Laboratoire Paul Painlevé

Université de Lille

Université de Lille

École doctorale **MADIS-631**

Unité de recherche **Laboratoire Paul Painlevé, UMR CNRS 8524**

Thèse présentée par **Justine Fasquel**

Soutenue le **12 juillet 2022**

En vue de l'obtention du grade de docteur de l'Université de Lille

Discipline **Mathématiques pures et appliquées**

Geometry and new rational $\mathcal{W}$-algebras
Géométrie et nouvelles $\mathcal{W}$-algèbres rationnelles

Thèse dirigée par	Anne Moreau	directrice
	Tomoyuki Arakawa	co-directeur

Composition du jury

Rapporteurs	Dražen Adamović	University of Zagreb	
	Reimundo Heluani	IMPA	
Examinateurs	David Hernandez	Université de Paris	président du jury
	Dimitri Markouchevitch	Université de Lille	
	Michela Varagnolo	Université Cergy Paris	
	Huafeng Zhang	Université de Lille	
Invité	David Ridout	University of Melbourne	
Directeurs de thèse	Anne Moreau	Université Paris-Saclay	
	Tomoyuki Arakawa	RIMS, Kyoto University	

Laboratoire Paul Painlevé

Université de Lille

Université de Lille

École doctorale **MADIS-631**

Unité de recherche **Laboratoire Paul Painlevé, UMR CNRS 8524**

Thèse présentée par **Justine Fasquel**

Soutenue le **12 juillet 2022**

En vue de l'obtention du grade de docteur de l'Université de Lille

Discipline **Mathématiques pures et appliquées**

Geometry and new rational $\mathcal{W}$-algebras

Géométrie et nouvelles $\mathcal{W}$-algèbres rationnelles

Thèse dirigée par

Anne Moreau	directrice
Tomoyuki Arakawa	co-directeur

Composition du jury

Rapporteurs	Dražen Adamović	University of Zagreb	
	Reimundo Heluani	IMPA	
Examinateurs	David Hernandez	Université de Paris	président du jury
	Dimitri Markouchevitch	Université de Lille	
	Michela Varagnolo	Université Cergy Paris	
	Huafeng Zhang	Université de Lille	
Invité	David Ridout	University of Melbourne	
Directeurs de thèse	Anne Moreau	Université Paris-Saclay	
	Tomoyuki Arakawa	RIMS, Kyoto University	

Université de Lille

Doctoral School **MADIS-631**

University Department **Laboratoire Paul Painlevé, UMR CNRS 8524**

Thesis defended by **Justine Fasquel**

Defended on **12th July, 2022**

In order to become Doctor from Université de Lille

Academic Field **Pure Mathematics Applied Mathematics**

Geometry and new rational $\mathcal{W}$-algebras

Géométrie et nouvelles $\mathcal{W}$-algèbres rationnelles

Thesis supervised by	Anne Moreau	Supervisor
	Tomoyuki Arakawa	Co-Supervisor

Committee members

Referees	Dražen Adamović	University of Zagreb	
	Reimundo Heluani	IMPA	
Examiners	David Hernandez	Université de Paris	Committee President
	Dimitri Markouchevitch	Université de Lille	
	Michela Varagnolo	Université Cergy Paris	
	Huafeng Zhang	Université de Lille	
Guest	David Ridout	University of Melbourne	
Supervisors	Anne Moreau	Université Paris-Saclay	
	Tomoyuki Arakawa	RIMS, Kyoto University	

Keywords: vertex algebras, $\mathcal{W}$-algebras, rationality, representation theory, Lie algebras, nilpotent orbits

Mots clés : algèbres vertex, $\mathcal{W}$-algèbres, rationalité, théorie des représentations, algèbres de Lie, orbites nilpotentes

This thesis has been prepared at

Laboratoire Paul Painlevé, UMR CNRS 8524

Université de Lille
Cité Scientifique - Bâtiment M2
F-59655 VILLENEUVE d'ASCQ
France

Web Site https://math.univ-lille.fr/

Geometry and new rational $\mathcal{W}$-algebras

Abstract

Affine $\mathcal{W}$-algebras form a rich one-parameter family of vertex algebras associated with nilpotent elements of simple Lie algebras. These complex algebraic structures appear in several areas of physic and mathematics. Because of their recent construction, numerous aspects of the theory of $\mathcal{W}$-algebras remain unknown.
In this thesis, we study $\mathcal{W}$-algebras associated with nilpotent elements of Lie algebras of small ranks. We prove the rationality of a new family of $\mathcal{W}$-algebras, describe their set of simple modules and study other geometrical aspects. We describe new associated varieties of vertex algebras. The geometry of these objects often reflects some important algebraic properties of the vertex algebras. For some particular values of the parameter, called collapsing levels, we also get new remarkable isomorphisms of $\mathcal{W}$-algebras.

Keywords: vertex algebras, $\mathcal{W}$-algebras, rationality, representation theory, Lie algebras, nilpotent orbits

Géométrie et nouvelles $\mathcal{W}$-algèbres rationnelles

Résumé

Les $\mathcal{W}$-algèbres affines forment une famille riche d'algèbres vertex à un paramètre associées à un élément nilpotent d'une algèbre de Lie simple. Ce sont des structures algébriques complexes qui apparaissent dans plusieurs domaines de la physique et des mathématiques. Du fait de leur construction récente, de nombreux aspects de la théorie des $\mathcal{W}$-algèbres restent méconnus.
Dans cette thèse, nous étudions des $\mathcal{W}$-algèbres associées à des éléments nilpotents d'algèbres de Lie de petits rangs. Nous démontrons la rationalité d'une nouvelle famille de $\mathcal{W}$-algèbres, décrivons l'ensemble des modules simples sur ces dernières et étudions d'autres aspects géométriques. Nous décrivons de nouvelles variétés associées à des algèbres vertex. La géométrie de ces objets reflète souvent des propriétés algébriques importantes des algèbres vertex. Pour certaines valeurs particulières du paramètre, appelées niveaux d'effondrement, nous obtenons également de nouveaux isomorphismes remarquables de $\mathcal{W}$-algèbres.

Mots clés : algèbres vertex, $\mathcal{W}$-algèbres, rationalité, théorie des représentations, algèbres de Lie, orbites nilpotentes

Laboratoire Paul Painlevé, UMR CNRS 8524
Université de Lille – Cité Scientifique - Bâtiment M2 – F-59655 VILLENEUVE d'ASCQ – France

Remerciements

J'aimerais profiter de cette opportunité pour exprimer ma gratitude aux nombreuses personnes qui m'ont accompagnée dans cette aventure. Ces quelques mots sont bien peu de choses comparés à toute l'aide et le soutien que j'ai reçus.

Mes premières pensées vont à ma directrice de thèse, Anne Moreau. Chère Anne, tu m'as fait découvrir le monde de la recherche, et tu m'as amenée à m'intéresser à un sujet si vaste et fascinant qu'il m'est désormais impossible de m'en détourner. Merci de m'avoir accordé ta confiance et d'avoir partagé ta passion et ton enthousiasme.

Bien sûr, je pense aussi à mon co-directeur, Tomoyuki Arakawa, qui a su me faire profiter de son immense savoir. Thank you so much for patiently listening to me as I told about my not always clear progress and for giving me a boost whenever I was at a standstill. Anne and you are wonderful mathematicians and mentors: I hope to have the chance to work with you again in the future.

C'est un très grand honneur pour moi que Dražen Adamović et Reimundo Heluani aient accepté d'être rapporteurs de cette thèse. J'ai un profond respect pour leurs recherches et les questions qui les intéressent ont motivé certaines parties de mon travail. I am very grateful to Dražen for his invitation to Zagreb in October 2020 and I wanted to let him know that I really appreciated the discussions that ensued this visit.

Je remercie David Hernandez, Dimitri Markouchevitch, Michela Varagnolo et Huafeng Zhang pour avoir accepté de faire partie de mon jury de thèse.

Je veux également remercier tous les membres du laboratoire Painlevé pour m'avoir accueillie et apporté leur aide durant ces trois années de thèse. Merci pour les nombreuses discussions, mathématiques ou non. Une pensée affectueuse pour la merveilleuse équipe formée d'Aurore, Catherine, Christelle, Fatima, Frédérique, Ludivine, Omar, Sabine, Sébastien, Soledad et tant d'autres. Vous nous accueillez toujours dans la bonne humeur et, sans vous, la vie au laboratoire serait bien compliquée.

Un grand merci à tous ceux qui ont organisé les conférences auxquelles j'ai participé pendant ma thèse et où j'ai tant appris, aux personnes que j'y ai rencontrées et avec qui j'ai pu discuter, à celles qui m'ont donné l'opportunité de présenter mon travail. En particulier, mon séjour à l'IMPA lors du semestre dernier a été mathématiquement très stimulant. Je suis très reconnaissante aux organisateurs du semestre thématique "Vertex and Chiral Algebras" et au Collège doctoral qui ont rendu ce voyage possible.

Merci à tous ceux qui se sont intéressés de près ou de loin à mon travail ; à Thomas Creutzig, Naoki Genra et David Ridout pour avoir pris le temps de discuter de mes futurs projets ; à Autun Milas dont l'expertise a permis de déceler et corriger une typo dans le calcul de mes OPEs.

Je ne saurais oublier mes camarades doctorants et jeunes docteurs : Chris, Gabriel, Imane, Iacopo, Ivan, Jules, Nicola, Thibaut, Thomas, Tino, Vincent, etc. Merci pour l'atmosphère chaleureuse que vous avez créée et qui me donne envie d'aller travailler tous les matins. Merci aux anciens du bureau 318–320, Angelo et Jimena, qui m'ont très gentiment accueillie et intégrée à mon arrivée. Mention spéciale à Ivan qui aura occupé le bureau à côté du mien depuis le tout premier jour : la rigueur de ton travail fût une grande source de motivation.

Je remercie du fond du cœur tous mes amis, les anciens comme les nouveaux, ceux qui m'accompagnent depuis longtemps ou depuis quelques mois, ceux que j'ai perdus de vue et ceux que je n'ai pas encore rencontrés. Merci à Alice pour tout le chemin parcouru à ses côtés, pour les sorties à la patinoire et les longs après-midis à se remémorer notre enfance ; à Elsa, un morceau de Savoie à Lille, rien ne vaut ta joie de vivre et nos promenades du week-end pour oublier les soucis du travail ; à Cécile, Zozo et Vincent pour leurs encouragements, leurs précieux conseils et leur petit brin de folie.

Je sais que la liste s'allonge mais il me reste à remercier les personnes les plus importantes, celles sans qui je n'existerais pas : ma famille. Au-delà de la recherche et des mathématiques, mes années à Lille m'ont permis de passer plus de temps avec mes grands-mères, mes oncles et tantes, et mes cousins, cousines. Je vous remercie de m'avoir fait confiance et soutenue.

Maman, papa, vous avez toujours veillé à ce que je ne manque de rien et m'avez donné tous les moyens nécessaires pour réussir. Quant à toi, Clémentine, la distance nous a rapprochées et je ne peux imaginer meilleure confidente pour partager mes danses de la joie, mes peines, mes doutes et mes frustrations. Merci d'être toujours là pour moi. C'est une immense fierté de vous avoir pour parents et pour sœur. Je vous aime tendrement.

Contents

Introduction

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, f a nilpotent element of $\mathfrak{g}$, and $k \in \mathbb{C}$ a complex number. The *affine* $\mathcal{W}$*-algebra* $\mathcal{W}^k(\mathfrak{g}, f)$ associated with $(\mathfrak{g}, f)$ is a certain vertex algebra obtained from the quantized Drinfeld-Sokolov reduction of the *universal affine vertex algebra* $V^k(\mathfrak{g})$ (see Sect. 1.2). The $\mathcal{W}$-algebras can be regarded as affinizations of finite $\mathcal{W}$-algebras introduced by Premet [89]. They can also be considered as generalizations of infinite-dimensional Lie algebras such as Kac-Moody or Virasoro algebras.

First $\mathcal{W}$-algebras appeared in the 90s in the works of Zamolodchikov and Fateev-Lukyanov on the two-dimensional conformal field theory in physic. The construction of $\mathcal{W}$-algebras that we know today was introduced by Feigin and Frenkel [58] for f a principal nilpotent element. Then the definition was extended for general nilpotent elements by Kac, Roan and Wakimoto [73]. The theory of $\mathcal{W}$-algebras is also related with integrable systems [72], the two-dimensional conformal field theory, the geometric Langlands program [23, 60, 65], and the 4d/2d duality [20, 34, 35, 92] in physics.

Since the area is relatively recent, there are a lot of unknown aspects and open ambitious questions to investigate. In this thesis, we study the algebraic structure, geometry and representation theory of certain $\mathcal{W}$-algebras associated with simple Lie algebras of small ranks. From these examples, we observe phenomena that we suspect to be general. Thus, we expect that certain techniques we develop can be extended to study larger families of $\mathcal{W}$-algebras. For instance, we wish to extend the reasoning presented in Chap. 4 to study simple modules of $\mathcal{W}_k(\mathfrak{sp}_{2n}, f_{\min})$ at admissible levels when $n \geqslant 2$.

Affine $\mathcal{W}$-algebras can be described using their strong generators and the relations between them. These relations are called Operator Product Expansions (OPEs). In general, any vertex algebra admits such a description. In the context of $\mathcal{W}$-algebras, even if the construction of the generators is theoretically well known only a few examples were explicitly computed [41, 76]. Our first contribution is the computation of OPEs of all $\mathcal{W}$-algebras associated with Lie algebras of rank two (Appx. A, B and C)[1].

The explicit descriptions we obtain are the starting point to prove precise statements. For special values of the parameter k, called *collapsing levels* (see Chap. 2), we deduce remarkable isomorphisms of simple vertex algebras (Tables 2.2 and 2.3). Collapsing levels have interesting applications to the representation theory of affine vertex algebras [10, 95]. They have been fully classified when f belongs to the minimal nilpotent orbit of $\mathfrak{g}$ [8, 9]. For other nilpotent elements, [22] provides many new collapsing levels when k is *admissible* (see Sect. 1.2.1) and conjectures it is an exhaustive list. The study of OPEs allows achieving the classification of collapsing levels of $\mathcal{W}$-algebras associated with Lie algebras of rank 2. We thank Prof. Dražen Adamović whose remarks on OPEs of $\mathcal{W}(\mathfrak{sp}_4, f_{\text{subreg}})$ initiate this axis of research.

The nicest (*conformal*) vertex algebras are those which are both *rational* and *lisse* (see

[1] Due to limited computational resources, OPEs of $\mathcal{W}$-algebras associated with principal nilpotent elements of $\mathfrak{sp}_4$ and G_2 remain incomplete for the moment.

Sect. 1.1.3 and 1.1.4). The rationality means the complete reducibility of $\mathbb{Z}_{\geqslant 0}$-graded modules while the lisse property – we say also the C_2-cofiniteness – is detectable from a geometric object related to vertex algebra called *associated variety*. Both imply a finiteness condition on dimensions of graded components of *positive energy representations* of the vertex algebra. It has been conjectured by Zhu [97] that rational vertex algebras are also lisse. Lisse and rational vertex algebras generate rational conformal field theories. The rich class of $\mathcal{W}$-algebras gives rise to many new examples of rational vertex algebras. Nonetheless, it is in general extremely difficult to study the rationality of the simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$ of $\mathcal{W}^k(\mathfrak{g}, f)$. The rationality has been conjectured by Kac-Wakimoto [78] and Arakawa [17] for a large family of $\mathcal{W}$-algebras called *exceptional* (see Sect. 1.3.2). Particular cases of this conjecture were previously established by Arakawa [16, 18], Creutzig-Linshaw [41], and Arakawa-van Ekeren [21]. We prove the rationality of the exceptional $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$ associated with f_{subreg}, a subregular nilpotent element of $\mathfrak{sp}_4$ (type $C_2 = B_2$) (Theorem 3.1). These $\mathcal{W}$-algebras can be viewed as the "easiest" exceptional $\mathcal{W}$-algebras not covered by the previous works. It is an analogue for the type C of the Bershadsky-Polyakov vertex algebra studied by Arakawa [16]. However, we cannot use directly method of [16] because the reduction function is not exact in this case. To encounter the difficulty, we exploit certain techniques of [21].

The increasing number of new results in the last few years shows that the classification of rational $\mathcal{W}$-algebras is a major topical problem. After we proved the previous result, Creutzig and Linshaw [43] gave a proof of the rationality of a larger class of lisse $\mathcal{W}$-algebras – which covers a part of our cases. More recently McRae [86] gave a general conceptual proof of the conjecture of Kac-Wakimoto and Arakawa asserting that all exceptional $\mathcal{W}$-algebras are rational. These articles are important advances for the classification of rational $\mathcal{W}$-algebras. Nevertheless, the problem is far from being completely solved because exceptional $\mathcal{W}$-algebras do not provide an exhaustive list of rational ones. Indeed, there exist examples of rational but non-exceptional $\mathcal{W}$-algebras. For instance, [80] provides examples related to the exceptional series of Deligne. Certain $\mathcal{W}$-algebras associated with principal or subregular nilpotent elements at non-exceptional levels are also rational [43, 79].

As a by-product of our proof, we obtain an explicit description of the simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules when k is admissible (Proposition 3.2.5). One uses this description to compute the characters of simple modules (Sect. 3.5). They allow to deduce the fusion rules of the simple $\mathcal{W}$-algebras. Moreover, we show that the component group of the nilpotent orbit $\mathbb{O}_{\text{subreg}}$ acts non-trivially on the finite set of the simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules (Theorem 3.4.3). As far as we know, it is the first example of rational $\mathcal{W}$-algebras whose corresponding component group satisfies this property.

When $f = f_{\min}$ is a minimal nilpotent element of $\mathfrak{sp}_4$, the rationality of exceptional $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$ has been proved by Creutzig-Linshaw [43]. Nonetheless, we give an explicit description of the simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$-modules (Proposition 4.2.3). In fact, we show that $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$ is isomorphic to the reduction of the simple affine vertex algebra $H^0_{f_{\min}}(L_k(\mathfrak{sp}_4))$ (Theorem 4.1). Thus, we prove new cases of the conjecture of Kac-Roan-Wakimoto [73, 78] holding that the vertex algebra $H^0_f(L_k(\mathfrak{g}))$ is simple provided it is non-zero. This conjecture has been verified in many cases [12, 14].

The geometry of associated varieties reflects some algebraic properties of the vertex algebras [26, 30]. However, their description is an open question in most cases. The associated variety $X_{V^k(\mathfrak{g})}$ of the affine vertex algebra $V^k(\mathfrak{g})$ is $\mathfrak{g}^*$. Therefore, that of its simple quotient $L_k(\mathfrak{g})$ is a subvariety of $\mathfrak{g}^*$, conical and invariant under the action of the adjoint group of $\mathfrak{g}$. When k is a positive integer, $X_{L_k(\mathfrak{g})} = 0$ [53], and when it is an admissible level, it corresponds to the closure of a certain nilpotent orbit of $\mathfrak{g}$ depending on k [17]. In [28], Arakawa and Moreau provide additional examples of this form coming from the Deligne exceptional series. They also provide examples of associated varieties corresponding to Dixmier sheet closures [29]. This proves that associated varieties of simple affine vertex algebras are not necessarily contained in the nilpotent cone. More examples of this type are computed in [24] but the associated variety of $L_k(\mathfrak{g})$ remains unknown in general. According to [25], it is equal to $\mathfrak{g}$ if and only if $V^k(\mathfrak{g})$ is simple. All the previous examples correspond to closures of a Jordan class of a nilpotent or a semisimple element of $\mathfrak{g}$. Based on the work of Adamović, Perše and Vukorepa [7], we compute the associated variety of $L_{-5/2}(\mathfrak{sl}_4)$. It is also the closure of a Jordan class, but associated with an element of $\mathfrak{sl}_4$ which is neither nilpotent nor semisimple (Theorem 5.1).

In addition, when f is an element in the nilpotent orbit corresponding to the partition $(2, 2)$ of $\mathfrak{sl}_4$, [6] and [45] independently proved that the $\mathcal{W}$-algebra $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)$ is isomorphic to the *coset* $\mathrm{Com}(M(1), S(2))$ of the Heisenberg vertex algebra $M(1)$ in a $\beta\gamma$-system of rank two (see Sect. 5.2). This description allows us to compute the associated variety of the $\mathcal{W}$-algebra $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)$ (Theorem (i)).

Cosets, as well as *orbifolds*, are among of the most common ways to construct new vertex algebras from old ones. Cosets, also called commutants, were introduced by Frenkel and Zhu [63], while orbifolds are originated from physics [49, 50]. Both appear in the Gaoitto-Rapčák triality conjectures [64] recently proved by Creutzig and Linshaw [43, 44]. Let U be a vertex subalgebra of a vertex algebra V and G a reductive subgroup of V-automorphisms. It is expected that the coset $\mathrm{Com}(U, V)$ and the orbifold V^G inherit certain good properties from those of V, such as rationality or C_2-cofiniteness. One can wonder, when G is reductive and V is strongly finitely generated, if V^G is also strongly finitely generated, or equivalently, if its *Zhu's C_2-algebra* R_{V^G} is a finitely generated Poisson algebra (see Sect. 1.1.4). This problem is known as the vertex algebra Hibert's problem because of its similarities with the Hilbert's theorem stating that if G is a reductive group and V a finite G-module then the ring $\mathbb{C}[V]^G$ is finitely generated. It has a positive solution in many cases including all affine vertex algebras and $\mathcal{W}$-algebras [42, 44]. Considering this question, Lian and Linshaw [84] compare the Zhu's C_2-algebra R_{V^G} and the invariants of the Zhu's C_2-algebra $(R_V)^G$. They are not isomorphic in general, but in many cases one recovers an isomorphism by taking their reduced rings. So, if V and V^G are strongly generated, the C_2-cofiniteness of V would imply that of V^G.

When U is an affine vertex algebra, the coset $\mathrm{Com}(U, V)$ is identifiable to a certain orbifold of V [42]. An analogue of the vertex algebra Hilbert's problem can be formulate in this context. Creutzig and al. developed the theory for cosets of a Heisenberg vertex algebra inside a larger vertex algebra in [45], but the general case remains largely unknown. If we consider a vertex subalgebra U of a vertex algebra V, the inclusion induces a Poisson algebra morphism $R_U \to R_V$. If this morphism is injective, R_U can be viewed as a subalgebra of R_V and one defines $\mathrm{Com}(R_U, R_V)$ as the commutant of the image of R_U in R_V (see Sect. 5.2). Then one can compare it to the Zhu's C_2-algebra $R_{\mathrm{Com}(U,V)}$. Again, both Poisson algebras are not isomorphic in general. One wonders which kind of relations hold at level of their reduced rings. Whereas cosets take a important place in the theory of vertex algebras and $\mathcal{W}$-algebra, their Zhu's C_2-algebras has not been investigate a lot to our knowledge. We thank Prof. Andrew Linshaw for indicating considerations formulated for vertex algebras orbifolds [84, Sect. 13] which are similar for cosets.

In Sect. 5.2, we consider the coset of the Heisenberg vertex algebra of rank 1 embedded in a $\beta\gamma$-system of rank 2. We check that, even if their spectra are not equal, they still are closely related. More precisely, the spectrum of the commutant of Zhu's C_2-algebras is the normalization of the associated scheme of $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)$ (Theorem (iii)). Conjecturally, the associated variety of $\mathcal{W}_k(\mathfrak{g}, f)$ is equal to the intersection of the Slodowy slice $\mathscr{S}_f$ with $X_{L_k(\mathfrak{g})}$. Thus, using the description of the associated variety of $L_{-5/2}(\mathfrak{sl}_4)$, we conjecture an algebraic realization of the associated variety $X_{\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)}$ (Conjecture 5.3.2).

Several questions naturally raise from our work (see Chap. 6). We implemented an algorithm in the computer algebra system `GAP` in order to compute conformal weights of $\mathcal{W}$-algebras (Appx. D). This algorithm also provides representatives of nilpotent orbits in Lie algebras, and it is helpful to construct strong generators of $\mathcal{W}$-algebras. We expect to use it to detect similarities in – *a priori* – different $\mathcal{W}$-algebras. In particular, we would like to explore possible relations between $\mathcal{W}$-algebras with same conformal weights and central charge. Our examples brings us to think that there exist certain "dualities" analogous to the Feigin-Frenkel duality [58]. Let ${}^L\mathfrak{g}$ be the dual Langlands of $\mathfrak{g}$. The Feigin-Frenkel duality holds isomorphisms between principal $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{g}, f)$ and $\mathcal{W}^\ell({}^L\mathfrak{g}, f)$ for some particular levels k and ℓ. Recently, dualities of this type have been obtained for subregular $\mathcal{W}$-algebras in type A and B and principal $\mathcal{W}$-superalgebras [40, 46].

The thesis is structured as follows. In Chap. 1, we recall basic definitions and properties of vertex algebras and $\mathcal{W}$-algebras. We introduce the associated objects which play a major role in the general theory, such as Zhu's algebra, Zhu's C_2-algebra and associated variety. The Chap. 2 is dedicated to the algebraic structure of $\mathcal{W}$-algebras. We study the OPEs of $\mathcal{W}$-algebras associated with Lie algebras of rank two, and we classify their collapsing levels. Main results of Chap. 3 are published in [56]. We prove the rationality of exceptional $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{sp}_4, f_{\mathrm{subreg}})$ associated with a subregular nilpotent element of $\mathfrak{sp}_4$. In addition, we provide a complete classification of their simple modules. We present a similar classification for exceptional $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$ associated with a minimal nilpotent element of $\mathfrak{sp}_4$ in Chap. 4. In Chap. 5, we compute the associated varieties of $L_{-5/2}(\mathfrak{sl}_4)$ and $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)$, where f is a nilpotent element corresponding to the partition $(2, 2)$. We also look at the commutant of the Zhu's C_2-algebras $\mathrm{Com}(R_{M(1)}, R_{S(2)})$ and we compare its spectrum to the associated spectrum of the coset $\mathrm{Com}(M(1), S(2))$. Finally, in Chap. 6, we introduce some problems related to the present work. We hope to study them in the future.

Chapter 1

Vertex algebras

Vertex algebras have been introduced in 1986 by Borcherds [37] who was motivated by the construction of the Mooshine module due to Frenkel-Lepowsky-Meurman. In the last decades, they have generated an intense interest because of their numerous applications. They play a crucial role in the representation theory of infinite-dimensional Lie algebras. Moreover, they are deeply connected with several areas of physics including integrable systems and two-dimensional conformal field theory. They have applications in algebraic geometry with the study of moduli spaces. Recent developments also relate vertex algebras to quantum groups through Kazhdan-Lusztig correspondence. We refer to [31, 61, 71] as more complete references about the theory of vertex algebras.

In this chapter, we first recall basic definitions and general properties of vertex algebras (Sect. 1.1). We also introduce important tools to study vertex algebras and their modules, such as Zhu's algebra, Zhu's C_2-algebra and associated variety. A family of vertex algebras, called affine vertex algebras, is significant in the study of $\mathcal{W}$-algebras. We review them briefly in Sect. 1.2. Finally in Sect. 1.3, we give an overview of the construction of $\mathcal{W}$-algebras.

1.1 Vertex algebras operators

1.1.1 Definition of vertex algebras and OPEs

Let V be a complex vector space. We denote by $(\mathrm{End}\ V)[[z,z^{-1}]]$ the set of all Laurent series in the variable z with coefficients in $\mathrm{End}\ V$. For any series $a(z)\in(\mathrm{End}\ V)[[z,z^{-1}]]$, we write

$$a(z) := \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$$

such that $a_{(n)} = \mathrm{Res}_{z=0}\, a(z)z^n$.

A series $a(z)\in(\mathrm{End}\ V)[[z,z^{-1}]]$ is called a *field* on V if for all $b\in V$, $a(z)b := \sum_{n\in\mathbb{Z}} a_{(n)}bz^{-n-1}$ belongs to $V((z))$, that is $a_{(n)}b = 0$ for n large enough. Denote $\mathcal{F}(V)$ the set of fields on V.

Definition 1.1.1. *A* vertex algebra *is a vector space V equipped with a distinguished vector $|0\rangle\in V$ and two linear maps: a translation operator $T: V\to V$ and a vertex operator*

$$V\to(\mathrm{End}\ V)[[z,z^{-1}]],\quad a\mapsto Y(a,z) := a(z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1},$$

satisfying the following axioms:

- *for all* $a \in V$*,* $a(z)$ *is a field on* V*,*
- *(vacuum axiom)* $|0\rangle(z) = \mathrm{id}_V$ *and, for all* $a \in V$*,* $a(z)|0\rangle \in a + zV[[z]]$*,*
- *(translation axiom)* $T|0\rangle = 0$ *and, for all* $a \in V$*,* $[T, a(z)] = \partial_z a(z)$ *and* $(Ta)(z) = \partial_z a(z)$*,*
- *(locality axiom) for all* $a, b \in V$*,* $(z-w)^N [a(z), b(w)] = 0$ *for a sufficiently large* N *depending on* a *and* b*.*

Remark 1.1.2. The translation axiom implies that for all $a \in V$ and $n \in \mathbb{Z}$,

$$[T, a_{(n)}] = -na_{(n-1)} \qquad \text{and} \qquad (Ta)_{(n)} = -na_{(n-1)}.$$

In particular, we directly deduce that $Ta = a_{(-2)}|0\rangle$.

Remark 1.1.3. The vacuum axiom defines a correspondence between the spaces of states V and the fields $\mathcal{F}(V)$. From the field $a(z)$, we can recover $a = \lim_{z \to 0} a(z)|0\rangle = a_{(-1)}|0\rangle$. Moreover, since $a(z)|0\rangle \in V[[z]]$, $a_{(n)}|0\rangle = 0$ for $n \geqslant 0$. It implies that the vertex operator $Y(\cdot, z)$ is injective.

Because the product of two fields on a vertex algebra V does not make sense in general, we define the *normally ordered product* of two fields $a(z), b(z) \in \mathcal{F}(V)$ by

$$: a(z)b(z) : \stackrel{\text{def}}{=} a(z)_+ b(z) + b(z)a(z)_-,$$

where $a(z)_+ := \sum_{n<0} a_{(n)} z^{-n-1}$ and $a(z)_- := \sum_{n \geqslant 0} a_{(n)} z^{-n-1}$. This product is not associative, and we define the normally ordered product of multiple fields by induction:

$$: a^1(z) a^2(z) \dots a^n(z) : = : a^1(z) : a^2(z) \dots a^n(z) :: .$$

The vertex algebra V is *strongly generated* [73] by a family of fields $\{a^i(z)\}_{i \in I}$ if any field of V is a linear combination of normally ordered products of the fields $\{a^i(z)\}_{i \in I}$ and their derivatives. This means that, as a vector space, V is spanned by

$$a^{i_1}_{(-n_1)} \dots a^{i_s}_{(-n_s)} |0\rangle \tag{1.1}$$

with $s \geqslant 0$, $n_r \geqslant 1$, and $i_r \in I$. If V is freely strongly generated by the fields $\{a^i(z)\}_{i \in I}$, the set of monomials (1.1) where the sequence of pairs $(i_1, n_1), \dots, (i_r, n_r)$ is decreasing in the lexicographical order is a basis of V called a *Poincaré-Birkhoff-Witt (PBW) basis*. Then, the structure of V is completely determined by the relations among the fields $a^i(z)$, $i \in I$, or, equivalently, the Lie brackets in $\mathrm{End}(V)$ among the $a^i_{(n)}$:

Proposition 1.1.4 ([71]). *Consider two fields* $a(z), b(z) \in \mathcal{F}(V)$*. The following assertions are equivalent:*

(i) it exists some $N \in \mathbb{Z}_{\geqslant 0}$ *such that* $(z-w)^N [a(z), b(w)] = 0$*,*

(ii) there exist $c_0(w), \dots c_{N-1}(w) \in \mathcal{F}(V)$ *such that*

$$[a(z), b(w)] = \sum_{n=0}^{N-1} c_n(w) \frac{1}{n!} \partial_w^n \delta(z-w),$$

where $\delta(z-w)$ *refers to the formal delta-function* $\sum_{n \in \mathbb{Z}} w^n z^{-n-1} \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$*,* *(iii) there exist* $c_0(w), \dots c_{N-1}(w) \in \mathcal{F}(V)$ *such that*

$$a(z)b(w) = \sum_{n=0}^{N-1} c_n(w)\tau_{z,w}\left(\frac{1}{(z-w)^{n+1}}\right) + :a(z)b(w):$$

and

$$b(w)a(z) = \sum_{n=0}^{N-1} c_n(w)\tau_{w,z}\left(\frac{1}{(z-w)^{n+1}}\right) + :a(z)b(w):,$$

where $\tau_{z,w}\left(\frac{1}{z-w}\right) = \sum_{n\geqslant 0} w^n z^{-n-1}$ *and* $\tau_{w,z}\left(\frac{1}{z-w}\right) = -\sum_{n\geqslant 0} z^n w^{-n-1}$.

By abuse of notation, we write

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c_j(w)}{(z-w)^{j+1}} \tag{1.2}$$

for the relations of Proposition 1.1.4 (iii). The relation (1.2) is called the *operator product expansion* (OPE) of $a(z)$ and $b(w)$. It is equivalent to the data, for all $m, n \in \mathbb{Z}$, of the Lie brackets

$$[a_{(m)}, b_{(n)}] = \sum_{j=0}^{N-1} \binom{m}{j} (c_j)_{(m+n-j)},$$

where for $j \geqslant 0$,

$$\binom{m}{j} = \frac{m(m-1)\cdots(m-j+1)}{j(j-1)\cdots 1}.$$

One naturally defines a *morphism* between two vertex algebras $(V, |0\rangle, T, Y) \to (V', |0\rangle', T', Y')$ to be a linear map $\phi : V \to V'$ mapping $|0\rangle$ to $|0\rangle'$ such that, for any $a, b \in V$,

$$\phi(Ta) = T'\phi(a) \qquad \text{and} \qquad \phi(Y(a,z)b) = Y'(\phi(a), z)\phi(b).$$

The previous construction can easily be extended to a superspace $V = V_0 \oplus V_1$, and gives rise to the notion of *vertex superalgebra*.

1.1.2 Conformal vertex algebras

Virasoro vertex algebras

Let $\mathrm{Vir} = \mathbb{C}((t))\partial_t \oplus \mathbb{C}C$ be the *Virasoro Lie algebra* with the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n+m,0}C \qquad \text{and} \qquad [C, \mathrm{Vir}] = 0,$$

where $L_n = -t^{n+1}\partial_t$ for $n \in \mathbb{Z}$. Given any $c \in \mathbb{C}$, let

$$\mathrm{Vir}^c = U(\mathrm{Vir}) \otimes_{U(\mathbb{C}[[t]]\partial_t \oplus \mathbb{C}C)} \mathbb{C}_c,$$

where $\mathbb{C}_c$ is the one-dimensional representation of $\mathbb{C}[[t]]\partial_t \oplus \mathbb{C}C$ on which $\mathbb{C}[[t]]\partial_t$ acts trivially and C acts as the multiplication by c. The space Vir^c admits a PBW basis of the form

$$L_{-n_1} \dots L_{-n_m}|0\rangle,$$

where $|0\rangle$ is the image of $1 \otimes 1$ in Vir^c and $n_1 \geqslant \ldots \geqslant n_m \geqslant 2$.

Let $T = L_{-1}$ and $L(z) := (L_{-2}|0\rangle)(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$. These data define a unique structure of vertex algebra on Vir^c which is called the *Virasoro vertex algebra* with central charge c. The OPE of the generating field $L(z)$ with itself is given by

$$L(z)L(w) \sim \frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{(z-w)}.$$

Conformal structure of vertex algebras

A vertex algebra V is called *conformal* if there exists a vector $\omega \in V$, called a *conformal vector*, such that $\omega(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ satisfies the following conditions

(a) $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}c_V$, where c_V is some constant called the *central charge* of V,

(b) L_0 acts semisimply on V,

(c) $T = L_{-1}$ is the translation operator on V.

For a conformal vertex algebra V, we set $V_\Delta = \{a \in V \mid L_0 a = \Delta a\}$ so that

$$V = \bigoplus_{\Delta\in\mathbb{C}} V_\Delta.$$

For $a \in V_\Delta$, Δ is called the *conformal weight* of a. We denote it by $\Delta_a := \Delta$. If $a \in V$ is homogeneous of conformal weight Δ_a, we set $a_n = a_{(n+\Delta_a-1)}$ so that

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-\Delta_a}, \tag{1.3}$$

which is a more standard notation in physics. Let $a \in V$ be a homogeneous element of conformal weight Δ_a. The OPE between the conformal field $L(z)$ and $a(w)$ is given by

$$L(z)a(w) \sim \frac{\partial a(w)}{(z-w)} + \frac{\Delta_a a(w)}{(z-w)^2} + o\left(\frac{1}{(z-w)^2}\right).$$

We say that the field $a(z)$ is *primary* when

$$L(z)a(w) \sim \frac{\partial a(w)}{(z-w)} + \frac{\Delta_a a(w)}{(z-w)^2},$$

or, equivalently,

$$[L_m, a_{(n)}] = -na_{(m+n)},$$

for all $m, n \in \mathbb{Z}$.

If V is a $\mathbb{Z}$-graded conformal vertex algebra such that $V_\Delta = 0$ for Δ small enough, then V is called a *vertex operator algebra* (VOA).

Example 1.1.5. By definition, the Virasoro vertex algebra Vir^c is conformal with central charge c and conformal vector $\omega = L_{-2}|0\rangle$.

Remark 1.1.6. If V is a VOA of central charge c, there is an injective vertex algebra morphism $\mathrm{Vir}^c \hookrightarrow V$ via the identification $L \mapsto \omega$.

1.1.3 Modules over vertex algebras and Zhu's correspondence

Definition 1.1.7. *Let V be a vertex algebra. A* module M *over V is a vector space together with a linear map,*

$$V \to (\operatorname{End} M)[[z, z^{-1}]], \quad a \mapsto Y_M(a,z) := a^M(z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1},$$

satisfying the following axioms:

- *for all $a \in V$, $a^M(z) \in \mathcal{F}(M)$,*
- $|0\rangle(z) = \mathrm{id}_M$,
- $(Ta)^M(z) = \partial_z a^M(z)$, *for all $a \in V$,*
- *for all $a, b \in V$ and $m, n \in \mathbb{Z}$,*

$$\sum_{j\geqslant 0} \binom{m}{j} (a_{(n+j)}b)^M_{(m+k-j)} = \sum_{j\geqslant 0} (-1)^j \binom{n}{j} (a^M_{(m+n-j)} b^M_{(k+j)} - (-1)^n b^M_{(n+k-j)} a^M_{(m+j)}).$$

This implies in particular that V is a module over itself.

The usual notions of submodules, quotients and ideals naturally arise in the context of vertex algebras. In particular, a module which has only two submodules, 0 and itself, is said *simple*. A vertex algebra V is *simple* if it is simple as V-module. Moreover, all ideals of a vertex algebra are two-sided ideals. Hence, if I is an ideal of the vertex algebra V, the quotient V/I inherits a natural vertex algebra structure.

Let V be a conformal vertex algebra and M be a V-module. The module M is called a *positive energy representation* if L_0 acts semisimply on M with spectrum bounded from below, that is,

$$M = \bigoplus_{\Delta\in\chi+\mathbb{Z}_{\geqslant 0}} M_\Delta,$$

where $M_\Delta = \{m \in M \mid L_0 m = \Delta m\}$, and $M_\chi \neq 0$.. Let $M_{\mathrm{top}} := M_\chi$ be the *top degree component* of M. For all homogeneous vector $a \in V$, we have

$$a^M_{(n)} M_\Delta \subset M_{\Delta+\Delta_a-n-1},$$

for all $n \in \mathbb{Z}$ and $\Delta \in \mathbb{C}$. This defines a grading on M.

A conformal vertex algebra V is *rational* if any $\mathbb{Z}_{\geqslant 0}$-graded module is completely reducible, that is, isomorphic to a direct sum of simple V-modules. The rationality condition implies that V has finitely many simple $\mathbb{Z}_{\geqslant 0}$-graded modules and that the graded components of each of these $\mathbb{Z}_{\geqslant 0}$-graded modules are finite dimensional [51].

Zhu's algebra and Zhu's correspondence

To any vertex operator algebra V one associates the quotient space $\mathcal{A}(V) := V/(V \circ V)$, where $V \circ V$ is the subspace of V generated by the elements

$$a \circ b := \operatorname{Res}_{z=0} \left(a(z) \frac{(z+1)^{\Delta_a}}{z^2} b \right),$$

where $a \in V_{\Delta_a}$ and $b \in V$. For all $a, b \in V$ with a homogeneous, we define the bilinear operation

$$a * b := \operatorname{Res}_{z=0}\left(a(z)\frac{(z+1)^{\Delta_a}}{z}b\right).$$

Then, $(\mathcal{A}(V), *)$ is an associative algebra called the *Zhu's algebra* of V [97].

The following theorem, known as Zhu's theorem or Zhu's correspondence, illustrates how important this algebra is for the representation theory of vertex algebras.

Theorem 1.1.8 (Zhu's correspondence [97]). *If $M = \bigoplus_{\Delta \in \chi + \mathbb{Z}_{\geqslant 0}} M_\Delta$ is a positive energy representation of V, then M_{top} is a representation of $\mathcal{A}(V)$ whose action is described as follows. For any $a \in V$ homogeneous, the image $[a]$ of a in $\mathcal{A}(V)$ acts on M_{top} as $a_{(\Delta_a - 1)}$. The correspondence $M \mapsto M_{top}$ gives a bijection between the set of isomorphism classes of irreducible positive energy representations of V and that of simple $\mathcal{A}(V)$-modules.*

1.1.4 C_2-cofiniteness condition and lisse vertex algebras

Let M be a module over a vertex algebra V. There is a natural filtration on M, called Li filtration [83], defined by $F^0 M = M$ and

$$F^p M = \operatorname{Span}_{\mathbb{C}}\{a_{(-i-1)}b \mid a \in V,\, b \in F^{p-i}M,\, i \geqslant 1\},$$

for all $p \geqslant 1$. The subspace $F^1 M$, usually denoted $C_2(M)$, is spanned by the elements $a_{(-2)}m$ where $a \in V$ and $m \in M$. Set $R_M := M/C_2(M)$. In particular, the quotient space R_V is naturally endowed with a structure of Poisson algebra defined by

$$1 = \overline{|0\rangle}, \qquad \bar{a}.\bar{b} = \overline{a_{(-1)}b}, \qquad \text{and} \qquad \{\bar{a}, \bar{b}\} = \overline{a_{(0)}b},$$

where $\bar{a}$ denotes the image of $a \in V$ in the quotient R_V. The algebra R_V is called the *Zhu's C_2-algebra* of V. If M is a V-module then, R_M is a Poisson module over R_V. The V-module M is said *finitely strongly generated* if R_M is finitely generated over R_V. It is *C_2-cofinite* if R_M is finite dimensional. A vertex algebra V is *C_2-cofinite* if it is C_2-cofinite as a module over itself. Such vertex algebras are also called *lisse*.

The *associated scheme* $\tilde{X}_V$ and the *associated variety* X_V of V [15] are respectively the scheme and reduced scheme of the Zhu's C_2-algebra R_V:

$$\tilde{X}_V := \operatorname{Spec} R_V \qquad \text{and} \qquad X_V := \operatorname{Specm} R_V.$$

The vertex algebra V is lisse if and only if $\dim X_V = 0$. If V is lisse then, all its simple modules are positive energy representations with finite dimensional graded components [1]. These objects are very powerful tools to check the potential rationality of a vertex algebra. Indeed, the previous implication is close from the rationality condition and Zhu conjectured [97] that rational vertex algebras are lisse. The conjecture is still open even if Ai and Lin [11] recently gave a counterexample in the context of vertex superalgebras.

Similarly, we can construct the associated scheme $\tilde{X}_M$ and the associated variety X_M of a V-module M. They are Poisson subschemes of $\tilde{X}_V$ and X_V respectively.

Remark 1.1.9. In [54], van Ekeren and Heluani propose a uniform construction of the Zhu's algebra and Zhu's C_2-algebra associated to a vertex algebra.

1.2 Affine vertex algebras

1.2.1 Affine Kac-Moody algebras and admissible weights

Let $\mathfrak{g}$ be a complex simple Lie algebra with adjoint group G. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a triangular decomposition with a Cartan subalgebra $\mathfrak{h}$. Set Δ the root system of $(\mathfrak{g}, \mathfrak{h})$ and fix Δ_+ a set of positive roots. Let $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ be the corresponding set of simple roots. Denote θ the highest positive root. For any root $\alpha \in \Delta$, we define the coroot $\alpha^\vee = 2\alpha/(\alpha|\alpha)$, where $(\ |\ ) = \frac{1}{2h^\vee} \times \kappa_\mathfrak{g}$ is the normalized invariant inner product of $\mathfrak{g}$ satisfying $(\theta|\theta) = 2$, $h^\vee$ the dual Coxeter number, and $\kappa_\mathfrak{g}$ is the Killing form of $\mathfrak{g}$. Let $\Delta^\vee$ be the set of coroots. Let $Q = \sum_{\alpha\in\Delta} \mathbb{Z}\alpha$ be the root lattice and $Q^\vee = \sum_{\alpha\in\Delta} \mathbb{Z}\alpha^\vee$ be the coroot lattice. Let P be the weight lattice of $\mathfrak{g}$ and $P^\vee$ be the coweight lattice. The fundamental weights $\{\varpi_i\}_{1\leqslant i\leqslant\ell}$ form a basis of P dual to the basis of $Q^\vee$ formed by the simple coroots $\{\alpha_i^\vee\}_{1\leqslant i\leqslant\ell}$.

Let $\widetilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$ be the *extended affine Kac-Moody algebra* with the commutation relations:

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}(x|y)K, \qquad [D, xt^n] = -nxt^n, \qquad \text{and} \qquad [K, \widetilde{\mathfrak{g}}] = 0,$$

for all $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$, where xt^n stands for $x \otimes t^n$ and $\delta_{i,j}$ stands for the Kronecker symbol. Let $\widetilde{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$ be the standard triangular decomposition, that is, $\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$ is a Cartan subalgebra of $\widetilde{\mathfrak{g}}$, $\widehat{\mathfrak{n}}_+ = \mathfrak{n}_+ \oplus t\mathfrak{g}[t]$ and $\widehat{\mathfrak{n}}_- = \mathfrak{n}_- \oplus t^{-1}\mathfrak{g}[t^{-1}]$.

Set $\widehat{\mathfrak{g}} = [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ and $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \subset \widehat{\mathfrak{g}}$, so that $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$. The Cartan subalgebra $\widetilde{\mathfrak{h}}$ is equipped with a bilinear form extending that on $\mathfrak{h}$ by

$$(K|D) = 1, \qquad \text{and} \qquad (\mathfrak{h}|\mathbb{C}K \oplus \mathbb{C}D) = (K|K) = (D|D) = 0.$$

We write Λ_0 and δ for the elements of $\widetilde{\mathfrak{h}}^*$ orthogonal to $\mathfrak{h}^*$ and dual to K and D respectively. We have the (real) root systems

$$\begin{aligned}\widehat{\Delta}^{\mathrm{re}} &= \{\alpha + n\delta \mid n \in \mathbb{Z}, \alpha \in \Delta\} = \widehat{\Delta}^{\mathrm{re}}_+ \sqcup (-\widehat{\Delta}^{\mathrm{re}}_+),\\ \widehat{\Delta}^{\mathrm{re}}_+ &= \{\alpha + n\delta \mid \alpha \in \Delta_+,\ n \geqslant 0\} \sqcup \{-\alpha + n\delta \mid \alpha \in \Delta_+,\ n > 0\}.\end{aligned}$$

The affine Weyl group $\widehat{W}$ is generated by reflections r_α with $\alpha \in \widehat{\Delta}^{\mathrm{re}}$. For $\alpha \in \mathfrak{h}^*$, the translation $t_\alpha : \widetilde{\mathfrak{h}}^* \to \widetilde{\mathfrak{h}}^*$ is defined by

$$t_\alpha(\lambda) = \lambda + \lambda(K)\alpha - \left[(\alpha|\lambda) + \frac{|\alpha|^2}{2}\lambda(K)\right]\delta.$$

For $\alpha \in Q^\vee$, we have $t_\alpha \in \widehat{W}$. In fact, $\widehat{W} \cong W \ltimes t_{Q^\vee}$, where $t_{Q^\vee} := \{t_\alpha \mid \alpha \in Q^\vee\}$, and W is the Weyl group of $\mathfrak{g}$. The extended affine Weyl group, which is the group of isometries of $\widehat{\Delta}$, is $\widetilde{W} = W \ltimes t_P$, where $t_P := \{t_\alpha \mid \alpha \in P\}$.

The *category* $\mathcal{O}$ is be the full subcategory of $U(\widehat{\mathfrak{g}})$-Mod whose objects are the $\widehat{\mathfrak{h}}$-diagonizable modules

$$M = \bigoplus_{\lambda\in\widehat{\mathfrak{h}}^*} M_\lambda,$$

where each M_λ is finite dimensional and so that there exists a finite subset $\{\lambda_i\}_{1\leqslant i\leqslant s} \subset \widehat{\mathfrak{h}}^*$ such that $M_\lambda \neq 0$ implies $\lambda - \lambda_i \in \sum_{j=1}^{\ell} \mathbb{Z}_{\geqslant 0}\alpha_j$ for at least one λ_i. For $\lambda \in \widehat{\mathfrak{h}}^*$, $M(\lambda)$ denotes the Verma module of $\widehat{\mathfrak{g}}$ with highest weight λ, and $L(\lambda)$ is the unique simple quotient of $M(\lambda)$. All simple modules of the category $\mathcal{O}$ are with the form $L(\lambda)$ for some $\lambda \in \widehat{\mathfrak{h}}^*$. Let $\mathcal{O}_k$ be the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$ at level k [70]. The simple objects of $\mathcal{O}_k$ are the irreducible highest weight representations $L(\lambda)$, $\lambda \in \widehat{\mathfrak{h}}^*$ such that $\lambda(K) = k$. For $\lambda \in \mathfrak{h}^*$, denote $\widehat{L}_k(\lambda)$ the simple $\widehat{\mathfrak{g}}$-module with highest weight $\hat{\lambda} = \lambda + k\Lambda_0$. For a weight $\lambda \in \widehat{\mathfrak{h}}^*$, the corresponding *integral root system* is

$$\widehat{\Delta}(\lambda) = \{\alpha \in \widehat{\Delta}^{\mathrm{re}} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\},$$

where $\alpha^\vee = 2\alpha/(\alpha|\alpha)$ as usual. Let $\widehat{W}(\lambda)$ be the corresponding *integral Weyl group* generated by the reflections r_α, $\alpha \in \widehat{\Delta}(\lambda)$.

A weight $\lambda \in \widehat{\mathfrak{h}}^*$ is said to be *admissible* [75] (equivalently, we say that $L(\lambda)$ is *admissible*) if

- λ is regular dominant, that is, $\langle \lambda + \widehat{\rho}, \alpha^\vee \rangle > 0$ for all $\alpha \in \widehat{\Delta}_+(\lambda) = \widehat{\Delta}(\lambda) \cap \widehat{\Delta}^{\mathrm{re}}_+$,
- $\mathbb{Q}\widehat{\Delta}^{\mathrm{re}} = \mathbb{Q}\widehat{\Delta}(\lambda)$.

Here, $\widehat{\rho} = \rho + h^\vee \Lambda_0$ with $\rho = \sum_{\alpha \in \Delta_+} \alpha/2$. A complex number k is *admissible* if $\lambda = k\Lambda_0$ is an admissible weight.

Proposition 1.2.1 ([78]). *A complex number k is admissible if and only if it is of the form*

$$k = -h^\vee + \frac{p}{q},$$

with $p, q \in \mathbb{Z}_{>0}$, $(p, q) = 1$ and either $p \geqslant h^\vee$ if $(r^\vee, q) = 1$, or $p \geqslant h$ if $(r^\vee, q) = r^\vee$. Here ,$r^\vee$ is the lacety of $\mathfrak{g}$, that is, $r^\vee = 1$ if $\mathfrak{g}$ has type $A_\ell, D_\ell, E_6, E_7, E_8$, $r^\vee = 2$ if $\mathfrak{g}$ has type B_ℓ, C_ℓ, F_4 and $r^\vee = 3$ if $\mathfrak{g}$ has type G_2.

For k an admissible number, let Pr^k be the set of weights $\lambda \in \mathfrak{h}^*$ such that $\hat{\lambda} = \lambda + k\Lambda_0$ is admissible and there exists $y \in \widetilde{W}$ such that $\widehat{\Delta}(\hat{\lambda}) = y(\widehat{\Delta}(k\Lambda_0))$. For $\lambda \in \mathrm{Pr}^k$, if $y \in \widetilde{W}$ satisfies $y(\widehat{\Delta}(k\Lambda_0)_+) \subset \widehat{\Delta}^{\mathrm{re}}_+$ and $\widehat{\Delta}(\lambda) = y(\widehat{\Delta}(k\Lambda_0))$ then, $\widehat{W}(\lambda) = y\widehat{W}(k\Lambda_0)y^{-1}$. The weights of Pr^k are said *principal admissible* if $(r^\vee, q) = 1$ and *coprincipal admissible* if $(r^\vee, q) = r^\vee$. For $\lambda \in \mathrm{Pr}^k$, consider

$$J_\lambda := \mathrm{Ann}_{U(\mathfrak{g})} L(\lambda),$$

the annihilating ideal of $L(\lambda)$ in the enveloping algebra $U(\mathfrak{g})$. For a nilpotent orbit $\mathbb{O}$ in $\mathfrak{g}$, denote $\mathrm{Pr}^k_{\mathbb{O}}$ the subset of Pr^k consisting of principal or coprincipal admissible weights λ such that $\mathrm{Var}(J_\lambda) = \overline{\mathbb{O}}$. For $y \in W$ and $\lambda \in \mathfrak{h}^*$, let $y \circ \lambda = y(\lambda + \rho) - \rho$. Set $[\mathrm{Pr}^k] := \mathrm{Pr}^k/\sim$ and $[\mathrm{Pr}^k_{\mathbb{O}}] := \mathrm{Pr}^k_{\mathbb{O}}/\sim$ where $\lambda \sim \mu$ if $\mu \in W \circ \lambda$.

1.2.2 Affine vertex algebras associated with a simple Lie algebra

Given any $k \in \mathbb{C}$, let

$$V^k(\mathfrak{g}) := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts by 0 and K acts as a multiplication by k. It is the representation of $\widehat{\mathfrak{g}}$ induced from $\mathbb{C}_k$ and a highest weight representation with highest weight $k\Lambda_0$. Hence, it is a representation *of level* k of $\widehat{\mathfrak{g}}$, i.e. K acts as $k\,$id on it.

As a vector space,

$$V^k(\mathfrak{g}) \simeq U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]).$$

Let $\{x^1, \dots, x^d\}$ be an ordered basis of $\mathfrak{g}$. By the Poincaré-Birkhoff-Witt theorem, $V^k(\mathfrak{g})$ has a PBW basis which consists in the monomials

$$x^{i_1}_{(-n_1)} \dots x^{i_m}_{(-n_m)}|0\rangle, \tag{1.4}$$

where $|0\rangle$ is the image of $1 \otimes 1$ in $V^k(\mathfrak{g})$, $x_{(n)} = xt^n$ for all $n \in \mathbb{Z}$, $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_m > 0$, and $i_j \leqslant i_{j+1}$ if $n_j = n_{j+1}$.

There is a unique vertex algebra structure on $V^k(\mathfrak{g})$ such that $|0\rangle$ is the image of $1 \otimes 1$ in $V^k(\mathfrak{g})$ and

$$x(z) := (x_{(-1)}|0\rangle)(z) = \sum_{n \in \mathbb{Z}} x_{(n)} z^{-n-1}$$

for all $x \in \mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is considered as a subspace of $V^k(\mathfrak{g})$ through the embedding $x \in \mathfrak{g} \hookrightarrow x_{(-1)}|0\rangle \in V^k(\mathfrak{g})$. The vertex algebra $V^k(\mathfrak{g})$ is called the *universal affine vertex algebra* associated with $\mathfrak{g}$ at level k. Its structure is defined by induction for any monomial of the basis (1.4) by:

$$Y(x^{i_1}_{(-n_1)} \dots x^{i_m}_{(-n_m)}|0\rangle, z) = \frac{1}{(n_1 - 1)! \dots (n_m - 1)!} : \partial_z^{n_1 - 1} x^{i_1}(z) \dots \partial_z^{n_m - 1} x^{i_m}(z) : .$$

Moreover, for $x, y \in \mathfrak{g}$, the OPE between the corresponding fields on $V^k(\mathfrak{g})$ is given by

$$x(z)y(w) \sim \frac{[x,y](w)}{(z-w)} + \frac{(x|y)k}{(z-w)^2}.$$

When the level k is non-critical, that is, $k \neq -h^\vee$, the affine vertex algebra $V^k(\mathfrak{g})$ is conformal by Sugawara construction. Its central charge is

$$c_{V^k(\mathfrak{g})} = \frac{k \dim \mathfrak{g}}{k + h^\vee},$$

and the Sugawara conformal vector is defined by

$$\omega = \frac{1}{2(k + h^\vee)} \sum_{i=1}^{d} x^i_{(-1)} x^{i*}_{(-1)} |0\rangle,$$

where $\{x^{i*}\}_i$ is the dual basis of $\{x^i\}_i$ with respect to $(\ |\ )$.

Modules of affine vertex algebras

The universal affine vertex algebra $V^k(\mathfrak{g})$ plays a crucial role in the representation theory of $\widehat{\mathfrak{g}}$. A $\widehat{\mathfrak{g}}$-module M of level k is *smooth* if $x^M(z) \in \mathcal{F}(M)$ for all $x \in \mathfrak{g}$. By definition, $V^k(\mathfrak{g})$-modules correspond exactly with smooth $\widehat{\mathfrak{g}}$-modules of level k.

Let $L_k(\mathfrak{g})$ be the unique simple graded quotient of $V^k(\mathfrak{g})$. It inherits a structure of vertex algebra from the one of $V^k(\mathfrak{g})$. It is called the *simple affine vertex algebra associated with* $\mathfrak{g}$ *at level* k. As a representation of $\widehat{\mathfrak{g}}$, $L_k(\mathfrak{g}) \cong L(k\Lambda_0)$.

Moreover, for all $k \in \mathbb{C}$, the Zhu's algebra of $V^k(\mathfrak{g})$ is isomorphic to the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Hence, $\mathcal{A}(L_k(\mathfrak{g}))$ is a quotient of $U(\mathfrak{g})$:

$$\mathcal{A}(L_k(\mathfrak{g})) \simeq U(\mathfrak{g})/I_k,$$

where I_k is a certain two-side ideal of $U(\mathfrak{g})$. Thus, $L(\hat{\lambda})$ is a $L_k(\mathfrak{g})$-module if and only if J_λ contains I_k. When k is admissible, this happens if and only if $\lambda \in \mathrm{Pr}^k$ [19].

Associated varieties of affine vertex algebras

For any graded quotient V of $V^k(\mathfrak{g})$, the Zhu's C_2-algebra of V is given by

$$R_V = V/t^{-2}\mathfrak{g}[t^{-1}]V.$$

In particular, $R_{V^k(\mathfrak{g})} \simeq \mathbb{C}[\mathfrak{g}^*]$. Hence, $X_{V^k(\mathfrak{g})} = \mathfrak{g}^*$. It follows that $X_{L_k(\mathfrak{g})}$ is a subvariety of $\mathfrak{g}^* \simeq \mathfrak{g}$, G-invariant and conical. The associated variety $X_{L_k(\mathfrak{g})}$ is difficult to compute in general. It is known that $X_{L_k(\mathfrak{g})} = \{0\}$, i.e. $L_k(\mathfrak{g})$ is lisse, if and only if $L(k\Lambda_0)$ is an integrable representation of $\widehat{\mathfrak{g}}$, that is $k \in \mathbb{Z}_{\geqslant 0}$ [53]. Furthermore, when k is admissible, we have the following result:

Proposition 1.2.2 ([17]). *If $k = -h^\vee + p/q$ is an admissible level for $\mathfrak{g}$, then $X_{L_k(\mathfrak{g})}$ is the closure of some nilpotent orbit $\mathbb{O}_q$ which only depends on q.*

The nilpotent orbit $\mathbb{O}_q$ is explicitly described in [17, Tables 2–10].

1.2.3 Universal affine vertex algebras

The previous construction of vertex algebra is generalizable to any Lie algebra $\mathfrak{a}$ endowed with a symmetric invariant bilinear form κ. Let $\widehat{\mathfrak{a}} = \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$ be the Kac-Moody Lie algebra of $\mathfrak{a}$ with commutation relations

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}\kappa(x, y)\mathbf{1}, \qquad \text{and} \qquad [\mathbf{1}, \widehat{\mathfrak{a}}] = 0$$

for all $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. The vector space

$$V^\kappa(\mathfrak{a}) = U(\widehat{\mathfrak{a}}) \otimes_{U(\mathfrak{a}[t]\oplus\mathbb{C}\mathbf{1})} \mathbb{C},$$

where $\mathbb{C}$ is the one-dimensional representation of $\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1}$ on which $\mathfrak{a}[t]$ acts by 0 and $\mathbf{1}$ acts as identity, is a $\mathbb{Z}_{\geqslant 0}$-graded vertex algebra called the *universal affine vertex algebra associated with $\mathfrak{a}$ and κ.*

Example 1.2.3. Let $\mathfrak{a} \simeq \mathbb{C}$ be a one-dimensional abelian Lie algebra, and κ any non-degenerate bilinear form on $\mathfrak{a}$. Then $V^\kappa(\mathfrak{a})$ is called the *Heisenberg vertex algebra*, denoted by $M(1)$. If $b \in \mathfrak{a}\backslash\{0\}$, then $b(z)$ strongly generates $M(1)$ and

$$b(z)b(w) \sim \frac{1}{(z-w)^2}.$$

1.3 The BRST reduction and $\mathcal{W}$-algebras

The following construction of the $\mathcal{W}$-algebra is due to Kac, Roan and Wakimoto [73]. We refer to [73] and [12, 76] as general references for the construction of $\mathcal{W}$-algebras.

1.3.1 The complex $\mathscr{C}(\mathfrak{g}, f, k)$ and the BRST reduction

Let f be a nilpotent element of $\mathfrak{g}$ that we embed into an $\mathfrak{sl}_2$-triple (e, h, f) given by the Jacobson-Morosov theorem. We have

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad \text{and} \qquad [e, f] = h.$$

The semisimple element $x_0 := h/2$ induces an $\frac{1}{2}\mathbb{Z}$-gradation on $\mathfrak{g}$,

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j, \tag{1.5}$$

where $\mathfrak{g}_j = \{y \in \mathfrak{g} \mid [x_0, y] = jy\}$. Set $\mathfrak{g}_{\geqslant 0} := \bigoplus_{j \geqslant 0} \mathfrak{g}_j$ and $\mathfrak{g}_{>0} := \bigoplus_{j>0} \mathfrak{g}_j$. We define similarly $\mathfrak{g}_{\leqslant 0}$ and $\mathfrak{g}_{<0}$. Choose a basis $\{e_\alpha\}_{\alpha \in S_j}$ of each $\mathfrak{g}_j$. One can assume that $\mathfrak{h} \subset \mathfrak{g}_0$ and that the root system Δ is compatible with the grading induced by x_0. Then we can pick each e_α to be a root vector if $j \neq 0$, and for $j = 0$, either e_α is a root vector or e_α belongs to the Cartan subalgebra $\mathfrak{h}$ [55]. Set $S = \sqcup_j S_j$, $S_+ = \sqcup_{j>0} S_j$ and let $\{e^\alpha\}_{\alpha \in S_+}$ be the dual basis in $\mathfrak{g}^*_{>0}$ to $\{e_\alpha\}_{\alpha \in S_+}$.

The x_0-grading is *good* if it satisfies the decomposition (1.5). In the following, we always consider good gradings. We refer to [55] for additional details and classification of good gradings in simple Lie algebras.

Neutral free fermions

The nilpotent element f belongs to $\mathfrak{g}_{-1}$ and defines a skew-symmetric bilinear form on $\mathfrak{g}_{1/2}$:

$$\langle a, b \rangle = (f | [a, b])$$

for all $a, b \in \mathfrak{g}_{1/2}$. Consider the vector space $(\mathfrak{g}_{1/2}, \langle\ ,\ \rangle)$, and let $\widehat{\mathfrak{g}}_{1/2} = \mathfrak{g}_{1/2}[t, t^{-1}]$. For $\alpha \in S_{1/2}$ and $n \in \mathbb{Z}$, $\Phi_\alpha(n)$ denotes $e_\alpha t^n$. Set

$$\Phi_\alpha(z) = \sum_{n \in \mathbb{Z}} \Phi_\alpha(n) z^{-n-1}.$$

The set $\{\Phi_\alpha\}_{\alpha \in S_{1/2}}$ gives a basis of $\mathfrak{g}_{1/2}$. Let $\{\Phi^\alpha\}_{\alpha \in S_{1/2}}$ be the corresponding dual basis with respect to $\langle\ ,\ \rangle$. Then the *vertex algebra of neutral free fermions* $\mathcal{F}_{\mathrm{ne}}$ is the vertex algebra strongly generated by the *neutral free fermions* $\{\Phi_\alpha(z), \Phi^\alpha(z)\}_{\alpha \in S_{1/2}}$ satisfying relations

$$[\Phi_\alpha(m), \Phi^\beta(n)] = \delta_{\alpha,\beta}\delta_{m+n,-1}, \qquad \text{and} \qquad [\Phi_\alpha(m), \Phi_\beta(n)] = [\Phi^\alpha(m), \Phi^\beta(n)] = 0$$

for all $\alpha, \beta \in S_{1/2}$ and $m, n \in \mathbb{Z}$.

Moreover, the field

$$L^{\mathrm{ne}}(z) := \frac{1}{2} \sum_{\alpha \in S_{1/2}} :(\partial \Phi^\alpha(z)) \Phi_\alpha(z):$$

defines a structure of conformal vertex algebra on $\mathcal{F}_{\mathrm{ne}}$ whose central charge is given by

$$c_{\mathrm{ne}} = -\frac{1}{2} \dim \mathfrak{g}_{1/2}.$$

All fields $\Phi_\alpha(z)$ and $\Phi^\alpha(z)$ on $\mathcal{F}_{\mathrm{ne}}$ are primary with respect to L^{ne} and have conformal weight 1/2.

Charged free fermions

Let $\mathcal{C}\ell(\mathfrak{g}_{>0}) = \mathfrak{g}_{>0}[t, t^{-1}] \oplus \mathfrak{g}^*_{>0}[t, t^{-1}]$ the Clifford affinization of $\mathfrak{g}_{>0}$ endowed with the symmetric bilinear form $\langle\ ,\ \rangle$ defined by

$$\langle x t^m, \psi t^n \rangle = \delta_{m+n,0} \psi(x), \qquad \text{and} \qquad \langle x t^m, y t^n \rangle = \langle \psi t^m, \phi t^n \rangle = 0$$

for $x, y \in \mathfrak{g}_{>0}$ and $\psi, \phi \in \mathfrak{g}^*_{>0}$. We write $\varphi_\alpha(m)$ for $e_\alpha t^m \in \mathcal{C}\ell(\mathfrak{g}_{>0})$ and $\varphi^\alpha(m)$ for $e^\alpha t^m \in \mathcal{C}\ell(\mathfrak{g}_{>0})$. Then $\mathcal{C}\ell(\mathfrak{g}_{>0})$ is the associative superalgebra with

- odd generators: $\varphi_\alpha(m)$, $\varphi^\alpha(n)$ for all $m, n \in \mathbb{Z}$ and $\alpha \in S_+$,
- relations: $[\varphi_\alpha(m), \varphi_\beta(n)] = [\varphi^\alpha(m), \varphi^\beta(n)] = 0$ and $[\varphi_\alpha(m), \varphi^\beta(n)] = \delta_{\alpha,\beta}\delta_{m+n,0}$,

where the parity of $\varphi_\alpha(m)$ and $\varphi^\alpha(n)$ is reverse to e_α. Since elements of $\mathfrak{g}$ are purely even, this means that $\varphi_\alpha(m)$ and $\varphi^\alpha(n)$ are odd.

Define the *charged fermion Fock space* associated with $\mathfrak{g}_{>0}$ as

$$\mathcal{F}(\mathfrak{g}_{>0}) := \frac{\mathcal{C}\ell(\mathfrak{g}_{>0})}{\sum\limits_{\substack{m \geqslant 0 \\ \alpha \in S_+}} \mathcal{C}\ell(\mathfrak{g}_{>0})\varphi_\alpha(m) + \sum\limits_{\substack{n \geqslant 1 \\ \alpha \in S_+}} \mathcal{C}\ell(\mathfrak{g}_{>0})\varphi^\alpha(n)} \cong \bigwedge\big(\varphi_\alpha(m)\big)_{\substack{m<0 \\ \alpha \in S_+}} \otimes \bigwedge\big(\varphi^\alpha(n)\big)_{\substack{n \leqslant 0 \\ \alpha \in S_+}},$$

where $\bigwedge(a_i)_{i\in I}$ denotes the exterior algebra with generators $\{a_i\}_{i\in I}$. It is an irreducible $\mathcal{C}\ell(\mathfrak{g}_{>0})$-module, and as $\mathbb{C}$-vector spaces we have

$$\mathcal{F}(\mathfrak{g}_{>0}) \cong \bigwedge(\mathfrak{g}^*_{>0}[t^{-1}]) \otimes \bigwedge(\mathfrak{g}_{>0}[t^{-1}]t^{-1}).$$

There is a unique vertex superalgebra structure on $\mathcal{F}(\mathfrak{g}_{>0})$ such that the image of 1 is the vacuum $|0\rangle$ and

$$Y(\varphi_\alpha(-1)|0\rangle, z) = \varphi_\alpha(z) := \sum_{n\in\mathbb{Z}} \varphi_\alpha(n) z^{-n-1},$$
$$Y(\varphi^\alpha(0)|0\rangle, z) = \varphi^\alpha(z) := \sum_{n\in\mathbb{Z}} \varphi^\alpha(n) z^{-n}$$

for all $\alpha \in S_+$. We denote by $\mathcal{F}_{\mathrm{ch}}$ this vertex algebra called the *charged free fermion vertex algebra* associated with $\mathcal{C}\ell(\mathfrak{g}_{>0})$. The fields $\{\varphi_\alpha\}_{\alpha\in S_+} \cup \{\varphi^\alpha\}_{\alpha\in S_+}$ strongly generate $\mathcal{F}_{\mathrm{ch}}$.

Let a conformal field on $\mathcal{F}_{\mathrm{ch}}$ be

$$L^{\mathrm{ch}}(z) := -\sum_{\alpha\in S_+} m_\alpha : \varphi^\alpha(z)(\partial\varphi_\alpha(z)) : + \sum_{\alpha\in S_+} (1-m_\alpha) : (\partial\varphi^\alpha(z))\varphi_\alpha(z) :,$$

where $[x_0, e_\alpha] = m_\alpha e_\alpha$ for all $\alpha \in S_+$. The central charge of L^{ch} is given by

$$c_{\mathrm{ch}} = -\sum_{\alpha\in S_+} (12m_\alpha^2 - 12m_\alpha + 2),$$

and the fields φ_α and φ^α are primary with respect to L^{ch} with conformal weight $1 - m_\alpha$ and m_α respectively.

The vertex algebra $\mathcal{F}_{\mathrm{ch}}$ has the charge decomposition

$$\mathcal{F}_{\mathrm{ch}} = \bigoplus_{m\in\mathbb{Z}} \mathcal{F}^m_{\mathrm{ch}}, \tag{1.6}$$

defined by the relations $\texttt{charge}\,\varphi_\alpha(z) = -\texttt{charge}\,\varphi^\alpha(z) = -1$ for $\alpha \in S_+$.

Reduction of the complex $\mathscr{C}(\mathfrak{g}, f, k)$

Let $k \in \mathbb{C}$ and set the vertex algebra

$$\mathscr{C}(\mathfrak{g}, f, k) = V^k(\mathfrak{g}) \otimes \mathcal{F}(\mathfrak{g}, f),$$

where $\mathcal{F}(\mathfrak{g}, f) = \mathcal{F}_{\mathrm{ch}} \otimes \mathcal{F}_{\mathrm{ne}}$. Define $\texttt{charge}\, V^k(\mathfrak{g}) = \texttt{charge}\, \mathcal{F}_{\mathrm{ne}} = 0$. Then (1.6) induces the charge decompositions

$$\mathcal{F}(\mathfrak{g}, f) = \bigoplus_{m \in \mathbb{Z}} \mathcal{F}(\mathfrak{g}, f)_m \quad \text{and} \quad \mathscr{C}(\mathfrak{g}, f, k) = \bigoplus_{m \in \mathbb{Z}} \mathscr{C}_m,$$

where $\mathscr{C}_m := \mathscr{C}(\mathfrak{g}, f, k)_m$, $m \in \mathbb{Z}$.

Following [73], set

$$d_{\mathrm{st}}(z) = \sum_{\alpha \in S_+} :e_\alpha(z)\varphi^\alpha(z): -\frac{1}{2} \sum_{\alpha,\beta,\gamma \in S_+} c^\gamma_{\alpha,\beta} :\varphi_\gamma(z)\varphi^\alpha(z)\varphi^\beta(z):, \tag{1.7}$$

$$d_{\mathrm{nd}}(z) = \sum_{\alpha \in S_+} (f|e_\alpha)\varphi^\alpha(z) + \sum_{\alpha \in S_{1/2}} :\varphi^\alpha(z)\Phi_\alpha(z):, \tag{1.8}$$

where $c^\gamma_{\alpha,\beta}$ is the structure constant defined by $[e_\alpha, e_\beta] = \sum_\gamma c^\gamma_{\alpha,\beta} e_\gamma$. Let $d(z) := d_{\mathrm{st}}(z) + d_{\mathrm{nd}}(z)$. The field $d(z)$ does not depend on the choice of the basis. One has $[d(z), d(w)] = 0$. Since $d(z)$ is odd, $d^2_{(0)} = 0$. Moreover, $[d_{(0)}, \mathscr{C}_m] \subset \mathscr{C}_{m+1}$. Thus, $(\mathscr{C}(\mathfrak{g}, f, k), d_{(0)})$ is a $\mathbb{Z}$-graded cohomology complex. The zero-th cohomology of this complex is a vertex algebra denoted by $\mathcal{W}^k(\mathfrak{g}, f)$:

$$\mathcal{W}^k(\mathfrak{g}, f) := H^0(\mathscr{C}(\mathfrak{g}, f, k), d_{(0)}).$$

This construction is a particular case of BRST reduction usually referred as the Drinfeld-Sokolov reduction of $V^k(\mathfrak{g})$. We denote this reduction functor $H^\bullet_f(?) := H^\bullet_{BRST}(?)$ in the following. We briefly write

$$\mathcal{W}^k(\mathfrak{g}, f) = H^0_f(V^k(\mathfrak{g})).$$

The vertex algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is called the *(affine) $\mathcal{W}$-algebra associated with $\mathfrak{g}$ and f at the level k*. Its simple graded quotient is denoted by $\mathcal{W}_k(\mathfrak{g}, f)$. Historically, $\mathcal{W}$-algebras were firstly defined for principal (or regular) nilpotent elements of Lie algebras [58]. The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g})$ always refers to the principal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f_{\mathrm{reg}})$.

Conformal vector and generating fields of $\mathcal{W}$-algebras

Provided that $k \neq -h^\vee$ is non-critical, $\mathcal{W}^k(\mathfrak{g}, f)$ inherits a structure of conformal vertex algebra from those of $V^k(\mathfrak{g})$, $\mathcal{F}_{\mathrm{ch}}$ and $\mathcal{F}_{\mathrm{ne}}$. Define the conformal field on $\mathcal{W}^k(\mathfrak{g}, f)$ by

$$L(z) = L^{\mathfrak{g}}(z) + \frac{d}{dz}x(z) + L^{\mathrm{ch}}(z) + L^{\mathrm{ne}}(z),$$

with central charge

$$\begin{aligned} c_k &= c_{V^k(\mathfrak{g})} - 12k(x|x) + c_{\mathrm{ch}} + c_{\mathrm{ne}} \\ &= \frac{k \dim \mathfrak{g}}{k+h^\vee} - 12k(x|x) - \sum_{\alpha\in S_+}(12m_\alpha^2 - 12m_\alpha + 2) - \frac{1}{2}\dim \mathfrak{g}_{1/2} \\ &= \dim \mathfrak{g}_0 - \frac{1}{2}\dim \mathfrak{g}_{1/2} - \frac{12}{k+h^\vee}|\rho - (k+h^\vee)x|^2. \end{aligned} \tag{1.9}$$

Given $a \in \mathfrak{g}_{-j}$, introduce the field of $\mathscr{C}(\mathfrak{g}, f, k)$

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a(n) z^{-n-1} = a(z) - \sum_{\alpha,\beta\in S_+} c^\alpha_{a,\beta} : \varphi_\alpha(z)\varphi^*_\beta(z) :,$$

where $[a, e_\beta] = \sum_{\alpha\in S} c^\alpha_{a,\beta} e_\alpha$. This field has conformal weight $1 + j$ with respect to L. The fields $\{J^a\}_{a\in\mathfrak{g}}$ play an important role in the structure of the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$. Let $\mathfrak{g}^f$ be the centralizer of f in $\mathfrak{g}$, and set $\mathfrak{g}^f_j := \mathfrak{g}^f \cap \mathfrak{g}_j$ for all $j \in \frac{1}{2}\mathbb{Z}$. By the theory of $\mathfrak{sl}_2$, we have

$$\mathfrak{g}^f = \bigoplus_{j\leqslant 0} \mathfrak{g}^f_j. \tag{1.10}$$

Theorem 1.3.1 ([76]). *For each $a \in \mathfrak{g}^f_{-j}$, $j \geqslant 0$, there exists a field $J^{\{a\}}(z)$ of $\mathcal{W}^k(\mathfrak{g}, f)$ of conformal weight $1 + j$ with respect to L such that $J^{\{a\}}(z) - J^a(z)$ is a linear combination of normally order products of the fields $J^b(z)$, where $b \in \mathfrak{g}_{-s}$, $0 \leqslant s < j$ and their derivatives.*

Let $\{a_i\}_{i\in I}$ be a basis of $\mathfrak{g}^f$ compatible with the graduation (1.10). Then $\mathcal{W}^k(\mathfrak{g}, f)$ is freely strongly generated by the fields $\{J^{\{a_i\}}\}_{i\in I}$.

In practice, to construct $J^{\{a\}}(z)$ from $J^a(z)$, we write a linear combination of fields as in the theorem and find coefficients so that the field $J^{\{a\}}(z)$ is $d_{(0)}$-closed. In [76], Kac and Wakimoto give formulas to construct strong generators with conformal weights 1, 1/2 and 0, and they compute OPEs between them. This provides in particular an explicit description of all $\mathcal{W}$-algebras associated with minimal nilpotent elements. Indeed, if f belongs to the minimal nilpotent orbit $\mathbb{O}_{\min}$ then the grading (1.5) induced by x is *minimal*:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1,$$

with $\mathfrak{g}_{-1} = \mathbb{C}f$ and $\mathfrak{g}_1 = \mathbb{C}e$.

In fact, there is a general algorithm to construct the strong generators of $\mathcal{W}^k(\mathfrak{g}, f)$ whatever their conformal weights. This algorithm is introduces in Chap. 2 when f induces an *even* good grading on $\mathfrak{g}$, i.e. when the x_0-grading (1.5) is a $\mathbb{Z}$-grading. Using the computer program Mathematica and the package [94], we compute OPEs between the strong generators. Appx. A, B and C provide the description of $\mathcal{W}$-algebras associated with Lie algebras of rank two.[1]

[1] The complexity of computations exponentially increases with values of the conformal weights. Due to the limited computational power of the machine, the OPEs of $\mathcal{W}^k(\mathfrak{sp}_4)$ and $\mathcal{W}^k(G_2)$ remain incomplete for the moment.

1.3.2 Representation theory of $\mathcal{W}$-algebras

Let M be a $V^k(\mathfrak{g})$-module, that is, a smooth $\widehat{\mathfrak{g}}$-module of level k. It can be extended into a $\mathcal{W}^k(\mathfrak{g}, f)$-module as follows. Set the $\mathscr{C}(\mathfrak{g}, f, k)$-module

$$\mathscr{C}(M) := M \otimes \mathcal{F}(\mathfrak{g}, f).$$

Set `charge` $M = 0$, then the $\mathscr{C}(\mathfrak{g}, f, k)$-module $\mathscr{C}(M)$ admits a charge decomposition inherited from the one of $\mathscr{C}(\mathfrak{g}, f, k)$:

$$\mathscr{C}(M) = \bigoplus_{m \in \mathbb{Z}} \mathscr{C}(M)_m.$$

Hence, $(\mathscr{C}(M), d_{(0)})$ is a $\mathscr{C}(\mathfrak{g}, f, k)$-module complex, and its cohomology $H_f^\bullet(M) := \bigoplus_{i \in \mathbb{Z}} H_f^i(M)$ is a direct sum of $\mathcal{W}^k(\mathfrak{g}, f)$-modules. In particular, it defines a functor

$$V^k(\mathfrak{g})\text{-Mod} \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, \qquad M \mapsto H_f^0(M). \tag{1.11}$$

Let KL_k be the full subcategory of $\mathcal{O}_k$ consisting of $\widehat{\mathfrak{g}}$-modules on which $\mathfrak{g}$ acts locally finitely. Then for $M \in \mathrm{KL}_k$, $H_f^i(M) = 0$ if $i \neq 0$, and the restriction of the previous functor

$$\mathrm{KL}_k \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, \qquad M \mapsto H_f^0(M), \tag{1.12}$$

is exact [17].

Associated varieties and lisse $\mathcal{W}$-algebras

It is known that the associated variety of $\mathcal{W}^k(\mathfrak{g}, f)$ is the *Slodowy slice* $\mathscr{S}_f := f + \mathfrak{g}^e$, where $\mathfrak{g}^e$ is the centralizer of e in $\mathfrak{g}$ [48]. Its Poisson structure comes from the one of $\mathfrak{g}^*$ by Hamiltonian reduction. The Li filtration defined in Sect. 1.1.4 is compatible with the functor (1.12). Hence, for any finitely generated module $M \in \mathrm{KL}_k$, $R_{H_f^0(M)} \simeq H_f^0(R_M)$ as Poisson modules over $R_{\mathcal{W}^k(\mathfrak{g},f)} = \mathbb{C}[\mathscr{S}_f]$ [48]. Moreover, $\tilde{X}_{H_f^0(M)} = \tilde{X}_M \times_{\mathfrak{g}^*} \mathscr{S}_f$ [17]. In particular, the associated variety of $H_f^0(L_k(\mathfrak{g}))$ is isomorphic to the intersection

$$X_{L_k(\mathfrak{g})} \cap \mathscr{S}_f.$$

Furthermore, $H_f^0(L_k(\mathfrak{g}))$ is a quotient of $\mathcal{W}^k(\mathfrak{g}, f) = H_f^0(V^k(\mathfrak{g}))$. Thus, the simple $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is a quotient of $H_f^0(L_k(\mathfrak{g}))$, provided that the latter is non-zero. We deduce that $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse whenever $X_{L_k(\mathfrak{g})}$ is the closure of the nilpotent orbit $\mathbb{O}$ of f in $\mathfrak{g}$. Indeed, if $f \in \mathbb{O}$, we have $\overline{\mathbb{O}} \cap \mathscr{S}_f = \{f\}$. Consequently, we have the following result:

Proposition 1.3.2 ([17]). *Assume that $k = -h^\vee + p/q$ is admissible for $\mathfrak{g}$ and pick $f \in \mathbb{O}_q$, with $\overline{\mathbb{O}_q}$ the associated variety of $L_k(\mathfrak{g})$ (cf. Proposition 1.2.2). Then $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse.*

Pairs (f, k) with $k = -h^\vee + p/q$ admissible and $f \in \mathbb{O}_q$ are called *exceptional* [17]. This extends the original notion of exceptional pair given by Kac-Wakimoto [78]. If (f, k) is exceptional, the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is said *exceptional*. It has been recently proved that exceptional $\mathcal{W}$-algebras are rational. We give more details about this result in the following.

Finite $\mathcal{W}$-algebras and Zhu's algebra $\mathcal{A}(\mathcal{W}^k(\mathfrak{g}, f))$

We can extend the construction of the Zhu's algebra introduced in Sect. 1.1.3 to any module M of a vertex algebra V setting $\mathcal{A}(M) = M/(V \circ M)$ [63]. Then $\mathcal{A}(M)$ is a bimodule over $\mathcal{A}(V)$.

Recall that $\mathcal{A}(V^k(\mathfrak{g})) \simeq U(\mathfrak{g})$. For $M \in \mathrm{KL}_k$, $\mathcal{A}(M)$ belongs to $\mathcal{HC}(\mathfrak{g})$, the category of Harish-Chandra bimodules. Hence, $\mathcal{A}(M)$ is a finitely generated $U(\mathfrak{g})$-bimodule on which the adjoint action of $\mathfrak{g}$ is locally finite. For any Harish-Chandra $U(\mathfrak{g})$-bimodule M, one defines an analogue of the Drinfeld-Sokolov reduction $H_f^0(M)$ which is naturally a $H_f^0(U(\mathfrak{g}))$-bimodule [18]. Using the exactness of Zhu's functor, we get $H_f^0(\mathcal{A}(M)) \simeq \mathcal{A}(H_f^0(M))$ for any module $M \in \mathrm{KL}_k$. In particular,

$$\mathcal{A}(\mathcal{W}^k(\mathfrak{g}, f)) := H_f^0(U(\mathfrak{g})) = U(\mathfrak{g}, f),$$

is the *finite $\mathcal{W}$-algebra* [89] associated with $f \in \mathfrak{g}$ [48]. The finite $\mathcal{W}$-algebra $U(\mathfrak{g}, f)$ is the enveloping algebra of the Slodowy slice $\mathscr{S}_f$. Moreover, $U(\mathfrak{g}, f) \simeq \mathrm{End}_{U(\mathfrak{g})}(Y)^{\mathrm{op}}$ where $Y = U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi$ is induced from a one-dimensional representation $\mathbb{C}_\chi$ over a certain nilpotent subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ of dimension $\dim \mathfrak{m} = \frac{1}{2} \dim \mathbb{O}_f$. We refer to [13, 47, 48] for a precise construction of $U(\mathfrak{g}, f)$.

In [21], Arakawa and van Ekeren prove the semisimplicity of the Zhu's algebra $\mathcal{A}(H_f^0(L_k(\mathfrak{g})))$ when (f, k) is an exceptional pair. It implies the semisimplicity of the quotient $\mathcal{A}(\mathcal{W}_k(\mathfrak{g}, f))$ of $\mathcal{A}(H_f^0(L_k(\mathfrak{g})))$. Recently, McRae showed [86, Theorem 5.10] that a VOA V is rational if its Zhu's algebra $\mathcal{A}(V)$ is semisimple, proving in particular the conjecture of Kac-Wakimoto and Arakawa.

Theorem 1.3.3 (McRae [86], conjectured by Kac-Wakimoto [76] and Arakawa [17]). *If (f, k) is an exceptional pair then $\mathcal{W}_k(\mathfrak{g}, f)$ is rational.*

Ramond twisted representation and "$-$"-reduction functor

Previously we mentioned that the functor (1.12) is exact. However, in order to recover all the irreducible positive energy representations of $\mathcal{W}^k(\mathfrak{g}, f)$ – which are in correspondence with simple $\mathcal{A}(\mathcal{W}^k(\mathfrak{g}, f))$-modules – we need to extend this functor to the whole category $\mathcal{O}_k$. Unfortunately, the functor

$$\mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, \qquad M \mapsto H_f^0(M), \tag{1.13}$$

is not exact in general. It is if f belongs to the minimal nilpotent orbit of $\mathfrak{g}$ [12]. One modifies the functor (1.13) into a "$-$"-reduction functor $H_{f,-}^0(?)$ defined in [62] (see also [14, 78]).

Let σ_R be the automorphism of the complex $\mathscr{C}(\mathfrak{g}, f, k)$ defined by $\sigma_R = \mathrm{e}^{2\pi i x_0}$. It fixes the vector d and thus, induces an automorphism of $\mathcal{W}^k(\mathfrak{g}, f)$ [77]. If M is a $V^k(\mathfrak{g})$-module then $H_{f,-}^0(M)$ carries a structure of a *Ramond twisted representation* of $\mathcal{W}^k(\mathfrak{g}, f)$ coming from the σ_R-twisted action on $\mathscr{C}(M)$:

$$u \cdot m = \widehat{w}_0 \widehat{t}_{-x_0}(u) \cdot m, \tag{1.14}$$

where the isomorphim $\widehat{t}_{-x_0}$ is given by [78]:

$$\begin{aligned}
\widehat{t}_{-x_0} : J^{\{e_\alpha\}}(n)^R &\mapsto J^{\{e_\alpha\}}(n + \alpha(x_0)), && (\alpha \in \Delta),\\
J^{\{h\}}(n)^R &\mapsto J^{\{h\}}(n) + \delta_{n,0}(x_0|h)K, && (h \in \mathfrak{h}),\\
K^R &\mapsto K,\\
D^R &\mapsto D - x_0(0),\\
\varphi_\alpha(n)^R &\mapsto \varphi_\alpha(n + \alpha(x_0)), && (\alpha \in S_+ \cup S_-),\\
\Phi_\alpha(n)^R &\mapsto \Phi_\alpha(n + 1/2), && (\alpha \in S_{1/2}),
\end{aligned}$$

where φ_α stands for $\varphi^{-\alpha}$ when $\alpha \in S_-$, and $\widehat{w}_0$ is the lift of the longest element w_0 of the Weyl group W such that

$$\begin{aligned} &\widehat{w}_0(J^{\{e_\alpha\}}(n)) = c_{w_0(\alpha)} J^{\{e_{w_0(\alpha)}\}}(n), && (\alpha \in \Delta),\\ &\widehat{w}_0(\varphi_\alpha(n)) = c_{w_0(\alpha)} \varphi_{w_0(\alpha)}(n), && (\alpha \in S_+),\\ &\widehat{w}_0(\varphi_{-\alpha}(n)) = c_{w_0(\alpha)}^{-1} \varphi_{-w_0(\alpha)}(n), && (\alpha \in S_+), \end{aligned}$$

with $c_\alpha \in \mathbb{C}^*$.

Let $\mathcal{O}_{0,k}$ be the full subcategory of the category of left $\widehat{\mathfrak{g}}$-modules of level k consisting in the objects M such that

- M admits a weight space decomposition with respect to the action of $\widehat{\mathfrak{h}}$,
- there is a finite subset of weights $\{\mu_1, \ldots, \mu_n\} \subset \mathfrak{h}^*$ such that $M = \bigoplus_{\mu \in \cup_i \mu_i - Q_+} M^\mu$, where $Q_+ = \sum_{\alpha \in \Delta_+} \mathbb{Z}_{\geqslant 0}\alpha$,
- for each $d \in \mathbb{C}$, M_d is a direct sum of finite dimensional $\mathfrak{g}_0$-modules.

The functor

$$H^0_{f,-}(?) : \mathcal{O}_{0,k} \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}_R,$$

is exact.

Note that a $\mathcal{W}$-algebra is generally $\frac{1}{2}\mathbb{Z}_{\geqslant 0}$-graded, as well as its irreducible representations whereas Ramond twisted representations are $\mathbb{Z}_{\geqslant 0}$-graded [78]. Also, when $\mathcal{W}^k(\mathfrak{g}, f)$ is $\mathbb{Z}_{\geqslant 0}$-graded – for instance when f is even – a Ramond twisted module is the same as an untwisted module. In fact, the finite $\mathcal{W}$-algebra $U(\mathfrak{g}, f)$ appears as the Zhu's algebra of $\mathcal{W}^k(\mathfrak{g}, f)$ when we consider Ramond twisted representations rather than untwisted ones in Zhu's correspondence. The same holds for the simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$.

For a simple $\mathcal{A}(\mathcal{W}^k(\mathfrak{g}, f))$-module E, denote $\mathbf{L}(E)$ the corresponding irreducible Ramond twisted representation of $\mathcal{W}^k(\mathfrak{g}, f)$. The module $\mathbf{L}(E)$ is the unique simple quotient of the Verma module,

$$\mathbf{M}(E) := U(\mathcal{W}^k(\mathfrak{g}, f)) \otimes_{U(\mathcal{W}^k(\mathfrak{g}, f))_{\geqslant 0}} E,$$

where $U(\mathcal{W}^k(\mathfrak{g}, f))$ is the Ramond twisted current algebra of $\mathcal{W}^k(\mathfrak{g}, f)$ [21, 63].

Characters of highest weight representations

Let M be a highest weight $\widehat{\mathfrak{g}}$-module of level $k \neq -h^\vee$ with highest weight $\hat{\lambda} \in \widehat{\mathfrak{h}}^*$ and highest weight vector $v(\hat{\lambda})$. If $v(\hat{\lambda})$ is not in the image of $d_{(0)}$, its image $\tilde{v}(\hat{\lambda})$ in $H^0_f(M)$ generates a non-zero $\mathcal{W}^k(\mathfrak{g}, f)$-module. Moreover, the eigenvalue of $\tilde{v}(\hat{\lambda})$ for L_0 is given by

$$\frac{(\hat{\lambda}|\hat{\lambda} + 2\widehat{\rho})}{2(k + h^\vee)} - (x + D|\hat{\lambda}),$$

and for $h \in \mathfrak{h}^f$, its eigenvalue for $J_0^{\{h\}}$ is $\hat{\lambda}(h)$.

The character of a $\widehat{\mathfrak{g}}$-module M is defined as the formal serie

$$\mathrm{ch}_M(\tau, z, u) := \mathrm{tr}_M\, \mathrm{e}^{2\pi i(z - \tau D + uK)},$$

where $z \in \mathfrak{h}$ and $\tau, u \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$. In the following we fix $u = 0$ and consider

$\mathrm{ch}_M(\tau, z) := \mathrm{ch}_M(\tau, z, u)$. In particular, it was proved by Kac-Wakimoto [74] that for $\lambda \in \mathrm{Pr}^k$,

$$\mathrm{ch}_{\widehat{L}_k(\lambda)} = \frac{1}{\widehat{R}} \sum_{w \in \widehat{W}(\hat{\lambda})} \epsilon(w)\, \mathrm{e}^{w \circ \hat{\lambda}},$$

where $\widehat{R} = \prod_{\alpha \in \widehat{\Delta}_+} (1 - e^{-\alpha})^{\mathrm{mult}\alpha}$ is the Weyl denominator for $\widehat{\mathfrak{g}}$ and $w \circ \hat{\lambda} = w(\hat{\lambda} + \widehat{\rho}) - \widehat{\rho}$ is the "dot"-action extended to $\widehat{W}$. This character can be considered as a meromorphic function of $(\tau, z) \in \mathcal{H} \times \mathfrak{h}$, with $\mathcal{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\, \tau > 0\}$,

$$\mathrm{ch}_{\widehat{L}_k(\lambda)}(\tau, z) = \frac{1}{\widehat{R}} \sum_{w \in \widehat{W}(\hat{\lambda})} \epsilon(w)\, \mathrm{e}^{-2\pi i (w \circ \hat{\lambda} | \tau D - z)}\, .$$

Following [73], let $\mathrm{ch}_{H_f^\bullet(M)}$ be the Euler-Poincaré character of $H_f^\bullet(M)$:

$$\mathrm{ch}_{H_f^\bullet(M)}(q, h) = \sum_{j \in \mathbb{Z}} (-1)^j \,\mathrm{ch}_{H_f^j(M)}(q, z) = \sum_{j \in \mathbb{Z}} (-1)^j \,\mathrm{tr}_{H_f^j(M)}\, q^{L_0}\, \mathrm{e}^{2\pi i J_0^{\{h\}}}, \tag{1.15}$$

with $q = e^{2\pi i \tau}$ and $h \in \mathfrak{h}^f$.

Theorem 1.3.4 ([73]). *Let M be the highest weight $\widehat{\mathfrak{g}}$-module of level $k \neq h^\vee$ with highest weight $\hat{\lambda}$. Suppose that ch_M extends to a meromorphic function on $\mathcal{H} \times \mathfrak{h}$ with at most simple poles at the hyperplane $T_\alpha = \{h \in \widehat{\mathfrak{h}} \mid \alpha(h) = 0\}$, $\alpha \in \widehat{\Delta}^{re}$. Then*

$$\begin{aligned}\mathrm{ch}_{H_f^\bullet(M)}(q, h) = {} & \frac{q^{\frac{(\hat{\lambda}|\hat{\lambda}+2\widehat{\rho})}{2(k+h^\vee)}}}{\prod_{j=1}^\infty (1 - q^j)^{\dim \mathfrak{h}}} (\widehat{R}\, \mathrm{ch}_M(\tau, -\tau x + h)) \\ & \times \prod_{n=1}^\infty \prod_{\alpha \in \Delta_+, (\alpha|x_0)=0} (1 - q^{n-1}\, \mathrm{e}^{-2\pi i (\alpha|h)})^{-1} (1 - q^n\, \mathrm{e}^{2\pi i(\alpha|h)})^{-1} \\ & \times \prod_{n=1}^\infty \prod_{\alpha \in \Delta_+, (\alpha|x_0)=1/2} (1 - q^{n-1/2}\, \mathrm{e}^{2\pi i (\alpha|h)})^{-1},\end{aligned}$$

where $h \in \mathfrak{h}^f$.

Chapter 2

OPEs and collapsing levels of $\mathcal{W}$-algebras associated with simple Lie algebras of rank 2

In this chapter, we present the concrete construction of $\mathcal{W}$-algebras in terms of strong generators and relations (OPEs) due to [76]. To simplify the theory, we assume that f admits an *even* good grading on $\mathfrak{g}$ (see Sect. 1.3.1). In particular, this covers all cases in type A ($\mathfrak{sl}_n$) and cases where f is *even*, i.e. its Dynkin grading is a $\mathbb{Z}$-grading, such as principal or subregular nilpotent elements. Then the derivation $d_{(0)}$ splits into two derivations which define a structure of bicomplex over $\mathscr{C}(\mathfrak{g}, f, k)$. The computation of the strong generators of $\mathcal{W}^k(\mathfrak{g}, f)$ is simpler and we recall an algorithm to construct them in Sect. 2.1. The structure of bicomplex remains valid for any nilpotent element but explicit computations are more difficult when the grading is not even. We consider several examples of this type in our work.

Even if the construction of the strong generators is known, the computation of OPEs remains difficult. Only few examples were explicitly computed so far [41, 76]. We compute generators and OPEs for $\mathcal{W}$-algebras associated to Lie algebras of rank 2 using the Mathematica package [94]. Results of our computations are presented in Appx. A, B, and C. The computational complexity increases exponentially with the value of conformal weights. As a consequence, due to limited capacity of the machine, the OPEs for principal $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{sp}_4)$ and $\mathcal{W}^k(G_2)$ remain incomplete for the moment.

For particular values of the level k, said *collapsing*, the simple $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is isomorphic to another vertex algebra – for instance, an affine vertex algebra. Collapsing levels of minimal $\mathcal{W}$-algebras have been completely classified in [8, 9] using OPEs. However, it is difficult to predict which levels collapse for other nilpotent elements, since OPEs are not known. Few particular cases have been computed explicitly [6, 7]. Recently, Arakawa, van Ekeren and Moreau [22] provided a large – and conjecturally complete – family of *admissible* collapsing levels. In Sect. 2.2, we study conditions to obtain certain isomorphisms of $\mathcal{W}$-algebras (Propositions 2.2.2 and 2.2.7). We provide several new examples of generalized collapsing levels which are not admissible (Tables 2.2 and 2.3). We thank Prof. Dražen Adamović for drawing our attention to new collapsing levels in $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$.

2.1 Explicit construction of generating fields of $\mathcal{W}$-algebras

2.1.1 Structure of bicomplex over $\mathscr{C}(\mathfrak{g}, f, k)$

We keep the notations of Chap. 1. Let f be a nilpotent element of $\mathfrak{g}$ admitting an even good grading. The grading (1.5), defined by $x_0 = \frac{h}{2}$ on $\mathfrak{g}$, turns to a $\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j. \tag{2.1}$$

As a consequence, there is no neutral part in the complex $\mathscr{C}(\mathfrak{g}, f, k) = V^k(\mathfrak{g}) \otimes \mathcal{F}(\mathfrak{g}_{>0})$. Let $\mathscr{C}^+$ be the vertex subalgebra of $\mathscr{C}(\mathfrak{g}, f, k)$ generated by the fields $\varphi_\alpha(z)$ and $J^{e_\alpha}(z)$, $\alpha \in S_+$, and $\mathscr{C}^-$ be the vertex subalgebra generated by $\varphi^\alpha(z)$, $\alpha \in S_+$, and $J^{e_\alpha}(z)$, $e_\alpha \in \mathfrak{g}_{\leqslant 0}$. Obviously, as vector space, $\mathscr{C}(\mathfrak{g}, f, k) = \mathscr{C}^+ \otimes \mathscr{C}^-$. Moreover, $\mathscr{C}^+$ and $\mathscr{C}^-$ inherit cohomological grading from the one of $\mathscr{C}(\mathfrak{g}, f, k)$:

$$\mathscr{C}^+ = \bigoplus_{i \leqslant 0} \mathscr{C}_i^+, \qquad \text{and} \qquad \mathscr{C}^- = \bigoplus_{i \geqslant 0} \mathscr{C}_i^-.$$

Hence,

$$\mathcal{W}^k(\mathfrak{g}, f) = \bigoplus_{m+n=0} H^m(\mathscr{C}^+, d) \otimes H^n(\mathscr{C}^-, d).$$

Recall that $d(z) = d_{\mathrm{st}}(z) + d_{\mathrm{nd}}(z)$. Because of the even grading, the second sum in $d_{\mathrm{nd}}(z)$ disappears and we get

$$d_{\mathrm{st}}(z) = \sum_{\alpha \in S_+} : e_\alpha(z)\varphi^\alpha(z) : -\frac{1}{2} \sum_{\alpha,\beta,\gamma \in S_+} c_{\alpha,\beta}^{\gamma} : \varphi_\gamma(z)\varphi^\alpha(z)\varphi^\beta(z) :, \tag{2.2}$$

$$d_{\mathrm{nd}}(z) = \sum_{\alpha \in S_+} (f|e_\alpha)\varphi^\alpha(z). \tag{2.3}$$

Set $d_{\mathrm{st}}(z) = \sum_{n \in \mathbb{Z}} d_{\mathrm{st}}(n) z^{-n-1}$ and $d_{\mathrm{nd}}(z) = \sum_{n \in \mathbb{Z}} d_{\mathrm{nd}}(n) z^{-n}$. Then the fields $d_{\mathrm{st}}(z)$ and $d_{\mathrm{nd}}(z)$ respectively correspond to the vectors $d_{\mathrm{st},(0)}$ and $d_{\mathrm{nd},(0)}$ given by

$$d_{\mathrm{st},(0)} = d_{\mathrm{st}}(0) = \sum_{\substack{\alpha \in S_+ \\ n \in \mathbb{Z}}} : e_\alpha(-n)\varphi^\alpha(n) : -\frac{1}{2} \sum_{\substack{\alpha,\beta,\gamma \in S_+ \\ k+l+m=0}} c_{\alpha,\beta}^{\gamma} : \varphi_\gamma(k)\varphi^\alpha(l)\varphi^\beta(m) :,$$

$$d_{\mathrm{nd},(0)} = d_{\mathrm{nd}}(1) = \sum_{\alpha \in S_+} (f|e_\alpha)\varphi^\alpha(1).$$

By abuse of notation, denote d, d_{st}, d_{nd} for $d_{(0)}$, $d_{\mathrm{st},(0)}$, $d_{\mathrm{nd},(0)}$. By direct calculation, we check that d_{st} and d_{nd} are two derivations which commute with each other:

$$d_{\mathrm{st}}^2 = d_{\mathrm{nd}}^2 = [d_{\mathrm{st}}, d_{\mathrm{nd}}] = 0.$$

On the one hand, for $n \in \mathbb{Z}$ and $\alpha \in S_+$, we have the relations

$$d_{\mathrm{st}}(\varphi_\alpha) = J^{e_\alpha}, \qquad d_{\mathrm{nd}}(\varphi_\alpha) = (f|e_\alpha)|0\rangle, \qquad d_{\mathrm{st}}(J^{e_\alpha}) = d_{\mathrm{nd}}(J^{e_\alpha}) = 0.$$

Consequently, the cohomology $H^\bullet(\mathscr{C}^+, d)$ vanishes for non-zero degrees and $H^0(\mathscr{C}^+, d) = \mathbb{C}|0\rangle$. Thus,

$$\mathcal{W}^k(\mathfrak{g}, f) = H^0(\mathscr{C}^-, d).$$

On the other hand, for all $e_\alpha \in \mathfrak{g}_{\leqslant 0}$ and $\gamma \in S_+$, we have

$$\begin{aligned}
d_{\mathrm{st}}(J^{e_\alpha}) &= -\sum_{\beta\in S_+,[e_\alpha,e_\beta]\in\mathfrak{g}_{\leqslant 0}} :\varphi^\beta J^{[e_\alpha,e_\beta]}: + \sum_{\beta\in S_+}(k(e_\alpha|e_\beta)+\mathrm{tr}_{\mathfrak{g}_{>0}}(\mathrm{ad}\,e_\alpha)(\mathrm{ad}\,e_\beta))\partial\varphi^\beta,\\
d_{\mathrm{nd}}(J^{e_\alpha}) &= \sum_{\beta\in S_+}([f,e_\alpha]|e_\beta)\varphi^\beta,\\
d_{\mathrm{st}}(\varphi^\gamma) &= -\frac{1}{2}\sum_{\alpha,\beta\in S_+} c^\gamma_{\alpha,\beta}\varphi^\alpha\varphi^\beta,\\
d_{\mathrm{nd}}(\varphi^\gamma) &= 0.
\end{aligned}$$

Let consider first the cohomology induced by d_{nd} on $\mathscr{C}^-$. We deduce from the relations above that all degrees of the cohomology $H^\bullet(\mathscr{C}^-, d_{\mathrm{nd}})$ vanish except the zero-th. Besides, $H^0(\mathscr{C}^-, d_{\mathrm{nd}})$ is a vertex algebra strongly generated by the fields $J^a(z)$, where $a \in \mathfrak{g}^f$ is a linear combination of elements $e_\alpha \in \mathfrak{g}_{\leqslant 0}$.

Define a $\mathbb{Z}^2$-grading on $\mathscr{C}^- = \bigoplus_{m,n\in\mathbb{Z}} \mathscr{C}^-_{m,n}$ by

$$\mathrm{bideg}\,|0\rangle = (0,0), \qquad \mathrm{bideg}\,J^a = (-j,j)\ (a\in\mathfrak{g}_{-j}), \qquad \text{and} \qquad \mathrm{bideg}\,\varphi^\alpha = (j,-j+1)\ (e_\alpha\in\mathfrak{g}_j).$$

Since $d_{\mathrm{st}}(\mathscr{C}^-_{m,n}) \subset \mathscr{C}^-_{m,n+1}$ and $d_{\mathrm{nd}}(\mathscr{C}^-_{m,n}) \subset \mathscr{C}^-_{m+1,n}$, the direct sum $\mathscr{C}^-$ admits a structure of bicomplex:

$$\begin{array}{ccccccccc}
 & & & & \vdots & & \vdots & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & \cdots & \xrightarrow{d_{\mathrm{st}}} & \mathscr{C}^-_{-1,0} & \xrightarrow{d_{\mathrm{st}}} & \mathscr{C}^-_{-1,1} & \longrightarrow & \cdots\\
 & & & & \downarrow d_{\mathrm{nd}} & & \downarrow d_{\mathrm{nd}} & & \\
\cdots & \xrightarrow{d_{\mathrm{st}}} & \mathscr{C}^-_{0,-1} & \xrightarrow{d_{\mathrm{st}}} & \mathscr{C}^-_{\mathbf{0,0}} & \xrightarrow{d_{\mathrm{st}}} & \mathscr{C}^-_{0,1} & \longrightarrow & \cdots\\
 & & \downarrow d_{\mathrm{nd}} & & \downarrow d_{\mathrm{nd}} & & \downarrow d_{\mathrm{nd}} & & \\
\cdots & \xrightarrow{d_{\mathrm{st}}} & \mathscr{C}^-_{\mathbf{1,-1}} & \xrightarrow{d_{\mathrm{st}}} & \mathscr{C}^-_{1,0} & \xrightarrow{d_{\mathrm{st}}} & \mathscr{C}^-_{1,1} & \longrightarrow & \cdots\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Furthermore, $H^0(\mathscr{C}^-, d_{\mathrm{nd}}) \subset \bigoplus_{m\in\mathbb{Z}_{\geqslant 0}} \mathscr{C}^-_{m,-m}$. Then d_{st} induces a zero differential on $H^0(\mathscr{C}^-, d_{\mathrm{nd}})$ and all higher differentials of the spectral sequence vanish.

The two derivations, d_{st} and d_{nd}, preserve the conformal grading induced by L_0. In practice, for $a \in \mathfrak{g}^f_{-j}$ ($j \geqslant 0$), the strong generator $J^{\{a\}}$ of $\mathcal{W}^k(\mathfrak{g}, f)$ of conformal weight $1+j$ is construct by extending the close field J^a with respect to d_{nd} into a close field with respect to d. Initially $d_{\mathrm{nd}}(J^a) = 0$. If $d_{\mathrm{st}}(J^a) = 0$ then set $J^{\{a\}} := J^a$. Else, we look for a field X^a_1 of conformal weight $1+j$ such that $d_{\mathrm{st}}(J^a) = -d_{\mathrm{nd}}(X^a_1)$. The two sides of the equality must have the same bidegree for the $\mathbb{Z}^2$-grading on $\mathscr{C}^-$. Hence,

$$\begin{aligned}
\mathrm{bideg}\,X^a_1 &= \mathrm{bideg}(d_{\mathrm{nd}}(X^a_1)) + (1,0)\\
&= \mathrm{bideg}\,d_{\mathrm{st}}(J^a) + (1,0)\\
&= \mathrm{bideg}\,J^a + (0,-1) + (1,0) = (-j+1, j-1).
\end{aligned}$$

As a consequence, $X_1^a \in \mathscr{C}^-_{-j+1,j-1}$ is a linear combination of normally ordered products of fields J^{e_α}, $e_\alpha \in \mathfrak{g}_{-j+1}$ and their derivatives. Differentiating a field increases its conformal weight of 1 but does not impact the bidegree, and the conformal weight of a normally ordered product of fields is the sum of the conformal weights of each of them. If $d_{\mathrm{nd}}(X_1^a) = 0$ then $J^{\{a\}} := J^a + X_1^a$, else iterate and look for X_2^a such that $d_{\mathrm{st}}(X_1^a) = -d_{\mathrm{nd}}(X_2^a)$, etc. In this way, we construct a sequence of fields $(X_i^a)_{i\geqslant 0}$ of conformal weight $j+1$ such that

$$X_0^a = J^a, \qquad X_i^a \in \mathscr{C}^-_{-j+i,j-i}, \qquad \text{and} \qquad d_{\mathrm{st}}(X_i^a) = -d_{\mathrm{nd}}(X_{i+1}^a).$$

In the worst case, $d_{\mathrm{st}}(X_j^a) = 0$ because $\mathscr{C}^-_{m,n} = 0$ for $m > 0$.

2.1.2 Example: construction of the strong generators of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\mathrm{subreg}})$

In this paragraph, consider $\mathfrak{g}$ to be the simple Lie algebra $\mathfrak{sp}_4$ that we may realize as the set of 4-size square matrices x such that $x^T J_4 + J_4 x = 0$, where J_4 is the anti-diagonal matrix given by

$$J_4 = \begin{pmatrix} 0 & U_2 \\ -U_2 & 0 \end{pmatrix}, \qquad \text{where } U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We make the standard choice that $\mathfrak{h}$ is the set of diagonal matrices of $\mathfrak{g}$. Nilpotent orbits of $\mathfrak{g} = \mathfrak{sp}_4$ are parameterized by the partitions of 4 such that the number of parts of each odd number is even (see, for instance, [39, Theorem 5.1.3]). Thus, there are four nilpotent orbits in $\mathfrak{g} = \mathfrak{sp}_4$ corresponding to the following partitions: (4), (2^2), $(2,1^2)$, (1^4). They correspond respectively to the principal, subregular, minimal and zero nilpotent orbits of $\mathfrak{g}$, with respective dimensions 8, 6, 4, 0. Write $\Pi = \{\alpha_1, \alpha_2\}$ a set of simple roots for the root system Δ of $(\mathfrak{g}, \mathfrak{h})$ such that α_1 is a long root and α_2 is short. Then $\Delta_+ = \{\alpha_1, \alpha_2, \eta, \theta\}$, with $\eta := \alpha_1 + \alpha_2$ and $\theta := \alpha_1 + 2\alpha_2$ is the highest positive root.

The centralizer of $e_{-\eta}$ is four-dimensional generated by $e_{-\eta}, e_{-\alpha_1}, e_{-\theta}, h_2$, where $h_i := \alpha_i^\vee \in (\mathfrak{h}^*)^* \cong \mathfrak{h}$, for $i = 1, 2$. Hence, $e_{-\eta}$ belongs to the subregular nilpotent orbit of $\mathfrak{g}$. In this section, we fix

$$f := e_{-\eta} = f_{\mathrm{subreg}}.$$

Setting $e := e_\eta$ and $h := [e, f] = 2h_1 + h_2$ we get the $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$. Let $x_0 := \frac{h}{2} = \varpi_1$. The nilpotent element f is even and we have

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

Moreover, $\mathfrak{g}^f = \mathfrak{g}^f_{-1} \oplus \mathfrak{g}^f_0$, with $\mathfrak{g}^f_{-1} = \mathbb{C}f \oplus \mathbb{C}e_{-\alpha_1} \oplus \mathbb{C}e_{-\theta}$ and $\mathfrak{g}^f_0 = \mathbb{C}h_2$. It follows from Theorem 1.3.1 that $\mathcal{W}^k(\mathfrak{sp}_4, f)$ is strongly generated by the fields $J^{\{f\}}$, $J^{\{e_{-\alpha_1}\}}$, $J^{\{e_{-\theta}\}}$ and $J^{\{h_2\}}$. We detail their construction in the following.

Remark 2.1.1. The smallest Levi subalgebra of $\mathfrak{sp}_4$ containing f has semisimple type A_1 with basis $h_1, h_2, e_{\pm\eta}$ (it is the centralizer in $\mathfrak{sp}_4$ of h_2). Hence, f has *Levi type*. It means that f is a principal nilpotent element of the minimal Levi subalgebra that contains it.

First of all, because of the decomposition of $\mathfrak{g}$, there are only two possibilities for the bidegree of the fields J^a, $a \in \mathfrak{g}_{\leqslant 0}$. We classify these fields and certain of their derivatives in the Table 2.1 depending on their bidegrees as well as their conformal weights.

bideg	conformal weight	fields
$(-1,1)$	2	J^f, $J^{e_{-\alpha_1}}$, $J^{e_{-\theta}}$
$(0,0)$	2	∂J^{h_1}, ∂J^{h_2}, $\partial J^{e_{\alpha_2}}$, $\partial J^{e_{-\alpha_2}}$
$(0,0)$	1	J^{h_1}, J^{h_2}, $J^{e_{\alpha_2}}$, $J^{e_{-\alpha_2}}$

Table 2.1 – Bidegrees and conformal weights

By direct computation, $d_{\mathrm{st}}(J^{h_2}) = d_{\mathrm{nd}}(J^{h_2}) = 0$, so we pick for $J^{\{h_2\}}$ any multiple of J^{h_2}. For normalization reasons, set

$$J = J^{\{h_2\}} := \frac{1}{2} J^{h_2} = J^{h_2/2} = J^{\alpha_2}.$$

On the contrary of J^{h_2}, the field J^f is not vanished by the derivation d_{st}. We look for a field X_1^f such that $d_{\mathrm{st}}(J^f) = -d_{\mathrm{nd}}(X_1^f)$. Using bidegree and conformal weight considerations, we get that X_1^f is a linear combination of fields of bidegree $(0,0)$ and conformal weight 2 (see Table 2.1):

$$X_1^f = -\frac{1}{2}\left(: J^{e_{\alpha_2}} J^{e_{-\alpha_2}} : + : (J^{h_1})^2 : + : J^{h_1} J^{h_2} : + (3+2k)\partial J^{h_1}\right).$$

Moreover, since $d_{\mathrm{nd}}(X_1^f) = 0$, $J^{\{f\}} := J^f + X_1^f$. We have similar computations for the rest of the generators:

$$\begin{aligned} G^+ = J^{\{e_{-\alpha_1}\}} &:= J^{e_{-\alpha_1}} + \frac{1}{2}(: J^{h_1} J^{e_{\alpha_2}} : + (k+2)\partial J^{e_{\alpha_2}}), \\ G^- = J^{\{e_{-\theta}\}} &:= J^{e_{-\theta}} + \frac{1}{2}(: J^{h_1} J^{e_{-\alpha_2}} : + : J^{h_2} J^{e_{-\alpha_2}} : + (k+2)\partial J^{e_{-\alpha_2}}). \end{aligned}$$

The field $J^{\{f\}}$ is a conformal vector but its central charge differs from the Sugawara's 1.9:

$$-\frac{(k+1)(43+46k+12k^2)}{2}.$$

To remedy this, we slightly modify the field $J^{\{f\}}$ into a field L with the same conformal weight. We write L as a linear combination of fields which act semisimply – in particular $J^{\{f\}}$ and fields $J^{\{a\}}$, $a \in \mathfrak{h}$ – their derivatives and normally ordered products:

$$L = -\frac{1}{(3+k)} J^{\{f\}} + \frac{(1+k)}{(3+k)} \partial J^{\{h_2\}} + \frac{1}{(3+k)} : {J^{\{h_2\}}}^2 : .$$

2.2 Particular isomorphisms of $\mathcal{W}$-algebras associated with rank-two Lie algebras

Let f be a nilpotent element of $\mathfrak{g}$ embedded into an $\mathfrak{sl}_2$-triple (e,h,f). The centralizer in $\mathfrak{g}$ of this triple, denoted $\mathfrak{g}^\natural$, is the intersection of centralizers of at least two elements of the triple. In particular, $\mathfrak{g}^\natural = \mathfrak{g}_0^f$. It is a reductive Lie subalgebra of $\mathfrak{g}$, hence $\mathfrak{g}^\natural = \mathfrak{z}(\mathfrak{g}^\natural) \oplus [\mathfrak{g}^\natural, \mathfrak{g}^\natural]$ with $\mathfrak{z}(\mathfrak{g}^\natural)$ the center of $\mathfrak{g}^\natural$, and

$$[\mathfrak{g}^\natural, \mathfrak{g}^\natural] = \bigoplus_{i=1}^{s} \mathfrak{g}_i^\natural,$$

where $\mathfrak{g}_1^\natural, \dots, \mathfrak{g}_s^\natural$ are simple Lie algebras. Set

$$V^{k^\natural}(\mathfrak{g}^\natural) := M(1)^{\otimes d} \otimes \bigotimes_{i=1}^{s} V^{k_i^\natural}(\mathfrak{g}_i^\natural),$$

where $M(1)$ denotes the Heisenberg vertex algebra of central charge 1, d is a non negative integer and $k_1^\natural, \dots, k_s^\natural$ are complex numbers all depending on k. We refer to [9] or [22] for the precise relations of dependence between d, k_i and k.

According to [76], the vertex algebra $V^{k^\natural}(\mathfrak{g}^\natural)$ is a vertex subalgebra of $\mathcal{W}^k(\mathfrak{g}, f)$ via the embedding

$$\iota : V^{k^\natural}(\mathfrak{g}^\natural) \hookrightarrow \mathcal{W}^k(\mathfrak{g}, f).$$

Denote $\mathcal{V}^k(\mathfrak{g}^\natural) := \iota(V^{k^\natural}(\mathfrak{g}^\natural))$ and $\mathcal{V}_k(\mathfrak{g}^\natural)$ the image of $\mathcal{V}^k(\mathfrak{g}^\natural)$ by the canonical projection $\mathcal{W}^k(\mathfrak{g}, f) \twoheadrightarrow \mathcal{W}_k(\mathfrak{g}, f)$. The level k is said to be *collapsing* [9] if $\mathcal{W}_k(\mathfrak{g}, f) \simeq \mathcal{V}_k(\mathfrak{g}^\natural)$. Equivalently, k is a collapsing level if and only if there is a surjective vertex algebra morphism

$$\mathcal{W}^k(\mathfrak{g}, f) \twoheadrightarrow L_{k^\natural}(\mathfrak{g}^\natural),$$

where $L_{k^\natural}(\mathfrak{g}^\natural) := \bigotimes_{i=0}^{s} L_{k_i^\natural}(\mathfrak{g}_i^\natural)$, or if and only if $\mathcal{W}_k(\mathfrak{g}, f)$ is isomorphic to $L_{k^\natural}(\mathfrak{g}^\natural)$. More generally, we said that k is a *generalized collapsing level* if the $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is isomorphic to another $\mathcal{W}$-algebra $\mathcal{W}_{k'}(\mathfrak{g}', f')$.

Collapsing levels have remarkable applications to representation theory of affine vertex algebras. They play a important role in the study of the category KL_k [2, 10]. In [8, 9], Adamović and al. give a complete classification of collapsing levels for $\mathcal{W}$-algebras associated with a minimal nilpotent element using OPEs. Few particular cases have also been computed explicitly for other nilpotent elements [6, 7]. It is difficult to predict which levels collapse in general since OPEs are unkown. Recently, Arakawa, van Ekeren and Moreau [22] proposed a different approach using associative varieties and Slodowy slices. Its provides a large family of admissible collapsing levels. We use the explicit description of $\mathcal{W}$-algebras associated with Lie algebras of rank two to obtain a complete list of collapsing levels and some other remarkable isomorphisms of vertex algebras (Tables 2.2 and 2.3). Before getting into the heart of the matter, let state the following useful lemma:

Lemma 2.2.1. *Let V be a vertex algebra and $a \in V$. For $b \in V$, denote*

$$a(z)b(w) \sim \sum_{j=0}^{N} \frac{c_j(w)}{(z-w)^{j+1}}.$$

Then for all $j \in [\![0, N]\!]$, c_j belongs to the ideal generated by a.

Proof. This follows directly from the definition of a vertex algebra ideal. Let I be the ideal generated by a. Then I is a two-sided ideal of V, T-invariant such that $a(z)b \in I((z))$ for any $b \in V$. As a consequence, $a(z)b(w) \in I((z))((w))$ and $c_j \in I$. □

2.2.1 Trivial simple $\mathcal{W}$-algebras

Consider the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ associated with a nilpotent element $f \in \mathfrak{g}$. Denote $\mathcal{I}_k$ its maximal ideal – which depends on the level k. We have $\mathcal{W}_k(\mathfrak{g}, f) = \mathcal{W}^k(\mathfrak{g}, f)/\mathcal{I}_k$. At non-critical level, it follows from the conformal structure of $\mathcal{W}^k(\mathfrak{g}, f)$ that the vertex algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is trivial, i.e. isomorphic to $\mathbb{C}$, if and only if the conformal vector generates $\mathcal{I}_k$. This happens only if the central charge c_k is zero, but this necessary condition is not sufficient. For instance, the conformal vector L of $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$ does not belong to the maximal ideal when $k = -1$ whereas $c_{-1} = 0$. Indeed, according to the OPEs described in Appx. B.1 and Lemma 2.2.1, if L were in $\mathcal{I}_{-1}$ then J would be in $\mathcal{I}_{-1}$ as well as the vacuum.

Proposition 2.2.2. *Let $J_1, \dots, J_n$ be a set of strong generators of $\mathcal{W}^k(\mathfrak{g}, f)$ with conformal weights $w_1, \dots, w_n$ respectively. The simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$ is trivial if and only if for any $1 \leqslant i, j \leqslant n$, the pole of degree $w_i + w_j$ of the OPE $J_i(z)J_j(w)$ vanishes.*

Proof. For $1 \leqslant i, j \leqslant n$, set

$$J_i(z)J_j(w) \sim \sum_{m=0}^{w_i+w_j} \frac{c_{i,j}^m(w)}{(z-w)^m}.$$

For all m, the field $c_{i,j}^m$ has conformal weight $w_i + w_j - m$. Hence, it is a multiple of the vacuum if and only if $m = w_i + w_j$. If $c_{i,j}^{w_i+w_j} = 0$ for all $1 \leqslant i, j \leqslant n$ then the vacuum does not belong to the ideal I generated by $J_1, \dots, J_n$. Hence $I = \mathcal{I}$. In particular, the conformal vector of $\mathcal{W}^k(\mathfrak{g}, f)$ generates $\mathcal{I}$ and $\mathcal{W}_k(\mathfrak{g}, f) \simeq \mathbb{C}$. Else, assume there exists $c_{i,j}^{w_i+w_j} \neq 0$, by Lemma 2.2.1, J_i and J_j do not belong to $\mathcal{I}$. The simple quotient is not trivial. □

Remark 2.2.3. If the condition of Proposition 2.2.2 is satisfied, the central charge vanishes. Indeed, it is a multiple of the maximal pole of the OPE of the conformal vector with itself.

Consequently, the descriptions of $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{g}, f)$, where $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{sp}_4$, or G_2 (Appx. A, B, C), allow to classify all pairs (f, k) such that the simple $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is trivial.

Corollary 2.2.4. 1. *Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$, then $\mathcal{W}_k(\mathfrak{g}, f) \simeq \mathbb{C}$ if and only if*

$$(f, k) \in \{(f_{\min}, -3/2), (f_{\text{reg}}, -5/3), (f_{\text{reg}}, -9/4)\}.$$

2. *Consider the Lie algebra $\mathfrak{g} = \mathfrak{sp}_4$, then $\mathcal{W}_k(\mathfrak{g}, f) \simeq \mathbb{C}$ if and only if*

$$(f, k) \in \{(f_{\min}, -1/2), (f_{\text{reg}}, -13/6), (f_{\text{reg}}, -12/5)\}.$$

3. *Consider the Lie algebra $\mathfrak{g} = G_2$, then $\mathcal{W}_k(\mathfrak{g}, f) \simeq \mathbb{C}$ if only if*

$$(f, k) \in \{(f_{\min}, -5/3), (f_{\text{subreg}}, -17/6), (f_{\text{subreg}}, -2), (f_{\text{reg}}, -41/12), (f_{\text{reg}}, -24/7)\}.$$

Proof. The complete classification of trivial simple $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{g}, f)$ follows directly from the OPEs (Appx. A, B, C) and Proposition 2.2.2 except for $\mathfrak{g} = G_2$ and $f = f_{\text{reg}}$. The central charge of $\mathcal{W}^k(G_2)$ vanishes if and only if $k = -41/12$ or $k = -24/7$. So, $\mathcal{W}_k(G_2)$ cannot be trivial outside these two levels. Moreover, by [22, Theorem 10.10], $\mathcal{W}_{-41/12}(G_2) \simeq \mathcal{W}_{-24/7}(G_2) \simeq \mathbb{C}$. □

Most of the pairs listed in the Corollary 2.2.4 have already been studied in [9, 22]. It is not the case of $\mathcal{W}_{-2}(G_2, f_{\text{subreg}})$. It provides a new example of rational $\mathcal{W}$-algebra at non admissible level.

Corollary 2.2.5. *The simple $\mathcal{W}$-algebra $\mathcal{W}_{-2}(G_2, f_{\text{subreg}})$ is rational and lisse.*

In another direction, the principal $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{sl}_3)$, $\mathcal{W}_k(\mathfrak{sp}_4)$ and $\mathcal{W}_k(G_2)$, and the subregular $\mathcal{W}$-algebra $\mathcal{W}_k(G_2, f_{\text{subreg}})$ are one-dimensional if and only if $c_k = 0$. More generally, we conjecture the following:

Conjecture 2.2.6. *If f is distinguished, i.e. $\mathfrak{g}_0^f = 0$, then $\mathcal{W}_k(\mathfrak{g}, f) \simeq \mathbb{C}$ if and only if $c_k = 0$.*

We check this conjecture in many cases. In particular, we verify in [22] that, when f is distinguished and k is an admissible level which vanishes the central charge, it is a collapsing level, i.e. $\mathcal{W}_k(\mathfrak{g}, f) \simeq \mathbb{C}$.

2.2.2 Simple quotients isomorphic to Virasoro vertex algebras

From now on, we consider non-trivial simple $\mathcal{W}$-algebras. Hence, the conformal vector does not belong to the maximal ideal of $\mathcal{W}^k(\mathfrak{g}, f)$. If the other strong generators are in the maximal ideal then $\mathcal{W}_k(\mathfrak{g}, f)$ is isomorphic to the Virasoro vertex algebra Vir_{c_k} (see Sect. 1.1.2).

Proposition 2.2.7. *Let $J_1, \dots, J_n, L$ be a free family of strong generators of $\mathcal{W}^k(\mathfrak{g}, f)$ with L a conformal vector. Assume $J_1, \dots, J_n$ have conformal weights $w_1, \dots, w_n$ respectively. The simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$ is isomorphic to Vir_{c_k} if and only if for any $1 \leqslant i, j \leqslant n$ the pole of degree $w_i + w_j$ of the OPE $J_i(z)J_j(w)$ vanishes and the pole of degree $w_i + w_j - 2$ does not depend on L.*

Proof. The simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$ is isomorphic to Vir_{c_k} if and only if it is strongly generated only by the conformal vector L. Hence, $\mathcal{W}_k(\mathfrak{g}, f) \simeq \mathrm{Vir}_{c_k}$ if and only if $J_1, \dots, J_n \in \mathcal{I}_k$ and $L \notin \mathcal{I}_k$. For $1 \leqslant i, j \leqslant n$, set

$$J_i(z)J_j(w) \sim \sum_{m=0}^{w_i+w_j} \frac{c_{i,j}^m(w)}{(z-w)^m}.$$

The field $c_{i,j}^m$ is a multiple of the vacuum if and only if $m = w_i + w_j$ and a linear combination of vector of weight 2 – in particular the vector L – if and only if $m = w_i + w_j - 2$. As a consequence, if $c_{i,j}^{w_i+w_j} = 0$ and L does not appeared in the linear combinations $c_{i,j}^{w_i+w_j-2}$, the vectors $J_1, \dots, J_n$ belongs to $\mathcal{I}_k$ whereas $|0\rangle$ and L do not.

Conversely, assume $\mathcal{W}_k(\mathfrak{g}, f) \simeq \mathrm{Vir}_{c_k}$. Then $\mathcal{I}$ is generated by all strong generators except the conformal vector. Suppose there are $1 \leqslant i, j \leqslant n$ and $c \in \mathbb{C}^*$ such that

$$c_{i,j}^{w_i+w_j-2} = cL + R \in \mathcal{I}_k,$$

where R is a field of $\mathcal{W}^k(\mathfrak{g}, f)$ which does not depend on L. The field R is a linear combination of $J_1, \dots, J_n$, their derivatives and normally ordered products. Thus, $R \in \mathcal{I}$ and $L \in \mathcal{I}_k$. Then $\mathcal{W}_k(\mathfrak{g}, f)$ is trivial whence a contradiction. □

Remark 2.2.8. The previous criterion is very convenient when one considers principal $\mathcal{W}$-algebras because they are strongly generated only by two fields. It can also be useful when one of the strong generator – denote it J – acts semisimply on graded component of the $\mathcal{W}$-algebra, that is

$$J(z)G(w) \sim \frac{w_J(G)}{(z-w)} G(w),$$

with $w_J(G) \in \mathbb{C}$, for any strong generator G distinct from J and L. Then, if $w_J(G) \neq 0$ for all G, J generates $\mathcal{I}_k$. In many cases, J will be a strong generator corresponding to a semisimple element of $\mathfrak{g}$.

Corollary 2.2.9.
1. *The simple $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sl}_3)$ is isomorphic to Vir_{c_k} if and only if $k = -4/3$ or $k = -12/5$.*
2. *Consider the Lie algebra $\mathfrak{g} = \mathfrak{sp}_4$, then $\mathcal{W}_k(\mathfrak{g}, f) \simeq \mathrm{Vir}_{c_k}$ if and only if*

- $f = f_{\mathrm{subreg}}$ *and* $k = -2$*, or*
- $f = f_{\mathrm{reg}}$ *and* $k \in \{-3/2, -8/3, -11/5, -11/6, -18/7, -19/8\}$.

3. *The simple* $\mathcal{W}$*-algebra* $\mathcal{W}_k(G_2, f_{\tilde{A}_1})$ *is isomorphic to* Vir_{c_k} *if and only if* $k = -3/2$.
4. *The simple* $\mathcal{W}$*-algebra* $\mathcal{W}_k(G_2, f_{\mathrm{subreg}})$ *is isomorphic to* Vir_{c_k} *if and only if* $k = -10/3$.

Remark 2.2.10. Recall [15] that the Virasoro vertex algebra Vir_c is rational if and only if it is lisse, that is if and only if

$$c = 1 - \frac{6(p-q)^2}{pq},$$

for $p, q \in \mathbb{Z}_{\geqslant 2}$, $(p,q) = 1$. In the examples listed in Corollary 2.2.9, when the pair (k, f) is exceptional then $\mathcal{W}_k(\mathfrak{g}, f) \simeq \mathrm{Vir}_{c_k}$ is rational. On the contrary, when it is not, the simple $\mathcal{W}$-algebras are neither lisse nor rational: $\mathcal{W}_{-2}(\mathfrak{sp}_4, f_{\mathrm{subreg}})$, $\mathcal{W}_{-3/2}(\mathfrak{sp}_4)$, $\mathcal{W}_{-8/3}(\mathfrak{sp}_4)$ and $\mathcal{W}_{-10/3}(G_2, f_{\mathrm{subreg}})$ are neither lisse nor rational.

2.2.3 Other collapsing levels

For several additional particular simple $\mathcal{W}$-algebras, OPEs give a complete or partial description of the maximal ideal. For instance, we recover some isomorphisms from [8] and [22, Theorem 10.10]:

$$\mathcal{W}_{-2}(\mathfrak{sp}_4, f_{\min}) \simeq M(1), \qquad \mathcal{W}_{-2}(\mathfrak{sp}_4, f_{\min}) \simeq L_{-3/2}(\mathfrak{sl}_2),$$
$$\mathcal{W}_{-4/3}(G_2, f_{\min}) \simeq L_1(\mathfrak{sl}_2), \qquad \mathcal{W}_{-17/6}(G_2, f_{\tilde{A}_1}) \simeq L_{-4/3}(\mathfrak{sl}_2).$$

We are also able to determine other collapsing levels.

Proposition 2.2.11. *We have the following isomorphisms:*

$$\mathcal{W}_{-10/3}(G_2, f_{\tilde{A}_1}) \simeq L_{-11/6}(\mathfrak{sl}_2), \qquad \mathcal{W}_{-2}(G_2, f_{\tilde{A}_1}) \simeq L_{-1/2}(\mathfrak{sl}_2), \qquad \mathcal{W}_{-1}(\mathfrak{sp}_4, f_{\mathrm{subreg}}) \simeq M(1).$$

Proof. Assume $(\mathfrak{g}, f)$ is one of the pairs appearing in Proposition 2.2.11. Let $\{a_i\}_{i \in I}$ be a basis of $\mathfrak{g}^f$ compatible with the gradation (1.10) and $\{J^{\{a_i\}}\}_{i \in I}$ be the strong generators of $\mathcal{W}^k(\mathfrak{g}, f)$ constructed as in Theorem 1.3.1. Using the OPEs, one checks that if $a_i \notin \mathfrak{g}^\natural$, then the field $J^{\{a_i\}}$ generates a non trivial ideal. Hence, it belongs to the maximal ideal $\mathcal{I}$ of $\mathcal{W}^k(\mathfrak{g}, f)$. Using Lemma 2.2.1, we get relations between the conformal vector L and the fields $J^{\{a_i\}}$, $a_i \in \mathfrak{g}^\natural$. The latter generate a affine vertex algebra and L can be written as a multiple of the Sugawara conformal vector of $V^{k^\natural}(\mathfrak{g}^\natural)$ (see Sect. 1.2.2). Since $\mathcal{W}_k(\mathfrak{g}, f)$ is simple, it is then isomorphic to $L_{k^\natural}(\mathfrak{g}^\natural)$ for some level $k^\natural$. The level $k^\natural$ is determined using the equality of central charge:

$$c_k = c_{L_{k^\natural}(\mathfrak{g}^\natural)} = \frac{k^\natural \dim \mathfrak{g}^\natural}{k^\natural + h^\vee_{\mathfrak{g}^\natural}}.$$

□

Remark 2.2.12. The OPEs of $\mathcal{W}^k(G_2, f_{\mathrm{subreg}})$ admit simplifications at level $k = -16/5$ (see Appx. C.3). Indeed, the strong generator F belongs to the maximal ideal $\mathcal{I}_{-16/5}$. This induces relations between the three other strong generators and their derivatives in the simple $\mathcal{W}$-algebra. We have not identify $\mathcal{W}_{-16/5}(G_2, f_{\mathrm{subreg}})$ with another vertex algebra for the moment.

We summarize in Tables 2.2 and 2.3 the (generalized) collapsing levels of $\mathcal{W}$-algebras associated with simple Lie algebras of rank two we mentioned in Sect. 2.2.

$\mathcal{W}_k(\mathfrak{g},f)$	$\mathfrak{g}^\natural$	$L_{k^\natural}(\mathfrak{g}^\natural)$	k or $(k,k^\natural)$ (when it is relevant)
$\mathcal{W}_k(\mathfrak{sl}_3,f_{\min})$	$\mathbb{C}$	$M(1)$	-1
$\mathcal{W}_k(\mathfrak{sl}_3)$	0	$\mathbb{C}$	$-\frac{5}{3}$, $-\frac{9}{4}$
$\mathcal{W}_k(\mathfrak{sp}_4,f_{\min})$	$\mathfrak{sl}_2$	$L_{k^\natural}(\mathfrak{sl}_2)$	$(-2,-\frac{3}{2})$
$\mathcal{W}_k(\mathfrak{sp}_4,f_{\text{subreg}})$	$\mathbb{C}$	$M(1)$	-1
$\mathcal{W}_k(\mathfrak{sp}_4)$	0	$\mathbb{C}$	$-\frac{13}{6}$, $-\frac{12}{5}$
$\mathcal{W}_k(G_2,f_{\min})$	$\mathfrak{sl}_2$	$L_{k^\natural}(\mathfrak{sl}_2)$	$(-\frac{4}{3},1)$
$\mathcal{W}_k(G_2,f_{\tilde{A}_1})$	$\mathfrak{sl}_2$	$L_{k^\natural}(\mathfrak{sl}_2)$	$(-\frac{17}{6},-\frac{4}{3})$, $(-\frac{10}{3},-\frac{11}{6})$, $(-2,-\frac{1}{2})$
$\mathcal{W}_k(G_2,f_{\text{subreg}})$	0	$\mathbb{C}$	$-\frac{17}{6}$, -2
$\mathcal{W}_k(G_2)$	0	$\mathbb{C}$	$-\frac{41}{12}$, $-\frac{24}{7}$

Table 2.2 – Collapsing levels for $\mathcal{W}$-algebras of rank two.

$\mathcal{W}_k(\mathfrak{g},f)$	$\mathcal{V}$	k or (k,c_k) (when it is relevant)
$\mathcal{W}_k(\mathfrak{sl}_3,f_{\min})$	$\mathbb{C}$	$-\frac{3}{2}$
$\mathcal{W}_k(\mathfrak{sl}_3)$	Vir_{c_k}	$(-\frac{4}{3},-\frac{22}{5})$, $(-\frac{12}{5},-\frac{22}{5})$
$\mathcal{W}_k(\mathfrak{sp}_4,f_{\min})$	$\mathbb{C}$	$-\frac{1}{2}$
$\mathcal{W}_k(\mathfrak{sp}_4,f_{\text{subreg}})$	Vir_{c_k}	$(-2,-2)$
$\mathcal{W}_k(\mathfrak{sp}_4)$	Vir_{c_k}	$(-\frac{3}{2},-24)$, $(-\frac{8}{3},-24)$, $(-\frac{11}{5},\frac{1}{2})$, $(-\frac{11}{6},\frac{68}{7})$, $(-\frac{18}{7},\frac{68}{7})$, $(-\frac{19}{8},\frac{1}{2})$
$\mathcal{W}_k(G_2,f_{\min})$	$\mathbb{C}$	$-\frac{5}{3}$
$\mathcal{W}_k(G_2,f_{\tilde{A}_1})$	Vir_{c_k}	$(-\frac{3}{2},-\frac{22}{5})$
$\mathcal{W}_k(G_2,f_{\text{subreg}})$	Vir_{c_k}	$(-\frac{10}{3},-24)$

Table 2.3 – Generalized collapsing levels for $\mathcal{W}$-algebras of rank two.

Chapter 3

Rationality of exceptional $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$

Classification of rational $\mathcal{W}$-algebras is a major problem of the theory. In this chapter, we prove the rationality of a new family of $\mathcal{W}$-algebras: exceptional $\mathcal{W}$-algebras associated with subregular nilpotent elements of $\mathfrak{sp}_4$. It proves new cases of Kac-Wakimoto and Arakawa's conjecture (see Sect. 1.3.2). This conjecture has been intensively studied in the recent years and particular cases was previously established by Arakawa [16, 18], Creutzig-Linshaw [41] and Arakawa-van Ekeren [21]. The case of exceptional $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$ can be consider as the "easiest" not covered by the works mentioned above.

The main results of the chapter have been published in [56]. In particular, we recall the precise statement of the main theorem:

Theorem 3.1. *Let $f = f_{\text{subreg}}$ be a subregular nilpotent element of $\mathfrak{g} = \mathfrak{sp}_4$. Then the exceptional $\mathcal{W}$-algebras $\mathcal{W}_{-3+p/3}(\mathfrak{g}, f)$, with $(p, 3) = 1$, $p \geqslant 3$, and the exceptional $\mathcal{W}$-algebras $\mathcal{W}_{-3+p/4}(\mathfrak{g}, f)$, with $(p, 2) = 1$, $p \geqslant 4$, are rational (and lisse). Moreover, Proposition 3.2.5 gives a complete classification of their simple modules.*

The chapter is organized as follows. In Sect. 3.1, we recall briefly the relations between strong generators of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$ and look at positive energy representations with finite dimensional top component. When (f, k) is an exceptional pair, all possible simple $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules satisfy this finiteness condition. Using the twist action introduced in [82], we show that simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules are in finite number. We describe them in Sect. 3.2.

The Sect. 3.3 is devoted to the proof of the main Theorem 3.1. The $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$ can be viewed as an analogous of the Bershadsky-Polyakov vertex algebra in type C. The later corresponds to the $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sl}_3, f_{\min})$ studied in [16]. In $\mathfrak{sl}_3$, minimal nilpotent elements are also subregular, whence $\mathcal{W}_k(\mathfrak{sl}_3, f_{\min}) \simeq \mathcal{W}_k(\mathfrak{sl}_3, f_{\text{subreg}})$. However, contrary to the case where f is minimal, the functor $H^0_f(?)$ appearing in the construction of $\mathcal{W}$-algebras is not exact (see Sect.1.3.2). As a consequence, we cannot use directly the methods of [16] to prove the rationality of exceptional $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$. To encounter the difficulty, we exploit certain techniques of [21].

The two last sections give applications to the explicit description of the simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules we obtained as a by-product of our proof. In Sect. 3.4, we show that the component group of the nilpotent orbit $\mathbb{O}_{\mathrm{subreg}}$ acts non-trivially on the finite set of the simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\mathrm{subreg}})$-modules (Theorem 3.4.3). Moreover, one uses the description to compute the characters of simple modules (Sect. 3.5). Part of the simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\mathrm{subreg}})$-modules comes from the reduction of certain highest weight representations of $\widehat{\mathfrak{sp}_4}$. We conjecture that this holds for all the simple modules (Conjecture 3.5.7).

3.1 Actions of strong generators on positive energy representations of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\mathrm{subreg}})$

From now on, $\mathfrak{g} = \mathfrak{sp}_4$ and $f = f_{\mathrm{subreg}}$. We saw in Chap. 2, that the centralizer of f is four-dimensional. Hence, the vertex algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is freely strongly generated by the fields $J(z)$, $G^{\pm}(z)$ and $L(z)$. We refer to Sect. 2.1.2 for the description of these generators and Appx. B.2 for OPEs between them. By construction, provided $k \neq -3$, the field $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is a conformal vector of $\mathcal{W}^k(\mathfrak{g}, f)$ with central charge

$$c_k := -\frac{2(9+16k+6k^2)}{3+k}.$$

It gives $J(z)$, $G^+(z)$ and $G^-(z)$ the conformal weights 1, 2 and 2, respectively. Following (1.3) we write

$$J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}, \qquad G^{\pm}(z) = \sum_{n\in\mathbb{Z}} G^{\pm}_n z^{-n-2}, \qquad L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}.$$

The monomials

$$J_{n_1}\dots J_{n_j} L_{m_1}\dots L_{m_t} G^-_{p_1}\dots G^-_{p_g} G^+_{q_1}\dots G^+_{q_h}|0\rangle, \tag{3.1}$$

with $n_1 \leqslant \ldots \leqslant n_j \leqslant -1$, $m_1 \leqslant \ldots m_t \leqslant -2$, $p_1 \leqslant \ldots \leqslant p_g \leqslant -2$, and $q_1 \leqslant \ldots \leqslant q_h \leqslant -2$ form a PBW basis of the vertex algebra $\mathcal{W}^k(\mathfrak{g}, f)$.

From the OPEs, we deduce the commutation relations for all $m, n \in \mathbb{Z}$:

$$\begin{aligned}
[J_m, J_n] =& (2+k)m\delta_{m+n,0},\\
[J_m, G^{\pm}_n] =& \pm G^{\pm}_{m+n},\\
[L_m, L_n] =& \frac{c_k}{12}(m^3-m)\delta_{n+m,0} + (m-n)L_{m+n},\\
[L_m, G^{\pm}_n] =& (m-n)G^{\pm}_{m+n},\\
[L_m, J_n] =& -nJ_{m+n},\\
[G^+_m, G^-_n] =& -\frac{(1+k)(2+k)^2}{2}(m^3-m)\delta_{m+n,0} + \frac{(2+k)(3+k)}{2}(m-n)L_{m+n}\\
&+ \left(\frac{3(1+k)(2+k)}{2}(m+1)(n+1) - \frac{(5+4k+k^2)}{2}(m+n+1)(m+n+2)\right) J_{m+n}\\
&- (3+2k)(m+1)(J^2)_{m+n} + (3+k)(LJ)_{m+n} - (J^3)_{m+n} - (3+2k)(J\partial J)_{m+n},
\end{aligned}$$

where

$$\sum_{n\in\mathbb{Z}}(J^2)_n z^{-n-2} \stackrel{\mathrm{def.}}{=} :J(z)^2:, \quad \sum_{n\in\mathbb{Z}}(LJ)_n z^{-n-3} \stackrel{\mathrm{def.}}{=} :L(z)J(z):, \quad \sum_{n\in\mathbb{Z}}(J\partial J)_n z^{-n-3} \stackrel{\mathrm{def.}}{=} :J(z)\partial J(z):.$$

Let M be an irreducible positive energy representation of $\mathcal{W}^k(\mathfrak{g}, f)$, with $M_{\text{top}} = M_\chi$, $\chi \in \mathbb{C}$ (see Sect. 1.1.3). Using the commutation relations, we notice that the submodule N of M generated by all J_0-eigenvectors of M is stable by $L_n, G_n^\pm, J_n$, $n \in \mathbb{Z}$. Hence, $N = M$ because M is simple.

Lemma 3.1.1. *Let M be an irreducible positive energy representation of $\mathcal{W}^k(\mathfrak{g}, f)$, with $M_{top} = M_\chi$, $\chi \in \mathbb{C}$. Suppose that M_{top} is finite dimensional. Then there is a vector $v \in M$ such that $L_0 v = \chi v$, $J_0 v = \xi v$ for some $\xi \in \mathbb{C}$, and such that the relations below hold:*

$$\begin{aligned} J_n v &= 0 \qquad \text{for } n > 0,\\ L_n v &= 0 \qquad \text{for } n > 0,\\ G_n^+ v &= 0 \qquad \text{for } n > 0,\\ G_n^- v &= 0 \qquad \text{for } n \geqslant 0. \end{aligned}$$

Moreover, $M = \bigoplus_{\substack{a \in \xi + \mathbb{Z}\\ d \in \chi + \mathbb{Z}_{\geqslant 0}}} M_{a,d}$, where $M_{a,d} = \{m \in M \mid J_0 m = am,\ L_0 m = dm\}$, $\dim M_{\xi,\chi} = 1$ and $M_{top} = M_\chi$ is spanned by the vectors $(G_0^+)^i v$ for $i \geqslant 0$.

Proof. Since J_0 and L_0 commute, the action of J_0 preserves each $M_{\chi+n}$, $n \in \mathbb{Z}_{\geqslant 0}$. Moreover, J_0 is semisimple over M and each $M_{\chi+n}$. Then M can be written

$$M = \bigoplus_{\substack{(a,d) \in \mathbb{C}^2\\ d \in \chi + \mathbb{Z}_{\geqslant 0}}} M_{a,d}.$$

As M_{top} is finite dimensional, there is a vector $v \in M_{\text{top}}$ such that $J_0 v = \xi v$, $\xi \in \mathbb{C}$ and $\xi - n$ is not an eigenvalue of J_0 in M_{top} for all $n \in \mathbb{Z}_{>0}$. The relations of the lemma result from the following equations. Let $n \in \mathbb{Z}$,

$$\begin{aligned} J_0 J_n v &= \xi J_n v, & L_0 J_n v &= (\chi - n) J_n v,\\ J_0 L_n v &= \xi L_n v, & L_0 L_n v &= (\chi - n) L_n v,\\ J_0 G_n^\pm v &= (\xi \pm 1) G_n^\pm v, & L_0 G_n^\pm v &= (\chi - n) G_n^\pm v. \end{aligned}$$

It ensues from those relations that all the eigenvalues of J_0 are in $\xi + \mathbb{Z}$. Hence,

$$M = \bigoplus_{\substack{a \in \xi + \mathbb{Z}\\ d \in \chi + \mathbb{Z}_{\geqslant 0}}} M_{a,d}.$$

We explain the relation $G_n^- v = 0$, $n \geqslant 0$, the others are obtained similarly. Since M is a positive energy representation of $\mathcal{W}^k(\mathfrak{g}, f)$, for $n > 0$, $\chi - n$ is not an eigenvalue of L_0, whence $G_n^- v = 0$. Besides, $G_0^- v \in M_{\text{top}}$ and the choice of v implies that $\xi - 1$ is not a eigenvalue of J_0 in M_{top}. Hence $G_0^- v = 0$. Finally, the vectors $(G_0^+)^i v$, $i \geqslant 0$, are the only ones attached to the eigenvalue χ for L_0 so they span M_{top}. Moreover, $J_0 (G_0^+)^i v = (\xi + i) G_0^+ v$ for $i \geqslant 0$. As a consequence, $M_{\xi,\chi} = \mathbb{C} v$. □

For $(\xi, \chi) \in \mathbb{C}^2$, let $L(\xi, \chi)$ be the irreducible representation of $\mathcal{W}^k(\mathfrak{g}, f)$ generated by a vector $v = |\xi, \chi\rangle$ satisfying the relations of Lemma 3.1.1. According to Lemma 3.1.1, $|\xi, \chi\rangle$ is uniquely defined up to nonzero scalar, so the notation is legitimate. Zhu's correspondence ensures that such $L(\xi, \chi)$ does exist and is unique up to isomorphism of $\mathcal{W}^k(\mathfrak{g}, f)$-modules (see, for example, [31]).

Remark 3.1.2. When k is an admissible level appearing in the Theorem 3.1, the $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$ is lisse. As a consequence, its simple modules are positive energy representations with finite dimensional top component (see Sect. 1.1.4). Hence, they are of the form $L(\xi,\chi)$ with $(\xi,\chi)\in\mathbb{C}^2$.

Since $L(\xi,\chi)_{\chi,\xi}$ is one-dimensional and $G_0^-G_0^+|\xi,\chi\rangle$ belongs to $L(\xi,\chi)_{\chi,\xi}$, it is proportional to $|\xi,\chi\rangle$:

$$\begin{aligned}G_0^-G_0^+|\xi,\chi\rangle &= \underbrace{G_0^+G_0^-|\xi,\chi\rangle}_{0} + [G_0^-,G_0^+]|\xi,\chi\rangle \\ &= \left(-\frac{k^2+k-4}{2}J_0+(3+2k)(J^2)_0-(3+k)(LJ)_0+(J^3)_0+(3+2k)(J\partial J)_0\right)|\xi,\chi\rangle \\ &= \left(-\frac{k^2+k-4}{2}J_0+(3+2k)J_0^2-(3+k)(L_0J_0+J_0)+J_0^3-(3+2k)J_0^2\right)|\xi,\chi\rangle \\ &= g(\xi,\chi)|\xi,\chi\rangle,\end{aligned}$$

where

$$g(\xi,\chi) = -\frac{1}{2}\xi\left(2+3k+k^2-2\xi^2+6\chi+2k\chi\right).$$

For $i\geqslant 1$ and $m\geqslant 0$,

$$J_m(G_0^+)^{i-1}|\xi,\chi\rangle = \delta_{m,0}(\xi+i-1)(G_0^+)^{i-1}|\xi,\chi\rangle,\quad L_m(G_0^+)^{i-1}|\xi,\chi\rangle = \delta_{m,0}\chi(G_0^+)^{i-1}|\xi,\chi\rangle.$$

We deduce that,

$$G_0^-(G_0^+)^i|\xi,\chi\rangle = G_0^+G_0^-(G_0^+)^{i-1}|\xi,\chi\rangle + g(\xi+i-1,\chi)(G_0^+)^{i-1}|\xi,\chi\rangle.$$

Hence, by induction, for $i\geqslant 1$,

$$G_0^-(G_0^+)^i|\xi,\chi\rangle = i\,h_i(\xi,\chi)(G_0^+)^{i-1}|\xi,\chi\rangle, \tag{3.2}$$

where

$$\begin{aligned}h_i(\xi,\chi) &= \frac{1}{i}\sum_{m=0}^{i-1} g(\xi+m,\chi) \\ &= \frac{(2\xi+i-1)}{4}(-2-i+i^2-3k-k^2-2\xi+2i\xi+2\xi^2-6\chi-2k\chi).\end{aligned}$$

Proposition 3.1.3. *Suppose that $L(\xi,\chi)_{top}$ is n-dimensional. Then $h_n(\xi,\chi)=0$.*

Proof. If $\dim L(\xi,\chi)_{\text{top}} = n$ then $(G_0^+)^n|\xi,\chi\rangle = 0$ and $(G_0^+)^{n-1}|\xi,\chi\rangle \neq 0$. It results from (3.2) that $h_n(\xi,\chi)=0$. □

3.2 Twist-action over simple $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules

Following the ideas of [16], we introduce the twist-action ψ described in [82]. Let us define

$$\Delta(-J,z) := z^{-J_0}\exp\left(\sum_{m=1}^{\infty}(-1)^{m+1}\frac{-J_m}{mz^m}\right).$$

For $a \in \mathcal{W}^k(\mathfrak{g}, f)$,

$$\Delta(-J, z)a = z^{-J_0}\left(\sum_{n=0}^{\infty} \frac{X^n}{n!} a\right),$$

where $X = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{-J_m}{mz^m}$ and z^{-J_0} is defined by $z^{-J_0}a = z^{-c}a$ if $J_0 a = ca$. For a field $a(z)$, set the twist-field:

$$\sum_{n\in\mathbb{Z}} \psi(a_{(n)})z^{-n-1} := Y(\Delta(-J,z)a, z) = \sum_{n=0}^{\infty} Y(z^{-J_0}\frac{X^n}{n!}a, z).$$

For any $\mathcal{W}^k(\mathfrak{g}, f)$-module M, the space $\psi(M)$ denotes the $\mathcal{W}^k(\mathfrak{g}, f)$-module obtained by twisting the action of $\mathcal{W}^k(\mathfrak{g}, f)$ as $a_{(n)} \mapsto \psi(a_{(n)})$. The following relations are obtained by applying the ψ-action to the strong generators of $\mathcal{W}^k(\mathfrak{g}, f)$:

$$\begin{aligned}
\psi(J_n) &= J_n - (2+k)\delta_{n,0},\\
\psi(L_n) &= L_n - J_n + \frac{(2+k)}{2}\delta_{n,0},\\
\psi(G_n^+) &= G_{n-1}^+,\\
\psi(G_n^-) &= G_{n+1}^-.
\end{aligned}$$

Proposition 3.2.1. *Assume that* $\dim L(\xi,\chi)_{top} = i$ *and* $\dim \psi(L(\xi,\chi))_{top} = j$. *Then*

$$\psi(L(\xi,\chi)) \simeq L(\xi + (i-1) - (2+k)\,,\, \chi - \xi - (i-1) + \frac{(2+k)}{2}),$$

and

$$\psi^2(L(\xi,\chi)) \simeq L(\xi + i + j - 6 - 2k\,,\, \chi - 2\xi - 2i - j + 7 + 2k).$$

Proof. For all $m \geqslant 0$, we have

$$\begin{aligned}
\psi(J_0)(G_0^+)^m|\xi,\chi\rangle &= (\xi + m - (2+k))(G_0^+)^m|\xi,\chi\rangle,\\
\psi(L_0)(G_0^+)^m|\xi,\chi\rangle &= (\chi - (\xi+m) + \frac{(2+k)}{2})(G_0^+)^m|\xi,\chi\rangle.
\end{aligned}$$

Since the smallest eigenvalue associated with the $\psi(L_0)$-action is attached to the vector $(G_0^+)^{i-1}|\xi,\chi\rangle$, we get

$$\psi(L(\xi,\chi)) \simeq L(\xi + (i-1) - (2+k)\,,\, \chi - \xi - (i-1) + \frac{(2+k)}{2}),$$

and, by induction,

$$\begin{aligned}
\psi^2(L(\xi,\chi)) &\simeq \psi(L(\xi + (i-1) - (2+k)\,,\, \chi - \xi - (i-1) + \frac{(2+k)}{2}))\\
&\simeq L(\xi + (i-1) + (j-1) - 2(2+k)\,,\, \chi - 2\xi - 2(i-1) - (j-1) + 2(2+k)). \quad \square
\end{aligned}$$

Remark 3.2.2. For all $m, n \in \mathbb{Z}_{\geqslant 0}$,

$$\begin{aligned}
\psi^2(J_0)(G_{-1}^+)^m(G_0^+)^n|\xi,\chi\rangle &= (\xi + n + m - 2(2+k))(G_{-1}^+)^m(G_0^+)^n|\xi,\chi\rangle,\\
\psi^2(L_0)(G_{-1}^+)^m(G_0^+)^n|\xi,\chi\rangle &= (\chi - 2\xi - 2n - m + 2(2+k))(G_{-1}^+)^m(G_0^+)^n|\xi,\chi\rangle.
\end{aligned}$$

Proposition 3.2.3. *Suppose that* $\dim L(\xi,\chi)_{top} = i$, $\dim \psi(L(\xi,\chi))_{top} = j$ *and* $\dim \psi^2(L(\xi,\chi))_{top} = l$.

(a) *If* $k = -3 + p/3$ *with* $(p,3) = 1$, $p \geqslant 3$, *then* $(\xi,\chi,l) = (\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}, l_{i,j}^{(s)})$ *with* $s \in \{1,2,3\}$, *where*

$$\begin{cases} \xi_{i,j}^{(1)} = \dfrac{1-i}{2}, \\ \chi_{i,j}^{(1)} = \dfrac{13-6i+i^2-12j+2ij+2j^2+6k-2ik-4jk}{4(3+k)}, \\ l_{i,j}^{(1)} = 9-i-j+3k, \end{cases}$$
$$\begin{cases} \xi_{i,j}^{(2)} = \dfrac{7-2i-j+2k}{2}, \\ \chi_{i,j}^{(2)} = \dfrac{31-12i+2i^2-12j+2ij+j^2+18k-4ik-4jk+2k^2}{4(3+k)}, \\ l_{i,j}^{(2)} = i, \end{cases}$$
$$\begin{cases} \xi_{i,j}^{(3)} = \dfrac{4-i-j+k}{2}, \\ \chi_{i,j}^{(3)} = \dfrac{4+i^2-6j+j^2-2jk-k^2}{4(3+k)}, \\ l_{i,j}^{(3)} = 9-i-j+3k. \end{cases}$$

(b) *If* $k = -3 + p/4$ *with* $(p,2) = 1$, $p \geqslant 4$, *then* $(\xi,\chi,l) = (\xi_{i,j}^{(s')}, \chi_{i,j}^{(s')}, l_{i,j}^{(s')})$ *with* $s \in \{1,2\}$, *where*

$$\begin{cases} \xi_{i,j}^{(1')} = \dfrac{1-i}{2}, \\ \chi_{i,j}^{(1')} = \dfrac{13-6i+i^2-12j+2ij+2j^2+6k-2ik-4jk}{4(3+k)}, \\ l_{i,j}^{(1')} = 12-i-2j+4k, \end{cases}$$
$$\begin{cases} \xi_{i,j}^{(2')} = \dfrac{7-2i-j+2k}{2}, \\ \chi_{i,j}^{(2')} = \dfrac{31-12i+2i^2-12j+2ij+j^2+18k-4ik-4jk+2k^2}{4(3+k)}, \\ l_{i,j}^{(2')} = i. \end{cases}$$

Proof. By solving the system of equations

$$\begin{cases} h_i(\xi,\chi) = 0, \\ h_j(\xi + (i-1) - (2+k), \chi - \xi - (i-1) + \dfrac{(2+k)}{2}) = 0, \\ h_l(\xi + (i-1) + (j-1) - 2(2+k), \chi - 2\xi - 2(i-1) - (j-1) + 2(2+k)) = 0, \end{cases}$$

we find nine triples (ξ, χ, l) in term of i, j and k:

$$\begin{cases} \xi_{i,j}^{(1)} = \dfrac{1-i}{2}, \\ \chi_{i,j}^{(1)} = \dfrac{13-6i+i^2-12j+2ij+2j^2+6k-2ik-4jk}{4(3+k)}, \\ l_{i,j}^{(1)} = 9-i-j+3k, \end{cases}$$

$$\begin{cases} \xi_{i,j}^{(1')} = \dfrac{1-i}{2}, \\ \chi_{i,j}^{(1')} = \dfrac{13-6i+i^2-12j+2ij+2j^2+6k-2ik-4jk}{4(3+k)}, \\ l_{i,j}^{(1')} = 12-i-2j+4k, \end{cases}$$

$$\begin{cases} \xi_{i,j}^{(1'')} = \dfrac{1-i}{2}, \\ \chi_{i,j}^{(1'')} = \dfrac{13-6i+i^2-12j+2ij+2j^2+6k-2ik-4jk}{4(3+k)}, \\ l_{i,j}^{(1'')} = 3-j+k, \end{cases}$$

$$\begin{cases} \xi_{i,j}^{(2)} = \dfrac{7-2i-j+2k}{2}, \\ \chi_{i,j}^{(2)} = \dfrac{31-12i+2i^2-12j+2ij+j^2+18k-4ik-4jk+2k^2}{4(3+k)}, \\ l_{i,j}^{(2)} = i, \end{cases}$$

$$\begin{cases} \xi_{i,j}^{(2')} = \dfrac{7-2i-j+2k}{2}, \\ \chi_{i,j}^{(2')} = \dfrac{31-12i+2i^2-12j+2ij+j^2+18k-4ik-4jk+2k^2}{4(3+k)}, \\ l_{i,j}^{(2')} = 6-j+2k, \end{cases}$$

$$\begin{cases} \xi_{i,j}^{(2'')} = \dfrac{7-2i-j+2k}{2}, \\ \chi_{i,j}^{(2'')} = \dfrac{31-12i+2i^2-12j+2ij+j^2+18k-4ik-4jk+2k^2}{4(3+k)}, \\ l_{i,j}^{(2'')} = 6-i-j+2k, \end{cases}$$

$$\begin{cases} \xi_{i,j}^{(3)} = \dfrac{4-i-j+k}{2}, \\ \chi_{i,j}^{(3)} = \dfrac{4+i^2-6j+j^2-2jk-k^2}{4(3+k)}, \\ l_{i,j}^{(3)} = 9-i-j+3k, \end{cases}$$

$$\begin{cases} \xi_{i,j}^{(3')} = \dfrac{4-i-j+k}{2}, \\ \chi_{i,j}^{(3')} = \dfrac{31-12i+2i^2-12j+2ij+j^2+18k-4ik-4jk+2k^2}{4(3+k)}, \\ l_{i,j}^{(3')} = 6-i+2k, \end{cases}$$

$$\begin{cases} \xi_{i,j}^{(3'')} = \dfrac{4-i-j+k}{2}, \\ \chi_{i,j}^{(3'')} = \dfrac{31-12i+2i^2-12j+2ij+j^2+18k-4ik-4jk+2k^2}{4(3+k)}, \\ l_{i,j}^{(3'')} = 3-j+k. \end{cases}$$

Since l is the dimension of $\psi^2(L(\xi,\chi))_{\mathrm{top}}$, it must be a positive integer. If $k=-3+p/3$, with $(p,3)=1$, $p\geqslant 3$, the three triples described in the first part of the proposition are the only ones among the solutions of the system corresponding to this restrictive condition. Similarly, if $k=-3+p/4$, $(p,2)=1$, $p\geqslant 4$, we find that only two triples satisfy the condition. □

Proposition 3.2.4. (a) *Let $k=-3+p/3$ with $(p,3)=1$, $p\geqslant 3$ then $(G^+_{-2})^{p-2}|0\rangle$ belongs to the maximal ideal of $\mathcal{W}^k(\mathfrak{g},f)$.*

(b) *Let $k=-3+p/4$ with $(p,2)=1$, $p\geqslant 4$ then $(G^+_{-2})^{p-3}|0\rangle$ belongs to the maximal ideal of $\mathcal{W}^k(\mathfrak{g},f)$.*

Proof. (a) For $i=j=1$, we have $l_{1,1}^{(1)}=p-2$. Since $\xi_{1,1}^{(1)}=\chi_{1,1}^{(1)}=0$ and $L_0|0\rangle=J_0|0\rangle=0$ the correspondence $|0\rangle\mapsto|\xi_{1,1}^{(1)},\chi_{1,1}^{(1)}\rangle$ yields the isomorphism

$$\mathcal{W}_k(\mathfrak{g},f)\simeq L(\xi_{1,1}^{(1)},\chi_{1,1}^{(1)}).$$

Moreover, $\psi^2(\mathcal{W}_k(\mathfrak{g},f))_{\mathrm{top}}$ is at most $(p-2)$-dimensional because

$$h_{p-2}(-2(2+k),2(2+k))=0.$$

Hence, $(G^+_{-2})^{p-2}|0\rangle=\psi^2((G^+_0)^{p-2})|0\rangle=0$.

(b) The argument is the same as in the previous case with $l_{1,1}^{(1')}=p-3$ and $\xi_{1,1}^{(1')}=\chi_{1,1}^{(1')}=0$. □

We are now in a position to state the main result of this section.

Proposition 3.2.5. *Let M be a simple $\mathcal{W}_k(\mathfrak{g},f)$-module.*

(a) *If $k=-3+p/3$ with $(p,3)=1$, $p\geqslant 3$, then the $\mathcal{W}_k(\mathfrak{g},f)$-module M is isomorphic to $L(\xi_{i,j}^{(s)},\chi_{i,j}^{(s)})$ for $\xi_{i,j}^{(s)}$ and $\chi_{i,j}^{(s)}$ as in Proposition 3.2.3(a) with $1\leqslant i\leqslant p-2$, $1\leqslant j\leqslant p-i-1$ and $s\in\{1,2,3\}$.*

(b) *If $k=-3+p/4$ with $(p,2)=1$, $p\geqslant 4$, then the $\mathcal{W}_k(\mathfrak{g},f)$-module M is isomorphic to $L(\xi_{i,j}^{(s')},\chi_{i,j}^{(s')})$ for $\xi_{i,j}^{(s')}$ and $\chi_{i,j}^{(s')}$ as in Proposition 3.2.3(b) with $1\leqslant i\leqslant p-3$ and $1\leqslant j\leqslant (p-i-1)/2$ if $s=1$ or $1\leqslant i\leqslant p-3$ and $1\leqslant j\leqslant p-2i-1$ if $s=2$.*

Proof. (a) Since M is a simple $\mathcal{W}_k(\mathfrak{g},f)$-module, there exist $\xi,\chi\in\mathbb{C}$ such that $M\cong L(\xi,\chi)$. By Proposition 3.2.4, $G^+(z)^{p-2}=0$. Hence,

$$:G^+(z)^{p-2}:\stackrel{\mathrm{def}}{=}\sum_{n\in\mathbb{Z}}((G^+)^{p-2})_n z^{-n-2}=0.$$

In particular, $(G^+_0)^{p-2}|\xi,\chi\rangle=((G^+)^{p-2})_0|\xi,\chi\rangle=0$. As a consequence, $L(\xi,\chi)_{\mathrm{top}}$ is at most $(p-2)$-dimensional. By Proposition 3.2.3(a), since $\psi^2(L(\xi,\chi))$ is a simple $\mathcal{W}_k(\mathfrak{g},f)$-module, there exist $1\leqslant i,j\leqslant p-2$ and $s\in\{1,2,3\}$ such that $\xi=\xi_{i,j}^{(s)}$ and $\chi=\chi_{i,j}^{(s)}$. In the same way,

$\psi^4(L(\xi,\chi))$ is a simple $\mathcal{W}_k(\mathfrak{g}, f)$-module and there are $1 \leqslant l, m \leqslant p-2$ and $r \in \{1,2,3\}$ such that $\psi^2(\xi) := \xi+(i-1)+(j-1)-2(2+k) = \xi^{(r)}_{l,m}$ and $\psi^2(\chi) := \chi-2\xi-2(i-1)-(j-1)+2(2+k) = \chi^{(r)}_{l,m}$. The ψ^2-action permutes the three forms of the eigenvalues ξ and χ:

$$\psi^2(L(\xi^{(1)}_{i,j}, \chi^{(1)}_{i,j})) \simeq L(\xi^{(2)}_{p-i-j,i}, \chi^{(2)}_{p-i-j,i}),$$
$$\psi^2(L(\xi^{(2)}_{i,j}, \chi^{(2)}_{i,j})) \simeq L(\xi^{(3)}_{i,p-i-j}, \chi^{(3)}_{i,p-i-j}),$$
$$\psi^2(L(\xi^{(3)}_{i,j}, \chi^{(3)}_{i,j})) \simeq L(\xi^{(1)}_{p-i-j,j}, \chi^{(1)}_{p-i-j,j}).$$

The condition $j \leqslant p-i-1$ comes from $1 \leqslant l, m \leqslant p-2$.

(b) The argument is quite similar. By Proposition 3.2.3(b), $G^+(z)^{p-3} = 0$ and $L(\xi,\chi)_{\text{top}}$ is at most $(p-3)$-dimensional. Moreover, since $\psi^2(L(\xi,\chi))$ and $\psi^4(L(\xi,\chi))$ are simple $\mathcal{W}_k(\mathfrak{g}, f)$-modules, $\xi = \xi^{(s')}_{i,j}$, $\chi = \chi^{(s')}_{i,j}$, $\psi^2(\xi) = \xi^{(r')}_{l,m}$ and $\psi^2(\chi) = \chi^{(r')}_{l,m}$ with $1 \leqslant i,j,l,m \leqslant p-3$, $r, s \in \{1,2\}$. On the contrary of the first case, the ψ^2-action preserves the form of the eigenvalues ξ and χ:

$$\psi^2(L(\xi^{(1')}_{i,j}, \chi^{(1')}_{i,j})) \simeq L(\xi^{(1')}_{p-i-2j,j}, \chi^{(1')}_{p-i-2j,j}),$$
$$\psi^2(L(\xi^{(2')}_{i,j}, \chi^{(2')}_{i,j})) \simeq L(\xi^{(2')}_{i,p-2i-j}, \chi^{(2')}_{i,p-2i-j}).$$

If $s = 1$, the condition $p-i-2j \geqslant 1$ implies $j \leqslant \frac{p-i-1}{2}$, and if $s = 2$, we get $j \leqslant p-2i-1$ with the same argument. □

Remark 3.2.6. The simple $\mathcal{W}_k(\mathfrak{g}, f)$-modules $L(\xi^{(s)}_{i,j}, \chi^{(s)}_{i,j})$ of Proposition 3.2.5 are all mutually non-isomorphic since their highest weights are distinct.

Remark 3.2.7. For $k = -3 + p/3$, with $(p,3) = 1$, $p \geqslant 3$, or $k = -3 + p/4$, with $(p,2) = 1$, $p \geqslant 4$, the application ψ is a bijection of the set of the simple $\mathcal{W}_k(\mathfrak{g}, f)$-modules $L(\xi^{(s)}_{i,j}, \chi^{(s)}_{i,j})$ described in Proposition 3.2.5 over itself of inverse ψ^5 if k is principal admissible, and ψ^3 otherwise. We describe below the $L(\xi^{(s)}_{i,j}, \chi^{(s)}_{i,j})$ orbits under the ψ-action:

- if $k = -3 + p/3$ with $(p,3) = 1$, $p \geqslant 3$, then

$$L(\xi^{(1)}_{i,j}, \chi^{(1)}_{i,j}) \xrightarrow{\psi} L(\xi^{(3)}_{j,p-i-j}, \chi^{(3)}_{j,p-i-j}) \xrightarrow{\psi} L(\xi^{(2)}_{p-i-j,i}, \chi^{(2)}_{p-i-j,i}) \xrightarrow{\psi} L(\xi^{(1)}_{i,p-i-j}, \chi^{(1)}_{i,p-i-j})$$
$$\xrightarrow{\psi} L(\xi^{(3)}_{p-i-j,j}, \chi^{(3)}_{p-i-j,j}) \xrightarrow{\psi} L(\xi^{(2)}_{j,i}, \chi^{(2)}_{j,i})) \xrightarrow{\psi} L(\xi^{(1)}_{i,j}, \chi^{(1)}_{i,j}),$$

- if $k = -3 + p/4$ with $(p,2) = 1$, $p \geqslant 4$, then

$$L(\xi^{(1')}_{i,j}, \chi^{(1')}_{i,j}) \xrightarrow{\psi} L(\xi^{(2')}_{j,p-i-2j}, \chi^{(2')}_{j,p-i-2j}) \xrightarrow{\psi} L(\xi^{(1')}_{p-i-2j,j}, \chi^{(1')}_{p-i-2j,j})$$
$$\xrightarrow{\psi} L(\xi^{(2')}_{j,i}, \chi^{(2')}_{j,i}) \xrightarrow{\psi} L(\xi^{(1')}_{i,j}, \chi^{(1')}_{i,j}).$$

3.3 Proof of the rationality of $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$ at admissible levels

This section is devoted to the proof of the Theorem 3.1. Unfortunately, contrary to the case where $f = f_{min}$ is minimal, the functor $H^0_f(?)$ appearing in the construction of $\mathcal{W}$-algebras (see Sect. 1.3.2) is not exact for $f = f_{\mathrm{subreg}}$. To encounter the difficulty, we exploit techniques developed in [21], when f is an even nilpotent element. We show that simple modules described in Proposition 3.2.5 exist and that there are no nontrivial extension between them.

For the moment, let f be an even nilpotent of a simple Lie algebra $\mathfrak{g}$. Set

$$P_{0,+} := \{\lambda \in \widehat{\mathfrak{h}},\ \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geqslant 0} \text{ for all } \alpha \in \Delta_{0,+}\},$$

where $\Delta_{0,+} = \Delta_0 \cap \Delta+$ with Δ_0 the root system of $(\mathfrak{g}_0, \mathfrak{h})$. Recall that $J_\lambda \subset U(\mathfrak{g})$ is the annihilating ideal of the simple $\mathfrak{g}$-module $L(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$ (see Sect. 1.2.1). The quotient $H^0_f(U(\mathfrak{g})/J_\lambda)$ is a quotient algebra of the finite $\mathcal{W}$-algebra $U(\mathfrak{g}, f) = H^0_f(U(\mathfrak{g}))$. For $\lambda \in P_{0,+} \cap \mathrm{Pr}^k$ such that $\dim L(\lambda)$ is maximal, $H^0_f(U(\mathfrak{g})/J_\lambda)$ has a unique simple module denoted by E_{J_λ} [21, Theorem 7.7]. Let $\mathbf{L}(E_{J_\lambda}) = H^0_{f,-}(\widehat{L}_k(\lambda))$ be the irreducible Ramond twisted $\mathcal{W}_k(\mathfrak{g}, f)$-module attached to E_{J_λ} (see Sect. 1.3.2).

Theorem 3.3.1 ([21]). *Let $k = -h^\vee + p/q$ be an admissible number for $\mathfrak{g}$ and pick $f \in \mathbb{O}_q$. Let $\lambda \in P_{0,+} \cap \mathrm{Pr}^k$. Then*

$$H^0_f(\widehat{L}_k(\lambda)) \simeq \mathbf{L}(E_{J_{\lambda - \frac{p}{q}x_0}}).$$

In particular,

$$\mathcal{W}_k(\mathfrak{g}, f) \simeq H^0_f(L_k(\mathfrak{g})) \simeq \mathbf{L}(E_{J_{-\frac{p}{q}x_0}}).$$

In the case $\mathfrak{g} = \mathfrak{sp}_4$ and $f = f_{\mathrm{subreg}} = e_{-\eta}$, we have $\Delta_{0,+} = \{\alpha_2\}$, whence

$$P_{0,+} = \{\lambda \in \mathfrak{h}^* \colon \langle \lambda, \alpha_2^\vee \rangle \in \mathbb{Z}_{\geqslant 0}\} = \mathbb{C}\varpi_1 + \mathbb{Z}_{\geqslant 0}\varpi_2.$$

According Proposition 3.2.5, when $k = -3 + p/3$, with $(p, 3) = 1$, $p \geqslant 3$, or $k = -3 + p/4$, with $(p, 2) = 1$, $p \geqslant 4$, if M is a simple $\mathcal{W}_k(\mathfrak{g}, f)$-module, then it is isomorphic to $L(\xi^{(s)}_{i,j}, \chi^{(s)}_{i,j})$ with $\xi^{(s)}_{i,j}$ and $\chi^{(s)}_{i,j}$ as described in Proposition 3.2.3. The following assertion ensures the reverse.

Proposition 3.3.2. *Let k, $\xi^{(s)}_{i,j}$ and $\chi^{(s)}_{i,j}$ be as in Proposition 3.2.5, then $L(\xi^{(s)}_{i,j}, \chi^{(s)}_{i,j})$ is a simple $\mathcal{W}_k(\mathfrak{g}, f)$-module.*

Proof. The computation depends on whether k is principal admissible of coprincipal admissible, but the argument is very similar in both cases. We only detail the case $k = -3 + p/3$, with $(p, 3) = 1$, $p \geqslant 3$. For $1 \leqslant i \leqslant p - 2$ and $1 \leqslant j \leqslant p - i - 1$, set $\lambda_{i,j} = (j - 1)\varpi_1 + (i - 1)\varpi_2$. We check that $\lambda_{i,j} \in P_{0,+} \cap \mathrm{Pr}^k$. By Theorem 3.3.1,

$$H^0_f(\widehat{L}_k(\lambda_{i,j})) \simeq \mathbf{L}(E_{J_{\lambda_{i,j} - \frac{p}{3}x_0}}).$$

The minimal L_0-eigenvalue $h_{\lambda_{i,j} - \frac{p}{3}x_0}$, called the *conformal dimension* of $\mathbf{L}(E_{J_{\lambda_{i,j} - \frac{p}{3}x_0}})$, is given by [21, (7.4)]:

$$\begin{aligned} h_{\lambda_{i,j} - \frac{p}{3}x_0} &= \frac{(\lambda_{i,j} - \frac{p}{3}x_0 | \lambda_{i,j} - \frac{p}{3}x_0 + 2\rho)}{2(k + h^\vee)} - \frac{k + h^\vee}{2}|x_0|^2 + (x_0|\rho) \\ &= \frac{-15 + 3i^2 + 6ij + 6j^2 + 6p - 2ip - 4jp}{4p} = \chi^{(1)}_{i,j}. \end{aligned}$$

Besides, using the identification between Ramond twisted and non-twisted representation (1.14) for $J(z) = J^{\{\alpha_2\}}(z)$, we get that $\widehat{w}_0 \widehat{t}_{-x_0} J^{\{\alpha_2\}}(z)^R = -J^{\{\alpha_2\}}(z)$. We deduce that the lowest

J_0-eigenvalue is

$$(\lambda_{i,j} - \frac{p}{3}x_0| - \frac{\alpha_2^\vee}{2}) = \frac{1-i}{2} = \xi_{i,j}^{(1)}.$$

In conclusion,

$$\mathbf{L}(E_{J_{\lambda_{i,j}-\frac{p}{3}x_0}}) \simeq L(\xi_{i,j}^{(1)}, \chi_{i,j}^{(1)}). \tag{3.3}$$

Thus, $L(\xi_{i,j}^{(1)}, \chi_{i,j}^{(1)})$ is a simple $\mathcal{W}_k(\mathfrak{g}, f)$-module for all $1 \leqslant i \leqslant p-2$ and $1 \leqslant j \leqslant p-i-1$. If $s = 2$ or 3, using the ψ-action on the module $L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})$, it always comes down to a module $L(\xi_{i',j'}^{(1)}, \chi_{i',j'}^{(1)})$. As a consequence, $L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})$ is a simple module of $\mathcal{W}_k(\mathfrak{g}, f)$ too. □

Lemma 3.3.3. *Suppose that there is a nontrivial extension of $\mathcal{W}_k(\mathfrak{g}, f)$-modules,*

$$0 \longrightarrow L(\xi, \chi) \xrightarrow{\iota} M \xrightarrow{\pi} L(\xi', \chi') \longrightarrow 0.$$

Then L_0 acts locally finitely on M.

Proof. Suppose there is a nontrivial extension

$$0 \longrightarrow L(\xi, \chi) \xrightarrow{\iota} M \xrightarrow{\pi} L(\xi', \chi') \longrightarrow 0.$$

Since $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse, $L := L(\xi, \chi)$ and $L' := L(\xi', \chi')$ are L_0-diagonalizable and the L_0-eigenspaces are finite dimensional. Let $m \in M$. Since $\pi(m) \in L'$, there exist $w_1, \ldots, w_s \in L'$ and $\mu_1, \ldots, \mu_s \in \mathbb{C}$ such that $L_0 w_j = \mu_j w_j$ for all $1 \leqslant j \leqslant s$ and $\prod_{j=1}^s (L_0 - \mu_j \operatorname{id})\pi(m) = 0$. Then

$$\pi(\prod_{j=1}^s (L_0 - \mu_j \operatorname{id})m) = 0.$$

As a consequence, $\prod_{j=1}^s (L_0 - \mu_j \operatorname{id})m \in \operatorname{im} \iota$. Let $m_1 \in L$ such that $\iota(m_1) = \prod_{j=1}^s (L_0 - \mu_j \operatorname{id})m$. As before, there are $v_1, \ldots, v_r \in L$ and $\nu_1, \ldots, \nu_r \in \mathbb{C}$ such that $\prod_{i=1}^r (L_0 - \nu_i \operatorname{id})m_1 = 0$. Then

$$\iota(\prod_{i=1}^r (L_0 - \nu_i \operatorname{id})m_1) = 0 = \prod_{i=1}^r (L_0 - \nu_i \operatorname{id})\iota(m_1) = \prod_{i=1}^r (L_0 - \nu_i \operatorname{id}) \prod_{j=1}^s (L_0 - \mu_j \operatorname{id})m.$$

Hence, m belongs to some L_0-stable finite dimensional vector subspace of

$$\bigoplus_{i=1}^r \ker(L_0 - \nu_i \operatorname{id}) \oplus \bigoplus_{j=1}^s \ker(L_0 - \mu_j \operatorname{id}).$$

□

Lemma 3.3.4. *If there exists a nontrivial extension of $\mathcal{W}_k(\mathfrak{g}, f)$-modules*

$$0 \longrightarrow L(\xi, \chi) \xrightarrow{\iota} M \xrightarrow{\pi} L(\xi', \chi') \longrightarrow 0,$$

then χ and χ' coincide modulo $\mathbb{Z}$.

Proof. Suppose that there is a nontrivial extension

$$0 \longrightarrow L(\xi, \chi) \xrightarrow{\iota} M \xrightarrow{\pi} L(\xi', \chi') \longrightarrow 0.$$

As in the previous proof, set $L := L(\xi, \chi)$ and $L' := L(\xi', \chi')$. For $d \in \mathbb{C}$, let M_d be the generalized L_0-eigenspace of M attached to the eigenvalue d. Set $M[d] := \bigoplus_{d' \in d + \mathbb{Z}} M_{d'}$. It is a $\mathcal{W}_k(\mathfrak{g}, f)$-submodule of M. Then $M = \bigoplus_{\substack{d\in\mathbb{C},\\ 0\leqslant \mathrm{Re}(d)<1}} M[d]$ is a direct sum decomposition of the $\mathcal{W}_k(\mathfrak{g}, f)$-modules of M. For any d, the previous decomposition induces the following exact sequence

$$0 \longrightarrow L[d] \longrightarrow M[d] \longrightarrow L'[d] \longrightarrow 0. \tag{3.4}$$

Assume $\chi - \chi' \notin \mathbb{Z}$. Since $L[d] = 0$ if $d - \chi \notin \mathbb{Z}$, and $L[d'] = 0$ if $d' - \chi' \notin \mathbb{Z}$, we get that $M = M[\chi] \oplus M[\chi']$. Taking $d = \chi$ and $d = \chi'$ in (3.4) we get

$$0 \longrightarrow L[\chi] \longrightarrow M[\chi] \longrightarrow 0,$$
$$0 \longrightarrow M[\chi'] \longrightarrow L'[\chi'] \longrightarrow 0.$$

Finally, $M = L[\chi] \oplus L'[\chi'] = L \oplus L'$ since L and L' are simple modules. So the sequence $0 \to L \to M \to L' \to 0$ splits, whence a contradiction. □

Proposition 3.3.5. *Suppose that either $k = -3 + p/3$ with $(p, 3) = 1$, $p \geqslant 3$, or $k = -3 + p/4$ with $(p, 2) = 1$, $p \geqslant 4$. Then*

$$\mathrm{Ext}^1_{\mathcal{W}^k(\mathfrak{g},f)\text{-Mod}}(L(\xi^{(s)}_{i,j}, \chi^{(s)}_{i,j}), L(\xi^{(s')}_{i',j'}, \chi^{(s')}_{i',j'})) = 0,$$

where $\xi^{(s)}_{i,j}$ and $\chi^{(s)}_{i,j}$ are described in Proposition 3.2.5.

Proof. Assume $k = -3+p/3$ with $(p, 3) = 1$, $p \geqslant 3$. It clearly appears that for all $1 \leqslant i, i' \leqslant p-2$, $1 \leqslant j \leqslant p - i - 1$ and $1 \leqslant j' \leqslant p - i' - 1$, the differences $\chi^{(2)}_{i,j} - \chi^{(1)}_{i',j'}$ and $\chi^{(3)}_{i,j} - \chi^{(1)}_{i',j'}$ are not integers. According to Lemma 3.3.4, any extension

$$0 \longrightarrow L(\xi^{(s)}_{i,j}, \chi^{(s)}_{i,j}) \longrightarrow M \longrightarrow L(\xi^{(s')}_{i',j'}, \chi^{(s')}_{i',j'}) \longrightarrow 0,$$

where exactly one of s, s' is equal to 1 is trivial. Applying ψ we deduce that if $s \neq s'$ then

$$\mathrm{Ext}^1_{\mathcal{W}^k(\mathfrak{g},f)\text{-Mod}}(L(\xi^{(s)}_{i,j}, \chi^{(s)}_{i,j}), L(\xi^{(s')}_{i',j'}, \chi^{(s')}_{i',j'})) = 0.$$

Suppose that $s = s'$. Using the ψ-action, we can assume that $s = s' = 1$. According to (3.3), since $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse, it suffices to show that there is no nontrivial extension

$$0 \longrightarrow \mathbf{L}(E_{J_{\lambda_{i,j}-\frac{p}{3}x_0}}) \xrightarrow{\iota} M \xrightarrow{\pi} \mathbf{L}(E_{J_{\lambda_{i',j'}-\frac{p}{3}x_0}}) \longrightarrow 0. \tag{3.5}$$

Set $L_{i,j} := \mathbf{L}(E_{J_{\lambda_{i,j}-\frac{p}{3}x_0}})$ and $L_{i',j'} := \mathbf{L}(E_{J_{\lambda_{i',j'}-\frac{p}{3}x_0}})$. If $\chi^{(1)}_{i,j} = \chi^{(1)}_{i',j'}$, since the Zhu algebra $\mathcal{A}(\mathcal{W}_k(\mathfrak{g}, f))$ is semisimple, the sequence

$$0 \longrightarrow (L_{i,j})_{\mathrm{top}} \longrightarrow M_{\mathrm{top}} \longrightarrow (L_{i',j'})_{\mathrm{top}} \longrightarrow 0$$

of $\mathcal{A}(\mathcal{W}_k(\mathfrak{g}, f))$-modules splits. Applying the Zhu induction functor we get that (3.5) splits.

Let us suppose $\chi^{(1)}_{i,j} > \chi^{(1)}_{i',j'}$. Set $M_{i',j'} := \mathbf{M}(E_{J_{\lambda_{i',j'}-\frac{p}{3}x_0}})$. Let v_+ be a primitive vector of $M_{i',j'}$ and $v \in M$ be such that $\pi(v)$ is the image of v_+ in $L_{i',j'}$. We have $\pi((L_0 - \chi^{(1)}_{i',j'}\,\mathrm{id})v) = (L_0 - \chi^{(1)}_{i',j'}\,\mathrm{id})\pi(v) = 0$. Hence, $(L_0 - \chi^{(1)}_{i',j'}\,\mathrm{id})v \in \mathrm{im}\,\iota$. Since $\mathrm{im}\,\iota \simeq L_{i,j}$ and $\chi^{(1)}_{i,j} > \chi^{(1)}_{i',j'}$, $(L_0 - \chi^{(1)}_{i',j'}\,\mathrm{id})v \neq 0$. As a consequence, there exists an injective $\mathcal{W}^k(\mathfrak{g}, f)$-module homomorphism $f : M_{i',j'} \to M$ such that the diagram commutes:

$$\begin{array}{ccc} & & M_{i',j'} \\ & \overset{f}{\swarrow} & \downarrow \\ M & \twoheadrightarrow & L_{i',j'} \end{array}$$

Let us suppose that the sequence (3.5) does not split. The module $f(M_{i',j'})$ is a submodule of M. As a consequence, since $L_{i,j} \simeq \iota(L_{i,j})$ is simple, either $\iota(L_{i,j}) \subset f(M_{i',j'})$ or $\iota(L_{i,j}) \oplus f(M_{i',j'})$. However, if $\iota(L_{i,j}) \oplus f(M_{i',j'})$ then the sequence

$$0 \longrightarrow L_{i,j} \longrightarrow \iota(L_{i,j}) \oplus f(M_{i',j'}) \longrightarrow L_{i',j'} \longrightarrow 0$$

splits contradicting the fact that the sequence (3.5) does not split. Hence, $\iota(L_{i,j}) \subset f(M_{i',j'})$. Let $m \in M$. Since π is surjective, it exists $m_1 \in M_{i',j'}$ such that $\pi(m) = \pi \circ f(m_1)$. Thus, $m - f(m_1) \in \ker \pi$. We get $m \in \iota(L_{i,j}) \subset f(M_{i',j'})$. Therefore, f is surjective. It implies that M is isomorphic to $M_{i',j'}$ as $\mathcal{W}^k(\mathfrak{g}, f)$-modules. Hence, $[M_{i',j'} : L_{i,j}] \neq 0$. By [21, Theorem 7.6], this happens only if it exists $\mu \in P_{0,+}$ such that $[\widehat{M}_k(\lambda_{i',j'} - \frac{p}{3}x_0) : \widehat{L}_k(\mu - \frac{p}{3}x_0)] \neq 0$, where $\widehat{M}_k(\lambda_{i',j'} - \frac{p}{3}x_0)$ is the Verma module of $\mathfrak{g}$ with highest weight $\hat{\lambda}_{i',j'} - \frac{p}{3}x_0$, and $E_{J_{\lambda_{i,j} - \frac{p}{3}x_0}}$ is a direct summand of $H_0^{\text{Lie}}(L(\mu - \frac{p}{3}x_0))$[1]. The first condition implies that $\mu \in W \circ \lambda_{i',j'}$ and we get $\lambda_{i,j} \in W \circ \mu$ from the second one. Hence, $\lambda_{i,j} \in W \circ \lambda_{i',j'}$. Since $\hat{\lambda}_{i,j}$ and $\hat{\lambda}_{i',j'}$ are both dominant they are equal, and $\lambda_{i,j} = \lambda_{i',j'}$ contradicting $\chi_{i,j}^{(1)} > \chi_{i',j'}^{(1)}$. Finally if $\chi_{i,j}^{(1)} < \chi_{i',j'}^{(1)}$ by applying the duality functor to (3.5) we are back to the previous case $\chi_{i,j}^{(1)} > \chi_{i',j'}^{(1)}$.

The argument for the coprincipal case is the same. □

Propositions 3.2.5 and 3.3.2 give the complete classifications of simple $\mathcal{W}_k(\mathfrak{g}, f)$-modules when k is an admissible level appearing in Theorem 3.1. When k is principal admissible, we list exactly $3(p-1)(p-2)/2$ irreducible representations, and $(p-1)(p-3)/2$ when k is coprincipal admissible.

Example 3.3.6. Let $k = -\frac{5}{3}$. There exist nine simple $\mathcal{W}_{-\frac{5}{3}}(\mathfrak{g}, f)$-modules. We describe below the two orbits under the action of ψ:

$$\begin{aligned} L(0,0) &\xrightarrow{\psi} L\left(-\frac{1}{3}, \frac{1}{6}\right) \xrightarrow{\psi} L\left(-\frac{2}{3}, \frac{2}{3}\right) \\ &\xrightarrow{\psi} L\left(0, \frac{1}{2}\right) \xrightarrow{\psi} L\left(-\frac{1}{3}, \frac{2}{3}\right) \\ &\xrightarrow{\psi} L\left(\frac{1}{3}, \frac{1}{6}\right) \xrightarrow{\psi} L(0,0), \end{aligned}$$

$$L\left(-\frac{1}{2}, \frac{7}{16}\right) \xrightarrow{\psi} L\left(\frac{1}{6}, \frac{5}{48}\right) \xrightarrow{\psi} L\left(-\frac{1}{6}, \frac{5}{48}\right) \xrightarrow{\psi} L\left(-\frac{1}{2}, \frac{7}{16}\right).$$

A vertex algebra V is *positive* [21] if every irreducible V-modules besides V itself has positive conformal dimension. In our setting, $\mathcal{W}_k(\mathfrak{g}, f)$ is positive if $\chi_{i,j}^{(s)} \geqslant 0$ for all k, s, i, j as in Proposition 3.2.5. We observe the vertex algebras $W_{-\frac{5}{3}}(\mathfrak{g}, f)$, $\mathcal{W}_{-\frac{4}{3}}(\mathfrak{g}, f)$, $\mathcal{W}_{-\frac{7}{4}}(\mathfrak{g}, f)$ and $\mathcal{W}_{-\frac{5}{4}}(\mathfrak{g}, f)$ are positive.

If V is *unitary* ([52, Sect. 2]) then it is unitary as a module of the Virasoro subalgebra generated by the conformal vector as well. This forces the conformal dimension to be non-negative. In particular, it is a positive vertex algebra. We expect the following:

[1] Here, $H_\bullet^{\text{Lie}}$ denotes the usual Lie algebra homology functor and $M \mapsto H_0^{\text{Lie}}(M)$ defines a correspondence between the subcategory of the category $\mathcal{O}$ of $\mathfrak{g}$-modules which are integrable as $\mathfrak{g}_0$-modules and the category of the finite dimensional representations of $U(\mathfrak{g}, f)$ [14, Sect. 5].

Conjecture 3.3.7. *At level $k \in \{-\frac{5}{3}, -\frac{4}{3}, -\frac{7}{4}, -\frac{5}{4}\}$, the vertex algebra $\mathcal{W}_k(\mathfrak{sp}_4, f_{\rm subreg})$ is unitary.*

Remark 3.3.8. For the other admissible levels as in Theorem 3.1, that is for $k = -3 + p/3$ with $(p, 3) = 1$ and $p \geqslant 6$, or $k = -3 + p/4$ with $(p, 2) = 1$ and $p \geqslant 8$, the vertex algebra $\mathcal{W}_k(\mathfrak{sp}_4, f_{\rm subreg})$ is not positive because $\chi^{(1)}_{1,2} = -1 + \frac{6}{p}$ and $\chi^{(1')}_{1,2} = -1 + \frac{8}{p}$ are negative.

3.4 Action of the component group of $\mathbb{O}_{\rm subreg}$ over simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\rm subreg})$-modules

Let $\mathfrak{g}^\natural$ be the centralizer of the $\mathfrak{sl}_2$-triple $\{e, h, f\}$. As we saw in Sect. 2.2, it is a Lie subalgebra of $\mathfrak{g}$. We denote by $G^\natural$ the stabilizer in the adjoint group G of $\{e, h, f\}$. This group is not necessary connected, so denote $(G^\natural)^\circ$ its identity component. The *component group* of $G^\natural$ is the quotient group:

$$A(\mathbb{O}_f) := G^\natural/(G^\natural)^\circ.$$

The component group of $G^\natural$ coincides with the one of G^f the centralizer of f in G:

$$A(\mathbb{O}_f) \simeq G^f/(G^f)^\circ.$$

We refer to [39, Chap.3] for general facts on the component group $A(\mathbb{O}_f)$.

Realize $\mathfrak{sp}_4$ as in Sect. 2.1.2. The adjoint group of $\mathfrak{sp}_4$ is the quotient $PSp_4 = Sp_4/\{\pm I_4\}$ where Sp_4 is the set of 4-size square matrices g such that $gJ_4g^T = J_4$ and I_n denotes the n-size square identity matrix. Fix an $\mathfrak{sl}_2$-triple $\{e, h, f\}$ such that $f \in \mathbb{O}_{\rm subreg}$. One may assume that

$$f = \begin{pmatrix} 0 & 0 \\ -I_2 & 0 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad h = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

Then the centralizer $\mathfrak{g}^f$ is generated by the matrices f, f_1, f_θ and h_2 where

$$f_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_\theta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad h_2 = \mathrm{diag}(1, -1, 1, -1).$$

Moreover, the stabilizer in G of the $\mathfrak{sl}_2$-triple $\{e, h, f\}$ is given by

$$G^\natural = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, AU_2A^T = U_2 \right\} \simeq O(2),$$

where $U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence, $A(\mathbb{O}_{\rm subreg}) \simeq O(2)/SO(2) = \{I_4, U\} \simeq \mathbb{Z}/2\mathbb{Z}$, where $U = \begin{pmatrix} U_2 & 0 \\ 0 & U_2 \end{pmatrix}$. Then U acts on $\mathfrak{g}^f$ by

$$U.f = f, \qquad U.h_2 = -h_2, \qquad U.f_1 = f_\theta \qquad \text{and} \qquad U.f_\theta = f_1.$$

It induces an automorphism Φ of $\mathcal{W}^k(\mathfrak{g}, f)$ defined by

$$\Phi(L) = L, \qquad \Phi(J) = -J, \qquad \text{and} \qquad \Phi(G^\pm) = G^\mp.$$

Remark 3.4.1. For $U \in G^f$, denote $\bar{U}$ its image in the component group $A(\mathbb{O}_{\text{subreg}})$. Setting the convention that $J^{\{cx\}} = cJ^{\{x\}}$ for any strong generator $J^{\{x\}}$ and $c \in \mathbb{C}$, we notice that $A(\mathbb{O}_{\text{subreg}})$ acts on $\mathcal{W}^k(\mathfrak{g}, f)$ by

$$\bar{U}.J^{\{x\}} = J^{\{U.x\}},$$

where $U.x = UxU^{-1}$ for all $U \in G^f$.

For any nilpotent element f in a simple Lie algebra $\mathfrak{g}$. The correspondence of the strong generators with a good basis of the centralizer $\mathfrak{g}^f$ leads us to strongly believe that the action of the component group $A(\mathbb{O}_f)$ on $\mathfrak{g}^f$ induces an action on $\mathcal{W}^k(\mathfrak{g}, f)$ defined as above.

Remark 3.4.2. We can describe all automorphisms of $\mathcal{W}^k(\mathfrak{g}, f)$ using OPEs. Indeed, an automorphism of $\mathcal{W}^k(\mathfrak{g}, f)$ is an isomorphism of conformal vertex algebra. It preserves OPEs, the conformal vector and the conformal weights. Thus, any automorphism of $\mathcal{W}(\mathfrak{sp}_4, f_{\text{subreg}})$ is of the form Φ_α^+ or Φ_α^-, $\alpha \in \mathbb{C}^*$, where

$$\Phi_\alpha^\pm(J) = \pm J, \quad \Phi_\alpha^\pm(G^+) = \alpha G^\pm \quad \text{and} \quad \Phi_\alpha^\pm(G^-) = \frac{1}{\alpha} G^\mp.$$

With this notation we have $\Phi = \Phi_1^-$.

Theorem 3.4.3. *Suppose that k is an admissible level. The automorphism Φ induces an action on the set of simple $\mathcal{W}_k(\mathfrak{g}, f)$-modules:*

$$\Phi(L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})) \simeq L(-(\xi_{i,j}^{(s)} + i - 1), \chi_{i,j}^{(s)}).$$

Moreover, Φ is an involution (i.e. $\Phi^2 = \mathrm{id}$) which preserves pairwise the simple modules of the first form. If k is principal admissible then Φ induces a one-to-one correspondence between the modules of the second form and those of the third form. If k is coprincipal admissible, Φ induces an action of the simple modules of the second form (remember that there is only two forms of simple modules in this case).

Proof. Let $L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})$ be a simple module of $\mathcal{W}_k(\mathfrak{g}, f)$. Since Φ fixes L, the L_0-weights of $L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})$ remain unchanged and so the top-component of the simple module under the action of $\Phi(\mathcal{W}^k(\mathfrak{g}, f))$ is still spanned by the vectors $(G_0^+)^m|\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}\rangle$, $0 \leqslant m \leqslant i-1$. Moreover,

$$-J_0(G_0^+)^m|\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}\rangle = -(\xi + m)(G_0^+)^m|\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}\rangle$$

for all m. Then the smallest eigenvalue of $-J_0$ in the top-component is $-(\xi + i - 1)$ associated with the vector $(G_0^+)^{i-1}|\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}\rangle$. Hence,

$$\Phi(L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})) \simeq L(-(\xi_{i,j}^{(s)} + i - 1), \chi_{i,j}^{(s)}).$$

If $s = 1, 1'$ then $\xi_{i,j}^{(s)} = \frac{1-i}{2}$ and $-(\xi_{i,j}^{(s)} + i - 1) = \xi_{i,j}^{(s)}$. Thus, $\Phi(L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})) \simeq L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})$ for $s = 1, 1'$. Moreover, suppose that k is principal admissible then

$$\Phi(L(\xi_{i,j}^{(2)}, \chi_{i,j}^{(2)})) \simeq L(\xi_{i,p-i-j}^{(3)}, \chi_{i,p-i-j}^{(3)}) \quad \text{and} \quad \Phi(L(\xi_{i,p-i-j}^{(3)}, \chi_{i,p-i-j}^{(3)})) \simeq L(\xi_{i,j}^{(2)}, \chi_{i,j}^{(2)}),$$

and, if k is coprincipal admissible,

$$\Phi(L(\xi_{i,j}^{(2')}, \chi_{i,j}^{(2')})) \simeq L(\xi_{i,p-2i-j}^{(2')}, \chi_{i,p-2i-j}^{(2')}).$$

In particular, Φ is an involution. □

Remark 3.4.4. It seems that exceptional $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{sp}_4, f_{\mathrm{subreg}})$ are the first known examples of rational $\mathcal{W}$-algebras such that the corresponding component group acts non-trivially on the set of simple $\mathcal{W}_k(\mathfrak{g}, f)$-modules.

Example 3.4.5. Let $k = -5/3$. We described in Example 3.3.6 the nine simple $\mathcal{W}_{-5/3}(\mathfrak{g}, f)$-modules. There Φ fixes the modules $L(0,0)$, $L\left(0, \frac{1}{2}\right)$ and $L\left(-\frac{1}{2}, \frac{7}{16}\right)$ and permutes the others as follows:

$$L\left(-\frac{1}{3}, \frac{1}{6}\right) \overset{\Phi}{\leftrightarrow} L\left(\frac{1}{3}, \frac{1}{6}\right),$$
$$L\left(-\frac{2}{3}, \frac{2}{3}\right) \overset{\Phi}{\leftrightarrow} L\left(-\frac{1}{3}, \frac{2}{3}\right),$$
$$L\left(\frac{1}{6}, \frac{5}{48}\right) \overset{\Phi}{\leftrightarrow} L\left(-\frac{1}{6}, \frac{5}{48}\right).$$

3.5 Character of admissible highest weight modules

From now one k denotes a admissible level of the form $k = -3 + p/3$ with $(p, 3) = 1$, $p \geqslant 3$ or $k = -3 + p/4$ with $(p, 2) = 1$, $p \geqslant 4$.

The intersection $\mathfrak{h}^f = \mathfrak{g}^f \cap \mathfrak{h}$ is one-dimensional, generated by $\alpha_2^\vee$ viewed as an element of $\mathfrak{h}$, and $J = J^{\{\alpha_2\}}$. Following [73], for any simple $\mathcal{W}_k(\mathfrak{g}, f)$-modules $L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})$ described in Proposition 3.2.5, one associates the formal character

$$\mathrm{ch}_{L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})}(q, z) := \mathrm{tr}_{L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})}\, q^{L_0} z^{J_0} = \sum_{\substack{d \in \chi_{i,j}^{(s)} + \mathbb{Z}_{\geqslant 0},\\ a \in \xi_{i,j}^{(s)} + \mathbb{Z}}} \dim L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})_{a,d}\, q^d z^a,$$

where $q = e^{2\pi i \tau}$ and $z = e^{-2\pi i \nu}$, $\nu \in \mathbb{C}$.

Recall from the proof of Proposition 3.3.2 that the simple modules $L(\xi_{i,j}^{(1)}, \chi_{i,j}^{(1)})$ (if k is principal) and $L(\xi_{i,j}^{(1')}, \chi_{i,j}^{(1')})$ (if k is coprincipal) corresponds to some highest weight $\widehat{\mathfrak{g}}$-modules

$$L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}) \simeq H_f^0(\widehat{L}_k(\lambda_{i,j}^{(s)})), \tag{3.6}$$

with $s = 1, 1'$ and $\lambda_{i,j}^{(s)} := \lambda_{i,j} = (j-1)\varpi_1 + (i-1)\varpi_2 \in P_{0,+}$.

Proposition 3.5.1. (a) *Let $k = -3 + p/3$ with $(p, 3) = 1$, $p \geqslant 3$, and $\xi_{i,j}^{(1)}$ and $\chi_{i,j}^{(1)}$ be as in Proposition 3.2.5(a) such that $L(\xi_{i,j}^{(1)}, \chi_{i,j}^{(1)})$ is a simple $\mathcal{W}_k(\mathfrak{g}, f)$-module. Then*

$$\mathrm{ch}_{L(\xi_{i,j}^{(1)}, \chi_{i,j}^{(1)})}(q, z) = \frac{q^{\chi_{i,j}^{(1)} - \xi_{i,j}^{(1)} + j - 1}}{\prod_{n \in \mathbb{Z}_{>0}} (1 - q^n)^2} \prod_{n \in \mathbb{Z}_{>0}} (1 - q^{n-1} z)^{-1} (1 - q^n z^{-1})^{-1}$$
$$\times \sum_{w \in \widehat{W}(k\Lambda_0)} \epsilon(w) q^{-(w \circ \hat{\lambda}_{i,j} | D + x_0)} z^{-(w \circ \hat{\lambda}_{i,j} | \frac{\alpha_2^\vee}{2})}.$$

(b) *Let $k=-3+p/4$ with $(p,2)=1$, $p\geqslant 4$, and $\xi_{i,j}^{(1')}$ and $\chi_{i,j}^{(1')}$ be as in Proposition 3.2.5(b) such that $L(\xi_{i,j}^{(1')},\chi_{i,j}^{(1')})$ is a simple $\mathcal{W}_k(\mathfrak{g},f)$-module. Then*

$$\mathrm{ch}_{L(\xi_{i,j}^{(1')},\chi_{i,j}^{(1')})}(q,z)=\frac{q^{\chi_{i,j}^{(1')}-\xi_{i,j}^{(1')}+j-1}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2}\prod_{n\in\mathbb{Z}_{>0}}(1-q^{n-1}z)^{-1}(1-q^nz^{-1})^{-1}$$
$$\times\sum_{w\in\widehat{W}(k\Lambda_0)}\epsilon(w)q^{-(w\circ\hat{\lambda}_{i,j}|D+x_0)}z^{-(w\circ\hat{\lambda}_{i,j}|\frac{\alpha_2^\vee}{2})}.$$

Proof. It is an immediate consequence of [76, Theorem 6.2] that for $\lambda\in P_{0,+}$, $H_f^i(\widehat{L}_k(\lambda))=0$ for all $i\neq 0$. Hence, $\mathrm{ch}_{H_f^\bullet(\widehat{L}_k(\lambda_{i,j}^{(s)}))}=\mathrm{ch}_{H_f^0(\widehat{L}_k(\lambda_{i,j}^{(s)}))}$. According to the isomorphism (3.6) and Theorem 1.3.4, we get that for $s=1,1'$,

$$\begin{aligned}
\mathrm{ch}_{L(\xi_{i,j}^{(s)},\chi_{i,j}^{(s)})}(q,z)&=\mathrm{ch}_{H_f^\bullet(\widehat{L}_k(\lambda_{i,j}^{(s)}))}(q,-\nu\frac{\alpha_2^\vee}{2})\\
&=\frac{q^{\frac{(\hat{\lambda}_{i,j}|\hat{\lambda}_{i,j}+2\widehat{\rho})}{2(k+h^\vee)}}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2}\prod_{n\in\mathbb{Z}_{>0}}(1-q^{n-1}z)^{-1}(1-q^nz^{-1})^{-1}\widehat{R}\,\mathrm{ch}_{\widehat{L}_k(\lambda_{i,j}^{(s)})}(q,z)\\
&=\frac{q^{\chi_{i,j}^{(s)}+(x_0|\hat{\lambda}_{i,j})}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2}\prod_{n\in\mathbb{Z}_{>0}}(1-q^{n-1}z)^{-1}(1-q^nz^{-1})^{-1}\\
&\quad\times\sum_{w\in\widehat{W}(\hat{\lambda}_{i,j})}\epsilon(w)e^{2\pi i(w\circ\hat{\lambda}_{i,j}|-\tau D-\tau x_0-\nu\frac{\alpha_2^\vee}{2})}\\
&=\frac{q^{\chi_{i,j}^{(s)}+(x_0|\hat{\lambda}_{i,j})}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2}\prod_{n\in\mathbb{Z}_{>0}}(1-q^{n-1}z)^{-1}(1-q^nz^{-1})^{-1}\\
&\quad\times\sum_{w\in\widehat{W}(k\Lambda_0)}\epsilon(w)q^{-(w\circ\hat{\lambda}_{i,j}|D+x_0)}z^{-(w\circ\hat{\lambda}_{i,j}|\frac{\alpha_2^\vee}{2})}.
\end{aligned}$$
□

Using the ψ-action, it is possible to get a similar expression for the other simple $\mathcal{W}_k(\mathfrak{g},f)$-modules. Indeed, since ψ is an isomorphism of vector spaces, it sends an eigenspace $(L(\xi_{i,j}^{(s)},\chi_{i,j}^{(s)}))_{d,a}$ to an eigenspace $(\psi(L(\xi_{i,j}^{(s)},\chi_{i,j}^{(s)})))_{d',a'}$. Because the minimal eigenvalues for $\psi(L_0)$ and $\psi(J_0)$ are uniquely determined by $\chi_{i,j}^{(s)}$ and $\xi_{i,j}^{(s)}$, we denote without ambiguity

$$\psi(\chi_{i,j}^{(s)})=\chi_{i,j}^{(s)}-\xi_{i,j}^{(s)}-(i-1)+\frac{(2+k)}{2},$$
$$\psi(\xi_{i,j}^{(s)})=\xi_{i,j}^{(s)}+(i-1)-(2+k).$$

Then $L(\psi(\xi_{i,j}^{(s)}),\psi(\chi_{i,j}^{(s)}))=\psi(L(\xi_{i,j}^{(s)},\chi_{i,j}^{(s)}))$.

Lemma 3.5.2. *Let $\xi_{i,j}^{(s)}$ and $\chi_{i,j}^{(s)}$ be such that $L(\xi_{i,j}^{(s)},\chi_{i,j}^{(s)})$ is an irreducible $\mathcal{W}_k(\mathfrak{g},f)$-module (see Proposition 3.2.5). For $n\in\mathbb{Z}_{\geqslant 0}$ and $a\in\mathbb{Z}$,*

$$\dim(L(\psi(\xi_{i,j}^{(s)}),\psi(\chi_{i,j}^{(s)})))_{\psi(\chi_{i,j}^{(s)})+n,\psi(\xi_{i,j}^{(s)})+a}=\dim(L(\xi_{i,j}^{(s)},\chi_{i,j}^{(s)}))_{\chi_{i,j}^{(s)}+n+a,\xi_{i,j}^{(s)}+a+i-1}.$$

Proof. Let $m \in (L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}))_{\chi_{i,j}^{(s)}+n, \xi_{i,j}^{(s)}+a}$. Then

$$\psi(L_0)m = (\chi_{i,j}^{(s)} - \xi_{i,j}^{(s)} + n - a + \frac{2+k}{2})m = (\psi(\chi_{i,j}^{(s)}) + n - a + (i-1))m,$$
$$\psi(J_0)m = (\xi_{i,j}^{(s)} + a - (2+k))m = (\psi(\xi_{i,j}^{(s)}) + a - (i-1))m.$$

Hence, $(L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}))_{\chi_{i,j}^{(s)}+n, \xi_{i,j}^{(s)}+a} \simeq (L(\psi(\xi_{i,j}^{(s)}), \psi(\chi_{i,j}^{(s)}))_{\psi(\chi_{i,j}^{(s)})+n-a+(i-1), \psi(\xi_{i,j}^{(s)})+a-(i-1)}$. □

Remark 3.5.3. By induction we get that for all $n \in \mathbb{Z}_{\geqslant 0}, a \in \mathbb{Z}$, if $k = -3 + p/3$, $(p,3) = 1$, $p \geqslant 3$, then

$$\begin{aligned}(L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}))_{\chi_{i,j}^{(s)}+n, \xi_{i,j}^{(s)}+a} &\simeq (\psi^6(L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})))_{\chi_{i,j}^{(s)}+n, \xi_{i,j}^{(s)}+a} \\ &\simeq (L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})))_{\chi_{i,j}^{(s)}+6\xi_{i,j}^{(s)}+n+6a+18(2+k), \xi_{i,j}^{(s)}+a+6(2+k)},\end{aligned}$$

and if $k = -3 + p/4$, $(p,2) = 1$, $p \geqslant 4$, then

$$\begin{aligned}(L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}))_{\chi_{i,j}^{(s)}+n, \xi_{i,j}^{(s)}+a} &\simeq (\psi^4(L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})))_{\chi_{i,j}^{(s)}+n, \xi_{i,j}^{(s)}+a} \\ &\simeq (L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})))_{\chi_{i,j}^{(s)}+4\xi_{i,j}^{(s)}+n+4a+8(2+k), \xi_{i,j}^{(s)}+a+4(2+k)}.\end{aligned}$$

Proposition 3.5.4. *Let $\xi_{i,j}^{(s)}$ and $\chi_{i,j}^{(s)}$ be such that $L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})$ is a simple $\mathcal{W}_k(\mathfrak{g}, f)$-module (see Proposition 3.2.5). Then,*

$$\mathrm{ch}_{\psi(L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}))}(q,z) = q^{\frac{2+k}{2}} z^{-(2+k)} \mathrm{ch}_{L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})}(q, q^{-1}z).$$

Proof. Using Lemma 3.5.2 we get

$$\begin{aligned}\mathrm{ch}_{\psi(L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}))}(q,z) &= \sum_{\substack{n \in \mathbb{Z}_{\geqslant 0} \\ a \in \mathbb{Z}}} \dim(\psi(L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})))_{\psi(\chi_{i,j}^{(s)})+n, \psi(\xi_{i,j}^{(s)})+a} q^{\psi(\chi_{i,j}^{(s)})+n} z^{\psi(\xi_{i,j}^{(s)})+a} \\ &= \sum_{\substack{n \in \mathbb{Z}_{\geqslant 0} \\ a \in \mathbb{Z}}} \dim(L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}))_{\chi_{i,j}^{(s)}+n+a, \xi_{i,j}^{(s)}+a+i-1} q^{\chi_{i,j}^{(s)} - \xi_{i,j}^{(s)} - (i-1) + \frac{2+k}{2} + n} z^{\xi_{i,j}^{(s)} + (i-1) - (2+k) + a} \\ &= q^{\frac{2+k}{2}} z^{-(2+k)} \mathrm{ch}_{L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})}(q, q^{-1}z).\end{aligned}$$

□

Corollary 3.5.5. (a) *Let $k = -3 + p/3$ with $(p,3) = 1$, $p \geqslant 3$, $s = 2, 3$, and $\xi_{i,j}^{(s)}$ and $\chi_{i,j}^{(s)}$ be as in Proposition 3.2.5(a) such that $L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})$ is a simple $\mathcal{W}_k(\mathfrak{g}, f)$-module. Then*

$$\begin{aligned}\mathrm{ch}_{L(\xi_{i,j}^{(2)}, \chi_{i,j}^{(2)})}(q,z) &= \frac{q^{\chi_{i,j}^{(2)} + 2p - 2i - \frac{j+1}{2}} z^{-\frac{2p}{3}}}{\prod_{n \in \mathbb{Z}_{>0}} (1-q^n)^2} \prod_{n \in \mathbb{Z}_{>0}} (1 - q^{n-1}z)^{-1} (1 - q^n z^{-1})^{-1} \\ &\quad \times \sum_{w \in \widehat{W}(k\Lambda_0)} \epsilon(w) q^{-(w \circ \hat{\lambda}_{j,p-i-j} | D + x_0 - \alpha_2^\vee)} z^{-(w \circ \hat{\lambda}_{j,p-i-j} | \frac{\alpha_2^\vee}{2})} \\ &= \frac{q^{\chi_{i,j}^{(2)} + 2p - 2i - \frac{j}{2}} z^{-\frac{2p}{3} + \frac{1}{2}}}{\prod_{n \in \mathbb{Z}_{>0}} (1-q^n)^2} \prod_{n \in \mathbb{Z}_{>0}} (1 - q^{n-1}z)^{-1} (1 - q^n z^{-1})^{-1}\end{aligned}$$

$$\times \sum_{\substack{w\in W\\ \eta\in Q^\vee}} \epsilon(wt_{3\eta}) q^{-(w(\lambda_{j,p-i-j}+\rho+p\eta)|x_0-\alpha_2^\vee)+3(\eta|\lambda_{j,p-i-j}+\rho)+\frac{3}{2}|\eta|^2 p} z^{-(w(\lambda_{j,p-i-j}+\rho+p\eta)|\frac{\alpha_2^\vee}{2})},$$

and

$$\begin{aligned}\mathrm{ch}_{L(\xi^{(3)}_{i,j},\chi^{(3)}_{i,j})}(q,z) &= -\frac{q^{\chi^{(3)}_{i,j}+p-j-1} z^{-\frac{p}{3}}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2} \prod_{n\in\mathbb{Z}_{>0}} (1-q^{n-1}z)^{-1}(1-q^n z^{-1})^{-1}\\ &\quad \times \sum_{w\in \widehat{W}(k\Lambda_0)} \epsilon(w) q^{-(w\circ\hat{\lambda}_{p-i-j,i}|D+x_0-\frac{\alpha_2^\vee}{2})} z^{-(w\circ\hat{\lambda}_{p-i-j,i}|\frac{\alpha_2^\vee}{2})}\\ &= -\frac{q^{\chi^{(3)}_{i,j}+p-j} z^{-\frac{p}{3}+\frac{1}{2}}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2} \prod_{n\in\mathbb{Z}_{>0}} (1-q^{n-1}z)^{-1}(1-q^n z^{-1})^{-1}\\ &\quad \times \sum_{\substack{w\in W\\ \eta\in Q^\vee}} \epsilon(wt_{3\eta}) q^{-(w(\lambda_{p-i-j,i}+\rho+p\eta)|x_0-\frac{\alpha_2^\vee}{2})+3(\eta|\lambda_{p-i-j,i}+\rho)+\frac{3}{2}|\eta|^2 p} z^{-(w(\lambda_{p-i-j,i}+\rho+p\eta)|\frac{\alpha_2^\vee}{2})}.\end{aligned}$$

(b) *Let $k=-3+p/4$ with $(p,2)=1$, $p\geqslant 4$, and $\xi^{(2')}_{i,j}$ and $\chi^{(2')}_{i,j}$ be as in Proposition 3.2.5(b) such that $L(\xi^{(2')}_{i,j},\chi^{(2')}_{i,j})$ is a simple $\mathcal{W}_k(\mathfrak{g},f)$-module. Then*

$$\begin{aligned}\mathrm{ch}_{L(\xi^{(2')}_{i,j},\chi^{(2')}_{i,j})}(q,z) &= -\frac{q^{\chi^{(2')}_{i,j}+i+j-2} z^{\frac{p}{4}}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2} \prod_{n\in\mathbb{Z}_{>0}} (1-q^{n-1}z)^{-1}(1-q^n z^{-1})^{-1}\\ &\quad \times \sum_{w\in \widehat{W}(k\Lambda_0)} \epsilon(w) q^{-(w\circ\hat{\lambda}_{j,i}|D+x_0+\frac{\alpha_2^\vee}{2})} z^{-(w\circ\hat{\lambda}_{j,i}|\frac{\alpha_2^\vee}{2})}\\ &= -\frac{q^{\chi^{(2')}_{i,j}+i+j} z^{\frac{p}{4}+\frac{1}{2}}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2} \prod_{n\in\mathbb{Z}_{>0}} (1-q^{n-1}z)^{-1}(1-q^n z^{-1})^{-1}\\ &\quad \times \sum_{\substack{w\in W\\ \eta\in Q^\vee}} \epsilon(wt_{4\eta}) q^{-(w(\lambda_{j,i}+\rho+p\eta)|x_0+\frac{\alpha_2^\vee}{2})+4(\eta|\lambda_{j,i}+\rho)+2|\eta|^2 p} z^{-(w(\lambda_{j,i}+\rho+p\eta)|\frac{\alpha_2^\vee}{2})}.\end{aligned}$$

Proof. If k is principal admissible then the irreducible module $L(\xi^{(3)}_{i,j},\chi^{(3)}_{i,j})$ is isomorphic to $\psi(L(\xi^{(1)}_{p-i-j,i},\chi^{(1)}_{p-i-j,i}))$ whereas $L(\xi^{(2)}_{i,j},\chi^{(2)}_{i,j})$ is isomorphic to $\psi^2(L(\xi^{(1)}_{j,p-i-j},\chi^{(1)}_{j,p-i-j}))$. Hence,

$$\mathrm{ch}_{L(\xi^{(3)}_{i,j},\chi^{(3)}_{i,j})}(q,z) = q^{\frac{2+k}{2}} z^{-(2+k)}\, \mathrm{ch}_{L(\xi^{(1)}_{p-i-j,i},\chi^{(1)}_{p-i-j,i})}(q,q^{-1}z),$$

and

$$\begin{aligned}\mathrm{ch}_{L(\xi^{(2)}_{i,j},\chi^{(2)}_{i,j})}(q,z) &= q^{\frac{2+k}{2}} z^{-(2+k)}\, \mathrm{ch}_{L(\xi^{(3)}_{p-i-j,i},\chi^{(3)}_{p-i-j,i})}(q,q^{-1}z)\\ &= q^{2(2+k)} z^{-2(2+k)}\, \mathrm{ch}_{L(\xi^{(1)}_{j,p-i-j},\chi^{(1)}_{j,p-i-j})}(q,q^{-2}z).\end{aligned}$$

If k is coprincipal admissible then the irreducible module $L(\xi^{(1')}_{j,i},\chi^{(1')}_{j,i})$ is isomorphic to $\psi(L(\xi^{(2')}_{i,j},\chi^{(2')}_{i,j}))$.

Hence,

$$\mathrm{ch}_{L(\xi^{(2')}_{i,j},\chi^{(2')}_{i,j})}(q,z) = q^{\frac{2+k}{2}} z^{(2+k)}\, \mathrm{ch}_{L(\xi^{(1')}_{j,i},\chi^{(1')}_{j,i})}(q,qz).$$

Moreover, if $A(q,z)$ denotes the product

$$\frac{\prod_{n\in\mathbb{Z}_{>0}}(1-q^{n-1}z)^{-1}(1-q^n z^{-1})^{-1}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2},$$

then $A(q,q^{-1}z) = -qz^{-1}A(q,z)$, $A(q,q^{-2}z) = q^3z^{-2}A(q,z)$ and $A(q,qz) = -zA(q,z)$. This remark allows to complete the computation. □

We have already showed that the simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules with the form $L(\xi^{(1)}_{i,j},\chi^{(1)}_{i,j})$ (if k is principal admissible) and $L(\xi^{(1')}_{i,j},\chi^{(1')}_{i,j})$ (if k is coprincipal admissible) are isomorphic to the zero-th homology of the $H^{\bullet}_f$-reduction of certain highest weight $\widehat{\mathfrak{g}}$-modules. We expect that this result holds for all simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules. More precisely, we expect that for any simple $\mathcal{W}_k(\mathfrak{g}, f)$-module $L(\xi^{(s)}_{i,j},\chi^{(s)}_{i,j})$, it exists an admissible weight $\lambda^{(s)}_{i,j}\in\widehat{\mathfrak{h}}^*$ such that

$$H^l_f(\widehat{L}_k(\lambda^{(s)}_{i,j})) = \begin{cases} L(\xi^{(s)}_{i,j},\chi^{(s)}_{i,j}) & \text{if } l=0,\\ 0 & \text{otherwise.}\end{cases}$$

If such a weight exists then it satisfies the following equation:

$$\begin{cases} \xi^{(s)}_{i,j} = \Lambda^{(s)}_{i,j}(-\frac{\alpha_2^\vee}{2}) \mod \mathbb{Z},\\ \chi^{(s)}_{i,j} = \dfrac{(\Lambda^{(s)}_{i,j}|\Lambda^{(s)}_{i,j}+2\rho)}{2(k+h^\vee)} - \Lambda^{(s)}_{i,j}(x_0) \mod \mathbb{Z}.\end{cases} \tag{3.7}$$

Solving this system for all pairs $(\xi^{(s)}_{i,j},\chi^{(s)}_{i,j})$, we get candidates for the weights $\lambda^{(s)}_{i,j}\in\mathbb{Q}\varpi_1+\mathbb{Q}\varpi_2$ which answer to the conjecture. For all $1\leqslant i\leqslant p-2$ and $1\leqslant j\leqslant p-i-1$,

$$\lambda^{(2)}_{i,j}\in\{\Lambda^{(2)+}_{i,j},\Lambda^{(2)-}_{i,j}\},\text{ where } \Lambda^{(2)\pm}_{i,j} = \frac{-6-6i-3j+4p\pm(3j-2p)}{6}\varpi_1 + (2i+j-2p/3-1)\varpi_2,$$

$$\lambda^{(3)}_{i,j}\in\{\Lambda^{(3)+}_{i,j},\Lambda^{(3)-}_{i,j}\},\text{ where } \Lambda^{(3)\pm}_{i,j} = \frac{-6-3i-3j+3p\pm(3j-3i-p)}{6}\varpi_1 + (i+j-p/3-1)\varpi_2,$$

and for $1\leqslant i\leqslant p-3$ and $1\leqslant j\leqslant p-2i-1$,

$$\lambda^{(2')}_{i,j}\in\{\Lambda^{(2')+}_{i,j},\Lambda^{(2')-}_{i,j}\},\quad\text{where } \Lambda^{(2')\pm}_{i,j} = \frac{-4-4i-2j+2p\pm(2j-p)}{4}\varpi_1 + (2i+j-p/2-1)\varpi_2$$

Among all of them we keep only the admissible ones. As a consequence, we set

$$\begin{aligned}\lambda^{(2)}_{i,j} &= \Lambda^{(2)+}_{i,j} = (-i+p/3-1)\varpi_1 + (2i+j-2p/3-1)\varpi_2\\ \lambda^{(3)}_{i,j} &= \Lambda^{(3)+}_{i,j} = (-i+p/3-1)\varpi_1 + (i+j-p/3-1)\varpi_2,\\ \text{and}\quad \lambda^{(2')}_{i,j} &= \Lambda^{(2')+}_{i,j} = (-i+p/4-1)\varpi_1 + (2i+j-p/2-1)\varpi_2.\end{aligned}$$

Proposition 3.5.6. (a) *Let $k=-3+p/3$ with $(p,3)=1$, $p\geqslant 3$, $s=2,3$, and $\xi^{(s)}_{i,j}$ and $\chi^{(s)}_{i,j}$*

be as in Proposition 3.2.5(a) so that $L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})$ *is a simple* $\mathcal{W}_k(\mathfrak{g}, f)$*-module. Then*

$$\mathrm{ch}_{H_f^{\bullet}(\widehat{L}_k(\lambda_{i,j}^{(2)}))}(q,z) = \mathrm{ch}_{L(\xi_{i,j}^{(2)},\chi_{i,j}^{(2)})}(q,z), \tag{3.8}$$

and

$$\mathrm{ch}_{H_f^{\bullet}(\widehat{L}_k(\lambda_{i,j}^{(3)}))}(q,z) = \mathrm{ch}_{L(\xi_{i,j}^{(3)},\chi_{i,j}^{(3)})}(q,z). \tag{3.9}$$

(b) *Let* $k = -3 + p/4$ *with* $(p,2) = 1$, $p \geqslant 4$, *and* $\xi_{i,j}^{(2')}$ *and* $\chi_{i,j}^{(2')}$ *be as in Proposition 3.2.5(b) such that* $L(\xi_{i,j}^{(2')}, \chi_{i,j}^{(2')})$ *is a simple* $\mathcal{W}_k(\mathfrak{g}, f)$*-module. Then*

$$\mathrm{ch}_{H_f^{\bullet}(\widehat{L}_k(\lambda_{i,j}^{(2')}))}(q,z) = \mathrm{ch}_{L(\xi_{i,j}^{(2')},\chi_{i,j}^{(2')})}(q,z). \tag{3.10}$$

Proof. We only detail the proof of (3.8). By Theorem 1.3.4,

$$\mathrm{ch}_{H_f^{\bullet}(\widehat{L}_k(\lambda_{i,j}^{(2)}))}(q,z) = \frac{q^{\chi_{i,j}^{(2)}+\frac{j-3}{2}}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2} \prod_{n\in\mathbb{Z}_{>0}} (1-q^{n-1}z)^{-1}(1-q^n z^{-1})^{-1}$$
$$\times \sum_{w\in\widehat{W}(\hat{\lambda}_{i,j}^{(2)})} \epsilon(w) q^{-(w(\hat{\lambda}_{i,j}^{(2)}+\widehat{\rho})-\widehat{\rho}|D+x_0)} z^{-(w(\hat{\lambda}_{i,j}^{(2)}+\widehat{\rho})-\widehat{\rho}|\frac{\alpha_2^\vee}{2})}.$$

Moreover, $\widehat{\Delta}(\hat{\lambda}_{i,j}^{(2)}) = y\left(\widehat{\Delta}(k\Lambda_0)\right)$ where $y = -r_{\alpha_1}r_{\alpha_2}t_{-\varpi_1^\vee} \in \widehat{W}$ and $\widehat{W}(\hat{\lambda}_{i,j}^{(2)}) = \{ywy^{-1} \mid w \in \widehat{W}(k\Lambda_0)\}$. Hence,

$$\mathrm{ch}_{H_f^{\bullet}(\widehat{L}_k(\lambda_{i,j}^{(2)}))}(q,z) = \frac{q^{\chi_{i,j}^{(2)}+\frac{j-3}{2}}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2} \prod_{n\in\mathbb{Z}_{>0}} (1-q^{n-1}z)^{-1}(1-q^n z^{-1})^{-1}$$
$$\times \sum_{w\in\widehat{W}(k\Lambda_0)} \epsilon(w) q^{-(y_2wy_2^{-1}(\hat{\lambda}_{i,j}^{(2)}+\widehat{\rho})-\widehat{\rho}|D+x_0)} z^{-(y_2wy_2^{-1}(\hat{\lambda}_{i,j}^{(2)}+\widehat{\rho})-\widehat{\rho}|\frac{\alpha_2^\vee}{2})}$$
$$= \frac{q^{\chi_{i,j}^{(2)}+2p-2i-\frac{j}{2}} z^{-\frac{2p}{3}+\frac{1}{2}}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2} \prod_{n\in\mathbb{Z}_{>0}} (1-q^{n-1}z)^{-1}(1-q^n z^{-1})^{-1}$$
$$\times \sum_{\substack{w\in W\\ \eta\in Q^\vee}} \epsilon(wt_{3\eta}) q^{((r_{\alpha_1}r_{\alpha_2}-2\,\mathrm{id})w(-(\lambda_{j,p-i-j}+\rho)+p\eta)|\varpi_1)-3(\eta|\lambda_{j,p-i-j}+\rho)+\frac{3}{2}|\eta|^2 p}$$
$$\times z^{(r_{\alpha_1}r_{\alpha_2}w(-(\lambda_{j,p-i-j}+\rho)+p\eta)||\frac{\alpha_2^\vee}{2})}$$
$$= \frac{q^{\chi_{i,j}^{(2)}+2p-2i-\frac{j}{2}} z^{-\frac{2p}{3}+\frac{1}{2}}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2} \prod_{n\in\mathbb{Z}_{>0}} (1-q^{n-1}z)^{-1}(1-q^n z^{-1})^{-1}$$
$$\times \sum_{\substack{w\in W\\ \eta\in Q^\vee}} \epsilon(wt_{-3\eta}) q^{-((\mathrm{id}+2r_{\alpha_1}r_{\alpha_2})w(\lambda_{j,p-i-j}+\rho+p\eta)|\varpi_1)+3(\eta|\lambda_{j,p-i-j}+\rho)+\frac{3}{2}|\eta|^2 p}$$
$$\times z^{-(w(\lambda_{j,p-i-j}+\rho+p\eta)||\frac{\alpha_2^\vee}{2})}.$$

In addition, for all $w \in W$, $(-r_{\alpha_1}r_{\alpha_2}w(\lambda_{j,p-i-j}+\rho+p\eta)|\varpi_1) = (w(\lambda_{j,p-i-j}+\rho+p\eta)|\frac{\alpha_2^\vee}{2})$. As a consequence,

$$\begin{aligned}\mathrm{ch}_{H_f^\bullet(\widehat{L}_k(\lambda_{i,j}^{(2)}))}(q,z) &= \frac{q^{\chi_{i,j}^{(2)}+2p-2i+\frac{j}{2}} z^{-\frac{2p}{3}+\frac{1}{2}}}{\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^2} \prod_{n\in\mathbb{Z}_{>0}}(1-q^{n-1}z)^{-1}(1-q^n z^{-1})^{-1} \\ &\times \sum_{\substack{w\in W\\ \eta\in Q^\vee}} \epsilon(wt_{3\eta}) q^{-(w(\lambda_{j,p-i-j}+\rho+p\eta)|x_0-\alpha_2^\vee)+3(\eta|\lambda_{j,p-i-j}+\rho)+\frac{3|\eta|^2}{2}p} z^{-(w(\lambda_{j,p-i-j}+\rho+p\eta)|\frac{\alpha_2^\vee}{2})} \\ &= \mathrm{ch}_{L(\xi_{i,j}^{(2)},\chi_{i,j}^{(2)})}(q,z).\end{aligned}$$

Formulas (3.9) and (3.10) are obtained with similar computations using that

$$\begin{aligned} &\widehat{\Delta}(\hat{\lambda}_{i,j}^{(3)}) = y\left(\widehat{\Delta}(k\Lambda_0)\right), \qquad \text{with } y = -t_{-(\varpi_1^\vee+\varpi_2^\vee)/2}, \\ \text{and} \quad &\widehat{\Delta}(\hat{\lambda}_{i,j}^{(2')}) = y\left(\widehat{\Delta}(k\Lambda_0)\right), \qquad \text{with } y = r_{\alpha_1} t_{-\varpi_1^\vee/2}. \end{aligned}$$

□

We are in a position to formulate a conjecture.

Conjecture 3.5.7. *Let k be an admissible level. Then all the simple $\mathcal{W}_k(\mathfrak{g}, f)$-modules $L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)})$ as in Proposition 3.2.5 are obtained from the Drinfeld-Sokolov reduction of a highest weight $\widehat{\mathfrak{g}}$-modules. More precisely,*

(a) *If $k = -3 + p/3$ with $(p, 3) = 1$, $p \geqslant 3$, and $\xi_{i,j}^{(s)}$ and $\chi_{i,j}^{(s)}$ be as in Proposition 3.2.5(a), then*

$$H_f^l(\widehat{L}_k(\lambda_{i,j}^{(s)})) = \begin{cases} L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}) & \text{if } l = 0, \\ 0 & \text{else.} \end{cases}$$

(b) *If $k = -3 + p/4$ with $(p, 2) = 1$, $p \geqslant 4$, and $\xi_{i,j}^{(s)}$ and $\chi_{i,j}^{(s)}$ be as in Proposition 3.2.5(b), then*

$$H_f^l(\widehat{L}_k(\lambda_{i,j}^{(s)})) = \begin{cases} L(\xi_{i,j}^{(s)}, \chi_{i,j}^{(s)}) & \text{if } l = 0, \\ 0 & \text{else.} \end{cases}$$

Chapter 4

On the simple modules of exceptional $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$

In the previous chapter we proved the rationality of exceptional $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$, an analogous of the Bershadsky-Polyakov vertex algebra $\mathcal{W}_k(\mathfrak{sl}_3, f_{\min})$ in type C. Another analogous is the $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$. The rationality of these exceptional $\mathcal{W}$-algebras has been proved recently by Creutzig-Linshaw [43] and also appears as a particular case of the conjecture of Kac-Wakimoto and Arakawa proved in all generality by McRae [86]. However, both conceptual proves do not provide the set of simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$-modules. In this chapter, we give a explicit realization of these modules as highest weight representations.

Since $f := f_{\min}$ is not an even nilpotent element, the usual conformal vector induces a $\frac{1}{2}\mathbb{Z}$-grading on $\mathcal{W}^k(\mathfrak{sp}_4, f_{\min})$. Twisting it, we recover a conformal even grading of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\min})$ (Sect. 4.1). At exceptional levels, we deduce that any positive energy representation is a highest weight module depending on two complex parameters. However, the structure of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\min})$ is more complicated to study than the one of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$, so we need to change our approach to the problem. Let say a few words on the strategy implemented in this chapter. Since $\mathcal{W}_k(\mathfrak{g}, f)$ is a quotient of $H^0_f(L_k(\mathfrak{g}))$, simple $\mathcal{W}_k(\mathfrak{g}, f)$-modules are also simple $H^0_f(L_k(\mathfrak{g}))$-modules. In Sect. 4.2, we describe the set of simple $H^0_{f_{\min}}(L_k(\mathfrak{sp}_4))$-modules using Zhu's correspondence and the classification provided in [21]. We get an exhaustive list of candidates for the irreducible positive energy representations of $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$. Moreover, they correspond to certain highest weight representations of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\min})$. The realization of the simple $H^0_{f_{\min}}(L_k(\mathfrak{sp}_4))$-modules is the key point to prove the simplicity of this vertex algebra (Theorem 4.3.2). The simplicity of $H^0_f(L_k(\mathfrak{g}))$ when it is non zero is a long-standing conjecture of Kac-Roan-Wakimoto [73, 78]. It has been verified in many cases [12, 14]. The classification of simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$-modules follows.

Theorem 4.1. *Let $f = f_{\min}$ be a minimal nilpotent element of $\mathfrak{g} = \mathfrak{sp}_4$. Then, for admissible level $k = -3 + p/2$, $(p, 2) = 1$, $p \geqslant 4$, $H^0_f(L_k(\mathfrak{g})) \simeq \mathcal{W}_k(\mathfrak{g}, f)$. Moreover, Proposition 4.2.3 provides a complete classification of their simple modules.*

The realization of simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$-modules as highest weight representation is similar to the one presented in Chap. 3. Zhu's correspondence emphasizes their correspondence with to Ramond twisted representations. Furthermore, since the functor $H^0_{f_{\min}}(?)$ is exact (see Sect. 1.3.2), any simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$-module corresponds to the reduction of a certain highest weight representation of $\widehat{\mathfrak{g}}$. We conjecture (Conjecture 4.3.4), that each Ramond twisted representations corresponds to the reduction of a certain highest weight representation of $\widehat{\mathfrak{g}}$.

4.1 Twist conformal vector and even grading

In this chapter, fix $f := f_{\min} = e_{-\theta}$ that we embed in an $\mathfrak{sl}_2$-triple (e, h, f) where $h = \theta$. The semisimple element $x_0 = \theta/2$ induces odd gradings on $\mathfrak{g}$ and $\mathfrak{g}^f$:

$$\mathfrak{g} = \mathbb{C}f \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathbb{C}e,$$

and

$$\mathfrak{g}^f = \mathbb{C}f \oplus \mathfrak{g}^f_{-1/2} \oplus \mathfrak{g}^f_0.$$

As a consequence, the usual conformal vector L defines a $\frac{1}{2}\mathbb{Z}$-grading on the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$. To recover a conformal $\mathbb{Z}$-grading, we twist the field corresponding to L using the semisimplicity of J. Set

$$\tilde{L}(z) = L(z) + \frac{1}{2}\partial J(z).$$

For all $m \in \mathbb{Z}$, we have

$$\tilde{L}_m = L_m - \frac{m+1}{2} J_m.$$

This defines a conformal vector of $\mathcal{W}^k(\mathfrak{g}, f)$ which induces a $\mathbb{Z}_{\geqslant 0}$-grading. Its central charge is given by

$$\tilde{c}_k = -\frac{6(2+k)(1+2k)}{3+k}.$$

The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is $\mathbb{Z}_{\geqslant 0}$-graded with respect to $\tilde{L}_0$. Since conformal weights of J, F^+, F^-, G^+ and G^- are respectively 1, 0, 2, 1, and 2, we have:

$$J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}, \quad F^+(z) = \sum_{n\in\mathbb{Z}} F^+_n z^{-n}, \quad F^-(z) = \sum_{n\in\mathbb{Z}} F^-_n z^{-n-2}$$
$$G^+(z) = \sum_{n\in\mathbb{Z}} G^+_n z^{-n-1}, \quad G^-(z) = \sum_{n\in\mathbb{Z}} G^-_n z^{-n-2}, \quad \tilde{L}(z) = \sum_{n\in\mathbb{Z}} \tilde{L}_n z^{-n-2}.$$

We have the following commutation relations:

$$\begin{aligned}
[J_m, J_n] &= (1+2k)m\delta_{m+n,0},\\
[J_m, F^\pm_n] &= \pm 2F^\pm_{m+n},\\
[F^+_m, F^-_n] &= \frac{1+2k}{2}(m-1)\delta_{m+n,0} + J_{m+n},\\
[J_m, G^\pm_n] &= \pm G^\pm_{m+n},\\
[G^\pm_m, F^\mp_n] &= G^\mp_{m+n}\\
[\tilde{L}_m, \tilde{L}_n] &= \frac{\tilde{c}_k}{12}(m^3-m)\delta_{n+m,0} + (m-n)\tilde{L}_{m+n},\\
[\tilde{L}_m, F^+_n] &= -(m+n)F^+_{m+n},\\
[\tilde{L}_m, F^-_n] &= (m-n)F^-_{m+n},\\
[\tilde{L}_m, G^+_n] &= -nG^+_{m+n},
\end{aligned}$$

$$
\begin{aligned}
[\tilde{L}_m, G^-_n] &= (m-n)G^-_{m+n},\\
[\tilde{L}_m, J_n] &= -\frac{(1+2k)m(m+1)}{2}\delta_{m+n,0} - nJ_{m+n},\\
[G^\pm_m, G^\pm_n] &= \pm 2(2+k)(m-n)F^\pm_{m+n},\\
[G^+_m, G^-_n] &= (1+2k)(2+k)m(m-1)\delta_{m+n,0} + \left(2(2+k)m - (3+2k)(m+n+1)\right) J_{m+n}\\
&\quad - 2(3+k)\tilde{L}_{m+n} + 4(F^+F^-)_{m+n} + (J^2)_{m+n}.
\end{aligned}
$$

where

$$
\sum_{n\in\mathbb{Z}} (J^2)_n z^{-n-2} \overset{\text{def}}{=} :J(z)^2:, \qquad \text{and} \qquad \sum_{n\in\mathbb{Z}} (F^+F^-)_n z^{-n-2} \overset{\text{def}}{=} :F^+(z)F^-(z):.
$$

Similarly to the case where f is a subregular nilpotent element of $\mathfrak{sp}_4$ (see Chap. 3), we can describe irreducible positive energy representations with finite dimensional top component as highest weight modules depending on two parameters.

Lemma 4.1.1. *Let M be an irreducible positive energy representation of $\mathcal{W}^k(\mathfrak{g}, f)$ with respect to the conformal vector $\tilde{L}_0$, with $M_{top} = M_\chi$, $\chi \in \mathbb{C}$. Suppose that M_{top} is finite dimensional. Then there exist $\xi \in \mathbb{C}$ and a vector $v \in M$ such that $\tilde{L}_0 v = \chi v$, $J_0 v = \xi v$ and the below relations hold:*

$$
\begin{aligned}
J_n v &= 0 \quad \text{for} \quad n > 0,\\
\tilde{L}_n v &= 0 \quad \text{for} \quad n > 0,\\
F^+_n v &= 0 \quad \text{for} \quad n > 0,\\
F^-_n v &= 0 \quad \text{for} \quad n \geqslant 0,\\
G^+_n v &= 0 \quad \text{for} \quad n > 0,\\
G^-_n v &= 0 \quad \text{for} \quad n \geqslant 0.
\end{aligned}
$$

Moreover, $M = \bigoplus_{\substack{a\in\xi+\mathbb{Z}\\ d\in\chi+\mathbb{Z}_{\geqslant 0}}} M_{a,d}$, *where* $M_{a,d} = \{m \in M \mid J_0 m = am,\ \tilde{L}_0 m = dm\}$, $\dim M_{\xi,\chi} = 1$ *and $M_{top} = M_\chi$ is spanned by the vectors $(F^+_0)^m (G^+_0)^n v$, $m, n \geqslant 0$.*

Proof. The proof is identical to the demonstration of Lemma 3.1.1. □

In the following, $L(\xi,\chi)$ denotes the irreducible representation of $\mathcal{W}^k(\mathfrak{g}, f)$ generated by the vector $|\xi,\chi\rangle$ following the previous construction. Since $L(\xi,\chi)_{\text{top}}$ is spanned by vectors $(F^+_0)^m (G^+_0)^n |\xi,\chi\rangle$, $m,n \in \mathbb{Z}_{\geqslant 0}$, for any non negative integer N, the subspace $L(\xi,\chi)_{\chi,\xi+N}$ is spanned by

$$
\begin{cases}
(F^+_0)^{N'-\ell}(G^+_0)^{2\ell}|\xi,\chi\rangle,\ (\ell = 0,\dots,N') & \text{if } N = 2N',\\
(F^+_0)^{N'-\ell}(G^+_0)^{2\ell+1}|\xi,\chi\rangle,\ (\ell = 0,\dots,N') & \text{if } N = 2N'+1.
\end{cases}
$$

We have no guaranty that the previous family form basis of the vector space $L(\xi,\chi)_{\chi,\xi+N}$. As a consequence, if we assume that $L(\xi,\chi)_{\text{top}}$ is finite dimensional, its dimension does not allow to recover a basis of the vector space. In the current state, the argument of dimension we used in the subregular case (see Sect. 3.1) fails and we need to find another approach.

4.2 Irreducible positive energy representations of $H_f^0(L_k(\mathfrak{g}))$

Let f be any nilpotent element in a simple Lie algebra $\mathfrak{g}$. Recall that irreducible positive energy representations of $H_f^0(L_k(\mathfrak{g}))$ are in one-to-one correspondence with simple module of its Zhu's algebra (see Sect. 1.1.3). It is more convenient to work in $\mathcal{A}(H_f^0(L_k(\mathfrak{g})))$. In [21], Arakawa and van Ekeren prove that, for an admissible level k, isomorphism classes of simple $\mathcal{A}(H_f^0(L_k(\mathfrak{g})))$-modules are parameterized by $[\mathrm{Pr}_\circ^k] = \mathrm{Pr}_\circ^k/\sim$, where

$$\mathrm{Pr}_\circ^k = \{\lambda \in \mathrm{Pr}^k \mid |\Delta(\lambda)| = \dim \mathcal{N} - \dim \overline{\mathbb{O}}_q\},$$

with $\Delta(\lambda) = \{\alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\}$. Precisely, we have the following statement:

Theorem 4.2.1 ([21]). *Let $k = -h^\vee + p/q$ be a admissible level such that $f \in \mathbb{O}_q$ then*

$$\mathcal{A}(H_f^0(L_k(\mathfrak{g}))) \simeq \prod_{\lambda \in [\mathrm{Pr}_\circ^k]} \left(\prod_{E \in \mathrm{Fin}_{J_\lambda}(U(\mathfrak{g},f))} E \otimes E^* \right),$$

where $\mathrm{Fin}_{J_\lambda}(U(\mathfrak{g}, f))$ denotes the set of isomorphism classes of finite dimensional simple $U(\mathfrak{g}, f)$-module E such that $\mathrm{Ann}_{U(\mathfrak{g})}(Y \otimes_{U(\mathfrak{g},f)} E) = J_\lambda$ with $U(\mathfrak{g}, f) \simeq \mathrm{End}_{U(\mathfrak{g})}(Y)^{\mathrm{op}}$ (see Sect. 1.3.2). Then a complete set of isomorphism classes of simple $\mathcal{A}(H_f^0(L_k(\mathfrak{g})))$-modules is given by the disjoint union of isomorphism classes of $\mathrm{Fin}_{J_\lambda}(U(\mathfrak{g}, f))$ for λ through $[\mathrm{Pr}_\circ^k]$.

On the other hand, when $f = f_{\min}$ is a minimal nilpotent element of $\mathfrak{g}$, for each primitive ideal $\mathcal{I}$ of the enveloping algebra $U(\mathfrak{g})$ satisfying $\mathcal{VA}(\mathcal{I}) = \overline{\mathbb{O}}_{\min}$, there exists a unique (up to isomorphism) finite-dimensional simple $U(\mathfrak{g}, f)$-module $M_{\mathcal{I}}$ such that $\mathcal{I} = \mathrm{Ann}_{U(\mathfrak{g})}(Y \otimes_{U(\mathfrak{g},f)} M_{\mathcal{I}})$ [90, Theorem 1.2]. When $\mathcal{I} = J_\lambda$, $\lambda \in \mathfrak{h}^*$, denote E_{J_λ} the corresponding $U(\mathfrak{g}, f_{\min})$-module. Then

$$\mathcal{A}(H_f^0(L_k(\mathfrak{g}))) \simeq \prod_{\lambda \in [\mathrm{Pr}_\circ^k]} E_{J_\lambda} \otimes E_{J_\lambda}^*,$$

and isomorphism classes of its simple modules are in one-to-one correspondence with $[\mathrm{Pr}_\circ^k]$. Denote $\mathbf{L}(E_{J_\lambda})$ be the corresponding simple $H_f^0(L_k(\mathfrak{g}))$-module through the Zhu's correspondence.

For the rest of this chapter, let $\mathfrak{g} = \mathfrak{sp}_4$, $f = f_{\min}$ and consider an admissible level

$$k = -3 + \frac{p}{2}, \qquad (p, 2) = 1,\ p \geqslant 4, \tag{4.1}$$

then (f, k) is an exceptional pair and the $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse [17].

Lemma 4.2.2. *Let $f = f_{\min}$ be a minimal nilpotent element of $\mathfrak{g} = \mathfrak{sp}_4$. Then, for $k = -3+p/2$, $(p, 2) = 1$, $p \geqslant 4$, a complete set of representatives of the equivalent classes $[\mathrm{Pr}_\circ^k]$ is given by*

$$[\mathrm{Pr}_\circ^k] = \left\{ (\lambda_1 - 1/2)\varpi_1 + \lambda_2\varpi_2 \mid 0 \leqslant \lambda_1, \lambda_2, \lambda_1 + \lambda_2 \leqslant \frac{p-5}{2} \right\}$$

Proof. Let $k = -3 + p/2$ with $p \geqslant 4$ and $(p, 2) = 1$. For such level,

$$\begin{aligned} \mathrm{Pr}_\circ^k &= \{\lambda \in \mathrm{Pr}^k \mid |\Delta(\lambda)| = 4\} \\ &= \{(\lambda_1 - 1/2)\varpi_1 + \lambda_2\varpi_2 \in \mathrm{Pr}^k \mid \lambda_1, \lambda_2 \in \mathbb{Z}\}. \end{aligned}$$

Using that $(\lambda_1 - 1/2)\varpi_1 + \lambda_2\varpi_2 + k\Lambda_0$ is admissible we get the following inequalities:

$$\begin{aligned} -\frac{p-1}{2} \leqslant \lambda_1 \leqslant \frac{p-3}{2}, \\ 0 \leqslant \lambda_2 \leqslant p-2, \\ -\frac{p+1}{2} \leqslant \lambda_1 + \lambda_2 \leqslant \frac{p-5}{2}, \\ -1 \leqslant 2\lambda_1 + \lambda_2 \leqslant p-3. \end{aligned}$$

In particular, $-\frac{p-1}{2} \leqslant \lambda_1 \leqslant \frac{p-5}{2}$. Moreover, for any weight $\lambda = (\lambda_1 - 1/2)\varpi_1 + \lambda_2\varpi_2 \in \mathrm{Pr}^k_\circ$, we have

$$r_1 \circ (\lambda) = \underbrace{(-(\lambda+1) - 1/2)}_{\geqslant 0}\varpi_1 + (2\lambda_1 + \lambda_2 + 1)\varpi_2.$$

As a consequence, we can consider only weights λ such that $\lambda_1 \geqslant 0$. Other conditions follow. □

Since the vertex algebra $H^0_f(L_k(\mathfrak{g}))$ is a quotient of $\mathcal{W}^k(\mathfrak{g}, f)$, we can regard simple $H^0_f(L_k(\mathfrak{g}))$-module as simple $\mathcal{W}^k(\mathfrak{g}, f)$-modules. We deduce from Lemma 4.1.1, that they correspond to highest weight representations $L(\xi, \chi)$ for certain $(\xi, \chi) \in \mathbb{C}^2$.

Proposition 4.2.3. *Let $f = f_{\min}$ be a minimal nilpotent element of $\mathfrak{g} = \mathfrak{sp}_4$ and $k = -3 + p/2$, $(p, 2) = 1$, $p \geqslant 4$. Then, for $\lambda = (\lambda_1 - 1/2)\varpi_1 + \lambda_2\varpi_2 \in [\mathrm{Pr}^k_\circ]$,*

$$\mathbf{L}(E_{J_\lambda}) \simeq L(\chi_\lambda, \xi_\lambda),$$

with

$$\xi_\lambda = \frac{p-5}{2} - (\lambda_1 + \lambda_2), \quad \text{and} \quad \chi_\lambda = \frac{4\lambda_1^2 + 2\lambda_2^2 + 4\lambda_1\lambda_2 + 8\lambda_1 + 3\lambda_2 - 5 - p(p-6)}{4p}.$$

Moreover,

$$\dim L(\xi_\lambda, \chi_\lambda)_{top} = \frac{(\lambda_2 + 1)(2\lambda_1 + \lambda_2 + 2)}{2}.$$

Proof. Let $\lambda \in [\mathrm{Pr}^k_\circ]$. Since $H^0_f(L_k(\mathfrak{g}))$ is a quotient of $\mathcal{W}^k(\mathfrak{g}, f)$, $\mathbf{L}(E_{J_\lambda})$ is a simple $\mathcal{W}^k(\mathfrak{g}, f)$-module. From Lemma 4.1.1, it exists $\chi_\lambda, \xi_\lambda \in \mathbb{C}$ such that

$$\mathbf{L}(E_{J_\lambda}) \simeq L(\xi_\lambda, \chi_\lambda).$$

We determine values of χ_λ and ξ_λ using the image $E_{J_\lambda} \simeq L(\xi_\lambda, \chi_\lambda)_{\mathrm{top}}$ in the finite $\mathcal{W}$-algebra $U(\mathfrak{g}, f)$. According to [90, Theorem 7.1], there exists a unique highest weight module $L_{U(\mathfrak{g},f)}(\lambda, c)$ such that

$$E_{J_\lambda} \simeq L_{U(\mathfrak{g},f)}(\lambda, c),$$

where the Casimir element Ω of $U(\mathfrak{g})$ acts on $L_{U(\mathfrak{g},f)}(\lambda, c)$ as $c\,\mathrm{id}$ with $c = (\lambda|\lambda + 2\rho)$. It follows from the construction of L that its image $[L]$ in the finite $\mathcal{W}$-algebra acts on E_{J_λ} as the multiplication by

$$c_L = \frac{c}{2(k + h^\vee)} - \frac{1}{2}(\lambda|\theta).$$

Moreover, $[J]$ acts semisimply on $L_{U(\mathfrak{g},f)}(\lambda, c)$ and the eigenvalues of $[J]$ lie in $\{\widehat{\lambda}(\theta^\vee - \delta) - \mathbb{Z}_{\geqslant 0}\}$. As a consequence, J_0 acts semisimply on the top component of the corresponding $H^0_f(L_k(\mathfrak{g}))$-module

$L(\xi_\lambda, \chi_\lambda)_{\mathrm{top}}$ with eigenvalues in $\{-\widehat{\lambda}(\theta^\vee - \delta) + \mathbb{Z}_{\geqslant 0}\}$. Hence,

$$\xi_\lambda = -\widehat{\lambda}(\theta^\vee - \delta).$$

Finally, $\tilde{L}_0 = L_0 - \frac{1}{2}J_0$ acts on $L(\xi_\lambda, \chi_\lambda)_{\mathrm{top}}$ as the multiplication by

$$\chi_\lambda = c_L - \frac{1}{2}\xi_\lambda.$$

In addition, by [90, Theorem 6.2], for $\lambda \in [\mathrm{Pr}^k_\circ]$, the dimension of $L(\xi_\lambda, \chi_\lambda)_{\mathrm{top}}$ is given by

$$\dim L(\xi_\lambda, \chi_\lambda)_{\mathrm{top}} = \prod_{\alpha \in \Delta^s_+} \frac{(\lambda + \rho|\alpha)}{(\lambda_0 + \rho|\alpha)},$$

where Δ^s_+ is the set of all short positive roots and $\lambda_0 = -1//2\varpi_1$. □

Since the weight $\lambda \in \mathfrak{h}^*$ uniquely determines the pair $(\chi_\lambda, \xi_\lambda)$, we denote $|\lambda\rangle := |\xi_\lambda, \chi_\lambda\rangle$.

Example 4.2.4. For $p = 5$, the only simple $H^0_f(L_k(\mathfrak{g}))$-module is $L(0, 0)$.
For $p = 7$, there are three simple $H^0_f(L_k(\mathfrak{g}))$-modules: $L(1, -3/7)$, $L(0, 0)$, and $L(0, -1/7)$.

4.3 Classification of simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$-modules

For any admissible level $k = -3 + p/2$, $(p, 2) = 1$, $p \geqslant 4$, Proposition 4.2.3 together with Lemma 4.2.2 give a realization of any simple $H^0_{f_{\min}}(L_k(\mathfrak{sp}_4))$-module $\mathbf{L}(E_{J_\lambda})$ as a highest weight representation $L(\chi_\lambda, \xi_\lambda)$. We use this realization to prove that the simplicity of the vertex algebra $H^0_{f_{\min}}(L_k(\mathfrak{sp}_4))$.

In the previous example (Example 4.2.4), χ_λ is never positive. In fact, given any admissible weight k of the form (4.1), it is a general fact that for all $\lambda \in [\mathrm{Pr}^k_\circ]$, χ_λ is a non positive rational number.

Lemma 4.3.1. *Let $f = f_{\min}$ be a minimal nilpotent element of $\mathfrak{g} = \mathfrak{sp}_4$ and $k = -3 + p/2$, $(p, 2) = 1$, $p \geqslant 4$. For all $\lambda \in [\mathrm{Pr}^k_\circ]$, $\chi_\lambda \leqslant 0$.*

Proof. Let $\lambda = (\lambda_1 - 1/2)\varpi_1 + \lambda_2\varpi_2 \in [\mathrm{Pr}^k_\circ]$, one verifies that

$$\chi_\lambda = -\frac{(p-5)(p-1)}{4p} + \frac{(\lambda_1 + \lambda_2)^2 + \lambda_1(\lambda_1 + 4) + 3\lambda_2}{2p}.$$

Moreover, λ_2 can be upper bounded using that $\lambda_1 + \lambda_2 \leqslant (p-5)/2$:

$$\begin{aligned} \chi_\lambda &\leqslant -\frac{(p-5)(p-1)}{4p} + \frac{(p-5)^2}{8} + \frac{3(p-5) + 2\lambda_1(\lambda_1+1)}{4p} \\ &\leqslant \frac{1}{4p}\left(-(p-5)(p-1) + \frac{(p-5)^2}{2} + 3(p-5) + \frac{(p-5)(p-1)}{2}\right) \\ &\leqslant 0. \end{aligned}$$

□

It follows that $\mathcal{W}_k(\mathfrak{g}, f)$ is semisimple.

Theorem 4.3.2. *Let $f = f_{\min}$ be a minimal nilpotent element of $\mathfrak{g} = \mathfrak{sp}_4$ and $k = -3 + p/2$, $(p, 2) = 1$, $p \geqslant 4$. The vertex algebra $H^0_f(L_k(\mathfrak{g}))$ is simple.*

Proof. Let N be a simple submodule of $H_f^0(L_k(\mathfrak{g}))$. According to Proposition 4.2.3, $N \simeq L(\xi_\lambda, \chi_\lambda)$ for a certain $\lambda \in [\mathrm{Pr}_\circ^k]$, whence $N = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} N_{\chi_\lambda + n}$. Moreover, $H_f^0(L_k(\mathfrak{g}))$ is $\mathbb{Z}_{\geqslant 0}$-graded by L_0

$$H_f^0(L_k(\mathfrak{g})) = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} H_f^0(L_k(\mathfrak{g}))_n.$$

Each graded component of N is contained in a graded component of $H_f^0(L_k(\mathfrak{g}))$. In particular, there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that $N_{\mathrm{top}} \subset H_f^0(L_k(\mathfrak{g}))_n$, thus $\chi_\lambda = n$. However, by Lemma 4.3.1, $\chi_\lambda \leqslant 0$. As a consequence, $\chi_\lambda = 0$ and $N_{\mathrm{top}} = H_f^0(L_k(\mathfrak{g}))_0 = \mathbb{C}|0\rangle$. Thus, $N = H_f^0(L_k(\mathfrak{g}))$ is simple. □

Since $\mathcal{W}_k(\mathfrak{g}, f)$ is a quotient of $H_f^0(L_k(\mathfrak{g}))$, both vertex algebras are isomorphic. This proves additional case of Kac-Roan-Wakimoto conjecture [73, 78]. Then we obtain a complete classification of simple $\mathcal{W}_k(\mathfrak{g}, f)$-modules.

Corollary 4.3.3. *Let $k = -3 + p/2$, $(p, 2) = 1$, $p \geqslant 4$, then*

$$\mathcal{W}_k(\mathfrak{g}, f) \simeq H_f^0(L_k(\mathfrak{g})).$$

Moreover, the complete set of irreducible positive energy representations of $\mathcal{W}_k(\mathfrak{g}, f)$ is

$$\{L(\xi_\lambda, \chi_\lambda) \mid \lambda \in [\mathrm{Pr}_\circ^k]\},$$

where ξ_λ and χ_λ have been described in Proposition 4.2.3.

From the proof of Proposition 4.2.3, note that $\chi_\lambda = h_{\lambda - \frac{p}{2}x_0} - \frac{1}{2}\xi_\lambda$, where

$$h_{\lambda - \frac{p}{2}x_0} = \frac{|\lambda+\rho|^2 - |\rho|^2}{2(k+h^\vee)} - \frac{k+h^\vee}{2}|x_0|^2 + (x_0, \rho).$$

In the case where f is an *even* nilpotent element, this correspond to the conformal dimension of the Ramond twisted module $\mathbf{L}\left(E_{J_{\lambda - \frac{p}{q}x_0}}\right)$ [21]. Moreover it has been showed that when λ satisfy certain conditions, this Ramond twisted module is isomorphic to the reduction $H_f^0(\widehat{L}_k(\lambda))$ of the highest weight $\widehat{\mathfrak{g}}$-module $\widehat{L}_k(\lambda)$. This draws us to think about a possible similar construction for odd nilpotent elements.

Since $\mathcal{W}_k(\mathfrak{g}, f)$ is a simple module over itself, there is a weight $\lambda \in [\mathrm{Pr}_\circ^k]$ such that

$$\mathcal{W}_k(\mathfrak{g}, f) \simeq L(\xi_\lambda, \chi_\lambda).$$

Let $\lambda = k\varpi_1 \in [\mathrm{Pr}_\circ^k]$. We have $\xi_\lambda = \chi_\lambda = 0$. The correspondence $|0\rangle \mapsto |\lambda\rangle$ yields the isomorphism

$$\mathcal{W}_k(\mathfrak{g}, f) \simeq L(\xi_\lambda, \chi_\lambda) \simeq \mathbf{L}(E_{J_{k\varpi_1}}).$$

Moreover, $\mathcal{W}_k(\mathfrak{g}, f) \simeq H_f^0(L_k(\mathfrak{g})) \simeq H_f^0(L(k\Lambda_0))$. The isomorphisms imposes an additional condition on the possible correspondence. More precisely, we conjecture that:

Conjecture 4.3.4. *Let $f = f_{\min}$ be a minimal nilpotent element of $\mathfrak{g} = \mathfrak{sp}_4$ and $k = -3 + p/2$, $(p, 2) = 1$, $p \geqslant 4$. For $\lambda \in [\mathrm{Pr}_\circ^k]$, $\mathbf{L}(E_{J_\lambda}) \simeq H_f^0(\widehat{L}_k(\mu_\lambda))$ for a certain μ_λ depending on λ, k and x_0 and satisfying $\mu_{k\varpi_1} = 0$.*

This conjecture is also supported by the exactness of the functor $H_{f_{\min}}^0(?) : \mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g}, f_{\min})$-Mod [12].

Chapter 5

Associated varieties of affine vertex algebras and $\mathcal{W}$-algebras associated with $\mathfrak{sl}_4$ at level $k = -5/2$

In this chapter we present interesting and quite surprising computations of associated varieties. We start by computing the associated variety of a simple affine vertex algebra $L_k(\mathfrak{sl}_4)$ at the non admissible level $k = -5/2$. We obtain that it is the adherence of some Jordan class.

Theorem 5.1. *The associated variety of $L_{-5/2}(\mathfrak{g})$ is the closure of the Jordan class of $x_0 \in \mathfrak{g}$,*

$$X_{L_{-5/2}(\mathfrak{g})} = \overline{J_G(x_0)},$$

where

$$x_0 = 3h_1 + 2h_2 + h_3 + e_{\varepsilon_2-\varepsilon_4} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Let $x \in \mathfrak{g}$ and denote by $x = x_s + x_n$ its Jordan decomposition with x_s and x_n the semisimple and nilpotent components of x respectively. The centralizer $\mathfrak{g}^{x_s}$ is a reductive Lie algebra and we consider its center $\mathfrak{z}(\mathfrak{g}^{x_s})$. Let $\mathfrak{z}(\mathfrak{g}^{x_s})^{\mathrm{reg}}$ be the set of y in $\mathfrak{z}(\mathfrak{g}^{x_s})$ such that $\mathfrak{g}^y$ has the minimal dimension. We define $J_G(x)$ the *Jordan class* of x to be the G-invariant, irreducible and locally closed subset of $\mathfrak{g}$:

$$J_G(x) := G.(\mathfrak{z}(\mathfrak{g}^{x_s})^{\mathrm{reg}} + x_n).$$

We refer to [93] for general facts about Jordan classes.

It is known that there are several families of pairs $(\mathfrak{g}, k)$ such that $X_{L_k(\mathfrak{g})}$ is the closure of some Jordan class. So, our example is not the first of this type. For instance, when k is a positive integer or an admissible level, the associated variety of $L_k(\mathfrak{g})$ is known to be the closure of some nilpotent orbit of $\mathfrak{g}$ [17, 53], that is the Jordan class of a nilpotent element of $\mathfrak{g}$. Arakawa and Moreau provided additional examples coming from the Deligne exceptional series that are also of this form [28]. Moreover, in [29], they showed that Dixmier sheet closures – i.e. closures of Jordan classes of semisimple elements – can also appear as associated varieties of simple affine vertex algebras. Recently, it has been proved [25] that the affine vertex algebra $V^k(\mathfrak{g})$ is simple if and only if the associated variety $X_{L_k(\mathfrak{g})}$ is $\mathfrak{g}^* \simeq \mathfrak{g}$. This corresponds to the regular sheet, another particular case of closure of Jordan class. More examples of this type have been computed [24].

The particularity of our result is that the associated variety $X_{L_{-5/2}(\mathfrak{sl}_4)}$ is the closure of a Jordan class of an element in $\mathfrak{sl}_4$ which is neither semisimple nor nilpotent. Sect. 5.1 is devoted to prove Theorem 5.1. Part of the computation is based on the article of Adamović, Perše and Vukorepa [7].

We would like to relate the previous computation to the description of the associated varieties of $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{sl}_4, f)$ at level $k = -5/2$. Since the subregular nilpotent orbit $\mathbb{O}_{(3,1)}$ is included in $X_{L_{-5/2}(\mathfrak{sl}_4)}$ (Proposition 5.1.4), $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{subreg}})$ is lisse, i.e. its associated variety is reduced to $\{f_{\text{subreg}}\}$. We study the case where f is an element in the nilpotent orbit of $\mathfrak{sl}_4$ corresponding to partition $(2, 2)$ in Sect. 5.2. It has been independently obtained by [6] and [45] that, for such a nilpotent element f, the $\mathcal{W}$-algebra $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)$ is isomorphic to the *coset* – or commutant – $\mathrm{Com}(M(1), S(2))$ of the Heisenberg vertex algebra $M(1)$ viewed as a subalgebra of the $\beta\gamma$-system of rank two $S(2)$. Thank to the works [6, 45] we can explicitly compute the associated variety of the $\mathcal{W}$-algebra $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)$ (Theorem 5.2(i)).

Cosets play a major role in the theory of vertex algebras. They were introduced in [63] and become an usual way to construct new vertex algebras from old ones. They appear in the Gaoitto-Rapčák triality conjectures [64] which have been recently proved by Creutzig and Linshaw [43, 44]. Let U be a vertex subalgebra of a vertex algebra V. The coset $\mathrm{Com}(U, V)$ is the commutant of U in V. It is expected that $\mathrm{Com}(U, V)$ inherits certain properties from U and V, such as rationality and C_2-cofiniteness. The case where U is a Heisenberg vertex algebra has been study by Creutzig and al. [45], but the general case remains largely unknown.

If we consider a vertex subalgebra U of a vertex algebra V, the inclusion induces a Poisson algebra morphism $\varphi : R_U \to R_V$ which is not necessary injective. It does not always make sense to define the Poisson commutant of the Zhu's C_2-algebras. Nonetheless, when φ is injective R_U can be viewed as a subalgebra of R_V. Then, we can consider the commutant $\mathrm{Com}(R_U, R_V)$ of the image of R_U in R_V and compare it to the Zhu's C_2-algebra $R_{\mathrm{Com}(U,V)}$. Only a few is known about the structure of cosets. Then it is difficult to compute the Zhu's C_2-algebra $R_{\mathrm{Com}(U,V)}$. Provided it is well-defined, it is sometimes simpler to compute the commutant of Zhu's C_2-algebras $\mathrm{Com}(R_U, R_V)$. There is an Poisson algebra morphism $R_{\mathrm{Com}(U,V)} \to R_V$ coming from the inclusion $\mathrm{Com}(U, V) \subset V$. Clearly, the image of $R_{\mathrm{Com}(U,V)}$ is contained in $\mathrm{Com}(R_U, R_V)$.

This holds for the Heisenberg vertex algebra $M(1)$ embedded in the $\beta\gamma$-system $S(2)$ of rank 2, and more generally, embedded in a $\beta\gamma$-system $S(n)$ of rank n (Sect. 5.2). Thus, we compare the algebras $R_{\mathrm{Com}(M(1),S(2))}$ and $\mathrm{Com}(R_{M(1)}, R_{S(2)})$. Note that, in general, if R_V is normal – it is the case when V is the $\beta\gamma$-system $S(n)$ – so is the commutant $\mathrm{Com}(R_U, R_V)$ (Lemma 5.2.1). On the contrary, $R_{\mathrm{Com}(U,V)}$ has no apparent reason to be normal. We cannot hope an equality between the two spectra in general. In particular, the associated scheme of $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)$ is not the same as the spectrum $\mathrm{Spec}\,\mathrm{Com}(R_U, R_V)$. However, we notice that the latter is the normalization of the previous one.

Theorem 5.2. *The Zhu's C_2-algebra $R_{\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)}$ is a subalgebra of the commutant of Zhu's C_2-algebras $\mathrm{Com}(R_{M(1)}, R_{S(2)})$. Moreover, these Poisson algebras satisfy the following properties:*

(i) $\tilde{X}_{\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)}$ *is a three-dimensional reduced, irreducible and not normal scheme,*

(ii) $\mathrm{Spec}\,\mathrm{Com}(R_{M(1)}, R_{S(2)})$ *is a three-dimensional reduced, irreducible and normal scheme,*

(iii) $\mathrm{Spec}\,\mathrm{Com}(R_{M(1)}, R_{S(2)})$ *is the normalization of* $\tilde{X}_{\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)}$.

Finally, in Sect. 5.3, using that the associated variety of $H^0_f(L_k(\mathfrak{g}))$ is the intersection of the associated variety of $L_k(\mathfrak{g})$ with the Slodowy slice $\mathscr{S}_f$ [17], and the explicit computation we obtained for $X_{L_{-5/2}(\mathfrak{sl}_4)}$, we are able to compute the associated variety of $H^0_f(L_{-5/2}(\mathfrak{sl}_4))$. Conjecturally, the latter is isomorphic to the simple $\mathcal{W}$-algebra $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)$ and so we should obtain an algebraic realization of the associated variety $X_{\mathcal{W}_{-5/2}(\mathfrak{sl}_4,f)}$ (Conjecture 5.3.2).

5.1 Associated variety of $L_{-5/2}(\mathfrak{sl}_4)$

We consider $\mathfrak{g} := \mathfrak{sl}_4$, the vector space of four-size square matrices with trace zero endowed with the Lie bracket $[a, b] = ab - ba$ for all $a, b \in \mathfrak{sl}_4$. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the usual triangular decomposition, where the Cartan subalgebra $\mathfrak{h}$ consists in the diagonal matrices in $\mathfrak{g}$ and $\mathfrak{n}_+$ corresponds to the upper triangular matrices. For $1 \leqslant i \leqslant 3$, write $\alpha_i = \epsilon_i - \epsilon_{i+1}$ the simple roots of $(\mathfrak{g}, \mathfrak{h})$, Δ the corresponding root system and $h_i = \alpha_i^\vee$ the corresponding coroots. Denote ϖ_i, $1 \leqslant i \leqslant 3$, the fundamental weights. For $\alpha \in \Delta_+$ a positive root, let e_α and f_α be root vectors corresponding to α and $-\alpha$ respectively.

Recall that a vector v in $V^k(\mathfrak{g})$ is *singular* if it is annihilated by all raising operators in $V^k(\mathfrak{g})$, i.e. $\widehat{\mathfrak{n}}.v = 0$. Singular vectors depend on the complex k, and for a fixed level k it is in general difficult to determine one. They play an important role in the description of the simple quotient of the affine vertex algebra $L_k(\mathfrak{g})$ and its associated objects. For instance, thank to the singular vector v of $V^{-5/2}(\mathfrak{sl}_4)$ provided in [7] we compute the associated variety of $L_{-5/2}(\mathfrak{sl}_4)$. We prove (see Theorem 5.1) that it is the closure of the Jordan class of the element

$$x_0 = 3h_1 + 2h_2 + h_3 + e_{\varepsilon_2-\varepsilon_4} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which is neither nilpotent nor semisimple. Let $x_s = \mathrm{diag}(3, -1, -1, -1)$ and $x_n = x_0 - x_s$ be the Jordan decomposition of x_0. Since $\mathfrak{z}(\mathfrak{g}^{x_s})$ is one-dimensional, we easily describe the Jordan class of x_0:

$$J_G(x_0) = G.\mathbb{C}^* x_0.$$

The rest of this section is devoted to the proof of Theorem 5.1.

In [7], the authors give a singular vector v of $V^{-5/2}(\mathfrak{sl}_4)$ and prove that it generates the maximal ideal in $V^{-5/2}(\mathfrak{sl}_4)$. Hence,

$$L_{-5/2}(\mathfrak{sl}_4) = V^{-5/2}(\mathfrak{sl}_4)/\langle v \rangle.$$

From this equality, we can deduce $\mathcal{A}(L_{-5/2}(\mathfrak{sl}_4)) \simeq U(\mathfrak{g})/\langle v' \rangle$ where v' is the image of $[v] \in \mathcal{A}(V^{-5/2}(\mathfrak{sl}_4))$ through the isomorphism $\mathcal{A}(V^{-5/2}(\mathfrak{sl}_4)) \simeq U(\mathfrak{g})$. Similarly, $R_{L_{-5/2}(\mathfrak{sl}_4)} \simeq \mathcal{S}(\mathfrak{g})/\mathcal{I}_W$, where $\mathcal{I}_W$ is the ideal of $\mathcal{S}(\mathfrak{g})$ generated by W, the $\mathfrak{g}$-module generated by v'', where v'' is the image of $\overline{v} \in R_{V^{-5/2}(\mathfrak{g})} = V^{-5/2}(\mathfrak{g})/C_2(V^{-5/2}(\mathfrak{g}))$ with the identification of the Zhu's C_2-algebra of $V^{-5/2}(\mathfrak{g})$ with the symmetric algebra $\mathcal{S}(\mathfrak{g})$. Explicit formulas for v, v' and v'' appear in [7, Sect. 3]. We recall below formula of v'' which will be useful to prove Theorem 5.1.

Proposition 5.1.1 ([7]). *The Zhu's C_2-algebra $R_{L_{-5/2}(\mathfrak{sl}_4)}$ is isomorphic to the quotient $\mathcal{S}(\mathfrak{g})/\mathcal{I}_W$*

where W is the $\mathfrak{g}$-module generated by the following vector v'':

$$\begin{aligned}
v'' &= 2e_{\varepsilon_3-\varepsilon_4}e^2_{\varepsilon_2-\varepsilon_3}e_{\varepsilon_1-\varepsilon_2} + \frac{2}{3}(h_1h_2 - h_1h_3 + h_2^2 + h_2h_3)e_{\varepsilon_2-\varepsilon_4}e_{\varepsilon_1-\varepsilon_3}\\
&+ \frac{2}{3}(-h_1h_2 - 2h_1h_3 - h_2^2 - h_2h_3)e_{\varepsilon_2-\varepsilon_3}e_{\varepsilon_1-\varepsilon_4}\\
&+ \frac{2}{3}\Big(- f_{\varepsilon_1-\varepsilon_2}e_{\varepsilon_1-\varepsilon_2} + 2f_{\varepsilon_1-\varepsilon_3}e_{\varepsilon_1-\varepsilon_3} + 2f_{\varepsilon_1-\varepsilon_4}e_{\varepsilon_1-\varepsilon_4} + 2f_{\varepsilon_2-\varepsilon_3}e_{\varepsilon_2-\varepsilon_3}\\
&\quad + 2f_{\varepsilon_2-\varepsilon_4}e_{\varepsilon_2-\varepsilon_4} - f_{\varepsilon_3-\varepsilon_4}e_{\varepsilon_3-\varepsilon_4}\Big)e_{\varepsilon_2-\varepsilon_4}e_{\varepsilon_1-\varepsilon_3}\\
&+ \frac{2}{3}\Big(f_{\varepsilon_1-\varepsilon_2}e_{\varepsilon_1-\varepsilon_2} - 2f_{\varepsilon_1-\varepsilon_3}e_{\varepsilon_1-\varepsilon_3} - 2f_{\varepsilon_1-\varepsilon_4}e_{\varepsilon_1-\varepsilon_4} - 2f_{\varepsilon_2-\varepsilon_3}e_{\varepsilon_2-\varepsilon_3}\\
&\quad - 2f_{\varepsilon_2-\varepsilon_4}e_{\varepsilon_2-\varepsilon_4} + f_{\varepsilon_3-\varepsilon_4}e_{\varepsilon_3-\varepsilon_4}\Big)e_{\varepsilon_2-\varepsilon_3}e_{\varepsilon_1-\varepsilon_4}\\
&- 2h_3e_{\varepsilon_1-\varepsilon_2}e_{\varepsilon_2-\varepsilon_3}e_{\varepsilon_2-\varepsilon_4} + 2h_1e_{\varepsilon_3-\varepsilon_4}e_{\varepsilon_2-\varepsilon_3}e_{\varepsilon_1-\varepsilon_3} + 2h_3f_{\varepsilon_1-\varepsilon_2}e_{\varepsilon_1-\varepsilon_4}e_{\varepsilon_1-\varepsilon_3}\\
&- 2f_{\varepsilon_1-\varepsilon_2}e_{\varepsilon_3-\varepsilon_4}e^2_{\varepsilon_1-\varepsilon_3} - 2h_1f_{\varepsilon_3-\varepsilon_4}e_{\varepsilon_2-\varepsilon_4}e_{\varepsilon_1-\varepsilon_4} - 2f_{\varepsilon_3-\varepsilon_4}e_{\varepsilon_1-\varepsilon_2}e^2_{\varepsilon_2-\varepsilon_4} + 2f_{\varepsilon_3-\varepsilon_4}f_{\varepsilon_1-\varepsilon_2}e^2_{\varepsilon_1-\varepsilon_4}.
\end{aligned}$$

Since the singular vector v generates the maximal ideal of $V^{-5/2}(\mathfrak{sl}_4)$, we obtain an explicit description of the simple quotient $L_{-5/2}(\mathfrak{sl}_4)$ and its Zhu's C_2-algebra. By definition, the associated variety of $L_{-5/2}(\mathfrak{sl}_4)$ is the reduced spectrum $\mathrm{Specm}(R_{L_{-5/2}(\mathfrak{sl}_4)})$, that is the zero locus in $\mathfrak{g}^* \simeq \mathfrak{g}$ of the kernel of the map

$$\begin{aligned}
\mathbb{C}[\mathfrak{g}^*] &\longrightarrow R_{L_{-5/2}(\mathfrak{sl}_4)} = L_{-5/2}(\mathfrak{sl}_4)/t^{-2}\mathfrak{g}[t^{-1}]L_{-5/2}(\mathfrak{sl}_4)\\
x_1 \dots x_n &\longmapsto \overline{(x_1t^{-1})\dots(x_nt^{-1})|0\rangle} \mod t^{-2}\mathfrak{g}[t^{-1}]L_{-5/2}(\mathfrak{sl}_4),
\end{aligned}$$

where $x_1, \leqslant, x_n \in \mathfrak{g}$.

Lemma 5.1.2. *Let $H = \{3h_1 + 2h_2 + h_3, -h_1 + 2h_2 + h_3, -h_1 - 2h_2 + h3, -h_1 - 2h_2 - 3h_3\}$. Then,*

$$X_{L_{-5/2}(\mathfrak{g})} \cap \mathfrak{h} = \cup_{\lambda \in H}\mathbb{C}\lambda.$$

Proof. Let $h \in X_{L_{-5/2}(\mathfrak{g})} \cap \mathfrak{h}$. We identify h with its dual through the symmetric bilinear form $(.|.)$. Since $\mathcal{I}_W$ is invariant under the adjoint action, we have in particular

$$\begin{cases}
(\mathrm{ad}_{f_{\varepsilon_1-\varepsilon_4}}\, \mathrm{ad}_{f_{\varepsilon_2-\varepsilon_3}}\, v'')(h) = 0\\
(\mathrm{ad}_{f_{\varepsilon_2-\varepsilon_4}}\, \mathrm{ad}_{f_{\varepsilon_1-\varepsilon_3}}\, v'')(h) = 0\\
(\mathrm{ad}_{f_{\varepsilon_3-\varepsilon_4}}\, \mathrm{ad}_{f_{\varepsilon_1-\varepsilon_3}}\, \mathrm{ad}_{f_{\varepsilon_2-\varepsilon_3}}\, v'')(h) = 0.
\end{cases}$$

The solutions of this system are precisely the set of lines generated by the elements in H : $\cup_{\lambda\in H}\mathbb{C}\lambda$. Let prove the converse inclusion. Note that all the semisimple elements of H are in the same semisimple orbit under the action of the adjoint group G. As a consequence, since $X_{L_{-5/2}(\mathfrak{g})}$ is a G-invariant cone, it suffices to show that $3h_1 + 2h_2 + h_3$ belongs to the associated variety. By [7, Lemma 4.1], the zero weight space of W is generated by the following polynomials:

$$\begin{aligned}
p_1 :=& -\frac{5}{2}h_2 - \frac{7}{2}h_1h_2 + \frac{3}{2}h_1h_3 - \frac{7}{2}h_2h_3 - \frac{31}{6}{h_2}^2 - \frac{13}{3}h_1{h_2}^2 - {h_1}^2h_2 - 2h_1h_2h_3\\
&- \frac{10}{3}{h_2}^3 + h_1{h_3}^2 + {h_1}^2h_3 - \frac{13}{3}{h_2}^2h_3 - h_2{h_3}^2 - \frac{4}{3}h_1{h_2}^3 - \frac{2}{3}{h_1}^2{h_2}^2\\
&+ \frac{2}{3}{h_1}^2{h_3}^2 - \frac{4}{3}h_1{h_2}^2h_3 - \frac{2}{3}{h_2}^4 - \frac{4}{3}{h_2}^3h_3 - \frac{2}{3}{h_2}^2{h_3}^2,
\end{aligned}$$

and

$$\begin{aligned} p_2 :=& \frac{5}{2}h_2 + \frac{7}{2}h_1h_2 + \frac{7}{2}h_2h_3 + \frac{31}{6}{h_2}^2 + \frac{13}{3}h_1{h_2}^2 + {h_1}^2h_2 + \frac{16}{3}h_1h_2h_3 \\ &+ \frac{10}{3}{h_2}^3 + \frac{13}{3}{h_2}^2h_3 + h_2{h_3}^2 + \frac{4}{3}h_1{h_2}^3 + \frac{2}{3}{h_1}^2{h_2}^2 + \frac{4}{3}{h_1}^2h_2h_3 \\ &+ \frac{8}{3}h_1{h_2}^2h_3 + \frac{4}{3}h_1h_2{h_3}^2 + \frac{2}{3}{h_2}^4 + \frac{4}{3}{h_2}^3h_3 + \frac{2}{3}{h_2}^2{h_3}^2. \end{aligned}$$

Since for $1 \leqslant i \leqslant 3$,

$$\varpi_i = (\varepsilon_1 + \ldots + \varepsilon_i) - \frac{i}{4}(\varepsilon_1 + \ldots + \varepsilon_4),$$

the element $3h_1 + 2h_2 + h_3$ is identified with $4\varpi_1$ through $(.|.)$. We easily check that $p_1(4\varpi_1) = p_2(4\varpi_1) = 0$. Hence, $3h_1 + 2h_2 + h_3 \in X_{L_{-5/2}(\mathfrak{g})}$. □

Given a nilpotent element f of $\mathfrak{sl}_4$, let $\chi_f = (f|\cdot) \in \mathfrak{g}^*$. Choose a Lagrangian subspace $\mathcal{L} \subset \mathfrak{g}_{1/2}$ and set

$$\mathfrak{m} = \mathcal{L} \oplus \bigoplus_{j \geqslant 1} \mathfrak{g}_j, \qquad \text{and} \qquad J_\chi = \sum_{x \in \mathfrak{m}} \mathcal{S}(\mathfrak{g})(x - \chi(x)).$$

Lemma 5.1.3 ([29]). *Let I be an $\operatorname{ad}\mathfrak{g}$-invariant ideal of $\mathbb{C}[\mathfrak{g}^*]$. Then $\overline{\mathbb{O}_f} \not\subset \operatorname{Var}(I)$ if and only if*

$$\mathbb{C}[\mathfrak{g}^*] = I + J_\chi,$$

where $\operatorname{Var}(I)$ is the zero locus of I in $\mathfrak{g}^*$.

Applying this result to a subregular and a principal nilpotent element of $\mathfrak{g}$ successively, one proves the following result.

Proposition 5.1.4. *The associated variety of $L_{-5/2}(\mathfrak{g})$ contains some nilpotent elements. More precisely,*

$$\mathbb{O}_{\text{subreg}} \subset X_{L_{-5/2}(\mathfrak{g})} \qquad \text{and} \qquad \mathbb{O}_{\text{reg}} \not\subset X_{L_{-5/2}(\mathfrak{g})}.$$

Proof. The first inclusion follows from [7, Lemma 3.8]. Focus on the second part of the proposition. Let $\chi_{\text{reg}} = (f_{\text{reg}}|.)$ with $f_{\text{reg}} = f_{\varepsilon_1-\varepsilon_2} + f_{\varepsilon_2-\varepsilon_3} + f_{\varepsilon_3-\varepsilon_4}$ a regular nilpotent element of $\mathfrak{sl}_4$. Then

$$v'' = 1 \mod J_{\chi_{\text{reg}}}.$$

Hence, $I_W + J_{\chi_{\text{reg}}} = \mathbb{C}[\mathfrak{g}^*]$. By Lemma 5.1.3, this implies $\mathbb{O}_{\text{reg}} \not\subset X_{L_{-5/2}(\mathfrak{g})}$. □

Let $\mathfrak{p}$ be a parabolic subalgebra with Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical. Given a nilpotent orbit $\mathbb{O}_\mathfrak{l}$ in the Levi subalgebra $\mathfrak{l}$, it is shown in [85] that there is a unique nilpotent orbit $\mathbb{O}$ in $\mathfrak{g}$ such that $\mathbb{O} \cap (\mathbb{O}_\mathfrak{l} \oplus \mathfrak{n})$ is dense in $\mathbb{O}_\mathfrak{l} \oplus \mathfrak{n}$. This orbit only depends on $\mathfrak{l}$ and not on the parabolic subgroup $\mathfrak{p}$. It is said *induced* from $\mathbb{O}_\mathfrak{l}$ and denoted $\operatorname{Ind}_\mathfrak{l}^\mathfrak{g}(\mathbb{O}_\mathfrak{l})$. Not all nilpotent orbit of $\mathfrak{g}$ are induced from one of $\mathfrak{l}$. We refer to [39, Chap. 7] for basic results on induced nilpotent orbits. For a generic element $x \in \mathfrak{g}$ with Jordan decomposition $x = x_s + x_n$, one can consider the induced nilpotent orbit $\operatorname{Ind}_{\mathfrak{g}^{x_s}}^\mathfrak{g}(\mathbb{O}_{x_n})$ of x_n in the centralizer $\mathfrak{g}^{x_s}$. Recall the following result from [38].

Theorem 5.1.5 ([38]). *1. If $x = x_n + x_s$ is a non nilpotent element of $\mathfrak{g}$. Then $\operatorname{Ind}_{\mathfrak{g}^{x_s}}^\mathfrak{g}(\mathbb{O}_{x_n})$ is the unique nilpotent orbit contained in the closure $C(x) := \overline{G.\mathbb{C}^* x}$ of the G-invariant cone generated by x whose dimension is $\dim G.x$. Furthermore, $C(x) \cap \mathcal{N} = \overline{\operatorname{Ind}_{\mathfrak{g}^{x_s}}^\mathfrak{g}(\mathbb{O}_{x_n})}$.*

2. *Conversely, if $\mathbb{O}$ is an induced nilpotent orbit, there exists a non nilpotent element $x \in \mathfrak{g}$ such that $C(x) \cap \mathcal{N} = \overline{\mathbb{O}}$.*

We are now in position to prove Theorem 5.1.

Proof of Theorem 5.1. Let $x \in X_{L_{-5/2}(\mathfrak{g})}$ and write $x = x_s + x_n$ its Jordan decomposition with x_s semisimple and x_n nilpotent. Since $X_{L_{-5/2}(\mathfrak{g})}$ is G-invariant, one can assume that x_s belongs to $\mathfrak{h}$. If $x_n = 0$ then $x = x_s \in G.\mathbb{C}^*(3h_1 + 2h_2 + h_3) \subset \overline{G.\mathbb{C}^* x_0}$.

Assume $x_n \neq 0$. If x is nilpotent, i.e. $x_s = 0$, then $x \in \overline{\mathbb{O}}_{\text{subreg}}$. Besides, $\mathbb{O}_{\text{subreg}}$ is the induced nilpotent orbit of $e_{\varepsilon_2-\varepsilon_4} \in \mathfrak{g}^{3h_1+2h_2+h_3}$ in $\mathfrak{g}$. Thus

$$\overline{\mathbb{O}}_{\text{subreg}} \subset \overline{G.\mathbb{C}^*(3h_1 + 2h_2 + h_3 + e_{\varepsilon_2-\varepsilon_4})}.$$

Else, if x is not nilpotent, we embed $x_n = e$ in an $\mathfrak{sl}_2$-triple (e, h, f) with $h \in \mathfrak{h}$. Let the one-parameter subgroup $\rho : \mathbb{C}^* \to G$ generated by ad_h:

$$\rho(t).y = t^i y, \quad \text{for all } y \in \mathfrak{g},\ [h, y] = iy.$$

Then for all $t \in \mathbb{C}^*$,

$$\rho(t).x = x_s + t^2 x_n \in X_{L_{-5/2}(\mathfrak{g})}.$$

Since the associated variety $X_{L_{-5/2}(\mathfrak{g})}$ is a closed G-invariant cone, we get that x_s and x_n belong to $X_{L_{-5/2}(\mathfrak{g})}$. By Lemma 5.1.2, it exists $c \in \mathbb{C}^*$ such that $x_s = c\lambda$ with $\lambda \in H$. Since all the elements in H are G-conjugated and the associated variety is conical, one can assume $\lambda = 3h_1 + 2h_2 + h_3$ and $c = 1$. Then $x_n \in \mathfrak{g}^{x_s} \cap \mathcal{N}$. Let $\mathbb{O}$ be the nilpotent orbit of x_n in $\mathfrak{g}^{x_s}$. By Theorem 5.1.5, the induced nilpotent orbit $\mathrm{Ind}_{\mathfrak{g}^{x_s}}^{\mathfrak{g}}(\mathbb{O})$ of $\mathbb{O}$ in $\mathfrak{g}$ is included in $\overline{G.\mathbb{C}^* x}$, so in the associated variety. Since

$$\mathfrak{g}^{x_s} = \mathfrak{h} \oplus \mathbb{C}e_{\varepsilon_2-\varepsilon_3} \oplus \mathbb{C}e_{\varepsilon_3-\varepsilon_4} \oplus \mathbb{C}e_{\varepsilon_2-\varepsilon_4} \oplus \mathbb{C}f_{\varepsilon_2-\varepsilon_3} \oplus \mathbb{C}f_{\varepsilon_3-\varepsilon_4} \oplus \mathbb{C}f_{\varepsilon_2-\varepsilon_4}$$

with $e_{\varepsilon_2-\varepsilon_4} = [e_{\varepsilon_2-\varepsilon_3}, e_{\varepsilon_3-\varepsilon_4}]$, $\mathfrak{g}^{x_s} \simeq \mathfrak{sl}_3 \times \mathbb{C}$, and we get the following possibilities:

(1) $\mathbb{O} = 0$ and $\mathrm{Ind}_{\mathfrak{g}^{x_s}}^{\mathfrak{g}}(\mathbb{O}) = \mathbb{O}_{\text{min}}$,

(2) $\mathbb{O} = \mathbb{O}_{(2,1)}$ and $\mathrm{Ind}_{\mathfrak{g}^{x_s}}^{\mathfrak{g}}(\mathbb{O}) = \mathbb{O}_{\text{subreg}}$,

(3) $\mathbb{O} = \mathbb{O}_{(3)}$ and $\mathrm{Ind}_{\mathfrak{g}^{x_s}}^{\mathfrak{g}}(\mathbb{O}) = \mathbb{O}_{\text{reg}}$.

The condition (3) cannot happen since $\mathbb{O}_{\text{reg}} \not\subset X_{L_{-5/2}(\mathfrak{g})}$ and condition (1) was already treated as x is semisimple. Focus on condition (2). Let $G^{x_s} \subset G$ the adjoint group of $\mathfrak{g}^{x_s}$. Since $\mathbb{O}_{(2,1)} = G^{x_s}.e_{\varepsilon_1-\varepsilon_4}$, there exists $y \in G^{x_s}$ such that $x_n = y.e_{\varepsilon_1-\varepsilon_4}$. Hence,

$$x = x_s + y.e_{\varepsilon_1-\varepsilon_4} = y.(x_s + e_{\varepsilon_1-\varepsilon_4}) \in \overline{G.\mathbb{C}^* x_0}.$$

In order to prove the converse inclusion, it suffices to show that $x = x_s + e_{\varepsilon_2-\varepsilon_4}$, where $x_s = 3h_1 + 2h_2 + h_3$ belongs to the associated variety. Assume it does not. One deduces from the previous discussion that $X_{L_{-5/2}(\mathfrak{g})} = \overline{G.\mathbb{C}^* x_s} \cup \overline{\mathbb{O}}_{\text{subreg}}$. Then the intersection between the associated variety and the Slodowy slice $\mathscr{S}_{f_{\text{subreg}}}$ is reduced to a point. Indeed, according to Theorem 5.1.5,

$$\overline{G.\mathbb{C}^* x_s} \cap \mathcal{N} = \overline{\mathrm{Ind}_{\mathfrak{g}^{x_s}}^{\mathfrak{g}}(0)} = \overline{\mathbb{O}}_{\text{min}}.$$

Thus, the intersection $\overline{G.\mathbb{C}^* x_s} \cap \mathscr{S}_{f_{\text{subreg}}}$ is empty and $\overline{\mathbb{O}}_{\text{subreg}} \cap \mathscr{S}_{f_{\text{subreg}}} = \{f_{\text{subreg}}\}$. However, according to [7, Proposition 5.3], $H^0_{f_{\text{subreg}}}(L_{-5/2}(\mathfrak{g})) \simeq M(1)$, so this intersection is one-dimensional, whence $x \in X_{L_{-5/2}(\mathfrak{g})}$.

Let show that $G.\mathbb{C}^*x_0 = J_G(x_0)$. Recall [93] that

$$J_G(x_0) = G.(\mathfrak{z}(\mathfrak{g}^{3h_1+2h_2+h_3})^{\text{reg}} + e_{\varepsilon_1-\varepsilon_4}) = G.(\mathfrak{z}(\mathfrak{g}^{x_0})^{\text{reg}}),$$

where $\mathfrak{z}(\mathfrak{g}^x)^{\text{reg}} = \{y \in \mathfrak{g}, \mathfrak{g}^y = \mathfrak{g}^x\}$. Since $x_0 \in \mathfrak{z}(\mathfrak{g}^{x_0})^{\text{reg}}$ and $J_G(x_0)$ is G-invariant, $G.\mathbb{C}^*x_0 \subset J_G(x_0)$. Conversely, let $y \in J(x_0)$. The set $\mathfrak{z}(\mathfrak{g}^{3h_1+2h_2+h_3})^{\text{reg}}$ is one-dimensional generated by $3h_1 + 2h_2 + h_3$, and one can assume $y = \alpha(3h_1 + 2h_2 + h_3) + e_{\varepsilon_1-\varepsilon_4} \in G.(\mathfrak{z}(\mathfrak{g}^{x_0})^{\text{reg}})$ for some $\alpha \in \mathbb{C}^*$. We embed $e_{\varepsilon_1-\varepsilon_4}$ in an $\mathfrak{sl}_2$-triple whose the semisimple element h belongs to $\mathfrak{h}$. Again, consider the one-parameter subgroup $\rho : \mathbb{C}^* \to G$ generated by ad_h. For all $t \in \mathbb{C}^*$,

$$\rho(t)y = \alpha(3h_1 + 2h_2 + h_3) + t^2 e_{\varepsilon_1-\varepsilon_4} \in J_G(x_0).$$

In particular, for $\beta = \alpha^{1/2}$ we get that $\rho(\beta)y \in G.\mathbb{C}^*x_0$. Hence, $y \in G.\mathbb{C}^*x_0$. □

Remark 5.1.6. The closure of the Jordan class $\overline{J_G(x_0)}$ is not normal. Indeed, if it were, the quotient $\overline{J_G(x_0)}//G$ would be normal. It is not the case according to the classification of normality of quotients of closures of decomposition classes of Richardson [91]. We thank Prof. Giovanna Carnovale and Francesco Esposito for informing us about this.

5.2 Commutant of Zhu's C_2-algebras

Let U be a vertex subalgebra of the vertex algebra V. We define the *coset* [63] of U in V to be the vertex subalgebra of V

$$\mathrm{Com}(U, V) := \{a \in V \mid a_{(n)}b = 0, \text{ for all } b \in U \text{ and } n \geqslant 0\}.$$

In particular, we consider in the following the coset of the Heisenberg algebra $M(1)$ in the $\beta\gamma$-system $S(n)$ of rank $n \geqslant 1$.

The *$\beta\gamma$-system* $S(n)$ of rank n is the vertex algebra strongly generated by fields $\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n$ satisfying the only non-trivial OPEs

$$\beta_i(z)\gamma_i(w) \sim \frac{1}{z-w}.$$

The Heisenberg vertex algebra $M(1)$ can be viewed as a vertex subalgebra of $S(n)$ sending its strong generator on the field

$$h =: \beta_1\gamma_1 : + \dots + : \beta_n\gamma_n : .$$

Hence, we can consider the coset $\mathrm{Com}(M(1), S(n))$.

Recall that the associated variety of $S(n)$ is $R_{S(n)} \simeq \mathbb{C}[T^*\mathbb{C}^n] \simeq \mathbb{C}[p_1, \dots, p_n, q_1, \dots, q_n]$ with Poisson bracket

$$\{p_i, q_j\} = \delta_{i,j}, \qquad \text{and} \qquad \{p_i, p_j\} = \{q_i, q_j\} = 0,$$

for all $i, j = 1, \dots, n$. On the other hand, setting $\bar{h} = \sum_{i=1}^n p_i q_i$, we get that the Zhu's C_2-algebra of the Heisenberg algebra $M(1)$ is $R_{M(1)} = \mathbb{C}[\bar{h}]$, a subalgebra of $R_{S(n)}$. Hence, we can define the commutant $\mathrm{Com}(R_{M(1)}, R_{S(n)})$ which is the Poisson commutant of $\bar{h}$ in $\mathbb{C}[p_1, \dots, p_n, q_1, \dots, q_n]$.

It has been independently proved by [6] and [45] that, when $n = 2$, the coset $\mathrm{Com}(M(1), S(2))$ is isomorphic to the $\mathcal{W}$-algebra $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)$ where $f := f_{\varepsilon_1-\varepsilon_3} + f_{\varepsilon_2-\varepsilon_4}$ is a nilpotent element corresponding to the partition $(2, 2)$ of 4. Using the explicit description presented in [6] and [45], we compute the associated variety of $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)$ and compare it to the commutant of Zhu's C_2-algebras $M(1)$ in $S(2)$ (see Theorem 5.2).

The proof of Theorem 5.2 is the guiding principle of this section. It allows to introduce a more general reflexion on the possible link between Zhu's C_2-algebras of cosets $\mathrm{Com}(U, V)$ and commutants of Zhu's C_2-algebras $\mathrm{Com}(R_U, R_V)$. Similar considerations have been formulated in [84, Sect. 13] in the context of vertex algebras *orbifolds*, that are vertex subalgebras invariant under the action of a reductive group of automorphisms.

5.2.1 Commutants of Zhu's C_2-algebras and Zhu's C_2-algebras of cosets

Recall (Sect. 1.1.4), that for any vertex algebra V the Zhu's C_2-algebra R_V corresponds to the quotient $V/C_2(V)$, where $C_2(V)$ is spanned by the elements $a_{(-2)}b$, $a, b \in V$. A morphism of vertex algebras $f : U \to V$ induces a morphism of Poisson algebras $\varphi : R_U \to R_V$ such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & V \\ \downarrow & & \downarrow \\ R_U & \underset{\varphi}{\dashrightarrow} & R_V \end{array}$$

Consider the commutant of the Poisson algebra $\varphi(R_U)$ in R_V

$$\begin{aligned} \mathrm{Com}(\varphi(R_U), R_V) &= \{\bar{a} \in R_V \mid \{\bar{a}, \phi(\tilde{b})\} = 0, \text{ for all } b \in U\} \\ &= \{\bar{a} \in R_V \mid \{\bar{a}, \overline{f(b)}\} = 0, \text{ for all } b \in U\}. \end{aligned}$$

In particular, when U is a vertex subalgebra of the vertex algebra V, the inclusion induces a map φ as above. Clearly, φ is injective if and only if $U \cap C_2(V) = C_2(U)$. If so, R_U can be viewed as a subalgebra of R_V.

On another hand, the inclusion $\mathrm{Com}(U, V) \subset V$ gives rise to a map $\phi : R_{\mathrm{Com}(U,V)} \to R_V$ which is injective if and only if $\mathrm{Com}(U, V) \cap C_2(V) = C_2(\mathrm{Com}(U, V))$. Assume that is the case. Then $R_{\mathrm{Com}(U,V)}$ can also be viewed as a subalgebra of R_V. By abuse of notation, we identify $R_{\mathrm{Com}(U,V)}$ with its image through ϕ. Then $R_{\mathrm{Com}(U,V)} = \{\bar{a} \in R_V \mid a_{(n)}b = 0, \text{ for all } b \in U \text{ and } n \geqslant 0\}$, and $R_{\mathrm{Com}(U,V)} \subset \mathrm{Com}(R_U, R_V)$. This inclusion induces a dominant morphism of schemes

$$\operatorname{Spec} \mathrm{Com}(R_U, R_V) \longrightarrow \tilde{X}_{\mathrm{Com}(U,V)}.$$

There is apparently no reason for $\tilde{X}_{\mathrm{Com}(U,V)}$ or $\operatorname{Spec} \mathrm{Com}(R_U, R_V)$ to be normal. It would be interesting to determine conditions on the vertex algebras U and V which induce the normality in either or both spectra. Note that $\mathrm{Com}(R_U, R_V)$ is normal if R_V is so.

Lemma 5.2.1. *Let $(A, \{.,.\})$ be a normal Poisson algebra and B a Poisson subalgebra of A. Then the Poisson commutant of B in A,*

$$\mathrm{Com}(B, A) = \{a \in A \mid \{a, b\} = 0, \textit{ for all } b \in B\},$$

is normal.

Proof. Set $C = \mathrm{Com}(B, A)$. Let $f \in \operatorname{Frac} C$, $n \geqslant 1$ and $a_0, \dots, a_{n-1} \in C$ such that

$$f^n + a_{n-1}f^{n-1} + \ldots + a_1 f + a_0 = 0$$

is a minimal degree equation in C. Since $\operatorname{Frac} C \subset \operatorname{Frac} A$ and A is normal, f belongs to A. We need to show that f belongs to C. Set $f = p/q$, $p, q \in C$, $q \neq 0$. In particular, $qf = p \in C$. Let

$b \in B$, using the Leibniz's rule we get

$$\{b, p\} = \{b, qf\} = q\{b, f\} + \{b, q\}f.$$

Since $p, q \in C$, $\{b, p\} = \{b, q\} = 0$, hence $q\{b, f\} = 0$. Because A is an integral domain, we deduce that $\{b, f\} = 0$ for any $b \in B$. As a consequence, $f \in C$. The integral closure of C in $\operatorname{Frac} C$ is C itself and C is normal. □

Corollary 5.2.2. *For any positive integer n, since $R_{S(n)}$ is normal, so is $\operatorname{Com}(R_{M(1)}, R_{S(n)})$.*

The commutant $\operatorname{Com}(R_{M(1)}, R_{S(n)})$ is the Poisson commutant of $\bar{h}$ in $\mathbb{C}[p_1, \dots, p_n, q_1, \dots, q_n]$. It is generated by monomials $p_i q_j$ with $i, j = 1, \dots, n$. Let $\psi : \mathbb{C}^{2n} \to \mathbb{C}^{n^2}$ defined by

$$(p_1, \dots, p_n, q_1, \dots, q_n) \mapsto (p_1q_1, \dots p_1q_n, p_2q_1, \dots p_2q_n, \dots, p_nq_1, \dots, p_nq_n).$$

The comorphism

$$\psi^* : \mathbb{C}[x_{1,1}, \dots, x_{1,n}, x_{2,1}, \dots, x_{2,n}, \dots, x_{n,1}, \dots, x_{n,n}] \to \mathbb{C}[p_1, \dots, p_n, q_1, \dots, q_n],$$

has kernel the ideal I_n generated by $(x_{i,j}x_{k,l} - x_{i,l}x_{k,j})_{\substack{1\leqslant i<k\leqslant n, \\ 1\leqslant j<l\leqslant n}}$. Hence,

$$\operatorname{Com}(R_{M(1)}, R_{S(n)}) \simeq \mathbb{C}[x_{1,1}, \dots, x_{1,n}, x_{2,1}, \dots, x_{2,n}, \dots, x_{n,1}, \dots, x_{n,n}]/I_n. \tag{5.1}$$

with non trivial Poisson brackets $\{x_{i,j}, x_{k,l}\} = \delta_{j,k}x_{i,l} - \delta_{i,l}x_{k,j}$.

5.2.2 Proof of Theorem 5.2

Using the previous computations, we are now in position to prove all assertions of the Theorem 5.2

(ii) First, considering $n = 2$ in (5.1), we get that $\operatorname{Com}(R_{M(1)}, R_{S(2)}) \simeq \mathbb{C}[x, y, z, t]/(xy - zt)$ with non trivial Poisson brackets $\{x, z\} = -z$, $\{x, t\} = t$, $\{y, z\} = z$, $\{y, t\} = -t$ and $\{z, t\} = y - x$. It follows directly that $\operatorname{Spec}\operatorname{Com}(R_{M(1)}, R_{S(2)})$ is a reduced and irreducible variety of dimension three. It is normal by Corollary 5.2.2.

(i) Using the explicit description of $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)$ as the coset of $M(1)$ in $S(2)$ given in [6] we get that $R_{\mathcal{W}_{-5/2}(\mathfrak{sl}_4,f)}$ is the subalgebra of $\mathbb{C}[p_1, p_2, q_1, q_2]$ generated by

$$\{p_1q_2, \quad p_2q_1, \quad p_1q_1 - p_2q_2, \quad p_1q_2(p_1q_1 + 2p_2q_2), \quad p_2q_1(p_1q_1 + 2p_2q_2), \quad (p_1q_1)^2 - (p_2q_2)^2\}.$$

Consider $\psi : \mathbb{C}^4 \to \mathbb{C}^6$ defined by

$$(p_1, p_2, q_1, q_2) \mapsto (p_1q_2, p_2q_1, p_1q_1 - p_2q_2, p_1q_2(p_1q_1 + 2p_2q_2), p_2q_1(p_1q_1 + 2p_2q_2), (p_1q_1)^2 - (p_2q_2)^2).$$

We denote by $K_{\mathcal{W}}$ the kernel of the comorphism $\psi^* : \mathbb{C}[e_1, e_2, e_3, e_4, e_5, e_6] \to \mathbb{C}[p_1, p_2, q_1, q_2]$, which is a prime ideal. Then $R_{\mathcal{W}_{-5/2}(\mathfrak{sl}_4,f)} = \mathbb{C}[e_1, e_2, e_3, e_4, e_5, e_6]/K_{\mathcal{W}}$, and we conclude that $\tilde{X}_{\mathcal{W}_{-5/2}(\mathfrak{sl}_4,f)}$ is reduced, irreducible of dimension three but not normal.

(iii) Let prove that $\operatorname{Com}(R_{M(1)}, R_{S(2)})$ is isomorphic to the integral closure of $R_{\mathcal{W}_{-5/2}(\mathfrak{sl}_4,f)}$ denoted by F. By direct computation, we have $F = \mathbb{C}[e_0, e_1, e_2, e_3, e_4, e_5, e_6]/I_F$ where I_F is the ideal generated by

$$e_2e_4 - e_1e_5, \quad e_0e_3 + e_6, \quad 3e_0e_2 + e_2e_3 + 2e_5, \quad 3e_0e_1 + e_1e_3 + 2e_4, \quad e_0^2 - 4e_1e_2 - e_3^2.$$

Set $\phi : \mathbb{C}^4 \to \mathbb{C}^7$ mapping

$$(x, y, z, t) \longmapsto (-x - y, z, t, x - y, z(x + 2y), t(x + 2y)x^2 - y^2).$$

The kernel of the comorphism $\phi^* : \mathbb{C}[e_0, e_1, e_2, e_3, e_4, e_5, e_6] \to \mathbb{C}[x, y, z, t]$ is the ideal K_C of $\mathbb{C}[e_0, e_1, e_2, e_3, e_4, e_5, e_6]$ generated by

$$e_2e_4 - e_1e_5, \quad e_0e_3 + e_6, \quad 3e_0e_2 + e_2e_3 + 2e_5, \quad 3e_0e_1 + e_1e_3 + 2e_4.$$

We check that $\mathbb{C}[e_0, e_1, e_2, e_3, e_4, e_5, e_6]/K_C \simeq \mathbb{C}[x, y, z, t]$, and so there is a surjective map

$$\mathbb{C}[e_0, e_1, e_2, e_3, e_4, e_5, e_6]/K_C \to \mathbb{C}[x, y, z, t]/(xy - zt).$$

The image of $e_0^2 - 4e_1e_2 - e_3^2$ through ϕ^* is $xy - zt$ so $\mathbb{C}[x, y, z, t]/(xy - zt) \simeq \mathbb{C}[e_0, e_1, e_2, e_3, e_4, e_5, e_6]/\tilde{K}_C$ where $\tilde{K}_C$ is the ideal generated by K_C and $e_0^2 - 4e_1e_2 - e_3^2$. Clearly, $I_F = \tilde{K}_C$.

5.3 Associated variety of $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{(2,2)})$

Assume the vertex algebra $H_f^0(L_k(\mathfrak{g}))$ is non zero. Since the $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is the simple quotient of $\mathcal{W}^k(\mathfrak{g}, f)$, it is a quotient of $H_f^0(L_k(\mathfrak{g}))$. As a consequence, the associated variety $X_{\mathcal{W}_k(\mathfrak{g},f)}$ is a subvariety of $X_{H_f^0(L_k(\mathfrak{g}))}$. It has been proved by Arakawa [17] that the latter coincides with the intersection of $X_{L_k(\mathfrak{g})}$ with the Slodowy slice of f.

By [67], we know that the intersection $X_{L_{-5/2}(\mathfrak{g})} \cap \mathscr{S}_f$ is equidimensional of dimension three. Moreover, one verifies that $X_{L_{-5/2}(\mathfrak{g})} \cap \mathscr{S}_f$ contains the following element

$$v = f + (4e_{\varepsilon_2 - \varepsilon_4} + h_1 + h_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Proposition 5.3.1. *Let $G^\natural$ be the stabilizer of the $\mathfrak{sl}_2$-triple $\{e, h, f\}$ in the adjoint group $G = SL_4$. For all $t \in \mathbb{C}^*$, define $\tilde{\rho}(t) := t^2\rho(t)$, where ρ is the one-parameter subgroup introduced in the proof of Theorem 5.1:*

$$\tilde{\rho}(t).v = \begin{pmatrix} t^2 & 0 & 0 & 0 \\ 0 & -t^2 & 0 & 4t^4 \\ 1 & 0 & t^2 & 0 \\ 0 & 1 & 0 & -t^2 \end{pmatrix}.$$

Then $\overline{G^\natural.\tilde{\rho}(\mathbb{C}^*)v}$ is a three-dimensional irreducible component of $X_{H_f^0(L_k(\mathfrak{g}))}$.

Proof. Since the centralizer $\mathfrak{g}^f$ is generated by the elements $e_{\varepsilon_1-\varepsilon_2} + e_{\varepsilon_3-\varepsilon_4}$, $e_{\varepsilon_2-\varepsilon_3}$, $e_{\varepsilon_1-\varepsilon_3}$, $e_{\varepsilon_2-\varepsilon_4}$, $e_{\varepsilon_1-\varepsilon_4}$, $f_{\varepsilon_1-\varepsilon_2} + f_{\varepsilon_3-\varepsilon_4}$ and $h_1 + h_3$, one easily checks that $\tilde{\rho}(t).v \in \mathscr{S}_f$ for all $t \in \mathbb{C}^*$. Moreover, $G^\natural$ stabilizes f and $\mathfrak{g}^f$, thus $\overline{G^\natural.\tilde{\rho}(\mathbb{C}^*)v} \subset \mathscr{S}_f$. Besides, v and x are conjugated in G and $X_{L_{-5/2}(\mathfrak{g})}$ is G-invariant – and so $G^\natural$-invariant. Hence, $\overline{G^\natural.\tilde{\rho}(\mathbb{C}^*)v} \subset X_{L_{-5/2}(\mathfrak{g})}$.

The stabilizer of the $\mathfrak{sl}_2$-triple $\{e, h, f\}$ in the adjoint group $G = \mathrm{SL}_4$ is

$$G^\natural = \{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \det A = \pm 1\}.$$

Furthermore, the stabilizer $G^\natural(v)$ of v in $G^\natural$ is one-dimensional, given by

$$G^\natural(v) = \{\mathrm{diag}(a, \frac{1}{a}, a, \frac{1}{a}) \mid a \in \mathbb{C}^*\} \cup \{\mathrm{diag}(a, -\frac{1}{a}, a, -\frac{1}{a}) \mid a \in \mathbb{C}^*\}.$$

As a consequence, $\{(g,t) \in G^\natural \times \mathbb{C}^* \mid g.(\tilde{\rho}(t)v) = v\} = G^\natural(v) \times \{1\}$ is also one-dimensional. Hence, $\overline{G^\natural.\tilde{\rho}(\mathbb{C}^*)v}$ has dimension 3.

From the previous description, the group $G^\natural$ is not connected. Let $(G^\natural)^\circ$ be the identity component. Set $g \in G^\natural \backslash (G^\natural)^\circ$ and $d = \mathrm{diag}\,(1,-1,1,-1) \in G^\natural(v)$, $d \notin (G^\natural)^\circ$. For all $t \in \mathbb{C}^*$, $d.(\tilde{\rho}(t)v) = \tilde{\rho}(t)v$. As a consequence, $g.(\tilde{\rho}(\mathbb{C}^*)v) = (gd).(\tilde{\rho}(\mathbb{C}^*)v) \subset (G^\natural)^\circ.\tilde{\rho}(\mathbb{C}^*)v$. Hence, $\overline{G^\natural.\tilde{\rho}(\mathbb{C}^*)v} = \overline{(G^\natural)^\circ.\tilde{\rho}(\mathbb{C}^*)v}$ and so it is a irreducible component of $X_{L_{-5/2}(\mathfrak{g})} \cap \mathscr{S}_f$. □

Conjecturally, $H_f^0(L_{-5/2}(\mathfrak{sl}_4))$ is simple and isomorphic to the $\mathcal{W}$-algebra $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)$. As a consequence, the associated schemes $\tilde{X}_{\mathcal{W}_{-5/2}(\mathfrak{sl}_4,f)}$ and $\tilde{X}_{H_f^0(L_{-5/2}(\mathfrak{sl}_4))} = \tilde{X}_{L_{-5/2}(\mathfrak{sl}_4)} \times_{\mathfrak{sl}_4} \mathscr{S}_f$ should be isomorphic. In regard with this, the latter should be irreducible. We formulate the following conjecture.

Conjecture 5.3.2. *The variety $\widetilde{X}_{C(2)}$ is isomorphic to $\overline{G^\natural.\tilde{\rho}(\mathbb{C}^*)v}$ where v is described in Proposition 5.3.1.*

Another evidence to support this conjecture comes from Remark 5.1.6: since $\tilde{X}_{L_{-5/2}(\mathfrak{sl}_4)}$ is not normal, it is also the case of $\tilde{X}_{H_f^0(L_{-5/2}(\mathfrak{sl}_4))}$ because the Slodowy slice $\mathscr{S}_f$ is transversal. Hence, both $\tilde{X}_{\mathcal{W}_{-5/2}(\mathfrak{sl}_4,f)}$ and $\overline{G^\natural.\tilde{\rho}(\mathbb{C}^*)v}$ are reduced, irreducible of dimension three and not normal.

An idea to prove Conjecture 5.3.2 is to compute the characters of vertex algebras $H_f^0(L_{-5/2}(\mathfrak{sl}_4))$ and $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f)$. If they are isomorphic both characters should be the same. We plan to go back to this topic in the future.

Chapter 6

Open problems and future works

We already mentioned some questions we expect to work on in the future (see Conjectures 2.2.6 and 5.3.2, and Sect. 5.2.1). In this chapter, we present several additional problems and projects.

Our work on the classification of simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules has paved the way for many interesting questions and applications (Sect. 6.1). We would like to use the OPEs to study in depth the representation theory of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$ and its consequences on $\mathcal{W}$-superalgebras $\mathcal{W}^k(\mathfrak{osp}_{N|2n})$ attached to the complex orthosymplectic Lie superalgebra $\mathfrak{osp}_{N|2n}$ with $N = 1, 2$. For instance, the latter can be written as a coset involving the $\mathcal{W}$-algebra $\mathcal{W}^{k'}(\mathfrak{so}_{2n+1}, f_{\text{subreg}})$. More precisely we have [40]

$$\mathcal{W}^{k'}(\mathfrak{osp}_{2|2n}) \simeq \operatorname{Com}\left(\pi_{\tilde{H}_1}, \mathcal{W}^k(\mathfrak{so}_{2n+1}, f_{\text{subreg}}) \otimes V_{\mathbb{Z}}\right), \tag{6.1}$$

where k and k' are non-critical levels satisfying a certain relation. Since $\mathfrak{sp}_4 \simeq \mathfrak{so}_5$, we could use (6.1) to obtain rationality of new families of $\mathcal{W}$-superalgebras. We plan to use techniques developed by Creutzig and Linshaw [43] to study this kind of structure.

Also, we wish to extend the methods we develop to classify simple modules of exceptional $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$ to other families of rational $\mathcal{W}$-algebras (Sect. 6.2). Since simple $\mathcal{W}_k(\mathfrak{g}, f)$-modules can be viewed as simple $H^0_f(L_k(\mathfrak{g}))$-modules and we plan to study this vertex algebra in a first place. The set of simple modules of the corresponding Zhu's algebra $\mathcal{A}(H^0_f(L_k(\mathfrak{g})))$ is easy to describe using [21, Theorem 4.2]. Moreover, Premet's results on finite $\mathcal{W}$-algebras associated with minimal nilpotent elements [90] make minimal $\mathcal{W}$-algebras good candidates for a first generalization.

The problem presented in Sect. 6.3 is directly related with observations of OPEs of the $\mathcal{W}$-algebra $\mathcal{W}^k(G_2, f_{\text{subreg}})$. We saw in Chap. 2 that, at non admissible level $k = -2$, the simple $\mathcal{W}$-algebra $\mathcal{W}_{-2}(G_2, f_{\text{subreg}})$ is isomorphic to $\mathbb{C}$ (see Corollary 2.2.4). Hence, it is lisse, so its associated variety is reduced to the point $\{f_{\text{subreg}}\}$. As a consequence, f_{subreg} belongs to $X_{H^0_f(L_{-2}(G_2))}$ which corresponds to the intersection $X_{L_{-2}(G_2)} \cap \mathscr{S}_{f_{\text{subreg}}}$, viewed as topological varieties [17]. It follows that the associated variety $X_{L_{-2}(G_2)}$ contains the subregular nilpotent orbit. We conjecture (Conjecture 6.3.2) that it is exactly the closure of the subregular nilpotent orbit $\mathbb{O}_{\text{subreg}}$ of G_2.

Finally, Sect. 6.4 introduces a longer term project dealing with links between certain vertex algebras obtained from $\mathcal{W}$-algebras of type B_n and C_n, $n \geqslant 2$. These two Lie algebras are Langlands duals of each other. It is known [59] that the principal $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{g})$ at non critical level k is isomorphic to the principal $\mathcal{W}$-algebra of its Langlands dual ${}^L\mathfrak{g}$ at a certain level ℓ:

$$\mathcal{W}^k(\mathfrak{g}) \simeq \mathcal{W}^\ell({}^L\mathfrak{g}).$$

This isomorphism is known as the *Feigin-Frenkel duality*. Recently, similar dualities between subregular $\mathcal{W}$-algebras of type A_n and B_n, and principal $\mathcal{W}$-superalgebras $\mathcal{W}^k(\mathfrak{sl}_{1|n})$ and $\mathcal{W}^k(\mathfrak{osp}_{2|2n})$ have been proved by Creutzig, Genra and Nakatsuka [40]. The aim of our project is to extend these results to a larger family of $\mathcal{W}$-algebras. Our candidates are pairs $\{(f,k),(f',k')\}$, where f and f' are nilpotent elements of B_n and C_n respectively, and k and k' are two non critical levels, such that the $\mathcal{W}$-algebras $\mathcal{W}^k(B_n,f)$ and $\mathcal{W}^{k'}(C_n,f')$ have the same conformal weights and central charge (Tables 6.1–6.4).

6.1 Classification of simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules at other levels

As a by-product of our proof of the rationality of $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$ at admissible levels $k=-3+p/q$ with $q=3,4$, we explicit the set of simple modules. We verify that part of our modules are originated from Drinfeld-Sokolov reduction of highest weight $\widehat{\mathfrak{g}}$-modules. We expect that other modules are also with this form. The characters of these modules provide candidates for the corresponding highest weights. Finally, we expect (Conjecture 3.5.7) that for any simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$-module $L(\xi,\chi)$, there is an admissible weight $\lambda_{\xi,\chi}\in\mathfrak{h}^*$ such that:

$$L(\xi,\chi) \simeq H^0_{f_{\text{subreg}}}(\widehat{L}_k(\lambda_{\xi,\chi})).$$

Similar results hold for several other examples of rational $\mathcal{W}$-algebras, for instance when the Lie algebra is $\mathfrak{sl}_n$ [16, 21]. We embed irreducible representations $L(\xi,\chi)$ in $H^0_{f_{\text{subreg}}}(L_k(\mathfrak{sp}_4))$. This module should coincide with a certain Ramond twisted representation. We determine the latter through Zhu's correspondence comparing the conformal dimensions and highest weights.

Moreover, we intend to study the set of simple $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$-module at integer levels. For instance we wish to classify simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules for certain levels k not rational *a priori*: when k is a positive integer or an admissible level not of the form appearing in Theorem 3.1. This project is a joint work with Prof. Dražen Adamović.

The minimal and subregular nilpotent orbits in $\mathfrak{sl}_3$ coincide. In addition the structure of the Bershadsky-Polyakov vertex algebra $\mathcal{W}^k(\mathfrak{sl}_3, f_{\min})$ is relatively close from the one of the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$. Hence, the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$ can be viewed as a natural analogue to $\mathcal{W}^k(\mathfrak{sl}_3, f_{\min})$ for the type C. The latter has been studied in depth during the last few years [3, 4, 5, 57]. As a first step, we intend to extend the method developed in [5] to classify irreducible representations of $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$. By direct computation, we determine two singular vectors of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$ when k is an integer bigger than -1 or when $k=-3+(m+2)/3$, with $m\geqslant 1$.

Lemma 6.1.1. *Let $k=m-2$ or $k=-3+(m+2)/3$ with m a positive integer. Then*

$$(G^+_{-2})^m|0\rangle,\ (G^-_{-2})^m|0\rangle$$

are singular vectors of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$.

We plan to project these singular vectors in the Zhu's algebra $\mathcal{A}(\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}}))$, and use Zhu's correspondence to find a necessary condition on the form of the simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules. We expect to construct an infinite set of non-isomorphic simple modules as it was done in [5].

The classification of the simple modules when k is an admissible level not of the previous forms is probably more complicated to obtain. This has been treated in the context of the Bershadsky-Polyakov vertex algebra using minimal quantum Hamiltonian reduction [57]. We also plan to study this technique to determine if it can be relevant in our case.

For any levels k, the Bershadsky-Polyakov vertex algebra $\mathcal{W}^k(\mathfrak{sl}_3, f_{\min})$ is a vertex subalgebra of $\mathcal{W}^k(\mathfrak{sl}_3) \otimes \Pi(0)$, where $\Pi(0)$ is a certain lattice vertex algebra [3]. The realization still holds for simple quotients when $2k+3 \notin \mathbb{Z}_{\geqslant 0} \cup \{-3\}$, and we have an embedding

$$\mathcal{W}_k(\mathfrak{sl}_3, f_{\min}) \hookrightarrow \mathcal{W}_k(\mathfrak{sl}_3) \otimes \Pi(0).$$

The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$ admits a similar realization. For generic k, $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$ is a vertex subalgebra of $\mathcal{W}^k(\mathfrak{sp}_4) \otimes \Pi(0)$ [32]. As for the case of the Bershadsky-Polyakov vertex algebra, we expect this embedding to be preserve at certain levels when we consider the simple quotients. More precisely, we should be able to prove that for a non admissible level $k \neq -3$,

$$\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}}) \hookrightarrow \mathcal{W}_k(\mathfrak{sp}_4) \otimes \Pi(0).$$

We can later use this realization to transfer the classification of irreducible $\mathcal{W}_k(\mathfrak{sp}_4)$-representations to the classification of simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules.

In a different direction, applications to the representation theory of the principal $\mathcal{W}$-superalgebra associated with the Lie superalgebra $\mathfrak{osp}_{1|2n}$ have been brought to our attention during the publication process of article [56]. It seems that, using [43], we can deduce the rationality of $\mathcal{W}_s(\mathfrak{osp}_{1|2n})$ from the one of $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$ for a certain level s depending on k. More precisely, from Theorem 3.1, we obtain

Theorem 6.1.2. *The $\mathcal{W}$-superalgebra $\mathcal{W}_s(\mathfrak{osp}_{1|2n})$ is rational when*

$$k = -3 + \frac{2(n+2)}{3}, \qquad \text{and} \qquad s = -(n+\frac{1}{2}) + \frac{n+2}{2n+1}.$$

Since $\mathfrak{sp}_4 \simeq \mathfrak{so}_5$, we can also deduce properties on principal $\mathcal{W}$-superalgebras associated with the Lie superalgebra $\mathfrak{osp}_{2|2n}$ using the Kazama-Suzuki type coset isomorphisms [40]:

$$\begin{aligned} \mathcal{W}_{k'}(\mathfrak{osp}_{2|2n}) &\simeq \operatorname{Com}\left(\pi_{\tilde{H}_1}, \mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\text{subreg}}) \otimes V_{\mathbb{Z}}\right), \\ \mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\text{subreg}}) &\simeq \operatorname{Com}\left(\pi_{\tilde{H}_2}, \mathcal{W}_{k'}(\mathfrak{osp}_{2|2n}, f_{\text{subreg}}) \otimes V_{\mathbb{Z}\sqrt{-1}}\right), \end{aligned} \tag{6.2}$$

where $V_{\mathbb{Z}}$ and $V_{\mathbb{Z}\sqrt{-1}}$ are two lattice vertex superalgebras, and k and k' are non critical levels satisfying $2(k+2n-1)(k'+n) = 1$. From the classification of simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\text{subreg}})$-modules, we classify irreducible representations of $\mathcal{W}_{k'}(\mathfrak{osp}_{2|2n})$. It is then possible to compute their characters and deduce the *fusion rules* of $\mathcal{W}_{k'}(\mathfrak{osp}_{2|2n})$. They encode the decomposition of the tensor product of two irreducible representations of $\mathcal{W}_{k'}(\mathfrak{osp}_{2|2n})$ in the fusion algebra. Only a few is known about fusion rules of $\mathcal{W}$-algebras [21, 62] and those of $\mathcal{W}$-superalgebras are even more mysterious. So, it would be interesting to have new explicit examples. We thank Naoki Genra for suggesting this problem.

6.2 Classification of simple $\mathcal{W}_k(G_2, f_{\min})$ and $\mathcal{W}_k(\mathfrak{sp}_{2n}, f_{\min})$-modules

Another natural project is to extend the techniques presented in this manuscript to larger families of $\mathcal{W}$-algebras. First, we plan to classify simple modules of minimal $\mathcal{W}$-algebras $\mathcal{W}_k(G_2, f_{\min})$ and $\mathcal{W}_k(\mathfrak{sp}_{2n}, f_{\min})$, when $n \geqslant 2$. Indeed, when (f, k) is an exceptional pair (see Sect. 1.3.2), these $\mathcal{W}$-algebras are rational [43, 86]. However, current proofs of the rationality do not provide an explicit description of the simple modules. In order to classify the simple modules, we will adapt techniques used in Chap. 4. The $\mathcal{W}$-algebras $\mathcal{W}_k(G_2, f_{\min})$ and $\mathcal{W}_k(\mathfrak{sp}_{2n}, f_{\min})$ do not admit a good even grading but we expect to recover even conformal weights by twisting the conformal vector. We expect then that simple modules of $\mathcal{W}_k(\mathfrak{g}, f_{\min})$ at admissible levels are highest weight modules of the form

$$L(\chi, \xi_1, \dots, \xi_\ell),$$

where $\chi, \xi_1, \dots, \xi_\ell \in \mathbb{C}$ are the smallest eigenvalues corresponding to the twisted conformal vector $\tilde{L}$ and field $J^{\{x_1\}}, \dots, J^{\{x_\ell\}}$ defined as in Lemmas 3.1.1 and 4.1.1. Here $\{x_1, \dots, x_\ell\}$ is a basis of $\mathfrak{g}^{f_{\min}} \cap \mathfrak{h}$ respecting the grading induced by $x_0 = 1/2h_{\min}$. Hence,

$$\ell = \begin{cases} 1, & \text{if } \mathfrak{g} = G_2, \\ n-1, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2n}. \end{cases}$$

The modules $L(\chi, \xi_1, \dots, \xi_\ell)$ are also simple $H^0_{f_{\min}}(L_k(\mathfrak{g}))$-modules. According Zhu's correspondence, Arakawa and van Ekeren's article [21, Theorem 4.2] and Premet's result on minimal finite $\mathcal{W}$-algebras [90], we deduce that the latter are in a finite number. The rest of the proof of the classification of simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\min})$-module when k is an admissible level with denominator $q = 2$, should be generalizable too. As a consequence, simple modules $L(\chi, \xi_1, \dots, \xi_\ell)$ would be isomorphic to certain Ramond twisted modules $\mathbf{L}(E_{J_\lambda})$ of $H^0_f(L_k(\mathfrak{g}))$, with $\lambda \in [\mathrm{Pr}^k_\circ]$. In Chap. 3, we saw that part of simple $\mathcal{W}_k(\mathfrak{sp}_4, f_{\mathrm{subreg}})$-modules $L(\xi, \chi)$ are isomorphic to a Ramond twisted module $\mathbf{L}(E_{J_{\lambda - \frac{p}{q}x_0}})$, where the weight λ is admissible and depends on χ and ξ.

According to [21], when f admits a good even grading and λ satisfies certain conditions, the Ramond twisted module $\mathbf{L}(E_{J_{\lambda - \frac{p}{q}x_0}})$ is the Drinfeld Sokolov reduction of the $\widehat{\mathfrak{g}}$-module $H^0_f(\widehat{L}(\lambda))$. Minimal nilpotent elements do not admit an even grading in general. Nonetheless, certain coincidences emphasized in Sect. 4.3 lead us to believe that we can extend this correspondence in this context (Conjecture 4.3.4). So, if we manage to free ourselves from the even parity condition in the minimal case, we hope later to generalize and standardize the construction to all good gradings.

Note that, for the moment, we cannot extend all our techniques to $\mathcal{W}_k(G_2, f_{\mathrm{subreg}})$. Indeed certain tools require having a generator – different from the conformal vector – which acts semisimply. Such a generator arises from a semisimple element in the Lie algebra commuting with f. There is no such element in $\mathcal{W}^k(G_2, f_{\mathrm{subreg}})$ since the subregular nilpotent orbit is distinguished (i.e. $\mathfrak{g}^\natural = 0$). We are still thinking about how to adapt our techniques in this case.

6.3 Associated variety of $L_{-2}(G_2)$

In Chap. 2, we give examples of new collapsing levels. In particular, we observe (Corollary 2.2.4) that at non admissible level $k = -2$, when f is a subregular nilpotent element of G_2,

$$\mathcal{W}_{-2}(G_2, f) \simeq \mathbb{C}.$$

As a consequence, the $\mathcal{W}$-algebra $\mathcal{W}_{-2}(G_2, f)$ is lisse and its associated variety is

$$X_{\mathcal{W}_{-2}(G_2,f)} = \{f\}.$$

It follows, since $X_{\mathcal{W}_{-2}(G_2,f)}$ contains the intersection $X_{L_{-2}(G_2)} \cap \mathscr{S}_f$ (see Sect. 1.3.2), that the closure of the subregular nilpotent orbit of G_2 is included in the associated variety $X_{L_{-2}(G_2)}$.

According to the long-standing conjecture of Kac-Wakimoto [78], provided $H^0_f(L_k(\mathfrak{g}))$ is non zero, it is isomorphic to $\mathcal{W}_k(\mathfrak{g}, f)$. In particular, if the conjecture holds for $k = -2$ and $\mathfrak{g} = G_2$, we are able to prove the reverse inclusion.

Proposition 6.3.1. *Let f be a subregular nilpotent element of G_2. If $\mathcal{W}_{-2}(G_2, f) \simeq H^0_f(L_{-2}(G_2))$, then the associated variety of the simple affine vertex algebra $L_{-2}(G_2)$ is the closure of the subregular nilpotent orbit of G_2:*

$$X_{L_{-2}(G_2)} = \overline{\mathbb{O}}_f.$$

Proof. Suppose $\mathcal{W}_{-2}(G_2, f) \simeq H^0_f(L_{-2}(G_2)$. On the one hand, the associated variety of $H^0_f(L_{-2}(G_2))$ is the intersection of $X_{L_{-2}(G_2)}$ with the Slodowy slice $\mathscr{S}_f$ [17], whence

$$X_{H^0_f(L_{-2}(G_2)} = X_{L_{-2}(G_2)} \cap \mathscr{S}_f = \{f\}.$$

As a consequence, the associated variety $X_{L_{-2}(G_2)}$ contains f. Moreover, it is closed and G-invariant. Hence,

$$\overline{\mathbb{O}}_f \subset X_{L_{-2}(G_2)}.$$

On another hand, the associated variety $X_{L_{-2}(G_2)}$ is included in the nilpotent cone of G_2. Indeed, suppose there exists a non nilpotent element $x \in X_{L_{-2}(G_2)}$. Denote $x = x_n + x_s$ its Jordan decomposition with x_n nilpotent and $x_s \neq 0$ semisimple. Then the closure $C(x) = \overline{G.\mathbb{C}^* x}$ of the adjoint orbit of x is included in the associated variety:

$$C(x) \subset X_{L_{-2}(G_2)}.$$

According to Theorem 5.1.5, $C(x)$ contains the induced nilpotent orbit $\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{x_s}}(\mathbb{O}_{x_n})$ from the adjoint orbit of x_n in $\mathfrak{g}^{x_s}$. The only induced nilpotent orbits in G_2 are the regular and subregular orbits [68, 87], so $C(x)$ strictly contains the subregular nilpotent orbit. The variety $C(x)$ is G-invariant, reduced and irreducible. Thus [67, Corollary 1.3.8],

$$\dim(C(x) \cap \mathscr{S}_f) = \dim C(x) - \dim \overline{\mathbb{O}}_f > 0 = \dim(X_{L_{-2}(G_2)} \cap \mathscr{S}_f),$$

whence a contradiction. At this point, we have,

$$\overline{\mathbb{O}}_f \subset X_{L_{-2}(G_2)} \subset \mathcal{N}.$$

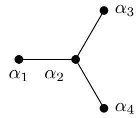
Thus, $X_{L_{-2}(G_2)}$ is the closure of the regular or the subregular nilpotent orbit of G_2. However, the intersection between the nilpotent cone and the Slodowy slice $\mathscr{S}_f$ is two-dimensional. Hence, $X_{L_{-2}(G_2)} = \overline{\mathbb{O}}_f$. □

In view of this, it is reasonable to conjecture the following.

Conjecture 6.3.2. *Let f be a subregular nilpotent element of G_2, then*

$$X_{L_{-2}(G_2)} = \overline{\mathbb{O}}_f.$$

In addition, the conjecture is coherent with the strong connection between Lie algebras G_2 and D_4. Consider the Lie algebra D_4. Its Dynkin diagram is given by



where $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, 2, 3$ and $\alpha_4 = \epsilon_3 + \epsilon_4$ are the simple roots of D_4. The symmetric group $\mathfrak{S}_3$ can be viewed as a subgroup of the group of automorphisms of the Dynkin diagram of D_4. It is generated by a translation τ and a 3-cycle σ satisfying

$$\tau(\epsilon_3 - \epsilon_4) = \epsilon_3 + \epsilon_4,$$
$$\sigma(\epsilon_1 - \epsilon_2) = \epsilon_3 + \epsilon_4, \ \ \sigma(\epsilon_3 + \epsilon_4) = \epsilon_3 - \epsilon_4.$$

The symmetric group $\mathfrak{S}_3$ acts on the set of simple roots, so on the corresponding root vectors. Finally, it acts on D_4 and the Lie algebra $D_4^{\mathfrak{S}_3}$ of the $\mathfrak{S}_3$-invariants in D_4 is isomorphic to G_2 [81]. The embedding $G_2 \overset{\iota}{\hookrightarrow} D_4$ induces a comorphism $\iota^* : D_4^* \to G_2^*$ given by the restriction. We get a G_2-module map $\pi : D_4 \to G_2$ by identifying each Lie algebra with its dual through the Killing form. According to [81], π maps the closure of the minimal nilpotent orbit $\mathbb{O}_{\text{min}}^{D_4}$ in D_4 to the one of the subregular nilpotent orbit $\mathbb{O}_{\text{subreg}}^{G_2}$ of G_2:

$$\pi(\overline{\mathbb{O}_{\text{min}}^{D_4}}) = \overline{\mathbb{O}_{\text{subreg}}^{G_2}}.$$

By a fortunate coincidence, $X_{L_{-2}(D_4)}$ is the closure of the minimal nilpotent orbit of D_4 [28]:

$$X_{L_{-2}(D_4)} = \overline{\mathbb{O}}_{\text{min}}.$$

So, if Conjecture 6.3.2 is true, we have

$$X_{L_{-2}(D_4^{\mathfrak{S}_3})} \simeq X_{L_{-2}(G_2)} \simeq \overline{\mathbb{O}_{\text{subreg}}^{G_2}} = \pi(\overline{\mathbb{O}_{\text{min}}^{D_4}}) = \pi(X_{L_{-2}(D_4)}).$$

Furthermore, the action of $\mathfrak{S}_3$ on D_4 induces an action on the affine vertex algebra $V^{-2}(D_4)$. This action is preserved by taking the simple quotient $L_{-2}(D_4)$. This should be verifiable considering the three singular vectors given in [28] which generate the maximal ideal of $V^{-2}(D_4)$. Then we can consider the orbifold $L_{-2}(D_4)^{\mathfrak{S}_3}$, i.e. the vertex subalgebra of the $\mathfrak{S}_3$-invariants in $L_{-2}(D_4)$. It would be interesting to investigate the link between the vertex algebras $L_{-2}(D_4)^{\mathfrak{S}_3}$ and $L_{-2}(G_2)$. We will use the description of $L_{-2}(D_4)$ in [33] to compute directly $L_{-2}(D_4)^{\mathfrak{S}_3}$. The comparison of their associated varieties echoes questions introduced in Sect. 5.2 and [84, Sect. 13]. Consider a vertex algebra V and let G be a finite group of V-automorphisms. Consider the orbifold V^G. Then the action of G on V leads to an action of G on R_V. The inclusion $V^G \hookrightarrow V$ induces a morphism of Poisson algebras $R_{V^G} \to R_V$ whose image is clearly in $(R_V)^G$. In general R_{V^G} and $(R_V)^G$ are non isomorphic [27] but sometimes one recovers a isomorphism at level of reduced rings, so at level of reduced schemes. It would be interesting to know whether the G-action and the associated variety functor commute. Indeed, if $X_{V^G} \simeq (X_V)^G$, the orbifold V^G would inherit certain properties, as C_2-cofiniteness, from the ones of V.

6.4 Duality of Lie algebras and duality of $\mathcal{W}$-algebras

In the early 90s, Feigin and Frenkel [59] showed that the principal $\mathcal{W}$-algebras associated to a Lie algebra $\mathfrak{g}$ and its Langlands dual ${}^L\mathfrak{g}$ are isomorphic

$$\mathcal{W}^k(\mathfrak{g}) \simeq \mathcal{W}^\ell({}^L\mathfrak{g}), \tag{6.3}$$

when k and ℓ are non critical level satisfying

$$r^\vee(k+h^\vee)(\ell+{}^L h^\vee)=1, \tag{6.4}$$

where $r^\vee$ is the lacing number of $\mathfrak{g}$, and $h^\vee$ and ${}^L h^\vee$ are the dual Coxeter numbers of $\mathfrak{g}$ and ${}^L\mathfrak{g}$ respectively. This result also holds for $\mathcal{W}$-superalgebras $\mathcal{W}^k(\mathfrak{osp}_{1|2n}, f_{\mathrm{reg}})$ [66].

Thanks to a `GAP` program (see Appx. D), we compute conformal weights of $\mathcal{W}$-algebras. We discover some pairs of nilpotent elements (f_1, f_2) in Lie algebras $(\mathfrak{g}_1, \mathfrak{g}_2)$ associated to the same list of conformal weights. In most cases, $\mathfrak{g}_1 = B_n$ and $\mathfrak{g}_2 = C_n$. Notice that these two Lie algebras are Langlands duals of each other. In the following, we call a *dual pair*, a pair $(f_1, f_2) \in (B_n, C_n)$ such that the corresponding $\mathcal{W}$-algebras have the same conformal weights. Because of Feigin-Frenkel duality, principal nilpotent elements in B_n and C_n always have the same conformal weights. For small values of n ($n \leqslant 11$), we list below the pairs $(f_1, f_2) \in (B_n, C_n)$ which have same conformal weights (see Tables 6.1–6.4). For $n = 7, 9, 10$, there are not dual pairs distinct from the principal nilpotent elements $(f_{\mathrm{reg}}^{B_n}, f_{\mathrm{reg}}^{C_n})$.

Motivated by the idea to generalize (6.3), we check for which non critical levels (k_1, k_2) the $\mathcal{W}$-algebras $\mathcal{W}^{k_1}(\mathfrak{g}_1, f_1)$ and $\mathcal{W}^{k_2}(\mathfrak{g}_2, f_2)$ have the same central charge. From our examples, we get a relation between k_1 and k_2 similar to (6.4):

$$r^\vee \lambda_{f_1,f_2}(k_1+h^\vee)(k_2+{}^L h^\vee)=\dim\mathfrak{g},$$

where λ_{f_1,f_2} is a positive integer depending on f_1 and f_2. When f_1 and f_2 are principal nilpotent elements of B_n and C_n respectively, $\lambda_{f_1,f_2} = \dim\mathfrak{g}$ and we recover (6.4).

After discussing with Naoki Genra, it appears that the $\mathcal{W}$-algebras $\mathcal{W}^{k_1}(\mathfrak{g}_1, f_1)$ and $\mathcal{W}^{k_2}(\mathfrak{g}_2, f_2)$ satisfying the previous conditions are non isomorphic in general. Indeed, another invariant of the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is the radical of the centralizer $\mathfrak{g}^f$. Those of centralizers $\mathfrak{g}_1^{f_1}$ and $\mathfrak{g}_2^{f_2}$ are not always isomorphic. However, it is still possible that the $\mathcal{W}$-algebras $\mathcal{W}^{k_1}(\mathfrak{g}_1, f_1)$ and $\mathcal{W}^{k_2}(\mathfrak{g}_2, f_2)$ are closely related. Recently, more Feigin-Frenkel type dualities have been found involving cosets [40, 43]. We hope that our pairs are, for example, cosets of each other. If so, we should obtain relations similar to (6.2).

A preliminary step would be to understand how we obtain pairs (f_1, f_2) to predict their apparition. Moreover, we would like to find a rigorous interpretation of the constant λ_{f_1,f_2}. In the same time, we will investigate on the simplest examples applying the usual techniques. The *screening operators* [66] are the main tool to prove Feigin-Frenkel type dualities in [40]. We hope to explore this topic in collaboration with Naoki Genra.

conformal weights	orbit partition in B_5	orbit partition in C_5	λ_{f_1,f_2}
$1^{13}, (\frac{3}{2})^{12}, 2^6$	$2^4, 1^3$	$2^3, 1^4$	1
$1^3, (\frac{3}{2})^2, 2^2, (\frac{5}{2})^2, 3^3, (\frac{7}{2})^2, 4$	$4^2, 3$	$4, 3^2$	6
$1, 2^6, 3^3, 4^3$	$5, 3^2$	$4^2, 2$	7
$2^3, 3, 4^3, 5, 6$	$7, 3, 1$	$6, 4$	15
$2, 4, 6, 8, 10$	11 (principal)	10 (principal)	55

Table 6.1 – Dual pairs (f_1, f_2) in (B_5, C_5)

conformal weights	orbit partition in B_6	orbit partition in C_6	λ_{f_1,f_2}
$1^6, (\frac{3}{2})^6, 2^7, (\frac{5}{2})^6, 3^3$	$3^3, 2^2$	$3^2, 2^3$	3
$1^2, 2^{10}, 3^5, 4^3$	$5, 3^2, 1^2$	$4^2, 2^2$	6
$2, 4, 6, 8, 10, 12$	13 (principal)	12 (principal)	78

Table 6.2 – Dual pairs (f_1, f_2) in (B_6, C_6)

conformal weights	orbit partition in B_8	orbit partition in C_8	λ_{f_1,f_2}
$1^{31}, (3/2)^{30}, 2^{15}$	$2^6, 1^5$	$2^5, 1^6$	1
$1^3, (3/2)^2, 2^2, (5/2)^2, 3^3, (7/2)^2, 4^2, (9/2)^2, 5^3, (11/2)^2, 6^1$	$6^2, 5$	$6, 5^2$	15
$1, 2^6, 3^3, 4^6, 5^3, 6^3$	$7, 5^2$	$6^2, 4$	16
$2^4, 3^2, 4^5, 5^2, 6^3, 7, 8$	$9, 5, 3$	$8, 6, 2$	24
$2^2, 3, 4^3, 5, 6^3, 7, 8^2, 10$	$11, 5, 1$	$10, 6$	40
$2, 4, 6, 8, 10, 12, 14, 16$	17 (principal)	16 (principal)	136

Table 6.3 – Dual pairs (f_1, f_2) in (B_8, C_8)

conformal weights	orbit partition in B_{11}	orbit partition in C_{11}	λ_{f_1,f_2}
$1^{57}, (3/2)^{56}, 2^{28}$	$2^8, 1^7$	$2^7, 1^8$	1
$1^{28}, (3/2)^{36}, 2^{30}, (5/2)^{12}, 3^3$	$3^3, 2^4, 1^6$	$3^2, 2^6, 1^4$	2
$1^{20}, (3/2)^{20}, 2^{21}, (5/2)^{20}, 3^{10}$	$3^5, 2^4$	$3^4, 2^5$	3
$1^{14}, (3/2)^{16}, 2^{20}, (5/2)^{14}, 3^{12}, (7/2)^4, 4$	$4^2, 3^2, 2^4, 1$	$4, 3^4, 2^2, 1^2$	4
$1^{13}, 2^{36}, 3^{18}, 4^6$	$5, 3^5, 1^3$	$4^3, 2^5$	5
$1^{13}, 2^{15}, 3^{21}, 4^9, 5^3$	$5^3, 3, 1^5$	$5^2, 3^4$	8
$1^6, (3/2)^6, 2^7, (5/2)^6, 3^6, (7/2)^6, 4^7, (9/2)^6, 5^3$	$5^3, 4^2$	$5^2, 4^3$	10
$1^2, 2^{13}, 3^{10}, 4^{12}, 5^5, 6^3$	$7, 5^2, 3^2$	$6^2, 4^2, 2$	13
$1^3, (3/2)^2, 2^2, (5/2)^2, 3^3, (7/2)^2, 4^2, (9/2)^2, 5^3, (11/2)^2, 6^2, (13/2)^2, 7^3, (15/2)^2, 8$	$8^2, 7$	$8, 7^2$	28
$1, 2^6, 3^3, 4^6, 5^3, 6^6, 7^3, 8^3$	$9, 7^2$	$8^2, 6$	29
$2^4, 3^2, 4^6, 5^3, 6^5, 7^2, 8^3, 9, 10$	$11, 7, 5$	$10, 8, 4$	37
$2^3, 3, 4^4, 5^2, 6^4, 7^2, 8^3, 9, 10^2, 12$	$13, 7, 3$	$12, 8, 2$	53
$2^2, 4^3, 5, 6^3, 7^1, 8^3, 9, 10^2, 11, 12, 14$	$15, 7, 1$	$14, 8$	77
$2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22$	23 (principal)	22 (principal)	253

Table 6.4 – Dual pairs (f_1, f_2) in (B_{11}, C_{11})

Bibliography

[1] Toshiyuki Abe, Geoffrey Buhl, and Chongying Dong. “Rationality, Regularity, and C_2-Cofiniteness”. In: *Trans. Amer. Math. Soc.* 356 (2004), pp. 3391–3402.

[2] Dražen Adamović, Pierluigi Moseneder Frajria, and Paolo Papi. *On the Semisimplicity of the Category KL_k for Affine Lie Superalgebras.* 2021. arXiv: 2107.12105 [math.RT].

[3] Dražen Adamović, Kazuya Kawasetsu, and David Ridout. “A Realisation of the Bershadsky–Polyakov Algebras and Their Relaxed Modules”. In: *Lett. Math. Phys.* 111.38 (2021).

[4] Dražen Adamović and Ana Kontrec. “Classification of Irreducible Modules for Bershadsky–Polyakov Algebra at Certain Levels”. In: *J. Algebra Appl.* 20.6 (2020).

[5] Dražen Adamović and Ana Kontrec. “Bershadsky-Polyakov Vertex Algebras at Positive Integer Levels and Duality”. In: *Transform. Groups* (2022).

[6] Dražen Adamović, Antun Milas, and Michael Penn. “On Certain $\mathcal{W}$-Algebras of Type $\mathcal{W}_k(sl_4, f)$”. In: *Contemp. Math.* 768 (2021), pp. 151–165.

[7] Dražen Adamović, Ozren Perše, and Ivana Vukorepa. “On the Representation Theory of the Vertex Algebra $L_{-5/2}(sl(4))$”. In: *Commun. Contemp. Math.* (2021).

[8] Dražen Adamović, Victor Kac, Pierluigi Möseneder Frajria, Paolo Papi, and Ozren Perše. “Conformal Embeddings of Affine Vertex Algebras in Minimal $\mathcal{W}$-Algebras II: Decompositions”. In: *Jpn. J. Math.* 12 (2017), pp. 261–315.

[9] Dražen Adamović, Victor Kac, Pierluigi Möseneder Frajria, Paolo Papi, and Ozren Perše. “Conformal Embeddings of Affine Vertex Algebras in Minimal $\mathcal{W}$-Algebras I: Structural Results”. In: *J. Algebra* 500 (2018), pp. 117–152.

[10] Dražen Adamović, Victor Kac, Pierluigi Möseneder Frajria, Paolo Papi, and Ozren Perše. “An Application of Collapsing Levels to the Representation Theory of Affine Vertex Algebras”. In: *Int. Math. Res. Not.* 13 (2020), pp. 4103–4143.

[11] Chunrui Ai and Xingjun Lin. “Module Category and C_2-Cofiniteness of Affine Vertex Operator Superalgebras”. In: *J. Algebra* 595 (2022), pp. 145–164.

[12] Tomoyuki Arakawa. “Representation Theory of Superconformal Algebras and the Kac-Roan-Wakimoto Conjecture”. In: *Duke Math. J.* 130 (2005), pp. 435–478.

[13] Tomoyuki Arakawa. “Representation Theory of $\mathcal{W}$-Algebras”. In: *Invent. math.* 169 (2007), pp. 219–320.

[14] Tomoyuki Arakawa. “Representation Theory of $\mathcal{W}$-Algebras II”. In: Exploring New Structures and Natural Constructions in Mathematical Physics. Vol. 61. Adv. Stud. Pure Math. Math. Soc. Japan, 2011, pp. 51–90.

[15] Tomoyuki Arakawa. “A Remark on the C_2-Cofiniteness Condition on Vertex Algebras”. In: *Math. Z.* 270 (2012), pp. 559–575.

[16] Tomoyuki Arakawa. "Rationality of Bershadsky-Polyakov Vertex Algebras". In: *Comm. Math. Phys.* 323 (2013), pp. 627–633.

[17] Tomoyuki Arakawa. "Associated Varieties of Modules over Kac-Moody Algebras and C_2-Cofiniteness of $\mathcal{W}$-Algebras". In: 22 (2015), pp. 11605–11666.

[18] Tomoyuki Arakawa. "Rationality of $\mathcal{W}$-Algebras: Principal Nilpotent Cases". In: *Ann. Math.* 182 (2015), pp. 565–604.

[19] Tomoyuki Arakawa. "Rationality of Admissible Affine Vertex Algebras in the Category $\mathcal{O}$". In: *Duke Math. J.* 165 (2016), pp. 67–93.

[20] Tomoyuki Arakawa. *Chiral Algebras of Class $\mathcal{S}$ and Moore-Tachikawa Symplectic Varieties.* 2018. arXiv: 1811.01577 [math.RT].

[21] Tomoyuki Arakawa and Jethro van Ekeren. "Rationality and Fusion Rules of Exceptional $\mathcal{W}$-Algebras". In: *J. Eur. Math. Soc.* (To appear). arXiv: 1905.11473 [math.RT].

[22] Tomoyuki Arakawa, Jethro van Ekeren, and Anne Moreau. *Singularities of Nilpotent Slodowy Slices and Collapsing Levels of $\mathcal{W}$-Algebras.* 2022. arXiv: 2102.13462 [math.RT].

[23] Tomoyuki Arakawa and Edward Frenkel. "Quantum Langlands Duality of Representations of $\mathcal{W}$-Algebras". In: *Compos. Math.* 155 (2019), pp. 2235–2262.

[24] Tomoyuki Arakawa, Vyacheslav Futorny, and Libor Křižka. *Work in Progress.*

[25] Tomoyuki Arakawa, Cuipo Jiang, and Anne Moreau. "Simplicity of Vacuum Modules and Associated Varieties". In: *J. Ec. Polytech. - Math* 8 (2021), pp. 169–191.

[26] Tomoyuki Arakawa and Kazuya Kawasetsu. "Quasi-Lisse Vertex Algebras and Modular Linear Differential Equations". In: *Lie Groups, Geometry, and Representation Theory: A Tribute to the Life and Work of Bertram Kostant.* Ed. by Victor Kac and Vladimir Popov. Progress in Mathematics. Cham: Springer International Publishing, 2018, pp. 41–57.

[27] Tomoyuki Arakawa and Andrew Linshaw. "Singular Support of a Vertex Algebra and the Arc Space of Its Associated Scheme". In: *Representations and Nilpotent Orbits of Lie Algebraic Systems.* Ed. by Maria Gorelik, Vladimir Hinich, and Anna Melnikov. Vol. 330. Progress in Mathematics. Cham: Springer International Publishing, 2019, pp. 1–17.

[28] Tomoyuki Arakawa and Anne Moreau. "Joseph Ideals and Lisse Minimal $\mathcal{W}$-Algebras". In: *J. Inst. Math. Jussieu* 17 (2015), pp. 397–417.

[29] Tomoyuki Arakawa and Anne Moreau. "Sheets and Associated Varieties of Affine Vertex Algebras". In: *Adv. Math.* 320 (2017), pp. 157–209.

[30] Tomoyuki Arakawa and Anne Moreau. "Arc Spaces and Chiral Symplectic Cores". In: *Publ. Res. Inst. Math. Sci.* 57 (2021), pp. 795–829.

[31] Tomoyuki Arakawa and Anne Moreau. *Arc Spaces and Vertex Algebras.* CEMPI, Lecture Series. Springer, In preparation.

[32] Christopher Beem and Carlo Meneghelli. "Geometric Free Field Realization for the Genus-Two Class $\mathcal{S}$ Theory of Type $\mathfrak{a}_1$". In: *Phys. Rev. D* 104 (2021).

[33] Christopher Beem, Carlo Meneghelli, and Leonardo Rastelli. "Free Field Realizations from the Higgs Branch". In: *J. High Energ. Phys.* 58 (2019).

[34] Christopher Beem, Wolfger Peelaers, Leonardo Rastelli, and BaltC. van Rees. "Chiral Algebras of Class $\mathcal{S}$". In: *J. High Energ. Phys.* 20.5 (2015).

[35] Christopher Beem, Madalena Lemos, Pedro Liendo, Wolfger Peelaers, Leonardo Rastelli, and Balt C. van Rees. "Infinite Chiral Symmetry in Four Dimensions". In: *Comm. Math. Phys.* 336 (2015), pp. 1359–1433.

[36] Michael Bershadsky. “Conformal Field Theories via Hamiltonian Reduction”. In: *Comm. Math. Phys.* 139 (1991), pp. 71–82.

[37] Richard Borcherds. “Vertex Algebras, Kac-Moody Algebras, and the Monster”. In: *Proc. Natl. Acad. Sci. U.S.A.* 83 (1986), pp. 3068–3071.

[38] Jean-Yves Charbonnel and Anne Moreau. “The Index of Centralizers of Elements of Reductive Lie Algebras”. In: *Doc. Math.* 15 (2010), pp. 387–422.

[39] David Collingwood and William McGovern. *Nilpotent Orbits in Semisimple Lie Algebra: An Introduction*. Vol. 65. New York: Van Nostrand Reinhold Co., 1993.

[40] Thomas Creutzig, Naoki Genra, and Shigenori Nakatsuka. “Duality of Subregular $\mathcal{W}$-Algebras and Principal $\mathcal{W}$-Superalgebras”. In: *Adv. Math.* 383.107685 (2021).

[41] Thomas Creutzig and Andrew Linshaw. “Cosets of the $\mathcal{W}^k(\mathfrak{sl}_4, f_{subreg})$-Algebra”. In: *Contemp. Math.* 711 (2018), pp. 105–117.

[42] Thomas Creutzig and Andrew Linshaw. “Cosets of Affine Vertex Algebras inside Larger Structures”. In: *J. Algebra* 517 (2019), pp. 396–438.

[43] Thomas Creutzig and Andrew Linshaw. *Trialities of Orthosymplectic $\mathcal{W}$-Algebras*. 2021. arXiv: 2102.10224 [math.RT].

[44] Thomas Creutzig and Andrew Linshaw. “Trialities of $\mathcal{W}$-Algebras”. In: *Cambridge J. Math.* 10.1 (2022), pp. 69–194.

[45] Thomas Creutzig, Shashank Kanade, Andrew R. Linshaw, and David Ridout. “Schur-Weyl Duality for Heisenberg Cosets”. In: *Transform. Groups* 24 (2019), pp. 301–354.

[46] Thomas Creutzig, Naoki Genra, Shigenori Nakatsuka, and Ryo Sato. *Correspondences of Categories for Subregular $\mathcal{W}$-Algebras and Principal $\mathcal{W}$-Superalgebras*. 2021. arXiv: 2104.00942 [math.RT].

[47] Alessandro D’Andrea, Corrado De Concini, Alberto De Sole, Reimundo Heluani, and Victor Kac. “Three Equivalent Definitions of Finite $\mathcal{W}$-algebras”. In: *appendix to [48]* (2006).

[48] Alberto De Sole and Victor Kac. “Finite vs Affine $\mathcal{W}$-Algebras”. In: 1 (2006), pp. 137–261.

[49] Lance Dixon, Jeffrey Harvey, Cumrun Vafa, and Edward Witten. “Strings on Orbifolds”. In: *Nucl. Phys. B.* 261 (1985), pp. 678–686.

[50] Lance Dixon, Jeffrey Harvey, Cumrun Vafa, and Edward Witten. “Strings on Orbifolds (II)”. In: *Nucl. Phys. B.* 274.2 (1986), pp. 285–314.

[51] Chongying Dong, Haisheng Li, and Geoffrey Mason. “Twisted Representations of Vertex Operator Algebras”. In: *Math. Ann.* 310 (1998), pp. 571–600.

[52] Chongying Dong and Xingjun Lin. “Unitary Vertex Operator Algebras”. In: *J. Algebra* 397 (2014), pp. 252–277.

[53] Chongying Dong and Geoffrey Mason. “Integrability of C_2-Cofinite Vertex Operator Algebras”. In: *Int. Math. Res. Not.* 80468 (2006).

[54] Jethro van Ekeren and Reimundo Heluani. “A Short Construction of the Zhu Algebra”. In: *J. Algebra* 528 (2019), pp. 85–95.

[55] Alexander Elashvili and Victor Kac. “Classification of Good Gradings of Simple Lie Algebras”. In: *Lie Groups and Invariant Theory*. Ernest Vinberg Editor. Vol. 213. Amer. Math. Soc. Transl. Ser. 2. American Mathematical Society, 2005, pp. 85–104.

[56] Justine Fasquel. “Rationality of the Exceptional $\mathcal{W}$-Algebras $\mathcal{W}_k(\mathfrak{sp}_4, f_{subreg})$ Associated with Subregular Nilpotent Elements of $\mathfrak{sp}_4$”. In: *Commun. Math. Phys.* 390 (2022), pp. 33–65.

[57] Zachary Fehily, Kazuya Kawasetsu, and David Ridout. “Classifying Relaxed Highest-Weight Modules for Admissible-Level Bershadsky–Polyakov Algebras”. In: *Commun. Math. Phys.* 385 (2021), pp. 859–904.

[58] Boris Feigin and Edward Frenkel. “Quantization of the Drinfeld-Sokolov Reduction”. In: *Phys. Lett. B* 246 (1990), pp. 75–81.

[59] Boris Feigin and Edward Frenkel. “Duality in $\mathcal{W}$-Algebras”. In: 1991.6 (1991), pp. 75–82.

[60] Edward Frenkel. *Langlands Correspondence for Loop Groups*. Vol. 103. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2007.

[61] Edward Frenkel and David Ben-Zvi. *Vertex Algebras and Algebraic Curves*. Second. Vol. 88. Mathematical Surveys and Monographs. Amer. Math. Soc., 2004.

[62] Edward Frenkel, Victor Kac, and Minoru Wakimoto. “Characters and Fusion Rules for $\mathcal{W}$-Algebras via Quantized Drinfeld-Sokolov Reduction”. In: *Comm. Math. Phys.* 147 (1992), pp. 295–328.

[63] Igor Frenkel and Yongchang Zhu. “Vertex Operator Algebras Associated to Representations of Affine and Virasoro Algebras”. In: *Duke Math. J.* 66 (1992), pp. 123–168.

[64] Davide Gaiotto and Miroslav Rapčák. “Vertex Algebras at the Corner”. In: *J. High Energ. Phys.* 2019.160 (2019).

[65] Dennis Gaitsgory. *Quantum Langlands Correspondence*. 2016. arXiv: 1601.05279 [math.AG].

[66] Naoki Genra. “Screening Operators for $\mathcal{W}$-Algebras”. In: *Sel. Math. New Ser.* 23 (2017), pp. 2157–2202.

[67] Victor Ginzburg. “Harish-Chandra Bimodules for Quantized Slodowy Slices”. In: *Represent. Theory* 13.12 (2009), pp. 236–271.

[68] Willem de Graaf and Alexander Elashvili. “Induced Nilpotent Orbits of the Simple Lie Algebras of Exceptional Type”. In: *Georgian Math. J.* 16 (2009), pp. 257–278.

[69] Willem de Graaf and GAP Team. *SLA, Computing with Simple Lie Algebras, Version 1.5.3*. 2019.

[70] Victor Kac. *Infinite-Dimensional Lie Algebras*. 3rd ed. Cambridge: Cambridge University Press, 1990.

[71] Victor Kac. *Vertex Algebras for Beginners*. Vol. 10. University Lecture Series. American Mathematical Society, 1998.

[72] Victor Kac. *Introduction to Vertex Algebras, Poisson Vertex Algebras, and Integrable Hamiltonian PDE*. Ed. by Filippo Callegaro, Giovanna Carnovale, Fabrizio Caselli, Corrado De Concini, and Alberto De Sole. Vol. 19. Springer INdAM Series. Cham: Springer International Publishing, 2017.

[73] Victor Kac, Shi-Shyr Roan, and Minoru Wakimoto. “Quantum Reduction for Affine Superalgebras”. In: *Comm. Math. Phys.* 241 (2003), pp. 307–342.

[74] Victor Kac and Minoru Wakimoto. “Modular Invariant Representations of Infinite-Dimensional Lie Algebras and Superalgebras”. In: *Proc. Natl. Acad. Sci. U.S.A.* 85.14 (1988), pp. 4956–4960.

[75] Victor Kac and Minoru Wakimoto. “Classification of Modular Invariant Representations of Affine Algebras”. In: *Infinite-Dimensional Lie Algebras and Groups (Luminy-Marseille, 1988).* Vol. 7. Adv. Ser. Math. Phys. Teaneck, NJ: World Sci. Publ., 1989, pp. 138–177.

[76] Victor Kac and Minoru Wakimoto. “Quantum Reduction and Representation Theory of Superconformal Algebras”. In: *Adv. Math.* 185 (2004), pp. 400–458.

[77] Victor Kac and Minoru Wakimoto. “Quantum Reduction in the Twisted Case”. In: *Infinite Dimensional Algebras and Quantum Integrable Systems*. Ed. by Petr Kulish, Nenad Manojlovich, and Henning Samtleben. Progress in Mathematics. Basel: Birkhäuser, 2005, pp. 89–131.

[78] Victor Kac and Minoru Wakimoto. “On Rationality of $\mathcal{W}$-Algebras”. In: *Transf. Groups* 13 (2008), pp. 671–713.

[79] Shashank Kanade and Andrew Linshaw. “Universal Two-Parameter Even Spin $\mathcal{W}_\infty$-Algebra”. In: *Adv. Math.* 355.106774 (2019).

[80] Kazuya Kawasetsu. “$\mathcal{W}$-Algebras with Non-Admissible Levels and the Deligne Exceptional Series”. In: 2018.3 (2018), pp. 641–676.

[81] Thierry Levasseur and S. Paul Smith. “Primitive Ideals and Nilpotent Orbits in Type G_2”. In: *J. Algebra* 114 (1988), pp. 81–105.

[82] Haisheng Li. “The Physics Superselection Principle in Vertex Operator Algebra Theory”. In: *J. Algebra* 196 (1997), pp. 436–457.

[83] Haisheng Li. “Abelianizing Vertex Algebras”. In: *Commun. Math. Phys.* 259 (2005), pp. 391–411.

[84] Bong H. Lian and Andrew Linshaw. *Vertex Algebras and Commutative Algebras*. 2021. arXiv: 2107.03243 [math.QA].

[85] George Lusztig and Nicolas Spaltenstein. “Induced Unipotent Classes”. In: *J. Lond. Math. Soc.* 19 (1979), pp. 41–52.

[86] Robert McRae. *On Rationality for C_2-Cofinite Vertex Operator Algebras*. 2021. arXiv: 2108.01898 [math.QA].

[87] Anne Moreau and Rupert Yu. “Jet Schemes of the Closure of Nilpotent Orbits”. In: *Pacific J. Math.* 281 (2016), pp. 137–183.

[88] Alexander Polyakov. “Gauge Transformations and Diffeomorphisms”. In: *Int. J. Mod. Phys. A* 5.5 (1990), pp. 833–842.

[89] Alexander Premet. “Special Transverse Slices and Their Enveloping Algebras”. In: *Adv. Math.* 170 (2002), pp. 1–55.

[90] Alexander Premet. “Enveloping Algebras of Slodowy Slices and the Joseph Ideal”. In: *J. Eur. Math. Soc.* 9 (2007), pp. 487–543.

[91] Roger Richardson. “Normality of G-Stable Subvarieties of a Semisimple Lie Algebra”. In: *Algebraic Groups Utrecht 1986*. Ed. by Arjeh Cohen, Wim Hesselink, Wilberd vander Kallen, and Jan Strooker. Vol. 1271. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1987, pp. 243–264.

[92] Jaewon Song, Dan Xie, and Wenbin Yan. “Vertex Operator Algebras of Argyres-Douglas Theories from $M5$-Branes”. In: *J. High Energ. Phys.* 12 (2017).

[93] Patrice Tauvel and Rupert Yu. *Lie Algebras and Algebraic Groups*. Springer Monographs in Mathematics. Berlin, Heidelberg: Springer, 2005.

[94] Krivono Thielemans. “A Mathematica™ package for Computing Operator Product Expansions”. In: *Int. J. Mod. Phys. C* 2 (1991), pp. 787–798.

[95] Dan Xie and Wenbin Yan. “$\mathcal{W}$-Algebras, Cosets and VOAs for 4d $\mathcal{N}=2$ SCFTs from $M5$ Branes”. In: *J. High Energ. Phys.* 4.76 (2021).

[96] Alexandre Zamolodchikov. “Infinite Additional Symmetries in Two-Dimensional Conformal Quantum Field Theory”. In: *Theor. Math. Phys.* 65.3 (1985), pp. 1205–1213.

[97] Yongchang Zhu. “Modular Invariance of Characters of Vertex Operator Algebras”. In: *J. Amer. Math. Soc.* 9 (1996), pp. 237–302.

Appendix A

$\mathcal{W}$-algebras associated with nilpotent elements of $\mathfrak{sl}_3$

The simple Lie algebra $\mathfrak{g} := \mathfrak{sl}_3$ of type A_2 may be realized as the set of 3-size square matrices with trace zero. As a Cartan subalgebra $\mathfrak{h}$ fix the set of diagonal matrices of $\mathfrak{g}$. The Lie algebra $\mathfrak{sl}_3$ admits three nilpotent orbits which are parametrized by the partitions of 3: (3), $(2,1)$, (1^3). They correspond respectively to the principal, the minimal and the trivial nilpotent orbits. Recall that in this particular case, the minimal nilpotent orbit is also the subregular one. Let $\Pi = \{\alpha_1, \alpha_2\}$ be a set of simple roots for the root system Δ corresponding with our choice of $\mathfrak{h}$. We denote $h_i := \alpha_i^\vee$, for $i = 1, 2$. The dual Coxeter number of $\mathfrak{sl}_3$ is $h^\vee = 3$. In the following, we give the OPEs between the strong generators of $\mathcal{W}^k(\mathfrak{sl}_3, f_{\min})$ and $\mathcal{W}^k(\mathfrak{sl}_3)$ at level $k \neq -3$. These two $\mathcal{W}$-algebras are among the first to have been described explicitly. They are respectively called the Bershadsky-Polyakov vertex algebra [36, 88] and the Zamolodchikov vertex algebra [96].

A.1 Generators and OPEs of the Bershadsky-Polyakov vertex algebra $\mathcal{W}^k(\mathfrak{sl}_3, f_{\min})$

In [16], Arakawa give an explicit description of the Bershadsky-Polyakov vertex algebra which corresponds to the $\mathcal{W}$-algebra associated with a minimum nilpotent element of $\mathfrak{sl}_3$.

The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_3, f_{\min})$ is strongly generated by the fields $J(z)$, $G^\pm(z)$ and $L(z)$ corresponding respectively to vectors $h_1 - h_2$, $e_{-\alpha_1}$, $e_{-\alpha_2}$ and $f := f_{\min} = e_{-\alpha_1} + e_{-\alpha_2}$ which form a basis of the centralizer $\mathfrak{g}^f$. These fields satisfy the following relations:

$$\begin{aligned}
J(z)J(w) &\sim \frac{(3+2k)}{(z-w)^2},\\
J(z)G^\pm(w) &\sim \pm\frac{1}{(z-w)}G^\pm(w),\\
L(z)L(w) &\sim \frac{c_k}{2(z-w)^4} + \frac{2}{(z-w)^2}L(w) + \frac{1}{(z-w)}\partial L(w),\\
L(z)G^\pm(w) &\sim \frac{3}{2(z-w)^2}G^\pm(w) + \frac{1}{(z-w)}\partial G^\pm(w),
\end{aligned}$$

$$
\begin{aligned}
L(z)J(w) \sim& \frac{1}{(z-w)^2}J(w) + \frac{1}{(z-w)}\partial J(w),\\
G^{\pm}(z)G^{\pm}(w) \sim& 0,\\
G^{+}(z)G^{-}(w) \sim& -\frac{(1+k)(3+2k)}{(z-w)^3} + \frac{3(1+k)}{(z-w)^2}J(w)\\
&+\frac{1}{(z-w)}\left(-(3+k)L(w) + 3 : J(w)^2 : -\frac{3(1+k)}{2}\partial J(w)\right),
\end{aligned}
$$

where

$$
c_k = -\frac{(1+3k)(3+2k)}{3+k}.
$$

A.2 Generators and OPEs of the Zamolodtchikov vertex algebra $\mathcal{W}^k(\mathfrak{sl}_3)$

Set $f := f_{\mathrm{reg}} = 2(e_{-\alpha_1} + e_{-\alpha_2})$, $e = 2(e_{\alpha_1} + e_{\alpha_2})$ and $h = 2(h_1 + h_2)$. The centralizer of f in $\mathfrak{g}$ is two-dimensional:

$$
\mathfrak{g}^f = \mathbb{C}f \oplus \mathbb{C}e_{-\theta},
$$

with $e_{-\theta} \in \mathfrak{g}_{-3}$.

The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_3)$ is strongly generated by the fields $L(z)$ and $W(z)$ satisfying the OPEs:

$$
\begin{aligned}
L(z)L(w) \sim& \frac{c_k}{2(z-w)^4} + \frac{2}{(z-w)^2}L(w) + \frac{1}{(z-w)}\partial L(w),\\
L(z)W(w) \sim& \frac{4}{(z-w)^2}W(w) + \frac{1}{(z-w)}\partial W(w),\\
W(z)W(w) \sim& \frac{w_k c_k}{3(z-w)^6} + \frac{2w_k}{(z-w)^4}L(w) + \frac{w_k}{2(z-w)^3}\partial L(w)\\
&+\frac{1}{(z-w)^2}\left(\frac{2(3+k)^2}{3} : L(w)^2 : -\frac{3(3+k)^2(2+k)^2}{4}\partial^2 L(w)\right)\\
&+\frac{1}{(z-w)}\left(\frac{2(3+k)^3}{3} : L(w)\partial L(w) : -\frac{(3+k)^2(18+14k+3k^2)}{18}\partial^3 L(w)\right),
\end{aligned}
$$

where

$$
c_k = -\frac{2(5+3k)(9+4k)}{3+k},
$$

and

$$
w_k = -\frac{(3+k)^2(4+3k)(12+5k)}{6}.
$$

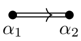
Appendix B

$\mathcal{W}$-algebras associated with nilpotent elements of $\mathfrak{sp}_4$

The simple Lie algebra $\mathfrak{g} := \mathfrak{sp}_4$ may be realized as the set of 4-size square matrices x such that $x^T J_4 + J_4 x = 0$, where J_4 is the anti-diagonal matrix given by

$$J_4 = \begin{pmatrix} 0 & U_2 \\ -U_2 & 0 \end{pmatrix}, \qquad \text{where } U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We make the standard choice that $\mathfrak{h}$ is the set of diagonal matrices of $\mathfrak{g}$. Nilpotent orbits of $\mathfrak{g}$ are parameterized by the partitions of 4 such that the number of parts of each odd number is even [39, Theorem 5.1.3]. Thus there are four nilpotent orbits in $\mathfrak{g}$ corresponding to the following partitions: (4), (2^2), $(2, 1^2)$, (1^4). They correspond, respectively, to the principal, the subregular, the minimal and the zero nilpotent orbits of $\mathfrak{g}$, with respective dimensions 8, 6, 4, 0.

Write $\Pi = \{\alpha_1, \alpha_2\}$ a set of simple roots for the root system Δ of $(\mathfrak{g}, \mathfrak{h})$ such that α_1 is a long root and α_2 is short. Then $\Delta_+ = \{\alpha_1, \alpha_2, \eta, \theta\}$, with $\eta := \alpha_1 + \alpha_2$, and $\theta := \alpha_1 + 2\alpha_2$ the highest positive root. We denote $h_i := \alpha_i^\vee \in (\mathfrak{h}^*)^* \cong \mathfrak{h}$, for $i = 1, 2$. The Lie algebra of $\mathfrak{sp}_4$ is of type $B_2 \simeq C_2$. The previous choice of root system fixes the Dynkin diagram  .

The dual Coxeter number of $\mathfrak{sp}_4$ is $h^\vee = 3$. In the following, we give the OPEs between strong generators of $\mathcal{W}$-algebras associated with nilpotent elements of $\mathfrak{sp}_4$ at level $k \neq -3$.

B.1 Generators and OPEs of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\min})$

Let $f := f_{\min} = e_{-\theta}$ be a minimal nilpotent element of $\mathfrak{sp}_4$, $e = e_\theta$ and $h = [e, f] = \theta$. Then (e, h, f) is an $\mathfrak{sl}_2$-triple. The centralizer of f is six-dimensional generated by $e_{\alpha_1}, e_{-\alpha_1}, e_{-\alpha_2}, e_{-\eta}, e_{-\theta}, h_1$. We have the decomposition

$$\mathfrak{g}^f = \mathfrak{g}^f_{-1} \oplus \mathfrak{g}^f_{-1/2} \oplus \mathfrak{g}^f_0,$$

where

$$\mathfrak{g}^f_{-1} = \mathbb{C}f, \qquad \mathfrak{g}^f_{-1/2} = \mathbb{C}e_{-\alpha_2} \oplus \mathbb{C}e_{-\eta}, \qquad \text{and} \qquad \mathfrak{g}^f_0 = \mathbb{C}e_{\alpha_1} \oplus \mathbb{C}h_1 \oplus \mathbb{C}e_{-\alpha_1} \simeq \mathfrak{sl}_2.$$

The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sp}_4, f_\theta)$ is strongly generated by the fields $J(z)$, $G^\pm(z)$, $F^\pm(z)$, and $L(z)$

satisfying the following OPEs:

$$
\begin{aligned}
J(z)J(w) \sim& \frac{(1+2k)}{(z-w)^2},\\
J(z)F^{\pm}(w) \sim& \pm \frac{2}{(z-w)} F^{\pm}(w),\\
F^{\pm}(z)F^{\pm}(w) \sim& 0,\\
F^{+}(z)F^{-}(w) \sim& \frac{(1+2k)}{2(z-w)^2} + \frac{1}{(z-w)} J(w),\\
J(z)G^{\pm}(w) \sim& \pm \frac{1}{(z-w)} G^{\pm}(w),\\
G^{\pm}(z)F^{\pm}(w) \sim& 0,\\
G^{\pm}(z)F^{\mp}(w) \sim& \frac{1}{(z-w)} G^{\mp}(w),\\
L(z)L(w) \sim& \frac{c_k}{2(z-w)^4} + \frac{2}{(z-w)^2} L(w) + \frac{1}{(z-w)} \partial L(w),\\
L(z)G^{\pm}(w) \sim& \frac{3}{2(z-w)^2} G^{\pm}(w) + \frac{1}{(z-w)} \partial G^{\pm}(w),\\
L(z)F^{\pm}(w) \sim& \frac{1}{(z-w)^2} F^{\pm}(w) + \frac{1}{(z-w)} \partial F^{\pm}(w),\\
L(z)J(w) \sim& \frac{1}{(z-w)^2} J(w) + \frac{1}{(z-w)} \partial J(w),\\
G^{\pm}(z)G^{\pm}(w) \sim& \pm \frac{4(2+k)}{(z-w)^2} F^{\pm}(w) \pm \frac{2(2+k)}{(z-w)} \partial F^{\pm}(w),\\
G^{+}(z)G^{-}(w) \sim& \frac{2(1+2k)(2+k)}{(z-w)^3} + \frac{2(2+k)}{(z-w)^2} J(w)\\
&+ \frac{1}{(z-w)} \left(-2(3+k)L(w) + 4 : F^{+}(w)F^{-}(w) : + : J(w)^2 : + k\partial J(w)\right),
\end{aligned}
$$

where

$$c_k = -\frac{3(k+1)(2k+1)}{3+k}.$$

B.2 Generators and OPEs of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$

Let $f := f_{\text{subreg}} = e_{-\eta}$ be a subregular nilpotent element in $\mathfrak{sp}_4$. Setting $e := e_\eta$ and $h := 2h_1 + h_2$, we get an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. The centralizer of f is four-dimensional generated by $e_{-\eta}, e_{-\alpha_1}, e_{-\theta}, h_2$. Moreover,

$$\mathfrak{g}^f = \mathfrak{g}^f_{-1} \oplus \mathfrak{g}^f_0,$$

where

$$\mathfrak{g}^f_{-1} = \mathbb{C}f \oplus \mathbb{C}e_{-\alpha_1} \oplus \mathbb{C}e_{-\theta} \qquad \text{, and} \qquad \mathfrak{g}^f_0 = \mathbb{C}h_2.$$

The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sp}_4, f)$ is strongly generated by the fields $J(z)$, $G^{\pm}(z)$ and $L(z)$ satisfying the OPEs:

$$J(z)J(w) \sim \frac{(2+k)}{(z-w)^2},$$

$$
\begin{aligned}
J(z)G^{\pm}(w) \sim& \pm \frac{1}{(z-w)}G^{\pm}(w),\\
L(z)L(w) \sim& \frac{c_k}{2(z-w)^4} + \frac{2}{(z-w)^2}L(w) + \frac{1}{(z-w)}\partial L(w),\\
L(z)G^{\pm}(w) \sim& \frac{2}{(z-w)^2}G^{\pm}(w) + \frac{1}{(z-w)}\partial G^{\pm}(w),\\
L(z)J(w) \sim& \frac{1}{(z-w)^2}J(w) + \frac{1}{(z-w)}\partial J(w),\\
G^{\pm}(z)G^{\pm}(w) \sim& 0,\\
G^{+}(z)G^{-}(w) \sim& -\frac{3(1+k)(2+k)^2}{(z-w)^4} - \frac{3(1+k)(2+k)}{(z-w)^3}J(w)\\
&+\frac{1}{(z-w)^2}\left((2+k)(3+k)L(w) - (3+2k) :J(w)^2: - \frac{3(1+k)(2+k)}{2}\partial J(w)\right)\\
&+\frac{1}{(z-w)}\left((3+k) :L(w)J(w): + \frac{(3+k)(2+k)}{2}\partial L(w) - :J(w)^3:\right.\\
&\qquad \left. -(3+2k) :J(w)\partial J(w): - \frac{(5+4k+k^2)}{2}\partial^2 J(w)\right),
\end{aligned}
$$

where

$$c_k = -\frac{2(9+16k+6k^2)}{3+k}.$$

B.3 Generators and OPEs of $\mathcal{W}^k(\mathfrak{sp}_4)$

Let $f := f_{\text{reg}} = 4e_{-\alpha_1} + 3e_{-\alpha_2}$, $e := 4e_{\alpha_1} + 3e_{\alpha_2}$ and $h := 4h_1 + 3h_2$. Then f belongs to the regular nilpotent orbit of $\mathfrak{sp}_4$ and the centralizer of f is

$$\mathfrak{g}^f = \mathbb{C}f \oplus \mathbb{C}e_{-\theta},$$

with $e_{-\theta} \in \mathfrak{g}_{-3}$.

The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sp}_4)$ is strongly generated by the fields $L(z)$ and $W(z)$ satisfying the OPEs:

$$
\begin{aligned}
L(z)L(w) \sim& \frac{c_k}{2(z-w)^4} + \frac{2}{(z-w)^2}L(w) + \frac{1}{(z-w)}\partial L(w),\\
L(z)W(w) \sim& \frac{4}{(z-w)^2}W(w) + \frac{1}{(z-w)}\partial W(w),\\
W(z)W(w) \sim& \frac{w_k c_k}{4(z-w)^8} + \frac{2w_k}{(z-w)^6}L(w) + \frac{w_k}{(z-w)^5}\partial L(w)\\
&+\frac{3(3+k)(3+2k)(8+3k)}{(z-w)^4}\big(30(17+12k+2k^2)W(z)\\
+(3+k)(11+5k)&\ (11+6k)(18+7k)(19+8k)(-7(3+k) :L(w)^2: + \frac{(162+139k+30k^2)}{2}\partial^2 L(w))\Big)\\
&+\frac{1}{(z-w)^3}(\ldots) + \frac{1}{(z-w)^2}(\ldots) + \frac{1}{(z-w)}(\ldots),
\end{aligned}
$$

where

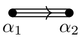
$$c_k = -\frac{2(12+5k)(13+6k)}{3+k},$$

and

$$w_k = (3+k)^2(3+2k)(8+3k)(11+5k)(11+6k)(18+7k)(19+8k)(747+674k+150k^2).$$

Appendix C

$\mathcal{W}$-algebras associated with nilpotent elements of G_2

Consider the Lie algebra $\mathfrak{g} := G_2$ of rank 2. Let $\Pi = \{\alpha_1, \alpha_2\}$ be a set of simple roots for the root system Δ of $\mathfrak{g}$ with α_1 the short root and α_2 the long one. Then $\Delta_+ = \{\alpha_1, \alpha_2, \eta_1, \eta_2, \eta_3, \theta\}$, with $\eta_i := i\alpha_1 + \alpha_2$ for $i = 1, 2, 3$, and $\theta := 3\alpha_1 + 2\alpha_2$ the highest positive root. Denote $h_i := \alpha_i^\vee \in (\mathfrak{h}^*)^* \cong \mathfrak{h}$, for $i = 1, 2$. The previous choice of root system fixes the Dynkin diagram of G_2 

.

The Lie algebra $\mathfrak{g}$ has four non-zero nilpotent orbits, the three canonical nilpotent orbits ($\mathbb{O}_{\min}$, $\mathbb{O}_{\text{subreg}}$, and $\mathbb{O}_{\text{reg}}$) of dimension 6, 10 and 12, and an additional nilpotent orbit of dimension 8, denoted $\tilde{A}_1$ [39, Chap. 8].

The dual Coxeter number of G_2 is $h^\vee = 4$. In the following, we give the OPEs between the strong generators of $\mathcal{W}$-algebras associated with nilpotent element of G_2 provided $k \neq -4$.

C.1 Generators and OPEs of $\mathcal{W}^k(G_2, f_{\min})$

Let $f := f_{\min} = e_{-\theta}$ be a minimal nilpotent element of G_2, and $h := h_1 + 2h_2$. The centralizer of f is eight-dimensional, generated by $\{e_{\alpha_1}, e_{-\alpha_1}, e_{-\alpha_2}, e_{-\eta_1}, e_{-\eta_2}, e_{-\eta_3}, f, h_1\}$. The minimal grading on $\mathfrak{g}$ induces the decomposition of the centralizer of f:

$$\mathfrak{g}^f = \mathbb{C}f \oplus \mathfrak{g}^f_{-1/2} \oplus \mathfrak{g}^f_0,$$

where

$$\mathfrak{g}^f_{-1} = \mathbb{C}f, \qquad \mathfrak{g}^f_{-1/2} = \mathbb{C}e_{-\alpha_2} \oplus \bigoplus_{i=1}^{3} \mathbb{C}e_{-\eta_i}, \qquad \text{and} \qquad \mathfrak{g}^f_0 = \mathbb{C}e_{\alpha_1} \oplus \mathbb{C}e_{-\alpha_1} \oplus \mathbb{C}h_1 \simeq \mathfrak{sl}_2.$$

The $\mathcal{W}$-algebra $\mathcal{W}^k(G_2, f_{\min})$ is strongly generated by the fields $J(z)$, $F^\pm(z)$, $G^\pm(z)$, $W^\pm(z)$, and $L(z)$ satisfying the OPEs:

$$L(z)L(w) \sim \frac{c_k}{2(z-w)^4} + \frac{2}{(z-w)^2}L(w) + \frac{1}{(z-w)}\partial L(w),$$

$$
\begin{aligned}
L(z)J(w) &\sim \frac{1}{(z-w)^2}J(w) + \frac{1}{(z-w)}\partial J(w),\\
L(z)F^{\pm}(w) &\sim \frac{1}{(z-w)^2}F^{\pm}(w) + \frac{1}{(z-w)}\partial F^{\pm}(w),\\
L(z)G^{\pm}(w) &\sim \frac{3}{2(z-w)^2}G^{\pm}(w) + \frac{1}{(z-w)}\partial G^{\pm}(w),\\
L(z)W^{\pm}(w) &\sim \frac{3}{2(z-w)^2}W^{\pm}(w) + \frac{1}{(z-w)}\partial W^{\pm}(w),\\
J(z)J(w) &\sim \frac{2(5+3k)}{(z-w)^2},\\
J(z)F^{\pm}(w) &\sim \pm\frac{2}{(z-w)}F^{\pm}(w),\\
J(z)G^{\pm}(w) &\sim \pm\frac{3}{(z-w)}G^{\pm}(w),\\
J(z)W^{\pm}(w) &\sim \pm\frac{1}{(z-w)}W^{\pm}(w),\\
F^{\pm}(z)F^{\pm}(w) &\sim G^{\pm}(z)G^{\pm}(w) \sim F^{\pm}(z)G^{\pm}(w) \sim 0,\\
F^{+}(z)F^{-}(w) &\sim \frac{(5+3k)}{(z-w)^2} + \frac{1}{(z-w)}J(w),\\
G^{+}(z)G^{-}(w) &\sim \frac{2(4+3k)(5+3k)}{9(z-w)^3} + \frac{(4+3k)}{3(z-w)^2}J(w)\\
&\quad + \frac{1}{(z-w)}\left(-(4+k)L(w) + \frac{2}{3} : F^{+}(w)F^{-}(w) : + \frac{1}{3} : J(w)^2 : + \frac{(2+3k)}{6}\partial J(w)\right),\\
F^{\pm}(z)G^{\mp}(w) &\sim \frac{1}{(z-w)}W^{\mp}(w),\\
W^{\pm}(z)W^{\pm}(w) &\sim \pm\frac{4(4+3k)}{3(z-w)^2}F^{\pm}(w) \pm \frac{2(4+3k)}{3(z-w)}\partial F^{\pm}(w),\\
W^{+}(z)W^{-}(w) &\sim -\frac{2(4+3k)(5+3k)}{3(z-w)^3} - \frac{(4+3k)}{3(z-w)^2}J(w)\\
&\quad + \frac{1}{(z-w)}\left(3(4+k)L(w) - \frac{10}{3} : F^{+}(w)F^{-}(w) : -\frac{1}{3} : J(w)^2 : + \frac{(2-k)}{2}\partial J(w)\right),\\
F^{\pm}(z)W^{\pm}(w) &\sim \frac{3}{(z-w)}G^{\pm}(w),\\
F^{\pm}(z)W^{\mp}(w) &\sim \frac{2}{(z-w)}W^{\pm}(w),\\
G^{\pm}(z)W^{\pm}(w) &\sim \pm\frac{2}{3(z-w)} : F^{\pm}(w)^2 :,\\
G^{\pm}(z)W^{\mp}(w) &\sim \mp\frac{2(4+3k)}{3(z-w)^2}F^{\pm}(w) + \frac{1}{(z-w)}\left(-\frac{2}{3} : J(w)F^{\pm}(w) : \mp \frac{(2+3k)}{3}\partial F^{\pm}(w)\right),
\end{aligned}
$$

where

$$c_k = -\frac{2k(5+3k)}{4+k}.$$

C.2 Generators and OPEs of $\mathcal{W}^k(G_2, f_{\tilde{A}_1})$

Set $f := f_{\tilde{A}_1} = e_{-\eta_2}$ and $h := 2h_1 + 3h_2$. Then f is a nilpotent element in $\tilde{A}_1$. A basis of the vector space $\mathfrak{g}^f$ is given by $\{e_{\alpha_2}, e_{-\alpha_2}, f, e_{-\eta_3}, e_{-\theta}, h_2\}$. Moreover, we have the decomposition

$$\mathfrak{g}^f = \mathfrak{g}^f_{-3/2} \oplus \mathfrak{g}^f_{-1} \oplus \mathfrak{g}^f_0,$$

where

$$\mathfrak{g}^f_{-1} = \mathbb{C}f, \qquad \mathfrak{g}^f_{-3/2} = \mathbb{C}e_{-\eta_3} \oplus \mathbb{C}e_{-\theta}, \qquad \text{and} \qquad \mathfrak{g}^f_0 = \mathbb{C}e_{\alpha_2} \oplus \mathbb{C}h_2 \oplus \mathbb{C}e_{-\alpha_2} \simeq \mathfrak{sl}_2.$$

The $\mathcal{W}$-algebra $\mathcal{W}^k(G_2, f)$ is strongly generated by the fields $J(z)$, $F^{\pm}(z)$, $G^{\pm}(z)$, and $L(z)$ satisfying the OPEs:

$$\begin{aligned}
L(z)L(w) &\sim \frac{c_k}{2(z-w)^4} + \frac{2}{(z-w)^2}L(w) + \frac{1}{(z-w)}\partial L(w),\\
L(z)J(w) &\sim \frac{1}{(z-w)^2}J(w) + \frac{1}{(z-w)}\partial J(w),\\
L(z)F^{\pm}(w) &\sim \frac{1}{(z-w)^2}F^{\pm}(w) + \frac{1}{(z-w)}\partial F^{\pm}(w),\\
L(z)G^{\pm}(w) &\sim \frac{5}{2(z-w)^2}G^{\pm}(w) + \frac{1}{(z-w)}\partial G^{\pm}(w),\\
J(z)J(w) &\sim \frac{(3+2k)}{(z-w)^2},\\
J(z)F^{\pm}(w) &\sim \pm\frac{2}{(z-w)}F^{\pm}(w),\\
J(z)G^{\pm}(w) &\sim \pm\frac{1}{(z-w)}G^{\pm}(w),\\
F^{\pm}(z)F^{\pm}(w) &\sim G^{\pm}(z)F^{\pm}(w) \sim 0,\\
F^{+}(z)F^{-}(w) &\sim \frac{(3+2k)}{2(z-w)^2} + \frac{1}{(z-w)}J(w),\\
F^{\pm}(z)G^{\mp}(w) &\sim \frac{1}{(z-w)}G^{\pm}(w),\\
G^{\pm}(z)G^{\pm}(w) &\sim \mp\frac{2(2+k)(10+3k)(17+6k)}{(z-w)^4}F^{\pm}(w) \mp \frac{(2+k)(10+3k)(17+6k)}{(z-w)^3}\partial F^{\pm}(w)\\
&\quad + \frac{1}{(z-w)^2}\big(\pm 2(4+k)(16+5k) :L(w)F^{\pm}(w): \mp 16(3+k) :F^{+}(w)F^{\pm}(w)F^{-}(w):\\
&\qquad \mp 4(3+k) :F^{\pm}(w)J(w)^2: -(44 \mp 48 + (24 \mp 16)k + 3k^2) :F^{\pm}(w)\partial J(w):\\
&\qquad +(2+k)(10+3k) :J(w)\partial F^{\pm}(w): \mp 2(3+k)(38 \mp 4 + 20k + 3k^2)\partial^2 F^{\pm}(w)\big)\\
&\quad + \frac{1}{(z-w)}\big(\pm(4+k)(16+5k) :L(w)\partial F^{\pm}(w): \mp 16(3+k) :F^{+}(w)F^{-}(w)\partial F^{\pm}(w):\\
&\qquad \pm (4+k)(16+5k) :\partial L(w)F^{\pm}(w): \mp 8(3+k) :F^{\pm}(w)^2\partial F^{\mp}(w):\\
&\qquad \mp 4(3+k) :F^{\pm}(w)J(w)\partial J(w): \mp 2(3+k) :J(w)^2\partial F^{\pm}(w):\\
&\qquad + \frac{44+24k+3k^2}{2}(:J(w)\partial^2 F^{\pm}(w): - :F^{\pm}(w)\partial^2 J(w):)
\end{aligned}$$

$$\pm 4(2\mp 1)(3+k):\partial J(w)\partial F^{\pm}(w):\mp\frac{940+736k+195k^2+18k^3}{12}\partial^3F^{\pm}(w)\Big),$$

$$\begin{aligned}
G^+(z)G^-(w)\sim&\frac{(2+k)(3+2k)(10+3k)(17+6k)}{(z-w)^5}+\frac{(2+k)(10+3k)(17+6k)}{(z-w)^4}J(w)\\
&+\frac{1}{(z-w)^3}\Big(-(4+k)(3+2k)(16+5k)L(w)\\
&\quad+(38+34k+7k^2)(4:F^+(w)F^-(w):+:J(w)^2:)\\
&\quad+\frac{188+256k+119k^2+18k^3}{2}\partial J(w)\Big)\\
&+\frac{1}{(z-w)^2}\Big(-(4+k)(16+5k):L(w)J(w):+8(3+k):F^+(w)F^-(w)J(w):\\
&\quad+(8+36k+11k^2):F^+(w)\partial F^-(w):+(144+100k+17k^2):F^-(w)\partial F^+(w):\\
&\quad+2(3+k):J(w)^3:+(26+30k+7k^2):J(w)\partial J(w):\\
&\quad-\frac{(4+k)(3+2k)(16+5k)}{2}\partial L(w)+(3+k)(42+20k+3k^2)\partial^2J(w)\Big)\\
&+\frac{1}{(z-w)}\Big(\frac{3(4+k)^2}{2}:L(w)^2:-\frac{3(2+k)(3+k)(4+k)}{2}\partial^2L(w)\\
&\quad-\frac{(4+k)(16+5k)}{2}:\partial L(w)J(w):-\frac{(4+k)(8+5k)}{2}:L(w)\partial J(w):\\
&\quad+\frac{396+332k+90k^2+9k^3}{12}\partial^3J(w)-8(4+k):L(w)F^+(w)F^-(w):\\
&\quad-2(4+k):L(w)J(w)^2:+\frac{116+72k+15k^2}{2}:\partial F^+(w)\partial F^-(w):\\
&\quad+\frac{(2+k)(26+15k)}{8}:\partial J(w)^2:+\frac{96+52k+9k^2}{4}:J(w)\partial^2J(w):\\
&\quad+2(27+19k+3k^2):F^-(w)\partial^2F^+(w):+(26+14k+3k^2):F^+(w)\partial^2F^-(w):\\
&\quad+(7+3k):J(w)^2\partial J(w):+4(5+k):J(w)F^-(w)\partial F^+(w):\\
&\quad+4(1+k):J(w)F^+(w)\partial F^-(w):+4(k-1):F^+(w)F^-(w)\partial J(w):\\
&\quad+8:F^+(w)^2F^-(w)^2:+4:F^+(w)F^-(w)J(w)^2:+\frac{1}{2}:J(w)^4:\Big),
\end{aligned}$$

where

$$c_k=-\frac{(92+81k+18k^2)}{4+k}.$$

C.3 Generators and OPEs of $\mathcal{W}^k(G_2,f_{\text{subreg}})$

Set $f:=f_{\text{subreg}}=e_{-\alpha_2}+e_{-\eta_2}$, $e:=e_{\alpha_2}+e_{\eta_2}$, and $h:=2h_1+4h_2$. The centralizer of f in $\mathfrak{g}$ is four-dimensional:

$$\mathfrak{g}^f=\mathfrak{g}^f_{-2}\oplus\mathfrak{g}^f_{-1},$$

where

$$\mathfrak{g}^f_{-2}=\mathbb{C}e_{-\theta},\qquad\text{and}\qquad\mathfrak{g}^f_{-1}=\mathbb{C}f\oplus\mathbb{C}e_{-\alpha_2}\oplus\mathbb{C}(e_{-\eta_1}-3e_{-\eta_3}).$$

The $\mathcal{W}$-algebra $\mathcal{W}^k(G_2,f)$ is strongly generated by the fields $L(z)$, $G^{\pm}(z)$ and $F(z)$ satisfying the OPEs:

$$
\begin{aligned}
L(z)L(w) \sim & \frac{c_k}{2(z-w)^4} + \frac{2}{(z-w)^2}L(w) + \frac{1}{(z-w)}\partial L(w), \\
L(z)G^{\pm}(w) \sim & \frac{2}{(z-w)^2}G^{\pm}(w) + \frac{1}{(z-w)}\partial G^{\pm}(w), \\
L(z)F(w) \sim & \frac{3}{(z-w)^2}F(w) + \frac{1}{(z-w)}\partial F(w), \\
G^+(z)F(w) \sim & \frac{2(2+k)(16+5k)}{(z-w)^3}G^-(w) + \frac{(2+k)(16+5k)}{2(z-w)^2}\partial G^-(w) \\
& + \frac{1}{(z-w)}\left(2 : G^+(w)G^-(w) : -2(4+k) : L(w)G^-(w) : +2\partial F(w) + \frac{(2+k)^2}{2}\partial^2 G^-(w)\right), \\
G^-(z)F(w) \sim & \frac{2(2+k)(16+5k)}{(z-w)^3}G^+(w) + \frac{(2+k)(16+5k)}{2(z-w)^2}\partial G^+(w) \\
& + \frac{1}{(z-w)}\left(- : G^+(w)^2 : -2(4+k) : L(w)G^+(w) : - : G^-(w)^2 : + \frac{(2+k)^2}{2}\partial^2 G^+(w)\right), \\
F(z)F(w) \sim & -\frac{(2+k)(10+3k)(16+5k)(4+k)c_k}{2(z-w)^6} \\
& - \frac{3(2+k)(4+k)(10+3k)(16+5k)}{(z-w)^4}L(w) - \frac{3(2+k)(4+k)(10+3k)(16+5k)}{2(z-w)^3}\partial L(w) \\
& + \frac{1}{(z-w)^2}\big(-(8+3k) : G^+(w)^2 : +2(4+k)^2(10+3k) : L(w)^2 : \\
& +(8+3k) : G^-(w)^2 : - \frac{3(2+k)(4+k)(8+3k)(10+3k)}{4}\partial^2 L(w)\Big) \\
& + \frac{1}{(z-w)}\big(-(8+3k) : G^+(w)\partial G^+(w) : +2(4+k)^2(10+3k) : L(w)\partial L(w) : \\
& +(8+3k) : G^-(w)\partial G^-(w) : - \frac{(2+k)(4+k)(4+3k)(10+3k)}{6}\partial^3 L(w)\Big), \\
G^{\pm}(z)G^{\pm}(w) \sim & \pm\frac{(10+3k)(4+k)c_k}{2(z-w)^4} + \frac{1}{(z-w)^2}\left(\pm 2(4+k)(10+3k)L(w) - 4(3+k)G^+(w)\right) \\
& + \frac{1}{(z-w)}\left(\pm(4+k)(10+3k)\partial L(w) - 2(3+k)\partial G^+(w)\right), \\
G^+(z)G^-(w) \sim & \frac{4(3+k)}{(z-w)^2}G^-(w) + \frac{1}{(z-w)}\left(-2F(w) + 2(3+k)\partial G^-(w)\right),
\end{aligned}
$$

where

$$c_k = -\frac{4(k+2)(17+6k)}{4+k}.$$

C.4 Generators and OPEs of $\mathcal{W}^k(G_2)$

Let $f := f_{\text{reg}} = 6e_{-\alpha_1} + 10e_{-\alpha_2}$ and $h := 6h_1 + 10h_2$. Then f is a regular nilpotent element of G_2 and the centralizer of f is

$$\mathfrak{g}^f = \mathbb{C}f \oplus \mathbb{C}e_{-\theta},$$

where $[h, e_{-\theta}] = -10e_{-\theta}$.

The $\mathcal{W}$-algebra $\mathcal{W}^k(G_2)$ is strongly generated by the fields $L(z)$ and $W(z)$ satisfying the OPEs:

$$\begin{aligned}
L(z)L(w) &\sim \frac{c_k}{2(z-w)^4} + \frac{2}{(z-w)^2}L(w) + \frac{1}{(z-w)}\partial L(w),\\
L(z)W(w) &\sim \frac{6}{(z-w)^2}W(w) + \frac{1}{(z-w)}\partial W(w),\\
W(z)W(w) &\sim \frac{1}{(z-w)^{12}}(\dots) + \frac{1}{(z-w)^{11}}(\dots) + \frac{1}{(z-w)^{10}}(\dots)\\
&+ \frac{1}{(z-w)^{9}}(\dots) + \frac{1}{(z-w)^{8}}(\dots) + \frac{1}{(z-w)^{7}}(\dots)\\
&+ \frac{1}{(z-w)^{6}}(\dots) + \frac{1}{(z-w)^{5}}(\dots) + \frac{1}{(z-w)^{4}}(\dots)\\
&+ \frac{1}{(z-w)^{3}}(\dots) + \frac{1}{(z-w)^{2}}(\dots) + \frac{1}{(z-w)}(\dots),
\end{aligned}$$

where

$$c_k = -\frac{2(12k+41)(7k+24)}{4+k}.$$

Appendix D

GAP commands to compute conformal weights of $\mathcal{W}$-algebras

Let L be a simple Lie algebra and e be a element in the nilpotent orbit N of L. This appendix introduces several commands in the language of the computer algebra system GAP to compute the conformal weights of the $\mathcal{W}$-algebra associated with the nilpotent element e in L. These functions require the package SLA [69] which make easier computations of various data of simple Lie algebras. The following commands implemented in the package SLA are very useful for our computations:

- `NilpotentOrbits(L)` returns the list of all non trivial nilpotent orbits of L,
- `NilpotentOrbit(L,wd)` returns the nilpotent orbit of L whose the weighted Dynkin diagram corresponds to the list wd,
- `OrbitPartition(N)` returns the partition of the nilpotent orbit N when L is a classical Lie algebra,
- `WeightedDynkinDiagram(N)` returns the weighted Dynkin digrams of the nilpotent orbit N,
- `WeightedDynkinDiagram(L,e)` returns the weighted Dynkin digrams of the nilpotent orbit containing e,
- `SL2Triple(N)` returns an $\mathfrak{sl}_2$-triple whose the nilpotent elements are in N.

D.1 Computation of conformal weights of a $\mathcal{W}$-algebra

In this section, we present our main function `ConformalWeightsN(L,N)`. It returns the list of conformal weights of the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathtt{e},\mathtt{L})$, where e is a element in the nilpotent orbit N. This function uses two auxiliary functions:

- `characteristic(L,N)` returns the list of eigenvalues of $\mathrm{ad}(h)$ where h is a semisimple element of an $\mathfrak{sl}_2$-triple whose nilpotent elements are in the nilpotent orbit N,
- `extract(l,i)` returns the number of occurrences of i in the list l. In particular, when l is the list `characteristic(L,N)`, the function returns the dimension of the eigenspace of $\mathrm{ad}(h)$ corresponding with the eigenvalue i.

We have also constructed two variations of the function `ConformalWeightsN(L,N)` depending on the data we know:

- `ConformalWeights(L)` returns all lists of conformal weights corresponding to nilpotent orbits of `L`,
- `ConformalWeightsE(L,e)` returns the list of conformal weights corresponding to a nilpotent element `e` in `L`.

```
characteristic := function(L,N)
        local R,tC,pR;
        R:=RootSystem(L);
        pR:=PositiveRoots(R);
        tC:=Inverse(TransposedMat(CartanMatrix(R)));
        return List([1..Length(pR)], j->
                        tC*pR[j]*WeightedDynkinDiagram(N)); end;

extract := function(l,i)
        local n,c,j;
        n:=Length(l);
        c:=0;
        for j in [1..n] do if l[j]=i then c:=c+1; fi; od;
        return c; end;

ConformalWeightsN := function(L,N)
        local Caract,n,N_max,Even_Part,Odd_Part,j,
                Eigenvalues_list,hL;
        Caract:=characteristic(L,N);
        n:=Length(Caract);
        Sort(Caract);
        N_max:=Caract[n];
        hL:=ChevalleyBasis(L)[3];
        Even_Part:=[[0/2+1,extract(Caract,0)*2+Length(hL)
                        -extract(Caract,2)]];
        Odd_Part:=[];
        for j in [1..N_max] do
                Add(Even_Part,[(2*j)/2+1, extract(Caract,2*j)
                        -extract(Caract,2*j+2)]);
                Add(Odd_Part,[(2*j-1)/2+1, extract(Caract,2*j-1)
                        -extract(Caract,2*j+1)]); od;
        Eigenvalues_list:=Concatenation(Even_Part,Odd_Part);
        Sort(Eigenvalues_list,function(v,w) return v[2]<w[2];end);
        while Eigenvalues_list[1][2]=0 do
                Remove(Eigenvalues_list,1); od;
        Sort(Eigenvalues_list,function(v,w) return v[1]<w[1];end);
        return Eigenvalues_list; end;
```

Example D.1.1. The conformal weights of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$ are $(1, 2^3)$. This means that the $\mathcal{W}$-algebra is strongly generated by one field of conformal weight 1 and three of conformal weight 2.

```
gap> L:=SimpleLieAlgebra("B",2,Rationals);
<Lie algebra of dimension 10 over Rationals>
gap> N:=NilpotentOrbits(L)[2];
<nilpotent orbit in Lie algebra of type B2>
gap> ConformalWeightsN(L,N);
[ [ 1, 1 ], [ 2, 3 ] ]
```

```
ConformalWeights := function(L)
        local Orbits,N;
        Orbits:=NilpotentOrbits(L);
        N:=Length(Orbits);
        return List([1..N],i->ConformalWeightsN(L,Orbits[i]));end;
```

Example D.1.2. To obtain the conformal weights of all $\mathcal{W}$-algebras associated with $\mathfrak{sp}_4$ one uses `ConformalWeights`.

```
gap> L:=SimpleLieAlgebra("B",2,Rationals);
<Lie algebra of dimension 10 over Rationals>
gap> ConformalWeights(L);
[ [ [ 1, 3 ], [ 3/2, 2 ], [ 2, 1 ] ], [ [ 1, 1 ], [ 2, 3 ] ],
[ [ 2, 1 ], [ 4, 1 ] ] ]
```

```
ConformalWeightsE := function(L,e)
        local N;
        N:=NilpotentOrbit(L,WeightedDynkinDiagram(L,e));
        return ConformalWeightsN(L,N); end;
```

Example D.1.3. The conformal weights of $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{subreg}})$ also correspond to the conformal weights of the nilpotent element $\mathtt{e} = e_{\alpha_1} + e_\theta$.

```
gap> L:=SimpleLieAlgebra("B",2,Rationals);
<Lie algebra of dimension 10 over Rationals>
gap> Ch:=ChevalleyBasis(L);
[ [ v.1, v.2, v.3, v.4 ], [ v.5, v.6, v.7, v.8 ], [ v.9, v.10 ] ]
gap> e:=Ch[1][1]+Ch[1][4];
v.3
gap> ConformalWeightsE(L,e);
[ [ 1, 1 ], [ 2, 3 ] ]
```

D.2 Applications to find nilpotent orbits sharing same conformal weights

Let f be a nilpotent element in $\mathfrak{g}$ and $\{x_1, \dots, x_n\}$ be a basis of the centralizer $\mathfrak{g}^f$ with respect to the grading of h. The list of conformal weights of f encodes the h-eigenvalues of $x_1, \dots, x_n$. It contains a lot of additional information. For instance, given a list of conformal weights `CW`, one easily deduces dimensions of the corresponding nilpotent orbit (`DimensionN(CW)`) and Lie algebra (`DimensionL(CW)`).

```
DimensionL := function(CW)
        local i,dim;
        dim:=0;
        for i in [1..Length(CW)] do
        dim:=dim+((CW[i][1]-1)*2+1)*CW[i][2]; od;
        return dim; end;

DimensionN := function(CW)
        return DimensionL(CW)-Sum(CW)[2]; end;
```

We use the previous programs to determine if a list of half-integers can be the list of conformal weights of a $\mathcal{W}$-algebra. In order to do that, we use several auxiliary functions. Indeed, to reduce successively the number of candidates, we start comparing the dimensions of Lie algebras and then ones of nilpotent orbits:

- `PossibleLie(d)` returns the list of simple Lie algebras of dimension `d`,
- `OrbitsFixedDimension(L,d)` returns the list of nilpotent orbits in `L` whose dimension is `d`.

Then we obtain a list of candidate orbits which satisfy the previous dimensional conditions (`PossibleOrbits(CW)`). We conclude comparing the conformal weights of the candidates to the initial list:

- `OrbitsCW(CW)` returns the list of nilpotent orbits in all simple Lie algebras whose conformal weights is `CW`,
- `OrbitsCWE(CW)` returns the list of nilpotent orbits `N` in all simple Lie algebras and a nilpotent element f in `N` such as the list of conformal weights of $\mathcal{W}^k(\mathfrak{g}, f)$ is `CW`.

```
PossibleLie := function(d)
        local Lie,exceptionnal_Lie,classical_Lie,i;
        exceptionnal_Lie:=[[["E",6],78],[["E",7],133],
                [["E",8],248],[["F",4],52],[["G",2],14]];
        Lie:=[];
        for i in [1..Length(exceptionnal_Lie)] do
                if exceptionnal_Lie[i][2]=d
                then Add(Lie,exceptionnal_Lie[i][1]); fi; od;
        classical_Lie:=[-1+Sqrt(1+d),(-1+Sqrt(1+8*d))/4,
                (1+Sqrt(1+8*d))/4];
        if classical_Lie[1] in Integers
        then Add(Lie,["A",classical_Lie[1]]); fi;
        if classical_Lie[2] in Integers
        then Append(Lie,[["B",classical_Lie[2]],
                ["C",classical_Lie[2]]]); fi;
        if classical_Lie[3] in Integers and classical_Lie[3]>3
        then Add(Lie,["D",classical_Lie[3]]); fi;
        return Lie; end;

OrbitsFixedDimension := function(L,d)
        local Orbits,N,i,GoodOrbits;
```

```
        Orbits:=NilpotentOrbits(L);
        N:=Length(Orbits);
        GoodOrbits:=[];
        for i in [1..N] do
        if DimensionN(ConformalWeightsN(L,Orbits[i]))=d
        then Add(GoodOrbits,[Orbits[i],i]); fi; od;
        return GoodOrbits; end;

PossibleOrbits := function(CW)
        local dimL,dimN,Lie,i,GoodOrbits,Orbits,L;
        dimL:=DimensionL(CW);
        dimN:=DimensionN(CW);
        Lie:=PossibleLie(dimL);
        GoodOrbits:=[];
        for i in [1..Length(Lie)] do
                L:=Lie[i];
                Orbits:=OrbitsFixedDimension(
                        SimpleLieAlgebra(L[1],L[2],Rationals),
                                dimN);
                if Orbits<>[] then
                Add(GoodOrbits,[L,Orbits]); fi; od;
        return GoodOrbits; end;

OrbitsCW := function(CW)
        local PO,GoodOrbits,i,j,L;
        PO:=PossibleOrbits(CW);
        GoodOrbits:=[];
        for i in [1..Length(PO)] do
                L:=SimpleLieAlgebra(
                        PO[i][1][1],PO[i][1][2],Rationals);
                for j in [1..Length(PO[i][2])] do
                        if CW=ConformalWeightsN(L,PO[i][2][j][1])
                        then Add(GoodOrbits,
                                [PO[i][1],PO[i][2][j]]);fi;od;od;
        return GoodOrbits; end;
```

Example D.2.1. Two nilpotent orbits have conformal weights $(1^{13}, 3/2^{12}, 2^6)$: the second nilpotent orbit in the list `NilpotentOrbits(B5)` and the third one in `NilpotentOrbits(C5)`.

```
gap> OrbitsCW([ [ 1, 13 ], [ 3/2, 12 ], [ 2, 6 ] ]);
[[[ "B", 5 ], [ <nilpotent orbit in Lie algebra of type B5>, 2 ]],
[[ "C", 5 ], [ <nilpotent orbit in Lie algebra of type C5>, 3 ]]]
```

```
OrbitsECW := function (CW)
        local Orbits,OrbitsE,i;
        Orbits:=OrbitsCW(CW);
        OrbitsE:=[];
        for i in [1..Length(Orbits)] do
                Add(OrbitsE,[Orbits[i][2],
```

```
                        SL2Triple(Orbits[i][2][1])]); od;
        return OrbitsE; end;
```

Example D.2.2. The command `OrbitsECW([ [ 1, 13 ], [ 3/2, 12 ], [ 2, 6 ] ])` returns the same result as `OrbitsCW([ [ 1, 13 ], [ 3/2, 12 ], [ 2, 6 ] ])` with the additional data of an $\mathfrak{sl}_2$-triple whose nilpotent elements are representatives of the nilpotent orbit.

```
gap> OrbitsECW([ [ 1, 13 ], [ 3/2, 12 ], [ 2, 6 ] ]);
[ [ [ <nilpotent orbit in Lie algebra of type B5>, 2 ],
[v.47+v.48, v.51+(2)*v.52+(3)*v.53+(4)*v.54+(2)*v.55, v.22+v.23]],
[[ <nilpotent orbit in Lie algebra of type C5>, 3 ],
[v.47+v.48, v.51+(2)*v.52+(3)*v.53+(3)*v.54+(3)*v.55, v.22+v.23]]]
```

Finally, we compare conformal weights of $\mathcal{W}$-algebras associated with two simple Lie algebras `L1` and `L2`. The function `SharedWeights(L1,L2)` returns the lists of conformal weights which correspond simultaneously to a nilpotent orbit of `L1` and one of `L2`. This command has been used to compute examples appearing in Tables 6.1-6.4.

```
SharedWeights := function(L1,L2)
        local OrbitsL1,OrbitsL2,N1,N2,CW1,CW2,i,j,CW;
        OrbitsL1:=NilpotentOrbits(L1);
        N1:=Length(OrbitsL1);
        OrbitsL2:=NilpotentOrbits(L2);
        N2:=Length(OrbitsL2);
        CW1:=ConformalWeights(L1);
        CW2:=ConformalWeights(L2);
        CW:=[];
        for i in [1..N1] do
                for j in [1..N2] do
                        if CW1[i]=CW2[j]
                        then Add(CW,[i,j,CW1[i]]); fi; od; od;
        return CW; end;
```

Example D.2.3. Some nilpotent orbits in B_5 and C_5 have the same conformal weights. The command `SharedWeights(B5,C5)` returns a list whose each element is a triple: the first and second terms correspond respectively to the index of the nilpotent orbits in `NilpotentOrbits(B5)` and `NilpotentOrbits(C5)`, and the third item of the triple is their list of conformal weights.

```
gap> B5:=SimpleLieAlgebra("B",5,Rationals);
<Lie algebra of dimension 55 over Rationals>
gap> C5:=SimpleLieAlgebra("C",5,Rationals);
<Lie algebra of dimension 55 over Rationals>
gap> S:=SharedWeights(B5,C5);
[ [ 2, 3, [[ 1, 13 ],[ 3/2, 12 ],[ 2, 6 ]] ],
[ 10, 13, [[ 1, 3 ],[ 3/2, 2 ],[ 2, 2 ],[ 5/2, 2 ],[ 3, 3 ],
[ 7/2, 2 ],[ 4, 1 ]] ],
[ 14, 15, [[ 1, 1 ],[ 2, 6 ],[ 3, 3 ],[ 4, 3 ]] ],
[ 18, 20, [[ 2, 3 ],[ 3, 1 ],[ 4, 3 ],[ 5, 1 ],[ 6, 1 ]] ],
[ 20, 23, [[ 2, 1 ],[ 4, 1 ],[ 6, 1 ],[ 8, 1 ],[ 10, 1 ]] ] ]
gap> OrbitPartition(NilpotentOrbits(B5)[2]);
```

```
[ 2, 2, 2, 2, 1, 1, 1 ]
gap> OrbitPartition(NilpotentOrbits(C5)[3]);
[ 2, 2, 2, 1, 1, 1, 1 ]
```

Geometry and new rational $\mathcal{W}$-algebras

Abstract

Affine $\mathcal{W}$-algebras form a rich one-parameter family of vertex algebras associated with nilpotent elements of simple Lie algebras. These complex algebraic structures appear in several areas of physic and mathematics. Because of their recent construction, numerous aspects of the theory of $\mathcal{W}$-algebras remain unknown.
In this thesis, we study $\mathcal{W}$-algebras associated with nilpotent elements of Lie algebras of small ranks. We prove the rationality of a new family of $\mathcal{W}$-algebras, describe their set of simple modules and study other geometrical aspects. We describe new associated varieties of vertex algebras. The geometry of these objects often reflects some important algebraic properties of the vertex algebras. For some particular values of the parameter, called collapsing levels, we also get new remarkable isomorphisms of $\mathcal{W}$-algebras.

Keywords: vertex algebras, $\mathcal{W}$-algebras, rationality, representation theory, Lie algebras, nilpotent orbits

---

Géométrie et nouvelles $\mathcal{W}$-algèbres rationnelles

Résumé

Les $\mathcal{W}$-algèbres affines forment une famille riche d'algèbres vertex à un paramètre associées à un élément nilpotent d'une algèbre de Lie simple. Ce sont des structures algébriques complexes qui apparaissent dans plusieurs domaines de la physique et des mathématiques. Du fait de leur construction récente, de nombreux aspects de la théorie des $\mathcal{W}$-algèbres restent méconnus.
Dans cette thèse, nous étudions des $\mathcal{W}$-algèbres associées à des éléments nilpotents d'algèbres de Lie de petits rangs. Nous démontrons la rationalité d'une nouvelle famille de $\mathcal{W}$-algèbres, décrivons l'ensemble des modules simples sur ces dernières et étudions d'autres aspects géométriques. Nous décrivons de nouvelles variétés associées à des algèbres vertex. La géométrie de ces objets reflète souvent des propriétés algébriques importantes des algèbres vertex. Pour certaines valeurs particulières du paramètre, appelées niveaux d'effondrement, nous obtenons également de nouveaux isomorphismes remarquables de $\mathcal{W}$-algèbres.

Mots clés : algèbres vertex, $\mathcal{W}$-algèbres, rationalité, théorie des représentations, algèbres de Lie, orbites nilpotentes

---

Laboratoire Paul Painlevé, UMR CNRS 8524
Université de Lille – Cité Scientifique - Bâtiment M2 – F-59655 VILLENEUVE d'ASCQ – France