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Étude fine de processus multifractionnaires non classiques.

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Résumé: Les processus stochastiques multifractionnaires sont des généralisations naturelles des mouvements brownien et brownien fractionnaire. Leur caractéristique essentielle est que leurs propriétés locales peuvent être prescrites via un paramètre fonctionnel et peuvent donc changer significativement d'un point à un autre. Le mouvement brownien multifractionnaire et d'autres processus multifractionnaires *classiques* sont construits en remplaçant le paramètre de Hurst constant d'un processus fractionnaire par une fonction qui dépend de la variable qui permet d'indexer le processus. Une importante idée nouvelle est que le paramètre fonctionnel (déterministe ou aléatoire) de tels processus peut être dépendant de la variable d'intégration associée à l'intégrale stochastique qui représente le processus; un tel processus est alors dit multifractionnaire *non classique*. Ces processus *non classiques* sont plus complexes à étudier et il n'est pas certain que les méthodes usuelles s'adaptent à ce nouveau contexte. Un objectif important de cette thèse est de réussir à déterminer les exposants de Hölder local et ponctuel de ces processus *non classiques* pour un événement universel qui ne dépend pas du point considéré. Un autre objectif est l'estimation statistique de leur paramètre de Hurst (qui est parfois aléatoire) à partir d'une trajectoire discrétisée. Enfin, la question de la simulation de tels processus *non classiques* est également étudiée.

Mots clés : Autosimilarité, Paramètre de Hurst aléatoire, Mouvement brownien fractionnaire, Processus Gaussiens, Processus multifractionnaires, Régularité Hölderienne.

Thorough study of non-classical multifractional processes.

Abstract: Multifractional processes are natural generalisations of Brownian motion and fractional Brownian motion. Their essential feature is that their local properties can be prescribed via a functional parameter and can therefore change significantly from one point to another. Multifractional Brownian motion and other *classical* multifractional processes are constructed by replacing the constant Hurst parameter of a fractional process by a function that depends on the variable which indexes the process. An important new idea is that the functional parameter (deterministic or random) of such processes can depend on the integration variable associated with the stochastic integral that represents the process; such a process is then said to be *non-classical* multifractional. These *non-classical* processes are more complex to study and it is not clear that the usual methods fit this new context. An important objective of this thesis is to determine the local and pointwise Hölder exponents of these *non-classical* processes for a universal event that does not depend on the location. Another objective is the statistical estimation of their Hurst parameter (which is sometimes random) from a discretized trajectory. Finally, the question of the simulation of such *non-classical* processes is also presented.

Keywords : Self-similarity, Random Hurst parameter, Fractional Brownian motion, Gaussian processes, Multifractionnal processes, Hölder regularity.

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Chapter 1

Introduction et contexte de la thèse

On considère un espace de probabilité complet filtré $\left(\Omega, (\mathcal{F}_s)_{s\in\mathbb{R}}, \mathcal{F}, \mathbb{P}\right)$ et $\{B(s)\}_{s\in\mathbb{R}}$ un $(\mathcal{F}_s)$-mouvement brownien (voir [LL12]), c'est-à-dire un processus gaussien à valeurs réelles et à trajectoires continues qui vérifie:
- Pour tout $s \in \mathbb{R},\ B(s)$ est $\mathcal{F}_s$-mesurable.
- Si $s \le t,\ B(t) - B(s)$ est indépendant de la tribu $\mathcal{F}_s$.
- Si $s \le t$, la loi de $B(t) - B(s)$ est identique à celle de $B(t-s) - B(0)$.

D'autre part, $L^2(\mathbb{R})$ désignera l'espace des fonctions de carré intégrable sur $\mathbb{R}$ et à valeurs réelles.

1.1 Le mouvement brownien fractionnaire

1.1.A Du mouvement brownien au mouvement brownien fractionnaire

En 1827, le biologiste Robert Brown découvre ce qui sera appelé bien plus tard le mouvement brownien. Lors d'expériences sur des particules de pollen en suspension sur l'eau, il observe que ces particules ont un mouvement constant et anarchique mais il est alors incapable d'expliquer l'origine de ce mouvement. Il pense tout d'abord que cette propriété est propre aux particules de pollen mais il retrouve plus tard ces étranges mouvements avec d'autres types de particules comme des particules de poussière. Ses observations sont ensuite publiées en 1828 dans [Bro28]. C'est en 1900 qu'est formulée pour la première fois une théorie mathématique du mouvement brownien dans la thèse [Bac00] de Louis Bachelier. Puis, c'est en 1905 dans [Ein05] que Albert Einstein donne pour la première fois une explication physique et statistique du mouvement brownien comme conséquence des nombreux chocs des particules étudiées avec les molécules d'eau. Ils mettent tous deux en évidence plusieurs propriétés du mouvement brownien à savoir des accroissements indépendants et gaussiens de variance proportionnelle au temps écoulé ainsi que des trajectoires continues. Cependant, personne n'avait alors prouvé rigoureusement l'existence mathématique d'un processus qui satisferait les propriétés fondamentales du mouvement brownien. Norbert Wiener montre en 1923 dans [Wie23] l'existence d'un tel processus $\{B(t)\}_{t\in\mathbb{R}}$ appelé mouvement brownien ou alors processus de Wiener en l'honneur de ce dernier.

A l'instar de la loi normale que l'on rencontre dans un grand nombre de phénomènes physiques, biologiques et sociaux, on retrouve la loi du mouvement brownien dans de nombreuses situations; en effet, la loi du mouvement brownien apparaît comme limite de certaines marches aléatoires à travers, par exemple, le théorème central limite fonctionnel de Donsker. C'est cette propriété fondamentale qui fait du mouvement brownien un objet mathématique très utile pour de nombreux domaines comme, par exemple, en finance où il est omniprésent. En 1951, l'hydrologiste Harold E. Hurst met en exergue dans [Hur51] des corrélations dans les données des crues annuelles du fleuve du Nil; les accroissements du mouvement brownien étant toujours indépendants, il n'est donc pas un bon candidat pour décrire ce genre de phénomène. Il faut donc trouver un nouveau type de processus qui permettra de rendre compte de la propriété de *longue mémoire* présente notamment dans les données de Hurst et qui induit une corrélation d'accroissements dans les données.

C'est Benoît Mandelbrot qui reconnaît cette propriété de *longue mémoire* dans un autre processus stochastique alors appelé *Wiener Helix*, introduit en 1940 par Andreï Kolmogorov dans [Kol40] pour ses travaux sur les espaces de Hilbert. Mandelbrot et John W. Van Ness démontrent de nombreuses propriétés de ce processus dans leur article [MN68] et popularisent le nom "mouvement brownien fractionnaire". Le mouvement brownien fractionnaire de paramètre de Hurst $H \in \,]0,1[$, noté $\{B_H(t)\}_{t\in\mathbb{R}}$, est un processus gaussien centré dont la fonction de covariance est donnée par

$$\forall t, s \in \mathbb{R},\ \mathbb{E}\left[B_H(t)B_H(s)\right] = \frac{c(H)}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \tag{1.1.1}$$

où $c(H)$ est une constante déterministe strictement positive. Dans le même article [MN68], Mandelbrot et Van Ness montrent que le mouvement brownien fractionnaire de paramètre de Hurst $H \in \,]0,1[$ peut être représenté, à une constante multiplicative près, par l'intégrale de Wiener suivante

$$\forall t \in \mathbb{R},\ B_H(t) = \frac{1}{\Gamma(H+1/2)}\int_{\mathbb{R}}\left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\right)dB(s), \tag{1.1.2}$$

où Γ représente la fonction Gamma d'Euler et

$$\forall x, y \in \mathbb{R},\ (x)_+^y := \begin{cases} x^y & \text{si } x > 0 \\ 0 & \text{si } x \leq 0. \end{cases}$$

En particulier, lorsque $H = 1/2$, on retrouve le mouvement brownien.

Le mouvement brownien fractionnaire possède deux propriétés fondamentales; la première est appelée *auto-similarité* et signifie que le mouvement brownien fractionnaire est invariant en loi par changement d'échelle à un facteur multiplicatif près. Mathématiquement, elle s'écrit

$$\forall a > 0,\ \{B_H(at)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{a^H B_H(t)\right\}_{t\in\mathbb{R}}, \tag{1.1.3}$$

où $\stackrel{d}{=}$ désigne l'égalité au sens des lois finis dimensionnelles. Cette propriété d'*auto-similarité* caractérise les fractales et fait du mouvement brownien fractionnaire un objet fractal. La deuxième propriété est la *stationnarité des accroissements* qui signifie que la loi des accroissements du mouvement brownien fractionnaire est invariante par translation et s'écrit

$$\forall \tau \in \mathbb{R},\ \{B_H(t+\tau) - B_H(\tau)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{B_H(t)\}_{t\in\mathbb{R}}. \tag{1.1.4}$$

Réciproquement, un processus gaussien centré qui obéit à ces propriétés est aussi un mouvement brownien fractionnaire. Cette fois-ci, en dehors du cas particulier $H = 1/2$, les accroissements du mouvement brownien fractionnaire ne sont plus indépendants et font de lui un outil plus général que le mouvement brownien pour décrire des situations du réel.

1.1.B Régularité des trajectoires du mouvement brownien fractionnaire

Notons dès à présent que les trajectoires du mouvement brownien fractionnaire sont (à modification près), avec probabilité 1, continues partout mais nulles part dérivables. Pour quantifier la régularité d'une fonction continue partout mais nulle part dérivable, on introduit les espaces de Hölder.

Définition 1.1.1. Soit I un intervalle compact de $\mathbb{R}$ et $\beta \in [0,1]$. On dit qu'une fonction continue $f : I \to \mathbb{R}$ est β-hölderienne sur I si

$$\sup_{\substack{t,t' \in I \\ t \neq t'}} \frac{|f(t) - f(t')|}{|t - t'|^\beta} < +\infty.$$

On note $\mathcal{C}^\beta(I)$ l'espace des fonctions β-hölderiennes sur I. Notons que

$$\mathcal{C}^{\beta_2}(I) \subset \mathcal{C}^{\beta_1}(I), \text{ pour tous } 0 \leq \beta_1 \leq \beta_2 \leq 1.$$

On introduit également une notion ponctuelle des espaces de Hölder.

Définition 1.1.2. Soient $\tau \in \mathbb{R}$ et $\beta \in [0,1]$. L'espace de Hölder ponctuel $\mathcal{C}^\beta(\tau)$ est l'ensemble des fonctions continues f définies sur un voisinage de τ et à valeurs dans $\mathbb{R}$ telles que

$$\exists \delta > 0, \sup_{0<|t-\tau|\leq\delta} \frac{|f(t) - f(\tau)|}{|t - \tau|^\beta} < +\infty.$$

Notons que

$$\mathcal{C}^{\beta_2}(\tau) \subset \mathcal{C}^{\beta_1}(\tau), \text{ pour tous } 0 \leq \beta_1 \leq \beta_2 \leq 1.$$

A partir ces deux espaces, on peut définir deux exposants qui quantifient la régularité locale et la régularité ponctuelle en un point $\tau \in \mathbb{R}$ d'une fonction continue et non dérivable.

Définition 1.1.3. Soient J un intervalle ouvert de $\mathbb{R}$ et $f : J \to \mathbb{R}$ une fonction continue. On appelle *exposant de Hölder local* de f au point $\tau \in J$ et on note $\widetilde{\alpha}_f(\tau)$ le réel défini par

$$\widetilde{\alpha}_f(\tau) := \sup \left\{ \beta_f(I), \ I \text{ intervalle compact inclus dans } J \text{ et tel que } \tau \in \mathring{I} \right\},$$

où $\mathring{I}$ désigne l'intérieur de I et $\beta_f(I)$ est *l'exposant de Hölder global* de f sur I défini par

$$\beta_f(I) := \sup \left\{ \beta \in [0,1], \ f \in \mathcal{C}^\beta(I) \right\}. \tag{1.1.5}$$

On appelle *exposant de Hölder ponctuel* de f au point $\tau \in J$ et on note $\alpha_f(\tau)$ le réel défini par

$$\alpha_f(\tau) := \sup \left\{ \beta \in [0,1], \ f \in \mathcal{C}^\beta(\tau) \right\}.$$

Remarques 1.1.4. • On peut montrer que pour tout point $\tau \in \mathbb{R}$ et pour toute fonction continue f définie au voisinage du point τ, on a

$$\alpha_f(\tau) = \sup\left\{ r \in [0,1],\ \limsup_{t\to\tau} \frac{|f(t)-f(\tau)|}{|t-\tau|^r} < +\infty \right\},$$

et

$$\widetilde{\alpha}_f(\tau) = \sup\left\{ \widetilde{r} \in [0,1],\ \limsup_{(t',t'')\to(\tau,\tau)} \frac{|f(t')-f(t'')|}{|t'-t''|^{\widetilde{r}}} < +\infty \right\}.$$

• On a toujours l'inégalité $\widetilde{\alpha}_f(\tau) \leq \alpha_f(\tau)$. L'inégalité est parfois stricte; citons par exemple la fonction "chirp" $q : [-1,1] \to \mathbb{R}$ définie par

$$q(x) = \begin{cases} |x| \sin\left|\frac{1}{x}\right| & \text{si } x \neq 0 \\ 0 & \text{si } x = 0. \end{cases}$$

Cette fonction oscille beaucoup mais avec une faible amplitude et on a l'inégalité stricte $\widetilde{\alpha}_q(0) = 1/2 < \alpha_q(0) = 1$.

Plus ces exposants sont proches de 1 et plus la fonction f est régulière au voisinage du point considéré. Dans le contexte des processus stochastiques, le théorème de Kolmogorov-Chentsov (voir [KS87,KS07]) suivant est un outil souvent utilisé pour obtenir une minoration des exposants de Hölder local et ponctuel.

Théorème 1.1.5. ***(Kolmogorov-Chentsov)*** *Soient I un intervalle compact de $\mathbb{R}$ et $\{X(t)\}_{t\in I}$ un processus stochastique tel que pour des constantes $\delta > 0$ et $\varepsilon > 0$ on ait*

$$\exists c(I) > 0,\ \forall t,t' \in I,\ \mathbb{E}|X(t)-X(t')|^\delta \leq c(I)|t-t'|^{1+\varepsilon}.$$

Alors, $\{X(t)\}_{t\in I}$ admet une modification continue dont les trajectoires sont dans $\mathcal{C}^\gamma(I)$ pour tout $\gamma < \varepsilon/\delta$.

Par la suite, on identifiera toujours un processus stochastique avec sa modification continue. En appliquant le théorème de Kolmogorov-Chentsov au mouvement brownien fractionnaire, on obtient une minoration des exposants de Hôlder local et ponctuel.

Lemme 1.1.6. ***(équivalence des moments gaussiens)*** *Pour tout $p > 0$, il existe une constante $c(p) > 0$ telle que, pour toute variable aléatoire normale centrée X, on a*

$$\mathbb{E}\left(|X|^p\right) = c(p)\left(\mathbb{E}(|X|^2)\right)^{p/2}.$$

Preuve. Le lemme est trivial si X est de variance nulle donc on peut supposer $\sqrt{E(|X|^2)} > 0$. La variable aléatoire $\frac{X}{\sqrt{E(|X|^2)}}$ suit une loi normale centrée réduite. Soit Y une variable aléatoire qui suit une loi $\mathcal{N}(0,1)$, on peut écrire pour tout $p > 0$

$$\mathbb{E}\left(\left|\frac{X}{\sqrt{E(|X|^2)}}\right|^p\right) = \mathbb{E}(|Y|^p) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |y|^p e^{-y^2/2} dy$$

et donc

$$\mathbb{E}\left(|X|^p\right) = \left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |y|^p e^{-y^2/2} dy\right)\left(\mathbb{E}(|X|^2)\right)^{p/2}.$$

□

Proposition 1.1.7. *Il existe un événement universel Ω_* de probabilité 1, tel que*

$$\alpha_{B_H}(\tau,\omega) \geq \widetilde{\alpha}_{B_H}(\tau,\omega) \geq H\,, \quad \textit{pour tout } (\tau,\omega) \in \mathbb{R}\times\Omega_*.$$

Preuve. En utilisant la définition (1.1.1), on obtient pour tous $t, t' \in \mathbb{R}$

$$\mathbb{E}\left[\left|B_H(t) - B_H(t')\right|^2\right] = \mathbb{E}\left[B_H(t)B_H(t)\right] + \mathbb{E}\left[B_H(t')B_H(t')\right] - 2\mathbb{E}\left[B_H(t)B_H(t')\right] = c(H)|t-t'|^{2H}. \tag{1.1.6}$$

D'après le Lemme 1.1.6 et l'égalité (1.1.6), pour tout $n \geq 1$, il existe une constante $c(2n) > 0$ telle que

$$\forall t, t' \in \mathbb{R},\ \mathbb{E}\left[\left|B_H(t) - B_H(t')\right|^{2n}\right] = c(2n)c(H)^n|t-t'|^{2Hn}.$$

Ainsi pour tout entier $n > \frac{1}{2H}$, on peut appliquer le théorème de Kolmogorov-Chentsov avec $\delta := 2n$ et $\varepsilon := 2Hn - 1$ sur l'intervalle $I(n) := [-n, n]$. Il existe donc un événement $\Omega(n)$ de probabilité 1 tel que

$$\forall \omega \in \Omega(n),\ \forall \tau \in\ ]-n, n[,\ \widetilde{\alpha}_{B_H}(\tau,\omega) \geq \beta_{B_H}\Big(I(n),\omega\Big) \geq \frac{2Hn-1}{2n}.$$

Le résultat est ainsi démontré pour $\Omega_* := \bigcap_{n>1/(2H)} \Omega(n)$. □

Montrer d'une fonction continue qu'elle est dérivable partout semble nettement moins difficile que de montrer qu'elle est nulle part dérivable. Les propriétés d'irrégularités sont souvent plus complexes à obtenir et la majoration des exposants de Hölder du mouvement brownien fractionnaire n'échappe pas à cette règle. On peut néanmoins obtenir une majoration pour un événement qui dépend du point $\tau \in \mathbb{R}$ considéré en utilisant les deux propriétés fondamentales du mouvement brownien fractionnaire.

Proposition 1.1.8. *Pour tout point $\tau \in \mathbb{R}$, il existe un événement $\Omega_*(\tau)$ de probabilité 1, tel que*

$$\widetilde{\alpha}_{B_H}(\tau,\omega) \leq \alpha_{B_H}(\tau,\omega) \leq H\,, \quad \textit{pour tout } \omega \in \Omega_*(\tau).$$

Preuve. Soit $\tau \in \mathbb{R}$ fixé. Pour tous entiers $n, M \geq 1$ et réel $\varepsilon > 0$, on pose

$$\Omega(n, M, \varepsilon) := \left\{\sup_{\frac{1}{n}\leq\delta\leq 1}\left|\frac{B_H(\tau+\delta) - B_H(\tau)}{\delta^{H+\varepsilon}}\right| \geq M\right\}.$$

Fixons $M \geq 1$ et $\varepsilon > 0$. En utilisant la propriété d'accroissements stationnaires (1.1.4), on obtient

$$\forall n \geq 1,\ \mathbb{P}(\Omega(n,M,\varepsilon)) \geq \mathbb{P}\left(\left|B_H(\tau+n^{-1}) - B_H(\tau)\right| n^{H+\varepsilon} \geq M\right) = \mathbb{P}\left(\left|B_H(n^{-1})\right| n^{H+\varepsilon} \geq M\right).$$

Puis en utilisant la propriété d'auto-similarité (1.1.3), on obtient

$$\forall n \geq 1,\ \mathbb{P}(\Omega(n,M,\varepsilon)) \geq \mathbb{P}\left(|B_H(1)|\, n^{\varepsilon} \geq M\right) = 1 - \mathbb{P}\left(|B_H(1)| < M n^{-\varepsilon}\right).$$

La variable aléatoire $B_H(1)$ suit une loi normale de variance strictement positive donc

$$\mathbb{P}(\Omega(n,M,\varepsilon)) \geq 1 - \mathbb{P}\left(|B_H(1)| < M n^{-\varepsilon}\right) \underset{n\to+\infty}{\longrightarrow} 1.$$

On pose $\Omega_{**}(M,\varepsilon) := \bigcup_{n\geq 1} \Omega(n,M,\varepsilon)$ qui est de probabilité 1 par continuité croissante. En particulier, on a

$$\forall \omega \in \Omega_{**}(M,\varepsilon),\ \exists N \geq 1,\ \forall n \geq N,\ \sup_{\frac{1}{n}\leq\delta\leq 1} \left| \frac{B_H(\tau+\delta,\omega) - B_H(\tau,\omega)}{\delta^{H+\varepsilon}} \right| \geq M$$

et donc

$$\forall \omega \in \Omega_{**}(M,\varepsilon),\ \sup_{\delta\in]0,1]} \left| \frac{B_H(\tau+\delta,\omega) - B_H(\tau,\omega)}{\delta^{H+\varepsilon}} \right| \geq M.$$

En posant $\Omega_* := \bigcap_{\varepsilon\in\mathbb{Q}_*^+} \bigcap_{M\geq 1} \Omega_{**}(M,\varepsilon)$ on obtient donc

$$\forall \omega \in \Omega_*,\ \forall \varepsilon > 0,\ \sup_{\delta\in]0,1]} \left| \frac{B_H(\tau+\delta,\omega) - B_H(\tau,\omega)}{\delta^{H+\varepsilon}} \right| = +\infty,$$

c'est donc que

$$\forall \omega \in \Omega_*,\ \alpha_{B_H}(\tau,\omega) \leq H.$$

□

Théorème 1.1.9. *Pour tout point $\tau \in \mathbb{R}$, il existe un événement $\Omega_*(\tau)$ de probabilité 1, tel que*

$$\alpha_{B_H}(\tau,\omega) = \widetilde{\alpha}_{B_H}(\tau,\omega) = H\,, \quad \text{pour tout } \omega \in \Omega_*(\tau).$$

Remarque 1.1.10. Il est important de souligner que dans le Théorème 1.1.9 l'événement $\Omega_*(\tau)$ dépend du point τ considéré, et ne permet donc pas d'obtenir un événement universel de probabilité 1 valable pour tous les points $\tau \in \mathbb{R}$ en même temps. Passer d'une détermination des exposants de Hölder (local et ponctuel) valable pour tout τ presque sûrement à une détermination presque sûrement pour tout τ est une tâche plus difficile et met en jeu des techniques plus complexes. Obtenir les exposants de Hölder pour un événement universel donne beaucoup plus d'informations; on réussit par exemple à déterminer le spectre des singularités du mouvement brownien fractionnaire B_H défini par

$$\forall \alpha \in \mathbb{R},\ D(\alpha) := \dim_{\text{Haus}} \{\tau \in \mathbb{R},\ \alpha_{B_H}(\tau) = \alpha\},$$

où $\dim_{\text{Haus}}$ désigne la dimension de Hausdorff (voir [Fal90]). En particulier, le Théorème 1.1.11 assure que le mouvement brownien fractionnaire est un objet monofractal. Le spectre des singularités joue un rôle central en analyse multifractale où il est omniprésent (voir par exemple [Jaf99, Bal14, AJT07]).

On sait depuis longtemps que le résultat reste vrai pour un événement universel de probabilité 1 ([Ber72, Xia97]).

Théorème 1.1.11. *Il existe un événement universel* Ω_* *de probabilité 1, tel que*

$$\alpha_{B_H}(\tau,\omega) = \widetilde{\alpha}_{B_H}(\tau,\omega) = H\,, \quad \textit{pour tout } (\tau,\omega) \in \mathbb{R}\times\Omega_*.$$

Nous présentons dans la suite deux méthodes pour obtenir le Théorème 1.1.11 ; la première repose sur les temps locaux et la deuxième sur les ondelettes. On peut également trouver une autre démonstration de ce résultat en utilisant la frontière 2-microlocale du mouvement brownien fractionnaire dans [HLV09].

1.1.B.a Temps locaux et principe de Berman

Dans cette sous section, I est un intervalle compact de $\mathbb{R}$ et $\{Z(t)\}_{t\in I}$ est un processus stochastique arbitraire dont les trajectoires sont des fonctions boréliennes.

Définition 1.1.12. Fixons $\omega\in\Omega$ et un borélien $T\in\mathcal{B}(I)$. La mesure d'occupation associée à $Z(\cdot,\omega)$ à la "période de temps" T est la mesure $\mu_T(\bullet,\omega)$ définie sur $\mathcal{B}(\mathbb{R})$ par

$$\forall A\in\mathcal{B}(\mathbb{R}),\ \mu_T(A,\omega) := \lambda\left(\{t\in T,\ Z(t,\omega)\in A\}\right),$$

où λ désigne la mesure de Lebesgue. La quantité $\mu_T(A,\omega)$ peut être interprétée comme la durée t passée par la trajectoire $Z(\cdot,\omega)$ dans le borélien A durant la période de temps T.

Définition 1.1.13. Soient $T\in\mathcal{B}(I)$, le temps local sur T du processus $\{Z(t)\}_{t\in I}$ est bien défini uniquement si, pour $\mathbb{P}$ presque tout $\omega\in\Omega$, la mesure d'occupation $\mu_T(\bullet,\omega)$ est absolument continue par rapport à la mesure de Lebesgue λ. Dans ce cas, le temps local est noté $L(\bullet,T,\omega)$ et il est défini comme la dérivée au sens de Radon-Nikodym de $\mu_T(\bullet,\omega)$ par rapport à λ

$$L(\bullet,T,\omega) := \frac{d\mu_T(\bullet,\omega)}{d\lambda}.$$

Donc, pour toute fonction borélienne positive f sur $\mathbb{R}$ on a

$$\int_{\mathbb{R}} f(x)d\mu_T(x,\omega) = \int_{\mathbb{R}} f(x)L(x,T,\omega)\lambda(dx).$$

La proposition suivante (voir par exemple [Aya19]) va nous assurer l'existence des temps locaux pour le mouvement brownien fractionnaire.

Proposition 1.1.14. *Supposons que* $\{Z(t)\}_{t\in I}$ *soit un processus gaussien centré. Alors, une condition suffisante pour avoir l'existence de* $L(\bullet,T)$ *est*

$$\int_T\left(\int_T \mathbb{E}\left[(Z(t)-Z(s))^2\right]^{-1/2} dt\right) ds < +\infty.$$

De plus, quand cette condition est vérifiée, $L(\bullet,T)$ *est de carré intégrable par rapport à* (x,ω).

Dans le cas du mouvement brownien fractionnaire, on suppose que T est un intervalle compact de $\mathbb{R}$ et on utilise (1.1.6) pour obtenir

$$\int_T \left(\int_T \mathbb{E}\left[(B_H(t)-B_H(s))^2\right]^{-1/2} dt \right) ds \leq c(H)^{-1/2} \int_T \left(\int_T |t-s|^{-H} dt \right) ds$$
$$= c(H)^{-1/2} \int_T \left(\int_{T-s} |t|^{-H} dt \right) ds$$

avec la convention $T - s := \{x - s,\ x \in T\}$. On pose $\nu(T) := \sup\{|x|,\ x \in T\}$ et on obtient

$$\int_T \left(\int_{T-s} |t|^{-H} dt \right) ds \leq \int_T \left(\int_{-2\nu(T)}^{2\nu(T)} |t|^{-H} dt \right) ds = \lambda(T) \int_{-2\nu(T)}^{2\nu(T)} |t|^{-H} dt < +\infty,$$

ce qui assure l'existence du temps local $L(\bullet, T)$ pour tout intervalle compact T.

Les temps locaux peuvent permettre de mesurer l'irrégularité d'un processus stochastique à travers le principe de Berman. Ce principe peut être reformulé de la façon suivante: plus les trajectoires des temps locaux $\{L(x,T),\ (x,T) \in \mathbb{R} \times \mathcal{B}(I)\}$ sont régulières et plus les trajectoires du processus stochastique associé $\{Z(t)\}_{t \in I}$ sont irrégulières. Il est souvent moins difficile de montrer une propriété de régularité qu'une propriété d'irrégularité et donc le principe de Berman fournit une importante stratégie pour obtenir des majorations des exposants de Hölder local et ponctuel.

Définition 1.1.15. On suppose l'existence du temps local $L(\bullet, I)$, le théorème de Radon-Nikodym assure alors l'existence des temps locaux $L(\bullet, T)$ pour tout $T \in \mathcal{B}(I)$ ([Rud86]). En particulier, c'est le cas pour pour les intervalles de la forme $T = I(s)$ avec

$$\forall s \in I,\ I(s) := \{x \leq s,\ x \in I\}.$$

On dit que le processus stochastique $\{Z(t)\}_{t \in I}$ a des temps locaux bicontinus si le champ stochastique $\{L(x, I(s)),\ (x,s) \in \mathbb{R} \times I\}$ admet une modification continue. Dans ce cas, on note $\{\mathcal{L}(x, I(s)),\ (x,s) \in \mathbb{R} \times I\}$ cette modification continue.

Théorème 1.1.16. ***(Principe de Berman)*** *On suppose que le processus $\{Z(t)\}_{t \in I}$ est à trajectoires continues avec des temps locaux bicontinus. De plus, on suppose qu'il existe un événement universel Ω_0 de probabilité 1 sur lequel on a*

$$\forall \omega \in \Omega_0,\ \forall \tau \in \overset{\circ}{I},\ \limsup_{\rho \to 0^+} \left\{ \sup_{x \in \mathbb{R}} \mathcal{L}\big(x, [\tau - \rho, \tau + \rho], \omega\big) \rho^{\theta(\tau) - 1} \right\} < +\infty, \tag{1.1.7}$$

où $\theta(\cdot)$ est une fonction à valeurs dans $]0,1[$. Alors, on a

$$\widetilde{\alpha}_Z(\tau, \omega) \leq \alpha_Z(\tau, \omega) \leq \theta(\tau), \quad \textit{pour tout } (\tau, \omega) \in \overset{\circ}{I} \times \Omega_0.$$

Preuve. On fixe $\omega \in \Omega_0$, $\tau \in \overset{\circ}{I}$ et $\rho > 0$ suffisamment petit pour avoir $[\tau - \rho, \tau + \rho] \subset I$. On introduit ensuite l'oscillation de $Z(\cdot, \omega)$ sur ce même intervalle

$$\mathrm{Osc}_Z([\tau - \rho, \tau + \rho], \omega) := \sup_{t \in [\tau - \rho, \tau + \rho]} Z(t, \omega) - \inf_{t \in [\tau - \rho, \tau + \rho]} Z(t, \omega).$$

La continuité de $Z(\cdot,\omega)$ assure que cette oscillation existe et est finie. On introduit l'intervalle compact

$$J(\tau,\rho,\omega) := \left[\inf_{t\in[\tau-\rho,\tau+\rho]} Z(t,\omega), \sup_{t\in[\tau-\rho,\tau+\rho]} Z(t,\omega)\right].$$

Observons que la mesure $\mu_{[\tau-\rho,\tau+\rho]}(\bullet,\omega)$ est supportée par $J(\tau,\rho,\omega)$ et donc $\mathcal{L}(x,[\tau-\rho,\tau+\rho],\omega) = 0$ pour tout $x \notin J(\tau,\rho,\omega)$. On a donc

$$\begin{aligned}\mu_{[\tau-\rho,\tau+\rho]}(\mathbb{R},\omega) = \int_{\mathbb{R}} \mathcal{L}(x,[\tau-\rho,\tau+\rho],\omega)dx &= \int_{J(\tau,\rho,\omega)} \mathcal{L}(x,[\tau-\rho,\tau+\rho],\omega)dx \\ &\le \lambda(J(\tau,\rho,\omega)) \times \sup_{x\in\mathbb{R}} \mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big) \\ &\le \mathrm{Osc}_Z([\tau-\rho,\tau+\rho],\omega) \times \sup_{x\in\mathbb{R}} \mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big), \end{aligned} \tag{1.1.8}$$

avec

$$\mu_{[\tau-\rho,\tau+\rho]}(\mathbb{R},\omega) = \lambda\left(\{t\in[\tau-\rho,\tau+\rho],\ Z(t,\omega)\in\mathbb{R}\}\right) = 2\rho. \tag{1.1.9}$$

En combinant (1.1.8) et (1.1.9), on obtient

$$2 \le \Big(\mathrm{Osc}_Z([\tau-\rho,\tau+\rho],\omega)\rho^{-\theta(\tau)}\Big) \times \Big(\sup_{x\in\mathbb{R}} \mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big)\rho^{\theta(\tau)-1}\Big)$$

et donc, (1.1.7) assure que, pour tout $\varepsilon>0$, on a

$$\limsup_{\rho\to 0^+} \Big\{\rho^{-(\theta(\tau)+\varepsilon)}\mathrm{Osc}_Z([\tau-\rho,\tau+\rho],\omega)\Big\} = +\infty. \tag{1.1.10}$$

D'autre part, en utilisant l'inégalité triangulaire, on obtient pour tout $\varepsilon>0$

$$\begin{aligned}\rho^{-(\theta(\tau)+\varepsilon)}\mathrm{Osc}_Z([\tau-\rho,\tau+\rho],\omega) &= \rho^{-(\theta(\tau)+\varepsilon)} \sup_{t_1,t_2\in[\tau-\rho,\tau+\rho]} |Z(t_1,\omega)-Z(t_2,\omega)| \\ &\le 2\rho^{-(\theta(\tau)+\varepsilon)} \sup_{t\in[\tau-\rho,\tau+\rho]} |Z(t,\omega)-Z(\tau,\omega)|\end{aligned}$$

et donc, grâce à (1.1.10) on trouve finalement pour tout $\varepsilon>0$

$$\limsup_{\rho\to 0^+} \Big\{\rho^{-(\theta(\tau)+\varepsilon)} \sup_{t\in[\tau-\rho,\tau+\rho]} |Z(t,\omega)-Z(\tau,\omega)|\Big\} = +\infty.$$

□

Remarques 1.1.17. • La condition (1.1.7) peut être interprétée comme une condition de régularité sur $\{\mathcal{L}(x,I(s))\}_{(x,s)\in\mathbb{R}\times I}$ de la variable s uniformément en la variable x. En effet, on peut écrire

$$\forall x\in\mathbb{R},\ \mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big) := \mathcal{L}\big(x,I(\tau+\rho),\omega\big) - \mathcal{L}\big(x,I(\tau-\rho),\omega\big),$$

et la continuité sur $\mathbb{R}\times I$ de la fonction à support compact $(x,s)\mapsto\mathcal{L}(x,I(s))$ assure l'existence de $\sup_{x\in\mathbb{R}} \mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big) = \sup_{x\in J(\tau,\rho,\omega)} \mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big)$ avec

$$\limsup_{\rho\to 0^+} \Big\{\sup_{x\in\mathbb{R}} \mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big)\Big\} = 0. \tag{1.1.11}$$

La condition (1.1.7) est donc une condition de convergence plus rapide que (1.1.11) qui provient de la continuité de $\{\mathcal{L}(x, I(s))\}_{(x,s)\in\mathbb{R}\times I}$.
• Pour obtenir une propriété de régularité sur les temps locaux du type (1.1.7), il est souvent nécessaire d'avoir des expressions explicites des fonctions caractéristiques associées aux lois finis dimensionnelles du processus Z. Les méthodes de temps locaux deviennent donc peu efficaces dans le contexte de processus stochastiques dont les fonctions caractéristiques des lois finis dimensionnelles n'ont pas de forme explicite; ce qui est le cas de tous les processus multifractionnaires *non classiques* étudiés dans cette thèse (voir section 1.3).

Une méthode alternative à celles des temps locaux est celle des ondelettes dont on retrouve différentes variantes dans un certain nombre d'articles sur l'étude de la régularité et de l'irrégularité locale de processus stochastiques.

1.1.B.b La méthode d'ondelettes et ses variantes

Une base d'ondelettes est une base hilbertienne de $L^2(\mathbb{R})$ formée de fonctions à valeurs réelles $(\psi_{j,k})_{j,k\in\mathbb{Z}}$ générées par translations et dilatations d'une même fonction $\psi \in L^2(\mathbb{R})$ appelée ondelette mère avec

$$\forall x \in \mathbb{R},\ \psi_{j,k}(x) := 2^{j/2}\psi\left(2^j x - k\right).$$

En particulier, toute fonction $f \in L^2(\mathbb{R})$ peut s'écrire

$$f = \sum_{j=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} a_{j,k}\psi_{j,k}, \tag{1.1.12}$$

où la convergence à lieu dans $L^2(\mathbb{R})$ et les $(a_{j,k})_{j,k\in\mathbb{Z}}$ sont des coefficients d'ondelettes définis par

$$a_{j,k} := \langle f, \psi_{j,k}\rangle = \int_{\mathbb{R}} f(t)\psi_{j,k}(t)dt. \tag{1.1.13}$$

L'intérêt des ondelettes dans l'analyse de la régularité locale transparaît par exemple dans un résultat de Stéphane Jaffard (voir [Jaf91, Jaf04]): si f est hölderienne sur $\mathbb{R}$ et si ψ est une ondelette suffisamment régulière, on a la caractérisation suivante de l'exposant de Hölder ponctuel en un point $\tau \in \mathbb{R}$

$$\alpha_f(\tau) = \liminf_{j\to+\infty}\inf_{k\in\mathbb{Z}} \frac{\log|2^{j/2}a_{j,k}|}{\log(2^{-j} + |\tau - k2^{-j}|)}.$$

Cette égalité est une conséquence du Théorème 1.1.18 suivant.

On suppose que l'ondelette mère ψ est dérivable et d'intégrale nulle (c'est-à-dire que son premier moment est nul). De plus, ψ et sa fonction dérivée ψ' sont à décroissance rapide à l'infini, plus précisément on a

$$\forall i \in \{0,1\},\ \forall m \in \mathbb{N},\ \sup_{x\in\mathbb{R}} |\psi^{(i)}(x)|(1+|x|)^m < +\infty. \tag{1.1.14}$$

Ces conditions sur ψ sont classiques et souvent imposées aux ondelettes.

Théorème 1.1.18. *Soient $f \in L^2(\mathbb{R})$ bornée et globalement ε-hölderienne (avec $\varepsilon > 0$), $\alpha \in]0,1[$ et $x_0 \in \mathbb{R}$. Si $\alpha_f(x_0) > \alpha$ alors il existe une constante $c > 0$ telle que*

$$\forall j,k \in \mathbb{Z},\ |a_{j,k}| \leq c 2^{-(\alpha+1/2)j} \left(1 + |2^j x_0 - k|^\alpha\right). \tag{1.1.15}$$

Réciproquement, si (1.1.15) est vérifiée alors il existe une constante $c' > 0$ telle que pour tout $x \in \mathbb{R}$ avec $|x - x_0| \leq 1$ on a

$$|f(x) - f(x_0)| \leq c'|x - x_0|^\alpha \log\left(\frac{2}{|x - x_0|}\right)$$

et donc $\alpha_f(x_0) \geq \alpha$.

On donne ici une démonstration de l'implication directe; la preuve de la réciproque qui est bien plus complexe est présentée dans [Jaf04].

Preuve. On fixe $x_0 \in \mathbb{R}$ et on pose $d_{j,k} := 2^{j/2} a_{j,k}$ pour simplifier les calculs. Puisque ψ est d'intégrale nulle, on peut écrire

$$\forall j,k \in \mathbb{Z},\ d_{j,k} = 2^j \int_{\mathbb{R}} f(x)\psi(2^j x - k)dx = 2^j \int_{\mathbb{R}} (f(x) - f(x_0))\psi(2^j x - k)dx.$$

Puisque f est bornée avec $\alpha_f(x_0) > \alpha$, la quantité $c_1 := \sup\limits_{x\in\mathbb{R}} \frac{|f(x)-f(x_0)|}{|x-x_0|^\alpha}$ existe et est finie. On obtient ensuite en utilisant l'inégalité triangulaire, l'inégalité $(a+b)^\alpha \leq a^\alpha + b^\alpha$ valable pour $a, b \geq 0$ et (1.1.14) avec $i = 0$ et $m = 2$

$$|d_{j,k}| \leq 2^j c_1 \int_{\mathbb{R}} \frac{|x - x_0|^\alpha}{(1 + |2^j x - k|)^2} dx \leq 2^j c_1 \int_{\mathbb{R}} \frac{|x - k2^{-j}|^\alpha + |x_0 - k2^{-j}|^\alpha}{(1 + |2^j x - k|)^2} dx.$$

Finalement, en effectuant le changement de variable $t = 2^j x - k$, on trouve

$$\begin{aligned} |d_{j,k}| \leq 2^{-\alpha j} c_1 \int_{\mathbb{R}} \frac{|t|^\alpha + |2^j x_0 - k|^\alpha}{(1+|t|)^2} dt &\leq 2^{-\alpha j} c_1 \left(\int_{\mathbb{R}} \frac{|t|^\alpha}{(1+|t|)^2} dt + |2^j x_0 - k|^\alpha \int_{\mathbb{R}} \frac{1}{(1+|t|)^2} dt \right) \\ &\leq c_2 2^{-\alpha j} \left(1 + |2^j x_0 - k|^\alpha\right) \end{aligned}$$

où $c_2 > 0$ est une constante.

□

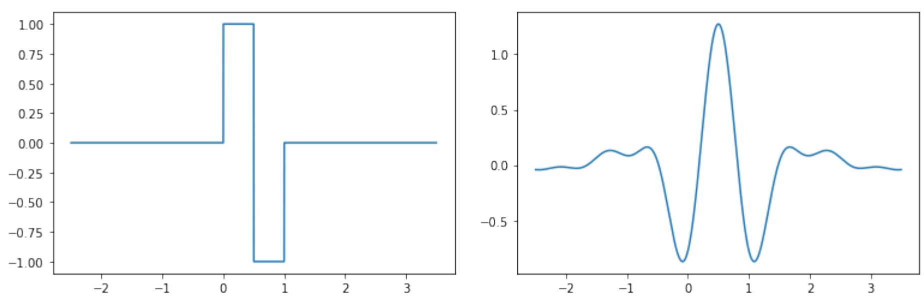
L'exemple le plus simple de bases d'ondelettes, appelée parfois de façon humoristique "ondelette du pauvre", est celle de Haar introduite en 1909 ([Haa10]) avec pour ondelette mère

$$\forall t \in \mathbb{R},\ \psi(t) := \begin{cases} 1 & \text{si } 0 \leq t < 1/2 \\ -1 & \text{si } 1/2 \leq t < 1 \\ 0 & \text{sinon.} \end{cases}$$

Une autre catégorie bien connue de bases d'ondelettes sont les bases de Meyer. Grâce à une telle base, différents types de représentations du mouvement brownien fractionnaire en séries aléatoires ont pu être obtenues dans l'article [MST99]. L'une de ces représentations est la suivante:

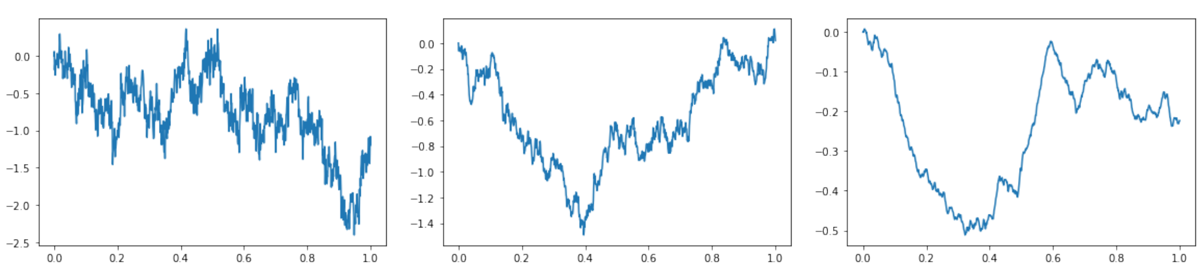
$$\forall t \in \mathbb{R},\ B_H(t) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} 2^{-jH} \left(\Psi_H(2^j t - k) - \Psi_H(-k) \right) \varepsilon_{j,k},$$

où Ψ_H désigne la primitive fractionnaire (à gauche) d'ordre $H + 1/2$ d'une l'ondelette mère de Meyer ψ et où les $\varepsilon_{j,k}$ sont des variables aléatoires indépendantes de même loi $\mathcal{N}(0,1)$. Cette représentation en série aléatoire permet d'avoir une preuve alternative du Théorème 1.1.11 et permet de façon indirecte de simuler le mouvement brownien fractionnaire.



L'ondelette mère de Haar à gauche et une ondelette mère de Meyer à droite.

Ces deux ondelettes sont diamétralement opposées: l'ondelette de Haar est irrégulière alors que l'ondelette de Meyer est très régulière ce qui la rend très commode pour utiliser les techniques d'analyse qui nécessitent une certaine régularité comme les intégrations par parties. Néanmoins, les coefficients d'ondelettes de Haar s'approximent plus facilement numériquement (voir Remarque 1.3.4), ce qui peut lui donner de l'intérêt du point de vue des applications. Notons également que plusieurs nouveaux résultats de cette thèse sont obtenus en utilisant la base de Haar (voir Chapitre 7). Signalons tout de même, qu'il existe d'autres méthodes que celles des séries d'ondelettes pour simuler le mouvement brownien fractionnaire et nombre d'entre elles sont présentées dans [Coe00]. Dans ce même article, Jean-François Coeurjolly présente aussi plusieurs estimateurs du paramètre de Hurst H.



Simulations du mouvement brownien fractionnaire
($H = 0.3$ à gauche, $H = 0.5$ au milieu, $H = 0.8$ à droite).

Ces simulations illustrent le Théorème 1.1.11: plus le paramètre de Hurst H est proche de 1, plus les trajectoires sont lisses et régulières; à l'inverse, plus le paramètre de Hurst H est proche de 0 et plus les trajectoires sont rugueuses et irrégulières.

1.2 Processus multifractionnaires classiques

La régularité locale du mouvement brownien fractionnaire étant prescrite par le paramètre de Hurst H (comme le montre le Théorème 1.1.11), elle ne peut évoluer d'un point à un autre et reste la même tout le long de la trajectoire. Le mouvement brownien fractionnaire ne serait donc pas un modèle bien adapté pour pouvoir rendre compte de nombreux phénomènes (cours de la bourse, encéphalogrammes, génération de montagnes artificielles, etc) qui nécessitent que la rugosité des trajectoires puissent changer d'un point à un autre.

1.2.A Le mouvement brownien multifractionnaire

Pour palier à cette limitation du mouvement brownien fractionnaire dont la régularité locale ne peut évoluer d'un point à un autre, Romain-François Peltier et Jacques Lévy Véhel ainsi que Albert Benassi, Stéphane Jaffard et Daniel Roux introduisent de manière indépendante un nouveau type de processus gaussien dit multifractionnaire, au milieu des années 90, dans les articles [PL95] et [BJR97]. L'idée est de remplacer le paramètre de Hurst H constant du mouvement brownien fractionnaire par une fonction déterministe $t \mapsto H(t)$ à valeurs dans $]0,1[$, qui dépend de la variable t qui permet d'indexer le processus. Soulignons que cette idée de faire dépendre le paramètre de Hurst de l'indice du processus t caractérise les processus multifractionnaires dits *classiques*.

En reprenant la définition (1.1.2), on peut ainsi définir un nouveau processus stochastique gaussien $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$, appelé mouvement brownien multifractionnaire, par la relation suivante

$$\forall t \in \mathbb{R},\ \mathcal{M}(t) := \frac{1}{\Gamma(H(t)+1/2)} \int_{\mathbb{R}} \left((t-s)_+^{H(t)-1/2} - (-s)_+^{H(t)-1/2}\right) dB(s). \tag{1.2.1}$$

Bien qu'elle soit plus complexe que celle du mouvement brownien fractionnaire, il est toujours possible de calculer explicitement la fonction de covariance du mouvement brownien multifractionnaire ([ACLV00]).

Proposition 1.2.1. *Pour tous $t, s \in \mathbb{R}$, on a*

$$\mathbb{E}\left(\mathcal{M}(t)\mathcal{M}(s)\right) = D(H(t),H(s))\left(|t|^{H(t)+H(s)} + |s|^{H(t)+H(s)} - |t-s|^{H(t)+H(s)}\right),$$

avec

$$D(x,y) := \frac{\sqrt{\Gamma(2x+1)\Gamma(2y+1)\sin(\pi x)\sin(\pi y)}}{2\Gamma(x+y+1)\sin(\pi(x+y)/2)}.$$

Lorsque la fonction $H(\cdot)$ est discontinue ou pas suffisamment régulière, les exposants de Hölder local et ponctuel au point $\tau \in \mathbb{R}$ ne coïncident pas avec la valeur de $H(\cdot)$ au point τ comme le montre par exemple la proposition suivante (voir [Aya19]).

Proposition 1.2.2. *Supposons que la fonction de Hurst $H(\cdot)$ soit discontinue en un point $\tau \in \mathbb{R}^*$, alors il existe un événement $\Omega_*(\tau)$ de probabilité 1 tel que, pour tout $\omega \in \Omega_*(\tau)$ la trajectoire $\mathcal{M}(\cdot,\omega)$ est discontinue au point τ.*

Néanmoins, lorsque la fonction $H(\cdot)$ est suffisamment régulière, la régularité locale du mouvement brownien multifractionnaire peut évoluer d'un point à un autre et les exposants de Hölder coïncident avec les valeurs prises part $H(\cdot)$ ([PL95, BJR97]).

Théorème 1.2.3. *Supposons que la condition (C) suivante soit satisfaite par la fonction* $H : \mathbb{R} \to]0,1[$

$$(C): \ \widetilde{\alpha}_H(\tau) > H(\tau), \ \textit{pour tout } \tau \in \mathbb{R}.$$

Alors, pour tout $\tau \in \mathbb{R}$*, il existe un événement* $\Omega_*(\tau)$ *de probabilité 1, tel que*

$$\alpha_{\mathcal{M}}(\tau,\omega) = \widetilde{\alpha}_{\mathcal{M}}(\tau,\omega) = H(\tau)\,, \quad \textit{pour tout } \omega \in \Omega_*(\tau).$$

Dix ans après, Antoine Ayache, Stéphane Jaffard et Murad S. Taqqu démontrent avec des méthodes d'ondelettes dans l'article [AJT07] que le résultat reste vrai pour un événement universel de probabilité 1. On peut retrouver une autre démonstration de ce résultat à partir de la frontière 2-microlocale du mouvement brownien multifractionnaire dans [Her04, HLV09].

Théorème 1.2.4. *Supposons que la condition (C) suivante soit satisfaite par la fonction* $H : \mathbb{R} \to]0,1[$

$$(C): \ \widetilde{\alpha}_H(\tau) > H(\tau), \ \textit{pour tout } \tau \in \mathbb{R}.$$

Alors, il existe un événement universel Ω_* *de probabilité 1, tel que*

$$\alpha_{\mathcal{M}}(\tau,\omega) = \widetilde{\alpha}_{\mathcal{M}}(\tau,\omega) = H(\tau)\,, \quad \textit{pour tout } (\tau,\omega) \in \mathbb{R} \times \Omega_*.$$

Remarque 1.2.5. La condition (C) est par exemple vérifiée lorsque la fonction $H(\cdot)$ est lipschitzienne.

Ce nouveau processus $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ ne conserve pas les deux propriétés fondamentales d'auto-similarité et d'accroissements stationnaires du mouvement brownien fractionnaire; néanmoins, on retrouve localement une propriété d'auto-similarité asymptotique.

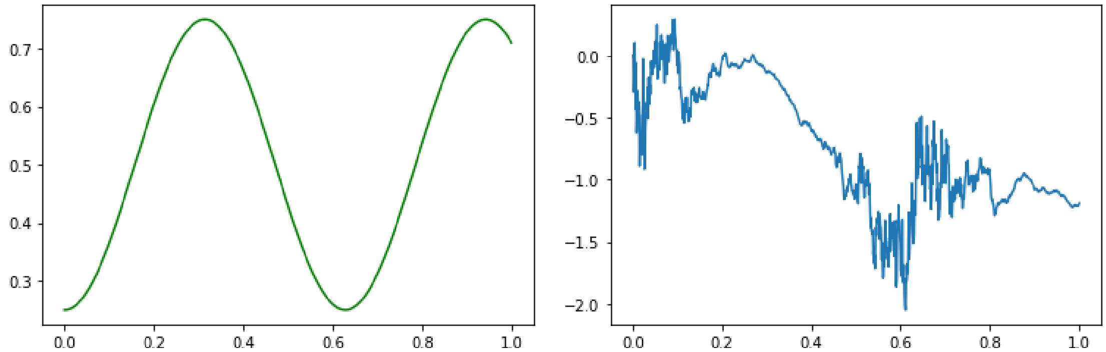
Théorème 1.2.6. *Supposons que la condition (C) suivante soit satisfaite par la fonction* $H : \mathbb{R} \to]0,1[$

$$(C): \ \widetilde{\alpha}_H(\tau) > H(\tau), \ \textit{pour tout } \tau \in \mathbb{R}.$$

Alors en tout point $\tau \in \mathbb{R}$*, le mouvement brownien multifractionnaire* $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ *est fortement localement asymptotiquement auto-similaire d'exposant* $H(\tau)$ *et le mouvement brownien fractionnaire* $\left\{B_{H(\tau)}(t)\right\}_{t\in\mathbb{R}}$ *est le processus tangent. En somme, il existe une constante strictement positive* $c(\tau)$ *telle que pour tout réel* $T > 0$*, le processus stochastique* $\left\{\lambda^{-H(\tau)}\left(\mathcal{M}(\tau+\lambda t) - \mathcal{M}(\tau)\right)\right\}_{t\in\mathbb{R}}$ *converge en loi vers* $\left\{c(\tau)B_{H(\tau)}(t)\right\}_{t\in\mathbb{R}}$ *quand* $\lambda \to 0^+$ *dans l'espace de Banach des fonctions continues* $\mathcal{C}([-T,T], ||.||_\infty)$*.*

Sous la condition (C), on a donc pour tout $t \in \mathbb{R}$ un mouvement brownien fractionnaire de paramètre de Hurst $H(t)$ qui est tangent à la trajectoire du mouvement brownien multifractionnaire au point t.

Il existe quelques méthodes pour simuler le mouvement brownien multifractionnaire; une méthode de simulation est d'ailleurs présentée par Peltier et Lévy Véhel dans leur article fondateur [PL95].



Simulation du mouvement brownien multifractionnaire
(la fonction de Hurst $H(\cdot)$ à gauche et une trajectoire de $\mathcal{M}$ associée à droite)

On sait aussi estimer $H(t_0)$, la valeur de la fonction de Hurst $H(\cdot)$ en un point arbitraire et fixé $t_0 \in [0,1]$, à partir de l'observation d'une réalisation discrétisée $\left\{\mathcal{M}\left(\frac{k}{N}\right)\right\}_{0\leq k\leq N}$ (où N est un entier assez grand) du mouvement brownien multifractionnaire (voir [AL04, BS13a, BCI98, Coe05, Coe06]). L'estimateur est construit à partir des variations discrètes généralisées d'ordre $L \geq 1$ notées $(d_{N,k})_{0\leq k\leq N-L}$ avec

$$\forall k \in \{0,\dots,N-L\}\,,\ d_{N,k} := \sum_{q=0}^{L} a_q \mathcal{M}\left(\frac{k+q}{N}\right)$$

où

$$\forall q \in \{0,\dots,L\}\,,\ a_q := (-1)^{L-q}\binom{L}{q}.$$

Par exemple, lorsque $L=1$ on retrouve un accroissement classique

$$\forall k \in \{0,\dots,N-1\}\,,\ d_{N,k} = \mathcal{M}\left(\frac{k}{N}\right) - \mathcal{M}\left(\frac{k+1}{N}\right), \tag{1.2.2}$$

et lorsque $L=2$ on obtient un accroissement entre deux accroissements

$$\forall k \in \{0,\dots,N-2\}\,,\ d_{N,k} = \left[\mathcal{M}\left(\frac{k}{N}\right) - \mathcal{M}\left(\frac{k+1}{N}\right)\right] - \left[\mathcal{M}\left(\frac{k+1}{N}\right) - \mathcal{M}\left(\frac{k+2}{N}\right)\right].$$

Pour comprendre comment ces variations peuvent permettent d'estimer les valeurs de $H(\cdot)$, on se place dans le cas $L=1$ et on suppose que l'entier N est assez grand pour faire l'approximation $H((k+1)/N) \approx H(k/N)$. En combinant (1.2.2) avec la Proposition 1.2.1, on obtient l'approximation

$$\mathbb{E}|d_{N,k}|^2 \approx N^{-2H(k/N)}.$$

La quantité $H(t_0)$ est donc intimement reliée aux valeurs des $(d_{N,k})^2$ situés dans un voisinage de t_0. Fixons un réel $a \in\,]0,1/2[$. Pour tout entier $N \geq 2$, on définit le voisinage discret du point $t_0 \in [0,1]$, noté $\nu_N(t_0)$, par

$$\nu_N(t_0) := \left\{k \in \{0,\dots,N\}\,,\ \left|\frac{k}{N} - t_0\right| \leq N^{-a}\right\}.$$

On désigne ensuite par $V_N(t_0)$ le carré de la moyenne quadratique des $(d_{N,k})_k$ au voisinage du point t_0

$$V_N(t_0) := \frac{1}{\#(\nu_N(t_0))} \sum_{k\in\nu_N(t_0)} (d_{N,k})^2 .$$

C'est à partir des $V_N(t_0)$ que l'on peut construire un estimateur consistant de $H(t_0)$. Heuristiquement, $V_N(t_0)$ va suivre une loi forte des grands nombres et on va obtenir pour N assez grand et $k \in \nu_N(t_0)$

$$V_N(t_0) \approx \mathbb{E}|d_{N,k}|^2 \approx N^{-2H(k/N)} \approx N^{-2H(t_0)}.$$

Ainsi, on parvient aux estimations

$$H(t_0) \approx \frac{\log(V_N(t_0))}{-2\log(N)} \text{ et } H(t_0) \approx 2^{-1}\log\left(\frac{V_N(t_0)}{V_{2N}(t_0)}\right).$$

Le théorème suivant ([BCI98, Coe05, Coe06]) met en lumière la convergence de ce second estimateur. Cet estimateur a l'avantage de faire intervenir un quotient entre deux moyennes quadratiques et donc d'être invariant quand on multiplie le processus $\mathcal{M}$ par une constante.

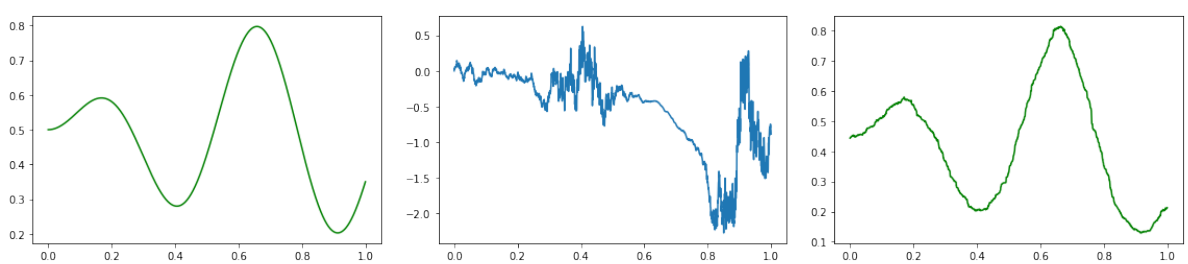
Théorème 1.2.7. *On suppose que $L \geq 2$ et que la condition (C) suivante est satisfaite par la fonction $H : \mathbb{R} \to]0,1[$*

$$(C) : \ \widetilde{\alpha}_H(t) > H(t), \ \text{pour tout } t \in \mathbb{R}.$$

Alors pour tout $t_0 \in [0,1]$

$$\widehat{H}_N(t_0) := 2^{-1}\log\left(\frac{V_N(t_0)}{V_{2N}(t_0)}\right)$$

est un estimateur fortement consistant de $H(t_0)$; c'est-à-dire que $\widehat{H}_N(t_0)$ tend presque sûrement vers $H(t_0)$ quand N tend vers $+\infty$.



Estimation de la fonction de Hurst du mouvement brownien multifractionnaire ($H(\cdot)$ à gauche, une trajectoire $\mathcal{M}$ associée au milieu et l'estimation de $H(\cdot)$ à droite)

Remarques 1.2.8. • La condition $L \geq 2$ permet de décorréler les accroissements et obtenir un théorème central limite pour $\widehat{H}_N(t_0)$.

• L'estimateur donné par le Théorème 1.2.7 converge point par point et ne converge à priori pas uniformément sur tout l'intervalle $[0,1]$. Il semble plus intéressant d'obtenir une convergence uniforme; c'est le cas dans l'article [AH17], où est construit, dans le cadre non gaussien du

mouvement multifractionaire stable linéaire, un estimateur de la fonction déterministe de Hurst qui converge uniformément à une vitesse que l'on peut estimer.
• Notons également que la preuve du Théorème 1.2.7 qui est donnée dans la littérature, que nous avons citée, repose essentiellement sur des méthodes gaussiennes qui nécessitent de disposer d'une estimation précise de la covariance des accroissements généralisés $d_{N,k}$ du mouvement brownien multifractionnaire.

Signalons que dans le Chapitre 6 de cette thèse nous construisons un estimateur qui converge presque sûrement, au sens de la norme uniforme, vers la fonction de Hurst aléatoire d'un processus multifractionnaire *non classique*. L'une des difficulté pour obtenir ce résultat est que ce dernier processus est non gaussien, en outre sa fonction de covariance est beaucoup plus complexe que celle du mouvement brownien multifractionnaire classique.

1.2.B Un premier type de processus multifractionnaire avec exposant aléatoire

Le fait que dans la définition (1.2.1) la fonction de Hurst $H(\cdot)$ soit déterministe est restrictif pour les applications. Par exemple, en finance, l'exposant de Hölder ponctuel peut être interprété comme le poids que donnent les investisseurs aux prix du passé pour prendre leurs décisions (voir [BPP12]) et il n'y a pas de raison à priori pour que ce poids soit déterministe. Il peut donc être intéressant de réussir à définir une nouvelle classe de processus multifractionnaires avec un paramètre fonctionnel aléatoire. Cependant, il n'est pas possible de simplement remplacer la fonction déterministe $H(\cdot)$ par un processus stochastique $\{S(t)\}_{t\in\mathbb{R}}$ dans la définition du mouvement brownien multifractionnaire car l'intégrale (1.2.1) ne serait plus définie au sens d'Itô.

En 2003, George C. Papanicolaou et Knut Solna suggèrent dans [PS03] de remplacer la fonction de Hurst déterministe $H(\cdot)$ du mouvement brownien multifractionnaire par un processus stochastique $\{S(t)\}_{t\in\mathbb{R}}$ suffisamment régulier, à accroissements stationnaires et indépendant du mouvement brownien $\{B(s)\}_{s\in\mathbb{R}}$. Cependant, leur idée ne se généralise pas du tout au cas général où $\{S(t)\}_{t\in\mathbb{R}}$ peut dépendre du mouvement brownien $\{B(s)\}_{s\in\mathbb{R}}$ car l'intégrale (1.2.1) ne serait plus correctement définie.

En 2005, Antoine Ayache et Murad S. Taqqu définissent dans [AT05] un nouveau processus multifractionnaire avec une fonction de Hurst aléatoire à partir d'un processus stochastique $\{S(t)\}_{t\in[0,1]}$ à valeurs dans un compact $[a,b]\subset\,]0,1[$. Ils introduisent tout d'abord le champ gaussien $\{B_H(t)\}_{(t,H)\in[0,1]\times[a,b]}$ défini par

$$\forall(t,H)\in[0,1]\times[a,b],\ B_H(t):=\int_{\mathbb{R}}\left((t-s)_+^{H-1/2}-(-s)_+^{H-1/2}\right)dB(s) \tag{1.2.3}$$

et qui à H fixé correspond, à un facteur multiplicatif près, à la définition (1.1.2) du mouvement brownien fractionnaire. Ils obtiennent ensuite la convergence uniforme en (t,H) pour la représentation en série aléatoire suivante du gaussien défini par (1.2.3) suivante

$$\forall(t,H)\in[0,1]\times[a,b],\ B_H(t):=\sum_{j=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty}a_{j,k}(t,H)\varepsilon_{j,k},$$

où $\{\varepsilon_{j,k}\}_{j,k\in\mathbb{Z}}$ sont des variables aléatoires indépendantes de même loi $\mathcal{N}(0,1)$, et où les coefficients déterministes $a_{j,k}(t,H)$ sont donnés par la relation

$$\forall (t,H) \in [0,1]\times[a,b],\ \forall j,k\in\mathbb{Z},\ a_{j,k}(t,H) := \int_{\mathbb{R}} \frac{e^{it\xi}-1}{i\xi|\xi|^{H-1/2}}\widehat{\psi}_{j,k}(\xi)d\xi. \tag{1.2.4}$$

Notons que $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ désigne une base d'ondelettes de Lemarié-Meyer de $L^2(\mathbb{R})$. Le nouveau processus $\{Z(t)\}_{t\in[0,1]}$ peut alors être défini par la relation

$$\forall t\in[0,1],\ Z(t) := \sum_{j=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} a_{j,k}(t,S(t))\varepsilon_{j,k}. \tag{1.2.5}$$

Ce processus fait parti d'une nouvelle classe de processus multifractionnaires, appelée MPRE *(Multifractional Processes with Random Exponent).*

L'étude d'un tel processus avec un paramètre fonctionnel aléatoire est bien plus complexe que celle du mouvement brownien multifractionnaire. En effet, contrairement au mouvement brownien multifractionnaire ce nouveau processus n'est pas gaussien; de plus, sa loi et sa fonction de covariance ne sont pas explicites, et comme nous l'avons vu avec les exemples du mouvement brownien fractionnaire et du mouvement brownien multifractionnaire, avoir accès à une formule explicite de la fonction de covariance d'un processus est un outil précieux pour l'étudier.

Dans le même article, Ayache et Taqqu déterminent les exposants de Hölder ponctuel et local du processus $\{Z(t)\}_{t\in[0,1]}$ dans le cas général où ce dernier n'est à priori pas indépendant du mouvement brownien $\{B(s)\}_{s\in\mathbb{R}}$.

Théorème 1.2.9. *Supposons que la condition (C) suivante soit satisfaite presque-sûrement par le processus stochastique* $\{S(t)\}_{t\in[0,1]}$

$$(C):\ \widetilde{\alpha}_S(t) > S(t),\ \textit{pour tout}\ t\in\mathbb{R}.$$

où $\beta_S([0,1])$ *désigne l'exposant de Hölder global de* S *(voir (1.1.5)) sur l'intervalle* $[0,1]$*. Alors, pour tout* $\tau\in\mathbb{R}$*, il existe un événement* $\Omega_*(\tau)$ *de probabilité 1, tel que*

$$\widetilde{\alpha}_Z(\tau,\omega) = \alpha_Z(\tau,\omega) = S(\tau,\omega)\,,\quad \textit{pour tout}\ \omega\in\Omega_*(\tau).$$

Le premier ingrédient pour démontrer ce théorème est une majoration des coefficients aléatoires $(\varepsilon_{j,k})_{j,k\in\mathbb{Z}}$, présents dans (1.2.5), au moyen du lemme suivant.

Lemme 1.2.10. *Il existe une variable aléatoire* $C>0$ *avec des moments finis de tout ordre et un événement* Ω_{**} *de probabilité 1 tels que*

$$\forall\omega\in\Omega_{**},\ \forall j,k\in\mathbb{Z},\ |\varepsilon_{j,k}(\omega)| \le C(\omega)\sqrt{\log(2+|j|)}\sqrt{\log(2+|k|)}.$$

Remarque 1.2.11. Ce lemme est très commode pour s'affranchir des $(\varepsilon_{j,k})_{j,k\in\mathbb{Z}}$ lorsque l'on cherche, au moyen de la série aléatoire, à majorer la valeur absolu d'un accroissement du processus Z. Signalons au passage que des résultats analogues au Lemme 1.2.10 jouent un rôle important dans le Chapitre 5 et le Chapitre 7 de cette thèse.

Le second ingrédient de la preuve du Théorème 1.2.9 est l'estimation des coefficients $(a_{j,k}(t,S(t)))_{j,k\in\mathbb{Z}}$ définis à (1.2.4). L'ondelette $\psi_{j,k}$ de Lemarié-Meyer est très régulière et permet donc de faire tendre ces coefficients très vite vers 0 quand $k\to\infty$ en effectuant des intégrations par parties. Soulignons tout de même que bien que ces coefficients tendent très vite vers 0, ils ne sont pas explicites ce qui rend la définition (1.2.5) peu adaptée pour simuler le processus.

La méthode du temps local, que nous avons décrite précédemment, ne semble par permettre d'étudier l'irrégularité de ce processus Z puisque l'on ne sait pas si ce processus admet un temps local, de plus on ne dispose pas de formules explicites pour les fonctions caractéristiques de ses lois finis dimensionnelles (voir Remarques 1.1.17).
De plus, un autre défaut considérable du processus Z est qu'il n'a pas de représentation sous forme d'intégrale d'Itô, il est donc difficile d'utiliser les outils performants du calcul stochastique d'Itô pour l'étudier.
Dans un autre registre, une condition de régularité (C) est toujours nécessaire pour le Théorème 1.2.9 et on aimerait l'affaiblir pour s'autoriser des paramètres fonctionnels moins réguliers.

1.3 Processus multifractionnaires non classiques

1.3.A Les processus multifractionnaires de Surgailis

C'est en 2008, dans son article [Sur08], que Donatas Surgailis va introduire un nouveau type de processus multifractionnaire. Ces processus sont construits à partir d'une fonction réelle continue $\alpha(\cdot)$ à valeurs dans $]-1/2,1/2[$ qui pourra être interprétée comme $H(\cdot)-1/2$ où $H(\cdot)$ est la fonction de Hurst. Les processus de Surgailis $\{X(t)\}_{t\in\mathbb{R}}$ et $\{Y(t)\}_{t\in\mathbb{R}}$ sont donnés par les relations

$$X(t) := \int_{\mathbb{R}} \left(\int_0^t \frac{1}{\Gamma(\alpha(\tau))} (\tau - s)_+^{\alpha(\tau)-1} e^{H_-(s,\tau)} d\tau \right) dB(s)$$

et

$$Y(t) := \int_{\mathbb{R}} \frac{1}{\Gamma(1+\alpha(s))} \left((t-s)_+^{\alpha(s)} e^{-H_+(s,t)} - (-s)_+^{\alpha(s)} e^{-H_+(s,0)} \right) dB(s),$$

avec pour tous $s<t$

$$H_-(s,t) := \int_s^t \frac{\alpha(u)-\alpha(t)}{t-u} du,\ \ H_+(s,t) := \int_s^t \frac{\alpha(s)-\alpha(v)}{v-s} dv.$$

Ces processus proviennent d'opérateurs d'intégration et de dérivation multifractionnaires et sont plus complexes que le mouvement brownien multifractionnaire *classique*. Une importante idée nouvelle qui transparaît déjà est que le paramètre fonctionnel de tels processus peut ne pas dépendre de la variable t qui indexe le processus, mais être dépendant de la variable d'intégration s associée à l'intégrale stochastique qui représente le processus. Un tel processus stochastique où le paramètre fonctionnel dépend de la variable d'intégration est dit multifractionnaire *non classique*.

Surgailis montre que ces deux processus sont bien définis lorsque $\alpha(\cdot)$ vérifie les trois conditions suivantes. La première condition est une condition de Dini uniforme suivante

$$(1):\ \sup_{t\in\mathbb{R}}\int_{-1}^{1}\frac{|\alpha(t)-\alpha(t+u)|}{|u|}du<+\infty.$$

La deuxième condition est la suivante

$$(2):\ \inf_{u\in\mathbb{R}}\alpha(u)>0 \text{ pour } X,\ \inf_{u\in\mathbb{R}}\alpha(u)>-1/2 \text{ pour } Y.$$

La troisième condition est une majoration de la moyenne généralisée supérieure de Césaro

$$(3):\ \overline{\alpha}_{\sup}:=\limsup_{t-s\to+\infty}\frac{1}{t-s}\int_{s}^{t}\alpha(u)du<1/2. \tag{1.3.1}$$

Soulignons que contrairement au mouvement brownien multifractionnaire *classique*, la condition (1.3.1) permet au paramètre fonctionnel $\alpha(\cdot)$ de prendre parfois de grandes valeurs. Surgailis démontre également au sens faible (c'est-à-dire au sens des lois finis dimensionnelles) l'auto-similarité locale asymptotique de ces processus.

Théorème 1.3.1. *Soit $\tau\in\mathbb{R}$, tel que $\alpha(\tau)\in\,]0,1/2[$ dans le cas de X et $\alpha(\tau)\in\,]-1/2,1/2[\setminus\{0\}$ dans le cas de Y. Supposons que la condition (D) suivante soit satisfaite par la fonction $\alpha(\cdot)$*

$$(D):\ |\alpha(\tau+h)-\alpha(\tau)|\underset{h\to0}{=}o\left(\frac{1}{|\log(h)|}\right).$$

Alors les processus X et Y sont faiblement localement asymptotiquement auto-similaires au point τ d'ordre $H(\tau)=\alpha(\tau)+1/2$. En somme, il existe une constante strictement positive $c(\tau)$ telle que le processus $\left\{\lambda^{-H(\tau)}(X(\tau+\lambda t)-X(\tau))\right\}_{t\in\mathbb{R}}$ converge au sens des lois finis dimensionnelles vers le mouvement brownien fractionnaire $\left\{c(\tau)B_{H(\tau)}(t)\right\}_{t\in\mathbb{R}}$ quand $\lambda\to0^+$ (de même pour Y).

Remarque 1.3.2. La condition (D) sur le paramètre $\alpha(\cdot)$ des processus de Surgailis est beaucoup plus faible que la condition (C) (présente par exemple dans le Théorème 1.2.6) sur la fonction de Hurst $H(\cdot)$ du mouvement brownien multifractionnaire.

Faire dépendre la fonction de Hurst de la variable d'intégration pourrait ne pas sembler très judicieux à prime abord. En effet, il n'est pas certain que la méthode d'ondelettes qui fonctionnait bien pour les processus multifractionnaires *classiques* puisse d'adapter à ce nouveau contexte. Pour illustrer cette affirmation, reprenons les coefficients d'ondelettes $(a_{j,k}(t,H))_{j,k\in\mathbb{Z}}$ définis à (1.2.4)

$$a_{j,k}(t,H):=\int_{\mathbb{R}}\frac{e^{it\xi}-1}{i\xi|\xi|^{H-1/2}}\widehat{\psi}_{j,k}(\xi)d\xi.$$

Lorsque H est constante ou dépend de la variable t, on montre que ces coefficients tendent très vite vers 0 quand $k\to+\infty$ en effectuant des intégrations par parties. Cependant, si on fait dépendre H de la variable d'intégration ξ alors on peut plus difficilement intégrer par parties; les coefficients convergent lentement et on ne réussit plus à déterminer les exposants de Hölder.

Aucun résultat sur la régularité locale des processus de Surgailis n'apparaît alors explicitement dans la littérature. D'autre part, Surgailis reconnaît lui même dans l'article [BS13a] qu'il sera bien plus complexe d'estimer le paramètre fonctionnel (comme le fait le Théorème 1.2.7) de ses processus et plus généralement, des processus multifractionnaires *non classiques* car leur fonction de covariance devient beaucoup plus complexe.

Sous certaines hypothèses sur la fonction $\alpha(\cdot)$, nous apportons une réponse à chacun de ces deux problèmes dans le Chapitre 3 de cette thèse. En particulier, nous déterminons les exposants de Hölder locaux et ponctuels des processus de Surgailis, pour un événement universel qui ne dépend pas du point considéré, en montrant que la différence entre chacun des deux processus de Surgailis et le mouvement brownien multifractionnaire est plus régulière que ce dernier. De plus, nous montrons que l'estimateur fortement consistant $\widehat{H}_N(t_0)$ du Théorème 1.2.7 fonctionne également pour les processus de Surgailis.

1.3.B Une nouvelle classe de processus multifractionnaires non classiques avec exposant aléatoire

1.3.B.a Le Riemann-Liouville MPRE

C'est en s'inspirant plus ou moins de l'idée introduite dix ans plus tôt par Surgailis que Antoine Ayache, Céline Esser et Julien Hamonier construisent dans l'article [AEH18] un nouveau processus multifractionnaire *non classique* au moyen de l'intégrale d'Itô, dont le paramètre fonctionnel n'est plus une fonction déterministe $H(\cdot)$ mais un processus stochastique dont l'indice est la variable d'intégration. De cette manière, le paramètre fonctionnel peut être adapté à la filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ à laquelle est associé le mouvement brownien $\{B(s)\}_{s\in\mathbb{R}}$. Pour cela, ils considèrent uniquement la partie haute-fréquence du mouvement brownien fractionnaire, aussi appelé processus de Riemann-Liouville fractionnaire, notée $\{R_H(t)\}_{t\in[0,1]}$, qui gouverne la régularité des trajectoires du mouvement brownien fractionnaire, et qui est définie pour $H \in \,]0,1[$ par

$$\forall t \in [0,1],\ R_H(t) := \int_0^1 (t-s)_+^{H-1/2} dB(s). \tag{1.3.2}$$

Ils remplacent ensuite le paramètre de Hurst constant H par un processus stochastique $\{H(s)\}_{s\in[0,1]}$ adapté à la filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ et à valeurs dans un compact déterministe $[\underline{H},\overline{H}] \subset \,]1/2,1[$. Ce nouveau processus $\{X(t)\}_{t\in[0,1]}$ est donc défini par

$$\forall t \in [0,1],\ X(t) := \int_0^1 K_t(s) dB(s)$$

avec

$$K_t(s) := (t-s)_+^{H(s)-1/2}. \tag{1.3.3}$$

Ils décomposent ensuite le noyau (1.3.3) dans la base de Haar de $L^2([0,1])$ qui est composée des fonctions

$$\mathcal{U} := \mathbb{1}_{[0,1[} \text{ et } h_{j,k} := 2^{j/2}\left(\mathbb{1}_{[2^{-j}k,2^{-j}(k+1/2)[} - \mathbb{1}_{[2^{-j}(k+1/2),2^{-j}(k+1)[}\right),\ j \in \mathbb{N} \text{ et } k \in \left\{0,\dots,2^j-1\right\}, \tag{1.3.4}$$

et obtiennent une représentation de $\{X(t)\}_{t\in[0,1]}$ en série aléatoire, donnée par le théorème suivant.

Théorème 1.3.3. *Supposons que pour certaines constantes* $\rho \in]1/2, 1]$ *et* $c > 0$ *on a*

$$\forall x, y \in [0,1],\ \mathbb{E}\left(|H(x) - H(y)|^2\right) \leq c|x-y|^{2\rho}.$$

Alors, il existe un événement universel Ω_{**} *de probabilité 1, tel que*

$$\forall \omega \in \Omega_{**},\ X(t,\omega) = \langle K_t(\cdot,\omega), \mathcal{U}\rangle \eta_0(\omega) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \langle K_t(\cdot,\omega), h_{j,k}\rangle \varepsilon_{j,k}(\omega),$$

où la convergence est uniforme en t *sur* $[0,1]$ *et où*

$$\eta_0 := \int_0^1 \mathcal{U}(s)dB(s) = B(1) - B(0)$$

et

$$\varepsilon_{j,k} := \int_0^1 h_{j,k}(s)dB(s) = 2^{j/2}\left(2B(2^{-(j+1)}(2k+1)) - B(2^{-j}k) - B(2^{-j}(k+1))\right)$$

sont des variables aléatoires de même loi $\mathcal{N}(0,1)$.

Remarque 1.3.4. Dans ce cadre, l'un des intérêts de la base de Haar ici est qu'elle donne des coefficients que l'on peut facilement estimer. En effet, en combinant (1.3.3) et (1.3.4) on obtient

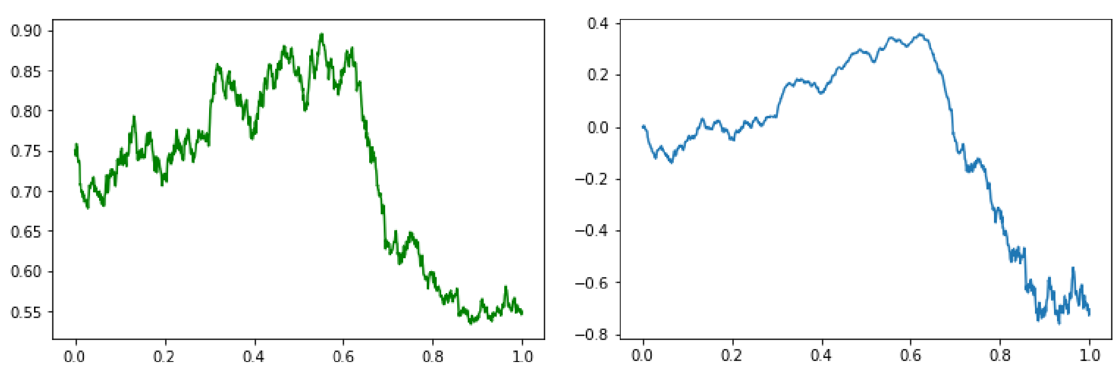
$$\langle K_t, h_{j,k}\rangle = 2^{j/2}\left(\int_{k2^{-j}}^{(k+1/2)2^{-j}} (t-s)_+^{H(s)-1/2} ds - \int_{(k+1/2)2^{-j}}^{(k+1)2^{-j}} (t-s)_+^{H(s)-1/2} ds\right).$$

On fait ensuite l'approximation $H(s) \approx H(k2^{-j})$ lorsque $s \in [k2^{-j}, (k+1)2^{-j}[$ pour avoir

$$\langle K_t, h_{j,k}\rangle \approx 2^{j/2}\left(\int_{k2^{-j}}^{(k+1/2)2^{-j}} (t-s)_+^{H(k2^{-j})-1/2} ds - \int_{(k+1/2)2^{-j}}^{(k+1)2^{-j}} (t-s)_+^{H(k2^{-j})-1/2} ds\right)$$

$$\langle K_t, h_{j,k}\rangle \approx \frac{2^{j/2}}{H(k2^{-j})+1/2}\left((t-k2^{-j})_+^{H(k2^{-j})+1/2} - (t-(k+1)2^{-j})_+^{H(k2^{-j})+1/2}\right).$$

Les coefficients $\langle K_t, h_{j,k}\rangle$ peuvent donc être approximés facilement ce qui permet au même article [AEH18] d'introduire deux méthodes de simulation du processus $\{X(t)\}_{t\in[0,1]}$.



Une réalisation $H(\cdot,\omega)$ *à gauche et la trajectoire associée* $X(\cdot,\omega)$ *à droite*

Cependant, les conditions du Théorème 1.3.3 sont toujours nécessaires et ils ne parviennent pas à obtenir une méthode de simulation de X dans un cadre plus général.
Toutefois, nous présentons dans le Chapitre 5 de cette thèse, une méthode de simulation du processus $\{X(t)\}_{t\in[0,1]}$ dans le cas où le processus stochastique $\{H(s)\}_{s\in[0,1]}$ est à valeurs dans un compact déterministe de $]0,1[$ et à trajectoires non continues.

Par ailleurs, grâce à cette décomposition de X via la base de haar, Ayache, Esser et Hamonier obtiennent une minoration des exposants de Hölder.

Théorème 1.3.5. *Supposons que les trajectoires de $\{H(s)\}_{s\in[0,1]}$ soient presque-sûrement hölderiennes d'ordre $\gamma > 1/2$ et que pour certaines constantes $\rho \in]1/2,1]$ et $c > 0$ on ait*

$$\forall x, y \in [0,1],\ \mathbb{E}\left(|H(x) - H(y)|^2\right) \leq c|x-y|^{2\rho}.$$

Alors, il existe un événement universel Ω_* de probabilité 1, tel que

$$\alpha_X(\tau,\omega) \geq \widetilde{\alpha}_X(\tau,\omega) \geq H(\tau,\omega)\,, \quad \textit{pour tout } (\tau,\omega) \in [0,1]\times\Omega_*.$$

Remarque 1.3.6. La condition d'hölderiennité du Théorème 1.3.5 est parfois plus faible que la condition (C) des processus multifractionnaires *classiques*, c'est par exemple le cas lorsque $\underline{H} \geq \gamma$.

1.3.B.b Récents résultats sur une classe de processus plus générale que le Riemann Liouville MPRE

Récemment dans [LMS21], Dennis Loboda, Fabian Mies et Ansgar Steland ont proposé un nouvel angle d'attaque pour étudier la régularité du Riemann Liouville MPRE ainsi qu'une vaste classe de processus qui le généralise. En effet, les processus stochastiques qu'ils étudient sont de la forme

$$\forall t \geq 0,\ X(t) := \int_{-\infty}^{t} g(t,s)dB(s) \tag{1.3.5}$$

où pour tout réel $t \geq 0$ fixé, l'intégrande aléatoire $\{g(t,s)\}_{s\in]-\infty,t[}$ est un processus adapté à la filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ et vérifiant $\int_{-\infty}^{t}|g(t,s)|^2ds < +\infty$ presque-sûrement; ce qui garantit l'existence de l'intégrale d'Itô dans (1.3.5). En outre, pour pouvoir étudier la régularité de X, Loboda, Mies et Steland imposent à g plusieurs conditions supplémentaires qui sont présentées dans [LMS21] ainsi que dans le Chapitre 4 de cette thèse; citons tout de même deux exemples de processus g qui vérifient ces conditions:

$$\forall (t,s) \in \mathbb{R}_+\times\mathbb{R},\ g_1(t,s) := (t-s)_+^{H(s)-1/2} - (-s)_+^{H(s)-1/2}$$

et

$$\forall (t,s) \in \mathbb{R}_+\times\mathbb{R},\ g_2(t,s) := (t-s)_+^{H(s)-1/2}e^{-\lambda(t-s)},$$

où $\lambda > 0$ est fixé et $\{H(s)\}_{s\in\mathbb{R}}$ est un processus stochastique adapté à la filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ et à valeurs dans $[\underline{H}, \overline{H}] \subset]0,1[$. Le processus g_1 correspond à l'intégrande du mouvement brownien multifractionnaire (1.2.1) où on a remplacé $H(t)$ par le processus stochastique $\{H(s)\}_{s\in\mathbb{R}}$. Le processus g_2 correspond quant à lui au noyau du processus de Matérn ([LSEO17]).

Leur idée directrice est de tirer du fait que ces processus disposent d'une représentation via l'intégrale d'Itô pour leur appliquer certains outils performants du calcul stochastique d'Itô. Ils commencent par établir une judicieuse généralisation du Théorème 1.1.5 *(Kolmogorov-Chentsov)* de sorte à ce que ce théorème devienne un outil bien adapté pour la minoration des exposants de Hölder aléatoires.

Théorème 1.3.7. *Soient $T > 0$ ainsi que $\{Y(t)\}_{t\in[0,T]}$ à valeurs réelles et $\{a(t)\}_{t\in[0,T]}$ à valeurs dans $]0,1[$ deux processus stochastiques. Le processus $\{a(t)\}_{t\in[0,T]}$ est à trajectoires semi-continues inférieurement et pour tout ouvert ou fermé $B \subset [0,T]$ la fonction $\underline{a}_B := \inf_{t\in B} a(t)$ est mesurable et ne s'annule pas. Supposons que pour un certain $p > 0$, il existe $\varepsilon > 0$ et une constante $C(p,\epsilon)$ tels que pour tous $t, t' \in [0,T]$ avec $|t-t'| \leq \varepsilon$ on ait*

$$\mathbb{E}\left|\frac{Y(t)-Y(t')}{|t-t'|^{\underline{a}_{[t,t']}}}\right|^p \leq C(p,\varepsilon)|t-t'|.$$

Alors $\{Y(t)\}_{t\in[0,T]}$ admet une modification dont les trajectoires sont, sur tout intervalle fermé $B \subset [0,T]$, $\underline{a}_B$-hölderiennes. Ainsi, il existe un événement universel Ω_* de probabilité 1 tel que

$$\alpha_Y(\tau,\omega) \geq \widetilde{\alpha}_Y(\tau,\omega) \geq a(\tau,\omega)\,, \quad \text{pour tout } (\tau,\omega) \in]0,T[\times\Omega_*.$$

Remarque 1.3.8. Lorsque a est constante et déterministe, on retrouve le Théorème 1.1.5 avec $\delta = p$ et $\varepsilon = ap$.

Corollaire 1.3.9. *Soient $T > 0$ ainsi que $\{Y(t)\}_{t\in[0,T]}$ et $\{a(t)\}_{t\in[0,T]}$ comme dans le Théorème 1.3.7. Supposons que pour tous $p > 1$ et $\delta > 0$, il existe un $\varepsilon > 0$ et une constante $C(p,\varepsilon,\delta)$ tels que pour tous $t, t' \in [0,T]$ avec $|t-t'| \leq \varepsilon$ on ait*

$$\mathbb{E}\left|\frac{Y(t)-Y(t')}{|t-t'|^{\underline{a}_{[t,t']}-\delta}}\right|^p \leq C(p,\varepsilon,\delta).$$

Alors $\{Y(t)\}_{t\in[0,T]}$ admet une modification dont les trajectoires sont, sur tout intervalle fermé $B \subset [0,T]$, $\underline{a}_B$-hölderiennes. Ainsi, il existe un événement universel Ω_* de probabilité 1 tel que

$$\alpha_Y(\tau,\omega) \geq \widetilde{\alpha}_Y(\tau,\omega) \geq a(\tau,\omega)\,, \quad \text{pour tout } (\tau,\omega) \in]0,T[\times\Omega_*.$$

Pour appliquer le Corollaire 1.3.9, il faut être capable de majorer les moments des accroissements de Y. Pour cette raison, l'inégalité de Burkholder-Davis-Gundy (voir [Mao07, Pro05]) donné dans la proposition suivante joue un rôle tout à fait fondamental dans l'article [LMS21].

Proposition 1.3.10. *Soit $p \in [1,+\infty[$ arbitraire et fixé. Il existe une constante déterministe universelle $a(p)$ pour laquelle on a le résultat suivant: pour tout processus stochastique $\{f(s)\}_{s\in\mathbb{R}}$ qui est $(\mathcal{F}_s)_{s\in\mathbb{R}}$-adapté qui vérifie presque-sûrement $\int_{-\infty}^{+\infty} |f(s)|^2\,ds < +\infty$, on a*

$$\mathbb{E}\left(\Big|\int_{-\infty}^{+\infty} f(s)\,dB(s)\Big|^p\right) \leq a(p)\mathbb{E}\left(\left(\int_{-\infty}^{+\infty} |f(s)|^2\,ds\right)^{p/2}\right), \tag{1.3.6}$$

où $\int_{-\infty}^{+\infty} f(s)\,dB(s)$ désigne l'intégrale d'Itô de f sur $\mathbb{R}$.

Remarque 1.3.11. L'inégalité de Burkholder-Davis-Gundy est à mettre en parallèle avec l'équivalence des moments gaussiens donnée par le Lemme 1.1.6. En effet, cette inégalité donne une majoration des moments de tout ordre $p \geq 1$ au moyen du moment d'ordre 2.

Dans la suite de cette partie, et bien que leurs résultats restent valables pour une classe de processus plus vaste (voir [LMS21] ou Chapitre 4), on supposera dans un soucis de simplicité que le processus $\{X(t)\}_{t\geq 0}$ est de la forme

$$\forall t \geq 0,\ X(t) := \int_{\mathbb{R}} \left((t-s)_+^{H(s)-1/2} - (-s)_+^{H(s)-1/2} \right) dB(s), \tag{1.3.7}$$

où $\{H(s)\}_{s\in\mathbb{R}}$ est un processus stochastique adapté à la filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ et à valeurs dans $[\underline{H}, \overline{H}] \subset]0,1[$. En combinant l'inégalité (1.3.6) et le Corollaire 1.3.9, l'article [LMS21] parvient à obtenir minorations des exposants de Hölder local et ponctuel données par la théorème suivant.

Théorème 1.3.12. *S'il existe une fonction déterministe croissante et continue $\mu : \mathbb{R}_+ \mapsto \mathbb{R}_+$ avec $\mu(0) = 0$ telle que presque sûrement*

$$\forall s', s'' \in \mathbb{R},\ |H(s') - H(s'')| \leq \mu(|s' - s''|), \tag{1.3.8}$$

alors il existe un événement universel Ω_1^* de probabilité 1 tel que

$$\alpha_X(\tau,\omega) \geq \widetilde{\alpha}_X(\tau,\omega) \geq H(\tau,\omega)\,, \quad \text{pour tout } (\tau,\omega) \in \mathbb{R}_+^* \times \Omega_1^*.$$

Sinon, il existe un événement universel Ω_2^* de probabilité 1 tel que

$$\alpha_X(\tau,\omega) \geq \widetilde{\alpha}_X(\tau,\omega) \geq \underline{H}\,, \quad \text{pour tout } (\tau,\omega) \in \mathbb{R}_+^* \times \Omega_2^*. \tag{1.3.9}$$

Remarques 1.3.13. • Le fait que la fonction μ doit être déterministe dans la condition (1.3.8) semble être restrictif. Cependant, il est possible de relâcher cette hypothèse de fonction déterministe avec une procédure de localisation via des temps d'arrêts (voir section 4.3). Notons aussi que cette condition (1.3.8) est beaucoup plus faible que la condition (C) (par exemple dans le Théorème 1.2.4) du mouvement brownien multifractionnaire.
• Les inégalités (1.3.9) sont obtenues même dans le cas où H est discontinue ce qui implique la continuité des trajectoires du processus X, contrairement au mouvement brownien multifractionnaire classique qui était discontinu aux points de discontinuité de la fonction de Hurst (voir Proposition 1.2.2).

De plus, Loboda, Mies et Steland réussissent à obtenir une majoration des exposants de Hölder, qui est cependant valable pour un événement qui dépend du point considéré, à partir de la forte convergence locale asymptotique du processus X. Plus précisément :

Théorème 1.3.14. *On suppose qu'il existe une fonction déterministe croissante et continue $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ telle que presque sûrement*

$$(D):\ \forall s', s'' \in \mathbb{R},\ |H(s') - H(s'')| \leq \mu(|s'-s''|) \underset{|s-s'|\to 0}{=} o\left(\frac{1}{|\log(|s-s'|)|}\right).$$

Alors en tout point $\tau > 0$, le processus X est fortement localement asymptotiquement auto-similaire d'ordre $H(\tau)$. En somme, pour tous réels $a < 0 < b$ fixés le processus $\left\{\lambda^{-H(\tau)}(X(\tau+\lambda t) - X(\tau))\right\}_{t\in[a,b]}$ converge en loi vers un mouvement brownien fractionnaire $\left\{\int_{\mathbb{R}} \left((t-s)_+^{H(\tau)-1/2} - (-s)_+^{H(\tau)-1/2}\right) d\widetilde{B}(s)\right\}_{t\in[a,b]}$ quand $\lambda \to 0^+$ dans l'espace des fonctions continues $\mathcal{C}([a,b])$ et où $\left\{\widetilde{B}(s)\right\}_{s\in[a,b]}$ est un mouvement brownien indépendant de $H(\tau)$.

Corollaire 1.3.15. *Sous les mêmes conditions que le Théorème 1.3.14: pour tout point $\tau \in \mathbb{R}$, il existe un événement $\Omega_*(\tau)$ de probabilité 1, tel que*

$$\widetilde{\alpha}_X(\tau,\omega) \leq \alpha_X(\tau,\omega) \leq H(\tau,\omega)\,, \quad \textit{pour tout } \omega \in \Omega_*(\tau). \tag{1.3.10}$$

Théorème 1.3.16. *Sous les mêmes conditions que le Théorème 1.3.14: pour tout point $\tau \in \mathbb{R}$, il existe un événement $\Omega_*(\tau)$ de probabilité 1, tel que*

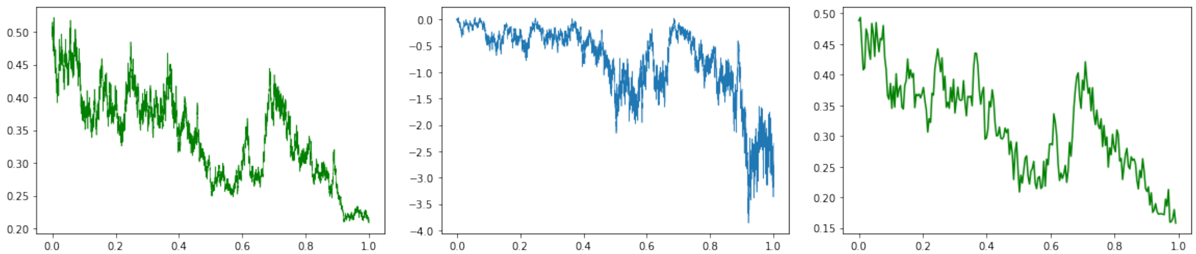
$$\widetilde{\alpha}_X(\tau,\omega) = \alpha_X(\tau,\omega) = H(\tau,\omega)\,, \quad \textit{pour tout } \omega \in \Omega_*(\tau). \tag{1.3.11}$$

Remarque 1.3.17. La condition de régularité (D) qui apparaît dans l'énoncé du Théorème 1.3.14 est beaucoup plus faible que la condition (C) du Théorème 1.2.4.

Bien que ce nouveau Théorème 1.3.16 constitue un progrès considérable dans l'étude de la régularité locale des MPRE, aucun résultat n'est alors connu pour un événement universel qui ne dépend pas du point considéré (voir Remarque 1.1.10). Sous une faible hypothèse de régularité de Hölder sur les trajectoires de H, nous proposons un tel résultat dans le Chapitre 4. La démonstration de ce nouveau résultat repose sur une stratégie de minoration des oscillations locales du processus; historiquement, ce sont les méthodes du temps local qui permettaient d'obtenir de telles inégalités (voir Théorème 1.1.16) mais ces méthodes sont difficilement utilisables dans le contexte du processus X qui est *non classiques* (voir Remarques 1.1.17). Notre méthode s'inspire de celle développée par Antoine Ayache en 2020 dans l'article [Aya20] pour le processus d'Hermite, bien qu'elle fut significativement modifiée pour s'adapter au contexte de ces processus multifractionnaires *non classiques* avec un paramètre fonctionnel aléatoire.

D'autre part, comme nous avons déjà eu l'occasion de le mentionner, nous donnons dans le Chapitre 5 une méthode de simulation du MPRE Riemann-Liouville (1.3.2) sous des hypothèses bien plus faibles que celles de l'article [AEH18]. Cette méthode de simulation consiste pour $s \in [k2^{-j}, (k+1)2^{-j}[$ à approximer $H(s)$ par sa valeur moyenne sur ce même intervalle dyadique. L'inégalité de Burkholder-Davis-Gundy (1.3.6) est un élément clé de nos démonstrations.

Enfin, nous proposons dans le Chapitre 6 un estimateur uniforme du paramètre H du processus (1.3.7). C'est à notre connaissance la première fois qu'est mis en exergue un estimateur qui converge au sens de la norme uniforme d'un paramètre fonctionnel aléatoire.



Estimation du paramètre fonctionnel aléatoire
(une réalisation $H(\cdot,\omega)$ à gauche, la trajectoire $X(\cdot,\omega)$ associée au milieu et l'estimation de $H(\cdot,\omega)$ à droite)

Chapter 2

Introduction and background

Let us consider a complete filtered probability space $\left(\Omega, (\mathcal{F}_s)_{s\in\mathbb{R}}, \mathcal{F}, \mathbb{P}\right)$ and $\{B(s)\}_{s\in\mathbb{R}}$ a $(\mathcal{F}_s)$-Brownian motion ([LL12]), i.e. a real-valued Gaussian process with continuous paths which satisfies:
- For all $s \in \mathbb{R},\ B(s)$ is $\mathcal{F}_s$-measurable.
- If $s \leq t,\ B(t) - B(s)$ is independent of the σ-algebra $\mathcal{F}_s$.
- If $s \leq t$, the distribution of $B(t) - B(s)$ is the same as the distribution of $B(t-s) - B(0)$.

On the other hand, $L^2(\mathbb{R})$ denotes the space of square-integrable functions on $\mathbb{R}$ and with real values.

2.1 Fractional Brownian motion

2.1.A From Brownian motion to fractional Brownian motion

In 1827, the biologist Robert Brown discovered what would later be called Brownian motion. During his experiments on pollen particles suspended in water, he observed that these particles had a constant and anarchic movement, but he was unable to explain the origin of this movement. At first he thought that this property was specific to pollen particles, but later he found these strange movements with other types of particles such as dust particles. His observations were published in 1828 in [Bro28]. It was in 1900 that a mathematical theory of Brownian motion was formulated for the first time in the Louis Bachelier's thesis [Bac00]. Then, it was in 1905 in [Ein05] that Albert Einstein gave for the first time a physical and statistical explanation of Brownian motion as a consequence of the multiple shocks of the particles studied with the water molecules. They both highlighted several properties of the Brownian motion, namely independent and Gaussian increments with variance proportional to the elapsed time and also continuous paths. However, no one had yet rigorously proven the mathematical existence of a process that would satisfy the fundamental properties of the Brownian motion. Norbert Wiener showed in 1923 in [Wie23] the existence of such a process $\{B(s)\}_{s\in\mathbb{R}}$ called Brownian motion or Wiener process in his honour.

As the Gaussian distribution, can be found in a hudge number of phenomena (physical, biological and social), the distribution of the Brownian motion can be found in many situations;

indeed, the distribution of the Brownian motion appears as a limit of some random walks, for example, through the Donsker's functional central limit theorem. It is this fundamental property that makes the Brownian motion a very useful mathematical object in many domains, such as finance, where it is omnipresent. In 1951, the hydrologist Harold E. Hurst pointed out in [Hur51] some correlations in the annual flood data of the Nile River; since the increments of the Brownian motion are always independent, it is therefore not a good candidate to describe this kind of phenomenon. It is thus necessary to find a new type of process which will make possible to take into account of the property of *long memory* which is present in particular in the Hurst's data and which induces a correlation of increments in the data.

It was Benoît Mandelbrot who recognised this property of *long memory* in another stochastic process called *Wiener Helix* and introduced in 1940 by Andreï Kolmogorov in [Kol40] for his work on Hilbert spaces. Mandelbrot and John W. Van Ness demonstrated many properties of this process in their paper [MN68] and popularised the name "fractional Brownian motion". The fractional Brownian motion of Hurst parameter $H \in (0,1)$, denoted by $\{B_H(t)\}_{t\in\mathbb{R}}$, is a centred Gaussian process whose covariance function is given by

$$\forall t,s \in \mathbb{R},\ \mathbb{E}\left[B_H(t)B_H(s)\right] = \frac{c(H)}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \tag{2.1.1}$$

where $c(H) > 0$ is a deterministic constant. In the same article [MN68], Mandelbrot and Van Ness showed that the fractional Brownian motion of Hurst parameter $H \in (0,1)$ can be represented, up to a multiplicative constant, by the following Wiener integral

$$\forall t \in \mathbb{R},\ B_H(t) = \frac{1}{\Gamma(H+1/2)} \int_{\mathbb{R}} \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\right) dB(s), \tag{2.1.2}$$

where Γ denotes the Gamma function and

$$\forall x,y \in \mathbb{R},\ (x)_+^y := \begin{cases} x^y & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

In particular, when $H = 1/2$, one obtains the Brownian motion.

The fractional Brownian motion has two fundamental properties; the first one is called *self-similarity* and means that the fractional Brownian motion is scaling invariant, up to a multiplicative constant. Mathematically, it can be written as

$$\forall a > 0,\ \{B_H(at)\}_{t\in\mathbb{R}} \overset{d}{=} \left\{a^H B_H(t)\right\}_{t\in\mathbb{R}}, \tag{2.1.3}$$

where $\overset{d}{=}$ denotes the equality in the sense of finite-dimensional distributions. This property of *self-similarity* characterises fractals and makes the fractional Brownian motion a fractal object. The second property is *stationary increments* which means that the fractional Brownian motion is translation invariant and is written

$$\forall \tau \in \mathbb{R},\ \{B_H(t+\tau) - B_H(\tau)\}_{t\in\mathbb{R}} \overset{d}{=} \{B_H(t)\}_{t\in\mathbb{R}}. \tag{2.1.4}$$

Reversely, any centred Gaussian process which satisfies these two properties is (*self-similarity* and *stationary increments*) is a fractional Brownian motion. Let us emphasize that, except for the very particular case $H = 1/2$, increments of the fractional Brownian motion are no longer independent which makes it to be a more adapted tool than Brownian motion.

2.1.B Regularity of the paths of the fractional Brownian motion

Let us first recall that paths of the fractional Brownian motion are (for a modification), with probability 1, continuous everywhere but nowhere differentiable. To quantify the regularity of a function continuous everywhere but differentiable nowhere, we introduce Hölder spaces.

Definition 2.1.1. Let I be a compact interval of $\mathbb{R}$ and $\beta \in [0,1]$. A continuous function $f : I \to \mathbb{R}$ is said to be β Hölder function on I if

$$\sup_{\substack{t,t' \in I \\ t \neq t'}} \frac{|f(t) - f(t')|}{|t - t'|^\beta} < +\infty.$$

Let $\mathcal{C}^\beta(I)$ be the space of β-Hölder functions on I. Observe that

$$\mathcal{C}^{\beta_2}(I) \subset \mathcal{C}^{\beta_1}(I), \text{ for all } 0 \leq \beta_1 \leq \beta_2 \leq 1.$$

We also introduce a pointwise variant of Hölder spaces.

Definition 2.1.2. Let $\tau \in \mathbb{R}$ and $\beta \in [0,1]$. The pointwise Hölder space $\mathcal{C}^\beta(\tau)$ is the set of continuous functions f defined on a neighbourhood of τ and with values in $\mathbb{R}$ such that

$$\exists \delta > 0, \sup_{0 < |t - \tau| \leq \delta} \frac{|f(t) - f(\tau)|}{|t - \tau|^\beta} < +\infty.$$

Notice that

$$\mathcal{C}^{\beta_2}(\tau) \subset \mathcal{C}^{\beta_1}(\tau), \text{ for all } 0 \leq \beta_1 \leq \beta_2 \leq 1.$$

From these two spaces, we can define two exponents which quantify the local regularity and the pointwise regularity at some point $\tau \in \mathbb{R}$ of a continuous and non-differentiable function.

Definition 2.1.3. Let J be an open interval in $\mathbb{R}$ and $f : J \to \mathbb{R}$ a continuous function. *The local Hölder exponent* of f at point $\tau \in J$ is denoted by $\widetilde{\alpha}_f(\tau)$ and defined as

$$\widetilde{\alpha}_f(\tau) := \sup\left\{\beta_f(I),\ I \text{ compact interval included in } J \text{ and such that } \tau \in \mathring{I}\right\},$$

where $\mathring{I}$ denotes the interior of I and $\beta_f(I)$ is *the global Hölder exponent* of f on I defined as

$$\beta_f(I) := \sup\left\{\beta \in [0,1],\ f \in \mathcal{C}^\beta(I)\right\}. \tag{2.1.5}$$

The local Hölder exponent of f at point $\tau \in J$ is denoted by $\alpha_f(\tau)$ and defined as

$$\alpha_f(\tau) := \sup\left\{\beta \in [0,1],\ f \in \mathcal{C}^\beta(\tau)\right\}.$$

Remarks 2.1.4. • It can be shown that for any point τ and for any continuous function f defined in a neighbourhood of the point τ, one has

$$\alpha_f(\tau) = \sup\left\{r \in [0,1],\ \limsup_{t \to \tau} \frac{|f(t) - f(\tau)|}{|t - \tau|^r} < +\infty\right\},$$

and

$$\widetilde{\alpha}_f(\tau) = \sup \left\{ \widetilde{r} \in [0,1], \ \limsup_{(t',t'')\to(\tau,\tau)} \frac{|f(t') - f(t'')|}{|t'-t''|^{\widetilde{r}}} < +\infty \right\}.$$

• The inequality $\widetilde{\alpha}_f(\tau) \leq \alpha_f(\tau)$ always holds true. This inequality is sometimes strict; for example the "chirp" function $q : [-1,1] \to \mathbb{R}$ defined by

$$q(x) = \begin{cases} |x| \sin \left| \frac{1}{x} \right| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

This function oscillates a lot but with a small amplitude and one gets the following strict inequality $\widetilde{\alpha}_q(0) = 1/2 < \alpha_q(0) = 1$.

The closer these exponents are to 1, the smoother the function f is. In framework of stochastic processes, the following Kolmogorov-Chentsov theorem (see [KS87,KS07]) is often used to obtain a lower bound for local and pointwise Hölder exponents.

Theorem 2.1.5. ***(Kolmogorov-Chentsov)*** *Let I be a compact interval of $\mathbb{R}$ and $\{X(t)\}_{t\in I}$ a stochastic process such that for some constants $\delta > 0$ and $\varepsilon > 0$ one has*

$$\exists c(I) > 0, \ \forall t, t' \in I, \ \mathbb{E}|X(t) - X(t')|^\delta \leq c(I)|t-t'|^{1+\varepsilon}.$$

Then, there is a continuous modification of $\{X(t)\}_{t\in I}$ whose paths are in $\mathcal{C}^\gamma(I)$ for all $\gamma < \varepsilon/\delta$.

In all what follows, a stochastic process will always be identified with its continuous modification. By applying the Kolmogorov-Chentsov theorem to the fractional Brownian motion, one obtains a lower bound for its local and pointwise Hôlder exponents.

Lemma 2.1.6. ***(equivalence of Gaussian moments)*** *For all $p > 0$, there exists a constant $c(p) > 0$ such that, for any centred normal random variable X, one has*

$$\mathbb{E}\left(|X|^p\right) = c(p) \left(\mathbb{E}(|X|^2)\right)^{p/2}.$$

Proof. The lemma is trivial if X has zero variance thus one assumes that $\sqrt{E(|X|^2)} > 0$. The random variable $\frac{X}{\sqrt{E(|X|^2)}}$ has a $\mathcal{N}(0,1)$ distribution. Let Y be a random variable which also has a $\mathcal{N}(0,1)$ distribution, we can write for all $p > 0$

$$\mathbb{E}\left(\left| \frac{X}{\sqrt{E(|X|^2)}} \right|^p \right) = \mathbb{E}(|Y|^p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |y|^p e^{-y^2/2} dy,$$

then

$$\mathbb{E}\left(|X|^p\right) = \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |y|^p e^{-y^2/2} dy \right) \left(\mathbb{E}(|X|^2)\right)^{p/2}.$$

□

Proposition 2.1.7. *There is a universal event Ω_* of probability 1, such that*

$$\alpha_{B_H}(\tau,\omega) \geq \widetilde{\alpha}_{B_H}(\tau,\omega) \geq H\,, \quad \textit{for all } (\tau,\omega) \in \mathbb{R}\times\Omega_*.$$

Proof. Using definition (2.1.1), one obtains for all $t, t' \in \mathbb{R}$

$$\mathbb{E}\left[\left|B_H(t) - B_H(t')\right|^2\right] = \mathbb{E}\left[B_H(t)B_H(t)\right] + \mathbb{E}\left[B_H(t')B_H(t')\right] - 2\mathbb{E}\left[B_H(t)B_H(t')\right] = c(H)|t-t'|^{2H}. \tag{2.1.6}$$

According to Lemma 2.1.6 and equality (2.1.6), for all $n \geq 1$, there exists a constant $c(2n) > 0$ such that

$$\forall t, t' \in \mathbb{R},\ \mathbb{E}\left[\left|B_H(t) - B_H(t')\right|^{2n}\right] = c(2n)c(H)^n|t-t'|^{2Hn}.$$

Thus for any integer $n > \frac{1}{2H}$, we can apply the Kolmogorov-Chentsov theorem with $\delta := 2n$ and $\varepsilon := 2Hn - 1$ on the interval $I(n) := [-n, n]$. Thus, there is an event $\Omega(n)$ of probability 1 such that

$$\forall \omega \in \Omega(n),\ \forall \tau \in\,]-n,n[,\ \widetilde{\alpha}_{B_H}(\tau,\omega) \geq \beta_{B_H}\Big(I(n),\omega\Big) \geq \frac{2Hn-1}{2n}.$$

The result is proved for $\Omega_* := \bigcap_{n>1/(2H)} \Omega(n)$. □

Proving that a continuous function is differentiable everywhere seems less difficult than proving that it is differentiable nowhere. Irregularity properties are often more complex to derive. Nevertheless, one can obtain an upper bound for the pointwise and local Hölder exponents on an event which depends on the point $\tau \in \mathbb{R}$ by using the two fundamental properties of the fractional Brownian motion.

Proposition 2.1.8. *For all $\tau \in \mathbb{R}$, there exists an event $\Omega_*(\tau)$ of probability 1, such that*

$$\widetilde{\alpha}_{B_H}(\tau,\omega) \leq \alpha_{B_H}(\tau,\omega) \leq H\,, \quad \textit{for all } \omega \in \Omega_*(\tau).$$

Proof. Let $\tau \in \mathbb{R}$ be fixed. For all integers $n, M \geq 1$ and real number $\varepsilon > 0$, we introduce the event

$$\Omega(n,M,\varepsilon) := \left\{\sup_{\frac{1}{n}\leq\delta\leq 1}\left|\frac{B_H(\tau+\delta) - B_H(\tau)}{\delta^{H+\varepsilon}}\right| \geq M\right\}.$$

Let us fix $M \geq 1$ and $\varepsilon > 0$. Using the property of stationary increments (2.1.4), one gets

$$\forall n \geq 1,\ \mathbb{P}(\Omega(n,M,\varepsilon)) \geq \mathbb{P}\left(\left|B_H(\tau+n^{-1}) - B_H(\tau)\right| n^{H+\varepsilon} \geq M\right) = \mathbb{P}\left(\left|B_H(n^{-1})\right| n^{H+\varepsilon} \geq M\right).$$

Then using the self-similarity property (2.1.3), one obtains

$$\forall n \geq 1,\ \mathbb{P}(\Omega(n,M,\varepsilon)) \geq \mathbb{P}\left(|B_H(1)|\, n^{\varepsilon} \geq M\right) = 1 - \mathbb{P}\left(|B_H(1)| < Mn^{-\varepsilon}\right).$$

The random variable $B_H(1)$ has a Gaussian distribution with a non-zero variance thus

$$\mathbb{P}(\Omega(n,M,\varepsilon)) \geq 1 - \mathbb{P}\left(|B_H(1)| < Mn^{-\varepsilon}\right) \underset{n\to+\infty}{\longrightarrow} 1.$$

Let set $\Omega_{**}(M,\varepsilon) := \bigcup_{n\geq 1} \Omega(n,M,\varepsilon)$. The latter event is still of probability 1 by the continuity of probability for an increasing sequence of events. In particular, one has that

$$\forall \omega \in \Omega_{**}(M,\varepsilon),\ \exists N \geq 1,\ \forall n \geq N,\ \sup_{\frac{1}{n}\leq\delta\leq 1} \left| \frac{B_H(\tau+\delta,\omega) - B_H(\tau,\omega)}{\delta^{H+\varepsilon}} \right| \geq M,$$

and consequently that

$$\forall \omega \in \Omega_{**}(M,\varepsilon),\ \sup_{\delta\in]0,1]} \left| \frac{B_H(\tau+\delta,\omega) - B_H(\tau,\omega)}{\delta^{H+\varepsilon}} \right| \geq M.$$

Next, letting Ω^* be the event of probability 1 defined as $\Omega_* := \bigcap_{\varepsilon\in\mathbb{Q}_*^+} \bigcap_{M\geq 1} \Omega_{**}(M,\varepsilon)$ one gets that

$$\forall \omega \in \Omega_*,\ \forall \varepsilon > 0,\ \sup_{\delta\in]0,1]} \left| \frac{B_H(\tau+\delta,\omega) - B_H(\tau,\omega)}{\delta^{H+\varepsilon}} \right| = +\infty,$$

and consequently that

$$\forall \omega \in \Omega_*,\ \alpha_{B_H}(\tau,\omega) \leq H.$$

□

Theorem 2.1.9. *For all $\tau \in \mathbb{R}$, there exists an event $\Omega_*(\tau)$ of probability 1, such that*

$$\alpha_{B_H}(\tau,\omega) = \widetilde{\alpha}_{B_H}(\tau,\omega) = H\,, \quad \textit{for all } \omega \in \Omega_*(\tau).$$

Remark 2.1.10. It is important to emphasize that in Theorem 2.1.9 the event $\Omega_*(\tau)$ depends on the point τ. To go from a determination of Hölder exponents (local and pointwise) for all τ almost surely to a determination almost surely for all τ is a difficult task and brings us into more complex techniques. Determining the Hölder exponents on a universal event of probability 1 not depending on τ gives much more informations; as for instance informations on the singularity spectrum of fractional Brownian motion B_H defined as

$$\forall \alpha \in \mathbb{R},\ D(\alpha) := \dim_{\text{Haus}} \{\tau \in \mathbb{R},\ \alpha_{B_H}(\tau) = \alpha\}\,,$$

where $\dim_{\text{Haus}}$ denotes the Hausdorff dimension (see [Fal90]). In particular, the following Theorem 2.1.11 shows that fractional Brownian motion is a monofractal object. We mention in passing that singularity spectrum plays a fundamental role in multifractal analysis (see for example [Jaf99, Bal14, AJT07]).

The result remains true for a universal event of probability 1 ([Ber72, Xia97]).

Theorem 2.1.11. *There is a universal event Ω_* of probability 1, such that*

$$\alpha_{B_H}(\tau,\omega) = \widetilde{\alpha}_{B_H}(\tau,\omega) = H\,, \quad \textit{for all } (\tau,\omega) \in \mathbb{R}\times\Omega_*.$$

In the following, we present two methods to obtain the Theorem 1.1.11; the first one is based on local times and the second one on the wavelets methodology. Another proof of this result can be found by using 2-microlocal analysis (see [HLV09]).

2.1.B.a Local times and Berman's principle

In this subsection, I denotes a compact interval of $\mathbb{R}$ and $\{Z(t)\}_{t\in I}$ is an arbitrary stochastic process whose paths are Borel measurable functions.

Definition 2.1.12. Let us fix $\omega \in \Omega$ and a Borel set $T \in \mathcal{B}(I)$. The occupation measure associated to $Z(\cdot,\omega)$ at "the period of time" T is the measure $\mu_T(\bullet,\omega)$ defined on $\mathcal{B}(\mathbb{R})$ as

$$\forall A \in \mathcal{B}(\mathbb{R}),\ \mu_T(A,\omega) := \lambda\left(\{t \in T,\ Z(t,\omega) \in A\}\right),$$

where λ denotes the Lebesgue measure. The quantity $\mu_T(A,\omega)$ can be interpreted "as the amount of time" spent by the path $Z(\cdot,\omega)$ in the Borel set A during the period of time T.

Definition 2.1.13. Let $T \in \mathcal{B}(I)$, the local time on T of the process $\{Z(t)\}_{t\in I}$ is well defined only if, for $\mathbb{P}$ almost all $\omega \in \Omega$, the occupation measure $\mu_T(\bullet,\omega)$ is absolutely continuous with respect to the Lebesgue measure λ. In this case, the local time is denoted by $L(\bullet,T,\omega)$ and it is defined as the Radon-Nikodym derivative of $\mu_T(\bullet,\omega)$ with respect to λ

$$L(\bullet,T,\omega) := \frac{d\mu_T(\bullet,\omega)}{d\lambda}.$$

Thus, for any Borel measurable nonnegative function f on $\mathbb{R}$ one has

$$\int_{\mathbb{R}} f(x)d\mu_T(x,\omega) = \int_{\mathbb{R}} f(x)L(x,T,\omega)\lambda(dx).$$

The following proposition (see for example [Aya19]) allows to prove the existence of local times for the fractional Brownian motion.

Proposition 2.1.14. *Assume that $\{Z(t)\}_{t\in I}$ is a centred Gaussian process. Then, a sufficient condition for having the existence of $L(\bullet,T)$ and its square integrability in (x,ω) is that*

$$\int_T \left(\int_T \mathbb{E}\left[(Z(t)-Z(s))^2\right]^{-1/2} dt\right) ds < +\infty.$$

In the fractional Brownian motion case, one assumes that T is a compact interval of $\mathbb{R}$ and one uses (2.1.6) to obtain

$$\begin{aligned}\int_T \left(\int_T \mathbb{E}\left[(B_H(t)-B_H(s))^2\right]^{-1/2} dt\right) ds &\le c(H)^{-1/2}\int_T \left(\int_T |t-s|^{-H}dt\right) ds \\ &= c(H)^{-1/2}\int_T \left(\int_{T-s} |t|^{-H}dt\right) ds\end{aligned}$$

with the convention $T-s := \{x-s,\ x\in T\}$. Then setting $\nu(T) := \sup\{|x|,\ x \in T\}$, it follows that

$$\int_T \left(\int_{T-s} |t|^{-H}dt\right) ds \le \int_T \left(\int_{-2\nu(T)}^{2\nu(T)} |t|^{-H}dt\right) ds = \lambda(T)\int_{-2\nu(T)}^{2\nu(T)} |t|^{-H}dt < +\infty,$$

which implies the existence of the local time $L(\bullet,T)$ for any compact interval T.

Thanks to the Berman's principle, can be used for obtaining results on irregularity of a stochastic process. Roughly speaking, this principle can be formulated in the following way: the more the paths of the local times $\{L(x,T),\ (x,T)\in\mathbb{R}\times\mathcal{B}(I)\}$ are regular and the more the paths of the associated stochastic process $\{Z(t)\}_{t\in I}$ are irregular. It is often less difficult to show a regularity property than an irregularity property, thus, Berman's principle provides an important strategy for obtaining an upper bound of the local and pointwise Hölder exponents of stochastic processes.

Definition 2.1.15. Assume the existence of the local time $L(\bullet,I)$, the Radon-Nikodym theorem then ensures the existence of the local times $L(\bullet,T)$ for any $T\in\mathcal{B}(I)$ ([Rud86]). In particular, this is the case for the intervals $T=I(s)$ where

$$\forall s\in I,\ \ I(s):=\{x\leq s,\ x\in I\}\,.$$

The stochastic process $\{Z(t)\}_{t\in I}$ is said to have a jointly continuous local time on (the compact interval) I, if the stochastic field $\{L(x,I(s)),\ (x,s)\in\mathbb{R}\times I\}$ has a continuous modification. When it exists, we denote this continuous modification by $\{\mathcal{L}(x,I(s)),\ (x,s)\in\mathbb{R}\times I\}$.

Theorem 2.1.16. ***(Berman's principle)*** *Assume that $\{Z(t)\}_{t\in I}$ has continuous paths as well as $\{\mathcal{L}(x,I(s)),\ (x,s)\in\mathbb{R}\times I\}$ a jointly continuous local time on I. Moreover, assume that there exists an event Ω_0 of probability 1 on which one has*

$$\forall\omega\in\Omega_0,\ \forall\tau\in\overset{\circ}{I},\ \limsup_{\rho\to0^+}\left\{\sup_{x\in\mathbb{R}}\mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big)\rho^{\theta(\tau)-1}\right\}<+\infty, \tag{2.1.7}$$

where $\theta(\cdot)$ is a function with values in $(0,1)$. Then, one gets

$$\widetilde{\rho}_Z(\tau,\omega)\leq\rho_Z(\tau,\omega)\leq\theta(\tau)\,,\quad \textit{for all } (\tau,\omega)\in\overset{\circ}{I}\times\Omega_0.$$

Proof. Let us fix $\omega\in\Omega_0,\ \tau\in\overset{\circ}{I}$ and $\rho>0$ small enough so that $[\tau-\rho,\tau+\rho]\subset I$. Then, one denotes by $\mathrm{Osc}_Z([\tau-\rho,\tau+\rho],\omega)$ the oscillation of $Z(\cdot,\omega)$ on the interval $[\tau-\rho,\tau+\rho]$ defined as

$$\mathrm{Osc}_Z([\tau-\rho,\tau+\rho],\omega):=\sup_{t\in[\tau-\rho,\tau+\rho]}Z(t,\omega)-\inf_{t\in[\tau-\rho,\tau+\rho]}Z(t,\omega).$$

The continuity of $Z(\cdot,\omega)$ ensures that this oscillation exists and is finite. Let us introduce the compact interval

$$J(\tau,\rho,\omega):=\left[\inf_{t\in[\tau-\rho,\tau+\rho]}Z(t,\omega),\ \sup_{t\in[\tau-\rho,\tau+\rho]}Z(t,\omega)\right].$$

Observe that the measure $\mu_{[\tau-\rho,\tau+\rho]}(\bullet,\omega)$ is supported by $J(\tau,\rho,\omega)$ then $\mathcal{L}(x,[\tau-\rho,\tau+\rho],\omega)=0$ for all $x\notin J(\tau,\rho,\omega)$. One gets

$$\begin{aligned}\mu_{[\tau-\rho,\tau+\rho]}(\mathbb{R},\omega)&=\int_{\mathbb{R}}\mathcal{L}(x,[\tau-\rho,\tau+\rho],\omega)dx=\int_{J(\tau,\rho,\omega)}\mathcal{L}(x,[\tau-\rho,\tau+\rho],\omega)dx\\&\leq\lambda(J(\tau,\rho,\omega))\times\sup_{x\in\mathbb{R}}\mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big)\\&\leq\mathrm{Osc}_Z([\tau-\rho,\tau+\rho],\omega)\times\sup_{x\in\mathbb{R}}\mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big),\end{aligned}\tag{2.1.8}$$

where

$$\mu_{[\tau-\rho,\tau+\rho]}(\mathbb{R},\omega) = \lambda\left(\{t \in [\tau-\rho,\tau+\rho],\ Z(t,\omega) \in \mathbb{R}\}\right) = 2\rho. \tag{2.1.9}$$

Putting together (2.1.8) and (2.1.9), one obtains

$$2 \leq \Big(\mathrm{Osc}_Z([\tau-\rho,\tau+\rho],\omega)\rho^{-\theta(\tau)}\Big) \times \Big(\sup_{x\in\mathbb{R}} \mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big)\rho^{\theta(\tau)-1}\Big)$$

thus, (2.1.7) ensures that, for all $\varepsilon > 0$, one has

$$\limsup_{\rho\to 0^+} \Big\{\rho^{-(\theta(\tau)+\varepsilon)}\mathrm{Osc}_Z([\tau-\rho,\tau+\rho],\omega)\Big\} = +\infty. \tag{2.1.10}$$

On the other hand, using triangle inequality, one gets for all $\varepsilon > 0$

$$\begin{aligned}\rho^{-(\theta(\tau)+\varepsilon)}\mathrm{Osc}_Z([\tau-\rho,\tau+\rho],\omega) &= \rho^{-(\theta(\tau)+\varepsilon)} \sup_{t_1,t_2\in[\tau-\rho,\tau+\rho]} |Z(t_1,\omega)-Z(t_2,\omega)| \\ &\leq 2\rho^{-(\theta(\tau)+\varepsilon)} \sup_{t\in[\tau-\rho,\tau+\rho]} |Z(t,\omega)-Z(\tau,\omega)|\end{aligned}$$

thus, thanks to (2.1.10), one obtains

$$\forall\varepsilon > 0,\ \limsup_{\rho\to 0^+} \Big\{\rho^{-(\theta(\tau)+\varepsilon)} \sup_{t\in[\tau-\rho,\tau+\rho]} |Z(t,\omega)-Z(\tau,\omega)|\Big\} = +\infty.$$

□

Remarks 2.1.17. • The condition (2.1.7) can be interpreted as a condition of regularity on $\{\mathcal{L}(x,I(s))\}_{(x,s)\in\mathbb{R}\times I}$ with respect to the variable s uniformly in the variable x. Indeed, we can write

$$\forall x \in \mathbb{R},\ \mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big) := \mathcal{L}\big(x,I(\tau+\rho),\omega\big) - \mathcal{L}\big(x,I(\tau-\rho),\omega\big),$$

and the continuity on $\mathbb{R}\times I$ of the function with compact support $(x,s) \mapsto \mathcal{L}(x,I(s))$ ensures the existence of $\sup_{x\in\mathbb{R}} \mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big) = \sup_{x\in J(\tau,\rho,\omega)} \mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big)$ and also that

$$\limsup_{\rho\to 0^+} \Big\{\sup_{x\in\mathbb{R}} \mathcal{L}\big(x,[\tau-\rho,\tau+\rho],\omega\big)\Big\} = 0. \tag{2.1.11}$$

The condition (2.1.7) is therefore a faster convergence condition than (2.1.11) which comes from the continuity of $\{L(x,I(s))\}_{(x,s)\in\mathbb{R}\times I}$.
• To obtain a regularity property for local times of the type (2.1.7), it is often necessary to have explicit expressions for the characteristic functions associated with the finite-dimensional distributions of the process Z. Local time methods become inefficient in the context of stochastic processes whose characteristic functions of the dimensional-finite distributons do not have an explicit form; which is the case for all the multifractional processes studied in this thesis (see section 2.3).

An alternative method to local times is the wavelets methodology. Different variants of the wavelets methodology can be found in a number of articles on the study of the regularity and local irregularity of stochastic processes.

2.1.B.b The wavelets methodology and its variants

A wavelet basis is a Hilbertian basis of $L^2(\mathbb{R})$ formed of real-valued functions $(\psi_{j,k})_{j,k\in\mathbb{Z}}$ generated by translations and dilations of the same function $\psi \in L^2(\mathbb{R})$ called the mother wavelet with

$$\forall x \in \mathbb{R},\ \psi_{j,k}(x) := 2^{j/2}\psi\left(2^j x - k\right).$$

In particular, any function $f \in L^2(\mathbb{R})$ can be expressed as

$$f = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} a_{j,k}\psi_{j,k}, \tag{2.1.12}$$

where the convergence holds in $L^2(\mathbb{R})$ and the wavelet coefficients $(a_{j,k})_{j,k\in\mathbb{Z}}$ are defined by

$$a_{j,k} := \langle f, \psi_{j,k}\rangle = \int_{\mathbb{R}} f(t)\psi_{j,k}(t)dt. \tag{2.1.13}$$

One of the main interests of wavelets in the analysis of local regularity can be seen, through an important result of Stéphane Jaffard (see [Jaf91, Jaf04]): if f is a Hölder continuous function on $\mathbb{R}$ and ψ a regular enough wavelet, one gets the following characterisation of the pointwise Hölder exponent at point $\tau \in \mathbb{R}$

$$\alpha_f(\tau) = \liminf_{j\to+\infty} \inf_{k\in\mathbb{Z}} \frac{\log|2^{j/2}a_{j,k}|}{\log(2^{-j} + |\tau - k2^{-j}|)}.$$

This equality is a consequence of the following Theorem 2.1.18.

It is assumed that the mother wavelet ψ is differentiable and has zero integral (i.e. its first moment vanishes). Moreover, ψ and its derivative function ψ' are rapidly decreasing at infinity, more precisely

$$\forall i \in \{0,1\},\ \forall m \in \mathbb{N},\ \sup_{x\in\mathbb{R}} |\psi^{(i)}(x)|(1+|x|)^m < +\infty. \tag{2.1.14}$$

These conditions on ψ are classical and satisfied by many wavelets.

Theorem 2.1.18. *Let $f \in L^2(\mathbb{R})$ be bounded and globally ε Hölder continuous function (with ε arbitrarily small), $\alpha \in (0,1)$ and $x_0 \in \mathbb{R}$. If $\alpha_f(x_0) > \alpha$ then there exists a constant $c > 0$ such that*

$$\forall j,k \in \mathbb{Z},\ |a_{j,k}| \le c2^{-(\alpha+1/2)j}\left(1 + |2^j x_0 - k|^\alpha\right). \tag{2.1.15}$$

Reversely, if (2.1.15) is satisfied then there exists a constant $c' > 0$ such that for all $x \in \mathbb{R}$ with $|x - x_0| \le 1$ one has

$$|f(x) - f(x_0)| \le c'|x - x_0|^\alpha \log\left(\frac{2}{|x - x_0|}\right)$$

thus $\alpha_f(x_0) \ge \alpha$.

We only give a proof of the first part of the theorem, the proof of the second part is much more complex and we refer to [Jaf04] for it.

Proof. Let us fix $x_0 \in \mathbb{R}$ and set $d_{j,k} := 2^{j/2} a_{j,k}$. Since the first moment of ψ vanishes, we have

$$\forall j,k \in \mathbb{Z},\ d_{j,k} = 2^j \int_{\mathbb{R}} f(x)\psi(2^j x - k)dx = 2^j \int_{\mathbb{R}} (f(x) - f(x_0))\psi(2^j x - k)dx.$$

Since f is bounded with $\alpha_f(x_0) > \alpha$, the quantity $c_1 := \sup_{x\in\mathbb{R}} \frac{|f(x)-f(x_0)|}{|x-x_0|^\alpha}$ exists and is finite. Using the triangle inequality, the inequality $(a+b)^\alpha \le a^\alpha + b^\alpha$ for $a,b \ge 0$ and (2.1.14) with $i = 0$ and $m = 2$, one obtains

$$|d_{j,k}| \le 2^j c_1 \int_{\mathbb{R}} \frac{|x-x_0|^\alpha}{(1+|2^j x - k|)^2} dx \le 2^j c_1 \int_{\mathbb{R}} \frac{|x - k2^{-j}|^\alpha + |x_0 - k2^{-j}|^\alpha}{(1+|2^j x - k|)^2} dx.$$

Finally, using the change of variable $t = 2^j x - k$, one gets

$$\begin{aligned} |d_{j,k}| &\le 2^{-\alpha j} c_1 \int_{\mathbb{R}} \frac{|t|^\alpha + |2^j x_0 - k|^\alpha}{(1+|t|)^2} dt \le 2^{-\alpha j} c_1 \left(\int_{\mathbb{R}} \frac{|t|^\alpha}{(1+|t|)^2} dt + |2^j x_0 - k|^\alpha \int_{\mathbb{R}} \frac{1}{(1+|t|)^2} dt \right) \\ &\le c_2 2^{-\alpha j} \left(1 + |2^j x_0 - k|^\alpha\right) \end{aligned}$$

where $c_2 > 0$ is a constant.

□

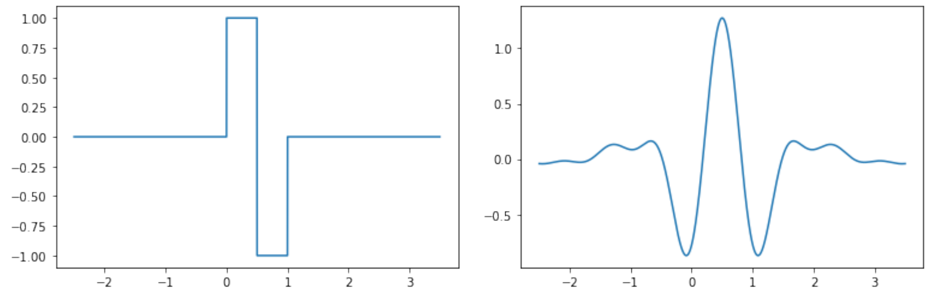
The simplest example of wavelet basis is that of Haar introduced in 1909 ([Haa10]) where the mother wavelet ψ is defined as

$$\forall t \in \mathbb{R},\ \psi(t) := \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \\ 0 & \text{else.} \end{cases}$$

Another well known class of wavelet bases are the Meyer bases. One can derive from such a basis, different types of representations for fractional Brownian motion in terms of random series. These representations have been introduced in the seminal article [MST99]. One of them is the following:

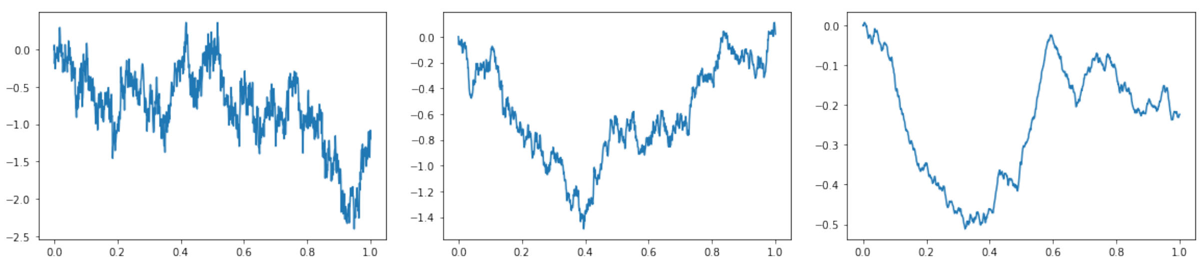
$$\forall t \in \mathbb{R},\ B_H(t) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} 2^{-jH} \left(\Psi_H(2^j t - k) - \Psi_H(-k)\right) \varepsilon_{j,k},$$

where Ψ_H denotes the fractional primitive (on the left) of order $H + 1/2$ of a Meyer wavelet ψ and where the $\varepsilon_{j,k}$ are independent standard (i.e. $\mathcal{N}(0,1)$) Gaussian random variables. This representation in random series allows to have an alternative proof of the Theorem 2.1.11 and allows, in an indirect way, to simulate the fractional Brownian motion.



The Haar wavelet on the left and a Meyer wavelet on the right.

These two wavelets are completely opposed: the Haar wavelet is irregular whereas the Meyer wavelet is very regular which makes it very convenient for using techniques that require regularity such as integrations by parts for instance. Nevertheless, the Haar basis still has an advantage, namely the corresponding coefficient can be computed explicitely or approximated in a numerical way without great difficulty (see Remark 2.3.4). Let us also emphasize note that several new results of our thesis are obtained thanks to the Haar basis (see Chapter 7).
Before ending this subsection, we mention that the wavelet method is one among many other methods for simulating fractional Brownian motion. Several of them are discribed in the article [Coe00] which also presents several statistical estimators of the Hurst parameter H.



Simulations of the fractional Brownian motion
($H = 0.3$ on the left, $H = 0.5$ in the middle, $H = 0.8$ on the right).

These simulations illustrate the result provided by Theorem 2.1.11: the closer the Hurst parameter H is to 1, the smoother and more regular the paths are; on the other hand, the closer the Hurst parameter H is to 0 and the more rough and irregular the paths are.

2.2 Classical multifractional processes

The local regularity the fractional Brownian motion is prescribed by the constant Hurst parameter H (as shown by the Theorem 2.1.11). Thus, its local regularity cannot evolve from one point to another and it remains the same all along the path. Therefore, fractional Brownian motion does not seem to be a well adaped model for describing many phenomena (stock market prices, encephalograms, generation of artificial mountains, etc) which require that roughness of paths changes from one point to another.

2.2.A Multifractional Brownian motion

To overcome this limitation of fractional Brownian motion coming from the fact its regularity cannot change from point to point, Romain-François Peltier and Jacques Lévy Véhel as well as Albert Benassi, Stéphane Jaffard and Daniel Roux independently introduced a new type of Gaussian process, called multifractional process, in the middle of the 1990s, in the articles [PL95] and [BJR97]. The idea is to replace the constant Hurst parameter H of fractional Brownian motion by a deterministic function $t \mapsto H(t)$ with values in $(0,1)$, which depends on the variable t consisting in the index of the process. Let us emphasize that this idea of making the Hurst parameter depends on the index of the process t characterizes the multifractional processes called *classical*.

Using the definition (2.1.2), they defined a new Gaussian stochastic process $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$, called multifractional Brownian motion, by the following relation

$$\forall t \in \mathbb{R},\ \mathcal{M}(t) := \frac{1}{\Gamma(H(t)+1/2)} \int_{\mathbb{R}} \left((t-s)_+^{H(t)-1/2} - (-s)_+^{H(t)-1/2} \right) dB(s). \tag{2.2.1}$$

Even if it is more complex than that of the fractional Brownian motion, it is still possible to compute the covariance function of the multifractional Brownian motion ([ACLV00]).

Proposition 2.2.1. *For all $t, s \in \mathbb{R}$, one has*

$$\mathbb{E}\left(\mathcal{M}(t)\mathcal{M}(s)\right) = D(H(t),H(s)) \left(|t|^{H(t)+H(s)} + |s|^{H(t)+H(s)} - |t-s|^{H(t)+H(s)} \right),$$

where

$$D(x,y) := \frac{\sqrt{\Gamma(2x+1)\Gamma(2y+1)\sin(\pi x)\sin(\pi y)}}{2\Gamma(x+y+1)\sin(\pi(x+y)/2)}.$$

When the function $H(\cdot)$ is discontinuous or not fairly regular, the local and pointwise Hölder exponents at the point τ do not coincide with the value of $H(\cdot)$ at the point τ as shown for example by the following proposition (see [Aya19]).

Proposition 2.2.2. *Assume that the Hurst function $H(\cdot)$ is discontinuous at some point $\tau \in \mathbb{R}^*$, then there exists an event $\Omega_*(\tau)$ of probability 1 such that, for any $\omega \in \Omega_*(\tau)$ the path $\mathcal{M}(\cdot,\omega)$ is discontinuous at the point τ.*

Nevertheless, when the function $H(\cdot)$ is sufficiently regular, the local regularity of the multifractional Brownian motion can evolve from one point to another and the Hölder exponents coincide with the values taken by $H(\cdot)$ ([PL95, BJR97]).

Theorem 2.2.3. *Assume that the following condition (C) is satisfied by the function $H : \mathbb{R} \to (0,1)$*

$$(C):\ \widetilde{\alpha}_H(\tau) > H(\tau),\ \text{for all } \tau \in \mathbb{R}.$$

Then, for all $\tau \in \mathbb{R}$, there exists an event $\Omega_*(\tau)$ of probability 1, such that

$$\alpha_{\mathcal{M}}(\tau,\omega) = \widetilde{\alpha}_{\mathcal{M}}(\tau,\omega) = H(\tau)\,, \quad \text{for all } \omega \in \Omega_*(\tau).$$

Ten years later, Antoine Ayache, Stéphane Jaffard and Murad S. Taqqu demonstrate with wavelets methods in the article [AJT07] that the result remains true for a universal event of probability 1. Another demonstration of this result can be found from 2-microlocal analysis (see [Her04, HLV09]).

Theorem 2.2.4. *Assume that the following condition (C) is satisfied by the function* $H : \mathbb{R} \to (0, 1)$

$$(C): \ \widetilde{\alpha}_H(\tau) > H(\tau), \ \textit{for all } \tau \in \mathbb{R}.$$

Then, there exists a universal event Ω_* *of probability 1, such that*

$$\alpha_{\mathcal{M}}(\tau, \omega) = \widetilde{\alpha}_{\mathcal{M}}(\tau, \omega) = H(\tau)\,, \quad \textit{for all } (\tau, \omega) \in \mathbb{R} \times \Omega_*.$$

Remark 2.2.5. Condition (C) is for example satisfied when the function $H(\cdot)$ is a Lipschitz function.

This new process $\{M(t)\}_{t\in\mathbb{R}}$ does not preserve the two fundamental properties of self-similarity and stationary increments of the fractional Brownian motion; nevertheless, there are a locally property of asymptotic self-similarity.

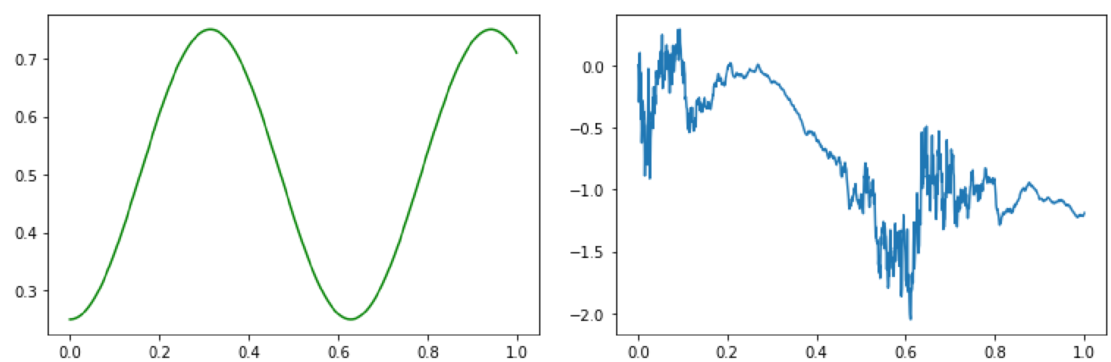
Theorem 2.2.6. *Assume that the following condition (C) is satisfied by the function* $H : \mathbb{R} \to (0, 1)$

$$(C): \ \widetilde{\alpha}_H(\tau) > H(\tau), \ \textit{for all } \tau \in \mathbb{R}.$$

Then, at every fixed point $\tau \in \mathbb{R}$, *the stochastic process* $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ *is strongly locally asymptotically self-similar of exponent* $H(\tau)$. *This means that, for some positive deterministic constant* $c(\tau)$ *and for any fixed positive real number* T, *when* $\lambda \to 0_+$, *the stochastic process* $\left\{\lambda^{-H(\tau)}(\mathcal{M}(\tau + \lambda t) - \mathcal{M}(\tau))\right\}_{t\in\mathbb{R}}$ *converges in distribution to* $\{c(\tau)B_{H(\tau)}(t)\}_{t\in\mathbb{R}}$ *in the Banach space of the real-valued continuous functions* $\mathcal{C}([-T, T], ||.||_\infty)$.

Under condition (C), for all point t there is a fractional Brownian motion of Hurst parameter $H(t)$ which is tangent to the path of the multifractional Brownian motion at point t.

There are few methods for simulating the multifractional Brownian motion; a simulation method is presented by Peltier and Lévy Véhel in their paper [PL95].



Simulation of the multifractional Brownian motion
(the Hurst function $H(\cdot)$ *on the left and an associated path of* M *on the right)*

We can also estimate $H(t_0)$, the value of the Hurst function $H(\cdot)$ at an arbitrary and fixed point $t_0 \in [0,1]$, from the observation of a discretised realisation $\left\{\mathcal{M}\left(\frac{k}{N}\right)\right\}_{0\leq k\leq N}$ (where N is a large enough integer) of the multifractional Brownian motion (see [AL04, BS13a, BCI98, Coe05, Coe06]). The estimator is constructed from a generalized discrete variations of order $L \geq 1$ denoted by $(d_{N,k})_{0\leq k\leq N-L}$ and defined as

$$\forall k \in \{0,\ldots,N-L\}\,,\ d_{N,k} := \sum_{q=0}^{L} a_q \mathcal{M}\left(\frac{k+q}{N}\right)$$

where

$$\forall q \in \{0,\ldots,L\}\,,\ a_q := (-1)^{L-q}\binom{L}{q}.$$

For example, when $L=1$ one gets a classical increment

$$\forall k \in \{0,\ldots,N-1\}\,,\ d_{N,k} = \mathcal{M}\left(\frac{k}{N}\right) - \mathcal{M}\left(\frac{k+1}{N}\right), \tag{2.2.2}$$

and when $L=2$ one obtains an increment between two increments

$$\forall k \in \{0,\ldots,N-2\}\,,\ d_{N,k} = \left[\mathcal{M}\left(\frac{k}{N}\right) - \mathcal{M}\left(\frac{k+1}{N}\right)\right] - \left[\mathcal{M}\left(\frac{k+1}{N}\right) - \mathcal{M}\left(\frac{k+2}{N}\right)\right].$$

To understand how these variations can be used to estimate the values of $H(\cdot)$, let us assume that $L=1$ and the integer N is large enough to make the approximation $H((k+1)/N) \approx H(k/N)$. By combining (2.2.2) with Proposition 2.2.1, on obtains the approximation

$$\mathbb{E}|d_{N,k}|^2 \approx N^{-2H(k/N)}.$$

The quantity $H(t_0)$ is thus closely related to the values of $(d_{N,k})^2$ located in a neighbourhood of t_0. Let us fix a real $a \in (0,1/2)$. For any integer $N \geq 2$, the discrete neighbourhood of the point $t_0 \in [0,1]$, denoted by $\nu_N(t_0)$, is defined as

$$\nu_N(t_0) := \left\{k \in \{0,\ldots,N\}\,,\ \left|\frac{k}{N} - t_0\right| \leq N^{-a}\right\}.$$

Then, let us denote by $V_N(t_0)$ the square of the mean square of $(d_{N,k})_k$ in the neighbourhood of the point t_0

$$V_N(t_0) := \frac{1}{\#(\nu_N(t_0))}\sum_{k\in\nu_N(t_0)} (d_{N,k})^2\,.$$

It is from $V_N(t_0)$ that the consistent estimator of $H(t_0)$ can be constructed. Heuristically, $V_N(t_0)$ will follow a strong law of large numbers thus, for N big enough and $k \in \nu_N(t_0)$ one gets the approximation

$$V_N(t_0) \approx \mathbb{E}|d_{N,k}|^2 \approx N^{-2H(k/N)} \approx N^{-2H(t_0)}.$$

This leads us to the following estimates

$$H(t_0) \approx \frac{\log(V_N(t_0))}{-2\log(N)} \text{ and } H(t_0) \approx 2^{-1} \log\left(\frac{V_N(t_0)}{V_{2N}(t_0)}\right).$$

The following theorem ([BCI98, Coe05, Coe06]) highlights the convergence of this second estimator. This estimator has the advantage of being invariant when the process $\mathcal{M}$ is multiplied by a constant.

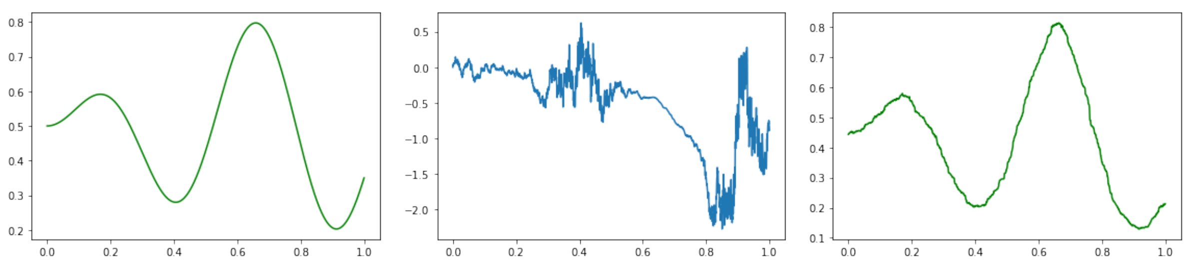
Theorem 2.2.7. *Assume that $L \geq 2$ and the following condition (C) is satisfied by the function $H : \mathbb{R} \to (0,1)$*

$$(C): \ \widetilde{\alpha}_H(t) > H(t), \text{ for all } t \in \mathbb{R}.$$

Then, for all $t_0 \in [0,1]$

$$\widehat{H}_N(t_0) := 2^{-1} \log\left(\frac{V_N(t_0)}{V_{2N}(t_0)}\right)$$

is a strongly consistent estimator of $H(t_0)$; that is, $\widehat{H}_N(t_0)$ converges almost surely to $H(t_0)$ when N goes to $+\infty$.



Estimation of the Hurst function of multifractional Brownian motion ($H(\cdot)$ on the left, an associated path of M in the middle and the estimation of $H(\cdot)$ on the right)

Remarks 2.2.8. • The condition $L \geq 2$ allows to decorrelate the increments and to obtain a central limit theorem for $\widehat{H}_N(t_0)$.
• The estimator given by the Theorem 2.2.7 converges point by point and does not converge a priori uniformly on the whole interval $[0,1]$. It seems more interesting to obtain a uniform convergence; this is the case in the article [AH17], where an estimator of the deterministic Hurst function is constructed, in the non-Gaussian framework of linear stable multifractional motion, which converges uniformly at a speed that can be estimated.
• It should be noted that the proof of the Theorem 2.2.7 which is given in the literature, relies essentially on Gaussian methods which require to have a precise estimate of the covariance of the generalized increments $d_{N,k}$ of the multifractional Brownian motion.

Let us mention that in the Chapter 6 of this thesis, we construct an estimator which converges almost surely, in the sense of the uniform norm, to the random Hurst function of a *non-classical* multifractional process. One of the difficulties is that this process is non-Gaussian, and its covariance function is much more complex than that of the classical multifractional Brownian motion.

2.2.B A first type of multifractional process with random exponent

The fact that in the definition (2.2.1) the Hurst function $H(\cdot)$ is deterministic may seem to be restrictive for applications. For example, in finance, the pointwise Hölder exponent can be interpreted as the weight that investors give to past prices to make their decisions (see [BPP12]) and there is no reason for this weight to be deterministic. It may therefore be interesting to define a new class of multifractional processes with a random functional parameter. However, it is not possible to simply replace the deterministic function $H(\cdot)$ by a stochastic process $\{S(t)\}_{t\in\mathbb{R}}$ in the definition of the multifractional Brownian motion because the integral (2.2.1) would no longer be defined in the Itô sense.

In 2003, George C. Papanicolaou and Knut Solna suggest in [PS03] to replace the deterministic Hurst function $H(\cdot)$ of the multifractional Brownian motion by a stochastic process $\{S(t)\}_{t\in\mathbb{R}}$ which is regular enough, with stationary increments and independent of the Brownian motion $\{B(s)\}_{s\in\mathbb{R}}$. However, their idea does not generalise to the general case where $\{S(t)\}_{t\in\mathbb{R}}$ can depend on the Brownian motion $\{B(s)\}_{s\in\mathbb{R}}$ because the integral (2.2.1) would no longer be correctly defined.

In 2005, Antoine Ayache and Murad S. Taqqu define in [AT05] a new multifractional process with a random Hurst function from a stochastic process $\{S(t)\}_{t\in[0,1]}$ with values in a compact set $[a,b]\subset(0,1)$. They first introduce the Gaussian field $\{B_H(t)\}_{(t,H)\in[0,1]\times[a,b]}$ defined as

$$\forall (t,H)\in[0,1]\times[a,b],\ B_H(t) := \int_{\mathbb{R}} \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\right) dB(s) \tag{2.2.3}$$

and which at a fixed H corresponds, up to a multiplicative factor, to the definition (2.1.2) of the fractional Brownian motion. Then, they obtain the uniform convergence in (t,H) of the following random series representation

$$\forall (t,H)\in[0,1]\times[a,b],\ B_H(t) := \sum_{j=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} a_{j,k}(t,H)\varepsilon_{j,k},$$

where $\{\varepsilon_{j,k}\}_{j,k\in\mathbb{Z}}$ are independent random variables of the same distribution $\mathcal{N}(0,1)$, and where the deterministic coefficients $a_{j,k}(t,H)$ are given by the relation

$$\forall (t,H)\in[0,1]\times[a,b],\ \forall j,k\in\mathbb{Z},\ a_{j,k}(t,H) := \int_{\mathbb{R}} \frac{e^{it\xi}-1}{i\xi|\xi|^{H-1/2}}\widehat{\psi}_{j,k}(\xi)d\xi. \tag{2.2.4}$$

Notice that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ denotes a Lemarié-Meyer wavelet basis of $L^2(\mathbb{R})$. The new process $\{Z(t)\}_{t\in[0,1]}$ can be defined by the following relation

$$\forall t\in[0,1],\ Z(t) := \sum_{j=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} a_{j,k}(t,S(t))\varepsilon_{j,k}. \tag{2.2.5}$$

This process is part of a new class of multifractional processes, called MPRE *(Multifractional Processes with Random Exponent)*.

The study of such a process with a random functional parameter is much more complex than that of the multifractional Brownian motion. Indeed, unlike the multifractional Brownian motion, this new process is not Gaussian; moreover, its distributions and covariance function are not explicit, and as we have seen with the examples of the fractional Brownian motion and the multifractional Brownian motion, having access to an explicit formula of the covariance function of a process is a precious benefit to study it.

In the same paper, Ayache and Taqqu obtain the pointwise and local Hölder exponents of the process $\{Z(t)\}_{t\in[0,1]}$ in the general case where the process may be dependent on the Brownian motion $\{B(s)\}_{s\in\mathbb{R}}$.

Theorem 2.2.9. *Assume that the following condition (C) is satisfied almost surely by the stochastic process* $\{S(t)\}_{t\in[0,1]}$

$$(C):\ \widetilde{\alpha}_S(t) > S(t),\ \textit{for all } t\in\mathbb{R},$$

where $\beta_S([0,1])$ *denotes the global Hölder exponent of* S *(voir (2.1.5)) on the interval* $[0,1]$*. Then, for all* $\tau\in\mathbb{R}$*, there exists an event* $\Omega_*(\tau)$ *of probability 1, such that*

$$\widetilde{\alpha}_Z(\tau,\omega) = \alpha_Z(\tau,\omega) = S(\tau,\omega)\,, \quad \textit{for all } \omega\in\Omega_*(\tau).$$

The first ingredient to prove this theorem is an upper bound of the random coefficients $(\varepsilon_{j,k})_{j,k\in\mathbb{Z}}$ in (2.2.5), given by the following lemma.

Lemma 2.2.10. *There exists a random variable* $C>0$ *with finite moments of any order and an event* Ω_{**} *of probability 1 such that*

$$\forall\omega\in\Omega_{**},\ \forall j,k\in\mathbb{Z},\ |\varepsilon_{j,k}(\omega)| \le C(\omega)\sqrt{\log(2+|j|)}\sqrt{\log(2+|k|)}.$$

Remark 2.2.11. This lemma is very convenient in order to get rid of the coefficients $(\varepsilon_{j,k})_{j,k\in\mathbb{Z}}$, when one seeks to find an upper bound for the absolute value of an increment of the process Z by using the series representation (2.2.5) . Let us mention in passing that similar results play an important role in the Chapter 5 and the Chapter 7 of this thesis.

The second ingredient of the proof of the Theorem 2.2.9 is the estimation of the coefficients $(a_{j,k}(t,S(t)))_{j,k\in\mathbb{Z}}$ defined at (2.2.4). The Lemarié-Meyer wavelet $\psi_{j,k}$ is very regular and thus allows to make these coefficients converge very quickly to 0 when $k\to\infty$ by making multiple integrations by parts. Let us underline that although these coefficients converge very quickly to 0, they are not explicit which makes the definition (2.2.5) not very adapted to simulate the process.

The local times method, which we have described previously, does not seem to allow us to study the irregularity of this process Z since we do not know if this process admits a local time, moreover we do not have explicit formulas for the characteristic functions of its finite-dimensional distributions (see Remarks 2.1.17).
Moreover, another considerable shortcoming of the Z process is that it has not Itô integral representation, so it is difficult to use the powerful tools of the Itô calculus to study it.
On the other hand, a regularity condition (C) is always necessary for the Theorem 2.2.9 and one would like to weaken it to allow less regular functional parameters.

2.3 Non-classical multifractional processes

2.3.A Surgailis multifractional processes

It is in 2008, in his article [Sur08], that Donatas Surgailis has introduced a new type of multifractional process. These processes are constructed from a continuous real function $\alpha(\cdot)$ with values in $(-1/2, 1/2)$ which can be interpreted as $H(\cdot) - 1/2$ where $H(\cdot)$ is the Hurst function. The Surgailis multifractional processes $\{X(t)\}_{t\in\mathbb{R}}$ and $\{Y(t)\}_{t\in\mathbb{R}}$ are defined as

$$X(t) := \int_{\mathbb{R}} \left(\int_0^t \frac{1}{\Gamma(\alpha(\tau))} (\tau - s)_+^{\alpha(\tau)-1} e^{H_-(s,\tau)} d\tau \right) dB(s)$$

and

$$Y(t) := \int_{\mathbb{R}} \frac{1}{\Gamma(1+\alpha(s))} \left((t-s)_+^{\alpha(s)} e^{-H_+(s,t)} - (-s)_+^{\alpha(s)} e^{-H_+(s,0)} \right) dB(s),$$

where for all $s < t$

$$H_-(s,t) := \int_s^t \frac{\alpha(u) - \alpha(t)}{t-u} du, \ H_+(s,t) := \int_s^t \frac{\alpha(s) - \alpha(v)}{v-s} dv.$$

These processes come from multifractional integration and derivation operators and are more complex than the *classical* multifractional Brownian motion. An important new idea is that the functional parameter of such processes may not depend on the variable t that indexes the process, but be dependent on the integration variable s. Such a stochastic process where the functional parameter depends on the integration variable is said to be *non-classical* multifractional.

Surgailis shows that these two processes are well defined when $\alpha(\cdot)$ satisfies the following three conditions. The first condition is a uniform Dini condition as follows

$$(1): \ \sup_{t\in\mathbb{R}} \int_{-1}^1 \frac{|\alpha(t) - \alpha(t+u)|}{|u|} du < +\infty.$$

The second condition is

$$(2): \ \inf_{u\in\mathbb{R}} \alpha(u) > 0 \text{ for } X, \ \inf_{u\in\mathbb{R}} \alpha(u) > -1/2 \text{ for } Y.$$

The third condition is an upper bound for the generalized mean of Cesaro

$$(3): \ \overline{\alpha}_{\sup} := \limsup_{t-s\to+\infty} \frac{1}{t-s} \int_s^t \alpha(u) du < 1/2. \tag{2.3.1}$$

Let us emphasize that contrary to the multifractional Brownian motion, the condition (2.3.1) allows the functional parameter $\alpha(\cdot)$ to sometimes take large values. Surgailis also demonstrates in a weak sense (i.e. in the sense of finite-dimensional distributions) the local asymptotic self-similarity of these processes.

Theorem 2.3.1. *Let $\tau \in \mathbb{R}$, such that $\alpha(\tau) \in (0, 1/2)$ in the case of X and $\alpha(\tau) \in (-1/2, 1/2) \setminus \{0\}$ in the case of Y. Assume that the following condition (D) is satisfied for the function $\alpha(\cdot)$*

$$(D): \ |\alpha(\tau+h) - \alpha(\tau)| \underset{h\to 0}{=} o\left(\frac{1}{|\log(h)|}\right).$$

Then the processes X and Y are weakly locally asymptotically self-similar at the point τ of order $H(\tau) = \alpha(\tau) + 1/2$. This means that, for some positive deterministic constant $c(\tau)$, when $\lambda \to 0^+$, the stochastic process $\left\{\lambda^{-H(\tau)}(X(\tau + \lambda t) - X(\tau))\right\}_{t\in\mathbb{R}}$ converges in the sense of finite-dimensional distributions to $\left\{c(\tau)B_{H(\tau)}(t)\right\}_{t\in\mathbb{R}}$ (as well as for Y).

Remark 2.3.2. The condition (D) on $\alpha(\cdot)$ is much weaker than the condition (C) (for example in Theorem 2.2.6) on $H(\cdot)$.

Making the Hurst function depends on the variable of integration might not seem very judicious at beginning. Indeed, it is not certain that the wavelet method that worked well for *classical* multifractional processes can be adapted to this new context. To illustrate this statement, let us take the wavelet coefficients $(a_{j,k}(t, H))_{j,k}$ defined at (2.2.4) as

$$a_{j,k}(t, H) := \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{i\xi|\xi|^{H-1/2}} \widehat{\psi}_{j,k}(\xi) d\xi.$$

When H is constant or depends on the variable t, we can prove that these coefficients converge very quickly to 0 when $k \to +\infty$ by using integrations by parts. However, if H depends on the variable of integration ξ then it is more difficult to integrate by parts; the coefficients converge slowly and we no longer succeed in determining the Hölder exponents. No results on the local regularity of the Surgailis processes then appears explicitly in the literature. On the other hand, Surgailis himself admits in the article [BS13a] that it will be much more complex to estimate the functional parameter (as the Theorem 2.2.7 does) of his processes and more generally, of the *non-classical* multifractional processes because their covariance function becomes much more complex.

Under some assumptions on $\alpha(\cdot)$, we give an answer to each of these two problems in Chapter 3 of this thesis. In particular, we determine the local and pointwise Hölder exponents of the Surgailis processes, for a universal event that does not depend on the location, by showing that the difference between each of the two Surgailis processes and the multifractional Brownian motion is more regular than the latter. Moreover, we show that the strongly consistent estimator $\widehat{H}_N(t_0)$ of the Theorem 2.2.7 also works for the Surgailis processes.

2.3.B A new class of non-classical multifractional processes with random exponent

2.3.B.a Riemann-Liouville MPRE

It is by being more or less inspired by the idea introduced ten years earlier by Surgailis that Antoine Ayache, Céline Esser and Julien Hamonier has constructed in the article [AEH18] a new *non-classical* multifractional process by using the Itô integral. The functional parameter is no longer a deterministic function $H(\cdot)$ but a stochastic process indexed by the integration variable. By this way, the functional parameter can be adapted to the filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ to which the Brownian motion $\{B(s)\}_{s\in\mathbb{R}}$ is associated. To do so, they only considered the high-frequency part of the fractional Brownian motion, also called fractional Riemann-Liouville process, denoted by $\{R_H(t)\}_{t\in[0,1]}$ which governs the regularity of the paths of the fractional Brownian motion, and defined for $H \in (0,1)$ by

$$\forall t \in [0,1],\ R_H(t) := \int_0^1 (t-s)_+^{H-1/2} dB(s). \tag{2.3.2}$$

Then, they replaced the constant Hurst parameter H by a stochastic process $\{H(s)\}_{s\in[0,1]}$ adapted to the filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ and with values in a deterministic compact $[\underline{H}, \overline{H}] \subset (1/2, 1)$. This new process $\{X(t)\}_{t\in[0,1]}$ is thus defined as

$$\forall t \in [0,1],\ X(t) := \int_0^1 K_t(s) dB(s),$$

where

$$K_t(s) := (t-s)_+^{H(s)-1/2}. \tag{2.3.3}$$

Then, they decomposed the kernel (2.3.3) into the Haar basis of $L^2([0,1])$ which is composed of the functions

$$\mathcal{U} := \mathbb{1}_{[0,1[} \text{ et } h_{j,k} := 2^{j/2}\left(\mathbb{1}_{[2^{-j}k, 2^{-j}(k+1/2)[} - \mathbb{1}_{[2^{-j}(k+1/2), 2^{-j}(k+1)[}\right),\ j \in \mathbb{N} \text{ and } k \in \left\{0, \dots, 2^j - 1\right\}, \tag{2.3.4}$$

and they obtained a random series representation of $\left\{X(t)\right\}_{t\in[0,1]}$, given by the following theorem.

Theorem 2.3.3. *Assume that for some constants $\rho \in (1/2, 1]$ and $c > 0$ one has*

$$\forall x, y \in [0,1],\ \mathbb{E}\left(|H(x) - H(y)|^2\right) \le c|x-y|^{2\rho}.$$

Then, there exists a universal event Ω_{**} of probability 1, such that

$$\forall \omega \in \Omega_{**},\ X(t,\omega) = \langle K_t(\cdot,\omega), \mathcal{U}\rangle \eta_0(\omega) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \langle K_t(\cdot,\omega), h_{j,k}\rangle \varepsilon_{j,k}(\omega),$$

where the convergence is uniform in t on $[0,1]$ and

$$\eta_0 := \int_0^1 \mathcal{U}(s) dB(s) = B(1) - B(0)$$

and

$$\varepsilon_{j,k} := \int_0^1 h_{j,k}(s) dB(s) = 2^{j/2}\left(2B(2^{-(j+1)}(2k+1)) - B(2^{-j}k) - B(2^{-j}(k+1))\right)$$

are Gaussian random variables of the same distribution $\mathcal{N}(0,1)$.

Remark 2.3.4. In this context, one of the interests of the Haar basis is that it gives coefficients that can be easily estimated. Indeed, by combining (2.3.3) and (2.3.4) one obtains

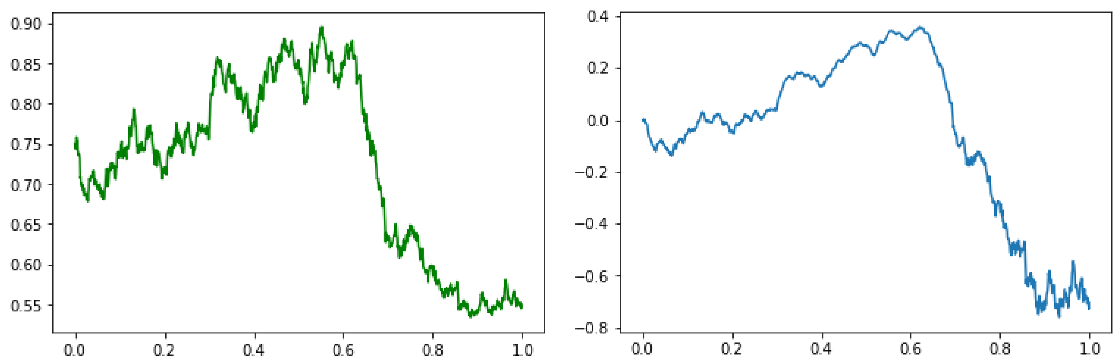
$$\langle K_t, h_{j,k}\rangle = 2^{j/2}\left(\int_{k2^{-j}}^{(k+1/2)2^{-j}} (t-s)_+^{H(s)-1/2} ds - \int_{(k+1/2)2^{-j}}^{(k+1)2^{-j}} (t-s)_+^{H(s)-1/2} ds\right).$$

Then, we do the approximation $H(s) \approx H(k2^{-j})$ when $s \in [k2^{-j}, (k+1)2^{-j})$ to get

$$\langle K_t, h_{j,k}\rangle \approx 2^{j/2}\left(\int_{k2^{-j}}^{(k+1/2)2^{-j}} (t-s)_+^{H(k2^{-j})-1/2} ds - \int_{(k+1/2)2^{-j}}^{(k+1)2^{-j}} (t-s)_+^{H(k2^{-j})-1/2} ds\right)$$

$$\langle K_t, h_{j,k}\rangle \approx \frac{2^{j/2}}{H(k2^{-j})+1/2}\left((t-k2^{-j})_+^{H(k2^{-j})+1/2} - (t-(k+1)2^{-j})_+^{H(k2^{-j})+1/2}\right).$$

The coefficients $\langle K_t, h_{j,k}\rangle$ can thus be approximated easily and the same article [AEH18] has introduced two methods of simulation of the process $\{X(t)\}_{t\in[0,1]}$.



A realization $H(\cdot,\omega)$ on the left and the associated path $X(\cdot,\omega)$ on the right

However, the conditions of Theorem 2.3.3 are still necessary and they fail to provide a method of simulating X in a more general case.
Nevertheless, we present a simulation method of the process $\{X(t)\}_{t\in[0,1]}$ in Chapter 5 of this thesis, in the case where the stochastic process $\{H(s)\}_{s\in[0,1]}$ has values in a deterministic compact of $(0,1)$ and non-continuous paths.

Moreover, thanks to this decomposition of X via the Haar basis, Ayache, Esser and Hamonier obtain a lower bound for the Hölder exponents.

Theorem 2.3.5. *Assume that the paths of $\{H(s)\}_{s\in[0,1]}$ are almost surely Hölder continuous functions of order $\gamma > 1/2$ and for some constants $\rho \in (1/2, 1]$ and $c > 0$ one has*

$$\forall x, y \in [0,1],\ \mathbb{E}\left(|H(x)-H(y)|^2\right) \le c|x-y|^{2\rho}.$$

Then, there exists a universal event Ω_* of probability 1, such that

$$\alpha_X(\tau,\omega) \ge \widetilde{\alpha}_X(\tau,\omega) \ge H(\tau,\omega), \quad \text{for all } (\tau,\omega) \in [0,1]\times\Omega_*.$$

Remark 2.3.6. The Hölder condition of the Theorem 2.3.5 is sometimes weaker than the condition (C) of the *classical* multifractional Brownian motion; for example when $\underline{H} \ge \gamma$.

2.3.B.b Recent results on a more general class of processes than the Riemann Liouville MPRE

Recently in [LMS21] Dennis Loboda, Fabian Mies and Ansgar Steland have proposed a new strategy to study the local regularity of the Riemann Liouville MPRE as well as a large class of processes that generalize it. Indeed, the stochastic processes they study are of the form

$$\forall t \geq 0,\ X(t) := \int_{-\infty}^{t} g(t,s)dB(s) \tag{2.3.5}$$

where for all fixed $t \geq 0$, the stochastic process $\{g(t,s)\}_{s\in]-\infty,t[}$ is adapted to the filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ and satisfies almost surely $\int_{-\infty}^{t} |g(t,s)|^2 ds < +\infty$. Moreover, in order to study the regularity of X, Loboda, Mies and Steland have added several conditions on g which are specified in [LMS21] as well as in Chapter 4 of this thesis; let us give two examples of processes which satisfy these conditions:

$$\forall (t,s) \in \mathbb{R}_+ \times \mathbb{R},\ g_1(t,s) := (t-s)_+^{H(s)-1/2} - (-s)_+^{H(s)-1/2}$$

and

$$\forall (t,s) \in \mathbb{R}_+ \times \mathbb{R},\ g_2(t,s) := (t-s)_+^{H(s)-1/2} e^{-\lambda(t-s)},$$

where $\lambda > 0$ is fixed and $\{H(s)\}_{s\in\mathbb{R}}$ is a stochastic process adapted to the filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ and with values in $[\underline{H}, \overline{H}] \subset (0,1)$. The process g_1 corresponds to the integrated function of the multifractional Brownian motion (2.2.1) where the deterministic function $H(t)$ is replaced by the stochastic process $\{H(s)\}_{s\in\mathbb{R}}$. On the other hand, the process g_2 is the kernel of the Matérn process ([LSEO17]).

Their main idea is to take advantage of the fact that these processes have a representation via the Itô integral to apply to them some powerful tools of the Itô calculus. They started by proving a judicious generalization of the Theorem 2.1.5 such that this theorem becomes a good tool to obtain a lower bound for the random Hölder exponents.

Theorem 2.3.7. *Let $T > 0$ be a fixed real number. One introduces two stochastic processes $\{Y(t)\}_{t\in[0,T]}$ with real values and $\{a(t)\}_{t\in[0,T]}$ with values in $(0,1)$. The process $\{a(t)\}_{t\in[0,T]}$ is lower semicontinuous and for any open or closed set $B \subset [0,T]$ the function $\underline{a}_B := \inf_{t\in B} a(t)$ is measurable and is zero nowhere. Assume that, for some constant $p > 0$, there exists $\varepsilon > 0$ and a constant $C(p,\epsilon)$ such that for all $t, t' \in [0,T]$ with $|t-t'| \leq \varepsilon$ one has*

$$\mathbb{E}\left|\frac{Y(t) - Y(t')}{|t-t'|^{\underline{a}_{[t,t']}}}\right|^p \leq C(p,\varepsilon)|t-t'|.$$

Thus, $\{Y(t)\}_{t\in[0,T]}$ has a modification with Hölder continuous paths of order $\underline{a}_B$ on any interval $B \subset [0,T]$. Then, there exists a universal event Ω_* with probability 1 such that

$$\alpha_Y(\tau,\omega) \geq \widetilde{\alpha}_Y(\tau,\omega) \geq a(\tau,\omega)\,, \quad \textit{for all } (\tau,\omega) \in\ (0,T) \times \Omega_*.$$

Remark 2.3.8. When a is constant and deterministic, one finds the Theorem 2.1.5 with $\delta = p$ and $\varepsilon = ap$.

Corollary 2.3.9. *Let $T > 0$ be a fixed real number and let us consider the processes $\{Y(t)\}_{t\in[0,T]}$ and $\{a(t)\}_{t\in[0,T]}$ as in the Theorem 2.3.7. Asume that, for all real numbers $p > 1$ and $\delta > 0$, there exists $\varepsilon > 0$ and a constant $C(p,\varepsilon,\delta)$ such that for all $t, t' \in [0,T]$ with $|t - t'| \leq \varepsilon$ one has*

$$\mathbb{E}\left|\frac{Y(t) - Y(t')}{|t-t'|^{\underline{a}_{[t,t']}-\delta}}\right|^p \leq C(p,\varepsilon,\delta).$$

Thus, $\{Y(t)\}_{t\in[0,T]}$ has a modification with Hölder continuous paths of order $\underline{a}_B$ on any interval $B \subset [0,T]$. Then, there exists a universal event Ω_* with probability 1 such that

$$\alpha_Y(\tau,\omega) \geq \widetilde{\alpha}_Y(\tau,\omega) \geq a(\tau,\omega)\,, \quad \text{for all } (\tau,\omega) \in \ (0,T) \times \Omega_*.$$

To use the Corollary 2.3.9, one must be able to find an upper bound for the moments of the increments of Y. For this reason, the Burkholder-Davis-Gundy inequality (see [Mao07, Pro05]) given in the following proposition plays a fundamental role in the article [LMS21].

Proposition 2.3.10. *Let $p \in [1,+\infty)$ be arbitrary and fixed. There is a universal deterministic finite constant $a(p)$ for which the following result holds: for any $(\mathcal{F}_s)_{s\in\mathbb{R}}$-adapted stochastic process $f = \{f(s)\}_{s\in\mathbb{R}}$ satisfying almost surely $\int_{-\infty}^{+\infty} |f(s)|^2\, ds < +\infty$, one has*

$$\mathbb{E}\left(\Big|\int_{-\infty}^{+\infty} f(s)\, dB(s)\Big|^p\right) \leq a(p)\mathbb{E}\left(\left(\int_{-\infty}^{+\infty} |f(s)|^2\, ds\right)^{p/2}\right), \tag{2.3.6}$$

where $\int_{-\infty}^{+\infty} f(s)\, dB(s)$ denotes the Itô integral of f on $\mathbb{R}$.

Remark 2.3.11. The Burkholder-Davis-Gundy inequality looks like the equivalence of Gaussian moments given by the Lemma 2.1.6. Indeed, this inequality gives an upper bound for the moments of any order $p \geq 1$ from the moment of order 2.

In the following, we will assume for the sake of simplicity (although their results remain true for a larger class of processes (see [LMS21] or Chapter 4)) that the process $\{X(t)\}_{t\geq 0}$ is of the form

$$\forall t \geq 0,\ X(t) := \int_{\mathbb{R}} \left((t-s)_+^{H(s)-1/2} - (-s)_+^{H(s)-1/2}\right) dB(s), \tag{2.3.7}$$

where $\{H(s)\}_{s\in\mathbb{R}}$ is a stochastic process adapted to the filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ and with values in $[\underline{H}, \overline{H}] \subset (0,1)$. Putting together the inequality (2.3.6) and the Corollary 2.3.9, the article [LMS21] obtains a lower bound for the local and pointwise Hölder exponents given by the following theorem.

Theorem 2.3.12. *If there exists an increasing and continuous deterministic function $\mu : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\mu(0) = 0$ such that almost surely*

$$\forall s', s'' \in \mathbb{R},\ |H(s') - H(s'')| \leq \mu(|s' - s''|), \tag{2.3.8}$$

then there exists a universal event Ω_1^* of probability 1 such that

$$\alpha_X(\tau,\omega) \geq \widetilde{\alpha}_X(\tau,\omega) \geq H(\tau,\omega)\,, \quad \text{for all } (\tau,\omega) \in \mathbb{R}_+^* \times \Omega_1^*.$$

Otherwise, there exists a universal event Ω_2^* of probability 1 such that

$$\alpha_X(\tau,\omega) \geq \widetilde{\alpha}_X(\tau,\omega) \geq \underline{H}\,, \quad \text{for all } (\tau,\omega) \in \mathbb{R}_+^* \times \Omega_2^*. \tag{2.3.9}$$

Remarks 2.3.13. • The fact that the function μ must be deterministic in the condition (2.3.8) seems to be restrictive. However, it is possible to relax this assumption with a localization procedure via stopping times (see section 4.3). Note also that this condition (2.3.8) is much weaker than the condition (C) (e.g. in the Theorem 2.2.4) of the multifractional Brownian motion.
• The inequalities (2.3.9) are obtained even in the case where H is discontinuous which implies the continuity of the paths of the process X in contrast to the classical multifractional Brownian motion, which was discontinuous at the points of discontinuity of the Hurst function (see Proposition 2.2.2)

Moreover, Loboda, Mies and Steland obtained an upper bound for the Hölder exponents for an event depending on the location. This result is obtained from the strong local asymptotic convergence of the process X. More precisely:

Theorem 2.3.14. *Assume that there exists a deterministic increasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that almost surely*

$$(D): \ \forall s', s'' \in \mathbb{R}, \ |H(s') - H(s'')| \le \mu(|s' - s''|) \underset{|s-s'|\to 0}{=} o\left(\frac{1}{|\log(|s-s'|)|}\right).$$

Then, at any point $\tau > 0$, the process X is strongly locally asymptotically self-similar of order $H(\tau)$. Roughly speaking, for all fixed real numbers $a < 0 < b$ the process $\left\{\lambda^{-H(\tau)}(X(\tau+\lambda t) - X(\tau))\right\}_{t\in[a,b]}$ converges in distribution to a fractional Brownian motion $\left\{\int_{\mathbb{R}} \left((t-s)_+^{H(\tau)-1/2} - (-s)_+^{H(\tau)-1/2}\right) d\widetilde{B}(s)\right\}_{t\in[a,b]}$ when $\lambda \to 0^+$ in the space of continuous functions $\mathcal{C}([a,b])$ and where $\left\{\widetilde{B}(s)\right\}_{s\in[a,b]}$ denotes a Brownian motion independent on $H(\tau)$.

Corollary 2.3.15. *Under the same assumptions as the Theorem 2.3.14: for all $\tau \in \mathbb{R}$, there exists a universal event $\Omega_*(\tau)$ of probability 1, such that*

$$\widetilde{\alpha}_X(\tau,\omega) \le \alpha_X(\tau,\omega) \le H(\tau,\omega), \quad \text{for all } \omega \in \Omega_*(\tau). \tag{2.3.10}$$

Theorem 2.3.16. *Under the same assumptions as the Theorem 2.3.14: for all $\tau \in \mathbb{R}$, there exists a universal event $\Omega_*(\tau)$ of probability 1, such that*

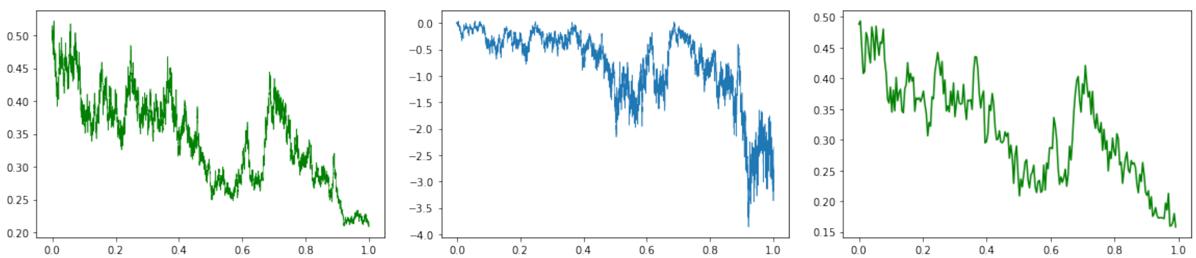
$$\widetilde{\alpha}_X(\tau,\omega) = \alpha_X(\tau,\omega) = H(\tau,\omega), \quad \text{for all } \omega \in \Omega_*(\tau). \tag{2.3.11}$$

Remark 2.3.17. The regularity condition (D) that appears in the statement of Theorem 2.3.14 is much weaker than condition (C) of Theorem 2.2.4.

This new Theorem 2.3.16 constitutes a considerable progress in the study of the local regularity of MPREs. Nevertheless, no result is known for a universal event which does not depend on the location (see Remark 2.1.10). Under a weak assumption of Hölder regularity on H paths, we give such a result in Chapter 4. The proof of this new result relies on a strategy of minimizing the local oscillations of the process; historically, it is the local time methods which allowed to obtain such inequalities (see Theorem 2.1.16) but these methods are hardly usable in the context of the process X (see Remarks 2.1.17). Our method is inspired by the one developed by Antoine Ayache in 2020 in the paper [Aya20] for the Hermite process, although it has been significantly modified to fit the context of these *non-classical* multifractional processes with a random functional parameter.

On the other hand, as we have already had the opportunity to mention, we give in Chapter 5 a simulation method of the Riemann-Liouville MPRE (2.3.2) under much weaker assumptions than those of the article [AEH18]. This simulation method consists for each $s \in [k2^{-j}, (k+1)2^{-j})$ to approximate $H(s)$ by its mean value on this same dyadic interval. The Burkholder-Davis-Gundy inequality (2.3.6) is a key element of our demonstrations.

Finally, we give in the Chapter 6 a uniform estimator of the parameter H of the process (2.3.7). To our knowledge, this is the first time that an estimator that converges in the sense of the uniform norm of a random functional parameter is obtained.



Estimation of the random functional parameter
(a realization $H(\cdot,\omega)$ on the left, the associated path $X(\cdot,\omega)$ in the middle and the estimate of $H(\cdot,\omega)$ on the right)

Chapter 3

On local path behavior of Surgailis multifractional processes

This chapter is a restatement of the article [AB22b].

3.1 Introduction and statement of the main results

Let $H \in (0,1)$, the fractional Brownian motion Field (FBF) of Hurst parameter H, which is also called multivariate fractional Brownian, is a real-valued centred continuous Gaussian field on $\mathbb{R}^N$ denoted by $\{B_H(t)\}_{t\in\mathbb{R}^N}$ having, for all $t', t'' \in \mathbb{R}^N$, the covariance:

$$\mathrm{Cov}(B_H(t'), B_H(t'')) = \mathbb{E}(B_H(t')B_H(t'')) = c(H)\big(|t'|^{2H} + |t''|^{2H} - |t'-t''|^{2H}\big), \tag{3.1.1}$$

where $c(H)$ is a positive constant only depending on H, and $|\cdot|$ is the Euclidian norm. where $c(H)$ is a positive constant only depending on H, and $|\cdot|$ is the Euclidian norm. One refers to e.g. Chapter 1 of the book [Aya19] for a detailed presentation FBF. Notice that the equality (3.1.1) can equivalently be expressed as: $\mathbb{E}|B_H(t') - B_H(t'')|^2 = 2c(H)|t'-t''|^{2H}$, for every $t', t'' \in \mathbb{R}^N$; thus the existence of a modification of $\{B_H(t)\}_{t\in\mathbb{R}^N}$ with Hölder continuous paths on any bounded subset of $\mathbb{R}^N$ results from the equivalence of Gaussian moments and the well-known Kolmogorov-Chentsov Hölder continuity theorem (see e.g. [KS87, Kho02] and Proposition 3.2.1 in the beginning of the next section). Also, notice that, up to a multiplicative constant, $\{B_H(t)\}_{t\in\mathbb{R}^N}$ is in distribution the unique Gaussian field which satisfies the following three fundamental properties: *self-similaritity* that is for all fixed positive real number a one has $\{B_H(at)\}_{t\in\mathbb{R}^N} \overset{d}{=} \{a^H B_H(t)\}_{t\in\mathbb{R}^N}$, where the symbol $\overset{d}{=}$ means equality of finite-dimensional distributions; *stationarity of increments*, that is for each fixed $\mathbf{t} \in \mathbb{R}^N$ one has $\{B_H(\mathbf{t}+t) - B_H(\mathbf{t})\}_{t\in\mathbb{R}^N} \overset{d}{=} \{B_H(t)\}_{t\in\mathbb{R}^N}$; and *isotropy*, that is for every fixed orthogonal matrix Q of size N one has $\{B_H(Qt)\}_{t\in\mathbb{R}^N} \overset{d}{=} \{B_H(t)\}_{t\in\mathbb{R}^N}$. Though FBF is a useful model, a serious limitation of it comes from the fact that local behavior of its paths does not change from point to point. More precisely, roughness of paths of a continuous nowhere differentiable real-valued stochastic field $\{Z(t)\}_{t\in\mathbb{R}^N}$ around some fixed point $\tau \in \mathbb{R}^N$ is usually measured through the *pointwise Hölder exponent at* τ

$$\alpha_Z(\tau) := \sup\left\{ r \in [0,1]\,;\ \limsup_{t\to\tau} \frac{|Z(t)-Z(\tau)|}{|t-\tau|^r} < +\infty \right\}, \tag{3.1.2}$$

or through the *local Hölder exponent at* τ

$$\widetilde{\alpha}_Z(\tau) := \sup\left\{\widetilde{r} \in [0,1]\,;\ \limsup_{(t',t'')\to(\tau,\tau)} \frac{|Z(t')-Z(t'')|}{|t'-t''|^{\widetilde{r}}} < +\infty\right\}. \tag{3.1.3}$$

Observe that, one always has that

$$\widetilde{\alpha}_Z(\tau) \le \alpha_Z(\tau)\,, \quad \text{for all } \tau \in \mathbb{R}^N. \tag{3.1.4}$$

Local roughness of paths of the FBF $\{B_H(t)\}_{t\in\mathbb{R}^N}$ does not change from point to point since it is known (see for instance [Xia97, Ber72, Ber73, Pit78, Aya19]) that there exists a universal event Ω_* of probability 1 such that one has

$$\widetilde{\alpha}_{B_H}(\tau,\omega) = \alpha_{B_H}(\tau,\omega) = H\,, \quad \text{for all } (\tau,\omega) \in \mathbb{R}^N \times \Omega_*.$$

In order to overcome this limitation of FBF, it has been proposed in [BJR97, PL95] to replace its constant Hurst parameter H by $H(t)$, where $H(\cdot)$ denotes a continuous function on $\mathbb{R}^N$ with values in some compact interval included in $(0,1)$. This idea has led to Multifractional Brownian motion Field $\{\mathcal{M}(t)\}_{t\in\mathbb{R}^N}$ which is more commonly called multivariate Multifractional Brownian motion (MBM). It is usually assumed that the continuous function $H(\cdot)$ satisfies the condition:

$$\widetilde{\alpha}_H(\tau) > H(\tau)\,, \quad \text{for all } \tau \in \mathbb{R}^N, \tag{3.1.5}$$

where $\widetilde{\alpha}_H(\tau)$ denotes the local Hölder exponent of $H(\cdot)$ at τ.

Remarks 3.1.1.

(*i*) Under the condition (3.1.5), it was shown (see [AJT07, Aya19]) that there exists a universal event Ω_{**} of probability 1 such that one has

$$\widetilde{\alpha}_{\mathcal{M}}(\tau,\omega) = \alpha_{\mathcal{M}}(\tau,\omega) = H(\tau)\,, \quad \text{for all } (\tau,\omega) \in \mathbb{R}^N \times \Omega_{**}. \tag{3.1.6}$$

(*ii*) Also, under this same condition (3.1.5), it was shown (see [BJR97, Aya19]) that, at every fixed point $\tau \in \mathbb{R}^N$, the stochastic field $\{\mathcal{M}(t)\}_{t\in\mathbb{R}^N}$ is *strongly locally asymptotically self-similar* of exponent $H(\tau)$. This means that, for some positive deterministic constant $c(\tau)$ and for any fixed positive real number T, when $\lambda \to 0_+$, the stochastic field $\big\{\lambda^{-H(\tau)}(\mathcal{M}(\tau+\lambda u) - \mathcal{M}(\tau))\big\}_{u\in\mathbb{R}^N}$ converges in distribution to $\{c(\tau)B_{H(\tau)}(u)\}_{u\in\mathbb{R}^N}$ in $\mathcal{C}\big([-T,T]^N\big)$ the Banach space of the real-valued continuous functions on the cube $[-T,T]^N$ equipped with the uniform norm.

(*iii*) When one drops the condition (3.1.5), then (3.1.6) fails to be satisfied except maybe in some very particular cases. Namely, under very weak conditions, [Her06] showed that, for each $\tau \in \mathbb{R}^N \setminus \{0\}$, one has almost surely

$$\widetilde{\alpha}_{\mathcal{M}}(\tau) = H(\tau) \wedge \widetilde{\alpha}_H(\tau) \quad \text{and} \quad \alpha_{\mathcal{M}}(\tau) = H(\tau) \wedge \alpha_H(\tau)\,; \tag{3.1.7}$$

later [Aya13, ca15] proved that the second equality in (3.1.7) only holds on an event of probability 1 which depends on τ.

(*iv*) When one drops the condition (3.1.5), then it can be shown that $\{\mathcal{M}(t)\}_{t\in\mathbb{R}^N}$ fails to satisfy the local-self similarity property (*ii*) except maybe in some very particular cases.

From now on, we assume that $N = 1$. Since we are mainly concerned with the *non-classical* Gaussian multifractional processes introduced by Surgailis in his article [Sur08], it is convenient to use the same notations as in this article. Therefore, we denote by $\alpha(\cdot)$ the continuous function on $\mathbb{R}$ defined as:

$$\alpha(x) = H(x) - 1/2\,, \quad \text{for all } x \in \mathbb{R}. \tag{3.1.8}$$

Similarly to the article [Sur08], we always suppose that $\alpha(\cdot)$ satisfies the uniform Dini condition:

$$\sup_{t\in\mathbb{R}} \int_{-1}^{1} \frac{|\alpha(t) - \alpha(t+u)|}{|u|}\, du < +\infty\,. \tag{3.1.9}$$

Yet, in contrast with the latter article, our main results require us to impose to $\alpha(\cdot)$ to be with values in a compact interval $[\alpha_{\inf}, \alpha_{\sup}]$ included in $(0, 1/2)$ and to satisfy the other condition:

$$\widetilde{\alpha}_\alpha(\tau) > \alpha(\tau) + 1/2\,, \quad \text{for all } \tau \in \mathbb{R}, \tag{3.1.10}$$

which is in fact nothing else than the usual condition (3.1.5) with $N = 1$ expressed in terms of $\alpha(\cdot)$.

Remark 3.1.2. The two conditions (3.1.9) and (3.1.10) hold typically when $\alpha(\cdot)$ is an arbitrary function with values in any compact interval included in $(0, 1/2)$ which is Hölder continuous of any order $\beta \in [1/2, 1]$ on the whole real line; in other words, for all $x', x'' \in \mathbb{R}$, one has $|\alpha(x') - \alpha(x'')| \le c|x' - x''|^\beta$, where c denotes a finite constant not depending on x' and x''.

We are now going to precisely define the *classical* Gaussian MBM $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ initially introduced in [PL95], and the two *non-classical* Surgailis Gaussian multifractional processes $\{X(t)\}_{t\in\mathbb{R}}$ and $\{Y(t)\}_{t\in\mathbb{R}}$ constructed in [Sur08]. To this end, we make use of the usual convention:

$$\text{for all } (y,\theta) \in \mathbb{R}^2, \text{ one has } (y)_+^\theta := y^\theta \text{ if } y > 0 \text{ and } (y)_+^\theta = 0 \text{ else.} \tag{3.1.11}$$

The *classical* MBM $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ with continuous paths is defined, for each $t \in \mathbb{R}$, through the Wiener integral:

$$\mathcal{M}(t) := \int_{\mathbb{R}} \frac{1}{\Gamma(1+\alpha(t))} \left((t-s)_+^{\alpha(t)} - (-s)_+^{\alpha(t)} \right) dB(s)\,, \tag{3.1.12}$$

where $\{B(s)\}_{s\in\mathbb{R}}$ denotes the standard Wiener process and $\Gamma(\cdot)$ the well-known "Gamma" function, recall that

$$\Gamma(x) := \int_0^{+\infty} y^{x-1}\, e^{-y}\, dy\,, \quad \text{for each } x \in (0, +\infty).$$

When $\alpha(\cdot)$ is with values in $(0, 1/2)$, then it easily follows from (3.1.11), (3.1.12), and the equality $\Gamma(1+x) = x\Gamma(x)$, for all $x \in (0, +\infty)$, that, for every $t \in \mathbb{R}$, one has

$$\mathcal{M}(t) = \int_{\mathbb{R}} \left(\int_0^t \frac{1}{\Gamma(\alpha(t))} (\tau - s)_+^{\alpha(t)-1} d\tau \right) dB(s)\,. \tag{3.1.13}$$

The two *non-classical* Surgailis multifractional processes $\{X(t)\}_{t\in\mathbb{R}}$ and $\{Y(t)\}_{t\in\mathbb{R}}$ with continuous paths are, for every $t \in \mathbb{R}$, defined through the Wiener integrals:

$$X(t) := \int_{\mathbb{R}} \left(\int_0^t \frac{1}{\Gamma(\alpha(\tau))} (\tau - s)_+^{\alpha(\tau)-1} e^{H_-(s,\tau)} d\tau \right) dB(s) \tag{3.1.14}$$

and

$$Y(t) := \int_{\mathbb{R}} \frac{1}{\Gamma(1+\alpha(s))} \Big((t-s)_+^{\alpha(s)} e^{-H_+(s,t)} - (-s)_+^{\alpha(s)} e^{-H_+(s,0)} \Big)\, dB(s)\,, \tag{3.1.15}$$

where, for all real numbers s and t satisfying $s < t$, one has set

$$H_-(s,t) := \int_s^t \frac{\alpha(u)-\alpha(t)}{t-u} du \quad \text{and} \quad H_+(s,t) := \int_s^t \frac{\alpha(s)-\alpha(v)}{v-s} dv. \tag{3.1.16}$$

Remark 3.1.3. Some of the results concerning $\{X(t)\}_{t\in\mathbb{R}}$ and $\{Y(t)\}_{t\in\mathbb{R}}$ which were obtained in [Sur08] are the following.

(i) The process $\{X(t)\}_{t\in\mathbb{R}}$ can be defined under the very weak 3 conditions: (3.1.9), $\alpha_{\text{inf}} := \inf_{u\in\mathbb{R}} \alpha(u) > 0$ and $\overline{\alpha}_{\text{sup}} < 1/2$, where

$$\overline{\alpha}_{\text{sup}} := \limsup_{t-s\to+\infty} \frac{1}{t-s} \int_s^t \alpha(u)du\,. \tag{3.1.17}$$

Moreover, for some finite constant c and for every $t', t'' \in \mathbb{R}$ satisfying $|t'-t''| \le 1$, one has $\mathbb{E}|X(t')-X(t'')|^2 \le c|t'-t''|^{2\alpha_{\text{inf}}+1}$; thus one can derive from the equivalence of Gaussian moments and the Kolmogorov-Chentsov Hölder continuity theorem that there exists a modification of $\{X(t)\}_{t\in\mathbb{R}}$ whose paths are on each compact interval Hölder continuous functions of any order strictly less than $\alpha_{\text{inf}} + 1/2$.

(ii) Assume that $\tau \in \mathbb{R}$ is a fixed point such that $\alpha(\tau) \in (0, 1/2)$ and

$$\big|\alpha(t) - \alpha(\tau)\big| = o\Big(\big|\log|t-\tau|\big|^{-1} \Big) \quad \text{when } |t-\tau| \to 0. \tag{3.1.18}$$

Then, at this point τ, the process $\{X(t)\}_{t\in\mathbb{R}}$ is *weakly locally asymptotically self-similar* of exponent $H(\tau) = \alpha(\tau) + 1/2$. This means that, for some positive deterministic constant $c(\tau)$, when $\lambda \to 0_+$, the finite-dimensional distributions of the centred Gaussian process $\big\{\lambda^{-H(\tau)}(X(\tau+\lambda u) - X(\tau))\big\}_{u\in\mathbb{R}}$ converge to those of the fractional Brownian motion $\{c(\tau)B_{H(\tau)}(u)\}_{u\in\mathbb{R}}$.

(iii) The process $\{Y(t)\}_{t\in\mathbb{R}}$ can be defined under the very weak 3 conditions: (3.1.9), $\alpha_{\text{inf}} > -1/2$ and $\overline{\alpha}_{\text{sup}} < 1/2$ (see (3.1.17)). Moreover, for some finite constant c and for every $t', t'' \in \mathbb{R}$ satisfying $|t'-t''| \le 1$, one has $\mathbb{E}|Y(t')-Y(t'')|^2 \le c|t'-t''|^{2\alpha_{\text{inf}}+1}$; thus there exists a modification of $\{Y(t)\}_{t\in\mathbb{R}}$ whose paths are on each compact interval Hölder continuous functions of any order strictly less than $\alpha_{\text{inf}} + 1/2$.

(iv) Assume that $\tau \in \mathbb{R}$ is a fixed point such that $\alpha(\tau) \in (-1/2, 1/2) \setminus \{0\}$ and (3.1.18) holds. Then, at this point τ, the process $\{Y(t)\}_{t\in\mathbb{R}}$ is *weakly locally asymptotically self-similar* of exponent $H(\tau) = \alpha(\tau) + 1/2$.

The following proposition easily results from Hölder continuity of paths of $\{X(t)\}_{t\in\mathbb{R}}$, part ($ii$) of Remark 3.1.3 and Theorem 1.74 in [Aya19].

Proposition 3.1.4. *Let $\tau \in \mathbb{R}$ be a point satisfying the same conditions as in part (ii) of Remark 3.1.3. Then $\alpha_X(\tau)$, the pointwise Hölder exponent of $\{X(t)\}_{t\in\mathbb{R}}$ at τ, satisfies*

$$\alpha_X(\tau) = H(\tau),$$

where the equality holds on an event of probability 1 which a priori depends on τ.

The following proposition easily results from Hölder continuity of paths of $\{Y(t)\}_{t\in\mathbb{R}}$, part (iv) of Remark 3.1.3 and Theorem 1.74 in [Aya19].

Proposition 3.1.5. *Let $\tau \in \mathbb{R}$ be a point satisfying the same conditions as in part (iv) of Remark 3.1.3. Then $\alpha_Y(\tau)$, the pointwise Hölder exponent of $\{Y(t)\}_{t\in\mathbb{R}}$ at τ, satisfies*

$$\alpha_Y(\tau) = H(\tau),$$

where the equality holds on an event of probability 1 which a priori depends on τ.

Remark 3.1.6. Combining Remarks 3.1.1 (iii) and (iv) and Remarks 3.1.3 (ii) and (iv) with Propositions 3.1.4 and 3.1.5, it turns out that when the condition (3.1.10) (i.e. the condition (3.1.5)) fails to be satisfied for some point τ_0, then the two processes $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ and $\{X(t)\}_{t\in\mathbb{R}}$, as well as the two processes $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ and $\{Y(t)\}_{t\in\mathbb{R}}$, have in general a different local path behavior in a neighborhood of τ_0.

From now on, we will always suppose that the following hypothesis (A) holds.

Hypothesis (A)**:** *$\alpha(\cdot)$ is a continuous function on $\mathbb{R}$ with values in $[\alpha_{\inf}, \alpha_{\sup}] \subset (0, 1/2)$ which satisfies the two conditions (3.1.9) and (3.1.10).*

In the framework of the hypothesis (A), it is natural to seek to compare the *classical* MBM $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ with each one of the two *non-classical* multifractional processes $\{X(t)\}_{t\in\mathbb{R}}$ and $\{Y(t)\}_{t\in\mathbb{R}}$. This leads us to introduce the two centred Gaussian processes with continuous paths $\{R(t)\}_{t\in\mathbb{R}}$ and $\{D(t)\}_{t\in\mathbb{R}}$ defined, for all $t \in \mathbb{R}$, as:

$$R(t) := X(t) - \mathcal{M}(t) \tag{3.1.19}$$

and

$$D(t) := Y(t) - \mathcal{M}(t). \tag{3.1.20}$$

The following two theorems are the two main results of our article. Roughly speaking they show that $\{R(t)\}_{t\in\mathbb{R}}$ and $\{D(t)\}_{t\in\mathbb{R}}$ are locally more regular than $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$. Thus, it turns out that under the hypothesis (A) (in which the assumption (3.1.10) on $\alpha(\cdot)$ or (3.1.5) on $H(\cdot)$ plays a crucial role), $\{X(t)\}_{t\in\mathbb{R}}$ *and* $\{Y(t)\}_{t\in\mathbb{R}}$ *have exactly the same local path behavior as* $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$.

Theorem 3.1.7. *Assume that the hypothesis (A) is satisfied. Then, there exists a universal event Ω' of probability 1 such that one has*

$$\widetilde{\alpha}_R(\tau,\omega) > \alpha(\tau) + 1/2 = H(\tau) = \alpha_{\mathcal{M}}(\tau,\omega)\,, \quad \textit{for all } (\tau,\omega) \in \mathbb{R}\times\Omega'. \tag{3.1.21}$$

Theorem 3.1.8. *Assume that the hypothesis (A) is satisfied. Then, there exists a universal event Ω'' of probability 1 such that one has*

$$\widetilde{\alpha}_D(\tau,\omega) > \alpha(\tau) + 1/2 = H(\tau) = \alpha_{\mathcal{M}}(\tau,\omega)\,, \quad \textit{for all } (\tau,\omega) \in \mathbb{R}\times\Omega''. \tag{3.1.22}$$

It easily follows from Theorem 3.1.7, (3.1.19), (3.1.6), (3.1.2), (3.1.3) and (3.1.4) that:

Corollary 3.1.9. *Assume that the hypothesis (A) is satisfied and that Ω' is the same event of probability 1 as in Theorem 3.1.7. Then, one has*

$$\alpha_X(\tau,\omega) = \widetilde{\alpha}_X(\tau,\omega) = \alpha(\tau) + 1/2 = H(\tau)\,, \quad \textit{for all } (\tau,\omega) \in \mathbb{R}\times\Omega'.$$

It easily follows from Theorem 3.1.8, (3.1.20), (3.1.6), (3.1.2), (3.1.3) and (3.1.4) that:

Corollary 3.1.10. *Assume that the hypothesis* (A) *is satisfied and that* Ω'' *is the same event of probability* 1 *as in Theorem 3.1.8. Then, one has*

$$\alpha_Y(\tau,\omega) = \widetilde{\alpha}_Y(\tau,\omega) = \alpha(\tau) + 1/2 = H(\tau)\,, \quad \textit{for all } (\tau,\omega) \in \mathbb{R}\times\Omega''.$$

Corollary 3.1.11. *Assume that the hypothesis* (A) *is satisfied. Then, at every fixed point* $\tau \in \mathbb{R}$*, the stochastic process* $\{X(t)\}_{t\in\mathbb{R}}$ *is* strongly locally asymptotically self-similar *of exponent* $H(\tau) = \alpha(\tau) + 1/2$*. More precisely, for some positive constant* $c(\tau)$ *and for any fixed positive real number* T*, when* $\lambda \to 0_+$*, the stochastic process* $\big\{\lambda^{-H(\tau)}(X(\tau+\lambda u) - X(\tau))\big\}_{u\in\mathbb{R}}$ *converges in distribution to* $\{c(\tau)B_{H(\tau)}(u)\}_{u\in\mathbb{R}}$ *in* $\mathcal{C}\big([-T,T]\big)$ *the Banach space of the real-valued continuous functions over the compact* $[-T,T]$ *equipped with the uniform norm.*

Proof. It follows from (3.1.19) that, for each fixed positive real number λ, the stochastic process $\big\{\lambda^{-H(\tau)}(X(\tau+\lambda u)-X(\tau))\big\}_{u\in\mathbb{R}}$ can be expressed as the sum of the two processes $\big\{\lambda^{-H(\tau)}(\mathcal{M}(\tau+\lambda u)-\mathcal{M}(\tau))\big\}_{u\in\mathbb{R}}$ and $\big\{\lambda^{-H(\tau)}(R(\tau+\lambda u)-R(\tau))\big\}_{u\in\mathbb{R}}$. One already knows from [BJR97, Aya19] that the process $\big\{\lambda^{-H(\tau)}(\mathcal{M}(\tau+\lambda u)-\mathcal{M}(\tau))\big\}_{u\in\mathbb{R}}$ converges in distribution to $\{c(\tau)B_{H(\tau)}(u)\}_{u\in\mathbb{R}}$ in $\mathcal{C}\big([-T,T]\big)$ when $\lambda \to 0_+$. Thus, for proving the corollary it is enough to show that the process $\big\{\lambda^{-H(\tau)}(R(\tau+\lambda u) - R(\tau))\big\}_{u\in\mathbb{R}}$, viewed as a random variable with values in the space $\mathcal{C}\big([-T,T]\big)$, converges almost surely to 0 in this space when $\lambda \to 0_+$. The latter fact results from Theorem 3.1.7, (3.1.4) and (3.1.2) which entail that, for each $\omega \in \Omega'$ (the same event of probability 1 as in Theorem 3.1.7), there are 3 positive finite constants $C_0(\omega)$, $\varepsilon_0(\omega)$ and $\eta_0(\omega)$ such that, for all real number v satisfying $|v| \le \eta_0(\omega)$, one has $\big|R(\tau+v,\omega) - R(\tau,\omega)\big| \le C_0(\omega)|v|^{H(\tau)+\varepsilon_0(\omega)}$. □

Corollary 3.1.12. *Assume that the hypothesis* (A) *is satisfied. Then, at every fixed point* $\tau \in \mathbb{R}$*, the stochastic process* $\{Y(t)\}_{t\in\mathbb{R}}$ *is* strongly locally asymptotically self-similar *of exponent* $H(\tau) = \alpha(\tau) + 1/2$.

The proof of Corollary 3.1.12 is skipped since it is very similar to that of Corollary 3.1.11 except that Theorem 3.1.8 and (3.1.20) have to be used instead of Theorem 3.1.7 and (3.1.19).

Before ending the present section, let us emphasize that Theorem 3.1.7 (resp. Theorem 3.1.8) is the main ingredient for proving strong consistency of a statistical estimator of $\alpha(\tau)$ derived from the observation of a discretized path of the Surgailis multifractional process X (resp. Y) in a neighborhood of any fixed point $\tau \in \mathbb{R}$. From now on and till the end of the present section one denotes by S either X or Y, and one assumes that, for each integer N large enough, a discretized realization $\big\{S(\frac{k}{N}), S(\frac{k+1}{N}), S(\frac{k+2}{N})\big\}_{k\in\nu_N(\tau)}$ is observed; notice that the set of indices $\nu_N(\tau)$, which can be viewed as "discrete neighborhood of τ", is defined as $\nu_N(\tau) := \big\{k \in \mathbb{Z},\, |\frac{k}{N} - \tau| \le N^{-a}\big\}$, where $a \in (0, 1/2)$ is a fixed parameter allowing control of width $\nu_N(\tau)$. As explained in Remark 5 of [BS13a], the study of the covariance function of the process S is significantly more difficult than that of the classical MBM $\mathcal{M}$. This is why, as far as we know, the problem of finding from a discrete realization of S a consistent statistical estimator of $\alpha(\tau)$ remains open. The following theorem provides a solution to this problem when the hypothesis (A) holds.

Theorem 3.1.13. *For all integer N large enough one denotes by $\#(\nu_N(\tau))$ the cardinality of $\nu_N(\tau)$ and one sets*

$$V_N^S(\tau) := \frac{1}{\#(\nu_N(\tau))} \sum_{k\in\nu_N(\tau)} \left(S\left(\frac{k}{N}\right) - 2S\left(\frac{k+1}{N}\right) + S\left(\frac{k+2}{N}\right)\right)^2 , \tag{3.1.23}$$

where $\tau \in \mathbb{R}$ is any fixed point and S is either the process X or Y. Then, under the hypothesis (A),

$$\widehat{\alpha}_N(\tau) := (\log 4)^{-1} \log\left(\frac{V_N^S(\tau)}{V_{2N}^S(\tau)}\right) - \frac{1}{2} \tag{3.1.24}$$

is a strongly consistent estimator of $\alpha(\tau)$; that is $\widehat{\alpha}_N(\tau)$ converges almost surely to $\alpha(\tau)$ when N goes to $+\infty$.

Proof. For the sake of simplicity, we assume that $S = X$; the proof can be done in a similar way when $S = Y$. The empirical mean $V_N^{\mathcal{M}}(\tau)$ (resp. $V_N^R(\tau)$) is defined by replacing in the empirical mean (3.1.23) the Surgailis multifractional process S by the classical MBM $\mathcal{M}$ (resp. the process R defined through (3.1.19)). It is known (see for instance [AL04, BS13a]) that

$$\frac{V_N^{\mathcal{M}}(\tau)}{\mathbb{E}\big(V_N^{\mathcal{M}}(\tau)\big)} \xrightarrow[N\to+\infty]{a.s.} 1 \tag{3.1.25}$$

and

$$\frac{\mathbb{E}\big(V_N^{\mathcal{M}}(\tau)\big)}{c(\tau)N^{-2(\alpha(\tau)+1/2)}} \xrightarrow[N\to+\infty]{} 1, \tag{3.1.26}$$

where $c(\tau)$ is a positive constant. Observe that it results from (3.1.24) and (3.1.26) that for proving the theorem it is enough to show that

$$\frac{V_N^S(\tau)}{\mathbb{E}\big(V_N^{\mathcal{M}}(\tau)\big)} \xrightarrow[N\to+\infty]{a.s.} 1\,. \tag{3.1.27}$$

Next, notice that in view of (3.1.25), one can get (3.1.27) by proving that

$$\frac{\left|\sqrt{V_N^S(\tau)} - \sqrt{V_N^{\mathcal{M}}(\tau)}\right|}{\sqrt{\mathbb{E}\big(V_N^{\mathcal{M}}(\tau)\big)}} \xrightarrow[N\to+\infty]{a.s.} 0\,. \tag{3.1.28}$$

Combining the triangle inequality with (3.1.19) and the definitions of $V_N^S(\tau)$, $V_N^{\mathcal{M}}(\tau)$ and $V_N^R(\tau)$ (see (3.1.23)), one obtains that

$$\left|\sqrt{V_N^S(\tau)} - \sqrt{V_N^{\mathcal{M}}(\tau)}\right| \le \sqrt{V_N^R(\tau)}\,. \tag{3.1.29}$$

Moreover, using the definition of $\sqrt{V_N^R(\tau)}$, Theorem 3.1.7 and (3.1.3) one has, for some $\widetilde{\varepsilon}_0 > 0$,

$$N^{\alpha(\tau)+1/2+\widetilde{\varepsilon}_0} \sqrt{V_N^R(\tau)} \xrightarrow[N\to+\infty]{a.s.} 0\,. \tag{3.1.30}$$

Finally, putting together (3.1.29), (3.1.30) and (3.1.26) it follows that (3.1.28) is satisfied. □

Remark 3.1.14. It is worth mentioning that a careful inspection of the proof of Theorem 3.1.8 shows that it remains valid when the condition $[\alpha_{\text{inf}}, \alpha_{\text{sup}}] \subset (0, 1/2)$ in the hypothesis (A) is weakened to $[\alpha_{\text{inf}}, \alpha_{\text{sup}}] \subset (-1/2, 1/2)$. Thus, Corollaries 3.1.10 and 3.1.12 as well as Theorem 3.1.13 for $S = Y$ also remain valid under this weaker assumtion.

3.2 Proof of Theorem 3.1.7

Let us first point out that the proof of Theorems 3.1.7 mainly relies on the following proposition which is a classical result derived from the equivalence of Gaussian moments and the well-known Kolmogorov-Chentsov Hölder continuity theorem (see e.g. [KS87, Kho02]).

Proposition 3.2.1. *Let $\{Z(t)\}_{t\in\mathbb{R}}$ be a real-valued Gaussian process with continuous* [1] *paths. Suppose that, for some compact interval $I\subset\mathbb{R}$ and for some constants $c>0$ and $\zeta\in(0,1]$, the inequality*

$$\mathbb{E}|Z(t')-Z(t'')|^2\le c|t'-t''|^{2\zeta}$$

holds for all $t',t''\in I$. Then, with probability 1, the paths of $\{Z(t)\}_{t\in\mathbb{R}}$ satisfy on I a uniform Hölder condition of any order $\beta\in(0,\zeta)$. More precisely, there exists Ω_I' an event of probability 1, which a priori depends on I, such that one has

$$\sup_{t',t''\in I}\frac{|Z(t',\omega)-Z(t'',\omega)|}{|t'-t''|^{\beta}}<+\infty\,,\quad \textit{for all } (\omega,\beta)\in\Omega_I'\times(0,\zeta). \tag{3.2.1}$$

Remark 3.2.2. For proving Theorem 3.1.7 it is enough to show that, for all fixed $t_0\in\mathbb{R}$, there are 3 constants $\varepsilon_{t_0}>0$, $\eta_{t_0}>0$ and $c_{t_0}\ge 0$, which may depend on t_0, such that

$$\mathbb{E}|R(t')-R(t'')|^2\le c_{t_0}\,|t'-t''|^{2(\alpha(t_0)+1/2+3\varepsilon_{t_0})}\,,\quad \text{for all } t',t''\in I(t_0,\eta_{t_0}/2), \tag{3.2.2}$$

where $I(t_0,\eta_{t_0}/2):=[t_0-\eta_{t_0}/2,t_0+\eta_{t_0}/2]$. Indeed, in view of Proposition 3.2.1, it results from (3.2.2) that there exists Ω'_{t_0} an event of probability 1 such that

$$\sup_{t',t''\in I(t_0,\eta_{t_0}/2)}\frac{|R(t',\omega)-R(t'',\omega)|}{|t'-t''|^{\alpha(t_0)+1/2+2\varepsilon_{t_0}}}<+\infty\,,\quad \text{for all } \omega\in\Omega'_{t_0}. \tag{3.2.3}$$

Observe that, using the continuity at t_0 of the function $\alpha(\cdot)$, one can choose η_{t_0} small enough so that

$$|\alpha(t)-\alpha(t_0)|\le\varepsilon_{t_0}\,,\quad \text{for all } t\in I(t_0,\eta_{t_0}/2). \tag{3.2.4}$$

Next, let $\mathring{I}(t_0,\eta_{t_0}/2)$ be the open interval, containing t_0 and included in $I(t_0,\eta_{t_0}/2)$, defined as $\mathring{I}(t_0,\eta_{t_0}/2):=(t_0-\eta_{t_0}/2,t_0+\eta_{t_0}/2)$. One clearly has that $\mathbb{R}=\bigcup_{t_0\in\mathbb{R}}\mathring{I}(t_0,\eta_{t_0}/2)$. Therefore, the local compactness of $\mathbb{R}$ implies that

$$\mathbb{R}=\bigcup_{m\in\mathbb{N}}\mathring{I}(t_{0,m},\eta_{t_{0,m}}/2)\,, \tag{3.2.5}$$

for some sequence $(t_{0,m})_{m\in\mathbb{N}}$ of real numbers. Next, one denotes by Ω' the event of probabiliy 1 defined as

$$\Omega':=\Big(\bigcap_{m\in\mathbb{N}}\Omega_{t_{0,m}}\Big)\cap\Omega_{**}\,,$$

[1]For the sake of simplicity, we can make this continuity assumption since we already know that the paths of the Gaussian processes $\{R(t)\}_{t\in\mathbb{R}}$ and $\{D(t)\}_{t\in\mathbb{R}}$, defined through (3.1.19) and (3.1.20), are continuous functions and we are interested in their local Hölder regularity. Notice that when the assumption of continuity of the paths of $\{Z(t)\}_{t\in\mathbb{R}}$ is dropped then (3.2.1) holds for a well-chosen modification $\{\widetilde{Z}(t)\}_{t\in I}$ of $\{Z(t)\}_{t\in I}$.

where Ω_{**} is the same event of probability 1 as in (3.1.6) with $N = 1$. Let $(\tau, \omega) \in \mathbb{R} \times \Omega'$ be arbitrary and fixed. One knows from (3.2.5) that there are $\widetilde{m}(\tau) \in \mathbb{N}$ and $\widetilde{\eta}(\tau) \in (0,1)$ such that

$$I\big(\tau, \widetilde{\eta}(\tau)\big) \subset \mathring{I}\big(t_{0,\widetilde{m}(\tau)}, \eta_{t_{0,\widetilde{m}(\tau)}}/2\big)\,. \tag{3.2.6}$$

Thus, it follows from (3.2.6), (3.2.4) and (3.2.3) that

$$\begin{aligned}&\sup_{t',t''\in I(\tau,\widetilde{\eta}(\tau))} \frac{|R(t',\omega)-R(t'',\omega)|}{|t'-t''|^{\alpha(\tau)+1/2+\varepsilon_{t_{0,\widetilde{m}(\tau)}}}}\\ &\le \sup_{t',t''\in I(t_{0,\widetilde{m}(\tau)},\eta_{t_{0,\widetilde{m}(\tau)}}/2)} \frac{|R(t',\omega)-R(t'',\omega)|}{|t'-t''|^{\alpha(t_{0,\widetilde{m}(\tau)})+1/2+2\varepsilon_{t_{0,\widetilde{m}(\tau)}}}} < +\infty\,. \end{aligned}\tag{3.2.7}$$

Finally, (3.2.7), (3.1.3), (3.1.6), and (3.1.8) imply that (3.1.21) is satisfied.

From now on, the goal is to prove that, for any fixed $t_0 \in \mathbb{R}$, the inequality (3.2.2) holds. To this end, one will make an extensive use of the following lemma borrowed from [Sur08].

Lemma 3.2.3. *One assumes that the continuous function $\alpha(\cdot)$ satisfies the condition (3.1.9). Then, for all fixed (strictly) positive real numbers ε and ν, there is a constant C (which depends on ε and ν) such that the inequalities*

$$(t-s)^{\alpha(t)} e^{H_-(s,t)} \le C\,(t-s)^{\overline{\alpha}_{\sup}+\varepsilon} \tag{3.2.8}$$

and

$$(t-s)^{\alpha(s)} e^{-H_+(s,t)} \le C\,(t-s)^{\overline{\alpha}_{\sup}+\varepsilon} \tag{3.2.9}$$

hold for all real numbers s and t satisfying $t - s \ge \nu$. Recall that the exponent $\overline{\alpha}_{\sup} \in [\alpha_{\inf}, \alpha_{\sup}]$ was defined in (3.1.17).

Notice that, one knows from (3.1.10) that the open interval $(\alpha(t_0)+1/2, \widetilde{\alpha}_\alpha(t_0))$ is non-empty. Let $\gamma \in (\alpha(t_0)+1/2, \widetilde{\alpha}_\alpha(t_0))$ be arbitrary and fixed. Then, one can derive from the definition of local Hölder exponent (see (3.1.3)) that there are two constants $k_\alpha \ge 0$ and $\delta \in (0, 1/2]$, such that one has

$$|\alpha(x)-\alpha(y)| \le k_\alpha |x-y|^\gamma\,, \quad \text{for all } x, y \in I(t_0, 2\delta) := [t_0 - 2\delta, t_0 + 2\delta]. \tag{3.2.10}$$

Remark 3.2.4. In all the sequel, one assumes that the two arbitrary and fixed positive real numbers ε and η are small enough so that they satisfy the following four conditions:

$$\varepsilon < 8^{-1} \times \min\big\{\alpha_{\inf}\,, 1/2-\alpha_{\sup}\,, \gamma - (\alpha(t_0)+1/2)\big\}, \tag{3.2.11}$$

$$\eta < \delta/2 \le 1/4\,, \tag{3.2.12}$$

$$\eta^{2\varepsilon} \le \delta/2 - \eta/2 \tag{3.2.13}$$

and

$$|\alpha(x)-\alpha(y)| \le \varepsilon\,, \quad \text{for all } x, y \in I(t_0, 2\eta). \tag{3.2.14}$$

Observe that a straightforward consequence of (3.2.10) and (3.2.12) is that

$$|\alpha(x)-\alpha(y)| \le k_\alpha |x-y|^\gamma\,, \quad \text{for all } x, y \in I(t_0, 2\eta). \tag{3.2.15}$$

Remark 3.2.5. In all the sequel, $t_0 \in \mathbb{R}$ is arbitrary and fixed; one denotes by t and $t+h$, where $h \in (0,1]$, two arbitrary real numbers belonging to the interval $I(t_0, \eta/2)$, and one sets

$$\sigma_R^2(t,h) := \mathbb{E}|R(t+h) - R(t)|^2. \tag{3.2.16}$$

It can easily be seen that for proving (3.2.2) it is enough to show that there exists a positive finite constant C_0 not depending on t and h such that one has

$$\sigma_R^2(t,h) \le C_0\, h^{2(\alpha(t_0)+1/2+\varepsilon)}\,. \tag{3.2.17}$$

Using (3.2.16), (3.1.19), (3.1.13), (3.1.14) and the isometry property of Wiener integral, one gets that

$$\begin{aligned}
\sigma_R^2(t,h) = \int_{\mathbb{R}} \Big[\int_0^{t+h} \frac{1}{\Gamma(\alpha(t+h))} (\tau - s)_+^{\alpha(t+h)-1} d\tau - \int_0^t \frac{1}{\Gamma(\alpha(t))} (\tau - s)_+^{\alpha(t)-1} d\tau \\
- \int_t^{t+h} \frac{1}{\Gamma(\alpha(\tau))} (\tau - s)_+^{\alpha(\tau)-1} e^{H_-(s,\tau)} d\tau \Big]^2 ds \\
= \int_{\mathbb{R}} \Big[\int_0^{t+h} \Big(\frac{1}{\Gamma(\alpha(t+h))} (\tau - s)_+^{\alpha(t+h)-1} - \frac{1}{\Gamma(\alpha(t))} (\tau - s)_+^{\alpha(t)-1} \Big) d\tau \\
+ \int_t^{t+h} \Big(\frac{1}{\Gamma(\alpha(t))} (\tau - s)_+^{\alpha(t)-1} - \frac{1}{\Gamma(\alpha(\tau))} (\tau - s)_+^{\alpha(\tau)-1} e^{H_-(s,\tau)} \Big) d\tau \Big]^2 ds.
\end{aligned}$$

Then, it follows from the inequality

$$(u+v)^2 \le 2u^2 + 2v^2\,, \quad \text{for all } (u,v) \in \mathbb{R}^2, \tag{3.2.18}$$

that

$$\sigma_R^2(t,h) \le 2\lambda_1(t,h) + 2\lambda_2(t,h)\,, \tag{3.2.19}$$

where

$$\lambda_1(t,h) := \int_{\mathbb{R}} \Big[\int_0^{t+h} \Big(\frac{1}{\Gamma(\alpha(t+h))} (\tau - s)_+^{\alpha(t+h)-1} d\tau - \frac{1}{\Gamma(\alpha(t))} (\tau - s)_+^{\alpha(t)-1} \Big) d\tau \Big]^2 ds \tag{3.2.20}$$

and

$$\lambda_2(t,h) := \int_{\mathbb{R}} \Big[\int_t^{t+h} \Big(\frac{1}{\Gamma(\alpha(t))} (\tau - s)_+^{\alpha(t)-1} - \frac{1}{\Gamma(\alpha(\tau))} (\tau - s)_+^{\alpha(\tau)-1} e^{H_-(s,\tau)} \Big) d\tau \Big]^2 ds. \tag{3.2.21}$$

The following lemma provides an appropriate upper bound for $\lambda_1(t,h)$.

Lemma 3.2.6. *There is a constant C_1, not depending on t and h, such that*

$$\lambda_1(t,h) \le C_1 h^{2\gamma}. \tag{3.2.22}$$

Proof. Let us set

$$\lambda_1^1(t,h) := \int_{-\infty}^{t+h} \left(\int_0^{t+h} \left(\frac{1}{\Gamma(\alpha(t+h))} - \frac{1}{\Gamma(\alpha(t))} \right) (\tau - s)_+^{\alpha(t+h)-1} d\tau \right)^2 ds \tag{3.2.23}$$

and

$$\lambda_1^2(t,h) := \int_{-\infty}^{t+h} \left(\int_0^{t+h} \frac{1}{\Gamma(\alpha(t))} \left((\tau - s)_+^{\alpha(t+h)-1} - (\tau - s)_+^{\alpha(t)-1} \right) d\tau \right)^2 ds\,. \tag{3.2.24}$$

Then one can derive from (3.2.20), (3.2.23), (3.2.24) and (3.2.18) that

$$\lambda_1(t,h) \le 2\left(\lambda_1^1(t,h) + \lambda_1^2(t,h)\right). \tag{3.2.25}$$

Let us first show that one has for some constant c_0, not depending on t and h,

$$\lambda_1^1(t,h) \le c_0 h^{2\gamma}. \tag{3.2.26}$$

Applying on the interval $[\alpha(t)\wedge\alpha(t+h), \alpha(t)\vee\alpha(t+h)] \subseteq [\alpha_{\text{inf}}, \alpha_{\text{sup}}] \subset (0,1/2)$ the mean value theorem to the infinitely differentiable positive function $x \mapsto 1/\Gamma(x)$, and using the inequality (3.2.15), one obtains that

$$\left| \frac{1}{\Gamma(\alpha(t+h))} - \frac{1}{\Gamma(\alpha(t))} \right| \le c_1 h^\gamma\,, \tag{3.2.27}$$

where the positive constant c_1 does not depend on t and h. Then combining (3.2.23) and (3.2.27) with the inequality $\alpha(t+h) \ge \alpha_{\text{inf}}$ one gets that

$$\begin{aligned}
\lambda_1^1(t,h) &\le c_1^2\, h^{2\gamma} \int_{-\infty}^{t+h} \left(\int_0^{t+h} (\tau - s)_+^{\alpha(t+h)-1} d\tau \right)^2 ds \\
&\le \frac{c_1^2 h^{2\gamma}}{\alpha(t+h)^2} \int_{-\infty}^{t+h} \left((t+h-s)^{\alpha(t+h)} - (-s)_+^{\alpha(t+h)} \right)^2 ds \\
&\le \frac{c_1^2 h^{2\gamma}}{\alpha_{\text{inf}}^2} \int_0^{+\infty} \left(s^{\alpha(t+h)} - (s-t-h)_+^{\alpha(t+h)} \right)^2 ds.
\end{aligned} \tag{3.2.28}$$

Next, one studies two cases $t+h \ge 0$ and $t+h < 0$. In the case where $t+h \ge 0$ one has

$$\begin{aligned}
&\int_0^{+\infty} \left(s^{\alpha(t+h)} - (s-t-h)_+^{\alpha(t+h)} \right)^2 ds \\
&= \int_0^{t+h} s^{2\alpha(t+h)} ds + \int_0^{+\infty} \left((s+t+h)^{\alpha(t+h)} - s^{\alpha(t+h)} \right)^2 ds \\
&= \frac{(t+h)^{2\alpha(t+h)+1}}{2\alpha(t+h)+1} + (t+h)^{2\alpha(t+h)+1} \int_0^{+\infty} \left((s+1)^{\alpha(t+h)} - s^{\alpha(t+h)} \right)^2 ds.
\end{aligned} \tag{3.2.29}$$

Next, observe that, for each fixed real number $s \ge 1$, by applying on the interval $[s, s+1]$ the mean value theorem to the infinitely differentiable function $x \mapsto x^{\alpha(t+h)}$, one obtains that

$$\left|(s+1)^{\alpha(t+h)} - s^{\alpha(t+h)}\right| \le s^{\alpha_{\text{sup}}-1}\,, \quad \text{for all } s \ge 1.$$

Therefore, one has that

$$\int_0^{+\infty}\left((s+1)^{\alpha(t+h)}-s^{\alpha(t+h)}\right)^2 ds \le \int_0^1 ds + \int_1^{+\infty} s^{2\alpha_{\sup}-2}\,ds = c_2. \tag{3.2.30}$$

Next, one denotes by c_0 the constant defined as:

$$c_0 := \frac{c_1^2}{\alpha_{\inf}^2}\Big((|t_0|+\eta)^{2\alpha_{\inf}+1}+(|t_0|+\eta)^{2\alpha_{\sup}+1}\Big)\left(\frac{1}{2\alpha_{\inf}+1}+c_2\right).$$

Then, using (3.2.28), (3.2.29), (3.2.30) and the fact $t+h\in I(t_0,\eta/2):=[t_0-\eta/2,t_0+\eta/2]$, it follows that (3.2.26) is satisfied. Let us now turn to the case where $t+h<0$. In this case one has

$$\begin{aligned}&\int_0^{+\infty}\left(s^{\alpha(t+h)}-(s-t-h)_+^{\alpha(t+h)}\right)^2 ds\\&=(-(t+h))^{2\alpha(t+h)+1}\int_0^{+\infty}\left((s+1)^{\alpha(t+h)}-s^{\alpha(t+h)}\right)^2 ds\\&\le c_2\Big((|t_0|+\eta)^{2\alpha_{\inf}+1}+(|t_0|+\eta)^{2\alpha_{\sup}+1}\Big).\end{aligned} \tag{3.2.31}$$

Thus combining (3.2.28) and (3.2.31) it turns out that (3.2.26) is satisfied in this case as well.

Let us now prove that one has for some constant c_3, not depending on t and h,

$$\lambda_1^2(t,h)\le c_3 h^{2\gamma}. \tag{3.2.32}$$

Let c_4 be a positive constant, only depending on ε, such that one has

$$\big|\log(x)\big|\le c_4\big(x\vee x^{-1}\big)^{\varepsilon}, \quad \text{for all } x\in(0,+\infty). \tag{3.2.33}$$

Moreover, let $\Gamma_{\inf}$ be the positive constant defined as

$$\Gamma_{\inf} := \inf_{z\in(0,+\infty)}\Gamma(z)>0. \tag{3.2.34}$$

One mentions in passing that $\Gamma_{\inf}$ is larger than 1/2. Using (3.2.24), the mean value theorem, (3.2.33), (3.2.15) and (3.2.11) one obtains that

$$\begin{aligned}\lambda_1^2(t,h)&\le\frac{c_4^2\,|\alpha(t+h)-\alpha(t)|^2}{\Gamma_{\inf}^2}\int_{-\infty}^{t+h}\left(\int_0^{t+h}\Big((\tau-s)_+^{\alpha_{\inf}-1-\varepsilon}+(\tau-s)_+^{\alpha_{\sup}-1+\varepsilon}\Big)d\tau\right)^2 ds\\&\le\left[\frac{c_4^2\,k_\alpha^2}{(\alpha_{\inf}-\varepsilon)^2\Gamma_{\inf}^2}\int_0^{+\infty}\left(s^{\alpha_{\inf}-\varepsilon}-(s-t-h)_+^{\alpha_{\inf}-\varepsilon}+s^{\alpha_{\sup}+\varepsilon}-(s-t-h)_+^{\alpha_{\sup}+\varepsilon}\right)^2 ds\right]h^{2\gamma}.\end{aligned} \tag{3.2.35}$$

In the case where $t+h \geq 0$, one has

$$\begin{aligned}
&\int_0^{+\infty} \Big(s^{\alpha_{\inf}-\varepsilon} - (s-t-h)_+^{\alpha_{\inf}-\varepsilon} + s^{\alpha_{\sup}+\varepsilon} - (s-t-h)_+^{\alpha_{\sup}+\varepsilon} \Big)^2 ds \\
&= \int_0^{t+h} \Big(s^{\alpha_{\inf}-\varepsilon} + s^{\alpha_{\sup}+\varepsilon} \Big)^2 ds \\
&\qquad + \int_0^{+\infty} \Big((s+t+h)^{\alpha_{\inf}-\varepsilon} - s^{\alpha_{\inf}-\varepsilon} + (s+t+h)^{\alpha_{\sup}+\varepsilon} - s^{\alpha_{\sup}+\varepsilon} \Big)^2 ds \\
&\leq 2 \int_0^{|t_0|+\eta} \Big(s^{2(\alpha_{\inf}-\varepsilon)} + s^{2(\alpha_{\sup}+\varepsilon)} \Big) ds + 2 \int_0^{+\infty} \Big((s+t+h)^{\alpha_{\inf}-\varepsilon} - s^{\alpha_{\inf}-\varepsilon} \Big)^2 ds \\
&\qquad + 2 \int_0^{+\infty} \Big((s+t+h)^{\alpha_{\sup}+\varepsilon} - s^{\alpha_{\sup}+\varepsilon} \Big)^2 ds \\
&\leq 4\big(1+|t_0|+\eta\big)^2 + 2(t+h)^{2(\alpha_{\inf}-\varepsilon)+1} \int_0^{+\infty} \Big((s+1)^{\alpha_{\inf}-\varepsilon} - s^{\alpha_{\inf}-\varepsilon} \Big)^2 ds \\
&\qquad + 2(t+h)^{2(\alpha_{\sup}+\varepsilon)+1} \int_0^{+\infty} \Big((s+1)^{\alpha_{\sup}+\varepsilon} - s^{\alpha_{\sup}+\varepsilon} \Big)^2 ds \\
&\leq c_5 \,,
\end{aligned} \tag{3.2.36}$$

where c_5 is the finite constant, not depending on t and h, defined as:

$$\begin{aligned}
c_5 := 4\big(1+|t_0|+\eta\big)^2 \bigg(1 + &\int_0^{+\infty} \Big((s+1)^{\alpha_{\inf}-\varepsilon} - s^{\alpha_{\inf}-\varepsilon} \Big)^2 ds \\
&+ \int_0^{+\infty} \Big((s+1)^{\alpha_{\sup}+\varepsilon} - s^{\alpha_{\sup}+\varepsilon} \Big)^2 ds \bigg) .
\end{aligned}$$

In the case where $t+h < 0$, one has

$$\begin{aligned}
&\int_0^{+\infty} \Big(s^{\alpha_{\inf}-\varepsilon} - (s-t-h)_+^{\alpha_{\inf}-\varepsilon} + s^{\alpha_{\sup}+\varepsilon} - (s-t-h)_+^{\alpha_{\sup}-\varepsilon} \Big)^2 ds \\
&\leq 2(-(t+h))^{2(\alpha_{\inf}-\varepsilon)+1} \int_0^{+\infty} \Big((s+1)^{\alpha_{\inf}-\varepsilon} - s^{\alpha_{\inf}-\varepsilon} \Big)^2 ds \\
&\qquad + 2(-(t+h))^{2(\alpha_{\sup}+\varepsilon)+1} \int_0^{+\infty} \Big((s+1)^{\alpha_{\sup}+\varepsilon} - s^{\alpha_{\sup}+\varepsilon} \Big)^2 ds \\
&\leq c_5 \,.
\end{aligned} \tag{3.2.37}$$

Thus, (3.2.35), (3.2.36) and (3.2.37) entail that (3.2.32) holds.

Finally, combining (3.2.26) and (3.2.32) with (3.2.25), one gets (3.2.22). □

Let us now focus on $\lambda_2(t,h)$ defined in (3.2.21). Using the inequality

$$(u+v+w)^2 \le 3(u^2+v^2+w^2), \quad \text{for all } (u,v,w)\in\mathbb{R}^3, \tag{3.2.38}$$

and (3.2.34) one has that

$$\lambda_2(t,h) \le 3\lambda_2^1(t,h) + 3\,\Gamma_{\inf}^{-2}\,\lambda_2^2(t,h) + 3\,\Gamma_{\inf}^{-2}\,\lambda_2^3(t,h), \tag{3.2.39}$$

where

$$\lambda_2^1(t,h) := \int_{\mathbb{R}} \left(\int_t^{t+h} \left(\frac{1}{\Gamma(\alpha(t))} - \frac{1}{\Gamma(\alpha(\tau))} \right) (\tau-s)_+^{\alpha(t)-1} d\tau \right)^2 ds\,, \tag{3.2.40}$$

$$\lambda_2^2(t,h) := \int_{\mathbb{R}} \left(\int_t^{t+h} \left| (\tau-s)_+^{\alpha(t)-1} - (\tau-s)_+^{\alpha(\tau)-1} \right| e^{H_-(s,\tau)} d\tau \right)^2 ds \tag{3.2.41}$$

and

$$\lambda_2^3(t,h) := \int_{\mathbb{R}} \left(\int_t^{t+h} (\tau-s)_+^{\alpha(t)-1} \left| e^{H_-(s,\tau)} - 1 \right| d\tau \right)^2 ds\,. \tag{3.2.42}$$

In view of (3.2.39), our next goal is to obtain three lemmas which will allow us to conveniently bound from above $\lambda_2^1(t,h)$, $\lambda_2^2(t,h)$ and $\lambda_2^3(t,h)$.

Lemma 3.2.7. *There is a constant C_2^1, not depending on t and h, such that*

$$\lambda_2^1(t,h) \le C_2^1 h^{2(\gamma+\alpha_{\inf})+1}.$$

Proof. Similarly to (3.2.27), it can be shown that there is a constant c_1, not depending on t, h and τ, such that, for all $\tau\in[t,t+h]$, one has

$$\left| \frac{1}{\Gamma(\alpha(t))} - \frac{1}{\Gamma(\alpha(\tau))} \right| \le c_1 |t-\tau|^\gamma \le c_1 h^\gamma\,. \tag{3.2.43}$$

Next combining (3.2.40) and (3.2.43) one gets that

$$\lambda_2^1(t,h) \le \left[c_1^2 \int_{\mathbb{R}} \left(\int_t^{t+h} (\tau-s)_+^{\alpha(t)-1} d\tau \right)^2 ds \right] h^{2\gamma}. \tag{3.2.44}$$

Moreover, one has

$$\begin{aligned}
\int_{\mathbb{R}} \left(\int_t^{t+h} (\tau-s)_+^{\alpha(t)-1} d\tau \right)^2 ds &= \frac{1}{\alpha(t)^2} \left(\int_0^{+\infty} \left((s+h)^{\alpha(t)} - s^{\alpha(t)} \right)^2 ds + \int_0^h s^{2\alpha(t)} ds \right) \\
&\le \frac{h^{2\alpha_{\inf}+1}}{\alpha_{\inf}^2} \left(\int_0^{+\infty} \left((s+1)^{\alpha(t)} - s^{\alpha(t)} \right)^2 ds + \frac{1}{2\alpha_{\inf}+1} \right) \\
&\le \frac{h^{2\alpha_{\inf}+1}}{\alpha_{\inf}^2} \left(c_2 + \frac{1}{2\alpha_{\inf}+1} \right),
\end{aligned} \tag{3.2.45}$$

where the constant c_2, which does not depend on t and h, is the same constant as in (3.2.30). Finally combining (3.2.44) and (3.2.45) one obtains the lemma. □

Lemma 3.2.8. *There is a constant C_2^2, not depending on t and h, such that*

$$\lambda_2^2(t,h) \le C_2^2 h^{2\gamma}.$$

Proof. First observe that using the mean value theorem, (3.2.14), (3.2.33) and (3.2.15) , one has for all $\tau \in [t, t+h]$ and $s < \tau$

$$\left|(\tau - s)^{\alpha(t)-1} - (\tau - s)^{\alpha(\tau)-1}\right| \le c_1 h^\gamma \left((\tau - s)^{\alpha(t_0)-1-2\varepsilon} + (\tau - s)^{\alpha(t_0)-1+2\varepsilon}\right), \tag{3.2.46}$$

where the constant c_1 does not depend on t and h. Next, one sets

$$K_-(t_0) := \sup\left\{|H_-(a,b)|\,,\ \ (a,b) \in I(t_0,\delta) \text{ and } a < b\right\}. \tag{3.2.47}$$

Observe that $K_-(t_0)$ is a finite constant. Indeed, one can derive from (3.1.16) and (3.2.10) that, for all $(a,b) \in I(t_0,\delta)$ satisfying $a < b$, one has

$$|H_-(a,b)| \le \int_a^b \frac{|\alpha(u) - \alpha(b)|}{b-u} du \le k_\alpha \int_a^b \frac{du}{(b-u)^{1-\gamma}} = k_\alpha \int_0^{2\delta} \frac{dv}{v^{1-\gamma}} < +\infty.$$

It results from (3.2.41), (3.2.46), (3.2.47), the inequalities $(\tau - s)_+ \le 4\eta \le 1$, for all $(\tau, s) \in I(t_0, 2\eta)^2 \subset I(t_0,\delta)^2$ (see (3.2.12)), and the inequality

$$\tau - s \ge \eta/2\,, \quad \text{for all } (\tau,s) \in \mathbb{R}^2 \text{ s.t. } \tau \in [t,t+h] \subseteq I(t_0,\eta/2) \text{ and } s \le t+h-3\eta/2\,, \tag{3.2.48}$$

that

$$\begin{aligned}
\lambda_2^2(t,h) &\le c_1^2\, h^{2\gamma} \int_{-\infty}^{t+h} \left(\int_t^{t+h} \left((\tau - s)_+^{\alpha(t_0)-1-2\varepsilon} + (\tau - s)_+^{\alpha(t_0)-1+2\varepsilon} \right) e^{H_-(s,\tau)} d\tau \right)^2 ds \\
&\le c_1^2 \Bigg[4e^{2K_-(t_0)} \int_{t+h-3\eta/2}^{t+h} \left(\int_t^{t+h} (\tau - s)_+^{\alpha(t_0)-1-2\varepsilon} d\tau \right)^2 ds \\
&\qquad + \left(1 + (2/\eta)^{4\varepsilon}\right)^2 \int_{-\infty}^{t+h-3\eta/2} \left(\int_t^{t+h} (\tau - s)^{\alpha(t_0)-1+2\varepsilon} e^{H_-(s,\tau)} d\tau \right)^2 ds \Bigg] h^{2\gamma}\,.
\end{aligned} \tag{3.2.49}$$

Let us now prove that each one of the two integrals in the right-hand side of the last inequality can be bounded from above by a finite constant not depending on t and h. In view of (3.2.11) one has

$$\begin{aligned}
&\int_{t+h-3\eta/2}^{t+h} \left(\int_t^{t+h} (\tau - s)_+^{\alpha(t_0)-1-2\varepsilon} d\tau \right)^2 ds \\
&\le \frac{1}{(\alpha_{\inf} - 2\varepsilon)^2} \int_{t+h-3\eta/2}^{t+h} \left((t+h-s)^{\alpha(t_0)-2\varepsilon} - (t-s)_+^{\alpha(t_0)-2\varepsilon} \right)^2 ds \\
&\le \frac{1}{(\alpha_{\inf} - 2\varepsilon)^2} \int_{t+h-3\eta/2}^{t+h} (t+h-s)^{2\alpha(t_0)-4\varepsilon} ds \\
&= \frac{1}{(\alpha_{\inf} - 2\varepsilon)^2} \int_0^{3\eta/2} z^{2\alpha(t_0)-4\varepsilon} dz := c_2\,.
\end{aligned} \tag{3.2.50}$$

On the other hand, one can derive from (3.2.48), Lemma 3.2.3 (with $\nu = \eta/2$) and (3.2.14) that

$$\begin{aligned}
&\int_{-\infty}^{t+h-3\eta/2}\left(\int_t^{t+h}(\tau-s)^{\alpha(t_0)-1+2\varepsilon}e^{H_-(s,\tau)}d\tau\right)^2 ds\\
&=\int_{-\infty}^{t+h-3\eta/2}\left(\int_t^{t+h}(\tau-s)^{\alpha(t_0)-\alpha(\tau)}(\tau-s)^{\alpha(\tau)-1+2\varepsilon}e^{H_-(s,\tau)}d\tau\right)^2 ds\\
&\le C^2\int_{-\infty}^{t+h-3\eta/2}\left(\int_t^{t+h}\Big((\tau-s)^{\overline{\alpha}_{\sup}-1+4\varepsilon}+(\tau-s)^{\overline{\alpha}_{\sup}-1+2\varepsilon}\Big)d\tau\right)^2 ds\le c_3\,,
\end{aligned}\tag{3.2.51}$$

where C is the same finite constant as in (3.2.8), and c_3 is the finite constant not depending on t and h defined as:

$$c_3 := \left(\frac{C}{\overline{\alpha}_{\sup}}\right)^2\int_0^{+\infty}\Big((s+1)^{\overline{\alpha}_{\sup}+4\varepsilon}-s^{\overline{\alpha}_{\sup}+4\varepsilon}+(s+1)^{\overline{\alpha}_{\sup}+2\varepsilon}-s^{\overline{\alpha}_{\sup}+2\varepsilon}\Big)^2 ds.$$

Finally, putting together (3.2.49), (3.2.50) and (3.2.51) one obtains the lemma. □

Lemma 3.2.9. *There is a constant C_2^3, not depending on t and h, such that*

$$\lambda_2^3(t,h)\le C_2^3 h^{2(\alpha(t_0)+1/2+\varepsilon)}.$$

Proof. One can derive from (3.2.42) that

$$\lambda_2^3(t,h)=\mu_1(t,h)+\mu_2(t,h)+\mu_3(t,h),\tag{3.2.52}$$

where

$$\mu_1(t,h):=\int_{-\infty}^{t-\delta/2}\left(\int_t^{t+h}(\tau-s)^{\alpha(t)-1}\left|e^{H_-(s,\tau)}-1\right|d\tau\right)^2 ds\,,\tag{3.2.53}$$

$$\mu_2(t,h):=\int_{t-\delta/2}^{t-h^{2\varepsilon}}\left(\int_t^{t+h}(\tau-s)^{\alpha(t)-1}\left|e^{H_-(s,\tau)}-1\right|d\tau\right)^2 ds\,,\tag{3.2.54}$$

and

$$\mu_3(t,h):=\int_{t-h^{2\varepsilon}}^{t+h}\left(\int_t^{t+h}(\tau-s)_+^{\alpha(t)-1}\left|e^{H_-(s,\tau)}-1\right|d\tau\right)^2 ds\,.\tag{3.2.55}$$

Observe that one knows from (3.2.13) and the inequalities $0<h\le\eta$, that $t-h^{2\varepsilon}>t-\delta/2$. Using (3.2.53), (3.2.18), Lemma 3.2.3 with $\nu=\delta/2$, (3.2.14), the mean value theorem and

(3.2.11), one obtains that

$$
\begin{aligned}
\mu_1(t,h) &\le 2\int_{-\infty}^{t-\delta/2}\left(\int_t^{t+h}(\tau-s)^{\alpha(t)-1}e^{H_-(s,\tau)}d\tau\right)^2 ds + 2\int_{-\infty}^{t-\delta/2}\left(\int_t^{t+h}(\tau-s)^{\alpha(t)-1}d\tau\right)^2 ds\\
&\le 2C^2(2/\delta)^{4\varepsilon}\int_{-\infty}^{t-\delta/2}\left(\int_t^{t+h}(\tau-s)^{\overline{\alpha}_{\sup}+2\varepsilon-1}d\tau\right)^2 ds + 2\int_{-\infty}^{t-\delta/2}\left(\int_t^{t+h}(\tau-s)^{\alpha(t)-1}d\tau\right)^2 ds\\
&\le \frac{2C^2(2/\delta)^{4\varepsilon}}{(\overline{\alpha}_{\sup}+2\varepsilon)^2}\int_{\delta/2}^{+\infty}\left((s+h)^{\overline{\alpha}_{\sup}+2\varepsilon}-s^{\overline{\alpha}_{\sup}+2\varepsilon}\right)^2 ds + \frac{2}{\alpha(t)^2}\int_{\delta/2}^{+\infty}\left((s+h)^{\alpha(t)}-s^{\alpha(t)}\right)^2 ds\\
&\le \left[2C^2(2/\delta)^{4\varepsilon}\int_{\delta/2}^{+\infty}s^{2\overline{\alpha}_{\sup}+4\varepsilon-2}ds + 2\int_{\delta/2}^{+\infty}s^{2\alpha_{\sup}-2}ds + 2\int_{\delta/2}^{+\infty}s^{2\alpha_{\inf}-2}ds\right]h^2.
\end{aligned}
\tag{3.2.56}
$$

Next, observe that it follows from (3.2.12), (3.2.13) and the inclusion $[t,t+h]\subseteq I(t_0,\eta/2)$ that $[t-\delta/2, t-h^{2\varepsilon}]\subset I(t_0,\delta)$. Thus, using (3.2.54), (3.2.47), the mean value theorem and (3.2.11), one gets that

$$
\begin{aligned}
\mu_2(t,h) &\le (e^{K_-(t_0)}+1)^2\int_{t-\delta/2}^{t-h^{2\varepsilon}}\left(\int_t^{t+h}(\tau-s)^{\alpha(t)-1}d\tau\right)^2 ds\\
&\le \frac{(e^{K_-(t_0)}+1)^2}{\alpha(t)^2}\int_{h^{2\varepsilon}}^{\delta/2}\left((s+h)^{\alpha(t)}-s^{\alpha(t)}\right)^2 ds \le \left(e^{K_-(t_0)}+1\right)^2 h^2\int_{h^{2\varepsilon}}^{\delta/2}s^{2\alpha(t)-2}ds\\
&\le \frac{\left(e^{K_-(t_0)}+1\right)^2}{1-2\alpha_{\sup}}h^{2-2\varepsilon} \le \frac{\left(e^{K_-(t_0)}+1\right)^2}{1-2\alpha_{\sup}}h^{2(\alpha(t_0)+1/2+\varepsilon)}.
\end{aligned}
\tag{3.2.57}
$$

In order to bound from above $\mu_3(t,h)$, one denotes by $E_-(t_0)$ the finite constant defined as:

$$
E_-(t_0) := \sup\left\{\left|\frac{e^x-1}{x}\right|,\ x\in\mathbb{R} \text{ and } 0<|x|\le K_-(t_0)\right\}. \tag{3.2.58}
$$

Observe that the inclusion $[t-h^{2\varepsilon}, t+h]\subset I(t_0,\delta)$, (3.2.47), (3.2.58), (3.1.16) and (3.2.10) entail that, for all $(s,\tau)\in[t-h^{2\varepsilon},t+h]\times[t,t+h]$ satisfying $s<\tau$, one has

$$
\left|e^{H_-(s,\tau)}-1\right| \le E_-(t_0)\,|H_-(s,\tau)| \le E_-(t_0)\,k_\alpha\int_s^\tau(\tau-u)^{\gamma-1}du = \frac{E_-(t_0)\,k_\alpha}{\gamma}(\tau-s)^\gamma\,. \tag{3.2.59}
$$

Thus, one can derive from (3.2.55) and (3.2.59) that

$$
\mu_3(t,h) \le \left(\frac{E_-(t_0)\,k_\alpha}{\gamma(\gamma+\alpha_{\inf})}\right)^2\int_{t-h^{2\varepsilon}}^{t+h}\left((t+h-s)^{\gamma+\alpha(t)}-(t-s)_+^{\gamma+\alpha(t)}\right)^2 ds\,. \tag{3.2.60}
$$

Moreover, standard computations, the mean value theorem, the inequality $\gamma>1/2$ and the inequalities $0 < \alpha_{\inf} \le \alpha_{\sup}$ allow to show that

$$\begin{aligned}
&\int_{t-h^{2\varepsilon}}^{t+h} \Big((t+h-s)^{\gamma+\alpha(t)} - (t-s)_+^{\gamma+\alpha(t)} \Big)^2 ds \\
&\le \int_0^{h^{2\varepsilon}} \Big((h+s)^{\gamma+\alpha(t)} - s^{\gamma+\alpha(t)} \Big)^2 ds + \int_0^h s^{2(\gamma+\alpha(t))} ds \\
&\le h^2(\gamma+\alpha(t))^2 \int_0^{h^{2\varepsilon}} \Big((s+1)^{2(\gamma+\alpha(t)-1)} + s^{-2(1-\gamma-\alpha(t))} \Big) ds + h^{2(\gamma+\alpha(t))+1} \\
&\le h^2(\gamma+\alpha_{\sup}+1)^2 \bigg[\int_0^1 \Big((s+1)^{2(\gamma+\alpha_{\sup}-1)} + s^{-2(1-\gamma-\alpha_{\inf})} \Big) ds + 1 \bigg]. \qquad (3.2.61)
\end{aligned}$$

Then (3.2.60) and (3.2.61) entail that, for some constant c_1 not depending on t and h, one has

$$\mu_3(t,h) \le c_1 h^2 \,. \qquad (3.2.62)$$

Finally, putting together (3.2.52), (3.2.56), (3.2.57), (3.2.62) and (3.2.11) one obtains the lemma. □

We are now in a position to prove the inequality (3.2.17).

Remark 3.2.10. Putting together (3.2.39) and (3.2.11) with Lemmas 3.2.7, 3.2.8 and 3.2.9 one obtains that

$$\lambda_2(t,h) \le C_2\, h^{2(\alpha(t_0)+1/2+\varepsilon)} \,, \qquad (3.2.63)$$

where the constant $C_2 := 3(C_2^1 + \Gamma_{\inf}^{-2} C_2^2 + \Gamma_{\inf}^{-2} C_2^3)$. Thus, it results from (3.2.19), Lemma 3.2.6, (3.2.63) and (3.2.11) that

$$\sigma_R^2(t,h) \le 2(C_1+C_2) h^{2(\alpha(t_0)+1/2+\varepsilon)} \,,$$

which shows that (3.2.17) is satisfied.

3.3 Proof of Theorem 3.1.8

Remark 3.3.1. By arguing as in Remark 3.2.2 it turns out that for proving Theorem 3.1.8 it is enough to show that, for all fixed $t_0 \in \mathbb{R}$, there are 3 constants $\varepsilon_{t_0} > 0$, $\eta_{t_0} > 0$ and $c_{t_0} \ge 0$, which may depend on t_0, such that

$$\mathbb{E}|D(t') - D(t'')|^2 \le c_{t_0}\, |t'-t''|^{2(\alpha(t_0)+1/2+3\varepsilon_{t_0})} \,, \quad \text{for all } t', t'' \in I(t_0, \eta_{t_0}/2). \qquad (3.3.1)$$

Remark 3.3.2. In all the sequel, one assumes that $t_0 \in \mathbb{R}$ is arbitrary and fixed and that $\delta \in (0, 1/2]$ is the same as in (3.2.10). Also, one assumes that the two arbitrary and fixed positive real numbers ε and η are small enough so that they satisfy the 3 conditions (3.2.11), (3.2.12) and (3.2.14).

Remark 3.3.3. In all the sequel one denotes by t and $t+h$, where $h \in (0,1]$, two arbitrary real numbers belonging to the interval $I(t_0, \eta/2)$, and one sets

$$\sigma_D^2(t,h) := \mathbb{E}|D(t+h) - D(t)|^2. \tag{3.3.2}$$

It can easily be seen that for proving (3.3.1) it is enough to show that there exists a positive finite constant $\widetilde{C}_0$ not depending on t and h such that one has

$$\sigma_D^2(t,h) \le \widetilde{C}_0\, h^{2(\alpha(t_0)+1/2+\varepsilon)}\,. \tag{3.3.3}$$

Using (3.3.2), (3.1.20), (3.1.15), (3.1.12), the isometry property of Wiener integral, and (3.2.18) one gets that

$$\sigma_D^2(t,h) \le 2\Lambda_0(t,h) + 2\Lambda_1(t,h) + \Lambda_2(t,h) + \Lambda_3(t,h)\,, \tag{3.3.4}$$

where

$$\Lambda_0(t,h) := \int_{-\infty}^{t-\eta} \Bigg(\frac{1}{\Gamma(1+\alpha(t+h))}\Big((t+h-s)^{\alpha(t+h)} - (-s)_+^{\alpha(t+h)}\Big) \tag{3.3.5}$$

$$- \frac{1}{\Gamma(1+\alpha(t))}\Big((t-s)^{\alpha(t)} - (-s)_+^{\alpha(t)}\Big)\Bigg)^2 ds\,,$$

$$\Lambda_1(t,h) := \int_{-\infty}^{t-\eta} \left(\frac{1}{\Gamma(1+\alpha(s))}(t+h-s)^{\alpha(s)} e^{-H_+(s,t+h)} - \frac{1}{\Gamma(1+\alpha(s))}(t-s)^{\alpha(s)} e^{-H_+(s,t)} \right)^2 ds\,, \tag{3.3.6}$$

$$\Lambda_2(t,h) := \int_{t-\eta}^{t} \Bigg(\frac{1}{\Gamma(1+\alpha(s))}(t+h-s)^{\alpha(s)} e^{-H_+(s,t+h)} - \frac{1}{\Gamma(1+\alpha(s))}(t-s)^{\alpha(s)} e^{-H_+(s,t)} \tag{3.3.7}$$

$$- \frac{1}{\Gamma(1+\alpha(t+h))}(t+h-s)^{\alpha(t+h)} + \frac{1}{\Gamma(1+\alpha(t))}(t-s)^{\alpha(t)} \Bigg)^2 ds$$

and

$$\Lambda_3(t,h) := \int_{t}^{t+h} \left(\frac{1}{\Gamma(1+\alpha(s))}(t+h-s)^{\alpha(s)} e^{-H_+(s,t+h)} - \frac{1}{\Gamma(1+\alpha(t+h))}(t+h-s)^{\alpha(t+h)} \right)^2 ds. \tag{3.3.8}$$

The following lemma provides an appropriate upper bound for $\Lambda_0(t,h)$.

Lemma 3.3.4. *There is a constant C_0, not depending on t and h, such that*

$$\Lambda_0(t,h) \le C_0 h^{2\gamma}. \tag{3.3.9}$$

Proof. It follows from (3.3.5), (3.2.20), (3.2.18), (3.2.34) and the equality $\Gamma(x+1) = x\Gamma(x)$, for all $x \in (0,+\infty)$, that

$$\Lambda_0(t,h) \le 2\lambda_1(t,h) + 2\Gamma_{\inf}^{-2}\,\widetilde{\Lambda}_0(t,h)\,, \tag{3.3.10}$$

where

$$\widetilde{\Lambda}_0(t,h) := \int_{-\infty}^{t-\eta} \Big((t+h-s)^{\alpha(t)} - (t-s)^{\alpha(t)} \Big)^2 ds\,.$$

Moreover, using the mean value theorem, one has that

$$\widetilde{\Lambda}_0(t,h) \le c\, h^2\,, \tag{3.3.11}$$

where the finite constant $c := \int_\eta^{+\infty} \big(s^{2(\alpha_{\sup}-1)} + s^{2(\alpha_{\inf}-1)} \big) ds$. Finally, putting together (3.3.10), (3.2.22) and (3.3.11), one gets (3.3.9). □

The following lemma provides an appropriate upper bound for $\Lambda_1(t,h)$.

Lemma 3.3.5. *There is a constant C_1, not depending on t and h, such that*

$$\Lambda_1(t,h) \le C_1 h^2. \tag{3.3.12}$$

Proof. One can derive from (3.3.6), (3.2.34) and (3.2.18) that

$$\Lambda_1(t,h) \le 2\,\Gamma_{\inf}^{-2} \Big(\Lambda_1^1(t,h) + \Lambda_1^2(t,h) \Big)\,, \tag{3.3.13}$$

where

$$\Lambda_1^1(t,h) = \int_{-t+\eta}^{+\infty} (t+h+s)^{2\alpha(-s)} \Big(e^{-H_+(-s,t+h)} - e^{-H_+(-s,t)} \Big)^2 ds \tag{3.3.14}$$

and

$$\Lambda_1^2(t,h) = \int_{-t+\eta}^{+\infty} e^{-2H_+(-s,t)} \Big((t+h+s)^{\alpha(-s)} - (t+s)^{\alpha(-s)} \Big)^2 ds\,. \tag{3.3.15}$$

Let us first focus on $\Lambda_1^1(t,h)$. It can easily be seen that

$$e^{-H_+(-s,t+h)} - e^{-H_+(-s,t)} = e^{-H_+(-s,t+h)} \Big(1 - e^{H_+(-s,t+h) - H_+(-s,t)} \Big) \tag{3.3.16}$$

and that

$$\sup_{|x|\le M_0} \left| \frac{1-e^x}{x} \right| < +\infty\,, \quad \text{for each fixed } M_0 \in (0,+\infty). \tag{3.3.17}$$

Moreover, in view of (3.1.16) and the fact that $\alpha(\cdot)$ is with values in $[\alpha_{\inf}, \alpha_{\sup}] \subset (0,1/2)$ one has, for all real number $s \ge -t+\eta$, that

$$|H_+(-s,t+h) - H_+(-s,t)| = \left| \int_0^h \frac{\alpha(-s) - \alpha(v+t)}{v+t+s} dv \right| \le \frac{h}{2(s+t)}. \tag{3.3.18}$$

Then, putting together (3.3.14), (3.3.16), (3.3.17) and (3.3.18), one obtains, for some finite constant c_1 not depending on t and h, that

$$\begin{aligned} \Lambda_1^1(t,h) &\le \int_{-t+\eta}^{+\infty} (t+h+s)^{2\alpha(-s)} \Big(e^{-H_+(-s,t+h)} - e^{-H_+(-s,t)} \Big)^2 ds \\ &\le c_1 h^2 \int_{-t+\eta}^{+\infty} \frac{(t+h+s)^{2\alpha(-s)} e^{-2H_+(-s,t+h)}}{(s+t)^2} ds. \end{aligned}$$

Thus, using (3.2.9), in which t and s are replaced by $t+h$ and $-s$, and the fact that $h \in (0,1]$, one gets that

$$\Lambda_1^1(t,h) \leq \left[c_1 C^2 \int_\eta^{+\infty} \frac{(s+1)^{2\overline{\alpha}_{\sup}+2\varepsilon}}{s^2} ds \right] h^2 \,. \tag{3.3.19}$$

Notice that (3.2.11) and the inequality $\overline{\alpha}_{\sup} \leq \alpha_{\sup}$ imply that the integral in the right-hand side of (3.3.19) is finite.

On the other hand, it follows from (3.3.15), the mean value theorem, and (3.2.9) that

$$\Lambda_1^2(t,h) \leq h^2 \int_{-t+\eta}^{+\infty} e^{-2H_+(-s,t)} \, (t+s)^{2\alpha(-s)-2} ds \leq \left[C^2 \int_\eta^{+\infty} s^{2\overline{\alpha}_{\sup}+2\varepsilon-2} ds \right] h^2. \tag{3.3.20}$$

Finally, putting together (3.3.19) and (3.3.20) with (3.3.13) one gets (3.3.12). □

In order to derive an appropriate upper bound for $\Lambda_2(t,h)$, defined in (3.3.7), let us express it as:

$$\Lambda_2(t,h) = \Lambda_2^1(t,h) + \Lambda_2^2(t,h) \,, \tag{3.3.21}$$

where

$$\begin{aligned} \Lambda_2^1(t,h) := \int_{t-\eta}^{t-\eta h^{1/2}} \bigg(& \frac{1}{\Gamma(1+\alpha(s))} (t+h-s)^{\alpha(s)} e^{-H_+(s,t+h)} - \frac{1}{\Gamma(1+\alpha(s))} (t-s)^{\alpha(s)} e^{-H_+(s,t)} \\ & - \frac{1}{\Gamma(1+\alpha(t+h))} (t+h-s)^{\alpha(t+h)} + \frac{1}{\Gamma(1+\alpha(t))} (t-s)^{\alpha(t)} \bigg)^2 ds \end{aligned} \tag{3.3.22}$$

and

$$\begin{aligned} \Lambda_2^2(t,h) := \int_{t-\eta h^{1/2}}^{t} \bigg(& \frac{1}{\Gamma(1+\alpha(s))} (t+h-s)^{\alpha(s)} e^{-H_+(s,t+h)} - \frac{1}{\Gamma(1+\alpha(s))} (t-s)^{\alpha(s)} e^{-H_+(s,t)} \\ & - \frac{1}{\Gamma(1+\alpha(t+h))} (t+h-s)^{\alpha(t+h)} + \frac{1}{\Gamma(1+\alpha(t))} (t-s)^{\alpha(t)} \bigg)^2 ds. \end{aligned} \tag{3.3.23}$$

The following lemma provides an appropriate upper bound for $\Lambda_2^1(t,h)$ defined in (3.3.22).

Lemma 3.3.6. *There is a constant C_2^1, not depending on t and h, such that*

$$\Lambda_2^1(t,h) \leq C_2^1 \big(h^{2\gamma} + h^{2(\alpha(t_0)+1/2)+7\varepsilon} \big) \,. \tag{3.3.24}$$

Proof. It follows from (3.3.22), (3.2.38) and (3.2.34) that

$$\Lambda_2^1(t,h) \leq 3 \Big(\Gamma_{\inf}^{-2} \, \Lambda_2^{1,1}(t,h) + \Gamma_{\inf}^{-2} \, \Lambda_2^{1,2}(t,h) + \Lambda_2^{1,3}(t,h) \Big) \,, \tag{3.3.25}$$

where

$$\Lambda_2^{1,1}(t,h) := \int_{t-\eta}^{t-\eta h^{1/2}} (t+h-s)^{2\alpha(s)} \left(e^{-H_+(s,t+h)} - e^{-H_+(s,t)}\right)^2 ds\,, \tag{3.3.26}$$

$$\Lambda_2^{1,2}(t,h) := \int_{t-\eta}^{t-\eta h^{1/2}} \left((t+h-s)^{\alpha(s)} - (t-s)^{\alpha(s)}\right)^2 e^{-2H_+(s,t)} ds \tag{3.3.27}$$

and

$$\Lambda_2^{1,3}(t,h) := \int_{t-\eta}^{t-\eta h^{1/2}} \left(\frac{1}{\Gamma(1+\alpha(t+h))}(t+h-s)^{\alpha(t+h)} - \frac{1}{\Gamma(1+\alpha(t))}(t-s)^{\alpha(t)}\right)^2 ds. \tag{3.3.28}$$

Let us first focus on $\Lambda_2^{1,1}(t,h)$. One clearly has that

$$\left(e^{-H_+(s,t+h)} - e^{-H_+(s,t)}\right)^2 = e^{-2H_+(s,t+h)} \left(1 - e^{H_+(s,t+h)-H_+(s,t)}\right)^2. \tag{3.3.29}$$

Moreover, in view of the inclusions $[t,t+h] \subset I(t_0,\eta/2)$ and $[t-\eta, t-\eta h^{1/2}] \subset I(t_0, 2\eta)$, one can derive from (3.1.16), (3.2.15) and the mean value theorem that one has, for all $s \in [t-\eta, t-\eta h^{1/2}]$,

$$\begin{aligned} |H_+(s,t+h) - H_+(s,t)| &\le \int_t^{t+h} \frac{|\alpha(s)-\alpha(v)|}{v-s} dv \\ &\le k_\alpha \gamma^{-1}\Big((t+h-s)^\gamma - (t-s)^\gamma\Big) \le k_\alpha\, \eta^{\gamma-1}\, h^{(1+\gamma)/2} \end{aligned} \tag{3.3.30}$$

and

$$\begin{aligned} |H_+(s,t+h)| &\le \int_s^{t+h} \frac{|\alpha(s)-\alpha(v)|}{v-s} dv \\ &\le k_\alpha \int_s^{t+h} (v-s)^{\gamma-1} dv = k_\alpha \gamma^{-1}\,(t+h-s)^\gamma \le k_\alpha \gamma^{-1}\,(\eta+1)^\gamma. \end{aligned} \tag{3.3.31}$$

Thus, putting together (3.3.26), (3.3.29), (3.3.30), (3.3.17), (3.3.31) and $\alpha(s) \in [\alpha_{\inf}, \alpha_{\sup}]$, one gets, for some (finite) constant c_1 not depending on t and h, that

$$\Lambda_2^{1,1}(t,h) \le c_1 h^{1+\gamma}\,. \tag{3.3.32}$$

As regards $\Lambda_2^{1,2}(t,h)$ defined in (3.3.27), one can derive from (3.3.31), the mean value theorem, the inclusion $[t-\eta, t-\eta h^{1/2}] \subset I(t_0, 2\eta)$, (3.2.14) and (3.2.11) that

$$\Lambda_2^{1,2}(t,h) \le c_2 h^2 \int_{t-\eta}^{t-\eta h^{1/2}} (t-s)^{2(\alpha(t_0)-\varepsilon-1)} ds \le c_2' h^{2(\alpha(t_0)+1/2)+7\varepsilon}\,, \tag{3.3.33}$$

where c_2 and c_2' are two (finite) constants not depending on t and h.

As regards $\Lambda_2^{1,3}(t,h)$ defined in (3.3.28), one can derive from (3.2.38), and (3.2.34) that

$$\begin{aligned} \Lambda_2^{1,3}(t,h) \le 3 &\int_{t-\eta}^{t-\eta h^{1/2}} \left(\frac{1}{\Gamma(1+\alpha(t+h))} - \frac{1}{\Gamma(1+\alpha(t))}\right)^2 (t+h-s)^{2\alpha(t+h)} ds \\ &+ 3\,\Gamma_{\inf}^{-2} \int_{t-\eta}^{t-\eta h^{1/2}} \left((t+h-s)^{\alpha(t+h)} - (t-s)^{\alpha(t+h)}\right)^2 ds \\ &+ 3\,\Gamma_{\inf}^{-2} \int_{t-\eta}^{t-\eta h^{1/2}} \left((t-s)^{\alpha(t+h)} - (t-s)^{\alpha(t)}\right)^2 ds\,. \end{aligned} \tag{3.3.34}$$

Rather similarly to (3.2.26), it can be shown that, for some constant c_3 not depending on t and h, one has

$$\int_{t-\eta}^{t-\eta h^{1/2}} \left(\frac{1}{\Gamma(1+\alpha(t+h))} - \frac{1}{\Gamma(1+\alpha(t))}\right)^2 (t+h-s)^{2\alpha(t+h)} ds \le c_3 h^{2\gamma}\,. \tag{3.3.35}$$

Rather similarly to (3.3.33), it can be shown that for some constant c_4 not depending on t and h, one has

$$\int_{t-\eta}^{t-\eta h^{1/2}} \Big((t+h-s)^{\alpha(t+h)} - (t-s)^{\alpha(t+h)}\Big)^2 ds \le c_4 h^{2(\alpha(t_0)+1/2)+7\varepsilon}\,. \tag{3.3.36}$$

Rather similarly to (3.2.35), it can be shown that for some constant c_5 not depending on t and h, one has

$$\int_{t-\eta}^{t-\eta h^{1/2}} \Big((t-s)^{\alpha(t+h)} - (t-s)^{\alpha(t)}\Big)^2 ds \le c_5 h^{2\gamma}\,. \tag{3.3.37}$$

Finally, putting together (3.3.25), (3.3.32), (3.3.33), (3.3.34), (3.3.35), (3.3.36), (3.3.37) and the fact that $\gamma \in (1/2, 1)$, it follows that (3.3.24) holds. □

The following lemma provides an appropriate upper bound for $\Lambda_2^2(t,h)$ defined in (3.3.23).

Lemma 3.3.7. *There is a constant C_2^2, not depending on t and h, such that*

$$\Lambda_2^2(t,h) \le C_2^2\, h^{2(\alpha(t_0)+1/2)+6\varepsilon}\,. \tag{3.3.38}$$

Proof. Using (3.3.23) and (3.2.18) one has that

$$\Lambda_2^2(t,h) \le 2\Big(\Lambda_2^{2,0}(t,h) + \Lambda_2^{2,1}(t,h)\Big)\,, \tag{3.3.39}$$

where, for $j=0$ or $j=1$,

$$\Lambda_2^{2,j}(t,h) := \int_{t-\eta h^{1/2}}^{t} \left(\frac{1}{\Gamma(1+\alpha(t+jh))}(t+jh-s)^{\alpha(t+jh)} - \frac{1}{\Gamma(1+\alpha(s))}(t+jh-s)^{\alpha(s)} e^{-H_+(s,t+jh)}\right)^2 ds\,. \tag{3.3.40}$$

In view of (3.3.39), in order to show that (3.3.38) is satisfied it is enough to prove that, for each $j \in \{0,1\}$, the following inequality, in which c denotes a (finite) constant not depending on t and h, holds

$$\Lambda_2^{2,j}(t,h) \le c\, h^{2(\alpha(t_0)+1/2)+6\varepsilon}\,. \tag{3.3.41}$$

It follows from (3.3.40), (3.2.38) and (3.2.34) that

$$\Lambda_2^{2,j}(t,h) \le 3\Big(\varphi_{1,j}(t,h) + \Gamma_{\inf}^{-2}\, \varphi_{2,j}(t,h) + \Gamma_{\inf}^{-2}\, \varphi_{3,j}(t,h)\Big)\,, \tag{3.3.42}$$

where

$$\varphi_{1,j}(t,h) := \int_{t-\eta h^{1/2}}^{t} \left(\frac{1}{\Gamma(1+\alpha(t+jh))} - \frac{1}{\Gamma(1+\alpha(s))} \right)^2 (t+jh-s)^{2\alpha(t+jh)} ds \,, \tag{3.3.43}$$

$$\varphi_{2,j}(t,h) := \int_{t-\eta h^{1/2}}^{t} (t+jh-s)^{2\alpha(t+jh)} \big(1 - e^{-H_+(s,t+jh)}\big)^2 ds \tag{3.3.44}$$

and

$$\varphi_{3,j}(t,h) := \int_{t-\eta h^{1/2}}^{t} \Big((t+jh-s)^{\alpha(t+jh)} - (t+jh-s)^{\alpha(s)} \Big)^2 e^{-2H_+(s,t+jh)} ds \,. \tag{3.3.45}$$

It results from (3.3.43), the mean value theorem, (3.2.15), (3.2.12), (3.2.14) and (3.2.11) that

$$\begin{aligned} \varphi_{1,j}(t,h) &\le c_1 \int_{t-\eta h^{1/2}}^{t} (t+jh-s)^{2\gamma+2\alpha(t+jh)} ds \\ &\le c_1' h^{1/2(1+2\gamma+2\alpha(t_0)-2\varepsilon)} \le c_1' h^{2(\alpha(t_0)+1/2)+7\varepsilon} \,, \end{aligned} \tag{3.3.46}$$

where c_1 and c_1' are two (finite) constants not depending on t and h. Next, observe that, similarly to (3.3.31), it can be shown that there exists a (finite) constant c_2, not depending on t, h and s, such that, for all $s \in [t-\eta h^{1/2}, t]$, one has

$$|H_+(s,t+jh)| \le c_2 \, h^{\gamma/2} \,. \tag{3.3.47}$$

Thus, one can derive from (3.3.44), (3.3.47), (3.3.17), (3.2.12), (3.2.14) and (3.2.11) that

$$\varphi_{2,j}(t,h) \le c_3 \, h^{\gamma} \int_{t-\eta h^{1/2}}^{t} (t+jh-s)^{2\alpha(t+jh)} ds \le c_3' h^{2(\alpha(t_0)+1/2)+7\varepsilon} \,, \tag{3.3.48}$$

where c_3 and c_3' are two (finite) constants not depending on t and h. Next, using (3.3.45), (3.3.47), the mean value theorem, (3.2.15), (3.2.33), (3.2.14) and (3.2.11), one gets that

$$\begin{aligned} \varphi_{3,j}(t,h) &\le c_4 h^{\gamma} \int_{t-\eta h^{1/2}}^{t} (t+jh-s)^{2\alpha(t_0)-4\varepsilon} ds \\ &\le c_4' h^{1/2(1+2\gamma+2\alpha(t_0)-4\varepsilon)} \le c_4' h^{2(\alpha(t_0)+1/2)+6\varepsilon} \,, \end{aligned} \tag{3.3.49}$$

where c_4 and c_4' are two (finite) constants not depending on t and h. Finally, putting together (3.3.42), (3.3.46), (3.3.48) and (3.3.49), one obtains (3.3.41). □

The following lemma provides an appropriate upper bound for $\Lambda_3(t,h)$ defined in (3.3.8).

Lemma 3.3.8. *There is a constant C_3, not depending on t and h, such that*

$$\Lambda_3(t,h) \le C_3 h^{2+14\varepsilon} \,. \tag{3.3.50}$$

Proof. It follows from (3.3.8), (3.2.38) and (3.2.34) that

$$\Lambda_3(t,h) \le 3\Big(\Lambda_3^1(t,h) + \Gamma_{\inf}^{-2} \, \Lambda_3^2(t,h) + \Gamma_{\inf}^{-2} \, \Lambda_3^3(t,h) \Big) \,, \tag{3.3.51}$$

where

$$\Lambda_3^1(t,h) := \int_0^h \left(\frac{1}{\Gamma(1+\alpha(s+t))} - \frac{1}{\Gamma(1+\alpha(t+h))}\right)^2 (h-s)^{2\alpha(s+t)} e^{-2H_+(s+t,t+h)} ds\,, \quad (3.3.52)$$

$$\Lambda_3^2(t,h) := \int_0^h (h-s)^{2\alpha(t+s)} \left(e^{-H_+(s+t,t+h)} - 1\right)^2 ds \quad (3.3.53)$$

and

$$\Lambda_3^3(t,h) := \int_0^h \left((h-s)^{\alpha(t+s)} - (h-s)^{\alpha(t+h)}\right)^2 ds\,. \quad (3.3.54)$$

Observe that, similarly to (3.3.31), it can be shown that there exists a (finite) constant c_1, not depending on t, h and s, such that, for all $s \in [0,h]$, one has

$$|H_+(s+t,t+h)| \le c_1\, h^\gamma\,. \quad (3.3.55)$$

One can derive from (3.3.52), (3.3.55), the mean value theorem and (3.2.15) that

$$\Lambda_3^1(t,h) \le c_2 h^{2\gamma} \int_0^h (h-s)^{2\alpha(s+t)} ds \le c_2 h^{2(\gamma+\alpha_{\inf})+1}\,, \quad (3.3.56)$$

where c_2 is a (finite) constant not depending on t and h. Moreover, it follows from (3.3.53), (3.3.55) and (3.3.17) that

$$\Lambda_3^2(t,h) \le c_3 h^{2\gamma} \int_0^h (h-s)^{2\alpha(t+s)} ds \le c_3 h^{2(\gamma+\alpha_{\inf})+1}\,, \quad (3.3.57)$$

where c_3 is a (finite) constant not depending on t and h. Furthermore, using (3.3.54), the mean value theorem, (3.2.15), (3.2.33) and (3.2.11) one has that

$$\Lambda_3^3(t,h) \le c_4 h^{2\gamma} \int_0^h s^{2(\alpha_{\inf}-\varepsilon)} ds \le c_4 h^{2(\gamma+\alpha_{\inf}-\varepsilon)+1}\,, \quad (3.3.58)$$

where c_4 is a (finite) constant not depending on t and h. Finally, putting together (3.3.51), (3.3.56), (3.3.57), (3.3.58), the inequality $2\gamma > 1$ and the inequality $\alpha_{\inf} > 8\varepsilon$ (see (3.2.11)), one obtains (3.3.50). □

We are now in a position to prove the inequality (3.3.3).

Remark 3.3.9. Putting together (3.3.4), (3.3.9), (3.3.12), (3.3.21), (3.3.24), (3.3.38), (3.3.50), the fact that $h \in (0,1]$, the fact that $\gamma \in (1/2,1)$ and (3.2.11), it follows that (3.3.3) is satisfied.

Chapter 4

Moving average Multifractional Processes with Random Exponent: Lower bounds for local oscillations

This chapter is a restatement of the article [AB22a]. The Appendix 4.3 is unpublished.

4.1 Introduction and statement of the main result

Roughly speaking multifractional stochastic processes (see e.g. [Aya19]) are continuous real-valued stochastic processes with non-stationary increments which extend the well-known fractional Brownian motion (see e.g. [EM02, ST94]); yet, in contrast with it, their local path roughness can be prescribed via a functional Hurst parameter and thus can change from point to point. For a generic multifractional process $Y = \{Y(t)\}_{t \in \mathbb{R}_+}$, path roughness in a neighborhood of any arbitrary fixed point $\tau \in \mathbb{R}_+^* := \mathbb{R}_+ \setminus \{0\} = (0, +\infty)$ is usually measured through $\alpha_Y(\tau)$, the pointwise Hölder exponent at τ, defined as:

$$\alpha_Y(\tau) := \sup\left\{\alpha \in [0,1] : \limsup_{r \to 0_+} r^{-\alpha}\, \mathrm{Osc}_Y(\tau, r) < +\infty\right\}, \tag{4.1.1}$$

where, for all real number $r > 0$ small enough,

$$\mathrm{Osc}_Y(\tau, r) := \sup\left\{|Y(t') - Y(t'')| : (t', t'') \in [\tau - r, \tau + r]^2\right\} \tag{4.1.2}$$

is the oscillation of the process Y on the interval $[\tau - r, \tau + r] \subset \mathbb{R}_+^*$. When a stochastic process is the functional Hurst parameter of Y, then Y is said to be a Multifractional Process with Random Exponent (MPRE). Such kind of process turned out to be useful in financial time series modeling. Indeed, it has been shown in the literature (see for instance [BPP12, BPP13, BP14]) that MPRE allows to replicate main stylized facts (non-Gaussianity, volatility clustering and so on) of financial time series, and it provides a rational for the trading mechanism (for instance its pointwise Hölder exponent at a given point can be viewed as a weight that investors assign to the past prices in taking their trading decisions). A long time ago, [AT05] introduced a first type of MPRE which is given by a random wavelet series, but unfortunately cannot be represented through the usual Itô integral. In order to avoid the latter drawback, another type of MPRE was introduced in the last few years in [AEH18], and was generalized very recently in [LMS21].

Let $(\Omega, (\mathcal{F}_s)_{s\in\mathbb{R}}, \mathcal{F}, \mathbb{P})$ be a complete filtered probability space, and let $B = \{B(s)\}_{s\in\mathbb{R}}$ be a standard Brownian motion with respect to the filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$, the MPRE $X = \{X(t)\}_{s\in\mathbb{R}_+}$ studied in [LMS21] is defined, for each fixed $t \in \mathbb{R}_+$, as the Itô integral:

$$X(t) := \int_{-\infty}^{t} g(t,s)\, dB(s)\,, \tag{4.1.3}$$

where, for each fixed $t \in \mathbb{R}_+$, the stochastic process $\{g(t,s)\}_{s\in(-\infty,t)}$ is adapted to the filtration $(\mathcal{F}_s)_{s\in(-\infty,t)}$ and satisfies almost surely $\int_{-\infty}^{t} |g(t,s)|^2\, ds < +\infty$, which guarantees the existence of Itô integral in (4.1.3). In order to show that X has a modification with continuous paths and to conveniently bound from below its pointwise Hölder exponents, a well adapted extension of the Kolmogorov-Chentsov Hölder continuity Theorem has been derived in [LMS21] (see Theorem 2.1 and Corollary 2.2 in [LMS21]). Thus, the latter article has established that such a nice modification of X exists as soon as the associated integrand g satisfies the 3 conditions (A), (B) and (C) that we are now going to give.

Condition (A). For each fixed $s \in \mathbb{R}$, the random function $t \mapsto g(t,s)$ is differentiable on the open interval $(s \vee 0, +\infty)$. Moreover, there exist $(\mathcal{F}_s)_{s\in\mathbb{R}}$-adapted processes $\{H(s)\}_{s\in\mathbb{R}}$, $\{L(s)\}_{s\in\mathbb{R}}$ and $\{R(s)\}_{s\in\mathbb{R}}$, such that $H(s) \in (0,1)$, $L(s) > 0$, $R(s) > 1/2$, and it holds, for all $t \in \mathbb{R}_+^*$,

$$\begin{aligned}
|g(t,s)| &\le L(s)|t-s|^{H(s)-1/2}\,, \quad \text{for every } s \in (t-1,t),\\
|\partial_t g(t,s)| &\le L(s)|t-s|^{H(s)-3/2}\,, \quad \text{for every } s \in (t-1,t),\\
|\partial_t g(t,s)| &\le L(s)|t-s|^{-R(s)}\,, \quad \text{for every } s \in (-\infty,t-1].
\end{aligned}$$

Condition (B). There are deterministic real numbers $\underline{H}$, $\overline{H}$, $\overline{L}$ and $\underline{R}$, such that, for all $s \in \mathbb{R}$, one has

$$0 < \underline{H} \le H(s) \le \overline{H} < 1\,, \quad 0 < L(s) \le \overline{L} \quad \text{and} \quad R(s) \ge \underline{R} > 1/2\,. \tag{4.1.4}$$

Condition (C). One has, for all $(s', s'') \in [-1, +\infty)^2$,

$$\big|H(s') - H(s'')\big| \le \mu\big(|s'-s''|\big)\,, \tag{4.1.5}$$

where μ is some deterministic function from $\mathbb{R}_+$ to $\mathbb{R}_+$ which is continuous and increasing and satisfies $\mu(0) = 0$.

Among many other things, the following theorem has been obtained in [LMS21].

Theorem 4.1.1. [LMS21] *Assume that the conditions (A), (B) and (C) hold, then the process X has a modification with continuous paths which is identified with X from now on. Moreover, its pointwise Hölder exponents satisfy*

$$\mathbb{P}\big(\forall\, \tau \in \mathbb{R}_+^*,\, \alpha_X(\tau) \ge H(\tau)\big) = 1\,. \tag{4.1.6}$$

In other words, there exists Ω_1^* a universal event of probability 1 not depending on τ such that one has

$$\alpha_X(\tau,\omega) \ge H(\tau,\omega)\,, \quad \textit{for all } (\tau,\omega) \in \mathbb{R}_+^* \times \Omega_1^*. \tag{4.1.7}$$

In order to obtain the further result on pointwise Hölder exponents of X stated below, the article [LMS21] has assumed that the integrand g satisfies the following additional condition (A*) in which $\{L(s)\}_{s\in\mathbb{R}}$ denotes the same process as in Condition (A).

Condition (A*). One has, for all $t\in\mathbb{R}_+^*$,

$$\begin{aligned}\left|g(t,s)-\sigma(s)(t-s)^{H(s)-1/2}\right| &\le L(s)|t-s|^{H(s)-1/2+\rho}, \quad \text{for every } s\in(t-1,t),\\ \left|\partial_t g(t,s)-\partial_t\sigma(s)(t-s)^{H(s)-1/2}\right| &\le L(s)|t-s|^{H(s)-3/2+\rho}, \quad \text{for every } s\in(t-1,t),\end{aligned}$$

where ρ is some positive deterministic real number which satisfies $\overline{H}+\rho<1$ and does not depend on t and s, and where $\{\sigma(s)\}_{s\in\mathbb{R}}$ is a $(\mathcal{F}_s)_{s\in\mathbb{R}}$-adapted process, not depending on t, which is continuous on $[-1,+\infty)$ and satisfies

$$0<|\sigma(s)|\le L(s)\,, \quad \text{for all } s\in[-1,+\infty). \tag{4.1.8}$$

Theorem 4.1.2. [LMS21] *Under the conditions (A), (A*), (B), (C) and the additional condition*

$$\lim_{\varepsilon\to 0_+}\mu(\varepsilon)\log(\varepsilon)=0\,, \tag{4.1.9}$$

one has

$$\mathbb{P}\big(\alpha_X(\tau)=H(\tau)\big)=1\,, \quad \textit{for all } \tau\in\mathbb{R}_+^*. \tag{4.1.10}$$

Let us point out that the keystone of the proofs of Theorems 4.1.1 and 4.1.2 given in [LMS21] is the important Burkholder-Davis-Gundy inequality (see for instance [Mao07, Pro05]) as formulated in the following proposition:

Proposition 4.1.3. *Let $p\in[1,+\infty)$ be arbitrary and fixed. There is a universal deterministic finite constant $a(p)$ for which the following result holds: for any $(\mathcal{F}_s)_{s\in\mathbb{R}}$-adapted stochastic process $f=\{f(s)\}_{s\in\mathbb{R}}$ satisfying almost surely $\int_{-\infty}^{+\infty}|f(s)|^2\,ds<+\infty$, one has*

$$\mathbb{E}\bigg(\Big|\int_{-\infty}^{+\infty}f(s)\,dB(s)\Big|^p\bigg)\le a(p)\mathbb{E}\bigg(\Big(\int_{-\infty}^{+\infty}|f(s)|^2\,ds\Big)^{p/2}\bigg), \tag{4.1.11}$$

where $\int_{-\infty}^{+\infty}f(s)\,dB(s)$ denotes the Itô integral of f on $\mathbb{R}$.

Remark 4.1.4. In fact the additional information brought by Theorem 4.1.2 with respect to Theorem 4.1.1 is that, for each fixed $\tau\in\mathbb{R}_+^*$, there exists $\widetilde{\Omega}(\tau)$ an event of probability 1 which a priori *depends on* τ, such that

$$\alpha_X(\tau,\omega)\le H(\tau,\omega)\,, \quad \text{for all } \omega\in\widetilde{\Omega}(\tau). \tag{4.1.12}$$

Notice that, in view of (4.1.1) the inequality (4.1.12) can equivalently be expressed as follows in terms of the local oscillations of X in the vicinity of τ:

$$\limsup_{r\to 0_+}\Big(r^{-H(\tau,\omega)-\theta}\,\mathrm{Osc}_X(\tau,r,\omega)\Big)=+\infty\,, \quad \text{for all } (\theta,\omega)\in\mathbb{R}_+^*\times\widetilde{\Omega}(\tau). \tag{4.1.13}$$

The main goal of this chapter is to show that when the condition (C) is strengthened to the condition (C*) given below then a significantly more strong result than (4.1.13) holds, namely:

Theorem 4.1.5. *Suppose that the conditions (A), (A*), (B) and (C*) are satisfied. Then, there exists Ω_2^* a universal event of probability 1 not depending on τ such that one has*

$$\liminf_{r\to 0_+}\Big(r^{-H(\tau,\omega)-\theta}\,\mathrm{Osc}_X(\tau,r,\omega)\Big)=+\infty\,,\quad \textit{for all } (\theta,\tau,\omega)\in\mathbb{R}_+^*\times\mathbb{R}_+^*\times\Omega_2^*. \tag{4.1.14}$$

Condition (C*). There are two deterministic constants $\kappa\in(0,+\infty)$ and $\gamma\in(0,1)$ such that one has, for all $(s',s'')\in[-1,+\infty)^2$,

$$\big|H(s')-H(s'')\big|+\big|\sigma(s')-\sigma(s'')\big|\le\kappa\,|s'-s''|^{\gamma}\,. \tag{4.1.15}$$

It is worth mentioning that a straightforward consequence of Theorems 4.1.1 and 4.1.5 and (4.1.1) is that:

Corollary 4.1.6. *Suppose that the conditions (A), (A*), (B) and (C*) hold. Let Ω_1^* and Ω_2^* be the two universal events of probability 1 which were introduced in Theorems 4.1.1 and 4.1.5. Then $\Omega^*:=\Omega_1^*\cap\Omega_2^*$ is a universal event of probability 1 such that*

$$\alpha_X(\tau,\omega)=H(\tau,\omega)\,,\quad \textit{for all } (\tau,\omega)\in\mathbb{R}_+^*\times\Omega^*. \tag{4.1.16}$$

Remarks 4.1.7. (i) The conditions (B), (C) and (C*) might seem restrictive, yet they can be relaxed by a localization procedure via stopping times (see for instance Section 4.4.1 in [JP12]) which is explained in the setting of the MPRE X in Section 3 of [LMS21] and also in Appendix 4.3 of our thesis in a much more detailed way. In particular, it is important to mention that Theorem 4.1.5 remains true when the condition (4.1.15) only holds for all s' and s'' belonging to every fixed compact interval $I\subset[-1,+\infty)$, and the constant κ in it is no longer a deterministic real number but a random variable (with finite values) which may depend on I.

(ii) Let $\{H(s)\}_{s\in\mathbb{R}}$ be an arbitrary $(\mathcal{F}_s)_{s\in\mathbb{R}}$-adapted process with values in the deterministic interval $[\underline{H},\overline{H}]\subset(0,1)$ (see the condition (B)) and which satisfies the condition (C*). A natural example of an integrand g_1 for which all the conditions of Theorem 4.1.5 hold is given by:

$$g_1(t,s):=(t-s)_+^{H(s)-1/2}-(-s)_+^{H(s)-1/2}\,,\quad \text{for all } (t,s)\in\mathbb{R}_+\times\mathbb{R},$$

with the convention that, for each $(y,\alpha)\in\mathbb{R}^2$, one has

$$y_+^{\alpha}:=\begin{cases} y^{\alpha}, \text{if } y>0,\\ 0, \text{else;}\end{cases} \tag{4.1.17}$$

observe that in the very particular case where all the random variables $H(s)$, $s\in\mathbb{R}$, are equal to a same deterministic constant, the stochastic process X_1 associated to g_1 via (4.1.3) reduces to a fractional Brownian. Another example of an integrand g_2 satisfying all the conditions of Theorem 4.1.5 is provided by:

$$g_2(t,s):=(t-s)_+^{H(s)-1/2}e^{-\lambda(t-s)}\,,\quad \text{for all } (t,s)\in\mathbb{R}_+\times\mathbb{R},$$

where $\lambda\in\mathbb{R}_+^*$ is an arbitrary fixed parameter; one mentions that the stochastic process X_2 associated to g_2 via (4.1.3) is the so called multifractional Matérn process which was introduced in [LMS21].

(iii) Usually, in the literature relevant lower bounds for local oscillations of a stochastic process Y valid on a universal event of probability 1 not depending on the location (that is the point τ in our setting) are obtained via the classical strategy (see e.g. [Ber70, Ber72, Ber73, Xia97, Xia06, Xia13]) which consists in showing that the local time associated to Y is regular in the set variable uniformly in the space variable. For deriving such a regularity result for the local time explicit and exploitable formulas for the characteristic functions of the finite-dimensional distributions of Y need to be available. Unfortunately, such formulas are not available for the MPRE X. Thus, for proving Theorem 4.1.5 we make use of another strategy which is to some extent reminiscent of that introduced in the last few years in [Aya20] in the framework of Hermite processes. However several significant modifications of the strategy of [Aya20] are needed since there is a wide difference between the latter self-similar chaotic processes with stationary increments and the MPRE X. Among other things the Burkholder-Davis-Gundy inequality provided by Proposition 4.1.3 plays a crucial role in our proof of Theorem 4.1.5.

(iv) In the framework of multifractal analysis (see e.g. [Jaf99, Bal14, AJT07]) one is very often interested in determining singularity spectra of sample paths of stochastic processes. For a generic process $Y = \{Y(t)\}_{t\in\mathbb{R}_+}$ with nowhere differentiable paths the singularity spectrum $\rho_Y = \{\rho_Y(\alpha)\}_{\alpha\in[0,1]}$ is defined, for each $\omega \in \Omega$ (the underlying probability space) and $\alpha \in [0,1]$, as the Hausdorff dimension (see e.g. [Fal90]) of the level set $\big\{\tau \in \mathbb{R}_+^* : \alpha_Y(\tau,\omega) = \alpha\big\}$, where $\alpha_Y(\tau)$ is the pointwise Hölder exponent of Y. Thus, in contrast with (4.1.10) the more strong result (4.1.16) draws close connections between the singularity spectrum of the MPRE X and its functional random Hurst parameter H.

4.2 Proof of the main result

Let σ and H be the same processes as in the conditions (A) and (A*). One assumes that the condition (B) is satisfied and one denotes by $Z = \{Z(t)\}_{t\in\mathbb{R}_+}$ and $R = \{R(t)\}_{t\in\mathbb{R}_+}$ the two processes defined, for all $t \in \mathbb{R}_+$, as:

$$Z(t) := \int_{-1}^{t} \sigma(s)(t-s)^{H(s)-1/2}\, dB(s) = \int_{-\infty}^{t} \sigma(s)(t-s)^{H(s)-1/2}\, \mathbb{1}_{[-1,t)}(s)\, dB(s) \tag{4.2.1}$$

and

$$R(t) := X(t) - Z(t)\,. \tag{4.2.2}$$

One can easily derive from Theorem 4.1.1 applied to Z that:

Lemma 4.2.1. *Suppose that the conditions (B) and (C) hold, then the process Z has a modification with continuous paths which is identified with Z from now on. Moreover, its pointwise Hölder exponents satisfy*

$$\mathbb{P}\big(\forall\, \tau \in \mathbb{R}_+^*,\ \alpha_Z(\tau) \geq H(\tau)\big) = 1\,. \tag{4.2.3}$$

On the other hand, combining (4.1.3), (4.2.2) and (4.2.1), for all $t \in \mathbb{R}_+$, one has $R(t) = \int_{-\infty}^{t} \widetilde{g}(t,s)\, dB(s)$, where $\widetilde{g}(t,s) := g(t,s) - \sigma(s)(t-s)^{H(s)-1/2}\, \mathbb{1}_{[-1,t)}(s)$, for every $s \in (-\infty, t)$. Moreover, one can derive from the conditions (A) and (A*), imposed to g, that $\widetilde{g}$ satisfies a slightly modified version of the condition (A) in which, for all $t \in \mathbb{R}_+^*$ and $s \in (t-1,t)$, the exponent $H(s)$ is replaced by $H(s) + \rho$. Thus, similarly to Theorem 4.1.1 it can be shown that:

Lemma 4.2.2. *Assume that the conditions (A), (A*), (B) and (C) hold, then the process R has a modification with continuous paths which is identified with R from now on. Moreover, its pointwise Hölder exponents satisfy*

$$\mathbb{P}\big(\forall\,\tau \in \mathbb{R}_+^*,\, \alpha_R(\tau) \ge H(\tau)+\rho\big) = 1\,. \tag{4.2.4}$$

Remark 4.2.3. In view of Lemma 4.2.2, (4.1.1) and (4.1.2), in order to show that Theorem 4.1.5 holds, it is enough to prove it in the particular case where $X = Z$. On the other hand, in view of the fact that the interval $\mathbb{R}_+^* := (0, +\infty)$ can be expressed as the countable union of the open, bounded and overlapping intervals $\big(2^{-1}q, 1+2^{-1}q\big)$, $q \in \mathbb{Z}_+$, it is enough to prove the theorem for every $\tau \in \big(2^{-1}q, 1+2^{-1}q\big)$, the nonnegative integer q being arbitrary and fixed. For the sake of simplicity, we will only prove it when $q = 0$, that is $\tau \in (0,1)$; its proof can be done in a similar way for any other q.

For any integers $j \ge 2$ and $k \in \{0, \dots, 2^j - 1\}$, one denotes by $\Delta(j,k)$ the increment of the process Z such that

$$\Delta(j,k) := Z(d_{j,k+1}) - Z(d_{j,k})\,, \tag{4.2.5}$$

where $d_{j,k+1}$ and $d_{j,k}$ are the two dyadic numbers on the interval $[0,1]$ defined as:

$$d_{j,k+1} := (k+1)/2^j \quad \text{and} \quad d_{j,k} := k/2^j\,. \tag{4.2.6}$$

Observe that, in view of (4.2.1) and (4.1.17), the increment $\Delta(j,k)$ can be expressed as:

$$\Delta(j,k) = \int_{-1}^{d_{j,k+1}} \sigma(s)\Big((d_{j,k+1}-s)^{H(s)-1/2} - (d_{j,k}-s)_+^{H(s)-1/2}\Big)\,dB(s)\,. \tag{4.2.7}$$

In all the sequel the parameter $b \in (0, 1/2)$ is arbitrary and fixed. For each integer $j \ge 2$, one sets

$$e_j := \lfloor 2^{jb} \rfloor\,, \tag{4.2.8}$$

where $\lfloor \cdot \rfloor$ is the integer part function, and one denotes by $\mathcal{L}^j$ the non-empty finite set of positive integers defined as:

$$\mathcal{L}^j := \mathbb{N} \cap \big[1, (2^j/e_j) - 1\big]\,. \tag{4.2.9}$$

Observe that the cardinality of $\mathcal{L}^j$ satisfies, for some positive finite constant c not depending on j,

$$\mathrm{Card}(\mathcal{L}^j) \le c\, 2^{j(1-b)}\,. \tag{4.2.10}$$

Also, observe that, for any $l \in \mathcal{L}^j$, the random variable $\Delta(j, le_j)$, defined through (4.2.7) with $k = le_j$, can be expressed as

$$\Delta(j, le_j) = \widetilde{\Delta}(j, le_j) + \breve{\Delta}(j, le_j)\,, \tag{4.2.11}$$

where

$$\widetilde{\Delta}(j, le_j) := \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \sigma(s)\Big((d_{j,le_j+1}-s)^{H(s)-1/2} - (d_{j,le_j}-s)_+^{H(s)-1/2}\Big)\,dB(s) \tag{4.2.12}$$

and

$$\breve{\Delta}(j, le_j) := \int_{-1}^{d_{j,(l-1)e_j+1}} \sigma(s)\Big((d_{j,le_j+1}-s)^{H(s)-1/2} - (d_{j,le_j}-s)^{H(s)-1/2}\Big)\,dB(s)\,. \tag{4.2.13}$$

Let us now focus on the study of asymptotic behavior of the random variables $\big|\breve{\Delta}(j, le_j)\big|$ when j goes to $+\infty$.

Lemma 4.2.4. *There is $\Omega_3^*(b)$ an event of probability 1 depending on b on which one has*

$$\limsup_{j\to+\infty}\sup_{l\in\mathcal{L}^j} 2^{j(H(d_{j,le_j})+b(1-\overline{H})/2)}\left|\breve{\Delta}(j,le_j)\right| = 0\,. \tag{4.2.14}$$

For showing that Lemma 4.2.4 holds one needs two preliminary results. In order to state them one first has to introduce some additional notations. Let $\eta\in(0,1)$ be arbitrary and fixed and such that

$$5\kappa\eta^\gamma < b(1-\overline{H})\,, \tag{4.2.15}$$

where κ, γ, b and $\overline{H}$ are as in (4.1.15), (4.2.8) and (4.1.4). One assumes that the integer $\overline{J}_0\geq 6$ is chosen so that one has $e_j/2^j\leq 2^{-j(1-b)}<\eta/2$, for all integer $j\geq\overline{J}_0$. Then, one can derive from (4.2.13) that

$$\breve{\Delta}(j,le_j) = \breve{\Delta}'(j,le_j)+\breve{\Delta}''(j,le_j)\,, \tag{4.2.16}$$

where

$$\breve{\Delta}'(j,le_j) := \int_{-1}^{d_{j,le_j}-\eta}\sigma(s)\Big((d_{j,le_j+1}-s)^{H(s)-1/2}-(d_{j,le_j}-s)^{H(s)-1/2}\Big)\,dB(s) \tag{4.2.17}$$

and

$$\breve{\Delta}''(j,le_j) := \int_{d_{j,le_j}-\eta}^{d_{j,(l-1)e_j+1}}\sigma(s)\Big((d_{j,le_j+1}-s)^{H(s)-1/2}-(d_{j,le_j}-s)^{H(s)-1/2}\Big)\,dB(s)\,. \tag{4.2.18}$$

Lemma 4.2.5. *There is a finite deterministic constant $c>0$ such that, for all integer $j\geq\overline{J}_0$, one has*

$$\sup_{l\in\mathcal{L}^j}\int_{-1}^{d_{j,le_j}-\eta}|\sigma(s)|^2\Big|(d_{j,le_j+1}-s)^{H(s)-1/2}-(d_{j,le_j}-s)^{H(s)-1/2}\Big|^2\,ds\leq c\,2^{-2j}\,. \tag{4.2.19}$$

Proof. of Lemma 4.2.5 Using the mean value Theorem, (4.2.6), (4.1.8), and (4.1.4), one gets, for all $j\geq\overline{J}_0$, $l\in\mathcal{L}^j$ and $s\in[-1,d_{j,le_j}-\eta]$, that

$$|\sigma(s)|^2\Big|(d_{j,le_j+1}-s)^{H(s)-1/2}-(d_{j,le_j}-s)^{H(s)-1/2}\Big|^2\leq\overline{L}^{\,2}\,\eta^{2(\underline{H}-\overline{H})}\,(d_{j,le_j}-s)^{2\overline{H}-3}\,2^{-2j}\,.$$

Thus setting $c:=\overline{L}^{\,2}\,\eta^{2(\underline{H}-\overline{H})}\int_\eta^2 x^{2\overline{H}-3}\,dx<+\infty$, it follows that (4.2.19) holds. □

Lemma 4.2.6. *There is a finite deterministic constant $c>0$ such that, for all integer $j\geq\overline{J}_0+2b^{-1}$, one has*

$$\sup_{l\in\mathcal{L}^j}\int_{d_{j,le_j}-\eta}^{d_{j,(l-1)e_j+1}}2^{2jH(d_{j,le_j}-\eta)}\,|\sigma(s)|^2\Big|(d_{j,le_j+1}-s)^{H(s)-1/2}-(d_{j,le_j}-s)^{H(s)-1/2}\Big|^2\,ds$$
$$\leq c\,2^{-2j(4b(1-\overline{H})/5)}\,. \tag{4.2.20}$$

Proof. of Lemma 4.2.6 One can derive from the mean value Theorem, (4.2.6), (4.1.8), (4.1.4), (4.2.8) and (4.1.15) that, for all $j\geq\overline{J}_0+2b^{-1}$, $l\in\mathcal{L}^j$ and $s\in[d_{j,le_j}-\eta,d_{j,(l-1)e_j+1}]$,

$$|\sigma(s)|^2\Big|(d_{j,le_j+1}-s)^{H(s)-1/2}-(d_{j,le_j}-s)^{H(s)-1/2}\Big|^2\leq\overline{L}^{\,2}\,(d_{j,le_j}-s)^{2H(d_{j,le_j}-\eta)-2\kappa\eta^\gamma-3}\,2^{-2j}\,,$$

and consequently that

$$\begin{aligned}
&\int_{d_{j,le_j}-\eta}^{d_{j,(l-1)e_j+1}} |\sigma(s)|^2 \Big|(d_{j,le_j+1}-s)^{H(s)-1/2}-(d_{j,le_j}-s)^{H(s)-1/2}\Big|^2\,ds \\
&\le \overline{L}^{\,2}\, 2^{-2j} \int_{d_{j,le_j}-\eta}^{d_{j,(l-1)e_j+1}} (d_{j,le_j}-s)^{2H(d_{j,le_j}-\eta)-2\kappa\eta^\gamma-3}\,ds \\
&\le \frac{\overline{L}^{\,2}\,2^{-2j}}{2(1+\kappa\eta^\gamma-H(d_{j,le_j}-\eta))}\Big(2^{-j}(\lfloor 2^{jb}\rfloor-1)\Big)^{2(H(d_{j,le_j}-\eta)-\kappa\eta^\gamma-1)} \\
&\le \frac{4^{\kappa\eta^\gamma+1}\,\overline{L}^{\,2}\,2^{-2j}}{2(1-\overline{H})}\times 2^{-2j(1-b)(H(d_{j,le_j}-\eta)-\kappa\eta^\gamma-1)} \\
&\le \frac{4^{\kappa\eta^\gamma+1}\,\overline{L}^{\,2}}{2(1-\overline{H})}\times 2^{-2j(H(d_{j,le_j}-\eta)-(1-b)\kappa\eta^\gamma+b(1-\overline{H}))} \\
&\le \frac{16\overline{L}^{\,2}}{2(1-\overline{H})}\times 2^{-2j(H(d_{j,le_j}-\eta)+4b(1-\overline{H})/5)}\,, \qquad (4.2.21)
\end{aligned}$$

where the last inequality follows from (4.2.15). Finally, (4.2.20) results from (4.2.21). □

We are now ready to prove Lemma 4.2.4.

Proof. of Lemma 4.2.4 It easily follows (4.2.16) and the triangle inequality that, for all integer $j \ge \overline{J}_0 + 2b^{-1}$, one has

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{l\in\mathcal{L}^j} 2^{jH(d_{j,le_j})}\,\big|\breve{\Delta}(j,le_j)\big| > 2^{-j(8b(1-\overline{H})/15)+1}\Big) \\
&\le \mathbb{P}\Big(\sup_{l\in\mathcal{L}^j} 2^{jH(d_{j,le_j})}\,\big|\breve{\Delta}'(j,le_j)\big| > 2^{-j(8b(1-\overline{H})/15)}\Big) \qquad (4.2.22)\\
&\qquad\qquad + \mathbb{P}\Big(\sup_{l\in\mathcal{L}^j} 2^{jH(d_{j,le_j})}\,\big|\breve{\Delta}''(j,le_j)\big| > 2^{-j(8b(1-\overline{H})/15)}\Big)\,.
\end{aligned}$$

In order to conveniently bound each one of the two probabilities in the right-hand side of (4.2.22), one denotes by p a fixed real number such that

$$p > \frac{15}{b(1-\overline{H})} > 15\,. \qquad (4.2.23)$$

One can derive from (4.1.4), (4.2.10), the Markov inequality, (4.1.11), (4.2.17) and (4.2.19) that

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{l\in\mathcal{L}^j} 2^{jH(d_{j,le_j})}\,\big|\breve{\Delta}'(j,le_j)\big| > 2^{-j(8b(1-\overline{H})/15)}\Big) \le \mathbb{P}\Big(\sup_{l\in\mathcal{L}^j}\big|\breve{\Delta}'(j,le_j)\big| > 2^{-j(\overline{H}+8b(1-\overline{H})/15)}\Big) \\
&\le \sum_{l\in\mathcal{L}^j}\mathbb{P}\Big(\big|\breve{\Delta}'(j,le_j)\big| > 2^{-j(\overline{H}+8b(1-\overline{H})/15)}\Big) \le 2^{jp(\overline{H}+8b(1-\overline{H})/15)}\sum_{l\in\mathcal{L}^j}\mathbb{E}\Big(\big|\breve{\Delta}'(j,le_j)\big|^p\Big) \\
&\le c_1 2^{j((1-b)-p(1-\overline{H}-8b(1-\overline{H})/15))} \le c_1 2^{j(1-p(7(1-\overline{H})/15))}\,, \qquad (4.2.24)
\end{aligned}$$

where c_1 is a constant not depending on j. Moreover, it results from (4.1.15), (4.2.15), (4.2.10), the Markov inequality, (4.1.11), (4.2.18) and (4.2.20) that

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{l\in\mathcal{L}^j} 2^{jH(d_{j,le_j})}\,\big|\breve{\Delta}''(j,le_j)\big| > 2^{-j(8b(1-\overline{H})/15))}\Big)\\
&\le \mathbb{P}\Big(\sup_{l\in\mathcal{L}^j} 2^{jH(d_{j,le_j}-\eta)+j\kappa\eta^{\gamma}}\,\big|\breve{\Delta}''(j,le_j)\big| > 2^{-j(8b(1-\overline{H})/15)}\Big)\\
&\le \mathbb{P}\Big(\sup_{l\in\mathcal{L}^j} \big|2^{jH(d_{j,le_j}-\eta)}\,\breve{\Delta}''(j,le_j)\big| > 2^{-j(11b(1-\overline{H})/15)}\Big)\\
&\le \sum_{l\in\mathcal{L}^j}\mathbb{P}\Big(\big|2^{jH(d_{j,le_j}-\eta)}\,\breve{\Delta}''(j,le_j)\big| > 2^{-j(11b(1-\overline{H})/15)}\Big)\\
&\le 2^{jp(11b(1-\overline{H})/15)}\sum_{l\in\mathcal{L}^j}\mathbb{E}\Big(\big|2^{jH(d_{j,le_j}-\eta)}\,\breve{\Delta}''(j,le_j)\big|^{p}\Big)\\
&\le c_2 2^{j((1-b)-pb(1-\overline{H})/15)}\,,
\end{aligned} \tag{4.2.25}$$

where c_2 is a constant not depending on j. Next, putting together (4.2.22), (4.2.23), (4.2.24) and (4.2.25), one obtains that

$$\sum_{j=2}^{+\infty}\mathbb{P}\Big(\sup_{l\in\mathcal{L}^j} 2^{jH(d_{j,le_j})}\,\big|\breve{\Delta}(j,le_j)\big| > 2^{-j(8b(1-\overline{H})/15)+1}\Big) < +\infty.$$

Thus, one can derive from the Borel-Cantelli Lemma that there is $\Omega_3^*(b)$ an event of probability 1 depending b on which one has

$$\sup_{j\ge 2}\sup_{l\in\mathcal{L}^j} 2^{j(H(d_{j,le_j})+8b(1-\overline{H})/15)}\,\big|\breve{\Delta}(j,le_j)\big| < +\infty\,. \tag{4.2.26}$$

Finally, it is clear that (4.2.26) implies that (4.2.14) is satisfied on $\Omega_3^*(b)$. □

Let us now focus on the study of asymptotic behavior of the random variables $\big|\widetilde{\Delta}(j,le_j)\big|$ (see (4.2.12)) when j goes to $+\infty$. First, one needs to introduce some additional notations. For each fixed integer $j\ge 6$, the integer $M_j\ge 1$ denotes the integer part of $j^{-1}\big((2^j/e_j)-2\big)$, that is

$$M_j := \lfloor j^{-1}\big((2^j/e_j)-2\big)\rfloor\,. \tag{4.2.27}$$

Moreover, $(U_m^j)_{m\in\{0,1,\dots,M_j\}}$ denotes the subdivision of the interval $\big[1,(2^j/e_j)-1\big]$ by the M_j+1 points such that:

$$U_{M_j}^j := (2^j/e_j)-1 \quad\text{and}\quad U_m^j := 1+mj\,,\ \text{for all } m\in\{0,1,\dots,M_j-1\}; \tag{4.2.28}$$

notice that

$$j \le U_{M_j}^j - U_{M_j-1}^j < 2j\,. \tag{4.2.29}$$

For all integers $j\ge 6$ and $m\in\{1,\dots,M_j\}$, let $\mathcal{L}_m^j$ be the non-empty finite set of positive integers defined as:

$$\mathcal{L}_m^j := \mathbb{N}\cap\big[U_{m-1}^j, U_m^j\big]\,; \tag{4.2.30}$$

observe that, (4.2.9), (4.2.28), (4.2.29) and (4.2.30) entail that

$$\mathcal{L}^j = \bigcup_{m=1}^{M_j}\mathcal{L}_m^j \tag{4.2.31}$$

and

$$j < \mathrm{Card}(\mathcal{L}_m^j) \le 2j\,, \quad \text{for all } m \in \{1,\dots,M_j\}. \tag{4.2.32}$$

For every integers $j \ge 6$, $m \in \{1,\dots,M_j\}$ and $l \in \mathcal{L}_m^j$, the 3 random variables $\widehat{\Delta}_m(j,le_j)$, $\widehat{\Delta}'_m(j,le_j)$ and $\widehat{\Delta}''_m(j,le_j)$ are defined as:

$$\widehat{\Delta}_m(j,le_j) := \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \sigma(\zeta_{j,m})\Big((d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j}-s)_+^{H(\zeta_{j,m})-1/2}\Big)\,dB(s)\,, \tag{4.2.33}$$

$$\widehat{\Delta}'_m(j,le_j) := \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \big(\sigma(s)-\sigma(\zeta_{j,m})\big)\Big((d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j}-s)_+^{H(\zeta_{j,m})-1/2}\Big)\,dB(s) \tag{4.2.34}$$

and

$$\begin{aligned}\widehat{\Delta}''_m(j,le_j) := \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \sigma(s)\bigg(&\Big((d_{j,le_j+1}-s)^{H(s)-1/2} - (d_{j,le_j}-s)_+^{H(s)-1/2}\Big)\\ &-\Big((d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j}-s)_+^{H(\zeta_{j,m})-1/2}\Big)\bigg)\,dB(s)\,,\end{aligned} \tag{4.2.35}$$

where the dyadic number $\zeta_{j,m}$ is defined as:

$$\zeta_{j,m} := d_{j,(U^j_{m-1}-1)e_j+1}\,. \tag{4.2.36}$$

Notice that, in view of (4.2.12), (4.2.33), (4.2.34) and (4.2.35), one has

$$\widetilde{\Delta}(j,le_j) = \widehat{\Delta}_m(j,le_j) + \widehat{\Delta}'_m(j,le_j) + \widehat{\Delta}''_m(j,le_j)\,, \quad \text{for all } l \in \mathcal{L}_m^j. \tag{4.2.37}$$

The following lemma allows to understand the reason for which one has introduced the random variables $\widehat{\Delta}_m(j,le_j)$, $l \in \mathcal{L}_m^j$.

Lemma 4.2.7. *For all integers $j \ge 6$ and $m \in \{1,\dots,M_j\}$, and for each finite sequence $(z_l)_{l\in\mathcal{L}_m^j}$ of real numbers, one has, almost surely,*

$$\begin{aligned}&\mathbb{E}\bigg(\exp\Big(i\sum_{l\in\mathcal{L}_m^j} z_l\,\widehat{\Delta}_m(j,le_j)\Big)\bigg|\mathcal{F}_{\zeta_{j,m}}\bigg) \\ &= \exp\bigg(-2^{-1}\sum_{l\in\mathcal{L}_m^j} z_l^2\int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}}\Big|\sigma(\zeta_{j,m})\Big((d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2} \\ &\qquad\qquad -(d_{j,le_j}-s)_+^{H(\zeta_{j,m})-1/2}\Big)\Big|^2\,ds\bigg)\,.\end{aligned} \tag{4.2.38}$$

Notice that (4.2.38) means that, conditionally to the sigma-algebra $\mathcal{F}_{\zeta_{j,m}}$, the random variables $\widehat{\Delta}_m(j,le_j)$, $l \in \mathcal{L}_m^j$, have independent centred Gaussian distributions with variances $\mathbb{E}\big(|\widehat{\Delta}_m(j,le_j)|^2\,\big|\mathcal{F}_{\zeta_{j,m}}\big)$, $l \in \mathcal{L}_m^j$, satisfying

$$\begin{aligned}&\mathbb{E}\big(|\widehat{\Delta}_m(j,le_j)|^2\,\big|\mathcal{F}_{\zeta_{j,m}}\big) \\ &= \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}}\Big|\sigma(\zeta_{j,m})\Big((d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j}-s)_+^{H(\zeta_{j,m})-1/2}\Big)\Big|^2\,ds \\ &\ge 2^{-1}\,|\sigma(\zeta_{j,m})|^2\,2^{-2jH(\zeta_{j,m})}\,.\end{aligned} \tag{4.2.39}$$

Proof. of Lemma 4.2.7 The main idea of the proof consists in the observation that, for all integers $j \geq 6$, $m \in \{1, \dots, M_j\}$ and $l \in \mathcal{L}_m^j$, the Brownian motion B in (4.2.33) can be replaced by the Brownian motion $W_{j,m} = \{W_{j,m}(x)\}_{x\in\mathbb{R}_+} := \{B(x+\zeta_{j,m}) - B(\zeta_{j,m})\}_{x\in\mathbb{R}_+}$ which is independent on the sigma-algebra $\mathcal{F}_{\zeta_{j,m}}$. Therefore $W_{j,m}$ is independent on the integrand in (4.2.33), denoted by $K_{j,m,l}$, which is $\mathcal{F}_{\zeta_{j,m}}$-measurable. Having made this observation the proof becomes classical: it can be done in a standard way by approximating, for each fixed $l \in \mathcal{L}_m^j$, the integrand $K_{j,m,l} = \{K_{j,m,l}(s)\}_{s\in[d_{j,(l-1)e_j+1}, d_{j,le_j+1}]}$ by a sequence $\big(K^n_{j,m,l}\big)_{n\in\mathbb{N}} = \big(\{K^n_{j,m,l}(s)\}_{s\in[d_{j,(l-1)e_j+1}, d_{j,le_j+1}]}\big)_{n\in\mathbb{N}}$ of elementary processes of the form:

$$K^n_{j,m,l}(s) = \sum_{p=0}^{q-1} A_p \mathbb{1}_{[t_p,t_{p+1})}(s)\,,$$

where the random variables A_p, $0 \leq p < q$, are $\mathcal{F}_{\zeta_{j,m}}$-measurable, and the finite sequence $(t_p)_{0\leq p\leq q}$ is a subdivision of the interval $[d_{j,(l-1)e_j+1}, d_{j,le_j+1}]$. □

For every fixed integer $j \geq 6$, let $(\Lambda_{j,m})_{m\in\{1,\dots,M_j\}}$, $(\breve{\Lambda}_{j,m})_{m\in\{1,\dots,M_j\}}$, $(\widetilde{\Lambda}_{j,m})_{m\in\{1,\dots,M_j\}}$, $(\widehat{\Lambda}_{j,m})_{m\in\{1,\dots,M_j\}}$, $(\widehat{\Lambda}'_{j,m})_{m\in\{1,\dots,M_j\}}$ and $(\widehat{\Lambda}''_{j,m})_{m\in\{1,\dots,M_j\}}$ be the 6 finite sequences of nonnegative finite random variables defined, for all $m \in \{1, \dots, M_j\}$, as:

$$\Lambda_{j,m} := \sup\Big\{\big|\Delta(j, le_j)\big| \,:\, l \in \mathcal{L}_m^j\Big\}\,, \tag{4.2.40}$$

$$\breve{\Lambda}_{j,m} := \sup\Big\{\big|\breve{\Delta}(j, le_j)\big| \,:\, l \in \mathcal{L}_m^j\Big\}\,, \tag{4.2.41}$$

$$\widetilde{\Lambda}_{j,m} := \sup\Big\{\big|\widetilde{\Delta}(j, le_j)\big| \,:\, l \in \mathcal{L}_m^j\Big\}\,, \tag{4.2.42}$$

$$\widehat{\Lambda}_{j,m} := \sup\Big\{\big|\widehat{\Delta}_m(j, le_j)\big| \,:\, l \in \mathcal{L}_m^j\Big\}\,, \tag{4.2.43}$$

$$\widehat{\Lambda}'_{j,m} = \sup\Big\{\big|\widehat{\Delta}'_m(j, le_j)\big| \,:\, l \in \mathcal{L}_m^j\Big\}\,, \tag{4.2.44}$$

and

$$\widehat{\Lambda}''_{j,m} := \sup\Big\{\big|\widehat{\Delta}''_m(j, le_j)\big| \,:\, l \in \mathcal{L}_m^j\Big\}\,. \tag{4.2.45}$$

The main ingredient of the proof of Theorem 4.1.5 is the following proposition.

Proposition 4.2.8. *There exists $\Omega_2^*(b)$ an event of probability 1 depending on b on which one has*

$$\liminf_{j\to+\infty} \inf_{1\leq m\leq M_j} 2^{jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|^{-1} \Lambda_{j,m} \geq 1/4\,. \tag{4.2.46}$$

For proving Proposition 4.2.8 one needs several preliminary results.

Lemma 4.2.9. *Let $\Omega_3^*(b)$ be the same event of probability 1 as in Lemma 4.2.4. Then, one has on $\Omega_3^*(b)$*

$$\limsup_{j\to+\infty} \sup_{1\leq m\leq M_j} 2^{j(H(\zeta_{j,m})+b(1-\overline{H})/2)} \breve{\Lambda}_{j,m} = 0\,. \tag{4.2.47}$$

Proof. of Lemma 4.2.9 It follows from (4.2.31), (4.1.15), (4.2.36), (4.2.8), (4.2.6), (4.2.30), (4.2.28), (4.2.29) and (4.2.41) that, for all integer $j \geq 6$, one has

$$\begin{aligned}
&\sup_{l\in\mathcal{L}^j} 2^{j(H(d_{j,le_j})+b(1-\overline{H})/2)} \big|\breve{\Delta}(j,le_j)\big| = \sup_{1\leq m\leq M_j} \sup_{l\in\mathcal{L}^j_m} 2^{j(H(d_{j,le_j})+b(1-\overline{H})/2)} \big|\breve{\Delta}(j,le_j)\big| \\
&\geq 2^{-2\kappa\, j^{\gamma+1}\, 2^{-j(1-b)\gamma}} \sup_{1\leq m\leq M_j} 2^{j(H(\zeta_{j,m})+b(1-\overline{H})/2)}\, \breve{\Lambda}_{j,m} \\
&\geq c_0 \sup_{1\leq m\leq M_j} 2^{j(H(\zeta_{j,m})+b(1-\overline{H})/2)}\, \breve{\Lambda}_{j,m}\,, \qquad (4.2.48)
\end{aligned}$$

where the strictly positive constant $c_0 := \inf_{j\in\mathbb{N}} 2^{-2\kappa\, j^{\gamma+1}\, 2^{-j(1-b)\gamma}} > 0$. Then combining (4.2.14) and (4.2.48) one obtains (4.2.47). □

Lemma 4.2.10. *There exists $\Omega_4^*(b)$ an event of probability 1 depending on b on which one has*

$$\liminf_{j\to+\infty} \inf_{1\leq m\leq M_j} 2^{jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|^{-1} \widehat{\Lambda}_{j,m} \geq 1/4\,. \qquad (4.2.49)$$

Proof. of Lemma 4.2.10 Let $\Omega_4^*(b)$ be the event defined as:

$$\Omega_4^*(b) := \bigcup_{J=6}^{+\infty} \bigcap_{j=J}^{+\infty} \bigcap_{m=1}^{M_j} \Big\{\omega\in\Omega\,:\, 2^{jH(\zeta_{j,m},\omega)} |\sigma(\zeta_{j,m},\omega)|^{-1} \widehat{\Lambda}^j_m(\omega) \geq 1/4\Big\}\,. \qquad (4.2.50)$$

In order to show that the lemma holds, it is enough to prove that $\mathbb{P}(\Omega_4^*(b)) = 1$ which is equivalent to prove that

$$\mathbb{P}(\Omega\setminus\Omega_4^*(b)) = 0\,. \qquad (4.2.51)$$

Notice that, in view of (4.2.50) the event $\Omega\setminus\Omega_4^*(b)$ can be expressed as:

$$\Omega\setminus\Omega_4^*(b) := \bigcap_{J=6}^{+\infty} \bigcup_{j=J}^{+\infty} \bigcup_{m=1}^{M_j} \Big\{\omega\in\Omega\,:\, \widehat{\Lambda}^j_m(\omega) < 4^{-1}\, 2^{-jH(\zeta_{j,m},\omega)} |\sigma(\zeta_{j,m},\omega)|\Big\}\,.$$

Thus, one knows from the Borel-Cantelli Lemma that in order to derive (4.2.51) it is enough to prove that

$$\sum_{j=6}^{+\infty} \mathbb{P}\Big(\bigcup_{m=1}^{M_j} \big\{\omega\in\Omega\,:\, \widehat{\Lambda}^j_m(\omega) < 4^{-1}\; 2^{-jH(\zeta_{j,m},\omega)} |\sigma(\zeta_{j,m},\omega)|\big\}\Big) < +\infty\,. \qquad (4.2.52)$$

One clearly has, for every $j \geq 6$,

$$\begin{aligned}
&\mathbb{P}\Big(\bigcup_{m=1}^{M_j} \big\{\omega\in\Omega\,:\, \widehat{\Lambda}^j_m(\omega) < 4^{-1}\; 2^{-jH(\zeta_{j,m},\omega)} |\sigma(\zeta_{j,m},\omega)|\big\}\Big) \\
&\leq \sum_{m=1}^{M_j} \mathbb{P}\Big(\widehat{\Lambda}^j_m < 4^{-1}\; 2^{-jH(\zeta_{j,m},\omega)} |\sigma(\zeta_{j,m},\omega)|\Big) \qquad (4.2.53) \\
&= \sum_{m=1}^{M_j} \mathbb{P}\Big(\bigcap_{l\in\mathcal{L}^j_m} \Big\{\omega\in\Omega\,:\, \big|\widehat{\Delta}(j,le_j,\omega)\big| < 4^{-1}\; 2^{-jH(\zeta_{j,m},\omega)} |\sigma(\zeta_{j,m},\omega)|\Big\}\Big)\,,
\end{aligned}$$

where the last equality follows from (4.2.43). Moreover, using the fact that, for each $m \in \{1,\dots,M_j\}$,

$$\begin{aligned}
&\mathbb{P}\Big(\bigcap_{l\in\mathcal{L}_m^j}\Big\{\omega\in\Omega\,:\,\big|\widehat{\Delta}(j,le_j,\omega)\big|<4^{-1}\,2^{-jH(\zeta_{j,m},\omega)}\,|\sigma(\zeta_{j,m},\omega)|\Big\}\Big)\\
&=\mathbb{E}\Big(\prod_{l\in\mathcal{L}_m^j}\mathbb{1}_{\big\{|\widehat{\Delta}(j,le_j)|<4^{-1}\,2^{-jH(\zeta_{j,m})}\,|\sigma(\zeta_{j,m})|\big\}}\Big)\\
&=\mathbb{E}\bigg(\mathbb{E}\Big(\prod_{l\in\mathcal{L}_m^j}\mathbb{1}_{\big\{|\widehat{\Delta}(j,le_j)|<4^{-1}\,2^{-jH(\zeta_{j,m})}\,|\sigma(\zeta_{j,m})|\big\}}\,\Big|\mathcal{F}_{\zeta_{j,m}}\Big)\bigg)
\end{aligned}$$

and Lemma 4.2.7, one gets that

$$\begin{aligned}
&\mathbb{P}\Big(\bigcap_{l\in\mathcal{L}_m^j}\Big\{\omega\in\Omega\,:\,\big|\widehat{\Delta}(j,le_j,\omega)\big|<4^{-1}\,2^{-jH(\zeta_{j,m},\omega)}\,|\sigma(\zeta_{j,m},\omega)|\Big\}\Big)\\
&=\mathbb{E}\bigg(\prod_{l\in\mathcal{L}_m^j}\Big(2\pi\,\mathbb{E}\big(|\widehat{\Delta}_m(j,le_j)|^2\,\big|\mathcal{F}_{\zeta_{j,m}}\big)\Big)^{-1/2}\\
&\qquad\times\int_{\mathbb{R}}\exp\Big(-2^{-1}\Big(\mathbb{E}\big(|\widehat{\Delta}_m(j,le_j)|^2\,\big|\mathcal{F}_{\zeta_{j,m}}\big)\Big)^{-1}x^2\Big)\,\mathbb{1}_{\big\{|x|<4^{-1}\,2^{-jH(\zeta_{j,m})}\,|\sigma(\zeta_{j,m})|\big\}}\,dx\bigg)\\
&\le\mathbb{E}\bigg(\prod_{l\in\mathcal{L}_m^j}\pi^{-1/2}\,|\sigma(\zeta_{j,m})|^{-1}\,2^{jH(\zeta_{j,m})}\int_{\mathbb{R}}\mathbb{1}_{\big\{|x|<4^{-1}\,2^{-jH(\zeta_{j,m})}\,|\sigma(\zeta_{j,m})|\big\}}\,dx\bigg)\\
&\le\big(2^{-1}\,\pi^{-1/2}\big)^{\mathrm{Card}(\mathcal{L}_m^j)}<\big(2^{-1}\,\pi^{-1/2}\big)^{j}\,,
\end{aligned}\tag{4.2.54}$$

where the first inequality and the last inequality respectively follow from (4.2.39) and (4.2.32). Finally, putting together (4.2.53), (4.2.54) and (4.2.27), one obtains for every $j\ge 6$, that

$$\mathbb{P}\Big(\bigcup_{m=1}^{M_j}\big\{\omega\in\Omega\,:\,\widehat{\Lambda}_m^j(\omega)<4^{-1}\,2^{-jH(\zeta_{j,m},\omega)}\,|\sigma(\zeta_{j,m},\omega)|\big\}\Big)\le M_j\big(2^{-1}\,\pi^{-1/2}\big)^j\le\pi^{-j/2}\,,$$

which shows that (4.2.52) holds. □

In view of Lemma 4.2.10, the following two lemmas show that $\sup_{1\le m\le M_j}\,2^{jH(\zeta_{j,m})}\,\widehat{\Lambda}'_{j,m}$ and $\sup_{1\le m\le M_j}\,2^{jH(\zeta_{j,m})}\,\widehat{\Lambda}''_{j,m}$ are almost surely asymptotically negligible with respect to $\inf_{1\le m\le M_j}\,2^{jH(\zeta_{j,m})}\,|\sigma(\zeta_{j,m})|^{-1}\,\widehat{\Lambda}_{j,m}$ when j goes to $+\infty$.

Lemma 4.2.11. *There is $\Omega_5^*(b)$ an event of probability 1 depending on b on which one has*

$$\limsup_{j\to+\infty}\sup_{1\le m\le M_j}2^{j(H(\zeta_{j,m})+(1-b)\gamma/2)}\,\widehat{\Lambda}'_{j,m}=0\,.\tag{4.2.55}$$

Lemma 4.2.12. *There is $\Omega_6^*(b)$ an event of probability 1 depending on b on which one has*

$$\limsup_{j\to+\infty}\sup_{1\le m\le M_j}2^{j(H(\zeta_{j,m})+(1-b)\gamma/2)}\,\widehat{\Lambda}''_{j,m}=0\,.\tag{4.2.56}$$

Before proving Lemmas 4.2.11 and 4.2.12 let us give the proof of Proposition 4.2.8.

Proof. of Proposition 4.2.8 First notice that the event of probability 1 $\Omega_2^*(b)$ is defined as: $\Omega_2^*(b) := \bigcap_{q=3}^{6} \Omega_q^*(b)$, where the $\Omega_q^*(b)$, $3 \le q \le 6$, are the same events of probability 1 as in Lemmas 4.2.9, 4.2.10, 4.2.11 and 4.2.12. Using (4.2.11), (4.2.40), (4.2.41), (4.2.42) and the triangle inequality, one obtains, for all integers $j \ge 6$ and $m \in \{1, \dots, M_j\}$, that $\Lambda_{j,m} \ge \widetilde{\Lambda}_{j,m} - \breve{\Lambda}_{j,m}$. Thus, combining the latter inequality with the fact that (see the first inequality in (4.1.8))

$$|\sigma|_{\inf} := \inf_{s\in[0,1]} |\sigma(s)| > 0\,, \tag{4.2.57}$$

one gets that

$$\begin{aligned}&\inf_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\,|\sigma(\zeta_{j,m})|^{-1}\,\Lambda_{j,m}\\ &\ge \inf_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\,|\sigma(\zeta_{j,m})|^{-1}\,\widetilde{\Lambda}_{j,m} - |\sigma|_{\inf}^{-1} \sup_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\,\breve{\Lambda}_{j,m}\,.\end{aligned}$$

Then, one can derive from Lemma 4.2.9 that the following inequality holds on $\Omega_2^*(b)$:

$$\liminf_{j\to+\infty} \inf_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\,|\sigma(\zeta_{j,m})|^{-1}\,\Lambda_{j,m} \ge \liminf_{j\to+\infty} \inf_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\,|\sigma(\zeta_{j,m})|^{-1}\,\widetilde{\Lambda}_{j,m}\,. \tag{4.2.58}$$

Moreover, (4.2.37), (4.2.42), (4.2.43), (4.2.44), (4.2.45) and the triangle inequality entail, for all integer $j \ge 6$ and $m \in \{1, \dots, M_j\}$, that $\widetilde{\Lambda}_{j,m} \ge \widehat{\Lambda}_{j,m} - \widehat{\Lambda}'_{j,m} - \widehat{\Lambda}''_{j,m}$. Thus, combining the latter inequality with (4.2.57) one obtains that

$$\begin{aligned}&\inf_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\,|\sigma(\zeta_{j,m})|^{-1}\,\widetilde{\Lambda}_{j,m}\\ &\ge \inf_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\,|\sigma(\zeta_{j,m})|^{-1}\,\widehat{\Lambda}_{j,m}\\ &\qquad -|\sigma|_{\inf}^{-1}\Big(\sup_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\,\widehat{\Lambda}'_{j,m} + \sup_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\,\widehat{\Lambda}''_{j,m}\Big)\,.\end{aligned}$$

Then, one can derive from Lemmas 4.2.10, 4.2.11 and 4.2.12 that the following inequality holds on $\Omega_2^*(b)$:

$$\liminf_{j\to+\infty} \inf_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\,|\sigma(\zeta_{j,m})|^{-1}\,\widetilde{\Lambda}_{j,m} \ge 1/4\,. \tag{4.2.59}$$

Finally, (4.2.46) results from (4.2.58) and (4.2.59). □

For proving Lemmas 4.2.11 and 4.2.12 one needs some preliminary results.

Lemma 4.2.13. *Let $\overline{J}_1 \ge 6$ be a fixed integer such that*

$$\kappa\, j^{\gamma}\, 2^{-j(1-b)\gamma} \le 4^{-1}\underline{H} \quad \text{for all } j \ge \overline{J}_1. \tag{4.2.60}$$

There is a finite deterministic constant $c > 0$ such that, for all integers $j \ge \overline{J}_1$ and $m \in \{1, \dots, M_j\}$, one has

$$\begin{aligned}&2^{2jH(\zeta_{j,m})} \sup_{l\in\mathcal{L}_m^j} \Bigg\{ \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \big|\sigma(s)\big|^2 \Big| \Big((d_{j,le_j+1}-s)^{H(s)-1/2} - (d_{j,le_j}-s)_+^{H(s)-1/2} \Big)\\ &\qquad\qquad - \Big((d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j}-s)_+^{H(\zeta_{j,m})-1/2} \Big) \Big|^2 ds \Bigg\}\\ &\le c\, j^{2(1+\gamma)}\, 2^{-2j(1-b)\gamma}\,.\end{aligned} \tag{4.2.61}$$

Proof. of Lemma 4.2.13 Throughout the proof the integers $j \ge \overline{J}_1$, $m \in \{1,\dots,M_j\}$ and $l \in \mathcal{L}_m^j$ are arbitrary and fixed. Let $s \in [d_{j,(l-1)e_j+1}, d_{j,le_j})$ be arbitrary and fixed; using the mean value Theorem, (4.1.15), the inequalities

$$0 < (d_{j,le_j} - s) < (d_{j,le_j+1} - s) < 1,$$

(4.2.36), (4.2.8), (4.2.6), (4.2.30), (4.2.28), (4.2.29) and the triangle inequality one obtains, for some real number β belonging to interval $[H(s) \wedge H(\zeta_{j,m}), H(s) \vee H(\zeta_{j,m})]$, that

$$\begin{aligned}
&\Big|\Big((d_{j,le_j+1}-s)^{H(s)-1/2} - (d_{j,le_j}-s)^{H(s)-1/2}\Big) \\
&\qquad -\Big((d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j}-s)^{H(\zeta_{j,m})-1/2}\Big)\Big| \\
&\le \Big|(d_{j,le_j+1}-s)^{\beta-1/2}\log(d_{j,le_j+1}-s) - (d_{j,le_j}-s)^{\beta-1/2}\log(d_{j,le_j}-s)\Big|\,|H(s)-H(\zeta_{j,m})| \\
&\le 2\kappa\, j^{\gamma}\, 2^{-j(1-b)\gamma}\Big|(d_{j,le_j+1}-s)^{\beta-1/2}\log(d_{j,le_j+1}-s) - (d_{j,le_j}-s)^{\beta-1/2}\log(d_{j,le_j}-s)\Big| \\
&\le 2\kappa\, j^{\gamma}\, 2^{-j(1-b)\gamma}\Big((d_{j,le_j+1}-s)^{H(\zeta_{j,m})-2\kappa j^{\gamma} 2^{-j(1-b)\gamma}-1/2}\,|\log(d_{j,le_j+1}-s)| \\
&\qquad +(d_{j,le_j}-s)^{H(\zeta_{j,m})-2\kappa j^{\gamma} 2^{-j(1-b)\gamma}-1/2}\,|\log(d_{j,le_j}-s)|\Big)\mathbb{1}_{[d_{j,le_j-1},d_{j,le_j})}(s) \\
&\quad +2\kappa\, j^{\gamma}\, 2^{-j(1+(1-b)\gamma)}(d_{j,le_j}-s)^{H(\zeta_{j,m})-2\kappa j^{\gamma} 2^{-j(1-b)\gamma}-3/2}\big(1+|\log(d_{j,le_j}-s)|\big)\mathbb{1}_{[d_{j,(l-1)e_j+1},d_{j,le_j-1})}(s)\,.
\end{aligned} \tag{4.2.62}$$

Moreover, let us notice that

$$c_1 := \sup_{j\in\mathbb{N}} 2^{2\kappa j^{\gamma+1} 2^{-j(1-b)\gamma}} < +\infty\,. \tag{4.2.63}$$

One can derive from (4.2.62), (4.1.8), (4.1.4), the change of variables $u = 2^j(s - d_{j,le_j})$ and $v = -u$, (4.2.63), and (4.2.60) that

$$\begin{aligned}
&\int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j}} \big|\sigma(s)\big|^2\Big|\Big((d_{j,le_j+1}-s)^{H(s)-1/2} - (d_{j,le_j}-s)^{H(s)-1/2}\Big) \\
&\qquad -\Big((d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j}-s)^{H(\zeta_{j,m})-1/2}\Big)\Big|^2 ds \\
&\le 4\kappa^2\,\overline{L}^2\, j^{2\gamma}\, 2^{-j(1+2(1-b)\gamma)}\int_{-1}^{0}\Big((2^{-j}-2^{-j}u)^{H(\zeta_{j,m})-2\kappa j^{\gamma} 2^{-j(1-b)\gamma}-1/2}|\log(2^{-j}-2^{-j}u)| \\
&\qquad +(-2^{-j}u)^{H(\zeta_{j,m})-2\kappa j^{\gamma} 2^{-j(1-b)\gamma}-1/2}|\log(-2^{-j}u)|\Big)^2 du \\
&\quad +4\kappa^2\,\overline{L}^2\, j^{2\gamma}\, 2^{-j(3+2(1-b)\gamma)}\int_{-\infty}^{-1}(-2^{-j}u)^{2H(\zeta_{j,m})-4\kappa j^{\gamma} 2^{-j(1-b)\gamma}-3}\big(1+|\log(-2^{-j}u)|\big)^2 du \\
&\le c_2\, j^{2(1+\gamma)}\, 2^{-2j(H(\zeta_{j,m})+(1-b)\gamma)}\int_0^1\Big((1+v)^{H(\zeta_{j,m})-2\kappa j^{\gamma} 2^{-j(1-b)\gamma}-1/2}\big(1+|\log(1+v)|\big) \\
&\qquad +v^{H(\zeta_{j,m})-2\kappa j^{\gamma} 2^{-j(1-b)\gamma}-1/2}\big(1+|\log(v)|\big)\Big)^2 dv \\
&\quad +c_2\, j^{2(1+\gamma)}\, 2^{-2j(H(\zeta_{j,m})+(1-b)\gamma)}\int_1^{+\infty} v^{2H(\zeta_{j,m})-4\kappa j^{\gamma} 2^{-j(1-b)\gamma}-3}\big(2+|\log(v)|\big)^2 dv \\
&\le c_3\, j^{2(1+\gamma)}\, 2^{-2j(H(\zeta_{j,m})+(1-b)\gamma)}\,,
\end{aligned} \tag{4.2.64}$$

where the constant finite constant $c_2 := 4\kappa^2 \,\overline{L}^{\,2} c_1^2$ and the finite constant

$$c_3 := c_2 \int_0^1 \Big((1+v)^{\overline{H}-1/2}\big(1+|\log(1+v)|\big) + v^{\underline{H}/2-1/2}\big(1+|\log(v)|\big)\Big)^2\, dv$$
$$+ c_2 \int_1^{+\infty} v^{2\overline{H}-3}\big(2+|\log(v)|\big)^2\, dv\,.$$

Similarly to (4.2.64) it can be shown that

$$\int_{d_{j,le_j}}^{d_{j,le_j+1}} \big|\sigma(s)\big|^2 \Big| (d_{j,le_j+1}-s)^{H(s)-1/2} - (d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2}\Big|^2\, ds$$
$$\le c_4\, j^{2(1+\gamma)}\, 2^{-2j(H(\zeta_{j,m})+(1-b)\gamma)}\,, \tag{4.2.65}$$

where the finite constant

$$c_4 := c_2 \int_0^1 (1-u)^{\underline{H}-1}\big(2+|\log(1-u)|\big)^2\, du\,.$$

Finally, (4.2.61) results from (4.1.17), (4.2.64) and (4.2.65). □

Lemma 4.2.14. *There are two finite deterministic constants $0 < c' < c''$ such that, for all integers $j \ge 6$, $m \in \{1,\dots,M_j\}$ and $l \in \mathcal{L}_m^j$, one has*

$$c' 2^{-2jH(\zeta_{j,m})} \le \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \Big| (d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j}-s)_+^{H(\zeta_{j,m})-1/2}\Big|^2\, ds \le c'' 2^{-2jH(\zeta_{j,m})}\,. \tag{4.2.66}$$

Proof. of Lemma 4.2.14 In view of (4.2.6), it is clear that

$$\int_{d_{j,le_j}}^{d_{j,le_j+1}} (d_{j,le_j+1}-s)^{2H(\zeta_{j,m})-1}\, ds = \frac{2^{-2jH(\zeta_{j,m})}}{2H(\zeta_{j,m})}\,. \tag{4.2.67}$$

Thus, setting $c' := (2\overline{H}\,)^{-1} > 2^{-1}$ and noticing that (see (4.1.17))

$$\int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \Big| (d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j}-s)_+^{H(\zeta_{j,m})-1/2}\Big|^2\, ds$$
$$= \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j}} \Big| (d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j}-s)^{H(\zeta_{j,m})-1/2}\Big|^2\, ds$$
$$+ \int_{d_{j,le_j}}^{d_{j,le_j+1}} (d_{j,le_j+1}-s)^{2H(\zeta_{j,m})-1}\, ds\,, \tag{4.2.68}$$

it clearly follows from (4.2.67) and (4.1.4) that the first inequality in (4.2.66) is satisfied. Thus, from now on, our goal is to show that the second inequality in (4.2.66) holds. Using (4.2.6), the change of variable $x = 2^j(d_{j,le_j}-s)$, and standard computations, one gets that

$$\int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j}} \Big| (d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j}-s)^{H(\zeta_{j,m})-1/2}\Big|^2\, ds$$
$$\le 2^{-2jH(\zeta_{j,m})} \int_0^{+\infty} \Big| (x+1)^{H(\zeta_{j,m})-1/2} - x^{H(\zeta_{j,m})-1/2}\Big|^2\, dx\,. \tag{4.2.69}$$

One can easily derive from (4.1.4) that

$$\int_0^1 \left|(x+1)^{H(\zeta_{j,m})-1/2} - x^{H(\zeta_{j,m})-1/2}\right|^2 dx \le 2\int_0^1 (x+1)^{2\overline{H}-1}\, dx + 2\int_0^1 x^{2\underline{H}-1}\, dx\,. \tag{4.2.70}$$

Moreover, it follows from the mean value Theorem and (4.1.4) that

$$\int_1^{+\infty} \left|(x+1)^{H(\zeta_{j,m})-1/2} - x^{H(\zeta_{j,m})-1/2}\right|^2 dx \le \int_1^{+\infty} x^{2\overline{H}-3}\, dx\,. \tag{4.2.71}$$

Finally, letting c'' be the finite constant defined as

$$c'' := (2\underline{H}\,)^{-1} + 2\int_0^1 (x+1)^{2\overline{H}-1}\, dx + 2\int_0^1 x^{2\underline{H}-1}\, dx + \int_1^{+\infty} x^{2\overline{H}-3}\, dx\,,$$

one can derive from (4.1.4) and (4.2.67) to (4.2.71) that the second inequality in (4.2.66) holds. □

Lemma 4.2.15. *There is a finite deterministic constant $c>0$ such that, for all integers $j\ge 6$ and $m\in\{1,\ldots,M_j\}$, one has*

$$\begin{aligned} &2^{2jH(\zeta_{j,m})} \sup_{l\in\mathcal{L}_m^j} \Bigg\{ \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \left|\sigma(s)-\sigma(\zeta_{j,m})\right|^2 \\ &\qquad\qquad \times \left|(d_{j,le_j+1}-s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j}-s)_+^{H(\zeta_{j,m})-1/2}\right|^2 ds \Bigg\} \\ &\le c\, j^{2\gamma}\, 2^{-2j(1-b)\gamma}\,. \end{aligned} \tag{4.2.72}$$

Proof. of Lemma 4.2.15 Let $j\ge 6$, $m\in\{1,\ldots,M_j\}$, $l\in\mathcal{L}_m^j$ and $s\in[d_{j,(l-1)e_j+1}, d_{j,le_j+1})$ be arbitrary and fixed. It follows from (4.1.15), (4.2.36), (4.2.8), (4.2.6), (4.2.30), (4.2.28) and (4.2.29) that

$$\left|\sigma(s)-\sigma(\zeta_{j,m})\right| \le 2\kappa\, j^{\gamma}\, 2^{-j(1-b)\gamma}\,. \tag{4.2.73}$$

Thus combining (4.2.73) and the second inequality in (4.2.66) one obtains (4.2.72). □

We are now ready to prove Lemmas 4.2.11 and 4.2.12.

Proof. of Lemma 4.2.11 Let p be a fixed real number such that

$$p > 3/\gamma > 1\,. \tag{4.2.74}$$

It follows from (4.2.44), the Markov inequality, (4.1.11), (4.2.34), (4.2.72), (4.2.27), (4.2.8) and (4.2.32), that one has, for all integer $j\ge 6$,

$$\begin{aligned} &\mathbb{P}\Big(\sup_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\,\widehat{\Lambda}'_{j,m} > 2^{-2j(1-b)\gamma/3}\Big) \le \sum_{m=1}^{M_j} \mathbb{P}\Big(2^{jH(\zeta_{j,m})}\,\widehat{\Lambda}'_{j,m} > 2^{-2j(1-b)\gamma/3}\Big) \\ &\le \sum_{m=1}^{M_j} \sum_{l\in\mathcal{L}_m^j} \mathbb{P}\Big(2^{jH(\zeta_{j,m})}\,\big|\widehat{\Delta}'_m(j,le_j)\big| > 2^{-2j(1-b)\gamma/3}\Big) \\ &\le 2^{2jp(1-b)\gamma/3} \sum_{m=1}^{M_j} \sum_{l\in\mathcal{L}_m^j} \mathbb{E}\Big(\big|2^{jH(\zeta_{j,m})}\,\widehat{\Delta}'_m(j,le_j)\big|^p\Big) \le c_1 M_j\, \mathrm{Card}(\mathcal{L}_m^j)\, j^{p\gamma}\, 2^{-jp(1-b)\gamma/3} \\ &\le c_2\, j^{p\gamma}\, 2^{-j(1-b)(p\gamma/3-1)}\,, \end{aligned} \tag{4.2.75}$$

where c_1 and c_2 are two finite constants not depending on j. Next, combining (4.2.75) and (4.2.74), one obtains that

$$\sum_{j=6}^{+\infty} \mathbb{P}\Big(\sup_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\, \widehat{\Lambda}'_{j,m} > 2^{-2j(1-b)\gamma/3}\Big) < +\infty.$$

Thus, one can derive from the Borel-Cantelli Lemma that there is $\Omega_5^*(b)$ an event of probability 1 depending b on which one has

$$\sup_{j\ge 6}\sup_{1\le m\le M_j} 2^{j(H(\zeta_{j,m})+2(1-b)\gamma/3)}\, \widehat{\Lambda}'_{j,m} < +\infty\,. \tag{4.2.76}$$

Finally, it is clear that (4.2.76) implies that (4.2.55) is satisfied on $\Omega_5^*(b)$. □

Proof. of Lemma 4.2.12 Let p be a fixed real number satisfying (4.2.74). It follows from (4.2.45), the Markov inequality, (4.1.11), (4.2.35), (4.2.61), (4.2.27), (4.2.8) and (4.2.32), that one has, for all integer $j\ge \overline{J}_1$ (see (4.2.60)),

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\, \widehat{\Lambda}''_{j,m} > 2^{-2j(1-b)\gamma/3}\Big) \le \sum_{m=1}^{M_j} \mathbb{P}\Big(2^{jH(\zeta_{j,m})}\, \widehat{\Lambda}''_{j,m} > 2^{-2j(1-b)\gamma/3}\Big)\\
&\le \sum_{m=1}^{M_j}\sum_{l\in\mathcal{L}_m^j} \mathbb{P}\Big(2^{jH(\zeta_{j,m})}\, \big|\widehat{\Delta}''_m(j,le_j)\big| > 2^{-2j(1-b)\gamma/3}\Big)\\
&\le 2^{2jp(1-b)\gamma/3}\sum_{m=1}^{M_j}\sum_{l\in\mathcal{L}_m^j} \mathbb{E}\Big(\big|2^{jH(\zeta_{j,m})}\, \widehat{\Delta}''_m(j,le_j)\big|^p\Big) \le c_1 M_j\, \mathrm{Card}(\mathcal{L}_m^j)\, j^{p(1+\gamma)}\, 2^{-jp(1-b)\gamma/3}\\
&\le c_2\, j^{p(1+\gamma)}\, 2^{-j(1-b)(p\gamma/3-1)}\,,
\end{aligned} \tag{4.2.77}$$

where c_1 and c_2 are two finite constants not depending on j. Next, combining (4.2.77) and (4.2.74), one obtains that

$$\sum_{j=\overline{J}_1}^{+\infty} \mathbb{P}\Big(\sup_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\, \widehat{\Lambda}''_{j,m} > 2^{-2j(1-b)\gamma/3}\Big) < +\infty.$$

Thus, one can derive from the Borel-Cantelli Lemma that there is $\Omega_6^*(b)$ an event of probability 1 depending b on which one has

$$\sup_{j\ge \overline{J}_1}\sup_{1\le m\le M_j} 2^{j(H(\zeta_{j,m})+2(1-b)\gamma/3)}\, \widehat{\Lambda}''_{j,m} < +\infty\,. \tag{4.2.78}$$

Finally, it is clear that (4.2.78) implies that (4.2.56) is satisfied on $\Omega_6^*(b)$. □

We are now in position to complete the proof of Theorem 4.1.5.

End of the proof of Theorem 4.1.5 In view of Remark 4.2.3 we will only prove that (4.1.14) holds in the case where X is the process Z defined through (4.2.1) and $\tau\in(0,1)$. For any integer $j\ge 6$, one sets

$$l_j(\tau) := \lfloor 2^j\tau/e_j\rfloor\,, \tag{4.2.79}$$

where $\lfloor \cdot \rfloor$ denotes as usual the integer part function and e_j is as in (4.2.8). It is clear that $l_j(\tau)$ satisfies

$$0 \le \tau - \frac{l_j(\tau)e_j}{2^j} < \frac{e_j}{2^j} \le 2^{-j(1-b)}\,. \tag{4.2.80}$$

Moreover, it easily follows from (4.2.79), the inequalities $0 < b < 1/2$, $0 < \tau < 1$, (4.2.9) and (4.2.8) that there exists a positive integer $\overline{J}_2(\tau) \ge 6$, such that, for all integer $j \ge \overline{J}_2(\tau)$, one has

$$l_j(\tau) \in \mathcal{L}^j \tag{4.2.81}$$

and $[\tau - 12j2^{-j(1-b)}, \tau + 12j2^{-j(1-b)}] \subset [\tau - 2^{-j(1-2b)+1}, \tau + 2^{-j(1-2b)+1}] \subset (0,1)$. Next observe that in view of (4.2.81) and (4.2.31), there is $m_j(\tau) \in \{1, \dots, M_j\}$ such that

$$l_j(\tau) \in \mathcal{L}^j_{m_j(\tau)}\,. \tag{4.2.82}$$

Thus, one can derive from (4.2.82), (4.2.30), (4.2.28) and (4.2.29) that

$$\big|l_j(\tau) - l\big| < 2j\,, \quad \text{for all } l \in \mathcal{L}^j_{m_j(\tau)}. \tag{4.2.83}$$

Next, combining (4.2.83), (4.2.80), (4.2.8) and the triangle inequality, one gets that

$$\Big|\tau - \frac{le_j}{2^j}\Big| < \frac{(2j+1)e_j}{2^j} \le \frac{3je_j}{2^j} \le 3j2^{-j(1-b)}\,, \quad \text{for all } l \in \mathcal{L}^j_{m_j(\tau)}, \tag{4.2.84}$$

and

$$\Big|\tau - \frac{le_j+1}{2^j}\Big| < \frac{(2j+1)e_j+1}{2^j} \le \frac{3je_j}{2^j} \le 3j2^{-j(1-b)}\,, \quad \text{for all } l \in \mathcal{L}^j_{m_j(\tau)}. \tag{4.2.85}$$

Next it follows from (4.2.40), (4.2.5), (4.2.6), (4.1.2), (4.2.84), (4.2.85), (4.2.80), (4.2.82), (4.2.30), (4.2.36), (4.2.8), (4.2.28), (4.2.29) and (4.1.15) that

$$\begin{aligned} 2^{jH(\tau)}\operatorname{Osc}_Z\big(\tau\,, 3j2^{-j(1-b)}\big) &\ge 2^{jH(\tau)}\Lambda^j_{m_j(\tau)} \ge 2^{-j|H(\tau)-H(\zeta_{j,m_j(\tau)})|}\, 2^{jH(\zeta_{j,m_j(\tau)})}\Lambda^j_{m_j(\tau)} \\ &\ge 2^{-j\kappa(2j+1)^\gamma 2^{-j\gamma(1-b)}} \inf_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\Lambda^j_m \ge c_1 \inf_{1\le m\le M_j} 2^{jH(\zeta_{j,m})}\Lambda^j_m\,, \end{aligned} \tag{4.2.86}$$

where the deterministic constant $c_1 := \inf_{j\in\mathbb{N}} 2^{-j\kappa(2j+1)^\gamma 2^{-j\gamma/2}} > 0$ (recall that $b \in (0,1/2)$). Thus, one can derive from (4.2.86), Proposition 4.2.8, (4.2.57) and (4.1.2) that the following inequalities hold on $\Omega_2^*(b)$ (the event of probability 1 introduced in Proposition 4.2.8):

$$\liminf_{j\to+\infty} 2^{jH(\tau)}\operatorname{Osc}_Z\big(\tau\,, 2^{-j(1-2b)}\big) \ge \liminf_{j\to+\infty} 2^{jH(\tau)}\operatorname{Osc}_Z\big(\tau\,, 3j2^{-j(1-b)}\big) \ge 4^{-1}\, c_1|\sigma|_{\inf}\,.$$

Hence, for all $\omega \in \Omega_2^*(b)$, there exists an integer $j_3 = j_3(\tau,\omega) \ge \overline{J}_2(\tau)$ such that, for all integer $j \ge j_3$, one has

$$2^{jH(\tau,\omega)}\operatorname{Osc}_Z\big(\tau\,, 2^{-j(1-2b)}, \omega\big) \ge 5^{-1}\, c_1|\sigma|_{\inf}(\omega)\,. \tag{4.2.87}$$

Next, let ρ be an arbitrary positive real number such $\rho \le 2^{-j_3}$, one sets

$$j^*(\rho) := \lfloor -\log_2 \rho \rfloor\,. \tag{4.2.88}$$

One clearly has that $j^*(\rho) \ge j_3$ and

$$2^{j^*(\rho)} \le \rho^{-1} < 2^{j^*(\rho)+1}\,. \tag{4.2.89}$$

Then, (4.2.87), (4.2.88), (4.2.89) and (4.1.2) imply that, for all $\rho \in [j_3, +\infty)$, one has

$$\rho^{-H(\tau,\omega)}\,\mathrm{Osc}_Z\big(\tau\,,(2\rho)^{1-2b},\omega\big) \geq 2^{j^*(\rho)H(\tau,\omega)}\,\mathrm{Osc}_Z\big(\tau\,,2^{-j^*(\rho)(1-2b)},\omega\big) \geq 5^{-1}\,c_1|\sigma|_{\inf}(\omega)\,.$$

Thus, for all $r \in [2j_3, +\infty)$, one has

$$r^{-H(\tau,\omega)}\,\mathrm{Osc}_Z\big(\tau\,,r^{1-2b},\omega\big) \geq 10^{-1}\,c_1|\sigma|_{\inf}(\omega)\,. \tag{4.2.90}$$

It clearly follows from (4.2.90) that

$$\liminf_{r\to 0_+} r^{-H(\tau)}\,\mathrm{Osc}_Z\big(\tau\,,r^{1-2b}\big) \geq 10^{-1}\,c_1|\sigma|_{\inf}\,,\quad \text{on } \Omega_2^*(b),$$

which amounts to saying that

$$\liminf_{r\to 0_+} r^{-H(\tau)/(1-2b)}\,\mathrm{Osc}_Z(\tau\,,r) \geq 10^{-1}\,c_1|\sigma|_{\inf}\,,\quad \text{on } \Omega_2^*(b). \tag{4.2.91}$$

Finally, denoting by Ω_2^* the event of probability 1 defined as $\Omega_2^* := \bigcap_{b\in(0,1/2)\cap\mathbb{Q}} \Omega_2^*(b)$, where $\mathbb{Q}$ is the countable set of the rational numbers, it results from (4.2.91), the inequality $c_1|\sigma|_{\inf} > 0$ and the equality $\lim_{b\to 0_+} H(\tau)/(1-2b) = H(\tau)$ that (4.1.14) holds when $X = Z$ and $\tau \in (0,1)$. □

4.3 Appendix : Localization procedure via stopping times

The purpose of this appendix is to explain how the condition (C*) of the Section 4.1 can be relaxed with a localization procedure via stopping times. For simplifying our presentation we focus on the stochastic process $\{Z(t)\}_{t\in[-1,1]}$ defined as

$$\forall t \in [-1,1],\ Z(t) := \int_{-1}^{t} (t-s)_+^{H(s)-1/2}\,dB(s)$$

where the stochastic process $\{H(s)\}_{s\in[-1,1]}$ is $(\mathcal{F}_s)_{s\in[-1,1]}$-adapted and with values in $[\underline{H},\overline{H}] \subset (0,1)$. Moreover, we suppose that there is a deterministic real number $\gamma_0 \in (0,1]$ such that the paths of $\{H(s)\}_{s\in[-1,1]}$ are Hölder continuous functions of order γ_0. Thus, there exists a random variable C, such that

$$\forall \omega \in \Omega,\ \forall s,s' \in [-1,1],\ |H(s,\omega) - H(s',\omega)| \leq C(\omega)|s-s'|^{\gamma_0}. \tag{4.3.1}$$

Since C depends on ω, the condition (C^*) are not a priori satisfied by the stochastic process H. Roughly speaking the idea behind the localization procedure is to stop the process Z by using stopping times so that C becomes a deterministic constant.

Let us fix an arbitrary real number $\gamma \in (0,\gamma_0)$ and introduce for all real number $S \in [-1,1]$ the random variable

$$C_S := \sup_{-1\leq s_1<s_2\leq S} \frac{|H(s_1)-H(s_2)|}{|s_1-s_2|^{\gamma}} = \sup_{s_1,s_2\in[-1,S]\cap\mathbb{Q}} \frac{|H(s_1)-H(s_2)|}{|s_1-s_2|^{\gamma}} \tag{4.3.2}$$

where the last equality follows from Hölder continuity of paths of the process $\{H(s)\}_{s\in[-1,1]}$ (see (4.3.1)). Then, for each fixed integer $n \geq 1$, one sets

$$S_n := \inf\{S \in [-1,1],\ C_S \geq n\} \tag{4.3.3}$$

with the convention that $\inf \emptyset = 1$. Observe that inequality (4.3.1) implies that

$$\forall \omega \in \Omega,\ \forall S \in [-1,1],\ C_S(\omega) \leq C(\omega)$$

thus,

$$\exists n_0(\omega) \in \mathbb{N},\ \forall n \geq n_0(\omega),\ S_n(\omega) = 1. \tag{4.3.4}$$

Lemma 4.3.1. *For all $\omega \in \Omega$, the non decreasing function $S \mapsto C_S(\omega)$ is continuous and zero at -1.*

The following two lemmas show that S_n is a stopping time with respect to the filtration $(\mathcal{F}_s)_{s\in[-1,1]}$.

Lemma 4.3.2. *For all $S \in [-1,1]$, the random variable C_S is $\mathcal{F}_S$-measurable.*

Proof. For all fixed $a \in \mathbb{R}$, one has

$$\begin{aligned}
(C_S)^{-1}\Big((a,+\infty)\Big) &= \left\{ \sup_{-1\leq s_1\leq s_2\leq S} \frac{|H(s_1)-H(s_2)|}{|s_1-s_2|^\gamma} > a \right\} \\
&= \left\{ \exists (s_1,s_2) \in [-1,S]^2,\ \frac{|H(s_1)-H(s_2)|}{|s_1-s_2|^\gamma} > a \right\} \\
&= \bigcup_{s_1,s_2\in[-1,S]} \left\{ \frac{|H(s_1)-H(s_2)|}{|s_1-s_2|^\gamma} > a \right\} \\
&= \bigcup_{s_1,s_2\in[-1,S]\cap\mathbb{Q}} \left\{ \frac{|H(s_1)-H(s_2)|}{|s_1-s_2|^\gamma} > a \right\} \in \mathcal{F}_S.
\end{aligned}$$

□

Lemma 4.3.3. *For all $n \in \mathbb{N}$, S_n is a stopping time with respect to the filtration $(\mathcal{F}_s)_{s\in[-1,1]}$.*

Proof. Let $n \in \mathbb{N}$ be arbitrary. Using the fact $S \mapsto C_S$ is a continuous non decreasing random function on $[-1,1]$ which vanishes at $S=-1$, one can derive from (4.3.3) that
$\{S_n \leq -1\} = \emptyset,\ \{S_n \leq t\} = \Omega$ and for all real number $s \in (-1,1)$

$$\{S_n > s\} = \{C_s < n\} = (C_s)^{-1}\Big((-\infty,n)\Big) \in \mathcal{F}_s,$$

thus, S_n is a stopping time. □

Observe that, using (4.3.2),

$$\forall n \in \mathbb{N},\ \forall \omega \in \Omega,\ \forall s,s' \in [-1,1],\ |H(s\wedge S_n(\omega),\omega) - H(s'\wedge S_n(\omega),\omega)| \leq n|s-s'|^\gamma,$$

thus the stochastic process $\{H(s\wedge S_n)\}_{s\in[-1,1]}$ satisfies the condition (C^*). Then, it seems natural to introduce the process $\Big\{\widetilde{Z}_n(t)\Big\}_{t\in[-1,1]}$ defined as

$$\forall t \in [-1,1],\ \widetilde{Z}_n(t) := \int_{-1}^t (t-s)_+^{H(s\wedge S_n)-1/2}\, dB(s).$$

Theorem 4.1.5 and Corollary 4.1.6 are applicable to the process $\widetilde{Z}_n$, thus its trajectories are continuous and there exists a universal event Ω^* of probability 1 such that

$$\alpha_{\widetilde{Z}_n}(\tau,\omega) = H(\tau \wedge S_n(\omega),\omega)\,, \quad \text{for all } (n,\tau,\omega) \in \mathbb{N}\times(0,1)\times\Omega^*. \tag{4.3.5}$$

The motivation of the localization procedure consists in proving that the paths $\widetilde{Z}_n(\omega)$ and $Z(\omega)$ are the same for almost all ω, for n big enough depending on ω. This result is given by the following theorem.

Proposition 4.3.4. *There is a universal event Ω' of probability 1 such that*

$$\forall \omega \in \Omega',\ \exists N(\omega) \in \mathbb{N},\ \forall n \geq N(\omega),\ \forall t \in [-1,1],\ \widetilde{Z}_n(t,\omega) = Z(t,\omega).$$

Thus, using (4.3.5) and (4.3.4), there is a universal event $\Omega^{**} := \Omega^* \cap \Omega'$ such that

$$\alpha_Z(\tau,\omega) = \lim_{n\to+\infty} H(\tau \wedge S_n(\omega),\omega) = H(\tau,\omega)\,, \quad \textit{for all } (\tau,\omega) \in (0,1)\times\Omega^{**}.$$

Let us define the stochastic field $\{Y(u,v)\}_{(u,v)\in[-1,1]^2}$

$$\forall (u,v) \in [-1,1]^2,\ Y(u,v) := \int_{-1}^{u} (v-s)_+^{H(s)-1/2} dB(s) = \int_{-1}^{1} K(u,v,s) dB(s)$$

where

$$\forall (u,v,s) \in [-1,1]^3,\ K(u,v,s) := (v-s)_+^{H(s)-1/2} \mathbb{1}_{[-1,u)}(s).$$

Observe that

$$\forall t \in [-1,1],\ Z(t) = Y(t,t).$$

Lemma 4.3.5. *There is a deterministic constant $c_0 > 0$, such that*

$$\forall u,v,u',v' \in [-1,1],\ \int_{-1}^{1} \Big(K(u',v',s) - K(u,v,s)\Big)^2 ds \leq c_0 ||(u'-u, v'-v)||^{2\min(\underline{H},1/2)}.$$

Proof. We can suppose that $u' = u + h_1$ and $v' = v + h_2$ where $h_1 \in \mathbb{R}$ and $h_2 \in \mathbb{R}_+$. One has

$$\begin{aligned}&\int_{-1}^{1} \Big(K(u+h_1,v+h_2,s) - K(u,v,s)\Big)^2 ds \\ &\leq 2\int_{-1}^{1} \Big(K(u+h_1,v+h_2,s) - K(u,v+h_2,s)\Big)^2 ds + 2\int_{-1}^{1} \Big(K(u,v+h_2,s) - K(u,v,s)\Big)^2 ds\end{aligned} \tag{4.3.6}$$

where

$$\int_{-1}^{1} \Big(K(u+h_1,v+h_2,s) - K(u,v+h_2,s)\Big)^2 ds = \left|\int_{u}^{u+h_1} (v+h_2-s)_+^{2H(s)-1} ds\right|. \tag{4.3.7}$$

One can assume $\min(u, u+h_1) < v+h_2$ since the latter integral is zero in the opposite case. If $\underline{H} \geq 1/2$, one gets

$$\left| \int_u^{u+h_1} (v+h_2-s)_+^{2H(s)-1} ds \right| \leq \left| \int_u^{u+h_1} 2^{2\overline{H}-1} ds \right| \leq 2|h_1| \tag{4.3.8}$$

and if $\underline{H} < 1/2$, one obtains

$$\begin{aligned} \left| \int_u^{u+h_1} (v+h_2-s)_+^{2H(s)-1} ds \right| &\leq 2^{2\overline{H}-1} \left| \int_u^{u+h_1} (2^{-1}(v+h_2-s))_+^{2\underline{H}-1} ds \right| \\ &\leq 4 \left| \int_u^{u+h_1} (v+h_2-s)_+^{2\underline{H}-1} ds \right| \\ &= 4 \left| \int_{\min(u,v+h_2)}^{\min(u+h_1,v+h_2)} (v+h_2-s)^{2\underline{H}-1} ds \right| \\ &\leq 4 \left| \int_{\min(u,v+h_2)}^{\min(u+h_1,v+h_2)} (\min(u+h_1,v+h_2)-s)^{2\underline{H}-1} ds \right| \\ &= \frac{2}{\underline{H}} \left|\min(u+h_1,v+h_2) - \min(u,v+h_2)\right|^{2\underline{H}} \\ &\leq \frac{2}{\underline{H}} |h_1|^{2\underline{H}}. \end{aligned} \tag{4.3.9}$$

On the other hand, one has

$$\begin{aligned} &\int_{-1}^1 \Big(K(u,v+h_2,s) - K(u,v,s)\Big)^2 ds = \int_{-1}^u \Big((v+h_2-s)_+^{H(s)-1/2} - (v-s)_+^{H(s)-1/2}\Big)^2 ds \\ &\leq \int_{-1}^v \Big((v+h_2-s)^{H(s)-1/2} - (v-s)^{H(s)-1/2}\Big)^2 ds + \int_v^{v+h_2} (v+h_2-s)^{2H(s)-1} ds. \end{aligned} \tag{4.3.10}$$

Using the change of variable $u := (v-s)h_2^{-1}$, one obtains

$$\begin{aligned} &\int_{-1}^v \Big((v+h_2-s)^{H(s)-1/2} - (v-s)^{H(s)-1/2}\Big)^2 ds \\ &\leq h_2^{2\underline{H}} \int_0^{+\infty} \Big((u+1)^{H(v-h_2u)-1/2} - u^{H(v-h_2u)-1/2}\Big)^2 du \leq c(\underline{H}, \overline{H}) h_2^{2\underline{H}}. \end{aligned} \tag{4.3.11}$$

Moreover, one has

$$\int_v^{v+h_2} (v+h_2-s)^{2H(s)-1} ds \leq \frac{1}{2\underline{H}} h_2^{2\underline{H}}. \tag{4.3.12}$$

Putting together (4.3.6), (4.3.7), (4.3.8), (4.3.9), (4.3.10), (4.3.11) and (4.3.12), there is a constant $c_0 > 0$, such that

$$\int_{-1}^1 \Big(K(u+h_1,v+h_2,s) - K(u,v,s)\Big)^2 ds \leq c_0 ||(h_1,h_2)||^{2\underline{H}}.$$

□

Lemma 4.3.6. *There is a universal event Ω'' of probability 1 such that for all $\beta \in (0, \min(1/2, \underline{H}))$, there exists a random variable C for which one has*

$$\forall \omega \in \Omega',\ \forall u, v, u', v' \in [-1,1],\ |Y(u',v',\omega) - Y(u,v,\omega)| \leq C(\omega)||(u'-u, v'-v)||^{\beta}.$$

Proof. Using Burkholder-Davis-Gundy inequality and Lemma 4.3.5, for all $p > \frac{2}{\min(1/2,\underline{H})}$ and $u', v', u, v \in [-1,-1]$ one gets

$$\begin{aligned}
\mathbb{E}\Big|Y(u',v') - Y(u,v)\Big|^p &= \mathbb{E}\left(\Big|\int_{-1}^{1}\Big(K(u',v',s) - K(u,v,s)\Big)^2 ds\Big|^p\right) \\
&\leq a(p)\mathbb{E}\left(\Big(\int_{-1}^{1}\Big(K(u',v',s) - K(u,v,s)\Big)^2 ds\Big)^{p/2}\right) \\
&\leq a(p){c_0}^{p/2}||(u'-u, v'-v)||^{p\min(1/2,\underline{H})}.
\end{aligned}$$

Using Kolmogorov-Chentsov theorem, there is an event $\Omega(p)$ of probability 1, such that the paths of the process $\{Y(u,v)\}_{(u,v)\in[-1,1]^2}$ are Hölder functions of order $\dfrac{p\min(1/2,\underline{H}) - 2}{p}$. The lemma is proved for $\Omega'' := \bigcap_{p=\lfloor \min(1/2,\underline{H})^{-1}\rfloor+2}^{+\infty} \Omega(p)$.

□

Using the Haar basis of $L^2([-1,1])$, it can be shown that the function $f_{j,n}(t,\cdot)$ defined for all fixed $t \in [-1,1]$, $n \in \mathbb{N}$ and $j \in \mathbb{N}$ as

$$\forall s \in [-1,1],\ f_{j,n}(t,s) := \sum_{k=-2^j}^{2^j-1}\left(2^j\int_{k2^{-j}}^{(k+1)2^{-j}} (t-x)_+^{H(x\wedge S_n)-1/2} dx\right)\mathbb{1}_{[k2^{-j},(k+1)2^{-j})}(s)$$

converges in $L^2([-1,1])$ to the function $s \mapsto f_n(t,s) = (t-s)_+^{H(s\wedge S_n)-1/2}$ which means

$$\forall n \geq 1,\ \forall t \in [-1,1],\ \int_{-1}^{1}|f_{j,n}(t,s) - f_n(t,s)|^2 ds \underset{j\to+\infty}{\longrightarrow} 0. \tag{4.3.13}$$

Observe that

$$\forall n \in \mathbb{N},\ \forall t \in [-1,1],\ \widetilde{Z}_n(t) = \int_{-1}^{1} f_n(t,s) dB(s). \tag{4.3.14}$$

Let us now introduce the function $g_{j,n}(t,\cdot)$ defined for all fixed $t \in [-1,1]$, $n \in \mathbb{N}$ and $j \in \mathbb{N}$, as

$$\forall s \in [-1,1],\ g_{j,n}(t,s) := \sum_{k=-2^j}^{2^j-1}\left(2^j\int_{k2^{-j}}^{(k+1)2^{-j}} (t-x)_+^{H(k2^{-j}\wedge S_n)-1/2} dx\right)\mathbb{1}_{[k2^{-j},(k+1)2^{-j})}(s). \tag{4.3.15}$$

Lemma 4.3.7. *Let us fix a real number* $\beta < \min(\underline{H}, \gamma)$. *There is a deterministic constant* $\widetilde{c}_1 > 0$ *such that for all* $j \geq 1$ *one has almost surely*

$$\forall n \geq 1, \quad \sup_{t\in[-1,1]} \int_{-1}^{1} |f_{j,n}(t,s) - g_{j,n}(t,s)|^2 ds \leq \widetilde{c}_1 n^2 2^{-2\beta j}.$$

Proof. One has for all $t \in [-1,1]$

$$\int_{-1}^{1} |f_{j,n}(t,s) - g_{j,n}(t,s)|^2 ds = \sum_{k=-2^j}^{2^j-1} \sigma_{j,n}(t,k)$$

where

$$\sigma_{j,n}(t,k) := 2^j \left(\int_{k2^{-j}}^{(k+1)2^{-j}} \left((t-x)^{H(k2^{-j}\wedge S_n)-1/2} - (t-x)^{H(x\wedge S_n)-1/2} \right) dx \right)^2. \tag{4.3.16}$$

Let us first focus on terms such that $k \leq d_j(t) - 2$ where $d_j(t) := \lfloor t2^j \rfloor$. In this case, for all $x \in [k2^{-j}, (k+1)2^{-j}]$, using mean value theorem and definitions (4.3.3) and (4.3.2), one gets for all $\varepsilon > 0$

$$\begin{aligned}
&|(t-x)^{H(k2^{-j}\wedge S_n)-1/2} - (t-x)^{H(x\wedge S_n)-1/2}| \\
&\leq \left| H(k2^{-j}\wedge S_n) - H(x\wedge S_n) \right| \left(\frac{t-x}{2} \right)^{\underline{H}-1/2-\varepsilon} 2^{\overline{H}-1/2-\varepsilon} \\
&\leq n|k2^{-j}\wedge S_n - x\wedge S_n|^{\gamma} (t-x)^{\underline{H}-1/2-\varepsilon} 2^{\overline{H}-\underline{H}} \\
&\leq 2n(x-k2^{-j})^{\gamma} (t-x)^{\underline{H}-1/2-\varepsilon}.
\end{aligned} \tag{4.3.17}$$

Combining (4.3.16) and (4.3.17), for $k \leq t2^j - 2$ one obtains

$$\begin{aligned}
\sigma_{j,n}(t,k) &\leq \frac{4n^2}{\underline{H}+1/2-\varepsilon} 2^{(1-2\gamma)j} \left((t-k2^{-j})^{\underline{H}+1/2-\varepsilon} - (t-(k+1)2^{-j})^{\underline{H}+1/2-\varepsilon} \right)^2 \\
&\leq 8n^2 \left(\underline{H}+1/2-\varepsilon\right)^2 2^{-(1+2\gamma)j} (t-(k+1)2^{-j})^{2\underline{H}-1-2\varepsilon}.
\end{aligned} \tag{4.3.18}$$

In the case $\underline{H} \leq 1/2$, for $\varepsilon < \frac{\underline{H}}{2}$ one gets

$$\begin{aligned}
\sum_{k=-2^j}^{d_j(t)-2} (t-(k+1)2^{-j})^{2\underline{H}-1-2\varepsilon} &\leq \sum_{k=-2^j}^{d_j(t)-2} \int_{k+1}^{k+2} (t-s2^{-j})^{2\underline{H}-1-2\varepsilon} ds \leq 2^j \int_{-1}^{t} (t-u)^{2\underline{H}-1-2\varepsilon} du \\
&\leq 2^j \int_0^2 (v^{\underline{H}-1}+1) dv,
\end{aligned} \tag{4.3.19}$$

and in the case $\underline{H} > 1/2$, for $\varepsilon < \frac{\underline{H}-1/2}{2}$, one obtains

$$\begin{aligned}
\sum_{k=-2^j}^{d_j(t)-2} (t-(k+1)2^{-j})^{2\underline{H}-1-2\varepsilon} &\leq \sum_{k=-2^j}^{d_j(t)-2} \int_{k}^{k+1} (t-s2^{-j})^{2\underline{H}-1-2\varepsilon} ds \leq 2^j \int_{-1}^{t} (t-u)^{2\underline{H}-1-2\varepsilon} du \\
&\leq 2^j \int_0^2 \left(v^{\underline{H}-1/2} + v^2 \right) dv
\end{aligned} \tag{4.3.20}$$

Putting together (4.3.18), (4.3.19) and (4.3.20) one gets

$$\sum_{k=-2^j}^{d_j(t)-2} \sigma_{j,n}(t,k) \le c_2 n^2 2^{-2\gamma j} \tag{4.3.21}$$

where $c_2 := 2c_1^2 \int_0^2 v^{\underline{H}-1} dv$. Moreover, using (4.3.16) and (4.3.17), one has

$$\sum_{k=d_j(t)-1}^{d_j(t)} \sigma_{j,n}(t,k) \le 8 \times 2^j \left(2^{\overline{H}-\underline{H}} \int_{t-2^{-j+1}}^{t} (t-x)^{\underline{H}-1/2} dx \right)^2 \le \frac{16 \times 2^{1+2\overline{H}}}{(\underline{H}+1/2)^2} 2^{-2j\underline{H}}. \tag{4.3.22}$$

The lemma is proved by combining (4.3.21) and (4.3.22). □

Let us introduce

$$\forall n \in \mathbb{N},\ \forall t \in [-1,1],\ Z_n(t) := Y(t \wedge S_n, t) = \int_{-1}^{t} (t-s)^{H(s\wedge S_n)-1/2} \mathbb{1}_{[-1,S_n[}(s) dB(s).$$

We already know that the paths of $(\widetilde{Z}_n(t))_t$ are continuous. Moreover, the paths of $(Z_n(t))_t$ are also continuous thanks to Lemma 4.3.6.

Let the event $A_n := \{S_n = 1\}$, then one has

$$\forall n \in \mathbb{N},\ \forall \omega \in A_n,\ \forall t \in [-1,1],\ Z_n(t,\omega) = Y(t,t,\omega) = Z(t,\omega).$$

Also notice that (4.3.4) implies that $\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} A_n \right) = 1$.

Lemma 4.3.8. *For all $n \in \mathbb{N}$ and $t \in [-1,1]$, there is an event $B_{n,t} \subset A_n$ satisfying $P(A_n \backslash B_{n,t}) = 0$ and*

$$\forall \omega \in B_{n,t},\ \widetilde{Z}_n(t,\omega) = Z_n(t,\omega) = Z(t,\omega).$$

Proof. It results from Lemma 4.3.7 and (4.3.13) that, for each fixed $n \in \mathbb{N}$,

$$\forall t \in [-1,1],\ \int_{-1}^{1} |g_{j,n}(t,s) - f_n(t,s)|^2 ds \xrightarrow[j\to+\infty]{a.s.} 0. \tag{4.3.23}$$

Then, (4.3.23) implies

$$\forall t \in [-1,1],\ \int_{-1}^{1} g_{j,n}(t,s) dB(s) \xrightarrow[j\to+\infty]{\mathcal{P}} \int_{-1}^{1} f_n(t,s) dB(s)$$

where the convergence holds in probability. Then, one can derive from (4.3.14) and (4.3.15)

$$\sum_{k=-2^j}^{2^j-1} \left(2^j \int_{k2^{-j}}^{(k+1)2^{-j}} (t-x)_+^{H(k2^{-j}\wedge S_n)-1/2} dx \right) \Big(B((k+1)2^{-j}) - B(k2^{-j}) \Big) \xrightarrow[j\to+\infty]{\mathcal{P}} \widetilde{Z}_n(t). \tag{4.3.24}$$

Observe that it clearly results from (4.3.23) that

$$\forall t \in [-1,1],\ \int_{-1}^{1} |g_{j,n}(t,s)\mathbb{1}_{[-1,S_n)}(s) - f_n(t,s)\mathbb{1}_{[-1,S_n)}(s)|^2 ds \xrightarrow[j\to+\infty]{a.s.} 0 \tag{4.3.25}$$

thus, one has for all $t \in [-1,1]$

$$\sum_{k=-2^j}^{2^j-1} \left(2^j \int_{k2^{-j}}^{(k+1)2^{-j}} (t-x)_+^{H(k2^{-j}\wedge S_n)-1/2} dx \right) \left(\int_{-1}^{1} \mathbb{1}_{[k2^{-j},(k+1)2^{-j})}(s)\mathbb{1}_{[-1,S_n)}(s) dB(s) \right) \xrightarrow[j\to+\infty]{\mathcal{P}} Z_n(t).$$

Since

$$\int_{-1}^{1} \mathbb{1}_{[k2^{-j},(k+1)2^{-j})}(s)\mathbb{1}_{[-1,S_n)}(s) dB(s) = B((k+1)2^{-j} \wedge S_n) - B(k2^{-j} \wedge S_n)$$

one obtains the following convergence for all $t \in [-1,1]$

$$\sum_{k=-2^j}^{2^j-1} \left(2^j \int_{k2^{-j}}^{(k+1)2^{-j}} (t-x)_+^{H(k2^{-j}\wedge S_n)-1/2} dx \right) \left(B((k+1)2^{-j} \wedge S_n) - B(k2^{-j} \wedge S_n) \right) \xrightarrow[j\to+\infty]{\mathcal{P}} Z_n(t). \tag{4.3.26}$$

Putting together (4.3.24) and (4.3.26) and the fact that $S_n = 1$ on the event A_n, the lemma is proved. □

We are now in position to prove Theorem 4.3.4.

Proof of Theorem 4.3.4. A consequence of Lemma 4.3.8 is that for all $n \in \mathbb{N}$, one has

$$\forall j \in \mathbb{N},\ \bigcap_{k=-2^j}^{2^j-1} B_{n,k2^{-j}} \subset A_n \text{ and } \mathbb{P}\left(\bigcap_{k=-2^j}^{2^j-1} B_{n,k2^{-j}} \right) = P(A_n)$$

since the intersection contains a finite number of $(B_{n,t})_t$. Then, by continuity property of the probability one obtains

$$\bigcap_{j\in\mathbb{N}} \bigcap_{k=-2^j}^{2^j-1} B_{n,k2^{-j}} \subset A_n \text{ and } \mathbb{P}\left(\bigcap_{j\in\mathbb{N}} \bigcap_{k=-2^j}^{2^j-1} B_{n,k2^{-j}} \right) = P(A_n). \tag{4.3.27}$$

Let us introduce for all $n \in \mathbb{N}$ the event

$$B_n := \bigcap_{j\in\mathbb{N}} \bigcap_{k=-2^j}^{2^j-1} B_{n,k2^{-j}} \subset A_n. \tag{4.3.28}$$

Since the sequencer $(S_n)_{n\in\mathbb{N}}$ is non decreasing, for all $N \in \mathbb{N}$, one has

$$A_N = \bigcup_{n=1}^{N} A_n. \tag{4.3.29}$$

Then putting together (4.3.27), (4.3.29) and (4.3.28), one obtains

$$\forall N \in \mathbb{N},\, \mathbb{P}\left(A_N\right) = \mathbb{P}\left(\bigcup_{n=1}^{N} B_n\right)$$

thus by continuity property, one obtains

$$\mathbb{P}\left(\bigcup_{N\in\mathbb{N}} \bigcup_{n=1}^{N} B_n\right) = \lim_{N\to+\infty} \mathbb{P}\left(\bigcup_{n=1}^{N} B_n\right) = \lim_{N\to+\infty} \mathbb{P}\left(A_N\right) = \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = 1.$$

Let us introduce $\mathbb{D}$ the set of dyadic numbers and $\Omega' := \bigcup_{n\in\mathbb{N}} B_n$. Then one has

$$\forall \omega \in \Omega',\ \exists N(\omega) \in \mathbb{N},\ \forall n \geq N(\omega),\ \forall t \in \mathbb{D} \cap [-1,1],\ \widetilde{Z}_n(t,\omega) = Z_n(t,\omega) = Z(t,\omega).$$

Thus, the continuity property on these three processes implies that

$$\forall \omega \in \Omega',\ \exists N(\omega) \in \mathbb{N},\ \forall n \geq N(\omega),\ \forall t \in [-1,1],\ \widetilde{Z}_n(t,\omega) = Z_n(t,\omega) = Z(t,\omega).$$

□

Chapter 5

Simulation of the Riemann-Liouville MPRE

5.1 Introduction

In the previous Chapter 4, we studied the local regularity of a class of processes with random exponent introduced in [AEH18] and generalized in [LMS21]. A central example of such a process is the process $\{Z(t)\}_{t\in\mathbb{R}_+}$ defined as

$$\forall t \geq 0,\ Z(t) := \int_{-\infty}^{t} \left((t-s)_+^{H(s)-1/2} - (-s)_+^{H(s)-1/2}\right) dB(s).$$

This process generalizes the fractional Brownian motion: the constant Hurst parameter H is replaced by $\{H(s)\}_{s\in\mathbb{R}}$ a stochastic process adapted to the filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ (to which the Brownian motion $\{B(s)\}_{s\in\mathbb{R}}$ is associated) and with values in a deterministic compact $[\underline{H}, \overline{H}] \subset (0,1)$.

We know from Chapter 4 that, under the general assumption that the paths of H are Hölder continuous functions of an arbitrary order $\gamma > 0$, one has for a universal event Ω^* of probability 1

$$\rho_Z(\tau,\omega) = H(\tau,\omega)\,, \quad \text{for all } (\tau,\omega) \in \mathbb{R}_+^* \times \Omega^*, \tag{5.1.1}$$

where $\rho_Z(\tau)$ denotes the pointwise Hölder exponent of Z at point τ. This equality (5.1.1) makes the process Z to be an interesting model for applications. Thus, simulation of this process is an important issue.

The following lemma shows that low-frequency part $\left\{\dot{Z}(t)\right\}_{t\in[0,1]}$ defined as

$$\forall t \in [0,1],\ \dot{Z}(t) := \int_{-\infty}^{0} \left((t-s)_+^{H(s)-1/2} - (-s)_+^{H(s)-1/2}\right) dB(s) \tag{5.1.2}$$

does not impact the local regularity of the full process $\{Z(t)\}_{t\in[0,1]}$.

Lemma 5.1.1. *There is a universal event Ω' of probability 1 such that , for all $t \in (0,1]$ and $\omega \in \Omega'$, one has*

$$\rho_{\dot{Z}}(t,\omega) = \widetilde{\rho}_{\dot{Z}}(t,\omega) = 1 \tag{5.1.3}$$

and thus

$$\rho_{Z-\dot{Z}}(\tau,\omega) = H(\tau,\omega)\,, \quad \textit{for all } (\tau,\omega) \in (0,1)\times(\Omega^*\cap\Omega'). \tag{5.1.4}$$

Proof. Let us fix $\varepsilon > 0$, $t \in [\varepsilon, 1)$ and $h > 0$ such that $t + h \in (0, 1]$. Using Burkholder-Davis-Gundy inequality, for all $p \geq 2$, there is a universal constant $a(p)$ such that

$$\begin{aligned}\mathbb{E}\Big|\dot{Z}(t+h) - \dot{Z}(t)\Big|^p &= \mathbb{E}\left(\Big|\int_0^{+\infty}\big((t+h+s)^{H(-s)-1/2} - (t+s)^{H(-s)-1/2}\big)dB(s)\Big|^p\right)\\ &\leq a(p)\mathbb{E}\left(\Big(\int_0^{+\infty}\Big((t+h+s)^{H(-s)-1/2} - (t+s)^{H(-s)-1/2}\Big)^2 ds\Big)^{p/2}\right).\end{aligned} \tag{5.1.5}$$

Moreover, one can derive from the mean value theorem that

$$\Big((t+h+s)^{H(-s)-1/2} - (t+s)^{H(-s)-1/2}\Big)^2 \leq \varepsilon^{\underline{H}-\overline{H}}h^2(t+s)^{2\overline{H}-3}$$

thus

$$\int_0^{+\infty}\Big((t+h+s)^{H(-s)-1/2} - (t+s)^{H(-s)-1/2}\Big)^2 ds \leq h^2\varepsilon^{\underline{H}-\overline{H}}\int_0^{+\infty}(\varepsilon+s)^{\underline{H}-3}ds = \frac{\varepsilon^{\overline{H}+\underline{H}-2}}{2-2\overline{\overline{H}}}h^2. \tag{5.1.6}$$

Combining (5.1.6) and (5.1.5), one gets for all $p \geq 2$

$$\mathbb{E}\Big|\dot{Z}(t+h) - \dot{Z}(t)\Big|^p \leq a(p)\left(\frac{\varepsilon^{2\underline{H}-2}}{2-2\overline{\overline{H}}}\right)^{p/2} h^p.$$

Using Kolmogorov-Chentsov theorem, there is an event $\Omega(\varepsilon, p)$ of probability 1 such that for all $\omega \in \Omega(\varepsilon, p)$ the function $t \mapsto Z(t,\omega)$ is Hölder continuous of any order $\beta \in [0, (p-1)/p)$ on the interval $[\varepsilon, 1]$. Thus, the lemma is proved for $\Omega' := \bigcap_{p\in\mathbb{N}} \Omega(1/p, p)$. □

Since the low-frequency part does not impact the local regularity of $\{Z(t)\}_{t\in[0,1]}$, we will focus on the simulation of the Riemann-Liouville MPRE $\{X(t)\}_{t\in[0,1]}$ defined as

$$\forall t \in [0,1],\ X(t) := Z(t) - \dot{Z}(t) = \int_0^1 (t-s)_+^{H(s)-1/2}dB(s).$$

The Riemann-Liouville MPRE $\{X(t)\}_{t\in[0,1]}$ has been introduced in 2018 by Ayache, Esser and Hamonier in the article [AEH18]. We presented it in the subsection 2.3.B.a of our thesis.
Let us introduce the two following notations :

$$\forall u, v \in \mathbb{R},\ \mathcal{L}(u,v) := (u)_+^{v-1/2}$$

and

$$\forall t, s \in [0,1],\ K(t,s) := \mathcal{L}(t-s, H(s)) = (t-s)_+^{H(s)-1/2}. \tag{5.1.7}$$

Ayache, Esser and Hamonier decomposed the kernel (5.1.7) into the Haar basis of $L^2([0,1])$ which is composed of the functions

$$\mathcal{U} := \mathbb{1}_{[0,1)} \text{ and } h_{j,k} := 2^{j/2}\left(\mathbb{1}_{[2^{-j}k,2^{-j}(k+1/2))} - \mathbb{1}_{[2^{-j}(k+1/2),2^{-j}(k+1))}\right),\ j \in \mathbb{N} \text{ and } k \in \left\{0,\ldots,2^j-1\right\},$$

and they obtained a random series representation of $\left\{X(t)\right\}_{t\in[0,1]}$ under the following condition

$$(C_0) : \underline{H} > 1/2 \text{ and } \exists \rho > 1/2,\ \exists c > 0,\ \forall x, y \in [0,1],\ \mathbb{E}\left(|H(x)-H(y)|^2\right) \le c|x-y|^{2\rho}. \tag{5.1.8}$$

Observe that condition (C_0) and the Kolmogorov-Chentsov theorem imply the continuity of the paths of $\{H(s)\}_{s\in[0,1]}$.

Theorem 5.1.2. *Assume that condition (5.1.8) is satisfied. Then, there exists a universal event Ω_{**} of probability 1, such that*

$$\forall \omega \in \Omega_{**},\ X(t,\omega) = \langle K_t(\cdot,\omega),\mathcal{U}\rangle \eta_0(\omega) + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1} \langle K_t(\cdot,\omega), h_{j,k}\rangle \varepsilon_{j,k}(\omega), \tag{5.1.9}$$

where the convergence is uniform in t on $[0,1]$ and

$$\eta_0 := \int_0^1 \mathcal{U}(s)dB(s) = B(1) - B(0)$$

and

$$\varepsilon_{j,k} := \int_0^1 h_{j,k}(s)dB(s) = 2^{j/2}\left(2B(2^{-(j+1)}(2k+1)) - B(2^{-j}k) - B(2^{-j}(k+1))\right)$$

are Gaussian random variables of the same distribution $\mathcal{N}(0,1)$.

Using representation (5.1.9), they proposed a simulation method for X which consists in approximating for all fixed $t \in [0,1]$ the quantity $K(t,s)$ by $K(t,k2^{-J})$ for every s in the dyadic interval $[k2^{-J},(k+1)2^{-J})$. Roughly speaking, $K(t,s)$ is approximated by

$$\forall J \ge 1,\ \widetilde{K}_J(t,s) := \sum_{k=0}^{2^J-1}\left(K(t,k2^{-J})\mathbb{1}_{[k2^{-J},(k+1)2^{-J})}(s)\right),$$

then $X(t)$ is approximated by

$$\forall J \ge 1,\ \widetilde{X}_J(t) := \int_0^1 \widetilde{K}_J(t,s)dB(s) = \sum_{k=0}^{2^J-1}\left((t-k2^{-J})_+^{H(k2^{-J})-1/2}\left(B((k+1)2^{-J}) - B(k2^{-J})\right)\right).$$

They proved in [AEH18] the uniformly convergence of $\widetilde{X}_J$ to X on the interval $[0,1]$ when $J \to +\infty$.

Proposition 5.1.3. *Under the condition (C_0), there exists an event Ω_* of probability 1 such that*

$$\forall \omega \in \Omega_*, \quad \lim_{J \to +\infty} \left\{ \sup_{t \in [0,1]} \left| \widetilde{X}_J(t,\omega) - X(t,\omega) \right| \right\} = 0.$$

On another hand, for all fixed $t \in [0,1]$ they approximated the quantity $K(t,s) = \mathcal{L}(t-s, H(s))$ by the mean value of the function $s \mapsto \mathcal{L}(t-s, H(k2^{-J}))$ on each dyadic interval $[k2^{-J}, (k+1)2^{-J})$. Roughly speaking, $K(t,s)$ is approximated by

$$\forall J \geq 1, \ \widehat{K}_J(t,s) := \sum_{k=0}^{2^J-1} \left(2^J \int_{k2^{-J}}^{(k+1)2^{-J}} \mathcal{L}(t-u, H(k2^{-J}))du \right) \mathbb{1}_{[k2^{-J},(k+1)2^{-J})}(s), \tag{5.1.10}$$

then $X(t)$ is approximated by

$$\begin{aligned} \forall J \geq 1, \ \widehat{X}_J(t) &:= \int_0^1 \widehat{K}_J(t,s)dB(s) \\ &= \sum_{k=0}^{2^J-1} \left(2^J \int_{k2^{-J}}^{(k+1)2^{-J}} \mathcal{L}(t-u, H(k2^{-J}))du \right) \left(B((k+1)2^{-J}) - B(k2^{-J}) \right) \end{aligned} \tag{5.1.11}$$

where

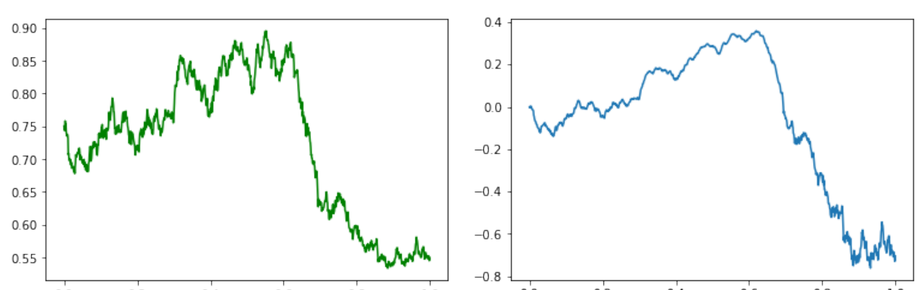
$$\begin{aligned} &2^J \int_{k2^{-J}}^{(k+1)2^{-J}} \mathcal{L}(t-u, H(k2^{-J}))du \\ &= \frac{2^J}{H(k2^{-J})+1/2} \left((t-k2^{-J})_+^{H(k2^{-J})+1/2} - (t-(k+1)2^{-J})_+^{H(k2^{-J})+1/2} \right). \end{aligned}$$

Observe that in contrast to the processes $\widetilde{X}_J$, the processes $\widehat{X}_J$ have continuous paths even if $\underline{H} < 1/2$.

Using Proposition 5.1.3, Ayache, Esser and Hamonier also proved the uniform convergence of $\widehat{X}_J$ to X on the interval $[0,1]$ when $J \to +\infty$.

Proposition 5.1.4. *Under the condition (C_0), there exists an event Ω_* of probability 1 such that*

$$\forall \omega \in \Omega_*, \quad \lim_{J \to +\infty} \left\{ \sup_{t \in [0,1]} \left| \widehat{X}_J(t,\omega) - X(t,\omega) \right| \right\} = 0.$$



A realization $H(\cdot,\omega)$ on the left and the associated path $X(\cdot,\omega)$ on the right. (Simulation obtained using Proposition 5.1.4)

5.2 Statement of the main result

The main result of this chapter shows that Proposition 5.1.4 remains true under the following condition (α) which is much weaker than the condition (C_0) given at (5.1.8). Moreover, we obtain a rate of convergence for the simulation method. Let $\mathbb{D}_J$ the set of dyadic numbers of order J defined as:

$$\forall J \geq 1,\ \mathbb{D}_J := \left\{k2^{-J} \text{ where } k \in \left\{0, \dots, 2^J - 1\right\}\right\} \subset [0,1)$$

and the quantity $\mu(J)$

$$\forall J \geq 1,\ \mu(J) := \sqrt{2^J \sup_{t\in\mathbb{D}_J} \int_0^{2^{-J}} |H(t) - H(t+u)|^2 du}. \tag{5.2.1}$$

The condition (α) is given by

$$(\alpha) : \exists \gamma > 0,\ \limsup_{J\to+\infty} 2^{\gamma J}\mu(J) = 0. \tag{5.2.2}$$

It is worth noticing that this condition (α) does not imply the continuity of the paths of H. For instance, the condition (α) is satisfied by all the the stochastic processes of the form

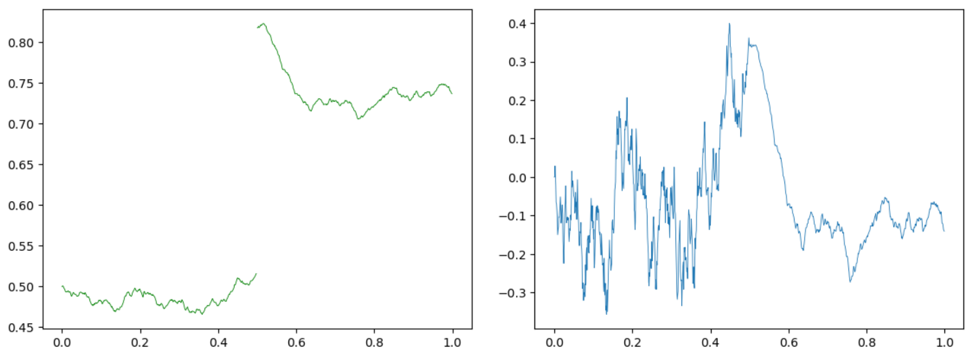
$$H(s,\omega) = H_0(s,\omega) + l_1(s,\omega) + l_2(s,\omega)$$

where:
- the paths of H_0 are Hölder functions of same deterministic order $\gamma > 0$,
- the paths of l_1 are right continuous step functions whose discontinuity points (which are in finite number) belong to $\mathbb{D} := \bigcup_{J>1} \mathbb{D}_J$,
- the paths of l_2 satisfy $\text{supp}(l_2) \cap \mathbb{D} = \emptyset$ and $\lambda(\text{supp}(l_2)) = 0$ where λ denotes the Lebesgue measure.

The simulation method relies on the following Theorem 5.2.1.

Theorem 5.2.1. *Under the condition (5.2.2), there exists an event Ω_* of probability 1 such that for all $\beta < \min(\gamma, \underline{H}, 1/2)$, one has*

$$\forall \omega \in \Omega_*,\ \lim_{J\to+\infty} \left\{2^{\beta J} \sup_{t\in[0,1]} \left|\widehat{X}_J(t,\omega) - X(t,\omega)\right|\right\} = 0.$$



A realization $H(\cdot,\omega)$ with discontinuity at $s = 1/2$ on the left and the associated path $X(\cdot,\omega)$ on the right. (Simulation obtained using Theorem 5.2.1)

In the following, one uses the convention that $\sum_{k=a}^{b} u_k$ is zero if $b < a$. Moreover, since $[\underline{H}, \overline{H}] \subseteq [\min(\underline{H}, 1/2), \overline{H}]$, we will always assume in the following proofs that $\underline{H} \leq 1/2$.

5.3 Proof for the dyadic indices

Lemma 5.3.1. *There is a deterministic constant $\widehat{c_1} > 0$ such that for all $J \geq 1$ and $\beta < \underline{H}$ one has almost surely*

$$\sup_{t\in\mathbb{D}_J} \int_0^1 \left|K(t,s) - \widehat{K}_J(t,s)\right|^2 ds \leq \widehat{c_1} \left(\mu(J)^2 + 2^{-2\beta J}\right). \tag{5.3.1}$$

Proof. Let us fix $J \geq 1$, $t \in \mathbb{D}_J \setminus \{0\}$ and introduce the integer $k_{t,J} := t2^J \geq 1$ such that $t = k_{t,J}2^{-J}$. Observe that, one has

$$\int_0^1 \left|K(t,s) - \widehat{K}_J(t,s)\right|^2 ds = \sum_{k=0}^{k_{t,J}-1} \sigma_J(t,k) \tag{5.3.2}$$

where

$$\begin{aligned}\sigma_J(t,k) &:= \int_{k2^{-J}}^{(k+1)2^{-J}} \left(\mathcal{L}(t-s, H(s)) - 2^J \int_{k2^{-J}}^{(k+1)2^{-J}} \mathcal{L}(t-u, H(k2^{-J}))du\right)^2 ds \\ &= \int_{k2^{-J}}^{(k+1)2^{-J}} 2^{2J} \left(\int_{k2^{-J}}^{(k+1)2^{-J}} \left((t-s)_+^{H(s)-1/2} - (t-u)_+^{H(k2^{-J})-1/2}\right) du\right)^2 ds. \end{aligned} \tag{5.3.3}$$

Using mean value theorem, one has for all $s, u \leq t$ and $\varepsilon > 0$

$$\begin{aligned}&\left|(t-s)^{H(s)-1/2} - (t-u)^{H(k2^{-J})-1/2}\right| \\ &\leq \left|(t-s)^{H(s)-1/2} - (t-s)^{H(k2^{-J})-1/2}\right| + \left|(t-s)^{H(k2^{-J})-1/2} - (t-u)^{H(k2^{-J})-1/2}\right| \\ &\leq c_1\Big(|H(k2^{-J}) - H(s)|(t-s)^{\underline{H}-1/2-\varepsilon} + |u-s| \min(t-s, t-u)^{H(k2^{-J})-3/2}\Big)\end{aligned} \tag{5.3.4}$$

where $c_1 > 0$ is a constant. Combining (5.3.2) and (5.3.3) one obtains

$$\int_0^1 \left|K(t,s) - \widehat{K}_J(t,s)\right|^2 ds \le 2\left(\lambda_1(t,J) + \lambda_2(t,J)\right) \tag{5.3.5}$$

where

$$\lambda_1(t,J) := \sum_{k=0}^{k_{t,J}-1} \int_{k2^{-J}}^{(k+1)2^{-J}} \left|(t-s)^{H(s)-1/2} - (t-s)^{H(k2^{-J})-1/2}\right|^2 ds$$

and

$$\lambda_2(t,J) := \sum_{k=0}^{k_{t,J}-1} \int_{k2^{-J}}^{(k+1)2^{-J}} 2^{2J} \left(\int_{k2^{-J}}^{(k+1)2^{-J}} \left|(t-s)^{H(k2^{-J})-1/2} - (t-u)^{H(k2^{-J})-1/2}\right| du\right)^2 ds. \tag{5.3.6}$$

Upper bound for $\lambda_1(t,J)$: Using mean value theorem, there is a constant $c_1 > 0$ such that for all $s < t$ and $\varepsilon \in (0, \underline{H}/2)$ one has

$$\left|(t-s)^{H(s)-1/2} - (t-s)^{H(k2^{-J})-1/2}\right| \le c_1 |H(k2^{-J}) - H(s)|(t-s)^{\underline{H}-1/2-\varepsilon}$$

thus

$$\lambda_1(t,J) \le c_1 \sum_{k=0}^{k_{t,J}-2} \int_{k2^{-J}}^{(k+1)2^{-J}} |H(k2^{-J}) - H(s)|^2 (t-s)^{2\underline{H}-1-2\varepsilon} ds + (2\overline{H})^2 \int_{t-2^{-J}}^{t} (t-s)^{2\underline{H}-1-2\varepsilon} ds \tag{5.3.7}$$

with

$$\int_{t-2^{-J}}^{t} (t-s)^{2\underline{H}-1-2\varepsilon} ds = \frac{1}{2\underline{H}-2\varepsilon} 2^{-2J(\underline{H}-\varepsilon)} \le \frac{1}{\underline{H}} 2^{-2J(\underline{H}-\varepsilon)}. \tag{5.3.8}$$

Using definition (5.2.1), one gets for some constant $c_2 \ge 0$

$$\begin{aligned} &\sum_{k=0}^{k_{t,J}-2} \int_{k2^{-J}}^{(k+1)2^{-J}} |H(k2^{-J}) - H(s)|^2 (t-s)^{2\underline{H}-1-2\varepsilon} ds \\ &\le \sum_{k=0}^{k_{t,J}-2} \left((t-(k+1)2^{-J})^{2\underline{H}-1-2\varepsilon} \int_{k2^{-J}}^{(k+1)2^{-J}} |H(k2^{-J}) - H(s)|^2 ds \right) \\ &\le c_2 2^{-J} \mu(J)^2 \int_1^{k_{t,J}} (t-u2^{-J})^{2\underline{H}-1-2\varepsilon} du \\ &= \frac{c_2}{2\underline{H}-2\varepsilon} \mu(J)^2 (t-2^{-J})^{2\underline{H}-2\varepsilon} \le \frac{c_2}{\underline{H}} \mu(J)^2. \end{aligned} \tag{5.3.9}$$

Combining (5.3.7), (5.3.8) and (5.3.9), one obtains for some constant $c_3 \ge 0$

$$\lambda_1(t,J) \le c_3 \left(2^{-2(\underline{H}-\varepsilon)J} + \mu(J)^2\right). \tag{5.3.10}$$

Upper bound for $\lambda_2(t,J)$: Using mean value theorem, there is a constant $c_1' > 0$ such that for all $s, u < t$ one has

$$\left|(t-s)^{H(k2^{-J})-1/2} - (t-u)^{H(k2^{-J})-1/2}\right| \leq c_1'|u-s| \min(t-s, t-u)^{H(k2^{-J})-3/2},$$

thus, for all $k \leq k_{t,J} - 3$, one has

$$\begin{aligned}
&\int_{k2^{-J}}^{(k+1)2^{-J}} 2^{2J} \left(\int_{k2^{-J}}^{(k+1)2^{-J}} |u-s| \min(t-s,t-u)^{H(k2^{-J})-3/2} du \right)^2 ds \\
&\leq \int_{k2^{-J}}^{(k+1)2^{-J}} \left(\int_{k2^{-J}}^{(k+1)2^{-J}} (t-(k+1)2^{-J})^{H(k2^{-J})-3/2} du \right)^2 ds \\
&\leq 2^{-3J}(t-(k+1)2^{-J})^{2\underline{H}-3}. \qquad (5.3.11)
\end{aligned}$$

Moreover, one gets

$$\begin{aligned}
\sum_{k=0}^{k_{t,J}-3} (t-(k+1)2^{-J})^{2\underline{H}-3} \leq \sum_{k=0}^{k_{t,J}-3} \int_{k+1}^{k+2} (t-s2^{-J})^{2\underline{H}-3} ds = \int_1^{k_{t,J}-1} (t-s2^{-J})^{2\underline{H}-3} ds \\
\leq \frac{1}{2-2\underline{H}} 2^{-J(2\underline{H}-3)}. \qquad (5.3.12)
\end{aligned}$$

Let us now focus on the terms where $k \in \{k_{t,J}-2, k_{t,J}-1\}$. Using triangle inequality and the inequality $(a+b)^2 \leq 2(a^2+b^2)$, one obtains for $\delta \in \{0,1\}$

$$\begin{aligned}
&\int_{t-(\delta+1)2^{-J}}^{t-\delta 2^{-J}} 2^{2J} \left(\int_{t-(\delta+1)2^{-J}}^{t-\delta 2^{-J}} \left|(t-s)^{H(t-2^{-J})-1/2} - (t-u)^{H(t-2^{-J})-1/2}\right| du \right)^2 ds \\
&\leq 2^{2J+1} \Bigg(\int_{t-(\delta+1)2^{-J}}^{t-\delta 2^{-J}} \left(\int_{t-(\delta+1)2^{-J}}^{t-\delta 2^{-J}} (t-s)^{H(t-2^{-J})-1/2} du \right)^2 ds \\
&+ \int_{t-(\delta+1)2^{-J}}^{t-\delta 2^{-J}} \left(\int_{t-(\delta+1)2^{-J}}^{t-\delta 2^{-J}} (t-u)^{H(t-2^{-J})-1/2} du \right)^2 ds \Bigg). \qquad (5.3.13)
\end{aligned}$$

Next, one has

$$\int_{t-(\delta+1)2^{-J}}^{t-\delta 2^{-J}} \left(\int_{t-(\delta+1)2^{-J}}^{t-\delta 2^{-J}} (t-s)^{H(t-2^{-J})-1/2} du \right)^2 ds \leq \frac{((\delta+1)2^{-J})^{2+2\underline{H}}}{2\underline{H}} \qquad (5.3.14)$$

and

$$\int_{t-(\delta+1)2^{-J}}^{t-\delta 2^{-J}} \left(\int_{t-(\delta+1)2^{-J}}^{t-\delta 2^{-J}} (t-u)^{H(t-2^{-J})-1/2} du \right)^2 ds \leq \frac{((\delta+1)2^{-J})^{2+2\underline{H}}}{(\underline{H}+1/2)^2}. \qquad (5.3.15)$$

Putting together (5.3.13), (5.3.14) and (5.3.15), one gets for some constant $c_4 > 0$

$$\int_{t-(\delta+1)2^{-J}}^{t-\delta 2^{-J}} 2^{2J} \left(\int_{t-(\delta+1)2^{-J}}^{t-\delta 2^{-J}} \left|(t-s)^{H(t-2^{-J})-1/2} - (t-u)^{H(t-2^{-J})-1/2}\right| du \right)^2 ds \leq c_4 2^{-J(2\underline{H})}. \qquad (5.3.16)$$

Then, putting together (5.3.6), (5.3.11), (5.3.12) and (5.3.16), one obtains for some constant $c_5 > 0$

$$\lambda_2(t,J) \le c_5 2^{-J(2\underline{H})}. \tag{5.3.17}$$

Finally, it follows from (5.3.5), (5.3.10) and (5.3.17), that one has for some constant $c_6 > 0$

$$\int_0^1 \left| K(t,s) - \widehat{K}_J(t,s) \right|^2 ds \le c_6 \left(2^{-2(\underline{H}-\varepsilon)J} + \mu(J)^2 \right).$$

□

Lemma 5.3.2. *For all $\beta < \min(\gamma, \underline{H})$, one has almost surely*

$$\lim_{J\to+\infty} \left\{ 2^{\beta J} \sup_{t\in\mathbb{D}_J} \left| X(t) - \widehat{X}_J(t) \right| \right\} = 0. \tag{5.3.18}$$

Proof. Let us fix a real number $\beta < \min(\gamma, \underline{H})$. Let us fix an integer p such that $p > \frac{1}{\min(\gamma,\underline{H})-\beta}$, a real number $\varepsilon > 0$ and $J \in \mathbb{N}$, one obtains using Markov's inequality

$$\begin{aligned}
\mathbb{P}\left(2^{\beta J} \sup_{t\in\mathbb{D}_J} \left| X(t) - \widehat{X}_J(t) \right| > \varepsilon \right) &\le \sum_{t\in\mathbb{D}_J} \mathbb{P}\left(2^{\beta J} \left| X(t) - \widehat{X}_J(t) \right| > \varepsilon \right) &(5.3.19)\\
&\le \varepsilon^{-p}\, 2^{p\beta J} \sum_{t\in\mathbb{D}_J} \mathbb{E}\left| X(t) - \widehat{X}_J(t) \right|^p &(5.3.20)\\
&\le 2^J \varepsilon^{-p}\, 2^{p\beta J} \sup_{t\in\mathbb{D}_J} \mathbb{E}\left| X(t) - \widehat{X}_J(t) \right|^p. &(5.3.21)
\end{aligned}$$

Now using Burkholder-Davis-Gundy inequality and Lemma 5.3.1 for $\beta' \in (\beta, \underline{H})$, one obtains for all $t \in \mathbb{D}_J$

$$\begin{aligned}
\mathbb{E}\left| X(t) - \widehat{X}_J(t) \right|^p &= \mathbb{E}\left(\left| \int_0^1 \left(K(t,s) - \widehat{K}_J(t,s) \right) dB(s) \right|^p \right)\\
&\le a(p)\mathbb{E}\left(\left(\int_0^1 \left| K(t,s) - \widehat{K}_J(t,s) \right|^2 ds \right)^{p/2} \right) &(5.3.22)\\
&\le a(p)\widehat{c_1}^{p/2} (\mu(J)^2 + 2^{-2\beta' J})^{p/2}.
\end{aligned}$$

Combining (5.3.19) and (5.3.22) one obtains

$$\mathbb{P}\left(2^{\beta J} \sup_{t\in\mathbb{D}_J} \left| X(t) - \widehat{X}_J(t) \right| > \varepsilon \right) \le \left(\varepsilon^{-p} a(p) \widehat{c_1}^{p/2} \right) 2^J 2^{p\beta J} (\mu(J)^2 + 2^{-2\beta' J})^{p/2}. \tag{5.3.23}$$

Using (5.3.23) and (5.2.2), one obtains

$$\sum_{J=1}^{+\infty} \mathbb{P}\left(2^{\beta J} \sup_{t\in\mathbb{D}_J} \left| X(t) - \widehat{X}_J(t) \right| > \varepsilon \right) < +\infty.$$

Using Borel-Cantelli lemma, for all $\varepsilon > 0$, there is an event $\Omega(\varepsilon)$ of probability 1 such that

$$\forall \omega \in \Omega(\varepsilon),\ \exists N(\omega,\varepsilon) \in \mathbb{N},\ \forall J \geq N(\omega,\varepsilon),\ 2^{\beta J} \sup_{t\in\mathbb{D}_J} \left| X(t,\omega) - \widehat{X}_J(t,\omega)\right| \leq \varepsilon.$$

Let us now introduce $\Omega' := \bigcap_{p\in\mathbb{N}} \Omega(1/p)$ which is still of probability 1 and such a way one has

$$\forall \omega \in \Omega',\ \lim_{J\to+\infty} \left\{ 2^{\beta J} \sup_{t\in\mathbb{D}_J} \left| X(t,\omega) - \widehat{X}_J(t,\omega)\right| \right\} = 0.$$

□

5.4 Proof of Theorem 5.2.1

Lemma 5.4.1. *There is a random variable $\widehat{C}$, such that, for all $t', t'' \in [0,1]$ and $J \in \mathbb{N}$, one has almost surely*

$$\left|\widehat{X}_J(t'') - \widehat{X}_J(t')\right| \leq \widehat{C}\sqrt{J+1}\left(2^{J(1-\underline{H})}|t''-t'| + 2^{J/2}|t''-t'|^{1/2+\underline{H}}\right). \tag{5.4.1}$$

Proof. Let us fix $t'', t' \in [0,1]$ and $J \geq 1$. Using definitions (5.1.10) and (5.1.11), one has

$$\widehat{X}_J(t'') - \widehat{X}_J(t') = \sum_{k=0}^{2^J-1} c_{J,k}(t'',t')\varepsilon_{J,k} \tag{5.4.2}$$

where

$$c_{J,k}(t'',t') := 2^{J/2}\left(\int_{k2^{-J}}^{(k+1)2^{-J}} \left((t''-u)_+^{H(k2^{-J})-1/2} - (t'-u)_+^{H(k2^{-J})-1/2}\right) du\right) \tag{5.4.3}$$

and

$$\varepsilon_{J,k} := 2^{J/2}(B((k+1)2^{-J}) - B(k2^{-J})) \overset{d}{=} 2^{J/2}B(2^{-J}) \overset{d}{=} \mathcal{N}(0,1). \tag{5.4.4}$$

Using Lemma A.23 in [Aya19], there is a random variable C_*, such that for all $J \geq 1$ and $k \in \{0,\ \ldots,\ 2^J-1\}$, one has almost surely

$$|\varepsilon_{J,k}| \leq C_*\sqrt{J+1}. \tag{5.4.5}$$

We can suppose without any restriction that $t'' > t'$. Let $k_{t',J}$ be the nonnegative integer defined as $k_{t',J} := \lfloor t'2^J \rfloor$. One sets

$$\mu_1(t'',t',J) := \sum_{k=0}^{k_{t',J}-2} |c_{J,k}(t'',t')|, \qquad \mu_2(t'',t',J) := \sum_{k=k_{t',J}-1}^{k_{t',J}} |c_{J,k}(t'',t')| \tag{5.4.6}$$

and

$$\mu_3(t'',t',J) := \sum_{k=k_{t',J}+1}^{2^J-1} |c_{J,k}(t'',t')|. \tag{5.4.7}$$

Upper bound for $\mu_1(t'', t', J)$: Applying the mean value theorem to the function

$$x \mapsto (x - k2^{-J})^{H(k2^{-J})+1/2} - (x - (k+1)2^{-J})^{H(k2^{-J})+1/2}$$

on the interval $[t', t'']$, one obtains for all $k \le k_{t',J} - 1$

$$\begin{aligned}
&\int_{k2^{-J}}^{(k+1)2^{-J}} \Big((t''-u)_+^{H(k2^{-J})-1/2} - (t'-u)_+^{H(k2^{-J})-1/2} \Big)\, du \\
&= \frac{1}{H(k2^{-J})+1/2} \Bigg((t'' - k2^{-J})^{H(k2^{-J})+1/2} - (t'' - (k+1)2^{-J})^{H(k2^{-J})+1/2} \\
&- (t' - k2^{-J})^{H(k2^{-J})+1/2} + (t' - (k+1)2^{-J})^{H(k2^{-J})+1/2} \Bigg) \\
&= (t''-t') \Bigg((c(t'',t') - k2^{-J})^{H(k2^{-J})-1/2} - (c(t'',t') - (k+1)2^{-J})^{H(k2^{-J})-1/2} \Bigg)
\end{aligned} \tag{5.4.8}$$

where $c(t'', t') \in (t', t'')$. Then applying the mean value theorem to the function

$$x \mapsto (c(t'', t') - x)^{H(k2^{-J})-1/2}$$

on the interval $[k2^{-J}, (k+1)2^{-J}]$, one obtains for $k \le k_{t',J} - 2$

$$\begin{aligned}
&(c(t'',t') - k2^{-J})^{H(k2^{-J})-1/2} - (c(t'',t') - (k+1)2^{-J})^{H(k2^{-J})-1/2} \\
&= 2^{-J}(H(k2^{-J}) - 1/2)\Big(c(t'',t') - c'(t'',t') \Big)^{H(k2^{-J})-3/2}
\end{aligned} \tag{5.4.9}$$

where $c'(t'', t') \in \ (k2^{-J}, (k+1)2^{-J})$. Next combining (5.4.3), (5.4.8), (5.4.9) and the fact that for all $k \le k_{t',J} - 2$ one has $c(t'', t') - c'(t'', t') \ge t' - (k+1)2^{-J}$, one obtains that

$$|c_{J,k}(t'', t')| \le 2^{-J/2}(t''-t')\Big(t' - (k+1)2^{-J}\Big)^{\underline{H}-3/2}.$$

Therefore one gets that

$$\sum_{k=0}^{k_{t',J}-3} |c_{J,k}(t'',t')| \le 2^{-J/2}(t''-t') \int_1^{k_{t',J}-1} \Big(t' - u2^{-J}\Big)^{\underline{H}-3/2} du \le 2^{J(1-\underline{H})}(t''-t') \tag{5.4.10}$$

and

$$|c_{J,k_{t',J}-2}(t'',t')| \le 2^{-J/2}(t''-t')(2^{-J})^{\underline{H}-3/2} = 2^{J(1-\underline{H})}(t''-t'). \tag{5.4.11}$$

Finally (5.4.10) and (5.4.11) entail that

$$\mu_1(t'', t', J) = \sum_{k=0}^{k_{t',J}-2} |c_{J,k}(t'',t')| \le 2 \times 2^{J(1-\underline{H})}(t''-t'). \tag{5.4.12}$$

Upper bound for $\mu_2(t'',t',J)$: Let us now focus on $c_{J,k_{t',J}-1}(t'',t')$. Using definition (5.4.3) and the change of variable $v=(t'-u)(t''-t')^{-1}$, one gets

$$\begin{aligned} c_{J,k_{t',J}-1}(t'',t') &= 2^{J/2}\int_{(k_{t',J}-1)2^{-J}}^{k_{t',J}2^{-J}} \left((t''-u)^{H((k_{t',J}-1)2^{-J})-1/2}-(t'-u)^{H((k_{t',J}-1)2^{-J})-1/2}\right)du \\ &= 2^{J/2}(t''-t')^{1/2+H((k_{t',J}-1)2^{-J})} \\ &\times \int_{(t'-D_J(t'))(t''-t')^{-1}}^{(t'-D_J(t')+2^{-J})(t''-t')^{-1}} \left((v+1)^{H((k_{t',J}-1)2^{-J})-1/2}-v^{H((k_{t',J}-1)2^{-J})-1/2}\right)dv \end{aligned} \tag{5.4.13}$$

where $D_J(t'):=\lfloor t'2^J\rfloor 2^{-J}=k_{t',J}2^{-J}$. If $H((k_{t',J}-1)2^{-J})\leq 1/2$ one obtains

$$|c_{J,k_{t',J}-1}(t'',t')|\leq 2^{J/2}(t''-t')^{1/2+\underline{H}}\int_0^{+\infty}\left|(v+1)^{\underline{H}-1/2}-v^{\underline{H}-1/2}\right|dv. \tag{5.4.14}$$

If $H((k_{t',J}-1)2^{-J})>1/2$, using (5.4.3) and (5.4.8), one obtains

$$\begin{aligned} &|c_{J,k_{t',J}-1}(t'',t')| \\ &\leq 2^{J/2}\Big((t''-t')(t'-D_J(t')+2^{-J})^{H((k_{t',J}-1)2^{-J})-1/2}+(t''-t')(t'-D_J(t'))^{H((k_{t',J}-1)2^{-J})-1/2}\Big) \\ &\leq 2^{J/2+1}(t''-t'). \end{aligned} \tag{5.4.15}$$

Thus, using (5.4.13), (5.4.14) and (5.4.15) one obtains for some constant $c_1\geq 0$

$$|c_{J,k_{t',J}-1}(t'',t')|\leq c_1\left(2^{J/2}(t''-t')^{1/2+\underline{H}}+2^{J/2}(t''-t')\right). \tag{5.4.16}$$

For the term $c_{J,k_{t',J}}(t'',t')$, one has

$$\begin{aligned} c_{J,k_{t',J}}(t'',t') &= 2^{J/2}\int_{D_J(t')}^{t'}\left((t''-u)^{H(D_J(t'))-1/2}-(t'-u)^{H(D_J(t'))-1/2}\right)du \\ &\quad + 2^{J/2}\int_{t'}^{D_J(t')+2^{-J}}(t''-u)_+^{H(D_J(t'))-1/2}du. \end{aligned} \tag{5.4.17}$$

One gets

$$\begin{aligned} &\left|\int_{D_J(t')}^{t'}\left((t''-u)^{H(D_J(t'))-1/2}-(t'-u)^{H(D_J(t'))-1/2}\right)du\right| \\ &\leq \frac{1}{\underline{H}+1/2}\left(\left|(t''-D_J(t'))^{H(D_J(t'))+1/2}-(t'-D_J(t'))^{H(D_J(t'))+1/2}\right|+(t''-t')^{\underline{H}+1/2}\right). \end{aligned} \tag{5.4.18}$$

If $H(D_J(t'))>1/2$, using mean value theorem one obtains

$$\begin{aligned} \left|(t''-D_J(t'))^{H(D_J(t'))+1/2}-(t'-D_J(t'))^{H(D_J(t'))+1/2}\right| &\leq \frac{3}{2}(t''-t')(t''-D_J(t'))^{H(D_J(t'))-1/2} \\ &\leq \frac{3}{2}(t''-t'). \end{aligned} \tag{5.4.19}$$

Else if $H(D_J(t')) \leq 1/2$, using the inequality $|a^\beta - b^\beta| \leq |a-b|^\beta$ where $\beta \in (0,1)$ and $a, b \in (0, +\infty)$ one obtains

$$\Big|(t'' - D_J(t'))^{H(D_J(t'))+1/2} - (t' - D_J(t'))^{H(D_J(t'))+1/2}\Big| \leq (t''-t')^{\underline{H}+1/2} \tag{5.4.20}$$

Moreover, one has

$$\int_{t'}^{D_J(t')+2^{-J}} (t''-u)_+^{H(D_J(t'))-1/2} du \leq \int_{t'}^{t''} (t''-u)^{H(D_J(t'))-1/2} du \leq \frac{1}{\underline{H}+1/2}(t''-t')^{\underline{H}+1/2}. \tag{5.4.21}$$

Combining (5.4.17), (5.4.18), (5.4.19), (5.4.20) and (5.4.21), there is a constant $c_2 \geq 0$ such that

$$|c_{J,k_{t',J}}(t'',t')| \leq c_2 \left(2^{J/2}(t''-t')^{\underline{H}+1/2} + 2^{J/2}(t''-t')\right). \tag{5.4.22}$$

Combining (5.4.16) and (5.4.22), one obtains

$$\mu_2(t'',t',J) \leq (c_1+c_2)\left(2^{J/2}(t''-t')^{\underline{H}+1/2} + 2^{J/2}(t''-t')\right). \tag{5.4.23}$$

Upper bound for $\mu_3(t'',t',J)$: Using definition (5.4.3), one gets

$$\begin{aligned}\mu_3(t'',t',J) &= \sum_{k=k_{t',J}+1}^{2^J-1} |c_{J,k}(t'',t')| = 2^{J/2} \sum_{k=k_{t',J}+1}^{2^J-1} \int_{k2^{-J}}^{(k+1)2^{-J}} (t''-u)_+^{H(k2^{-J})-1/2} du \\ &\leq 2^{J/2} \int_{\min(D_J(t')+2^{-J},t'')}^{t''} (t''-u)^{\underline{H}-1/2} du \leq 2^{J/2+1}(t''-t')^{\underline{H}+1/2}.\end{aligned} \tag{5.4.24}$$

Combining (5.4.12), (5.4.23) and (5.4.24), there is a constant $c_3 \geq 0$ such that

$$\sum_{k=0}^{2^J-1} |c_{J,k}(t'',t')| \leq c_3\left(2^{J/2}(t''-t')^{\underline{H}+1/2} + 2^{J(1-\underline{H})}(t''-t') + 2^{J/2}(t''-t')\right). \tag{5.4.25}$$

Using (5.4.1), (5.4.5) and (5.4.25), one gets

$$|\widehat{X}_J(t'') - \widehat{X}_J(t')| \leq C_* c_3 \sqrt{J+1}\left(2^{J/2}(t''-t')^{\underline{H}+1/2} + (t''-t')2^{J(1-\underline{H})} + 2^{J/2}(t''-t')\right) \tag{5.4.26}$$

and the lemma is proved. □

Lemma 5.4.2. *There is a universal event Ω' of probability 1 such that, for all $\beta \in (0, \underline{H})$, there is a random variable C such that, for all t', t'' in $[0,1]$ and $\omega \in \Omega'$ one has*

$$\Big|X(t'',\omega) - X(t',\omega)\Big| \leq C(\omega)|t''-t'|^\beta. \tag{5.4.27}$$

Proof. For all $t'', t' \in [0,1]$ such that $t' < t''$, one has

$$\int_0^1 \big|K(t'',s) - K(t',s)\big|^2 ds = \int_0^1 \Big((t''-s)_+^{H(s)-1/2} - (t'-s)_+^{H(s)-1/2}\Big)^2 ds$$
$$= \int_{t'}^{t''} (t''-s)^{2H(s)-1} ds + \int_0^{t'} \Big((t''-t'+v)^{H(t'-v)-1/2} - v^{H(t'-v)-1/2}\Big)^2 dv. \tag{5.4.28}$$

One has

$$\int_{t'}^{t''} (t''-s)^{2H(s)-1} ds \le \int_{t'}^{t''} (t''-s)^{2\underline{H}-1} ds = \frac{1}{2\underline{H}}(t''-t')^{2\underline{H}} \tag{5.4.29}$$

and using the change of variable $u := v(t''-t')^{-1}$, one gets

$$\int_0^{t'} \Big((t''-t'+v)^{H(t'-v)-1/2} - v^{H(t'-v)-1/2}\Big)^2 dv$$
$$\le (t''-t')^{2\underline{H}} \int_0^{t'(t''-t')^{-1}} \Big((u+1)^{H(t'-(t''-t')u)-1/2} - u^{H(t'-(t''-t')u)-1/2}\Big)^2 du. \tag{5.4.30}$$

Next, using the mean value theorem, one obtains

$$\int_1^{+\infty} \Big((u+1)^{H(t'-(t''-t')u)-1/2} - u^{H(t'-(t''-t')u)-1/2}\Big)^2 du \le \int_1^{+\infty} u^{2\underline{H}-3} du. \tag{5.4.31}$$

On the other hand, using the inequality $|a^\beta - b^\beta| \le |a-b|^\beta$ where $\beta \in (0,1)$ and $a, b \in (0,+\infty)$ one gets

$$\int_0^1 \Big((u+1)^{H(t'-(t''-t')u)-1/2} - u^{H(t'-(t''-t')u)-1/2}\Big)^2 du \le 1. \tag{5.4.32}$$

Combining (5.4.28), (5.4.29), (5.4.30), (5.4.31) and (5.4.32), there is a constant $c_1 \ge 0$ such that

$$\int_0^1 \big|K(t'',s) - K(t',s)\big|^2 ds \le c_1 (t''-t')^{2\underline{H}}. \tag{5.4.33}$$

Now using Burkholder-Davis-Gundy inequality, one obtains for all $t, t' \in [0,1]$ and $p > \frac{1}{\underline{H}}$

$$\mathbb{E}\Big|X(t'') - X(t')\Big|^p = \mathbb{E}\bigg(\Big|\int_0^1 \big(K(t'',s) - K(t',s)\big) dB(s)\Big|^p\bigg)$$
$$\le a(p)\mathbb{E}\bigg(\Big(\int_0^1 \big|K(t'',s) - K(t',s)\big|^2 ds\Big)^{p/2}\bigg)$$
$$\le a(p) {c_1}^{p/2} |t''-t'|^{p\underline{H}}. \tag{5.4.34}$$

Using Kolmogorov-Chentsov theorem, there is an event $\Omega(p)$ of probability 1, such that the paths of the process $\{X(t)\}_{t\in[0,1]}$ are Hölder continuous functions of order $\dfrac{p\underline{H}-1}{p}$. The lemma is proved for $\Omega' := \bigcap_{p=\lfloor \underline{H}^{-1}\rfloor+1}^{+\infty} \Omega(p)$. □

We are now in position to prove the Theorem 5.2.1.

Proof of Theorem 5.2.1. For all $J \geq 1$, one has

$$\sup_{t\in[0,1]} \left| X(t) - \widehat{X}_J(t) \right| \leq \sup_{t\in\mathbb{D}_J} \left| X(t) - \widehat{X}_J(t) \right| + \sup_{t\in(0,1)} \left| \widehat{X}_J(t) - \widehat{X}_J(D_J(t)) \right| + \sup_{t\in(0,1)} \left| X(t) - X(D_J(t)) \right| \tag{5.4.35}$$

where $D_J(t) := \lfloor t2^J \rfloor 2^{-J}$. Since $|t - D_J(t)| < 2^{-J}$ for all $t \in (0,1)$, the theorem is now a consequence of Lemma 5.3.2, Lemma 5.4.1 and Lemma 5.4.2.

□

Chapter 6

Uniformly and strongly consistent estimation for the random Hurst function of a multifractional process

This chapter is a restatement of the preprint [AB].

6.1 Introduction and background

Throughout the article the underlying probability space is denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. It is endowed with a complete filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$, and $\{B(s)\}_{s\in\mathbb{R}}$ is a standard Brownian Motion with respect to this filtration. fractional Brownian motion (FBM) of constant Hurst parameter $H \in (0,1)$, denoted by $\{B_H(t)\}_{t\in\mathbb{R}}$, is a very classical centred self-similar Gaussian process with stationary increments. It has continuous (sample) paths and it can be defined, for all $t \in \mathbb{R}$, through the non-anticipative moving average Wiener integral:

$$B_H(t) := \int_{-\infty}^{t} \left((t-s)^{H-1/2} - (-s)_+^{H-1/2} \right) dB(s)\,, \tag{6.1.1}$$

with the convention that, for each $(y,b) \in \mathbb{R}^2$, one has

$$y_+^b := \begin{cases} y^b, \text{if } y > 0, \\ 0, \text{else.} \end{cases} \tag{6.1.2}$$

One refers to the two well-known books [EM02, ST94] for a detailed presentation of FBM and many other related topics. FBM is a widespread model in signal processing (see e.g. [DOT03]). Unfortunately, in many situations, it does not fit very well to modeling of erratic real-life signals since it lacks of flexibility. An important drawback of FBM model comes from the fact that local roughness of its paths is not allowed to change from point to point. More precisely, for a generic stochastic process $Y = \{Y(t)\}_{t\in\mathbb{R}}$ with continuous and nowhere differentiable paths, their roughness in a neighborhood of any arbitrary fixed point $\tau \in \mathbb{R}$ is usually measured through $\alpha_Y(\tau)$ and $\widetilde{\alpha}_Y(\tau)$, the pointwise Hölder exponent and local Hölder exponent of Y at τ, defined, for all $\omega \in \Omega$, as:

$$\alpha_Y(\tau,\omega) := \sup\left\{ a \in [0,1] : \limsup_{t\to\tau} \frac{|Y(t,\omega) - Y(\tau,\omega)|}{|t-\tau|^a} < +\infty \right\} \tag{6.1.3}$$

and

$$\widetilde{\alpha}_Y(\tau,\omega) := \sup\left\{\widetilde{a}\in[0,1] : \limsup_{(t',t'')\to(\tau,\tau)} \frac{|Y(t',\omega)-Y(t'',\omega)|}{|t'-t''|^{\widetilde{a}}} < +\infty\right\}. \tag{6.1.4}$$

For a given $\omega\in\Omega$, the more close to zero are $\alpha_Y(\tau,\omega)$ and $\widetilde{\alpha}_Y(\tau,\omega)$, the more rough is the path $t\mapsto Y(t,\omega)$ in the vicinity of τ. In the case of FBM $\{B_H(t)\}_{t\in\mathbb{R}}$, path roughness remains everywhere the same because one has (see for instance [Xia97])

$$\mathbb{P}\big(\forall\,\tau\in\mathbb{R}\,,\ \alpha_{B_H}(\tau)=\widetilde{\alpha}_{B_H}(\tau)=H\big)=1\,. \tag{6.1.5}$$

In order to overcome the latter limitation of FBM, multifractional processes have started to be constructed and studied since the mid 1990s. One refers to the recent book [Aya19] for a detailed presentation of such processes and their connections to wavelet methods. The paradigmatic example of them is the Classical Multifractional Brownian motion (CMBM) which was introduced independently in the two pioneering articles [BJR97, PL95]. CMBM is a Gaussian process with non-stationary increments and continuous paths. It is obtained by replacing the constant Hurst parameter H in a stochastic integral representation of FBM (as for instance the moving average representation (6.1.1)) by a deterministic continuous function $t\mapsto H(t)$, with values in an arbitrary compact interval $[\underline{H},\overline{H}]\subset(0,1)$, which depends on the time variable t, that is the index of the process. The latter function is called the Hurst function. Under a local Hölder condition on it, the articles [BJR97, PL95] have shown that this deterministic function can be used for prescribing local path roughness of CMBM paths which is thus allowed to change from point to point in a deterministic way. Namely, for any point $\tau\in\mathbb{R}$ at which the local Hölder exponent $\widetilde{\alpha}_H(\tau)$ of the function $t\mapsto H(t)$ satisfies the inequality

$$H(\tau)<\widetilde{\alpha}_H(\tau)\,, \tag{6.1.6}$$

one has, almost surely,

$$\alpha_{\mathrm{CMBM}}(\tau)=\widetilde{\alpha}_{\mathrm{CMBM}}(\tau)=H(\tau)\,, \tag{6.1.7}$$

where $\alpha_{\mathrm{CMBM}}(\tau)$ and $\widetilde{\alpha}_{\mathrm{CMBM}}(\tau)$ are the pointwise Hölder exponent and the local Hölder exponent of the CMBM at τ. Even if the Gaussian CMBM is a more flexible model than FBM, it still has some limitations, a major one of them is that the two exponents $\alpha_{\mathrm{CMBM}}(\tau)$ and $\widetilde{\alpha}_{\mathrm{CMBM}}(\tau)$ are deterministic quantities since the Hurst function $t\mapsto H(t)$ itself is deterministic. The difficulty for overcoming the latter limitation of the CMBM comes from the fact that one can not replace the Hurst parameter H in (6.1.1), or in another stochastic integral representation of FBM, by a random variable $H(t)$ without imposing to it to be (stochastically) independent of the Brownian Motion $\{B(s)\}_{s\in\mathbb{R}}$ generating the stochastic integral. Indeed, when this very restrictive independence condition is dropped, then the stochastic integral, in which H is substituted by $H(t)$, is no longer well-defined. Therefore, the two articles [AT05, AJT07] have proposed to use a random wavelet series representation of FBM, instead of a stochastic integral representation of it, in order to be allowed to make this substitution. The multifactional process with random Hurst function obtained in this way is called, in our present article, the Wavelet Multifractional Process with Random Exponent (WMPRE). It is a non-Gaussian process with non-stationary increments and continuous paths. Thanks to wavelet methods, the paper [AT05] has shown that, under the condition (6.1.6), the two fundamental equalities (6.1.7), relating $H(\cdot)$ to local path roughness, can be extended to the WMPRE. Moreover, the latter result has been significantly strengthened in the paper [AJT07] in which it has been established that these two fundamental equalities are even valid on a universal event of probability 1 not depending on τ, and for a much more general class of multifractional processes.

It worth mentioning that several articles (see for instance [BPP12, BPP13, BP14]) have pointed out that multifractional processes with random Hurst function are good candidates for modeling of financial time series. Indeed, they allow to replicate main stylized facts (non-Gaussianity, volatility clustering and so on) of such time series. Moreover, analysis of evolution over time of their random pointwise and local Hölder exponents can provide explanations for trading mechanisms over financial markets. For instance, at a given time one or the other of these two exponents can be viewed as a weight that investors assign to past prices in taking their trading decisions.

As we have already mentioned, there are significant difficulties in construction and study of multifractional processes with random Hurst functions. Even if the WMPRE, constructed a long time ago in [AT05], is a first breakthrough in this area, it does not at all clear how this process can be represented via Itô integral and how Itô calculus can be applied in its framework. In the last few years, another type of non-Gaussian multifractional process with random Hurst function having a natural representation via Itô intergral was introduced in [AEH18]. It has non-stationary increments and continuous paths. It is called the Moving Average Multifractional Process with Random Exponent (MAMPRE) in our present article. In contrast with the WMPRE for which the random Hurst function depends on the time variable t, in the case of the MAMPRE this function depends on the integration variable s. Indeed, the MAMPRE, denoted by $\{X(t)\}_{t\in\mathbb{R}}$, is obtained by substituting to the constant Hurst parameter H in (6.1.1) a stochastic process $\{H(s)\}_{s\in\mathbb{R}}$ with continuous paths, indexed by the integration variable s, which is adapted to the filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ and satisfies

$$0 < \underline{H} \le H(s) \le \overline{H} < 1\,, \quad \text{for all } s \in \mathbb{R}, \tag{6.1.8}$$

for some deterministic constants $\underline{H}$ and $\overline{H}$ belonging to the open interval $(0, 1)$. More formally, the MAMPRE $\{X(t)\}_{t\in\mathbb{R}}$ is defined, for all $t \in \mathbb{R}$, as the Itô integral:

$$X(t) := \int_{-\infty}^{t} \Big((t-s)^{H(s)-1/2} - (-s)_+^{H(s)-1/2} \Big)\, dB(s)\,. \tag{6.1.9}$$

Recently, in the article [LMS21], under a very weak global regularity condition on paths of the process $\{H(s)\}_{s\in\mathbb{R}}$, for all $\tau \in \mathbb{R}$, the two fundamental equalities (6.1.7), relating $H(\cdot)$ to local path roughness, have been extended to MAMPRE. A short time later, the article [AB22a] has shown that they are even valid on a universal event of probability 1 not depending on τ, as soon as paths of $\{H(s)\}_{s\in\mathbb{R}}$ are, on each compact interval, Hölder functions of any arbitrarily small deterministic order $\gamma > 0$. Notice that in contrast with the condition (6.1.6) there is no need to impose to γ to be greater than $H(\tau)$.

As we have already emphasized, local roughness of paths of a multifractional process is governed by its deterministic or random Hurst function $H(\cdot)$. For that reason statistical estimation of values of $H(\cdot)$ is an important issue both form a practical point of view and from a theoretical one. Many articles have dealt with this issue in the case where $H(\cdot)$ is deterministic (see e.g. [Dan20, LP17, AH17, AH15, BS13b, LAM$^+$11, LG13, Coe05, Coe06, Lac04, ALV04, BBCI00, BCI98]). However, statistical estimation of $H(\cdot)$ when it is random remains an open problem. A major difficulty in it is that few information is available on finite-dimensional distributions of multifractional process with random exponent. Another one is that the dependence structure of such a process is very complex. The main goal of our present article is to propose a solution for this problem in the framework of the MAMPRE $\{X(t)\}_{t\in\mathbb{R}}$, defined through (6.1.9), under a weak local Hölder condition on paths of the stochastic process $\{H(s)\}_{s\in\mathbb{R}}$.

Let us describe in a more precise way the main contribution of our present article. Similarly to the previous literature on statistical estimation of $H(\cdot)$, we assume that, on the interval $[0,1]$, the discrete realization: $\big\{X(k/N) : k \in \{0,\dots,N\}\big\}$ of the MAMPRE $\{X(t)\}_{t\in\mathbb{R}}$ is available for all integer N large enough; notice that our main result can be extended without great difficulty to the general case where the interval $[0,1]$ is replaced by any other compact interval with non-empty interior. Also, we suppose that, for some deterministic constants $\gamma \in (0,1)$ and $\rho \in (0,+\infty)$ paths of $\{H(s)\}_{s\in\mathbb{R}}$ satisfy, on the interval $[-1,1]$, the Hölder condition:

$$\big|H(s') - H(s'')\big| \le \rho\,|s' - s''|^{\gamma}\,, \quad \text{for all } (s',s'') \in [-1,1]^2. \tag{6.1.10}$$

Then, we construct from generalized quadratic variations associated with $\big\{X(k/N) : k \in \{0,\dots,N\}\big\}$ and $\big\{X(k/QN) : k \in \{0,\dots,QN\}\big\}$, the integer $Q \ge 2$ being arbitrary and fixed, a continuous piecewise linear random function on $[0,1]$, denoted by $\widetilde{H}_N(\cdot)$, which provides a uniformly and strongly consistent estimator of the whole random Hurst function $H(\cdot)$ on $[0,1]$. More precisely we show that, when N goes to $+\infty$, the uniform norm $\sup_{s\in[0,1]}\big|H(s) - \widetilde{H}_N(s)\big|$ converges almost surely to zero at the rate $N^{-\beta}$, where the positive exponent β belongs to some known interval. It is worth noticing that such kind of strong consistency result in uniform norm is rather unusual in literature on statistical estimation of functions.

Remark 6.1.1. It might seem restrictive to impose to the positive constant ρ, in the Hölder condition (6.1.10), to be deterministic. In fact, thanks to a localization procedure via stopping times (see for instance Section 4.4.1 in [JP12]) which is explained in the setting of MAMPRE in Section 3 of [LMS21], Theorem 6.2.2 (our main result) remains valid when ρ is an almost surely finite random variable. Moreover, a careful inspection of the proof of this same theorem shows that it also remains valid when the interval $[-1,1]$ in (6.1.10) is replaced by any other compact interval of the form $[-b,1]$, where b is a fixed arbitrarily small positive real number.

The remaining of our article is organized as follows. In section 6.2, the way of construction via generalized quadratic variations of X of the estimator $\widetilde{H}_N(\cdot)$ is precisely explained, our main result is stated, and $\widetilde{H}_N(\cdot)$ is tested on simulated data. In section 6.3, basically it is shown that generalized quadratic variations of X can be simplified since some parts of them are negligible for our purpose. The goal of section 6.4 is to precisely determine their asymptotic behavior when N goes to $+\infty$. At last, section 6.5 is devoted to complete the proof of our main result.

6.2 Statement of the main result and simulations

In order to state our main result, first we need to introduce several notations. From now till the end of our article, the integer $L \ge 2$ is arbitrary and fixed. The coefficients $a_0, a_1, \dots, a_L$ are defined, for every $l \in \{0,\dots,L\}$, as:

$$a_l := (-1)^{L-l}\binom{L}{l} := (-1)^{L-l}\frac{L!}{l!\,(L-l)!}\,. \tag{6.2.1}$$

Observe that one can derive from (6.2.1) that the finite sequence of real numbers $(a_l)_{0\le l\le L}$ has exactly L vanishing first moments; that is, for all $q \in \{0,\dots,L-1\}$, one has

$$\sum_{l=0}^{L} l^q a_l = 0 \quad \text{(with the convention } 0^0 = 1\text{), and} \quad \sum_{l=0}^{L} l^L a_l \neq 0. \tag{6.2.2}$$

For each integer N large enough, the estimator $\{\widetilde{H}_N(s)\}_{s\in[0,1]}$ for paths of the stochastic process $\{H(s)\}_{s\in[0,1]}$ is built from generalized quadratic variations of the MAMPRE X (see (6.1.9)) associated with its generalized increments $d_{N,k}$, $0\le k\le N-L$, defined, for all $k\in\{0,\dots,N-L\}$, as:

$$d_{N,k} = \sum_{l=0}^{L} a_l X\big((k+l)/N\big). \tag{6.2.3}$$

For any compact interval, with non-empty interior, $I \subseteq [0,1]$, the generalized quadratic variation of X on I is denoted by $V_N(I)$ and defined as the empirical mean:

$$V_N(I) := |\nu_N(I)|^{-1} \sum_{k\in\nu_N(I)} |d_{N,k}|^2\,, \tag{6.2.4}$$

where the finite set of indices

$$\nu_N(I) := \big\{k \in \{0,\dots,N-L\} : k/N \in I\big\}, \tag{6.2.5}$$

and $|\nu_N(I)|$ is the cardinality of $\nu_N(I)$. Observe that $|\nu_N(I)|$ does not really depend on the position of I, but mainly on $\lambda(I)$, the Lebesgue measure of this interval. Indeed, it can easily be seen that one has

$$N\lambda(I) - L - 1 < |\nu_N(I)| \le N\lambda(I) + 1; \tag{6.2.6}$$

thus, as soon as $N \ge 2(L+1)\lambda(I)^{-1}$, one gets that

$$N\lambda(I)/2 < |\nu_N(I)| \le 7N\lambda(I)/6. \tag{6.2.7}$$

Before giving a formal definition of the estimator $\{\widetilde{H}_N(s)\}_{s\in[0,1]}$, let us explain, in a few sentences, its way of construction. Let $(\theta_N)_N$ be an arbitrary sequence of real numbers on the interval $(0,1/2]$ which converges to zero at a convenient rate (see (6.2.10) and (6.2.11)), when N goes to $+\infty$. One splits the interval $[0,1]$ into a finite sequence $(\mathcal{I}_{N,n})_{0\le n<\lfloor\theta_N^{-1}\rfloor}$ of compact subintervals with the same length θ_N (except the last one $\mathcal{I}_{N,\lfloor\theta_N^{-1}\rfloor-1}$ having a length lying between θ_N and $2\theta_N$), where $\lfloor\theta_N^{-1}\rfloor$ is the integer part of θ_N^{-1}. Then, for any fixed integer $Q\ge 2$, the estimator $\{\widetilde{H}_N(s)\}_{s\in[0,1]} = \{\widetilde{H}^Q_{N,\theta_N}(s))\}_{s\in[0,1]}$ is obtained as the linear interpolation between the $\lfloor\theta_N^{-1}\rfloor+1$ random points having the coordinates

$$\big(0,\widehat{H}^Q_N(\mathcal{I}_{N,0})\big),\dots,\big((\lfloor\theta_N^{-1}\rfloor-1)\theta_N,\widehat{H}^Q_N(\mathcal{I}_{N,\lfloor\theta_N^{-1}\rfloor-1})\big),\big(1,\widehat{H}^Q_N(\mathcal{I}_{N,\lfloor\theta_N^{-1}\rfloor-1})\big),$$

where, for all $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$,

$$\widehat{H}^Q_N(\mathcal{I}_{N,n}) := \min\left\{\max\left\{\log_{Q^2}\left(\frac{V_N(\mathcal{I}_{N,n})}{V_{QN}(\mathcal{I}_{N,n})}\right),0\right\},1\right\}\,. \tag{6.2.8}$$

Notice that, for every $x \in (0,+\infty)$, $\log_{Q^2}(x) := \log(x)/\log(Q^2)$, with the convention that log is the Napierian logarithm. Also, notice that the ordinate of the last point is assumed to be the same as that of the previous one. This weak assumption comes from the fact that the set of the indices t of the process X has been restricted to the interval $[0,1]$; it does not significantly alter the results on the estimation of $H(\cdot)$ on this interval. Let us now define the estimator $\widetilde{H}_N(\cdot) = \widetilde{H}^Q_{N,\theta_N}(\cdot)$ in a formal and very precise way.

Definition 6.2.1. Assume that the integer $L \geq 2$ is arbitrary and fixed, and that the integer N_0 is defined as

$$N_0 := \min\left\{N \in \mathbb{N} : 0 < 9(L+1)\,N^{-1}(\log N)^2 \leq 1\right\}. \tag{6.2.9}$$

Observe that (6.2.9) implies that $N_0 > 9(L+1) \geq 27$. Let $(\theta_N)_{N\geq N_0}$ be an arbitrary sequence of real numbers belonging to the interval $(0,1/2]$ and satisfying, for all integer $N \geq N_0$,

$$\theta_N \leq \kappa\, N^{-\mu} \tag{6.2.10}$$

and

$$\theta_N \geq \kappa' N^{-\mu'} + 4(L+1)\,N^{-1}(\log N)^2\,, \tag{6.2.11}$$

where $\kappa > 0$, $\mu \in (0,1)$, $\kappa' > 0$ and $\mu' \in [\mu,1)$ are four constants not depending on N. For each $n \in \left\{0,\ldots,\lfloor\theta_N^{-1}\rfloor - 1\right\}$, we denote by $\mathcal{I}_{N,n}$ the compact subinterval of $[0,1]$, defined as:

$$\mathcal{I}_{N,n} := \left[n\theta_N,(n+1)\theta_N\right] \text{ when } n < \lfloor\theta_N^{-1}\rfloor - 1, \text{ and } \mathcal{I}_{N,\lfloor\theta_N^{-1}\rfloor-1} := \left[(\lfloor\theta_N^{-1}\rfloor - 1)\theta_N, 1\right]. \tag{6.2.12}$$

Observe that it follows from (6.2.7), (6.2.12) and (6.2.11) that, for all integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0,\ldots\lfloor\theta_N^{-1}\rfloor - 1\}$, the cardinality $|\nu_{QN}(\mathcal{I}_{N,n})|$ of $\nu_{QN}(\mathcal{I}_{N,n})$ satisfies

$$QN\theta_N/2 < |\nu_{QN}(\mathcal{I}_{N,n})| \leq 7QN\theta_N/3\,, \tag{6.2.13}$$

which in particular implies that $\nu_{QN}(\mathcal{I}_{N,n})$ is non-empty. At last, for all fixed integer $Q \geq 2$ and for every integer $N \geq N_0$, we denote by $\{\widetilde{H}^Q_{N,\theta_N}(s)\}_{s\in[0,1]}$ the stochastic process with continuous piecewise linear paths, defined as:

$$\widetilde{H}^Q_{N,\theta_N}(s) := \widehat{H}^Q_N\big(\mathcal{I}_{N,\lfloor\theta_N^{-1}\rfloor-1}\big), \text{ for all } s \in \mathcal{I}_{N,[\theta_N^{-1}]-1}, \tag{6.2.14}$$

and, for every $n \in \{0,\ldots,[\theta_N^{-1}] - 2\}$ and $s \in \mathcal{I}_{N,n}$, as:

$$\widetilde{H}^Q_{N,\theta_N}(s) := \big(1 - \theta_N^{-1}(s - n\theta_N)\big)\widehat{H}^Q_N\big(\mathcal{I}_{N,n}\big) + \theta_N^{-1}(s - n\theta_N)\widehat{H}^Q_N\big(\mathcal{I}_{N,n+1}\big)\,, \tag{6.2.15}$$

where $\widehat{H}^Q_N\big(\mathcal{I}_{N,n}\big)$ is defined through (6.2.8) for all $n \in \{0,\ldots,[\theta_N^{-1}] - 1\}$.

Let us now state the main result of our article.

Theorem 6.2.2. *Assume that the conditions (6.1.8), (6.1.10), (6.2.10) and (6.2.11) hold. Let β be an arbitrary real number satisfying*

$$0 < \beta < \min\left\{\gamma\big((1-\delta)\wedge\mu\big),\, \delta(L - \overline{H}) + \underline{H} - \overline{H},\, 2^{-1}(1-\mu')\right\},$$

where $L \geq 2$ is as in (6.2.2), and where δ is an arbitrary fixed real number such that

$$\frac{\overline{H} - \underline{H}}{L - \overline{H}} < \delta < 1\,.$$

Then, one has almost surely, for all $Q \in \mathbb{N}$,

$$\lim_{N\to+\infty}\left\{N^\beta \sup_{s\in[0,1]}\left|H(s) - \widetilde{H}^Q_{N,\theta_N}(s)\right|\right\} = 0\,. \tag{6.2.16}$$

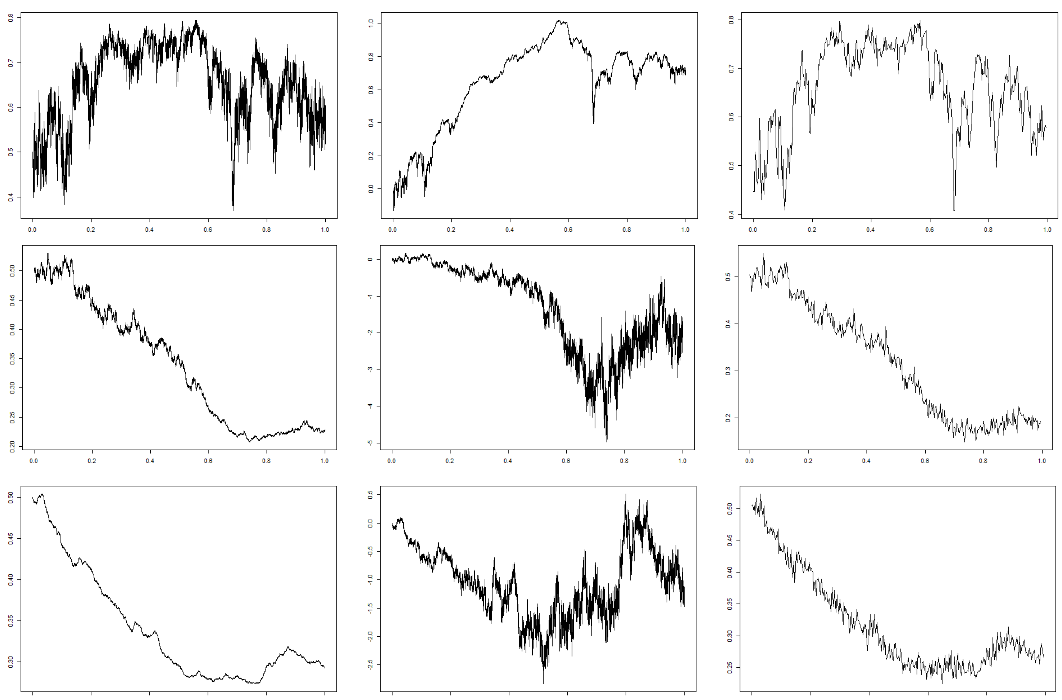
We mention in passing that $x \wedge y := \min\{x, y\}$, for all $(x, y) \in \mathbb{R}^2$. It is noteworthy that a major ingredient of the proof of Theorem 6.2.2 is the important Burkholder-Davis-Gundy inequality (see for instance [Mao07, Pro05]) as formulated in the following proposition:

Proposition 6.2.3. *Let $p \in [1, +\infty)$ be arbitrary and fixed. There is a universal deterministic finite constant $a(p)$ for which the following result holds: for any $(\mathcal{F}_s)_{s\in\mathbb{R}}$-adapted stochastic process $f = \{f(s)\}_{s\in\mathbb{R}}$ satisfying almost surely $\int_{-\infty}^{+\infty} |f(s)|^2\, ds < +\infty$, one has*

$$\mathbb{E}\Big(\Big|\int_{-\infty}^{+\infty} f(s)\, dB(s)\Big|^p\Big) \leq a(p)\mathbb{E}\bigg(\Big(\int_{-\infty}^{+\infty} |f(s)|^2\, ds\Big)^{p/2}\bigg), \tag{6.2.17}$$

where $\int_{-\infty}^{+\infty} f(s)\, dB(s)$ denotes the Itô integral of f on $\mathbb{R}$.

The statistical estimator of random Hurst functions, introduced in Definition 6.2.1, has been tested in the following simulations:



The random Hurst functions $H^1(s) := \psi(B_{0.3}(s))$, $H^2(s) := \psi(B_{0.55}(s))$ and $H^3(s) := \psi(B_{0.75}(s))$ have been successively simulated on the interval $[0, 1]$ in the first column. The corresponding MAMPREs X^1, X^2 and X^3 have been simulated on the same interval in the second column, by using a simulation method, relying on the Haar wavelet basis, which is rather similar to that introduced in [AEH18]. The estimated versions of these three random Hurst functions, via the statistical estimator introduced in Definition 6.2.1 with $L = 2$, $Q = 2$, $\theta_N = N^{-0.6}$ and $N = 2^{14}$ have been simulated on the interval $[0, 1]$ in the third column. Notice that ψ is the deterministic function from $\mathbb{R}$ into the interval $[0.1, 0.9] \subset (0, 1)$, defined, for all $x \in \mathbb{R}$, as

$\psi(x) := 0.8(\pi^{-1}\arctan(x) + 0.5) + 0.1$. Also, notice that $\{B_{0.3}(s)\}_s$, $\{B_{0.55}(s)\}_s$ and $\{B_{0.75}(s)\}_s$ are FBMs (see (6.1.1)) which are adapted to the filtration $(\mathcal{F}_s)_s$ and whose Hurst parameters are respectively equal to 0.3, 0.55 and 0.75.

In view of the simulations the statistical estimator of random Hurst functions, introduced in Definition 6.2.1, seems to work fairly well. Indeed, the simulations show that it allows to reconstruct random Hurst functions in a rather precise way, even when they are very erratic, as for instance the random Hurst function $H^1(s)$.

Before ending the present section let us point out that:

Remark 6.2.4. From now till the end of the article we always assume that the four conditions (6.1.8), (6.1.10), (6.2.10) and (6.2.11) hold, without mentioning it explicitly in the statements of the intermediate results, obtained in the remaining sections, which will allow us to prove Theorem 6.2.2.

6.3 Negligible parts of generalized quadratic variations of X

In view of (6.2.8), (6.2.14) and (6.2.15), for proving Theorem 6.2.2 it is useful to study, for any fixed positive integer Q, asymptotic behavior of the generalized quadratic variations $V_{QN}(\mathcal{I}_{N,n})$, $n \in \{0, \ldots, \lfloor \theta_N^{-1} \rfloor - 1\}$, when N goes to $+\infty$. A first difficulty in this matter is that the domains of integration of the Itô integrals representing the generalized increments $d_{N,k}$ are unbounded intervals. Indeed, one can derive from (6.1.9), (6.2.3), (6.2.2), (6.1.2) and easy computations, that, for all $k \in \{0, \ldots, N-L\}$,

$$d_{N,k} = \int_{-\infty}^{N^{-1}(k+L)} N^{-(H(s)-1/2)}\, \Phi\big(Ns - k, H(s)\big)\, dB(s)\,, \tag{6.3.1}$$

where Φ is the real-valued deterministic function defined, for all $(u,v) \in \mathbb{R} \times (0,1)$, as:

$$\Phi(u,v) := \sum_{l=0}^{L} a_l (l-u)_+^{v-1/2}. \tag{6.3.2}$$

Roughly speaking, our first goal will be to show that $d_{N,k}$ can be expressed as the sum of an Itô integral over a well-chosen bounded interval and another term which is negligible in some sense. From now till the end of our article, we assume that $\delta \in (0,1)$ is arbitrary and fixed, and that, for every integer $N \geq N_0$, $e_N = e_N(\delta)$ is the positive integer defined as:

$$e_N := \lfloor N^\delta \rfloor. \tag{6.3.3}$$

Then we can derive from (6.3.1), that, for all $k \in \{0, \ldots, N-L\}$,

$$d_{N,k} = \widetilde{d}^{\,\delta}_{N,k} + \breve{d}^{\,\delta}_{N,k}\,, \tag{6.3.4}$$

where

$$\widetilde{d}^{\,\delta}_{N,k} := \int_{N^{-1}(k-e_N+L)}^{N^{-1}(k+L)} N^{-(H(s)-1/2)}\, \Phi\big(Ns - k, H(s)\big)\, dB(s) \tag{6.3.5}$$

and

$$\breve{d}^{\,\delta}_{N,k} := \int_{-\infty}^{N^{-1}(k-e_N+L)} N^{-(H(s)-1/2)}\, \Phi\big(Ns - k, H(s)\big)\, dB(s)\,. \tag{6.3.6}$$

Definition 6.3.1. For any $\delta \in (0,1)$ and integers $Q \geq 1$, $N \geq N_0$ and $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$, the generalized quadratic variations $\widetilde{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$ and $\breve{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$ are defined as:

$$\widetilde{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) = |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} |\widetilde{d}^{\delta}_{QN,k}|^2 \tag{6.3.7}$$

and

$$\breve{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) = |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} |\breve{d}^{\delta}_{QN,k}|^2. \tag{6.3.8}$$

Basically, the following lemma shows that the generalized quadratic variations $\breve{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$, $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$, are negligible when N goes to $+\infty$. In other words, when N goes to $+\infty$, the asymptotic behavior of $V_{QN}(\mathcal{I}_{N,n})$ is similar to that of the "less complicated" generalized quadratic variation $\widetilde{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$.

Lemma 6.3.2. *Let $\underline{H}$, $\overline{H}$ and L be as in (6.1.8) and (6.2.2). Let $\delta \in (0,1)$ be arbitrary and fixed. One has almost surely*

$$\limsup_{N \to +\infty} \Big\{ N^{\beta} \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \breve{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) \Big\} = 0\,, \quad \text{for all } Q \in \mathbb{N} \text{ and } \beta < 2\delta(L - \overline{H}) + 2\underline{H}. \tag{6.3.9}$$

For proving Lemma 6.3.2 one needs the following lemma whose proof will be given in the sequel.

Lemma 6.3.3. *Let c be the same constant as in (6.3.15). For all $p \in [1, +\infty)$, $\delta \in (0,1)$, and integers $Q \geq 1$ and $N \geq (3L)^{1/\delta} + N_0$, the following inequality is satisfied:*

$$\max_{0 \leq k \leq N-L} \mathbb{E}\big(|\breve{d}^{\delta}_{QN,k}|^{2p}\big) \leq c^{2p}\, a(2p) N^{-2p(\delta(L-\overline{H})+\underline{H})}\,, \tag{6.3.10}$$

where $a(2p)$ is the same constant only depending on p as in Proposition 6.2.3.

Proof of Lemma 6.3.2. Let $Q \in \mathbb{N}$ and b a fixed real number such that

$$\beta < b < 2\delta(L - \overline{H}) + 2\underline{H}\,, \tag{6.3.11}$$

where β is as in (6.3.9). Let $p \in [1, +\infty)$ be fixed and such that

$$p(2\delta(L - \overline{H}) + 2\underline{H} - b) > 2\,. \tag{6.3.12}$$

Using (6.3.8), Markov inequality, the fact that $z \mapsto |z|^p$ is a convex function on $\mathbb{R}$, (6.3.10) and (6.2.11), one gets that

$$\begin{aligned}
&\mathbb{P}\Big(N^b \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \breve{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) > 1\Big) \leq \sum_{n=0}^{\lfloor \theta_N^{-1} \rfloor - 1} \mathbb{P}\Big(N^b\, \breve{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) > 1\Big) \\
&\leq N^{pb} \sum_{n=0}^{\lfloor \theta_N^{-1} \rfloor - 1} \mathbb{E}\Big(\big|\breve{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|^p\Big) \leq N^{pb} \sum_{n=0}^{\lfloor \theta_N^{-1} \rfloor - 1} |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} \mathbb{E}\big(|\breve{d}^{\delta}_{QN,k}|^{2p}\big) \\
&\leq c_1 \lfloor \theta_N^{-1} \rfloor N^{-p(2\delta(L-\overline{H})+2\underline{H}-b)} \leq c_1 N^{1-p(2\delta(L-\overline{H})+2\underline{H}-b)}\,,
\end{aligned} \tag{6.3.13}$$

where $c_1 > 0$ is a constant not depending on N and Q. Next, combining (6.3.12) and (6.3.13), one obtains that

$$\sum_{N=N_0}^{+\infty} \mathbb{P}\Big(N^b \max_{0\le n<\lfloor \theta_N^{-1}\rfloor} \breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) > 1\Big) < +\infty\,.$$

Thus, it results from the Borel-Cantelli Lemma that one has, almost surely,

$$\sup_{N\ge N_0} \Big\{ N^b \max_{0\le n<\lfloor \theta_N^{-1}\rfloor} \breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \Big\} < +\infty\,. \tag{6.3.14}$$

Finally, (6.3.11) and (6.3.14) imply that (6.3.9) holds. □

Let us now focus on the proof of Lemma 6.3.3. It mainly relies on Proposition 6.2.3 and the following proposition.

Proposition 6.3.4. *Let $\underline{H}$ and $\overline{H}$ be as in (6.1.8), and let Φ be the function introduced in (6.3.2). One has*

$$c := \sup\Big\{ \big(1+L+|u|\big)^{L+1/2-\overline{H}} \big|\Phi(u,v)\big| \,:\, (u,v)\in(-\infty,-2L]\times[\underline{H},\overline{H}] \Big\} < +\infty \tag{6.3.15}$$

and

$$c' := \sup\left\{ \frac{\big(1+L+|u|\big)^{L+1/2-\overline{H}}}{\log\big(1+L+|u|\big)} \big|(\partial_v\Phi)(u,v)\big| \,:\, (u,v)\in(-\infty,-2L]\times[\underline{H},\overline{H}] \right\} < +\infty. \tag{6.3.16}$$

Proof of Proposition 6.3.4. Combining (6.3.2) and (6.1.2) one gets, for all $(u,v)\in(-\infty,-L)\times(0,1)$, that

$$\Phi(u,v) = |u|^{v-1/2}\sum_{l=0}^{L} a_l f(lu^{-1},v) \tag{6.3.17}$$

and

$$(\partial_v\Phi)(u,v) = |u|^{v-1/2}\log(|u|)\sum_{l=0}^{L} a_l f(lu^{-1},v) + |u|^{v-1/2}\sum_{l=0}^{L} a_l (\partial_v f)(lu^{-1},v), \tag{6.3.18}$$

where f is the C^∞ function on $(-1,1)\times(-2,2)$ defined, for all $(y,v)\in(-1,1)\times(-2,2)$, as $f(y,v)=(1-y)^{v-1/2}$. Then noticing that when u belongs to $(-\infty,-2L]$ one equivalently has that $z=u^{-1}$ belongs to $\big[-2^{-1}L^{-1},0\big)\subset\big(-L^{-1},L^{-1}\big)$, one can easily derive from (6.3.17), (6.3.18) and Lemma 6.3.5 below that (6.3.15) and (6.3.16) are satisfied. □

Lemma 6.3.5. *Assume that y_0 and v_0 are two arbitrary and fixed positive real numbers. Let φ be an arbitrary real-valued C^∞ function on $(-y_0,y_0)\times(-v_0,v_0)$ and let g be the C^∞ function on $\big(-L^{-1}y_0,L^{-1}y_0\big)\times(-v_0,v_0)$ defined, for all $(z,v)\in\big(-L^{-1}y_0,L^{-1}y_0\big)\times(-v_0,v_0)$, as:*

$$g(z,v) := \sum_{l=0}^{L} a_l\varphi(lz,v).$$

Then, one has, for every $(z,v)\in\big[-2^{-1}L^{-1}y_0,2^{-1}L^{-1}y_0\big]\times\big[-2^{-1}v_0,2^{-1}v_0\big]$,

$$\big|g(z,v)\big|\le c|z|^L,$$

where c *is the finite constant defined as*

$$c:=(L!)^{-1}\sup\big\{|(\partial_z^L g)(z,v)|:(z,v)\in\big[-2^{-1}L^{-1}y_0,2^{-1}L^{-1}y_0\big]\times\big[-2^{-1}v_0,2^{-1}v_0\big]\big\}.$$

Proof of Lemma 6.3.5. Assume that $v\in(-v_0,v_0)$ is arbitrary and fixed. Applying Taylor formula to the function $z\mapsto g(z,v)$ it follows, for all $z\in\big[-2^{-1}L^{-1}y_0,2^{-1}L^{-1}y_0\big]$, that

$$g(z,v)=\Big(\sum_{q=0}^{L-1}\frac{(\partial_z^q g)(0,v)}{q!}z^q\Big)+\frac{(\partial_z^L g)(\theta,v)}{L!}z^L\,, \tag{6.3.19}$$

where $\theta\in\big(-2^{-1}L^{-1}y_0,2^{-1}L^{-1}y_0\big)$. Next, observe that, for each $z\in\big(-L^{-1}y_0,L^{-1}y_0\big)$ and $q\in\mathbb{N}$, one has

$$(\partial_z^q g)(z,v)=\sum_{l=0}^{L}l^q a_l(\partial_y^q\varphi)(lz,v)\,.$$

Therefore, one gets that

$$(\partial_z^q g)(0,v)=(\partial_y^q\varphi)(0,v)\big(\sum_{l=0}^{L}l^q a_l\big).$$

Then, in view of (6.2.2), it turns out that $(\partial_z^q g)(0,v)=0$, for all $q\in\{0,\dots,L-1\}$. Finally, combining the latter equality with (6.3.19), one obtains the lemma. □

We are now ready to prove Lemma 6.3.3.

Proof of Lemma 6.3.3. Using (6.3.6), (6.2.17), (6.1.8), (6.3.15) and (6.3.3) one gets, for all $p\in[1,+\infty)$, $\delta\in(0,1)$, and integers $Q\ge1$, $N\ge(3L)^{1/\delta}+N_0$ and $k\in\{0,\dots,N-L\}$, that

$$\begin{aligned}
&\mathbb{E}\big(|\breve{d}_{QN,k}^{\delta}|^{2p}\big)\le a(2p)\mathbb{E}\bigg(\Big(\int_{-\infty}^{(QN)^{-1}(k-e_{QN}+L)}(QN)^{-2H(s)+1}\,\big|\Phi\big(QNs-k,H(s)\big)\big|^2\,ds\Big)^p\bigg)\\
&\le c^{2p}\,a(2p)N^{-2p\underline{H}}\Big(QN\int_{-\infty}^{(QN)^{-1}(k-e_{QN}+L)}\big(1+L+k-QNs\big)^{2\overline{H}-2L-1}\,ds\Big)^p\\
&\le c^{2p}\,a(2p)N^{-2p\underline{H}}\,(1+e_{QN})^{2p(\overline{H}-L)}\le c^{2p}\,a(2p)N^{2p(\delta(\overline{H}-L)-\underline{H})}\,,
\end{aligned}$$

which proves that (6.3.10) holds. □

Roughly speaking, so far we have shown that, when N goes to $+\infty$, the asymptotic behavior of $V_{QN}(\mathcal{I}_{N,n})$ is similar to that of $\widetilde{V}_{QN}^{\delta}(\mathcal{I}_{N,n})$ defined in (6.3.7). There is still a difficulty in the study of the latter behavior. Basically, it comes from the $H(s)$ which figures in (6.3.5). It is convenient to replace $H(s)$ by a well-chosen random variable not depending on s. This is the main motivation behind the following definition.

Definition 6.3.6. For every $\delta \in (0,1)$ and integers $N \geq N_0$ and $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$, one sets

$$\zeta_{N,n} := n\theta_N - N^{-(1-\delta)}\,; \tag{6.3.20}$$

moreover, for each $Q \in \mathbb{N}$, the generalized quadratic variations $\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})$ and $\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n})$ are defined as:

$$\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) = |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} |\widehat{d}_{QN,k}^{\delta,n}|^2 \tag{6.3.21}$$

and

$$\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) = |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} |\overline{d}_{QN,k}^{\delta,n}|^2\,, \tag{6.3.22}$$

where, for all $k \in \nu_{QN}(\mathcal{I}_{N,n})$,

$$\widehat{d}_{QN,k}^{\delta,n} := \int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)} (QN)^{-(H(\zeta_{N,n})-1/2)}\, \Phi\big(QNs-k, H(\zeta_{N,n})\big)\, dB(s) \tag{6.3.23}$$

and

$$\begin{aligned}\overline{d}_{QN,k}^{\delta,n} &:= \widetilde{d}_{QN,k}^{\delta} - \widehat{d}_{QN,k}^{\delta,n} \\ &= \int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)} \Big((QN)^{-(H(s)-1/2)}\, \Phi\big(QNs-k, H(s)\big) \\ &\qquad -(QN)^{-(H(\zeta_{N,n})-1/2)}\, \Phi\big(QNs-k, H(\zeta_{N,n})\big)\Big)\, dB(s)\,.\end{aligned} \tag{6.3.24}$$

Basically, the following lemma shows that the generalized quadratic variations $\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n})$, $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$, are negligible when N goes to $+\infty$. In other words, when N goes to $+\infty$, the asymptotic behavior of $\widetilde{V}_{QN}^{\delta}(\mathcal{I}_{N,n})$ (and consequently that of $V_{QN}(\mathcal{I}_{N,n})$) is similar to that of the "less complicated" generalized quadratic variation $\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})$.

Lemma 6.3.7. *Let γ and μ be as in (6.1.10) and (6.2.10). Let $\delta \in (0,1)$ be arbitrary and fixed. One has almost surely*

$$\limsup_{N \to +\infty} \Big\{ N^{\beta} \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} N^{2H(\zeta_{N,n})}\, \overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \Big\} = 0\,, \quad \text{for all } Q \in \mathbb{N} \text{ and } \beta < 2\gamma((1-\delta) \wedge \mu). \tag{6.3.25}$$

In order to show that Lemma 6.3.7 holds, one needs the following lemma.

Lemma 6.3.8. *For any fixed $p \in [1, +\infty)$, $\delta \in (0,1)$ and $Q \in \mathbb{N}$, there exists a finite constant c, which depends on p, δ and Q, such that, for all integer $N \geq N_0$, one has*

$$\max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \max_{k \in \nu_{QN}(\mathcal{I}_{N,n})} \mathbb{E}\Big(\big| N^{H(\zeta_{N,n})}\, \overline{d}_{QN,k}^{\delta,n} \big|^{2p} \Big) \leq c (\log N)^{2p} N^{-2p\gamma((1-\delta)\wedge\mu)}\,. \tag{6.3.26}$$

Proof of Lemma 6.3.8. One assumes that $p \in [1, +\infty)$, $\delta \in (0,1)$, and $Q \in \mathbb{N}$ are arbitrary and fixed. It follows from (6.3.24), (6.2.17), the inequality $Q^{2H(\zeta_{N,n})} \geq 1$, the inequality

$$|x+y|^2 \leq 2\big(|x|^2 + |y|^2\big), \quad \text{for all } (x,y) \in \mathbb{R}^2, \tag{6.3.27}$$

and the convexity on $\mathbb{R}_+$ of the function $z \mapsto z^p$ that, for all integers $N \geq N_0$, $n \in \{0, \ldots, \lfloor \theta_N^{-1} \rfloor - 1\}$ and $k \in \nu_{QN}(\mathcal{I}_{N,n})$, one has

$$\begin{aligned}
&\mathbb{E}\Big(\big| N^{H(\zeta_{N,n})}\, \overline{d}_{QN,k}^{\delta,n} \big|^{2p} \Big) \\
&\leq a(2p)\mathbb{E}\Bigg(\bigg(QN \int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)} \Big| (QN)^{(H(\zeta_{N,n})-H(s))}\, \Phi\big(QNs-k, H(s)\big) \\
&\qquad\qquad -\Phi\big(QNs-k, H(\zeta_{N,n})\big) \Big|^2 ds \bigg)^p \Bigg) \\
&\leq 2^{2p-1} a(2p) \Big(\mathbb{E}\big((A_{QN,k}^{\delta,n})^p \big) + \mathbb{E}\big((B_{QN,k}^{\delta,n})^p \big) \Big)\,, \qquad (6.3.28)
\end{aligned}$$

where

$$A_{QN,k}^{\delta,n} := QN \int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)} \big| (QN)^{(H(\zeta_{N,n})-H(s))} - 1 \big|^2 \big| \Phi\big(QNs-k, H(s)\big) \big|^2 ds \qquad (6.3.29)$$

and

$$B_{QN,k}^{\delta,n} := QN \int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)} \Big| \Phi\big(QNs-k, H(s)\big) - \Phi\big(QNs-k, H(\zeta_{N,n})\big) \Big|^2 ds\,. \qquad (6.3.30)$$

Next, using the mean value Theorem, (6.1.10), (6.3.20), the fact that $(QN)^{-1}k \in \mathcal{I}_{N,n}$ (see (6.2.5) and (6.2.12)), (6.3.3) and (6.2.10), for all $s \in \big[(QN)^{-1}(k - e_{QN} + L), (QN)^{-1}(k+L)\big]$, one gets that

$$\begin{aligned}
\big| (QN)^{(H(\zeta_{N,n})-H(s))} - 1 \big| &\leq c_1 \rho \exp\big(c_1 \rho (\log QN) N^{-\gamma((1-\delta)\wedge\mu)} \big) (\log QN) N^{-\gamma((1-\delta)\wedge\mu)} \\
&\leq \exp\big(c_2 \rho \big) (\log N) N^{-\gamma((1-\delta)\wedge\mu)}\,, \qquad (6.3.31)
\end{aligned}$$

where the two deterministic finite constants c_1 and c_2 are defined as: $c_1 := (2\kappa + L + 1)^\gamma$ and $c_2 := c_1\big(2\log(3+Q) + \sup_{N \geq N_0} (\log QN) N^{-\gamma((1-\delta)\wedge\mu)}\big)$. Next, putting together (6.3.29), (6.3.31), the change of variable $u = QNs - k$, (6.3.15) and (6.3.2), one obtains that

$$\begin{aligned}
A_{N,k}^{\delta,n} &\leq \exp\big(2c_2\rho \big) (\log N)^2 N^{-2\gamma((1-\delta)\wedge\mu)} \int_{-\infty}^{L} \big| \Phi\big(u, H((QN)^{-1}(u+k))\big) \big|^2 du \\
&\leq c_3 \exp\big(2c_2\rho \big) (\log N)^2 N^{-2\gamma((1-\delta)\wedge\mu)}, \qquad (6.3.32)
\end{aligned}$$

where the deterministic finite constant

$$c_3 := c_4^2 \int_{-\infty}^{-2L} \big(1+L+|u|\big)^{2\overline{H}-2L-1} du + \int_{-2L}^{L} \Big(\sum_{l=0}^{L} |a_l| \big((l-u)_+^{\underline{H}-1/2} + (l-u)_+^{\overline{H}-1/2} \big) \Big)^2 du, \quad (6.3.33)$$

c_4 being the constant c in (6.3.15). Next, notice that, in view of (6.3.3), one can assume without any restriction that N is big enough so that $L - e_{QN} \leq L - e_N < -2L$. Then, it follows from

(6.3.30), the change of variable $u = QNs - k$ and (6.3.2) that

$$
\begin{aligned}
B^{\delta,n}_{QN,k} &= \int_{L-e_{QN}}^{L} \Big|\Phi\big(u, H((QN)^{-1}(u+k))\big) - \Phi\big(u, H(\zeta_{N,n})\big)\Big|^2 du \\
&= \int_{L-e_{QN}}^{-2L} \Big|\Phi\big(u, H((QN)^{-1}(u+k))\big) - \Phi\big(u, H(\zeta_{N,n})\big)\Big|^2 du \\
&\quad + \int_{-2L}^{0} \Big(\sum_{l=0}^{L} a_l (l-u)^{H((QN)^{-1}(u+k))-1/2} - \sum_{l=0}^{L} a_l (l-u)^{H(\zeta_{N,n})-1/2} \Big)^2 du \\
&\quad + \sum_{p=0}^{L-1} \int_{p}^{p+1} \Big(\sum_{l=p+1}^{L} a_l (l-u)^{H((QN)^{-1}(u+k))-1/2} - \sum_{l=p+1}^{L} a_l (l-u)^{H(\zeta_{N,n})-1/2} \Big)^2 du \,.
\end{aligned}
$$

Thus, one can derive from the mean value Theorem, (6.1.10), (6.3.20), the fact that $(QN)^{-1}k \in \mathcal{I}_{N,n}$, (6.3.3), (6.2.10), (6.3.16) and (6.1.8) that

$$
B^{\delta,n}_{QN,k} \le c_5 \rho^2 N^{-2\gamma((1-\delta)\wedge\mu)} \,, \tag{6.3.34}
$$

where the deterministic finite constant

$$
\begin{aligned}
c_5 \;:=\; & c_7^2\, c_6^2 \int_{-\infty}^{-2L} \big(1 + L + |u|\big)^{2\overline{H}-2L-1} \log^2\big(1 + L + |u|\big)\, du \\
& + c_6^2 \int_{-2L}^{0} \Big(\sum_{l=0}^{L} |a_l| \big((l-u)^{\underline{H}-1/2} + (l-u)^{\overline{H}-1/2}\big) \big|\log(l-u)\big| \Big)^2 du \\
& + c_6^2 \sum_{p=0}^{L-1} \int_{p}^{p+1} \Big(\sum_{l=p+1}^{L} |a_l| \big((l-u)^{\underline{H}-1/2} + (l-u)^{\overline{H}-1/2}\big) \big|\log(l-u)\big| \Big)^2 du \,,
\end{aligned}
$$

$c_6 := (2\kappa + 2)^\gamma$ and c_7 being the constant c' in (6.3.16). Finally, putting together (6.3.28), (6.3.32) and (6.3.34), one obtains (6.3.26). □

We are now ready to prove Lemma 6.3.7.

Proof of Lemma 6.3.7. Let $Q \in \mathbb{N}$ and b be a fixed real number such that

$$
\beta < b < 2\gamma((1-\delta)\wedge\mu) \,, \tag{6.3.35}
$$

where β is as in (6.3.25). Let $p \in [1, +\infty)$ be fixed and such that

$$
p(2\gamma((1-\delta)\wedge\mu) - b) > 2 \,. \tag{6.3.36}
$$

Using (6.3.22), Markov inequality, the fact that $z \mapsto |z|^p$ is a convex function on $\mathbb{R}$, (6.3.26) and

(6.2.11), one gets that

$$
\begin{aligned}
&\mathbb{P}\Big(N^b \max_{0\le n<\lfloor\theta_N^{-1}\rfloor} N^{2H(\zeta_{N,n})}\,\overline{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})>1\Big) \le \sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1} \mathbb{P}\Big(N^{b+2H(\zeta_{N,n})}\,\overline{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})>1\Big)\\
&\le N^{pb}\sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1}\mathbb{E}\Big(\big|N^{2H(\zeta_{N,n})}\,\overline{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|^p\Big)\\
&\le N^{pb}\sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1}|\nu_{QN}(\mathcal{I}_{N,n})|^{-1}\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})}\mathbb{E}\Big(\big|N^{H(\zeta_{N,n})}\,\overline{d}_{QN,k}^{\,\delta,n}\big|^{2p}\Big)\\
&\le c_1\lfloor\theta_N^{-1}\rfloor(\log N)^{2p}N^{-p(2\gamma((1-\delta)\wedge\mu)-b)}\le c_1(\log N)^{2p}N^{1-p(2\gamma((1-\delta)\wedge\mu)-b)}\,,
\end{aligned}
\tag{6.3.37}
$$

where $c_1>0$ is a constant not depending on N. Next, combining (6.3.36) and (6.3.37), one obtains that

$$
\sum_{N=N_0}^{+\infty}\mathbb{P}\Big(N^b \max_{0\le n<\lfloor\theta_N^{-1}\rfloor} N^{2H(\zeta_{N,n})}\,\overline{V}_{N}^{\,\delta}(\mathcal{I}_{N,n})>1\Big)<+\infty\,.
$$

Thus, it results from the Borel-Cantelli Lemma that one has, almost surely,

$$
\sup_{N\ge N_0}\Big\{N^b \max_{0\le n<\lfloor\theta_N^{-1}\rfloor} N^{2H(\zeta_{N,n})}\,\overline{V}_{N}^{\,\delta}(\mathcal{I}_{N,n})\Big\}<+\infty\,. \tag{6.3.38}
$$

Finally, (6.3.35) and (6.3.38) imply that (6.3.25) holds. □

6.4 Asymptotic behavior of generalized quadratic variation of X

The main goal of the present section is to prove the following lemma.

Lemma 6.4.1. *Let $\underline{H}$, $\overline{H}$, γ, L, μ and μ' be as in (6.1.8), (6.1.10), (6.2.2), (6.2.10) and (6.2.11). Let β be an arbitrary real number satisfying*

$$
0<\beta<\min\Big\{\gamma\big((1-\delta)\wedge\mu\big),\,\delta(L-\overline{H})+\underline{H}-\overline{H},\,2^{-1}(1-\mu')\Big\}\,, \tag{6.4.1}
$$

where δ is an arbitrary fixed real number such that

$$
\frac{\overline{H}-\underline{H}}{L-\overline{H}}<\delta<1\,. \tag{6.4.2}
$$

Then, one has almost surely, for all $Q\in\mathbb{N}$,

$$
\limsup_{N\to+\infty}\left\{N^\beta \max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}-1\right|\right\}=0\,, \tag{6.4.3}
$$

where $V_{QN}(\mathcal{I}_{N,n})$, $\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})$ and $\zeta_{N,n}$ are defined through (6.2.4), (6.3.21) and (6.3.20). Notice that $\mathbb{E}(\,\cdot\,|\mathcal{F}_{\zeta_{N,n}})$ is the conditional expectation operator with respect to the sigma-algebra $\mathcal{F}_{\zeta_{N,n}}$.

The proof of Lemma 6.4.1, which will be given at the end of the present section, relies on Lemma 6.3.2, Lemma 6.3.7, and the following crucial lemma.

Lemma 6.4.2. *Let $\mu' \in [\mu, 1) \subset (0,1)$ be as in (6.2.11). Let $\delta \in (0,1)$ be arbitrary and fixed. One has almost surely*

$$\limsup_{N\to+\infty}\left\{N^{\beta(1-\mu')}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\Big|\mathcal{F}_{\zeta_{N,n}}\Big)}-1\right|\right\}=0\,,\quad \textit{for all } Q\in\mathbb{N} \textit{ and } \beta<1/2. \tag{6.4.4}$$

In order to show that Lemma 6.4.2 holds, one needs some preliminary results.

Lemma 6.4.3. *Let $\delta \in (0,1)$ be arbitrary and fixed. For all integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \dots, \lfloor\theta_N^{-1}\rfloor - 1\}$, and for each finite sequence $(z_k)_{k\in\nu_{QN}(\mathcal{I}_{N,n})}$ of real numbers, one has, almost surely,*

$$\begin{aligned}&\mathbb{E}\bigg(\exp\Big(i\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})} z_k\,\widehat{d}_{QN,k}^{\delta,n}\Big)\Big|\mathcal{F}_{\zeta_{N,n}}\bigg) \\ &=\exp\Bigg(\\ &\quad -2^{-1}\int_{\zeta_{N,n}}^{1}\Big|\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})} z_k\mathbb{1}_{\mathcal{D}_{QN,k}}(s)(QN)^{-(H(\zeta_{N,n})-1/2)}\,\Phi\big(QNs-k,H(\zeta_{N,n})\big)\Big|^2 ds\Bigg),\end{aligned} \tag{6.4.5}$$

where $\zeta_{N,n}$ is an in (6.3.20), and $\mathbb{1}_{\mathcal{D}_{QN,k}}$ is the indicator function of the interval

$$\mathcal{D}_{QN,k} := \big[(QN)^{-1}(k-e_{QN}+L),(QN)^{-1}(k+L)\big]\,. \tag{6.4.6}$$

Notice that (6.4.5) means that, for each $n \in \{0, \dots, \lfloor\theta_N^{-1}\rfloor - 1\}$, conditionally to the sigma-algebra $\mathcal{F}_{\zeta_{N,n}}$, the random vector $(\widehat{d}_{QN,k}^{\delta,n})_{k\in\nu_{QN}(\mathcal{I}_{N,n})}$ has a centred Gaussian distribution with covariance matrix $\big(\mathbb{E}(\widehat{d}_{QN,k'}^{\delta,n}\,\widehat{d}_{QN,k''}^{\delta,n}|\mathcal{F}_{\zeta_{N,n}})\big)_{k',k''\in\nu_{QN}(\mathcal{I}_{N,n})}$ such that, for every $k', k'' \in \nu_{QN}(\mathcal{I}_{N,n})$,

$$\begin{aligned}&\mathbb{E}\big(\widehat{d}_{QN,k'}^{\delta,n}\,\widehat{d}_{QN,k''}^{\delta,n}\big|\mathcal{F}_{\zeta_{N,n}}\big) \\ &=(QN)^{1-2H(\zeta_{N,n})} \\ &\quad\times\int_{\zeta_{N,n}}^{1}\mathbb{1}_{\mathcal{D}_{QN,k'}}(s)\mathbb{1}_{\mathcal{D}_{QN,k''}}(s)\Phi\big(QNs-k',H(\zeta_{N,n})\big)\Phi\big(QNs-k'',H(\zeta_{N,n})\big)\,ds\,.\end{aligned} \tag{6.4.7}$$

Proof of Lemma 6.4.3. First observe that one can derive from (6.3.23), (6.2.5), (6.2.12), (6.3.20) and (6.4.6) that, for all integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \dots, \lfloor\theta_N^{-1}\rfloor - 1\}$, and for each finite sequence $(z_k)_{k\in\nu_{QN}(\mathcal{I}_{N,n})}$ of real numbers, one has

$$\begin{aligned}&\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})} z_k\,\widehat{d}_{QN,k}^{\delta,n} \\ &=\int_{\zeta_{N,n}}^{1}\Big(\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})} z_k\mathbb{1}_{\mathcal{D}_{QN,k}}(s)(QN)^{-(H(\zeta_{N,n})-1/2)}\,\Phi\big(QNs-k,H(\zeta_{N,n})\big)\Big)\,dB(s)\,.\end{aligned} \tag{6.4.8}$$

The main idea of the proof of this lemma consists in the observation that the Brownian motion B in (6.4.8) can be replaced by the Brownian motion $W_{N,n} = \{W_{N,n}(x)\}_{x\in\mathbb{R}_+} := \{B(x+\zeta_{N,n}) - B(\zeta_{N,n})\}_{x\in\mathbb{R}_+}$ which is independent of the sigma-algebra $\mathcal{F}_{\zeta_{N,n}}$. Therefore $W_{N,n}$ is independent of the integrand in (6.4.8), denoted by $K_{N,n}$, which is $\mathcal{F}_{\zeta_{N,n}}$-measurable. Having made this observation the proof becomes classical: it can be done in a standard way by approximating the integrand $K_{N,n} = \{K_{N,n}(s)\}_{s\in[\zeta_{N,n},1]}$ by a sequence $\big(K^j_{N,n}\big)_{j\in\mathbb{N}} = \big(\{K^j_{N,n}(s)\}_{s\in[\zeta_{N,n},1]}\big)_{j\in\mathbb{N}}$ of elementary processes of the form:

$$K^j_{N,n}(s) = \sum_{p=0}^{q-1} A_p \mathbb{1}_{[t_p,t_{p+1})}(s)\,,$$

where the random variables A_p, $0 \le p < q$, are $\mathcal{F}_{\zeta_{N,n}}$-measurable, and the finite sequence $(t_p)_{0\le p\le q}$ is a subdivision of the interval $[\zeta_{N,n}, 1]$. □

Roughly speaking, the following lemma shows that $\mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big)$ behaves in the same way as $(QN)^{-2H(\zeta_{N,n})}$

Lemma 6.4.4. *For every $\delta \in (0,1)$ one has almost surely, for all integers $Q \in \mathbb{N}$, $N \ge N_0$ and $n \in \{0,\dots,\lfloor\theta_N^{-1}\rfloor - 1\}$, that*

$$\mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big) = (QN)^{-2H(\zeta_{N,n})} \int_{L-e_{QN}}^{L} \Big|\Phi\big(u, H(\zeta_{N,n})\big)\Big|^2 du \tag{6.4.9}$$

and consequently that

$$c'(QN)^{-2H(\zeta_{N,n})} \le \mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big) \le c''(QN)^{-2H(\zeta_{N,n})}\,, \tag{6.4.10}$$

where c' and c'' are two finite, deterministic and strictly positive constants not depending on δ, Q, N and n.

Proof of Lemma 6.4.4. Let $\delta \in (0,1)$ be arbitrary and fixed. One can derive from (6.4.7), (6.4.6) and the change of variable $u = (QN)s - k$ that one has almost surely, for integers $Q \in \mathbb{N}$, $N \ge N_0$, $n \in \{0,\dots,\lfloor\theta_N^{-1}\rfloor - 1\}$ and $k \in \nu_{QN}(\mathcal{I}_{N,n})$, that

$$\mathbb{E}\big(|\widehat{d}^{\,\delta,n}_{QN,k}|^2\big|\mathcal{F}_{\zeta_{N,n}}\big) = (QN)^{-2H(\zeta_{N,n})} \int_{L-e_{QN}}^{L} \Big|\Phi\big(u, H(\zeta_{N,n})\big)\Big|^2 du\,. \tag{6.4.11}$$

Thus combining (6.3.21) and (6.4.11) one obtains (6.4.9). Then notice that (6.3.2), (6.1.2), (6.2.1), (6.3.3), (6.3.15) and (6.1.8) entail that

$$\int_{L-e_{QN}}^{L} \Big|\Phi\big(u, H(\zeta_{N,n})\big)\Big|^2 du \ge \int_{L-1}^{L} (L-u)^{2H(\zeta_{N,n})-1}\, du \ge c' \tag{6.4.12}$$

and

$$\int_{L-e_{QN}}^{L} \Big|\Phi\big(u, H(\zeta_{N,n})\big)\Big|^2 du \le \int_{-\infty}^{L} \Big|\Phi\big(u, H(\zeta_{N,n})\big)\Big|^2 du \le c''\,, \tag{6.4.13}$$

where the strictly positive constant $c' := (2\overline{H})^{-1}$ and the constant c'' is equal to the constant c_3 defined in (6.3.33). Finally, putting together (6.4.9), (6.4.12) and (6.4.13) one gets (6.4.10). □

Remark 6.4.5. The integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$ are arbitrary and fixed. One denotes by $\mathcal{G}$ a Gaussian Hilbert space on $\mathbb{R}$ spanned by a centred real-valued Gaussian vector $(G_k)_{k \in \nu_{QN}(\mathcal{I}_{N,n})}$ whose distribution is equal to the conditional distribution of the random vector $(\widehat{d}_{QN,k}^{\delta,n})_{k \in \nu_{QN}(\mathcal{I}_{N,n})}$ with respect to the sigma-algebra $\mathcal{F}_{\zeta_{N,n}}$ (see Lemma 6.4.3) for some given arbitrary value of the random variable $H(\zeta_{N,n})$. Then, for the same given value of $H(\zeta_{N,n})$, the conditional distribution with respect to $\mathcal{F}_{\zeta_{N,n}}$ of the random variable $\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\big)$ (see (6.3.21)) is equal to the distribution of the random variable $|\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} \big(|G_k|^2 - \mathbb{E}(|G_k|^2)\big)$. Since the latter random variable belongs to $\overline{\mathcal{P}}_2(\mathcal{G})$ the second order chaos associated to $\mathcal{G}$ (see Definition 2.1 on page 17 in [Jan97]), one knows from Theorem 5.10 on page 62 in [Jan97] that, for any fixed $q \in \mathbb{N}$, there exists a universal deterministic finite constant $\widehat{c}(q)$, only depending on q, such that

$$\mathbb{E}\left(\Big| |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} \big(|G_k|^2 - \mathbb{E}(|G_k|^2)\big) \Big|^{2q} \right) \leq \widehat{c}(q) \left(\mathbb{E}\left(\Big| |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} \big(|G_k|^2 - \mathbb{E}(|G_k|^2)\big) \Big|^{2} \right) \right)^q .$$

Therefore, one has

$$\mathbb{E}\left(\Big| \widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\big) \Big|^{2q} \Big| \mathcal{F}_{\zeta_{N,n}} \right) \leq \widehat{c}(q) \left(\mathbb{E}\left(\Big| \widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\big) \Big|^{2} \Big| \mathcal{F}_{\zeta_{N,n}} \right) \right)^q . \tag{6.4.14}$$

Also, notice that one can derive from Theorem 3.9 on page 26 in [Jan97] that

$$\mathbb{E}\Big(\big(|G_{k'}|^2 - \mathbb{E}(|G_{k'}|^2)\big)\big(|G_{k''}|^2 - \mathbb{E}(|G_{k''}|^2)\big) \Big) = 2\big(\mathbb{E}(G_{k'} G_{k''})\big)^2, \quad \text{for all } k', k'' \in \nu_{QN}(\mathcal{I}_{N,n}),$$

which implies that

$$\mathbb{E}\left(\Big(|\widehat{d}_{QN,k'}^{\delta,n}|^2 - \mathbb{E}\big(|\widehat{d}_{QN,k'}^{\delta,n}|^2 \big| \mathcal{F}_{\zeta_{N,n}}\big) \Big) \Big(|\widehat{d}_{QN,k''}^{\delta,n}|^2 - \mathbb{E}\big(|\widehat{d}_{QN,k''}^{\delta,n}|^2 \big| \mathcal{F}_{\zeta_{N,n}}\big) \Big) \Big| \mathcal{F}_{\zeta_{N,n}} \right) = 2\Big(\mathbb{E}\big(\widehat{d}_{QN,k'}^{\delta,n}\, \widehat{d}_{QN,k''}^{\delta,n} | \mathcal{F}_{\zeta_{N,n}}\big) \Big)^2, \quad \text{for all } k', k'' \in \nu_{QN}(\mathcal{I}_{N,n}). \tag{6.4.15}$$

Lemma 6.4.6. *Let $\delta \in (0,1)$ be arbitrary and fixed. There exists a finite deterministic constant c, not depending on δ, such that one has almost surely, for all integers $Q \in \mathbb{N}$, $N \geq N_0$, $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$ and $k', k'' \in \nu_{QN}(\mathcal{I}_{N,n})$, that*

$$\big| \mathbb{E}(\widehat{d}_{QN,k'}^{\delta,n}\, \widehat{d}_{QN,k''}^{\delta,n} | \mathcal{F}_{\zeta_{N,n}}) \big| \leq c\, (QN)^{-2H(\zeta_{N,n})} \big(1 + |k' - k''|\big)^{-(L - \overline{H})} . \tag{6.4.16}$$

Proof of Lemma 6.4.6. Let $\delta \in (0,1)$ be arbitrary and fixed. The integers $Q \in \mathbb{N}$, $N \geq N_0$, $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$ and $k', k'' \in \nu_N(\mathcal{I}_{N,n})$ are assumed to be arbitrary; moreover one can assume without any restriction that $k'' \geq k'$. One can derive from (6.4.7) and the change of variable $QNs - k'$ that

$$\big| \mathbb{E}(\widehat{d}_{QN,k'}^{\delta,n}\, \widehat{d}_{QN,k''}^{\delta,n} | \mathcal{F}_{\zeta_{N,n}}) \big| \leq (QN)^{-2H(\zeta_{N,n})} \int_{\mathbb{R}} \Big| \Phi\big(u, H(\zeta_{N,n})\big) \Phi\big(u + k' - k'', H(\zeta_{N,n})\big) \Big| \, du . \tag{6.4.17}$$

One denotes by c_1 the finite deterministic constant c_3 defined in (6.3.33) which does not depend on δ, Q, N, n, $H(\zeta_{N,n})$, k' and k''. Using (6.4.17), the Cauchy-Scharwz inequality, (6.3.2) and (6.1.2), one gets that

$$\left|\mathbb{E}(\widehat{d}_{N,k'}^{\delta,n}\,\widehat{d}_{N,k''}^{\delta,n}|\mathcal{F}_{\zeta_{N,n}})\right| \le (QN)^{-2H(\zeta_{N,n})}\int_{-\infty}^{L}\left|\Phi\big(u,H(\zeta_{N,n})\big)\right|^2 du \le c_1(QN)^{-2H(\zeta_{N,n})} \quad (6.4.18)$$

and

$$\begin{aligned}&\left|\mathbb{E}(\widehat{d}_{QN,k'}^{\delta,n}\,\widehat{d}_{QN,k''}^{\delta,n}|\mathcal{F}_{\zeta_{N,n}})\right|\\ &\le (QN)^{-2H(\zeta_{N,n})}\int_{-\infty}^{L}\left|\Phi\big(u,H(\zeta_{N,n})\big)\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\right|du\\ &= (QN)^{-2H(\zeta_{N,n})}\Bigg(\int_{2^{-1}(k'-k'')}^{L}\left|\Phi\big(u,H(\zeta_{N,n})\big)\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\right|du\\ &\qquad+\int_{-\infty}^{2^{-1}(k'-k'')}\left|\Phi\big(u,H(\zeta_{N,n})\big)\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\right|du\Bigg)\\ &\le \sqrt{c_1}(QN)^{-2H(\zeta_{N,n})}\Bigg(\sqrt{\int_{2^{-1}(k'-k'')}^{L}\left|\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\right|^2du} \qquad (6.4.19)\\ &\qquad+\sqrt{\int_{-\infty}^{2^{-1}(k'-k'')}\left|\Phi\big(u,H(\zeta_{N,n})\big)\right|^2du}\Bigg).\end{aligned}$$

Next observe that, under the condition that

$$k'-k''\le -4L\,, \quad (6.4.20)$$

one clearly has $2^{-1}(k'-k'')\le -2L$, and thus one can derive from (6.3.15) that

$$\begin{aligned}&\int_{-\infty}^{2^{-1}(k'-k'')}\left|\Phi\big(u,H(\zeta_{N,n})\big)\right|^2du \qquad (6.4.21)\\ &\le c_2^2\int_{-\infty}^{2^{-1}(k'-k'')}\big(1+L-u\big)^{2\overline{H}-2L-1}du \le c_3\big(1+L+k''-k'\big)^{-2(L-\overline{H})},\end{aligned}$$

where c_2 is the finite deterministic constant c in (6.3.15) and $c_3 := 2^{2(L-\overline{H})-1}(L-\overline{H})^{-1}c_2^2$. Also, observe that under the condition (6.4.20), for all $u\in\big[2^{-1}(k'-k''),L\big]$, one has $u+k'-k''\le -3L$, and thus one can derive from (6.3.15) that

$$\begin{aligned}&\int_{2^{-1}(k'-k'')}^{L}\left|\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\right|^2du \le c_2^2\int_{-\infty}^{L}\big(1+L-u+k''-k'\big)^{2\overline{H}-2L-1}du\\ &\le \frac{c_2^2}{2(L-\overline{H})}\big(1+k''-k'\big)^{-2(L-\overline{H})} \le c_3\big(1+k''-k'\big)^{-2(L-\overline{H})}. \qquad (6.4.22)\end{aligned}$$

Finally setting $c := c_1(4L)^{L-\overline{H}} + 2\sqrt{c_1c_3}$, it follows from (6.4.18), (6.4.19), (6.4.21) and (6.4.22) that (6.4.16) is satisfied. □

We are now ready to prove Lemma 6.4.2.

Proof of Lemma 6.4.2. Let $\delta \in (0,1)$ be arbitrary and fixed, and let b be a fixed real number such that

$$\beta < b < 1/2\,, \tag{6.4.23}$$

where β is as in (6.4.4). Let $q \in \mathbb{N}$ be fixed and big enough so that

$$q(1-\mu')(1-2b) > \mu' + 1\,, \tag{6.4.24}$$

where $\mu' \in (0,1)$ is as in (6.2.11). Using Markov inequality one obtains, for all integers $Q \in \mathbb{N}$ and $N \geq N_0$, that

$$\begin{aligned}
&\mathbb{P}\left(N^{b(1-\mu')} \max_{0 \le n < \lfloor \theta_N^{-1} \rfloor} \left| \frac{\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \right| > 1 \right) \\
&\le \sum_{n=0}^{\lfloor \theta_N^{-1} \rfloor - 1} \mathbb{P}\left(N^{b(1-\mu')} \left| \frac{\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \right| > 1 \right) \\
&\le N^{2qb(1-\mu')} \sum_{n=0}^{\lfloor \theta_N^{-1} \rfloor - 1} \mathbb{E}\left(\left| \frac{\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \right|^{2q} \right).
\end{aligned} \tag{6.4.25}$$

Moreover, the expectations in (6.4.25) can be expressed, for all $n \in \{0, \ldots, \lfloor \theta_N^{-1} \rfloor - 1\}$, as:

$$\begin{aligned}
&\mathbb{E}\left(\left| \frac{\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \right|^{2q} \right) \\
&= \mathbb{E}\left(\left(\mathbb{E}\Big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big) \right)^{-2q} \left| \widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\big) \right|^{2q} \right) \\
&= \mathbb{E}\left(\left(\mathbb{E}\Big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big) \right)^{-2q} \mathbb{E}\left(\left| \widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\big) \right|^{2q} \middle| \mathcal{F}_{\zeta_{N,n}} \right) \right).
\end{aligned} \tag{6.4.26}$$

On the other hand, it follows from (6.3.21), (6.4.15), (6.4.16) and the inequality $L - \overline{H} > 1$ that, for all integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \ldots, \lfloor \theta_N^{-1} \rfloor - 1\}$, one has

$$
\begin{aligned}
&\mathbb{E}\Big(\Big|\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})-\mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big)\Big|^2\Big|\mathcal{F}_{\zeta_{N,n}}\Big)\\
&=\big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-2}\sum_{k',k''\in\nu_{QN}(\mathcal{I}_{N,n})}\mathbb{E}\Big(\Big(|\widehat{d}^{\delta,n}_{QN,k'}|^2-\mathbb{E}\big(|\widehat{d}^{\delta,n}_{QN,k'}|^2\big|\mathcal{F}_{\zeta_{N,n}}\big)\Big)\\
&\qquad\qquad\times\Big(|\widehat{d}^{\delta,n}_{QN,k''}|^2-\mathbb{E}\big(|\widehat{d}^{\delta,n}_{QN,k''}|^2\big|\mathcal{F}_{\zeta_{N,n}}\big)\Big)\Big|\mathcal{F}_{\zeta_{N,n}}\Big)\\
&=2\big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-2}\sum_{k',k''\in\nu_{QN}(\mathcal{I}_{N,n})}\Big(\mathbb{E}\big(\widehat{d}^{\delta,n}_{QN,k'}\,\widehat{d}^{\delta,n}_{QN,k''}\big|\mathcal{F}_{\zeta_{N,n}}\big)\Big)^2\\
&\le 2c_1^2\big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-2}(QN)^{-4H(\zeta_{N,n})}\sum_{k',k''\in\nu_{QN}(\mathcal{I}_{N,n})}\big(1+|k'-k''|\big)^{-2(L-\overline{H})}\\
&\le c_2\big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-1}(QN)^{-4H(\zeta_{N,n})}\,,
\end{aligned}
\tag{6.4.27}
$$

where c_1 denotes the constant c in (6.4.16) and $c_2 := 4c_1^2\sum_{j=1}^{+\infty} j^{-2(L-\overline{H})} < +\infty$. Next, putting together (6.4.26), the first inequality in (6.4.10), (6.4.14), (6.4.27), the first inequality in (6.2.13) and (6.2.11), one gets, for all integers $Q\in\mathbb{N}$, $N\ge N_0$ and $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$, that

$$
\mathbb{E}\left(\left|\frac{\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}-1\right|^{2q}\right)\le c_3\big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-q}\le c_4 N^{-q(1-\mu')}\,,
\tag{6.4.28}
$$

where c_3 and c_4 are two deterministic finite constants not depending on Q, N and n. Then, one can derive from (6.4.25), (6.4.28) and (6.2.11) that, for all integers $Q\in\mathbb{N}$ and $N\ge N_0$,

$$
\begin{aligned}
&\mathbb{P}\left(N^{b(1-\mu')}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\frac{\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}-1\right|>1\right)\\
&\le c_4\,N^{2qb(1-\mu')}\theta_N^{-1}N^{-q(1-\mu')}\le\frac{c_4}{\kappa'}\,N^{\mu'-q(1-\mu')(1-2b)}\,.
\end{aligned}
\tag{6.4.29}
$$

Thus, it follows from (6.4.24) and (6.4.29) that

$$
\sum_{N=N_0}^{+\infty}\mathbb{P}\left(N^{b(1-\mu')}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\frac{\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}-1\right|>1\right)<+\infty\,.
$$

Then the Borel-Cantelli Lemma entails that one has almost surely

$$
\sup_{N\ge N_0}\left\{N^{b(1-\mu')}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\frac{\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}-1\right|\right\}<+\infty\,.
\tag{6.4.30}
$$

Finally, combining (6.4.23) and (6.4.30) one gets (6.4.4). □

We are now ready to prove Lemma 6.4.1.

Proof of Lemma 6.4.1. First observe that, for all integers $Q \in \mathbb{N}$ and $N \geq N_0$, one has

$$
\begin{aligned}
&\max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \right| \\
&= \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)}} - 1 \right| \left| \sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)}} + 1 \right| \\
&\leq \left(\max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)}} - 1 \right| \right)^2 \\
&\qquad\qquad\qquad + 2 \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)}} - 1 \right|,
\end{aligned}
$$

where δ is an arbitrary real number satisfying (6.4.2). Thus, in order to prove that (6.4.3) holds, it is enough to show almost surely that

$$
\limsup_{N \to +\infty} \left\{ N^{\beta} \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)}} - 1 \right| \right\} = 0 \,. \tag{6.4.31}
$$

Let us point out that throughout this proof β denotes an arbitrary fixed positive real number satisfying (6.4.1). Next observe that (6.2.4), (6.3.4), (6.3.24), (6.3.8), (6.3.21), (6.3.22) and the triangle inequality imply, for all integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$, that

$$
\begin{aligned}
&\sqrt{\frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)}} - \sqrt{\frac{\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)}} - \sqrt{\frac{\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)}} \\
&\leq \sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)}} \\
&\leq \sqrt{\frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)}} + \sqrt{\frac{\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)}} + \sqrt{\frac{\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\Big)}}
\end{aligned}
$$

and consequently that

$$\begin{aligned}
&\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-1\right| \\
&\le \max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\sqrt{\frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-1\right| + \max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\sqrt{\frac{\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} \\
&\qquad + \max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\sqrt{\frac{\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} \\
&\le \max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\sqrt{\frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-1\right| + c_1 N^{\overline{H}}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\sqrt{\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}
\end{aligned} \tag{6.4.32}$$

where c_1 is a deterministic finite constant not depending on N. Notice that the last inequality in (6.4.32) results from (6.4.10) and (6.1.8). It clearly follows from (6.4.1), (6.4.2), Lemma 6.3.2 and Lemma 6.3.7 that one has almost surely

$$\limsup_{N\to+\infty}\left\{N^{\beta+\overline{H}}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\sqrt{\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}\right\}=0$$

and

$$\limsup_{N\to+\infty}\left\{N^{\beta}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\sqrt{N^{2H(\zeta_{N,n})}\,\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}\right\}=0\,.$$

Thus, in view of (6.4.32), in order to show that (6.4.31) holds, it is enough to prove that

$$\limsup_{N\to+\infty}\left\{N^{\beta}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\sqrt{\frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-1\right|\right\}=0\,. \tag{6.4.33}$$

Combining (6.4.1) with Lemma 6.4.2 and the inequality $|\sqrt{z}-1|\le|z-1|$, for every $z\in\mathbb{R}_+$, one gets (6.4.33). □

6.5 Final steps of the proof of Theorem 6.2.2

Lemma 6.5.1. *Let $\overline{H}$, γ, L and μ be as in (6.1.8), (6.1.10), (6.2.2) and (6.2.10). Let β be an arbitrary real number satisfying*

$$0<\beta<\min\Big\{\gamma\big((1-\delta)\wedge\mu\big),\,2\delta(L-\overline{H})\Big\}\,, \tag{6.5.1}$$

where $\delta\in(0,1)$ is arbitrary and fixed. Then, one has almost surely, for all $i\in\{0,1\}$ and $Q\in\mathbb{N}$,

$$\limsup_{N\to+\infty}\left\{N^{\beta}\sup_{s\in[0,1]}\left|\frac{\mathbb{E}\Big(\widehat{V}_{N}^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}-Q^{2H(s)}\right|\right\}=0\,, \tag{6.5.2}$$

where, for every $s \in [0,1]$,

$$n_0(N,s) := \begin{cases} \lfloor \theta_N^{-1} s \rfloor & \text{if } s \in \big[0, (\lfloor \theta_N^{-1} \rfloor - 1)\theta_N\big), \\ \lfloor \theta_N^{-1} \rfloor - 1 & \text{if } s \in \big[(\lfloor \theta_N^{-1} \rfloor - 1)\theta_N, 1\big], \end{cases} \tag{6.5.3}$$

and

$$n_1(N,s) := \begin{cases} \lfloor \theta_N^{-1} s \rfloor + 1 & \text{if } s \in \big[0, (\lfloor \theta_N^{-1} \rfloor - 1)\theta_N\big), \\ \lfloor \theta_N^{-1} \rfloor - 1 & \text{if } s \in \big[(\lfloor \theta_N^{-1} \rfloor - 1)\theta_N, 1\big]. \end{cases} \tag{6.5.4}$$

Proof of Lemma 6.5.1. One can derive from (6.4.9) and (6.1.8), that one has almost surely, for each real number $s \in [0,1]$ and integers $i \in \{0,1\}$, $Q \in \mathbb{N}$ and $N \geq N_0$, that

$$\begin{aligned} & \left| \frac{\mathbb{E}\Big(\widehat{V}_N^{\delta}(\mathcal{I}_{N,n_i(N,s)}) \big| \mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n_i(N,s)}) \big| \mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)} - Q^{2H(s)} \right| \\ & = \left| Q^{2H(\zeta_{N,n_i(N,s)})} \frac{\int_{L-e_N}^{L} \big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2 \, du}{\int_{L-e_{QN}}^{L} \big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2 \, du} - Q^{2H(s)} \right| \\ & \leq U_{i,N}(s) + V_{i,N}(s) \,, \end{aligned} \tag{6.5.5}$$

where

$$U_{i,N}(s) := Q^{2\overline{H}} \left| Q^{2(H(\zeta_{N,n_i(N,s)}) - H(s))} - 1 \right| \frac{\int_{L-e_N}^{L} \big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2 \, du}{\int_{L-e_{QN}}^{L} \big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2 \, du} \tag{6.5.6}$$

and

$$\begin{aligned} V_{i,N}(s) & := Q^{2\overline{H}} \left| \frac{\int_{L-e_N}^{L} \big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2 \, du}{\int_{L-e_{QN}}^{L} \big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2 \, du} - 1 \right| \\ & = Q^{2\overline{H}} \frac{\int_{L-e_{QN}}^{L-e_N} \big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2 \, du}{\int_{L-e_{QN}}^{L} \big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2 \, du} \,. \end{aligned} \tag{6.5.7}$$

Next observe that, one can derive from the mean value Theorem, (6.1.8), (6.1.10), (6.3.20), (6.5.3), (6.5.4), (6.2.10) and (6.3.3) that

$$\begin{aligned} & \left| Q^{2(H(\zeta_{N,n_i(N,s)}) - H(s))} - 1 \right| \\ & \leq 2(\log Q) \exp\Big(2(\log Q)\overline{H}\Big) \big| H(\zeta_{N,n_i(N,s)}) - H(s) \big| \\ & \leq \rho(2\kappa+1)^{\gamma} \log(Q^2) Q^{2\overline{H}} N^{-\gamma((1-\delta)\wedge\mu)} \,. \end{aligned} \tag{6.5.8}$$

Moreover, it easily follows (6.3.3) that

$$\frac{\int_{L-e_N}^{L} \big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2 \, du}{\int_{L-e_{QN}}^{L} \big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2 \, du} \leq 1 \,. \tag{6.5.9}$$

Thus, combining (6.5.6), (6.5.8) and (6.5.9), one gets, that, for all $N \geq N_0$,

$$\sup_{s\in[0,1]} U_{i,N}(s) \leq \rho(2\kappa+1)^{\gamma} \log(Q^2) Q^{4\overline{H}} N^{-\gamma((1-\delta)\wedge\mu)} . \tag{6.5.10}$$

Next observe that similarly to (6.4.12) it can be shown that, for all real number $s \in [0,1]$ and integers $i \in \{0,1\}$, $Q \in \mathbb{N}$ and $N \geq N_0$, one has

$$\int_{L-e_{QN}}^{L} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du \geq (2\overline{H})^{-1} .$$

Thus, one can derive from (6.5.7) that, for all real number $s \in [0,1]$ and integers $i \in \{0,1\}$, $Q \in \mathbb{N}$ and $N \geq N_0$,

$$V_{i,N}(s) \leq 2\overline{H} Q^{2\overline{H}} \int_{-\infty}^{L-e_N} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du . \tag{6.5.11}$$

Notice that there is no restriction to assume that $N \geq (3L)^{1/\delta} + N_0$ which implies that $L - e_N < -2L$. Then, using (6.3.15), one gets that

$$\begin{aligned} &\int_{-\infty}^{L-e_N} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du \\ &\leq c_1^2 \int_{-\infty}^{L-e_N} \big(1+L-u\big)^{2\overline{H}-2L-1} du \leq \frac{c_1^2}{2(L-\overline{H})} N^{-2\delta(L-\overline{H})} , \end{aligned} \tag{6.5.12}$$

where c_1 denotes the finite and deterministic constant c in (6.3.15) which does not depend on N and $\zeta_{N,n_i(N,s)}$. Then combining (6.5.11) and (6.5.12), one obtains, for all integers $i \in \{0,1\}$ and $N \geq (3L)^{1/\delta} + N_0$, that

$$\sup_{s\in[0,1]} V_{i,N}(s) \leq 2\overline{H} Q^{2\overline{H}} \frac{c_1^2}{2(L-\overline{H})} N^{-2\delta(L-\overline{H})} . \tag{6.5.13}$$

Finally, putting together (6.5.1), (6.5.5), (6.5.10) and (6.5.13) it follows that (6.5.2) holds. □

Lemma 6.5.2. *Let β be an arbitrary real number satisfying the condition (6.4.1), where δ is an arbitrary fixed real number satisfying the condition (6.4.2). Then, one has almost surely, for all $i \in \{0,1\}$ and $Q \in \mathbb{N}$,*

$$\limsup_{N\to+\infty} \left\{ N^{\beta} \sup_{s\in[0,1]} \left| \frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})} - Q^{2H(s)} \right| \right\} = 0 , \tag{6.5.14}$$

where $n_0(N,s)$ and $n_1(N,s)$ are as in (6.5.3) and (6.5.4).

Proof of Lemma 6.5.2. First observe that, for each real number $s \in [0,1]$ and integers $i \in \{0,1\}$,

$Q \in \mathbb{N}$ and $N \geq N_0$, one has

$$\begin{aligned}
&\left|\frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})} - Q^{2H(s)}\right| \\
&= \left|R_N^i(s) S_{Q,N}^i(s) \big(Z_{Q,N}^i(s)\big)^{-1} - Q^{2H(s)}\right| \\
&\leq Q^{2H(s)} \left|R_N^i(s)\big(Z_{Q,N}^i(s)\big)^{-1} - 1\right| + \frac{R_N^i(s)}{Z_{Q,N}^i(s)} \left|S_{Q,N}^i(s) - Q^{2H(s)}\right| \\
&\leq Q^{2\overline{H}} \left|R_N^i(s) - 1\right| + Q^{2\overline{H}} \frac{R_N^i(s)}{Z_{Q,N}^i(t)} \left|Z_{Q,N}^i(s) - 1\right| + \frac{R_N^i(s)}{Z_{Q,N}^i(s)} \left|S_{Q,N}^i(s) - Q^{2H(s)}\right|,
\end{aligned} \tag{6.5.15}$$

where

$$R_N^i(s) := \frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{\mathbb{E}\Big(\widehat{V}_N^{\delta}(\mathcal{I}_{N,n_i(N,s)}) \big| \mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}, \tag{6.5.16}$$

$$S_{Q,N}^i(s) := \frac{\mathbb{E}\Big(\widehat{V}_N^{\delta}(\mathcal{I}_{N,n_i(N,s)}) \big| \mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n_i(N,s)}) \big| \mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)} \tag{6.5.17}$$

and

$$Z_{Q,N}^i(s) := \frac{V_{QN}(\mathcal{I}_{N,n_i(N,s)})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n_i(N,s)}) \big| \mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}. \tag{6.5.18}$$

Recall that δ is an arbitrary fixed real number satisfying (6.4.2). Next notice that one knows from (6.4.1), (6.5.16), (6.5.18) and Lemma 6.4.1 that, one has almost surely, for all $i \in \{0,1\}$ and $Q \in \mathbb{N}$,

$$\sup_{s\in[0,1]} \left|R_N^i(s) - 1\right| = o\big(N^{-\beta}\big) \tag{6.5.19}$$

and

$$\sup_{s\in[0,1]} \left|Z_N^i(s) - 1\right| = o\big(N^{-\beta}\big). \tag{6.5.20}$$

Moreover, it results from (6.5.19) that, almost surely,

$$\sup_{N\geq N_0} \sup_{s\in[0,1]} R_N^i(s) < +\infty, \tag{6.5.21}$$

and it follows from (6.5.20) that, almost surely,

$$\inf_{s\in[0,1]} Z_N^i(s) \geq 1/2, \quad \text{for all } N \text{ big enough.} \tag{6.5.22}$$

On the other hand, one knows from (6.4.1), (6.5.17) and Lemma 6.5.1 that, one has almost surely, for all $i \in \{0,1\}$ and $Q \in \mathbb{N}$,

$$\sup_{s\in[0,1]} \left|S_N^i(s) - Q^{2H(s)}\right| = o\big(N^{-\beta}\big). \tag{6.5.23}$$

Finally, putting together (6.5.15) and (6.5.19) to (6.5.23) one obtains (6.5.14). □

We are now in position to complete the proof of Theorem 6.2.2.

End of the proof of Theorem 6.2.2. One can derive from (6.2.14), (6.2.15), (6.2.8), (6.5.3), (6.5.4) and (6.1.8) that, for all integer $Q \geq 2$ and $N \geq N_0$, one has

$$\begin{aligned} &\sup_{s\in[0,1]} \left| H(s) - \widetilde{H}^{Q}_{N,\theta_N}(s) \right| \\ &\leq \sum_{i=0}^{1} \sup_{s\in[0,1]} \left| \log_{Q^2}\left(Q^{2H(s)}\right) - \log_{Q^2}\left(\frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})}\right) \right| . \end{aligned} \tag{6.5.24}$$

Next observe that one knows from (6.5.14) and (6.1.8) that, one has almost surely, for all N large enough,

$$\inf_{s\in[0,1]} \frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})} \geq 2^{-1} Q^{2\underline{H}} .$$

Thus, one can derive (6.5.24) and the mean value Theorem that one has almost surely, for all N large enough,

$$\begin{aligned} &\sup_{s\in[0,1]} \left| H(s) - \widetilde{H}^{Q}_{N,\theta_N}(s) \right| \\ &\leq \frac{2Q^{-2\underline{H}}}{\log(Q^2)} \sum_{i=0}^{1} \sup_{s\in[0,1]} \left| Q^{2H(s)} - \frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})} \right| . \end{aligned} \tag{6.5.25}$$

Then, (6.5.25) and Lemma 6.5.2 imply that (6.2.16) holds. □

Chapter 7

Optimality of series representation of the fractional Brownian motion via Haar basis

7.1 Introduction and statement of the main result

An important property of continuous Gaussian processes is that they can be represented as random series. The following theorem gives such a representation (see [LT91, Lif95])

Theorem 7.1.1. *Let $\{X(t)\}_{t\in K}$ be a real-valued, continous and centred Gaussian process where $K \subset \mathbb{R}$ is a compact set. Then, the process X can almost surely be represented as an uniformly convergent random series of the form*

$$\forall t \in K,\ X(t) = \sum_{n=0}^{+\infty} \varepsilon_n f_n(t), \tag{7.1.1}$$

where the real-valued random variables ε_n are independent with same distribution $\mathcal{N}(0,1)$ and the deterministic real-valued functions f_n are continuous on K.

In the case of the Brownian motion $\{B(t)\}_{t\in[0,1]}$, an example is given by the following series representation obtained via the trigonometric system

$$\forall t \in [0,1],\ B(t) = \varepsilon_0 \times t + \sqrt{2}\sum_{n=1}^{+\infty} \varepsilon_n \frac{\sin(\pi n t)}{\pi n}$$

where $(\varepsilon_n)_{n\geq 0}$ is a standard normal independent sequence.

In order to simulate the process X, one must replace the series (7.1.1) by a finite sum; since there may exist many series representations of the type (7.1.1), it seems natural to look for the ones for which the tail of the series

$$u_m := \mathbb{E}\left(\sup_{t\in K}\left|\sum_{n=m}^{+\infty} \varepsilon_n f_n(t)\right|\right)$$

converge to zero as fast as possible when m goes to $+\infty$. This lead us to introduce the sequence $(l_m(X))_{m\in\mathbb{N}}$ defined as

$$l_m(X) := \inf\left\{\mathbb{E}\left(\sup_{t\in K}\left|\sum_{n=m}^{+\infty}\varepsilon_n f_n(t)\right|\right) \;:\; X(t) = \sum_{n=1}^{+\infty}\varepsilon_n f_n(t)\right\}.$$

The real number $l_m(X)$ is called the *m-th l-approximation number of X*. A representation of the type (7.1.1) is said to be *optimal* if

$$\mathbb{E}\left(\sup_{t\in K}\left|\sum_{n=m}^{+\infty}\varepsilon_n f_n(t)\right|\right) \underset{m\to+\infty}{=} O\left(l_m(X)\right).$$

Kühn and Linde obtained in [KL02] sharp estimates of the asymptotic behavior of the sequence $l_m(X)$ in the case where X is a fractional Brownian motion B_H or more generally a fractional Brownian sheet. The following theorem provides their estimates in the case of the fractional Brownian motion $\{B_H(t)\}_{t\in[0,1]}$ of Hurst parameter $H \in (0,1)$.

Theorem 7.1.2. *There is two deterministic constants $0 < c_1 \leq c_2$ such that*

$$\forall m \geq 2,\ \ c_1 m^{-H}\sqrt{\log(m)} \leq l_m(B_H) \leq c_2 m^{-H}\sqrt{\log(m)}.$$

Thus, a representation $B_H = \sum_{n=0}^{+\infty}\varepsilon_n f_n$ is optimal *if*

$$\mathbb{E}\left(\sup_{t\in[0,1]}\left|\sum_{n=m+1}^{+\infty}\varepsilon_n f_n(t)\right|\right) \underset{m\to+\infty}{=} O\left(m^{-H}\sqrt{\log(m)}\right). \tag{7.1.2}$$

Several optimal representations are known for the fractional Brownian motion (see [GS03, Nda18]). An important example of such an optimal representation has been optained by Ayache and Taqqu in [AT03] via regular wavelets. Suppose that ψ is either a Lemarié-Meyer or Daubechies mother wavelet with at least $N \geq 2$ first vanishing moments; Meyer, Sellan and Taqqu proved in [MST99] that the fractional Brownian motion $\{B_H(t)\}_{t\in[0,1]}$ can be represented as the following almost surely uniformly convergent series

$$\forall t \in [0,1],\ \ B_H(t) = \sum_{j=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} 2^{-jH}\left(\Psi_H(2^j t - k) - \Psi_H(-k)\right)\varepsilon_{j,k}, \tag{7.1.3}$$

where Ψ_H is the factional primitive of ψ of order $H + 1/2$ defined via its Fourier transform as follows

$$\widehat{\Psi}_H(\xi) := (i\xi)^{-H-1/2}\widehat{\psi}(\xi),$$

and where the $\varepsilon_{j,k}$ are independent random variables $\mathcal{N}(0,1)$. Then, Ayache and Taqqu proved the optimality of the series representation (7.1.3) when the mother wavelet ψ is regular enough.

In this chapter, we will study the optimality of the representation of the fractional Brownian motion via the Haar basis $\{(h_{j,k})_{j,k\in\mathbb{Z}}\}$. Recall that, for all $j,k\in\mathbb{Z}$, the function $h_{j,k}$ is defined as

$$\forall s\in\mathbb{R},\ h_{j,k}(s) := 2^{j/2}h(2^j s-k),$$

where h is the mother wavelet defined as

$$h := \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}.$$

One of the main interest of the Haar basis here is that the coefficients of the decomposition of a function in it can be computed explicitely (see Remark 2.3.4). Yet, in contrast to the previous mother wavelet ψ, the function h is very discontinuous and for that reason the results of Ayache and Taqqu do not cover it.

The fractional Brownian motion can be decomposed into two parts

$$\forall t\in[0,1],\ B_H(t) = \dot{B}_H(t) + R_H(t), \tag{7.1.4}$$

where R_H is called the high-frequency part of the fractional Brownian motion (or Riemann-Liouville process) and defined as

$$\forall t\in[0,1],\ R_H(t) := \int_0^{+\infty} (t-s)_+^{H-1/2}dB(s), \tag{7.1.5}$$

and $\dot{B}_H$ is called the low-frequency part of the fractional Brownian motion and defined as

$$\forall t\in[0,1],\ \dot{B}_H(t) := \int_{\mathbb{R}} K(t,s)dB(s) \tag{7.1.6}$$

where

$$\forall (t,s)\in[0,1]\times\mathbb{R},\ K(t,s) := \Big((t-s)^{H-1/2} - (-s)^{H-1/2}\Big)\mathbb{1}_{(-\infty,0)}(s).$$

In the article [AL09], Ayache and Linde decomposed the kernel of the high-frequency part (7.1.5) in the Haar basis. Using this decomposition, they proved that the equality (7.1.2) is satisfied for the Haar series representation of the high-frequency part of the fractional Brownian motion (7.1.5). Nevertheless, the optimality of the Haar series representation of the fractionnal Brownian motion remained an open question.

By decomposing the kernel K in the Haar basis, one gets the following equality in $L^2(\mathbb{R})$

$$\forall t\in[0,1],\ \dot{B}_H(t) = \sum_{j=-\infty}^{+\infty}\sum_{k=1}^{+\infty} b_{j,k}(t)\varepsilon_{j,k} \tag{7.1.7}$$

where for all $j,k\in\mathbb{Z}$ one has $b_{j,k}(t) := \langle K(t,\cdot), h_{-j,-k}\rangle$ and $\varepsilon_{j,k} := \int_{-\infty}^0 h_{-j,-k}(s)dB(s)$.

The main result of this chapter shows that the equality (7.1.2) holds for the Haar series representation of the low-frequency part of the fractional Brownian motion (7.1.7). A consequence on this new result is the optimality of the Haar series representation for the fractional Brownian motion.

For all real numbers $a \leq b$ and real sequence $(u_k)_{k\in\mathbb{Z}}$ we will use the following convention

$$\sum_{k=a}^{b} u_k := \sum_{k\in[a,b]\cap\mathbb{Z}} u_k.$$

Our new result is given by the following theorem.

Theorem 7.1.3. *Let us fix $q \in (H,1)$ and $p := \lfloor (H+(q-H)(1-H)) \min(1-H,H)^{-1} \rfloor + 1$. For all $t \in [0,1]$ and $J \geq 0$, one defines*

$$S_J(t) := \sum_{j=-pJ}^{pJ} \sum_{k=1}^{2^{qJ}} b_{j,k}(t)\varepsilon_{j,k}, \tag{7.1.8}$$

and

$$R_J(t) := \dot{B}_H(t) - S_J(t).$$

There is a random variable C of finite moments of any order such that

$$\forall J \geq 2, \ \sup_{t\in[0,1]} |R_J(t)| \leq C\sqrt{J\log(J)}2^{-J(H+(q-H)(1-H))}. \tag{7.1.9}$$

The number of terms in (7.1.8) is $\rho_J := (2pJ+1)\lfloor 2^{qJ} \rfloor$. For all $m \geq 1$, there exists an integer $J(m) \geq 0$ such that

$$\rho_{J(m)} \leq m < \rho_{J(m)+1}. \tag{7.1.10}$$

Notice that inequalities $m < \rho_{J(m)+1}$ and $q < 1$ imply that there exists an integer $m_0 \geq 2$ such that for all $m \geq m_0$, one has

$$m < (2p(J(m)+1)+1)2^{q(J(m)+1)} \leq 2^{J(m)}$$

thus

$$\forall m \geq m_0, \ \log_2(m) \leq J(m). \tag{7.1.11}$$

Let $\varphi : \mathbb{N} \to \mathbb{Z} \times \mathbb{N}$ a bijection such that for all $J \geq 0$, one has

$$n \leq \rho_J \implies \varphi(n) \in [-pJ, pJ] \times [1, \lfloor 2^{qJ} \rfloor]. \tag{7.1.12}$$

For all $(j,k) \in \mathbb{Z} \times \mathbb{N}$, one defines

$$b(j,k) := b_{j,k} \text{ and } \varepsilon(j,k) := \varepsilon_{j,k}.$$

Moreover, for all $n \geq 1$ one introduces

$$\overline{b}_n := b(\varphi(n)) \text{ and } \overline{\varepsilon}_n := \varepsilon(\varphi(n))$$

and for all $m \geq 1$ and $t \in [0,1]$, one defines

$$\mathcal{S}_m(t) := \sum_{n=1}^{m} \overline{b}_n(t)\overline{\varepsilon}_n \text{ and } \mathcal{R}_m(t) := \dot{B}_H(t) - \mathcal{S}_m(t) = \sum_{n=m+1}^{+\infty} \overline{b}_n(t)\overline{\varepsilon}_n.$$

Observe that, using (7.1.12) one has

$$\forall m \geq 1,\ \forall t \in [0,1],\ R_{J(m)}(t) = \mathcal{R}_{\rho_{J(m)}}(t). \tag{7.1.13}$$

Lemma 7.1.4. *The real sequence* $\left(\mathbb{E}\left(\sup_{t\in[0,1]} |\mathcal{R}_m(t)|\right)\right)_{m\geq 1}$ *is decreasing.*

Proof. Let us introduce for all $m \geq 1$ the σ-algebra $\mathcal{F}(m)$ defined as

$$\mathcal{F}(m) := \sigma(\overline{\varepsilon}_n,\ n > m).$$

For all $m_2 > m_1 \geq 1$ one has

$$\mathbb{E}\left(\sup_{t\in[0,1]} |\mathcal{R}_{m_1}(t)|\right) = \mathbb{E}\left(\sup_{t\in[0,1]} \left|\sum_{n=m_1+1}^{+\infty} \overline{b}_n(t)\overline{\varepsilon}_n\right|\right) = \mathbb{E}\left(\mathbb{E}\left[\sup_{t\in[0,1]} \left|\sum_{n=m_1+1}^{+\infty} \overline{b}_n(t)\overline{\varepsilon}_n\right| \middle| \mathcal{F}(m_2)\right]\right) \tag{7.1.14}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,1]} \left|\sum_{n=m_1+1}^{+\infty} \overline{b}_n(t)\overline{\varepsilon}_n\right| \middle| \mathcal{F}(m_2)\right] \geq \sup_{t\in[0,1]} \mathbb{E}\left[\left|\sum_{n=m_1+1}^{+\infty} \overline{b}_n(t)\overline{\varepsilon}_n\right| \middle| \mathcal{F}(m_2)\right]. \tag{7.1.15}$$

Since the centered Gaussian variables $(\overline{\varepsilon}_n)_{n\in\mathbb{N}}$ are independents, one has

$$\mathbb{E}\left[\left|\sum_{n=m_1+1}^{+\infty} \overline{b}_n(t)\overline{\varepsilon}_n\right| \middle| \mathcal{F}(m_2)\right] = \left|\sum_{n=m_2+1}^{+\infty} \overline{b}_n(t)\overline{\varepsilon}_n\right| = |\mathcal{R}_{m_2}(t)|\,. \tag{7.1.16}$$

Combining (7.1.14), (7.1.15) and (7.1.16) one obtains

$$\mathbb{E}\left(\sup_{t\in[0,1]} |\mathcal{R}_{m_1}(t)|\right) \geq \mathbb{E}\left(\sup_{t\in[0,1]} |\mathcal{R}_{m_2}(t)|\right).$$

□

Notice that, using Lemma 7.1.4, (7.1.10) and (7.1.13), for all $m \geq 1$ one obtains

$$\mathbb{E}\left(\sup_{t\in[0,1]} |\mathcal{R}_m(t)|\right) \leq \mathbb{E}\left(\sup_{t\in[0,1]} \left|\mathcal{R}_{\rho_{J(m)}}(t)\right|\right) = \mathbb{E}\left(\sup_{t\in[0,1]} \left|R_{J(m)}(t)\right|\right). \tag{7.1.17}$$

Then, inequalities (7.1.9) and (7.1.17) imply that there exists a deterministic constant $c > 0$ such that for all $m \geq 1$ with $J(m) \geq 2$, one has

$$\mathbb{E}\left(\sup_{t\in[0,1]} |\mathcal{R}_m(t)|\right) \leq c\sqrt{J(m)\log(J(m))}2^{-J(m)(H+(q-H)(1-H))}. \tag{7.1.18}$$

Let us introduce the bijection $f : \mathbb{R}_+ \to [1, +\infty)$ defined as $f(x) := (2px + 1)\lfloor 2^{qx} \rfloor$ such that $J(m) = \lfloor f^{-1}(m) \rfloor$ for all $m \geq 1$ where f^{-1} denotes the inverse function of f. Notice that $f(x) \underset{+\infty}{\sim} 2px2^{qx}$ and $f^{-1}(m) \underset{+\infty}{\sim} J(m)$ thus $m \underset{+\infty}{\sim} 2pJ(m)2^{qJ(m)}$. One obtains using basic computations

$$J(m) \underset{+\infty}{\sim} (q\log(2))^{-1}\log(m). \tag{7.1.19}$$

Putting together (7.1.18) with (7.1.19) and (7.1.11), there exists a deterministic constant $c' > 0$ such that for all $m \geq m_0$ one has

$$\mathbb{E}\left(\sup_{t\in[0,1]} |\mathcal{R}_m(t)|\right) \leq c'\sqrt{\log(m)\log(\log(m))} \times m^{-(H+(q-H)(1-H))},$$

which implies

$$\mathbb{E}\left(\sup_{t\in[0,1]} |\mathcal{R}_m(t)|\right) \underset{m\to+\infty}{=} O\left(m^{-H}\sqrt{\log(m)}\right). \tag{7.1.20}$$

Remark 7.1.5. Theorem 7.1.3 does not imply the optimality of the Haar series representation of the low-frequency part of the fractional Brownian motion. Indeed, Belinsky and Linde proved in [BL02] that $l_m(\dot{B}_H)$ tends to zero exponentially which is much stronger than inequality (7.1.20).

7.2 Proof of Theorem 7.1.3

In this section, all the randoms variables are of finite moments of any order.

For all integer $J \geq 0$ and real number $t \in [0,1]$, let us define $R_J^1(t)$ and $R_J^2(t)$ as

$$R_J^1(t) := \sum_{j=1}^{pJ}\sum_{k=2^{qJ}+1}^{+\infty} |b_{j,k}(t)\varepsilon_{j,k}| + \sum_{j=pJ+1}^{+\infty}\sum_{k=1}^{+\infty} |b_{j,k}(t)\varepsilon_{j,k}| \tag{7.2.1}$$

and

$$R_J^2(t) := \sum_{j=0}^{pJ}\sum_{k=2^{qJ}+1}^{+\infty} |b_{-j,k}(t)\varepsilon_{-j,k}| + \sum_{j=pJ+1}^{+\infty}\sum_{k=1}^{+\infty} |b_{-j,k}(t)\varepsilon_{-j,k}|. \tag{7.2.2}$$

Notice that, for all $J \geq 2$, one has $R_J = R_J^1 + R_J^2$.

7.2.A Preliminary lemmas

Observe that, using the substitution $u := 2^{-j}s + k$, one obtains for all $(j,k) \in \mathbb{Z} \times \mathbb{N}$

$$\begin{aligned} b_{j,k}(t) &= 2^{-j/2} \int_{-\infty}^{0} \Big((t-s)^{H-1/2} - (-s)^{H-1/2} \Big) h(2^{-j}s + k) ds \\ &= 2^{j/2} \int_{-\infty}^{k} \Big((t-(u-k)2^j)^{H-1/2} - (-(u-k)2^j)^{H-1/2} \Big) h(u) du \\ &= 2^{jH} \int_{0}^{1} \Big((2^{-j}t + k - u)^{H-1/2} - (k-u)^{H-1/2} \Big) h(u) du \\ &= 2^{jH} \int_{0}^{1/2} g_{j,k}(t,u) du \end{aligned} \tag{7.2.3}$$

where

$$g_{j,k}(t,u) = (2^{-j}t + k - u)^{H-1/2} - (k-u)^{H-1/2} - (2^{-j}t + k - u - 1/2)^{H-1/2} + (k - u - 1/2)^{H-1/2}.$$

Lemma 7.2.1. *There is a constant $a_1 > 0$ such that, for all $j \in \mathbb{Z}$ and $k \geq 2$, one has*

$$\sup_{t \in [0,1]} |b_{j,k}(t) - b_{j,k+1}(t)| \leq a_1 2^{-j(1-H)} k^{H-7/2}.$$

Proof. Let us fix $t \in (0,1]$. Using mean value theorem to the function $g_{1,j,k} : x \mapsto (x + k - u)^{H-1/2} - (x + k - u - 1/2)^{H-1/2}$ on the interval $[0, 2^{-j}t]$ one gets

$$g_{j,k}(t,u) = 2^{-j}t \times g'_{1,j,k}(c_1) \tag{7.2.4}$$

where $c_1 \in (0, 2^{-j}t)$. Applying the mean value theorem to the function $g_{2,j,k} : x \mapsto (x + k - u - 1/2)^{H-3/2}$ on the interval $[c_1, c_1 + 1/2]$, one obtains

$$\begin{aligned} g'_{1,j,k}(c_1) &= (H-1/2)(g_{2,j,k}(c_1 + 1/2) - g_{2,j,k}(c_1)) = 2^{-1}(H-1/2) g'_{2,j,k}(c_2) \\ &= 2^{-1}(H-1/2)(H-3/2)(c_2 + k - u - 1/2)^{H-5/2} \end{aligned} \tag{7.2.5}$$

where $c_2 \in (c_1, c_1 + 1/2)$; notice that c_2 (and c_1) depends on j and k. Combining (7.2.4) and (7.2.5), one obtains for all $j \in \mathbb{Z}$ and $k \geq 1$

$$g_{j,k}(t,u) = t2^{-j-1}(H-1/2)(H-3/2)(c_2 + k - u - 1/2)^{H-5/2} \tag{7.2.6}$$

and

$$g_{j,k+1}(t,u) = t2^{-j-1}(H-1/2)(H-3/2)(c'_2 + k - u + 1/2)^{H-5/2} \tag{7.2.7}$$

where $c_2, c'_2 \in (0, 2^{-j}t + 1/2)$. Using (7.2.3), one gets for $k \geq 2$

$$(b_{j,k}(t) - b_{j,k+1}(t)) = 2^{jH} \int_{0}^{1/2} (g_{j,k}(t,u) - g_{j,k+1}(t,u)) \, du. \tag{7.2.8}$$

Combining (7.2.6), (7.2.7) and the mean value theorem applied to the function $x \mapsto (x + k - u)^{H-5/2}$ on the interval $[\min(c_2 - 1/2, c'_2 + 1/2), \max(c_2 - 1/2, c'_2 + 1/2)]$, one obtains

$$g_{j,k}(t,u) - g_{j,k+1}(t,u) = 2^{-j-1}t(H-1/2)(H-3/2)(H-5/2)(c_3 + k - u)^{H-7/2} \tag{7.2.9}$$

where $c_3 \in (-1/2, t2^{-j}+1)$. Since $t \in [0,1]$, $H \in (0,1)$, (7.2.8) and (7.2.9) one obtains for $k \geq 2$

$$\begin{aligned}|b_{j,k}(t) - b_{j,k+1}(t)| &\leq \frac{15}{16} 2^{-j(1-H)} \int_0^{1/2} (k-u-1/2)^{H-7/2} du \\ &\leq \frac{15}{16} 2^{-j(1-H)} k^{H-7/2} \int_0^{1/2} \left(1 - \frac{u+1/2}{k}\right)^{H-7/2} du \\ &\leq a_1 2^{-j(1-H)} k^{H-7/2}\end{aligned}$$

where $a_1 := \dfrac{15}{16} \displaystyle\int_0^{1/2} \left(1 - \frac{u+1/2}{2}\right)^{H-7/2} du.$ □

Lemma 7.2.2. *There is a constant $a_2 > 0$ such that, for all $j \geq 0$ and $k \geq 2$, one has*

$$\sup_{t\in[0,1]} |b_{-j,k}(t) - b_{-j,k+1}(t)| \leq a_2 2^{-jH} k^{H-5/2}.$$

Proof. Let us fix $t \in (0,1]$. Using (7.2.3), for all $j \geq 0$ and $k \geq 2$, one has

$$b_{-j,k}(t) = 2^{-jH} \int_0^{1/2} g_{-j,k}(t,u) du. \tag{7.2.10}$$

Using the mean value theorem on the interval $[0,1/2]$ to the function $x \mapsto (2^j t + k - u - x)^{H-1/2} - (k-u-x)^{H-1/2}$, one obtains

$$g_{-j,k}(t,u) = 2^{-1}(H-1/2)\left((2^j t + k - u - c_1)^{H-3/2} - (k-u-c_1)^{H-3/2}\right) \tag{7.2.11}$$

where $c_1 \in (0,1/2)$ depends on j and k. In the same way, there is a constant $c_1' \in (0,1/2)$ such that

$$g_{-j,k+1}(t,u) = 2^{-1}(H-1/2)\left((2^j t + k + 1 - u - c_1')^{H-3/2} - (k+1-u-c_1')^{H-3/2}\right). \tag{7.2.12}$$

For all $k \geq 2$, thanks to the mean value theorem applied on the interval $[k-c_1, k+1-c_1']$ to the function $x \mapsto (2^j t + x - u)^{H-3/2} - (x-u)^{H-3/2}$, one obtains

$$g_{-j,k}(t,u) - g_{-j,k+1}(t,u) = 2^{-1}(H-1/2)(H-3/2)\left((2^j t + c_2 - u)^{H-5/2} - (c_2-u)^{H-5/2}\right)$$

where $c_2 \in (k-c_1, k+1-c_1')$, thus

$$|g_{-j,k}(t,u) - g_{-j,k+1}(t,u)| = \frac{3}{8}(c_2-u)^{H-5/2} \leq \frac{3}{8}(k-u-1/2)^{H-5/2}. \tag{7.2.13}$$

Combining (7.2.10) and (7.2.13) one obtains

$$\begin{aligned}|b_{-j,k}(t) - b_{-j,k+1}(t)| &\leq \frac{3}{8} 2^{-jH} \int_0^{1/2} (k-u-1/2)^{H-5/2} du \\ &\leq a_2 2^{-jH} k^{H-5/2}\end{aligned}$$

where $a_2 := \dfrac{3}{8} \displaystyle\int_0^{1/2} \left(1 - \frac{u+1/2}{2}\right)^{H-5/2} du.$ □

7.2.B Upper bound for R_J^1

One introduces two integers $K_2 > K_1 \geq 2$ and the sequence $\eta_{j,k} := \sum_{m=1}^{k} \varepsilon_{j,m}$ for $j \geq 1$ and $k \geq 1$. Using Abel transform, one gets for $t \in [0,1]$ and $j \geq 1$

$$\sum_{k=K_1}^{K_2} b_{j,k}(t)\varepsilon_{j,k} = (b_{j,K_2}(t)\eta_{j,K_2} - b_{j,K_1}(t)\eta_{j,K_1-1}) + \sum_{k=K_1}^{K_2-1} (b_{j,k}(t) - b_{j,k+1}(t))\eta_{j,k}. \tag{7.2.14}$$

Lemma 7.2.3. *There is a random variable $C_1' > 0$ such that, for all integers $j \geq 1$ and $K_1 \geq 2$, one has*

$$\sup_{t\in[0,1]} \sum_{k=K_1}^{+\infty} \left|(b_{j,k}(t) - b_{j,k+1}(t))\eta_{j,k}\right| \leq C_1' 2^{-j(1-H)} K_1^{H-2}\sqrt{\log(3+j+K_1)}.$$

Proof. Using Lemma A.23 in [Aya19], there is a random variable C such that one has almost surely

$$\forall j \geq 1,\ \forall k \geq 1,\ \left|\frac{\eta_{j,k}}{\sigma(\eta_{j,k})}\right| \leq C\sqrt{\log(3+j+k)} \tag{7.2.15}$$

where $\sigma(Y)$ denotes the standard deviation of a random variable Y. Since the random variables $(\varepsilon_{j,k})_{j,k\in\mathbb{Z}}$ are independents, one obtains

$$\forall j \geq 1,\ \forall k \geq 1,\ |\eta_{j,k}| \leq C\sqrt{k}\sqrt{\log(3+j+k)}. \tag{7.2.16}$$

Combining Lemma 7.2.1 with (7.2.16), for all integers $j \geq 1$, $K_2 > K_1 \geq 2$ and $t \in (0,1]$ one gets

$$\sum_{k=K_1}^{K_2} \left|(b_{j,k}(t) - b_{j,k+1}(t))\eta_{j,k}\right| \leq a_1 C 2^{-j(1-H)} \sum_{k=K_1}^{K_2} k^{H-3}\sqrt{\log(3+j+k)}. \tag{7.2.17}$$

For all fixed real number $a \geq 3$, the function $x \mapsto x^{H-3}\sqrt{\log(a+x)}$ is decreasing in $[0,+\infty)$ since its derivative function is negative. Thus, one gets

$$\sum_{k=K_1}^{K_2} k^{H-3}\sqrt{\log(3+j+k)} \leq \sum_{k=K_1}^{K_2} \int_{k-1}^{k} x^{H-3}\sqrt{\log(3+j+x)}dx \leq \int_{K_1-1}^{+\infty} x^{H-3}\sqrt{\log(3+j+x)}dx. \tag{7.2.18}$$

By integration by parts, one has

$$\begin{aligned}
&\int_{K_1-1}^{+\infty} x^{H-3}\sqrt{\log(3+j+x)}dx \\
&= \frac{1}{H-2}\left[x^{H-2}\sqrt{\log(3+j+x)}\right]_{K_1-1}^{+\infty} + \frac{1}{2(2-H)}\int_{K_1-1}^{+\infty} \frac{x^{H-2}}{(3+j+x)\sqrt{\log(3+j+x)}}dx \\
&\leq \frac{1}{2-H}(K_1-1)^{H-2}\sqrt{\log(2+j+K_1)} + \frac{1}{2(2-H)}\int_{K_1-1}^{+\infty} x^{H-3}dx \\
&\leq \frac{1}{2-H}(K_1-1)^{H-2}\sqrt{\log(2+j+K_1)} + \frac{1}{2(2-H)^2}(K_1-1)^{H-2} \\
&\leq c_1 K_1^{H-2}\sqrt{\log(3+j+K_1)}
\end{aligned} \tag{7.2.19}$$

where c_1 is a constant only depending on H. Combining (7.2.17), (7.2.18) and (7.2.19), the lemma is proved when K_2 goes to $+\infty$. □

Lemma 7.2.4. *There is a random variable $C_1 > 0$ such that for all integers $j \geq 1$, $J \geq 2$ and real number $a \in [J^{-1}, 1)$ one has*

$$\sup_{t\in[0,1]} \sum_{k=2^{aJ}}^{+\infty} |b_{j,k}(t)\varepsilon_{j,k}| \leq 8C_1 2^{-j(1-H)} 2^{-a(2-H)J} \sqrt{aJ\log(3+j)}.$$

Proof. For all $t \in [0,1]$, $j \geq 1$ and $K_2 \geq 3$, using (7.2.3), (7.2.4), (7.2.5) and (7.2.16) one obtains

$$\begin{aligned} |b_{j,K_2}(t)\eta_{j,K_2}| &\leq \frac{3C}{8} 2^{-j(1-H)} \sqrt{K_2}\sqrt{\log(3+j+K_2)} \int_0^{1/2} (K_2-u-1/2)^{H-5/2} du \\ &\leq c_1 C 2^{-j(1-H)} K_2^{H-2} \sqrt{\log(3+j+K_2)} \end{aligned} \tag{7.2.20}$$

where c_1 is a constant only depending on H. In particular, one obtains

$$\forall t \in [0,1],\ \forall j \geq 1,\ b_{j,K_2}(t)\eta_{j,K_2} \underset{K_2\to+\infty}{\longrightarrow} 0. \tag{7.2.21}$$

In the same way as (7.2.20), one obtains for $K_1 \geq 2$

$$|b_{j,K_1}(t)\eta_{j,K_1-1}| \leq c_2 C 2^{-j(1-H)} K_1^{H-2} \sqrt{\log(2+j+K_1)} \tag{7.2.22}$$

where c_2 is a constant only depending on H. Combining (7.2.22), (7.2.21), Lemma 7.2.3 and (7.2.14), there is a random variable C_1 such that for all $j \geq 1$ and $K_1 \geq 2$ one has almost surely

$$\sup_{t\in[0,1]} \sum_{k=K_1}^{+\infty} |b_{j,k}(t)\varepsilon_{j,k}| \leq C_1 2^{-j(1-H)} K_1^{H-2} \sqrt{\log(3+j+K_1)}. \tag{7.2.23}$$

For all integers $j \geq 1$, $J \geq 2$ and real numbers $a \in [J^{-1}, 1)$, $t \in [0,1]$ one obtains by using the substitution $K_1 = \lfloor 2^{aJ} \rfloor$ in (7.2.23)

$$\begin{aligned} \sum_{k=2^{aJ}}^{+\infty} |b_{j,k}(t)\varepsilon_{j,k}| \leq \sum_{k=\lfloor 2^{aJ}\rfloor}^{+\infty} |b_{j,k}(t)\varepsilon_{j,k}| &\leq C_1 2^{-j(1-H)} (\lfloor 2^{aJ} \rfloor)^{H-2} \sqrt{\log(3+j+\lfloor 2^{aJ}\rfloor)} \\ &\leq 4C_1 2^{-j(1-H)} 2^{-a(2-H)J} \sqrt{\log(3+j+2^{aJ})} \end{aligned} \tag{7.2.24}$$

since $\sup_{x\geq 1} \left(\frac{\lfloor x \rfloor}{x}\right)^{H-2} \leq 2^{2-H} \leq 4$. Moreover, one has

$$\begin{aligned} \sqrt{\log(3+j+2^{aJ})} &= \sqrt{\log(3+j)\log(3+2^{aJ})} \\ &\leq \sqrt{\log(3+j)}\sqrt{a\log(2)J + \log(4)} \\ &\leq \sqrt{3\log(2)}\sqrt{aJ\log(3+j)} \leq 2\sqrt{aJ\log(3+j)}. \end{aligned} \tag{7.2.25}$$

Combining (7.2.24) and (7.2.25), for all $a \in [J^{-1}, 1)$, $t \in (0,1]$ and all integers $j \geq 1$ and $J \geq 2$ one obtains

$$\sum_{k=2^{aJ}}^{+\infty} |b_{j,k}(t)\varepsilon_{j,k}| \leq 8C_1 2^{-j(1-H)} 2^{-a(2-H)J} \sqrt{aJ\log(3+j)}.$$

□

Lemma 7.2.5. *There is a random constant $C_0 \geq 0$ such that for all integer $J \geq \max((2-H)H^{-1}, 2)$ one has*

$$\sup_{t\in[0,1]} R_J^1(t) \leq C_0 \left(2^{-q(2-H)J}\sqrt{J} + 2^{-pJ(1-H)}\sqrt{\log(3+pJ)} + 2^{-pJ/2}\sqrt{\log(3+pJ)}\right).$$

Proof. Let us fix an integer $J \geq \max((2-H)H^{-1}, 2)$. Using Lemma 7.2.4 with $a := q$, for all $t \in [0,1]$ one obtains

$$\sum_{j=1}^{pJ}\sum_{k=2^{qJ}+1}^{+\infty} |b_{j,k}(t)\varepsilon_{j,k}| \leq 8C_1 2^{-q(2-H)J}\sqrt{J}\sum_{j=1}^{pJ}\left(2^{-j(1-H)}\sqrt{\log(3+j)}\right)$$
$$\leq C_2 2^{-q(2-H)J}\sqrt{J} \tag{7.2.26}$$

where $C_2 := 8C_1 \sum_{j=1}^{+\infty}\left(2^{-j(1-H)}\sqrt{\log(3+j)}\right)$. Thanks to Lemma 7.2.4 with $a = J^{-1}$ one gets for all $t \in [0,1]$

$$\sum_{j=pJ+1}^{+\infty}\sum_{k=2}^{+\infty} |b_{j,k}(t)\varepsilon_{j,k}| \leq 8C_1 \sum_{j=1}^{+\infty}\left(2^{-(j+pJ)(1-H)}\sqrt{\log(3+j+pJ)}\right). \tag{7.2.27}$$

Notice that, for all $j \geq 1$ one has

$$\log(3+j+pJ) \leq \log(3+pJ)\log(3+j). \tag{7.2.28}$$

Combining (7.2.27) and (7.2.28), one obtains

$$\sum_{j=pJ+1}^{+\infty}\sum_{k=2}^{+\infty} |b_{j,k}(t)\varepsilon_{j,k}| \leq 8C_1 \times 2^{-pJ(1-H)}\sqrt{\log(3+pJ)}\sum_{j=1}^{+\infty}\left(2^{-j(1-H)}\sqrt{\log(3+j)}\right)$$
$$\leq C_3 2^{-pJ(1-H)}\sqrt{\log(3+pJ)} \tag{7.2.29}$$

where $C_3 := 8C_1 \sum_{j=1}^{+\infty}\left(2^{-j(1-H)}\sqrt{\log(3+j)}\right)$.

We still have to manage the terms $\sum_{j=pJ+1}^{+\infty} |b_{j,1}(t)\varepsilon_{j,1}|$ which correspond to the case $k=1$. Let us start by dealing with the case $H > 1/2$; using (7.2.3) and the mean value theorem, one obtains for all $j \geq 1$ and $t \in [0,1]$

$$|b_{j,1}(t)| \leq 2^{qJ}\int_0^1 \left|(2^{-j}t+1-u)^{H-1/2} - (1-u)^{H-1/2}\right| du \leq c_1 2^{-j(1-H)} \tag{7.2.30}$$

where $c_1 := |H-1/2|\int_0^1 (1-u)^{H-3/2}du$. Moreover, using Lemma A.23 in [Aya19], there is a random variable C, such that

$$\forall j \geq 1,\ \forall k \geq 1,\ |\varepsilon_{j,k}| \leq C\sqrt{\log(3+|j|+|k|)}. \tag{7.2.31}$$

Combining (7.2.30) and (7.2.31) one gets for $H > 1/2$ and $j \geq 1$

$$|b_{j,1}(t)\varepsilon_{j,1}| \leq c_1 C 2^{-j(1-H)}\sqrt{\log(4+j)} \leq 2c_1 C 2^{-j(1-H)}\sqrt{\log(3+j)}$$

thus, using inequality (7.2.28), one gets for some random variable C_4

$$\sum_{j=pJ+1}^{+\infty} |b_{j,1}(t)\varepsilon_{j,1}| \leq C_4 2^{-pJ(1-H)}\sqrt{\log(3+pJ)}. \tag{7.2.32}$$

For the case $H \leq 1/2$, using the definition of $b_{j,1}(t)$ one has for all $t \in (0,1]$

$$|b_{j,1}(t)| \leq 2^{-j/2}\int_{-\infty}^{0} \left|(t-s)^{H-1/2} - (-s)^{H-1/2}\right| ds \leq 2^{-j/2}\int_{-\infty}^{0} \left((-s)^{H-1/2} - (1-s)^{H-1/2}\right) ds$$

thus, similarly to (7.2.32) one gets for some random variable C_5

$$\sum_{j=pJ+1}^{+\infty} |b_{j,1}(t)\varepsilon_{j,1}| \leq C_5 2^{-pJ/2}\sqrt{\log(3+pJ)}. \tag{7.2.33}$$

The lemma is proved by combining definition (7.2.1) with (7.2.26), (7.2.29), (7.2.32) and (7.2.33). □

7.2.C Upper bound for R_J^2

One introduces two integers $K_2 > K_1 \geq 2$ and the sequence $\mu_{j,k} := \sum_{m=1}^{k} \varepsilon_{-j,m}$ for $j \geq 0$ and $k \geq 1$. Using Abel transform, one gets for $t \in (0,1]$ and $j \geq 0$

$$\sum_{k=K_1}^{K_2} b_{-j,k}(t)\varepsilon_{-j,k} = (b_{-j,K_2}(t)\mu_{j,K_2} - b_{-j,K_1}(t)\mu_{j,K_1-1}) + \sum_{k=K_1}^{K_2-1} (b_{-j,k}(t) - b_{-j,k+1}(t))\mu_{j,k}. \tag{7.2.34}$$

Lemma 7.2.6. *There is a random variable $C_2' > 0$ such that, for all $j \geq 0$ and $K_1 \geq 2$, one has*

$$\sup_{t\in[0,1]} \sum_{k=K_1}^{+\infty} \left|(b_{-j,k}(t) - b_{-j,k+1}(t))\mu_{j,k}\right| \leq C_2' 2^{-jH} K_1^{H-1}\sqrt{\log(3+j+K_1)}.$$

Proof. Using Lemma A.23 in [Aya19], there is a random variable C such that

$$\forall j \geq 0,\ \forall k \geq 1,\ |\mu_{j,k}| \leq C\sqrt{k}\sqrt{\log(3+|j|+|k|)}. \tag{7.2.35}$$

Using Lemma 7.2.2 and (7.2.35), the rest of the proof is analagous to the proof of Lemma 7.2.3. □

Lemma 7.2.7. *There is a random variable $C_2 > 0$ such that for all integers $j \geq 1,\ J \geq 2$ and real number $a \in [J^{-1}, 1)$ one has*

$$\sup_{t\in[0,1]} \sum_{k=2^{aJ}}^{+\infty} |b_{-j,k}(t)\varepsilon_{-j,k}| \leq 4C_2 2^{-jH} 2^{-a(1-H)J}\sqrt{aJ\log(3+j)} \tag{7.2.36}$$

Proof. Using (7.2.10), (7.2.11) and (7.2.35), one obtains for all $j \geq 0$, $K_2 > 2$ and $t \in [0,1]$

$$|b_{-j,K_2}(t)\mu_{j,K_2}| \leq C 2^{-jH-2}\sqrt{K_2}\sqrt{\log(3+j+K_2)}\int_0^{1/2}(K_2-u-1/2)^{H-3/2}du$$
$$\leq c_1 C 2^{-jH}\sqrt{\log(3+j+K_2)}K_2^{H-1} \quad (7.2.37)$$

where c_1 is a constant only depending on H. In particular, one obtains

$$\forall t \in [0,1],\ \forall j \geq 0,\ b_{j,K_2}(t)\mu_{j,K_2} \underset{K_2\to+\infty}{\longrightarrow} 0. \quad (7.2.38)$$

In the same way as (7.2.37), it could be shown that

$$|b_{-j,K_1}(t)\mu_{j,K_1-1}| \leq a_3' C 2^{-jH} K_1^{H-1}\sqrt{\log(2+j+K_1)} \quad (7.2.39)$$

where a_3' is a constant only depending on H. Using (7.2.34), Lemma 7.2.6, (7.2.38) and (7.2.39), there is a random variable C_2 such that for all $j \geq 0$ and $K_1 \geq 2$ one has almost surely

$$\sum_{k=K_1}^{+\infty}|b_{-j,k}(t)\varepsilon_{-j,k}| \leq C_2 2^{-jH}K_1^{H-1}\sqrt{\log(3+j+K_1)}. \quad (7.2.40)$$

For all integers $j \geq 0$, $J \geq 2$ and real numbers $a \in [J^{-1},1)$, $t \in [0,1]$ one obtains by using the substitution $K_1 = \lfloor 2^{aJ}\rfloor$ in (7.2.40)

$$\sum_{k=2^{aJ}}^{+\infty}|b_{-j,k}(t)\varepsilon_{j,k}| \leq \sum_{k=\lfloor 2^{aJ}\rfloor}^{+\infty}|b_{j,k}(t)\varepsilon_{j,k}| \leq C_2 2^{-jH}(\lfloor 2^{aJ}\rfloor)^{H-1}\sqrt{\log(3+j+\lfloor 2^{aJ}\rfloor)}$$
$$\leq 2C_2 2^{-jH}2^{-a(1-H)J}\sqrt{\log(3+j+2^{aJ})}$$

since $\sup_{x\geq 1}\left(\frac{\lfloor x\rfloor}{x}\right)^{H-1} \leq 2^{1-H} \leq 2$. Then, using (7.2.25), one obtains

$$\sum_{k=2^{aJ}}^{+\infty}|b_{-j,k}(t)\varepsilon_{j,k}| \leq 4C_2 2^{-jH}2^{-a(1-H)J}\sqrt{aJ\log(3+j)}.$$

□

Lemma 7.2.8. *There is a random variable $C_3 > 0$ such that for all integers $j \geq 1$, $J \geq 2$ and real number $a \in [J^{-1},1)$ one has*

$$\sup_{t\in[0,1]}\sum_{k=2^{aJ}}^{+\infty}|b_{-j,k}(t)\varepsilon_{-j,k}| \leq 8C_3 2^{j(1-H)}2^{-a(2-H)J}\sqrt{aJ\log(3+j)}. \quad (7.2.41)$$

Proof. This proof is exactly the same as the Lemma 7.2.4 by replacing j with $-j$. □

Lemma 7.2.9. *There is a random variable $C_0' \geq 0$ such that for all integer $J \geq 2$ one has*

$$\sup_{t\in[0,1]} R_J^2(t) \leq C_0'\left(2^{-J(H+(q-H)(1-H))}\sqrt{J\log(3+JH)} + 2^{-pJH}\sqrt{\log(3+pJ)}\right).$$

Proof. Using Lemma 7.2.7 and Lemma 7.2.8 with $a := q$, for all $t \in [0,1]$ and $J \geq 2$, one gets

$$\sum_{j=0}^{pJ} \sum_{k=2^{qJ}+1}^{+\infty} |b_{-j,k}(t)\varepsilon_{-j,k}| = \sum_{j=0}^{\lfloor JH \rfloor} \sum_{k=2^{qJ}+1}^{+\infty} |b_{-j,k}(t)\varepsilon_{-j,k}| + \sum_{j=\lfloor JH \rfloor+1}^{pJ} \sum_{k=2^{qJ}+1}^{+\infty} |b_{-j,k}(t)\varepsilon_{-j,k}|$$

$$\leq 8C_3 2^{-q(2-H)J}\sqrt{J} \sum_{j=0}^{\lfloor JH \rfloor} \left(2^{j(1-H)}\sqrt{\log(3+j)}\right) + 4C_2 2^{-q(1-H)J}\sqrt{J} \sum_{j=\lfloor JH \rfloor+1}^{pJ} \left(2^{-jH}\sqrt{\log(3+j)}\right)$$

$$\leq C_6\sqrt{J}\sqrt{\log(3+JH)}\left(2^{-J(q(2-H)-H(1-H))} + 2^{-J(q(1-H)+H^2)}\right) \tag{7.2.42}$$

where C_6 is a random variable. Observe that, since $q > H$ one has

$$q(2-H) - H(1-H) > q(1-H) + H^2 = H + (q-H)(1-H),$$

thus one obtains

$$\sum_{j=0}^{pJ} \sum_{k=2^{qJ}+1}^{+\infty} |b_{-j,k}(t)\varepsilon_{-j,k}| \leq C_6'\sqrt{J\log(3+JH)}2^{-J(H+(q-H)(1-H))}, \tag{7.2.43}$$

where C_6' is a random variable. Using (7.2.36) with $a = J^{-1}$ one gets for all $t \in [0,1]$

$$\sum_{j=pJ+1}^{+\infty} \sum_{k=2}^{+\infty} |b_{-j,k}(t)\varepsilon_{-j,k}| \leq 8C_2 \sum_{j=1}^{+\infty} 2^{-(j+pJ)H}\sqrt{\log(3+j+pJ)}. \tag{7.2.44}$$

Combining (7.2.44) and (7.2.28), one obtains

$$\sum_{j=pJ+1}^{+\infty} \sum_{k=2}^{+\infty} |b_{-j,k}(t)\varepsilon_{-j,k}| \leq 8C_2 \times 2^{-pHJ}\sqrt{\log(3+pJ)} \sum_{j=1}^{+\infty} 2^{-jH}\sqrt{\log(3+j)}$$

$$\leq C_7 2^{-pHJ}\sqrt{\log(3+pJ)} \tag{7.2.45}$$

where $C_7 := 8C_2 \sum_{j=1}^{+\infty} 2^{-jH}\sqrt{\log(3+j)}$. We still have to manage the terms $\sum_{j=pJ+1}^{+\infty} |b_{-j,1}(t)\varepsilon_{j,-1}|$ which correspond to the case $k = 1$. Using (7.2.3), for all $j \geq 0$ and $t \in [0,1]$ one gets

$$b_{-j,1}(t) = 2^{-jH}\int_0^1 \left((2^j t + 1 - u)^{H-1/2} - (1-u)^{H-1/2}\right)h(u)du \tag{7.2.46}$$

$$= 2^{-jH}\int_0^{1/2} g_{-j,1}(t,u)du. \tag{7.2.47}$$

In the case $H \leq 1/2$, using (7.2.46), for all $j \geq 0$ and $t \in [0,1]$ one gets

$$|b_{-j,1}(t)| \leq 2^{-jH}\int_0^1 \left((2^j t + 1 - u)^{H-1/2} + (1-u)^{H-1/2}\right)du \leq \left(2\int_0^1 (1-u)^{H-1/2}du\right)2^{-jH}. \tag{7.2.48}$$

In the case $H > 1/2$, using (7.2.11) and (7.2.47), for all $j \geq 0$ and $t \in [0,1]$ one gets

$$|b_{-j,1}(t)| \leq 2^{-jH-2} \int_0^{1/2} \Big((2^j t + 1/2 - u)^{H-3/2} + (1/2-u)^{H-3/2} \Big) du$$
$$\leq \Big(2^{-1} \int_0^{1/2} (1/2-u)^{H-3/2} du \Big) 2^{-jH}. \tag{7.2.49}$$

Combining (7.2.48), (7.2.49) and (7.2.31), there is a random variable $C_8 \geq 0$ such that

$$|b_{-j,1}(t)\varepsilon_{j,-1}| \leq C_8 2^{-jH} \sqrt{\log(4+j)}.$$

thus, similary to (7.2.45) one obtains for some random variable $C_9 \geq 0$

$$\sum_{j=pJ+1}^{+\infty} |b_{-j,1}(t)\varepsilon_{j,-1}| \leq C_9 2^{-pHJ} \sqrt{\log(3+pJ)}. \tag{7.2.50}$$

By combining (7.2.43), (7.2.45) and (7.2.50), the lemma is proved.

□

7.2.D Proof of the theorem

We are now in position to prove the Theorem 7.1.3.

Proof of Theorem 7.1.3. Observe that $q(2-H) = H + (q-H)(1-H) + (q-H^2)$. Thus by combining Lemma 7.2.5 and Lemma 7.2.9 with triangle inequality, there is a random variable $C_0'' \geq 0$ and an integer $J_0 \geq 2$ such that

$$\forall J \geq J_0, \ \sup_{t\in[0,1]} R_J(t) \leq \sup_{t\in[0,1]} R_J^1(t) + \sup_{t\in[0,1]} R_J^2(t)$$
$$\leq C_0'' \Big(2^{-J(H+(q-H)(1-H))} \sqrt{J \log(3+JH)} + 2^{-p\min(1-H,H)J} \sqrt{\log(3+pJ)} \Big).$$

Since $p > (H + (q-H)(1-H)) \min(1-H,H)^{-1}$, the theorem is proved.

□

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