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ZHOU Shuang

ETUDES SUR LA SOMMABILITE DE LA SOLUTION FORMELLE DE L'EQUATION DE LA CHALEUR AVEC UNE CONDITION INITIALE SINGULIERE ET SUR DES FONCTIONS INTEGRALES DU TYPE MORDELL

Thèse dirigée par **CHEN Hua (Prof. Wuhan University) / ZHANG Changgui**

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Jury :
Président : XU Chaojiang
Membres : ZHANG Changgui
CHEN Hua
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ZHAO Huijiang
CHEN Wenyi
SAULOY Jacques

Studies on Summability of Formal Solution to a Cauchy Problem and on Integral Functions of Mordell's Type

Dr. Candidate: Zhou Shuang

Supervisor: Prof. Chen Hua & Prof. Zhang Changgui

School of Mathematics and Statistics

Wuhan University

&

UFR de Mathématiques Pures et Appliquées

Université des Sciences et Technologies de Lille

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Abstract

From formal solutions of ordinary or partial differential equations, one may give different sums by different summation processes. This phenomenon occurs for functional equations such as difference or q-difference equations. In this thesis, we shall consider the heat equation with a singular initial condition $\varphi(z) = \frac{1}{1-e^z}$, $z \in \mathbb{C} \setminus \{2\pi\mathbb{Z}i\}$. The aim is to give three sums of a divergent formal solution to this Cauchy problem: Borel-sum based on known results in [26] and two q-Borel sums obtained by means of Heat kernel and Jacobi theta function respectively (cf. [50] and [42, 51]) and establish relations among them. More specifically, this thesis consists of the following six chapters.

In Chapter 1, we introduce some known results on summability of formal solutions, state our problem and main conclusions, and recall how to solve Cauchy problem for the real Heat equation with Heat kernel.

In Chapter 2, we introduce the classic Borel-Laplace summation and show the theorem on the finely Borel sum of divergent solutions of the complex Heat equation by Lutz, Miyake and Sch$\ddot{a}$fke (cf. [26]), and obtain the finely Borel sum of the formal solution to our problem.

In Chapter 3, we introduce the so-called Gq-summation (cf. [50]). By variable substitutions, we can transfer the divergent formal solution to our Cauchy problem into a q-series. Then we obtain a q-Borel sum based on Heat kernel and compare the sum function defined in the previous sections.

In Chapter 4, we firstly prove some properties of the Jacobi theta function and introduce a method of summation based on Jacobi theta function (cf. [51]), and then get the other q-Borel sum of the q-series.

In Chapter 5, we study integral functions which have been considered

by Riemann, Kronecker, Lerch, Hardy, Ramanujan, Mordell and many other mathematicians. We say that Mordell's theorem (cf. [34, 35]) about these integrals can be deduced from one of our main theorems. And we can apply our ideas mentioned above to the more general cases.

In Chapter 6, we sum up in a few sentences and provide some unsolved problems.

Key Words: Gevrey asymptotic expansion, Borel summability, q-difference equation, q-Gevrey asymptotic expansion, Gq-summability, Heat kernel, Jacobi theta function, Mordell's theorem.

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Chapter 1 Introduction

§1.1 Background and basic notations

§1.1.1 Background

Asymptotic expansions have been an important and very successful tool to understand the structure of solutions of ordinary and partial differential (or difference) equations. By now the classical part of this theory has been presented in many books on differential equations in the complex plane or related topics, by such distinguished authors as Wasow [47], Sibuya [44], and many others. Personally, one of the most important results in this context is the Main Asymptotic Existence theorem: it states that to every formal solution of a differential equation, and every sector in the complex plane of sufficiently small opening, one can find a solution of the equation having the formal one as its asymptotic expansion. This solution, in general, is not uniquely determined, and the proofs given for this theorem do not provide a truly efficient way to compute such a solution, say, in terms of the formal solution. In fact, to prove this result, even for linear, but in particular non-linear equations, and to determine sharp bounds for the opening of the sector is not an easy task and many researchers have made contribution to the proofs; see, e.g., Ramis and Sibuya's paper on Hukuhara domains [40] of 1989, or Jurkat's discussion of asymptotic sectors [25].

In the general theory of asymptotic expansions, the analogue to the Main Asymptotic Existence theorem is usually called Ritt's theorem, and is much easier to prove: given any formal power series and any sector of arbitrary (but finite) opening on the Riemann surface of the logarithm, there exists a function, analytic in this sector and having the formal power series as its asymptotic expansion. This function is never uniquely determined - not even when the power series converges. To overcome this non-uniqueness, Watson (cf. [48, 49]) in 1911/12, and Nevanlinna (cf. [36]) in 1918, introduced a special kind of asymptotic expansions, now commonly called *Gevrey asymptotic expansions of order* $k > 0$. These have the property that analogue to Ritt's theorem holds for sectors of opening up to π/k, in which cases the function again is not uniquely determined. However, if the opening is large than π/k, a function which has a given formal power series as its Gevrey asymptotic expansion of order $k > 0$ may not exist, but if it does, then it is uniquely determined. In case of existence, the function can be represented as Lapalce transform of another function, which is analytic at the origin, and whose power series expansion is explicitly given in terms of the formal power series.

In 1978/80, Ramis (cf. [38, 39]) introduced his notion of k-*summability* of formal power series. Applying this to linear systems of (meromorphic) ODE, he proved that every formal solution to every such equation can be factored into a finite product of power series (times some explicit functions), so that each factor is k-*summable*, with k depending upon the factor. This factorization of formal solutions is not truly effective, so that this result did not really give a way to compute the resulting function from the formal series. Then Ecalle (cf. [15, 16]) presented a way to achieve this computation, introducing his definition of *multisummability*. In a way, his method differs from Ramis' definition of k-summability by cleverly enlarging the class of functions to which Laplace transform, in some weak form, can be applied.

Here to better understand this theory, we give the most simple example: the Euler series

$$\hat{y}(x) = \sum_{n\geq 0}(-1)^n n! x^{n+1},$$

which is a divergent formal solution to the following linear ODE of first order

$$x^2y' + y = x.$$

The power series is Gevrey of order 1 and 1-summable (or Borel summable). We know that its Borel sum is defined by

$$\int_0^{\infty e^{id}} \frac{e^{-\xi/x}}{1+\xi} d\xi$$

over an open sector

$$S(d,\pi) := \{x \in \mathbb{C}^* \mid |\arg x - d| < \frac{\pi}{2}\},$$

for any direction $d \in (-\pi, \pi)$. The integral function admits the Euler series as its Gevrey asymptotic expansion of order 1 and is also a solution of the above ODE.

By now, it is well known that every formal solution to an ordinary differential equation are multisummable (see Braaksma [2], Basler [7] and Ramis and Sibuya [41]). The multisummbility index can be calculated by means of the Newton polygon associated to the linearized equation along the formal power series solution. Further, the Gevrey type expansion solutions and summability or multisummability have be studied for other functional equations: finite difference equations, q-difference equations, in particular partial differential equations, etc. But for partial differential equation this problem is more complicated. Only a few results are obtained about the summability of the divergent formal series. The first result on summability of formal solutions was from Lutz, Miyake and Schäfke (cf. [26]). They showed that the formal solution to the Cauchy problem for the 1-dimensional homogeneous complex heat equation is 1-summable (or Borel summable) in a sector of direction θ and opening angle more than π if and only if the Cauchy data can be analytically continued to infinity in a domain consisting two sectors of directions $\theta/2$ and $\pi + \theta/2$, and the continuation is of exponential growth of order at most 2. In their case, the Newton polygon admits a unique strictly positive slope 1. Analogous result for more general initial data was given by Balser (cf. [4]). The multidimensional homogeneous heat equation was investigated by Balser and Malek (cf. [6]) and by Michalik (cf. [30, 31]). And Hibino [22], Ichinobe [23, 24], $\bar{O}$uchi [37] and others studied more general classes of linear partial differential equations. In [9, 10, 11, 18, 27], Gérard, Tahara, Chen, Luo and Zhang considered Cauchy problems of a class of nonlinear first order or even higher order, PDEs with irregular singularities on the complex domain $\mathbb{C}_t \times \mathbb{C}_x$. They proved respectively that the corresponding formal solutions are holomorphic, or have a calculable Gevry index, or are k-summable, when these PDEs satisfies different conditions.

In the last fifteen years, analogous summation theories for q-difference equations have been developed (cf. [50, 29, 51, 42, 43, 12, 46, 52]). Trjitzinsky, Ramis, Sauloy, Zhang and so on have done a lot of work in this field. For a calss of singular q-difference equations, their formal solutions can be expected to be q-*Gevrey of order* (*optimal*) s, introduced by Bézivin [8]. Ramis and Zhang gave some notions of q-Gevery asymptotic expansion and Gq-summability. Di Vizio and Zhang (cf. [13]) replaced, in the classical Laplace integral, the exponential function by using a q-exponential function or Jacobi theta function. Then they got four different q-Borel sums of a q-deformation of the Euler series, by using a usual integral or a discrete q-analogue. And they also studied the relations between the different kind of q-Borel sums considered in the literature. As they said, their work is a first step towards the proof of a general result for a divergent solution of a q-difference equations, having a Newton polygon with more than one slope. Later on, Ramis, Sauloy and Zhang classified analytically isoformal q-difference equations based on the theory of summation.

§1.1.2 Basic notations

We use the following notations:

$\mathbb{R}^n$: the set of n-dimensional real vectors;

$\mathbb{R}_+$: the set of nonnegative real numbers;

$\mathbb{C}$: the set of complex numbers;

$\mathbb{C}^*$: the set of complex numbers except for $\{0\}$;

$\mathbb{N}$: the set of natural numbers;

$\mathbb{N}^*$: the set of natural numbers except for $\{0\}$;

$\mathbb{Z}$: the set of of integers;

$\tilde{\mathbb{C}}^*$: the Riemann surface of the logarithm;

$C(\mathbb{R}^n)$: the set of continuous functions on $\mathbb{R}^n$;

$C^\infty(\mathbb{R}^n)$: the set of smooth functions on $\mathbb{R}^n$;

$L^\infty(\mathbb{R}^n)$: the set of essentially bounded measurable functions on $\mathbb{R}^n$.

The definitions of other notations appeared later will given in corresponding chapters.

§1.2 Problems and main results

From formal solutions of ordinary or partial differential equations, one may give different solutions by different summation processes. This phenomenon occurs for functional equations such as difference or q-difference equations. Our motivation is to reveal the relationship existing between different sums of a divergent power series which is related to a Cauchy problem of the Heat equation and which may also be viewed as a solution to a singular q-difference equation. And we also want to use our results in more complicated cases.

More precisely, let us consider the following Cauchy problem for the complex heat equation

$$\begin{cases} \partial_\tau u - \partial_z^2 u = 0, \\ u(0,z) = \varphi(z) = \frac{1}{1-e^z}, \end{cases} \tag{1.1}$$

where $(\tau, z) \in \mathbb{C} \times \mathbb{C}$, and $\varphi(z)$ is defined on $\mathbb{C} \setminus \{2\pi\mathbb{Z}i\}$ $(:= \mathbb{C} \setminus \{2k\pi i \mid k = 0, \pm 1, \pm 2, \cdots\})$. It has a unique formal power series solution

$$\hat{u}(\tau, z) := \sum_{n=0}^{\infty} \frac{\varphi^{(2n)}(z)}{n!} \tau^n, \tag{1.2}$$

which is divergent in τ and holomorphic in z. On the other hand, if we consider the series expansion $\hat{\varphi}(z) := \sum\limits_{n=0}^{\infty} e^{nz}$ of $\varphi(z)$ in e^z, then Problem (1.1) has a formal solution

$$\sum_{n=0}^{\infty} e^{n^2\tau + nz},$$

which takes the form of $\sum\limits_{n=0}^{\infty} q^{-n^2} x^n$ and then satisfies the q-difference equation

$$\frac{x}{q} y(\frac{x}{q^2}) - y(x) = -1, \tag{1.3}$$

provided that $q = e^{-\tau}$ and $x = e^z$. If suppose that $\mathrm{Re}(\tau) > 0$, then $0 < |q| < 1$ and the q-series

$$\hat{y}(x, q) := \sum_{n=0}^{\infty} q^{-n^2} x^n \tag{1.4}$$

is divergent for all $x \in \mathbb{C}^*$. Utilizing the results from [50] and [42, 51] on the singular q-difference equation, one can give two different q-Borel sums of $\hat{y}(x,q)$: $f_\alpha(x,q)$ and $g_\lambda(x,q)$ (see (3.2) and (4.3)), by which one shall get two sums of $\hat{u}(\tau, z)$, denoted by $F_\alpha(\tau, z)$ and $G_\lambda(\tau, z)$ (see (3.5) and (4.5)).

Following the classical Borel summation method with respective to τ, one can obtain the Borel sum of the power series $\hat{u}(\tau, z)$ (cf. [26]). Here, we only give the definition of the fine Borel sum $U_k(\tau, z)$ (see (2.11)) in the direction of the real axis, where $\mathrm{Re}(\tau) > 0$ and

$$z \in \Omega_k := \{z \in \mathbb{C} \mid 2k\pi < \mathrm{Im}(z) < 2(k+1)\pi\} \text{ for some } k \in \mathbb{Z}.$$

Then our main results are summarized as follows.

Proposition A (**Proposition 2.3.4**) *Given $k \in \mathbb{Z}$. For $Re(\tau) > 0$ and $z \in \Omega_k$, we define*

$$U_k(\tau, z) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4\tau}} \frac{1}{1 - e^{z+s}} ds.$$

Then we have

(i) $U_k(\tau, z)$ is holomorphic over $\{\tau \in \mathbb{C} \mid Re(\tau) > 0\} \times \Omega_k$.

(ii) $U_k(\tau, z)$ is the unique solution of (1.1) which admits $\hat{u}(\tau, z) = \sum_{n=0}^{\infty} \frac{\varphi^{(2n)}(z)}{n!}\tau^n$ as asymptotic expansion in the sense: for some compact subset O' of $\{\tau \in \mathbb{C} \mid Re(\tau) > 0\}$ and some compact subset Ω_k' of Ω_k, there exist positive constants C, K and δ, such that the following asymptotic estimates hold:

$$\left| U_k(\tau, z) - \sum_{n=0}^{N-1} \frac{\varphi^{(2n)}(z)}{n!} \tau^n \right| \leq C K^N N! \, |\tau|^N, \qquad (\tau, z) \in O' \times \Omega_k',$$

for all $N = 1, 2, 3, \ldots$.

Theorem B (**Theorem 3.2.4**) *For any given $k \in \mathbb{Z}$, we have $U_k(\tau, z) = F_\alpha(\tau, z)$ where $\alpha \in (2k\pi, 2(k+1)\pi)$, $Re(\tau) > 0$ and $z \in \Omega_k$.*

Theorem C (**Theorem 5.2.1**) *The following relation holds for all $\alpha \in (-2\pi, 0)$, $\lambda \in \mathbb{C}^* \setminus \{q^{2\mathbb{Z}}\}$ and $x \in \mathbb{C}^* \setminus \{-\lambda q^{2\mathbb{Z}+1}\}$:*

$$f_\alpha(x, q) = g_\lambda(x, q) - i\sqrt{\frac{\pi}{\log 1/q}} e^{\frac{(\log x)^2}{4 \log q}} g_{\lambda^*}(x^*, q^*),$$

where q^*, x^* and λ^* are the modular variables defined by

$$q^* = e^{\pi^2/\log q}, \quad x^* = e^{-\pi i \frac{\log x}{\ln q}}, \quad \lambda^* = e^{-\pi i \frac{\log \lambda}{\log q}}.$$

Replacing λ by $\frac{1}{q}e^{\pi i}$ in Theorem B, one can get the following Mordell's result in [35]:

Corollary D (**Corollary 5.2.2**) (**Mordell's theorem**) *Let f be the integral function of x defined by the series and θ_{11} be the following Jacobi theta function*

$$if(x,\omega) = \sum_{m\ odd}^{\pm\infty} \frac{(-1)^{\frac{1}{2}(m-1)} q^{\frac{1}{4}m^2} e^{m\pi i x}}{1+q^m},$$

$$i\theta_{11}(x,\omega) = \sum_{m\ odd}^{\pm\infty} (-1)^{\frac{1}{2}(m-1)} q^{\frac{1}{4}m^2} e^{m\pi i x}.$$

Then

$$\int_{-\infty}^{\infty} \frac{e^{\pi i \omega t^2 - 2\pi t x}}{e^{2\pi t}-1} dt = \frac{f(\frac{x}{\omega}, -\frac{1}{\omega}) + i\omega f(x,\omega)}{\omega \theta_{11}(x,\omega)},$$

where the path of integration may be taken as a straight line parallel to the real axis of t and below it at a distance less than unity.

Finally, we shall treat some more general integral functions

$$I_k(\nu,\chi) = I_k(\nu,\chi;\omega) = -\frac{1}{\sqrt{(-i\omega)}} e^{-\frac{\pi\chi^2}{i\omega}} \int_{-\infty}^{\infty} \frac{e^{\frac{\pi t^2}{i\omega} + \frac{2\pi\chi t}{\omega}}}{(e^{2\pi t} - e^{2\pi\nu})^k} dt,$$

where $k \in \mathbb{N}$, $\nu \in \mathbb{C}$ with $\mathrm{Im}(\nu) \in (-1, 0]$ and the path of integration may be any straight line parallel to the real axis of t and just below the point ν at a distance less then unity, i.e., $(-\infty + \nu - i\epsilon, \infty + \nu - i\epsilon)$, $\epsilon \in (0,1)$, and obtain similar conclusions.

§1.3 Real Heat equation

In this section, we will recall how to get the Heat kernel on the real space and to structure a C^∞ solution to Cauchy problem for the real heat equation.

§1.3.1 Fourier transform

Definition 1.3.1. *The set of rapidly decreasing functions is defined by*

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \mid \sup_{\mathbb{R}^n} \left|\varsigma^\alpha D^\beta f(\varsigma)\right| < +\infty, \forall \alpha, \beta\}.$$

Definition 1.3.2. *Let $f(\varsigma) \in \mathcal{S}(\mathbb{R}^n)$. Define its Fourier transform by*

$$\hat{f}(\xi) = F(f)(\xi) = \int_{\mathbb{R}^n} e^{-i\varsigma\cdot\xi} f(\varsigma) d\varsigma,$$

where $\varsigma \cdot \xi = \sum_{j=1}^{n} \varsigma_j \xi_j$.

Obviously, the above integral on the right hand is convergent. Moreover, we can prove that $\hat{f}(\xi) \in \mathcal{S}(\mathbb{R}^n)$, and the mapping $F : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ has the following properties:

Theorem 1.3.3.

(1) For any $\alpha \in \mathbb{R}$ and $\forall f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$F(\alpha f + g) = \alpha F(f) + F(g).$$

(2) For any $\alpha \in \mathbb{R}$ and $\forall f \in \mathcal{S}(\mathbb{R}^n)$,

$$F(D^\alpha f)(\xi) = \xi^\alpha \hat{f}(\xi),$$

$$D_\xi^\alpha \hat{f}(\xi) = F[(-x)^\alpha](\xi).$$

So we know that the mapping $F : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is linear and continuous. In fact, it is a linear isomorphism, so the inverse mapping F^{-1} exists and continues. Before giving the definition of F^{-1}, we need the following lemma:

Lemma 1.3.4. *For the Gauss function $e^{-|\varsigma|^2/2}$, its Fourier transform*

$$\int_{\mathbb{R}^n} e^{-i\varsigma\cdot\xi} e^{-|\varsigma|^2/2} d\varsigma = (2\pi)^{n/2} e^{-|\xi|^2/2}.$$

Furthermore, we have

Theorem 1.3.5. *The mapping $F : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ has a continuous inverse mapping $F^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$;$\hat{g} \mapsto g$, where*

$$g(\varsigma) = F^{-1}(g)(\varsigma) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\varsigma\cdot\xi}\hat{g}(\xi)d\xi.$$

A linear continuous functional defined on the space $\mathcal{S}(\mathbb{R}^n)$ is called the *temperate distribution*, and the set of temperate distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$. We can define the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ by transferring the Fourier transform into the test function space. Firstly, for a "good" function $f \in \mathcal{S}(\mathbb{R}^n)$, we define

$$\begin{aligned} \langle \hat{f}, g\rangle &= \int\int e^{-i\varsigma\cdot\xi} f(\varsigma)d\varsigma g(\xi)d\xi \\ &= \langle f, \hat{g}\rangle, \qquad \forall g \in \mathcal{S}(\mathbb{R}^n). \end{aligned}$$

Then, similarly,

Definition 1.3.6. *Let $f(\varsigma) \in \mathcal{S}'(\mathbb{R}^n)$. Define its Fourier transform $\hat{f}(\xi) \in \mathcal{S}'(\mathbb{R}^n)$ as follows:*

$$\langle \hat{f}(\xi), g(\xi)\rangle = \langle f(\varsigma), \hat{g}(\varsigma)\rangle, \qquad \forall g \in \mathcal{S}(\mathbb{R}^n).$$

If only take the Fourier transform with respect to the space variable ς, it is called the *partial Fourier transform*, i.e.,

$$F_\varsigma(p)(t,\xi) = \tilde{p}(t,\xi) = \int_{\mathbb{R}^n} e^{-i\varsigma\cdot\xi} p(t,\varsigma)d\varsigma,$$

for $p(t,\varsigma) \in \mathcal{S}(\mathbb{R}^{1+n}_{t,\varsigma})$. Apparently, we have $\tilde{p}(t,\xi) \in \mathcal{S}(\mathbb{R}^{1+n}_{t,\xi})$, i.e., $F_\varsigma : \mathcal{S}(\mathbb{R}^{1+n}_{t,\varsigma}) \to \mathcal{S}(\mathbb{R}^{1+n}_{t,\xi})$. Then, as before, we can also define

$$\langle \tilde{p}(t,\xi), q(t,\xi)\rangle = \langle p(t,\varsigma), \tilde{q}(t,\varsigma)\rangle, \qquad \forall q \in \mathcal{S}(\mathbb{R}^{1+n}_{t,\xi}),$$

for $p(t,\varsigma) \in \mathcal{S}'(\mathbb{R}^{1+n}_{t,\varsigma})$. And $\tilde{p}(t,\xi) \in \mathcal{S}'(\mathbb{R}^{1+n}_{t,\xi})$.

Here we introduce the Dirac delta function $\delta(\varsigma)$ on $\mathbb{R}^n$:

$$\delta(\varsigma) = \begin{cases} 0, & \text{if } \varsigma \neq 0, \\ \infty, & \text{if } \varsigma = 0, \end{cases}$$

and it satisfies

$$\int_{\mathbb{R}^n} \delta(\varsigma) d\varsigma = 1.$$

Lemma 1.3.7. *The partial Fourier transform of Dirac delta function $\delta(t,\varsigma)$ on $\mathbb{R}^{1+n}_{t,\varsigma}$: $\tilde{\delta}(t,\xi) = 1 \cdot \delta(t)$.*

Proof. In fact, for any $q(t,\xi) \in \mathcal{S}(\mathbb{R}^{1+n}_{t,\xi})$,

$$\begin{aligned} \langle \tilde{\delta}(t,\xi), q(t,\xi) \rangle &= \langle \delta(t,\varsigma), \tilde{q}(t,\varsigma) \rangle = \tilde{q}(0,0) \\ &= \int_{\mathbb{R}^n} e^{-i\varsigma\cdot\xi} q(t,\xi) d\xi \mid_{(t,\varsigma)=(0,0)} \\ &= \int_{\mathbb{R}^n} q(0,\xi) d\xi \\ &= \langle 1 \cdot \delta(t), q(t,\xi) \rangle. \end{aligned}$$

So

$$\tilde{\delta}(t,\xi) = 1 \cdot \delta(t).$$

This completes the proof. ⊛

§1.3.2 Derivation of the fundamental solution on $(0,\infty) \times \mathbb{R}^n$

Suppose $E(t,\varsigma)$ to be the generalized function that is the fundamental solution of the Heat equation on $(0,\infty) \times \mathbb{R}^n$. So

$$(\partial_t - \Delta_\varsigma) E(t,\varsigma) = \delta(t,\varsigma). \tag{1.5}$$

We take the Fourier transform with respect to the space variable ς on both sides of (1.5) and note $\tilde{E}(t,\xi) := F_x(E)(t,\xi)$. Then by Lemma 1.3.7

$$\frac{d\tilde{E}(t,\xi)}{dt} + |\xi|^2 \tilde{E}(t,\xi) = 1 \cdot \delta(t). \tag{1.6}$$

Multiply the both sides of (1.6) by $e^{|\xi|^2 t}$ and notice that $f(\varsigma)\delta(\varsigma) = f(0)\delta(\varsigma)$. Then

$$\frac{d}{dt}(e^{|\xi|^2 t}\tilde{E}(t,\xi)) = \delta(t).$$

Because of integrability, we have $\lim_{t\to-\infty} e^{|\xi|^2 t} d\tilde{E} = 0$. Then

$$e^{|\xi|^2 t}\tilde{E}(t,\xi) \;=\; \int_{-\infty}^{t} \delta(\tau)d\tau := H(t),$$

$$\tilde{E}(t,\xi) = H(t)e^{-|\xi|^2 t}.$$

By the inverse Fourier transform with respect to ξ and Lemma 1.3.4, we obtain

$$E(t,\varsigma) = (4\pi t)^{-\frac{n}{2}} H(t) e^{-\frac{|\varsigma|^2}{4t}}.$$

Here with the appearance of the function $H(t)$, when $t < 0$, $E(t,\varsigma) \equiv 0$. It means that making use of initial data, we only solve Cauchy problem of the Heat equation when $t > 0$.

In the case of the real space, we can employ the fundamental solution E to fashion a solution to the Cauchy problem

$$\begin{cases} \partial_t u - \Delta_\varsigma u = 0, \\ u(0,\varsigma) = \psi(\varsigma), \end{cases} \tag{1.7}$$

where $(t,\varsigma) \in (0,\infty) \times \mathbb{R}^n$, and $\psi(\varsigma)$ is defined on $\mathbb{R}^n$.

It has been proved that the convolution

$$\begin{aligned} u(t,\varsigma) &= \int_{\mathbb{R}^n} E(t,\varsigma-\mu)\psi(\mu)d\mu & (1.8) \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|\varsigma-\mu|^2}{4t}} \psi(\mu)d\mu \qquad (t > 0, \varsigma \in \mathbb{R}^n) & (1.9) \end{aligned}$$

is the solution of (1.7).

Theorem 1.3.8. *(Solution of Cauchy problem) Assume* $\psi \in C(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$, *and define* u *by (1.9). Then* *(i)* $u \in C^\infty((0,\infty)\times\mathbb{R}^n)$,

(ii) $\partial_t u(t,\varsigma) - \Delta_\varsigma u(t,\varsigma) = 0$ *(*$t>0, \varsigma\in\mathbb{R}^n$*)*,

(iii) $\lim\limits_{(t,\varsigma)\to(0,\varsigma^0)} u(t,\varsigma) = \psi(\varsigma^0)$ *for each point* $\varsigma^0\in\mathbb{R}^n$.

Proof. 1. Since the function $\frac{1}{t^{n/2}}e^{-\frac{|\varsigma|^2}{4t}}$ is infinitely differentiable, with uniformly bounded derivatives of all orders, on $[\delta,\infty)\times\mathbb{R}^n$ for each $\delta>0$, we see that $u\in C^\infty((0,\infty)\times\mathbb{R}^n)$. Furthermore

$$\begin{aligned}\partial_t u - \Delta_\varsigma u &= \int_{\mathbb{R}^n}[(\partial_t E - \Delta_\varsigma E)(t,\varsigma-\mu)]\psi(\mu)d\mu \\ &= 0 \qquad (t>0,\varsigma\in\mathbb{R}^n),\end{aligned}$$

since E itself solves the Heat equation.

2. Fixed $\varsigma^0\in\mathbb{R}^n$, $\epsilon>0$. Choose $\delta>0$ such that

$$\left|\psi(\mu)-\psi(\varsigma^0)\right| < \epsilon \qquad \text{if } \left|\mu-\varsigma^0\right| < \delta, \mu\in\mathbb{R}^n. \tag{1.10}$$

Then if $|\varsigma-\varsigma^0|<\frac{\delta}{2}$, we have, according to the following lemma,

$$\begin{aligned}\left|u(t,\varsigma)-\psi(\varsigma^0)\right| &= \left|\int_{\mathbb{R}^n} E(t,\varsigma-\mu)[\psi(\mu)-\psi(\varsigma^0)]d\mu\right| \\ &\le \int_{B(\varsigma^0,\delta)} E(t,\varsigma-\mu)\left|\psi(\mu)-\psi(\varsigma^0)\right|d\mu \\ &\quad + \int_{\mathbb{R}^n-B(\varsigma^0,\delta)} E(t,\varsigma-\mu)\left|\psi(\mu)-\psi(\varsigma^0)\right|d\mu \\ &=: I+J.\end{aligned}$$

Now

$$I \le \epsilon\int_{\mathbb{R}^n} E(t,\varsigma-\mu)d\mu = \epsilon,$$

owing to (1.10) and the following lemma. Furthermore, if $|\varsigma-\varsigma^0|\le\frac{\delta}{2}$ and $|\mu-\varsigma^0|\ge\delta$, then

$$\begin{aligned}\left|\mu-\varsigma^0\right| &\le \left|\mu-\varsigma^0\right| + \frac{\delta}{2} \\ &\le \left|\mu-\varsigma^0\right| + \frac{1}{2}\left|\mu-\varsigma^0\right|.\end{aligned}$$

Thus $|\mu-\varsigma| \geq \frac{1}{2}|\mu-\varsigma^0|$. Consequently

$$\begin{aligned} J &\leq 2 \parallel \psi \parallel_{L^\infty} \int_{\mathbb{R}^n - B(\varsigma^0,\delta)} E(t,\varsigma-\mu)d\mu \\ &\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n - B(\varsigma^0,\delta)} e^{-\frac{|\varsigma-\mu|}{4t}} d\mu \\ &\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n - B(\varsigma^0,\delta)} e^{-\frac{\left|\varsigma^0-\mu\right|}{16t}} d\mu \\ &= \frac{C}{t^{n/2}} \int_\delta^\infty e^{-\frac{r^2}{16t}} r^{n-1} dr \to 0, \qquad \text{as } t \to 0^+. \end{aligned}$$

Hence if $|\varsigma-\varsigma^0| \leq \frac{\delta}{2}$ and $t>0$ is small enough, $|u(t,\varsigma)-\psi(\varsigma^0)| < 2\epsilon$. ⊛

Lemma 1.3.9. *(Integral of fundamental solution) For each time $t>0$,*

$$\int_{\mathbb{R}^n} E(t,\varsigma)d\varsigma = 1.$$

For more details, refer to Evans [17].

§1.4 Contents

The organization of the thesis is as follows:

In Chapter 1, we introduce some known results on summability of formal solutions to ordinary differential equations, partial differential equations and q-difference equations. After stating our problem and main conclusions, we recall how to solve Cauchy problem for the real Heat equation with Heat kernel.

In Chapter 2, we introduce the classic Borel-Laplace summation and show the theorem on the finely Borel sum of divergent solutions of the complex Heat equation by Lutz, Miyake and Schäfke (cf. [26]). And making use of it, we obtain the Borel sum of the formal solution to our problem.

Next, in Chapter 3, we introduce the so-called Gq-summation which is used to treat q-Gevrey power series of order 1 (cf. [50]). By variable substitutions, we can transfer the divergent formal solution to our Cauchy problem into a q-series. Then a q-Borel sum based on Heat kernel will be obtained. And also we compare the sum functions defined in the previous sections and get one of our main theorems.

In Chapter 4, we firstly prove some properties of the Jacobi theta function and introduce a method of summation based on Jacobi theta function (cf. [51]). Then we get the other sum of the q-series, which is entirely uniform but admits a spiral of simple poles.

In Chapter 5, after giving some motivations of the study of a integral function from Mordell's point of view, we say that his theorem can be deduced from our other main theorem on the relations of the two sums of the q-series as a corollary. And we can apply our ideas mentioned above to the more general cases.

Finally, in Chapter 6, we sum up in a few sentences and list some problems in consideration.

Chapter 2 Classical Borel-Laplace Summation

In this chapter, we will get the fine Borel sum of the divergent solution $\hat{u}(\tau, z)$ defined by (1.2).

§2.1 Borel summability

Here we summarize the fundamentals on Gevrey asymptotic expansions and the Borel summability for Gevrey type formal power series without the proofs, since they are essentially the same with one variable cases studied by many authors (see [39, 28, 3]). In fact, the only difference with them is that we consider the formal power series over the ring of analytic functions, not over just the complex numbers.

§2.1.1 Formal power series

$\mathcal{O}(r)$ is the ring of analytic functions on $B(r) := \{z \in \mathbb{C} \mid |z| \leq r\}$. $\mathcal{O}(r)[[\tau]]$ is the ring of formal power series in τ over the ring $\mathcal{O}(r)$, and we define $\mathcal{O}[[\tau]]$ by

$$\mathcal{O}[[\tau]] := \bigcup_{r>0} \mathcal{O}(r)[[\tau]].$$

An element $\hat{u}(\tau, z) \in \mathcal{O}[[\tau]]$ is written as

$$\hat{u}(\tau, z) = \sum_{n=0}^{\infty} u_n(z)\tau^n, \qquad u_n(z) \in \mathcal{O}(r) \text{ for some } r > 0.$$

§2.1.2 Gevrey formal power series

$\mathcal{O}(r)[[\tau]]_1$, which is called to be Gevrey of order one, is the subring of $\mathcal{O}(r)[[\tau]]$ whose coefficient satisfy the following inequalities for some positive constants C and K,

$$\max_{|z|\leq r}|u_n(z)| \leq CK^n n!, \qquad \text{for } n = 0, 1, 2, \dots.$$

Also we define $\mathcal{O}[[\tau]]_1$ by

$$\mathcal{O}[[\tau]]_1 := \bigcup_{r>0} \mathcal{O}(r)[[\tau]]_1.$$

§2.1.3 Sectorial domain

For $\theta \in \mathbb{R}$, $\alpha > 0$ and $0 < T \leq +\infty$, we denote by $S(\theta, \alpha; T)$ a sectorial domain defined by

$$S(\theta, \alpha; T) := \{\tau \in \mathbb{C} | \, |\arg(\tau) - \theta| < \alpha/2, 0 < |r| < T\}.$$

Here θ, α and T are called the direction, opening angle, and radius of the sectorial domain $S(\theta, \alpha; T)$, respectively. If the radius T is not so important to identify, we will sometimes suppress it and denote the sector by $S(\theta, \alpha)$ for simplicity. A sectorial domain S' is called a proper subsector of $S(\theta, \alpha; T)$ if its closure is contained in $S(\theta, \alpha; T) \bigcup \{0\}$.

§2.1.4 Gevrey asymptotic expansion

Let $u(\tau, z)$ be analytic on $\bigcap_{\alpha'<\alpha} S(\theta, \alpha') \times B(r(\alpha'))$, where $r(\alpha')$ may tend to 0 as $\alpha' \to \alpha$. Then $\hat{u}(\tau, z) \in \mathcal{O}[[\tau]]_1$ is called a *Gevrey asymptotic expansion* of $u(\tau, z)$ as $\tau \to 0$ in $S(\theta, \alpha)$ or shortly in $S(\theta, \alpha)$ if for any proper subsector $S' \in S(\theta, \alpha; T)$ (with sufficiently small radius), there exist positive constants C, K and $0 < r_1 < r$, such that $\hat{u}(\tau, z) \in \mathcal{O}(r_1)[[\tau]]_1$ and

$$\max_{|z|\leq r_1}\left|u(\tau, z) - \sum_{n=0}^{N-1} u_n(z)\tau^n\right| \leq CK^N N! \, |\tau|^N, \tag{2.1}$$

$$\tau \in S', N = 1, 2, 3, \dots.$$

The relation is denoted by

$$u(\tau, z) \sim_1 \hat{u}(\tau, z), \text{ in } S(\theta, \alpha).$$

An analytic function $u(\tau, z)$ is said to be Gevrey asymptotic expandable in $S(\theta, \alpha)$ if it has a Gevrey asymptotic expansion $\hat{u}(\tau, z) \in \mathcal{O}[[\tau]]_1$.

Let us denote by $\mathcal{A}^1(S(\theta, \alpha))$ the set of analytic functions which are Gevrey asymptotic expandable in $S(\theta, \alpha)$. We denote a mapping $J^{(1)}$ by

$$J^{(1)} : \mathcal{A}^1(S(\theta, \alpha)) \to \mathcal{O}[[\tau]]_1, \tag{2.2}$$

where $J^{(1)}(u(\tau, z)) = \hat{u}(\tau, z)$ is the Gevrey asymptotic expansion $\hat{u}(\tau, z)$ of $u(\tau, z)$.

Now the following result is known as an analogue of Borel-Ritt's theorem for Gevrey asymptotic expansions (cf. [39, 28, 3]).

Theorem 2.1.1.

(1) The mapping $J^{(1)}$ defined by (2.2) is surjective but is not injective for any $\theta \in \mathbb{R}$ and α with $\alpha \leq \pi$.

(2) For any α with $\alpha > \pi$, the mapping $J^{(1)}$ is not surjective but is injective for any $\theta \in \mathbb{R}$.

§2.1.5 Formal Borel transform

To get an element $u(\tau, z) \in (J^{(1)})^{-1}(\hat{u})$ for $\hat{u} \in \mathcal{O}[[\tau]]_1$, an effective way is to introduce the *formal Borel transform* $v(s, z)$ of $\hat{u}(\tau, z)$ defined by

$$v(s, z) := \sum_{n=0}^{\infty} u_n(z) \frac{s^n}{n!}.$$

By the definition of $\mathcal{O}[[\tau]]_1$, $v(s, z)$ is analytic in a neighborhood of the origin $(s, z) = (0, 0)$, and so we assume that it is analytic on $\{|s| < r\} \times \{|z| < r\}$ $(\exists r > 0)$. Now for any $\theta \in \mathbb{R}$, we fix a positive constant ρ such that $0 < \rho < r$, and we define

$$u^{(\theta,\rho)}(\tau, z) := \frac{1}{\tau} \int_0^{\rho e^{i\theta}} e^{-s/\tau} v(s, z) ds, \qquad |\arg(\tau) - \theta| < \pi/2.$$

Then it can be proved (cf. [3]) that $J^{(1)}(u^{(\theta,\rho)}(\tau,z)) = \hat{u}(\tau,z)$ in $S(\theta,\pi)$. Here, the arbitrariness of ρ shows the non uniqueness of functions $u(\tau,z)$ such that $J^{(1)}(u) = \hat{u}$ in $S(\theta,\pi)$.

§2.1.6 Borel summability

According to the assertion (2) of Theorem 2.1.1, we know that if the opening angle of the sector S is large than π, for $\hat{u}(\tau,z) \in J^{(1)}(\mathcal{A}^1(S))$, there exists $u(\tau,z) \in \mathcal{A}^1(S)$ such that $J^{(1)}(u) = \hat{u}$ is unique. Then we say that $u(\tau,z)$ is called the *Borel sum* of $\hat{u}(\tau,z)$, and $\hat{u}(\tau,z)$ is said to be *Borel summable* in S. The Borel summability of $\hat{u}(\tau,z) \in \mathcal{O}[[\tau]]_1$ can be characterized (with respect to its formal Borel transform) as follows:

Theorem 2.1.2. *A formal series $\hat{u}(\tau,z) \in \mathcal{O}[[\tau]]_1$ is Borel summable in $S(\theta,\alpha)$ $(\alpha > \pi)$ if and only if its Borel transform $v(s,z)$ is analytic on $S(\theta,\alpha-\pi;\infty)\times B(r)$ and satisfies a growth condition of exponential type as $s \to \infty$ in $S(\theta,\alpha-\pi;\infty)$, i.e., for any proper subsection $S' \subset S(\theta,\alpha-\pi;\infty)$ of infinite radius there exists $0 < r_1 < r$ such that*

$$\max_{|z|\le r_1} |v(s,z)| \le Ce^{\delta|s|}, \qquad s \in S'$$

for some positive constants C and δ. Finally, the Borel sum $(J^{(1)})^{-1}(\hat{u})$ in $S(\theta,\alpha)$ is represented by the Laplace integral,

$$u^{\varphi}(\tau,z) := \frac{1}{\tau}\int_0^{\infty e^{i\varphi}} e^{-s/\tau} v(s,z)ds, \tag{2.3}$$

where the path of integration is taken over the ray $e^{i\varphi}\mathbb{R}_+ = \{re^{i\varphi} | r \ge 0\}$ for φ such that $|\varphi-\theta| < \frac{\alpha-\pi}{2}$.

§2.1.7 Fine Borel summability

In the definition of Gevrey asymptotic expansion, we always take a proper subsector $S' \subset S(\theta,\alpha)$. In the case of crucial value of $\alpha = \pi$, Nevanlinna [36] has given a refined form of asymptotic expansion which corresponds to taking open disks instead of subsectors as follows. For $\theta \in \mathbb{R}$ and $T > 0$ we define an open disk $O(\theta, T)$ by

$$O(\theta, T) := \{\tau \in \mathbb{C} |\, \left|\tau - Te^{i\theta}\right| < T\}.$$

Then we say that $\hat{u}(\tau, z) \in \mathcal{O}[[\tau]]_1$ is *finely Borel summable* in a direction θ if there exists an analytic function $u(\tau, z)$ on $O(\theta, T) \times B(r)$ $(\exists T, r > 0)$ such that for some $0 < T' < T$ the following inequalities

$$\max_{|z| \le r} \left| u(\tau, z) - \sum_{n=0}^{N-1} u_n(z)\tau^n \right| \le CK^N N! \left|\tau\right|^N, \qquad (2.4)$$
$$\tau \in O(\theta, T'), N = 1, 2, 3, \dots,$$

hold for some positive comstants C and K. This relation is denoted by

$$u(\tau, z) \sim_1 \hat{u}(\tau, z), \qquad \text{finely in the direction } \theta.$$

For a modern treatment of fine Borel summability, see Malgrange [28].

As in the Borel summable case, it is proved that if $u(\tau, z) \sim_1 0$ finely in the direction θ we have $u(\tau, z) \equiv 0$ (cf. [28]). Hence for $\hat{u}(\tau, z) \in \mathcal{O}[[\tau]]_1$, if there exists $u(\tau, z)$ such that the relation (2.4) holds, it is called the *fine Borel sum* in the direction θ of $\hat{u}(\tau, z)$ (see [28]).

To characterize the fine Borel summability we need to define a set $E_+(\theta, w)$ by

$$E_+(\theta, w) := \{s \in \mathbb{C} | dist(s, e^{i\theta}\mathbb{R}_+) < w\}.$$

Now the fine Borel summability is characterized as follows.

Theorem 2.1.3. *The formal power series $\hat{u}(\tau, z) \in \mathcal{O}[[\tau]]_1$ is finely Borel summable in a direction θ if and only if its formal Borel transform $v(s, z)$ is analytic on $E_+(\theta, w) \times B(r)$ for some $w > 0$ and $r > 0$, and satisfies a growth condition of exponential type as $s \to \infty$ in $E_+(\theta, w)$, that is,*

$$\max_{|z| \le r} |v(s, z)| \le Ce^{\delta|s|}, \qquad s \in E_+(\theta, w).$$

The fine Borel sum $u^\theta(\tau, z)$ in the direction θ of $\hat{u}(\tau, z)$ is obtained by the expression (2.3).

For the proof, see Malgrange [28].

§2.1.8 Function satisfying $J^{(1)}(u) = 0$

The following criterion for analytic functions with 0 Gevrey asymptotic expansion is used in the latter sections.

Proposition 2.1.4. *In order that $u(\tau, z) \sim_1 0$ in $S(\theta, \alpha)$ it is necessary and sufficient that for any proper subsector $S' \subset S(\theta, \alpha)$ there exist positive constants r_1, C and δ such that*

$$\max_{|z| \leq r_1} |u(\tau, z)| \leq Ce^{-\delta/|\tau|}, \qquad \tau \in S'.$$

In the case of fine Borel summability in a direction θ, S' should be replaced by $O(\theta, T)$ for some $T > 0$, and the inequality implies that $u \equiv 0$ by Watson's lemma (cf. [28]).

§2.2 Properties of Borel summability

Let $r > 0$ and $\theta \in \mathbb{R}$ be given. By

$$\mathcal{O}(r)\{\tau\}_{1,\theta}$$

we denote the set of all $\hat{u}(\tau, z) \in \mathcal{O}(r)[[\tau]]_1$ that are Borel summable in $S = S(\theta, \alpha)$ with $\alpha > \pi$. And we shall write the Borel sum of $\hat{u}(\tau, z)$ as $\mathcal{S}_{1,\theta}(\hat{u})(\tau, z)$. Also we define $\mathcal{O}\{\tau\}_{1,\theta}$ by

$$\mathcal{O}\{\tau\}_{1,\theta} := \bigcup_{r>0} \mathcal{O}(r)\{\tau\}_{1,\theta}.$$

This set is a differential algebra over the ring of analytic functions. In this thesis, we only need to use the following properties:

Theorem 2.2.1. *For fixed but arbitrary $\theta \in \mathbb{R}$, we have:*

(i) If $\hat{f}$, $\hat{g} \in \mathcal{O}\{\tau\}_{1,\theta}$, then $\hat{f}+\hat{g}$, $\hat{f}\hat{g} \in \mathcal{O}\{\tau\}_{1,\theta}$, and

$$\mathcal{S}_{1,\theta}(\hat{f}+\hat{g}) = \mathcal{S}_{1,\theta}(\hat{f}) + \mathcal{S}_{1,\theta}(\hat{g}),$$

$$\mathcal{S}_{1,\theta}(\hat{f}\hat{g}) = (\mathcal{S}_{1,\theta}(\hat{f}))(\mathcal{S}_{1,\theta}(\hat{g})).$$

(ii) If $\hat{f} \in \mathcal{O}\{\tau\}_{1,\theta}$, then $\frac{\partial}{\partial\tau}\hat{f}$, $\int_0^\tau \hat{f}(w,z)dw \in \mathcal{O}\{\tau\}_{1,\theta}$, and

$$\mathcal{S}_{1,\theta}(\frac{\partial}{\partial\tau}\hat{f}) = \frac{\partial}{\partial\tau}\mathcal{S}_{1,\theta}(\hat{f}),$$

$$\mathcal{S}_{1,\theta}(\int_0^\tau \hat{f}(w,z)dw) = \int_0^\tau \mathcal{S}_{1,\theta}(\hat{f})(w,z)dw,$$

with $\int_0^\tau \hat{f}(w,z)dw$ denoting the termwise integrated formal power series.

(iii) If $\hat{f} \in \mathcal{O}\{\tau\}_{1,\theta}$, then $\frac{\partial}{\partial z}\hat{f} \in \mathcal{O}\{\tau\}_{1,\theta}$, and

$$\mathcal{S}_{1,\theta}(\frac{\partial}{\partial z}\hat{f}) = \frac{\partial}{\partial z}\mathcal{S}_{1,\theta}(\hat{f}).$$

Proof. It follows the following lemmas. $\circledast$

Lemma 2.2.2. *Given a sector S, suppose that*

$$f(\tau,z) \sim_1 \hat{f}(\tau,z), \ \textit{in } S,$$

$$g(\tau,z) \sim_1 \hat{g}(\tau,z), \ \textit{in } S.$$

Then

$$f(\tau,z)+g(\tau,z) \sim_1 \hat{f}(\tau,z)+\hat{g}(\tau,z), \ \textit{in } S,$$

$$f(\tau,z)g(\tau,z) \sim_1 \hat{f}(\tau,z)g(\tau,z), \ \textit{in } S.$$

Lemma 2.2.3. *Given a sector S, suppose that*

$$f(\tau,z) \sim_1 \hat{f}(\tau,z), \ \textit{in } S.$$

Then

$$\frac{\partial}{\partial\tau}f(\tau,z) \sim_1 \frac{\partial}{\partial\tau}\hat{f}(\tau,z), \ \textit{in } S,$$

$$\int_0^\tau f(\tau,w)dw \sim_1 \int_0^\tau \hat{f}(w,z)dw,\ \ in\ S,$$
$$\frac{\partial}{\partial z} f(\tau,z) \sim_1 \frac{\partial}{\partial z}\hat{f}(\tau,z),\ \ in\ S.$$

Balser in [3, 5] gave similar conclusions in the case of formal power series over the complex numbers, and they are still true about the formal power series over the ring of analytic functions.

Remark 2.2.4. *We say that in the case of fine Borel summability in a direction θ, these properties remain valid.*

§2.3 Fine Borel sum of $\hat{u}(\tau,z)$

In [26], Lutz, Miyake and Schäfke studied a normalized Cauchy problem for linear partial differential equations in two variables τ and z with constant coefficients. The Cauchy problem has formal solutions that are power series in the variable τ, with coefficients that are holomorphic functions of z in a disc about the origin. Explicitly, they considered

$$\begin{cases} \partial_\tau u - \partial_z^2 u = 0, \\ u(0,z) = \phi(z), \end{cases} \tag{2.5}$$

where $(\tau,z) \in \mathbb{C}\times\mathbb{C}$, and $\phi(z)$ is assumed to be analytic in a neighborhood of the origin.

And they have proved

Theorem 2.3.1. *Let $v(s,z)$ be the formal Borel transform of the formal solution $\hat{u}(\tau,z) = \sum_{n=0}^\infty \frac{\phi^{(2n)}(z)}{n!}\tau^n$ of (2.5). Then the following two statements are equivalent:*

(i) $v(s,z)$ is analytic on $E_+(\theta,w)\times B(r)$ for some positive constants w and r, and satisfies the exponential type growth condition as $s\to\infty$ in $E_+(\theta,w)$.

(ii) The Cauchy data $\phi(z)$ is analytic on $\Omega(\theta/2,\omega)$ for some positive ω, and satisfies the growth condition of exponential order at most 2 as $z \to \infty$ in $\Omega(\theta/2,\omega)$, that is, the following inequality holds for some positive constants C and δ,

$$|\phi(z)| \leq Ce^{\delta|z|^2}, \qquad z \in \Omega(\theta/2,\omega).$$

Here,

$$\Omega(\theta/2,\omega) = \{z \in \mathbb{C} \mid dist(z, e^{i\theta/2}\mathbb{R}) < \omega\}.$$

Remark 2.3.2. *Actually, $v(s,z)$ is analytic on $E_+(\theta,w') \times \Omega(\theta/2,\omega')$ for small w' and ω', and satisfies the growth condition*

$$|v(s,z)| \leq Ce^{A|s|+B|z|^2}, \text{ on } E_+(\theta,w') \times \Omega(\theta/2,\omega')$$

for some positive constants A, B and C. This implies that the Borel sum $u^\theta(\tau,z)$ in the direction θ, which is given by (2.3), is analytic on $O(\theta,T) \times \Omega(\theta/2,\omega'')$ for some positive constants T and ω''.

Theorem 2.3.3. *Suppose the formal solution $\hat{u}(\tau,z) := \sum_{n=0}^{\infty} u_n(z)\tau^n$ to be finely Borel summable in a direction θ with the Cauchy data $u_0(z) = \phi(z)$, and $u^\theta(\tau,z)$ be its fine Borel sum defined on $O(\theta,T) \times \Omega(\theta/2,\omega)$. Then we have:*

(i) (Gevrey asymptotic estimates) For some $T' < T$, $\omega' < \omega$, there exist positive constants C, K and δ such that the following asymptotic estimates hold:

$$\left| u^\theta(\tau,z) - \sum_{n=0}^{N-1} u_n(z)\tau^n \right| \leq CK^N N! e^{\delta|z|^2} |\tau|^N, \qquad (2.6)$$

$$(\tau,z) \in O(\theta,T') \times \Omega(\theta/2,\omega'),$$

for all $N = 1,2,3,\ldots$.

(ii) (integral expression) The fine Borel sum $u^\theta(\tau,z)$ has the following integral expression involving the Heat Kernel $e^{-\zeta^2/(4\tau)}/\sqrt{4\pi\tau}$:

$$u^\theta(\tau,z) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty e^{i\theta/2}}^{\infty e^{i\theta/2}} e^{-\zeta^2/(4\tau)} \phi(z+\zeta) d\zeta. \qquad (2.7)$$

Proof. (i) By Theorem 2.3.1, the Cauchy data $\phi(z)$ is analytic on $\Omega(\theta/2, \omega)$ for some positive constant ω, and satisfies there a growth condition of exponential order at most 2. We recall that the fine Borel sum $u^\theta(\tau, z)$ has an integral expression (cf. (2.3))

$$u^\theta(\tau, z) = \frac{1}{\tau} \int_0^{\infty e^{i\theta}} e^{-s/\tau} v(s, z) ds, \tag{2.8}$$

where $v(s, z)$ is the formal Borel transform of $\hat{u}(\tau, z)$, and satisfies (from Remark 2.3.2) the following inequality

$$|v(s, z)| \leq C e^{A|s| + B|z|^2}, \qquad (s, z) \in E_+(\theta, w') \times \Omega(\theta/2, \omega'),$$

for from positive constants A, B, C, w' and ω'. Hence by Cauchy's integral formula,

$$\left| \frac{\partial^n v}{\partial s^n}(s, z) \right| \leq C' K^n n! e^{A|s| + B|z|^2},$$
$$(s, z) \in E_+(\theta, w'') \times \Omega(\theta/2, \omega'),$$

for any $w'' < w'$ and suitable positive constants C' and K.

Now by the repeated use of integration by parts in (2.8), we have

$$u^\theta(\tau, z) - \sum_{n=0}^{N-1} u_n(z) \tau^n = u_N(z) \tau^N + \tau^N \int_0^{\infty e^{i\theta}} e^{-s/\tau} \frac{\partial^{N+1}}{\partial s^{N+1}} v(s, z) ds,$$

for τ such that $\mathrm{Re}(e^{i\theta}/\tau) > 2A$, and $z \in \Omega(\theta/2, \omega')$. Here, $u_n(z) = (\partial^n v/\partial s^n)(0, z)$.

By restricting τ as above, the integral part in the above equality is estimated by

$$\begin{aligned} C' |\tau|^N K^N (N+1)! e^{B|z|^2} \int_0^\infty e^{-Ar} dr &= C' A^{-1} |\tau|^N K^N (N+1)! e^{B|z|^2} \\ &\leq C'' K'^N N! e^{B|z|^2} |\tau|^N, \end{aligned}$$

for some positive constants C'' and K'. Here we notice that

$$\{\tau \mid \mathrm{Re}(e^{i\theta}/\tau) > 2A)\} = \{\tau \mid \left|\tau - e^{i\theta}/A\right| < 1/A\} = O(\theta, 1/A).$$

Combining the results above, we have the desired asymptotic estimates (2.6).

(ii) Because $\hat{u}(\tau, z)$ is finely Borel summable in a direction θ, by Theorem 2.3.1, $\phi(z)$ is analytic on $\Omega(\theta/2, \omega)$ for some positive ω, and satisfies the growth condition of exponential order at most 2 as $z \to \infty$ in $\Omega(\theta/2, \omega)$. Then we can deduce that $v(s, z)$ has the integral expression

$$v(s,z) = \frac{1}{2\pi i}\int_\Gamma \frac{\phi(z+\zeta)}{\sqrt{\zeta^2 - 4s}}d\zeta, \tag{2.9}$$

where Γ is any simple closed piecewise smooth curve surrounding the segment joining two points $\{\pm 2\sqrt{s}\}$ $(s \neq 0)$. We restrict s to the ray $\{re^{i\theta} \mid r \geq 0\}$, and put $\sqrt{s} := \sqrt{r}e^{i\theta/2}$.

Let $r > 0$ and $\zeta = \xi + i\eta$, where $|\xi| < 2\sqrt{r}$. Then by noticing

$$\sqrt{(\xi + i0)^2 - 4r} = i\sqrt{4r - \xi^2}, \qquad \sqrt{(\xi - i0)^2 - 4r} = -i\sqrt{4r-\xi^2},$$

we have

$$\begin{aligned} v(s,z) &= v(re^{i\theta}, z) \\ &= \frac{1}{\pi}\int_{-2\sqrt{r}}^{2\sqrt{r}} \frac{\phi(z+\xi e^{i\theta/2})}{\sqrt{4r-\xi^2}}d\xi, \qquad r > 0. \end{aligned}$$

Then by substituting this into (2.8), we have

$$\begin{aligned} u^\theta(\tau, z) &= \frac{e^{i\theta}}{\pi\tau}\int_0^\infty \exp(-\frac{e^{i\theta}}{\tau}r)dr\int_{2\sqrt{r}}^{-2\sqrt{r}} \frac{\phi(z+\xi e^{i\theta/2})}{\sqrt{4r-\xi^2}}d\xi \\ &= \frac{e^{i\theta}}{\pi\tau}\int_{-\infty}^\infty \phi(z+\xi e^{i\theta/2})d(\xi e^{i\theta/2})\int_{\xi^2/4}^\infty \exp(-\frac{e^{i\theta}}{\tau}r)\frac{dr}{4r-\xi^2}. \end{aligned}$$

Now the desired integral expression (2.7) is a consequence of the following calculations:

$$\begin{aligned} &\int_{\xi^2/4}^\infty \exp(-\frac{e^{i\theta}}{\tau}r)\frac{dr}{\sqrt{4r-\xi^2}} \qquad (x = \sqrt{4r-\xi^2}) \\ =\ & \frac{1}{2}\exp(-\frac{(\xi e^{i\theta/2})^2}{4\tau})\int_0^\infty \exp(-\frac{e^{i\theta}}{4\tau}x^2)dx \qquad (y = xe^{i\theta/2}/(2\sqrt{r})) \\ =\ & e^{-i\theta/2}\frac{\sqrt{\pi\tau}}{2}\exp(-\frac{(\xi e^{i\theta/2})^2}{4\tau}). \end{aligned}$$

This completes the proof. ⊛

Back to our explicit Problem (1.1). Since the initial value $\varphi(z)$ is not analytic at the origin, one cannot directly apply results in [26] to get the fine Borel sum of $\hat{u}(\tau, z) = \sum\limits_{n=0}^{\infty} \frac{\varphi^{(2n)}(z)}{n!}\tau^n$ in the direction of the real axis. But using variable transformation

$$Z = z - (2k+1)\pi i,$$

one can transfer the strip domain Ω_k with $k \in \mathbb{Z}$, into the domain

$$\tilde{\Omega} := \{Z \in \mathbb{C} \mid -\pi < \mathrm{Im}(Z) < \pi\}$$

which contains the origin $Z = 0$. Hence, Problem (1.1) has the equivalent form:

$$\begin{cases} \partial_\tau u - \partial_Z^2 u = 0, \\ u(0, Z) = \frac{1}{1-e^{Z+(2k+1)\pi i}}. \end{cases} \tag{2.10}$$

Apparently, the initial condition $u(0, Z)$ is analytic on $\tilde{\Omega}$. By Theorem 2.3.3, Problem (2.10) has a solution $u_k(\tau, Z)$ defined by

$$u_k(\tau, Z) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4\tau}} \frac{1}{1 - e^{Z+s+(2k+1)\pi i}} ds,$$

where $\mathrm{Re}(\tau) > 0$ and $Z \in \tilde{\Omega}$.

Therefore, one can get a solution of (1.1):

$$\begin{aligned} U_k(\tau, z) &= u_k(\tau, z - (2k+1)\pi i) \\ &= \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4\tau}} \frac{1}{1 - e^{z+s}} ds, \end{aligned}$$

where $\mathrm{Re}(\tau) > 0$ and $z \in \Omega_k$.

Proposition 2.3.4. *Given $k \in \mathbb{Z}$. For Re$(\tau) > 0$ and $z \in \Omega_k$, we define*

$$U_k(\tau, z) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4\tau}} \frac{1}{1 - e^{z+s}} ds. \tag{2.11}$$

Then we have

(i) $U_k(\tau, z)$ *is holomorphic over* $\{\tau \in \mathbb{C} \mid Re(\tau) > 0\} \times \Omega_k$.

(ii) $U_k(\tau, z)$ *is the unique solution of (1.1) which admits* $\hat{u}(\tau, z) = \sum_{n=0}^{\infty} \frac{\varphi^{(2n)}(z)}{n!}\tau^n$ *as asymptotic expansion in the sense: for some compact subset* O' *of* $\{\tau \in \mathbb{C} \mid Re(\tau) > 0\}$ *and some compact subset* Ω_k' *of* Ω_k*, there exist positive constants* C, K *and* δ*, such that the following asymptotic estimates hold:*

$$\left| U_k(\tau, z) - \sum_{n=0}^{N-1} \frac{\varphi^{(2n)}(z)}{n!}\tau^n \right| \leq CK^N N! \, |\tau|^N , \qquad (2.12)$$

$$(\tau, z) \in O' \times \Omega_k',$$

for all $N = 1, 2, 3, \ldots$.

Proof. (i) Obviously, the integral is absolutely convergent for $\{\tau \in \mathbb{C} \mid \mathrm{Re}(\tau) > 0\} \times \Omega_k$, and so $U_k(\tau, z)$ is holomorphic.

(ii) According to the process of derivation, we know that

$$U_k(\tau, z) \sim_1 \hat{u}(\tau, z), \qquad \text{finely in the direction of the real axis.}$$

So from Theorem 2.2.1,

$$\partial_\tau U_k - \partial_z^2 U_k \sim_1 \partial_\tau \hat{u} - \partial_z^2 \hat{u} = 0, \qquad \text{finely in the direction of the real axis.}$$

Then by Watson's lemma, we have

$$\partial_\tau U_k - \partial_z^2 U_k \equiv 0,$$

that is to say, $U_k(\tau, z)$ is a solution of the heat equation. In the same time, we can prove that

$$\lim_{\tau \to 0} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{s^2}{4\tau}} = \delta(s).$$

Therefore, we know that $\lim\limits_{\tau \to 0} U_k(\tau, z) = \frac{1}{1-e^z} = \varphi(z)$.

Notice that our initial data $\varphi(z)$ satisfies not only the growth condition of exponential order at most 2, but also the following growth estimate for some positive constant C,

$$|\varphi(z)| = \left|\frac{1}{1-e^z}\right| < C, \qquad z \in \Omega_k^{'}.$$

So, similarly to the proof of the assertion (i) of Theorem 2.3.3, we can get (2.12). The result is proved. ⊛

Remark 2.3.5. *In fact, we can derive the fine Borel sum of $\hat{u}(\tau, z)$ in the direction θ with $-\pi < \theta < \pi$ by a similar approach. But notice that then the variable τ is defined over $\{\tau \in \mathbb{C} \mid |\arg \tau - \theta| < \pi/2\}$ and z over the oblique strip $\Omega_k^{\theta/2} := \{z \in \mathbb{C} \mid 2k\pi < Im(z) - tan(\frac{\theta}{2})Re(z) < 2(k+1)\pi\}$ for $k \in \mathbb{Z}$.*

Chapter 3 Sums Based on Heat Kernel

In this chapter, we introduce a new method summation which is called Gq-summation. Similarly to the theory of Borel summation, we shall give definitions of q-Gevrey asymptotic expansion of order 1 in a direction, q-Borel transform and q-Laplace transfom. Then we will make use of them to get one q-Borel sum $F_\alpha(\tau, z)$ of $\sum\limits_{n=0}^{\infty} e^{n^2\tau+nz}$. And finally, we obtain one of our main theorem.

§3.1 Gq-summation

Let q be a real number with $0 < q < 1$.

Here we firstly present a new notion of asymptotics, which is used to treat the case of a formal solution belonging to the class of q-Gevrey power series of order 1.

We use the notation $\mathbb{C}[[x]]$ to be all the formal power series in the variable x, and denote by $\mathbb{C}\{x\}$ the subspace of $\mathbb{C}[[x]]$, in which the radius of convergence of the power series is positive.

And define

$$d_\alpha = \{x \in \tilde{\mathbb{C}}^* : \arg x = \alpha\} \text{ for } \alpha \in \mathbb{R},$$

$$\tilde{D}(0; R) = \{x \in \tilde{\mathbb{C}}^* : \mid x \mid < R\} \text{ for } R > 0,$$

$$\tilde{S}(\alpha, \varepsilon) = \{x \in \tilde{\mathbb{C}}^* : \mid \arg x - \alpha \mid < \varepsilon/2\} \text{ with } \alpha \in \mathbb{R},\ \varepsilon > 0.$$

Denote the principle determination of the logarithm by log and $\frac{\log}{\ln q}$ by $\log_q$. Namely, for $x \in \tilde{\mathbb{C}}^*$

$$\log x := \ln |x| + i \arg x,$$

and

$$\begin{aligned}\log_q x &= \frac{\ln|x|}{\ln q} + i\frac{\arg x}{\ln q} \\ &:= \log_q|x| + i\arg_q x.\end{aligned}$$

Let $a \in \mathbb{C}$. Then the notation x^a is representing the function $e^{a\log x}$: $\tilde{\mathbb{C}}^* \to \tilde{\mathbb{C}}^*$.

We denote by $\tilde{\mathbb{O}}$ the class of functions defined and analytic on $\tilde{D}(0;R)$ with $R > 0$ arbitrary.

Definition 3.1.1. *Let $\hat{f} = \sum\limits_{n\geq 0} a_n x^n \in \mathbb{C}[[x]]$. We define formal q-Borel transform of $\hat{f}$ by the series*

$$\hat{\mathcal{B}}_{q;1}\hat{f}(\xi) := \sum_{n\geq 0} a_n q^{n^2}\xi^n.$$

Definition 3.1.2. *We will say that $\hat{f}(x)$ is q-Gevrey of order 1 if and only if $\hat{\mathcal{B}}_{q;1}\hat{f}$ has a positive radius of convergence in the $\xi-$plane. Denote the set of q-Gevrey series of order 1 by $\mathbb{C}[[x]]_{q;1}$.*

Remark 3.1.3. *In this definition and in the subsequent definitions, we take q^{n^2} as a whole and so name them with order 1.*

Definition 3.1.4. *Furthermore, we denote by $\mathbb{C}\{x\}^{\alpha}_{q;1}$ the subset of q-Gevrey series whose formal q-Borel transform can be analytically extended to an analytic function with a q-exponential growth of order 1 as $\mid \xi \mid \to \infty$ in an open sector $\tilde{S}(\alpha,\varepsilon)$ with $\varepsilon > 0$ arbitrary, if the analytic function is written as $\varphi(\xi)$, then that is to say, there exist $\mu \in \mathbb{R}$ and $C > 0$, such that*

$$|\varphi(\xi)| \leq C\,|\xi|^{\mu}\, q^{-\frac{1}{4}\log_q^2|\xi|},\ \ for\ \ |\xi| \to \infty\ in\ \tilde{S}(\alpha,\varepsilon).$$

For such a function $\varphi(\xi)$, we can define its q-Laplace transform in the direction d_α as follows,

$$\mathcal{L}^{\alpha}_{q;1}\varphi(x) := \frac{1}{\sqrt{4\pi\ln 1/q}}\int_{d_\alpha} q^{\frac{1}{4}(\log_q\frac{x}{\xi})^2}\varphi(\xi)\frac{d\xi}{\xi}. \tag{3.1}$$

Remark 3.1.5. *It is clear that* $\mathbb{C}\{x\} \subset \mathbb{C}\{x\}_{q;1}^{\alpha} \subset \mathbb{C}[[x]]_{q;1}$.

Definition 3.1.6. *Let* $\alpha \in \mathbb{R}$, $f \in \tilde{\mathbb{O}}$ *and* $\hat{f} = \sum_{n\geq 0} a_n x^n \in \mathbb{C}[[x]]$. *We will say that* f *admits* $\hat{f}$ *as q-Gevrey asymptotic expansion of order 1 in a direction* d_α *if and only if there exist* $C > 0$, $A > 0$, *such that for all* $N \in \mathbb{N}^*$

$$\mid f(x) - \sum_{n=0}^{N-1} a_n x^n \mid < C A^N q^{-(N^2 + \frac{1}{4}\arg_q^2(xe^{-\alpha i}))} |x|^N ,$$

where $x \in \tilde{D}(0;R)$ *with* $R > 0$ *small enough. In this case, we note* $f \sim_{q;1}^{\alpha} \hat{f}$.

Remark 3.1.7. *We can deduce that the q-Gevrey asymptotic expansion of order 1* $\hat{f}$ *is unique, and that* $\hat{f} \in \mathbb{C}[[x]]_{q;1}$.

Denote by $\mathbb{A}_{q;1}^{\alpha}$ the subset of $\tilde{\mathbb{O}}$, in which the function has q-Gevrey asymptotic expansion of order 1 in a direction d_α. Then we conclude that the mapping

$$J_{q;1}^{\alpha} : \ \mathbb{A}_{q;1}^{\alpha} \longrightarrow \mathbb{C}[[x]]_{q;1}$$

is surjective.

Definition 3.1.8. *Let* $\alpha \in \mathbb{R}$, $f \in \tilde{\mathbb{O}}$ *and* $\hat{f} = \sum_{n\geq 0} a_n x^n \in \mathbb{C}[[x]]$. *We will say that* $\hat{f}$ *is Gq-summable of order 1 in a direction* d_α, *and* f *is its Gq-sum if there exist* $\varepsilon > 0$, $C > 0$, $A > 0$, *such that for all* $N \in \mathbb{N}^*$ *and* $\beta \in]\alpha-\varepsilon, \alpha+\varepsilon[$

$$\mid f(x) - \sum_{n=0}^{N-1} a_n x^n \mid < C A^N q^{-(N^2 + \frac{1}{4}\arg_q^2(xe^{-\beta i}))} |x|^N ,$$

where $x \in \tilde{D}(0;R)$ *with* $R > 0$ *small enough.*

Remark 3.1.9. *We can deduce that in this case* $\hat{f} \in \mathbb{C}\{x\}_{q;1}^{\alpha}$ *and the Gq-sum of* $\hat{f}$ *is unique, which is denoted by* $\mathcal{S}_{q;1}^{\alpha}\hat{f}$.

Lemma 3.1.10. *Let* $\alpha \in \mathbb{R}$ *and* $\hat{f} \in \mathbb{C}_{q;1}^{\alpha}\{x\}$. *Suppose* φ *be the sum function of* $\hat{\mathcal{B}}_{q;1}\hat{f}$, *which can be analytically extended and satisfy the growth condition. Then the Gq-sum* $\mathcal{S}_{q;1}^{\alpha}\hat{f}$ *can be represented by means of the q-Laplace transform of* φ *in the direction* d_α *defined by (3.1).*

§3.2 q-Borel sums of $\sum_{n=0}^{\infty} e^{n^2\tau+nz}$

Using the Gq-summation method introduced in the last section, we can give a Gq-sum $f_\alpha(x,q)$ of $\hat{y}(x,q)$ defined by (1.4). On the other hand, the function f_α can also be formulated in terms of Fourier analysis, namely, from the Gaussian integral, it follows that

$$q^{-n^2} = \frac{1}{\sqrt{4\pi \ln 1/q}} \int_{-\infty}^{\infty} e^{\frac{t^2}{4\ln q}+nt} dt,$$

so the power series $\sum_{n=0}^{\infty} q^{-n^2} x^n$ may be associated with the integral of the type:

$$\frac{1}{\sqrt{4\pi \ln 1/q}} \int_{-\infty}^{\infty} \frac{e^{\frac{t^2}{4\ln q}}}{1 - xe^t} dt,$$

which gives rise to the integral function f_α.

So, if $\alpha \in \mathbb{R}\backslash 2\pi\mathbb{Z}$, we can define

$$\begin{aligned} f_\alpha(x,q) &= \frac{1}{\sqrt{4\pi \ln 1/q}} \int_{-\infty}^{\infty} \frac{e^{\frac{t^2}{4\ln q}}}{1 - e^{t+\log x}} dt \\ &= \frac{1}{\sqrt{4\pi \ln 1/q}} \int_{-\infty+\alpha i}^{\infty+\alpha i} \frac{e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1 - e^{\xi}} d\xi, \end{aligned}$$

for all $x \in \tilde{\mathbb{C}}^*$.

Proposition 3.2.1. *For any $\alpha \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ and any x over the Riemann surface of the logarithm $\tilde{\mathbb{C}}^*$, we define*

$$f_\alpha(x,q) = \frac{1}{\sqrt{4\pi \ln 1/q}} \int_{-\infty+\alpha i}^{\infty+\alpha i} \frac{e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1 - e^{\xi}} d\xi. \tag{3.2}$$

Then we have

(i) $f_\alpha(x,q)$ is holomorphic over $\tilde{\mathbb{C}}^*$, and if α and β belong to a common interval of the set $\{\alpha \in \mathbb{R} \setminus 2\pi\mathbb{Z}\}$, then $f_\alpha(x,q) = f_\beta(x,q)$.

(ii) $f_\alpha(xe^{-2\pi i}, q) - f_\alpha(x,q) = i\sqrt{\frac{\pi}{\ln 1/q}}e^{\frac{(\log x - 2\pi k_\alpha i)^2}{4\ln q}}$*, where* $2\pi k_\alpha$ *is the integer between* α *and* $\alpha + 2\pi$*.*

(iii) $f_\alpha(x,q)$ *is the unique solution of (1.3) which admits* $\sum_{n=0}^{\infty} q^{-n^2}x^n$ *as q-Gevrey asymptotic expansion at* $x = 0$ *along the direction* $(0,\ \infty e^{\alpha i})$*.*

Proof. (i) By means of variable substitutions, we write

$$\begin{aligned}
|f_\alpha(x,q)| &= \frac{1}{\sqrt{4\pi \ln 1/q}}\int_{-\infty}^{\infty}\left|\frac{e^{\frac{(\log x - R - \alpha i)^2}{4\ln q}}}{1 - e^{R+\alpha i}}\right| dR \quad (\xi = R + \alpha i) \\
&\leq \frac{1}{\sqrt{4\pi \ln 1/q}}\int_{-\infty}^{\infty}\frac{e^{\mathrm{Re}(\frac{(\log x - R - \alpha i)^2}{4\ln q})}}{|e^R - 1|}dR \\
&= \frac{e^{-\frac{(\arg x - \alpha)^2}{4\ln q}}}{\sqrt{4\pi\ln 1/q}}\int_{-\infty}^{\infty}\frac{e^{\frac{(\ln|x| - R)^2}{4\ln q}}}{|e^R - 1|}dR.
\end{aligned}$$

With $0 < q < 1$ and the Gaussian integral, we conclude that $f_\alpha(x,q)$ is holomorphic over $\tilde{\mathbb{C}}^*$.

Suppose that $\alpha > \beta$ and $\alpha,\ \beta \in (2k\pi, 2(k+1)\pi)$ for some $k \in \mathbb{Z}$. Then

$$f_\alpha(x,q) - f_\beta(x,q) = \frac{1}{\sqrt{4\pi\ln 1/q}}\left(\int_{-\infty+\alpha i}^{\infty+\alpha i}\frac{e^{\frac{(\log x - \xi)^2}{4\ln q}}}{1 - e^\xi}d\xi - \int_{-\infty+\beta i}^{\infty+\beta i}\frac{e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1 - e^\xi}d\xi\right).$$

Consider the contour integral

$$J_1 = \oint_{\mathcal{C}_1}\frac{e^{\frac{(\log x - \xi)^2}{4\ln q}}}{1 - e^\xi}d\xi,$$

where $\mathcal{C}_1$ is the oriented rectangle, whose vertexes are at the points $R + \alpha i$, $R + \beta i$, $-R + \beta i$, $-R + \alpha i$ (*some fixed* $R \in (0, \infty)$), in a clockwise direction.

Because the contour $\mathcal{C}_1$ encloses none of the singularities of the integrand, by Cauchy's theorem we have

$$\begin{aligned}
J_1 &= \left(\int_{-R+\alpha i}^{R+\alpha i} + \int_{R+\alpha i}^{R+\beta i} + \int_{R+\beta i}^{-R+\beta i} + \int_{-R+\beta i}^{-R+\alpha i}\right)\frac{e^{\frac{(\log x - \xi)^2}{4\ln q}}}{1 - e^\xi}d\xi \\
&= 0.
\end{aligned}$$

However,

$$\begin{aligned}\Big|\int_{R+\alpha i}^{R+\beta i}\frac{e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1-e^{\xi}}d\xi\Big| &= \Big|\int_{\alpha}^{\beta}\frac{e^{\frac{(\log x-R-yi)^2}{4\ln q}}}{1-e^{R+yi}}idy\Big| \quad (\xi=R+iy)\\ &\le \int_{\beta}^{\alpha}\frac{e^{\frac{(\ln|x|-R)^2-(\arg x-y)^2}{4\ln q}}}{e^R-1}dy\\ &= \frac{e^{\frac{(\ln|x|-R)^2}{4\ln q}}}{e^R-1}\int_{\beta}^{\alpha}e^{\frac{-(\arg x-y)^2}{4\ln q}}dy\to 0,\quad as\ R\to\infty,\end{aligned}$$

and in the same way,

$$\Big|\int_{-R+\beta i}^{-R+\alpha i}\frac{e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1-e^{\xi}}d\xi\Big|\to 0,\quad as\ R\to\infty.$$

Hence, putting all of these together, we have, as $R\to\infty$,

$$\Big(\int_{-R+\alpha i}^{R+\alpha i}+\int_{R+\beta i}^{-R+\beta i}\Big)\frac{e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1-e^{\xi}}d\xi\to 0.$$

(ii) For $\alpha\in\mathbb{R}\backslash 2\pi\mathbb{Z}$, we have

$$\begin{aligned}f_\alpha(xe^{-2\pi i},q)-f_\alpha(x,q) &= f_{\alpha+2\pi}(x,q)-f_\alpha(x,q)\\ &= \frac{1}{\sqrt{4\pi\ln 1/q}}\Big(\int_{-\infty+(\alpha+2\pi)i}^{\infty+(\alpha+2\pi)i}\frac{e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1-e^{\xi}}d\xi\\ &\quad -\int_{-\infty+\alpha i}^{\infty+\alpha i}\frac{e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1-e^{\xi}}d\xi\Big).\end{aligned}$$

Consider

$$J_2=\oint_{\mathcal{C}_2}\frac{e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1-e^{\xi}}d\xi,$$

where the contour $\mathcal{C}_2$ is the oriented rectangle with vertexes at the points $R'+(\alpha+2\pi)i$, $R'+\alpha i$, $-R'+\alpha i$, $-R'+(\alpha+2\pi)i$ $(some\ fixed\ R'\in(0,\infty))$, in a clockwise direction.

Notice that $\mathcal{C}_2$ encloses only the singularity $2\pi k_\alpha i$ of the integrand. Then similarly, let $R'\to\infty$ and use Cauchy residue theorem to find that

$$\Big(\int_{-\infty+(\alpha+2\pi)i}^{\infty+(\alpha+2\pi)i}+\int_{\infty+\alpha i}^{-\infty+\alpha i}\Big)\frac{e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1-e^{\xi}}d\xi=-2\pi iRes\Big(\frac{e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1-e^{\xi}};2\pi k_\alpha i\Big).$$

So

$$f_\alpha(xe^{-2\pi i}, q) - f_\alpha(x, q) = i\sqrt{\frac{\pi}{\ln 1/q}}e^{\frac{(\log x - 2\pi k_\alpha i)^2}{4\ln q}}.$$

(iii) We can show that $f_\alpha(x, q)$ is a solution of the q-difference equation (1.3):

$$\begin{aligned}
\frac{x}{q}f_\alpha(\frac{x}{q^2}, q) - f_\alpha(x, q) &= \frac{1}{\sqrt{4\pi\ln 1/q}}(\int_{-\infty+\alpha i}^{\infty+\alpha i}\frac{x}{q}\frac{e^{\frac{(\log x - \xi - 2\ln q)^2}{4\ln q}}}{1-e^\xi}d\xi \\
&\quad - \int_{-\infty+\alpha i}^{\infty+\alpha i}\frac{e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1-e^\xi}d\xi) \\
&= \frac{1}{\sqrt{4\pi\ln 1/q}}\int_{-\infty+\alpha i}^{\infty+\alpha i}(\frac{e^\xi e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1-e^\xi}d\xi - \frac{e^{\frac{(\log x-\xi)^2}{4\ln q}}}{1-e^\xi})d\xi \\
&= -\int_{-\infty}^{\infty}\frac{e^{\frac{(\log x-\eta-\alpha i)^2}{4\ln q}}}{\sqrt{4\pi\ln 1/q}}d\eta \qquad (\xi = \eta + \alpha i) \\
&= -1,
\end{aligned}$$

where the last equality arises from the following integral:

$$\frac{1}{\sqrt{4\pi\ln 1/q}}\int_{-\infty+bi}^{\infty+bi}e^{\frac{(a+\eta)^2}{4\ln q}}d\eta = 1,$$

for $a, b \in \mathbb{R}$. This integral can be obtained by means of Cauchy's theorem and Lemma 1.3.9.

Due to the assertions (i) and (ii), we only need to consider the asymptotic behavior of $f_\alpha(x, q)$ with $\alpha \in (0, 2\pi)$.

Let $N \in \mathbb{N}^*$. Denote formal q-Borel transform of $\hat{y}$ by φ, i.e., $\varphi(\xi) = \sum_{n=0}^{\infty}\xi^n$, which is convergent in the unit disc and admits $\frac{1}{1-\xi}$ as its analytic continuation outsides the unit disc. For all $n \in \mathbb{N}^*$ and $\xi \neq 1$, we have

$$\frac{1}{1-\xi} = \varphi_n(\xi) + \psi_n(\xi),$$

with $\varphi_n(\xi) = 1 + \xi + \cdots + \xi^{n-1}$ and $\psi_n(\xi) = \frac{\xi^n}{1-\xi}$.

Notice that from the Gaussian integral, for all $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, and $x \in \tilde{\mathbb{C}}^*$

$$\mathcal{L}_{q;1}^{\alpha}(\xi^n) = q^{-n^2}x^n.$$

So we have for $\alpha \in (0, 2\pi)$

$$\begin{aligned}
f_\alpha(x,q) - \sum_{n=0}^{N-1} q^{-n^2} x^n &= \frac{1}{\sqrt{4\pi \ln 1/q}} \int_0^{\infty e^{\alpha i}} q^{\frac{1}{4}(\log_q(\frac{x}{\xi}))^2} \frac{1}{1-\xi} \frac{d\xi}{\xi} - \sum_{n=0}^{N-1} q^{-n^2} x^n \\
&= \mathcal{L}_{q;1}^{\alpha}(\varphi_N(\xi) + \psi_N(\xi)) - \sum_{n=0}^{N-1} q^{-n^2} x^n \\
&= \mathcal{L}_{q;1}^{\alpha}(\psi_N(\xi)) \\
&= \frac{1}{\sqrt{4\pi \ln 1/q}} \int_0^{\infty e^{\alpha i}} q^{\frac{1}{4}\log_q^2(\frac{x}{\xi})} \psi_N(\xi) \frac{d\xi}{\xi}.
\end{aligned}$$

Let D_α be the distance between the point 1 and the radial d_α, i.e., $D_\alpha = dist(\{1\}; d_\alpha)$. Then with the Gaussian integral

$$\begin{aligned}
\left| f_\alpha(x,q) - \sum_{n=0}^{N-1} q^{-n^2} x^n \right| &= \left| \frac{1}{\sqrt{4\pi \ln 1/q}} \int_0^{\infty e^{\alpha i}} q^{\frac{1}{4}\log_q^2(\frac{x}{\xi})} \frac{\xi^N}{1-\xi} \frac{d\xi}{\xi} \right| \\
&\leq \frac{q^{-\frac{1}{4}\arg_q^2(xe^{-\alpha i})}}{\sqrt{4\pi \ln 1/q}} \int_0^{\infty} q^{\frac{1}{4}(\log_q^2 \frac{|x|}{|\xi|})} \frac{|\xi|^{N-1}}{|1-\xi|} d|\xi| \\
&\leq \frac{q^{-\frac{1}{4}\arg_q^2(xe^{-\alpha i})}}{D_\alpha \sqrt{4\pi \ln 1/q}} \int_0^{\infty} q^{\frac{1}{4}(\log_q^2 \frac{|x|}{r})} r^{N-1} dr \\
&\leq C q^{-(N^2 + \frac{1}{4}\arg_q^2(xe^{-\alpha i}))} |x|^N .
\end{aligned}$$

This completes the proof. ⊛

Let $f_-(x,q)$ be the function associated with $f_\alpha(x,q)$ for $\alpha \in (-2\pi, 0)$, then

$$f_-(x,q) = \frac{1}{\sqrt{4\pi \ln 1/q}} \int_{-\infty}^{\infty} \frac{e^{\frac{(\log x - \xi)^2}{4\ln q}}}{1-e^{\xi}} d\xi, \tag{3.3}$$

where the path of integration may be taken as either the real axis of ξ intended by the lower half of a small circle described about the origin as center, denoted by the path $(-\infty,\ \underline{0},\ \infty)$, or as a straight line parallel to the real axis of ξ and below it at a distance less than 2π.

Remark 3.2.2. *Since*

$$f_{\alpha+2\pi}(x,q) = f_\alpha(xe^{-2\pi i}, q),$$

we get

$$f_-(xq^{-2\pi i}) - f_-(x,q) = i\sqrt{\frac{\pi}{\ln 1/q}}e^{\frac{(\log x)^2}{4\ln q}},$$

where the function given in the right hand is infinitely small or said flat as $x \to 0$ *and is the solution of the homogeneous* q*-difference equation associated with (1.3):*

$$\frac{x}{q}y(\frac{x}{q^2}) - y(x) = 0.$$

On the other hand, the relation connecting $f_-(xq^{-2\pi i})$ *and* $f_-(x,q)$ *can be seen as a Stokes phenomenon in the singular direction argument* $0 \in 2\pi\mathbb{Z}$ *in the Riemann surface of the logarithm function.*

Remark 3.2.3. *Here we suppose that* q *is a real parameter with* $0 < q < 1$*. But* q *is also a analytic parameter, so by the analytical continuation principle, all the conclusions remain true for a complex parameter* q *with* $0 < |q| < 1$*. Correspondingly, the range of definition of* τ *can be taken as the complex half-plane* $\{\tau \in \mathbb{C} \mid Re(\tau) > 0\}$*.*

Therefore, we can get one q-Borel sum $F_\alpha(\tau,z)$ of $\sum\limits_{n=0}^{\infty} e^{n^2\tau+nz}$:

$$\begin{aligned} F_\alpha(\tau,z) &= f_\alpha(e^z, e^{-\tau}) && (3.4)\\ &= \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty+\alpha i}^{\infty+\alpha i} \frac{e^{-\frac{(z-\xi)^2}{4\tau}}}{1-e^\xi}d\xi, && (3.5) \end{aligned}$$

for $\alpha \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, $\mathrm{Re}(\tau) > 0$ and $z \in \mathbb{C}$.

Theorem 3.2.4. *For any given* $k \in \mathbb{Z}$*, we have* $U_k(\tau,z) = F_\alpha(\tau,z)$*, where* $\alpha \in (2k\pi, 2(k+1)\pi)$*,* $Re(\tau) > 0$ *and* $z \in \Omega_k$*.*

Proof. For fixed k, one have

$$\begin{aligned} U_k(\tau,z) &= \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{+\infty} e^{-\frac{s^2}{4\tau}}\frac{1}{1-e^{z+s}}ds \\ &= \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty+\mathrm{Im}(z)i}^{+\infty+\mathrm{Im}(z)i} e^{-\frac{(\xi-z)^2}{4\tau}}\frac{1}{1-e^\xi}d\xi, \end{aligned}$$

where $z \in \Omega_k$.

We know that putting different $z \in \Omega_k$, the integral path would change, but the integrand has no singularity along any $(-\infty + \mathrm{Im}(z)i, +\infty + \mathrm{Im}(z)i)$. Therefore, by the assertion (i) of Proposition 3.2.1, one conclude that

$$U_k(\tau, z) = F_\alpha(\tau, z) \text{ for any } \alpha \in (2k\pi, 2(k+1)\pi).$$

It proves our theorem. ⊛

Remark 3.2.5. *By the assertion (ii) of Proposition 3.2.1 and Theorem 3.2.4, one can get the analytic continuation of $U_k(\tau, z)$ and*

$$U_{k+1}(\tau, z) - U_k(\tau, z - 2\pi i) = i\sqrt{\frac{\pi}{\tau}}e^{-\frac{[z-2(k+1)\pi i]^2}{4\tau}} \textit{ for } z \in \Omega_{k+1}.$$

In fact, from the assertion (ii) of Proposition 3.2.1, we have

$$F_\alpha(\tau, z - 2\pi i) - F_\alpha(\tau, z) = i\sqrt{\frac{\pi}{\tau}}e^{-\frac{(z-2\pi k_\alpha i)^2}{4\tau}},$$

where $2\pi k_\alpha$ is the integer between α and $\alpha + 2\pi$. And by the above theorem,

$$U_{k+1}(\tau, z) - U_k(\tau, z - 2\pi i) = F_{\alpha+2\pi}(\tau, z) - F_\alpha(\tau, z),$$

for any $\alpha \in (2k\pi, 2(k+1)\pi)$ and $z \in \Omega_{k+1}$.

Chapter 4 Sums Based on Jacobi Theta Function

In this chapter, we shall define a sum of the q-series $\hat{y}(x,q)$ defined by (1.4) in a different procedure, which is formed by Jacobi theta function.

§4.1 Preparation

From the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi},$$

we can find that if, for convergence, α has positive real part, then for $\beta \in \mathbb{C}$

$$\int_{-\infty}^{\infty} e^{-\alpha t^2+\beta t} dt = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/(4\alpha)},$$

where the root is taken with a positive real part, and the path of integration is either the real axis or a line parallel to the real axis.

Lemma 4.1.1. *For $Re(t) > 0$,*

$$\sum_{n=-\infty}^{\infty} e^{-(n+a)^2 \pi t} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-n^2\pi/t} e^{2\pi i n a}.$$

Proof. Denote the left side by $f(a)$ and notice that f has period 1. Expand f as a Fourier series

$$f(a) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi i n a},$$

where $C_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) e^{-2\pi i n x} dx$, $n = 0, \pm 1, \pm 2, \cdots$.

Then

$$C_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} e^{-(m+x)^2 \pi t} e^{-2\pi i n x} dx$$

$$\begin{aligned}
&= \sum_{m=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-(m+x)^2\pi t} e^{-2\pi i n x} dx \\
&= \sum_{m=-\infty}^{\infty} \int_{-\frac{1}{2}+m}^{\frac{1}{2}+m} e^{-\pi t y^2} e^{-2\pi i n y} dy \\
&= \int_{-\infty}^{\infty} e^{-\pi t y^2} e^{-2\pi i n y} dy \\
&= \frac{1}{\sqrt{t}} e^{-n^2\pi/t}.
\end{aligned}$$

The result is proved. ⊛

Now write

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n(n-1)\cdots(n-k)}{k!} = \frac{n!}{k!(n-k)!},$$

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}),$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q)\cdots(1-q^n)}{(1-q)\cdots(1-q^k)(1-q)\cdots(1-q^{n-k})} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Lemma 4.1.2. *For* $|x| < 1$, $|q| < 1$,

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty},$$

where $(a;q)_\infty = \prod_{k=0}^{\infty} (1-aq^k)$.

First Proof. Let

$$f_a(x) = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k.$$

Apply the q-difference operator $\triangle_q$ to both sides. Then

$$\begin{aligned}
\frac{f_a(x) - f_a(qx)}{x} &= \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} (1-q^k) x^{k-1} \\
&= (1-a) \sum_{k=1}^{\infty} \frac{(aq;q)_{k-1}}{(q;q)_{k-1}} x^{k-1}
\end{aligned}$$

$$
\begin{aligned}
&= (1-a)\sum_{k=0}^{\infty}\frac{(aq;q)_k}{(q;q)_k}x^k \\
&= (1-a)f_{aq}(x),
\end{aligned}
$$

or

$$f_a(x) - f_a(qx) = (1-a)xf_{aq}(x). \tag{4.1}$$

Now consider

$$
\begin{aligned}
f_a(x) - f_{aq}(x) &= \sum_{k=0}^{\infty}\frac{(aq;q)_{k-1}}{(q;q)_k}(1-a-1+aq^k)x^k \\
&= -axf_{aq}(x),
\end{aligned}
$$

or

$$f_a(x) = (1-ax)f_{aq}(x). \tag{4.2}$$

Eliminate $f_{aq}(x)$ from (4.1) and (4.2) to get

$$f_a(x) = \frac{1-ax}{1-x}f_a(qx).$$

Iterate this relation n times and let $n \to \infty$ to arrive at

$$
\begin{aligned}
f_a(x) &= \frac{(ax;q)_n}{(x;q)_n}f_a(q^nx) \\
&= \frac{(ax;q)_\infty}{(x;q)_\infty}f_a(0) \\
&= \frac{(ax;q)_\infty}{(x;q)_\infty}.
\end{aligned}
$$

This proves the lemma. ⊛

Second Proof. The infinite product $\frac{(ax;q)_\infty}{(x;q)_\infty}$ is uniformly and absolutely convergent for fixed a and q in $|x| \le 1-\epsilon$ and so represents an analytic function in $|x| < 1$. Consider its Taylor expansion in $|x| < 1$,

$$F(x) = \frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{n=0}^{\infty}A_nx^n.$$

Clearly,

$$F(x) = \frac{(1-ax)}{(1-x)} F(qx).$$

This implies

$$(1-x)\sum_{n=0}^{\infty} A_n x^n = (1-ax)\sum_{n=0}^{\infty} A_n q^n x^n.$$

Equate the coefficients of x^n on both sides. Then

$$\begin{aligned} A_n &= \frac{1-aq^{n-1}}{1-q^n} A_{n-1} \\ &= \frac{(a;q)_n}{(q;q)_n}. \end{aligned}$$

This completes the second proof. ⊛

Remark 4.1.3. *Set $a = q^{-N}$ in above lemma. Then*

$$\sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} x^k = (x;q)_N = (1-x)\cdots(1-xq^{N-1}).$$

§4.2 Jacobi theta function

In this section, we give some important properties of Jacobi theta function

$$\theta(x,q) = \theta_{00}(\chi,\omega) = \sum_{n\in\mathbb{Z}} q^{n^2} x^n,$$

where we put $q = e^{\pi i\omega}$ with $\mathrm{Im}(\omega) > 0$ and $x = e^{-2\pi i\chi}$ with $\chi \in \mathbb{C}$. Then $0 < |q| < 1$.

By simply observing, we have that $\theta(x,q) = \theta(\frac{1}{x}, q)$.

Lemma 4.2.1. *(i)The functional equation:*

$$qz\theta(q^2x,q) = \theta(x,q).$$

(ii)The modular relation:

$$\theta(x,q)=\sqrt{\frac{\pi}{\log 1/q}}e^{-\frac{(\log x)^2}{4\log q}}\theta(x^*,q^*),$$

where $0<|q|<1$ *and where* $q^*=e^{\frac{\pi^2}{\log q}}$ *and* $x^*=e^{-\pi i\frac{\log x}{\log q}}$ *are modular variables.*

Proof. (i) By definition,

$$\begin{aligned}\theta(q^2x,q) &= \sum_{n\in\mathbb{Z}}q^{n^2}(q^2x)^n\\ &= \frac{1}{qx}\sum_{n\in\mathbb{Z}}q^{(n+1)^2}x^{n+1}\\ &= \frac{1}{qx}\theta(x,q).\end{aligned}$$

(ii) Firstly, notice that the modular variable can be written as

$$q^*=e^{\frac{\pi^2}{\log q}}=e^{-\frac{\pi i}{\omega}},\qquad x^*=e^{-\pi i\frac{\log x}{\log q}}=e^{2\pi i\frac{\chi}{\omega}}.$$

Then

$$\theta(x,q)=\theta(\chi,\omega)=\sum_{n\in\mathbb{Z}}e^{(n^2\pi i\omega-2n\pi i\chi)},$$

$$\begin{aligned}\theta(x^*,q^*) &= \theta(\frac{\chi}{\omega},-\frac{1}{\omega})\\ &= \sum_{n\in\mathbb{Z}}e^{(-n^2\pi i/\omega+2n\pi i\chi/\omega)}\\ &= \sum_{n\in\mathbb{Z}}e^{-\frac{\pi i}{\omega}(n^2-2n\chi)}\\ &= e^{\frac{\pi i\chi^2}{\omega}}\sum_{n\in\mathbb{Z}}e^{-\frac{\pi i}{\omega}(n-\chi)^2}.\end{aligned}$$

According to Lemma 4.1.1, we have

$$\sum_{n\in\mathbb{Z}}e^{-\frac{\pi i}{\omega}(n-\chi)^2}=\sqrt{(-i\omega)}\sum_{n\in\mathbb{Z}}e^{n^2\pi i\omega}e^{-2n\pi i\chi},$$

so

$$\theta(\frac{\chi}{\omega},-\frac{1}{\omega})=\sqrt{(-i\omega)}e^{\frac{\pi i\chi^2}{\omega}}\theta(\chi,\omega).$$

That is to say,

$$\theta(x^*, q^*) = \sqrt{\frac{\log 1/q}{\pi}} e^{\frac{(\log x)^2}{4\log q}} \theta(x, q).$$

This completes the proof. ⊛

Lemma 4.2.2. *For $0 < |q| < 1$ and $x \in \mathbb{C}^*$,*

$$(x;q)_\infty (\frac{q}{x};q)_\infty (q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} x^k.$$

Proof. Take $N = 2n$ in Remark 4.1.3 to obtain

$$(x;q)_{2n} = \sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q (-1)^{k+n} q^{(k+n)(k+n-1)/2} x^{k+n}.$$

Then replace x by xq^{-n} and rewrite $(xq^{-n};q)_{2n}$ as

$$(xq^{-n};q)_n (x;q)_n = (-1)^n x^n q^{-n^2+n(n-1)/2} (\frac{q}{x};q)_n (x;q)_n.$$

The above identity then becomes

$$(\frac{q}{x};q)_n (x;q)_n = \sum_{k=-n}^{n} \frac{(q;q)_{2n} (-1)^k q^{k(k-1)/2} x^k}{(q;q)_{n+k} (q;q)_{n-k}}.$$

When let $n \to \infty$, this gives

$$(\frac{q}{x};q)_\infty (x;q)_\infty = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} x^k}{(q;q)_\infty}.$$

This limiting process can be justified by Tannery's theorem. The result here is called the *triple product identity*. ⊛

Remark 4.2.3. *When replacing q with q^2 and x with $-qx$, we get Jacobi's triple product formula*

$$\theta(x, q) = \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + xq^{2n+1})(1 + \frac{q^{2n+1}}{x}).$$

From now on, we assume that $\omega \in i\mathbb{R}_+ \backslash \{0\}$, and then

$$q = e^{\pi i \omega} \in (0, 1).$$

Lemma 4.2.4. *For $\epsilon > 0$, define*

$$\Omega_\epsilon = \bigcap_{n\in\mathbb{Z}} \{x \in \mathbb{C}^* : \left|x + q^{2n+1}\right| > \epsilon q^{2n+1}\}.$$

Then there exists $C > 0$ such that for all $\epsilon > 0$ small enough

$$|\theta(x,q)| \geq C\epsilon\theta(|x|, q), \text{ where } x \in \Omega_\epsilon.$$

Proof. In order to make sure that $\Omega_\epsilon \neq \emptyset$, it is sufficient that $\epsilon \leq 1$. Furthermore, we suppose that $\epsilon < \frac{1-q^2}{1+q^2}$ so that Ω_ϵ is connected.

By substituting qx for x, the region Ω_ϵ admits Γ_ϵ as fundamental domain:

$$\Gamma_\epsilon = \{x \in \mathbb{C}^* : \left|x + q^{-1}\right| > \epsilon q^{-1}, \frac{2q}{(1+q^2)} < |x| \leq \frac{2}{q(1+q^2)}\}.$$

Recalling that $qx\theta(q^2x, q) = \theta(x, q)$, then we have

$$\frac{|\theta(q^2x,q)|}{\theta(q^2|x|, q)} = \frac{|\theta(x,q)|}{\theta(|x|, q)}.$$

Therefore, we can prove this lemma with Γ_ϵ instead of Ω_ϵ.

The function $\frac{|\theta(x,q)|}{\theta(|x|,q)}$ is continuous and nonzero, so it could attain its minimum C on the compact ring $\{x \in \mathbb{C}^* : \frac{2q}{(1+q^2)} \leq |x| \leq \frac{2}{q(1+q^2)}\}$.

Then with Jacobi's triple product formula in Remark 4.2.3, we get the result. ⊛

If one want to know more properties on Jacobi theta function, one can refer to [1].

§4.3 Other q-Borel sums of $\sum_{n=0}^{\infty} e^{n^2\tau+nz}$

In this section, we will present another sum $g_\lambda(x,q)$ of $\hat{y}(x,q)$.

Without confusion, we will denote Jacobi theta function $\theta(x,q) = \sum_{n\in\mathbb{Z}} q^{n^2}x^n$ by $\theta(x)$ for simplicity.

From the functional equation in Lemma 4.2.1, we deduce that

$$\theta(q^{2n}x) = q^{-n^2}x^{-n}\theta(x),$$

so, for any $\lambda \in \mathbb{C}^*$

$$\sum_{n\in\mathbb{Z}} \frac{1}{\theta(\lambda q^{2n})} = \sum_{n\in\mathbb{Z}} \frac{q^{n^2}\lambda^n}{\theta(\lambda)} = 1,$$

provided that $\theta(\lambda q^{2n}) \neq 0$ for all integers n.

From Jacobi's triple product formula in Remark 4.2.3, we deduce that $\theta(\lambda q^{2n}) \neq 0$ holds for all $\lambda \in \mathbb{C}^*\backslash(-q^{2\mathbb{Z}+1})$.

Because of the above deduction, we can get, for any integer $m \in \mathbb{Z}$ and any $\lambda \in \mathbb{C}^*\backslash(-q^{2\mathbb{Z}+1})$,

$$\sum_{\xi\in\lambda q^{2\mathbb{Z}}} \frac{\xi^m}{\theta(\xi)} = q^{-m^2} \sum_{n\in\mathbb{Z}} \frac{q^{(m+n)^2}\lambda^{m+n}}{\theta(\lambda)} = q^{-m^2}.$$

Thus the divergent power series $\sum_{n\geq 0} q^{-n^2}x^n$ may be written as a double series and by this way we are led to $\sum_{\xi\in\lambda q^{2\mathbb{Z}}} \frac{1}{1-\xi x}\frac{1}{\theta(\xi)}$.

So, if $\lambda \in \mathbb{C}^*\backslash q^{2\mathbb{Z}}$, we can define

$$\begin{aligned} g_\lambda(x,q) &= \sum_{\xi\in\lambda q^{2\mathbb{Z}}} \frac{1}{1-\xi}\frac{1}{\theta(\frac{\xi}{x})} \\ &= \frac{1}{\theta(\frac{\lambda}{x})} \sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{1-\lambda q^{2n}}(\frac{\lambda}{x})^n, \end{aligned}$$

for all $x \in \mathbb{C}^*\backslash(-\lambda q^{2\mathbb{Z}+1})$.

Proposition 4.3.1. *If $\lambda \in \mathbb{C}^* \setminus q^{2\mathbb{Z}}$, we define*

$$g_\lambda(x,q) = \frac{1}{\theta(\frac{\lambda}{x})} \sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{1-\lambda q^{2n}}(\frac{\lambda}{x})^n, \tag{4.3}$$

for all $x \in \mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$. Then we have

(i) g_λ is holomorphic over $\mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$ and admits $(-\lambda q^{2\mathbb{Z}+1})$ as a set of simples poles.

(ii) $g_\lambda(xe^{2\pi i}, q) = g_\lambda(x, q)$.

(iii) $g_\lambda(x, q)$ *is the unique solution of (1.3) which admits the power series* $\sum_{n=0}^{\infty} q^{-n^2} x^n$ *as asymptotic expansion in the sense: for* $x \to 0$ *in* $\mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$*, there exist* $C > 0$ *and* $A > 0$*, such that for all* $N \in \mathbb{N}^*$ *and* $\epsilon > 0$ *small enough*

$$| \, g_\lambda(x, q) - \sum_{n=0}^{N-1} q^{-n^2} x^n \, | \leq \frac{C}{\epsilon} A^N q^{-N^2} \, |x|^N \, ,$$

where $x \in \mathbb{C}^* \setminus \bigcup_{n \in \mathbb{Z}} \{q^{2n-1}x : |x + \lambda| \leq \epsilon\}$.

Proof. Firstly, in view of the definition of $g_\lambda(x, q)$, we know that $g_\lambda(x, q) = g_{\lambda q^{2m}}(x, q)$ for all $m \in \mathbb{Z}$. So it is reasonable to give this proof with $q \leq |\lambda| < 1$.

(i) Note that

$$\begin{aligned} g_\lambda(x, q) &= \frac{1}{\theta(\frac{\lambda}{x})} \Big[\sum_{n \geq 0} \frac{q^{n^2}}{1 - \lambda q^{2n}} (\frac{\lambda}{x})^n + \sum_{n < 0} \frac{q^{n^2}}{1 - \lambda q^{2n}} (\frac{\lambda}{x})^n \Big] \\ &:= \frac{1}{\theta(\frac{\lambda}{x})} [g^{n \geq 0}(x) + g^{n < 0}(x)]. \end{aligned}$$

Because $\left|\frac{1}{1-x}\right| \leq \frac{1}{1-|x|}$, for $0 < |x| < 1$ and $\left|\frac{1}{1-x}\right| \leq \frac{1}{|x|-1}$, for $|x| > 1$, $g^{n \geq 0}(x)$ or $g^{n<0}(x)$ represents respectively a function analytic on $\mathbb{C}^* \bigcup \{\infty\}$ or $\mathbb{C}$.

Notice that $g_\lambda(x, q)$ is a ratio of two analytic functions whose singularities occur at the zeroes of $\theta(\frac{\lambda}{x})$, and from Jacobi's triple product formula, we know that $(-\lambda q^{2\mathbb{Z}+1})$ is the only set of simple zeroes for $\theta(\frac{\lambda}{x})$. So $g_\lambda(x, q)$ is holomorphic over $\mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$ and $(-\lambda q^{2\mathbb{Z}+1})$ is the set of its simple poles.

(ii) By definition,

$$g_\lambda(xe^{2\pi i}) = \frac{1}{\theta(\frac{\lambda}{xe^{2\pi i}})} \sum_{n \in \mathbb{Z}} \frac{q^{n^2}}{1 - \lambda q^{2n}} (\frac{\lambda}{xe^{2\pi i}})^n = g_\lambda(x).$$

(iii) It is obvious that

$$\begin{aligned}
\frac{x}{q}g_\lambda(\frac{x}{q^2},q)-g_\lambda(x,q) &= \frac{x}{q}\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{1}{1-\xi}\frac{1}{\theta(\frac{q^2\xi}{x})}-\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{1}{1-\xi}\frac{1}{\theta(\frac{\xi}{x})}\\
&= \sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{\xi}{1-\xi}\frac{1}{\theta(\frac{\xi}{x})}-\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{1}{1-\xi}\frac{1}{\theta(\frac{\xi}{x})}\\
&= -\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{1}{\theta(\frac{\xi}{x})},\ where\ x\in\mathbb{C}^*\backslash(-\lambda q^{2\mathbb{Z}+1})\\
&= -1.
\end{aligned}$$

So $g_\lambda(x,q)$ satisfies the q-difference equation (1.3).

Finally let us consider the asymptotic behavior of $g_\lambda(x,q)$ as $x\to 0$ in some compact subset D_ϵ of $\mathbb{C}^*\backslash\bigcup_{n\in\mathbb{Z}}\{q^{2n+1}x\ :\ |x+\lambda|\ \leq\ \epsilon\}$ with $\epsilon\in(0,\frac{(1-q^2)|\lambda|}{1+q^2})$.

For all $n\in\mathbb{N}^*$ and $\xi\neq 1$, we note that

$$\frac{1}{1-\xi}=\varphi_n(\xi)+\psi_n(\xi),$$

with $\varphi_n(\xi)=1+\xi+\cdots+\xi^{n-1}$ and $\psi_n(\xi)=\frac{\xi^n}{1-\xi}$.

Obviously, there exist $C>0,\ A>0$, independent on n, such that

$$|\varphi_n(\xi)|\leq CA^n\,|\xi|^n,\ if\ \ |\xi|\geq 1 \text{ and } |\psi_n(\xi)|\leq CA^n\,|\xi|^n,\ if\ \ |\xi|<1.$$

And we have

$$\begin{aligned}
&g_\lambda(x,q)-\sum_{n=0}^{N-1}q^{-n^2}x^n\\
=&\sum_{\xi\in\lambda q^{2\mathbb{Z}}}(\varphi_N(\xi)+\psi_N(\xi))\frac{1}{\theta(\frac{\xi}{x})}-\sum_{n=0}^{N-1}q^{-n^2}x^n\\
=&\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{\psi_N(\xi)}{\theta(\frac{\xi}{x})}\\
=&\frac{1}{\theta(\frac{\lambda}{x})}\left[\sum_{n\geq 0}\psi_N(\lambda q^{2n})q^{n^2}(\frac{\lambda}{x})^n+\sum_{n<0}\psi_N(\lambda q^{2n})q^{n^2}(\frac{\lambda}{x})^n\right]
\end{aligned}$$

$$:= \quad I + J.$$

With Lemma 4.2.4, we can deduce that for all $x \in D_\epsilon$

$$\begin{aligned} |I| &< \frac{CA^N |\lambda|^N}{\epsilon\theta(\left|\frac{\lambda}{x}\right|)} \sum_{n\geq 0} \left|\frac{q^{2N}\lambda}{x}\right|^n q^{n^2} \\ &< \frac{CA^N |\lambda|^N}{\epsilon\theta(\left|\frac{\lambda}{x}\right|)} \theta(q^{2N}\left|\frac{\lambda}{x}\right|) \\ &= \frac{C}{\epsilon} A^N q^{-N^2} |x|^N, \end{aligned}$$

where C and A are positive constants independent on N.

In order to estimate the contribution coming from J, we write

$$\begin{aligned} J &= \frac{g^{n<0}(x)}{\theta(\frac{\lambda}{x})} - \sum_{n<0} \frac{\varphi_N(q^{2n}\lambda)}{\theta(\frac{q^{2N}\lambda}{x})} \\ &:= J_1 + J_2. \end{aligned}$$

Since $g^{n<0}(z)$ is bounded on any compact subset of $\mathbb{C}$, we have by Lemma 4.2.4,

$$|J_1| < \frac{C}{\epsilon} \frac{1}{\theta(\left|\frac{\lambda}{x}\right|)},$$

for $x \in D_\epsilon$.

Furthermore, for $N \in \mathbb{N}^*$

$$\begin{aligned} |J_1| &< \frac{C}{\epsilon} q^{-N^2} (\left|\frac{\lambda}{x}\right|)^{-N} \frac{1}{\theta(q^{2N}\left|\frac{\lambda}{x}\right|)} \\ &< \frac{C}{\epsilon} (\frac{1}{q})^N q^{-N^2} |x|^N. \end{aligned}$$

It remains to consider J_2. Remember that $|\varphi_n(\xi)| \leq CA^n |\xi|^n$, if $|\xi| > 1$. Then similarly to the case of I, there exist $C > 0$ and $A > 0$, independent on $N \in \mathbb{N}^*$ and $x \in D_\epsilon$, such that

$$|J_2| < \frac{C}{\epsilon} A^N q^{-N^2} |x|^N.$$

Combining all these relations together, we obtain the estimation of $\left|g_\lambda(x,q) - \sum_{n=0}^{N-1} q^{-n^2}x^n\right|$, which completes the proof. ⊛

Remark 4.3.2. *Since $\lambda \mapsto g_\lambda(x,q)$ is left invariant by $\lambda \mapsto q\lambda$, one can calculate the cocycle $g_\lambda - g_\mu$ as follows:*

$$g_\lambda(x,q) - g_\mu(x,q) = \frac{K(\lambda,\mu,x)}{\theta(x)},$$

where $\lambda,\ \mu \in \mathbb{C}^*\backslash q^{2\mathbb{Z}}$, $z \in \mathbb{C}^*\backslash((-\lambda q^{2\mathbb{Z}+1})\bigcup(-\mu q^{2\mathbb{Z}+1}))$ and

$$K(\lambda,\mu,x) = \frac{(q^2;q^2)_\infty^3\theta(-\frac{\lambda}{\mu}q)\theta(\frac{\lambda\mu}{x})\theta(\frac{1}{x})}{\theta(-q\lambda)\theta(-\frac{\mu}{q})\theta(\lambda x)\theta(\frac{\mu}{x})}.$$

Such elliptic cocycles play the role of Stokes multipliers and allow to classify the corresponding q-difference equation.

Remark 4.3.3. *In our results, q is assumed to belong to the interval $(0,1)$. By the standard argument of analytical continuation, the formulae remain valid for all $q = e^{\pi i\omega}$ with $Im(\omega) > 0$.*

Therefore, we can get the other q-Borel sum $G_\lambda(\tau,z)$ of $\sum_{n=0}^{\infty} e^{n^2\tau+nz}$:

$$\begin{aligned} G_\lambda(\tau,z) &= g_\lambda(e^z, e^{-\tau}) && (4.4)\\ &= \frac{1}{\theta(\frac{\lambda}{e^z})}\sum_{n\in\mathbb{Z}}\frac{e^{-n^2\tau}}{1-\lambda e^{-2n\tau}}(\frac{\lambda}{e^z})^n, && (4.5)\end{aligned}$$

for $\lambda \in \mathbb{C}^* \setminus q^{2\mathbb{Z}}$, $\mathrm{Re}(\tau) > 0$ and $z \in \tilde{\mathbb{C}}^* \setminus (\log(-\lambda q^{2\mathbb{Z}+1}))$.

Chapter 5 Generalization of Mordell's Theorem

In this chapter, we would like to prove a natural generalization of Mordell's theorem by means of Stokes analysis on two sums of $\hat{y}(x,q)$ defined by (1.4). And we shall generalize our result.

§5.1 Mordell's theorem

Mordell begun his paper (cf. [35]) with the following observation: Professor Siegel in a memoir recently published dealing with the manuscripts left by Riemann has pointed out that Riemann dealt with some integrals of the type

$$I=\int_{-\infty}^{\infty}\frac{e^{ax^2+bx}}{e^{cx}+d}dx$$

in his researches on the zetafunction. Not only can the usual functional equation be thus found, but also an asymptotic formula is obtained for the zetafunction of which the first term gives the well known approximate functional equation due to Hardy and Littlewood (cf. [19, 20, 21])$\cdots$.

The general integral or particular cases have also been considered by Kronecker, Lerch, Hardy, Ramanujan, van der Corput and Mordell himself. By taking $a=\pi i\omega$ and $b=-2\pi\chi$, it has been already proved in [34] that the general integral I can be reduced to two standard forms. The first is the integral

$$\int_{-\infty}^{\infty}\frac{e^{\pi i\omega t^2-2\pi t\chi}}{e^{2\pi t}-1}dt,$$

while the second is the integral

$$\int_{-\infty}^{\infty}\frac{e^{\pi i\omega t^2-2\pi t\chi}}{e^{2\pi\omega t}-1}dt.$$

The path of these integrals will be explained later.

On the other hand, as said by Mordell himself in [35], the starting point of his investigations was the theory of the positive, definite binary quadratic form

$$ax^2 + 2hxy + by^2,$$

where a, h, b are integers, so that the determinant of the form is

$$h^2 - ab = -D < 0, \ \ say.$$

Let $F(D)$ be the number of uneven classes of forms of given determinant $-D$, that is, classes of forms in which a and b are not both even.

Until now, the formulae for the class number have been more nearly two centuries old. Dirichlet in 1839 proved that when $-D$ is negative and has no squared factors > 1,

$$F(D) = \frac{2}{\pi}\sqrt{D}((\frac{-D}{1}) + \frac{1}{3}(\frac{-D}{3}) + \frac{1}{5}(\frac{-D}{5}) + \cdots).$$

Mordell in [33] published his results which states that for all $-D < 0$,

$$\frac{F(D)}{\sqrt{D}} = \frac{N(n)}{1} - \frac{N(3)}{3} + \frac{N(5)}{5} - \cdots,$$

where $N(n)$ is the number of solutions mod n of the congruence

$$x^2 \equiv D \ (\mathrm{mod}\ n).$$

Let $q = e^{\pi i \omega}$ with $\mathrm{Im}(\omega) > 0$ and let

$$\Omega(\omega) = \sum_{n=1}^{\infty} F(n) q^n$$

be the generating function for $F(n)$. Mordell in [32] discovered the a simple generating function for $\Omega(\omega)$:

$$\Omega(\omega) = \frac{i}{4\pi}\frac{f_{01}'(0)}{\theta_{01}},$$

where $f_{01}(\chi)$ denotes the unique integral function defined by the functional equations

$$\begin{cases} f_{01}(\chi+1) = f_{01}(\chi), \\ f_{01}(\chi+\omega) + f_{01}(\chi) = \theta_{01}(\chi), \end{cases}$$

and where $\theta_{01} = \theta_{01}(0,\omega)$, $\theta_{01}(\chi,\omega)$ being one of the four Jacobi functions:

$$\theta_{01}(\chi,\omega) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2n\pi i\chi}.$$

In order to express "modular" relations connecting $\Omega(\omega)$ and $\Omega(-\frac{1}{\omega})$, Mordell used the integrals $\int_{-\infty}^{\infty} \frac{te^{\pi i\omega t^2}}{e^{2\pi t}\pm 1}dt$:

$$\int_{-\infty}^{\infty} \frac{te^{\pi i\omega t^2}}{e^{2\pi t}-1}dt = -2\Omega(\omega) + \frac{2}{\omega^2}\sqrt{(-i\omega)}\Omega(-\frac{1}{\omega}) + \frac{1}{4}\theta_{00}^3(0,\omega), \tag{5.1}$$

$$\int_{-\infty}^{\infty} \frac{te^{\pi i\omega t^2}}{e^{2\pi t}+1}dt = \sum_{n=1}^{\infty}(-1)^n F(4n-1) q^{\frac{1}{4}(4n-1)} + \frac{2}{\omega^2}\sqrt{(-i\omega)}\sum_{n=1}^{\infty}(-1)^{n-1}F(n)q_1^n, \tag{5.2}$$

where $q_1 = e^{-\pi i/\omega}$ and $\theta_{00}(\chi,\omega)$ being another Jacobi functions:

$$\theta_{00}(\chi,\omega) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\pi i\chi}.$$

So Mordell published his paper [35] about the definite integral $\int_{-\infty}^{\infty} \frac{e^{\pi i\omega t^2 - 2\pi t\chi}}{e^{2\pi t}-1}dt$ in 1933:

Theorem 5.1.1. *(Mordell,1933) Let $Im(\omega) > 0$. Let f be the integral function of χ defined as follows:*

$$if(\chi,\omega) = \sum_{m\ odd}^{\pm\infty} \frac{(-1)^{\frac{1}{2}(m-1)} q^{\frac{1}{4}m^2} e^{m\pi i\chi}}{1+q^m}.$$

Let θ_{11} be the following Jacobi theta function:

$$i\theta_{11}(\chi,\omega) = \sum_{m\ odd}^{\pm\infty} (-1)^{\frac{1}{2}(m-1)} q^{\frac{1}{4}m^2} e^{m\pi i\chi}.$$

Then

$$\int_{-\infty}^{\infty} \frac{e^{\pi i\omega t^2 - 2\pi t\chi}}{e^{2\pi t}-1}dt = \frac{f(\frac{\chi}{\omega}, -\frac{1}{\omega}) + i\omega f(\chi,\omega)}{\omega\theta_{11}(\chi,\omega)}, \tag{5.3}$$

where the path of integration may be taken as either the real axis of t indented by the lower half of a small circle described about the origin as center, say the path $(-\infty, \underline{0}, \infty)$, or as a straight line parallel to the real axis of t and below it at a distance less than unity. Such a path may be denoted by $P_{0,-1}$.

Notice that the relation (5.1) follows on differentiating both sides of (5.3) with respect to χ and putting $\chi = 0$.

In the above theorem, the integral function f can be uniquely defined by two equations such as

$$\begin{cases} f(\chi+1) + f(\chi) = 0, \\ f(\chi+\omega) + f(\chi) = \theta_{11}(\chi). \end{cases}$$

And

$$\begin{cases} \theta_{11}(\chi+1) = -\theta_{11}(\chi), \\ \theta_{11}(\chi+\omega) = -e^{-\pi i(2\chi+\omega)}\theta_{11}(\chi). \end{cases}$$

So, if set $g(\chi) = \frac{f(\chi)}{\theta_{11}(\chi)}$, then $g(\chi)$ satisfies:

$$\begin{cases} g(\chi+1) = g(\chi), \\ e^{-\pi i(2\chi+\omega)}g(\chi+\omega) - g(\chi) = -1. \end{cases}$$

§5.2 Comparison between two sums of $\sum\limits_{n=0}^{\infty} q^{-n^2}x^n$

In this section, we shall compare the functions defined in the previous chapters and give a natural generalization of Mordell's theorem such as

Theorem 5.2.1. *The following relation holds for all $\lambda \in \mathbb{C}^* \setminus \{q^{2\mathbb{Z}}\}$ and $x \in \mathbb{C}^* \setminus \{-\lambda q^{2\mathbb{Z}+1}\}$:*

$$f_-(x,q) = g_\lambda(x,q) - i\sqrt{\frac{\pi}{\log 1/q}}\, e^{\frac{(\log x)^2}{4\log q}}\, g_{\lambda^*}(x^*, q^*),$$

where $f_-(x,q)$ is defined by (3.3) and $g_\lambda(x,q)$ by (4.3), and q^*, x^* and λ^* are the modular variables defined by

$$q^* = e^{\pi^2/\log q}, \quad x^* = e^{-\pi i \frac{\log x}{\log q}}, \quad \lambda^* = e^{-\pi i \frac{\log \lambda}{\log q}}.$$

Proof. To prove the result, we use a simple fact that if y_1 and y_2 are two solutions of (1.3), then $y_1 - y_2$ will be a solution of the associated homogeneous equation $\frac{x}{q} y(\frac{x}{q^2}) - y(x) = 0$ and be flat or asymptotically zero.

Let us consider

$$h_\lambda(x,q) := \frac{1}{i}\sqrt{\frac{\log 1/q}{\pi}} e^{-\frac{(\log x)^2}{4\log q}} (f_-(x,q) - g_\lambda(x,q)),$$

where $\lambda \in \mathbb{C}^* \setminus \{q^{2\mathbb{Z}}\}$ and $x \in \mathbb{C}^* \setminus \{-\lambda q^{2\mathbb{Z}+1}\}$.

Owing to Proposition 3.2.1 and Proposition 4.3.1, we have

$$\begin{cases} \dfrac{x}{q} f_-(\dfrac{x}{q^2}, q) - f_-(x,q) = -1, \\ f_-(ze^{-2\pi i}, q) - f_-(z,q) = i\sqrt{\dfrac{\pi}{\log 1/q}} e^{\frac{(\log z)^2}{4\log q}}, \end{cases}$$

and

$$\begin{cases} \dfrac{x}{q} g_\lambda(\dfrac{x}{q^2}, q) - g_\lambda(x,q) = -1, \\ g_\lambda(xe^{2\pi i}, q) - g_\lambda(x,q) = 0. \end{cases}$$

So we can find the following relations:

$$\begin{cases} h_\lambda(\dfrac{x}{q^2}, q) = h_\lambda(x,q), \\ e^{-\frac{\pi i}{\log q}\log x} e^{-\frac{\pi^2}{\log q}} h_\lambda(xe^{-2\pi i}, q) - h_\lambda(x,q) = 1. \end{cases}$$

If set $q^* = e^{\frac{\pi^2}{\log q}}$ and $x^* = e^{-\pi i \frac{\log x}{\log q}}$, then

$$(\frac{x}{q^2})^* = x^* e^{2\pi i}, \ (xe^{-2\pi i})^* = \frac{x^*}{q^{*2}}.$$

Now consider

$$h^*_\lambda(x^*, q) := -h_\lambda(x,q).$$

It follows that

$$\begin{cases} h_\lambda^*(x^* e^{2\pi i}, q) = h_\lambda^*(x^*, q), \\ \dfrac{x^*}{q^*} h_\lambda^*(\dfrac{x^*}{q^{*2}}, q) - h_\lambda^*(x^*, q) = -1. \end{cases}$$

Because $h_\lambda(x,q)$ has simple poles $(-\lambda q^{2\mathbb{Z}+1})$, $h_\lambda^*(x^*,q)$ admits simple poles in the q-spiral $(-\lambda^* q^{*2\mathbb{Z}+1})$, where $\lambda^* = e^{-\pi i \frac{\log \lambda}{\log q}}$. And observe that $h_\lambda^*(x^*,q)$ is holomorphic over $\mathbb{C}^* \backslash (-\lambda^* q^{*2\mathbb{Z}+1})$.

Therefore, we conclude that

$$h_\lambda^*(x^*, q) = g_{\lambda^*}(x^*, q^*),$$

and then get the theorem. ⊛

We say that the function $f_-(x,q)$ defined by (3.3) is a variant of the function given by the integral in Mordell's theorem. Namely, if we set $q = e^{\pi i \omega}$ and $x = e^{-2\pi i \chi}$, then

$$-\frac{1}{\sqrt{(-i\omega)}} e^{-\frac{\pi \chi^2}{i\omega}} \int_{-\infty}^{\infty} \frac{e^{\frac{\pi t^2}{i\omega} + \frac{2\pi t \chi}{\omega}}}{e^{2\pi t} - 1} dt = f_-(x,q),$$

so that one can write

$$-\sqrt{(-i\omega)} e^{\frac{\pi \chi^2}{i\omega}} \int_{-\infty}^{\infty} \frac{e^{\pi i \omega t^2 - 2\pi t \chi}}{e^{2\pi t} - 1} dt = f_-(x^*, q^*),$$

where $q^* = e^{-\frac{\pi i}{\omega}}$ and $x^* = e^{2\pi i \frac{\chi}{\omega}}$.

Corollary 5.2.2. *Replacing λ by $\frac{1}{q} e^{\pi i}$ in Theorem 5.2.1, one can get Mordell's theorem.*

In fact, recall that

$$-\sqrt{(-i\omega)} e^{\frac{\pi \chi^2}{i\omega}} \int_{-\infty}^{\infty} \frac{e^{\pi i \omega t^2 - 2\pi t \chi}}{e^{2\pi t} - 1} dt = f_-(x^*, q^*),$$

and by our main theorem

$$f_-(x^*, q^*) = g_{\lambda^*}(x^*, q^*) - i \sqrt{\frac{\pi}{\log 1/q^*}} e^{\frac{(\log x^*)^2}{4 \log q^*}} g_{\lambda^{**}}(x^{**}, q^{**}),$$

where $q^{**} = e^{\frac{\pi^2}{\log q^*}} = q$, $x^{**} = e^{-\pi i \frac{\log x^*}{\log q^*}} = \frac{1}{x}$ and $\lambda^{**} = e^{-\pi i \frac{\log \lambda^*}{\log q^*}} = \frac{1}{\lambda}$.

Therefore, we can find the following generalization of Mordell's theorem:

$$-\sqrt{(-i\omega)}e^{\frac{\pi\chi^2}{i\omega}} \int_{-\infty}^{\infty} \frac{e^{\pi i\omega t^2 - 2\pi t\chi}}{e^{2\pi t} - 1} dt = g_{\lambda^*}(x^*, q^*) - i\sqrt{\frac{\pi}{\log 1/q^*}} e^{\frac{(\log x^*)^2}{4\log q^*}} g_{\frac{1}{\lambda}}(\frac{1}{x}, q). \quad (5.4)$$

If $\lambda = \frac{1}{q}e^{\pi i}$, then $\lambda^* = q^* e^{\pi i}$ and $\lambda^{**} = qe^{-\pi i}$.

Remember that $q = e^{\pi i\omega}$ and $x = e^{-2\pi i\chi}$, so $q^* = e^{-\frac{\pi i}{\omega}}$ and $x^* = e^{2\pi i \frac{\chi}{\omega}}$.

Substituting all these variables in the above identity, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{e^{\pi i\omega t^2 - 2\pi t\chi}}{e^{2\pi t} - 1} dt \\
= \;& -\frac{1}{\sqrt{(-i\omega)}} e^{-\frac{\pi\chi^2}{i\omega}} \frac{1}{\theta^*(\frac{\lambda^*}{x^*})} \sum_{n\in\mathbb{Z}} \frac{q^{*n^2}}{1 - \lambda^* q^{*2n}} (\frac{\lambda^*}{x^*})^n \\
& + i\frac{1}{\theta(\frac{x}{\lambda})} \sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{1 - \frac{1}{\lambda}q^{2n}} (\frac{x}{\lambda})^n, \qquad where\ \theta^*(\frac{\lambda^*}{x^*}) = \theta(\frac{\lambda^*}{x^*}, q^*) \\
= \;& -\frac{1}{\sqrt{(-i\omega)}} e^{-\frac{\pi\chi^2}{i\omega}} \frac{1}{\theta^*(\frac{-q^*}{x^*})} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{*(n^2+n)}}{1 + q^{*(2n+1)}} x^{*(-n)} \\
& + i\frac{1}{\theta(-qx)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{(n^2+n)}}{1 + q^{(2n+1)}} x^n \\
= \;& \frac{-\frac{1}{\sqrt{(-i\omega)}} e^{-\frac{\pi\chi^2}{i\omega}}}{\sum\limits_{m\ odd}^{\pm\infty} (-1)^{\frac{1}{2}(m-1)} q^{*\frac{m^2}{4}} x^{*\frac{1}{2}(m-1)}} \sum_{m\ odd}^{\pm\infty} \frac{(-1)^{\frac{1}{2}(m-1)} q^{*\frac{m^2}{4}} x^{*\frac{1}{2}(m-1)}}{1 + q^{*m}} \\
& + \frac{i}{\sum\limits_{m\ odd}^{\pm\infty} (-1)^{\frac{1}{2}(m-1)} q^{\frac{m^2}{4}} x^{\frac{1}{2}(m-1)}} \sum_{m\ odd}^{\pm\infty} \frac{(-1)^{\frac{1}{2}(m-1)} q^{\frac{m^2}{4}} x^{\frac{1}{2}(m-1)}}{1 + q^m} \\
= \;& -\frac{1}{\sqrt{(-i\omega)}} e^{-\frac{\pi\chi^2}{i\omega}} \frac{f(\frac{\chi}{\omega}, -\frac{1}{\omega})}{\theta_{11}(\frac{\chi}{\omega}, -\frac{1}{\omega})} + i\frac{f(\chi,\omega)}{\theta_{11}(\chi,\omega)}.
\end{aligned}$$

Recall that

$$\theta_{11}(\frac{\chi}{\omega}, -\frac{1}{\omega}) = -i\sqrt{(-i\omega)}e^{-\frac{\pi\chi^2}{i\omega}} \theta_{11}(\chi,\omega).$$

So

$$\int_{-\infty}^{\infty} \frac{e^{\pi i\omega t^2 - 2\pi t\chi}}{e^{2\pi t} - 1} dt = \frac{1}{i\sqrt{(-i\omega)^2}} \frac{f(\frac{\chi}{\omega}, -\frac{1}{\omega})}{\theta_{11}(\frac{\chi}{\omega}, -\frac{1}{\omega})} + i\frac{f(\chi,\omega)}{\theta_{11}(\chi,\omega)}$$

$$= \quad \frac{f(\frac{\chi}{\omega}, -\frac{1}{\omega}) + i\omega f(\chi,\omega)}{\omega\theta_{11}(\chi,\omega)}.$$

§5.3 More general cases

Here we still assume that $q = e^{\pi i\omega}$ with $\mathrm{Im}(\omega) > 0$. Then $0 < |q| < 1$.

In practical researches, only for a few ideal problems we know the exact solutions, many parts of which are represented by integral functions or special functions. To get some useful scientific conclusions from the exact solutions, and then apply them to the specific engineering designs, researchers often need to calculate concrete numerical solutions by approximate method. So people want to establish some relations between series and integral functions or special functions. In this section, we will explain how to treat the more general situation such as

$$\int_{-\infty}^{\infty} e^{\frac{\pi t^2}{i\omega}+\frac{2\pi\chi t}{\omega}} \Phi(e^{2\pi t})dt,$$

where Φ denotes an analytic function over the complex plane perhaps excepted a countable number of singular points. Under some reasonable assumptions, we can put the meromorphic function Φ into partial fraction decomposition and we only need to consider the integral of the type

$$\int_{-\infty}^{\infty} \frac{e^{\frac{\pi t^2}{i\omega}+\frac{2\pi\chi t}{\omega}}}{(e^{2\pi t}-e^{2\pi\nu})^k}dt,$$

where $k \in \mathbb{N}$ and $\nu \in \mathbb{C}$. In particular, if $\Phi(\xi) = \frac{1}{(\xi;q)_\infty}$, then we meet Ramanujan's entire function, which is also called a q-Airy function in the literature.

First of all, we introduce the following notation:

$$I(\nu,\chi) = I(\nu,\chi;\omega) = -\frac{1}{\sqrt{(-i\omega)}}e^{-\frac{\pi\chi^2}{i\omega}} \int_{-\infty}^{\infty} \frac{e^{\frac{\pi t^2}{i\omega}+\frac{2\pi\chi t}{\omega}}}{e^{2\pi t}-e^{2\pi\nu}}dt,$$

where $\nu \in \mathbb{C}$ with $\mathrm{Im}(\nu) \in (-1, 0]$ and the path of integration may be any straight line parallel to the real axis of t and just below the point ν at a distance less then unity, i.e., $(-\infty + \nu - i\epsilon, \infty + \nu - i\epsilon)$, $\epsilon \in (0, 1)$.

By making use of a suitable change of variables in the above integral, it follows that, for any generic value of $(\lambda, x) \in \mathbb{C}^* \times \tilde{\mathbb{C}}^*$,

$$\begin{aligned}
I(\nu, \chi) &= e^{-2\pi\nu} I(0, \chi + \frac{\nu}{i}) \\
&= e^{-2\pi\nu} \sum_{\xi \in \lambda q^{2\mathbb{Z}}} \frac{1}{1 - e^{-2\pi\nu}\xi} \frac{1}{\theta(\frac{\xi}{x})} \\
&\quad - \frac{i}{\sqrt{(-i\omega)}} e^{-\frac{\pi(\chi + \frac{\nu}{i})^2}{i\omega} - 2\pi\nu} \sum_{\xi \in \lambda^* q^{*2\mathbb{Z}}} \frac{1}{1 - e^{\frac{2\pi\nu}{\omega}}\xi} \frac{1}{\theta^*(\frac{\xi}{x^*})} \\
&:= A(\lambda, \nu, \chi) - \frac{i}{\sqrt{(-i\omega)}} e^{-\frac{\pi(\chi + \frac{\nu}{i})^2}{i\omega} - 2\pi\nu} B(\lambda, \nu, \chi) \\
&= A(\lambda, \nu, \chi) - i\sqrt{\frac{\pi}{\log 1/q}} e^{\frac{(\log x + 2\pi\nu)^2}{4\log q} - 2\pi\nu} B(\lambda, \nu, \chi) \\
&:= J(\lambda, \nu, x) = J(\nu, x),
\end{aligned}$$

where, as before,

$$x = e^{-2\pi i\chi},\ x^* = e^{2\pi i\frac{\chi}{\omega}},$$

$$q = e^{\pi i\omega},\ q^* = e^{-\pi i\frac{1}{\omega}},$$

$$\lambda^* = e^{-\pi i\frac{\log\lambda}{\log q}},\ \theta^*(x) = \theta(x, q^*).$$

We can easily check that

$$e^{-2\pi i\chi - \pi i\omega} A(\lambda, \nu, \chi + \omega) - e^{2\pi\nu} A(\lambda, \nu, \chi) = -1,$$

$$B(\lambda, \nu, \chi + \omega) - B(\lambda, \nu, \chi) = 0,$$

and also,

$$e^{-2\pi i\chi - \pi i\omega} I(\nu, \chi + \omega) - e^{2\pi\nu} I(\nu, \chi) = -1.$$

So instead of equation (1.3), in this section, let us consider a q-difference equation:

$$e^{-2\pi i\chi - \pi i\omega} y(\nu, \chi + \omega) - e^{2\pi\nu} y(\nu, \chi) = -1. \tag{5.5}$$

A straightforward formal computation shows that the above equation has a formal series solution which may be written as

$$\hat{y}(\nu,\chi)=\sum_{n\geq 0}e^{-2(n+1)\pi\nu}e^{-n^2\pi i\omega}e^{-2n\pi i\chi}.$$

Then consider the derivatives of $I(\nu,\chi)$ with respect to ν. So we define that for any $k\in\mathbb{N}^*$

$$I_k(\nu,\chi)=\frac{1}{k!}(\frac{e^{-2\pi\nu}}{2\pi}\frac{\partial}{\partial\nu})^k I(\nu,\chi)=-\frac{1}{\sqrt{(-i\omega)}}e^{-\frac{\pi\chi^2}{i\omega}}\int_{-\infty}^{\infty}\frac{e^{\frac{\pi t^2}{i\omega}+\frac{2\pi\chi t}{\omega}}}{(e^{2\pi t}-e^{2\pi\nu})^{k+1}}dt. \tag{5.6}$$

It follows that

$$e^{-2\pi i\chi-\pi i\omega}I_k(\nu,\chi+\omega)-e^{2\pi\nu}I_k(\nu,\chi)=I_{k-1}(\nu,\chi), \tag{5.7}$$

where $I_0(\nu,\chi):=I(\nu,\chi)$.

Combining the q-difference equation (5.5) with the recurrence relation (5.7), we can deduce that $I_k(\nu,\chi)$ satisfies the following q-difference equation of order $k+1$:

$$\sum_{m=0}^{k+1}(-1)^m\begin{pmatrix}k+1\\ m\end{pmatrix}\frac{e^{-2(k+1-m)\pi i\chi}}{e^{(k+1-m)^2\pi i\omega}}e^{2m\pi\nu}y_k(\nu,\chi+(k+1-m)\omega)=-1,$$

which also has a divergent formal solution:

$$\begin{aligned}\hat{y}_k(\nu,\chi)&=(-1)^k\sum_{n\geq 0}\begin{pmatrix}n+k\\ n\end{pmatrix}e^{-2(n+k+1)\pi\nu}e^{-n^2\pi i\omega}e^{-2n\pi i\chi}\\ &=(-1)^k\frac{1}{k!}\sum_{n\geq 0}(n+1)\cdots(n+k)e^{-2(n+k+1)\pi\nu}e^{-n^2\pi i\omega}e^{-2n\pi i\chi}\\ &=\frac{1}{k!}(\frac{e^{-2\pi\nu}}{2\pi}\frac{\partial}{\partial\nu})^k\hat{y}(\nu,\chi).\end{aligned}$$

Therefore, similarly to the analysis in the preceding chapters, we know that $\hat{y}_k(\nu,\chi)$ has two sums of different forms, one of which is the integral function $I_k(\nu,\chi)$ defined by (5.6). And we also know that $I_k(\nu,\chi)$ can be represented by factorial series expansion $J_k(\nu,x)$:

$$\begin{aligned}J_k(\nu,x) &= \frac{1}{k!}(\frac{e^{-2\pi\nu}}{2\pi}\frac{\partial}{\partial\nu})^k J(\nu,x)\\ &= \frac{1}{k!}(\frac{e^{-2\pi\nu}}{2\pi}\frac{\partial}{\partial\nu})^k(A(\lambda,\nu,\chi))\\ &\quad -i\sqrt{\frac{\pi}{\log 1/q}}\frac{1}{k!}(\frac{e^{-2\pi\nu}}{2\pi}\frac{\partial}{\partial\nu})^k(e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-2\pi\nu}B(\lambda,\nu,\chi)),\end{aligned}$$

where, as before,

$$A(\lambda,\nu,\chi)=e^{-2\pi\nu}\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{1}{1-e^{-2\pi\nu}\xi}\frac{1}{\theta(\frac{\xi}{x})},$$

and

$$B(\lambda,\nu,\chi)=\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{1-e^{\frac{2\pi\nu}{\omega}}\xi}\frac{1}{\theta^*(\frac{\xi}{x^*})}.$$

Examples. *Here we only present explicit formulae for $k=1,2$ as two examples, i.e.,*

$$\begin{aligned}& I_1(\nu,\chi)\\ =\;& -\frac{1}{\sqrt{(-i\omega)}}e^{-\frac{\pi\chi^2}{i\omega}}\int_{-\infty}^{\infty}\frac{e^{\frac{\pi t^2}{i\omega}+\frac{2\pi\chi t}{\omega}}}{(e^{2\pi t}-e^{2\pi\nu})^2}dt\\ =\;& J_1(\nu,x)\\ =\;& (\frac{e^{-2\pi\nu}}{2\pi}\frac{\partial}{\partial\nu})J(\nu,x)\\ =\;& -e^{-4\pi\nu}\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{1}{1-e^{-2\pi\nu}\xi}\frac{1}{\theta(\frac{\xi}{x})}+\frac{e^{-4\pi\nu}}{2\pi}\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{\partial}{\partial\nu}(\frac{1}{1-e^{-2\pi\nu}\xi})\frac{1}{\theta(\frac{\xi}{x})}\\ & -i\sqrt{\frac{\pi}{\log 1/q}}(\frac{\log x+2\pi\nu}{2\log q}-1)e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-4\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{1-e^{\frac{2\pi\nu}{\omega}}\xi}\frac{1}{\theta^*(\frac{\xi}{x^*})}\\ & -i\sqrt{\frac{\pi}{\log 1/q}}\frac{e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-4\pi\nu}}{2\pi}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{\partial}{\partial\nu}(\frac{1}{1-e^{\frac{2\pi\nu}{\omega}}\xi})\frac{1}{\theta^*(\frac{\xi}{x^*})}\\ =\;& -e^{-4\pi\nu}\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{1}{1-e^{-2\pi\nu}\xi}\frac{1}{\theta(\frac{\xi}{x})}-e^{-4\pi\nu}\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{e^{-2\pi\nu}\xi}{(1-e^{-2\pi\nu}\xi)^2}\frac{1}{\theta(\frac{\xi}{x})}\end{aligned}$$

$$
\begin{aligned}
& -i\sqrt{\frac{\pi}{\log 1/q}}(\frac{\log x+2\pi\nu}{2\log q}-1)e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-4\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{1-e^{\frac{2\pi\nu}{\omega}}\xi}\frac{1}{\theta^*(\frac{\xi}{x^*})} \\
& -i\sqrt{\frac{\pi}{\log 1/q}}\frac{e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-4\pi\nu}}{\omega}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{e^{\frac{2\pi\nu}{\omega}}\xi}{(1-e^{\frac{2\pi\nu}{\omega}}\xi)^2}\frac{1}{\theta^*(\frac{\xi}{x^*})} \\
= \; & -e^{-4\pi\nu}\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{1}{(1-e^{-2\pi\nu}\xi)^2}\frac{1}{\theta(\frac{\xi}{x})} \\
& -i\sqrt{\frac{\pi}{\log 1/q}}(\frac{\log x+2\pi\nu}{2\log q}-1-\frac{1}{\omega})e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-4\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{1-e^{\frac{2\pi\nu}{\omega}}\xi}\frac{1}{\theta^*(\frac{\xi}{x^*})} \\
& -\frac{i}{\omega}\sqrt{\frac{\pi}{\log 1/q}}e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-4\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{(1-e^{\frac{2\pi\nu}{\omega}}\xi)^2}\frac{1}{\theta^*(\frac{\xi}{x^*})},
\end{aligned}
$$

$$
\begin{aligned}
& I_2(\nu,\chi) \\
= \; & -\frac{1}{\sqrt{(-i\omega)}}e^{-\frac{\pi\chi^2}{i\omega}}\int_{-\infty}^{\infty}\frac{e^{\frac{\pi t^2}{i\omega}+\frac{2\pi\chi t}{\omega}}}{(e^{2\pi t}-e^{2\pi\nu})^3}dt \\
= \; & J_2(\nu,x) \\
= \; & \frac{1}{2}(\frac{e^{-2\pi\nu}}{2\pi}\frac{\partial}{\partial\nu})J_1(\nu,x) \\
= \; & e^{-6\pi\nu}\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{1}{(1-e^{-2\pi\nu}\xi)^2}\frac{1}{\theta(\frac{\xi}{z})}-\frac{e^{-6\pi\nu}}{4\pi}\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{\partial}{\partial\nu}(\frac{1}{(1-e^{-2\pi\nu}\xi)^2})\frac{1}{\theta(\frac{\xi}{x})} \\
& -\frac{i}{4\log q}\sqrt{\frac{\pi}{\log 1/q}}e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-6\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{1-e^{\frac{2\pi\nu}{\omega}}\xi}\frac{1}{\theta^*(\frac{\xi}{x^*})} \\
& -i\sqrt{\frac{\pi}{\log 1/q}}(\frac{\log x+2\pi\nu}{2\log q}-1-\frac{1}{\omega})(\frac{\log x+2\pi\nu}{4\log q}-1) \\
& \qquad e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-6\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{1-e^{\frac{2\pi\nu}{\omega}}\xi}\frac{1}{\theta^*(\frac{\xi}{x^*})} \\
& -\frac{i}{4\pi}\sqrt{\frac{\pi}{\log 1/q}}(\frac{\log x+2\pi\nu}{2\log q}-1-\frac{1}{\omega}) \\
& \qquad e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-6\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{\partial}{\partial\nu}(\frac{1}{1-e^{\frac{2\pi\nu}{\omega}}\xi})\frac{1}{\theta^*(\frac{\xi}{x^*})} \\
& -\frac{i}{\omega}\sqrt{\frac{\pi}{\log 1/q}}(\frac{\log x+2\pi\nu}{4\log q}-1)e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-6\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{(1-e^{\frac{2\pi\nu}{\omega}}\xi)^2}\frac{1}{\theta^*(\frac{\xi}{x^*})} \\
& -\frac{i}{4\pi\omega}\sqrt{\frac{\pi}{\log 1/q}}e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-6\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{\partial}{\partial\nu}(\frac{1}{(1-e^{\frac{2\pi\nu}{\omega}}\xi)^2})\frac{1}{\theta^*(\frac{\xi}{x^*})}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & e^{-6\pi\nu}\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{1}{(1-e^{-2\pi\nu}\xi)^2}\frac{1}{\theta(\frac{\xi}{x})}+e^{-6\pi\nu}\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{e^{-2\pi\nu}\xi}{(1-e^{-2\pi\nu}\xi)^3}\frac{1}{\theta(\frac{\xi}{x})}\\
& -\frac{i}{4\log q}\sqrt{\frac{\pi}{\log 1/q}}e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-6\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{1-e^{\frac{2\pi\nu}{\omega}}\xi}\frac{1}{\theta^*(\frac{\xi}{x^*})}\\
& -i\sqrt{\frac{\pi}{\log 1/q}}\left(\frac{\log x+2\pi\nu}{2\log q}-1-\frac{1}{\omega}\right)\left(\frac{\log x+2\pi\nu}{4\log q}-1\right)\\
& \qquad e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-6\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{1-e^{\frac{2\pi\nu}{\omega}}\xi}\frac{1}{\theta^*(\frac{\xi}{x^*})}\\
& -\frac{i}{2\omega}\sqrt{\frac{\pi}{\log 1/q}}\left(\frac{\log x+2\pi\nu}{2\log q}-1-\frac{1}{\omega}\right)\\
& \qquad e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-6\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{e^{\frac{2\pi\nu}{\omega}}\xi}{(1-e^{\frac{2\pi\nu}{\omega}}\xi)^2}\frac{1}{\theta^*(\frac{\xi}{x^*})}\\
& -\frac{i}{\omega}\sqrt{\frac{\pi}{\log 1/q}}\left(\frac{\log x+2\pi\nu}{4\log q}-1\right)e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-6\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{(1-e^{\frac{2\pi\nu}{\omega}}\xi)^2}\frac{1}{\theta^*(\frac{\xi}{x^*})}\\
& -\frac{i}{\omega^2}\sqrt{\frac{\pi}{\log 1/q}}e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-6\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{e^{\frac{2\pi\nu}{\omega}}\xi}{(1-e^{\frac{2\pi\nu}{\omega}}\xi)^3}\frac{1}{\theta^*(\frac{\xi}{x^*})}\\
= \quad & e^{-6\pi\nu}\sum_{\xi\in\lambda q^{2\mathbb{Z}}}\frac{1}{(1-e^{-2\pi\nu}\xi)^3}\frac{1}{\theta(\frac{\xi}{x})}\\
& -i\sqrt{\frac{\pi}{\log 1/q}}\left[\left(\frac{\log x+2\pi\nu}{2\log q}-1-\frac{1}{\omega}\right)\left(\frac{\log x+2\pi\nu}{4\log q}-1-\frac{1}{2\omega}\right)+\frac{1}{4\log q}\right]\\
& \qquad e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-6\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{1-e^{\frac{2\pi\nu}{\omega}}\xi}\frac{1}{\theta^*(\frac{\xi}{x^*})}\\
& -\frac{i}{2\omega}\sqrt{\frac{\pi}{\log 1/q}}\left(\frac{\log x+2\pi\nu}{\log q}-3-\frac{3}{\omega}\right)\\
& \qquad e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-6\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{(1-e^{\frac{2\pi\nu}{\omega}}\xi)^2}\frac{1}{\theta^*(\frac{\xi}{x^*})}\\
& -\frac{i}{\omega^2}\sqrt{\frac{\pi}{\log 1/q}}e^{\frac{(\log x+2\pi\nu)^2}{4\log q}-6\pi\nu}\sum_{\xi\in\lambda^* q^{*2\mathbb{Z}}}\frac{1}{(1-e^{\frac{2\pi\nu}{\omega}}\xi)^3}\frac{1}{\theta^*(\frac{\xi}{x^*})}.
\end{aligned}
$$

Chapter 6 Summary and Unsolved Problems

In our thesis, We obtain three types of sums of a divergent formal solution to Cauchy problem (1.1) and build two relations between them. One of them is Theorem 3.2.4. The other one is Theorem 5.2.1, which is deduced from a Stokes analysis on the singular q-difference equation (1.3). The formula connecting two fundamental solutions of (1.3) is expressed in terms of elliptic functions and Gaussian functions (i.e., Heat kernel). And it is also a generalization of Mordell's theorem. In the last section, with the same mode of thought we get some surprising identities on q-series, which are deduced from statements in Theorem 5.2.1.

But we still have some problems under consideration. On the one hand, for the q-Borel sum $F_\alpha(\tau, z)$ defined by (3.5), in view of Theorem 3.2.4 we know it is also a solution of (1.1); but for the q-Borel sum $G_\lambda(\tau, z)$ defined by (4.5), because the properties of this summation method need to be further studied, we do not say that it is a solution of (1.1). On the other hand, for the q-series $\hat{y}(x, q)$ defined by (1.4), according to [13] we belief that there are more different kinds of q-Borel sums and the relations among them is unknown. And in this thesis, by variable substitution, the Cauchy problem (1.1) of the Heat equation, which is a reaction-diffusion equation, corresponds to the q-difference equation (1.3). So, are there more correspondences between reaction-diffusions and q-difference equations? If there are, do these correspondences imply more essential connections between the two types of equations? And in the Riemann-Siegel formula, we can find that the main integral part is similar to our considered integral function in (5.4). Then naturally, we think about applying the idea used in this thesis towards the study of the Riemann-Siegel integral formula(cf. [14, 45])?

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RESUME

Dans cette Thèse, nous considérons dans le plan complexe l'équation de la chaleur avec la condition initiale singulière u(0,z)=1/(1-exp(z)). Ce problème de Cauchy possède une unique solution formelle série entière, laquelle peut être sommée par des procédés de sommation différents. Le but est d'établir des relations existant entre les différentes sommes ainsi étudiées: d'une part la somme de Borel de celle-ci et, de l'autre, deux versions q-analogues de la somme de Borel qui sont obtenues respectivement avec le noyau de la chaleur et la fonction thêta de Jacobi. Notre analyse sur le phénomène de Stokes correspondant nous conduit à une généralisation d'un résultat de Mordell sur le nombre de classes des formes quadratiques binaires définies et positives.

Mots-clés : Développement asymptotique - Borel-sommable - Équation aux q-différences - Développement asymptotique q-Gevrey - Gq-sommable - Noyau de chaleur - Fonction thêta de Jacobi - Théorème de Mordell.

