Laboratoire
Paul Painlevé

Université de Lille

École doctorale SPI-MADIS

Doctorat Université de Lille

THÈSE

pour obtenir le grade de docteur délivré par

Université de Lille

Spécialité doctorale "Mathématiques"

présentée et soutenue publiquement par

Obayda ASSAAD

le 28 octobre 2021

Solutions des équations différentielles stochastiques : analyse asymptotique par la méthode de Malliavin-Stein et estimation statistique.

Directeur de thèse : **Ciprian Tudor**

Jury

M. Podolskij Mark,	Professeur, Université du Luxembourg	Rapporteur
M. Réveillac Anthony,	Professeur, INSA de Toulouse	Rapporteur
M. Ayache Antoine,	Professeur, Université de Lille	Examinateur
Mme. Comte Fabienne,	Professeur, Université de Paris-Descartes	Examinateur, Présidente du Jury
M. Tudor Ciprian,	Professeur, Université de Lille	Directeur de thèse

Table des matières

0.1 Résumé

Cette thèse porte essentiellement sur une étude analytique et statistique des équations différentielles stochastiques (EDS). La grande souplesse du Calcul de Malliavin et de la méthode de Stein-Malliavin permet de considérer un large panorama d'EDS. Toute la thèse se placera dans cette vision variationnelle et asymptotique. Ainsi seront considérées les équations différentielles aux dérivées partielles stochastiques (EDPS) suivantes : l'équation des ondes avec un bruit gaussien fractionnaire en temps et blanc en espace, l'équation de Burger avec un bruit blanc en temps et espace, l'équation de la chaleur fractionnaire avec un bruit blanc en temps et colorié en espace et finalement les équations de Langevin avec un bruit non gaussien de type Hermite et Hermite généralisé.

Dans un premier temps, nous étudierons les variations quadratiques de l'équation des ondes par une décomposition en ondelettes de la solution qui permet un contrôle de régularité ainsi que les variations quadratiques du processus de Hermite-Ornstein–Uhlenbeck, solution de l'équation de Langevin perturbée par le processus de Hermite, pour obtenir des résultats asymptotiques de convergence et de contrôle en loi. Ces résultats nous permettrons de définir un estimateur pour le paramètre de Hurst et d'étudier ses propriétés. Par ergodicité nous donnerons aussi un estimateur pour le paramètre de diffusion.

Dans un second temps nous décomposerons la solution de l'équation de Burger en somme de deux processus, l'un qui s'identifie avec la solution de l'équation de la chaleur linéaire et l'autre correspondant au terme non linéaire. Nous montrerons par une analyse fine du noyau et de sa dérivée que ce dernier est plus régulier et qu'ainsi il n'affecte pas les p-variations de la solution. En regardant la solution en des temps discrets ou en des points discrets nous pourrons alors estimer le paramètre de drift par l'étude des 2 et 4 variations. Dans chaque cas nous étudierons le consistance forte et son erreur pour la convergence L^p.

La troisième partie portera sur l'étude asymptotique de la moyenne spatiale sur la sphère de la solution de l'équation de la chaleur fractionnaire avec un bruit multiplicatif général, qui inclut le très populaire bruit blanc temps-espace et le non moins populaire bruit blanc en temps et colorié en espace dont la covariance spatiale est donnée par le noyau de Riesz. On montre que correctement renormalisée, la moyenne converge en variation totale vers une loi gaussienne et qu'elle vérifie un théorème central limite fonctionnel.

Finalement, la dernière partie est consacrée à l'étude de l'intégrale stochastique par rapport au processus de Hermite généralisé, processus non gaussien dont le coefficient d'auto-similarité est défini sur tout l'intervalle $(0, 1)$. Nous pouvons définir une intégrale de type Wiener et Riemann-Stieljes, ce qui permet de regarder le processus de Hermite-Ornstein–Uhlenbeck généralisé et de montrer que ces deux intégrales coïncident dans ce cas. Nous montrerons également que la solution converge (dans un sens à définir) quand le drift tend vers zéro vers le processus de Hermite généralisé.

0.2 Abstract

This thesis focuses on an analytical and statistical study of stochastic differential equations (SDE). The great flexibility of the Malliavin calculus and the Stein-Malliavin method allows a large panorama of EDS to be considered. All the thesis will be placed in this variational and asymptotic vision. Thus the following stochastic partial differential equations (SPDE) will be considered : the wave equation with fractional Gaussian noise in time and white in space, Burger's equation with white noise in time and space, the fractional heat equation with a white noise in time and colored in space and finally the Langevin equation with a generalized non-Gaussian Hermite noise. First, we will study the quadratic variations of the equation of waves by a wavelet decomposition of the solution which allows a regularity check as well as the quadratic variations of the Hermite-Ornstein–Uhlenbeck process, solution of the Langevin equation disturbed by the Hermite process, to obtain asymptotics results for the convergence and control in law. These results will allow us to define an estimator for the Hurst parameter and to study its properties. By ergodicity we will also give an estimator for the diffusion parameter. Secondly, we will decompose the solution of Burger's equation into sum of two processes, one that identifies with the solution of the heat equation linear and the other corresponding to the nonlinear term. We will show by an analysis of the kernel and its derivative that the latter is more regular and thus does not affect the p-variations of the solution. By looking at the solution in discrete times or in discrete points we can then estimate the drift parameter by studying the variations of order two (in space) and of order four (in time). In each case we will study the strong consistency and its error for Lp convergence. The third part will focus on the asymptotic study of the spatial mean on the sphere of the solution of the fractional heat equation with a general multiplicative noise, which includes the very popular time-space white noise and the equally popular white noise white in time and colored in space, the spatial covariance of which is given by the kernel of Riesz. We show that correctly renormalized, the mean converges in total variation towards a Gaussian law and that it satisfies a functional central limit theorem. Finally, the last part is devoted to the study of the stochastic integral with respect to the generalized Hermite process, a non-Gaussian process whose self-similarity index is defined over the entire interval (0, 1). We can define an integral of the Wiener and Riemann-Stieljes type, which allows us to look at the generalized Hermite- Ornstein – Uhlenbeck and to show that these two integrals coincide in this case (where it makes sense to say that). We will also show that the solution converges (in a sense to be defined) when the drift tends zero towards the generalized Hermite process.

Chapitre 1

Introduction et présentation des résultats obtenus

Avant de rappeler le cadre général dans lequel toutes nos équations différentielles partielles étudiées vont se placer nous proposons une petite introduction historico-mathématique qui pourrait servir de support intuitif aux considérations abstraites. Nous tenons à préciser que le présent document ne cherchera pas à discuter de l'effectivité empirique des modèles stochastiques, sujet intéressant mais nous prenons l'idée que les mathématiques se contiennent elles-mêmes. Pour une analogie avec les concepts physiques nous renvoyons au prélude du livre de D. Koshnevisan [1].

1.1 Préliminaires aux résultats

1.1.1 Forces stochastiques

On note $C_c^\infty(\mathbb{R}^d)$ l'espace des fonctions continuement différentiables à support compact. Considérons le problème de Cauchy pour l'équation de la chaleur dans sa forme classique. Soit $g \in C_c^\infty(\mathbb{R}^d)$, le problème de Cauchy s'écrit

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) - \Delta u(t,x) = 0, \quad (x,t) \in \mathbb{R}^d \times (0,\infty) \\ u(x,0) = g(x), \quad x \in \mathbb{R}^d \end{cases} . \tag{1.1}$$

En utilisant la transformée de Fourier il est facile de voir que (1.1) admet pour solution

$$u(x,t) = (G(t,\cdot) \star g)(x,t) := \int_{\mathbb{R}^d} G(t,x-y) g(y) dy$$

où $G(t,x)$ est le noyau gaussien que nous définirons à travers sa transformée de Fourier $\mathcal{F}(G(t,\cdot))(\xi) = e^{-|\xi|^2 t}$ pour tout $t \geq 0$ et $|\cdot|$ est la norme euclidienne.

La deuxième étape mathématique est d'ajouter une force déterministe (qui correspond à une volonté de décrire au mieux le système physique associé) c'est-à-dire de considérer le problème non homogène suivant, avec $g, f \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) - \Delta u(t,x) = f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x), \quad x \in \mathbb{R}^n \end{cases}. \tag{1.2}$$

De la même façon en prenant la transformée de Fourier, et par quelques calculs dont la méthode s'appelle le principe de Duhamel, on montre que la solution $u(t,x)$ s'écrit

$$u(x,t) = (G(t,\cdot) \star g)(x,t) + \int_0^t (G(t-s,\cdot) \star f(\cdot,s))(x)ds.$$

Le problème qui s'est posé historiquement a été de chercher à donner un sens mathématique à (1.2) quand f est remplacée par une force stochastique, autrement dit f devient $W = W(x,t)$ un processus stochastique à deux paramètres. Des considérations mathématiques sur des expériences sur le mouvement des particules ont amené les chercheurs à considérer que W devrait satisfaire les propriétés suivantes :

- Pour tout x, t, $W(x,t)$ est une variable gaussienne centrée.
- Qu'il n'y a pas de corrélation spatio-temporelle, autrement dit on a

$$\mathbb{E}(W(t,x)W(s,y)) = \delta_0(t-s)\delta_0(x-y). \tag{1.3}$$

Un tel processus n'est pas bien défini car δ_0 n'est pas une fonction scalaire, mais selon le point de vue une mesure ou une forme linéaire.

Du point de vue de la mesure, cela suggère l'interprétation $(t,x) \to [0,t] \times [0,x]$ et le calcul suivant

$$\begin{aligned} \mathbb{E}(W(t,x)W(s,y)) &= \mathbb{E}(W([0,t] \times [0,x])W([0,s] \times [0,y])) \\ &= \int_{\mathbb{R}_+^2} \mathbf{1}_{[0,t]\times[0,x]} \mathbf{1}_{[0,s]\times[0,y]} dudz \\ &= \min(t,s)\min(x,y). \end{aligned}$$

Cette interprétation possible nous donne le mouvement brownien de degré 2, connu surtout dans sa dénomination anglo-américaine « Brownian Sheet ».

Par extension linéaire on voit qu'on peut, au moins formellement définir le processus pour les fonctions $h \in \mathcal{H} = \mathbb{L}^2(\mathbb{R}^2, \nu = \mathbf{1}_{\mathbb{R}_+^2} dx)$ et ainsi on définit un processus gaussien $W = \{W(h),\ \mathcal{H}\}$ tel que pour tout $h_1, h_2 \in \mathcal{H}$

$$\mathbb{E}(W(h_1)W(h_2)) = \langle h_1, h_2 \rangle_{\mathcal{H}}.$$

La procédure peut s'étendre en toute généralité pour un espace de Hilbert quelconque, c'est ce qu'on nomme les processus isonormaux que nous définirons plus loin et qui sont à la base de la vision actualisée du Calcul de Malliavin.

En prenant $\mathcal{H} = \mathbb{L}^2(\mathbb{R}^2, \nu = \mathbf{1}_{\mathbb{R}^+}(t)dtdx)$ nous aurons plutôt

$$\begin{aligned} \mathbb{E}(W(t,x)W(t,s)) &= \mathbb{E}(W([0,t]\times A)W([0,s]\times B)) \\ &= \min(t,s)\lambda(A\cap B) \end{aligned} \tag{1.4}$$

pour A, B deux boréliens de mesures finies et λ désigne la mesure de Lesbesgue sur $\mathbb{R}^d$.

Remarque 1 *Ce processus gaussien 1.4 s'appelle* ***le bruit blanc en temps et en espace****. Le lien extrinsèque qui permet de faire le pont entre la mesure de Lesbesgue et la distribution de dirac est le fait que la mesure spectrale de $\delta_0(dx)$ est la mesure de Lebesgue $\mu(dx) = dx$. Ce lien sera explicité à la section 1.1.4.3.*

▷ Un processus sur $\mathbb{R}^d$ sera appelé un champ conformément à l'analogie avec le concept de champ de vecteurs en géométrie.

On peut définir en toute généralité un bruit blanc pour n'importe quelle mesure μ qui est $\sigma-$finie et pour tout borélien A tel que $\mu(A) < \infty$. On note $\mathcal{B}_b(A)$ l'ensemble de tels boréliens.

Définition 1 *Un bruit blanc de support μ est un champ gaussien $(W(A); A \in \mathcal{B}_b(\mathbb{R}^d))$ défini sur un espace de probabilité $(\Omega, \mathcal{F}, P)$ tel que $\mu(A) = \mathbb{E}(W(A)) = 0$ et*

$$\mathbb{E}(W(A)W(B)) = \mu(A\cap B).$$

L'existence d'un bruit blanc est triviale car il suffit de vérifier le caractère positif de la fonction de covariance.

Remarque 2 *Un tel processus n'est pas, presque-sûrement, une mesure mais une mesure vectorielle. Autrement dit $A \mapsto W(A)$ de $\mathcal{B}_b(A) \to L^2(\Omega, \mathcal{F}, P)$ est une mesure vectorielle, mais pour tout $\omega \in \Omega$, $\omega \mapsto W(A)(\omega)$ n'est pas une mesure $\sigma-$finie. Cette remarque est importante à avoir à l'esprit car elle permet de comprendre pourquoi nous introduirons la notion de mesure de martingale à la section 1.1.2.*

Considérons maintenant (toujours de manière formelle), l'équation de la chaleur stochastique non homogène avec un tel bruit :

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) - \Delta u(t,x) = W(x,t), \quad (t,x)\in\mathbb{R}^d\times(0,\infty) \\ \qquad\qquad\qquad\qquad u(x,0) = 0, \quad x\in\mathbb{R}^d \end{cases}. \tag{1.5}$$

Nous avons alors formellement l'écriture suivante pour la solution

$$u(x,t) = \int_0^t (G(t-s,\cdot) \star W(\cdot,s))(x)ds = \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)W(s,y)dsdy.$$

Cependant on voit qu'une telle écriture demande à être rigoureusement définie et donc nous sommes dans la nécessité de définir une intégrale stochastique de la forme

$$\int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)W(ds,dy), \tag{1.6}$$

en interprétant $W(s,y)dsdy = W(ds,dy)$ [1], intégrale que nous appellerons intégrale de Wiener et dont la construction est classique et nous renvoyons le lecteur au livre [1] déjà cité. Voir cependant la section 1.1.3 sur le mouvement brownien fractionnaire qui suit le même chemin conceptuel.

Remarque 3 *Une telle intégrale est un processus gaussien centré qui vérifie la propriété d'isométrie i.e.*

$$\mathbb{E}(\int_0^t \int_{\mathbb{R}^d} f(s,y)W(ds,dy) \int_0^t \int_{\mathbb{R}^d} g(s,y)W(ds,dy)) = \langle f,g\rangle_{\mathcal{H}}.$$

Fort de cette remarque faisons le calcul suivant

$$\begin{aligned}\mathbb{E}(|u(t,x)|^2) &= \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)^2 dsy \\ &= \frac{1}{(4\pi)^d} \int_0^t \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{2s}} \frac{dyds}{s^d} \\ &= C_d \int_0^t \frac{1}{s^{d/2}} ds \quad \text{avec } C_d > 0.\end{aligned}$$

Supposons $d \geq 2$ alors $u(t,x)$ n'existe pas car u doit être un processus gaussien donc tous ses moments sont finis.

▷ Il nous faut alors définir une solution pour l'équation de la chaleur en toute dimension, ceci nous amène à définir un bruit avec des corrélations. Par conséquent, nous devrons alors définir une intégrale qui permet finalement de gérer ces corrélations. C'est ici que nous allons rencontrer la théorie de Walsh-Dalang.

Considérons, toujours à un niveau conceptuel plus élevé, l'équation de la chaleur non linéaire et non homogène suivante :

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) - \Delta u(t,x) = \sigma(u(s,y))W(x,t), \quad (x,t) \in \mathbb{R}^d \times (0,\infty) \\ u(x,0) = 0, \quad x \in \mathbb{R}^d \end{cases}. \tag{1.7}$$

1. Conformément à cette interprétation on rencontre dans la littératrue la notation $\dot{W}$ pour rappeler l'idée infinitésimale de la dérivée $df(x) = f dx$.

On écrira alors

$$u(x,t)=\int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)\sigma(u(s,y))W(s,y)dsdy.$$

▷ Pour pouvoir donner un sens à cette écriture il faut pouvoir définir une intégrale stochastique de la forme

$$\int h\Phi W(ds,dy) \tag{1.8}$$

avec Φ un processus stochastique.

Avant de discuter de cette intégrale, continuons notre lancée.

À ce stade nous avons considéré l'opérateur de la chaleur $L=\frac{\partial}{\partial t}-\Delta$, or un opérateur non moins célèbre est celui des ondes $L=\frac{\partial^2}{\partial t^2}-\Delta$.

Nous parlerons que de l'équation suivante dont il sera question dans le chapitre 1.

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(t,x) &= \Delta u(t,x)+W^H(t,x), \;\; t\in(0,T],\; T>0,\; x\in\mathbb{R}^d \\ u(0,x) &= 0, \quad x\in\mathbb{R}^d \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x\in\mathbb{R}^d. \end{cases} \tag{1.9}$$

Ne disons rien encore sur la nature du bruit, et considérons d'abord la solution fondamentale de (1.9) i.e. la solution de

$$\frac{\partial^2}{\partial t^2}u(t,x)-\Delta u(t,x)=0. \tag{1.10}$$

On a, toujours par la méthode de la transformée de Fourier,

$$\mathcal{F}(G(t,\cdot)(\xi)=\frac{\sin(t|\xi|)}{|\xi|}.$$

On peut en déduire que

$$\begin{aligned} G(t,x) &= \mathbf{1}_{\{|x|<1\}} \quad \textbf{si} \quad d=1 \\ G(t,x) &= \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}\mathbf{1}_{\{|x|<1\}} \quad \textbf{si} \quad d=2 \\ G(t,x) &= \frac{1}{4\pi}\sigma_t \quad \textbf{si} \quad d=3 \end{aligned}$$

où σ_t est la mesure de surface sur la sphère. En dimension $d\geq 4$ nous avons que $G(t,\cdot)$ est une distribution a support compact dans $\mathbb{R}^d$.

Ainsi pour $d\geq 3$, si nous voulons chercher à donner un sens à la solution $u(t,x)$ sous forme intégrale, nous devrions définir une intégrale radicalement différente des exemples vus précédemment.

Conclusion Ainsi nous devons être capables de répondre aux questions suivantes.

▷ Comment définir l'équation de la chaleur stochastique en toute dimension ?

▷ Comment définir une intégrale stochastique qui prend en compte un bruit avec des corrélations ?

▷ Comment définir une intégrale stochastique pour des intégrandes qui sont des mesures ou des distributions ?

Nous avons déjà donné des éléments de réponse mais la section suivante va permettre de donner un cadre général qui unifie (presque) toutes ces possibilités.

1.1.2 Cadre général des solutions pour les équations différentielles aux dérivées partielles stochastiques

Dans la section précédente nous avons montré comment chaque considération nous amène à définir un niveau conceptuel toujours plus élevé. Nous allons donc poser le cadre le plus général possible qui va nous permettre d'inclure (presque) tous les résultats obtenus durant cette thèse. On considère donc l'équation stochastique aux dérivées partielles suivante

$$Lu(t,x) = \sigma(u(t,x))W(t,x) + b((u(t,x)) \qquad (1.11)$$

avec certaines conditions initiales à déterminer au cas par cas. L est un opérateur différentiel, σ est une fonction lipschitzienne et b une fonction à valeur réelles et W est notre bruit stochastique.

Deux visions s'affronteront qui auront une certaine unité dans le Calcul de Malliavin à travers la notion de processus isonormal déjà discuté. En effet,

1. On peut considérer un bruit blanc en temps et colorié ou non en espace : on considère alors la théorie de Walsh-Dalang.
2. On peut aussi considérer un bruit colorié en temps et colorié ou non en espace : cette vision s'inclut soit dans le cadre des intégrales de Wiener pour les intégrandes déterministes, soit dans l'opérateur divergence avec le Calcul de Malliavin pour des intégrandes aléatoires. Nous ne parlerons pas de la dernière possibilité ici, mais nous renvoyons le lecteur au chapitre 5 de l'introduction [2] écrite par Raluca M. Balan, ainsi que les références citées.

Parlons d'abord du premier cas.

Walsh-Dalang Avant de définir notre processus gaussien, voici quelques considérations. Soit $\mathcal{C}_c^\infty(\mathbb{R}^{d+1})$ l'espace des fonctions infiniment différentiables à support compact. Soit $\mathcal{S}(\mathbb{R}^{d+1})$ l'espace de Schwartz. Pour tout $\phi \in \mathcal{S}(\mathbb{R}^d)$ la transformée de Fourier de ϕ est définie par

$$\mathcal{F}(\phi)(\xi) = \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle}\phi(x)dx.$$

Soit

$$\int_0^\infty dt \int_{R^d} \Lambda(dx)(\phi(t) \star \tilde{\psi}(t))(x) = J(\phi, \psi),$$

où $\tilde{\psi}(t,x) = \psi(t,-x)$, et Λ est une mesure positive tempérée i.e. $T_\Lambda \in \mathcal{S}'(\mathbb{R}^d)$ où

$$T_\Lambda : \phi \to \int_{\mathbb{R}^d} \phi(x)\Lambda(dx),\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^{d+1}).$$

On suppose aussi que J est positive, on dit aussi que Λ est définie positive. On montre alors que Λ est symétrique et donc que la fonctionnelle J définit un produit scalaire sur $\mathcal{C}_c^\infty(\mathbb{R}^{d+1}) \times \mathcal{C}_c^\infty(\mathbb{R}^{d+1})$ et qu'il existe une mesure tempérée μ sur $\mathbb{R}^d$ telle que sa transformée de Fourier soit Λ.

Ainsi on peut définir notre bruit de la façon suivante :

$$W = \Big\{ W(\phi),\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^{d+1}) \Big\} \tag{1.12}$$

un processus gaussien de covariance

$$\begin{aligned} \mathbb{E}(W(\phi)W(\Psi)) &= \int_0^\infty dt \int_{R^d} \Lambda(dx)(\phi(t) \star \tilde{\psi}(t))(x) \\ &= \int_0^\infty dt \int_{\mathbb{R}^d} \mu(d\xi)\mathcal{F}\phi(t)(\xi)\overline{\mathcal{F}\psi(t)(\xi)}. \end{aligned}$$

Nous dirons que le bruit est blanc en temps et colorié en espace de corrélation Λ. Voici deux choix typiques (qui seront utilisés aux chapitres 4,5 et 6).

- ✠ $\Lambda(dx) = f(x)dx$ donne pour covariance

$$\mathbb{E}(W(\phi)W(\Psi)) = \int_0^\infty dt \int_{R^d} dx f(x)\phi(t,x)\psi(t,x)$$

- ✠ $\Lambda(dx) = \delta_0 dx$ donne le bruit blanc temps-espace avec

$$\mathbb{E}(W(\phi)W(\Psi)) = \int_0^\infty dt \int_{R^d} dx \phi(t,x)\psi(t,x)$$

Donnons seulement les étapes et les idées de la construction de l'intégrale de Walsh et comment celle-ci nous permet d'intégrer nos bruits stochastiques [2]. Les énoncés exactes et preuves se trouvent dans [106].

Soit un espace de probabilité $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

1. On définit la notion de mesure de martingale $\Big(M_t(A),\ t \in \mathbb{R}^+,\ A \in \mathcal{B}_b(\mathbb{R}^d)\Big)$ sur $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ avec les hypothèses suivantes.

2. Rappelons nous de la remarque 2

- $t \mapsto M_t(A)$ est une martingale.
- $A \mapsto M_t(A)$ est une $L^2(\Omega, \mathbb{F}, P)-$mesure.
- Soit Q une mesure signée. La covariance s'écrit

$$\mathbb{E}(M_t(A)M_t(B)) = Q([0,t] \times A \times B).$$

2. On définit la filtration $(\mathcal{F}_t^0, t \geq 0)$ par

$$\mathcal{F}_t^0 = \sigma\left(M_s(A) : 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R})\right) \vee \mathcal{N}, \quad t \geq 0$$

où $\mathcal{N}$ est la tribu engendrée par les ensembles de mesures nulles. On pose $\mathcal{F}_t := \wedge_{s>t}\mathcal{F}_t^0$ pour $t \geq 0$.

Dans toute la suite on fixe donc l'espace $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, P)$.

3. On commence par les processus simples de la forme $f(t,x,\omega) = X(\omega)\mathbf{1}_{(a,b]}(t)\mathbf{1}_A(x)$ pour X une variable aléatoire bornée $\mathcal{F}_a-$mesurable,$A \in \mathcal{B}_b(\mathbb{R}),\ 0 \leq a < b \leq T$, et ainsi définir l'intégrale

$$\int_0^t \int_{\mathbb{R}^d} f(s,y)M(ds,dy) = (M_{\min(t,b)}(A) - M_{\min(t,b}(A))X.$$

On a l'isométrie suivante

$$\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} f(s,y)M(ds,dy)\right)^2\right] = \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^{2d}} f(s,y)f(s,y)Q(ds,dx,dy)\right)^2\right]. \tag{1.13}$$

C'est ici que des difficultés techniques apparaissent car pour étendre l'intégrale pour des processus plus généraux on cherche à passer la limite : on a besoin d'unicité. Ceci nous amène à définir une nouvelle notion (qui fut introduite par Walsh dans [106]).

4. On définit donc la notion de martingale worthy qui est une mesure de martingale tel qu'il existe une mesure signée positive K sur $\mathcal{B}([0,\infty) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d)$ telle que

$$Q([0,t] \times A \times B) \leq K([0,t] \times A \times B).$$

5. L'intégrale de Walsh

$$\int_{[0,T]} \int_A X(t,x)M(dt,dx)$$

est alors définie pour tous les processus réels prévisibles $X = \{X(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ tels que

$$\mathbb{E}\left(\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |X(t,x)||X(t,y)|K(dt,dx,dy)\right) < \infty.$$

6. Soit W notre bruit 1.12. On peut définir une mesure de martingale worthy de façon standard.

   6.1. On prend une suite de fonctions $(\phi_n, n \geq 1) \subset \mathcal{C}_0^\infty(\mathbb{R}^{d+1})$ qui tend vers le graphe d'un rectangle R et on définit $W(R)$ comme la limite $L^2(\Omega)$ des $W(\phi_n)$.

   6.2. On étend $R \to W(R)$ en une mesure $L^2(\Omega)$ qui nous permet de définir $M_t(A) = W([0,t] \times A)$.

   6.3. Le processus $M = \{M_t(A), t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^{d+1})\}$ est une mesure de martingale worthy pour la tribu engendré par les variables $M_s(A)$ pour tout $0 \leq s \leq t$ et $A \in \mathcal{B}_b(\mathbb{R}^{d+1})$. La mesure de covariance est donnée par

$$\mathbb{E}(M_t(A)M_t(B)) = t \int_{\mathbb{R}^d} \Lambda(dx)(\mathbf{1}_A \star \tilde{\mathbf{1}}_B)(x).$$

7. Ainsi la théorie de Walsh nous permet ainsi d'intégrer alors tous les processus réels prévisibles $X = \{X(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ tels que

$$\mathbb{E}\left(\int_0^T ds \int_{\mathbb{R}^d} \Lambda(dx)(|X(s,\cdot)| \star |\tilde{X}(s,\cdot)|)(x)\right) < \infty. \tag{1.14}$$

Cette vision est suffisante pour l'équation de la chaleur linéaire en toutes dimensions et pour l'équation des ondes en dimension $1, 2$ car nous l'avons vu à partir de la dimension 3 la solution fondamentale n'est plus une fonction mais une mesure pour $d = 3$ et une distribution pour $d \geq 4$. C'est dans cette optique qu'on peut étendre la construction de la l'intégrale de Walsh qui nous donne la théorie de Walsh-Dalang.

Nous nous placerons directement dans le cadre hilbertien car c'est celui-ci qui nous intéresse pour utiliser le Calcul de Malliavin.

Ainsi on peut définir l'intégrale

$$\int_0^T \int_{\mathbb{R}}^d G(s,y)Z(s,y)W(ds,dy) \tag{1.15}$$

avec comme hypothèses

- Z est un processus stochastique prévisible à valeur dans un espace de Hilbert $\mathcal{A}$ doté de son produit scalaire $\langle \cdot, \cdot \rangle$ tel que

  1.
$$\sup_{(s,y) \in \mathbb{R}^d \times [0,T]} \mathbb{E}(\|Z(s,y)\|_{\mathcal{A}}^2) < \infty, \tag{1.16}$$

  2. et pour tout $j \geq 0,\ s \in [0,T], x, y \in \mathbb{R}^d$

$$\mathbb{E}(Z^j(s,x)Z^j(s,y)) = \mathbb{E}(Z^j(s,0)Z^j(s,y-x))$$

  avec $Z^j(s,y) := \langle Z(s,y), e_j \rangle_{\mathcal{A}}$ où $(e_j,\ j \geq 0)$ est une base orthonormale de $\mathcal{A}$.

- $t \mapsto G(t, \cdot)$ est une fonction à valeur dans l'espace des distributions i.e le dual de l'espace de Schwartz noté $\mathcal{S}'(\mathbb{R}^d)$.
- W est un bruit blanc en temps et colorié en espace.

L'intégrale sera bien définie sous l'hypothèse fondamentale

$$\int_0^T \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G(t, \cdot)(\xi)|^2 < \infty. \tag{1.17}$$

On peut donc définir la solution de (1.11) en prenant $Z(s, y) := \sigma(u(s, y))$, $\mathcal{A} = \mathbb{R}$ et en faisant d'abord les hypothèses suivantes.

Hypothèse 1 (Condition de Dalang) *La solution $G(\cdot, \star)$ de $Lu = 0$ est une fonction déterministe en $\cdot$ à valeur dans l'espace dual de l'espace de Schwartz $\mathcal{S}'(\mathbb{R}^d)$ tels que*

$$\int_0^T \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G(t, \cdot)(\xi)|^2 < \infty \tag{1.18}$$

et G est une mesure positive de la forme $G(t, dx)dt$ telle que

$$\sup_{t \in [0,T]} G(t, \mathbb{R}^d) < \infty. \tag{1.19}$$

Définition 2 *Une solution de (1.11) est un processus $u = (u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d)$ tel que*

1. *Pour tout $(t, x) \in [0, T] \times \mathbb{R}$, $u(t, x)$ est $\mathcal{F}_t$ mesurable*
2. *u est bimesurable i.e. mesurable par rapport à $[\imath, \mathcal{T}] \times \mathbb{R} \times \mathcal{F}$*
3. *Pour tout t, x on a*

$$\sup_{(t,x) \in \mathbb{R}^d \times [0,T]} \mathbb{E}(|u(t, x)|^2) < \infty. \tag{1.20}$$

$$\mathbb{E}(u(t, x)u(t, y)) = \mathbb{E}(u(t, 0)u(t, x - y)) \tag{1.21}$$

4. *Pour tout $(t, x) \in [0, T] \times \mathbb{R}$, u satisfait*

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y)\sigma(u(s, y))W(ds, dy) \tag{1.22}$$

$$+ \int_0^t \int_{\mathbb{R}^d} b(u(t - s, x - y))G(s, dy) \tag{1.23}$$

Avec cette définition on a le théorème d'existence et d'unicité, ainsi que de régularité. Pour une preuve voir l'article [44].

Théorème 1 (Existence, unicité et régularité) *On suppose que σ est une fonction lipschitzienne, que la condition de Dalang (1) est vérifiée. Alors l'équation (1.11) a une unique solution au sens de la définition 2. On a aussi que pour tout $p \geq 2$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}(|u(t,x)|^p) < \infty.$$

De plus, u est L^p continue en (t,x) et faiblement stationnaire en x pour tout $t > 0$.

Proposition 1 (Condition de Dalang en pratique) *Soit L l'opérateur des ondes pour la dimension $d \in \{1,2,3\}$ ou de la chaleur pour tout $d \geq 1$. Dans l'article ([44]) Dalang montre qu'une condition nécessaire et suffisante pour l'existence d'une solution est*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty. \tag{1.24}$$

Cette théorie bien que très puissante ne permet pas de considérer des corrélations temporelles[3]. Le cas qui nous intéresse, comme nous l'avons déjà dit, est une corrélation fractionnaire en temps. La section suivante est dédiée à ce type de bruit et nous permettra d'embrayer sur les processus chaotiques plus généraux.

1.1.3 Du mouvement brownien fractionnaire aux processus chaotiques

Nous étudierons comme déjà dit en introduction l'équation des ondes stochastiques avec un bruit fractionnaire en temps et blanc en espace au chapitre 1.

Nous définissons dans un premier temps le mouvement brownien fractionnaire et rappelons les propriétés essentielles. Nous renvoyons aux livres [4] et [3] pour les preuves.

Définition 3 *Soit $H \in (0,1]$. Un mouvement brownien fractionnaire de paramètre H est une processus gaussien centré $B^H = (B^H_t)_{t\geq 0}$ de covariance*

$$R_H(t,s) := \mathbb{E}[B^H_t B^H_s] := \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1.25}$$

Remarque 4 *On sait d'après le théorème de Kolomogorov qu'un processus gaussien existe si la covariance est de type positif. La restriction de H est nécessaire car on montre que le processus fractionnaire existe si et seulement si $H \in (0,1]$. Le même théorème nous donne que B^H est höldérien pour tout $\alpha \in (0,H)$.*

Remarque 5 *Nous pouvons définir le mouvement brownien fractionnaire sur $\mathbb{R}$ simplement en prenant les valeurs absolues i.e.*

$$\mathbb{E}[B^H_t B^H_s] := \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

Voir [5] page 47 pour les détails.

3. La preuve de cette assertion dans notre cas précis sera donnée à la section suivante.

Voici une liste non exhaustive de ses propriétés.

1. Pour tout $a > 0$ on a $(a^{-H} B^H_{at})_{t\geq 0} \stackrel{L}{=} (B^H_t)_{t\geq 0}$.

2. Pour tout $h > 0$ on a $(B^H_{t+h} - B^H_h)_{t\geq 0} \stackrel{L}{=} (B^H_t)_{t\geq 0}$.

3. Pour tout $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, B^H n'est pas une semimartingale .

4. Soit r la fonction d'autocovariance i.e. $r(n) := \mathbb{E}(B^H_1(B^H_{n+1} - B^H_n))$.

   4.1. On montre que si $H \in (0, \frac{1}{2})$ alors $\sum_{n\in\mathbb{Z}} |r(n)| < \infty$, on dit que le processus est à mémoire courte.

   4.2. Par contre si $H \in (\frac{1}{2}, 1)$ alors $\sum_{n\in\mathbb{Z}} |r(n)| = \infty$. On dit dans ce cas que B^H est un processus à longue mémoire.

5. Pour tout $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, B^H n'est pas un processus de Markov.

▷ Le troisième point est de la plus haute importance pour nous car si on cherche à définir notre bruit de la façon suivante

$$\mathbb{E}(W^H_t(A) W^H_s(B)) = R^H(t, s)\lambda(A \cap B), \text{ pour tout } t, s \geq 0, A, B \in \mathfrak{B}_b(\mathbb{R}^d) \quad (1.26)$$

avec comme notation $W_t(A) = W([0, t]\times A)$, alors $\Big(W_t(A),\ t \in \mathbb{R}^+,\ A \in \mathcal{B}_b(\mathbb{R}^d)\Big)$ n'est pas une mesure de martingale worthy et donc on ne peut pas utiliser l'approche qui a été définie plus haut pour considérer la solution de notre équation des ondes 1.9.

Mais nous pouvons construire, au moins pour les intégrandes déterministes, l'intégrale de Wiener.

La procédure est standard mais robuste car on verra qu'elle s'adapte pour les processus de Hermite dont nous aurons besoin dans le chapitre 1 et 3.

L'heuristique est la suivante, pour une démonstration complète nous renvoyons le lecteur aux livres déjà cités sans oublier l'article [73]. On cherche à définir les deux intégrales (la première pour les équations différentielles stochastiques et la deuxième pour les équations différentielles aux dérivées partielles stochastiques) suivantes

$$\int_0^T f(t) dB^H(t) \quad \text{ou} \quad \int_0^T \int_{\mathbb{R}^d} G(s, y) W^H(ds, dy). \quad (1.27)$$

Les deux procèdent exactement de la même façon.

L'intégrale $\int_0^T f(t)dB^H(t)$

Soit $\mathcal{E}$ l'ensemble des fonctions f élémentaires définies par

$$f(x) = \sum_{k=1}^{n} f_k 1_{x_k, x_{k+1}}(x) \quad \text{avec} \quad f_k \in \mathbb{R}, 0 \leq x_k \leq x_{k+1} \leq T. \tag{1.28}$$

Pour $f \in \mathcal{E}$, on définit la variable aléatoire, qui sera notre intégrale, suivante

$$I_T^H(f) = \sum_{k=1}^{n} f_k \left(B^H(x_{k+1}) - B^H(x_k) \right). \tag{1.29}$$

Il est alors naturel de passer à toutes les fonctions $L^2(\Omega)$ par un passage à la limite.

Remarque 6 *La procédure s'adapte facilement sur $\mathbb{R}^+$.*

Remarque 7 *Par une transformation d'Abel, on remarque que toute combinaison linéaire de B^H s'écrit comme une intégrale fractionnaire. On a donc l'égalité d'ensemble*

$$Vect\{B^H(t), t \in [0,T]\} = \{I_T^H(f), f \in \mathcal{E}\}.$$

Donc par passage à l'adhérence dans $L^2(\Omega)$ on a

$$\mathcal{H}_T^H := \overline{Vect\{B^H(t), t \in [0,T]\}} = \overline{\{I_T^H(f), f \in \mathcal{E}\}}.$$

L'espace $\mathcal{H}_T^H$ est l'espace gaussien engendré par B^H. Cette espace joue un rôle historique fondamental, notamment pour son aspect géométrique.

Pour l'intégrale dans sa généralité on a le théorème suivant dont on trouvera la preuve dans [5] page 370.

Théorème 2 *Soient $\mathcal{E}$ l'espace des fonctions élémentaires, I_T^H la variable aléatoire définie par (1.29), $H \in (0,1]$. Soit $\mathcal{C}$ l'ensemble des fonctions définies sur $[0,T]$ tel que*

- *Il existe un produit scalaire sur $\mathcal{C}$ noté $\langle f, g \rangle_{\mathcal{C}}$ pour tout $f, g \in \mathcal{C}$,*
- *$\mathcal{E} \subset \mathcal{C}$ et $\langle f, g \rangle_{\mathcal{C}} = \mathbb{E}(I_T^H(f) I_T^H(g))$ pour tout $f, g \in \mathcal{E}$,*
- *$\mathcal{E}$ est dense dans $\mathcal{C}$.*

Alors il existe une isométrie de $\mathcal{C}$ dans un sous espace vectoriel de $\mathcal{H}_T^H$ qui étend $f \mapsto \mathcal{I}_T^H$ pour $f \in \mathcal{E}$. $\mathcal{C}$ est isométrique à $\mathcal{H}_T^H$ si et seulement si $\mathcal{C}$ est complet.

Si nous cherchons à définir l'intégrale comme décrit dans 2 il nous faut alors un espace muni d'un produit scalaire. Faisons le calcul pour $H > \frac{1}{2}$, qui est le cas qui nous intéresse. On a alors pour la fonction de covariance du mouvement brownien fractionnaire (1.25) avec $t, s \in [0,T]$

$$R_H(t,s) \quad = \quad H(2H-1) \int_0^t du \int_0^s dv |u-v|^{2H-2} \tag{1.30}$$

$$= \quad c_H \int_0^T \int_0^T \mathbf{1}_{[0,t]}\mathbf{1}_{[0,s]}|u-v|^{2H-2}dudv.$$

Nous pouvons donc considérer l'espace associé

$$R_H = \{f : [0,T] \to \mathbb{R} : \int_{[0,T]^2} f(u)f(v)||u-v|^{2H-2}dudv < \infty\} \tag{1.31}$$

muni du produit scalaire

$$\langle f, g\rangle_{R_H,1} = (2H-1)H \int_{[0,T]^2} f(u)g(v)|u-v|^{2H-2}dudv. \tag{1.32}$$

Remarque 8 *Nous pouvons aussi considérer l'espace*

$$|R_H| = \{f : [0,T] \to \mathbb{R} : \int_{[0,T]^2} |f(u)||f(v)|||u-v|^{2H-2}dudv < \infty\} \tag{1.33}$$

muni du produit scalaire

$$\langle f, g\rangle_{R_H,1} = (2H-1)H \int_{[0,T]^2} f(u)g(v)|u-v|^{2H-2}dudv. \tag{1.34}$$

On alors peut montrer qu'on a pour $H > 1/2$, $L_2([0,T]) \subset L_{\frac{1}{H}}([0,T]) \subset |R_H| \subset R_H$ *et que l'espace* $(|R_H|, \langle f, g\rangle_{R_H,1})$ *n'est pas complet mais que* $(R_H, \langle f, g\rangle_{R_H,1})$ *l'est et coincide avec un espace fractionnaire où des distributions apparaissent. Nous renvoyons le lecteur au chapitre* 5 *du livre de David Nualart pour un panomara assez exhaustif .*

Résumons nous dans la proposition suivante.

Proposition 2 *Soient* $H \in (\frac{1}{2}, 1)$ *et* $f, g \in R_H$ *alors* $I_T^H(f)$ *a un sens et on a*

$$\mathbb{E}I_T^H(f)I_T^H(g) = (2H-1)H \int_0^T \int_0^T f(u)g(v)|u-v|^{2H-2}dudv.$$

Remarque 9 *La représentation (1.30) ne fonctionne pas pour* $H < 1/2$ *car l'intégrale diverge. Dans ce cas il faudrait parler de l'espace de Cameron-Martin associé au mBf, ces considérations sont profondes mais nous amèneraient trop loin. Pour comprendre les difficultés, la lecture du document [17] écrit par L. Decreusefond est un impératif.*

Remarque 10 *Bien que le caractère hilbertien est fondamental pour pouvoir appliquer les outils du Calcul de Malliavin, comme nous avons que* $|R_H|$ *est identifiable (par le théorème 2) à un sous-espace de* $\mathcal{H}_T^H$ *nous pouvons aussi travailler avec* $|R_H|$.

L'intégrale $\int_0^T \int_{\mathbb{R}^d} G(s,y)W^H(ds,dy)$

Compte tenu de notre volonté de faire du Calcul de Malliavin, du fait que la covariance 1.26 ne définit pas une martingale, et du théorème 2, on cherche à donner un sens à l'intégrale par rapport à notre bruit fractionnaire W_t^H en définissant l'espace de Hilbert associé H_W à notre bruit.

Soit $\mathcal{E}_2$ l'espace des fonctions élémentaires sur les ensembles $[0,t] \times A$ avec $t \in (0,T], A \in \mathcal{B}_d(\mathbb{R}^d)$, et on prend l'adhérence de $\mathcal{E}_2$ muni du produit scalaire

$$\begin{aligned}\langle \mathbf{1}_{[0,t]\times A}, \mathbf{1}_{[0,s]\times B}\rangle_{H_W} &:= \mathbb{E}(W_t^H(A)W_s^H(B)) \\ &= (2H-1)H\lambda(A\cap B)\int_0^t\int_0^s |u-v|^{2H-2}dudv.\end{aligned}$$

C'est notre espace de Hilbert gaussien $\mathcal{H}_W$. Puis on étend par densité car $\mathcal{E}_2$ est dense dans $\mathcal{H}_W$ et $\mathbf{1}_{[0,t]\times A} \mapsto W_t^H(A)$ est une isométrie. Ainsi on définit une isométrie de $\mathcal{H}_W$ vers $L^2(\Omega)$ dont on note l'extension

$$W^H(f) := \int_0^T \int_{\mathbb{R}^d} f(t,x)W^H(dt,dx), \tag{1.35}$$

et nous obtenons l'isométrie explicite suivante

$$\begin{aligned}\mathbb{E}(W^H(f)W^H(g)) &= \langle f,g\rangle_{\mathcal{H}_W} \\ &= (2H-1)H\int_0^T\int_0^T\int_{\mathbb{R}^d} f(u,x)g(v,x)|u-v|^{2H-2}dxdudv.\end{aligned}$$

Revenons à notre équation des ondes 1.9 i.e.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + W^H(t,x), \;\; t\in(0,T],\; T>0,\; x\in\mathbb{R}^d \\ u(0,x) = 0, \quad x\in\mathbb{R}^d \\ \frac{\partial u}{\partial t}(0,x) = 0, \quad x\in\mathbb{R}^d. \end{cases}$$

Nous voyons donc qu'on peut définir la solution et que celle-ci existe si et seulement si

$$(s,y)\mapsto G(t-s,x-y)\in\mathcal{H}_W. \tag{1.36}$$

▷ Comme nous l'avons dit G n'est plus une fonction pour $d \geq 3$ et c'est pourquoi nous avons considérer l'espace $\mathcal{H}_W$ qui est un espace fractionnaire de type Sobolev.

En fait cette procédure hilbertienne fonctionne de la même façon pour les bruits définis par la théorie de Walsh-Dalang et nous avons donc un cadre qui les unit, voir la section 1.1.4.3.

1.1.3.1 Intégrales multiples et processus de Hermite

Notre volonté est maintenant de pouvoir généraliser le mouvement brownien fractionnaire, c'est-à-dire de conserver ses propriétés fondamentales tout en s'affranchissant du caractère gaussien, qui reste une restriction en soi.

Nous pouvons donc définir la notion d'intégrale multiple.

Pour généraliser le mouvement brownien fractionnaire au sens des intégrales multiples, la proposition suivante est l'élément déclencheur de notre récit.

Soit $W = \{W_t, t \in \mathbb{R}\}$ le processus de Wiener sur $\mathbb{R}$.

Théorème 3 *Le processus $\tilde{B}^H = \{\tilde{B}^H_t, t \in \mathbb{R}\}$ défini par*

$$\tilde{B}^H_t := C_H \int_{\mathbb{R}} \left((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}} \right) dW_u, \ H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \tag{1.37}$$

est un mouvement brownien fractionnaire sur $\mathbb{R}$

où $x_+ = x\mathbf{1}_{(0,\infty)}(x)$. Pour une preuve voir la proposition 2.6.5 page 48 dans [5].

Remarque 11 *Prenons $H \in (0, \frac{1}{2})$ alors (1.37) peut s'écrire*

$$\tilde{B}^H_t := (H - \frac{1}{2}) C_H \int_{\mathbb{R}} \left(\int_0^t (s-u)_+^{H-\frac{3}{2}} ds \right) dW_u, \ H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right). \tag{1.38}$$

La formule (1.38) va nous permettre de définir nos processus de Hermite. En effet, considérons la fonction

$$f_t(u_1, u_2) := (t-u_1)_+^{(1-\frac{1-H}{2})-\frac{3}{2}} (t-u_2)_+^{(1-\frac{1-H}{2})-\frac{3}{2}}.$$

alors $f_t \in L^2(\mathbb{R}^2)$. On sait que l'intégrale de Wiener classique est définie pour les fonctions $L^2(\mathbb{R})$ et ainsi si on souhaite écrire

$$\int_{\mathbb{R}^2} f_t(u_1, u_2) dW_{u_1} dW_{u_2}. \tag{1.39}$$

Il nous faut définir proprement l'intégrale multiple pour le processus de Wiener sur $\mathbb{R}$.

Nous renvoyons le lecteur à la section 1.5 du chapitre 1 de [6] pour une construction de l'intégrale multiple avec toutes les preuves.

On supposera donc que le lecteur est familier avec cette construction et ainsi on peut définir le processus de Hermite comme suit. Pour toutes les considérations qui vont suivre nous renvoyons au livre [100].

Définition 4 (Processus de Hermite) *Soient d un entier et*

$$H \in \left(\frac{1}{2}, 1\right).$$

Posons

$$H_0 = 1 - \frac{1-H}{d} \in \left(1 - \frac{1}{2q}, 1\right).$$

Le processus de Hermite $Z_{t\in\mathbb{R}}^{(q,H)}$ *d'ordre* q *est défini par*

$$Z_t^{(q,H)} := \alpha_{q,H_0} \int_{\mathbb{R}^d} \left(\int_0^t \prod_{j=1}^{q} (s-u_i)_+^{H_0-\frac{3}{2}} ds \right) B(du_1) \dots B(du_n). \tag{1.40}$$

La constante α_{q,H_0} est une constante de renormalisation. On dit que le processus 4 est standard si $\mathbb{E}(Z_H^q(t))^2 = 1$. L'intégrale $\int_0^t$ est interprété comme $-\int_t^0$ si $t < 0$.

Nous appellerons l'intégrande le noyau du processus qui est la fonction

$$f_t(u_1, \dots, u_q) = \int_0^t \prod_{j=1}^{q} (s-u_i)_+^{H_0-\frac{3}{2}} ds. \tag{1.41}$$

On peut montrer les propriétés suivantes.

1. Le processus de Hermite est bien défini i.e. $\|f_t\|_{L^2(\mathbb{R}^d)} < \infty$.
2. Le processus est à accroissement stationnaire, auto-similaire d'ordre H et tous ses moments sont finis.
3. On peut alors en déduire que la processus est höldérien continue d'ordre $\delta < H$.
4. La fonction de covariance s'écrit

$$R(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}). \tag{1.42}$$

- En fait tout les processus auto-similaires d'ordre $H \in (0,1)$ à accroissements stationnaires sur $\mathbb{R}$ ont la même covariance, voir la proposition 2.5.6 page 45 de [5].
- Pour $q = 1$ on a le mouvement brownien fractionnaire mais quelle différence conceptuelle existe-t-il pour $q \geq 2$? **Les processus définis ne sont plus gaussiens** voir le théorème 10. Il s'avère que sous des conditions techniques les processus de Hermite apparaissent comme des limites en loi d'une large classe de séries aléatoires : c'est ce qu'on appelle le théorème non central limite. Voir la section 5.3 dans [5].

Dans cette thèse nous rencontrerons les processus de Hermite par l'étude de l'équation de Langevin avec un bruit Hermite, dont le but sera l'estimation du paramètre de Hurst H et ainsi que le processus de Hermite généralisé car ce dernier nous permet de nous affranchir de la condition $H > \frac{1}{2}$. Nous définirons ce dernier à la partie 1.4.

1.1.3.2 De l'équation de Langevin au processus de Ornstein-Uhlenbeck

Au début du XX-ième siècle l'étude d'une particule libre émergée dans fluide a amené Langevin à établir l'équation suivante

$$\frac{dv(t)}{dt} = -fv(t) + \frac{F(t)}{m} \tag{1.43}$$

où m est la masse de la particule, $f > 0$ le coefficient de friction et F la force résultante des chocs de la particule avec les molécules du fluide. Étant donné le très grand nombre de molécules, les physiciens et mathématiciens de l'époque ont adopté un point de vue statistique pour F.

Disons simplement qu'en faisant des hypothèses stochastiques banales sur F i.e $\mathbb{E}(F) = 0$, fonction de covariance C connue, stationnaire tel que $C(\tau) = C\delta(\tau)$ (où C est une constante physique descriptible) et en écrivant la solution par la méthode de la variation de la constante (comme dans la section 1.1.1)

$$v(t) = v_0 e^{-ft} + \frac{1}{m}\int_0^t F(s)e^{-f(t-s)}ds \tag{1.44}$$

on obtient, formellement,

$$\mathbb{E}(v(t)) = v_0 e^{-ft} \quad \text{et} \quad \sigma_v^2 = \frac{C}{f}\left(1 - e^{-2ft}\right). \tag{1.45}$$

Dans notre langage moderne on dit que v est une variable aléatoire normale d'espérance $v_0 e^{-ft}$ et de variance $\frac{C}{f}\left(1 - e^{-2ft}\right)$. J.L. Doob en 1937 dans son article [7] montra que (dont le calcul est évident)

$$\mathbb{E}\left((v(t+s) - v(s))^2\right) \sim_{t\to 0} ct, \tag{1.46}$$

pour c une certaine constante, ainsi la particule a une accélération infinie.

Dans le même article Doob montre que

$$\tilde{v}(t) := \sqrt{t}v(\frac{1}{2f}\log(t)) \quad t > 0 \tag{1.47}$$

est un mouvement brownien standard.

Cet événement marque la naissance de la théorie des équations différentielles stochastiques dont l'équation de Langevin s'écrit aujourd'hui

$$dX(t) = -fX(t)dt + dB(t), \tag{1.48}$$

Ainsi la solution s'écrit comme intégrale de Wiener par rapport au mouvement brownien

$$X(t) = \int_0^t e^{-f(t-s)}dB(s). \tag{1.49}$$

X est ce qu'on appelle le processus de **Ornstein-Uhlenbeck**.

Au vu des considérations précédentes sur le mouvement brownien fractionnaire il est naturel de vouloir généraliser l'équation (1.48) en considérant

$$X_t = \xi - f\int_0^t X_s ds + \sigma B_t^H \quad t \geq 0 \tag{1.50}$$

où $\xi \in L^0(\Omega)$. La solution de 1.50 s'écrit

$$Y_t^{(H,\xi)} := e^{-ft}\left(\xi + \sigma\int_0^t e^{fu} dB_u^H\right), \tag{1.51}$$

où l'intégrale est au sens de Wiener ou de Stieljes.

Ce travail a été fait dans [8]. Le processus Y s'appelle le processus de Ornstein-Uhlenbeck fractionnaire. C'est une processus gaussien. Si on suppose que la condition initiale est $\xi = \sigma\int_{-\infty}^0 e^{fu} dB_u^H$ on obtient un processus stationnaire car on a

$$Y_t^{(H,\xi)} = \sigma\int_{-\infty}^t e^{-f(t-u)} dB_u^H. \tag{1.52}$$

On peut dès lors considérer de même l'équation avec un bruit défini par le processus de Hermite 4.

$$X_t = \sigma - f\int_0^t X_s ds + Z_t^{(H,q)} \quad t \geq 0 \tag{1.53}$$

avec $Z_t^{(H,q)}$ notre processus de Hermite d'ordre q. Cette fois-ci notre solution *devrait* s'écrire

$$Y_t^{(H,\xi)} := e^{-ft}\left(\xi + \sigma\int_0^t e^{fu} dZ_u^{(H,q)}\right). \tag{1.54}$$

Nous devons donc définir une intégrale de Wiener pour les processus de Hermite. La méthode est la même que pour le mouvement brownien fractionnaire, nous renvoyons le lecteur à l'article [11] pour les détails. Faisons malgré tout quelques remarques dont quelques-unes seront nécessaire pour la partie 3.

Remarque 12 *1. L'intégrale $\int_{\mathbb{R}} g(u) dZ_u^{(H,q)}$ est Wiener intégrable si*

$$g \in \{f : \mathbb{R} \to \mathbb{R};\ \int_{\mathbb{R}}\int_{\mathbb{R}} f(u)f(v)|u-v|^{2H-2} dudv < \infty\}$$

1.1. Nous avons en fait

$$\int_{\mathbb{R}} f(u) dZ_u^{(q,H)} = \int_{\mathbb{R}^q} (Jf)(y_1, ..., y_q) dB(y_1)\dots dB(y_q)$$

avec, pour tout $y_1, ..., y_q \in \mathbb{R}$,

$$(Jf)(y_1,\dots,y_q) = d(q,H)\int_{\mathbb{R}} f(u)(u-y_1)_+^{-(\frac{1}{2}+\frac{1-H}{q})}\dots(u-y_q)_+^{-(\frac{1}{2}+\frac{1-H}{q})} du.$$

1.2. Nous avons donc l'isométrie suivante

$$\mathbb{E}\int_{\mathbb{R}} f(u)dZ_u^{(q,H)}\int_{\mathbb{R}} g(u)dZ_u^{(q,H)} = H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} dudvf(u)f(v)|u-v|^{2H-2}.$$

2. Le processus suivant est alors bien défini

$$Y_t^{(H,\xi)} = \sigma\int_{-\infty}^{t} e^{-f(t-u)}dZ_u^{(H,q)}. \quad (1.55)$$

et

2.1. c'est un processus non gaussien, stationnaire de covariance

$$E(Y_t^{(H,\xi)}Y_s^{(H,\xi)}) = \sigma^2\int_{-\infty}^{t}\int_{-\infty}^{s} e^{-f(t-u)}e^{-f(s-v)}|u-v|^{2H-2}dvdu < \infty. \quad (1.56)$$

3. En fait l'intégrale peut s'interpréter comme une intégrale de Riemann-Stieljes. Ce lien ne sera utilisé pour nous que réellement dans la partie 3 *donc pour ne pas alourdir l'introduction nous en parlerons le moment venu.*

Conclusion Après toutes ces considérations, il ne faut pas oublier notre but : étudier nos solutions. Nous les avons définies de manière progressive, bien qu'assez formellement, mais notre objectif est de pouvoir les étudier. Le cadre qui va nous permettre de parler de régularité, de convergence, d'estimation est le Calcul de Malliavin.

Cependant nous ne résistons pas à énoncer le théorème suivant, qui nous sera d'ailleurs utile au chapitre 1 pour estimer le paramètre H, qui établit l'ergodicité des accroissements discrétisés des intégrales-multiples de « type » Hermite.

Ergodicité des chaos de type Hermite Soient $q \geq 1$ et $0 < H < 1$. On suppose qu'il existe une suite de noyaux $Q_t^{(q)} : \mathbb{R}^q \to \mathbb{R}$ pour $t \geq 0$ telle que

- Pour tout $t \geq 0$ on a $Q_t^{(q)} \in L^2(\mathbb{R}^q, \lambda_q)$.
- Pour tout $s \in [0,t), c > 0$ on a

$$Q_t^{(q)}(y_1,\dots,y_q) - Q_s^{(q)}(y_1,\dots,y_q) = Q_{t-s}^{(q)}(y_1-s,\dots,y_q-s),$$

et

$$Q_{ct}^{(q)}(cy_1,\dots,cy_q) = c^{H-\frac{k}{2}}Q_t^{(q)}(y_1,\dots,y_q),$$

égalité qu'on suppose vraie pour presque tout $(y_1,\dots,y_q)$.

Définissons maintenant le chaos d'ordre q associé à ce noyau. Autrement dit soit

$$X_t^{(q)} = \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} Q_t^{(q)}(y_1,\dots,y_q)B(dy_1)\cdots B(dy_q)) = I_q(Q_t^{(q)}), \quad t \geq 0. \quad (1.57)$$

Voici le théorème central (voir le livre [91] chapitre 8)

Théorème 4 *Le processus $X = \{X_t^{(q)}\}_{t\geq 0}$ est un processus auto-similaire d'ordre H, à accroissements stationnaires et le processus discrétisé $\tilde{X} = \{X^{(q)}(i) - X^{(q)}(i-1)\}_{i\in\mathbb{N}}$ est un processus stationnaire et* **ergodique**.

Remarque 13 *Nous avons appelé ce type de noyau « type » Hermite car les conditions que le processus 1.57 doit vérifier nous semble amener nécessairement à un noyau de la forme "$\int \prod$".*

1.1.4 Du Calcul de Malliavin à la méthode de Stein-Malliavin

1.1.4.1 Calcul de Malliavin

Pour toute cette section on renvoie aux livres [77] et [10]. Cependant on peut lire dans la thèse de Hélène Halconruy [16] une très belle introduction aux concepts et à l'histoire du Calcul de Malliavin et ses différentes applications. Le concept central est celui de processus isonormal. Soit $\mathcal{H}$ un espace de Hilbert muni d'un produit scalaire $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Soit le processus isonormal $X = \{X(h) : h \in \mathcal{H}\}$ ce qui signifie que X est un processus gaussien défini sur un espace $(\Omega, \mathcal{F}, P)$ tel que $\mathbb{E}(X(h)X(g)) = \langle f, g\rangle_{\mathcal{H}}$.

Pour tout $q \geq 1$ soit $\mathcal{H}_q$ le $q-$ième chaos de Wiener de X : c'est le sous espace vectoriel fermé de $L^2(\Omega, \mathcal{F}, P)$ engendré par les variables aléatoires $\{H_q(X(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}}\} = 1$ où H_q est le $q-$ième polynôme de Hermite défini par

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}(e^{-\frac{x^2}{2}}). \tag{1.58}$$

On a par exemple $H_1(x) = x,\ H_2(x) = x^2 - 1,\ H_3(x) = x^3 - 3x$. Ceci nous permet de définir l'intégrale multiple. Soit $q \geq 1$ on définit l'application $H_q(X(h)) = I_q(h^{\otimes q})$ qui s'étend en application linéaire isométrique du produit tensoriel symétrique $\mathcal{H}^{\odot q}$ de norme $\|\cdot\|_{\mathcal{H}^{\odot q}} = \sqrt{q!}\|\cdot\|_{\mathcal{H}^{\otimes}}$ vers $\mathcal{H}_q$. On obtient donc pour tout $f, g \in \mathcal{H}^{\odot p}$ et $p, q \geq 1$

$$\mathbb{E}(I_p(f)I_q(g)) = \delta_{p,q} \times p!\langle f, g\rangle_{\mathcal{H}^{\otimes}} \tag{1.59}$$

où $\delta_{p,q}$ est le produit de Kronecker.

Nous avons la décomposition en chaos de Wiener pour toute fonction $F \in L^2(\Omega, \mathcal{F}, \sigma(\{X\}), P)$

$$F = E(F) + \sum_{q=1}^{\infty} I_q(f_q) \tag{1.60}$$

où $f_q \in \mathcal{H}^{\odot q}, q \geq 1$ sont entièrement déterminés par F.

Définissons maintenant les concepts purs du Calcul de Malliavin. Soit $\mathcal{S}$ l'ensemble des fonctions

$$f(X(h_1), \ldots, X(h_d))$$

où $F = f \in \mathcal{C}_b^\infty(\mathbb{R}^d)$. La dérivée de Malliavin de F est l'élément de $L^2(\Omega, \mathcal{H})$ défini par

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n))h_i. \tag{1.61}$$

Par itération on définit la $p-$ième dérivée comme élément de $L^2(\Omega, \mathcal{H}^{\odot p})$ pour tout $p \geq 1$. À titre d'exemple pour $p = 2$ on a

$$D^2F = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(W(h_1), \dots, W(h_n))h_i \otimes h_j. \tag{1.62}$$

On peut alors définir notre norme de Sobolev. Soit $\mathbb{D}^{p,q}$ où $q, p \geq 1$ l'adhérence de $\mathcal{S}$ pour la norme

$$\|F\|_{\mathbb{D}^{q,p}}^p = \mathbb{E}(|F|^p) + \sum_{i=1}^{q} \mathbb{E}(\|D^i F\|_{\mathcal{H}^{\otimes i}}^p). \tag{1.63}$$

L'espace $\mathbb{D}^{p,q}$ est appelé en anglais "The Gaussian Sobolev space". Prenons $\mathcal{H} = L^2(A, \mathcal{A}, \mu)$ avec μ une mesure non atomique. Alors tout élément $u \in \mathcal{H}$ s'écrit comme un processus $u = \{u_t,\ t \in A\}$ et la dérivée de Malliavin de $F = I_q(f)$ est le processus $\{D_t F, t \in A\}$ tel que

$$D_t F = q I_{q-1}(f(\cdot, t)). \tag{1.64}$$

La dérivée de Malliavin se comporte bien comme une dérivée, c'est évident mais nous avons en plus la règle de dérivation pour les fonctions composées suivante. Soit $\phi : \mathbb{R}^n \to \mathbb{R}$ une fonction lipschitzienne et soit $F = (F_1, \dots, F_n)$ un vecteur aléatoire dans $\mathbb{D}^{1,q}$ pour $q > 1$. Alors si F est absolument continue par rapport à la mesure de Lebesgue alors $\phi(F) \in \mathbb{D}^{1,q}$ et

$$D\phi(F) = \sum_{k=1}^{n} \frac{\partial \phi}{\partial x_i}(F) DF_i. \tag{1.65}$$

Quelques remarques s'imposent.

Remarque 14 1. *On peut montrer que la dérivée de Malliavin ne dépend pas de l'écriture de $F \in \mathcal{S}$.*

2. *On pourrait définir la notion de dérivée de Malliavin pour les courbes sur l'espace tangent de l'espace de Wiener. Cette notion géométrique est élégante mais nous oblige à parler de continuité alors qu'en toute généralité l'espace $(\Omega, \mathcal{F}, P)$ n'est pas topologique. On peut même montrer que certaines solutions d'EDS ne sont pas continues, ce qu'on nomme "l'Itô map".*[4]

3. *Pour étendre la notion pour les fonctions $L^p(\Omega)$ on montre que l'opérateur est fermable. Cette notion est nécessaire ici car même si $\mathcal{S}$ est dense dans $L^2(\Omega)$, il n'y*

4. À ce sujet on peut d'ailleurs consulter la thèse de Hélène Halconruy

aucune garantie qu'étant donné deux suites F_n et G_n qui convergent vers F dans $L^p(\Omega)$ il en est de même pour les suites DF_n et DG_n. Le caractère fermable donne l'unicité d'une certaine façon.

$D^p : \mathbb{D}^{p,2} \to L^2(\Omega, \mathcal{H}^p)$ est donc un opérateur à valeur dans un espace de Hilbert. On peut alors définir l'adjoint de D.

Définition 5 *Soit $p \geq 1$ un entier.*

1.

$$\text{Dom } \delta^p = \{u \in L^2(\Omega, \mathcal{H}^{\otimes p}); \quad |\mathbb{E}(\langle D^p F, u\rangle_{\mathcal{H}^{\otimes p})})| \leq c\sqrt{\mathbb{E}(F^2)} \text{ pour tout } F \in \mathbb{D}^{p,2}\}$$

2. *Soit $u \in$ Dom δ^p alors $\delta^p(u)$ est l'unique élément de $L^2(\Omega)$ défini par la formule de dualité*

$$\mathbb{E}(F\delta^p(u)) = \mathbb{E}(\langle D^p F, u\rangle_{\mathcal{H}^{\otimes p})}). \tag{1.66}$$

L'opérateur ainsi défini s'appelle l'opérateur de **divergence**. L'existence est une conséquence évidente du théorème de représentation de Riesz.

Remarque 15 *La formule (1.66) peut-être vue comme une **intégration par parties**. Le lemme de Stein qui jouera un rôle fondamental dans la méthode de Stein-Malliavin est d'ailleurs contenu. Soit $F = f(W(h))$ pour f suffisamment régulière alors on a $\delta(h) = W(h) \sim \mathcal{N}(0, \|h\|^2_{\mathcal{H}})$ et (1.66) se lit*

$$\mathbb{E}(f(W(h))W(h)) = \mathbb{E}(f'(W(h)))\|h\|^2_{\mathcal{H}}.$$

En fait l'opérateur de divergence est extrêmement puissant, il nous permet de définir une notion d'intégrale pour des processus non adaptés, ceci permet par exemple de définir des solutions aux SPDE dont la formulation mild (c'est la seule définition de solution que nous considérons ici) n'est pas possible. Nous renvoyons le lecteur à l'introduction [2] de Raluca Balan déjà citée ainsi qu'à l'excellent article [9] dont nous reparlons au résumé du chapitre 2.

On a une définition équivalente de l'intégrale multiple comme suit.

Définition 6 *Soit $p \geq 1$ et $f \in \mathcal{H}^{\otimes p}$. L'intégrale multiple de f est définie par $I_p(f) = \delta^p(f)$.*

Voir l'exercice 2.7.6 page 37 de [81] pour l'équivalence entre les deux visions, ainsi que le théorème 2.7.7 page 39.

La formule technique suivante joue un rôle fondamental dans la méthode de Stein-Malliavin ainsi que dans les calculs car elle permet de "linéariser".

Théorème 5 (Formule produit) *Soient* $p, q \geq 1$ *et* $f \in \mathcal{H}^{\odot p}$ *et* $g \in \mathcal{H}^{\odot p}$ *alors*

$$I_p(f)I_q(g) = \sum_{k=0}^{\min(p,q)} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \tilde{\otimes}_r g) \tag{1.67}$$

où $f\tilde{\otimes}_r g$ *désigne la contraction d'ordre* r.

Nous aurons aussi besoin à plusieurs reprises de l'hypercontractivité qui montre que dans un chaos toutes les normes $L^p(\Omega)$ sont équivalentes.

Théorème 6 (Hypercontractivité) *Soient* $p, q \geq 1$. *Alors si* $Y = I_p(f)$ *alors il existe une constante* $0 < k(p,q) < \infty$ *telle que*

$$\mathbb{E}(|Y|^q)^{\frac{1}{q}} \leq k(p,q)\mathbb{E}(Y^2)^{\frac{1}{2}}. \tag{1.68}$$

Ces deux derniers théorèmes nous seront très utiles pour nos estimations statistiques car nous utiliserons une approche variationnelle par le Calcul de Malliavin.

1.1.4.2 Méthode de Stein-Malliavin

Le but de cette section est de présenter le théorème du quatrième moment qui nous permet de contrôler la convergence en loi vers une loi gaussienne à travers des bornes effectives issues du Calcul de Malliavin et de donner un critère de convergence vers une telle loi.

Nous n'allons pas rentrer dans les détails des définitions, théorèmes qui suivent, mais simplement dans les idées qui permettent d'arriver au théorème 9 à travers des heuristiques techniques et conceptuelles. L'exposé rigoureux de tout cela se trouve parfaitement bien écrit dans le livre de Ivan Nourdin et Giovanni Peccati i.e. [10]. On se donne $F \in L^2(\Omega)$. On suppose que $\mathbb{E}(F) = 0$ pour rendre la lecture plus facile.

- On part de la décomposition en chaos pour $F \in L^2(\Omega)$: $F = \sum_{p=1}^{\infty} I_p(f_p)$.
- On définit dès lors l'opérateur L : $LF = -\sum_{p=1}^{\infty} I_p(f_p)$.
- Prenons le cas où F est une intégrale multipli i.e. $F = I_p(f)$ alors on a $DI_p(f) = pI_{p-1}(f)$ donc $\delta(DI_p(f)) = pI_p(f) = -LI_p$. Ainsi on a $\delta(DF) = -LF$. Par approximation le résultat est vrai pour tout $F \in L^2(\Omega)$.
- On définit ensuite l'opérateur $L^{-1}F = -\sum_{p=1}^{\infty} \frac{1}{p} I_p(f_p)$ et on obtient dès lors $LL^{-1}F = F$.

Ceci nous donne le théorème suivant.

Théorème 7 *Pour tout* $F, G \in \mathbb{D}^{1,2}, g \in C^1$ *dont la dérivée est bornée*

$$\mathbb{E}(Fg(G)) = \mathbb{E}\left(g'(G)\langle DG, -DL^{-1}F\rangle_{\mathcal{H}}\right). \tag{1.69}$$

Proof: On a d'après les quatre points précédents et la formule de dualité (1.66)

$$\begin{aligned}\mathbb{E}(Fg(G)) &= \mathbb{E}(LL^{-1}F \times g(G)) \\ &= \mathbb{E}(\delta(-DL^{-1}F)g(G)) \\ &= \mathbb{E}(\langle Dg(G), -DL^{-1}F\rangle_{\mathcal{H}}) \\ &= \mathbb{E}(g'(G)\langle D(G), -DL^{-1}F\rangle_{\mathcal{H}}).\end{aligned}$$

Partons maintenant de l'idée suivante : Soient F_∞ une variablé aléatoire et $\mathcal{L}_\infty$ un opérateur qui agit sur une classe de fonction $\mathcal{F}$ qui caractérise la loi i.e.

$$\forall f \in \mathcal{F} \quad \mathbb{E}\mathcal{L}_\infty f(F) \approx 0 \to F \sim F_\infty.$$

Il s'avère que pour la loi gaussienne cette caractérisation est possible et donnée par le *lemme de Stein* qui nous dit de prendre

$$\begin{aligned}\mathcal{F} &= \{f : \text{absolument continue et } f' \in L^1(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}})\} \\ \mathcal{L}_\infty &= f'(x) - xf(x).\end{aligned}$$

Avant d'aller plus loin, énonçons clairement ce lemme.

Lemme 1 *1. Soit W une variable aléatoire. Alors $W \overset{loi}{=} Z \sim \mathcal{N}(0,1)$ si et seulement si $E(f'(W) - Wf(W)) = 0$ pour tout $f \in \mathcal{F}$.*

2. Soit d l'une des distances suivantes d_{Kol}, d_{TV}, d_W respectivement la distance de Kolmogorov, la distance en variation totale, de Wasserstein (voir l'appendice C.2 dans [10] pour les définitions ainsi que les espaces associés $\mathcal{F}_d$) alors on a

$$d(Y,Z) \leq \sup_{f\in\mathcal{F}_d} |E\left(f'(Y) - Yf(Y)\right)|. \tag{1.70}$$

En utilisant ce lemme et l'intégration par parties (1.69) on peut énoncer le théorème suivant,

Théorème 8 *Soient $Z \sim \mathcal{N}(0,1)$ et $F \in \mathbb{D}^{1,2}$ tels que $\mathbb{E}(F) = 0, \mathbb{E}(F^2) = 1$. Alors on a*

$$d_W(F,Z) \leq \left(\mathbb{E}\left(1 - \langle DF, -DL^{-1}F\rangle_{\mathcal{H}}\right)^2\right)^{1/2} \tag{1.71}$$

et si F est absolument continue par rapport à la mesure de Lebesgue on a

$$\begin{aligned}d_{Kol}(F,Z) &\leq \left(\mathbb{E}\left(1 - \langle DF, -DL^{-1}F\rangle_{\mathcal{H}}\right)^2\right)^{1/2} &(1.72)\\ d_{TV}(F,Z) &\leq 2\left(\mathbb{E}\left(1 - \langle DF, -DL^{-1}F\rangle_{\mathcal{H}}\right)^2\right)^{1/2}. &(1.73)\end{aligned}$$

Proof: On prend $g = f$ et $G = F$ dans l'intégration par parties (1.69) pour $g = f$ et $G = F$ et on applique l'inégalité de Cauchy-Schwarz.

- ▷ On a donc un lien intrinsèque entre le contrôle d'une loi vers une loi gaussienne $\mathcal{N}(0,1)$ et la dérivée de Malliavin pour $F \in \mathbb{D}^{1,2}$.
- ▷ En fait les bornes dans le théorème 8 sont calculables explicitement quand $F = I_q(f)$. En effet dans ce cas on a $\langle DF, -DL^{-1}F\rangle_{\mathcal{H}} = \frac{1}{q}\|DF\|^2_{\mathcal{H}}$ car $L^{-1}(F) = -\frac{1}{q}F$. Il suffit alors de calculer cette dernière quantité qui se fait à l'aide de la formule produit (5). Voir le lemme 5.2.4 page 95 dans [10].
- ▷ Nous verrons à la partie 2 (cf la proposition 8) qu'on peut considérer à la place de $\langle DF, -DL^{-1}F\rangle_{\mathcal{H}}$ la quantité $\langle DF, v\rangle_{\mathcal{H}}$ quand $F = \delta(v)$.

Nous pouvons donc énoncer le théorème fondamental suivant

Théorème 9 (Théorème du quatrième moment) *Soit d l'une des distances de Wasserstein, de Kolmogorov, variation totale et Forter-Mourier. Si $F_n = I_p(f_n)$ est une suite de chaos d'ordre $p \geq 2$ associée à un processus isonormal $\mathcal{H}$ telle que $\mathbb{E}(F_n^2) \to_{n\to\infty} \sigma^2 > 0$ alors*

$$d(F_n, Z) \leq c \times \left(\sqrt{Var(\|DF_n\|^2_{\mathcal{H}})} + |\mathbb{E}(F_n^2) - \sigma^2|\right). \tag{1.74}$$

Ainsi, $F_n \overset{n\to\infty}{\to} \mathcal{N}(0,1)$ si et seulement $\|DF_n\|^2_{\mathcal{H}}$ converge en $L^2(\Omega)$ vers $p\sigma^2$.

Pour nous ce qui nous intéresse c'est la dérivée de Malliavin, cependant au montre que la condition de convergence est équivalente à $\mathbb{E}(F_n^4) \overset{n\to\infty}{\to} 3$.

Le théorème suivant montre que les chaos d'ordre $q \geq 2$ ne sont plus gaussiens, nous le plaçons ici car c'est bien une conséquence des calculs du théorème précédent qui nous donne le résultat.

Théorème 10 (Caractère non gaussien des chaos) *Soit $q \geq 2$ et $f \in \mathcal{H}^{\odot q}$ telle que $\mathbb{E}(I_q(f)^2) = \sigma^2 > 0$. Alors $\mathbb{E}(I_q(f)^4) > 3\sigma^4$. En particulier la loi de $I_q(f)$, $q \geq 2$ n'est pas gaussienne.*

1.1.4.3 Unité des équations différentielles dans le Calcul de Malliavin

Reprenons maintenant nos considérations sur l'équation (1.11). Nous avons vu en 1.1.3 qu'à partir du bruit fractionnaire en temps et blanc en espace on pouvait définir un cadre hilbertien et un processus isonormal, concept fondamental pour utiliser le Calcul de Malliavin ainsi que la méthode de Stein-Malliavin afin d'obtenir nos résultats comme des théorèmes centraux limites et pouvoir faire de l'estimation statistique.

Soit $\mathcal{E}$ l'espace préhilbertien constitué des fonctions $\phi \in \mathcal{S}(\mathbb{R}^d)$ muni du produit scalaire

$$\langle \phi, \psi\rangle_{\mathcal{E}} = \int_{\mathbb{R}^d} \Gamma(dx)(\phi \star \tilde{\psi})(x) = \int_{\mathbb{R}^d} \mu(d\xi)\mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi}(\xi).$$

Soit $\mathcal{H}$ le complété de $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}})$. Soit $\mathcal{H}_T = L^2([0,T],\mathcal{H})$ notre espace de Hilbert. Il est clair que $\mathcal{H}$ et $\mathcal{H}_T$ peuvent contenir des distributions. On définit alors notre processus isonormal par

$$W(h) := \int_0^T \int_{\mathbb{R}^d} h(s,x)W(ds,dx) \quad \text{pour} \quad h \in \mathcal{H}_T. \tag{1.75}$$

On a donc l'isométrie

$$\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{R}^d} h(s,x)W(ds,dx)\right)^2\right] = \|h\|^2_{\mathcal{H}_T}.$$

Exemple 1 *Supposons que Λ est absolument continue par rapport à la mesure de Lebesgue i.e. $\Lambda(dx) = f(x)dx$. On a donc la covariance suivante*

$$\mathbb{E}(W(\phi)W(\Psi)) = \int_0^\infty dt \int_{\mathbb{R}^{2d}} dxdyf(x-y)\phi(t,x)\psi(t,x). \tag{1.76}$$

Un choix très important qui nous servira à la partie 2 est le cas où f est donné par le noyau de Riesz. Autrement dit $\forall x \in \mathbb{R}^d,\ f(x) = \frac{1}{|x|^\beta}$ où $\beta \in (0, \min(d,2))$.

Exemple 2 *Le deuxième exemple très important est le bruit blanc en temps et en espace donné par $\Lambda(dx) = \delta_0 dx$ avec pour covariance $\mathbb{E}(W(\phi)W(\Psi)) = \int_0^\infty dt \int_{\mathbb{R}^{2d}} dx\phi(t,x)\psi(t,x)$. Dans le cas où $\mathcal{H}_T$ est un espace de fonctions, l'intégrale de Walsh ainsi définie coïncident avec l'intégrale de type Wiener.*

Une fois qu'on a défini le cadre isonormal, on souhaite naturellement connaître la régularité de notre solution et la dériver.

Dans [79] les auteurs montrent le théorème suivante

Théorème 11 *On suppose que G vérifie (1), que b, σ sont des fonctions C^1 dont la dérivée est lipschitzienne. Alors pour tout $(t,x) \in [0,T] \times \mathbb{R}^d$, $u(t,x) \in \mathbb{D}^{1,p}$ pour tout $p \in [1,\infty)$. De plus la dérivée de Malliavin $Du(t,x)$ est un $\mathcal{H}_T$ processus qui vérifie l'équation différentielle stochastique suivante*

$$\begin{aligned} D_r u(t,x) &= \sigma(u(r,*))G(t-r,x-*) \\ &+ \int_r^t \int_{\mathbb{R}^d} G(t-s,x-y)\sigma'(u(s,y))D_r u(s,y)W(ds,dy) \\ &+ \int_r^t \int_{\mathbb{R}^d} b'(u(s,x-y))D_r u(s,x-y)G(t-s,dy)ds. \end{aligned}$$

Lien entre l'intégrale de Walsh et l'opérateur de divergence.

Par le fait qu'on ait $D(FG) = D(F)G + D(G)F$ pour tout $F, G \in \mathbb{D}^{1,2}$ on déduit que

$$\mathbb{E}[\langle D(FG), h\rangle_{\mathcal{H}_T}] = -\mathbb{E}[\langle FD(G), h\rangle_{\mathcal{H}_T}] + [X(h)FG]. \tag{1.77}$$

- Donc pour tout $h \in \mathcal{H}_T$ on a par unicité de l'opérateur divergence, la formule d'intégration par parties (1.66) et en prenant $G = 1$ dans (1.77) $\delta(h) = W(h)$. Ainsi

$$\delta(h) = \int_0^T \int_{\mathbb{R}^d} h(s,x)W(ds,dx).$$

- Maintenant considérons un processus prévisible [5] $(X(s,y), s \geq 0, y \in \mathbb{R}^d)$ tel que

$$\mathbb{E}\left[\|X\|_{\mathfrak{H}}^2\right] < \infty,$$

on sait définir l'intégrale de Walsh de la forme

$$\int_0^\infty \int_{\mathbb{R}^d} X(s,y)W(ds,dy)$$

et qui vérifie donc l'isométrie

$$\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{R}^d} X(s,y)W(ds,dy)\right)^2\right] = \mathbb{E}\left[\|X\|_{\mathfrak{H}}^2\right].$$

Dans ces conditions on a $X \in \text{Dom}\ \delta$ et $\delta(X)$ coincide avec l'intégrale de Walsh i.e. on a

$$\delta(X) = \int_0^\infty \int_{\mathbb{R}^d} X(s,y)W(ds,dy).$$

La preuve utilise le fait suivant

$$\delta(Fu) = F\delta u - \langle DF, u\rangle_{\mathcal{H}_T}$$

dès que $Fu \in \text{Dom}\ \delta$ qui est une conséquence de (1.77). En effet prenons ϕ un processus simple qui s'écrit donc $\phi(t,x,\omega) = G(\omega)\mathbf{1}_{(a,b]}(t))\mathbf{1}_A(x)$ où A est un borélien de $\mathbb{R}$ et $G \in L^0(\Omega)$, bornée et $\mathcal{F}_a-$mesurable. On suppose de plus que $G \in \mathbb{D}^{1,2}$ alors on a $f \in \text{Dom}\ \delta$ et

$$\delta(\phi) = GX(\mathbf{1}_{(a,b]}(\cdot))\mathbf{1}_A(*))$$

car $DG = 0$ p.s. (voir la proposition 1.2.8 page 34 de [77]). On finit par approximation.

1.2 Partie 1 : Inférence statistique des équations différentielles stochastiques : approche variationnelle

Dans cette partie tous les espaces de probabilités sont filtrés de la façon formelle suivante. Soit $W = \{W_t(A),\ A \in \mathcal{B}_d(\mathbb{R}),\ t \geq 0\}$ notre bruit stochastique définit sur un espace de probabilité $(\Omega, \mathcal{F}, P)$ alors on définit la filtration $(\mathcal{F}_t^0, t \geq 0)$ par

$$\mathcal{F}_t^0 = \sigma\left(W_s(A) : 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R})\right) \vee \mathcal{N}, \quad t \geq 0$$

5. Rappel : On suppose que $(X(s,y), s \geq 0, y \in \mathbb{R}^d)$ est un processus bimesurable adapté à la filtration engendrée par les $W(h)$.

où $\mathcal{N}$ est la tribu engendrée pa les ensembles de mesures nulles. On pose $\mathcal{F}_t := \wedge_{s>t}\mathcal{F}_t^0$ pour $t \geq 0$.

Dans toute la suite on se donnera toujours un tel espace $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, P)$.

Dans cette partie nous allons nous intéresser à l'estimation statistique par une approche variationnelle.
Avant de discuter de la stratégie nous allons poser le cadre : On considère ici trois équations différentielles l'équation des ondes stochastique avec bruit fractionnaire en temps et blanc en espace, le processus de Hermite-Ornstein-Uhlenbeck, l'équation de Burger stochastique avec un bruit blanc.
Une fois le cadre posé, nous discuterons de la stratégie statistique utilisée et ensuite nous résumerons les résultats obtenus pour chaque cas.

1.2.1 Le cadre théorique

L'équation des ondes fractionnaire

Elle a déjà été discutée en 1.1.3 et s'écrit

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + W^H(t,x), & t \in (0,T],\ T>0,\ x \in \mathbb{R}^d \\ u(0,x) = 0, & x \in \mathbb{R}^d \\ \frac{\partial u}{\partial t}(0,x) = 0, & x \in \mathbb{R}^d \end{cases} \tag{1.78}$$

avec $W^H = \{W_t^H(A);\ t \in [0,T],\ A \in \mathcal{B}_b(\mathbb{R}^d)\}$ notre processus gaussien centré sur l'espace de probabilité complet $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, P)$. de covariance fractionnaire en temps et blanc en espace i.e. on a

$$\mathbb{E}(W_t^H(A)W_s^H(B)) = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big)\lambda(A \cap B), \tag{1.79}$$

pour tout $t, s \geq 0, A, B \in \mathcal{B}_b(\mathbb{R}^d)$.

Nous savons que la solution de (1.78) s'écrit pour

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) W^H(\mathrm{d}s, \mathrm{d}y), \quad t \geq 0, x \in \mathbb{R}, \tag{1.80}$$

où $G(t,\cdot)$ est notre solution fondamentale et que l'intégrale est l'intégrale de Wiener dont la construction a été explicitée à la même sous-section. On sait que l'intégrale 1.80 est bien définie si et seulement si $g_{t,x} = (s,y) \mapsto G(t-s, x-y) \in \mathcal{H}_W$.

On sait alors qu'à partir de notre espace de Hilbert $\mathcal{H}_W$ on peut lui associer un processus isonormal et ainsi notre intégrale multiples $I_q,\ q \geq 1$. On a donc

$$u(t,x) = I_1(g_{t,x}),$$

en particulier notre processus est gaussien.

Dans [66] les auteurs montrent les propriétés suivantes.

Théorème 12 *1. La solution 1.80 de l'équation des ondes existe si et seulement si $d < 2H+1$ et dans ce cas*

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\mathbb{E}(|u(t,x)|^p) < \infty.$$

Ainsi comme $H \in (\frac{1}{2}, 1)$ alors $d = 1$ ou $d = 2$.

2. $x \mapsto u(t,x)$ est höldérien d'ordre $\delta \in (0, H - \frac{d-1}{2})$

3. On a pour $d = 1$ la fonction de covariance

$$\begin{aligned}\mathbb{E}\left(u(t,x)u(t,y)\right) &= \frac{1}{2}\left(c_H|y-x|^{2H+1} - t\frac{|y-x|^{2H}}{2} + \frac{t^{2H+1}}{2H+1}\right)1_{\{|y-x|<t\}} \\ &\quad + \frac{(2t-|y-x|)^{2H+1}}{8(2H+1)}1_{\{t\le|y-x|<2t\}}.\end{aligned} \tag{1.81}$$

On en déduit :

- *$(u(t,x))_{t\geq 0}$ est un processus auto-similaire d'ordre $H + \frac{1}{2}$.*
- *$(u(t,x))_{x\in\mathbb{R}}$ est un processus stationnaire.*
- *Comme le processus $(u(t,x))_{x\in\mathbb{R}}$ est gaussien on a*

$$u(t,x) \sim \frac{t^{H+\frac{1}{2}}}{\sqrt{2(2H+1)}}Z \quad \text{où} \quad Z \sim \mathcal{N}(0,1).$$

Équation de Burger avec bruit blanc

Celle-ci s'écrit comme suit.

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) &= \theta\frac{\partial^2}{\partial x^2}u(t,x) - \frac{1}{2}\frac{\partial}{\partial x}u(t,x)^2 + \dot{W}(t,x), \quad t\in[0,T],\ T>0,\ x\in\mathbb{R} \\ u(0,x) &= u_0, \quad x\in\mathbb{R}\end{cases} \tag{1.82}$$

où cette fois notre bruit est un bruit blanc i.e. $(W(A), A \in \mathcal{B}_b([0,T]\times\mathbb{R}))$ est un champs gaussien de covariance

$$\mathbb{E}W(A)W(B) = \lambda(A\cap B) \quad \text{pour tous } A, B \in \mathcal{B}_b([0,T]\times\mathbb{R}), \tag{1.83}$$

et $\theta > 0$ est notre coefficient de diffusion à estimer.

Pour la solution de 1.82 elle s'écrit

$$u_1(t,x) := u(t,x) = \int_{\mathbb{R}} G_t(x-y)u_0(y)dy + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G_{t-s}(x-y)u(s,y)^2 dyds + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)W(ds,dy) \tag{1.84}$$

où G_t est notre noyau gaussien

$$G_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \qquad (t,x) \in [0,T] \times \mathbb{R}. \tag{1.85}$$

L'intégrale stochastique peut-être vue comme une intégrale de Wiener ou comme l'intégrale de Walsh, les deux notions coïncident ici comme nous l'avons montré dans 1.1.4.3.

- Dans l'article [71] les auteurs montrent que l'équation 1.82 admet une unique solution dès que la condition initiale u_0 est continue et bornée $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Et aussi que pour tout $p \geq 2$ il existe une constante $C = C(T,p) > 0$ telle que (voir e.g. [71], Proposition 5.2)

$$\sup_{t\in[0,T], x\in\mathbb{R}} \mathbf{E}\,|u(t,x)|^p \leq C. \tag{1.86}$$

  Dans note travail on supposera $u_0 = 0$ par simplicité mais nos résultats restent vrais pour u_0 suffisamment régulier.

Nous devons remarquer, car cela sera le point de départ sur notre travail sur l'équation de Burger, que la solution 1.84 se décompose en deux processus

$$u(t,x) = X(t,x) + Y(t,x) \tag{1.87}$$

où

$$X(t,x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)W(ds,dy) \tag{1.88}$$

et

$$Y(t,x) = \frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G_{t-s}(x-y)u(s,y)^2 dyds. \tag{1.89}$$

Le champ $X = (X(t,x),\ t \in [0,T], x \in \mathbb{R})$ est en fait la solution de l'équation de la chaleur avec bruit blanc et $X(0,x) = 0$ pour tout $x \in \mathbb{R}$ i.e.

$$\frac{\partial}{\partial t} X(t,x) = \Delta X(t,x) + W(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}. \tag{1.90}$$

Rappelons les propriétés de X dont nous aurons besoin dans la suite

Théorème 13 *1. (Continuité hölderienne) : pour tout $s,t \in [0,T]$, $x,y \in \mathbb{R}$, et $p \geq 1$ il existe $C = C(p,T) > 0$ telle que*

$$\mathbb{E}\,|X(t,x) - X(s,y)|^p \leq C\left(|t-s|^{\frac{p}{4}} + |x-y|^{\frac{p}{2}}\right). \tag{1.91}$$

En particulier X est Hölder continue d'ordre $\delta \in (0, \frac{1}{4})$ en temps et d'ordre $\delta \in (0, \frac{1}{2})$ en espace.

2. *(Bornitude dans L^p) : Pour tout $p \geq 1$ il existe $C = C(p,T) > 0$ telle que*

$$\sup_{t\in[0,T],x\in\mathbb{R}} \mathbb{E}|X(t,x)|^p \leq C < \infty.$$

3. *(Variation quadratique spatiale) : Soient $t \in (0,T]$, $A_1 < A_2$ et*

$$x_i = A_1 + \frac{i}{N}(A_2 - A_1), \quad i = 0, ..., N \tag{1.92}$$

une partition de $[A_1, A_2]$. Alors

$$\sum_{i=0}^{N-1} (X(t,x_{i+1}) - X(t,x_i))^2 \to_{N\to\infty} \frac{1}{2}(A_2 - A_1) \textit{ dans } L^2(\Omega). \tag{1.93}$$

4. *(Variation quadratique temporelle) : Soit $x \in \mathbb{R}$, $A_1 < A_2$ et*

$$t_i = A_1 + \frac{i}{N}(A_2 - A_1), \quad i = 0, ..., N \tag{1.94}$$

une partition de $[A_1, A_2]$. Alors

$$\sum_{i=0}^{N-1} (X(t_{i+1},x) - X(t_i,x))^4 \to_{N\to\infty} \frac{3}{\pi}(A_2 - A_1) \textit{ dans } L^2(\Omega). \tag{1.95}$$

Les prorpriétés (1.93) and (1.95) sont prouvées dans [88], Proposition 3.1 et Proposition 3.2 respectivement. La continuité höldérienne et la bornitude dans L^p se trouvent, par exemple, dans [100].

Processus de Hermite-Ornstein-Uhlenbeck Nous avons déjà introduit le processus de Hermite-Ornstein-Uhlenbeck à la section 2.5. Nous nous permettons alors d'aller plus vite dans la présentation.

Soit $Z_t^{(q,H)}$ notre processus de Hermite.

$$dX_t = -X_t dt + \sigma dZ_t^{(q,H)}, \qquad t \in [0,T] \tag{1.96}$$

avec comme condition initiale $X_0 = \xi \in L^2(\Omega)$ et $\sigma > 0$. De même on a vu que la solution s'écrivait dans le cas non stationnaire

$$X_t = e^{-t}\left(\xi + \sigma \int_0^t e^u dZ_u^{(q,H)}\right) \tag{1.97}$$

où l'intégrale est au sens de Wiener ou de Stieljes, dont on a vu qu'elles coïncident dans ce cas. On supposera que ξ est déterministe dans notre situation et donc sans perte de généralité $\xi = 0$.

Dans le cas stationnaire i.e. $\xi = \sigma \int_{-\infty}^0 e^u dZ_u^{(q,H)}$, la solution s'écrit pour tout $t \in [0,T]$

$$X_t = \sigma \int_{-\infty}^t e^{-(t-s)} dZ_s^{(q,H)}. \tag{1.98}$$

1.2.2 Le cadre opérationnel

Notre but : L'estimation ! Nous chercherons dans (1.80) à estimer H, dans 1.84 à estimer θ et dans 1.97 à estimer H et σ. Bien-sûr au delà de l'estimation ce qui nous intéresse également ce sont ses propriétés de stabilité et d'asymptoticité.

Outil de travail Le point central de cette partie sera l'étude des variations de nos solutions, c'est à dire l'étude de

$$V_N(X) = \sum_{i=0}^{N-1} (X(t_{i+1}, x) - X(t_i, x))^2 \quad \text{ou} \quad \tilde{V}_N(X) = \frac{1}{N} \sum_{i=0}^{N-1} \left[\frac{(X(t_{i+1}, x) - X(t_i, x))^2}{\mathbb{E}(X(t_{i+1}, x) - X(t_i, x))^2} - 1 \right] \tag{1.99}$$

où $0 = t_0 < t_1 < \cdots < t_n = T$ est une partition de l'interval $[0, T]$.

Comme nous l'avons dit plus haut on peut choisir aussi une partition spatiale $(x_i; i = 0, \ldots, N-1)$.

On s'intéresse au comportement asymptotique de V_N quand $N \to \infty$. $X = (X_t)_{t \in T}$ sera notre décomposition en ondelette de la solution des ondes 1.80, notre solution de l'équation de Burger 1.84 et pour finir notre processus HOU 1.97.

Stratégie Pour pouvoir estimer nos paramètres nous devons faire des choix d'observations. En effet, nos équations différentielles aux dérivées partielles décrivent des conditions spatio-temporelles donc nous pouvons faire le choix d'observer nos solutions dans l'espace ou dans le temps. On supposera donc qu'on observe nos solutions soit dans des moments spatiaux $x_i : i = 1, \ldots, N$ en fixant le paramètre temps ou en fixant la variable espace et en des moments temporels $t_i : i = 1, \ldots, N$.

Une fois ce choix fait, il reste à choisir notre outil pour parvenir à estimer nos paramètres et à étudier ses propriétés. Notre approche est variationnelle car celle-ci s'étudie naturellement dans le Calcul de Malliavin.

Le point de vue qu'on adoptera pour étudier ses quantités est le Calcul de Malliavin et plus particulièrement la décomposition en chaos de nos intégrales stochastiques. En effet on a vu (section 1.1.4.3) que pour toute solution on peut définir une processus isonormal $W = (W(\phi),\ \phi \in \mathcal{H}))$ et définir l'intégrale stochastique multiple I_q d'ordre q associé.

Nous résumerons le tavail sur le cas des ondes et le procesus de Hermite-Ornstein-Uhlenbeck d'abord car on utilisera les variations centrées $\tilde{V}_N$ et en second temps le cas de l'équation de Burger car nous utiliserons V_N.

Exemple pour l'estimation de H Afin de fixer les idées précédentes et qu'elles soient claires dans l'esprit des résultats que nous obtiendrons, prenons le cas où $X = (X_t^{q,H})_{t \in [0,1]}$ le processus de Hermite d'ordre $q \geq 1$.

Soit

$$S_N = \frac{1}{N}\sum_{1}^{N}\left(X\left(\frac{i}{N}\right) - X\left(\frac{i-1}{N}\right)\right)^2.$$

On a $\mathbb{E}(S_N) = N^{-2H}$. On va estimer H en remplaçant S_N par $E(S_N)$ et donc on définit une suite

$$\hat{H}_N = -\frac{\log(S_N)}{2\log(N)}.$$

La question est de savoir si $\hat{H}_N$ est un bon estimateur (consistance faible, forte, convergence en loi). Pour ce faire on remarque que

$$1 + V_N = S_N N^{2H}$$

et donc heuristiquement on aimerait avoir

$$V_N \overset{N\to\infty}{\sim} \log(1+V_N) = -2(\hat{H}_N - H)\log(N)$$

si V_N tend vers 0 (ce qui sera le cas par régularité). On voit donc que le comportement asymptotique de V_N est intiment relié à celui de $\hat{H}_N$ vers H quand $N \to \infty$. Prenons le cas où $q = 1$ pour aller plus loin c-a-d le cas du mouvement brownien fractionnaire.

Exemple avec le cas du mouvement brownien fractionnnaire

Soit $B^H = (B_t^H)_{t\in[0,1]}$ le mouvement brownien fractionnaire sur $[0,1]$ et considérons donc $\tilde{V}_N$ avec $t_i = \frac{1}{N}$, $i = 1,\ldots,N$ i.e.

$$\begin{aligned}\tilde{V}_N(B^H) &= \frac{1}{N}\sum_{i=0}^{N-1}\left[\frac{(B^H(t_{i+1}) - B^H(t_i))^2}{\mathbb{E}(B^H(t_{i+1}) - B^H(t_i))^2} - 1\right] \\ &= \frac{1}{N}\sum_{i=0}^{N-1}\left[N^{2H}(B^H_{t_{i+1}} - B^H_{t_i})^2 - 1\right].\end{aligned}$$

On a de plus $B^H_{t_{i+1}} - B^H_{t_i} = I_1(A_i)$ où $A_i, N := A_i = \mathbf{1}_{(i/N,(i+1)/N)}$ pour tout $i = 0,\ldots,N-1$. Ainsi par la formule produit (5) on linéarise $\tilde{V}_N$ et on a

$$\tilde{V}_N(B^H) = N^{2H-1} I_2\left(\sum_{i=0}^{N-1} A_i \otimes_1 A_i\right).$$

Par isométrie on a

$$\mathbb{E}|V_N(B^H)|^2 = 2N^{4H-4}\sum_{i=0}^{N-1}\sum_{i=0}^{N-1}|\langle A_i, A_i\rangle_{\mathcal{H}}|^2$$

où $\mathcal{H}$ est l'espace de Hilbert canonique associé au mouvement fractionnaire, ainsi on a

$$\begin{aligned}\langle A_i, A_i\rangle_{\mathcal{H}} &= \mathbb{E}\left[\left(B_{(i+1)/N} - B_{i/N}\right)\right)\left(B_{(j+1)/N} - B_{j/N}\right))\right] \\ &= \frac{1}{2}\left(2\left|\frac{i-j}{N}\right|^{2H} - \left|\frac{i-j-1}{N}\right|^{2H} - \left|\frac{i-j+1}{N}\right|^{2H}\right).\end{aligned}$$

On voit précisément que la convergence L^2 de $\tilde{V}_N$ est liée à la fonction d'autocariance ρ du mouvement brownien fractionnaire : soit $k \in \mathbb{Z}$ on a

$$\rho_H(k) = \frac{1}{2}(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}).$$

On voit apparaitre des conditions de régularité sur H i.e. si $H \geq 3/4$ la série $\sum_k k^{4H-4}$ diverge.

Pour $H < 3/4$ on montre (voir [100]) que

$$\mathbb{E}(F_N^2) := \mathbb{E}(C_H\sqrt{N}\tilde{V}_N)^2 \overset{N\to\infty}{\to} 1.$$

On montre ensuite que $\|DF_k\|_{\mathcal{H}}^2 \to 2$ en $L^2(\Omega)$ quand $k \to \infty$ et ainsi d'après le théorème des quatre moments 9 on sait que F_N converge vers une loi gaussienne $\mathcal{N}(0,1)$ et

$$d(\mathcal{L}(F_N), \mathcal{N}(0,1)) \leq cN^{2H-\frac{3}{2}}.$$

Cette étape est la plus technique car elle fait intervenir des sommes de chaos et des contractions qu'il faut analyser en détails. Les difficultés sont similaires à celle du théorème de Breuer-Major (voir le théorème 7.2.4 dans [81]).

- Ces conditions techniques sont très délicates : pour $H > 3/4$ le mouvement brownien fractionnaire apparait comme "variationnellement" non régulier en L^2. La convergence L^2 est une condition intrinsèque pour espérer une convergence vers une loi normale d'après le théorème des quatre moments 9.

- Il s'avère que pour $H > 3/4$ que $\tilde{V}_N$ (correctement renormalisée) ne converge pas en loi vers une gausienne mais vers une variable aléatoire de Rosenblatt. Voir le théorème 5.5 dans [100] pour les détails.

Conclusion Ces considérations ne nous aurons pas amenés trop loin car les idées seront dans le même esprit :

1. Estimer nos paramètres par les variations.

2. Étudier les variations par le Calcul de Mallaivin et la méthode de Stein-Malliavin en sachant que des considérations de régularité sur H apparaitrons.

1.2.3 Résumé du chapitre 2

Avant de présenter nos résultats nous devons (encore) présenter quelques outils car dans ce travail nous avons analysé la solution par une décomposition en ondelettes.

Ondelettes Soit Ψ notre ondelette i.e. une fonction continue à support compact dans $[0,1]$ telle qu'il existe un entier $Q \geq 1$ such that

$$\int_{\mathbb{R}} t^p \Psi(t)dt = 0 \text{ for } p = 0, 1, .., Q-1 \text{ and } \int_{\mathbb{R}} t^Q \Psi(t)dt \neq 0. \tag{1.100}$$

Soient $a > 0$, $i = 1, .., N_a$ (avec $N_a = [N/a] - 1$) le coefficient d'ondelette

$$d(t,a,i) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \Psi\left(\frac{x}{a} - i\right) u(t,x)dx = \sqrt{a} \int_{\mathbb{R}} \Psi(x) u(t, a(x+i))dx \tag{1.101}$$

et

$$\tilde{d}(t,a,i) = \frac{d(t,a,i)}{(\mathbf{E}(d(t,a,i))^2)^{\frac{1}{2}}}.$$

On définit maintenant notre *variation d'ondelette* dans l'espace de la solution par

$$V_N(t,a) = \frac{1}{N_a} \sum_{i=1}^{N_a} \left(\tilde{d}(t,a,i)^2 - 1\right). \tag{1.102}$$

Stratégie

- La première chose à remarquer est que d'après la structure de la covariance de la solution (1.81) on remarque que le temps va jouer un rôle qui va dépendre de la structure des observations spatiales avec $|x-y|$: il nous faudra considérer deux situations le cas où t est fixé et le cas où t "bouge".

Ainsi nous étudierons les deux cas suivants.

$$a = a_N = N^\alpha \text{ avec } 0 < \alpha < 1 \text{ et } t = t_N = N^\beta \text{ avec } \beta \geq 1 \tag{1.103}$$

et

$$a = a_N = N^\alpha \text{ avec } 0 < \alpha < 1 \text{ et } t > 0 \text{ est fixé} \tag{1.104}$$

Ici nous donnons les résultats qui nous permettrons d'estimer H par la suite et d'étudier ses propriétés.

Le cas où t varie

On se place dans le cas 1.103.

Au lieu de présenter les résultats un par un de manière standard nous préférons présenter les idées stratégiques étape par étape jusqu'à l'estimation de H. Nous renvoyons au chapitre 2 pour une présentation académique où les résultats sont énoncés dès le début.

1. On part de la structure de la covariance de la solution et on calcule la covariance pour le coefficient d'ondelette. On a

$$\mathbb{E}d(t_N, a_N, i)d(t_N, a_N, j) = \frac{c_H}{2} a_N^{2H+2} g_{H+\frac{1}{2}}(i-j) - \frac{t_N}{4} a_N^{2H+1} g_H(i-j) \qquad (1.105)$$

   où

$$g_H(k) = \int_{\mathbb{R}} \int_{\mathbb{R}} dxdy \Psi(x)\Psi(y)|x-y+k|^{2H}. \qquad (1.106)$$

   On remarque que si $i = j$ la variance est asymptotiquement une puissance de H i.e. $\mathbb{E}d(t_N, a_N, i))^2 \sim N^{\beta+\alpha(2H+1)}$. On voit déjà apparaitre notre puissance de H.

2. Pour pouvoir estimer la convergence L^2 de V_N le point clé est le fait que notre ondelette régularise notre solution. En effet on montre que pour tout $H \in \left(\frac{1}{2}, \frac{3}{2}\right)$ et k assez grand

$$|g_H(k)| \leq C_{\Psi,H,Q} k^{4H-4Q}$$

   où $C_{\Psi,H,Q} > 0$ ne dépend pas de k. On obtient donc le même phénoème que pour le mouvement brownien fractionnaire : on doit considérer $Q \geq 2$ ou $Q = 1, H < \frac{3}{4}$ pour la convergence de la série $\sum_{k\in\mathbb{Z}} g_H^2(k)$

3. Ceci permet de montrer que pour $Q \geq 2$ ou $Q = 1, H < \frac{3}{4}$ on a

$$N^{1-\alpha}\mathbb{E}V_N(t_N, a_N)^2 \to_{N\to\infty} \frac{2}{K_{\Psi,H}^2} \sum_{k\in\mathbb{Z}} g_H(k)^2 := K_{0,\Psi,H} \qquad (1.107)$$

4. En ayant en tête le théorème des quatre moments 9 on normalise les variations i.e. soit $F_N = K_{0,\Psi,H}^{-1/2} N^{\frac{1-\alpha}{2}} V_N(t_N, a_N)$, on a donc $\mathbb{E}F_N^2 \to_{N\to\infty} 1$.

5. D'après le théorème des quatre moments on a que $(F_N)_{N\geq 1}$ converge vers une loi gaussienne standard si et seulement si $\|DF_N\|_{\mathcal{H}}^2$ converge vers 2 dans $L^2(\Omega)$ quand $N \to \infty$. Et dans ce cas

$$d(F_N, Z) \leq c \left(\sqrt{Var\left(\|DF_N\|_{\mathcal{H}}^2\right)} + \mathbf{E}\|DF_N\|_{\mathcal{H}}^2 - 2 \right)$$

6. Cette partie est la plus délicate comme déjà dit mais la preuve est au final standard comme pour le théorème de Breuer–Major déjà cité.

7. On obtient alors le théorème suivant pour $c > 0$ une constante générique.

   Théorème 14 *Soit* F_N *. Alors* $(F_N)_{N\geq 1}$ *converge en loi vers* $Z \sim N(0,1)$ *et*

$$d(F_N, Z) \leq cN^{\frac{\alpha-1}{2}}.$$

8. Pour estimer H à différentes échelles on utilise le théorème des quatre moments vectorielle (voir [85]).

   On montre que le vecteur aléatoire d-dimensionnel $\left(N^{\frac{1-\alpha}{2}} V_n(t_N, La_N)\right)_{L=1,..,d}$ converge en loi, quand $N \to \infty$, vers le vecteur gaussien centré de covariance $(\Gamma_{L_1,L_2})_{L_1,L_2=1,..,d}$.

Le cas où t est fixé

On se place dans le cas 1.104.

1. Si on cherche de même à calculer la covariance pour le coefficient d'ondelette il nous faut choisir une ondelette explicite. On fait le choix de prendre l'ondelette de Haar :

$$\Psi(x) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1, & \frac{1}{2} \leq t < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{1.108}$$

2. Dans ce cas on montre que $\mathbb{E}d^2(t, a_N, i) \to_{N\to\infty} \frac{1}{2(H+1)} t^{2H+2}$. Ici la différence est grande car t est fixé et donc une log-log régression ne nous permettra pas d'estimer H.

3. Pour un tel choix de Ψ on montre en fait que $\mathbb{E}d(t, a_N, i)d(t, a_N, j)$ sont très peu corrélées : pour $|i - j| \geq 2$ elles sont nulles.

4. Les calculs sont donc plus faciles et on obtient que

$$N^{1-\alpha}\mathbf{E}V_N(t, a_N)^2 \to_{N\to\infty} 2. \tag{1.109}$$

5. Soit $G_N =: \frac{1}{\sqrt{2}} N^{\frac{1-\alpha}{2}} V(t, a_N)$ alors on montre que

$$d(G_N, Z) \leq C\left(\frac{1}{N^{\frac{1-\alpha}{2}}} + \frac{1}{N^{2\alpha}}\right),$$

   ainsi que la convergence en loi vers Z.

Estimation de H

Quand $t = t_N$. Nous avons vu que pour le cas où $t = t_N$ nous avions que $\mathbb{E}d(t_N, a_N, i))^2 \sim N^{\beta+\alpha(2H+1)}$ et ainsi la méthode d'estimation pour H est la même que celle décrit pour le mouvement brownien fractionnaire.

Remarque 16 *On suppose qu'on observe notre solution $u(t_N, x_i)$ pour $i = 1, \ldots, N$ pour un certain temps t_N. Bien que notre solution soit observée, rien nous dit qu'on puisse observer $d(t_N, a_N, i)$. Dans le chapitre 2 on discrétise nos intégrales et on montre qu'on peut changer nos intégrales par nos discrétisations dans les théorèmes précédents.*

Notons $\widehat{H}_N$ un tel estimateur. Nous renvoyons à la section 2.5 pour les détails car nous pensons que celle-ci peut se lire directement après les considérations précédentes.

Disons simplement qu'on obtient le théorème suivant

Théorème 15 *Soit $\widehat{H}_N$ notre estimateur (donné par (2.90)). Alors $\widehat{H}_N$ est fortement consistant et on a*

$$2\alpha(\log N)N^{\frac{1-\alpha}{2}}\left(\widehat{H}_N - H\right) \to^{(d)} N\left(0, \Gamma(1, ..., 1)^T\right)$$

où la matrice Γ est définie par (2.82).

Quand t est fixé. Nous avons vu dans ce cas qu'on ne pouvait pas espérer faire une log-log régression car on a

$$\mathbf{E}S_N(t, a_N) = \to_{N\to\infty} \frac{1}{2(H+1)} t^{2H+2}(H) := K_{1,t}.$$

Par contre on peut définir $\widehat{H}_N$ comme l'unique solution de

$$S_N(t, a_N) - K_{1,t}(x) = 0$$

dans l'intervalle $\left[\frac{1}{2}, 1\right]$.

On montre de même le théorème suivant.

Proposition 3 *L'estimateur $\widehat{H}_N$ est fortement consistant et de plus on a*

$$N^{\frac{1-\alpha}{2}}(\widehat{H}_N - H) \to_{N\to\infty}^{(d)} N\left(0, 2K_{1,t}(H)^2\left(\frac{\partial}{\partial H}K_{1,t}^{-1}(H)\right)^2\right).$$

1.2.4 Résumé du chapitre 3

Variations de HOU Pour notre processus X de Hermite-Ornstein-Uhlenbeck on cherche à estimer H et σ en supposant qu'on observe X en des temps discrets $\frac{i}{N}$, $i = 1, \ldots, N$. On utilisera les variations $\tilde{V}_N$ qu'on redéfinit dans notre cadre

$$\begin{aligned}\tilde{V}_N(X) &= \frac{1}{N}\sum_{i=0}^{N-1}\left[\frac{(X_{t_{i+1}}-X_{t_i})^2}{\sigma^2\mathbf{E}\left(Z^{(q,H)}_{t_{i+1}}-Z^{(q,H)}_{t_i}\right)^2}-1\right]\\ &= \frac{1}{N}\sum_{i=0}^{N-1}\left[\frac{N^{2H}}{\sigma^2}(X_{t_{i+1}}-X_{t_i})^2-1\right].\end{aligned}$$

Il s'avère qu'on connaît déjà le comportement de $\tilde{V}_N(Z^{(q,H)})$ (voir le théorème 3.12).

L'égalité suivante montre qu'on peut espérer à partir des propriétés des variations du processus de Hermite $\tilde{V}_N(Z^{(q,H)})$ obtenir celles de $\tilde{V}_N(X)$. On a

$$\int_0^T X_t dt = Z^{(q,H)}_t - X_T.$$

Donc en prenant $Y_t = -\int_0^t X_s ds$ on peut espérer obtenir un théorème de convergence en loi pour $\tilde{V}_N(X)$ si Y_t est assez régulier en L^p et les corrélations entre Y et $Z^{(q,H)}$ sont faibles. On montre que c'est le cas et on a donc le théorème suivant

Proposition 4 *1. Si $H \in \left(\frac{1}{2}, 1\right)$ et $q \geq 2$. Alors*

$$K_q N^{\frac{2-2H}{q}} V_N(X) \to_{N\to\infty} Z^{(2,H')}_1 \text{ in } L^2(\Omega).$$

où $Z^{(2,H')}_1$ est un processus de Rosenblatt d'ordre d'auto-similarité $H' = \frac{2(H-1)}{q} + 1$.

2. Si $q = 1$ et $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$ alors

$$K_{1,1}\sqrt{N} V_N(X) \to^{(d)}_{N\to\infty} N(0,1).$$

Les constantes $K_{1,1}$ et K_q sont des constantes explicites qui sont celles du même théorème pour $V_N(Z^{(q,H)})$.

Remarque 17 *On montre aussi que pour N grand on a pour $q = 1$ et $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$ la borne pour la distance de Kolmogorov suivante*

$$d_{Kol}(K_{1,1}\sqrt{N}V_N(X), \mathcal{N}(0,1)) \leq CN^{H-\frac{3}{4}}.$$

Ceci nous donne l'intuition a posteriori que pour $H > \frac{3}{4}$ nous aurons des corrélations fortes et que le résultat n'est plus vrai.

La proposition précédente est vraie pour le cas stationnaire ainsi que non stationnaire, les preuves sont similaires bien que techniquement différentes car dans le premier cas on intègre sur $(-\infty, 0)$.

Estimation de H L'estimation de H_N suit exactement le même esprit que pour le mouvement brownien fractionnaire :

$$\widehat{H}_N = \frac{-\log(S_N)}{2\log(N)}$$

et on montre le théorème de régularité suivant

Proposition 5 *L'estimateur $\widehat{H}_N$ est fortement consistent i.e. $\widehat{H}_N$ converge presque sûrement vers H quand $N \to \infty$. De plus, pour $q \geq 2$, on a*

$$K_q N^{\frac{2-2H}{q}} \left[2\log(N)\left(H - \widehat{H}_N\right) - \log(\sigma^2)\right] \to^{(d)}_{N\to\infty} Z_1^{(2,H')},$$

tandis que pour $q = 1$ et $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$ on a

$$K_{1,1}\sqrt{N}\left[2\log(N)\left(H - \widehat{H}_N\right) - \log(\sigma^2)\right] \to^{(d)}_{N\to\infty} N(0,1).$$

Remarque 18 *Notre estimateur $\widehat{H}_N$ ne dépend pas de σ mais ce dernier affecte en échelle notre limite en loi. Si on suppose σ connu on montre que pour $q \geq 2$ on a*

$$2K_q \log(N) N^{\frac{2-2H}{q}} \left(H - \widetilde{H}_N\right) \to^{(d)}_{N\to\infty} Z_1^{(2,H')}.$$

Estimation de σ Supposons cette fois que H est connu : peut-on estimer σ ? La réponse dans le cas $q = 1$ est standard, on utilise les $p - variations$ de la solution. C'est ici que le théorème 4 pour les chaos de type Hermite va nous servir. En effet d'après ce théorème la suite stationnaire $\left(Z_{i+1}^{(q,H)} - Z_i^{(q,H)}, i \geq 1\right)$ est ergodique donc on a par auto-similarité et le théorème ergodique

$$\sum_{i=0}^{N-1} \left|Z_{t_{i+1}}^{(q,H)} - Z_{t_i}^{(q,H)}\right|^{\frac{1}{H}} \to_{N\to\infty} \mathbb{E}\left|Z_1^{(q,H)}\right|^{\frac{1}{H}} \text{ en probabilité.}$$

Par une application presque directe de l'inégalité de Minkovski on obtient que

$$\sum_{i=0}^{N-1} |X_{t_{i+1}} - X_{t_i}|^{\frac{1}{H}} \to_{N\to\infty} \sigma^{\frac{1}{H}}\mathbf{E}\left|Z_1^{(q,H)}\right|^{\frac{1}{H}} \text{ en probabilité.}$$

On définit donc l'estimateur $\widehat{\sigma}_N$ par

$$\widehat{\sigma}_N := m(q,H)^{-H}\left(\sum_{i=0}^{N-1} |X_{t_{i+1}} - X_{t_i}|^{\frac{1}{H}}\right)^H \tag{1.110}$$

avec $m(q,H) = \mathbb{E}\left|Z_1^{(q,H)}\right|^{\frac{1}{H}}$.

On obtient donc que $\widehat{\sigma}_N \to_{N\to\infty} \sigma$ en probabilité.

Remarque 19 *La convergence en loi est une problème non résolu intrigant car on ne connaît pas le comportement asymptotique des $\frac{1}{H}$−variations du processus de Hermite.*

1.2.5 Résumé du chapitre 4

Notre but est d'estimer notre paramètre $\theta > 0$ en supposant que la solution u_θ de (1.82) est observée en des temps discrets ou des points discrets. Nous avons vu 1.87 que la solution u_θ peut se décomposer en deux processus $X(t,x) + Y(t,x)$.

Pour estimer θ la première idée (similaire à celle dans [88]) est de faire un changement de variable $v_\theta(t,x) = u_\theta\left(\frac{t}{\theta}, x\right)$ on obtient alors par la propriété d'auto-similarité du bruit que

$$u_\theta\left(\frac{t}{\theta}, x\right) = \theta^{-1/2} X(t,x) + \theta^{-1} Y_\theta(t,x)$$

avec

$$Y_\theta(t,x) = \frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G_{t-s}(x-y) u_\theta\left(\frac{s}{\theta}, y\right) ds.$$

Estimation de θ quand on observe $(u_\theta(t,x_i), i = 0,..,N)$ On considère alors notre variation quadratique

$$S_{N,t}(*) = \sum_{i=0}^{N-1} \left(*(t,x_{i+1}) - *(t,x_i)\right)^2.$$

On a

$$\begin{aligned} S_{N,t}(v_\theta) &= \theta^{-1} S_{N,t}(X) + \theta^{-2} S_{N,t}(Y_\theta) \\ &\quad + 2\theta^{-\frac{3}{2}} \sum_{i=0}^{N-1} \left(X(t,x_{i+1}) - X(t,x_i)\right)\left(Y_\theta(t,x_{i+1}) - Y_\theta(t,x_i)\right). \end{aligned}$$

Pour le premier on a d'après (1.93) que

$$\theta^{-1} S_{N,t}(X) \to_{N\to\infty} \theta^{-1} \frac{1}{2}(A_2 - A_1) \text{ dans } L^2(\Omega).$$

Il reste à montrer que les deux autres quantités sont négligeables au sens de $L^1(\Omega)$. On montre en fait que le processus Y est plus régulier que le processus X. C'est contenu dans la proposition suivante.

Proposition 6 *Pour tout $t \in [0,T]$, $x \to Y(t,x)$ est Hölder continue d'ordre δ pour tout $\delta \in \left(0, \frac{2}{\beta} - 1\right)$ et pour tout $\beta \in \left(1, \frac{3}{2}\right)$ tandis que $x \to u(t,x)$ est Hölder continue d'ordre δ pour tout $\delta \in \left(0, \frac{1}{2}\right)$.*

On prouve finalement que

$$S_{N,t}(v_\theta) \to_{N\to\infty} \theta^{-1} \frac{1}{2}(A_2 - A_1) \text{ dans } L^1(\Omega).$$

On définit donc notre estimateur pour tout $t > 0$ et $N \geq 1$ par

$$\widehat{\theta}_{N,t} = \frac{A_2 - A_1}{2 S_{N,t}(u_\theta)}. \tag{1.111}$$

Remarque 20 *(1.111) est calculable à partir de nos observations* $u(t, x_i), i = 0, 1, \ldots, N$.

On montre ensuite les propriétés suivantes

Proposition 7
1. *Pour tout* $t > 0$*, l'estimateur* $\widehat{\theta}_{N,t}$ *définit par (1.111) est consistant, i.e.* $\widehat{\theta}_{N,t}$ *converge en probabilité vers* θ *quand* $N \to \infty$.
2. *L'estimateur* $\widehat{\theta}_{N,t}$ *est fortement consistent, i.e.* $\widehat{\theta}_{N,t}$ *converge presque sûrement vers* θ *quand* $N \to \infty$*. De plus pour tout* $p \geq 2$*, on a*

$$\mathbf{E}\left|\widehat{\theta}_{N,t} - \theta\right|^p \leq CN^{-\frac{p}{2}}$$

pour N *grand.*

Estimation de θ quand on observe $(u_\theta(t_i, x), i = 0, .., N)$. Cette fois-ci on considère les 4− variations

$$T_{N,x}(X) = \sum_{i=0}^{N-1} \left(X(t_{i+1}, x) - X(t_i, x)\right)^4, \quad N \geq 1. \tag{1.112}$$

Le cheminement est le même. Pour $x \in \mathbb{R}$ on aura l'estimateur

$$\widehat{\theta}_{N,x} = \frac{3}{\pi}(A_2 - A_1)\frac{1}{T_{N,x}(u_\theta)}. \tag{1.113}$$

La proposition 7 est stricto-sensu la même en remplaçant $\widehat{\theta}_{N,t}$ par $\widehat{\theta}_{N,x}$.

1.3 Partie 2 : Fluctuations gaussiennes pour l'équation de la chaleur fractionnaire

vant d'introduire nos résultats commençons par situer le contexte. Tout commence avec l'article [12] où les auteurs ont établi la convergence en variation totale de la moyenne spatiale de la solution de l'équation de la chaleur non linéaire vers une loi gaussienne, ainsi qu'un théorème central limite fonctionnel. Considérons donc l'équation étudiée

$$\frac{\partial}{\partial t}u(t, x) = \frac{1}{2}\Delta u(t, x) + \sigma(u(t, x))W(x, t), \quad (x, t) \in \mathbb{R} \times (0, \infty) \tag{1.114}$$

où $u(0, x) = 1$, W est un bruit blanc de support, λ la mesure de Lebesgue (voir la définition 1 et l'étude de l'équation de Burger dans la partie 1) et σ est une fonction lipschitzienne telle que $\sigma(1) \neq 0$.[6] Nous savons (d'après le théorème 1) que l'équation a une unique solution adaptée à la filtration engendrée par W tel que $\mathbb{E}(|u(t, x)|^2) < \infty$ et satisfait

$$u(t, x) = 1 + \int_0^t \int_{\mathbb{R}} G(t - s, x - y)\sigma(u(s, y))W(ds, dy) \tag{1.115}$$

6. Si $\sigma(1) = 0$ alors pour tout t, x on a $u(t, x) = 0$.

où l'intégrale a été étudiée dans la section 1.1.2, et $G(t,x) = (2\pi t)^{-\frac{1}{2}} e^{-x^2/(2t)}$ est le noyau gaussien.

Les auteurs se sont donc intéressés à la quantité suivante

$$F_R(t) := \frac{1}{\sigma_R}\left(\int_{-R}^{R} u(t,x)dx - 2R\right) \tag{1.116}$$

où $R > 0$ et $\sigma_R^2 := Var\left(\int_{-R}^{R} u(t,x)dx\right)$.

Nous savons (cf la section 1.1.4.3) que la solution u peut s'écrire

$$u(t,x) = 1 + \delta(G(t-\cdot, x-\star)u(\cdot,\star)) \tag{1.117}$$

où δ est l'opérateur de divergence associé à notre processus isonormal $\mathcal{H}$ où ici $\mathcal{H} = L^2(\mathbb{R}^+ \times \mathbb{R})$.

L'idée pour majorer la distance en variation totale est de reprendre la formule d'intégration par partie (1.69) dans le cas spécifique où $F = \delta(v)$. On obtient alors la proposition suivante (voir la proposition 2.2 dans [12]) dont nous rappelons la preuve car celle-ci est élégante de simplicité.

Proposition 8 *Soit $F = \delta(v)$ tel que v soit une variable aléatoire dans $\mathcal{H}$ et qui soit dans $Dom\delta$. On suppose que $\mathbb{E}(F^2) = 1$ et $F \in \mathbb{D}^{1,2}$. Soit $Z \sim \mathcal{N}(0,1)$. Alors on a*

$$d_{TV}(F,Z) \leq 2\sqrt{Var\langle DF, v\rangle_{\mathcal{H}}}. \tag{1.118}$$

On rappelle que

$$d_{\mathrm{TV}}(F,Z) := \sup\Big\{P(F\in A) - P(Z\in A) \,:\, A\subset\mathbb{R} \ \text{ est un borélien}\Big\}.$$

Proof: On a par intégrations par parties

$$\begin{aligned}\mathbb{E}[Ff(F)] &= \mathbb{E}[\delta(v)f(F)] = \mathbb{E}\left[\langle v, Df(F)\rangle_{\mathcal{H}}\right] \\ &= \mathbb{E}\left[\langle v, f'(F)DF\rangle_{\mathcal{H}}\right] = \mathbb{E}\left[f'(F)\langle v, DF\rangle_{\mathcal{H}}\right].\end{aligned}$$

Donc, par le théorème 8 on obtient

$$\begin{aligned} d_{TV} &\leq \sup_{f\in\mathcal{F}_{TV}} \left|\mathbb{E}\left[f'(F) - Ff(F)\right]\right| \\ &= \sup_{f\in\mathcal{F}_{TV}} \left|\mathbb{E}\left[f'(F)(1-\langle v, DF\rangle_{\mathcal{H}})\right]\right| \\ &\leq 2|\mathbb{E}\left[1-\langle v, DF\rangle_{\mathcal{H}}\right] \\ &\leq \sqrt{Var\langle DF, v\rangle_{\mathcal{H}}} \end{aligned}$$

où la dernière inégalité est une conséquence de l'inégalité de Cauchy-Shwarz et du fait que $\mathbb{E}\langle DF, v\rangle_{\mathcal{H}} = \mathbb{E}[F\delta(v)] = \mathbb{E}(F^2) = 1$ et donc $\mathbb{E}\left[1-\langle v, DF\rangle_{\mathcal{H}}\right]^2 = Var\langle DF, v\rangle_{\mathcal{H}}$.

Remarque 21 *Il est évident que la proposition est la même pour les autres distances d'après le même théorème 8.*

Les étapes pour parvenir à obtenir une borne pour 1.116 ont donc été les suivantes

- On écrit $F_R(t) = \delta(v_{t,R})$ avec
$$v_{t,R}(s,y) = \sigma(u(s,y))\mathbf{1}_{[0,t]}(s)\frac{1}{\sigma_R}\int_{-R}^{R} G(t-s,x-y)dx$$
.
- On doit estimer
$$\langle DF_R(t), v_{t,R}\rangle_{\mathcal{H}} = \frac{1}{\sigma_R^2}\int_0^t\int_{\mathbb{R}}\int_{-R}^{R}\int_{-R}^{R} G_{t-s}(x-y)\sigma(u(s,y))D_{s,y}u(t,x')dxdx'dyds. \tag{1.119}$$
La solution est bien différentiable au sens de Malliavin, cf le théorème 11. Deux sous-étapes donc :
  - On obtient par des calculs standards que $\sigma_R^2 \sim R$: le point clé de cette étape est la propriété de semi-groupe du noyau gaussien c'est-à-dire pour tout $x, x', y \in \mathbb{R},\ t, s \in [0,T]$,
$$\int_{\mathbb{R}} G(t,x'-y)G(s,y-x)dx = G(t+s,x'-x).$$
  - Estimer la partie intégrale est la plus technique et fait appel à deux gros outils du calcul d'Itô comme la formule de Clark-Ocone et l'inégalité de Burkholder mais surtout le point clé est de réussir à borner la dérivée de Malliavin. Dans [12]) le lemme 5.1 montre que pour tout $0 < s < r \le t$ et $y, z \in \mathbb{R}$
$$\|D_{s,y}u(r,z)\|_p \le C_{t,p}G(t-s,z-y). \tag{1.120}$$

En parallèle à cette estimation, par des considérations multidimensionnelles et de tension un théorème limite central fonctionnel est prouvé. Voici les deux principaux résultats que les auteurs obtiennent.

Théorème 16 *Soient $u(t,x)$ la solution de l'équation (1.114) et F_R la moyenne spatiale (1.116). Soit $\xi(s) := \mathbb{E}(\sigma(u(s,y))^2), s \ge 0$. Alors on a*
$$d_{TV}(F_R(t), Z) \le \frac{C}{\sqrt{R}} \tag{1.121}$$
$$\left(\frac{1}{\sqrt{R}}\left(\int_{-R}^{R} u(t,x)dx - 2R\right)\right)_{t\in[0,T]} \overset{R\to\infty}{\Longrightarrow} \left(\int_0^t \sqrt{2\xi(s)}dB_s\right)_{t\in[0,T]}. \tag{1.122}$$
La convergence a lieu au sens des processus dans l'espace des fonctions continues $C([0,T])$.

L'histoire continue car ayant considéré un bruit blanc la solution n'existe qu'en dimension 1, donc il est logique de considérer un bruit blanc en temps et colorié en espace. Dans [13] les auteurs considèrent la même équation avec W de covariance

$$\mathbb{E}(W(t,s)W(s,y)) = \delta_0(t-s)|x-y|^{-\beta} \tag{1.123}$$

où $0 < \beta < \min(d,2)$ avec d la dimension de l'espace. [7]

Étant donné que nous allons considérer l'équation de la chaleur en tout dimension avec un bruit similaire on présente les résultats de l'article [13] . Soit B_R la boule euclidienne de rayon $R > 0$. On remplace simplement l'intervalle $[-R, R]$ par la boule dans les définitions précedentes et les intégrales $\int_{\mathbb{R}} \to \int_{\mathbb{R}^d}$. Voici la forme du théorème précédent en toute dimension pour un tel bruit (1.123).

Théorème 17 *Soit u la solution de l'équation de la chaleur avec un bruit blanc en temps et riesz en espace (1.123) alors il existe une constante $C = C(t, \beta)$ telle que*

$$d_{TV}\left(\frac{1}{\sigma_R}\int_{B_R}(u(t,x)-1)\,dx, Z\right) \leq CR^{-\beta/2} \tag{1.124}$$

et si $k_\beta := \int_{B_1^2} |x_1 - x_2|^{-\beta} dx_1 dx_2$ et $\eta(s) := \mathbb{E}(\sigma(u(s,y)))$ les auteurs obtiennent le théorème central limite suivante

$$\left(R^{\frac{\beta}{2}-d}\int_{B_R}(u(t,x)-1)\,dx\right)_{t\in[0,T]} \overset{R\to\infty}{\Longrightarrow} \left(\sqrt{k_\beta}\int_0^t \eta(s)dB_s\right)_{t\in[0,T]}. \tag{1.125}$$

Une remarque s'impose naturellement.

Remarque 22 *Contrairement au cas du bruit blanc en temps et en espace on trouve dans la limite fonctionnelle non pas $\sqrt{\mathbb{E}(\sigma(u(s,y))^2)}$ mais $\mathbb{E}(\sigma(u(s,y)))$. Ceci s'explique par le fait que quand on ajoute de la corrélation, l'isométrie ne nous donne plus une intégrande au carré, mais un produit. Se rappeler la formule de covariance (1.76). Pour ensuite approximer le produit $\mathbb{E}[\sigma(u(s,y))\sigma(u(s,z))]$ les auteurs utilisent (et nous utiliserons la même chose) la formule de Clark-Ocone*

$$\sigma(u(s,y)) = \mathbb{E}[\sigma(u(s,y))] + \int_0^s \int_{\mathbb{R}^d} E\left[D_{r,\gamma}(\sigma(u(s,y)))|\mathcal{F}_r\right] W(dr,dy). \tag{1.126}$$

On montre ensuite que

$$\mathbb{E}[\sigma(u(s,y))\sigma(u(s,z))] \sim \mathbb{E}(\sigma(u(s,y)))^2$$

par des majorations techniques.

▷ Dans le présent travail que nous allons présenter, nous avons considéré un bruit blanc en temps et colorié en espace donné par le potentiel de Riesz. Mais surtout l'originalité est que nous n'allons pas considérer l'opérateur Laplacien classique mais l'opérateur Laplacien fractionnaire qui est intimement relié au potentiel de Riesz.

7. Beaucoup d'autres considérations ont été étudiées depuis, nous renvoyons au dernière article en date [9] sur ce sujet avec toutes les références citées.

Laplacien fractionnaire Considérons l'équation de Poisson sur $\mathbb{R}^d$:

$$\Delta u = f$$

pour f suffisamment régulière. On montre que la solution fondamentale (pour $d \geq 3$) est donnée par la potentiel de Newton

$$G(x) := \frac{1}{d(d-2)\omega_d} \frac{1}{|x|^{d-2}}$$

où ω_d est le volume de la boule en dimension d. Ainsi on peut montrer que

$$I_2(f)(x) = \frac{1}{d(d-2)\omega_d} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2}} dy$$

est solution de l'équation de Poisson. On montre de même que cette dernière est solution au sens des distributions. Ce qui est intéressant c'est qu'on a pour $f \in \mathcal{S}(\mathbb{R}^d)$,

$$I_2(-\Delta f) = f.$$

Autrement dit I_2 est l'inverse de $-\Delta$.

On cherche donc à généraliser cette identité et le potentiel de Newton devient le potentiel de Riesz. Soit $0 < \beta < d$ le potentiel de Riesz est la quantité suivante

$$I_\beta(f)(x) = C_{d,\beta} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\beta}} dy$$

et on aimerait obtenir un opérateur L telle qu'on ait l'identité $I_\beta(Lf) = f$. Un tel opérateur est l'opérateur laplacien fractionnaire dont la définition la plus naturelle (à nos yeux) est celle donnée par la transformée de Fourier.

Définition 7 *L'opérateur L est défini par l'identité suivante, avec $\alpha \in (0, 2]$,*

$$\mathcal{F}(Lf)(\xi) = -|\xi|^\alpha \mathcal{F}(f)(\xi), \tag{1.127}$$

pour $f \in L^p$, $p \in [1, 2]$. On note $L = -(-\Delta)^{\alpha/2}$.

La définition est équivalente si on définissait l'opérateur comme l'inverse du potentiel de Riesz. Voir l'article [14] où de multiples définitions équivalentes sont données. Nous pouvons dés lors considérer notre équation de la chaleur fractionnaire.

1.3.1 Résumé du chapitre 5

Dans le chapitre 5 nous avons donc considéré l'équation de la chaleur fractionnaire suivante

$$\frac{\partial u}{\partial t}(t,x) = -(-\Delta)^{\frac{\alpha}{2}} u(t,x) + \sigma(u(t,x))W(t,x), \quad t \geq 0, x \in \mathbb{R}^d \tag{1.128}$$

avec $u(0,x) \equiv 1$. On suppose que σ est une fonction lipschitzienne telle que $\sigma(1) \neq 0$ et $-(-\Delta)^{\frac{\alpha}{2}}$ le laplacien fractionnaire. Le bruit est le même que nous avons rencontré dans la section DIRE i.e. W est un processus gaussien centré de covariance

$$\mathbb{E}[W(t,x)W(s,y)] = \delta_0(t-s)\gamma(x-y). \tag{1.129}$$

Comme accoutumé la solution de (1.128) s'écrit pour nous

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_\alpha(t-s, x-y)\sigma(u(s,y))W(ds,dy), \tag{1.130}$$

où l'intégrale est au sens de Dalang-Walsh.

Le noyau G_α est défini par sa transformée de Fourier

$$(\mathcal{F}G_\alpha(t,\cdot))(\xi) = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d, t \geq 0. \tag{1.131}$$

Remarque 23 *Pour $\alpha = 2$ on retrouve le noyau gaussien dont G_2 se calcule explicitement, pour $\alpha = 1$ c'est le noyau de Poisson. Il n'y a pas de formule explicite pour $\alpha \neq 1, 2$. Cette remarque nous fait comprendre les difficultés techniques inhérentes pour estimer asymptotiquement nos intégrales.*

Le premier résultat que nous montrons est l'existence et l'unicité de la solution.

Théorème 18 *On suppose que la transformée de Fourier $\widehat{\gamma} = \mathcal{F}\gamma$ satisfait la condition de Dalang fractionnaire*

$$\int_{\mathbb{R}^d} \frac{\widehat{\gamma}(d\xi)}{\beta + |\xi|^\alpha} < \infty, \tag{1.132}$$

pour $\beta > 0$. Ainsi l'équation (1.128) admet une unique solution (1.130). De plus on a pour tout $p \geq 1$ et pour tout $T > 0$

$$\sup_{t\in(0,T], y\in\mathbb{R}^d} \mathbb{E}[|u(t,x)|^p] < \infty.$$

La preuve est classique au sens où on définit l'itération de Picard associée à (1.128) i.e. pour tout $n \geq 1$ on considère

$$u_{n+1}(t,x) = u_0(t,x) + \int_0^t \int_{\mathbb{R}^d} G_\alpha(t-s, x-y)\sigma(u_n(s,y))W(ds,dy), \quad t \geq 0, x \in \mathbb{R}^d.$$

Puis on montre que u_n converge uniformément en L^p et que la limite satisfait (1.130) grâce à un lemme de type Grönwall fractionnaire et par des majorations usuelles. Pour la preuve voir le chapitre 5.

Pour obtenir notre borne en variation totale et un théorème limite central fonctionnel nous ferons l'hypothèse suivante pour les corrélations :

Hypothèse 2 *On suppose que γ est donné par le potentiel de Riesz $\gamma = I_{d-\beta}\mu$*[8] *où $0 < \beta \leq d$ et μ est une mesure finie symétrique. On suppose de plus qu'au moins une des trois conditions suivante est vérifiée*

(i) $\beta < \alpha \wedge d$.

(ii) $\beta = d = 1$ *and* $\alpha > 1$.

(iii) $\beta = d \geq \alpha$ *et* $\mu = \gamma$ *est absolument continue i.e.* $d\gamma(x) = \gamma(x)dx$*, avec* $\gamma \in L^r(\mathbb{R}^d)$ *pour* $r > \frac{d}{\alpha}$*. Par ailleurs, si* $r > 2$*,on suppose (1.132).*

Notre premier résultat va dans le sens du premier point du théorème 17.

Distance en variation totale de la moyenne spatiale

Théorème 19 *On suppose que γ satisfait l'un des points de 2 et soit u la solution de l'équation de la chaleur fractionnaire (1.130). Alors pour tout $t > 0$ il existe une constante C qui ne dépend que de t, α, σ, et de la covariance γ, telle que*

$$d_{\mathrm{TV}}\left(\frac{1}{\sigma_R}\int_{B_R}[u(t,x)-1]\,dx,\ Z\right) \leq CR^{-\frac{\beta}{2}},$$

où $Z \sim N(0,1)$ et $\sigma_R^2 = \mathrm{Var}\big(\int_{B_R}[u(t,x)-1]\,dx\big) \sim R^{2d-\beta}$, quand $R \to \infty$.

On remarque que selon nos hypothèses 2 on retrouve pour $\beta = d = 1$ le cas du bruit blanc en temps et en espace, et de même on retrouve la même borne que dans le cas non fractionnaire. Ainsi la constante α ne joue pas explicitement de rôle dans la distance même si β et α sont interdépendants pour avoir l'existence et l'unicité de la solution (1.130).

Remarque 24 *La condition $\sigma(1) \neq 0$ implique en fait que $\sigma_R > 0$, $\forall R > 0$. Cela a été remarqué dans [50, Lemme 3.4] et la preuve s'adapte facilement à notre travail.*

Théorème central limite fonctionnel Notre second théorème est le théorème limite central qui correspond à l'idée du deuxième point du théorème 17.

Théorème 20 *Soit γ telle que l'hypothèse 2 soit satisfaite et soit u la solution de l'équation de la chaleur fractionnaire (1.130). Alors*

$$\left\{R^{\frac{\beta}{2}-d}\int_{B_R}[u(t,x)-1]\,dx\right\}_{t\in[0,T]} \Rightarrow \left\{\int_0^t \varrho(s)dB_s\right\}_{t\in[0,T]},$$

quand $R \to \infty$, où B est un mouvement brownien standard, la convergence a lieu au sens des processus dans $C([0,T])$, et $\varrho(s)$ est donné par

- *Si $\beta < d$ alors $\varrho(s) = \sqrt{\mu(\mathbb{R}^d)\int_{B_1^2}|x-x'|^{-\beta}dxdx'}\,\mathbb{E}[\sigma(u(s,y))]$.*

---

8. Le potentiel est défini par $I_{d-\beta}(x) = \int_{\mathbb{R}^d}|x-y|^{-\beta}d\mu(y)$.

- *Si* $\beta = d$ *alors* $\varrho(s) = \sqrt{|B_1| \int_{\mathbb{R}^d} \mathbb{E}\left[\sigma(u(s,0))\sigma(u(s,z))\right] d\mu(z)}$.

Remarque 25 *Nous ne l'avons pas encore dit mais il est temps : le fait que* $\rho(s)$ *ne dépend pas de la variable spatiale est une conséquence de la stationnarité en espace de la solution i.e. pour tout* $t \geq 0$ *les loi finies dimensionnelles de* $\{u(t, x+y)\}_{x \in \mathbb{R}^d}$ *ne dépendent pas de* $y \in \mathbb{R}^d$. *On peut se référer au lemme* 7.1 *de l'article [37] pour une preuve détaillée.*

Techniques pour les démonstrations

Nous avons déjà explicité, en début de section, la stratégie générale pour obtenir le théorème centrale limite 20 et la borne en variation totale (19). Disons, malgré tout, quelques mots sur les techniques utilisées et les différences par rapport au cas non fractionnaire.

L'étape la plus technique est de majorer la dérivée de Malliavin en norme p. Dans l'article [15], section 3 lemme 3.1, un résultat général a été établi pour l'équation de la chaleur non fractionnaire sur $\mathbb{R}^d$. La preuve utilise le fait magique suivant

$$G_2(s,x)G_2(t-s,y) = G_2\left(\frac{s(t-s)}{t}, \frac{sy-(t-s)x}{x}\right) G_2(t, x+y).$$

Or pour $\alpha \geq 2$, le noyau G_α n'est pas donné explicitement et une telle égalité semble impossible à prouver. Par contre le noyau conserve la propriété de semi-groupe et une propriété d'échelle qui permet de montrer que

$$K'_\alpha \frac{t^{-\frac{d}{\alpha}}}{\left(1+|t^{-\frac{1}{\alpha}}x|\right)^{d+\alpha}} \leq |G_\alpha(t,x)| \leq K_\alpha \frac{t^{-\frac{d}{\alpha}}}{\left(1+|t^{-\frac{1}{\alpha}}x|\right)^{d+\alpha}}.$$

Cette inégalité nous donne une forme d'intégrablité et de convergence asymptotique, couplée avec l'élégante inégalité suivante

$$\int_{\mathbb{R}^d} f(y)\left[g * \gamma\right](y) dy \leq C \|f\|_{L^{2q}(\mathbb{R}^d)} \|g\|_{L^{2q}(\mathbb{R}^d)}$$

où γ vérifie (2) et pour $2q \in \left(1, \frac{2d}{2d-\alpha} \wedge \frac{d+\alpha}{d}\right)$ ainsi que pour tout $f, g \in L^{2q}(\mathbb{R}^d)$,

nous arriverons à borner la dérivée de Malliavin ainsi qu'estimer asymptotiquement nos intégrales. Bien-sûr nous utiliserons aussi les concepts puissants, déjà rappelés, du calcul d'Itô comme la formule de Clark-Ocone et l'inégalité de Burkholder. Les détails sont dans le chapitre 5.

1.4 Partie 3 : Du processus de Hermite généralisé au processus de Orstein-Uhlenbeck rugueux

Le processus de Hermite est un processus auto-similaire d'ordre $H > 1/2$, à accroissements stationnaires comme nous l'avons vu. Cette classe de processus peut-être vue comme des processus réguliers car ils sont à longue mémoire bien que non-gaussiens. La volonté de s'affranchir de cette restriction a été marqué par une volonté abstraite, cependant dans l'article [57] les auteurs montrent que la volatilité est rugueuse. Autrement dit en pratique financière, la modélisation par des processus devraient pouvoir considérer des processus auto-similaires d'ordre H petit, le mouvement brownien fractionnaire pour $H < 1/2$ peut donc être considéré, nous dirons qu'un tel processus est rugueux. Cependant il semble que pour d'autres données financière (comme le Bitcoin) le caractère gaussien ne "colle" pas. Nous renvoyons le lecteur à la lecture de 6.1 du chapitre 4 pour une description plus précise.

Nous proposerons donc un processus rugueux non-gaussien : le processus de Orstein-Uhlenbeck rugueux !

1.4.1 Résumé du chapitre 6

Dans ce chapitre nous étudions le processus de Orstein-Uhlenbeck associé au processus de Hermite généralisé dont voici la définition.

Définition 8 (Processus de Hermite généralisé) *Soit* $\left(X_t^{(q,H,\beta)}\right)_{t\geq 0}$ *le processus stohastique donné par*

$$X_t^{(q,H,\beta)} = d(q,H,\beta)\int_{\mathbb{R}^q}\left[\int_{\mathbb{R}} g_t^\beta(u)(u-y_1)_+^{-(\frac{1}{2}+\frac{1-H}{q})}\dots(u-y_q)_+^{-(\frac{1}{2}+\frac{1-H}{q})}du\right]dB(y_1)\dots dB(y_q) \tag{1.133}$$

pour tout $t \geq 0$ *et la fonction* g_t^β *est égale à*

$$g_t^\beta(u) = (t-u)_+^\beta - (-u)_+^\beta \text{ if } \beta \neq 0 \tag{1.134}$$

et $g_t^\beta(u) = \mathbf{1}_{[0,t]}(u)$ *if* $\beta = 0$.

La constante $d(q,H,\beta)$ est notre constante de renormalisation i.e. telle que $\mathbf{E}(X_1^{(q,H,\beta)})^2 = 1$.

En vertu de la définition des intégrales multiples on peut donc écrire $X_t^{(q,H,\beta)}$

$$X_t^{(q,H,\beta)} = I_q(L_t^\beta)$$

où L_t^β est le noyau

$$L_t^\beta(y_1,...,y_q) = d(q,H,\beta)\int_{\mathbb{R}} g_t^\beta(u)(u-y_1)_+^{-(\frac{1}{2}+\frac{1-H}{q})}\dots(u-y_q)_+^{-(\frac{1}{2}+\frac{1-H}{q})}du \tag{1.135}$$

pour tout $y_1,\dots,y_q \in \mathbb{R}$. D'après la proposition 3.25 dans [24] on a pour tout $t \geq 0$, $L_t^\beta \in L^2(\mathbb{R}^q)$ si

$$0 < H + \beta < 1 \text{ and } H \in \left(\frac{1}{2}, 1\right) \tag{1.136}$$

ou de manière équivalente

$$-1 < -H < \beta < 1 - H < \frac{1}{2}.$$

Ce que nous supposerons toujours dans la suite.

- Le processus de Hermite généralisé est auto-similaire d'ordre $H+\beta$ et à accroissement stationnaire. En particulier sa fonction de covariance s'écrit

$$\mathbb{E}X_t^{(q,H,\beta)}X_s^{(q,H,\beta)} = \frac{1}{2}\left(t^{2(H+\beta)} + s^{2(H+\beta)} - |t-s|^{2(H+\beta)}\right) =: R^{H+\beta}(t,s).$$

- Nous avons donc un processus non gaussien où le coefficient d'autosimilarité est compris dans tout l'intervalle $(0,1)$.
- Le processus est höldérien continue de paramètre $\delta \in (0, H+\beta)$

D'après notre remarque 12 on a donc la proposition suivante.

Proposition 9 *Soit $X^{(q,H,\beta)}$ donné par (1.133) et $Z^{(q,H)}$ le processus de Hermite classique. Alors pour tout $t \geq 0$*

$$X_t^{(q,H,\beta)} = \frac{d(q,H,\beta)}{d(q,H)} \int_{\mathbb{R}} g_t^{\beta}(u) dZ_u^{(q,H)}.$$

Notre but est d'étudier l'équation de Langevin avec un tel processus comme bruit, autrement dit on considère

$$dY_t = -\alpha(Y_t - m)dt + \sigma dX_t^{(q,H,\beta)}, \quad t \geq 0 \tag{1.137}$$

avec condition initiale $Y_0 \in L^0(\mathbb{R})$ et $\alpha, \sigma > 0$, $m \in \mathbb{R}$. On écrit donc, comme dans 1.1.3.2,

$$Y_t = Y_0 e^{-\alpha t} + m(1-e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-s)} dX_s^{(q,H,\beta)}.$$

Il nous faut donc donner un sens à l'intégrale par rapport à $X_s^{(q,H,\beta)}$.

C'est ici que des considérations techniques apparaissent. En effet, on souhaite définir l'intégrale de Wiener, on considère donc une fonction simple

$$f(t) = \sum_{i=0}^{N-1} \lambda_i 1_{[t_i,t_{i+1})}(t)$$

où $\lambda_i \in \mathbb{R}, N \geq 1$ and $0 < t_1 < \ldots t_N$. Notre intégrale élémentaire est donc

$$\int_{\mathbb{R}} f(u) dX_u^{(q,H,\beta)} = \sum_{i=0}^{N-1} \lambda_i \left(X_{t_{i+1}}^{(q,H,\beta)} - X_{t_i}^{(q,H,\beta)}\right).$$

Faisons maintenant le calcul suivant

$$\begin{aligned} &\int_{\mathbb{R}} f(u) dX_u^{(q,H,\beta)} \\ =\ & d(q,H,\beta) \sum_{i=0}^{N-1} \lambda_i \int_{\mathbb{R}^q} dB(y_1 \ldots dB(y_q) \int_{\mathbb{R}} (g_{t_{i+1}}^{\beta}(u) - (g_{t_i}^{\beta}(u)) \prod_{j=1}^{q} (u-y_j)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} du \end{aligned}$$

$$= \quad d(q,H,\beta)\int_{\mathbb{R}^q} dB(y_1 \dots dB(y_q)\int_{\mathbb{R}} du \left(\int_{\mathbb{R}} f(s) d_s g_s^\beta(u)\right)\prod_{j=1}^{q}(u-y_j)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)}$$
$$= \quad d(q,H,\beta)\int_{\mathbb{R}^q} dB(y_1 \dots dB(y_q)\left(\int_{\mathbb{R}} du\beta \int_{\mathbb{R}} f(s)(s-u)_+^{\beta-1} ds\right)\prod_{j=1}^{q}(u-y_j)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)}$$

Malheureusement l'intégrale $\int_{\mathbb{R}} f(s)(s-u)_+^{\beta-1} ds$ est finie seulement pour $\beta > 0$. Cette intégrale s'appelle l'intégrale de Riemann-Liouville qu'on note $\mathcal{I}_-^\beta(f)$. Donc si on souhaite définir une intégrale de Wiener nous devons nous restreindre à $\beta > 0$.

Dans l'article [8] les auteurs ont montré que l'intégrale de Wiener pour le Orstein-Uhlenbeck fractionnaire coïncide avec l'intégrale de Stieljes grâce à la régularité de $x \mapsto e^x$. Nous pouvons faire la même chose ici de manière plus générale.

Intégrale de Riemann-Stieltjes Soit f une fonction C^1 définie sur $[a,b]$ $(-\infty \le a < b < \infty)$ telle que l'intégrale de Riemann suivante

$$\int_a^b X_u^{(q,H,\beta)} f'(u) du \text{ existe} \tag{1.138}$$

et

$$\lim_{u \to a} f(u) X_u^{(q,H,\beta)} := L_a \text{ existe.} \tag{1.139}$$

Alors l'intégrale de Riemann-Stieltjes de f par rapport à $X^{(q,H,\beta)}$ notée

$$\int_a^b f(u) d_{RS} X_u^{(q,H,\beta)}$$

existe et on a l'intégration par partie suivante

$$\int_a^b f(u) d_{RS} X_u^{(q,H,\beta} = f(b) X_b^{(q,H,\beta)} - L_a - \int_a^b f'(u) X_u^{(q,H,\beta)} du.$$

Retour à la solution On peut donc donner un sens à

$$Y_t = Y_0 e^{-\alpha t} + m(1-e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-s)} dX_s^{(q,H,\beta)}. \tag{1.140}$$

en toute généralité pour β si les conditions (1.138) et (1.139) sont vérifiées, ce qui est le cas car notre processus est un processus höldérien d'order $\delta \in (0, H+\beta)$. Donc Y_t est bien solution de 1.137. De plus, quand $Y_0 = \sigma \int_{-\infty}^0$ on a

$$Y_t = m(1-e^{-\alpha t}) + \sigma \int_{-\infty}^t e^{-\alpha(t-s)} d_{RS} X_s^{(q,H,\beta)}. \tag{1.141}$$

Remarque 26 *Pour $\beta > 0$ l'intégrale de Wierner et de Riemann-Stieltjes coïncident là où leurs domaines de définition coïncident. Voir la proposition 38.*

On peut énoncer quelques propriétés de (1.140).

Proposition 10 *Soit $(Y_t)_{t\geq 0}$ donné par (1.140). Alors*

1. *Si $m = 0$, le processus $(Y_t)_{t\geq 0}$ est stationnaire, i.e. pour tout $h > 0$, $(Y_t)_{t\geq 0}$ et $(Y_{t+h})_{t\geq 0}$ ont même loi.*
2. *Soit $t > 0$. Alors $Cov(Y_{t+h}, Y_t)$ est asymptotiquement équivalent quand $h \to \infty$ à $h^{2(H+\beta)}$.*
3. *$(Y_t)_{t\geq 0}$ est höldérien continu d'ordre δ pour tout $\delta \in (0, H+\beta)$.*

Comme dans [57], on montre que si le drift est proche de 0 alors notre processus de Hermite-Ornstein-Uhlenbeck généralisé se comporte comme un Hermite généralisé, nous montrons aussi une propriété d'échelle. La preuve du premier point ci-dessous est une conséquence de l'hypercontractivité des chaos (voir la preuve de 40).

Proposition 11

- *Pour tout $T > 0$ et $p \geq 1$,*

$$\sup_{t\in[0,T]} \mathbb{E}\left|Y_t - Y_0 - \sigma X_t^{(q,H,\beta)}\right|^p \to_{\alpha\to 0} 0.$$

- *Pour tout $t > 0, 0 < \Delta < T$ et $p \geq 1$,*

$$\mathbb{E}\,|Y_{t+\Delta} - Y_t|^p \to_{\alpha\to 0} \sigma^p \mathbb{E}|X_1^{(q,H,\beta)}|^p |X_{t+\Delta}^{(q,H,\beta)} - X_t^{(q,H,\beta)})|^{p(H+\beta)}.$$

Première partie

Inférence statistique des équations différentielles stochastiques : approche variationnelle

Chapitre 2

Wavelet analysis for the solution to the wave equation with fractional noise in time and white noise in space

This article is published on ESAIM : PS Volume 25, 2021 Page(s) 220 - 257. DOI : https ://doi.org/10.1051/ps/2021009

Joint work with Ciprian Tudor

Abstract Via Malliavin calculus, we analyze the limit behavior in distribution of the spatial wavelet variation for the solution to the stochastic linear wave equation with fractional Gaussian noise in time and white noise in space. We propose a wavelet-type estimator for the Hurst parameter of the this solution and we study its asymptotic properties.

2.1 Introduction

In mathematical statistics, the parameter estimation for stochastic (partial) differential equations constitutes a topic of wide interest (see, among many others, the monographs or surveys [69], [40] or [89]). In the last decades, the statistical inference for stochastic models driven by fractional Brownian motion and related processes also became a popular topic, due to the developments of the stochastic calculus for fractional processes (see, again among many others, [67], [90], [102]). A common characteristic of the above mentioned references is that they analyze estimators for the drift parameter or for the diffusion coefficient for standard fractional stochastic (partial) differential equations and very few works studied the problem of the estimation of the Hurst parameter of the driving noise (see [92], [66], [98]).

In our work, we will consider the linear stochastic wave equation (2.1) driven by a fractional-white Gaussian noise (i.e. a Gaussian noise that behaves as a fractional Brownian motion in time and as a white noise in space) and we construct and analyze statistical estimators for the Hurst index of the solution, based on the discrete observations of the solution in space and time. The stochastic partial differential equation (2.1) constitutes a model for an infinite vibrating string (under an ideal context, with uniform mass, neglecting the air resistance etc) perturbed by a random force which behaves as a fractional Brownian motion in time and as a Wiener process in space. For related works on the stochastic wave equation, we refer, among many others, to [23], [45], [100]. The value $u(t,x)$ modelizes the vertical displacement from the x-axis of the string at time t and at position x (in a coordinate system with x on the horizontal line and u on the vertical line). The displacement of the string is clearly affected by the random force and in particular by its Hurst parameter H. This influence of the Hurst parameter appears in several aspects, such as the probability distribution of the solution to (2.1) or the regularity of its sample paths. Indeed, for fixed $x \in \mathbb{R}$, the process u is self-similar of order $H+\frac{1}{2}$ in time and its paths are Hölder continuous of order $\delta \in (0,H)$ in space and the same Hölder continuity holds with respect to the time variable (see e.g. [100]). The Hurst parameter also characterizes other properties of the solution, such as the hitting times, the Hausdorff dimension or the regularity of its local times (see e.g. [47]). Therefore, the estimation of this parameter is of interest.

We propose a wavelet-type estimator defined via the decomposition of the observed process in a wavelet basis. The wavelet estimators have been intensively used in order to identify the Hurst paramter of the fractional Brownian motion and related processes (see e.g. [18], [25], [27], [51], [75]). Such estimators have in general several advantages : they are robust and computationally efficient, they are based on the log-log regression of the empirical variance onto several scales and this regression is useful for goodness-of-fit of the model, they offer flexibility on the choice of the wavelet basis etc.

Let $(u(t,x), t \geq 0, x \in \mathbb{R})$ be the solution to the wave equation with fractional-white additive noise. Here we used a wavelet decomposition of the solution to the wave equation (2.1) with respect to its space variable by assuming that the time variable is fixed. That is, we consider a "mother wavelet" Ψ with Q vanishing moments ($Q \geq 1$) and we define the wavelet coefficient $d(t,a,i) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \Psi\left(\frac{x}{a}\right) u(t,x)dx$ with $t > 0$ fixed and the scale $a > 0$. The wavelet variation, denoted $V_N(t,a)$ in the sequel, is defined by (2.10) by taking the sum of the centered and renormalized squared wavelet coefficients. By analyzing the asymptotic behavior of the wavelet variation $V_N(t,a)$ as $N \to \infty$, we are able to construct, via a log-log regression of the empirical variance onto several scales, an estimator for the Hurst parameter of the solution to (2.1) and to analyze its asymptotic behavior. The asymptotic behavior of the estimator is strongly connected to the asymptotic behavior of the wavelet variation $V_N(a)$. The time t also plays a role. For practical purposes, it would be convenient to estimate H by assuming that the solution is observed at a fixed time and at discrete points in space. On the other hand, as we will notice later, in the case of fixed time the empirical variance does not behave as a power function whose exponent is a linear function of H and the log-log regression argument cannot be applied. The relation between the wavelet variance and the Hurst index is more complex and we construct our estimator by analyzing this connection.

The techniques that we use to study the limit behavior in distribution of the wavelet variation are based on the Malliavin calculus and Stein method. We employ the recent Stein-Malliavin theory (see e.g. [81]) in order to prove that this sequence satisfies a Central Limit Theorem (CLT) and to derive the rate of convergence for this limit theorem. As mentioned above, we distinguish two situations : when the time t varies with N (i.e $t = N^{\beta}$ with $\beta > 0$) or when the time t is fixed (and in this case we restrict to the case of the Haar wavelet). We will see that in these two situations, the behavior of the wavelet variation is pretty different, although it always satisfies a CLT (with a different rate of convergence). We deduce the limit behavior of the associated Hurst parameter estimators, via a log-log regression of the empirical variance. We also notice that we use *spatial* wavelet variation to estimate the Hurst parameter of the solution, although this parameter appears in the time covariance of the noise and it characterizes the self-similarity of the solution in time.

We organized our paper in the following way : Section 2 contains some preliminaries on the wave equation with fractional-colred noise and on wavelets. In Section 3 we derive the correlation structure of the wavelet coefficients while in Section 4 we analyze the magnitude of the L^2 -norm of the wavelet variation. In Section 5 we included the proof of the Central Limit Theorem for the wavelet variation as well as the Berry-Essén bound for this limit theorem. Section 6 is devoted to discretized of the wavelet variation and the construction and the asymptotic study of the wavelet-type estimator for the Hurst parameter of the solution to the stochastic wave equation.

2.2 Preliminaries

Let us start by presenting some basic facts on the solution to the wave equation with additive fractional-colored noise and on the wavelet analysis.

2.2.1 The solution to the wave equation

Let $(u(t,x), t \geq 0, x \in \mathbb{R}^n)$ be the solution to the wave equation with fractional-white noise

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + \dot{W}^H(t,x), \quad t \in (0,T], \ T>0, \ x \in \mathbb{R}^n \\ u(0,x) &= 0, \quad x \in \mathbb{R}^n \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

Here Δ is the Laplacian on $\mathbb{R}^n$, $n \geq 1$ and $W^H = \{W_t^H(A); \ t \in [0,T], \ A \in \mathcal{B}_b(\mathbb{R}^n)\}$ is a real valued centered Gaussian field, over a given complete filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$, whose covariance function is

$$\mathbf{E}(W_t^H(A)W_s^H(B)) = R^H(t,s)\lambda(A \cap B), \text{ for every } t,s \geq 0, A, B \in \mathfrak{B}_b(\mathbb{R}^n), \tag{2.2}$$

where λ is the d-dimensional Lebesgue measure, $\mathfrak{B}_b(\mathbb{R}^n)$ is the set of the λ-bounded Borel subsets of $\mathbb{R}^n$ and R^H is the covariance function of the fBm with Hurst parameter $H \in (0,1)$ given by

$$R^H(t,s) := \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big), \quad s,t \geq 0. \tag{2.3}$$

Throughout this work, we will assume $H \in \left(\frac{1}{2}, 1\right)$.

The solution of the equation (2.1) is understood in the mild sense, that is, it is defined as a square-integrable centered field $u = (u(t,x);\ t \in [0,T], x \in \mathbb{R}^n)$ defined by

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y) W^H(\mathrm{d}s, \mathrm{d}y), \quad t \geq 0, x \in \mathbb{R}^n, \tag{2.4}$$

where G_1 is the fundamental solution to the wave equation and the integral in (2.4) is a Wiener integral with respect to the Gaussian process W^H. Recall that for $n = 1$ (we will later restrict to this situation in our work) we have, for every $t \geq 0$ and $x \in \mathbb{R}$,

$$G_1(t,x) = \frac{1}{2}\mathbb{1}_{\{|x|<t\}}. \tag{2.5}$$

We refer to e.g. [45] (when $H = \frac{1}{2}$) and to e.g. [23] (for $H \in \left(\frac{1}{2}, 1\right)$) for the definition and basic properties of the solution. The solution (2.4) is well-defined in dimension $n = 1$ for every $H \in \left(\frac{1}{2}, 1\right)$ (see e.g. [100]) and we have an explicit formula for its spatial covariance which will be a key ingredient in our study (see [66])

$$\begin{aligned}\mathbf{E}\left(u(t,x)u(t,y)\right) &= \frac{1}{2}\left(c_H|y-x|^{2H+1} - t\frac{|y-x|^{2H}}{2} + \frac{t^{2H+1}}{2H+1}\right)1_{\{|y-x|<t\}} \\ &\quad + \frac{(2t-|y-x|)^{2H+1}}{8(2H+1)}1_{\{t\leq|y-x|<2t\}}\end{aligned} \tag{2.6}$$

with $c_H = \frac{4H-1}{4(2H+1)}$. When $t > 1$ and $|x-y| \leq 1$, this expression reduces to

$$\mathbf{E}\left(u(t,x)u(t,y)\right) = \frac{1}{2}\left(c_H|y-x|^{2H+1} - t\frac{|y-x|^{2H}}{2} + \frac{t^{2H+1}}{2H+1}\right). \tag{2.7}$$

We notice that the solution is stationary in space while it has a scaling property in time (it is actually self-similar in time of order $H + \frac{1}{2}$). The sample paths of the solution are Hölder continuous in time and in time of order $\delta \in (0, H)$ (see e.g. [100]).

2.2.2 Wavelets

Let Ψ be a continuous function with support in $[0,1]$ such that its first Q moments vanish i.e. there exists and integer $Q \geq 1$ such that

$$\int_{\mathbb{R}} t^p \Psi(t) dt = 0 \text{ for } p = 0, 1, .., Q-1 \text{ and } \int_{\mathbb{R}} t^Q \Psi(t) dt \neq 0. \tag{2.8}$$

The function Ψ is usually called *mother wavelet*. Define for $a > 0$, $i = 1, .., N_a$ (with $N_a = [N/a] - 1$)

$$d(t,a,i) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \Psi\left(\frac{x}{a} - i\right) u(t,x)dx = \sqrt{a} \int_{\mathbb{R}} \Psi(x) u(t, a(x+i))dx \tag{2.9}$$

and

$$\tilde{d}(t,a,i) = \frac{d(t,a,i)}{(\mathbf{E}(d(t,a,i))^2)^{\frac{1}{2}}}.$$

Also define *the wavelet variation* in space of the solution (2.4) by

$$V_N(t,a) = \frac{1}{N_a} \sum_{i=1}^{N_a} \left(\tilde{d}(t,a,i)^2 - 1 \right). \tag{2.10}$$

We will study the asymptotic behavior, as $N_a \to \infty$, of the wavelet variation $V_N(t,a)$. In applications, the parameter a, which is called *scale*, will depend on N and it is usually assumed that $a = a_N \to_{N\to\infty} \infty$.

Given the covariance of the solution to the wave equation (see formula (2.6)), it is clear that the time t will play an important role, depending on its position with respect to the spatial increment $|x - y|$.

We will consider two situations : the *fixed time case,* i.e. the time $t > 0$ is fixed, and the *moving time case,* when the time depends on N and it tends to infinity as $N \to \infty$. The first situation would be more convenient for applications to parameter estimation, since it means that the solution is observed only at a fixed time. Nevertheless, in this case the wavelet variation does not provide an explicit estimator since the usual log-log regression procedure to construct an wavelet estimator based on $V_N(t,a)$ leads to a more complicated equation in H. A slightly different argument is then used for fixed time.

We will start with the moving time situation. We will assume

$$a = a_N = N^\alpha \text{ with } 0 < \alpha < 1 \text{ and } t = t_N = N^\beta \text{ with } \beta \geq 1. \tag{2.11}$$

The choice of such time t will be explained later, it allows to reduce the expression of the correlation of the wavelet coefficients. Then, we will consider the situation when the time is fixed, i.e. we suppose

$$a = a_N = N^\alpha \text{ with } 0 < \alpha < 1 \text{ and } t > 0 \text{ is fixed.} \tag{2.12}$$

In this second case, in order to have a precise estimate on the wavelet coefficient and on the empirical variance $\mathbf{E}V_N(t,a)$, we need to restrict to a particular case of wavelet system (the Haar wavelet).

2.3 Main results

In this section we will state our main theoretical results. Their proofs are postponed to Section 2.4. These results give the asymptotic behavior as $N \to \infty$ of the wavelet variation $V_N(t,a)$ given by (2.10) as well as the limit behavior in distribution of the renormalized wavelet variation. We will show that, in both moving time and fixed time cases, the magnitude of the variance of $V_N(t,a)$ as $N \to \infty$ is the same and the renormalized wavelet variation satisfies a Central Limit Theorem. We also evaluate the rate of convergence to the normal distribution, which varies in the two cases under consideration.

2.3.1 The moving time case

Let us start by treating the situation when the time t depends on N, i.e. we assume (2.11). In this case, we obtain the following renormalization of the wavelet variation.

Proposition 12 *Let $V_N(t,a)$ be given by (2.10). Assume $Q \geq 2$ or $Q = 1, H < \frac{3}{4}$. Let a_N, t_N be given by (2.11). Then*

$$N^{1-\alpha}\mathbf{E}V_N(t_N,a_N)^2 \to_{N\to\infty} \frac{2}{K^2_{\Psi,H}} \sum_{k\in\mathbb{Z}} g_H(k)^2 := K_{0,\Psi,H} \tag{2.13}$$

with g_H given by

$$g_H(k) = \int_{\mathbb{R}}\int_{\mathbb{R}} dxdy\Psi(x)\Psi(y)|x-y+k|^{2H}. \tag{2.14}$$

and $K_{\Psi,H}$ given by, for $H \in \left(\frac{1}{2}, \frac{3}{2}\right)$

$$K_{\Psi,H} = -\int_{\mathbb{R}}\int_{\mathbb{R}} dxdy\Psi(x)\Psi(y)|x-y|^{2H}. \tag{2.15}$$

Notice that the above integral (2.15) is finite because the support of the mother wavelet Ψ is included in the interval $[0,1]$ and $2H > 0$. We assume, as in [25], that $K_{\Psi,H} > 0$ (which is satisfied by a large choice of the mother wavelet Ψ). The results in Section 2.4 show also that the series in the right-hand side of (2.13) is convergent.

Let us denote, for every $N \geq 1$

$$F_N = K_{0,\Psi,H}^{-\frac{1}{2}} N^{\frac{1-\alpha}{2}} V_N(t_N,a_N) \tag{2.16}$$

with $V_N(t_N,a_N)$ defined in (2.10), $K_{0,\Psi,H}$ from (2.13) and suppose that assumption (2.11) is verified. From Proposition 12

$$\mathbf{E}F_N^2 \to_{N\to\infty} 1.$$

We will obtain the following result. We denote below by c, C generic strictly positive constants that may change from line to line. By d we denote the distance between distributions of random variable and below it can be each of the following distances : Kolmogorov, total variation, Wasserstein or Fortet-Mourier (see [81]).

Theorem 1 *Let F_N be given by (2.16). Then the sequence $(F_N)_{N\geq 1}$ converges in distribution to a standard normal random variable $Z \sim N(0,1)$ and*

$$d(F_N, Z) \leq cN^{\frac{\alpha-1}{2}}.$$

We can also prove a multidimensional central limit theorem for the wavelet variation considered at different scales. This will be used in order to estimate the Hust parameter of the solution to the wave equation in the next section.

Theorem 2 *Let $V_N(t,a)$ be given by (2.10) and assume (2.11). Let $d \geq 1$. Then the d-dimensional random vector $\left(N^{\frac{1-\alpha}{2}} V_n(t_N, La_N)\right)_{L=1,..,d}$ converges in distribution, as $N \to \infty$, to a centered d-dimensional Gaussian vector with covariance matrix $(\Gamma_{L_1,L_2})_{L_1,L_2=1,..,d}$ where*

$$\Gamma_{L_1,L_2} = \frac{32}{K^2_{\Psi,H}} \frac{1}{(L_1L_2)^{2H+1}} C(L_1, L_2, H) \tag{2.17}$$

with $C(L_1, L_2, H)$ given by

$$C(L_1, L_2, H) = \lim_{N\to\infty} N^{1-\alpha} \sum_{i=1}^{N_{L_1 a_N}} \sum_{j=1}^{N_{L_2 a_N}} (g_{L_1,L_2,H}(i,j))^2 \tag{2.18}$$

where

$$g_{L_1,L_2,H}(i,j) = \int_{\mathbb{R}} \int_{\mathbb{R}} dxdy \Psi(x)\Psi(y)|L_1x - L_2y + L_1 - L_2j|^{2H}.$$

It follows from our proofs in Section 2.4 that the right-hand side of (2.18) converges.

2.3.2 The fixed time case

If t is fixed, we can prove the following approximation result for the variance of the wavelet variation.

Proposition 13 *If $V_N(t,a)$ is given by (2.10) and (2.12), (2.30) hold true, we have for every $t > 0$*

$$N^{1-\alpha}\mathbf{E}V_N(t, a_N)^2 \to_{N\to\infty} 2. \tag{2.19}$$

By Proposition 13, we have the following renormalization of the wavelet variation

$$G_N =: \frac{1}{\sqrt{2}} N^{\frac{1-\alpha}{2}} V(t, a_N), \tag{2.20}$$

i.e. $\mathbf{E}G_N^2 \to_{N\to\infty} 1$. We will show below that the renormalized wavelet variation satisfies a CLT also when the time is fixed.

Theorem 3 *The sequence* $(G_N)_{N\geq 1}$ *given by (2.20) converges in distribution to* $Z \sim N(0,1)$ *and for* N *large enough*

$$d(G_N, Z) \leq C\left(\frac{1}{N^{\frac{1-\alpha}{2}}} + \frac{1}{N^{2\alpha}}\right).$$

Let us make a short discussion around the above statements.

Remark 1
- *We notice that the renormalization of (2.10) is of the same order in both cases (fixed time or moving time) although, as we will see in Section 2.4, the correlation structure of the wavelet coefficient is different.*
- *The wavelet variation (2.10) satisfies a CLT both in the moving or fixed time cases. On the other hand, the behavior of this sequence is pretty different in these two cases. While for fixed time, this sequence basically behaves as a sum of independent random variables (see also Remark 1), in the moving time case there is a non-trivial correlation between all the summands that compose* $V_N(t,a)$.
- *The rate of convergence of the sequence (2.20) to the normal distribution varies upon* $\alpha \in (0,1)$ *: when* $\alpha \in \left(0, \frac{1}{5}\right)$, *we have* $d(G_N, Z) \leq c\frac{1}{N^{2\alpha}}$ *while for* $\alpha \in \left(\frac{1}{5}, 1\right)$, *one has* $d(G_N, Z) \leq c\frac{1}{N^{\frac{1-\alpha}{2}}}$. *Theorem 3 also suggests that if the scale* a *is constant (i.e.* $\alpha = 0$*) the sequence* $V_N(t,a)$ *does not satisfy a CLT.*

2.4 Proofs

This part contains the proofs of the theoretical results stated in Section 2.3.

2.4.1 The correlation structure of the wavelet coefficient

The behavior of the wavelet variation (2.10) will depend on the behavior of the variance of the wavelet coefficient $\mathbf{E}d(t,a,i)^2$ and of the correlation between the wavelet coefficients, i.e. $\mathbf{E}d(t,a,i)d(t,a,j)$ with $i \neq j$. We will start by analyzing the behavior of these quantities in both cases (2.11) and (2.12).

Let $d(t,a,i)$ be given by (2.9) with $t > 0, a > 0$ and $i = 1, .., N_a$. We will use the following notation throughout our work

$$D(t,a) := \mathbf{E}d(t,a,i)^2 \tag{2.21}$$

for every $t > 0, a > 0$ and $i = 1, .., N_a$. Notice that, due to the stationarity of the process $(u(t,x), x \in \mathbb{R})$, the quantity $\mathbf{E}d(t,a,i)^2$ does not depend on i.

Let $t > 0, a > 0$. For every $i, j = 1, ..., N_a$ we have from the covariance formula (2.6)

$$\mathbf{E}d(t,a,i)d(t,a,j) = a\int_{\mathbb{R}}\int_{\mathbb{R}} dxdy\Psi(x)\Psi(y)\mathbf{E}u(t,a(x+i))u(t,a(x+j)) \tag{2.22}$$

$$= a \int_{\mathbb{R}} \int_{\mathbb{R}} dxdy \Psi(x)\Psi(y) \left[\frac{c_H}{2} a^{2H+1} |x-y+i-j|^{2H+1} \right.$$
$$\left. - \frac{t}{4} a^{2H} |x-y+i-j|^{2H} + \frac{t^{2H+1}}{2(2H+1)} \right] 1_{\{|x-y+i-j|<\frac{t}{a}\}}$$
$$+ a \int_{\mathbb{R}} \int_{\mathbb{R}} dxdy \Psi(x)\Psi(y) \frac{(2t-a|x-y+i-j|)^{2H+1}}{8(2H+1)} 1_{\{\frac{t}{a} \leq |x-y+i-j| < 2\frac{t}{a}\}}.$$

We will see below that the above expression will simplify under assumption (2.11).

2.4.2 The moving time case

First, we assume that we work under the assumption (2.11). We start by studying the variance of the wavelet coefficient. Let us recall the notation $K_{\Psi,H}$ from (2.15).

We have the following result.

Lemma 1 *Assume (2.11). Consider the wavelet coefficient $d(t,a,i)$ defined by (2.9) and its variance $D(t,a)$ given by (2.21). Then*

$$\frac{1}{N^{\beta+(2H+1)\alpha}} D(t_N, a_N) \rightarrow_{N\to\infty} \frac{1}{4} K_{\Psi,H}$$

with $K_{\Psi,H}$ from (2.15).

Proof : From the assumption (2.11) and the property (2.8) of the function Ψ, using also $|x-y| \leq 1$ (which implies that $|x-y+i-j| \leq t_N = N^\beta$ by (2.11)), the last two summands in (2.22) vanish and we obtain

$$\mathbf{E} d(t_N, a_N, i) d(t_N, a_N, j) \quad (2.23)$$
$$= a_N \int_{\mathbb{R}} \int_{\mathbb{R}} dxdy \Psi(x)\Psi(y) \left[\frac{c_H}{2} a_N^{2H+1} |x-y+i-j|^{2H+1} - a_N^{2H} \frac{t_N}{4} |x-y+i-j|^{2H} \right].$$

Let us take $i = j$ in (2.23). We have

$$D(t_N, a_N) = \mathbf{E} d(t_N, a_N, i)^2 = -\frac{c_H}{2} K_{\Psi,H+\frac{1}{2}} N^{(2H+2)\alpha} + \frac{1}{4} K_{\Psi,H} N^{\beta+(2H+1)\alpha}. \quad (2.24)$$

Since $\beta + (2H+1)\alpha > (2H+2)\alpha$ (because $\beta > \alpha$) we obtain the conclusion. ∎

Let us now study the correlation (2.23) with $i \neq j$. We can write

$$\mathbf{E} d(t_N, a_N, i) d(t_N, a_N, j) = \frac{c_H}{2} a_N^{2H+2} g_{H+\frac{1}{2}}(i-j) - \frac{t_N}{4} a_N^{2H+1} g_H(i-j) \quad (2.25)$$

with the notation $g_H(k)$ from (2.14). Notice that for every $k \in \mathbb{Z}$ we have $g_H(k) = g_H(-k)$ for $k \in \mathbb{Z}$. The analysis of the quantity $g_H(k)$ for k large, will give the asymptotics of the correlation (2.25). Recall that the integer $Q \geq 1$ is fixed by (2.8).

Lemma 2 *Let g_H be given by (2.14). Then for k large enough, we have for every $H \in \left(\frac{1}{2}, \frac{3}{2}\right)$*

$$|g_H(k)| \leq C_{\Psi,H,Q} k^{4H-4Q}$$

where $C_{\Psi,H,Q}$ is a strictly positive constant not depending on k.

Proof : Using the following asymptotic expansion at $z = 0$

$$(1+z)^{2H} = 1 + 2Hz + ... + \frac{2H(2H-1)...(2H-2Q)}{(2Q-1)!} z^{2Q-1} + C_{H,Q}(1+\theta_z)^{2H-2Q} z^{2Q}$$

where θ_z is a point located between 0 and z, we can write, for k large enough, if $C_{H,Q}$ is a constant depending only of H and Q,

$$\begin{aligned} g_H(k) &= k^{2H} \int_{\mathbb{R}} \int_{\mathbb{R}} dxdy \Psi(x)\Psi(y) \left(1 + \frac{x-y}{k}\right)^{2H} \\ &= C_{H,Q} k^{2H} \int_{\mathbb{R}} \int_{\mathbb{R}} dxdy \Psi(x)\Psi(y) \left(\frac{x-y}{k}\right)^{2Q} (1+\theta_{x,y,k})^{2H-2Q} \end{aligned}$$

where we used (2.8) and we denoted by $\theta_{x,y,k}$ a point located between 0 and $\frac{x-y}{k}$. Since $|x-y| \leq 1$, we have for $k \geq 2$

$$\frac{1}{2} \leq |1 + \theta_{x,y,k}| \leq \frac{3}{2}.$$

We deduce that , for k large

$$|g_H(k)| \leq C_{H,Q} 2^{2Q-2H} k^{2H-2Q} \int_{\mathbb{R}} \int_{\mathbb{R}} dxdy |\Psi(x)\Psi(y)||x-y|^{2Q} = C_{\Psi,H,Q} k^{2H-2Q}$$

using the fact that the support of Ψ is included in the interval $[0, 1]$. ■

Lemma 3 *Let g_H be given by (2.14). Denote, for $a > 0$ and $N \geq 1$*

$$g_{N,H}(a) = \sum_{i,j=1}^{N_a} g_H(i-j)^2. \tag{2.26}$$

Then, for every $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$ (if $Q = 1$) and for every $H \in \left(\frac{1}{2}, 1\right)$ (if $Q \geq 2$)

$$\frac{1}{N_a} g_{N,H}(a) \to_{N_a \to \infty} \sum_{k \in \mathbb{Z}} g_H(k)^2. \tag{2.27}$$

Moreover, for every $H \in \left(\frac{1}{2}, 1\right)$ and for every $Q \geq 1$, for N large enough

$$\frac{1}{N_a} |g_{N,H+\frac{1}{2}}(a)| \leq \begin{cases} C_{\Psi,H,Q} & \text{if } Q \geq 2, \\ C_{\Psi,H,Q} N_a^{4H-1} & \text{if } Q = 1 \end{cases} \tag{2.28}$$

and

$$\frac{1}{N_a} \left| \sum_{i,j=1}^{N_a} g_H(i-j) g_{H+\frac{1}{2}}(i-j) \right| \leq \begin{cases} C_{\Psi,H,Q} & \text{if } Q \geq 2, \\ C_{\Psi,H,Q} N_a^{4H-2} & \text{if } Q = 1. \end{cases} \tag{2.29}$$

Proof : We can write

$$\frac{1}{N_a} g_{N,H}(a) = \sum_{k\in\mathbb{Z}} g_H(k)^2 1_{\{|k|\le N_a\}} \frac{N_a - |k|}{N_a}.$$

By the dominated convergence theorem and Lemma 2 we clearly have

$$\frac{1}{N_a} g_{N,H}(a) \to_{N_a\to\infty} \sum_{k\in\mathbb{Z}} g_H(k)^2.$$

Note that the series $\sum_{k\in\mathbb{Z}} g_H(k)^2$ is convergent due to Lemma 2. Now

$$\begin{aligned} \frac{1}{N_a} |g_{N,H+\frac{1}{2}}(a)| &= \left| \sum_{k\in\mathbb{Z}} g_{H+\frac{1}{2}}(k)^2 1_{\{|k|\le N_a\}} \frac{N_a - |k|}{N_a} \right| \\ &\le \sum_{|k|\le N_a} g_{H+\frac{1}{2}}(k)^2 \le C \sum_{|k|\le N_a} k^{4H+2-4Q} \end{aligned}$$

again by Lemma 2. The series $\sum_{k\in\mathbb{Z}} k^{4H+2-4Q}$ is convergent when $Q \ge 2$ and for $Q = 1$ and $H > \frac{1}{2}$, the sequence $\sum_{|k|\le N_a} k^{4H+2-4Q}$ behaves as $C_{H,Q} N_a^{4H-1}$. This implies the estimate (2.28). A similar argument gives (2.29), because from Lemma 2

$$\begin{aligned} \frac{1}{N_a} \left| \sum_{i,j=1}^{N_a} g_H(i-j) g_{H+\frac{1}{2}}(i-j) \right| &= \left| \sum_{k\in\mathbb{Z}} g_H(k) g_{H+\frac{1}{2}}(k) 1_{\{|k|\le N_a\}} \frac{N_a - |k|}{N_a} \right| \\ &\le C_{\Psi,H,Q} \sum_{|k|\le N_a} k^{4H+1-4Q}. \end{aligned}$$

■

2.4.2.1 The fixed time case

Let us assume $t > 0$ is fixed, i.e. we assume (2.12). As before, we use the notation

$$D(t, a_N) = \mathbf{E} d(t, a_N, i)^2$$

for $i = 1, .., N_{a_N}$, with $a_N = N^\alpha$, $0 < \alpha < 1$. We start by estimating the behavior of $D(t, a_N)$ as $N \to \infty$. It is impossible to get the exact behavior of this quantity for an arbitrary function Ψ. Therefore, in the sequel we will choose the function Ψ to be the mother wavelet of the Haar system, i.e.

$$\Psi(x) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{2.30}$$

Proposition 14 *Let Ψ be given by (2.30) and assume (2.12). For every $t > 0$ and for N large enough*

$$D(t, a_N) = K_{1,t}(H) + K_{2,t}(H)\frac{1}{N^\alpha}$$

with

$$K_{1,t}(H) = \frac{1}{2(H+1)} t^{2H+2}. \tag{2.31}$$

and $K_{2,t}(H) = \sum_{j=1}^{4} K_{j,2,t}(H)$ for $K_{j,2,t}(H)$, $j = 1,..,4$ given by (2.38), (2.39), (2.40) and (2.41). In particular,

$$D(t, a_N) \to_{N\to\infty} K_{1,t}(H).$$

Proof : From (2.22) we have

$$D(t, a_N) = I_{1,t,N} + I_{2,t,N} + I_{3,t,N} + I_{4,t,N}$$

with

$$I_{1,t,N} = \frac{c_H}{2} N^{\alpha(2H+2)} A_{H+\frac{1}{2},N}, \quad I_{2,t,N} = -\frac{t}{4} N^{\alpha(2H+1)} A_{H,N}, \tag{2.32}$$

$$I_{3,t,N} = \frac{t^{2H+1}}{2(2H+1)} N^\alpha \int_{\mathbb{R}}\int_{\mathbb{R}} dxdy \Psi(x)\Psi(y) 1_{\{|x-y|<\frac{t}{N^\alpha}\}} \tag{2.33}$$

$$I_{4,t,N} = N^\alpha \frac{1}{8(2H+1)} B_{H,N} \tag{2.34}$$

where we used the notation

$$A_{H,N} := \int_{\mathbb{R}}\int_{\mathbb{R}} dxdy \Psi(x)\Psi(y)|x-y|^{2H} 1_{\{|x-y|<\frac{t}{N^\alpha}\}} \tag{2.35}$$

and

$$B_{H,N} = \int_0^1\int_0^1 dxdy \Psi(x)\Psi(y)\left(2t - N^\alpha|x-y|\right)^{2H+1} 1_{\{\frac{t}{N^\alpha}\le|x-y|<2\frac{t}{N^\alpha}\}}. \tag{2.36}$$

To obtain the speed of convergence of $I_{1,t,N}$ and $I_{2,t,N}$, we need to study the sequence $A_{H,N}$ defined by (2.35). Clearly, $A_{H,N}$ converges to zero as $N \to \infty$ but we need to analyze how fast this sequence goes to zero. We have

$$\begin{aligned}
A_{H,N} &= 2\int_0^1 dx \int_0^x dy \Psi(x)\Psi(y)(x-y)^{2H} 1_{\{x-y<\frac{t}{N^\alpha}\}} \\
&= 2\int_0^1 dx \int_{(x-tN^{-\alpha})\vee 0}^x dy \Psi(x)\Psi(y)(x-y)^{2H} \\
&= 2\int_0^{tN^{-\alpha}} dx \int_0^x dy \Psi(x)\Psi(y)(x-y)^{2H} + 2\int_{tN^{-\alpha}}^1 dx \int_{x-tN^{-\alpha}}^x dy \Psi(x)\Psi(y)(x-y)^{2H}.
\end{aligned}$$

Let us chose N large enough such that

$$\frac{t}{N^\alpha} < \frac{1}{2}.$$

We will have, with Ψ from (2.30),

$$\begin{aligned} A_{H,N} &= 2\int_0^{tN^{-\alpha}} dx \int_0^x dy(x-y)^{2H} + 2\int_{tN^{-\alpha}}^{\frac{1}{2}} dx \int_{x-tN^{-\alpha}}^x dy(x-y)^{2H} \\ &\quad -2\int_{\frac{1}{2}}^1 dx \int_{x-tN^{-\alpha}}^x dy\Psi(y)(x-y)^{2H} \end{aligned}$$

and by separating the integral dy in the last term above upon $x = tN^{-\alpha}$ less or bigger than one-half we will obtain

$$\begin{aligned} A_{H,N} &= 2\int_0^{tN^{-\alpha}} dx \int_0^x dy(x-y)^{2H} + 2\int_{tN^{-\alpha}}^{\frac{1}{2}} dx \int_{x-tN^{-\alpha}}^x dy(x-y)^{2H} \\ &\quad -2\int_{\frac{1}{2}}^{\frac{1}{2}+tN^{-\alpha}} dx \int_{x-tN^{-\alpha}}^{\frac{1}{2}} dy(x-y)^{2H} + 2\int_{\frac{1}{2}}^{\frac{1}{2}+tN^{-\alpha}} dx \int_{\frac{1}{2}}^x dy(x-y)^{2H} \\ &\quad +2\int_{\frac{1}{2}+tN^{-\alpha}}^1 dx \int_{x-tN^{-\alpha}}^x dy(x-y)^{2H}. \end{aligned}$$

This gives

$$\begin{aligned} A_{H,N} &= \frac{2}{2H+1}\left[\frac{1}{2H+2}\left(\frac{t}{N^\alpha}\right)^{2H+2} + \left(\frac{t}{N^\alpha}\right)^{2H+1}\left(\frac{1}{2}-\frac{t}{N^\alpha}\right) - \left(\frac{t}{N^\alpha}\right)^{2H+1}\frac{t}{N^\alpha}\right. \\ &\quad \left. +\frac{1}{2H+2}\left(\frac{t}{N^\alpha}\right)^{2H+2} + \frac{1}{2H+2}\left(\frac{t}{N^\alpha}\right)^{2H+2} + \left(\frac{t}{N^\alpha}\right)^{2H+1}\left(\frac{1}{2}-\frac{t}{N^\alpha}\right)\right] \\ &= \frac{2}{2H+1}\left[3\frac{1}{2H+2}\left(\frac{t}{N^\alpha}\right)^{2H+2} + \left(\frac{t}{N^\alpha}\right)^{2H+1}\left(1-3\frac{t}{N^\alpha}\right)\right] \\ &= -\frac{6}{2H+2}\left(\frac{t}{N^\alpha}\right)^{2H+2} + \frac{2}{2H+1}\left(\frac{t}{N^\alpha}\right)^{2H+1}. \end{aligned} \tag{2.37}$$

Consequently, we obtain from (2.37) the following behavior for the summand $I_{1,t,N}$ in (2.32)

$$I_{1,t,N} = \frac{c_H}{2}N^{\alpha(2H+2)}A_{H+\frac{1}{2},N} = K_{1,1,t}(H) + K_{1,2,t}(H)\frac{1}{N^\alpha} \tag{2.38}$$

with

$$K_{1,1,t}(H) = \frac{c_H}{2H+2}t^{2H+2} \text{ and } K_{1,2,t}(H) = \frac{-3c_H}{2H+3}t^{2H+3}.$$

The second summand $I_{2,t,N}$ gives, using (2.37)

$$I_{2,t,N} = -\frac{t}{4}N^{\alpha(2H+1)}A_{H,N} = K_{2,1,t}(H) + K_{2,2,t}(H)\frac{1}{N^\alpha} \tag{2.39}$$

with

$$K_{2,1,t}(H) = -\frac{1}{2(2H+1)}t^{2H+2} \text{ and } K_{2,2,t}(H) = \frac{3}{2(2H+2)}t^{2H+3}.$$

Let us now calculate the term $I_{3,t,N}$ defined in (2.34). We can write

$$I_{3,t,N} = N^{\alpha}\frac{t^{2H+1}}{2(2H+1)}2\int_0^1 dx \int_0^x dy\Psi(x)\Psi(y)1_{\{x-y<\frac{t}{N^\alpha}\}}$$

and since (this is the same calculation as for $A_{H,N}$ without the factor $(x-y)^{2H}$)

$$2\int_0^1 dx \int_0^x dy\Psi(x)\Psi(y)1_{\{x-y<\frac{t}{N^\alpha}\}} = \frac{2t}{N^\alpha} - 3\left(\frac{t}{N^\alpha}\right)^2$$

we obtain

$$I_{3,t,N} = K_{3,1,t}(H) + K_{3,2,t}(H)\frac{1}{N^\alpha} \tag{2.40}$$

with

$$K_{3,1,t}(H) = \frac{1}{(2H+1)}t^{2H+2} \text{ and } K_{3,2,t}(H) = \frac{-3}{2(2H+1)}t^{2H+3}.$$

Let us regard the last summand $I_{4,t,N}$ in (2.34). With $B_{H,N}$ given by (2.36)

$$\begin{aligned}
B_{H,N} &= \int_0^1\int_0^1 dxdy\Psi(x)\Psi(y)\left(2t - N^\alpha|x-y|\right)^{2H+1} 1_{\{\frac{t}{N^\alpha}\le|x-y|<2\frac{t}{N^\alpha}\}}\\
&= 2\int_0^1\int_0^x dxdy\Psi(x)\Psi(y)\left(2t - N^\alpha(x-y)\right)^{2H+1} 1_{\{\frac{t}{N^\alpha}\le x-y<2\frac{t}{N^\alpha}\}}\\
&= 2\int_0^1 dx\int_{(x-2tN^{-\alpha})\vee 0}^{x-tN^{-\alpha}} dy\Psi(x)\Psi(y)\left(2t - N^\alpha(x-y)\right)^{2H+1}\\
&= 2\int_0^{2tN^{-\alpha}} dx\int_0^{x-tN^{-\alpha}} dy\Psi(x)\Psi(y)\left(2t - N^\alpha(x-y)\right)^{2H+1}\\
&\quad +2\int_{2tN^{-\alpha}}^1 dx\int_{x-2tN^{-\alpha}}^{x-tN^{-\alpha}} dy\Psi(x)\Psi(y)\left(2t - N^\alpha(x-y)\right)^{2H+1} := B_{1,H,N}+B_{2,H,N}.
\end{aligned}$$

We estimate separately the summands $B_{1,H,N}$ and $B_{2,H,N}$. First, notice that we can choose N large enough so that $\frac{t}{N^\alpha} < \frac{1}{4}$ and therefore $\frac{2t}{N^\alpha} < \frac{1}{2}$. We then get

$$B_{1,H,N} = 2\frac{t^{2H+3}}{(2H+2)N^{2\alpha}}(2 - \frac{2^{2H+3}}{2H+3})$$

while for $B_{2,H,N}$ we have

$$B_{2,H,N} = 2\int_{2tN^{-\alpha}}^{1/2} dx\int_{x-2tN^{-\alpha}}^{x-tN^{-\alpha}} dy\left(2t - N^\alpha(x-y)\right)^{2H+1}$$

$$
\begin{aligned}
& -2\int_{1/2}^{1/2+tN^{-\alpha}} dx \int_{x-2tN^{-\alpha}}^{x-tN^{-\alpha}} dy\,(2t-N^{\alpha}(x-y))^{2H+1} \\
& +2\int_{1/2+tN^{-\alpha}}^{1} dx \int_{x-2tN^{-\alpha}}^{x-tN^{-\alpha}} dy\,(2t-N^{\alpha}(x-y))^{2H+1} \\
= \; & 2\frac{t^{2H+2}}{N^{\alpha}(2H+2)}\left(1-\frac{4t}{N^{\alpha}}\right).
\end{aligned}
$$

By putting together the above computations, we obtain

$$
I_{4,t,N} := \frac{1}{8(2H+1)} N^{\alpha} B_{H,N} = K_{4,1,t}(H) + K_{4,2,t}(H)\frac{1}{N^{\alpha}} \tag{2.41}
$$

with

$$
K_{4,1,t}(H) = \frac{t^{2H+2}}{8(H+1)(2H+1)} \text{ and } K_{4,2,t}(H) = -\frac{t^{2H+3}}{8(H+1)(2H+1)}\left(2+\frac{2^{2H+3}}{2H+3}\right).
$$

From (2.38), (2.39), (2.40) and (2.41) we obtain the conclusion. In particular, concerning the constant $K_{1,t}(H)$ which is needed in the sequel

$$
K_{1,t}(H) = t^{2H+2}\left(\frac{c_H}{2H+2} - \frac{1}{2(2H+1)} + \frac{1}{2H+1} + \frac{1}{8(2H+1)(H+1)}\right) = \frac{1}{2H+2}t^{2H+2}
$$

by using the expression of c_H in (2.6). ■

We also need to analyze $\mathbf{E}d(t,a_N,i)d(t,a_N,j)$ when $|i-j|=1$. Only this correlation coefficient will be needed for the renormalization of the sequence (2.10).

Proposition 15 *Let $d(t,a,i)$ be given by (2.9) and assume (2.12) and (2.30). Then for every $t>0, N\geq 1$*

$$
\mathbf{E}d(t,a_N,i)d(t,a_N,i+1) = L_t(H)\frac{1}{N^{\alpha}}
$$

with $L_t(H)$ from (2.49).

Proof : We have

$$
\mathbf{E}d(t,a_N,i)d(t,a_N,j) = f_{H,N}(i-j)
$$

where (recall $a_N = N^{\alpha}$)

$$
\begin{aligned}
& f_{H,N}(k) \\
= \; & a_N \int_{\mathbb{R}}\int_{\mathbb{R}} dxdy\Psi(x)\Psi(y)\left[\frac{c_H}{2}a_N^{2H+1}|x-y+k|^{2H+1}\right. \\
& \left. -\frac{t}{4}a_N^{2H}|x-y+k|^{2H} + \frac{t^{2H+1}}{2(2H+1)}\right] 1_{\{|x-y+k|<\frac{t}{a_N}\}}
\end{aligned}
$$

$$+a_N\int_{\mathbb{R}}\int_{\mathbb{R}} dxdy\Psi(x)\Psi(y)\frac{(2t-a_N|x-y+k|)^{2H+1}}{8(2H+1)}1_{\{\frac{t}{a_N}\leq |x-y+k|<2\frac{t}{a_N}\}}. \quad (2.42)$$

Hence

$$\mathbf{E}d(t,a_N,i)d(t,a_N,i+1)=f_{H,N}(1).$$

We can write, via (2.22)

$$f_{H,N}(1)=J_{1,t,N}+J_{2,t,N}+J_{3,t,N}+J_{4,t,N}$$

with

$$J_{1,t,N}=\frac{c_H}{2}N^{\alpha(2H+2)}C_{H+\frac{1}{2},N},\quad J_{2,t,N}=\frac{-t}{4}N^{\alpha(2H+1)}C_{H,N} \quad (2.43)$$

$$J_{3,t,N}=N^{\alpha}\frac{t^{2H+1}}{2(2H+1)}\int_0^1\int_0^1 dxdy\Psi(x)\Psi(y)1_{\{x-y+1<\frac{t}{N^{\alpha}}\}} \quad (2.44)$$

$$J_{4,t,N}=N^{\alpha}\int_0^1\int_0^1 dxdy\Psi(x)\Psi(y)\frac{(2t-N^{\alpha}|x-y+1|)^{2H+1}}{8(2H+1)}1_{\{\frac{t}{a_N}\leq |x-y+1|<2\frac{t}{a_N}\}} \quad (2.45)$$

where

$$C_{H,N}=\int_0^1\int_0^1 dxdy\Psi(x)\Psi(y)(x-y+1)^{2H}1_{\{x-y+1<\frac{t}{N^{\alpha}}\}}.$$

We have, if N is such that $\frac{t}{N^{\alpha}}<\frac{1}{2}$,

$$\begin{aligned}
C_{H,N} &= \int_0^1 dx\int_{x+1-tN^{-\alpha}}^1 dy\Psi(x)\Psi(y)(x-y+1)^{2H}\\
&= -\int_0^{\frac{t}{N^{\alpha}}}\int_{x+1-tN^{-\alpha}}^1 dy(x-y+1)^{2H}\\
&= \left(\frac{1}{(2H+1)(2H+2)}-\frac{1}{2H+1}\right)\left(\frac{t}{N^{\alpha}}\right)^{2H+2}.
\end{aligned}$$

Therefore

$$J_{1,t,N}=K_{5,1,t}(H)\frac{1}{N^{\alpha}} \quad (2.46)$$

with

$$K_{5,1,t}(H)=\frac{c_Ht^{2H+3}}{2}\left(\frac{1}{(2H+2)(2H+3)}-\frac{1}{2H+2}\right).$$

For the second term $J_{2,t,N}$ in (2.43), it is immediate to see that

$$J_{2,t,N}=\frac{-t}{4}N^{\alpha(2H+1)}C_{H,N}=K_{6,1,t}(H)\frac{1}{N^{\alpha}} \quad (2.47)$$

with

$$K_{6,1,t}(H) = \frac{-t^{2H+3}}{4}\left(\frac{1}{(2H+1)(2H+2)} - \frac{1}{2H+1}\right).$$

The third summand (2.44) gives

$$\begin{aligned} J_{3,t,N} &= N^{\alpha}\frac{t^{2H+1}}{2(2H+1)}\int_0^1\int_0^1 dxdy\Psi(x)\Psi(y)1_{\{x-y+1<\frac{t}{N^{\alpha}}\}} \\ &= -N^{\alpha}\frac{t^{2H+1}}{2(2H+1)}\int_0^{\frac{t}{N^{\alpha}}} dx\int_{x+1-tN^{-\alpha}}^1 dy = \frac{t^{2H+3}}{4(2H+1)}\frac{1}{N^{\alpha}} =: K_{7,1,t}(H)\frac{1}{N^{\alpha}}. \end{aligned}$$

Finally, concerning the summand $J_{4,t,N}$ in (2.45), if $2t/N^{\alpha} < \frac{1}{2}$,

$$\begin{aligned} J_{4,t,N} &= N^{\alpha}\int_{\mathbb{R}}\int_{\mathbb{R}} dxdy\Psi(x)\Psi(y)\frac{(2t-N^{\alpha}|x-y+1|)^{2H+1}}{8(2H+1)}1_{\{\frac{t}{a_N}\leq|x-y+1|<2\frac{t}{a_N}\}} \\ &= -\frac{1}{8(2H+1)}N^{\alpha}\int_0^{tN^{-\alpha}} dx\int_{x+1-2tN^{-\alpha}}^{x+1-tN^{-\alpha}} dy(2t-N^{\alpha}(x-y+1))^{2H+1}. \end{aligned}$$

We obtain

$$J_{4,t,N} = K_{8,1,t}(H)\frac{1}{N^{\alpha}} \text{ with } K_{8,1,t}(H) = \frac{-t^{2H+3}}{8(2H+2)(2H+1)}. \tag{2.48}$$

Consequently,

$$f_{H,N}(1) = L_t(H)\frac{1}{N^{\alpha}} \text{ with } L_t(H) = K_{5,1,t}(H) + K_{6,1,t}(H) + K_{7,1,t}(H) + K_{8,1,t}(H). \tag{2.49}$$

■

2.4.3 Renormalization of the wavelet variation

In order to analyze the asymptotic behavior of the wavelet variation (2.10), we will use the chaotic expression of $V_N(t,a)$. We will work with multiple stochastic integrals with respect to the fractional-white noise W^H.

Let $\mathcal{E}$ denote the space of all linear combinations of indicator functions $1_{[0,t]\times A}$ with $t \geq 0$ and $A \in \mathfrak{B}_b(\mathbb{R})$ (the bounded Borel subsets of $\mathbb{R}$). Let $\mathcal{H}$ be the completion of $\mathcal{E}$ with respect to the inner product

$$\langle 1_{[0,t]\times A}, 1_{[0,s]\times A}\rangle = \mathbf{E}(W_t^H(A)W_s^H(B)) = R^H(t,s)\lambda(A\cap B), \text{ for every } t,s\geq 0, A,B\in\mathfrak{B}_d(\mathbb{R}^n).$$

In particular (see [21])

$$\langle\varphi,\psi\rangle_{\mathcal{H}} = H(2H-1)\int_0^t\int_0^s dv_1dv_2|v_1-v_2|^{2H-2}\int_{\mathbb{R}} dx\varphi(v_1,x)\psi(v_2,x)$$

for every $\varphi, \psi \in \mathcal{H}$ such that $\int_0^t \int_0^s dv_1 dv_2 |v_1 - v_2|^{2H-2} \int_{\mathbb{R}} dx |\varphi(v_1, x)\psi(v_2, x)| < \infty$.

Let I_q be the multiple stochastic integral of order q with respect to the isonormal process $(W(\varphi), \varphi \in \mathcal{H})$ (see the Appendix or [22]). Then

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G_1(t-s, x-y) W^H(ds, dy) = I_1(g_{t,x})$$

where

$$g_{t,x}(s,y) = G_1(t-s, x-y)$$

and therefore the wavelet coefficient $d(t,a,i)$ given by (2.9) can be written as

$$d(t,a,i) = I_1(f_{t,a,i}) \text{ with } f_{t,a,i}(s,y) = \sqrt{a} \int_{\mathbb{R}} \psi(x) g_{t,a(x+i)}(s,y) dx \text{ for every } s > 0, y \in \mathbb{R}. \tag{2.50}$$

Then, by the product formula for multiple stochastic integrals (2.99), we have, for every $t > 0, a > 0$ and $N \geq 1$

$$\begin{aligned} V_N(t,a) &= \frac{1}{N_a} \sum_{i=1}^{N_a} \left(\frac{I_2(f_{t,a,i}^{\otimes 2}) + \mathbf{E}d(t,a,i)^2}{\mathbf{E}d(t,a,i)^2} - 1 \right) \\ &= \frac{1}{N_a D(t,a)} \sum_{i=1}^{N_a} I_2(f_{t,a,i}^{\otimes 2}) \end{aligned} \tag{2.51}$$

with $f_{t,a,i}$ given by (2.50).

Let us compute the L^2-norm of the random variable $V_N(t,a)$ given by (2.10). By using the isometry formula for multiple integrals (2.98),

$$\begin{aligned} \mathbf{E}V_N(t,a)^2 &= \frac{2}{D(t,a)^2 N_a^2} \sum_{i,j=1}^{N_a} \langle f_{t,a,i}, f_{t,a,j} \rangle_{\mathcal{H}}^2 \\ &= \frac{2}{D(t,a)^2 N_a^2} \sum_{i,j=1}^{N_a} \left(\mathbf{E}d(t,a,i) d(t,a,j) \right)^2. \end{aligned} \tag{2.52}$$

Again we study the behavior of (2.52) as $N \to \infty$ when t varies with N and when t is fixed.

2.4.4 The moving time case : Proof of Proposition 12

Assume (2.11) and let us prove the limit theorem (2.13). The formula (2.52) becomes

$$\mathbf{E}V_N(t_N, a_N)^2 = \frac{2}{N_{a_N}^2} D(t_N, a_N)^{-2} \sum_{i,j=1}^{N_{a_N}} \left[-\frac{t_N}{4} a_N^{2H+1} g_H(i-j) + \frac{c_H}{2} a_N^{2H+2} g_{H+\frac{1}{2}}(i-j) \right]^2$$

with g_H given by (2.14). Thus, with $g_{N,H}$ defined by (2.26),

$$\begin{aligned}\mathbf{E}V_N(t_N,a_N)^2 &= \frac{2}{N_{a_N}^2}D(t_N,a_N)^{-2}\\ &\times\left[\frac{t_N^2}{16}a_N^{4H+2}g_{N,H}(a_N)+\frac{c_H^2}{4}a_N^{4H+4}g_{N,H+\frac{1}{2}}(a_N)\right.\\ &\left.-\frac{t_N c_H}{4}a_N^{4H+3}\sum_{i,j=1}^{N_{a_N}}g_H(i-j)g_{H+\frac{1}{2}}(i-j)\right].\end{aligned}$$

We will use the notation $f_N \sim g_N$ which in our work means that the sequences f_N and g_N have the same limit as $N \to \infty$.

Under assumption (2.11), we can estimate $\mathbf{E}V_N(t,a)^2$ as follows

$$\begin{aligned}\mathbf{E}V_N(t_N,a_N)^2 &\sim \frac{32}{K_{\Psi,H}^2}N^{2(\alpha-1)}D(t_N,a_N)^{-2}\left[\frac{1}{16}N^{2\beta+(4H+2)\alpha}g_{N,H}(a_N)\right.\\ &\left.+\frac{c_H^2}{4}N^{\alpha(4H+4)}g_{N,H+\frac{1}{2}}(a_N)-\frac{c_H}{4}N^{\beta+\alpha(4H+3)}\sum_{i,j=1}^{N_{a_N}}g_H(i-j)g_{H+\frac{1}{2}}(i-j)\right]\\ &:= v_{1,N}+v_{2,N}+v_{3,N}. \qquad (2.53)\end{aligned}$$

Let us estimate the three summands above. By the estimate of $D(t_N,a_N)^{-2}$ in Lemma 1 and by (2.27), we have

$$\begin{aligned}v_{1,N} &= \frac{2}{K_{\Psi,H}^2}N^{2(\alpha-1)}D(t_N,a_N)^{-2}N^{2\beta+(4H+2)\alpha}N^{1-\alpha}g_{N,H}(N^\alpha)\\ &\sim \frac{2}{K_{\Psi,H}^2}N^{2(\alpha-1)}N^{-2\beta-2\alpha(2H+1)}N^{2\beta+(4H+2)\alpha}N^{1-\alpha}g_{N,H}(N^\alpha)\\ &\sim \frac{2}{K_{\Psi,H}^2}\sum_{k\in\mathbb{Z}}g_H(k)^2N^{\alpha-1}\end{aligned}$$

with $K_{\Psi,H}$ from (2.15). Consequently

$$N^{1-\alpha}v_{1,N}\to_{N\to\infty}\frac{2}{K_{\Psi,H}^2}\sum_{k\in\mathbb{Z}}g_H(k)^2. \qquad (2.54)$$

Let us look at the term $v_{2,N}$. By (2.28),

$$v_{2,N} \le C_{\Psi,H}N^{\alpha-1}N^{-2\beta+2\alpha}\times\begin{cases}C_{\Psi,H,Q} & \text{if } Q\ge 2,\\ C_{\Psi,H,Q}N_a^{4H-1} & \text{if } Q=1,\end{cases}$$

$$\leq \quad C_{\Psi,H}N^{\alpha-1}\begin{cases} C_{\Psi,H,Q}N^{-2\beta+2\alpha} & \text{if } Q\geq 2,\\ C_{\Psi,H,Q}N^{\alpha-1}N^{\alpha(3-4H)-2\beta+4H-1} & \text{if } Q=1, H<\frac{3}{4}.\end{cases}$$

Thus

$$N^{1-\alpha}v_{2,N}\leq C_{\Psi,H,Q}\begin{cases} N^{-2\beta+2\alpha} & \text{if } Q\geq 2,\\ N^{\alpha(3-4H)-2\beta+4H-1} & \text{if } Q=1, H<\frac{3}{4}.\end{cases}\rightarrow_{N\to\infty} 0 \tag{2.55}$$

because $\alpha<1<\beta$, $\alpha(3-4H)<0$ and $-2\beta+4H-1<-2\beta+2<0$. Finally we look at $v_{3,N}$. We can write

$$\begin{aligned} v_{3,N} &\leq C_{\Psi,H}N^{\alpha-1}N^{\alpha-\beta}\times\begin{cases} C_{\Psi,H,Q} & \text{if } Q\geq 2,\\ C_{\Psi,H,Q}N_{a_N}^{4H-2} & \text{if } Q=1,\end{cases}\\ &\leq C_{\Psi,H}N^{\alpha-1}\begin{cases} C_{\Psi,H,Q}N^{\alpha-\beta} & \text{if } Q\geq 2,\\ C_{\Psi,H,Q}N^{\alpha-1}N^{\alpha(3-4H)-\beta+4H-2} & \text{if } Q=1, H<\frac{3}{4}.\end{cases}\end{aligned}$$

Thus

$$N^{1-\alpha}v_{3,N}\leq C_{\Psi,H,Q}\begin{cases} N^{\alpha-\beta} & \text{if } Q\geq 2,\\ N^{\alpha(3-4H)-\beta+4H-2} & \text{if } Q=1, H<\frac{3}{4}.\end{cases}\rightarrow_{N\to\infty} 0 \tag{2.56}$$

since $\alpha<\beta$, $\alpha(3-4H)<0$ and $-\beta+4H-1<-\beta+1<0$.

The bounds (2.54), (2.55), (2.56) lead to the desired conclusion. ■

2.4.4.1 The fixed time case : Proof of Proposition 13

If t is fixed, we can prove the approximation result (2.19). We have

$$\begin{aligned} \mathbf{E}V_N(t,a_N)^2 &= \frac{2}{N_{a_N}^2}\sum_{i,j=1}^{N_a}\frac{(\mathbf{E}d(t,a_N,i)d(t,a_N,j))^2}{\mathbf{E}d(t,a,i)^2\mathbf{E}d(t,a,j)^2}\\ &= \frac{2}{N_{a_N}^2}D(t,a_N)^{-2}\sum_{i,j=1}^{N_a}(\mathbf{E}d(t,a_N,i)d(t,a_N,j))^2\\ &= \frac{2}{N_{a_N}^2}D(t,a_N)^{-2}\sum_{i,j=1}^{N_a}(f_{H,N}(i-j))^2\end{aligned}$$

with $f_{H,N}$ given by (2.42). Notice that $f_{H,N}(k)=f_{H,N}(-k)$ and

$$f_{H,N}(k)=0 \text{ if } |k|\geq 2$$

by chosing N large enough. Since can be seen via (2.42), since the function Ψ has support included in $[0,1]$. Therefore

$$\mathbf{E}V_N(t,a_N)^2 = \frac{2}{N_{a_N}^2} D(t,a_N)^{-2}\left[N_a(f_{H,N}(0))^2 + 2(N_a-1)(f_{H,N}(1))^2\right]. \tag{2.57}$$

We have $f_{H,N}(0) = D(t,a_N)$ and $f_{H,N}(1)$ was computed before. Using (2.49), (2.57) can be written as follows

$$\mathbf{E}V_N(t,a_N)^2 = \frac{2}{N^{2(1-\alpha)}} D(t,a_N)^{-2}\left(N^{1-\alpha}D(t,a_N)^2 + \frac{2(N^{1-\alpha}-1)L_t^2(H)}{D(t,a_N)^2 N^{2\alpha}}\right) \tag{2.58}$$

with $L_t(H)$ given by (2.49). Then

$$\mathbf{E}V_N(t,a_N)^2 \sim \left(\frac{2}{N^{1-\alpha}} + \frac{4L_t(H)^2}{K_{1,t}(H)^2}\frac{1}{N^{1+\alpha}}\right)$$

and the conclusion follows. ■

Remark 2 *As already noticed in Remark 1, the renormalization of (2.10) is of the same order in both cases (fixed time or moving time) although the correlation structure of the wavelet coefficient is different. On the other hand, in the fixed time case, the diagonal term of* $\mathbf{E}V_N(t,a_N)^2$ *is dominant for the behavior of this quantity as* $N\to\infty$ *(here is only one non-diagonal term which does not contribute to the limit), while when* t *increases with* N*, all the diagonal and non-diagonal terms have contribution to the limit.*

2.4.5 Central Limit Theorem and rate of convergence

We will show that, both in the moving time and fixed time cases, the renormalized wavelet variation satisfies a central limit theorem if $Q\geq 2$ or $Q=1, H<\frac{3}{4}$.

Our main tool is the following result (see Theorem 5.2.6 and Corollary 5.2.10 in [81]). By d we denote the distance between distributions of random variable and below it can be each of the following distances : Kolmogorov, total variation, Wasserstein or Fortet-Mourier (see [81]).

Theorem 4 *Let* $(F_N)_{N\geq 1}$ *be a sequence of random variables in the qth Wiener chaos* ($q\geq 1$) *with respect to an isonormal process indexed by the Hilbert space* $\mathcal{H}$. *Assume that* $\mathbf{E}F_N^2 \to_{N\to\infty} \sigma^2 > 0$. *Then the sequence* $(F_N)_{N\geq 1}$ *converges in law to the standard normal random variable* Z *if and only if* $\|DF_N\|_{\mathcal{H}}^2$ *converges in* $L^2(\Omega)$ *as* $N\to\infty$ *to* $q\sigma^2$. *In this case*

$$d(F_N,Z) \leq C\left(\sqrt{Var\left(\|DF_N\|_{\mathcal{H}}^2\right)} + \left|\mathbf{E}F_N^2 - \sigma^2\right|\right).$$

2.4.6 The moving time : Proof of Theorems 1 and 2

Consider the sequence $(F_N)_{N\geq 1}$ given by (2.16) and recall that from Proposition 12 we know that

$$\mathbf{E}F_N^2 \to_{N\to\infty} 1.$$

Also, by (2.51) we have the following chaos expansion of F_N, for every $N \geq 1$, with $f_{t,a,i}$ given by (2.50),

$$F_N = K_{0,\Psi,H}^{-\frac{1}{2}} N^{\frac{\alpha-1}{2}} I_2\left(\sum_{i=1}^{N_{a_N}} \frac{f_{t_N,a_N,i}^{\otimes 2}}{\mathbf{E}(d(t_N,a_N,i)^2}\right) = K_{0,\Psi,H}^{-\frac{1}{2}} N^{\frac{\alpha-1}{2}} D(t_N,a_N)^{-1} I_2\left(\sum_{i=1}^{N_{a_N}} f_{t_N,a_N,i}^{\otimes 2}\right). \tag{2.59}$$

So, F_N is an element of the second Wiener chaos (with respect to the Gaussian noise W^H introduced in Section 2.4.3) for every $N \geq 1$ and we may apply Theorem 4 in order to check its asymptotic behavior in distribution.

Proof of Theorem 1 : By taking the Malliavin derivative with respect to fractional-white noise W^H in (2.59) (see formula (2.100)),

$$DF_N = 2K_{0,\Psi,H}^{-\frac{1}{2}} N^{\frac{\alpha-1}{2}} D(t_N,a_N)^{-1} \sum_{i=1}^{N_{a_N}} I_1(f_{t_N,a_N,i}) f_{t_N,a_N,i}$$

and, if $\mathcal{H}$ is the Hilbert space associated with the fractional-white Gaussian noise (see the beginning of Section 2.4.3),

$$\begin{aligned}
\|DF_N\|_{\mathcal{H}}^2 &= 4K_{0,\Psi,H}^{-1} N^{\alpha-1} D(t_N,a_N)^{-2} \sum_{i,j=1}^{N_{a_N}} I_1(f_{t,a,i}) I_1(f_{t,a,j}) \langle f_{t,a,i} f_{t,a,j}\rangle_{\mathcal{H}} \\
&= 4K_{0,\Psi,H}^{-1} N^{\alpha-1} D(t_N,a_N)^{-2} \sum_{i,j=1}^{N_{a_N}} I_2(f_{t,a,i}\otimes f_{t,a,j}) \langle f_{t,a,i} f_{t,a,j}\rangle_{\mathcal{H}} \\
&\quad +4K_{0,\Psi,H}^{-1} N^{\alpha-1} D(t_N,a_N)^{-2} \sum_{i,j=1}^{N_{a_N}} \langle f_{t,a,i} f_{t,a,j}\rangle_{\mathcal{H}}^2 \\
&= 4K_{0,\Psi,H}^{-1} N^{\alpha-1} D(t_N,a_N)^{-2} \sum_{i,j=1}^{N_{a_N}} I_2(f_{t,a,i}\otimes f_{t,a,j}) \langle f_{t,a,i} f_{t,a,j}\rangle_{\mathcal{H}} + \mathbf{E}\|DF_N\|_{\mathcal{H}}^2
\end{aligned}$$

with $f_{t,a,i}$ from (2.50). Notice that, since F_N belongs to the second Wiener chaos, we have $\mathbf{E}\|DF_N\|_{\mathcal{H}}^2 = 2\mathbf{E}F_N^2 \to_{N\to\infty} 2$. By Theorem 4,

$$d(F_N,Z) \leq c\left(\sqrt{Var\left(\|DF_N\|_{\mathcal{H}}^2\right)} + \mathbf{E}\|DF_N\|_{\mathcal{H}}^2 - 2\right)$$

so

$$d(F_N, Z) \le c(\sqrt{T_{1,N}} + T_{2,N})$$

where

$$T_{1,N} = \mathbf{E}\left(4K_{0,\Psi,H}^{-1} N^{\alpha-1} D(t_N, a_N)^{-2} \sum_{i,j=1}^{N_{a_N}} I_2(f_{t,a,i} \otimes f_{t,a,j}) \langle f_{t,a,i} f_{t,a,j}\rangle_{\mathcal{H}}\right)^2 \tag{2.60}$$

and

$$T_{2,N} = \mathbf{E}\|DF_N\|_{\mathcal{H}}^2 - 2 = 4K_{0,\Psi,H}^{-1} N^{\alpha-1} D(t_N, a_N)^{-2} \sum_{i,j=1}^{N_{a_N}} \langle f_{t,a,i} f_{t,a,j}\rangle_{\mathcal{H}}^2 - 2. \tag{2.61}$$

Let us first estimate $T_{2,N}$. Since

$$\langle f_{t_N,a_N,i} f_{t_N,a_N,j}\rangle_{\mathcal{H}} = \mathbf{E}d(t_N, a_N, i) d(t_N, a_N, j)$$

we can write, as in (2.53)

$$\begin{aligned}
T_{2,N} &= 4K_{0,\Psi,H}^{-1} N^{\alpha-1} D(t_N, a_N)^{-2} \left[\frac{1}{16} N^{2\beta+(4H+2)\alpha} g_{N,H}(a_N)\right. \\
&\quad \left. + \frac{c_H^2}{4} N^{\alpha(4H+4)} g_{N,H+\frac{1}{2}}(a_N) - \frac{c_H}{4} N^{\beta+\alpha(4H+3)} \sum_{i,j=1}^{N_{a_N}} g_H(i-j) g_{H+\frac{1}{2}}(i-j)\right] - 2 \\
&:= T_{2,1,N} + T_{2,2,N} + T_{2,3,N}.
\end{aligned}$$

First, we analyze the term $T_{2,1,N}$. We have

$$\begin{aligned}
T_{2,1,N} &= 4K_{0,\Psi,H}^{-1} N^{\alpha-1} D(t_N, a_N)^{-2} \frac{1}{16} N^{2\beta+(4H+2)\alpha} g_{N,H}(a_N) - 2 \\
&= \frac{1}{4} N^{2\beta+(4H+2)\alpha} K_{0,\Psi,H}^{-1} D(t_N, a_N)^{-2} \sum_{k\in\mathbb{Z}} g_H(k)^2 1_{\{|k|\le N_{a_N}\}} \frac{N_{a_N} - |k|}{N_{a_N}} - 2 \\
&= \frac{1}{4} K_{0,\Psi,H}^{-1} N^{2\beta+(4H+2)\alpha} D(t_N, a_N)^{-2} \sum_{k\in\mathbb{Z}} g_H(k)^2 \\
&\quad + \frac{1}{4} K_{0,\Psi,H}^{-1} N^{2\beta+(4H+2)\alpha} D(t_N, a_N)^{-2} \\
&\quad \times \left(\sum_{k\in\mathbb{Z}} g_H(k)^2 1_{\{|k|\le N_{a_N}\}} \frac{N_{a_N} - |k|}{N_{a_N}} - \sum_{k\in\mathbb{Z}} g_H(k)^2\right) - 2
\end{aligned}$$

and since by (2.24)

$$D(t_N, a_N) = \mathbf{E}d(t_N, a_N, i)^2 = -\frac{c_H}{2} K_{\Psi,H+\frac{1}{2}} N^{(2H+2)\alpha} + \frac{1}{4} K_{\Psi,H} N^{\beta+(2H+1)\alpha},$$

we obtain

$$\begin{aligned} T_{2,1,N} &= \frac{1}{4}K_{0,\Psi,H}^{-1}N^{2\beta+(4H+2)\alpha}\left(\frac{1}{4}K_{\Psi,H}N^{\beta+(2H+1)\alpha}\right)^{-2}\sum_{k\in\mathbb{Z}}g_H(k)^2 \\ &+\frac{1}{4}K_{0,\Psi,H}^{-1}N^{2\beta+(4H+2)\alpha} \\ &\times\left[\left(-\frac{c_H}{2}K_{\Psi,H+\frac{1}{2}}N^{(2H+2)\alpha}+\frac{1}{4}K_{\Psi,H}N^{\beta+(2H+1)H}\right)^{-2}-\left(\frac{1}{4}K_{\Psi,H}N^{\beta+(2H+1)\alpha}\right)^{-2}\right]\sum_{k\in\mathbb{Z}}g_H(k)^2 \\ &+\frac{1}{4}K_{0,\Psi,H}^{-1}N^{2\beta+(4H+2)\alpha}D(t_N,a_N)^{-2}\left(\sum_{k\in\mathbb{Z}}g_H(k)^2 1_{\{|k|\le N_{a_N}\}}\frac{N_{a_N}-|k|}{N_{a_N}}-\sum_{k\in\mathbb{Z}}g_H(k)^2\right)-2. \end{aligned}$$

The first term in the above expression vanishes with 2 so it remains

$$\begin{aligned} T_{2,1,N} &= \frac{1}{4}K_{0,\Psi,H}^{-1}N^{2\beta+(4H+2)\alpha} \\ \times & \left[\left(-\frac{c_H}{2}K_{\Psi,H+\frac{1}{2}}N^{(2H+2)\alpha}+\frac{1}{4}K_{\Psi,H}N^{\beta+(2H+1)\alpha}\right)^{-2}-\left(\frac{1}{4}K_{\Psi,H}N^{\beta+(2H+1)\alpha}\right)^{-2}\right]\sum_{k\in\mathbb{Z}}g_H(k)^2 \\ &+\frac{1}{4}K_{0,\Psi,H}^{-1}N^{2\beta+(4H+2)\alpha}D(t_N,a_N)^{-2}\left(\sum_{k\in\mathbb{Z}}g_H(k)^2 1_{\{|k|\le N_{a_N}\}}\frac{N_{a_N}-|k|}{N_{a_N}}-\sum_{k\in\mathbb{Z}}g_H(k)^2\right). \end{aligned}$$

We have the following bound for the first summand in $T_{2,1,N}$

$$\begin{aligned} &\left[\left(-\frac{c_H}{2}K_{\Psi,H+\frac{1}{2}}N^{(2H+2)\alpha}+\frac{1}{4}K_{\Psi,H}N^{\beta+(2H+1)\alpha}\right)^{-2}-\left(\frac{1}{4}K_{\Psi,H}N^{\beta+(2H+1)\alpha}\right)^{-2}\right] \\ \le\ & cN^{\alpha-\beta}N^{2\beta+(4H+2)\alpha}. \end{aligned}$$

To obtain a bound for the second term in the expression of $T_{2,1,N}$, we write $\sum_{k\in\mathbb{Z}}g_H(k)^2 = \sum_{k\in\mathbb{Z}}g_H(k)^2 1_{\{|k|\le N_{a_N}\}}+\sum_{k\in\mathbb{Z}}g_H(k)^2 1_{\{|k|>N_{a_N}\}}$ and using the fact that $|g_H|$ is bounded by $|k|^{4H-4Q}$ we get that

$$\begin{aligned} &D(t_N,a_N)^{-2}\left(\sum_{k\in\mathbb{Z}}g_H(k)^2 1_{\{|k|\le N_{a_N}\}}\frac{N_{a_N}-|k|}{N_{a_N}}-\sum_{k\in\mathbb{Z}}g_H(k)^2\right) \\ \le\ & cD(t_N,a_N)^{-2}N_{a_N}^{8H-8Q+2} = cN^{-2\beta-2\alpha(2H+1)+(1-\alpha)(8H-8Q+2)}. \end{aligned}$$

Therefore

$$T_{2,1,N}\le cN^{\alpha-\beta}. \tag{2.62}$$

For $T_{2,2,N}$ we have by (2.28) and Lemma 1

$$\begin{aligned}
|T_{2,2,N}| &\leq cD(t_N,a_N)^{-2}N^{\alpha-1}N^{\alpha(4H+4)}g_{H,\frac{1}{2}}(a_N) \\
&\leq D(t_N,a_N)^{-2}N^{\alpha(4H+4)}\begin{cases} C, \text{ if } Q\geq 2 \\ N^{(1-\alpha)(4H-1)}, \text{ if } Q=1 \end{cases} \\
&\leq c\begin{cases} N^{2\alpha-2\beta} \text{ if } Q\geq 2 \\ N^{2\alpha-2\beta}N^{(1-\alpha)(4H-1)} \text{ if } Q=1. \end{cases}
\end{aligned}$$

Regarding $T_{2,3,N}$ we use (2.28) and Lemma 1 to get that

$$\begin{aligned}
|T_{2,3,N}| &\leq cD(t_N,a_N)^{-2}N^{\alpha-1}N^{\alpha(4H+3)}\sum_{i=1}^{N_{a_N}} g_H(i-j)g_{H+\frac{1}{2}}(i-j) \\
&\leq D(t_N,a_N)^{-2}N^{\alpha(4H+3)}\begin{cases} C, \text{ if } Q\geq 2 \\ N^{(1-\alpha)(4H-2)}, \text{ if } Q=1 \end{cases} \\
&\leq c\begin{cases} N^{\alpha-\beta} \text{ if } Q\geq 2 \\ N^{\alpha-\beta}N^{(1-\alpha)(4H-2)} \text{ if } Q=1. \end{cases}
\end{aligned}$$

Combining (2.62), (2.63) and (2.63), we have the following bound for (2.61) :

$$|T_{2,N}| \leq \begin{cases} CN^{\alpha-\beta} \text{ if } Q\geq 2 \\ N^{\alpha-\beta}N^{(1-\alpha)(4H-2)} \text{ if } Q=1 \end{cases} \tag{2.63}$$

Concerning $T_{1,N}$, by the isometry of multiple integrals

$$\begin{aligned}
T_{1,N} &= Var(\|DF_N\|_H^2) = \mathbf{E}\left(|\|DF_N\|_H^2 - \mathbf{E}\|DF_N\|_H^2\right) \\
&= \mathbf{E}\left(4K_{0,\Psi,H}^{-1}N^{\alpha-1}D(t_N,a_N)^{-2}\sum_{i,j=1}^{N_{a_N}} I_2(f_{t_N,a_N,i}\otimes f_{t_N,a_N,j})\langle f_{t_N,a_N,i};f_{t_N,a_N,j}\rangle_{\mathcal{H}}\right)^2 \\
&= 16K_{0,\Psi,H}^{-2}N^{2(\alpha-1)}D(t_N,a_N)^{-4} \\
&\quad \mathbf{E}\left(\sum_{i,j,k,l=1}^{N_{a_N}} I_2(f_{t_N,a_N,i}\otimes f_{t_N,a_N,j})I_2(f_{t_N,a_N,k}\otimes f_{t_N,a_N,l})\langle f_{t_N,a_N,i};f_{t_N,a_N,j}\rangle_{\mathcal{H}}\langle f_{t_N,a_N,k};f_{t_N,a_N,l}\rangle_{\mathcal{H}}\right) \\
&= 32K_{0\Psi,H}^{-2}N^{2(\alpha-1)}D(t_N,a_N)^{-4} \\
&\quad \times\sum_{i,j,k,l=1}^{N_{a_N}}\langle f_{t,a,i};f_{t,a,j}\rangle_{\mathcal{H}}\langle f_{t,a,k};f_{t,a,l}\rangle_{\mathcal{H}}\langle f_{t_N,a_N,i};f_{t_N,a_N,k}\rangle_{\mathcal{H}}\langle f_{t,a,j};f_{t,a,l}\rangle_{\mathcal{H}}
\end{aligned}$$

Recall that for all integers p,q we have (see relation (2.25)) $\langle f_{t,a,p}; f_{t,a,q}\rangle = \frac{c_H}{2} a_N^{2H+2} g_{H+\frac{1}{2}}(p-q) - \frac{t_N}{4} a_N^{2H+1} g_H(p-q)$. Hence,

$$\begin{aligned} T_{1,N} &= 32K_{0,\Psi,H}^{-2} N^{2(\alpha-1)} D(t_N,a_N)^{-4} \sum_{i,j,k,l=1}^{N_{a_N}} \left[\frac{c_H}{2} a_N^{2H+2} g_{H+\frac{1}{2}}(i-j) - \frac{t_N}{4} a_N^{2H+1} g_H(i-j)\right] \\ &\quad \times \left[\frac{c_H}{2} a_N^{2H+2} g_{H+\frac{1}{2}}(k-l) - \frac{t_N}{4} a_N^{2H+1} g_H(k-l)\right] \left[\frac{c_H}{2} a_N^{2H+2} g_{H+\frac{1}{2}}(i-k) - \frac{t_N}{4} a_N^{2H+1} g_H(i-k)\right] \\ &\quad \times \left[\frac{c_H}{2} a_N^{2H+2} g_{H+\frac{1}{2}}(j-l) - \frac{t_N}{4} a_N^{2H+1} g_H(j-l)\right] \\ &= \sum_{i=1}^{5} T_{1,i,N}. \end{aligned}$$

Above we denoted by $T_{1,1,N}$ a product that depends only on functions g_H and $T_{1,2,N}$ is a product that depends only on one function g_H (see (2.14)) and three functions $g_{H+\frac{1}{2}}$. $T_{1,3,N}$ is the product of two functions g_H and two $g_{H+\frac{1}{2}}$. As well, $T_{1,4,N}$ contains three g_H and one $g_{H+\frac{1}{2}}$ while $T_{1,5,N}$ contains the four functions $g_{H+\frac{1}{2}}$. To study each term we will use the same technique of proof of the Theorem 7.3.1 in [81] or Lemma 3 in [66]. Let $g_{H,N}(k) := |g_H(k)| 1_{\{|k|\le N_a\}}$ (this is not the same as $g_{N,H}$ in (2.26)) and $g_{H+\frac{1}{2},N}(k) := |g_{H+\frac{1}{2}}(k)| 1_{\{|k|\le N_a\}}$.

Let $u,v:\mathbb{Z}\to\mathbb{R}$ be two sequences, we define their convolution by

$$(u*v)(k) = \sum_{n\in\mathbb{Z}} u(n)v(k-n).$$

We will need Young's inequality, which can be written, for $s,p,q\ge 1$ such that $\frac{1}{s}+1=\frac{1}{p}+\frac{1}{q}$ as

$$\|u*v\|_{l^s(\mathbb{Z})} \le \|u\|_{l^p(\mathbb{Z})} \|v\|_{l^q(\mathbb{Z})}.$$

Now we can estimate each term $T_{1,i,N}$, for $i=1,..,5$. First

$$\begin{aligned} T_{1,1,N} &= 32K_0^{-2} N^{2(\alpha-1)} \frac{t_N^4}{4^4} a_N^{4(2H+1)} D(t_N,a_N)^{-4} \\ &\quad \times \sum_{i,j,k,l=1}^{N_{a_N}} g_H(i-j) g_H(k-l) g_H(i-k) g_H(j-l) \\ &\le C N^{\alpha(8H+6)-2+4\beta} D(t_N,a_N)^{-4} \end{aligned}$$

$$\times \sum_{j,k=1}^{N_{a_N}} \sum_{i,l\in\mathbb{Z}} g_{H,N}(i-j) g_{H,N}(k-l) g_{H,N}(i-k) g_{H,N}(j-l)$$
$$= CN^{\alpha(8H+6)-2+4\beta} D(t_N, a_N)^{-4}$$
$$\sum_{j,k=1}^{N_{a_N}} (g_{H,N} * g_{H,N})(k-j)^2$$
$$= CN^{\alpha(8H+6)-2+4\beta} N^{1-\alpha} D(t_N, a_N)^{-4} \|g_{H,N} * g_{H,N}\|^2_{l^2(\mathbb{Z})}$$
$$\leq CN^{\alpha(8H+5)-1+4\beta} D(t_N, a_N)^{-4} \|g_{H,N}\|^4_{l^{\frac{4}{3}}(\mathbb{Z})}.$$

where the last inequality follows from Young inequality

$$T_{1,1,N} \leq CN^{\alpha(8H+5)-1+4\beta} D(t_N, a_N)^{-4} \left(\sum_{k=-N}^{N} |g_H(k)|^{4/3} \right)^3.$$

The series $\sum_{k\in\mathbb{Z}} |g_H(k)|^{4/3}$ converges because $|g_H(k)|$ is bounded, for k large, by $C|k|^{4H-4Q}$, see Lemma 3. Finally,

$$T_{1,1,N} \leq CN^{\alpha-1}. \tag{2.64}$$

Next, we have

$$T_{1,2,N} = -CN^{2(\alpha-1)} D(t_N, a_N)^{-4} N^{\beta+\alpha(8H+7)} \sum_{i,j,k,l=1}^{N_{a_N}} g_H(i-j) g_{H+\frac{1}{2}}(k-l) g_{H+\frac{1}{2}}(i-k) g_{H+\frac{1}{2}}(j-l).$$

We apply the same idea as above

$$|T_{1,2,N}| \leq CN^{2(\alpha-1)} D(t_N, a_N)^{-4} N^{\beta} N^{\alpha(8H+7)} \sum_{i,j,k,l=1}^{N_{a_N}} g_H(i-j) g_{H+\frac{1}{2}}(k-l) g_{H+\frac{1}{2}}(i-k) g_{H+\frac{1}{2}}(j-l)$$
$$\leq CN^{\beta-2+\alpha(8H+9)} D(t_N, a_N)^{-4} \sum_{j,k=0}^{N_{a_N}} \sum_{i,l\in\mathbb{Z}} g_{H,N}(i-j) g_{H+\frac{1}{2},N}(k-l) g_{H+\frac{1}{2},N}(i-k) g_{H+\frac{1}{2},N}(j-l)$$
$$\leq CN^{\beta-1+\alpha(8H+8)} D(t_N, a_N)^{-4} \sum_{k\in\mathbb{Z}} (g_{H,N} * g_{H+\frac{1}{2},N})(k) (g_{H+\frac{1}{2},N} * g_{H+\frac{1}{2},N})(k)$$
$$\leq CN^{\beta-1+\alpha(8H+8)} D(t_N, a_N)^{-4} \|g_{H,N} * g_{H+\frac{1}{2},N}\|_{l^2(\mathbb{Z})} \|g_{H+\frac{1}{2},N} * g_{H+\frac{1}{2},N}\|_{l^2(\mathbb{Z})}$$
$$\leq CN^{\beta-1+\alpha(8H+8)} D(t_N, a_N)^{-4} \|g_{H,N}\|_{l^{\frac{4}{3}}(\mathbb{Z})} \|g_{H+\frac{1}{2},N}\|^3_{l^{\frac{4}{3}}(\mathbb{Z})}$$
$$= CN^{\beta-1+\alpha(8H+8)} D(t_N, a_N)^{-4} \left(\sum_{k=-N}^{N} |g_{H,N}|^{\frac{4}{3}} \right)^{\frac{4}{3}} \left(\sum_{k=-N}^{N} |g_{H+\frac{1}{2},N}|^{\frac{4}{3}} \right)^{\frac{9}{4}}$$

$$\leq \quad CN^{4\alpha-3\beta-1}\left(\sum_{k=-N}^{N}|g_{H,N}|^{\frac{4}{3}}\right)^{\frac{3}{4}}\left(\sum_{k=-N}^{N}|g_{H+\frac{1}{2},N}|^{\frac{4}{3}}\right)^{\frac{9}{4}}.$$

Now, by Lemma 3, $g_{H+\frac{1}{2}}(k)$ is bounded for k large by $C|k|^{4H+2-4Q}$ and this yields to

$$\sum_{k=-N}^{N}|g_{H+\frac{1}{2},N}|^{\frac{4}{3}} \leq \begin{cases} O(1) & \text{if } Q\geq 2 \text{ and } Q=1, H<\frac{5}{16} \\ O(\log(N)) & \text{if } Q=1, H=\frac{5}{16} \\ O(N^{\frac{16H-5}{3}}) & \text{if } Q=1, H>\frac{5}{16}. \end{cases}$$

Recall that $\frac{1}{2}<H<1$, hence

$$T_{1,2,N}\leq C\begin{cases} N^{4\alpha-3\beta-1} & \text{if } Q\geq 2 \\ N^{\frac{3(16H-5)}{4}+4\alpha-3\beta-1} & \text{if } Q=1, H\in(\frac{1}{2},\frac{3}{4}). \end{cases} \tag{2.65}$$

In the same way,

$$\begin{aligned} T_{1,3,N} &= CN^{2(\alpha-1)}N^{\alpha(8H+6)+2\beta}D(t_N,a_N)^{-4}\sum_{i,j,k,l=1}^{N_{a_N}} g_H(i-j)g_H(k-l)g_{H+\frac{1}{2}}(i-k)g_{H+\frac{1}{2}}(j-l) \\ &\leq CN^{3\alpha-2\beta-1}\left(\sum_{k=-N}^{N}|g_{H,N}|^{\frac{4}{3}}\right)^{\frac{3}{2}}\left(\sum_{k=-N}^{N}|g_{H+\frac{1}{2},N}|^{\frac{4}{3}}\right)^{\frac{3}{2}}. \end{aligned}$$

Using the previous case we have

$$T_{1,3,N}\leq C\begin{cases} N^{3\alpha-2\beta-1} & \text{if } Q\geq 2 \\ N^{\frac{16H-5}{2}+3\alpha-2\beta-1} & \text{if } Q=1, H\in(\frac{1}{2},\frac{3}{4}) \end{cases} \tag{2.66}$$

and as well,

$$\begin{aligned} T_{1,4,N} = &= CN^{2(\alpha-1)}N^{\alpha(8H+5)+3\beta}D(t_N,a_N)^{-4}\sum_{i,j,k,l=1}^{N_{a_N}} g_H(i-j)g_H(k-l)g_H(i-k)g_{H+\frac{1}{2}}(j-l) \\ &\leq CN^{2\alpha-\beta-1}\left(\sum_{k=-N}^{N}|g_{H,N}|^{\frac{4}{3}}\right)^{\frac{9}{4}}\left(\sum_{k=-N}^{N}|g_{H+\frac{1}{2},N}|^{\frac{4}{3}}\right)^{\frac{3}{4}} \end{aligned}$$

which implies that

$$T_{1,4,N}\leq C\begin{cases} N^{2\alpha-\beta-1} & \text{if } Q\geq 2 \\ N^{\frac{16H-5}{4}+\alpha-\beta-1} & \text{if } Q=1, H\in(\frac{1}{2},\frac{3}{4}) \end{cases}. \tag{2.67}$$

It remains to look at $T_{1,5,N}$,

$$\begin{aligned} T_{1,5,N} = &= CN^{2(\alpha-1)}N^{4\alpha(2H+2)}D(t_N,a_N)^{-4}\sum_{i,j,k,l=1}^{N_{a_N}} g_{H+\frac{1}{2}}(i-j)g_{H+\frac{1}{2}}(k-l)g_{H+\frac{1}{2}}(i-k)g_{H+\frac{1}{2}}(j-l) \\ &\leq CN^{5\alpha-4\beta-1}\left(\sum_{k=-N}^{N}|g_{H+\frac{1}{2},N}|^{\frac{4}{3}}\right)^3 \end{aligned}$$

and this gives us

$$T_{1,5,N} \leq \begin{cases} N^{5\alpha-4\beta-1} & \text{if } Q \geq 2 \\ N^{16H+5\alpha-4\beta-6} & \text{if } Q=1, H \in (\frac{1}{2},\frac{3}{4}). \end{cases}$$

Combining (34) – (38) we see that $T_{1,1,N}$ is the biggest term and finally we get a simple estimate for $T_{1,N}$

$$T_{1,N} \leq CN^{\alpha-1}. \tag{2.68}$$

By (2.63) and (2.68), we obtain the conclusion. ■

Proof of Theorem 2 : By Theorem 1 (applied to the scale La_N, $L=1,..,d$), we know that the each component of the vector $\left(N^{1-\alpha}V_n(t_N,La_N)\right)_{L=1,..,d}$ converges in distribution, as $N\to\infty$, to a centered Gaussian random variable. By the main result in [85], it suffices to show that for every $L_1,L_2=1,..,d$

$$N^{1-\alpha}\mathbf{E}V_N(t_N,L_1a_N)V_N(t_N,L_2a_N)$$

converges as $N\to\infty$ to Γ_{L_1,L_2}. We have

$$\begin{aligned} &N^{1-\alpha}\mathbf{E}V_N(t_N,L_1a_N)V_N(t_N,L_2a_N) \\ = &\frac{2}{D(t_N,L_1a_N)D(t_N,L_2a_N)}N^{1-\alpha}\sum_{i=1}^{N_{L_1a_A}}\sum_{j=1}^{N_{L_2a_A}}\left(\mathbf{E}d(t_N,L_1a_N,i)d(t_N,L_2a_N,j)\right)^2 \end{aligned}$$

where

$$\mathbf{E}d(t_N,L_1a_N,i)d(t_N,L_2a_N,j) = \frac{c_H}{2}a_N^{2H+2}g_{L_1,L_2,H+\frac{1}{2}}(i,j) - \frac{t_N}{4}a_N^{2H+1}g_{L_1,L_2,H}(i,j)$$

and

$$g_{L_1,L_2,H}(i,j) = \int_{\mathbb{R}}\int_{\mathbb{R}} dxdy\Psi(x)\Psi(y)|L_1x-L_2y+L_1-L_2j|^{2H}.$$

We know (see Section 3.2 in [27] or Proposition 2.3 in [25]), that for every i, j large and for $H \in \left(\frac{1}{2}, \frac{3}{2}\right)$,

$$|g_{L_1,L_2,H}(i,j)| \leq C(1 + |L_1 i - L_2 j|^{2H-2Q}) \tag{2.69}$$

and if $H \in \left(\frac{1}{2}, 1\right)$,

$$N^{1-\alpha} \sum_{i=1}^{N_{L_1 a_N}} \sum_{j=1}^{N_{L_2 a_N}} (g_{L_1,L_2,H}(i,j))^2 \to_{N\to\infty} C(L_1, L_2, H) \tag{2.70}$$

with $C(L_1, L_2, H)$ being explicit constant. From Lemma 1, (2.70) and (2.69) we obtain the conclusion by following the proof of Proposition 12. ■

2.4.6.1 The case of fixed time : Proof of Theorem 3

Now consider the random sequence $(G_N)_{N\geq 1}$ defined by (2.20). It satisfies $\mathbf{E}G_N^2 \to 1$ as $N \to \infty$ and it admits the following chaos expansion

$$G_N \quad =: \quad \frac{1}{\sqrt{2}} N^{\frac{1-\alpha}{2}} V(t, a_N) = \frac{1}{\sqrt{2}} N^{\frac{1-\alpha}{2}} D(t, a_N)^{-1} I_2 \left(\sum_{i=1}^{N} f_{t,a_N,i}^{\otimes 2} \right). \tag{2.71}$$

From (2.71) and (2.100), the Malliavin derivative of G_N writes as

$$DG_N = \frac{1}{\sqrt{2}} N^{\frac{1-\alpha}{2}} d(t, a_N)^{-2} I_1 \left(\sum_{i=1}^{N} f_{t,a_N,i} \right) f_{t,a_N,i}$$

so

$$\begin{aligned} & \|DG_N\|_{\mathcal{H}}^2 \\ = \quad & 2N^{\alpha-1} D(t, a_N)^{-2} \sum_{i,j=1}^{N_{a_N}} I_2(f_{t,a_N,i} \otimes f_{t,a_N,i}) f_{t,a_N,j} + 2N^{\alpha-1} D(t, a_N)^{-2} \sum_{i,j=1}^{N} \langle (f_{t,a_N,i}; f_{t,a_N,j} \rangle)^2 \\ = \quad & 2N^{\alpha-1} D(t, a_N)^{-2} \sum_{i,j=1}^{N_{a_N}} I_2(f_{t,a_N,i} \otimes f_{t,a_N,i}) f_{t,a_N,j} + \mathbf{E}\|DG_N\|^2. \end{aligned}$$

From Theorem 4,

$$\begin{aligned} d(G_N, Z) \quad & \leq \quad c\left(\sqrt{Var\left(\|DG_N\|_{\mathcal{H}}^2\right)} + \mathbf{E}\|DG_N|_{\mathcal{H}}^2 - 2 \right) \\ & =: \quad c(\sqrt{T_{1,N}} + T_{2,N}) \end{aligned}$$

We analysis first $T_{1,N}$

$$\begin{aligned}
T_{1,N} &= 8N^{2(\alpha-1)}D(t,a_N)^{-4}\sum_{i,j,k,l=1}^{N_{a_N}} f_{H,N}(i-j)f_{H,N}(k-l)f_{H,N}(i-k)f_{H,N}(j-l) \\
&\le CN^{\alpha-1}D(t,a_N)^{-4}\|\tilde{f}_{H,N}\|^4_{l^{4/3}(\mathbb{Z})} \\
&\le CN^{\alpha-1}D(t,a_N)^{-4}\left(\sum_{k=-N_{a_N}}^{N_{a_N}} |f_{H,N}(k)|^{4/3}\right)^3 \\
&= CN^{\alpha-1}D(t,a_N)^{-4}\left(2\sum_{k=1}^{N_{a_N}} |f_{H,N}(k)|^{4/3} + |f_{H,N}(0)|^{4/3}\right)^3 \\
&= CN^{\alpha-1}D(t,a_N)^{-4}\left(2|f_{H,N}(1)|^{4/3} + |f_{H,N}(0)|^{4/3}\right)^3 \\
&= CN^{\alpha-1}D(t,a_N)^{-4}\left(2\frac{|K_{3,t}|^{4/3}}{N^{\frac{4\alpha}{3}}} + |D(t,a_N)|^{4/3}\right)^3 \\
&\le CN^{\alpha-1}D(t,a_N)^{-4}\left(8\frac{|K_{3,t}|^4}{N^{4\alpha}} + 4|D(t,a_n)|^4\right) \le C\frac{1}{N^{1-\alpha}}.
\end{aligned}$$

For $T_{2,N}$ we can write

$$\begin{aligned}
T_{2,N} &= 2N^{\alpha-1}D(t,a_N)^{-2}\sum_{i,j=1}^{N_{a_N}} \langle (f_{t,a_N,i}; f_{t,a_N,j}\rangle)^2 - 2 \\
&= 2N^{\alpha-1}D(t,a_N)^{-2}\left(N_{a_N} f_{H,N}(0)^2 + 2(N_{a_N}-1)f_{H,N}(1)^2\right) - 2 \\
&= 4N^{\alpha-1}D(t,a_N)^{-2}(N_{a_N}-1)\frac{K_3^2}{N^{2\alpha}} \le \frac{C}{N^{2\alpha}}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
d(G_N,Z) &\le c\left(\sqrt{Var\left(\|DF_N\|^2_{\mathcal{H}}\right)} + \mathbf{E}\|DF_N|^2_{\mathcal{H}} - 2\right) \\
&= c\left(\frac{1}{N^{\frac{1-\alpha}{2}}} + \frac{1}{N^{2\alpha}}\right).
\end{aligned}$$

■

2.5 Estimation of the Hurst parameter

We will apply our theoretical results in Section 2.3 in order to construct an estimator for the Hurst parameter of the solution to the stochastic wave equation (2.1). The estimator will be constructed by using the wavelet variation (2.10). We will assume that the solution is observed at discrete points in space $x_i = i, i = 1, .., N$ and at a certain time t (fixed or depending on N). Different estimators (but all of them constructed via the wavelet variation) are obtained in these two situations treated in our work (moving or fixed time). While for fixed time the logarithm of the variance of the wavelet coefficient depends linearly on H (Lemma 2), a linear log-log regression will give the explicit form of the estimator. For fixed time, this variance of the wavelet coefficient have a more complex dependence on the Hurst parameter (Proposition 14) and a different argument will be employed.

2.5.1 The moving time case

First we introduce a discrete version of the wavelet variation (2.10). Then we define an estimator in terms of the discrete wavelet variation and we prove its asymptotic properties.

2.5.2 Discretization of the wavelet variation

We will use an estimator constructed by using the wavelet variation (2.10) or more precisely, by using its discretized version defined below. Notice that the wavelet coefficient $d(t, a, i)$ is defined as a continuous integral (see (2.9)) and it cannot be observed directly when the process u is observed. Therefore, by approximating the integral in (2.9) by Riemann sums, we define the discrete wavelet coeffcient, for $a > 0, t > 0$,

$$e_N(t, a, i) = \frac{1}{\sqrt{a}} \sum_{k=1}^{N} \Psi\left(\frac{k}{a} - i\right) u(t, k). \quad N \geq 1.$$

Since Ψ has its support contained in the interval $[0, 1]$, the above coefficient can also expressed as

$$e_N(t, a, i) = \frac{1}{\sqrt{a}} \sum_{k=0}^{[a]} \Psi\left(\frac{k}{a}\right) u(t, k + ai). \tag{2.72}$$

Let us also define the discrete version of the wavelet variation by setting

$$\widehat{V}_N(t, a) = \frac{1}{N_a} \sum_{i=1}^{N_a} \left(\tilde{e}(t, a, i)^2 - 1\right) \tag{2.73}$$

with, see notation (2.21),

$$\tilde{e}_N(t, a, i) = \frac{e_N(t, a, i)}{(\mathbf{E}(d(a, t, i))^2)^{\frac{1}{2}}} = \frac{1}{\sqrt{D(t, a)}} e_N(t, a, i).$$

In a first step, we will show that the sequence $\widehat{V}_N(t_N, a_N)$ has the same limit behavior in distribution as $V_N(t_N, a_N)$ when N goes to infinity. We need to assume some differentiability of the mother wavelet (several examples satisfy this assumption, among others the Daubechies wavelet or the mexican hat wavelet, see [25] or [27]).

Proposition 16 *Suppose that* $\Psi \in C^m(\mathbb{R})$ *with* $m > \frac{H\beta}{\alpha}$. *Assume (2.11) with* $\alpha \in \left(\frac{1}{2}, 1\right)$ *and let* $V_N(t_N, a_N), \widehat{V}_N(t_N, a_N)$ *be given by (2.10) and (2.73) respectively. Then*

$$\mathbf{E}\left|N^{\frac{1-\alpha}{2}}\left(V_N(t_N, a_N) - \widehat{V}_N(t_N, a_N)\right)\right| \to_{N\to\infty} 0.$$

Proof : We start by estimating the difference between the coefficient $d(t_N, a_N, i)$ and its discrete counterpart $e_N(t_N, a_N, i)$ with $i = 1, ..., N_{a_N}$ and with t_N, a_N as in (2.11). Let us compute the $L^2(\Omega)$-norm of this difference. We write

$$\mathbf{E}\left(d(t_n, a_N, i) - e_N(t_N, a_N, i)\right)^2 = \mathbf{E}d(t_N, a_N, i)^2 - 2\mathbf{E}d(t_N, a_N, i)e_N(t_N, a_N, i) + \mathbf{E}e_N(t_N, a_N, i)^2.$$

The first summand $\mathbf{E}d(t_N, a_N, i)^2$ has already been computed in (2.23). Let us compute the other two terms. For $N \geq 1$ and $i = 1, ..., N_{a_N}$ we have from the covariance formula (2.6)

$$\begin{aligned} \mathbf{E}e_N(t_N, a_N, i)^2 &= \frac{1}{a_N}\sum_{k,l=0}^{[a_N]}\Psi\left(\frac{k}{a_N}\right)\Psi\left(\frac{l}{a_N}\right)\mathbf{E}u(t_N, k + a_N i)u(t_N, l + a_N i) \qquad (2.74)\\ &= \frac{1}{a_N}\sum_{k,l=0}^{[a_N]}\Psi\left(\frac{k}{a_N}\right)\Psi\left(\frac{k}{a_N}\right)\left[\frac{c_H}{2}|k-l|^{2H+1} - \frac{t_N}{4}|k-l|^{2H} + \frac{t_N^{2H+1}}{2(2H+1)}\right]. \end{aligned}$$

We used the fact that $|k - l| \leq a_N = N^\alpha < t_N = N^\beta$ under (2.11), so the last summand in (2.6) vanishes. We also have from (2.6), (2.9) and (2.72)

$$\begin{aligned} &\mathbf{E}d(t_N, a_N, i)e_N(t_N, a_N, i) \qquad (2.75)\\ &= \sum_{k=0}^{[a_N]}\Psi\left(\frac{k}{a_N}\right)\int_{\mathbb{R}} dx\Psi(x)\mathbf{E}u(t_N, k + a_N i)u(t_N, a_N(x+i))\\ &= \sum_{k=0}^{[a_N]}\Psi\left(\frac{k}{a_N}\right)\int_{\mathbb{R}} dx\Psi(x)\left[\frac{c_H}{2}|k - a_N x|^{2H+1} - \frac{t_N}{4}|k - a_N x|^{2H} + \frac{t_N^{2H+1}}{2(2H+1)}.\right] \end{aligned}$$

Via (2.23), (2.74) and (2.75),

$$\begin{aligned} &\mathbf{E}\left(d(t_n, a_N, i) - e_N(t_N, a_N, i)\right)^2\\ &= \frac{c_H}{2}a_N^{2H+2}\left(\frac{1}{a_N^2}\sum_{k,l=0}^{[a_N]}\Psi\left(\frac{k}{a_N}\right)\Psi\left(\frac{l}{a_N}\right)\left|\frac{k-l}{a_N}\right|^{2H+1} + \int_{\mathbb{R}}\int_{\mathbb{R}} dxdy\Psi(x)\Psi(y)|x-y|^{2H+1}\right.\\ &\left.-2\frac{1}{a_N}\sum_{k=0}^{[a_N]}\Psi\left(\frac{k}{a_N}\right)\int_{\mathbb{R}} dx\Psi(x)\left|k - \frac{a_N}{x}\right|^{2H+1}\right)\\ &+\frac{c_H}{2}a_N^{2H+1}\left(\sum_{k,l=0}^{[a_N]}\Psi\left(\frac{k}{a_N}\right)\Psi\left(\frac{l}{a_N}\right)\left|\frac{k-l}{a_N}\right|^{2H} + \int_{\mathbb{R}}\int_{\mathbb{R}} dxdy\Psi(x)\Psi(y)|x-y|^{2H}\right. \end{aligned}$$

$$- 2\frac{1}{a_N}\sum_{k=0}^{[a_N]}\Psi\left(\frac{k}{a_N}\right)\int_{\mathbb{R}} dx\Psi(x)\left|k-\frac{a_N}{x}\right|^{2H}\Bigg)$$

$$+\frac{t_N^{2H+1}}{2(2H+1)}\frac{1}{a_N}\sum_{k,l=0}^{[a_N]}\Psi\left(\frac{k}{a_N}\right)\Psi\left(\frac{l}{a_N}\right). \tag{2.76}$$

Now we use the following bounds (we refer to [25] for their proofs, see also [27]) for N large

$$\left|\frac{1}{a_N^2}\sum_{k,l=0}^{[a_N]}\Psi\left(\frac{k}{a_N}\right)\Psi\left(\frac{l}{a_N}\right)\left|\frac{k-l}{a_N}\right|^{2H} - \int_{\mathbb{R}}\int_{\mathbb{R}} dxdy\Psi(x)\Psi(y)|x-y|^{2H}\right| \le C\frac{1}{a_N}, \tag{2.77}$$

and

$$\left|\frac{1}{a_N}\sum_{k=0}^{[a_N]}\Psi\left(\frac{k}{a_N}\right)\int_{\mathbb{R}} dx\Psi(x)\left|k-\frac{a_N}{x}\right|^{2H} - \int_{\mathbb{R}}\int_{\mathbb{R}} dxdy\Psi(x)\Psi(y)|x-y|^{2H}\right| \le C\frac{1}{a_N} \tag{2.78}$$

and, for Ψ of class $C^m(\mathbb{R})$,

$$\left|\frac{1}{a_N}\sum_{k=0}^{[a_N]}\Psi\left(\frac{k}{a_N}\right)\right| \le C\frac{1}{a_N^m}. \tag{2.79}$$

with $C>0$ not depending on N. By using the inequalities (2.77), (2.78) and (2.79) in (2.76), we obtain

$$\mathbf{E}\left(d(t_n,a_N,i)-e_N(t_N,a_N,i)\right)^2 \le C\left[a_N^{2H+1}+t_Na_N^{2H}+t_N^{2H+1}a_N^{1-2m}\right]$$
$$\le C\left[N^{(2H+1)\alpha}+N^{\beta+2H\alpha}+N^{(2H+1)\beta+\alpha(1-2m)}\right]. \tag{2.80}$$

For the renormalized coefficients, we have the estimate

$$\mathbf{E}\left(\tilde{d}(t_n,a_N,i)-\tilde{e}_N(t_N,a_N,i)\right)^2 = D(t_N,a_N)^{-1}\mathbf{E}\left(d(t_n,a_N,i)-e_N(t_N,a_N,i)\right)^2$$
$$\le CN^{-\beta+(2H+1)\alpha}\left[N^{(2H+1)\alpha}+N^{\beta+2H\alpha}+N^{(2H+1)\beta+\alpha(1-2m)}\right]$$
$$= C\left[N^{-\beta}+N^{-\alpha}+N^{2H\beta-\alpha(2m+2H)}\right].$$

If $m>\frac{H\beta}{\alpha}$, then for all $i=1,...,N_{a_N}$,

$$\mathbf{E}\left(\tilde{d}(t_n,a_N,i)-\tilde{e}_N(t_N,a_N,i)\right)^2 \le CN^{-\alpha}. \tag{2.81}$$

Finally, we regard the $L^1(\Omega)$-norm of the difference $V_N(t_N,a_N)-\widehat{V}_N(t_N,a_N)$. By using Cauchy-Scwarz inequality as proceeding as in the the proof of Lemma 1 in [27], we can write, with $C_1, C_2>0$,

$$\mathbf{E}\left|V_N(t_N,a_N)-\widehat{V}_N(t_N,a_N)\right|$$

$$\leq C_1 \left(\frac{1}{N_{a_N}} \sum_{i=1}^{N_{a_N}} \mathbf{E} \left(\widetilde{d}(t_N, a_N, i) - \widetilde{e}_N(t_N, a_N, i) \right)^2 \right)^{\frac{1}{2}}$$
$$\times \left(C_2 + 2 \frac{1}{N_{a_N}} \sum_{i=1}^{N_{a_N}} \mathbf{E} \left(\widetilde{d}(t_N, a_N, i) - \widetilde{e}_N(t_N, a_N, i) \right)^2 \right)^{\frac{1}{2}}$$

and by (2.81),

$$\mathbf{E} \left| V_N(t_N, a_N) - \widehat{V}_N(t_N, a_N) \right| \leq C N^{-\frac{\alpha}{2}} \left(C + \frac{1}{N^{\alpha}} \right)^{\frac{1}{2}} \leq C N^{-\frac{\alpha}{2}}.$$

Consequently,

$$\mathbf{E} \left| N^{\frac{1-\alpha}{2}} \left(V_N(t_N, a_N) - \widehat{V}_N(t_N, a_N) \right) \right| \leq C N^{\frac{1-\alpha}{2}} N^{-\frac{\alpha}{2}} = C N^{\frac{1}{2} - \alpha}$$

and the conclusion is obtained since $\alpha > \frac{1}{2}$. ■

As a consequence of the above result, the discrete wavelet variation $\widehat{V}_N(t_N, a_N)$ has the same limit in law as $V_N(t_N, a_N)$.

Corollary 1 *Let the assumptions in Proposition 16 prevail and let $\widehat{V}_N$ be given by (2.73). Then the d-dimensional random vector $\left(N^{\frac{1-\alpha}{2}} \widehat{V}_N(t_N, La_N) \right)_{L=1,..,d}$ converges in distribution, as $N \to \infty$, to a centered d-dimensional Gaussian vector with covariance matrix $\Gamma = (\Gamma_{L_1,L_2})_{L_1,L_2=1,..,d}$ where*

$$\Gamma_{L_1,L_2} = \frac{32}{K_{\Psi,H}^2} \frac{1}{(L_1 L_2)^{2H+1}} C(L_1, L_2, H) \tag{2.82}$$

with $C(L_1, L_2, H)$ in (2.70).

Proof : The proof immediately follows from Theorem 2 and Proposition 16. ■

2.5.3 The definition of the estimator

Let us denote, for $t > 0, a > 0$

$$S_N(t, a) := \frac{1}{N_a} \sum_{i=1}^{N_a} d(t, a, i)^2. \tag{2.83}$$

Notice that the sequence $S_N(t, a)$ is related to the wavelet variation $V_N(t, a)$ in (2.10) as follows

$$S_N(t, a) = D(t, a) \left(V_N(t, a) + 1 \right) \tag{2.84}$$

with $D(t,a)$ defined by (2.21). Let $d \geq 1$ and assume (2.11). By taking the expectation in (2.83) (note that $\mathbf{E}V_N(t_N, a_N) = 0$), we have, for every $L = 1, ..., d$, by using Lemma 1,

$$N^{-\beta-(2H+1)\alpha}\mathbf{E}S_N(t_N, La_N) = N^{-\beta-(2H+1)\alpha}D(t_N, La_N) \underset{N\to+\infty}{\to} \frac{1}{4}K_{\Psi,H}L^{2H+1} > 0$$

with $K_{\Psi,H}$ from Lemma 1. We write the above relation for $t_N = t_{La_N}$ (which means that we replace $t_N = N^\beta$ by $L^\beta N^{\alpha\beta}$ in (2.23). To do this, we will assume that in the sequel $\alpha\beta > 1$ and with this assumption all our theoretical results (such as Theorem 2) can be applied. So

$$N^{-\alpha\beta-(2H+1)\alpha}\mathbf{E}S_N(t_{La_N}, La_N) = N^{-\alpha\beta-(2H+1)\alpha}D(t_{La_N}, La_N) \underset{N\to+\infty}{\to} \frac{1}{4}K_{\Psi,H}L^{\beta+2H+1} > 0. \tag{2.85}$$

The above relation (2.85) implies, for $N \geq 1$,

$$\begin{aligned}
\log \mathbf{E}(S_N(t_{La_N}, La_N)) &= \log(D(t_{La_N}, La_N) \\
&= (\beta+2H+1)\alpha \log N + (\beta+2H+1)\log L + \log\left(\frac{1}{4}K_{\Psi,H}\right) + \log(1+\varepsilon_N) \\
&= (\beta+2H+1)(\alpha\log N + \log L) + \log\left(\frac{1}{4}K_{\Psi,H}\right) + \log(1+\varepsilon_N) \\
&= (\beta+2H+1)\log(La_N) + \log\left(\frac{1}{4}K_{\Psi,H}\right) + \log(1+\varepsilon_N) \qquad (2.86)
\end{aligned}$$

where $(\varepsilon_N)_{N\geq 1}$ is a a deterministic sequence defined by, for every $N \geq 1$,

$$\begin{aligned}
\varepsilon_N &= \left(\frac{1}{4}K_{\Psi,H}L^{2H+1}\right)^{-1} N^{-\beta-(2H+1)\alpha}\mathbf{E}S_N(t_N, La_N) \\
&= \left(\frac{1}{4}K_{\Psi,H}L^{2H+1}\right)^{-1} N^{-\alpha\beta-(2H+1)\alpha}D(t_{La_N}, La_N) \to_{N\to\infty} 0.
\end{aligned}$$

Let us also introduce the discretized counterpart of $S_N(t,a)$, i.e.

$$\widehat{S}_N(t,a) := \frac{1}{N_a}\sum_{i=1}^{N_a} e_N(t,a,i)^2 = D(t_{a_N}, a_N)(1+\widehat{V}_N(t_{a_N}, a_N)) \tag{2.87}$$

with e_N from (2.72).

The above relation (2.86) inspires the following definition of the estimator for the Hurst parameter

$$2\widehat{H}_N + \beta + 1 = (X^TX)^{-1}X^TY \tag{2.88}$$

where ^{T} denotes the transpose and with the notation

$$X = \log(La_N)_{L=1,..,d} = (\alpha\log N + logL)_{L=1,..,d} \text{ and } Y = \left(\log\widehat{S}_N(t_{La_N}, La_N)\right)_{L=1,..,d}. \tag{2.89}$$

Equivalently, we have

$$\widehat{H}_N = \frac{1}{2}(X^T X)^{-1} X^T Y - \frac{1}{2} = \frac{1}{2}\sum_{L=1}^{d} \frac{\log \widehat{S}_N(t_{La_N}, La_N)\log(La_N)}{\sum_{L=1}^{d}(\log(La_N))^2} - \frac{1}{2} - \frac{\beta}{2} \tag{2.90}$$

with X, Y from (2.89).

Remark 3 *1. The following heuristics leads to the expression (2.88) :*

- *we approximate* $\mathbf{E}(S_N(t_{La_N}, La_N)$ *by* $S_N(t_{La_N}, La_N)$ *and then* $\log S_N(t_{La_N}, a_N)$ *by* $\log \widehat{S}_N(t_{La_N}, a_N)$.
- *In the expression*

$$\log(\widehat{S}_N(t_{La_N}, La_N)) \quad \sim \quad (\beta + 2H + 1)\log(La_N) + \log\left(\frac{1}{4}K_{\Psi,H}\right) + \log(1 + \varepsilon_N)$$

we use a log-log regression of $(\log(S_N(t_{La_N}, La_N))_{L=1,..,d}$ *on* $(\log La_N)_{L=1,..,d} = (\alpha + \log L)_{L=1,..,d}$ *i.e. we minimize with respect to* $H \in \left(\frac{1}{2}, 1\right)$ *the function*

$$f(H) = \sum_{L=1}^{d} \left(\log S_N(t_{La_N}, La_N) - (2H + 1 + \beta)\log(La_N)\right)^2$$

2. *Notice that the estimator (2.90) is expressed in terms of the sequence (2.87) which depends on the discrete wavelet coefficients* e_N. *Therefore, the estimator can be computed from the data, that is, from the observations* $u(t, k), k = 1, 2, ..., N$ *with* $t = t_{a_N} = N^{\alpha\beta}$ *with* $\alpha\beta > 1$. *So, if we have at our disposal* N *observations in space, one needs to be able to observe them at time* $N^{\alpha\beta}$. *Recall that in practice our wave equation describes the vertical displacement of a vibrating string under a random force. This means that the observation time of the vibrating time should be sufficiently long and it is related to the number of spatial observations.*

Using Theorem 2, we can deduce the limit behavior of the estimator $\widehat{H}_N$.

Theorem 5 *Consider the estimator* $\widehat{H}_N$ *given by (2.90). Let the assumptions in Proposition 16 prevail. Assume also* $\alpha\beta > 1$. *Then the estimator (2.90) is stronly consistent and*

$$2\alpha(\log N)N^{\frac{1-\alpha}{2}}\left(\widehat{H}_N - H\right) \to^{(d)} N\left(0, \Gamma(1, ..., 1)^T\right) \tag{2.91}$$

with the matrix Γ *defined by (2.82).*

Proof : From (2.87) and (2.86), we have for large enough N,

$$\log \widehat{S}_N(t_{La_N}, La_N) \quad = \quad \log D(t_{La_N}, La_N) + \log\left(1 + \widehat{V}_N(t_{La_N}, La_N)\right)$$

$$= (\beta + 2H + 1)\log(La_N) + \log\left(\frac{1}{4}K_{\Psi,H}\right) + \log\left(1 + \widehat{V}_N(t_{La_N}, La_N)\right) + \log(1 + \varepsilon_N).$$

By plugging the above relation into (2.90), we obtain, for N large,

$$\begin{aligned}\beta + 2\widehat{H}_N + 1 &= \beta + 2H + 1 + \frac{\log\left(\frac{1}{4}K_{\Psi,H}\right)\sum_{L=1}^{d}\log(La_N)}{\sum_{L=1}^{d}(\log(La_N))^2} + \log(1 + \widehat{V}_N(t_{La_N}, La_N))\frac{\sum_{L=1}^{d}\log(La_N)}{\sum_{L=1}^{d}(\log(La_N))^2} \\ &\quad + \log(1+\varepsilon_N)\frac{\sum_{L=1}^{d}\log(La_N)}{\sum_{L=1}^{d}(\log(La_N))^2}\end{aligned}$$

and so

$$\begin{aligned}2(\widehat{H}_N - H) &= \frac{\log\left(\frac{1}{4}K_{\Psi,H}\right)\sum_{L=1}^{d}\log(La_N)}{\sum_{L=1}^{d}(\log(La_N))^2} + \log(1 + \widehat{V}_N(t_{La_N}, La_N))\frac{\sum_{L=1}^{d}\log(La_N)}{\sum_{L=1}^{d}(\log(La_N))^2} \\ &\quad + \log(1+\varepsilon_N)\frac{\sum_{L=1}^{d}\log(La_N)}{\sum_{L=1}^{d}(\log(La_N))^2}\end{aligned}$$

Note that $\widehat{V}_N(t_{a_N}, a_N)$ converges to zero almost surely as $N \to \infty$, this is a consequence of Proposition 12 and of a standard Borel-Cantelli argument, see e.g. [100]. Therefore, as ε_N tends to zero we get that $\widehat{H}_N \to_{N\to\infty} H$ almost surely and by using Theorem 2 we obtain the convergence (2.91). ■

2.5.4 Estimation when the time is fixed

Assume now that the time t is fixed, as in (2.12). We would like to estimate the parameter H of the mild solution (2.4) based on the observation of the solution at a fied time and at discrete points in space. The result in Proposition 14 shows that the behavior of the wavelet coefficient is not a power-function with exponent depending on H and their relationship is more complex. Actually

$$\mathbf{E}d(t, a_N, i)^2 = \frac{1}{N_{a_N}}\left[K_{1,t}(H) + K_{2,t}(H)N^{-\alpha}\right] \to_{N\to\infty} K_{1,t}(H) \qquad (2.92)$$

with $K_{1,t}(H), K_{2,t}(H)$ from Proposition 14. Therefore the log-log regression argument employed above cannot work when the time is fixed. We proposed and alternative method via the analysis of the constant $K_{1,t}(H)$.

Consider the sequence S_N given by (2.83) and assume now (2.12). By Proposition 14,

$$\mathbf{E}S_N(t, a_N) = D(t, a_N) = K_{1,t}(H) + K_{2,t}(H)N^{-\alpha} \to_{N\to\infty} K_{1,t}(H) \qquad (2.93)$$

with $K_{1,t}(H) = \frac{1}{2(H+1)}t^{2H+2}$, see (2.31). By approximating, as usual, $\mathbf{E}S_N(t, a_N)$ by $S_N(t, a_N)$, we can say that for N large enough, $S_N(t, a_N)$ is close to $K_{1,t}(H)$.

Lemma 4 *For some $t > 0$ sufficiently large (not depending on N), the equation*

$$S_N(t, a_N) - K_{1,t}(x) = 0 \tag{2.94}$$

has a unique solution in the interval $\left[\frac{1}{2}, 1\right]$.

Proof : Consider the function

$$f_{N,t}(H) = S_N(t, a_N) - K_{1,t}(H)$$

with $H \in \left[\frac{1}{2}, 1\right]$. We have

$$f'_{N,t}(x) = -\frac{\partial}{\partial H} K_{1,t}(H) = -f_1(H) t^{2H+2} \log t + f_2(H) t^{2H+2}$$

with

$$f_1(H) = \frac{1}{2(H+1)} > 0$$

and $f_2(H) = \frac{1}{2(H+1)^2}$ for $H \in \left[\frac{1}{2}, 1\right]$. When $t \to \infty$, this derivative behaves as $f_1(H) t^{2H+2} \log t$ so it is positive by choosing a suitable time t large enough. Consequently, the function $f_{N,t}$ is invertible on $\left[\frac{1}{2}, 1\right]$ and the conclusion follows. ■

We will assume in the sequel that t is large enough in order to ensures the existence and uniqueness of the solution to (2.94).

Definition 1 *We define $\widehat{H}_N$ to be the unique solution of the equation (2.94).*

We derive the asymptotic properties of the estimator $\widehat{H}_N$.

Proposition 17 *The estimator $\widehat{H}_N$ from Definition 1 is strongly consistent. Moreover, it satisfies the following limit behavior in distribution*

$$N^{\frac{1-\alpha}{2}} (\widehat{H}_N - H) \to^{(d)}_{N \to \infty} N\left(0, 2K_{1,t}(H)^2 \left(\frac{\partial}{\partial H} K_{1,t}^{-1}(H)\right)^2\right). \tag{2.95}$$

Proof : By Lemma 4, for some $t > 0$

$$S_N(t, a_N) = K_{1,t}(\widehat{H}_N)$$

and from (2.84),

$$S_N(t, a_N) = D(t, a_N)(1 + V_N(t, a_N)).$$

Thus

$$K_{1,t}(\widehat{H}_N) = D(t, a_N)(1 + V_N(t, a_N)).$$

We let $N \to \infty$ above. Since $V_N(t, a_N)$ tends to zero almost surely and $D(t, a_N)$ converges to $K_{1,t}(H)$ as $N \to \infty$, we get

$$\lim_{N\to\infty} K_{1,t}(\widehat{H}_N) = K_{1,t}(H) \text{ almost surely.} \tag{2.96}$$

By the proof of Lemma 4, we deduce that $K_{1,H}$ is invertible on $\left[\frac{1}{2}, 1\right]$ and its inverse is continuously differentiable on this interval. By applying $K_{1,t}^{-1}$ to (2.96) we deduce that $\widehat{H}_N \to_{N\to\infty} H$ almost surely.

Let us show that the estimator is asymptotically normal. Indeed, from (2.84)

$$\frac{1}{D(t, a_N)} \left(S_N(t, a_N) - D(t, a_N)\right) = V_N(t, a_N)$$

and consequently, in distribution, by Theorem 3

$$\frac{1}{\sqrt{2}} N^{\frac{1-\alpha}{2}} \frac{1}{D(t, a_N)} \left(S_N(t, a_N) - D(t, a_N)\right) \to_{N\to\infty} N(0, 1).$$

Given the asymptotic behavior of $D(t, a_N)$ (see 2.93)), we can write

$$N^{\frac{1-\alpha}{2}} \left(K_{1,t}(\widehat{H}_N) - K_{1,t}(H)\right) \to^{(d)}_{N\to\infty} N(0, 2K_{1,t}(H)^2).$$

By using the delta-method with the continuously differentiable function $K_{1,t}^{-1}$ on $\left[\frac{1}{2}, 1\right]$, we obtain (2.95). ■

Let us end this statistical inference part with some comments :

Remark 4 *1. There exists alternative ways to estimate the Hurst parameter for t fixed by exploiting the relation (2.92). But using this result for different fixed times $t_1, ..., t_d > 0$, and by applying a nonlinear least-squares regression of $S_N(t_i, a_N)$ on t_i with $i = 1, ..., d$ we can obtain another estimator for H (which should be, in principle, consistent and asymptotically normal). Being related to nonlinear regression, it analysis could be more complex but probabably we can avoid the restriction of t large, assumed in Proposition 17.*

2. The estimator from Definition 1 is based on the variation $S_N(t, a, i)$ which is written in terms of the continuous wavelet transforms $d(t, a, i)$ and the of the inverse function $K_{1,t}^{-1}$. While the inverse can be (at least numerically) computed, the wavelet coefficients are not directly computed from the observations $u(t, k), k = 1, ..., N$. An approach to compute them is use Riemann sums approximations, as in (2.72). Another possibility is to use a pyramidal multiresolution algorithm (see for example the survey [26]).

3. Other methods (not based on wavelets) to estimate the Hurst parameter based on observations of (2.4) at fixed time are obtained via the generalized spatial variations of the solution (see e.g. [66] or [92]).

2.6 Appendix

The basic tools from the analysis on Wiener space are presented in this section. We will focus on some elementary facts about multiple stochastic integrals. We refer to [77] for a complete review on the topic.

Consider $\mathcal{H}$ a real separable infinite-dimensional Hilbert space with its associated inner product $\langle .,. \rangle_{\mathcal{H}}$, and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}}$ for every $\varphi, \psi \in \mathcal{H}$. Denote by I_q the qth multiple stochastic integral with respect to B, which is an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}}\|\cdot\|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by :

$$H_q(x) = (-1)^q \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^q}{\mathrm{d}x^q}\left(\exp\left(-\frac{x^2}{2}\right)\right), \; x \in \mathbb{R}. \tag{2.97}$$

The isometry of multiple integrals can be written as follows : for $p, \; q \geq 1$, $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$

$$\mathbf{E}\Big(I_p(f)I_q(g)\Big) = \begin{cases} q!\langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}^{\otimes q}} & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases} \tag{2.98}$$

We have the following product formula : if $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$, then

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}\left(f \otimes_r g\right) \tag{2.99}$$

where $f \otimes_r g$ denotes the contraction of order $r = 0, 1, .., p \wedge q$. We denote by D the Malliavin derivative operator that acts on cylindrical random variables of the form $F = g(B(\varphi_1), \ldots, B(\varphi_n))$, where $n \geq 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support and $\varphi_i \in \mathcal{H}$ in the following way

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i.$$

The operator D is closable and it can be extended to the closure of the set of cylindrical random variables (denotes $\mathbb{D}^{1,2}$) with respect to the norm

$$\|F\|_{1,2}^2 := \mathbf{E}|F|^2 + \mathbf{E}\|DF\|_{\mathcal{H}}^2.$$

If $F = I_p(f)$ with $f \in \mathcal{H}^{\odot p}$ and $p \geq 1$, then

$$DF = pI_{p-1}\left(f(\cdot, *)\right) \tag{2.100}$$

where " $*$ " stands for $p-1$ variables.

Chapitre 3

Parameter identification for the Hermite Ornstein-Uhlenbeck process

This article is published on Stat Inference Stoch Process 23, 251–270 (2020). https ://doi.org/10.1007/s11203-020-09219-z

Joint work with Ciprian Tudor

Abstract By using the analysis on Wiener chaos, we study the behavior of the quadratic variations of the Hermite Ornstein–Uhlenbeck process, which is the solution to the Langevin equation driven by a Hermite process. We apply our results to the identification of the Hurst parameter of the Hermite Ornstein–Uhlenbeck process.

3.1 Introduction

The Hermite processes are non-Gaussian extensions of the fractional Brownian motion (fBm). They share many properties with fBm, such as self-similarity, stationarity of the increments or pathwise regularity, and this fact makes them potential good models for various phenomena. The recent monographs [87] and [100] offer a rather complete description of these stochastic processes. The stochastic analysis with respect to the Hermite processes has started to develop in the last years and several types of stochastic (partial) differential equations with Hermite noise have been considered (see e.g. [30], [41], [42], [43], [56], [84], [93], [94], [95], [99]).

Let $T > 0$. Our purpose is to discuss the estimation of the parameters for the following model

$$dX_t = -aX_t dt + \sigma dZ_t^{(q,H)}, \quad t \in [0, T] \tag{3.1}$$

with initial condition $X_0 = \xi \in L^2(\Omega)$ and $a \in \mathbb{R}, \sigma > 0$. The noise $Z^{(q,H)}$ in (3.1) is a Hermite process of order $q \geq 1$ with self-similarity parameter $H \in \left(\frac{1}{2}, 1\right)$ (see the next section for its definition and properties).

The statistical inference for systems with Hermite noise is at its beginning. The estimation of the Hurst parameter of the Hermite process itself has been treated in [27], [39], [103]. A first step to estimate the parameter of a stochastic equation driven by a Hermite process has been done in [84], where the authors constructed and analyzed estimators for the drift parameter a of the equation (3.1) based on the continuous time observation of the process $(X_t)_{t\geq 0}$.

Here, we will focus on the identification of the other two parameters that appear in (3.1), i.e. the Hurst parameter H and the diffusion parameter σ. We aim at constructing estimators for these parameters from the observation of the process X at discrete times $\frac{i}{N}$, with $i = 0, 1, \ldots, N$. We will follow a well-known approach, based on the quadratic (or higher order) variations of the solution X (see [100] and the references therein). Our estimators are expressed in terms of the quadratic (and generalized) variations of the solution to the Langevin equation (3.1) and their limit behavior will be strongly connected with the limit behavior of these variations. Therefore, we start by analyzing the sequence $V_N(X)$ given by (3.11). In order to do this, we will use the fact that the solution to (3.1) is a Wiener integral with respect to the Hermite process $Z^{(q,H)}$ and consequently it can be written as a multiple stochastic integral of order $q \geq 1$ with respect to the Wiener process. We will also use the limit behavior of the variations of the random noise of (3.1) which has been studied in [39], [103] and which will actually induce the limit behavior of the quadratic variation of the solution. Our techniques are mainly based on the properties of the random variables in Wiener chaos. Our results are new for every order $q \geq 1$. For $q = 1$, a related result has been obtained in [54], where the case of a stochastic equation driven an additive fBm has been treated. Notice that our result for $q = 1$ is not covered by the findings in [54], since in this reference a bounded drift is considered.

We discuss several situations : the case when the initial condition ξ in (3.1) is deterministic (this is called the nonstationary case, because the law of the corresponding process X is not stationary) and the situation when the initial value of the Langevin equation is a particular random variable which makes the solution a stationary process. Even if many calculations are similar in both cases, the fact that in the stationary case we work with integrals over intervals with infinite Lebesgue measure requires different arguments in the proofs.

We organized the paper as follows. Section 2 contains the definition and the basic properties of the Hermite processes and of the Wiener integrals with respect to them. In Section 3 we recall the basic facts about the solution to (3.1) and we study the asymptotic behavior of its quadratic variations. In Section 4 we discuss the estimation of the Hurst parameter of this solution. We provide a consistent estimator for H (which does not depend on σ) and we give its limit behavior in law. We also discuss the estimation of the parameter σ by assuming that H is known. The last section is the appendix where we recall the main facts on multiple stochastic integrals.

3.2 Hermite processes and Wiener-Hermite Integrals

In this preliminary part, we introduce the Hermite processes and the Wiener-Hermite integrals.

3.2.1 Hermite processes

We will denote by $(Z_t^{(q,H)})_{t\geq 0}$ the Hermite process of order $q \geq 1$ and with self-similarity index $H \in \left(\frac{1}{2}, 1\right)$. It lives in the qth Wiener chaos and it is defined as a multiple Wiener-Itô integral, for every $t \geq 0$,

$$Z_t^{(q,H)} = d(q,H) \int_{\mathbb{R}} dB(y_1) \dots \int_{\mathbb{R}} dB(y_q) \left(\int_0^t (s-y_1)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \dots (s-y_q)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} ds \right) \tag{3.2}$$

where $x_+^\alpha = x^\alpha 1_{(0,\infty)}(x)$, $d(q,H)$ is a normalizing positive constant chosen such that $\mathbf{E}\left(Z_1^{(q,H)}\right)^2 = 1$ and $(B(y))_{y\in\mathbb{R}}$ is a Wiener process with time interval $\mathbb{R}$ (see the Appendix for the definition and basic properties of multiple stochastic integrals). The process (3.2) is H-self-similar and it has stationary increments and long memory. Its trajectories are Hölder continuous of order $\delta \in (0,H)$. We will denote by L_t the kernel of the Hermite process, i.e.

$$L_t(y_1,...,y_q) = d(q,H) \int_0^t (s-y_1)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \dots (s-y_q)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} ds. \tag{3.3}$$

For every $t \geq 0$, one can show that L_t belongs to $L^2(\mathbb{R}^q)$ and this ensures the existence of the multiple integral (3.2). A random variable with the same law as $Z_1^{(q,H)}$ is called *Hermite random variable* with Hurst parameter H while for $q = 2$, it is usually called *Rosenblatt random variable*. These random variables have non Gaussian probability distributions, which are absolutely continuous with respect to the Lebesgue measure, with a rather complex structure. While some information is known for the Roseblatt distribution, very few facts are known for the law of general Hermite variables. In particular, the covariance of the Hermite process $Z^{(q,H)}$ is the same for all integer $q \geq 1$ and it coincides with the covariance of the fBm, i.e.

$$\mathbf{E} Z_t^{(q,H)} Z_s^{(q,H)} = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \geq 0.$$

The class of Hermite processes includes the fractional Brownian motion which is obtained for $q = 1$ and the Rosenblatt process ($q = 2$). The fBm is the only Gaussian Hermite process. The Hermite process is non-Gaussian if $q \geq 2$. These processes have been widely studied since the seventies (see the monographs [87], [100] and the references therein).

3.2.2 Wiener-Hermite integrals

Consider the Hilbert space $\mathcal{H}$ associated with the fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$, that is,

$$\mathcal{H} = \left\{ f : \mathbb{R} \to \mathbb{R} : \|f\|_{\mathcal{H}}^2 < +\infty \right\} \tag{3.4}$$

where

$$\|f\|_{\mathcal{H}}^2 := H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)f(v)|u-v|^{2H-2} dudv.$$

For $f \in \mathcal{H}$, one can construct Wiener integrals with respect to the Hermite process (see [73]). This object, called in the sequel as *Wiener-Hermite integral*, will be denoted by

$$\int_{\mathbb{R}} f(s) dZ_s^{(q,H)}.$$

It is well-defined for $f \in \mathcal{H}$ and it is the element of the qth Wiener chaos given by

$$\int_{\mathbb{R}} f(s) dZ_s^{(q,H)} = I_q(Jf) \tag{3.5}$$

where I_q denotes the multiple integral of order q with respect to the standard Brownian motion $(B(y))_{y \in \mathbb{R}}$ and $Jf \in L^2(\mathbb{R}^q)$ is given by

$$Jf(y_1, ..., y_q) = d(H,q) \int_{\mathbb{R}} du f(u)(u-y_1)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \dots (u-y_q)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)}. \tag{3.6}$$

The Wiener-Hermite integral satisfies the following isometry

$$\mathbf{E}\left(\int_{\mathbb{R}} f(s) dZ_s^{(q,H)} \int_{\mathbb{R}} g(s) dZ_s^{(q,H)} \right) = \langle f, g \rangle_{\mathcal{H}} \tag{3.7}$$

where, for every $f, g \in \mathcal{H}$ such that $\int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||g(v)||u-v|^{2H-2} dudv < \infty$,

$$\langle f, g \rangle_{\mathcal{H}} = H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)g(v)|u-v|^{2H-2} dudv.$$

Notice that the isometry formula (3.7) holds for any $q \geq 1$. Consequently, for every $q \geq 1$ the variance of the Wiener-Hermite integral coincides with the variance of the Wiener integral with respect to the fractional Brownian motion. On the other hand, the law of the object (3.5) is much more complicated when $q \geq 2$.

3.3 The Hermite Ornstein-Uhlenbeck process and its quadratic variation

Let $T > 0$ and let $\left(Z_t^{(q,H)}\right)_{t \in [0,T]}$ be a Hermite process of order $q \geq 1$ with self-similarity order $H \in \left(\frac{1}{2}, 1\right)$.

The *Hermite Ornstein-Uhlenbeck process (HOU process in the sequel)* $(X_t)_{t\in[0,T]}$ is defined as the unique solution of the Langevin equation (3.1) with initial condition $X_0 = \xi \in L^2(\Omega)$ and $a \in \mathbb{R}, \sigma > 0$. When ξ is deterministic, we will call the process $(X_t)_{t\in[0,T]}$ as *the nonstationary HOU process* while for ξ given by (3.31), we will use the name *stationary HOU process*. This is related to the stationarity of finite dimensional distributions of these stochastic processes which will be discussed in the following section.

Our purpose is to estimate the Hurst parameter H and the diffusion parameter σ from the discrete observations of the process X on the time interval $[0,1]$. Since we are not concerned with the estimation of the drift parameter (which has been treated in [84]), we will choose $a = 1$ in the sequel.

We will separately treat the stationary and the nonstationary cases. Generally speaking, several of our results follow easier in the nonstationary case, due to the fact that we work on a interval with finite Lebesgue measure, while some arguments in the proofs need more care in the stationary case.

3.3.1 The (nonstationary) HOU process

If we assume the initial value ξ in (3.1) is deterministic, it will play no role in our computations and we can assume without loss of generality that $\xi = 0$. In this case, the explicit of the solution to (3.1) is for every $t \in [0,T]$ (recall that we assumed $a = 1$)

$$X_t = \sigma e^{-t} \int_0^t e^u dZ_u^{(q,H)}. \tag{3.8}$$

The stochastic integral with respect to the Hermite process $Z^{(q,H)}$ is well-defined both in the Wiener and in the Riemann-Stieltjes sense (see [38], [73], [84], [93], [94]). The HOU process lives in the qth Wiener chaos and for every $t \in [0,T]$, the random variable X_t can be written as a multiple stochastic integral, i.e.

$$\begin{aligned} X_t &= \sigma d(q,H) \int_{\mathbb{R}^q} \left(\int_0^t e^{-(t-u)} \prod_{i=1}^q (u-y_i)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} du \right) dB(y_1)\dots dB(y_q) \\ &:= I_q(h_t) \end{aligned}$$

where $d(q,H)$ from (3.2), where we used the notation

$$h_t(y_1,..,y_q) = \sigma d(q,H) \int_0^t e^{-(t-u)} \prod_{i=1}^q (u-y_i)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} du \tag{3.9}$$

for every $t \in [0,T]$ and for every $y_1,\dots,y_q \in \mathbb{R}$.

Several properties of the HOU process have been obtained in the literature. Let us recall that this process has the same covariance for every $q \geq 1$, which is given by

$$\mathbf{E} X_t X_s = \sigma^2 H(2H-1) \int_0^t \int_0^s e^{-(t-u)-(s-v)} |u-v|^{2H-2} dv du, \quad s,t \in [0,T].$$

It follows from [84], [94] that, with some $C > 0$,

$$\sup_{t\in[0,T]} \mathbf{E}|X_t|^2 < C.$$

By the hypercontractivity property (3.50) of the random variables in Wiener chaos, we also get, for every $p \geq 2$

$$\sup_{t\in[0,T]} \mathbf{E}|X_t|^p < C \tag{3.10}$$

with $C > 0$ depending only on p and T.

As mentioned before, our purpose is to use the quadratic variation of the process (3.8) in order to identify its Hurst parameter. Let us consider the partition of the unit interval $[0,1]$ given by $t_i = \frac{i}{N}$ for $i = 0, ..., N$ for $N \geq 1$. We will assume that the process X is observed at times t_i and we will define estimators for the parameters H and σ in (3.1) based on these observations. Define, for every $N \geq 1$, the sequence of (centered and renormalized) quadratic variations

$$V_N(X) = \frac{1}{N}\sum_{i=0}^{N-1}\left[\frac{(X_{t_{i+1}} - X_{t_i})^2}{\sigma^2\mathbf{E}\left(Z^{(q,H)}_{t_{i+1}} - Z^{(q,H)}_{t_i}\right)^2} - 1\right] = \frac{1}{N}\sum_{i=0}^{N-1}\left[\frac{N^{2H}}{\sigma^2}(X_{t_{i+1}} - X_{t_i})^2 - 1\right]. \tag{3.11}$$

We aim at finding the limit behavior, as $N \to \infty$, of the sequence $V_N(X)$. We will benefit from the behavior of the quadratic variation of the noise of (3.1) which is well-known : while for $q = 1$, this is the famous Breuer-Major theorem (see e.g. [31], [81]), for $q \geq 2$, it has been obtained in [103] and [39]. Let us recall these results. By "$\to^{(d)}$" we denote the convergence in distribution and by $N(0,1)$ we indicate the standard normal law.

Theorem 6 *Assume $H \in \left(\frac{1}{2}, 1\right)$ and $q \geq 1$ integer. Let $V_N(Z^{(q,H)})$ be given by*

$$V_N(Z^{(q,H)}) = \frac{1}{N}\sum_{i=0}^{N-1}\left[\frac{\left(Z^{(q,H)}_{t_{i+1}} - Z^{(q,H)}_{t_i}\right)^2}{\mathbf{E}\left(Z^{(q,H)}_{t_{i+1}} - Z^{(q,H)}_{t_i}\right)^2} - 1\right]. \tag{3.12}$$

Then

1. *If $q = 1$ and $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$,*

$$K_{1,1}\sqrt{N}V_N(Z^{(1,H)}) \to^{(d)}_{N\to\infty} N(0,1). \tag{3.13}$$

2. *If $q \geq 2$ or $q = 1$ and $H > \frac{3}{4}$, then*

$$K_q N^{\frac{2-2H}{q}} V_N(Z^{(q,H)}) \to_{N\to\infty} Z_1^{(2,H')} \text{ in } L^2(\Omega) \tag{3.14}$$

where $Z_1^{(2,H')}$ is a Rosenblatt random variable with Hurst paramter $H' = \frac{2(H-1)}{q} + 1$. The constants $K_{1,1}, K_i, i = 1, .., q$ are explicit, we refer to [81], [39] for their expression.

Actually the result at point 1. above holds for every $H \in (0, \frac{3}{4})$ (and even for $H = \frac{3}{4}$ under a different renormalization) but this case will be not discussed in our work.

We will prove in the sequel that $V_N(X)$ keeps the same behavior as the quadratic variations of the noise of (3.1). This means that the drift process $(Y_t)_{t\in[0,T]}$ given by

$$Y_t = -\int_0^t X_s ds, \qquad t \in [0,T] \tag{3.15}$$

does not affect the behavior of $V_N(X)$. This is due to the regularity of Y and to the fact that the correlation between the increments of Y and $Z^{(q,H)}$ is weak enough.

Proposition 18 *Let $V_N(X)$ be given by (3.11).*

1. *Assume $H \in \left(\frac{1}{2}, 1\right)$ and $q \geq 2$. Then, with $K_q, Z_1^{(2,H')}$ from (3.14)*

$$K_q N^{\frac{2-2H}{q}} V_N(X) \to_{N\to\infty} Z_1^{(2,H')} \text{ in } L^2(\Omega).$$

2. *If $q = 1$ and $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$,*

$$K_{1,1}\sqrt{N} V_N(X) \to^{(d)}_{N\to\infty} N(0,1).$$

with $K_{1,1}$ from (3.13).

Proof : Assume first $q \geq 2$. We decompose $V_N(X)$ as follows :

$$\begin{aligned} V_N(X) &= \frac{1}{N}\sum_{i=0}^{N-1}\left[N^{2H}\left(Z_{t_{i+1}}^{(q,H)} - Z_{t_i}^{(q,H)}\right)^2 - 1\right] + \frac{1}{\sigma^2}\frac{1}{N}\sum_{i=0}^{N-1} N^{2H}\left(Y_{t_{i+1}} - Y_{t_i}\right)^2 \\ &\quad + \frac{2}{\sigma}\frac{1}{N}\sum_{i=0}^{N-1} N^{2H}\left(Y_{t_{i+1}} - Y_{t_i}\right)\left(Z_{t_{i+1}}^{(q,H)} - Z_{t_i}^{(q,H)}\right) \\ &= V_N(Z^{(q,H)}) + T_{1,N} + T_{2,N} \end{aligned} \tag{3.16}$$

with

$$T_{1,N} = \frac{1}{\sigma^2} N^{2H-1} \sum_{i=0}^{N-1} \left(Y_{t_{i+1}} - Y_{t_i}\right)^2 \tag{3.17}$$

and

$$T_{2,N} = \frac{2}{\sigma} N^{2H-1} \sum_{i=0}^{N-1} \left(Y_{t_{i+1}} - Y_{t_i}\right)\left(Z_{t_{i+1}}^{(q,H)} - Z_{t_i}^{(q,H)}\right). \tag{3.18}$$

In 93.16), the limit of the sequence $V_N(Z^{(q,H)})$ is known from Proposition 18. We will prove that the other terms does not contribute to the limit. That is, we show that, for $i = 1, 2$ and for every $p \geq 1$

$$N^{\frac{2-2H}{q}} T_{i,N} \to_{N\to\infty} 0 \text{ in } L^p(\Omega). \tag{3.19}$$

The summand $T_{1,N}$ can be easily estimated, by using Hölder inequality. Indeed, for every $p \geq 1$,

$$\begin{aligned}
\mathbf{E}|T_{1,N}|^p &= \sigma^{-2p} N^{(2H-1)p} \mathbf{E} \left| \sum_{i=0}^{N-1} \left(\int_{t_i}^{t_{i+1}} X_s ds \right)^2 \right|^p \leq \sigma^{-2p} N^{(2H-1)p} N^{(p-1)} \mathbf{E} \sum_{i=0}^{N-1} \left(\int_{t_i}^{t_{i+1}} X_s ds \right)^{2p} \\
&\leq \sigma^{-2p} N^{(2H-1)p} N^{-p} \mathbf{E} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |X_s|^{2p} ds \leq C N^{(2H-2)p} \qquad (3.20)
\end{aligned}$$

where we used (3.10). Consequently (3.19) holds true for $i = 1$, since for every $p \geq 1$

$$\mathbf{E} \left| N^{\frac{2-2H}{q}} T_{1,N} \right|^p \leq C N^{(2H-2)(1-\frac{1}{q})p} \to_{N \to \infty} 0.$$

To analyze the term $T_{2,N}$, we need to use its Wiener chaos decomposition (actually, a direct proof based on Hölder inequality can be done only for $q \geq 3$). We can write, for every $N \geq 1$ and for every $i = 0, ..., N$,

$$Y_{t_{i+1}} - Y_{t_i} = I_q(h_{i,N})$$

where $h_{i,N} = h_{t_{i+1}} - h_{t_i}$ (h_t is given in (3.9)), i.e.

$$h_{i,N}(y_1, \ldots, y_q) = \sigma d(q, H) \int_{t_i}^{t_{i+1}} ds \int_0^s du e^{-(s-u)} \prod_{l=1}^{q} (u - y_l)_+^{-(\frac{1}{2} + \frac{1-H}{q})} \qquad (3.21)$$

and

$$Z_{t_{i+1}}^{(q,H)} - Z_{t_i}^{(q,H)} = I_q(\ell_{i,N})$$

with (recall that L_t is the kernel of the Hermite process, see (3.3))

$$\ell_{i,N}(y_1, \ldots, y_q) = L_{t_{i+1}}(y_1, \ldots, y_q) - L_{t_i}(y_1, \ldots, y_q) \qquad (3.22)$$

for every $y_1, ..., y_q \in \mathbb{R}$. In this way, via the product formula for multiple integrals (3.49),

$$\begin{aligned}
T_{2,N} &= \frac{2}{\sigma} N^{2H-1} \sum_{i=0}^{N-1} I_q(h_{i,N}) I_q(\ell_{i,N}) \\
&= \frac{2}{\sigma} N^{2H-1} \sum_{i=0}^{N-1} \sum_{r=0}^{q} r! (C_q^r)^2 I_{2q-2r}(h_{i,N} \otimes_r \ell_{i,N}) := \sum_{r=0}^{q} T_{2,N}^{(r)}
\end{aligned}$$

with, for $r = 0, .., q$,

$$T_{2,N}^{(r)} = \frac{2}{\sigma} N^{2H-1} r! \binom{r}{q}^2 \sum_{i=0}^{N-1} I_{2q-2r}(h_{i,N} \otimes_r \ell_{i,N}). \qquad (3.23)$$

We will obtain (3.19) for $i = 2$ if we show that

$$\mathbf{E}|N^{\frac{2-2H}{q}} T_{2,N}^{(r)}|^2 \to_{N \to \infty} 0 \qquad (3.24)$$

via the hypercontractivity property (3.50). Assume $r = q$. Notice that $h_{i,N}, \ell_{i,N}$ are symmetric functions. Below, we denote by c, C generic strictly positive constants not depending on N and that are allowed to change from one line to another. We have

$$T_{2,N}^{(q)} = cN^{2H-1} \sum_{i=0}^{N-1} \langle h_{i,N}, \ell_{i,N} \rangle_{L^2(\mathbb{R}^q)}$$

with $h_{i,N}, \ell_{i,N}$ given by (3.21), (3.22) respectively. Thus $T_{2,N}^{(q)}$ is a deterministic sequence and

$$\begin{aligned} |T_{2,N}^{(q)}| &= cN^{2H-1} \sum_{i=0}^{N-1} \int_{\mathbb{R}^q} dy_1 ... dy_q \int_{t_i}^{t_{i+1}} ds \int_0^s du e^{-(s-u)} \\ &\times \prod_{j=1}^{q} (u - y_j)_+^{-(\frac{1}{2}+\frac{1-H}{q})} \int_{t_i}^{t_{i+1}} dv \prod_{j=1}^{q} (v - y_j)_+^{-(\frac{1}{2}+\frac{1-H}{q})}. \end{aligned}$$

By using the formula (see [100] Lemma 3.1) : if $a \in (0, \frac{1}{2})$ and β is the Beta function,

$$\int_{\mathbb{R}} (u - y)_+^{a-1} (v - y)_+^{a-1} dy = \beta(a, 1 - 2a) |u - v|^{2a-1}. \tag{3.25}$$

we obtain

$$\begin{aligned} |T_{2,N}^{(q)}| &= cN^{2H-1} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} ds \int_0^s du e^{-(s-u)} \int_{t_i}^{t_{i+1}} dv |u - v|^{2H-2} \\ &\leq cN^{2H-1} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} ds \int_0^1 du \int_{t_i}^{t_{i+1}} dv |u - v|^{2H-2} \leq CN^{2H-2} \end{aligned} \tag{3.26}$$

and (3.24) holds for $r = q$ because we assumed $q \geq 2$.

Now assume $1 \leq r \leq q - 1$. We study the sequence

$$T_{2,N}^{(r)} = cN^{2H-1} I_{2q-2r} \left(\sum_{i=0}^{N-1} h_{i,N} \otimes_r \ell_{i,N} \right)$$

where, again via (3.25) and Fubini,

$$\begin{aligned} (h_{i,N} \otimes_r \ell_{i,N})(y_1, ..., y_{2q-2r}) &= cN^{2H-1} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} ds \int_0^s du e^{-(s-u)} \int_{t_i}^{t_{i+1}} dv |u - v|^{(2H-2)\frac{r}{q}} \\ & (u - y_1)_+^{-(\frac{1}{2}+\frac{1-H}{q})} ... (u - y_{q-r})_+^{-(\frac{1}{2}+\frac{1-H}{q})} \\ & \times (v - y_{q-r+1})_+^{-(\frac{1}{2}+\frac{1-H}{q})} ... (v - y_{2q-2r})_+^{-(\frac{1}{2}+\frac{1-H}{q})} \end{aligned}$$

for every $y_1, ..., y_{2q-2r} \in \mathbb{R}$. By isometry (see (3.47))

$$\begin{aligned}
\mathbf{E}\left|T_{2,N}^{(r)}\right|^2 &= cN^{4H-2}\|\sum_{i=0}^{N-1} h_{i,N}\tilde{\otimes}_r \ell_{i,N}\|^2_{L^2(\mathbb{R}^{2q-2r})} \leq cN^{4H-2}\|\sum_{i=0}^{N-1} h_{i,N}\otimes_r \ell_{i,N}\|^2_{L^2(\mathbb{R}^{2q-2r})} \\
&= CN^{4H-2}\sum_{i,j=0}^{N-1} \langle h_{i,N}\otimes_r \ell_{i,N}, h_{j,N}\otimes_r \ell_{j,N}\rangle_{L^2(\mathbb{R}^{2q-2r})} \\
&= CN^{4H-2}\sum_{i,j=0}^{N-1}\int_{t_i}^{t_{i+1}} ds\int_0^s du e^{-(s-u)}\int_{t_i}^{t_{i+1}} dv\int_{t_j}^{t_{j+1}} ds'\int_0^{s'} du' e^{-(s'-u')}\int_{t_j}^{t_{j+1}} dv' \\
&\quad \times|u-v|^{(2H-2)\frac{r}{q}}|u'-v'|^{(2H-2)\frac{r}{q}}|u-u'|^{(2H-2)\frac{(q-r)}{q}}|v-v'|^{(2H-2)\frac{(q-r)}{q}} \qquad (3.27)
\end{aligned}$$

where we used again (3.25). Now, we majorize the exponential function by 1 and the integral over $[0,s]\times[0,s']$ by the integral over $[0,1]^2$. We will obtain, for every $r = 1, ..., q-1$,

$$\begin{aligned}
\mathbf{E}\left|T_{2,N}^{(r)}\right|^2 &\leq CN^{4H-2}\sum_{i,j=0}^{N-1}\int_0^1 du\int_0^1 du'\int_{t_i}^{t_{i+1}} dv\int_{t_j}^{t_{j+1}} dv' \\
&\quad \times|u-v|^{(2H-2)\frac{r}{q}}|u'-v'|^{(2H-2)\frac{r}{q}}|u-u'|^{(2H-2)\frac{(q-r)}{q}}|v-v'|^{(2H-2)\frac{(q-r)}{q}} \\
&\leq CN^{4H-4} \qquad (3.28)
\end{aligned}$$

and consequently (3.24) holds since (see e.g. [39])

$$\int_{[0,1]^4} dudu'dvdv'|u-v|^{(2H-2)\frac{r}{q}}|u'-v'|^{(2H-2)\frac{r}{q}}|u-u'|^{(2H-2)\frac{(q-r)}{q}}|v-v'|^{(2H-2)\frac{(q-r)}{q}} < \infty$$

and since (3.28) and (3.26) give, for every $p \geq 2$

$$\mathbf{E}|T_{2,N}|^p \leq CN^{(2H-2)p}. \qquad (3.29)$$

Assume $q = 1$ and $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$. In this case we still have the decomposition (3.16) and the estimates (3.20), (3.26) and (3.28) (with $r = 0$). These bounds clearly imply, via (3.50), that for every $p \geq 1$

$$\mathbf{E}\left|\sqrt{N}T_{i,N}\right|^p \leq CN^{(2H-\frac{3}{2})p} \to_{N\to\infty} 0 \text{ for } i = 1,2 \qquad (3.30)$$

for $H < \frac{3}{4}$, with $T_{1,N}, T_{2,N}$ given by (3.17), (3.18) respectively. ■.

Remark 5 • *The case $H > \frac{3}{4}$ and $q = 1$ is not covered by our result in Proposition 18. Actually, after inspecting the estimates in the proof of this result, one can guess that the summand $T_{2,N}$ (which involves the correlation of the increments of Y and $Z^{(1,H)}$) may not converge to zero and may affect the limit distribution.*

- *In the case* $q = 1, H \in (\frac{1}{2}, \frac{3}{4})$, *it is also possible to estimate the Wasserstein and the Kolmogorov distance between the law of* $K_{1,1}\sqrt{N}V_N(X)$ *and the standard normal law. Recall that the Wasserstein (respectively Kolmogorov) distance, denoted* d_W *(resp.* d_{Kol}*), between the laws of two random variables* X *and* Y *are defined as (by* $\|f\|_{Lip}$ *we denote the Lipschitz norm of the function* f*)*

$$d_W(X,Y) = \sup_{\|f\|_{Lip}\leq 1} |\mathbf{E}f(X) - \mathbf{E}f(Y)|$$

and

$$d_{Kol}(X,Y) = \sup_{z\in\mathbb{R}} |\mathbf{P}(X \leq z) - \mathbf{P}(Y \leq z)|\,.$$

By triangle's inequality and the properties of the Wasserstein distance,

$$\begin{aligned} & d_W\left(K_{1,1}\sqrt{N}V_N(X), N(0,1)\right) \\ & \leq d_W\left(K_{1,1}\sqrt{N}V_N(Z^{(1,H)}), N(0,1)\right) + d_W\left(K_{1,1}\sqrt{N}V_N(X), K_{1,1}\sqrt{N}V_N(Z^{(1,H)})\right) \\ & \leq d_W\left(K_{1,1}\sqrt{N}V_N(Z^{(1,H)}), N(0,1)\right) + \mathbf{E}|T_{1,N} + T_{2,N}| \end{aligned}$$

with $T_{1,N}, T_{2,N}$ *given by (3.17), (3.18). We know (see e.g. [100]) that*

$$d_W\left(K_{1,1}\sqrt{N}V_N(Z^{(1,H)}), N(0,1)\right) \leq cN^{2H-\frac{3}{2}}$$

for N *large while from (3.30) the same bound holds for* $\mathbf{E}|T_{1,N} + T_{2,N}|$*. Therefore, for* N *sufficiently large,*

$$d_W\left(K_{1,1}\sqrt{N}V_N(X), N(0,1)\right) \leq cN^{2H-\frac{3}{2}}.$$

Using the well known fact (see the Appendix C in [81]) $d_{Kol}(X,Y) \leq 2\sqrt{d_W(X,Y)}$*, we deduce that for large enough* N*,*

$$d_{Kol}\left(K_{1,1}\sqrt{N}V_N(X), N(0,1)\right) \leq CN^{H-\frac{3}{4}}.$$

Proposition 18 also shows (together with (3.50)) that the sequence $V_N(X)$ converges to zero in $L^p(\Omega)$ ($p \geq 1$) as $N \to \infty$. Via a Borel-Cantelli argument, we can easily obtain the almost sure convergence of $V_N(X)$ to zero as $N \to \infty$. This will be needed later for the estimation of the Hurst parameter.

Corollary 2 *Let* $V_N(X)$ *be given by (3.11). Then for every* $q \geq 2$ *(if* $H \in \left(\frac{1}{2}, 1\right)$*) and for* $q = 1$ *(if* $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$*), the sequence* $V_N(X)$ *converges to zero almost surely as* $N \to \infty$*.*

Proof : Let $q \geq 2$ and take $\gamma \in (0, 1-H)$. Then for every $p \geq 1$, from Proposition 18

$$P(V_N(X) \geq N^{-\gamma}) \leq CN^{-\gamma p}\mathbf{E}|V_N|^p \leq cN^{p(H+\gamma-1)}.$$

By choosing p large enough we will have that $\sum_{N\geq 1} P(V_N(X) \geq N^{-\gamma})$ is convergent and the Borel-Cantelli lemma gives the conclusion.

For $q = 1$ and $H \in \left(0, \frac{3}{4}\right)$, we proceed similarly by taking $\gamma \in \left(0, \frac{1}{2}\right)$. ■

3.3.2 The stationary case

The stationary Hermite Ornstein-Uhlenbeck process, denoted in the sequel by $(X_{0,t})_{t\in[0,T]}$, is defined as the solution of the Langevin (3.1) with initial value

$$\xi = \sigma \int_{-\infty}^{0} e^{u} dZ_u^{(q,H)}. \tag{3.31}$$

Thus we have, for every $t \in [0,T]$,

$$X_{0,t} = \sigma \int_{-\infty}^{t} e^{-(t-s)} dZ_s^{(q,H)}. \tag{3.32}$$

As in the nonstationary case, the above integral with respect to $Z^{(q,H)}$ exists both in the Wiener and in the Riemann-Stieljes sense (see [38], [93] or [94]). Moreover, $(X_{0,t})_{t\in[0,T]}$ is a stationary process, and it can be expressed as

$$\begin{aligned} X_{0,t} &= \sigma \int_{\mathbb{R}^q} \left(\int_{-\infty}^{t} e^{-(t-u)} (u-y_1)_+^{-\left(\frac{1}{2}+\frac{1-H}{2}\right)} \dots (u-y_q)_+^{-\left(\frac{1}{2}+\frac{1-H}{2}\right)} du \right) dB(y_1)\dots dB(y_q) \\ &= I_q(g_t) \end{aligned} \tag{3.33}$$

where, for every $y_1, ..., y_q \in \mathbb{R}$,

$$g_t(y_1,...,y_q) = \sigma \int_{-\infty}^{t} e^{-(t-u)} (u-y_1)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \dots (u-y_q)_+^{-\left(\frac{1}{2}+\frac{1-H}{2}\right)} du. \tag{3.34}$$

We also know that for every $t \geq 0$, with $C > 0$ not depending on t,

$$\sup_{s\in[0,t]} \mathbf{E} X_{0,s}^2 \leq C$$

and by the hypercontractivity (3.50), for every $p \geq 2$

$$\sup_{s\in[0,t]} \mathbf{E} |X_{0,s}|^p \leq C. \tag{3.35}$$

We will show that its renormalized quadratic variation has the same behavior as in the stationary case. Although many calculations are similar (and we will not insist on them), several steps of the proof of Proposition 18 need additional arguments, due to the fact that we consider integral over intervals of infinite Lebesgue measure.

Let us denote by $V_N(X_0)$ the centered renormalized quadratic variation of the stationary HOU process, which is similarly defined as (3.11), i.e.

$$V_N(X_0) = \frac{1}{N} \sum_{i=0}^{N-1} \left[\frac{(X_{0,t_{i+1}} - X_{0,t_i})^2}{\sigma^2 \mathbf{E}\left(Z_{t_{i+1}}^{(q,H)} - Z_{t_i}^{(q,H)}\right)^2} - 1 \right] = \frac{1}{N} \sum_{i=0}^{N-1} \left[\frac{N^{2H}}{\sigma^2} (X_{0,t_{i+1}} - X_{0,t_i})^2 - 1 \right]. \tag{3.36}$$

Proposition 19 *Let X_0 be given by (3.32).*

1. *Assume $H \in \left(\frac{1}{2}, 1\right)$ and $q \geq 2$. Let $V_N(X_0)$ be given by (3.36). Then, with $K_q, Z_1^{(2,H')}$ from (3.14)*

$$K_q N^{\frac{2-2H}{q}} V_N(X_0) \to_{N\to\infty} Z_1^{(2,H')} \text{ in } L^2(\Omega).$$

2. *If $q = 1$ and $H \in \left(0, \frac{3}{4}\right)$,*

$$K_{1,1}\sqrt{N} V_N(X_0) \to^{(d)} N(0,1).$$

with $K_{1,1}$ from (3.13).

Proof : We can write, for every $N \geq 1$,

$$V_N(X_0) = V_N(Z^{(q,H)}) + U_{1,N} + U_{2,N}$$

with $V_N(Z^{(q,H)})$ given by (3.12) and with

$$U_{1,N} = \sigma^{-2} N^{2H-1} \sum_{i=0}^{N-1} \left(Y_{0,t_{i+1}} - Y_{0,t_i}\right)^2$$

and

$$U_{2,N} = 2\sigma^{-1} N^{2H-1} \sum_{i=0}^{N-1} \left(Y_{0,t_{i+1}} - Y_{0,t_i}\right)\left(Z_{t_{i+1}}^{(q,H)} - Z_{t_i}^{(q,H)}\right).$$

Let $q \geq 2$. The convergence in $L^p(\Omega)$ of $U_{1,N}$ follows easily, since for every $p \geq 1$, via Hölder's inequality and (3.35)

$$\begin{aligned} \mathbf{E}|U_{1,N}|^p &= \sigma^{-2p}\left(N^{2H-1}\right)^p \mathbf{E}\left|\sum_{i=0}^{N-1}\left(\int_{t_i}^{t_{i+1}} X_{0,s}ds\right)^2\right|^p \\ &\leq C\left(N^{2H-1}\right)^p N^{-p} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} X_{0,s}^{2p} ds \leq C N^{(2H-2)p} \end{aligned}$$

and this implies

$$\mathbf{E}|N^{\frac{2-2H}{q}} U_{1,N}|^p \leq C\left(N^{(2H-2)\left(1-\frac{1}{q}\right)}\right)^p.$$

which goes to zero as $N \to \infty$ since $q \geq 2$.

For the term $A_{2,N}$ we use again the Wiener chaos decomposition to write

$$Y_{0,t_{i+1}} - Y_{0,t_i} = I_q(g_{i,N}) \text{ with } g_{i,N} = g_{t_{i+1}} - g_{t_i}$$

with g_t given by (3.34) (it is symmetric). Thus, by the product formula (3.49)

$$U_{2,N} = 2N^{2H-1}\sum_{i=0}^{N-1}\sum_{r=0}^{q} r!\binom{r}{q}^2 I_{2q-2r}(g_{i,N}\otimes_r \ell_{i,N}) := \sum_{r=0}^{q} S_{2,N}^{(r)}$$

with

$$S_{2,N}^{(r)} = 2N^{2H-1}\sum_{i=0}^{N-1} r!\binom{r}{q}^2 I_{2q-2r}(g_{i,N}\otimes_r \ell_{i,N}). \tag{3.37}$$

We will show that for every $r = 0, .., q$

$$\mathbf{E}|N^{\frac{2-2H}{q}} S_{2,N}^{(r)}|^2 \to_{N\to\infty} 0. \tag{3.38}$$

If $r = q$, then by (3.25)

$$\begin{aligned}
|S_{2,N}^{q)}| &= cN^{2H-1}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} ds \int_{-\infty}^{s} du \int_{t_i}^{t_{i+1}} dv e^{-(s-u)}|u-v|^{2H-2} \\
&= cN^{2H-1}\sum_{i=0}^{N-1}\int_{-\infty}^{t_{i+1}} du \int_{t_i}^{t_{i+1}} dv |u-v|^{2H-2}\int_{t_i\vee u}^{t_{i+1}} ds e^{-(s-u)}
\end{aligned}$$

and by splitting the integral $\int_{-\infty}^{t_{i+1}}$ as $\int_{-\infty}^{t_i} + \int_{t_i}^{t_{i+1}}$, we obtain

$$\begin{aligned}
|S_{2,N}^{(q)}| &\leq cN^{2H-1}\sum_{i=0}^{N-1}\int_{-\infty}^{t_i} du \int_{t_i}^{t_{i+1}} dv |u-v|^{2H-2}\left(e^{-(t_i-u)} - e^{-(t_{i+1}-u)}\right) \\
&\quad + cN^{2H-1}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} du \int_{t_i}^{t_{i+1}} dv |u-v|^{2H-2}\left(1 - e^{-(t_{i+1}-u)}\right) \\
&\leq cN^{2H-1}\sum_{i=0}^{N-1}\left(e^{-t_i} - e^{-t_{i+1}}\right)\int_{-\infty}^{1} du e^{u}\int_{t_i}^{t_{i+1}} dv |u-v|^{2H-2} \\
&\quad + cN^{2H-1}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} du \int_{t_i}^{t_{i+1}} dv |u-v|^{2H-2} \\
&\leq CN^{2H-1}\left(\frac{1}{N} + N^{-2H+1}\right) \leq cN^{2H-2}
\end{aligned}$$

and (3.38) is obtained.

For $0 \leq r \leq q-1$, we can write, as for (3.27),

$$\begin{aligned}
\mathbf{E}\left|S_{2,N}^{(r)}\right|^2 &= cN^{4H-2}\|\sum_{i=0}^{N-1} g_{i,N}\tilde{\otimes}_r \ell_{i,N}\|^2 \leq cN^{4H-2}\|\sum_{i=0}^{N-1} g_{i,N}\otimes_r \ell_{i,N}\|^2_{L^2(\mathbb{R}^{2q-2r})} \\
&= CN^{4H-2}\sum_{,ji=0}^{N-1}\langle g_{i,N}\otimes_r \ell_{i,N}, g_{j,N}\otimes_r \ell_{k,N}\rangle_{L^2(\mathbb{R}^{2q-2r})}
\end{aligned}$$

$$
\begin{aligned}
&= CN^{4H-2}\sum_{i,j=0}^{N-1}\int_{t_i}^{t_{i+1}} ds \int_{-\infty}^{s} du e^{-(s-u)} \int_{t_i}^{t_{i+1}} dv \int_{t_j}^{t_{j+1}} ds' \int_{-\infty}^{s'} du' e^{-(s'-u')} \int_{t_j}^{t_{j+1}} dv' \\
&\quad \times |u-v|^{(2H-2)\frac{r}{q}} |u'-v'|^{(2H-2)\frac{r}{q}} |u-u'|^{(2H-2)\frac{(q-r)}{q}} |v-v'|^{(2H-2)\frac{(q-r)}{q}} \\
&= CN^{4H-2}\sum_{i,j=0}^{N-1}\int_{-\infty}^{t_{i+1}} du \int_{t_i}^{t_{i+1}} dv \int_{-\infty}^{t_{j+1}} du' \int_{t_j}^{t_{j+1}} dv' \\
&\quad \times |u-v|^{(2H-2)\frac{r}{q}} |u'-v'|^{(2H-2)\frac{r}{q}} |u-u'|^{(2H-2)\frac{(q-r)}{q}} |v-v'|^{(2H-2)\frac{(q-r)}{q}} \\
&\quad \times \int_{t_i\vee u}^{t_{i+1}} ds \int_{t_j\vee u'}^{t_{j+1}} ds' e^{-(s-u)} e^{-(s'-u')}
\end{aligned}
$$

and by calculating first the integrals ds and ds' and by splitting the integrals du and du' into two parts, we get

$$
\begin{aligned}
\mathbf{E}\left|S_{2,N}^{(r)}\right|^2 &= CN^{4H-2}\sum_{i,j=0}^{N-1}\int_{t_i}^{t_{i+1}} du \int_{t_i}^{t_{i+1}} dv \int_{t_j}^{t_{j+1}} du' \int_{t_j}^{t_{j+1}} dv' \\
&\quad \times |u-v|^{(2H-2)\frac{r}{q}} |u'-v'|^{(2H-2)\frac{r}{q}} |u-u'|^{(2H-2)\frac{(q-r)}{q}} |v-v'|^{(2H-2)\frac{(q-r)}{q}} \\
&\quad \times \left(1-e^{-(t_{i+1}-u)}\right)\left(1-e^{-(t_{j+1}-u)}\right) \\
&\quad + CN^{4H-2}\sum_{i,j=0}^{N-1}\int_{-\infty}^{t_i} du \int_{t_i}^{t_{i+1}} dv \int_{-\infty}^{t_j} du' \int_{t_j}^{t_{j+1}} dv' \\
&\quad \times |u-v|^{(2H-2)\frac{r}{q}} |u'-v'|^{(2H-2)\frac{r}{q}} |u-u'|^{(2H-2)\frac{(q-r)}{q}} |v-v'|^{(2H-2)\frac{(q-r)}{q}} \\
&\quad \times \left(e^{-(t_i-u)}-e^{-(t_{i+1}-u)}\right)\left(e^{-(t_j-u)}-e^{-(t_{j+1}-u)}\right) \\
&\quad + CN^{4H-2}\sum_{i,j=0}^{N-1}\int_{t_i}^{t_{i+1}} du \int_{t_i}^{t_{i+1}} dv \int_{-\infty}^{t_j} du' \int_{t_j}^{t_{j+1}} dv' \\
&\quad \times |u-v|^{(2H-2)\frac{r}{q}} |u'-v'|^{(2H-2)\frac{r}{q}} |u-u'|^{(2H-2)\frac{(q-r)}{q}} |v-v'|^{(2H-2)\frac{(q-r)}{q}} \\
&\quad \times \left(1-e^{-(t_{i+1}-u)}\right)\left(e^{-(t_j-u)}-e^{-(t_{j+1}-u)}\right) \\
&:= s_{1,N}+s_{2,N}+s_{3,N}.
\end{aligned}
$$

For the summand $s_{1,N}$, we majorize $1-e^{-(t_{i+1}-u)}$ by $1-e^{-(t_{i+1}-t_i)} \leq c\frac{1}{N}$ when $u \in (t_i, t_{i+1})$. In this way

$$
\begin{aligned}
s_{1,N} &\leq CN^{4H-4}\sum_{i,j=0}^{N-1}\int_{-\infty}^{t_i} du \int_{t_i}^{t_{i+1}} dv \int_{-\infty}^{t_j} du' \int_{t_j}^{t_{j+1}} dv' \\
&\quad \times |u-v|^{(2H-2)\frac{r}{q}} |u'-v'|^{(2H-2)\frac{r}{q}} |u-u'|^{(2H-2)\frac{(q-r)}{q}} |v-v'|^{(2H-2)\frac{(q-r)}{q}}
\end{aligned}
$$

and by the change of variables $\tilde{u} = N(u - \frac{i}{N})$ (and with similar operation for the other variables),

$$s_{1,N} \leq c\frac{1}{N^2}\int_{[0,1]^4}|u-v|^{(2H-2)\frac{r}{q}}|u'-v'|^{(2H-2)\frac{r}{q}}|u-u'|^{(2H-2)\frac{(q-r)}{q}}|v-v'|^{(2H-2)\frac{(q-r)}{q}} = c\frac{1}{N^2}.$$

For the term $s_{2,N}$, we bound $e^{-(t_i-u)} - e^{-(t_{i+1}-u)}$ by $c\frac{1}{N}e^u$ and the integrals over $(-\infty, t_i)$ and $(-\infty, t_j)$ by integrals over $(-\infty, 1)$. So

$$\begin{aligned} s_{2,N} &\leq CN^{4H-4}\sum_{i,j=0}^{N-1}\int_{-\infty}^{1}e^u du\int_{t_i}^{t_{i+1}}dv\int_{-\infty}^{1}du'\int_{t_j}^{t_{j+1}}dv' \\ &\quad\times|u-v|^{(2H-2)\frac{r}{q}}|u'-v'|^{(2H-2)\frac{r}{q}}|u-u'|^{(2H-2)\frac{(q-r)}{q}}|v-v'|^{(2H-2)\frac{(q-r)}{q}} \\ &\leq cN^{4H-4} \end{aligned}$$

because the integral $\int_{-\infty}^{1}e^u du\int_{-\infty}^{1}e^{u'}du'\int_0^1 dv\int_0^1 dv'[\cdots]$ is finite. Similarly, with the same estimates as above, we found $s_{3,N} \leq cN^{4H-4}$ so

$$\mathbf{E}\left|S_{2,N}^{(r)}\right|^2 \leq cN^{4H-4}$$

which implies (3.38) since $q \geq 2$. ■

3.3.3 Adjusted variations in the case q=2

For $q \geq 2$, the limit distribution of the quadratic variation is a non-Gaussian random variable. In the case $q = 2$ (i.e. the random noise of (3.1) is a Rosenblatt process), it is also possible to "adjust" the sequence $V_N(X)$, by substracting its limit, to get a Gaussian limit in distribution.

Let us recall the following result from [103].

Proposition 20 *Assume $H \in \left(\frac{1}{2}, \frac{2}{3}\right)$ and let $V_N(Z^{(2,H)})$ be defined by (3.12). Then*

$$\frac{\sqrt{N}}{K_{2,2}}\left[V_N(Z^{(2,H)}) - \frac{\sqrt{K_2}}{N^{1-H}}Z_1^{(2,H)}\right] \to_{N\to\infty}^{(d)} N(0,1).$$

The constant K_2 appears in (3.14) while the explicit expression of $K_{2,2}$ can be found in [103].

We will show that a similar result can be obtained for the HOU process.

Proposition 21 *Assume $H \in \left(\frac{1}{2}, \frac{2}{3}\right)$. Then, with $K_2, K_{2,2}$ as above,*

$$\frac{\sqrt{N}}{K_{2,2}}\left[V_N(X) - \frac{\sqrt{K_2}}{N^{1-H}}Z_1^{(2,H)}\right] \to_{N\to\infty}^{(d)} N(0,1).$$

Proof : We can write

$$V_N(X) - \frac{\sqrt{K_2}}{N^{1-H}} Z_1^{(2,H)} \quad = \quad V_N(Z^{(q,H)}) - \frac{\sqrt{K_2}}{N^{1-H}} Z_1^{(2,H)} + T_{1,N} + T_{2,N}$$

with $T_{i,N}, i = 1, 2$ given by (3.17), (3.18) respectively. From the estimates (3.20) and (3.29), it follows easily that for $i = 1, 2$

$$\mathbf{E}\left(\sqrt{N}|T_{i,N}|\right)^2 \to_{N\to\infty} 0$$

because $H < \frac{2}{3}$. ■

A similar result can be proved analogously in the stationary case (i.e. for the process (3.32)).

3.4 Estimation of the parameters

The quadratic variation estimators are known to be well-suited to identify the scaling parameter for self-similar process. Although the HOU process is not self-similar, we can adapt the standard procedure to estimate its Hurst parameter. Notice that the model (3.1) (assuming $a = 1$) includes two parameters : H and σ. We will discuss the estimation of both H and σ. First, we propose an estimator for H (without assuming that σ is known) and then we estimate σ via the $\frac{1}{H}$-variations of the solution to (3.1), by assuming that H is known.

We will focus on the nonstationary case, that is, we will estimate the parameter H of the process (3.8) from its the discrete observations. The stationary case can be treated similarly, without particular difficulties.

3.4.1 Estimation of the Hurst parameter

The idea to construct an estimator for the Hurst parameter H in terms of the observation of the solution to (3.1) at times $X_{\frac{i}{N}}$, $i = 0, 1, \ldots, N$ is standard : one starts with an evaluation of $\mathbf{E}S_N$ with

$$S_N = \frac{1}{N} \sum_{i=0}^{N-1} \left(X_{t_{i+1}} - X_{t_i}\right)^2$$

yielding

$$\mathbf{E}S_N = \sigma^2 N^{-2H} + \sigma^2 N^{-2H} \left(\mathbf{E}T_{1,N} + \mathbf{E}T_{2,N}\right)$$

with $T_{1,N}, T_{2,N}$ given by (3.17), (3.18) respectively. From the proof of Proposition 18, the last two summands converge to zero as $N \to \infty$. Indeed, by using the estimates (3.20) and (3.29) we have for $i = 1, 2$,

$$N^{-2H}\mathbf{E}\left|T_{i,N}\right| \leq N^{-2H}\left(\mathbf{E}\left|T_{i,N}\right|^2\right)^{\frac{1}{2}} \leq CN^{-2} \to_{N\to\infty} 0.$$

Consequently, we have (in the sequel $a_N \sim b_N$ means that the sequences a_N and b_N have the same behavior at infinity)

$$\mathbf{E}S_N \sim \sigma^2 N^{-2H}$$

and by taking the logarithm and by approximating $\mathbf{E}S_N$ by S_N

$$\log(S_N) \sim -2H\log(N) + \log(\sigma^2) \sim -2H\log(N). \tag{3.39}$$

This leads to a natural estimator

$$\widehat{H}_N = \frac{-\log(S_N)}{2\log(N)}. \tag{3.40}$$

From Proposition 18 and Corollary 2 we immediately get the asymptotic behavior of the above estimator. The constants $K_{1,1}, K_q$ and the random variable $Z_1^{(2,H')}$ are those from Proposition 18.

Proposition 22 *The estimator (3.40) is strongly consistent, i.e. $\widehat{H}_N$ converges almost surely to H as $N \to \infty$. Moreover, for $q \geq 2$*

$$K_q N^{\frac{2-2H}{q}} \left[2\log(N)\left(H - \widehat{H}_N\right) - \log(\sigma^2)\right] \to^{(d)}_{N\to\infty} Z_1^{(2,H')}$$

while for $q = 1$ and $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$,

$$K_{1,1}\sqrt{N}\left[2\log(N)\left(H - \widehat{H}_N\right) - \log(\sigma^2)\right] \to^{(d)}_{N\to\infty} N(0,1).$$

Proof : For $N \geq 1$, we have

$$1 + V_N(X) = \frac{N^{2H}}{\sigma^2} S_N$$

and by taking the logarithm and using Corollary 2, almost surely,

$$V_N(X) \sim \log(1 + V_N(X)) = 2H\log(N) + \log(S_N) - \log(\sigma^2). \tag{3.41}$$

By combining (3.41) with (3.40), we can write for N large, almost surely,

$$H - \widehat{H}_N \sim \frac{V_N(X) + \log(\sigma^2)}{2\log N}. \tag{3.42}$$

The strong consistency of $\widehat{H}_N$ (that is, $H - \widehat{H}_N$ converges almost surely to zero as $N \to \infty$) follows from (3.42) and Corollary 2 while the limit distribution is obtained from Proposition 18. ■

Remark 6 *The estimator (3.40) does not depend on σ so we can assume that this coefficient is unknown. On the other hand, when σ is known, we can also use relation (3.39) to define $\widetilde{H}_N = \frac{\log(\sigma^2)-\log(S_N)}{2\log(N)}$. This estimator will satisfy the limit theorem (when $q \geq 2$)*

$$2K_q \log(N) N^{\frac{2-2H}{q}} \left(H - \widetilde{H}_N\right) \to^{(d)}_{N\to\infty} Z_1^{(2,H')}.$$

3.4.2 Estimation of σ

In this last section, let us discuss the estimation of the diffusion paramter σ by assuming that H is known. Actually it can also be estimated from the discrete observation of the process X, via generalized variations.

Let us first state the following result concerning the variations of the Hermite process.

Proposition 23 *If $Z^{(q,H)}$ is a Hermite process of order $q \geq 1$, then*

$$\sum_{i=0}^{N-1} \left| Z^{(q,H)}_{t_{i+1}} - Z^{(q,H)}_{t_i} \right|^{\frac{1}{H}} \to_{N\to\infty} \mathbf{E} \left| Z^{(q,H)}_1 \right|^{\frac{1}{H}} \text{ in probability.}$$

Proof : By self-similarity, the random variable $\sum_{i=0}^{N-1} \left| Z^{(q,H)}_{t_{i+1}} - Z^{(q,H)}_{t_i} \right|^{\frac{1}{H}}$ has the same distribution as

$$\frac{1}{N} \sum_{i=0}^{N-1} \left| Z^{(q,H)}_{i+1} - Z^{(q,H)}_{i} \right|^{\frac{1}{H}}.$$

The sequence $\left(Z^{(q,H)}_{i+1} - Z^{(q,H)}_{i}, i \geq 1 \right)$ is stationary and ergodic (for the ergodicity, see Theorem 8.3.1 in [91]). Consequently, by the ergodic theorem,

$$\frac{1}{N} \sum_{i=0}^{N-1} \left| Z^{(q,H)}_{i+1} - Z^{(q,H)}_{i} \right|^{\frac{1}{H}} \to_{N\to\infty} \mathbf{E} \left| Z^{(q,H)}_1 \right|^{\frac{1}{H}}$$

almost surely. Thus $\sum_{i=0}^{N-1} \left| Z^{(q,H)}_{t_{i+1}} - Z^{(q,H)}_{t_i} \right|^{\frac{1}{H}}$ converges in law to $\mathbf{E} \left| Z^{(q,H)}_1 \right|^{\frac{1}{H}}$. Since the limit is a constant, the convergence also holds in probability. ■

The previous property can be easily transfered to the solution to (3.1).

Proposition 24 *Let X be the solution to (3.1). Then*

$$\sum_{i=0}^{N-1} \left| X_{t_{i+1}} - X_{t_i} \right|^{\frac{1}{H}} \to_{N\to\infty} \sigma^{\frac{1}{H}} \mathbf{E} \left| Z^{(q,H)}_1 \right|^{\frac{1}{H}} \text{ in probability.}$$

Proof : By Minkovski's inequality,

$$\sigma \left(\sum_{i=0}^{N-1} \left| Z^{(q,H)}_{t_{i+1}} - Z^{(q,H)}_{t_i} \right|^{\frac{1}{H}} \right)^H - \left(\sum_{i=0}^{N-1} \left| Y_{t_{i+1}} - Y_{t_i} \right|^{\frac{1}{H}} \right)^H \leq \left(\sum_{i=0}^{N-1} \left| X_{t_{i+1}} - X_{t_i} \right|^{\frac{1}{H}} \right)^H$$

$$\leq \left(\sum_{i=0}^{N-1} \left| Y_{t_{i+1}} - Y_{t_i} \right|^{\frac{1}{H}} \right)^H + \sigma \left(\sum_{i=0}^{N-1} \left| Z^{(q,H)}_{t_{i+1}} - Z^{(q,H)}_{t_i} \right|^{\frac{1}{H}} \right)^H.$$

On the other hand,

$$\mathbf{E}\sum_{i=0}^{N-1}|Y_{t_{i+1}}-Y_{t_i}|^{\frac{1}{H}} \le N^{1-\frac{1}{H}}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\mathbf{E}\,|X_s|^{\frac{1}{H}} \le CN^{1-\frac{1}{H}}.$$

Consequently, $\left(\sum_{i=0}^{N-1}|Y_{t_{i+1}}-Y_{t_i}|^{\frac{1}{H}}\right)^H$ converges to zero in $L^1(\Omega)$ as $N\to\infty$ and the conclusion follows. ■

Consequently, define the estimator $\widehat{\sigma}_N$

$$\widehat{\sigma}_N := m(q,H)^{-H}\left(\sum_{i=0}^{N-1}|X_{t_{i+1}}-X_{t_i}|^{\frac{1}{H}}\right)^H \tag{3.43}$$

with $m(q,H)=\mathbf{E}\left|Z_1^{(q,H)}\right|^{\frac{1}{H}}$. Clearly, by Proposition 23

$$\widehat{\sigma}_N \to_{N\to\infty} \sigma$$

in probability, so $\widehat{\sigma}_N$ is consistent.

Remark 7
- *The limit behavior of $\widehat{\sigma}_N$ is an interesting open problem, it depends on the limit behavior of the $\frac{1}{H}$-variations of the Hermite process which is not known.*
- *We can also prove, by adapting Propositions 23 and 24, that*

$$N^{H-1}\sum_{i=0}^{N-1}|X_{t_{i+1}}-X_{t_i}| \to_{N\to\infty} \sigma\mathbf{E}\left|Z_1^{(q,H)}\right| \text{ in probability.} \tag{3.44}$$

thus (3.44) will provide another estimator for σ (also depending on H),

$$\widetilde{\sigma}_N = \left(\mathbf{E}\left|Z_1^{(q,H)}\right|\right)^{-1}\sum_{i=0}^{N-1}|X_{t_{i+1}}-X_{t_i}| \tag{3.45}$$

A plug-in of (3.40) into (3.45) will give an estimator for σ when H is unknown.

3.5 Appendix : Multiple stochastic integrals

Here, we shall only recall some elementary facts; our main reference is [77]. Consider $\mathcal{H}$ a real separable infinite-dimensional Hilbert space with its associated inner product $\langle .,. \rangle_{\mathcal{H}}$, and $(B(\varphi),\varphi\in\mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega,\mathfrak{F},\mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right)=\langle\varphi,\psi\rangle_{\mathcal{H}}$, for every $\varphi,\psi\in\mathcal{H}$. Denote by I_q the qth multiple stochastic integral with respect to B. This I_q is actually an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}}\|\cdot\|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by :

$$H_q(x) = (-1)^q \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^q}{\mathrm{d}x^q}\left(\exp\left(-\frac{x^2}{2}\right)\right), \; x \in \mathbb{R}. \tag{3.46}$$

The isometry of multiple integrals can be written as : for $p,\ q \geq 1$, $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$,

$$\mathbf{E}\Big(I_p(f)I_q(g)\Big) = \begin{cases} q!\langle \tilde{f}, \tilde{g}\rangle_{\mathcal{H}^{\otimes q}} & \text{if } p = q \\ 0 & \text{otherwise} \end{cases} \tag{3.47}$$

where $\tilde{f}$ denotes the canonical symmetrization of f and it is defined by :

$$\tilde{f}(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)}),$$

in which the sum runs over all permutations σ of $\{1, \ldots, q\}$. It also holds that :

$$I_q(f) = I_q(\tilde{f}).$$

In the particular case when $\mathcal{H} = L^2(T, \mathcal{B}(T), \mu)$, the rth contraction $f \otimes_r g$ is the element of $\mathcal{H}^{\otimes(p+q-2r)}$, which is defined by :

$$\begin{aligned} &(f \otimes_r g)(s_1, \ldots, s_{p-r}, t_1, \ldots, t_{q-r}) \\ &= \textstyle\int_{T^r} \mathrm{d}u_1 \ldots \mathrm{d}u_r f(s_1, \ldots, s_{p-r}, u_1, \ldots, u_r) g(t_1, \ldots, t_{q-r}, u_1, \ldots, u_r), \end{aligned} \tag{3.48}$$

for every $f \in L^2(T^p)$, $g \in L^2(T^q)$ and $r = 1, \ldots, p \wedge q$. The $f \tilde{\otimes}_l g$ we denote the symmetrization of the contraction $f \otimes_l g$.

The product for two multiple integrals can be expanded into a sum of multiple integrals (see [77]) : if $f \in L^2(T^n)$ and $g \in L^2(T^m)$ are symmetric functions, then it holds that

$$I_n(f)I_m(g) = \sum_{l=0}^{m \wedge n} l! C_m^l C_n^l I_{m+n-2l}(f \otimes_l g). \tag{3.49}$$

Another useful property of finite sums of multiple integrals is the hypercontractivity. Namely, if $F = \sum_{k=0}^{n} I_k(f_k)$ with $f_k \in \mathcal{H}^{\otimes k}$ then

$$\mathbf{E}|F|^p \leq C_p \left(\mathbf{E}F^2\right)^{\frac{p}{2}}. \tag{3.50}$$

for every $p \geq 2$.

Chapitre 4

Pathwise analysis and parameter estimation for the stochastic Burgers equation

This article is published on Bulletin des Sciences Mathématiques,Volume 170, 2021, 102995,ISSN 0007-4497. https ://doi.org/10.1016/j.bulsci.2021.102995.

Joint work with Ciprian A. Tudor

Abstract We analyze the solution to the stochastic Burgers equation with additive space-time white noise. We show that this solution can be expressed as a sum of a random field that solves the stochastic heat equation with additive space-time white noise and a more regular random field. We apply these results in order to estimate the drift parameter of solution to the Burgers equation by exploiting the behavior of the p-variations of the solution in time and in space.

4.1 Introduction

We study the following stochastic partial differential equation (SPDE in the sequel) By using the analysis on Wiener chaos, we study the behavior of the quadratic variations of the Hermite Ornstein–Uhlenbeck process, which is the solution to the Langevin equation driven by a Hermite process. We apply our results to the identification of the Hurst parameter of the Hermite Ornstein–Uhlenbeck process

$$\frac{\partial}{\partial t}u(t,x) = \theta\frac{\partial^2}{\partial x^2}u(t,x) - \frac{1}{2}\frac{\partial}{\partial x}u(t,x)^2 + \dot{W}(t,x), \qquad (t,x) \in [0,T] \times \mathbb{R} \tag{4.1}$$

with deterministic initial $u(0,x) = u_0(x)$ for every $x \in \mathbb{R}$ and where $\theta > 0$ is the parameter to be estimated.

When $\theta = 1$, the equation (4.1) is the well-known stochastic Burgers equation with additive space-time white noise, which has been widely studied in the literature. It is one of the most famous singular stochastic partial differential equations. For the deterministic Burgers equation (i.e. there is no random noise in (4.1)) we refer to the monograph [32] and the references therein. The Burgers equation with a random forcing term has been also studied by several authors (see among others [28], [49], [48], [60], [70], [71], [76]) due to its applications in various fields ranging from statistical physics, cosmology, and fluid dynamics to engineering. Several aspects of this equation have been analyzed, on bounded or unbounded spatial domain, including the existence and uniqueness of the solution, its Feynman-Kac representation, the pathwise regularity of the solution in time and in space or the absolute continuity of the law of this solution.

Our purpose is to make a first step for the statistical inference of the stochastic Burgers equation. While the parameter estimation for stochastic differential equations, or stochastic partial differential equations, constitutes a dynamic and modern research direction (the reader may consult the site https ://sites.google.com/prod/view/stats4spdes/ for a list of publications related to these topics), no results are available, as far as we know, for the particular case of the Burgers equation. We aim at estimating the drift parameter $\theta > 0$ in (4.1) based on the discrete observation of the solution to (4.1) on a finite interval. We will consider two situations : when the time t is fixed and the solution is observed discretely in space or when the space variable is fixed and the solution is observed at discrete points in time. In both situations, our estimators are constructed via the p-variations of the solution. We will show that the solution to the Burgers equation admits a quadratic variation in space and a quartic variation in time and we will use this fact to obtain the asymptotic properties (strong consistency, approximation error in L^p-norm) of the estimators. The idea behind the proofs is to decompose the solution to (4.1) into a sum of two stochastic processes, one of them coinciding with the solution to the linear stochastic heat equation (whose properties are well-known) and a second term which contains the nonlinear coefficient $g(x) = \frac{1}{2}x^2$. Based on a sharp analysis of the heat kernel and of its derivative, we will actually show that this second term is more regular than the first one and so it does not affect the p-variations of the solution.

Our work is organized as follows : Section 2 contains some preliminaries on the solutions to the stochastic Burgers and heat equations. In Section 3 we make a detailed analysis of the term which contains the nonlinearity of the Burgers equations while in Section 4 we apply the results to the parameter estimation for the drift of the Burgers equation. We construct two estimators, one based on the observations of the solution to (4.1) at discrete points in space and the second one defined via the observation of this solution at discrete times. For both estimators, we study the (strong) consistency and the approximation error in L^p- norm.

4.2 Preliminaries

In this paragraph we recall some basic facts concerning the solutions to the stochastic Burgers and heat equations. We will assume in this part $\theta = 1$ in (4.1).

4.2.1 Stochastic Burgers equation

We start by introducing the Gaussian noise that governs the SPDE (4.1). Denote by $\mathcal{B}_b([0,T]\times\mathbb{R})$ the set of bounded Borels subsets of $[0,T]\times\mathbb{R}$ and consider a centered Gaussian field $(W(A), A\in\mathcal{B}_b([0,T]\times\mathbb{R}))$ on a complete probability space $(\Omega,\mathcal{F},P)$ with covariance

$$\mathbf{E}W(A)W(B) = \lambda(A\cap B) \text{ for every } A,B\in\mathcal{B}_b([0,T]\times\mathbb{R}) \tag{4.2}$$

where λ stands for the Lebesgue measure. For $\varphi\in L^2([0,T]\times\mathbb{R})$, we denote by

$$W(\varphi) = \int_0^T\int_{\mathbb{R}}\varphi(s,y)W(ds,dy)$$

the Wiener integral with respect to the Gaussian white noise W. The family $(W(\varphi),\varphi\in L^2([0,T]\times\mathbb{R}))$ represents a Gaussian family of random variables and we have the isometry

$$\mathbf{E}W(\varphi)^2 = \int_0^T\int_{\mathbb{R}}\varphi^2(s,y)dyds \text{ for every } \varphi\in L^2([0,T]\times\mathbb{R}).$$

Let us now recall some basic facts related to the solution to the stochastic Burgers equation. This solution is understood in its mild (or evolutive) form. For $\theta=1$, the solution to (4.1) can be expressed as

$$\begin{aligned} u_1(t,x) := u(t,x) &= \int_{\mathbb{R}} G_t(x-y)u_0(y)dy + \frac{1}{2}\int_0^t\int_{\mathbb{R}}\frac{\partial}{\partial y}G_{t-s}(x-y)u(s,y)^2dyds \\ &\quad + \int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)W(ds,dy) \end{aligned} \tag{4.3}$$

for every $(t,x)\in[0,T]\times\mathbb{R}$, where $W(ds,dy)$ denotes the Wiener stochastic integral with respect to the Gaussian noise W and the Green kernel G is the fundamental solution associated to the heat equation which is defined by

$$G_t(x) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}, \quad (t,x)\in[0,T]\times\mathbb{R}. \tag{4.4}$$

It is known (see e.g. [71]) that the SPDE (4.1) admits a unique solution if the initial value u_0 is a continuous and bounded function in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Moreover, the solution satisfies (see e.g. [71], Proposition 5.2) for every $p\geq 2$

$$\sup_{t\in[0,T],x\in\mathbb{R}}\mathbf{E}\,|u(t,x)|^p \leq C \tag{4.5}$$

where $C > 0$ may depend on T and p. In the sequel, we will use the notation C to indicate a generic strictly positive constant which may change througout the paper. We will assume for simplicity $u_0 = 0$, but all our results can be also obtained for a regular enough initial value u_0.

4.2.2 On the solution to the linear stochastic heat equation with additive white noise

Our analysis of the solution to the SPDE (4.1) will be done by decomposing it as

$$u(t,x) = X(t,x) + Y(t,x) \tag{4.6}$$

where for $t \in [0,T], x \in \mathbb{R}$,

$$X(t,x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)W(ds,dy) \tag{4.7}$$

and

$$Y(t,x) = \frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G_{t-s}(x-y)u(s,y)^2 dyds. \tag{4.8}$$

The random field $(X(t,x), t \in [0,T], x \in \mathbb{R})$ is actually the solution to the stochastic heat equation with additive space-time white noise W defined by (4.2), i.e. it solves (in the mild sense)

$$\frac{\partial}{\partial t} X(t,x) = \Delta X(t,x) + \dot{W}(t,x), \qquad (t,x) \in [0,T] \times \mathbb{R} \tag{4.9}$$

with vanishing intial condition $X(0,x) = 0$ for every $x \in \mathbb{R}$. The equation (4.9) has been widely studied in the literature. Let us recall some properties of its solution, that are needed in the sequel.

- (Joint Hölder continuity) : For every $s,t \in [0,T]$, $x,y \in \mathbb{R}$, and $p \geq 1$ with $C = C(p,T) > 0$
$$\mathbf{E}\,|X(t,x) - X(s,y)|^p \leq C\left(|t-s|^{\frac{p}{4}} + |x-y|^{\frac{p}{2}}\right). \tag{4.10}$$
In particular X is Hölder continuous of order $\delta \in (0, \frac{1}{4})$ in time and of order $\delta \in (0, \frac{1}{2})$ in space.

- (Moment estimates) : For every $p \geq 1$
$$\sup_{t\in[0,T], x\in\mathbb{R}} \mathbf{E}|X(t,x)|^p \leq C = C(p,T) < \infty.$$

- (Spatial quadratic variation) : Let $t \in (0,T]$ be fixed, $A_1 < A_2$ and
$$x_i = A_1 + \frac{i}{N}(A_2 - A_1), \quad i = 0, ..., N \tag{4.11}$$

be a partition of the interval $[A_1, A_2]$. Then

$$\sum_{i=0}^{N-1} (X(t, x_{i+1}) - X(t, x_i))^2 \to_{N\to\infty} \frac{1}{2}(A_2 - A_1) \text{ in } L^2(\Omega). \tag{4.12}$$

- (Temporal quartic variation) : Let $x \in \mathbb{R}$ be fixed and $A_1 < A_2$. Let

$$t_i = A_1 + \frac{i}{N}(A_2 - A_1), \quad i = 0, ..., N \tag{4.13}$$

be a partition of the interval $[A_1, A_2]$. Then

$$\sum_{i=0}^{N-1} (X(t_{i+1}, x) - X(t_i, x))^4 \to_{N\to\infty} \frac{3}{\pi}(A_2 - A_1) \text{ in } L^2(\Omega). \tag{4.14}$$

The properties (4.12) and (4.14) are proven in e.g. [88], Proposition 3.1 and Proposition 3.2 respectively. The Hölder continuity and the moment estimates can be found in e.g. [100].

4.3 Analysis of the process Y

In order to have a complete picture on the pathwise regularity of the solution to (4.1), we will study the regularity of the sample paths of the process Y defined by (4.8) both in time and in space. This is will be done via some sharp estimates of the Green kernel (4.4) and of its derivative with respect to its space variable. Notice that the random field Y contains the nonlinearity that appears in the expression of the stochastic Burgers equation.

4.3.1 Analysis in space variable

Let us start by regarding the space regularity of the random field (4.8). We need the following lemma which concerns the spatial increment of the Green kernel (4.4).

Lemma 5 *Let G be given by (4.4). Then for every $t \in [0, T], x \in \mathbb{R}$ and $h > 0$, we have*

$$I_{t,x}(h) := \int_{\mathbb{R}} \left(\int_0^t \left| \frac{\partial}{\partial y} G_{t-s}(x + h - y) - \frac{\partial}{\partial y} G_{t-s}(x - y) \right|^{\beta} ds \right)^{\frac{1}{\beta}} dy \leq C h^{\frac{2}{\beta} - 1}$$

for every $\beta \in \left(1, \frac{3}{2}\right)$, with $C = C_{T,\beta} > 0$.

Proof : Fix $t \in [0, T]$ and $x \in \mathbb{R}$. Since

$$\frac{\partial}{\partial y} G_t(x - y) = \frac{1}{2\sqrt{4\pi}} t^{-\frac{3}{2}} e^{-\frac{(x-y)^2}{4t}} (x - y) \tag{4.15}$$

we have

$$\begin{aligned} I_{t,x}(h) &= C\int_{\mathbb{R}} dy \left(\int_0^t \left|s^{-\frac{3}{2}}\left((x+h-y)e^{-\frac{(x+h-y)^2}{4s}} - (x-y)e^{-\frac{(x-y)^2}{4s}}\right)\right|^\beta ds\right)^{\frac{1}{\beta}} \\ &= C\int_{\mathbb{R}} dz \left(\int_0^t s^{-\frac{3}{2}\beta}\left|(z+h)e^{-\frac{(z+h)^2}{4s}} - ze^{-\frac{z^2}{4s}}\right|^\beta ds\right)^{\frac{1}{\beta}} \end{aligned}$$

and by change of variables $z = h\tilde{z}$ and $s = h^2\tilde{s}$ we get

$$\begin{aligned} I_{t,x}(h) &= Ch^{\frac{2}{\beta}-1}\int_{\mathbb{R}} dz \left(\int_0^{th^{-2}} ds \times s^{-\frac{3}{2}\beta}\left|(z+1)e^{-\frac{(z+1)^2}{4s}} - ze^{-\frac{z^2}{4s}}\right|^\beta\right)^{\frac{1}{\beta}} \\ &\leq Ch^{\frac{2}{\beta}-1}\int_{\mathbb{R}} dz \left(\int_0^{\infty} ds \times s^{-\frac{3}{2}\beta}\left|(z+1)e^{-\frac{(z+1)^2}{4s}} - ze^{-\frac{z^2}{4s}}\right|^\beta\right)^{\frac{1}{\beta}} \\ &= Ch^{\frac{2}{\beta}-1}\int_{\mathbb{R}} dz J(z)^{\frac{1}{\beta}} \end{aligned}$$

where we used the notation

$$J(z) = \int_0^\infty dss^{-\frac{3}{2}\beta}\left|(z+1)e^{-\frac{(z+1)^2}{4s}} - ze^{-\frac{z^2}{4s}}\right|^\beta.$$

Let us calculate $J(z)$. We use the notation $U_s(z) = ze^{-\frac{z^2}{4s}}$ and we write

$$U_s(z+1) - U_s(z) = \int_0^1 \frac{\partial}{\partial z}U_s(z+\theta)d\theta.$$

Since $\frac{\partial}{\partial z}U_s(z) = e^{-\frac{z^2}{4s}} - \frac{z^2}{2s}e^{-\frac{z^2}{4s}}$, we obtain, by using the assumption $\beta > 1$,

$$\begin{aligned} J(z) &= \int_0^\infty ds \times s^{-\frac{3}{2}\beta}\left|\int_0^1 d\theta e^{-\frac{(z+\theta)^2}{4s}} - \int_0^1 d\theta \frac{(z+\theta)^2}{2s}e^{-\frac{(z+\theta)^2}{4s}}\right|^\beta \\ &\leq C\int_0^\infty ds \times s^{-\frac{3}{2}\beta}\left(\int_0^1 d\theta e^{-\frac{\beta(z+\theta)^2}{4s}} + \int_0^1 d\theta \left(\frac{(z+\theta)^2}{2s}\right)^\beta e^{-\frac{\beta(z+\theta)^2}{4s}}\right) \\ &:= J_1(z) + J_1(z) \end{aligned}$$

with

$$J_1(z) = \int_0^\infty ds \quad s^{-\frac{3}{2}\beta}\int_0^1 d\theta e^{-\frac{\beta(z+\theta)^2}{4s}} \text{ and } J_2(z) = \int_0^\infty ds \quad s^{-\frac{3}{2}\beta}\int_0^1 d\theta \left(\frac{(z+\theta)^2}{2s}\right)^\beta e^{-\frac{\beta(z+\theta)^2}{4s}}.$$

By calculating first the integral ds throught the change of variables $\frac{(z+\theta)^2}{s} = a$ we obtain

$$\begin{aligned} J_1(z) &= \int_0^1 d\theta(z+\theta)^{2-3\beta} \int_0^\infty a^{\frac{3\beta}{2}-2} e^{-\frac{\beta a}{4}} da = C \int_0^1 d\theta (z+\theta)^{2-3\beta} \\ &= C\left((z+1)^{3-3\beta} - z^{3-3\beta}\right). \end{aligned}$$

We proceed in the same way for $J_2(z)$ and we obtain

$$J_2(z) \le C\left((z+1)^{3-3\beta} - z^{3-3\beta}\right).$$

Therefore, for $t \in [0,T], x \in \mathbb{R}, h > 0$,

$$I_{t,x}(h) \le C \int_{\mathbb{R}} dz \left((z+1)^{3-3\beta} - z^{3-3\beta}\right)^{\frac{1}{\beta}}.$$

The above integrand behaves as $z^{\frac{3}{\beta}-3}$ around zero and it is integrable because $\beta < \frac{3}{2}$. Around infinity, $\left((z+1)^{3-3\beta} - z^{3-3\beta}\right)^{\frac{1}{\beta}}$ behaves as $z^{\frac{2}{\beta}-3}$ and the integral converges because $\beta > 1$. ■

We will deduce an useful estimate for the $L^2(\Omega)$-norm of the spatial increment of the process Y.

Proposition 25 *For $t \in [0,T], x \in \mathbb{R}$, denote*

$$Y(t,x) = \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G_{t-s}(x-y) u(s,y)^2 dyds$$

where $(u(t,x), t \in [0,T], x \in \mathbb{R})$ is a random field satisfying (4.5). Then for every $h > 0, t \in [0,T], x \in \mathbb{R}$ and for every $p \ge 2$

$$\mathbf{E}\left|Y(t,x+h) - Y(t,x)\right|^p \le C h^{p(\frac{2}{\beta}-1)} \tag{4.16}$$

for any $\beta \in \left(1, \frac{3}{2}\right)$.

Proof : For $t \in [0,T]$ and $x \in \mathbb{R}$, we have by applying Hölder's inequality to the integral ds

$$\begin{aligned} |Y(t,x+h) - Y(t,x)| &= \left| \int_0^t \int_{\mathbb{R}} \left(\frac{\partial}{\partial y} G_{t-s}(x-y) - \frac{\partial}{\partial y} G_{t-s}(x-y) \right) u(s,y)^2 dyds \right| \\ &\le \int_{\mathbb{R}} dy \left(\int_0^t ds \left| \frac{\partial}{\partial y} G_{t-s}(x-y) - \frac{\partial}{\partial y} G_{t-s}(x-y) \right|^\beta \right)^{\frac{1}{\beta}} \\ &\quad \times \left(\int_0^t ds (u(s,y)^2)^{\frac{\beta-1}{\beta}} \right)^{\frac{\beta}{\beta-1}} \end{aligned}$$

$$\leq C\int_{\mathbb{R}} dy \left(\int_0^t ds \left|\frac{\partial}{\partial y}G_{t-s}(x-y) - \frac{\partial}{\partial y}G_{t-s}(x-y)\right|^{\beta}\right)^{\frac{1}{\beta}} \int_0^t u(s,y)^2 ds.$$

By Minkovski's inequality, for every $p \geq 2$

$$\|Y(t,x+h) - Y(t,x)\|_p \leq \int_{\mathbb{R}} dy \left(\int_0^t ds \left|\frac{\partial}{\partial y}G_{t-s}(x-y) - \frac{\partial}{\partial y}G_{t-s}(x-y)\right|^{\beta}\right)^{\frac{1}{\beta}} \left\|\int_0^t dsu(s,y)^2\right\|_p.$$

By (4.5) we have

$$\left\|\int_0^t dsu(s,y)^2\right\|_p \leq C\left(\int_0^T \mathbf{E}u(s,y)^{2p} ds\right)^{\frac{1}{p}} \leq C$$

and thus

$$\begin{aligned}\|Y(t,x+h) - Y(t,x)\|_p &\leq C\int_{\mathbb{R}} dy \left(\int_0^t ds \left|\frac{\partial}{\partial y}G_{t-s}(x-y) - \frac{\partial}{\partial y}G_{t-s}(x-y)\right|^{\beta}\right)^{\frac{1}{\beta}} \\ &\leq Ch^{\frac{2}{\beta}-1}.\end{aligned}$$

■

From the above result, we deduce the sample paths regularity of the solution to the stochastic Burgers equation.

Corollary 3 *Let $(Y(t,x),(t,x) \in [0,T] \times \mathbb{R})$ be given by (4.8) and let $(u(t,x),(t,x) \in [0,T] \times \mathbb{R})$ be the solution to the Burgers equation (4.3). Then, for fixed $t \in [0,T]$, the sample path $x \to Y(t,x)$ is Hölder continuous of order δ for every $\delta \in \left(0, \frac{2}{\beta} - 1\right)$ and for any $\beta \in \left(1, \frac{3}{2}\right)$, while the sample path $x \to u(t,x)$ is Hölder continuous of order δ for every $\delta \in \left(0, \frac{1}{2}\right)$.*

Proof : It is an immediate consequence of relation (4.10), Proposition 25 and of the Kolmogorov continuity criterion. ■

Remark 8 *Notice that for $\beta \in \left(1, \frac{4}{3}\right)$, the process Y is more regular in space than the Gaussian process X given by (4.7), although it contains the singularity of the Burgers equation. When β is close to one then Y is almost Lipschitz with respect to the space variable. Similar estimates for the Green kernel of the heat equation on a bounded spatial domain have been obtained in [76].*

4.3.2 Analysis in time variable

We study here the temporal increment of the random field (4.8). Again we start with a technical lemma involving the Green kernel (4.4).

Lemma 6 *1. For every $h > 0$, and for every $t \in [0,T], x \in \mathbb{R}$, we have*

$$J_{t,x}(h) := \int_{\mathbb{R}} \left(\int_0^t \left| \frac{\partial}{\partial y} G_{t+h-s}(x-y) - \frac{\partial}{\partial y} G_{t-s}(x-y) \right|^\beta ds \right)^{\frac{1}{\beta}} dy \leq C h^{\frac{1}{\beta}-\frac{1}{2}} \quad (4.17)$$

for every $\beta \in \left(1, \frac{4}{3}\right)$.

2. For every $h > 0$, and for every $t \in [0,T], x \in \mathbb{R}$,

$$K_{t,x}(h) := \int_t^{t+h} \int_{\mathbb{R}} \left| \frac{\partial}{\partial y} G_{t+h-s}(x-y) \right|^\beta dyds \leq C h^{\frac{3}{2}-\beta} \quad (4.18)$$

for every $\beta \in \left(0, \frac{3}{2}\right)$.

Proof : By (4.15),

$$\begin{aligned} J_{t,x}(h) &= C \int_{\mathbb{R}} \left(\int_0^t \left| (t+h-s)^{-\frac{3}{2}} e^{-\frac{(x-y)^2}{4(t+h-s)}} (x-y) - (t-s)^{-\frac{3}{2}} e^{-\frac{(x-y)^2}{4(t-s)}} (x-y) \right|^\beta ds \right)^{\frac{1}{\beta}} dy \\ &= \int_{\mathbb{R}} dz \times |z| \left(\int_0^t \left| (s+h)^{-\frac{3}{2}} e^{-\frac{z^2}{4(s+h)}} - s^{-\frac{3}{2}} e^{-\frac{z^2}{4s}} \right|^\beta ds \right)^{\frac{1}{\beta}} \end{aligned}$$

and with the change of variables $s = h\tilde{s}$ and $z = \sqrt{h}\tilde{z}$ we arrrive at the expression

$$\begin{aligned} J_{t,x}(h) &= C h^{\frac{1}{\beta}-\frac{1}{2}} \int_{\mathbb{R}} |z| \left(\int_0^{\frac{t}{h}} \left| (s+1)^{-\frac{3}{2}} e^{-\frac{z^2}{4(s+1)}} - s^{-\frac{3}{2}} e^{-\frac{z^2}{4s}} \right|^\beta ds \right)^{\frac{1}{\beta}} dz \\ &\leq C h^{\frac{1}{\beta}-\frac{1}{2}} \int_{\mathbb{R}} |z| \left(\int_0^{\infty} \left| (s+1)^{-\frac{3}{2}} e^{-\frac{z^2}{4(s+1)}} - s^{-\frac{3}{2}} e^{-\frac{z^2}{4s}} \right|^\beta ds \right)^{\frac{1}{\beta}} dz \\ &= C h^{\frac{1}{\beta}-\frac{1}{2}} \int_0^{\infty} z \left(\int_0^{\infty} \left| (s+1)^{-\frac{3}{2}} e^{-\frac{z^2}{4(s+1)}} - s^{-\frac{3}{2}} e^{-\frac{z^2}{4s}} \right|^\beta ds \right)^{\frac{1}{\beta}} dz. \end{aligned}$$

We need to show that the quantity

$$A_{t,x} := \int_0^{\infty} z \left(\int_0^{\infty} \left((s+1)^{-\frac{3}{2}} e^{-\frac{z^2}{4(s+1)}} - s^{-\frac{3}{2}} e^{-\frac{z^2}{4s}} \right)^\beta ds \right)^{\frac{1}{\beta}} dz < \infty.$$

Denote

$$F_z(t) = t^{-\frac{3}{2}} e^{-\frac{z^2}{4t}} \text{ with } \frac{\partial}{\partial t} F_z(t) = -\frac{3}{2} t^{-\frac{5}{2}} e^{-\frac{z^2}{4t}} - \frac{1}{4} z^2 t^{-\frac{7}{2}} e^{-\frac{z^2}{4t}}$$

and write

$$F_z(t+1) - F_z(t) = \int_0^1 d\theta \frac{\partial}{\partial t} F_z(t+\theta).$$

We can bound the quantity $A_{t,x}$ as follows

$$\begin{aligned} A_{t,x} &\leq C \int_0^\infty z \left[\int_0^\infty ds \left(\int_0^1 d\theta (s+\theta)^{-\frac{5}{2}} e^{-\frac{z^2}{4(s+\theta)}} + \int_0^1 d\theta (s+\theta)^{-\frac{7}{2}} e^{-\frac{z^2}{4(s+\theta)}} z^2 \right)^\beta \right]^{\frac{1}{\beta}} dz \\ &\leq C \int_0^\infty z \left[\int_0^\infty ds \int_0^1 d\theta (s+\theta)^{-\frac{5\beta}{2}} e^{-\frac{\beta z^2}{4(s+\theta)}} + \int_0^\infty ds \int_0^1 d\theta (s+\theta)^{-\frac{7\beta}{2}} e^{-\frac{\beta z^2}{4(s+\theta)}} z^{2\beta} \right]^{\frac{1}{\beta}} dz. \end{aligned}$$

We perform the change of variable $\frac{z^2}{s+\theta} = a$ for the integral ds. In this way,

$$\begin{aligned} A_{t,x} &\leq C \int_0^\infty z \left[z^{2-5\beta} \int_0^1 d\theta \int_0^{\frac{z^2}{\theta}} a^{\frac{5\beta}{2}-2} e^{-\frac{a\beta}{4}} da \right]^{\frac{1}{\beta}} dz \\ &= C \int_0^\infty z^{\frac{2}{\beta}-4} \left[\int_0^1 d\theta \int_0^{\frac{z^2}{\theta}} a^{\frac{5\beta}{2}-2} e^{-\frac{a\beta}{4}} da \right]^{\frac{1}{\beta}} dz \\ &\leq C \int_{z\leq 1} z^{\frac{2}{\beta}-4} \left[\int_0^1 d\theta \int_0^{\frac{z^2}{\theta}} a^{\frac{5\beta}{2}-2} e^{-\frac{a\beta}{4}} da \right]^{\frac{1}{\beta}} dz \\ &\quad + C \int_{z>1} z^{\frac{2}{\beta}-4} \left[\int_0^1 d\theta \int_0^{\frac{z^2}{\theta}} a^{\frac{5\beta}{2}-2} e^{-\frac{a\beta}{4}} da \right]^{\frac{1}{\beta}} dz. \end{aligned}$$

For the integral over the region $z \geq 1$, we majorize $\int_0^{\frac{z^2}{\theta}} da \times a^{\frac{5\beta}{2}-2} e^{-\frac{a\beta}{4}} da$ by the integral $\int_0^\infty da \times a^{\frac{5\beta}{2}-2} e^{-\frac{a\beta}{4}} da = C$ and then

$$\int_{z>1} z^{\frac{2}{\beta}-4} \left[\int_0^1 d\theta \int_0^{\frac{z^2}{\theta}} da \times a^{\frac{5\beta}{2}-2} e^{-\frac{a\beta}{4}} da \right]^{\frac{1}{\beta}} dz \leq C \int_{z>1} z^{\frac{2}{\beta}-4} dz < \infty$$

for $\beta > 1 > \frac{2}{3}$.

The integral over the region $z \leq 1$ is treated as follows. We separate the integral $d\theta$ into the sum of the integrals over $[0, z^2]$ and $[z^2, 1]$, i.e.

$$\int_{z\leq 1} z^{\frac{2}{\beta}-4} \left[\int_0^1 d\theta \int_0^{\frac{z^2}{\theta}} da \times a^{\frac{5\beta}{2}-2} e^{-\frac{a\beta}{4}} da \right]^{\frac{1}{\beta}} dz$$

$$= \int_{z\leq 1} z^{\frac{2}{\beta}-4}\left[\int_0^{z^2} d\theta \int_0^{\frac{z^2}{\theta}} da \times a^{\frac{5\beta}{2}-2} e^{-\frac{a\beta}{4}} da\right]^{\frac{1}{\beta}} dz + \int_{z\leq 1} z^{\frac{2}{\beta}-4}\left[\int_{z^2}^{1} d\theta \int_0^{\frac{z^2}{\theta}} da \times a^{\frac{5\beta}{2}-2} e^{-\frac{a\beta}{4}} da\right]^{\frac{1}{\beta}} dz.$$

Now, when $\theta \in [0, z^2]$, we bound $\int_0^{\frac{z^2}{\theta}} da$ by $\int_0^{\infty} da$ and when $\theta \in [z^2, 1]$, we interchange the integrals $d\theta$ and da and we calculate the integral $d\theta$. Then

$$\begin{aligned}
&\int_{z\leq 1} z^{\frac{2}{\beta}-4}\left[\int_0^{1} d\theta \int_0^{\frac{z^2}{\theta}} a^{\frac{5\beta}{2}-2} e^{-\frac{a\beta}{4}} da\right]^{\frac{1}{\beta}} dz \\
\leq\; & C\int_{z\leq 1} z^{\frac{4}{\beta}-4} dz + \int_{z\leq 1} z^{\frac{2}{\beta}-4}\int_0^{1} a^{\frac{5\beta}{2}-2} e^{-\frac{a\beta}{4}} \int_{z^2}^{\frac{z^2}{a}} d\theta \\
=\; & C\int_{z\leq 1} z^{\frac{4}{\beta}-4} dz + \int_{z\leq 1} z^{\frac{4}{\beta}-4} dz \int_0^{1} a^{\frac{5\beta}{2}-3} e^{-\frac{a\beta}{4}}(1-a) \leq C\int_{z\leq 1} z^{\frac{4}{\beta}-4} dz
\end{aligned}$$

and this is convergent for $\beta < \frac{4}{3}$. So the bound (4.17) is obtained. To deal with (4.18), we write

$$\begin{aligned}
K_{t,x}(h) &= C\int_0^h ds (h-s)^{-\frac{3}{2}\beta} \int_0^{\infty} dy \times y^{\beta} e^{-\frac{y^2\beta}{4(s-h)}} \\
&= C\int_0^h ds (h-s)^{-\frac{3}{2}\beta} (h-s)^{\frac{\beta+1}{2}} \int_0^{\infty} dz \times e^{-\frac{z^2\beta}{4}} = C\int_0^h ds (h-s)^{\frac{1}{2}-\beta} \leq Ch^{\frac{3}{2}-\beta}
\end{aligned}$$

where we used $\beta < \frac{3}{2}$. ■

Let us regard the increment in time of the process Y.

Proposition 26 *For $t \in [0,T], x \in \mathbb{R}$, denote*

$$Y(t,x) = \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G_{t-s}(x-y) u(s,y)^2 dy ds$$

where $(u(t,x), t \in [0,T], x \in \mathbb{R})$ is a random field satisfying (4.5). Then for every $t \in [0,T], x \in \mathbb{R}$ and for any $h \in [0, T-t]$ and $p \geq 1$

$$\mathbf{E}\left|Y(t+h,x) - Y(t,x)\right|^p \leq Ch^{p(\frac{1}{\beta}-\frac{1}{2})} \tag{4.19}$$

for every $\beta \in \left(1, \frac{4}{3}\right)$.

Proof : We write, with $\beta \in \left(1, \frac{4}{3}\right)$

$$|Y(t+h,x) - Y(t,x)| \leq \int_t^{t+h} \int_{\mathbb{R}} \left|\frac{\partial}{\partial y} G_{t+h-s}(x-y)\right| u(s,y)^2 dy ds$$

$$
\begin{aligned}
&+\int_0^t \int_{\mathbb{R}} \left| \frac{\partial}{\partial y} G_{t+h-s}(x-y) - \frac{\partial}{\partial y} G_{t-s}(x-y) \right| u(s,y)^2 dyds \\
\leq\ & \int_t^{t+h} \int_{\mathbb{R}} \left| \frac{\partial}{\partial y} G_{t+h-s}(x-y) \right| u(s,y)^2 dyds \\
&+C \int_{\mathbb{R}} \left(\int_0^t \left(\frac{\partial}{\partial y} G_{t+h-s}(x-y) - \frac{\partial}{\partial y} G_{t-s}(x-y) \right)^{\beta} ds \right)^{\frac{1}{\beta}} dy \int_0^t u(s,y)^2 ds
\end{aligned}
$$

and by Minkovski inequality and (4.5)

$$
\begin{aligned}
||Y(t+h,x) - Y(t,x)||_p \leq\ & \int_t^{t+h} \int_{\mathbb{R}} \left| \frac{\partial}{\partial y} G_{t+h-s}(x-y) \right| \|u(s,y)^2\|_p dyds \\
&+C \int_{\mathbb{R}} \left(\int_0^t \left(\frac{\partial}{\partial y} G_{t+h-s}(x-y) - \frac{\partial}{\partial y} G_{t-s}(x-y) \right)^{\beta} ds \right)^{\frac{1}{\beta}} dy \left\| \int_0^t u(s,y)^2 ds \right\|_p \\
\leq\ & \int_t^{t+h} \int_{\mathbb{R}} \left| \frac{\partial}{\partial y} G_{t+h-s}(x-y) \right| dyds \\
&+C \int_{\mathbb{R}} \left(\int_0^t \left(\frac{\partial}{\partial y} G_{t+h-s}(x-y) - \frac{\partial}{\partial y} G_{t-s}(x-y) \right)^{\beta} ds \right)^{\frac{1}{\beta}} dy.
\end{aligned}
$$

By (4.18) with $\beta = 1$ and (4.17) in Lemma 6, we get

$$
||Y(t+h,x) - Y(t,x)||_p \leq C(h^{\frac{1}{2}} + h^{\frac{1}{\beta} - \frac{1}{2}}) \leq Ch^{\frac{1}{\beta} - \frac{1}{2}}
$$

for h small enough and $\beta > 1$. ■

We apply the previous result to get the regularity in time to the solution to the stochastic Burgers equation (4.3).

Corollary 4 *Let $(u(t,x), (t,x) \in [0,T] \times \mathbb{R})$ be the solution to the stochastic Burgers equation (4.1) and let $(Y(t,x), (t,x) \in [0,T] \times \mathbb{R})$ be given by (4.8). Then, for every $x \in \mathbb{R}$, the sample path $t \to Y(t,x)$ is Hölder continuous of order δ for every $\delta \in \left(0, \frac{1}{\beta} - \frac{1}{2}\right)$ and for any $\beta \in \left(1, \frac{4}{3}\right)$ and the sample path $t \to u(t,x)$ is Hölder continuous of order δ for any $\delta \in \left(0, \frac{1}{4}\right)$ and for any $\beta \in \left(1, \frac{4}{3}\right)$.*

Remark 9 *Again, as in the case of the spatial regularity, the process Y from (4.8) is more regular in time than the process X in (4.7) and the solution to the Burgers equation (4.3) keeps the regularity of X, the solution to the heat equation. The Hölder regularity for the stochastic Burgers equation with multiplicative noise can be found in [71].*

4.4 Estimation of the drift parameter

In this section, we will construct and analyze two estimators for the drift parameter $\theta > 0$ in (4.1) based on the discrete observations of the solution.

4.4.1 On the solution to the parametrized stochastic Burgers equation

Consider the SPDE (4.1) with $\theta > 0$ and with vanishing initial condition. Since the Green kernel associated to the heat equation $\frac{\partial}{\partial t}u_\theta(t,x) = \theta\frac{\partial^2}{\partial x^2}u_\theta(t,x)$ is $G_{\theta t}(x)$ with G given by (4.4), the mild solution to (4.1) can be written as (from now on, we will use the notation u_θ for the mild solution to (4.1)

$$\begin{aligned} u_\theta(t,x) &= \frac{1}{2}\int_0^t\int_{\mathbb{R}}\frac{\partial}{\partial y}G_{\theta(t-s)}(x-y)u_\theta(s,y)^2dyds \\ &\quad + \int_0^t\int_{\mathbb{R}}G_{\theta(t-s)}(x-y)W(ds,dy), \quad (t,x)\in[0,T]\times\mathbb{R}. \end{aligned} \tag{4.20}$$

The purpose is to estimate the parameter $\theta > 0$ based on the observation of the solution to (4.1) at discrete points in space. The following result is very useful. It allows to move the parameter θ before the noise term, and then to use the quadratic or quartic variations of the solution in order to estimate it. A similar idea has been used in [88].

Proposition 27 *Denote by, for $(t,x)\in[0,T]\times\mathbb{R}$,*

$$v_\theta(t,x) = u_\theta\left(\frac{t}{\theta},x\right). \tag{4.21}$$

Then v_θ solves the SPDE

$$\frac{\partial}{\partial t}v_\theta(t,x) = \frac{\partial^2}{\partial x^2}v_\theta(t,x) - \frac{1}{2}\theta^{-1}\frac{\partial}{\partial x}v_\theta(t,x)^2 + \theta^{-\frac{1}{2}}\sigma(v_\theta(t,x))\dot{\widetilde{W}}(t,x) \tag{4.22}$$

with $v_\theta(0,x) = u_0(x) = 0$ for every $x\in\mathbb{R}$, where $\widetilde{W}$ is a space-time white noise.

Proof : By (4.20), we have

$$\begin{aligned} v_\theta(t,x) &= \frac{1}{2}\int_0^{\frac{t}{\theta}}\int_{\mathbb{R}}\frac{\partial}{\partial y}G_{t-\theta s}(x-y)u_\theta(s,y)^2dyds + \int_0^{\frac{t}{\theta}}\int_{\mathbb{R}}G_{t-\theta s}W(ds,dy) \\ &= \frac{1}{2}\theta^{-1}\int_0^t\int_{\mathbb{R}}\frac{\partial}{\partial y}G_{t-s}(x-y)u_\theta\left(\frac{s}{\theta},y\right)^2dyds + \theta^{-\frac{1}{2}}\int_0^t\int_{\mathbb{R}}G_{t-s}(x-y)W(d(\frac{s}{\theta}),dy) \\ &= \frac{1}{2}\theta^{-1}\int_0^t\int_{\mathbb{R}}\frac{\partial}{\partial y}G_{t-s}(x-y)v_\theta(s,y)^2dyds + \theta^{-\frac{1}{2}}\int_0^t\int_{\mathbb{R}}G_{t-s}(x-y)\widetilde{W}(ds,dy) \end{aligned} \tag{4.23}$$

The conclusion follows by noticing that for every $\theta > 0$, the random field $\left(\widetilde{W}(s,y) = \theta^{-\frac{1}{2}}W(\frac{s}{\theta},y), s\in[0,T], y\in\mathbb{R}\right)$ is a space-time white-noise, due to the scaling property of W in time. ■

4.4.2 An estimator for the drift parameter based on spatial observations

We will define two estimators for the drift parameter θ in (4.1), one based on the discrete observation of the solutions (4.20) in space and a second one by assuming that this solution is observed at discrete times. We will start with the first case, i.e. we assume that the solution is observed on a spatial interval.

Let $A_1 < A_2$ and and assume that we have at our disposal the observations $(u_\theta(t, x_i), i = 0, .., N)$ with x_i given by (4.11) and $t > 0$ fixed. Given a random field $(X(t, x), (t, x) \in [0, T] \times \mathbb{R})$, we define

$$S_{N,t}(X) = \sum_{i=0}^{N-1} (X(t, x_{i+1}) - X(t, x_i))^2 \tag{4.24}$$

for any $t > 0$ and $N \geq 1$.

We first establish the asymptotic behavior of $S_{N,t}(u_\theta)$, u_θ being the solution to the (parametrized) stochastic Burgers equation.

Proposition 28 *Let u_θ be given by (4.20). Then, for every $t > 0$,*

$$S_{N,t}(u_\theta) \to_{N\to\infty} \frac{1}{2}(A_2 - A_1)\theta^{-1} \text{ in } L^1(\Omega).$$

Proof : By the definition (4.21) of the process v_θ, we can see that for every $t > 0, N \geq 1$,

$$S_{N,\theta t}(v_\theta) = S_{N,t}(u_\theta).$$

It therefore suffices to study the limit as $N \to \infty$ of the sequence $S_{N,t}(v_\theta)$ with fixed $t > 0$. Notice that by (4.23), we can decompose v_θ as

$$v_\theta(t, x) = \theta^{-\frac{1}{2}} X(t, x) + \theta^{-1} Y_\theta(t, x) \tag{4.25}$$

with X given by (4.7) and

$$Y_\theta(t, x) = \frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G_{t-s}(x - y) v_\theta(s, y)^2 dyds = \frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G_{t-s}(x - y) u_\theta\left(\frac{s}{\theta}, y\right)^2 dyds. \tag{4.26}$$

In particular, pour $\theta = 1$, Y_1 coincide with the processes Y defined by (4.8) (and actually, the process X in (4.25) coincides in distributions with the process X in (4.7)). Now we write

$$S_{N,t}(v_\theta) = \theta^{-1} S_{N,t}(X) + \theta^{-2} S_{N,t}(Y_\theta) + 2\theta^{-\frac{3}{2}} \sum_{i=0}^{N-1} (X(t, x_{i+1}) - X(t, x_i))(Y_\theta(t, x_{i+1}) - Y_\theta(t, x_i)).$$

By (4.12),

$$S_{N,t}(X) \to_{N\to\infty} \frac{1}{2}(A_2 - A_1) \text{ in } L^2(\Omega).$$

Concerning the spatial quadratic variations of Y_θ, we have, by Proposition 25,

$$\mathbf{E}S_{N,t}(Y_\theta) = \sum_{i=0}^{N-1} \mathbf{E}\left(Y_\theta(t,x_{i+1}) - Y_\theta(t,x_i)\right)^2 \leq C\theta^{-2}N^{3-\frac{4}{\beta}} \to_{N\to\infty} 0$$

since $\beta < \frac{4}{3}$. This implies that $S_{N,t}(Y_\theta)$ converges to zero in $L^1(\Omega)$ as $N \to \infty$. Let

$$S_{N,t}(X, Y_\theta) := \sum_{i=0}^{N-1} \left(X(t,x_{i+1}) - X(t,x_i)\right)\left(Y_\theta(t,x_{i+1}) - Y_\theta(t,x_i)\right)$$

for every $N \geq 1, t > 0$. Using Cauchy-Schwarz inequality and the bound (4.10) and (4.16)

$$\begin{aligned} \mathbf{E}|S_{N,t}(X,Y_\theta)| &\leq \sum_{i=0}^{n-1} \left(\mathbf{E}\left|X(t,x_{i+1}) - X(t,x_i)\right|^2\right)^{\frac{1}{2}} \left(\mathbf{E}\left|Y_\theta(t,x_{i+1}) - Y_\theta(t,x_i)\right|^2\right)^{\frac{1}{2}} \\ &\leq C\theta^{-\frac{3}{2}} \sum_{i=0}^{N-1} N^{-\frac{1}{2}} N^{1-\frac{2}{\beta}} = C\theta^{-\frac{3}{2}} N^{\frac{3}{2}-\frac{2}{\beta}} \to_{N\to\infty} 0 \end{aligned}$$

by choosing $\beta < \frac{4}{3}$. ■

Define the estimator, for $t > 0$ and $N \geq 1$,

$$\widehat{\theta}_{N,t} = \frac{A_2 - A_1}{2S_{N,t}(u_\theta)}. \tag{4.27}$$

Notice that (4.27) can be directly computed from the observations $u(t,x_i), i = 0,1,...,N$ with x_i from (4.11).

By Proposition 28, we have the following result.

Proposition 29 *For every $t > 0$, the estimator $\widehat{\theta}_{N,t}$ defined by (4.27) is consistent, i.e. $\widehat{\theta}_{N,t}$ converges in probability to θ as $N \to \infty$.*

We will deduce below some other asymptotic properties of the estimator (4.27), including the strong consistency and the approximation error. We start by proving the almost sure convergence of the quadratic variations of u_θ.

Proposition 30 *Let u_θ be given by (4.20) and let $S_{N,t}(u_\theta)$ be defined as in (4.24). Then, for every $t > 0$,*

$$S_{N,t}(u_\theta) \to_{N\to\infty} \frac{1}{2}(A_2 - A_1)\theta^{-1} \text{ almost surely.}$$

Proof : For simplicity, we assume $\theta = 1$. We decompose $S_{N,t}(u_1)$ as in the proof of Proposition 28

$$S_{N,t}(u_1) - \frac{1}{2}(A_2 - A_1)$$

$$= \quad S_{N,t}(X) - \frac{1}{2}(A_2 - A_1) + 2\sum_{i=0}^{N-1}\left(X(t,x_{i+1}) - X(t,x_i)\right)\left(Y(t,x_{i+1}) - Y(t,x_i)\right) + S_{N,t}(Y)$$

with X, Y given by (4.7), (4.8) respectively. We show first that the sequence $S_{N,t}(X) - \frac{1}{2}(A_2 - A_1)$ converges almost surely to 0 as $N \to \infty$. Let us use the notation

$$g(N) = \mathbf{E}\left(X(t,x_{i+1}) - X(t,x_i)\right)^2. \tag{4.28}$$

Notice that g_N does not depend on i, due to the stationarity of the solution to the heat equation in space (see e.g. [20]). We have

$$\begin{aligned} S_{N,t}(X) - \frac{1}{2}(A_2 - A_1) &= \sum_{i=0}^{N-1}\left[\left(X(t,x_{i+1}) - X(t,x_i)\right)^2 - g(N)\right] \\ &\quad + Ng(N) - \frac{1}{2}(A_2 - A_1) \end{aligned}$$

We recall the following estimates (see relations (9) and (11) in [88]) :

$$|Ng(N) - \frac{1}{2}(A_2 - A_1)| \le C\frac{1}{N} \tag{4.29}$$

and

$$\mathbf{E}(X(t,x_{i+1} - X(t,x_i))(X(t,x_{j+1} - X(t,x_j)) \le C\frac{1}{N^2} \tag{4.30}$$

with C denoting again an universal constant (it does not depend on i, N). By (4.30), we know that

$$\begin{aligned} &\mathbf{E}\left(\sum_{i=0}^{N-1}\left[\left(X(t,x_{i+1}) - X(t,x_i)\right)^2 - g(N)\right]\right)^2 \\ &= 2\sum_{i,j=0}^{N-1}\left(\mathbf{E}(X(t,x_{i+1} - X(t,x_i))(X(t,x_{j+1} - X(t,x_j))\right)^2 \le C\frac{1}{N}. \end{aligned} \tag{4.31}$$

Thus, for every $p \ge 2$, by (4.31) and (4.29)

$$\begin{aligned} \mathbf{E}\left|S_{N,t}(X) - \frac{1}{2}\right|^{2p} &\le C\left(\mathbf{E}\left(\sum_{i=0}^{N-1}\left[\left(X(t,x_{i+1}) - X(t,x_i)\right)^2 - g(N)\right]\right)^{2p} + |Ng(N) - \frac{1}{2}|^{2p}\right) \\ &\le C\left(\left[\mathbf{E}\left(\sum_{i=0}^{N-1}\left[\left(X(t,x_{i+1}) - X(t,x_i)\right)^2 - g(N)\right]\right)^2\right]^p + |Ng(N) - \frac{1}{2}|^{2p}\right) \\ &\le C\frac{1}{N^p}. \end{aligned} \tag{4.32}$$

We used the well-known fact the quantity $t_N := \sum_{i=0}^{N-1}\left[(X(t,x_{i+1})-X(t,x_i))^2-g(N)\right]$ is an element in the second Wiener chaos (see e.g. [100]) and so $\mathbf{E}|t_N|^{2p} \leq C_p(\mathbf{E}|t_N|^2)^p$, by the hypercontractibity property of the Wiener chaos, see (4.44).

Let $\gamma > 0$. By Markov's inequality,

$$P\left(\left|S_{N,t}(X)-\frac{1}{2}(A_2-A_1)\right| \geq N^{-\gamma}\right) \leq N^{2\gamma p}\mathbf{E}\left|S_{N,t}(X)-\frac{1}{2}\right|^{2p} \leq CN^{(2\gamma-1)p}.$$

By chosing $\gamma \in \left(0,\frac{1}{2}\right)$ and p large enough, we notice that the series

$$\sum_{N\geq 1} P\left(\left|S_{N,t}(X)-\frac{1}{2}(A_2-A_1)\right| \geq N^{-\gamma}\right) \leq C\sum_{N\geq 1} N^{(2\gamma-1)p}$$

is convergent. By applying Borel-Cantelli lemma, we get the almost sure convergence of $S_{N,t}(u_1)$ to $\frac{1}{2}(A_2-A_1)$ as $N\to\infty$.

Next, we prove that the sequence $a_N := \sum_{i=0}^{N-1}\left(X(t,x_{i+1})-X(t,x_i)\right)\left(Y(t,x_{i+1})-Y(t,x_i)\right)$ converges almost surely to zero as $N\to\infty$. Notice that for every $p\geq 2$, via Hölder's inequality, (4.10) and (4.16)

$$\begin{aligned}
\mathbf{E}|a_N|^p &\leq N^{p-1}\mathbf{E}\sum_{i=0}^{N-1}|X(t,x_{i+1})-X(t,x_i)|^p\,|Y(t,x_{i+1})-Y(t,x_i)|^p\\
&\leq N^{p-1}\sum_{i=0}^{N-1}\left(\mathbf{E}\,|X(t,x_{i+1})-X(t,x_i)|^{2p}\right)^{\frac{1}{2}}\left(\mathbf{E}\,|Y(t,x_{i+1})-Y(t,x_i)|^{2p}\right)^{\frac{1}{2}}\\
&\leq CN^pN^{-\frac{p}{2}}N^{-p(\frac{2}{\beta}-1)} = N^{p(\frac{3}{2}-\frac{2}{\beta})}. \qquad (4.33)
\end{aligned}$$

We can write, for fixed $\gamma > 0$,

$$P\left(a_N \geq N^{-\gamma}\right) \leq N^{\gamma p}\mathbf{E}|a_N|^p \leq N^{p(\frac{3}{2}-\frac{2}{\beta}+\gamma)}$$

and, for $\beta \in (1,\frac{4}{3})$, $\gamma > 0$ close enough to zero and p sufficiently large,

$$\sum_{N\geq 1} P\left(a_N \geq N^{-\gamma}\right) < \infty$$

so a_N converges almost surely to zero as $N\to\infty$ by Borel-Cantelli lemma.

Finally, we show that $S_{N,t}(Y)$ also converges to zero almost surely as $N\to\infty$. For $\gamma > 0$ and $p\geq 2$, we have by (4.16)

$$\begin{aligned}
&\sum_{N\geq 1} P\left(S_{N,t}(Y)\geq N^{-\gamma}\right) \leq N^{\gamma p}\mathbf{E}|S_{N,t}(Y)|^p\\
\leq\ &\sum_{N\geq 1} N^{\gamma p}N^{p-1}\sum_{i=0}^{N-1}\mathbf{E}\,|(Y(t,x_{i+1})-Y(t,x_i)|^{2p}
\end{aligned}$$

$$\leq \quad C\sum_{N\geq 1} N^{p(\gamma+1)}N^{2p(1-\frac{2}{\beta})} = C\sum_{N\geq 1} N^{p(\gamma+3-\frac{2}{\beta})} \tag{4.34}$$

and the last series converges for $\beta \in (1, \frac{3}{2})$, $\gamma > 0$ close to zero and p large.

■

We state and prove our result concerning the strong consistency and the approximation error in L^p-norm for the estimator (4.27).

Theorem 7 *The estimator $\widehat{\theta}_{N,t}$ is strongly consistent, i.e. $\widehat{\theta}_{N,t}$ converges almost surely to θ as $N \to \infty$. Moreover, for every $p \geq 2$,*

$$\mathbf{E}\left|\widehat{\theta}_{N,t} - \theta\right|^p \leq CN^{-\frac{p}{2}}$$

for N large enough.

Proof : We have

$$\widehat{\theta}_{N,t} - \theta = \frac{\frac{1}{2}\theta - S_{N,t}(u_\theta)}{S_{N,t}(u_\theta)}.$$

This relation gives the strong consistency of the estimator, via Proposition 30. Also, by the same result, $S_{N,t}(u_\theta) \geq C$ almost surely for N large. Hence

$$\begin{aligned} \mathbf{E}\left|\widehat{\theta}_{N,t} - \theta\right|^p &\leq C\mathbf{E}\left|\frac{1}{2}\theta - S_{N,t}(u_\theta)\right|^p \\ &\leq CN^{-\frac{p}{2}} \end{aligned}$$

by using the estimates (4.32) and (4.33) in the proof of Proposition 30 and the assumption $\beta > 1$. ■

4.4.3 An estimator for the drift parameter based on temporal observations

Let us now assume that the spatial variable $x \in \mathbb{R}$ is fixed and let $t_i, i = 0, .., N$ be the partition of the interval $[A_1, A_2]$ given by (4.13). For every $x \in \mathbb{R}$ and for any random field $(X(t,x), (t,x) \in [0,T] \times \mathbb{R})$, we define its quartic variation in time

$$T_{N,x}(X) = \sum_{i=0}^{N-1} \left(X(t_{i+1}, x) - X(t_i, x)\right)^4, \quad N \geq 1. \tag{4.35}$$

We start again by proving the asymptotic behavior of the quartic variation of the solution (4.20).

Proposition 31 *Let u_θ be given by (4.20). Then, for every fixed $x \in \mathbb{R}$,*

$$T_{N,x}(u_\theta) \to_{N\to\infty} \frac{3}{\pi}\theta^{-1}(A_2 - A_1) \text{ in } L^1(\Omega). \tag{4.36}$$

Proof : Fix $x \in \mathbb{R}$. We show first that

$$T_{N,x}(v_\theta) \to_{N\to\infty} \theta^{-2}\frac{3}{\pi}(A_2 - A_1) \text{ in } L^1(\Omega). \tag{4.37}$$

Using the decomposition of v_θ as

$$v_\theta(t,x) = \theta^{-\frac{1}{2}}X(t,x) + \theta^{-1}Y_\theta(t,x)$$

with X the solution to the stochastic heat equation (4.7) and Y_θ given by (4.26), we will write $T_{N,x}(v_\theta)$ as follows

$$\begin{aligned} T_{N,x}(v_\theta) &= \theta^{-2}T_{N,x}(X) + 4\theta^{-\frac{5}{2}}\sum_{i=0}^{N-1}\left(X(t_{i+1},x) - X(t_i,x)\right)^3\left(Y_\theta(t_{i+1},x) - Y_\theta(t_i,x)\right) \\ &\quad +6\theta^{-3}\sum_{i=0}^{N-1}\left(X(t_{i+1},x) - X(t_i,x)\right)^2\left(Y_\theta(t_{i+1},x) - Y_\theta(t_i,x)\right)^2 \\ &\quad +4\theta^{-\frac{7}{2}}\sum_{i=0}^{N-1}\left(X(t_{i+1},x) - X(t_i,x)\right)\left(Y_\theta(t_{i+1},x) - Y_\theta(t_i,x)\right)^3 + T_{N,x}(Y_\theta) \end{aligned} \tag{4.38}$$

We know (see (4.14)) that the first summand $\theta^{-2}T_{N,x}(X_\theta)$ converges to $\theta^{-2}\frac{3}{\pi}(A_2 - A_1)$. We will prove that the other terms converge to zero in $L^1(\Omega)$ as $N \to \infty$. To this end, we will use the bounds (4.10) and (4.19). Indeed, by Hölder's inequality and these two estimates,

$$\begin{aligned} &\mathbf{E}\left|\sum_{i=0}^{N-1}\left(X(t_{i+1},x) - X(t_i,x)\right)^3\left(Y_\theta(t_{i+1},x) - Y_\theta(t_i,x)\right)\right| \\ \leq\ & \sum_{i=0}^{N-1}\left[\mathbf{E}\left(X(t_{i+1},x) - X(t_i,x)\right)^6\right]^{\frac{1}{2}}\left[\mathbf{E}\left(Y_\theta(t_{i+1},x) - Y_\theta(t_i,x)\right)^2\right]^{\frac{1}{2}} \\ \leq\ & CNN^{-\frac{3}{4}}N^{\frac{1}{2}-\frac{1}{\beta}} = CN^{\frac{3}{4}-\frac{1}{\beta}} \to_{N\to\infty} 0 \end{aligned}$$

for $\beta < \frac{4}{3}$. Also,

$$\begin{aligned} &\mathbf{E}\left|\sum_{i=0}^{N-1}\left(X(t_{i+1},x) - X(t_i,x)\right)^2\left(Y_\theta(t_{i+1},x) - Y_\theta(t_i,x)\right)^2\right| \\ \leq\ & \sum_{i=0}^{N-1}\left[\mathbf{E}\left(X(t_{i+1},x) - X(t_i,x)\right)^4\right]^{\frac{1}{2}}\left[\mathbf{E}\left(Y_\theta(t_{i+1},x) - Y_\theta(t_i,x)\right)^4\right]^{\frac{1}{2}} \\ \leq\ & CN^{\frac{3}{4}-\frac{1}{\beta}} \to_{N\to\infty} 0 \end{aligned}$$

for $\beta < \frac{4}{3}$. Similarly,

$$\mathbf{E}\left|\sum_{i=0}^{N-1}\left(X(t_{i+1},x)-X(t_i,x)\right)\left(Y_\theta(t_{i+1},x)-Y_\theta(t_i,x)\right)^3\right| \leq CN^{\frac{9}{4}-\frac{3}{\beta}} \to_{N\to\infty} 0$$

and

$$\mathbf{E}|T_{N,x}(Y_\theta)| \leq CN^{3-\frac{4}{\beta}} \to_{N\to\infty} 0.$$

Let us deduce now the limit in $L^1(\Omega)$ of the sequence $T_{N,x}(u_\theta)$. We can write, for $x\in\mathbb{R}$, $t,\theta>0$,

$$T_{N,x}(u_\theta) = \sum_{i=0}^{N-1}\left(u_\theta(t_{i+1},x)-u_\theta(t_i,x)\right)^4 = \sum_{i=0}^{N-1}\left(v_\theta(\theta t_{i+1},x)-v_\theta(\theta t_i,x)\right)^4$$

and since $(\theta t_i, i=0,...,N)$ constitutes a partition of the interval $[\theta A_1,\theta A_2]$, we deduce (4.36) from (4.37). ■

For $x\in\mathbb{R}$ define the estimator

$$\widehat{\theta}_{N,x} = \frac{3}{\pi}(A_2-A_1)\frac{1}{T_{N,x}(u_\theta)}. \tag{4.39}$$

This estimator can also be computed from the data $u(t_i,x), i=0,...,N$ with fixed space variable $x\in\mathbb{R}$.

Proposition 32 *The estimator (4.39) is consistent, i.e. it converges in probability to θ as $N\to\infty$.*

Proof : It suffices to apply Proposition 31. ■

As for the spatial estimator, we obtain the strong consistency of the estimator (4.39) and its approximation error. These results will be deduced from the almost sure convergence of the quartic variation $T_{N,x}(u_\theta)$.

Proposition 33 *If u_θ is given by (4.20) and $T_{N,x}(u_\theta)$ by (4.35). Then*

$$T_{N,x}(u_\theta) \to_{N\to\infty} \frac{3}{\pi}(A_2-A_1) \text{ almost surely.}$$

Proof : Again, let us assume $\theta=1$. Using the decomposition (4.38) of the process $u=u_1=v_1$, it suffices to show the right-hand side of (4.38) go to $\frac{3}{\pi}(A_2-A_1)$ almost surely. We will actually show that $T_{N,x}(X)-\frac{3}{\pi}(A_2-A_1)$ tends to zero almost surely and the other terms in the right-hand side of (4.38) go to zero almost surely as $N\to\infty$. This will based again on a Borel-Cantelli argument.

To deal with $T_{N,x}(X)-\frac{3}{\pi}(A_2-A_1)$, we first compute its L^p-norm. By Proposition 3.2 in [88], we know that

$$\mathbf{E}\left|T_{n,x}(X)-\frac{3}{\pi}(A_2-A_1)\right|^2\leq C\frac{1}{N}.$$

Let us notice that the random variable $T_{N,x}(X)-\frac{3}{\pi}(A_2-A_1)$ can be decomposed into a finite sum of Wiener chaoses. Indeed, let I_q denote the multiple integral of order $q\geq 1$ with respect to the Gaussian process $(X(t,x),t\in[0,T])$ with $x\in\mathbb{R}$ fixed (see the Appendix for the definition and basic properties of the multiple stochastic integrals). Then

$$X(t_{i+1},x)-X(t_i,x)=I_1(1_{[t_i,t_{i+1})})$$

and by the product formula (4.43)

$$(X(t_{i+1},x)-X(t_i,x))^4=I_4\left(1^{\otimes 4}_{[t_i,t_{i+1})}\right)+4I_2\left(1^{\otimes 2}_{[t_i,t_{i+1})}\otimes_1 1^{\otimes 2}_{[t_i,t_{i+1})}\right)+\mathbf{E}\left(X(t_{i+1},x)-X(t_i,x)\right)^4.$$

This implies

$$T_{N,x}(X)-\frac{3}{\pi}(A_2-A_1)=I_4(f_{N,1,x})+I_2(f_{N,2,x})$$

with

$$f_{N,1,x}=\sum_{i=0}^{N-1}1^{\otimes 4}_{[t_i,t_{i+1})}\text{ and }f_{N,2.x}=4\sum_{i=0}^{N-1}1^{\otimes 2}_{[t_i,t_{i+1})}\otimes_1 1^{\otimes 2}_{[t_i,t_{i+1})}.$$

So for every $p\geq 2$, using the hypercontractivity property of the Wiener chaos (4.44)

$$\mathbf{E}\left|T_{n,x}(X)-\frac{3}{\pi}(A_2-A_1)\right|^p\leq CN^{-\frac{p}{2}}.$$

Therefore, for $\gamma>0$,

$$P\left(\left|T_{n,x}(X)-\frac{3}{\pi}(A_2-A_1)\right|\geq N^{-\gamma}\right)\leq N^{\gamma p}\mathbf{E}\left|T_{n,x}(X)-\frac{3}{\pi}(A_2-A_1)\right|^p\leq CN^{p(\gamma-\frac{1}{2})}$$

and for p large and $0<\gamma<\frac{1}{2}$ close enough to zero, we find

$$\sum_{N\geq 1}P\left(\left|T_{n,x}(X)-\frac{3}{\pi}(A_2-A_1)\right|\geq N^{-\gamma}\right)\leq C\sum_{N\geq 1}N^{p(\gamma-\frac{1}{2})}<\infty$$

which ensure the almost sure convergence to zero of the sequence $T_{n,x}(X)-\frac{3}{\pi}(A_2-A_1)$ as $N\to\infty$. A similar proof can be done for the almost sure convergence of the other terms in (4.38), by using the estimates (obtained via Hölder's inequality), (4.10) and (4.19),

$$\mathbf{E}\left|\sum_{i=0}^{N-1}\left(X(t_{i+1},x)-X(t_i,x)\right)^3\left(Y(t_{i+1},x)-Y(t_i,x)\right)\right|^p\leq CN^{p(\frac{3}{4}-\frac{1}{\beta})},$$

$$\mathbf{E}\left|\sum_{i=0}^{N-1}\left(X(t_{i+1},x)-X(t_i,x)\right)^2\left(Y(t_{i+1},x)-Y(t_i,x)\right)^2\right|^p\leq CN^{p(\frac{3}{4}-\frac{1}{\beta})},$$

$$\mathbf{E}\left|\sum_{i=0}^{N-1}\left(X(t_{i+1},x)-X(t_i,x)\right)\left(Y(t_{i+1},x)-Y(t_i,x)\right)^3\right|^p \leq CN^{p(\frac{9}{4}-\frac{3}{\beta})},$$

and

$$\mathbf{E}\left|\sum_{i=0}^{N-1}\left(Y(t_{i+1},x)-Y(t_i,x)\right)^4\right|^p \leq CN^{p(3-\frac{4}{\beta})}.$$

■

We deduce from the above result the asymptotic properties of the estimator (4.39).

Theorem 8 *For every $x \in \mathbb{R}$, the estimator $\widehat{\theta}_{N,x}$ defined by (4.39) is strongly consistent, i.e. $\widehat{\theta}_{N,x}$ converges almost surely to θ as $N \to \infty$. Moreover, for every $p \geq 2$,*

$$\mathbf{E}\left|\widehat{\theta}_{N,x}-\theta\right|^p \leq CN^{-\frac{p}{2}}$$

for N large enough.

Proof : It follows from the relation

$$\widehat{\theta}_{N,x}-\theta = \frac{\frac{3}{\pi}(A_2-A_1)-T_{N,x}(u_\theta)}{T_{N,x}(u_\theta)}$$

and from Proposition 33, as in the proof of Theorem 7. ■.

4.5 Appendix : Multiple stochastic integrals

We have used the multiple stochastic integrals and the Wiener chaos conly in the proof of Propositions 30 and (33). We recall some elementary facts concerning these mathematical object. We refer to the main reference [77] for more details. Consider $\mathcal{H}$ a real separable infinite-dimensional Hilbert space with its associated inner product $\langle .,. \rangle_{\mathcal{H}}$, and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}}$, for every $\varphi, \psi \in \mathcal{H}$. Denote by I_q the qth multiple stochastic integral with respect to B. This I_q is actually an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}}\|\cdot\|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by :

$$H_q(x) = (-1)^q \exp\left(\frac{x^2}{2}\right)\frac{\mathrm{d}^q}{\mathrm{d}x^q}\left(\exp\left(-\frac{x^2}{2}\right)\right), \; x \in \mathbb{R}. \tag{4.40}$$

The isometry of multiple integrals can be written as : for $p,\ q \geq 1$, $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$,

$$\mathbf{E}\Big(I_p(f)I_q(g)\Big) = \begin{cases} q!\langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}^{\otimes q}} & \text{if } p=q \\ 0 & \text{otherwise} \end{cases} \tag{4.41}$$

where $\tilde{f}$ denotes the canonical symmetrization of f and it is defined by :

$$\tilde{f}(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)}),$$

in which the sum runs over all permutations σ of $\{1, \ldots, q\}$. It also holds that :

$$I_q(f) = I_q(\tilde{f}).$$

In the particular case when $\mathcal{H} = L^2(T, \mathcal{B}(T), \mu)$, the rth contraction $f \otimes_r g$ is the element of $\mathcal{H}^{\otimes(p+q-2r)}$, which is defined by :

$$\begin{aligned} &(f \otimes_r g)(s_1, \ldots, s_{p-r}, t_1, \ldots, t_{q-r}) \\ &= \textstyle\int_{T^r} \mathrm{d}u_1 \ldots \mathrm{d}u_r f(s_1, \ldots, s_{p-r}, u_1, \ldots, u_r) g(t_1, \ldots, t_{q-r}, u_1, \ldots, u_r), \end{aligned} \tag{4.42}$$

for every $f \in L^2(T^p)$, $g \in L^2(T^q)$ and $r = 1, \ldots, p \wedge q$. The $f \tilde{\otimes}_l g$ we denote the symmetrization of the contraction $f \otimes_l g$.

The product for two multiple integrals can be expanded into a sum of multiple integrals (see [77]) : if $f \in L^2(T^n)$ and $g \in L^2(T^m)$ are symmetric functions, then it holds that

$$I_n(f) I_m(g) = \sum_{l=0}^{m \wedge n} l! C_m^l C_n^l I_{m+n-2l}(f \otimes_l g). \tag{4.43}$$

Another useful property of finite sums of multiple integrals is the hypercontractivity. Namely, if $F = \sum_{k=0}^{n} I_k(f_k)$ with $f_k \in \mathcal{H}^{\otimes k}$ then

$$\mathbf{E}|F|^p \leq C_p \left(\mathbf{E} F^2 \right)^{\frac{p}{2}}. \tag{4.44}$$

for every $p \geq 2$.

Deuxième partie

Fluctuations gaussienne pour l'équation de la chaleur fractionnaire

Chapitre 5

Quantitative normal approximations for the stochastic fractional heat equation.

This article is published on Quantitative normal approximations for the stochastic fractional heat equation. Stoch PDE : Anal Comp (2021). https ://doi.org/10.1007/s40072-021-00198-7.

Joint work with David Nualart, Ciprian A. Tudor and Lauri Viitasaari

Abstract In this article we present a quantitative central limit theorem for the stochastic fractional heat equation driven by a a general Gaussian multiplicative noise, including the cases of space–time white noise and the white-colored noise with spatial covariance given by the Riesz kernel or a bounded integrable function. We show that the spatial average over a ball of radius R converges, as R tends to infinity, after suitable renormalization, towards a Gaussian limit in the total variation distance. We also provide a functional central limit theorem. As such, we extend recently proved similar results for stochastic heat equation to the case of the fractional Laplacian and to the case of general noise.

5.1 Introduction

In this article we consider the stochastic fractional heat equation

$$\frac{\partial u}{\partial t}(t,x) = -(-\Delta)^{\frac{\alpha}{2}} u(t,x) + \sigma(u(t,x))\dot{W}(t,x), \qquad t \geq 0, x \in \mathbb{R}^d \tag{5.1}$$

with initial condition $u(0,x) \equiv 1$. Here σ is assumed to be a Lipschitz continuous function with the property $\sigma(1) \neq 0$ and $-(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplace operator.

Fractional Laplace operator can be viewed as a generalization of spatial derivatives and classical Sobolev spaces into fractional order derivatives and fractional Sobolev spaces, and together with the associated equations it has numerous applications in different fields including fluid dynamics, quantum mechanics, and finance to simply name a few. For detailed discussions and different equivalent formal definitions, see [55] and the references therein.

In the present article we provide a general existence and uniqueness result to equation (5.1) that covers many different choices of the (Gaussian) random perturbation $\dot{W}$. We note that in this context, existence and uniqueness of the solution to (5.1) can be deduced from general results of [52] (although in [52] it was assumed that the spatial covariance of $\dot{W}$ has a spectral density). However, for the reader's convenience we present and prove the existence and uniqueness in our particular setting. Our main contribution is in providing quantitative limit theorems in a general context. These results cover three different important situations : when $\dot{W}$ is a standard space-time white noise, when $\dot{W}$ is a white-colored noise, i.e. a Gaussian field that behaves as a Wiener process in time and it has a non-trivial spatial covariance given by the Riesz kernel of order $\beta < \min(\alpha, d)$, and when $\dot{W}$ is a white-colored noise with spatial covariance given by an integrable and bounded function γ.

Our results continue the line of research initiated in [63] and [64] where a similar problem for the stochastic heat equation on $\mathbb{R}$ (or $\mathbb{R}^d$, respectively) driven by a space-time white noise (or spatial covariance given by the Riesz kernel, respectively) was considered. As such, we extend the results presented in [63] and [64] as the main theorems of [63] and [64] can be recovered from ours by simply plugging in $\alpha = 2$. Proof-wise our methods are similar to those of these two references. However, we stress that in our case we do not have fine properties of the heat kernel at our disposal, and hence one has to be more careful in the computations. In particular, our main contribution is the bound for the norm of the Malliavin derivative (cf. Proposition 4) that differs from the classical Laplacian case. Moreover, we provide a general approach how such bounds can be achieved, based on the boundedness properties of the convolution operator with the spatial covariance γ (see Proposition 2) together with the semigroup property and some integrability of the Green kernel.

On a related literature, we also mention [50] studying the case of stochastic wave equation on $\mathbb{R}^d$. In this article, the driving noise was assumed to be Gaussian multiplicative noise that is white in time and colored in space such that the correlation in the space variable is described by the Riesz kernel. As such, our results complements the above mentioned works studying the stochastic heat and wave equation.

The rest of the paper is organised as follows. In Section 5.2 we describe and discuss our main results. In particular, we provide the existence and uniqueness result for the solution, and provide quantitative central limit theorems for the spatial average in the mentioned particular cases. In Section 5.3 we recall some preliminaries, including some basic facts on Stein's method and Malliavin calculus that are used to prove our results, together with some basic facts on the Green kernel related to the fractional heat equation, and a key inequality proved in Proposition 2. Proofs of our main results are provided in

Section 5.4 and Section 5.5.

5.2 Main results

In this section we introduce and discuss our main results concerning equation (5.1). Throughout the article, we assume that $\dot{W}$ is a centered Gaussian noise with a covariance given by

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)\gamma(x-y), \tag{5.2}$$

where δ_0 denotes the Dirac delta function and γ is a nonnegative and nonnegative definite symmetric measure. The spectral measure $\widehat{\gamma}(d\xi)$ is defined through the Fourier transform of the measure γ :

$$\widehat{\gamma}(d\xi) = (\mathcal{F}\gamma)\,(d\xi) := \int_{\mathbb{R}^d} e^{-i\langle \xi, y\rangle} d\gamma(y).$$

The existence of the solution to (5.1) is guaranteed if a fractional version (5.5) of Dalang's condition is satisfied. In particular, this is the case on all examples mentioned in the introduction.

We next introduce the Green kernel (or fundamental solution) associated to the operator $-(-\Delta)^{\frac{\alpha}{2}}$, where $\alpha \in (0,2]$. This kernel, denoted in the sequel by G_α, is defined via its Fourier transform

$$(\mathcal{F}G_\alpha(t,\cdot))\,(\xi) = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d, t \geq 0 \tag{5.3}$$

for $\alpha > 0$ (here and in the sequel, $|\cdot|$ denotes the Euclidean norm). While explicit formulas for $G_\alpha(t,x)$ are known only in the special cases $\alpha = 1$ (the Poisson kernel) and $\alpha = 2$ (the heat kernel), the kernel $G_\alpha(t,x)$ admits many desirable properties. Some of them that are suitable for our purposes are recorded in Section 5.3.

Similarly to the classical stochastic heat equation case, the solution to the stochastic equation (5.1) can be expressed in terms of G_α. That is, the mild solution is a measurable random field $\Big(u(t,x), t \geq 0, x \in \mathbb{R}^d\Big)$ which satisfies

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_\alpha(t-s, x-y)\sigma(u(s,y))W(ds,dy), \tag{5.4}$$

where the stochastic integral is understood in the Walsh sense [106]. The following existence and uniqueness result holds. Taking into account that the claim follows as a special case of [52, Theorem 1.2] provided that $\hat{\gamma}(d\xi)$ is absolutely continuous, the result and condition (5.5) are not at all surprising.

Theorem 9 *Suppose that the Fourier transform $\widehat{\gamma} = \mathcal{F}\gamma$ satisfies the fractional Dalang's condition :*

$$\int_{\mathbb{R}^d} \frac{\widehat{\gamma}(d\xi)}{\beta + |\xi|^\alpha} < \infty, \tag{5.5}$$

for some (and hence for all) $\beta > 0$. Then equation (5.1) admits a unique mild solution given by (5.4)*. Moreover, for any $p \geq 1$ and any $T > 0$ we have*

$$\sup_{t\in(0,T],y\in\mathbb{R}^d} \mathbb{E}[|u(t,x)|^p] < \infty.$$

Remark 10 *We present our results only in the case of the initial condition $u(0,x) \equiv 1$ which makes our presentation and notation easier. We stress however, that with a little bit of extra effort our results could be extended to cover more general initial conditions. Actually, our existence result can be generalised to cover even the cases of initial conditions given by measures (satisfying certain suitable conditions). Indeed, then the mild solution is given by*

$$u(t,x) = \int_{\mathbb{R}^d} G_\alpha(t,x-y)u_0(dy) + \int_0^t \int_{\mathbb{R}^d} G_\alpha(t-s,x-y)\sigma(u(s,y))W(ds,dy),$$

where $u_0(dy)$ denotes the initial measure. In this case, one needs to require conditions on $u_0(dy)$ as well, in addition to (5.5)*. In particular, the integral $\int_{\mathbb{R}^d} G_\alpha(t,x-y)u_0(dy)$ above should exists. For a detailed exposure on the topic in the case of the stochastic heat equation ($\alpha = 2$), we refer to [36]. Similarly, in the spirit of [64, Corollary 3.3], our approximation results can be generalised to the case of $u(0,x) = f(x)$ with suitable assumptions on the function f, once a comparison principle is established.*

Throughout the article, for a function f and a (signed) measure μ we denote by $f * \mu$ the convolution defined by

$$(f * \mu)(y) = \int_{\mathbb{R}^d} f(y-x)d\mu(x), \tag{5.6}$$

provided it exists. If μ is absolutely continuous, then $d\mu(x) = \mu(x)dx$ for some function μ and we recover the classical convolution for integrable functions

$$(f * \mu)(y) = \int_{\mathbb{R}^d} f(y-x)\mu(x)dx.$$

If μ can be viewed as a function, the well-known Young convolution inequality states that for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ with $1 \leq p, q \leq r \leq \infty$, we have

$$\|f * \mu\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|\mu\|_{L^q(\mathbb{R}^d)}. \tag{5.7}$$

In particular, this gives us, for any $p \geq 1$,

$$\|f * \mu\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|\mu\|_{L^1(\mathbb{R}^d)}. \tag{5.8}$$

More generally, if μ is a finite measure, a simple mollification argument shows that (5.8) remains valid with $\|\mu\|_{L^1(\mathbb{R}^d)}$ replaced by $\mu(\mathbb{R}^d)$ (see, e.g. [29, Proposition 3.9.9]). Finally, by $I_{d-\beta}$ we denote the Riesz potential defined by, for $0 < \beta < d$,

$$(I_{d-\beta}f)(x) = \int_{\mathbb{R}^d} f(y)|x-y|^{-\beta}dy = (K_{d-\beta} * f)(x),$$

where $K_{d-\beta}(y) = |y|^{-\beta}$. More generally, Riesz potential $I_{d-\beta}\mu$ with respect to a measure μ is defined through the convolution

$$(I_{d-\beta}\mu)(x) = (K_{d-\beta} * \mu)(x) = \int_{\mathbb{R}^d} |x-y|^{-\beta} d\mu(y).$$

In order to simplify our notation, we also define $I_{d-\beta}$ for $\beta = d$ simply as an identity operator.

We also provide approximation results for the spatial average over an Euclidean ball of radius R, denoted by B_R. For these purposes we require some more refined information on the covariance γ instead of the general condition (5.5).

Assumption 1 *We assume that γ is given by the Riesz potential $\gamma = I_{d-\beta}\mu$, where $0 < \beta \leq d$ and μ is a finite symmetric measure. Moreover, one of the following conditions holds :*

(i) $\beta < \alpha \wedge d$.

(ii) $\beta = d = 1$ *and* $\alpha > 1$.

(iii) $\beta = d \geq \alpha$ *and* $\mu = \gamma$ *is absolutely continuous, i.e.* $d\gamma(x) = \gamma(x)dx$, *with* $\gamma \in L^r(\mathbb{R}^d)$ *for some* $r > \frac{d}{\alpha}$. *In addition, if* $r > 2$, *we impose Dalang's condition* (5.5).

Remark 11 *Condition $\beta < \alpha$ in Case (i) implies that Dalang's condition* (5.5) *is satisfied. Indeed, we recall that a Fourier transform $\widehat{\mu}$ of a finite measure μ is a bounded continuous function. Consequently, by recalling the convolution theorem $\widehat{f * \mu} = \widehat{f}\widehat{\mu}$ and the fact that the Riesz potential $I_{d-\beta}$ is a Fourier multiplier, we obtain*

$$\widehat{\gamma}(d\xi) = c_{d,\beta}|\xi|^{\beta-d}\widehat{\mu}(\xi)d\xi, \tag{5.9}$$

from which we deduce (5.5). *Dalang's condition* (5.5) *clearly holds in Case (ii). Finally, in Case (iii) we can deduce* (5.5) *from the Hausdorff-Young inequality if $r \leq 2$.*

Remark 12 *By carefully examining our proof one can see that our results remains valid provided that $\gamma = I_{d-\beta}\mu$ satisfies Dalang's condition and the statement in Proposition 2 holds for suitable number* $2q$.

Case (ii) covers the case of the space-time white noise, where γ is given by the Dirac delta $\gamma(y) = \delta_0(y)$. The case $\gamma(y) = |y|^{-\beta}$ corresponds to the noise with spatial correlation given by the Riesz kernel, studied in the heat equation case $\alpha = 2$ in [64]. In our terminology, this is included in Case (i) where $\gamma = I_{d-\beta}\delta_0$.

Recall that the total variation distance between random variables (or associated probability distributions) is given by

$$d_{\mathrm{TV}}(F, Z) := \sup\Big\{P(F \in A) - P(Z \in A) \,:\, A \subset \mathbb{R} \text{ Borel sets}\Big\}. \tag{5.10}$$

Our first main results concern the following two quantitative central limit theorems for the spatial average of the solution.

Theorem 10 *Let γ satisfy Assumption 1 and let u be the solution to the stochastic fractional heat equation (5.1). Then for every $t > 0$ there exists a constant C, depending solely on t, α, σ, and the covariance γ, such that*

$$d_{\mathrm{TV}}\left(\frac{1}{\sigma_R}\int_{B_R}[u(t,x)-1]\,dx,\ Z\right) \leq CR^{-\frac{\beta}{2}},$$

where $Z \sim N(0,1)$ is a standard normal random variable, and $\sigma_R^2 = \mathrm{Var}\left(\int_{B_R}[u(t,x)-1]\,dx\right) \sim R^{2d-\beta}$, as $R \to \infty$.

Remark 13 *While we have stated our result concerning only a ball B_R centered at the origin, we stress that with exactly the same arguments, one can replace B_R with some other body $RA_0 = \{Ra : a \in A_0\}$. This affects only the normalization constants. Moreover, as in the heat case (cf. [64, Remark 3]), one can allow the center of the ball a_R to vary in R as well. This fact follows easily from the stationarity.*

Following the spirit of the mentioned references, we also provide functional version of Theorem 10.

Theorem 11 *Let γ satisfy Assumption 1 and let u be the solution to the stochastic fractional heat equation (5.1). Then*

$$\left\{R^{\frac{\beta}{2}-d}\int_{B_R}[u(t,x)-1]\,dx\right\}_{t\in[0,T]} \Rightarrow \left\{\int_0^t \varrho(s)dY_s\right\}_{t\in[0,T]},$$

as $R \to \infty$, where Y is a standard Brownian motion, the weak convergence takes place on the space of continuous functions $C([0,T])$, and $\varrho(s)$ is given by;

- *If $\beta < d$, then $\varrho(s) = \sqrt{\mu\left(\mathbb{R}^d\right)\int_{B_1^2}|x-x'|^{-\beta}dxdx'}\,\mathbb{E}[\sigma(u(s,y))]$.*
- *If $\beta = d$, then $\varrho(s) = \sqrt{|B_1|\int_{\mathbb{R}^d}\mathbb{E}\left[\sigma(u(s,0))\sigma(u(s,z))\right]d\mu(z)}$.*

Note that ϱ depends on α through the the solution $u(s,y)$, see Remark 15.

Remark 14 *We prove later (see Lemma 10) that*

$$\int_{\mathbb{R}^d}\mathbb{E}\left[\sigma(u(s,0))\sigma(u(s,z))\right]d\mu(z) \geq \left[\mathbb{E}[\sigma(u(s,y))]\right]^2.$$

Under our initial condition $u(0,x) \equiv 1$, we may hence apply the arguments of [50, Lemma 3.4] to see the equivalence

$$\sigma(1) = 0 \Leftrightarrow \sigma_R = 0, \forall R > 0 \Leftrightarrow \sigma_R = 0 \text{ for some } R > 0 \Leftrightarrow \lim_{R\to\infty}\sigma_R^2 R^{\beta-2d} = 0.$$

Hence $\sigma(1) \neq 0$ is a natural condition that guarantees $\sigma_R > 0$ for all $R > 0$. Note also that $\sigma(1) \neq 0$ is necessary to exclude the trivial solution $u(t,x) \equiv 1$ by using the Picard iteration.

Example 1

Suppose $\mu = \delta_0$ and let $\beta = d = 1$ and $\alpha > 1$. This case corresponds to the space-time white noise, and now

$$\varrho(s) = \sqrt{|B_1| \int_{\mathbb{R}^d} \mathbb{E}\left[\sigma(u(s,0))\sigma(u(s,z))\right] d\mu(z)} = \sqrt{2\mathbb{E}\sigma^2(u(s,0))}.$$

In the case $\alpha = 2$, we thus recover the results of [63].

Example 2 *Suppose $\beta < d$ and let $\mu = \delta_0$. This case corresponds to the white-colored case with the spatial covariance given by the Riesz kernel. Now*

$$\varrho(s) = \sqrt{\int_{B_1^2} |x - x'|^{-\beta} dx dx'} \mathbb{E}[\sigma(u(s,y))]$$

and consequently, for $\alpha = 2$ we obtain the results of [64].

Remark 15 *We emphasis that the additional parameters associated to the fractional operator (i.e. α) does not affect the above results, except for the constant quantities through the solution u. Indeed, the renormalization rate and the total variation distance are, up to multiplicative constants, the same as in the case $\alpha = 2$ corresponding to the classical stochastic heat equation. Similarly, the limiting normal distribution in Theorem 10 and the limiting time-changed Brownian motion in Theorem 11 are the same as in the case $\alpha = 2$. Since G_α is intimately connected to a stable Lévy process, this might appear surprising as one might expect stable limiting laws. However, the Gaussian form of the limiting distribution is connected to the Gaussian nature of the noise $\dot{W}$, while the Green kernel $G_\alpha(t,x)$ (associated to a stable process) is simply a deterministic function that has suitable scaling in the time variable t and sufficient integrability in the spatial variable x. In contrast, one could expect stable limiting law, even in the case $\alpha = 2$ when G_α is the (Gaussian) heat kernel, if the noise $\dot{W}$ is driven by a suitable Lévy process.*

5.3 Preliminaries

In this section we present some preliminaries that are required for the proofs of our main theorems. In particular, we recall some facts on Malliavin calculus and Stein's method together with some basic properties of the fractional Green kernel. Finally, in Proposition 2 we present a basic inequality that allows us to derive a bound for the Malliavin derivative.

5.3.1 Malliavin calculus and Stein's method

We start by introducing the Gaussian noise that governs the stochastic fractional heat equation (5.1).

Denote by $C_c^\infty\left([0,\infty)\times\mathbb{R}^d\right)$ the class of C^∞ functions on $[0,\infty)\times\mathbb{R}^d$ with compact support. We consider a Gaussian family of centered random variables

$$\left(W(\varphi), \varphi \in C_c^\infty\left([0,\infty)\times\mathbb{R}^d\right)\right)$$

on some complete probability space $(\Omega, \mathcal{F}, P)$ such that

$$\mathbb{E}[W(\varphi)W(\psi)] = \int_0^\infty \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(s,y)\psi(s,y')\gamma(y-y')dydy'ds := \langle \varphi, \psi\rangle_{\mathfrak{H}}. \tag{5.11}$$

We stress again that, in general, γ is not a function, and hence (5.11) should be understood as

$$\mathbb{E}[W(\varphi)W(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} \varphi(s,y)\left[\psi(s,\cdot)*\gamma\right](y)dyds. \tag{5.12}$$

We denote by $\mathfrak{H}$ the Hilbert space defined as the closure of $C_c^\infty\left([0,\infty)\times\mathbb{R}^d\right)$ with respect to the inner product (5.11). By density, we obtain an isonormal process $(W(\varphi), \varphi \in \mathfrak{H})$, which consists of a Gaussian family of centered random variable such that, for every $\varphi, \psi \in \mathfrak{H}$,

$$\mathbb{E}[W(\varphi)W(\Psi)] = \langle \varphi, \psi\rangle_{\mathfrak{H}}.$$

The Gaussian family $(W(\varphi), \varphi \in \mathfrak{H})$ is usually called a white-colored noise because it behaves as a Wiener process with respect to the time variable $t \in [0,\infty)$ and it has a spatial covariance given by the measure γ.

Let us introduce the filtration associated to the random noise W. For $t>0$, we denote by $\mathcal{F}_t$ the sigma-algebra generated by the random variables $W(\varphi)$, with $\varphi \in \mathfrak{H}$ having its support included in the set $[0,t]\times\mathbb{R}^d$. For every random field $(X(s,y), s\geq 0, y\in\mathbb{R}^d)$, jointly measurable and adapted with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, satisfying

$$\mathbb{E}\left[\|X\|_{\mathfrak{H}}^2\right] < \infty,$$

we can define stochastic integrals with respect to W of the form

$$\int_0^\infty \int_{\mathbb{R}^d} X(s,y)W(ds,dy)$$

in the sense of Dalang-Walsh (see [44] and [106]). This integral satisfies the Itô-type isometry

$$\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{R}^d} X(s,y)W(ds,dy)\right)^2\right] = \mathbb{E}\left[\|X\|_{\mathfrak{H}}^2\right]. \tag{5.13}$$

The Dalang-Walsh integral also satisfies the following version of the Burkholder-Davis-Gundy inequality : for any $t\geq 0$ and $p\geq 2$,

$$\left\|\int_0^\infty \int_{\mathbb{R}^d} X(s,y)W(ds,dy)\right\|_p^2$$

$$\leq c_p \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|X(s,y)X(s,y')\|_{\frac{p}{2}} \gamma(y-y') dy dy' ds. \tag{5.14}$$

Let us next describe the basic tools from Malliavin calculus needed in this work. We introduce $C_p^\infty(\mathbb{R}^n)$ as the space of smooth functions with all their partial derivatives having at most polynomial growth at infinity, and $\mathcal{S}$ as the space of simple random variables of the form

$$F = f(W(h_1), \dots, W(h_n)),$$

where $f \in C_p^\infty(\mathbb{R}^n)$ and $h_i \in \mathfrak{H}$, $1 \leq i \leq n$. Then the Malliavin derivative DF is defined as $\mathfrak{H}$-valued random variable

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n)) h_i\,. \tag{5.15}$$

For any $p \geq 1$, the operator D is closable as an operator from $L^p(\Omega)$ into $L^p(\Omega; \mathfrak{H})$. Then $\mathbb{D}^{1,p}$ is defined as the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = \left(\mathbb{E}[|F|^p] + \mathbb{E}(\|DF\|_{\mathfrak{H}}^p)\right)^{1/p}\,.$$

The adjoint operator δ of the derivative is defined through the duality formula

$$\mathbb{E}(\delta(u)F) = \mathbb{E}(\langle u, DF\rangle_{\mathfrak{H}}), \tag{5.16}$$

valid for any $F \in \mathbb{D}^{1,2}$ and any $u \in \operatorname{Dom}\delta \subset L^2(\Omega; \mathfrak{H})$. The operator δ is also called the Skorokhod integral since, in the case of the standard Brownian motion, it coincides with an extension of the Itô integral introduced by Skorokhod (see, e.g. [59,82]). In our context, any adapted random field X which is jointly measurable and satisfies (5.13) belongs to the domain of δ, and $\delta(X)$ coincides with the Walsh integral :

$$\delta(X) = \int_0^\infty \int_{\mathbb{R}} X(s,y) W(ds,dy).$$

This allows us to represent the solution $u(t,x)$ to (5.1) as a Skorokhod integral.

The proofs of our main results are based on Malliavin-Stein approach, introduced by Nourdin and Peccati in [80] (see also the book [81]). In particular, we apply the following result to obtain rate of convergence in the total variation distance (see [105] and also [63,78]).

Proposition 1 *If F is a centered random variable in the Sobolev space $\mathbb{D}^{1,2}$ with unit variance such that $F = \delta(v)$ for some $\mathfrak{H}$-valued random variable v belonging to the domain of δ, then, with $Z \sim N(0,1)$,*

$$d_{\mathrm{TV}}(F,Z) \leq 2\sqrt{\operatorname{Var}\left(\langle DF, v\rangle_{\mathfrak{H}}\right)}. \tag{5.17}$$

5.3.2 On fractional Green kernel

We recall some useful properties of the kernel G_α defined through (5.3). For details, we refer to [33, 35, 46].

1. For every $t > 0$, $G_\alpha(t, \cdot)$ is the density of a d-dimensional Lévy stable process at time t. In particular, we have
$$\int_{\mathbb{R}^d} G_\alpha(t,x)dx = 1. \tag{5.18}$$

2. For every t, the kernel $G_\alpha(t,x)$ is real valued, positive, and symmetric in x.

3. The operator G_α satisfies the semigroup property, i.e.
$$G_\alpha(t+s,x) = \int_{\mathbb{R}^d} G_\alpha(t,z)G_\alpha(s,x-z)dz \tag{5.19}$$
for $0 < s < t$ and $x \in \mathbb{R}^d$.

4. G_α is infinitely differentiable with respect to x, with all the derivatives bounded and converging to zero as $|x| \to \infty$. Moreover, we have the scaling property
$$G_\alpha(t,x) = t^{-\frac{d}{\alpha}} G_\alpha(1, t^{-\frac{1}{\alpha}} x). \tag{5.20}$$

5. There exist two constants $0 < K'_\alpha < K_\alpha$ such that
$$K'_\alpha \frac{1}{(1+|x|)^{d+\alpha}} \le |G_\alpha(1,x)| \le K_\alpha \frac{1}{(1+|x|)^{d+\alpha}} \tag{5.21}$$
for all $x \in \mathbb{R}^d$. Together with the scaling property, this further translates into
$$K'_\alpha \frac{t^{-\frac{d}{\alpha}}}{\left(1+|t^{-\frac{1}{\alpha}}x|\right)^{d+\alpha}} \le |G_\alpha(t,x)| \le K_\alpha \frac{t^{-\frac{d}{\alpha}}}{\left(1+|t^{-\frac{1}{\alpha}}x|\right)^{d+\alpha}}. \tag{5.22}$$

5.3.3 A basic inequality

The following proposition contains an inequality that plays a fundamental role in the proof of the estimates of the p-norm of the Malliavin derivative.

Proposition 2 *Suppose that the covariance γ satisfies Assumption 1. Then, there exists a number $2q \in \left(1, \frac{2d}{2d-\alpha} \wedge \frac{d+\alpha}{d}\right)$ such that for any functions $f, g \in L^{2q}(\mathbb{R}^d)$ we have*
$$\int_{\mathbb{R}^d} f(y)\,[g * \gamma]\,(y)dy \le C\|f\|_{L^{2q}(\mathbb{R}^d)}\|g\|_{L^{2q}(\mathbb{R}^d)}. \tag{5.23}$$

Remark 16 *The requirement $2q < \frac{2d}{2d-\alpha}$ ensures that*

$$\kappa = \frac{2d}{\alpha}\left(1 - \frac{1}{2q}\right) < 1, \tag{5.24}$$

while the requirement $2q < \frac{d+\alpha}{d}$ ensures that $G^{\frac{1}{2q}}(1,x)$ is integrable. Note also that $\frac{d+\alpha}{d} \le \frac{2d}{2d-\alpha}$ only if $d \le \alpha$. Since $\alpha \le 2$, this can happen only in the one-dimensional case $d = 1$ or in the heat case $\alpha = 2$ and $d = 1, 2$. In the latter however, $G^{\frac{1}{2q}}(1,x)$ is integrable regardless of the value $2q$ and consequently, our results can be applied in that case as well under a condition $2q \in \left(1, \frac{2d}{2d-2}\right)$.

Proof: [Proof of Proposition 2] We decompose the proof into the three possible cases from Assumption 1 :

Case (i) : Taking $2q = \frac{2d}{2d-\beta}$ (recall $\beta < \alpha \wedge d$) and using Hölder's inequality, we obtain

$$\int_{\mathbb{R}^{2d}} f(x)[g * \gamma](x)dx \le \|f\|_{L^{2q}(\mathbb{R}^d)} \|g * \gamma\|_{L^{2q/(2q-1)}(\mathbb{R}^d)}.$$

Notice that $g * \gamma = (I_{d-\beta} g) * \mu$. Therefore, it follows from (5.8) that

$$\|g * \gamma\|_{L^{2q/(2q-1)}(\mathbb{R}^d)} \le \mu(\mathbb{R}^d) \|I_{d-\beta} g\|_{L^{2q/(2q-1)}(\mathbb{R}^d)}.$$

We then conclude the proof using the fact that $2q = \frac{2d}{2d-\beta}$ and applying the following Hardy-Littlewood-Sobolev inequality (see e.g. [72] and references therein) : for $1 < p < r < \infty$ satisfying $\frac{1}{r} = \frac{1}{p} - \frac{d-\beta}{d}$, we have

$$\|I_{d-\beta} g\|_{L^r(\mathbb{R}^d)} \le C \|g\|_{L^p(\mathbb{R}^d)}. \tag{5.25}$$

Case (ii) : Suppose $\beta = d$. By Young's inequality (5.8) and Hölder's inequality, we get

$$\int_{\mathbb{R}^d} f(y)\, [g * \gamma]\, (y) dy \le \|f\|_{L^2(\mathbb{R}^d)} \|g * \gamma\|_{L^2(\mathbb{R}^d)} \le C \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}.$$

Consequently, one can always choose $q = 1$ in (5.23). However, then $2q < \frac{2d}{2d-\alpha} \wedge \frac{d+\alpha}{d}$ only if $\alpha > d$. Taking into account the fact $\alpha \in (0, 2]$, this forces $d = 1$ and $\alpha > 1$. In conclusion, in the one-dimensional case and for $\alpha > 1$ we obtain the estimate (5.23) with $q = 1$, which completes the proof of Case (ii).

Case (iii) : Let $\beta = d \ge \alpha$ and suppose that γ is absolutely continuous with a density $\gamma \in L^r(\mathbb{R}^d)$, where $r > \frac{d}{\alpha}$. In this case we choose $2q = \frac{2r}{2r-1}$. Clearly $2q > 1$. Moreover, condition $r > \frac{d}{\alpha}$ implies $2q < \frac{2d}{2d-\alpha}$ and $\frac{2d}{2d-\alpha} \le \frac{d+\alpha}{d}$ because $d \ge \alpha$. Finally, Hölder's inequality and Young's inequality (5.7) gives us

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) g(y) \gamma(x-y) dx dy \le \|f\|_{L^{2q}(\mathbb{R}^d)} \|g\|_{L^{2q}(\mathbb{R}^d)} \|\gamma\|_{L^r(\mathbb{R}^d)}.$$

5.4 Proof of Theorem 9

For $t \geq 0$, we denote

$$I(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_\alpha(t,y) G_\alpha(t,y') \gamma(y-y') dy' dy. \tag{5.26}$$

Taking the Fourier transform and using (5.3), we see that $I(t)$ can equally be given by

$$I(t) = \int_{\mathbb{R}^d} e^{-2t|\xi|^\alpha} \widehat{\gamma}(d\xi). \tag{5.27}$$

Suppose now that γ satisfies (5.5). For $\beta > 0$, we define a function $\Upsilon(\beta)$ by

$$\Upsilon(\beta) := \int_0^\infty e^{-\beta t} I(t) dt = \int_{\mathbb{R}^d} \frac{\widehat{\gamma}(d\xi)}{\beta + 2|\xi|^\alpha}.$$

Clearly, Υ is non-negative, decreasing in β, and $\lim_{\beta\to\infty} \Upsilon(\beta) = 0$.

Before proving Theorem 9 we introduce the following technical lemma that can be viewed as a fractional version of [36, Lemma 2.5].

Lemma 7 *Let $I(t)$ be given by (5.26) and, for given $\iota > 0$, let h_n be defined recursively by $h_0(t) = 1$, and for $n \geq 1$*

$$h_n(t) = \iota \int_0^t h_{n-1}(s) I(t-s) ds.$$

Then for any $p \geq 1$ and any fixed $T < \infty$, the series

$$H(\iota, p, t) := \sum_{n\geq 0} [h_n(t)]^{\frac{1}{p}} \tag{5.28}$$

converges uniformly in $t \in [0, T]$.

Proof: By the same argument as in the proof of [36, Lemma 2.5], we get, for any $\beta > 0$, that

$$\int_0^\infty e^{-\beta t} h_n(t) dt = \frac{1}{\beta} \left(\iota \int_0^\infty e^{-\beta t} I(t) dt \right)^n = \frac{1}{\beta} \left[\iota \Upsilon(\beta)\right]^n.$$

By choosing β large enough, we have $\iota\Upsilon(\beta) \leq 1/2$ and, as in [36], by choosing the smallest such β this gives us $H(\iota, 1, t) \leq \exp(Ct)$ for some constant C depending on Υ and ι. Similarly, for the general case $p > 1$ we may apply Hölder inequality to get

$$\int_0^\infty e^{-\beta t} h_n^{1/p}(t) dt \leq \beta^{-\frac{1}{q}} \left(\int_0^\infty e^{-\beta t} h_n(t) dt \right)^{\frac{1}{p}} = \frac{1}{\beta} \left[\iota \Upsilon(\beta)\right]^{\frac{n}{p}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Hence, similar arguments show that $H(\iota, p, t) \leq \exp(Ct)$ and, in particular, that the series in (5.28) converges. Equipped with Lemma 7, we are now able to prove Theorem 9.

Proof: [Proof of Theorem 9] Define the standard Picard iterations by setting $u_0(t,x) = 1$ and, for $n \geq 1$,

$$u_{n+1}(t,x) = u_0(t,x) + \int_0^t \int_{\mathbb{R}^d} G_\alpha(t-s, x-y)\sigma(u_n(s,y))W(ds,dy), \qquad t \geq 0, x \in \mathbb{R}^d.$$

By induction, we can easily show that for every $n \geq 0$, $u_n(t,x)$ is well-defined and, for every $p \geq 2$ and $\beta > 0$, we have

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[e^{-p\beta t}\,|u_n(t,x)|^p\right] < \infty. \tag{5.29}$$

This in turn shows that

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[|u_n(t,x)|^p\right] < \infty. \tag{5.30}$$

To see (5.29), we first observe that it is clearly true for $n = 0$. Suppose now that it holds for some n. We have

$$e^{-\beta t}u_{n+1}(t,x) = e^{-\beta t}u_0(t,x) + \int_0^t e^{-\beta t} \int_{\mathbb{R}^d} G_\alpha(t-s, x-y)\sigma(u_n(s,y))W(ds,dy)$$

and, for every $p \geq 2$, by using (5.13) and (5.14),

$$\begin{aligned}
\mathbb{E}\left[e^{-p\beta t}\,|u_{n+1}(t,x)|^p\right] &\leq C\Bigg(1 + \Big\| \int_0^t e^{-\beta t} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_\alpha(t-s,x-y)G_\alpha(t-s,x-y') \\
&\qquad \times\sigma(u_n(s,y))\sigma(u_n(s,y'))\gamma(y-y')dy'dy\Big\|_{\frac{p}{2}}^{\frac{p}{2}}\Bigg) \\
&\leq C\Bigg[1 + \Bigg(\int_0^t e^{-\beta(t-s)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_\alpha(t-s,x-y)G_\alpha(t-s,x-y') \\
&\qquad \times e^{-\beta s}\left|\left|\sigma(u_n(s,y))\sigma(u_n(s,y'))\right|\right|_{\frac{p}{2}}\gamma(y-y')dy'dy\Bigg)\Bigg]^{\frac{p}{2}}.
\end{aligned}$$

By using the Lipschitz assumption on σ and the induction hypothesis we get

$$\mathbb{E}\left[e^{-p\beta s}|\sigma(u_n(s,y))|^p\right] \leq C\left(1 + \sup_{s\in[0,T], y\in\mathbb{R}^d} \mathbb{E}\left[e^{-p\beta s}|u_n(s,y)|^p\right]\right) < \infty.$$

Hence

$$\sup_{s\in[0,T]} \sup_{y,y'\in\mathbb{R}^d} e^{-\beta s}\left|\left|\sigma(u_n(s,y))\sigma(u_n(s,y'))\right|\right|_{\frac{p}{2}} < \infty$$

and we obtain

$$\begin{aligned}
\mathbb{E}\left[e^{-p\beta t}\,|u_{n+1}(t,x)|^p\right] &\leq C\left(1 + \textstyle\int_0^t e^{-\beta(t-s)}I(t-s)ds\right)^{\frac{p}{2}} \\
&\leq C\left(1 + \textstyle\int_0^T e^{-\beta s}I(s)ds\right)^{\frac{p}{2}} < \infty.
\end{aligned}$$

Applying similar arguments together with Hölder's inequality for

$$H_n(t) := \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[|u_{n+1}(t,x) - u_n(t,x)|^p\right]$$

gives us

$$\begin{aligned} H_n(t) \quad &\leq C\Big[\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_\alpha(t-s,x-y)G_\alpha(t-s,x-y') \left|\left|\sigma(u_n(s,y)) - \sigma(u_{n-1}(s,y))\right|\right|_p \\ &\qquad \times \left|\left|\sigma(u_n(s,y')) - \sigma(u_{n-1}(s,y'))\right|\right|_p \gamma(y-y')dy'dyds\Big]^{\frac{p}{2}} \\ &\leq C\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_\alpha(t-s,x-y)G_\alpha(t-s,x-y')H_{n-1}(s)\gamma(y-y')dy'dyds \\ &\leq C\int_0^t I(t-s)H_{n-1}(s)ds. \end{aligned}$$

By standard arguments, it suffices to consider the case of an equality. In this case, it follows from Lemma 7 that $\sum_{n\geq 1} H_n(t)^{\frac{1}{p}}$ converges uniformly on $[0,T]$. Consequently, the sequence u_n converges in $L^p(\Omega)$, uniformly on $[0,T]\times\mathbb{R}^d$, and its limit satisfies (5.4). The uniqueness follows in a similar way, and the stationarity of the solution with respect to the space variable is a consequence of the proof of Lemma 18 in [44].

5.5 Proofs of Theorem 10 and Theorem 11

In this section we prove Theorems 10 and 11. The key ingredient for the proofs is to bound the Malliavin derivative of the solution to (5.1) by a quantity involving the Green kernel associated to the fractional operator (5.3). Once a suitable bound is established, it suffices to study the asymptotic variance and follow the ideas presented in [63,64]. We divide this section into four subsections. In the first one we study the (bound for the) Malliavin derivative of the solution, and in the second we study the correct normalization rate. The last two subsections are devoted to the proofs of Theorem 10 and Theorem 11.

5.5.1 Bound for the Malliavin derivative

We begin by providing a linear equation for the Malliavin derivative of the solution. The claim follows from (5.4), and the proof is rather standard. For this reason we omit the details.

Proposition 3 *Let u be the mild solution to (5.1). Then for every $t\in(0,T]$, $p\geq 2$ and $x\in\mathbb{R}^d$, the random variable $u(t,x)$ belongs to the Sobolev space $\mathbb{D}^{1,p}$ and its Malliavin derivative satisfies*

$$\begin{aligned} &D_{r,z}u(t,x) = G_\alpha(t-r,x-z)\sigma(u(r,z)) \\ &+\int_r^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)\Sigma(s,y)D_{r,z}u(s,y)W(ds,dy), \end{aligned} \tag{5.31}$$

where $\Sigma(r,z)$ is an adapted and bounded (uniformly with respect to r and z) stochastic process that coincides with $\sigma'(u(r,z))$ whenever σ is differentiable.

The following result provides a bound for the p-norm of the Malliavin derivative of the solution.

Proposition 4 *Suppose that γ satisfies Assumption 1 and recall (see (5.24)) that $\kappa = \frac{2d}{\alpha}\left(1 - \frac{1}{2q}\right)$, where q is from Proposition 2. Then for every $0 < s < t < T$, for every $x, y \in \mathbb{R}^d$, and for every $p \geq 2$ we have*

$$\|D_{s,y}u(t,x)\|_p \leq c(t-s)^{-\frac{\kappa}{2}} G_\alpha^{\frac{1}{2q}}(t-s, x-y).$$

Proposition 4 is based on the following lemma which proof is postponed to the appendix.

Lemma 8 *Suppose that γ satisfies Assumption 1 and assume that $g : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is non-negative function satisfying, for every $t \in [0,T]$ and $x \in \mathbb{R}^d$,*

$$\begin{aligned} g(t,x)^2 \quad &\leq \quad G_\alpha(t,x)^2 + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_\alpha(t-s, x-y) G_\alpha(t-s, x-y') \\ &\qquad \times g(s,y) g(s,y') \gamma(y-y') dy' dy ds. \end{aligned} \tag{5.32}$$

Then

$$g(t,x) \leq ct^{-\frac{\kappa}{2}} G_\alpha^{\frac{1}{2q}}(t,x), \tag{5.33}$$

where $\kappa = \frac{2d}{\alpha}\left(1 - \frac{1}{2q}\right)$ and q is from Proposition 2.

Remark 17 *By carefully examining the proof of Lemma 8, one observes that the statement remains valid as long as, for $2q$ determined through Proposition 2 and depending solely on the covariance γ, we have*

$$G_\alpha(t,x) \leq Ct^{-\kappa q} G_\alpha(t,x)$$

for some constant C and parameter $\kappa < 1$, and G_α satisfies the semigroup property (5.19). This encodes the required connection between the density G_α of the associated stable process and the covariance γ. Indeed, the above requirements means that improved integrability induced by the convolution with γ is sufficient to compensate low integrability (or scaling in the time variable t) of the kernel G_α. As such, we could consider more general Green kernels G in place of G_α in Lemma 8. For example, G can be taken to be a density of more general Lévy process. In this case, we obtain Proposition 4 provided that G satisfies the above condition.

Proof: [Proof of Proposition 4] In a standard way we can show that, for every $t \in (0,T]$ and $x \in \mathbb{R}$, the random variable $u(t,x)$ belongs to the Sobolev space $\mathbb{D}^{1,p}$ for all $p \geq 2$ and its Malliavin derivative satisfies (5.31). Moreover, using the Burkholder-Davis-Gundy inequality (5.14) we obtain that, for any $p \geq 2$,

$$\|D_{r,z}u(t,x)\|_p^2 \quad \leq \quad C_p G_\alpha(t-r, x-z)^2$$

$$+ \quad C_p \int_r^t \int_{\mathbb{R}} \int_{\mathbb{R}} G_\alpha(t-s, x-y) G_\alpha(t-s, x-y') \\ \times \|D_{r,z}u(s,y)\|_p \|D_{r,z}u(s,y)\|_p \gamma(y-y') dy' dy ds.$$

To conclude the proof, it suffices to apply Lemma 8 with $\theta = t - r, \eta = x - z$, and

$$g(\theta, \eta) = \|D_{r,z}u(\theta + r, \eta + z)\|_p.$$

For later use we also record the following simple technical fact.

Lemma 9 *Suppose* $2q \in \left(1, \frac{2d}{2d-\alpha} \wedge \frac{d+\alpha}{d}\right)$. *Then*

$$\int_{\mathbb{R}^d} G_\alpha^{\frac{1}{2q}}(r-s, \eta) d\eta = C(r-s)^{\frac{\kappa}{2}},$$

where κ *is defined in (5.24).*

Proof: By the scaling property (5.20) we get

$$\int_{\mathbb{R}^d} G_\alpha^{\frac{1}{2q}}(r-s, \eta) d\eta = (r-s)^{\frac{\kappa}{2}} \int_{\mathbb{R}^d} G_\alpha^{\frac{1}{2q}}(1, \eta) d\eta$$

where, by (5.21),

$$\int_{\mathbb{R}^d} G_\alpha^{\frac{1}{2q}}(1, \eta) d\eta \le C \int_{\mathbb{R}^d} (1 + |\eta|)^{-\frac{d+\alpha}{2q}} d\eta < \infty$$

since $\frac{d+\alpha}{2q} > d$.

5.5.2 Asymptotic behavior of the covariance

Let us use the following notation. For fixed $t > 0$, we define

$$G_R(t) := \int_{B_R} [u(t,x) - 1] \, dx \text{ and } \varphi_R(t,y) := \int_{B_R} G_\alpha(t, x-y) dx. \tag{5.34}$$

The constant k_β, for $\beta \le d$, is defined by

$$k_d = |B_1| \text{ and } k_\beta := \int_{B_1^2} |x - x'|^{-\beta} dx dx', \quad \beta < d. \tag{5.35}$$

Set

$$\Psi(s,z) = \mathbb{E}[\sigma(u(s,0))\sigma(u(s,z))] \tag{5.36}$$

and

$$\theta_\alpha(s) = \mathbb{E}[\sigma(u(s,y))]. \tag{5.37}$$

When $\beta = d$, we put

$$\nu_\alpha(s) = \left(\int_{\mathbb{R}^d} \Psi(s,z)d\mu(z)\right)^{\frac{1}{2}} \tag{5.38}$$

and following lemma justifies the fact that ν_α is well-defined. The proof is postponed to the end of this subsection.

Lemma 10 *Suppose that γ satisfies Assumption 1 with $\beta = d$ and let Ψ be given by* (5.36). *Then for every $s \geq 0$ we have*

$$\int_{\mathbb{R}^d} \Psi(s,z)d\mu(z) \geq 0.$$

In particular, ν_α given by (5.38) *is well-defined. Moreover, for every $s \in [0,T]$ we have*

$$\nu_\alpha^2(s) \geq \theta_\alpha^2(s).$$

The following theorem provides us the correct renormalization as well as the limiting covariance.

Theorem 12 *Suppose that γ satisfies Assumption 1. Then*

$$\lim_{R\to\infty} R^{\beta-2d}\mathbb{E}(G_R(t)G_R(r)) = k_\beta \int_0^{t\wedge r} \left[\mu(\mathbb{R}^d)\theta_\alpha^2(s)\mathbf{1}_{\beta<d} + \nu_\alpha^2(s)\mathbf{1}_{\beta=d}\right] ds.$$

Before we proceed to the proof of Theorem 12, we present a couple of technical lemmas.

Lemma 11 *Suppose that γ satisfies Assumption 1. Then for any bounded function $s \mapsto \theta(s)$ we have, as $R \to \infty$,*

$$R^{\beta-2d}\int_0^t \theta(s)\int_{\mathbb{R}^{2d}} \varphi_R(t-s,y)\varphi_R(t-s,y')\gamma(y-y')dy'dyds \to k_\beta\mu(\mathbb{R}^d)\int_0^t \theta(s)ds,$$

where k_β is defined in (5.35).

Proof: Recall that, writing formally by (5.12), we have

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} \varphi_R(t-s,y)\varphi_R(t-s,y')\gamma(y-y')dy'dy\\ =\ &\int_{\mathbb{R}^d} \varphi_R(t-s,y)\left[\varphi_R(t-s,\bullet) * I_{d-\beta} * \mu\right](y)dy.\end{aligned}$$

Since clearly $\varphi_R(t-s,\bullet) \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, it follows from Young's inequality (5.8) and Hardy-Littlewood-Sobolev's inequality (5.25) that $\varphi_R(t-s,\bullet) * I_{d-\beta} * \mu \in L^2(\mathbb{R}^d)$. Hence we obtain, by taking a Fourier transform and using Plancherel's theorem, that

$$\textstyle\int_{\mathbb{R}^d} \varphi_R(t-s,y)\left[\varphi_R(t-s,\bullet) * I_{d-\beta} * \mu\right](y)dy$$

$$= \tfrac{c_{d,\beta}}{(2\pi)^d} \int_{\mathbb{R}^d} |[\widehat{\varphi}_R(t-s,\bullet)](\xi)|^2 \, |\xi|^{\beta-d}\widehat{\mu}(\xi)d\xi,$$

where $c_{d,\beta} = 1$ for $\beta = d$. By recalling that

$$\left| \int_{B_R} e^{-i\langle x,\xi\rangle} dx \right|^2 = (2\pi R)^d |\xi|^{-d} J^2_{\frac{d}{2}}(R|\xi|),$$

where $J_{\frac{d}{2}}$ denotes the Bessel function of the first kind of order $d/2$, we obtain

$$|[\widehat{\varphi}_R(t-s,\bullet)](\xi)|^2 = (2\pi R)^d |\xi|^{-d} J^2_{\frac{d}{2}}(R|\xi|) e^{-2(t-s)|\xi|^\alpha}$$

leading to

$$\begin{aligned} & \frac{c_{d,\beta}}{(2\pi)^d} \int_{\mathbb{R}^d} |[\widehat{\varphi}_R(t-s,\bullet)](\xi)|^2 \, |\xi|^{\beta-d}\widehat{\mu}(\xi)d\xi \\ = \quad & c_{d,\beta} \int_{\mathbb{R}^d} R^d |\xi|^{-d} J^2_{\frac{d}{2}}(R|\xi|) e^{-2(t-s)|\xi|^\alpha} |\xi|^{\beta-d}\widehat{\mu}(\xi)d\xi \\ = \quad & c_{d,\beta} R^{2d-\beta} \int_{\mathbb{R}^d} |\xi|^{-d} J^2_{\frac{d}{2}}(|\xi|) e^{-2(t-s)R^{-\alpha}|\xi|^\alpha} |\xi|^{\beta-d}\widehat{\mu}\left(\frac{\xi}{R}\right) d\xi. \end{aligned}$$

Since $\widehat{\mu} \in L^\infty(\mathbb{R}^d)$, we have $\sup_{R>0} e^{-2(t-s)R^{-\alpha}|\xi|^\alpha}\widehat{\mu}\left(\frac{\xi}{R}\right) < \infty$. Moreover, since $J^2_{\frac{d}{2}}(|\xi|) = O(|\xi|^{-1})$ as $|\xi| \to \infty$ and $J^2_{\frac{d}{2}}(|\xi|) \sim c_d|\xi|^d$ as $|\xi| \to 0$ (here we have used standard Landau notation $O(|\xi|)$ and $f \sim g$ if $\frac{f}{g} \to 1$), we have $\int_{\mathbb{R}^d} |\xi|^{\beta-2d} J^2_{\frac{d}{2}}(|\xi|)d\xi < \infty$. This, together with the boundedness of $\theta(s)$, allows us to use the dominated convergence theorem and therefore, as $R \to \infty$,

$$\begin{aligned} & R^{\beta-2d} \int_0^t \theta(s) \int_{\mathbb{R}^{2d}} \varphi_R(t-s,y)\varphi_R(t-s,y')\gamma(y-y')dy'dyds \\ \to \quad & c_{d,\beta} \int_0^t \theta(s)ds \int_{\mathbb{R}^d} |\xi|^{\beta-2d} J^2_{\frac{d}{2}}(|\xi|)\widehat{\mu}(0)d\xi. \end{aligned}$$

The result now follows from $\widehat{\mu}(0) = \mu(\mathbb{R}^d)$ together with the fact that

$$c_{d,\beta} \int_{\mathbb{R}^d} |\xi|^{\beta-2d} J^2_{\frac{d}{2}}(|\xi|)d\xi = \int_{B_1^2} |x_1 - x_2|^{-\beta} dx_1 dx_2 \tag{5.39}$$

for $\beta < d$ and, for $\beta = d$, we have

$$\int_{\mathbb{R}^d} |\xi|^{-d} J^2_{\frac{d}{2}}(|\xi|)d\xi = |B_1|. \tag{5.40}$$

Indeed, the validity of (5.40) can be seen from

$$\int_{\mathbb{R}^d} \mathbf{1}_{B_1}(x)dx = \int_{\mathbb{R}^d} \mathbf{1}^2_{B_1}(x)dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \widehat{\mathbf{1}_{B_1}}(\xi) \right|^2 d\xi = \int_{\mathbb{R}^d} |\xi|^{-d} J^2_{\frac{d}{2}}(|\xi|)d\xi$$

while the validity of (5.39) can be seen from

$$\begin{aligned}\int_{B_1^2} |x_1 - x_2|^{-\beta} dx_1 dx_2 &= \int_{\mathbb{R}^d} \mathbf{1}_{B_1}(x_1) \left[I_{d-\beta} \mathbf{1}_{B_1} \right](x_1) dx_1 \\ &= \frac{c_{d,\beta}}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \widehat{\mathbf{1}_{B_1}}(\xi) \right|^2 |\xi|^{\beta - d} d\xi \\ &= c_{d,\beta} \int_{\mathbb{R}^d} |\xi|^{\beta - 2d} J^2_{\frac{d}{2}}(|\xi|) d\xi.\end{aligned}$$

This completes the proof.

Lemma 12 *Suppose that γ satisfies Assumption 1 and $\beta < d$. Then*

$$\lim_{|z|\to\infty} \sup_{0\le s\le t} \left| \Psi(s,z) - \theta_\alpha^2(s) \right| = 0.$$

Proof: As in the proof of Theorem 3.1 in [64], we can write, via the Clark-Ocone formula,

$$\Psi(s, y - y') - \theta_\alpha^2(s) = T(s, y, y'),$$

where

$$|T(s,y,y')| \le C \int_0^s \int_{\mathbb{R}^{2d}} \|D_{r,z} u(s,y)\|_2 \|D_{r,z'} u(s,y')\|_2 \gamma(z - z') dz' dz dr.$$

Hence, by applying Proposition 4, we obtain the estimate

$$\begin{aligned}|T(s,y,y')| \le\ & C \int_0^s (s-r)^{-\kappa} \int_{\mathbb{R}^{2d}} G_\alpha^{\frac{1}{2q}}(s-r, y-z) G_\alpha^{\frac{1}{2q}}(s-r, y'-z') \\ & \times \gamma(z-z') dz' dz dr =: T_1(s,y,y').\end{aligned}$$

We prove the claim by an argument based on uniform integrability. We know that $\gamma = K_{d-\beta} * \mu$. Therefore,

$$\begin{aligned}T_1(s,y,y') =\ & C \int_0^s (s-r)^{-\kappa} \int_{\mathbb{R}^{3d}} G_\alpha^{\frac{1}{2q}}(s-r, y-z) G_\alpha^{\frac{1}{2q}}(s-r, y'-z') \\ & \times |z - z' - w|^{-\beta} dz' dz d\mu(w) dr,\end{aligned}$$

where $2q = \frac{2d}{2d-\beta}$. Making the change of variables $u = s - r$, $\xi = y - z$ and $\xi' = y - z'$, we can write

$$\begin{aligned}T_1(s,y,y') =\ & C \int_0^s u^{-\kappa} \int_{\mathbb{R}^{3d}} G_\alpha^{\frac{1}{2q}}(u,\xi) G_\alpha^{\frac{1}{2q}}(u,\xi') \\ & \times |y - y' - \xi - \xi' - w|^{-\beta} d\xi' d\xi d\mu(w) du.\end{aligned}$$

For any fixed $\xi, \xi', w \in \mathbb{R}^d$, clearly, $|y - y' - \xi - \xi' - w|^{-\beta}$ tends to zero as $|y - y'|$ tends to infinity. Taking into account that

$$\int_0^s u^{-\kappa} \int_{\mathbb{R}^{3d}} G_\alpha^{\frac{1}{2q}}(u,\xi) G_\alpha^{\frac{1}{2q}}(u,\xi') d\xi' d\xi d\mu(w) du < \infty,$$

to show that $\lim_{|y-y'|\to\infty} T_1(s,y,y') = 0$, it suffices to check that

$$I := \int_0^s u^{-\kappa} \int_{\mathbb{R}^{3d}} G_\alpha^{\frac{1}{2q}}(u,\xi) G_\alpha^{\frac{1}{2q}}(u,\xi') |y - y' - \xi - \xi' - w|^{-\beta'} d\xi' d\xi d\mu(w) du < \infty$$

for some $\beta' > \beta$. Making a change of variables, we can write

$$I = \int_0^s u^{-\kappa} \int_{\mathbb{R}^{3d}} G_\alpha^{\frac{1}{2q}}(u, y-z) G_\alpha^{\frac{1}{2q}}(u, y'-z') |z - z' - w|^{-\beta'} dz' dz d\mu(w) du.$$

Appying Hölder's and Hardy-Littlewood-Sobolev's inequality (5.25) yields

$$I \le C \int_0^s u^{-\kappa} du \left(\int_{\mathbb{R}^d} G_\alpha^{\frac{2d-\beta}{2d-\beta'}}(u,x) dx \right)^{\frac{2d-\beta'}{d}},$$

which is finite since β' is close to β. This concludes the proof.

Proof: [Proof of Theorem 12] For notational simplicity, we only consider the case $r = t$ while the case of general $t, r \in [0,T]$ follows in a similar way. Using (5.4) and (5.34), we can write

$$G_R(t) = \int_0^t \varphi_R(t-s,y) \sigma(u(s,y)) W(ds,dy).$$

Hence, by (5.13), we get

$$\mathbb{E}[G_R^2(t)] = \int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(t-s,y) \varphi_R(t-s,y') \Psi(s, y'-y) \gamma(y-y') dy' dy ds.$$

Let us begin with the case $\beta < d$. In view of Lemma 11 together with the boundedness of $\theta_\alpha^2(s)$, it suffices to show that

$$\begin{aligned} T_R &:= R^{\beta-2d} \int_0^t \int_{\mathbb{R}^{2d}} \left[\Psi(s, y-y') - \theta_\alpha^2(s) \right] \varphi_R(t-s,y) \varphi_R(t-s,y') \\ &\quad \times \gamma(y-y') dy' dy ds \to 0. \end{aligned} \tag{5.41}$$

Now by Lemma 12 we know that for every $\varepsilon > 0$ there exists $K > 0$ such that, for every $s \in [0,t]$ and every y, y' with $|y - y'| \ge K$,

$$\left| \Psi(s, y-y') - \theta_\alpha^2(s) \right| \le \varepsilon. \tag{5.42}$$

By using $\gamma = I_{d-\beta} * \mu$, we split $T_R = T_{R,1} + T_{R,2}$, where

$$\begin{aligned} T_{R,1} &= R^{\beta-2d} \int_0^t \int_{\mathbb{R}^{3d}} \varphi_R(t-s,y) \varphi_R(t-s,y') \left[\Psi(s, y-y') - \theta_\alpha^2(s) \right] \\ &\quad \times |y - y' - w|^{-\beta} 1_{|y-y'| \le K} d\mu(w) dy' dy ds \end{aligned}$$

and

$$T_{R,2} = R^{\beta-2d}\int_0^t\int_{\mathbb{R}^{3d}}\varphi_R(t-s,y)\varphi_R(t-s,y')\left[\Psi(s,y-y')-\theta_\alpha^2(s)\right] \times|y-y'-w|^{-\beta}1_{|y-y'|\geq K}d\mu(w)dy'dyds.$$

On the region $|y'-y|\leq K, 0\leq s\leq T$ the quantity $\Psi(s,y-y')-\theta_\alpha^2(s)$ is uniformly bounded. Using also the semigroup property and (5.18) allows us to estimate

$$\begin{aligned}T_{R,1} &\leq CR^{\beta-2d}\int_0^t\int_{\mathbb{R}^{3d}}\int_{B_R^2}G_\alpha(t-s,x-y)G_\alpha(t-s,x-y')\\&\qquad\times|y-y'-w|^{-\beta}1_{|y-y'|\leq K}dx'dxdy'dyd\mu(w)ds\\&= CR^{\beta-2d}\int_0^t\int_{\mathbb{R}^{2d}}\int_{B_R^2}G_\alpha(2(t-s),x-x'-y')|y'-w|^{-\beta}\\&\qquad\times 1_{|y'|\leq K}dx'dxdy'd\mu(w)ds\\&\leq CR^{\beta-d}\int_{\mathbb{R}^{2d}}|y-w|^{-\beta}1_{|y|\leq K}dyd\mu(w)\to 0\end{aligned}$$

as $R\to\infty$, since clearly here we have

$$\int_{\mathbb{R}^{2d}}|y-w|^{-\beta}1_{|y|\leq K}dyd\mu(w)<\infty.$$

For the term $T_{R,2}$, we apply (5.42) to get

$$\begin{aligned}T_{R,2} &\leq \varepsilon C_\alpha R^{\beta-2d}\int_0^t\int_{\mathbb{R}^{3d}}\int_{B_R^2}G_\alpha(t-s,x-y)G_\alpha(t-s,x'-y')\\&\quad\times|y-y'-w|^{-\beta}dx'dxdy'dyd\mu(w)ds.\end{aligned}$$

The change of variables $x-y=\theta$, $x'-y'=\theta'$, $x_1=R\xi_1$ and $x'=R\xi'$ yields

$$\begin{aligned}T_{R,2} &\leq \varepsilon C_\alpha\textstyle\int_0^t\int_{\mathbb{R}^{3d}}\int_{B_1^2}G_\alpha(t-s,\theta)G_\alpha(t-s,\theta')\\&\quad\times|\xi-\xi'-R^{-1}\theta+R^{-1}\theta'-w|^{-\beta}d\xi'd\xi d\theta d\theta' d\mu(w)ds,\end{aligned}$$

which is bounded by $C\varepsilon$ because $\sup_{z\in\mathbb{R}^d}\int_{B_1}|y-z|^{-\beta}dy<\infty$. Since $\varepsilon>0$ is arbitrary, the desired limit (5.41) follows. This verifies the claim for the case $\beta<d$.

Let next $\beta=d$. Since for a fixed $s>0$, the function $y\mapsto\Psi(s,y)$ is a bounded function and now $\gamma=\mu$ is a finite measure, we may regard $\tilde{\gamma}_s(dy)=\Psi(s,y)\gamma(dy)$ as a signed measure. Considering positive and negative parts separately, we may use exactly the same arguments as in the proof of Lemma 11 and get

$$\begin{aligned}R^{-d}\mathbb{E}[G_R^2(t)] &= R^{-d}\int_0^t\int_{\mathbb{R}^{2d}}\varphi_R(t-s,y)\varphi_R(t-s,y')\Psi(s,y'-y)\gamma(y-y')dy'dyds\\&\to \int_0^t\widehat{\tilde{\gamma}_s}(0)\int_{\mathbb{R}^d}|\xi|^{-d}J_{\frac{d}{2}}^2(|\xi|)d\xi ds,\end{aligned}$$

where now

$$\widehat{\gamma_s}(0) = \int_{\mathbb{R}^d} \Psi(s,z)d\gamma(z) = \nu_\alpha^2(s).$$

This verifies the claim for $\beta = d$ as well, and hence the proof is completed. We end this subsection by proving Lemma 10.

Proof: [Proof of Lemma 10] Denote

$$T(s,y) := \Psi(s,y) - \theta_\alpha^2(s).$$

Since $T(s,y)$ is also a bounded function, we may follow the proofs of Theorem 12 and Lemma 11 to obtain

$$\begin{aligned} & R^{-d}\int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(t-s,y)\varphi_R(t-s,y')\Psi(s,y'-y)\gamma(y-y')dy'dyds \\ = \; & R^{-d}\int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(t-s,y)\varphi_R(t-s,y')T(s,y'-y)\gamma(y-y')dy'dyds \\ + \; & R^{-d}\int_0^t \theta_\alpha^2(s)\int_{\mathbb{R}^{2d}} \varphi_R(t-s,y)\varphi_R(t-s,y')\gamma(y-y')dy'dyds, \end{aligned}$$

where now, as $R\to\infty$,

$$\begin{aligned} & R^{-d}\int_0^t \theta_\alpha^2(s)\int_{\mathbb{R}^{2d}} \varphi_R(t-s,y)\varphi_R(t-s,y')\gamma(y-y')dy'dyds \\ \to \; & \mu\left(\mathbb{R}^d\right)|B_1|\int_0^t \theta_\alpha^2(s)ds \end{aligned}$$

and

$$\begin{aligned} & R^{-d}\int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(t-s,y)\varphi_R(t-s,y')T(s,y'-y)\gamma(y-y')dy'dyds \\ \to |B_1| & \int_0^t T(\widehat{s,\bullet)\gamma}(\bullet)(0)ds. \end{aligned}$$

By the very definition, we have

$$T(s,y) := \Psi(s,y) - \theta_\alpha^2(s) = \mathrm{Cov}\left[\sigma(u(s,y))\sigma(u(s,0))\right]$$

and since $T(s,y)$ and $\gamma(y)$ are both covariances, they are positive semidefinite. Consequently, the product $T(s,y)\gamma(y)$ is again a covariance. It follows that

$$T(\widehat{s,\bullet)\gamma}(\bullet)(\xi) \geq 0$$

for all $\xi \in \mathbb{R}^d$ and, in particular, for $\xi = 0$. Now

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} \Psi(s,z)d\mu(z)dt &= \int_0^t \nu_\alpha^2(s)ds \\ &= \int_0^t T(\widehat{s,\bullet)\gamma}(\bullet)(0)ds + \int_0^t \theta_\alpha^2(s)ds. \end{aligned}$$

The claim follows from this together with the observations $\Psi(0,z) = \theta_\alpha^2(0)$ for all $z \in \mathbb{R}^d$ and $T(\widehat{s,\bullet)\gamma}(\bullet)(0) \geq 0$ for all $s \geq 0$.

5.5.3 Proof of Theorem 10

We start with the following result that we will utilise in the case $\beta < d$.

Lemma 13 *Suppose that* $0 < \beta < \alpha < 2 \wedge d$. *For every* $t > 0$ *we have*

$$\int_{\mathbb{R}^d} G_\alpha(t, x-y)|y|^{-\beta} dy \le C_{\beta,\alpha}|x|^{-\beta}. \tag{5.43}$$

Proof: Using the estimate (5.22), we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} G_\alpha(t,x-y)|y|^{-\beta}dy \\
&\le C\int_{\mathbb{R}^d} \frac{t^{-\frac{d}{\alpha}}}{1+|(x-y)t^{-\frac{1}{\alpha}}|^{\alpha+d}}|y|^{-\beta}dy \\
&= C\int_{|y|<\frac{|x|}{2}} \frac{t^{-\frac{d}{\alpha}}}{1+|(x-y)t^{-\frac{1}{\alpha}}|^{\alpha+d}}|y|^{-\beta}dy + C\int_{|y|\ge\frac{|x|}{2}} \frac{t^{-\frac{d}{\alpha}}}{1+|(x-y)t^{-\frac{1}{\alpha}}|^{\alpha+d}}|y|^{-\beta}dy \\
&\le C\int_{|y|<\frac{|x|}{2}} \frac{t^{-\frac{d}{\alpha}}}{1+|xt^{-\frac{1}{\alpha}}|^{\alpha+d}}|y|^{-\beta}dy + C|x|^{-\beta}\int_{|y|\ge\frac{|x|}{2}} \frac{t^{-\frac{d}{\alpha}}}{1+|(x-y)t^{-\frac{1}{\alpha}}|^{\alpha+d}}dy \\
&\le C\frac{t^{-\frac{d}{\alpha}}}{1+|xt^{-\frac{1}{\alpha}}|^{\alpha+d}}|x|^{-\beta+d} + C|x|^{-\beta}.
\end{aligned}$$

The estimate (5.43) follows from this, because one can show that

$$\sup_{x\in\mathbb{R}^d}\sup_{t>0}\frac{t^{-\frac{d}{\alpha}}|x|^d}{1+|xt^{-\frac{1}{\alpha}}|^{\alpha+d}} < \infty.$$

Proof: [Proof of Theorem 10] Let φ_R be given by (5.34). By the same arguments as in the proof of [64, Theorem 1.1] (see pp. 7178-7180), using Proposition 1, Theorem 12 and Proposition 3, we get $d_{TV}(F_R, Z) \le 2(A_1 + A_2)$, where

$$\begin{aligned}
A_1 \;\le\; & CR^{\beta-2d}\int_0^t \left(\int_0^s (s-r)^{-\kappa}\int_{\mathbb{R}^{6d}} \varphi_R(t-s,y)\varphi_R(t-s,y')\varphi_R(t-s,\tilde{y})\right. \\
& \times\varphi_R(t-s,\tilde{y}')G_\alpha^{\frac{1}{2q}}(s-r,y-z)G_\alpha^{\frac{1}{2q}}(s-r,\tilde{y}-z')\gamma(y-y') \\
& \left.\times\gamma(\tilde{y}-\tilde{y}')\gamma(z-z')dydy'd\tilde{y}d\tilde{y}'dzdz'dr\right)^{1/2} ds
\end{aligned} \tag{5.44}$$

and

$$\begin{aligned}
A_2 \;\le\; & CR^{\beta-2d}\int_0^t \left(\int_s^t (r-s)^{-\kappa}\int_{\mathbb{R}^{6d}} \varphi_R(t-r,z)\varphi_R(t-r,\tilde{z})\varphi_R(t-s,y')\right. \\
& \times\varphi_R(t-s,\tilde{y}')G_\alpha^{\frac{1}{2q}}(r-s,z-y)G_\alpha^{\frac{1}{2q}}(r-s,\tilde{z}-\tilde{y}) \\
& \left.\times\gamma(y-y')\gamma(\tilde{y}-\tilde{y}')\gamma(z-\tilde{z})dydy'd\tilde{y}d\tilde{y}'dzd\tilde{z}dr\right)^{1/2} ds.
\end{aligned}$$

We begin with the case $\beta = d$ that is simpler. For the term A_1 in this case, we use the trivial bound $\varphi_R(t-s,y')\varphi_R(t-s,\tilde{y})\varphi_R(t-s,\tilde{y}') \leq 1$, integrate in the variables y' and $\tilde{y}'$, perform the change of variables $y \mapsto y-z$ and $\tilde{y} \mapsto \tilde{y}-z$ in the integrals with respect to $y, \tilde{y}$, and then integrate with respect to z', z, and finally with respect to y and $\tilde{y}'$. Together with Lemma 9, this leads to

$$\begin{aligned}
A_1 &\leq CR^{-d} \int_0^t \left(\int_0^s (s-r)^{-\kappa} \int_{\mathbb{R}^{4d}} \varphi_R(t-s,y) G_\alpha^{\frac{1}{2q}}(s-r,y-z) \right.\\
&\quad \left. \times G_\alpha^{\frac{1}{2q}}(s-r,\tilde{y}-z')\gamma(z-z')dyd\tilde{y}dzdz'dr \right)^{1/2} ds \\
&= CR^{-d} \int_0^t \left(\int_0^s (s-r)^{-\kappa} \int_{\mathbb{R}^{4d}} \varphi_R(t-s,y+z) G_\alpha^{\frac{1}{2q}}(s-r,y) \right.\\
&\quad \left. \times G_\alpha^{\frac{1}{2q}}(s-r,\tilde{y})\gamma(z-z')dzdz'dyd\tilde{y}dr \right)^{1/2} ds \\
&\leq CR^{-\frac{d}{2}} \int_0^t \left(\int_0^s (s-r)^{-\kappa} \int_{\mathbb{R}^{2d}} G_\alpha^{\frac{1}{2q}}(s-r,y) G_\alpha^{\frac{1}{2q}}(s-r,\tilde{y}) dyd\tilde{y}dr \right)^{1/2} ds \\
&\leq CR^{-\frac{d}{2}}.
\end{aligned}$$

Treating the term A_2 with similar arguments completes the proof for the case $\beta = d$. Suppose next $\beta < d$ and let us again first treat the term A_1. We can bound A_1 as follows

$$\begin{aligned}
A_1 \quad &\leq CR^{\beta-2d} \int_0^t \Bigg(\int_0^s (s-r)^{-\kappa} \int_{B_R^4} \int_{\mathbb{R}^{9d}} G_\alpha(t-s,x_1-y) G_\alpha(t-s,x_2-y') \\
&\quad \times G_\alpha(t-s,x_3-\tilde{y}) G_\alpha(t-s,x_4-\tilde{y}') G_\alpha^{\frac{1}{2q}}(s-r,z-y) G_\alpha^{\frac{1}{2q}}(s-r,\tilde{z}-\tilde{y}) \\
&\quad \times |y-y'-w_1|^{-\beta} |\tilde{y}-\tilde{y}'-w_2|^{-\beta} |z-\tilde{z}-w_3|^{-\beta} \\
&\quad \times dydy'd\tilde{y}d\tilde{y}'dzd\tilde{z}d\mu(w_1)d\mu(w_2)d\mu(w_3)dx_1dx_2dx_3dx_4dr \Bigg)^{1/2} ds.
\end{aligned}$$

The change of variables $x_1 - y = \theta_1$, $x_2 - y' = \theta_2$, $x_3 - \tilde{y} = \theta_3$, $x_4 - \tilde{y}' = \theta_4$, $z - y = \eta_1$ and $\tilde{z} - \tilde{y} = \eta_2$, yields

$$\begin{aligned}
A_1 \quad &\leq CR^{\beta-2d} \int_0^t \Bigg(\int_0^s (s-r)^{-\kappa} \int_{B_R^4} \int_{\mathbb{R}^{9d}} G_\alpha(t-s,\theta_1) G_\alpha(t-s,\theta_2) \\
&\quad \times G_\alpha(t-s,\theta_3) G_\alpha(t-s,\theta_4) G_\alpha^{\frac{1}{2q}}(s-r,\eta_1) G_\alpha^{\frac{1}{2q}}(s-r,\eta_2) \\
&\quad \times |x_1-x_2+\theta_2-\theta_1-w_1|^{-\beta} |x_3-x_4+\theta_4-\theta_3-w_2|^{-\beta} \\
&\quad \times |x_1-x_3-\theta_1+\theta_4+\eta_1-\eta_2-w_3|^{-\beta} \\
&\quad \times d\theta_1 d\theta_2 d\theta_3 d\theta_4 d\eta_1 d\eta_2 d\mu(w_1)d\mu(w_2)d\mu(w_3)dx_1dx_2dx_3dx_4dr \Bigg)^{1/2} ds.
\end{aligned}$$

Integrating in the variables θ_2 and θ_3 and using the estimate (5.43), we can write

$$A_1 \leq CR^{\beta-2d}\int_0^t \Bigg(\int_0^s (s-r)^{-\kappa} \int_{B_R^4}\int_{\mathbb{R}^{7d}} G_\alpha(t-s,\theta_1) \times G_\alpha(t-s,\theta_4) G_\alpha^{\frac{1}{2q}}(s-r,\eta_1) G_\alpha^{\frac{1}{2q}}(s-r,\eta_2) \times |x_1-x_2-\theta_1-w_1|^{-\beta}|x_3-x_4+\theta_4-w_2|^{-\beta} \times |x_1-x_3-\theta_1+\theta_4+\eta_1-\eta_2-w_3|^{-\beta} \times d\theta_1 d\theta_4 d\eta_1 d\eta_2 d\mu(w_1)d\mu(w_2)d\mu(w_3)dx_1dx_2dx_3dx_4dr\Bigg)^{1/2} ds.$$

The change of variables $x_i = R\xi_i$, $i=1,2,3,4$ yields

$$A_1 \leq CR^{-\beta/2}\int_0^t \Bigg(\int_0^s (s-r)^{-\kappa} \int_{B_1^4}\int_{\mathbb{R}^{7d}} G_\alpha(t-s,\theta_1) \times G_\alpha(t-s,\theta_4) G_\alpha^{\frac{1}{2q}}(s-r,\eta_1) G_\alpha^{\frac{1}{2q}}(s-r,\eta_2) \times |\xi_1-\xi_2-R^{-1}[\theta_1-w_1]|^{-\beta}|\xi_3-\xi_4+R^{-1}[\theta_4-w_2]|^{-\beta} \times |\xi_1-\xi_3+R^{-1}[-\theta_1+\theta_4+\eta_1-\eta_2-w_3]|^{-\beta} \times d\theta_1 d\theta_4 d\eta_1 d\eta_2 d\mu(w_1)d\mu(w_2)d\mu(w_3)d\xi_1d\xi_2d\xi_3d\xi_4dr\Bigg)^{1/2} ds.$$

Taking into account that

$$\sup_{z\in\mathbb{R}^d}\int_{B_1}|x+z|^{-\beta}dx < \infty,$$

and that, by Lemma 9,

$$\int_{\mathbb{R}^d} G_\alpha^{\frac{1}{2q}}(s-r,\eta)d\eta = C(s-r)^{\frac{\kappa}{2}},$$

we conclude that

$$A_1 \leq CR^{-\beta/2}.$$

Treating the term A_2 similarly verifies the case $\beta < d$ as well, completing the whole proof.

5.5.4 Proof of Theorem 11

In order to prove 11 it suffices to prove tightness and the convergence of the finite dimensional distributions. For the latter we can proceed as in [64] together with the arguments of the proof of Theorem 10. The tightness is ensured by the following result and Kolmogorov's criterion.

Proposition 5 *Let $u(t,x)$ be the solution to* (5.1)*. Then for any $0 \leq s < t \leq T$ and any $p \geq 1$ there exists a constant $C = C(p,T)$ such that*

$$\mathbb{E}\left(\left|\int_{B_R} u(t,x)dx - \int_{B_R} u(s,x)dx\right|^p\right) \leq CR^{(d-\frac{\beta}{2})p}(t-s)^{\frac{p}{2}}.$$

Proof: Let $\Theta_{x,t,s}$ be given by

$$\Theta_{x,t,s}(r,y) = G_\alpha(t-r,x-y)\mathbf{1}_{\{r\le t\}} - G_\alpha(s-r,x-y)\mathbf{1}_{\{r\le s\}}.$$

We have, for $0<s<t<T$,

$$\int_{B_R} u(t,x)dx - \int_{B_R} u(s,x)dx = \int_0^T \int_{\mathbb{R}^d}\int_{B_R} \Theta_{x,t,s}(r,y)\sigma(u(r,y))dxW(dr,dy).$$

Now Burkholder inequality implies that, for every $p\ge 1$,

$$\begin{aligned} &\mathbb{E}\left(\left|\textstyle\int_{B_R} u(t,x)dx - \int_{B_R} u(s,x)dx\right|^p\right) \\ \le\ & C_{p,T}\left(\textstyle\int_0^T\int_{\mathbb{R}^{2d}}\left(\int_{B_R^2}\Theta_{x,t,s}(r,y)\Theta_{x',t,s}(r,y')dx'dx\right)\gamma(y-y')dydy'dr\right)^{\frac{p}{2}}. \end{aligned}$$

Hence it remains to show that

$$\begin{aligned} K_R(t,s) &:= \int_0^T\int_{\mathbb{R}^{2d}}\left(\int_{B_R^2}\Theta_{x,t,s}(r,y)\Theta_{x',t,s}(r,y')dx'dx\right)\gamma(y-y')dydy'dr \\ &\le CR^{2d-\beta}(t-s). \end{aligned} \tag{5.45}$$

By taking the Fourier transform, we obtain $K_R(t,s)\le C(I_1+I_2)$, where

$$I_1 = \int_0^s\int_{\mathbb{R}^d} R^d|\xi|^{-d}J^2_{\frac{d}{2}}(R|\xi|)\left|e^{-(t-r)|\xi|^\alpha} - e^{-(s-r)|\xi|^\alpha}\right|^2\widehat{\gamma}(\xi)d\xi dr$$

and

$$I_2 = \int_s^t\int_{\mathbb{R}^d} R^d|\xi|^{-d}J^2_{\frac{d}{2}}(R|\xi|)e^{-2(t-r)|\xi|^\alpha}\widehat{\gamma}(\xi)d\xi dr.$$

Using $e^{-2(t-r)|\xi|^\alpha}\le 1$ and

$$\left|e^{-(t-r)|\xi|^\alpha} - e^{-(s-r)|\xi|^\alpha}\right|^2 \le C(t-s)$$

leads to

$$\begin{aligned} I_1+I_2 &\le C(t-s)\int_{\mathbb{R}^d} R^d|\xi|^{-d}J^2_{\frac{d}{2}}(R|\xi|)\widehat{\gamma}(\xi)d\xi \\ &= C(t-s)R^{2d-\beta}\int_{\mathbb{R}^d}|\xi|^{\beta-2d}J^2_{\frac{d}{2}}(|\xi|)d\xi. \end{aligned}$$

This concludes the proof.

Theorem 11 follows by the arguments of the proof of [64, Theorem 1.3] together with Proposition 5. Details that, despite being rather lengthy, are directly based on the same arguments that we have used above, and for this reason they are left to the reader.

5.6 Appendix : proof of Lemma 8

Proof: [Proof of Lemma 8] As in [34] (see the proofs of Lemmas 2.4 and 3.1), it suffices to prove the bound (5.33) in the case when (5.32) is an equality. Let $g_n, n \geq 0$ be a sequence defined iteratively by setting $g_0(t,x) = G_\alpha(t,x)$, and for $n \geq 0$

$$\begin{aligned} g_{n+1}(t,x)^2 &= G_\alpha(t,x)^2 + \int_0^t \int_{\mathbb{R}^{2d}} G_\alpha(t-s,x-y) G_\alpha(t-s,x-y') \\ &\quad \times g_n(s,y) g_n(s,y') \gamma(y-y') dy' dy ds. \end{aligned}$$

Denote $\kappa = \frac{2d}{\alpha} - \frac{d}{q\alpha}$. We prove by induction that for every $n \geq 0$,

$$g_n(t,x)^2 \leq C \sum_{j=0}^n \frac{\Gamma^j(1-\kappa)}{\Gamma((j+1)(1-\kappa))} t^{j(1-\kappa)-\kappa} G_\alpha^{\frac{1}{q}}(t,x). \tag{5.46}$$

For $n = 0$, taking into account that $\alpha + d \geq \frac{\alpha+d}{2q}$, $\frac{\kappa}{2} = \frac{d}{\alpha} - \frac{d}{2q\alpha}$, and $2q > 1$, we can use the estimate (5.22),

$$\begin{aligned} g_0(t,x) &= G_\alpha(t,x) \leq C \frac{t^{-\frac{d}{\alpha}}}{(1+|t^{-\frac{1}{\alpha}} x|)^{\alpha+d}} \\ &\leq C \frac{t^{-\frac{d}{\alpha}}}{(1+|t^{-\frac{1}{\alpha}} x|)^{(\alpha+d)/2q}} \\ &\leq C t^{-\frac{\kappa}{2}} G_\alpha^{\frac{1}{2q}}(t,x). \end{aligned} \tag{5.47}$$

Hence (5.46) is true for $n = 0$.

Suppose that (5.46) holds for n. Denoting $c_j = \frac{\Gamma^j(1-\kappa)}{\Gamma\left((j+1)(1-\kappa)\right)}$ and by the induction hypothesis,

$$\begin{aligned} g_{n+1}(t,x)^2 &\leq G_\alpha(t,x)^2 + \int_0^t \int_{\mathbb{R}^{2d}} G_\alpha(t-s,x-y) G_\alpha(t-s,x-y') \\ &\quad \times \sum_{j=0}^n c_j s^{j(1-\kappa)-\kappa} G_\alpha^{\frac{1}{2q}}(s,y) G_\alpha^{\frac{1}{2q}}(s,y') \gamma(y-y') dy' dy ds \\ &=: G_\alpha(t,x)^2 + \sum_{j=0}^n c_j I_j. \end{aligned} \tag{5.48}$$

The inequality (5.23) with $g(y) = f(y) = G_\alpha(t-s,x-y) G_\alpha^{\frac{1}{2q}}(s,y)$ allows us to estimate

$$I_j \leq C \int_0^t s^{j(1-\kappa)-\kappa} \left(\int_{\mathbb{R}^d} G_\alpha^{2q}(t-s,x-y) G_\alpha(s,y) dy \right)^{\frac{1}{q}} ds.$$

The scaling and asymptotic properties of the kernel G_α imply that

$$G_\alpha^{2q}(t-s,x-y) \le C\frac{(t-s)^{-\frac{2qd}{\alpha}}}{(1+|(t-s)^{-\frac{1}{\alpha}}(x-y)|)^{2q(\alpha+d)}}.$$

Taking into account that $(\alpha+d)2q \ge \alpha+d$, we obtain

$$\begin{aligned} G_\alpha^{2q}(t-s,x-y) \quad &\le C\frac{(t-s)^{-\frac{2qd}{\alpha}}}{(1+|(t-s)^{-\frac{1}{\alpha}}(x-y)|)^{\alpha+d}} \\ &\le C(t-s)^{-\kappa q}G_\alpha(t-s,x-y). \end{aligned}$$

Therefore, by the semigroup property

$$\begin{aligned} I_j \quad &\le C\int_0^t s^{j(1-\kappa)-\kappa}(t-s)^{-\kappa}G_\alpha^{\frac{1}{q}}(t-s+s,x)ds \\ &= CG_\alpha^{\frac{1}{q}}(t,x)\int_0^t s^{j(1-\kappa)-\kappa}(t-s)^{-\kappa}ds \\ &= CG_\alpha^{\frac{1}{q}}(t,x)t^{(j+1)(1-\kappa)-\kappa}\frac{\Gamma(1-\kappa)\Gamma((j+1)(1-\kappa))}{\Gamma((j+2)(1-\kappa))}. \end{aligned} \tag{5.49}$$

Substituting (5.49) and (5.47) into (5.48) yields

$$\begin{aligned} g_{n+1}(t,x)^2 \quad &\le Ct^{-\kappa}G_\alpha^{\frac{1}{q}}(t,x) \\ &+C\sum_{j=0}^n c_j G_\alpha^{\frac{1}{q}}(t,x)t^{(j+1)(1-\kappa)-\kappa}\frac{\Gamma(1-\kappa)\Gamma((j+1)(1-\kappa))}{\Gamma((j+2)(1-\kappa))} \\ &= CG_\alpha^{\frac{1}{q}}(t,x)\sum_{j=0}^{n+1} c_{j-1}t^{j(1-\kappa)-\kappa}\frac{\Gamma(1-\kappa)\Gamma(j(1-\kappa))}{\Gamma((j+1)(1-\kappa))} \\ &= CG_\alpha^{\frac{1}{q}}(t,x)\sum_{j=0}^{n+1} t^{j(1-\kappa)-\kappa}\frac{\Gamma^j(1-\kappa)}{\Gamma((j+1)(1-\kappa))}. \end{aligned}$$

Finally, it follows from (5.46)

$$\begin{aligned} g(t,x) \quad &= \quad \lim_{n\to\infty} g_n(t,x) \le C\left(\sum_{j=0}^{\infty}\frac{\Gamma^j(1-\kappa)}{\Gamma((j+1)(1-\kappa))}t^{j(1-\kappa)-\kappa}G_\alpha(t,x)\right)^{1/2} \\ &\le \quad Ct^{-\frac{\kappa}{2}}G_\alpha^{\frac{1}{2q}}(t,x). \end{aligned}$$

This finishes the proof.

Troisième partie

Processus de Hermite généralisé

Chapitre 6

Generalized Wiener-Hermite integrals and rough non-Gaussian Ornstein-Uhlenbeck process

This article is accepted on Stochastics.

Joint work with Charles-Philippe Diez and Ciprian A. Tudor

6.1 Introduction

The Hermite processes constitutes a class of self-similar processes with stationary increments. The Hermite process is characterized by its order (an integer $q \geq 1$) and its Hurst parameter (or self-similarity index) H. The Hermite process of order $q = 1$ is nothing else than the fractional Brownian motion (fBm in the sequel) which is the only Gaussian Hermite process and it is well-defined for every $H \in (0, 1)$. For $q \geq 2$ the Hermite processes are non-Gaussian and they are defined only when the Hurst parameter belongs to the interval $\left(\frac{1}{2}, 1\right)$. This implies that these stochastic processes have smooth sample paths, which are Hölder continuous of order δ for any $\delta \in (0, H)$, with $H > \frac{1}{2}$. They also always exhibits long-range dependence (or long memory). These nice properties make them good models for various phenomena in physics, hydrology or economics. For a more complete exposition on Hermite processes and their applications, we refer to the monographs [87] or [100], which treat them in extenso. On the other hand, since for $q \geq 2$ their self-similarity index is always contained in the interval $\left(\frac{1}{2}, 1\right)$, they do not provide a good model for applications where non-gaussianity together short memory or rough sample paths are required. Such an example of application comes from the mathematical finance. During the last years, researchers from this area seemed to arrived to the conclusion that in many markets the volatility is rough and it should be modeled by a stochastic processes with rough trajectories (see, among many others [57], [62]). As a proxi for volatility, it is often chosen the fractional Brownian motion with small Hurst parameter, close to zero, or related processes such as the fractional Ornstein-Uhlenbeck process (see again [57], among others). This seems to be be a well-fitted model for $S\&P$ and other indices. For other markets (such as Crude Oil or Gold futures), besides rough sample paths and scaling property, the empirical data appears to indicate in addition a non-Gaussian character (see e.g. [19] and [101] for the market of some precious metals or [96] for the Bitcoin, see also the survey [53]). To modelize such phenomena, a self-similar non-Gaussian process with rough sample paths seems to be more appropiate. Our purpose is to propose such a stochastic processes, namely the generalized Hermite processes. It has been introduced in [74] and recently revisited by [24]. The generalized Hermite process is a self-similar process with stationary increments, including the Hermite process as a particular case, but it offers more flexibility for models, since its self-similarity index belongs to the whole interval $(0, 1)$, and therefore, upon the values of this index, it can have rough or smooth paths, or long or short memory.

Our aim is to make a deeper study of this class of stochastic processes. First we show that they can be expressed as Wiener integrals with respect to the standard Hermite process, by obtaining in this way some useful identities involving fractional integrals and derivatives. Then we aim at making a first step to construct a calculus with respect to the generalized Hermite process. We define Wiener integrals with respect to these processes and this allows to define an associated Ornstein-Uhlenbeck process, which will be called as generalized Hermite Ornstein-Uhlenbeck process (GHOU process). We will show that the GHOU process is well-defined, in Wiener or pathwise sense, upon the values of the parameters that apear in its expression. We will also give other properties for this process (its probability distribution, its sample paths) and describe its behavior when its drift parameter is small.

We organized our paper as follows. Section 2 contains the definition and basic properties of the standard and generalized Hermite processes as well as a new link between these two stochastic process (i.e. the generalized Hemite process is actually a Wiener integral with respect to the standard Hermite process). In Section 2 we construct the generalized Wiener-Hermite integral while in Section 4 we define and analyze the Ornstein-Uhlenbeck process associated with the generalized Hermite process. The last section is the Appendix where we included the basics of multiple stochastic integrals and fractional calculus.

6.2 Preliminaries

Let us start by introducing the definition and the basic properties of the standard and generalized Hermite processes. We will also notice that the generalized Hermite process can be expressed as a Wiener integral with respect to the standard Wiener process and this fact will be used in the sequel.

6.2.1 Hermite and generalized Hermite processes : Definition and basic properties

Let us start by introducing the standard Hermite process. We will denote by $(Z_t^{(q,H)})_{t\geq 0}$ the Hermite process of order $q \geq 1$ and with self-similarity index $H \in \left(\frac{1}{2}, 1\right)$. It lives in the qth Wiener chaos and it is defined as a multiple Wiener-Itô integral, for every $t \geq 0$,

$$Z_t^{(q,H)} = d(q,H) \int_{\mathbb{R}} dB(y_1) \dots \int_{\mathbb{R}} dB(y_q) \left(\int_0^t (s-y_1)_+^{-(\frac{1}{2}+\frac{1-H}{q})} \dots (s-y_q)_+^{-(\frac{1}{2}+\frac{1-H}{q})} ds \right) \tag{6.1}$$

where $x_+^\alpha = x^\alpha \mathbf{1}_{(0,\infty)}(x)$, $d(q,H)$ is a normalizing positive constant chosen such that $\mathbf{E}\left(Z_H^q(1)\right)^2 = 1$ and $(B(y))_{y\in\mathbb{R}}$ is a Wiener process with time interval $\mathbb{R}$. The stochastic integral in (6.1) is a multiple integral with respect to the Wiener process $(B(y))_{y\in\mathbb{R}}$ (its definition and properties are recalled in the Appendix). We can also write, for every $t \geq 0$,

$$Z_t^{(q,H)} = I_q(L_t)$$

with the kernel

$$L_t(y_1,..,y_q) = d(q,H) \int_0^t ds (s-y_1)_+^{-(\frac{1}{2}+\frac{1-H}{q})} \dots (s-y_q)_+^{-(\frac{1}{2}+\frac{1-H}{q})}, \ \ y_1,...,y_q \in \mathbb{R}.$$

We know that for every $t \geq 0$ and $H \in \left(\frac{1}{2}, 1\right)$, the function L_t belongs to $L^2(\mathbb{R}^q)$ and consequently the random variable $Z_t^{(q,H)}$ is well-defined. The constant $d(q,H)$, which will play a role in the sequel, is given by (see e.g. [100])

$$d(q,H)^2 = \frac{H(2H-1)}{q! B\left(\frac{1}{2}+\frac{1-H}{q}, \frac{2-2H}{q}\right)^q} \tag{6.2}$$

where B denote the beta function $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ for $a,b > 0$. We recall (see [87] or [100]) that the Hermite process (6.1) is H-self-similar with stationary increments and its sample paths are Hölder continuous of order δ for every $\delta \in (0,H)$. Since $H > \frac{1}{2}$, it always exhibits long-range dependence.

Consider the stochastic process $\left(X_t^{(q,H,\beta)}\right)_{t\geq 0}$ given by

$$X_t^{(q,H,\beta)} = d(q,H,\beta) \int_{\mathbb{R}^q} \left[\int_{\mathbb{R}} g_t^\beta(u)(u-y_1)_+^{-(\frac{1}{2}+\frac{1-H}{q})} \dots (u-y_q)_+^{-(\frac{1}{2}+\frac{1-H}{q})} du \right] dB(y_1)\dots dB(y_q) \tag{6.3}$$

for every $t \geq 0$ where the function g_t^β is given by

$$g_t^\beta(u) = (t-u)_+^\beta - (-u)_+^\beta \text{ if } \beta \neq 0 \tag{6.4}$$

and $g_t^\beta(u) = \mathbf{1}_{[0,t]}(u)$ if $\beta = 0$.

Remark 18 *In [24], the authors used the expression* $g_t^\beta(u) = \frac{1}{\beta}\left[(t-u)_+^\beta - (-u)_+^\beta\right]$ *if* $\beta \neq 0$. *If* $\beta = 0$, *then* $X^{(q,H,0)}$ *coincides with the standard Hermite process.*

The constant $d(q,H,\beta)$ in (6.3) will be also chosen such that $\mathbf{E}(X_1^{(q,H,\beta)})^2 = 1$. Its explicit expression will be deduced later, see relation (6.14).

We can also write, for every $t \geq 0$, the random variable $X_t^{(q,H,\beta)}$ as a multiple Wiener -Itô integral

$$X_t^{(q,H,\beta)} = I_q(L_t^\beta)$$

where we denote by L_t^β the kernel of the generalized Hermite process

$$L_t^\beta(y_1,...,y_q) = d(q,H,\beta)\int_{\mathbb{R}} g_t^\beta(u)(u-y_1)_+^{-(\frac{1}{2}+\frac{1-H}{q})} \dots (u-y_q)_+^{-(\frac{1}{2}+\frac{1-H}{q})} du \tag{6.5}$$

for every $y_1,\dots,y_q \in \mathbb{R}$. As the standard Hermite process, the generalized Hermite process lives in the qth Wiener chaos.

It has been shown in Proposition 3.25 in [24] that for every $t \geq 0$, $L_t^\beta \in L^2(\mathbb{R}^q)$ if

$$0 < H+\beta < 1 \text{ and } H \in \left(\frac{1}{2}, 1\right) \tag{6.6}$$

or equivalently

$$-1 < -H < \beta < 1-H < \frac{1}{2}.$$

We will assume (6.6) throughout in the sequel.

The process $\left(X_t^{(q,H,\beta)}\right)_{t\geq 0}$ is $H+\beta$ -selfsimilar and it has stationary increments. In particular, its covariance is given by

$$\mathbf{E}X_t^{(q,H,\beta)}X_s^{(q,H,\beta)} = \frac{1}{2}\left(t^{2(H+\beta)} + s^{2(H+\beta)} - |t-s|^{2(H+\beta)}\right) =: R^{H+\beta}(t,s).$$

This implies that for $q=1$, the process $X^{(1,H,\beta)}$ coincides with the fractional Brownian motion with Hurst index $H+\beta \in (0,1)$.

Moreover, that $0 < s < t$, we have, due to the self-similarity and to the stationarity of the increments, for every $0 \leq s \leq t$ and $p \geq 1$,

$$\mathbf{E}\left|X_t^{(q,H,\beta)} - X_s^{(q,H,\beta)}\right|^p = \mathbf{E}|X_1^{(q,H,\beta)}|^p|t-s|^{p(H+\beta)}.$$

Consequently, by the Kolmogorov continuity criterion, the paths of the generalized Hermite process are Hölder continuous of order δ for any $\delta \in (0, H+\beta)$.

The generalized Hermite process has the same properties as the standard Hermite process. But, since its self-similarity index $H+\beta$ belongs to the whole interval $(0,1)$, it can have rough sample paths or short memory when $H+\beta$ is close to zero. It provides an interesting (and maybe the only) example of a self-similar non-Gaussian process with stationary increments, short memory and rough trajectories. Notice that the index $H+\beta$ is allowed to be $\frac{1}{2}$ by assumption (6.6). This case provide an interesting example (and also probably the only) of a self-similar processes with stationary and non-correlated increments, with the same covariance as the Brownian motion.

6.2.2 The generalized Hermite process as a Wiener-Hermite integral

We need to recall the construction of the Wiener integral with respect to the (standard) Hermite process $Z^{(q,H)}$ (see [73]). The Wiener integral $\int_{\mathbb{R}} f(u) dZ_u^{(q,H)}$ (we will call is *Wiener-Hermite integral*) is well-defined for every $f \in \mathcal{H}_H$, the class of measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(u) f(v) |u-v|^{2H-2} du dv < \infty.$$

This space is not complete and may contain distributions, see [86]. The Wiener-Hermite integral is given by

$$\begin{aligned} \int_{\mathbb{R}} f(u) dZ_u^{(q,H)} &= d(q,H) \int_{\mathbb{R}^q} \left(\int_{\mathbb{R}} f(u)(u-y_1)_+^{-(\frac{1}{2}+\frac{1-H}{q})} \dots (u-y_q)_+^{-(\frac{1}{2}+\frac{1-H}{q})} du \right) dB(y_1)\dots dB(y_q) \\ &= \int_{\mathbb{R}^q} (Jf)(y_1,...,y_q) dB(y_1)\dots dB(y_q) \end{aligned} \quad (6.7)$$

with, for every $y_1,...,y_q \in \mathbb{R}$,

$$(Jf)(y_1,\dots,y_q) = d(q,H) \int_{\mathbb{R}} f(u)(u-y_1)_+^{-(\frac{1}{2}+\frac{1-H}{q})} \dots (u-y_q)_+^{-(\frac{1}{2}+\frac{1-H}{q})} du. \quad (6.8)$$

The Wiener integral satisfies the following isometry

$$\mathbf{E} \int_{\mathbb{R}} f(u) dZ_u^{(q,H)} \int_{\mathbb{R}} g(u) dZ_u^{(q,H)} = H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} du dv f(u) f(v) |u-v|^{2H-2} := \|f\|_{\mathcal{H}_H}^2. \quad (6.9)$$

From the definition of the generalized Hermite process and by the expression of the Wiener-Hermite integral, we obtain the following simple, but important result.

Proposition 34 *Let $X^{(q,H,\beta)}$ be given by (6.3) and $Z^{(q,H)}$ by (6.1). Then for every $t>0$,*

$$X_t^{(q,H,\beta)} = \frac{d(q,H,\beta)}{d(q,H)} \int_{\mathbb{R}} g_t^\beta(u) dZ_u^{(q,H)}. \quad (6.10)$$

Proof : Let us show first that the integral from the right-hand side above is well-defined. We need to check that for every $t \geq 0$

$$I_t := \int_{\mathbb{R}} \int_{\mathbb{R}} du dv g_t^\beta(u) g_t^\beta(v) |u-v|^{2H-2} < \infty.$$

This quantity can be written as

$$I_t = 2\int_{\mathbb{R}} dv g_t^{\beta}(v) \int_v^{\infty} du g_t^{\beta}(u)(u-v)^{2H-2} = 2\int_{\mathbb{R}} dv g_t^{\beta}(v) \int_0^{\infty} dx g_t^{\beta}(v+x) x^{2H-2}$$

and it follows from Proposition 3.25 in [24] that the above integral is finite under condition (6.6). The equality (6.10) is a direct consequence of the definition of the Wiener-Hermite integral (6.7). ■

If $q = 1$, we retrieve a result in [65], which is called a transformation formula for the fractional Brownian motion. That is, it has been shown in Corollary 5.7 in [65] that for every H, β such that $H \in (0,1)$ and $0 < H + \beta < 1$, the stochastic process

$$B_t^{H+\beta} := A(H,\beta) \int_{\mathbb{R}} g_t^{\beta}(u) dB_u^H \tag{6.11}$$

is a fBm with Hurst parameter $H + \beta$. Here

$$A(H,\beta)^2 = \left[H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} dudv \left((1+u)_+^{\beta} - u_+^{\beta} \right) \left((1+v)_+^{\beta} - v_+^{\beta} \right) |u-v|^{2H-2} \right]^{-1}. \tag{6.12}$$

The fact that the right-hand side of (6.11) is a fBm with index $H + \beta$ can be also prove directly. Indeed, $\left(A(H,\beta) \int_{\mathbb{R}} g_t^{\beta}(u) dB_u^H \right)_{t \geq 0}$ is a Gaussian process and for every $0 \leq s \leq t$, due to the isometry (6.9),

$$\begin{aligned}
& \mathbf{E} \left(A(H,\beta) \int_{\mathbb{R}} (g_t^{\beta}(u) - g_s^{\beta}(u)) dB_u^H \right)^2 \\
= \; & A(H,\beta)^2 H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} dudv \left[(t-u)_+^{\beta} - (s-u)_+^{\beta} \right] \left[(t-v)_+^{\beta} - (s-v)_+^{\beta} \right] |u-v|^{2H-2} \\
= \; & A(H,\beta)^2 H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} dudv \left[(t-s+u)_+^{\beta} - u_+^{\beta} \right] \left[(t-s+v)_+^{\beta} - v_+^{\beta} \right] |u-v|^{2H-2} \\
= \; & A(H,\beta)^2 H(2H-1) |t-s|^{2H+2\beta} \int_{\mathbb{R}} \int_{\mathbb{R}} dudv \left((1+u)_+^{\beta} - u_+^{\beta} \right) \left((1+v)_+^{\beta} - v_+^{\beta} \right) |u-v|^{2H-2} \\
= \; & |t-s|^{2H+2\beta}
\end{aligned}$$

where we used (6.12).

Remark 19 *There are various ways to express the constant $A(H,\beta)$. For instasnce, from relation (2.4) in [65] we have*

$$A(H,\beta)^2 = \frac{\Gamma(2H+2\beta+1)\sin(\pi(H+\beta))}{\Gamma(2H+1)\sin(\pi H)\Gamma(1+\beta)^2} = \frac{\Gamma(2H+2\beta+1)\Gamma(H)\Gamma(1-H)}{\Gamma(H+\beta)\Gamma(1-H-\beta)\Gamma(2H+1)\Gamma(1+\beta)^2}. \tag{6.13}$$

In particular, the constant $A(H,\beta)$ is finite.

From (6.11) we will deduce the expression of the constant $d(q,H,\beta)$ which appears in the definition of the generalized Hermite process. Actually

$$d(q,H,\beta)^2 = d(q,H)^2 A(H,\beta)^2 = \frac{H(2H-1)A^2(H,\beta)}{q!B\left(\frac{1}{2}-\frac{1-H}{q},\frac{2-2H}{q}\right)^q}. \tag{6.14}$$

This can be deduced by using the Wiener isometry (6.9) in (6.10) and (6.11). On one hand in (6.10)

$$\begin{aligned} R^{H+\beta}(t,s) &= \frac{d(q,H,\beta)^2}{d(q,H)^2}\langle g_t^\beta(u), g_s^\beta(u)\rangle_{\mathcal{H}_H} \\ &= \frac{d(q,H,\beta)^2}{d(q,H)^2} H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} dudv g_t^\beta(u) g_s^\beta(v)|u-v|^{2H-2} \end{aligned}$$

while in (6.11), for every $s,t\geq 0$,

$$R^{H+\beta}(t,s) = A(H,\beta)^2\langle g_t^\beta(u), g_s^\beta(u)\rangle_{\mathcal{H}_H} = A(H,\beta)^2 H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} dudv g_t^\beta(u) g_s^\beta(v)|u-v|^{2H-2}. \tag{6.15}$$

Assume $\beta>0$ and recall that H is fixed in the interval $\left(\frac{1}{2},1\right)$. We have (see relation (3.16) in [65])

$$\left(\mathcal{I}_-^\beta 1_{[0,t]}\right)(u) = \frac{1}{\Gamma(\beta+1)} g_t^\beta(u) \tag{6.16}$$

(here $\mathcal{I}_-^\beta$ is the right-sided Riemann-Liouville fractional integral, see (6.45)), so (6.11) can be read as

$$\int_{\mathbb{R}} 1_{[0,t]}(u) dB_u^{H+\beta} = A(H,\beta)\Gamma(\beta+1)\int_{\mathbb{R}}\left(\mathcal{I}_-^\beta 1_{[0,t]}\right)(u) dB_u^H. \tag{6.17}$$

Let us give an extension of relation (6.17) to more general functions.

Proposition 35 *Assume $\beta>0$ and $H\in\left(\frac{1}{2},1\right)$. If $f\in L^1(\mathbb{R})$ such that $\mathcal{I}_-^\beta f\in\mathcal{H}_H$ then $f\in\mathcal{H}_{H+\beta}$ and*

$$\int_{\mathbb{R}} f(u) dB_u^{H+\beta} = A(H,\beta)\Gamma(\beta+1)\int_{\mathbb{R}}\left(\mathcal{I}_-^\beta f\right)(u) dB_u^H \tag{6.18}$$

with $B^{H+\beta}$ from (6.11).

Proof : Take $f\in L^1(\mathbb{R})$ with $\mathcal{I}_-^\beta f\in\mathcal{H}_H$ and let us show that $f\in\mathcal{H}_{H+\beta}$. Notice that $H+\beta>\frac{1}{2}$.

We claim that, for every $s,t\geq 0$,

$$(H+\beta)(2(H+\beta)-1)|t-s|^{2(H+\beta)-2} = A(H,\beta)^2 H(2H-1)\beta^2\int_{\mathbb{R}}\int_{\mathbb{R}} dudv (t-u)_+^{\beta-1}(s-v)_+^{\beta-1}|u-v|^{2H-2}. \tag{6.19}$$

This can be seen by differentiating $\frac{\partial^2}{\partial t \partial s}$ in (6.15) but it can be also proved directly, via the change of variables $u = s - |t-s|\left(\frac{1}{z}-1\right), v = s - |t-s|\left(\frac{1}{y}-1\right)$ (to get the right-constant in (6.19), we need to use the expression (6.13) of the constant $A(H,\beta)$ and various properties of the gamma function).

We can write (the norm in $\mathcal{H}_{H+\beta}$ is defined in (6.9))

$$\|f\|^2_{\mathcal{H}+\beta} = (H+\beta)(2(H+\beta)-1)\int_{\mathbb{R}}\int_{\mathbb{R}} dudv|u-v|^{2(H+\beta)-2} f(u)f(v)$$

and by plugging (6.19) into the above relation,

$$\begin{aligned}
&\|f\|^2_{\mathcal{H}+\beta}\\
&= A(H,\beta)^2 H(2H-1)\beta^2 \int_{\mathbb{R}}\int_{\mathbb{R}} dudv f(u)f(v)\int_{\mathbb{R}}\int_{\mathbb{R}} dsds'(u-s)_+^{\beta-1}(v-s')_+^{\beta-1}|s-s'|^{2H-2}\\
&= H(2H-1)A(H,\beta)^2\Gamma(\beta)^2\int_{\mathbb{R}}\int_{\mathbb{R}} dsds'|s-s'|^{2H-2}(\mathcal{I}^{\beta}_{-}f)(s)(\mathcal{I}^{\beta}_{-}f)(s')\\
&= A(H,\beta)^2\Gamma(\beta+1)^2\|\mathcal{I}^{\beta}_{-}f\|^2_{\mathcal{H}_H} < \infty \qquad (6.20)
\end{aligned}$$

where we used the definition of the fractional integral (6.45). Now, relation (6.18) is true for simple functions due to (6.17). Since the set of simple functions $\mathcal{E}$ is dense in $\mathcal{H}_{H+\beta}$, we take $(f_n)_{n\geq 1}\subset \mathcal{E}$ such that $\|f_n - f\|_{\mathcal{H}_{H+\beta}} \to_{n\to\infty} 0$. We have for every $n\geq 1$

$$\int_{\mathbb{R}} f_n(u)dB_u^{H+\beta} = A(H,\beta)\Gamma(\beta+1)\int_{\mathbb{R}}\left(\mathcal{I}^{\beta}_{-}f_n\right)(u)dB^H_u. \qquad (6.21)$$

The left-hand side converges in $L^2(\Omega)$ as $n\to\infty$ to $\int_{\mathbb{R}} f(u)dB_u^{H+\beta}$ from the properties of the Wiener integral with respect to the fBm (see e.g. [86]) while the right-hand side converges in $L^2(\Omega)$ as $n\to\infty$ to $A(H,\beta)\Gamma(\beta+1)\int_{\mathbb{R}}\left(\mathcal{I}^{\beta}_{-}f\right)(u)dB^H_u$ because

$$\begin{aligned}
&A(H,\beta)^2\Gamma(\beta+1)^2\mathbf{E}\left[\int_{\mathbb{R}}\left(\mathcal{I}^{\beta}_{-}f_n\right)(u)dB^H_u - \int_{\mathbb{R}}\left(\mathcal{I}^{\beta}_{-}f\right)(u)dB^H_u\right]^2\\
&= A(H,\beta)^2\Gamma(\beta+1)^2\mathbf{E}\left(\int_{\mathbb{R}}\left(\mathcal{I}^{\beta}_{-}(f_n-f)\right)(u)dB^H_u\right)^2 = A(H,\beta)^2\Gamma(\beta+1)^2\|\mathcal{I}^{\beta}_{-}(f_n-f)\|^2_{\mathcal{H}_H}\\
&= \|f_n - f\|^2_{\mathcal{H}_{H+\beta}} \to_{n\to\infty} 0
\end{aligned}$$

where for the last equality we used (6.20).

■

Remark 20 *With $B^{H+\beta}$ from (6.11), the equality (6.18) is pathwise. If one considers an arbitrary fBm with Hurst parameter $H+\beta$ in the right-hand side of (6.18), we get equality in distribution.*

An immediate consequence of the above result is the following identity, which uses the isometry of the fractional Wiener integrals.

Corollary 5 *Assume* $H \in \left(\frac{1}{2}, 1\right), \beta > 0$ *and* $f, g \in L^1(\mathbb{R})$ *with* $\mathcal{I}_-^\beta f, \mathcal{I}_-^\beta g \in \mathcal{H}_H$*. Then*

$$(H+\beta)(2(H+\beta)-1)\int_{\mathbb{R}}\int_{\mathbb{R}} dudv f(u)g(v)|u-v|^{2(H+\beta)-2}$$
$$= A(H,\beta)^2\Gamma(\beta+1)^2\int_{\mathbb{R}}\int_{\mathbb{R}} dudv(\mathcal{I}_-^\beta f)(u)(\mathcal{I}_-^\beta g)(v)|u-v|^{2H-2}.$$

We will use Proposition 35 and Corollary 5 in order to define and to analyze the properties of the Wiener integral with respect to the generalized Hermite process $X^{(q,H,\beta)}$ and in particular to study the Ornstein-Uhlenbeck process associated to $X^{(q,H,\beta)}$.

Let us make a short discussion for the case $\beta < 0$. In this case the counterpart of the relation (6.16) is, if $\beta \in \left(-\frac{1}{2}, 1\right)$ (see Lemma 3.1 in [86])

$$g_t^\beta(u) = \Gamma(\beta+1)(\mathbf{D}_-^{-\beta}1_{[0,t]}(u)) \tag{6.22}$$

for any $t > 0$ and $u \in \mathbb{R}$. Therefore, the relation (6.11) can be written in this case ($\mathbf{D}$ being the Marchaud fractional derivative recalled in Section 6.4.1.2)

$$B_t^{H+\beta} = A(H,\beta)\Gamma(\beta+1)\int_{\mathbb{R}}(\mathbf{D}_-^{-\beta}1_{[0,t]}(u))dB_u^H.$$

It is possible to extend the above relation by replacing the indicator function by a regular enough function f in a bigger class as in Proposition 35 (for instance, if $\mathbf{D}_-^{-\beta}f$ exists and belongs to $\mathcal{H}_H$). On the other hand, these conditions are hardly satisfied by usual functions (in particular for the exponential function that appears in the expression of the Ornstein-Uhlenbeck process in Section 6.4). Therefore, for $\beta < 0$, we will construct the Wiener integral with respect to the generalized Hermite process via a pathwise approach instead of using Proposition 35 and Corollary 5.

6.3 Wiener integral with respect to the generalized Hermite process

We define here the generalized Wiener-Hermite integral. We start with a short discuss which motivates the definition of the integral and we give several properies of this Wiener integral.

6.3.1 The definition : the motivation

In order to motivate the definition of the Wiener integral with respect to the generalized Hermite process, let us start with a simple case. Consider f a step function written as

$$f(t) = \sum_{i=0}^{N-1}\lambda_i 1_{[t_i,t_{i+1})}(t) \tag{6.23}$$

with $\lambda_i \in \mathbb{R}, N \geq 1$ and $0 < t_1 < \ldots t_N$. For such a step function, it is natural to set

$$\int_{\mathbb{R}} f(u) dX_u^{(q,H,\beta)} = \sum_{i=0}^{N-1} \lambda_i \left(X_{t_{i+1}}^{(q,H,\beta)} - X_{t_i}^{(q,H,\beta)} \right). \tag{6.24}$$

We can also observe that the integral defined above defines an isometry between the set of step functions and $L^2(\Omega)$. Indeed,

$$\begin{aligned} \mathbf{E} \left(\int_{\mathbb{R}} f(u) dX_u^{(q,H,\beta)} \right)^2 &= \sum_{i,j=0}^{N-1} \lambda_i \lambda_j \mathbf{E} \left(X_{t_{i+1}}^{(q,H,\beta)} - X_{t_i}^{(q,H,\beta)} \right) \left(X_{t_{j+1}}^{(q,H,\beta)} - X_{t_j}^{(q,H,\beta)} \right) \\ &= \sum_{i,j=0}^{N-1} \lambda_i \lambda_j \left(R^{H+\beta}(t_{i+1}, t_{j+1}) - R^{H+\beta}(t_{i+1}, t_j) - R^{H+\beta}(t_i, t_{j+1}) + R^{H+\beta}(t_i, t_j) \right) \\ &= \langle f, f \rangle_{\mathcal{H}_{H+\beta}}. \end{aligned}$$

Note that the above definition holds for functions f defined on $\mathbb{R}_+$ onto $\mathbb{R}$. This is due to the fact that the generalized Hermite process (as the standard Hermite process too) is defined only for the time variable $t \in \mathbb{R}_+$. (If we work only with functions defined on $\mathbb{R}_+$, we can define the Wiener integral $\int_{\mathbb{R}} f(u) dX_u^{(q,H,\beta)}$ by isometry, using the fact that the step functions are dense in $\mathcal{H}_{H+\beta}$ on $\mathbb{R}_+$.)

In order to find a more explicit expression of the integral $\int_{\mathbb{R}} f(u) dX_u^{(q,H,\beta)}$, let us notice for f as above in (6.23), for $\beta > 0$,

$$\begin{aligned} & \int_{\mathbb{R}} f(u) dX_u^{(q,H,\beta)} \\ = \; & d(q,H,\beta) \sum_{i=0}^{N-1} \lambda_i \int_{\mathbb{R}^q} dB(y_1 \ldots dB(y_q) \int_{\mathbb{R}} (g_{t_{i+1}}^{\beta}(u) - (g_{t_i}^{\beta}(u)) \prod_{j=1}^{q} (u - y_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} du \\ = \; & d(q,H,\beta) \int_{\mathbb{R}^q} dB(y_1 \ldots dB(y_q) \int_{\mathbb{R}} du \left(\int_{\mathbb{R}} f(s) d_s g_s^{\beta}(u) \right) \prod_{j=1}^{q} (u - y_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \\ = \; & d(q,H,\beta) \int_{\mathbb{R}^q} dB(y_1 \ldots dB(y_q) \left(\int_{\mathbb{R}} du \beta \int_{\mathbb{R}} f(s)(s-u)_+^{\beta-1} ds \right) \prod_{j=1}^{q} (u - y_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \\ = \; & d(q,H,\beta) \beta \Gamma(\beta) \int_{\mathbb{R}^q} dB(y_1 \ldots dB(y_q) \int_{\mathbb{R}} du (\mathcal{I}_-^{\beta} f)(u) \prod_{j=1}^{q} (u - y_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)}. \end{aligned}$$

We used the definition of the fractional operator (6.45) and the notation $d_s g_s^{\beta}(u)$ means the integral with respect to the function $s \to g_s^{\beta}(u)$.

By the expression (6.7) of the Wiener integral with respect to the standard Hermite process, we can also write

$$\int_{\mathbb{R}} f(u) dX_u^{(q,H,\beta)} \quad = \quad \frac{d(q,H,\beta)}{d(q,H)} \Gamma(\beta+1) \int_{\mathbb{R}} (\mathcal{I}_-^{\beta} f)(u) dZ_u^{(q,H)}$$

$$= \quad A(H,\beta)\Gamma(\beta+1)\int_{\mathbb{R}}(\mathcal{I}_-^{\beta}f)(u)dZ_u^{(q,H)}$$

with $A(H,\beta), d(q,H), d(q,H,\beta)$ given by (6.12), (6.2), (6.14) respectively.

This motivates the definition of the Wiener integral with respect to the generalized Hermite process.

Definition 2 *Let $\beta > 0$ and $H \in \left(\frac{1}{2}, 1\right)$. For $f \in L^1(\mathbb{R})$ such that $\mathcal{I}_-^{\beta} f \in \mathcal{H}_H$ we define,*

$$\int_{\mathbb{R}} f(u)dX_u^{(q,H,\beta)} = A(H,\beta)\Gamma(\beta+1)\int_{\mathbb{R}}(\mathcal{I}_-^{\beta}f)(u)dZ_u^{(q,H)}. \tag{6.25}$$

where the right-hand side is a Wiener integral with respect to the (standard) Hermite process (see (6.7)) and $A(H,\beta)$ is given by (6.12).

By Proposition 35, the Wiener integral in the right-hand side of (6.25) is well-defined, since $f \in \mathcal{H}_{H+\beta}$. We will call the integral defined above *the generalized Wiener-Hermite integral.*

6.3.1.1 The isometry

This integral satisfies the following property.

Proposition 36 *Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfying the assumption in Proposition 35. Then we have*

$$\begin{aligned} &\mathbf{E}\int_{\mathbb{R}} f(u)dX_u^{(q,H,\beta)}\int_{\mathbb{R}} g(u)dX_u^{(q,H,\beta)} = \langle f,g\rangle_{\mathcal{H}_{H+\beta}} \\ = \quad &(H+\beta)(2(H+\beta)-1)\int_{\mathbb{R}}\int_{\mathbb{R}} dudv f(u)g(v)|u-v|^{2(H+\beta)-2}. \end{aligned}$$

Proof : Notice that $\beta > 0$ implies $H+\beta > 0$. We will Proposition 35 and Corollary 5 to get by the isometry of the Wiener-Hermite integral

$$\begin{aligned} &\mathbf{E}\int_{\mathbb{R}} f(u)dX_u^{(q,H,\beta)}\int_{\mathbb{R}} g(u)dX_u^{(q,H,\beta)} \\ = \quad &\left(A(H,\beta)\Gamma(\beta+1)\right)^2 H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} dudv(\mathcal{I}_-^{\beta}f)(u)(\mathcal{I}_-^{\beta}g)(v)|u-v|^{2H-2} \\ = \quad &(H+\beta)(2(H+\beta)-1)\int_{\mathbb{R}}\int_{\mathbb{R}} dudv f(u)g(v)|u-v|^{2(H+\beta)-2}. \end{aligned}$$

■

6.3.1.2 Spectral representation

Let $\widehat{W}$ be a complex-valued Gaussian random spectral measure that satisfies $\mathbf{E}\widehat{W}(A) = 0, \mathbf{E}\left[\widehat{W}(A)\overline{\widehat{W}(B)}\right] = \lambda(A\cap B), \widehat{W}(A) = \overline{\widehat{W}(-A)}$ and $\widehat{W}(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \widehat{W}(A_i)$, for disjoint Borel sets that have finite Lebesgue measure, denoted by λ. The real and imaginary parts of $\widehat{W}(A)$ are independent Gaussian random variables with mean 0 and variance $\frac{\lambda(A)}{2}$. Let $\overline{L}^2 := L^2(\mathbb{R}^q;\mathbb{C})$ be the set of complex-valued functions on $\mathbb{R}^q$ such that

$$g(-x_1,...,-x_q) = \overline{g(x_1,...,x_q)}$$

for every $x_i \in \mathbb{R}$, $i = 1,..q$ and

$$\|g\|^2_{\overline{L}^2} = \int_{\mathbb{R}^q} |g(x_1,...,x_q)|^2 dx_1...dx_q < \infty.$$

For $g \in L^2(\mathbb{R}^q;\mathbb{C})$, the multiple stochastic integral $\widehat{I}_q(g)$ with respect to the Gaussian measure $\widehat{W}$ is defined via (as the standard multiple stochastic integral, see the Appendix) an isometry between $L^2(\mathbb{R}^q;\mathbb{C})$ and $L^2(\Omega)$, i.e.

$$\mathbf{E}\left[\widehat{I}_p(f)\widehat{I}_q(g)\right] = \begin{cases} q! < \widetilde{f},\widetilde{g} >_{\overline{L}^2} & ,\text{if} \quad q = p. \\ 0 & ,\text{if} \quad q \neq p. \end{cases} . \tag{6.26}$$

The integral $\widehat{I}_q(g)$ will be called in the sequel *multiple stochastic integral of spectral type* or *of Fourier type*). We refer also to [74] for more details.

We have the following connection between the two types of multiple integral (see [97], Lemma 6.1) : if $f \in L^2(\mathbb{R}^q)$, then

$$I_q(f) \overset{(d)}{=} (2\pi)^{-\frac{q}{2}} \widehat{I}_q(\widehat{f}) \tag{6.27}$$

where $\overset{(d)}{=}$ means equality in distribution and $\widehat{f}$ denotes the Fourier transform of f, i.e.

$$\widehat{f}(\lambda) = \int_{\mathbb{R}} f(y)e^{i\lambda y}dy, \qquad \lambda \in \mathbb{R}.$$

The Wiener integral with respect to the standard Hermite process admits the following spectral representation (see [93]). Assume $f \in \mathcal{H}_H$. Then

$$\int_{\mathbb{R}^d} f(s)dZ_s^{(q,H)} \overset{(d)}{=} C_{0,q}\int_{\mathbb{R}}\dots\int_{\mathbb{R}} d\widehat{W}(z_1)\dots d\widehat{W}(z_q)\widehat{f}(z_1+\ldots+z_q)|z_1|^{-\frac{1}{2}+\frac{1-H}{q}}\dots|z_q|^{-\frac{1}{2}+\frac{1-H}{q}} \tag{6.28}$$

with

$$C_{0,q} = (2\pi)^{\frac{-q}{2}} d(q,H)\Gamma\left(\frac{1}{2} - \frac{1-H}{q}\right)^q . \tag{6.29}$$

From the above result we deduce immediately the spectral representation for the generalized Wiener-Hermite integral.

Proposition 37 *Let $X^{(q,H,\beta)}$ be a generalized Hermite process given by (6.3). Assume $\beta > 0$ and $f \in L^1(\mathbb{R})$ such that $\mathcal{I}^{\beta}_{-} f \in \mathcal{H}_H$. Then*

$$\int_{\mathbb{R}} f(u) dX_u^{(q,H,\beta)} \overset{(d)}{=} C_{0,q} A(H,\beta) \frac{\Gamma(\beta+1)}{\Gamma(\beta)} c_\beta$$
$$\times \int_{\mathbb{R}^q} \left(\int_{\mathbb{R}} \widehat{f}(z_1 + \ldots + z_q) |z_1 + \ldots z_q|^{-\beta} |z_1|^{-\frac{1}{2} + \frac{1-H}{q}} \ldots |z_q|^{-\frac{1}{2} + \frac{1-H}{q}} \right) d\widehat{W}(z_1) \ldots d\widehat{W}(z_q)$$

with $c_\beta = \int_0^\infty u^{\beta-1} e^{-i\lambda u} du$.

Proof : Let us compute the Fourier transform of the function $\mathcal{I}^{\beta}_{-} f$ where $f \in L^1(\mathbb{R})$ such that $\mathcal{I}^{\beta}_{-} f \in \mathcal{H}_H$. For every $\lambda \in \mathbb{R}$,

$$\begin{aligned} \left(\widehat{\mathcal{I}^{\beta}_{-} f} \right)(\lambda) &= \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}} du e^{i\lambda u} \left(\int_u^\infty ds f(s) (s-u)^{\beta-1} \right) \\ &= \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}} ds f(s) e^{i\lambda s} \int_0^\infty du e^{-i\lambda u} u^{\beta-1} du \\ &= \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}} ds f(s) e^{i\lambda s} c_\beta |\lambda|^{-\beta} = \frac{1}{\Gamma(\beta)} c_\beta |\lambda|^{-\beta} \widehat{f}(\lambda). \end{aligned}$$

The conclusion then follows from (6.28) and the definition of the generalized Hermite Wiener integral (6.25). ■

6.3.1.3 Wiener integral in the Riemann-Stieltjes sense

It is possible to give a pathwise definition of the Wiener-Hermite integral by using the Hölder regularity of the simple paths of the generalized Hermite process. The definition follows the steps from the fBm case in [38]. We consider a continuously differentiable function f defined on $[a,b]$ $(-\infty \le a < b < \infty)$ such that the Riemann integral

$$\int_a^b X_u^{(q,H,\beta)} f'(u) du \text{ exists} \tag{6.30}$$

and

$$\lim_{u \to a} f(u) X_u^{(q,H,\beta)} := L_a \text{ exists.} \tag{6.31}$$

Then the Riemann-Stieltjes integral of f with respect to $X^{(q,H,\beta)}$, denoted

$$\int_a^b f(u) d_{RS} X_u^{(q,H,\beta)},$$

exist and we have

$$\int_a^b f(u) d_{RS} X_u^{(q,H,\beta} = f(b) X_b^{(q,H,\beta)} - L_a - \int_a^b f'(u) X_u^{(q,H,\beta)} du. \tag{6.32}$$

This follows from Theorem 2.21 in [104].

Proposition 38 *Assume $\beta > 0$. Let $f \in L^1(\mathbb{R})$ with $\mathcal{I}_-^{\beta} f \in \mathcal{H}_H$ satisfying (6.30) and (6.31). Then*

$$\int_a^b f(u) dX_u^{(q,H,\beta)} = \int_a^b f(u) d_{RS} X_u^{(q,H,\beta)}. \tag{6.33}$$

Proof : By hypothesis, the integrals in both sides of (6.33) are well-defined and they clearly coincide if f is a simple function, their expression being given by (6.24). Then it suffices to approximate f in $\mathcal{H}_{H+\beta}$ as in the proof of Proposition 35. ■

6.4 Generalized Hermite Ornstein-Uhlenbeck process

We define the generalized Hermite Ornstein-Uhlenbeck process as the solution to the Langevin equation

$$dY_t = -\alpha(Y_t - m)dt + \sigma dX_t^{(q,H,\beta)}, \quad t \geq 0 \tag{6.34}$$

with initial value $Y_0 \in L^0(\mathbb{R})$ and $\alpha, \sigma > 0$, $m \in \mathbb{R}$. The above equation can be solved in the Riemann-Stieltjes sense and it admits the explicit solution

$$Y_t = Y_0 e^{-\alpha t} + m(1 - e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-s)} d_{RS} X_s^{(q,H,\beta)}.$$

When $Y_0 = \sigma \int_{-\infty}^0 e^{\alpha s} d_{RS} X_s^{(q,H,\beta)}$, then

$$Y_t = m(1 - e^{-\alpha t}) + \sigma \int_{-\infty}^t e^{-\alpha(t-s)} d_{RS} X_s^{(q,H,\beta)}. \tag{6.35}$$

The process $(Y_t)_{t \geq 0}$ given by (6.35) will be called the *generalized Hermite Ornstein-Uhlenbeck process.* For $\beta = 0$, this stochastic process has been considered in several works, see e.g. [58], [84], [94] or [99].

For every β satisfying (6.6), the above integral is well-defined in the Riemann - Stieltjes sense. This follows from Proposition A1 in [38] as in the case when the noise of the Langevin equation is the fBm with Hurst index $H \in (0, 1)$. Indeed, the proof only uses the fact that the trajectories of the driving noise are Hölder continuous of any order $\delta \in (0, H + \beta)$. It also follows from Proposition A1 in [38] that the process (6.35) verifies (6.34).

Let us notice that, when $\beta > 0$, the solution $(Y_t)_{t \geq 0}$ also admits the following Wiener-Hermite integral representation, for every $t \geq 0$

$$Y_t = m(1 - e^{-\alpha t}) + \sigma \int_{-\infty}^t e^{-\alpha(t-s)} dX_s^{(q,H,\beta)}. \tag{6.36}$$

Indeed, the Wiener integral $\int_{-\infty}^t e^{-\alpha(t-s)} dX_s^{(q,H,\beta)}$ is well-defined since for every $t \geq 0$, the function $f(u) = e^{\alpha u} 1_{(-\infty,t)}(u)$ belongs to $L^1(\mathbb{R})$, and by a standard argument, it also belongs to $\mathcal{H}_{H+\beta}$ because

$$\int_{-\infty}^{t}\int_{\infty} dudv f(u)f(v)|u-v|^{2(H+\beta)-2} = 2\int_{-\infty}^{t} due^{\alpha u}\int_{-\infty}^{u} dve^{\alpha v}(u-v)^{2(H+\beta)-2}$$
$$= 2\int_{0}^{\infty} dve^{-\alpha v}v^{2(H+\beta)-2}\int_{-\infty}^{t} due^{2\alpha u} = \frac{1}{\alpha}e^{2\alpha t}\int_{0}^{\infty} dve^{-\alpha v}v^{2(H+\beta)-2} < \infty.$$

Relation (6.20) also implies that $\mathcal{I}_{-}^{\beta} f \in \mathcal{H}_H$. Then, the representations (6.35) and (6.36) coincide due to Proposition 38.

Let us list some properties of the generalized Hermite Ornstein-Uhlenbeck process, which follows directly from [38].

Proposition 39 *Let $(Y_t)_{t\geq 0}$ be given by (6.35). Then*

1. *If $m = 0$, the process $(Y_t)_{t\geq 0}$ is stationary, i.e. for every $h > 0$, the processes $(Y_t)_{t\geq 0}$ et $(Y_{t+h})_{t\geq 0}$ have the same finite-dimensional distributions.*
2. *For fixed $t > 0$, $Cov(Y_{t+h}, Y_t)$ behaves, when $h \to \infty$, as $h^{2(H+\beta)}$.*
3. *The sample paths of $(Y_t)_{t\geq 0}$ are Hölder continuous of order δ for every $\delta \in (0, H+\beta)$.*

Proof : Point 1. is an immediate consequence of the stationarity of the increments of $X^{(q,H,\beta)}$ since for $t, h > 0$,

$$\begin{aligned} Y_{t+h} &= \sigma\int_{-\infty}^{t+h} e^{\alpha(t+h-s)} d_{RS}X_s^{(q,H,\beta)} = \sigma\int_{-\infty}^{t} e^{-\alpha(t-s)} d_{RS}X_{s-h}^{(q,H,\beta)} \\ &\equiv^{(d)} \sigma\int_{-\infty}^{t} e^{-\alpha(t-s)} d_{RS}X_s^{(q,H,\beta)} = Y_t \end{aligned}$$

where $\equiv^{(d)}$ stands for the equivalence of finite -dimensional distributions. Point 2. is proved in Theorem 2.3 in [38] while point 3. can be deduced easily from (6.34). ■

In the case when the driving noise in (6.34) is the fractional Brownian motion (i.e. $\beta = 0$ and $q = 1$), the corresponding Ornstein-Uhlenbeck process is taken as a proxi for the log-volatility is financial applications. It is showed in [57] that when the drift parameter α in (6.34) is close to zero, then the fractional Ornstein -Uhlenbeck process behaves as the fBm. The next property shows that, when α is small, the generalized Hermite Ornstein-Uhlenbeck process also behaves as the driving noise of the Langevin equation (6.34).

Proposition 40 *For every $T > 0$ and $p \geq 1$,*

$$\sup_{t\in[0,T]} \mathbf{E}\left|Y_t - Y_0 - \sigma X_t^{(q,H,\beta)}\right|^p \to_{\alpha\to 0} 0. \tag{6.37}$$

Proof : We have by (6.35),

$$Y_t - Y_0 = m(1-e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-s)} d_{RS} X_s^{(q,H,\beta)}$$

and via the integration by parts formula (6.32) we get

$$Y_t - Y_0 = m(1-e^{-\alpha t}) - \alpha\sigma \int_0^t e^{-\alpha(t-s)} X_s^{(q,H,\beta)} ds + \sigma X_s^{(q,H,\beta)} \tag{6.38}$$

Now, for $p \geq 1$,

$$\mathbf{E}\left|Y_t - Y_0 - \sigma X_t^{(q,H,\beta)}\right|^p \leq C_p \left|m(1-e^{-\alpha t})\right|^p + C_p \alpha\sigma \mathbf{E}\left|\int_0^t e^{-\alpha(t-s)} X_s^{(q,H,\beta)} ds\right|^p.$$

Clearly for every $t \in [0,T]$, $|(1-e^{-\alpha t})|^p \leq \left|(1-e^{-\alpha T})\right|^p \to 0$ as $\alpha \to 0$ and notice that the random variable $\int_0^t e^{-\alpha(t-u)} X_u^{(q,H,\beta)} du$ belongs to the qth Wiener chaos. Since for every $p \geq 1$ we have by hypercontractivity (relation (6.44)

$$\mathbf{E}\left|\int_0^t e^{-\alpha(t-s)} X_s^{(q,H,\beta)} ds\right|^p \leq C_p \left[\mathbf{E}\left|\int_0^t e^{-\alpha(t-s)} X_s^{(q,H,\beta)} ds\right|^2\right]^{\frac{p}{2}}$$

to obtain (6.37) it suffices to show that

$$\sup_{t\in[0,T]} \mathbf{E}\left|\int_0^t e^{-\alpha(t-s)} X_s^{(q,H,\beta)} ds\right|^2 < C_T. \tag{6.39}$$

Now,

$$\begin{aligned}
\mathbf{E}\left|\int_0^t e^{-\alpha(t-s)} X_s^{(q,H,\beta)} ds\right|^2 &= \int_0^t \int_0^t dudv e^{-\alpha(t-u)} e^{-\alpha(t-v)} \mathbf{E} X_u^{(q,H,\beta)} X_v^{(q,H,\beta)} \\
&= \int_0^t \int_0^t dudv e^{-\alpha(t-u)} e^{-\alpha(t-v)} R^{H+\beta}(u,v) \leq \int_0^T \int_0^T dudv e^{-\alpha u} e^{-\alpha v} R^{H+\beta}(t+u,t+v) \\
&\leq 3T^{2H} \int_0^T \int_0^T dudv e^{-\alpha u} e^{-\alpha v} = C_T.
\end{aligned}$$

■

The GROU process also keeps (asymptotically when the drift parameter α tends to zero) the scaling property of the generalized Hermite process $X^{(q,H,\beta)}$.

Corollary 6 *For every $t > 0, 0 < \Delta < T$ and $p \geq 1$,*

$$\mathbf{E}\left|Y_{t+\Delta} - Y_t\right|^p \to_{\alpha\to 0} \sigma^p \mathbf{E}|X_1^{(q,H,\beta)}|^p \Delta^{p(H+\beta)}.$$

Proof : Fix $t > 0, 0 < \Delta < T$. By relation (6.38), we can write

$$\begin{aligned} Y_{t+\Delta} - Y_t &= -me^{-\alpha t}(1 - e^{\alpha\Delta}) - \alpha\sigma \int_0^t \left(e^{-\alpha(t-s+\Delta)} - e^{-\alpha(t-s)}\right) X_s^{(q,H,\beta)} ds \\ & -\alpha\sigma \int_t^{t+\Delta} e^{-\alpha(t-s+\Delta)} X_s^{(q,H,\beta)} ds + \sigma(X_{t+\Delta}^{(q,H,\beta)} - X_t^{(q,H,\beta)}). \end{aligned}$$

For every $p \geq 1$, $|me^{-\alpha t}(1 - e^{\alpha\Delta})|^p$ converges to zero as α tends to zero. Now we prove that show that

$$\sup_{t\in[0,T]} \mathbf{E} \left| \int_0^t \left(e^{-\alpha(t-s+\Delta)} - e^{-\alpha(t-s)}\right) X_s^{(q,H,\beta)} ds \right|^p < C_T$$

and

$$\sup_{t\in[0,T]} \mathbf{E} \left| \int_t^{t+\Delta} e^{-\alpha(t-s+\Delta)} X_s^{(q,H,\beta)} ds \right|^p < C_T.$$

This follows easily as in the proof of Proposition 40 and by (6.44), since

$$\begin{aligned} & \mathbf{E} \left| \int_0^t \left(e^{-\alpha(t-s+\Delta)} - e^{-\alpha(t-s)}\right) X_s^{(q,H,\beta)} ds \right|^2 \\ = & \int_0^t \int_0^t dudv \left(e^{-\alpha(t-u+\Delta)} - e^{-\alpha(t-u)}\right) \left(e^{-\alpha(t-v+\Delta)} - e^{-\alpha(t-v)}\right) \mathbf{E} X_u^{(q,H,\beta)} X_v^{(q,H,\beta)} \\ = & \int_0^t \int_0^t dudv \left(e^{-\alpha(t-u+\Delta)} - e^{-\alpha(t-u)}\right) \left(e^{-\alpha(t-v+\Delta)} - e^{-\alpha(t-v)}\right) R_{H+\beta}(u, v) \\ \leq & \int_0^T \int_0^T dudv (e^{-\alpha(u+\Delta)} - e^{-\alpha u})(e^{-\alpha(v+\Delta)} - e^{-\alpha v}) R_{H+\beta}(t-u, t-v) \leq C_T \qquad (6.40) \end{aligned}$$

and

$$\begin{aligned} & \mathbf{E} \left| \int_t^{t+\Delta} e^{-\alpha(t-s+\Delta)} X_s^{(q,H,\beta)} ds \right|^2 = \int_t^{t+\Delta} \int_t^{t+\Delta} e^{-\alpha(t-u+\Delta)} e^{-\alpha(t-v+\Delta)} R^{H+\beta}(u, v) \\ = & \int_0^\Delta \int_0^\Delta dudv e^{-\alpha(\Delta-u)} e^{-\alpha(\Delta-v)} R^{H+\beta}(t+u, t+v) \leq C_T. \qquad (6.41) \end{aligned}$$

The estimates (6.40) and (6.41) indicate that for t, Δ as above, the random variable $Y_{t+\Delta} - T_t$ converges in $L^p(\Omega)$ (and so in law) to $\sigma(X_{t+\Delta}^{(q,H,\beta)} - X_t^{(q,H,\beta)})$. Since this variable belongs to the qth Wiener chaos we have directly the convergence of moments and to finish we use the scaling property of $X^{(q,H,\beta)}$ ■

6.4.1 Appendix

We recall here the basic definition for multiple stochastic integrals and for the fractional integral and derivative.

6.4.1.1 Multiple Wiener-Iô integrals

Here, we shall only recall some elementary facts ; our main reference is [77]. Consider $\mathcal{H}$ a real separable infinite-dimensional Hilbert space with its associated inner product $\langle .,. \rangle_{\mathcal{H}}$, and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}}$, for every $\varphi, \psi \in \mathcal{H}$. Denote by I_q the qth multiple stochastic integral with respect to B. This I_q is actually an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}}\|\cdot\|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H},\ \|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by :

$$H_q(x) = (-1)^q \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^q}{\mathrm{d}x^q}\left(\exp\left(-\frac{x^2}{2}\right)\right),\ x \in \mathbb{R}. \tag{6.42}$$

The isometry of multiple integrals can be written as : for $p,\ q \geq 1$, $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$,

$$\mathbf{E}\Big(I_p(f)I_q(g)\Big) = \begin{cases} q!\langle \tilde{f}, \tilde{g}\rangle_{\mathcal{H}^{\otimes q}} & \text{if } p = q \\ 0 & \text{otherwise} \end{cases} \tag{6.43}$$

where $\tilde{f}$ denotes the canonical symmetrization of f and it is defined by :

$$\tilde{f}(x_1, \ldots, x_q) = \frac{1}{q!}\sum_{\sigma \in \mathcal{S}_q} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)}),$$

in which the sum runs over all permutations σ of $\{1, \ldots, q\}$. It also holds that :

$$I_q(f) = I_q(\tilde{f}).$$

We need to recall the hypercontractivity propery of random variables in Wiener chaos. If $F = \sum_{k=0}^{n} I_k(f_k)$ with $f_k \in \mathcal{H}^{\otimes k}$ then

$$\mathbf{E}|F|^p \leq C_p\left(\mathbf{E}F^2\right)^{\frac{p}{2}}. \tag{6.44}$$

for every $p \geq 2$.

6.4.1.2 Fractional calculus

If $f : \mathbb{R} \to \mathbb{R}$ belongs to $L^1(\mathbb{R})$, we define the right-sided Riemann-Liouville fractional integral operator of order $\alpha > 0$ by

$$\left(\mathcal{I}_{-}^{\alpha} f\right)(s) = \frac{1}{\Gamma(\alpha)}\int_s^{\infty} f(u)(u-s)^{\alpha-1}du \text{ for } s \in \mathbb{R}. \tag{6.45}$$

The above fractional integral is well-defined for $f \in L^1(\mathbb{R})$, see e.g. [65].

If $0 < \alpha < 1$, we define the right-sided Marchaud fractional derivative operator of $f : \mathbb{R} \to \mathbb{R}$ by

$$\left(\mathbf{D}_{-}^{\alpha} f\right)(s) = \lim_{\varepsilon \searrow 0} (\mathbf{D}_{-,\varepsilon}^{\alpha} f)(s), \qquad s \in \mathbb{R} \tag{6.46}$$

where

$$(\mathbf{D}_{-,\varepsilon}^{\alpha} f)(s) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{\varepsilon}^{\infty} (f(s) - f(u+s)) u^{-\alpha-1} du.$$

Bibliographie

[1] D. Khoshnevisan : Analysis of Stochastic Partial Differentiel Equations. CBMS, Number 119.

[2] A gentle introduction to SPDEs : the random field approach by Raluca M. Balan `https://arxiv.org/abs/1812.02812`

[3] Yuliva Mishura : Sotchasti Calculus for Fractional Brownian Motion and Related Processes. Lecture Notes in Mathematics. Springer. Number 1929.

[4] Ivan Nourdin : Selected Aspects of Fractional Brownian Motion. Bocconi and Springer Series.

[5] Vladas Pipiras and Murad S. Taqqu : Long-Range Dependence and Self-Similarity. Cambridge Series in Statistical and Probabilitis Mathematics.

[6] Markus Kunze : An Introduction to Malliavin Calculus. Lecture Notes. 2013.

[7] J. L. Doob : The Brownian Movement and Stochastic Equations. Annals of Mathematics. Second Series, Vol. 43, No. 2 (Apr., 1942), pp. 351-369 (19 pages).

[8] Patrick Cheridito et Hideyuki Kawaguchi : Fractional Ornstein-Uhlenbeck

[9] The hyperbolic Anderson model : Moment estimates of the Malliavin derivatives and applications (with David Nualart, Lluis Quer-Sardanyons and Guangqu Zheng), e-print : arXiv2101.10957

[10] Ivan Nourdin et Giovanni Peccati : Normal Approximations with Malliavin Calculus : From Stein's Method to Universality. Cambridge Tracts in Mathematics.

[11] Makoto Maejima1 et Ciprian A. Tudor : Wiener integrals with respect to the Hermite process and a Non-Central Limit Theorem.

[12] Huang J., Nualart D. and Viitasaari L. (2020) : A central limit theorem for the stochastic heat equation. Stochastic Processes and Their Applications. 130, Issue 12, 7170-7184.

[13] J. Huang, D. Nualart, L. Viitasaari and G. Zheng (2020) : Gaussian fluctuations for the stochastic heat equation with colored noise. Stochastics and Partial Differential Equations : Analysis and Computations volume 8, pages 402–421

[14] M. Kwasnicki : Ten equivalent definitions of the fractional Laplace operator

[15] Le Chen, Jingyu Huang "Comparison principle for stochastic heat equation on Rd," The Annals of Probability, Ann. Probab. 47(2), 989-1035, (March 2019)

[16] Hélène Halconruy : Calcul de Malliavin et structures de Dirichlet pour des variables aléatoires indépendantes `https://www.theses.fr/2020IPPAT016`

[17] Laurent Decreusefond : STOCHASTIC INTEGRATION WITH RESPECT TO FRACTIONAL BROWNIAN MOTION

[18] P. Abry, P. Flandrin, M. S. Taqqu and D. Veitch (2003) : *Self-similarity and long range dependence through the wavelet lens.* In P. Doukhan, G. Oppenheim, M.S. Taqqu editors, Long-range dependence : Theory and Applications, Birhäuser.

[19] M. Alfeus and C. Nikitopoulos (2020) : Forecasting commodity markets volatility : HAR or Rough ? Preprint.

[20] H. Araya and C. A. Tudor (2021) : *Asymptotic expansion for the quadratic variations of the solution to the heat equation with additive white noise.* Stoch. Dyn. 21 (2), 2150010, 23 pp.

[21] R. Balan and C. A. Tudor (2008) : *The stochastic heat equation with fractional-colored noise : existence of the solution.* ALEA Lat. Am. J. Probab. Math. Stat. 4, 57-87.

[22] R. Balan and C. A. Tudor (2010) : *Stochastic heat equation with multiplicative fractional-colored noise.* J. Theoret. Probab. 23 (3), 834-870.

[23] R.M. Balan, C.A. Tudor (2010) : *The stochastic wave equation with Fractional colored noise : a random field approach.* Stoch. Process. Appl. 120, 2468-2494.

[24] S. Bai and M. Taqqu (2014) : *Generalized Hermite processes, discrete chaos and limit theorems.* Stochastic Process. Appl. 124 (4), 1710-1739.

[25] J-M. Bardet (2002) : *Statistical study of the wavelet analysis of fractional Brownian motion.* IEEE Trans. Inform. Theory 48, 991-999.

[26] J-M. Bardet, G. Lang, G. Oppenheim, Georges, A. Philippe, S. Stoev and M. S.Taqqu, Murad (2003) : *Semi-parametric estimation of the long-range dependence parameter : a survey.* Theory and applications of long-range dependence, 557-577, Birkhäuser Boston, Boston, MA.

[27] J-M. Bardet et C. A. Tudor (2010) : *A wavelet analysis of the Rosenblatt process : chaos expansion and estimation of the self-similarity parameter.* Stochastic Process. Appl. 120 (120), 2331-2362.

[28] L. Bertini, N. Cancrini, C. Jona-Lasinio (1994) : *The stochastic Burgers equation.* Comm. Math. Phys. 165 (2), 211-232.

[29] V. Bogachev. *Measure Theory.* Springer-Verlag, Berlin, 2007.

[30] S. Bonaccorsi and C. A. Tudor (2011) : *Dissipative stochastic evolution equations driven by general Gaussian and non-Gaussian noise.* J. Dynam. Differential Equations 23(4), 791-816.

[31] P. Breuer and P. Major (1983) : *Central limit theorem for non-linear functionals of Gaussian fields.* J. Mult. Anal. 13, 425-441.

[32] J. M. Burgers (1974) : The nonlinear Diffusion Equation. Reidel, Dordrecht.

[33] L. Chen and R. Dalang. Moments, intermittency and growth indices for nonlinear stochastic fractional heat equation. *Stoch. Partial Differ. Equ. Anal. Comput.*, 3(3) : 360-397, 2015.

[34] L. Chen and J. Huang. Comparison principle for stochastic heat equation on $\mathbb{R}^d$. *Ann. Probab.*, 47(2) : 989-1035, 2018.

[35] L. Chen, Y. Hu and D. Nualart. Regularity and strict positivity of densities for the nonlinear stochastic heat equation. To appear in : *Mem. Amer. Math. Soc.*, 2018.

[36] L. Chen and K. Kunwoo. Nonlinear stochastic heat equation driven by spatially colored noise : moments and intermittency. *Acta Math. Sci.*, 39 : 645-668, 2019.

[37] L. Chen, D. Khoshnevisan, F. Pu and D. Nualart Spatial ergodicity for SPDEs via a Poincaré-type inequalities `https://arxiv.org/abs/1907.11553`

[38] P. Cheridito, H. Kawaguchi and M. Maejima (2003) : *Fractional Ornstein-Uhlenbeck processes.* Electronic Journal of Probability 8, 1-14.

[39] A. Chronopoulou, C.A. Tudor and F. Viens (2011) : *Self-similarity parameter estimation and reproduction property for non-Gaussian Hermite processes.* Commun. Stoch. Anal. 5 (1), 161-185.

[40] I. Cialenco (2018) : *Statistical inference for SPDEs : an overview.* Stat. Inference Stoch. Process. 21, no. 2, 309-329.

[41] P. Coupek (2018) : *Limiting measure and stationarity of solutions to stochastic evolution equations with Volterra noise.* Stoch. Anal. Appl. 36, no. 3, 393-412.

[42] P. Coupek and B. Maslowski (2017) : *Stochastic evolution equations with Volterra noise.* Stoch. Proc. Appl., 127, 877-900.

[43] P. Coupek, B. Maslowski and M. Ondrejat (2018) : *Lp-valued stochastic convolution integral driven by Volterra noise.* Stochastics and Dynamics, 18 (6), 1850048.

[44] R. Dalang. Extending the martingale measure stochastic integral with applications to spatially homogeneous S.P.D.E.'s. *Electron. J. Probab.*, 4(6), 29 pp., 1999.

[45] R. C. Dalang and M. Sanz-Solé (2009) : *Hölder-Sobolev regularity of the solution to the stochastic wave equation in dimension three.* Memoirs of the American Mathematical Society 199(931), 1-70.

[46] L. Debbi and M. Dozzi. On the solutions of nonlinear stochastic fractional partial differential equations in one spatial dimension. *Stoch. Proc. Appl.*, 115 : 1761-1781, 2005.

[47] J. Clarke De la Cerda, C.A. Tudor (2014) : *Hitting probabilities for the stochastic wave equation with fractional colored noise.* Rev. Mat. Iberoam. 30 (2), 685-709.

[48] A. Dermoune (1994) : *Around the stochastic Burgers equation.* Stochastic Anal. Appl. 15 (3), 295-311.

[49] G. Da Prato, A. Debussche and R. Temam (1994) : *Stochastic Burgers equation.* NoDEA Nonlinear Differential Equations Appl. 1(4), 389-402.

[50] F. Delgado-Vences, D. Nualart and G. Zheng. A central limit theorem for the stochastic wave equation with fractional noise. *Ann. Inst. H. Poincaré Probab. Statist.*, 56(4) : 3032-3042, 2020.

[51] P. Flandrin (1992) : *Wavelet analysis and synthesis of fractional Brownian motion.* IEEE Trans. Inform. Theory, 38, 910-917.

[52] M. Foondun and D. Khoshnevisan. On the stochastic heat equation with spatially-colored random forcing. *Trans. Amer. Math. Soc.*, 365 : 409-458, 2013.

[53] J. Gatheral (2006) : The volatility surface : A practitioner's guide, volume 357. John Wiley and Sons, 2006.

[54] J. Gairing, P. Imkeller, R. Shevchebko and C. A. Tudor (2019) : *Hurst index estimation in stochastic differential equations driven by fractional Brownian motion.* To appear in *Journal of Theoretical Probability.*

[55] N. Garofalo. *Fractional thoughts.* In : New Developments in the Analysis of Nonlocal Operators, Contemp. Math., 723 : 1-135, Amer. Math. Soc., Providence, RI, 2019.

[56] M. Gubinelli, P. Imkeller and N. Perkowski (2016) : *A Fourier analytic approach to pathwise stochastic integration.* Electron. J. Probab. 21, Paper No. 2, 37 pp.

[57] J. Gatheral, T. Jaisson and M. Rosenbaum (2018) : *Volatility is rough.* Quantitative Finance 18 (6).

[58] V. Garino, I. Nourdin, D. Nualart and M. Salamat (2020) : *Limit theorems for integral functionals of Hermite-driven processes.* Preprint.

[59] B. Gaveau and P. Trauber. L'intégrale stochastique comme opérateur de divergence dans l'espace founctionnel. *J. Funct. Anal.*, 46 : 230-238, 1982.

[60] I. Gyongy (1998) : *Existence and uniqueness results for semilinear stochastic partial differential equations.* Stochastic Process. Appl. 73 (2), 271-299.

[61] I. Gyongy, and D. Nualart (1999) : *On the stochastic Burgers' equation in the real line.* Ann. Probab. 27 (2)2, 782-802.

[62] B. Horvath, A. Jacquier and P. Tankov (2020) : *Volatility options in rough volatility models.* SIAM J. Financial Math. 11 (2), 437–469.

[63] J. Huang, D. Nualart and L. Viitasaari. A central limit theorem for the stochastic heat equation. *Stochastic Process. Appl.*, 130 : 7170-7184, 2020.

[64] J. Huang, D. Nualart, L. Viitasaari and G. Zheng. Gaussian fluctuations for the stochastic heat equation with colored noise. *Stoch. Partial Differ. Equ. Anal. Comput.*, 8 : 402-421, 2020.

[65] C. Jost (2006) : *Transformation formulas for fractional Brownian motion.* Stochastic Process. Appl. 116, 1341-1357.

[66] M. Khalil and C. A. Tudor (2018) : *Correlation structure, quadratic variations and parameter estimation for the solution to the wave equation with fractional noise.* Electronic Journal of Statistics, 12 (2), 3639-3672.

[67] M. Kleptsyna and A. Le Breton (2002) : *Statistical analysis of the fractional Ornstein-Uhlenbeck type process.* Statistical Inference for Stochastic Processes, 5, 229-241.

[68] T. Komatsu. On the martingale problem for generators of stable processes with perturbations. *Osaka. J. Math.*, 21 : 113-132, 1984.

[69] Yu. A. Kutoyants (2004) : Statistical Inference for Ergodic Diffusion Processes. Springer Series in Statistics. Springer.

[70] J.A Leon, D Nualart and R Pettersson (2000 : *The stochastic Burgers equation : finite moments and smoothness of the density.* Infinite Dimensional Analysis, Quantum Probability and Related Topics, 3, 363-385.

[71] P. Lewis and D. Nualart (2018) : *Stochastic Burgers' equation on the real line : regularity and moment estimates.* Stochastics 90, 1053-1086.

[72] E.H. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math.*, 118 : 349-374, 1983.

[73] M. Maejima and C. A. Tudor (2007) : *Wiener Integrals with respect to the Hermite process and a Non-Central Limit Theorem.* Stoch. Anal. Appl. 25 (5), 1043-1056.

[74] P. Major (1981) : *Multiple Wiener-Itô integrals.* Lecture Notes in Mathematics, 849, Springer, Berlin.

[75] Y. Meyer, F. Sellan, M.S. Taqqu (2008) : *Wavelets, generalized white noise and fractional integration : the synthesis of fractional Brownian motion.* J. Fourier Anal. Appl. 5, 465-494.

[76] P-L. Morien (1999) : *On the density for the solution of a Burgers-type SPDE.* Ann. Inst. H. Poincaré Probab. Statist. 35 (4), 459-482.

[77] D. Nualart (2006) : Malliavin Calculus and Related Topics. Second Edition. Springer.

[78] D. Nualart and E. Nualart. *Introduction to Malliavin Calculus.* IMS Textbooks, Cambridge University Press, 2018.

[79] Nualart, D., Quer-Sardanyons, L. Existence and Smoothness of the Density for Spatially Homogeneous SPDEs. Potential Anal 27, 281–299 (2007). `https://doi.org/10.1007/s11118-007-9055-3`

[80] I. Nourdin and G. Peccati. Stein's method on Wiener chaos. *Probab. Theory Rel.*, 145(1) :75-118, 2009.

[81] I. Nourdin and G. Peccati (2012) : Normal Approximations with Malliavin Calculus From Stein's Method to Universality. Cambridge University Press.

[82] D. Nualart and E. Pardoux. Stochastic calculus with anticipating integrands. *Probab. Theory Related Fields*, 78 : 535-581, 1988.

[83] D. Nualart and G. Peccati (2005) : *Central limit theorems for sequences of multiple stochastic integrals.* Ann. Probab. 33, 177-193.

[84] I. Nourdin and T. T. Diu Tran (2019) : *Statistical inference for Vasicek-type model driven by Hermite processes.* Stochastic Process. Appl. 129 (10), 3774-3791.

[85] G. Peccati and C.A. Tudor (2004) : *Gaussian limits for vector-valued multiple stochastic integrals.* Séminaire de Probabilités, XXXIV, 247-262.

[86] V. Pipiras and M. S. Taqqu (2000) : *Integration questions related to fractional Brownian motion.* Probab. Theory Relat. Fields, 118, 251-291.

[87] V. Pipiras and M. Taqqu (2017) : Long-range dependence and self-similarity. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press.

[88] J. Pospisil and R. Tribe (2007) : *Parameter estimates and exact variations for stochastic heat equation s driven by space-time white noise.* Stoch. Anal. Appl. 25(3), 593–611.

[89] B.L. S. Praksa Rao (1999) : Semimartingales and their Statistical Inference. Chapman and Hall.

[90] B. L. S. Prakasa Rao (2010) : Statistical inference for fractional diffusion processes. Wiley Series in Probability and Statistics, Chichester, John Wiley & Sons.

[91] G. Samorodnitsky (2016) : Stochastic processes and long range dependence. Springer Series in Operations Research and Financial Engineering. Springer, Cham.

[92] R. Shevchenko, M. Slaoui and C. A. Tudor (2020) : *Generalized k-variations and Hurst parameter estimation for the fractional wave equation via Malliavin calculus.* J. Statist. Plann. Inference 207, 155-180.

[93] M. Slaoui and C. A. Tudor (2020) : *The linear stochastic heat equation with Hermite noise.* Infin. Dimens. Anal. Quantum Probab. Relat. Top. 22(3), 1950022, 23 pp.

[94] M. Slaoui and C. A. Tudor (2019) : *Behavior with respect to the Hurst index of the Wiener Hermite integrals and application to SPDEs.* J. Math. Anal. Appl. 479 (1), 350-383.

[95] M. Slaoui and C. A. Tudor (2019) : *On the linear stochastic heat equation with Hermite noise.* To appear in *Infinite Dimensional Analysis and Quantum Probability.*

[96] T. Takaishi (2020) : *Rough volatility of Bitcoin.* Finance Research Letters, 32, 101379.

[97] M. S. Taqqu (1979) : *Convergence of integrated processes of arbitrary Hermite rank.* Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 50, 53-83.

[98] S. Torres, C.A. Tudor and F. Viens (2014) : *Quadratic variations for the fractional-colored stochastic heat equation.* Electronic Journal of Probability, 19, paper 76, 51 pages.

[99] T. T. Diu Tran (2018) : *Non-central limit theorems for quadratic functionals of Hermite-driven long memory moving average processes.* Stoch. Dyn. 18 (4), 1850028, 18 pp.

[100] C. A. Tudor (2013) : Analysis of variations for self-similar processes. A stochastic calculus approach. Probability and its Applications (New York). Springer, Cham.

[101] W. Ma (2017) : *Heavy -tailed distributions, Volatility Clustering and Asset Price of the Precious Metal.* International Journal of Academic Research in Business and Social Sciences, 7(7), 686-692

[102] C. A. Tudor and F. G. Viens (2007). *Statistical aspects of the fractional stochastic calculus*, Ann. Statist. 35 (3) 1183-1212.

[103] C.A. Tudor and F. Viens (2008) : *Variations and estimators through Malliavin calculus.* Ann. Probab. 37 (6), 2093-2134.

[104] R. L. Wheeden and A. Zygmund (1977) : Measure and Integral. Marcel Dekker, New York-Basel.

[105] D. Nualart and H. Zhou. Total variation estimates in the Breuer-Major theorem. To appear in *Ann. Inst. H. Poincaré Probab. Statist.*, 2018.

[106] J. B. Walsh. *An introduction to stochastic partial differential equations.* In : École d'Été de Probabilités de Saint-Flour, XIV—1984, 265-439. Lecture Notes in Math. 1180, Springer, Berlin, 1986.