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On functional equations of p-adic L-functions for GL_2

Sur les équations fonctionnelles des fonctions L p-adiques pour GL_2

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"Có thể tôi không phải là người giỏi nhất, nhưng tôi phải là người có ích nhất, tôi phải là người dám đứng ra nhận những nhiệm vụ nặng nề nhất, những công việc mà không ai dám làm.

Này chàng trai, đã đến lúc trở lại và chết vì đất nước của cậu rồi!"

On functional equations of p-adic L-functions for GL_2

Abstract

For a regular, non-critical, p-refinement of a cohomological cuspidal automorphic representation of GL_2 over a totally real number field, we prove a functional equation of its attached p-adic L-function. We obtain it from the interpolation formula between p-adic and complex L-functions and the functional equation of L-functions. We use this functional equation to prove the Trivial Zero Conjecture at the central critical point.

On the other hand, we develop a theory of overconvergent modular symbols with values in p-adic distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ inspired by Stevens' overconvergent modular symbols and an idea of Colmez with the hope that one can obtain some functional equations of p-adic L-functions involving the transformation $z \mapsto \frac{1}{z}$ on $\mathbb{P}^1(\mathbb{Q}_p)$.

Sur les équations fonctionnelles des fonctions L p-adiques pour GL_2

Résumé

Pour un p-raffinement non-critique et régulier d'une représentation automorphe cuspidale cohomologique de GL_2 sur un corps de nombres totalement réel, nous prouvons l'équation fonctionnelle de sa fonction L p-adique attachée. Nous obtenons cela grâce à la formule d'interpolation reliant ses valeurs à la fonction L complexe et l'équation fonctionnelle de cette dernière. Comme application nous démontrons la Conjecture du Zéro Trivial au point critique central.

D'autre part, nous développons une théorie de symboles modulaires surconvergents à valeurs dans des distributions p-adiques sur $\mathbb{P}^1(\mathbb{Q}_p)$ inspirée par les symboles modulaires surconvergents de Stevens et une idée de Colmez, dans l'espoir d'obtenir une preuve purement p-adique de l'équation fonctionnelle des fonctions L p-adiques, faisant intervenir la transformation $z \mapsto \frac{1}{z}$ sur $\mathbb{P}^1(\mathbb{Q}_p)$.

Contents

Introduction

The theory of p-adic L-functions plays an important role in number theory. For a p-adic representation V of the absolute Galois group $\mathrm{G}_{\mathbb{Q}}$ such that $V_p = V_{|\mathrm{G}_{\mathbb{Q}_p}}$ is semi-stable and V is critical in the sense of Deligne, let $D \subset \mathcal{D}_{\mathrm{st}}(V_p)$ be a regular submodule in the sense of Perrin-Riou [PR]. Coates and Perrin-Riou conjectured a p-adic L-function $L_p(V, D, s)$ satisfying an interpolation formula related to special values of the complex L-function attached to V with corrected Euler factors at p.

For a p-refinement $\tilde{\pi}$ of an algebraic cuspidal automorphic representations π of a reductive group over a number field F, Iwasawa theory tries to relate the properties of the attached p-adic L-function $L_p(\tilde{\pi}, s)$ to the arithmetic of the restriction of V_π to the p-adic cyclotomic extension of F, where V_π is the conjectured p-adic Galois representation of G_F attached to π. In the pioneer works of Amice-Vélu, Vishik and Mazur-Tate-Teitelbaum (see [AV], [Vis], [MTT]) they associated a p-adic L-function to normalized eigenforms of non-critical slope. These p-adic L-functions satisfy an interpolation formula related to special values of complex L-functions. The functional equation of p-adic L-functions is deduced from the functional equation of L-functions involving the twist by the matrix $\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ which corresponds to the transformation $z \mapsto \frac{-1}{z}$.

For a modular elliptic curve E over $\mathbb{Q}$, Mazur, Tate and Teitelbaum constructed in [MTT] a p-adic L-function $L_p(E, s)$ having a trivial zero at $s = 1$ if E has split multiplicative reduction at p. Moreover, they stated a p-adic analogue of the Birch-Swinnerton-Dyer conjecture relating the analytic properties of $L_p(E, s)$, namely the order of vanishing at $s = 1$ and the Fourier coefficients, and the arithmetic properties of E such as the rank of its rational points. After that, Stevens gave an another construction of p-adic L-functions attached to modular forms of non-critical slope using his theory of overconvergent modular symbols (see [PS11]). The theory of overconvergent modular symbols was generalized by Ash-Stevens and Urban to the theory of overconvergent cohomology in [AS08] and [Urb]. Barrera-Dimitrov-Jorza applied this idea in [BDJ] to construct p-adic L-functions for cuspidal automorphic representations of GL_2 over a totally real number field having an arbitrary cohomological weight.

The goal of this thesis is to study functional equations of p-adic L-functions attached to modular forms for both classical and Hilbert modular forms. A p-adic L-function is a p-adic distribution on certain p-adic space which is an open compact subset of $\mathbb{Q}_p^d$ for some d. This leads us to the study of p-adic distributions. While the most popular p-adic distributions are those on $\mathbb{Z}_p$ or $\mathbb{Z}_p^\times$, one of our contributions is the study of p-adic distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ inspired by an idea of Colmez. Contrary to $\mathbb{Z}_p$, the space $\mathbb{P}^1(\mathbb{Q}_p)$ admits the transformation $z \mapsto \frac{-1}{z}$ occurring in the functional equation of L-functions, providing an additional motivation for the study of p-adic distributions on $\mathbb{P}^1(\mathbb{Q}_p)$.

Let L be a finite extension of $\mathbb{Q}_p$ and k be an integer. We denote by $\mathcal{A}_k(\mathbb{P}^1, L)$ the space of L-valued functions on $\mathbb{P}^1(\mathbb{Q}_p)$ which are locally analytic on $\mathbb{Q}_p$ and meromorphic at infinity with a pole of order $\leq k$ (see Definition 2.1.1). The space $\mathcal{D}_k(\mathbb{P}^1, L)$ of L-valued distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ is defined as the continuous L-dual of $\mathcal{A}_k(\mathbb{P}^1, L)$ and is endowed with a right weight k action of $\mathrm{GL}_2(\mathbb{Q}_p)$ (see (2.5)). If $k \geq 0$, let $\mathcal{V}_k^\dagger(L)$ be the L-dual of the space $\mathcal{P}_k^\dagger(L)$ of locally polynomial functions of degree $\leq k$ on $\mathbb{P}^1(\mathbb{Q}_p)$ with coefficients in L. The natural inclusion $\mathcal{P}_k^\dagger(L) \to \mathcal{A}_k(\mathbb{P}^1, L)$ induces the dual map $\rho_k : \mathcal{D}_k(\mathbb{P}^1, L) \to \mathcal{V}_k^\dagger(L)$ which is equivariant for the action of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Stevens' overconvergent modular symbols are elements of the space $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))$ of modular symbols on a congruence subgroup $\Gamma_0 \subset \Gamma_0(p)$ with values in the space $\mathcal{D}_k(\mathbb{Z}_p, L)$ of L-valued p-adic distributions on $\mathbb{Z}_p$ (see §3.1.1 for the definition of abstract modular symbols). Here $\mathcal{D}_k(\mathbb{Z}_p, L)$ is endowed with a right weight k action of a monoid $\Sigma_0(p)$ in $\mathrm{SL}_2(\mathbb{Z})$ containing the group $\Gamma_0(p)$ and $\left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right)$ (see §1.2.1). It is also natural to consider modular symbols with values in $\mathcal{D}_k(\mathbb{P}^1, L)$.

The overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{Z}_p, L)$ or $\mathcal{D}_k(\mathbb{P}^1, L)$ are endowed with a right action of the Hecke operator U_p given by the double coset $\Gamma_0 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0$. The basic tool of Stevens' construction is a control theorem allowing to lift classical eigensymbols of small U_p-slope to overconvergent ones with values in $\mathcal{D}_k(\mathbb{Z}_p, L)$.

Consider the exact sequence

$$0 \to \mathcal{D}(\mathbb{Z}_p) \overset{\text{ext}}{\to} \mathcal{D}_k(\mathbb{P}^1) \overset{\text{res}}{\to} \mathcal{D}(\mathbb{Z}_p) \to 0, \quad (1)$$

where the map ext is the extension map of distributions, the map res is the composition of the restriction map of distributions from $\mathbb{P}^1(\mathbb{Q}_p)$ to $\mathbb{P}^1(\mathbb{Q}_p)\backslash\mathbb{Z}_p$ followed by the isomorphism of distributions on $\mathbb{P}^1(\mathbb{Q}_p)\backslash\mathbb{Z}_p$ and $\mathbb{Z}_p$ induced by the transformation $z \mapsto \frac{1}{pz}$. It turns out that the map ext is equivariant for the action of $\Sigma_0(p)$, so $\mathcal{D}(\mathbb{Z}_p)$ is endowed with an another action of $\Sigma_0(p)$ for which the map res : $\mathcal{D}_{\mathrm{k}}(\mathbb{P}^1) \to \mathcal{D}(\mathbb{Z}_{\mathrm{p}})$ is $\Sigma_0(p)$-equivariant. The exact sequence (1) induces the exact sequence of modular symbols:

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)) \overset{\text{ext}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1)) \overset{\text{res}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)),$$

where the map ext is U_p-equivariant. Then we can equip $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ a right operator V_p for which the map res : $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_{\mathrm{k}}(\mathbb{P}^1)) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_{\mathrm{k}}(\mathbb{Z}_{\mathrm{p}}))$ is equivariant for the action of U_p on the left space and V_p on the right space. It turns out that V_p is induced by the double coset $\Gamma_0 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0$. Although $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \notin \Sigma_0(p)$, the action of $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ on $\mathcal{D}_k(\mathbb{Z}_p)$ is defined similarly to that of $\Sigma_0(p)$ (see 3.5).

For a rational number $h \in \mathbb{Q}$ we use the superscript $\leq h$ (resp. $V_p \leq h$) to denote the subspace of modular symbols of U_p (resp. V_p)-slope $\leq h$, which is the subspace where every eigenvalue of the corresponding operator has p-adic valuation $\leq h$ (see Definition 3.1.2). We define similarly if $\leq$ is replaced by $<$. The overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{Z}_p, L)$ and $\mathcal{D}_k(\mathbb{P}^1, L)$ are related by the following theorem:

Theorem 0.0.1 (Corollary 3.3.11)**.** *For $k \in \mathbb{Z}\backslash\{0\}$, there is an exact sequence of modular symbols:*

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L)) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L)) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L)) \to 0$$

which is equivariant for the U_p-action on the first two spaces and the V_p-action on the last space. The restriction on the $\leq h$-slope subspace is also exact:

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{\leq h} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))^{\leq h} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{V_p \leq h} \to 0,$$

and similar if $\leq h$ is replaced by $< h$.

If $k = 0$, the last space 0 in the above exact sequences is replaced by L.

We construct subspaces of $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))$ which are finite dimensional and U_p-stable.

Let $D(\infty, 1) = \{z \in \mathbb{P}^1(\mathbb{Q}_p), v_p(z) \leq -1\}$, where v_p is the usual p-adic valuation. Let $\mathcal{D}_k(D(\infty, 1), L) \subset \mathcal{D}_k(\mathbb{P}^1, L)$ be the subspace of distributions supported in $D(\infty, 1)$, endowed with the induced weight k action of matrices. There is also an operator V_p acting on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty, 1), L))$ defined as above.

For $h, h' \in \mathbb{Q}$, denote by $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))^{(V_p \leq h')}$ the subspace of $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))$ consisting of modular symbols Φ such that the restriction $\Phi_{|D(\infty,1)} \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty, 1), L))$ has V_p-slope $\leq h'$, and putting

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))^{U_p \leq h, (V_p \leq h')} = \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))^{U_p \leq h} \cap \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))^{(V_p \leq h')}.$$

Here is the main theorem of the first part about overconvergent modular symbols:

Main Theorem 0.0.2 (Theorem 3.3.23)**.** *The subspace $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{U_p \leq h, (V_p \leq h')}$ of $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))$ is finite dimensional and U_p-stable for any $k \in \mathbb{Z}$ and $h, h' \in \mathbb{Q}$, and similar if $\leq$ is replaced by $<$.*

Moreover, if $k \in \mathbb{N}^*$ and $0 \leq h \leq k+1$, there is an exact sequence:

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{\leq h} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))^{U_p \leq h, (V_p \leq h')} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{k+1-h \leq U_p \leq h'} \to 0,$$

while if $k = 0$ the last space 0 is replaced by L.

In particular, there is an exact sequence:

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{\leq k+1} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))^{U_p \leq k+1, (V_p \leq k+1)} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{\leq k+1} \to 0$$

for any $k \in \mathbb{N}^*$ and if $k = 0$ the last space 0 is replaced by L.

Meanwhile try to improve the interpolation approach, we plan to pursue this result to construct and prove functional equations of p-adic L-functions in future works. The exact sequence in the above theorem can be seen as an analogue of Stevens' control theorem for $\mathbb{P}^1(\mathbb{Q}_p)$.

The second part of the thesis is devoted to prove the functional equation of p-adic L-functions by using the interpolation formula and prove the trivial zero conjecture at central critical points as an application.

Let F be a totally real number field of different $\mathfrak{d}$. For each fractional ideal $\mathfrak{f}$ of F, we choose an element $\varpi_\mathfrak{f}$ in the ring of finite adeles of F such that $\varpi_{v\mathfrak{f}} = \varpi_v \cdot \varpi_\mathfrak{f}$ for any finite place v, where ϖ_v is a uniformizer of the ring of integers $\mathcal{O}_v$ of F_v. For each finite place v of F dividing a prime number l, denote by q_v the residue degree of v and δ_v the valuation at v of the different $\mathfrak{d}$, and consider the additive character $\psi_v : F_v \to \mathbb{C}^\times$ given by the composition of the trace map from F_v to $\mathbb{Q}_l$ and the character $\psi_{0,l} : \mathbb{Q}_l \to \mathbb{C}^\times$ given by the value of $\exp(2\pi i \cdot)$ at the l-non integer part of $\mathbb{Q}_l$. More explicitly,

$$\psi_{0,l}\Big(\sum_{i=n}^{+\infty} a_i l^i\Big) = e^{2\pi i \sum_{i=n}^{-1} a_i l^i},$$

where $n \in \mathbb{Z}, a_i \in \mathbb{N}, 0 \leq a_i < l$.

Let $\mathrm{Gal}_{p\infty}$ be the Galois group of the maximal abelian extension of F which is unramified outside p and ∞. Denote by $\mathrm{Gal}_{\mathrm{cyc}}$ the Galois group of the cyclotomic $\mathbb{Z}_p$-extension $F_{\mathrm{cyc}} \subset F(\mu_{p^\infty})$ of F. Let ω_p be the Teichmüller lift of the cyclotomic character $\chi_{\mathrm{cyc}} : \mathrm{Gal}_{p\infty} \to \mathrm{Gal}(F(\mu_{p^\infty})/F) \to \mathbb{Z}_p^\times$. Then the character $\langle\cdot\rangle_p = \chi_{\mathrm{cyc}}\omega_p^{-1} : \mathrm{Gal}_{p\infty} \to 1 + 2p\mathbb{Z}_p$ factors through $\mathrm{Gal}_{\mathrm{cyc}}$.

Let π be a cuspidal automorphic representation of GL_2 over F of tame conductor $\mathfrak{n}$ and cohomological weight $(k, \mathrm{w}) := \left(\frac{\mathrm{w}+k_\sigma-2}{2}, \frac{\mathrm{w}+2-k_\sigma}{2}\right)_{\sigma\in\Sigma} \in (\mathbb{Z}^2)^\Sigma$ with $\Sigma = \mathrm{Hom}(F, \mathbb{C})$, where $k_\sigma \geq 2$ and $k_\sigma \equiv \mathrm{w} \pmod 2$ for any $\sigma \in \Sigma$. We assume further that

$$\pi \text{ has central character } \omega_\pi = \omega^2|\cdot|_F^{\mathrm{w}} \text{ with w even, and } \pi_v \text{ is not supercuspidal for all } v|p, \tag{2}$$

where ω is a finite order character of $\mathrm{Gal}_{p\infty}$. The condition on the central character of π implies that the twist $\pi \otimes \omega^{-1}|\cdot|_F^{-\mathrm{w}/2}$ of π is self-dual.

We choose a regular p-refinement $\tilde{\pi} = (\pi, \{\nu_v\}_{v|p})$ of π, i.e., choose a character ν_v of $F_v^\times$ which appears as a one dimensional sub of the Weil-Deligne representation attached to π_v via the local Langlands correspondence for $\mathrm{GL}_2(F_v)$, for each place v of F above p. Assume that $\tilde{\pi}$ is non-critical in the sense of Definition 5.1.8, then by [BDJ, (4.2)] we can attach a p-adic L-function $\mathcal{L}_p(\tilde{\pi}, \cdot)$ which is a distribution on $\mathrm{Gal}_{p\infty}$. This p-adic L-function is the specialization of a multi-variable p-adic L-function $\mathcal{L}_p \in \mathcal{D}(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{X}(\tilde{\pi})))$, where $\mathcal{X}(\tilde{\pi})$ is a family of non-critical p-refinements of cohomological cuspidal automorphic representations indexed by a neighborhood of (k, w) in the weight space (see the paragraph after Theorem 5.1.12).

Let St_p denote the set of places v above p such that π_v is a special representation. We put

$$\tilde{\varepsilon}_{\pi\otimes\omega^{-1}} = \varepsilon\Big(\pi \otimes \omega^{-1}, \frac{1-\mathrm{w}}{2}\Big) \prod_{v\in\mathrm{St}_p, \nu_v\omega_v^{-1}\text{unramified}} \varepsilon\Big(\pi_v \otimes \omega_v^{-1}, \frac{1-\mathrm{w}}{2}, \psi_v\Big) \in \{\pm 1\}.$$

We are ready to state our main theorem of the second part about the functional equation of p-adic L-functions attached to automorphic representations of GL_2, which is a generalization of [BDJ, Theorem 6.4].

Main Theorem 0.0.3 (Theorem 5.3.5)**.** *Suppose π satisfies* (2)*. Then the sign $\tilde{\varepsilon}_{\pi_\lambda\otimes\omega^{-1}}$ of $\tilde{\pi}_\lambda$ is independent of the cohomological weight $\lambda \in \mathcal{X}(\tilde{\pi})$. For any $\lambda \in \mathcal{X}(\tilde{\pi})$, any continuous character $f : \mathrm{Gal}_{\mathrm{cyc}} \to L^\times$ and any finite order character $\chi : \mathrm{Gal}_{p\infty} \to L^\times$, one has*

$$\mathcal{L}_p(\lambda, \chi \cdot f) = \tilde{\varepsilon}_{\pi\otimes\omega^{-1}} \cdot (\chi\omega\omega_p^{\mathrm{w}/2} f)(-\varpi_\mathfrak{n})\langle\mathfrak{n}\rangle_p^{\mathrm{w}_\lambda/2}\mathcal{L}_p(\lambda, \omega^{-2}\chi_{\mathrm{cyc}}^{-\mathrm{w}_\lambda}(\chi \cdot f)^{-1}).$$

We give an application of the above formula to the trivial zero conjecture. The cyclotomic p-adic L-function attached to $\tilde{\pi}$ is defined as

$$L_p(\tilde{\pi}, s) = \mathcal{L}_p(\tilde{\pi}, \omega^{-1}\omega_p^{-\mathrm{w}/2}\langle\cdot\rangle_p^{s-1}), \quad \text{where } s \in \mathcal{O}_{\mathbb{C}_p}.$$

Let E be the set of places $v \in \mathrm{St}_p$ such that $\nu_v\omega_v^{-1}$ is unramified and $\varepsilon\left(\pi_v \otimes \omega_v^{-1}, \frac{1-\mathrm{w}}{2}, \psi_v\right) = -1$.

The following result is a generalization of [BDJ, Theorem 7.1] under our less stringent hypothesis (2).

Theorem 0.0.4 (Theorem 6.0.1)**.** *Suppose π satisfies* (2)*. The p-adic L-function $L_p(\tilde{\pi}, s)$ has order of vanishing at least $e = |E|$ at $\frac{2-\mathrm{w}}{2}$ and*

$$\frac{L_p^{(e)}\big(\tilde{\pi}, \frac{2-\mathrm{w}}{2}\big)}{e!} = \mathcal{L}(\widetilde{\pi \otimes \omega^{-1}}) \frac{\omega(\varpi_{\mathfrak{d}}) L\big(\pi \otimes \omega^{-1}, \frac{1-\mathrm{w}}{2}\big)}{\mathrm{N}_{F/\mathbb{Q}}^{\mathrm{w}/2}(i\mathfrak{d}) \Omega_{\tilde{\pi}}^{\omega_\infty \omega_{p,\infty}^{\mathrm{w}/2}}} \cdot 2^{|\{v \in \mathrm{St}_p \setminus E,\, \nu_v \omega_v^{-1} \text{ is unramified}\}|} \times$$
$$\times \prod_{v | p, \pi_v \otimes \omega_v^{-1} \text{ is unramified}} \left(1 - \frac{q_v^{-\mathrm{w}/2}}{(\nu_v \omega_v^{-1})(\varpi_v)}\right)^2 \prod_{v|p, c_{\nu_v \omega_v^{-1}} > 0} q_v^{-\frac{\mathrm{w} \cdot c_{\nu_v \omega_v^{-1}} + \delta_v}{2}} (\nu_v \omega_v^{-1})(\varpi_v)^{\delta_v} \tau(\nu_v \omega_v^{-1}, \psi_v),$$

where $\mathcal{L}(\widetilde{\pi \otimes \omega^{-1}})$ is the Fontaine-Mazur $\mathcal{L}$-invariant (see [BDJ, Definition 5.3]), $\Omega_{\tilde{\pi}}^{\cdot}$ is the Betti-Whittaker period defined in [BDJ, Definition 1.14], $\omega_{p,\infty}$ is the sign character on $(F \otimes_{\mathbb{Q}} \mathbb{R})^\times$, the symbol $c_{\nu_v \omega_v^{-1}}$ denotes the conductor of $\nu_v \omega_v^{-1}$ and $\tau(\nu_v \omega_v^{-1}, \psi_v)$ is the Gauss sum of $\nu_v \omega_v^{-1}$ with respect to ψ_v.

The thesis is structured as follows. In the first chapter we recall the definition and basic properties of p-adic distributions on open compact subsets of $\mathbb{Q}_p^d$, especially p-adic distributions on $\mathbb{Z}_p$ and $\mathbb{Z}_p^\times$. In Chapter 2 we study p-adic distributions on $\mathbb{P}^1(\mathbb{Q}_p)$. Chapter 3 is devoted to overconvergent modular symbols with values in distributions on $\mathbb{Z}_p$ or $\mathbb{P}^1(\mathbb{Q}_p)$, and the proof of Theorems 0.0.1, 0.0.2. In Chapter 4 we review p-adic L-functions attached to modular forms and the well-known functional equation of those functions (see Proposition 4.2.4). In Chapter 5 we prove our Main Theorem 0.0.3 about functional equation of p-adic L-functions attached to automorphic representations of GL_2. Finally, Chapter 6 is devoted to prove Theorem 0.0.4 about the trivial zero conjecture at central critical points.

Notations

In this thesis, let p be a prime number. Let $\bar{\mathbb{Q}}_p$ denote the algebraic closure of the field $\mathbb{Q}_p$ of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of $\bar{\mathbb{Q}}_p$ for p-adic norm. Let $v_p : \mathbb{C}_p \to \mathbb{Q} \cup \{+\infty\}$ be the normalized p-adic valuation such that $v_p(p) = 1$, and denote by $|\cdot|_p$ the corresponding p-adic norm on $\mathbb{C}_p$ defined by $|\cdot|_p = p^{-v_p(\cdot)}$.

We fix an embedding $\iota_p : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. Let L denote a finite extension of $\mathbb{Q}_p$.

For $a \in \mathbb{Q}_p$ or $\mathbb{C}_p$ and $r \in |\mathbb{C}_p^\times|_p$, denote by $D(a, r)$ (resp. $B(a, r)$) the closed (resp. open) disc of center a and radius r.

Chapter 1

p-adic distributions on open compact subsets of $\mathbb{Q}_p^d$

We consider p-adic distributions in the first two chapters which are the central object of this thesis. In this chapter, we study p-adic distributions on p-adic spaces which are open compact subsets of $\mathbb{Q}_p^d$ for some $d \in \mathbb{N}^*$. We start by defining the general notion of distributions on an open compact subset of $\mathbb{Q}_p^d$ in Section 1.1. In Section 1.2 we investigate distributions on $\mathbb{Z}_p$ which give the foundations for our consideration of distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ in Chapter 2.

1.1 Generalities

Throughout this section, let X be an open compact subset of $\mathbb{Q}_p^d$ for $d \in \mathbb{N}^*$. Roughly speaking, a p-adic distribution on X is the continuous dual of the space of locally analytic functions on X.

For a finite extension L of $\mathbb{Q}_p$, let $\mathcal{A}(X, L)$ denote the space of locally analytic functions on X with values in L. If the role of L is less important, we will omit it from notations, e.g., we write simply $\mathcal{A}(X)$ for $\mathcal{A}(X, L)$. The set $\mathcal{A}(X)$ is naturally an L-vector space.

For each $r \in |\mathbb{C}_p^\times|_p = p^{\mathbb{Q}}$, putting

$$D[X, r] = \{z \in \mathbb{C}_p^d, \exists\, a \in X \text{ such that } |z - a|_p \leq r\},$$

where the norm $|\cdot|_p$ on $\mathbb{Q}_p^d$ is the maximal of the norms of components. Then $D[X, r]$ is the union in $\mathbb{C}_p^d$ of closed polydiscs of radius r and center in X. Denote by $\mathcal{A}(X)[r]$ the space of functions $f \in \mathcal{A}(X)$ such that for every $a \in X$, the function f can be extended to a power series

$$\sum_{i \in \mathbb{N}^d} \alpha_i(a)(z_1 - a_1)^{i_1} ... (z_d - a_d)^{i_d} \tag{1.1}$$

converging on the closed polydisc $D(a, r)(\mathbb{C}_p) = \{z \in \mathbb{C}_p^d, |z - a|_p \leq r\}$ in $\mathbb{C}_p^d$, where $\alpha_i(a) \in L$. In other words, $f \in \mathcal{A}(X)[r]$ if f can be extended to an analytic function on $D[X, r]$. In particular, if $f \in \mathcal{A}(X)[r]$, then f is analytic on every closed polydisc of radius r in X. We say that f is r-analytic on X if $f \in \mathcal{A}(X)[r]$.

Since X is compact, $D[X, r]$ is covered by finitely many closed polydiscs of radius r and center in X. We define the supremum norm $\| \cdot \|_r$ on $\mathcal{A}(X)[r]$ by

$$\|f\|_r = \sup_{z \in D[X, r]} |f(z)|_p\,. \tag{1.2}$$

If f has expansion (1.1) on each closed polydisc $D(a, r)(\mathbb{C}_p)$ with $a \in X$, then it is well-known that

$$\|f\|_r = \sup_{a \in X, i \in \mathbb{N}^d} |\alpha_i(a)|_p\, r^{i_1 + ... + i_d}. \tag{1.3}$$

The space $\mathcal{A}(X)[r]$ endowed with this norm is an L-Banach algebra.

We denote by $\mathcal{D}(X)[r]$ the continuous L-dual of $\mathcal{A}(X)[r]$, endowed with the dual norm which we also denote by $\|\cdot\|_r$. Then $\mathcal{D}(X)[r]$ has the structure of Banach L-vector space.

If $f \in \mathcal{A}(X)[r_1]$ and $r_1 > r_2$ in $|\mathbb{C}_p^\times|_p$, since f is analytic on $D[X, r_1]$, its restriction on $D[X, r_2]$ is also analytic, so $f \in \mathcal{A}(X)[r_2]$. We get a natural map $\mathcal{A}(X)[r_1] \to \mathcal{A}(X)[r_2]$. This map is clearly continuous, injective and norm-decreasing, hence it induces the continuous dual map $\mathcal{D}(X)[r_2] \to \mathcal{D}(X)[r_1]$ which is also norm-decreasing.

Lemma 1.1.1. *For an open compact subset X of $\mathbb{Q}_p^d$ and for any $r_1 > r_2$ in $|\mathbb{C}_p^\times|_p$, the inclusion map $\mathcal{A}(X)[r_1] \to \mathcal{A}(X)[r_2]$ has dense image, so the dual map $\mathcal{D}(X)[r_2] \to \mathcal{D}(X)[r_1]$ is injective.*

Proof. Since X is open compact in $\mathbb{Q}_p^d$, we can write X as a disjoint union of finitely many closed polydiscs X_i of radius r_1 in $\mathbb{Q}_p^d$. A function in $\mathcal{A}(X)[r_1]$ is then determined uniquely by its restrictions on these polydiscs X_i. So the Banach space $\mathcal{A}(X)[r_1]$ is homeomorphic to the product of $\mathcal{A}(X_i)[r_1]$'s. Hence we can assume that X is a closed polydisc of radius r_1 in $\mathbb{Q}_p^d$. By affine transformations, every bounded closed polydisc in $\mathbb{Q}_p^d$ is homeomorphic to $\mathbb{Z}_p^d$, which is the unit closed polydisc. Therefore, we can assume that $r_1 = 1$ and $X = \mathbb{Z}_p^d$.

Since $\mathcal{A}(\mathbb{Z}_p^d)[1]$ contains all polynomials, it suffices to show that the set of polynomials in d variables is dense in $\mathcal{A}(\mathbb{Z}_p^d)[r_2]$. Let $f \in \mathcal{A}(\mathbb{Z}_p^d)[r_2]$. Regarding f as a function on $D[\mathbb{Z}_p^d, r_2] = D[\mathbb{Z}_p, r_2]^d$ and writing

$$f(z) = \sum_{a=(a_1,\dots,a_d)} (\mathbb{1}_{D(a,r_2)(\mathbb{C}_p)} f_a)(z) = \sum_{a=(a_1,\dots,a_d)} \mathbb{1}_{D(a_1,r_2)(\mathbb{C}_p)}(z_1) \cdots \mathbb{1}_{D(a_d,r_2)(\mathbb{C}_p)}(z_d) f_a(z_1, \dots, z_d),$$

where a runs through a finite set of representatives of $\mathbb{Z}_p^d$ for which the discs $D(a, r_2)$ cover $\mathbb{Z}_p^d$, and f_a is the restriction of f on $D(a, r_2)(\mathbb{C}_p)$. By (1.1), each function f_a can be approximated by polynomials in the variables $z_1, \dots, z_d$ which are linear combinations of products of the form $(z_1 - a_1)^{n_1} \cdots (z_d - a_d)^{n_d}$ for $n_1, \dots, n_d \in \mathbb{N}$. Therefore, f is the limit in $\mathcal{A}(\mathbb{Z}_p^d)[r_2]$ of a sequence of functions which are linear combinations of products of the form $\mathbb{1}_{D(a_1,r_2)(\mathbb{C}_p)}(z_1)(z_1 - a_1)^{n_1} \cdots \mathbb{1}_{D(a_d,r_2)(\mathbb{C}_p)}(z_d)(z_d - a_d)^{n_d}$. Since the space of polynomials in one variable is dense in $\mathcal{A}(\mathbb{Z}_p)[r_2]$ for all $r_2 \in |\mathbb{C}_p^\times|_p = p^{\mathbb{Q}}$ (if $r_2 \in p^{-\mathbb{N}}$, a result of Amice proved in [Colm, Théorème I.4.7] says that the binomial functions $z \mapsto \binom{z}{n}, n \in \mathbb{N}$ form an orthogonal basis of $\mathcal{A}(\mathbb{Z}_p)[r_2]$), we deduce that each function $\mathbb{1}_{D(a_i,r_2)(\mathbb{C}_p)}(z_i)(z_i - a_i)^{n_i}$ can be approximated by polynomials in one variable. Therefore, f is the limit in $\mathcal{A}(\mathbb{Z}_p^d)[r_2]$ of a sequence of polynomials in d variables. The injectivity of the dual map $\mathcal{D}(X)[r_2] \to \mathcal{D}(X)[r_1]$ is clear. □

Lemma 1.1.2. *For an open compact subset X of $\mathbb{Q}_p^d$, the inclusion map $\mathcal{A}(X)[r_1] \to \mathcal{A}(X)[r_2]$ is compact for any $r_1 > r_2$ in $|\mathbb{C}_p^\times|_p$. The dual map $\mathcal{D}(X)[r_2] \to \mathcal{D}(X)[r_1]$ is also compact.*

Proof. We can assume that $d = 1$. The compactness of the map on functions follows from [Bel, Lemma V.1.20]. The compactness of the map on distributions is implied from Schauder's lemma (see [Schn, Lemma 16.4]). □

Since X is compact, the space $\mathcal{A}(X)$ is the increasing union of the Banach spaces $\mathcal{A}(X)[r]$ when r decreases to 0, we then equip $\mathcal{A}(X)$ the locally convex final topology (see [Schn, §5E]). This is the finest locally convex topology for which all the inclusion maps $\mathcal{A}(X)[r] \to \mathcal{A}(X)$ are continuous.

Definition 1.1.3. *Let $\mathcal{D}(X)$ denote the continuous L-dual of $\mathcal{A}(X)$. We call it the space of p-adic distributions on X with values in L. The value of a distribution $\mu \in \mathcal{D}(X)$ at a function $f \in \mathcal{A}(X)$ is also written by $\int_X f(z)\mu(z)$ or $\int_X f(z)d\mu(z)$.*

If $Y \subset X$ is an open compact subset and if μ is a distribution on X, we denote by $\mu_{|Y}$ the restriction of μ on Y, i.e., $\mu_{|Y}$ is the distribution on Y given by

$$\int_Y f(z)\mu_{|Y}(z) := \int_X \mathbb{1}_Y(z) f(z)\mu(z)$$

for $f \in \mathcal{A}(Y)$, where $\mathbb{1}_Y$ is the characteristic function of Y.

The inclusion maps $\mathcal{A}(X)[r] \to \mathcal{A}(X)$ induce the dual maps from $\mathcal{D}(X)$ to $\mathcal{D}(X)[r]$, so $\mathcal{D}(X)$ is endowed with a family of norms $\{\|\cdot\|_r\}$ for $r \in |\mathbb{C}_p^\times|_p$.

Proposition 1.1.4. *The family of norms $\{\|\cdot\|_r\}$ makes $\mathcal{D}(X)$ into a Fréchet space. Moreover, $\mathcal{D}(X)$ is canonically isomorphic (as topological vector spaces) to the projective limit of $\mathcal{D}(X)[r]$'s endowed with its locally convex inductive limit topology. The natural maps $\mathcal{D}(X) \to \mathcal{D}(X)[r_2] \to \mathcal{D}(X)[r_1]$ are injective for any $r_1 > r_2$ in $|\mathbb{C}_p^\times|_p$.*

Proof. The first two statements are direct applications of the conclusions ii., iii. in [Schn, Prop. 16.10]. The assumptions of that proposition are satisfied since $\mathcal{A}(X)$ is an increasing union of $\mathcal{A}(X)[r]$'s when r decreases to 0, and the inclusion maps $\mathcal{A}(X)[r_1] \to \mathcal{A}(X)[r_2]$ for $r_1 > r_2$ are all compact by Lemma 1.1.2.

The natural map $\mathcal{D}(X)[r_2] \to \mathcal{D}(X)[r_1]$ for $r_1 > r_2$ is injective by Lemma 1.1.1. Since $\mathcal{D}(X)$ is the projective limit of $\mathcal{D}(X)[r]$'s for $r \in |\mathbb{C}_p^\times|_p$ and the transition maps are injective, the map $\mathcal{D}(X) \to \mathcal{D}(X)[r]$ is injective for any r. □

Definition 1.1.5. *For $u \geq 0$, a distribution $\mu \in \mathcal{D}(X)$ is said to be u-admissible (or u-tempered) or of order (of growth) $\leq u$ if there exists a constant $C > 0$ such that $\|\mu\|_r \leq Cr^{-u}$ as $r \to 0^+$. Equivalently, μ has order $\leq u$ if there is $C > 0$ such that for any $r \in |\mathbb{C}_p^\times|_p$ with $r \leq 1$ and any $f \in \mathcal{A}(X)[r]$, we have*

$$|\mu(f)|_p \leq Cr^{-u}\|f\|_r. \tag{1.4}$$

A distribution of order ≤ 0 is called a measure. The set of u-admissible distributions on X is denoted by $\mathcal{D}(X)_{\leq u}$.

Lemma 1.1.6. *A distribution $\mu \in \mathcal{D}(X)$ has order $\leq u$ if and only if there exists $C > 0$ such that μ satisfies (1.4) for any $r \in p^{-\mathbb{N}}$ and any $f \in \mathcal{A}(X)[r]$.*

Proof. Suppose μ satisfies (1.4) for any $r \in p^{-\mathbb{N}}$. Consider $r \in |\mathbb{C}_p^\times|_p$ with $r \leq 1$. Take $n \in \mathbb{N}$ such that $p^{-n-1} < r \leq p^{-n}$. If $f \in \mathcal{A}(X)[r]$, then $f \in \mathcal{A}(X)[p^{-n-1}]$, so

$$|\mu(f)|_p \leq Cp^{(n+1)u}\|f\|_{p^{-n-1}} \leq Cp^{(n+1)u}\|f\|_r \leq Cp^u r^{-u}\|f\|_r$$

(the first inequality follows from (1.4)). Therefore, $\|\mu\|_r \leq Cp^u r^{-u}$ for any $r \leq 1$, so μ has order $\leq u$. □

1.2 p-adic distributions on $\mathbb{Z}_p$

In this section we turn our attention to distributions on $\mathbb{Z}_p$ which is an open compact subset of $\mathbb{Q}_p$. We introduce an action of the monoid $\Sigma_0(p)$ on distributions in Subsection 1.2.1. In Subsection 1.2.2, we discuss admissible distributions and prove an useful lemma on a criterion for the vanishing of admissible distributions on $\mathbb{Z}_p^\times$ which will be used in the proof of the functional equation of p-adic L-functions attached to automorphic representations in Chapter 5 (see Lemma 1.2.5). The rest of the section is devoted to the Amice-Vélu and Mellin transforms of distributions on $\mathbb{Z}_p$.

In this section let k be an integer .

1.2.1 Actions of $\Sigma_0(p)$

We set

$$\Sigma_0(p) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_p) : p \nmid a, p \mid c, ad - bc \neq 0 \right\}.$$

For a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathrm{\mathbb{Q}_p})$, we put

$$\gamma^* = \det(\gamma) \cdot \gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Define the right weight k action of $\Sigma_0^*(p) = \{\gamma^*, \gamma \in \Sigma_0(p)\}$ on $\mathcal{A}(\mathbb{Z}_p)$ by

$$f_{|_k\gamma^*}(z) = (a - cz)^k f\Big(\frac{dz - b}{a - cz}\Big), \tag{1.5}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p)$ and $f \in \mathcal{A}(\mathbb{Z}_p)$. Note that $a - cz \in \mathbb{Z}_p^\times$ since $p|c$ and $p \not| a$.

Then $\Sigma_0(p)$ acts on $\mathcal{A}(\mathbb{Z}_p)$ on the left and on $\mathcal{D}(\mathbb{Z}_p)$ on the right, given by

$$\gamma \cdot_k f = f_{|_k\gamma^*} = (a - cz)^k f\Big(\frac{dz - b}{a - cz}\Big), \tag{1.6}$$

$$\mu_{|_k\gamma}(f) = \mu(\gamma \cdot_k f) = \mu(f_{|_k\gamma^*}) = \mu\Big((a - cz)^k f\Big(\frac{dz - b}{a - cz}\Big)\Big), \tag{1.7}$$

where $\mu \in \mathcal{D}(\mathbb{Z}_p), f \in \mathcal{A}(\mathbb{Z}_p), \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p)$. We denote $\mathcal{A}_k(\mathbb{Z}_p)$ (resp. $\mathcal{D}_k(\mathbb{Z}_p)$) the space $\mathcal{A}(\mathbb{Z}_p)$ (resp. $\mathcal{D}(\mathbb{Z}_p)$) endowed with the above weight k action of $\Sigma_0(p)$.

For $k \in \mathbb{N}$, let $\mathcal{P}_k$ denote the space of polynomials of degree $\leq k$ in one variable with coefficients in L, and denote by $\mathcal{V}_k$ its L-dual. Since $\mathcal{P}_k$ is embedded in $\mathcal{A}_k(\mathbb{Z}_p)$ which is stable by the action (1.6) of $\Sigma_0(p)$, the natural map $\mathcal{D}_k(\mathbb{Z}_p) \to \mathcal{V}_k$ given by the restriction of distributions to $\mathcal{P}_k$ is $\Sigma_0(p)$-equivariant, where $\mathcal{V}_k$ is endowed with action (1.7). Note that the action of $\Sigma_0(p)$ on $\mathcal{P}_k$ and $\mathcal{V}_k$ can be extended to an action of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Proposition 1.2.1 ([Bel, Prop. and Def. V.4.3])**.** *The formula* (1.6) *defines a continuous weight k action of $\Sigma_0(p)$ on $\mathcal{A}(\mathbb{Z}_p)[r]$ for any $r \in |\mathbb{C}_p^\times|_p$. If p^n divides exactly* $\det\gamma$ *and $p^n r < p$, then for any $f \in \mathcal{A}(\mathbb{Z}_p)[r]$, the function $\gamma \cdot_k f$ belongs to $\mathcal{A}(\mathbb{Z}_p)[p^n r]$. Moreover, $\|\gamma \cdot_k f\|_{p^n r} = \|f\|_r$. In particular, if* $\det\gamma = 1$ *and $r < p$, then γ acts by isometry on $\mathcal{A}(\mathbb{Z}_p)[r]$. We denote by $\mathcal{A}_k(\mathbb{Z}_p)[r]$ the space $\mathcal{A}(\mathbb{Z}_p)[r]$ endowed with this weight k action of* $\Sigma_0(p)$.

Corollary 1.2.2 ([Bel, Proposition V.4.5])**.** *The formula* (1.7) *defines a continuous weight k action of $\Sigma_0(p)$ on $\mathcal{D}(\mathbb{Z}_p)[r]$ for any $r \in |\mathbb{C}_p^\times|_p$. Assume $r < p$. Let $\mu \in \mathcal{D}(\mathbb{Z}_p)[r]$, $\gamma \in \Sigma_0(p)$ such that $p^n | \det\gamma$. Then $\mu_{|_k\gamma} \in \mathcal{D}(\mathbb{Z}_p)[r/p^n]$. If p^n divides exactly* $\det\gamma$*, then moreover $\left\|\mu_{|_k\gamma}\right\|_{r/p^n} = \|\mu\|_r$. In particular, if* $\det\gamma = 1$*, then γ acts by isometry on $\mathcal{D}(\mathbb{Z}_p)[r]$. We denote by $\mathcal{D}_k(\mathbb{Z}_p)[r]$ the space $\mathcal{D}(\mathbb{Z}_p)[r]$ endowed with this weight k action of* $\Sigma_0(p)$.

Remark 1.2.3. For $r_1 > r_2$ in $|\mathbb{C}_p^\times|_p$, the inclusion maps $\mathcal{A}(\mathbb{Z}_p)[r_1] \to \mathcal{A}(\mathbb{Z}_p)[r_2] \to \mathcal{A}(\mathbb{Z}_p)$ and its duals $\mathcal{D}(\mathbb{Z}_p) \to \mathcal{D}(\mathbb{Z}_p)[r_2] \to \mathcal{D}(\mathbb{Z}_p)[r_1]$ are $\Sigma_0(p)$-equivariant.

1.2.2 Admissible distributions

We now investigate more deeply admissible distributions on $\mathbb{Z}_p$. Let $u \geq 0$ and given a distribution $\mu \in \mathcal{D}(\mathbb{Z}_p)$ of order $\leq u$ (see Definition 1.1.5). For any $a \in \mathbb{Z}_p, n, j \in \mathbb{N}$, in (1.4) if we take f to be the function $\mathbb{1}_{a+p^n\mathbb{Z}_p} \cdot (z-a)^j$ which belongs to $\mathcal{A}(\mathbb{Z}_p)[p^{-n}]$, then there exists $C > 0$ such that

$$\left|\mu(\mathbb{1}_{a+p^n\mathbb{Z}_p} \cdot (z-a)^j)\right|_p \leq C p^{n(u-j)}.$$

Conversely, if there is $C > 0$ such that μ satisfies this inequality for any $a \in \mathbb{Z}_p, n, j \in \mathbb{N}$, then μ has order $\leq u$.

The following result says that an admissible distribution on $\mathbb{Z}_p$ is uniquely determined by its values on locally polynomial functions of bounded degree (the bound depends only on the order of distribution).

Theorem 1.2.4 (Vishik, Amice-Vélu)**.** *Let L be a finite extension of $\mathbb{Q}_p$ and $u \geq 0$.*

i) Let $\mu \in \mathcal{D}(\mathbb{Z}_p)$ be a distribution of order $\leq u$ with values in L and N be an integer greater or equal to the integral part of u. Then μ is uniquely determined by the linear forms for $a \in \mathbb{Z}_p$ and $n \in \mathbb{N}$:

$$i_{\mu,a+p^n\mathbb{Z}_p} : \mathcal{P}_N(L) \to L$$
$$P \mapsto \int_{a+p^n\mathbb{Z}_p} P(z)d\mu(z).$$

Here $\mathcal{P}_N(L)$ is the space of polynomials of degree less than N with coefficients in L.

ii) Conversely, suppose we are given, for every disc $a+p^n\mathbb{Z}_p$ in $\mathbb{Z}_p$, a linear form $i_{a+p^n\mathbb{Z}_p} : \mathcal{P}_N(L) \to L$ satisfying the additivity relation (for all $a \in \mathbb{Z}_p, n \in \mathbb{N}$):

$$i_{a+p^n\mathbb{Z}_p} = \sum_{i=0}^{p-1} i_{a+p^n i+p^{n+1}\mathbb{Z}_p},$$

and suppose there exist constants $C > 0$ and $u \geq 0$ such that for every $a \in \mathbb{Z}_p, j, n \in \mathbb{N}$ with $j \leq N$:

$$\left|i_{a+p^n\mathbb{Z}_p}\left((z-a)^j\right)\right|_p \leq C p^{n(u-j)}.$$

Then there exists a unique distribution μ on $\mathbb{Z}_p$ of order $\leq u$ such that $i_{\mu,a+p^n\mathbb{Z}_p} = i_{a+p^n\mathbb{Z}_p}$, and for any $n \in \mathbb{N}$ one has

$$\|\mu\|_{p^{-n}} \leq C p^{nu}.$$

Proof. See [Bel, Theorem V.2.13]. □

We now prove a useful lemma about a criterion for the vanishing of admissible distributions on $\mathbb{Z}_p^\times$, this is a generalization of [Vis, Lemma 2.10] (compare with the part i) of the above theorem).

Lemma 1.2.5. *Let $u \geq 0$, let μ be a distribution of order $\leq u$ on $\mathbb{Z}_p^\times$ such that $\mu(\chi z^j) = 0$ for any integer $0 \leq j \leq u$ and for all but finitely many Dirichlet characters $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$. Then $\mu = 0$.*

Proof. Denote $k = [u]$ the integer part of u. Suppose that $\mu(\chi z^j) = 0$ for $0 \leq j \leq k$ and for all characters χ of conductor $\geq p^{n_0+1}$ for some $n_0 \in \mathbb{N}$. Then for any $0 \leq j \leq k$ and any $n \geq n_0$, we have

$$\sum_{\chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee} \mu(\chi z^j) = \sum_{\mathrm{cond}(\chi) \leq p^n} \mu(\chi z^j) = \sum_{\mathrm{cond}(\chi) \leq p^{n_0}} \mu(\chi z^j) = \sum_{\chi \in (\mathbb{Z}/p^{n_0}\mathbb{Z})^\vee} \mu(\chi z^j), \tag{1.8}$$

where $(\mathbb{Z}/p^n\mathbb{Z})^\vee$ is the set of characters on $(\mathbb{Z}/p^n\mathbb{Z})^\times$ with values in $\mathbb{C}_p^\times$, and similar for $(\mathbb{Z}/p^{n_0}\mathbb{Z})^\vee$.

On the other hand,

$$\begin{aligned}\sum_{\chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee} \mu(\chi z^j) &= \mu\Big(\sum_{\chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee} \chi z^j\Big) = \mu\Big(\Big(\sum_{\chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee} \chi\Big) z^j\Big) \\ &= \mu\left(p^{n-1}(p-1)\mathbb{1}_{1+p^n\mathbb{Z}_p} \cdot z^j\right)\end{aligned}$$

since $\sum_{\chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee} \chi(z) = 0$ if $z \not\equiv 1 (\mathrm{mod}\, p^n)$ and $\sum_{\chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee} \chi(1) = |(\mathbb{Z}/p^n\mathbb{Z})^\times| = p^{n-1}(p-1)$. So for any $n \geq n_0$ we obtain

$$p^{n-1}(p-1)\mu\left(\mathbb{1}_{1+p^n\mathbb{Z}_p} \cdot z^j\right) = p^{n_0-1}(p-1)\mu\left(\mathbb{1}_{1+p^{n_0}\mathbb{Z}_p} \cdot z^j\right),$$

hence $\mu\left(\mathbb{1}_{1+p^n\mathbb{Z}_p} \cdot z^j\right) = p^{n_0-n}\mu\left(\mathbb{1}_{1+p^{n_0}\mathbb{Z}_p} \cdot z^j\right)$ for any $n \geq n_0$.

For each $a \in \mathbb{Z}_p^\times$, since $\mu\left(\chi\left(\frac{z}{a}\right) z^j\right) = \chi(a)^{-1}\mu(\chi z^j) = 0$ if $\mathrm{cond}(\chi) > p^{n_0}$, we replace χ by $\chi\left(\frac{z}{a}\right)$ in (1.8) to get

$$\mu\left(\mathbb{1}_{a+p^n\mathbb{Z}_p} \cdot z^j\right) = p^{n_0-n}\mu\left(\mathbb{1}_{a+p^{n_0}\mathbb{Z}_p} \cdot z^j\right) \tag{1.9}$$

for any $n \geq n_0$ and $0 \leq j \leq k$.

Since μ has order $\leq u$, there exists a constant $C > 0$ such that for any $a \in \mathbb{Z}_p^\times$ and $n, j \in \mathbb{N}$:

$$|\mu(\mathbb{1}_{a+p^n\mathbb{Z}_p} \cdot (z-a)^j)|_p \leq Cp^{n(u-j)}.$$

Take $j = k$, we have

$$|\mu(\mathbb{1}_{a+p^n\mathbb{Z}_p} \cdot (z-a)^k)|_p \leq Cp^{n(u-k)}.$$

Moreover, for any $a \in \mathbb{Z}_p^\times$ and $n \geq n_0$, we have

$$\begin{aligned}\mu\left(\mathbb{1}_{a+p^n\mathbb{Z}_p} \cdot (z-a)^k\right) &= \mu\Big(\mathbb{1}_{a+p^n\mathbb{Z}_p} \sum_{j=0}^{k} \binom{k}{j} (-a)^{k-j} z^j\Big) \\ &= \sum_{j=0}^{k} \binom{k}{j} (-a)^{k-j} p^{n_0-n}\mu\left(\mathbb{1}_{a+p^{n_0}\mathbb{Z}_p} \cdot z^j\right) \quad (\text{by } (1.9)) \\ &= p^{n_0-n}\mu\left(\mathbb{1}_{a+p^{n_0}\mathbb{Z}_p} \cdot (z-a)^k\right).\end{aligned}$$

So $|\mu(\mathbb{1}_{a+p^{n_0}\mathbb{Z}_p} \cdot (z-a)^k)|_p \leq Cp^{n_0+n(u-k-1)}$ for any $a \in \mathbb{Z}_p^\times$ and $n \geq n_0$. Since $u-k-1 < 0$, we infer that $\mu\left(\mathbb{1}_{a+p^{n_0}\mathbb{Z}_p} \cdot (z-a)^k\right) = 0$ for any $a \in \mathbb{Z}_p^\times$. Replace a by $a + p^{n_0}b$ for $b \in \mathbb{Z}_p$ with the note that $a + p^{n_0}b + p^{n_0}\mathbb{Z}_p = a + p^{n_0}\mathbb{Z}_p$, we obtain

$$\mu\left(\mathbb{1}_{a+p^{n_0}\mathbb{Z}_p} \cdot (z-a-p^{n_0}b)^k\right) = 0.$$

Therefore,

$$\begin{aligned}0 &= \mu\Big(\mathbb{1}_{a+p^{n_0}\mathbb{Z}_p} \sum_{j=0}^{k} \binom{k}{j} (-p^{n_0}b)^j (z-a)^{k-j}\Big) \\ &= \sum_{j=0}^{k} \binom{k}{j} (-p^{n_0})^j \mu\left(\mathbb{1}_{a+p^{n_0}\mathbb{Z}_p} \cdot (z-a)^{k-j}\right) \cdot b^j\end{aligned}$$

for any $b \in \mathbb{Z}_p$. Since the polynomial $\sum_{j=0}^{k} \binom{k}{j}(-p^{n_0})^j \mu\left(\mathbb{1}_{a+p^{n_0}\mathbb{Z}_p} \cdot (z-a)^{k-j}\right) X^j$ has infinitely many zeros $X \in \mathbb{Z}_p$, it must be zero. We deduce that

$$\mu\left(\mathbb{1}_{a+p^{n_0}\mathbb{Z}_p} \cdot (z-a)^j\right) = 0$$

for any $a \in \mathbb{Z}_p^\times$ and $0 \leq j \leq k$, so $\mu\left(\mathbb{1}_{a+p^{n_0}\mathbb{Z}_p} \cdot z^j\right) = 0$ for any $a \in \mathbb{Z}_p^\times$ and $0 \leq j \leq k$ since z^j is a linear combination of $(z-a)^i$ for $0 \leq i \leq j$. By (1.9), we obtain

$$\mu\left(\mathbb{1}_{a+p^{n}\mathbb{Z}_p} \cdot z^j\right) = 0$$

for any $a \in \mathbb{Z}_p^\times, n \geq n_0$ and $0 \leq j \leq k$, hence for any $n \in \mathbb{N}$ since every characteristic function $\mathbb{1}_{a+p^n\mathbb{Z}_p}$ for $0 \leq n < n_0$ is the sum of characteristic functions of smaller discs $b + p^m\mathbb{Z}_p$ for $m \geq n_0$. Therefore, μ vanishes on the space of locally polynomial functions of degree $\leq k$ on $\mathbb{Z}_p^\times$. Since μ has order $\leq u$, we conclude that $\mu = 0$ by Theorem 1.2.4i). □

1.2.3 Amice transform

Given $\mu \in \mathcal{D}(\mathbb{Z}_p)$, we associate its Amice transform $A_\mu(T)$ which is a formal power series defined by

$$A_\mu(T) = \sum_{n=0}^{+\infty} \left(\int_{\mathbb{Z}_p} \binom{z}{n} d\mu(z) \right) T^n = \int_{\mathbb{Z}_p} (1+T)^z d\mu(z).$$

We now describe explicitly the set of Amice transforms of all elements in $\mathcal{D}(\mathbb{Z}_p)$.

Denote by $\mathcal{R}$ the L-algebra of formal power series with coefficients in L which converge on the open unit disc of $\mathbb{C}_p$.

Proposition 1.2.6. *The map $\mu \mapsto A_\mu(T)$ is an isomorphism of L-vector spaces $\mathcal{D}(\mathbb{Z}_p)$ and $\mathcal{R}$.*

Proof. Given $\mu \in \mathcal{D}(\mathbb{Z}_p)$, we prove $A_\mu(T) \in \mathcal{R}$.

Let $T_0 \in \mathbb{C}_p$ such that $|T_0|_p < 1$, then $v_p(T_0) > 0$. Take $h \in \mathbb{N}$ such that $v_p(T_0) > \frac{1}{p^h(p-1)}$. For $n \in \mathbb{N}$, the binomial function $z \mapsto \binom{z}{n}$ has p^{-h}-norm $\left|[\frac{n}{p^h}]!\right|_p^{-1}$ by [Colm, Théorème I.4.7], where $[\cdot]$ denotes the integer part. In the view of μ as a distribution in $\mathcal{D}(\mathbb{Z}_p)[p^{-h}]$, there exists $C > 0$ such that $|\mu(f)|_p \leq C\|f\|_{p^{-h}}$ for any $f \in \mathcal{A}(\mathbb{Z}_p)[p^{-h}]$. Therefore,

$$\left|\int_{\mathbb{Z}_p} \binom{z}{n} d\mu(z)\right|_p = \left|\mu\left(\binom{z}{n}\right)\right|_p \leq C\left\|\binom{z}{n}\right\|_{p^{-h}} = C\left|[\frac{n}{p^h}]!\right|_p^{-1} = Cp^{v_p\left([\frac{n}{p^h}]!\right)}.$$

Since $v_p\left([\frac{n}{p^h}]!\right) = \frac{[\frac{n}{p^h}] - S_p\left([\frac{n}{p^h}]\right)}{p-1} \leq \frac{n}{p^h(p-1)}$ with $S_p\left([\frac{n}{p^h}]\right)$ is the sum of digits of expansion of $[\frac{n}{p^h}]$ in the base p, we obtain

$$\left|\int_{\mathbb{Z}_p} \binom{z}{n} d\mu(z)\right|_p \leq Cp^{\frac{n}{p^h(p-1)}}.$$

So $\left|\left(\int_{\mathbb{Z}_p} \binom{z}{n} d\mu(z)\right) T_0^n\right|_p \leq Cp^{\frac{n}{p^h(p-1)}} |T_0|_p^n = Cp^{n\left(\frac{1}{p^h(p-1)} - v_p(T_0)\right)} \underset{n \to +\infty}{\longrightarrow} 0$, hence $\mathcal{A}_\mu(T)$ converges at T_0. We conclude that $\mathcal{A}_\mu(T) \in \mathcal{R}$.

For each $h \in \mathbb{N}$ the binomial functions $\binom{z}{n}$ with $n \in \mathbb{N}$ form an orthogonal basis of $\mathcal{A}(\mathbb{Z}_p)[p^{-h}]$ by ibid., so μ is determined by its values at these functions. Hence the map $\mu \mapsto \mathcal{A}_\mu(T)$ is injective.

Conversely, given an element $A(T) = \sum_{n=0}^{+\infty} a_n T^n \in \mathcal{R}$, we prove that the linear form μ_A on $\mathcal{A}(\mathbb{Z}_p)$ given by $\mu_A\left(\binom{z}{n}\right) = a_n$ for each $n \in \mathbb{N}$ is a distribution, i.e., μ_A is continuous on $\mathcal{A}(\mathbb{Z}_p)$. Since $\mathcal{A}(\mathbb{Z}_p)$ has the locally convex final topology induced by the inclusion maps $\mathcal{A}(\mathbb{Z}_p)[r] \to \mathcal{A}(\mathbb{Z}_p)$ for $r \in |\mathbb{C}_p^\times|_p$, it suffices to show that μ_A is continuous on $\mathcal{A}(\mathbb{Z}_p)[p^{-h}]$ for any $h \in \mathbb{N}$. Since the set of functions $[\frac{n}{p^h}]!\binom{z}{n}$ for $n \in \mathbb{N}$ is an orthonormal basis of $\mathcal{A}(\mathbb{Z}_p)[p^{-h}]$ by ibid., we need to show the existence of a constant $C > 0$ (depends on h) such that

$$\left|\mu_A\left([\frac{n}{p^h}]!\binom{z}{n}\right)\right|_p = \left|[\frac{n}{p^h}]!\right|_p |a_n|_p \leq C$$

for all $n \in \mathbb{N}$. This is equivalent to

$$|a_n|_p \leq C\big|[\frac{n}{p^h}]!\big|_p^{-1} = Cp^{\frac{[\frac{n}{p^h}]-S_p([\frac{n}{p^h}])}{p-1}}. \tag{1.10}$$

Since the formal power series $\sum_{n=0}^{+\infty} a_n T^n$ converges in the open unit disc of $\mathbb{C}_p$, for any $r \in |\mathbb{C}_p^\times|_p$ with $r < 1$, we have $|a_n|_p \cdot r^n \to 0$ as $n \to +\infty$, so $|a_n|_p = o(r^{-n})$ for any $r < 1$. Take $r \in |\mathbb{C}_p^\times|_p$ such that $1 < r^{-1} < p^{\frac{1}{2}\frac{1}{p^h(p-1)}}$, we deduce the existence of a constant C satisfying (1.10). $\square$

We now find the set of Amice transforms of admissible distributions on $\mathbb{Z}_p$. For $u \geq 0$, denote by $\mathcal{R}_{\leq u} \subset \mathcal{R}$ the subset consisting of formal power series $\sum_{n=0}^{+\infty} a_n T^n$ such that $|a_n|_p = O(n^u)$.

Proposition 1.2.7. *A distribution $\mu \in \mathcal{D}(\mathbb{Z}_p)$ is of order $\leq u$ if and only if its Amice transform belongs to $\mathcal{R}_{\leq u}$.*

Proof. Let $\mu \in \mathcal{D}(\mathbb{Z}_p)$. By Lemma 1.1.6 and [Colm, Théorème I.4.7], μ has order $\leq u$ if and only if there exists $C > 0$ such that for any $n, h \in \mathbb{N}$:

$$\Big|\mu\left([\frac{n}{p^h}]!\binom{z}{n}\right)\Big|_p \leq Cp^{hu}. \tag{1.11}$$

Suppose μ has order $\leq u$. Then there is $C > 0$ such that

$$\Big|\mu\left(\binom{z}{n}\right)\Big|_p \leq C\big|[\frac{n}{p^h}]!\big|_p^{-1} p^{hu} = Cp^{\frac{[\frac{n}{p^h}]-S_p([\frac{n}{p^h}])}{p-1}} p^{hu} \leq Cp^{\frac{n}{p^h(p-1)}+hu}$$

for any $n, h \in \mathbb{N}$. Take $h = [\frac{\log n}{\log p}]$ we infer that $\big|\mu\left(\binom{z}{n}\right)\big|_p = O(n^u)$. Therefore, $A_\mu(T) \in \mathcal{R}_{\leq u}$.

Conversely, suppose $\mathcal{A}_\mu(T) \in \mathcal{R}_{\leq u}$. Take $C_1 > 0$ such that $\big|\mu\left(\binom{z}{n}\right)\big|_p \leq C_1 n^u$ for all $n \in \mathbb{N}$. The real function $x \in \mathbb{R}^+ \mapsto g(x) = \frac{n}{2x(p-1)} + u\frac{\log x}{\log p}$ attains its minimum at $x = \frac{n \log p}{2u(p-1)}$ with the value $u\frac{\log n}{\log p} + C_2$ for $C_2 = \frac{u}{\log p} + \frac{u}{\log p}[\log\log p - \log(2u(p-1))]$. For $n, h \in \mathbb{N}$, we have

$$\begin{aligned}
\Big|\mu\left([\frac{n}{p^h}]!\binom{z}{n}\right)\Big|_p &\leq C_1\big|[\frac{n}{p^h}]!\big|_p \cdot n^u = C_1 \cdot p^{-\frac{[\frac{n}{p^h}]-S_p([\frac{n}{p^h}])}{p-1}} n^u \leq C_1 \cdot p^{-\frac{\frac{n}{p^h}-1-S_p([\frac{n}{p^h}])}{p-1}} n^u \\
&= C_1 \cdot p^{\frac{1}{p-1}} \cdot p^{-\frac{n}{2p^h(p-1)}} \cdot p^{\frac{S_p([\frac{n}{p^h}])-\frac{n}{2p^h}}{p-1}} \cdot p^{u\frac{\log n}{\log p}} \\
&= C_1 \cdot p^{\frac{1}{p-1}} \cdot p^{\frac{S_p([\frac{n}{p^h}])-\frac{n}{2p^h}}{p-1}} \cdot p^{uh-g(p^h)+u\frac{\log n}{\log p}} \\
&\leq C_1 \cdot p^{\frac{1}{p-1}} \cdot p^{\frac{S_p([\frac{n}{p^h}])-\frac{n}{2p^h}}{p-1}} \cdot p^{uh-C_2}
\end{aligned}$$

(the last inequality comes from the fact $g(p^h) \geq u\frac{\log n}{\log p} + C_2$ for any $h \in \mathbb{N}$). Since $S_p(a) - \frac{a}{2} \leq \frac{p-1}{2}$ for any $a \in \mathbb{N}$, we get $\big|\mu\left([\frac{n}{p^h}]!\binom{z}{n}\right)\big|_p \leq Cp^{hu}$ with $C = C_1 \cdot p^{\frac{1}{p-1}+\frac{1}{2}-C_2}$, for any $n, h \in \mathbb{N}$. Since μ satisfies (1.11), it has order $\leq u$. $\square$

1.2.4 Mellin transform

We now study the $\mathbb{Z}_p^\times$-part (i.e. the restriction to $\mathbb{Z}_p^\times$) of distributions on $\mathbb{Z}_p$. Since $\mathbb{Z}_p^\times$ is compact and abelian, any locally analytic function on $\mathbb{Z}_p^\times$ is an infinite linear combination of characters $\mathbb{Z}_p^\times \to \mathbb{C}_p^\times$, so the restriction to $\mathbb{Z}_p^\times$ of a distribution on $\mathbb{Z}_p$ is uniquely determined by its values on the set of characters $\mathbb{Z}_p^\times \to \mathbb{C}_p^\times$.

Definition 1.2.8. *The (p-adic) weight space $\mathcal{W}$ is the rigid analytic space* $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{G}_m)$ *of continuous group homomorphisms from $\mathbb{Z}_p^\times$ to the multiplicative group $\mathbb{G}_m$. If A is a Banach algebra over $\mathbb{Q}_p$, the set of A-points of $\mathcal{W}$ is given by*

$$\mathcal{W}(A) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, A^\times).$$

We describe the rigid analytic structure of $\mathcal{W}$ by considering its $\mathbb{C}_p$-points. Let $\mathbf{q} = p$ if $p \neq 2$ and $\mathbf{q} = 4$ if $p = 2$. There is a canonical isomorphism

$$\begin{aligned} \mathbb{Z}_p^\times &\cong (\mathbb{Z}/\mathbf{q}\mathbb{Z})^\times \times (1 + \mathbf{q}\mathbb{Z}_p) \\ z &\mapsto (\omega(z), \langle z \rangle_p), \end{aligned}$$

where $\omega(z)$ is the Teichmüller lift of z mod p, that is the unique element in $\mathbb{Z}_p^\times$ congruent to z modulo p such that $\omega(z)^{\phi(\mathbf{q})} = 1$, and $\langle z \rangle_p = z/\omega(z)$.

From this isomorphism, every continuous character $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ is given by a character $\chi_1 : (\mathbb{Z}/\mathbf{q}\mathbb{Z})^\times \to \mathbb{C}_p^\times$ and a continuous character $\chi_2 : 1 + \mathbf{q}\mathbb{Z}_p \to \mathbb{C}_p^\times$. There are finitely many choices of χ_1. We consider χ_2. The multiplicative group $1 + \mathbf{q}\mathbb{Z}_p$ is isomorphic to the additive group $\mathbb{Z}_p$ by the p-adic logarithm map $\log_p$, so $1 + \mathbf{q}\mathbb{Z}_p$ is procyclic with a generator γ. Hence χ_2 is determined uniquely by its image at γ. Since χ_2 is continuous and $\gamma^{p^n} \to 1$ as $n \to +\infty$, it follows that $\chi_2(\gamma)^{p^n} \to 1$ and this condition is equivalent to $|\chi_2(\gamma) - 1|_p < 1$ (see [Schi, Theorem 32.2]). So χ_2 corresponds to an element in the open disc $B(1,1)(\mathbb{C}_p)$. Therefore, $\mathcal{W}(\mathbb{C}_p)$ is the finite disjoint union of components indexed by characters $(\mathbb{Z}/\mathbf{q}\mathbb{Z})^\times \to \mathbb{C}_p^\times$, each of them is homeomorphic to $B(1,1)(\mathbb{C}_p)$. This identification depends on the choice of uniformizer $\gamma \in 1 + \mathbf{q}\mathbb{Z}_p$.

Lemma 1.2.9 ([Bel, Theorem V.3.4])**.** *Any continuous character $\mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ is locally analytic as a function on $\mathbb{Z}_p^\times$.*

We can see each p-adic distribution on $\mathbb{Z}_p^\times$ as a function on $\mathcal{W}$.

Definition 1.2.10. *If $\mu \in \mathcal{D}(\mathbb{Z}_p)$ with values in L, we call its p-adic Mellin transform the function M_μ on $\mathcal{W}(L)$ given by: if $\chi \in \mathcal{W}(L) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, L^\times)$, putting*

$$M_\mu(\chi) = \int_{\mathbb{Z}_p^\times} \chi(z) d\mu_{|\mathbb{Z}_p^\times}(z).$$

We say that a distribution $\mu \in \mathcal{D}(\mathbb{Z}_p)$ has support in $\mathbb{Z}_p^\times$ if $\mu = \mu_{|\mathbb{Z}_p^\times}$, i.e. $\mu_{|p\mathbb{Z}_p} = 0$. The following result says that the map $\mu \mapsto M_\mu$ gives a characterization of distributions on $\mathbb{Z}_p$ supported in $\mathbb{Z}_p^\times$.

Proposition 1.2.11 (Vishik, Amice-Vélu)**.** *The p-adic Mellin transform gives a homeomorphism of Fréchet spaces between the space of L-valued distributions on $\mathbb{Z}_p$ with support in $\mathbb{Z}_p^\times$ and the space $\mathcal{R}'$ of L-analytic functions on the weight space $\mathcal{W}(L)$.*

Proof. This is Proposition II.2.2 in [AV], we explain here why Mellin transforms are analytic functions on the weight space. Let $\mu \in \mathcal{D}(\mathbb{Z}_p)$ with support in $\mathbb{Z}_p^\times$. In the view of the rigid analytic structure of $\mathcal{W}$, consider an arbitrary component $B(1,1)(L)$ of $\mathcal{W}(L)$ indexed by a character $\kappa : (\mathbb{Z}/\mathbf{q}\mathbb{Z})^\times \to L^\times$, for each element x belonging to this component, denote by χ_x the unique character $1 + \mathbf{q}\mathbb{Z}_p \to L^\times$ given by $\chi_x(\gamma) = x$, where γ is a fixed generator of $1 + \mathbf{q}\mathbb{Z}_p$. Consider the function $x \in B(1,1)(L) \mapsto M_\mu(x) = \int_{\mathbb{Z}_p^\times} (\kappa\chi_x)(z) d\mu_{|\mathbb{Z}_p^\times}(z)$. For every $R \in |\mathbb{C}_p^\times|_p$ such that $R < 1$, we show that $M_\mu(x)$ is analytic on the closed disc $D(1,R) \subset L$. We apply the following lemma.

Lemma 1.2.12. *The expression $\exp_p(\log_p(x)\log_p(z)/\log_p(\gamma))$ defines an analytic function in two variables $x \in L^\times, z \in \mathbb{Z}_p^\times$ such that $|x-1|_p < 1, |z-1|_p < 1, |\log_p(x)\log_p(z)|_p < p^{-1/(p-1)}|\log_p(\gamma)|_p$. Moreover, for x fixed in L satisfying $|x - 1|_p < 1$, we get an analytic function in z which equals χ_x for $z \in \mathbb{Z}_p^\times$ sufficiently close to 1.*

Proof. See the proof of [Bel, Theorem V.3.4]. □

From this lemma, we deduce that the function $(\kappa\chi_x)(z)$ is analytic on $x \in D(1,R) \subset L$ and on z belonging to closed discs of radius p^{-m} in $\mathbb{Z}_p^\times$ for $m \in \mathbb{N}$ big enough depending only on R (note that $|\log_p(z)|_p = |z - 1|_p$ if $|z - 1|_p < p^{\frac{-1}{p-1}}$ by [Bel, (V.3.2)]). We have

$$\int_{\mathbb{Z}_p^\times} (\kappa\chi_x)(z) d\mu_{|\mathbb{Z}_p^\times}(z) = \sum_{a=1}^{p^m-1} \int_{a+p^m\mathbb{Z}_p} (\kappa\chi_x)(z) d\mu_{|\mathbb{Z}_p^\times}(z) = \sum_{a=1}^{p^m-1} \kappa(a) \int_{a+p^m\mathbb{Z}_p} \chi_x(z) d\mu_{|\mathbb{Z}_p^\times}(z).$$

For each $1 \leq a \leq p^m - 1$ with $(a,p) = 1$, since the function $(x,z) \mapsto \chi_x(z)$ is analytic on $x \in D(1,R) \subset L$ and on $z \in a + p^m\mathbb{Z}_p \subset \mathbb{Z}_p^\times$, writing the restriction of $\chi_x(z)$ on $a + p^m\mathbb{Z}_p$ by $\sum_{n=0}^{+\infty} \alpha_{n,a}(z)(x-1)^n$ where

$\alpha_{n,a}(z)$ is analytic on $a+p^m\mathbb{Z}_p$ (seen as a function in $\mathcal{A}(\mathbb{Z}_p^\times)[p^{-m}]$ by extending by zero outside $a+p^m\mathbb{Z}_p$) such that $\|\alpha_{n,a}\|_{p^{-m}}R^n \to 0$ when $n \to +\infty$ for any a. Then

$$M_\mu(x) = \sum_{a=1}^{p^m-1} \Big[\kappa(a) \sum_{n=0}^{+\infty} \Big(\int_{a+p^m\mathbb{Z}_p} \alpha_{n,a}(z) d\mu_{|\mathbb{Z}_p^\times}(z)\Big)(x-1)^n\Big] = \sum_{n=0}^{+\infty} b_n (x-1)^n$$

for $x \in D(1,R)$, where $b_n = \sum_{a=1}^{p^m-1} \kappa(a) \int_{a+p^m\mathbb{Z}_p} \alpha_{n,a}(z) d\mu_{|\mathbb{Z}_p^\times}(z)$. It follows that

$$|b_n|_p \cdot R^n \leq \|\mu\|_{p^{-m}} \cdot \max\{\|\alpha_{n,a}\|_{p^{-m}} : 1 \leq a \leq p^m - 1\} \cdot R^n \to 0 \quad \text{when } n \to +\infty,$$

so the Mellin transform M_μ of μ is analytic on $D(1,R)$. Moreover,

$$\begin{aligned} \|M_\mu(x)_{|D(1,R)}\|_R &= \sup_{n\in\mathbb{N}} |b_n|_p \cdot R^n \leq \|\mu\|_{p^{-m}} \cdot \sup\{\|\alpha_{n,a}\|_{p^{-m}} R^n : 1 \leq a \leq p^m - 1, n \in \mathbb{N}\} \\ &= O(\|\mu\|_{p^{-m}}) \end{aligned}$$

for any distribution μ, where $R \in |\mathbb{C}_p^\times|_p$ with $R < 1$ is arbitrary and $m \in \mathbb{N}$ big enough depending on R. Hence the map $\mu \mapsto M_\mu$ is continuous between Fréchet spaces. □

We describe the Mellin transform of admissible distributions. Let $u \geq 0$ serving as the order of growth.

Definition 1.2.13. *An analytic function $f(x) = \sum_{n=0}^{+\infty} a_n(x-1)^n$ on $B(1,1)$ has order $\leq u$ if $|a_n|_p = O(n^u)$. A function on the weight space $\mathcal{W}$ has order $\leq u$ if its restrictions to components of $\mathcal{W}$ isomorphic to $B(1,1)$ have order $\leq u$.*

If $u \in \mathbb{N}$, then an analytic function f on $B(1,1)$ has order $\leq u$ if and only if $\sup_{|x-1|_p \leq R} |f(x)|_p = O(\sup_{|x-1|_p\leq R} |\log_p(x)^u|_p)$ as $R \to 1^-$. The following result is the content of Proposition II.2.4 in [AV], see [Vis, Theorem 2.3] for the proof of one direction.

Proposition 1.2.14 (Vishik, Amice-Vélu)**.** *A distribution μ on $\mathbb{Z}_p^\times$ has order $\leq u$ if and only if M_μ has order $\leq u$.*

Chapter 2

p-adic distributions on $\mathbb{P}^1(\mathbb{Q}_p)$

In this chapter, we consider p-adic distributions on the topological space $\mathbb{P}^1(\mathbb{Q}_p)$ which is no longer an open compact subset of $\mathbb{Q}_p^d$ for some d as in Chapter 1. The consideration of $\mathbb{P}^1(\mathbb{Q}_p)$ is more general than that of $\mathbb{Z}_p$ since $\mathbb{P}^1(\mathbb{Q}_p)$ can be seen as a gluing of two copies of $\mathbb{Z}_p$ overlapping on $\mathbb{Z}_p^\times$, where one copy is the neighborhood $\mathbb{Z}_p$ of 0 and the other one corresponds to the neighborhood $D(\infty, 0) := \{z \in \mathbb{P}^1(\mathbb{Q}_p), v_p(z) \leq 0\}$ of ∞, which is isomorphic to $\mathbb{Z}_p$ via the transformation $z \mapsto \frac{1}{z}$. We begin by introducing the definition of certain functions and distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ in Section 2.1. Then we define an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on these functions and distributions, and establish some results about an exact sequence and the zeroth homology group of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ involving distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ insprired by those for distributions on $\mathbb{Z}_p$ (see Proposition 2.2.4 and Theorem 2.2.11) in Section 2.2. We finish the chapter by discussing the notion of admissible distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ in Section 2.3.

We fix an integer k throughout this chapter. We extend the p-adic valuation v_p to $\mathbb{P}^1(\mathbb{Q}_p)$ by defining $v_p(\infty) = -\infty$.

2.1 Definition and the first results

In this section, we define various spaces of functions and distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ due to Pierre Colmez. We will see that these spaces of functions and distributions have a natural structure of locally convex topology which makes them into Fréchet spaces.

Definition 2.1.1 (Colmez)**.** *An L-valued function f on $\mathbb{P}^1(\mathbb{Q}_p)$ is said to be meromorphic at infinity with a pole of order $\leq k$ if f is of the form $f(z) = \sum\limits_{i=-\infty}^{k} a_i z^i$ on a neighborhood of ∞, where the sum $\sum\limits_{i=-\infty}^{-1} a_i z^i$ converges in this neighborhood. In other words, f is meromorphic at infinity with a pole of order $\leq k$ if the function $z^k f(\frac{1}{z})$ converges on a neighborhood of* 0.

Let $\mathcal{A}_k(\mathbb{P}^1, L)$ denote the space of L-valued functions f on $\mathbb{P}^1(\mathbb{Q}_p)$ such that f is locally analytic on $\mathbb{Q}_p$ and meromorphic with a pole of order $\leq k$ at ∞. If the role of L is less important, we will omit it from notations, e.g., we write simply $\mathcal{A}_k(\mathbb{P}^1)$ for $\mathcal{A}_k(\mathbb{P}^1, L)$.

Remark 2.1.2. The condition $\sum\limits_{i=-\infty}^{-1} a_i z^i$ converges in a neighborhood of ∞ means there is some $R > 0$ such that $\sum\limits_{i=-\infty}^{-1} a_i z^i$ converges in $\{z \in \mathbb{C}_p, |z|_p \geq R\}$, this is equivalent to $|a_i|_p R^i \to 0$ as $i \to -\infty$ (so that the function $\sum\limits_{i=-\infty}^{-1} a_i \left(\frac{1}{z}\right)^i = \sum\limits_{n=1}^{+\infty} a_{-n} z^n$ converges in $\{z \in \mathbb{C}_p, |z|_p \leq \frac{1}{R}\}$).

Definition 2.1.3. *For each $r \in |\mathbb{C}_p^\times|_p$ with $r < 1$, denote $\mathcal{A}_k(\mathbb{P}^1)[r] \subset \mathcal{A}_k(\mathbb{P}^1)$ the subspace of functions f such that f is analytic on every closed disc of radius r in $\mathbb{Q}_p$ and f is of the form $\sum\limits_{i=-\infty}^{k} a_i z^i$ on the neighborhood $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r}\}$ of ∞.*

Remark 2.1.4. If $r \in |\mathbb{C}_p^\times|_p, r < 1$ and if $a \in \mathbb{Q}_p$ such that $|a|_p < \frac{1}{r}$ (resp. $|a|_p \geq \frac{1}{r}$), then the closed disc $D(a,r)$ in $\mathbb{Q}_p$ is contained in $\{z \in \mathbb{Q}_p, |z|_p < \frac{1}{r}\}$ (resp. $\{z \in \mathbb{Q}_p, |z|_p \geq \frac{1}{r}\}$) since if $z \in D(a,r)$, then $|z-a|_p \leq r < \frac{1}{r}$, so $|z|_p < \frac{1}{r}$ if $|a|_p < \frac{1}{r}$, and $|z|_p = |a|_p \geq \frac{1}{r}$ if $|a|_p \geq \frac{1}{r}$.

Proposition 2.1.5. *For $r \in |\mathbb{C}_p^\times|_p, r < 1$, a function f belongs to $\mathcal{A}_k(\mathbb{P}^1)[r]$ if and only if f is analytic on every closed disc of radius r in $\{z \in \mathbb{Q}_p, |z|_p < \frac{1}{r}\}$ and f is of the form $\sum_{i=-\infty}^{k} a_i z^i$ converging on $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r}\}$.*

Proof. It suffices to prove that if $f(z) = \sum_{i=-\infty}^{k} a_i z^i$ for $z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r}$ with $r < 1$, then f is analytic on every closed disc of radius r in $\{z \in \mathbb{Q}_p, |z|_p \geq \frac{1}{r}\}$.

We can assume that $k = -1$. Consider a closed disc $D(a,r)$ in $\{z \in \mathbb{Q}_p, |z|_p \geq \frac{1}{r}\}$. For $i \in \mathbb{Z}_{<0}$, the function z^i is analytic on $D(a,r)$ with the Taylor expansion

$$\begin{aligned} z^i &= \Big(\frac{1}{(z-a)+a}\Big)^{-i} = \Big(\frac{1}{a(\frac{z-a}{a}+1)}\Big)^{-i} = \Big(\frac{1}{a}\sum_{n=0}^{+\infty}\Big(-\frac{z-a}{a}\Big)^n\Big)^{-i} \\ &= a^i \sum_{n=0}^{+\infty} \binom{n-i-1}{-i-1}\Big(-\frac{z-a}{a}\Big)^n, \end{aligned} \tag{2.1}$$

note that $|\frac{z-a}{a}|_p \leq \frac{r}{1/r} = r^2 < 1$. We need the folowing lemma given in [Was, page 53].

Lemma 2.1.6. *Let $P_m(X) = \sum_{n=0}^{+\infty} a_{n,m}X^n, m = 1, 2, ...$ be a sequence of power series which converge in a fixed subset D of $\mathbb{C}_p$ and suppose*

i) $a_{n,m} \to a_{n,0}$ as $m \to +\infty$ for each n, and

ii) for each $X \in D$ and every $\varepsilon > 0$ there exists an $n_0 = n_0(X, \varepsilon)$ such that $|a_{n,m}X^n|_p < \varepsilon$ for all $n \geq n_0$ and uniformly in m.

Then $\lim_{m\to+\infty} P_m(X) = P_0(X) = \sum_{n=0}^{+\infty} a_{n,0}X^n$ for all $X \in D$.

Applying this lemma for $P_m(z) = \sum_{i=-m}^{-1} a_i z^i, m = 1, 2, ...$ with the expansion (2.1) of z^i on the set $D = D(a,r)$. The coefficient of $(z-a)^n$ in the expansion of $P_m(z)$ is $a_{n,m} = \sum_{i=-m}^{-1} a_i a^i \binom{n-i-1}{-i-1}\left(-\frac{1}{a}\right)^n$. For fixed n, the sequence $\{a_{n,m}\}_{m=1}^{\infty}$ converges to $\sum_{i=-\infty}^{-1} a_i a^i \binom{n-i-1}{-i-1}\left(-\frac{1}{a}\right)^n$ as $m \to +\infty$ since $\left|a_i a^i \binom{n-i-1}{-i-1}\right|_p \leq |a_i a^i|_p \to 0$ as $i \to -\infty$ (the limit is 0 since the sum $\sum_{i=-\infty}^{-1} a_i z^i$ converges on $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r}\}$). The condition i) of Lemma 2.1.6 is satisfied.

Fix $z \in D(a,r)$ and $\varepsilon > 0$, we have

$$|a_{n,m}(z-a)^n|_p = \Big|\sum_{i=-m}^{-1} a_i a^i \binom{n-i-1}{-i-1}\Big(-\frac{z-a}{a}\Big)^n\Big|_p \leq \sup_{i<0}\Big\{|a_i|_p\Big(\frac{1}{r}\Big)^i\Big\}\Big|\frac{z-a}{a}\Big|_p^n < \varepsilon$$

for n big enough and for every m since $|\frac{z-a}{a}|_p \leq \frac{r}{1/r} = r^2 < 1$ and $\sup_{i<0} |a_i|_p \left(\frac{1}{r}\right)^i < +\infty$. The condition ii) of Lemma 2.1.6 is satisfied. We deduce that the function $\sum_{i=-\infty}^{-1} a_i z^i = \lim_{m\to+\infty} P_m(z)$ is analytic on $D(a,r)$. $\square$

If $f \in \mathcal{A}_k(\mathbb{P}^1)[r]$, then f is of the form $\sum_{i=-\infty}^{k} a_i z^i$ on $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r}\}$, so the function $g(z) = z^k f\left(\frac{1}{z}\right)$ is analytic on the closed disc $D(0,r) \subset \mathbb{Q}_p$. We identify the restriction of f to $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq$

$\frac{1}{r}\}$ with the function g on $D(0,r)$. On the open disc $B(0,\frac{1}{r})\subset\mathbb{Q}_p$, the function f is r-analytic. There is a canonical isomorphism of vector spaces:

$$\begin{aligned}\mathcal{A}_k(\mathbb{P}^1)[r] &\cong \mathcal{A}(D(0,r))[r]\times\mathcal{A}\Big(B\Big(0,\frac{1}{r}\Big)\Big)[r] \\ f &\mapsto \Big(z^k f\Big(\frac{1}{z}\Big)_{|D(0,r)}, f_{|B(0,\frac{1}{r})}\Big).\end{aligned} \tag{2.2}$$

We give $\mathcal{A}_k(\mathbb{P}^1)[r]$ the r-supremum norm $\|\cdot\|_r$ induced by that of the Banach space in the right hand side of this isomorphism, i.e., for $f\in\mathcal{A}_k(\mathbb{P}^1)[r]$ we define

$$\|f\|_r=\max\Big\{\Big\|z^k f\Big(\frac{1}{z}\Big)_{|D(0,r)}\Big\|_r, \big\|f_{|B(0,\frac{1}{r})}\big\|_r\Big\}. \tag{2.3}$$

Then $\mathcal{A}_k(\mathbb{P}^1)[r]$ becomes a Banach space. Let $\mathcal{D}_k(\mathbb{P}^1)[r]$ be the continuous L-dual of $\mathcal{A}_k(\mathbb{P}^1)[r]$ endowed with the dual norm which we still denote by $\|\cdot\|_r$, so $\mathcal{D}_k(\mathbb{P}^1)[r]$ is also a Banach space.

Proposition 2.1.7. *If $r_1,r_2\in|\mathbb{C}_p^\times|_p$ such that $1>r_1>r_2$, then $\mathcal{A}_k(\mathbb{P}^1)[r_1]\subset\mathcal{A}_k(\mathbb{P}^1)[r_2]$ and the inclusion map $\mathcal{A}_k(\mathbb{P}^1)[r_1]\to\mathcal{A}_k(\mathbb{P}^1)[r_2]$ is continuous. The space $\mathcal{A}_k(\mathbb{P}^1)$ is the union of the subspaces $\mathcal{A}_k(\mathbb{P}^1)[r]$ for $r\in|\mathbb{C}_p^\times|_p, r<1$.*

Proof. Suppose $f\in\mathcal{A}_k(\mathbb{P}^1)[r_1]$, then f is r_1-analytic on $\mathbb{Q}_p$ and f is of the form $\sum_{i=-\infty}^{k} a_i z^i$ on $\{z\in\mathbb{P}^1(\mathbb{Q}_p), |z|_p\geq\frac{1}{r_1}\}$, hence on $\{z\in\mathbb{P}^1(\mathbb{Q}_p), |z|_p\geq\frac{1}{r_2}\}$ since $\frac{1}{r_2}>\frac{1}{r_1}$. The condition f is r_1-analytic on $\mathbb{Q}_p$ implies that f is r_2-analytic on $\mathbb{Q}_p$. Therefore, $f\in\mathcal{A}_k(\mathbb{P}^1)[r_2]$. So $\mathcal{A}_k(\mathbb{P}^1)[r_1]$ is contained in $\mathcal{A}_k(\mathbb{P}^1)[r_2]$.

We prove $\|f\|_{r_2}\leq\frac{1}{r_2^k}\|f\|_{r_1}$ for any $f\in\mathcal{A}_k(\mathbb{P}^1)[r_1]$. Firstly,

$$\Big\|z^k f\Big(\frac{1}{z}\Big)_{|D(0,r_2)}\Big\|_{r_2}\leq\Big\|z^k f\Big(\frac{1}{z}\Big)_{|D(0,r_1)}\Big\|_{r_1}$$

since $r_2<r_1$. Secondly, on the open disc $B\big(0,\frac{1}{r_1}\big)$ of $\mathbb{Q}_p$, we have

$$\big\|f_{|B(0,\frac{1}{r_1})}\big\|_{r_2}\leq\|f_{|B(0,\frac{1}{r_1})}\|_{r_1}.$$

Finally, if $a\in\mathbb{Q}_p$ such that $\frac{1}{r_1}\leq|a|_p<\frac{1}{r_2}$ and if $z\in\mathbb{C}_p$ with $|z-a|_p\leq r_2$, then $|z-a|_p\leq r_2<1<\frac{1}{r_1}\leq|a|_p$, so $|z|_p=|a|_p$, hence $\frac{1}{r_1}\leq|z|_p<\frac{1}{r_2}$. We get

$$\begin{aligned}\|f_{|\{z\in\mathbb{Q}_p,\frac{1}{r_1}\leq|z|_p<\frac{1}{r_2}\}}\|_{r_2} &= \sup\Big\{|f(z)|_p : z\in\mathbb{C}_p, \exists\, a\in\mathbb{Q}_p \text{ such that } \frac{1}{r_1}\leq|a|_p<\frac{1}{r_2}, |z-a|_p\leq r_2\Big\} \\ &\leq \sup\Big\{|f(z)|_p : z\in\mathbb{C}_p, \frac{1}{r_1}\leq|z|_p<\frac{1}{r_2}\Big\} = \sup\Big\{\Big|f\Big(\frac{1}{z}\Big)\Big|_p : z\in\mathbb{C}_p, r_2<|z|_p\leq r_1\Big\} \\ &\leq \frac{1}{r_2^k}\sup\Big\{\Big|z^k f\Big(\frac{1}{z}\Big)\Big|_p : z\in\mathbb{C}_p, r_2<|z|_p\leq r_1\Big\} \\ &\leq \frac{1}{r_2^k}\sup\Big\{\Big|z^k f\Big(\frac{1}{z}\Big)\Big|_p : z\in\mathbb{C}_p, |z|_p\leq r_1\Big\} = \frac{1}{r_2^k}\Big\|z^k f\Big(\frac{1}{z}\Big)_{|D(0,r_1)}\Big\|_{r_1}.\end{aligned}$$

Combining all of the cases considered above we obtain $\|f\|_{r_2}\leq\max(1,r_2^{-k})\|f\|_{r_1}=r_2^{-k}\|f\|_{r_1}$. We deduce that the inclusion map $\mathcal{A}_k(\mathbb{P}^1)[r_1]\to\mathcal{A}_k(\mathbb{P}^1)[r_2]$ is continuous.

Now consider $f\in\mathcal{A}_k(\mathbb{P}^1)$. There exists $r'\in|\mathbb{C}_p^\times|_p, r'<1$ such that f is of the form $\sum_{i=-\infty}^{k} a_i z^i$ on $\{z\in\mathbb{P}^1(\mathbb{Q}_p), |z|_p\geq\frac{1}{r'}\}$, so f is analytic on every closed disc of radius r' contained in this set by Proposition 2.1.5. Since f is locally analytic on $\mathbb{Q}_p$ and the set $\{z\in\mathbb{Q}_p, |z|_p<\frac{1}{r'}\}$ is compact, there exists $0<r<r'$ such that f is analytic on every closed disc of radius r in $\{z\in\mathbb{Q}_p, |z|_p<\frac{1}{r'}\}$. So f is analytic on every closed disc of radius r in $\mathbb{Q}_p$. Moreover, $f(z)=\sum_{i=-\infty}^{k} a_i z^i$ on $\{z\in\mathbb{P}^1(\mathbb{Q}_p), |z|_p\geq\frac{1}{r}\}$ since $\frac{1}{r}>\frac{1}{r'}$. We deduce that $f\in\mathcal{A}_k(\mathbb{P}^1)[r]$. Therefore, $\mathcal{A}_k(\mathbb{P}^1)$ is the union of the subspaces $\mathcal{A}_k(\mathbb{P}^1)[r]$ for $r<1$. ∎

Definition 2.1.8. *We give $\mathcal{A}_k(\mathbb{P}^1)$ the locally convex final topology defined by the increasing union $\mathcal{A}_k(\mathbb{P}^1) = \bigcup_{r<1} \mathcal{A}_k(\mathbb{P}^1)[r]$ when r decreases to 0 (see [Schn, §5E]). This is the finest locally convex topology for which all the inclusion maps $\mathcal{A}_k(\mathbb{P}^1)[r] \to \mathcal{A}_k(\mathbb{P}^1)$ are continuous. We denote $\mathcal{D}_k(\mathbb{P}^1)$ the continuous L-dual of $\mathcal{A}_k(\mathbb{P}^1)$ and call it the space of L-valued p-adic distributions on $\mathbb{P}^1(\mathbb{Q}_p)$.*

Since the inclusion map $\mathcal{A}_k(\mathbb{P}^1)[r_1] \to \mathcal{A}_k(\mathbb{P}^1)[r_2]$ is continuous for any $r_1, r_2 \in |\mathbb{C}_p^\times|_p$ such that $1 > r_1 > r_2$, it induces the continuous L-dual map $\mathcal{D}_k(\mathbb{P}^1)[r_2] \to \mathcal{D}_k(\mathbb{P}^1)[r_1]$ on distributions on $\mathbb{P}^1(\mathbb{Q}_p)$.

Proposition 2.1.9. *The inclusion map $\mathcal{A}_k(\mathbb{P}^1)[r_1] \to \mathcal{A}_k(\mathbb{P}^1)[r_2]$ is compact for any $r_1, r_2 \in |\mathbb{C}_p^\times|_p$ such that $1 > r_1 > r_2$. The dual map $\mathcal{D}_k(\mathbb{P}^1)[r_2] \to \mathcal{D}_k(\mathbb{P}^1)[r_1]$ is also compact.*

Proof. By isomorphism (2.2) every function $f \in \mathcal{A}_k(\mathbb{P}^1)[r_1]$ is identified with a pair of r_1-analytic functions (f_∞, f_0), where $f_\infty(z) = z^k f\left(\frac{1}{z}\right)$ is defined on $D(0, r_1)$ and f_0 is the restriction of f to the open disc $B(0, \frac{1}{r_1})$. The r_1-norm of f is defined by the maximal of the r_1-norms of these two functions.

Consider a bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}_k(\mathbb{P}^1)[r_1]$. Since the sequence $\left\{z^k f_n\left(\frac{1}{z}\right)\right\}_{n\in\mathbb{N}}$ is analytic and bounded on $D(0, r_1)$ for r_1-norm, by Lemma 1.1.2, the sequence $\left\{z^k f_n\left(\frac{1}{z}\right)_{|D(0,r_2)}\right\}_{n\in\mathbb{N}}$ of the restriction to $D(0, r_2)$ is relatively compact for r_2-norm. On $\{z \in \mathbb{Q}_p, |z|_p < \frac{1}{r_1}\}$ the sequence $\{f_n\}$ is bounded for r_1-norm. By Remark 2.1.4, if $a \in \mathbb{Q}_p$ such that $\frac{1}{r_1} \le |a|_p < \frac{1}{r_2}$, and if $z \in \mathbb{C}_p$ with $|z - a|_p \le r_1$, then $\frac{1}{r_1} \le |z|_p < \frac{1}{r_2}$. On $\{z \in \mathbb{Q}_p, \frac{1}{r_1} \le |z|_p < \frac{1}{r_2}\}$, we have

$$\begin{aligned}
&\sup\{\|f_{n|\{z\in\mathbb{Q}_p, \frac{1}{r_1}\le|z|_p<\frac{1}{r_2}\}}\|_{r_1} : n \in \mathbb{N}\} \\
&= \sup\left\{|f_n(z)|_p : n \in \mathbb{N}, z \in \mathbb{C}_p, \exists\, a \in \mathbb{Q}_p \text{ such that } \frac{1}{r_1} \le |a|_p < \frac{1}{r_2}, |z - a|_p \le r_1\right\} \\
&\le \sup\left\{|f_n(z)|_p : n \in \mathbb{N}, z \in \mathbb{C}_p, \frac{1}{r_1} \le |z|_p < \frac{1}{r_2}\right\} = \sup\left\{\left|f_n\left(\frac{1}{z}\right)\right|_p : n \in \mathbb{N}, z \in \mathbb{C}_p, r_2 < |z|_p \le r_1\right\} \\
&\le \frac{1}{r_2^k} \sup\left\{\left|z^k f_n\left(\frac{1}{z}\right)\right|_p : n \in \mathbb{N}, z \in \mathbb{C}_p, r_2 < |z|_p \le r_1\right\} \\
&\le \frac{1}{r_2^k} \sup\left\{\left|z^k f_n\left(\frac{1}{z}\right)\right|_p : n \in \mathbb{N}, z \in \mathbb{C}_p, |z|_p \le r_1\right\} = \frac{1}{r_2^k} \sup\left\{\left\|z^k f_n\left(\frac{1}{z}\right)_{|D(0,r_1)}\right\|_{r_1} : n \in \mathbb{N}\right\} < +\infty.
\end{aligned}$$

So the sequence $\{f_n\}$ is bounded for r_1-norm on $\{z \in \mathbb{Q}_p, |z|_p < \frac{1}{r_2}\}$. By Lemma 1.1.2, the sequence of restriction of f_n's to $\{z \in \mathbb{Q}_p, |z|_p < \frac{1}{r_2}\}$ is relatively compact for r_2-norm. We conclude that $\{f_n\}$ is relatively compact as a sequence in $\mathcal{A}_k(\mathbb{P}^1)[r_2]$. Hence the inclusion map $\mathcal{A}_k(\mathbb{P}^1)[r_1] \to \mathcal{A}_k(\mathbb{P}^1)[r_2]$ is compact. The compactness of the dual map on distributions follows from Schauder's lemma. □

Proposition 2.1.10. *For any $r_1, r_2 \in |\mathbb{C}_p^\times|_p$ such that $r_1 > r_2$, the inclusion map $\mathcal{A}_k(\mathbb{P}^1)[r_1] \to \mathcal{A}_k(\mathbb{P}^1)[r_2]$ has dense image.*

Proof. Let $f \in \mathcal{A}_k(\mathbb{P}^1)[r_2]$ and $\varepsilon > 0$. Since f is r_2-analytic on $\mathbb{Q}_p$, it is r_2-analytic on the open disc $B\big(0, \frac{1}{r_1}\big)$ of $\mathbb{Q}_p$. By Lemma 1.1.1, there exists an r_1-analytic function g_1 on $B\big(0, \frac{1}{r_1}\big)$ such that $\big\|g_1 - f_{|B(0,\frac{1}{r_1})}\big\|_{r_2} < \varepsilon$. We use the following lemma.

Lemma 2.1.11. *The transformation $z \mapsto \frac{1}{z}$ is a homeomorphism between $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \ge \frac{1}{r_1}\}$ and $\{z \in \mathbb{Q}_p, |z|_p \le r_1\}$. It maps $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \ge \frac{1}{r_2}\}$ onto $\{z \in \mathbb{Q}_p, |z|_p \le r_2\}$ and maps each closed disc $D(a, r_2) \subset \{z \in \mathbb{Q}_p, \frac{1}{r_1} \le |z|_p < \frac{1}{r_2}\}$ onto the closed disc $D\big(\frac{1}{a}, \frac{r_2}{|a|_p^2}\big)$ contained in $\{z \in \mathbb{Q}_p, r_2 < |z|_p \le r_1\}$, where $r_1, r_2 \in |\mathbb{C}_p^\times|_p$ such that $1 > r_1 > r_2$. We have the similar statements if we take the variable z in $\mathbb{C}_p$, not only in $\mathbb{Q}_p$.*

Proof. Suppose $a \in \mathbb{Q}_p$ such that $\frac{1}{r_1} \le |a|_p < \frac{1}{r_2}$ and $z \in D(a, r_2)$ ($z \in \mathbb{C}_p$ or $\mathbb{Q}_p$). By Remark 2.1.4, $|z|_p = |a|_p$. We have

$$\left|\frac{1}{z} - \frac{1}{a}\right|_p = \frac{|z - a|_p}{|az|_p} = \frac{|z - a|_p}{|a|_p^2} \le \frac{r_2}{|a|_p^2},$$

so $\frac{1}{z} \in D\big(\frac{1}{a}, \frac{r_2}{|a|_p^2}\big)$.

Conversely, if $x \in D\big(\frac{1}{a}, \frac{r_2}{|a|_p^2}\big)$, let $z = \frac{1}{x}$, we prove $z \in D(a, r_2)$. Since $|\frac{1}{z} - \frac{1}{a}|_p \leq \frac{r_2}{|a|_p^2} < |\frac{1}{a}|_p$ ($r_2 < 1 < \frac{1}{r_1} \leq |a|_p$), we deduce that $|\frac{1}{z}|_p = |\frac{1}{a}|_p$, so $|z|_p = |a|_p$. Therefore,

$$\frac{r_2}{|a|_p^2} \geq \Big|\frac{1}{z} - \frac{1}{a}\Big|_p = \frac{|z-a|_p}{|az|_p} = \frac{|z-a|_p}{|a|_p^2},$$

hence $|z-a|_p \leq r_2$ and $z \in D(a, r_2)$. The lemma is proven. □

Returning to the proof of the proposition. Since f is of the form $\sum_{i=-\infty}^{k} a_i z^i$ on $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r_2}\}$, the function $z^k f\big(\frac{1}{z}\big)$ is analytic on the closed disc $D(0, r_2)$ of $\mathbb{Q}_p$.

Since the set $\{z \in \mathbb{Q}_p, \frac{1}{r_1} \leq |z|_p < \frac{1}{r_2}\}$ is compact, it is the disjoint union of finitely many closed discs $D(a, r_2)$ of radius r_2. By the lemma, since f is analytic on each such $D(a, r_2)$ (f is r_2-analytic on $\mathbb{Q}_p$), the function $z^k f\left(\frac{1}{z}\right)$ is analytic on $D\big(\frac{1}{a}, \frac{r_2}{|a|_p^2}\big)$.

By the lemma, the closed disc $D(0, r_1)$ of $\mathbb{Q}_p$ is partitioned by the closed disc $D(0, r_2)$ and finitely many closed discs $D\big(\frac{1}{a}, \frac{r_2}{|a|_p^2}\big)$, where the closed discs $D(a, r_2)$ form a partition of $\{z \in \mathbb{Q}_p, \frac{1}{r_1} \leq |z|_p < \frac{1}{r_2}\}$. We have a family of analytic functions $\{z^k f\left(\frac{1}{z}\right)_{|D(0,r_2)}, z^k f\left(\frac{1}{z}\right)_{|D\left(\frac{1}{a}, \frac{r_2}{|a|_p^2}\right)}\}_a$ on the components of this partition of $D(0, r_1)$. Since $D(0, r_1)$ is an open compact subset of $\mathbb{Q}_p$, by an argument similar to Lemma 1.1.1, there exists an analytic function h on $D(0, r_1)$ such that

$$\Big\|\Big(h(z) - z^k f\Big(\frac{1}{z}\Big)\Big)_{|D(0,r_2)}\Big\|_{r_2} < \varepsilon \quad \text{and} \quad \Big\|\Big(h(z) - z^k f\Big(\frac{1}{z}\Big)\Big)_{|D\left(\frac{1}{a}, \frac{r_2}{|a|_p^2}\right)}\Big\|_{\frac{r_2}{|a|_p^2}} < r_2^k \varepsilon.$$

Putting $g_2(z) = z^k h\left(\frac{1}{z}\right)$, then g_2 is of the form $\sum_{i=-\infty}^{k} b_i z^i$ on $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r_1}\}$. For every closed disc $D(a, r_2)$ contained in $\{z \in \mathbb{Q}_p, \frac{1}{r_1} \leq |z|_p < \frac{1}{r_2}\}$, we have

$$\begin{aligned}
\|(g_2 - f)_{|D(a,r_2)}\|_{r_2} &= \sup_{z \in \mathbb{C}_p, |z-a|_p \leq r_2} |g_2(z) - f(z)|_p = \sup_{z \in \mathbb{C}_p, |z - \frac{1}{a}|_p \leq \frac{r_2}{|a|_p^2}} \Big|g_2\Big(\frac{1}{z}\Big) - f\Big(\frac{1}{z}\Big)\Big|_p \\
&\leq \frac{1}{r_2^k} \cdot \sup_{z \in \mathbb{C}_p, |z - \frac{1}{a}|_p \leq \frac{r_2}{|a|_p^2}} \Big|z^k g_2\Big(\frac{1}{z}\Big) - z^k f\Big(\frac{1}{z}\Big)\Big|_p \\
&= \frac{1}{r_2^k} \Big\|\Big(h(z) - z^k f\Big(\frac{1}{z}\Big)\Big)_{|D\left(\frac{1}{a}, \frac{r_2}{|a|_p^2}\right)}\Big\|_{\frac{r_2}{|a|_p^2}} < \varepsilon.
\end{aligned}$$

Let g be the function on $\mathbb{P}^1(\mathbb{Q}_p)$ given by $g = g_1$ on $B(0, \frac{1}{r_1}) \subset \mathbb{Q}_p$ and $g = g_2$ on $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r_1}\}$. Then $g \in \mathcal{A}_k(\mathbb{P}^1)[r_1]$ and

$$\begin{aligned}
&\|g - f\|_{r_2} \\
&= \max\Big\{\Big\|\Big(z^k g\Big(\frac{1}{z}\Big) - z^k f\Big(\frac{1}{z}\Big)\Big)_{|D(0,r_2)}\Big\|_{r_2}, \|(g - f)_{|D(a,r_2)}\|_{r_2} : D(a, r_2) \subset \Big\{z \in \mathbb{Q}_p, |z|_p < \frac{1}{r_2}\Big\}\Big\} \\
&= \max\Big\{\Big\|\Big(h(z) - z^k f\Big(\frac{1}{z}\Big)\Big)_{|D(0,r_2)}\Big\|_{r_2}, \|(g_2 - f)_{|D(a,r_2)}\|_{r_2} : D(a, r_2) \subset \Big\{z \in \mathbb{Q}_p, \frac{1}{r_1} \leq |z|_p < \frac{1}{r_2}\Big\}, \\
&\qquad , \|(g_1 - f)_{|B(0, \frac{1}{r_1})}\|_{r_2}\Big\} \\
&< \varepsilon.
\end{aligned}$$

Therefore, $\mathcal{A}_k(\mathbb{P}^1)[r_1]$ is dense in $\mathcal{A}_k(\mathbb{P}^1)[r_2]$. □

Corollary 2.1.12. *The dual map* $\mathcal{D}_k(\mathbb{P}^1)[r_2] \to \mathcal{D}_k(\mathbb{P}^1)[r_1]$ *is injective for any* $r_1, r_2 \in |\mathbb{C}_p^\times|_p$ *such that* $1 > r_1 > r_2$.

Proof. It is immediate from Proposition 2.1.10. □

The inclusion maps $\mathcal{A}_k(\mathbb{P}^1)[r] \to \mathcal{A}_k(\mathbb{P}^1)$ for $r \in |\mathbb{C}_p^\times|_p, r < 1$ induce the dual maps from $\mathcal{D}_k(\mathbb{P}^1)$ to $\mathcal{D}_k(\mathbb{P}^1)[r]$, so $\mathcal{D}_k(\mathbb{P}^1)$ is endowed with a family of norms $\{\|\cdot\|_r : r \in |\mathbb{C}_p^\times|_p, r < 1\}$, where $\mathcal{D}_k(\mathbb{P}^1)[r]$ is endowed with the dual norm of $\|\cdot\|_r$ on $\mathcal{A}_k(\mathbb{P}^1)[r]$ defined by (2.3).

Corollary 2.1.13. *This family of norms makes $\mathcal{D}_k(\mathbb{P}^1)$ into a Fréchet space. Moreover, $\mathcal{D}_k(\mathbb{P}^1)$ is canonically isomorphic (as topological vector spaces) to the projective limit of $\mathcal{D}_k(\mathbb{P}^1)[r]$'s, endowed with its locally convex inductive limit topology (see [Schn, §5D]). The natural maps $\mathcal{D}_k(\mathbb{P}^1) \to \mathcal{D}_k(\mathbb{P}^1)[r_2] \to \mathcal{D}_k(\mathbb{P}^1)[r_1]$ are injective for any $1 > r_1 > r_2$ in $|\mathbb{C}_p^\times|_p$.*

Proof. The first two statements are direct applications of the conclusions ii., iii. in [Schn, Prop. 16.10]. The assumptions of that proposition are satisfied since $\mathcal{A}_k(\mathbb{P}^1)$ is an increasing union of $\mathcal{A}_k(\mathbb{P}^1)[r]$'s when r decreases to 0 by Proposition 2.1.7, and the inclusion maps $\mathcal{A}_k(\mathbb{P}^1)[r_1] \to \mathcal{A}_k(\mathbb{P}^1)[r_2]$ are compact for all $1 > r_1 > r_2$ by Proposition 2.1.9. The injectivity of the map between distributions is implied from Corollary 2.1.12. ☐

2.2 Actions of $\mathrm{GL}_2(\mathbb{Q}_p)$ and further results

In this section, after defining an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathcal{A}_k(\mathbb{P}^1)$ and $\mathcal{D}_k(\mathbb{P}^1)$ (see (2.4) and (2.5)), we set up some exact sequences involving functions and distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ (see Lemma 2.2.2 and Proposition 2.2.4) which will be used to study overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{P}^1)$ in Chapter 3. We compute the zeroth homology group of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ with values in $\mathcal{D}_k(\mathbb{P}^1)$ in Theorem 2.2.11. The results in this section are analogous versions for $\mathbb{P}^1(\mathbb{Q}_p)$ of those for $\mathbb{Z}_p$.

One of the advantages of the space $\mathbb{P}^1(\mathbb{Q}_p)$ is that it admits an action of any matrix in $\mathrm{GL}_2(\mathbb{Q}_p)$ via linear fractional transformations. The weight k action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathcal{A}_k(\mathbb{P}^1)$ and $\mathcal{D}_k(\mathbb{P}^1)$ is defined similarly to (1.5), (1.7):

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$, $f \in \mathcal{A}_k(\mathbb{P}^1)$ and $\mu \in \mathcal{D}_k(\mathbb{P}^1)$, we set

$$f_{|_k\gamma}(z) = (cz+d)^k f\Big(\frac{az+b}{cz+d}\Big), \tag{2.4}$$

$$\mu_{|_k\gamma}(f) = \mu(f_{|_k\gamma^*}) = \mu\Big((a-cz)^k f\Big(\frac{dz-b}{a-cz}\Big)\Big), \tag{2.5}$$

where $\gamma^* = \det\gamma \cdot \gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Let us consider the derivatives of functions in $\mathcal{A}_k(\mathbb{P}^1)$.

Lemma 2.2.1. *If $k \in \mathbb{N}$ and $f \in \mathcal{A}_k(\mathbb{P}^1)[r]$ for $r \in |\mathbb{C}_p^\times|_p, r < 1$, then its $(k+1)$-th derivative $\frac{d^{k+1}f}{dz^{k+1}}$ belongs to $\mathcal{A}_{-k-2}(\mathbb{P}^1)[r]$. The map $\left(\frac{d}{dz}\right)^{k+1} : \mathcal{A}_k(\mathbb{P}^1)[r] \to \mathcal{A}_{-k-2}(\mathbb{P}^1)[r]$ is continuous.*

Proof. Suppose $f \in \mathcal{A}_k(\mathbb{P}^1)[r]$ for $r \in |\mathbb{C}_p^\times|_p, r < 1$. Since f is locally analytic on $\mathbb{Q}_p$, it is C^∞-differentiable on $\mathbb{Q}_p$. The function f is of the form

$$f(z) = \sum_{i=-\infty}^{k} a_i z^i$$

on $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \ge \frac{1}{r}\}$, where the coefficients a_i satisfy the condition $|a_i|_p\left(\frac{1}{r}\right)^i \to 0$ as $i \to -\infty$ (see Remark 2.1.2)). Then $\frac{d^{k+1}f}{dz^{k+1}} = d^{k+1}(\sum_{i<0} a_i z^i)$ for $z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \ge \frac{1}{r}$. The function

$$\sum_{i<0} i(i-1)...(i-k)\, a_i z^{i-k-1}$$

is convergent on $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \ge \frac{1}{r}\}$ since

$$|i(i-1)...(i-k)\, a_i|_p\Big(\frac{1}{r}\Big)^{i-k-1} \le |a_i|_p\Big(\frac{1}{r}\Big)^{i}\Big(\frac{1}{r}\Big)^{-k-1} \to 0 \text{ as } i \to -\infty.$$

By the proof of Proposition 2.1.7, the function $\sum_{i<0} a_i z^i$ has the Taylor expansion

$$\sum_{n=0}^{+\infty}\Big(\sum_{i<0} a_i a^i \binom{n-i-1}{-i-1}\Big)\big(-\frac{z-a}{a}\big)^n$$

around each point $a \in \mathbb{Q}_p$ with $|a|_p \geq \frac{1}{r}$, hence $\frac{d^{k+1}f}{dz^{k+1}}$ and $\sum\limits_{i<0} i(i-1)...(i-k)\, a_i z^{i-k-1}$ have the same Taylor expansion around such those points a. We deduce that

$$\frac{d^{k+1}f}{dz^{k+1}} = \sum_{i<0} i(i-1)...(i-k)\, a_i z^{i-k-1}$$

on $\{z \in \mathbb{Q}_p, |z|_p \geq \frac{1}{r}\}$. On the other hand, $\frac{d^{k+1}f}{dz^{k+1}}$ is r-analytic on $\mathbb{Q}_p$ since f is. Therefore, $\frac{d^{k+1}f}{dz^{k+1}} \in \mathcal{A}_{-k-2}(\mathbb{P}^1)[r]$.

We compare the r-norms of f and $\frac{d^{k+1}f}{dz^{k+1}}$. We have

$$\begin{aligned}\Big\|z^{-k-2} \cdot \frac{d^{k+1}f}{dz^{k+1}}\Big(\frac{1}{z}\Big)_{|D(0,r)}\Big\|_r &= \sup_{i<0} |i...(i-k)a_i|_p \Big(\frac{1}{r}\Big)^{i+1} \leq \sup_{i<0} |a_i|_p \Big(\frac{1}{r}\Big)^{i+1} \\ &\leq \sup_{i\leq k} |a_i|_p \Big(\frac{1}{r}\Big)^{i+1} = \Big(\frac{1}{r}\Big)^{k+1} \Big\|z^k f\Big(\frac{1}{z}\Big)_{|D(0,r)}\Big\|_r.\end{aligned}$$

On each closed disc $D(a,r)$ inside $B(0,\frac{1}{r}) \subset \mathbb{Q}_p$, if f has Taylor expansion $\sum\limits_{n=0}^{+\infty} \alpha_n (z-a)^n$, then $\frac{d^{k+1}f}{dz^{k+1}} = \sum\limits_{n=k+1}^{+\infty} n(n-1)...(n-k)\alpha_n(z-a)^{n-k-1}$ on $D(a,r)$. Therefore,

$$\Big\|\frac{d^{k+1}f}{dz^{k+1}}_{|D(a,r)}\Big\|_r = \sup_{n\geq k+1} |n...(n-k)\alpha_n|_p \cdot r^{n-k-1} \leq \sup_{n\geq 0} |\alpha_n|_p \cdot r^{n-k-1} = r^{-k-1}\|f_{|D(a,r)}\|_r.$$

We conclude that $\big\|\frac{d^{k+1}f}{dz^{k+1}}\big\|_r \leq \frac{1}{r^{k+1}}\|f\|_r$ for any $f \in \mathcal{A}_k(\mathbb{P}^1)[r]$, hence the map $\big(\frac{d}{dz}\big)^{k+1} : \mathcal{A}_k(\mathbb{P}^1)[r] \to \mathcal{A}_{-k-2}(\mathbb{P}^1)[r]$ is continuous. □

Consider the map $\big(\frac{d}{dz}\big)^{k+1} : \mathcal{A}_k(\mathbb{P}^1, L) \to \mathcal{A}_{-k-2}(\mathbb{P}^1, L)$ for $k \in \mathbb{N}$. Its kernel is obviously the space $\mathcal{P}_k^{\dagger}(L)$ of locally polynomial functions of degree $\leq k$ on $\mathbb{P}^1(\mathbb{Q}_p)$ with coefficients in L. We obtain the following complex which is left exact for each $k \in \mathbb{N}$:

$$0 \to \mathcal{P}_k^{\dagger}(L) \xrightarrow{i} \mathcal{A}_k(\mathbb{P}^1, L) \xrightarrow{(\frac{d}{dz})^{k+1}} \mathcal{A}_{-k-2}(\mathbb{P}^1, L) \to 0, \tag{2.6}$$

where i is the inclusion map. It turns out that this complex is exact.

Lemma 2.2.2. *The sequence* (2.6) *is exact, i.e., the map* $\big(\frac{d}{dz}\big)^{k+1} : \mathcal{A}_k(\mathbb{P}^1) \to \mathcal{A}_{-k-2}(\mathbb{P}^1)$ *is surjective. Moreover, this map is continuous and open.*

Proof. Let $g \in \mathcal{A}_{-k-2}(\mathbb{P}^1)$, then $g \in \mathcal{A}_{-k-2}(\mathbb{P}^1)[r]$ for some $r \in |\mathbb{C}_p^{\times}|_p, r < 1$. Take $0 < r' < r$ arbitrary. We construct a function f in $\mathcal{A}_k(\mathbb{P}^1)[r']$ such that $\frac{d^{k+1}f}{dz^{k+1}} = g$.

If $a \in \mathbb{Q}_p$ such that $|a|_p < \frac{1}{r'}$, and if $z \in D(a,r)$, then $|z-a|_p \leq r < \frac{1}{r} < \frac{1}{r'}$, so $|z|_p < \frac{1}{r'}$, hence the closed disc $D(a,r)$ in $\mathbb{Q}_p$ is contained in $B(0,\frac{1}{r'})$. The disc $B(0,\frac{1}{r'})$ is the disjoint union of closed discs $D(a,r')$ for $a \in \mathbb{Q}_p$ with $|a|_p < \frac{1}{r'}$. The restriction of g on $D(a,r)$ for each such a is analytic, writing the Taylor expansion of g on $D(a,r)$ by

$$g_{|D(a,r)}(z) = \sum_{n=0}^{+\infty} \alpha_n (z-a)^n,$$

where α_n satisfies $|\alpha_n|_p \cdot r^n \to 0$ when $n \to +\infty$. Then the function

$$f_{a,r'} = \sum_{n=k+1}^{+\infty} \frac{\alpha_{n-k-1}}{n(n-1)...(n-k)} (z-a)^n$$

is analytic on $D(a,r')$ since

$$\Big|\frac{\alpha_{n-k-1}}{n(n-1)...(n-k)}\Big|_p (r')^n \leq n...(n-k)\Big(\frac{r'}{r}\Big)^{n-k-1} |\alpha_{n-k-1}|_p \cdot r^{n-k-1}(r')^{k+1} \underset{n\to+\infty}{\longrightarrow} 0.$$

Moreover, the $(k+1)$-th derivative of $f_{a,r'}$ equals g on $D(a,r')$. We define the restriction of f on $B(0,\frac{1}{r'})$ by putting $f=f_{a,r'}$ on each disc $D(a,r')\subset B(0,\frac{1}{r'})$.

On $\{z\in\mathbb{P}^1(\mathbb{Q}_p),|z|_p\geq\frac{1}{r}\}$ the function g is of the form $\sum_{i\leq -k-2} a_iz^i$, where a_i satisfies $|a_i|_p\left(\frac{1}{r}\right)^i\to 0$ when $i\to-\infty$. We define the restriction of f on $\{z\in\mathbb{P}^1(\mathbb{Q}_p),|z|_p\geq\frac{1}{r'}\}$ by

$$\sum_{i<0}\frac{a_{i-k-1}}{i(i-1)...(i-k)}z^i.$$

This function is convergent on $\{z\in\mathbb{P}^1(\mathbb{Q}_p),|z|_p\geq\frac{1}{r'}\}$ since

$$\left|\frac{a_{i-k-1}}{i(i-1)...(i-k)}\right|_p\left(\frac{1}{r'}\right)^i\leq(-i)(-i+1)...(-i+k)\left(\frac{r}{r'}\right)^{i-k-1}|a_{i-k-1}|_p\left(\frac{1}{r}\right)^{i-k-1}\left(\frac{1}{r'}\right)^{k+1}\underset{i\to-\infty}{\longrightarrow}0.$$

It is obvious that $\frac{d^{k+1}f}{dz^{k+1}}=g=\sum_{i\leq -k-2}a_iz^i$ on $\{z\in\mathbb{Q}_p,|z|_p\geq\frac{1}{r'}\}$. So $\frac{d^{k+1}f}{dz^{k+1}}=g$ on $\mathbb{Q}_p$. Moreover, $f\in\mathcal{A}_k(\mathbb{P}^1)[r']\subset\mathcal{A}_k(\mathbb{P}^1)$. The surjectivity is proven.

The map $\left(\frac{d}{dz}\right)^{k+1}:\mathcal{A}_k(\mathbb{P}^1)[r]\to\mathcal{A}_{-k-2}(\mathbb{P}^1)$ is continuous for any $r\in|\mathbb{C}_p^\times|_p, r<1$ since it is the composition of the continuous map $\left(\frac{d}{dz}\right)^{k+1}:\mathcal{A}_k(\mathbb{P}^1)[r]\to\mathcal{A}_{-k-2}(\mathbb{P}^1)[r]$ (by Lemma 2.2.1) and the inclusion map $\mathcal{A}_{-k-2}(\mathbb{P}^1)[r]\to\mathcal{A}_{-k-2}(\mathbb{P}^1)$, which is also continuous. Hence the inductive limit of these maps, $\left(\frac{d}{dz}\right)^{k+1}:\mathcal{A}_k(\mathbb{P}^1)\to\mathcal{A}_{-k-2}(\mathbb{P}^1)$, is continuous by [Schn, Lemma 5.1i.].

For the openness of $\left(\frac{d}{dz}\right)^{k+1}$, we need to show that it sends any neighborhood of 0 in $\mathcal{A}_k(\mathbb{P}^1)$ to a neighborhood of 0 in $\mathcal{A}_{-k-2}(\mathbb{P}^1)$. A neighborhood of 0 in $\mathcal{A}_k(\mathbb{P}^1)$ is by definition a subset A such that $A\cap\mathcal{A}_k(\mathbb{P}^1)[r']$ is a neighborhood of 0 in $\mathcal{A}_k(\mathbb{P}^1)[r']$, for any $r'\in|\mathbb{C}_p^\times|_p, r'<1$. Then A contains an open ball of center 0 and radius R in $\mathcal{A}_k(\mathbb{P}^1)[r']$. For any $r,r'\in|\mathbb{C}_p^\times|_p$ such that $r'<r<1$, and any $g\in\mathcal{A}_{-k-2}(\mathbb{P}^1)[r]$, we have shown the existence of a function $f\in\mathcal{A}_k(\mathbb{P}^1)[r']$ such that $\frac{d^{k+1}f}{dz^{k+1}}=g$. Moreover, it is easy to see that there exists a constant $C=C(r,r')>0$ depending only on r and r' such that

$$\|f\|_{r'}\leq C\|g\|_r.$$

If $\|g\|_r<R/C$, then $\|f\|_{r'}<R$, so $f\in A$. Therefore, the image of A by $\left(\frac{d}{dz}\right)^{k+1}$ contains the open ball of center 0 and radius R/C in $\mathcal{A}_{-k-2}(\mathbb{P}^1)[r]$. Hence the image of A is open in $\mathcal{A}_{-k-2}(\mathbb{P}^1)$. □

In the view of the exact sequence (2.6), the subspace $\mathcal{P}_k^\dagger(L)$ of $\mathcal{A}_k(\mathbb{P}^1,L)$ is stable by the weight k action of $\mathrm{GL}_2(\mathbb{Q}_p)$, defined by (2.4). We equip $\mathcal{P}_k^\dagger(L)$ the topology and the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ inherited from $\mathcal{A}_k(\mathbb{P}^1,L)$. Then the inclusion $\mathcal{P}_k^\dagger(L)\to\mathcal{A}_k(\mathbb{P}^1,L)$ is continuous and $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant. We want to make the second map of (2.6), i.e. the map $\left(\frac{d}{dz}\right)^{k+1}$, is also $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant.

Lemma 2.2.3. *For each $k\in\mathbb{N}$, the exact sequence*

$$0\to\mathcal{P}_k^\dagger(L)\xrightarrow{i}\mathcal{A}_k(\mathbb{P}^1,L)\xrightarrow{\left(\frac{d}{dz}\right)^{k+1}}\mathcal{A}_{-k-2}(\mathbb{P}^1,L)\otimes\det{}^{k+1}\to 0 \tag{2.7}$$

is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant, where $\otimes\det^{k+1}$ means the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ is twisted by $\det^{k+1}$.

Proof. We follow the calculations in the proof of [Bel, Lemma V.4.13]. Although in ibid. it is made for analytic functions on $\mathbb{Z}_p$ and we are considering functions on $\mathbb{P}^1(\mathbb{Q}_p)$, but the actions of matrices on both types of functions are the same. □

Dualizing the exact sequence (2.7), we get a complex of $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant maps. In fact, this dual complex is exact.

Proposition 2.2.4. *For each $k\in\mathbb{N}$, there is a canonical $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant exact sequence:*

$$0\to\mathcal{D}_{-k-2}(\mathbb{P}^1,L)\otimes\det{}^{k+1}\xrightarrow{\theta_k}\mathcal{D}_k(\mathbb{P}^1,L)\xrightarrow{\rho_k}\mathcal{V}_k^\dagger(L)\to 0$$

which is the L-dual of (2.7), where $\mathcal{V}_k^\dagger(L)$ is the L-dual of $\mathcal{P}_k^\dagger(L)$ endowed with the weight k action of $\mathrm{GL}_2(\mathbb{Q}_p)$ defined similarly to (2.5).

Proof. The injectivity of θ_k is obvious. The surjectivity of ρ_k is an application of Hahn-Banach's theorem (see [Schn, Corollary 9.4]), applicable here since the spaces $\mathcal{P}_k^\dagger(L)$ and $\mathcal{A}_k(\mathbb{P}^1, L)$ are locally convex, and the base field L is spherically complete. We prove the exactness in the middle. Let $\mu \in \mathcal{D}_k(\mathbb{P}^1, L)$ such that $\rho_k(\mu) = 0$. Then $\mu(\mathcal{P}_k^\dagger(L)) = 0$, so μ induces an L-linear form μ' on $\mathcal{A}_{-k-2}(\mathbb{P}^1, L)$ such that $\mu = \mu' \circ \left(\frac{d}{dz}\right)^{k+1}$. We show that the form μ' is continuous. If $L_0 \subset L$ is an open subset, then $(\mu')^{-1}(L_0) = \left(\frac{d}{dz}\right)^{k+1}(\mu^{-1}(L_0))$ is open in $\mathcal{A}_{-k-2}(\mathbb{P}^1, L)$ since $\mu^{-1}(L_0)$ is open in $\mathcal{A}_k(\mathbb{P}^1, L)$ and the map $\left(\frac{d}{dz}\right)^{k+1}$ is open by Lemma 2.2.2. It follows that $\mu' \in \mathcal{D}_{-k-2}(\mathbb{P}^1, L)$ and $\mu = \theta_k(\mu') \in \operatorname{Im}\theta_k$. □

For $k \in \mathbb{Z}$, let $\mathcal{A}_k^{(-1)}(\mathbb{P}^1) \subset \mathcal{A}_k(\mathbb{P}^1)$ denote the subspace of functions f such that f is of the form $\sum_{i \le k, i \neq -1} a_i z^i$ in a neighborhood of ∞, endowed with the induced topology. Note that $\mathcal{A}_k^{(-1)}(\mathbb{P}^1) = \mathcal{A}_k(\mathbb{P}^1)$ if $k \le -2$.

Proposition 2.2.5. *Let Δ be the operator on $\mathcal{A}_k(\mathbb{P}^1)$ given by $\Delta(f) = f(z+1) - f(z)$. Then the image of Δ is the subspace $\mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1)$.*

Proof. Let $f \in \mathcal{A}_k(\mathbb{P}^1)$, we prove $\Delta(f) \in \mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1)$. There exists $r \in |\mathbb{C}_p^\times|_p, r < 1$ such that $f \in \mathcal{A}_k(\mathbb{P}^1)[r]$. Since f is r-analytic in $\mathbb{Q}_p$, so is $\Delta(f)$. Suppose $f(z) = \sum_{i=-\infty}^{k} a_i z^i$ for $z \in \mathbb{P}^1(\mathbb{Q}_p)$ with $|z|_p \ge \frac{1}{r}$. We have

$$\Delta\Big(\sum_{i \le k} a_i z^i\Big) = \Delta\Big(\sum_{i \le \min\{-1,k\}} a_i z^i\Big) + \Delta\Big(\sum_{i \le k, i \ge 0} a_i z^i\Big),$$

where we make the convention $\Delta\Big(\sum_{i \le k, i \ge 0} a_i z^i\Big) = 0$ if $k < 0$. If $k \ge 0$, then $\Delta\Big(\sum_{i \le k, i \ge 0} a_i z^i\Big)$ is a polynomial of degree $\le k-1$. If $i \le -1$, then for $z \in \mathbb{P}^1(\mathbb{Q}_p)$ with $|z|_p \ge \frac{1}{r}$, we have

$$\begin{aligned}\Delta(a_i z^i) &= a_i(z+1)^i - a_i z^i = a_i z^i \Big(\frac{z}{z+1}\Big)^{-i} - a_i z^i = a_i z^i \Big(\frac{1}{1+z^{-1}}\Big)^{-i} - a_i z^i \\ &= a_i z^i \Big(\sum_{n=0}^{+\infty} (-z^{-1})^n\Big)^{-i} - a_i z^i = a_i z^i \Big(\sum_{n=0}^{+\infty} \binom{n-i-1}{-i-1} (-z^{-1})^n\Big) - a_i z^i = \sum_{j<i} b_{i,j} z^j,\end{aligned}$$

where $b_{i,j} = a_i \binom{-j-1}{-i-1}(-1)^{i-j}$. It follows that $|b_{i,j}|_p \le |a_i|_p$ for all j. We get

$$\begin{aligned}\Delta\Big(\sum_{i \le \min\{-1,k\}} a_i z^i\Big) &= \sum_{i \le \min\{-1,k\}} \Delta(a_i z^i) = \lim_{n \to +\infty} \sum_{-n \le i \le \min\{-1,k\}} \Delta(a_i z^i) \\ &= \lim_{n \to +\infty} \sum_{j < \min\{-1,k\}} \Big(\sum_{\max\{-n, j+1\} \le i \le \min\{-1,k\}} b_{i,j}\Big) z^j \qquad (2.8)\end{aligned}$$

for $z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \ge \frac{1}{r}$. Let us apply an argument similar to Lemma 2.1.6, where we consider the functions around ∞ with expansion by negative powers. For each j, the sequence

$$\sum_{\max\{-n, j+1\} \le i \le \min\{-1,k\}} b_{i,j} \quad \text{tends to} \quad \sum_{j+1 \le i \le \min\{-1,k\}} b_{i,j} \quad \text{as} \quad n \to +\infty.$$

Moreover, if we fix $z \in \mathbb{P}^1(\mathbb{Q}_p)$ such that $|z|_p \ge \frac{1}{r}$ and $\varepsilon > 0$, then for small j, we have

$$\Big|\Big(\sum_{\max\{-n, j+1\} \le i \le \min\{-1,k\}} b_{i,j}\Big) z^j\Big|_p \le \max_{j < i < 0}\{|a_i|_p\}\Big(\frac{1}{r}\Big)^j < \varepsilon$$

for all n (since $|a_i|_p \left(\frac{1}{r}\right)^i \to 0$ as $i \to -\infty$, and $\left(\frac{1}{r}\right)^j < \left(\frac{1}{r}\right)^i$ for any $j < i$ since $\frac{1}{r} > 1$). Therefore, by the spirit of Lemma 2.1.6 applying to the limit in (2.8), we obtain

$$\Delta\Big(\sum_{i \le \min\{-1,k\}} a_i z^i\Big) = \sum_{j < \min\{-1,k\}} \Big(\sum_{j+1 \le i \le \min\{-1,k\}} b_{i,j}\Big) z^j$$

for $z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r}$. Hence

$$\Delta(f) = \sum_{j<\min\{-1,k\}} \Big(\sum_{j+1\leq i\leq \min\{-1,k\}} b_{i,j} \Big) z^j + \Delta\Big(\sum_{i\leq k, i\geq 0} a_i z^i \Big) \tag{2.9}$$

in $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r}\}$. We deduce that $\Delta(f) \in \mathcal{A}^{(-1)}_{k-1}(\mathbb{P}^1)[r]$, so $\Delta(f) \in \mathcal{A}^{(-1)}_{k-1}(\mathbb{P}^1)$.

Let us prove that $\Delta : \mathcal{A}_k(\mathbb{P}^1) \to \mathcal{A}^{(-1)}_{k-1}(\mathbb{P}^1)$ is surjective. Let $g \in \mathcal{A}^{(-1)}_{k-1}(\mathbb{P}^1)$, then $g \in \mathcal{A}^{(-1)}_{k-1}(\mathbb{P}^1)[r]$ for some $r \in |\mathbb{C}_p^\times|_p, r < 1$. Fix $0 < r' < r$ with $r' = p^{-n_0}$ for some $n_0 \in \mathbb{N}^*$, we show the existence of a function $f \in \mathcal{A}_k(\mathbb{P}^1)[r']$ such that $\Delta(f) = g$.

Defining a relation $\sim$ on the family of closed discs of radius r' in $B(0, \frac{1}{r'}) \subset \mathbb{Q}_p$ as follows: for two discs D_1, D_2, we denote $D_1 \sim D_2$ if there exist $a \in B(0, \frac{1}{r'})$ and $h \in \mathbb{Z}$ such that $D_1 = D(a, r')$ and $D_2 = D(a + h, r')$. This is equivalent to $|z_2 - z_1 - h|_p \leq r'$ for any $z_1 \in D_1, z_2 \in D_2$. So this is an equivalence relation. There are exactly p^{n_0} closed discs in each equivalence class, namely $D(a, r'), D(a+1, r'), ..., D(a+p^{n_0}-1, r')$ for some $a \in B(0, \frac{1}{r'})$. To construct f on $B(0, \frac{1}{r'})$, we construct f on closed discs in each such class. Note that if $a \in B(0, \frac{1}{r'})$, then $a + \mathbb{Z} \subset B(0, \frac{1}{r'})$ since $|\mathbb{Z}|_p \leq 1 < \frac{1}{r'}$. If f is constructed and analytic on $D(a, r')$, then f is automatically determined on $D(a + 1, r')$ by $f(z) = f(z-1) + g(z-1)$ for $z \in D(a+1, r')$ (note that $z - 1 \in D(a, r')$ if $z \in D(a+1, r')$). Similar, f is automatically determined on every other disc in this class, given by $f(z) = f(z-1) + g(z-1) = f(z-2) + g(z-2) + g(z-1)$ for $z \in D(a+2, r')$, and so on. Therefore, it suffices to construct f analytic on $D(a, r')$ satisfying

$$\begin{aligned} f(z+p^{n_0}) &= f(z+p^{n_0}-1) + g(z+p^{n_0}-1) = ... \\ &= f(z) + g(z) + g(z+1) + ... + g(z+p^{n_0}-1) = f(z) + g_0(z), \end{aligned}$$

where $g_0(z) = g(z) + g(z+1) + ... + g(z+p^{n_0}-1)$ is analytic on $D(a, r)$. Note that $z + p^{n_0} \in D(a, r')$ if $z \in D(a, r')$.

Suppose g_0 has Taylor expansion $g_0(z) = \sum\limits_{n=0}^{+\infty} \alpha_n (z-a)^n$ in $D(a, r)$, where α_n satisfies $|\alpha_n|_p \cdot r^n \to 0$ when $n \to +\infty$. We need the following result.

Lemma 2.2.6. *We can write g_0 under the form $g_0(z) = \sum\limits_{n=0}^{+\infty} \beta_n (z-a)^{[n]}$ in $D(a, r)$ for β_n satisfies $|\beta_n|_p \cdot r^n \to 0$ when $n \to +\infty$, where $z^{[n]} := z(z-p^{n_0})...(z-(n-1)p^{n_0})$ (if $n = 0$ we make the convention $z^{[0]} = 1$). Moreover, the r-norm of g_0 on $D(a, r)$ equals $\sup\{|\beta_n|_p \cdot r^n \mid n \in \mathbb{N}\}$. Conversely, if g_0 is of this form, then g_0 is analytic on $D(a, r)$.*

Proof. Consider $z \in D(a, r)$. For each $n \in \mathbb{N}$, the polynomial $(z-a)^n$ is the sum of $(z-a)^{[n]}$ and a linear combination of $(z-a)^j$ for $1 \leq j \leq n-1$ with coefficient divisible by $(p^{n_0})^{n-j}$. The induction on n yields an expansion of $(z-a)^n$ in terms of $(z-a)^{[j]}$ for $0 \leq j \leq n$ with integer coefficients:

$$(z-a)^n = \sum_{j=0}^{n} \gamma_{n,j} (z-a)^{[j]}$$

with $\gamma_{n,n} = 1$ and $\gamma_{n,j} \in \mathbb{Z}$ is divisible by $(p^{n_0})^{n-j}$ for all j. We have

$$g_0(z) = \sum_{n=0}^{+\infty} \alpha_n (z-a)^n = \lim_{N\to+\infty} \sum_{n=0}^{N} \alpha_n (z-a)^n = \lim_{N\to+\infty} \sum_{j=0}^{N} \Big(\sum_{n=j}^{N} \alpha_n \gamma_{n,j} \Big) (z-a)^{[j]}. \tag{2.10}$$

Let us apply an argument similar to Lemma 2.1.6. For $j \in \mathbb{N}$ fixed, the sequence $\sum\limits_{n=j}^{N} \alpha_n \gamma_{n,j}$ converges to $\sum\limits_{n=j}^{+\infty} \alpha_n \gamma_{n,j}$ when $N \to +\infty$ since

$$|\alpha_n \gamma_{n,j}|_p \leq |\alpha_n|_p (p^{-n_0})^{n-j} = |\alpha_n|_p \cdot (r')^{n-j} \leq |\alpha_n|_p \cdot r^{n-j} \to 0 \text{ when } n \to +\infty.$$

Moreover, for every $z \in D(a, r)$ and $\varepsilon > 0$, if $j \in \mathbb{N}$ is big and $N \geq j$, then

$$\begin{aligned} \Big| \Big(\sum_{n=j}^{N} \alpha_n \gamma_{n,j} \Big) (z-a)^{[j]} \Big|_p &\leq \max_{j\leq n\leq N} \{|\alpha_n \gamma_{n,j}|_p\} \cdot |(z-a)^{[j]}|_p \leq \max_{n\geq j} \{|\alpha_n|_p (r')^{n-j}\} \cdot r^j \\ &\leq \max_{n\geq j} \{|\alpha_n|_p \cdot r^n\} < \varepsilon, \end{aligned}$$

here the second inequaliy is deduced from $|\gamma_{n,j}|_p \leq (p^{-n_0})^{n-j} = (r')^{n-j}$ and $|(z-a)^{[j]}|_p \leq r^j$ for any $j \in \mathbb{N}$. By the spirit of Lemma 2.1.6 applied to the limit (2.10), we obtain

$$g_0(z) = \sum_{j=0}^{+\infty} \Big(\sum_{n=j}^{+\infty} \alpha_n \gamma_{n,j} \Big) (z-a)^{[j]} = \sum_{j=0}^{+\infty} \beta_j (z-a)^{[j]},$$

where $\beta_j = \sum\limits_{n=j}^{+\infty} \alpha_n \gamma_{n,j}$ satisfying

$$|\beta_j|_p \cdot r^j \leq \sup_{n \geq j} \{|\alpha_n \gamma_{n,j}|_p\} \cdot r^j \leq \sup_{n \geq j} \{|\alpha_n|_p (r')^{n-j}\} \cdot r^j \leq \sup_{n \geq j} \{|\alpha_n|_p \cdot r^n\} \to 0 \text{ when } j \to +\infty.$$

It follows that $\sup\{|\beta_j|_p \cdot r^j \mid j \in \mathbb{N}\} \leq \sup\{|\alpha_n|_p \cdot r^n \mid n \in \mathbb{N}\}$.

Conversely, if $g_0(z) = \sum\limits_{n=0}^{+\infty} \beta_n (z-a)^{[n]}$ for $z \in D(a,r)$, then by the same argument we deduce that g_0 is analytic in $D(a,r)$ with Taylor expansion $\sum\limits_{n=0}^{+\infty} \alpha_n (z-a)^n$ for the coefficients α_n satisfying $\sup\{|\alpha_n|_p \cdot r^n \mid n \in \mathbb{N}\} \leq \sup\{|\beta_j|_p \cdot r^j \mid j \in \mathbb{N}\}$. Therefore,

$$\|g_{0|D(a,r)}\|_r = \sup\{|\alpha_n|_p \cdot r^n \mid n \in \mathbb{N}\} = \sup\{|\beta_j|_p \cdot r^j \mid j \in \mathbb{N}\}.$$

The lemma is proven. □

Writing $g_0(z) = \sum\limits_{n=0}^{+\infty} \beta_n (z-a)^{[n]}$ in $D(a,r)$ where $|\beta_n|_p \cdot r^n \underset{n \to +\infty}{\longrightarrow} 0$ as in the above lemma. We put

$$f(z) = \sum_{n=1}^{+\infty} \frac{\beta_{n-1}(z-a)^{[n]}}{np^{n_0}}$$

in $D(a,r')$, then f satisfies $f(z+p^{n_0}) = f(z) + g_0(z)$ for $z \in D(a,r)$ since $(z+p^{n_0}-a)^{[n]} - (z-a)^{[n]} = np^{n_0}(z-a)^{[n-1]}$ for any $n \in \mathbb{N}^*$. The analyticity of f on $D(a,r')$ is deduced from Lemma 2.2.6 and from

$$\Big|\frac{\beta_{n-1}}{np^{n_0}}\Big|_p (r')^n \leq np^{n_0} |\beta_{n-1}|_p (r')^n = |\beta_{n-1}|_p \cdot r^{n-1} n \Big(\frac{r'}{r}\Big)^{n-1} \underset{n \to +\infty}{\longrightarrow} 0,$$

since $|\beta_{n-1}|_p \cdot r^{n-1} \underset{n \to +\infty}{\longrightarrow} 0$ and $n\big(\frac{r'}{r}\big)^{n-1} \underset{n \to +\infty}{\longrightarrow} 0$. Moreover,

$$\begin{aligned}\|f_{|D(a,r')}\|_{r'} &= \sup_{n \geq 1} \Big|\frac{\beta_{n-1}}{np^{n_0}}\Big|_p (r')^n \leq \sup_{n \geq 1} \Big\{ n \Big(\frac{r'}{r}\Big)^{n-1} \Big\} \cdot \sup_{n \geq 1} |\beta_{n-1}|_p \cdot r^{n-1} \\ &\leq \sup_{n \geq 1} \Big\{ n \Big(\frac{r'}{r}\Big)^{n-1} \Big\} \|g_{0|D(a,r)}\|_r \leq \sup_{n \geq 1} \Big\{ n \Big(\frac{r'}{r}\Big)^{n-1} \Big\} \|g_{|B(0,\frac{1}{r'})}\|_r,\end{aligned}$$

for any closed disc $D(a,r')$ inside $B\big(0,\frac{1}{r'}\big)$. Hence

$$\|f_{|B(0,\frac{1}{r'})}\|_{r'} \leq \sup_{n \geq 1} \Big\{ n \Big(\frac{r'}{r}\Big)^{n-1} \Big\} \|g_{|B(0,\frac{1}{r'})}\|_r. \tag{2.11}$$

It remains to construct f on $X_\infty = \{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r'}\}$. The function g is of the form

$$g(z) = \sum_{i \leq k-1, i \neq -1} b_i z^i$$

on the set $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r}\}$ containing X_∞, where b_i satisfies $|b_i|_p \big(\frac{1}{r}\big)^i \to 0$ when $i \to -\infty$. The topological space X_∞ is isomorphic to $D(0,r')$ via the transformation $z \mapsto \frac{1}{z}$. Setting $g_\infty(z) = g\big(\frac{1}{z}\big)$ for $z \in D(0,r)$ and $f_\infty(z) = f\big(\frac{1}{z}\big)$ for $z \in D(0,r')$, then

$$\begin{aligned}g_\infty(z) &= \sum_{i \leq k-1, i \neq -1} b_i z^{-i} = \sum_{0 \leq i \leq k-1} b_i z^{-i} + \sum_{i \leq -2} b_i z^{-i} = \sum_{0 \leq n \leq k-1} b_n z^{-n} + \sum_{n \geq 2} b_{-n} z^n \\ &=: \sum_{0 \leq n \leq k-1} b_n z^{-n} + \sum_{n \geq 2} \tilde{b}_n z^n =: g_\infty^-(z) + g_\infty^+(z),\end{aligned}$$

where $g_\infty^-(z) = \sum_{0\le n\le k-1} b_n z^{-n}$, $g_\infty^+(z) = \sum_{n\ge 2} \tilde{b}_n z^n$ and $\tilde{b}_n = b_{-n}$ for $n \ge 2$. We want to construct a meromorphic function f_∞ on $D(0,r')$ with order of vanishing $\mathrm{ord}_0(f_\infty) \ge -k$ such that

$$g_\infty(z) = g\Big(\frac{1}{z}\Big) = f\Big(\frac{1}{z}+1\Big) - f\Big(\frac{1}{z}\Big) = f_\infty\Big(\frac{z}{z+1}\Big) - f_\infty(z) \ \text{ for } z \in D(0,r').$$

Putting $R(z) = \frac{z}{z+1}$, the above condition rewrites

$$f_\infty(R(z)) - f_\infty(z) = g_\infty(z) = g_\infty^-(z) + g_\infty^+(z) \ \text{ for } z \in D(0,r'). \tag{2.12}$$

By induction, for each $n \in \mathbb{N}^*$, the composition $R^{\circ n}(z)$ equals $\frac{z}{nz+1}$, so $\frac{1}{R^{\circ n}(z)} = \frac{1}{z} + n$. We make the convention $R^{\circ 0}(z) = 1$. Since $g_\infty^-(z)$ is a polynomial of $\frac{1}{z}$ of degree $\le k-1$, we can write $g_\infty^-(z)$ in the form

$$g_\infty^-(z) = \sum_{n=0}^{k-1} c_n^- \Big(\frac{1}{z}+1\Big)\Big(\frac{1}{z}+2\Big)...\Big(\frac{1}{z}+n\Big) = \sum_{n=0}^{k-1} \frac{c_n^-}{R(z)R^{\circ 2}(z)...R^{\circ n}(z)}. \tag{2.13}$$

We then put $f_\infty^-(z) = \sum_{n=0}^{k-1} \frac{c_n^-}{n+1}\frac{1}{zR(z)...R^{\circ n}(z)}$ for $z \in D(0,r')$. It follows that

$$f_\infty^-(R(z)) - f_\infty^-(z) = g_\infty^-(z) \tag{2.14}$$

for any $z \in D(0,r')$ since $\frac{1}{R^{\circ(n+1)}(z)} - \frac{1}{z} = n+1$. The function $f_\infty^-(z)$ is a linear combination of negative powers of z of degree between $-k$ and -1. The following lemma will be used in the proof of Proposition 2.2.9.

Lemma 2.2.7. *The r-norm on $D(0,r)$ of the polynomial $z^{k-1}g_\infty^-(z)$, where $g_\infty^-(z)$ is written in the form* (2.13)*, equals*

$$\max_{0\le n\le k-1} |c_n^-|_p \cdot r^{k-1-n}.$$

Proof. Since $R^{\circ n}(z) = \frac{z}{nz+1}$ and since $|nz|_p \le |z|_p \le r < 1$ if $z \in D(0,r) \subset \mathbb{C}_p$, it follows that $|R^{\circ n}(z)|_p = |z|_p$ for any $z \in D(0,r) \subset \mathbb{C}_p$ and any $n \in \mathbb{N}$. Hence

$$\begin{aligned} \|(z^{k-1}g_\infty^-(z))_{|D(0,r)}\|_r &\le \sup_{z\in\mathbb{C}_p, |z|_p\le r, 0\le n\le k-1} \frac{|c_n^-|_p \cdot |z|_p^{k-1}}{|R(z)...R^{\circ n}(z)|_p} \\ &= \sup_{z\in\mathbb{C}_p, |z|_p\le r, 0\le n\le k-1} |c_n^-|_p \cdot |z|_p^{k-1-n} = \max_{0\le n\le k-1} |c_n^-|_p \cdot r^{k-1-n}. \end{aligned} \tag{2.15}$$

Returning to the expansion $g_\infty^-(z) = \sum_{0\le n\le k-1} b_n z^{-n} = \sum_{0\le n\le k-1} b_n \big(\frac{1}{z}\big)^n$ of g_∞^-. For $n \in \mathbb{N}$, since $\big(\frac{1}{z}\big)^n$ is the sum of $\big(\frac{1}{z}+1\big)\big(\frac{1}{z}+2\big)...\big(\frac{1}{z}+n\big)$ and a polynomial of $\frac{1}{z}$ of degree $\le n-1$ with integer coefficients, by induction on k we can show that all the coefficients c_n^- in (2.13) are integer linear combinations of the coefficients $b_n, b_{n+1}, ..., b_{k-1}$. It follows that

$$|c_n^-|_p \le \max\{|b_n|_p, |b_{n+1}|_p, ..., |b_{k-1}|_p\}$$

for any $0 \le n \le k-1$. We get

$$\begin{aligned} &\max_{0\le n\le k-1} |c_n^-|_p \cdot r^{k-1-n} \le \max_{0\le n\le k-1} \max\{|b_n|_p, |b_{n+1}|_p, ..., |b_{k-1}|_p\} \cdot r^{k-1-n} \\ &\le \max_{0\le n\le k-1} \max_{n\le i\le k-1} |b_i|_p \cdot r^{k-1-i} = \max_{0\le n\le k-1} |b_n|_p \cdot r^{k-1-n} = \|(z^{k-1}g_\infty^-(z))_{|D(0,r)}\|_r. \end{aligned}$$

Combining with (2.15) we get the desired formula. □

By Lemma 2.2.7 and the construction of $f_\infty^-(z)$, we have

$$\begin{aligned} \|(z^k f_\infty^-(z))_{|D(0,r')}\|_{r'} &= \max_{0\le n\le k-1} \Big|\frac{c_n^-}{n+1}\Big|_p \cdot (r')^{k-1-n} \\ &\le \max_{0\le n\le k-1} \Big\{(n+1)\Big(\frac{r'}{r}\Big)^{k-1-n}\Big\} \cdot \max_{0\le n\le k-1} \Big\{|c_n^-|_p \cdot r^{k-1-n}\Big\} \\ &= \max_{0\le n\le k-1} \Big\{(n+1)\Big(\frac{r'}{r}\Big)^{k-1-n}\Big\} \|(z^{k-1}g_\infty^-(z))_{|D(0,r)}\|_r. \end{aligned} \tag{2.16}$$

Lemma 2.2.8. *The function g_∞^+ on $D(0,r)$ can be written under the form*

$$g_\infty^+(z) = \sum_{n=2}^{+\infty} c_n^+ zR(z)...R^{\circ(n-1)}(z),$$

where c_n^+ satisfies $|c_n^+|_p \cdot r^n \to 0$ when $n \to +\infty$. The r-norm of g_∞^+ on $D(0,r)$ equals $\sup\{|c_n^+|_p \cdot r^n \mid n \geq 2\}$. *Conversely, if g_∞^+ is of this form, then it is analytic in $D(0,r)$ with order of vanishing* $\mathrm{ord}_0(g_\infty^+) \geq 2$.

Proof. Recall that

$$g_\infty^+(z) = \sum_{n\geq 2} \tilde{b}_n z^n,$$

where $\tilde{b}_n = b_{-n}$ satisfying $|\tilde{b}_n|_p \cdot r^n \to 0$ when $n \to +\infty$. We have seen that $R^{\circ n}(z) = \frac{z}{1+nz} = z\sum_{j=0}^{+\infty}(-nz)^j$ in $D(0,r)$ for each $n \in \mathbb{N}$, so $R^{\circ n}(z)$ is analytic in $D(0,r)$ with $\mathrm{ord}_0(R^{\circ n}(z)) = 1$. Hence the function $zR(z)...R^{\circ(n-1)}(z)$ is analytic in $D(0,r)$ with order of vanishing n at 0, for any $n \geq 2$. Since z^n is the sum of $zR(z)...R^{\circ(n-1)}(z)$ and an infinite linear combination of $z^{n+1}, z^{n+2}, ...$ with integer coefficients, by induction on $j \geq 2$, we can write g_∞^+ in the form

$$g_\infty^+(z) = \sum_{n=2}^{j} c_n^+ zR(z)...R^{\circ(n-1)}(z) + \sum_{n=j+1}^{+\infty} d_{n,j} z^n, \tag{2.17}$$

where the coefficients $c_n^+, d_{n,j}$ are integer linear combinations of $\tilde{b}_2, ..., \tilde{b}_n$. It follows that $|c_n^+|_p, |d_{n,j}|_p \leq \max\{|\tilde{b}_2|_p, ..., |\tilde{b}_n|_p\}$. Then for any $z \in D(0,r)$ and any $2 \leq j \leq n-1$, we have

$$|d_{n,j} z^n|_p \leq \max\{|\tilde{b}_2|_p, ..., |\tilde{b}_n|_p\} \cdot r^n \underset{n\to+\infty}{\longrightarrow} 0,$$

since $|\tilde{b}_n|_p \cdot r^n$ and r^n tend to 0 when $n \to +\infty$. Therefore, by (2.17) we deduce

$$g_\infty^+(z) = \lim_{j\to+\infty} \sum_{n=2}^{j} c_n^+ zR(z)...R^{\circ(n-1)}(z) = \sum_{n=2}^{+\infty} c_n^+ zR(z)...R^{\circ(n-1)}(z),$$

and $|c_n^+|_p \cdot r^n \leq \max\{|\tilde{b}_2|_p, ..., |\tilde{b}_n|_p\} \cdot r^n \underset{n\to+\infty}{\longrightarrow} 0$. We also have

$$\sup\{|c_n^+|_p \cdot r^n \mid n \geq 2\} \leq \sup\{|\tilde{b}_n|_p \cdot r^n \mid n \geq 2\} = \|(g_\infty^+)_{|D(0,r)}\|_r. \tag{2.18}$$

Since $|R^{\circ n}(z)|_p = |z|_p$ for any $z \in \mathbb{C}_p$ with $|z|_p \leq r$, we get

$$\|(g_\infty^+)_{|D(0,r)}\|_r = \sup_{z\in\mathbb{C}_p, |z|_p\leq r} |g_\infty^+(z)|_p \leq \sup_{z\in\mathbb{C}_p, |z|_p\leq r, n\geq 2} |c_n^+ zR(z)...R^{\circ(n-1)}(z)|_p \leq \sup_{n\geq 2} |c_n^+|_p \cdot r^n.$$

Combining with (2.18) we obtain $\|(g_\infty^+)_{|D(0,r)}\|_r = \sup_{n\geq 2} |c_n^+|_p \cdot r^n$.

The inverse statement of the lemma is implied by Lemma 2.1.6 with the note that the function $zR(z)...R^{(n-1)}(z)$ is analytic in $D(0,r)$ with order of vanishing n at 0, for any $n \geq 1$. □

Writing $g_\infty^+(z) = \sum_{n=2}^{+\infty} c_n^+ zR(z)...R^{\circ(n-1)}(z)$ for $z \in D(0,r)$ as in Lemma 2.2.8. We then define

$$f_\infty^+(z) = \sum_{n=1}^{+\infty} c_{n+1}^+ \frac{zR(z)...R^{\circ(n-1)(z)}}{-n}$$

for $z \in D(0,r')$. This function satisfies

$$f_\infty^+(R(z)) - f_\infty^+(z) = g_\infty^+(z) \tag{2.19}$$

since $R^{\circ n}(z) - z = -nzR^{\circ n}(z)$ for any $n \in \mathbb{N}$ (recall that $R^{\circ n}(z) = \frac{z}{nz+1}$, so $\frac{1}{R^{\circ n}(z)} = n + \frac{1}{z}$, hence $\frac{1}{R^{\circ n}(z)} - \frac{1}{z} = n$). The function $f_\infty^+(z)$ is analytic in $D(0,r')$ since

$$\left|\frac{c_{n+1}^+}{-n}\right|_p (r')^n \leq |c_{n+1}^+|_p \cdot n(r')^n = |c_{n+1}^+|_p \cdot r^{n+1} \cdot n\left(\frac{r'}{r}\right)^n r^{-1} \underset{n\to+\infty}{\longrightarrow} 0,$$

with the note that $|c_n^+|_p \cdot r^n \underset{n\to+\infty}{\longrightarrow} 0$ as in the proof of Lemma 2.2.8. It follows that

$$\begin{aligned}\|(z^k f_\infty^+(z))_{|D(0,r')}\|_{r'} &= (r')^k \sup_{n\geq 1} \Big|\frac{c_{n+1}^+}{-n}\Big|_p (r')^n \leq (r')^k r^{-1} \sup_{n\geq 1}\Big\{n\Big(\frac{r'}{r}\Big)^n\Big\} \cdot \sup_{n\geq 1} |c_{n+1}^+|_p \cdot r^{n+1} \\ &= \Big(\frac{r'}{r}\Big)^k \sup_{n\geq 1}\Big\{n\Big(\frac{r'}{r}\Big)^n\Big\} \cdot \|(z^{k-1} g_\infty^+(z))_{|D(0,r)}\|_r. \end{aligned} \tag{2.20}$$

From (2.14) and (2.19), the function f_∞ defined on $D(0,r')$ by $f_\infty(z) = f_\infty^-(z) + f_\infty^+(z)$ satisfies the condition (2.12). By the construction, $f_\infty(z)$ is meromorphic of order of vanishing $\mathrm{ord}_0(f_\infty) \geq -k$. The function f on X_∞ is defined by $f(z) = f_\infty(\frac{1}{z})$, then $\Delta(f) = g$ on X_∞ and $f \in \mathcal{A}_k(\mathbb{P}^1, L)[r']$. The proposition is proven. □

Proposition 2.2.9. *The map $\Delta : \mathcal{A}_k(\mathbb{P}^1) \to \mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1)$ is continuous and open for any $k \in \mathbb{Z}$.*

Proof. For the continuity of Δ, by the proof of Lemma 2.2.2, it suffices to show that the map $\Delta : \mathcal{A}_k(\mathbb{P}^1)[r] \to \mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1)[r]$ is continuous for any $r \in |\mathbb{C}_p^\times|_p, r < 1$ (by the proof of Proposition 2.2.5 we know that Δ maps $\mathcal{A}_k(\mathbb{P}^1)[r]$ into $\mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1)[r]$).

Let $f \in \mathcal{A}_k(\mathbb{P}^1)[r]$. We compare the r-norms of f and $\Delta(f)$. We have

$$\begin{aligned}\|\Delta(f)_{|B(0,\frac{1}{r})}\|_r &= \sup\{|(\Delta f)(z)|_p : z \in \mathbb{C}_p, \exists\, a \in \mathbb{Q}_p \text{ such that } |a|_p < \frac{1}{r}, |z-a|_p \leq r\} \\ &= \sup\{|f(z+1) - f(z)|_p : z \in \mathbb{C}_p, \exists\, a \in \mathbb{Q}_p \text{ such that } |a|_p < \frac{1}{r}, |z-a|_p \leq r\} \\ &\leq \sup\{|f(z)|_p : z \in \mathbb{C}_p, \exists\, a \in \mathbb{Q}_p \text{ such that } |a|_p < \frac{1}{r}, |z-a|_p \leq r\} = \|f_{|B(0,\frac{1}{r})}\|_r. \end{aligned} \tag{2.21}$$

Suppose f is of the form $\sum\limits_{i=-\infty}^{k} a_i z^i$ in $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r}\}$, then by (2.9) we know

$$\begin{aligned}\Delta(f) &= \sum_{j<\min\{-1,k\}} \Big(\sum_{j+1\leq i\leq \min\{-1,k\}} b_{i,j}\Big) z^j + \sum_{j\geq 0, j\leq k-1} \Big(\sum_{i\geq j, i\leq k} a_i \binom{i}{j} - a_j\Big) z^j \\ &= \sum_{j\leq k-1, j\neq -1} b'_j z^j \end{aligned}$$

in $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq \frac{1}{r}\}$, where $b_{i,j} = a_i\binom{-j-1}{-i-1}(-1)^{i-j}$, $b'_j = \sum\limits_{j+1\leq i\leq \min\{-1,k\}} b_{i,j}$ if $j < -1$ and $b'_j = \sum\limits_{i\geq j, i\leq k} a_i\binom{i}{j} - a_j$ if $0 \leq j \leq k-1$. It follows that $|b'_j|_p \leq \max\limits_{j\leq i\leq k} |a_i|_p$. Then

$$\begin{aligned}\Big\|z^{k-1}\cdot(\Delta f)\Big(\frac{1}{z}\Big)_{|D(0,r)}\Big\|_r &= \sup_{j\leq k-1, j\neq -1} |b'_j|_p \cdot r^{k-1-j} \leq \sup_{j\leq k-1, j\neq -1}\big\{\max_{j\leq i\leq k} |a_i|_p \cdot r^{k-1-j}\big\} \\ &\leq \sup_{i\leq k} |a_i|_p \cdot r^{k-1-i} = r^{-1}\Big\|z^k f\Big(\frac{1}{z}\Big)_{|D(0,r)}\Big\|_r. \end{aligned} \tag{2.22}$$

Combining (2.21) and (2.22) we deduce that $\|\Delta(f)\|_r \leq \|f\|_r$. Therefore Δ is continuous.

We show the openness of Δ. In the proof of Proposition 2.2.5, for each $g \in \mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1)[r]$ and $r' \in |\mathbb{C}_p^\times|_p, r' < r < 1$ we have constructed a function $f \in \mathcal{A}_k(\mathbb{P}^1)[r']$ such that $\Delta(f) = g$. By (2.11) we have

$$\begin{aligned}\|f_{|B(0,\frac{1}{r'})}\|_{r'} &\leq \sup_{n\geq 1}\Big\{n\Big(\frac{r'}{r}\Big)^{n-1}\Big\}\|g_{|B(0,\frac{1}{r'})}\|_r \\ &= \sup_{n\geq 1}\Big\{n\Big(\frac{r'}{r}\Big)^{n-1}\Big\} \cdot \max\{\|g_{|B(0,\frac{1}{r})}\|_r, \|g_{|\{z\in\mathbb{Q}_p, \frac{1}{r}\leq |z|_p < \frac{1}{r'}\}}\|_r\}. \end{aligned}$$

Since

$$\begin{aligned}
&\|g_{|\{z\in\mathbb{Q}_p,\frac{1}{r}\le|z|_p<\frac{1}{r'}\}}\|_r\} = \sup\Big\{|g(z)|_p : z\in\mathbb{C}_p, \exists\, a\in\mathbb{Q}_p \text{ such that } \frac{1}{r}\le|a|_p<\frac{1}{r'}, |z-a|_p\le r\Big\}\\
&\le \sup\Big\{|g(z)|_p : z\in\mathbb{C}_p, \frac{1}{r}\le|z|_p<\frac{1}{r'}\Big\} \le \Big(\frac{1}{r'}\Big)^{k-1}\sup\Big\{\Big|\frac{g(z)}{z^{k-1}}\Big|_p : z\in\mathbb{C}_p, \frac{1}{r}\le|z|_p<\frac{1}{r'}\Big\}\\
&\le \Big(\frac{1}{r'}\Big)^{k-1}\sup\Big\{\Big|\frac{g(z)}{z^{k-1}}\Big|_p : z\in\mathbb{C}_p, |z|_p\ge\frac{1}{r}\Big\}\\
&= \Big(\frac{1}{r'}\Big)^{k-1}\sup\Big\{\Big|z^{k-1}g\Big(\frac{1}{z}\Big)\Big|_p : z\in\mathbb{C}_p, |z|_p\le r\Big\} = \Big(\frac{1}{r'}\Big)^{k-1}\Big\|z^{k-1}g\Big(\frac{1}{z}\Big)_{|D(0,r)}\Big\|_r,
\end{aligned}$$

we deduce that

$$\|f_{|B(0,\frac{1}{r'})}\|_{r'} \le \sup_{n\ge1}\Big\{n\Big(\frac{r'}{r}\Big)^{n-1}\Big\}\cdot\max\Big\{\Big(\frac{1}{r'}\Big)^{k-1},1\Big\}\|g\|_r. \tag{2.23}$$

On $X_\infty = \{z\in\mathbb{P}^1(\mathbb{Q}_p), |z|_p\ge\frac{1}{r'}\}$ the function f is constructed by $f(z) = f_\infty\big(\frac{1}{z}\big)$ and $f_\infty(z) = f_\infty^-(z)+f_\infty^+(z)$, where the function $f_\infty^-(z)$ (resp. $f_\infty^+(z)$) satisfies (2.14) (resp. (2.19)), and $g_\infty^-(z)$ (resp. $g_\infty^+(z)$) is the negative (resp. stricly positive)-powers part of $g_\infty(z) = g\big(\frac{1}{z}\big)$. The inequalities (2.16) and (2.20) yield

$$\begin{aligned}
&\Big\|z^kf\Big(\frac{1}{z}\Big)_{|D(0,r')}\Big\|_{r'} = \|(z^kf_\infty(z))_{|D(0,r')}\|_{r'} = \max\{\|(z^kf_\infty^-(z))_{|D(0,r')}\|_{r'}, \|(z^kf_\infty^+(z))_{|D(0,r')}\|_{r'}\}\\
&\le \max\Big\{\max_{0\le n\le k-1}\Big\{(n+1)\Big(\frac{r'}{r}\Big)^{k-1-n}\Big\}, \Big(\frac{r'}{r}\Big)^k\sup_{n\ge1}\Big\{n\Big(\frac{r'}{r}\Big)^n\Big\}\Big\}\cdot\|(z^{k-1}g_\infty(z))_{|D(0,r)}\|_r\\
&= \max\Big\{\max_{0\le n\le k-1}\Big\{(n+1)\Big(\frac{r'}{r}\Big)^{k-1-n}\Big\}, \Big(\frac{r'}{r}\Big)^k\sup_{n\ge1}\Big\{n\Big(\frac{r'}{r}\Big)^n\Big\}\Big\}\cdot\Big\|z^{k-1}g\Big(\frac{1}{z}\Big)_{|D(0,r)}\Big\|_r.
\end{aligned}\tag{2.24}$$

From (2.23) and (2.24) we infer that there exists a constant $C>0$ depending only on r, r', k such that $\|f\|_{r'}\le C\|g\|_r$ for any $g\in\mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1)[r]$ and for $f\in\Delta^{-1}(g)\cap\mathcal{A}_k(\mathbb{P}^1)[r']$ determined by g.

Returning to the openness of Δ. Let A be a neighborhood of 0 in $\mathcal{A}_k(\mathbb{P}^1)$. We prove that $\Delta(A)$ is a neighborhood of 0 in $\mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1)$. It is equivalent to show that $\Delta(A)\cap\mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1)[r]$ is a neighborhood of 0 in $\mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1)[r]$ for any $r\in|\mathbb{C}_p^\times|_p, r<1$. Let $r<1$ and choose $0<r'<r$ such that $r'=p^{-n_0}$ for some $n_0\in\mathbb{N}^*$. Since A is a neighborhood of 0 in $\mathcal{A}_k(\mathbb{P}^1)[r']$, there is a number $R>0$ such that A contains an open ball of center 0 and radius R in $\mathcal{A}_k(\mathbb{P}^1)[r']$. If $g\in\mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1)[r]$ such that $\|g\|_r<R/C$, then there is a function $f\in\mathcal{A}_k(\mathbb{P}^1)[r']$ such that $\Delta(f)=g$ and $\|f\|_{r'}\le C\|g\|_r$. It follows that $\|f\|_{r'}<R$, so $f\in A$, hence $g\in\Delta(A)$. Therefore, $\Delta(A)$ contains the open ball of center 0 and radius R/C in $\mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1)[r]$, hence $\Delta(A)$ is a neighborhood of 0 in $\mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1)[r]$. The openness of Δ is proven. ☐

Proposition 2.2.5 yields the following exact sequence for each $k\in\mathbb{Z}$:

$$0\to\ker\Delta\xrightarrow{i}\mathcal{A}_k(\mathbb{P}^1,L)\xrightarrow{\Delta}\mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1,L)\to0,$$

where i is the inclusion map. Dualizing this exact sequence yields the following complex

$$0\to\mathcal{D}_{k-1}^{(-1)}(\mathbb{P}^1,L)\xrightarrow{\Delta^*}\mathcal{D}_k(\mathbb{P}^1,L)\xrightarrow{i^*}\mathrm{Hom}_{\mathrm{cont}}(\ker\Delta,L^\times)\to0, \tag{2.25}$$

where $\mathcal{D}_{k-1}^{(-1)}(\mathbb{P}^1,L)$ is the continuous L-dual of $\mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1,L)$. It turns out that this dual complex is exact.

Lemma 2.2.10. *If $k\in\mathbb{Z}_{<0}$, there is a canonical exact sequence*

$$\begin{aligned}
0\to\mathcal{D}_{k-1}^{(-1)}(\mathbb{P}^1,L)\xrightarrow{\Delta^*}\mathcal{D}_k(\mathbb{P}^1,L)\xrightarrow{i^*}&\prod_{D(\alpha,1)\subset\mathbb{Q}_p}L\to0\\
\mu\mapsto&\,(\mu(\mathbb{1}_{D(\alpha,1)}))_{D(\alpha,1)\subset\mathbb{Q}_p}.
\end{aligned}\tag{2.26}$$

If $k\in\mathbb{Z}_{\ge0}$, the above exact sequence is replaced by

$$\begin{aligned}
0\to\mathcal{D}_{k-1}^{(-1)}(\mathbb{P}^1,L)\xrightarrow{\Delta^*}\mathcal{D}_k(\mathbb{P}^1,L)\xrightarrow{i^*}&\,L\times\prod_{D(\alpha,1)\subset\mathbb{Q}_p}L\to0\\
\mu\mapsto&\,\big(\mu(\mathbb{1}_{D_\infty}),(\mu(\mathbb{1}_{D(\alpha,1)}))_{D(\alpha,1)\subset\mathbb{Q}_p}\big),
\end{aligned}\tag{2.27}$$

where D_∞ is any open neighborhood of ∞ in $\mathbb{P}^1(\mathbb{Q}_p)$.

Proof. We show that the sequence (2.25) is exact. The injectivity of Δ^* is obvious. The exactness in the middle follows from the openness of $\Delta : \mathcal{A}_k(\mathbb{P}^1, L) \to \mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1, L)$ by Proposition 2.2.9 (see the proof of Proposition 2.2.4). The surjectivity of i^* is a consequence of [Schn, Corollary 9.4], where the field L here is spherically complete since it is a finite extension of $\mathbb{Q}_p$.

Let us determine the kernel of $\Delta : \mathcal{A}_k(\mathbb{P}^1, L) \to \mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1, L)$. Let $f \in \ker\Delta$, then $f(z+1) = f(z)$ for any $z \in \mathbb{P}^1(\mathbb{Q}_p)$, so $f(z+\mathbb{Z}) = f(z)$ for all $z \in \mathbb{P}^1(\mathbb{Q}_p)$. Since f is continuous, it follows that $f(z+\mathbb{Z}_p) = f(z)$ for any $z \in \mathbb{P}^1(\mathbb{Q}_p)$, so f is constant in every closed disc of radius 1 in $\mathbb{Q}_p$. We prove that f is constant in a neighborhood of ∞. Let $f_\infty(z) = f\left(\frac{1}{z}\right)$, defined in a neighborhood of 0. Then f_∞ is meromorphic of order of vanishing $\mathrm{ord}_0(f_\infty) \geq -k$. By (2.13) and Lemma 2.2.8, we can write f_∞ in the form

$$\sum_{n=0}^{k-1} \frac{c_n^-}{zR(z)R^{\circ 2}(z)...R^{\circ n}(z)} + \sum_{n=0}^{+\infty} c_n^+ zR(z)...R^{\circ(n-1)}(z),$$

where $R(z) = \frac{z}{z+1}$. Since $f_\infty(R(z)) = f_\infty(z)$, $\frac{1}{R^{\circ(n+1)}(z)} - \frac{1}{z} = n+1$ and $R^{\circ n}(z) - z = -nzR^{\circ n}(z)$, it follows that $c_n^- = 0$ for any $0 \leq n \leq k-1$ and $c_n^+ = 0$ for any $n \geq 1$. Therefore, f_∞ is constant in a neighborhood of 0, so f is constant in a neighborhood of ∞. Since $\mathrm{ord}_0(f_\infty) \geq -k$, if $k \geq 0$ this constant can be arbitrary, while if $k < 0$ it must be 0.

In summary, if $k < 0$, the kernel of $\Delta : \mathcal{A}_k(\mathbb{P}^1, L) \to \mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1, L)$ has the basis

$$\{\mathbb{1}_{D(\alpha,1)}\}_{D(\alpha,1)\subset\mathbb{Q}_p}$$

consisting of characteristic functions of all closed discs of radius 1 in $\mathbb{Q}_p$, while if $k \geq 0$ the basis of $\ker\Delta$ has one more function, that is the characteristic function of D_∞. Note that if $f = c \in L$ in the neighborhood $\{z \in \mathbb{P}^1(\mathbb{Q}_p), |z|_p \geq R\}$ of ∞ for $R > 1$ big enough, then the restriction of f on that neighborhood equals the function

$$c \cdot \mathbb{1}_{D_\infty} - \sum_{D(\alpha,1)\subset\{z\in D_\infty, |z|_p<R\}} c \cdot \mathbb{1}_{D(\alpha,1)}.$$

If μ is an L-valued continuous linear form on $\ker\Delta$, then μ is uniquely determined by the values $\mu(\mathbb{1}_{D(\alpha,1)})$ in L for $D(\alpha,1) \subset \mathbb{Q}_p$ if $k < 0$, or one more value $\mu(\mathbb{1}_{D_\infty})$ in L if $k \geq 0$. Since the functions $\mathbb{1}_{D(\alpha,1)}$ do not belong simultaneously to $\mathcal{A}_k(\mathbb{P}^1, L)[r]$ for any $r \in |\mathbb{C}_p^\times|_p, r < 1$, it follows that the values $\mu(\mathbb{1}_{D(\alpha,1)})$ for $D(\alpha,1) \subset \mathbb{Q}_p$ can be chosen arbitrarily. This explains the appearance of the last space in the sequences (2.26), (2.27). □

The zeroth homology group of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ with values in p-adic distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ is computed in the following result:

Theorem 2.2.11. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup containing the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.*

i) For any $k \in \mathbb{Z}\backslash\{0\}$, one has $H_0(\Gamma, \mathcal{D}_k(\mathbb{P}^1, L)) = 0$.

ii) If $\Gamma_1(N) \cap \Gamma_1(p^r) \subset \Gamma \subset \Gamma_0(N) \cap \Gamma_1(p^r)$ for $N, r \in \mathbb{N}^*$ with $(N,p) = 1$, letting $c_r = p^{[\frac{r}{2}]} + p^{r-[\frac{r}{2}]-1}$ where $[\cdot]$ denotes the integral part, then

$$H_0(\Gamma, \mathcal{D}_0(\mathbb{P}^1, L)) = L^{c_r}.$$

If $\Gamma_1(N) \cap \Gamma_0(p^r) \subset \Gamma \subset \Gamma_0(Np^r)$ for $N, r \in \mathbb{N}^*$ with $(N,p) = 1$, then

$$H_0(\Gamma, \mathcal{D}_0(\mathbb{P}^1, L)) = \begin{cases} L^{2r} & \text{if } p \neq 2 \text{ or } r = 1, \\ L^3 & \text{if } p = r = 2, \\ L^{2r-2} & \text{if } p = 2 \text{ and } r \geq 3. \end{cases}$$

Finally, $H_0(\mathrm{SL}_2(\mathbb{Z}), \mathcal{D}_0(\mathbb{P}^1, L)) = L$.

Proof. Since the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ belongs to Γ, its inverse $\gamma_0 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ also belongs to Γ. Recall that $H_0(\Gamma, V) = V/IV$ (resp. $V/V_{|I}$) for a left (resp. right) Γ-module V, where I is the augmentation ideal of Γ generated by elements $\gamma - 1$ for $\gamma \in \Gamma$. The inclusion $\mathcal{A}_{k-1}^{(-1)}(\mathbb{P}^1, L) \to \mathcal{A}_k(\mathbb{P}^1, L)$ induces the dual map

$$\mathcal{D}_k(\mathbb{P}^1, L) \to \mathcal{D}_{k-1}^{(-1)}(\mathbb{P}^1, L) \tag{2.28}$$

which is surjective by [Schn, Corollary 9.4]. We consider the following cases:

- $k < 0$. Combining the surjective map (2.28) with the exact sequence (2.26) yields the following exact sequence:

$$\mathcal{D}_k(\mathbb{P}^1, L) \xrightarrow{\Delta^*} \mathcal{D}_k(\mathbb{P}^1, L) \xrightarrow{i^*} \prod_{D(\alpha,1)\subset\mathbb{Q}_p} L \to 0$$
$$\mu \mapsto (\mu(\mathbb{1}_{D(\alpha,1)}))_{D(\alpha,1)\subset\mathbb{Q}_p},$$

where $\Delta^* : \mathcal{D}_k(\mathbb{P}^1, L) \to \mathcal{D}_k(\mathbb{P}^1, L)$ is the composition of (2.28) and the map $\Delta^* : \mathcal{D}_{k-1}^{(-1)}(\mathbb{P}^1, L) \to \mathcal{D}_k(\mathbb{P}^1, L)$. Then $\Delta^* : \mathcal{D}_k(\mathbb{P}^1, L) \to \mathcal{D}_k(\mathbb{P}^1, L)$ is given by the right weight k action of $\gamma_0 - 1$. We get the isomorphism

$$\mathcal{D}_k(\mathbb{P}^1, L)/\mathcal{D}_k(\mathbb{P}^1, L)_{|_k\gamma_0-1} \cong \prod_{D(\alpha,1)\subset\mathbb{Q}_p} L \tag{2.29}$$
$$\mu \mapsto (\mu(\mathbb{1}_{D(\alpha,1)}))_{D(\alpha,1)\subset\mathbb{Q}_p}.$$

Let $(x_\alpha) \in \prod_{D(\alpha,1)\subset\mathbb{Q}_p} L$. Choose a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ such that $p|c \neq 0$. For $\mu \in \mathcal{D}_k(\mathbb{P}^1, L)$, we have

$$\begin{aligned} \mu_{|_k\gamma-1}(\mathbb{1}_{D(\alpha,1)}) &= \mu((a-cz)^k \mathbb{1}_{D(\alpha,1)}(\gamma^{-1}z)) - \mu(\mathbb{1}_{D(\alpha,1)}) \\ &= \mu((a-cz)^k \mathbb{1}_{\gamma(D(\alpha,1))}(z)) - \mu(\mathbb{1}_{D(\alpha,1)}), \end{aligned} \tag{2.30}$$

where γ acts on $\mathbb{P}^1(\mathbb{Q}_p)$ by the linear fractional transformation.

The family of characteristic functions $\{\mathbb{1}_{D(\alpha,1)} \mid D(\alpha,1) \subset \mathbb{Q}_p\}$ is linear independent since the discs $D(\alpha,1)$ are pairwise disjoint, it spans the subspace V_1 of $\mathcal{A}_k(\mathbb{P}^1, L)$. The family of functions

$$\{(a-cz)^k \mathbb{1}_{\gamma(D(\alpha,1))}(z) \mid D(\alpha,1) \subset \mathbb{Q}_p\}$$

is also linear independent, it spans the subspace V_2 of $\mathcal{A}_k(\mathbb{P}^1, L)$. We show that the sum $V_1 + V_2$ is direct, i.e., $V_1 \cap V_2 = \{0\}$. Let f be a function in $V_1 \cap V_2$. Since $f \in V_1$, it has only finitely many values. Since $f \in V_2$, it has the form

$$f(z) = \sum_{D(\alpha,1)\subset\mathbb{Q}_p} t_\alpha (a-cz)^k \mathbb{1}_{\gamma(D(\alpha,1))}(z)$$

for $t_\alpha \in L$ and $t_\alpha = 0$ for all but finitely many α. Since the sets $\gamma(D(\alpha,1))$ are pairwise disjoint, if the coefficients t_α are not simultaneously 0, then the function f would have infinitely many values since $c \neq 0, k \neq 0$. Therefore, all the coefficients t_α are 0, so $f = 0$. Hence $V_1 \cap V_2 = \{0\}$.

Defining the L-linear form μ_0 on $V_1 \oplus V_2$ by

$$\mu_{0|V_1} = 0, \quad \mu_0((a-cz)^k \mathbb{1}_{\gamma(D(\alpha,1))}(z)) = x_\alpha \text{ for any } D(\alpha,1) \subset \mathbb{Q}_p.$$

We will see that every linear form on $V_1 \oplus V_2$ is continuous, so we can extend μ_0 to a continuous L-linear form μ on $\mathcal{A}_k(\mathbb{P}^1, L)$ by [Schn, Corollary 9.4]. Then $\mu \in \mathcal{D}_k(\mathbb{P}^1, L)$ and $\mu_{|_k\gamma-1}(\mathbb{1}_{D(\alpha,1)}) = x_\alpha$ for any $D(\alpha,1) \subset \mathbb{Q}_p$ by (2.30). We deduce that the image of $\mathcal{D}_k(\mathbb{P}^1, L)_{|_k\Gamma-1}$ by (2.29) is all of $\prod_{D(\alpha,1)\subset\mathbb{Q}_p} L$. Therefore, $H_0(\Gamma, \mathcal{D}_k(\mathbb{P}^1, L)) = 0$.

To see that every linear form on $V_1 \oplus V_2$ is continuous, we need the following lemma:

Lemma 2.2.12. *Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$ such that $c \neq 0$. For any $\alpha \in \mathbb{Q}_p$ such that $v_p(\alpha) < -v_p(c)$, the linear fractional transformation of γ maps the disc $D(\alpha,1) \subset \mathbb{Q}_p$ onto the disc $D\big(\gamma\alpha, \frac{1}{|c|_p^2|\alpha|_p^2}\big)$ which is contained in $\{z \in \mathbb{Q}_p, v_p(z) = -v_p(c)\}$.*

Proof. Since $v_p(\alpha) < -v_p(c) \leq 0 \leq v_p(\mathbb{Z}_p)$, we infer that $|\alpha|_p > 1$ and the disc $D(\alpha,1) = \alpha + \mathbb{Z}_p$ is contained in $\{z \in \mathbb{Q}_p, v_p(z) = v_p(\alpha)\}$. If $v_p(z) = v_p(\alpha)$, then $v_p(cz) = v_p(c) + v_p(\alpha) < 0 \leq v_p(d)$, so $v_p(cz+d) = v_p(cz)$. Since $\gamma \in \Gamma_0(p)$, it follows that $p|c$, so $p \nmid a$, hence $v_p(az) = v_p(z)$ for any z.

Therefore, if $v_p(z) = v_p(\alpha)$, then $v_p(az) = v_p(\alpha) < -v_p(c) \le 0 \le v_p(b)$, so $v_p(az+b) = v_p(az)$. We get

$$v_p(\gamma z) = v_p\Big(\frac{az+b}{cz+d}\Big) = v_p(az+b) - v_p(cz+d) = v_p(az) - v_p(cz) = -v_p(c)$$

if $v_p(z) = v_p(\alpha)$. Therefore, γ maps $D(\alpha,1)$ into the set $\{z \in \mathbb{Q}_p, v_p(z) = -v_p(c)\}$. We have

$$\begin{aligned}\gamma z - \gamma\alpha &= \frac{az+b}{cz+d} - \frac{a\alpha+b}{c\alpha+d} = \frac{z-\alpha}{(cz+d)(c\alpha+d)} = \frac{z-\alpha}{c^2(z+\frac{d}{c})(\alpha+\frac{d}{c})}\\ &= \frac{z-\alpha}{c^2(\alpha+\frac{d}{c})(z-\alpha+\alpha+\frac{d}{c})} = \frac{1}{c^2(\alpha+\frac{d}{c})(1+\frac{\alpha+\frac{d}{c}}{z-\alpha})}.\end{aligned} \tag{2.31}$$

Since $v_p(\alpha) < -v_p(c)$ and $v_p(\frac{d}{c}) = v_p(d) - v_p(c) \ge -v_p(c)$, so $v_p(\alpha) < v_p(\frac{d}{c})$, it follows that $v_p(\alpha + \frac{d}{c}) = v_p(\alpha)$, hence $|\alpha + \frac{d}{c}|_p = |\alpha|_p$. If $z \in D(\alpha,1)$, then $|z-\alpha|_p \le 1$, so

$$\Big|\frac{\alpha+\frac{d}{c}}{z-a}\Big|_p = \frac{|\alpha|_p}{|z-\alpha|_p} \ge |\alpha|_p > 1.$$

Hence

$$\Big|1+\frac{\alpha+\frac{d}{c}}{z-a}\Big|_p = \Big|\frac{\alpha+\frac{d}{c}}{z-a}\Big|_p = \frac{|\alpha|_p}{|z-\alpha|_p}.$$

Therefore,

$$|\gamma z - \gamma\alpha|_p = \frac{|z-\alpha|_p}{|c|_p^2|\alpha|_p^2} \le \frac{1}{|c|_p^2|\alpha|_p^2}.$$

It follows that $\gamma z \in D\big(\gamma\alpha, \frac{1}{|c|_p^2|\alpha|_p^2}\big)$. So γ maps the disc $D(\alpha,1)$ into the disc $D\big(\gamma\alpha, \frac{1}{|c|_p^2|\alpha|_p^2}\big)$.

If $x \in D\big(\gamma\alpha, \frac{1}{|c|_p^2|\alpha|_p^2}\big)$, then $x - \gamma\alpha = \frac{1}{c^2(\alpha+\frac{d}{c})y}$ for some $y \in \mathbb{Q}_p$ such that $|y|_p \ge |\alpha|_p$ (note that $|\alpha+\frac{d}{c}|_p = |\alpha|_p$). Writing $y = 1 + \frac{\alpha+\frac{d}{c}}{z-\alpha}$ for $z = \alpha + \frac{\alpha+\frac{d}{c}}{y-1} \in D(\alpha,1)$ (since $|y|_p \ge |\alpha|_p > 1$, it follows that $|y-1|_p = |y|_p$). By (2.31) we see that $x - \gamma\alpha = \gamma z - \gamma\alpha$, so $x = \gamma z$. Therefore, $\gamma(D(\alpha,1))$ equals $D\big(\gamma\alpha, \frac{1}{|c|_p^2|\alpha|_p^2}\big)$. ☐

By the lemma, if $\alpha \to \infty$, then the radius of the disc $\gamma(D(\alpha,1))$ tends to 0, so there does not exist $r \in |\mathbb{C}_p^\times|_p, r < 1$ such that the functions $(a-cz)^k \mathbb{1}_{\gamma(D(\alpha,1))}$ belong to $\mathcal{A}_k(\mathbb{P}^1, L)[r]$ for infinitely many discs $D(\alpha,1)$. Therefore, the values on the functions $(a-cz)^k\mathbb{1}_{\gamma(D(\alpha,1))}(z)$ indexed by the discs $D(\alpha,1)$ of a continuous form μ_0 on V_2 can be arbitrary.

We resume with the proof of the theorem.

- $k > 0$. By (2.27), the isomorphism (2.29) is replaced by

$$\begin{aligned}\mathcal{D}_k(\mathbb{P}^1, L)/\mathcal{D}_k(\mathbb{P}^1, L)_{|_k\gamma_0 - 1} &\cong L \times \prod_{D(\alpha,1)\subset\mathbb{Q}_p} L\\ \mu &\mapsto \big(\mu(\mathbb{1}_{D_\infty}), (\mu(\mathbb{1}_{D(\alpha,1)}))_{D(\alpha,1)\subset\mathbb{Q}_p}\big),\end{aligned} \tag{2.32}$$

where D_∞ is any open neighborhood of ∞ in $\mathbb{P}^1(\mathbb{Q}_p)$. Let $(x_\infty, (x_\alpha)) \in L \times \prod_{D(\alpha,1)\subset\mathbb{Q}_p} L$. Fix a matrix $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma \cap \Gamma_0(p)$ such that $c \neq 0$. If $\mu \in \mathcal{D}_k(\mathbb{P}^1, L)$, we have

$$\begin{aligned}\mu_{|_k\gamma^{-1}}(\mathbb{1}_{D_\infty}) &= \mu((a-cz)^k \mathbb{1}_{D_\infty}(\gamma^{-1}z)) - \mu(\mathbb{1}_{D_\infty})\\ &= \mu((a-cz)^k\mathbb{1}_{\gamma(D_\infty)}(z)) - \mu(\mathbb{1}_{D_\infty}).\end{aligned} \tag{2.33}$$

Let $V_1' \subset \mathcal{A}_k(\mathbb{P}^1, L)$ be the subspace with the basis $\{\mathbb{1}_{D(\alpha,1)} \,|\, D(\alpha,1) \subset \mathbb{Q}_p\} \cup \{\mathbb{1}_{D_\infty}\}$. Let $V_2' \subset \mathcal{A}_k(\mathbb{P}^1, L)$ be the subspace with the basis

$$\{(a-cz)^k\mathbb{1}_{\gamma(D(\alpha,1))}(z) \,|\, D(\alpha,1) \subset \mathbb{Q}_p\} \cup \{(a-cz)^k\mathbb{1}_{\gamma(D_\infty)}(z)\}.$$

Similar to the above case, the sum $V_1' + V_2'$ in $\mathcal{A}_k(\mathbb{P}^1, L)$ is direct. We define the continuous L-linear form μ_0' on $V_1' \oplus V_2'$ by

$$\mu'_{0|V_1'} = 0, \quad \mu_0'((a-cz)^k \mathbb{1}_{\gamma(D_\infty)}(z)) = x_\infty, \quad \mu_0'((a-cz)^k \mathbb{1}_{\gamma(D(\alpha,1))}(z)) = x_\alpha.$$

Extending μ_0' to a continuous L-linear form μ' on $\mathcal{A}_k(\mathbb{P}^1, L)$ by [Schn, Corollary 9.4]. Then $\mu' \in \mathcal{D}_k(\mathbb{P}^1, L)$ and

$$\mu'_{|_k\gamma-1}(\mathbb{1}_{D_\infty}) = x_\infty, \quad \mu'_{|_k\gamma-1}(\mathbb{1}_{D(\alpha,1)}) = x_\alpha$$

for any $D(\alpha, 1) \subset \mathbb{Q}_p$ by (2.30) and (2.33). So the image of $\mathcal{D}_k(\mathbb{P}^1, L)_{|_k\Gamma-1}$ by (2.32) is all of $L \times \prod_{D(\alpha,1)\subset\mathbb{Q}_p} L$. Therefore, $H_0(\Gamma, \mathcal{D}_k(\mathbb{P}^1, L)) = 0$.

- $k = 0$. Similar to (2.32) we have the isomorphism

$$\mathcal{D}_0(\mathbb{P}^1, L)/\mathcal{D}_0(\mathbb{P}^1, L)_{|0,\gamma_0-1} \cong L \times \prod_{D(\alpha,1)\subset\mathbb{Q}_p} L \tag{2.34}$$

$$\mu \mapsto \big(\mu(\mathbb{1}_{D(\infty,2r)}), (\mu(\mathbb{1}_{D(\alpha,1)}))_{D(\alpha,1)\subset\mathbb{Q}_p}\big),$$

where for each integer r' we set $D(\infty, r') = \{z \in \mathbb{P}^1(\mathbb{Q}_p), v_p(z) \leq -r'\}$. The set of closed discs of radius 1 in $\mathbb{Q}_p$ consists of $\mathbb{Z}_p = D(0, 1)$ and the discs $D(\alpha, 1) = \alpha + \mathbb{Z}_p$ for $v_p(\alpha) \leq -1$. If $v_p(\alpha) = -n \leq -1$ for $n \in \mathbb{N}^*$, then $D(\alpha, 1)$ is contained in $\{z \in \mathbb{Q}_p, v_p(z) = -n\}$. The set $\{z \in \mathbb{Q}_p, v_p(z) = -n\}$ is partitioned by discs $D(p^{-n}\beta, 1)$ where $\beta \in \mathbb{Z}_p^\times$ runs through a complete set of representatives of $(\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times$.

Let $\Gamma \subset \Gamma_0(p)$ be a congruence subgroup. The weight 0 action of a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ on $\mathcal{D}_0(\mathbb{P}^1, L)$ is given by

$$\mu_{|0,\gamma}(f) = \mu(f(\gamma^{-1}z)),$$

where $\mu \in \mathcal{D}_0(\mathbb{P}^1, L)$ and $f \in \mathcal{A}_0(\mathbb{P}^1, L)$. Since $\Gamma \subset \Gamma_0(p) \subset \Sigma_0(p)$ (see §1.2.1), every matrix in Γ preserves $\mathbb{Z}_p$. We have

$$\mu_{|0,\gamma-1}(\mathbb{1}_{\mathbb{Z}_p}) = \mu(\mathbb{1}_{\gamma(\mathbb{Z}_p)}) - \mu(\mathbb{1}_{\mathbb{Z}_p}) = 0 \tag{2.35}$$

for any $\gamma \in \Gamma$ and $\mu \in \mathcal{D}_0(\mathbb{P}^1, L)$. For the image of other closed discs of radius 1 in $\mathbb{Q}_p$ by elements of Γ , we need the following lemmas:

Lemma 2.2.13. *Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$ with $c \neq 0$. Inside the set $\{z \in \mathbb{Q}_p, v_p(z) = -v_p(c)\}$, there is exactly one closed disc of radius 1 (which is $D(-\frac{d}{c}, 1)$) such that its image by γ is the neighborhood $D(\infty, 2v_p(c)) := \{z \in \mathbb{P}^1(\mathbb{Q}_p), v_p(z) \leq -2v_p(c)\}$ of ∞, other closed discs of radius 1 are mapped either onto closed discs of radius $\geq p^2$ in $\{z \in \mathbb{Q}_p, -2v_p(c) < v_p(z) < -v_p(c)\}$ or onto closed discs of radius 1 in $\{z \in \mathbb{Q}_p, v_p(z) = -v_p(c)\}$ by γ.*

Proof. Since $p|c$, we have $(a, p) = (d, p) = 1$, so $v_p(\frac{-d}{c}) = -v_p(c) < 0$, hence $D(\frac{-d}{c}, 1) = \frac{-d}{c} + \mathbb{Z}_p \subset \{z \in \mathbb{Q}_p, v_p(z) = -v_p(c)\}$. We show that γ maps $D(\frac{-d}{c}, 1)$ onto $D(\infty, 2v_p(c))$. Let $z \in D(\frac{-d}{c}, 1)$, then $z = -\frac{d}{c} + z'$ for some $z' \in \mathbb{Z}_p$. We have

$$\gamma z = \frac{az+b}{cz+d} = \frac{az' - \frac{1}{c}}{cz'} = \frac{a}{c} - \frac{1}{c^2 z'}.$$

Since $v_p(\frac{a}{c}) = -v_p(c)$ and $v_p(\frac{1}{c^2z'}) = -2v_p(c) - v_p(z') \leq -2v_p(c) < -v_p(c)$, it follows that $v_p(\gamma z) = v_p(\frac{1}{c^2z'}) \leq -2v_p(c)$. We deduce that if z' runs through $\mathbb{Z}_p$, then γz runs through $\{v_p(\cdot) \leq -2v_p(c)\}$. Therefore, γ maps $D(-\frac{d}{c}, 1)$ onto $D(\infty, 2v_p(c))$.

Consider a disc $D(\alpha, 1) \subset \{z \in \mathbb{Q}_p, v_p(z) = -v_p(c)\}$ different from $D(\frac{-d}{c}, 1)$, then $\gamma(D(\alpha, 1))$ is disjoint from $D(\infty, 2v_p(c))$, hence $\gamma(D(\alpha, 1))$ is contained in $\{z \in \mathbb{Q}_p, v_p(z) > -2v_p(c)\}$. Let

$z \in D(\alpha,1)$. Since $v_p(z) = -v_p(c)$, it follows that $v_p(cz) = 0$, so $v_p(cz+d) \geq 0$. On the other hand, since $v_p(az) = v_p(z) = -v_p(c) < 0 \leq v_p(b)$, we have $v_p(az+b) = v_p(az) = -v_p(c)$. Therefore,

$$v_p(\gamma z) = v_p\Big(\frac{az+b}{cz+d}\Big) = v_p(az+b) - v_p(cz+d) \leq v_p(az+b) = -v_p(c).$$

So γ maps $D(\alpha,1)$ into the set $\{z \in \mathbb{Q}_p, v_p(z) \leq -v_p(c)\}$. Moreover, since $z \in D(\alpha,1) = \alpha + \mathbb{Z}_p$, we have

$$cz + d \in c\alpha + d + c\mathbb{Z}_p = c\alpha + d + p^{v_p(c)}\mathbb{Z}_p. \tag{2.36}$$

If $c\alpha+d \in p\mathbb{Z}_p$, then $cz+d \in p\mathbb{Z}_p$, so $v_p(\gamma z) < -v_p(c)$, hence the image of $D(\alpha,1)$ by γ is contained in $\{z \in \mathbb{Q}_p, -2v_p(c) < v_p(z) < -v_p(c)\}$. Writing $z = \alpha + z'$ for $z' \in \mathbb{Z}_p$, we have

$$\begin{aligned}\gamma z - \gamma\alpha &= \frac{z-\alpha}{(cz+d)(c\alpha+d)} = \frac{z'}{(cz'+c\alpha+d)(c\alpha+d)}\\ &= \frac{1}{c\alpha+d}\cdot\frac{1}{c+\frac{c\alpha+d}{z'}} = \frac{1}{c(c\alpha+d)}\cdot\frac{1}{1+\frac{\alpha+\frac{d}{c}}{z'}}.\end{aligned}$$

Since $\alpha \notin D(-\frac{d}{c},1)$, we have $v_p(\alpha+\frac{d}{c}) < 0$, so $v_p(\frac{\alpha+\frac{d}{c}}{z'}) < 0 = v_p(1)$, hence $v_p(1+\frac{\alpha+\frac{d}{c}}{z'}) = v_p(\frac{\alpha+\frac{d}{c}}{z'}) \leq v_p(\alpha+\frac{d}{c})$. Moreover, the set $\{1+\frac{\alpha+\frac{d}{c}}{z'} \mid z' \in \mathbb{Z}_p\}$ equals the set $\{z'' \in \mathbb{Q}_p, v_p(z'') \leq v_p(\alpha+\frac{d}{c})\}$. Therefore, $\gamma(D(\alpha,1)) = D(\gamma\alpha, p^{2v_p(c\alpha+d)})$ is the closed disc of radius $\geq p^2$ if $c\alpha+d \in p\mathbb{Z}_p$.

If $c\alpha+d \in \mathbb{Z}_p^\times$, then $cz+d \in \mathbb{Z}_p^\times$ by (2.36), so $v_p(\gamma z) = -v_p(c)$, hence the image of $D(\alpha,1)$ by γ is contained in $\{z \in \mathbb{Q}_p, v_p(z) = -v_p(c)\}$. Since

$$\gamma z - \gamma\alpha = \frac{z-\alpha}{(cz+d)(c\alpha+d)}$$

and $cz+d, c\alpha+d \in \mathbb{Z}_p^\times$, it follows that $|\gamma z - \gamma\alpha|_p = |z-\alpha|_p \leq 1$, so $\gamma z \in D(\gamma\alpha,1)$, hence $\gamma(D(\alpha,1)) \subset D(\gamma\alpha,1)$. Similar, by considering $\gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$, we get $\gamma^{-1}(D(\gamma\alpha,1)) \subset D(\alpha,1)$ since $-c(\gamma\alpha)+a = (c\alpha+d)^{-1} \in \mathbb{Z}_p^\times$, so $D(\gamma\alpha,1) \subset \gamma(D(\alpha,1))$. We deduce that $\gamma(D(\alpha,1)) = D(\gamma\alpha,1) \subset \{z \in \mathbb{Q}_p, v_p(z) = -v_p(c)\}$ if $c\alpha+d \in \mathbb{Z}_p^\times$. ☐

Lemma 2.2.14. *Let $r \in \mathbb{N}_{\geq 2}$ and $n \in \mathbb{N}^*$ such that $-r+1 \leq -n \leq -1$. Let $\gamma \in \Gamma_0(p^r)$. The linear fractional transformation of γ on $\mathbb{P}^1(\mathbb{Q}_p)$ has the following properties:*

i) γ permutes the family of closed discs of radius 1 in $\{z \in \mathbb{Q}_p, v_p(z) = -n\}$. Moreover, if $\alpha_0 \in \mathbb{Z}_p^\times$, then γ maps the disc $D(p^{-n}\alpha_0,1)$ onto the disc $D(p^{-n}\beta_0,1)$ for some $\beta_0 \in \mathbb{Z}_p^\times$ such that $\frac{\alpha_0}{\beta_0}$ is a square modulo p^{r-n} (note that $r-n>0$ since $-n \geq -r+1$). In particular, if $-[\frac{r}{2}] \leq -n \leq -1$, then $\frac{\alpha_0}{\beta_0}$ is a square modulo p^n.

Conversely, for any $\alpha_0, \beta_0 \in \mathbb{Z}_p^\times$ such that $\frac{\alpha_0}{\beta_0}$ is a square modulo p^{r-n}, there exists a matrix $\gamma \in \Gamma(N) \cap \Gamma_0(p^r)$ mapping the disc $D(p^{-n}\alpha_0,1)$ onto the disc $D(p^{-n}\beta_0,1)$, where $N \in \mathbb{N}^*$ such that $(N,p)=1$. If moreover $-[\frac{r}{2}] \leq -n \leq -1$, then $r-n \geq n$ and we can reduce to the condition $\frac{\alpha_0}{\beta_0}$ is a square modulo p^n.

ii) If moreover $\gamma \in \Gamma_1(p^r)$, then γ preserves closed discs of radius 1 in $\{z \in \mathbb{Q}_p, v_p(z) = -n\}$ if $-[\frac{r}{2}] \leq -n \leq -1$ with $r \geq 2$, and γ maps a disc $D(p^{-n}\alpha_0,1)$ for $\alpha_0 \in \mathbb{Z}_p^\times$ onto the disc $D(p^{-n}\beta_0,1)$ for some $\beta_0 \in \mathbb{Z}_p^\times$ congruence to α_0 modulo p^{r-n} if $-r+1 \leq -n \leq -[\frac{r}{2}]-1$ with $r \geq 3$.

Conversely, for any $\alpha_0, \beta_0 \in \mathbb{Z}_p^\times$ such that $\alpha_0 \equiv \beta_0 \pmod{p^{r-n}}$ with $-r+1 \leq -n \leq -[\frac{r}{2}]-1$, there is a matrix $\gamma \in \Gamma(Np^r)$ mapping the disc $D(p^{-n}\alpha_0,1)$ onto the disc $D(p^{-n}\beta_0,1)$, where $N \in \mathbb{N}^*$ such that $(N,p)=1$. Note that the disc $D(p^{-n}\beta_0,1)$ depends only on the congruence class of β_0 modulo p^n, and $r-n<n$ if $-n \leq -[\frac{r}{2}]-1$.

Proof. i) Consider $r,n \in \mathbb{N}^*$ such that $-r+1 \leq -n \leq -1$ and $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(p^r)$. If $\alpha \in \mathbb{Q}_p$ such that $v_p(\alpha) = -n$, then $v_p(\alpha) \leq -1 < 0$, so the disc $D(\alpha,1)$ is contained in

$\{z \in \mathbb{Q}_p, v_p(z) = -n\}$. If $z \in \mathbb{Q}_p$ such that $v_p(z) = -n$, then $v_p(cz) = v_p(c)+v_p(z) \geq r-n \geq 1$. Since $\gamma \in \Gamma_0(p^r)$ and $r \geq 1$, it follows that $p|c$, so $(a,p) = (d,p) = 1$, hence $cz + d \in \mathbb{Z}_p^\times$ and $v_p(az) = v_p(z) = -n \leq -1 < 0 \leq v_p(b)$. Therefore, $v_p(az + b) = v_p(az) = -n$. We get

$$v_p(\gamma z) = v_p\Big(\frac{az + b}{cz + d}\Big) = v_p(az + b) - v_p(cz + d) = -n.$$

Therefore, γ preserves the set $\{z \in \mathbb{Q}_p, v_p(z) = -n\}$. We have

$$\gamma z - \gamma\alpha = \frac{z - \alpha}{(cz + d)(c\alpha + d)}.$$

We have seen that $v_p(cz + d) = v_p(c\alpha + d) = 0$ if $v_p(z) = v_p(\alpha) = -n$, so $\gamma z \in D(\gamma\alpha, 1)$ if $z \in D(\alpha, 1)$, hence $\gamma(D(\alpha, 1)) \subset D(\gamma\alpha, 1)$. Similar, $\gamma^{-1}(D(\gamma\alpha, 1)) \subset D(\alpha, 1)$. Therefore, $\gamma(D(\alpha, 1)) = D(\gamma\alpha, 1)$. We conclude that γ permutes the family of closed discs of radius 1 in $\{z \in \mathbb{Q}_p, v_p(z) = -n\}$ for any n such that $-r + 1 \leq -n \leq -1$.

Consider $\alpha_0 \in \mathbb{Z}_p^\times$, then γ maps $D(p^{-n}\alpha_0, 1)$ onto $D(\gamma(p^{-n}\alpha_0), 1) = \gamma(p^{-n}\alpha_0) + \mathbb{Z}_p$. We have

$$\gamma(p^{-n}\alpha_0) = \frac{ap^{-n}\alpha_0 + b}{cp^{-n}\alpha_0 + d} = p^{-n}\frac{a\alpha_0 + bp^n}{cp^{-n}\alpha_0 + d} \in p^{-n}\frac{a\alpha_0}{cp^{-n}\alpha_0 + d} + \mathbb{Z}_p, \tag{2.37}$$

since $cp^{-n}\alpha_0 + d \in \mathbb{Z}_p^\times$. Therefore, $\gamma(D(p^{-n}\alpha_0, 1)) = D(p^{-n}\beta_0, 1)$ for $\beta_0 = \frac{a\alpha_0}{cp^{-n}\alpha_0+d}$. Since $p^r|c$, it follows that $cp^{-n}\alpha_0 + d \equiv d \pmod{p^{r-n}}$. On the other hand, since $ad = 1 + bc \equiv 1 \pmod{p^r}$, we have $cp^{-n}\alpha_0 + d \equiv a^{-1} \pmod{p^{r-n}}$. Therefore, $\beta_0 \equiv a^2\alpha_0 \pmod{p^{r-n}}$, so $\frac{\alpha_0}{\beta_0}$ is a square modulo p^{r-n}. If $-[\frac{r}{2}] \leq -n \leq -1$, then $r - n \geq n$, so $\frac{\alpha_0}{\beta_0}$ is a square modulo p^n.

Conversely, suppose $\alpha_0, \beta_0 \in \mathbb{Z}_p^\times$ such that $\frac{\alpha_0}{\beta_0}$ is a square modulo p^{r-n}. By Chinese remainder theorem, since $(N, p) = 1$, we can choose $a \in \mathbb{Z} \cap \mathbb{Z}_p^\times$ such that $a \equiv 1 \pmod N$ and $a^2 \equiv \frac{\beta_0}{\alpha_0} \pmod{p^{r-n}}$. Since $\frac{a\alpha_0}{\beta_0} \equiv \frac{1}{a} \pmod{p^{r-n}}$, letting $\frac{a\alpha_0}{\beta_0} = \frac{1}{a} + xp^{r-n}$ for $x \in \mathbb{Z}_p$. Since $(a, Np) = 1$, we can choose $c \in \mathbb{Z}$ such that $c \equiv p^r\frac{x}{\alpha_0} \pmod{p^{n+r}}$, $c \equiv 0 \pmod N$ and $c \equiv 1 \pmod a$. Then $Np^r|c$ and $(a, c) = 1$. Hence $(a, Np^rc) = 1$. Taking $d \in \mathbb{Z}$ such that $ad \equiv 1 \pmod{Np^rc}$. Letting $b = \frac{ad-1}{c} \in Np^r\mathbb{Z}$, then the matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $\Gamma(N) \cap \Gamma_0(p^r)$.

By (2.37), γ maps $D(p^{-n}\alpha_0, 1)$ onto $D(p^{-n}\frac{a\alpha_0}{cp^{-n}\alpha_0+d}, 1)$. By the construction, $p^n|(x - cp^{-r}\alpha_0)$, so $p^r|(xp^{r-n} - cp^{-n}\alpha_0)$, hence $\frac{a\alpha_0}{\beta_0} \equiv \frac{1}{a} + cp^{-n}\alpha_0 \pmod{p^r}$. Since $p^r|c$ and $c|(ad-1)$, it follows that $\frac{1}{a} \equiv d \pmod{p^r}$. Therefore, $\frac{a\alpha_0}{\beta_0} \equiv d + cp^{-n}\alpha_0 \pmod{p^r}$. So $\frac{a\alpha_0}{cp^{-n}\alpha_0+d} \equiv \beta_0 \pmod{p^r}$, hence mod p^n since $r > n$. We obtain that $p^{-n}\frac{a\alpha_0}{cp^{-n}\alpha_0+d} \in p^{-n}\beta_0 + \mathbb{Z}_p$. We get

$$D\Big(p^{-n}\frac{a\alpha_0}{cp^{-n}\alpha_0 + d}, 1\Big) = p^{-n}\frac{a\alpha_0}{cp^{-n}\alpha_0 + d} + \mathbb{Z}_p = p^{-n}\beta_0 + \mathbb{Z}_p = D(p^{-n}\beta_0, 1).$$

We conclude that γ maps the disc $D(p^{-n}\alpha_0, 1)$ onto $D(p^{-n}\beta_0, 1)$.

If $-[\frac{r}{2}] \leq -n \leq -1$ and $\frac{\alpha_0}{\beta_0}$ is a square modulo p^n, we construct as above accept the number a is chosen so that $a^2 \equiv \frac{\beta_0}{\alpha_0} \pmod{p^n}$ and the number c satisfies $c \equiv p^{2n}\frac{x}{\alpha_0} \pmod{p^{n+r}}$, where $x \in \mathbb{Z}_p$ is given by $\frac{a\alpha_0}{\beta_0} = \frac{1}{a} + xp^n$. Then $cp^{-n}\alpha_0 \equiv p^n x \pmod{p^r}$, so $cp^{-n}\alpha_0 \equiv 0 \pmod{p^n}$ since $r > n$, hence $\frac{a\alpha_0}{cp^{-n}\alpha_0+d} \equiv \frac{a\alpha_0}{d} \equiv a^2\alpha_0 \equiv \beta_0 \pmod{p^n}$. Therefore, $p^{-n}\frac{a\alpha_0}{cp^{-n}\alpha_0+d} \in p^{-n}\beta_0 + \mathbb{Z}_p$.

ii) Suppose $\gamma \in \Gamma_1(p^r)$ and $\alpha \in \mathbb{Q}_p$ such that $v_p(\alpha) = -n$. We have

$$\gamma\alpha - \alpha = \frac{a\alpha + b}{c\alpha + d} - \alpha = \frac{\alpha(-c\alpha + a - d) + b}{c\alpha + d}. \tag{2.38}$$

We have seen that $c\alpha + d \in \mathbb{Z}_p^\times$. Since $a \equiv d \equiv 1 \pmod{p^r}$, $v_p(a - d) \geq r$. Since $v_p(-c\alpha) = v_p(c) + v_p(\alpha) \geq r - n$, it follows that $v_p(-c\alpha + a - d) \geq r - n$, so $v_p(\alpha(-c\alpha + a - d)) \geq r - 2n$. Hence $v_p(\gamma\alpha - \alpha) \geq \min(r - 2n, 0)$. Therefore, $\gamma\alpha \equiv \alpha \pmod{p^{\min(r-2n,0)}}$.

If $-[\frac{r}{2}] \leq -n \leq -1$, then $r - 2n \geq 0$, so $\gamma\alpha - \alpha \in \mathbb{Z}_p$, hence

$$\gamma(D(\alpha, 1)) = D(\gamma\alpha, 1) = \gamma\alpha + \mathbb{Z}_p = \alpha + \mathbb{Z}_p = D(\alpha, 1).$$

If $-r+1 \leq -n \leq -[\frac{r}{2}]-1$, then $r-2n<0$, so $\gamma\alpha \equiv \alpha \,(\mathrm{mod}\, p^{r-2n})$. If $\alpha = p^{-n}\alpha_0$ for $\alpha_0 \in \mathbb{Z}_p^\times$, then $\gamma\alpha = p^{-n}\beta_0$ for some $\beta_0 \in \mathbb{Z}_p^\times$ such that $\beta_0 \equiv \alpha_0 \,(\mathrm{mod}\, p^{r-n})$. The image of the disc $D(p^{-n}\alpha_0, 1)$ by γ is the disc $D(\gamma\alpha, 1)$ which is $D(p^{-n}\beta_0, 1)$.
Conversely, consider $-r+1 \leq -n \leq -[\frac{r}{2}]-1$ and $\alpha_0, \beta_0 \in \mathbb{Z}_p^\times$ such that $\alpha_0 \equiv \beta_0 \,(\mathrm{mod}\, p^{r-n})$. In the proof of part i), we have constructed a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) \cap \Gamma_0(p^r)$ mapping the disc $D(p^{-n}\alpha_0, 1)$ onto the disc $D(p^{-n}\beta_0, 1)$, where $b \in Np^r\mathbb{Z}$ and the entry a is chosen so that $a \equiv 1 \,(\mathrm{mod}\, N)$ and $a^2 \equiv \frac{\beta_0}{\alpha_0} \,(\mathrm{mod}\, p^{r-n})$. Since $\alpha_0 \equiv \beta_0 \,(\mathrm{mod}\, p^{r-n})$, we can choose a such that $a \equiv 1 \,(\mathrm{mod}\, p^r)$, then $\gamma \in \Gamma(Np^r)$.

The lemma is proven. □

We resume with the proof of the theorem. Consider the congruence subgroup Γ such that $\Gamma_1(N) \cap \Gamma_1(p^r) \subset \Gamma \subset \Gamma_0(N) \cap \Gamma_1(p^r)$, where $(N,p)=1$ and $r \geq 1$. For $\mu \in \mathcal{D}_0(\mathbb{P}^1, L), \gamma \in \Gamma$ and $\alpha \in \mathbb{Q}_p$, we have

$$\begin{aligned} \mu_{|0,\gamma-1}(\mathbb{1}_{D(\alpha,1)}) &= \mu(\mathbb{1}_{\gamma(D(\alpha,1))}) - \mu(\mathbb{1}_{D(\alpha,1)}), \\ \mu_{|0,\gamma-1}(\mathbb{1}_{D(\infty,2r)}) &= \mu(\mathbb{1}_{\gamma(D(\infty,2r))}) - \mu(\mathbb{1}_{D(\infty,2r)}). \end{aligned} \tag{2.39}$$

Since $\gamma \in \Gamma \subset \Gamma_1(p^r)$, by part ii) of Lemma 2.2.14, we have

$$\mu_{|0,\gamma-1}(\mathbb{1}_{D(\alpha,1)}) = 0 \tag{2.40}$$

if $-[\frac{r}{2}] \leq v_p(\alpha) =: -n \leq -1$, and for any $\alpha_0 \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times$,

$$\sum_{\beta_0 \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times, \beta_0 \equiv \alpha_0 \,(\mathrm{mod}\, p^{r-n})} \mu_{|0,\gamma-1}(\mathbb{1}_{D(p^{-n}\beta_0,1)}) = 0 \tag{2.41}$$

if $-r+1 \leq -n \leq -[\frac{r}{2}]-1$, since γ permutes the family of closed discs $D(p^{-n}\beta_0, 1)$ for $\beta_0 \equiv \alpha_0 \,(\mathrm{mod}\, p^{r-n})$. Since $\mathbb{Z}_p$ and the set $\{z \in \mathbb{Q}_p, -r+1 \leq v_p(z) \leq -1\}$ are invariant by γ by Lemma 2.2.14i), it follows that γ preserves $D(\infty, r)$ which is the disjoint union of $D(\infty, 2r)$ and closed discs of radius 1 in $\{z \in \mathbb{Q}_p, -2r < v_p(z) \leq -r\}$. So

$$\begin{aligned} \mu_{|0,\gamma-1}(\mathbb{1}_{D(\infty,2r)}) + \sum_{D(\alpha,1) \subset \{z \in \mathbb{Q}_p, -2r < v_p(z) \leq -r\}} \mu_{|0,\gamma-1}(\mathbb{1}_{D(\alpha,1)}) = \mu_{|0,\gamma-1}(\mathbb{1}_{D(\infty,r)}) = \\ = \mu(\mathbb{1}_{\gamma(D(\infty,r))}) - \mu(\mathbb{1}_{D(\infty,r)}) = 0. \end{aligned} \tag{2.42}$$

Therefore, by (2.35), (2.40), (2.41), (2.42), the image via (2.34) of the subspace of $\mathcal{D}_0(\mathbb{P}^1, L)$ generated by distributions $\mu_{|0,\gamma-1}$ for $\mu \in \mathcal{D}_0(\mathbb{P}^1, L)$ and $\gamma \in \Gamma$ is contained in the space of all $(x_\infty, (x_\alpha)) \in L \times \prod_{D(\alpha,1) \subset \mathbb{Q}_p} L$ such that

$$x_\infty + \sum_{D(\alpha,1) \subset \{z \in \mathbb{Q}_p, -2r < v_p(z) \leq -r\}} x_\alpha = 0, \tag{2.43}$$

$$x_\alpha = 0 \quad \text{if } D(\alpha,1) = \mathbb{Z}_p \text{ or } -\Big[\frac{r}{2}\Big] \leq v_p(\alpha) \leq -1, \tag{2.44}$$

$$\sum_{\beta_0 \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times, \beta_0 \equiv \alpha_0 \,(\mathrm{mod}\, p^{r-n})} x_{p^{-n}\beta_0} = 0 \quad \text{for all } \alpha_0 \in (\mathbb{Z}_p/p^{r-n}\mathbb{Z}_p)^\times \tag{2.45}$$

and all $n \in \mathbb{N}^*$ such that $-r+1 \leq -n \leq -[\frac{r}{2}]-1$. We show that this image is all of such $(x_\infty, (x_\alpha))$'s.

Since the set $\{z \in \mathbb{Q}_p, -r+1 \leq v_p(z) \leq -1\}$ and $\mathbb{Z}_p$ are stable by the action of $\Gamma_0(p^r)$ by Lemma 2.2.14i), the complement $D(\infty, r)$ in $\mathbb{P}^1(\mathbb{Q}_p)$ is also stable by $\Gamma_0(p^r)$. Therefore, the values of $\mathcal{D}_0(\mathbb{P}^1, L)_{|0,\Gamma-1}$ at functions supported in $\{z \in \mathbb{Q}_p, -r+1 \leq v_p(z) \leq -1\}$ (resp. $D(\infty, r)$) depend only on the restrictions of $\mathcal{D}_0(\mathbb{P}^1, L)$ on these subsets of $\mathbb{P}^1(\mathbb{Q}_p)$.

We partition $D(\infty, r)$ by $D(\infty, 2r)$ and the two following families:

$$\begin{aligned} \mathcal{F}_1 &= \big\{D(\alpha,1) \,|\, D(\alpha,1) \subset \{z \in \mathbb{Q}_p, -2r < v_p(z) < -r\}\big\}, \\ \mathcal{F}_2 &= \big\{D(\alpha,1) \,|\, D(\alpha,1) \subset \{z \in \mathbb{Q}_p, v_p(z) = -r\}\big\}. \end{aligned}$$

Putting $\mathcal{F}_3 = \{D(\alpha,1) \,|\, D(\alpha,1) \subset \{z \in \mathbb{Q}_p, v_p(z) \leq -2r\}\}$. The family of closed discs of radius 1 in $\{z \in \mathbb{Q}_p, v_p(z) \leq -r\}$ is $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

The characteristic functions of $D(\infty, 2r)$ and all discs in $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ are linear independent in $\mathcal{A}_0(\mathbb{P}^1, L)$ since all discs in $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ are pairwise disjoint. Let E_0 denote the subspace of $\mathcal{A}_0(\mathbb{P}^1, L)$ with the basis consisting of these characteristic functions. Let $E \subset \mathcal{A}_0(\mathbb{P}^1, L)$ be the subspace consisting of functions supported in $D(\infty, r)$, then $E_0 \subset E$. Note that every L-linear form on E_0 is automatically continuous since the families $\mathcal{F}_1, \mathcal{F}_2$ are finite and every subspace $\mathcal{A}_0(\mathbb{P}^1, L)[\tilde{r}]$ of $\mathcal{A}_0(\mathbb{P}^1, L)$ for $\tilde{r} \in |\mathbb{C}_p^\times|_p, \tilde{r} < 1$ does not contain an infinite number of discs in $\mathcal{F}_3$.

Fix a disc $D_0 \in \mathcal{F}_2$.

For each disc $D \in \mathcal{F}_1$, take a matrix $\gamma_D \in \Gamma(N) \cap \Gamma_1(p^r) \subset \Gamma$ mapping D into D_0 as in Lemma 2.2.15i) below. Then $\gamma_D^{-1}(D_0)$ contains D. So the image of $D(\infty, 2r)$ and every disc in $\mathcal{F}_1 \cup (\mathcal{F}_2 \backslash \{D_0\}) \cup \mathcal{F}_3$ by γ_D^{-1} are disjoint from D. Defining the L-linear form μ'_D on E_0 by

$$\mu'_D(\mathbb{1}_{D(\infty,2r)}) = 0, \quad \mu'_D(\mathbb{1}_D) = -1,$$
$$\mu'_D(\mathbb{1}_{D(\alpha,1)}) = 0 \quad \text{for any } D(\alpha,1) \in \mathcal{F}_3 \cup \mathcal{F}_2 \cup \mathcal{F}_1 \backslash \{D\}.$$

By [Schn, Corollary 9.4], we can extend μ'_D to a continuous L-linear form μ_D on E such that μ_D is 0 outside D. Then

$$(\mu_D)_{|0,\gamma_D^{-1}-1}(\mathbb{1}_{D(\infty,2r)}) = \mu_D(\mathbb{1}_{\gamma_D^{-1}(D(\infty,2r))}) - \mu_D(\mathbb{1}_{D(\infty,2r)}) = 0 - 0 = 0,$$
$$(\mu_D)_{|0,\gamma_D^{-1}-1}(\mathbb{1}_D) = \mu_D(\mathbb{1}_{\gamma_D^{-1}(D)}) - \mu_D(\mathbb{1}_D) = 0 - (-1) = 1,$$
$$(\mu_D)_{|0,\gamma_D^{-1}-1}(\mathbb{1}_{D_0}) = \mu_D(\mathbb{1}_{\gamma_D^{-1}(D_0)}) - \mu_D(\mathbb{1}_{D_0}) = -1 - 0 = -1,$$
$$(\mu_D)_{|0,\gamma_D^{-1}-1}(\mathbb{1}_{D(\alpha,1)}) = \mu_D(\mathbb{1}_{\gamma_D^{-1}(D(\alpha,1))}) - \mu_D(\mathbb{1}_{D(\alpha,1)}) = 0 - 0 = 0$$

for any $D(\alpha,1) \in (\mathcal{F}_1 \backslash \{D\}) \cup (\mathcal{F}_2 \backslash \{D_0\}) \cup \mathcal{F}_3$.

The L-linear form μ'_∞ on E_0 given by

$$\mu'_\infty(\mathbb{1}_{D(\infty,2r)}) = -1, \quad \mu'_\infty(\mathbb{1}_{D(\alpha,1)}) = 0 \quad \text{for any } D(\alpha,1) \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3.$$

can be extended to a continuous L-linear form μ_∞ on E such that $\mu_\infty = 0$ outside $D(\infty, 2r)$ by [Schn, Corollary 9.4]. Writing the center of the disc D_0 by $p^{-r}x_0$ for x_0 belongs to a congruence class of $\mathbb{Z}_p^\times$ modulo p^r. Defining $c_0 = Np^r c'_0$ for some $c'_0 \in \mathbb{Z} \cap \mathbb{Z}_p^\times$ such that $c'_0 \equiv -N^{-1}x_0^{-1} \pmod{p^r}$. Then

$$\frac{-1}{c_0} = \frac{-1}{Np^r c'_0} = p^{-r}\frac{-1}{Nc'_0}.$$

Since $\frac{-1}{Nc'_0} \equiv x_0 \pmod{p^r}$, it follows that $D(\frac{-1}{c_0}, 1) = D(p^{-r}x_0, 1) = D_0$. Let $\gamma_\infty \in \Gamma(N) \cap \Gamma_1(p^r)$ such that its lower left (resp. right) entry is c_0 (resp. 1). By Lemma 2.2.13, the image of D_0 by γ_∞ is $D(\infty, 2r)$. So the image by γ_∞ of $D(\infty, 2r)$ and every disc in $\mathcal{F}_1 \cup (\mathcal{F}_2 \backslash \{D_0\}) \cup \mathcal{F}_3$ are disjoint from $D(\infty, 2r)$. We obtain

$$(\mu_\infty)_{|0,\gamma_\infty-1}(\mathbb{1}_{D(\infty,2r)}) = \mu_\infty(\mathbb{1}_{\gamma_\infty(D(\infty,2r))}) - \mu_\infty(\mathbb{1}_{D(\infty,2r)}) = 0 - (-1) = 1,$$
$$(\mu_\infty)_{|0,\gamma_\infty-1}(\mathbb{1}_{D_0}) = \mu_\infty(\mathbb{1}_{\gamma_\infty(D_0)}) - \mu_\infty(\mathbb{1}_{D_0}) = \mu_\infty(\mathbb{1}_{D(\infty,2r)}) - \mu_\infty(\mathbb{1}_{D_0}) = -1 - 0 = -1,$$
$$(\mu_\infty)_{|0,\gamma_\infty-1}(\mathbb{1}_{D(\alpha,1)}) = \mu_\infty(\mathbb{1}_{\gamma_\infty(D(\alpha,1))}) - \mu_\infty(\mathbb{1}_{D(\alpha,1)}) = 0 - 0 = 0$$

for any $D(\alpha,1) \in \mathcal{F}_1 \cup (\mathcal{F}_2 \backslash \{D_0\}) \cup \mathcal{F}_3$.

For each disc $D' \in \mathcal{F}_2 \backslash \{D_0\}$, taking a matrix $\gamma_{D'} \in \Gamma(N) \cap \Gamma_1(p^r)$ mapping D' onto D_0 as in Lemma 2.2.15ii) below. Defining the L-linear form μ'_0 on E_0 by

$$\mu'_0(\mathbb{1}_{D(\infty,2r)}) = 0, \quad \mu'_0(\mathbb{1}_{D_0}) = 1,$$
$$\mu'_0(\mathbb{1}_{D(\alpha,1)}) = 0 \quad \text{for any } D(\alpha,1) \in \mathcal{F}_1 \cup (\mathcal{F}_2 \backslash \{D_0\}) \cup \mathcal{F}_3.$$

Extending μ'_0 to a continuous L-linear form μ_0 on E such that $\mu_0 = 0$ outside D_0 by [Schn, Corollary 9.4]. Since $\gamma_{D'}(D') = D_0$, the image by $\gamma_{D'}$ of $D(\infty, 2r)$ and every disc in $\mathcal{F}_1 \cup (\mathcal{F}_2 \backslash \{D'\}) \cup \mathcal{F}_3$ are disjoint from D_0. We obtain

$$\begin{aligned}
&(\mu_0)_{|_{0,\gamma_{D'}-1}}(\mathbb{1}_{D(\infty,2r)}) = \mu_0(\mathbb{1}_{\gamma_{D'}(D(\infty,2r))}) - \mu_0(\mathbb{1}_{D(\infty,2r)}) = 0-0=0,\\
&(\mu_0)_{|_{0,\gamma_{D'}-1}}(\mathbb{1}_{D'}) = \mu_0(\mathbb{1}_{\gamma_{D'}(D')}) - \mu_0(\mathbb{1}_{D'}) = \mu_0(\mathbb{1}_{D_0}) - \mu_0(\mathbb{1}_{D'}) = 1-0=1,\\
&(\mu_0)_{|_{0,\gamma_{D'}-1}}(\mathbb{1}_{D_0}) = \mu_0(\mathbb{1}_{\gamma_{D'}(D_0)}) - \mu_0(\mathbb{1}_{D_0}) = 0-1=-1,\\
&(\mu_0)_{|_{0,\gamma_{D'}-1}}(\mathbb{1}_{D(\alpha,1)}) = \mu_0(\mathbb{1}_{\gamma_{D'}(D(\alpha,1))}) - \mu_0(\mathbb{1}_{D(\alpha,1)}) = 0-0=0
\end{aligned}$$

for any $D(\alpha,1) \in \mathcal{F}_1 \cup (\mathcal{F}_2\backslash\{D_0, D'\}) \cup \mathcal{F}_3$.

For each $(x_\alpha) \in \prod_{D(\alpha,1)\in\mathcal{F}_3} L$, defining the L-linear form ${\mu'}^{\mathcal{F}_3}_{(x_\alpha)}$ on E_0 by

$$\begin{aligned}
&{\mu'}^{\mathcal{F}_3}_{(x_\alpha)}(\mathbb{1}_{D(\infty,2r)}) = 0, \quad {\mu'}^{\mathcal{F}_3}_{(x_\alpha)}(\mathbb{1}_{D(\alpha,1)}) = 0 \text{ for any } D(\alpha,1)\in\mathcal{F}_1\cup\mathcal{F}_2,\\
&{\mu'}^{\mathcal{F}_3}_{(x_\alpha)}(\mathbb{1}_{D(\alpha,1)}) = -x_\alpha \text{ for any } D(\alpha,1)\in\mathcal{F}_3,
\end{aligned}$$

and extending it to a continuous L-linear form $\mu^{\mathcal{F}_3}_{(x_\alpha)}$ on E such that $\mu^{\mathcal{F}_3}_{(x_\alpha)} = 0$ outside $D(\infty, 2r)$ by [Schn, Corollary 9.4]. Take a matrix $\gamma_{\mathcal{F}_3} \in \Gamma(N) \cap \Gamma_1(p^r)$ such that the p-adic valuation of its lower left entry is r. Then $\gamma_{\mathcal{F}_3} \in \Gamma$. The set $D(\infty, 2r)$ and all discs in $\mathcal{F}_1 \cup \mathcal{F}_3$ are mapped by $\gamma_{\mathcal{F}_3}$ into the set $\{z \in \mathbb{Q}_p, v_p(z) = -r\}$ by Lemma 2.2.12. The discs in $\mathcal{F}_2$ are mapped by $\gamma_{\mathcal{F}_3}$ either onto $D(\infty, 2r)$ or into the set $\{z \in \mathbb{Q}_p, -2r < v_p(z) \le -r\}$ by Lemma 2.2.13. Therefore,

$$\mu^{\mathcal{F}_3}_{(x_\alpha)}(\mathbb{1}_{\gamma_{\mathcal{F}_3}(D(\infty,2r))}) = \mu^{\mathcal{F}_3}_{(x_\alpha)}(\mathbb{1}_{\gamma_{\mathcal{F}_3}(D(\alpha,1))}) = 0 \text{ for any } D(\alpha,1)\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3.$$

We get

$$\begin{aligned}
&(\mu^{\mathcal{F}_3}_{(x_\alpha)})_{|_{0,\gamma_{\mathcal{F}_3}-1}}(\mathbb{1}_{D(\infty,2r)}) = \mu^{\mathcal{F}_3}_{(x_\alpha)}(\mathbb{1}_{\gamma_{\mathcal{F}_3}(D(\infty,2r))}) - \mu^{\mathcal{F}_3}_{(x_\alpha)}(\mathbb{1}_{D(\infty,2r)}) = 0-0=0,\\
&(\mu^{\mathcal{F}_3}_{(x_\alpha)})_{|_{0,\gamma_{\mathcal{F}_3}-1}}(\mathbb{1}_{D(\alpha,1)}) = \mu^{\mathcal{F}_3}_{(x_\alpha)}(\mathbb{1}_{\gamma_{\mathcal{F}_3}(D(\alpha,1))}) - \mu^{\mathcal{F}_3}_{(x_\alpha)}(\mathbb{1}_{D(\alpha,1)}) = 0-0=0 \ \ \forall D(\alpha,1)\in\mathcal{F}_1\cup\mathcal{F}_2,\\
&(\mu^{\mathcal{F}_3}_{(x_\alpha)})_{|_{0,\gamma_{\mathcal{F}_3}-1}}(\mathbb{1}_{D(\alpha,1)}) = \mu^{\mathcal{F}_3}_{(x_\alpha)}(\mathbb{1}_{\gamma_{\mathcal{F}_3}(D(\alpha,1))}) - \mu^{\mathcal{F}_3}_{(x_\alpha)}(\mathbb{1}_{D(\alpha,1)}) = 0-(-x_\alpha)=x_\alpha \ \ \forall D(\alpha,1)\in\mathcal{F}_3.
\end{aligned}$$

Let $\mathcal{F}_4$ be the family of closed discs of radius 1 in $\{z \in \mathbb{Q}_p, -r+1 \le v_p(z) \le -1\}$. Let $F_0 \subset \mathcal{A}_0(\mathbb{P}^1, L)$ be the subspace with the basis consisting of characteristic functions of all discs in $\mathcal{F}_4$. Let $F \subset \mathcal{A}_0(\mathbb{P}^1, L)$ denote the subspace of functions supported in $\{z \in \mathbb{Q}_p, -r+1 \le v_p(z) \le -1\}$. Then $F_0 \subset F$. Every L-linear form on F_0 is automatically continuous since F_0 is finite dimensional.

For $n \in \mathbb{N}^*$ with $-r+1 \le -n \le -[\frac{r}{2}]-1$, fix a representative $\alpha_0 \in \mathbb{Z}_p^\times$ of each congruence class in $(\mathbb{Z}_p/p^{r-n}\mathbb{Z}_p)^\times$. For each $\beta_0 \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times$ such that $\beta_0 \equiv \alpha_0 \,(\mathrm{mod}\, p^{r-n})$ and $\beta_0 \not\equiv \alpha_0 \,(\mathrm{mod}\, p^n)$ (so that $D(p^{-n}\beta_0, 1) \ne D(p^{-n}\alpha_0, 1)$), take a matrix $\gamma_{\beta_0} \in \Gamma(Np^r) \subset \Gamma$ mapping $D(p^{-n}\beta_0, 1)$ onto $D(p^{-n}\alpha_0, 1)$ as in Lemma 2.2.14ii). Then the image by γ_{β_0} of every disc in $\mathcal{F}_4$ different from $D(p^{-n}\beta_0, 1)$ is a disc in $\mathcal{F}_4$ different from $D(p^{-n}\alpha_0, 1)$ by Lemma 2.2.14. Defining the L-linear form μ'_{α_0} on F_0 by

$$\mu'_{\alpha_0}(\mathbb{1}_{D(p^{-n}\alpha_0,1)}) = 1, \quad \mu'_{\alpha_0}(\mathbb{1}_{D(\alpha,1)}) = 0 \text{ for any } D(\alpha,1)\in\mathcal{F}_4\backslash\{D(p^{-n}\alpha_0,1)\}.$$

Extending μ'_{α_0} to a continuous L-linear form μ_{α_0} on F by [Schn, Corollary 9.4]. We have

$$\begin{aligned}
&(\mu_{\alpha_0})_{|_{0,\gamma_{\beta_0}-1}}(\mathbb{1}_{D(p^{-n}\beta_0,1)}) = \mu_{\alpha_0}(\mathbb{1}_{D(p^{-n}\alpha_0,1)}) - \mu_{\alpha_0}(\mathbb{1}_{D(p^{-n}\beta_0,1)}) = 1-0=1,\\
&(\mu_{\alpha_0})_{|_{0,\gamma_{\beta_0}-1}}(\mathbb{1}_{D(p^{-n}\alpha_0,1)}) = \mu_{\alpha_0}(\mathbb{1}_{\gamma_{\beta_0}(D(p^{-n}\alpha_0,1))}) - \mu_{\alpha_0}(\mathbb{1}_{D(p^{-n}\alpha_0,1)}) = 0-1=-1,\\
&(\mu_{\alpha_0})_{|_{0,\gamma_{\beta_0}-1}}(\mathbb{1}_{D(\alpha,1)}) = \mu_{\alpha_0}(\mathbb{1}_{\gamma_{\beta_0}(D(\alpha,1))}) - \mu_{\alpha_0}(\mathbb{1}_{D(\alpha,1)}) = 0-0=0
\end{aligned}$$

for any $D(\alpha,1) \in \mathcal{F}_4\backslash\{D(p^{-n}\alpha_0, 1), D(p^{-n}\beta_0, 1)\}$.

Now let $(x_\infty, (x_\alpha)) \in L \times \prod_{D(\alpha,1)\subset\mathbb{Q}_p} L$ satisfying the conditions (2.43), (2.44), (2.45). The conditions (2.43) rewrites $x_\infty + \sum_{D(\alpha,1)\in\mathcal{F}_1\cup\mathcal{F}_2} x_\alpha = 0$. Consider the distribution $\mu \in \mathcal{D}_0(\mathbb{P}^1, L)$ defined by

$$\mu = (\mu^{\mathcal{F}_3}_{(x_\alpha)})_{|_{0,\gamma_{\mathcal{F}_3}-1}} + x_\infty(\mu_\infty)_{|_{0,\gamma_\infty-1}} + \sum_{D\in\mathcal{F}_1} x_D(\mu_D)_{|_{0,\gamma_D^{-1}-1}} + \sum_{D'\in\mathcal{F}_2\backslash\{D_0\}} x_{D'}(\mu_0)_{|_{0,\gamma_{D'}-1}}$$

on $D(\infty, r)$, and

$$\mu = \sum_{n=[\frac{r}{2}]+1}^{r-1} \sum_{\alpha_0 \in (\mathbb{Z}_p/p^{r-n}\mathbb{Z}_p)^\times} \sum_{\beta_0 \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times, \beta_0 \equiv \alpha_0 \pmod{p^{r-n}}, \beta_0 \not\equiv \alpha_0 \pmod{p^n}} x_{p^{-n}\beta_0} (\mu_{\alpha_0})_{|0,\gamma_{\beta_0}-1}$$

on $\{z \in \mathbb{Q}_p, -r+1 \leq v_p(z) \leq -1\}$, and $\mu = 0$ on $\mathbb{Z}_p$. Then $\mu \in \mathcal{D}_0(\mathbb{P}^1, L)_{|0,\Gamma-1}$ and it is checked easily that

$$\mu(\mathbb{1}_{D(\infty,2r)}) = x_\infty, \quad \mu(\mathbb{1}_{D(\alpha,1)}) = x_\alpha \text{ for any } D(\alpha, 1) \subset \mathbb{Q}_p.$$

We conclude that the image of the subspace of $\mathcal{D}_0(\mathbb{P}^1, L)$ genereted by $\mathcal{D}_0(\mathbb{P}^1, L)_{|0,\Gamma-1}$ via (2.34) is all of such $(x_\infty, (x_\alpha))$'s. For each $n \in \mathbb{N}^*$, every closed disc of radius 1 in $\{z \in \mathbb{Q}_p, v_p(z) = -n\}$ is of the form $D(p^{-n}\beta, 1)$ for β runs through a complete set of representatives of $(\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times$. The dimension of $H_0(\Gamma, \mathcal{D}_0(\mathbb{P}^1, L))$ if $\Gamma_1(N) \cap \Gamma_1(p^r) \subset \Gamma \subset \Gamma_0(N) \cap \Gamma_1(p^r)$ is thus

$$1 + 1 + \sum_{n=1}^{[\frac{r}{2}]} |(\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times| + \sum_{n=[\frac{r}{2}]+1}^{r-1} |(\mathbb{Z}_p/p^{r-n}\mathbb{Z}_p)^\times| = p^{[\frac{r}{2}]} + p^{r-[\frac{r}{2}]-1}.$$

If $\Gamma_1(N) \cap \Gamma_0(p^r) \subset \Gamma \subset \Gamma_0(Np^r)$, by Lemma 2.2.14i), if $n \in \mathbb{N}^*$ with $-[\frac{r}{2}] \leq -n \leq -1$, then Γ permutes the family of discs $D(p^{-n}\beta_0, 1)$ for $\beta_0 \in \mathbb{Z}_p^\times$ such that $\frac{\beta_0}{\alpha_0}$ is a square modulo p^n, for each one of two equivalence classes up to square multiple of congruence classes $\alpha_0 \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times$ if $p \neq 2$ or $p = 2$ and $n > 1$, while if $p = 2$ and $n = 1$ there is only one class $\alpha_0 \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times$; if $-r+1 \leq -n \leq -[\frac{r}{2}] - 1$, then Γ permutes the family of discs $D(p^{-n}\beta_0, 1)$ for $\beta_0 \in \mathbb{Z}_p^\times$ such that $\frac{\beta_0}{\alpha_0}$ is a square modulo p^{r-n}, for each one of two equivalence classes up to square multiple of congruence classes $\alpha_0 \in (\mathbb{Z}_p/p^{r-n}\mathbb{Z}_p)^\times$ if $p \neq 2$ or $p = 2$ and $-n > -r+1$, or only one equivalence class if $p = 2$ and $-n = -r+1$. Note that the group $(\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$ is cyclic of order $p^{n-1}(p-1)$ for any $n \in \mathbb{N}^*$, and this order is even if and only if $p \neq 2$ or $p = 2$ and $n > 1$.

Combining with (2.35) and (2.42) we deduce that the image via (2.34) of the subspace of $\mathcal{D}_0(\mathbb{P}^1, L)$ generated by $\mathcal{D}_0(\mathbb{P}^1, L)_{|0,\Gamma-1}$ is contained in the space of all $(x_\infty, (x_\alpha)) \in L \times \prod_{D(\alpha,1)\subset\mathbb{Q}_p} L$ such that

$$x_\alpha = 0 \quad \text{if } D(\alpha, 1) = \mathbb{Z}_p, \quad x_\infty + \sum_{D(\alpha,1)\subset\{z\in\mathbb{Q}_p, -2r<v_p(z)\leq -r\}} x_\alpha = 0, \tag{2.46}$$

$$\sum_{\beta_0 \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times, \frac{\beta_0}{\alpha_0} \text{ is square} \pmod{p^n}} x_{p^{-n}\beta_0} = 0 \tag{2.47}$$

for any equivalence class up to square multiple of congruence classes $\alpha_0 \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times$ and any $n \in \mathbb{N}^*$ such that $-[\frac{r}{2}] \leq -n \leq -1$ if $r \geq 2$, and

$$\sum_{\beta_0 \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times, \frac{\beta_0}{\alpha_0} \text{ is square} \pmod{p^{r-n}}} x_{p^{-n}\beta_0} = 0 \tag{2.48}$$

for any equivalence class up to square multiple of congruence classes $\alpha_0 \in (\mathbb{Z}_p/p^{r-n}\mathbb{Z}_p)^\times$ and any $n \in \mathbb{N}^*$ such that $-r+1 \leq -n \leq -[\frac{r}{2}] - 1$ if $r \geq 3$.

By the converse part of Lemma 2.2.14i) we can explain as above to deduce that the image via (2.34) of the subspace generated by $\mathcal{D}_0(\mathbb{P}^1, L)_{|0,\Gamma-1}$ is all of $(x_\infty, (x_\alpha))$ satisfying the conditions (2.46), (2.47), (2.48). Therefore, the dimension of $H_0(\Gamma, \mathcal{D}_0(\mathbb{P}^1, L))$ if $p \neq 2$ is

$$1 + 1 + \sum_{n=1}^{[\frac{r}{2}]} 2 + \sum_{n=[\frac{r}{2}]+1}^{r-1} 2 = 2r,$$

while if $p = 2$ the dimension is

$$1 + 1 + \big(1 + \sum_{n=2}^{[\frac{r}{2}]} 2\big) + \big(1 + \sum_{n=[\frac{r}{2}]+1}^{r-2} 2\big) = \begin{cases} 2 & \text{if } r = 1, \\ 3 & \text{if } r = 2, \\ 2r-2 & \text{if } r \geq 3. \end{cases}$$

If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then $\Gamma \supset \Gamma_0(p)$. By (2.34) we have the following isomorphism:

$$\mathcal{D}_0(\mathbb{P}^1, L)/\mathcal{D}_0(\mathbb{P}^1, L)_{|_0\gamma_0-1} \cong L \times \prod_{D(\alpha,1)\subset\mathbb{Q}_p} L \tag{2.49}$$
$$\mu \mapsto \big(\mu(\mathbb{1}_{D(\infty,2)}), (\mu(\mathbb{1}_{D(\alpha,1)}))_{D(\alpha,1)\subset\mathbb{Q}_p}\big).$$

By (2.46), the image of the subspace of $\mathcal{D}_0(\mathbb{P}^1, L)$ generated by $\mathcal{D}_0(\mathbb{P}^1, L)_{|_0,\Gamma_0(p)-1}$ by (2.49) is the space of all $(x_\infty, (x_\alpha)) \in L \times \prod_{D(\alpha,1)\subset\mathbb{Q}_p} L$ such that

$$x_{\mathbb{Z}_p} = x_\infty + \sum_{D(\alpha,1)\subset\{z\in\mathbb{Q}_p, v_p(z)=-1\}} x_\alpha = 0, \tag{2.50}$$

where $x_{\mathbb{Z}_p}$ corresponds to the disc $\mathbb{Z}_p = D(0,1)$. Since $\mathbb{P}^1(\mathbb{Q}_p)$ is partitioned by $\mathbb{Z}_p, D(\infty, 2)$ and the set $\{z \in \mathbb{Q}_p, v_p(z) = -1\}$, for any $\mu \in \mathcal{D}_0(\mathbb{P}^1, L)$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\mu_{|_0,\gamma-1}(\mathbb{1}_{\mathbb{Z}_p}) + \mu_{|_0,\gamma-1}(\mathbb{1}_{D(\infty,2)}) + \sum_{D(\alpha,1)\subset\{z\in\mathbb{Q}_p,v_p(z)=-1\}} \mu_{|_0,\gamma-1}(\mathbb{1}_{D(\alpha,1)})$$
$$= \mu_{|_0,\gamma-1}(\mathbb{1}_{\mathbb{P}^1(\mathbb{Q}_p)}) = \mu(\mathbb{1}_{\gamma(\mathbb{P}^1(\mathbb{Q}_p))}) - \mu(\mathbb{1}_{\mathbb{P}^1(\mathbb{Q}_p)}) = 0.$$

Therefore, the image of the subspace of $\mathcal{D}_0(\mathbb{P}^1, L)$ generated by $\mathcal{D}_0(\mathbb{P}^1, L)_{|_0,\mathrm{SL}_2(\mathbb{Z})-1}$ by (2.49) is contained in the space of all $(x_\infty, (x_\alpha)) \in L \times \prod_{D(\alpha,1)\subset\mathbb{Q}_p} L$ such that

$$x_{\mathbb{Z}_p} + x_\infty + \sum_{D(\alpha,1)\subset\{z\in\mathbb{Q}_p,v_p(z)=-1\}} x_\alpha = 0, \tag{2.51}$$

and this image contains all $(x_\infty, (x_\alpha))$ satisfying (2.50). Since the image of $\mathbb{Z}_p$ by a matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ can be different from $\mathbb{Z}_p$ (e.g. $\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$), we can define an L-linear form μ on the subspace of $\mathcal{A}_0(\mathbb{P}^1, L)$ generated by the characteristic functions of $\mathbb{Z}_p$ and $\gamma(\mathbb{Z}_p)$ so that $\mu_{|_0,\gamma-1}(\mathbb{1}_{\mathbb{Z}_p}) = \mu(\mathbb{1}_{\gamma(\mathbb{Z}_p)}) - \mu(\mathbb{1}_{\mathbb{Z}_p}) \neq 0$, and extending μ to a continuous L-linear form (which we still denote by μ) on $\mathcal{A}_0(\mathbb{P}^1, L)$ by [Schn, Corollary 9.4] to have a distribution $\mu \in \mathcal{D}_0(\mathbb{P}^1, L)$ such that $\mu_{|_0,\gamma-1}(\mathbb{1}_{\mathbb{Z}_p}) \neq 0$. We conclude that the image of the subspace of $\mathcal{D}_0(\mathbb{P}^1, L)$ generated by $\mathcal{D}_0(\mathbb{P}^1, L)_{|_0,\mathrm{SL}_2(\mathbb{Z})-1}$ by (2.49) is the space of all $(x_\infty, (x_\alpha)) \in L \times \prod_{D(\alpha,1)\subset\mathbb{Q}_p} L$ satisfying (2.51). So the dimension of $H_0(\mathrm{SL}_2(\mathbb{Z}), \mathcal{D}_0(\mathbb{P}^1, L))$ is 1.

We finish the proof of theorem by proving the following lemma:

Lemma 2.2.15. *Let $N, r \in \mathbb{N}^*$ with $(N, p) = 1$.*

i) For any closed disc $D_0 \subset \{z \in \mathbb{Q}_p, v_p(z) = -r\}$ of radius 1 and any closed disc $D \subset \{z \in \mathbb{Q}_p, -2r < v_p(z) < -r\}$ of radius 1, there exists a matrix $\gamma \in \Gamma(N) \cap \Gamma_1(p^r)$ such that γ maps D into D_0.

ii) For any two closed discs D_0, D of radius 1 in $\{z \in \mathbb{Q}_p, v_p(z) = -r\}$, there exists a matrix $\gamma \in \Gamma(N) \cap \Gamma_1(p^r)$ mapping D onto D_0.

Proof. i) Let $D = D(p^{-r-n}x, 1)$ and $D_0 = D(p^{-r}y, 1)$, where $x, y \in \mathbb{Z}_p^\times$ and $n \in \mathbb{N}$ such that $0 < n < r$. We choose $c' \in \mathbb{Z}$ such that

$$c' \equiv \frac{-p^n}{x} + \frac{1}{y} \pmod{p^r}, \qquad c' \equiv 0 \pmod{N}.$$

Putting $c = p^r c'$, then $c \equiv 0 \pmod{Np^r}$. Taking $a, d \in \mathbb{Z}$ such that $a \equiv d \equiv 1 \pmod{Nc}$, then $a \equiv d \equiv 1 \pmod{Np^r}$ and $ad \equiv 1 \pmod{Nc}$. Letting $b = \frac{ad-1}{c} \in N\mathbb{Z}$, then the matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $\Gamma(N) \cap \Gamma_1(p^r)$.

We check that γ maps D into D_0. We have

$$\gamma(p^{-r-n}x) = \frac{ap^{-r-n}x + b}{cp^{-r-n}x + d} = \frac{ax + p^{r+n}b}{cx + p^{r+n}d}.$$

Since $v_p(cx) = v_p(c) = r < r+n = v_p(p^{r+n}d)$, we have $v_p(cx+p^{r+n}d) = v_p(cx) = r$, so $\frac{p^{r+n}b}{cx+p^{r+n}d} \in p^n\mathbb{Z}_p \subset \mathbb{Z}_p$. Therefore,

$$D(\gamma(p^{-r-n}x),1) = \gamma(p^{-r-n}x) + \mathbb{Z}_p = \frac{ax}{cx+p^{r+n}d} + \mathbb{Z}_p.$$

Since

$$\frac{ax}{cx+p^{r+n}d} = \frac{ax}{p^rc'x+p^{r+n}d} = p^{-r}\frac{ax}{c'x+p^nd}$$

and since

$$\frac{ax}{c'x+p^nd} \equiv \frac{x}{c'x+p^n} \equiv y \pmod{p^r}$$

by the construction of c', we infer that $D(\gamma(p^{-r-n}x),1) = p^{-r}y+\mathbb{Z}_p = D_0$. By Lemma 2.2.12, the radius of $\gamma(D)$ is strictly less than 1, so

$$\gamma(D) \subset D(\gamma(p^{-r-n}x),1) = D_0.$$

ii) Suppose $D = D(p^{-r}x,1)$ and $D_0 = D(p^{-r}y,1)$ for $x,y \in \mathbb{Z}_p^\times$. Choosing $c' \in \mathbb{Z}$ such that $c' \equiv y^{-1} - x^{-1} \pmod{p^r}$ and $c' \equiv 0 \pmod{N}$. Putting $c = p^rc'$, then $c \equiv 0 \pmod{Np^r}$. Taking $a,d \in \mathbb{Z}$ such that $a \equiv d \equiv 1 \pmod{Nc}$, then $a \equiv d \equiv 1 \pmod{Np^r}$ and $ad \equiv 1 \pmod{Nc}$. Letting $b = \frac{ad-1}{c} \in N\mathbb{Z}$, then the matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $\Gamma(N) \cap \Gamma_1(p^r)$.

Since $c(p^{-r}x)+d \equiv c'x+1 \equiv xy^{-1} \pmod{p^r}$, we deduce that $c(p^{-r}x)+y \in \mathbb{Z}_p^\times$. By the proof of Lemma 2.2.13 we deduce that γ maps D onto the disc $D(\gamma(p^{-r}x),1)$. We have

$$\gamma(p^{-r}x) = \frac{ap^{-r}x+b}{cp^{-r}x+d} = p^{-r}\frac{ax}{c'x+d} + \frac{b}{cp^{-r}x+d} \in p^{-r}\frac{ax}{c'x+d} + \mathbb{Z}_p$$

since $cp^{-r}x+d \in \mathbb{Z}_p^\times$. On the other hand, since $\frac{ax}{c'x+d} \equiv ay \equiv y \pmod{p^r}$, it follows that $\gamma(p^{-r}x) \in p^{-r}y+\mathbb{Z}_p$. Therefore, $\gamma(D) = p^{-r}y+\mathbb{Z}_p = D_0$.

The lemma follows. □

The theorem is proven. □

2.3 Admissible distributions on $\mathbb{P}^1(\mathbb{Q}_p)$

If $\mu \in \mathcal{D}_k(\mathbb{P}^1)$, then μ is uniquely determined by two distributions μ_1,μ_2 on $\mathbb{Z}_p$, where μ_1 is the restriction of μ on $\mathbb{Z}_p$ and μ_2 is defined by

$$\mu_2(f) = \int_{D(\infty,0)} z^k f\left(\frac{1}{z}\right) d\mu(z),$$

where $D(\infty,0) = \{z \in \mathbb{P}^1(\mathbb{Q}_p), v_p(z) \le 0\}$. These two distributions are related by

$$\mu_{1|\mathbb{Z}_p^\times}(f) = \mu_{2|\mathbb{Z}_p^\times}\left(z^k f\left(\frac{1}{z}\right)\right)$$

for any analytic function f on $\mathbb{Z}_p^\times$.

Definition 2.3.1. *For $u \ge 0$, we say that a distribution $\mu \in \mathcal{D}_k(\mathbb{P}^1)$ is u-admissible or of order $\le u$ if the distributions μ_1,μ_2 defined above have order $\le u$ as distributions on $\mathbb{Z}_p$. We denote the set of u-admissible distributions in $\mathcal{D}_k(\mathbb{P}^1)$ by $\mathcal{D}_k(\mathbb{P}^1)_{\le u}$.*

Chapter 3

Overconvergent modular symbols

In this chapter, we study overconvergent modular symbols introduced by Glenn Stevens which provide one with a powerful tool to construct p-adic L-functions attached to modular forms, as will be discussed in the next chapter. In Section 3.1, we introduce the general notion of modular symbols. Then we define the action of Hecke operators on modular symbols, especially the important operator U_p, as well as the notions of slopes and slope decompositions which are considered mainly for U_p. In Section 3.2 we investigate classical and overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{Z}_p)$, and their relation via the specialization map. We state the slope decompositions for the latter modular symbols. Section 3.3 is devoted to study overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{P}^1)$, which is one of the innovations of this thesis.

We fix an integer k in this chapter.

3.1 Abstract modular symbols

3.1.1 Modular symbols and Hecke operators

The notion of modular symbols is defined in [AS86].

Let $\Delta_0 = \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ denote the abelian group of divisors of degree 0 on $\mathbb{P}^1(\mathbb{Q})$. The linear fractional transformations of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathbb{P}^1(\mathbb{Q})$ gives Δ_0 the structure of $\mathbb{Z}[\mathrm{GL}_2(\mathbb{Q})]$-module.

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and V be a right $\mathbb{Z}[\Gamma]$-module. The group of additive homomorphisms $\mathrm{Hom}(\Delta_0, V)$ is endowed with the structure of a right Γ-module given by

$$(\phi_{|\gamma})(D) := \phi(\gamma D)_{|\gamma}$$

for $\phi \in \mathrm{Hom}(\Delta_0, V), \gamma \in \Gamma$ and $D \in \Delta_0$.

Definition 3.1.1. *A group homomorphism $\phi \in \mathrm{Hom}(\Delta_0, V)$ is called a V-valued modular symbol on Γ if $\phi_{|\gamma} = \phi$ for any $\gamma \in \Gamma$, i.e.,*

$$\phi(\gamma D) = \phi(D)_{|\gamma^{-1}} \text{ for all } \gamma \in \Gamma, D \in \Delta_0.$$

We denote by $\mathrm{Symb}_\Gamma(V)$ the space of all V-valued modular symbols on Γ.

If V is an $R[\Gamma]$-module for R a commutative ring with identity, then $\mathrm{Symb}_\Gamma(V)$ has the natural structure of R-module.

Let $\mathbb{H}$ denote the Poincaré upper half plane and $\mathcal{V}$ the locally constant sheaf on the modular curve $\mathbb{H}/\Gamma$ associated to V. By [AS86, Proposition 4.2], there is a canonical isomorphism

$$\mathrm{Symb}_\Gamma(V) \cong H^1_c(\mathbb{H}/\Gamma, \mathcal{V})$$

if the order of any torsion element of Γ acts invertibly on V.

If $\Sigma \subset \mathrm{SL}_2(\mathbb{Z})$ is a monoid acting on V and containing the group Γ and a matrice s, we define the action of the Hecke operator $[\Gamma s \Gamma]$ on $\mathrm{Symb}_\Gamma(V)$ on setting:

$$\phi_{|[\Gamma s \Gamma]} = \sum_i \phi_{|s_i}, \text{ where } \Gamma s \Gamma = \bigsqcup_i \Gamma s_i.$$

If $\begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix} \in \Sigma$ for a prime number l, then we denote by T_l the Hecke operator acting on $\mathrm{Symb}_\Gamma(V)$ given by the double coset $\Gamma \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix} \Gamma$. For example, if $\Gamma = \Gamma_0(N)$ and $l \not| N$, then

$$\phi_{|T_l} = \phi_{|\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}} + \sum_{a=0}^{l-1} \phi_{|\begin{pmatrix} 1 & a \\ 0 & l \end{pmatrix}}.$$

If l divides the level of Γ, we write U_l instead of T_l and we have

$$\phi_{|U_l} = \sum_{a=0}^{l-1} \phi_{|\begin{pmatrix} 1 & a \\ 0 & l \end{pmatrix}}.$$

The Hecke operator U_p will play an important role in this thesis.

If $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ normalizes Γ, then the operator $T_\infty := \left[\Gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma\right]$ acts on $\mathrm{Symb}_\Gamma(V)$ by the action of this matrix and this action is an involution. When 2 acts invertibly on V, we have a decomposition

$$\mathrm{Symb}_\Gamma(V) = \mathrm{Symb}_\Gamma(V)^+ \oplus \mathrm{Symb}_\Gamma(V)^-$$

into ± 1-eigenspaces for T_∞-action, given by $\phi = \phi^+ + \phi^-$ for $\phi \in \mathrm{Symb}_\Gamma(V)$, where

$$\phi^\pm = \frac{1}{2}\left(\phi \pm \phi_{|\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\right).$$

The operator T_∞ commutes with all Hecke operators T_l and U_l, since the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ commutes with $\begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix}$ for any l. Therefore, the subspaces $\mathrm{Symb}_\Gamma(V)^\pm$ are stable by the actions of T_l and U_l.

3.1.2 Slopes and slope decompositions

Definition 3.1.2 (Slopes)**.** *Let h be a rational number.*

i) We say that a polynomial $P(X) \in L[X]$ has slope $\leq h$ (resp. $< h$) if all its roots in $\bar{\mathbb{Q}}_p$ have p-adic valuation $\leq h$ (resp. $< h$). The polynomial P is said to be of finite slope if it has slope $\leq h$ for some $h \in \mathbb{Q}$ (i.e., $P(0) \neq 0$).

ii) If M is an L-vector space with an endomorphism named U acting on it, we define its subspace of vectors of slope $\leq h$ (resp. $< h$), denoted by $M^{U \leq h}$ (resp. $M^{U<h}$), as the sum of the subspaces $\ker P(U)$ *where P runs among monic polynomials of slope $\leq h$ (resp. $< h$). A vector in M is said to be of finite slope if it has slope $\leq h$ for some $h \in \mathbb{Q}$. Slope of a U-eigenvector is defined by the p-adic valuation of its eigenvalue.*

The notion of slope decompositions is introduced by Ash and Stevens. We follow [Urb] for its definition.

Definition 3.1.3 (Slope decompositions)**.** *Let M be a vector space over L and let U be a linear endomorphism of M. A $\leq h$-slope decomposition of M with respect to U is a direct sum decomposition $M = M_1 \oplus M_2$ such that*

i) M_1 and M_2 are stable under the action of U.

ii) M_1 is finite dimensional over L.

iii) The characteristic polynomial of U on M_1 has slope $\leq h$.

iv) For any polynomial $Q \in L[X]$ of slope $\leq h$, the restriction of $Q(U)$ to M_2 is an invertible endomorphism of M_2.

We denote $M^{U \leq h}$ for M_1 and $M^{U>h}$ for M_2. The subspace $M^{U \leq h}$ is defined in Definition 3.1.2ii). Note that the subspace $M^{U>h}$ is bigger than the subspace of vectors of finite slope $> h$.

By a generalization of a result of Serre (see [Ser, Proposition 12] and [Buz, Theorem 3.3]), we infer that any compact endomorphism of a Banach space admits a $\leq h$-slope decomposition for any h, or more general for any compact operator on a compact Fréchet space, in the sense of [Urb, §2.3.12].

The notions of slopes and slope decompositions will be considered mainly for the operator U_p on classical and overconvergent modular symbols. When we talk about the slope of a modular symbol without mentioning operator acting on it, we mean U_p-slope, e.g., we write $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\leq h}$ for $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{U_p\leq h}$.

3.2 Classical and overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{Z}_p)$

In this section, we define the classical and overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{Z}_p)$, endowed with an action of U_p, as well as the specialization map between them. This map gives us an isomorphism when restricted to the subspace of small slope (see Theorem 3.2.1). We formulate the slope decompositions of overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{Z}_p)$ in Proposition 3.2.3.

Throughout this section, let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup of level prime to p. We put $\Gamma_0 = \Gamma \cap \Gamma_0(p)$.

For $k \in \mathbb{N}$, since $\Gamma_0 \subset \Sigma_0(p)$, it acts on the space $\mathcal{D}_k(\mathbb{Z}_p)$ of p-adic distributions on $\mathbb{Z}_p$, and on the L-dual $\mathcal{V}_k$ of the space $\mathcal{P}_k \subset L[X]$ of polynomials of degree $\leq k$ with coefficients in L, by the weight k actions defined by (1.7). The space of modular symbols with values in $\mathcal{V}_k$ (resp. $\mathcal{D}_k(\mathbb{Z}_p)$) is called the space of classical (resp. overconvergent) modular symbols.

The natural inclusion $\mathcal{P}_k \to \mathcal{A}_k(\mathbb{Z}_p)$ induces the $\Sigma_0(p)$-equivariant dual map $\rho_k : \mathcal{D}_k(\mathbb{Z}_p) \to \mathcal{V}_k$, then induces the map

$$\rho_k : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k)$$

which is equivariant for Hecke operators and which we call the specialization map.

Theorem 3.2.1 (Stevens' control theorem)**.** *The specialization map*

$$\rho_k : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k)$$

is surjective. Moreover, its restriction to the subspace of U_p-slope $< k+1$ is an isomorphism

$$\rho_k : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<k+1} \xrightarrow{\cong} \mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k)^{<k+1}.$$

Proof. See [PS11, Theorem 5.1] for surjectivity, see [Ste, Theorem 7.1] or [PS13, Theorem 5.4] for the isomorphism on small slope subspace. ☐

A well-known result of Manin says that Δ_0 is a finite $\mathbb{Z}[\Gamma]$-module for any finite index subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ (see [Man]), so Δ_0 is finite $\mathbb{Z}[\Gamma_0]$-module. For each $r \in |\mathbb{C}_p^\times|_p, r < p$, since Γ_0 acts isometrically on $\mathcal{D}_k(\mathbb{Z}_p)[r]$ by Corollary 1.2.2, we get the space $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$. The action of $\Sigma_0(p)$ on $\mathcal{D}_k(\mathbb{Z}_p)[r]$ induces the action of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$. The r-norm $\|\cdot\|_r$ on $\mathcal{D}(\mathbb{Z}_p)[r]$ induces the r-norm $\|\cdot\|_r$ on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ given by

$$\|\Phi\|_r = \sup_{D\in\Delta_0} \|\Phi(D)\|_r \,,$$

note that $\|\Phi(\gamma D)\|_r = \|\Phi(D)\|_r$ for any $\gamma \in \Gamma_0$ and $D \in \Delta_0$ since $\Phi(\gamma D) = \Phi(D)_{|\gamma^{-1}}$ and Γ_0 acts by isometry on $\mathcal{D}_k(\mathbb{Z}_p)[r]$. Then $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ becomes a Banach space and it can be embedded to a product of finite copies of $\mathcal{D}(\mathbb{Z}_p)[r]$ indexed by a finite family of generators of $\mathbb{Z}[\Gamma_0]$-module Δ_0. Since $\mathcal{D}_k(\mathbb{Z}_p)$ is the projective limit of $\mathcal{D}_k(\mathbb{Z}_p)[r]$'s for $r \in |\mathbb{C}_p^\times|_p$, where the transition maps $\mathcal{D}_k(\mathbb{Z}_p)[r_2] \to \mathcal{D}_k(\mathbb{Z}_p)[r_1]$ are injective, compact (by Lemmas 1.1.1, 1.1.2) and $\Sigma_0(p)$-equivariant for any $r_2 < r_1$, we have

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)) = \varprojlim_{r\in|\mathbb{C}_p^\times|_p, r<p} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$$

and the transition maps are injective, compact and U_p-equivariant. So $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ has the structure of a Fréchet space endowed with the family of norms $\{\|\cdot\|_r : r \in |\mathbb{C}_p^\times|_p, r < p\}$.

Proposition 3.2.2. *The Hecke oparator U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ is compact for any $r \in |\mathbb{C}_p^\times|_p, r < p$.*

Proof. Recall the action of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$:

$$\Phi_{|U_p}(D) = \sum_{a=0}^{p-1} \Phi_{|\gamma_a}(D) = \sum_{a=0}^{p-1} \Phi(\gamma_a D)_{|_k\gamma_a},$$

where $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r]), D \in \Delta_0$ and $\gamma_a = \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$.

Since γ_a has determinant p and $\Phi(\gamma_a D) \in \mathcal{D}_k(\mathbb{Z}_p)[r]$, the distribution $\Phi(\gamma_a D)_{|_k\gamma_a}$ belongs to $\mathcal{D}_k(\mathbb{Z}_p)[r/p]$ by Corollary 1.2.2. Therefore, $\Phi_{|U_p} \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r/p])$. In other words, the operator U_p factors through

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r]) \xrightarrow{U_p} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r/p]) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r]),$$

where the second map is induced by the dual map $\mathcal{D}_k(\mathbb{Z}_p)[r/p] \to \mathcal{D}_k(\mathbb{Z}_p)[r]$. Since the second map is compact, U_p is compact as an endomorphism of $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$. □

Proposition 3.2.3. *For each $h \in \mathbb{Q}_{\geq 0}$, there is a $\leq h$-slope decomposition*

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)) = \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\leq h} \oplus \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{>h}$$

and similar for $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$, where the space $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\leq h}$ is defined in Definition 3.1.2ii). Moreover,

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\leq h} \cong \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])^{\leq h}$$

for any $r \in |\mathbb{C}_p^\times|_p, r < p$.

Proof. It follows by slope decompositions for compact operators on compact Fréchet spaces proved in [Urb, Lemma 2.3.13]. Note that the Fréchet space $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ is compact and the Hecke operator U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ is compact by Proposition 3.2.2, in the sense of [Urb, §2.3.12]. The last isomorphism follows from [Urb, Lemma 2.3.13]. □

3.3 Overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{P}^1)$

As mentioned at the beginning of Chapter 2, the consideration of $\mathbb{P}^1(\mathbb{Q}_p)$ promises a more general framework than that of $\mathbb{Z}_p$, so it is useful to study overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{P}^1)$.

We establish an exact sequence involving overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{P}^1)$ in Proposition 3.3.2. Like the case of $\mathcal{D}_k(\mathbb{Z}_p)$ in Section 3.2, there is also a U_p-operator acting on these overconvergent modular symbols. The difference is that this operator U_p no longer admits the slope decompositions, even the subspaces of bounded above slope are no longer finite dimensional (see Corollary 3.3.15). While proving this result, we are led to the existence of a new operator V_p acting on overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{Z}_p)$ on the right, and we realize that the composition $V_p \circ U_p$ equals $p^{k+1}\mathrm{Id}$ on these modular symbols (see Proposition 3.3.12). From this identity, we derive some corollaries involving the operators U_p, V_p (see Corollaries 3.3.13, 3.3.15, 3.3.17, 3.3.18). We define some finite dimensional U_p-stable subspaces of overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{P}^1)$ arising in an exact sequence of modular symbols in Theorem 3.3.23.

In this section, we consider the congruence subgroup $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$ for N prime to p. We put $\Gamma_0 = \Gamma \cap \Gamma_0(p)$. Denote $D(\infty, 1) = \{z \in \mathbb{P}^1(\mathbb{Q}_p), v_p(z) \leq -1\}$, the complement of $\mathbb{Z}_p$ in $\mathbb{P}^1(\mathbb{Q}_p)$.

Lemma 3.3.1. *For any finite index subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and short exact sequence*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

of Γ-modules such that the order of any torsion element of Γ acts invertibly on V_i's, there is a canonical long exact sequence

$$0 \to \mathrm{Symb}_\Gamma(V_1) \to \mathrm{Symb}_\Gamma(V_2) \to \mathrm{Symb}_\Gamma(V_3) \to H_0(\Gamma, V_1) \to H_0(\Gamma, V_2) \to H_0(\Gamma, V_3) \to 0.$$

Proof. We have seen that the space of modular symbols is isomorphic to the first cohomology group with compact support of the modular curve. The result follows from the long exact sequence of cohomology with compact support associated to a short exact sequence of Γ-modules and the fact that

$$H_0(\Gamma, V_i) \cong H_0(\mathbb{H}/\Gamma, \mathcal{V}_i) \cong H_c^2(\mathbb{H}/\Gamma, \mathcal{V}_i)$$

by Poincaré's duality, where $\mathcal{V}_i$ is the locally constant sheaf associated to the module V_i for $i = 1, 2, 3$. Note that $H_c^3(\mathbb{H}/\Gamma, \mathcal{V}_1) = 0$ since the real dimension of $\mathbb{H}/\Gamma$ is 2. □

Proposition 3.3.2. *For any $k \in \mathbb{N}$, there is an exact sequence compatible with Hecke operators:*

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_{-k-2}(\mathbb{P}^1, L))(k+1) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L)) \xrightarrow{\rho_k} \mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k^\dagger(L)) \to 0$$

induced by the exact sequence in Proposition 2.2.4, where $(k+1)$ means the action of a Hecke operator $[\Gamma_0 s \Gamma_0]$ for $s \in \mathrm{GL}_2(\mathbb{Q}_p)$ is twisted by $(\det s)^{k+1}$.

Proof. We apply Lemma 3.3.1 for the short exact sequence of Γ_0-modules in Proposition 2.2.4 with the note that $H_0(\mathcal{D}_{-k-2}(\mathbb{P}^1, L)) = 0$ by Theorem 2.2.11. □

We denote by $\mathcal{A}_k(D(\infty,1))$ the subspace of $\mathcal{A}_k(\mathbb{P}^1)$ consisting of functions supported in $D(\infty, 1)$. A function $f \in \mathcal{A}_k(\mathbb{P}^1)$ belongs to $\mathcal{A}_k(D(\infty,1))$ if $f_{|\mathbb{Z}_p} = 0$. Since $\Gamma_0(p)$ preserves $\mathbb{Z}_p$, the weight k action of $\Gamma_0(p)$ on $\mathcal{A}_k(\mathbb{P}^1)$ stabilizes $\mathcal{A}_k(D(\infty,1))$, so it induces a weight k action on $\mathcal{A}_k(D(\infty,1))$. We set $\mathcal{D}_k(D(\infty,1))$ the continuous dual of $\mathcal{A}_k(D(\infty,1))$, endowed with the right weight k action of $\Gamma_0(p)$ defined similarly to that on $\mathcal{D}_k(\mathbb{P}^1)$ (see (2.5)). We get the space $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty,1)))$. The restriction map

$$\begin{aligned} \mathrm{res} : \mathcal{D}_k(\mathbb{P}^1) &\to \mathcal{D}_k(D(\infty,1)) \\ \mu &\mapsto \mu_{|D(\infty,1)} \end{aligned}$$

on $D(\infty,1)$ is $\Gamma_0(p)$-equivariant since it is the dual of the inclusion map $\mathcal{A}_k(D(\infty,1)) \to \mathcal{A}_k(\mathbb{P}^1)$.

Consider the exact sequence of p-adic distributions

$$0 \to \mathcal{D}_k(\mathbb{Z}_p) \xrightarrow{\mathrm{ext}} \mathcal{D}_k(\mathbb{P}^1) \xrightarrow{\mathrm{res}} \mathcal{D}_k(D(\infty,1)) \to 0, \tag{3.1}$$

where the first map is the extension map given by $\mathrm{ext}(\mu)(f) = \mu(f_{|\mathbb{Z}_p})$ for $\mu \in \mathcal{D}_k(\mathbb{Z}_p)$ and $f \in \mathcal{A}_k(\mathbb{P}^1)$.

Lemma 3.3.3. *The exact sequence (3.1) is equivariant with the weight k actions of $\Sigma_0(p)$, where the weight k action of $\Sigma_0(p)$ on $\mathcal{D}_k(D(\infty,1))$ is extended from the action of $\Gamma_0(p)$ by*

$$\mu_{|_k\gamma}(f) = \mu(f_{|_k\gamma^*}) = \mu\left(f_{|_k \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}\right) = \mu\Big(\mathbb{1}_{\gamma(D(\infty,1))}(z) \cdot (a-cz)^k f\Big(\frac{dz-b}{a-cz}\Big)\Big), \tag{3.2}$$

where $\mu \in \mathcal{D}_k(D(\infty,1)), \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p), f \in \mathcal{A}_k(D(\infty,1))$. Here we abuse the notation $|_k$ for the above actions on μ and f since they are defined almost similarly to (1.5), (1.7).

Proof. For $\mu_0 \in \mathcal{D}_k(\mathbb{Z}_p), f \in \mathcal{A}_k(\mathbb{P}^1)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p)$, we have

$$\begin{aligned} \mathrm{ext}(\mu_{0|_k\gamma})(f) &= (\mu_{0|_k\gamma})(f_{|\mathbb{Z}_p}) = \mu_0\Big((a-cz)^k f\Big(\frac{dz-b}{a-cz}\Big)_{|\mathbb{Z}_p}\Big) \\ &= \mathrm{ext}(\mu_0)\Big((a-cz)^k f\Big(\frac{dz-b}{a-cz}\Big)\Big) = \mathrm{ext}(\mu_0)_{|_k\gamma}(f). \end{aligned}$$

So $\mathrm{ext}(\mu_{0|_k\gamma}) = \mathrm{ext}(\mu_0)_{|_k\gamma}$. The map ext is $\Sigma_0(p)$-equivariant. Since $\mathcal{D}_k(D(\infty,1))$ is isomorphic to the quotient $\mathcal{D}_k(\mathbb{P}^1)/\mathrm{ext}(\mathcal{D}_k(\mathbb{Z}_p))$ by (3.1), there is an action of $\Sigma_0(p)$ on $\mathcal{D}_k(D(\infty,1))$ such that the map res is $\Sigma_0(p)$-equivariant. We check that this action is given by (3.2).

Let $\mu \in \mathcal{D}_k(\mathbb{P}^1), f \in \mathcal{A}_k(D(\infty,1))$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p)$. We have

$$\begin{aligned} \mathrm{res}(\mu_{|_k\gamma})(f) &= (\mu_{|_k\gamma})(\mathbb{1}_{D(\infty,1)} \cdot f) = \mu\Big((a-cz)^k \mathbb{1}_{D(\infty,1)}(\gamma^{-1}z) \cdot f\Big(\frac{dz-b}{a-cz}\Big)\Big) \\ &= \mu\Big((a-cz)^k \mathbb{1}_{\gamma(D(\infty,1))}(z) \cdot f\Big(\frac{dz-b}{a-cz}\Big)\Big). \end{aligned}$$

We consider the condition $z \in \gamma(D(\infty,1))$. Since $D(\infty,1)$ is the complement of $\mathbb{Z}_p$ in $\mathbb{P}^1(\mathbb{Q}_p)$, it is equivalent to $z \notin \gamma(\mathbb{Z}_p)$. Since

$$\gamma^{-1}(\mathbb{Z}_p) = \gamma^*(\mathbb{Z}_p) = \frac{d\mathbb{Z}_p - b}{a - c\mathbb{Z}_p} \subset \mathbb{Z}_p \qquad \text{(note that } p|c, p \not| a\text{)},$$

it follows that $\mathbb{Z}_p \subset \gamma(\mathbb{Z}_p)$, so $z \notin \mathbb{Z}_p$, hence $z \in D(\infty,1)$. Therefore, $\gamma(D(\infty,1)) \subset D(\infty,1)$. We get $\mathbb{1}_{\gamma(D(\infty,1))} = \mathbb{1}_{D(\infty,1)} \cdot \mathbb{1}_{\gamma(D(\infty,1))}$. We obtain

$$\operatorname{res}(\mu_{|_k\gamma})(f) = \operatorname{res}(\mu)\Big(\mathbb{1}_{\gamma(D(\infty,1))}(z) \cdot (a-cz)^k f\Big(\frac{dz-b}{a-cz}\Big)\Big) = \operatorname{res}(\mu)_{|_k\gamma}(f).$$

The lemma follows. □

Remark 3.3.4. The condition $z \in \gamma(D(\infty,1)) \subset D(\infty,1)$ in (3.2) ensures that $\frac{dz-b}{a-cz} \in D(\infty,1)$.

The action (3.2) of $\Sigma_0(p)$ on $\mathcal{D}_k(D(\infty,1))$ induces the right action of U_p on $\operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty,1)))$.

Corollary 3.3.5. *For any $k \in \mathbb{Z}\backslash\{0\}$, there is a U_p-equivariant exact sequence of modular symbols:*

$$0 \to \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p,L)) \overset{\text{ext}}{\to} \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1,L)) \overset{\text{res}}{\to} \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty,1),L)) \to 0.$$

If $k=0$, the last space 0 in this exact sequence is replaced by L:

$$0 \to \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_0(\mathbb{Z}_p,L)) \overset{\text{ext}}{\to} \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_0(\mathbb{P}^1,L)) \overset{\text{res}}{\to} \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_0(D(\infty,1),L)) \to L.$$

Proof. We apply Lemma 3.3.1 for the short exact sequence (3.1) which is $\Sigma_0(p)$-equivariant by Lemma 3.3.3, with the note that $H_0(\Gamma_0,\mathcal{D}_k(\mathbb{Z}_p,L)) = 0$ for any $k \in \mathbb{Z}\backslash\{0\}$ and $H_0(\Gamma_0,\mathcal{D}_0(\mathbb{Z}_p,L)) = L$ by [PS13, Lemma 5.2]. □

It is obvious that the functor which takes subspace of slope $\leq h$ (or $< h$) is left exact, so we get a left exact sequence of $\leq h$-slope modular symbols for U_p-operator from the exact sequence in the above corollary for each $h \in \mathbb{Q}$ and $k \in \mathbb{Z}$:

$$0 \to \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p,L))^{\leq h} \overset{\text{ext}}{\to} \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1,L))^{\leq h} \overset{\text{res}}{\to} \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty,1),L))^{\leq h},$$

and similar if $\leq h$ is replaced by $< h$.

Proposition 3.3.6. *For any $h \in \mathbb{Q}$ and $k \in \mathbb{Z}\backslash\{0\}$, the map*

$$\operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1,L))^{\leq h} \overset{\text{res}}{\to} \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty,1),L))^{\leq h}$$

is surjective. Therefore, for $k \in \mathbb{Z}\backslash\{0\}$, there is a U_p-equivariant exact sequence of $\leq h$-slope modular symbols:

$$0 \to \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p,L))^{\leq h} \overset{\text{ext}}{\to} \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1,L))^{\leq h} \overset{\text{res}}{\to} \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty,1),L))^{\leq h} \to 0.$$

If $k=0$, the last space 0 in this exact sequence is replaced by L. We have the same results if $\leq h$ is replaced by $< h$.

Proof. Consider $k \neq 0$. Let $\Phi_1 \in \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty,1),L))^{\leq h}$. There is a polynomial $P(X) \in L[X]$ of slope $\leq h$ such that $\Phi_{1|P(U_p)} = 0$. By Corollary 3.3.5, there exists $\Phi \in \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1,L))$ such that $\operatorname{res}(\Phi) = \Phi_1$.

Since the map res is U_p-equivariant, we have $\Phi_{|P(U_p)} \in \ker(\operatorname{res}) = \operatorname{Im}(\operatorname{ext})$, so there is $\Phi_0 \in \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p,L))$ such that $\Phi_{|P(U_p)} = \operatorname{ext}(\Phi_0)$.

Recall the $\leq h$-slope decomposition:

$$\operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p,L)) = \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p,L))^{\leq h} \oplus \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p,L))^{>h}.$$

Writing $\Phi_0 = \Psi_0 + \Theta_0$ where Ψ_0 has slope $\leq h$ and $\Theta_0 \in \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p,L))^{>h}$. There exists a polynomial $Q(X) \in L[X]$ of slope $\leq h$ such that $\Psi_{0|Q(U_p)} = 0$. We have

$$\begin{aligned}\Phi_{|(PQ)(U_p)} &= (\Phi_{|P(U_p)})_{|Q(U_p)} = \operatorname{ext}(\Phi_0)_{|Q(U_p)} \\ &= \operatorname{ext}((\Psi_0+\Theta_0)_{|Q(U_p)}) = \operatorname{ext}(\Theta_{0|Q(U_p)}).\end{aligned}$$

Since PQ is of slope $\leq h$, $(PQ)(U_p)$ acts invertibly on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{>h}$ which contains $\Theta_{0|Q(U_p)}$, so there is $\Xi_0 \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{>h}$ such that $\Xi_{0|(PQ)(U_p)} = \Theta_{0|Q(U_p)}$. We get

$$(\Phi - \mathrm{ext}(\Xi_0))_{|(PQ)(U_p)} = 0,$$

so $\Phi - \mathrm{ext}(\Xi_0) \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))^{\leq h}$. Since $\mathrm{Im}\,(\mathrm{ext}) = \mathrm{Ker}\,(\mathrm{res})$, we have

$$\mathrm{res}(\Phi - \mathrm{ext}(\Xi_0)) = \mathrm{res}(\Phi) = \Phi_1.$$

So the map res restricted on the subspace of $\leq h$-slope modular symbols is surjective. The exactness for $k \neq 0$ is followed from Corollary 3.3.5. For $k = 0$, the proof is the same. □

Consider the canonical isomorphism $\iota : \mathcal{A}(\mathbb{Z}_p) \to \mathcal{A}_k(D(\infty, 1))$ given by

$$\iota(f)(z) = z^k f\Big(\frac{1}{Npz}\Big) \text{ for } f \in \mathcal{A}(\mathbb{Z}_p), z \in D(\infty, 1) \text{ (note that } (N,p) = 1).$$

Taking continuous duals induces an isomorphism of Fréchet spaces:

$$\begin{aligned} \hat{\iota} : \mathcal{D}_k(D(\infty, 1)) &\xrightarrow{\cong} \mathcal{D}(\mathbb{Z}_p) \\ \mu &\mapsto \hat{\iota}(\mu) : f \mapsto \mu\Big(z^k f\Big(\frac{1}{Npz}\Big)\Big), \end{aligned} \tag{3.3}$$

where $\mu \in \mathcal{D}_k(D(\infty, 1))$, $f \in \mathcal{A}(\mathbb{Z}_p)$ and $z \in D(\infty, 1)$. So $\mathcal{D}(\mathbb{Z}_p)$ is endowed with a weight k action of $\Sigma_0(p)$ induced from the action (3.2) on $\mathcal{D}_k(D(\infty, 1))$. We determine this action.

Let $\tau_{Np} = \begin{pmatrix} 0 & 1 \\ Np & 0 \end{pmatrix}$. Define the right weight k action $\cdot_k$ of $\Sigma_0(p)$ on $\mathcal{D}(\mathbb{Z}_p)$ by

$$\begin{aligned} (\mu \cdot_k \gamma)(f) &:= (\mu_{|_k \tau_{Np}\gamma\tau_{Np}^{-1}})(f) = \mu_{\big|_k \begin{pmatrix} d & c/(Np) \\ bNp & a \end{pmatrix}}(f) = \mu\Big(f_{\big|_k \begin{pmatrix} a & -c/(Np) \\ -bNp & d \end{pmatrix}}\Big) \\ &= \mu\left(\mathbb{1}_{\frac{c}{aNp}+(ad-bc)\mathbb{Z}_p}(z) \cdot (d - bNpz)^k f\Big(\frac{az - c/(Np)}{d - bNpz}\Big)\right), \end{aligned} \tag{3.4}$$

where $\mu \in \mathcal{D}(\mathbb{Z}_p), \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p)$ and $f \in \mathcal{A}(\mathbb{Z}_p)$. Here we abuse the notation $|_k$ for the above actions on μ and f since they are defined almost similarly to (1.5), (1.7) (the condition $z \in \frac{c}{aNp} + (ad - bc)\mathbb{Z}_p$ for the action on f ensures that $\frac{az-c/(Np)}{d-bNpz} \in \mathbb{Z}_p$).

Denote $\mathcal{D}'_k(\mathbb{Z}_p)$ the space $\mathcal{D}(\mathbb{Z}_p)$ endowed with the action $\cdot_k$ of $\Sigma_0(p)$.

Remark 3.3.7. The appearance of the Atkin-Lehner operator in (3.4) suggests that we are on correct track, for the eventual future goal to prove functional equations with the aid of p-adic distributions on $\mathbb{P}^1(\mathbb{Q}_p)$.

Lemma 3.3.8. *The isomorphism* (3.3) *is* $\Sigma_0(p)$*-equivariant for the action* (3.2) *on* $\mathcal{D}_k(D(\infty, 1))$ *and the action* (3.4) *on* $\mathcal{D}'_k(\mathbb{Z}_p)$.

Proof. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p), \mu \in \mathcal{D}_k(D(\infty, 1))$ and $f \in \mathcal{A}(\mathbb{Z}_p)$, we have

$$\begin{aligned} \hat{\iota}(\mu_{|_k\gamma})(f) &= (\mu_{|_k\gamma})\Big(z^k f\Big(\frac{1}{Npz}\Big)\Big) = \mu\left(\mathbb{1}_{\gamma(D(\infty,1))}(z) \cdot (a - cz)^k \Big(\frac{dz - b}{a - cz}\Big)^k f\Big(\frac{a - cz}{Np(dz - b)}\Big)\right) \\ &= \mu\left(\mathbb{1}_{\gamma(D(\infty,1))}(z) \cdot z^k \Big(d - \frac{bNp}{Npz}\Big)^k \cdot f\left(\frac{\frac{a}{Npz} - \frac{c}{Np}}{d - \frac{bNp}{Npz}}\right)\right) \\ &=: \mu\Big(z^k g\Big(\frac{1}{Npz}\Big)\Big) = \hat{\iota}(\mu)(g), \end{aligned}$$

where $g(z) = \mathbb{1}_{\gamma(D(\infty,1))}(\frac{1}{Npz}) \cdot (d - bNpz)^k \cdot f(\frac{az-c/(Np)}{d-bNpz})$ for $z \in \mathbb{Z}_p$.

We simplify the condition $\frac{1}{Npz} \in \gamma(D(\infty,1)) = \mathbb{P}^1(\mathbb{Q}_p)\backslash\gamma(\mathbb{Z}_p)$. We have

$$\frac{1}{Npz} \notin \gamma(\mathbb{Z}_p) \Leftrightarrow \gamma^{-1}\Big(\frac{1}{Npz}\Big) \notin \mathbb{Z}_p \Leftrightarrow v_p\Big(\frac{\frac{d}{Npz}-b}{a-\frac{c}{Npz}}\Big) < 0 \Leftrightarrow v_p\Big(\frac{d-bNpz}{aNpz-c}\Big) < 0$$
$$\Leftrightarrow v_p\Big(\frac{aNpz-c}{d-bNpz}\Big) > 0 \Leftrightarrow \frac{Npz-c/a}{d-bNpz} \in p\mathbb{Z}_p \text{ (since } p \nmid a) \Leftrightarrow \frac{bNpz-bc/a}{bNpz-d} \in bp\mathbb{Z}_p$$
$$\Leftrightarrow 1+\frac{d-bc/a}{bNpz-d} \in bp\mathbb{Z}_p \Leftrightarrow \frac{d-bc/a}{bNpz-d} \in -1+bp\mathbb{Z}_p \Leftrightarrow bNpz-d \in \frac{d-bc/a}{-1+bp\mathbb{Z}_p}.$$

If $x \in -1+bp\mathbb{Z}_p$, then $x=-1+bpy$ for $y \in \mathbb{Z}_p$. Since $|bpy|_p < 1$, we have

$$\frac{1}{x} = \frac{1}{-1+bpy} = \frac{-1}{1-bpy} = -(1+bpy+(bpy)^2+...) \in -1+bp\mathbb{Z}_p.$$

So $\frac{1}{-1+bp\mathbb{Z}_p} \subset -1+bp\mathbb{Z}_p$, hence $-1+bp\mathbb{Z}_p \subset \frac{1}{-1+bp\mathbb{Z}_p}$. Therefore, $\frac{1}{-1+bp\mathbb{Z}_p} = -1+bp\mathbb{Z}_p$. We obtain

$$bNpz-d \in \Big(d-\frac{bc}{a}\Big)(-1+bp\mathbb{Z}_p) = \frac{bc}{a}-d+\Big(d-\frac{bc}{a}\Big)bp\mathbb{Z}_p.$$

We deduce that $z \in \frac{c}{aNp}+(ad-bc)\mathbb{Z}_p \subset \mathbb{Z}_p$ since $p|c, p \nmid aN$. So

$$g(z) = \mathbb{1}_{\frac{c}{aNp}+(ad-bc)\mathbb{Z}_p} \cdot (d-bNpz)^k \cdot f\Big(\frac{az-c/(Np)}{d-bNpz}\Big) = f_{|_k \begin{pmatrix} a & -c/(Np) \\ -bNp & d \end{pmatrix}}.$$

We obtain

$$\hat{\imath}(\mu_{|_k\gamma})(f) = \hat{\imath}(\mu)(g) = \hat{\imath}(\mu)_{|_k \begin{pmatrix} d & c/(Np) \\ bNp & a \end{pmatrix}}(f) = \hat{\imath}(\mu)_{|_k \tau_{Np}\gamma\tau_{Np}^{-1}}(f).$$

Therefore, $\hat{\imath}(\mu_{|_k\gamma}) = \hat{\imath}(\mu)_{|_k\tau_{Np}\gamma\tau_{Np}^{-1}} = \hat{\imath}(\mu)\cdot_k\gamma$. Note that τ_{Np} normalizes Γ_0. □

We define as usual the Hecke operator U_p acting on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}'_k(\mathbb{Z}_p))$ on the right induced from the action $\cdot_k$ of $\Sigma_0(p)$ on $\mathcal{D}'_k(\mathbb{Z}_p)$. We have

$$(\Phi\cdot U_p)(D) = \sum_{a=0}^{p-1}(\Phi\cdot\gamma_a)(D) = \sum_{a=0}^{p-1}\Phi(\gamma_a D)\cdot_k\gamma_a = \sum_{a=0}^{p-1}\Phi(\gamma_a D)_{|_k\tau_{Np}\gamma_a\tau_{Np}^{-1}}$$

for $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}'_k(\mathbb{Z}_p))$ and $D \in \Delta_0$.

Corollary 3.3.9. *For $k \in \mathbb{Z}\backslash\{0\}$, there is a U_p-equivariant exact sequence of modular symbols:*

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p,L)) \overset{\text{ext}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1,L)) \overset{\text{res}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}'_k(\mathbb{Z}_p,L)) \to 0.$$

The restriction on the $\leq h$-slope subspace is also exact:

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p,L))^{\leq h} \overset{\text{ext}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1,L))^{\leq h} \overset{\text{res}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}'_k(\mathbb{Z}_p,L))^{\leq h} \to 0,$$

and similar if $\leq h$ is replaced by $< h$.

If $k=0$, the last space 0 in the above exact sequences is replaced by L.

Proof. This is immediate from Corollary 3.3.5, Proposition 3.3.6 and Lemma 3.3.8. □

Since $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \tau_{Np}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\tau_{Np}^{-1}$ and $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \in \Sigma_0(p)$, by (3.4), there is a right weight k action of $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ on $\mathcal{D}_k(\mathbb{Z}_p)$ given by

$$\mu_{|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}(f) = \mu\Big(f_{|_k \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}}\Big) = \mu\Big(p^k\mathbb{1}_{p\mathbb{Z}_p}(z)\cdot f\Big(\frac{z}{p}\Big)\Big), \tag{3.5}$$

where $\mu \in \mathcal{D}_k(\mathbb{Z}_p)$ and $f \in \mathcal{A}_k(\mathbb{Z}_p)$. This action is compatible with the action of $\Gamma_0(p)$. Let V_p be the double coset operator $\left[\Gamma_0 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0\right]$ acting on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ on the right.

Let $\tau_{Np} = \begin{pmatrix} 0 & 1 \\ Np & 0 \end{pmatrix}$ and $\gamma_a = \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$. Since τ_{Np} normalizes Γ_0, there is a decomposition:

$$\begin{aligned}\Gamma_0 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0 &= \Gamma_0 \tau_{Np} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \tau_{Np}^{-1} \Gamma_0 = \tau_{Np} \Gamma_0 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0 \tau_{Np}^{-1} \\ &= \bigsqcup_{a=0}^{p-1} \tau_{Np} \Gamma_0 \gamma_a \tau_{Np}^{-1} = \bigsqcup_{a=0}^{p-1} \Gamma_0 \cdot \tau_{Np} \gamma_a \tau_{Np}^{-1}. \end{aligned} \tag{3.6}$$

Therefore, V_p acts on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ by $\phi_{|V_p} = \sum_{a=0}^{p-1} \phi_{|\tau_{Np}\gamma_a\tau_{Np}^{-1}}$.

Lemma 3.3.10. *There is a canonical isomorphism:*

$$\begin{aligned}\mathrm{Symb}_{\Gamma_0}(\mathcal{D}'_k(\mathbb{Z}_p)) &\overset{\cong}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)) \\ \Phi &\mapsto \Psi : D \mapsto \Phi(\tau_{Np}^{-1} D) \quad \text{for } D \in \Delta_0\end{aligned}$$

which is equivariant for the action of U_p on the left hand side and the action of V_p on the right hand side. Therefore, for any $h \in \mathbb{Q}$,

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}'_k(\mathbb{Z}_p))^{\le h} \cong \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \le h}.$$

Proof. For $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}'_k(\mathbb{Z}_p))$, since Φ is Γ_0-invariant, for any $D \in \Delta_0$ and $\gamma \in \Gamma_0$, we have

$$\Phi(D) = (\Phi \cdot \gamma)(D) = \Phi(\gamma D) \cdot_k \gamma = \Phi(\gamma D)_{|_k \tau_{Np}\gamma\tau_{Np}^{-1}}.$$

Replacing γ by $\tau_{Np}^{-1}\gamma\tau_{Np}$ yields

$$\Phi(D) = \Phi(\tau_{Np}^{-1}\gamma\tau_{Np} D)_{|_k \gamma}.$$

For each $D \in \Delta_0$, let $\Psi(D) = \Phi(\tau_{Np}^{-1} D)$, then

$$\Psi(\tau_{Np} D) = \Psi(\gamma \tau_{Np} D)_{|_k \gamma}.$$

Replacing D by $\tau_{Np}^{-1} D$, we get $\Psi(D) = \Psi(\gamma D)_{|_k \gamma} = \Psi_{|\gamma}(D)$ for any $D \in \Delta_0$ and $\gamma \in \Gamma_0$, hence $\Psi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$. Therefore, the correspondence $\Phi \mapsto \Psi$ is an isomorphism between $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}'_k(\mathbb{Z}_p))$ and $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$.

Let Ψ_1 be the image of $\Phi \cdot U_p$ under the correspondence. We show that $\Psi_1 = \Psi_{|V_p}$. For any $D \in \Delta_0$, we have

$$\begin{aligned}\Psi_1(D) = (\Phi \cdot U_p)(\tau_{Np}^{-1} D) &= \sum_{a=0}^{p-1} \Phi(\gamma_a \tau_{Np}^{-1} D) \cdot_k \gamma_a = \sum_{a=0}^{p-1} \Phi(\gamma_a \tau_{Np}^{-1} D)_{|_k \tau_{Np}\gamma_a\tau_{Np}^{-1}} \\ &= \sum_{a=0}^{p-1} \Psi(\tau_{Np}\gamma_a\tau_{Np}^{-1} D)_{|_k \tau_{Np}\gamma_a\tau_{Np}^{-1}} = \sum_{a=0}^{p-1} \Psi_{|\tau_{Np}\gamma_a\tau_{Np}^{-1}}(D) = \Psi_{|V_p}(D).\end{aligned}$$

The lemma is proven. ☐

Corollary 3.3.11. *For $k \in \mathbb{Z}\backslash\{0\}$, there is an exact sequence of modular symbols:*

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L)) \overset{\text{ext}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L)) \overset{\text{res}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L)) \to 0 \tag{3.7}$$

which is equivariant for the U_p-action on the first two spaces and the V_p-action on the last space. The restriction on the $\le h$-slope subspaces for the corresponding operators is also exact:

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{\le h} \overset{\text{ext}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))^{\le h} \overset{\text{res}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{V_p \le h} \to 0, \tag{3.8}$$

and similar if $\le h$ is replaced by $< h$.

If $k = 0$, the last space 0 in the above exact sequences is replaced by L.

Proof. This follows by Corollary 3.3.9 and Lemma 3.3.10. □

The operators U_p and V_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ are related by the following result:

Proposition 3.3.12. *For any $k \in \mathbb{Z}$ and $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$, we have*

$$\Phi_{|U_pV_p} := (\Phi_{|U_p})_{|V_p} = p^{k+1}\Phi.$$

Proof. For any $D \in \Delta_0, f \in \mathcal{A}_k(\mathbb{Z}_p)$, we have

$$\begin{aligned}
\Phi_{|U_pV_p}(D)(f) &= \sum_{a,b=0}^{p-1} \Phi_{|_k\begin{pmatrix}1 & a\\ 0 & p\end{pmatrix}|_k\begin{pmatrix}p & 0\\ bNp & 1\end{pmatrix}}(D)(f) \\
&= \sum_{a,b=0}^{p-1} \Phi_{|_k\begin{pmatrix}1 & a\\ 0 & p\end{pmatrix}}\left(\begin{pmatrix}p & 0\\ bNp & 1\end{pmatrix} D\right)\left(\mathbb{1}_{p\mathbb{Z}_p}(z)\cdot(-bNpz+p)^k f\Big(\frac{z}{-bNpz+p}\Big)\right) \\
&= \sum_{a,b=0}^{p-1} \Phi\left(\begin{pmatrix}1 & a\\ 0 & p\end{pmatrix}\begin{pmatrix}p & 0\\ bNp & 1\end{pmatrix} D\right)_{|_k\begin{pmatrix}1 & a\\ 0 & p\end{pmatrix}}\left(\mathbb{1}_{p\mathbb{Z}_p}(z)\cdot(-bNpz+p)^k f\Big(\frac{z}{-bNpz+p}\Big)\right) \\
&= \sum_{a,b=0}^{p-1} \Phi\left(\begin{pmatrix}p+abNp & a\\ bNp^2 & p\end{pmatrix} D\right)\left(\mathbb{1}_{p\mathbb{Z}_p}(pz-a)\cdot(-bNp(pz-a)+p)^k f\Big(\frac{pz-a}{-bNp(pz-a)+p}\Big)\right).
\end{aligned}$$

Since the function $\mathbb{1}_{p\mathbb{Z}_p}(pz-a)$ on $\mathbb{Z}_p$ is nonzero if and only if $a = 0$, we have

$$\begin{aligned}
\Phi_{|U_pV_p}(D)(f) &= \sum_{b=0}^{p-1} \Phi\left(\begin{pmatrix}p & 0\\ bNp^2 & p\end{pmatrix} D\right)\left((-bNp^2z+p)^k f\Big(\frac{pz}{-bNp^2z+p}\Big)\right) \\
&= \sum_{b=0}^{p-1} \Phi\left(\begin{pmatrix}1 & 0\\ bNp & 1\end{pmatrix} D\right)\left(p^k(-bNpz+1)^k f\Big(\frac{z}{-bNpz+1}\Big)\right) \\
&= \sum_{b=0}^{p-1} \Phi\left(\begin{pmatrix}1 & 0\\ bNp & 1\end{pmatrix} D\right)\left(p^k f_{|_k\begin{pmatrix}1 & 0\\ -bNp & 1\end{pmatrix}}\right) \\
&= \sum_{b=0}^{p-1} p^k\Phi\left(\begin{pmatrix}1 & 0\\ bNp & 1\end{pmatrix} D\right)_{|_k\begin{pmatrix}1 & 0\\ bNp & 1\end{pmatrix}}(f) = \sum_{b=0}^{p-1} p^k\Phi_{|\begin{pmatrix}1 & 0\\ bNp & 1\end{pmatrix}}(D)(f).
\end{aligned}$$

Since $\begin{pmatrix}1 & 0\\ bNp & 1\end{pmatrix} \in \Gamma_1(N)\cap\Gamma_0(p) \subset \Gamma_0$ for all b, we have $\Phi_{|\begin{pmatrix}1 & 0\\ bNp & 1\end{pmatrix}} = \Phi$ for all b, so $\Phi_{|U_pV_p} = p^{k+1}\Phi$. □

Corollary 3.3.13. *The operator U_p is injective on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$. If $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ is annihilated by $P(U_p)$ for some polynomial $P \in L[X]$, then Φ is annihilated by $Q(V_p)$ for the polynomial $Q(X) = X^nP(\frac{p^{k+1}}{X})$, where n is the degree of P. In particular, if α is an eigenvalue of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$, then $\alpha \neq 0$ and $\frac{p^{k+1}}{\alpha}$ is an eigenvalue of V_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$.*

Proof. U_p is injective on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ since $V_p \circ U_p = p^{k+1}\mathrm{Id}$ is injective on it.

Suppose P splits in $\bar{\mathbb{Q}}_p$ as $P(X) = \prod_{i=1}^{n}(X-\alpha_i)$ (we can assume that P is monic). Regarding Φ as an element of $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)) \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p$, we have

$$\Phi_{|(U_p-\alpha_1)\dots(U_p-\alpha_n)} = 0.$$

Let $\Phi_1 = \Phi_{|(U_p-\alpha_1)\dots(U_p-\alpha_{n-1})}$, then $\Phi_{1|(U_p-\alpha_n)} = 0$. So

$$0 = \Phi_{1|(U_p-\alpha_n)V_p} = \Phi_{1|(p^{k+1}-\alpha_nV_p)}.$$

Let $\Phi_2 = \Phi_{|(U_p-\alpha_1)\ldots(U_p-\alpha_{n-2})}$, then $\Phi_1 = \Phi_{2|(U_p-\alpha_{n-1})}$ and $\Phi_{2|(U_p-\alpha_{n-1})(p^{k+1}-\alpha_n V_p)} = 0$. So

$$\begin{aligned} 0 &= \Phi_{2|(U_p-\alpha_{n-1})(p^{k+1}-\alpha_n V_p)V_p} = \Phi_{2|(U_p-\alpha_{n-1})V_p(p^{k+1}-\alpha_n V_p)} \\ &= \Phi_{2|(p^{k+1}-\alpha_{n-1}V_p)(p^{k+1}-\alpha_n V_p)}. \end{aligned}$$

Repeating this process, we get $\Phi_{|(p^{k+1}-\alpha_1 V_p)\ldots(p^{k+1}-\alpha_n V_p)} = 0$. Therefore, Φ is annihilated by $Q(V_p)$ where Q is the polynomial given by

$$Q(X) = \prod_{i=1}^{n}(p^{k+1}-\alpha_i X) = X^n \prod_{i=1}^{n}\Big(\frac{p^{k+1}}{X}-\alpha_i\Big) = X^n P\Big(\frac{p^{k+1}}{X}\Big).$$

If α is an eigenvalue of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$, then $\alpha \neq 0$ since U_p is injective and there exists $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))\backslash\{0\}$ such that $\Phi_{|U_p} = \alpha\Phi$. Since Φ is annihilated by $P(U_p)$ where P is the polynomial $X-\alpha$, Φ is also annihilated by $Q(V_p)$ where $Q(X) = p^{k+1}-\alpha X$. Therefore, $\frac{p^{k+1}}{\alpha}$ is an eigenvalue of V_p for the eigensymbol Φ. □

Corollary 3.3.14. *The endomorphism V_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ is open.*

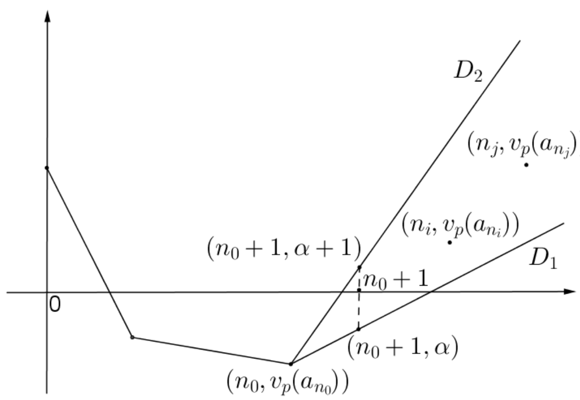
Proof. By [Schn, Prop. 8.2 and Prop. 8.6], every surjective continuous map between Fréchet spaces is open. On the other hand, the identity $V_p \circ U_p = p^{k+1}\mathrm{Id}$ on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ of Proposition 3.3.12 implies that V_p is surjective on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$. We get the conclusion. □

Corollary 3.3.15. *For any $h \in \mathbb{Q}$ and $k \in \mathbb{Z}$, we have*

$$\dim\big(\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))^{\leq h}\big) = \dim\big(\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{V_p \leq h}\big) = +\infty.$$

Proof. Since U_p acts compactly on the Fréchet space $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ in the sense of [Urb, §2.3.12], by the spectral theory of compact operators, U_p has only a finite or countable number of eigenvalues on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$. In the latter case, the sequence of these eigenvalues tends to 0, so the sequence of its slope tends to $+\infty$. We show that this case happens. The slope of U_p-eigenvalues can be determined by the Newton polygon of the characteristic power series of U_p.

Let $P(X) = \det(1-XU_p) =: \sum_{n=0}^{\infty} a_n X^n$ be the characteristic power series of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$. Suppose that the Newton polygon of $P(X)$ has a finite number of edges with a unique infinite edge D_1, as in the below image. Denote $(n_0, v_p(a_{n_0}))$ the coordinates of the endpoint of D_1, where $n_0 \in \mathbb{N}$. Let (n_0+1, α) be the point on D_1 of abscissa n_0+1. Let D_2 denote the ray of endpoint $(n_0, v_p(a_{n_0}))$ and containing the point $(n_0+1, \alpha+1)$. By the definition of Newton polygons, there is a sequence of points $\{(n_i, v_p(a_{n_i}))\}_i$ lying between the rays D_1 and D_2 such that $n_i \to +\infty$ when $i \to +\infty$, since otherwise every point $(n, v_p(a_n))$ will lie above the ray D_2 for n big enough, so D_1 will not be an edge of the Newton polygon of $P(X)$. Since the points $(n_i, v_p(a_{n_i}))$ lie below the ray D_2, the values $v_p(a_{n_i})$ are bounded above linearly on n_i, i.e., there are positive constants a, b such that $v_p(a_{n_i}) \leq an_i + b$ for all i. We show that this is impossible.



By [Wan, Lemma 3.1], there is a lower bound:

$$v_p(a_n) \geq \frac{p-1}{p+1}\Big(\sum_{i=0}^{l} im_i + (l+1)(n-d_l)\Big) - n$$

for $d_l \leq n < d_{l+1}$, where d_i is the dimension of the space $\mathcal{M}_{k+2+i(p-1)}(\Gamma_1(N))$ of classical modular forms of weight $k+2+i(p-1)$ and level $\Gamma_1(N)$, and $m_i = d_i - d_{i-1}$ for $i > 0$. Note that the set of U_p-eigenvalues of overconvergent modular forms of weight $k+2$ is the same as that of overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{Z}_p)$. Consider $d_l \leq n < d_{l+1}$, we have

$$v_p(a_n) \geq \frac{p-1}{p+1}\Big((l+1)n - \sum_{i=0}^{l} d_i\Big) - n \geq \frac{p-1}{p+1}\Big(ln - \sum_{i=0}^{l-1} d_i\Big) - n.$$

The dimension of a space of modular forms of given weight and level is given in [DS, Theorem 3.5.1 and Theorem 3.6.1]. In particular, there exist positive constants c, d depend only on the level N such that $ck - d \leq \dim \mathcal{M}_k(\Gamma_1(N)) \leq ck + d$ for any $k \geq 0$. So

$$c(k+2+i(p-1)) - d \leq d_i \leq c(k+2+i(p-1)) + d$$

for any $i \geq 0$. We get

$$\begin{aligned} v_p(a_n) &\geq \frac{p-1}{p+1}\left[ln - c\Big(l(k+2) + (p-1)\frac{l(l-1)}{2}\Big) - ld\right] - n \\ &= \frac{l(p-1)}{p+1}\Big(n - c(k+2) - \frac{c(p-1)(l-1)}{2} - d\Big) - n. \end{aligned}$$

Since $n \geq d_l \geq c(k+2+l(p-1)) - d$, we have $c(p-1)(l-1) \leq n - c(k+2) + d$. So

$$v_p(a_n) \geq \frac{l(p-1)}{p+1} \cdot \frac{n - c(k+2) - 3d}{2} - n.$$

Since $n < d_{l+1} \leq c(k+2+(l+1)(p-1)) + d$, we have $l \geq \frac{n-c(k+2)-c(p-1)-d}{c(p-1)}$. Hence

$$v_p(a_n) \geq \frac{n - c(k+2) - c(p-1) - d}{c(p+1)} \cdot \frac{n - c(k+2) - 3d}{2} - n > an + b$$

for n big enough. This is a contradiction. So the Newton polygon of $P(X)$ has infinitely many edges. If a finite edge of the Newton polygon of $P(X)$ has length M and slope λ (the length of a finite edge of a Newton polygon is the absolute value of the difference of the abscissas of its endpoints), then $P(X)$ has exactly M roots of valuation $-\lambda$ (counting with multiplicity), so U_p has exactly M eigenvalues of p-adic valuation λ on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$. Therefore, U_p has infinitely many eigenvalues on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ with arbitrarily large slope. We deduce from Corollary 3.3.13 that V_p has infinitely many eigenvalues on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ with arbitrarily small slope. Therefore, $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \leq h}$ is infinite dimensional, then so is $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))^{\leq h}$ by (3.8). ☐

Remark 3.3.16. The above corollary is also true for the congruence subgroup $\Gamma_1(N\mathbf{q})$ instead of $\Gamma_0 = \Gamma_1(N) \cap \Gamma_0(p)$. The crucial point is that the slope of eigenvalues of U_p on $\mathrm{Symb}_{\Gamma_1(N\mathbf{q})}(\mathcal{D}_k(\mathbb{Z}_p))$ can be arbitrarily large, which can be deduced from [Cole, Proposition I4] and [PS13, Corollary 7.4].

Corollary 3.3.17. *Let* $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<\infty}$ *denote the subspace of* $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ *of modular symbols of finite* U_p*-slope. The space* $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<\infty}$ *is stable by the actions of* U_p *and* V_p*. Moreover, the operators* U_p *and* V_p*, seen as endomorphisms on* $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<\infty}$*, are isomorphisms and satisfy* $U_p \circ V_p = V_p \circ U_p = p^{k+1}\mathrm{Id}$.

Proof. If $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<\infty}$, then there exists a polynomial $P(X)$ of finite slope such that $\Phi_{|P(U_p)} = 0$. We have

$$(\Phi_{|U_p})_{|P(U_p)} = (\Phi_{|P(U_p)})_{|U_p} = 0,$$

so $\Phi_{|U_p} \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<\infty}$. Therefore, $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<\infty}$ is stable by U_p.

Consider Φ and P as above, since $P(X)$ has finite slope, $P(0) \neq 0$. Writing $P(X) = a(1 + XQ(X))$, where $a \neq 0$ is the constant coefficient of P. Then

$$0 = \Phi_{|P(U_p)} = a(\Phi + \Phi_{|Q(U_p)U_p}),$$

so $\Phi = -\Phi_{|Q(U_p)U_p}$. Hence

$$\Phi_{|V_p} = -(\Phi_{|Q(U_p)})_{|U_p|V_p} = -p^{k+1}\Phi_{|Q(U_p)} \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<\infty}$$

by Proposition 3.3.12 and $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<\infty}$ is U_p-stable. Therefore, $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<\infty}$ is stable by V_p.

By the same method, the subspace $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\leq h}$ is stable under the action of U_p and V_p for any $h \in \mathbb{Q}$. Since U_p is injective on the finite dimensional vector space $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\leq h}$ by Corollary 3.3.13, U_p is an isomorphism on it. Combining with the identity $V_p \circ U_p = p^{k+1}\mathrm{Id}$ of Proposition 3.3.12, we deduce that V_p is also an isomorphism and $U_p \circ V_p = p^{k+1}\mathrm{Id}$ on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\leq h}$. Since $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<\infty}$ is the union of $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\leq h}$'s, it follows that U_p and V_p are isomorphisms on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<\infty}$ and $U_p \circ V_p = V_p \circ U_p = p^{k+1}\mathrm{Id}$ on it. □

For $h \in \mathbb{Q}$, by Corollary 3.3.13, the space of modular symbols of finite U_p-slope $\geq k+1-h$ in $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ is contained in the space $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \leq h}$. If we impose the condition on the upper bound for U_p-slope on both spaces, then we get an equality. We have the following result:

Corollary 3.3.18. *For $h, h' \in \mathbb{Q}$ such that $k+1-h' \leq h$, we have*

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{U_p \leq h, V_p \leq h'} = \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{k+1-h' \leq U_p \leq h},$$

where $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{U_p \leq h, V_p \leq h'} = \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{U_p \leq h} \cap \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \leq h'}$ *and we denote by* $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{k+1-h' \leq U_p \leq h}$ *the subspace of* $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ *consisting of modular symbols of U_p-slope between $k+1-h'$ and h.*

Proof. We have seen that the right hand side is contained in the left hand side. We prove the opposite inclusion.

Let $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{U_p \leq h, V_p \leq h'}$. Then there is a polynomial $P(X)$ of slope $\leq h'$ such that $\Phi_{|P(V_p)} = 0$. By Corollary 3.3.17, $U_p \circ V_p = p^{k+1}\mathrm{Id}$ on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{U_p \leq h}$. Using the method in the proof of Corollary 3.3.13, the equality $\Phi_{|P(V_p)} = 0$ yields $\Phi_{|Q(U_p)} = 0$ for the polynomial $Q(X) = X^n P\left(\frac{p^{k+1}}{X}\right)$, where n is the degree of P. The polynomial $Q(X)$ has finite slope $\geq k+1-h'$.

Since $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{U_p \leq h}$, there is a polynomial $R(X)$ of slope $\leq h$ such that $\Phi_{|R(U_p)} = 0$.

Let $S(X) = \gcd(Q(X), R(X))$, then S is of slope between $k+1-h'$ and h since $S|Q, S|R$. We have $\Phi_{|S(U_p)} = 0$ since $\Phi_{|Q(U_p)} = \Phi_{|R(U_p)} = 0$ and there exist polynomials R_1, R_2 such that $S = R_1 Q + R_2 R$. Therefore, $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{k+1-h' \leq U_p \leq h}$. □

The following lemma will be very useful.

Lemma 3.3.19. *If $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ for $r \in |\mathbb{C}_p^\times|_p$ with $r \leq p^{-1}$, then $\Phi_{|V_p} \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[pr])$ and $\left\|\Phi_{|V_p}\right\|_{pr} \leq p^{-k}\left\|\Phi\right\|_r$.*

Proof. The modular symbol $\Phi_{|V_p}$ is defined by

$$\begin{aligned}\Phi_{|V_p}(D)(f) &= \sum_{a=0}^{p-1} \Phi_{\left|\begin{pmatrix} p & 0 \\ aNp & 1 \end{pmatrix}\right.}(D)(f) \\ &= \sum_{a=0}^{p-1} \Phi\left(D_a\right)\left(\mathbb{1}_{p\mathbb{Z}_p}(z) \cdot (-aNpz + p)^k f\Big(\frac{z}{p(-aNz+1)}\Big)\right),\end{aligned}$$

where $D \in \Delta_0, D_a = \begin{pmatrix} p & 0 \\ aNp & 1 \end{pmatrix} D, f \in \mathcal{A}(\mathbb{Z}_p)$ (we will determine the radius of convergence of f).

We set $g_a(z) = \mathbb{1}_{p\mathbb{Z}_p}(z) \cdot (-aNpz + p)^k f(\frac{z}{p(-aNz+1)})$.

If $z \in p\mathbb{Z}_p$, then $-aNz + 1 \in \mathbb{Z}_p^\times$, so $|-aNz+1|_p = 1$. The transformation $z \mapsto \frac{z}{p(-aNz+1)}$ maps a closed disc of radius r in $p\mathbb{Z}_p = D(0, p^{-1})$ to a closed disc of radius pr in $\mathbb{Z}_p$ since

$$\Big|\frac{z}{p(-aNz+1)} - \frac{e}{p(-aNe+1)}\Big|_p = \Big|\frac{z-e}{p(-aNz+1)(-aNe+1)}\Big|_p = p|z-e|_p \quad \text{for any } z, e \in p\mathbb{Z}_p.$$

It also maps a closed disc of radius r in $\{z \in \mathbb{C}_p, |z|_p \leq p^{-1}\}$ to a closed disc of radius pr in $\mathbb{C}_p$. Therefore, if $f \in \mathcal{A}(\mathbb{Z}_p)[pr]$, then $f(\frac{z}{p(-aNz+1)}) \in \mathcal{A}(\mathbb{Z}_p)[r]$ and $\|f(\frac{z}{p(-aNz+1)})\|_r = \|f\|_{pr}$ by (1.2). We deduce that

$g_a \in \mathcal{A}(\mathbb{Z}_p)[r]$ and $\|g_a\|_r = p^{-k}\|f\|_{pr}$ for any a if $f \in \mathcal{A}(\mathbb{Z}_p)[pr]$, since $|-aNz+1|_p = 1$ for any $z \in \mathbb{C}_p$ such that $|z|_p \leq p^{-1}$. So $\Phi_{|V_p} \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[pr])$ and

$$\begin{aligned}\left\|\Phi_{|V_p}\right\|_{pr} &= \sup_{D\in\Delta_0}\ \sup_{f\in\mathcal{A}(\mathbb{Z}_p)[pr]\backslash\{0\}} \frac{\left|\Phi_{|V_p}(D)(f)\right|_p}{\|f\|_{pr}} \\ &\leq \max_{0\leq a\leq p-1}\ \sup_{D\in\Delta_0}\ \sup_{f\in\mathcal{A}(\mathbb{Z}_p)[pr]\backslash\{0\}} \frac{|\Phi(D_a)(g_a)|_p}{p^k\|g_a\|_r} \\ &\leq p^{-k}\|\Phi\|_r.\end{aligned}$$

The lemma follows. ☐

Corollary 3.3.20. *If $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ is a V_p-eigensymbol of slope $\leq h$, then $\|\Phi\|_{r/p^n} \geq p^{n(k-h)}\|\Phi\|_r$ for any $r \in |\mathbb{C}_p^\times|_p$ with $r \leq 1$ and $n \in \mathbb{N}$.*

Proof. Let $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ be a V_p-eigensymbol of eigenvalue α such that $v_p(\alpha) \leq h$. For $r \leq 1$, by the above lemma, we have

$$\|\Phi\|_r = \|\alpha^{-n}\Phi_{|V_p^n}\|_r \leq |\alpha^{-n}|_p \cdot p^{-nk}\|\Phi\|_{r/p^n} \leq p^{n(h-k)}\|\Phi\|_{r/p^n}.$$

Therefore, $\|\Phi\|_{r/p^n} \geq p^{n(k-h)}\|\Phi\|_r$. ☐

Proposition 3.3.21. *There are decompositions:*

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)) = \ker(V_p) \oplus \mathrm{Im}(U_p),$$
$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r]) = \ker(V_p) \oplus \mathrm{Im}(U_p)$$

for each $r \in |\mathbb{C}_p^\times|_p$ with $r < p^{-1}$, where U_p and V_p are endomorphisms of $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ in the first decomposition and U_p (resp. V_p) is the map from $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ to $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ (resp. $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[pr])$) in the second decomposition. Moreover, the kernel of V_p in the second decomposition is infinite dimensional.

Proof. By Lemma 3.3.19, V_p maps $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ into $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[pr])$ for any $r < p^{-1}$.

The decomposition $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)) = \ker(V_p) \oplus \mathrm{Im}(U_p)$ is given by

$$\Phi = p^{-(k+1)}(\Phi_{|(p^{k+1}-V_pU_p)} + \Phi_{|V_pU_p}),$$

where $\Phi_{|(p^{k+1}-V_pU_p)} = p^{k+1}\Phi - (\Phi_{|V_p})_{|U_p} \in \ker(V_p)$ since $V_p \circ U_p = p^{k+1}\mathrm{Id}$ by Proposition 3.3.12, and $\Phi_{|V_pU_p} = (\Phi_{|V_p})_{|U_p} \in \mathrm{Im}(U_p)$. The sum is direct since if $\Phi = (\Phi_0)_{|U_p} \in \mathrm{Im}(U_p)$ and if $\Phi \in \ker(V_p)$, then

$$0 = \Phi_{|V_p} = (\Phi_0)_{|U_pV_p} = p^{k+1}\Phi_0,$$

so $\Phi_0 = 0$, hence $\Phi = 0$. The decomposition for $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ is proven similarly.

Suppose $\ker(V_p)$ is finite dimensional in $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$, then $\mathrm{Im}(U_p)$ has finite codimension in $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ by the decomposition. We infer that $\mathrm{Im}(U_p)$ is closed in the Banach space $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ by [Abr, Corollary 2.17]. So $\mathrm{Im}(U_p)$ is itself a Banach space. The map U_p from $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ to $\mathrm{Im}(U_p)$ is therefore open by the open mapping theorem. It is also compact by Proposition 3.2.2, so the image of the open unit disc is open and relatively compact in $\mathrm{Im}(U_p)$. By Riesz's theorem, we deduce that $\mathrm{Im}(U_p)$ is finite dimensional. Hence $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ is finite dimensional since U_p is injective by Corollary 3.3.13, then so is $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ since the natural map $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ is injective. This is a contradiction. Therefore, $\ker(V_p)$ is infinite dimensional, where V_p is seen as a map on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ for $r < p^{-1}$. ☐

Proposition 3.3.22. *For $k \in \mathbb{Z}\backslash\{0\}$, the set of nonzero eigenvalues of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))$ is the union of the set of eigenvalues of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))$ and the set of nonzero eigenvalues of V_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))$.*

If 0 is an eigenvalue of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))$, then 0 is an eigenvalue of V_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))$

Proof. We omit L from the notations for simplicity. Since U_p is injective on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ by Corollary 3.3.13, every eigenvalue of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ is nonzero.

Let α be an eigenvalue of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))$ associated with an eigensymbol Φ. Consider the map $\mathrm{res} : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1)) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ in (3.7). Since it is equivariant for the action of U_p on the left hand side and that of V_p on the right hand side, we have

$$\mathrm{res}(\Phi)_{|V_p} = \alpha \cdot \mathrm{res}(\Phi).$$

If $\mathrm{res}(\Phi) \neq 0$, then $\mathrm{res}(\Phi)$ is an eigensymbol of V_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ with eigenvalue α. Otherwise, $\Phi \in \ker(\mathrm{res}) = \mathrm{Im}(\mathrm{ext})$, so Φ is an eigensymbol of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ with eigenvalue α since ext is injective and U_p-equivariant. If $\alpha = 0$, then the former case follows since U_p is injective on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$.

Conversely, if α is an eigenvalue of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ associated with an eigensymbol Φ_0, then α is an eigenvalue of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))$ associated with the eigensymbol $\mathrm{ext}(\Phi_0)$.

Now let α be a nonzero eigenvalue of V_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$. Let $\Phi_1 \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ be a V_p-eigensymbol with eigenvalue α, then $\Phi_1 \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \leq v_p(\alpha)}$. Since $\mathrm{res} : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{\leq v_p(\alpha)} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \leq v_p(\alpha)}$ is surjective by (3.8), there is $\Phi_2 \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{\leq v_p(\alpha)}$ such that $\mathrm{res}(\Phi_2) = \Phi_1$. We have

$$\mathrm{res}(\Phi_{2|U_p - \alpha}) = \mathrm{res}(\Phi_2)_{|V_p - \alpha} = \Phi_{1|V_p - \alpha} = 0,$$

so $\Phi_{2|U_p-\alpha} \in \ker(\mathrm{res}) = \mathrm{Im}(\mathrm{ext})$. Let $\Phi_{2|U_p-\alpha} = \mathrm{ext}(\Phi_3)$ for $\Phi_3 \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\leq v_p(\alpha)}$ (since Φ_2 has slope $\leq v_p(\alpha)$). If α is an eigenvalue of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$, then α is an eigenvalue of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))$. If not, then $U_p - \alpha$ is an injective endomorphism of the finite dimensional space $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\leq v_p(\alpha)}$, so it is an isomorphism. Then there exists $\Phi_4 \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\leq v_p(\alpha)}$ such that $\Phi_3 = \Phi_{4|U_p-\alpha}$. We have $\Phi_{2|U_p-\alpha} = \mathrm{ext}(\Phi_4)_{|U_p-\alpha}$, hence $\Phi_2 - \mathrm{ext}(\Phi_4) \in \ker(U_p - \alpha)$. On the other hand, $\Phi_2 - \mathrm{ext}(\Phi_4) \neq 0$ since

$$\mathrm{res}(\Phi_2 - \mathrm{ext}(\Phi_4)) = \mathrm{res}(\Phi_2) = \Phi_1 \neq 0.$$

Therefore, α is an eigenvalue of U_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))$ associated to $\Phi_2 - \mathrm{ext}(\Phi_4)$. □

For any $h \in \mathbb{Q}$, the subspace of modular symbols of slope $\leq h$ (or $< h$) of $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ is finite dimensional and U_p-stable. We find finite dimensional U_p-stable subspaces of $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))$.

Recall the map

$$\begin{aligned} \mathrm{res} : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1)) &\to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty,1))) \\ \Phi &\mapsto \Phi_{|D(\infty,1)} : D \mapsto \Phi(D)_{|D(\infty,1)} \quad (D \in \Delta_0) \end{aligned}$$

induced by the restriction map on $D(\infty,1)$ between distributions. The matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ acts on $\mathcal{D}_k(D(\infty,1))$ on the right by

$$\mu_{\big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}(f) = \mu\left(f_{\big|_k \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}}\right) = \mu\left(p^k f\left(\frac{z}{p}\right)\right),$$

where $\mu \in \mathcal{D}_k(D(\infty,1))$, $f \in \mathcal{A}_k(D(\infty,1))$. This action is compatible with the action of $\Gamma_0(p)$, so we can define the double coset operator $V_p = \left[\Gamma_0 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0\right]$ acting on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty,1)))$ by

$$\Phi_{|V_p} = \sum_{a=0}^{p-1} \Phi_{|\tau_{Np}\gamma_a\tau_{Np}^{-1}} = \sum_{a=0}^{p-1} \Phi_{|\tau_{Np}^{-1}\gamma_a\tau_{Np}} \qquad \text{(see (3.6))},$$

note that $\tau_{Np}^{-1} = \frac{1}{Np}\tau_{Np}$.

For $h, h' \in \mathbb{Q}$, denote by $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{(V_p \leq h')}$ the subspace of $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))$ consisting of modular symbols Φ such that $\Phi_{|D(\infty,1)} \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty,1)))^{V_p \leq h'}$, and put

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{U_p \leq h, (V_p \leq h')} = \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{U_p \leq h} \cap \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{(V_p \leq h')}.$$

We define similarly if $\leq$ is replaced by $<$.

Theorem 3.3.23. *For $k \in \mathbb{Z}$ and $h, h' \in \mathbb{Q}$, the subspace $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{U_p \le h, (V_p \le h')}$ of $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))$ is finite dimensional and U_p-stable, and similar if $\le$ is replaced by $<$.*

Moreover, if $k \in \mathbb{N}^*$ and $0 \le h \le k+1$, there is an exact sequence:

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{\le h} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))^{U_p \le h, (V_p \le h')} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{k+1-h \le U_p \le h'} \to 0,$$

while if $k = 0$ the last space 0 is replaced by L.

In particular, there is an exact sequence:

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{\le k+1} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1, L))^{U_p \le k+1, (V_p \le k+1)} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{\le k+1} \to 0$$

for any $k \in \mathbb{N}^*$ and if $k = 0$ the last space 0 is replaced by L.

Proof. We drop L from the notations for simplicity. We prove for the case $k \neq 0$, the case $k = 0$ is proven similarly. Recall the exact sequence (3.8):

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\le h} \overset{\mathrm{ext}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{\le h} \overset{\mathrm{res}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \le h} \to 0,$$

it induces an exact sequence

$$\begin{aligned} 0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\le h} &\overset{\mathrm{ext}}{\to} \mathrm{res}^{-1}(\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \le h, U_p \le h'}) \\ &\overset{\mathrm{res}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \le h, U_p \le h'} \to 0, \end{aligned} \tag{3.9}$$

note that $\mathrm{ext}(\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\le h}) = \ker(\mathrm{res}) \subset \mathrm{res}^{-1}(\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \le h, U_p \le h'})$ and

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \le h, U_p \le h'} = \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{k+1-h \le U_p \le h'}$$

by Corollary 3.3.18. Since the spaces $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{\le h}$ and $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{k+1-h \le U_p \le h'}$ are finite dimensional, so is the space $\mathrm{res}^{-1}(\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \le h, U_p \le h'})$.

The subspace $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{k+1-h \le U_p \le h'}$ is V_p-stable in $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ since V_p is an automorphism on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<\infty}$ given by $V_p = p^{k+1} U_p^{-1}$ by Corollary 3.3.17. Since the map

$$\mathrm{res} : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1)) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$$

is equivariant for the action of U_p on the left hand side and that of V_p on the right hand side by Corollary 3.3.11, we deduce that $\mathrm{res}^{-1}(\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \le h, U_p \le h'})$ is U_p-stable in $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))$.

The map $\mathrm{res} : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1)) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ is the composition of

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1)) \overset{\mathrm{res}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty, 1))) \overset{\simeq}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}'_k(\mathbb{Z}_p)) \overset{\simeq}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)),$$

where the first map is induced by the restriction map on $D(\infty, 1)$ between distributions, the second is induced from the isomorphism $\hat{\iota}$ in (3.3) between distributions, and the last is given in Lemma 3.3.10. Denote by $\theta : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty, 1))) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ the composition of the last two isomorphisms. By construction, the isomorphism θ is given by

$$\theta(\Phi)(D)(f) = \Phi(\tau_{Np}^{-1} D)\Big(z^k f\Big(\frac{1}{Npz}\Big)\Big)$$

for $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty, 1))), D \in \Delta_0, f \in \mathcal{A}_k(\mathbb{Z}_p), z \in D(\infty, 1)$. For any such Φ, D, f, we have

$$\begin{aligned} \theta(\Phi_{|V_p})(D)(f) &= \Phi_{|V_p}(\tau_{Np}^{-1} D)\Big(z^k f\Big(\frac{1}{Npz}\Big)\Big) \\ &= \sum_{a=0}^{p-1} \Phi(\tau_{Np}^{-1}\gamma_a \tau_{Np}\tau_{Np}^{-1} D)_{|_k \tau_{Np}^{-1}\gamma_a \tau_{Np}}\Big(z^k f\Big(\frac{1}{Npz}\Big)\Big) \\ &= \sum_{a=0}^{p-1} \Phi(\tau_{Np}^{-1}\gamma_a D)\Big(z^k f\Big(\frac{1}{Nz} - a\Big)\Big) \\ &= \sum_{a=0}^{p-1} \theta(\Phi)(\gamma_a D)(f(pz - a)) = \theta(\Phi)_{|U_p}(D)(f). \end{aligned}$$

Therefore, $\theta(\Phi_{|V_p}) = \theta(\Phi)_{|U_p}$ for any $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(D(\infty,1)))$. We deduce that a modular symbol Φ in $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))$ belongs to $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{(V_p \le h')}$ if and only if $\mathrm{res}(\Phi) \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ has U_p-slope $\le h'$. Therefore, the inverse image of $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{V_p \le h, U_p \le h'}$ under res in $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{\le h}$ is $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{U_p \le h, (V_p \le h')}$. We conclude that $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{U_p \le h, (V_p \le h')}$ is finite dimensional and U_p-stable in $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))$.

The desired exact sequence is exactly (3.9), where we need the condition $0 \le h \le k+1$ since $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<0} = 0$, note that U_p has norm ≤ 1 on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)[r])$ for any $r \in |\mathbb{C}_p^\times|_p$ by [Bel, Lemma V.5.4]. Taking $h = h' = k+1$ yields the special case. $\square$

The problem of finding subspaces of $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))$ which are U_p-stable and finite dimensional leads to the following question:

Question 3.3.24. *Is the subspace* $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{h_1 \le U_p \le h_2}$ *of* $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))$ *finite dimensional for* $h_1 \le h_2$*? Note that by Corollary 3.3.15 and* (3.7)*, if we only impose one bound (upper or lower) for the* U_p*-slope, then we would not have a finite dimensional subspace.*

Proposition 3.3.25. *If* $h, h' \in \mathbb{Q}$ *such that* $h + h' \le k$*, then* $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)_{\le h})^{V_p < h'} = 0$*. In particular,* $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)_{\le k/2})^{V_p < k/2} = 0$*. Recall that* $\mathcal{D}_k(\mathbb{Z}_p)_{\le h}$ *is the space of* h*-admissible distributions on* $\mathbb{Z}_p$*.*

Proof. By Lemma 3.3.19, the action of V_p on $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ stabilizes the subspace $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)_{\le h})$ since $\left\|\Phi_{|V_p}\right\|_r \le p^{-k} \left\|\Phi\right\|_{r/p} = \mathrm{O}((r/p)^{-h}) = \mathrm{O}(r^{-h})$ as $r \to 0^+$ if $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)_{\le h})$.

It suffices to show that if $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)_{\le h})$ is a V_p-eigensymbol of slope $< h'$, then $\Phi = 0$. Let α be the eigenvalue of Φ. By Corollary 3.3.20, for any $n \in \mathbb{N}$, we have

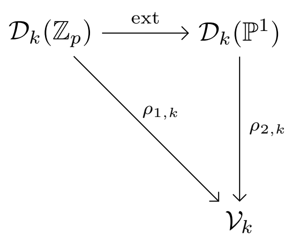
$$\|\Phi\|_{1/p^n} \ge p^{n(k - v_p(\alpha))} \|\Phi\|_1 .$$

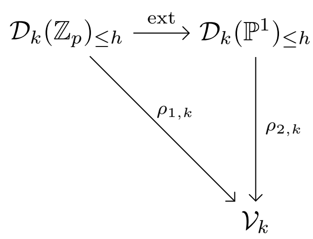
On the other hand, since $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)_{\le h})$, we have

$$\|\Phi\|_{1/p^n} = \mathrm{O}(p^{nh}) \text{ as } n \to +\infty.$$

Since $k - v_p(\alpha) > k - h' \ge h$, we deduce that $\|\Phi\|_1 = 0$, so $\Phi = 0$. $\square$

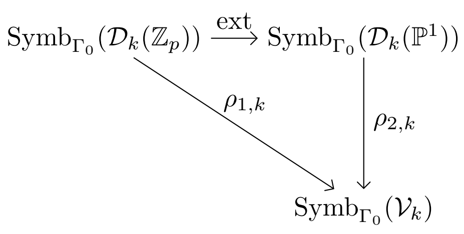
For $k \in \mathbb{N}$, we have the following commutative diagrams which are compatible with the actions of $\Sigma_0(p)$:





where in the first diagram, the map ext is given in (3.1), the maps $\rho_{1,k}, \rho_{2,k}$ are the duals of the inclusions $\mathcal{P}_k \to \mathcal{A}_k(\mathbb{Z}_p)$ and $\mathcal{P}_k \to \mathcal{A}_k(\mathbb{P}^1)$, respectively; while the second diagram is induced from the first by restricting on the subspace of h-admissible distributions.

Taking modular symbols, we get the following commutative diagram which is U_p-equivariant:



Take modular symbols of slope $< h$, we get the U_p-equivariant commutative diagram

$$\begin{array}{ccccc}
\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<h} & \xrightarrow{\mathrm{ext}} & \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1)_{\leq h})^{<h} & \hookrightarrow & \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{<h} \\
 & \searrow^{\rho_{1,k}} & \downarrow{\scriptstyle \rho_{2,k}} & \swarrow_{\rho_{2,k}} & \\
 & & \mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k)^{<h} & &
\end{array}$$

(by [PS11, Lemma 6.2], modular symbols in $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<h}$ take values in the space of h-admissible distributions, so the map ext takes the space $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<h}$ into $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1)_{\leq h})^{<h}$).

Proposition 3.3.26. *The specialization maps*

$$\rho_{2,k} : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1)) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k),$$
$$\rho_{2,k} : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1))^{<h} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k)^{<h}$$

are surjective for all h.

Proof. The map $\rho_{1,k} : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)) \to \mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k)$ is surjective by [PS11, Theorem 5.1].

The map $\rho_{1,k} : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<h} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k)^{<h}$ is surjective by [Bel, Proposition V.5.17].

Since $\rho_{1,k} = \rho_{2,k} \circ \mathrm{ext}$, $\rho_{2,k}$ is also surjective. □

Proposition 3.3.27. *The specialization map $\rho_{2,k} : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1)_{\leq h})^{<h} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k)^{<h}$ is an isomorphism for any $k \in \mathbb{N}$ and $h \leq \frac{k}{2}$.*

Proof. Applying Proposition 3.3.25 and the left exact sequence

$$0 \to \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<h} \overset{\mathrm{ext}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1)_{\leq h})^{<h} \overset{\mathrm{res}}{\to} \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)_{\leq h})^{V_p<h}$$

similar to (3.8), we get the isomorphism

$$\mathrm{ext} : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<h} \cong \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{P}^1)_{\leq h})^{<h}.$$

On the other hand, the specialization map $\rho_{1,k} : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{<h} \to \mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k)^{<h}$ is isomorphic by Stevens' control theorem, since $h \leq \frac{k}{2} < k+1$. We get the desired result since $\rho_{1,k} = \rho_{2,k} \circ \mathrm{ext}$. □

Chapter 4

Functional equation of p-adic L-functions attached to modular forms

In this chapter, we follow Stevens' construction of p-adic L-functions attached to cuspidal normalized eigenforms of non-critical slope via his theory of overconvergent modular symbols with values in $\mathcal{D}_k(\mathbb{Z}_p)$ studied in Chapter 3. These p-adic L-functions satisfy an interpolation property related to special values of L-functions of modular forms. From this interpolation formula and the functional equation of L-functions of modular forms given in [Shi, Theorem 3.66], we deduce a functional equation of these p-adic L-functions (see Proposition 4.2.4). In Section 4.1 we recall Stevens' construction of p-adic L-functions attached to modular forms. In Section 4.2 we prove a functional equation of these functions. The results in this chapter are well-known and proved by an independent work of the author, so the readers can skip it and go to the next chapter if they wish.

We fix a natural number k and a positive integer N prime to p in this chapter. Denote by $\mathbb{H}$ the Poincaré upper half plane.

4.1 Stevens' construction of p-adic L-functions

4.1.1 Some preparatory results

In this subsection, let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup of level N such that $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ normalizes Γ (e.g. $\Gamma = \Gamma_1(N)$). Let $\mathcal{S}_{k+2}(\Gamma)$ denote the space of cuspidal modular forms of weight $k+2$ and level Γ. Denote by $\mathrm{GL}_2^+(\mathbb{Q})$ the subgroup of matrices of positive determinant in $\mathrm{GL}_2(\mathbb{Q})$.

Recall the right weight m action of $\mathrm{GL}_2^+(\mathbb{Q})$ on a modular form f for each $m \in \mathbb{Z}$:

$$f_{|_m \gamma}(z) = (\det\gamma)^{m-1}(cz+d)^{-m} f\Big(\frac{az+b}{cz+d}\Big),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$ and $z \in \mathbb{H}$ (compare with the action $|_k$ of matrices on p-adic valued functions on $\mathbb{Z}_p$ or $\mathbb{P}^1(\mathbb{Q}_p)$ in (1.5),(2.4)).

Modular forms and classical modular symbols are related via Eichler-Shimura map.

Lemma 4.1.1. *There is a canonical map*

$$\begin{aligned} &\mathcal{S}_{k+2}(\Gamma) \to \mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C})) \\ &f \mapsto \phi_f : D \in \Delta_0 \mapsto \Big(P \in \mathcal{P}_k(\mathbb{C}) \mapsto \int_D f(z)P(z)dz\Big) \end{aligned}$$

which is compatible with all Hecke operators $[\Gamma s \Gamma]$ for $s \in \mathrm{GL}_2^+(\mathbb{Q})$, where the integral $\int_D f(z)P(z)dz$ is defined by:

If $D = \sum_{i=1}^n (\{b_i\} - \{a_i\})$ for $a_i, b_i \in \mathbb{P}^1(\mathbb{Q})$, then we define $\int_D f(z)P(z)dz = \sum_{i=1}^n \int_{a_i}^{b_i} f(z)P(z)dz$. Here $\int_{a_i}^{b_i}$ is the integral along the geodesic from a_i to b_i inside $\bar{\mathbb{H}} := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$.

Proof. We follow the computations in the proof of [Bel, Lemma IV.3.1]. □

In the view of the decomposition $\mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C})) = \mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))^+ \oplus \mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))^-$, if $f \in \mathcal{S}_{k+2}(\Gamma)$, then the modular symbol ϕ_f in the above lemma is decomposed by $\phi_f^+ + \phi_f^-$, where $\phi_f^\pm \in \mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))^\pm$ is given by

$$\phi_f^\pm(D)(P) = \frac{1}{2}\Big(\int_D f(z)P(z)dz \pm \int_{\bar{D}} f(z)P(-z)dz\Big)$$

for $D \in \Delta_0$ and $P \in \mathcal{P}_k(\mathbb{C})$, where $\bar{D} = \sum_i(\{-b_i\} - \{-a_i\})$ if $D = \sum_i(\{b_i\} - \{a_i\})$.

Lemma 4.1.2. *The map*

$$\mathcal{S}_{k+2}(\Gamma) \to \mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$$
$$f \mapsto \phi_f^\pm$$

is compatible with the Hecke operators T_l and U_l.

Proof. Recall that $\phi_f^\pm = \frac{1}{2}(\phi_f \pm \phi_{f|T_\infty})$, where T_∞ acts via the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The assertion is clear since the map $f \mapsto \phi_f$ is compatible with any double coset operator by Lemma 4.1.1 and the fact that T_∞ commutes with T_l and U_l for any l. □

Now let $f \in \mathcal{S}_{k+2}(\Gamma)$ be a normalized eigenform. Let $\mathcal{H}$ denote the polynomial ring over $\mathbb{Z}$ in the variables indexed by the Hecke operators T_l for $l \nmid N$ and U_l for $l|N$. Denote by $\lambda : \mathcal{H} \to \mathbb{C}$ the ring homomorphism associated to the system of Hecke eigenvalues of f, i.e., $\lambda(T)$ is the eigenvalue of T on f for any Hecke operator T generating $\mathcal{H}$.

We let $\mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))^\pm[\lambda]$ be the subspace of $\mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))^\pm$ consisting of simultaneous Hecke eigensymbols of eigenvalues given by λ.

Lemma 4.1.3. *The dimension of* $\mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))^\pm[\lambda]$ *is 1.*

Proof. See [Bel, Lemma IV.4.7]. □

This lemma is also true if we restrict coefficients to the number field K_f generated by Fourier coefficients of f.

Lemma 4.1.4. *The dimension over K_f of* $\mathrm{Symb}_\Gamma(\mathcal{V}_k(K_f))^\pm[\lambda]$ *is 1.*

Proof. See [Bel, Lemma IV.4.8]. □

By Lemmas 4.1.2 and 4.1.3, the symbol $\phi_f^\pm$ is a basis of $\mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))^\pm[\lambda]$. By Lemma 4.1.4, there exists a complex number $\Omega_f^\pm \in \mathbb{C}^\times$ such that

$$\phi_f^\pm/\Omega_f^\pm \in \mathrm{Symb}_\Gamma(\mathcal{V}_k(K_f))^\pm[\lambda].$$

We refer to $\Omega_f^\pm$ as the periods of f. They are determined up to multiplication by elements of $K_f^\times$.

If f has the Fourier expansion $\sum_{n=1}^\infty a_n q^n$ with $q = e^{2\pi i z}$ for $z \in \mathbb{H}$, we define its L-function by

$$L(f,s) = \sum_{n=1}^\infty \frac{a_n}{n^s} \qquad \text{for } s \in \mathbb{C}.$$

This function is convergent if $\mathrm{Re}(s)$ is big enough, and it can be extended to an entire function on $\mathbb{C}$. If $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ is a primitive Dirichlet character, we define the twisted L-function

$$L(f,\chi,s) = \sum_{n=1}^\infty \frac{\chi(n)a_n}{n^s}.$$

Then $L(f,\chi,s)$ is the L-function of the modular form f_χ given by

$$f_\chi(z) = \frac{1}{\tau(\chi^{-1})} \sum_{a (\mathrm{mod}\, m)} \chi^{-1}(a) f\Big(z + \frac{a}{m}\Big),$$

where $\tau(\chi^{-1})$ is the Gauss sum of χ^{-1}.

4.1.2 Construction of p-adic L-functions

A p-adic L-function is a p-adic distribution on a p-adic space. The p-adic L-functions attached to modular forms will be distributions on $\mathbb{Z}_p^\times$.

In this subsection, let $\Gamma_0 = \Gamma_1(N) \cap \Gamma_0(p)$. Denote by $\mathcal{S}_{k+2}(N, \epsilon)$ the space of cuspidal modular forms of weight $k+2$, level N and nebentypus ϵ, for a Dirichlet character ϵ of $(\mathbb{Z}/N\mathbb{Z})^\times$.

We start with a normalized eigenform $f \in \mathcal{S}_{k+2}(N, \epsilon)$. Let a_p be the p-th Fourier coefficient of f, which is also the T_p-eigenvalue of f.

Let $\alpha, \beta \in \mathbb{C}$ be two complex roots of the polynomial

$$X^2 - a_p X + \epsilon(p) p^{k+1},$$

and defining the p-stabilizations (or p-refinements)

$$f_\alpha(z) = f(z) - \beta f(pz),$$
$$f_\beta(z) = f(z) - \alpha f(pz)$$

of f. The functions f_α and f_β belong to the space $\mathcal{S}_{k+2}(Np, \epsilon)$, where the character ϵ of $(\mathbb{Z}/Np\mathbb{Z})^\times$ is the lift of the character ϵ of $(\mathbb{Z}/N\mathbb{Z})^\times$. They are also eigenforms of the same Hecke eigenvalues as f for all operators T_l, U_l for $l \neq p$. For the operator U_p, we have

Lemma 4.1.5. *The forms f_α and f_β are eigenforms for U_p of eigenvalues α and β, respectively.*

Proof. See [Bel, Lemma V.7.2]. □

We see α and β as elements of $\mathbb{C}_p$ via the embedding $\iota_p : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. Since α, β are algebraic integers (since a_p is), they belong to the ring of integers $\mathcal{O}_{\mathbb{C}_p}$ of $\mathbb{C}_p$. Therefore, the p-adic valuations of α and β satisfy

$$v_p(\alpha) \geq 0, v_p(\beta) \geq 0,$$
$$v_p(\alpha) + v_p(\beta) = k+1.$$

We infer that $0 \leq v_p(\alpha), v_p(\beta) \leq k+1$. If $v_p(\alpha) < k+1$, the form f_α is said to be of non-critical slope. Otherwise, we say that it has critical slope. It is always true that at least one of the forms f_α, f_β is of non-critical slope. We choose one, say f_α.

Stevens' construction of the p-adic L-function attached to f_α is as follows:

Step 1: We attach to f_α the symbols $\phi^\pm_{f_\alpha}/\Omega^\pm_{f_\alpha} \in \mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k(K_{f,\epsilon}))^\pm$ which have the same U_p-eigenvalue α as f_α by Lemmas 4.1.2 and 4.1.5, where $K_{f,\epsilon}$ is the number field generated by Fourier coefficients of f and the values of ϵ. Let L be a finite extension of $\mathbb{Q}_p$ containing the image of $K_{f,\epsilon}$ under the embedding ι_p. We see $\phi^\pm_{f_\alpha}/\Omega^\pm_{f_\alpha}$ as an element of $\mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k(L))^\pm$. Since $v_p(\alpha) < k+1$, we have

$$\phi^\pm_{f_\alpha}/\Omega^\pm_{f_\alpha} \in \mathrm{Symb}_{\Gamma_0}(\mathcal{V}_k(L))^{<k+1}.$$

Step 2: By Stevens' control theorem (Theorem 3.2.1), there is a unique overconvergent modular symbol $\Phi^\pm_{f_\alpha} \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p, L))^{<k+1}$ whose specialization is the symbol $\phi^\pm_{f_\alpha}/\Omega^\pm_{f_\alpha}$. Let $\Phi_{f_\alpha} = \Phi^+_{f_\alpha} + \Phi^-_{f_\alpha}$. We define the p-adic L-function of f_α by

$$L_p(f_\alpha, \cdot) = \Phi_{f_\alpha}(\{\infty\} - \{0\})_{|\mathbb{Z}_p^\times} \in \mathcal{D}(\mathbb{Z}_p^\times, L).$$

Then $L_p(f_\alpha)$ is a function on the weight space $\mathcal{W}(L)$. If $\chi : \mathbb{Z}_p^\times \to L^\times$ is a continuous character, we can see that $\Phi^\pm_{f_\alpha}(\{\infty\} - \{0\})_{|\mathbb{Z}_p^\times}(\chi) = 0$ if $\chi(-1) = \mp 1$. Therefore,

$$L_p(f_\alpha, \chi) = \Phi^\pm_{f_\alpha}(\{\infty\} - \{0\})_{|\mathbb{Z}_p^\times}(\chi)$$

for $\pm = \chi(-1)$.

Since $\Phi_{f_\alpha} \in \mathrm{Symb}_\Gamma(\mathcal{D}_k(\mathbb{Z}_p, L))^{<k+1}$, by [PS11, Lemma 6.2] we deduce that $L_p(f_\alpha)$ has growth $< k+1$. The p-adic L-function $L_p(f_\alpha, \cdot)$ satisfies the following interpolation property:

Theorem 4.1.6. *Let $f \in \mathcal{S}_{k+2}(N, \epsilon)$ be a normalized eigenform. Let α be one of two roots of the polynomial $X^2 - a_p X + \epsilon(p) p^{k+1}$ such that $v_p(\alpha) < k+1$, where a_p is the p-th Fourier coefficient of f.*

Then for any finite order character $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ of conductor p^n and any integer j such that $0 \le j \le k$, we have

$$L_p(f_\alpha, \chi z^j) = e_p(f,\alpha,\chi,j)\frac{p^{n(j+1)}j!}{\alpha^n(-2\pi i)^{j+1}\tau\left(\chi^{-1}\right)\Omega_{f_\alpha}^{\chi(-1)(-1)^j}}L\left(f,\chi^{-1},j+1\right), \tag{4.1}$$

where $e_p(f,\alpha,\chi,j) = 1$ if χ is non trivial and $e_p(f,\alpha,\mathbb{1},j) = (1-\alpha^{-1}\epsilon(p)p^{k-j})(1-\alpha^{-1}p^j)$.

Proof. See [Bel, Corollary V.7.10]. □

4.2 Functional equation of p-adic L-functions attached to modular forms

4.2.1 The operator W_N on modular forms

Let $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. The matrix W_N normalizes the group $\Gamma_1(N)$ since

$$W_N \begin{pmatrix} a & b \\ c & d \end{pmatrix} W_N^{-1} = \begin{pmatrix} d & -c/N \\ -Nb & a \end{pmatrix}.$$

Hence the double coset operator $[\Gamma_1(N)W_N\Gamma_1(N)]$ on modular forms of level $\Gamma_1(N)$ is given by the action of W_N. The square of this action on modular forms of weight $k+2$ is the multiplication by $(-N)^k$.

Lemma 4.2.1. *If T is a Hecke operator T_l or a diamond operator on cusp forms of level $\Gamma_1(N)$, then the adjoint T^* of T equals $W_N T W_N^{-1}$. Here the space of cusp forms has the structure of an inner product space by Petersson scalar product.*

Proof. If $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ is diagonal, since W_N normalizes $\Gamma_1(N)$, we have

$$W_N \cdot \Gamma_1(N)\gamma\Gamma_1(N)\cdot W_N^{-1} = \Gamma_1(N)\cdot W_N\gamma W_N^{-1}\cdot\Gamma_1(N) = \Gamma_1(N)\gamma^*\Gamma_1(N),$$

where $\gamma^* = \det(\gamma)\cdot\gamma^{-1}$. By [DS, Proposition 5.5.2(b)] we infer that the operator $W_N[\Gamma_1(N)\gamma\Gamma_1(N)]W_N^{-1}$ on cusp forms of level $\Gamma_1(N)$ is adjoint of $[\Gamma_1(N)\gamma\Gamma_1(N)]$ if γ is diagonal. In particular, adjoint of the Hecke operator T_l is $W_N T_l W_N^{-1}$ for any prime l. By [DS, Proposition 5.5.2(a)] we deduce that adjoint of the diamond operator $\langle a\rangle$ is $W_N\langle a\rangle W_N^{-1}$ for any $a \in (\mathbb{Z}/N\mathbb{Z})^\times$. □

If f is a complex function on $\mathbb{H}$, define the function $f_\rho : \mathbb{H} \to \mathbb{C}$ by $f_\rho(z) = \overline{f(-\bar{z})}$.

Lemma 4.2.2. *If f is a modular form, then f_ρ is also a modular form of the same weight and level as f. If f has nebentypus ϵ, then the nebentypus of f_ρ is ϵ^{-1}. The form f_ρ shares the same properties to be cuspidal, normalized, eigenform, old form, newform as f. If f has Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n q^n$, then $f_\rho(z) = \sum_{n=0}^{\infty} \bar{a}_n q^n$.*

Proof. It follows from the identity

$$(f_\rho)_{\big|_m \begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \left(f_{\big|_m \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}}\right)_\rho \tag{4.2}$$

for any $m \in \mathbb{Z}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$. The Fourier expansion of f_ρ is clear. □

Proposition 4.2.3 ([Bel, Proposition IV.3.24])**.** *If f is a new form in $\mathcal{S}_{k+2}(\Gamma_1(N))$, then*

$$f_{|_{k+2}W_N} = W(f) f_\rho,$$

where $W(f) \in K_f^\times$ satisfies $|W(f)| = N^{k/2}$.

If f has moreover trivial nebentypus, then $f_{|_{k+2}W_N} = W(f)f$ and $W(f) = \pm N^{k/2}$.

Proof. We follow the proof in loc. cit. Let ϵ be the nebentypus of f. We write $f_{|W_N}$ for $f_{|_{k+2}W_N}$.

By Lemma 4.2.2, f_ρ is a newform in $\mathcal{S}_{k+2}(N,\epsilon^{-1})$.

By [DS, Theorem 5.5.3], on the inner product space $\mathcal{S}_{k+2}(\Gamma_1(N))$, the diamond operators $\langle l\rangle$ and the Hecke operators T_l for prime $l\nmid N$ have adjoints

$$\langle l\rangle^* = \langle l\rangle^{-1}, T_l^* = \langle l\rangle^{-1}T_l.$$

So, for $l\nmid N$, f is an eigenform of $\langle l\rangle^*$ with eigenvalue $\epsilon^{-1}(l)$ and of T_l^* with eigenvalue $\epsilon^{-1}(l)a_l$, where a_l is the l-th Fourier coefficient of f. From the identity of Petersson scalar products $(f_{|T_l^*},f)=(f,f_{|T_l})$, we have $\epsilon^{-1}(l)a_l=\bar{a}_l$. By Lemma 4.2.1, for $T=T_l$ or $T=\langle l\rangle$, it follows that f is an eigenform of $W_NTW_N^{-1}$ with eigenvalue λ_T given by $\lambda_T=\epsilon^{-1}(l)$ (resp. $\bar{a}_l$) if $T=\langle l\rangle$ (resp. T_l), for $l\nmid N$. Therefore, $f_{|W_N}$ is an engenform of any Hecke operator T_l and diamond operator with the same eigenvalues as f_ρ by Lemma 4.2.2.

We show that $f_{|W_N}$ is a newform. Let $g\in\mathcal{S}_{k+2}(\Gamma_1(N))$ is an old form. By [DS, Proposition 5.5.2(a)], we have

$$(f_{|W_N},g)=(f,g_{|(-W_N)})=(f,(-1)^kg_{|W_N})=(-1)^k(f,g_{|W_N}).$$

If $g\in\mathcal{S}_{k+2}(\Gamma_1(M))$ for a proper divisor M of N, then $g_{|W_M}\in\mathcal{S}_{k+2}(\Gamma_1(M))$, so $g_{|W_N}=\left(\frac{N}{M}\right)^{k+1}(g_{|W_M})\left(\frac{N}{M}\cdot\right)$ is an old form. If $g(z)=h(dz)$ for a modular form $h\in\mathcal{S}_{k+2}(\Gamma_1(N/d))$ and a divisor $d>1$ of N, then $g_{|W_N}=d^{-1}h_{|W_{N/d}}\in\mathcal{S}_{k+2}(\Gamma_1(N/d))$, so $g_{|W_N}$ is an old form. So $g_{|W_N}$ is an old form in any case, hence $(f,g_{|W_N})=0$ since f is a new form. Therefore, $(f_{|W_N},g)=0$ for any old form g, so $f_{|W_N}$ is a new form.

Since $f_{|W_N}$ and f_ρ are new forms in $\mathcal{S}_{k+2}(\Gamma_1(N))$ with the same eigenvalues for the Hecke operators T_l and diamond operators $\langle l\rangle$ with $l\nmid N$, by [DS, Theorem 5.8.2] we deduce that $f_{|W_N}=W(f)f_\rho$ for some constant $W(f)\in\mathbb{C}$. We have

$$\begin{aligned}(-N)^kf=f_{|W_N^2}&=(f_{|W_N})_{|W_N}=(W(f)f_\rho)_{|W_N}=W(f)\cdot(f_{|(-W_N)})_\rho \qquad \text{(by (4.2))}\\ &=W(f)\cdot(-1)^k(f_{|W_N})_\rho=W(f)\cdot(-1)^k(W(f)f_\rho)_\rho=|W(f)|^2(-1)^kf.\end{aligned}$$

Therefore, $|W(f)|=N^{k/2}$.

Finally, if f has trivial nebentypus, since $a_l\epsilon(l)^{-1}=\bar{a}_l$ for any prime $l\nmid N$ as above (this is still true if f is only an eigenform rather than a newform), we have $a_l=\bar{a}_l$ for any $l\nmid N$. By [DS, Theorem 5.8.2] we deduce that $f_\rho=f$ since they are newforms in $\mathcal{S}_{k+2}(\Gamma_1(N))$ with the same eigenvalues for T_l with $l\nmid N$ and the same (trivial) nebentypus. Hence $f_{|W_N}=W(f)f$. Since W_N^2 acts on $\mathcal{S}_{k+2}(\Gamma_1(N))$ by the multiplication by $(-N)^k=N^k$ (k is even since the nebentypus is trivial), we have $W(f)^2=N^k$, so $W(f)=\pm N^{k/2}$. □

4.2.2 Functional equation of p-adic L-functions

In this subsection, we prove a formula of functional equation relating the values of p-adic L-functions attached to a p-stabilization of f and $f_{|W_N}$, where $f\in\mathcal{S}_{k+2}(N,\epsilon)$ is a normalized eigenform. The idea is to use the interpolation formula (4.1) relating the values of p-adic and complex L-functions, and apply the functional equation of L-functions of modular forms given in [Shi, Theorem 3.66].

Let a_p be the p-th Fourier coefficient of f. As before, we choose a root α of the polynomial $X^2-a_pX+\epsilon(p)p^{k+1}$ such that $v_p(\alpha)<k+1$. Then we have defined the p-adic L-function $L_p(f_\alpha,\cdot)\in\mathcal{D}(\mathbb{Z}_p^\times)$.

By the proof of Proposition 4.2.3, $f_{|W_N}$ is a normalized eigenform in $\mathcal{S}_{k+2}(N,\epsilon^{-1})$ for the Hecke operators T_l and $\langle l\rangle$ for all l prime to N, and its p-th Fourier coefficient is $\bar{a}_p$ satisfying $\epsilon^{-1}(p)a_p=\bar{a}_p$. Consider the polynomial

$$X^2-\bar{a}_pX+\epsilon^{-1}(p)p^{k+1}=X^2-\epsilon^{-1}(p)a_pX+\epsilon^{-1}(p)p^{k+1}, \tag{4.3}$$

its set of roots is $\{\bar{\alpha},\bar{\beta}\}=\{\epsilon^{-1}(p)\alpha,\epsilon^{-1}(p)\beta\}$, where α,β are two roots of $X^2-a_pX+\epsilon(p)p^{k+1}$.

Choose a root u of (4.3) such that $v_p(u)<k+1$ (e.g. $u=\epsilon^{-1}(p)\alpha$), then we can attach the p-adic L-function $L_p((f_{|W_N})_u,\cdot)$ to the p-stabilization $(f_{|W_N})_u$ of $f_{|W_N}$ and this p-adic L-function has order of growth $<k+1$. The functional equation of p-adic L-functions states

Proposition 4.2.4. *Let $f\in\mathcal{S}_{k+2}(N,\epsilon)$ be a normalized eigenform. By choosing appropriate periods, for any finite order character $\chi:\mathbb{Z}_p^\times\to L^\times$ of conductor p^n and any integer $0\le j\le k$, where L is a sufficiently large finite extension of $\mathbb{Q}_p$, one has*

$$L_p(f_\alpha,\chi z^j)=-e_p(\alpha,u,\chi,j)N^{-k}\Big(\frac{\epsilon(p)u}{\alpha}\Big)^nL_p\left((f_{|W_N})_u,(Nz)^k\cdot(\chi z^j)\Big(-\frac{1}{Nz}\Big)\right),$$

where $e_p(\alpha,u,\chi,j)=1$ *if* χ *is non trivial and* $e_p(\alpha,u,\mathbb{1},j)=\frac{(1-\alpha^{-1}\epsilon(p)p^{k-j})(1-\alpha^{-1}p^j)}{(1-u^{-1}\epsilon^{-1}(p)p^j)(1-u^{-1}p^{k-j})}$.

In particular, if $u=\epsilon^{-1}(p)\alpha$, *then*

$$L_p(f_\alpha,\chi z^j)=-N^{-k}L_p\left((f_{|W_N})_{\epsilon^{-1}(p)\alpha},(Nz)^k\cdot(\chi z^j)\Big(-\frac{1}{Nz}\Big)\right).$$

Proof. Let

$$R(f,\chi,s)=\frac{\Gamma(s)}{(2\pi)^s}\left(Np^{2n}\right)^{s/2}L(f,\chi,s). \tag{4.4}$$

By (4.1), we have

$$L_p(f_\alpha,\chi z^j)=e_p(f,\alpha,\chi,j)\frac{p^{n(j+1)}(Np^{2n})^{-(j+1)/2}}{\alpha^n(-i)^{j+1}\tau(\chi^{-1})\Omega_{f_\alpha}^{\chi(-1)(-1)^j}}R\left(f,\chi^{-1},j+1\right). \tag{4.5}$$

Similar,

$$\begin{aligned}L_p((f_{|W_N})_u,\chi^{-1}z^{k-j})&=e_p(f_{|W_N},u,\chi,k-j)\frac{p^{n(k+1-j)}(Np^{2n})^{-(k+1-j)/2}}{u^n(-i)^{k+1-j}\tau(\chi)\Omega_{(f_{|W_N})_u}^{\chi(-1)(-1)^{k-j}}}\times\\&\times R\left(f_{|W_N},\chi,k+1-j\right).\end{aligned} \tag{4.6}$$

By [Shi, Theorem 3.66], we have

$$R(f,\chi^{-1},j+1)=i^{k+2}\epsilon(p)^n\chi^{-1}(N)\tau(\chi^{-1})^2p^{-n}N^{-k/2}R(f_{|W_N},\chi,k+1-j), \tag{4.7}$$

note that the weight $k+2$ action of $\mathrm{GL}_2^+(\mathbb{Q})$ on modular forms in this thesis differs by that in [Shi] by the multiple $N^{k/2}$.

From (4.5), (4.6), (4.7) and the identity $\tau(\chi)\tau(\chi^{-1})=\chi(-1)p^n$, we obtain

$$\begin{aligned}L_p(f_\alpha,\chi z^j)&=-\frac{e_p(f,\alpha,\chi,j)}{e_p(f_{|W_N},u,\chi,k-j)}\cdot\frac{\Omega_{(f_{|W_N})_u}^{\chi(-1)(-1)^{k-j}}}{\Omega_{f_\alpha}^{\chi(-1)(-1)^j}}\left(\frac{\epsilon(p)u}{\alpha}\right)^n(-N^{-1})^j\chi^{-1}(-N)\times\\&\quad\times L_p((f_{|W_N})_u,\chi^{-1}z^{k-j})\\&=-e_p(\alpha,u,\chi,j)\frac{\Omega_{(f_{|W_N})_u}^{\chi(-1)(-1)^{k-j}}}{\Omega_{f_\alpha}^{\chi(-1)(-1)^j}}N^{-k}\left(\frac{\epsilon(p)u}{\alpha}\right)^nL_p\left((f_{|W_N})_u,(Nz)^k\cdot(\chi z^j)\left(-\frac{1}{Nz}\right)\right).\end{aligned}$$

We show that the above quotient of periods can be removed for well-chosen periods. Choose an integer $0\le j'\le k$ such that $(-1)^{j'}=\chi(-1)(-1)^j$. By a theorem of Manin-Shokurov (see [Bel, Theorem IV.4.11]), we have

$$\frac{L(f_\alpha,j'+1)}{\Omega_{f_\alpha}^{\chi(-1)(-1)^j}(\pi i)^{j'+1}}\in K_{f_\alpha}\subset L,\ \ \frac{L((f_{|W_N})_u,k-j'+1)}{\Omega_{(f_{|W_N})_u}^{\chi(-1)(-1)^{k-j}}(\pi i)^{k-j'+1}}\in K_{(f_{|W_N})_u}\subset L$$

if L contains the image by ι_p of α and the fields $K_f,K_{f_{|W_N}}$. We get

$$\frac{\Omega_{(f_{|W_N})_u}^{\chi(-1)(-1)^{k-j}}}{\Omega_{f_\alpha}^{\chi(-1)(-1)^j}}\cdot\frac{L(f_\alpha,j'+1)}{(2\pi i)^{j'+1}}:\frac{L((f_{|W_N})_u,k-j'+1)}{(2\pi i)^{k-j'+1}}\in L.$$

By (4.4) and (4.7), we obtain

$$\frac{L(f_\alpha,j'+1)}{(2\pi i)^{j'+1}}:\frac{L((f_{|W_N})_u,k-j'+1)}{(2\pi i)^{k-j'+1}}\in L,$$

note that $L(f_\alpha,s)=(1+(\alpha-a_p)p^{-s})L(f,s)$ (since $L(f(pz),s)=p^{-s}L(f,s)$), and similar for $L((f_{|W_N})_u,s)$. We deduce that

$$\frac{\Omega_{(f_{|W_N})_u}^{\chi(-1)(-1)^{k-j}}}{\Omega_{f_\alpha}^{\chi(-1)(-1)^j}}\in L.$$

Since the periods $\Omega^{\pm}_{f_\alpha}, \Omega^{\pm}_{(f_{|W_N})_u}$ are determined up to multiplication by elements in $K^\times_{f_\alpha}, K^\times_{(f_{|W_N})_u}$, respectively, which are contained in L, we conclude that the quotient $\frac{\Omega^{\chi(-1)(-1)^{k-j}}_{(f_{|W_N})_u}}{\Omega^{\chi(-1)(-1)^j}_{f_\alpha}}$ in the functional equation can be removed. We get the desired formula.

If $u = \epsilon^{-1}(p)\alpha$, then $e_p(\alpha, u, \chi, j) = 1$ for any χ. □

Corollary 4.2.5. *For any locally analytic function $g : \mathbb{Z}_p^\times \to \mathbb{C}_p$, one has*

$$L_p(f_\alpha, g) = -N^{-k} L_p\left((f_{|W_N})_{\epsilon^{-1}(p)\alpha}, (Nz)^k g\Big(-\frac{1}{Nz}\Big)\right). \tag{4.8}$$

Proof. Since the expressions in the left and right hand sides of (4.8) are distributions on $\mathbb{Z}_p^\times$ of order $< k+1$ in the variable g by the construction, it suffices to prove for locally polynomial functions g of degree $\leq k$ by Theorem 1.2.4i). Therefore, we can assume that $g(z) = \mathbb{1}_{a+p^n\mathbb{Z}_p}(z) \cdot z^j$ for $a \in \mathbb{Z}_p^\times, n \in \mathbb{N}^*, j \in \mathbb{N}, 0 \leq j \leq k$. By Proposition 4.2.4, (4.8) is true for any g is of the form χz^j, where χ is a finite order character $\mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ and j is an integer such that $0 \leq j \leq k$. We show that the function $\mathbb{1}_{a+p^n\mathbb{Z}_p}$ is a finite linear combination of finite order characters $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$. We use the following result.

Lemma 4.2.6. *Denote by $(\mathbb{Z}/p^n\mathbb{Z})^\vee$ the group of characters on $(\mathbb{Z}/p^n\mathbb{Z})^\times \cong (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times$. For $a \in \mathbb{Z}_p^\times$ or $a \in \mathbb{Z}, (a,p) = 1$, we have*

$$\sum_{\chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee} \chi(a) = \begin{cases} p^{n-1}(p-1) & \text{, if } a \equiv 1 (\mathrm{mod}\, p^n), \\ 0 & \text{, otherwise.} \end{cases}$$

Proof. For every $\psi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee$, since $(\mathbb{Z}/p^n\mathbb{Z})^\vee = \{\chi\psi \,|\, \chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee\}$, we have

$$\sum_{\chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee} \chi(a) = \sum_{\chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee} (\chi\psi)(a) = \psi(a) \sum_{\chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee} \chi(a).$$

If $a \not\equiv 1 (\mathrm{mod}\, p^n)$, there exists a character ψ such that $\psi(a) \neq 1$, so $\sum_{\chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee} \chi(a) = 0$. If $a \equiv 1 (\mathrm{mod}\, p^n)$, then $\chi(a) = 1$ for any $\chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee$, so

$$\sum_{\chi \in (\mathbb{Z}/p^n\mathbb{Z})^\vee} \chi(a) = |(\mathbb{Z}/p^n\mathbb{Z})^\vee| = p^{n-1}(p-1).$$

The lemma follows. □

By the lemma, we have

$$\begin{aligned} \mathbb{1}_{a+p^n\mathbb{Z}_p}(z) = \mathbb{1}_{1+p^n\mathbb{Z}_p}\left(\frac{z}{a}\right) &= \frac{1}{p^{n-1}(p-1)} \sum_{\chi \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\vee} \chi\left(\frac{z}{a}\right) \\ &= \sum_{\chi \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\vee} \frac{\chi(z)}{p^{n-1}(p-1)\chi(a)}. \end{aligned}$$

The assertion is proven. □

Corollary 4.2.7. *If $f \in \mathcal{S}_{k+2}(N, \epsilon)$ is a new form, then*

$$L_p(f_\alpha, \chi z^j) = -e_p(\alpha, u, \chi, j) \cdot W(f) N^{-k} \left(\frac{\epsilon(p)u}{\alpha}\right)^n L_p\left((f_\rho)_u, (Nz)^k \cdot (\chi z^j)\left(-\frac{1}{Nz}\right)\right)$$

for any finite order character $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ of conductor p^n and any integer $0 \leq j \leq k$, where $W(f)$ is given in Proposition 4.2.3.

If moreover f has trivial nebentypus, then

$$L_p(f_\alpha, g) = -W(f)^{-1} L_p\left(f_\alpha, (Nz)^k g\Big(-\frac{1}{Nz}\Big)\right)$$

for any locally analytic function g on $\mathbb{Z}_p^\times$ with values in $\mathbb{C}_p$.

Proof. If f is a new form, then $f_{|W_N} = W(f)f_\rho$ by Proposition 4.2.3. The first formula is clear by Proposition 4.2.4.

If moreover f has trivial nebentypus (i.e. $\epsilon = 1$), then $\epsilon^{-1}(p)\alpha = \alpha$ and $f_{|W_N} = W(f)f$ with $W(f) = \pm N^{k/2}$. By Corollary 4.2.5, we have

$$L_p(f_\alpha, g) = -N^{-k}W(f) \cdot L_p\left(f_\alpha, (Nz)^k g\Big(-\frac{1}{Nz}\Big)\right)$$

for any locally analytic function $g : \mathbb{Z}_p^\times \to \mathbb{C}_p$. The second formula follows since $N^{-k}W(f) = W(f)^{-1}$. $\square$

Chapter 5

Functional equation of p-adic L-functions attached to automorphic representations of GL_2

Throughout this chapter, let F be a totally real number field and π be a cohomological cuspidal automorphic representation of GL_2 over F (see Definition 5.1.2), such that π_v is not supercuspidal for any place v of F dividing p. We choose a regular p-refinement $\tilde{\pi}$ of π, i.e., choosing a character ν_v of $F_v^\times$ which appears as a one dimensional sub of the Weil-Deligne representation attached to π_v via the local Langlands correspondence for $\mathrm{GL}_2(F_v)$, for each place v of F above p.

Assume that $\tilde{\pi}$ is non-critical (see Definition 5.1.8). Using the theory of automorphic symbols, Barrera, Dimitrov and Jorza in [BDJ] have constructed a p-adic L-function $\mathcal{L}_p(\tilde{\pi}, \cdot)$ as a distribution on the Galois group $\mathrm{Gal}_{p\infty}$ of the maximal abelian extension of F which is unramified outside p and infinite places, and they have shown an interpolation formula for this p-adic L-function attached to $\tilde{\pi}$ related to special values of complex L-function of π (see Theorem 5.3.2). Moreover, they proved the following functional equation:

Theorem 5.0.1 ([BDJ, Theorem 6.4])**.** *Let $\tilde{\pi}$ be a regular non-critical p-refinement of a cohomological self-dual cuspidal automorphic representation π of GL_2 over F with tame conductor $\mathfrak{n}$, such that π_v is Iwahori spherical for any place v above p. For any p-adic valued continuous character f on $\mathrm{Gal}_{\mathrm{cyc}}$ and any finite order character χ on $\mathrm{Gal}_{p\infty}$, one has*

$$\mathcal{L}_p(\tilde{\pi}, \chi \cdot f) = \tilde{\varepsilon}_\pi \cdot (\chi \cdot f)(-\varpi_\mathfrak{n})\mathcal{L}_p(\tilde{\pi}, (\chi \cdot f)^{-1}),$$

where $\varpi_\mathfrak{n}$ is a uniformizer corresponding to the ideal $\mathfrak{n}$ and $\tilde{\varepsilon}_\pi = \varepsilon\big(\pi, \frac{1}{2}\big) \prod_{v|p, \pi_v \text{ is special}} \varepsilon\big(\pi_v, \frac{1}{2}\big)$ is a product of ε-factors. Here $\mathrm{Gal}_{\mathrm{cyc}}$ denotes the Galois group of the cyclotomic $\mathbb{Z}_p$-extension $F_{\mathrm{cyc}} \subset F(\mu_{p^\infty})$ of F.

This chapter is devoted to prove a generalization of the above theorem where the hypothesis on the Iwahori sphericality of π_v for $v|p$ is relaxed to the case π_v is not supercuspidal and the central character of π is allowed to have the form ω^2 for a finite order character ω of $\mathrm{Gal}_{p\infty}$ (see Theorem 5.3.5). Similar to the proof in [BDJ] of the above theorem, we will prove its more general version not only for individual π but also for a family of automorphic representations in a neighborhood of π in the eigenvariety parametrized by cohomological weights. The key ingredient of the proof is the following generalization of [BDJ, Corollary 6.3]:

Theorem 5.0.2. *Suppose $\tilde{\pi}$ satisfies the hypotheses in Theorem 5.0.1 except that π_v is not supercuspidal for any place v above p (rather than Iwahori spherical). Then*

$$\mathcal{L}_p(\tilde{\pi}, \chi\langle\cdot\rangle_p^{r-1}) = \tilde{\varepsilon}_\pi \cdot \chi(-\varpi_\mathfrak{n}) \cdot \langle\mathfrak{n}\rangle_p^{r-1}\mathcal{L}_p(\tilde{\pi}, \chi^{-1}\langle\cdot\rangle_p^{1-r})$$

for any finite order character χ of $\mathrm{Gal}_{p\infty}$ with p-adic values and any integer r critical for the cohomological weight of π (see Definition 5.3.1), where $\langle\cdot\rangle_p = \chi_{\mathrm{cyc}}\omega_p^{-1} : \mathrm{Gal}_{p\infty} \to 1 + 2p\mathbb{Z}_p$ is the character on $\mathrm{Gal}_{p\infty}$ given by the quotient of the cyclotomic character χ_{cyc} by the Teichmüller lift ω_p of $\chi_{\mathrm{cyc}} \bmod p$.

Theorem 5.0.2 implies Theorem 5.0.1 for representations having very non-critical slope (see Definition 5.1.6), by applying [Vis, Theorem 2.3] and Lemma 1.2.5. We then deduce Theorem 5.0.1 from the continuity of p-adic L-functions when the weights vary, with the note that cohomological weights of very non-critical slope are Zariski dense in the weight space. The strategy to prove Theorem 5.0.2 is as follows: as before, we will prove this theorem for a family of automorphic representations in a neighborhood of π in the eigenvariety, so we can assume that $\tilde{\pi}$ has very non-critical slope. Firstly, we prove for characters χ such that χ_v is highly ramified (i.e. the conductor of χ_v is big enough) for all $v|p$. To do this, we apply the interpolation formula relating p-adic and complex L-functions of π given in [BDJ, Theorem 4.2], and we deduce from the functional equation $L(\pi, s) = \varepsilon(\pi, s)L(\pi^\vee, 1-s)$ of L-functions, where $\pi^\vee$ is the contragredient of π. Then we obtain the desired formula by applying Lemma 1.2.5.

Here is the outline of the chapter. In Section 5.1, we introduce the notion of cohomological and non-critical automorphic representations, and cite a result about the existence of families of such representations. In Section 5.2, we study ε-factors for GL_1 and GL_2. We finish the chapter with the functional equation given in Section 5.3.

Notations

In the sequel, let $d = [F : \mathbb{Q}]$ and we denote $\mathcal{O}_F$ the ring of integers of F and $\mathfrak{d}$ its different. Denote by $\mathbb{A}_F$ the ring of adeles of F and $\mathbb{A}_{F,f}$ the ring of finite adeles. Then $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F$ and $\mathbb{A}_{F,f} = \mathbb{A}_f \otimes_{\mathbb{Q}} F$, where $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ and $\mathbb{A}_f = \mathbb{A}_{\mathbb{Q},f}$. Let $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$.

For each finite place v, denote by q_v the cardinal of the residue field of F at v, and denote δ_v the valuation at v of the different $\mathfrak{d}$.

Let Σ be the set of infinite places of F, which are the embeddings of F into the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. Composing with the embedding $\iota_p : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ yields a partition $\Sigma = \sqcup_{v \in S_p} \Sigma_v$, where S_p is the set of places of F above p, and a place σ belongs to Σ_v if v is the kernel of the composition $\mathcal{O}_F \xrightarrow{\iota_p \circ \sigma} \bar{\mathbb{Z}}_p \twoheadrightarrow \bar{\mathbb{F}}_p$.

For each fractional ideal $\mathfrak{f}$ of F, we choose an element $\varpi_{\mathfrak{f}} \in \mathbb{A}_{F,f}^\times$ such that $\varpi_{v\mathfrak{f}} = \varpi_v \cdot \varpi_{\mathfrak{f}}$ for any finite place v, where ϖ_v is a uniformizer of the ring of integers $\mathcal{O}_v$ of F_v.

Recall that $\mathrm{Gal}_{p\infty}$ denotes the maximal abelian extension of F which is unramified outside p and infinity. By class field theory, there is a correspondence between finite order characters of $\mathrm{Gal}_{p\infty}$ and finite order Hecke characters of F which are unramified outside p. The cyclotomic character $\chi_{\mathrm{cyc}} : \mathrm{Gal}_{p\infty} \twoheadrightarrow \mathrm{Gal}(F(\mu_{p^\infty})/F) \to \mathbb{Z}_p^\times$ corresponds via class field theory to the character $\chi_{\mathrm{cyc}} : F_+^\times \backslash \mathbb{A}_{F,f}^\times \to \mathbb{Z}_p^\times$ mapping y to $\prod_{v \in S_p} \mathrm{N}_{F_v/\mathbb{Q}_p}(y_v)|y_f|_F$, where $F_+^\times$ is the set of totally positive elements of F. We extend the Teichmüller lift ω_p to $F_\infty^\times$ by the sign character. The character $\langle \cdot \rangle_p = \chi_{\mathrm{cyc}}\omega_p^{-1}$ can be seen as the projection on the Galois group $\mathrm{Gal}_{\mathrm{cyc}} = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$ of the cyclotomic $\mathbb{Z}_p$-extension $F_{\mathrm{cyc}} \subset F(\mu_{p^\infty})$ of F.

We consider the additive character $\psi : \mathbb{A}_F/F \to \mathbb{C}^\times$ given by the composition of the trace map of adeles from F to $\mathbb{Q}$ followed by the usual additive character ψ_0 on $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ given by $\psi_{0|\mathbb{R}} = \exp(-2\pi i \cdot)$ and $\psi_{0|\mathbb{Q}_l}$ is the value of $\exp(2\pi i \cdot)$ at the l-non integer part of $\mathbb{Q}_l$ for every prime number l. The conductor of ψ_v is $-\delta_v$ for any finite place v.

If χ_v is a character of $F_v^\times$ of conductor c_{χ_v} for a finite place v, we define its local Gauss sum by

$$\tau(\chi_v, \psi_v) = \int_{\varpi_v^{-c_{\chi_v} - \delta_v} \mathcal{O}_v^\times} \chi_v(x)\, \psi_v(x)\, d_v x$$

which is independent of the choice of uniformizer, where d_v is the Haar measure on F_v giving $\mathcal{O}_v$ volume $q_v^{-\delta_v/2}$. If χ_v is unramified, then $\tau(\chi_v, \psi_v) = \chi_v(\varpi_v)^{-\delta_v} q_v^{\delta_v/2}$. The Haar measure on $F_v^\times$ is considered to be $d_v^\times x = \frac{d_v x}{|x|_v}$.

For a Hecke character $\chi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ of conductor $\mathfrak{c}_\chi$, we define the global Gauss sum

$$\tau(\chi) = \prod_{v \nmid \infty} \tau(\chi_v, \psi_v) = \prod_{v | \mathfrak{c}_\chi} \tau(\chi_v, \psi_v) \prod_{v \nmid \mathfrak{c}_\chi \infty} \chi_v(\varpi_v)^{-\delta_v} q_v^{\delta_v/2}.$$

5.1 Families of cohomological cuspidal automorphic representations

As introduced in the beginning of the chapter, we will prove a functional equation of p-adic L-functions attached to a family of certain cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$. In this section, we consider a family of partial non-critical refinements of cohomological cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ which are not supercuspidal above p.

5.1.1 Cohomology of Hilbert modular varieties

Let $G = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathrm{GL}_2$. Let $K_\infty = O_2(F_\infty)F_\infty^\times$. Denote by K_∞^+ the connected component of the unit in K_∞, then $K_\infty^+ = SO_2(F_\infty)F_\infty^\times$.

For an open compact subgroup K of $G(\mathbb{A}_f)$, we define the Hilbert modular variety of level K as

$$Y_K = G(\mathbb{Q})\backslash G(\mathbb{A})/KK_\infty^+.$$

It projects to $\mathcal{C}_K^+ := F^\times\backslash\mathbb{A}_F^\times/\det(K)F_\infty^+$ by the determinant map. The fiber $Y_K[\alpha] = \det^{-1}([\alpha])$ of each class $[\alpha] \in \mathcal{C}_K^+$ has the structure of a symmetric space by the isomorphism

$$\Gamma_\alpha\backslash G_\infty^+/K_\infty^+ \cong Y_K[\alpha]$$
$$g_\infty \mapsto g_\infty \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix},$$

where $\Gamma_\alpha = G(\mathbb{Q}) \cap \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} K \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}^{-1} G_\infty^+$. Therefore, Y_K is a complex manifold.

In the sequel we assume that K is small enough such that for all $g \in G(\mathbb{A})$:

$$G(\mathbb{Q}) \cap gKK_\infty^+g^{-1} = F^\times \cap KF_\infty^\times. \tag{5.1}$$

Given a left K-module V such that

$$F^\times \cap KF_\infty^\times \text{ acts trivially on } V, \tag{5.2}$$

then the group $G(\mathbb{Q}) \cap gKK_\infty^+g^{-1}$ acts trivially on V. We get the local system $\mathcal{V}$:

$$G(\mathbb{Q})\backslash(G(\mathbb{A}) \times V)/KK_\infty^+ \to Y_K$$

defined by $\gamma(g,v)k = (\gamma gk, k^{-1}v)$, where $\gamma \in G(\mathbb{Q}), g \in G(\mathbb{A}), v \in V$ and $k \in KK_\infty^+$.

We also denote by $\mathcal{V}$ the sheaf of locally constant sections on Y_K.

5.1.2 Cohomological cuspidal automorphic representations

Let $B \subset G$ be the Borel subgroup of upper triangular matrices and T be the torus of diagonal matrices.

An element $k = \sum_{\sigma\in\Sigma} k_\sigma\sigma \in \mathbb{Z}[\Sigma]$ can be identified with a character of $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ as follows: for any $\mathbb{Q}$-algebra A, we define the character

$$x \in (F \otimes_\mathbb{Q} A)^\times \mapsto x^k = \prod_{\sigma\in\Sigma} \sigma(x)^{k_\sigma} \in A^\times.$$

Integral weights of G are characters of T of the form $\mathrm{diag}(a,d) \mapsto a^kd^{k'}$ for $(k,k') \in \mathbb{Z}[\Sigma]^2$. Such a weight is called dominant if $k_\sigma \geq k'_\sigma$ for any $\sigma \in \Sigma$. Each dominant weight (k,k') induces an irreducible algebraic representation of $G(A)$ given by

$$\bigotimes_{\sigma\in\Sigma}(\mathrm{Sym}^{k_\sigma-k'_\sigma} \otimes \det{}^{k'_\sigma})(A^2).$$

Suppose that (k,k') is dominant. For a $\mathbb{Q}$-algebra A, we define the representation $L_{k,k'}(A)$ of $G(A)$ consisting of polynomials P of degree at most $(k_\sigma - k'_\sigma)_{\sigma\in\Sigma}$ in the variables $z = (z_\sigma)_{\sigma\in\Sigma}$ with coefficients in A, where the action of $G(A) \cong \mathrm{GL}_2(A)^\Sigma$ is given by

$$P_{|\gamma}(z) = (\det\gamma)^{k'}(cz+d)^{k-k'}P\Big(\frac{az+b}{cz+d}\Big) \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(A). \tag{5.3}$$

The dual $L_{k,k'}^\vee(A) = \mathrm{Hom}_A(L_{k,k'}(A), A)$ is induced a left action of $G(A)$ given by

$$(\gamma\cdot\mu)(P) = \mu(P_{|(\det\gamma)^{-1}\cdot\gamma}), \tag{5.4}$$

where $\gamma \in G(A), \mu \in L_{k,k'}^\vee(A), P \in L_{k,k'}(A)$. It follows that

$$L_{k,k'}^\vee(A) \cong \bigotimes_{\sigma\in\Sigma}(\mathrm{Sym}^{k_\sigma-k'_\sigma} \otimes \det{}^{-k_\sigma})(A^2).$$

Definition 5.1.1. *A dominant weight of G is cohomological if it is of the form*

$$\Big(\frac{\mathrm{w}+k_\sigma-2}{2}, \frac{\mathrm{w}+2-k_\sigma}{2}\Big)_{\sigma\in\Sigma},$$

where $(k,\mathrm{w})\in\mathbb{Z}[\Sigma]\times\mathbb{Z}$ satisfying $k_\sigma\geq 2$ and $k_\sigma\equiv \mathrm{w}\,(\mathrm{mod}\,2)$ for any $\sigma\in\Sigma$. We will write simply (k,w) for this weight.

The left $G(A)$-module $L^\vee_{k,\mathrm{w}}(A)$ induces the sheaf $\mathcal{L}^\vee_{k,\mathrm{w}}(A)$ on Y_K. The condition (5.2) for $L^\vee_{k,\mathrm{w}}(A)$ is equivalent to

$$\mathrm{N}^{\mathrm{w}}_{F/\mathbb{Q}}(x)=1 \text{ for any } x\in F^\times\cap KF^\times_\infty.$$

If (k,w) is a cohomological weight and $\sigma\in\Sigma$, we consider the induction from B_σ to G_σ of the character which is trivial on the unipotent radical and given on T_σ by

$$\mathrm{diag}(a,d)\mapsto a^{(\mathrm{w}+k_\sigma-2)/2}d^{(\mathrm{w}+2-k_\sigma)/2}\Big|\frac{a}{d}\Big|^{1/2}.$$

It has a unique non-trivial finite dimensional quotient which is $L_{k_\sigma,\mathrm{w}}(\mathbb{C})$ and the kernel is denoted by $\pi_{k_\sigma,\mathrm{w}}$.

Definition 5.1.2. *We say that an automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ has cohomological weight (k,w) if $\pi_\infty\cong\bigotimes_{\sigma\in\Sigma}\pi_{k_\sigma,\mathrm{w}}$.*

Cohomological automorphic representations of weight (k,w) play an important role since they contribute to the cuspidal cohomology group $H^d_{\mathrm{cusp}}(Y_K,\mathcal{L}^\vee_{k,\mathrm{w}}(\mathbb{C}))$ in the following decomposition:

$$H^d_{\mathrm{cusp}}(Y_K,\mathcal{L}^\vee_{k,\mathrm{w}}(\mathbb{C}))=\bigotimes_\pi H^d(\mathfrak{g}_\infty,K^+_\infty,L^\vee_{k,\mathrm{w}}(\mathbb{C})\otimes\pi_\infty)\otimes\pi_f^K,$$

where π runs over all cuspidal automorphic representations of $G(\mathbb{A})$ and $\mathfrak{g}_\infty$ denotes the complexified Lie algebra of G_∞.

Definition 5.1.3. *Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ and let $v\in S_p$. We say that*

i) π_v is regular if either it is a twist of the Steinberg representation or it is a principal series representation associated to two different characters.

ii) If π_v is not supercuspidal, a refinement of π_v is a one dimensional sub ν_v of the Weil-Deligne representation attached to π_v via the local Langlands correspondence for $\mathrm{GL}_2(F_v)$.

iii) For $S\subset S_p$, an S-refinement of π is a pair $\tilde\pi_S=(\pi,\{\nu_v\}_{v\in S})$ where ν_v is a refinement of π_v for each $v\in S$. From now on, if $S=S_p$ we will omit it from the notations, e.g. we write $\tilde\pi$ for $\tilde\pi_{S_p}$ and call it a p-refinement of π.

5.1.3 Partial non-critical refinements and its families

We introduce the notion of partial non-critical refinements of automorphic representations of $G(\mathbb{A})$ which is crucial for the existence of p-adic L-functions of Hilbert cusp forms constructed in [BDJ]. We state a theorem about the existence of such families parametrized by cohomological weights. The definitions in this subsection follow [BDJ].

Let $S\subset S_p$. Let $\tilde\pi=(\pi,\{\nu_v\}_{v\in S_p})$ be a regular p-refinement of a cuspidal automorphic representation of cohomological weight (k,w) and tame conductor $\mathfrak{n}$ which is not supercuspidal above p.

For an ideal $\mathfrak{f}$ of $\mathcal{O}_F$ we consider the following open compact subgroups of $G(\hat{\mathbb{Z}})$:

$$K_0(\mathfrak{f})=\Big\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in G(\hat{\mathbb{Z}})\,|\,c\in\mathfrak{f}\Big\},\quad K_1(\mathfrak{f})=\Big\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in G(\hat{\mathbb{Z}})\,|\,c\in\mathfrak{f},d\in 1+\mathfrak{f}\Big\}.$$

The double coset operator $K_1(\mathfrak{n})xK_1(\mathfrak{n})=\sqcup_i K_1(\mathfrak{n})x_i$ for $x\in G(\mathbb{A}_f)$ acts on automorphic forms of level $K_1(\mathfrak{n})$ on the right by

$$g_{|K_1(\mathfrak{n})xK_1(\mathfrak{n})}(\cdot)=\sum_i g(\cdot\, x_i^{-1}).$$

For each finite place v of F, we define the Hecke operator T_v given by the double coset operator $K_1(\mathfrak{n})\begin{pmatrix}\varpi_v & 0\\ 0 & 1\end{pmatrix}K_1(\mathfrak{n})$ and for v not dividing $\mathfrak{n}$ we let $S_v = \left[K_1(\mathfrak{n})\begin{pmatrix}\varpi_v & 0\\ 0 & \varpi_v\end{pmatrix}K_1(\mathfrak{n})\right]$.

Consider $v \in S_p$. If π_v is a twist of the Steinberg representation we let $K_v = K_0(v)$, if π_v is a principal series attached to characters $\chi_{1,v}, \chi_{2,v}$ we let $K_v = K_0(v) \cap K_1(v^{m_v})$, where m_v is the conductor of $\chi_{1,v}/\chi_{2,v}$. Let

$$K'_v = \ker\,(K_v \xrightarrow{\det} \mathcal{O}_v^\times \xrightarrow{\nu_v} \mathbb{C}^\times).$$

For a uniformizer $\varpi_v \in \mathcal{O}_v$ and $\delta \in \mathcal{O}_v^\times$ we define the Hecke operators

$$U_{\varpi_v} = \left[K'_v\begin{pmatrix}\varpi_v & 0\\ 0 & 1\end{pmatrix}K'_v\right] \text{ and } U_\delta = \left[K'_v\begin{pmatrix}\delta & 0\\ 0 & 1\end{pmatrix}K'_v\right].$$

Remark 5.1.4. By the abuse of notation, the Hecke operator U_p on automorphic forms is defined by the matrix $\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}$, while we use the matrix $\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}$ for the action of U_p on p-adic distributions, since the matrices act on automorphic forms on the left and on distributions on the right.

We fix a prime ideal $\mathfrak{u}$ of F such that:

i) If K is an open compact subgroup of $G(\mathbb{A}_f)$ with $K_\mathfrak{u} = K_0(\mathfrak{u})$, then K satisfies (5.1),

ii) $\mathfrak{u}$ is unramified and $\pi_\mathfrak{u}$ is an unramified principal series with Hecke parameters $\alpha_\mathfrak{u} \neq \beta_\mathfrak{u}$.

The existence of $\mathfrak{u}$ follows from [Dim09, Lemma 2.1].

Definition 5.1.5. *Let E be a number field containing the Galois closure of F, the rationality field of π_f, the Hecke parameters of $\pi_\mathfrak{u}$ and the values of the characters ν_v for $v \in S_p$.*

Let $\mathfrak{m}_\pi$ be the maximal ideal corresponding to π_f of the Hecke algebra $\mathbb{T} = E[T_v, S_v \,|\, v \not| \,\mathfrak{n}\mathfrak{u}p]$.

For $S \subset S_p$, we consider the maximal ideal

$$\mathfrak{m}_{\tilde{\pi}_S} = (\mathfrak{m}_\pi, U_\mathfrak{u} - \alpha_\mathfrak{u}, U_{\varpi_v} - \nu_v(\varpi_v), U_\delta - \nu_v(\delta) \,|\, \delta \in \mathcal{O}_v^\times, v \in S)$$

of the Hecke algebra $\tilde{\mathbb{T}}_S = \mathbb{T}[U_\mathfrak{u}, U_{\varpi_v}, U_\delta \,|\, \delta \in \mathcal{O}_v^\times, v \in S]$.

We consider the following open compact subgroup of $G(\mathbb{A}_f)$:

$$K(\tilde{\pi}_S, \mathfrak{u}) = K_0(\mathfrak{u}) \prod_{v \notin S \cup \{\mathfrak{u}\}} K_1(v^{m_v}) \prod_{v \in S} K'_v,$$

where m_v is the conductor of π_v.

Definition 5.1.6. *The slope $h_{\tilde{\pi}_v}$ of $\tilde{\pi}_v = (\pi_v, \nu_v)$ is the p-adic valuation of*

$$\nu_v(\varpi_v) \prod_{\sigma \in \Sigma_v} \sigma(\varpi_v)^{(k_\sigma + \mathrm{w} - 2)/2}.$$

This slope is independent of the choice of uniformizer.

We say that $\tilde{\pi}_v$ has non-critical slope if $e_v h_{\tilde{\pi}_v} < \min_{\sigma \in \Sigma_v}(k_\sigma - 1)$, where e_v is the ramification index of p at v.

For $S \subset S_p$, we say that $\tilde{\pi}_S$ has non-critical slope if $\tilde{\pi}_v$ has non-critical slope for any $v \in S$.

We say that $\tilde{\pi}$ has very non-critical slope if

$$\sum_{v \in S_p} e_v h_{\tilde{\pi}_v} < \min_{\sigma \in \Sigma}(k_\sigma - 1).$$

Consider the monoid

$$\Lambda_S = \prod_{v \in S_p \setminus S} \mathrm{GL}_2(F_v) \prod_{v \in S} \mathrm{GL}_2(F_v) \cap \left(F_v^\times \begin{pmatrix}\mathcal{O}_v & \mathcal{O}_v\\ \varpi_v\mathcal{O}_v & \mathcal{O}_v^\times\end{pmatrix}\right)$$

which contains the partial Iwahori subgroup $I_S = \Lambda_S \cap G(\mathbb{Z}_p) = \prod_{v \in S_p \setminus S} \mathrm{GL}_2(\mathcal{O}_v) \prod_{v \in S} K_0(v)$.

For a finite extension L of $\mathbb{Q}_p$, we let

$$\mathcal{A}_{S,(k,\mathrm{w})} = \mathcal{A}(\mathcal{O}_{F,S}, L) \otimes_L \bigotimes_{\sigma \in \Sigma_{S_p \setminus S}} L_{k_\sigma,\mathrm{w}}(L)$$

be the space of L-valued analytic functions on $\mathcal{O}_F \otimes \mathbb{Z}_p$ which are polynomial of degree at most $(k_\sigma - 2)_{\sigma\in\Sigma_v}$ in the variables $(z_\sigma)_{\sigma\in\Sigma_v}$ for $v \in S_p\backslash S$. Denote by $\mathcal{D}_{S,(k,\mathrm{w})}$ its continuous L-dual.

We define a continuous right action of I_S on $\mathcal{A}_{S,(k,\mathrm{w})}$ given by

$$f_{|\gamma}(z) = (\det\gamma)^{((\mathrm{w}+2-k_\sigma)_{\sigma\in\Sigma})/2}(cz+d)^{k-2} f\Big(\frac{az+b}{cz+d}\Big),$$

where $f \in \mathcal{A}_{S,(k,\mathrm{w})}, \gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in I_S, z \in \mathcal{O}_F \otimes \mathbb{Z}_p$. We extend to an action of Λ_S by putting $f_{|\begin{pmatrix} \varpi_v^r & 0 \\ 0 & \varpi_v^s\end{pmatrix}}(z) = f(\varpi_v^{r-s}z)$ for all $v \in S$ and integers $r \geq s$.

It induces a continuous left action of Λ_S on $\mathcal{D}_{S,(k,\mathrm{w})}$, given by

$$(\gamma\cdot\mu)(f) = \mu(f_{|(\det\gamma)^{-1}\cdot\gamma}) \text{ for } \gamma \in \Lambda_S, \mu \in \mathcal{D}_{S,(k,\mathrm{w})}, f \in \mathcal{A}_{S,(k,\mathrm{w})}.$$

The natural inclusion $L_{k,\mathrm{w}}(L) \to \mathcal{A}_{S,(k,\mathrm{w})}$ induces the dual map $\rho_S : \mathcal{D}_{S,(k,\mathrm{w})} \to L^\vee_{k,\mathrm{w}}(L)$ which is equivariant for the action of I_S (see (5.3) and (5.4)) but it is not Λ_S-equivariant. More explicitly, for all $v \in S$ and $\mu \in \mathcal{D}_{S,(k,\mathrm{w})}$, one has

$$\rho_S\Big(\begin{pmatrix} \varpi_v & 0 \\ 0 & 1\end{pmatrix}\cdot\mu\Big) = \prod_{\sigma\in\Sigma_v} \sigma(\varpi_v)^{(\mathrm{w}+k_\sigma-2)/2}\begin{pmatrix} \varpi_v & 0 \\ 0 & 1\end{pmatrix}\cdot\rho_S(\mu).$$

Now let $K \subset G(\mathbb{A}_f)$ be an open compact subgroup satisfying (5.1) and $K_p \subset I_S$. The homomorphism ρ_S induces a homomorphism on the cohomology:

$$\rho_S : H^\cdot_c(Y_K, \mathcal{D}_{S,(k,\mathrm{w})}) \to H^\cdot_c(Y_K, \mathcal{L}^\vee_{k,\mathrm{w}}(L)) \tag{5.5}$$

which is equivariant for the action of U_{ϖ_v} on the left space and the action of the normalized Hecke operator $U^\circ_{\varpi_v} = \Big(\prod_{\sigma\in\Sigma_v}\sigma(\varpi_v)^{(\mathrm{w}+k_\sigma-2)/2}\Big)U_{\varpi_v}$ on the right space, for any place $v \in S$.

Let $h_S = (h_v)_{v\in S} \in \mathbb{Q}^S_{\geq 0}$. By [Urb, Lemma 2.3.13], the cohomology group $H^\cdot_c(Y_K, \mathcal{D}_{S,(k,\mathrm{w})})$ admits a $\leq h_S$-slope decomposition

$$H^\cdot_c(Y_K, \mathcal{D}_{S,(k,\mathrm{w})}) = H^\cdot_c(Y_K, \mathcal{D}_{S,(k,\mathrm{w})})^{\leq h_S} \oplus H^\cdot_c(Y_K, \mathcal{D}_{S,(k,\mathrm{w})})^{>h_S},$$

where $H^\cdot_c(Y_K, \mathcal{D}_{S,(k,\mathrm{w})})^{\leq h_S}$ denotes the subspace of elements having slope $\leq h_v$ with respect to U_{ϖ_v} for all $v \in S$.

Henceforth we consider $K = K(\tilde{\pi}_S, \mathfrak{u})$. The following theorem generalizes Stevens' control theorem for overconvergent modular symbols (see Theorem 3.2.1).

Theorem 5.1.7 ([BDJ, Theorem 2.7])**.** *Let $h_S = (h_v)_{v\in S} \in \mathbb{Q}^S_{\geq 0}$ be such that $e_v h_v < \min_{\sigma\in\Sigma_v}(k_\sigma - 1)$ for any $v \in S$. Then* (5.5) *induces an isomorphism of $\leq h_S$-slope subspaces*

$$\rho_S : H^\cdot_c(Y_K, \mathcal{D}_{S,(k,\mathrm{w})})^{\leq h_S} \to H^\cdot_c(Y_K, \mathcal{L}^\vee_{k,\mathrm{w}}(L))^{\leq h_S},$$

where we consider the operators $\{U_{\varpi_v}, v \in S\}$ on the left hand side and $\{U^\circ_{\varpi_v}, v \in S\}$ on the right hand side.

Definition 5.1.8. *We say that $\tilde{\pi}_S$ is non-critical if the localization*

$$\rho_S : H^\cdot_c(Y_K, \mathcal{D}_{S,(k,\mathrm{w})})_{\mathfrak{m}^\circ_{\tilde{\pi}_S}} \to H^\cdot_c(Y_K, \mathcal{L}^\vee_{k,\mathrm{w}}(L))_{\mathfrak{m}_{\tilde{\pi}_S}}$$

of (5.5) *is an isomorphism, where $\mathfrak{m}^\circ_{\tilde{\pi}_S}$ is the normalization of $\mathfrak{m}_{\tilde{\pi}_S}$ with the components $U_{\varpi_v} - \nu_v(\varpi_v)$ of $\mathfrak{m}_{\tilde{\pi}_S}$ for $v \in S$ are replaced by $U_{\varpi_v} - \nu_v(\varpi_v)\prod_{\sigma\in\Sigma_v}\sigma(\varpi_v)^{(\mathrm{w}+k_\sigma-2)/2}$. When $S = S_p$ we say that $\tilde{\pi}$ is non-critical.*

Corollary 5.1.9. *If $\tilde{\pi}_S$ has non-critical slope, then $\tilde{\pi}_S$ is non-critical.*

Proof. It is immediate from Theorem 5.1.7. □

Definition 5.1.10 (Weight space)**.** *Let $\mathcal{X}$ be the $(d+1)$-dimensional rigid analytic space over $\mathbb{Q}_p$ such that*

$$\mathcal{X}(\mathbb{C}_p) = \left\{\lambda \in \mathrm{Hom}_{\mathrm{cont}}(T(\mathbb{Z}_p), \mathbb{C}_p^\times) \mid \exists\, \mathrm{w}_\lambda \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times), \lambda\left(\begin{pmatrix} z & \\ & z \end{pmatrix}\right) = \mathrm{w}_\lambda(\mathrm{N}_{F/\mathbb{Q}}(z))\right\}.$$

We let $k_\lambda(z) = \lambda\left(\begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}\right)\mathrm{N}^2_{F/\mathbb{Q}}(z)$. There is a morphism

$$\begin{aligned} \mathcal{X}(\mathbb{C}_p) &\to \mathrm{Hom}_{\mathrm{cont}}((\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \times \mathbb{Z}_p^\times, \mathbb{C}_p^\times) \\ \lambda &\mapsto (k_\lambda, \mathrm{w}_\lambda). \end{aligned} \tag{5.6}$$

The space $\mathcal{X}$ contains all cohomological weights of G which are very Zariski dense in it.

Definition 5.1.11. *Fix a cohomological weight (k, w). Given a subset S of S_p, we let $\mathcal{X}_S$, resp. $\mathcal{X}'_S$ denote the rigid analytic subspaces of $\mathcal{X}$ consisting of weights which coincide with (k, w) on*

$$\prod_{v \in S_p \setminus S} \begin{pmatrix} \mathcal{O}_v^\times & 0 \\ 0 & \mathcal{O}_v^\times \end{pmatrix}, \text{ resp. } \prod_{v \in S_p \setminus S} \begin{pmatrix} \mathcal{O}_v^\times & 0 \\ 0 & 1 \end{pmatrix}.$$

We are ready to state a result about the existence of families of partial non-critical refinements indexing by cohomological weights.

Theorem 5.1.12 ([BDJ, Theorem 2.14(iii)])**.** *Suppose that $\tilde{\pi}_S$ is non-critical. There exists an affinoid neighborhood $\mathcal{U}_S$ of (k, w) in $\mathcal{X}_S$ such that we can attach to any cohomological weight $\lambda \in \mathcal{U}_S$ a non-critical S-refined weight λ cuspidal automorphic representation $\tilde{\pi}_{\lambda,S}$ of $G(\mathbb{A})$.*

By the above theorem, we can take an affinoid neighborhood $\mathcal{X}(\tilde{\pi})$ of (k, w) in the weight space $\mathcal{X}$ such that for any cohomological weight $\lambda \in \mathcal{X}(\tilde{\pi})$, $\tilde{\pi}_\lambda$ is non-critical, the map $\lambda \mapsto (k_\lambda, \mathrm{w}_\lambda)$ defined in (5.6) is injective on $\mathcal{X}(\tilde{\pi})$ and $\mathrm{w}_\lambda \circ \omega_p = \omega_p^{\mathrm{w}}$. By [BDJ, Lemma 5.1], we can further assume that the tame conductor of π_λ equals $\mathfrak{n}$ for every cohomological $\lambda \in \mathcal{X}(\tilde{\pi})$.

We consider the subset of $\mathcal{X}(\tilde{\pi})$ consisting of weights parametrized by the variables $((k_{\lambda,\sigma})_{\sigma\in\Sigma}, \mathrm{w}_\lambda)$ which correspond via (5.6) to characters of the form

$$z = ((z_v)_{v \in S_p}, z_0) \in \prod_{v \in S_p} \mathcal{O}_v^\times \times \mathbb{Z}_p^\times \mapsto (k, \mathrm{w})(z) \cdot \langle z_0 \rangle_p^{\mathrm{w}_\lambda - \mathrm{w}} \prod_{v \in S_p} \prod_{\sigma \in \Sigma_v} \sigma(\langle z_v \rangle_v)^{k_{\lambda,\sigma} - k_\sigma} \in \mathbb{C}_p^\times,$$

where $\langle \cdot \rangle_v : \mathcal{O}_v^\times \to 1 + \varpi_v \mathcal{O}_v$ is the natural projection map. We denote by $\mathcal{X}^{\mathrm{an}}(\tilde{\pi}) \subset \mathcal{X}(\tilde{\pi})$ a neighborhood of (k, w) in the space $\prod_{\sigma\in\Sigma}(k_\sigma + 2p\mathcal{O}_{\mathbb{C}_p}) \times (\mathrm{w} + \mathcal{O}_{\mathbb{C}_p})$ of these analytic weights.

5.2 ε-factors for GL_1 and GL_2

The ε-factors appear in the functional equation of L-functions of cuspidal automorphic representations. In this section, we recall and list some properties of ε-factors for GL_1 and GL_2.

Let χ be a Hecke character of F and π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Then $\chi = \bigotimes_v \chi_v$ and $\pi = \bigotimes_v \pi_v$ where v runs through all places of F and χ_v (resp. π_v) is the local component at v of χ (resp. π).

For a locally constant function Φ on $F_v^\times$ with compact support, we define its Fourier transform as

$$\hat{\Phi}(x) = \int_{F_v} \Phi(y)\psi_v(xy) d_v y.$$

It follows that $\hat{\hat{\Phi}}(x) = \Phi(-x)$. By [Tat67, Theorem 2.4.1], there is a meromorphic function $\gamma(\chi_v, s, \psi_v)$ with $s \in \mathbb{C}$ such that

$$\int_{F_v^\times} \hat{\Phi}(x)\chi_v^{-1}(x)\, |x|_v^{1-s}\, d_v^\times x = \gamma(\chi_v, s, \psi_v) \int_{F_v^\times} \Phi(x)\chi_v(x)\, |x|_v^s\, d_v^\times x$$

for any test function Φ. We define the local ε-factor $\varepsilon(\chi_v, s, \psi_v)$ by

$$\varepsilon(\chi_v, s, \psi_v) = \gamma(\chi_v, s, \psi_v)\frac{L(\chi_v, s)}{L(\chi_v^{-1}, 1-s)},$$

where the local L-functions for GL_1 are defined by

$$L(\chi_v, s) = \begin{cases} (1-\chi_v(\varpi_v)q_v^{-s})^{-1} & \text{if } v \not| \infty \text{ and } \chi_v \text{ is unramified,} \\ 1 & \text{if } v \not| \infty \text{ and } \chi_v \text{ is ramified,} \\ \pi^{-(s+t)/2}\,\Gamma\left(\frac{s+t}{2}\right) & \text{if } v \text{ is real and } \chi_v = |\cdot|_{\mathbb{R}}^t, \\ \pi^{-(s+t+1)/2}\,\Gamma\left(\frac{s+t+1}{2}\right) & \text{if } v \text{ is real and } \chi_v = \operatorname{sgn}|\cdot|_{\mathbb{R}}^t. \end{cases}$$

The global ε-factor of χ is

$$\varepsilon(\chi, s) = \prod_v \varepsilon(\chi_v, s, \psi_v).$$

By the same way, we can attach to π_v an ε-factor $\varepsilon(\pi_v, s, \psi_v)$ for any place v of F and attach to π a global ε-factor $\varepsilon(\pi, s)$.

We list some properties of local ε-factors for GL_1 and GL_2. Let v be a finite place of F. From now on we will use the symbol c (resp. $\mathfrak{c}$) to denote the local (resp. global) conductor of Hecke characters and automorphic representations of GL_2 over F (e.g. the conductor of χ_v (resp. π_v) will be denoted by c_{χ_v} (resp. c_{π_v})).

i) If χ_v is unramified, by the formulas (3.4.6), (3.2.6.3) and (3.2.6.1) in [Tat79], for any $s \in \mathbb{C}$ and any character μ_v on $F_v^\times$ of conductor c_{μ_v}, one has

$$\begin{aligned} \varepsilon(\pi_v \otimes \chi_v, s, \psi_v) &= \chi_v(\varpi_v)^{c_{\pi_v}+2\delta_v}\varepsilon(\pi_v, s, \psi_v), \\ \varepsilon(\chi_v\mu_v, s, \psi_v) &= \chi_v(\varpi_v)^{c_{\mu_v}+\delta_v}\varepsilon(\mu_v, s, \psi_v), \\ \varepsilon(\chi_v, s, \psi_v) &= \chi_v(\varpi_v)^{\delta_v}q_v^{-\delta_v(s-1/2)}. \end{aligned} \tag{5.7}$$

(Note that we take the Haar measure dx_v giving $\mathcal{O}_v$ volume $q_v^{-\delta_v/2}$ but in [Tat79], he considers the Haar measure giving $\mathcal{O}_v$ volume 1). Therefore, for any $t \in \mathbb{C}$, we have

$$\begin{aligned} \varepsilon(\pi_v, s+t, \psi_v) &= \varepsilon(\pi_v \otimes |\cdot|_v^t, s, \psi_v) = q_v^{-t(c_{\pi_v}+2\delta_v)}\varepsilon\left(\pi_v, s, \psi_v\right), \\ \varepsilon(\chi_v, s+t, \psi_v) &= \varepsilon(\chi_v\,|\cdot|_v^t, s, \psi_v) = q_v^{-t(c_{\chi_v}+\delta_v)}\varepsilon(\chi_v, s, \psi_v) \end{aligned} \tag{5.8}$$

for any π_v and any χ_v.

ii) For any χ_v, by [Tat79, (3.2.6.2)], one has

$$\varepsilon(\chi_v, s, \psi_v) = \int_{\varpi_v^{-c_{\chi_v}-\delta_v}\mathcal{O}_v^\times} \chi_v^{-1}(x)\,|x|_v^{-s}\,\psi_v(x)\,d_vx = q_v^{-s(c_{\chi_v}+\delta_v)}\tau(\chi_v^{-1}, \psi_v). \tag{5.9}$$

So $\tau(\chi_v\mu_v, \psi_v) = \chi_v(\varpi_v)^{-(c_{\mu_v}+\delta_v)}\tau(\mu_v, \psi_v)$ if χ_v is unramified and μ_v is any character of $F_v^\times$.

From the identity of local ε-factors (see [Schm, (7) and (12)]):

$$\begin{aligned} \varepsilon(\chi_v, s, \psi_v)\,\varepsilon(\chi_v^{-1}, 1-s, \psi_v) &= \chi_v(-1), \\ \varepsilon(\pi_v, s, \psi_v)\,\varepsilon(\pi_v^\vee, 1-s, \psi_v) &= \omega_{\pi_v}(-1) \end{aligned} \tag{5.10}$$

(where $\pi_v^\vee$ is the contragredient representation of π_v and ω_{π_v} is the central character of π_v), we get the identity of local Gauss sums:

$$q_v^{-c_{\chi_v}-\delta_v}\tau(\chi_v, \psi_v)\tau(\chi_v^{-1}, \psi_v) = \chi_v(-1). \tag{5.11}$$

The identities of global ε-factors for GL_1 and GL_2 state

$$\begin{aligned} \varepsilon(\chi, s)\,\varepsilon(\chi^{-1}, 1-s) &= 1, \\ \varepsilon(\pi, s)\,\varepsilon(\pi^\vee, 1-s) &= 1. \end{aligned} \tag{5.12}$$

iii) If χ_v is highly ramified, by [JS, Proposition 2.2], for all characters α, β of $F_v^\times$ such that $\alpha\beta = \omega_{\pi_v}$, one has

$$\varepsilon(\pi_v \otimes \chi_v, s, \psi_v) = \varepsilon(\chi_v\alpha, s, \psi_v)\cdot\varepsilon(\chi_v\beta, s, \psi_v). \tag{5.13}$$

In particular, $\varepsilon(\pi_v \otimes \chi_v, s, \psi_v) = \varepsilon(\chi_v\omega_{\pi_v}, s, \psi_v)\cdot\varepsilon(\chi_v, s, \psi_v)$ if χ_v is highly ramified.

5.3 Functional equation of p-adic L-functions

This section is devoted to the proof of a generalization of Theorem 5.0.1 (see Theorem 5.3.5). Throughout this section let $\tilde{\pi} = (\pi, \{\nu_v\}_{v\in S_p})$ be a regular non-critical p-refinement of a cuspidal automorphic representation π of $G(\mathbb{A})$ of cohomological weight (k, w) and tame conductor $\mathfrak{n}$ such that

$$\pi \text{ has central character } \omega_\pi = \omega^2|\cdot|_F^{\mathrm{w}} \text{ with w even, and } \pi_v \text{ is not supercuspidal for any } v\in S_p, \quad (5.14)$$

where ω is a finite order character of $\mathrm{Gal}_{p\infty}$ corresponding to a Hecke character which is unramified outside p.

By the assumption about ω_π, the twist $\pi\otimes\omega^{-1}\,|\cdot|_F^{-\mathrm{w}/2}$ is self-dual, its root number is given by

$$\varepsilon_{\pi\otimes\omega^{-1}} = \varepsilon\Big(\pi\otimes\omega^{-1}, \frac{1-\mathrm{w}}{2}\Big) \in \{\pm 1\}.$$

(the fact that $\varepsilon_{\pi\otimes\omega^{-1}} \in \{\pm1\}$ follows from (5.12) where we take $s=\frac{1}{2}$).

Denote by $\mathrm{St}_p \subset S_p$ the subset of places v such that π_v is the twist (by ν_v) of the unitary Steinberg representation . We let St_v denote the Steinberg representation at v. If $v \in S_p\backslash \mathrm{St}_p$, then $\pi_v = \pi(\nu_v\,|\cdot|_v^{1/2}, \alpha_v\,|\cdot|_v^{1/2})$ is a principal series, where $\alpha_v = \omega_{\pi_v}\nu_v^{-1}\,|\cdot|_v^{-1}$.

Let L be a finite extension of $\mathbb{Q}_p$ containing the image by ι_p of the number field E in Definition 5.1.5. The non-criticality of $\tilde{\pi}$ allows us to attach a p-adic L-function $\mathcal{L}_p(\tilde{\pi},\cdot)$ defined in [BDJ, (4.2)], which is a p-adic distribution on $\mathrm{Gal}_{p\infty}$ with values in L. This p-adic L-function is the specialization of a multi-variable p-adic L-function $\mathcal{L}_p \in \mathcal{D}(\mathrm{Gal}_{p\infty}, \mathcal{O}(\mathcal{X}(\tilde{\pi})))$ (see [BDJ, (4.8)]).

Definition 5.3.1. *We say that an integer r is critical for the cohomological weight (k,w) if*

$$0 \le r-1+\frac{\mathrm{w}+k_\sigma-2}{2} \le k_\sigma - 2 \text{ for all } \sigma\in\Sigma.$$

The p-adic L-function $\mathcal{L}_p(\tilde{\pi},\cdot)$ satisfies the following interpolation formula:

Theorem 5.3.2 ([BDJ, Theorem 4.2])**.** *Let χ be a finite order character of $\mathrm{Gal}_{p\infty}$ and for v dividing p denote by c_v the conductor of $\chi_v\nu_v$. Let r be a critical integer for (k,w). Letting $\mathrm{N}_{F/\mathbb{Q}}(i) = i^d$ and denote by $\Omega_{\tilde{\pi}}^{\cdot}$ the period defined in [BDJ, Definition 1.14], one has*

$$\mathcal{L}_p(\tilde{\pi}, \chi\cdot\chi_{\mathrm{cyc}}^{r-1}) = \frac{\mathrm{N}_{F/\mathbb{Q}}^{r-1}(i\mathfrak{d})\,\chi(\varpi_{\mathfrak{d}}^{-1})}{\Omega_{\tilde{\pi}}^{\chi_\infty\omega_{p,\infty}^{r-1}}} L\Big(\pi\otimes\chi, r-\frac{1}{2}\Big)\prod_{v\in S_p} E(\tilde{\pi}_v,\chi_v,r), \textit{ where}$$

$$E(\tilde{\pi}_v,\chi_v,s) = \begin{cases} q_v^{sc_v}(\chi_v\nu_v)(\varpi_v^{\delta_v})q_v^{-(c_v+\delta_v/2)}\tau(\chi_v\nu_v,\psi_v) & \textit{if } c_v\ge 1 \textit{ and } \chi_v\omega_{\pi_v}\nu_v^{-1} \textit{ ramified,}\\ \Big(1-\frac{(\chi_v\omega_{\pi_v}\nu_v^{-1})(\varpi_v)}{q_v^{s-1}}\Big)q_v^{sc_v}(\chi_v\nu_v)(\varpi_v^{\delta_v})\frac{\tau(\chi_v\nu_v,\psi_v)}{q_v^{(c_v+\delta_v/2)}} & \textit{if } c_v\ge 1 \textit{ and } \chi_v\omega_{\pi_v}\nu_v^{-1} \textit{ unramified,}\\ \Big(1-\frac{(\chi_v\omega_{\pi_v}\nu_v^{-1})(\varpi_v)}{q_v^{s-1}}\Big)\Big(1-\frac{q_v^{s-1}}{(\chi_v\nu_v)(\varpi_v)}\Big) & \textit{if } \pi_v\otimes\chi_v \textit{ is unramified,}\\ \Big(1-\frac{q_v^{s-1}}{(\chi_v\nu_v)(\varpi_v)}\Big) & \textit{otherwise.}\end{cases}$$

Note that the first and second expressions for the values of E-factors in the above theorem are multiples of those in [BDJ] by a multiple of $q_v^{c_v+\delta_v/2}$ since our Gauss sums differ from those in [BDJ] by the multiplication $q_v^{c_v+\delta_v/2}$.

We now prove a more general version of the formula comparing special values of $\mathcal{L}_p(\tilde{\pi},\cdot)$ given in [BDJ, Corollary 6.3]. Note that in ibid. they assume that π has trivial central character and π_v is Iwahori spherical for all $v|p$, while we assume only the condition (5.14).

Proposition 5.3.3. *Suppose π satisfies* (5.14)*. Let $\chi:\mathrm{Gal}_{p\infty}\to L^\times$ be a finite order character and let r be an integer critical for* (k,w)*. Then*

$$\mathcal{L}_p(\tilde{\pi},\chi\langle\cdot\rangle_p^{r-1}) = \tilde{\varepsilon}_{\pi\otimes\omega^{-1}}\cdot(\chi\omega\omega_p^{\mathrm{w}/2})(-\varpi_{\mathfrak{n}})\cdot\langle\mathfrak{n}\rangle_p^{r+\mathrm{w}/2-1}\mathcal{L}_p(\tilde{\pi},\chi^{-1}\omega^{-2}\omega_p^{-\mathrm{w}}\langle\cdot\rangle_p^{1-r-\mathrm{w}}), \quad (5.15)$$

where $\tilde{\varepsilon}_{\pi\otimes\omega^{-1}} = \varepsilon_{\pi\otimes\omega^{-1}}\cdot\prod_{v\in\mathrm{St}_p,\nu_v\omega_v^{-1}\mathrm{unramified}}\varepsilon\big(\pi_v\otimes\omega_v^{-1},\frac{1-\mathrm{w}}{2},\psi_v\big)\in\{\pm1\}.$

Proof. We will prove not only for individual $\tilde{\pi}$ but also for any $\tilde{\pi}_\lambda$ where $\lambda \in \mathcal{X}(\tilde{\pi})$ is cohomological. We remark that the slope is constant in the family and equals $\sum_{v\in S_p} e_v h_{\tilde{\pi}_v}$ (see Definition 5.1.6). Since the cohomological weights having very non-critical slope are Zariski dense in $\mathcal{X}(\tilde{\pi})$, we can assume that (k, w) is such a weight. Regarding both sides of (5.15) as functions of $\chi\langle\cdot\rangle_p^{r-1}$, since $\mathcal{L}_p(\tilde{\pi}, \cdot) \in \mathcal{D}(\mathrm{Gal}_{p\infty}, L)$ has growth at most $\sum_{v\in S_p} e_v h_{\tilde{\pi}_v} < \min_{\sigma\in\Sigma}(k_\sigma - 1)$, by Lemma 1.2.5 it suffices to prove for characters χ such that χ_v is highly ramified for any $v \in S_p$. Suppose that χ is such a character.

By the functional equation of the Jacquet-Langlands global L-functions:

$$L(\pi, s) = \varepsilon(\pi, s) L(\pi^\vee, 1-s)$$

(see [JL, Theorem 11.1]), we have

$$\begin{aligned} L\Big(\pi\otimes\chi, r-\frac{1}{2}\Big) &= \varepsilon\Big(\pi\otimes\chi, r-\frac{1}{2}\Big)\cdot L\Big(\pi^\vee\otimes\chi^{-1}, \frac{3}{2}-r\Big) \\ &= \varepsilon\Big(\pi\otimes\chi, r-\frac{1}{2}\Big)\cdot L\Big(\pi\otimes\omega_\pi^{-1}\chi^{-1}, \frac{3}{2}-r\Big) \\ &= \varepsilon\Big(\pi\otimes\chi, r-\frac{1}{2}\Big)\cdot L\Big(\pi\otimes\chi^{-1}\omega^{-2}, \frac{3}{2}-r-\mathrm{w}\Big). \end{aligned} \tag{5.16}$$

Combining with Theorem 5.3.2, we obtain

$$\begin{aligned} \mathcal{L}_p(\tilde{\pi}, \chi\chi_{\mathrm{cyc}}^{r-1}) &= \mathrm{N}_{F/\mathbb{Q}}^{2r-2+\mathrm{w}}(i\mathfrak{d})\cdot(\chi^2\omega^2)(\varpi_{\mathfrak{d}}^{-1})\varepsilon\Big(\pi\otimes\chi, r-\frac{1}{2}\Big)\times \\ &\times\prod_{v\in S_p}\frac{E\left(\tilde{\pi}_v, \chi_v, r\right)}{E\left(\tilde{\pi}_v, \chi_v^{-1}\omega_v^{-2}, 2-r-\mathrm{w}\right)}\mathcal{L}_p(\tilde{\pi}, \chi^{-1}\omega^{-2}\chi_{\mathrm{cyc}}^{1-r-\mathrm{w}}). \end{aligned} \tag{5.17}$$

For $v \in S_p$, since χ_v is highly ramified, it follows that the characters $\chi_v\nu_v$ and $\chi_v\omega_{\pi_v}\nu_v^{-1}$ are all ramified with the same conductor which equals the conductor c_{χ_v} of χ_v. So

$$\begin{aligned} E(\tilde{\pi}_v, \chi_v, r) &= q_v^{rc_{\chi_v}}(\chi_v\nu_v)(\varpi_v^{\delta_v})q_v^{-(c_{\chi_v}+\delta_v/2)}\tau(\chi_v\nu_v, \psi_v), \\ E(\tilde{\pi}_v, \chi_v^{-1}\omega_v^{-2}, 2-r-\mathrm{w}) &= q_v^{(2-r-\mathrm{w})c_{\chi_v}}(\chi_v^{-1}\omega_v^{-2}\nu_v)(\varpi_v^{\delta_v})q_v^{-(c_{\chi_v}+\delta_v/2)}\tau(\chi_v^{-1}\omega_v^{-2}\nu_v, \psi_v). \end{aligned}$$

Hence

$$\begin{aligned} &\frac{E(\tilde{\pi}_v, \chi_v, r)}{E(\tilde{\pi}_v, \chi_v^{-1}\omega_v^{-2}, 2-r-\mathrm{w})} = q_v^{(2r+\mathrm{w}-2)c_{\chi_v}}(\chi_v\omega_v)(\varpi_v)^{2\delta_v}\frac{\tau(\chi_v\nu_v, \psi_v)}{\tau(\chi_v^{-1}\omega_v^{-2}\nu_v, \psi_v)} \\ &= q_v^{(2r+\mathrm{w}-2)c_{\chi_v}}(\chi_v\omega_v)(\varpi_v)^{2\delta_v}\tau(\chi_v\nu_v, \psi_v)\tau(\chi_v\omega_v^2\nu_v^{-1}, \psi_v)q_v^{-(c_{\chi_v}+\delta_v)}(\chi_v\omega_v^2\nu_v^{-1})(-1) \qquad \text{(by (5.11))}. \end{aligned}$$

Since χ_v is highly ramified, by (5.9) and (5.13), we have

$$\tau(\chi_v\nu_v, \psi_v)\tau(\chi_v\omega_v^2\nu_v^{-1}, \psi_v) = \tau(\chi_v\omega_v, \psi_v)^2.$$

Therefore,

$$\prod_{v\in S_p}\frac{E(\tilde{\pi}_v, \chi_v, r)}{E(\tilde{\pi}_v, \chi_v^{-1}\omega_v^{-2}, 2-r-\mathrm{w})} = (\chi\nu^{-1})(-1)\mathrm{N}_{F/\mathbb{Q}}^{2r+\mathrm{w}-2}(\mathfrak{c}_\chi)\prod_{v\in S_p} q_v^{-(c_{\chi_v}+\delta_v)}(\chi_v^2\omega_v^2)(\varpi_v^{\delta_v})\tau(\chi_v\omega_v, \psi_v)^2. \tag{5.18}$$

Lemma 5.3.4. *Suppose π satisfies* (5.14). *Let χ be a character of* $\mathrm{Gal}_{p\infty}$ *such that χ_v is highly ramified for all $v \in S_p$. Denote by $\mathfrak{c}_\chi = \prod_{v\in S_p} v^{c_{\chi_v}}$ the conductor of χ, one has*

$$\begin{aligned} \varepsilon\Big(\pi\otimes\chi, r-\frac{1}{2}\Big) &= \tilde{\varepsilon}_{\pi\otimes\omega^{-1}}\cdot(\nu\omega^{-1})(-1)\cdot(\chi\omega)(\varpi_{\mathfrak{n}})\mathrm{N}_{F/\mathbb{Q}}^{1-r-\mathrm{w}/2}(\mathfrak{n})\cdot\mathrm{N}_{F/\mathbb{Q}}^{1-2r-\mathrm{w}}(\mathfrak{c}_\chi)\times \\ &\times\prod_{v\nmid p\infty} q_v^{(2-2r-\mathrm{w})\delta_v}(\chi_v\omega_v)(\varpi_v)^{2\delta_v}\prod_{v\in S_p} q_v^{(1-2r-\mathrm{w})\delta_v}\tau(\chi_v^{-1}\omega_v^{-1}, \psi_v)^2. \end{aligned}$$

Proof. For an infinite place v, since π_v is a discrete series, the value $\varepsilon(\pi_v, s, \psi_v)$ is independent of s and of twisting π_v by any character of $F_v^\times$ (see [Kna, (3.7)]). So

$$\varepsilon\Big(\pi_v\otimes\chi_v, r-\frac{1}{2}, \psi_v\Big) = \varepsilon\Big(\pi_v\otimes\omega_v^{-1}, \frac{1-\mathrm{w}}{2}, \psi_v\Big). \tag{5.19}$$

If v is a finite place not above p, then χ_v and ω_v are unramified since they are characters of $\mathrm{Gal}_{p\infty}$, so $\pi_v \otimes \omega_v^{-1}$ and π_v have the same conductor $c_{\pi_v\otimes\omega_v^{-1}} = c_{\pi_v}$. We have

$$\begin{aligned}
&\varepsilon\Big(\pi_v \otimes \chi_v, r-\frac{1}{2}, \psi_v\Big) = \varepsilon\Big(\pi_v \otimes \omega_v^{-1} \otimes \chi_v\omega_v, r-\frac{1}{2}, \psi_v\Big)\\
&= \varepsilon\Big(\pi_v \otimes \omega_v^{-1}, r-\frac{1}{2}, \psi_v\Big)(\chi_v\omega_v)(\varpi_v)^{c_{\pi_v}+2\delta_v} \text{ (by (5.7))}\\
&= \varepsilon\Big(\pi_v \otimes \omega_v^{-1}, \frac{1-\mathrm{w}}{2}, \psi_v\Big)q_v^{(1-r-\mathrm{w}/2)(c_{\pi_v}+2\delta_v)}(\chi_v\omega_v)(\varpi_v)^{c_{\pi_v}+2\delta_v} \text{ (by (5.8))}.
\end{aligned} \tag{5.20}$$

For $v \in S_p$, since χ_v is highly ramified, applying (5.13) with $\alpha = \omega|\cdot|_v^{\mathrm{w}}$ and $\beta = \omega$, we get

$$\begin{aligned}
\varepsilon\Big(\pi_v \otimes \chi_v, r-\frac{1}{2}, \psi_v\Big) &= \varepsilon\Big(\chi_v\omega_v, r-\frac{1}{2}+\mathrm{w}, \psi_v\Big)\varepsilon\Big(\chi_v\omega_v, r-\frac{1}{2}, \psi_v\Big)\\
&= q_v^{(1-2r-\mathrm{w})(c_{\chi_v}+\delta_v)}\tau(\chi_v^{-1}\omega_v^{-1}, \psi_v)^2 \qquad \text{(by (5.9))},
\end{aligned} \tag{5.21}$$

note that $c_{\chi_v\omega_v} = c_{\chi_v}$ since χ_v is highly ramified.

We claim that

$$\varepsilon\Big(\pi_v \otimes \omega_v^{-1}, \frac{1-\mathrm{w}}{2}, \psi_v\Big) = \begin{cases} (\nu_v\omega_v^{-1})(-1) & \text{if } v \in S_p\setminus \mathrm{St}_p \text{ or } v \in \mathrm{St}_p \text{ and } \nu_v\omega_v^{-1} \text{ is ramified,}\\ -(\nu_v\omega_v^{-1})(\varpi_v)q_v^{\mathrm{w}/2} & \text{if } v \in \mathrm{St}_p \text{ and } \nu_v\omega_v^{-1} \text{ is unramified.}\end{cases} \tag{5.22}$$

Indeed, if $v \in S_p\setminus\mathrm{St}_p$, then $\pi_v = \pi(\nu_v\,|\cdot|_v^{1/2}, \omega_{\pi_v}\nu_v^{-1}\,|\cdot|_v^{-1/2})$ is a principal series. So $\pi_v \otimes \omega_v^{-1} = \pi(\nu_v\omega_v^{-1}\,|\cdot|_v^{1/2}, \omega_v\nu_v^{-1}\,|\cdot|_v^{\mathrm{w}-1/2})$. By [JL, Proposition 3.5], we get

$$\begin{aligned}
\varepsilon\Big(\pi_v \otimes \omega_v^{-1}, \frac{1-\mathrm{w}}{2}, \psi_v\Big) &= \varepsilon\Big(\nu_v\omega_v^{-1}\,|\cdot|_v^{1/2}, \frac{1-\mathrm{w}}{2}, \psi_v\Big)\varepsilon\Big(\omega_v\nu_v^{-1}\,|\cdot|_v^{\mathrm{w}-1/2}, \frac{1-\mathrm{w}}{2}, \psi_v\Big)\\
&= \varepsilon\Big(\nu_v\omega_v^{-1}, 1-\frac{\mathrm{w}}{2}, \psi_v\Big)\varepsilon\Big(\omega_v\nu_v^{-1}, \frac{\mathrm{w}}{2}, \psi_v\Big) = (\nu_v\omega_v^{-1})(-1) \quad \text{(by (5.10))}.
\end{aligned}$$

Consider $v \in \mathrm{St}_p$, then $\pi_v \otimes \omega_v^{-1} = \nu_v\omega_v^{-1}\,\mathrm{St}_v$. If $\nu_v\omega_v^{-1}$ is ramified, by [JL, Proposition 3.6], we have

$$\begin{aligned}
\varepsilon\Big(\pi_v \otimes \omega_v^{-1}, \frac{1-\mathrm{w}}{2}, \psi_v\Big) &= \varepsilon\Big(\nu_v\omega_v^{-1}\,|\cdot|_v^{1/2}, \frac{1-\mathrm{w}}{2}, \psi_v\Big)\varepsilon\Big(\nu_v\omega_v^{-1}\,|\cdot|_v^{-1/2}, \frac{1-\mathrm{w}}{2}, \psi_v\Big)\\
&= \varepsilon\Big(\nu_v\omega_v^{-1}, 1-\frac{\mathrm{w}}{2}, \psi_v\Big)\varepsilon\Big(\nu_v^{-1}\omega_v, \frac{\mathrm{w}}{2}, \psi_v\Big) \qquad (\text{since } \nu_v^2 = \omega_{\pi_v} = \omega_v^2\,|\cdot|_v^{\mathrm{w}})\\
&= (\nu_v\omega_v^{-1})(-1) \qquad \text{(by (5.10))}.
\end{aligned}$$

If $v \in \mathrm{St}_p$ and $\nu_v\omega_v^{-1}$ is unramified, again by [JL, Proposition 3.6], we obtain

$$\begin{aligned}
\varepsilon\Big(\pi_v \otimes \omega_v^{-1}, \frac{1-\mathrm{w}}{2}, \psi_v\Big) &= \varepsilon\Big(\nu_v\omega_v^{-1}\,|\cdot|_v^{1/2}, \frac{1-\mathrm{w}}{2}, \psi_v\Big)\varepsilon\Big(\nu_v\omega_v^{-1}\,|\cdot|_v^{-1/2}, \frac{1-\mathrm{w}}{2}, \psi_v\Big)\frac{L\big(\nu_v^{-1}\omega_v\,|\cdot|_v^{-1/2}, \frac{1+\mathrm{w}}{2}\big)}{L\big(\nu_v\omega_v^{-1}\,|\cdot|_v^{-1/2}, \frac{1-\mathrm{w}}{2}\big)}\\
&= (\nu_v\omega_v^{-1})(-1)\cdot\frac{(1-(\nu_v^{-1}\omega_v)(\varpi_v)q_v^{-\mathrm{w}/2})^{-1}}{(1-(\nu_v\omega_v^{-1})(\varpi_v)q_v^{\mathrm{w}/2})^{-1}}\\
&= -(\nu_v\omega_v^{-1})(\varpi_v)q_v^{\mathrm{w}/2}.
\end{aligned}$$

The lemma follows from (5.19),(5.20), (5.21) and (5.22). □

From (5.17), (5.18) and Lemma 5.3.4, we obtain

$$\mathcal{L}_p(\tilde{\pi}, \chi\chi_{\mathrm{cyc}}^{r-1}) = \tilde{\varepsilon}_{\pi\otimes\omega^{-1}}\cdot(\chi\omega)(-\varpi_{\mathfrak{n}})\mathrm{N}_{F/\mathbb{Q}}^{1-r-\mathrm{w}/2}(-\mathfrak{n})\mathcal{L}_p(\tilde{\pi}, \chi^{-1}\omega^{-2}\chi_{\mathrm{cyc}}^{1-r-\mathrm{w}}),$$

where we used the formula $q_v^{-(c_{\chi_v}+\delta_v)}\tau(\chi_v\omega_v, \psi_v)\tau(\chi_v^{-1}\omega_v^{-1}, \psi_v) = (\chi_v\omega_v)(-1)$ for any $v \in S_p$ (see (5.11)) with the note that $c_{\chi_v\omega_v} = c_{\chi_v}$ since χ_v is highly ramified.

We get the desired formula (5.15) by replacing χ by $\chi\omega_p^{1-r}$ with the note that $\chi_{\mathrm{cyc}}\omega_p^{-1} = \langle\cdot\rangle_p$ is an even character. □

We are now ready to prove a functional equation of p-adic L-functions attached to automorphic representations.

Theorem 5.3.5. *The sign* $\tilde{\varepsilon}_{\pi_\lambda\otimes\omega^{-1}}$ *of* $\tilde{\pi}_\lambda$ *is independent of the cohomological weight* $\lambda \in \mathcal{X}(\tilde{\pi})$. *For any* $\lambda \in \mathcal{X}(\tilde{\pi})$, *any continuous character* $f : \mathrm{Gal}_{\mathrm{cyc}} \to L^\times$ *and any finite order character* $\chi : \mathrm{Gal}_{p\infty} \to L^\times$, *we have*

$$\mathcal{L}_p(\lambda, \chi\cdot f) = \tilde{\varepsilon}_{\pi\otimes\omega^{-1}} \cdot (\chi\omega\omega_p^{\mathrm{w}/2} f)(-\varpi_{\mathfrak{n}})\langle\mathfrak{n}\rangle_p^{\mathrm{w}_\lambda/2}\mathcal{L}_p(\lambda, \omega^{-2}\chi_{\mathrm{cyc}}^{-\mathrm{w}_\lambda}(\chi\cdot f)(\cdot)^{-1}). \tag{5.23}$$

Proof. We prove the assertion about the functional equation. We can assume that λ is cohomological having very non-critical slope since such weights are Zariski dense in $\mathcal{X}(\tilde{\pi})$. Fix χ and regard both sides of (5.23) as functions of f. Since the distributions $\mathcal{L}_p(\lambda, \chi\cdot)$ and $\mathcal{L}_p(\lambda, \omega^{-2}\chi_{\mathrm{cyc}}^{-\mathrm{w}_\lambda}\chi^{-1}\cdot)$ in $\mathcal{D}(\mathrm{Gal}_{\mathrm{cyc}}, L)$ have growth at most $\sum_{v\in S_p} e_v h_{\tilde{\pi}_v} < \min_{\sigma\in\Sigma}(k_{\lambda,\sigma} - 1)$, by [Vis, Theorem 2.3, Lemma 2.10] it suffices to check (5.23) for f is of the form $\chi'\langle\cdot\rangle_p^{r-1}$, where r is a critical integer for $(k_\lambda, \mathrm{w}_\lambda)$ and χ' is a finite order character of $\mathrm{Gal}_{\mathrm{cyc}}$. This is exactly formula (5.15) applied to $\chi\chi'$, except that $\tilde{\varepsilon}_{\pi\otimes\omega^{-1}}$ is replaced by $\tilde{\varepsilon}_{\pi_\lambda\otimes\omega^{-1}}$.

For the assertion about the sign, we follow the proof of [BDJ, Theorem 6.4]. The key ingredient is the following function:

$$\varepsilon(\lambda, s) = \frac{\langle\mathfrak{n}\rangle_p^{1-s-\mathrm{w}_\lambda/2}\mathcal{L}_p(\lambda, \chi\omega^{-1}\omega_p^{-\mathrm{w}/2}\langle\cdot\rangle_p^{s-1})}{\chi(-\varpi_{\mathfrak{n}})\mathcal{L}_p(\lambda, \chi^{-1}\omega^{-1}\omega_p^{-\mathrm{w}/2}\langle\cdot\rangle_p^{1-\mathrm{w}_\lambda-s})}$$

which is well-defined, non-identically zero and meromorphic in the variables $(\lambda, s) \in \mathcal{X}(\tilde{\pi}) \times \mathcal{O}_{\mathbb{C}_p}$. Moreover, by (5.23), $\varepsilon(\lambda, s) = \tilde{\varepsilon}_{\pi_\lambda\otimes\omega^{-1}} \in \{\pm 1\}$ for any cohomological weight $\lambda \in \mathcal{X}(\tilde{\pi})$ having very non-critical slope such that $\varepsilon(\lambda, s)$ is well-defined. The Zariski density of such weights deduces that $\varepsilon(\lambda, s)$ is constant with value $\tilde{\varepsilon} \in \{\pm 1\}$, independent of χ and λ. ☐

Definition 5.3.6. *The cyclotomic (resp. multi-variable) p-adic L-function attached to* $\tilde{\pi}$ *is defined by*

$$L_p(\tilde{\pi}, s) = \mathcal{L}_p(\tilde{\pi}, \omega^{-1}\omega_p^{-\mathrm{w}/2}\langle\cdot\rangle_p^{s-1}), \text{ resp. } L_p(\lambda, s) = \mathcal{L}_p(\lambda, \omega^{-1}\omega_p^{-\mathrm{w}/2}\langle\cdot\rangle_p^{s-1}), \text{ where } s \in \mathcal{O}_{\mathbb{C}_p}, \lambda \in \mathcal{X}(\tilde{\pi}).$$

By (5.23), we have

$$L_p(\tilde{\pi}_\lambda, s) = \tilde{\varepsilon}_{\pi\otimes\omega^{-1}}\langle\mathfrak{n}\rangle_p^{s+\mathrm{w}_\lambda/2-1}L_p(\tilde{\pi}_\lambda, 2 - \mathrm{w}_\lambda - s) \tag{5.24}$$

as analytic functions in s. By [BDJ, (6.5)], for $z \in \mathcal{O}_{\mathbb{C}_p}$ and $(k_\lambda, \mathrm{w}_\lambda) \in \mathcal{X}^{\mathrm{an}}(\tilde{\pi})$ such that $(k_\lambda, \mathrm{w}_\lambda\langle\cdot\rangle_p^{2z}) \in \mathcal{X}^{\mathrm{an}}(\tilde{\pi})$, one has

$$L_p((k_\lambda, \mathrm{w}_\lambda + 2z), s) = L_p((k_\lambda, \mathrm{w}_\lambda), s + z). \tag{5.25}$$

Proposition 5.3.7 ([BDJ, Remark 4.11])**.** *For any regular non-critical cohomological p-refinement* $\tilde{\pi}$ *and any finite order character* $\chi : \mathrm{Gal}_{p\infty} \to L^\times$, *one has*

$$\mathcal{L}_p(\widetilde{\pi\otimes\chi}, \cdot) = \mathcal{L}_p(\tilde{\pi}, \chi\cdot) \text{ in } \mathcal{D}(\mathrm{Gal}_{\mathrm{cyc}}, L).$$

Proof. We will prove this assertion for the family of weights in $\mathcal{X}(\tilde{\pi})$. By analyticity it suffices to prove for cohomological weights $\lambda \in \mathcal{X}(\tilde{\pi})$ such that $\tilde{\pi}_\lambda$ has very non-critical slope since such weights are very Zariski dense in $\mathcal{X}(\tilde{\pi})$. So we can assume that $\tilde{\pi}$ has such property.

Since the infinity part of π is a discrete series, and since the discrete series are invariant under twisting by the sign character, it follows that $\pi\otimes\chi$ has the same cohomological weight as π for any finite order character χ of $\mathrm{Gal}_{p\infty}$. Moreover, $\pi_v\otimes\chi_v$ has the same slope as π_v for any $v \in S_p$ since $\chi_v(\varpi_v)$ is a p-adic unit. By [Vis, Theorem 2.3, Lemma 2.10] it suffices to check

$$\mathcal{L}_p(\widetilde{\pi\otimes\chi}, \chi'\langle\cdot\rangle_p^{r-1}) = \mathcal{L}_p(\tilde{\pi}, \chi\chi'\langle\cdot\rangle_p^{r-1})$$

for any $r \in \mathbb{Z}$ critical for the weight of $\tilde{\pi}$ and any finite order character χ' of $\mathrm{Gal}_{\mathrm{cyc}}$. By [BDJ, Theorem 4.2], we have

$$\begin{aligned}
&\mathcal{L}_p(\widetilde{\pi\otimes\chi}, \chi'\langle\cdot\rangle_p^{r-1}) = \mathcal{L}_p(\widetilde{\pi\otimes\chi}, \chi'\omega_p^{1-r}\chi_{\mathrm{cyc}}^{r-1}) \\
&= \frac{\mathrm{N}_{F/\mathbb{Q}}^{r-1}(i\mathfrak{d})\cdot(\chi'\omega_p^{1-r})(\varpi_{\mathfrak{d}}^{-1})}{\Omega_{\widetilde{\pi\otimes\chi}}^{\chi'_\infty}} L\Big(\pi\otimes\chi\otimes\chi'\omega_p^{1-r}, r - \frac{1}{2}\Big)\prod_{v\in S_p} E(\widetilde{\pi_v\otimes\chi_v}, \chi'_v\omega_{p,v}^{1-r}, r) \\
&= \frac{\mathrm{N}_{F/\mathbb{Q}}^{r-1}(i\mathfrak{d})\cdot(\chi\chi'\omega_p^{1-r})(\varpi_{\mathfrak{d}}^{-1})}{\Omega_{\tilde{\pi}}^{\chi_\infty\chi'_\infty}} L\Big(\pi\otimes\chi\chi'\omega_p^{1-r}, r - \frac{1}{2}\Big)\prod_{v\in S_p} E(\tilde{\pi}_v, \chi_v\chi'_v\omega_{p,v}^{1-r}, r) \\
&= \mathcal{L}_p(\tilde{\pi}, \chi\chi'\omega_p^{1-r}\chi_{\mathrm{cyc}}^{r-1}) = \mathcal{L}_p(\tilde{\pi}, \chi\chi'\langle\cdot\rangle_p^{r-1}),
\end{aligned}$$

where we used the equality $E(\widetilde{\pi_v \otimes \chi_v}, \chi'_v \omega_{p,v}^{1-r}, r) = E(\tilde{\pi}_v, \chi_v \chi'_v \omega_{p,v}^{1-r}, r)$ which is obvious from the definition of E-factors and the equality $\Omega^{\chi'_\infty}_{\widetilde{\pi\otimes\chi}} = \chi_f(\varpi_{\mathfrak{d}})\Omega^{\chi_\infty\chi'_\infty}_{\tilde{\pi}}$ followed from [BDJ, Proposition 1.15]. □

Corollary 5.3.8. *For any $\lambda \in \mathcal{X}(\tilde{\pi})$, one has*

$$L_p(\tilde{\pi}_\lambda, s) = \mathcal{L}_p(\widetilde{\pi_\lambda \otimes \omega^{-1}}, \omega_p^{-\mathrm{w}/2}\langle\cdot\rangle_p^{s-1}). \tag{5.26}$$

Proof. This is obvious from the above proposition and the definition of cyclotomic p-adic L-functions. □

Remark 5.3.9. If $\omega_\pi = \omega^2|\cdot|^{\mathrm{w}}$ where ω is any Hecke character (not only characters of $\mathrm{Gal}_{p\infty}$), we take the right hand side of (5.26) for the definition of the cyclotomic p-adic L-functions $L_p(\tilde{\pi}_\lambda, s)$.

Chapter 6

The trivial zero conjecture at the central critical point

We keep the hypotheses in Chapter 5, §5.3.

Denote by $E \subset S_p$ the set of places v at which the local interpolation E-factor of $L_p(\tilde{\pi}, s)$ vanishes at the central point $\frac{2-\mathrm{w}}{2}$. By Corollary 5.3.8 and [BDJ, Corollary 6.2], E consists of places $v \in \mathrm{St}_p$ such that $\nu_v \omega_v^{-1}$ is unramified and $\varepsilon\left(\pi_v \otimes \omega_v^{-1}, \frac{1-\mathrm{w}}{2}, \psi_v\right) = -1$. Our main goal is to prove the following result:

Theorem 6.0.1 (Trivial zero conjecture at the central critical point)**.** *Suppose π satisfies* (5.14)*. The p-adic L-function $L_p(\tilde{\pi}, s)$ has order of vanishing at least $e = |E|$ at $\frac{2-\mathrm{w}}{2}$ and*

$$\frac{L_p^{(e)}\left(\tilde{\pi}, \frac{2-\mathrm{w}}{2}\right)}{e!} = \mathcal{L}(\widetilde{\pi \otimes \omega^{-1}}) \frac{\omega(\varpi_{\mathfrak{d}}) L\left(\pi \otimes \omega^{-1}, \frac{1-\mathrm{w}}{2}\right)}{\mathrm{N}_{F/\mathbb{Q}}^{\mathrm{w}/2}(i\mathfrak{d}) \Omega_{\tilde{\pi}}^{\omega_\infty \omega_{p,\infty}^{\mathrm{w}/2}}} \cdot 2^{|\{v \in \mathrm{St}_p \setminus E,\, \nu_v \omega_v^{-1} \text{ is unramified}\}|} \times$$

$$\times \prod_{v \in S_p, \pi_v \otimes \omega_v^{-1} \text{ is unramified}} \left(1 - \frac{q_v^{-\mathrm{w}/2}}{(\nu_v \omega_v^{-1})(\varpi_v)}\right)^2 \prod_{v \in S_p, c_{\nu_v \omega_v^{-1}} > 0} q_v^{-\frac{\mathrm{w} \cdot c_{\nu_v \omega_v^{-1}} + \delta_v}{2}} (\nu_v \omega_v^{-1})(\varpi_v)^{\delta_v} \tau(\nu_v \omega_v^{-1}, \psi_v),$$

where $\mathcal{L}(\widetilde{\pi \otimes \omega^{-1}})$ is the Fontaine-Mazur $\mathcal{L}$-invariant (see [BDJ, Definition 5.3]).

This is a generalization of [BDJ, Theorem 7.1]. By Corollary 5.3.8 and [BDJ, Proposition 1.15], we can assume that ω is trivial. From now on we suppose that $\omega = \mathbb{1}$.

Lemma 6.0.2 ([BDJ, Lemma 7.2])**.** *Let $S = S_p \backslash \{v\}$ for some $v \in \mathrm{St}_p$ such that π_v is an unramified twist of the Steinberg representation. After possibly shrinking $\mathcal{X}(\tilde{\pi})$, for any cohomological $\lambda \in \mathcal{X}_S \cap \mathcal{X}(\tilde{\pi})$, the local representation $\pi_{\lambda,v}$ is also an unramified twist of the Steinberg representation.*

Given $u \in 4\mathcal{O}_{\mathbb{C}_p}$ and $x = (x_v)_{v \in E} \in (2p\mathcal{O}_{\mathbb{C}_p})^E$. For any subset $S \subset E$ we let $x_S = (x_v)_{v \in S}$ and define $\lambda_{x,u}^S = (k_\lambda, \mathrm{w}_\lambda) \in \mathcal{X}^{\mathrm{an}}(\tilde{\pi})$ by

$$\mathrm{w}_\lambda = \mathrm{w} - u \text{ and } k_{\lambda,\sigma} = \begin{cases} k_\sigma & , \text{ for } \sigma \in \Sigma_{S_p \backslash E}, \\ k_\sigma + u & , \text{ for } \sigma \in \Sigma_{E \backslash S}, \\ k_\sigma + x_v & , \text{ for } \sigma \in \Sigma_S. \end{cases}$$

Letting $\mathbb{L}_p(x, u) = \langle \mathfrak{n} \rangle_p^{u/4} L_p(\lambda_{x,u}^E, \frac{2-\mathrm{w}}{2})$, (5.24) and (5.25) imply that

$$\mathbb{L}_p(x, -u) = \tilde{\varepsilon} \cdot \mathbb{L}_p(x, u), \text{ with } \tilde{\varepsilon} = (-1)^e \varepsilon_\pi.$$

Then we write $\mathbb{L}_p(x, u) = \sum_{i \geq 0} A_i(x) u^i$, where $A_i(x)$ is analytic in $(x_v)_{v \in E}$ and the sum runs over i even (resp. odd) if $\tilde{\varepsilon} = 1$ (resp. $\tilde{\varepsilon} = -1$). By (5.25), we have

$$L_p(\tilde{\pi}, s) = \langle \mathfrak{n} \rangle_p^{(2s+\mathrm{w}-2)/4} \mathbb{L}_p((0)_{v \in E}, 2 - \mathrm{w} - 2s).$$

Since $\lambda_{x,u}^S \in \mathcal{X}^{\mathrm{an}}(\tilde{\pi}) \cap \mathcal{X}'_{S \sqcup (S_p \backslash E)}$, we let

$$\mathbb{L}_S(x_S, u) = \langle \mathfrak{n} \rangle_p^{u/4} L_{S \sqcup (S_p \backslash E)}\left(\lambda_{x,u}^S, \mathbb{1}, \frac{2-\mathrm{w}}{2}\right), \tag{6.1}$$

where $L_{S\sqcup(S_p\setminus E)}$ is the improved p-adic L-function (see [BDJ, §4.3]).

By [BDJ] we know that $\mathbb{L}_p(x,u)=\mathbb{L}_E(x,u)$ and by [BDJ, Theorem 4.13(i)], for any $S\subset E$ we have

$$\mathbb{L}_p((x_S,(u)_{v\in E\setminus S}),u)=\mathbb{L}_S(x_S,u)\prod_{v\in E\setminus S}(1-\nu_v^{-1}(\lambda_{x,u}^S)(\varpi_v)q_v^{-\mathrm{w}/2}), \tag{6.2}$$

where $\nu_v(\lambda^S_{x,u})$ is the refinement at v of the weight $\lambda^S_{x,u}$.

Writing the power series expansion

$$A_i(x)=\sum_{n\in\mathbb{Z}^E_{\geq 0}}a_i(n)x^n,\text{ where } x^n=\prod_{v\in E}x_v^{n_v}\text{ with } n=(n_v)_{v\in E}.$$

For a multi-index $n=(n_v)_{v\in E}$ we denote $|n|=\sum\limits_{v\in E}n_v$ and $\|n\|=|\{v\in E\,|n_v\neq 0\}|$.

Proposition 6.0.3 ([BDJ, Proposition 7.5]). *i) If $\|n\|<e-i$, then $a_i(n)=0$.*

ii) For any $i<e$, we have $\sum\limits_{|n|=e-i}a_i(n)=0$.

The following result generalizes [BDJ, Lemma 7.7].

Lemma 6.0.4. *Keeping the hypotheses and notations of Theorem 6.0.1 and assuming in addition that ω is trivial (so that $\omega_\pi=|\cdot|_F^{\mathrm{w}}$), the analytic function $\mathbb{L}_p((u)_{v\in E},u)$ vanishes at $u=0$ to order at least e and*

$$\frac{(-2)^e}{e!}\cdot\frac{d^e}{du^e}\mathbb{L}_p((u)_{v\in E},u)|_{u=0}=\mathcal{L}(\tilde\pi)\cdot\frac{L\big(\pi,\frac{1-\mathrm{w}}{2}\big)}{\mathrm{N}_{F/\mathbb{Q}}^{\mathrm{w}/2}(i\mathfrak{d})\Omega_{\tilde\pi}^{\omega_{p,\infty}^{\mathrm{w}/2}}}\cdot 2^{|\{v\in\mathrm{St}_p\setminus E,c_{\nu_v}=0\}|}\times$$
$$\times\prod_{v\in S_p,\pi_v\text{ is unramified}}\Big(1-\frac{q_v^{-\mathrm{w}/2}}{\nu_v(\varpi_v)}\Big)^2\prod_{v\in S_p,c_{\nu_v}>0}q_v^{-\frac{\mathrm{w}\cdot c_{\nu_v}+\delta_v}{2}}\nu_v(\varpi_v)^{\delta_v}\tau(\nu_v,\psi_v).$$

Proof. Since $\lambda^{\varnothing}_{(0),0}=(k,\mathrm{w})$, by (5.22), for any $v\in E$ we have

$$\nu_v(\lambda^{\varnothing}_{(0),0})(\varpi_v)=q_v^{-\mathrm{w}/2}.$$

Combining with (6.2), we get

$$\mathbb{L}_p((u)_{v\in E},u)=\mathbb{L}_\varnothing(u)\prod_{v\in E}\Big(1-\frac{\nu_v(\lambda^\varnothing_{(0),0})(\varpi_v)}{\nu_v(\lambda^\varnothing_{(u),u})(\varpi_v)}\Big)=\mathbb{L}_\varnothing(u)\prod_{v\in E}\Big(\frac{\nu_v(\lambda^\varnothing_{(u),u})(\varpi_v)-\nu_v(\lambda^\varnothing_{(0),0})(\varpi_v)}{\nu_v(\lambda^\varnothing_{(u),u})(\varpi_v)}\Big).$$

Since each interpolation factor indexed by a place $v\in E$ vanishes at $u=0$, we deduce that the order of vanishing of $\mathbb{L}_p((u)_{v\in E},u)$ at $u=0$ is at least e. Differentiating e times at $u=0$, we deduce from (6.1) that

$$\frac{d^e}{du^e}\mathbb{L}_p((u)_{v\in E},u)|_{u=0}=e!L_{S_p\setminus E}\Big(\tilde\pi,\mathbb{1},\frac{2-\mathrm{w}}{2}\Big)\times$$
$$\times\prod_{v\in E}\frac{1}{\nu_v(k,\mathrm{w})(\varpi_v)}\frac{d}{du}\nu_v((k,\mathrm{w})+u((1)_{\sigma\in\Sigma_E},(0)_{\sigma\in\Sigma_{S_p\setminus E}},-1))(\varpi_v).$$

By [BDJ, Prop. 5.2, Def. 5.3], we obtain

$$\frac{d^e}{du^e}\mathbb{L}_p((u)_{v\in E},u)|_{u=0}=\mathcal{L}(\tilde\pi)\frac{e!}{(-2)^e}L_{S_p\setminus E}\Big(\tilde\pi,\mathbb{1},\frac{2-\mathrm{w}}{2}\Big). \tag{6.3}$$

By [BDJ, Theorem 4.13(ii)] and the definition of E-factors in Theorem 5.3.2, we have

$$L_{S_p\setminus E}\Big(\tilde\pi,\mathbb{1},\frac{2-\mathrm{w}}{2}\Big)=\frac{L\big(\pi,\frac{1-\mathrm{w}}{2}\big)}{\mathrm{N}_{F/\mathbb{Q}}^{\mathrm{w}/2}(i\mathfrak{d})\Omega_{\tilde\pi}^{\omega_{p,\infty}^{\mathrm{w}/2}}}\prod_{v\in S_p,c_{\nu_v}>0}q_v^{\frac{2-\mathrm{w}}{2}c_{\nu_v}}\nu_v(\varpi_v)^{\delta_v}q_v^{-c_{\nu_v}-\delta_v/2}\tau(\nu_v,\psi_v)\times$$
$$\times\prod_{v\in S_p,\pi_v\text{ is unramified}}\Big(1-\frac{q_v^{-\mathrm{w}/2}}{\nu_v(\varpi_v)}\Big)\prod_{v\in S_p\setminus E,\nu_v\text{ is unramified}}\Big(1-\frac{q_v^{-\mathrm{w}/2}}{\nu_v(\varpi_v)}\Big).$$

Consider $v \in S_p \backslash E$ such that ν_v is unramified. If $v \in \mathrm{St}_p \backslash E$, then $\varepsilon\big(\pi_v, \frac{1-\mathrm{w}}{2}, \psi_v\big) = 1$. By (5.22) we deduce that $\nu_v(\varpi_v) = -q_v^{-\mathrm{w}/2}$. If $v \in S_p \backslash \mathrm{St}_p$, then π_v is a principal series, hence π_v is an unramified principal series since ν_v and ω_{π_v} are unramified. Therefore,

$$L_{S_p \backslash E}\Big(\tilde{\pi}, \mathbb{1}, \frac{2-\mathrm{w}}{2}\Big) = \frac{L\big(\pi, \frac{1-\mathrm{w}}{2}\big)}{\mathrm{N}_{F/\mathbb{Q}}^{\mathrm{w}/2}(i\mathfrak{d})\Omega_{\tilde{\pi}}^{\omega_{p,\infty}^{\mathrm{w}/2}}} \cdot 2^{|\{v \in \mathrm{St}_p \backslash E, c_{\nu_v}=0\}|} \prod_{v \in S_p, c_{\nu_v}>0} q_v^{-\frac{\mathrm{w}\cdot c_{\nu_v}+\delta_v}{2}} \nu_v(\varpi_v)^{\delta_v} \tau(\nu_v, \psi_v) \times$$
$$\times \prod_{v \in S_p, \pi_v \text{ is unramified}} \Big(1 - \frac{q_v^{-\mathrm{w}/2}}{\nu_v(\varpi_v)}\Big)^2. \tag{6.4}$$

We get the desired formula of eth Taylor coefficient from (6.3) and (6.4). □

Proof of Theorem 6.0.1

Recall that we have assumed $\omega = \mathbb{1}$. Since $L_p(\tilde{\pi}, s) = \langle \mathfrak{n} \rangle_p^{(2s+\mathrm{w}-2)/4} \mathbb{L}_p((0)_{v \in E}, 2 - \mathrm{w} - 2s)$, we have

$$L_p^{(m)}(\tilde{\pi}, s)|_{s=\frac{2-\mathrm{w}}{2}} = \sum_{k=0}^{m} \binom{m}{k} \Big(\frac{1}{2}\log_p \langle \mathfrak{n} \rangle_p\Big)^{m-k} (-2)^k \frac{d^k}{du^k} \mathbb{L}_p((0)_{v \in E}, u)|_{u=0}.$$

The expansion $\mathbb{L}_p(x, u) = \sum_{i \geq 0} A_i(x) u^i$ yields $\frac{d^k}{du^k} \mathbb{L}_p((0)_{v \in E}, u)|_{u=0} = k! A_k((0)_{v \in E}) = k! a_k((0) v \in E)$. By Proposition 6.0.3 these derivatives vanish for any $k < e$, so the order of vanishing of $L_p(\tilde{\pi}, s)$ at $s = \frac{2-\mathrm{w}}{2}$ is at least e and

$$L_p^{(e)}(\tilde{\pi}, s)|_{s=\frac{2-\mathrm{w}}{2}} = (-2)^e \frac{d^e}{du^e} \mathbb{L}_p((0)_{v \in E}, u)|_{u=0} = (-2)^e e! A_e((0)_{v \in E}) = (-2)^e e! a_e((0)_{v \in E}).$$

Differentiating the power series expansion of $\mathbb{L}_p((u)_{v \in E}, u)$, we get

$$\frac{d^e}{du^e} \mathbb{L}_p((u)_{v \in E}, u)|_{u=0} = e! \sum_{i=0}^{e} \sum_{|n|=e-i} a_i(n).$$

By Proposition 6.0.3, we obtain

$$\frac{d^e}{du^e} \mathbb{L}_p((u)_{v \in E}, u)|_{u=0} = e! a_e((0)_{v \in E}).$$

Therefore,

$$L_p^{(e)}(\tilde{\pi}, s)|_{s=\frac{2-\mathrm{w}}{2}} = (-2)^e \frac{d^e}{du^e} \mathbb{L}_p((u)_{v \in E}, u)|_{u=0}.$$

Theorem 6.0.1 then follows from Lemma 6.0.4.

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