. *Suppose that* $(H_1)-(H_5)$ *hold and* $\rho_1^1 > \lambda_1^1,\ \lambda_2^1 > \rho_2^1,\ \omega_2^{14} \neq 0,\ \omega_2^{24} = 0$, *then*
(1). *for* $\omega_2\omega_2^{14} < 0$ (*resp.* $\omega_2\omega_2^{14} > 0$), $\omega_2^{14}\omega_2^{21}\delta_{\bar{u}}\delta_v < 0$, *system* (1) *has no double periodic orbit.*
(2). *for* $\omega_2\omega_2^{14} > 0,\ \omega_2^{14}\omega_2^{21}\delta_{\bar{u}}\delta_v > 0$, *system* (1) *has at most one double periodic orbit.*

(3). *for* $\omega_2\omega_2^{14} < 0,\ \omega_2^{14}\omega_2^{21}\delta_{\bar{u}}\delta_v > 0$, *if* $\rho_2^2/\rho_2^1 > \lambda_1^3/\lambda_1^1$, *then system* (1) *has exactly two double periodic orbit bifurcation surfaces when* $\lambda_1^3(\mu) > \bar{\lambda}_1^3(\mu)$ (*resp.* $\lambda_1^3(\mu) < \bar{\lambda}_1^3(\mu)$) *as* $\omega_2\omega_2^{21}\delta_{\bar{u}} < 0$ (*resp.* $\omega_2\omega_2^{21}\delta_{\bar{u}} > 0$);
system (1) *has a unique double periodic orbit bifurcation surface when* $\lambda_1^3(\mu) = \bar{\lambda}_1^3(\mu)$ *either* $\omega_2\omega_2^{21}\delta_{\bar{u}} < 0$ *or* $\omega_2\omega_2^{21}\delta_{\bar{u}} > 0$;
system (1) *has no double periodic orbit bifurcation surface when* $\lambda_1^3(\mu) < \bar{\lambda}_1^3(\mu)$ (*resp.* $\lambda_1^3(\mu) > \bar{\lambda}_1^3(\mu)$) *as* $\omega_2\omega_2^{21}\delta_{\bar{u}} < 0$ (*resp.* $\omega_2\omega_2^{21}\delta_{\bar{u}} > 0$).
The corresponding surface SN_1^h *and double positive zero point* $\bar{s}_1$ *are given by*

$$SN_1^h:\ \frac{1}{2}F(\mu)h^{2\lambda_1^1/(\lambda_1^3-\lambda_1^1)}+E(\mu)h^{\lambda_1^1/(\lambda_1^3-\lambda_1^1)}+W_1(\mu)+h.o.t. = -\omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}h^{\lambda_1^3/(\lambda_1^3-\lambda_1^1)}+h.o.t.,$$

$$\bar{s}_1 = h^{\lambda_1^1/(\lambda_1^3-\lambda_1^1)},$$

where h *are the positive solutions of Eq.(3.31).*

Remark 3.2. *In the above theorem, the tangency conditions for*

$$P(s_1,\mu) = Q(s_1,\mu),\ N_1(h) = N_2(h),$$

given by $\lambda_1^3(\mu) = \bar{\lambda}_1^3(\mu)$ *and* $\mu \in SN_1^h$ *are equivalent to the condition that* $M(s_1,\mu)$ *has a unique triple positive zero point, which corresponds to the triple periodic orbit for system* (1).

Theorem 3.11. *Suppose that* $(H_1)-(H_5)$ *hold and* $\rho_1^1 > \lambda_1^1, \lambda_2^1 > \rho_2^1$, $\rho_2^2/\rho_2^1 > \lambda_1^3/\lambda_1^1$, $\omega_2^{14} \neq 0,\ \omega_2^{24} = 0$, *then for* $\omega_2\omega_2^{14} < 0,\ \omega_2^{14}\omega_2^{21}\delta_{\bar{u}}\delta_v > 0$, *system* (1) *has a unique triple periodic orbit bifurcation surface*

$$\begin{aligned} SN_1^2:\quad & \tfrac{1}{2}F(\mu)\left[-\tfrac{(\lambda_1^1)^2\cdot\omega_2F(\mu)}{\lambda_1^3(\lambda_1^3-\lambda_1^1)\omega_2^{21}\delta_{\bar{u}}}\right]^{2\lambda_1^1/(\lambda_1^3-2\lambda_1^1)} + E(\mu)\left[-\tfrac{(\lambda_1^1)^2\cdot\omega_2F(\mu)}{\lambda_1^3(\lambda_1^3-\lambda_1^1)\omega_2^{21}\delta_{\bar{u}}}\right]^{\lambda_1^1/(\lambda_1^3-2\lambda_1^1)} \\ & +W_1(\mu)+h.o.t. = -\omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}\left[-\tfrac{(\lambda_1^1)^2\cdot\omega_2F(\mu)}{\lambda_1^3(\lambda_1^3-\lambda_1^1)\omega_2^{21}\delta_{\bar{u}}}\right]^{\lambda_1^3/(\lambda_1^3-2\lambda_1^1)} + h.o.t., \\ & \lambda_1^1F(\mu)\left[-\tfrac{(\lambda_1^1)^2\cdot\omega_2F(\mu)}{\lambda_1^3(\lambda_1^3-\lambda_1^1)\omega_2^{21}\delta_{\bar{u}}}\right]^{\lambda_1^1/(\lambda_1^3-2\lambda_1^1)} + \lambda_1^1E(\mu)+h.o.t. \\ & = -\omega_2^{-1}\omega_2^{21}\delta_{\bar{u}}\lambda_1^3\left[-\tfrac{(\lambda_1^1)^2\cdot\omega_2F(\mu)}{\lambda_1^3(\lambda_1^3-\lambda_1^1)\omega_2^{21}\delta_{\bar{u}}}\right]^{(\lambda_1^3-\lambda_1^1)/(\lambda_1^3-2\lambda_1^1)} + h.o.t.. \end{aligned}$$

when $\lambda_1^3(\mu) = \bar{\lambda}_1^3(\mu)$ *either* $\omega_2\omega_2^{21}\delta_{\bar{u}} < 0$ *or* $\omega_2\omega_2^{21}\delta_{\bar{u}} > 0$. *And the corresponding triple positive zero point is*

$$\bar{s}_1^{(2)} = \left[-\frac{(\lambda_1^1)^2\cdot\omega_2F(\mu)}{\lambda_1^3(\lambda_1^3-\lambda_1^1)\omega_2^{21}\delta_{\bar{u}}}\right]^{\lambda_1^1/(\lambda_1^3-2\lambda_1^1)} + h.o.t..$$

Proof. Obviously, $\bar{s}_1^{(2)}$ should satisfy Eq.(3.29)-(3.30) and $P''(s_1,\mu) = Q''(s_1,\mu)$. From $P''(s_1,\mu) = Q''(s_1,\mu)$, we have

$$\bar{s}_1^{(2)} = \left[-\frac{(\lambda_1^1)^2\cdot\omega_2F(\mu)}{\lambda_1^3(\lambda_1^3-\lambda_1^1)\omega_2^{21}\delta_{\bar{u}}}\right]^{\lambda_1^1/(\lambda_1^3-2\lambda_1^1)} + h.o.t..$$

By substituting it into Eq.(3.29)-(3.30), we obtain the triple periodic orbit bifurcation surface SN_1^2. □

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中文摘要

本文主要研究带有可加噪声扰动的随机微分方程的稳定性问题, 随机吸引子的存在性问题以及同异宿轨道分支问题. 全文内容共分两部分.

在第一部分, 首先考虑了可失掉免疫力的随机SIRS模型和带有分布时滞的随机SIRS模型. 在SIRS人口模型中, γ 代表康复人群失掉免疫力的比率. 对于$\gamma=0$ 的情况, 文章[**89**]中已有研究. 本文得到了可失掉免疫力的随机SIRS模型的随机稳定性条件. 对于不含有时滞的SIRS系统, 我们的条件 $0<\beta<\lambda+\mu-\frac{\sigma^2}{2}$ 推广了文献[**89**]中对$\gamma=0$的已有结果. 我们采用待定Lyapunov泛函的方法得到理论结果. 在此基础上,我们进行了三组数值模拟, 模拟结果证实了理论结果的可靠性, 同时我们猜测免疫能力丧失与否, 不会改变疾病消失平衡点稳定的临界值.

其次, 研究了定义在整个$\mathbb{R}^n$上的带有可加噪声的随机Ginzburg-Landau方程, 证明了当$\sqrt{3}\kappa\geq|\beta|$时, 由带有可加噪声的随机Ginzburg-Landau方程定义的随机动力系统ϕ在$\mathbb{L}^2(\mathbb{R}^n)$ 上有唯一的$\mathcal{D}$-随机吸引子. 考虑到无界区域上Sobolev嵌入的紧性不再成立, 通常的方法就不再能够保证解算子ϕ是一个紧算子, 从而给吸引子存在性的判断带来了很大的挑战. 在确定性方程中, 克服这个困难我们可以采用能量方程的办法. 然而对于定义在无界区域上的随机吸引子存在性的判断, 一直以来没有找到有效解决的办法. 本文通过尾数估计的办法首先得到解算子ϕ的渐进紧性, 再加上ϕ存在闭的随机吸引集, 从而证明了随机吸引子的存在性.

论文第二部分, 我们分别研究了带有倾斜翻转的3D同宿轨道分支, 扭曲双同宿环分支和带有轨道翻转的异维环分支问题. 在这部分, 我们研究光滑系统

$$\dot{z}=f(z)+g(z,\mu), \tag{32}$$

及其未扰动系统

$$\dot{z}=f(z), \tag{33}$$

其中 $z\in\mathbb{R}^{m+n+2},\ m\geq 0,n\geq 0,m+n>0,\ \mu\in\mathbb{R}^l,\ l\geq 2,\ 0\leq||\mu||\ll 1,\ g(z,0)=0.$

$||\cdot||$ 代表$\mathbb{R}^l$空间中由内积定义的模.

我们采用了局部活动坐标架的方法. 考虑到稳定叶层和不稳定叶层在同宿环小邻域内的动力学行为中扮演着强势的关键角色. 因此, 在一次近似的意义下, 由稳定流形和不稳定流形沿同宿或异宿轨道组成的切向量丛, 充分保留并展示了同宿轨道附近的动力学性质, 如几何不变性, 翻转性, 扭曲性, 扩张性和压缩性等. 因此, 巧妙的选取沿着同异宿轨道的切向量丛及其补空间丛中的向量丛组成活动坐标架, 不仅可以将系统化为较为简单的形式, 而且由此得到的Poincare映射和分支方程中的关键参数具有明确的几何和动力学意义.

首先研究带有倾斜翻转的非共振$3D$ 同宿环分支. 即, $m=0,n=1,l=2,f(0)=0.$

系统 (33) 的线性变分系统及其伴随系统分别为

$$\dot{z}=Df(r(t))z, \tag{34}$$
$$\dot{z}=-(Df(r(t)))^*z. \tag{35}$$

令 $r(t)=(r^x(t),r^y(t),r^v(t))$. 取 $T>0$ 充分大, 使得 $r(-T)=(\delta,0,0),\ r(T)=(0,\delta,\delta_v)$, 其中 $|\delta_v|=o(\delta)$, δ 充分小, 满足 $\{(x,y,v):\ |x|,|y|,|v|<2\delta\}\subset U$.

则系统 (34) 有一个基解矩阵 $Z(t)$ 满足

$$z(-T)=\begin{pmatrix}0&\omega_{21}&\omega_{31}\\0&0&\omega_{32}\\1&0&\omega_{33}\end{pmatrix},\qquad z(T)=\begin{pmatrix}\omega_{11}&0&0\\\omega_{12}&1&0\\\omega_{13}&\omega_{23}&1\end{pmatrix},$$

其中 $\mid\omega_{23}\mid\ll 1,\omega_{21}<0,\omega_{11}\neq 0,\omega_{32}\neq 0$. 系统 (35) 有一个基解矩阵 $\Phi(t)=(Z^{-1}(t))^*$, 其中 $\Phi(t)=(\phi_i^1(t),\phi_i^2(t),\phi_i^3(t))$. 记

$$M_j=\int_{-T}^{T}(\phi_j(t))^*g_\mu(r(t),0)\,\mathrm{d}t,j=1,3.$$

我们首先考虑不存在强倾斜翻转的同宿环分支, 即 $\omega_{33} \neq 0$. 这种情况下, 分支结果是唯一的, 即系统(32)或者保存原来的同宿轨道, 或者分支出唯一的周期轨. 其次, 在强倾斜翻转的情况下, 即当 $\omega_{33} = 0$ 时, 我们同样得到分支结果是唯一的, 即对于系统(32)来说, 或者同宿轨道保存, 或者分支出唯一的周期轨. 最后, 我们考虑带有强倾斜翻转的'弱型'同宿轨分支, 即当 $\omega_{33} = 0,\ \delta_v = 0$ 时. 我们得到了1-同宿轨分支曲面, 2重周期轨分支曲面及周期为2^{n-1}的周期轨道发生周期加倍的倍周期分支曲面P^{2^n}以及任意的2^n-同宿分支曲面H^{2^n}的存在性, $\forall\ n \in \mathbb{N}$. 同时, 为了更好的说明我们的分支结果, 给出了完整的分支图.

其次, 研究余维为2的扭曲双同宿环分支. 考虑C^r系统, 同时$m \geq 0, n \geq 0, m+n > 0,\ l \geq 2,\ f(0) = 0$. 与前面带有倾斜翻转的$3D$同宿环分支不同的是, 未扰动向量场(33)的退化性只来自双同宿环本身. 在其中一条轨道发生扭曲时, 我们得到了1-1双同宿轨道, 2-1 双同宿轨道, 2-1 右同宿轨道, 1-1 大同宿轨道, 2-1 大同宿轨道和2-1 大周期轨道的存在和唯一性问题. 在两条同宿轨道均发生扭曲时, 我们得到了1-1 双同宿轨道, 1-2 双同宿轨道, 2-1 双同宿轨道, 2-2 双同宿轨道, 2-1 大同宿轨道, 1-2 大同宿轨道, 2-2 大同宿轨道, 2-2 右同宿轨道, 2-2 大同宿轨道, 2-2 左同宿轨道和 2-2 大周期轨道的存在性和不存在性问题. 此处, 右(或者左)指相应的轨道围绕Γ_1(或者Γ_2)转的时间为无穷大, 而在另一条同宿轨道小邻域内转的时间为有限的. 大轨道指围绕整个双同宿环$\Gamma = \Gamma_1 \cup \Gamma_2$且在其小邻域内的轨道. p-q轨道指该轨道在$\Gamma = \Gamma_1 \cup \Gamma_2$得小邻域内围绕$\Gamma_1$转p圈, 围绕 Γ_2转q圈.

同时, 我们给出了具体的分支曲面及其存在区域. 并在由前两个Melnikov向量组成的2维子空间上画出了各种分支集.

最后, 研究带有轨道翻转的异维环分支. 关于异维环的课题研究, 是极具挑战性和难度的一个课题, 这主要体现在: 等维的异宿环的连接轨道的余维数是均匀分布的(一般为1), 而异维环的连接轨道的余维数是非均匀分布, 甚至是集中分布的, 并由此导致了分支方程的强退化性.

设 $z \in \mathbb{R}^4,\ \mu \in \mathbb{R}^l,\ l \geq 2$, 且未扰系统 (33) 有两个奇点 p_1, p_2, 即 $f(p_i) = 0,\ g(p_i, \mu) = 0,\ i = 1, 2$. 我们考虑带有轨道翻转的异维环分支, 给出了同宿轨道, 异宿轨道和周期轨道的存在性, 唯一性和不存在性问题. 同时, 我们也给出了同宿轨道和周期轨道共存的条件. 而保存下来的异维环和周期轨道不能共存, 分支出来的同宿环和保存下来的异维环也不可能共存. 此外, 计算出了2重周期轨和3重周期轨的分支曲面. 综合这些分支分析, 确立了各种分支曲面及其存在区域.

关键词: 活动坐标架; 倾斜翻转; 轨道翻转; 双同宿环; 大周期轨; 大同宿轨; 分支; 扭曲轨道; 异维环; 随机 SIRS 模型; Lyapunov 函数; 随机稳定; 随机 Ginzburg-Landau 方程; 随机动力系统; 拉回吸引子; 随机吸引子; 渐进紧

Stabilité stochastique, attracteur alléatoire et bifurcation d'orbites homocline et heterocline

Résumé

Cette thèse est consacré à l'étude de certaines équations différentielles stochastiques et la bifurcation des orbites homocline et heterocline. On présente les conditions pour la stabilité stochastique du modèle SIRS stochastique avec ou sans retard. Nous montrons que l'équation stochastique de Ginzburg-Landau avec perturbation aléatoire additive possède un unique D-attracteur aléatoire dans l'espace entier. Dans la seconde partie, en utilisant la méthode des coordonnées actives locales, on étudie la bifurcation dans trois cas de figure : la bifurcation d'orbite homocline non résonante en dimension 3 avec inclination-flip, la bifurcation d'orbites homocline doubles tordus de codimension 2, et la bifurcation de cycle heterodimensionnel dégénéré avec orbite-flip. Dans le premier cas nous montrons, pour le systeme perturbé, l'existence d'orbite 1-homocline, orbite 1-périodique, orbite 2^n-homocline et orbite 2^n-périodique. Dans le deuxieme cas, on montre des résultats de bifurcation sous la condition d'une orbite tordu ou les deux tordus. Dans le troisième situation, sous des hypotheses génériques, nous présentons des conditions pour l'existence, unicité, co-existence ou non-co-existence d'orbite homocline, d'orbite heterocline et d'orbite périodique. Dans tous les cas les surfaces de bifurcation sont obtenues et elles sont présentées dans le sous espace de dimension 2 engendré par les deux premiers vecteurs de Melnikov.

Stochastic stability, random attractor and bifurcation of homoclinic and heteroclinic orbit

Abstract

The thesis is devoted to the study of some stochastic differential equations and homoclinic and heteroclinic bifurcations. We present the stability conditions of the disease-free equilibrium for the stochastic SIRS model with or without distributed time delay. We show that the stochastic Ginzburg-Landau equation with additive noise on the entire n-dimensional space possesses a unique D-random attractor. In the second part, by employing the local active coordinates method, we study the bifurcations in three situations : the bifurcation of the non-resonant 3D homoclinic orbit with inclination-flip, codimension 2 bifurcation of twisted double homoclinic loops, and heterodimensional cycle bifurcation with orbit-flip. In the first case, we show, for the perturbed systems, the existence of 1-homoclinic orbit, 1-periodic orbit, 2^n-homoclinic orbit and 2^n-periodic orbit. In the second case, we obtain bifurcation results both under the condition of one twisted orbit and double twisted orbits. In the last case, under some generic hypotheses, we present conditions for the existence, uniqueness, coexistence or non-coexistence of the homoclinic orbit, heteroclinic orbit and periodic orbit. In all cases we figure out the bifurcation diagrams based on the existence region and they are presented on the 2-dimensional subspace spanned by the first two Melnikov vectors.

