Stability analysis, synchronization, and state observation of generalized Persidskii systems

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ABSTRACT

This work presents new results on input-to-output stability conditions, robust synchronization, and state estimation for generalized Persidskii systems in the presence of external input/disturbance, as well as input-to-state stability analysis of those dynamics with time delays. The thesis starts from the problem formulation followed by a brief introduction and state-of-the-art in Chapter 1. Preliminary definitions and auxiliary results are summarized in Chapter 2. Chapter 3 focuses on input-to-output stability conditions and their application to robust synchronization of generalized Persidskii models. The synchronization conditions are illustrated by the example of the neural Hindmarsh-Rose model. Chapter 4 considers a state observer designed for generalized Persidskii systems with nonlinear measurements, state disturbances, and output noise. The theory of input-to-output stability is applied to obtain robust stability and convergence conditions for the estimation error. Two applications to a perturbed two-mass spring-damper system and a multi-group susceptible-infected-susceptible model are provided to demonstrate the efficacy and performances of the proposed observer. In Chapter 5, the delay-dependent input-to-state stability and stabilization conditions for time-delay generalized Persidskii systems are studied and formulated in terms of state-dependent matrix inequalities. Numerical examples of opinion dynamics and a modified Lotka-Volterra model illustrate the proposed results.

RÉSUMÉ

Ce travail présente de nouveaux résultats sur les conditions de stabilité entrée-sortie, sur la synchronisation robuste et l'estimation d'état pour les systèmes Persidskii généralisés, en présence d'entrées/perturbations externes, ainsi que l'analyse de la stabilité entrée-état de ces dynamiques avec des retards. La thèse commence à partir de la formulation du problème, suivie d'une brève introduction et de l'état de l'art au chapitre 1. Les définitions préliminaires et les résultats auxiliaires sont résumés au chapitre 2. Le chapitre 3 se concentre sur les conditions de stabilité entrée-sortie et leur application à la synchronisation robuste de modèles de Persidskii généralisés. Les conditions de synchronisation sont illustrées par l'exemple du modèle neuronal de Hindmarsh-Rose. Le chapitre 4 considère un observateur d'état conçu pour les systèmes Persidskii généralisés avec des mesures non linéaires, des perturbations d'état et du bruit de sortie. La théorie de la stabilité entrée-sortie est appliquée pour obtenir des conditions de stabilité et de convergence robustes pour l'erreur d'estimation. Deux applications à un système ressort-amortisseur bimasse perturbé et à un modèle multigroupe susceptibles-infectés-susceptibles sont fournis pour démontrer l'efficacité et les performances de l'observateur proposé. Dans le chapitre 5, les conditions de stabilité et de stabilisation entrée-état dépendantes du retard pour les systèmes Persidskii généralisés à retard sont étudiées et formulées en termes d'inégalités matricielles dépendantes de l'état. Des exemples numériques de dynamique d'opinion et un modèle Lotka-Volterra modifié illustrent les résultats proposés.

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NOTATION

- $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ represent the sets of natural numbers, real numbers, and nonnegative real numbers, respectively. The symbols $|\cdot|$ and $\|\cdot\|$ denote the absolute value in $\mathbb{R}$ and the Euclidean norm on the Euclidean space $\mathbb{R}^n$ (and the induced matrix norm $\|A\|$ for a matrix $A \in \mathbb{R}^{m\times n}$), correspondingly.

- The identity matrix of dimension n is denoted by I_n, the n-dimensional all-ones vector by $\mathbb{1}_n$, and the $m\times n$ zero matrix by $\mathbf{O}_{m\times n}$. The Kronecker product is denoted by $\otimes$. Let $\mathbb{1}_{\mathcal{A}} : \mathcal{X} \to \{0,1\}$ denote the indicator function of a subset $\mathcal{A}$ of a set $\mathcal{X}$.

- For p, $n \in \mathbb{N}$ with $p \le n$, the notation $\overline{p,n}$ is used to represent the set $\{p, p+1, \ldots, n\}$.

- Let $(B_{i,j})_{i,j=p}^{n}$ denote the block matrix $\begin{bmatrix} B_{p,p} & \cdots & B_{p,n} \\ \vdots & \ddots & \vdots \\ B_{n,p} & \cdots & B_{n,n} \end{bmatrix}$.

- The $m\times n$ block diagonal matrix with matrices $v_i \in \mathbb{R}^{m_i\times n_i}$, $i \in \overline{1,N}$, along the main diagonal, where $m = \sum_{i=1}^{N} m_i$ and $n = \sum_{i=1}^{N} n_i$, is denoted by $\operatorname{diag}(v_1,\ldots,v_N)$. The set of diagonal matrices with nonnegative elements on the main diagonal is denoted by $\mathbb{D}_+^n \subset \mathbb{R}_+^{n\times n} = \{B \in \mathbb{R}^{n\times n} \mid B \ge 0\}$. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(A)$ denotes its maximal eigenvalue. For a matrix $B \in \mathbb{R}^{m\times n}$, let $\ker(B)$ stand for its kernel. For a complex number c, we use $\operatorname{Re}(c)$ to represent its real part.

- For a differentiable function $F : \mathbb{R}^n \to \mathbb{R}^m$ (or for $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ to be differentiable in the first argument) , we use $\frac{\partial F(x)}{\partial x}$ $\left(\text{or } \frac{\partial F(x,t)}{\partial x}\right)$ to denote the Jacobian matrix of F at a point $x \in \mathbb{R}^n$.

- For $\Delta := [t_1,t_2] \subset \mathbb{R}$ we denote by C_Δ^n the Banach space of continuous functions $\psi : \Delta \to \mathbb{R}^n$ with the norm $\|\psi\|_\Delta = \sup_{r\in\Delta}\|\psi(r)\|$. Denote by $\mathbb{W}_\Delta$ the Sobolev space of absolutely continuous functions $\phi : \Delta \to \mathbb{R}^n$ with the norm $\|\phi\|_{\mathbb{W}_\Delta} := \|\phi\|_\Delta + \|\dot{\phi}\|_\Delta < +\infty$, where $\dot{\phi}(\ell) = \frac{\partial \phi(\ell)}{\partial \ell}$, $\ell \in \Delta \subset \mathbb{R}$. For a Lebesgue measurable function $u\colon \mathbb{R}_+ \to \mathbb{R}^m$, define the norm $\|u\|_S = \operatorname{ess\,sup}_{t\in S}\|u(t)\|$ on a set $S \subseteq \mathbb{R}_+$. Let $\mathscr{L}_\infty^m$ be the Banach space of functions u with $\|u\|_\infty := \|u\|_{[0,\infty)} < +\infty$ and $\tilde{\mathscr{L}}_\Theta^m \subset \mathscr{L}_\infty^m$ be the space of functions taking values in a compact subset $\Theta \subset \mathbb{R}^m$.

- A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathscr{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathscr{K}_\infty$ if it also satisfies $\lim_{r\to\infty} \sigma(r) = \infty$. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathscr{K}\mathscr{L}$ if for fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s) \in \mathscr{K}$ and for fixed $r \in \mathbb{R}_+$, $\beta(r, \cdot)$ is a decreasing function with $\lim_{s\to\infty} \beta(r, s) = 0$.

- For a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, denote by $\nabla V(v) f(v)$ the Lie derivative of V along the vector field f evaluated at point $v \in \mathbb{R}^n$.

CHAPTER 1

General introduction

1.1 Background

The mathematical expression of physical dynamical systems usually takes two forms: an input-output (I/O) description, where the system itself is regarded as a map from inputs to outputs; or a state-space description depicting the system with trajectories in a metric space (or flows on proper manifolds) [42]. These descriptions can be exemplified by the convolution (or transfer function) and differential equation, respectively, which complement each other to form a foundation for system and control theory. When internal constraints (typically for the state of systems) are taken into account, the state-space description facilitates the designing for prescribed internal system modes and qualitative behavior of trajectories. Among the research directions of qualitative properties of systems, a prominent one is the stability analysis of dynamical systems, which is a complicated problem, especially in the nonlinear case and in the presence of external perturbations [140, 139, 57]. Stability theory in the 1970s can be classified as an I/O property (I/O stability in functional analysis terms) or as an internal property (for systems with a state-space model; for example, asymptotic stability (AS), Lyapunov stability) [41]. For the former one, a "system" is a causal operator f from a normed space to another normed space, and "stability" means that f maps bounded inputs into bounded outputs. Stronger requirements in this context may be: the operator f is bounded, or f is globally Lipschitz, to mention several cases. The latter framework is grounded on the internal model description in terms of differential equations. The I/O approach has great advantages in the robustness analysis of linear systems under nonlinear feedback and small nonlinear uncertainties (*e.g.*, small-gain theorem [148, 51]). It is worth indicating that under mild restrictions, the two stability notions (I/O and state-space stability) are equivalent in the case of linear systems. However, that does not necessarily hold for nonlinear systems. The notion of dissipativity [144, 143] was proposed for unifying I/O and state-space stability [124], whereas the well-known input-to-state stability (ISS) [125, 27] approach simplifies the statement of stability by combining I/O with a state-space component and a "nonlinear asymptotic gain" [41]. The main strength of ISS to I/O stability is that nonlinear asymptotic gain is useful to guarantee the norms of states are bounded by a function of the norms of inputs/disturbances, skipping the far too restrictive boundedness conditions on nonlinear operators. Nevertheless, in many realistic cases, engineers are interested in stabilizing the output values instead of the state. Therefore, under the essential consideration of boundedness of the state, the notion of input-to-output stability (IOS) was proposed [129, 130], which quantifies the boundedness and the convergence of an output signal for a nonlinear dynamical system in the presence of essentially bounded exogenous inputs. The ISS property [27] is a particular case of IOS when the whole state is considered as the system output. State-independent input-to-output stability (SIIOS) is an extension of IOS, for revealing an influence from (external) inputs to outputs on nonlinear systems by disregarding the value of the state [129]. Furthermore, robust output stability (ROS) deals with a "robust output gain margin" quantifying the magnitude of output feedback that the plant can endure with the maintenance of output stability [129, 50]. The IOS (SIIOS, ROS) theory provides necessary and sufficient conditions for IOS (SIIOS, ROS) in terms of the existence of corresponding Lyapunov functions (LFs) [130].

Roughly speaking, LFs for dynamical systems are continuous and positive definite functions that have continuous first derivatives and whose decay along the trajectories of the system can be used to establish stability properties of the systems [57, 38]. LFs give sufficient conditions for various conspicuous stability properties of control systems, for example: Lyapunov stability [77] and instability, and global or local AS (in some cases, LaSalle's invariance principle [63, 64] may be applied) of autonomous systems; uniform stability, (global) uniform asymptotic stability, and (global) exponential stability of general nonautonomous systems; exponential stability of linear time-varying systems; ISS [127] and IOS [129, 130] of nonlinear systems (autonomous and nonautonomous). The main drawback of this approach is the absence of constructive methods for design of LFs for a given nonlinear dynamical system. Such a problem has a well-known solution for linear systems, and a quadratic LF can be analyzed. For nonlinear dynamics that can be locally approximated by linear ones, the same LF candidate can be tested, or just eigenvalue analysis of linearization can be performed. Due to the complexity of nonlinear systems, most existing approaches for synthesizing Lyapunov functions for nonlinear dynamics involve various canonical forms such as Lur'e systems [37], Lipschitz dynamics, Persidskii systems [107], and homogeneous models, whose stability developments include: absolute stability [66], parametric absolute stability [141] and ISS [119] of Lur'e systems; global exponential stability of Lipschitz dynamics [102]; the necessary and sufficient conditions for absolute stability of Persidskii systems [107]; finite-time stability [17], ISS and integral ISS [15] of homogeneous models.

In this thesis, we focus our attention on a class of so-called *generalized Persidskii systems*, which have been extensively studied in the context of neural networks [32], power systems [47], infection dynamics [90], biological systems [83], and opinion dynamics [88] (for demonstrating examples, see, *e.g.* image classification [95], speech processing [78], short-circuit fault [47], automatic voltage regulator [48], Chua's circuits [80], Hindmarsh–Rose model [43], Lotka–Volterra equations [45]). The original form of this class of models was introduced in [107, 11, 55]. To address the problem that stability depends upon the system structure rather than the magnitude of its parameters, the authors in [114] presented the first work of ”qualitative stability” (or ”sign stability”), in which linear systems represented by ”sign stable matrices” admit a quadratic LF with a diagonal matrix. This motivated a theorem in [107], which also focuses on extending to a broader class of systems with a specific type of sign stable structure, followed by the study of [55], where a more general class of systems (under a feedback control) with one nonlinearity satisfying the positive infinite sector condition was proposed, as well as the absolute stability conditions for such a kind of systems. Recently the ISS, IOS, SIIOS, and ROS conditions have been established in [30, 83] for a generalized version (generalized Persidskii systems). In this work, we consider the challenging control problems of robust synchronization, state estimation, and time-delay effect in the stability of generalized Persidskii systems, whose technical particulars will be specifically investigated in chapters 3, 4, 5, respectively.

Synchronization

Synchronization is a complex phenomenon frequently observed in networked and interconnected systems. Formally, synchronization means diminishing the difference among the solutions of interconnected/networked systems. It has been extensively investigated in various fields, *e.g.*, neuroscience, robotics, communication security, and autonomous driving [99, 134, 109, 97, 18], to address practical problems, *e.g.*, cooperative schemes for multiple robot manipulators [132]; GPS disciplined oscillators are used to synchronize telecommunication networks with high time accuracy [70]. In the field of systems and control, the principal approaches to achieve synchronization for nonlinear systems are based on the passivity theory [108, 39], output regulation [23], incremental stability [7], Lyapunov approach [112], to mention a few recent results. Notice that Lur'e systems constitute a popular benchmark for testing these theories [135, 149, 58, 113]. Many neural models constitute popular benchmarks for studying synchronization (*e.g.*, Hindmarsh Rose and FitzHugh Nagumo models). They can be presented in the generalized Persidskii form, motivating the development of conditions for the emergence of synchronous movements for the considered class of systems, especially in the presence of external inputs, further investigated in this thesis.

State observation

The unmeasured state observation (or estimation) of dynamical systems is a fundamental engineering problem whose solution is needed in numerous control or monitor applications [57, 49]. Then it is naturally interesting to design state observation schemes for generalized Persidskii systems. State estimation means approximating the internal state of a physical system from measurements of the input and output of the considered plant. The estimation theory is relatively well developed nowadays [16], and many different observers were proposed in the literature for linear plants (*e.g.*, the most popular methods are the Kalman filter [52] or Luenberger observer [75]) or for nonlinear systems whose models can be approximated by linear ones under certain hypotheses [62, 36, 40]. For example, using the first-order approximations, the principal linear approaches can be extended to nonlinear plants (the extended/unscented Kalman filter [53, 120], moving horizon estimation [115]). Another related popular direction is focused on linear parameter-varying (LPV) or quasi-LPV representation of nonlinear dynamics that permits the utilization of linear estimation tools [94, 71]. Many differentiation algorithms are based on the state estimation techniques developed for a chain of integrators (a linear model) [117], as the seminal high-order sliding mode differentiator [69]. Methods based on linear approximations may lose their validity if the state estimation is demanded at large (or in the presence of a severe signal and parametric uncertainty). Then some canonical forms of nonlinear dynamical systems are considered, as in the cases mentioned above.

Time delay

The inclusion of time delays implies utilizing even more complex stability analysis approaches, *e.g.*, using Lyapunov–Krasovskii functionals [61] or Lyapunov-Razumikhin functions [116]. Besides these two methods, it is also possible to consider the comparison techniques [56]. Moreover, the appearance of lags usually requires a redesign of regulation or estimation algorithms since time delays may degrade the performance or even lead to instability of the system [34]. The Lyapunov–Razumikhin technique sometimes results in conservative results, but it can be applied to the case of bounded time-varying delays, while the Lyapunov–Krasovskii approach requires a bounded time derivative of time-varying delays (it is clear that in the case of constant time delays, this condition is satisfied). For time-delay systems with inputs, these methods have been extended following the ISS framework by using Lyapunov-Razumikhin functions [137] and Lyapunov-Krasovskii functional [103]. Nevertheless, there is still a lack of Lyapunov characterizations for time-delay systems [93]. On the other hand, for the mentioned canonical nonlinear systems, due to their possibly high nonlinear nature and uncertainties, it may be challenging to formulate ISS conditions for those dynamics with delays, even with constant time delays.

1.2 State of the art

In the spirit of the interests of canonical forms of nonlinear dynamics, numerous popular nonlinear systems have been proposed, among them $\dot{x} = Af(x)$ [11, 107] and $\dot{x} = A_0x + A_1f(x)$ [1, 76] (here, we assume that $x : \mathbb{R} \to \mathbb{R}^n$ is a time-dependent function, all matrices, A, A_0, *etc.*, have appropriate dimensions, and the nonlinearities have diagonal structure, $f(x) = \begin{bmatrix} f_1(x_1) & \dots & f_n(x_n) \end{bmatrix}^\top$, with each f_i, $i = \overline{1,n}$, belonging to a sector; details will be given later) attract our attention, since they can be used to model many physical dynamics. Particularly, the former is named a Persdiskii system, and the latter one has been extensively named as "Lur'e system" and has been widely investigated in the fields of absolute stability [145, 146] and dissipativity [131]. Inspired mainly by the study in [107], the so-called generalized Persidskii systems have been proposed [30], to generalize both, Persidskii and Lur'e systems, by considering multiple nonlinearities $\dot{x} = A_0x + \sum_{j=1}^{M} A_j f_j(x)$. As seen above, there is a close connection between (generalized) Persidskii systems and Lur'e models. The main advantage of generalized Persidskii systems is their breadth in modelings of practical systems (they can be utilized to represent more cases than Lur'e ones).

Stability analysis of Persidskii models, Lur'e dynamics, and generalized Persidskii systems are firmly based on Lyapunov theory. In generalized Persidskii systems, stability conditions can be formulated in terms of linear matrix inequalities under the assumption that the nonlinearities obey the sector boundedness condition. One of the main advantages of the proposed results is that all cross-terms in the Lyapunov function and its time derivative can be accurately treated rather than be regarded as a perturbation. An illustration of the advantages of these achievements: the ISS stability conditions for generalized Persidskii systems can be consulted in [30], in which a coordinate transformation method was applied to transfer the well-known population dynamics given by Lotka–Volterra equations to a system in the form of generalized Persidskii systems, then a relevant ISS analysis follows.

As far as we are aware, there are not sufficient works considering the conditions of synchronization of Persidskii-like systems (*e.g.*, the Persidskii model and its generalized version). One may check the synchronization results of Lur'e systems for recent relevant advances. The output synchronization [58] of Lur'e systems composed of a passive linear system and a static feedback nonlinearity usually requires passivity, which results in the necessary application of passivity-based approaches [133, 9, 111]. To overcome the limitations of the passivity-based methods, the algebraic connectivity of the considered network of systems is required to exceed a threshold. Also, in the presence of external disturbances, it is important to guarantee the state's boundedness and the synchronization's robustness (an internal-model-based controller can be a solution in such a case, see *e.g.* [28, 22]). Further methods for synchronizing Lur'e dynamics include impulsive control [72], Lyapunov method [136], robust synthesis of full static-state error feedback and dynamic-output error feedback [135], and sampled-data feedback control [73].

The observer design for the generalized Persidskii system was still a blank field to be investigated until a simple state observer was proposed [90]. Therefore, we start from a review of observer design for a similar class of systems: Lur'e dynamics. In [10], global convergent observers for Lur'e-like systems with monotonic nonlinearities were studied by applying ISS theory (*i.e.*, regarding noises as unknown deterministic inputs and the estimation error as the state, respectively), and full-order state observers for Lur'e-like systems with Lipschitz nonlinearities were also considered [3], by using ISS theory analogously. State observers for Lur'e systems with multivalued mappings in the feedback were proposed in [21] under a passivity condition on the linear part of the observation error dynamics. Also, the ISS method can be utilized to design full-order time-varying state observers for general nonlinear systems [4].

Although the generalization of the characterizations of ISS for ordinary differential equations [128] is far from being straightforward, there are many developments of ISS for delayed nonlinear systems. In [35], the sufficient conditions of ISS for systems with time-varying delays were presented, as well as the nonlinear matrix inequalities (NLMIs) approach-based ISS conditions for a class of systems with delayed state-feedback. The reference [79] studied ISS conditions for a type of delayed dynamics resulting from feedback delays. A connection between the exponential stability of a general unforced nonlinear time-delay system and its ISS was established in [147]. Recently, the characterizations of (local) ISS for a class of infinite-dimensional systems (take the form of a linear term and a nonlinearity defined on a space consisting of a Banach space of the state and a normed space of the input) were introduced [92, 91]. Further directions of ISS analysis of time-delay systems encompass: ISS conditions for general nonlinear systems with delayed impulses [65] and general nonlinear time-delay systems subject to delay-dependent impulse effects [68]; ISS analysis of general discrete time-delay systems by the Lyapunov–Razumikhin technique [67], to recap a few examples.

1.3 Gaps to fill

The generalized Persidskii systems allow many highly nonlinear physical and technical processes to be accurately modeled (*e.g.*, opinion dynamics, Hindmarsh-Rose model, (delayed) Lotka-Volterre equations for population dynamics). At the same time, there is a lack of analysis and design methodology for this class of models, as we can conclude from the existing literature. The closest results deal with Lur'e systems, which form a subclass of generalized Persdiskii dynamics. On another side, most control and estimation algorithms for Lur'e systems are based on the ISS property, while in practice more complex stability properties can be of interest, such as the IOS property. Therefore, in this work, we aim to bridge the results obtained for Lur'e systems with generalized Persidskii dynamics and extend all of them by using the theory of IOS.

Regardless of the breadth of generalized Persidskii systems presented in [30] (*i.e.*, they can be brought to numerous practical models), it is still beneficial to raise a more general framework to connect those related physical dynamics, under the restriction that their properties are reserved and to open the potential for unknown models. Therefore, in this study, another type (type II) of generalized Persidskii system, extended from one analyzed in [30], is considered to widen the possible range of applications, *i.e.*, to reflect the circumstances of, for example, non-square and non-identity weight matrices in neural networks.

The Lyapunov theory is a standard framework for stability analysis of nonlinear dynamics [57], and the absence of constructive tools for the selection of a LF for a general nonlinear system is its main drawback (thus, the mentioned canonical forms of nonlinear models with known compositions of LFs provide an answer to this weakness). Although in [30], a useful LF was employed for the ISS analysis of generalized Persidskii systems, there is still a lack of sufficient works dealing with the LF to justify its availability in various stability analyses, synchronization, and state observation of those dynamics or their time-delay version. To fill these gaps, in this thesis, the LF

$$V(x) = x^\top Px + 2\sum_{j=1}^{M}\sum_{i=1}^{n}\Lambda_i^j\int_0^{x_i} f_i^j(\tau)d\tau$$

and its variation (dedicated to the ISS analysis of time-delay generalized Persidskii systems) are considered, whose efficacy for the indicated control problems is proved.

Neither the IOS, SIIOS, and ROS properties of generalized Persidskii systems nor the IOS application to robust synchronization of these systems (with output having the same kernel as the synchronous mode) were previously investigated to the present. Even more, rare studies use the IOS technique to achieve robust synchronizations. It is also worth dealing with the unresolved synchronization problem of general linear systems admitting an upper bound for the norm of the input/disturbance that can be transformed into a study of robust synchronization of generalized Persidskii dynamics. Moreover, to work on the unfilled gaps of observer design for generalized Persidskii models may be useful in observer analysis for systems with continuous but non-Lipschitz nonlinearities, for instance, and may offer new insight for the state estimation of practical models. These gaps and the unresolved problem of the ISS analysis of generalized Persidskii systems with time delays motivate us to tackle the first works on the aforementioned problems. Due to the intrinsic complexity of generalized Persidskii models, the ISS of those time-delay models can be rather tricky. For this analysis, we select a Lyapunov–Krasovskii functional, including the terms dependent on $\dot{x}$ (hence, the stability analysis can be performed in a Sobolev space), and utilize the descriptor method, Jensen's inequality, the compensation principle, and other constructive methods to establish the feasible and solvable ISS sufficient conditions formulated in matrix inequalities. Also, an example of opinion dynamics with constant time delays elicits a problem: if a nonlinear feedback control can be applied to cancel nonlinearities leading to a linear controlled system? In this work, this is verified to be infeasible due to the demand for interactivity among agents, whereas the proposed ISS conditions (in Chapter 5) remain valid for the resulting closed-loop system in the generalized Persidskii form.

1.4 List of contributions

The contributions of my Ph.D. work deal with the development of several stability analysis conditions and control designs for generalized Persidskii systems:

1. IOS, SIIOS, and ROS analyses and the IOS application to robust synchronization [83, 80]
2. Observer design by the theories of IOS and SIIOS [90, 81]
3. ISS conditions for time-delay version [88, 82]
4. Convergence conditions for both continuous-time and discrete-time dynamics [87, 86]
5. Introduction of short-time stability (STS) notions and analysis of annular STS [84, 85]

CHAPTER 2

Preliminaries

This chapter presents an overview of basic concepts of dynamical systems, stability properties of general nonlinear systems and nonlinear retarded dynamics, synchronization of interconnected systems, and observer design for dynamical systems.

2.1 Dynamical systems

Since this thesis deals with control problems in dynamical systems, we initially consider a general definition of dynamical systems:

Definition 2.1. *[123] A tuple*

$$(I^{'}, \mathcal{M}, f) \tag{2.1}$$

is called a dynamical system, *where $I^{'}$ is the time interval; $\mathcal{M}$ is a manifold,* i.e., *locally a Banach space or Euclidean space; $f : \mathcal{M} \times I' \to \mathcal{M}$ is an evolution rule such that $f(.,t)$ is a diffeomorphism of $\mathcal{M}$ to itself.*

Remark 2.1. *There are many distinct generic definitions of dynamical systems, one of which, for instance, regards f as an action of a group $\big($or even of a semigroup,* e.g., $(\mathbb{R}_{+}, +)$ *for continuous-time$\big)$ $I^{'}$ on the* state space $\mathcal{M}$, i.e., *the* flow $f : \mathcal{M} \times I^{'} \to \mathcal{M}$, $(x,t) \to f(x,t)$ *[123, 121].*

A specific form of continuous-time dynamical systems can be represented by the ordinary differential equation

$$\dot{x}(t) = f(x(t), t), \quad x(t_0) = x_0, \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$; $t_0, t \in I^{'} \subset \mathbb{R}$; the function $f : \mathbb{R}^n \times I^{'} \to \mathbb{R}^n$. Here $\mathcal{M} = \mathbb{R}^n$.

In this study, the well-posedness of the dynamical system (2.2) is required, for which we present some fundamental results on the existence and uniqueness of the solution of (2.2) in the sequel. We first consider a generalization of the Picard-Lindelöf Theorem [57, 60] for (2.2):

Theorem 2.1 (**Local Existence and Uniqueness).** *[60] Let $\mathbb{B}$ be a Banach space, $\mathbb{S}$ be a region in $\mathbb{B}$, and $D = \mathbb{S} \times I^{'}$. Let the function $f : D \to \mathbb{B}$ be continuous for any $x \in \mathbb{S}$. Further let f satisfy*

$$\|f(x_1,t) - f(x_2,t)\|_{\mathbb{B}} \le c\|x_1 - x_2\|_{\mathbb{B}}, \quad \forall (x_1,t),(x_2,t) \in D,$$

where $c > 0$ is a constant and $\|\cdot\|_{\mathbb{B}}$ denotes the norm in $\mathbb{B}$. Then for each $(x_0,t_0) \in D$, there exists an interval $I \subset I^{'}$ with t_0 at its center such that (2.2) has a unique solution over the interval I.

The next theorem is involved in a global version of the statement of Theorem 2.1:

Theorem 2.2 (**Global Existence and Uniqueness).** *[60] Let $\mathbb{S} = \mathbb{B} = \mathbb{R}^n$ and f as in Theorem 2.1. Then for each $(x_0,t_0) \in \mathbb{R}^n \times I^{'}$, there exists an interval $I \subset I^{'}$ with t_0 at its center such that (2.2) has a unique solution over the interval I.*

Because of the conservativeness of the global Lipschitz condition, it would be useful to have a unique solution over the time interval $[t_0,\infty)$. The following theorem achieves that under a restriction on the solution of the system (2.2).

Theorem 2.3. *[57] Let $\mathbb{S} \subset \mathbb{B} = \mathbb{R}^n$, $I^{'} = [t_0,+\infty)$, $\mathbb{S}^{'}$ be a compact subset of $\mathbb{S}$, $x_0 \in \mathbb{S}^{'}$, and f be continuous in t and locally Lipschitz in x for all $(x,t) \in D$. Further let every solution of (2.2) lies entirely in $\mathbb{S}^{'}$. Then (2.2) has a unique solution for all $t \ge t_0$.*

Note that there are also other results on the existence of the solution of (2.2), *e.g.*, Peano Existence Theorem (it requires only continuity on f) [26] and Carathéodory's Existence Theorem (with even weaker conditions than continuity) [26], claiming less restrictive conditions on f.

2.2 Stability properties for general nonlinear systems

As emphasized above, stability plays a significant role in the performance analysis of dynamical systems. Furthermore, this study also concerns robustness problems and output properties in nonlinear dynamical systems. Therefore, let us consider a class of nonlinear systems in the presence of external disturbance/input and output:

$$\begin{aligned} \dot{x}(t) &= f(x(t),d(t)),\ \forall t \in \mathbb{R}_+, \quad \text{with } f(0,0) = 0,\ x_0 = x(0), \\ y(t) &= h(x(t)), \end{aligned} \tag{2.3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $d(t) \in \mathbb{R}^m$ is the external perturbation/input, $d \in \mathcal{L}_\infty^m$; $y(t) \in \mathbb{R}^p$ is the output vector. Moreover, $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz continuous function and $h\colon \mathbb{R}^n \to \mathbb{R}^p$ is a continuously differentiable function. For an initial state $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$, the corresponding solution of (2.3) is denoted by $x(t, x_0, d)$; for the values of $t \geq 0$ the solution exists, so the corresponding output is $y(t, x_0, d) = h(x(t, x_0, d))$.

Definition 2.2. *The system* (2.3) *is* forward complete *if for all $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$, the solution $x(t, x_0, d)$ is uniquely defined for all $t \geq 0$.*

In the rest of this work, to lighten the notation, the time-dependency of variables might remain implicitly understood, for instance we write x for $x(t)$. Let us give some stability definitions that will be used in the sequel.

Definition 2.3. *[130] A forward complete system* (2.3) *is said to be:*

1. practical input-to-output stable *(pIOS) if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $c \in \mathbb{R}_+$ such that*

$$\|y(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \gamma(\|d\|_\infty) + c, \ \forall t \geq 0$$

   for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$. The system is input-to-output stable *(IOS) if $c = 0$. When $y = x$, the IOS property is called* input-to-state stability *(ISS).*

2. output-Lagrange input-to-output stable *(OLIOS) if it is IOS and there exist $\sigma_1, \sigma_2 \in \mathcal{K}$ such that*

$$\|y(t, x_0, d)\| \leq \max\{\sigma_1(\|h(x_0)\|), \sigma_2(\|d\|_\infty)\}, \ \forall t \geq 0$$

   for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$.

3. state-independent input-to-output stable *(SIIOS) if there exist $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}$ such that*

$$\|y(t, x_0, d)\| \leq \beta(\|h(x_0)\|, t) + \gamma(\|d\|_\infty), \ \forall t \geq 0$$

   for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$.

4. robustly output stable *(ROS) if there exist a smooth function $\alpha \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that the system*

$$\dot{x} = \tilde{f}(x, \varsigma) := f(x, \varsigma\alpha(\|h(x)\|)) \tag{2.4}$$

   is forward complete, and the estimate

$$\|y_\alpha(t, x_0, \varsigma)\| \leq \beta(\|x_0\|, t), \ \forall t \geq 0$$

is satisfied for all $\varsigma \in \tilde{\mathscr{L}}^m_{\mathscr{C}}$*, where* $\mathscr{C} := \{\mu \in \mathbb{R}^m : \|\mu\| \le 1\}$*, and* $y_\alpha(t, x_0, \varsigma) = h(x(t, x_0, \varsigma))$ *denotes the output function of the system* (2.4)*.*

Definition 2.4. [130] *A forward complete system* (2.3) *is said to be* uniformly bounded-input-bounded-state stable *(UBIBS) if there exists* $\sigma \in \mathscr{K}$ *such that*

$$\|x(t, x_0, d)\| \le \max\{\sigma(\|x_0\|), \sigma(\|d\|_\infty)\},\ \forall t \ge 0$$

for all $x_0 \in \mathbb{R}^n$ *and* $d \in \mathscr{L}^m_\infty$*.*

The next definition is for the considered kinds of Lyapunov functions in this thesis.

Definition 2.5. [130] *For the system* (2.3)*, a smooth function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *is:*

1. *an IOS*-Lyapunov function *if there exist* $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$*,* $\chi \in \mathscr{K}$ *and* $\alpha_3 \in \mathscr{KL}$ *such that*

$$\begin{aligned} \alpha_1(\|h(x)\|) \le V(x) &\le \alpha_2(\|x\|), \\ V(x) \ge \chi(\|d\|) \quad &\Rightarrow \quad \nabla V(x) f(x,d) \le -\alpha_3(V(x), \|x\|) \end{aligned} \tag{2.5}$$

for all $x \in \mathbb{R}^n$ *and* $d \in \mathbb{R}^m$*.*

2. *an OLIOS*-Lyapunov function *if there exist* $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$*,* $\chi \in \mathscr{K}$ *and* $\alpha_3 \in \mathscr{KL}$ *such that*

$$\begin{aligned} \alpha_1(\|h(x)\|) \le V(x) &\le \alpha_2(\|h(x)\|), \\ V(x) \ge \chi(\|d\|) \quad &\Rightarrow \quad \nabla V(x) f(x,d) \le -\alpha_3(V(x), \|x\|) \end{aligned} \tag{2.6}$$

for all $x \in \mathbb{R}^n$ *and* $d \in \mathbb{R}^m$*.*

3. *an SIIOS*-Lyapunov function *if there exist* $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ *and* $\chi, \alpha_3 \in \mathscr{K}$ *such that*

$$\begin{aligned} \alpha_1(\|h(x)\|) \le V(x) &\le \alpha_2(\|h(x)\|), \\ V(x) \ge \chi(\|d\|) \quad &\Rightarrow \quad \nabla V(x) f(x,d) \le -\alpha_3(V(x)) \end{aligned} \tag{2.7}$$

for all $x \in \mathbb{R}^n$ *and* $d \in \mathbb{R}^m$*.*

4. *an ROS*-Lyapunov function *if there exist* $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$*,* $\chi \in \mathscr{K}$ *and* $\alpha_3 \in \mathscr{KL}$ *such that*

$$\begin{aligned} \alpha_1(\|h(x)\|) \le V(x) &\le \alpha_2(\|x\|), \\ \|h(x)\| \ge \chi(\|d\|) \quad &\Rightarrow \quad \nabla V(x) f(x,d) \le -\alpha_3(V(x), \|x\|) \end{aligned}$$

for all $x \in \mathbb{R}^n$ *and* $d \in \mathbb{R}^m$*.*

Theorem 2.4. [130] *A UBIBS system* (2.3) *is IOS (OLIOS, SIIOS, ROS) if and only if it admits an IOS (OLIOS, SIIOS, ROS)-Lyapunov function.*

Remark 2.2. *Note that for a sufficient condition of IOS, SIIOS, or ROS, the UBIBS requirement can be discarded provided that the system* (2.3) *is forward complete (or it possesses the unboundedness observability property [8]), and an IOS/SIIOS-Lyapunov function or a ROS-Lyapunov function satisfies* (2.5) *((2.6) in SIIOS case) and respectively,*

$$V(x) \geq \chi(\|d\|) \quad \Rightarrow \quad \nabla V(x) f(x,d) \leq -\alpha_3(V(x)) \tag{2.8}$$

or

$$\|h(x)\| \geq \chi(\|d\|) \quad \Rightarrow \quad \nabla V(x) f(x,d) \leq -\alpha_3(V(x))$$

for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$, some $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ and $\chi, \alpha_3 \in \mathscr{K}$.

2.3 Stability properties for nonlinear retarded systems

To provide preliminary descriptions for time-delay generalized Persidskii systems studied in Chapter 5, we then consider the nonlinear retarded dynamics [59, 35]:

$$\dot{x}(t) = f(x_t, d(t)),\ t \in \mathbb{R}_+, \tag{2.9}$$

where $x(t) \in \mathbb{R}^n$; $x_t \in \mathbb{W}_{[-\tau,0]}$ is the state function, $x_t(s) = x(t+s)$ for $s \in [-\tau, 0]$, $\tau > 0$ is a constant delay; $d(t) \in \mathbb{R}^m$ is the external input, $d \in \mathscr{L}_\infty^m$; $f : \mathbb{W}_{[-\tau,0]} \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous functional, $f(0,0) = 0$, and it ensures the existence and the uniqueness of solutions in forward time for the system (2.9). With the initial condition $x_0 \in \mathbb{W}_{[-\tau,0]}$ and the input $d \in \mathscr{L}_\infty^m$, such a unique solution is defined as $x(t, x_0, d)$, for which $x_t(s, x_0, d) = x(t+s, x_0, d)$, $s \in [-\tau, 0]$ denotes the corresponding state function.

We need a useful derivative for ISS analysis of (2.9): for a continuous functional $V : \mathbb{R} \times \mathbb{W}_{[-\tau,0]} \times C^n_{[-\tau,0]} \to \mathbb{R}_+$, we define the following derivative along the solutions of (2.9) [101]:

$$D^+V(t,\phi,\ell) = \limsup_{h \to 0^+} \frac{V(t+h, x_h(\phi,\ell), \dot{x}_h(\phi,\ell)) - V\left(t,\phi,\dot{\phi}\right)}{h},$$
$$x_h(\phi,\ell)(s) = \begin{cases} \phi(s+h), & s \in [-\tau, -h] \\ \phi(0) + (h+s)\cdot f(\phi,\ell), & s \in [-h, 0] \end{cases}$$

for any $\phi \in \mathbb{W}_{[-\tau,0]}$ and $\ell \in \mathbb{R}^m$.

Definition 2.6. *[103, 35] The system* (2.9) *is called* input-to-state stable *(ISS), if there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *such that*

$$\|x(t,x_0,d)\| \le \beta(\|x_0\|_{\mathbb{W}_{[-\tau,0]}},t)+\gamma(\|d\|_{[0,t)}), \quad \forall t \in \mathbb{R}_+$$

for all $x_0 \in \mathbb{W}_{[-\tau,0]}$ *and* $d \in \mathcal{L}_\infty^m$.

Definition 2.7. *[103, 35] The system* (2.9) *is said to possess the* asymptotic gain (AG) property*, if there exists* $\gamma \in \mathcal{K}$ *such that* $\limsup_{t\to+\infty}\|x(t,x_0,d)\| \le \gamma(\|d\|_\infty)$ *for all* $x_0 \in \mathbb{W}_{[-\tau,0]}$ *and* $d \in \mathcal{L}_\infty^m$.

Definition 2.8. *[103, 35] A continuous functional* $V : \mathbb{R}\times\mathbb{W}_{[-\tau,0]}\times C^n_{[-\tau,0]} \to \mathbb{R}_+$ *is called an* ISS Lyapunov-Krasovskii functional *(LKF) if there exist some* α_1, $\alpha_2 \in \mathcal{K}_\infty$, α_3, $\chi \in \mathcal{K}$ *such that*

$$\alpha_1(\|\phi(0)\|) \le V\left(t,\phi,\dot\phi\right) \le \alpha_2\left(\|\phi\|_{\mathbb{W}_{[-\tau,0]}}\right),$$
$$V\left(t,\phi,\dot\phi\right) \ge \chi(\|d\|) \quad \Rightarrow \quad D^+V(t,\phi,d) \le -\alpha_3\left(\|\phi\|_{\mathbb{W}_{[-\tau,0]}}\right)$$

for all $t \in \mathbb{R}_+$, $\phi \in \mathbb{W}_{[-\tau,0]}$ *and* $d \in \mathbb{R}^m$.

Theorem 2.5. *[103, 35] If the system* (2.9) *admits an ISS LKF, then it is ISS with* $\gamma = \alpha_1^{-1}\circ\chi$.

The existence of an LKF can also be necessary for ISS property under additional restrictions on continuity of f in (2.9) [104, 31], for instance, the function f is required to be Lipschitz on bounded sets in [54, 105].

Remark 2.3. *In this thesis stability definitions are given in Sobolev space. In many cases, it is technically proficient to use* $\dot x_t$ *as an argument of LKF (see,* e.g.*, [31, 33]), and in such a situation, the stability of the system is analyzed in a Sobolev space* $\mathbb{W}_{[-\tau,0]}$*. Nevertheless, for example, in [82]* $\dot x_t$ *is excluded, then the system* (2.9) *and definitions 2.6, 2.7, 2.8 save their meaning after substitution of* $C^n_{[-\tau,0]}$, $\|\phi\|_{[-\tau,0]}$ *in place of* $\mathbb{W}_{[-\tau,0]}$, $\|\phi\|_{\mathbb{W}_{[-\tau,0]}}$.

2.4 Generalized Persidskii systems

This work focuses on a class of so-called *generalized Persidskii systems*. In this section, two types of those systems are given. The first type was proposed for addressing modeling problems in power systems [47] and biological systems [83], to refer to a few cases, while the second kind was raised, mainly because, for example, in neural networks (see *e.g.* [46]), a weight matrix is usually non-square and not an identity matrix.

2.4.1 Type I

The following class of systems is the first type of generalized Persidskii systems:

$$\begin{aligned} \dot{x}(t) &= A_0x(t)+\sum_{j=1}^{M}A_jf^j(x(t))+d(t),\ \forall\, t\geq 0, \\ y(t) &= Cx(t), \end{aligned} \tag{2.10}$$

where $x(t) = [x_1(t)\dots x_n(t)]^\top \in \mathbb{R}^n$ is the state vector; $y(t) \in \mathbb{R}^p$ is the output signal; $C \in \mathbb{R}^{p\times n}$ with $C \neq 0$; $d(t) \in \mathbb{R}^n$ is the external perturbation, $d \in \mathscr{L}_\infty^n$; $f^j : \mathbb{R}^n \to \mathbb{R}^n$ with $f^j(x) = [f_1^j(x_1)\dots f_n^j(x_n)]^\top$, $j \in \overline{1,M}$ ($M \in \mathbb{N}\backslash\{0\}$) are continuous functions ensuring the existence of solutions of (2.10) in the forward time at least locally and $A_s \in \mathbb{R}^{n\times n}$, $s \in \overline{0,M}$.

In this thesis, it is assumed that if the upper limit of a summation or a sequence is smaller than the lower one, then the corresponding terms (or conditions) are omitted.

Following [107], the sector restrictions on f^j, $j \in \overline{1,M}$ are imposed:

Assumption 2.1. *For any $i \in \overline{1,n}$, $j \in \overline{1,M}$:*

$$\tau f_i^j(\tau) > 0, \quad \forall \tau \in \mathbb{R}\setminus\{0\}.$$

Assumption 2.1 states that all nonlinearities belong to a sector and may take zero values at zero only, and it is the main restriction on the class of systems given in (2.10).

For further use, we denote by the index $\varpi \in \overline{0,M}$, a positive integer such that for all $i \in \overline{1,n}$, $a \in \overline{1,\varpi}$:

$$\lim_{\tau\to\pm\infty} f_i^a(\tau) = \pm\infty$$

and by $\mu \in \overline{\varpi,M}$, a positive integer such that for all $i \in \overline{1,n}$, $b \in \overline{1,\mu}$:

$$\lim_{\nu\to\pm\infty}\int_0^\nu f_i^b(\tau)d\tau = +\infty.$$

The index $\varpi > 0$ characterizes the radially unbounded nonlinearities, and $\varpi = 0$ corresponds to the case when all nonlinearities are bounded (at least for negative or positive arguments). The index $\mu > 0$ selects the nonlinearities having unbounded integrals. Clearly, if $\varpi > 0$, then all radially unbounded nonlinearities also have unbounded integrals, thus $\mu \geq \varpi$ due to the introduced sector condition. Indexes ϖ and μ can be obtained after a proper re-indexing and decomposition of the f^j, and the featured restriction of (2.10) is formulated in Assumption 2.1 (the sector condition).

Remark 2.4. *The Lur'e models under the sector conditions [146, 66] may be presented in the form* (2.10) *under Assumption 2.1. The advantage of* (2.10) *over Lur'e dynamics is that all* *cross-terms between x_i and $f_i^j(x_i)$ appearing in the expressions of V and $\dot{V}$ can be accurately treated, rather than be considered as perturbations (see [30] or Chapter 3). The same analysis in the conventional form of Lur'e model can be less straightforward (especially for $M > 1$ and for $f_i^j(x_i)f_i^k(x_i)$ with $j \neq k \in \overline{1,M}$).*

2.4.2 Type II

Let us then consider another type of generalized Persidskii system:

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + \sum_{j=1}^{M} A_j f^j(H_jx(t)) + Dw(t),\\ y(t) &= \begin{bmatrix} C_0x(t) \\ C_1f^1(H_1x(t)) \\ \vdots \\ C_Mf^M(H_Mx(t)) \end{bmatrix} + v(t), \quad t \in \mathbb{R}_+,\end{aligned} \tag{2.11}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $A_0 \in \mathbb{R}^{n\times n}$, $A_j \in \mathbb{R}^{n\times k_j}$ and $H_j \in \mathbb{R}^{k_j\times n}$ for $j \in \overline{1,M}$; $f^j(\ell) = [f_1^j(\ell_1)\dots f_{k_j}^j(\ell_{k_j})]^\top$ with $\ell = [\ell_1 \dots \ell_{k_j}]^\top \in \mathbb{R}^{k_j}$ for $j \in \overline{1,M}$ are the functions ensuring the existence of the solutions of the system (2.11) in the forward time at least locally (to shorten further writing we define $k_0 = n$ and $H_0 = I_n$); $y(t) \in \mathbb{R}^z$ is the output available for measurements, $z = \sum_{s=0}^{M} z_s$ and $C_s \in \mathbb{R}^{z_s\times k_s}$ for $s \in \overline{0,M}$; $D \in \mathbb{R}^{n\times p}$; $w(t) \in \mathbb{R}^p, v(t) \in \mathbb{R}^z$ are the external perturbations, $w \in \mathscr{L}_\infty^p$, $v \in \mathscr{L}_\infty^z$.

Assumption 2.1 is also imposed on f^j, $j \in \overline{1,M}$ in the system (2.11). Similarly, under Assumption 2.1, with a reordering of nonlinearities and their decomposition, there exists an index $\varpi \in \overline{0,M}$ such that for all $a \in \overline{1,\varpi}$ and $i \in \overline{1,k_a}$, $\lim_{\nu\to\pm\infty} f_i^a(\nu) = \pm\infty$. Also, there exists $\mu \in \overline{\varpi,M}$ such that for all $b \in \overline{1,\mu}$, $i \in \overline{1,k_b}$, we have $\lim_{\nu\to\pm\infty} \int_0^\nu f_i^b(r)dr = +\infty$.

2.5 Stability conditions for generalized Persidskii systems

In this section, the conditions for checking IOS, ROS, and SIIOS properties of generalized Persidskii systems are formulated. These stability conditions are useful for the analyses of, for instance, robust synchronization and the performance of the proposed observers in chapters 3 and 4, respectively. The following theorem is one of the main results of this work.

Theorem 2.6. *[83] Let Assumption 2.1 be satisfied. If there exist $0 \leq P_1 = P_1^\top \in \mathbb{R}^{p\times p}$, $0 \leq P_2 = P_2^\top \in \mathbb{R}^{n\times n}$, $\Lambda^j = \mathrm{diag}(\Lambda_1^j, \dots, \Lambda_n^j) \in \mathbb{D}_+^n$ ($j \in \overline{1,M}$); $\Theta \in \mathbb{D}_+^n$; $\Psi \in \mathbb{D}_+^p$; $\Xi^k \in \mathbb{D}_+^n$ ($k \in \overline{0,M}$);*

$\{\Upsilon_{s,z}\}_{z=s+1}^{M} \subset \mathbb{D}_+^n$ *($s \in \overline{0, M-1}$) and* $0 < \Phi = \Phi^\top \in \mathbb{R}^{n\times n}$ *such that*

$$P_1 > 0 \quad or \quad P_2 > 0 \quad or \quad \sum_{j=1}^{\mu} \Lambda^j > 0; \tag{2.12}$$
$$P_2 \le \Theta; \quad Q = Q^\top = (Q_{a,\,b})_{a,\,b=1}^{M+2} \le 0,$$

where

$$P_C = C^\top P_1 C + P_2; \quad Q_{1,1} = A_0^\top P_C + P_C A_0 + \Xi^0 + C^\top \Psi C; \quad Q_{j+1,j+1} = A_j^\top \Lambda^j + \Lambda^j A_j + \Xi^j,\ j \in \overline{1, M},$$
$$Q_{1,j+1} = P_C A_j + A_0^\top \Lambda^j + \Upsilon_{0,j},\ j \in \overline{1,M}; \quad Q_{1,M+2} = P_C,$$
$$Q_{s+1,z+1} = A_s^\top \Lambda^z + \Lambda^s A_z + \Upsilon_{s,z},\ s \in \overline{1, M-1},\ z \in \overline{s+1, M}; \quad Q_{j+1,M+2} = \Lambda^j, j \in \overline{1,M}; \quad Q_{M+2,M+2} = -\Phi,$$

then a forward complete system (2.10) *is ROS if*

$$\Psi > 0; \quad \Theta + \sum_{j=1}^{M} \Lambda^j \le \xi \left(\sum_{k=0}^{M} \Xi^k + 2 \sum_{s=0}^{M-1} \sum_{z=s+1}^{M} \Upsilon_{s,z} \right),$$

or IOS if

$$P_1 \le \xi \Psi; \quad \Theta + \sum_{j=1}^{M} \Lambda^j \le \xi \left(\sum_{k=0}^{\varpi} \Xi^k + 2 \sum_{s=0}^{\varpi-1} \sum_{z=s+1}^{\varpi} \Upsilon_{s,z} \right) \tag{2.13}$$

for some $\xi > 0$.

Proof. Consider a candidate Lyapunov function

$$V(x) = x^\top P_C x + 2 \sum_{j=1}^{M} \sum_{i=1}^{n} \Lambda_i^j \int_0^{x_i} f_i^j(\tau) d\tau. \tag{2.14}$$

If $P_1 > 0$, then

$$y^\top P_1 y \le V(x) \le \alpha_2(\|x\|), \tag{2.15}$$
$$\text{with } \alpha_2(\tau) \le \lambda_{\max}(P_C)\tau^2 + 2nM \max_{i \in \overline{1,n}, j \in \overline{1,M}} \left\{ \Lambda_i^j \int_0^{\tau} f_i^j(\gamma) d\gamma \right\}$$

a function from class $\mathcal{K}_\infty$, so (2.5) is verified. If instead, $P_2 > 0$ or $\sum_{j=1}^{\mu} \Lambda^j > 0$ (see (2.12)), then $\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$ for a function $\alpha_1 \in \mathcal{K}_\infty$ (due to Assumption 2.1) and the definition of μ. Since $\|y\| \le \|C\| \|x\|$ with $C \ne 0$, then (2.5) is again satisfied. Consider the time derivative of V

calculated for (2.10) (denote $\dot{V} = \nabla V(x)\dot{x}$):

$$\begin{aligned}\dot{V} &= \dot{x}^\top P_C x + x^\top P_C \dot{x} + 2\sum_{j=1}^{M}\sum_{i=1}^{n} \Lambda_i^j f_i^j(x_i)\dot{x}_i \\ &= x^\top\left(A_0^\top P_C + P_C A_0\right)x + \left(\sum_{j=1}^{M} f^j(x)^\top A_j^\top\right)P_C x + x^\top P_C \sum_{j=1}^{M} A_j f^j(x) + 2x^\top P_C d \\ &\quad +2\sum_{j=1}^{M}\left(x^\top A_0^\top + d^\top + \left(\sum_{s=1}^{M} f^s(x)^\top A_s^\top\right)\right)\Lambda^j f^j(x).\end{aligned}$$

Therefore, under (2.12) we obtain

$$\begin{aligned}\dot{V} &= \begin{bmatrix} x \\ f^1(x) \\ \vdots \\ f^M(x) \\ d \end{bmatrix}^\top Q \begin{bmatrix} x \\ f^1(x) \\ \vdots \\ f^M(x) \\ d \end{bmatrix} - x^\top(C^\top \Psi C + \Xi^0)x \\ &\quad -\sum_{j=1}^{M} f^j(x)^\top \Xi^j f^j(x) - 2\sum_{j=1}^{M} x^\top \Upsilon_{0,j} f^j(x) - 2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} f^s(x)^\top \Upsilon_{s,z} f^z(x) + d^\top \Phi d \\ &\leq -x^\top(C^\top \Psi C + \Xi^0)x - 2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} f^s(x)^\top \Upsilon_{s,z} f^z(x) \\ &\quad -\sum_{j=1}^{M} f^j(x)^\top \Xi^j f^j(x) - 2\sum_{j=1}^{M} x^\top \Upsilon_{0,j} f^j(x) + d^\top \Phi d.\end{aligned}$$

If $\Psi > 0$, then under the restriction $\frac{1}{2}y^\top \Psi y \geq d^\top \Phi d$ we conclude that

$$\begin{aligned}\dot{V} &\leq -x^\top(\frac{1}{2}C^\top \Psi C + \Xi^0)x - \sum_{j=1}^{M} f^j(x)^\top \Xi^j f^j(x) \\ &\quad -2\sum_{j=1}^{M} x^\top \Upsilon_{0,j} f^j(x) - 2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} f^s(x)^\top \Upsilon_{s,z} f^z(x).\end{aligned}$$

Now we have to show that there exists $\alpha \in \mathcal{K}$ such that

$$\begin{aligned}\alpha(V(x)) &\leq x^\top(\frac{1}{2}C^\top \Psi C + \Xi^0)x + \sum_{j=1}^{M} f^j(x)^\top \Xi^j f^j(x) \\ &\quad +2\sum_{j=1}^{M} x^\top \Upsilon_{0,j} f^j(x) + 2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} f^s(x)^\top \Upsilon_{s,z} f^z(x), \qquad (2.16)\end{aligned}$$

which is true taking into account the form of V and if

$$P_1 \le \xi\Psi; \quad \Theta + \sum_{j=1}^{M} \Lambda^j \le \xi \left(\sum_{k=0}^{M} \Xi^k + 2 \sum_{s=0}^{M-1} \sum_{z=s+1}^{M} \Upsilon_{s,z} \right)$$

for some $\xi > 0$. The latter properties are imposed in the theorem (the first inequality can be verified since $\Psi > 0$). Hence,

$$\frac{1}{2} y^\top \Psi y \ge d^\top \Phi d \quad \Rightarrow \quad \dot{V} \le -\alpha(V).$$

Recalling Remark 2.2, by Theorem 2.4, we conclude that the system is ROS. To ensure the IOS property, if the function $\alpha \in \mathcal{K}_\infty$ in (2.16), then the property (2.8) can be guaranteed:

$$V \ge \alpha^{-1}(2d^\top \Phi d) \quad \Rightarrow \quad \dot{V} \le -\frac{1}{2}\alpha(V),$$

which according to Theorem 2.4 and Remark 2.2 is necessary to substantiate (the condition (2.5) has been already verified). The function α can be selected in the required class under the introduced conditions (2.13) since only the first ϖ nonlinearities and the quadratic term are radially unbounded. □

Remark 2.5. *When $M = 1$, which is the case of Lur'e systems, the matrix Q presented in the conditions of Theorem 2.6 can be expressed as:*

$$Q = \begin{bmatrix} A_0^\top P_C + P_C A_0 + \Xi^0 + C^\top \Psi C & P_C A_1 + A_0^\top \Lambda^1 + \Upsilon_{0,1} & P_C \\ A_1^\top P_C + \Lambda^1 A_0 + \Upsilon_{0,1} & A_1^\top \Lambda^1 + \Lambda^1 A_1 + \Xi^1 & \Lambda^1 \\ P_C & \Lambda^1 & -\Phi \end{bmatrix}.$$

Remark 2.6. *In the case that IOS conditions are verified with $P_2 > 0$ or $\sum_{j=1}^{\mu} \Lambda^j > 0$, the system is UBIBS, and the requirement on forward completeness stated in Theorem 2.6 can be dropped.*

Remark 2.7. *The Lyapunov function (2.14) was frequently used in the absolute stability theory [145, 146, 66].*

For the formulation of the conditions of OLIOS or SIIOS for the system (2.10), note that according to Definition 2.5, the difference between the corresponding Lyapunov functions is in the form of the function α_3 only (it can belong to the class $\mathcal{KL}$ or $\mathcal{K}$). As we can conclude from the proof of Theorem 2.6, within the applied framework, only $\alpha_3 \in \mathcal{K}$ can be obtained. Hence, we have to restrict our analysis to the SIIOS case and the following additional hypothesis is needed:

Assumption 2.2. *[83] For any $j \in \overline{1, \varpi}$:*

$$x^\top C^\top C f^j(x) > 0, \quad \forall x \in \mathbb{R}^n \setminus \{x \in \mathbb{R}^n : Cx = 0\}.$$

Assumption 2.2 assumes that all unbounded nonlinearities possess a kind of symmetry that Cf^j takes zero on the set where $y = 0$ only. Such a restriction is satisfied if, for example, $f_i^j(s) = f_1^j(s)$ for all $i \in \overline{2,n}$ and $j \in \overline{1,\varpi}$, and $C = \Gamma$ as in (3.3), under which Assumption 2.2 is essentially an incremental passivity condition for all nonlinearities [98, 151].

Theorem 2.7. *[83] Let assumptions 2.1 and 2.2 be satisfied and $C^\top C \in \mathbb{D}_+^n$. If there exist $0 < P_1 = P_1^\top \in \mathbb{R}^{p\times p}$; $\Lambda^j = \operatorname{diag}(\Lambda_1^j, \ldots, \Lambda_n^j) \in \mathbb{D}_+^n$ with $\ker(\Lambda^j) = \ker(C)$ $(j \in \overline{1,M})$; $\Xi^k \in \mathbb{D}_+^p$ $(k \in \overline{0,M})$; $\{\Upsilon_{s,z}\}_{z=s+1}^M \subset \mathbb{D}_+^p$ $(s \in \overline{0,M-1})$; and $0 < \Phi = \Phi^\top \in \mathbb{R}^{n\times n}$ such that*

$$Q \leq 0,$$

where the matrix Q is given in Theorem 2.6 under substitutions $\Psi \to 0$, $\Upsilon_{s,z} \to C^\top \Upsilon_{s,z} C$ for $s \in \overline{0,M-1}$ and $z \in \overline{s+1,M}$, $\Xi^k \to C^\top \Xi^k C$ for $k \in \overline{0,M}$ with $P_2 = 0$, then a forward complete system (2.10) is SIIOS if for some $\xi > 0$:

$$P_1 \leq \xi \Xi^0; \quad \sum_{j=1}^{M} \Lambda^j \leq \xi C^\top \left(\sum_{k=1}^{\varpi} \Xi^k + 2 \sum_{s=0}^{\varpi-1} \sum_{z=s+1}^{\varpi} \Upsilon_{s,z} \right) C. \tag{2.17}$$

Proof. Consider a candidate Lyapunov function

$$V(x) = x^\top C^\top P_1 C x + 2 \sum_{j=1}^{M} \sum_{i=1}^{n} \Lambda_i^j \int_0^{x_i} f_i^j(\tau) d\tau,$$

then it is straightforward that (2.6) is verified for any $P_1 > 0$ and due to the imposed conditions on the kernels of Λ^j and C. The derivative of V calculated for (2.10) under the assumptions of the theorem can be upper estimated as follows for $Q \leq 0$:

$$\begin{aligned} \dot{V} \quad \leq \quad & -x^\top C^\top \Xi^0 C x - \sum_{j=1}^{M} f^j(x)^\top C^\top \Xi^j C f^j(x) - 2 \sum_{j=1}^{M} x^\top C^\top \Upsilon_{0,j} C f^j(x) \\ & -2 \sum_{s=1}^{M-1} \sum_{z=s+1}^{M} f^s(x)^\top C^\top \Upsilon_{s,z} C f^z(x) + d^\top \Phi d \end{aligned}$$

and for the condition (2.17) there exists $\alpha \in \mathscr{K}_\infty$ such that

$$\dot{V} \leq -\alpha(V) + d^\top \Phi d,$$

which according to Theorem 2.4 and Remark 2.2 implies in our case SIIOS. □

The conditions of both theorems, 2.6 and 2.7, can be combined and also used for stability analysis (with $d(t)=0$ for all $t\geq 0$):

Corollary 2.1. *[83] Let assumptions 2.1 and 2.2 be satisfied and there exist* $0\leq P_1=P_1^\top\in\mathbb{R}^{p\times p}$*;* $0\leq P_2=P_2^\top\in\mathbb{R}^{n\times n}$*;* $\Xi^k\in\mathbb{D}_+^p$ *(*$k\in\overline{0,M}$*);* $\Lambda^j\in\mathbb{D}_+^n$ *(*$j\in\overline{1,M}$*);* $\{\Upsilon_{s,z}\}_{z=s+1}^M\subset\mathbb{D}_+^p$ *(*$s\in\overline{0,M-1}$*) such that*

$$P_2>0 \quad or \quad \sum_{q=1}^{\mu}\Lambda^q>0; \quad Q=Q^\top=(Q_{a,\,b})_{a,\,b=1}^{M+1}\leq 0,$$

where

$$P_C=C^\top P_1C+P_2; \quad Q_{1,1}=A_0^\top P_C+P_CA_0+C^\top\Xi^0C; \quad Q_{j+1,j+1}=A_j^\top\Lambda^j+\Lambda^jA_j+C^\top\Xi^jC,\ j\in\overline{1,M},$$
$$Q_{1,j+1}=P_CA_j+A_0^\top\Lambda^j+C^\top\Upsilon_{0,j}C,\ j\in\overline{1,M},$$
$$Q_{s+1,z+1}=A_s^\top\Lambda^z+\Lambda^sA_z+C^\top\Upsilon_{s,z}C,\ s\in\overline{1,M-1},\ z\in\overline{s+1,M},$$

and

$$\sum_{k=0}^{M}\Xi^k+2\sum_{s=0}^{M-1}\sum_{z=s+1}^{M}\Upsilon_{s,z}>0,$$

then the system (2.10) is UBIBS and $\lim_{t\to+\infty}\|y(t,x_0,0)\|=0$ *for all* $x_0\in\mathbb{R}^n$*.*

Proof. Consider the Lyapunov function (2.14) with $P_C=C^\top P_1C+P_2$. If $P_2>0$ or $\sum_{z=1}^{\mu}\Lambda^z>0$, then

$$\alpha_1(\|x\|)\leq V(x)\leq\alpha_2(\|x\|)$$

for some functions $\alpha_1,\alpha_2\in\mathscr{K}_\infty$, due to Assumption 2.1 and the definition of μ. Hence, such a Lyapunov function is positive definite and radially unbounded. The derivative of V calculated for (2.10) with $d(t)=0$ can be upper estimated as follows for $Q\leq 0$:

$$\dot V\leq -x^\top C^\top\Xi^0Cx-\sum_{j=1}^{M}f^j(x)^\top C^\top\Xi^jCf^j(x)$$
$$-2\sum_{j=1}^{M}x^\top C^\top\Upsilon_{0,j}Cf^j(x)-2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M}f^s(x)^\top C^\top\Upsilon_{s,z}Cf^z(x).$$

Since $\sum_{k=0}^{M}\Xi^k+2\sum_{s=0}^{M-1}\sum_{z=s+1}^{M}\Upsilon_{s,z}>0$ and due to assumptions 2.1 and 2.2, there exists a function $\alpha\in\mathscr{K}$ such that

$$\dot V\leq-\alpha(\|y\|).$$

The proven properties of V and the fact that $\dot V\leq 0$ implies that all solutions of (2.10) are bounded: $\|x(t,x_0,0)\|\leq\alpha_1\circ\alpha_2^{-1}(\|x_0\|)$ for all $x_0\in\mathbb{R}^n$ and all $t\geq 0$. Applying standard LaSalle arguments [57], we obtain $\lim_{t\to+\infty}\|y(t,x_0,0)\|=0$, for all $x_0\in\mathbb{R}^n$. □

Remark 2.8. *A minor modification of the conditions given in this section is needed if*

$$y(t) = \begin{bmatrix} C_0 x(t) \\ C_1 f^1(x(t)) \\ \vdots \\ C_M f^M(x(t)) \end{bmatrix}.$$

The concepts of IOS and SIIOS can be used for many analysis and design problems, e.g.*, for synchronization or estimation, and the former issue is considered below.*

2.6 Synchronization of dynamical systems

As the aforementioned stability notions are powerful tools for investigating *synchronization* problems in generalized Persidskii systems, it is necessary to first give a general definition of synchronization of the $N \geq 2$ dynamical systems [19]

$$\Sigma_\ell = \{I', U_\ell, X_\ell, Y_\ell, f_\ell, h_\ell\}, \quad \ell \in \overline{1,N}, \tag{2.18}$$

where I' is the common set of time instances; U_ℓ, X_ℓ, Y_ℓ are the sets of inputs, states, and outputs, respectively; the transition maps $f_\ell : X_\ell \times U_\ell \times I' \to X_\ell$; the output maps $h_\ell : X_\ell \times U_\ell \times I' \to Y_\ell$.

Let l functionals $G_j : \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_N \times I' \to \mathbb{R}$, $j \in \overline{1,l}$ be given, where $\mathcal{Y}_\ell$ are the sets of all functions from I' to Y_ℓ (*i.e.* $\mathcal{Y}_\ell = \{y : I' \to Y_\ell\}$), and define σ_τ as the *shift operator*, *i.e.* $\sigma_\tau : \mathcal{Y}_\ell \to \mathcal{Y}_\ell$, $(\sigma_\tau y)(t) = y(t+\tau)$ for all $y \in \mathcal{Y}_\ell$ and $t \in I'$.

We initially review the synchronization definitions in the case when all U_ℓ are singletons, *i.e.* inputs are not present and may be omitted in the formulation.

Definition 2.9. *[19] We say that the solutions $x_1(\cdot), \ldots, x_N(\cdot)$ of the dynamical systems $\Sigma_1, \ldots, \Sigma_N$ with initial conditions $x_1(t_0), \ldots, x_N(t_0)$ are*

1. synchronized with respect to the functionals $G_1, \ldots, G_l$ *if*

$$G_j\big(\sigma_{\tau_1} y_1(\cdot), \ldots, \sigma_{\tau_N} y_N(\cdot), t\big) = 0, \quad \forall j \in \overline{1,l}$$

for all $t \in I'$ and some $\tau_1, \ldots, \tau_N \in I'$, where $y_\ell(\cdot)$ denotes the output function of the system $\Sigma_\ell : y_\ell = h(x_\ell(t), t)$, $t \in I', \ell \in \overline{1,N}$.

2. asymptotically synchronized with respect to the functionals $G_1, \ldots, G_l$ *if*

$$\lim_{t \to \infty} G_j\big(\sigma_{\tau_1} y_1(\cdot), \ldots, \sigma_{\tau_N} y_N(\cdot), t\big) = 0, \quad \forall j \in \overline{1,l}$$

for some $\tau_1, \ldots, \tau_N \in I'$.

A possible extension of Definition 2.9 is to consider the time-varying shift operator defined as follows:

$$(\sigma_{\tau_\ell})y(t) = y(\tilde{t}_\ell(t)),$$

where $\tilde{t}_\ell : I' \longrightarrow I'$ are homeomorphisms such that

$$\lim_{t\to\infty}(\tilde{t}_\ell(t) - t) = \tau_\ell.$$

In practice, the synchronization of interconnected dynamical systems may be more interesting. For describing the potential interconnections between the systems, we suppose that input of each system Σ_ℓ can be composed of the output of the interconnection system

$$\Sigma_0 = \{I', U_0, X_0, Y_0, f_0, h_0\},$$

where the transition map $f_0 : X_0 \times U_0 \times I' \longrightarrow X_0$; the output map $h_0 : X_0 \times U_0 \times I' \longrightarrow Y_0$ with $U_0 = Y_1 \times Y_2 \times \cdots \times Y_N$ and $Y_0 = U_1 \times U_2 \times \cdots \times U_N$.

Definition 2.10. *[19] We say that the solutions* $x_0(\cdot), x_1(\cdot), \ldots, x_N(\cdot)$ *of the interconnected dynamical systems* $\Sigma_0, \Sigma_1, \ldots, \Sigma_N$ *with initial conditions* $x_0(t_0), x_1(t_0), \ldots, x_N(t_0)$ *are*

1. synchronized with respect to the functionals $G_1, \ldots, G_l$ *if*

$$G_j\left(\sigma_{\tau_0} y_0(\cdot), \sigma_{\tau_1} y_1(\cdot), \ldots, \sigma_{\tau_N} y_N(\cdot), t\right) = 0, \quad \forall j \in \overline{1,l}$$

for all $t \in I'$ *and some* $\tau_0, \tau_1, \ldots, \tau_N \in I'$*, where* $y_\ell(\cdot)$ *denotes the output function of the system* $\Sigma_\ell : y_\ell = h(x_\ell(t), t)$*,* $t \in I', \ell \in \overline{0,N}$.

2. asymptotically synchronized with respect to the functionals $G_1, \ldots, G_l$ *if*

$$\lim_{t\to\infty} G_j\left(\sigma_{\tau_0} y_0(\cdot), \sigma_{\tau_1} y_1(\cdot), \ldots, \sigma_{\tau_N} y_N(\cdot), t\right) = 0, \quad \forall j \in \overline{1,l}$$

for some $\tau_0, \tau_1, \ldots, \tau_N \in I'$.

In this thesis, we specialize in the problem of *controlled synchronization with respect to the functionals* $G_1, \ldots, G_l$ [19]. More specifically, we are interested in finding a simple control $U = U(x_0, x_1, \ldots, x_N)$ (the argument of t will be omitted) as a feedback function of the states $x_0, x_1, \ldots, x_N$ such that (robust) synchronization of the closed-loop system (the considered one is a diffusively coupled system) is realized.

Synchronization of diffusive coupled systems

As demonstrated above, we are mainly concerned with synchronization problems in closed-loop dynamical systems with a diffusive coupling scheme, providing the fact that there exist numerous coupling types in interconnected systems, *e.g.*, nearest-neighbor diffusive coupling and star coupling [100].

Consider a network of $N \geq 2$ diffusively coupled systems [100]:

$$\dot{x}_\ell(t) = f(x_\ell(t), t) + \sigma \sum_{\ell'=1}^{N} W_{\ell\ell'} g(x_{\ell'}(t) - x_\ell(t)), \quad \ell \in \overline{1, N}, \tag{2.19}$$

where $x_\ell(t) = [x_{\ell,1}(t) \dots x_{\ell,n}(t)]^\top \in \mathbb{R}^n$ is the state vector of the ℓ-th system; $\sigma > 0$ is the overall coupling strength; the function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ describes the isolated system; the function $g : \mathbb{R}^n \to \mathbb{R}^n$ depicts the diffusion-like interaction between systems; $W = (W_{\ell\ell'})_{\ell,\ell'=1}^{N} \in \mathbb{R}^{N \times N}$ is a matrix representing the interaction structure of the network.

For presenting an essential result on synchronization of the network (2.19), some definitions and assumptions are needed:

Definition 2.11. *[118] For the dynamical system* (2.1)*, a manifold $\mathcal{U} \subset \mathcal{M}$ is said to be* invariant *if for each $x(t_0) \in \mathcal{U}$ (t_0 is the initial time), the solution $t \to f^t(x(t_0))$, defined on its maximal interval of existence, has its image in $\mathcal{U}$.*

Definition 2.12. *[142] For the dynamical system* (2.1)*, a manifold $\mathcal{U} \subset \mathcal{M}$ is said to be* inflowing *if the flow is pointing strictly inward on the boundary of $\mathcal{U}$.*

Assumption 2.3. *[106] For the system* (2.19)*, the function f is continous, and there exists an inflowing invariant manifold $\mathcal{U}$ such that f is continuously differentiable in $\mathcal{U}$ with*

$$\left\| \frac{\partial f(x,t)}{\partial x} \right\| \leq c_f, \quad \forall t \in \mathbb{R}, x \in \mathcal{U}$$

for some $c_f > 0$.

Assumption 2.4. *[106] For the system* (2.19)*, the function g is continuously differentiable with $g(0) = 0$.*

Denote the (complex) eigenvalues of $\frac{\partial g(x)}{\partial x}|_{x=0}$ by β_i, $i \in \overline{1, n}$. Also, let λ_ℓ, $\ell \in \overline{1, N}$ denote the eigenvalues of the *Laplacian* $\mathcal{L}$ defined as:

$$\mathcal{L} = \operatorname{diag}\left(\sum_{\ell'=1}^{N} W_{1\ell'}, \dots, \sum_{\ell'=1}^{N} W_{N\ell'} \right) - W.$$

$\lambda_1 = 0$ is an eigenvalue of $\mathcal{L}$ and its multiplicity represents the number of connected components of the network.

The next assumption deals with the coupling and structural network properties.

Assumption 2.5. *[106] Assume that*

$$\gamma := \min_{2 \le \ell \le N; 1 \le i \le n} \mathrm{Re}(\lambda_\ell \beta_i) > 0.$$

The following main theorem gives the conditions for uniform exponential stability of the synchronization manifold $\mathcal{M}_s = \{x \in \mathcal{U} \mid x = x_1 = \cdots = x_N\}$ of the coupled systems (2.19):

Theorem 2.8 (**Synchronization).** *[106] Consider the network (2.19) of diffusively coupled systems satisfying assumptions 2.3, 2.4, 2.5. Then there exists $\rho = \rho\left(f, \frac{\partial g(x)}{\partial x} |_{x=0}\right)$ such that for all*

$$\sigma > \frac{\rho}{\gamma},$$

the network is locally uniformly synchronized. This means that there exist $\delta > 0$ and $c > 0$ such that if $x_\ell(t_0) \in \mathcal{U}$ and $\|x_\ell(t_0) - x_{\ell'}(t_0)\| \le \delta$ for any $\ell, \ell' \in \overline{1,N}$, then

$$\|x_\ell(t) - x_{\ell'}(t)\| \le c e^{-(\sigma\gamma - \rho)(t - t_0)} \|x_\ell(t_0) - x_{\ell'}(t_0)\|, \quad \forall t \ge t_0.$$

Remark 2.9. *Note that the value of $\rho = \rho\left(f, \frac{\partial g(x)}{\partial x} |_{x=0}\right)$ in Theorem 2.8 depends upon the bound c_f as in Assumption 2.3 and a number defined as: $\tilde{\kappa}\left(\frac{\partial g(x)}{\partial x} |_{x=0}\right) = \left\|\frac{\partial g(x)}{\partial x} |_{x=0}\right\| \left\|\left(\frac{\partial g(x)}{\partial x} |_{x=0}\right)^{-1}\right\|$. The estimate of the bounds for ρ in the case that $\frac{\partial g(x)}{\partial x} |_{x=0}$ is symmetric or $\frac{\partial g(x)}{\partial x} |_{x=0}$ is non-diagonalizable can be found in [106].*

As illustrated before, it is of interest to take into account the presence of small perturbation $d_\ell(x_\ell, t)$, $\ell \in \overline{1,N}$ verifying if the synchronization is stable (*i.e.*, trajectories starting near $\mathcal{M}_s$ remain in a neighbourhood of $\mathcal{M}_s$). Then we consider the perturbed coupled systems:

$$\dot{x}_\ell(t) = f_\ell(x_\ell(t), t) + \sigma \sum_{\ell'=1}^{N} W_{\ell\ell'} g(x_{\ell'}(t) - x_\ell(t)), \quad \ell \in \overline{1,N}, \tag{2.20}$$

where $f_\ell(x_\ell, t) = f(x_\ell, t) + d_\ell(x_\ell, t)$.

Theorem 2.9 (**Persistence).** *[106] Consider Theorem 2.8 and the perturbed network (2.20) of diffusively coupled systems satisfying assumptions 2.3, 2.4, 2.5 and assume that*

$$\sigma > \frac{\rho}{\gamma}$$

as in Theorem 2.8. Then there exist $\delta > 0$, $c > 0$ and $\epsilon_d > 0$ such that for all ϵ_0-perturbations satisfying

$$\|d_\ell(x,t)\| \le \epsilon_0 \le \epsilon_d, \quad \forall t \in \mathbb{R},\ x \in \mathcal{U},\ \ell \in \overline{1,N}$$

and initial conditions satisfying $\|x_\ell(t_0) - x_{\ell'}(t_0)\| \le \delta$ for any $\ell, \ell' \in \overline{1,N}$, the estimate

$$\|x_\ell(t) - x_{\ell'}(t)\| \le c e^{-(\sigma\gamma-\rho)(t-t_0)} \|x_\ell(t_0) - x_{\ell'}(t_0)\| + \frac{c\epsilon_0}{\sigma\gamma-\rho}, \quad \forall t \ge t_0$$

holds true.

By this section, the basic definitions of synchronization of dynamical systems were presented, as well as the synchronization results for diffusively coupled dynamics. In Chapter 3, we will show that for robust synchronization of generalized Persidskii systems, it is possible to skip the incorporation of the coupling and structural network properties, relax the imposed conditions in Assumption 2.3, extend the generality of the function g, and employ the IOS theory and linear matrix inequality to obtain conditions that can be verified more easily.

2.7 State observers for dynamical systems

The proposed stability analysis methods for generalized Persidskii systems can be used to study another important problem in those systems: observer design for state observation (or estimation), in the presence of external disturbance/input.

The main step in the design of an observer is the obligatory evaluation of convergence conditions of the state estimation error and its sturdiness to given classes of uncertainties, *i.e.*, analysis of the robust stability. To illustrate an important issue in the nonlinear state estimation, we consider a dynamical system:

$$\begin{aligned} \dot{x}(t) &= f(x(t), d(t)), \quad t \ge t_0 = 0, \\ y(t) &= h(x(t)), \end{aligned} \tag{2.21}$$

where $x(t) \in \mathbb{R}^n$, $d(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, the unknown input/disturbance and the measured output, respectively, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are known nonlinear functions. An *observer* for this system is often chosen as another nonlinear dynamics in a general form:

$$\begin{aligned} \dot{z}(t) &= r(z(t), y(t)), \\ \hat{x}(t) &= g(z(t)), \end{aligned} \tag{2.22}$$

where $z(t) \in \mathbb{R}^q$ is the state of the observer and $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$, $r : \mathbb{R}^q \times \mathbb{R}^p \to$

$\mathbb{R}^q$ and $g : \mathbb{R}^q \longrightarrow \mathbb{R}^n$ are functions to be selected so that the estimation error $e := x - \hat{x}$ dynamics is asymptotically stable for $d = 0$ (*i.e.*, $\lim_{t\to+\infty}\|e(t)\| = 0$) $\Big($or the manifold $\mathcal{M}_e = \{(x,\hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n \mid x = \hat{x}\}$ has the properties: i) $\mathcal{M}_e$ is invariant; ii) all trajectories $(x(t),\hat{x}(t))$ that start in a neighbourhood of $\mathcal{M}_e$ asymptotically converge to $\mathcal{M}_e$ [138]$\Big)$ and robustly stable in the presence of disturbances $d \neq 0$ (frequently, the input-to-state stability (ISS) framework [126, 27] is applied). For example, a popular choice is $r(z,y) = f(z,0) + \varrho(y - h(z))$ and $g(z) = z$, where the output injection term $\varrho : \mathbb{R}^p \to \mathbb{R}^n$ is properly adjusted. For this purpose, the estimation error dynamics is analyzed, and often it is assumed that it can be described by differential equations governed by exogenous disturbances/noises/input d, but independent of the system state x:

$$\dot{e}(t) = \ell(e(t), d(t)) \tag{2.23}$$

for some $\ell : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$, which is always the case for linear models $\Big($*i.e.*, if $f(x,d) = Ax + d$ and $h(x) = Cx$ for known matrices A and C of appropriate dimensions, then $r(z,y) = Az + L(y - Cz)$, $g(z) = z$ for an observer gain L, and $\ell(e,d) = (A - LC)e + d\Big)$. Such a representation can also be obtained for plants close to linear ones. Nevertheless, the independence of the dynamics of e in x can be a restrictive hypothesis, and in a general scenario, this differential equation has to take the following form:

$$\dot{e}(t) = \tilde{\ell}(e(t), x(t), d(t)) \tag{2.24}$$

with $\tilde{\ell} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$. Then the estimation error behavior has to be analyzed together with the observed system, and uniform stability or partial stability notions come to the attention (the same convergence and stability properties are required from e, but independently in x under imposed restrictions). One of the most popular concepts for robust partial stability analysis is given in the IOS theory [129, 130].

As indicated above, under the situation of free disturbance, the state of an observer is required to have asymptotic convergence to the state of the observed plant. This kind of observer can be formally defined as:

Definition 2.13. *[14] The system* (2.22) *is an* asymptotic observer *for the plant* (2.21) *if there exists* $Z_0 \subset \mathbb{R}^q$ *such that for any solution of* (2.21) *with* $x(0)$ *in a submanifold of an invariant manifold* $X_0 \subset \mathbb{R}^n$*, any solution of* (2.22) *with* $\hat{z}(0) \in Z_0$ *and* $y(t)$ *defined on* $[0, +\infty)$ *we have*

$$\lim_{t\to+\infty} \|\hat{x}(t) - x(t)\| = 0$$

with $\hat{x}(t) = g(z(t))$.

Furthermore, extending from the asymptotic observer, it is also of physical interest to consider the robustness of an observer (in the case that there is external disturbance/input to systems) as stated, which leads to the following definition:

Definition 2.14. *[14] If the system (2.22) is an asymptotic observer and admits an asymptotic gain in the presence of disturbances on the plant (2.21), then the observer (2.22) is a* robust observer. *More precisely, there exist $c \in \mathbb{R}_+ \cup \{+\infty\}$ and $\gamma \in \mathcal{K}$ such that, for any measurable disturbance $D(t) = \begin{bmatrix} d_x(t) \\ d_y(t) \end{bmatrix} \in \mathbb{R}^{n+p}$ such that $\|D\|_\infty < c$, and for any solution $x(t)$ to*

$$\dot{x}(t) = f(x(t),t) + d_x(t), \qquad y(t) = h(x(t),t) + d_y(t),\ t \in \mathbb{R}_+$$

with $x(0)$ in a submanifold of an invariant manifold $X_0 \subset \mathbb{R}^n$, any solution $\hat{z}(t)$ to (2.22) with $\hat{z}(0) \in Z_0$ and $y(t)$ defined on $[0,+\infty)$ we have

$$\limsup_{t \to +\infty} \|\hat{x}(t) - x(t)\| \leq \gamma(\|D\|_\infty).$$

Luenberger observer for linear systems

Consider linear time-invariant (LTI) systems of the following form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bd(t), \\ y(t) &= Cx(t), \end{aligned} \tag{2.25}$$

with known real matrices A, B, C and input d of appropriate dimensions, for which we have the following classical result [74]:

Theorem 2.10. *[16] If the system (2.25) satisfies the observability rank condition,* i.e., *the rank of*

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is equal to the dimension of x (or equivalently the condition that the pair (A, C) is observable), then there exists an observer of the form:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bd(t) + L(y(t) - C\hat{x}(t)) \tag{2.26}$$

with L such that $A - LC$ is Hurwitz, i.e., *all its eigenvalues have strictly negative real part.*

The observer (2.26) for (2.25) is so-called *Luenberger observer* [74].

Theorem 2.11. *[74] Consider the system (2.25) and suppose that there exists a matrix L such that $A-LC$ is Hurwitz. Then the state estimation error $e = x - \hat{x}$ from (2.26) converges exponentially to zero.*

Luenberger-like observer for nonlinear systems

In this work, we confine our attention to robust observer design for generalized Persidskii systems, which take a more general form than the following system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(x(t)) + D_1 w(t), \\ y(t) &= Cx(t) + D_2 v(t), \end{aligned} \tag{2.27}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $y(t) \in \mathbb{R}^p$ is the output signal; $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$ with $C \neq 0$, and the pair (A, C) is observable; $D_1 \in \mathbb{R}^{n\times q}$; $D_2 \in \mathbb{R}^{p\times z}$; $w(t) \in \mathbb{R}^q, v(t) \in \mathbb{R}^z$ are the external perturbations, $w \in \mathscr{L}_\infty^q$, $v \in \mathscr{L}_\infty^z$; the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous: there exists a constant $c_f > 0$ such that

$$\|f(x_1) - f(x_2)\| \leq c_f \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^n.$$

The work [3] proposes a robust Luenberger-like observer for (2.27) as follows:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t)) + L(y(t) - C\hat{x}(t)), \tag{2.28}$$

where $\hat{x}(t)$ is the state of the observer (2.28) and L is an observer gain to be designed.

Theorem 2.12. *[3] Consider the system (2.27), if there exist $0 < c$; $0 < P = P^\top \in \mathbb{R}^{n\times n}$ and $L \in \mathbb{R}^{n\times p}$ such that*

$$\begin{bmatrix} (A-LC)^\top P + P(A-LC) + cc_f^2 I_n & P \\ P & -\frac{c}{2} I_n \end{bmatrix} < 0, \tag{2.29}$$

then the observer (2.28) is ISS with respect to the estimation error $e = x - \hat{x}$.

Remark 2.10. *[3] The design of the observer gain is direct. The matrix inequality (2.29) can be expressed by the LMI under the setting $L = P^{-1}K$ for a matrix K. However, the requirement of global Lipschitz continuity for the nonlinearity is the main drawback (a large c_f may cause an issue of solving the LMI). On the other hand, L has to be selected to ensure that $A - LC$ is Hurwitz.*

In Chapter 4, for generalized Persidskii dynamics, a robust Luenberger-like observer scheme will be proposed including a copy of the system dynamics with a nonlinear output injection term, under relaxed continuity conditions on nonlinearities.

CHAPTER 3

Robust synchronization

The principal goals of this chapter are to apply the obtained conditions in Chapter 2 for the synchronization analysis in a family of systems as (2.10) and to design a robust nonlinear synchronization control in this framework.

The synchronization measure and an approach to study the synchronization of a family of generalized Persidskii systems are introduced in Section 3.1. A robust control design for synchronization of linear systems subject to highly nonlinear perturbations is presented in Section 3.2. The Hindmarsh-Rose model is considered as an example in Section 3.3 to examine the efficiency of our proposed results.

3.1 Robust synchronization of a family of generalized Persidskii systems

In this section, we consider an application of the previously proposed theory.

3.1.1 Family of generalized Persidskii systems

Consider a family of $N \geq 2$ systems of the following form:

$$\dot{x}_z(t) = A_{z,0}x_z(t) + \sum_{j=1}^{M} A_{z,j} f^j(x_z(t)) + B_z u_z(t) + d_z(t),\ z \in \overline{1,N}, \quad \forall t \geq 0, \tag{3.1}$$

where $x_z(t) = [x_{z,1}(t) \ldots x_{z,n}(t)]^\top \in \mathbb{R}^n$ is the state vector of a system, $A_{z,s} \in \mathbb{R}^{n\times n}$ for $s \in \overline{0,M}$, $B_z \in \mathbb{R}^{n\times r}$, $u_z(t) = [u_{z,1}(t) \ldots\ u_{z,r}(t)]^\top \in \mathbb{R}^r$ is the controlled input, and $d_z(t) \in \mathbb{R}^n$ is the external perturbation, $d_z \in \mathscr{L}_\infty^n$; $f^j(x_z(t)) = [f_1^j(x_{z,1}(t)) \ldots f_n^j(x_{z,n}(t))]^\top$ for $j \in \overline{1,M}$ are the functions ensuring existence of the solutions of the system (3.1) in the forward time at least locally. The sector restrictions on f^j, $j \in \overline{1,M}$ are imposed as in Assumption 2.1.

In this study, we consider the synchronization of a network of (3.1), *i.e.*, a system in the following form:

$$\dot{X}(t) = A_0 X(t) + \sum_{j=1}^{M} A_j F^j(X(t)) + BU(t) + d(t), \tag{3.2}$$

where $X(t) = [x_1(t)^\top \dots x_N(t)^\top]^\top \in \mathbb{R}^{Nn}$ is the state vector, $A_s = \mathrm{diag}(A_{1,s} \dots A_{N,s}) \in \mathbb{R}^{Nn \times Nn}$ for $s \in \overline{0,M}$, $B = \mathrm{diag}(B_1 \dots B_N) \in \mathbb{R}^{Nn \times Nr}$, $U(t) = [u_1(t)^\top \dots u_N(t)^\top]^\top \in \mathbb{R}^{Nr}$ is the controlled input, $d(t) = [d_1^\top(t) \cdots d_N^\top(t)]^\top \in \mathbb{R}^{Nn}$ is the common perturbation, $d \in \mathscr{L}_\infty^{Nn}$; $F^j(X(t)) = [f^j(x_1(t))^\top \dots f^j(x_N(t))^\top]^\top \in \mathbb{R}^{Nn}$ for $j \in \overline{1,M}$. Clearly, the functions F^j, $j \in \overline{1,M}$ also satisfy the sector condition. We denote the *consensus set* of (3.1) as

$$\mathscr{W} := \left\{ X \in \mathbb{R}^{Nn} \mid x_i = x_1 \ \text{ for } \ i \in \overline{2,N} \right\}$$

and we say that (3.2) is in the *synchronous mode* if $X(t) \in \mathscr{W}$, for all $t \geq 0$. To quantify the closeness of the system to the synchronous regime, we use a synchronization measure: a continuously differentiable function $\rho : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ such that

$$\rho(X) = 0 \quad \Rightarrow \quad X \in \mathscr{W}.$$

Notice that the presence of the disturbances d having all distinct components (in $\mathbb{R}^n$) does not allow the system to be in the synchronous mode.

Then the robust synchronization problem can be set: to design a feedback $U = U(X)$ that renders the system (3.2) to be IOS with respect to the output ρ and the input d. If d has all identical elements (in $\mathbb{R}^n$), then such a control U pushes (3.2) to the synchronous mode.

3.1.2 Conditions of synchronization

In this study, the synchronization measure $\rho(X)$ is defined as

$$\rho(X) = \Gamma X, \quad \text{where } \Gamma = \begin{bmatrix} -I_n & I_n & 0 & \cdots & 0 \\ 0 & -I_n & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I_n & I_n \\ I_n & 0 & \cdots & 0 & -I_n \end{bmatrix} \in \mathbb{R}^{Nn \times Nn}. \tag{3.3}$$

Note that due to properties of F^j, in the synchronization mode $\Gamma F^j(X) = 0$, for all $j \in \overline{1,M}$ and $X \in \mathscr{W}$, *i.e.*, an analog of Assumption 2.2 is satisfied for F^j, $j \in \overline{1,M}$.

The feedback to robustly synchronize the system (3.2) (to stabilize the system (3.2) in IOS sense) is selected in the form of diffusive coupling:

$$U = K_0 \Gamma X + \sum_{j=1}^{M} K_j \Gamma F^j(X) \tag{3.4}$$

with $K_s \in \mathbb{R}^{rN \times nN}$ for $s \in \overline{0, M}$ designed below.

Remark 3.1. *The control* (3.4) *can also be selected in the form of direct coupling [110]:*

$$U = K_0 X + \sum_{j=1}^{M} K_j F^j(X),$$

i.e.*, the coupling is diffusive if it is proportional to the synchronization measure* ρ *as in* (3.4)*, and it is direct if it is given in the form of a generic state feedback. Both types of coupling can be analyzed in the proposed framework, but for brevity the synchronization conditions are formulated below for the diffusive case only.*

Substituting the control (3.4) into the equations of the system (3.2) we obtain the following closed-loop dynamics:

$$\dot{X}(t) = \tilde{A}_0 X(t) + \sum_{j=1}^{M} \tilde{A}_j F^j(X(t)) + d(t), \quad Y(t) = \Gamma X(t), \tag{3.5}$$

where $\tilde{A}_s = A_s + BK_s\Gamma$ for $s \in \overline{0, M}$.

Clearly, the system (3.5) is in the form (2.10) and assumptions 2.1 and 2.2 are satisfied, then theorems 2.6 and 2.7 or Corollary 2.1 can be directly applied.

Corollary 3.1. *If the IOS conditions of Theorems 2.6 are satisfied under the substitution of* $p \to nN, n \to nN, C \to \Gamma, A_k \to \tilde{A}_k,\ k \in \overline{0, M}$*, then a forward complete system* (3.5) *is robustly synchronized.*

Proof. As we remarked above, assumptions 2.1 and 2.2 are verified by the system (3.5), and it is forward complete due to hypotheses of the corollary. Then, the IOS property guarantees boundedness of the synchronization error ρ in the presence of essentially bounded perturbations $d(t) \neq 0$, and asymptotic convergence of the synchronization error to zero for $d(t) = 0$ (that corresponds to the achievement of the synchronous mode). □

Corollary 3.2. *If the conditions of Corollary 2.1 are satisfied under the substitution of* $p \to nN, n \to nN, C \to \Gamma, A_k \to \tilde{A}_k,\ k \in \overline{0, M}$*, then the system* (3.5) *with* $d(t) = 0, \forall t \in \mathbb{R}_+$ *reaches the synchronous mode.*

Proof. It is a direct consequence of Corollary 2.1 since assumptions 2.1 and 2.2 hold. □

3.2 Robust synchronization of linear systems

Let us consider how the control gains $K_s \in \mathbb{R}^{rN\times nN}$ for $s \in \overline{0,M}$ can be designed to ensure synchronization.

For brevity, in this subsection, we consider the robust synchronization of two linear systems

$$\dot{x} = \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = Ax + Bu + d \tag{3.6}$$

where $x^1, x^2 \in \mathbb{R}^n$ are the states, $A \in \mathbb{R}^{2n\times 2n}$, $B \in \mathbb{R}^{2n\times m}$, $u \in \mathbb{R}^m$ is the controlled input, $d \in \mathbb{R}^{2n}$ is the external perturbation, and we assume two scenarios: either $d \in \mathscr{L}_\infty^{2n}$ or d is a nonlinear function of the state x admitting an upper bound

$$\|d\|^2 \le \sum_{i=1}^{2n} R_i^0|x_i|^2 + R_i^1|x_i|^{1+\zeta} + R_i^2|x_i|^{1+\pi}, \tag{3.7}$$

where $\zeta \in (0,1)$, $\pi > 1$ are growth parameters and $R^s = \mathrm{diag}(R_1^s \dots R_{2n}^s) \in \mathbb{D}_+^{2n}$ are given matrices for $s \in \overline{0,2}$. In the latter case (3.6) is a nonlinear system, and if $R^1 \neq 0$ or $R^2 \neq 0$, then a linear feedback cannot ensure robust synchronization of this system (in the sense of IOS), while Corollary 3.1 provides a tool for synchronization of the system (3.6) with such a disturbance.

For (3.6) we propose to use a feedback control in the form

$$u = K_0\Gamma x + K_1\Gamma f^1(x) + K_2\Gamma f^2(x),$$

where $K_0, K_1, K_2 \in \mathbb{R}^{m\times 2n}$ are the tuning gains,

$$\Gamma = \begin{bmatrix} -I_n & I_n \\ I_n & -I_n \end{bmatrix}$$

is the matrix defining synchronization measure for $N = 2$ (in such case we may take $\Gamma = [I_n \ -I_n]$ without losing generality), and f^1, f^2 are the functions following the imposed conditions of the system (2.10) and Assumption 2.1:

$$f_i^1(x_i) = |x_i|^\zeta \mathrm{sign}(x_i); \quad f_i^2(x_i) = |x_i|^\pi \mathrm{sign}(x_i),$$
$$f^j(x) = [f_1^j(x_1), \dots, f_{2n}^j(x_{2n})], \quad \forall i \in \overline{1,2n},\ j \in \{1,2\}.$$

Then the resulting closed-loop system is

$$\dot{x} = A_0 x + A_1 f^1(x) + A_2 f^2(x) + d, \tag{3.8}$$

where $A_0 = A + BK_0\Gamma$, $A_1 = BK_1\Gamma$ and $A_2 = BK_2\Gamma$. Using the same arguments as in subsection 3.1.2, we define the output function, or the synchronization measure, of (3.8) as

$$y(t) = \Gamma x(t).$$

Applying the Lyapunov function from Theorem 2.6:

$$V(x) = x^\top P_\Gamma x + 2\sum_{z=1}^{2}\sum_{j=1}^{2}\sum_{i=1}^{n}\Lambda_i^j \int_0^{x_i^z} f_i^j(\tau)d\tau, \tag{3.9}$$

where $P_\Gamma = \Gamma^\top P_1 \Gamma + P_2$, for the system (3.8) its derivative is calculated as

$$\begin{aligned}\dot{V}(x) = &\begin{bmatrix} x \\ f^1(x) \\ f^2(x) \\ d \end{bmatrix}^\top \mathcal{Q} \begin{bmatrix} x \\ f^1(x) \\ f^2(x) \\ d \end{bmatrix} - x^\top(\Xi^0 + \Gamma^\top\Psi\Gamma)x + \phi d^\top d \\ &- 2\sum_{j=1}^{2} x^\top \Upsilon_{0,j} f^j(x) + 2\sum_{j=1}^{2} x^\top (P_\Gamma A_j + A_0^\top \Lambda^j + \Upsilon_{0,j}) f^j(x),\end{aligned}$$

where $\Xi^0, \Upsilon_{0,j}, \Lambda^j$ are given in the formulation of Theorem 2.6, $\Phi = \phi I_{2n}$ and

$$\mathcal{Q} = \begin{bmatrix} A_0^\top P_\Gamma + P_\Gamma A_0 + \Xi^0 + \Gamma^\top\Psi\Gamma & 0 & 0 & P_\Gamma \\ 0 & A_1^\top\Lambda^1 + \Lambda^1 A_1 & A_1^\top\Lambda^2 + \Lambda^1 A_2 & \Lambda^1 \\ 0 & A_2^\top\Lambda^1 + \Lambda^2 A_1 & A_2^\top\Lambda^2 + \Lambda^2 A_2 & \Lambda^2 \\ P_\Gamma & \Lambda^1 & \Lambda^2 & -\phi I_{2n} \end{bmatrix}$$

for some $\phi > 0$. For the last term in $\dot{V}$, applying Young's inequality for all cross-terms out the main diagonal:

$$\begin{aligned} x_i|x_k|^\zeta \operatorname{sign}(x_k) &\le \frac{|x_i|^{1+\zeta}}{1+\zeta} + \frac{\zeta|x_k|^{1+\zeta}}{1+\zeta}, \\ x_i|x_k|^\pi \operatorname{sign}(x_k) &\le \frac{|x_i|^{1+\pi}}{1+\pi} + \frac{\pi|x_k|^{1+\pi}}{1+\pi} \end{aligned}$$

for any $i \neq k \in \overline{1, 2n}$, we obtain that if

$$\mathcal{Q} \le 0, \tag{3.10}$$

$$1_{2n}^\top[(1+\zeta)\delta(P_\Gamma A_1 + A_0^\top\Lambda^1) + \zeta\omega(P_\Gamma A_1 + A_0^\top\Lambda^1) + \omega^\top(P_\Gamma A_1 + A_0^\top\Lambda^1) + \Upsilon_{0,1}] \le 0, \tag{3.11}$$

$$1_{2n}^\top[(1+\pi)\delta(P_\Gamma A_2 + A_0^\top\Lambda^2) + \pi\omega(P_\Gamma A_2 + A_0^\top\Lambda^2) + \omega^\top(P_\Gamma A_2 + A_0^\top\Lambda^2) + \Upsilon_{0,2}] \le 0, \tag{3.12}$$

where $\delta(\mathscr{A})$ denotes the diagonal matrix having the diagonal elements of $\mathscr{A}$, and $\omega(\mathscr{A})$ has zero diagonal elements and absolute values of other elements of $\mathscr{A}$, then

$$x^\top(P_\Gamma A_j + A_0^\top \Lambda^j + \Upsilon_{0,j}) f^j(x) \le 0, \quad \forall j \in \overline{1,2},$$

hence, $\dot{V} \le -x^\top(\Xi^0 + \Gamma^\top \Psi \Gamma)x - 2\sum_{j=1}^{2} x^\top \Upsilon_{0,j} f^j(x) + \phi d^\top d$.

This allows us to present the main result of this section:

Theorem 3.1. *Given $K_0, K_1, K_2 \in \mathbb{R}^{m\times 2n}$; $\zeta \in (0,1)$ and $\pi > 1$, if there exist $0 \le P_1 = P_1^\top \in \mathbb{R}^{2n\times 2n}$; $0 \le P_2 = P_2^\top \in \mathbb{R}^{2n\times 2n}$; $\Lambda^j = \mathrm{diag}(\Lambda_1^j, \dots, \Lambda_{2n}^j) \in \mathbb{D}_+^{2n}$ ($j \in \overline{1,2}$); Θ, Ψ, Ξ^k, $\Upsilon_{s,z} \in \mathbb{D}_+^{2n}$ ($k \in \overline{0,2}$; $s \in \overline{0,1}$; $z \in \overline{s+1,2}$) and $\phi > 0$ such that*

$$P_1 > 0 \quad or \quad P_2 > 0 \quad or \quad \sum_{j=1}^{2} \Lambda^j > 0; \quad P_2 \le \Theta,$$

(3.10), (3.11) and (3.12) are satisfied, then a forward complete system (3.8) is IOS (robustly synchronized) if

$$P_1 \le \xi\Psi; \quad \Theta + \sum_{j=1}^{2} \Lambda^j \le \xi\left(\Xi^0 + 2\sum_{j=1}^{2} \Upsilon_{0,j}\right)$$

for some $\xi > 0$. If, additionally,

$$\Xi^0 + \Gamma^\top \Psi \Gamma > \phi R^0; \quad 2\Upsilon_{0,j} > \phi R^j, \ j = 1,2, \tag{3.13}$$

then for (3.7) the system is asymptotically reaching the synchronous mode.

Proof. Assume that there exists a function $\alpha \in \mathscr{K}_\infty$ such that

$$2\alpha(V) \le x^\top(\Xi^0 + \Gamma^\top \Psi \Gamma)x + 2\sum_{j=1}^{2} x^\top \Upsilon_{0,j} f^j(x),$$

then under the restriction $V(x) \ge \alpha^{-1}(\phi d^\top d)$, we get $\dot{V} \le -\alpha(V)$.

The selection of $\alpha \in \mathscr{K}_\infty$ follows the conditions:

$$P_1 \le \xi\Psi; \quad \Theta + \sum_{j=1}^{2} \Lambda^j \le \xi\left(\Xi^0 + 2\sum_{j=1}^{2} \Upsilon_{0,j}\right)$$

for some $\xi > 0$. The remaining steps repeat the proof of Theorem 2.6. If the perturbation d satisfies (3.7), *i.e.*, $d^\top d \le x^\top R^0 x + x^\top R^1 f^1(x) + x^\top R^2 f^2(x)$, then for (3.13) under the same conditions we get that $\dot{V} \leq -\epsilon\alpha(V)$ for some $\epsilon \in (0,1)$, implying global stability and convergence of the output Γx to zero. □

3.3 Application

The Hindmarsh-Rose (HR) model [43] is widely used to investigate chaotic behavior in isolated biological cells and neuronal dynamics (being a compact version of the general case [44]):

$$\begin{aligned} \dot{x}_1 &= ax_1^2 - x_1^3 - x_2 + x_3 + d, \\ \dot{x}_2 &= (a+\alpha)x_1^2 - x_2, \\ \dot{x}_3 &= \mu(bx_1 - x_3) + u, \end{aligned} \tag{3.14}$$

where $x = [x_1\ x_2\ x_3]^\top \in \mathbb{R}^3$ is the state, $d \in \mathbb{R}$ is the disturbance (equivalently applied current in experiments), $u \in \mathbb{R}$ is the control and $a, \alpha, \mu, b \in \mathbb{R}$. Let $\theta > \frac{1}{4}$ be an auxiliary parameter. Then the system (3.14) can be rewritten as

$$\dot{x} = \alpha_0 x + \alpha_1 f^1(x) + \alpha_2 f^2(x) + \tilde{b}u + \tilde{d}, \tag{3.15}$$

where

$$\tilde{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \tilde{d} = \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix}, \quad \alpha_0 = \begin{bmatrix} -a\theta & -1 & 1 \\ -(a+\alpha)\theta & -1 & 0 \\ \mu b & 0 & -\mu \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} -1-a & 0 & 0 \\ -a-\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\alpha_2 = \begin{bmatrix} a & 0 & 0 \\ a+\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad f^1(x) = \begin{bmatrix} x_1^3 \\ x_2^3 \\ x_3^3 \end{bmatrix}, \quad f^2(x) = \begin{bmatrix} x_1(x_1^2 + x_1 + \theta) \\ x_2(x_2^2 + x_2 + \theta) \\ x_3(x_3^2 + x_3 + \theta) \end{bmatrix},$$

the new nonlinearities f^1 and f^2 satisfy the sector condition given in Assumption 2.1. Let us set the number of systems in the family $N = 2$, $a = 2.8$, $d = 3.1$, $\alpha = 1.6$, $\mu = 10^{-3}$, $b = 9$ and $\theta = 0.3$. Therefore, the common dynamics of models (3.15) is

$$\dot{X} = A_0 X + A_1 F^1(X) + A_2 F^2(X) + BU + D, \tag{3.16}$$

where

$$X = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \in \mathbb{R}^6, \quad U = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \in \mathbb{R}^2, \quad F^j(X) = \begin{bmatrix} f^j(x^1) \\ f^j(x^2) \end{bmatrix}, \forall j \in \overline{1,2},$$
$$B = \begin{bmatrix} \tilde{b} & 0 \\ 0 & \tilde{b} \end{bmatrix}, \quad A_s = \begin{bmatrix} \alpha_s & 0 \\ 0 & \alpha_s \end{bmatrix}, \forall s \in \overline{0,2}, \quad D = \begin{bmatrix} \tilde{d}^1 \\ \tilde{d}^2 \end{bmatrix}$$

and $x^1, x^2 \in \mathbb{R}^3$ are the solutions of each of the couples HR models (3.15). Evidently, the system (3.16) is in the form (3.2). Consider a feedback control in the form (3.4), then U is a vector of scalar controls affecting the HR model to synchronize the system (3.16), we obtain the closed-loop system in the form (3.5)

$$\dot{X} = (A_0 + BK_0\Gamma)X + (A_1 + BK_1\Gamma)F^1(X) + (A_2 + BK_2\Gamma)F^2(X) + D.$$

The synchronization measure is selected as (3.3) with $\Gamma = \begin{bmatrix} I_3 & -I_3 \end{bmatrix}$. Let

$$K_0 = \begin{bmatrix} 0.4283 & 0.4820 & 0.1206 \\ 0.5895 & 0.2262 & 0.3846 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 0 & 0 & 0.5830 \\ 0 & 0 & 0.2518 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 & 0.2904 \\ 0 & 0 & 0.6171 \end{bmatrix},$$

then following Corollary 3.1, there exist matrices solving the proposed LMIs in Theorem 2.6. The norm of the difference $e := x^1 - x^2$ and the state trajectories x^1, x^2 of the closed-loop system with distinct initial states $x^1(0) = \begin{bmatrix} 0.12 & -0.21 & 0.80 \end{bmatrix}^\top$, $x^2(0) = \begin{bmatrix} 0.41 & 0.91 & 0.88 \end{bmatrix}^\top$ are shown in Fig. 3.1 and Fig. 3.2, respectively. The simulation results imply that the system (3.16) is synchronized by the feedback controller, while each separate system remains oscillating.

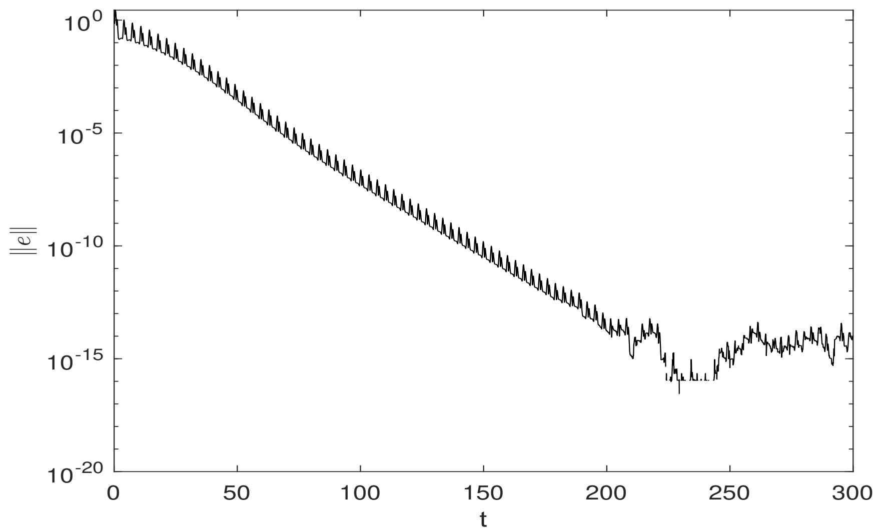

Figure 3.1: The norm of the synchronization error e versus the time t

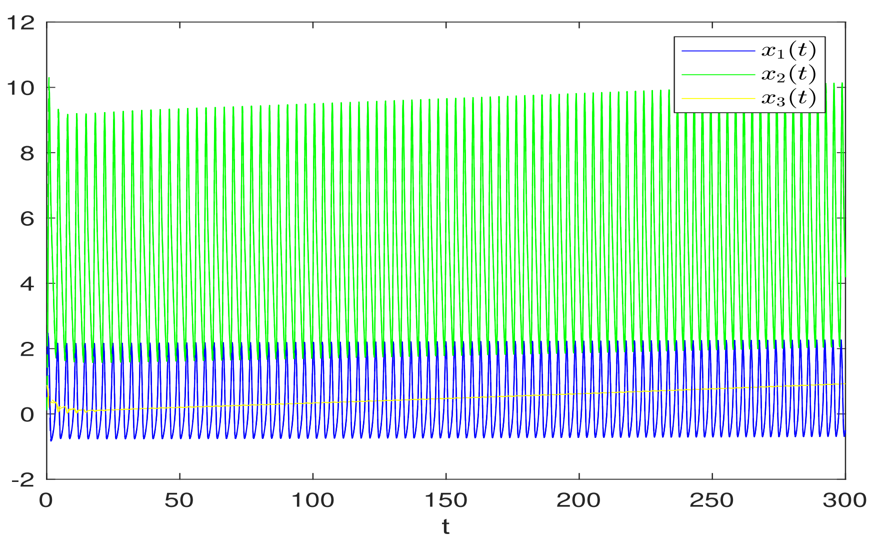

Figure 3.2: The state trajectories x^1 and x^2 versus the time t

CHAPTER 4

Nonlinear robust state estimation

4.1 Observer for generalized Persidskii systems

In this study, an observer for (2.11) is proposed in the following conventional form:

$$\begin{aligned}\dot{\hat{x}}(t) &= A_0\hat{x}(t)+\sum_{j=1}^{M}A_j f^j(H_j\hat{x}(t))+L(y(t)-\hat{y}(t)), \\ \hat{y}(t) &= \begin{bmatrix} C_0\hat{x}(t) \\ C_1 f^1(H_1\hat{x}(t)) \\ \vdots \\ C_M f^M(H_M\hat{x}(t)) \end{bmatrix},\end{aligned} \tag{4.1}$$

where $\hat{x}(t)\in\mathbb{R}^n$ is the estimation of the state $x(t)$; $L=[L_0\ L_1\ldots L_M]\in\mathbb{R}^{n\times z}$ is a matrix gain to be designed, with $L_s\in\mathbb{R}^{n\times z_s}$ for $s\in\overline{0,M}$.

Remark 4.1. *The observer design for Lur'e models using quadratic Lyapunov functions was considered in [10, 25, 24].*

As introduced above, the goal is to ensure asymptotic convergence of $\hat{x}$ to x in the case of no perturbations and boundedness of the estimates otherwise. To reach this objective, for this observer, we will analyze two cases of expression of dynamics of the estimation error $e=x-\hat{x}$ given in the introduction: (2.23) and (2.24). For the latter case, we will investigate IOS conditions for the common system (2.11), (4.1) for the output e, while in the former scenario, SIIOS conditions of the dynamics of e will be studied (in both cases, the inputs are represented by the disturbances w and v). Observability or detectability issues of (2.11) are not considered in this work, and they will not be related to the conditions of stability of e.

4.2 IOS analysis

Note that the output stability for the system (2.11), (4.1) is equivalent to a robust state synchronization of these two generalized Persidskii systems under the influence of perturbations w and v. Therefore, the synchronization method developed in [80] can be adopted. To this end, let us write the common dynamics of (2.11), (4.1):

$$\dot{X} = \tilde{A}_0 X + \sum_{j=1}^{M} \tilde{A}_j F^j(\tilde{H}_j X) + \mathscr{D}, \quad \mathscr{D} = \begin{bmatrix} Dw \\ Lv \end{bmatrix}, \tag{4.2}$$

where $X = [x^\top \ \hat{x}^\top]^\top \in \mathbb{R}^{2n}$ is the extended state; $\mathscr{D} \in \mathbb{R}^{2n}$ is the augmented disturbance and

$$\tilde{A}_s = \begin{bmatrix} A_s & \mathbf{O}_{n\times k_s} \\ L_s C_s & A_s - L_s C_s \end{bmatrix}, \ s \in \overline{0,M},$$

$$F^j(\tilde{H}_j X) = \begin{bmatrix} f^j(H_j x) \\ f^j(H_j \hat{x}) \end{bmatrix}, \ \tilde{H}_j = \begin{bmatrix} H_j & \mathbf{O}_{k_j\times n} \\ \mathbf{O}_{k_j\times n} & H_j \end{bmatrix}, \ j \in \overline{1,M}$$

with the output function given by the estimation error:

$$e = \Gamma X \text{ with } \Gamma := \begin{bmatrix} I_n & -I_n \end{bmatrix}.$$

Clearly the system (4.2) also yields the generalized Persidskii form.

We say that (4.1) is an observer for (2.11) if the common dynamics (4.2) is IOS with the inputs w, v (or $\mathscr{D}$) and the output e (this corresponds to the case (2.24) given in the preliminaries). The related conditions are as follows:

Theorem 4.1. *Let Assumption 2.1 be satisfied. Let there exist* $0 \le P_1 = P_1^\top \in \mathbb{R}^{n\times n}$; $0 \le P_2 = P_2^\top \in \mathbb{R}^{2n\times 2n}$; $\Lambda^j = \mathrm{diag}(\Lambda_1^j, \ldots, \Lambda_{2k_j}^j) \in \mathbb{D}_+^{2k_j}$ $(j \in \overline{1,M})$; $\Xi^s \in \mathbb{D}_+^{2k_s}$ $(s \in \overline{0,M})$, $\Upsilon_{0,s} \in \mathbb{D}_+^{2k_s}$ $(s \in \overline{1,M})$; $\{\Upsilon_{s,r}\}_{r=s+1}^{M} \subset \mathbb{D}_+^{2n}$ $(s \in \overline{1,M-1})$; $\Theta \in \mathbb{D}_+^{2n}$; $\Psi \in \mathbb{D}_+^{n}$; $\varrho \in \mathbb{R}$ *and* $0 < \Phi = \Phi^\top \in \mathbb{R}^{2n\times 2n}$ *such that*

$$P_1 > 0 \quad or \quad P_2^{11} - 2P_2^{12} + P_2^{22} + \varrho P_1 > 0 \quad or \quad \sum_{j=1}^{\mu} \tilde{\Lambda}^j + \varrho P_1 > 0,$$

$$P_2 \le \Theta; \quad Q = Q^\top = (Q_{a,b})_{a,b=1}^{M+2} \le 0, \tag{4.3}$$

where

$$P_2^{11}, P_2^{12}, P_2^{22} \in \mathbb{R}^{n\times n}; \quad P_2 = \begin{bmatrix} P_2^{11} & P_2^{12} \\ P_2^{12} & P_2^{22} \end{bmatrix},$$

$$\tilde{\Lambda}^j = H_j^\top \mathrm{diag}(\Lambda_1^j, \dots, \Lambda_{k_j}^j) H_j + H_j^\top \mathrm{diag}(\Lambda_{k_j+1}^j, \dots, \Lambda_{2k_j}^j) H_j,$$

$$Q_{1,1} = \tilde{A}_0^\top P_\Gamma + P_\Gamma \tilde{A}_0 + \Xi^0 + \Gamma^\top \Psi \Gamma; \quad P_\Gamma = \Gamma^\top P_1 \Gamma + P_2,$$

$$Q_{j+1,j+1} = \tilde{A}_j^\top \tilde{H}_j^\top \Lambda^j + \Lambda^j \tilde{H}_j \tilde{A}_j + \Xi^j, \ j \in \overline{1,M},$$

$$Q_{1,j+1} = P_\Gamma \tilde{A}_j + \tilde{A}_0^\top \tilde{H}_j^\top \Lambda^j + \tilde{H}_j^\top \Upsilon_{0,j}, \ j \in \overline{1,M},$$

$$Q_{s+1,r+1} = \tilde{A}_s^\top \tilde{H}_r^\top \Lambda^r + \Lambda^s \tilde{H}_s \tilde{A}_r + \tilde{H}_s^\top \tilde{H}_s \Upsilon_{s,r} \tilde{H}_r^\top \tilde{H}_r, \ s \in \overline{1,M-1}, r \in \overline{s+1,M},$$

$$Q_{1,M+2} = P_\Gamma; \quad Q_{M+2,M+2} = -\Phi; \quad Q_{j+1,M+2} = \Lambda^j \tilde{H}_j, \ j \in \overline{1,M},$$

and for some $\xi > 0$:

$$P_1 \le \xi \Psi; \quad \Theta + \sum_{j=1}^{M} \tilde{H}_j^\top \Lambda^j \tilde{H}_j \le \xi \left(\sum_{k=0}^{\varpi} \tilde{H}_k^\top \Xi^k \tilde{H}_k + 2 \sum_{r=1}^{\varpi} \tilde{H}_r^\top \Upsilon_{0,r} \tilde{H}_r + 2 \sum_{s=1}^{\varpi-1} \sum_{r=s+1}^{\varpi} \tilde{H}_s^\top \tilde{H}_s \Upsilon_{s,r} \tilde{H}_r^\top \tilde{H}_r \right). \quad (4.4)$$

Then a forward complete system (4.2) is IOS.

Proof. Consider a candidate Lyapunov function

$$V(X) = X^\top P_\Gamma X + 2 \sum_{j=1}^{M} \sum_{i=1}^{2k_j} \Lambda_i^j \int_0^{\tilde{H}_j^i X} F_i^j(\tau) d\tau, \quad (4.5)$$

where $\tilde{H}_j^i$ is the i^{th} row of the matrix $\tilde{H}_j$. Let us check the lower bound for V from the first condition in (2.5), which is valid if one of the following inequalities are satisfied:

$$X^\top P_2 X > 0; \quad X^\top \left(\sum_{j=1}^{\mu} \tilde{H}_j^\top \Lambda^j \tilde{H}_j \right) X > 0$$

under the constraints $X^\top \Gamma^\top P_1 \Gamma X = 0$, $\Gamma X \neq 0$, *i.e.*, the matrix P_2 or $\sum_{j=1}^{\mu} \tilde{H}_j^\top \Lambda^j \tilde{H}_j$ should be positive definite on the subset of $e \neq 0$ belonging to the kernel of P_1 (the summation is for μ terms since only unbounded nonlinearities are considered for radial unboundedness of V). Hence, if $P_1 > 0$, then the constraints are self-excluding, and the case $P_1 \geq 0$ is further considered. To simplify the formulation, define new coordinates $Z = SX$, $S = \begin{bmatrix} I_n & -I_n \\ I_n & I_n \end{bmatrix}$, then $e = \begin{bmatrix} I_n & \mathbf{O}_{n\times n} \end{bmatrix} Z$, S is not singular, and the above constraints take the form:

$$Z^\top \begin{bmatrix} I_n & \mathbf{O}_{n\times n} \end{bmatrix}^\top P_1 \begin{bmatrix} I_n & \mathbf{O}_{n\times n} \end{bmatrix} Z = 0; \quad \begin{bmatrix} I_n & \mathbf{O}_{n\times n} \end{bmatrix} Z \neq 0,$$

which can be equivalently rewritten with respect to the first component of Z (the error e), $e^\top P_1 e =$

0, $e \neq 0$, together with the conditions to check:

$$e^\top \left(P_2^{11} - 2P_2^{12} + P_2^{22}\right) e > 0; \quad e^\top \left(\sum_{j=1}^{\mu} \tilde{\Lambda}^j\right) e > 0,$$

where

$$P_2 = \begin{bmatrix} P_2^{11} & P_2^{12} \\ P_2^{12} & P_2^{22} \end{bmatrix} \text{ for } P_2^{11}, P_2^{12}, P_2^{22} \in \mathbb{R}^{n\times n},$$
$$\tilde{\Lambda}^j = H_j^\top \mathrm{diag}(\Lambda_1^j, \ldots, \Lambda_{k_j}^j) H_j + H_j^\top \mathrm{diag}(\Lambda_{k_j+1}^j, \ldots, \Lambda_{2k_j}^j) H_j.$$

Using Finsler's Lemma [29], these conditions follow the first LMI given in the formulation of the theorem. So, in such a case there are $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ such that

$$\alpha_1(\|e\|) \leq V(X) \leq \alpha_2(\|X\|), \tag{4.6}$$

where

$$\alpha_2(\tau) \leq \lambda_{\max}(P_\Gamma)\tau^2 + \left(\sum_{j=1}^{M} 4k_j\right) \max_{\substack{j\in\overline{1,M} \\ i\in\overline{1,2k_j}}} \left\{\Lambda_i^j \int_0^{\|\tilde{H}_j\|\tau} F_i^j(\gamma) d\gamma\right\},$$

then the first condition in (2.5) is verified.

Next, consider the derivative of V (denote $\dot{V} = \nabla V(X)\dot{X}$):

$$\begin{aligned}
\dot{V} &= \dot{X}^\top P_\Gamma X + X^\top P_\Gamma \dot{X} + 2\sum_{j=1}^{M} \dot{X}^\top \tilde{H}_j^\top \Lambda^j F^j(\tilde{H}_j X) \\
&= X^\top \left(\tilde{A}_0^\top P_\Gamma + P_\Gamma \tilde{A}_0\right) X + \left(\sum_{j=1}^{M} F^j(\tilde{H}_j X)^\top \tilde{A}_j^\top\right) P_\Gamma X \\
&\quad + X^\top P_\Gamma \sum_{j=1}^{M} \tilde{A}_j F^j(\tilde{H}_j X) + 2X^\top P_\Gamma \mathscr{D} + \\
&\quad 2\sum_{j=1}^{M} \left(X^\top \tilde{A}_0^\top \tilde{H}_j^\top \Lambda^j F^j(\tilde{H}_j X) + \mathscr{D}^\top \tilde{H}_j^\top \Lambda^j F^j(\tilde{H}_j X) + \right. \\
&\quad \left. \left(\sum_{s=1}^{M} F^s(\tilde{H}_s X)^\top \tilde{A}_s^\top\right) \tilde{H}_j^\top \Lambda^j F^j(\tilde{H}_j X)\right).
\end{aligned}$$

Therefore, under (4.3) we obtain

$$
\begin{aligned}
\dot{V} &= \begin{bmatrix} X \\ F^1(\tilde{H}_1X) \\ \vdots \\ F^M(\tilde{H}_MX) \\ \mathscr{D} \end{bmatrix}^\top Q \begin{bmatrix} X \\ F^1(\tilde{H}_1X) \\ \vdots \\ F^M(\tilde{H}_MX) \\ \mathscr{D} \end{bmatrix} - X^\top(\Gamma^\top\Psi\Gamma+\Xi^0)X \\
&\quad -\sum_{j=1}^{M} F^j(\tilde{H}_jX)^\top \Xi^j F^j(\tilde{H}_jX) - 2\sum_{j=1}^{M} X^\top \tilde{H}_j^\top \Upsilon_{0,j} F^j(\tilde{H}_jX) \\
&\quad -2\sum_{s=1}^{M-1}\sum_{r=s+1}^{M} F^s(\tilde{H}_sX)^\top \tilde{H}_s^\top \tilde{H}_s \Upsilon_{s,r} \tilde{H}_r^\top \tilde{H}_r F^r(\tilde{H}_rX) + \mathscr{D}^\top\Phi\mathscr{D} \\
&\le -X^\top(\Gamma^\top\Psi\Gamma+\Xi^0)X - \sum_{j=1}^{M} F^j(\tilde{H}_jX)^\top \Xi^j F^j(\tilde{H}_jX) \\
&\quad -2\sum_{j=1}^{M} X^\top \tilde{H}_j^\top \Upsilon_{0,j} F^j(\tilde{H}_jX) \\
&\quad -2\sum_{s=1}^{M-1}\sum_{r=s+1}^{M} F^s(\tilde{H}_sX)^\top \tilde{H}_s^\top \tilde{H}_s \Upsilon_{s,r} \tilde{H}_r^\top \tilde{H}_r F^r(\tilde{H}_rX) + \mathscr{D}^\top\Phi\mathscr{D}.
\end{aligned}
$$

Due to the form of the function V, there exists $\alpha \in \mathscr{K}_\infty$ such that

$$
\begin{aligned}
\alpha(V(X)) &\le X^\top(\Gamma^\top\Psi\Gamma+\Xi^0)X + \sum_{j=1}^{M} F^j(\tilde{H}_jX)^\top \Xi^j F^j(\tilde{H}_jX) \\
&\quad +2\sum_{j=1}^{M} X^\top \tilde{H}_j^\top \Upsilon_{0,j} F^j(\tilde{H}_jX) \\
&\quad +2\sum_{s=1}^{M-1}\sum_{r=s+1}^{M} F^s(\tilde{H}_sX)^\top \tilde{H}_s^\top \tilde{H}_s \Upsilon_{s,r} \tilde{H}_r^\top \tilde{H}_r F^r(\tilde{H}_rX)
\end{aligned}
$$

under the conditions (4.4), which have to be verified for some $\xi > 0$ (only the first ϖ nonlinearities and the quadratic term are radially unbounded). Finally, under the conditions of the theorem:

$$\dot{V} \le -\alpha(V) + \mathscr{D}^\top\Phi\mathscr{D}$$

for all $X \in \mathbb{R}^{2n}$ and $\mathscr{D} \in \mathbb{R}^{2n}$. Hence, the second relation in (2.5) can be recovered:

$$V \ge \alpha^{-1}(2\mathscr{D}^\top\Phi\mathscr{D}) \quad \Rightarrow \quad \dot{V} \le -\frac{1}{2}\alpha(V),$$

and the IOS property is guaranteed (if the right-hand side of the estimate for $\dot{V}$ is in the form of a function of class $\mathscr{K}$ (as above) and not of class $\mathscr{KL}$ (as in (2.5)), then UBIBS property can be omitted, and forward completeness is enough). □

We can require a stricter property for the nonlinearities of the system (2.11), which can be viewed as an incremental passivity condition [98, 150]:

Assumption 4.1. *For any $j \in \overline{1,M}$:*

$$X^\top \Gamma^\top \Gamma \tilde{H}_j^\top F^j(\tilde{H}_j X) > 0, \quad \forall X \in \mathbb{R}^{2n} \setminus \{Z \in \mathbb{R}^{2n} : \Gamma Z = 0\}.$$

Under these additional restrictions imposed on the system (4.2), a relaxed stability result can be obtained:

Corollary 4.1. *Let assumptions 2.1 and 4.1 be satisfied. If there exist $0 \le P_1 = P_1^\top \in \mathbb{R}^{n\times n}$; $0 \le P_2 = P_2^\top \in \mathbb{R}^{2n\times 2n}$; $\{\Xi^k\}_{k=0}^M$, $\{\Upsilon_{s,r}\}_{r=s+1}^M \subset \mathbb{D}_+^n$ ($s \in \overline{0, M-1}$), $\{\Lambda^j\}_{j=1}^M \subset \mathbb{D}_+^{2k_j}$ and $\varrho \in \mathbb{R}$ such that*

$$P_2 + \varrho \sum_{q=1}^{\mu} \tilde{H}_q^\top \Lambda^q \tilde{H}_q^\top > 0; \quad Q = Q^\top \le 0,$$

where

$$Q_{1,1} = \tilde{A}_0^\top P_\Gamma + P_\Gamma \tilde{A}_0 + \Gamma^\top \Xi^0 \Gamma; \quad P_\Gamma = \Gamma^\top P_1 \Gamma + P_2,$$
$$Q_{j+1,j+1} = \tilde{A}_j^\top \tilde{H}_j^\top \Lambda^j + \Lambda^j \tilde{H}_j \tilde{A}_j + \tilde{H}_j \Gamma^\top \Xi^j \Gamma \tilde{H}_j^\top, \ j \in \overline{1,M},$$
$$Q_{1,j+1} = P_\Gamma \tilde{A}_j + \tilde{A}_0^\top \tilde{H}_j^\top \Lambda^j + \tilde{H}_j \Gamma^\top \Upsilon_{0,j} \Gamma, \ j \in \overline{1,M},$$
$$Q_{s+1,r+1} = \tilde{A}_s^\top \tilde{H}_r^\top \Lambda^r + \Lambda^s \tilde{H}_s \tilde{A}_r + \tilde{H}_s \Gamma^\top \Upsilon_{s,r} \Gamma \tilde{H}_r^\top, \ s \in \overline{1, M-1}, \ r \in \overline{s+1, M},$$

and

$$\sum_{k=0}^{M} \Xi^k + 2\sum_{s=0}^{M-1} \sum_{r=s+1}^{M} \Upsilon_{s,r} > 0.$$

Then the system (4.2) with $\|\mathscr{D}\|_\infty = 0$ has globally bounded trajectories and $\lim_{t\to+\infty} \|e(t)\| = 0$.

Proof. Consider the Lyapunov function (4.5). By Finsler's Lemma [20], the first LMI of the corollary implies positive definiteness of V with respect to X, then

$$\alpha_1(\|X\|) \le V(X) \le \alpha_2(\|X\|)$$

for some functions $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$. Repeating the steps of the proof of Theorem 4.1, the derivative of V with $\mathscr{D}=0$ can be upper bounded as follows:

$$\begin{aligned}\dot V &\le -X^\top\Gamma^\top\Xi^0\Gamma X-\sum_{j=1}^{M}F^j(\tilde H_jX)^\top\tilde H_j\Gamma^\top\Xi^j\Gamma\tilde H_j^\top F^j(\tilde H_jX)\\ &\quad -2\sum_{j=1}^{M}X^\top\Gamma^\top\Upsilon_{0,j}\Gamma\tilde H_j^\top F^j(\tilde H_jX)\\ &\quad -2\sum_{s=1}^{M-1}\sum_{r=s+1}^{M}F^s(\tilde H_sX)^\top\tilde H_s\Gamma^\top\Upsilon_{s,r}\Gamma\tilde H_r^\top F^r(\tilde H_rX)\end{aligned}$$

under the condition $Q\le 0$. Since

$$\sum_{k=0}^{M}\Xi^k+2\sum_{s=0}^{M-1}\sum_{r=s+1}^{M}\Upsilon_{s,r}>0$$

and due to assumptions 2.1 and 4.1, there exists $\alpha\in\mathscr{K}$ such that $\dot V\le-\alpha(\|e\|)$, which implies boundedness of all solutions in (4.2) with $\|\mathscr{D}\|_\infty=0$. Applying LaSalle arguments [57], we obtain for all initial conditions, $\lim_{t\to+\infty}\|e(t)\|=0$. □

Previously, the observer gain L was assumed to be given. To find the gain as a solution of LMI, the next corollary considers an equivalent expression of (4.2):

$$\dot X=(\bar A_0+WL_0\bar C_0)X+K\bar F(\mathscr{H}X)+\begin{bmatrix}Dw\\ Lv\end{bmatrix}, \tag{4.7}$$

where

$$\begin{gathered}\bar A_s=\mathrm{diag}(A_s,A_s),\ s\in\overline{0,M};\quad W=\begin{bmatrix}\mathbf{O}_{n\times n} & I_n\end{bmatrix}^\top,\\ \bar F(\mathscr{H}X)=\begin{bmatrix}F^1(\tilde H_1X)\\ \vdots\\ F^M(\tilde H_MX)\end{bmatrix};\quad \mathscr{H}=\begin{bmatrix}\tilde H_1\\ \vdots\\ \tilde H_M\end{bmatrix};\quad \bar C_s=\begin{bmatrix}C_s & -C_s\end{bmatrix},\ s\in\overline{0,M},\\ K=[(\bar A_1+WL_1\bar C_1)\dots(\bar A_M+WL_M\bar C_M)].\end{gathered}$$

The conditions of Theorem 4.1 can then be expanded:

Corollary 4.2. *Let Assumption 2.1 be satisfied and $H_j=I_n$ for $j\in\overline{1,M}$. If there exist $0<P_1=P_1^\top\in\mathbb{R}^{n\times n}$; $0<P_2=P_2^\top\in\mathbb{R}^{2n\times 2n}$; $\{\Xi^j\}_{j=1}^{M}\subset\mathbb{D}_+^{2n}$, $0<\Phi_w=\Phi_w^\top\in\mathbb{R}^{p\times p}$, $0<\Phi_v=\Phi_v^\top\in\mathbb{R}^{z\times z}$ and $L_j\in\mathbb{R}^{n\times z_j}$ $(j\in\overline{1,M})$ such that*

$$G=G^\top=(G_{a,b})_{a,b=1}^{M+3}\le 0,$$

where

$$P_\Gamma = \Gamma^\top P_1 \Gamma + P_2; \quad G_{1,1} = P_\Gamma^{-1}\bar{A}_0^\top + \bar{A}_0 P_\Gamma^{-1} + P_\Gamma^{-1},$$
$$G_{j+1,j+1} = \bar{A}_j^\top + \bar{A}_j + \bar{C}_j^\top L_j^\top W^\top + W L_j \bar{C}_j + \Xi^j,\ j \in \overline{1,M},$$
$$G_{1,j+1} = \bar{A}_j + W L_j \bar{C}_j + P_\Gamma^{-1}\bar{A}_0^\top,\ j \in \overline{1,M},$$
$$G_{s+1,r+1} = \bar{A}_s^\top + \bar{C}_s^\top L_s^\top W^\top + \bar{A}_r + W L_r \bar{C}_r,\ s \in \overline{1,M-1},\ r \in \overline{s+1,M},$$
$$G_{s+1,M+2} = \begin{bmatrix} D^\top & \mathbf{O}_{p\times n} \end{bmatrix}^\top,\ s \in \overline{0,M},$$
$$G_{s+1,M+3} = \begin{bmatrix} \mathbf{O}_{z\times n} & \begin{bmatrix} \mathbf{O}_{n\times z_0} & L_1 & \dots & L_M \end{bmatrix}^\top \end{bmatrix}^\top,\ s \in \overline{0,M},$$
$$G_{M+2,M+2} = -\Phi_w; \quad G_{M+3,M+3} = -\Phi_v; \quad G_{M+2,M+3} = \mathbf{O}_{p\times z},$$

then a forward complete system (4.2) is IOS for the observer gain $L = \begin{bmatrix} \mathbf{O}_{n\times z_0} & L_1 & \dots & L_M \end{bmatrix}$.

Proof. Consider Theorem 4.1 and its proof under substitutions $H_j \to I_n (j \in \overline{1,M})$, $\Lambda^j \to I_{2n} (j \in \overline{1,M})$, $\Upsilon_{s,r} \to \mathbf{O}_{n\times n} (s \in \overline{0,M-1}, r \in \overline{s+1,M})$, $\Theta \to P_2$, $\Xi^0 \to P_2$, $\Psi \to P_1$, $\Phi \to \operatorname{diag}(\Phi_w,\Phi_v)$ and $L \to \begin{bmatrix} \mathbf{O}_{n\times z_0} & L_1 & \dots & L_M \end{bmatrix}$. Since $P_1 > 0$ and $P_2 > 0$, the relations from (2.5) about positive definiteness of the Lyapunov function V in (4.5) are satisfied. The time derivative of V with respect to (4.7) is:

$$\begin{aligned}
\dot{V} &= \begin{bmatrix} X \\ F^1(X) \\ \vdots \\ F^M(X) \\ w \\ v \end{bmatrix}^\top Q_0 \begin{bmatrix} X \\ F^1(X) \\ \vdots \\ F^M(X) \\ w \\ v \end{bmatrix} - X^\top P_\Gamma X \\
&\quad - \sum_{j=1}^{M} F^j(X)^\top \Xi^j F^j(X) + w^\top \Phi_w w + v^\top \Phi_v v \\
&\le -X^\top P_\Gamma X - \sum_{j=1}^{M} F^j(X)^\top \Xi^j F^j(X) + w^\top \Phi_w w + v^\top \Phi_v v,
\end{aligned}$$

where $Q_0 = E^\top G E$ and $E = \operatorname{diag}(P_\Gamma, I_{2n}, \dots, I_{2n}, I_p, I_z)$. Since $P_2 > 0$, there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\alpha(V(X)) \le X^\top P_\Gamma X + \sum_{j=1}^{M} F^j(X)^\top \Xi^j F^j(X).$$

Therefore, $V \ge \alpha^{-1}(2(w^\top \Phi_w w + v^\top \Phi_v v))$ implying $\dot{V} \le -\frac{1}{2}\alpha(V)$, so the IOS property is guaranteed. □

Thus, to calculate L as a solution of LMIs, the conditions of Corollary 4.2 introduce several additional restrictions to the ones proposed in Theorem 4.1, where positive definiteness of P_1, P_2, and substitution $\Lambda^j = I_{2n}$ for $j \in \overline{1, M}$ are the most constraining.

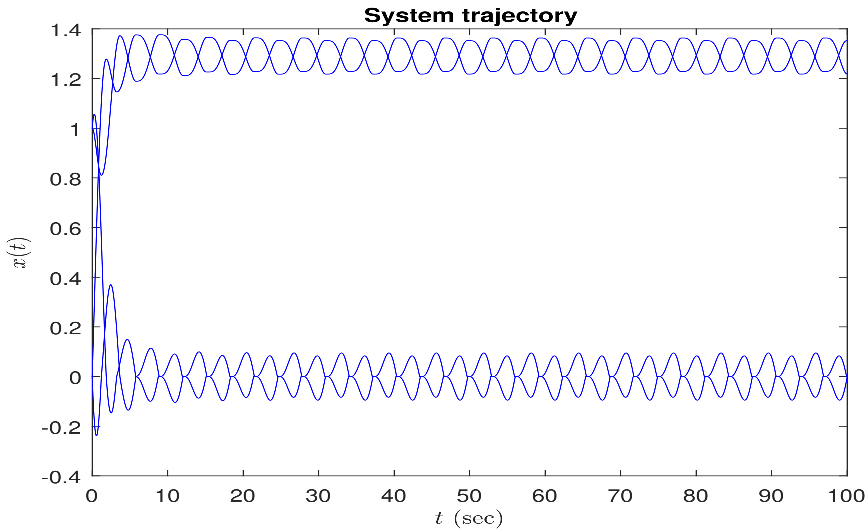

Figure 4.1: The system trajectory of the perturbed two-mass spring damper system

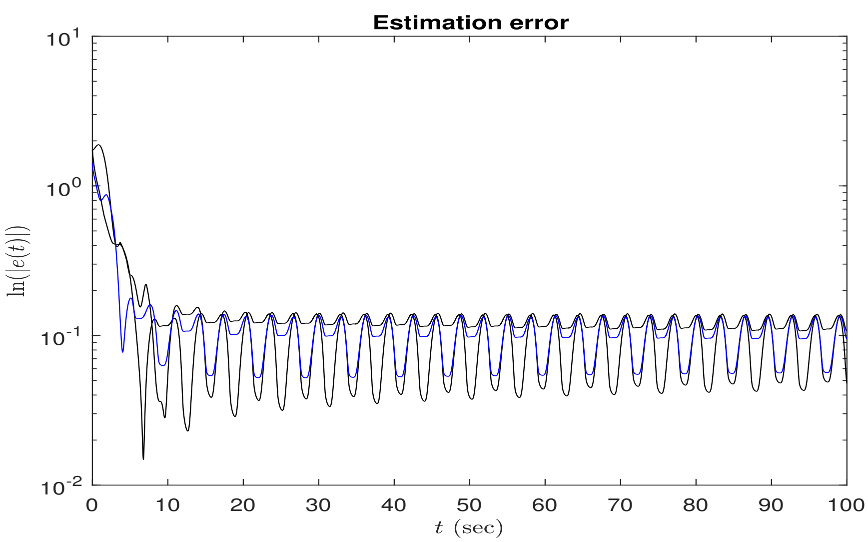

Figure 4.2: The logarithm of the norm of estimation error

Example 1. *Consider a perturbed two-mass spring damper system on a horizontal plane in the* *form of* (4.7) *from [6]:*

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -0.2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -0.4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2 \end{bmatrix},$$

$$C_0 = C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad \phi(s) = \min\{1, |s|^{0.1}\}\mathrm{sign}(s),$$

$$D = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad f^1(x) := \begin{bmatrix} \phi(x_1) \\ \phi(x_2) \\ \phi(x_3) \\ \phi(x_4) \end{bmatrix}, \quad F^1(X) := \begin{bmatrix} f^1(x) \\ f^1(\hat{x}) \end{bmatrix},$$

where $x \in \mathbb{R}^n$ for $n = 4$ is composed of the relative position of the first mass with its velocity and the same information for the second mass. It is assumed that the position of the first mass x_1 is measured, then we can also assume that $\phi(x_1)$ is available; $M = 1$ and the nonlinearity approximating the dry friction respects Assumption 2.1; for simulation $w(t) = 0.2\sin(t)$, $v(t) = 0$ for all $t \in \mathbb{R}_+$; $X = [x^\top \quad \hat{x}^\top]^\top \in \mathbb{R}^{2n}$ is the extended state. The selected observer gains are

$$L_0 = \begin{bmatrix} 1.1715 \\ 1.4461 \\ 0.4227 \\ 0.1555 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then the LMIs proposed in Theorem 4.1 are verified. A state trajectory of the system (2.11)*, and the logarithm of $|e(t)|$ in* (2.11)*,* (4.1) *for several initial conditions are presented in Fig. 4.1 and Fig. 4.2, respectively (the blue curve in Fig. 4.2 corresponds to the trajectory in Fig. 4.1). The simulation results illustrate that the behavior of the system* (2.11) *(it has nonlinearities in the state and in the output equations) is well estimated by the observer* (4.1)*.*

4.3 SIIOS analysis

To represent the dynamics of estimation error e for (2.11), (4.1) in the form (2.23), let us introduce a short-hand notation

$$\delta f^j(x,\hat{x}) := f^j(H_j x) - f^j(H_j \hat{x})$$

for all $j \in \overline{1, M}$ (further we will often skip the arguments $(x,\hat{x})$ of δf^j for brevity). Then the following properties are required in the sequel for nonlinear functions in (2.11) (a nonlinear version of quadratic constraints used in [10, 25, 24]):

Assumption 4.2. *Assume that there exist* S_0^j, S_1^j, S_2^j, $\Sigma_0^j, \Sigma_1^j, \Sigma_2^j \in \mathbb{D}_+^{k_j}$ *and* $S_3^{j,q}, \Sigma_3^{j,q} \in \mathbb{D}_+^n$ *with* $j, q \in \overline{1, M}$ *such that*

$$\begin{aligned}
(\delta f^j)^\top \delta f^j &\leq e^\top H_j^\top S_0^j H_j e + 2e^\top H_j^\top S_1^j(\delta f^j) \\
&\quad + 2e^\top H_j^\top S_2^j f^j(H_j e) + 2\sum_{q=1}^{M} (\delta f^j)^\top H_j S_3^{j,q} H_q^\top f^q(H_q e); \\
f^j(H_j e)^\top f^j(H_j e) &\leq e^\top H_j^\top \Sigma_0^j H_j e + 2e^\top H_j^\top \Sigma_1^j(\delta f^j) \\
&\quad + 2e^\top H_j^\top \Sigma_2^j f^j(H_j e) + 2\sum_{q=1}^{M} (\delta f^j)^\top H_j \Sigma_3^{j,q} H_q^\top f^q(H_q e)
\end{aligned}$$

for all $x, \hat{x} \in \mathbb{R}^n$ *with* $e = x - \hat{x}$.

Since the right-hand side of the first inequality above is proportional to the estimation error $H_j e$, this hypothesis implies Assumption 4.1 (further, we recall it in the formulations of all results to avoid confusion). The dynamics of e can be written as:

$$\dot{e} = \mathscr{A}_0 e + \sum_{j=1}^{M} \mathscr{A}_j \delta f^j + \mathscr{D}, \tag{4.8}$$

with $\mathscr{D} = Dw - Lv$ another auxiliary bounded input and $\mathscr{A}_s = A_s - L_s C_s$, $\forall s \in \overline{0, M}$. The error dynamics (4.8) can be interpreted as the autonomous system (2.23).

Theorem 4.2. *Let assumptions 2.1, 4.1 and 4.2 be satisfied. If there exist* $0 \leq P = P^\top \in \mathbb{R}^{n\times n}$; $\Xi^0 \in \mathbb{D}_+^n$; $\Lambda^j = \operatorname{diag}(\Lambda_1^j, \ldots, \Lambda_n^j)$, Γ_j, $\Omega_j \in \mathbb{D}_+^{k_j} (j \in \overline{1, M})$; $\{\Upsilon_{j,k}\}_{j,k=1}^M \subset \mathbb{D}_+^n$; $0 < \Phi = \Phi^\top \in \mathbb{R}^{n\times n}$; $\varrho \in \mathbb{R}$ *and* $\gamma, \eta > 0$ *such that*

$$P + \varrho \sum_{j=1}^{\mu} H_j^\top \Lambda^j H_j > 0; \quad \mathscr{Q} = \mathscr{Q}^\top = (\mathscr{Q}_{a,b})_{a,b=1}^4 \leq 0, \tag{4.9}$$

$$\Xi^0 - \gamma \sum_{j=1}^{M} H_j^\top S_0^j H_j - \eta \sum_{j=1}^{M} H_j^\top \Sigma_0^j H_j > 0,$$

$$\Gamma_j - \gamma S_1^j - \eta \Sigma_1^j \geq 0; \quad \Omega_j - \gamma S_2^j - \eta \Sigma_2^j \geq 0; \quad \Upsilon_{j,k} - \gamma S_3^{j,k} - \eta \Sigma_3^{j,k} \geq 0,$$

where

$$\mathcal{Q}_{1,1} = \mathscr{A}_0^\top P + P\mathscr{A}_0 + \Xi^0; \quad \mathcal{Q}_{2,2} = -\gamma I_{nM}; \quad \mathcal{Q}_{1,2} = P\mathscr{A} + \bar{\Gamma},$$
$$\mathcal{Q}_{1,3} = \mathscr{A}_0^\top \Lambda + \Omega; \quad \mathcal{Q}_{2,3} = \mathscr{A} \top \Lambda + \Upsilon; \quad \mathcal{Q}_{3,3} = -\eta I_{nM},$$
$$\mathcal{Q}_{1,4} = P; \quad \mathcal{Q}_{2,4} = \mathbf{O}_{nM\times n}; \quad \mathcal{Q}_{3,4} = \Lambda^\top; \quad \mathcal{Q}_{4,4} = -\Phi,$$
$$\mathscr{A} = \begin{bmatrix} \mathscr{A}_1 & \dots & \mathscr{A}_M \end{bmatrix}; \quad \bar{\Gamma} = \begin{bmatrix} H_1^\top \Gamma_1 & \dots H_M^\top \Gamma_M \end{bmatrix}; \quad \Upsilon = (H_j \Upsilon_{j,k} H_k^\top)_{j,k=1}^M,$$
$$\Lambda = \begin{bmatrix} H_1^\top \Lambda^1 & \dots H_M^\top \Lambda^M \end{bmatrix}; \quad \Omega = \begin{bmatrix} H_1^\top \Omega_1 & \dots H_M^\top \Omega_M \end{bmatrix},$$

then the system (2.11), (4.1) *is SIIOS with respect to the estimation error* e.

Proof. Consider a candidate Lyapunov function:

$$V(e) = e^\top P e + 2\sum_{j=1}^{M}\sum_{i=1}^{k_j} \Lambda_i^j \int_0^{H_j^i e_i} f_i^j(\tau)d\tau, \tag{4.10}$$

where H_j^i is the i^{th} row of the matrix H_j. Let us check the properties given in (2.7). Finsler's Lemma [20] and the first condition in (4.9) imply that the matrix $P + \sum_{j=1}^{\mu} H_j^\top \Lambda^j H_j$ is positive definite, which ensures required definiteness of V. The time derivative of V for (4.8) admits the following representation:

$$\begin{aligned}
\dot{V} &= \dot{e}^\top P e + e^\top P \dot{e} + 2\dot{e}^\top \sum_{j=1}^{M} H_j^\top \Lambda^j f^j(H_j e) \\
&= \begin{bmatrix} e \\ \delta f^1 \\ \vdots \\ \delta f^M \\ f^1(H_1 e) \\ \vdots \\ f^M(H_M e)) \end{bmatrix}^\top \tilde{\mathcal{Q}} \begin{bmatrix} e \\ \delta f^1 \\ \vdots \\ \delta f^M \\ f^1(H_1 e) \\ \vdots \\ f^M(H_M e)) \end{bmatrix} + 2\mathscr{D}^\top P e + 2\mathscr{D}^\top \sum_{j=1}^{M} H_j^\top \Lambda^j f^j(H_j e),
\end{aligned}$$

where

$$\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}^\top = (\tilde{\mathcal{Q}}_{a,b})_{a,b=1}^3,$$
$$\tilde{\mathcal{Q}}_{1,1} = \mathscr{A}_0^\top P + P\mathscr{A}_0; \quad \tilde{\mathcal{Q}}_{2,2} = \mathbf{O}_{\kappa\times\kappa},$$
$$\tilde{\mathcal{Q}}_{1,2} = P\mathscr{A}; \quad \tilde{\mathcal{Q}}_{1,3} = \mathscr{A}_0^\top \Lambda; \quad \tilde{\mathcal{Q}}_{2,3} = \mathscr{A} \top \Lambda; \quad \tilde{\mathcal{Q}}_{3,3} = \mathbf{O}_{\kappa\times\kappa}$$

with $\kappa=\sum_{j=1}^{M}k_j$. Therefore, using the matrices introduced in Theorem 4.2, we have:

$$\begin{aligned}
\dot V &= \begin{bmatrix} e \\ \delta f^1 \\ \vdots \\ \delta f^M \\ f^1(H_1e) \\ \vdots \\ f^M(H_Me) \\ \mathscr{D} \end{bmatrix}^\top \mathscr{Q} \begin{bmatrix} e \\ \delta f^1 \\ \vdots \\ \delta f^M \\ f^1(H_1e) \\ \vdots \\ f^M(H_Me) \\ \mathscr{D} \end{bmatrix} + \gamma\sum_{j=1}^{M}(\delta f^j)^\top \delta f^j + \eta\sum_{j=1}^{M} f^j(H_je)^\top f^j(H_je) \\
&\quad -e^\top \Xi^0 e - 2\sum_{j=1}^{M} e^\top H_j\Gamma_j(\delta f^j) - 2\sum_{j=1}^{M} e^\top H_j\Omega_j f^j(H_je) \\
&\quad -2\sum_{j=1}^{M}\sum_{k=1}^{M}(\delta f^j)^\top H_j\Upsilon_{j,k}H_k^\top f^k(H_ke) + \mathscr{D}^\top\Phi\mathscr{D}.
\end{aligned}$$

Since $\mathscr{Q}\le 0$ due to (4.9) and applying Assumption 4.2,

$$\begin{aligned}
\dot V &\le -e^\top(\Xi^0 - \gamma\sum_{j=1}^{M}H_j^\top S_0^j H_j - \eta\sum_{j=1}^{M}H_j^\top\Sigma_0^j H_j)e \\
&\quad -2\sum_{j=1}^{M} e^\top H_j^\top(\Gamma_j - \gamma S_1^j - \eta\Sigma_1^j)(\delta f^j) \\
&\quad -2\sum_{j=1}^{M} e^\top H_j^\top(\Omega_j - \gamma S_2^j - \eta\Sigma_2^j) f^j(H_je) \\
&\quad -2\sum_{j=1}^{M}\sum_{k=1}^{M}(\delta f^j)^\top H_j(\Upsilon_{j,k} - \gamma S_3^{j,k} - \eta\Sigma_3^{j,k})H_k^\top f^k(H_ke) \\
&\quad +\mathscr{D}^\top\Phi\mathscr{D}.
\end{aligned}$$

According to Theorem 2.4, to ensure the SIIOS property of (2.11), (4.8) the right-hand side of the above estimate should be a positive definite and radially unbounded function of the error e. The corresponding term, which can guarantee these characteristics, is $e^\top(\Xi^0 - \gamma\sum_{j=1}^{M}H_j^\top S_0^j H_j - \eta\sum_{j=1}^{M}H_j^\top\Sigma_0^j H_j)e$. Therefore, if the conditions of Theorem 4.2 are satisfied, then we can substantiate that the system (2.11), (4.8) is SIIOS with respect to estimation error e as desired. □

Remark 4.2. *Under an assumption that $H_j = I_n$ for $j\in\overline{1,M}$ and there exists $\alpha\in\mathscr{K}_\infty$ such that $\forall j\in\overline{1,M}$:*

$$(\delta f^j)^\top\delta f^j \ge \alpha(\|e\|);\quad f^j(e)^\top f^j(e)\ge\alpha(\|e\|)$$

for all $x, \hat{x} \in \mathbb{R}^n$ *with* $e = x - \hat{x}$ *(i.e., the functions* δf^j *and* f^j *are radially unbounded in terms of the estimation error* e*), the conditions* (4.9) *can be relaxed:*

$$P + \varrho \sum_{j=1}^{\mu} \Lambda^j > 0; \quad \mathcal{Q} \le 0,$$

$$\Xi^0 - \gamma \sum_{j=1}^{M} S_0^j - \eta \sum_{j=1}^{M} \Sigma_0^j \ge 0; \quad \Gamma_j - \gamma S_1^j - \eta \Sigma_1^j \ge 0,$$

$$\Omega_j - \gamma S_2^j - \eta \Sigma_2^j \ge 0; \quad \Upsilon_{j,k} - \gamma S_3^{j,k} - \eta \Sigma_3^{j,k} \ge 0,$$

$$\Xi^0 - \gamma \sum_{j=1}^{M} S_0^j - \eta \sum_{j=1}^{M} \Sigma_0^j + 2 \sum_{j=1}^{M} (\Gamma_j - \gamma S_1^j - \eta \Sigma_1^j)$$

$$+2 \sum_{j=1}^{M} (\Omega_j - \gamma S_2^j - \eta \Sigma_2^j) + 2 \sum_{j=1}^{M} \sum_{k=1}^{M} (\Upsilon_{j,k} - \gamma S_3^{j,k} - \eta \Sigma_3^{j,k}) > 0.$$

Remark 4.3. *If there exist* $\Delta_j = \mathrm{diag}(\Delta_j^1, \ldots, \Delta_j^{k_j}) \in \mathbb{D}_+^{k_j}$*,* $\Delta_j^i \in \{0,1\}$ *for* $i \in \overline{1, k_j}$ *and* $\Pi_j \in \mathbb{R}^{k_j \times z_0}$ *such that* $\Delta_j H_j = \Pi_j C_0$ *for* $j \in \overline{1, M}$ (i.e., *a part of the argument of the nonlinearity is measured by the linear components of the output), then the observer* (4.1) *can be extended:*

$$\begin{aligned} \dot{\hat{x}}(t) &= A_0 \hat{x}(t) + \sum_{j=1}^{M} A_j f^j (H_j \hat{x}(t)) + L(y(t) - \hat{y}(t)) \\ &\quad + \sum_{j=1}^{M} \delta_j H_j^\top \Delta_j f^j (\Pi_j y_0(t) - \Pi_j C_0 \hat{x}(t)), \end{aligned} \tag{4.11}$$

where $y_0(t) \in \mathbb{R}^{z_0}$ *is the first* z_0 *elements of the output* $y(t)$*,* $\delta_j > 0$ *are tuning parameters. The error dynamics* (4.8) *is:*

$$\dot{e} = \mathscr{A}_0 e + \sum_{j=1}^{M} \mathscr{A}_j \delta f^j - \sum_{j=1}^{M} \delta_j H_j^\top \Delta_j f^j (\Delta_j H_j e) + \mathscr{D}. \tag{4.12}$$

Clearly, the terms $-\delta_j H_j^\top \Delta_j f^j (\Delta_j H_j e)$ *are stabilizing and allow Assumption* 4.2 *to be relaxed (there is no need in the upper bound for the nonlinearities in these new items).*

In Theorem 4.2, the observer gains L_s, $s \in \overline{0, M}$ are assumed to be given. To find these gains as solutions of LMIs, the following equivalent representation of the error dynamics (4.8) will be used:

$$\dot{e} = (A_0 - L_0 C_0) e + (A - \bar{L} \bar{C}) \delta f + D w - L v, \tag{4.13}$$

where we define the block matrices

$$A = [A_1 \dots A_M]; \quad \bar{L} = [L_1 \dots L_M],$$
$$\bar{C} = \mathrm{diag}(C_1, \dots, C_M); \quad \delta f = \begin{bmatrix} \delta f^1 \\ \vdots \\ \delta f^M \end{bmatrix}.$$

We need (4.13) for compactness of notation in the next corollary:

Corollary 4.3. *Let assumptions 2.1, 4.1 and 4.2 with $S_2^j = S_3^{j,k} = \Sigma_2^j = \Sigma_3^{j,k} = \mathbf{O}_{n\times n}$ and $H_j = I_n$ for $j,k \in \overline{1,M}$ be satisfied. If there exist $0 < P = P^\top \in \mathbb{R}^{n\times n}$; $\Xi^0 \in \mathbb{D}_+^n$; $\{\Gamma_j\}_{j=1}^M \subset \mathbb{D}_+^n$; $U_s \in \mathbb{R}^{n\times z_s}$ ($s \in \overline{0,M}$); $0 < \Phi_w = \Phi_w^\top \in \mathbb{R}^{p\times p}$; $0 < \Phi_v = \Phi_v^\top \in \mathbb{R}^{z\times z}$ and $\gamma, \eta > 0$ such that*

$$P \le I_n; \quad \tilde{G} = \tilde{G}^\top = (\tilde{G}_{a,b})_{a,b=1}^5 \le 0, \tag{4.14}$$
$$\Xi^0 - \gamma \sum_{j=1}^M S_0^j - \eta \sum_{j=1}^M \Sigma_0^j > 0; \quad \Gamma_j - \gamma S_1^j - \eta \Sigma_1^j \ge 0,$$

where

$$\tilde{G}_{1,1} = A_0^\top P + PA_0 - C_0^\top U_0^\top - U_0 C_0 + \Xi^0,$$
$$\tilde{G}_{1,2} = PA - \bar{U}\bar{C} + \bar{\Gamma}; \quad \tilde{G}_{1,3} = A_0^\top PJ - C_0^\top U_0^\top J,$$
$$\tilde{G}_{1,4} = PD; \quad \tilde{G}_{1,5} = -U; \quad \tilde{G}_{2,2} = -\gamma I_{nM}; \quad \tilde{G}_{2,3} = A^\top PJ - \bar{C}^\top \bar{U}^\top J,$$
$$\tilde{G}_{2,4} = \mathbf{O}_{nM\times p}; \quad \tilde{G}_{2,5} = \mathbf{O}_{nM\times z}; \quad \tilde{G}_{3,3} = -\eta I_{nM},$$
$$\tilde{G}_{3,4} = J^\top PD; \quad \tilde{G}_{3,5} = -J^\top U; \quad \tilde{G}_{4,4} = -\Phi_w; \quad \tilde{G}_{4,5} = \mathbf{O}_{p\times z},$$
$$\tilde{G}_{5,5} = -\Phi_v; \quad \bar{U} = \begin{bmatrix} U_1 & \dots & U_M \end{bmatrix}; \quad U = [\ U_0 \quad \bar{U}\ ],$$
$$\bar{\Gamma} = \begin{bmatrix} \Gamma_1 & \dots & \Gamma_M \end{bmatrix}; \quad J = 1_M^\top \otimes I_n,$$

then the system (2.11), (4.1) is SIIOS with respect to the estimation error e with the observer gains $L_s = P^{-1}U_s$ for all $s \in \overline{0,M}$.

Proof. Consider a candidate Lyapunov function V from (4.10) with $H_j = \Lambda^j = I_n$ for $j \in \overline{1,M}$, which is positive definite and radially unbounded since $P > 0$, and whose time derivative for (4.13) admits the representation:

$$\begin{aligned}\dot{V} &= \dot{e}^\top P e + e^\top P \dot{e} + 2\sum_{j=1}^{M}\sum_{i=1}^{n} f_i^j(e_i)\dot{e}_i \\ &= \begin{bmatrix} e \\ \delta f^1 \\ \vdots \\ \delta f^M \\ f^1(e) \\ \vdots \\ f^M(e) \\ w \\ v \end{bmatrix}^\top \hat{\mathcal{Q}} \begin{bmatrix} e \\ \delta f^1 \\ \vdots \\ \delta f^M \\ f^1(e) \\ \vdots \\ f^M(e) \\ w \\ v \end{bmatrix} + \gamma \sum_{j=1}^{M} (\delta f^j)^\top \delta f^j + \eta \sum_{j=1}^{M} f^j(e)^\top f^j(e) \\ &\quad - e^\top \Xi^0 e - 2\sum_{j=1}^{M} e^\top \Gamma_j (\delta f^j) + w^\top \Phi_w w + v^\top \Phi_v v,\end{aligned}$$

where $\hat{\mathcal{Q}} \le \tilde{E}\tilde{G}\tilde{E}$, $\tilde{E} = \mathrm{diag}(I_n, I_{nM}, P^{-1}, \ldots, P^{-1}, I_p, I_z)$, by setting $U_s := PL_s$ for all $s \in \overline{0, M}$ and substituting $P^2 \le I_n$. Therefore, it holds that $\hat{\mathcal{Q}} \le 0$ if and only if $\tilde{G} \le 0$. If $\tilde{G} \le 0$ and applying Assumption 4.2 with $S_2^j = S_3^{j,k} = \Sigma_2^j = \Sigma_3^{j,k} = \mathbf{O}_{n\times n}$ for $j,k \in \overline{1, M}$, we have:

$$\begin{aligned}\dot{V} &\le -e^\top(\Xi^0 - \gamma\sum_{j=1}^{M} S_0^j - \eta\sum_{j=1}^{M}\Sigma_0^j)e \\ &\quad -2\sum_{j=1}^{M} e^\top(\Gamma_j - \gamma S_1^j - \eta\Sigma_1^j)(\delta f^j) + w^\top\Phi_w w + v^\top \Phi_v v.\end{aligned}$$

Following the proof of Theorem 4.2, the corresponding term guaranteeing SIIOS property is $e^\top(\Xi^0 - \gamma\sum_{j=1}^{M} S_0^j - \eta\sum_{j=1}^{M}\Sigma_0^j)e$. Therefore, if the conditions of (4.14) are satisfied, then the estimation error dynamics (2.11), (4.13) is SIIOS and the observer gains L_s, $s \in \overline{0, M}$ can be obtained as desired. □

Again the conditions of Corollary 4.3 are more restrictive than in Theorem 4.2 since it is assumed that $0 < P \le I_n$ and $\Lambda^j = I_n$ for $j \in \overline{1, M}$.

Example 2. *Consider a multi-group susceptible-infected-susceptible (SIS) model [89, 96]:*

$$\dot{x}(t) = \mathrm{diag}(\mathbb{1}_n - x(t))\Big(\beta A x(t) + w(t)\Big) - \gamma x(t), \qquad (4.15)$$

where $x(t) \in [0,1]^n$ represents infected populations in n groups, $\mathbb{1}_n \in \mathbb{R}^n$ is the vector of ones; $\beta > 0$ and $\gamma > 0$ are the infection and the recovery rates, respectively; $A \in [0,1]^{n\times n}$ is the adja-

cency matrix of infection transmission between groups; $w(t) \in [0,1]^n$ *corresponds to unmodelled cumulative infection receipt at each group. Following [96], assume that the infected population is measured in* $0 < p < n$ *groups:*

$$y(t) = Cx(t); \quad C = [I_p \ \mathbf{O}_{p\times(n-p)}]. \tag{4.16}$$

To represent this system in the form (2.11), consider a change of variables $z = \overline{\ln}(\mathbb{1}_n - x)$, $x = \mathbb{1}_n - \overline{e}^z$ *(here* $\overline{\ln} : \mathbb{R}^n \to \mathbb{R}^n$ *and* $\overline{\ln}([g_1 \ \dots \ g_n]^\top) = \begin{bmatrix}\ln(g_1) & \dots & \ln(g_n)\end{bmatrix}^\top$ *for* $g_1, \dots, g_n > 0$; $\overline{e}^{[g_1 \dots g_n]^\top} = \begin{bmatrix} e^{g_1} & \dots & e^{g_n}\end{bmatrix}^\top$ *for* $g_1, \dots, g_n \in \mathbb{R}$*), where application of an elementary function to a vector argument is understood elementwise, then*

$$\dot{z}(t) = \beta A f_1(z(t)) - \gamma f_2(z(t)) - w(t); \quad y(t) = -C f_1(z(t)),$$
$$f_1(z) = \overline{e}^z - \mathbb{1}_n; \quad f_2(z) = \mathbb{1}_n - \overline{e}^{\,-z}$$

and it is easy to check that assumptions 2.1 and 4.1 are satisfied. Assumption 4.2 holds locally with $S_3^{j,j} = I_n$ *and* $\Sigma_3^{j,j} = I_n$ *and all other matrices equal zero for* $j \in \{1,2\}$*. Then the state observation can be performed for a sufficiently small initial estimation error. Note that due to the form of the output, we obtain that* $\tilde{y} = Cz$ *is an auxiliary measured signal. The observer is taken in the form (4.11):*

$$\begin{aligned}\dot{\hat{z}}(t) &= \beta A f_1(\hat{z}(t)) - \gamma f_2(\hat{z}(t)) + L\Big(y(t) + C f_1(\hat{z}(t))\Big) \\ &\quad + \sum_{j=1}^{2} m_j C^\top C f_j\Big(x(t) - \hat{z}(t)\Big),\end{aligned}$$

where $\hat{z}(t) \in \mathbb{R}^n$ *is the estimate of* $z(t)$, $L \in \mathbb{R}^{n\times p}$ *is the observer gain to be selected,* $C f_1(x - \hat{z}) = \overline{e}^{\,\tilde{y} - C\hat{z}} - \mathbb{1}_p$ *and* $C f_2(x - \hat{z}) = \mathbb{1}_p - \overline{e}^{\,C\hat{z} - \tilde{y}}$ *are dependent on the measured information only,* $m_j \geq 0$ *are tuning parameters for* $j \in \{1,2\}$*. In the original coordinates the observer can be rewritten as follows:*

$$\begin{aligned}\dot{\hat{x}}(t) &= \operatorname{diag}\left(\mathbb{1}_n - \hat{x}(t)\right)\Big(L - m_2 C^\top \operatorname{diag}(\mathbb{1}_p - y(t))^{-1}\big) \times \\ &\quad (C\hat{x}(t) - y(t)) + \beta A \hat{x}(t)\Big) - \gamma \hat{x}(t) - m_1 C^\top \left(C\hat{x}(t) - y(t)\right),\end{aligned} \tag{4.17}$$

and it is straightforward to check that $\hat{x}(t) \in [0,1]^n$ *for all* $t > 0$ *provided that* $\hat{x}(0) \in [0,1]^n$ *and* $(m_1 + m_2)C^\top - L$ *is elementwise nonnegative.*

For $n = 15$*, we selected the pair of matrices* (A, C) *in (4.15),(4.16) to be observable (*A *is not symmetric), then the LMIs of Theorem 4.2 induced by the error dynamics (4.12) are verified.*

A state trajectory of (4.15) *is shown in Fig. 4.3, corresponding to the red curve in Fig. 4.4, representing the estimation error decay in logarithmic scale for different initial conditions. The error e converges to a vicinity of the origin proportional to the amplitude of disturbances (for simulation, w was chosen as a harmonic perturbation, and* $v \equiv 0$*. This application confirms the efficacy and the generality of the developed estimation framework.*

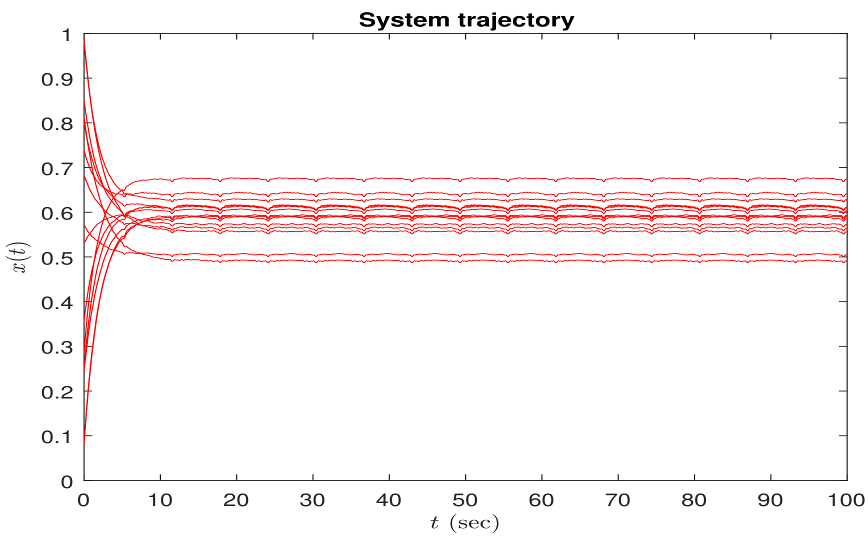

Figure 4.3: The system trajectory of (4.15) versus the time t

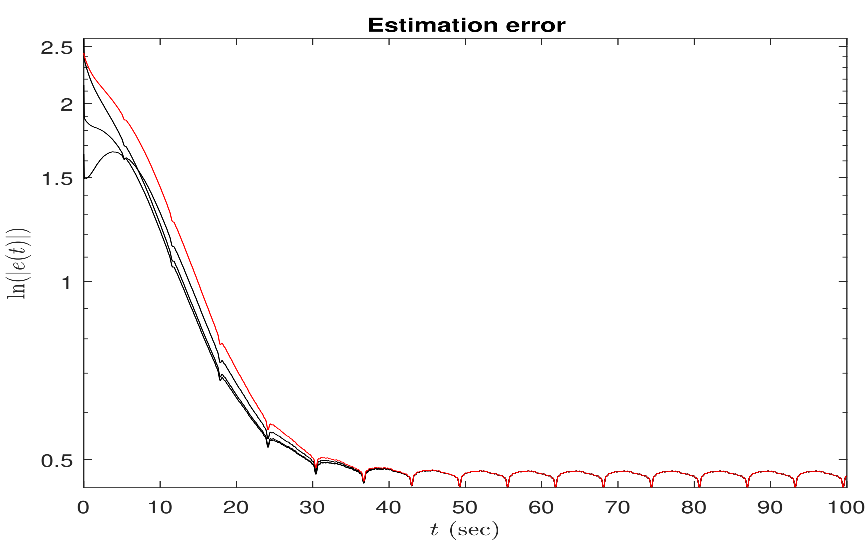

Figure 4.4: The logarithm of the norm of estimation error between (4.15) and (4.17)

CHAPTER 5

Delay-dependent input-to-state stability conditions

The main goal of this chapter is to consider the input-to-state stability of a class of nonlinear systems in generalized Persidskii form with constant time delays

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + \sum_{j=1}^{M} A_j f^j(x(t)) \\ &\quad + B_0 x(t-\tau_0) + \sum_{j=1}^{M} B_j f^j(x(t-\tau_j)) + d(t),\ t \in \mathbb{R}_+\end{aligned} \tag{5.1}$$

with $x(t) = [x_1(t) \dots x_n(t)]^\top \in \mathbb{R}^n$ is the current value of the state; $0 < \tau_s < +\infty$ are constant delays for $s \in \overline{0, M}$, $\tau = \max_{s \in \overline{0,M}} \tau_s$; $A_s, B_s \in \mathbb{R}^{n \times n}$ for $s \in \overline{0, M}$; the functions $f^j : \mathbb{R}^n \to \mathbb{R}^n$ have diagonal structure, $f^j(x) = [f_1^j(x_1) \dots f_n^j(x_n)]^\top$ for $j \in \overline{1, M}$, and ensure the existence of the solutions of the system (5.1) in forward time, at least locally; $d(t) \in \mathbb{R}^n$ is the external perturbation/input, $d \in \mathscr{L}_\infty^n$.

Assumption 5.1. *For any $i \in \overline{1, n}$ and $j,\ j' \in \overline{1, M}$, $z \in \overline{j+1, M}$, there exist $\eta_{0,j}^i$, $\eta_{1,jj'}^i$, $\eta_{2,jj'}^i$, $\eta_{3,jj'z}^i \geq 0$ such that*

$$\begin{aligned}2\int_0^{\mathrm{x}_i} f_i^j(s)ds &\leq \eta_{0,j}^i \mathrm{x}_i^2 + \sum_{j'=1}^{M} f_i^{j'}(\mathrm{x}_i)\Bigg(\eta_{1,jj'}^i f_i^{j'}(\mathrm{x}_i) \\ &\quad + 2\eta_{2,jj'}^i \mathrm{x}_i + 2\sum_{z=j+1}^{M} \eta_{3,jj'z}^i f_i^z(\mathrm{x}_i)\Bigg)\end{aligned}$$

for all $\mathrm{x} \in \mathbb{R}^n$.

Assumptions 2.1 and 5.1 are satisfied by many nonlinear functions: for polynomial ones, for example, it is sufficient to select $\eta_{2,jj'}^i \neq 0$. In the sequel, we denote the diagonal matrices:

$$\eta_{0,j} = \mathrm{diag}(\eta_{0,j}^1, \dots, \eta_{0,j}^n),\ \eta_{1,jj'} = \mathrm{diag}(\eta_{1,jj'}^1, \dots, \eta_{1,jj'}^n),$$
$$\eta_{2,jj'} = \mathrm{diag}(\eta_{2,jj'}^1, \dots, \eta_{2,jj'}^n),\ \eta_{3,jj'z} = \mathrm{diag}(\eta_{3,jj'z}^1, \dots, \eta_{3,jj'z}^n).$$

5.1 ISS analysis

In this section, we propose constructive conditions for verifying the ISS property of (5.1). This system is highly nonlinear, with multiple delays appearing in linear and nonlinear parts. The following theorem is the main result of this chapter, which formulates delay-dependent conditions based on a special ISS-LKF extending the previous results of [82].

Theorem 5.1. *Let assumptions 2.1 and 5.1 be satisfied and for given constants* $0 < w_0$, $0 < p_s < \delta_s$ $(s \in \overline{0,M})$ *and* $\rho \in \mathbb{R}$ *there exist* $0 \leq P = P^\top$, $\Phi = \Phi^\top \in \mathbb{R}^{n\times n}$; *sets* $\{R_s\}_{s=0}^M$, $\{S_s\}_{s=0}^M$, $\{\Xi_s\}_{s=0}^M \subset \mathbb{R}^{n\times n}$ *of symmetric nonnegative definite matrices;* $\{\Omega_j\}_{j=1}^M \subset \mathbb{R}^{n\times n}$; P_2, P_3, $P_4 \in \mathbb{R}^{n\times n}$; $\left\{\Lambda_j = \mathrm{diag}(\Lambda_j^1, \ldots, \Lambda_j^n)\right\}_{j=1}^M$, $\left\{\Upsilon_{s,r}\right\}_{0\leq s<r\leq M} \subset \mathbb{D}_+^n$ *such that*

$$P + \rho \sum_{j=1}^{\mu} \Lambda_j > 0, \tag{5.2}$$

$$Q = Q^\top = (Q_{a,b})_{a,b=1}^6 \leq 0, \tag{5.3}$$

$$\mathbb{1}_{\{0\}}(s) \cdot \Xi^0 + \mathbb{1}_{\overline{1,M}}(s) \cdot \Xi^s \geq \xi \left[\mathbb{1}_{\{0\}}(s) \left(P + \sum_{j=1}^M \Lambda_j \eta_{0,j} \right) + \mathbb{1}_{\overline{1,M}}(s) \cdot \Lambda_s \sum_{j'=1}^M \eta_{1,sj'} \right], \tag{5.4}$$

$$\mathbb{1}_{\{0\}}(s) \cdot \Upsilon_{0,j} + \mathbb{1}_{\overline{1,M}}(s) \cdot \Upsilon_{s,z} \geq \xi \left[\mathbb{1}_{\{0\}}(s) \cdot \Lambda_j \sum_{j'=1}^M \eta_{2,jj'} + \mathbb{1}_{\overline{1,M}}(s) \cdot \Lambda_s \sum_{j'=1}^M \eta_{3,sj'z} \right],$$

$$s \in \overline{0,M},\ j \in \overline{1,M},\ z \in \overline{s+1,M}$$

for some $\xi \in (0, w_0] \bigcap \left(\bigcap_{s\in\overline{0,M}} \left(0, \frac{\delta_s - p_s}{\delta_s \tau_s}\right]\right)$*, where*

$$Q_{1,1} = A_0^\top P_2 + P_2^\top A_0 + S_0 + \Xi^0 - p_0 R_0; \quad Q_{1,2} = P - P_2^\top + A_0^\top P_3,$$
$$Q_{1,3} = P_2^\top B_0 + p_0 R_0 + A_0^\top P_4; \quad Q_{1,4} = P_2^\top A + A_0^\top \Omega + \begin{bmatrix} \Upsilon_{0,1} & \dots & \Upsilon_{0,M} \end{bmatrix},$$
$$Q_{1,5} = P_2^\top B; \quad Q_{1,6} = P_2^\top,$$
$$Q_{2,2} = -P_3 - P_3^\top + \delta_0 \tau_0^2 R_0 + \sum_{j=1}^{M} \delta_j \tau_j^2 \frac{\partial f^j(x)^\top}{\partial x} R_j \frac{\partial f^j(x)}{\partial x},$$
$$Q_{2,3} = P_3^\top B_0 - P_4; \quad Q_{2,4} = P_3^\top A - \Omega + \Lambda; \quad Q_{2,5} = P_3^\top B; \quad Q_{2,6} = P_3^\top,$$
$$Q_{3,3} = -e^{-w_0\tau_0} S_0 - p_0 R_0 + 2P_4^\top B_0; \quad Q_{3,4} = B_0^\top \Omega + P_4^\top A,$$
$$Q_{3,5} = P_4^\top B; \quad Q_{3,6} = P_4^\top; \quad Q_{4,4} = Q_{4,4}^\top = (\widehat{Q}_{a,b})_{a,b=1}^M,$$
$$\widehat{Q}_{j,j} = A_j^\top \Omega_j + \Omega_j^\top A_j + \Xi^j + S_j - p_j R_j,\ j \in \overline{1,M},$$
$$\widehat{Q}_{s,z} = A_s^\top \Omega_z + \Omega_s^\top A_z + \Upsilon_{s,z},\ s \in \overline{1, M-1}, z \in \overline{s+1, M},$$
$$Q_{4,5} = \Omega^\top B + J; \quad Q_{4,6} = \Omega^\top,$$
$$Q_{5,5} = \mathrm{diag}(-e^{-w_0\tau_1} S_1, \dots, -e^{-w_0\tau_M} S_M) - J,$$
$$Q_{5,6} = \mathbf{O}_{nM\times n}; \quad Q_{6,6} = -\Phi,$$
$$A = \begin{bmatrix} A_1 & \dots & A_M \end{bmatrix}; \quad B = \begin{bmatrix} B_1 & \dots & B_M \end{bmatrix},$$
$$\Lambda = \begin{bmatrix} \Lambda_1 & \dots & \Lambda_M \end{bmatrix}; \quad \Omega = \begin{bmatrix} \Omega_1 & \dots & \Omega_M \end{bmatrix},$$
$$J = \mathrm{diag}(p_1 R_1, \dots, p_M R_M).$$

Then the system (5.1) *is ISS.*

Proof. We aim to check the conditions in Definition 2.8 for a LKF taken as follows:

$$\begin{aligned} V(x_t, \dot{x}_t) \quad = \quad & x(t)^\top P x(t) + \int_{t-\tau_0}^{t} e^{-w_0(t-s)} x(s)^\top S_0 x(s) ds \\ & + 2\sum_{j=1}^{M}\sum_{i=1}^{M} \Lambda_j^i \int_0^{x_i(t)} f_i^j(s) ds \\ & + \sum_{j=1}^{M} \int_{t-\tau_j}^{t} e^{-w_0(t-s)} f^j(x(s))^\top S_j f^j(x(s)) ds \\ & + \delta_0 \tau_0 \int_{-\tau_0}^{0} \int_{t+\theta}^{t} \dot{x}(s)^\top R_0 \dot{x}(s) ds d\theta \\ & + \sum_{j=1}^{M} \delta_j \tau_j \int_{t-\tau_j}^{t} \int_s^t \left(\dot{x}(r)^\top \frac{\partial f^j(x(r))^\top}{\partial x} R_j \frac{\partial f^j(x(r))}{\partial x} \dot{x}(r) \right) dr ds, \end{aligned} \tag{5.5}$$

which verifies the required lower (due to (5.2) and Finsler's lemma [29]) and upper (since all matrices are nonnegative definite) bounds given in Definition 2.8. The time derivative of V for (5.1) admits the following expression by using the descriptor method from [34, 33]:

$$
\begin{aligned}
&\dot{V}(t,x_t,\dot{x}_t)\\
&=\dot{x}(t)^\top Px(t)+x(t)^\top P\dot{x}(t)\\
&-w_0\int_{t-\tau_0}^{t} e^{-w_0(t-s)}x(s)^\top S_0x(s)ds+x(t)^\top S_0x(t)\\
&-e^{-w_0\tau_0}x(t-\tau_0)^\top S_0x(t-\tau_0)+2\dot{x}(t)^\top\sum_{j=1}^{M}\Lambda_j f^j(x(t))\\
&-w_0\sum_{j=1}^{M}\int_{t-\tau_j}^{t} e^{-w_0(t-s)}f^j(x(s))^\top S_j f^j(x(s))ds\\
&+\sum_{j=1}^{M} f^j(x(t))^\top S_j f^j(x(t))\\
&-\sum_{j=1}^{M} e^{-w_0\tau_j}f^j(x(t-\tau_j))^\top S_j f^j(x(t-\tau_j))\\
&+\delta_0\tau_0^2\dot{x}(t)^\top R_0\dot{x}(t)-\delta_0\tau_0\int_{t-\tau_0}^{t}\dot{x}^\top(s)R_0\dot{x}(s)ds\\
&+\sum_{j=1}^{M}\delta_j\tau_j^2\dot{x}(t)^\top\frac{\partial f^j(x(t))^\top}{\partial x}R_j\frac{\partial f^j(x(t))}{\partial x}\dot{x}(t)\\
&-\sum_{j=1}^{M}\delta_j\tau_j\int_{t-\tau_j}^{t}\dot{x}(s)^\top\frac{\partial f^j(x(s))^\top}{\partial x}R_j\frac{\partial f^j(x(s))}{\partial x}\dot{x}(s)ds\\
&+2\Big[x(t)^\top P_2^\top+\dot{x}(t)^\top P_3^\top+\sum_{j=1}^{M} f^j(x(t))^\top\Omega_j^\top\\
&+x(t-\tau_0)^\top P_4^\top\Big]\cdot\Big[A_0x(t)+\sum_{j=1}^{M}A_j f^j(x(t))\\
&+B_0x(t-\tau_0)+\sum_{j=1}^{M}B_j f^j(x(t-\tau_j))+d(t)-\dot{x}(t)\Big]
\end{aligned}
$$

$$
= \begin{bmatrix} x(t) \\ \dot{x}(t) \\ x(t-\tau_0) \\ F_1(x(t)) \\ \vdots \\ F_M(x(t)) \\ F_1(x(t-\tau_1)) \\ \vdots \\ F_M(x(t-\tau_M)) \\ d(t) \end{bmatrix}^\top Q \begin{bmatrix} x(t) \\ \dot{x}(t) \\ x(t-\tau_0) \\ F_1(x(t)) \\ \vdots \\ F_M(x(t)) \\ F_1(x(t-\tau_1)) \\ \vdots \\ F_M(x(t-\tau_M)) \\ d(t) \end{bmatrix}
$$

$$
-x(t)^\top \Xi^0 x(t) - \sum_{j=1}^{M} f^j(x(t))^\top \Xi^j f^j(x(t))
$$

$$
-2\sum_{j=1}^{M} x(t)^\top \Upsilon_{0,j} f^j(x(t))
$$

$$
-2\sum_{s=1}^{M-1} \sum_{z=s+1}^{M} f^s(x(t))^\top \Upsilon_{s,z} f^z(x(t))
$$

$$
-w_0 \int_{t-\tau_0}^{t} e^{-w_0(t-s)} x(s)^\top S_0 x(s) ds
$$

$$
-w_0 \sum_{j} \int_{t-\tau_j}^{t} e^{-w_0(t-s)} f^j(x(s))^\top S_j f^j(x(s)) ds
$$

$$
-(\delta_0 - p_0)\tau_0 \int_{t-\tau_0}^{t} \dot{x}(s)^\top R_0 \dot{x}(s) ds
$$

$$
-\sum_{j=1}^{M} (\delta_j - p_j)\tau_j \int_{t-\tau_j}^{t} \dot{x}(s)^\top \frac{\partial f^j(x(s))^\top}{\partial x} R_j \frac{\partial f^j(x(s))}{\partial x} \dot{x}(s) ds
$$

$$
+ d(t)^\top \Phi d(t)
$$

$$
\begin{aligned}
&\leq -x(t)^\top \Xi^0 x(t) - \sum_{j=1}^{M} f^j(x(t))^\top \Xi^j f^j(x(t)) \\
&\quad -2\sum_{j=1}^{M} x(t)^\top \Upsilon_{0,j} f^j(x(t)) - 2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} f^s(x(t))^\top \Upsilon_{s,z} f^z(x(t)) \\
&\quad - w_0 \int_{t-\tau_0}^{t} e^{-w_0(t-s)} x(s)^\top S_0 x(s) ds \\
&\quad - w_0 \sum_j \int_{t-\tau_j}^{t} e^{-w_0(t-s)} f^j(x(s))^\top S_j f^j(x(s)) ds \\
&\quad - (\delta_0 - p_0)\tau_0 \int_{t-\tau_0}^{t} \dot{x}^\top(s) R_0 \dot{x}(s) ds \\
&\quad - \sum_{j=1}^{M} (\delta_j - p_j)\tau_j \int_{t-\tau_j}^{t} \dot{x}(s)^\top \frac{\partial f^j(x(s))^\top}{\partial x} R_j \frac{\partial f^j(x(s))}{\partial x} \dot{x}(s) ds \\
&\quad + d(t)^\top \Phi d(t).
\end{aligned}
$$

Here the condition (5.3) and the Jensen's inequalities

$$
\begin{aligned}
&-p_0\tau_0 \int_{t-\tau_0}^{t} \dot{x}(s)^\top R_0 \dot{x}(s) ds \\
&\leq -[x(t) - x(t-\tau_0)]^\top \cdot p_0 R_0 \cdot [x(t) - x(t-\tau_0)],
\end{aligned}
$$

$$
\begin{aligned}
&-p_j\tau_j \int_{t-\tau_j}^{t} \dot{x}(s)^\top \frac{\partial f^j(x(s))^\top}{\partial x} R_j \frac{\partial f^j(x(s))}{\partial x} \dot{x}(s) ds \\
&\leq -\left[f^j(x(t)) - f^j(x(t-\tau_j))\right]^\top \cdot p_j R_j \cdot \left[f^j(x(t)) - f^j(x(t-\tau_j))\right]
\end{aligned}
$$

were utilized.

For $\xi \in (0, w_0] \bigcap \left(\bigcap_{s\in\overline{0,M}} \left(0, \frac{\delta_s - p_s}{\delta_s \tau_s}\right]\right)$, we have

$$
\begin{aligned}
\xi V(x_t, \dot{x}_t) =& \xi \left(x(t)^\top P x(t) + 2 \sum_{j=1}^{M} \sum_{i=1}^{n} \Lambda_j^i \int_0^{x_i(t)} f_i^j(s) ds \right) \\
&+ \xi \int_{t-\tau_0}^{t} e^{-w_0(t-s)} x(s)^\top S_0 x(s) ds \\
&+ \xi \sum_{j=1}^{M} \int_{t-\tau_j}^{t} e^{-w_0(t-s)} f^j(x(s))^\top S_j f^j(x(s)) ds \\
&+ \xi \delta_0 \tau_0 \int_{-\tau_0}^{0} \int_{t+\theta}^{t} \dot{x}^\top(s) R_0 \dot{x}(s) ds d\theta \\
&+ \xi \sum_{j=1}^{M} \delta_j \tau_j \int_{t-\tau_j}^{t} \int_{s}^{t} \left(\dot{x}(r)^\top \frac{\partial f^j(x(r))^\top}{\partial x} R_j \frac{\partial f^j(x(r))}{\partial x} \dot{x}(r) \right) dr ds \\
\leq & x(t)^\top \Xi^0 x(t) + \sum_{j=1}^{M} f^j(x(t))^\top \Xi^j f^j(x(t)) \\
&+ 2 \sum_{j=1}^{M} x(t)^\top \Upsilon_{0,j} f^j(x(t)) + 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^{M} f^s(x(t))^\top \Upsilon_{s,z} f^z(x(t)) \\
&+ w_0 \int_{t-\tau_0}^{t} e^{-w_0(t-s)} x(s)^\top S_0 x(s) ds \\
&+ w_0 \sum_{j=1}^{M} \int_{t-\tau_j}^{t} e^{-w_0(t-s)} f^j(x(s))^\top S_j f^j(x(s)) ds \\
&+ (\delta_0 - p_0) \tau_0 \int_{t-\tau_0}^{t} \dot{x}^\top(s) R_0 \dot{x}(s) ds \\
&+ \sum_{j=1}^{M} (\delta_j - p_j) \tau_j \int_{t-\tau_j}^{t} \dot{x}(s)^\top \frac{\partial f^j(x(s))^\top}{\partial x} R_j \frac{\partial f^j(x(s))}{\partial x} \dot{x}(s) ds
\end{aligned}
$$

due to the conditions (5.4), Assumption 5.1 and the relations

$$
\begin{aligned}
& \xi \delta_0 \tau_0 \int_{-\tau_0}^{0} \int_{t+\theta}^{t} \dot{x}^\top(s) R_0 \dot{x}(s) ds d\theta \\
& \leq (\delta_0 - p_0) \tau_0 \int_{t-\tau_0}^{t} \dot{x}^\top(s) R_0 \dot{x}(s) ds, \\
& \xi \delta_j \tau_j \int_{t-\tau_j}^{t} \int_{s}^{t} \left(\dot{x}(r)^\top \frac{\partial f^j(x(r))^\top}{\partial x} R_j \frac{\partial f^j(x(r))}{\partial x} \dot{x}(r) \right) dr ds \\
& \leq (\delta_j - p_j) \tau_j \int_{t-\tau_j}^{t} \dot{x}(s)^\top \frac{\partial f^j(x(s))^\top}{\partial x} R_j \frac{\partial f^j(x(s))}{\partial x} \dot{x}(s) ds.
\end{aligned}
$$

Under the restriction $\alpha(V) \geq d^\top \Phi d$ with $\alpha(s) = \frac{1}{2}\xi s$, it follows that

$$\dot{V}(t, x_t, \dot{x}_t) \leq -\frac{1}{2}\xi V(x_t, \dot{x}_t).$$

By Definition 2.8 and Theorem 2.5, we can substantiate that system (5.1) is ISS as desired. □

Note that the ISS LKF (5.5) used for the proof of Theorem 5.1 depends explicitly on the delays τ_s for $s \in \overline{0, M}$ due to the presence of the last two terms. The delays also appear and play an important role in the matrix inequality (5.3) of Theorem 5.1, which is nonlinear (or state-dependent) due to the term $\frac{\partial f^j(x)^\top}{\partial x} R_j \frac{\partial f^j(x)}{\partial x} \in \mathbb{D}_+^n$ ($j \in \overline{1, M}$) of $Q_{2,2}$. For practical verification of the matrix inequality (5.3), the following nonrestrictive conditions can be imposed on these terms:

Assumption 5.2. *There exist the sets $\mathcal{X}_j \subseteq \mathbb{R}^n$, $j \in \overline{1, M}$ with $\bigcap_{j \in \overline{1,M}} \mathcal{X}_j = \mathcal{X} \neq \emptyset$ and the matrices $\{\tilde{R}_j\}_{j=1}^M \subset \mathbb{D}_+^n$ such that*

$$\mathcal{X}_j \subseteq \left\{ \nu \in \mathbb{R}^n \middle| \frac{\partial f^j(\nu)^\top}{\partial \nu} R_j \frac{\partial f^j(\nu)}{\partial \nu} \leq \tilde{R}_j \right\}.$$

In the case $\mathcal{X} = \mathbb{R}^n$, the term $\frac{\partial f^j(x)^\top}{\partial x} R_j \frac{\partial f^j(x)}{\partial x}$ is bounded by some $\tilde{R}_j \in \mathbb{D}_+^n$ for all $x \in \mathbb{R}^n$ and $j \in \overline{1, M}$, which is the case of bounded nonlinearities, *e.g.*, $f^j(x) = \tanh(x)$.

Denote by $Q^\dagger$ the block matrix Q from Theorem 5.1 under the substitutions $\frac{\partial f^j(x)^\top}{\partial x} R_j \frac{\partial f^j(x)}{\partial x} \to \tilde{R}_j$ for $j \in \overline{1, M}$.

Corollary 5.1. *If Assumption 5.2 with $\mathcal{X} = \mathbb{R}^n$ and the conditions of Theorem 5.1 under the substitution $Q \leq 0 \to Q^\dagger \leq 0$ are satisfied, then the system (5.1) is ISS.*

Proof. Note that under Assumption 5.2

$$Q^\dagger \leq 0 \quad \Rightarrow \quad Q \leq 0,$$

then the conditions of Corollary 5.1 imply that all counterparts in Theorem 5.1 are verified, and the conclusion follows. □

If $\mathcal{X} \subset \mathbb{R}^n$, then similarly local ISS property can be established for the initial conditions inside $\mathcal{X}$ and properly bounded inputs d.

5.2 Stabilization

In this section, we design a feedback control to stabilize a system as (5.1) and study the ISS property of the resulting closed-loop system.

Consider a variation of (5.1):

$$\begin{aligned}\dot{x}(t) &= A_0x(t)+\sum_{j=1}^{M}A_jf^j(x(t))+B_0x(t-\tau_0)\\ &\quad+\sum_{j=1}^{M}B_jf^j(x(t-\tau_j))+Gu(t)+d(t),\end{aligned}\tag{5.6}$$

where all variables are defined as in (5.1), $G\in\mathbb{R}^{n\times q}$ and $u(t)\in\mathbb{R}^q$ is the control, which can be chosen for stabilization in the following general form:

$$\begin{aligned}u(t) &= K_{A,0}x(t)+\sum_{j=1}^{M}K_{A,j}f^j(x(t))\\ &\quad+K_{B,0}x(t-\tau_0)+\sum_{j=1}^{M}K_{B,j}f^j(x(t-\tau_j)),\end{aligned}\tag{5.7}$$

where $K_{A,s},\ K_{B,s}\in\mathbb{R}^{q\times n},\ s\in\overline{0,M}$. Such a form of the control keeps the closed-loop system in the class of generalized Persidskii models:

$$\begin{aligned}\dot{x}(t) &= \widetilde{A}_0x(t)+\sum_{j=1}^{M}\widetilde{A}_jf^j(x(t))+\widetilde{B}_0x(t-\tau_0)\\ &\quad+\sum_{j=1}^{M}\widetilde{B}_jf^j(x(t-\tau_j))+d(t),\end{aligned}\tag{5.8}$$

where $\widetilde{A}_s=A_s+GK_{A,s}$, $\widetilde{B}_s=B_s+GK_{B,s}$ for $s\in\overline{0,M}$.

Remark 5.1. *In the case that $K_{A,s},\ K_{B,s},\ s\in\overline{0,M}$ are given, we can directly formulate the results to analyze the input-to-state stability of the closed-loop system (5.8):*

If all conditions of Theorem 5.1 are satisfied under the substitutions $A_s\to\widetilde{A}_s,\ B_s\to\widetilde{B}_s$ for $s\in\overline{0,M}$, then the system (5.8) is ISS;

If Assumption 5.2 with $\mathcal{X}=\mathbb{R}^n$ and all conditions of Theorem 5.1 are satisfied under the substitutions $A_s\to\widetilde{A}_s,\ B_s\to\widetilde{B}_s$ for $s\in\overline{0,M}$, and $Q\le 0\to Q^\dagger\le 0$ ($Q^\dagger$ as in Corollary 5.1), then the system (5.8) is ISS.

By introducing additional mild hypotheses, we now state a theorem for designing the feedback gains $K_{A,s},\ K_{B,s},\ s\in\overline{0,M}$ that guarantee the ISS property of the system (5.8):

Theorem 5.2. *Let assumptions 2.1 and 5.1 be satisfied and let $0<w_0,\ 0<p_s<\delta_s$ ($s\in\overline{0,M}$) be given constants. If there exist matrices $0<\widetilde{P},\overline{P}\in\mathbb{D}^n_+$; $\left\{\overline{R}_s\right\}_{s=0}^{M}$, $\left\{\overline{\Lambda}_j=\mathrm{diag}(\overline{\Lambda}^1_j,\dots,\overline{\Lambda}^n_j)\right\}_{j=1}^{M}$,*

$\left\{\overline{\Upsilon}_{s,r}\right\}_{0\le s<r\le M}$, $\left\{\overline{S}_k\right\}_{k=0}^{M}$, $\left\{\overline{\Xi}^k\right\}_{k=0}^{M} \subset \mathbb{D}_+^n$; $0 < \Phi = \Phi^\top \in \mathbb{R}^{n\times n}$ *and* $\{U_k\}_{k=0}^M$, $\{L_k\}_{k=0}^M \subset \mathbb{R}^{q\times n}$ *such that*

$$\overline{Q} = \overline{Q}^\top = \left(\overline{Q}_{a,b}\right)_{a,b=1}^{6} \le 0,$$

$$\mathbb{1}_{\{0\}}(s)\cdot\overline{\Xi}^0 + \mathbb{1}_{\overline{1,M}}(s)\cdot\overline{\Xi}^s \ge \xi\left[\mathbb{1}_{\{0\}}(s)\cdot\left(\widetilde{P} + \sum_{j=1}^{M}\overline{\Lambda}_j\eta_{0,j}\right) + \mathbb{1}_{\overline{1,M}}(s)\cdot\overline{\Lambda}_s\sum_{j'=1}^{M}\eta_{1,sj'}\right], \tag{5.9}$$

$$\mathbb{1}_{\{0\}}(s)\cdot\overline{\Upsilon}_{0,j} + \mathbb{1}_{\overline{1,M}}(s)\cdot\overline{\Upsilon}_{s,z} \ge \xi\left[\mathbb{1}_{\{0\}}(s)\cdot\overline{\Lambda}_j\sum_{j'=1}^{M}\eta_{2,jj'} + \mathbb{1}_{\overline{1,M}}(s)\cdot\overline{\Lambda}_z\sum_{j'=1}^{M}\eta_{3,sj'z}\right],$$

$$s\in\overline{0,M},\ j\in\overline{1,M},\ z\in\overline{s+1,M}$$

for some $\xi \in (0, w_0] \bigcap \left(0, \frac{\delta_0 - p_0}{\delta_0\tau_0}\right]$, *where*

$$\overline{Q}_{1,1} = \overline{P}A_0^\top + U_0^\top G^\top + A_0\overline{P} + GU_0 + \overline{S}_0 + \overline{\Xi}^0 - p_0\overline{R}_0,$$

$$\overline{Q}_{1,2} = \widetilde{P} - \overline{P} + \overline{P}A_0^\top + U_0^\top G^\top; \quad Q_{1,3} = B_0\overline{P} + GL_0 + p_0\overline{R}_0 + \overline{P}A_0^\top + U_0^\top G^\top,$$

$$\overline{Q}_{1,4} = \begin{bmatrix} A_1\overline{P} + GU_1 + \overline{P}A_0^\top + U_0^\top G^\top + \overline{\Upsilon}_{0,1} & \dots & A_M\overline{P} + GU_M + \overline{P}A_0^\top + U_0^\top G^\top + \overline{\Upsilon}_{0,M}\end{bmatrix},$$

$$\overline{Q}_{1,5} = \begin{bmatrix} B_1\overline{P} + GL_1 & \dots & B_M\overline{P} + GL_M\end{bmatrix},$$

$$\overline{Q}_{1,6} = I_n; \quad Q_{2,2} = -2\overline{P} + \delta_0\tau_0^2\overline{R}_0; \quad Q_{2,3} = B_0\overline{P} + GL_0 - \overline{P},$$

$$\overline{Q}_{2,4} = \begin{bmatrix} A_1\overline{P} + GU_1 - \overline{P} + \overline{\Lambda}_1 & \dots & A_M\overline{P} + GU_M - \overline{P} + \overline{\Lambda}_M\end{bmatrix},$$

$$\overline{Q}_{2,5} = \begin{bmatrix} B_1\overline{P} + GL_1 & \dots & B_M\overline{P} + GL_M\end{bmatrix}; \quad Q_{2,6} = I_n,$$

$$\overline{Q}_{3,3} = -e^{-w_0\tau_0}\overline{S}_0 - p_0\overline{R}_0 + 2B_0\overline{P} + 2GL_0,$$

$$\overline{Q}_{3,4} = \begin{bmatrix} \overline{P}B_0^\top + L_0^\top G^\top + A_1P + GU_1 & \dots & \overline{P}B_0^\top + L_0^\top G^\top + A_MP + GU_M\end{bmatrix},$$

$$\overline{Q}_{3,5} = \begin{bmatrix} B_1P + GL_1 & \dots & B_MP + GL_M\end{bmatrix}; \quad \overline{Q}_{3,6} = \overline{P},$$

$$\overline{Q}_{4,4} = \overline{Q}_{4,4}^\top = (\hat{Q}'_{a,b})_{a,b=1}^M,$$

$$\hat{Q}'_{j,j} = \overline{P}A_j^\top + U_j^\top G^\top + A_j\overline{P} + GU_j + \overline{\Xi}^j + \overline{S}_j,\ j\in\overline{1,M},$$

$$\hat{Q}'_{s,z} = \overline{P}A_s^\top + U_s^\top G^\top + A_z\overline{P} + GU_z + \overline{\Upsilon}_{s,z},\ s\in\overline{1,M-1}, z\in\overline{s+1,M},$$

$$\overline{Q}_{4,5} = \begin{bmatrix} B_1\overline{P} + GL_1 & \cdots & B_M\overline{P} + GL_M \\ \vdots & \ddots & \vdots \\ B_1\overline{P} + GL_1 & \cdots & B_M + GL_M\overline{P}\end{bmatrix}; \quad \overline{Q}_{4,6} = \mathbb{1}_M \otimes I_n,$$

$$\overline{Q}_{5,5} = \mathrm{diag}(-e^{-w_0\tau_1}\overline{S}_1, \ldots, -e^{-w_0\tau_M}\overline{S}_M),$$

$$\overline{Q}_{5,6} = \mathbf{O}_{nM\times n}; \quad \overline{Q}_{6,6} = -\Phi.$$

Then the closed-loop system (5.8) *is ISS with feedback gains*

$$K_{A,s} = U_s \overline{P}^{-1},\ K_{B,s} = L_s \overline{P}^{-1}, s \in \overline{0,M}.$$

Proof. Using the prescribed properties of $\widetilde{P}$, $\overline{\Lambda}_j \left(j \in \overline{1,M}\right)$, $\overline{R}_s \left(s \in \overline{0,M}\right)$, $\overline{S}_k \left(k \in \overline{0,M}\right)$ and $\overline{P}$, select the LKF V given by (5.5) in the proof of Theorem 2 with: $P = \overline{P}^{-1}\widetilde{P}\overline{P}^{-1}$, $\Lambda_j = \overline{P}^{-1}\overline{\Lambda}_j\overline{P}^{-1}$, $R_0 = \overline{P}^{-1}\overline{R}_0\overline{P}^{-1}$, $S_k = \overline{P}^{-1}\overline{S}_k\overline{P}^{-1}$, $R_j = 0$ ($j \in \overline{1,M}$), then V verifies the positive definiteness requirements of Definition 2.8. Furthermore, consider the conditions and the proof of Theorem 5.1 and denote by $\widetilde{Q}$ the block matrix Q (in Theorem 5.1) under the substitutions $R_j \to 0$ for $j \in \overline{1,M}$, $(P_2^{-1}, P_3^{-1}, P_4^{-1}, \Omega_j^{-1}) \to (\overline{P}, \overline{P}, \overline{P}, \overline{P})$ for $j \in \overline{1,M}$, $A_s \to \widetilde{A}_s$, $B_s \to \widetilde{B}_s$ for $s \in \overline{0,M}$, define

$$\begin{aligned}
\overline{Q} &= H^\top \widetilde{Q} H, \\
H &= \mathrm{diag}(P_2^{-1}, P_3^{-1}, P_2^{-1}, \Omega_1^{-1}, \ldots, \Omega_M^{-1}, \Omega_1^{-1}, \ldots, \Omega_M^{-1}, I_n) \\
&= \mathrm{diag}(\underbrace{\overline{P}, \ldots, \overline{P}}_{2M+3}, I_n)
\end{aligned}$$

under the settings of

$$\begin{aligned}
\Xi^k &= \overline{P}^{-1}\overline{\Xi}^k\overline{P}^{-1},\ k \in \overline{0,M} \\
\Upsilon_{s,z} &= \overline{P}^{-1}\overline{\Upsilon}_{s,z}\overline{P}^{-1}, s \in \overline{0,M-1},\ z \in \overline{s+1,M},
\end{aligned}$$

by which we can deduce that

$$\overline{Q} \leq 0 \quad \Leftrightarrow \quad Q \leq 0$$

and the conditions (5.9) are equivalent to (5.4). This completes the proof. □

To find the control gains as solutions of LMIs in Theorem 5.2, more restrictive conditions are imposed than in Theorem 5.1 (or in Remark 5.1): the matrices $\widetilde{P}, \overline{\Lambda}_j, P_2, P_3, P_4$ are assumed to be diagonal and positive definite. In practice, Theorem 5.2 and Remark 5.1 can be applied iteratively: the former to find some guesses for $K_{A,s}$, $K_{B,s}$, $s \in \overline{0,M}$, while the latter to calculate more accurately the AGs from Definition 2.7 and to refine the restrictions on delays.

5.3 Applications

5.3.1 Application to opinion dynamics

For modeling opinion dynamics among a network, the following equation can be used [5, 12, 13]:

$$\begin{aligned}\dot{x}(t) &= -x(t) + \sum_{j=1}^{M} k_j A_j \overline{\tanh}(\alpha_j x(t)) \\ &+ \sum_{r=1}^{L} p_r B_r \overline{\tanh}(\beta_r x(t-\tau_r)) + Gu(t) + \varphi(t),\end{aligned} \tag{5.10}$$

where $x(t) \in \mathbb{R}^n$ is the opinion variable of n agents, and $\operatorname{sign}(x_i(t))$ $(i \in \overline{1,n})$ describes the qualitative stance toward a binary choice (the bigger $|x_i(t)|$, more extreme is the opinion of the agent i); $(M+L) \geq 2$ $(M, L \geq 1)$ is the number of social networks connecting the agents; $k_j, p_r > 0$ denote the social interaction strength among agents in the network, $j \in \overline{1,M}, r \in \overline{1,L}$; $\tau_r > 0$ is the time delay in the network $r \in \overline{1,L}$; $A_j, B_r \in \mathbb{R}^{n\times n}$ are the adjacency matrices, and $\alpha_j > 0$ or $\beta_r > 0$ characterizes the controversialness of the issue for j^{th} or r^{th} media; the function $\overline{\tanh} : \mathbb{R}^n \to \mathbb{R}^n$ and $\overline{\tanh}([g_1 \; \dots \; g_n]^\top) = \begin{bmatrix}\tanh(g_1) & \dots & \tanh(g_n)\end{bmatrix}^\top$ for $g_1, \dots, g_n \in \mathbb{R}$; $G \in \mathbb{R}^{n\times q}$; $u(t) \in \mathbb{R}^q$ is a controlling input for modifying the network connections among the agents (thus, it has to be of the form of (5.7), and any shape of control cannot be implemented); $\varphi(t) \in \mathbb{R}^n$ can be used to model the off-network influences on orientations of agents (*e.g.*, government communication). The detailed motivation for this model (for the case $M = 1$ and time-varying matrix A_1) is given in [12, 13]. The system under feedback control takes the form of (5.1), and assumptions 2.1, 5.1 are satisfied.

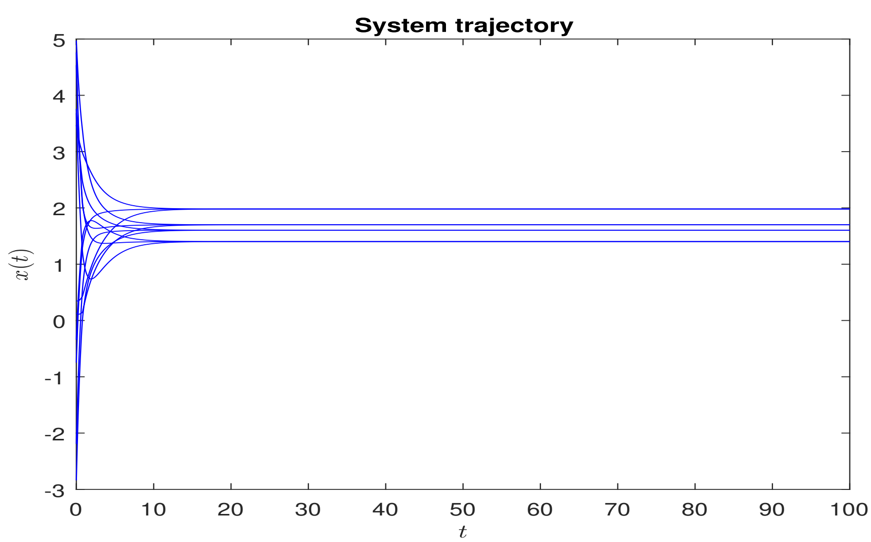

Figure 5.1: The trajectories of the controlled system (5.10) versus the time t

For illustration, let

$$n=4,\quad M=L=1,\quad A_1=\begin{bmatrix}0&1&1&1\\1&0&0&0\\1&0&0&0\\0&0&1&0\end{bmatrix},\quad B_1=\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&1&0&1\\1&0&0&0\end{bmatrix},$$

$$G=\begin{bmatrix}1&-1&0.2&1\\-1&0.5&1&-0.4\\1&-0.1&2&-1\\1&0.2&1&1\end{bmatrix},\quad k_1=1.8,\quad p_1=1.3,$$

$$\alpha_1=0.4,\quad \beta_1=0.2,\quad \tau_1=0.8,\quad \varphi(t)=\begin{bmatrix}1\\0.3\\0.7\\0.1\end{bmatrix},$$

$$u(t)=Z_1\overline{\tanh}(\alpha_1 x(t))+Z_2\overline{\tanh}(\beta_1 x(t-\tau_1)),$$

$$Z_1=\begin{bmatrix}0.3321&-0.0031&0.2004&-1.1101\\0.801&0.8805&-0.2584&-0.2578\\-0.9096&-0.1803&-0.4758&0.6053\\-0.5175&-0.6344&-0.0687&0.0637\end{bmatrix},$$

$$Z_2=\begin{bmatrix}0.3432&0.7509&-0.3315&0.2678\\-0.3231&-0.8566&0.1478&0.2328\\-0.2667&-1.1971&0.2777&-0.2342\\-0.6731&0.3070&0.1869&0.3786\end{bmatrix},$$

then the LMIs in Remark 5.1 are verified. The three sets of system trajectories ($x(t)\in\mathbb{R}^4$) with different initial conditions are presented in Fig. 5.1, which illustrate that all agents converge to a common decision under the chosen control. Simulations of the system (5.10) with $u=0$, $\varphi=0$ demonstrate pluralism of opinions in the uncontrolled network.

5.3.2 Application to a modified Lotka-Volterra model

In this subsection, a modified Lotka-Volterra (LV) dynamics is considered. Different versions of this model have been widely investigated in infectious diseases, biology, finance, to mention a few [45]. The basic model does not reflect some important phenomena, such as time delays and stable coexistence. Thus many modified LV models have been proposed. Among them, the following one considers population dynamics with several delays [122]:

$$\dot{x}(t) = \mathrm{diag}\{x(t)\}\left[r_0 + r(t) + A_1 x(t-\tau_1) + A_2 x(t-\tau_2)\right],\ t \in \mathbb{R}_+, \tag{5.11}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^\top \in \mathbb{R}_+^n$ contains the populations of n species; $r_0 \in \mathbb{R}^n$ models the birth and death rates; A_1, $A_2 \in \mathbb{R}^n$ represent the community matrices; τ_1, $\tau_2 > 0$ are delays corresponding to two different kinds of interactions between populations; the function $r : \mathbb{R}_+ \to \mathbb{R}^n$ is introduced to model the deviations of the rates from the nominal quantities $r_0 \in \mathbb{R}^n$.

Assuming the existence of a unique non-zero equilibrium point $x_e = \begin{bmatrix} x_e^1 & \ldots & x_e^n \end{bmatrix}^\top \in \mathbb{R}_+^n \setminus \{0\}$ for (5.11) with $r(t) = 0$, and defining

$$\rho(t) = \begin{bmatrix} \rho_1(t) \\ \vdots \\ \rho_n(t) \end{bmatrix} = \begin{bmatrix} \ln(x_1(t)) - \ln(x_e^1) \\ \vdots \\ \ln(x_n(t)) - \ln(x_e^n) \end{bmatrix},$$

we have

$$\begin{aligned} \dot{\rho}(t) \quad &= \quad A_1\,\mathrm{diag}(x_e) f^1(\rho(t-\tau_1)) \\ &\quad\ + A_2\,\mathrm{diag}(x_e) f^1(\rho(t-\tau_2)) + r(t), \end{aligned} \tag{5.12}$$

where

$$f^1(\rho) = \begin{bmatrix} e^{\rho_1} \\ \vdots \\ e^{\rho_n} \end{bmatrix} - \mathbb{1}_n.$$

It is clear that f^1 satisfies Assumption 2.1 and Assumption 5.1 with $\eta_{0,j}^i = \eta_{2,jj'}^i = 1$. The requirements of Assumption 5.2 are not satisfied globally. However, as in [2], due to assumed existence of the global equilibrium x_e, it is possible to show that for $r_0 + r(t) \geq r_{\min}$ all trajectories converge to a neighborhood of the steady state, so that $x(t) > 0$ for all $t \in \mathbb{R}_+$, which results in well-posedness of (5.12). The analysis can be next performed without taking into account the unbounded deviations of the state.

The simulation results are given for

$$n = 2, \quad \tau_1 = 0.001, \quad \tau_2 = 0.02, \quad A_1 = \begin{bmatrix} -0.6 & 0.4 \\ 0.5 & -0.6 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -0.3 & 0.8 \\ 0.6 & -0.9 \end{bmatrix}, \quad r_0 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}, \quad r(t) = \begin{bmatrix} -0.2\sin(t) \\ 0.1\cos(t) \end{bmatrix}.$$

The LMIs of Corollary 5.1 are verified, and the state trajectories are shown in Fig. 5.2 for two sets of initial conditions.

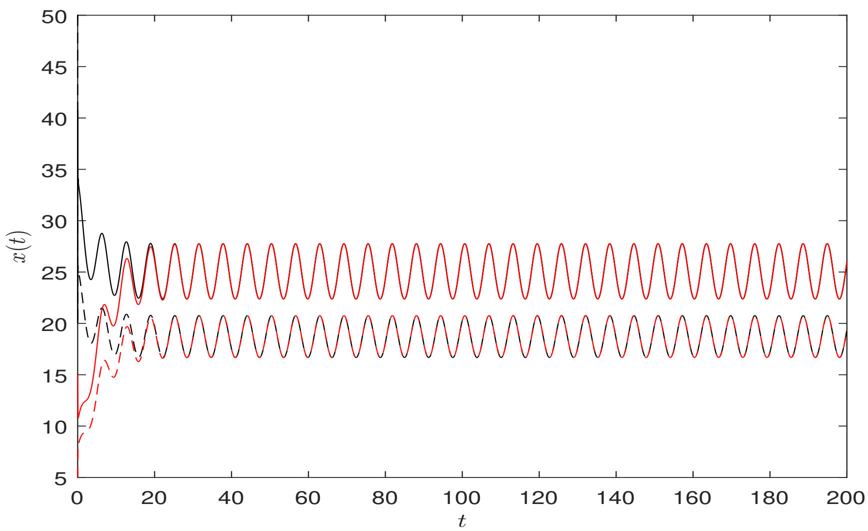

Figure 5.2: The state trajectories versus the time t for LV model

CHAPTER 6

Conclusion

In this thesis, we proposed IOS, SIIOS, and ROS conditions for generalized Persidskii systems. The conditions were obtained in the form of LMIs. This work can be mainly divided into three directions: robust synchronization and observer design for generalized Persidskii models, and ISS conditions for time-delay dynamics.

The third chapter dealt with applying general IOS theory to robust synchronization of a family of the considered dynamics, for which a synchronization measure was introduced so that the kernel of the output function is in the defined synchronous mode. The synchronization of linear systems admitting an upper bound of the disturbance/input was also considered. The proposed results were illustrated by a numerical example of Hindmarsh-Rose models of neurons.

The fourth chapter addressed the problem of robust state estimation for a class of generalized Persidskii systems. A simple observer was proposed containing a copy of the system dynamics with a nonlinear output injection term. Two sets of stability conditions were developed, establishing IOS and SIIOS properties of the common dynamics of the system and the observer with respect to the estimation error. Two examples were presented (two-mass and SIS models) to verify the effectiveness of our framework.

Ultimately, the last chapter presents ISS and stabilization conditions for generalized Persidskii dynamics with constant time delays. The formulated conditions are explicitly dependent on delays. Two conditions were formulated, for a given control and the design of feedback gains. The simulations of opinion dynamics and a modified Lotka-Volterra model were shown to illustrate the proposed results.

The future research directions include investigating the network structure's influence on the synchronization in generalized Persidskii systems, the design of adaptive or reduced-order observers, and ISS analysis for the considered systems with time-varying delays and the study of other practical applications.

Appendix A

List of publications

Submitted

- W. Mei, D. Efimov, and R. Ushirobira, ”Short-time stability notions for nonlinear systems,” *Submitted*.

Journal articles

- W. Mei, D. Efimov, R. Ushirobira, and E. Fridman, ”On delay-dependent conditions of ISS for generalized Persidskii systems,” *IEEE Transactions on Automatic Control*, 2023.
- W. Mei, R. Ushirobira, and D. Efimov, ”On nonlinear robust state estimation in generalized Persidskii systems,” *Automatica*, 2022.
- W. Mei, D. Efimov, R. Ushirobira, and A. Aleksandrov, ”On convergence conditions for generalized Persidskii systems,” *International Journal of Robust and Nonlinear Control*, 2022.
- W. Mei, D. Efimov, and R. Ushirobira, ”On input-to-output stability and robust synchronization of generalized Persidskii systems,” *IEEE Transactions on Automatic Control*, 2021.

Conference proceedings

- W. Mei, D. Efimov, and R. Ushirobira, ”Annular short-time stability of generalized Persidskii systems,” in *2022 61st IEEE Conference on Decision and Control (CDC)*, 2022.
- W. Mei, D. Efimov, and R. Ushirobira, ”Input-to-state stability of time-delay Persidskii systems,” in *2021 60th IEEE Conference on Decision and Control (CDC)*, 2021.

- W. Mei, D. Efimov, R. Ushirobira, and A. Aleksandrov, “Convergence conditions for Persidskii systems,” in *19th European Control Conference (ECC)*, 2021.
- W. Mei, D. Efimov, and R. Ushirobira, “Towards state estimation of Persidskii systems,” in *2020 59th IEEE Conference on Decision and Control (CDC)*, 2020.
- W. Mei, D. Efimov, and R. Ushirobira, “Feedback synchronization in Persidskii systems,” *IFAC-PapersOnLine*, 2020.

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