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Opérades de Koszul et homologie des algèbres en caractéristique positive

THÈSE

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par

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Avant-propos

Cette thèse se compose de quatre parties :

1. une introduction aux thèmes étudiés dans la thèse, suivie d'un résumé des travaux présentés dans les parties suivantes,
2. l'article *A Poincaré-Birkhoff-Witt criterion for Koszul operads*, manuscripta mathematica **131** (2010), pp. 87-110,
3. l'article *Gamma-homology of algebras over an operad*, Algebraic & Geometric Topology **10** (2010), pp. 1747-1780,
4. la prépublication *Obstruction theory for algebras over an operad*.

La première partie est en français alors que les trois chapitres suivants sont en anglais. Chaque partie est suivie de sa propre bibliographie.
Un bref résumé de la thèse (en français et en anglais) précède l'introduction détaillée.

Table des matières

Résumé

Cette thèse s'inscrit dans l'étude des catégories d'algèbres associées aux opérades. On développe des outils d'algèbre homologique et une méthode générale de classification (à homotopie près) des morphismes entre algèbres sur une opérade.

La dualité de Koszul des opérades, introduite par V. Ginzburg et M. Kapranov, permet de construire des théories homologiques appropriées pour des catégories d'algèbres associées à certaines bonnes opérades – les opérades de Koszul. On donne dans la première partie de cette thèse un critère effectif pour qu'une opérade soit de Koszul : on montre qu'une opérade, linéairement engendrée par une base, est de Koszul dès lors que l'on peut ordonner sa base de façon compatible avec la structure de composition opéradique – on parle alors d'opérade de Poincaré-Birkhoff-Witt.

La théorie originale de Ginzburg-Kapranov s'applique en caractéristique nulle seulement. On construit une théorie homologique adaptée - la Gamma-homologie - pour l'étude des catégories d'algèbres différentielles graduées associées à une opérade de Koszul en toute caractéristique. Cette théorie généralise la Gamma-homologie définie par A. Robinson et S. Whitehouse pour la catégorie des algèbres commutatives.

On montre que la Gamma-homologie opéradique contient l'obstruction à la réalisation de morphismes entre algèbres sur une opérade, ainsi que l'obstruction à la réalisation d'homotopies entre morphismes, et donne de la sorte un outil général pour classifier les morphismes entre algèbres sur une opérade.

Mots clés : Opérades, algèbres, dualité de Koszul, théories homologiques, théories d'obstruction.

Abstract

This thesis is concerned with the study of categories of algebras associated to operads. We develop tools of homological algebra and a general method to classify morphisms in the homotopy category of algebras over an operad.

The Koszul duality of operads, introduced by V. Ginzburg and M. Kapranov, allows us to construct suitable homology theories for categories of algebras associated to some good operads – the Koszul operads. We give in the first part of this thesis an effective criterion to prove that an operad is Kozul : we show that an operad, linearly generated by a basis, is Koszul as soon as we can order its basis in a way compatibly with the operadic composition structure – we call such operads Poincaré-Birkhoff-Witt operads.

The original theory of Ginzburg and Kapranov works in characteristic zero only. We construct a homology theory - the Gamma-homology - for the study of the categories of the differential graded algebras associated to a Koszul operad in any characteristic. This theory generalizes the Gamma-homology introduced by A. Robinson and S. Whitehouse for the category of commutative algebras.

We show that our Gamma-homology contains the obstruction to the realization of morphisms between algebras over an operad, and also the obstruction to the realization of homotopies between morphisms. We obtain in this way a general tool to classify morphisms between algebras over an operad.

Keywords : Operads, algebras, Koszul duality, homology theories, obstruction theories.

Chapitre 1

Introduction

On commence par rappeler les notions d'opérade, de construction bar et de dualité de Koszul. Une fois ces premières définitions données, on met en perspective et on motive les différents thèmes étudiés dans ce travail :
- Dualité de Koszul
- Gamma-homologie
- Homologie des algèbres sur les opérades
- Théories de l'obstruction

La fin de l'introduction consiste en un résumé détaillé des trois articles contenus dans cette thèse.

1.1 Brève introduction aux opérades et à la dualité de Koszul

Soit $\mathbb{K}$ un anneau. On se place dans la catégorie monoïdale symétrique des $\mathbb{K}$-modules différentiels gradués (qu'on abrégera en dg-modules). Les définitions suivantes se généralisent à de nombreuses catégories monoïdales symétriques, notamment les espaces topologiques ou les spectres. Les opérades définies ci-dessous sont parfois appelées opérades algébriques. On renvoie le lecteur au livre de Markl, Shnider et Stasheff [MSS] ainsi qu'aux articles de Ginzburg et Kapranov [GiK], de Getzler et Jones [GJ], et de Fresse [F3].

1.1.1 Σ_*-module

Un *Σ_*-module* P est une suite $\mathsf{P}(n)_{n\in\mathbb{N}}$ de dg-modules telle que chaque $\mathsf{P}(n)$ est muni d'une action du groupe symétrique à n éléments Σ_n.

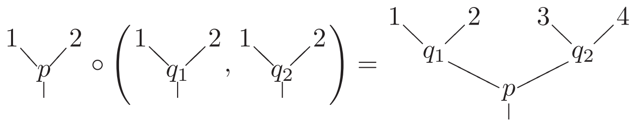
1.1.2 Opérade

Une *opérade* P est un Σ_*-module muni d'un élément unité $1 \in \mathsf{P}(1)$ et d'un produit de composition

$$\circ : \mathsf{P}(r) \otimes \mathsf{P}(n_1) \otimes \ldots \otimes \mathsf{P}(n_r) \to \mathsf{P}(n_1 + \ldots + n_r)$$

vérifiant des diagrammes d'associativité, d'unité et de compatibilité avec l'action du groupe symétrique.

Intuitivement, le dg-module $\mathsf{P}(n)$ est l'ensemble des opérations à n variables, et le produit de composition correspond à la composition de ces opérations. Il est pratique de représenter les opérations par des arbres étiquetés et les compositions par des greffes d'arbres :



L'exemple de base est l'opérade P des endomorphismes d'un $\mathbb{K}$-module V, définie par

$$\mathsf{P}(n) = Hom(V^{\otimes n}, V).$$

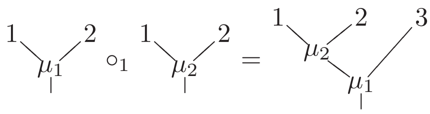
Une façon équivalente de définir une opérade est la donnée d'un monoïde dans la catégorie des Σ_*-modules. Le produit de ce monoïde est le produit de composition de l'opérade.

On dit qu'une opérade est *connexe* (par rapport à l'arité) si $\mathsf{P}(0) = 0$ et $\mathsf{P}(1) = \mathbb{K}.1$.

1.1.3 Produit de composition partielle

Le *produit de composition partielle* $p \circ_i q$ de p et q est la composition $p \circ (1, \ldots, 1, q, 1, \ldots, 1)$ où 1 est l'identité de l'opérade et où l'opération q est en $i$ème position.

Exemple d'une composition partielle



1.1.4 Algèbre sur une opérade

Une *algèbre sur une opérade* P est un dg-module A muni d'une action de P sur A, c'est-à-dire d'une famille d'applications Σ_n-équivariantes $\mathsf{P}(n) \otimes A^{\otimes n} \to A$ associatives par rapport au produit de l'opérade.

Les catégories d'algèbres usuelles sont associées à une opérade codant les opérations définissant ce type d'algèbre. Il existe ainsi une opérade As pour les algèbres associatives, et les algèbres sur l'opérade As sont les algèbres associatives. De même, il existe une opérade Com pour les algèbres commutatives (et associatives), une opérade Lie pour les algèbres de Lie, etc.

1.1.5 Opérade libre

L'*opérade libre* engendrée par un Σ_*-module M, notée $F(M)$, est caractérisée par la propriété universelle suivante : il existe un morphisme d'inclusion $M \subset F(M)$ et tout morphisme de Σ_*-modules $\phi : M \to \mathsf{P}$ s'étend en un unique morphisme d'opérades $\tilde{\phi} : F(M) \to \mathsf{P}$.

Explicitement, l'opérade libre $F(M)$ est réalisée par le dg-module engendré par les arbres planaires étiquetés par des éléments de M et quotienté par des isomorphismes venant de l'action du groupe symétrique.

La notion d'opérade libre permet notamment de donner des présentations par générateurs et relations. Les opérades usuelles ($\mathsf{As}, \mathsf{Com}, \mathsf{Lie}$ notamment) sont facilement définies ainsi :

– l'opérade As est engendrée par une opération binaire μ, qui vérifie la relation d'associativité $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$,
– l'opérade Com est engendrée par une opération binaire μ, qui vérifie la commutativité $\mu(a, b) = \mu(b, a)$ et l'associativité $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$,
– l'opérade Lie est engendrée par une opération binaire μ, qui vérifie l'anti-commutativité $\mu(a, b) = -\mu(b, a)$ et la relation de Jacobi $\mu(a, \mu(b, c)) + \mu(b, \mu(c, a)) + \mu(c, \mu(a, b)) = 0$,

– l'opérade Poisson associée aux algèbres de Poisson est engendrée par $M = \mathbb{K}\bullet \oplus \mathbb{K}[sgn][,]$, vérifiant les trois relations

$$a \bullet (b \bullet c) = (a \bullet b) \bullet c \text{ (Associativité)}$$

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0 \text{ (Jacobi)}$$

$$[a \bullet b, c] = a \bullet [b, c] + b \bullet [a, c] \text{ (Poisson)},$$

– l'opérade permutative, Perm, est engendrée par une opération binaire $\bullet$ qui vérifie la relation $(x \bullet y) \bullet z = x \bullet (y \bullet z) = x \bullet (z \bullet y)$.

1.1.6 Construction bar réduite

On rappelle que l'*idéal d'augmentation* $\bar{\mathsf{P}}$ d'une opérade P est le noyau de l'augmentation $\mathsf{P} \to I$, où I est l'opérade unité ($I(1) = \mathbb{K}$, $I(r) = 0$ sinon).

La *construction bar réduite* $B(\mathsf{P})$ est la coopérade quasi-libre définie par $F^c(\Sigma\bar{\mathsf{P}})$, la coopérade colibre engendrée par la suspension de l'idéal d'augmentation, munie d'une codérivation de torsion $\partial : F^c(\Sigma\bar{\mathsf{P}}) \to F^c(\Sigma\bar{\mathsf{P}})$ déterminée par les produits de composition partielle de P. Cette codérivation est ajoutée à la différentielle naturelle de $F^c(\Sigma\bar{\mathsf{P}})$ pour donner la différentielle de $B(\mathsf{P})$.

Cette construction généralise la notion habituelle de construction bar pour les groupes ou les algèbres.

On définit l'*homologie d'une opérade* comme étant l'homologie de sa construction bar.

1.1.7 Opérade munie d'un poids

On considère des $\mathbb{K}$-modules V munis d'une graduation en poids :

$$V = \bigoplus V_{(s)}.$$

Une opérade P est dite *munie d'un poids* si chaque $\mathsf{P}(n)$ est muni d'un poids et que ce poids est respecté par le produit de composition.

On dit qu'une opérade est *connexe* (par rapport au poids) si $\mathsf{P}_{(0)} = \mathbb{K}.1$.

L'opérade libre est naturellement munie d'un poids : le poids d'un tenseur homogène est défini comme le nombre d'éléments du tenseur. Si P est muni d'un poids, on en déduit un poids sur la construction bar réduite de P : le poids d'un tenseur homogène est défini comme la somme des poids des éléments du tenseur.

1.1.8 Dualité de Koszul

On dit qu'une opérade P (augmentée, munie d'un poids, connexe par rapport au poids) est *de Koszul* si $H_*(B_\bullet(\mathsf{P})_{(s)}) = 0$ pour $* \neq s$ (c'est-à-dire si l'homologie de sa construction bar est concentrée sur la diagonale $* = s$).

La *construction de Koszul* est définie par

$$K(\mathsf{P})_{(s)} := H_s(B_*(\mathsf{P})_{(s)}, \delta) = \ker(\delta : B_s(\mathsf{P})_{(s)} \to B_{s-1}(\mathsf{P})_{(s)}).$$

Par définition, $K(\mathsf{P})_{(s)}$ est concentrée en degré s. On observe que l'inclusion $K_d(\mathsf{P})_{(s)} \to B_d(\mathsf{P})_{(s)}$ est un morphisme de complexe de chaînes. L'opérade P est de Koszul si et seulement si le morphisme d'inclusion $K(\mathsf{P}) \to B(\mathsf{P})$ est un quasi-isomorphisme. On observe que $K(\mathsf{P})$ (parfois également noté $K\mathsf{P}$) est une coopérade. On l'appelle aussi *dual de Koszul* de P.

Cette notion généralise la dualité de Koszul des algèbres.

1.2 Historique et motivations

1.2.1 Dualité de Koszul pour les algèbres

La notion d'algèbre de Koszul a initialement été introduite dans le cadre des algèbres associatives, par Priddy [Pri], avec comme motivation l'étude de l'algèbre de Steenrod. Une algèbre de Koszul est une algèbre quadratique avec une bonne propriété homologique : pour une graduation donnée par la longueur des mots, l'homologie de la construction bar de l'algèbre est concentrée sur la diagonale. Il s'avère qu'un grand nombre d'algèbres quadratiques apparaissant dans la nature vérifient cette propriété. On trouve ainsi des algèbres de Koszul en géométrie algébrique, en théorie des représentations, en géométrie non-commutative, en topologie, en théorie des nombres et en algèbre non-commutative. On renvoie le lecteur à l'introduction du livre de Polishchuk et Positselski [PP] pour plus de détails.

Il est intéressant de noter que Priddy a introduit les algèbres de Koszul en même temps que la notion de base de Poincaré-Birkhoff-Witt (qu'on notera "PBW" dans toute la thèse). L'idée est de regarder une algèbre A via sa présentation par générateurs et relations, d'ordonner ces générateurs, et d'essayer de trouver une base monomiale de A (en tant que module) vérifiant deux propriétés par rapport à l'ordre :

1. Le produit de deux éléments de la base est un élément de la base, ou alors il se réécrit de façon unique comme somme d'éléments inférieurs.
2. Un monôme est dans la base si et seulement si tout sous-produit de ce monôme est dans la base.

Une telle base est appelée "base PBW" et assure que l'algèbre est de Koszul. Le nom "base de Poincaré-Birkhoff-Witt" provient du "théorème de Poincaré-Birkhoff-Witt" classique : si $\{X_i\}_{i\in I}$ forme une base de $\mathcal{G}$, les monômes $X_{i_1}\dots X_{i_n}$ pour $i_1 \leq \dots \leq i_n$ forment la base PBW de $U(\mathcal{G})$.

De plus, la notion de base PBW est fortement reliée à la notion de base de Gröbner, donc aux questions de réécriture. Une base PBW est en fait une base de Gröbner non-commutative quadratique. On voit ainsi qu'une algèbre possédant une base PBW n'est en fait pas vraiment différente (d'un point de vue homologique) de l'algèbre monomiale associée.

Il existe des algèbres de Koszul qui ne possèdent pas de base PBW. Le contre-exemple classique est une algèbre de Sklyanin de dimension 3, [PP, Section 4.11]. Néanmoins, le critère de Priddy s'est avéré commode à appliquer dans les cas usuels.

Il existe deux généralisations notables des algèbres de Koszul. La première consiste à regarder des algèbres avec des relations inhomogènes, c'est-à-dire où des termes de poids différents apparaissent dans une même relation. Ce problème a notamment été étudié par Braverman et Gaitsgory [BG]. La seconde généralisation consiste à regarder des relations homogènes, mais concentrées en un poids supérieur (3 par exemple). Cette dernière généralisation, appelée "N-Koszul", a été introduite par R. Berger dans [Ber] et a conduit à de nombreux développements par Green et Marcos d'une part, Cassidy et Shelton d'autre part.

1.2.2 Dualité de Koszul pour les opérades

Ginzburg et Kapranov ont introduit la notion d'opérade de Koszul en 1994 dans [GiK]. Comme dans le cas des algèbres, une opérade de Koszul est une opérade avec une bonne propriété homologique. On retrouve également que l'opérade est nécessairement quadratique. Cette théorie a été reprise et étendue ensuite dans de nombreux articles. Ainsi, la notion qui était initialement définie dans le cadre de la caractéristique nulle a été étendue pour des opérades définies sur un anneau quelconque par Fresse dans [F3]. De plus, dans le cadre des diopérades (resp. des propérades, des opérades colorées), la dualité de Koszul a été étudiée par Gan [Gan] (resp. Vallette [Val] et van der Laan [VdL]). La généralisation de la dualité de Koszul des opérades dans un cadre inhomogène a été étudiée également dans le cadre des opérades, initialement par Galvez-Carrillo, Tonks et Vallette [GCTV] pour l'opérade de Batalin-Vilkolisky, puis par Hirsch et Millès dans un cadre plus général incluant les propérades dans [HM]. Cependant, la notion d'opérade "N-Koszul" n'a pas encore été définie.

Les exemples les plus classiques d'opérades sont Koszul. Ainsi, les opérades As, Com, Lie sont Koszul. Il existe de nombreuses autres opérades intéressantes qui sont Koszul, comme $\mathsf{Poisson}$, $\mathsf{Gerstenhaber}$, Perm, PreLie notamment. L'exemple le plus notable d'opérade non Koszul est l'opérade $\mathsf{Anti-Ass}$, codant un produit binaire anti-associatif [GeK]. De même, l'opérade partiellement associative ternaire concentrée en degré pair n'est pas Koszul [MR] (il est intéressant de noter que par contre, elle est Koszul si les générateurs sont en degré impair).

La dualité de Koszul des opérades est particulièrement utile, servant notamment à obtenir des "petits" modèles quasi-libres d'opérades. En effet, on sait que la dualité cobar-bar permet d'obtenir un modèle quasi-libre $(B^c(B(\mathsf{P})), \partial) \xrightarrow{\sim} \mathsf{P}$. Cependant ce modèle est très grand (car on prend des arbres étiquetés par des arbres, eux-même étiquetés par P) et donc peu maniable. Dans le cas où l'opérade P est de Koszul, on peut se restreindre à prendre $(B^c(K(\mathsf{P})), \partial) \xrightarrow{\sim} \mathsf{P}$, qui est plus petit et donc plus maniable. Ces petits modèles quasi-libres permettent notamment de définir de petits complexes calculant l'homologie des algèbres sur l'opérade P. Ce point sera étudié dans le paragraphe 1.2.4.

D'autre part, lorsqu'on a une coopérade D vérifiant $(B^c(\mathsf{D}), \partial) \xrightarrow{\sim} \mathsf{P}$, on obtient une coaction de l'homologie de D sur l'homologie des P-algèbres. Dans le cas où P est de Koszul, ceci permet de munir la théorie (co)homologique d'une structure multiplicative.

Dans le cas où l'opérade P est de Koszul et donnée par générateurs et relations, la construction de Koszul $K\mathsf{P}$ (et donc l'homologie) se détermine facilement. Dans le cas binaire de dimension finie, la coopérade $K\mathsf{P}$ est le dual linéaire (à suspension opéradique près) de l'opérade $\mathsf{P}^!$ engendrée par les duaux des générateurs de P et quotientée par l'orthogonal des relations de P. On sait notamment que $(\mathsf{P}^!)^! = \mathsf{P}$, $\mathsf{As}^! = \mathsf{As}$, $\mathsf{Com}^! = \mathsf{Lie}$, $\mathsf{Pois}^! = \mathsf{Pois}$, $\mathsf{Perm}^! = \mathsf{PreLie}$.

Un point important est donc de réussir à montrer qu'une opérade est de Koszul. Il existe déjà différents critères dans la littérature :

- Ginzburg et Kapranov, dans le papier originel sur la dualité de Koszul des opérades [GiK], ont un premier résultat en caractéristique 0, disant qu'une opérade P est de Koszul si et seulement si l'homologie d'une P algèbre libre est triviale. Cependant, cette méthode suppose d'expliciter l'opérade et son dual de Koszul.
- Dans le cas ensembliste, Vallette a développé dans [Val2] un critère utilisant l'homologie des posets, notamment le fait qu'un poset est de Cohen-MacCaulay si et seulement si l'algèbre d'incidence associée est de Koszul. Cette méthode est donc restreinte au cas où l'opérade ou l'opérade duale est ensembliste.

La koszulité est stable par la suspension opéradique et par passage à la dualité. Markl a de plus montré que la composition de deux opérades de Koszul est de Koszul, à condition que cette composition s'écrive par une loi de distribution [Mar].

L'article "A Poincaré-Birkhoff-Witt criterion for Koszul operads" (chapitre 2 de cette thèse) donne un nouveau critère explicite, généralisant celui de Priddy, pour montrer qu'une opérade est de Koszul. Le point clé est de définir une notion adaptée de base PBW dans le contexte des opérades.

1.2.3 Gamma-homologie

La Γ-homologie est une théorie homologique pour les algèbres commutatives développée par Robinson et Whitehouse. Elle s'obtient en regardant les algèbres commutatives comme des algèbres E_∞. Elle correspond dans un cadre très général à d'autres théories homologiques définies sur des algèbres commutatives ou des S-algèbres. Notamment, elle s'identifie à l'homotopie stable des Γ-modules (d'où le nom de Γ-homologie), à l'homotopie stable des théories algébriques, à l'homologie d'André-Quillen topologique (habituellement notée TAQ) de Basterra [Bas] et à la notion d'homologie d'André-Quillen introduite par Goerss et Hopkins. Les définitions de ces théories homologiques, les preuves de certaines équivalences et de nombreuses références se trouvent dans l'article de Basterra et Richter [BR]. Lorsqu'on applique ces théories homologiques à des algèbres commutatives sur un corps de caractéristique nulle, on obtient toujours l'homologie de Harrison (éventuellement à décalage de degré près).

La catégorie Γ (parfois notée Γ^{op} est définie comme le squelette de la catégorie des ensembles finis pointés : les objets sont indexés par les entiers $[n] = \{0, 1, \ldots, n\}$ et les morphismes sont les applications envoyant 0 sur 0. Un Γ-espace est alors un foncteur de Γ dans la catégorie des ensembles simpliciaux pointés. Un Γ-module est un foncteur de Γ dans la catégorie des dg-modules sur un anneau commutatif k. La notion de Γ-espace a été introduite par Segal. Bousfield et Friedlander ont ensuite étendu cette définition, et ont muni la catégorie des Γ-espaces d'une structure modèle. Le produit smash introduit par Lydakis a permis de commencer l'étude des anneaux, modules et algèbres sur les Γ-espaces. Pour définir l'homologie d'un Γ-espace, on lui associe d'abord un spectre, puis on prend les groupes d'homotopie de ce spectre. La façon de définir la Γ-homologie des théories algébriques est tout à fait similaire.

Les définitions de l'homologie TAQ et d'André-Quillen s'obtiennent en généralisant les travaux d'André et de Quillen. Ces homologies sont respectivement dans le cadre des S-algèbres commutatives et des algèbres simpliciales sur une opérade simpliciale. Elles apparaissent via des foncteurs dérivés, et font intervenir des complexes cotangents dans le cadre de l'homotopie stable.

Le second article (chapitre 3 de la thèse) donne une généralisation de la Γ-homologie dans le contexte des algèbres sur les opérades. On explicite une généralisation du complexe de Robinson pour calculer la Γ-homologie dans le cadre des opérades de Koszul.

1.2.4 Théories homologiques pour les algèbres sur une opérade

Les théories homologiques associées aux types d'algèbres classiques ont été développées à partir des années 50. Pour les algèbres associatives, Hochschild a défini une théorie homologique. Ensuite, pour les algèbres commutatives, Harrison a défini une théorie homologique en caractéristique nulle, puis André et Quillen séparément l'ont inscrite dans un cadre plus général. Pour les algèbres de Lie, on dispose du complexe de Chevalley-Eilenberg pour calculer leur homologie.

Les théories homologiques associées aux catégories d'algèbres usuelles ont de nombreuses applications en théorie de l'homotopie.

- L'homologie de Hochschild des algèbres associatives contrôle leurs déformations [Fox].
- De façon générale dans une catégorie de modèle, l'homologie de Quillen (qui correspond à l'homologie des indécomposables d'un remplacement cofibrant) contrôle l'obstruction au relèvement des applications. Ceci a été par exemple étudié dans le cadre rationnel pour les algèbres sur une opérade de Koszul [Liv].
- La Gamma-homologie des algèbres commutatives donne une théorie de l'obstruction pour les structures E_∞ sur les spectres en anneaux [Rob].

Il est donc naturel de chercher à définir des théories homologiques pour des algèbres sur des opérades dans le cadre le plus général possible. Dans le cas où P est une opérade de Koszul sur un corps de caractéristique nulle, la théorie homologique associée aux P-algèbres est bien connue. Ginzburg et Kapranov l'ont définie en premier dans [GiK, Section 4.2]. Une définition avec coefficients (qui sont un module sur une certaine algèbre enveloppante) a été donnée par Balavoine [Bal]. D'autre part, Livernet a fait dans sa thèse [Liv] le lien entre la théorie homologique associée aux P-algèbres et l'homologie de Quillen venant de la structure modèle de la catégorie des P-algèbres, toujours dans le cadre P opérade de Koszul.

Pour obtenir des définitions plus générales (caractéristique positive, opérade pas nécessairement Koszul), on utilise à nouveau la notion de catégorie de modèle. Cependant, pour obtenir une structure modèle sur les P-algèbres, il est nécessaire d'imposer des hypothèses de cofibrance sur l'opérade P, qui sont toujours vérifiées en caractéristique nulle, mais qui sont déjà mises en défaut pour l'opérade Com en caractéristique positive. Ainsi, dans le cas où P est cofibrante en tant que Σ_*-module, on obtient une structure modèle sur les P-algèbres et donc une théorie homologique. Plus précisément, on n'obtient qu'une structure semi-modèle (les axiomes CM4 et CM5 ne sont vérifiés que si les domaines des applications sont cofibrants) dans le cas où on regarde des complexes de chaines $\mathbb{Z}$-gradués plutôt que $\mathbb{N}$-gradués.

Dans le second article (chapitre 3 de cette thèse), on cherche à définir l'homologie d'une algèbre sur une opérade dans le cadre le plus général possible : anneau de caractéristique quelconque, sans hypothèse de cofibrance sur P, complexes de chaines $\mathbb{Z}$-gradués.

1.2.5 Théories d'obstruction

Les problèmes de réalisation sont classiques en topologie algébrique. Ils prennent la forme suivante :

On se fixe une catégorie $\mathcal{C}$, avec une structure modèle (par exemple, les espaces topologiques, les spectres, les algèbres graduées sur une certaine opérade). On considère un foncteur d'homologie (ou d'homotopie) $H : \mathcal{C} \to \mathcal{A}$ à valeurs dans une catégorie "purement algébrique" (par exemple, les modules gradués, les algèbres graduées). On se pose les questions suivantes :

Q0 (réalisation des objets) : Soit a dans $\mathcal{A}$. Existe-t-il c tel que $H(c) = a$?

Q1 (réalisations des morphismes) : Soit $f : H(c_1) \to H(c_2)$. Existe-t-il $\phi : c_1 \to c_2$ tel que $H(\phi) = f$?

Q2 (unicité des réalisations) : Si $H(\phi_1) = H(\phi_2)$, existe-t-il une homotopie h entre ϕ_1 et ϕ_2 ?

L'exemple le plus classique a été étudié par Steenrod pour $\mathcal{C} = Top$ et $H = H^*_{\text{sing}}$. Steenrod a notamment mis en avant l'importance de la cohomologie et des opérations cohomologiques pour l'étude des questions de réalisation. Une solution aux questions Q1 et Q2 dans le cas des CW-complexes rationels nilpotents a été donnée par Halperin et Stasheff dans [HS]. Via les modèles minimaux, ils se ramènent à un problème d'obstruction dans un cadre algébrique. La catégorie $\mathcal{C}$ devient alors celle des algèbres commutatives graduées sur $\mathbb{Q}$. Le point clé de la méthode est alors de passer par des modèles filtrés. On peut également noter que la réalisation des objets est toujours possible, en utilisant l'adjonction entre le foncteur A_{PL} et le foncteur de réalisation.

Dans sa thèse [Liv], Livernet étudie les questions Q1 et Q2 dans le cadre rationnel pour $\mathcal{C}$ la catégorie des algèbres différentielles $\mathbb{N}$-graduées sur une opérade de Koszul P, $\mathcal{A}$ la catégories des algèbres $\mathbb{N}$-graduées sur l'opérade P, et H le foncteur d'homologie du dg-module sous-jacent à la P-algèbre. Ceci généralise le problème algébrique de Halperin et Stasheff, leur cadre correspondant en fait à $\mathsf{P} = \mathsf{Com}$.

Une autre généralisation possible est de continuer à regarder $\mathsf{P} = \mathsf{Com}$, mais de se placer dans le cadre des spectres. Ainsi, Robinson [Rob] étudie grâce à la Γ-homologie les questions d'existence et d'unicité d'une structure E_∞ sur un spectre en anneau.

Dans le troisième article (chapitre 4 de cette thèse), on s'intéresse à généraliser les résultats de Livernet, pour une opérade connexe sur un corps de caractéristique quelconque, dans le cadre des dg-modules $\mathbb{Z}$-gradués. Pour obtenir notre résultat, on considère des filtrations naturelles de certains remplacements cofibrants fonctoriels déduits de la dualité bar des opérades. L'idée consiste à construire les morphismes pas à pas en utilisant cette filtration et à identifier les obstructions comme des cocycles de Γ-homologie.

1.3 Résumé de l'article 1

Dans cet article, on développe un critère se basant sur une relation d'ordre sur les arbres, généralisant la notion de base PBW introduite par Priddy pour les algèbres. Dans le cas d'une algèbre donnée sous la forme $A = \mathbb{K}{<}x_1, \ldots, x_n{>}/I$, une base PBW consiste en un ensemble de représentants monomiaux d'une base de A tel que le produit d'éléments de la base reste dans la base ou se réduit en une somme d'éléments plus petits [1] (pour un certain ordre) dans la base. Une algèbre PBW est une algèbre munie d'une telle base. Le critère de Priddy affirme qu'une algèbre PBW est Koszul.

1.3.1 Introduction

On se place dans le contexte des modules différentiels gradués sur un corps de base $\mathbb{K}$. Etant donnée une opérade P de la forme $F(M)/(R)$ où M est le Σ_*-module des générateurs et R l'ensemble générateur des relations, on veut une condition suffisante disant que l'opérade P est de Koszul. On suppose que M n'a pas d'élément en arité 0 (ceci donnera une opérade réduite) et possède une base ordonnée notée $\mathcal{B}^M$.

1.3.2 Représentation planaire et base monomiale

On commence par définir la notion de représentation planaire et de shuffle pointé. L'opérade libre $F(M)$ est engendrée en tant que module par tous les arbres planaires avec les entrées étiquetées par 1 à n et les sommets étiquetés par les éléments de M, et on identifie certaines classes d'isomorphismes d'arbres. Pour définir une base monomiale de $F(M)$, il faut donc se débarasser de ce quotient. Pour cela, on définit un représentant unique de chaque classe via la représentation planaire : on conserve l'unique arbre vérifiant la propriété suivante :

> Pour tout sommet v de l'arbre, on associe à tout sommet v' situé immédiatement au-dessus de v le minimum des feuilles reliées à v'. Les sommets v' et les feuilles immédiatement au-dessus de v sont placés de gauche à droite par ordre croissant.

1. La notion de base PBW introduite pour les opérades est définie avec une ordre "inversé" par rapport à la notion originale : la réécriture éventuelle d'un produit est une somme d'éléments plus grands (au lieu de plus petits).

La base monomiale $\mathcal{B}^{F(M)}$ de $F(M)$ est alors donnée par les arbres en représentation planaire qu'on étiquette par la base de M.

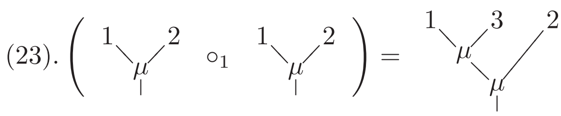
1.3.3 Exemple et contre-exemple d'arbres en représentation planaire

Le premier arbre est en représentation planaire et le second ne l'est pas.

1.3.4 Shuffle pointé

On définit ensuite la notion de *shuffle pointé d'une composition partielle* $\circ_i$: c'est une permutation préservant l'ordre des entrées de chaque tenseur arboré dans le produit de composition partielle et préservant l'entrée i. Plus précisément, pour α un tenseur arboré à s entrées et β un tenseur arboré à t entrées, une permutation $w \in \Sigma_{t+s-1}$ est un shuffle pointé si les ordres des entrées de α et de β sont inchangés dans la composition $w.(\alpha \circ_i \beta)$ et si le minimum des entrées de β dans la composition est i. Cette définition implique notamment que les entrées étiquetées de 1 à $i-1$ ne sont pas modifiées.

1.3.5 Exemple d'une composition partielle avec un shuffle pointé



Dans cet exemple, la transposition (23) est le seul shuffle pointé non trivial de la composition.

1.3.6 Condition sur l'ordre mis sur la base monomiale

La condition requise pour un ordre sur une base monomiale de $F(M)$ est la compatibilité avec les compositions partielles et les shuffles pointés.

Pour α, α' à m entrées et β, β' à n entrées, on a

$$\left\{ \begin{array}{l} \alpha \leq \alpha' \\ \beta \leq \beta' \end{array} \right. \Rightarrow \forall i, w.(\alpha \circ_i \beta) \leq w.(\alpha' \circ_i \beta'),\ \forall w \text{ shuffle pointé.}$$

1.3.7 Exemple d'ordre convenable

A un tenseur arboré α à n entrées, on commence par associer une suite $(\underline{a}_1, \ldots, \underline{a}_n)$ de n mots de la façon suivante : pour tout i, il existe un unique chemin monotone de sommets de la racine à la feuille i, et $\underline{a}_i$ est le mot composé (de gauche à droite) par les étiquettes de ce chemin (de bas en haut).

Si $\underline{a}$ et $\underline{b}$ sont deux mots, on compare d'abord la longueur des mots, et s'ils sont égaux, on compare les mots lexicographiquement (chaque lettre étant dans la base de M, qu'on avait supposée ordonnée).

On peut ainsi comparer deux tenseurs arborés avec le même nombre d'entrées α (associé à la suite $(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n)$) et β (associé à $(\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n)$), tels que $\alpha \neq \beta$, en comparant $\underline{a}_1$ avec $\underline{b}_1$, puis $\underline{a}_2$ avec $\underline{b}_2$, etc. Ceci définit l'ordre strict.

1.3.8 Définition d'une base PBW

On définit maintenant la notion de base de Poincaré-Birkhoff-Witt (qu'on abrège en base PBW).

Une *base PBW* de P est un ensemble $\mathcal{B}^{\mathsf{P}} \subset \mathcal{B}^{F(M)}$ d'éléments représentant une base du $\mathbb{K}$-module P, contenant l'unité, $\mathcal{B}^M$, et vérifiant les deux propriétés suivantes :

1. Pour α et β dans $\mathcal{B}^{\mathsf{P}}$ et pour w un shuffle pointé de la composition $\alpha \circ_i \beta$, soit $w.(\alpha \circ_i \beta)$ est dans $\mathcal{B}^{\mathsf{P}}$, soit les éléments $\gamma \in \mathcal{B}^{\mathsf{P}}$ qui apparaissent dans la réécriture de $w.(\alpha \circ_i \beta) = \sum_\gamma c_\gamma \gamma$ vérifient $\gamma > w.(\alpha \circ_i \beta)$ dans $F(M)$.
2. Un tenseur arboré α est dans $\mathcal{B}^{\mathsf{P}}$ si et seulement si pour toute arête interne de l'arbre sous-jacent, le tenseur arboré restreint à cette arête est dans $\mathcal{B}^{\mathsf{P}}$.

On observe que la condition 1 peut être restreinte au cas où α et β sont des corolles. Intuitivement, la condition 2 dit que la base est engendrée par ses éléments quadratiques, et la condition 1 correspond à une maximalité de la réécriture pour l'ordre lorsqu'on compose deux éléments de la base. On appelle *opérade PBW* une opérade possédant une base PBW.

On prouve alors le critère recherché :

1.3.9 Théorème (Théorème 2.3.10)**.** *Une opérade PBW, réduite et finiment engendrée est de Koszul.*

La démonstration se fait via une filtration, qu'on définit grâce à la notion d'ordre, et une suite spectrale.

1.3.10 Résultats complémentaires et exemples

On obtient de plus que le dual d'une opérade PBW est PBW, et notamment, on obtient explicitement une base de l'opérade duale. La théorie des opérades PBW fonctionne aussi dans le cadre non-symétrique.

Les exemples classiques d'opérades de Koszul (As, Com, Lie, $\mathsf{Poisson}$) sont en fait PBW. De plus, on peut montrer que l'opérade Perm (et donc PreLie également) possède une base PBW.

1.3.11 Applications

– Notre relation d'ordre a été par la suite affinée par Dotsenko et Khoroschkin [DK], et ils ont développé la théorie des bases de Gröbner pour les opérades. Les bases PBW sont en fait un cas particulier de bases de Gröbner. Ces dernières peuvent être (tout du moins dans le contexte non-gradué) recherchées à l'aide d'un ordinateur, l'analogue de l'algorithme de Buchberger dans le cas opéradique ayant été implémenté. La machinerie des bases de Gröbner facilite donc la recherche de bases PBW.
– Strohmayer [Str] a prouvé la koszulité des opérades de Nijenhuis en trouvant à la main une base PBW.
– Markl et Remm [MR] ont étudié la koszulité d'opérades $(2n+1)$-aires. Dans certains cas, ils ont utilisé l'existence de bases PBW pour démontrer que l'opérade considérée était de Koszul.

1.4 Résumé de l'article 2

On se place dans le contexte des modules différentiels $\mathbb{Z}$-gradués sur un corps $\mathbb{K}$, sans hypothèse sur la caractéristique.

1.4.1 Catégories de modèles mises en jeu

On rappelle qu'on dispose de structures de modèles sur les catégories suivantes :
– la catégorie $\mathcal{C}$ des dg-modules,
– la catégorie $\mathcal{M}$ des Σ_*-modules,
– la catégorie ${}_{\mathsf{P}}\mathcal{C}$ des P-algèbres, où P est une opérade Σ_*-cofibrante,
– la catégorie ${}_{\mathsf{P}}\mathcal{M}^0_{\mathsf{Q}}$ des P-Q-bimodules connexes (c'est-à-dire $M(0) = 0$), où P et Q sont des opérades.

Ces structures s'obtiennent à partir de la première par une méthode de transfert le long d'une adjonction. Dans le cas des P-algèbres et des P-Q-bimodules connexes, la structure obtenue n'est a priori qu'une structure semi-modèle (les axiomes MC4 et MC5 sont affaiblis en demandant que la source soit cofibrante), mais cela ne change rien à notre étude.

1.4.2 Rappels sur l'homologie des algèbres sur une opérade Σ_*-cofibrante

Soit Q une opérade Σ_*-cofibrante et B une algèbre sur Q.

L'algèbre enveloppante $U_{\mathsf{Q}}(B)$ est engendrée par des éléments $q(\diamond, b_1, \ldots, b_n)$, où $q \in \mathsf{Q}(n+1)$, $b_1, \ldots, b_n \in B$ et le symbole $\diamond$ correspond à une entrée libre, quotientée par les relations

$$p(\diamond, b_1, \ldots, b_{i-1}, q(b_i, \ldots, b_n), b_{n+1}, \ldots, b_m) = p \circ_{i+1} q(\diamond, b_1, \ldots, b_{i-1}, b_i, \ldots, b_m).$$

Le produit est donné par

$$p(\diamond, a_1, \ldots, a_n).q(\diamond, b_1, \ldots, b_m) = p \circ_1 q(\diamond, b_1, \ldots, b_m, a_1, \ldots, a_n).$$

Le module des (formes) différentielles de Kähler $\Omega_{\mathsf{Q}}(B)$ est un module à gauche sur $U_{\mathsf{Q}}(B)$ défini par

$$\mathrm{Hom}_{U_{\mathsf{Q}}(B)}(\Omega_{\mathsf{Q}}(B), F) = \mathrm{Der}_{\mathsf{Q}}(B, F)$$

pour tous les modules à gauche F sur $U_{\mathsf{Q}}(B)$, où $\mathrm{Der}_{\mathsf{Q}}(B, F)$ est le dg-module des Q-dérivations $B \to F$ (ne conservant pas nécessairement le degré) et où $\mathrm{Hom}_{U_{\mathsf{Q}}(B)}(\Omega_{\mathsf{Q}}(B), F)$ est le dg-module des homomorphismes des $U_{\mathsf{Q}}(B)$-modules à gauche entre $\Omega_{\mathsf{Q}}(B)$ et F.

Le module des différentielles de Kähler $\Omega_{\mathsf{Q}}(B)$ peut être vu comme le dg-module engendré par des éléments $q(b_1, \ldots, db_i, \ldots, b_n)$ où $q \in \mathsf{Q}(n)$, $b_1, \ldots, b_n \in B$ et d est un symbole formel de différentiation, quotienté par des relations de linéarité par rapport à d et des relations de compatibilité avec la composition opéradique.

Soit Q_B un remplacement cofibrant de B. Soit E un $U_{\mathsf{Q}}(Q_B)$-module à droite et F un $U_{\mathsf{Q}}(Q_B)$-module à gauche. L'homologie de B comme Q-algèbre à coefficients dans E est définie par

$$H_*^{\mathsf{Q}}(B, E) = H_*(E \otimes_{U_{\mathsf{Q}}(Q_B)} \Omega_{\mathsf{Q}}(Q_B)).$$

La cohomologie de B est définie par

$$H^*_{\mathsf{Q}}(B, F) = H^*(\mathrm{Hom}_{U_{\mathsf{Q}}(Q_B)}(\Omega_{\mathsf{Q}}(Q_B), F)).$$

1.4.3 Réduction du complexe

On veut réduire la taille de ces complexes calculant l'homologie. Pour cela, on a les lemmes suivants :

1.4.3.1 Lemme (Lemma 3.1.7.1). *Soit* ${}_{\mathsf{Q}}Res_{\mathsf{Q}} \xrightarrow{\sim} \mathsf{Q}$ *un remplacement cofibrant de* Q *dans la catégorie des* Q*-bimodules et* A *une* Q*-algèbre.*

On obtient un remplacement cofibrant $({}_{\mathsf{Q}}Res_{\mathsf{Q}} \circ_{\mathsf{Q}} A, \partial')$ *de* A *dans la catégorie des* Q*-algèbres, où* $\partial' = \partial \circ_{\mathsf{P}} A$.

1.4.3.2 Lemme (Lemma 3.2.1.1). *Si* Q_A *est une* Q*-algèbre quasi-libre* $Q_A = (\mathsf{Q}(C), \partial')$*, alors on a un isomorphisme de* $U_{\mathsf{Q}}(Q_A)$*-modules à gauche*

$$(U_{\mathsf{Q}}(Q_A) \otimes C, \partial'') \simeq \Omega_{\mathsf{Q}}(Q_A)$$

où la différentielle $\partial'' : U_{\mathsf{Q}}(Q_A) \otimes C \to U_{\mathsf{Q}}(Q_A) \otimes C$ *est un homomorphisme tordant de* $U_{\mathsf{Q}}(Q_A) \otimes C$*-modules induit par l'action de la différentielle tordante de* Q_A *sur* $U_{\mathsf{Q}}(Q_A) \otimes C$*.*

Soit Q une opérade Σ_*-cofibrante telle que l'on dispose d'un Q-bimodule quasi-libre $(\mathsf{Q} \circ N \circ \mathsf{Q}, \partial)$ faiblement équivalent à Q en tant que Q-bimodules avec N un Σ_*-module cofibrant. Soit B une algèbre sur Q.

On applique le Lemme 1.4.3.1 à $(\mathsf{Q} \circ N \circ \mathsf{Q}, \partial)$. On obtient un remplacement cofibrant $Q_B = (\mathsf{Q} \circ (N \circ B), \partial')$ de B. Le lemme 1.4.3.2 alors

$$H_*^{\mathsf{Q}}(B, E) = H_*(E \otimes N \circ B, \partial'').$$

1.4.4 Γ-homologie

Supposons maintenant que P est une opérade telle que l'on dispose d'un P-bimodule quasi-libre $(\mathsf{P} \circ M \circ \mathsf{P}, \partial)$ faiblement équivalent à P en tant que P-bimodules avec M un Σ_*-module cofibrant et ∂ une différentielle décomposable. Soit A une algèbre sur P.

1.4.4.1 Lemme (Lemma 3.2.3.1)**.** *Une équivalence faible de P-bimodules $(\mathsf{P}\circ M_1 \circ \mathsf{P}, \partial_1) \xrightarrow{\phi} (\mathsf{P}\circ M_2 \circ \mathsf{P}, \partial_2)$ (vérifiant les hypothèses ci-dessus) induit un quasi-isomorphisme $(E \otimes M_1 \circ A, \partial_1'') \to (E \otimes M_2 \circ A, \partial_2'')$.*

Ce lemme permet de montrer que l'homologie du complexe $(E \otimes M \circ A, \partial'')$ ne dépend pas du choix de M vérifiant les hypothèses précédentes. On définit alors la Γ-homologie de la P-algèbre A à coefficients dans E :

$$H\Gamma_*^{\mathsf{P}}(A, E) = H_*(E \otimes M \circ A, \partial'').$$

1.4.5 Théorème (Theorem 3.2.5)**.** *Si Q est un remplacement cofibrant de P, alors $H\Gamma_*^{\mathsf{P}} = H_*^{\mathsf{Q}}$. Notamment, si Q est une opérade cofibrante, alors $H\Gamma_*^{\mathsf{Q}} = H_*^{\mathsf{Q}}$.*

La Γ-cohomologie se définit de façon similaire.

1.4.6 Remplacement explicite de P en tant que P-bimodules

Dans le cas où P est binaire et de Koszul, on peut expliciter un choix de remplacement cofibrant de P dans la catégorie des P-bimodules. On choisit $M = K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)$, où le produit $\boxtimes$ est le produit tensoriel arité par arité. On rappelle que $K\mathsf{P}$ est la construction de Koszul et $C_*(E\Sigma_n)$ est la construction bar acyclique homogène du groupe symétrique, i.e. le module engendré en degré t par les $(t+1)$-uplets de permutations $\underline{w} = (w_0, \ldots, w_t)$ avec la différentielle δ telle que $\delta(\underline{w}) = \sum_i (-1)^i (w_0, \ldots, \widehat{w_i}, \ldots, w_t)$.

Il reste maintenant à définir la différentielle sur $\mathsf{P} \circ M \circ \mathsf{P}$. On commence par définir des applications entre bijections.

1.4.7 Applications entre bijections

Soit r un entier positif, $\underline{X}$ et $\underline{Y}$ deux ensembles ordonnés à r éléments.

On représente un élément w de $\mathrm{Bij}(\underline{X}, \underline{Y})$ par le tableau de valeurs :

$$w = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ w(x_1) & w(x_2) & \cdots & w(x_r) \end{pmatrix}.$$

Pour toute paire $\{i, j\} \subset \underline{Y}$, on forme la bijection

$$c_{i,j}^e(w) = \begin{pmatrix} x_1 & x_2 & \cdots & w^{-1}(i) & \cdots & \widehat{w^{-1}(j)} & \cdots & x_r \\ w(x_1) & w(x_2) & \cdots & e & \cdots & \widehat{j} & \cdots & w(x_r) \end{pmatrix}$$

si $w^{-1}(i) < w^{-1}(j)$ ou la bijection

$$c_{i,j}^e(w) = \begin{pmatrix} x_1 & x_2 & \cdots & w^{-1}(j) & \cdots & \widehat{w^{-1}(i)} & \cdots & x_r \\ w(x_1) & w(x_2) & \cdots & e & \cdots & \widehat{i} & \cdots & w(x_r) \end{pmatrix}$$

si $w^{-1}(j) < w^{-1}(i)$.

Les ordres permettent de considérer w comme un élément de Σ_r et $c_{i,j}^e(w)$ comme un élément de Σ_{r-1}.

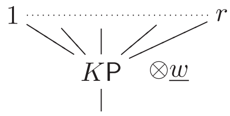
De même, pour tout élément i dans $\underline{Y}$, on forme la bijection

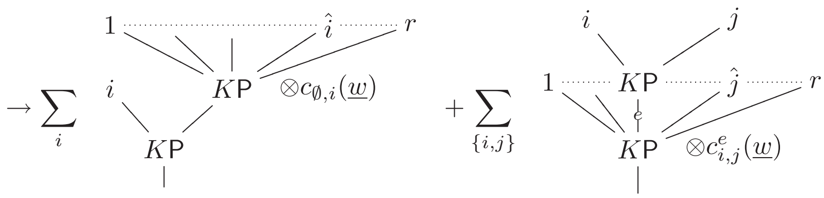
$$c_{\emptyset,i}(w) = \begin{pmatrix} x_1 & x_2 & \cdots & \widehat{w^{-1}(i)} & \cdots & x_r \\ w(x_1) & w(x_2) & \cdots & \widehat{i} & \cdots & w(x_r) \end{pmatrix}.$$

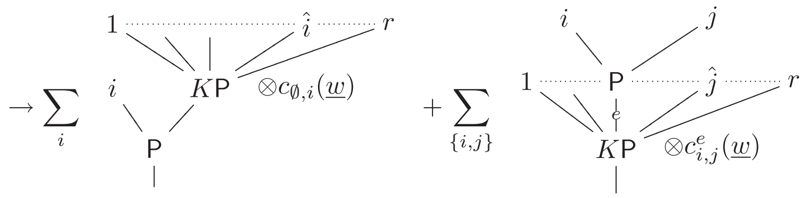
1.4.7.1 Lemme (Lemma 3.3.1.1)**.** *Les applications $c_{i,j}^e(w)$ et $c_{\emptyset,i}(w)$ sont compatibles avec l'action à gauche du groupe symétrique.*

1.4.8 Définition de la différentielle

On définit une différentielle $\Delta : M \to \mathsf{P} \circ M \circ \mathsf{P}$ par la composition :







La première flèche consiste à utiliser le coproduit de $K\mathsf{P}$ restreint à certaines composantes et les $c^e_{i,j}$. La seconde flèche est donnée par le morphisme tordant de Koszul $\kappa : K\mathsf{P} \to \mathsf{P}$.

1.4.8.1 Lemme (Lemma 3.3.2.1). *L'application Δ détermine une différentielle sur $\mathsf{P} \circ M \circ \mathsf{P}$.*

On définit de plus $\delta : M \to M$ par

$$\delta(\gamma \otimes \underline{w}) = (-1)^{|\gamma|} \gamma \otimes \delta(\underline{w}).$$

1.4.8.2 Lemme (Lemma 3.3.2.2). *L'application δ induit une différentielle sur $\mathsf{P} \circ M \circ \mathsf{P}$ qui anticommute avec Δ.*

1.4.9 Proposition (Proposition 3.3.6). *Le P-bimodule $(\mathsf{P} \circ (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ \mathsf{P}, \Delta + \delta)$ est un remplacement cofibrant de P.*

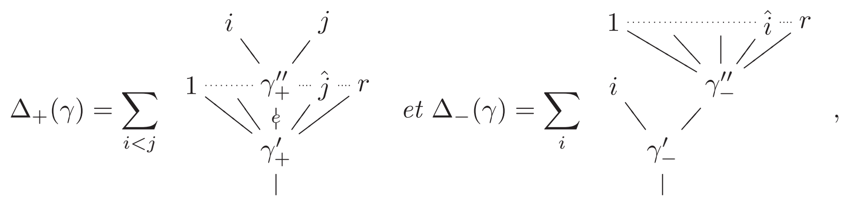
On peut donc utiliser $K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)$ pour calculer la Γ-homologie de P-algèbres.

1.4.10 Théorème (Theorem 3.3.7). *Soit P une opérade binaire de Koszul, A une P-algèbre et E un $U_{\mathsf{P}}(A)$-module à droite.*

$$H\Gamma^{\mathsf{P}}_*(A, E) = H_*(E \otimes (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ A, \partial'')$$

où ∂'' est la différentielle induite par $\Delta + \delta$ comme précédemment.

Explicitement, pour $x \in E$, $\gamma \in K\mathsf{P}$ tel que



$(w_0, \ldots, w_*) \in C_*(E\Sigma_r)$ et $a_1, \ldots, a_r$ dans A, on a :

$$\partial''(x \otimes \gamma \otimes (w_0, \ldots, w_*) \otimes (a_1, \ \ldots, a_r)) =$$

$$\sum_{i<j} \pm x \otimes \gamma'_+ \otimes (c^e_{i,j}(w_0), \ldots, c^e_{i,j}(w_*)) \otimes (a_1, \ \ldots, \kappa(\gamma''_+)(a_i, a_j), \ldots, \widehat{a_j}, \ldots a_r)$$

$$+ \sum_i \pm\kappa(\gamma'_-)(x, a_i) \otimes \gamma''_- \otimes (c_{\emptyset,i}(w_0), \ldots, c_{\emptyset,i}(w_*)) \otimes (a_1, \ldots, \widehat{a_i}, \ldots, a_r).$$

On obtient un théorème similaire sur la Γ-cohomologie.

1.4.11 Remarques

- Pour $\mathsf{P} = \mathsf{Lie}$, on obtient un complexe calculant l'homologie d'algèbres de Lie en caractéristique positive. Il pourra être intéressant d'utiliser cette nouvelle théorie homologique pour essayer de généraliser le théorème de Loday-Quillen-Tsygan, reliant la K-théorie additive et l'homologie cyclique.
- La restriction aux opérades engendrées par des opérations binaires dans l'écriture du complexe explicite peut en fait être omise. Elle est commode pour la définition des applications $c_{i,j}$, mais de telles applications peuvent être définies de façon plus générale. Néanmoins, la majorité des opérades étant engendrées par des opérations binaires, le complexe explicité ici est utilisable dans la grande majorité des cas.
- On a défini dans cet article $\mathsf{E} = C_*(E\Sigma_\bullet)$, qui est en fait l'opérade de Barratt-Eccles. On aurait également pu considérer le complexe normalisé associé, $N_*(E\Sigma_\bullet)$. Dans l'article 3, on doit utiliser ce dernier plutôt que le complexe formé par toutes les chaînes, pour définir une certaine action opéradique.

1.5 Résumé de l'article 3

On se place dans le contexte différentiel gradué sur un corps $\mathbb{K}$, sans hypothèse sur la caractéristique. Le but est d'étudier le problème suivant : soit $\tilde{\mathsf{P}}$ une opérade et A, B deux algèbres sur cette opérade. Peut-on toujours réaliser un morphisme entre H_*A et H_*B en un morphisme de A dans B dans la catégorie homotopique des $\tilde{\mathsf{P}}$-algèbres ? Cette réalisation est-elle unique ? On montre qu'il existe une suite d'obstructions vivant dans les deux premiers groupes de la Gamma-cohomologie opéradique de H_*A à coefficients dans H_*B.

Dans toute cette partie, on suppose l'opérade P graduée seulement, ou de façon équivalente différentielle graduée concentrée en degré 0 et avec une différentielle nulle.

1.5.1 Catégories de modèles mises en jeu

On rappelle qu'on dispose de structures de modèles sur les catégories suivantes :
- la catégorie $\mathcal{C}$ des dg-modules,
- la catégorie $\mathcal{M}$ des Σ_*-modules,
- la catégorie ${}_{\mathsf{P}}\mathcal{C}$ des P-algèbres, où P est une opérade Σ_*-cofibrante,
- la catégorie des opérades.

Ces structures s'obtiennent à partir de la première par une méthode de transfert le long d'une adjonction. Dans le cas des P-algèbres et des opérades, la structure obtenue n'est a priori qu'une structure semi-modèle, mais cela ne change rien à notre étude.

1.5.2 Rappels sur l'opérade de Barratt-Eccles

L'opérade de Barratt-Eccles E est définie par le complexe de chaînes normalisé $N_*(E\Sigma_\bullet)$. Son produit de composition est induit par la composition par blocs des permutations.

Cette opérade permet d'une part de donner un remplacement cofibrant explicite d'une opérade via la dualité cobar-bar, et d'autre part, elle nous permet de définir l'objet cylindre d'une P-algèbre. On rappelle que le produit $\boxtimes$ est le produit tensoriel arité par arité.

1.5.2.1 Fait ([BM]). *Soit* P *une opérade. L'opérade* $B^c(B(\mathsf{P}\boxtimes\mathsf{E}))$*, définie par l'application de la construction cobar-bar à* $\mathsf{P}\boxtimes\mathsf{E}$*, est un remplacement cofibrant de l'opérade* P.

Soit B une P-algèbre. Son objet cylindre dans la catégorie des P-algèbres est $B\otimes N^*(\Delta^1)$. Le complexe de chaines $N^*(\Delta^1)$ est défini par $\mathbb{K}.\underline{0}^\# \oplus \mathbb{K}.\underline{1}^\# \oplus \mathbb{K}.\underline{01}^\#$ où $\underline{0}^\#$, $\underline{1}^\#$ et $\underline{01}^\#$ sont le dual de la base des simplexes non-dégénérés. La différentielle ∂_N vérifie $\partial_N(\underline{01}^\#) = \underline{1}^\# - \underline{0}^\#$ et $\partial_N(\underline{0}^\#) = \partial_N(\underline{1}^\#) = 0$. Le complexe de chaines $N^*(\Delta^1)$ possède une action σ de l'opérade E, déterminée dans [BF]. On utilisera explicitement seulement l'action de degré 0. Le complexe $B\otimes N^*(\Delta^1)$ est alors muni d'une action de $\mathsf{P}\boxtimes\mathsf{E}$, que l'on peut resteindre à une action de P via une section $\rho : \mathsf{P}\to\mathsf{P}\boxtimes\mathsf{E}$. Une telle section ρ peut être définie explicitement car :
- la construction cobar-bar s'identifie à la construction cubique $W_\square$ de Boardman-Vogt,
- le complexe cubique possède une diagonale naturelle.

1.5.3 Explicitation d'un remplacement cofibrant d'une algèbre

On utilise ici la notion de coopérade et de cogèbre sur une coopérade.

Soit D une coopérade et A un dg-module.

Un résultat classique (*cf.* [GJ; F2]) affirme qu'une structure de $B^c(\mathsf{D})$-algèbre sur A (on note cette action α) est équivalente à la donnée d'une différentielle ∂_α sur $\mathsf{D}(A)$ s'annulant sur A. On dispose de formules explicites liant α et ∂_α.

Lorsqu'on se donne un morphisme d'opérades $B^c(\mathsf{D})\to\mathsf{Q}$, on a alors un foncteur qui à une D-cogèbre C associe une Q-algèbre quasi-libre $R_{\mathsf{Q}}(C) = (\mathsf{Q}(C),\partial)$, pour une certaine dérivation de torsion.

On applique cette construction à $\mathsf{D} = B(\mathsf{P}\boxtimes\mathsf{E})$, le morphisme $id : B^c(\mathsf{D})\to B^c(\mathsf{D}) = \tilde{\mathsf{P}}$ et la cogèbre $C = \mathsf{D}(A),\partial_\alpha)$ associée à une $\tilde{\mathsf{P}}$-algèbre A (avec l'action notée α). On a alors le résultat suivant :

1.5.3.1 Fait ([F2, Theorem 4.2.4]). *L'augmentation* $\epsilon : (\tilde{\mathsf{P}}(\mathsf{D}(A),\partial_\alpha),\partial)\to A$ *est une équivalence faible de* $\tilde{\mathsf{P}}$*-algèbres et* $(\tilde{\mathsf{P}}(\mathsf{D}(A),\partial_\alpha),\partial)$ *forme un remplacement cofibrant de* A.

L'intérêt majeur de ce remplacement cofibrant explicite est que $\mathsf{D}(A)$ est muni d'une graduation adéquate pour notre raisonnement ultérieur.

La coopérade $\mathsf{D} = B(\mathsf{P}\boxtimes\mathsf{E})$ est graduée par la somme du poids bar (*ie.* le nombre de tenseurs d'éléments de P) et du degré en E. On note $\mathsf{D}_{[k]}$ le k-ième étage de cette graduation. Ceci induit une graduation sur $\mathsf{D}(A)$.

1.5.4 Restriction des hypothèses

Le résultat suivant permet de transporter une structure de $\tilde{\mathsf{P}}$-algèbre (pour $\tilde{\mathsf{P}}$ cofibrant) en une structure sur son homologie :

1.5.4.1 Fait ([F4, Theorem C]). *Soit* $f : H \xrightarrow{\sim} A$ *une équivalence faible de dg-modules. Supposons que* A *est muni d'une action d'une opérade cofibrante* $\tilde{\mathsf{P}}$.

Alors H *hérite d'une structure d'une* $\tilde{\mathsf{P}}$*-algèbre telle que* $H \xleftarrow{\sim} \cdot \xrightarrow{\sim} A$ *où les morphismes sont des équivalences faible de* $\tilde{\mathsf{P}}$*-algèbres.*

Ceci permet de dire que pour deux $\tilde{\mathsf{P}}$-algèbres A et B, les morphismes de A vers B dans la catégorie homotopique des $\tilde{\mathsf{P}}$-algèbres sont les mêmes que ceux de H_*A vers H_*B. On s'est ainsi ramené à un problème entre algèbres ayant une différentielle nulle.

De plus, lorsque l'opérade $\tilde{\mathsf{P}}$ agit sur A, son homologie $H_*\tilde{\mathsf{P}} = \mathsf{P}$ agit sur H_*A. On note dans toute la suite α (resp. β) l'action de $\tilde{\mathsf{P}}$ sur A (resp. B). On note alors α_0 (resp β_0) l'action de P sur H_*A (resp. H_*B).

1.5.5 Problème de réalisation des morphismes

On se donne
- une opérade graduée P et son remplacement opéradique cofibrant explicite $\tilde{\mathsf{P}} = B^c(B(\mathsf{P}\boxtimes\mathsf{E}))$,
- deux $\tilde{\mathsf{P}}$-algèbres, A et B, avec des différentielles nulles,

– un morphisme de P-algèbres $f_0 : H_*A \to H_*B$ (où H_*A and H_*B sont munis de la structure induite en homologie).

On veut comprendre l'obstruction à l'existence d'un morphisme de A vers B dans la catégorie homotopique des $\tilde{\mathsf{P}}$-algèbres qui induit f_0 en homologie. On dit qu'un tel morphisme réalise f_0.

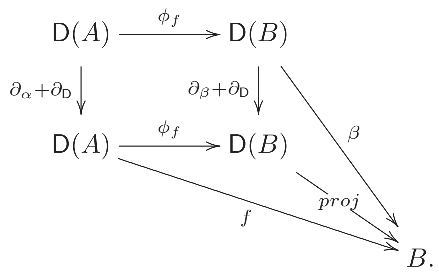
On cherche à construire un morphisme de D-cogèbres $\phi_f : (\mathsf{D}(A), \partial_\alpha) \to (\mathsf{D}(B), \partial_\beta)$ étendant f_0. Si on peut construire un tel ϕ_f, alors on obtient (par fonctorialité) un morphisme $\tilde{\mathsf{P}}\phi_f$ qui s'inscrit dans le diagramme suivant

$$\begin{array}{ccc} (\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_\alpha), \partial) & \xrightarrow{\tilde{\mathsf{P}}\phi_f} & (\tilde{\mathsf{P}}(\mathsf{D}(B), \partial_\alpha), \partial) \\ \downarrow \sim & & \downarrow \sim \\ A & & B \end{array} .$$

et on obtient un morphisme dans la catégorie homotopique qui réalise f_0.

On peut remarquer que tout morphisme dans la catégorie homotopique s'écrit sous cette forme.

Le morphisme ϕ_f doit faire commuter le diagramme suivant :



où la commutation du carré intérieur est équivalent à la commutation du carré extérieur, ce qui revient à l'équation suivante :

$$f \circ (\partial_{\mathsf{D}} + \partial_\alpha) = \beta \circ \phi_f. \tag{1.1}$$

On résout cette équation par récurrence, en utilisant la graduation précédemment définie sur $\mathsf{D}(A)$. On choisit $f_{[0]} = f_0$, ce qui est en fait imposé par le fait que ϕ_f doit étendre f_0.

On suppose ensuite les applications $f_{[k]}$ déjà construites pour $k < d$ et on essaie de construire $f_{[d]}(\gamma(a_1, \ldots, a_n))$ où γ est dans D et les $a_1, \ldots, a_n$ dans A. L'écriture des conditions obtenues par l'Equation (1.1) se réécrit sous la forme suivante :

$$\partial(f_{[d]}(\gamma(a_1, \ldots, a_n))) = \text{second membre connu}$$

où ∂ indique la différentielle de $\mathrm{Der}_{\tilde{\mathsf{P}}}(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_{\alpha_0}), (B, \beta_0))$. Le "second membre connu" correspond à des sommes arborées compliquées faisant intervenir des $f_{[k]}$ avec $k < d$ mais aucun $f_{[d]}$. L'obstruction à l'existence de $f_{[d]}$ vit donc dans le groupe de cohomologie $H^1 \mathrm{Der}_{\tilde{\mathsf{P}}}(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_{\alpha_0}), (B, \beta_0))$. Ce groupe s'identifie à $H\Gamma^1_{\mathsf{P}}(H_*A, H_*B)$. Le résultat s'annonce alors ainsi :

1.5.6 Théorème (Corollary 4.2.6)**.** *Si $H\Gamma^1_{\mathsf{P}}(H_*A, H_*B)$ est réduit à 0, alors tout $f_0 : H_*A \to H_*B$, morphisme de P-algèbres, se réalise par un morphisme de A vers B dans la catégorie homotopique des $\tilde{\mathsf{P}}$-algèbres.*

1.5.7 Problème de réalisation des homotopies

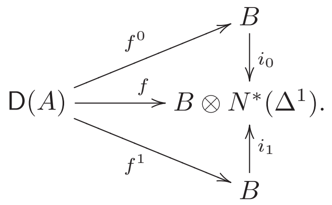
On se donne

– une opérade graduée P avec un remplacement opéradique cofibrant explicite $\tilde{\mathsf{P}} = B^c(B(\mathsf{P} \boxtimes \mathsf{E}))$,
– deux $\tilde{\mathsf{P}}$-algèbres, A et B, avec des différentielles nulles,
– deux morphismes $f^0, f^1 : \mathsf{D}(A) \to B$. Ils induisent des morphismes $\tilde{\mathsf{P}}\phi_{f^0}$ et $\tilde{\mathsf{P}}\phi_{f^1}$ de $\tilde{\mathsf{P}}\mathsf{D}(A)$ vers $\tilde{\mathsf{P}}\mathsf{D}(B)$, et donc deux morphismes de $\tilde{\mathsf{P}}$-algèbres de A vers B dans la catégorie homotopique. On suppose alors que ces morphismes homotopiques réalisent le même morphisme de P-algèbres $\psi : H_*A \to H_*B$.

La question qu'on étudie dans cette partie est la suivante : quelle est l'obstruction à l'égalité de ces morphismes dans la catégorie homotopique ?

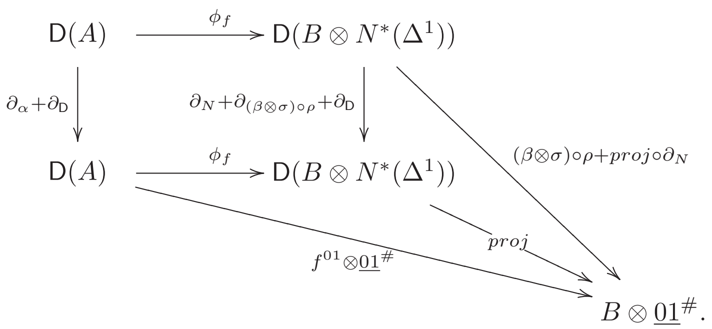
On rappelle que $B \otimes N^*(\Delta^1)$ est l'objet cylindre de B dans la catégories des $\tilde{\mathsf{P}}$-algèbres, et que l'action associée est notée $(\beta \otimes \sigma) \circ \rho$ où σ est l'action de E sur $N^*(\Delta^1)$ et ρ une section de $\tilde{\mathsf{P}}$ dans $\tilde{\mathsf{P}} \boxtimes \mathsf{E}$.

On veut construire un morphisme de D-cogèbres $\phi_f : (\mathsf{D}(A), \partial_\alpha) \to (\mathsf{D}(B \otimes N^*(\Delta^1)), \partial_{(\beta\otimes\sigma)\circ\rho})$ donnant une homotopie entre ϕ_{f^0} et ϕ_{f^1}. Sa restriction f doit s'inscrire dans le diagramme commutatif suivant :



Comme dans la partie précédente, on va construire ϕ_f par récurrence, et on fera apparaitre l'obstruction à la construction. Notre étude est très similaire à la précédente, si ce n'est qu'on doit considérer l'objet cylindre $B \otimes N^*(\Delta^1)$ plutôt que l'algèbre B.

Le morphisme ϕ_f doit faire commuter le diagramme suivant :



L'application f^{01} est définie par la commutation du triangle inférieur, c'est une restriction de f au but sur la composante $\underline{01}^\#$.

La commutation du carré intérieur est équivalente à la commutation du diagramme extérieur, ce qui revient à l'équation suivante :

$$(f^{01} \otimes \underline{01}^\#) \circ (\partial_\mathsf{D} + \partial_\alpha) = (\beta \otimes \sigma) \circ \rho \circ \phi_f + (f^1 - f^0) \otimes \underline{01}^\#. \tag{1.2}$$

Il suffit de construire f^{01}, ce que l'on fait par récurrence. On observe qu'en degré zéro, $\mathsf{D}_{[0]}(A)$ est réduit à A et que $f^1_{[0]} - f^0_{[0]} = \psi - \psi = 0$. On définit donc $f^{01}_{[0]} = 0$.

On suppose par f^{01} est défini pour des degrés inférieurs à un certain entier d positif. Les conditions obtenues dans l'Equation (1.2) se réécrivent comme précédemment sous la forme suivante :

$$\partial(f_{[d]}(\gamma, a_1, \ldots, a_n)) = \text{second membre connu}$$

où ∂ indique la différentielle de $\mathrm{Der}_{\tilde{\mathsf{P}}}(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_{\alpha_0}), (B, \beta_0) \otimes \underline{01}^\#)$. Le "second membre connu" correspond à des sommes arborées (encore plus compliquées) faisant intervenir des $f_{[k]}$ avec $k < d$. L'obstruction à l'existence de $f_{[d]}$ vit donc dans le groupe de cohomologie $H^1 \mathrm{Der}_{\tilde{\mathsf{P}}}(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_{\alpha_0}), (B, \beta_0) \otimes \underline{01}^\#)$. Ce groupe s'identifie à $H\Gamma^0_\mathsf{P}(H_*A, H_*B)$. On obtient alors :

1.5.8 Théorème (Corollary 4.3.5). *Si $H\Gamma^0_\mathsf{P}(H_*A, H_*B)$ est réduit à 0, alors tout $f_0 : H_*A \to H_*B$, morphisme de P-algèbres, se réalise par au plus un morphisme de A vers B dans la catégorie homotopique des $\tilde{\mathsf{P}}$-algèbres (ie. il y a unicité en cas d'existence).*

1.5.9 Remarques

- On peut raffiner cette théorie en identifiant les obstructions dans de la Γ-cohomologie bigraduée. On peut également travailler avec des modules $\mathbb{Z}/2$-gradués plutôt que $\mathbb{Z}$-gradués, ce qui impose ici d'utiliser le raffinement ci-dessus.
- Les premiers exemples que l'on comptera étudier sont les algèbres semi-simples et les superalgèbres, pour étudier leurs réalisations dans les contextes $\mathbb{Z}$-gradué et $\mathbb{Z}/2$-gradué. En caractéristique nulle, on sait notamment que $H^1_{\mathsf{Lie}}(sl_2(\mathbb{K})) = 0$. On s'attend à ce que ce ne soit pas le cas en caractéristique positive.
- Cette théorie d'obstruction donne en fait des informations sur le π_0 d'un ensemble simplicial $\mathrm{Map}_{\mathsf{P}}(A, B)_\bullet$. Cet ensemble simplicial est défini par $\mathrm{Map}(A, B)_\bullet = \mathrm{Mor}_{\tilde{\mathsf{P}}}(A, B^{\Delta^\bullet})$. Sous l'hypothèse A cofibrant, c'est un complexe de Kan, et on peut donc définir ses groupes d'homotopie.
  On a construit dans l'article 3 une filtration sur $\mathsf{D}(A)$. On peut en déduire, au niveau des algèbres quasi-libres, une suite de cofibrations :
  $$(\tilde{\mathsf{P}}(\mathsf{D}_{[0]}(A), \partial_\alpha), \partial) \to \ldots \to (\tilde{\mathsf{P}}(\mathsf{D}_{[0]}(A), \partial_\alpha), \partial) \to \ldots.$$
  Le foncteur $\mathrm{Map}(-, B)$ applique cette suite de cofibrations sur un tour de fibrations. On compte étudier la suite spectrale de Bousfield-Kan associée à cette tour, avec $\pi_* \mathrm{Map}(A, B)$ comme aboutissement. On conjecture que le terme E^2 de cette suite spectrale est donné par la Γ-cohomologie de H_*A à coefficients dans H_*B.

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Chapitre 2

A Poincaré-Birkhoff-Witt criterion for Koszul operads

The aim of this article is to give a criterion, generalizing the criterion introduced by Priddy for algebras, to prove that an operad is Koszul. We define the notion of Poincaré-Birkhoff-Witt basis in the context of operads. Then we show that an operad having a Poincaré-Birkhoff-Witt basis is Koszul. Besides, we obtain that the Koszul dual operad has also a Poincaré-Birkhoff-Witt basis.
We check that the classical examples of Koszul operads (commutative, associative, Lie, Poisson) have a Poincaré-Birkhoff-Witt basis. We also prove by our methods that new operads are Koszul.

The notion of an operad is used to model categories of algebras. An appropriate (co)homology theory is associated to each category of algebras associated to an operad. The Koszul duality of operads, introduced by Ginzburg and Kapranov [5], allows us to understand the structure of the (co)homology theory associated to some operads, the Koszul operads: when an operad is Koszul, we know exactly the multiplicative structure of the associated (co)homology and we have an explicit complex which allows us to determine practically the (co)homology of an algebra. Usual examples include the Hochschild complex for the associative operad $\mathcal{A}s$, the Chevalley-Eilenberg complex for the Lie operad $\mathcal{L}ie$, the Harrison complex for the commutative operad $\mathcal{C}om$. For further details on such applications, we refer the reader to the papers [4; 5] and the book [12].

The aim of this article is to give a criterion, generalizing the criterion introduced by Priddy for algebras in [14], to show that an operad is Koszul. This criterion relies on the existence of a basis, called Poincaré-Birkhoff-Witt (for short, we write PBW) basis, together with a suitable ordering.

In the case of an algebra $A = \mathbb{K} < x_1, \ldots, x_n > /I$, a PBW basis consists of a set of monomial representatives of a basis of A so that the product of basis elements remains in the basis or reduces to a sum of larger (for an appropriate order) elements in the basis. A PBW algebra is an algebra equipped with such a basis. Priddy's criterion asserts that a PBW algebra is Koszul. In the context of operads, we replace monomials by treewise compositions of generating operations and we adapt the order property. In this way, we generalize Priddy's definition to have an appropriate notion of a PBW basis for operads, and we prove that a PBW operad is Koszul. This gives an answer to a question asked by Kriz in his review [7] of the article of Ginzburg and Kapranov [5]. In the article, we prove also that the Koszul dual of a PBW operad is a PBW operad as well. Then we shall see that many usual operads (including commutative, Lie, associative) are PBW.

In Sections 1-2, we recall the definitions of the operadic bar construction and conventions on trees used in the definition of an operadic PBW basis. In Sections 3-4, we define the notion of a PBW operad and we prove that such an operad is Koszul. In Section 5, we prove that the Koszul dual operad of a PBW operad is PBW as well, with an explicit basis. In Section 6, we address the case of non-symmetric operads. To conclude the paper, we examine applications of the criterion to some examples.

Conventions. We are given a ground field $\mathbb{K}$, fixed once and for all, of any characteristic. We deal with differential graded modules over $\mathbb{K}$ (for short dg-modules), whose differential lowers degrees by 1. We only consider operads P equipped with a trivial differential, but possibly with a grading. In this context, the usual sign rule applies to the elements of P. A non-graded operad can be viewed as a graded operad concentrated in degree 0.

Remark. Our theorems remain true if $\mathbb{K}$ is a ring, provided that we restrict ourselves to objects formed by free $\mathbb{K}$-modules. But the basis conditions are in this context more difficult to check.

2.1 Bar construction and Koszul duality for operads

In this section, we recall the definition of the reduced bar construction and the definition of Koszul duality for operads. We refer the reader to [9; 12] for a comprehensive introduction to operads and to [5] for the original definition of the notion of a Koszul operad. This definition has been generalized by Getzler and Jones in [4] in the graded context, and by Getzler in [3] for operads with generators in all arities. We also refer the reader to the article of Fresse [2] for a generalization in the wider context of operads defined over a ring with generators in all arities and all degrees.

We follow the conventions of [2].

2.1.1 Augmentation ideal of an operad

The *identity operad* is defined by $I(1) = \mathbb{K}$ and $I(r) = 0$ for $r \neq 1$. An operad P equipped with a morphism $\epsilon : \mathsf{P} \to I$ is called an augmented operad. The augmentation ideal of P is $\bar{\mathsf{P}} = \ker \epsilon$. As ϵ is a retract of the identity morphism, we have a splitting $\mathsf{P} = I \oplus \bar{\mathsf{P}}$.

2.1.2 Reduced bar construction

Recall that the *suspension of a dg-module* M is the dg-module ΣM defined by $\mathbb{K}e \otimes M$, where e has degree 1 and a trivial differential. We have a natural identification $(\Sigma M)_d = M_{d-1}$. For a non-graded operad P, the module $\Sigma\bar{\mathsf{P}}(r)$ is equal to the module $\bar{\mathsf{P}}(r)$ in degree 1 and is zero in degree $* \neq 1$.

The *reduced bar construction* $B(\mathsf{P})$ is a quasi-cofree cooperad defined by $F^c(\Sigma\bar{\mathsf{P}})$, the cofree cooperad generated by the suspension of $\bar{\mathsf{P}}$. The bar construction $B(\mathsf{P})$ is equipped with a differential given by a coderivation $\partial : F^c(\Sigma\bar{\mathsf{P}}) \to F^c(\Sigma\bar{\mathsf{P}})$ which is determined by the partial composition products of P. Recall that $F^c(\Sigma\bar{\mathsf{P}})$ is generated by tensors $\bigotimes_{v \in V(\tau)} x_v$, where τ ranges over trees, the notation $V(\tau)$ refers to the set of vertices of τ and x_v is an element of $\Sigma\bar{\mathsf{P}}$ associated to each vertex. More details on this construction are given in Section 2.4.1.

We are interested in the homology of this bar complex. To calculate it, we use the notion of weight grading.

2.1.3 Modules equipped with a weight grading

We consider $\mathbb{K}$-modules V equipped with a weight grading: $V = \bigoplus V_{(s)}$. In the case of a dg-module V, the homogeneous components $V_{(s)}$ are supposed to be sub-dg-modules of V. A tensor product of modules equipped with a weight grading inherits a natural weight grading such that $(V \otimes W)_{(n)} = \bigoplus_{s+t=n} V_{(s)} \otimes W_{(t)}$.

2.1.4 Operads equipped with a weight grading

An operad P is *equipped with a weight grading* if each term $\mathsf{P}(n)$ is weight graded and the composition product $\mathsf{P} \circ \mathsf{P} \to \mathsf{P}$ preserves the weight grading. This condition asserts equivalently that the partial composition product of homogeneous elements $p \in \mathsf{P}_{(s)}(m)$ and $q \in \mathsf{P}_{(t)}(n)$ satisfy $p \circ_i q \in \mathsf{P}_{(s+t)}(m+n-1)$.

An operad equipped with a weight grading is called *connected* if

$$\mathsf{P}_{(0)}(r) = \begin{cases} \mathbb{K}.1 & \text{for } r = 1 \\ 0 & \text{otherwise.} \end{cases}$$

A connected operad is automatically augmented, the augmentation being the projection on the weight 0 component. We have $\bar{\mathsf{P}}_{(s)} = \mathsf{P}_{(s)}$ if $s \neq 0$.

In what follows, we will use the free operad $F(M)$. This operad has a natural weight grading which makes it a graded operad. Recall briefly that the free operad $F(M)$, like the cofree cooperad $F^c(M)$, is generated by tensors on trees $\bigotimes_{v \in V(\tau)} x_v$, representing formal compositions of operations. The weight of such a tensor in $F(M)$ is given by its number of factors x_v. Notice that the free operad is connected. We will go back to the construction of $F(M)$ in Section 2.2.7.

2.1.5 Homogeneous operadic ideals and quotients

A *homogeneous operadic ideal* is an operadic ideal I such that $I = \bigoplus I_{(s)}$, where $I_{(s)} = I \cap \mathsf{P}_{(s)}$.

We observe that the quotient of an operad equipped with a weight by a homogeneous ideal is naturally equipped with a weight. This assertion is an obvious generalization of a classical result for algebras.

2.1.6 Quadratic operads

A *quadratic operad* is an operad such that $\mathsf{P} = F(M)/I$, where $I = (R)$ is the operadic ideal generated by $R \subset F_{(2)}(M)$.

For the Koszul duality, we use $\overline{R} \subset F_{(2)}(M)$ the sub-Σ_*-module generated by $R \subset F_{(2)}(M)$. This sub-Σ_*-module generates the same operadic ideal $(R) = (\overline{R})$.

We will see that the elements of $(\overline{R})$ are represented by linear combinations of trees where one of the vertices is labelled by an element of $\overline{R}$ and the other vertices by elements of M.

A quadratic operad has a natural weight grading, induced by the weight grading of the free operad.

For a quadratic operad such that $M(0) = 0$, we have automatically

$$\mathsf{P}_{(0)}(r) = \begin{cases} \mathbb{K}.1, & \text{if } r = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We have a natural isomorphism $\mathsf{P}_{(1)}(r) = M(r)$. Moreover, we have $\mathsf{P}_{(2)} = \mathcal{F}_{(2)}(M)/\overline{R}$.

The operads associated respectively to the associative, commutative, and Lie algebras are quadratic.

2.1.7 Weight grading on the bar construction

If P is equipped with a weight grading, then $B(\mathsf{P})$ has an induced weight grading. Formally, we use that $B(\mathsf{P})$ is spanned by tensors $\bigotimes_v p_v$. The weight of such a tensor is the sum of the weight of the factors p_v, as defined in Section 2.1.3. The differential is homogeneous.

If we suppose that $\mathsf{P}_{(0)}$ is reduced to $\mathbb{K}.1$, then $\Sigma\bar{\mathsf{P}}_{(0)} = 0$. Hence the elements p_i which occur in the treewise tensors of $B(\mathsf{P})$ have a weight larger than 1. As a consequence, we have $B_d(\mathsf{P})_{(s)} = 0$ if $d > s$, where d is the bar degree, that is the number of suspensions.

2.1.8 Koszul operads

One says that a (connected, graded, equipped with weight) operad P is Koszul if $H_*(B_\bullet(\mathsf{P})_{(s)}) = 0$ for $* \neq s$ (in words if the homology of its bar construction is concentrated on the diagonal $* = s$).

The *Koszul construction* is defined by

$$K(\mathsf{P})_{(s)} := H_s(B_*(\mathsf{P})_{(s)}, \delta) = \ker(\delta : B_s(\mathsf{P})_{(s)} \to B_{s-1}(\mathsf{P})_{(s)}).$$

From the definition, $K(\mathsf{P})_{(s)}$ is concentrated in degree s. We observe that the inclusion $K_d(\mathsf{P})_{(s)} \to B_d(\mathsf{P})_{(s)}$ is a morphism of chain complexes. The operad P is Koszul if and only if the inclusion morphism $K(\mathsf{P}) \to B(\mathsf{P})$ is a quasi-isomorphism. We observe that $K(\mathsf{P})$ is a cooperad.

2.2 The language of trees

Trees allow us to represent graphically the elements of the free operad and of the bar construction. The goal of this section is to define the conventions used throughout the article to describe the structure of a tree.

2.2.1 Vertices and edges

An *n-tree* is an abstract oriented tree together with one *outgoing edge* (the root of the tree) and n *ingoing edges* (the entries of the tree) indexed by the set $\{1,\ldots,n\}$. Formally, an n-tree τ is determined by a set of *vertices* $V(\tau)$ and by a set of *edges* $e \in E(\tau)$ oriented from a source $s(e) \in V(\tau) \coprod \{1,\ldots,n\}$ to a target $t(e) \in V(\tau) \coprod \{0\}$, with the following conditions:

1. There is a unique edge $e \in E(\tau)$ such that $t(e) = 0$. We call this edge the *root*.
2. For every vertex $v \in V(\tau)$, there is a unique $e \in E(\tau)$ such that $s(e) = v$.
3. For every $i \in \{1,\ldots,n\}$, there is a unique e such that $s(e) = i$. This edge is the ith entry of the tree.
4. For every vertex v, there is a sequence of edges $e_1,\ldots,e_k$ such that $s(e_1) = v, t(e_i) = s(e_{i+1})$ for every $i \in [1, k-1]$ and $t(e_k) = 0$.

These conditions imply that the set $V(\tau) \coprod \{1,\ldots,n\}$ is equipped with a partial order so that $s(e) > t(e)$ for any edge e. The minimum of the order is 0. There is an associated partial order on edges.

The set $E'(\tau)$ of *internal edges* is the set $E(\tau)$ of edges minus the ingoing edges and the outgoing edge.

We call a *leaf* the source of an ingoing edge. We draw trees with leaves on top and the root at the bottom.

We say that a leaf i is *linked to a vertex* v if there is a monotonic path of edges between i and v. We assume also that a leaf i is linked to itself.

We define the entries of the vertex v by

$$I_v = \{s(e), e \in E(\tau) \text{ such that } t(e) = v\}.$$

Then a tree structure is determined by a partition of $V(\tau) \coprod \{1,\ldots,n\}$ of the form $\coprod_{v \in V(\tau) \coprod \{0\}} I_v$ (of course, not all partitions of this form are associated to a tree structure).

2.2.2 Tree isomorphisms

An *isomorphism of n-trees* $f : \tau \to \tau'$ is defined by two bijections

$$f_V : V(\tau) \to V(\tau') \text{ and } f_E : E(\tau) \to E(\tau')$$

which preserve the structure of the tree (the source and target of every edge). We can extend f_V by the identity on $\{1,\ldots,n\}$ to have the relation $I_{f_V(v)} = f_V(I_v)$ for every $v \in V(\tau) \coprod \{0\}$. The n-trees and their isomorphisms define a category.

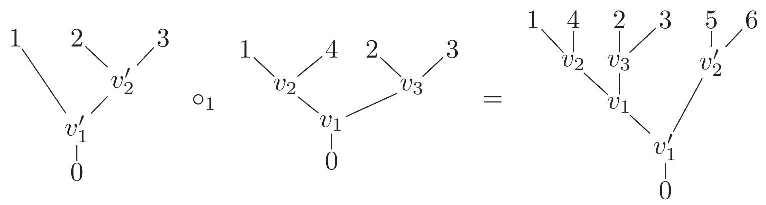


Figure 2.1: Example of a partial composition.

2.2.3 The Σ_*-category of trees

Let $T(n)$ be the category defined by the n-trees and their isomorphisms. This category has a weight splitting:

$$T(n) = \coprod_{r=0}^{\infty} T_{(r)}(n),$$

where $T_{(r)}(n)$ is the category formed by trees with r vertices.

We can generalize the construction of $T(n)$ by indexing the entries of an n-tree by a set $I = \{i_1, \ldots, i_n\}$ of n elements. We obtain the category $T(I)$ of I-trees. A bijection $u : I \to I'$ induces a functor $u_* : T(I) \to T(I')$ such that $u_*(T_{(r)}(I)) \subset T_{(r)}(I')$.

A permutation $w \in \Sigma_n$ (where Σ_n is the symmetric group on n elements) induces a functor from the category of n-trees to itself. Hence the symmetric group acts on n-trees.

2.2.4 Subtrees

A *subtree* σ of a tree τ is a tree (satisfying Conditions (1), (2), (3), (4) of Section 2.2.1) defined by subsets $V(\sigma) \subset V(\tau)$ and $E(\sigma) \subset E(\tau)$ satisfying Conditions (1'), (2') and (3') below. For convenience, s_τ and t_τ denote the source and the target in the tree τ.

(1') For all v in $V(\sigma)$ and for all e in $E(\tau)$, we have $(s(e) = v \text{ or } t(e) = v) \Rightarrow e \in E(\sigma)$.

(2') For all edges e in $E(\sigma)$, if $s_\tau(e)$ lies in $V(\sigma)$, then the source of e in the tree σ is $s_\tau(e)$, otherwise, the source of e in the tree σ is a leaf labelled by the minimum of the leaves linked to $s_\tau(e)$.

(3') For all edges e in $E(\sigma)$, if $t_\tau(e)$ lies in $V(\sigma)$, then the target of e in the tree σ is $t_\tau(e)$, otherwise the target of e in the tree σ is 0.

Graphically, a subtree corresponds to a connected part of the graph of the tree.

The subtree σ of a tree τ generated by an edge $e \in E'(\tau)$ is the tree τ_e such that $V(\tau_e) = \{s(e), t(e)\}$ and $E'(\tau_e) = \{e\}$. The ingoing edges relative to $s(e)$ and $t(e)$ are kept, and the outgoing edge is linked to $t(e)$. The leaves are labelled as specified above.

2.2.5 The operad of trees

One equips the sequence of categories $T(n)$ with the structure of an operad. The partial composition product

$$\circ_i : T_{(r)}(m) \times T_{(s)}(n) \to T_{(r+s)}(m + n - 1)$$

is defined as follows: for $\sigma \in T_{(r)}(m)$ and $\tau \in T_{(s)}(n)$, the composite tree $\sigma \circ_i \tau$ is obtained by grafting the root of τ to the ith entry of σ (*cf.* Figure 2.1).

2.2.6 The module of treewise tensors

A Σ_*-*module* $M = \{M(n)\}_{n\geq 0}$ is a sequence of graded $\mathbb{K}$-modules $M(n)$ equipped with an action of the symmetric group Σ_n.

Let M be a Σ_*-module. A *module of treewise tensors* $\tau(M)$ is associated to any tree τ.

Let v be a vertex of τ. Call n_v the cardinal of I_v. Let $M(I_v)$ be the $\mathbb{K}$-module generated by tensors $f \otimes_{\Sigma_{n_v}} x_v$ where $x_v \in M(n_v)$ and f is a bijection of the entries $\{1, \dots, n\}$ to the entries of x_v. One sets

$$\tau(M) = \bigotimes_{v\in V(\tau)} M(I_v).$$

Observe that this construction is functorial in τ : an isomorphism of trees $f : \tau \to \tau'$ induces a morphism $f_* : \tau(M) \to \tau'(M)$.

In practice, we see a treewise tensor as a tree with vertices labelled by elements of M, or equivalently a tensor product arranged on a tree.

Recall that a tree τ is called a corolla if it has only one vertex. For a corolla, we have an identification $\tau(M) \cong M(n)$ where n is the number of entries of τ.

2.2.7 The free operad

The free operad has an explicit expansion so that

$$F(M)(n) = \bigoplus_{\tau\in T(n)} \tau(M)/\cong .$$

In $F(M)$, the relation $\cong$ identifies treewise tensors which correspond to each other by an isomorphism. Explicitly, for $x \in \tau(M)$ and $x' \in \tau'(M)$, we have $x' \cong x$ if and only if $x' = f_*x$ for an isomorphism $f : \tau \to \tau'$.

In this representation, the weight grading of the free operad defined in Section 2.1.4 is given by the number of vertices of the tree.

2.2.8 Construction without quotient

Throughout the paper, we work with trees (called *reduced*) satisfying $I_v \neq \emptyset$ for every vertex v. A reduced tree has no automorphism except the identity. If $M(0) = 0$, then the free operad involves only treewise tensors $x \in \tau(M)$ where τ is reduced. An operad is called *reduced* if it is spanned by treewise tensors on reduced trees.

We are going to use that a reduced tree has a canonical planar representation. This representation is determined by an ordering of the entries of each vertex v.

We determine an order on I_v in the following way:

1. To every v' in I_v, we associate the minimum of the leaves linked to v'.
2. We place the vertices v' and the leaves directly linked to v from left to right above v in ascending order.

The order gives a bijection between $\{1, \dots, n_v\}$ and the entries of v. This bijection gives an isomorphim $M(I_v) \simeq M(n_v)$, for each $v \in V(\tau)$. As a consequence, for the module of treewise tensors $\tau(M)$, we obtain $\tau(M) \simeq \bigotimes_{v\in V(\tau)} M(n_v)$.

To obtain a canonical representation of elements of the free operad, we fix also a set $T'(n)$ of representatives of isomorphism classes of n-trees. The expansion of the free operad gives then:

$$F(M)(n) \simeq \bigoplus_{\tau\in T'(n)} \tau(M) \simeq \bigoplus_{\tau\in T'(n)} \Big\{ \bigotimes_{v\in V(\tau)} M(n_v)\Big\}.$$

2.3 The Poincaré-Birkhoff-Witt criterion

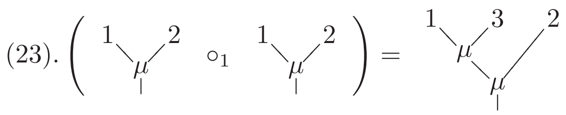
The aim of this section is to give the PBW criterion. We define the notion of a PBW basis for an operad, generalizing what Priddy did in the case of the algebras (*cf.* [14]).

2.3.1 A basis of treewise tensors and of the free operad

Let M be a Σ_*-module, with an ordered basis $\mathcal{B}^M$ (as a $\mathbb{K}$-module) and such that $M(0) = 0$. For every tree τ, we define a *monomial basis* $\mathcal{B}_\tau^{F(M)}$ of $\tau(M)$ in the following way. We use the planar representation of τ, giving an isomorphism $\tau(M) \cong \bigotimes_v M(n_v)$. An element $\bigotimes_v m_v$ belongs to $\mathcal{B}_\tau^{F(M)}$ if and only if each m_v is in $\mathcal{B}^M$. We set $\mathcal{B}^{F(M)} = \coprod_\tau \mathcal{B}_\tau^{F(M)}$.

A *pointed shuffle of a composition* $\alpha \circ_i \beta$ is a permutation preserving the order of the entries of each treewise tensor in the partial composition product and preserving the entry i. More explicitely, for α a treewise tensor with s entries and β a treewise tensor with t entries, a permutation $w \in \Sigma_{s+t-1}$ is a pointed shuffle if the orders of the entries of α and of β are the same as in the composition $w.(\alpha \circ_i \beta)$ and if the minimum of the entries of β in the composition is i. This definition implies that the entries labelled 1 to $i-1$ are not modified.

Example of a partial composition with a pointed shuffle.



In this example, the transposition (23) is the only non-trivial pointed shuffle of the composition.

2.3.2 Observation. *The basis* $\mathcal{B}^{F(M)}$ *is the only basis such that*

- $\mathcal{B}_\tau^{F(M)} = \mathcal{B}^{M(n)}$ *if* τ *is a corolla with* n *entries.*
- *For all* $\alpha \in \sigma(M), \beta \in \tau(M)$ *treewise tensors and* w *pointed shuffle, we have:*

$$w.(\alpha \circ_i \beta) \in \mathcal{B}_{w.(\sigma \circ_i \tau)}^{F(M)} \Leftrightarrow \alpha \in \mathcal{B}_\sigma^{F(M)} \text{ and } \beta \in \mathcal{B}_\tau^{F(M)}.$$

2.3.3 Order on the basis of the treewise tensors

We are choosing an order on the monomial basis of $F(M)(r)$ for every r in $\mathbb{N}$, satisfying the compatibility condition:

For α, α' with m entries and β, β' with n entries, we have

$$\begin{cases} \alpha \leq \alpha' \\ \beta \leq \beta' \end{cases} \Rightarrow \forall i, w.(\alpha \circ_i \beta) \leq w.(\alpha' \circ_i \beta'),\ \forall w \text{ pointed shuffle.}$$

2.3.4 Example of a suitable order

Let α be a treewise tensor with n entries. We associate a sequence of n words $(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n)$ to α in the following way: for all i, there exists a unique monotonic path of vertices from the root to i, and $\underline{a}_i$ is the word composed (from left to right) of the labels of these vertices (from bottom to top).

Recall that M has an ordered basis. If $\underline{a}$ and $\underline{b}$ are two words, we first compare the length of the words ($\underline{a} < \underline{b}$ if $l(\underline{a}) < l(\underline{b})$, where l is the length) and if they are equal, we compare them lexicographically (each letter being in M).

We can then compare two treewise tensors with the same number of entries α (associated to the sequence $(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n)$) and β (associated to $(\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n)$), such that $\alpha \neq \beta$, by comparing $\underline{a}_1$ with $\underline{b}_1$, then $\underline{a}_2$ with $\underline{b}_2$, etc. This defines a strict relation.

2.3.5 Proposition. *The order defined above satisfies the compatibility condition of Section 2.3.3.*

Proof. Let α and α' be treewise tensors with n entries, such that $\alpha \leq \alpha'$. Let β and β' be treewise tensors with m entries, such that $\beta \leq \beta'$.

Let $(\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n)$, resp. $(\underline{a}'_1, \underline{a}'_2, \dots, \underline{a}'_n)$, be the word sequence associated to α, resp. α'. Let $(\underline{b}_1, \underline{b}_2, \dots, \underline{b}_m)$, resp. $(\underline{b}'_1, \underline{b}'_2, \dots, \underline{b}'_m)$, be the word sequence associated to β, resp. β'.

The word sequence associated to the composite $\alpha \circ_i \beta$ has the form

$$(\underline{a}_1, \underline{a}_2, \dots, \underline{a}_i\underline{b}_1, \underline{a}_i\underline{b}_2, \dots, \underline{a}_i\underline{b}_m, \underline{a}_{i+1}, \dots, \underline{a}_n)$$

where $\underline{a}_i\underline{b}_j$ is the concatenation of $\underline{a}_i$ and $\underline{b}_j$. Similarly, the word sequence

$$(\underline{a}'_1, \underline{a}'_2, \dots, \underline{a}'_i\underline{b}'_1, \underline{a}'_i\underline{b}'_2, \dots, \underline{a}'_i\underline{b}'_m, \underline{a}'_{i+1}, \dots, \underline{a}'_n)$$

is associated to $\alpha' \circ_i \beta'$.

To begin with, note that the length of $\underline{a}_i\underline{b}_j$ is the sum of the length of $\underline{a}_i$ and $\underline{b}_j$.

We compare $\alpha \circ_i \beta$ and $\alpha' \circ_i \beta'$ as they have both $n+m-1$ entries. As $\alpha \leq \alpha'$, we have $(\underline{a}_1, \underline{a}_2, \dots, \underline{a}_{i-1}) \leq (\underline{a}'_1, \underline{a}'_2, \dots, \underline{a}'_{i-1})$. If the inequality is strict, we are done, as our order is lexicographical in the sequence. If the inequality is an equality, we look at $\underline{a}_i$ and $\underline{a}'_i$. If $\underline{a}_i < \underline{a}'_i$, then $\underline{a}_i\underline{b}_1 < \underline{a}'_i\underline{b}'_1$. If $\underline{a}_i = \underline{a}'_i$, comparing $\underline{a}_i\underline{b}_j$ with $\underline{a}'_i\underline{b}'_j$ is the same as comparing $\underline{b}'_j$ with $\underline{b}'_j$ for all j. So $(\underline{a}_i\underline{b}_1, \underline{a}_i\underline{b}_2, \dots, \underline{a}_i\underline{b}_m) \leq (\underline{a}'_i\underline{b}'_1, \underline{a}'_i\underline{b}'_2, \dots, \underline{a}'_i\underline{b}'_m)$. If the inequality is strict, we are done, else we have to look at the remainder of the sequence. As $\alpha \leq \alpha'$ and $(\underline{a}_1, \underline{a}_2, \dots, \underline{a}_i) = (\underline{a}'_1, \underline{a}'_2, \dots, \underline{a}'_i)$, we have $(\underline{a}_{i+1}, \dots, \underline{a}_n) \leq (\underline{a}'_{i+1}, \dots, \underline{a}'_n)$.

Finally we have $\alpha \circ_i \beta \leq \alpha' \circ_i \beta'$.

To show that $w.(\alpha \circ_i \beta) \leq w.(\alpha' \circ_i \beta')$ for all pointed shuffles w, we see how the pointed shuffles act on the sequence of words associated to a composition of treewise tensors. The pointed shuffles will induce a shuffle (in the usual meaning) between the set composed by the $\underline{a}_i\underline{b}_j$ for all j and the set composed by the $\underline{a}_j$ for $j > i$.

A shuffle preserves the order among each set it acts on and the order we have defined on the treewise tensors look at the associated words recursively. As a consequence, the order between $w.(\alpha \circ_i \beta)$ and $w.(\alpha' \circ_i \beta')$ will be the same as the one between $\alpha \circ_i \beta$ and $\alpha' \circ_i \beta'$. □

Remark. We call this order the *lexicographical order*. Another suitable order, the *reverse-length lexicographical order*, can be defined in a similar way. If $\underline{a}$ and $\underline{b}$ are two words, we first compare the length of the words ($\underline{a} > \underline{b}$ if $l(\underline{a}) < l(\underline{b})$, where l is the length) and if they are equal, we compare them lexicographically (each letter being in M). The proof of the compatibility condition of Section 2.3.3 is the same.

2.3.6 Restriction of a treewise tensor to a subtree

Let $\alpha = \bigotimes_{v \in \tau} m_v$ be a treewise tensor. The restriction of α to a subtree σ of τ is the tensor $\alpha_{|\sigma} = \bigotimes_{v \in V(\sigma)} m_v$ which gives an element of $\sigma(M)$.

We use this notion for a subtree $\sigma = \tau_e$ generated by an edge e (defined in Section 2.2.4).

2.3.7 Poincaré-Birkhoff-Witt basis

Let P be a reduced operad defined by $F(M)/(\overline{R})$. A *PBW basis* for P is a set $\mathcal{B}^{\mathsf{P}} \subset \mathcal{B}^{F(M)}$ of elements representing a basis of the $\mathbb{K}$-module P, containing 1, $\mathcal{B}^M$ and for every tree τ a subset $\mathcal{B}^{\mathsf{P}}_\tau$ of $\mathcal{B}^{F(M)}_\tau$, satisfying the conditions:

1. For $\alpha \in \mathcal{B}^{\mathsf{P}}_\sigma$, $\beta \in \mathcal{B}^{\mathsf{P}}_\tau$ and w a pointed shuffle, either $w.(\alpha \circ_i \beta)$ is in $\mathcal{B}^{\mathsf{P}}_{w.(\sigma \circ_i \tau)}$, or the elements of the basis $\gamma \in \mathcal{B}^{\mathsf{P}}$ which appear in the unique decomposition $w.(\alpha \circ_i \beta) \equiv \Sigma_\gamma c_\gamma \gamma$, satisfy $\gamma > w.(\alpha \circ_i \beta)$ in $F(M)$.

2. A treewise tensor α is in $\mathcal{B}^{\mathsf{P}}_\tau$ if and only if for every internal edge e of τ, the restricted treewise tensor $\alpha_{|\tau_e}$ is in $\mathcal{B}^{\mathsf{P}}_{\tau_e}$.

Remark. This definition generalizes Priddy's definition for algebras (*cf.* [14]). Recall that an algebra A is equivalent to an operad P_A such that $\mathsf{P}_A(r) = \begin{cases} A, & \text{for } r = 1, \\ 0, & \text{otherwise.} \end{cases}$

The algebra A has a PBW basis in Priddy's sense if and only if the operad P_A has a PBW basis in our sense.

2.3.8 Observation. *Condition 1 is equivalent to Condition 1':*

(1') For α in $\mathcal{B}^{F(M)}$, either $\alpha \in \mathcal{B}^{\mathsf{P}}$, or the elements of the basis $\gamma \in \mathcal{B}^{\mathsf{P}}$ which appear in the unique decomposition $\alpha \equiv \Sigma_\gamma c_\gamma \gamma$, satisfy $\gamma > \alpha$ in $F(M)$.

Proof. Condition 1' implies obviously Condition 1. For the conserve direction, we use an induction on the number of vertices in α in $\mathcal{B}^{F(M)}$ and observation 2.3.2. □

2.3.9 Proposition. *Assume that M is finitely generated. If Condition 1 is satisfied when α and β are corollas, and Condition 2 is satisfied, then Condition 1 is true for all α and β.*

Proof. Equivalently, we can say:

Let M be finitely generated. If Condition 1' is satisfied when α has only one internal edge and Condition 2 is satisfied, then Condition 1' is true for all α.

We prove this equivalent proposition.

Let α be in $\mathcal{B}^{F(M)}_\tau \setminus \mathcal{B}^{\mathsf{P}}_\tau$. Condition 2 implies that there exists an internal edge e such that $\alpha_{|\tau_e} \notin \mathcal{B}^{\mathsf{P}}_{\tau_e}$. By Condition 1', we can write $\alpha_{|\tau_e} \equiv \Sigma_\gamma c_\gamma \gamma$, where $\gamma > \alpha_{|\tau_e}$ in $F(M)$. We replace $\alpha_{|\tau_e}$ by $\Sigma_\gamma c_\gamma \gamma$ in α. This gives another representative of $\alpha \equiv \Sigma_{\gamma'} c_{\gamma'} \gamma'$ such that $\gamma' > \alpha$ (because the order is compatible with the partial composition product). If all γ' are in $\mathcal{B}^{\mathsf{P}}$, we are done. Otherwise, we iterate the method.

We get others representative of α as sums of treewise tensors, each time strictly larger. As the number of trees with a specified number of entries is finite and as the basis of M is also finite, then the number of treewise tensors with a specified number of entries is also finite. So the process stops after a finite number of steps, and the treewise tensors we get at the end are in $\mathcal{B}^{\mathsf{P}}$. □

2.3.10 Theorem. *A finitely generated reduced operad which has a PBW basis is Koszul.*

The proof of this statement is achieved in the next section. The finiteness assumption is satisfied by all usual examples and allows us to use standard spectral sequence arguments in our proof.

2.4 Proof of the Poincaré-Birkhoff-Witt criterion

To show this result, we describe more precisely $B(\mathsf{P})$ and a basis. Then we will use a filtration to study the homology of $E^0 B(\mathsf{P})(r)_\lambda$.

2.4.1 Explicit description of $B(\mathsf{P})$

By definition, $B(F(M))$ is equal to $\bigoplus_\sigma \sigma(F(M))$. Explicitly, a generator of $\sigma(F(M))$ corresponds to a tree σ labelled with trees labelled by elements of M, that is a treewise tensor composed of treewise tensors on M. We can represent it by a large tree τ labelled by elements of M and equipped with a splitting in subtrees τ_{comp}, that we can see as connected components. The τ_{comp} are separated by *cutting edges* which form a subset $D \subset E'(\tau)$. The union of the internal edges of the subtrees τ_{comp} form a set $S \subset E'(\tau)$ such that $S \coprod D = E'(\tau)$. We will work with S, the set of *marking edges*.

The marking edges S determine the decomposition of a treewise tensor α into $\bigotimes \alpha_{comp}$ where $\alpha_{comp} = \alpha_{|\tau_{comp}}$ are the factors in $F(M)$.

So we identify a basis element of $B(F(M))$ to a pair (α, S), with $\alpha \in \tau(M)$ (*cf.* Figure 2.2).

$$B(F(M)) \cong \bigoplus_{\tau,S} (\tau(M), S).$$

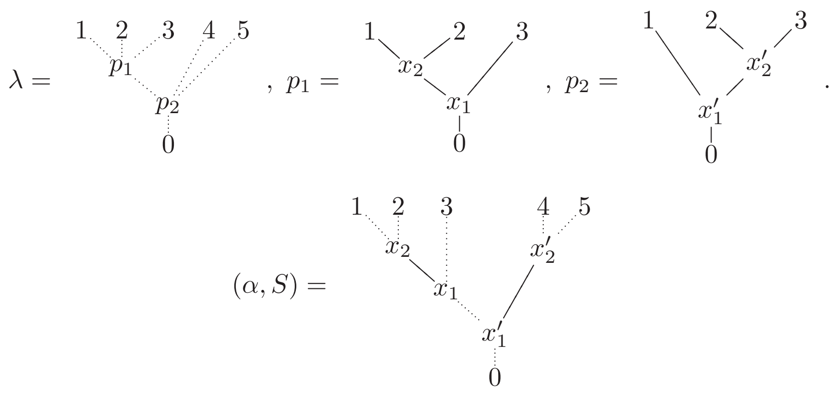


Figure 2.2: We identify an element of $B(F(M))$ with a pair (α, S). The first treewise tensor λ represents an element in $B(F(M))$, with p_1 and p_2 elements of $F(M)$. The edges in dots are the edges of the bar construction $B(F(M))$, the full edges are the edges of the free operad $F(M)$. The second line gives the result of the substitution. The set D of cutting edges is reduced to the dotted edge between x_1 and x'_1. The marked edges are the two full edges $x_2 - x_1$ and $x'_2 - x'_1$.

We examine now the differential structure of $B(F(M))$.

The differential δ is given by

$$\delta(\alpha, S) = \sum_{e \in E'(\tau) - S} \pm(\alpha, S \coprod \{e\})$$

for α a treewise tensor associated to the tree τ. The sign is determined by the usual rules of differential graded algebra. It is given by the permutations of tensors and suspensions which appear in the computation of the differential.

The operation $(\alpha, S) \mapsto (\alpha, S \coprod e)$ represents a partial composition at the edge e for the element in $B(F(M))$ represented by (α, S).

Notice that the differential changes only the marking and not $\tau(M)$. Hence $\delta(\bigoplus_S(\tau(M), S)) \subset \bigoplus_S(\tau(M), S)$.

2.4.2 Description and basis of $B(\mathsf{P})$

First, let $\mathcal{B}_\tau^{B(F(M))}$ be the natural basis of treewise tensors on the tree τ labelled with elements of $\mathcal{B}^{F(M)}$. Set also $\mathcal{B}^{B(F(M))} = \coprod_\tau \mathcal{B}_\tau^{B(F(M))}$.

As $\mathsf{P} = F(M)/(\overline{R})$, the reduced bar construction $B(\mathsf{P})$ is a quotient of $B(F(M))$. Two elements (α, S) and (α', S) are identified in $B(\mathsf{P})$ if and only if $S = S'$ and every factor α_{comp} is identified to α'_{comp} in P.

We define $\mathcal{B}^{B(\mathsf{P})}$, a set of elements in $B(F(M))$ representing a basis of $B(\mathsf{P})$, starting from the base $\mathcal{B}^{\mathsf{P}}$ as follows: an element (β, S) in $(\tau(M), S)$ is in $\mathcal{B}_\tau^{B(\mathsf{P})} \subset \mathcal{B}_\tau^{B(F(M))}$ if each of its factors $\beta_{\tau_{comp}}$ lies in $\mathcal{B}^{\mathsf{P}}_{\tau_{comp}}$. The element β is an element in $\mathcal{B}_\tau^{F(M)}$, the basis defined in Section 2.3.1.

Definition. An edge e is said to be *admissible* if the restricted treewise tensor $\alpha_{|\tau_e}$ is in $\mathcal{B}^{\mathsf{P}}$. The set Adm_α is the set of the admissible edges of α.

2.4.3 Observation. *We have an equivalence*

$$(\alpha, S) \in \mathcal{B}^{B(\mathsf{P})} \Leftrightarrow S \subset Adm_\alpha.$$

2.4.4 Filtration of $B(\mathsf{P})$

We use now that the monomial basis of $F(M)$ is equipped with a partial order satisfying the condition of Section 2.3.3. We associate to each λ a submodule $B(\mathsf{P})(r)_\lambda$ such that $\lambda \leq \mu \Rightarrow B(\mathsf{P})(r)_\lambda \supseteq B(\mathsf{P})(r)_\mu$. By abuse, we say that these submodules define a filtration of $B(P)$ indexed by the poset of the monoidal basis.

In practice, we forget the cutting and we use the order of the basis of $F(M) = \bigoplus_\tau \tau(M)$. Explicitly, an element $(\alpha, S) \in B(\mathsf{P})(r)$ is in $B(\mathsf{P})(r)_\lambda$ if and only if $\alpha \geq \lambda$. Hence

$$B(\mathsf{P})(r) = \bigcup_{\lambda \in I(r)} B(\mathsf{P})(r)_\lambda$$

where $I(r)$ is the monomial basis of $F(M)(r)$. We also set

$$E^0 B(\mathsf{P})(r)_\lambda = B(\mathsf{P})(r)_\lambda / \sum_{\mu > \lambda} B(\mathsf{P})(r)_\mu.$$

Observe that $B(\mathsf{P})(r)_\lambda$ is a subcomplex of $B(\mathsf{P})(r)$. In fact, the differential δ corresponds to a partial composition product, modifying the cutting (that we forget in the filtration). Condition 1 of a PBW basis insures that an element is sent on the sum of larger or equal elements. The differential d^0 induced by δ in the quotient preserves the factor $E^0 B(\mathsf{P})(r)_\lambda$, which is generated by the pairs (λ, S) which belong to $\mathcal{B}^{B(\mathsf{P})}$.

Remark that $d^0_e : (\lambda, S) \mapsto (\lambda, S \coprod \{e\})$, so we can write $d^0 = \sum_e \pm d^0_e$, taking the sum on the edges e such that $d^0_e(\lambda)$ remains in the basis.

2.4.5 Lemma. *An edge e is admissible if and only if $d^0_e(\lambda, S) \neq 0$.*

Proof. The differential d^0_e transforms a non-marked admissible edge into a marked admissible edge, by Condition 1 of a PBW basis.

Conversely, if $d^0_e(\lambda, S) \neq 0$, then by Condition 2 (converse direction), the edge is admissible. □

2.4.6 Homology of $E^0 B(\mathsf{P})(r)_\lambda$

The quotient $E^0 B(\mathsf{P})(r)_\lambda$ is generated by the pairs (λ, S) where S ranges over the subsets of Adm_λ, the set of admissible edges. The differential d^0_e sends (λ, S) to $(\lambda, \sum_{e \in Adm_\lambda - S} S \coprod \{e\})$.

Note that this combinatorial complex is the dual of the oriented complex $C_*(\Delta^{Adm_\lambda})_+$ of the simplex Δ^{Adm_λ} augmented over $\mathbb{K}$, with the augmentation term added in $C_*(\Delta^{Adm_\lambda})_+$. The inclusion of the summand spanned by $(\lambda, \emptyset)$ in $E^0 B(\mathsf{P})(r)_\lambda$ is dual to the augmentation $C_*(\Delta^{Adm_\lambda})_+ \to \mathbb{K}$.

If $Adm_\lambda = \emptyset$, then the complex is reduced to a unique generator $(\lambda, \emptyset)$. Every component τ_{comp} is reduced to a vertex (we cut on all edges). This implies that the weight of λ is equal to its degree.

If Adm_λ is not empty, then the homology is zero.

We deduce from these assertions that $H_* E^0 B(\mathsf{P})(r)_\lambda = 0$ when the degree is different from the weight. We want to deduce from this result that the homology of $B(\mathsf{P})(r)$ vanishes in degrees different from the weight. We can use a standard spectral sequence argument. We just have to fix a linear ordering refining the partial order of the monomial basis. We have then a genuine linear filtration of $B(\mathsf{P})(r)$ defined by

$$F_\lambda B(\mathsf{P})(r) = \sum_{\mu \trianglerighteq \lambda} B(\mathsf{P})(r)_\mu$$

where $\trianglerighteq$ refers to the linear ordering. Note that the subquotient of this filtration $F_\lambda B(\mathsf{P})(r)/F_{\lambda+1} B(\mathsf{P})(r)$ (where $\lambda + 1$ denotes the successor of λ) agrees with the subquotient $E^0 B(\mathsf{P})(r)_\lambda$.

The filtration is compatible with the weight. Hence the associated spectral sequence splits. For a fixed weight s, the spectral sequence is bounded. Therefore it converges to $H_* B(\mathsf{P})_{(s)}$. The homology of $B(\mathsf{P})_{(s)}$ is a subquotient of the page $E^1_{(s)}$ of the spectral sequence, and when the weight is different from the degree, this page is 0. We deduce that $H_* B(\mathsf{P}) = 0$ when the weight is different from the degree. This result achieves the proof of theorem 2.3.10. □

2.5 Result on the dual of a Poincaré-Birkhoff-Witt operad

In this section we consider a reduced quadratic operad $\mathsf{P} = F(M)/(R)$ such that M is a finitely generated Σ_*-module. Recall that the Koszul construction $K(\mathsf{P})$ defined in Section 2.1.8 is a cooperad, and its linear dual $K(\mathsf{P})^\#$ is an operad. The goal of this section is to prove the following result:

2.5.1 Theorem. *If P is a PBW operad, then the dual operad $K(\mathsf{P})^\#$ is also a PBW operad.*

We determine a basis of $K(\mathsf{P})^\#$, and we prove it defines a PBW basis.

2.5.2 Basis of $K(\mathsf{P})^\#$

Again, we fix a total order on $\mathcal{B}^{\mathsf{P}}$ which is a refinement of an order satisfying the condition of Section 2.3.3. From now on, the symbol $>$ refers to the total order. This is necessary for technical arguments. We need to form a genuine sequential filtration, but we will see at the end that the result of our construction does not depend on the choice of a particular order.

We work here with a fixed number of entries r and a fixed weight n. There is a finite number of trees with n vertices and r entries, so there is a finite basis of treewise tensors with n vertices labelled by elements of M and r entries. This finite totally ordered set of treewise tensors can be written symbolically $\Lambda_{n,r} = \{0 < 1 < \ldots < \lambda < \lambda + 1 < \ldots < \mu\}$.

Recall that an element $(\alpha, S) \in B(\mathsf{P})(r)$ is in $B(\mathsf{P})(r)_\lambda$ if and only if $\alpha \geq \lambda$. In what follows, we write $F_\lambda = B(\mathsf{P})_{(n)}(r)_\lambda$ (it is the component in weight n of the submodule $F_\lambda B(\mathsf{P})(r)$ defined in the previous section) and $E^0_\lambda = E^0 B(\mathsf{P})_{(n)}(r)_\lambda$.

We have a finite filtration of $B(\mathsf{P})_{(n)}(r)$:

$$B(\mathsf{P})_{(n)}(r) = F_0 \supseteq F_1 \supseteq \ldots \supseteq F_\lambda \supseteq F_{\lambda+1} \supseteq \ldots \supseteq F_\mu = E^0_\mu.$$

2.5.2.1 Lemma. *For every $\lambda \in \Lambda_{n,r}$, the homology $H_{n-1}F_\lambda$ is 0.*

Proof. We are using a decreasing induction.

For $\lambda = \mu$, we have $F_\mu = E^0_\mu$. We know that $H_{n-1}E^0_\mu = 0$ as the weight is different from the degree, so $H_{n-1}F_\mu$ is 0.

Suppose the result true for $\lambda + 1$. The long exact sequence in homology induced by $0 \to F_{\lambda+1} \to F_\lambda \to E^0_\lambda \to 0$ gives

$$\ldots \to H_{n-1}F_{\lambda+1} \to H_{n-1}F_\lambda \to H_{n-1}E^0_\lambda \to \ldots .$$

The first term is 0 by induction, and the third term is 0 because the weight is different from the degree. So $H_{n-1}F_\lambda = 0$. □

Another part of the long exact sequence gives the short exact sequence:

$$0 \to H_nF_{\lambda+1} \to H_nF_\lambda \to H_nE^0_\lambda \to 0.$$

The first term 0 comes from $H_{n+1}E^0_\lambda$ and the second one from $H_{n-1}F_\lambda$.

These short exact sequences can be put together in a diagram, where vertical arrows are Coker's:

$$\begin{array}{ccccccccc} H_nF_\mu & \longrightarrow & \cdots & \longrightarrow & H_nF_{\lambda+1} & \longrightarrow & H_nF_\lambda & \longrightarrow \cdots \longrightarrow & H_nF_0. \\ & & & & \downarrow & & \downarrow & & \\ & & & & H_nE^0_{\lambda+1} & & H_nE^0_\lambda & & \end{array}$$

Recall that $H_nF_\mu = H_n(E^0_\mu)$ and $H_nF_0 = H_n(B(\mathsf{P})_{(n)}(r))$. We dualize this diagram, using that $H_n(C^\#) = H_n(C)^\#$, and we get the following diagram, where vertical arrows are Ker's:

$$\begin{array}{ccccccccc} H_n(E^0_\mu)^\# & \longleftarrow & \cdots & \longleftarrow & H_n(F_{\lambda+1})^\# & \longleftarrow & H_n(F_\lambda)^\# & \longleftarrow \cdots \longleftarrow & \mathbb{K}(\mathsf{P})^\#_{(n)}(r). \\ & & & & \uparrow & & \uparrow & & \\ & & & & H_n(E^0_{\lambda+1})^\# & & H_n(E^0_\lambda)^\# & & \end{array}$$

We showed in Section 2.4.6 that

$$H_n(E^0_\lambda) = \begin{cases} \mathbb{K}, & \text{if } Adm_\lambda = \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have $K(\mathsf{P})^{\#}_{(n)}(r) = \bigoplus_\lambda \mathbb{K}$ where $Adm_\lambda = \emptyset$ with $\lambda \in \Lambda_{n,r}$. We can note it does not depend on the choice of the total order.

Recall $K(\mathsf{P})^{\#}$ is a quotient of $F(\Sigma^{-1}M^{\#})$.

2.5.3 Lemma. *A basis of the* $\mathbb{K}$*-module* $K(\mathsf{P})^{\#}$ *is represented by* $\{\lambda^{\#} \in F(\Sigma^{-1}(M^{\#})) \mid Adm_\lambda = \emptyset\}$.

Proof. This result is an obvious consequence of the description of the submodules $K(\mathsf{P})^{\#}_{(n)}(r)$ in the previous paragraph. □

These treewise tensors λ are determined by the following property: the restricted treewise tensor induced by any edge e is not in $\mathcal{B}^{\mathsf{P}}$.

2.5.4 A PBW basis of $K(\mathsf{P})^{\#}$

Ginzburg and Kapranov showed in [5] that $K(\mathsf{P})^{\#} = F(\Sigma^{-1}(M^{\#}))/(R')$, where $\overline{R'}$ is determined by the exact sequence

$$0 \to \overline{R'} \to F_{(2)}(\Sigma^{-1}(M^{\#})) \to K_{(2)}(\mathsf{P})^{\#} \to 0.$$

We have to determine $\overline{R'} \subset F_{(2)}(\Sigma^{-1}(M^{\#}))$ explicitely.

As $\overline{R}$ is characterized by $0 \to \overline{R} \to F_{(2)}(M) \to \mathsf{P}_{(2)} \to 0$, we have dually

$$0 \to \phi(\Sigma^{-2}\overline{R}^{\perp}) \to F_{(2)}(\Sigma^{-1}(M^{\#})) \to K_{(2)}(\mathsf{P})^{\#} \to 0,$$

where ϕ is the isomorphism between $\Sigma^{-2}F_{(2)}(M^{\#})$ and $F_{(2)}(\Sigma^{-1}(M^{\#}))$. We have to study $\phi(\Sigma^{-2}\overline{R}^{\perp})$.

The main problem will be the suspensions which induce signs. Signs are induced by the classical commutation rule $g \otimes f = (-1)^{|f|.|g|} f \otimes g$. The suspension has degree $+1$.

Recall $F_{(2)}(M)$ is the set of treewise tensors with exactly one internal edge. Its basis $\mathcal{B}^{F_{(2)}(M)}$ can be decomposed into $\{\alpha_i\}_{i\in I} \coprod \{\alpha_j\}_{j\in J}$ where $\forall i \in I, \alpha_i \notin \mathcal{B}^{\mathsf{P}}$ and $\forall j \in J, \alpha_j \in \mathcal{B}^{\mathsf{P}}$. The ideal generated by the relations is $\overline{R} = \mathrm{Span}\{\alpha_i - \sum_{j\in J} c_{ij}\alpha_j; i \in I\} \subset F_{(2)}(M)$. A classic result of linear algebra gives $\overline{R}^{\perp} = \mathrm{Span}\{\alpha_j^{\#} + \sum_{i\in I} c_{ij}\alpha_i^{\#}; j \in J\} \subset F_{(2)}(M^{\#})$.

For $x_1 \in M(n_1)$ and $x_2 \in M(n_2)$, the definition returns us the relation

$$\phi(\Sigma^{-2}w.(x_1^{\#} \circ_i x_2^{\#})) = \epsilon(w)(-1)^{|x_1|} w.(\Sigma^{-1}x_1^{\#} \circ_i \Sigma^{-1}x_2^{\#}),$$

where $\epsilon(w)$ denotes the signature of the permutation w.

As $\Sigma^{-2}\overline{R}^{\perp} \subset \Sigma^{-2}F_{(2)}(M^{\#})$, we have $\phi(\Sigma^{-2}\overline{R}^{\perp}) \subset F_{(2)}(\Sigma^{-1}(M^{\#}))$. We have the relation

$$\phi(\Sigma^{-2}\overline{R}^{\perp}) = \mathrm{Span}\{\phi(\Sigma^{-2}\alpha_j^{\#}) + \sum_{i\in I} c_{ij}\phi(\Sigma^{-2}\alpha_i^{\#}); j \in J\},$$

where we identify naturally $F_{(2)}(M^{\#})$ and $F_{(2)}(M)^{\#}$.

2.5.5 Theorem. *Consider the set* $\mathcal{B}^{\#}$ *formed by monomial treewise tensors* β $F(\Sigma^{-1}(M^{\#}))$*, such that every subtensor generated by an internal edge of* β *is in the set* $\phi(\Sigma^{-2}\alpha_i^{\#}), i \in I$. *This set* $\mathcal{B}^{\#}$ *forms a PBW basis of* $K(\mathsf{P})^{\#}$ *for the opposite order (denoted* $<^{\#}$*).*

Note $\mathcal{B}^{\#}$ is uniquely determined by the $\phi(\Sigma^{-2}\alpha_i^{\#}), i \in I$.

Proof. From the descriptions of $\overline{R}^{\perp}$ and of the basis of $K(\mathsf{P})^{\#}$, we observe that $\mathcal{B}^{\#}$ is a basis of the module $K(\mathsf{P})^{\#}$. Also, Condition 2 to be a PBW basis is true by definition.

Let us show Condition 1. Here signs and suspensions do not interfere. As the $\alpha_j, j \in J$ are the quadratic part of a PBW basis, we have $c_{ij} \neq 0$ for $\alpha_i < \alpha_j$. Hence we have $\alpha_i^{\#} >^{\#} \alpha_j^{\#}$ if $c_{ij} \neq 0$. As a consequence, Condition 1 is satisfied for tensors with only one internal edge (*cf.* 2.3.9). As $M^{\#}$ is finitely generated, this implies Condition 1 of a PBW basis. □

2.5.6 Remark

When the module M is non-graded, we identify M with a dg-object concentrated in degree 0.

In the original construction by Ginzburg and Kapranov [5], the Koszul dual $\mathsf{P}^!$ is only defined for quadratic operads generated by binary operations. The operad $\mathsf{P}^!$ is an operadic suspension of $K(\mathsf{P})^{\#}$. The presentation of $K(\mathsf{P})^{\#}$ becomes on this suspension $\mathsf{P}^! = F(M^{\#} \otimes sgn)/(R'')$, where $M^{\#} \otimes sgn$ is a twist of $M^{\#}$ by the signature. The generating operations of $K(\mathsf{P})^{\#}$ are put in degree 0 in $\mathsf{P}^!$ (at this point, we are using that the generating operations are binary and of degree 0).

Because of signs, the orthogonal (R'') is here $< \alpha_j^* + \sum_{i \in I} c_{ij} \alpha_i^*; j \in J >$, where

$$\alpha_k^* = \epsilon(w)(-1)^{|x_1|(n_2-1)}(-1)^{(i-1)(n_2-1)} w.(x_1^{\#} \circ_i x_2^{\#}) \in F_{(2)}(M^{\#})$$

if $\alpha_k = w.(x_1 \circ_i x_2)$. The operad $\mathsf{P}^!$ has also a PBW basis, whose quadratic part is composed of the treewise tensors $\alpha_i^*, i \in I$.

In the case of operads generated by binary operations, we will work with $\mathsf{P}^!$ rather than $K(\mathsf{P})^{\#}$, and determine the treewise tensors $\alpha_i^*, i \in I$ and the generating relations R''.

2.6 Case of non-symmetric operads

We obtain in a similar way a PBW criterion in the case of the non-symmetric operads.

2.6.1 Non-symmetric operads and planar trees

A non-symmetric operad is defined as an operad, but without the action of symmetric groups. For more details, we refer the reader to [12]. We can represent compositions in a non-symmetric operad by planar trees.

The planar structure of a tree is determined by a total order on every set of entries I_v for vertices $v \in V(\tau)$, as in the construction explained in Section 2.2.8. The planar structure induces a total order on the entries of the tree. When we work with non-symmetric operads, we always consider planar trees with a natural numerotation of the entries, the numeration preserving the order.

The non-symmetric free operad $F_{ns}(M)$ is associated to a non-symmetric module, a sequence of modules $M(n), n \in \mathbb{N}$ without an action of symmetric groups. We just replace abstract trees by planar trees in this construction.

2.6.2 Order on the treewise tensors

We define an order as in the symmetric case, the only difference being that we forget pointed shuffles.

Let M be a non-symmetric module, with an ordered basis $\mathcal{B}^M$. For every planar tree τ, we have a natural *monomial basis* $\mathcal{B}_\tau^{F(M)}$ of $\tau(M)$: an element of this basis is the tree τ labelled with elements of $\mathcal{B}^M$.

We choose an order on the monomial basis of $F_{ns}(M)(r)$ for every r in $\mathbb{N}$, satisfying the following condition:

For α, α' with m entries and β, β' with n entries, we have

$$\begin{cases} \alpha \leq \alpha' \\ \beta \leq \beta' \end{cases} \Rightarrow \forall i, \alpha \circ_i \beta \leq \alpha' \circ_i \beta'.$$

2.6.3 Non-symmetric PBW basis

We define this notion as in the symmetric case, but without pointed shuffles.

Let P be a non-symmetric operad, defined by $F_{ns}(M)/(R)$.

A *PBW basis* of P is a set $\mathcal{B}^{\mathsf{P}} \subset F_{ns}(M)$ of representatives of a base of the module P, containing 1, $\mathcal{B}^M$ and for all τ a subset $\mathcal{B}^{\mathsf{P}}_\tau$ of $\mathcal{B}^{F(M)}_\tau$, satisfying the following properties:

1. For $\alpha \in \mathcal{B}^{\mathsf{P}}_\sigma$, $\beta \in \mathcal{B}^{\mathsf{P}}_\tau$, either $\alpha \circ_i \beta$ is in $\mathcal{B}^{\mathsf{P}}_{\sigma\circ_i\tau}$, or the elements of the basis $\gamma \in \mathcal{B}^{\mathsf{P}}$ which appear in the unique decomposition $\alpha \circ_i \beta \equiv \Sigma_\gamma c_\gamma \gamma$, satisfy $\gamma > \alpha \circ_i \beta$ in $F(M)$.

2. A treewise tensor α is in $\mathcal{B}^{\mathsf{P}}_\tau$ if and only if for every internal edge e of τ, the restricted treewise tensor $\alpha_{|\tau_e}$ lies in $\mathcal{B}^{\mathsf{P}}_{\tau_e}$.

2.6.4 Symmetrization

The forgetful functor from Σ_*-modules to sequences of (non-symmetric) modules has a left adjoint $\_ \otimes \Sigma_*$. If P is a non-symmetric operad, then the associated Σ_*-module $\mathsf{P} \otimes \Sigma_*$ has a natural operad structure. For a free operad, we obtain $F_{ns}(M_{ns}) \otimes \Sigma_* = F(M_{ns} \otimes \Sigma_*)$.

We extend the order relation from $F_{ns}(M_{ns})$ to $F_{ns}(M_{ns}) \otimes \Sigma_*$, setting:

$$\alpha \otimes \sigma \leq \alpha' \otimes \sigma' \text{ if } \begin{cases} \sigma = \sigma' \\ \alpha \leq \alpha' \end{cases} .$$

We do not compare the elements if $\sigma \neq \sigma'$.

2.6.5 Lemma. *A symmetric PBW basis of* P *is given by the orbits of a non-symmetric PBW basis.*

Proof. Easy. □

As a corollary, we have:

2.6.6 Theorem. *A non-symmetric operad which has a non-symmetric PBW basis is Koszul, and the non-symmetric dual operad has a non-symmetric PBW basis, which can be explicitely determined from the other basis.* □

2.7 Examples

We know that the following operads are Koszul: commutative $\mathcal{C}$, associative $\mathcal{A}$ and Lie $\mathcal{L}ie$ (*cf.* Ginzburg and Kapranov [5]). We use our PBW criterion on these examples and on some other operads. To simplify notations, we write sometimes relations with treewise tensors in the operad, and sometimes in line in the associated algebra. We do not draw the root of the trees. In the examples with operads generated by binary operations, we will work with $\mathsf{P}^!$, and determine the treewise tensors $\alpha_i^*, i \in I$ and the generating relations R''. Otherwise we consider the dual $K(\mathsf{P})^\#$.

Recall that by Condition 2, a treewise tensor is in the basis if and only if every subtensor generated by an edge is in the basis. As a consequence, we specify only the quadratic part of the basis to determine the basis completely.

Verifications are omitted.

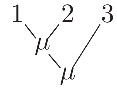
There are two main methods to find PBW bases:

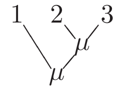
- We can start from a basis, and we need to find an order on M such that it is a PBW basis (we have to check whether it satisfies Conditions 1 and 2).

- We can start from an ordered basis of M, which forces the choice of the quadratic part (because of the relations). We then construct the set generated by this quadratic part (it satisfies Conditions 1 and 2 by construction) and we need to check whether it is a basis (as a $\mathbb{K}$-module).

2.7.1 The associative operad

In the binary case, the associative operad is generated by a single binary operation μ, which satisfies the associativity relation $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$.

For the lexicographical order, the quadratic part of a non-symmetric PBW basis is given by  .

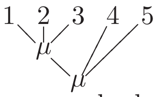
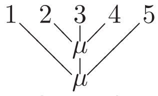
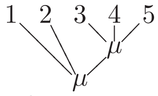
The dual operad also has a non-symmetric PBW basis, whose quadratic part is given by  .

The relation is still the associativity relation, so the associative operad is self-dual.

2.7.2 Generalization for higher associative operads

It is possible to generalize the notion of associativity for operations of arity larger than 2. For more details, we refer the reader to Gnedbaye [6]. The operads here are not generated by binary operations.

In the ternary case, one can define two types of associative operad. The totally associative operad satisfies $\mu(a, b, \mu(c, d, e)) = \mu(a, \mu(b, c, d), e) = \mu(a, b, \mu(c, d, e))$, while the partially associative operad satisfies the relation $\mu(a, b, \mu(c, d, e)) + \mu(a, \mu(b, c, d), e) + \mu(a, b, \mu(c, d, e)) = 0$.

For the lexicographical order, the quadratic part of a non-symmetric PBW basis of the totally associative operad is  . As a consequence, this operad is Koszul, and its dual $K(\mathsf{P})^{\#}$ is the partially associative operad where operations are in degree 1, with the quadratic part of a PBW basis composed by  and  for the reverse-length lexicographical order.

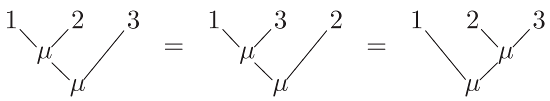
The same result can be shown for larger arities (with signs depending on the parity), *cf.* [6].

2.7.3 The commutative and Lie operads

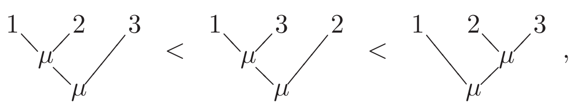
The commutative operad is generated by a single binary operation μ, which satisfies commutativity and associativity.

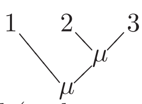
$$\mu(a, b) = \mu(b, a) \text{ and } \mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$$

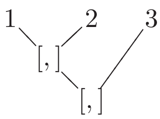
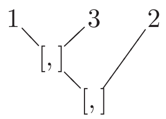
The relations in degree 2 are



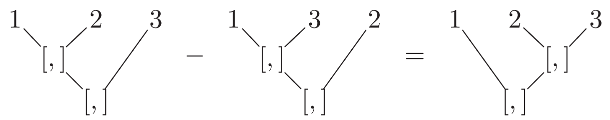
For the reverse-length lexicographical order, we get



We check easily that the maximal element  in the quadratic relations is the quadratic part of a PBW basis of the commutative operad (and as a consequence, this operad is Koszul).

The dual operad is also Koszul, and the quadratic part of a PBW basis consists of two treewise tensors  and  , where $[,]$ is the dual of μ and is anticommutative.

The relations in the dual operad is



We recognize the Jacobi relation. So the operad $\mathcal{L}ie$ is the dual operad of $\mathcal{C}$, and as a consequence is Koszul. Note that we retrieve the basis of Reutenauer in [15, Section 5.6.2].

2.7.4 The Poisson operad

The $\mathcal{P}oisson$ operad can be defined as $\mathcal{C}om \circ \mathcal{L}ie$.

Explicitely, it is generated by $M = \mathbb{K}\bullet \oplus \mathbb{K}[sgn][,]$, with the relations

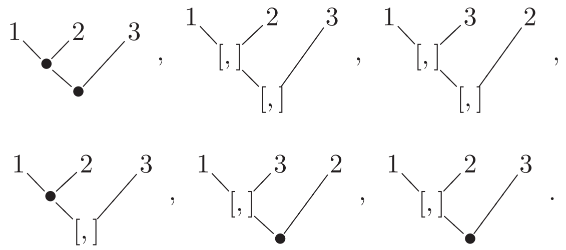
$$a \bullet (b \bullet c) = (a \bullet b) \bullet c \text{ (Associativity)}$$

$$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0 \text{ (Jacobi)}$$

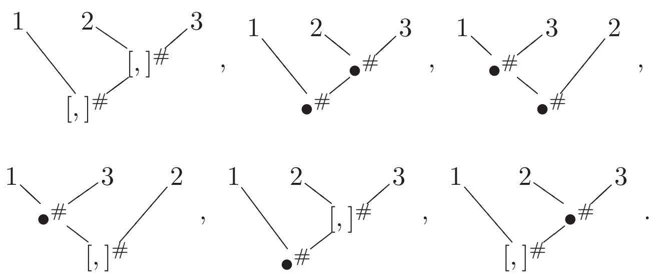
$$[a \bullet b, c] = a \bullet [b,c] + b \bullet [a,c] \text{ (Poisson)}$$

We use the lexicographical order and we set $\bullet > [,]$.

We already know the quadratic part of a PBW basis for $\mathcal{L}ie$ and $\mathcal{C}om$. After some calculations for the action of Σ_3 on the Poisson relation, we can determine the quadratic part of a PBW basis:



So the Poisson operad is Koszul and the quadratic part of a PBW basis of its dual is:



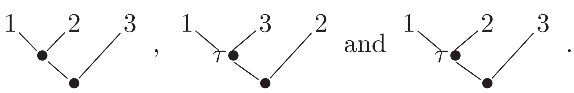
The operation $\bullet^{\#}$ is anticommutative and satisfies the Jacobi relation, while the operation $[,]^{\#}$ is commutative and associative. The two operations together satisfy a Poisson relation. So we have retrieved that $\mathcal{P}oisson$ is self-dual, which was already proved by Markl using distributive laws in [11].

2.7.5 The Perm and PreLie operads

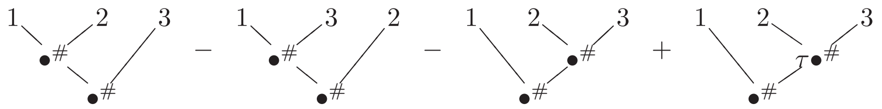
The $\mathcal{P}erm$ operad is defined by a single operation $\bullet$ satisfying: $(x \bullet y) \bullet z = x \bullet (y \bullet z) = x \bullet (z \bullet y)$.

Let τ be the transposite $(12) \in \Sigma_2$.

For the lexicographical order and $\bullet > \tau\bullet$, a PBW basis is given by



The duals of the nine quadratic treewise tensors in the complement are a PBW basis of the dual operad, and the relation ideal is generated by



$$= 0.$$

This relation is known to define the $\mathcal{P}reLie$ operad. So we have shown that $\mathcal{P}relie$ and $\mathcal{P}erm$ are Koszul and dual to each other. This was already proved by Chapoton and Livernet in [1].

2.7.6 The $m - Dend$ operad

A $\mathbb{K}$-vector space V is an m-dendriform algebra if it is equipped with m binary operations $\bullet_1, \ldots, \bullet_m : V^{\otimes 2} \longrightarrow V$ satisfying for all $x, y, z \in V$, and for all $2 \leq i \leq m-1$, the axioms

$$(x \prec y) \prec z = x \prec (y \star z), \qquad (x \prec y) \bullet_i z = x \bullet_i (y \succ z)\ \forall\, 2 \leq i \leq m-1,$$

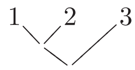
$$(x \succ y) \prec z = x \succ (y \prec z), \qquad (x \succ y) \bullet_i z = x \succ (y \bullet_i z)\ \forall\, 2 \leq i \leq m-1,$$

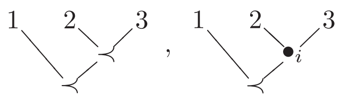
$$(x \star y) \succ z = x \succ (y \succ z), \qquad (x \bullet_i y) \prec z = x \bullet_i (y \prec z)\ \forall\, 2 \leq i \leq m-1,$$

$$(x \bullet_i y) \bullet_j z = x \bullet_i (y \bullet_j z) \quad \forall\, 2 \leq i < j \leq m-1,$$

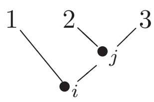
where $\bullet_1 := \succ$, $\bullet_m := \prec$ and $x \star y := x \prec y + x \succ y$.

We work with the associated operad, which was introduced by Leroux in [8]. He conjectured it was Koszul for $m > 2$. For $m = 2$, the operad is the classical dendriform operad, introduced by Loday, and is Koszul [10].

For the lexicographical order and $\bullet_i < \bullet_j$ if $i < j$, the quadratic part of a non-symmetric PBW basis is defined by all treewise tensors on  and the following tensors:



$$\forall\, 2 \leq i \leq m-1,$$



$$\forall\, 2 \leq j \leq i \leq m-1.$$

We have proved that the $m - Dend$ operad is Koszul, and so its dual $m - Tetra$ (calculated in [8]) is Koszul too.

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I am grateful to Benoit Fresse for many useful discussions on this subject. I would like to thank Muriel Livernet, Jean-Louis Loday, Martin Markl, Elisabeth Remm and Bruno Vallette for their comments. I also wish to express my gratitude to the referee for a careful reading of the manuscript.

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Chapitre 3

Gamma-homology of algebras over an operad

The purpose of this paper is to study generalizations of Gamma-homology in the context of operads. Good homology theories are associated to operads under appropriate cofibrancy hypotheses, but this requirement is not satisfied by usual operads outside the characteristic zero context. In that case, the idea is to pick a cofibrant replacement Q of the given operad P. We can apply to P-algebras the homology theory associated to Q in order to define a suitable homology theory on the category of P-algebras. We make explicit a small complex to compute this homology when the operad P is binary and Koszul. In the case of the commutative operad $\mathsf{P} = \mathsf{Com}$, we find an equivalent complex to the one introduced by Robinson for the Gamma-homology of commutative algebras.

The classical homology theories of commutative algebras (Harrison homology in the differential graded setting over a field of characteristic 0, cf. [Har], André-Quillen homology in the simplicial setting over a ring of any characteristic, cf. [Qui] and [And]) can be considered as homology theories associated to the commutative operad Com. There is another homology theory for commutative algebras, Γ-homology (Gamma-homology in plain words, also called topological André-Quillen), which has been introduced by Robinson and Whitehouse in [RW], and by Basterra in [Bas] (with a different point of view), to solve obstruction problems in homotopy theory. In the setting of [RW], Gamma-homology is defined as the homology theory associated to an E_∞-operad (a cofibrant replacement of Com). This new homology can be defined in the context of differential graded or simplicial context or in the context of spectra, and gives the same result in each case (cf. Mandell [Man]), in contrast with the usual André-Quillen homology.

The purpose of this paper is to study generalizations of Γ-homology in the context of operads.

Usual methods of homotopical algebra apply to the categories of algebras associated to operads which are cofibrant, or at least which fulfill sufficiently strong cofibrancy requirements. As a consequence, we have a good homology theory H_*^{Q} associated to any such operad Q. But many usual operads, like the commutative operad Com or the Lie operad Lie, do not fit this framework (unless we work with differential graded modules over a field of characteristic 0). In this situation, a natural idea is to pick a cofibrant replacement of the given operad P, let $\mathsf{Q} \xrightarrow{\sim} \mathsf{P}$, and to apply the homology H_*^{Q} to P-algebras in order to obtain a consistent homology theory on the category of P-algebras. We use the notation $H\Gamma_*^{\mathsf{P}} = H_*^{\mathsf{Q}}$ and the name Γ-homology to refer to this homology theory after observing that different choices of Q give the same result.

This generalizes the usual notion of Γ-homology where $\mathsf{P} = \mathsf{Com}$ and Q is an E_∞-operad. The homology H_*^{Q} associated to a cofibrant replacement of the operad Lie has also been used by Chataur, Rodriguez and Scherer in [CRS].

The problem is that the choice of a cofibrant replacement is satisfying in theory, but making such a cofibrant replacement explicit is often very difficult (especially when the ground ring is not a field of characteristic 0). We give a direct definition of $H\Gamma_*^{\mathsf{P}}$, which agrees with the initial one, but where the choice of an operadic cofibrant replacement is avoided. The idea is to use the model category on P-bimodules, which only needs mild assumptions on P. We show how to define a complex to determine $H\Gamma_*^{\mathsf{P}}$ from a choice of a cofibrant replacement of the operad P, not in the category of operads, but in the category of P-bimodules, the operad P being viewed as a bimodule over itself. The category of P-bimodules is easier to deal with than the category of operads.

In [Rob], Robinson makes explicit a small complex, analogous to Harrison's complex, which computes usual Γ-homology. In the case where the operad P is Koszul, we define an explicit complex to compute the Γ-homology associated to P. Recall that an operad is Koszul if we have a quasi-isomorphism between $(\mathsf{P} \circ K\mathsf{P} \circ \mathsf{P}, \partial)$ and P, where $K\mathsf{P}$ is the Koszul construction, defined by $K(\mathsf{P})_{(s)} := H_s(B_*(\mathsf{P})_{(s)}, \partial)$. In [Bal], Balavoine defined a complex computing H_*^{P} when working over a field of characteristic 0, using the Koszul construction. When we work over a ring of any characteristic, finding a complex is more complicated, as we need to resolve the symmetries in $K\mathsf{P}$. This can be done by tensoring the Koszul construction by the acyclic bar construction of the symmetric group. Finally, we get a small explicit complex computing the Γ-homology of P-algebras. In the case $\mathsf{P} = \mathsf{Com}$, we obtain a variant of Robinson's complex. As an illustration, we make our complex explicit in the case $\mathsf{P} = \mathsf{Lie}$.

We also define a cohomology theory $H\Gamma^*_{\mathsf{P}}$ associated to any operad P.

In Section 1, we recall the model category structures we use in the paper: dg-modules, Σ_*-modules, bimodules, algebras over operads. Most of the model structures we consider are defined by a transfer of structure. We make the cofibrations explicit in each case. In a second part, we recall the usual notion of homology for algebras over a cofibrant operad, and show how to reduce the complex when we are given a cofibrant replacement of bimodules. Then we make a similar construction of a reduced complex when the operad is not cofibrant. This leads us to the definition of Γ-homology of algebras over an operad (without any cofibrancy hypothesis). In Section 3, we construct an explicit complex for any binary Koszul operad P to compute Γ-homology. This complex is defined using the Koszul construction $K\mathsf{P}$ and the acyclic bar construction of the symmetric group.

Convention. We work in the differential graded setting. We take a category of differential graded modules (for short dg-modules) over a fixed base ring $\mathbb{K}$ as a base category (see Section 3.1.2 for details). We use the letter $\mathcal{C}$ to denote this category. When necessary, we assume tacitly that any dg-module, and more generally that any object defined over this base category, consists of projective modules over the ground ring.

We review the definition of the model category of Σ_*-modules underlying the category of operads in Section 3.1.4, the model category of bimodules in Section 3.1.5. All operads P will be assumed to be connected, in the sense that $\mathsf{P}(0) = 0$ and $\mathsf{P}(1) = \mathbb{K}$. All Σ_*-modules M, and more generally any object defined over the category of Σ_*-modules, will be assumed to be connected, that is $M(0) = 0$.

3.1 Model categories

We review here the model structures for the categories which are used in this paper. For general references on the subject, we refer the reader to the survey of Dwyer and Spalinksi [DS] and the books of Hirschhorn

[Hir] and Hovey [Hov]. For model structures in the operadic context, we refer to the articles of Hinich [Hin] and of Goerss and Hopkins [GH], and the book of Fresse [F1].

3.1.1 Transfer of structure

We use the notion of a pair of adjoint functors to transport model structures. Suppose we have an adjunction

$$F : \mathcal{X} \rightleftarrows \mathcal{A} : U$$

such that $\mathcal{X}$ is a cofibrantly generated model category and $\mathcal{A}$ is a category equipped with colimits and limits. We can then define classes of weak equivalences, fibrations and cofibrations in $\mathcal{A}$.

- The weak equivalences in $\mathcal{A}$ are morphisms f such that $U(f)$ is a weak equivalence in $\mathcal{X}$.
- The fibrations in $\mathcal{A}$ are morphisms f such that $U(f)$ is a fibration in $\mathcal{X}$.
- The cofibrations are the morphisms which have the left lifting property (in short, LLP) with respect to acyclic fibrations.

Under some technical hypotheses (cf. [Hir, Theorem 11.3.2]), a classical result says that $\mathcal{A}$ is equipped with a model structure given by the weak equivalences, fibrations and cofibrations above. Under weaker hypotheses (cf. [F1, Theorem 12.1.4]), the category $\mathcal{A}$ is equipped with a semi-model category, that is the lifting and factorization axioms only hold when the morphisms have a cofibrant domain. Semi-model categories will be enough for us here.

We can describe the generating (acyclic) cofibrations of the semi-model category $\mathcal{A}$ explicitly: they are the morphisms $F(i) : F(C) \to F(D)$ such that i ranges over the generating (acyclic) cofibrations of $\mathcal{X}$.

3.1.2 Model category structure for dg-modules

In this paper, the dg-modules we consider are $\mathbb{Z}$-graded modules endowed with a differential δ decreasing the degree by 1. The category of dg-modules is denoted by $\mathcal{C}$. The internal hom of this category is denoted by $\mathrm{Hom}_{\mathcal{C}}(C, D)$, for all $C, D \in \mathcal{C}$. This dg-module is spanned in degree d by the linear maps $f : C \to D$ which raises degrees by d. The differential of such a map in $\mathrm{Hom}_{\mathcal{C}}(C, D)$ is defined by its graded commutator with the internal differential of C and D. We adopt the terminology of homomorphisms to distinguish the elements of the dg-hom $\mathrm{Hom}_{\mathcal{C}}(C, D)$ from the actual morphisms of dg-modules, the linear maps which preserve gradings and commute with differentials.

The category of dg-modules is equipped with its usual model structure: The weak equivalences are the quasi-isomorphisms and the fibrations are degreewise surjective maps (cofibrations are characterized by the LLP with respect to acyclic fibrations).

Let $D_n = \mathbb{K}d_n \oplus \mathbb{K}c_{n-1}$ where d_n is a homogeneous element in degree n sent by the differential to c_{n-1} in degree $n-1$. Let C_n be $\mathbb{K}c_{n-1}$, submodule of D_n. The embeddings $C_n \to D_n$, $n \in \mathbb{Z}$, define a set of generating cofibrations in $\mathcal{C}$. The maps $0 \to D_n$ define generating acyclic cofibrations.

In what follows, the underlying dg-module of any object is tacitly assumed to be cofibrant.

3.1.3 Twisted dg-modules

In general, we assume that a dg-module C is equipped with a differential $\delta : C \to C$. We sometimes twist this internal differential by a cochain $\partial \in \mathrm{Hom}_{\mathcal{C}}(C, C)$ of degree -1 in order to get a new differential $\delta + \partial$. We assume the relation $\delta \circ \partial + \partial \circ \delta + \partial^2 = 0$, in order to obtain that $\delta + \partial$ satisfies $(\delta + \partial)^2 = 0$. We usually omit the internal differential δ in the notation: We write C for the module C with differential δ and write (C, ∂) to denote the module C with differential $\delta + \partial$.

We are going to define quasi-free objects (algebras over operads, bimodules), twisted objects (C, ∂) such that C is free with respect to an algebraic structure.

3.1.4 Model category structure for Σ_*-modules

We use the notation $\mathcal{M}$ for Σ_*-modules. We have an adjunction between the forgetful functor U (from the category $\mathcal{M}$ to the category of chain complexes) and the free Σ_*-module functor $\Sigma_* \otimes -$.

The transfer process of Section 3.1.1 gives us a model structure on Σ_*-modules where weak equivalences are morphisms all of whose components are weak equivalences of dg-modules and where fibrations are morphisms all of whose components are epimorphisms of dg-modules. Cofibrations are given by the LLP with respect to acyclic fibrations. Again, we can say more precisely which maps are cofibrations (cf. [F1, Prop. 11.4.A]). The generating cofibrations are given by tensor products

$$i \otimes F_r : C \otimes F_r \to D \otimes F_r$$

where $i : C \to D$ ranges over the generating cofibrations of dg-modules and $F_r, r \in \mathbb{N}$ denote the Σ_*-modules such that

$$F_r(n) = \begin{cases} \mathbb{K}[\Sigma_r], & \text{for } n = r, \\ 0, & \text{otherwise.} \end{cases}$$

We will use the composition product $\circ$ of Σ_*-modules. Recall that for a constant Σ_*-module N (such that $N(0) = C$ and $N(r) = 0$ for $r > 0$), the composition $M \circ C$ represents the application of a symmetric functor with coefficients in M to C:

$$M \circ C = \bigoplus_{r=1}^{+\infty} (M(r) \otimes C^{\otimes r})_{\Sigma_r}.$$

This object is denoted by $S(M, C)$ in the book [F1], but we use the notation $M \circ C$ in the paper.

In general, the composite $M \circ N$ is defined such that the associativity relation $M \circ (N \circ C) = (M \circ N) \circ C$ is satisfied for all constant Σ_*-modules C. The composition product $\circ$ is a monoidal product for the category of Σ_*-modules.

Recall that an operad is a Σ_*-module P equipped with an initial morphism $\mathsf{I} \to \mathsf{P}$ (where I is the unit Σ_*-module, with $\mathbb{K}$ in arity 1 and 0 everywhere else) and a composition product $\gamma : \mathsf{P} \circ \mathsf{P} \to \mathsf{P}$. As mentionned in the introduction, we assume that any operad P satisfies $\mathsf{P}(0) = 0$ and $\mathsf{P}(1) = \mathbb{K}$, so that the initial morphism of P is an isomorphism in arity 0 and in arity 1. We use the notation $\bar{\mathsf{P}}$ for the Σ_*-submodule of P formed by the components $\mathsf{P}(n)$ of arity $n > 1$ and trivial in arity 0 and in arity 1.

In what follows, we will often consider Σ_*-cofibrant operads, operads P such that the initial morphism $\mathsf{I} \to \mathsf{P}$ is a cofibration of Σ_*-modules.

3.1.5 Model category structure for bimodules over operads

Let P and Q be operads. Let ${}_{\mathsf{P}}\mathcal{M}^0_{\mathsf{Q}}$ be the category of connected (that is $M(0) = 0$) P-Q-bimodules in the sense of [F1]. We have an adjunction

$$\mathsf{P} \circ - \circ \mathsf{Q} : \mathcal{M} \rightleftarrows {}_{\mathsf{P}}\mathcal{M}^0_{\mathsf{Q}} : U,$$

where U is the forgetful functor.

The transfer process gives us a semi-model structure on P-Q-bimodules, where weak equivalences are morphisms all of whose components are weak equivalences of dg-modules and where fibrations are morphisms all of whose components are epimorphisms of dg-modules. Cofibrations are given by the LLP with respect to acyclic fibrations.

We now describe a particular class of cofibrant P-Q-bimodules that we will use extensively later.

3.1.6 Proposition. *Let P and Q be connected operads and M a cofibrant Σ_*-module.*

The quasi-free P-Q-bimodule $(\mathsf{P} \circ M \circ \mathsf{Q}, \partial)$ is cofibrant if the differential is decomposable (that is $\partial(M)$ has no component in $\mathsf{I} \circ M \circ \mathsf{I}$, or equivalently that each element in $\partial(M)$ is a sum of composites with at least a non-trivial element of P or of Q).

This result will be deduced from the following lemmas.

3.1.6.1 Lemma. *Let* Q *be a connected operad and* M *a cofibrant* Σ_**-module.*

The quasi-free Q*-module* $(M \circ \mathsf{Q}, \partial)$ *is cofibrant if the differential is decomposable (that is* $\partial(M)$ *has no component in* $M \circ \mathsf{I}$*).*

Proof. The complex $(M \circ \mathsf{Q}, \partial)$ is filtered by

$$ar_\lambda(M \circ \mathsf{Q}, \partial) = (ar_\lambda M \circ \mathsf{Q}, \partial)$$

where $ar_\lambda M(n) = M(n)$ if $n \leq \lambda$ and 0 otherwise, and where the differential is just the restriction of the differential on $(M \circ \mathsf{Q}, \partial)$.

Note that $\partial(ar_\lambda M) \subset ar_{\lambda-1}(M \circ \mathsf{Q})$.

We have the following pushout of right Q-modules:

$$\begin{array}{ccc}
(\partial M(n) \circ \mathsf{Q}, 0) & \longrightarrow & (ar_{n-1} M \circ \mathsf{Q}, \partial) \\
\downarrow & & \downarrow \\
(\partial M(n) \circ \mathsf{Q} \bigoplus M(n) \circ \mathsf{Q}, \partial) & \longrightarrow & (ar_n M \circ \mathsf{Q}, \partial)
\end{array}$$

The arrow on the left is a generating cofibration. Thus the arrow on the right is a cofibration too.

Thus $(M \circ \mathsf{Q}, \partial) = \operatorname{colim}_\lambda ar_\lambda(M \circ \mathsf{Q}, \partial)$ is a cofibrant right Q-module. □

3.1.6.2 Lemma. *Let* $N = (M \circ \mathsf{Q}, \partial)$ *be a right* Q*-module with the hypothesis of the above lemma. Let* P *be a connected operad.*

The quasi-free P*-*Q*-bimodule* $(\mathsf{P} \circ N, \partial)$ *is cofibrant if the differential is decomposable (that is* $\partial(N) \subset \bar{\mathsf{P}} \circ N$*, or equivalently* $\partial(N)$ *has no component in* $\mathsf{I} \circ N$*).*

Proof. First, note that $\partial(ar_\lambda N) \subset \bar{\mathsf{P}} \circ ar_{\lambda-1} N$. Therefore we can define a filtration by $ar_\lambda(\mathsf{P} \circ N, \partial) = (\mathsf{P} \circ ar_\lambda N, \partial)$.

Note that $\partial(ar_\lambda N) \subset ar_{\lambda-1}(\mathsf{P} \circ N)$. Using a similar argument as in the above proof, the obvious arrow $ar_{\lambda-1}(\mathsf{P} \circ N, \partial) \to ar_\lambda(\mathsf{P} \circ N, \partial)$ is a cofibration of P-Q-bimodules.

Thus $(\mathsf{P} \circ N, \partial) = \operatorname{colim}_\lambda ar_\lambda(\mathsf{P} \circ N, \partial)$ is a cofibrant P-Q-bimodule. □

The combination of these two lemmas proves Proposition 3.1.6. □

3.1.7 Model category structure for algebras over an operad

We have an adjunction between the forgetful functor U from P-algebras to dg-modules and the free P-algebra functor $\mathsf{P} \circ -$.

If P is Σ_*-cofibrant, the transfer process of Section 3.1.1 gives us a semi-model category on P-algebras, where weak equivalences are morphisms which are weak equivalences of dg-modules and where fibrations are morphisms which are epimorphisms of dg-modules. Cofibrations are given by the LLP with respect to acyclic fibrations.

The model category structure allows us to define the cofibrant replacement of a P-algebra A. It is a cofibrant P-algebra Q_A such that we have a weak equivalence of P-algebras $Q_A \xrightarrow{\sim} A$.

If we are given a cofibrant replacement ${}_{\mathsf{P}}Res_{\mathsf{P}} \xrightarrow{\sim} \mathsf{P}$ in the category of P-bimodules, we can easily make explicit a cofibrant replacement of a P-algebra A.

First we need to recall the definition of the relative composition product of P-modules. Suppose that M is a right P-module and A a P-algebra. We denote by $M \circ_P A$ the quotient of $M \circ A$ coequalizing the right action of P on M and the left action of P on A. When M is a P-bimodule, the relative composite $M \circ_P A$ inherits a P-algebra structure.

We can now give the result:

3.1.7.1 Lemma. *We get a cofibrant replacement* $({}_{\mathsf{P}}Res_{\mathsf{P}} \circ_{\mathsf{P}} A, \partial')$ *of* A *in the category of* P*-algebras, with* $\partial' = \partial \circ_{\mathsf{P}} A$*.*

Proof. The P-algebra $({}_{\mathsf{P}}Res_{\mathsf{P}} \circ_{\mathsf{P}} A, \partial')$ is cofibrant, following the same argument of the proof of Lemma 3.1.6.1. The P-bimodule ${}_{\mathsf{P}}Res_{\mathsf{P}}$ is cofibrant, thus it is cofibrant as a right P-module. The operad P is also cofibrant as a right P-module. As the functor $- \circ_{\mathsf{P}} A$ preserve weak equivalences between cofibrant objects (cf. [F1, Theorem 15.1.A]), we get that $({}_{\mathsf{P}}Res_{\mathsf{P}} \circ_{\mathsf{P}} A, \partial')$ is a cofibrant replacement of $\mathsf{P} \circ_{\mathsf{P}} A$. But $\mathsf{P} \circ_{\mathsf{P}} A = A$, thus $({}_{\mathsf{P}}Res_{\mathsf{P}} \circ_{\mathsf{P}} A, \partial')$ is a cofibrant replacement of A in the category of P-algebras. Explicitly, the differential ∂' is given by $\partial'(m \circ (a_1, \ldots, a_n)) = (\partial(m)) \circ (a_1, \ldots, a_n)$ where m lies in ${}_{\mathsf{P}}Res_{\mathsf{P}}$ and $\partial(m)$ in $\mathsf{P} \circ ({}_{\mathsf{P}}Res_{\mathsf{P}}) \circ \mathsf{P}$. Note that we use the structure of P-algebra of A on the right hand side to get an element of ${}_{\mathsf{P}}Res_{\mathsf{P}} \circ_{\mathsf{P}} A$. □

3.2 Gamma-homology of P-algebras

In this section, we recall the definition of the homology of Q-algebras for Q a Σ_*-cofibrant operad. In the differential graded setting over a field of characteristic 0, homology with trivial coefficients was defined by Ginzburg and Kapranov in [GK] and by Getzler and Jones in [GJ]. Homology with coefficients was defined by Balavoine in [Bal]. The extension to any category of dg-modules can be found in [Hin]. We adopt conventions of [F1] where these notions are reviewed. We define Γ-homology of P-algebras for any operad P, using bimodule resolutions. Then we prove the identity $H_*^{\mathsf{Q}} = H\Gamma_*^{\mathsf{P}}$ when Q is a Σ_*-cofibrant replacement of P.

3.2.1 Recollections on the homology of Q-algebras

We refer the reader to Section 4 of [F1] for the first definitions.

Let Q be a Σ_*-cofibrant operad, B an algebra over Q.

We denote by $U_{\mathsf{Q}}(B)$ the enveloping algebra of B and by $\Omega_{\mathsf{Q}}(B)$ the module of Kähler differentials of B.

The enveloping algebra $U_{\mathsf{Q}}(B)$ is spanned by elements $q(\diamond, b_1, \ldots, b_n)$, where $q \in \mathsf{Q}(n+1)$, $b_1, \ldots, b_n \in B$ and the symbol $\diamond$ denotes a free input, divided out by the relations

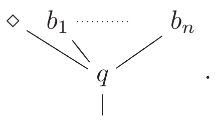
$$p(\diamond, b_1, \ldots, b_{i-1}, q(b_i, \ldots, b_n), b_{n+1}, \ldots, b_m) = p \circ_{i+1} q(\diamond, b_1, \ldots, b_{i-1}, b_i, \ldots, b_m).$$

The product is given by

$$p(\diamond, a_1, \ldots, a_n).q(\diamond, b_1, \ldots, b_m) = p \circ_1 q(\diamond, b_1, \ldots, b_m, a_1, \ldots, a_n).$$

We can put $\diamond$ at any place since the action of Σ_{n+1} on $\mathsf{Q}(n+1)$ allows us to permute the inputs of any operation $q \in \mathsf{Q}(n+1)$.

We represent graphically an element $q(\diamond, b_1, \ldots, b_n)$ by



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The module of Kähler differentials $\Omega_{\mathsf{Q}}(B)$ is a left module over $U_{\mathsf{Q}}(B)$ such that

$$\mathrm{Hom}_{U_{\mathsf{Q}}(B)}(\Omega_{\mathsf{Q}}(B), F) = \mathrm{Der}_{\mathsf{Q}}(B, F)$$

for all left modules F over $U_{\mathsf{Q}}(B)$, where $\mathrm{Der}_{\mathsf{Q}}(B, F)$ denotes the dg-module of Q-derivations $B \to F$ (not necessarily preserving the degree) and where $\mathrm{Hom}_{U_{\mathsf{Q}}(B)}(\Omega_{\mathsf{Q}}(B), F)$ is the dg-module of homomorphisms of left $U_{\mathsf{Q}}(B)$-modules between $\Omega_{\mathsf{Q}}(B)$ and F.

The module of Kähler differentials $\Omega_{\mathsf{Q}}(B)$ can be seen as the dg-module spanned by elements $q(b_1, \dots, db_i, \dots, b_n)$, where $q \in \mathsf{Q}(n)$, $b_1, \dots, b_n \in B$ and d denotes a formal differentiation symbol, divided out by the relations

$$p(b_1, \dots, q(b_i, \dots, b_n), b_{n+1}, \dots, db_j, \dots, b_m) = p \circ_i q(b_1, \dots, b_i, \dots, b_n, \dots, db_j, \dots, b_m), \text{for } i \neq j,$$

$$p(b_1, \dots, dq(b_i, \dots, b_n), b_{n+1}, \dots, \dots, \dots, b_m) = \sum_{j=i}^{n} p \circ_i q(b_1, \dots, b_i, \dots, db_j, \dots, b_n, \dots, b_m).$$

Let us now define the homology and the cohomology of an algebra over Q.

We choose Q_B a cofibrant replacement of B. Let E be a right $U_{\mathsf{Q}}(Q_B)$-module and F be a left $U_{\mathsf{Q}}(Q_B)$-module.

The homology of B as a Q-algebra with coefficients in E is defined by $H_*^{\mathsf{Q}}(B, E) = H_*(E \otimes_{U_{\mathsf{Q}}(Q_B)} \Omega_{\mathsf{Q}}(Q_B))$. In a similar way, the cohomology of B is defined by $H^*_{\mathsf{Q}}(B, F) = H^*(\mathrm{Hom}_{U_{\mathsf{Q}}(Q_B)}(\Omega_{\mathsf{Q}}(Q_B), F))$.

We will use the following usual lemma to reduce the complex appearing in the calculation of the homology and the cohomology.

3.2.1.1 Lemma. *If Q_A is a quasi-free Q-algebra $Q_A = (\mathsf{Q}(C), \partial')$, then we have an isomorphism of left $U_{\mathsf{Q}}(Q_A)$-modules*

$$(U_{\mathsf{Q}}(Q_A) \otimes C, \partial'') \simeq \Omega_{\mathsf{Q}}(Q_A)$$

where the differential $\partial'' : U_{\mathsf{Q}}(Q_A) \otimes C \to U_{\mathsf{Q}}(Q_A) \otimes C$ is a twisting homomorphism on $U_{\mathsf{Q}}(Q_A) \otimes C$-modules induced by the action of the twisting derivation of Q_A on $U_{\mathsf{Q}}(Q_A) \otimes C$ (see the detailed representation in Figure 3.2).

Proof. We begin by proving the result for a free algebra $Q_A = \mathsf{Q}(C)$.
First, we have $\mathrm{Der}_{\mathsf{Q}}(\mathsf{Q}(C), F) = \mathrm{Hom}_{\mathcal{C}}(C, F)$. To prove this identification, we define $\Phi : \mathrm{Der}_{\mathsf{Q}}(\mathsf{Q}(C), F) \to \mathrm{Hom}_{\mathcal{C}}(C, F)$ by $\Phi(\theta) = \theta_{|C} : C \to F$. This map is an isomorphism. The inverse map Φ^{-1} associates to any $f : C \to F$ the derivation θ_f such that $\theta_f(q(c_1, \dots, c_n)) = \pm \sum_i q(c_1, \dots, f(c_i), \dots, c_n)$ where the signs are induced by the usual Koszul rule.

We have $\mathrm{Hom}_{\mathcal{C}}(C, F) = \mathrm{Hom}_{U_{\mathsf{Q}}(Q_A)}(U_{\mathsf{Q}}(Q_A) \otimes C, F)$, which gives us the relation

$$\mathrm{Der}_{\mathsf{Q}}(Q_A, F) = \mathrm{Hom}_{U_{\mathsf{Q}}(Q_A)}(U_{\mathsf{Q}}(Q_A) \otimes C, F).$$

But $\Omega_{\mathsf{Q}}(Q_A)$ is defined by $\mathrm{Hom}_{U_{\mathsf{Q}}(Q_A)}(\Omega_{\mathsf{Q}}(Q_A), F) = \mathrm{Der}_{\mathsf{Q}}(Q_A, F)$. Thus Yoneda's lemma gives us an isomorphism Ψ of $U_{\mathsf{Q}}(Q_A)$-modules between $U_{\mathsf{Q}}(Q_A) \otimes C$ and $\Omega_{\mathsf{Q}}(Q_A)$.

The map $\Psi : U_{\mathsf{Q}}(Q_A) \otimes C \to \Omega_{\mathsf{Q}}(Q_A)$ associates to the element $q(\diamond, a_1, \dots, a_n) \otimes c$ the element $q(dc, a_1, \dots, a_n)$.

Its inverse $\Psi^{-1} : \Omega_{\mathsf{Q}}(Q_A) \to U_{\mathsf{Q}}(Q_A) \otimes C$ sends
$q(dq_0(\underline{c_0}), a_1, \dots, a_r) = \sum_i q \circ_1 q_0(c_1, \dots, dc_i, \dots, c_n, a_1, \dots, a_r)$
to $\sum_i q \circ_1 q_0(c_1, \dots, \diamond, \dots, c_n, a_1, \dots, a_r) \otimes c_i$, where $\underline{c_0} = (c_1, \dots, c_n)$.

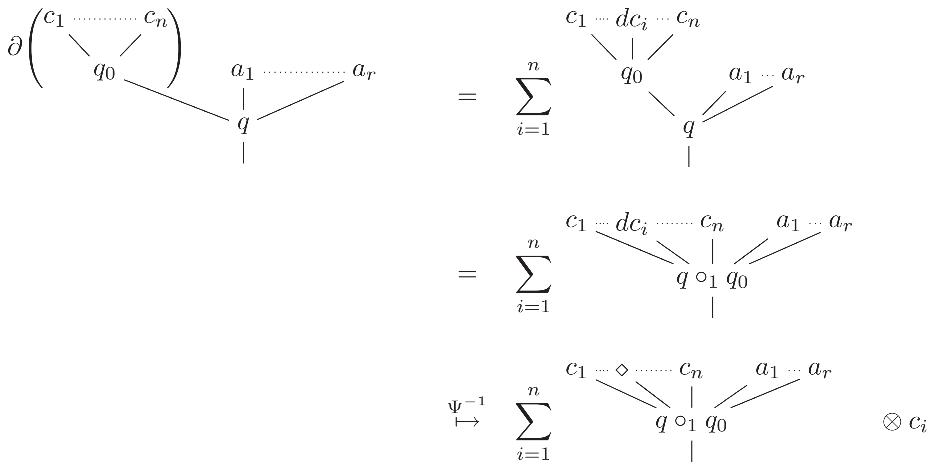
A graphical representation of the isomorphism Ψ^{-1} is given in Figure 3.1.

This morphism Ψ commutes with the internal differential of C.

We now consider a quasi-free Q-algebra $Q_A = (\mathsf{Q}(C), \partial')$ with a twisting differential ∂' and explain the twisting differential ∂'' we obtain on $U_{\mathsf{Q}}(Q_A) \otimes C$. A graphical representation of the twisting part of the differential is given in Figure 3.2.

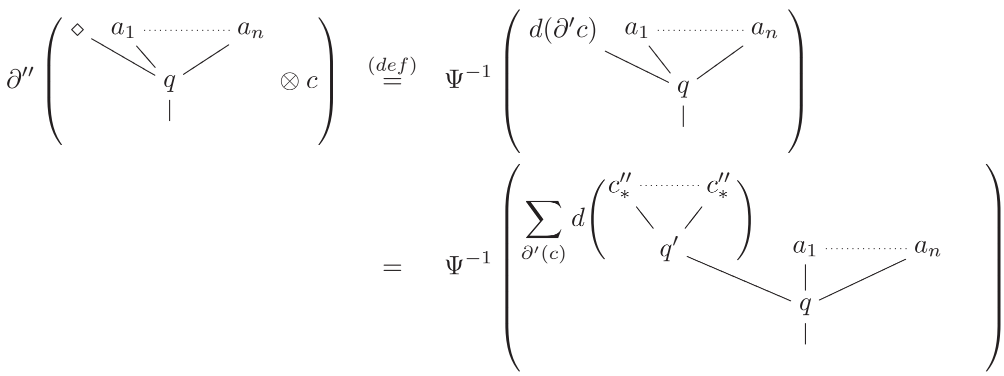
We consider an element $\omega = q(\diamond, a_1, \dots, a_n) \otimes c$ in $U_{\mathsf{Q}}(\mathsf{Q}(C)) \otimes C$. We compute

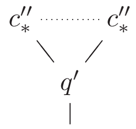
$$\begin{aligned} \partial'(\Psi(\omega)) &= \partial'(q(dc, a_1, \dots, a_n)) \\ &= \underbrace{q(d\partial' c, a_1, \dots, a_n)}_{\partial''(\Psi(w))} \quad + \quad \sum_i \underbrace{q(dc, a_1, \dots, \partial' a_i, \dots, a_n)}_{\Psi(q(\diamond, a_1, \dots, \partial' a_i, \dots, a_n) \otimes c)}. \end{aligned}$$



where the empty box is the i-th input of the tree.

Figure 3.1: A graphical representation of the inverse isomorphism Ψ^{-1}.



where $\partial'(c) = \sum_{\partial'(c)}$  and $a_1, \dots, a_n \in Q_A$.

By the identity of *Figure* 3.1, the last expression can be rewritten to give:

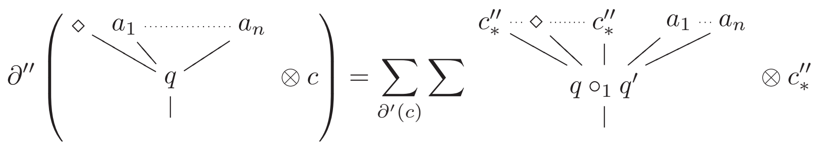


Figure 3.2: A graphical representation of the twisting differential in $U_{\mathsf{Q}}(Q_A) \otimes C$.

The second term of this sum is induced by the action of $\partial' : Q_A \to Q_A$. The image by Ψ^{-1} of the first term is computed in Figure 3.2. We denote $\Psi^{-1}\partial''(\Psi(\omega))$ by $\partial''(\omega)$.

To conclude, the twisting differential added to δ is the sum of $\partial' : Q_A \to Q_A$ and of ∂'' induced by the action of ∂' on C in $U_{\mathsf{Q}}(Q_A) \otimes C$. There are two equivalent ways to see the module $U_{\mathsf{Q}}(Q_A) \otimes C$ with its twisting differential: $(U_{\mathsf{Q}}(Q_A) \otimes C, \partial'')$ or $(U_{\mathsf{Q}}(Q(C)) \otimes C, \partial' + \partial'')$. □

Note that we have not used the cofibrancy hypothesis on Q in the proof of the lemma.

3.2.2 From quasi-free Q-bimodules to resolutions of an algebra

We suppose here that Q is a Σ_*-cofibrant operad. Let B be a Q-algebra and E a right $U_{\mathsf{Q}}(Q_B)$-module.

Suppose we have a quasi-free Q-bimodule $(\mathsf{Q} \circ N \circ \mathsf{Q}, \partial)$ weakly equivalent to Q as Q-bimodules, with N a cofibrant Σ_*-module satisfying $N(0) = 0$.

Applying Lemma 3.1.7.1, we get a cofibrant replacement $(\mathsf{Q} \circ N \circ B, \partial')$ of B in the category of Q-algebras, with $\partial' = \partial \circ_{\mathsf{Q}} B$.

This particular cofibrant replacement allows us to compute the homology and the cohomology of B as a Q-algebra using a smaller complex. We first get

$$H_*^{\mathsf{Q}}(B, E) = H_*(E \otimes_{U_{\mathsf{Q}}(\mathsf{Q} \circ N \circ B)} \Omega_{\mathsf{Q}}(\mathsf{Q} \circ N \circ B)).$$

By Lemma 3.2.1.1 (applied to $C = N \circ B$), this homology is identified with

$$H_*^{\mathsf{Q}}(B, E) = H_*(E \otimes N \circ B, \partial'')$$

where ∂'' is induced by ∂ in two steps explained in the proofs of Lemmas 3.1.7.1 and 3.2.1.1.

3.2.3 An analogous smaller complex for all operads

Let P be an operad and A an algebra over P.

Suppose we have a quasi-free P-bimodule $(\mathsf{P} \circ M \circ \mathsf{P}, \partial)$ weakly equivalent to P as a P-bimodule and such that

1. the Σ_*-module M is connected and cofibrant as a Σ_*-module;
2. the differential ∂ is decomposable, that is $\partial(M)$ has no component in $\mathsf{I} \circ M \circ \mathsf{I}$.

Under these hypotheses, Proposition 3.1.6 implies that $(\mathsf{P} \circ M \circ \mathsf{P}, \partial)$ is cofibrant as a P-bimodule.

Let $Q_A = (\mathsf{P} \circ M \circ A, \partial')$ be the P-algebra defined by the construction of Section 3.2.2 with the operad P instead of the operad Q. Form the dg-module $(E \otimes_{U_{\mathsf{P}}(Q_A)} \Omega_{\mathsf{P}}(Q_A), \partial'')$ associated to this P-algebra. We have again a map from Q_A to A, but this map is not a weak equivalence without a cofibrancy hypothesis on P. Nevertheless, with the result of Lemma 3.2.1.1, we can again reduce $(E \otimes_{U_{\mathsf{P}}(Q_A)} \Omega_{\mathsf{P}}(Q_A), \partial')$ to $(E \otimes M \circ A, \partial'')$.

Moreover, we have the following lemma of homology invariance:

3.2.3.1 Lemma. *A weak equivalence of* P*-bimodules* $(\mathsf{P} \circ M_1 \circ \mathsf{P}, \partial_1) \xrightarrow{\phi} (\mathsf{P} \circ M_2 \circ \mathsf{P}, \partial_2)$ *(both satisfying the above hypotheses (1) and (2)) induces a quasi-isomorphism* $(E \otimes M_1 \circ A, \partial_1'') \to (E \otimes M_2 \circ A, \partial_2'')$

Proof. We consider a filtration on $(E \otimes M_1 \circ A, \partial_1'')$ and then use a spectral argument.

We set $F_s(E \otimes M_1 \circ A) = \mathrm{Span}_{r \leq s}\{\xi \otimes m(a_1, \ldots, a_r)\}$. This complex is a subcomplex of $E \otimes M_1 \circ A$.

Let $\bar{\phi} : M_1 \to M_2$ denote the map $I \circ_{\mathsf{P}} \phi \circ_{\mathsf{P}} I$. The hypothesis (2) for M_1 and M_2 implies that $\bar{\phi}$ is the indecomposable part of ϕ, and is a trivial Σ_*-cofibration. We get that $\bar{\phi} \circ A : M_1 \circ A \to M_2 \circ A$ is a trivial cofibration of dg modules.

Abusing the notation, we let ϕ denote also $E \otimes \phi \circ A : E \otimes M_1 \circ A \to E \otimes M_2 \circ A$.

Let us now prove that $\phi(F_s(E \otimes M_1 \circ A)) \subseteq F_s(E \otimes M_2 \circ A)$ and that $E^0\phi = E \otimes \bar{\phi} \circ A$.

$$\begin{aligned}
& \phi(\xi \otimes m(a_1, \ldots, a_r)) \\
\overset{(1)}{=}\ & \xi \otimes \phi(m)(a_1, \ldots, a_r) \\
\overset{(2)}{=}\ & \xi \otimes \bar{\phi}(m)(a_1, \ldots, a_r) + \sum \xi \otimes p(y_1, \ldots, y_t)(q_1, \ldots, q_s)(\underline{a}) \\
\overset{(3)}{=}\ & \xi \otimes \bar{\phi}(m)(a_1, \ldots, a_r) + \sum \xi \otimes p(y_1(\underline{q_1}(\underline{a_1}), \ldots, y_t(\underline{q_t}(\underline{a_t}))) \\
\overset{(4)}{=}\ & \xi \otimes \bar{\phi}(m)(a_1, \ldots, a_r) + \sum \textstyle\sum_i \xi.u_i \otimes y_i(\underline{q_i}(\underline{a_i})).
\end{aligned}$$

Underlined elements denote sequences of elements. Equality (2) is just using the definition of $\bar{\phi}$ as the indecomposable part of ϕ. Equality (3) comes from the composition of the subtree above each y_i. In the equality (4), we use the isomorphism of Lemma 3.2.1.1, and $u_i = p(y_1(\underline{q_1}(\underline{a_1})), \ldots, \diamond, \ldots, y_t(\underline{q_t}(\underline{a_t})))$ with the hole in the ith position. The important thing to notice is that the arity of each y_i is smaller than r, as the differential is decomposable. This proves $\phi(F_s(E \otimes M_1 \circ A)) \subseteq F_s(E \otimes M_2 \circ A)$.

We now consider the associated graded complex $E^0_s(E \otimes M_1 \circ A) = F_s(E \otimes M_1 \circ A)/F_{r<s}(E \otimes M_1 \circ A)$.

$$E^0_s(E \otimes M_1 \circ A) = \mathrm{Span}\{\xi \otimes m(a_1, \ldots, a_s)\}.$$

The above calculation implies that $E^0\phi = E \otimes \bar{\phi} \circ A$.

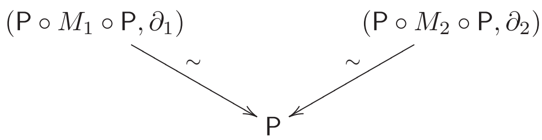
With this equality and as $\bar{\phi}$ is a trivial cofibration, we get that $E^1(\phi) = H_*(E \otimes \bar{\phi} \circ A)$ is an isomorphism. Moreover, the spectral sequence converges, as it is a homological spectral sequence with an increasing exhaustive filtration which is bounded below.

This result implies that $H_*(\phi)$ is an isomorphism. □

Thus we have the following result:

3.2.3.2 Lemma. *The homology of $(E \otimes M \circ A, \partial'')$ does not depend on the choice of the bimodule $(\mathsf{P} \circ M \circ \mathsf{P}, \partial)$ weakly equivalent to P (as a P-bimodule) such that hypotheses (1) and (2) are satisfied.*

Proof. Suppose we have the following configuration:



First the semi-model structure on P-bimodules gives a weak equivalence between $(\mathsf{P} \circ M_1 \circ \mathsf{P}, \partial_1)$ and $(\mathsf{P} \circ M_2 \circ \mathsf{P}, \partial_2)$ (as $(\mathsf{P} \circ M_1 \circ \mathsf{P}, \partial_1)$ is cofibrant). Then Lemma 3.2.3.1 implies that the induced arrow $(E \otimes M_1 \circ A, \partial''_1) \to (E \otimes M_2 \circ A, \partial''_2)$ is a quasi-isomorphism. □

3.2.4 Definition of Γ-homology

Let P be an operad, A an algebra over P and E a right $U_{\mathsf{P}}(A)$-module. Suppose we have a quasi-free P-bimodule $(\mathsf{P} \circ M \circ \mathsf{P}, \partial)$ weakly equivalent to P as a P-bimodule, satisfying hypotheses (1) and (2) of Section 3.2.3.

Define the Γ-homology of the P-algebra A with coefficients in E to be the homology of the small complex defined in Section 3.2.2:

$$H\Gamma^{\mathsf{P}}_*(A, E) = H_*(E \otimes M \circ A, \partial'').$$

Lemma 3.2.3.2 proves that the notion of Γ-homology is well defined, as it does not depend on the choice of the bimodule $(\mathsf{P} \circ M \circ \mathsf{P}, \partial)$.

Moreover:

3.2.5 Theorem. *Let* Q *be a* Σ_**-cofibrant replacement of* P.
For A *a* P*-algebra and* E *a right* $U_{\mathsf{P}}(A)$*-module, we have* $H\Gamma_*^{\mathsf{P}}(A,E) = H_*^{\mathsf{Q}}(A,E)$.

Proof. First, note that a P-algebra will also be a Q-algebra and E will also be a right $U_{\mathsf{Q}}(A)$-module. Suppose that we are given $(\mathsf{Q}\circ M\circ \mathsf{Q},\partial) \xrightarrow{\sim} \mathsf{Q}$ a cofibrant replacement as Q-bimodules with the hypotheses above. The functor $\mathsf{P} \circ_{\mathsf{Q}} - \circ_{\mathsf{Q}} \mathsf{P}$ induces a Quillen's adjunction, and therefore we get a weak equivalence $(\mathsf{P} \circ M \circ \mathsf{P},\partial) \xrightarrow{\sim} \mathsf{P}$ between quasi-free P-bimodules. Seeing A as a Q-algebra, we get $H_*^{\mathsf{Q}}(A,E) = H_*(E\otimes M\circ A,\partial'')$. But the right hand side is by definition $H\Gamma_*^{\mathsf{P}}(A,E)$, as long as the differential is the same. It is the case, as both differentials are induced by the initial differential of $\mathsf{Q}\circ M\circ \mathsf{Q}$. □

Thus the definition of homology by replacement of bimodules is equivalent to the natural definition by replacement of operads. Also, when the operad is Σ_*-cofibrant, we recover the usual notion of homology:

3.2.6 Corollary. *Let* Q *be a* Σ_**-cofibrant operad,* B *a* Q*-algebra and* E *a right* $U_{\mathsf{Q}}(B)$*-module. Then* $H\Gamma_*^{\mathsf{Q}}(B,E) = H_*^{\mathsf{Q}}(B,E)$.

3.2.7 Definition of Γ-cohomology

Let P be an operad, A an algebra over P and F a left $U_{\mathsf{P}}(A)$-module. Suppose we have a quasi-free P-bimodule $(\mathsf{P}\circ M\circ \mathsf{P},\partial)$ weakly equivalent to P as a P-bimodule, satisfying hypotheses (1) and (2) of Section 3.2.3.

When P is Σ_*-cofibrant, we can make a similar reduction of the complex $\mathrm{Hom}_{U_{\mathsf{P}}(Q_A)}(\Omega_{\mathsf{P}}(Q_A),F)$ computing cohomology. We take for Q_A the explicit cofibrant replacement $(\mathsf{P}\circ M\circ A,\partial')$ given by Lemma 3.1.7.1. We apply now Lemma 3.2.1.1 and we get $\mathrm{Hom}_{U_{\mathsf{P}}(Q_A)}((U_{\mathsf{P}}(Q_A)\otimes M\circ A,\partial''),F)$. By adjunction, this complex is just $(\mathrm{Hom}_{\mathcal{C}}(M\circ A,F),\partial'')$.

Following the same ideas as in Section 3.2.3, we consider this complex even when the operad P does not satisfy any cofibrancy hypothesis.

We define the Γ-cohomology of the P-algebra A with coefficients in F :

$$H^*_{\mathsf{P}}(A,F) = H^*(\mathrm{Hom}_{\mathcal{C}}(M\circ A,F),\partial'').$$

A lemma similar to Lemma 3.2.3.2 proves that this notion is well-defined. We recover also the usual notion of cohomology when P is Σ_*-cofibrant.

3.2.8 Theorem. *Let* Q *be a* Σ_**-cofibrant replacement of* P.
For A *a* P*-algebra and* F *a left* $U_{\mathsf{P}}(A)$*-module, we have* $H\Gamma^*_{\mathsf{P}}(A,F) = H^*_{\mathsf{Q}}(A,F)$.

3.2.9 Corollary. *Let* Q *be a* Σ_**-cofibrant operad,* B *a* Q*-algebra and* F *a left* $U_{\mathsf{Q}}(B)$*-module. Then* $H\Gamma^*_{\mathsf{Q}}(B,F) = H^*_{\mathsf{Q}}(B,F)$.

3.2.10 Remark.

If the ground ring $\mathbb{K}$ is a field of characteristic 0, then every operad P is Σ_*-cofibrant. Hence in that case Corollary 3.2.6 and Corollary 3.2.9 imply that our Γ-(co)homology agrees with the standard (co)homology of P-algebras.

3.3 Explicit complex à la Robinson

From now on, we assume that P is a connected binary (quadratic) Koszul operad. We define an explicit P-bimodule complex, using the Koszul construction $K\mathsf{P}$ and the bar construction of the symmetric group. Then we prove we can use this complex to compute Γ-homology of P-algebras. In the case $\mathsf{P} = \mathsf{Com}$, we retrieve the complex introduced by Robinson.

Before defining the P-bimodules involved in the complex, we construct maps which will be needed to define the differential.

3.3.1 Maps between bijections

Let r be a positive integer. Let $\underline{X}$ and $\underline{Y}$ be two ordered sets with r elements.

We represent an element w of $\mathrm{Bij}(\underline{X}, \underline{Y})$ by a table of values:

$$w = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ w(x_1) & w(x_2) & \cdots & w(x_r) \end{pmatrix}.$$

The ordering amounts to a fixed bijection between $\{1, \ldots, r\}$ and $\underline{X}$ (respectively $\underline{Y}$). We can use these bijections to identify elements of $\mathrm{Bij}(\underline{X}, \underline{Y})$ with permutation of $\{1, \ldots, r\}$.

For each pair $\{i, j\} \subset \underline{Y}$ and e a dummy variable, we form the bijection

$$c^e_{i,j}(w) = \begin{pmatrix} x_1 & x_2 & \cdots & w^{-1}(i) & \cdots & \widehat{w^{-1}(j)} & \cdots & x_r \\ w(x_1) & w(x_2) & \cdots & e & \cdots & \widehat{j} & \cdots & w(x_r) \end{pmatrix}$$

if $w^{-1}(i) < w^{-1}(j)$ or the bijection

$$c^e_{i,j}(w) = \begin{pmatrix} x_1 & x_2 & \cdots & w^{-1}(j) & \cdots & \widehat{w^{-1}(i)} & \cdots & x_r \\ w(x_1) & w(x_2) & \cdots & e & \cdots & \widehat{i} & \cdots & w(x_r) \end{pmatrix}$$

if $w^{-1}(j) < w^{-1}(i)$.

If $w^{-1}(i) < w^{-1}(j)$, we have removed the column where j is the image, and i has been replaced by e. The map $c^e_{i,j}(w)$ is a bijection from $\underline{X} \setminus \{w^{-1}(j)\}$ to $\underline{Y} \setminus \{i, j\} \coprod e$. In $\underline{X} \setminus \{w^{-1}(j)\}$, we consider the restriction of the order of $\underline{X}$. In $\underline{Y} \setminus \{i, j\} \coprod e$, we consider the restriction of the order in $\underline{Y}$ with e at the place of i. Note that the map $c^e_{i,j}(w)$ can be identified with an element of Σ_{r-1}.

In the case where $w^{-1}(j) < w^{-1}(i)$, we have removed the column where i is the image, and j has been replaced by e. The map $c^e_{i,j}(w)$ is a bijection from $\underline{X} \setminus \{w^{-1}(i)\}$ to $\underline{Y} \setminus \{i, j\} \coprod e$ and can be identified with an element of Σ_{r-1}.

For each element i in $\underline{Y}$, we form the bijection

$$c_{\emptyset,i}(w) = \begin{pmatrix} x_1 & x_2 & \cdots & \widehat{w^{-1}(i)} & \cdots & x_r \\ w(x_1) & w(x_2) & \cdots & \widehat{i} & \cdots & w(x_r) \end{pmatrix}.$$

Here we have only removed the column where i is the image.

The map $c_{\emptyset,i}(w)$ is a bijection from $\underline{X} \setminus \{w^{-1}(i)\}$ to $\underline{Y} \setminus \{i\}$. Again, it can be identified with an element of Σ_{r-1} by considering the induced orders. These maps $c^e_{i,j}$ and $c_{\emptyset,i}$ play different roles, but note that $c_{\emptyset,i}(w)$ is just $c^y_{y,i}(w)$ for any y in $\underline{Y}$ such that $w^{-1}(y) < w^{-1}(i)$.

3.3.1.1 Lemma. *Let σ be an element of $Bij(\underline{Y}) \simeq \Sigma_r$ and w an element of $Bij(\underline{X}, \underline{Y})$.*

The maps $c^e_{i,j}(w)$ and $c_{\emptyset,i}(w)$ are compatible with the action of the symmetric group on the left, that is $\bar{\sigma}.c^e_{i,j}(w) = c^e_{\sigma(i),\sigma(j)}(\sigma.w)$, where $\bar{\sigma}$ is the bijection fixing e induced by σ on $\underline{Y} \setminus \{i, j\} \coprod e$.

Proof. We prove the lemma in the case where $w^{-1}(i) < w^{-1}(j)$. The proof for the other case is obtained by permuting i and j.

We already know that

$$c^e_{i,j}(w) = \begin{pmatrix} x_1 & \cdots & w^{-1}(i) & \cdots & \widehat{w^{-1}(j)} & \cdots & x_r \\ w(x_1) & \cdots & e & \cdots & \widehat{j} & \cdots & w(x_r) \end{pmatrix}.$$

We get

$$\bar{\sigma}.c^e_{i,j}(w) = \begin{pmatrix} x_1 & \cdots & w^{-1}(i) & \cdots & \widehat{w^{-1}(j)} & \cdots & x_r \\ \sigma(w(x_1)) & \cdots & e & \cdots & \widehat{\sigma(j)} & \cdots & \sigma(w(x_r)) \end{pmatrix}.$$

On the right hand side, we have

$$\sigma.w = \begin{pmatrix} x_1 & \cdots & w^{-1}(i) & \cdots & w^{-1}(j) & \cdots & x_r \\ \sigma(w(x_1)) & \cdots & \sigma(i) & \cdots & \sigma(j) & \cdots & \sigma(w(x_r)) \end{pmatrix}$$

and then

$$c^e_{\sigma(i),\sigma(j)}(\sigma.w) = \begin{pmatrix} x_1 & \cdots & w^{-1}(i) & \cdots & \widehat{w^{-1}(j)} & \cdots & x_r \\ \sigma(w(x_1)) & \cdots & e & \cdots & \widehat{\sigma(j)} & \cdots & \sigma(w(x_r)) \end{pmatrix}.$$

□

We extend the definition of $c^e_{i,j}$ to sequences of bijections $\underline{w} = (w_0, \ldots w_n)$ by

$$c^e_{i,j}(\underline{w}) = (c^e_{i,j}(w_0), \ldots, c^e_{i,j}(w_n)).$$

3.3.2 Definition of the complex

We now define a Σ_*-module M involved in our explicit complex computing Γ-homology. We are given P a connected binary (quadratic) Koszul operad.

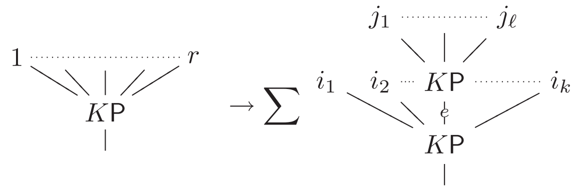
We consider $K\mathsf{P}$ the Koszul construction of P, defined by $K(\mathsf{P})_{(s)} := H_s(B_*(\mathsf{P})_{(s)})$. It is a cooperad, equipped with a differential, such that $(\mathsf{P} \circ K\mathsf{P} \circ \mathsf{P}, \partial)$ is quasi-isomorphic to P. For more details, we refer the reader to the initial article of Ginzburg and Kapranov [GK] or the article of Fresse [F3], of which we adopt the convention.

We also consider the chain complex $C_*(E\Sigma_\bullet)$ of the total space of the universal Σ_n-bundles in simplicial spaces, $n \in \mathbb{N}$. The chain complex $C_*(E\Sigma_n)$ is the acyclic homogeneous bar construction of the symmetric group Σ_n, the module spanned in degree t by the $(t+1)$-tuples of permutations $\underline{w} = (w_0, \ldots, w_t)$ together with the differential δ such that $\delta(\underline{w}) = \sum_i (-1)^i (w_0, \ldots, \widehat{w_i}, \ldots, w_t)$. We consider the left action of the symmetric group on this chain complex.

We define the Σ_*-module $M = K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)$ by $M(r) = K\mathsf{P}(r) \otimes C_*(E\Sigma_r)$. The action of the symmetric group is the diagonal action.

Now we construct a map $\Delta : M \to \mathsf{P} \circ M \circ \mathsf{P}$ which defines a twisting differential once extended by P-linearity on the right and as a P-derivation on the left.

Recall that the quadratic component of the cooperad coproduct of $K\mathsf{P}$ is given by the dual of the operadic composition in P :



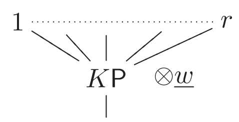
where the sum ranges over all partitions $\{i_1, \ldots, i_k\} \coprod \{j_1, \ldots, j_\ell\} = \{1, \ldots, r\}$ and e is a dummy variable. We define two restrictions of this coproduct:

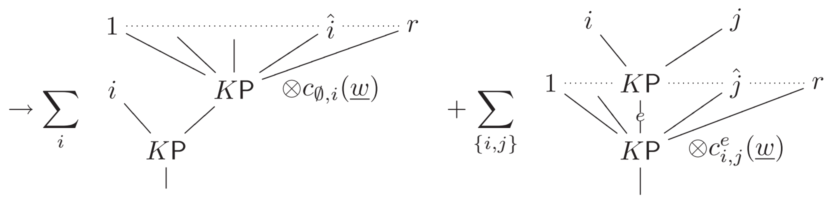
- Δ_- where we only keep the components of the differential where the set $\{i_1, \ldots, i_k\}$ is reduced to one index (when the element below in the composition is binary).

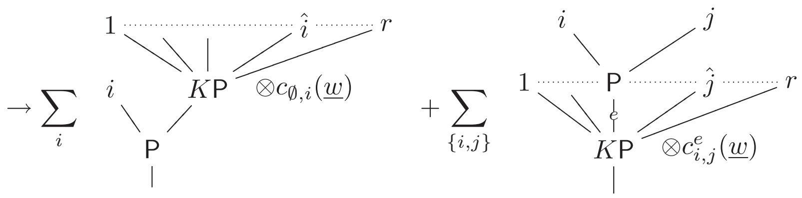
- Δ_+ where we only keep the components of the differential where the set $\{j_1, \ldots, j_\ell\}$ is composed of two indices (when the element above in the composition is binary).

Note that $\Delta_-(\gamma) = \Delta_+(\gamma)$ when γ is an element with three inputs.

We use this coproduct Δ on $K\mathsf{P}$ to define Δ on $K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)$ by the following composite:







The first arrow consists in using Δ_- and Δ_+ on $K\mathsf{P}$ and the $c^e_{i,j}$ defined in the previous paragraph. The second arrow comes from the twisting cochain $\kappa : K\mathsf{P} \to \mathsf{P}$ (which identifies elements of arity 2 in $K\mathsf{P}$ with elements of arity 2 in P).

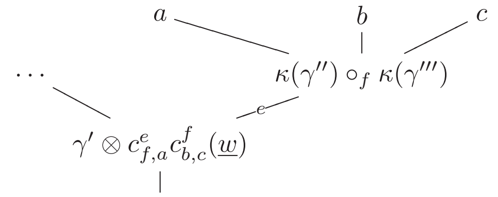
This construction defines Δ on representatives with the entries ordered from 1 to r. We apply Lemma 3.3.1.1 to extend this definition to $K\mathsf{P}$.

3.3.2.1 Lemma. *The map Δ determines a differential of Σ_*-modules on $\mathsf{P} \circ M \circ \mathsf{P}$.*

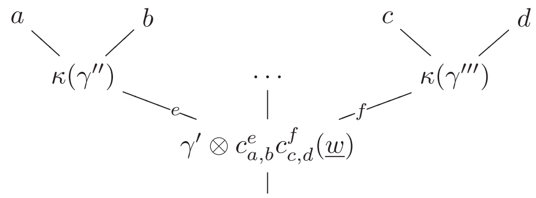
Proof. For an element $\gamma \in K\mathsf{P}(r)$ and $\underline{w}$ a sequence of permutations in Σ_r, we decompose $\Delta^2(\gamma \otimes \underline{w})$ in the sum of three terms: the part induced by $\Delta_+\Delta_+$, the part induced by $\Delta_-\Delta_-$ and the part induced by $\Delta_+\Delta_- + \Delta_-\Delta_+$.

The composite $\Delta_+\Delta_+$ yields terms of the form:

(I)



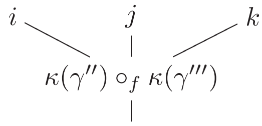
(II)



- Let $\{i < j < k\} = \{a, b, c\}$ denote the ordered subset formed by the triple $\{a, b, c\}$ in the indexing set. We can identify the permutation occuring in terms of the form (I):

$$c^e_{f,a}c^f_{b,c}(\underline{w}) = \begin{pmatrix} \cdots & w^{-1}(i) & \cdots & \widehat{w^{-1}(j)} & \cdots & \widehat{w^{-1}(k)} & \cdots \\ \cdots & e & \cdots & \widehat{j} & \cdots & \widehat{k} & \cdots \end{pmatrix}.$$

  Thus the result of the composite $c^e_{a,b}c^f_{c,d}$ only depends on $\{i < j < k\}$.

  The sum of the terms associated to a given triple $\{i < j < k\}$ is 0 because the sum of the compositions  cancels in P by construction of the Koszul dual (cf. [F3, Section 5.2]) and the sum of terms (I) is 0.

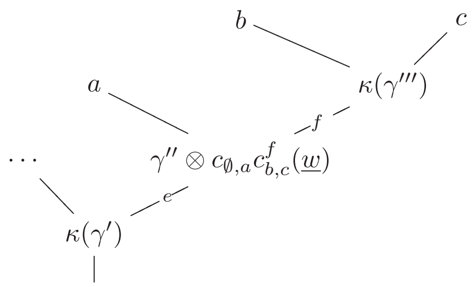
- For terms (II), we have the relation $c^e_{a,b}c^f_{c,d}(\underline{w}) = c^f_{c,d}c^e_{a,b}(\underline{w})$. By coassociativity of the coproduct in $K\mathsf{P}$, the terms (II) cancel each another. Note simply that a permutation of κ with a suspension produces a sign opposition.

Thus the part of Δ^2 induced by $\Delta_+\Delta_+$ is 0.

The cancellation of the part induced by $\Delta_-\Delta_-$ is similar to the proof of the cancellation of terms (I).

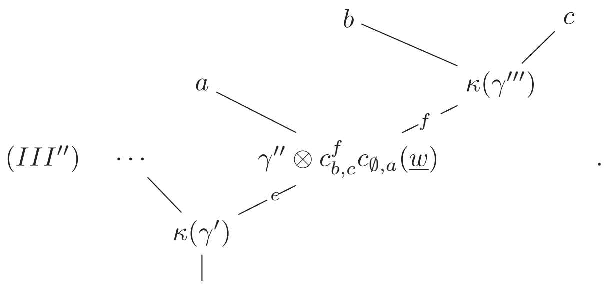
We now study the part induced by $\Delta_+\Delta_- + \Delta_-\Delta_+$. The composite $\Delta_-\Delta_+$ yields terms of the form:

(III')



,

while the composite $\Delta_+\Delta_-$ yields terms of the form:



But $c^f_{b,c} c_{\emptyset,a}(\underline{w}) = c_{\emptyset,a} c^f_{b,c}(\underline{w})$. So by coassociativity of the coproduct in $K\mathsf{P}$, we prove the cancellation of terms (III') with terms (III"). We use again that a permutation of κ with a suspension produces a sign opposition.

Thus the part of Δ^2 induced by $\Delta_+\Delta_- + \Delta_-\Delta_+$ is 0.

Finally, we have proved that $\Delta^2 = 0$.

Moreover, the map is compatible with the symmetric action by Lemma 3.3.1.1. □

We now consider the differential of the bar construction of the symmetric group and use it to define another differential δ on $\mathsf{P} \circ M \circ \mathsf{P}$.

For $\underline{w} = (w_0, \dots w_n)$, recall that $\delta(\underline{w}) = \sum_i (-1)^i (w_0, \dots, \widehat{w_i}, \dots, w_n)$.

We define the map δ on $K\mathsf{P} \boxtimes C_*(\Sigma_\bullet) \to K\mathsf{P} \boxtimes C_*(\Sigma_\bullet)$ by

$$\delta(\gamma \otimes \underline{w}) = (-1)^{|\gamma|} \gamma \otimes \delta(\underline{w}).$$

3.3.2.2 Lemma. *The map δ induces a differential on $\mathsf{P} \circ (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ \mathsf{P}$ that anticommutes with Δ.*

□

Putting all this together, we get:

3.3.3 Theorem. *We have defined a quasi-free dg P-bimodule $(\mathsf{P} \circ (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ \mathsf{P}, \Delta + \delta)$, where Δ and δ are both a differential.*

3.3.4 Homology of the complex

The goal of this paragraph is to prove that we have a quasi-isomorphism $(\mathsf{P} \circ (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ \mathsf{P}, \Delta + \delta) \xrightarrow{\sim} \mathsf{P}$ of P-bimodules.

First, we consider a dg-module morphism defined by:

$$\begin{cases} K\mathsf{P}(r) \otimes C_0(\Sigma_r) \to K\mathsf{P}(r) \\ K\mathsf{P}(r) \otimes C_{\geq 1}(\Sigma_r) \to 0. \end{cases}$$

The first part of the arrow just forgets the permutation.

This morphism induces a P-bimodule morphism $(\mathsf{P} \circ (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ \mathsf{P}, \Delta + \delta) \xrightarrow{\sim} (\mathsf{P} \circ K\mathsf{P} \circ \mathsf{P}, \partial)$, by extension by linearity on the right, and as a derivation on the left. We call ϵ this P-bimodule morphism. Note that Δ is sent to the usual differential ∂ of the Koszul construction with coefficients $K(\mathsf{P}, \mathsf{P}, \mathsf{P})$ (see [F3] for details about that Koszul contruction), while δ is sent to 0.

We will now use a spectral argument to show that $(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\Delta+\delta)$ is quasi-isomorphic to $(\mathsf{P}\circ K\mathsf{P}\circ\mathsf{P},\Delta)$.

We see $(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P})$ as a bimodule, with differentials Δ and δ. The first graduation is the bar degree r in $K\mathsf{P}$ and the second graduation is the number $*$ of permutations.

$$(E^0_{r,*},d^0)=(\mathsf{P}\circ(K\mathsf{P}_r\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\delta)$$

We now use that $C_*(E\Sigma_\bullet)$ is acyclic, that is $H_n(C_*(E\Sigma_\bullet))=\mathbb{K}$ if $n=0$ and 0 otherwise. We also use that the functors $\mathsf{P}\circ-$, $-\circ\mathsf{P}$ and $K\mathsf{P}_r\ \otimes-$ preserve quasi-isomorphims (for instance, cf.[F3, Theorem 2.1.15]).

Thus we get that $H_n(\mathsf{P}\circ(K\mathsf{P}_r\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\delta)=\mathsf{P}\circ K\mathsf{P}_r\circ\mathsf{P}$.

$$(E^1_{r,0},d^1)=(\mathsf{P}\circ K\mathsf{P}_r\circ\mathsf{P},\partial)$$

$$E^2_{r,0}=H_r(\mathsf{P}\circ K\mathsf{P}\circ\mathsf{P},\partial)$$

We know that the spectral sequence of a bicomplex (both graduations being bounded below) converges to the total homology of the bicomplex.

Thus $H_*(\mathsf{P}\circ(K\mathsf{P}\boxtimes C(E\Sigma_\bullet))\circ\mathsf{P},\Delta+\delta)=H_*(\mathsf{P}\circ K\mathsf{P}\circ\mathsf{P},\partial)$.

This proves that ϵ is a quasi-isomorphism $(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\Delta+\delta)\xrightarrow{\sim}(\mathsf{P}\circ K\mathsf{P}\circ\mathsf{P},\partial)$. We can compose it with the quasi-isomophism between $\mathsf{P}\circ K\mathsf{P}\circ\mathsf{P}$ and P, and finally this gives us a quasi-isomorphism $(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\Delta+\delta)\xrightarrow{\sim}\mathsf{P}$ of P-bimodules.

3.3.5 Back to Γ-homology

We now prove that the P-bimodule constructed in the previous paragraphs satisfies all the required hypotheses so we can use it to compute Γ-homology.

It has the form $\mathsf{P}\circ M\circ P$, with M a Σ_*-module such that $M(0)=0$. We now have to prove that $(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\Delta+\delta)$ is a cofibrant P-bimodule.

The P-bimodule $(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\Delta+\delta)$ can be seen as $(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet),\delta)\circ\mathsf{P},\Delta)$. The differential Δ is decomposable. We first prove that $(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet),\delta)$ is a cofibrant Σ_*-module, and then Proposition 3.1.6 will give us the result.

3.3.5.1 Lemma. *The Σ_*-module $(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet),\delta)$ is cofibrant.*

Proof. We consider the map f of Σ_*-modules $0\to(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet),\delta)$, which can be written as $f=(0\otimes id_{\Sigma_r})_{r\in\mathbb{N}}$. According to the description of generating cofibrations in Section 3.1.4, we have to prove that $0\to(K\mathsf{P}(r),0)$ is a cofibration of dg-modules. But $(K\mathsf{P}(r),0)$ is assumed to be free and its differential is 0. Hence the claim is immediate. □

Thus we have proved

3.3.6 Proposition. *The P-bimodule $(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\Delta+\delta)$ is cofibrant.*

Besides, we have seen in the previous paragraph that $(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\Delta+\delta)$ is weakly equivalent to P.

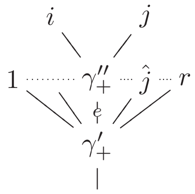
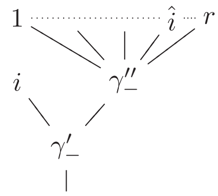
So we can use $K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet)$ to compute Γ-homology of algebras over P. Explicitly, we have:

3.3.7 Theorem. *Let P be a binary Koszul operad, A an algebra over P and E a right $U_{\mathsf{P}}(A)$-module.*

$$H\Gamma^{\mathsf{P}}_*(A,E)=H_*(E\otimes(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ A,\partial'')$$

where ∂'' is the differential induced by $\Delta+\delta$ in two steps, explained in the proofs of Lemmas 3.1.7.1 and 3.2.1.1.

Explicitly, for $x \in E$, $\gamma \in K\mathsf{P}$ such that

$\Delta_+(\gamma) = \sum_{i<j}$  *and* $\Delta_-(\gamma) = \sum_{i}$  ,

$(w_0, \ldots, w_*) \in C_*(E\Sigma_r)$ *and* $a_1, \ldots, a_r$ *in* A*, we have:*

$$\partial''(x \otimes \gamma \otimes (w_0, \ldots, w_*) \otimes (a_1, \ \ldots, a_r)) =$$

$$\sum_{i<j} \pm x \otimes \gamma'_+ \otimes (c^e_{i,j}(w_0), \ldots, c^e_{i,j}(w_*)) \otimes (a_1, \ \ldots, \kappa(\gamma''_+)(a_i, a_j), \ldots, \widehat{a_j}, \ldots a_r)$$

$$+ \sum_i \pm \kappa(\gamma'_-)(x, a_i) \otimes \gamma''_- \otimes (c_{\emptyset,i}(w_0), \ldots, c_{\emptyset,i}(w_*)) \otimes (a_1, \ldots, \widehat{a_i}, \ldots, a_r).$$

Signs are induced by the usual Koszul rule.

3.3.8 Examples

1. For $\mathsf{P} = \mathsf{Com}$, we have $K\mathsf{P} = (\Lambda\mathsf{Lie})^{\#}$ where Λ denotes the operadic suspension and $\#$ denotes the linear duality. Here we retrieve Robinson's complex (up to a usual isomorphism on the bar construction).

2. For $\mathsf{P} = \mathsf{Lie}$, we have $K\mathsf{P} = (\Lambda\mathsf{Com})^{\#}$. We denote γ_r the generator of $K\mathsf{P}$ in arity r (it is in degree $1 - r$). Let A be a Lie algebra concentrated in degree 0.

$$\partial''(x \otimes \gamma_r \otimes (w_0, \ldots, w_*) \otimes (a_1, \ \ldots, a_r)) =$$

$$\sum_{i<j} (-1)^j x \otimes \gamma_{r-1} \otimes (c^e_{i,j}(w_0), \ldots, c^e_{i,j}(w_*)) \otimes (a_1, \ \ldots, \kappa(\gamma_2)(a_i, a_j), \ldots, \widehat{a_j}, \ldots a_r)$$

$$+ \sum_i (-1)^{i-1} \kappa(\gamma_2)(x, a_i) \otimes \gamma_{r-1} \otimes (c_{\emptyset,i}(w_0), \ldots, c_{\emptyset,i}(w_*)) \otimes (a_1, \ldots, \widehat{a_i}, \ldots, a_r).$$

   Note that we find the same signs as in the complex of Chevalley-Eilenberg.

Similarly for the cohomology, we have the following theorem:

3.3.9 Theorem. *Let P be a binary Koszul operad, A an algebra over P and F a left $U_{\mathsf{P}}(A)$-module.*

$$H^*_{\mathsf{Q}}(A, F) = H^*(\mathrm{Hom}(K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet) \circ A, F), \partial'').$$

where ∂'' is the differential induced by $\Delta + \delta$ in two steps, explained in the proofs of Lemmas 3.1.7.1 and 3.2.1.1.

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Chapitre 4

Obstruction theory for algebras over an operad

The goal of this paper is to set up an obstruction theory in the context of algebras over an operad and in the framework of dg-modules over a field. Precisely, the problem we consider is the following: Suppose given two algebras A and B over an operad P and an algebra morphism from H_*A to H_*B. Can we realize this morphism as a morphism of P-algebras from A to B in the homotopy category ? Also, if the realization exists, is it unique in the homotopy category ?
We identify obstruction cocycles for this problem, and notice that they live in the first two groups of operadic Γ-cohomology.

We study in this paper a question of realization of morphisms in a category of algebras over an operad.

In general, a realization problem takes the following form. We fix a category $\mathcal{C}$ equipped with a model structure (for instance: Top, Sp, differential graded algebras over an operad). We have a homology (or homotopy) functor $H : \mathcal{C} \to \mathcal{A}$ with values in a purely algebraic category (for instance: graded modules, graded algebras). We ask the following questions:

Q0 (realization of objects): For any a in $\mathcal{A}$, can we find a c in $\mathcal{C}$ such that $H(c) = a$?

Q1 (realization of morphisms): For $f : H(c_1) \to H(c_2)$, does there exist $\phi : c_1 \to c_2$ such that $H(\phi) = f$?

Q2 (unicity of realizations): If $H(\phi_1) = H(\phi_2)$, does there exist a homotopy h between ϕ_1 and ϕ_2?

Generally, the obstructions to these existences can be interpreted as classes in some (co)homology theory.

The most classical example has been asked by Steenrod for $\mathcal{C} = Top$ and $H = H^*_{\text{sing}}$. A solution of this problem in the case of rational nilpotent CW-complexes has been given by Halperin and Stasheff in [HS]. They apply rational homotopy theory to reduce that topological realization problem to a realization problem in the category of differential graded commutative algebras. The obstructions then live in some Harrison cohomology groups. The obstruction theory of Blanc, Dwyer and Goerss [BDG] for the realizability of Π-algebras by a space, the theories of Robinson [Rob] and of Goerss and Hopkins [GH] for the realizability of an algebra by an E_∞-spectra are other fundamental examples of obstruction theory in homotopy.

We are here interested in the case $\mathcal{C} = {}_{\mathsf{P}}\mathrm{dgMod}_{\mathbb{K}}$, the category of algebras over a fixed operad P in the framework of differential graded modules over a field $\mathbb{K}$. The functor H is the homology of the underlying dg-module of an algebra over P. This homology inherits a $H_*\mathsf{P}$-algebra structure. The target category $\mathcal{A}$ consists of the graded $H_*\mathsf{P}$-algebras. The realization problem has been studied by Livernet in her thesis [Liv, Section 4] in the setting of $\mathbb{N}$-graded dg-modules and when the ground ring $\mathbb{K}$ is a field of characteristic 0. The obstruction classes live in some cohomology groups of a natural cohomology theory associated to P, generalizing Harrison cohomology for $\mathsf{P} = \mathsf{Com}$.

In this paper, we obtain an obstruction theory in the setting of $\mathbb{Z}$-graded dg-modules and when the ground ring $\mathbb{K}$ is any field. We can identify a sequence of obstructions lying in some cohomology groups. Precisely, we use the Γ-cohomology of algebras over an operad defined in [Hoff], which generalizes Robinson's Γ-homology, and we get the following theorems:

Theorem (Corollary 4.2.6). *Let P be a connected operad and let $\tilde{\mathsf{P}}$ be an operadic cofibrant replacement of P. Let A and B be two algebras over $\tilde{\mathsf{P}}$. Suppose given a P-algebra morphism $f : H_*A \to H_*B$ (where H_*A and H_*B have the structure induced in homology).*

The obstruction cocycles to the realization of f lie in $H\Gamma^1_{\mathsf{P}}(H_*A, H_*B)$. If $H\Gamma^1_{\mathsf{P}}(H_*A, H_*B)=0$, then there automatically exists a morphism ϕ in the homotopy category of $\tilde{\mathsf{P}}$-algebras such that $H_*\phi = f$

Theorem (Corollary 4.3.5). *Let P be a connected operad and let $\tilde{\mathsf{P}}$ be an operadic cofibrant replacement of P. Let A and B be two algebras over $\tilde{\mathsf{P}}$. Suppose given a P-algebra morphism $f : H_*A \to H_*B$ and two homotopy morphisms ϕ_1, ϕ_2 such that $H_*\phi_1 = H_*\phi_2 = f$.*

The obstruction cocycles to the unicity of the realizations in the homotopy category lie in the group $H\Gamma^0_{\mathsf{P}}(H_*A, H_*B)$. If $H\Gamma^0_{\mathsf{P}}(H_*A, H_*B)=0$, then $\phi_1 = \phi_2$ in the homotopy category of $\tilde{\mathsf{P}}$-algebras.

To obtain these theorems, the method is first to reduce our study to the case where the differentials of A and B are trivial. Then we use model category structures to make explicit cofibrant replacements of the algebras A and B. The crucial point of the proof is a natural filtration of the cooperad $B(\mathsf{P} \boxtimes \mathsf{E})$, which allows us to filter the cofibrant replacements. We construct step by step a map inducing the realization and identify the obstructions to this construction.

An important thing to notice in our theorems is that only the structures of P-algebras on H_*A and H_*B appear. So we do not need to know the full structures on A and B, but only a part of it.

In Section 1, we recall some results about operads and cooperads. In Section 2, we identify the obstructions to the realization. In the last section, we study the obstructions to the unicity up to homotopy of the realizations.

Convention

We work in the differential graded setting. We take as ground category the category of differential $\mathbb{Z}$-graded modules (for short dg-modules) over a fixed field $\mathbb{K}$.

All operads P will be assumed to be connected in the sense that $\mathsf{P}(0) = 0$ and $\mathsf{P}(1) = \mathbb{K}$.

4.1 Recollections

4.1.1 Model structures

We give references for the model structures of the categories which are used in this paper. For general references on the subject, we refer the reader to the survey of Dwyer and Spalinksi [DS] and the books of Hirschhorn [Hir] and Hovey [Hov]. For model structures in the operadic context, we refer to the articles of Hinich [Hin], of Berger and Moerdijk [BM1] and of Goerss and Hopkins [GH], and the book of Fresse [F1].

Just recall the following standard definitions:

1. The category of dg-modules is equipped with the model structure such that a morphism is a fibration (resp. a weak equivalence) if it is an epimorphism (resp. induces an isomorphism in homology).

2. The category of operads inherits a model structure where fibrations (resp. weak equivalences) are fibrations (resp. weak equivalences) of the underlying dg-modules.

3. The category of algebras over a cofibrant operad inherits a model structure where fibrations (resp. weak equivalences) are fibrations (resp. weak equivalences) of the underlying dg-modules.

In all cases, cofibrations are given by the LLP with respect to acyclic fibrations.

We usually call Σ_*-module the structure underlying an operad. It is defined by a collection of dg-modules $\{M(r)\}_{r\in\mathbb{N}}$ where each $M(r)$ is equipped with an action of the r-th symmetric group Σ_r. The category of Σ_*-modules also inherits a model structure such that fibrations (resp. weak equivalences) are fibrations (resp. weak equivalences) of the underlying dg-modules. We say that an operad is Σ_*-cofibrant if the underlying Σ_*-module is cofibrant. The category of algebras over a Σ_*-cofibrant operad can also be equipped with a semi-model structure, but we will not need this refinement.

We will use a cofibrant replacement of operads given by the cobar-bar duality, which can be found in the paper of Getzler and Jones [GJ] in characteristic 0, and the paper of Berger and Moerdijk [BM2, Section 8.5] in our more general context. We denote by B the bar construction of an operad, introduced in [GK], and by B^c the cobar construction, introduced in [GJ]. Recall that an element of the bar (or cobar) construction $B(\mathsf{P})$ can be seen as a tree labelled by elements of P. Thus the bar (and cobar) construction is equipped with a weight, given by the number of vertices of the tree representing an element. The operad E denotes the Barratt-Eccles operad, whose definition is recalled later in Section 4.1.3, and $\boxtimes$ denotes the arity-wise tensor product of Σ_*-modules (the tensor product such that $(\mathsf{P}\boxtimes\mathsf{E})(r) = \mathsf{P}(r)\otimes\mathsf{E}(r)$ for all $r\in\mathbb{N}$).

4.1.1.1 Fact ([BM2, Theorem 8.5.4])**.** *Let* P *be an operad.*
The operad $B^c(B(\mathsf{P}\boxtimes\mathsf{E}))$ *is a cofibrant replacement of the operad* P.

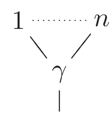
If Q is a cofibrant replacement of an operad P, working with algebras over Q is equivalent to working with algebras over $B^c(B(\mathsf{P}\boxtimes\mathsf{E}))$. In this paper, we always pick this particular cofibrant replacement.

4.1.2 Coalgebras over cooperads

Let D be a cooperad. In the following series of propositions, we recall how the structure of $B^c(\mathsf{D})$-algebra on A can be explicitely encoded in a quasi-cofree coalgebra $\mathsf{D}(A)$. We need precise formulas for our study.

These results have been first given in the preprint of Getzler and Jones [GJ]. But we use them in the wider context of $\mathbb{Z}$-graded modules and over a field of any characteristic, and we refer to [F2] for the generalization in the latter setting.

Let D be a cooperad and A a dg-module.

We may represent an element $\gamma\in\mathsf{D}(n)$ by a corolla with n inputs 

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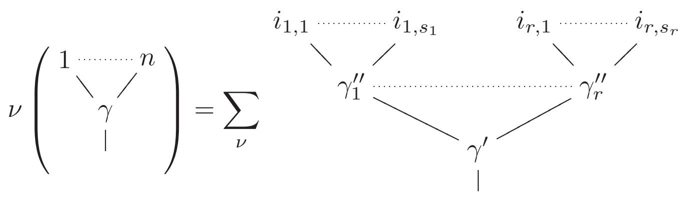
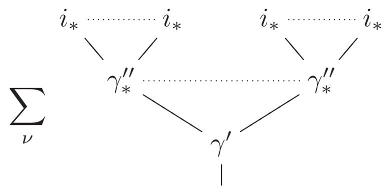
We consider the total coproduct and the quadratic coproduct of a cooperad structure, which send the element γ to a composed element arranged on a tree.

The total coproduct denoted by ν maps an element $\gamma\in\mathsf{D}$ to a sum of formal composites of elements represented by

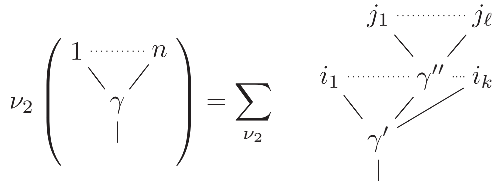


where γ', $\gamma''_1,\dots,\gamma''_r$ are elements of D and the entries form a multi-shuffle of $\{1,\dots,n\}$.

To avoid too many indices, we will write such a sum with the following form:



The quadratic coproduct of an element $\gamma \in \mathsf{D}$ is denoted by $\nu_2(\gamma)$ and represented by



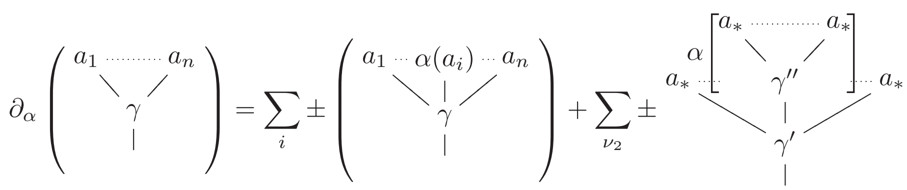
where γ' and the γ'' are elements of D and $\{i_1, \dots, i_k\} \coprod \{j_1, \dots, j_\ell\}$ run over partitions of $\{1, \dots, n\}$.

Let A be a dg-module, where the differential is denoted by d_A. Recall that $\mathsf{D}(A)$ is the cofree connected coalgebra given by

$$\mathsf{D}(A) = \bigoplus (\mathsf{D}(r) \otimes A^{\otimes r})_{\Sigma_r}.$$

The element $\gamma(a_1, \dots, a_r) \in \mathsf{D}(A)$ is associated to the tensor $\gamma \otimes (a_1, \dots, a_r)$. We represent an element in $\mathsf{D}(A)$ by a corolla with inputs indexed by elements of A.

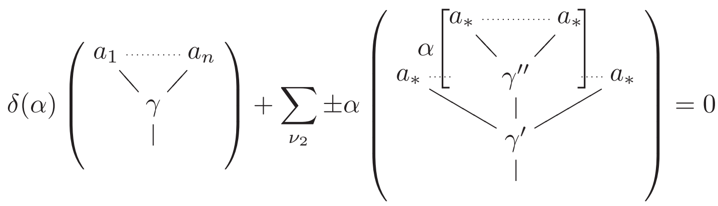
4.1.2.1 Proposition ([GJ, Proposition 2.14], [F2, Proposition 4.1.3])**.** *For a cofree coalgebra $\mathsf{D}(A)$, we have a bijective correspondance between D-coderivations $\partial : \mathsf{D}(A) \to \mathsf{D}(A)$ and homomorphisms $\alpha : \mathsf{D}(A) \to A$. The homomorphism α associated to a coderivation ∂ is given by the compositive with the canonical projection. Conversely, the coderivation ∂_α associated to α is determined by*



for every $\gamma(a_1, \dots, a_n)$ in $\mathsf{D}(A)$. The first term corresponds to α applied to $a_i \in A \subset \mathsf{D}(A)$. For the second term, we use the quadratic coproduct ν_2 and then apply α on the upper corolla which represents an element in $\mathsf{D}(A)$.

4.1.2.2 Proposition ([F2, Proposition 4.1.4])**.** *Let $\alpha : \mathsf{D}(A) \to A$ be a homomorphism of degree -1 such that $\alpha_{|A} = 0$.*

A D-coderivation of degree -1, $\partial_\alpha : \mathsf{D}(A) \to \mathsf{D}(A)$ so that $(\mathsf{D}(A), \partial_\alpha)$ defines a differential graded quasi-cofree coalgebra if and only if the homomorphism $\alpha : \mathsf{D}(A) \to A$ satisfies the relation



for every element $\gamma(a_1, \dots, a_n)$ in $\mathsf{D}(A)$, where $\delta(\alpha)$ denotes $d_A \circ \alpha \pm \alpha \circ \partial_\alpha$.

4.1.2.3 Proposition ([GJ, proposition 2.15], [F2, Proposition 4.1.5])**.** *A $B^c(\mathsf{D})$-algebra structure on a dg-module A is equivalent to a map $\alpha : \mathsf{D}(A) \to A$ which satisfies the equation of the previous paragraph and such that the restriction $\alpha_{|A}$ vanishes.*

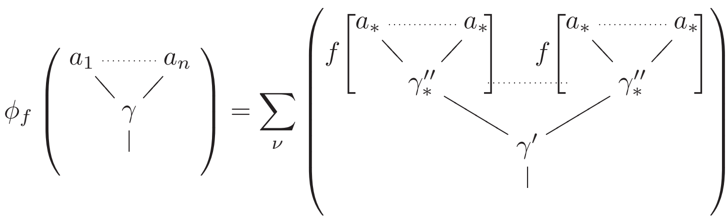
When we are given an operad morphism $B^c(\mathsf{D}) \to \mathsf{Q}$, we have a functor which, to any D-coalgebra C, associates a quasi-free Q-algebra $R_{\mathsf{Q}}(C) = (\mathsf{Q}(C), \partial)$ for some twisting differential ∂ (cf. [GJ] or [F2, Section 4.2.1]).

We apply this construction to $\mathsf{D} = B(\mathsf{P} \boxtimes \mathsf{E})$, the morphism $id : B^c(\mathsf{D}) \to B^c(\mathsf{D}) = \tilde{\mathsf{P}}$ and the coalgebra $C = (\mathsf{D}(A), \partial_\alpha)$ associated to a $\tilde{\mathsf{P}}$-algebra A (the action is denoted by α). We get the following result:

4.1.2.4 Proposition ([GJ, Theorem 2.19], [F2, Theorem 4.2.4])**.** *Let A be an algebra over $\tilde{\mathsf{P}}$ and let α denote the action. Let D denote $B(\mathsf{P} \boxtimes \mathsf{E})$. The augmentation $\epsilon : (\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_\alpha), \partial) \to A$ defines a weak equivalence and $(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_\alpha), \partial)$ forms a cofibrant replacement of A in the category of $\tilde{\mathsf{P}}$-algebras.*

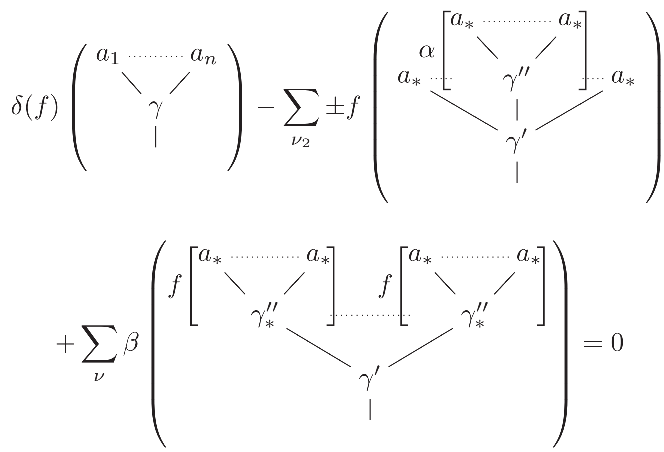
In this context, to study morphisms in the homotopy category of $\tilde{\mathsf{P}}$-algebras, we just have to study morphisms of quasi-cofree D-coalgebras. The following two propositions show how to reduce our study to the corestrictions of such morphisms.

4.1.2.5 Proposition ([F2, Observation 4.1.7])**.** *The homomorphisms $\phi : \mathsf{D}(A) \to \mathsf{D}(B)$ of degree 0 and commuting with coalgebra structures are in bijection with homomorphisms of dg-modules $f : \mathsf{D}(A) \to B$. The homomorphism f associated to ϕ is given by the composite of ϕ with the projection. Conversely, the homomorphism $\phi = \phi_f$ associated to f is determined by the formula*



for every element $\gamma(a_1, \ldots, a_n)$ in $\mathsf{D}(A)$. We use the total coproduct and we apply f to all upper corrolas.

4.1.2.6 Proposition ([F2, Proposition 4.1.8])**.** *The homomorphism of cofree coalgebras $\phi_f : \mathsf{D}(A) \to \mathsf{D}(B)$ associated to a homomorphism $f : \mathsf{D}(A) \to B$ defines a morphism between quasi-cofree coalgebras $(\mathsf{D}(A), \partial_\alpha) \to (\mathsf{D}(B), \partial_\beta)$ if and only if we have the identity*



for every element $\gamma(a_1, \ldots, a_n)$ in $\mathsf{D}(A)$.

4.1.3 The Barratt-Eccles operad and its action on cochains

Recall that an E_∞-operad is a Σ_*-cofibrant replacement of the commutative operad.

The Barratt-Eccles operad E is an example of E_∞-operad, defined by the normalized chain complex $\mathsf{E} = N_*(E\Sigma_\bullet)$, where $E\Sigma_n$ is the total space of the universal Σ_n-bundles in simplicial spaces. The chain complex $N_*(E\Sigma_n)$ is identified with the acyclic homogeneous bar construction of the symmetric group Σ_n, the module spanned in degree t by the $(t+1)$-tuples of permutations $\underline{w} = (w_0, \ldots, w_t)$ together with the differential δ such that $\delta(\underline{w}) = \sum_i (-1)^i (w_0, \ldots, \widehat{w_i}, \ldots, w_t)$. We consider the left action of the symmetric group on this chain complex.

The composition product of E is obtained using the composition product of permutations (which is just the insertion of a block). More precisely, for $\underline{w} = (w_0, \ldots, w_m) \in \mathsf{E}(r)$ and $\underline{w}' = (w'_0, \ldots, w'_n) \in \mathsf{E}(s)$, the composite $\underline{w} \circ_i \underline{w}' \in \mathsf{E}(r+s-1)$ is defined by

$$\underline{w} \circ_i \underline{w}' = \sum_{x_*, y_*} \pm (w_{x_0} \circ_i w'_{y_0}, \ldots, w_{x_{m+n}} \circ_i w'_{y_{m+n}})$$

where the sum ranges over the monotonic paths from $(0,0)$ to (m,n) in $\mathbb{N} \times \mathbb{N}$.

The operad E acts on $N^*(\Delta^1)$, according to the paper by Berger and Fresse [BF]. We denote this action by σ. For our purposes, we simply recall the action of the component of degree 0 of E. We have the equality of dg-modules $N^*(\Delta^1) = \mathbb{K}.\underline{0}^\# \oplus \mathbb{K}.\underline{1}^\# \oplus \mathbb{K}.\underline{01}^\#$ where $\underline{0}^\#$, $\underline{1}^\#$ and $\underline{01}^\#$ denote the dual of the basis of non-degenerate simplices. The differential ∂_N satisfies $\partial_N(\underline{01}^\#) = \underline{1}^\# - \underline{0}^\#$ and $\partial_N(\underline{0}^\#) = \partial_N(\underline{1}^\#) = 0$. The r-th component in degree 0 of E is actually Σ_r, and the identity of Σ_r acts on $N^*(\Delta^1)$ as follows:

- $id.(\underline{0}^\#, \ldots, \underline{0}^\#, \underline{01}^\#, \underline{1}^\#, \ldots, \underline{1}^\#) = \underline{01}^\#$
- $id.(\underline{0}^\#, \ldots, \underline{0}^\#) = \underline{0}^\#$
- $id.(\underline{1}^\#, \ldots, \underline{1}^\#) = \underline{1}^\#$
- $id.(u_1, \ldots, u_r) = 0$ otherwise.

The equivariance gives the action of the other permutations of Σ_r. We will not need the formula for the action of E in higher degrees.

4.1.4 The cylinder object of an algebra over an operad

Let Q be any cofibrant operad, for instance $\mathsf{Q} = B^c(B(\mathsf{P} \boxtimes \mathsf{E}))$. Let B be a Q-algebra, with the structure given by β. We recall in this section the results we need from [BF, Section 3.1].

The cylinder object of B in the category of Q-algebras is $B \otimes N^*(\Delta^1)$.

It is naturally endowed with the action $\beta \otimes \sigma$ of $\mathsf{Q} \boxtimes \mathsf{E}$:

$$(q \otimes \pi)(b_1 \otimes u_1, \ldots, b_r \otimes u_r) = q(b_1, \ldots, b_r) \otimes \pi(u_1, \ldots, u_r)$$

for $q \in \mathsf{Q}, \pi \in \mathsf{E}, (b_1, \ldots, b_r) \in B^r, (u_1, \ldots, u_r) \in N^*(\Delta^1)^r$. Fixing an operadic section $\rho : \mathsf{Q} \to \mathsf{Q} \boxtimes \mathsf{E}$ of the augmentation $\mathsf{Q} \boxtimes \mathsf{E} \to \mathsf{Q}$, we can see $B \otimes N^*(\Delta^1)$ as a Q-algebra. In Section 4.3.2, we will fix an explicit map ρ.

4.2 Realizations of morphisms

Suppose given

- an operad P with the canonical operadic cofibrant replacement $\tilde{\mathsf{P}} = B^c(B(\mathsf{P} \boxtimes \mathsf{E}))$,
- two algebras, A and B, over $\tilde{\mathsf{P}}$,
- a P-algebra morphism $f_0 : H_*A \to H_*B$ (where H_*A and H_*B have the structure induced in homology).

We want to understand the obstruction to the existence of a morphism $\phi : A \to B$ in the homotopy category of $\tilde{\mathsf{P}}$-algebras such that $H_*\phi = f_0$.

4.2.1 Outline of the study

We will proceed in the following way:

We first show in Section 4.2.2 that we can restrict our study to the case where the differentials of A and B are trivial, and we give some results concerning the structures induced in homology. We consider the cooperad $\mathsf{D} = B(\mathsf{P} \boxtimes \mathsf{E})$. In Section 4.2.4, we want to construct a D-coalgebra map $\phi_f : (\mathsf{D}(A), \partial_\alpha) \to (\mathsf{D}(B), \partial_\beta)$ extending f_0. We notice that the obstruction to the construction of ϕ_f lies in a certain cohomology group which can be identified with the first group of Γ-cohomology of H_*A with coefficients in H_*B. If ϕ_f can be constructed, then (as the construction $R_{\tilde{\mathsf{P}}}$ is functorial, see 4.1.2.4) we obtain $\tilde{\mathsf{P}}\phi_f$ which fits a diagram

$$\begin{array}{ccc} (\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_\alpha), \partial) & \xrightarrow{\tilde{\mathsf{P}}\phi_f} & (\tilde{\mathsf{P}}(\mathsf{D}(B), \partial_\alpha), \partial) \\ \downarrow \sim & & \downarrow \sim \\ A & & B \end{array} .$$

and we obtain a morphism from A to B in the homotopy category of $\tilde{\mathsf{P}}$-algebras.

4.2.2 Restriction of the hypotheses

We show here that we can reduce our study to the case where the differentials of A and B are trivial.

First, recall the following result concerning the transfer of structures:

4.2.2.1 Fact. *Let $f : A \xrightarrow{\sim} B$ be a weak equivalence of dg-modules. Suppose that B has an action of a cofibrant operad Q.*

Then A inherits the structure of a Q-algebra such that

1. *$A \xleftarrow{\sim} \cdot \xrightarrow{\sim} B$ where the morphisms are weak equivalences of Q-algebras,*
2. *$H_*(A \xleftarrow{\sim} \cdot \xrightarrow{\sim} B) = H_*f$.*

This result in the A_∞ context was already in Kadeishvili's work [Kad]. In our context, we refer to the result stated by Fresse [F4, Theorem A]. The second assertion is not made explicit in the theorem but follows immediately from the proof.

Let $H = H_*A$ be the homology of a Q-algebra A. The graded module H can be seen as a dg-module with a trivial differential, weakly equivalent to A as dg-modules. We fix a splitting $A_* = Z_*A \oplus B'_{*-1}A$, where Z_*A denote the cycles of A (and where $B'_{*-1}A$ is isomorphic to the boundaries $B_{*-1}A$). This yields a map $A \to Z_*A$, which induces a map $A \to H$ by composition with the projection $Z_*A \to H$. As we are working over a field, we can fix a section of dg-modules $s_A : H_*A \to Z_*A$ of the projection $Z_*A \twoheadrightarrow H_*A$, and thus a map $H \xrightarrow{\sim} A$.

The fact 4.2.2.1 implies that H inherits a structure of a Q-algebra such that $H \xleftarrow{\sim} \cdot \xrightarrow{\sim} A$, where the morphisms are weak equivalences of Q-algebras. This action of Q on H induces in homology an action of $H_*\mathsf{Q}$ on $H = H_*H$.

On the other hand, as H is the homology of the Q-algebra A, it inherits the structure of an algebra over $H_*\mathsf{Q}$.

4.2.2.2 Lemma. *The two actions of $H_*\mathsf{Q}$ on H defined above coincide.*

Proof. The zig-zag of Q-algebras $H \xleftarrow{\sim} \cdot \xrightarrow{\sim} A$ induces in homology the zig-zag of $H_*\mathsf{Q}$-algebras $H \xleftarrow{\cong} H_*(\cdot) \xrightarrow{\cong} H_*A$. By the second point of the fact, H (with the first action) and H_*A (with the second action) are equal as $H_*\mathsf{Q}$-algebras. □

Let B be another Q-algebra and $K = H_*B$ its homology. Let $\tilde{H}$ and $\tilde{K}$ be cofibrant replacements of H and K in the category of Q-algebras. We use the following identities

$$\mathrm{Hom}_{\mathrm{Ho}\,\mathsf{Q}-alg}(A, B) = \mathrm{Hom}_{\mathrm{Ho}\,\mathsf{Q}-alg}(H, K) = [\tilde{H}, \tilde{K}]_{\mathsf{Q}-alg}$$

where the notation $[-,-]$ refers to the homotopy classes, to restrict our study to the case of trivial differentials.

Let α denote the action of the operad Q on the dg-module A. We now make explicit the action α_1 of $H_*\mathsf{Q}$ on H_*A.

Let $Z_*\mathsf{Q}$ denote the cycles of Q. As before, we can consider a section of the homology $s_{\mathsf{Q}} : H_*\mathsf{Q} \to Z_*\mathsf{Q}$.

4.2.3 Observation. *The action α_1 can be determined by the commutation of the following diagram:*

$$\begin{array}{ccc}
H_*\mathsf{Q}(r) \otimes H_*A^{\otimes r} & \xrightarrow{\alpha_1} & H_*A\ , \\
\big\downarrow{\scriptstyle s_{\mathsf{Q}}\otimes(s_A)^{\otimes r}} & & \big\uparrow \\
Z_*\mathsf{Q}(r) \otimes Z_*A^{\otimes r} & \overset{\alpha}{\dashrightarrow} & Z_*A \\
\big\downarrow & & \big\downarrow \\
\mathsf{Q}(r) \otimes A^{\otimes r} & \xrightarrow{\alpha} & A.
\end{array}$$

where the dotted map is the restriction. The image of this restriction is included in the cycles of A.

We now consider the case where A is an algebra over $\mathsf{Q} = \tilde{\mathsf{P}} := B^c(B(\mathsf{P} \boxtimes \mathsf{E}))$, where P is a graded operad. We use the particular section $\mathsf{P} \hookrightarrow B^c(B(\mathsf{P}\boxtimes\mathsf{E}))$ given by the composite of the inclusion $\mathsf{P} \to \mathsf{P}\boxtimes\mathsf{E}$ (sending $p \in \mathsf{P}(r)$ to $p\otimes id_{\Sigma_r}$), with the obvious inclusions $\mathsf{P}\boxtimes\mathsf{E}$ to $B(\mathsf{P}\boxtimes\mathsf{E})$ and $B(\mathsf{P}\boxtimes\mathsf{E}) \to B^c(B(\mathsf{P}\boxtimes\mathsf{E}))$. The above paragraphs give an action of P on H_*A. If $\delta_A = 0$, then we identify A and H_*A, and thus we obtain the action α_1 of P on A.

4.2.4 Construction of the morphism of coalgebras

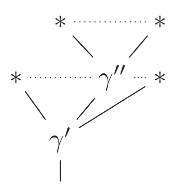
We can now study our problem. We are given

- a differential graded operad P such that δ_{P}=0,
- two algebras, A and B, over $\tilde{\mathsf{P}} = B^c(B(\mathsf{P} \boxtimes \mathsf{E}))$, with actions denoted by α and β, with trivial differentials,
- a P-algebra morphism $f_0 : (H_*A, \alpha_1) \to (H_*B, \beta_1)$.

In this section, we do not distinguish A (resp. B) and H_*A (resp. H_*B) as they are equal as dg-modules. We specify the structure (α or α_1, β or β_1) when we consider them as algebras over $\tilde{\mathsf{P}}$ or P.

We want to define a morphism ϕ_f of D-coalgebras from $(\mathsf{D}(A), \partial_\alpha)$ to $(\mathsf{D}(B), \partial_\beta)$ such that the first component for a certain graduation is f_0. Recall from Section 4.2.1 that such a morphism ϕ_f will induce a morphism from A to B in the homotopy category. The morphism $\phi_f : (\mathsf{D}(A), \partial_\alpha) \to (\mathsf{D}(B), \partial_\beta)$ will be the morphism induced by $f : \mathsf{D}(A) \to B$, as defined in Proposition 4.1.2.5.

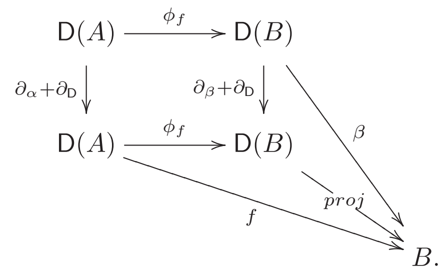
We use the graduation of $\mathsf{D} = B(\mathsf{P} \boxtimes \mathsf{E})$ given by the sum of the bar weight and the degree in E. This graduation of D induces a splitting $\mathsf{D}(A) = \bigoplus_d \mathsf{D}_{[d]}(A)$ (we do not take into account any degree of A or weight in A). The quadratic coproduct ν_2 on D sends $\gamma \in \mathsf{D}_{[d+1]}$ to composites



such that $\gamma' \in \mathsf{D}_{[p]}$, $\gamma'' \in \mathsf{D}_{[q]}$ and $p + q = d + 1$.

We want to construct the map f by induction on the degree. We notice that in degree zero, $\mathsf{D}_{[0]}(A)$ is reduced to A and thus we define $f_{[0]} = f_0$ (remember we want ϕ_f to realize f_0).

The morphism ϕ_f must fit the following commutative diagram:

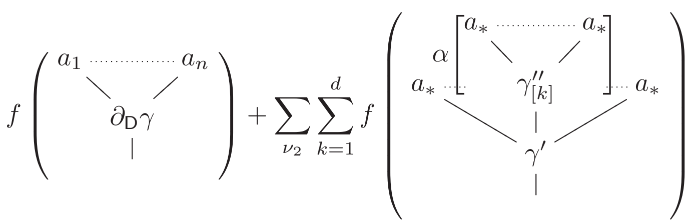


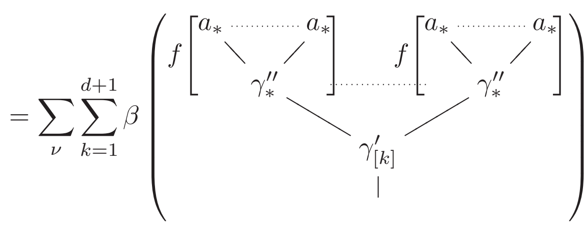
The triangle on the right obviously commutes. The commutation of the triangle on the left defines f, the restriction of ϕ_f at the target. The commutation of the exterior diagram is equivalent to the commutation of the inner square.

The commutation of this diagram is equivalent to the equation:

$$f \circ (\partial_\mathsf{D} + \partial_\alpha) = \beta \circ \phi_f. \tag{4.1}$$

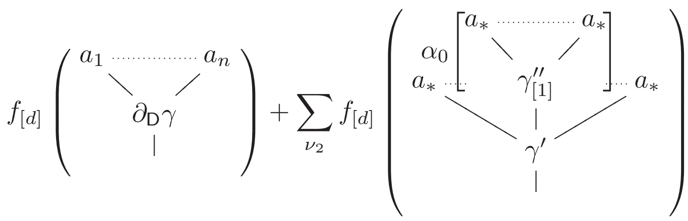
We now suppose that f is defined for degrees smaller than d and we consider an element $\gamma(a_1, \ldots, a_n)$ where γ lies in $\mathsf{D}_{[d+1]}$. For this element, Equation (4.1) is equivalent to

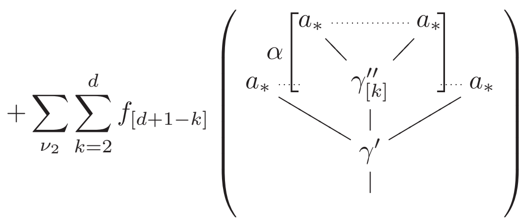




where $\gamma'_{[k]}$ and $\gamma''_{[k]}$ denote elements in $\mathsf{D}_{[k]}$.

Specifying the degrees of f and taking the terms for $k = 1$ out of the sums, we get:





$$= \beta_1 \left(\begin{array}{ccc} & f_{[d]}\left[\begin{array}{ccc} a_* & \cdots & a_* \\ & \diagdown \; \diagup & \\ & \gamma''_{[d]} & \end{array}\right] & \\ f_0 a_* \cdots & & \cdots f_0 a_* \\ & \gamma'_{[1]} & \\ & | & \end{array} \right) + \sum_{\nu} \sum_{k=2}^{d+1} \beta \left(\begin{array}{ccc} f\left[\begin{array}{ccc} a_* & \cdots & a_* \\ & \diagdown \; \diagup & \\ & \gamma''_* & \end{array}\right] & \cdots & f\left[\begin{array}{ccc} a_* & \cdots & a_* \\ & \diagdown \; \diagup & \\ & \gamma''_* & \end{array}\right] \\ & \gamma'_{[k]} & \\ & | & \end{array} \right).$$

The last sum of the left hand side and the last sum of the right hand side involve f in degrees smaller than d, while the three other terms involve f only in degree exactly d. The second and fourth terms involve respectively α_0 and β_0, as only the restricted structure matters for elements in degree 0.

Thus we write the above equation in the following form:

$$f_{[d]} \left(\begin{array}{ccc} a_1 & \cdots & a_n \\ & \diagdown \; \diagup & \\ & \partial_{\mathsf{D}}\gamma & \\ & | & \end{array} \right) + \sum_{\nu_2} f_{[d]} \left(\begin{array}{ccc} & \alpha_0\left[\begin{array}{ccc} a_* & \cdots & a_* \\ & \diagdown \; \diagup & \\ & \gamma''_{[1]} & \end{array}\right] & \\ a_* \cdots & & \cdots a_* \\ & \gamma' & \\ & | & \end{array} \right)$$

$$- \sum_{\nu} \beta_1 \left(\begin{array}{ccc} & f_{[d]}\left[\begin{array}{ccc} a_* & \cdots & a_* \\ & \diagdown \; \diagup & \\ & \gamma''_{[d]} & \end{array}\right] & \\ f_0 a_* \cdots & & \cdots f_0 a_* \\ & \gamma'_{[1]} & \\ & | & \end{array} \right)$$

$$= - \sum_{\nu_2} \sum_{k=2}^{d} f_{[d+1-k]} \left(\begin{array}{ccc} & \alpha\left[\begin{array}{ccc} a_* & \cdots & a_* \\ & \diagdown \; \diagup & \\ & \gamma''_{[k]} & \end{array}\right] & \\ a_* \cdots & & \cdots a_* \\ & \gamma' & \\ & | & \end{array} \right)$$

$$+ \sum_{\nu} \sum_{k=2}^{d+1} \beta \left(\begin{array}{ccc} f\left[\begin{array}{ccc} a_* & \cdots & a_* \\ & \diagdown \; \diagup & \\ & \gamma''_* & \end{array}\right] & \cdots & f\left[\begin{array}{ccc} a_* & \cdots & a_* \\ & \diagdown \; \diagup & \\ & \gamma''_* & \end{array}\right] \\ & \gamma'_{[k]} & \\ & | & \end{array} \right)$$

with f in degree d grouped in the left hand side and f in degrees smaller than d grouped in the right hand side.

According to our induction hypothesis, the right hand side is known. The left hand side can be identified with $\partial(f_{[d]}(\gamma))$ where ∂ is the differential in $\mathrm{Der}_{\tilde{\mathsf{P}}}(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_{\alpha_1}), (B, \beta_1))$ and $\gamma \in \mathsf{D}$ is identified with $1_{\tilde{\mathsf{P}}} \circ \gamma \in \tilde{\mathsf{P}}\mathsf{D}$. Note that these derivations are only for the restricted structures α_1 and β_1, and not the full structures α and β.

We have proved

4.2.5 Theorem. *If the cohomology group $H^1 \mathrm{Der}_{\tilde{\mathsf{P}}}(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_{\alpha_1}), (B, \beta_1))$ is equal to 0, we can construct $f_{[d]}$ (i.e. continue our induction), and hence ϕ_f answering the initial problem.*

We now relate this cohomology group with one group of Γ-homology:

4.2.6 Corollary. *The obstruction to the realization of morphisms lies in $H\Gamma^1_{\mathsf{P}}(H_*A, H_*B)$.*

Proof. The $\tilde{\mathsf{P}}$-algebra $\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_{\alpha_1})$ is nothing but a cofibrant replacement of (A, α_1) (cf. Proposition 4.1.2.4), so the cohomology $H^* \operatorname{Der}_{\tilde{\mathsf{P}}}(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_{\alpha_1}), (B, \beta_0))$ is the Γ-cohomology of the $\tilde{\mathsf{P}}$-algebra A with coefficients in B, for the actions α_1 and β_1. This cohomology is actually $H\Gamma^*_{\mathsf{P}}(H_*A, H_*B)$. □

4.2.7 Remarks.

- The d-th obstruction lies in $H^1 \operatorname{Der}_{\tilde{\mathsf{P}}}(\tilde{\mathsf{P}}(\mathsf{D}_{[d]}(A), \partial_{\alpha_1}), (B, \beta_1))$. Thus the total obstruction lies in $\bigoplus_d H^1 \operatorname{Der}_{\tilde{\mathsf{P}}}(\tilde{\mathsf{P}}(\mathsf{D}_{[d]}(A), \partial_{\alpha_1}), (B, \beta_1))$. Note that $\bigoplus_d H^1 \operatorname{Der}_{\tilde{\mathsf{P}}}(\tilde{\mathsf{P}}(\mathsf{D}_{[d]}(A), \partial_{\alpha_1}), (B, \beta_1))$ is included in $H\Gamma^1_{\mathsf{P}}(H_*A, H_*B)$ but has no reason to be equal.
- It is possible to work over a ring $\mathbb{K}$ instead of a field, but some additional assumptions are then necessary. We need to assume that all dg-modules over $\mathbb{K}$ are projective and that we are given sections of the maps: $H_*A \to A$ and $H_*B \to B$.

4.3 Homotopies

In this section, we consider the problem of unicity of realizations in the homotopy category. We are given

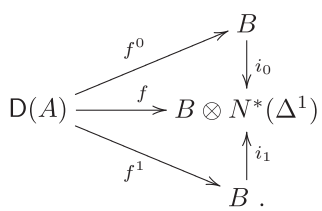
- an operad P with the canonical operadic cofibrant replacement $\tilde{\mathsf{P}} = B^c(B(\mathsf{P} \boxtimes \mathsf{E}))$
- two algebras over $\tilde{\mathsf{P}}$, (A, α) and (B, β),
- two morphisms $f^0, f^1 : \mathsf{D}(A) \to B$ realizing the same P-algebra morphism $\psi : H_*A \to H_*B$.

The morphisms f^0 and f^1 induce morphisms $\tilde{\mathsf{P}}\phi_{f^0}$ and $\tilde{\mathsf{P}}\phi_{f^1}$ from $\tilde{\mathsf{P}}\mathsf{D}(A)$ to $\tilde{\mathsf{P}}\mathsf{D}(B)$, and thus two morphisms of $\tilde{\mathsf{P}}$-algebras from A to B in the homotopy category. The question we want to study in this section is: what is the obstruction to the equality of these morphisms in the homotopy category? We show that the obstruction lies in a group of Γ-cohomology.

4.3.1 Outline of the study

We restrict our study to the case where the differentials of A and B are trivial. We consider the cooperad D defined by $B(\mathsf{P} \boxtimes \mathsf{E})$. We also consider the cylinder object $B \otimes N^*(\Delta^1)$ of B in the category of $\tilde{\mathsf{P}}$-algebras, whose action is denoted $(\beta \otimes \sigma) \circ \rho$, cf. Section 4.1.4. For this matter, we define an explicit section $\rho : \tilde{\mathsf{P}} \to \tilde{\mathsf{P}} \boxtimes \mathsf{E}$ in Section 4.3.2.

In Section 4.3.3, we wan to construct a D-coalgebra map $\phi_f : (\mathsf{D}(A), \partial_\alpha) \to (\mathsf{D}(B \otimes N^*(\Delta^1)), \partial_{(\beta\otimes\sigma)\circ\rho})$ giving a homotopy between ϕ_{f^0} and ϕ_{f^1}. Its restriction f must fit the following commutative diagram:



As in the previous section, we will construct ϕ_f by induction, and see the obstruction to the construction. Our study is very similar to the previous one, except we have to consider the cylinder object $B \otimes N^*(\Delta^1)$ instead of B itself.

4.3.2 Explicitation of a section

We define in this section an explicit operadic section $\rho : \tilde{\mathsf{P}} \to \tilde{\mathsf{P}} \boxtimes \mathsf{E}$.

Recall from [BM2] that the cobar-bar construction $B^c(B(-))$ can be identified with the cubical W-construction $W_\square(-)$. Markl and Shnider [MS] have constructed a diagonal on the W-construction: a map $W_\square(\mathsf{Q}) \xrightarrow{\Delta_{\mathsf{Q}}} W_\square(\mathsf{Q}) \boxtimes W_\square(\mathsf{Q})$ for any operad Q.

On the other hand, we easily observe that for any operads P and Q, we can identify $B^c(B(\mathsf{P} \boxtimes \mathsf{Q}))$ and $B^c(B(\mathsf{P})) \boxtimes B^c(B(\mathsf{Q}))$.

Combining these two facts, we can consider the composite :

$$\begin{array}{rcl} B^c(B(\mathsf{P} \boxtimes \mathsf{E})) & = & B^c(B(\mathsf{P})) \boxtimes B^c(B(\mathsf{E})) \\ & \stackrel{id \boxtimes \Delta_{\mathsf{E}}}{\to} & B^c(B(\mathsf{P})) \boxtimes B^c(B(\mathsf{E})) \boxtimes B^c(B(\mathsf{E})) \\ & = & B^c(B(\mathsf{P} \boxtimes \mathsf{E})) \boxtimes B^c(B(\mathsf{E})) \\ & \stackrel{id \boxtimes aug}{\to} & B^c(B(\mathsf{P} \boxtimes \mathsf{E})) \boxtimes \mathsf{E} \end{array}$$

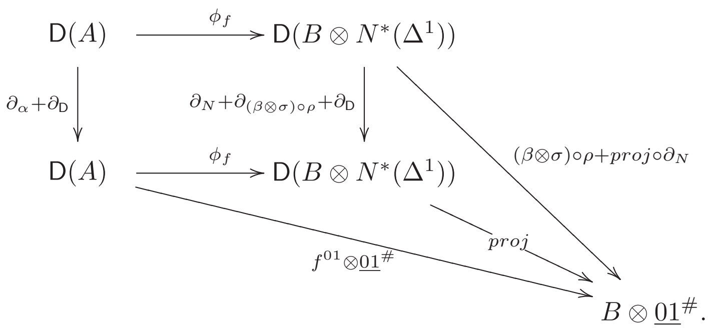
where aug denotes the augmentation $B^c(B(\mathsf{E})) \to \mathsf{E}$.

We denote this composite by $\rho : \tilde{\mathsf{P}} \to \tilde{\mathsf{P}} \boxtimes \mathsf{E}$.

4.3.3 Construction of the morphism of coalgebras

Suppose A and B are algebras over $\tilde{\mathsf{P}}$. The same argument as in Section 4.2.2 allows us to suppose their differentials are trivial. We use the same graduation as in Section 4.2.4.

The morphism ϕ_f must fit the following commutative diagram:



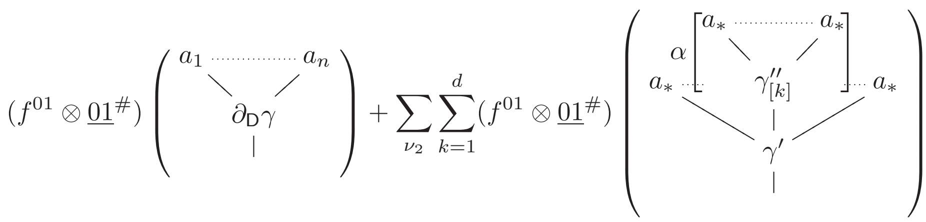
The triangle on the right obviously commutes. The commutation of the triangle on the left defines f^{01}, the restriction of f at the target in the component of $\underline{01}^{\#}$. The commutation of the exterior diagram is equivalent to the commutation of the inner square.

The commutation of this diagram is equivalent to the equation:

$$(f^{01} \otimes \underline{01}^{\#}) \circ (\partial_{\mathsf{D}} + \partial_\alpha) = (\beta \otimes \sigma) \circ \rho \circ \phi_f + (f^1 - f^0) \otimes \underline{01}^{\#}. \qquad (4.2)$$

We want to construct the map f^{01} by induction on the degree. We notice that in degree zero, $\mathsf{D}_{[0]}(A)$ is reduced to A and that $f^1_{[0]} - f^0_{[0]} = \psi - \psi = 0$. Thus we define $f^{01}_{[0]} = 0$.

We now suppose by induction that f^{01} is defined for degrees smaller than d and we consider an element $\gamma(a_1, \ldots, a_n)$ where γ lies in $\mathsf{D}_{[d+1]}$. For this element, Equation (4.2) is equivalent to



$$= \sum_{\substack{\nu \\ \epsilon_* \in \{0,1,01\}}} \sum_{k=1}^{d+1} (\beta \otimes \sigma) \circ \rho \left(\begin{array}{c} f^{\epsilon_1} \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon}_1^{\#} \quad \cdots\cdots \quad f^{\epsilon_r} \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon}_r^{\#} \\ \diagdown \qquad \diagup \\ \gamma'_{[k]} \\ | \end{array} \right)$$

$$+((f^1 - f^0) \otimes \underline{01}^{\#}) \left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \gamma \\ | \end{array} \right)$$

where $\gamma'_{[k]}$ and $\gamma''_{[k]}$ denote elements in $\mathsf{D}_{[k]}$.

The main difficulty in this equation (and the main difference with the equation of the previous section) comes from the term

$$\sum_{\nu} \sum_{k=1}^{d+1} (\beta \otimes \sigma) \circ \rho \left(\begin{array}{c} f^{\epsilon_1} \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon}_1^{\#} \cdots\cdots f^{\epsilon_r} \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon}_r^{\#} \\ \diagdown \qquad \diagup \\ \gamma'_{[k]} \\ | \end{array} \right).$$

If γ' is in $\mathsf{D}_{[k]}, k \geq 2$, then the maps f^{01} appearing in this term are applied to elements $\gamma''_{[\ell]}$ with $\ell \leq d-k$. Thus these terms are already known, according to the induction hypothesis.

If $\gamma' = p \otimes \pi$ is in $\mathsf{D}_{[1]}$, we first notice that $\rho(p \otimes \pi) = (p \otimes \pi) \otimes \pi$ for $p \otimes \pi \in \mathsf{P} \boxtimes \mathsf{E} \subset \tilde{\mathsf{P}}$. Then we can rewrite the term for $k = 1$ as

$$\beta \left(\begin{array}{c} f^{\epsilon_1} \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \cdots\cdots f^{\epsilon_r} \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \\ \diagdown \qquad \diagup \\ p \otimes \pi \\ | \end{array} \right) \otimes \sigma(\pi, \underline{\epsilon}_1^{\#}, \ldots, \underline{\epsilon}_r^{\#})$$

with p in P and π in E_0. Exactly one of the $\underline{\epsilon}^{\#}$ has to be $\underline{01}^{\#}$ so that this term arrives in $B \otimes \underline{01}^{\#}$ (cf. the description of the action of E_0 on $N^*(\Delta^1)$ in Section 4.1.3). Thus there is only one map f^{01} involved. If this map f^{01} is applied to an element $\gamma''_{[\ell]}$ with $\ell \leq d-1$, the term is known. If this map f^{01} is applied to an element $\gamma''_{[\ell]}$ with $\ell = d$, we know that all other γ'' must be in degree 0, and thus the f^{ϵ} applied to these γ'' are just ψ.

Thus we rewrite Equation (4.2) as

$$(f^{01} \otimes \underline{01}^{\#}) \left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \partial_{\mathsf{D}} \gamma \\ | \end{array} \right) + \sum_{\nu_2} (f^{01}_{[d]} \otimes \underline{01}^{\#}) \left(\begin{array}{c} \alpha \begin{bmatrix} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \gamma''_{[1]} \\ | \end{bmatrix} \\ a_* \cdots \qquad\qquad \cdots a_* \\ \diagdown \qquad \diagup \\ \gamma' \\ | \end{array} \right)$$

$$
-\sum_{\substack{\nu \\ \epsilon_* \in \{0,1\}}} (\beta\otimes\sigma)\circ\rho \left(\begin{array}{c} \psi\begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon_1}^{\#} \quad f^{01}_{[d]} \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{01}^{\#} \quad \psi \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon_r}^{\#} \\ \diagdown \qquad | \qquad \diagup \\ \gamma'_{[1]} \\ | \end{array} \right)
$$

$$
= \sum_{\substack{\nu \\ \epsilon_* \in \{0,1\}}} \sum_{\ell=0}^{d-1} (\beta\otimes\sigma)\circ\rho \left(\begin{array}{c} f^{\epsilon_1}\begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon_1}^{\#} \quad f^{01}_{[\ell]} \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{01}^{\#} \quad f^{\epsilon_r} \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon_r}^{\#} \\ \diagdown \qquad | \qquad \diagup \\ \gamma'_{[1]} \\ | \end{array} \right)
$$

$$
+ \sum_{\substack{\nu \\ \epsilon_* \in \{0,1,01\}}} \sum_{k=2}^{d+1} (\beta\otimes\sigma)\circ\rho \left(\begin{array}{c} f^{\epsilon_1}\begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon_1}^{\#} \quad \cdots \quad f^{\epsilon_r} \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon_r}^{\#} \\ \diagdown \qquad \diagup \\ \gamma'_{[k]} \\ | \end{array} \right)
$$

$$
-\sum_{\nu_2} \sum_{k=2}^{d} (f^{01}_{[d+1-k]} \otimes \underline{01}^{\#}) \left(\begin{array}{c} \alpha \begin{bmatrix} a_* \cdots a_* \\ \diagdown \quad \diagup \\ \gamma''_{[k]} \end{bmatrix} \\ a_* \cdots \qquad | \qquad \cdots a_* \\ \diagdown \qquad \diagup \\ \gamma' \\ | \end{array} \right) + ((f^1 - f^0)\otimes \underline{01}^{\#}) \left(\begin{array}{c} a_1 \cdots a_n \\ \diagdown \quad \diagup \\ \gamma \\ | \end{array} \right).
$$

The last big sum of the left hand side can be simplified. Actually, for a given $\gamma'_{[1]} = p\otimes\pi$, we have

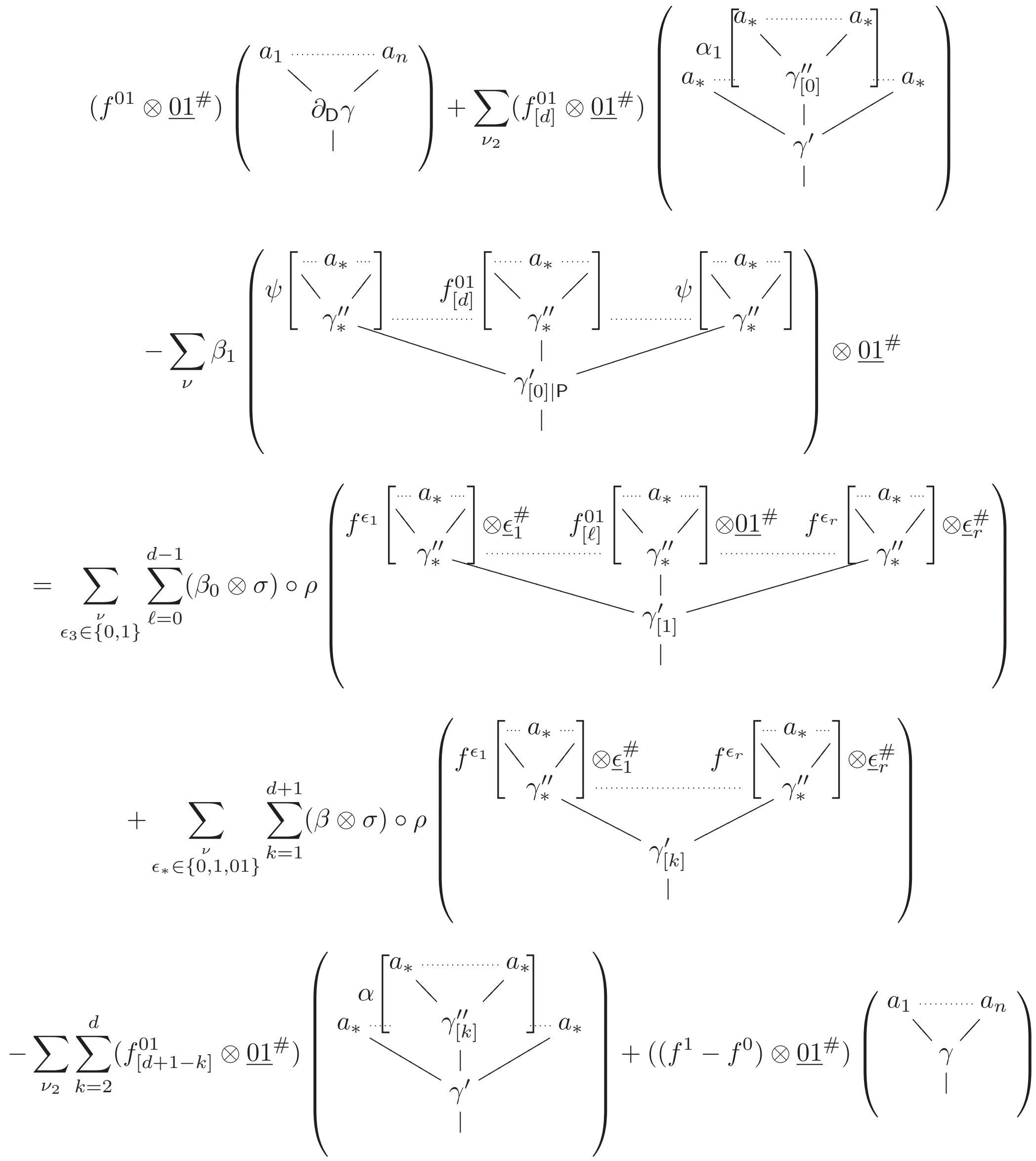
$$
(\beta\otimes\sigma)\circ\rho \left(\begin{array}{c} \psi\begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon_1}^{\#} \quad f^{01}_{[d]} \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{01}^{\#} \quad \psi \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon_r}^{\#} \\ \diagdown \qquad | \qquad \diagup \\ \gamma'_{[1]} \\ | \end{array} \right)
$$

$$
= \beta \left(\begin{array}{c} \psi\begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \cdots f^{01}_{[d]} \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \cdots \psi \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \\ \diagdown \qquad | \qquad \diagup \\ p\otimes\pi \\ | \end{array} \right) \otimes \sigma(\pi, \epsilon_1^{\#}, \ldots, \underline{01}^{\#}, \ldots, \epsilon_r^{\#})
$$

$$
= \beta_1 \left(\begin{array}{c} \psi\begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \cdots f^{01}_{[d]} \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \cdots \psi \begin{bmatrix} \cdots a_* \cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \\ \diagdown \qquad | \qquad \diagup \\ p \\ | \end{array} \right) \otimes \sigma(\pi, \epsilon_1^{\#}, \ldots, \underline{01}^{\#}, \ldots, \epsilon_r^{\#}).
$$

Only one choice of ϵ's will give a non-zero term: the one where after composition with the permutation π, the sequence is $(\underline{0}^{\#}, \ldots, \underline{0}^{\#}, \underline{01}^{\#}, \underline{1}^{\#}, \ldots, \underline{1}^{\#})$, according to the action of E_0 on $N^*(\Delta^1)$.

Thus we finally get



where $\gamma'_{[1]|\mathsf{P}}$ denotes the component in P of $\gamma'_{[1]} \in \mathsf{P} \boxtimes \mathsf{E}$.

All the terms in the right hand side are already known. The left hand side can be identified with $\partial(f^{01}_{[d]}(\gamma) \otimes \underline{01}^{\#})$ where ∂ is the differential in $\mathrm{Der}_{\tilde{\mathsf{P}}}(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_{\alpha_1}), (B \otimes \underline{01}^{\#}, \beta_1))$. Note that these derivations are only for the restricted structures α_1 and β_1, and not the full structures α and β.

We have proved

4.3.4 Theorem. *If the cohomology group $H^1 \mathrm{Der}_{\tilde{\mathsf{P}}}(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_{\alpha_1}), (B \otimes \underline{01}^{\#}, \beta_1))$ is equal to 0, we can construct $f^{01}_{[d]}$ (i.e. continue our induction), and hence ϕ_f answering the initial problem.*

We now relate this cohomology group with one group of Γ-cohomology:

4.3.5 Corollary. *The obstruction to the existence of a homotopy of two realizations of a morphism lies in $H\Gamma^0_{\mathsf{P}}(H_*A, H_*B)$.*

Proof. The proof is almost the same as the proof of Theorem 4.2.6. The only difference is that working with $B \otimes \underline{01}^{\#}$ instead of B creates a shift in the degree of the group of cohomology. □

Remarks similar to the ones in Section 4.2.7 for the realization of morphisms can also be stated for the unicity of the realizations.

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Résumé

Cette thèse s'inscrit dans l'étude des catégories d'algèbres associées aux opérades. On développe des outils d'algèbre homologique et une méthode générale de classification (à homotopie près) des morphismes entre algèbres sur une opérade.

La dualité de Koszul des opérades, introduite par V. Ginzburg et M. Kapranov, permet de construire des théories homologiques appropriées pour des catégories d'algèbres associées à certaines bonnes opérades – les opérades de Koszul. On donne dans la première partie de cette thèse un critère effectif pour qu'une opérade soit de Koszul : on montre qu'une opérade, linéairement engendrée par une base, est de Koszul dès lors que l'on peut ordonner sa base de façon compatible avec la structure de composition opéradique – on parle alors d'opérade de Poincaré-Birkhoff-Witt.

La théorie originale de Ginzburg-Kapranov s'applique en caractéristique nulle seulement. On construit une théorie homologique adaptée - la Gamma-homologie - pour l'étude des catégories d'algèbres différentielles graduées associées à une opérade de Koszul en toute caractéristique. Cette théorie généralise la Gamma-homologie définie par A. Robinson et S. Whitehouse pour la catégorie des algèbres commutatives.

On montre que la Gamma-homologie opéradique contient l'obstruction à la réalisation de morphismes entre algèbres sur une opérade, ainsi que l'obstruction à la réalisation d'homotopies entre morphismes, et donne de la sorte un outil général pour classifier les morphismes entre algèbres sur une opérade.

Mots-clés: Opérades, algèbres, dualité de Koszul, théories homologiques, théories d'obstruction.

Abstract

This thesis is concerned with the study of categories of algebras associated to operads. We develop tools of homological algebra and a general method to classify morphisms in the homotopy category of algebras over an operad.

The Koszul duality of operads, introduced by V. Ginzburg and M. Kapranov, allows us to construct suitable homology theories for categories of algebras associated to some good operads – the Koszul operads. We give in the first part of this thesis an effective criterion to prove that an operad is Kozul : we show that an operad, linearly generated by a basis, is Koszul as soon as we can order its basis compatibly with the operadic composition structure – we call such operads Poincaré-Birkhoff-Witt operads.

The original theory of Ginzburg and Kapranov works in characteristic zero only. We construct a homology theory - the Gamma-homology - for the study of the categories of the differential graded algebras associated to a Koszul operad in any characteristic. This theory generalizes the Gamma-homology introduced by A. Robinson and S. Whitehouse for the category of commutative algebras.

We show that our Gamma-homology contains the obstruction to the realization of morphisms between algebras over an operad, and also the obstruction to the realization of homotopies between morphisms. We obtain in this way a general tool to classify morphisms between algebras over an operad.

Keywords: Operads, algebras, Koszul duality, homology theories, obstruction theories.