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Étude de la Conjecture de Syracuse et des opérateurs de Bishop du point de vue de la dynamique linéaire

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Résumé

Nous nous intéressons dans cette thèse à deux célèbres problèmes ouverts, que sont la conjecture de Syracuse et le Problème du sous-espace invariant. Nous les étudierons du point de vue de la dynamique linéaire. La dynamique linéaire consiste en l'étude du comportement des itérés d'un opérateur linéaire continu agissant sur un espace de Banach ou de Fréchet. Cette théorie comprend notamment les notions de cyclicité, qui requiert l'existence d'orbites engendrant un sous-espace dense, ou d'hypercyclicité, qui requiert plus précisément l'existence d'orbites denses.

La conjecture de Syracuse affirme que les orbites de l'application de Collatz, qui agit sur les entiers, contiennent toutes le point 1. Afin d'adopter le point de vue de la dynamique linéaire, nous associons à l'application de Collatz un opérateur sur un espace de fonctions holomorphes et étudions ses propriétés dynamiques. Nous généralisons les résultats obtenus par Neklyudov et montrons notamment que cet opérateur est hypercyclique sans condition supplémentaire concernant l'application de Collatz.

Le problème du sous-espace invariant dans le cadre Hilbertien s'intéresse au fait que tout opérateur linéaire et continu, agissant sur un espace de Hilbert complexe, séparable et de dimension infinie, puisse admettre un sous-espace fermé invariant non-trivial. La famille des opérateurs de Bishop sur $L^2([0,1])$, dépendant d'un paramètre réel, a un intérêt particulier dans ce contexte, car certains de ces opérateurs pourraient être de potentiels contre-exemples au problème du sous-espace invariant. Nous étudions dans cette thèse la cyclicité des opérateurs de Bishop. Nous nous basons notamment sur l'étude par Chalendar et Partington de leur cyclicité dans le cas rationnel pour expliciter des paramètres irrationnels rendant l'opérateur de Bishop cyclique.

Abstract

We will be interested in this thesis in two famous open problems, which are the Collatz Conjecture and the Invariant Subspace Problem. We will study them from the point of view of linear dynamics. The linear dynamics consist in the study of the behavior of the iterates of a continuous linear operator acting on a Banach or Fréchet space. This theory includes in particular the notion of cyclicity, which requires the existence of orbits spanning a dense subspace, or the notion of hypercyclicity, which requires the existence of dense orbits.

The Collatz Conjecture claims that every orbit of the Collatz map, acting on integers, reaches the point 1. In order to study it from the point of view of linear dynamics, we associate to the Collatz map an operator on a space of holomorphic functions and determine its dynamical properties. We generalize the results obtained by Neklyudov and show in particular that this operator is hypercyclic without any additional condition on the Collatz map.

The Invariant Subspace Problem in the Hilbertian setting asks whether every linear continuous operator, acting on a separable, complex, infinite-dimensional Hilbert space, admits a non-trivial closed invariant subspace. The family of Bishop operators on $L^2([0,1])$, depending on a real parameter, was suggested as containing potential counterexamples to this problem. In this thesis, we study the cyclicity of Bishop operators. We rely in particular on a study by Chalendar and Partington of their cyclicity in the rational case to explicit irrational parameters making the Bishop operator cyclic.

Contents

Chapitre 1

Introduction (en français)

Les questions étudiées dans cette thèse s'inscrivent dans le domaine de la dynamique linéaire, et sont motivées par les deux problèmes ouverts célèbres que sont, d'une part, la Conjecture de Syracuse, et d'autre part le Problème du Sous-espace Invariant. Les résultats obtenus ont fait l'objet de deux publications :

— *Linear dynamics of an operator associated to the Collatz map*, accepté par les Proceedings of the American Mathematical Society, et qui fait l'objet du Chapitre 3 ;

— *A study of Bishop operators from the point of view of linear dynamics*, publié dans Journal of Mathematical Analysis and Applications **526** (2023), et qui fait l'objet du Chapitre 4.

La première partie de cette introduction présente le contexte de notre étude : après avoir rappelé quelques faits généraux sur la théorie des systèmes dynamiques, à la fois dans le cadre topologique et dans le cadre mesurable, nous présenterons quelques notions et résultats fondamentaux de dynamique linéaire, la dynamique linéaire consistant en l'étude des systèmes dynamiques donnés par l'action d'un opérateur linéaire continu sur un espace de Banach ou de Fréchet.

La deuxième partie de l'introduction débute par une brève présentation de la célèbre Conjecture de Syracuse : si T_0 est l'application de $\mathbf{N}$ dans lui-même définie par $T_0(n) = n/2$ si n est pair et $T_0(n) = 3n + 1$ si n est impair, cette conjecture affirme que l'orbite de n'importe quel entier $n \geq 1$ sous l'action de T_0 contient le point 1. Un des buts de cette thèse est d'étudier un opérateur $\mathcal{T}$ naturellement associé à T_0, dont l'adjoint a été introduit par Berg et Meinardus ([9]), et de relier ses propriétés dynamiques à celles de T_0. Nous généralisons certains résultats relatifs à la dynamique de $\mathcal{T}$ obtenus dans [36] par Neklyudov, et répondons également à certaines questions ouvertes de [36]. Ces résultats font l'objet de l'article *Linear dynamics of an operator associated to the Collatz map*, et seront exposés en détail dans le Chapite 3.

La troisième et dernière partie de l'introduction présente des résultats relatifs à la dynamique des opérateurs de Bishop. Cette famille $(T_\alpha)_{\alpha\in(0,1)}$ d'opérateurs sur $L^2([0,1])$, définis par $T_\alpha f(x) = xf(\{x + \alpha\})$, $f \in L^2([0,1])$, fait l'objet d'un intérêt particulier en théorie des opérateurs : malgré de nombreux travaux de Davie ([19]), Flattot ([22]), Chamizo, Gallardo-Gutiérrez, Monsalve-López et Ubis ([18]), entre autres, on ne sait pas si T_α possède un sous-espace invariant non-trivial pour tout irrationel $\alpha \in (0,1)$. Certains opérateurs T_α pourraient donc être de potentiels contre-exemples au Problème du

Sous-espace Invariant. Nous étudierons la dynamique des opérateurs T_α lorsque α est irrationnel. Notre résultat principal permet de montrer l'existence d'un ensemble G_δ dense de paramètres $\alpha \in (0,1)$ tels que T_α est cyclique, et fournit donc les premiers exemples de paramètres α irrationnels tels que T_α est cyclique. Notre étude s'appuie sur l'étude de la cyclicité des opérateurs T_α pour α rationnel menée par Chalendar et Partington dans [17]. Ces résultats font l'objet de l'article *A study of Bishop operators from the point of view of linear dynamics*, et seront exposés en détail dans le Chapitre 4.

Dans cette introduction, les résultats originaux sont énumérés de manière alphabétique (Proposition A, Théorème B, ...).

1.1 Systèmes dynamiques

Si X est un espace topologique, l'étude des propriétés dynamiques d'une transformation continue $T\colon X \to X$ est l'étude du comportement de ses itérées $T^n = T \circ T \circ \ldots \circ T$ où $n \in \mathbf{N}$. Si X est un espace vectoriel, par exemple normé, et si $T\colon X \to X$ est un opérateur continu, il peut sembler que le comportement de T sera prévisible du fait du cadre linéaire. Néanmoins Birkhoff, MacLane et Rolewicz ont découvert respectivement en 1929, 1952 et 1969 que des opérateurs linéaires pouvaient parfaitement admettre des orbites denses dans l'espace sur lequel ils agissent. Ces exemples, qui comprennent notamment l'opérateur de dérivation agissant sur l'ensemble des fonctions holomorphes sur le plan complexe $\mathbf{C}$, ont motivé la définition d'une théorie générale des propriétés dynamiques des opérateurs linéaires. La propriété d'admettre des orbites denses sera notamment appelée hypercyclicité.

Nous nous intéresserons dans un premier temps au cas où X est un espace topologique général et $T\colon X \to X$ une transformation continue, c'est-à-dire à la dynamique non-linéaire. En effet certaines des définitions ont un sens hors du cadre linéaire, comme par exemple l'existence d'une orbite dense ou la transitivité topologique. Cela permettra notamment de considérer des exemples de systèmes dynamiques plus variés. Réciproquement ces exemples pourront apparaître dans le cadre linéaire, comme nous le verrons avec la translation $R_\alpha\colon x \in [0,1] \mapsto \{x+\alpha\}$ et les opérateurs de Bishop. Nous donnerons également des définitions dans le cadre mesurable, lorsque X est un espace de probabilité et T une transformation préservant la mesure. Dans ce cas la notion d'ergodicité, qui peut être assimilée à celle de l'irréductibilité de l'étude d'une transformation, sera donnée et nous ferons le lien avec d'autres propriétés dynamiques, comme l'existence d'une orbite dense, grâce au théorème d'ergodicité ponctuel de Birkhoff. Enfin nous nous intéresserons à la dynamique linéaire quand X est un espace vectoriel et T un opérateur linéaire agissant sur X. Dans ce cadre, l'existence d'une orbite dense se nomme l'hypercyclicité. Le cadre linéaire permet également de définir les notions de supercyclicité et de cyclicité. Enfin lorsque l'on souhaite affiner la notion d'hypercyclicité, on peut s'intéresser à la fréquence à laquelle les orbites denses rencontrent chaque ouvert. Ceci donnera alors la notion de fréquente hypercyclicité. Puis nous donnerons un sens à la notion de chaos dans le cadre linéaire, qui comportera les opérateurs dont une partie dense de vecteurs admettent une orbite dense tandis qu'une autre partie dense de vecteurs admettent une orbite finie.

1.1.1 Dynamique non-linéaire

Nous commençons par présenter la notion de système dynamique, essentielle dans ce manuscrit. Nous renvoyons ici le lecteur au livre [26] pour plus d'informations à ce sujet.

Définition 1.1.1 ([26, Definition 1.1])**.** Un *système dynamique* est la donnée d'un espace métrique X et d'une application continue $T\colon X \to X$. Pour tout $x \in X$, l'ensemble $\{T^n(x) = T \circ T \circ \ldots \circ T(x); n \geq 0\}$ est noté $\mathrm{orb}(x, T)$ et est appelé *orbite de x sous l'action de T.*

De très nombreux exemples de systèmes dynamiques existent dans la littérature. Nous présentons notamment ici une première transformation très simple que nous retrouverons lors de notre étude des opérateurs de Bishop dans la section 1.3.

Exemple 1.1.2 ([26, Example 1.3 (f)])**.** Soit $\alpha \in \mathbf{R}$. L'espace métrique $[0, 1]$ muni de la métrique $|\cdot|$ ainsi que de l'application continue R_α définie par $R_\alpha(x) = \{x + \alpha\}$ pour tout $x \in [0, 1]$, où $\{\cdot\}$ désigne la partie fractionnaire d'un réel, est un système dynamique.

Celui-ci peut également être considéré sur le cercle unité $\mathbf{T} \subset \mathbf{C}$. On définit pour cela $S_\alpha(z) = e^{2i\pi\alpha}z$ pour tout $z \in \mathbf{T}$.

Une première propriété dynamique d'un système $T\colon X \to X$ que l'on peut étudier est celle d'admettre une orbite dense dans X.

Le comportement des itérés S_α^n, $n \geq 0$, dépend fondamentalement de la rationalité du paramètre α.

Proposition 1.1.3 ([26, Example 1.12 (c)])**.** Si α est rationnel, alors toutes les orbites de S_α sont finies, tandis que toutes les orbites de S_α sont denses dans $\mathbf{T}$ si α est irrationnel.

Démonstration. Si $\alpha = r/q$ est rationnel alors $S_\alpha^q(z) = e^{2i\pi r}z = z$ donc $\mathrm{orb}(z, S_\alpha) = \{z, e^{2i\pi r/q}z, \ldots, e^{2i\pi(q-1)r/q}z\}$ pour tout $z \in \mathbf{T}$.

Si α est irrationnel, par translation il suffit de montrer que l'orbite de 1 sous l'action de S_α est dense dans $\mathbf{T}$ car $\mathrm{orb}(z, S_\alpha) = \{e^{2i\pi n\alpha}z; n \geq 0\} = z \cdot \mathrm{orb}(1, S_\alpha)$. Notons $S_\alpha^n(1) = e^{2i\pi n\alpha} = z_\alpha^n$ et remarquons que si $\{z_\alpha^n; n \in \mathbf{Z}\}$ est dense dans $\mathbf{T}$ alors $\{z_\alpha^n; n \geq 0\}$ est dense dans $\mathbf{T}$. Si $\{z_\alpha^n; n \in \mathbf{Z}\}$ est dense, il existe une suite $(n_k)_{k\geq 0}$ d'entiers, où $(|n_k|)_{k\geq 0}$ est strictement croissante, telle que $z_\alpha^{n_k} \to 1$ quand $k \to +\infty$. Ainsi il existe une suite $(p_k)_{k\geq 0}$ d'entiers positifs strictement croissante telle que $z_\alpha^{p_k} \to 1$ quand $k \to +\infty$. En effet dans le cas où il existe une sous-suite $(m_k)_{k\geq 0}$ d'entiers négatifs de la suite $(n_k)_{k\geq 0}$, alors $(-m_k)_{k\geq 0}$ vérifie $z_\alpha^{-m_k} = \overline{z_\alpha^{m_k}} \to 1$ quand $k \to +\infty$. Ainsi $z_\alpha^{p_k+p} \to z_\alpha^p$ quand $k \to +\infty$ pour tout $p \in \mathbf{Z}$, ce qui implique que $\{z_\alpha^p; p \in \mathbf{Z}\} \subset \overline{\{z_\alpha^n; n \geq 0\}}$ et $\overline{\{z_\alpha^p; p \in \mathbf{Z}\}} \subset \overline{\{z_\alpha^n; n \geq 0\}}$.

Il suffit alors de montrer que $F = \overline{\{z_\alpha^n; n \in \mathbf{Z}\}}$ est dense dans $\mathbf{T}$. L'ensemble F est fermé et $S_\alpha(F) = F$. Supposons par l'absurde que $F \neq \mathbf{T}$, c'est-à-dire que l'ouvert $U = \mathbf{T}\backslash F$ est non-vide. Il existe une suite d'arcs ouverts $(I_k)_{k\geq 0}$ de $\mathbf{T}$ deux à deux disjoints telle que $U = \cup_{k\geq 0} I_k$. Si on note l_k la longueur de I_k pour tout $k \geq 0$, alors $\sum_{k\geq 0} l_k \leq 2\pi < +\infty$ et ainsi il existe $l_m = \max\{l_k; k \geq 0\}$. On sait que $S_\alpha(U) = U$, donc $S_\alpha^n(I_m) \subset U$ pour tout $n \in \mathbf{Z}$ et remarquons qu'on ne peut pas avoir $S_\alpha^n(I_m) \cap I_m \neq \emptyset$. En effet dans ce cas cela signifierait que $S_\alpha^n(I_m) \cup I_m$ est un arc ouvert de longueur strictement plus grande que l_m, à moins que $S_\alpha^n(I_m) = I_m$. Cela donnerait alors $z_\alpha^n = 1$, ce qui est impossible par irrationalité. Donc $S_\alpha^n(I_m) \cap I_m = \emptyset$ pour tout $n \in \mathbf{Z}$ et en particulier $S_\alpha^p(I_m) \cap S_\alpha^q(I_m) = \emptyset$ car $S_\alpha^{p-q}(I_m) \cap I_m = \emptyset$. Les arcs $(S_\alpha^n(I_m))_{n\geq 0}$ sont donc deux à deux disjoints et de longueur toutes égales à l_m, d'où $\sum_{n\geq 0} l_m \leq 2\pi < +\infty$, ce qui est absurde. Donc $F = \mathbf{T}$, ce qui conclut la preuve. □

Nous ne chercherons plus ici à montrer qu'une transformation admet une orbite dense en explicitant un élément dont l'orbite est dense. En revanche si l'on renforce les hypothèses faites sur l'espace topologique X, notamment si l'on suppose que l'espace X est polonais, la propriété d'admettre une orbite dense a été caractérisée par Birkhoff en 1920 grâce au théorème de transitivité. Afin de présenter ce théorème, nous donnons la définition de la transitivité topologique pour un système dynamique.

Définition 1.1.4 ([26, Definition 1.11])**.** Soit (X, T) un système dynamique. On dit que T est *topologiquement transitif* si pour tout couple (U, V) d'ouverts non-vides de X, il existe $n \geq 0$ tel que $T^n(U) \cap V \neq \emptyset$.

Pour tout couple (U, V) de sous-ensembles d'un espace métrique X et tout $n \geq 0$, remarquons que l'on a $T^n(U) \cap V \neq \emptyset$ si et seulement si $U \cap T^{-n}(V) \neq \emptyset$. Ainsi, si $T\colon X \to X$ est une transformation continue de X qui est inversible et dont l'inverse $T^{-1}\colon X \to X$ est continu, T est topologiquement transitif si et seulement si T^{-1} l'est.

Le cadre du théorème de transitivité de Birkhoff est celui des espaces polonais. Nous introduisons maintenant la notion d'ensemble co-maigre, qui jouera également un rôle important dans certains de nos énoncés ultérieurs.

Définition 1.1.5 ([29, Section 8.A])**.** Une distance d sur un espace topologique (X, τ) est dite *compatible avec* τ si la topologie induite par d sur X coïncide avec τ.

Un espace topologique (X, τ) est dit *polonais* s'il est séparable, c'est-à-dire qu'il admet un sous-ensemble dénombrable dense, et s'il est complètement métrisable, c'est-à-dire qu'il existe une métrique compatible d telle que (X, d) est complet.

On dit d'un sous-ensemble A d'un espace polonais X qu'il est un *sous-ensemble* G_δ *de* X s'il existe une suite d'ouverts $(U_n)_{n\geq 0}$ de X telle que $A = \cap_{n\geq 0} U_n$. On dit que A est un *sous-ensemble co-maigre de* X s'il contient un ensemble G_δ dense dans X. On dit au contraire que A est un *sous-ensemble maigre de* X si le complémentaire $X \backslash A$ est un sous-ensemble co-maigre de X.

Remarquons que si l'on veut qu'un système dynamique puisse admettre une orbite $\{T^n(x); n \geq 0\}$ dense dans X, l'hypothèse que X soit séparable est nécessaire.

Soit X un espace polonais et A un sous-ensemble de X. D'après le théorème de Baire, A est un ensemble G_δ dense dans X si et seulement s'il existe une suite $(U_n)_{n\geq 0}$ d'ouverts denses dans X telle que $A = \cap_{n\geq 0} U_n$. Ainsi en particulier une intersection dénombrable d'ensembles co-maigres de X est co-maigre dans X. Remarquons également qu'un ouvert U non-vide de X n'est pas maigre. En effet sinon le fermé $X \backslash U$ serait un ensemble co-maigre dans X, et en particulier dense, ce qui impliquerait que U est vide. Ainsi, les ensembles co-maigres sont les "gros" ensembles au sens de la catégorie de Baire.

Nous sommes maintenant en mesure de démontrer le théorème de transitivité de Birkhoff, qui caractérise l'existence d'une orbite dense grâce à la notion de transitivité topologique.

Théorème 1.1.6 (Transitivité topologique de Birkhoff, [12, §62])**.** *Soit X un espace polonais et $T\colon X \to X$ une application continue. Si T est topologiquement transitif, alors il* existe $x \in X$ dont l'orbite $\{T^n(x); n \geq 0\}$ sous l'action de T est dense dans X. Dans ce cas l'ensemble des éléments de X dont l'orbite sous l'action de T est dense dans X est un ensemble G_δ dense dans X.

Si de plus X n'admet pas de point isolé, alors la réciproque est également vraie.

Démonstration. Puisque X est métrisable et séparable, considérons une base d'ouverts non-vides $(U_k)_{k\geq 0}$ de X.

Supposons que T est topologiquement transitif. L'orbite d'un élément $x \in X$ sous l'action de T est dense dans X si et seulement si pour tout ouvert non-vide U de X, il existe $n \geq 0$ tel que $T^n(x) \in U$, c'est-à-dire si et seulement si pour tout $k \geq 0$, il existe $n \geq 0$ tel que $T^n(x) \in U_k$. Ainsi l'ensemble des éléments de X dont l'orbite sous l'action de T est dense dans X, noté ici A, est donné par $A = \cap_{k\geq 0} \cup_{n\geq 0} T^{-n}(U_k)$ et il suffit de montrer que A est un ensemble G_δ dense dans X. Soit $k \geq 0$, l'ensemble $\cup_{n\geq 0} T^{-n}(U_k)$ est un ouvert de X par continuité de T. De plus puisque T est topologiquement transitif, pour tout ouvert non-vide V de X il existe $n_0 \geq 0$ tel que $T^{n_0}(V) \cap U_k \neq \emptyset$, c'est-à-dire tel que $T^{-n_0}(U_k) \cap V \neq \emptyset$. Ainsi $(\cup_{n\geq 0} T^{-n}(U_k)) \cap V \neq \emptyset$, ce qui implique que l'ouvert $\cup_{n\geq 0} T^{-n}(U_k)$ est dense dans X pour tout $k \geq 0$. D'après le théorème de Baire A est alors un ensemble G_δ dense dans X, qui est donc en particulier non-vide.

Réciproquement supposons que X n'admet pas de point isolé et qu'il existe $x \in X$ dont l'orbite sous l'action de T est dense dans X. Soit un couple (U, V) d'ouverts non-vides de X. Puisque l'orbite de x sous l'action de T est dense dans X, il existe $n \geq 0$ tel que $T^n(x) \in U$. L'espace X n'ayant pas de point isolé, l'ensemble $\{T^k(x); k \geq 0\} \backslash \{x, T(x), \ldots, T^{n-1}(x)\}$ est encore dense dans X. Ainsi il existe $m \geq n$ tel que $T^m(x) \in V$, d'où le fait que $T^{m-n}(T^n(x)) = T^m(x)$ appartienne à $T^{m-n}(U) \cap V$, avec $m - n \geq 0$. On a alors $T^{m-n}(U) \cap V \neq \emptyset$, ce qui montre que T est topologiquement transitif. □

Dans le cadre d'un espace polonais sans point isolé, le théorème de transitivité de Birkhoff nous montre qu'il y a une dichotomie prononcée concernant l'existence d'une orbite dense. En effet soit un système dynamique $T : X \to X$ n'admet aucune orbite dense, soit il existe un ensemble G_δ d'éléments $x \in X$ dont l'orbite sous l'action de T est dense dans X.

1.1.2 Dynamique mesurable

Nous nous plaçons dans cette section dans le cadre mesurable. Soit $(X, \mathcal{F}, m)$ un espace de probabilité et $T : (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ une transformation mesurable. S'il existe un ensemble $B \in \mathcal{F}$ tel que $T^{-1}(B) = B$, alors l'étude de la transformation T peut se diviser en l'étude des transformations induites $T_{|B} : (B, \mathcal{F}_B, m) \to (B\, \mathcal{F}_B, m)$ et $T_{|X\backslash B} : (X\backslash B, \mathcal{F}_{X\backslash B}, m) \to (X\backslash B, \mathcal{F}_{X\backslash B}, m)$. Ceci induit a priori une simplification de l'étude de T, à moins que cette division ne soit triviale, ce qui est le cas si $m(B) = 0$ ou si $m(B) = 1$. Cette observation amène naturellement à la définition de la notion d'ergodicité d'une transformation, qui traduit une notion d'irréductibilité.

Définition 1.1.7 ([3, Definition 3.9])**.** Soit une transformation $T : (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$. On dit que :

(i) T est *mesurable* si $T^{-1}(B) \in \mathcal{F}$ pour tout $B \in \mathcal{F}$;

(ii) T *préserve la mesure* m si T est mesurable et si $m(T^{-1}(B)) = m(B)$ pour tout $B \in \mathcal{F}$;

(iii) T est *ergodique* si T préserve la mesure m et si pour tout $B \in \mathcal{F}$, l'égalité $T^{-1}(B) = B$ implique que $m(B) \in \{0, 1\}$.

Le théorème suivant permet de caractériser l'ergodicité d'une transformation. Cette caractérisation est notamment utile afin de lier l'ergodicité d'une transformation T et la mesure de l'ensemble des éléments $x \in X$ dont l'orbite sous l'action de T est dense dans X.

Théorème 1.1.8. Une transformation $T\colon (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ préservant la mesure m est ergodique si et seulement si $m(\cup_{n\geq 0} T^{-n}(B)) = 1$ pour tout $B \in \mathcal{F}$ tel que $m(B) > 0$.

Démonstration. Supposons que T est ergodique. Soit $B \in \mathcal{F}$ tel que $m(B) > 0$. Notons $C = \cap_{N\geq 0} \cup_{n\geq N} T^{-n}(B)$ et montrons que $T^{-1}(C) = C$. On a $T(x) \in C$ si et seulement si pour tout $N \geq 0$ il existe $n \geq N$ tel que $T^{n+1}(x) \in B$. Donc $T(x) \in C$ si et seulement si pour tout $N \geq 0$ il existe $n \geq N+1$ tel que $T^n(x) \in B$, c'est-à-dire si et seulement si $x \in C$. Par ergodicité, on a donc $m(C) \in \{0, 1\}$. Or $m(C) = \lim_{N\to+\infty} m(\cup_{n\geq N} T^{-n}(B))$ par décroissance de $(\cup_{n\geq N} T^{-n}(B))_{N\geq 0}$, d'où $m(C) \geq \liminf_{N\to+\infty} m(T^{-N}(B)) = m(B) > 0$ car T préserve la mesure m. Ainsi on a nécessairement $m(C) = 1$. En particulier $C \subset \cup_{n\geq 0} T^{-n}(B)$, d'où $m(\cup_{n\geq 0} T^{-n}(B)) = 1$.

Réciproquement soit $B \in \mathcal{F}$ tel que $T^{-1}(B) = B$. Si $m(B) > 0$, alors par hypothèse $m(\cup_{n\geq 0} T^{-n}(B)) = 1 = m(\cup_{n\geq 0} B) = m(B)$. Donc T est ergodique. □

Plaçons-nous dans le cas où X est un espace polonais sans point isolé et $\mathcal{F}$ est la σ-algèbre borélienne engendrée par les ouverts de X. Si on demande à la mesure de distinguer chaque ouvert de X, c'est-à-dire si $m(U) > 0$ pour tout ouvert U, alors l'ergodicité d'une transformation $T\colon (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ implique que l'ensemble $\cup_{n\geq 0} T^{-n}(U)$ rencontre tout ouvert non-vide V de X car sa mesure est pleine. En particulier T est topologiquement transitif. Cette observation nous permet de lier l'ergodicité à l'existence d'orbites denses.

Définition 1.1.9. Soient X un espace polonais et $\mathcal{F}$ la σ-algèbre des boréliens engendrée par les ouverts de X. Une mesure borélienne m est dite *de support plein* si $m(U) > 0$ pour tout ouvert U non-vide de X.

Nous pouvons maintenant montrer qu'une transformation ergodique relativement à une mesure de support plein admet un ensemble de vecteurs à orbite dense qui est de mesure pleine.

Théorème 1.1.10 ([26, Section 9.1])**.** Soient X un espace polonais sans point isolé, $\mathcal{F}$ la σ-algèbre des boréliens et m une mesure de probabilité borélienne de support plein. Soit $T\colon (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ une transformation continue et préservant la mesure. Si T est ergodique, alors T est topologiquement transitif et de plus

$$m(\{x \in X; \mathrm{orb}(x, T) \text{ est dense dans } X\}) = 1.$$

Démonstration. Soit $(U_k)_{k\geq 0}$ une base d'ouverts non-vides de X. Pour tout $x \in X$, $\mathrm{orb}(x, T)$ est dense dans X si et seulement si pour tout $k \geq 0$ il existe $n \geq 0$ tel que $T^n(x) \in U_k$. Donc $\{x \in X; \mathrm{orb}(x, T) \text{ est dense dans } X\} = \cap_{k\geq 0} \cup_{n\geq 0} T^{-n}(U_k)$. De plus, puisque m est de support plein, $m(U_k) > 0$ pour tout $k \geq 0$, et $m(\cup_{n\geq 0} T^{-n}(U_k)) = 1$ d'après le Théorème 1.1.8. Ainsi $m(\cap_{k\geq 0} \cup_{n\geq 0} T^{-n}(U_k)) = 1$, et en particulier T est topologiquement transitif d'après le Théorème 1.1.6. □

Un théorème ergodique est un énoncé qui exprime une relation entre les moyennes d'une fonction $f\colon X \to \mathbf{R}$ ou $\mathbf{C}$ sur les orbites de points de X sous l'action de T et la moyenne de la fonction f sur l'espace X. La relation typique est de la forme

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n(x)) \xrightarrow[N\to+\infty]{} \int_X f dm.$$

Plusieurs théorèmes ergodiques existent, chacun définissant le mode de convergence et la classe de fonctions sur laquelle il s'applique. Un premier théorème est le théorème ergodique en moyenne L^p de von Neumann.

Théorème 1.1.11 (Théorème ergodique L^p de von Neumann, [45, Corollaries 1.5 (ii)])**.** Soit $1 \leq p < +\infty$ et soit $T\colon (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ une transformation préservant la mesure. Pour tout $f \in L^p(X, \mathcal{F}, m)$, il existe $f^* \in L^p(X, \mathcal{F}, m)$ telle que $f^* \circ T = f^*$ presque partout et telle que

$$\left\| \frac{1}{N}\sum_{n=0}^{N-1} f\circ T^n - f^* \right\|_p \xrightarrow[N\to+\infty]{} 0.$$

Si de plus T est ergodique relativement à m, alors pour tout $f \in L^p(X, \mathcal{F}, m)$

$$\left\| \frac{1}{N}\sum_{n=0}^{N-1} f\circ T^n - \int_X f dm \right\|_p \xrightarrow[N\to+\infty]{} 0.$$

Birkhoff a démontré que si T est une transformation ergodique relativement à une mesure de probabilité m alors, pour toute fonction m-intégrable, sa moyenne en temps selon T et sa moyenne spatiale coïncident presque partout. Ce théorème fondamental s'énonce ainsi.

Théorème 1.1.12 (Théorème d'ergodicité ponctuel de Birkhoff, [45, Theorem 1])**.** Soit une transformation $T\colon (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ ergodique. Pour tout $f \in L^1(X, \mathcal{F}, m)$

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n(x)) \xrightarrow[N\to+\infty]{} \int_X f dm \quad \text{pour } m\text{-presque tout } x \in X.$$

Dans le cadre d'une transformation $T\colon X \to X$ ergodique où X est un espace polonais muni de sa σ-algèbre borélienne et d'une mesure de probabilité borélienne, ce théorème se révèle notamment très utile pour quantifier la fréquence avec laquelle l'orbite sous l'action de T de presque tout point de X rencontre chaque ouvert non-vide de l'espace. Cette propriété dynamique fait l'objet de la Définition 1.1.47 dans le cadre linéaire.

1.1.3 Dynamique linéaire

Nous nous plaçons maintenant dans le cadre où X est un espace de Banach ou de Fréchet et T est un opérateur linéaire et continu sur X. Le système (X, T) ainsi obtenu est un système dynamique linéaire.

Bien qu'on puisse à première vue penser qu'un opérateur sur un espace de Banach a un comportement stable et prévisible, cela n'est pas nécessairement le cas. Comme l'ont observé Birkhoff, MacLane et Rolewicz, il existe des opérateurs linéaires admettant des orbites denses dans l'espace sur lequel ils agissent. Cette propriété, appelée plus tard hypercyclicité, est l'un des deux ingrédients de la notion de chaos. En effet si l'on demande à un opérateur d'avoir à la fois des orbites denses et suffisamment d'orbites finies, on dit que cet opérateur est chaotique. Les exemples de Birkhoff, MacLane et Rolewicz vérifient d'ailleurs cette propriété.

Le cadre des espaces de Banach est suffisant pour notre étude. Cependant, les définitions de dynamique s'appliquent également dans le cadre plus général des espaces de Fréchet. Ceux-ci ne sont pas munis d'une norme mais plutôt d'une famille de semi-normes, ce qui permet d'inclure l'espace des fonctions holomorphes et donc les opérateurs de Birkhoff et MacLane qui sont respectivement les opérateurs de translation et l'opérateur de dérivation agissant sur l'espace des fonctions holomorphes.

Définition 1.1.13 ([26, Definition 2.3])**.** Soit X un espace vectoriel sur le corps $\mathbf{K} = \mathbf{R}$ ou $\mathbf{C}$. Une *semi-norme sur* X est une application $p\colon X \to \mathbf{R}_+$ telle que $p(x+y) \leq p(x)+p(y)$ et $p(\lambda x) = |\lambda|\, p(x)$ pour tous $x, y \in X$ et tout $\lambda \in \mathbf{K}$. On dit qu'une suite de semi-normes $(p_n)_{n\geq 0}$ sur X *sépare les points* si pour tout $x \in X$, les égalités $p_n(x) = 0$ pour tout $n \geq 0$ impliquent que $x = 0$.

On peut ainsi associer à une telle famille de semi-normes une distance qui définira la topologie d'un espace dit de Fréchet.

Définition 1.1.14 ([26, Definition 2.5])**.** Un *espace de Fréchet* est un espace vectoriel X, muni d'une famille de semi-normes $(p_n)_{n\geq 1}$ croissante et qui sépare les points, qui est complet pour la distance d définie par

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \min(1, p_n(x - y)) \quad \text{pour tous } x, y \in X.$$

La notion d'espace de Fréchet est bien une généralisation de celle d'espace de Banach ; il suffit de considérer toutes les semi-normes comme étant égales à la norme qui définit la topologie de l'espace de Banach.

Les espaces de Fréchet sont utiles pour travailler avec des espaces ne possédant pas de normes usuelles permettant de rendre l'espace complet. Tel est le cas de l'espace des fonctions holomorphes muni de la topologie de la convergence uniforme sur tout compact.

Exemple 1.1.15 ([26, Example 2.7 (b)])**.** L'espace complexe des fonctions entières $\mathrm{Hol}(\mathbf{C})$ muni de la famille croissante des semi-normes $(\|\cdot\|_{n,\infty}\colon f \in \mathrm{Hol}(\mathbf{C}) \mapsto \sup_{|z|\leq n} |f(z)|)_{n\geq 1}$ qui sépare les points est un espace de Fréchet.

Un espace de Fréchet ne possédant pas *a priori* de norme qui permette de parler d'opérateur borné, la proposition suivante permet de caractériser la continuité d'un opérateur relativement aux familles de semi-normes associées.

Proposition 1.1.16 ([26, Proposition 2.11])**.** Soient deux espaces de Fréchet X et Y, respectivement munis des familles de semi-normes $(p_n)_{n\geq 1}$ et $(q_n)_{n\geq 1}$. Un opérateur linéaire $T\colon X \to Y$ est continu si et seulement si pour tout $m \geq 1$, il existe $n \geq 1$ et $M > 0$ tels que $q_m(Tx) \leq Mp_n(x)$ pour tout $x \in X$.

Nous présentons maintenant les trois premiers exemples d'opérateurs linéaires admettant des orbites denses dans l'espace sur lequel ils agissent respectivement.

Exemple 1.1.17 ([26, Example 2.12 (b)])**.** L'opérateur de Birkhoff $T^{(a)}$, agissant sur $\mathrm{Hol}(\mathbf{C})$ et défini par $T^{(a)}f = f(\cdot + a)$ pour tout $f \in \mathrm{Hol}(\mathbf{C})$, avec $a \in \mathbf{C}$, est continu car $\|f(\cdot + a)\|_{n,\infty} \leq \|f\|_{n+\lceil|a|\rceil,\infty}$ pour tout $f \in \mathrm{Hol}(\mathbf{C})$ et tout $n \geq 1$.

Exemple 1.1.18 ([26, Example 2.12 (a)])**.** L'opérateur de MacLane D, agissant sur $\mathrm{Hol}(\mathbf{C})$ et défini par $Df = f'$ pour tout $f \in \mathrm{Hol}(\mathbf{C})$, est continu car la formule de Cauchy donne l'inégalité $\|f'\|_{n,\infty} \leq \|f\|_{n+1,\infty}$ pour tout $n \geq 1$ et tout $f \in \mathrm{Hol}(\mathbf{C})$.

Exemple 1.1.19 ([26, Example 2.12 (c)])**.** Soit $(e_n)_{n\geq 0}$ la base canonique de l'espace $\ell^2(\mathbf{N})$. L'opérateur de Rolewicz B, agissant sur $\ell^2(\mathbf{N})$ et défini par $Be_n = e_{n-1}$ pour tout $n \geq 1$ et $Be_0 = 0$, est continu car $\|Bx\|_2 \leq \|x\|_2$ pour tout $x \in \ell^2(\mathbf{N})$.

L'opérateur de décalage à gauche B défini ci-dessus fait l'objet d'une généralisation appelée décalage à gauche à poids B_ω. Cette généralisation se révèle très utile pour caractériser différentes propriétés dynamiques en fonction du poids ω et ainsi obtenir des exemples d'opérateurs admettant certaines propriétés dynamiques sans en vérifier d'autres.

Exemple 1.1.20 ([26, Example 4.9 (a)])**.** Soit $(e_n)_{n\geq 0}$ la base canonique de l'espace $\ell^2(\mathbf{N})$ et soit $\omega = (\omega_n)_{n\geq 0}$ une suite de complexes non-nuls appelée poids. L'opérateur de décalage à gauche à poids B_ω, agissant sur $\ell^2(\mathbf{N})$ et défini par $B_\omega e_n = \omega_n e_{n-1}$ pour tout $n \geq 1$ et $B_\omega e_0 = 0$, est continu si et seulement si la suite $(\omega_n)_{n\geq 0}$ est bornée car $\|B_\omega x\|_2 \leq \sup_{n\geq 0} |\omega_n| \, \|x\|_2$ pour tout $x \in \ell^2(\mathbf{N})$ et $\|B_\omega e_n\|_2 = |\omega_n| \, \|e_n\|_2$ pour tout $n \geq 1$.

Nous présentons également la classe des opérateurs diagonaux qui permettra notamment d'illustrer la notion de cyclicité.

Exemple 1.1.21. Soit $(e_n)_{n\geq 0}$ la base canonique de l'espace $\ell^2(\mathbf{N})$ et soit $(d_n)_{n\geq 0}$ une suite de nombres complexes. L'opérateur diagonal D, agissant sur $\ell^2(\mathbf{N})$ et défini par $De_n = d_n e_n$ pour tout $n \geq 0$, est continu si et seulement si $(d_n)_{n\geq 0}$ est bornée car $\|Dx\|_2 \leq \sup_{n\geq 0} |d_n| \, \|x\|_2$ pour tout $x \in \ell^2(\mathbf{N})$ et $\|De_n\|_2 = |d_n| \, \|e_n\|_2$ pour tout $n \geq 0$.

Nous nous placerons maintenant dans le cadre d'un espace de Banach (ou de Fréchet) X complexe, de dimension infinie et séparable. En effet, nous avons déjà pu remarquer que la séparabilité de l'espace était nécessaire pour supposer qu'un opérateur puisse admettre des orbites denses. Il s'avère de plus que si X est de dimension finie, aucun opérateur linéaire T sur X ne possède d'orbite dense. Ceci se remarque par exemple en identifiant X à $\mathbf{C}^n$ et en étudiant la décomposition de T en blocs de Jordan.

On note $\mathcal{B}(X)$ l'ensemble des opérateurs linéaires et continus agissant sur X, $\sigma(T) = \{\lambda \in \mathbf{C}; T - \lambda I$ n'est pas inversible dans $\mathcal{B}(X)\}$ le spectre d'un tel opérateur T et $\sigma_p(T)$ son spectre ponctuel qui est l'ensemble de ses valeurs propres.

Nous présentons la première propriété dynamique linéaire, l'hypercyclicité, qui requiert l'existence d'un vecteur dont l'orbite sous l'action de l'opérateur est dense.

Définition 1.1.22 ([13, Introduction]). Un opérateur $T \in \mathcal{B}(X)$ est dit *hypercyclique* s'il existe $x \in X$ tel que son orbite $\{T^n x; n \geq 0\}$ sous l'action de T est dense dans X. On dit que x est un *vecteur hypercyclique pour* T et on note $HC(T)$ l'ensemble des vecteurs hypercycliques pour T.

Il peut paraître étonnant que cette propriété ait son propre nom dans le cadre linéaire. Son origine provient en réalité de l'étude des vecteurs cycliques pour un opérateur, qui sont les vecteurs dont l'orbite engendre un sous-espace dense. Ceux-ci ne sont donc définis que dans un cadre linéaire. À partir des vecteurs cycliques ont alors été définis les vecteurs hypercycliques ci-dessus, puis les vecteurs supercycliques, qui sont ceux dont l'orbite projective est dense.

Un espace de Banach (ou de Fréchet) séparable étant polonais et sans point isolé, le théorème de transitivité de Birkhoff s'applique tout à fait pour les opérateurs hypercycliques.

Théorème 1.1.23 (Théorème de transitivité topologique de Birkhoff, [12, §62]). Un opérateur $T \in \mathcal{B}(X)$ est hypercyclique si et seulement si T est topologiquement transitif. Dans ce cas l'ensemble des vecteurs hypercycliques pour T est un ensemble G_δ dense dans X.

La dichotomie évoquée ci-dessus persiste pour les opérateurs linéaires, dont l'ensemble des vecteurs hypercycliques est soit vide, soit forme au contraire un ensemble G_δ dense dans X.

Le théorème de transitivité de Birkhoff permet notamment de démontrer l'hypercyclicité des opérateurs de Birkhoff, MacLane et Rolewicz.

Exemple 1.1.24 ([26, Example 2.20]). L'opérateur de Birkhoff $T^{(a)}$ est hypercyclique sur $\mathrm{Hol}(\mathbf{C})$ dès que $a \neq 0$.

Exemple 1.1.25 ([26, Example 2.21]). L'opérateur de MacLane D est hypercyclique sur $\mathrm{Hol}(\mathbf{C})$.

Soient deux ouverts U et V non-vides de $\mathrm{Hol}(\mathbf{C})$. Par densité des polynômes dans $\mathrm{Hol}(\mathbf{C})$, il existe deux polynômes $p \in U$ et $q \in V$ tels que $p(z) = \sum_{n=0}^{d} a_n z^n$ et $q(z) = \sum_{n=0}^{d} b_n z^n$. Considérons le polynôme r_k défini par $r_k(z) = p(z) + \sum_{n=0}^{d} n! b_n/(n+k)! \cdot z^{n+k}$ vérifiant $D^k r_k = q$ pour tout $k \geq d+1$. Or $\sup_{|z| \leq R} |r_k(z) - p(z)| \leq \sum_{n=0}^{d} n! \, |b_n| \, /(n+k)! R^{n+k} \to 0$ quand $k \to +\infty$ pour tout $R > 0$. Ainsi $(r_k)_{k \geq 0}$ converge uniformément sur tout compact vers p. Donc si k est suffisamment large alors $r_k \in U$ et $D^k r_k = q \in V$, d'où $D^k(U) \cap V \neq \emptyset$. Donc D est topologiquement transitif et donc hypercyclique.

Exemple 1.1.26 ([26, Example 2.22]). L'opérateur de Rolewicz λB est hypercyclique sur $\ell^2(\mathbf{N})$ si et seulement si $|\lambda| > 1$.

Si $|\lambda| \leq 1$ alors $\|(\lambda B)^n x\|_2 \leq |\lambda|^n \|x\|_2 \leq \|x\|_2$ pour tout $x \in \ell^2(\mathbf{N})$. Ainsi toute orbite sous l'action de λB est bornée, donc ne peut pas être dense dans $\ell^2(\mathbf{N})$.

Réciproquement soient deux ouverts U et V non-vides de $\ell^2(\mathbf{N})$ et notons $(e_n)_{n \geq 0}$ la base canonique de $\ell^2(\mathbf{N})$. Par densité des suites finies dans $\ell^2(\mathbf{N})$, il existe deux suites finies $x \in U$ et $y \in V$ telles que $x = \sum_{n=0}^{d} x_n e_n$ et $y = \sum_{n=0}^{d} y_n e_n$. Considérons la suite $z_k = \lambda^{-k} \cdot \sum_{n=0}^{d} y_n e_{n+k}$ vérifiant $(\lambda B)^k z_k = y$ et $(\lambda B)^k (x + z_k) = y$ si k est suffisamment large. De plus $\|x + z_k - x\|_2 = \|z_k\|_2 = |\lambda|^{-k} \|y\|_2 \to 0$ quand $k \to +\infty$. Donc si k est suffisamment large alors $x + z_k \in U$ et $(\lambda B)^k(x + z_k) = y \in V$, d'où $(\lambda B)^k(U) \cap V \neq \emptyset$. Ainsi λB est topologiquement transitif et donc hypercyclique si $|\lambda| > 1$.

L'hypercyclicité de l'opérateur de décalage à gauche à poids peut être caractérisée en fonction du produit des termes consécutifs du poids en question.

Exemple 1.1.27 ([26, Example 4.9 (a)])**.** Soit $\omega = (\omega_n)_{n\geq 0}$ une suite bornée de complexes non-nuls. L'opérateur B_ω est hypercyclique sur $\ell^2(\mathbf{N})$ si et seulement si $\sup_{n\geq 1} |\omega_1 \ldots \omega_n| = +\infty$.

Supposons que $\sup_{n\geq 1} |\omega_1 \ldots \omega_n| = +\infty$. Soient U et V deux ouverts non-vides de $\ell^2(\mathbf{N})$ et notons $(e_n)_{n\geq 0}$ la base canonique de $\ell^2(\mathbf{N})$. Par densité des suites finies dans $\ell^2(\mathbf{N})$, il existe deux suites finies $x \in U$ et $y \in V$ telles que $x = \sum_{n=0}^d x_n e_n$ et $y = \sum_{n=0}^d y_n e_n$. Considérons l'opérateur de décalage à droite à poids F_ω agissant sur $\ell^2(\mathbf{N})$ et défini par $F_\omega e_n = e_{n+1}/\omega_{n+1}$ pour tout $n \geq 0$, de sorte que $B_\omega F_\omega = I$ sur $\ell^2(\mathbf{N})$. Par hypothèse il existe $(m_k)_{k\geq 0}$ telle que $m_k \to +\infty$ et $|\omega_1 \ldots \omega_{m_k+d}| \to +\infty$ quand $k \to +\infty$. Alors $B_\omega^{m_k}(x + F_\omega^{m_k} y) = B_\omega^{m_k} x + y = y$ si k est suffisamment large. De plus

$$\|F_\omega^{m_k} y\|_2^2 = \sum_{n=0}^{d} \frac{|y_n|^2}{|\omega_{n+1} \ldots \omega_{n+m_k}|^2} \leq \max_{0\leq n\leq d} \frac{\|y\|_2^2}{|\omega_{n+1} \ldots \omega_{n+m_k}|^2},$$

or pour tout $0 \leq n \leq d$

$$\frac{1}{|\omega_{n+1} \ldots \omega_{n+m_k}|} = \frac{|\omega_1 \ldots \omega_n \cdot \omega_{m_k+n+1} \ldots \omega_{m_k+d}|}{|\omega_1 \ldots \omega_{m_k+d}|} \leq \frac{\|\omega\|_\infty^d}{|\omega_1 \ldots \omega_{m_k+d}|} \xrightarrow[k\to+\infty]{} 0.$$

Donc $F_\omega^{m_k} y \to 0$ quand $k \to +\infty$. Si k est suffisamment large alors $x + F_\omega^{m_k} y \in U$ et $B_\omega^{m_k}(x + F_\omega^{m_k} y) = y \in V$, d'où $B_\omega^{m_k}(U) \cap V \neq \emptyset$. Ainsi B_ω est topologiquement transitif et donc hypercyclique.

Réciproquement supposons que B_ω est hypercyclique. Alors il existe un vecteur $x = \sum_{n=0}^\infty x_n e_n \in \ell^2(\mathbf{N})$ hypercyclique pour B_ω, ce qui implique que pour tout $\varepsilon > 0$ il existe $N \geq 0$ tel que $\|B_\omega^N x - e_0\|_2 < \varepsilon$. Or

$$\varepsilon > \|B_\omega^N x - e_0\|_2 \geq |x_N \omega_1 \ldots \omega_N - 1| \geq 1 - |x_N| \, |\omega_1 \ldots \omega_N| \, ,$$

d'où l'inégalité $|x_N| \, |\omega_1 \ldots \omega_N| > 1 - \varepsilon$. En particulier avec $\varepsilon = 1/2$, on a l'inclusion $\{n \geq 0; \|B_\omega^n x - e_0\|_2 < 1/2\} \subset \{n \geq 0; |x_n| \, |\omega_1 \ldots \omega_n| > 1/2\}$. Or par densité de l'orbite $\mathrm{orb}(x, B_\omega)$ dans $\ell^2(\mathbf{N})$, l'ensemble $\{n \geq 0; \|B_\omega^n x - e_0\|_2 < 1/2\}$ est infini et contient donc une suite strictement croissante $(n_k)_{k\geq 0}$. Ainsi l'inégalité $|x_{n_k}| \, |\omega_1 \ldots \omega_{n_k}| > 1/2$ pour tout $k \geq 0$ et le fait que $x \in \ell^2(\mathbf{N})$ impliquent que $|\omega_1 \ldots \omega_{n_k}| \to +\infty$ quand $k \to +\infty$.

Dans la pratique, nous ne chercherons plus à montrer qu'un opérateur est hypercyclique en explicitant un vecteur hypercyclique ou en vérifiant directement qu'il est topologiquement transitif. En effet remarquons que pour montrer l'hypercyclicité des opérateurs de MacLane et Rolewicz, nous avons explicité un ensemble dense X_0 ainsi qu'un inverse à droite de T sur X_0, $S \colon X_0 \to X_0$, tel que $T^n x \to 0$ et $S^n x \to 0$ quand $n \to +\infty$ pour tout $x \in X_0$. Cette idée peut être généralisée, et conduit au Critère d'Hypercyclicité suivant, qui permet de donner des conditions suffisantes pour l'hypercyclicité qu'il est en pratique facile de vérifier.

Théorème 1.1.28 (Critère d'Hypercyclicité, [11, Theorem 2.3])**.** Soit $T \in \mathcal{B}(X)$. Supposons qu'il existe deux sous-ensembles X_0 et Y_0 denses de X, une suite $(n_k)_{k\geq 0}$ d'entiers strictement croissante et une suite $(S_{n_k} \colon Y_0 \to X)_{k\geq 0}$ d'applications tels que :

(i) $T^{n_k}x \to 0$ quand $k \to +\infty$ pour tout $x \in X_0$;

(ii) $S_{n_k}y \to 0$ quand $k \to +\infty$ pour tout $y \in Y_0$;

(iii) $T^{n_k}S_{n_k}y \to y$ quand $k \to +\infty$ pour tout $y \in Y_0$.

Alors T est hypercyclique.

Démonstration. Montrons que T est topologiquement transitif. Soient deux ouverts non-vides U et V de X, montrons qu'il existe $n \geq 0$ tel que $T^n(U) \cap V \neq \emptyset$. Par densité de X_0 et Y_0 dans X, il existe $x \in U \cap X_0$ et $y \in V \cap Y_0$. Par hypothèse $x + S_{n_k}y \to x$ quand $k \to +\infty$ et $T^{n_k}(x + S_{n_k}y) = T^{n_k}x + T^{n_k}S_{n_k}y \to y$ quand $k \to +\infty$. Ainsi il existe $K \geq 0$ tel que $x + S_{n_k}y \in U$ et $T^{n_k}(x + S_{n_k}y) \in V$ pour tout $k \geq K$. En particulier on a $T^{n_K}(x + S_{n_K}y) \in T^{n_K}(U) \cap V$, d'où $T^{n_K}(U) \cap V \neq \emptyset$. Donc T est topologiquement transitif, et ainsi hypercyclique d'après le Théorème 1.1.23. □

On peut remarquer qu'on n'a pas supposé que X_0 et Y_0 sont des sous-espaces vectoriels de X, ni que les $(S_{n_k})_{k\geq 0}$ sont des applications linéaires ou continues.

L'existence d'un opérateur hypercyclique ne vérifiant pas le Critère d'Hypercyclicité était alors encore une question ouverte quand Bès et Peris ont démontré que le Critère d'Hypercyclicité était équivalent à l'hypercyclicité de l'opérateur $T \oplus T \in \mathcal{B}(X \oplus X)$ défini par $T \oplus T(x, y) = (Tx, Ty)$ pour tout $(x, y) \in X \oplus X$ ([11, Theorem 2.3]).

Le Critère d'Hypercyclicité a pour origine le Critère de Kitai, qui supposait que la suite $(n_k)_{k\geq 0}$ était l'ensemble des entiers naturels et que l'opérateur T admettait un inverse à droite $S \colon Y_0 \to Y_0$. Ce dernier a été ensuite affiné par Gethner-Shapiro en le critère suivant.

Théorème 1.1.29 (Critère de Gethner-Shapiro, [23, Theorem 2.2] et [26, Theorem 3.10])**.** Soit $T \in \mathcal{B}(X)$. Supposons qu'il existe deux sous-ensembles X_0 et Y_0 denses de X, une suite $(n_k)_{k\geq 0}$ d'entiers strictement croissante et une application $S \colon Y_0 \to Y_0$ tels que :

(i) $T^{n_k}x \to 0$ quand $k \to +\infty$ pour tout $x \in X_0$;

(ii) $S^{n_k}y \to 0$ quand $k \to +\infty$ pour tout $y \in Y_0$;

(iii) $TSy = y$ pour tout $y \in Y_0$.

Alors T est hypercyclique.

Si le Critère de Gethner-Shapiro peut paraître strictement plus faible que le Critère d'Hypercyclicité, Peris a montré en 2001 que ces deux critères étaient en réalité équivalents.

Théorème 1.1.30 ([39, Theorem 2.3])**.** Un opérateur $T \in \mathcal{B}(X)$ satisfait les hypothèses du Critère d'Hypercyclicité si et seulement s'il satisfait celles du Critère de Gethner-Shapiro.

La définition de l'hypercyclicité, qui requiert l'existence d'une orbite dense, a donné le jour à la définition plus générale de la supercyclicité, qui requiert l'existence d'une orbite projective dense.

Définition 1.1.31 ([28, Definition §4])**.** Un opérateur $T \in \mathcal{B}(X)$ est dit *supercyclique* s'il existe $x \in X$ tel que l'ensemble $\{\lambda T^n x; \lambda \in \mathbf{C}, n \geq 0\}$ est dense dans X. On dit alors que x est un *vecteur supercyclique pour* T et on note $SC(T)$ l'ensemble des vecteurs supercycliques pour T.

Les opérateurs comme le décalage à gauche B de Rolewicz peuvent également motiver cette définition. En effet bien que B ne puisse pas être hypercyclique (B est une contraction, i.e. $\|B\| \leq 1$), il s'avère qu'il est tout de même supercyclique.

Tout comme dans le cas hypercyclique, un théorème de transitivité existe dans le cas supercyclique.

Théorème 1.1.32 ([4, Theorem 1.12])**.** Un opérateur $T \in \mathcal{B}(X)$ est supercyclique si et seulement si pour tout couple (U, V) d'ouverts non-vides de X, il existe $n \geq 0$ et $\lambda \in \mathbf{C}$ tels que $\lambda T^n(U) \cap V \neq \emptyset$. Dans ce cas l'ensemble $SC(T)$ des vecteurs supercycliques pour T est un ensemble G_δ dense dans X.

Démonstration. Supposons que T est supercyclique et soient deux ouverts U et V non-vides de X. Par supercyclicité il existe $x \in X$, $\lambda \in \mathbf{C}\backslash\{0\}$ et $n \geq 0$ tels que $\lambda T^n x \in U$. Puisque X n'admet pas de point isolé, l'ensemble $\{\mu T^m x; \mu \in \mathbf{C}, m \geq n\}$ est toujours dense dans X et il existe $\mu \in \mathbf{C}$ et $m \geq n$ tels que $\mu T^m x \in V$. On a alors $(\mu/\lambda)T^{m-n}(\lambda T^n x) = \mu T^m x \in (\mu/\lambda)T^{m-n}(U) \cap V$ avec $m - n \geq 0$, d'où $(\mu/\lambda)T^{m-n}(U) \cap V \neq \emptyset$.

Réciproquement montrons que T est supercyclique. Soit une base d'ouverts non-vides $(U_k)_{k\geq 0}$ de X. Un vecteur $x \in X$ est supercyclique pour T si et seulement si pour tout ouvert non-vide U de X il existe $\lambda \in \mathbf{C}$ et $n \geq 0$ tels que $\lambda T^n x \in U$, c'est-à-dire si et seulement si pour tout $k \geq 0$, il existe $\lambda \in \mathbf{C}$ et $n \geq 0$ tels que $\lambda T^n x \in U_k$. Ainsi l'ensemble $SC(T)$ des vecteurs supercycliques pour T est donné par $SC(T) = \cap_{k\geq 0} \cup_{\lambda\in\mathbf{C}, n\geq 0} (\lambda T^n)^{-1}(U_k)$ et il suffit de montrer que $SC(T)$ est un ensemble G_δ dense dans X. Soit $k \geq 0$, l'ensemble $\cup_{\lambda\in\mathbf{C}, n\geq 0}(\lambda T^n)^{-1}(U_k)$ est un ouvert de X par continuité de T. De plus pour tout ouvert V non-vide de X, par hypothèse il existe $\lambda_0 \in \mathbf{C}$ et $n_0 \geq 0$ tels que $\lambda_0 T^{n_0}(V) \cap U_k \neq \emptyset$, c'est-à-dire tels que $(\lambda_0 T^{n_0})^{-1}(U_k) \cap V \neq \emptyset$. Ainsi $(\cup_{\lambda\in\mathbf{C}, n\geq 0}(\lambda T^n)^{-1}(U_k)) \cap V \neq \emptyset$, ce qui implique que l'ouvert $\cup_{\lambda\in\mathbf{C}, n\geq 0}(\lambda T^n)^{-1}(U_k)$ est dense dans X pour tout $k \geq 0$. D'après le théorème de Baire $SC(T)$ est alors un ensemble G_δ dense dans X, qui est donc en particulier non-vide. □

Reprenons l'exemple de l'opérateur de décalage à poids B_ω. Il s'avère que celui-ci est toujours supercyclique, quelque soit le poids ω. Ceci permet de donner des exemples d'opérateurs T supercycliques tels que pour tout $\lambda \in \mathbf{C}$, λT n'est pas hypercyclique.

Exemple 1.1.33 ([4, Example 1.15])**.** L'opérateur de décalage à poids B_ω est supercyclique sur $\ell^2(\mathbf{N})$ pour tout poids ω borné. En particulier si $\omega_n = 1/n$ pour tout $n \geq 1$, alors B_ω est supercyclique et pour tout $\lambda \in \mathbf{C}$, λB_ω n'est pas hypercyclique.

En effet, soient deux ouverts U et V non-vides de $\ell^2(\mathbf{N})$. Par densité des suites finies dans $\ell^2(\mathbf{N})$, il existe deux suites finies $x \in U$ et $y \in V$. Considérons l'opérateur de décalage à droite à poids F_ω sur $\ell^2(\mathbf{N})$. Soit $k \geq 0$ tel que $B_\omega^k x = 0$. En considérant $\lambda \in \mathbf{C}$ un complexe non-nul tel que $x + \lambda^{-1}F_\omega^k y \in U$, on a alors $\lambda B_\omega^k(x + \lambda^{-1}F_\omega^k y) = \lambda B_\omega^k x + y = y$. Ainsi $\lambda B_\omega^k(U) \cap V \neq \emptyset$, d'où la supercyclicité de B_ω sur $\ell^2(\mathbf{N})$. De plus pour tout $\lambda \in \mathbf{C}$, $\lambda B_\omega = B_\delta$ où $\delta_n = \lambda\omega_n$ pour tout $n \geq 1$. Donc $\sup_{n\geq 1} |\delta_1 \ldots \delta_n| = \sup_{n\geq 1} |\lambda|^n / n! < +\infty$, ce qui implique que λB_ω n'est pas hypercyclique.

De même, le Critère d'Hypercyclicité s'adapte à la définition de la supercyclicité et permet d'obtenir un Critère de Supercyclicité.

Théorème 1.1.34 (Critère de Supercyclicité, [10, Lemma 3.1])**.** Soit $T \in \mathcal{B}(X)$. Supposons qu'il existe deux sous-ensembles X_0 et Y_0 denses de X, une suite $(n_k)_{k\geq 0}$ d'entiers strictement croissante, une suite $(\lambda_{n_k})_{k\geq 0}$ de complexes non-nuls et une suite $(S_{n_k} \colon Y_0 \to X)_{k\geq 0}$ d'applications tels que :

(i) $\lambda_{n_k} T^{n_k} x \to 0$ quand $k \to +\infty$ pour tout $x \in X_0$;

(ii) $\lambda_{n_k}^{-1} S_{n_k} y \to 0$ quand $k \to +\infty$ pour tout $y \in Y_0$;

(iii) $T^{n_k} S_{n_k} y \to y$ quand $k \to +\infty$ pour tout $y \in Y_0$.

Alors T est supercyclique.

Démonstration. Montrons que T vérifie la propriété du Théorème 1.1.32. Soient deux ouverts non-vides U et V de X, montrons qu'il existe $\lambda \in \mathbf{C}$ et $n \geq 0$ tels que $\lambda T^n(U) \cap V \neq \emptyset$. Par densité de X_0 et Y_0 dans X, il existe $x \in U \cap X_0$ et $y \in V \cap Y_0$. Par hypothèse $x + \lambda_{n_k}^{-1} S_{n_k} y \to x$ quand $k \to +\infty$ et $\lambda_{n_k} T^{n_k}(x + \lambda_{n_k}^{-1} S_{n_k} y) = \lambda_{n_k} T^{n_k} x + T^{n_k} S_{n_k} y \to y$ quand $k \to +\infty$. Ainsi il existe $K \geq 0$ tel que $x + \lambda_{n_k}^{-1} S_{n_k} y \in U$ et $\lambda_{n_k} T^{n_k}(x + \lambda_{n_k}^{-1} S_{n_k} y) \in V$ pour tout $k \geq K$. En particulier on a $\lambda_{n_K} T^{n_K}(x + \lambda_{n_K}^{-1} S_{n_K} y) \in \lambda_{n_K} T^{n_K}(U) \cap V$, d'où $\lambda_{n_K} T^{n_K}(U) \cap V \neq \emptyset$. Donc T est supercyclique d'après le Théorème 1.1.32. □

Dans le quatrième chapitre, nous voulons démontrer que les opérateurs de translation à poids ne peuvent pas vérifier le Critère de Supercyclicité. Pour cela nous aurons besoin de montrer que le Critère de Supercyclicité est en fait équivalent à un critère du type Gethner-Shapiro pour la supercyclicité, ce qui ne semble pas apparaître dans la littérature.

Nous présentons maintenant la propriété dynamique la plus faible qui est à l'origine de l'hypercyclicité, appelée la cyclicité. Celle-ci requiert l'existence d'une orbite engendrant un sous-espace dense. Cette notion a un lien fort avec le deuxième problème que l'on considérera ici, à savoir le problème du sous-espace invariant.

Définition 1.1.35 ([24, Introduction])**.** Un opérateur $T \in \mathcal{B}(X)$ est dit *cyclique* s'il existe $x \in X$ tel que le sous-espace $\mathrm{vect}[T^n x; n \geq 0]$ engendré par son orbite sous l'action de T est dense dans X. On dit que x est un *vecteur cyclique pour* T et on note $C(T)$ l'ensemble des vecteurs cycliques pour T.

Une version du théorème de transitivité topologique est bien connue dans le cadre de la cyclicité. Elle requiert néanmoins l'hypothèse que l'opérateur adjoint T^* n'admette pas de valeur propre afin d'assurer que l'opérateur $P(T)$ est à image dense dans X pour tout polynôme $P \in \mathbf{C}[\xi]$ non-nul.

Théorème 1.1.36. Soit un opérateur $T \in \mathcal{B}(X)$. Si pour tout couple (U, V) d'ouverts non-vides de X, il existe $P \in \mathbf{C}[\xi]$ tel que $P(T)(U) \cap V \neq \emptyset$, alors T est cyclique. Dans ce cas l'ensemble $C(T)$ des vecteurs cycliques pour T est un ensemble G_δ dense dans X.

De plus si $\sigma_p(T^*) = \emptyset$, alors la réciproque est vraie.

Démonstration. Supposons que pour tout couple (U, V) d'ouverts non-vides de X il existe $P \in \mathbf{C}[\xi]$ tel que $P(T)(U) \cap V \neq \emptyset$ et montrons que T est cyclique. Soit $(U_k)_{k\geq 0}$ une base d'ouverts non-vides de X. Un vecteur $x \in X$ est cyclique pour tout T si et seulement si pour tout ouvert U non-vide de X il existe $P \in \mathbf{C}[\xi]$ tel que $P(T)x \in U$, c'est-à-dire si et seulement si pour tout $k \geq 0$ il existe $P \in \mathbf{C}[\xi]$ tel que $P(T)x \in U_k$. Alors l'ensemble $C(T)$ des vecteurs cycliques pour T est donné par $C(T) = \cap_{k\geq 0} \cup_{P\in\mathbf{C}[\xi]} P(T)^{-1}(U_k)$ et il suffit de montrer que $C(T)$ est un ensemble G_δ dense dans X. Soit $k \geq 0$, l'ensemble $\cup_{P\in\mathbf{C}[\xi]} P(T)^{-1}(U_k)$ est un ouvert de X par continuité de T. De plus pour tout ouvert V non-vide de X, par hypothèse il existe $P_0 \in \mathbf{C}[\xi]$ tel que $P_0(T)(V) \cap U_k \neq \emptyset$, c'est-à-dire tel que $P_0(T)^{-1}(U_k) \cap V \neq \emptyset$. Ainsi $(\cup_{P\in\mathbf{C}[\xi]} P(T)^{-1}(U_k)) \cap V \neq \emptyset$, ce qui implique que l'ouvert $\cup_{P\in\mathbf{C}[\xi]} P(T)^{-1}(U_k)$ est dense dans X pour tout $k \geq 0$. D'après le théorème de Baire $C(T)$ est alors un ensemble G_δ dense dans X, qui est donc en particulier non-vide.

Réciproquement supposons que T est cyclique et que $\sigma_p(T^*) = \emptyset$ et considérons deux ouverts U et V non-vides de X. Par cyclicité il existe $x \in X$ et $P \in \mathbf{C}[\xi]$ tels que $P(T)x \in U$. De plus $P(T)$ a une image dense dans X puisque $\sigma_p(T^*) = \emptyset$. Ainsi il existe par continuité un ouvert W de X tel que $P(T)(W) \subset V$ et par cyclicité il existe $Q \in \mathbf{C}[\xi]$ tel que $Q(T)x \in W$. On a alors $Q(T)P(T)x = P(T)Q(T)x \in Q(T)(U) \cap V$, d'où $Q(T)(U) \cap V \neq \emptyset$. □

Ce théorème peut être illustré avec l'exemple d'un opérateur diagonal sur un espace de suites $\ell^2(\mathbf{N})$.

Exemple 1.1.37. Soit $(d_n)_{n\geq 0}$ une suite de nombres complexes bornée. L'opérateur D est cyclique sur $\ell^2(\mathbf{N})$ si et seulement si les complexes $(d_n)_{n\geq 0}$ sont deux à deux distincts.

En effet supposons qu'il existe $0 \leq i < j$ tels que $d_i = d_j$ et considérons $(e_n)_{n\geq 0}$ la base canonique de $\ell^2(\mathbf{N})$. Supposons par l'absurde que T est cyclique. Alors il existe $x = \sum_{n=0}^{\infty} x_n e_n \in \ell^2(\mathbf{N})$ un vecteur cyclique pour T. Pour tout $y \in \ell^2(\mathbf{N})$ il existe alors une suite $(P_k)_{k\geq 0}$ de polynômes, avec $P_k(\xi) = \sum_{m=0}^{m_k} a_{m,k}\xi^m$ pour tout $k \geq 0$, telle que $\|P_k(D)x - y\|_2 \to 0$ quand $k \to +\infty$. Or $P_k(D)x = \sum_{m=0}^{m_k} a_{m,k} D^m \sum_{n=0}^{\infty} x_n e_n = \sum_{n=0}^{\infty}(x_n \sum_{m=0}^{m_k} a_{m,k} d_n^m) e_n = \sum_{n=0}^{\infty} P_k(d_n) x_n e_n$ pour tout $k \geq 0$, d'où $|P_k(d_n)x_n - y_n| \leq \|P_k(D)x - y\|_2$ pour tout $n \geq 0$. En particulier $P_k(d_i)x_i \to y_i$ et $P_k(d_j)x_j \to y_j$ quand $k \to +\infty$. Puisque $d_i = d_j$, alors on a $P_k(d_i)x_i x_j \to y_i x_j$ et $P_k(d_i)x_j x_i \to y_j x_i$ quand $k \to +\infty$. Ainsi $y_i x_j = y_j x_i$ pour tout $y = (y_n)_{n\geq 0} \in \ell^2(\mathbf{N})$. On obtient alors $x_i = 0$ en prenant $y = e_j$. Le vecteur x ne peut donc pas être cyclique pour D puisque cela signifierait que $\|P(D)x - e_i\|_2 \geq |P(d_i)x_i - 1| = 1$ pour tout $P \in \mathbf{C}[\xi]$, d'où la contradiction.

Réciproquement supposons que $d_i \neq d_j$ dès que $i \neq j$ et considérons deux ouverts U et V non-vides de $\ell^2(\mathbf{N})$. Par densité des suites finies dans $\ell^2(\mathbf{N})$ il existe des suites finies $x \in U$ et $y \in V$ telles que $x = \sum_{n=0}^{d} x_n e_n$ et $y = \sum_{n=0}^{d} y_n e_n$. Remarquons qu'il existe une suite finie $x' = \sum_{n=0}^{d} x'_n e_n$ appartenant à U et vérifiant $x'_n \neq 0$ pour tout $n \in \{0, \ldots, d\}$. En effet soit $\varepsilon > 0$ tel que $\{z \in \ell^2(\mathbf{N}); \|z - x\|_2 < \varepsilon\} \subset U$. Considérons alors $u = \sum_{n\in A} e_n$ où $A = \{n \in \{0, \ldots, d\}; x_n = 0\}$ ainsi que $v = \varepsilon(2\|u\|_2)^{-1}u$. Donc $x' = x + v$ convient. On considère maintenant un polynôme interpolateur de Lagrange P tel que $P(d_i) = y_i/x'_i$ pour tout $i \in \{0, \ldots, d\}$. Alors $P(D)x' = \sum_{n=0}^{d} P(d_n)x'_n e_n = \sum_{n=0}^{d} y_n e_n = y$, ce qui implique que $P(D)(U) \cap V \neq \emptyset$. Donc D est cyclique.

Contrairement au Théorème 1.1.23, le Théorème 1.1.36 n'est pas une caractérisation de la cyclicité et il existe des exemples d'opérateurs cycliques dont l'ensemble des vecteurs cycliques n'est pas dense.

Exemple 1.1.38. L'opérateur de décalage à droite F sur $\ell^2(\mathbf{N})$ est cyclique et l'ensemble des vecteurs cycliques pour F n'est pas dense dans $\ell^2(\mathbf{N})$.

On identifie isométriquement $\ell^2(\mathbf{N})$ avec l'espace de Hardy

$$H^2(\mathbf{D}) = \left\{ f\colon z \mapsto \sum_{n=0}^{\infty} c_n z^n \in \mathrm{Hol}(\mathbf{D}); \|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |c_n|^2 < +\infty \right\}.$$

L'opérateur $F \in \mathcal{B}(\ell^2(\mathbf{N}))$ est naturellement identifié à l'opérateur de multiplication M_z défini sur $H^2(\mathbf{D})$ par $M_z f = zf$.

Si $f \in H^2(\mathbf{D})$ s'annule en un point $a \in \mathbf{D}$, alors $M_z^n f(a) = a^n f(a) = 0$ pour tout $n \geq 0$. Donc toutes les fonctions de $\mathrm{vect}[M_z^n f; n \geq 0]$ s'annulent en a. Or la convergence dans $H^2(\mathbf{D})$ implique la convergence uniforme sur tout compact et en particulier la convergence ponctuelle. Donc le sous-espace fermé engendré par l'orbite de f sous l'action de M_z ne comprend que des fonctions s'annulant en a, ce qui signifie que f ne peut pas être cyclique pour M_z.

Une fonction cyclique pour M_z ne s'annule donc pas sur $\mathbf{D}$. Or d'après le Théorème de Hurwitz, si une suite $(f_n)_{n\geq 0}$ de fonctions holomorphes sur $\mathbf{D}$ ne s'annulant pas sur $\mathbf{D}$ converge dans $H^2(\mathbf{D})$ vers une fonction $f \in H^2(\mathbf{D})$ alors f est identiquement nulle ou ne s'annule pas sur $\mathbf{D}$. Ainsi l'ensemble des fonctions cycliques pour M_z n'est pas dense dans $H^2(\mathbf{D})$.

L'existence d'orbites denses pour un opérateur linéaire montre qu'un tel opérateur peut présenter un comportement intéressant à étudier malgré le cadre linéaire dans lequel il agit. Lorsque l'opérateur possède simultanément une partie dense de vecteurs à orbite dense et une partie dense de vecteurs à orbite finie, il présente un comportement chaotique.

Définition 1.1.39 ([26, Definition 1.23])**.** Soit $T \in \mathcal{B}(X)$. On dit que $x \in X$ est un *point périodique de* T s'il existe $n \geq 1$ tel que $T^n x = x$. On note $\mathrm{Per}(T)$ l'ensemble des points périodiques de T.

Définition 1.1.40 ([26, Definition 2.29])**.** Un opérateur $T \in \mathcal{B}(X)$ est dit *chaotique* s'il est hypercyclique et si $\mathrm{Per}(T)$ est dense dans X.

Cette définition de chaos dans le cadre linéaire coïncide avec la notion de chaos au sens de Devaney ([26, Definition 2.29]). Que X soit un espace de Fréchet ou un espace de Banach, la topologie de X est induite par une distance d invariante par translation et le chaos au sens de Devaney requiert qu'un opérateur $T \in \mathcal{B}(X)$ soit hypercyclique, qu'il admette une partie dense de points périodiques et qu'il soit *sensible aux conditions initiales* au sens suivant : il existe $\delta > 0$ tel que pour tout $\varepsilon > 0$ et tout $x \in X$, il existe $y \in X$ et $n \geq 0$ vérifiant $d(x, y) < \varepsilon$ et $d(T^n x, T^n y) > \delta$.

Ces deux notions coïncident ici puisqu'un opérateur T hypercyclique est nécessairement sensible aux conditions initiales.

Proposition 1.1.41 ([26, Proposition 2.30])**.** *Soit d une distance invariante par translation compatible avec la topologie de X. Un opérateur $T \in \mathcal{B}(X)$ hypercyclique est sensible aux conditions initiales relativement à la distance d.*

Démonstration. Soient deux paramètres $\delta > 0$ et $\varepsilon > 0$ quelconques et soit un vecteur $x \in X$. Considérons les ouverts non-vides $U = \{z \in X; d(0, z) < \varepsilon\}$ et $V = \{z \in X; d(0, z) > \delta\}$. D'après le Théorème 1.1.23, il existe $n \geq 0$ et $z \in U$ tels que $T^n z \in V$. Le vecteur $y = x + z$ vérifie alors $d(x, y) = d(0, z) < \varepsilon$ et $d(T^n x, T^n y) = d(0, T^n z) > \delta$ par invariance par translation de d. □

Nous aurons l'occasion de constater que les valeurs propres et les vecteurs propres d'un opérateur peuvent jouer un rôle fondamental dans la détermination des propriétés dynamiques que vérifie l'opérateur. Pour introduire cela, remarquons que le chaos linéaire est fortement lié aux propriétés des vecteurs propres de l'opérateur associés à des valeurs propres qui sont des racines de l'unité.

Proposition 1.1.42 ([26, Proposition 2.33])**.** Soit $T \in \mathcal{B}(X)$. On a $\mathrm{Per}(T) = \mathrm{vect}[\ker(T - e^{i\alpha\pi}I); \alpha \in \mathbf{Q}]$.

Démonstration. Soit $\alpha = r/q \in \mathbf{Q}$ et $x \in \ker(T - e^{ir\pi/q}I)$. Alors $T^{2q}x = e^{i2r\pi}x = x$, donc $x \in \mathrm{Per}(T)$. Ainsi $\cup_{\alpha\in\mathbf{Q}} \ker(T - e^{i\alpha\pi}I) \subset \mathrm{Per}(T)$. Il suffit alors de montrer que $\mathrm{Per}(T)$ est un sous-espace vectoriel de X pour montrer l'inclusion $\mathrm{vect}[\ker(T - e^{i\alpha\pi}); \alpha \in \mathbf{Q}] \subset \mathrm{Per}(T)$. Soient $x, y \in \mathrm{Per}(T)$ et $\lambda \in \mathbf{C}$. Il existe $n, m \geq 1$ tels que $T^n x = x$ et $T^m y = y$. Alors $T^{nm}(\lambda x + y) = \lambda T^{nm}x + T^{nm}y = \lambda x + y$, donc $\lambda x + y \in \mathrm{Per}(T)$ et $\mathrm{Per}(T)$ est bien un sous-espace vectoriel de X.

Réciproquement soit $x \in \mathrm{Per}(T)$, il existe $N \geq 0$ tel que $T^N x = x$. Si on note $w = e^{i2\pi/N}$ alors le polynôme $P(\xi) = \xi^N - 1$ vérifie $\xi^N - 1 = (\xi - 1)(\xi - w)\dots(\xi - w^{N-1})$ et $P(T)x = T^N x - x = 0$. Considérons $P_k(\xi) = \prod_{i\in\{0,\dots,N-1\}\setminus\{k\}}(\xi - w^i)$ pour tout $k \in \{0, \dots, N-1\}$ et remarquons qu'il s'agit d'une base de l'espace des polynômes de degré inférieur ou égal à $N - 1$. En effet si $\sum_{k=0}^{N-1} \lambda_k P_k = 0$ alors $\sum_{k=0}^{N-1} \lambda_k P_k(w^i) = \lambda_i P_i(w^i) = 0$, d'où $\lambda_i = 0$, pour tout $i \in \{0, \dots, N-1\}$. Ainsi le polynôme constant égal à 1 s'écrit $1 = \sum_{k=0}^{N-1} \alpha_k P_k$, ce qui donne $I = \sum_{k=0}^{N-1} \alpha_k P_k(T)$ et $x = \sum_{k=0}^{N-1} \alpha_k P_k(T)x$. Or $(T - w^k I)P_k(T)x = P(T)x = 0$ donc $P_k(T)x \in \ker(T - w^k I)$ pour tout $k \in \{0, \dots, N-1\}$. Ainsi $x \in \mathrm{vect}[\ker(T - e^{i\alpha\pi}I); \alpha \in \mathbf{Q}]$, d'où $\mathrm{Per}(T) \subset \mathrm{vect}[\ker(T - e^{i\alpha\pi}I); \alpha \in \mathbf{Q}]$. □

Le critère suivant, dû à Godefroy et Shapiro, met en lumière de manière frappante le lien qu'il peut y avoir entre les vecteurs propres d'un opérateur et les propriétés dynamiques qu'il vérifie. En effet admettre suffisamment de vecteurs propres associés à des valeurs propres qui sont dans le disque unité ouvert et suffisamment de vecteurs propres associés à des valeurs propres qui sont en dehors du disque unité fermé permet déjà à un opérateur d'être hypercyclique. Si de plus c'est le cas pour les valeurs propres qui sont racines de l'unité, l'opérateur est en fait chaotique.

Théorème 1.1.43 (Critère de Godefroy-Shapiro, [26, Theorem 3.1])**.** Soit $T \in \mathcal{B}(X)$. Si les sous-espaces $X_+ = \mathrm{vect}[\ker(T - \lambda I); |\lambda| > 1]$ et $X_- = \mathrm{vect}[\ker(T - \lambda I); |\lambda| < 1]$ sont denses dans X, alors T est hypercyclique. De plus si $X_0 = \mathrm{vect}[\ker(T - e^{i\alpha\pi}I); \alpha \in \mathbf{Q}]$ est dense dans X, alors T est chaotique.

Démonstration. D'après le Théorème 1.1.23 et la Proposition 1.1.42, il suffit de montrer que T est topologiquement transitif sous les hypothèses de densité de X_+ et X_-. Soient deux ouverts U et V non-vides de X. Par densité de X_+ et X_- dans X, il existe $x \in X_- \cap U$ et $y \in X_+ \cap V$, où $x = \sum_{i=1}^d \alpha_i x_i$ et $y = \sum_{i=1}^d \beta_i y_i$ avec $Tx_i = \lambda_i x_i$, $Ty_i = \mu_i y_i$, $|\lambda_i| < 1$ et $|\mu_i| > 1$ pour tout $i \in \{1, \dots, d\}$. On note alors $a_n = \sum_{i=1}^d \alpha_i x_i + \sum_{i=1}^d (\beta_i/\mu_i^n) y_i \to x$ et $T^n(\sum_{i=1}^d \alpha_i x_i + \sum_{i=1}^d (\beta_i/\mu_i^n) y_i) = \sum_{i=1}^d \lambda_i^n \alpha_i x_i + \sum_{i=1}^d \beta_i y_i \to y$ quand $n \to +\infty$. Ainsi il existe $N \geq 0$ tel que $a_N \in U$ et $T^N(a_N) \in V$, d'où $T^N(U) \cap V \neq \emptyset$. Donc T est topologiquement transitif, et donc hypercyclique d'après le Théorème 1.1.23. Si de plus X_0 est dense, alors T est chaotique d'après la Proposition 1.1.42. □

Les vecteurs propres d'un opérateur jouent également un rôle important en théorie ergodique.

En effet si l'on munit X de la σ-algèbre borélienne $\mathcal{A}$ et d'une mesure de probabilité m, alors un opérateur $T \in \mathcal{B}(X)$ peut être considéré comme une transformation mesurable $T\colon (X, \mathcal{A}, m) \to (X, \mathcal{A}, m)$.

Pour illustrer cela, nous présentons la notion d'ensemble de vecteurs propres parfaitement engendrant.

Définition 1.1.44 ([3, Definition 3.1])**.** On dit qu'un opérateur $T \in \mathcal{B}(X)$ a un *ensemble de vecteurs propres associés à des valeurs propres unimodulaires parfaitement engendrant* s'il existe une mesure de probabilité continue σ sur le cercle unité $\mathbf{T}$ telle que les vecteurs propres de T associés à des valeurs propres $\mu \in A$ engendrent un sous-espace dense dans X pour tout sous-ensemble σ-mesurable A de $\mathbf{T}$ vérifiant $\sigma(A) = 1$.

Il est connu qu'un opérateur $T \in \mathcal{B}(X)$ admettant un ensemble de vecteurs propres associés à des valeurs propres unimodulaires parfaitement engendrant est ergodique relativement à une mesure dite Gaussienne. Nous en présentons ici la définition.

Définition 1.1.45 ([3, Definitions 3.13 and 3.14])**.** Soit $(\Omega, \mathcal{F}, P)$ un espace de probabilité. On dit qu'une fonction mesurable $f\colon (\Omega, \mathcal{F}, P) \to \mathbf{C}$ a une *distribution Gaussienne complexe et symétrique* si $\Re(f)$ et $\Im(f)$ ont des distributions Gaussiennes indépendantes et centrées et si elles ont la même variance.

On dit qu'une mesure de probabilité m sur $(X, \mathcal{A})$ est une *mesure Gaussienne* si la fonction $y \mapsto x^*(y)$ a une distribution Gaussienne complexe et symétrique pour tout $x^* \in X^* \backslash \{0\}$.

Théorème 1.1.46 ([6, Theorem 1.1])**.** Si un opérateur $T \in \mathcal{B}(X)$ admet un ensemble de vecteurs propres associés à des valeurs propres unimodulaires parfaitement engendrant, alors il existe une mesure Gaussienne de support plein m sur X telle que $T\colon (X, \mathcal{A}, m) \to (X, \mathcal{A}, m)$ est ergodique.

Nous savons qu'un opérateur hypercyclique admet une orbite rencontrant tous les ouverts non-vides. La notion de fréquente hypercyclicité s'intéresse à la fréquence à laquelle cette orbite rencontre de tels ouverts.

Définition 1.1.47 ([26, Definition 9.2])**.** Un opérateur $T \in \mathcal{B}(X)$ est dit *fréquemment hypercyclique* s'il existe $x \in X$ tel que $\liminf_{N\to+\infty} \operatorname{card}\{0 \leq n \leq N-1; T^n x \in U\}/N > 0$ pour tout ouvert U non-vide de X. On dit alors que x est un *vecteur fréquemment hypercyclique pour* T et on note $FHC(T)$ l'ensemble des vecteurs fréquemment hypercycliques pour T.

La notion de fréquente hypercyclicité admet son propre critère pratique. Pour le présenter, nous disons qu'une série $\sum_{n\geq 0} x_n$ est *inconditionnellement convergente dans* X si la série $\sum_{n\geq 0} x_{\varphi(n)}$ est convergente pour toute bijection $\varphi\colon \mathbf{N} \to \mathbf{N}$.

Théorème 1.1.48 (Critère de Fréquente Hypercyclicité, [26, Theorem 9.9])**.** Soit $T \in \mathcal{B}(X)$. Supposons qu'il existe un sous-ensemble X_0 dense de X et une application $S\colon X_0 \to X_0$ tels que pour tout $x \in X_0$:

(i) $\sum_{n\geq 0} T^n x$ converge inconditionnellement ;

(ii) $\sum_{n\geq 0} S^n x$ converge inconditionnellement ;

(iii) $TSx = x$.

Alors T est fréquemment hypercyclique.

Ce critère s'applique par exemple pour les multiples de l'opérateur de décalage à gauche λB si $|\lambda| > 1$.

Exemple 1.1.49 ([26, Example 9.15])**.** Si $|\lambda| > 1$ alors l'opérateur de décalage à gauche λB est fréquemment hypercyclique sur $\ell^2(\mathbf{N})$.

En effet considérons X_0 l'ensemble des suites finies dans $\ell^2(\mathbf{N})$ et l'application $S = \lambda^{-1}F$ où F est l'opérateur de décalage à droite sur $\ell^2(\mathbf{N})$ de sorte que $\lambda BS = I$ sur $\ell^2(\mathbf{N})$. Pour tout $x \in X_0$ la série $\sum_{n\geq 0}(\lambda B)^n x$ converge inconditionnellement car c'est une somme finie. De plus la série $\sum_{n\geq 0} S^n x = \sum_{n\geq 0} \lambda^{-n} F^n x$ est inconditionnellement convergente car elle est absolument convergente. Ainsi λB est fréquemment hypercyclique.

La fréquente hypercyclicité est également caractérisée pour les opérateurs de décalage à gauche à poids B_ω sur $\ell^p(\mathbf{N})$ où $1 \leq p < +\infty$. Ceci permet d'expliciter des opérateurs hypercycliques qui ne sont pas fréquemment hypercycliques.

Théorème 1.1.50 ([5, Theorem 4])**.** Soit $1 \leq p < +\infty$ et soit $\omega = (\omega_n)_{n\geq 0}$ une suite bornée de réels strictement positifs. L'opérateur B_ω agissant sur $\ell^p(\mathbf{N})$ est fréquemment hypercyclique si et seulement si $\sum_{n\geq 1} 1/(\omega_1 \dots \omega_n)^p < +\infty$.

Cette définition de fréquente hypercyclicité est parfaitement définie dans le cadre topologique, comme l'est celle de l'hypercyclicité. Elle a cependant un lien très naturel avec la théorie ergodique.

Ainsi si m est une mesure de probabilité borélienne de X de support plein et si T est ergodique relativement à m, alors T est également fréquemment hypercyclique et l'ensemble de ses vecteurs fréquemment hypercycliques est de mesure pleine pour la mesure m.

Théorème 1.1.51 ([4, Proposition 6.23])**.** Soient $\mathcal{F}$ la σ-algèbre des boréliens de X, m une mesure de probabilité borélienne de support plein de X et $T \in \mathcal{B}(X)$. Si $T\colon (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ est ergodique, alors T est fréquemment hypercyclique et $FHC(T)$ est de mesure pleine dans X.

Démonstration. Soit $(U_k)_{k\geq 0}$ une base d'ouverts non-vides de X. Soit $k \geq 0$. D'après le Théorème 1.1.12, avec $f = \mathbf{1}_{U_k}$, il existe $A_k \in \mathcal{F}$ tel que $m(A_k) = 1$ et tel que $(1/N)\sum_{n=0}^{N-1} \mathbf{1}_{U_k}(T^n x) \to \int_X \mathbf{1}_{U_k} dm$ quand $N \to +\infty$ pour tout $x \in A_k$. Ainsi $\mathrm{card}\{0 \leq n \leq N-1; T^n x \in U_k\}/N \to m(U_k) > 0$ quand $N \to +\infty$ pour tout $x \in A_k$. On considère $A = \cap_{k\geq 0} A_k$ vérifiant alors $m(A) = 1$. Soient $x \in A$ et U un ouvert non-vide de X. Il existe $k \geq 0$ tel que $U_k \subset U$ et $\liminf_{N\to+\infty} \mathrm{card}\{0 \leq n \leq N-1; T^n x \in U\}/N \geq \lim_{N\to+\infty} \mathrm{card}\{0 \leq n \leq N-1; T^n x \in U_k\}/N = m(U_k) > 0$. Donc T est fréquemment hypercyclique et $FHC(T)$ est de mesure pleine car $A \subset FHC(T)$. □

Dans la prochaine section, nous appliquerons ce théorème à un opérateur naturellement associé à la Conjecture de Syracuse.

1.2 Conjecture de Syracuse

Nous présentons dans cette partie quelques énoncés liés à la Conjecture de Syracuse, aussi connue sous le nom de Conjecture de Collatz. Celle-ci porte sur le comportement des orbites de l'application agissant sur les entiers suivante.

Définition 1.2.1. On définit l'application

$$\begin{array}{rccl} T_0\colon & \mathbf{N} & \to & \mathbf{N} \\ & n & \mapsto & \begin{cases} n/2 & \text{si } n \text{ est pair ;} \\ 3n+1 & \text{si } n \text{ est impair,} \end{cases} \end{array}$$

que l'on appelle *application de Collatz.* Pour tout $k \geq 0$, l'ensemble $\{T_0^n(k) = T_0 \circ T_0 \circ \ldots \circ T_0(k); n \geq 0\}$ est noté $\mathrm{orb}(k, T_0)$ et est appelé *orbite de k sous l'action de T_0*.

Remarquons que $T_0(1) = 4$, que $T_0(4) = 2$ et que $T_0(2) = 1$, ce qui incite à présenter la notion de cycle pour T_0.

Définition 1.2.2. On dit que le d-uplet $(n_1, \ldots, n_d)$ est un *cycle pour T_0* si $d > 1$, si $T_0(n_i) = n_{i+1}$ pour tout $i \in \{1, \ldots, d-1\}$ et si $T_0(n_d) = n_1$.

On dit que le cycle $(1, 4, 2)$ est le *cycle trivial* de T_0.

Remarquons également que les orbites de petites valeurs initiales k contiennent toutes l'entier 1. Par exemple $T_0(2) = 1$, $T_0^7(3) = 1$ et $T_0^{16}(7) = 1$. La Conjecture de Syracuse est une généralisation de cette remarque à l'orbite de tout entier $k \geq 1$.

Conjecture 1.2.3. Pour tout $k \geq 1$, l'orbite de k sous l'action de T_0 contient 1.

Cette conjecture semble trouver son origine dans des travaux de Lothar Collatz datant des années 1950. Celui-ci s'intéressait au comportement des itérés de certaines fonctions arithmétiques du même type. Elle est vérifiée pour les entiers $k \leq 10^9$ dans les années 1960, mais ne fait néanmoins l'objet d'articles publiés qu'à partir des années 1970.

Cette conjecture est toujours ouverte aujourd'hui, bien qu'elle ait fait l'objet de nombreux travaux faisant appel à des techniques issues de domaines différents. Nous renvoyons à la référence [31] de Lagarias pour une présentation globale du problème.

Dans le cadre de la conjecture, sachant que $3n+1$ est pair dès que n est impair, l'étude de l'application T_0 coïncide avec celle de l'application raccourcie suivante. Son utilisation permettra de simplifier certaines des expressions obtenues dans les calculs.

Définition 1.2.4. On définit l'application

$$\begin{array}{rccl} T\colon & \mathbf{N} & \to & \mathbf{N} \\ & n & \mapsto & \begin{cases} n/2 & \text{si } n \text{ est pair ;} \\ (3n+1)/2 & \text{si } n \text{ est impair.} \end{cases} \end{array}$$

La Conjecture de Syracuse est alors vraie pour T_0 si et seulement si elle l'est pour T. Le cycle $(1, 2)$ pour T est appelé *cycle trivial pour T*.

Explicitons les antécédents d'un entier quelconque par T, ce qui se révélera utile dans les preuves à venir. Pour tout $m \geq 1$, on a $T^{-1}(\{3m + r\}) = \{6m + 2r\}$ si $r \in \{0, 1\}$ et $T^{-1}(\{3m + 2\}) = \{2m + 1, 6m + 4\}$.

Présentons quelques résultats aujourd'hui connus concernant la Conjecture de Syracuse. Tout d'abord le nombre d'orbites connu vérifiant la conjecture a été significativement élargi. En effet on sait dorénavant ([37]) que la conjecture est vérifiée pour tout entier $k \leq 20 \times 2^{58} \approx 5.764 \times 10^{18}$.

L'existence d'un cycle non-trivial pour l'application de Collatz T permettrait de montrer que la conjecture est fausse, ce qui motive l'étude des cycles de T. Eliahou a montré ([20]) qu'un hypothétique cycle non-trivial aurait nécessairement une taille supérieure à 10 439 860 591.

Si l'on ne peut pas montrer directement que tout entier $k \geq 1$ admet une orbite sous l'action de T contenant 1, on peut néanmoins s'intéresser à la proportion des entiers ayant une telle propriété. Krasikov et Lagarias ont montré ([30]) que si K est suffisamment large, alors le nombre d'entiers $1 \leq k \leq K$ vérifiant la Conjecture de Syracuse est supérieur à $K^{0.84}$.

Les entiers vérifiant la conjecture ont également fait l'objet d'études plus approfondies et précises. Applegate et Lagarias ont notamment démontré dans ce sens ([1]) qu'une infinité d'entiers n a une orbite sous l'action de T qui atteint 1 après au moins $6.143 \log(n)$ itérations de T.

Enfin, Tao a récemment montré ([44]) plus généralement que "presque-toute" orbite sous l'action de T atteint des valeurs presque bornées. On entend par cela que si l'on fixe une fonction $f \colon \mathbf{N} \to \mathbf{R}$ telle que $f(k) \to +\infty$ quand $k \to +\infty$, alors la borne inférieure de l'orbite de $k \geq 1$ sous l'action de T est inférieure à $f(k)$ pour presque tout $k \geq 1$ au sens de la densité logarithmique : si on note A l'ensemble $\{k \geq 1; \inf_{n \geq 0} T^n(k) \leq f(k)\}$ alors

$$\frac{\sum_{n \in A \cap \{1,\ldots,N\}} 1/n}{\sum_{n \in \{1,\ldots,N\}} 1/n} \xrightarrow[N \to +\infty]{} 1.$$

Notre première contribution est une étude de la Conjecture de Syracuse du point de vue de la dynamique linéaire. Nous voulons alors associer à l'application de Collatz T un opérateur $\mathcal{T}$ agissant sur un espace à définir et lier certaines propriétés dynamiques de T à certaines propriétés dynamiques de $\mathcal{T}$. Cette démarche a pour origine les travaux de Berg et Meinardus ([9]).

En effet ces derniers ont caractérisé la Conjecture de Syracuse par le biais d'équations fonctionnelles. Plus précisément, ils ont montré que la conjecture était équivalente au fait que les solutions holomorphes sur $\mathbf{D}$ de l'équation

$$h(z^3) = h(z^6) + \frac{1}{3z} \sum_{i=0}^{2} \lambda^i h(\lambda^i z^2),$$

où $\lambda = e^{2i\pi/3}$, sont de la forme $h(z) = h_0 + h_1 z/(1-z)$. Neklyudov observe alors dans l'article [36] que cette caractérisation est équivalente au fait suivant : l'opérateur $\mathcal{F} \colon \mathrm{Hol}(\mathbf{D}) \to \mathrm{Hol}(\mathbf{D})$, défini pour toute fonction $f \in \mathrm{Hol}(\mathbf{D})$ telle que $f(z) = \sum_{n=0}^{\infty} c_n z^n$ par

$$\mathcal{F}f(z) = \sum_{n=0}^{\infty} c_{T(n)} z^n \quad \text{pour tout } z \in \mathbf{D},$$

a pour valeur propre 1 et celle-ci est de multiplicité égale à 2.

Notre étude s'appuie sur les travaux de Neklyudov ([36]), qui associe à l'application T un autre opérateur $\mathcal{T}$ défini sur l'espace de Bergman classique. En étudiant les propriétés dynamiques de l'opérateur $\mathcal{T}$, il est possible de lier le comportement des orbites de $\mathcal{T}$ avec celui des orbites de T.

Définition 1.2.5. L'espace

$$\left\{f\colon z \mapsto \sum_{n=0}^{\infty} c_n z^n \in \mathrm{Hol}(\mathbf{D}); \|f\|^2 = \sum_{n=0}^{\infty} \frac{\pi\,|c_n|^2}{n+1} < +\infty\right\}$$

est noté $\mathcal{B}^2$ et est appelé *espace de Bergman.*

Nous définissons maintenant l'opérateur $\mathcal{T}$, qui sera l'objet de notre étude dans le troisième chapitre de ce manuscrit.

Définition 1.2.6. On note $\mathcal{X}$ le quotient $\mathcal{B}^2/\mathrm{vect}[1, z, z^2]$ et on identifie canoniquement une fonction $f \in \mathcal{B}^2$ telle que $f(z) = \sum_{n=0}^{\infty} c_n z^n$ avec la fonction $\overline{f} \in \mathcal{X}$ telle que $\overline{f}(z) = \sum_{n=3}^{\infty} c_n z^n$. On note également $\mathcal{T}$ l'opérateur agissant sur $\mathcal{X}$ défini pour tout $f \in \mathcal{X}$ telle que $f(z) = \sum_{n=3}^{\infty} c_n z^n$ par

$$\mathcal{T} f(z) = \sum_{\{n\geq 3; T(n)\geq 3\}} c_n z^{T(n)} \quad \text{pour tout } z \in \mathbf{D}.$$

.

Puisque $T(0) = 0$, $T(1) = 2$ et $T(2) = 1$, on préfère faire agir $\mathcal{T}$ sur le quotient $\mathcal{X}$ plutôt que sur $\mathcal{B}^2$ afin d'éviter le point fixe 0 et le cycle trivial $(1, 2)$. L'opérateur $\mathcal{T}$ est alors bien défini sur l'espace $\mathcal{X}$ et y est continu, ce qui permet l'étude de ses propriétés dynamiques.

Proposition 1.2.7 ([36, Section 2])**.** L'opérateur $\mathcal{T}$ est bien défini et continu sur $\mathcal{X}$.

Démonstration. Pour tout $f \in \mathcal{X}$ telle que $f(z) = \sum_{n=3}^{\infty} c_n z^n$ on a

$$\mathcal{T} f(z) = \sum_{\{n\geq 3; T(n)\geq 3\}} c_n z^{T(n)} = \sum_{k=3}^{\infty} \left(\sum_{\{j\geq 3; T(j)=k\}} c_j \right) z^k$$

et par l'inégalité de Cauchy-Schwarz

$$\begin{aligned}
\|\mathcal{T}f\|^2 &= \sum_{k=3}^{\infty} \frac{\pi}{k+1} \left| \sum_{\{j\geq 3; T(j)=k\}} c_j \right|^2 \\
&= \sum_{k=3}^{\infty} \frac{\pi}{k+1} \left| \sum_{\{j\geq 3; T(j)=k\}} \frac{c_j}{\sqrt{j+1}} \sqrt{j+1} \right|^2 \\
&\leq \sum_{k=3}^{\infty} \frac{\pi}{k+1} \left(\sum_{\{j\geq 3; T(j)=k\}} \frac{|c_j|^2}{j+1} \right) \left(\sum_{\{j\geq 3; T(j)=k\}} j+1 \right) \\
&\leq \left(\sup_{k\geq 3} \frac{1}{k+1} \sum_{\{j\geq 3; T(j)=k\}} j+1 \right) \sum_{k=3}^{\infty} \sum_{\{j\geq 3; T(j)=k\}} \frac{\pi \, |c_j|^2}{j+1},
\end{aligned}$$

où

$$\begin{aligned}
\sup_{k\geq 3} \frac{1}{k+1} \sum_{\{j\geq 3; T(j)=k\}} j+1 &= \max_{r\in\{0,1,2\}} \left(\sup_{m\geq 1} \frac{1}{3m+r+1} \sum_{T(j)=3m+r} j+1 \right) \\
&= \max \left(\sup_{m\geq 1} \frac{6m+1}{3m+1}, \sup_{m\geq 1} \frac{6m+3}{3m+2}, \sup_{m\geq 1} \frac{6m+5+2m+2}{3m+3} \right) \\
&= \max \left(\frac{6}{3}, \frac{6}{3}, \frac{8}{3} \right) \\
&= \frac{8}{3}.
\end{aligned}$$

Donc $\|\mathcal{T}f\|^2 \leq 8/3 \cdot \|f\|^2$ pour tout $f \in \mathcal{X}$, ainsi $\mathcal{T}$ définit bien un opérateur continu agissant sur $\mathcal{X}$. □

Le théorème principal obtenu par Neklyudov concernant la dynamique de $\mathcal{T}$ s'énonce ainsi :

Théorème 1.2.8 ([36, Theorem 2.2])**.** Si T n'admet pas de cycle non-trivial, alors $\mathcal{T}$ est hypercyclique.

Démonstration. Il suffit de montrer que $\mathcal{T}$ vérifie le Théorème 1.1.28. On considère $X_0 = Y_0 = \{\sum_{k=3}^{d} c_k z^k; d \geq 3, c_3, \ldots, c_d \in \mathbf{D}\}$. Soit $k \geq 3$. Si T n'admet pas de cycle non-trivial, l'orbite de k sous l'action de T atteint 1 ou diverge vers l'infini. Si l'orbite atteint 1 alors il existe $N \geq 0$ tel que $T^n(k) \in \{1, 2\}$ pour tout $n \geq N$. Ainsi $\mathcal{T}^n z^k = z^{T^n(k)} = 0$ dans $\mathcal{X}$ pour tout $n \geq N$. Si l'orbite de k sous l'action de T diverge vers l'infini, alors

$$\|\mathcal{T}^n z^k\|^2 = \|z^{T^n(k)}\|^2 = \frac{\pi}{T^n(k)+1} \xrightarrow[n\to+\infty]{} 0.$$

On a donc prouvé que $\mathcal{T}^n z^k \to 0$ quand $n \to +\infty$ pour tout $k \geq 3$. Ainsi $\mathcal{T}^n x \to 0$ quand $n \to +\infty$ pour tout $x \in X_0$.

On considère l'application $S\colon Y_0 \to Y_0$ définie par $Sz^k = z^{2k}$ pour tout $k \geq 3$. Alors pour tout $k \geq 3$

$$\|S^n z^k\|^2 = \|z^{2^n k}\|^2 = \frac{\pi}{2^n k + 1} \xrightarrow[n \to +\infty]{} 0.$$

Ainsi $S^n y \to 0$ quand $n \to +\infty$ pour tout $y \in Y_0$. Finalement $\mathcal{T} S z^k = \mathcal{T} z^{2k} = z^k$ pour tout $k \geq 3$, donc $\mathcal{T} S y = y$ pour tout $y \in Y_0$. Ainsi $\mathcal{T}$ vérifie le Critère d'Hypercyclicité, ce qui donne l'hypercyclicité de $\mathcal{T}$ d'après le Théorème 1.1.28. □

L'hypothèse que T n'admet pas de cycle non-trivial provient du fait que l'orbite d'un entier $k \geq 1$ sous l'action de T peut admettre trois comportements. L'orbite peut atteindre le cycle trivial, ce qui signifie que k vérifie la Conjecture de Syracuse, ou l'orbite peut atteindre un cycle non-trivial, ou enfin l'orbite peut diverger vers l'infini. Ainsi, en éliminant l'éventualité de l'existence d'un cycle non-trivial pour T, Neklyudov a pu démontrer l'hypercyclicité de $\mathcal{T}$ grâce au Critère d'Hypercyclicité.

Nous avons déjà présenté dans la section 1.1.3 le rôle joué par les valeurs propres et les vecteurs propres en dynamique linéaire. Afin d'étudier la dynamique de $\mathcal{T}$, Neklyudov explicite certaines familles de vecteurs propres qui, sous certaines conditions, engendrent des sous-espaces denses.

Proposition 1.2.9 ([36, Theorem 2.3])**.** Pour tout $\mu \in \mathbf{C}$ tel que $0 \leq |\mu| < \sqrt{2}$, les fonctions

$$h_m(\mu, \cdot)\colon z \mapsto \sum_{n=0}^{\infty} \mu^n \left(z^{(6m+4)2^n} - z^{(2m+1)2^n}\right) \quad \text{et} \quad h_0(\mu, \cdot)\colon z \mapsto \sum_{n=0}^{\infty} \mu^n z^{2^{n+2}}$$

appartiennent à $\mathcal{X}$ et sont pour tout $m \geq 1$ des vecteurs propres de $\mathcal{T}$ associés à la valeur propre μ.

Démonstration. Pour tout $m \geq 1$ les ensembles $\{(6m+4)2^n; n \geq 0\}$ et $\{(2m+1)2^n; n \geq 0\}$ sont disjoints. Donc pour tout $\mu \in \mathbf{C}$ et tout $m \geq 1$

$$\|h_m(\mu, \cdot)\|^2 = \sum_{n=0}^{\infty} \frac{\pi\, |\mu|^{2n}}{(6m+4)2^n + 1} + \sum_{n=0}^{\infty} \frac{\pi\, |\mu|^{2n}}{(2m+1)2^n + 1} < +\infty$$

et

$$\|h_0(\mu, \cdot)\|^2 = \sum_{n=0}^{\infty} \frac{\pi\, |\mu|^{2n}}{2^{n+2} + 1} < +\infty$$

car $|\mu|^2 < 2$. De plus pour tout $m \geq 1$ et tout $z \in \mathbf{D}$

$$\begin{aligned}
\mathcal{T}h_m(\mu, \cdot)(z) &= \sum_{n=0}^{\infty} \mu^n \left(z^{T((6m+4)2^n)} - z^{T((2m+1)2^n)} \right) \\
&= \sum_{n=1}^{\infty} \mu^n \left(z^{(6m+4)2^{n-1}} - z^{(2m+1)2^{n-1}} \right) + z^{3m+2} - z^{3m+2} \\
&= \mu \sum_{n=0}^{\infty} \mu^n \left(z^{(6m+4)2^n} - z^{(2m+1)2^n} \right) \\
&= \mu h_m(\mu, z)
\end{aligned}$$

et

$$\mathcal{T}h_0(\mu, \cdot)(z) = \sum_{n=0}^{\infty} \mu^n z^{T(2^{n+2})} = \sum_{n=1}^{\infty} \mu^n z^{2^{n+1}} + z^2 = \mu \sum_{n=0}^{\infty} \mu^n z^{2^{n+2}} = \mu h_0(\mu, z).$$

□

Nous commençons par généraliser le cadre considéré par Neklyudov, et nous nous plaçons dans un espace de Bergman pondéré.

Définition 1.2.10. On note $\mathcal{B}^2_\omega$, où $\omega\colon \mathbf{N} \to (0, +\infty)$ est un poids strictement positif, l'espace

$$\left\{ f\colon z \mapsto \sum_{n=0}^{\infty} c_n z^n \in \mathrm{Hol}(\mathbf{D}); \|f\|^2_\omega = \sum_{n=0}^{\infty} \frac{|c_n|^2}{\omega(n)} < +\infty \right\},$$

que l'on appelle *espace de Bergman pondéré*. On notera $\mathcal{X}_\omega$ le quotient $\mathcal{B}^2_\omega / \mathrm{vect}[1, z, z^2]$ et $\mathcal{T}$ l'opérateur agissant sur $\mathcal{X}_\omega$ défini, pour tout $f \in \mathcal{X}_\omega$ telle que $f(z) = \sum_{n=3}^{\infty} c_n z^n$, par

$$\mathcal{T}f(z) = \sum_{\{n \geq 3; T(n) \geq 3\}} c_n z^{T(n)} \quad \text{pour tout } z \in \mathbf{D}.$$

On note ω_0 le poids défini par $\omega_0(n) = (n+1)/\pi$, de sorte que $\mathcal{X}_{\omega_0} = \mathcal{X}$, ce qui nous permettra de constater que nos résultats généralisent ceux obtenus par Neklyudov.

Notre premier résultat, démontré dans le troisième chapitre, caractérise la continuité de l'opérateur $\mathcal{T}$ agissant sur l'espace de Bergman à poids quotienté $\mathcal{X}_\omega$.

Proposition A. L'opérateur $\mathcal{T}$ est bien défini et continu sur $\mathcal{X}_\omega$ si et seulement si la suite

$$\left(\sum_{\{j \geq 3; T(j) = k\}} \frac{\omega(j)}{\omega(k)} \right)_{k \geq 3}$$

est bornée, c'est-à-dire si et seulement si les trois suites $(\omega(6m)/\omega(3m))_{m\geq 1}$, $(\omega(6m+2)/\omega(3m+1))_{m\geq 1}$ et $((\omega(6m+4)+\omega(2m+1))/\omega(3m+2))_{m\geq 1}$ sont bornées. Dans ce cas on a

$$\|\mathcal{T}\|_\omega^2 = \sup_{k\geq 3} \sum_{\{j\geq 3; T(j)=k\}} \frac{\omega(j)}{\omega(k)}$$
$$= \max\left\{\sup_{m\geq 1}\frac{\omega(6m)}{\omega(3m)}, \sup_{m\geq 1}\frac{\omega(6m+2)}{\omega(3m+1)}, \sup_{m\geq 1}\frac{\omega(6m+4)+\omega(2m+1)}{\omega(3m+2)}\right\}$$

et pour tout $n \geq 0$

$$\|\mathcal{T}^n\|_\omega^2 = \sup_{k\geq 3} \sum_{\{j\geq 3; T^n(j)=k\}} \frac{\omega(j)}{\omega(k)}.$$

Nous aurons besoin par la suite de conditions impliquant que les vecteurs propres $h_m(\mu,\cdot)$ explicités par Neklyudov ([36]) engendrent un sous-espace dense dans $\mathcal{X}_\omega$. Ceci est démontré dans le troisième chapitre sous la forme du théorème suivant.

Théorème B. Si ω is borné inférieurement, alors $h_m(\mu,\cdot)$ appartient à $\mathcal{X}_\omega$ pour tout $m \geq 0$ et tout $\mu \in \mathbf{D}$. De plus

$$\text{vect}[h_m(\mu,\cdot); m \geq 0, \mu \in \mathbf{D}]$$

est dense dans $\mathcal{X}_\omega$. C'est en particulier le cas si $\omega = \omega_0$.

En utilisant les mêmes arguments, on peut montrer plus précisément que si $h_m(\mu,\cdot)$, $m \geq 0$, appartient à $\mathcal{X}_\omega$ pour tout $\mu \in A$ où A est un disque ouvert et centré en 0, alors le sous-espace

$$\text{vect}[h_m(\mu,\cdot); m \geq 0, \mu \in \Delta]$$

est dense dans $\mathcal{X}_\omega$ dès que $\Delta \subset A$ admet un point d'accumulation dans A. Nous démontrons également d'une manière analogue que l'opérateur adjoint $\mathcal{T}^*$ n'admet pas de valeur propre.

Théorème C. Si ω is borné inférieurement, alors $\sigma_p(\mathcal{T}^*) = \emptyset$. C'est en particulier le cas si $\omega = \omega_0$.

Le fait que les vecteurs propres de $\mathcal{T}$ puissent engendrer un sous-espace dense nous permet de généraliser le Théorème 1.2.8. En effet nous sommes en mesure de montrer grâce au Critère d'Hypercyclicité que, sous une hypothèse assez faible sur le poids ω, $\mathcal{T}$ est hypercyclique. Il est important de remarquer ici que, au contraire de ce qui se produit dans le Théorème 1.2.8, nous n'avons besoin de faire aucune supposition supplémentaire concernant les cycles de l'application de Collatz T.

Théorème D. Si ω is borné inférieurement et si $\omega(k2^n) \to +\infty$ quand $n \to +\infty$ pour tout $k \geq 3$ alors $\mathcal{T}$ est hypercyclique sur $\mathcal{X}_\omega$. C'est en particulier le cas si $\omega = \omega_0$.

Nous donnons dans le troisième chapitre une condition sur le poids ω qui implique que l'opérateur $\mathcal{T}$ vérifie le Critère de Godefroy-Shapiro.

Théorème E. Si ω est borné inférieurement et s'il existe $\rho > 1$ tel que pour tout $k \geq 3$ la suite $(\rho^n/\omega(k2^n))_{n\geq 0}$ est bornée, alors $\mathcal{T}$ satisfait le Critère de Godefroy-Shapiro. C'est en particulier le cas si $\omega = \omega_0$.

Si on impose une condition supplémentaire sur le poids ω, les vecteurs propres $h_m(\mu, \cdot)$ forment en réalité un ensemble de vecteurs propres associés à des valeurs propres unimodulaires parfaitement engendrant, ce qui permet de montrer que l'opérateur $\mathcal{T}$ est ergodique relativement à une mesure de support plein.

Théorème F. Si ω est bornée inférieurement et si $\sum_{n=0}^{\infty} 1/\omega(k2^n) < +\infty$ pour tout $k \geq 3$, alors $\mathcal{T}$ est ergodique par rapport à une mesure Gaussienne de support plein sur $\mathcal{X}_\omega$. C'est en particulier le cas si $\omega = \omega_0$.

Puisqu'un opérateur ergodique relativement à une mesure de support plein est en particulier fréquemment hypercyclique, nous obtenons sous la même hypothèse sur le poids que $\mathcal{T}$ est fréquemment hypercyclique.

Théorème G. Si ω est bornée inférieurement et si $\sum_{n=0}^{\infty} 1/\omega(k2^n) < +\infty$ pour tout $k \geq 3$, alors $\mathcal{T}$ est fréquemment hypercyclique sur $\mathcal{X}_\omega$.

Enfin sous la même condition sur le poids ω, le Critère d'Hypercyclicité et le fait que les vecteurs propres $h_m(\mu, \cdot)$ associés à des valeurs propres $\mu \in \{e^{i\alpha\pi}; \alpha \in \mathbf{Q}\}$ engendrent un sous-espace dense dans $\mathcal{X}_\omega$ permettent de montrer que l'opérateur $\mathcal{T}$ est chaotique. Ceci permet de répondre affirmativement à une question posée par Neklyudov ([36, Introduction]).

Théorème H. Si ω est borné inférieurement et si $\sum_{n=0}^{\infty} 1/\omega(k2^n) < +\infty$, alors $\mathcal{T}$ est chaotique sur $\mathcal{X}_\omega$. C'est en particulier le cas si $\omega = \omega_0$.

1.3 Problème du sous-espace invariant et les opérateurs de Bishop

Nous commençons par présenter ici quelques résultats concernant le problème du sous-espace invariant. Ce problème motive l'étude de la dynamique des opérateurs de Bishop, qui fait l'objet du chapitre 4 de ce manuscrit.

Soit X un espace de Banach séparable, complexe et de dimension infinie.

Définition 1.3.1. Soit $T \in \mathcal{B}(X)$ un opérateur agissant sur X et F un sous-espace fermé de X. On dit que F est *non-trivial* si $F \neq \{0\}$ et $F \neq X$. On dit que F est *invariant pour* T si $T(F) \subset F$. On dit que F est *hyperinvariant pour* T si F est invariant pour tous les opérateurs commutant avec T.

Le problème du sous-espace invariant s'énonce alors ainsi.

Conjecture 1.3.2. Soit T un opérateur linéaire et continu agissant sur X. Existe-t-il un sous-espace fermé de X qui soit non-trivial et T-invariant ?

Nous renvoyons aux livres [40] de Radjavi et Rosenthal ainsi que [17] de Chalendar et Partington pour une présentation du problème du sous-espace invariant.

Ce célèbre problème de l'analyse fonctionnelle est ouvert depuis plus d'un demi-siècle, bien que des résultats (positifs ou négatifs) très substantiels aient été obtenus pour de nombreuses classes d'opérateurs ou d'espaces.

Von Neumann et Aronszajn et Smith ([2]) ont montré qu'un opérateur compact agissant sur un espace de Banach a toujours un sous-espace invariant non-trivial. Ce résultat a ensuite été généralisé par Lomonosov qui a démontré ([33]) qu'un opérateur commutant avec un opérateur compact (non-nul) a un sous-espace hyperinvariant non-trivial.

Néanmoins Enflo, puis Read, ont obtenu des contre-exemples au problème du sous-espace invariant dans le cas des espaces de Banach. Ainsi Enflo a montré dans [21] l'existence d'un espace de Banach X et d'un opérateur $T \in \mathcal{B}(X)$ n'admettant aucun sous-espace invariant non-trivial. Read a également construit dans [41] un opérateur agissant sur un espace de Banach n'admettant aucun sous-espace invariant non-trivial, puis a donné dans [42] un exemple d'un opérateur n'admettant aucun sous-espace invariant non-trivial sur l'espace de Banach des suites sommables $\ell^1(\mathbf{N})$. Ce sera alors le premier contre-exemple sur un espace de Banach classique. Dix ans plus tard il construira ([43]) des opérateurs T quasinilpotents, c'est-à-dire dont le rayon spectral $\lim_{n\to+\infty} \|T^n\|^{1/n}$ est nul, sur $\ell^1(\mathbf{N})$ qui n'ont pas de sous-espace invariant non-trivial.

Le problème est largement ouvert dans le cadre réflexif, et donc en particulier toujours non-résolu dans le cadre des espaces de Hilbert. Un des résultats positifs les plus spectaculaires dans le cadre Hilbertien est le Théorème de Brown-Chevreau-Pearcy ([14]) qui affirme qu'une contraction sur un espace de Hilbert dont le spectre contient le cercle unité admet un sous-espace invariant non-trivial.

Nous étudierons dans le cadre de cette thèse les opérateurs de Bishop. Il s'agit d'une famille d'opérateurs qui ont été proposés par Bishop comme de potentiels contre-exemples au problème du sous-espace invariant sur l'espace de Hilbert $L^2([0,1])$.

Définition 1.3.3 ([19])**.** Pour tout $\alpha \in [0,1]$, on appelle *opérateur de Bishop* l'opérateur T_α agissant sur $L^2([0,1])$ et défini pour tout $f \in L^2([0,1])$ par

$$T_\alpha f(x) = xf(\{x+\alpha\}) \quad \text{presque partout sur } [0,1],$$

où $\{\cdot\}$ désigne la partie fractionnaire d'un nombre réel.

Nous commençons par observer que si α est un rationnel r/q, T_α^q commute avec son adjoint et admet alors nécessairement un sous-espace invariant.

Définition 1.3.4 ([17, Definition 1.2.15])**.** Un opérateur $T \in \mathcal{B}(\mathcal{H})$ est dit *normal* si $TT^* = T^*T$.

Le théorème spectral permet de montrer (grâce au calcul fonctionnel borélien) qu'un tel opérateur admet un sous-espace invariant non-trivial.

Théorème 1.3.5 ([17, Theorem 3.4.1])**.** Un opérateur $T \in \mathcal{B}(\mathcal{H})$ normal qui n'est pas un multiple de l'identité admet un sous-espace hyperinvariant non-trivial.

Proposition 1.3.6 ([17, Section 5.4])**.** Soit $\alpha \in [0,1]$ un rationnel. L'opérateur T_α admet un sous-espace hyperinvariant non-trivial.

Démonstration. Soit $\alpha = r/q$ avec r et q premiers entre eux. Alors pour tout $f \in L^2([0,1])$ et pour presque tout $x \in [0,1]$

$$T_\alpha^q f(x) = x\{x+r/q\}\dots\{x+(q-1)r/q\}f(\{x+qr/q\}) = x\{x+1/q\}\dots\{x+(q-1)/q\}f(x).$$

Ainsi T_α^q est l'opérateur de multiplication, par w sur $L^2([0,1])$, M_w où $w\colon [0,1] \to [0,1]$ est défini par $w(x) = x\{x+1/q\}\dots\{x+(q-1)/q\}$. Ainsi l'opérateur T_α^q est normal et n'est pas un multiple de l'identité, il admet donc un sous-espace hyperinvariant non-trivial d'après le Théorème 1.3.5. Or un sous-espace hyperinvariant pour T_α^q est hyperinvariant pour T_α car un opérateur qui commute avec T_α commute nécessairement avec T_α^q. Ainsi T_α admet un sous-espace hyperinvariant non-trivial. □

Nous nous restreignons donc au cas où le paramètre α est irrationnel. Davie a montré ([19]) que si α n'est pas un nombre de Liouville, alors T_α admet un sous-espace hyperinvariant non-trivial. On rappelle qu'on dit qu'un irrationnel α est un *nombre de Liouville* s'il peut être bien approché par des rationnels, au sens suivant : il existe une suite $(r_n/q_n)_{n\geq 1}$ de rationnels telle que $|\alpha - r_n/q_n| < 1/q_n^n$ pour tout $n \geq 1$. Il peut être intéressant de noter qu'à ce stade les paramètres α dont on sait que T_α n'est pas un contre-exemple au problème du sous-espace invariant, à savoir les rationnels et les nombres qui ne sont pas de Liouville, sont de natures très différentes.

D'autres auteurs ont cherché ensuite à élargir cet ensemble de paramètres α tels que T_α admet un sous-espace hyperinvariant non-trivial en tentant d'exhiber certains nombres de Liouville α tels que T_α a un sous-espace (hyper)invariant non-trivial. Dans ce but, ils se sont intéressés à la façon dont un irrationnel est approché par des rationnels.

Nous présentons à cet effet les notions de fraction continue et de convergents, qui se révèlent particulièrement utiles pour approcher des irrationnels grâce à des rationnels. Nous renvoyons au livre de Bugeaud [15] afin d'approfondir cette notion.

Définition 1.3.7 ([15, Section 1.2])**.** Soient $n \geq 0$ et $(a_k)_{0\leq k\leq n}$ des entiers tels que $a_0 \in \mathbf{Z}$ et $a_k \geq 1$ pour tout $k \in \{1,\dots,n\}$. On note $[a_0; a_1, \dots, a_n]$ et on appelle *fraction continue finie* le rationnel

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_n}}}.$$

Nous appliquons à tout irrationnel α un algorithme qui permettra de lui associer une suite de fractions continues finies, que l'on appellera les convergents de α.

Définition 1.3.8 ([15, Definition 1.2])**.** Soit un irrationnel $x \in [0,1]$. Soient $a_0 \in \mathbf{Z}$ et $\xi_0 > 1$ tels que $x = a_0 + 1/\xi_0$. Soient $(a_k)_{k\geq 1}$ une suite d'entiers naturels non-nuls et $(\xi_k)_{k\geq 1}$ une suite de réels dans $(1,+\infty)$ telles que $\xi_k = a_{k+1} + 1/\xi_{k+1}$ pour tout $k \geq 0$. On définit les suites d'entiers $(r_n)_{n\geq 1}$ et $(q_n)_{n\geq 1}$ de la manière suivante : r_n et q_n sont premiers entre eux et $r_n/q_n = [a_0; a_1, \dots, a_n]$ pour tout $n \geq 0$. Les rationnels $(r_n/q_n)_{n\geq 1}$ sont appelés les *convergents de x*.

Le théorème suivant permet de constater que la suite des convergents vérifie certaines relations de récurrence, qui sont particulièrement utiles pour estimer l'erreur commise quand on approche un irrationnel par ses convergents.

Théorème 1.3.9 ([15, Theorem 1.3])**.** Soient $a_0 \in \mathbf{Z}$ et trois suites d'entiers $(a_n)_{n\geq 1}$, $(r_n)_{n\geq 1}$ et $(q_n)_{n\geq 1}$ telles que $a_n \geq 1$, telles que r_n et q_n sont premiers entre eux et telles que $r_n/q_n = [a_0; a_1, \dots, a_n]$ pour tout $n \geq 1$. Si on note $r_{-1} = 1$, $r_0 = a_0$, $q_{-1} = 0$ et $q_0 = 1$, alors $r_n = a_n r_{n-1} + r_{n-2}$ et $q_n = a_n q_{n-1} + q_{n-2}$ pour tout $n \geq 1$.

Démonstration. Procédons par récurrence. Si $n = 1$ on a $r_1/q_1 = [a_0; a_1] = a_0 + 1/a_1 = (a_1a_0 + 1)/a_1$, donc $r_1 = a_1a_0 + 1 = a_1r_0 + r_{-1}$ et $q_1 = a_1 = a_1q_0 + q_{-1}$ car $a_1a_0 + 1$ et a_1 sont premiers entre eux.

Supposons maintenant que la propriété est vraie jusqu'à un rang $n \geq 1$ pour toute suite $(b_k)_{k\geq 1}$ d'entiers où $b_0 \in \mathbf{Z}$ et $b_k \geq 1$ pour tout $k \geq 1$. Remarquons que pour tout $k \in \{2, \ldots, n+1\}$

$$\frac{r_k}{q_k} = [a_0; a_1, \ldots, a_k] = a_0 + \frac{1}{[a_1; a_2, \ldots, a_k]}.$$

Soient $r'_1/q'_1, \ldots, r'_n/q'_n$ les rationnels tels que $r'_j/q'_j = [a_1; a_2, \ldots, a_{j+1}]$ pour tout entier $j \in \{1, \ldots, n\}$ avec $r'_{-1} = 1$, $r'_0 = a_1$, $q'_{-1} = 0$ et $q'_0 = 1$. Ainsi pour tout $k \in \{2, \ldots, n+1\}$

$$\frac{r_k}{q_k} = a_0 + \frac{1}{[a_1; a_2, \ldots, a_k]} = a_0 + \frac{q'_{k-1}}{r'_{k-1}} = \frac{a_0r'_{k-1} + q'_{k-1}}{r'_{k-1}},$$

donc $r_k = a_0r'_{k-1} + q'_{k-1}$ et $q_k = r'_{k-1}$ car $a_0r'_{k-1} + q'_{k-1}$ et r'_{k-1} sont premiers entre eux. Par hypothèse de récurrence on a $r'_n = a_{n+1}r'_{n-1} + r'_{n-2}$ et $q'_n = a_{n+1}q'_{n-1} + q'_{n-2}$, d'où

$$\begin{aligned} r_{n+1} &= a_0r'_n + q'_n \\ &= a_0(a_{n+1}r'_{n-1} + r'_{n-2}) + a_{n+1}q'_{n-1} + q'_{n-2} \\ &= a_{n+1}(a_0r'_{n-1} + q'_{n-1}) + a_0r'_{n-2} + q'_{n-2} \\ &= a_{n+1}r_n + r_{n-1}. \end{aligned}$$

et

$$q_{n+1} = r'_n = a_{n+1}r'_{n-1} + r'_{n-2} = a_{n+1}q_n + q_{n-1}.$$

□

On peut estimer l'erreur faite en approximant un réel par ses convergents grâce aux relations que vérifient les convergents.

Théorème 1.3.10 ([15, Corollary 1.4])**.** Soient un irrationnel x et la suite de ses convergents $(r_n/q_n)_{n\geq 1} = ([a_0; a_1, \ldots, a_n])_{n\geq 1}$. Pour tout $n \geq 1$

$$\frac{1}{q_n(q_n + q_{n+1})} < \left| x - \frac{r_n}{q_n} \right| < \frac{1}{q_nq_{n+1}}.$$

Démonstration. Soit la suite $(\xi_n)_{n\geq 0}$ d'éléments de $(1, +\infty)$ telle que $x = a_0 + 1/\xi_0$ et $\xi_n = a_{n+1} + 1/\xi_{n+1}$ pour tout $n \geq 0$. Alors $x = [a_0; a_1, \ldots, a_n, \xi_n]$ pour tout $n \geq 1$.

Montrons que pour tout $n \geq 1$

$$x = [a_0; a_1, \ldots, a_n, \xi_n] = \frac{\xi_nr_n + r_{n-1}}{\xi_nq_n + q_{n-1}}.$$

On a tout d'abord

$$x = [a_0; a_1, \xi_1] = a_0 + \frac{1}{a_1 + 1/\xi_1} = \frac{a_0(a_1 + 1/\xi_1) + 1}{a_1 + 1/\xi_1} = \frac{\xi_1(a_1a_0 + 1) + a_0}{\xi_1a_1 + 1} = \frac{\xi_1r_1 + r_0}{\xi_1q_1 + q_0},$$

puis pour tout $n \geq 1$

$$\begin{aligned}
\frac{\xi_{n+1}r_{n+1}+r_n}{\xi_{n+1}q_{n+1}+q_n} &= \frac{\xi_{n+1}(a_{n+1}r_n+r_{n-1})+r_n}{\xi_{n+1}(a_{n+1}q_n+q_{n-1})+q_n} \\
&= \frac{(a_{n+1}+1/\xi_{n+1})r_n+r_{n-1}}{(a_{n+1}+1/\xi_{n+1})q_n+q_{n-1}} \\
&= \frac{\xi_n r_n + r_{n-1}}{\xi_n q_n + q_{n-1}} \\
&= \dots \\
&= \frac{\xi_1 r_1 + r_0}{\xi_1 q_1 + q_0} \\
&= x.
\end{aligned}$$

Ainsi

$$\begin{aligned}
q_n\left(x-\frac{r_n}{q_n}\right) &= q_n\frac{\xi_n r_n + r_{n-1}}{\xi_n q_n + q_{n-1}} - r_n \\
&= \frac{\xi_n r_n q_n + r_{n-1}q_n - \xi_n r_n q_n - r_n q_{n-1}}{\xi_n q_n + q_{n-1}} \\
&= \frac{r_{n-1}q_n - r_n q_{n-1}}{\xi_n q_n + q_{n-1}}.
\end{aligned}$$

Or pour tout $n \geq 1$

$$\begin{aligned}
r_{n-1}q_n - r_n q_{n-1} &= r_{n-1}(a_n q_{n-1} + q_{n-2}) - (a_n r_{n-1} + r_{n-2})q_{n-1} \\
&= -(r_{n-2}q_{n-1} - r_{n-1}q_{n-2}) \\
&= \dots \\
&= (-1)^n (r_{-1}q_0 - r_0 q_{-1}) \\
&= (-1)^n,
\end{aligned}$$

d'où

$$\left|x-\frac{r_n}{q_n}\right| = \frac{1}{q_n(\xi_n q_n + q_{n-1})}.$$

De plus $\xi_n = a_{n+1} + 1/\xi_{n+1} \in (a_{n+1}, a_{n+1}+1)$, donc

$$\left|x-\frac{r_n}{q_n}\right| > \frac{1}{q_n((a_{n+1}+1)q_n + q_{n-1})} = \frac{1}{q_n(q_n+q_{n+1})}$$

et

$$\left|x-\frac{r_n}{q_n}\right| < \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} = \frac{1}{q_n q_{n+1}}.$$

□

Flattot ([22]) est alors le premier à réussir à exhiber des nombres de Liouville α tels que T_α admet un sous-espace hyperinvariant non-trivial.

Théorème 1.3.11 ([22, Theorem 5.3]). Soit un irrationnel $\alpha \in [0,1]$. Supposons qu'il existe une suite de rationnels $(r_n/q_n)_{n\geq 0}$ telle que $1 \leq q_n \leq n\log(n)^{-3}$ et $|\alpha - r_n/q_n| \leq 1/q_n^2$ pour tout $n \geq 0$. Si de plus $|\alpha - r_n/q_n| \geq e^{-q_n^{1/3}}$ quand n est suffisamment large, alors T_α admet un sous-espace hyperinvariant non-trivial.

Plus récemment Chamizo, Gallardo-Gutiérrez, Monsalve-López et Ubis ([18]) ont généralisé le résultat de Flattot et obtenu une condition portant sur la croissance des dénominateurs des convergents d'un irrationnel α qui implique l'existence d'un sous-espace hyperinvariant non-trivial.

Théorème 1.3.12 ([18, Theorem 3.7]). Soit un irrationnel $\alpha \in [0,1]$ et soit $(r_n/q_n)_{n\geq 0}$ la suite de ses convergents. Si $\log(q_{n+1}) = O(q_n/\log(q_n)^3)$ alors T_α admet un sous-espace hyperinvariant non-trivial.

Conformément à l'approche de Davie qui a montré que T_α admettait un sous-espace hyperinvariant quand α n'était pas un nombre de Liouville, les hypothèses des généralisations obtenues ci-dessus portent sur la croissance des dénominateurs des convergents du nombre irrationnel α considéré.

Ces opérateurs de Bishop peuvent être de manière plus générale considérés comme des opérateurs de translation à poids.

Définition 1.3.13 ([22, Introduction]). Soit une fonction $\phi \in L^\infty([0,1])$ et soit $\alpha \in [0,1]$. On appelle respectivement *opérateur de multiplication* et *opérateur de translation* les opérateurs M_ϕ et U_α définis sur $L^2([0,1])$ par

$$M_\phi f = \phi f \quad \text{et} \quad U_\alpha f = f(\{\cdot + \alpha\}) \quad \text{pour tout } f \in L^2([0,1]).$$

On appelle alors *opérateur de translation à poids* l'opérateur $T_{\phi,\alpha}$ défini pour tout $f \in L^2([0,1])$ par

$$T_{\phi,\alpha} f(x) = M_\phi U_\alpha f(x) = \phi(x) f(\{x+\alpha\}) \quad \text{presque partout sur } [0,1].$$

Nous nous intéresserons dorénavant aux propriétés dynamiques des opérateurs de Bishop, et plus généralement des opérateurs de translation à poids. Cette étude est motivée par le problème du sous-espace invariant : en effet l'adhérence d'un sous-espace engendré par une orbite sous l'action d'un opérateur T est un sous-espace fermé et invariant par T. Ainsi un opérateur $T \in \mathcal{B}(X)$ n'admet aucun sous-espace invariant non-trivial si et seulement si le sous-espace engendré par l'orbite $\{T^n x; n \geq 0\}$ est dense dans X pour tout $x \in X\backslash\{0\}$. Cela signifie qu'un opérateur est un contre-exemple au problème du sous-espace invariant si et seulement si tout vecteur non-nul est cyclique pour cet opérateur.

Pour étudier les propriétés dynamiques des opérateurs de Bishop, nous les considérerons comme agissant sur $L^p([0,1])$ avec $1 < p < +\infty$. On notera m la mesure de Lebesgue sur $[0,1]$.

On remarque que T_α est une contraction, c'est-à-dire que $\|T_\alpha\| \leq 1$, pour tout $\alpha \in [0,1]$. Donc toutes les orbites sous l'action de T_α sont bornées, ce qui empêche T_α d'être hypercyclique.

Proposition I. Pour tout $\alpha \in [0,1]$, l'opérateur de Bishop T_α n'est pas hypercyclique.

Puisque $\|\mathcal{T}_{\phi,\alpha}\| = \|\phi\|_\infty$, le même argument montre la non-hypercyclicité des opérateurs de translation à poids $T_{\phi,\alpha}$ si $\|\phi\|_\infty \leq 1$.

Proposition J. Soient $\alpha \in [0,1]$ et $\phi \in L^\infty([0,1])$. Si $\|\phi\|_\infty \leq 1$ ou si $m(\{\phi = 0\}) > 0$, alors $T_{\phi,\alpha}$ n'est pas hypercyclique.

Dans le cas général, nous montrons dans le quatrième chapitre que les opérateurs de translation à poids ne vérifient pas le Critère d'Hypercyclicité, ce grâce à l'équivalence avec le Critère de Gethner-Shapiro.

Théorème K. Pour tout $\alpha \in [0,1]$ et tout $\phi \in L^\infty([0,1])$, l'opérateur $T_{\phi,\alpha}$ ne satisfait pas le Critère d'Hypercyclicité.

Le fait que le poids $\phi(x) = x$ intervenant dans la définition des opérateurs de Bishop est presque partout strictement positif ne leur permet pas d'être supercycliques. Pour le montrer, nous utilisons le théorème de supercyclicité positive, qui s'énonce ainsi.

Théorème 1.3.14 (Théorème de la supercyclicité positive, [4, Corollary 3.4])**.** Soit un opérateur $T \in \mathcal{B}(X)$ tel que $\sigma_p(T^*) = \emptyset$. Alors un vecteur $x \in X$ est supercyclique pour T si et seulement si $\{aT^n x; n \geq 0, a > 0\}$ est dense dans X.

Davie et Flattot ont montré qu'un opérateur de Bishop n'admet pas de valeur propre. Des ajustements de leurs arguments permettent d'également le montrer pour l'opérateur adjoint. On note p' le réel tel que $1/p + 1/p' = 1$.

Théorème 1.3.15 ([19, Theorem 2])**.** Pour tout $\alpha \in [0,1]$, l'opérateur $T_\alpha \in \mathcal{B}(L^p([0,1]))$ et l'opérateur $T_\alpha^* \in \mathcal{B}(L^{p'}([0,1]))$ n'admettent pas de valeur propre.

Nous montrons dans le quatrième chapitre qu'un opérateur de Bishop ne peut pas être supercyclique.

Théorème L. Pour tout $\alpha \in [0,1]$, l'opérateur de Bishop T_α n'est pas supercyclique.

La démarche se généralise au cas des opérateurs de translation à poids. La preuve de l'absence de valeur propre pour l'opérateur de Bishop permet de la transposer à ce contexte, à condition de supposer que le poids ϕ est convexe et strictement croissant.

Théorème 1.3.16 ([22, Theorem 2.1])**.** Soient $\alpha \in [0,1]$ et $\phi \in L^\infty([0,1])$ une fonction convexe strictement croissante telle que $\phi(0) = 0$. Alors les opérateurs de translation à poids $T_{\phi,\alpha} \in \mathcal{B}(L^p([0,1]))$ et $T_{\phi,\alpha}^* \in \mathcal{B}(L^{p'}([0,1]))$ n'admettent pas de valeur propre.

Ainsi en l'absence de valeur propre pour l'adjoint $T_{\phi,\alpha}^*$ et du fait de la positivité presque partout sur $[0,1]$ du poids ϕ, nous sommes en mesure de montrer que l'opérateur de translation à poids $T_{\phi,\alpha}$ ne peut pas non plus être supercyclique dans ce cas.

Théorème M. Soient $\alpha \in [0,1]$ et $\phi \in L^\infty([0,1])$ une fonction convexe strictement croissante telle que $\phi(0) = 0$. Alors l'opérateur de translation à poids $T_{\phi,\alpha}$ n'est pas supercyclique.

Dans le cas général des opérateurs de translation à poids, nous démontrons qu'ils ne satisfont jamais le Critère de Supercyclicité. Nous avons pour cela besoin d'un Critère du type Gethner-Shapiro pour la supercyclicité qui suppose l'existence d'un inverse à droite. En ajustant la preuve connue de l'équivalence entre le Critère d'Hypercyclicité et le Critère de Gethner-Shapiro, nous démontrons que le Critère de Supercyclicité est bien équivalent à un tel critère.

Théorème N. Un opérateur $T \in \mathcal{B}(X)$ satisfait les hypothèses du Critère de Supercyclicité si et seulement s'il satisfait les hypothèses suivantes : il existe X_0 et Y_0 deux sous-ensembles denses dans X, une suite d'entiers $(n_k)_{k\geq 1}$ strictement croissante, une suite $(\lambda_{n_k})_{k\geq 1}$ de complexes non-nuls et une application $S\colon Y_0 \to Y_0$ tels que :

(i) $\lambda_{n_k} T^{n_k} x \to 0$ quand $k \to +\infty$ pour tout $x \in X_0$;

(ii) $\lambda_{n_k}^{-1} S^{n_k} y \to 0$ quand $k \to +\infty$ pour tout $y \in Y_0$;

(iii) $TSy = y$ pour tout $y \in Y_0$.

Ainsi, il est possible de démontrer qu'aucun opérateur de translation à poids ne satisfait le Critère de Supercyclicité.

Théorème O. Pour tout $\alpha \in [0,1]$ et tout $\phi \in L^\infty([0,1])$, l'opérateur de translation à poids $T_{\phi,\alpha}$ ne satisfait pas le Critère de Supercyclicité.

Nous nous intéressons maintenant à l'étude de la cyclicité des opérateurs de Bishop T_α. Celle-ci a été notamment étudiée par Chalendar et Partington dans [16] et caractérisée dans le cas où α est rationnel. Nous présentons ici ces résultats.

Définition 1.3.17 ([17, Definition 5.4.1])**.** Soit un rationnel $\alpha = r/q$ où r et q sont premiers entre eux et $0 < r < q$. On définit la fonction $\Delta(f, r/q)$ presque partout sur $[0,1]$ par

$$\begin{vmatrix} f(t) & T_{r/q}f(t) & \dots & T_{r/q}^{q-1}f(t) \\ f(\{t+r/q\}) & T_{r/q}f(\{t+r/q\}) & \dots & T_{r/q}^{q-1}f(\{t+r/q\}) \\ \vdots & \vdots & & \vdots \\ f(\{t+(q-1)r/q\}) & T_{r/q}f(\{t+(q-1)r/q\}) & \dots & T_{r/q}^{q-1}f(\{t+(q-1)r/q\}) \end{vmatrix}.$$

La caractérisation obtenue par Chalendar et Partington montre qu'une fonction f est cyclique pour T_α si et seulement si $\Delta(f,\alpha)$ ne s'annule presque jamais.

Rappelons que m désigne ici la mesure de Lebesgue sur $[0,1]$.

Théorème 1.3.18 ([17, Theorem 5.4.4])**.** Soit un rationnel $\alpha = r/q$ où r et q sont premiers entre eux et $0 < r < q$. Une fonction $f \in L^p([0,1])$ est cyclique pour $T_{r/q}$ si et seulement si la fonction $\Delta(f, r/q)$ vérifie $m(\{t \in [0,1]; \Delta(f,r/q)(t) = 0\}) = 0$.

Nous pouvons à partir de cette caractérisation montrer dans le quatrième chapitre que tous les opérateurs de Bishop T_α, où α est rationnel, sont cycliques et partagent des vecteurs cycliques communs.

Théorème P. Toute fonction holomorphe sur un voisinage ouvert de $[0,1]$ telle que $f(0) \neq 0$ est cyclique pour T_α pour tout rationnel $\alpha \in [0,1]$.

En particulier la fonction $\mathbf{1}$ constante égale à 1 est une fonction cyclique commune à tous les opérateurs de Bishop T_α où α est rationnel.

Notre objectif est maintenant de prouver la cyclicité des opérateurs de Bishop T_α pour certains paramètres α irrationnels. Pour cela nous utilisons le théorème de Kuratowski-Ulam dont la preuve nécessite une présentation de la propriété de Baire.

On dit qu'un ensemble la vérifie s'il s'agit d'un ouvert, à un ensemble maigre près.

Définition 1.3.19 ([29, Definition 8.21])**.** Soit $\mathcal{X}$ un espace polonais. On dit d'un sous ensemble $A \subset \mathcal{X}$ qu'il *satisfait la propriété de Baire* s'il existe un ouvert U de $\mathcal{X}$ tel que $A\Delta U = (A\backslash U) \cup (U\backslash A)$ est un ensemble maigre dans $\mathcal{X}$.

Afin de démontrer le théorème de Kuratowski-Ulam, remarquons que la propriété de Baire est préservée par passage au complémentaire et qu'un ensemble co-maigre vérifie nécessairement la propriété de Baire.

Proposition 1.3.20 ([29, Section 8.F])**.** Soit $\mathcal{X}$ un espace polonais et soit A un sous-ensemble de $\mathcal{X}$. Le sous-ensemble A satisfait la propriété de Baire si et seulement si $\mathcal{X}\backslash A$ la satisfait. Si A est ensemble co-maigre dans $\mathcal{X}$, alors A satisfait la propriété de Baire.

Démonstration. Supposons que A satisfait la propriété de Baire. Il existe alors un ouvert U de $\mathcal{X}$ tel que $M = A\Delta U$ est un ensemble maigre dans $\mathcal{X}$. Ainsi $A = U\Delta M$ et

$$(\mathcal{X}\backslash\overline{U})\Delta(\partial U\Delta M) = \mathcal{X}\backslash(\overline{U}\Delta(\partial U\Delta M)) = \mathcal{X}\backslash((\overline{U}\Delta\partial U)\Delta M) = \mathcal{X}\backslash(U\Delta M) = \mathcal{X}\backslash A.$$

Or $\mathcal{X}\backslash\overline{U}$ est un ouvert de $\mathcal{X}$ et $\partial U\Delta M$ est un ensemble maigre dans $\mathcal{X}$ car $\partial U\Delta M \subset \partial U \cup M$ où ∂U est un fermé d'intérieur vide, donc un ensemble maigre dans $\mathcal{X}$. Ainsi $\mathcal{X}\backslash A$ satisfait la propriété de Baire.

De plus si A est un ensemble co-maigre dans $\mathcal{X}$, alors $A\Delta\mathcal{X} = \mathcal{X}\backslash A$ où $\mathcal{X}$ est un ouvert de $\mathcal{X}$ et $\mathcal{X}\backslash A$ est un ensemble maigre dans $\mathcal{X}$. Donc A satisfait la propriété de Baire. □

Nous pouvons maintenant montrer le théorème de Kuratowski-Ulam qui lie le caractère co-maigre d'un ensemble dans un espace produit avec celui des projections associées.

Théorème 1.3.21 (Kuratowski-Ulam, [29, Theorem 8.41])**.** Soient $\mathcal{X}$ et $\mathcal{Y}$ deux espaces polonais et soit un sous-ensemble $A \subset \mathcal{X} \times \mathcal{Y}$ satisfaisant la propriété de Baire. Alors A est un ensemble co-maigre dans $\mathcal{X} \times \mathcal{Y}$ si et seulement si

$$\{x \in \mathcal{X}; \{y \in \mathcal{Y}; (x, y) \in A\} \text{ est co-maigre dans } \mathcal{Y}\} \text{ est co-maigre dans } \mathcal{X}$$

et si et seulement si

$$\{y \in \mathcal{Y}; \{x \in \mathcal{X}; (x, y) \in A\} \text{ est co-maigre dans } \mathcal{X}\} \text{ est co-maigre dans } \mathcal{Y}.$$

Démonstration. Par symétrie des rôles joués par $\mathcal{X}$ et $\mathcal{Y}$, montrons la première équivalence. Pour tout $(x, y) \in \mathcal{X} \times \mathcal{Y}$ on considère les applications continues $\varphi_x \colon \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ et $\phi_y \colon \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ définies par $\varphi_x(z) = (x, z)$ pour tout $z \in \mathcal{Y}$ et par $\phi_y(z) = (z, y)$ pour tout $z \in \mathcal{X}$. Ainsi pour tout sous-ensemble $B \subset \mathcal{X} \times \mathcal{Y}$ et tout $(x, y) \in \mathcal{X} \times \mathcal{Y}$ on a $\{z \in \mathcal{Y}; (x, z) \in B\} = \varphi_x^{-1}(B)$ et $\{z \in \mathcal{X}; (z, y) \in B\} = \phi_y^{-1}(B)$.

Supposons que A est un ensemble co-maigre dans $\mathcal{X} \times \mathcal{Y}$. Par définition il existe une suite $(A_n)_{n\geq 0}$ d'ouverts denses dans $\mathcal{X} \times \mathcal{Y}$ telle que $\cap_{n\geq 0} A_n \subset A$. Pour tout $x \in \mathcal{X}$, d'après le théorème de Baire l'ensemble G_δ $\varphi_x^{-1}(\cap_{n\geq 0}A_n) = \cap_{n\geq 0}\varphi_x^{-1}(A_n)$ est dense dans $\mathcal{Y}$ si et seulement si $\varphi_x^{-1}(A_n)$ est dense dans $\mathcal{Y}$ pour tout $n \geq 0$ par continuité de φ_x. Donc

$$\left\{x \in \mathcal{X}; \varphi_x^{-1}\left(\bigcap_{n\geq 0} A_n\right) \text{ est dense}\right\} = \bigcap_{n\geq 0}\{x \in \mathcal{X}; \varphi_x^{-1}(A_n) \text{ est dense}\}$$

et

$$\bigcap_{n\geq 0}\{x \in \mathcal{X}; \varphi_x^{-1}(A_n) \text{ est dense}\} \subset \left\{x \in \mathcal{X}; \varphi_x^{-1}(A) \text{ est co-maigre}\right\}$$

car φ_x est continue pour tout $x \in \mathcal{X}$ et

$$\bigcap_{n\geq 0} \varphi_x^{-1}(A_n) = \varphi_x^{-1}\left(\bigcap_{n\geq 0} A_n\right) \subset \varphi_x^{-1}(A).$$

Il suffit dès lors de montrer que $\{x \in \mathcal{X}; \varphi_x^{-1}(A_n) \text{ est dense}\}$ est un ensemble co-maigre dans $\mathcal{X}$ pour tout $n \geq 0$. Fixons $n \geq 0$. L'espace $\mathcal{Y}$ étant séparable, on considère une base d'ouverts non-vides $(V_p)_{p\geq 0}$ de $\mathcal{Y}$ et on considère $U_p = \{x \in \mathcal{X}; \exists y \in V_p, (x,y) \in A_n\}$. Tout d'abord U_p est un ouvert de $\mathcal{X}$ car ϕ_y est continue pour tout $y \in \mathcal{Y}$ et

$$U_p = \bigcup_{y\in V_p} \phi_y^{-1}(A_n).$$

De plus U_p est dense dans $\mathcal{X}$. En effet, soit U un ouvert non-vide de $\mathcal{X}$. L'ensemble $U \times V_p$ est un ouvert non-vide de $\mathcal{X} \times \mathcal{Y}$, donc par densité de A_n il existe $(x,y) \in A_n \cap (U \times V_p)$. Ainsi $x \in U_p \cap U$, d'où la densité de U_p dans $\mathcal{X}$. Montrons l'inclusion

$$\bigcap_{p\geq 0} U_p \subset \{x \in \mathcal{X}; \varphi_x^{-1}(A_n) \text{ est dense}\}.$$

Soient $x \in \cap_{p\geq 0}U_p$ et $p \geq 0$. Il existe $y \in V_p$ tel que $(x,y) \in A_n$ puisque $x \in U_p$. Ainsi $y \in V_p \cap \varphi_x^{-1}(A_n)$, d'où $\varphi_x^{-1}(A_n) \cap V_p \neq \emptyset$ pour tout $p \geq 0$. Donc $\varphi_x^{-1}(A_n)$ est dense pour tout $n \geq 0$, ce qui indique que $\{x \in \mathcal{X}; \varphi_x^{-1}(A) \text{ est co-maigre}\}$ est co-maigre dans $\mathcal{X}$.

Réciproquement supposons que $\{x \in \mathcal{X}; \varphi_x^{-1}(A) \text{ est co-maigre}\}$ est un ensemble co-maigre dans $\mathcal{X}$. Supposons par l'absurde que A n'est pas co-maigre dans $\mathcal{X} \times \mathcal{Y}$ et notons $A' = (\mathcal{X} \times \mathcal{Y})\backslash A$. Alors A' n'est pas maigre et A' satisfait la propriété de Baire car A la satisfait d'après la Proposition 1.3.20. Ainsi il existe un ouvert B de $\mathcal{X} \times \mathcal{Y}$ et un ensemble maigre M dans $\mathcal{X} \times \mathcal{Y}$ tels que $A' = B\Delta M$. L'ouvert B ne peut pas être maigre car $A' \subset B \cup M$, ce qui ferait de A' un ensemble maigre dans $\mathcal{X} \times \mathcal{Y}$. Les espaces $\mathcal{X}$ et $\mathcal{Y}$ étant séparables, il existe un ouvert U de $\mathcal{X}$ et un ouvert V de $\mathcal{Y}$ non-vides tels que $U \times V \subset B$. Par hypothèse $C_1 = \{x \in \mathcal{X}; \varphi_x^{-1}(A') \text{ est maigre}\}$ est un ensemble co-maigre dans $\mathcal{X}$. De plus $M' = (\mathcal{X} \times \mathcal{Y})\backslash M$ est un ensemble co-maigre dans $\mathcal{X} \times \mathcal{Y}$, il satisfait donc la propriété de Baire. Ainsi d'après l'implication déjà démontrée $\{x \in \mathcal{X}; \varphi_x^{-1}(M') \text{ est co-maigre}\}$ est un ensemble co-maigre dans $\mathcal{X}$, ce qui implique que $C_2 = \{x \in \mathcal{X}; \varphi_x^{-1}(M) \text{ est maigre}\}$ est un ensemble co-maigre dans $\mathcal{X}$. Donc $C_1 \cap C_2$ est un ensemble co-maigre dans $\mathcal{X} \times \mathcal{Y}$. Il est donc en particulier dense et rencontre l'ouvert U. Ainsi il existe $x \in U$ tel que $\varphi_x^{-1}(A')$ et $\varphi_x^{-1}(M)$ sont des ensembles maigres dans $\mathcal{Y}$. Or

$A' = B\Delta M$ donc $\varphi_x^{-1}(A') = \varphi_x^{-1}(B)\Delta\varphi_x^{-1}(M)$. De plus $V \subset \varphi_x^{-1}(B)$ car $U \times V \subset B$, donc $V\backslash\varphi_x^{-1}(M) \subset \varphi_x^{-1}(B)\backslash\varphi_x^{-1}(M) \subset \varphi_x^{-1}(A')$. Ainsi $V \subset \varphi_x^{-1}(A') \cup \varphi_x^{-1}(M)$, ce qui implique que V est un ensemble maigre dans $\mathcal{Y}$. Ceci est absurde car V est un ouvert non-vide de $\mathcal{Y}$, ce qui montre que A est un ensemble co-maigre dans $\mathcal{X} \times \mathcal{Y}$. □

Le Théorème de Kuratowski-Ulam nous permet de démontrer qu'il existe des irrationnels α tels que T_α est cyclique, ce qui n'était pas connu jusqu'ici. Plus précisément, le théorème suivant est démontré dans le quatrième chapitre.

Théorème Q. L'ensemble $\{\alpha \in [0,1]; T_\alpha \text{ est cyclique}\}$ est un ensemble co-maigre dans $[0,1]$.

Celui-ci se démontre en prouvant que l'ensemble

$$\{(\alpha, f) \in [0,1] \times L^p([0,1]); f \text{ est cyclique pour } T_\alpha\}$$

est un sous-ensemble co-maigre dans $[0,1] \times L^p([0,1])$ et en lui appliquant le Théorème de Kuratowski-Ulam.

Le Théorème Q permet de montrer l'existence de paramètres α irrationnels tels que T_α est cyclique, mais ne fournit pas d'exemples explicites. Un deuxième but du quatrième chapitre est d'obtenir des tels exemples de paramètres α irrationnels.

En utilisant les estimées de l'erreur faite en approximant un irrationnel par ses convergents, nous montrons que T_α est cyclique dès que la suite des dénominateurs des convergents de α admet une infinité de trous suffisamment larges. Plus précisément, dans ce cas si f est une fonction cyclique commune à tous les opérateurs de Bishop T_β avec $\beta \in \mathbf{Q}$, alors f est également cyclique pour T_α.

Théorème R. Soit $f \in L^p([0,1])$ une fonction cyclique commune à tous les opérateurs T_α où $\alpha \in (0,1)$ est rationnel. Il existe une fonction $\psi_f \colon \mathbf{N} \to \mathbf{R}_+$ satisfaisant la propriété suivante : si $(r_n/q_n)_{n\geq 1}$ sont les convergents d'un nombre irrationnel $\alpha \in [0,1]$ et si pour tout $n \geq 0$ il existe $n_0 \geq n$ tel que $q_{n_0+1} > \psi_f(q_{n_0})$, alors f est cyclique pour T_α.

L'hypothèse ci-dessus peut demander une forte croissance des dénominateurs des convergents là où les hypothèses faites dans les Théorèmes 1.3.11 et 1.3.12 cherchent au contraire à contrôler cette croissance. Ceci s'explique par le fait que la cyclicité et le fait d'admettre un sous-espace invariant non-trivial sont des propriétés que l'on pourrait qualifier d'opposées : un opérateur n'admettant aucun sous-espace invariant non-trivial est un opérateur pour lequel tous les vecteurs non-nuls sont cycliques.

Nous cherchons maintenant à généraliser le Théorème R au cas des opérateurs de translation à poids $T_{\phi,\alpha}$. Pour ce faire, nous cherchons d'abord à obtenir une caractérisation de la cyclicité d'une fonction f pour $T_{\phi,\alpha}$, où α est rationnel, qui soit analogue à celle obtenue par Chalendar et Partington pour les opérateurs de Bishop T_α.

Définition 1.3.22. Soit $\phi \in L^p([0,1])$ et soit un rationnel $\alpha = r/q$ où r et q sont premiers entre eux et $0 < r < q$. On définit la fonction $\Delta_\phi(f, r/q)$ presque partout sur $[0,1]$ par

$$\begin{vmatrix} f(t) & T_{\phi,\alpha} f(t) & \dots & T_{\phi,\alpha}^{q-1} f(t) \\ f(\{t+r/q\}) & T_{\phi,\alpha} f(\{t+r/q\}) & \dots & T_{\phi,\alpha}^{q-1} f(\{t+r/q\}) \\ \vdots & \vdots & & \vdots \\ f(\{t+(q-1)r/q\}) & T_{\phi,\alpha} f(\{t+(q-1)r/q\}) & \dots & T_{\phi,\alpha}^{q-1} f(\{t+(q-1)r/q\}) \end{vmatrix}.$$

Nous pouvons effectivement généraliser la caractérisation obtenue par Chalendar et Partington et donnons la caractérisation suivante.

Théorème S. Soient $\phi \in L^\infty([0,1])$ et un rationnel $\alpha = r/q$ où r et q sont premiers entre eux et $0 < r < q$. Une fonction $f \in L^p([0,1])$ est cyclique pour $T_{\phi,\alpha}$ si et seulement si la fonction $\Delta_\phi(f, r/q)$ vérifie $m(\{t \in [0,1]; \Delta_\phi(f, r/q)(t) = 0\}) = 0$.

Grâce à cette caractérisation, et sous quelques conditions sur le poids ϕ, nous pouvons montrer qu'il y a des fonctions cycliques communes à tous les opérateurs de translation à poids $T_{\phi,\alpha}$, où α est rationnel.

Théorème T. Soit $\phi \in \mathcal{C}([0,1], \mathbf{R})$ une fonction strictement croissante, holomorphe sur un voisinage ouvert de $[0,1]$ telle que $\phi(0) = 0$. Toute fonction holomorphe sur un voisinage ouvert de $[0,1]$ telle que $f(0) \neq 0$ est cyclique pour $T_{\phi,\alpha}$ pour tout rationnel $\alpha \in [0,1]$.

En particulier la fonction $\mathbf{1}$ constante égale à 1 est une fonction cyclique commune à tous les opérateurs de translation à poids $T_{\phi,\alpha}$ où α est rationnel.

Sous ces mêmes conditions sur le poids ϕ et si la suite des dénominateurs des convergents d'un irrationnel α admet une infinité de trous suffisamment larges, nous démontrons que $T_{\phi,\alpha}$ est cyclique.

Théorème U. Soit $\phi \in \mathcal{C}([0,1], \mathbf{R})$ une fonction strictement croissante, holomorphe sur un voisinage ouvert de $[0,1]$ telle que $\phi(0) = 0$. Soit $f \in L^p([0,1])$ une fonction cyclique commune à tous les opérateurs $T_{\phi,\alpha}$ où $\alpha \in (0,1)$ est rationnel. Il existe un fonction $\psi_{\phi,f} \colon \mathbf{N} \to \mathbf{R}_+$ satisfaisant la propriété suivante : si $(p_n/q_n)_{n\geq 1}$ sont les convergents d'un nombre irrationnel $\alpha \in [0,1]$ et si pour tout $n \geq 0$ il existe $n_0 \geq n$ tel que $q_{n_0+1} > \psi_{\phi,f}(q_{n_0})$, alors f est cyclique pour $T_{\phi,\alpha}$.

Dans le Théorème R (ou le Théorème T), la fonction ψ_f (ou $\psi_{\phi,f}$) n'est pas explicite. Prenons le cas où $\alpha = 1/q$ et $f = \mathbf{1}$. Dans l'approche que nous avons suivie pour expliciter la fonction ψ_f, il est nécessaire de minorer sur $[0, 1/q)$ en valeur absolue le déterminant

$$\Delta(\mathbf{1}, 1/q)(x) = \begin{vmatrix} 1 & x & \dots & x\{x+1/q\}\dots\{x+(q-2)/q\} \\ 1 & \{x+1/q\} & \dots & \{x+1/q\}\{x+2/q\}\dots\{x+(q-1)/q\} \\ \vdots & \vdots & & \vdots \\ 1 & \{x+(q-1)/q\} & \dots & \{x+(q-1)/q\}x\dots\{x+(q-3)/q\} \end{vmatrix}.$$

Des simulations numériques semblent montrer que ce déterminant ne s'annule pas et est strictement monotone sur $[0, 1/q)$. Nous n'avons néanmoins pas réussi à le démontrer, ce qui nous a empêché d'expliciter la fonction ψ_f.

Chapter 2

Introduction (in english)

The questions studied in this thesis come from the field of linear dynamics, and are motivated by two famous open problems which are, on the one hand, the Collatz Conjecture, and on the other hand the Invariant Subspace Problem. The results obtained in this thesis resulted in the following two publications:

— *Linear dynamics of an operator associated to the Collatz map*, accepted by the Proceedings of the American Mathematical Society, which is the subject of Chapter 3;

— *A study of Bishop operators from the point of view of linear dynamics*, published in Journal of Mathematical Analysis and Applications **526** (2023), which is the subject of Chapter 4.

The first part of this introduction presents the context of our study: after recalling some general facts about dynamical systems theory, in topological spaces and in measurable spaces, we will present some notions and fondamental results of linear dynamics. Recall that linear dynamics consist in the study of dynamical systems given by the action of a linear and continuous operator acting on a Banach or Fréchet space.

The second part of the introduction starts with a short presentation of the famous Collatz Conjecture: if T_0 is the map acting on $\mathbf{Z}_+$ defined by $T_0(n) = n/2$ if n is even and $T_0(n) = 3n + 1$ if n is odd, this conjecture claims that the orbit of any integer $n \geq 1$ under the action of T_0 reaches the point 1. One of the goals of this thesis is to study an operator $\mathcal{T}$ naturally associated to T_0, whose adjoint has been introducted by Berg and Meinardus ([9]), and to link its dynamical properties to those of T_0. We generalize some results pertaining to the dynamics of $\mathcal{T}$ obtained in [36] by Neklyudov, and also answer some open questions from [36]. These results are the subjects of the article *Linear dynamics of an operator associated to the Collatz map*, and will be presented in detail in Chapter 3.

The third and last part of the introduction presents results related to the dynamics of Bishop operators. This family $(T_\alpha)_{\alpha\in(0,1)}$ of operators on $L^2([0, 1])$, defined by $T_\alpha f(x) = xf(\{x + \alpha\}), f \in L^2([0, 1])$, is particularly interesting in operator theory: despite very substantial work, notably of Davie ([19]), Flattot ([22]), Chamizo, Gallardo-Gutiérrez, Monsalve-López and Ubis ([18]), we do not know if T_α has a non-trivial invariant subspace for every irrational number $\alpha \in (0, 1)$. Some of the operators T_α could thus be potential counter-examples to the Invariant Subspace Problem. We will study the dynamics of operators T_α when α is irrational. Our main result shows the existence of a dense G_δ-set of parameters $\alpha \in (0, 1)$ such that T_α is cyclic, and thus provides the first examples of irrational numbers α such that T_α is cyclic. Our study relies on the study of the cyclicity of the operators T_α when α is rational, conducted by Chalendar and Partington in [17]. These results are the subjects of the article *A study of Bishop operators from the point of view of linear dynamics*, and will be presented in detail in Chapter 4.

In this introduction, the original results are alphabetically enumerated (Proposition A, Theorem B, ...).

2.1 Dynamical systems

If X is a topological space, studying the dynamical properties of a continuous transformation $T\colon X \to X$ consists in investigating the behavior of its iterates $T^n = T \circ T \circ \ldots \circ T$ where $n \geq 0$. If X is a vector space, for instance equipped with a norm, and if $T\colon X \to X$ is a continuous operator, one could think at first sight that the behavior of T is going to be predictable, because of the linear environment. Nonetheless Birkhoff, MacLane and Rolewicz discovered respectively in 1929, 1952 and 1969 linear operators admitting dense orbits in the space they act on. These examples, which notably include the differentiation operator on the space of holomorphic functions on the complex plane $\mathbf{C}$, motivated the definition of a general theory of dynamical properties of linear operators. A linear operator admitting vectors with dense orbits will be called hypercyclic.

We will be interested first in the case where X is a general topological space and $T\colon X \to X$ is a continuous transformation, that is to say, in non-linear dynamics. Indeed some definitions will not rely on linearity, as for instance the existence of a dense orbit or topological transitivity. This will notably allow us to enlarge the type of examples of dynamical systems which we consider. Conversely such examples could appear in the linear environment, like the translation $R_\alpha\colon x \in [0, 1] \mapsto \{x + \alpha\}$ which appears in the study of Bishop operators. We will also give definitions in the framework of measurable spaces, when X is a probability space and T is a measure-preserving transformation of X. In this case we will present the notion of ergodicity, which is linked to the study of the irreducibility of a transformation, and we will see correlations with other dynamical properties such as existence of a dense orbit, thanks to Birkhoff's Pointwise Ergodic Theorem. Eventually we will be interested in linear dynamics when X is a vector space and T is an operator acting on X. Then existence of a dense orbit is called hypercyclicity, and the linear environment also allows to define the notions of supercyclicity and cyclicity. If we wish to quantify in a more precise way the notion of hypercyclicity, we can look at the frequence at which an orbit intersects each open set. This will be called frequent hypercyclicity. We will also present the notion of chaos in a linear environment: chaotic operators are those which admit dense orbits for a dense set of vectors and finite orbits for another dense set of vectors.

2.1.1 Non-linear dynamics

Let us start by presenting the general notion of a dynamical system, which is essential in this manuscript. We refer the reader to the book [26] for more on this notion.

Definition 2.1.1 ([26, Definition 1.1])**.** A *dynamical system* consists of a metric space X

and a continuous map $T\colon X \to X$. For every $x \in X$, the set $\{T^n(x) = T \circ T \circ \ldots \circ T(x); n \geq 0\}$ is denoted by $\mathrm{orb}(x, T)$ and called *orbit of x under the action of T.*

Numerous examples of dynamical system exist in the literature. We present here a very simple first transformation which we will meet again in the study of Bishop operators involved in section 2.3.

Example 2.1.2 ([26, Example 1.3 (f)])**.** Let $\alpha \in \mathbf{R}$. The metric space $[0,1]$ endowed with the metric $|\cdot|$ and the continuous map R_α defined by $R_\alpha(x) = \{x + \alpha\}$ for every $x \in [0,1]$, where $\{\cdot\}$ denotes the fractional part of a real number, is a dynamical system.

The latter can be seen acting on the unit circle $\mathbf{T} \subset \mathbf{C}$. To do so we define $S_\alpha(z) = e^{2i\pi\alpha} z$ for every $z \in \mathbf{T}$.

A first dynamical property of a system $T\colon X \to X$ which will be studied is that of admitting a dense orbit in X.

The behavior of the iterates S_α^n, $n \geq 0$, essentially depends on the rationality of the parameter α.

Proposition 2.1.3 ([26, Example 1.12 (c)])**.** If α is a rational number, then every orbit of S_α is finite. Otherwise every orbit of S_α is dense in $\mathbf{T}$.

Proof. If $\alpha = r/q$ if a rational number then $S_\alpha^q(z) = e^{2i\pi r} z = z$ so we have $\mathrm{orb}(z, S_\alpha) = \{z, e^{2i\pi r/q} z, \ldots, e^{2i\pi(q-1)r/q} z\}$ for every $z \in \mathbf{T}$.

If α is an irrational number, it suffices to show that the orbit of 1 under the action of S_α is dense in $\mathbf{T}$ since $\mathrm{orb}(z, S_\alpha) = \{e^{2i\pi n\alpha} z; n \geq 0\} = z \cdot \mathrm{orb}(1, S_\alpha)$ for every $z \in \mathbf{T}$. Denote $S_\alpha^n(1) = e^{2i\pi n\alpha} = z_\alpha^n$ and remark that if $\{z_\alpha^n; n \in \mathbf{Z}\}$ is dense in $\mathbf{T}$ then $\{z_\alpha^n; n \geq 0\}$ is also dense in $\mathbf{T}$. If $\{z_\alpha^n; n \in \mathbf{Z}\}$ is dense in $\mathbf{T}$, there exists a sequence $(n_k)_{k\geq 0}$ of integers, where $(|n_k|)_{k\geq 0}$ is strictly increasing, such that $z_\alpha^{n_k} \to 1$ as $k \to +\infty$. Thus there exists a strictly increasing sequence $(p_k)_{k\geq 0}$ of positive integers such that $z_\alpha^{p_k} \to 1$ as $k \to +\infty$. Indeed if there exists a subsequence $(m_k)_{k\geq 0}$ of negative integers of $(n_k)_{k\geq 0}$, then $(-m_k)_{k\geq 0}$ satisfies $z_\alpha^{-m_k} = \overline{z_\alpha^{m_k}} \to 1$ as $k \to +\infty$. Hence $z_\alpha^{p_k+p} \to z_\alpha^p$ as $k \to +\infty$ for every $p \in \mathbf{Z}$, which implies that $\{z_\alpha^p; p \in \mathbf{Z}\} \subset \overline{\{z_\alpha^n; n \geq 0\}}$ and $\overline{\{z_\alpha^p; p \in \mathbf{Z}\}} \subset \overline{\{z_\alpha^n; n \geq 0\}}$.

It suffices now to show that $F = \overline{\{z_\alpha^n; n \in \mathbf{Z}\}}$ is dense in $\mathbf{T}$. The set F is closed and $S_\alpha(F) = F$. Suppose by contradiction that $F \neq \mathbf{T}$, that is to say the open set $U = \mathbf{T} \backslash F$ is non-empty. There exists a sequence of pairwise disjoint open arcs $(I_k)_{k\geq 0}$ of $\mathbf{T}$ such that $U = \cap_{k\geq 0} I_k$. If we denote by l_k the length of I_k for every $k \geq 0$, thus $\sum_{k\geq 0} l_k \leq 2\pi < +\infty$ and so there exists $l_m = \max\{l_k; k \geq 0\}$. We know that $S_\alpha(U) = U$, then $S_\alpha^n(I_m) \subset U$ for every $n \in \mathbf{Z}$ and remark that we can't have $S_\alpha^n(I_m) \cap I_m \neq \emptyset$. Indeed otherwise $S_\alpha^n(I_m) \cup I_m$ would either be an open ark whose length is strictly greater than l_m, or $S_\alpha^n(I_m) = I_m$. In this case it would implie that $z_\alpha^n = 1$, which is impossible since α is irrational. Thus $S_\alpha^n(I_m) \cap I_m = \emptyset$ for every $n \in \mathbf{Z}$ and in particular $S_\alpha^p(I_m) \cap S_\alpha^q(I_m) = \emptyset$ because $S_\alpha^{p-q}(I_m) \cap I_m = \emptyset$. The arcs $(S_\alpha^n(I_m))_{n\geq 0}$ are pairwise disjoint and of length l_m, hence $\sum_{n\geq 0} l_m \leq 2\pi < +\infty$, which is impossible. So $F = \mathbf{T}$, which concludes the proof. □

Henceforward we will not prove that a transformation $T\colon X \to X$ admits a dense orbit by expliciting an element whose orbit under the action of T is dense. If we restrict the assumptions made on the topological space X, notably if we suppose that X is a Polish space, this dynamical property has been characterized in 1920 thanks to the transitivity theorem. To introduce this theorem, we now give the definition of topologically transitive systems.

Definition 2.1.4 ([26, Definition 1.11])**.** Let (X, T) be a dynamical system. We say that T is *topologically transitive* if for every pair (U, V) of non-empty open sets in X there exists $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$.

For every pair (U, V) of subsets of a metric space X and every $n \geq 0$, remark that $T^n(U) \cap V \neq \emptyset$ if and only if $U \cap T^{-n}(V) \neq \emptyset$. Thus if $T\colon X \to X$ is a continuous transformation which has a continuous inverse $T^{-1}\colon X \to X$, then T is topologically transitive if and only if T^{-1} is.

The Birkhoff transitivity theorem applies on Polish spaces. The notion of co-meager sets will also play an important role in some of our statements.

Definition 2.1.5 ([29, Section 8.A])**.** A metric d on a topological space (X, τ) is said *compatible with* τ if the topology induced by d on X coincides with τ.

A topological space (X, τ) is said to be *a Polish space* if it is separable, that is to say it admits a countable dense subset, and if it is a completely metrizable space, that is to say there exists a metric d defining the topology τ such that (X, d) is complete.

We say that a subset A of a Polish space X is a G_δ-set in X if there exists a sequence $(U_n)_{n\geq 0}$ of open sets in X such that $A = \cap_{n\geq 0} U_n$. We say that A is a *co-meager set* in X if A contains a dense G_δ-set in X. On the contrary, we say that A is a *meager set* in X if the complement $X \backslash A$ is a co-meager set in X.

Remark that a dynamical system can admit a dense orbit $\{T^n(x); n \geq 0\}$ only in the case where X is separable.

Let X be a Polish space and let A a subset of X. By the Baire Category Theorem, A is a dense G_δ-set in X if and only if there exists a sequence $(U_n)_{n\geq 0}$ of dense open sets in X such that $A = \cap_{n\geq 0} U_n$. Hence in particular a countable intersection of co-meager sets is a co-meager set in X. Also remark that a non-empty open set U in X cannot be a meager set. Indeed otherwise the closed set $X \backslash U$ would be a co-meager set in X, hence would be dense in X, which would imply that U is an empty set. Thus, the co-meager sets are the "large" sets in the sense of the Baire Category Theorem.

We now have the tools to show the Birkhoff transitivity theorem, which characterizes the existence of a dense orbit thanks to the notion of topological transitivity.

Theorem 2.1.6 (Birkhoff transitivity theorem, [12, §62])**.** Let X be a Polish space and let $T\colon X \to X$ be a continuous map. If T is topologically transitive then there exists $x \in X$ whose orbit $\{T^n(x); n \geq 0\}$ under the action of T is dense in X. In this case the set of elements of X whose orbit under the action of T is dense in X is a dense G_δ-set in X.

Moreover if X has no isolated point, then the converse is also true.

Proof. Since X is metrizable and separable, consider a basis $(U_k)_{k\geq 0}$ of non-empty open sets in X.

Suppose that T is topologically transitive. The orbit of an element $x \in X$ under the action of T is dense in X if and only if for every non-empty open set U in X there exists

$n \geq 0$ such that $T^n(x) \in U$, that is to say if and only if for every $k \geq 0$ there exists $n \geq 0$ such that $T^n(x) \in U_k$. Hence the set of elements of X whose orbit under the action of T is dense in X, here denoted by A, is equal to $A = \cap_{k\geq 0} \cup_{n\geq 0} T^{-n}(U_k)$ and it suffices to show that A is a dense G_δ-set in X. Fix $k \geq 0$, the set $\cup_{n\geq 0} T^{-n}(U_k)$ is an open set in X by continuity of T. Besides since T is topologically transitive, for every non-empty open set V in X there exists $n_0 \geq 0$ such that $T^{n_0}(V) \cap U_k \neq \emptyset$, that is to say such that $T^{-n_0}(U_k) \cap V \neq \emptyset$, which implies that the open set $\cup_{n\geq 0} T^{-n}(U_k)$ is dense in X for every $k \geq 0$. By the Baire Category Theorem A is thus a dense G_δ-set in X, and so is in particular non-empty.

Conversely suppose that X has no isolated point and that there exists $x \in X$ whose orbit under the action of T is dense in X. Let (U, V) be a pair of non-empty open sets in X. By density of the orbit $\mathrm{orb}(x, T)$ there exists $n \geq 0$ such that $T^n(x) \in U$. Since X has no isolated point the set $\{T^k(x); k \geq 0\} \backslash \{x, T(x), \dots, T^{n-1}(x)\}$ is still dense in X. So there exists $m \geq n$ such that $T^m(x) \in V$, hence $T^{m-n}(T^n(x)) = T^m(x)$ belongs to $T^{m-n}(U) \cap V$ with $m - n \geq 0$. Thus $T^{m-n}(U) \cap V \neq \emptyset$, which proves that T is topologically transitive. □

In the case of a Polish space without any isolated point, the Birkhoff transitivity theorem shows that we have the following dichotomy concerning the existence of dense orbits: either a dynamical system $T: X \to X$ has no dense orbit, or it admits a dense G_δ-set of elements $x \in X$ whose orbit under its action is dense in X.

2.1.2 Measurable dynamics

We consider in this section the case of measurable spaces. Let $(X, \mathcal{F}, m)$ be a probability space and $T: (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ a measurable transformation. If there exists a set $B \in \mathcal{F}$ such that $T^{-1}(B) = B$, then the study of the transformation T can be reduced to the study of the two induced transformations $T_{|B}: (B, \mathcal{F}_B, m) \to (B, \mathcal{F}_B, m)$ and $T_{|X\backslash B}: (B, \mathcal{F}_{X\backslash B}, m) \to (B, \mathcal{F}_{X\backslash B}, m)$. Then the study is in principle easier, unless the restrictions are trivial, which is the case when $m(B) = 0$ or $m(B) = 1$. This observation naturally leads to the definition of the notion of ergodicity, which expresses a notion of irreducibility.

Definition 2.1.7 ([3, Definition 3.9])**.** Let $T: (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ be a transformation. We say that

(i) T is *measurable* if $T^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{F}$;

(ii) T *preserves the measure* m if T is measurable and if $m(T^{-1}(B)) = m(B)$ for every $B \in \mathcal{F}$;

(iii) T is ergodic if T preserves the measure m and if for every $B \in \mathcal{F}$, the fact that $T^{-1}(B) = B$ implies that $m(B) \in \{0, 1\}$.

The following theorem characterizes the ergodicity of a transformation. It will notably be very useful in order to deduce from the ergodicity of a transformation that the set of elements of X whose orbit under the action of T is dense in X has full measure.

Theorem 2.1.8. A transformation $T: (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ preserving the measure m is ergodic if and only if $m(\cup_{n\geq 0} T^{-n}(B)) = 1$ for every $B \in \mathcal{F}$ such that $m(B) > 0$.

Proof. Suppose that T is ergodic. Let $B \in \mathcal{F}$ such that $m(B) > 0$. Consider $C = \cap_{N\geq 0} \cup_{n\geq N} T^{-n}(B)$ and we will show that $T^{-1}(C) = C$. We have $T(x) \in C$ if and only if for every $N \geq 0$ there exists $n \geq N$ such that $T^{n+1}(x) \in B$. So $T(x) \in C$ if and only if for every $N \geq 0$ there exists $n \geq N+1$ such that $T^n(B) \in B$, that is to say if and only if $x \in C$. Then by ergodicity we have $m(C) \in \{0,1\}$. However $m(C) = \lim_{N\to+\infty} m(\cup_{n\geq N} T^{-n}(B))$ because the sequence $(\cup_{n\geq N} T^{-n}(B))_{N\geq 0}$ is non-increasing, hence $m(C) \geq \liminf_{N\to+\infty} m(T^{-N}(B)) = m(B) > 0$ since T preserves the measure m. Necessarily $m(C) = 1$. In particular $C \subset \cup_{n\geq 0} T^{-n}(B)$, thus $m(\cup_{n\geq 0} T^{-n}(B)) = 1$.

Conversely let $B \in \mathcal{F}$ such that $T^{-1}(B) = B$. If $m(B) > 0$ then $m(\cup_{n\geq 0} T^{-n}(B)) = 1 = m(\cup_{n\geq 0} B) = m(B)$ by assumption. So T is ergodic. □

Consider the case where X is a Polish space without any isolated point and where $\mathcal{F}$ is the Borel σ-algebra generated by open sets in X. If we assume that m puts mass on every open set in X, that is to say $m(U) > 0$ for every non-empty open set U in X, then the ergodicity of a transformation $T\colon (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ implies that the set $\cup_{n\geq 0} T^{-n}(U)$ intersects every non-empty set V of X since it is of full measure. In particular T is topologically transitive. This observation allows us to deduce the existence of dense orbits from the ergodicity of the transformation.

Definition 2.1.9. Let X be a Polish space and let $\mathcal{F}$ be its Borel σ-algebra. A Borel probability measure m is said to be *of full support* if $m(U) > 0$ for every non-empty open set U in X.

We can now show that an ergodic transformation with respect to a measure of full support admits a set of full measure of vectors whose orbit is dense.

Theorem 2.1.10 ([26, Section 9.1])**.** Let X be a Polish space without any isolated point, let $\mathcal{F}$ be its Borel σ-algebra and let m be a probability measure of full support on X. Let $T\colon (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ be a continuous transformation which preserves the measure m. If T is ergodic then T is topologically transitive and moreover

$$m(\{x \in X; \text{orb}(x, T) \text{ is dense in } X\}) = 1.$$

Proof. Let $(U_k)_{k\geq 0}$ be a basis of non-empty open sets in X. For every $x \in X$, $\text{orb}(x, T)$ is dense in X if and only if for every $k \geq 0$ there exists $n \geq 0$ such that $T^n(x) \in U_k$. So $\{x \in X; \text{orb}(x, T) \text{ is dense in } X\} = \cap_{k\geq 0} \cup_{n\geq 0} T^{-n}(U_k)$. Besides, $m(U_k) > 0$ for every $k \geq 0$ since m is of full measure, hence $m(\cup_{n\geq 0} T^{-n}(U_k)) = 1$ for every $k \geq 0$ by Theorem 2.1.8. Thus $m(\cap_{k\geq 0} \cup_{n\geq 0} T^{-n}(U_k)) = 1$, which implies that T is topologically transitive by Theorem 2.1.6. □

An ergodic theorem is a statement etablishing a relation between the time means of a function $f\colon X \to \mathbf{R}$ or $\mathbf{C}$ on the orbits of points of X under the action of T and the space mean of the function f on the space X. The typical relation is of the form

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) \xrightarrow[N\to+\infty]{} \int_X f dm.$$

Several ergodic theorems exist, depending on the mode of convergence and on the class of functions to which it applies. A first theorem is the von Neumann's L^p mean ergodic theorem.

Theorem 2.1.11 (von Neumann's L^p mean ergodic theorem, [45, Corollaries 1.5 (ii)])**.** Let $1 \leq p < +\infty$ and let $T\colon (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ be a preserving-measure transformation. For every $f \in L^p(X, \mathcal{F}, m)$, there exists $f^* \in L^p(X, \mathcal{F}, m)$ such that $f^* \circ T = f^*$ almost everywhere and

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n - f^* \right\|_p \xrightarrow[N \to +\infty]{} 0.$$

Moreover if T is ergodic with respect to m, then for every $f \in L^p(X, \mathcal{F}, m)$

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n - \int_X f dm \right\|_p \xrightarrow[N \to +\infty]{} 0.$$

Birkhoff showed that if T is an ergodic transformation with respect to a probability measure m then, for every m-integrable function, its time average with respect to T and its space average coincide almost everywhere. This fundamental theorem is stated as follows.

Theorem 2.1.12 (Birkhoff pointwise ergodic theorem, [45, Theorem 1])**.** Let T be an ergodic transformation acting on $(X, \mathcal{F}, m)$. For every $f \in L^1(X, \mathcal{F}, m)$

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) \xrightarrow[N \to +\infty]{} \int_X f dm \quad \text{for } m\text{-almost every } x \in X.$$

In the case of an ergodic transformation $T\colon X \to X$ where X is a Polish space endowed with its Borel σ-algebra and a Borel probability measure, this theorem can be very useful to quantify the rate at which the orbit under the action of T of almost every point of X visits each non-empty open set in the space. This property is the subject of the Definition 2.1.47 in the linear case.

2.1.3 Linear dynamics

We now consider the case where X is a Banach or Fréchet space and where T is a continuous and linear operator on X. The system (X, T) thus obtained is called a linear dynamical system.

Although at first sight we can think that an operator on a Banach space has a stable and predictable behavior, this is not necessarily the case. As observed by Birkhoff, MacLane and Rolewicz, there also exist linear operators admitting dense orbits in the space they act on. This property, later called hypercyclicity, will be one of the two ingredients for the notion of chaos. Indeed if we assume that an operator has dense orbits and sufficiently many finite orbits, we will call it a chaotic operator. Examples provided by Birkhoff, MacLane and Rolewicz will satisfy this property.

The case of Banach spaces will be sufficient for our study. However, the dynamical definitions will also apply in the more general case of Fréchet spaces. These spaces are not endowed with a norm, but with a family of semi-norms instead. It will include for instance the space of holomorphic functions on **C** and so the Birhoff and MacLane's operators (which respectively are the translation operator and the differentiation operator acting on holomorphic functions) are continuous operators on this Fréchet space.

Definition 2.1.13 ([26, Definition 2.3])**.** Let X be a vector space on $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. A *semi-norm on* X is a map $p\colon X \to \mathbf{R}_+$ such that $p(x+y) \leq p(x) + p(y)$ and $p(\lambda x) = |\lambda|\, p(x)$ for every $x, y \in X$ and every $\lambda \in \mathbf{K}$. A sequence of semi-norms $(p_n)_{n\geq 0}$ is said to be *separating* if for every $x \in X$, the equalities $p_n(x) = 0$ for every $n \geq 0$ imply that $x = 0$.

One can associate to such a family of semi-norms a metric, which will define the topology of a so-called Fréchet space.

Definition 2.1.14 ([26, Definition 2.5])**.** A *Fréchet space* is a vector space X, endowed with an increasing and separating family of semi-norms $(p_n)_{n\geq 1}$, which is complete with respect to the metric d defined by

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \min(1, p_n(x-y)) \quad \text{for every } x, y \in X.$$

The Fréchet spaces are generalizations of the Banach spaces. It suffices to consider the sequence where all the semi-norms are equal to a norm defining the topology of the Banach space.

Fréchet spaces are useful when working with spaces which do not admit norms which make them complete. Such is the case for the space of holomorphic functions on **C** endowed with the topology of uniform convergence on compact sets.

Example 2.1.15 ([26, Example 2.7 (b)])**.** The complex space of entire functions $\mathrm{Hol}(\mathbf{C})$ endowed with the increasing and separating family of semi-norms $(\|\cdot\|_{n,\infty}\colon f \in \mathrm{Hol}(\mathbf{C}) \to \sup_{|z|\leq n} |f(z)|)_{n\geq 1}$ is a Fréchet space.

Since a Fréchet space does not necessarily admit a norm thanks to which one could define the notion of bounded operator, the following proposition characterizes the continuity of an operator with respect to the families of semi-norms.

Proposition 2.1.16 ([26, Proposition 2.11])**.** Let X and Y be two Fréchet spaces respectively associated to the families of semi-norms $(p_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$. A linear operator $T\colon X \to Y$ is continuous if and only if for every $m \geq 1$ there exist $n \geq 1$ and $M > 0$ such that $q_m(Tx) \leq M p_n(x)$ for every $x \in X$.

We give now the first three examples of linear operators admitting dense orbits in the space they act on.

Example 2.1.17 ([26, Example 2.12 (b)])**.** The Birkhoff's operator $T^{(a)}$, acting on $\mathrm{Hol}(\mathbf{C})$ and defined by $T^{(a)}f = f(\cdot + a)$ for every $f \in \mathrm{Hol}(\mathbf{C})$ and with $a \in \mathbf{C}$, is continuous since $\|f(\cdot + a)\|_{n,\infty} \leq \|f\|_{n+\lceil |a| \rceil,\infty}$ for every $f \in \mathrm{Hol}(\mathbf{C})$ and for every $n \geq 1$.

Example 2.1.18 ([26, Example 2.12 (a)])**.** The MacLane's operator D, acting on $\mathrm{Hol}(\mathbf{C})$ and defined by $Df = f'$ for every $f \in \mathrm{Hol}(\mathbf{C})$, is continuous since it follows from the Cauchy's integral formula that $\|f'\|_{n,\infty} \leq \|f\|_{n+1,\infty}$ for every $f \in \mathrm{Hol}(\mathbf{C})$ and every $n \geq 1$.

Example 2.1.19 ([26, Example 2.12 (c)])**.** Let $(e_n)_{n\geq 0}$ be the canonical basis of the space $\ell^2(\mathbf{N})$. The Rolewicz's operator B, acting on $\ell^2(\mathbf{N})$ and defined by $Be_n = e_{n-1}$ for every $n \geq 1$ and $Be_0 = 0$, is continuous since $\|Bx\|_2 \leq \|x\|_2$ for every $x \in \ell^2(\mathbf{N})$.

The backward shift B defined above can be seen as a weighted backward shift B_ω. This generalization is useful in order to characterize dynamical properties in terms of conditions about the weight ω, and hence to obtain examples of operators satisfying some properties but not some other ones.

Example 2.1.20 ([26, Example 4.9 (a)])**.** Let $(e_n)_{n\geq 0}$ be the canonical basis of the space $\ell^2(\mathbf{N})$ and let $\omega = (\omega_n)_{n\geq 0}$ be a sequence of non-zero complex numbers called a weight. The weighted backward shift B_ω acting on $\ell^2(\mathbf{N})$ and defined by $B_\omega e_n = \omega_n e_{n-1}$ for every $n \geq 1$ and $B_\omega e_0 = 0$, is continuous if and only if the sequence $(\omega_n)_{n\geq 0}$ is bounded, since $\|B_\omega x\|_2 \leq \sup_{n\geq 0} |\omega_n| \, \|x\|_2$ for every $x \in \ell^2(\mathbf{N})$ and $\|B_\omega e_n\|_2 = |\omega_n| \, \|e_n\|_2$ for every $n \geq 1$.

We also introduce the class of diagonal operators which will illustrate below the notion of cyclicity.

Example 2.1.21. Let $(e_n)_{n\geq 0}$ be the canonical basis of $\ell^2(\mathbf{N})$ and let $(d_n)_{n\geq 0}$ be a sequence of complex numbers. The diagonal operator D, acting on $\ell^2(\mathbf{N})$ and defined by $Den = d_n e_n$ for every $n \geq 0$, is continuous if and only if $(d_n)_{n\geq 0}$ is bounded since $\|Dx\|_2 \leq \sup_{n\geq 0} |d_n| \, \|x\|_2$ for every $x \in \ell^2(\mathbf{N})$ and $\|De_n\|_2 = |d_n| \, \|e_n\|_2$ for every $n \geq 0$.

We will now work in the case of complex, separable, infinite-dimensional Banach (or Fréchet) spaces. Indeed, we already made the observation that X has to be separable in order to allow the existence of operators with dense orbits. Besides it appears that if X is finite-dimensional, no linear operator on X has a dense orbit. This can be seen by considering the operators on X as acting on $\mathbf{C}^n$, and by studying their Jordan blocks decomposition.

We denote by $\mathcal{B}(X)$ the set of continuous and linear operators acting on X, by $\sigma(T) = \{\lambda \in \mathbf{C}; T - \lambda I \text{ is not inversible in } \mathcal{B}(X)\}$ the spectrum of such an operator T, and by $\sigma_p(T)$ its point spectrum, which is the set of its eigenvalues.

We now introduce the first dynamical property, hypercyclicity, which requires the existence of a vector whose orbit under the action of an operator is dense in X.

Definition 2.1.22 ([13, Introduction])**.** An operator $T \in \mathcal{B}(X)$ is said to be *hypercyclic* if there exists $x \in X$ such that the orbit $\{T^n x; n \geq 0\}$ under the action of T is dense in X. We say that x is a *hypercyclic vector for* T and we denote by $HC(T)$ the set of hypercyclic vectors for T.

It can appear surprising that this property has its own name in the linear environment. It actually comes from the study of the cyclic vectors of an operator, which are vectors whose orbit spans a dense subspace and therefore are not defined in a non-linear environment. From this notion of cyclic vectors it is natural to define hypercyclic vectors, and then supercyclic vectors, whose projective orbit is dense.

A separable Banach (or Fréchet) space is Polish and does not admit any isolated point, so the Birkhoff's transitivity theorem applies to characterize hypercyclicity.

Theorem 2.1.23 (Birkhoff's transitivity theorem, [12, §62])**.** An operator $T \in \mathcal{B}(X)$ is hypercyclic if and only if T is topologically transitive. In this case the set of hypercyclic vectors for T is a dense G_δ-set in X.

The alternative seen above for vectors with dense orbits still holds for a linear operator. If there exists at least one hypercyclic vector, then such vectors form a dense G_δ set in X.

The Birkhoff's transitivity theorem notably allows to prove that Birkhoff, MacLane and Rolewicz's operators are hypercyclic.

Example 2.1.24 ([26, Example 2.20])**.** The Birkhoff's operator $T^{(a)}$ is hypercyclic on $\mathrm{Hol}(\mathbf{C})$ as soon as $a \neq 0$.

Example 2.1.25 ([26, Example 2.21])**.** The MacLane's operator D is hypercyclic on $\mathrm{Hol}(\mathbf{C})$.

Let U and V be two non-empty open sets in $\mathrm{Hol}(\mathbf{C})$. By density of polynomials in $\mathrm{Hol}(\mathbf{C})$, there exist two polynomials $p \in U$ and $q \in V$ such that $p(z) = \sum_{n=0}^{d} a_n z^n$ and $q(z) = \sum_{n=0}^{d} b_n z^n$. Consider the polynomial r_k defined by $r_k(z) = p(z) + \sum_{n=0}^{d} n! b_n/(n+k)! \cdot z^{n+k}$ satisfying $D^k r_k = q$ for every $k \geq d+1$. However $\sup_{|z| \leq R} |r_k(z) - p(z)| \leq \sum_{n=0}^{d} n! \, |b_n| \, /(n+k)! R^{n+k} \to 0$ as $k \to +\infty$ for every $R > 0$. Hence $(r_k)_{k \geq 0}$ converges uniformly on every compact set to p. So if k is sufficiently large then $r_k \in U$ and $D^k r_k = q \in V$, thus $D^k(U) \cap V \neq \emptyset$. Therefore D is topologically transitive and so hypercyclic.

Example 2.1.26 ([26, Example 2.22])**.** The Rolewicz's operator λB is hypercyclic on $\ell^2(\mathbf{N})$ if and only if $|\lambda| > 1$.

If $|\lambda| \leq 1$ then $\|(\lambda B)^n x\|_2 \leq |\lambda|^n \|x\|_2 \leq \|x\|_2$ for every $x \in \ell^2(\mathbf{N})$. Hence every orbit under the action of λB is bounded, so it cannot be dense in $\ell^2(\mathbf{N})$.

Conversely let U and V be two non-empty open sets in $\ell^2(\mathbf{N})$ et let $(e_n)_{n \geq 0}$ be the canonical basis of $\ell^2(\mathbf{N})$. By density of finite sequences in $\ell^2(\mathbf{N})$ there exist two finite sequences $x \in U$ and $y \in V$ such that $x = \sum_{n=0}^{d} x_n e_n$ and $y = \sum_{n=0}^{d} y_n e_n$. Consider the sequence $z_k = \lambda^{-k} \sum_{n=0}^{d} y_n e_{n+k}$ satisfying $(\lambda B)^k z_k = y$ and $(\lambda B)^k (x + z_k) = y$ if k is sufficiently large. Besides $\|x + z_k - x\|_2 = \|z_k\|_2 = |\lambda|^{-k} \|y\|_2 \to 0$ as $k \to +\infty$. Thus if k is sufficiently large then $x + z_k \in U$ and $(\lambda B)^k (x + z_k) = y \in V$, hence $(\lambda B)^k (U) \cap V \neq \emptyset$. So λB is topologically transitive and then hypercyclic if $|\lambda| > 1$.

The hypercyclicity of the weighted backward shift can be characterized in terms of products of consecutive terms of the weight.

Example 2.1.27 ([26, Example 4.9 (a)])**.** Let $\omega = (\omega_n)_{n \geq 0}$ be a bounded sequence of non-zero complex numbers. The operator B_ω is hypercyclic on $\ell^2(\mathbf{N})$ if and only if $\sup_{n \geq 1} |\omega_1 \ldots \omega_n| = +\infty$.

Suppose that $\sup_{n \geq 1} |\omega_1 \ldots \omega_n| = +\infty$. Let U and V be two non-empty open sets in $\ell^2(\mathbf{N})$ and let $(e_n)_{n \geq 0}$ be the canonical basis of $\ell^2(\mathbf{N})$. By density of finite sequences in $\ell^2(\mathbf{N})$ there exist two finite sequences $x \in U$ and $y \in V$ such that $x = \sum_{n=0}^{d} x_n e_n$ and $y = \sum_{n=0}^{d} y_n e_n$. Consider the weighted forward shift F_ω acting on $\ell^2(\mathbf{N})$ and defined by $F e_n = e_{n+1}/\omega_{n+1}$ for every $n \geq 0$, such that $B_\omega F_\omega = I$ on $\ell^2(\mathbf{N})$. By assumption there exists $(m_k)_{k \geq 0}$ such that $m_k \to +\infty$ and $|\omega_1 \ldots \omega_{m_k + d}| \to +\infty$ as $k \to +\infty$. Then

$B_\omega^{m_k}(x + F_\omega^{m_k} y) = B_\omega^{m_k} x + y = y$ if k is sufficiently large. Besides

$$\|F_\omega^{m_k} y\|_2^2 = \sum_{n=0}^{d} \frac{|y_n|^2}{|\omega_{n+1}\dots\omega_{n+m_k}|^2} \leq \max_{0\leq n\leq d} \frac{\|y\|_2^2}{|\omega_{n+1}\dots\omega_{n+m_k}|^2},$$

however for every $0 \leq n \leq d$

$$\frac{1}{|\omega_{n+1}\dots\omega_{n+m_k}|} = \frac{|\omega_1\dots\omega_n \cdot \omega_{m_k+n+1}\dots\omega_{m_k+d}|}{|\omega_1\dots\omega_{m_k+d}|} \leq \frac{\|\omega\|_\infty^d}{|\omega_1\dots\omega_{m_k+d}|} \xrightarrow[k\to+\infty]{} 0.$$

So $F_\omega^{m_k} y \to 0$ as $k \to +\infty$. Thus if k is sufficiently large then $x + F_\omega^{m_k} y \in U$ and $B_\omega^{m_k}(x + F_\omega^{m_k} y) = y \in V$, hence $B_\omega^{m_k}(U) \cap V \neq \emptyset$. So B_ω is topologically transitive and then hypercyclic.

Conversely suppose that B_ω is hypercyclic. Then there exists a hypercyclic vector $x = \sum_{n=0}^{\infty} x_n e_n \in \ell^2(\mathbf{N})$ for B_ω, which implies that for every $\varepsilon > 0$ there exists $N \geq 0$ such that $\|B_\omega^N x - e_0\|_2 < \varepsilon$. However

$$\varepsilon > \|B_\omega^N x - e_0\|_2 \geq |x_N \omega_1 \dots \omega_N - 1| \geq 1 - |x_N|\,|\omega_1\dots\omega_N|,$$

so $|x_N|\,|\omega_1\dots\omega_N| > 1-\varepsilon$. In particular with $\varepsilon = 1/2$, we just proved the inclusion $\{n \geq 0; \|B_\omega^n x - e_0\|_2 < 1/2\} \subset \{n \geq 0; |x_n|\,|\omega_1\dots\omega_n| > 1/2\}$. By density of the orbit $\mathrm{orb}(x, B_\omega)$ in $\ell^2(\mathbf{N})$, the set $\{n \geq 0; \|B_\omega^n x - e_0\|_2 < 1/2\}$ is infinite and then contains a strictly increasing sequence $(n_k)_{k\geq0}$. Thus the inequality $|x_{n_k}|\,|\omega_1\dots\omega_{n_k}| > 1/2$ for every $k \geq 0$ and the fact that $x \in \ell^2(\mathbf{N})$ imply that $|\omega_1\dots\omega_{n_k}| \to +\infty$ as $k \to +\infty$.

In the pratice, we will not prove anymore the hypercyclicity of an operator by expliciting a hypercyclic vector or by directly showing that it is topologically transitive. Indeed remark that in order to show that MacLane and Rolewicz's operators are hypercyclic, we have to find a dense set X_0 and a right inverse for T defined on X_0, $S\colon X_0 \to X_0$, such that $T^n x \to 0$ and $S^n x \to 0$ as $n \to +\infty$ for every $x \in X_0$. This idea can be generalized, and leads to the following Hypercyclicity Criterion, which provides us with sufficient and easily checked conditions to prove that an operator is hypercyclic.

Theorem 2.1.28 (Hypercyclicity Criterion, [11, Theorem 2.3])**.** Let $T \in \mathcal{B}(X)$. Suppose that there exist two dense subsets X_0 and Y_0 of X, a strictly increasing sequence $(n_k)_{k\geq0}$ and a sequence $(S_{n_k}\colon Y_0 \to X)_{k\geq0}$ of maps such that:

(i) $T^{n_k} x \to 0$ as $k \to +\infty$ for every $x \in X_0$;

(ii) $S_{n_k} y \to 0$ as $k \to +\infty$ for every $y \in Y_0$;

(iii) $T^{n_k} S_{n_k} y \to y$ as $k \to +\infty$ for every $y \in Y_0$.

Then T is hypercyclic.

Proof. Let us show that T is topologically transitive. Let U and V be two non-empty open sets in X, and we will show that there exists $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$. By density of X_0 and Y_0 in X, there exist $x \in U \cap X_0$ and $y \in V \cap Y_0$. By assumptions $x + S_{n_k} y \to x$ as $k \to +\infty$ and $T^{n_k}(x + S_{n_k} y) = T^{n_k} x + T^{n_k} S_{n_k} y \to y$ as $k \to +\infty$. Thus there exists $K \geq 0$ such that $x + S_{n_k} y \in U$ and $T^{n_k}(x + S_{n_k} y) \in V$ for every $k \geq K$. In particular we have $T^{n_K}(x + S_{n_K} y) \in T^{n_K}(U) \cap V$, hence $T^{n_K}(U) \cap V \neq \emptyset$. So T is topologically transitive, and then hypercyclic by Theorem 2.1.23. ☐

One can remark that we do not assume X_0 and Y_0 to be subspaces of X nor assume $(S_{n_k})_{k\geq 0}$ to be a sequence of linear or continuous maps.

The existence of a hypercyclic operator not satisfying the Hypercyclicity Criterion still was an open question when Bès and Peris showed that the Hypercyclicity Criterion is equivalent to the hypercyclicity of $T \oplus T \in \mathcal{B}(X \oplus X)$ defined by $T \oplus T(x, y) = (Tx, Ty)$ for every $(x, y) \in X \oplus Y$ ([11, Theorem 2.3]).

The Hypercyclicity Criterion is a generalization of the Kitai Criterion, which supposes that $(n_k)_{k\geq 0}$ is the whole sequence $(n)_{n\geq 0}$ and that T admits a right inverse $S\colon Y_0 \to Y_0$. This Kitai criterion had been previously extended by Gethner and Shapiro, who gave the following criterion.

Theorem 2.1.29 (Gethner-Shapiro Criterion, [23, Theorem 2.2] and [26, Theorem 3.10])**.** Let $T \in \mathcal{B}(X)$. Suppose that there exist two dense subsets X_0 and Y_0 of X, a strictly increasing sequence $(n_k)_{k\geq 0}$ and a map $S\colon Y_0 \to Y_0$ such that:

(i) $T^{n_k}x \to 0$ as $k \to +\infty$ for every $x \in X_0$;

(ii) $S_{n_k}y \to 0$ as $k \to +\infty$ for every $y \in Y_0$;

(iii) $TSy = y$ for every $y \in Y_0$.

Then T is hypercyclic.

Even if the Gethner-Shapiro Criterion seems to be strictly less general than the Hypercyclicity Criterion, Peris proved in 2001 that these criteria are actually equivalent.

Theorem 2.1.30 ([39, Theorem 2.3])**.** An operator $T \in \mathcal{B}(X)$ satisfies the assumptions of the Hypercyclicity Criterion if and only if it satisfies the Gethner-Shapiro Criterion.

The definition of hypercyclicity, which requires the existence of a dense orbit, led to the more general definition of the supercyclicity, which requires the existence of a dense projective orbit.

Definition 2.1.31 ([28, Definition §4])**.** An operator $T \in \mathcal{B}(X)$ is said to be *supercyclic* if there exists $x \in X$ such that $\{\lambda T^n x; \lambda \in \mathbf{C}, n \geq 0\}$ is dense in X. We say that x is a *supercyclic vector for* T and we denote by $SC(T)$ the set of supercyclic vectors for T.

Operators like the Rolewicz's backward shift B can also motivate this definition. Indeed even if B is not hypercyclic (B is a contraction, i.e. $\|B\| \leq 1$), it appears that it is supercyclic.

As in the hypercyclic case, a transitivity theorem exists for the notion of supercyclicity.

Theorem 2.1.32 ([4, Theorem 1.12])**.** An operator $T \in \mathcal{B}(X)$ is supercyclic if and only if for every pair (U, V) of non-empty open sets in X, there exist $n \geq 0$ and $\lambda \in \mathbf{C}$ such that $\lambda T^n(U) \cap V \neq \emptyset$. In this case the set $SC(T)$ of supercyclic vectors for T is a dense G_δ-set in X.

Proof. Suppose that T is supercyclic and let U and V be two non-empty open sets in X. By supercyclicity there exist $x \in X$, $\lambda \in C\backslash\{0\}$ and $n \geq 0$ such that $\lambda T^n x \in U$. Since X does not admit any isolated point, the set $\{\mu T^m x; \mu \in \mathbf{C}, m \geq n\}$ is still dense in X and there exist $\mu \in \mathbf{C}$ and $m \geq n$ such that $\mu T^m x \in V$. Then we have $(\mu/\lambda)T^{m-n}(\lambda T^n x) = \mu T^m x \in (\mu/\lambda)T^{m-n}(U) \cap V$ with $m - n \geq 0$, hence $(\mu/\lambda)T^{m-n}(U) \cap V \neq \emptyset$.

Conversely let us show that T is supercyclic. Let $(U_k)_{k\geq 0}$ be a basis of non-empty open sets in X. A vector $x \in X$ is supercyclic for T if and only if for every non-empty open set U in X there exist $\lambda \in \mathbf{C}$ and $n \geq 0$ such that $\lambda T^n x \in U$, that is to say if and only if for every $k \geq 0$ there exist $n \geq 0$ and $\lambda \in \mathbf{C}$ such that $\lambda T^n x \in U_k$. So the set $SC(T)$ of supercyclic vectors for T is given by $SC(T) = \cap_{k\geq 0} \cup_{\lambda\in\mathbf{C},n\geq 0} (\lambda T^n)^{-1}(U_k)$ and it suffices to show that $SC(T)$ is a dense G_δ-set in X. Fix $k \geq 0$, the set $\cup_{\lambda\in\mathbf{C},n\geq 0}(\lambda T^n)^{-1}(U_k)$ is an open set in X by continuity of T. Besides for every non-empty open set V in X, by assumption there exist $\lambda_0 \in \mathbf{C}$ and $n_0 \geq 0$ such that $\lambda_0 T^{n_0}(V) \cap U_k \neq \emptyset$, that is to say such that $(\lambda_0 T^{n_0})^{-1}(U_k) \cap V \neq \emptyset$. Thus $(\cup_{\lambda\in\mathbf{C},n\geq 0}(\lambda T^n)^{-1}(U_k)) \cap V \neq \emptyset$, which implies that the open set $\cup_{\lambda\in\mathbf{C},n\geq 0}(\lambda T^n)^{-1}(U_k)$ is dense in X for every $k \geq 0$. By the Baire Category Theorem the set $SC(T)$ is therefore a dense G_δ-set in X, and is not empty in particular. □

Let us now consider the case of weighted backward shift operators B_ω. It appears that B_ω is always supercyclic, for any weight ω. This allows us to give examples of supercyclic operators T such that for every $\lambda \in \mathbf{C}$, λT is not hypercyclic.

Example 2.1.33 ([4, Example 1.15]). The weighted backward shift operator B_ω is supercyclic on $\ell^2(\mathbf{N})$ for every bounded weight ω. In particular if $\omega_n = 1/n$ for every $n \geq 1$, then B_ω is supercyclic and for every $\lambda \in \mathbf{C}$, λB_ω is not hypercyclic.

Indeed let U and V be two non-empty open sets in $\ell^2(\mathbf{N})$. By the density of finite sequences in $\ell^2(\mathbf{N})$ there exist two finite sequences $x \in U$ and $y \in V$. Consider the weighted forward shift F_ω on $\ell^2(\mathbf{N})$. Fix $k \geq 0$ such that $B_\omega^k x = 0$. Fix a non-zero complex number λ such that $x + \lambda^{-1}F_\omega^k y \in U$, then $\lambda B_\omega^k(x + \lambda^{-1}F_\omega^k y) = \lambda B_\omega^k x + y = y$. So $\lambda B_\omega^k(U) \cap V \neq \emptyset$, hence B_ω is supercyclic on $\ell^2(\mathbf{N})$. Moreover for every $\lambda \in \mathbf{C}$, $\lambda B_\omega = B_\delta$ where $\delta_n = \lambda\omega_n$ for every $n \geq 1$. So $\sup_{n\geq 1} |\delta_1 \dots \delta_n| = \sup_{n\geq 1} |\lambda|^n /n! < +\infty$, which implies that λB_ω is not hypercyclic.

Similarly, the Hypercyclicity Criterion can be adapted into a Supercyclicity Criterion.

Theorem 2.1.34 (Supercyclicity Criterion, [10, Lemma 3.1]). Let $T \in \mathcal{B}(X)$ be an operator. Suppose that there exist two dense subsets X_0 and Y_0 of X, a strictly increasing sequence $(n_k)_{k\geq 0}$, a sequence $(\lambda_{n_k})_{k\geq 0}$ of non-zero complex numbers and a sequence $(S_{n_k} : Y_0 \to X)_{k\geq 0}$ of maps such that:

(i) $\lambda_{n_k} T^{n_k} x \to 0$ as $k \to +\infty$ for every $x \in X_0$;

(ii) $\lambda_{n_k}^{-1} S_{n_k} y \to 0$ as $k \to +\infty$ for every $y \in Y_0$;

(iii) $T^{n_k} S_{n_k} y \to y$ as $k \to +\infty$ for every $y \in Y_0$.

Then T is supercyclic.

Proof. Let us show that T satisfies the property of Theorem 2.1.32. Let U and V be two non-empty open sets in X. We will show that there exist $\lambda \in \mathbf{C}$ and $n \geq 0$ such that $\lambda T^n(U) \cap V \neq \emptyset$. By density of X_0 and Y_0 in X, there exist $x \in U \cap X_0$ and $y \in V \cap Y_0$. By our assumptions $x + \lambda_{n_k}^{-1} S_{n_k} y \to x$ as $k \to +\infty$ and $\lambda_{n_k} T^{n_k}(x + \lambda_{n_k}^{-1} S_{n_k} y) = \lambda_{n_k} T^{n_k} x + T^{n_k} S_{n_k} y \to y$ as $k \to +\infty$. Thus there exists $K \geq 0$ such that $x + \lambda_{n_k}^{-1} S_{n_k} y \in U$ and $\lambda_{n_k} T^{n_k}(x + \lambda_{n_k}^{-1} S_{n_k} y) \in V$ for every $k \geq K$. In particular $\lambda_{n_K} T^{n_K}(x + \lambda_{n_K}^{-1} S_{n_K} y) \in \lambda_{n_K} T^{n_K}(U) \cap V$, hence $\lambda_{n_K} T^{n_K}(U) \cap V \neq \emptyset$. So T is supercyclic by Theorem 2.1.32. □

In the fourth chapter, we will want to show that some weighted translation operators cannot satisfy the Supercyclicity Criterion. To do so we will need to prove that the Supercyclicity Criterion is actually equivalent to a Gethner-Shapiro-type Supercyclicity Criterion. This result does not seem to appear in the literature.

We now introduce the weakest dynamical property, from which hypercyclicity is derived, called cyclicity. It assumes the existence of an orbit spanning a dense subspace. This notion will have a direct link with the Invariant Subspace Problem.

Definition 2.1.35 ([24, Introduction])**.** An operator $T \in \mathcal{B}(X)$ is said to be *cyclic* if there exists $x \in X$ such that the vector space span$[T^n x; n \geq 0]$ spanneed by its orbit under the action of T is dense in X. We say that x is a *cyclic vector for* T and we denote by $C(T)$ the set of cyclic vectors for T.

A version of the transitivity theorem is well known is the cyclic case. Nevertheless it requires the additional assumption that the adjoint operator T^* does not admit any eigenvalue, which ensures that $P(T)$ has dense range in X for every non-zero polynomial $P \in \mathbf{C}[\xi]$.

Theorem 2.1.36. Let $T \in \mathcal{B}(X)$ be an operator. If for every pair (U, V) of non-empty open sets in X there exists $P \in \mathbf{C}[\xi]$ such that $P(T)(U) \cap V \neq \emptyset$, then T is cyclic. In this case the set $C(T)$ of cyclic vectors for T is a dense G_δ-set in X.

Moreover if $\sigma_p(T^*) = \emptyset$, then the converse is also true.

Proof. Suppose that for every pair (U, V) of non-empty open sets in X there exists $P \in \mathbf{C}[\xi]$ such that $P(T)(U) \cap V \neq \emptyset$. We will show that T is cyclic. Let $(U_k)_{k\geq 0}$ be a basis of non-empty open sets in X. A vector $x \in X$ is cyclic for T if and only if for every non-empty open set U in X there exists $P \in \mathbf{C}[\xi]$ such that $P(T)x \in U$, that is to say if and only if for every $k \geq 0$ there exists $P \in \mathbf{C}[\xi]$ such that $P(T)x \in U_k$. Thus the set $C(T)$ of cyclic vectors for T is given by $C(T) = \cap_{k\geq 0} \cup_{P\in\mathbf{C}[\xi]} P(T)^{-1}(U_k)$ and it suffices to show that $C(T)$ is a dense G_δ-set in X. Fix $k \geq 0$, the set $\cup_{P\in\mathbf{C}[\xi]} P(T)^{-1}(U_k)$ is an open set in X by continuity of T. Besides, for every non-empty open set V in X, by assumption there exists $P_0 \in \mathbf{C}[\xi]$ such that $P_0(T)(V) \cap U_k \neq \emptyset$, that is to say that $P_0(T)^{-1}(U_k) \cap V \neq \emptyset$. So $(\cup_{P\in\mathbf{C}[\xi]} P(T)^{-1}(U_k)) \cap V \neq \emptyset$, hence the open set $\cup_{P\in\mathbf{C}[\xi]} P(T)^{-1}(U_k)$ is dense in X for every $k \geq 0$. By the Baire Category Theorem the set $C(T)$ is a dense G_δ-set in X, and is not empty in particular.

Conversely suppose that T is cyclic and that $\sigma_p(T^*) = \emptyset$, and let U and V be two non-empty open sets in X. By cyclicity there exist $x \in X$ and $P \in \mathbf{C}[\xi]$ such that $P(T)x \in U$. Besides $P(T)$ has a dense range in X since $\sigma_p(T^*) = \emptyset$. Thus by continuity there exists an open set W in X such that $P(T)(W) \subset V$ and by cyclicity there exists $Q \in \mathbf{C}[\xi]$ such that $Q(T)x \in W$. Hence $Q(T)P(T)x = P(T)Q(T)x \in Q(T)(U) \cap V$, so $Q(T)(U) \cap V \neq \emptyset$. □

This theorem can be illustrated by the example of diagonal operators on the sequence space $\ell^2(\mathbf{N})$.

Example 2.1.37. Let $(d_n)_{n\geq 0}$ be a bounded sequence of complex numbers. The operator D is cyclic on $\ell^2(\mathbf{N})$ if and only if the complex numbers $(d_n)_{n\geq 0}$ are pairwise distinct.

Indeed suppose that there exist $0 \leq i < j$ such that $d_i = d_j$ and let $(e_n)_{n\geq 0}$ be the canonical basis of $\ell^2(\mathbf{N})$. Suppose by contradiction that T is cyclic. Thus there exists a cyclic vector $x = \sum_{n=0}^{\infty} x_n e_n \in \ell^2(\mathbf{N})$. For every $y \in \ell^2(\mathbf{N})$ there exists a sequence $(P_k)_{k\geq 0}$ of polynomials such that $P_k(\xi) = \sum_{m=0}^{m_k} a_{m,k}\xi^m$ for every $k \geq 0$ and $\|P_k(D)x - y\|_2 \to 0$ as $k \to +\infty$. However $P_k(D)x = \sum_{m=0}^{m_k} a_{m,k} D^m \sum_{n=0}^{\infty} x_n e_n = \sum_{n=0}^{\infty}(x_n \sum_{m=0}^{m_k} a_{m,k} d_n^m) e_n = \sum_{n=0}^{\infty} P_k(d_n) x_n e_n$ for every $k \geq 0$, hence $|P_k(d_n)x_n - y_n| \leq \|P_k(D)x - y\|_2$ for every $n \geq 0$. In particular $P_k(d_i)x_i \to y_i$ and $P_k(d_j)x_j \to y_j$ as $k \to +\infty$. Since $d_i = d_j$, we have $P_k(d_i)x_i x_j \to y_i x_j$ and $P_k(d_i)x_j x_i \to y_j x_i$ as $k \to +\infty$. Hence $y_i x_j = y_j x_i$ for every $(y_n)_{n\geq 0} \in \ell^2(\mathbf{N})$. Thus we obtain $x_i = 0$ by taking $y = e_j$. The vector x cannot be cyclic for D since it would mean that $\|P(D)x - e_i\|_2 \geq |P(d_i)x_i - 1| = 1$ for every $P \in \mathbf{C}[\xi]$, which is a contradiction.

Conversely suppose that $d_i \neq d_j$ if $i \neq j$ and let U and V be two non-empty open sets in $\ell^2(\mathbf{N})$. By density of finite sequences in $\ell^2(\mathbf{N})$ there exist two finite sequences $x \in U$ and $y \in V$ such that $x = \sum_{n=0}^{d} x_n e_n$ and $y = \sum_{n=0}^{d} y_n e_n$. Remark that there exists a finite sequence $x' = \sum_{n=0}^{d} x'_n e_n$ in U satisfying $x'_n \neq 0$ for every $n \in \{0, \ldots, d\}$. Indeed let $\varepsilon > 0$ such that $\{z \in \ell^2(\mathbf{N}); \|x - z\|_2 < \varepsilon\} \subset U$. Consider $u = \sum_{n\in A} e_n$ where $A = \{n \in \{0, \ldots, d\}; x_n = 0\}$ and $v = \varepsilon(2\|u\|_2)^{-1}u$. Then $x' = x + v$ is suitable. We consider now a Lagrange interpolating polynomial P such that $P(d_i) = y_i/x'_i$ for every $i \in \{0, \ldots, d\}$. Thus $P(D)x' = \sum_{n=0}^{d} P(d_n)x'_n e_n = \sum_{n=0}^{d} y_n e_n = y$, which implies that $P(D)(U) \cap V \neq \emptyset$. So D is cyclic.

Contrary to Theorem 2.1.23, Theorem 2.1.36 is not a characterization of cyclicity, and there exist examples of cyclic operators whose set of cyclic vectors is not dense.

Example 2.1.38. The forward shift F on $\ell^2(\mathbf{N})$ is cyclic and the set of cyclic vectors for F is not dense in $\ell^2(\mathbf{N})$.

The space $\ell^2(\mathbf{N})$ can be isometrically identified with the Hardy space

$$H^2(\mathbf{D}) = \left\{ f\colon z \mapsto \sum_{n=0}^{\infty} c_n z^n \in \mathrm{Hol}(\mathbf{C}); \|f\|_{H^2(\mathbf{D})}^2 = \sum_{n=0}^{\infty} |c_n|^2 < +\infty \right\}.$$

The operator $F \in \mathcal{B}(\ell^2(\mathbf{N}))$ is then naturally identified with the multiplication operator M_z defined on $H^2(\mathbf{D})$ by $M_z f = zf$, $f \in H^2(\mathbf{D})$.

If $f \in H^2(\mathbf{D})$ vanishes at a point $a \in \mathbf{D}$, then $M_z^n f(a) = a^n f(a) = 0$ for every $n \geq 0$. So every function in $\mathrm{span}[M_z^n f; n \geq 0]$ vanishes at a. However the convergence in $H^2(\mathbf{D})$ implies the convergence on every compact subset of $\mathbf{D}$, which implies the pointwise convergence on the open unit disk $\mathbf{D}$. Thus the closed subspace spanned by the orbit of f under the action of M_z only includes functions vanishing at a, which means that f cannot be cyclic for M_z.

A cyclic function for M_z does not vanish on $\mathbf{D}$. However by the Hurwitz's theorem, if a sequence $(f_n)_{n\geq 0}$ a holomorphic functions not vanishing on $\mathbf{D}$ converges to $f \in H^2(\mathbf{D})$ in $H^2(\mathbf{D})$, then either f is identically zero or f does not vanish on $\mathbf{D}$. Hence the set of cyclic functions for M_z cannot be dense in $H^2(\mathbf{D})$.

The existence of a dense orbit for an operator shows that such an operator can have interesting behavior in spite of the linear environment it acts on. When the operator simultaneously has a dense set of vectors whose orbit is dense and a dense set of vectors whose orbit is finite, it has a chaotic behavior.

Definition 2.1.39 ([26, Definition 1.23]). Let $T \in \mathcal{B}(X)$ be an operator. We say the $x \in X$ is a *periodic vector for* T if there exists $n \geq 1$ such that $T^n x = x$. We denote by $\mathrm{Per}(T)$ the set of periodic vectors for T.

Definition 2.1.40 ([26, Definition 2.29]). An operator $T \in \mathcal{B}(X)$ is said to be *chaotic* if it is hypercyclic and if $\mathrm{Per}(T)$ is dense in X.

This definition of linear chaos coincides with the notion of chaos in the sense of Devaney ([26, Definition 2.29]). If X is a Fréchet space or a Banach space, the topology of X is induced from a metric d that is invariant by translation, and the chaos in the sense of Devaney requires an operator $T \in \mathcal{B}(X)$ to be hypercyclic, to have a dense set of periodic points and to have sensitive dependence on initial conditions in the following sense: there exists $\delta > 0$ such that for every $\varepsilon > 0$ and every $x \in X$, there exist $y \in X$ and $n \geq 0$ such that $d(x, y) < \varepsilon$ and $d(T^n x, T^n y) > \delta$.

These notions coincide in the linear setting since a hypercyclic operator T necessarily has sensitive dependence on initial conditions.

Proposition 2.1.41 ([26, Proposition 2.30]). Let d be a metric which is compatible with the topology of X and which is invariant by translation. A hypercyclic operator $T \in \mathcal{B}(X)$ has sensitive dependence on initial conditions with respect to the metric d.

Proof. Let $\delta, \varepsilon > 0$ and let $x \in X$. Consider the non-empty open sets $U = \{z \in X; d(0, z) < \varepsilon\}$ and $V = \{z \in X; d(0, z) > \delta\}$. By Theorem 2.1.23, there exist $n \geq 0$ and $z \in U$ such that $T^n z \in V$. Since d is invariant by translation, the vector $y = x + z$ satisfies $d(x, y) = d(0, z) < \varepsilon$ and $d(T^n x, T^n y) = d(0, T^n z) > \delta$. $\square$

We will remark that eigenvalues and eigenvectors can play a crucial role in order to determine the dynamical properties of an operator. To introduce this fact, remark that linear chaos for an operator is strongly related to the properties of its eigenvectors associated to eigenvalues that are roots of unity.

Proposition 2.1.42 ([26, Proposition 2.33]). Let $T \in \mathcal{B}(X)$ be an operator. We have $\mathrm{Per}(T) = \mathrm{span}[\ker(T - e^{i\alpha\pi} I); \alpha \in \mathbf{Q}]$.

Proof. Let $\alpha = r/q \in \mathbf{Q}$ and let $x \in \ker(T - e^{ir\pi/q} I)$. Then $T^{2q} x = e^{i2r\pi} x = x$, so $x \in \mathrm{Per}(T)$. Hence $\cup_{\alpha \in \mathbf{Q}} \ker(T - e^{i\alpha\pi} I) \subset \mathrm{Per}(T)$. It thus suffices to show that $\mathrm{Per}(T)$ is a subspace of X to obtain the inclusion $\mathrm{span}[\ker(T - e^{i\alpha\pi} I); \alpha \in \mathbf{Q}] \subset \mathrm{Per}(T)$. Let $x, y \in \mathrm{Per}(T)$ and let λ be a complex number. there exist $n, m \geq 1$ such that $T^n x = x$ and $T^m y = y$. Hence $T^{nm}(\lambda x + y) = \lambda T^{nm} x + T^{nm} y = \lambda x + y$, so $\lambda x + y \in \mathrm{Per}(T)$ and $\mathrm{Per}(T)$ is indeed a subspace of X.

Conversely let $x \in \mathrm{Per}(T)$, there exists $N \geq 1$ such that $T^N x = x$. If we consider $w = e^{i2\pi/N}$ then the polynomial $P(\xi) = \xi^N - 1$ satisfies $\xi^N - 1 = (\xi - 1)(\xi - w) \ldots (\xi - w^{N-1})$ and $P(T)x = T^N x - x = 0$. Consider $P_k(\xi) = \prod_{i \in \{0, \ldots, N-1\} \setminus \{k\}} (\xi - w^k)$ for every $k \in \{0, \ldots, N-1\}$ and remark that they form a basis of the space of the polynomials whose degree is less or equal than $N - 1$. Indeed if $\sum_{k=0}^{N-1} \lambda_k P_k = 0$ then $\sum_{k=0}^{N-1} \lambda_k P_k(w^i) = \lambda_i P_i(w^i) = 0$, hence $\lambda_i = 0$ for every $i \in \{0, \ldots, N-1\}$. Thus the constant polynomial 1 is written as $1 = \sum_{k=0}^{N-1} \alpha_k P_k$, which gives $I = \sum_{k=0}^{N-1} \alpha_k P_k(T)$ and $x = \sum_{k=0}^{N-1} \alpha_k P_k(T) x$. However $(T - w^k I) P_k(T) x = P(T) x = 0$ therefore $P_k(T) x \in \ker(T - w^k I)$ for every $k \in \{0, \ldots, N-1\}$. Then $x \in \mathrm{span}[\ker(T - e^{i\alpha\pi} I); \alpha \in \mathbf{Q}]$, hence $\mathrm{Per}(T) \subset \mathrm{span}[\ker(T - e^{i\alpha\pi} I); \alpha \in \mathbf{Q}]$. $\square$

The following criterion, due to Godefroy and Shapiro, clearly highlights the relation between eigenvectors and the dynamical properties of an operator. Indeed if an operator admits enough eigenvectors associated to eigenvalues in the open unit disk and enough eigenvectors associated to eigenvalues outside of the closed unit disk, then it is necessarily hypercyclic. Moreover, if this is the case for eigenvectors associated to some eigenvalues on the unit circle, then the operator is actually chaotic.

Theorem 2.1.43 (Godefroy-Shapiro Criterion, [26, Theorem 3.1])**.** Let $T \in \mathcal{B}(X)$ be an operator. If the subspaces $X_+ = \text{span}[\ker(T - \lambda I); |\lambda| > 1]$ and $X_- = \text{span}[\ker(T - \lambda I); |\lambda| < 1]$ are dense in X, then T is hypercyclic. Moreover if $X_0 = \text{span}[\ker(T - e^{i\alpha\pi} I); \alpha \in \mathbf{Q}]$ is dense in X, then T is chaotic.

Proof. By Theorem 2.1.23 and Proposition 2.1.42, it suffices to show that T is topologically transitive under the assumptions that X_+ and X_- are dense in X. Let U and V be two non-empty open sets in X. By density of X_+ and X_- in X, there exist $x \in X_- \cap U$ and $y \in X_+ \cap V$, where $x = \sum_{i=1}^{d} \alpha_i x_i$ and $y = \sum_{i=1}^{d} \beta_i y_i$ with $Tx_i = \lambda_i x_i$, $Ty_i = \mu_i$, $|\lambda_i| < 1$ and $|\mu_i| > 1$ for every $i \in \{1, \dots, d\}$. Remark that $a_n = \sum_{i=1}^{d} \alpha_i x_i + \sum_{i=1}^{d} (\alpha_i/\mu_i^n) y_i \to x$ and $T^n(\sum_{i=1}^{d} \alpha_i x_i + \sum_{i=1}^{d} (\beta_i/\mu_i^n) y_i) = \sum_{i=1}^{d} \lambda_i^n \alpha_i x_i + \sum_{i=1}^{d} \beta_i y_i \to y$ as $n \to +\infty$. Hence there exists $N \geq 0$ such that $a_N \in U$ and $T^N(a_N) \in V$, so $T^N(U) \cap V \neq \emptyset$. So T is topologically transitive, and thus hypercyclic by Theorem 2.1.23. If moreover X_0 is dense in X, then T is chaotic by Proposition 2.1.42. □

The eigenvectors of an operator also play an important role in ergodic theory.

Indeed if X is endowed with its Borel σ-algebra $\mathcal{A}$ and with a probability measure m, then an operator $T \in \mathcal{B}(X)$ can be seen as a measurable transformation $T\colon (X, \mathcal{A}, m) \to (X, \mathcal{A}, m)$.

To illustrate this, we present the notion of a perfectly spanning set of eigenvectors.

Definition 2.1.44 ([3, Definition 3.1])**.** We say that an operator $T \in \mathcal{B}(X)$ has a *perfectly spanning set of eigenvectors associated to unimodular eigenvalues* if there exists a continuous probability measure σ on the unit disk $\mathbf{T}$ such that the eigenvectors of T associated to eigenvalues $\mu \in A$ span a dense subspace of X for every σ-measurable subset A of $\mathbf{T}$ satisfying $\sigma(A) = 1$.

It is known that if an operator $T \in \mathcal{B}(X)$ has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues, then it is ergodic with respect to a so-called Gaussian probability measure. We give here the definition.

Definition 2.1.45 ([3, Definitions 3.13 and 3.14])**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. We say that a measurable function $f\colon (\Omega, \mathcal{F}, P) \to \mathbf{C}$ has a *complex symmetric Gaussian distribution* if $\Re(f)$ and $\Im(f)$ have independent centered Gaussian distribution and the same variance. We say that a probabilty measure m on $(X, \mathcal{A})$ is a *Gaussian measure* if the function $y \mapsto x^*(y)$ has a complex symmetric Gaussian distribution for every $x^* \in X^* \backslash \{0\}$.

Theorem 2.1.46 ([6, Theorem 1.1])**.** If an operator $T \in \mathcal{B}(X)$ has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues, then there exists a Gaussian measure of full support m on X such that $T\colon (X, \mathcal{A}, m) \to (X, \mathcal{A}, m)$ is ergodic.

We know that a hypercyclic operator admits orbits intersecting every non-empty open set. The notion of frequent hypercyclicity is related to the rate at which a dense orbit intersects such open sets.

Definition 2.1.47 ([26, Definition 9.2])**.** An operator $T \in \mathcal{B}(X)$ is said to be *frequently hypercyclic* if there exists $x \in X$ such that $\liminf_{N\to+\infty} \operatorname{card}\{0 \le n \le N-1; T^n x \in U\}/N > 0$ for every non-empty open set U in X. We then say that x is a *frequently hypercyclic vector for* T and we denote by $FHC(T)$ the set of frequently hypercyclic vectors for T.

The notion of frequent hypercyclicity has its own practical criterion. To present it, we say that a series $\sum_{n\geq 0} x_n$ is unconditionally convergent in X if the series $\sum_{n\geq 0} x_{\varphi(n)}$ is convergent for every bijection $\varphi\colon \mathbf{Z}_+ \to \mathbf{Z}_+$.

Theorem 2.1.48 (Frequent Hypercyclicity Criterion, [26, Theorem 9.9])**.** Let $T \in \mathcal{B}(X)$. Suppose that there exist a dense subset X_0 of X and a map $S\colon X_0 \to X_0$ such that for every $x \in X_0$:

(i) $\sum_{n\geq 0} T^n x$ is unconditionally convergent;

(ii) $\sum_{n\geq 0} S^n x$ is unconditionally convergent;

(iii) $TSx = x$.

Then T is frequently hypercyclic.

This criterion can be applied to multiples of the backward shift λB if $|\lambda| > 1$.

Example 2.1.49 ([26, Example 9.15])**.** If $|\lambda| > 1$ then the backward shift λB is frequently hypercyclic on $\ell^2(\mathbf{N})$.

Indeed consider X_0 the set of finite sequences in $\ell^2(\mathbf{N})$ and the map $S = \lambda^{-1}F$ where F is the forward shift on $\ell^2(\mathbf{N})$ such that $\lambda BS = I$ on $\ell^2(\mathbf{N})$. For every $x \in X_0$ the series $\sum_{n\geq 0} B^n x$ is unconditionally convergent since it is a finite sum. Moreover the series $\sum_{n\geq 0} S^n x = \sum_{n\geq 0} \lambda^{-n} F^n x$ is unconditionally convergent since it is absolutely convergent. Hence λB is frequently hypercyclic.

The frequent hypercyclicity is also characterized for weighted backward shifts B_ω on $\ell^p(\mathbf{N})$ where $1 \le p < +\infty$. It allows us to obtain examples of hypercyclic operators that are not frequently hypercyclic.

Theorem 2.1.50 ([5, Theorem 4])**.** Let $1 \le p < +\infty$ and let $\omega = (\omega_n)_{n\geq 0}$ be a bounded sequence of positive real numbers. The operator B_ω acting on $\ell^p(\mathbf{N})$ is frequently hypercyclic if and only if $\sum_{n\geq 1} 1/(\omega_1 \ldots \omega_n)^p < +\infty$.

This notion of frequent hypercyclicity is well defined in topological spaces, just as hypercyclicity. It is however naturally linked to the ergodic theory.

If m is of full support and if T is ergodic with respect to m, then T is also frequently hypercyclic, and the set of frequently hypercyclic vectors has full measure with respect to m.

Theorem 2.1.51 ([4, Proposition 6.23])**.** Let $\mathcal{F}$ be the Borel σ-algebra of X, let m be a probability measure of full support on X and let $T \in \mathcal{B}(X)$ be an operator. If $T\colon (X, \mathcal{F}, m) \to (X, \mathcal{F}, m)$ is ergodic then T is frequently hypercyclic and $FHC(T)$ is of full measure in X.

Proof. Let $(U_k)_{k\geq 0}$ be a basis of non-empty open sets in X. Fix $k \geq 0$. By Theorem 2.1.12, with $f = \mathbf{1}_{U_k}$, there exists $A_k \in \mathcal{F}$ such that $m(A_k) = 1$ and satisfying $(1/N)\sum_{n=0}^{N-1} \mathbf{1}_{U_k}(T^n x) \to \int_X \mathbf{1}_{U_k} dm$ as $N \to +\infty$ for every $x \in A_k$. Hence $\text{card}\{0 \leq n \leq N-1; T^n x \in U_k\}/N \to m(U_k) > 0$ as $N \to +\infty$ for every $x \in A_k$. Consider $A = \cap_{k\geq 0} A_k$ satisfying $m(A) = 1$. Let $x \in A$ and let U be a non-empty open set in X. There exists $k \geq 0$ such that $U_k \subset U$ and $\liminf_{N\to+\infty} \text{card}\{0 \leq n \leq N-1; T^n x \in U\}/N \geq \lim_{N\to+\infty} \text{card}\{0 \leq n \leq N-1; T^n x \in U_k\}/N = m(U_k) > 0$. So T is frequently hypercyclic and $FHC(T)$ is a set of full measure since $A \subset FHC(T)$. □

In the following section, we will apply this theorem to an operator naturally associated to the Collatz Conjecture.

2.2 Collatz Conjecture

We present in this part some statements linked to the Collatz Conjecture, also known as the Syracuse Conjecture. It studies the behavior of the orbits of the following map acting on positive integers.

Definition 2.2.1. We define the map

$$\begin{array}{rccl} T_0 \colon & \mathbf{Z}_+ & \to & \mathbf{Z}_+ \\ & n & \mapsto & \begin{cases} n/2 & \text{if } n \text{ is even;} \\ 3n+1 & \text{if } n \text{ is odd,} \end{cases} \end{array}$$

which is called the *Collatz map.* For every $k \geq 0$, the set $\{T_0^n(k) = T_0 \circ T_0 \circ \ldots \circ T_0(k); n \geq 0\}$ is denoted $\text{orb}(k, T_0)$ and is called the *orbit of k under the action of* T_0.

Remark that $T_0(1) = 4$, that $T_0(4) = 2$ and that $T_0(2) = 1$, which leads to the notion of cycle for T_0.

Definition 2.2.2. We say that a d-tuples $(n_1, \ldots, n_d)$ is a *cycle for* T_0 if $d > 1$, if $T_0(n_i) = n_{i+1}$ for every $i \in \{1, \ldots, d-1\}$ and if $T_0(n_d) = n_1$. We call $(1, 4, 2)$ the *trivial cycle of* T_0.

Also remark that we can check that each orbit of a small integer k under the action of T_0 reaches 1. For instance $T_0(2) = 1$, $T_0^7(3) = 1$ and $T_0^{16}(7) = 1$. The Collatz Conjecture is a generalization of this observation to the orbit of every integer $k \geq 1$.

Conjecture 2.2.3. For every $k \geq 1$, the orbit of k under the action of T_0 reaches 1.

This conjecture seems to have appeared in the fifties in the work of Lothar Collatz. The latter was interested in the behavior of some arithmetic functions of the same type. It was checked for the integers $k \leq 10^9$ in the sixties but is nonetheless the subject of published papers only since the seventies.

This conjecture is still open today, although it is the subject of numerous papers which use technics from different fields. We refer to the Lagarias reference [31] for an overview about this question. We now present some observations, before introducing the results which we obtained in this thesis regarding this conjecture.

Since $3n + 1$ is even when n is odd, the study of the iterates of T_0 coincide with the study of the following shortcut map. Using it will make some computations simpler.

Definition 2.2.4. We consider the map

$$\begin{array}{rccl} T\colon & \mathbf{Z}_+ & \to & \mathbf{Z}_+ \\ & n & \mapsto & \begin{cases} n/2 & \text{if } n \text{ is even;} \\ (3n+1)/2 & \text{if } n \text{ is odd.} \end{cases} \end{array}$$

The Collatz Conjecture is true for T_0 if and only if it holds true for T. The cycle $(1, 2)$ for T is called the *trivial cycle for T*.

Let us explicit the preimages of an integer by T, which will be relevant in the forthcoming proof. For every $m \geq 1$, we have $T^{-1}(\{3m + r\}) = \{6m + 2r\}$ if $r \in \{0, 1\}$ and $T^{-1}(\{3m + 2\}) = \{2m + 1, 6m + 4\}$.

Let us now present some results which are known regarding this conjecture. First the number of orbits known to satisfy the conjecture has been enlarged. Indeed we now know ([37]) that the conjecture is satisfied for every $k \leq 20 \times 2^{58} \approx 5.764 \times 10^{18}$.

The existence of a non-trivial cycle for the Collatz map T would allow to say that the conjecture does not hold, which motivates the study of the cycles of T. Eliahou showed ([20]) that a hypothetical non-trivial cycle of T would necessarily have a length greater than 10 439 860 591.

If we cannot directly show that every $k \geq 1$ admits an orbit under the action of T reaching 1, we nonetheless can be interested in the proportion of integers having such a property. Krasikov and Lagarias showed ([30]) that the number of integers in $\{1, \dots, K\}$ satisfying the conjecture is greater than $K^{0.84}$ when K is sufficiently large.

The integers satisfying the conjecture have been more deeply and precisely studied. Applegate and Lagarias have proved ([1]) that there are infinitely many integers n whose orbit under the action of T reaches 1 after at least $6.143 \log(n)$ steps.

Eventually, Tao recently showed ([44]) that "almost-every" orbit under the action of T attains "almost bounded" values. That is to say if we fix $f\colon \mathbf{Z}_+ \to \mathbf{R}$ such that $f(k) \to +\infty$ as $k \to +\infty$, then the lower bound of $\mathrm{orb}(k, T)$ is less than $f(k)$ for almost every $k \geq 1$ in the logarithmic density sense: if $A = \{k \geq 1; \inf_{n \geq 0} T^n(k) \leq f(k)\}$ then

$$\frac{\sum_{n \in A \cap \{1,\dots,N\}} 1/n}{\sum_{n \in \{1,\dots,N\}} 1/n} \xrightarrow[N \to +\infty]{} 1.$$

Our first contribution is a study of the Collatz Conjecture from the point of view of linear dynamics. We want to associate to the Collatz map T an operator $\mathcal{T}$ (on a space to be defined) and to link some dynamical properties of T to some dynamical properties of $\mathcal{T}$. This method is first observed in the works of Berg and Meinardus ([9]).

Indeed they have characterized the Collatz Conjecture in terms of functional equations. More precisely, they showed that the Collatz Conjecture is equivalent to the fact that the holomorphic solutions on $\mathbf{D}$ of the equation

$$h(z^3) = h(z^6) + \frac{1}{3z} \sum_{i=0}^{2} \lambda^i h(\lambda^i z^2),$$

where $\lambda = e^{2i\pi/3}$, are of the form $h(z) = h_0 + h_1 z/(1 - z)$. Neklyudov remarks in the article [36] that this characterization is equivalent to the following fact: the operator

$\mathcal{F}\colon \mathrm{Hol}(\mathbf{D}) \to \mathrm{Hol}(\mathbf{D})$, defined for every $f \in \mathrm{Hol}(\mathbf{D})$ with $f(z) = \sum_{n=0}^{\infty} c_n z^n$ by

$$\mathcal{F}f(z) = \sum_{n=0}^{\infty} c_{T(n)} z^n \quad \text{for every } z \in \mathbf{D},$$

admits 1 as an eigenvalue and its multiplicity is equal to 2.

Our study rely on the Neklyudov's paper ([36]), which associates to the map T another operator $\mathcal{T}$ defined on the classical Bergman space. It is possible to link the behavior of the orbits of $\mathcal{T}$ to the behavior of the orbits of T.

Definition 2.2.5. The space

$$\left\{ f\colon z \mapsto \sum_{n=0}^{\infty} c_n z^n \in \mathrm{Hol}(\mathbf{D});\, \|f\|^2 = \sum_{n=0}^{\infty} \frac{\pi\, |c_n|^2}{n+1} < +\infty \right\}$$

is denoted by $\mathcal{B}^2$ and called the *Bergman space.*

We now define the operator $\mathcal{T}$, which will be the subject of our study in the third chapter of this manuscript.

Definition 2.2.6. We denote by $\mathcal{X}$ the quotient $\mathcal{B}^2/\mathrm{span}[1, z, z^2]$ and we canonically identify a function $f \in \mathcal{B}^2$ with $f(z) = \sum_{n=0}^{\infty} c_n z^n$ with the function $\overline{f} \in \mathcal{X}$ such that $\overline{f}(z) = \sum_{n=3}^{\infty} c_n z^n$. We also denote by $\mathcal{T}$ the operator acting on $\mathcal{X}$ defined for every $f \in \mathcal{X}$ with $f(z) = \sum_{n=3}^{\infty} c_n z^n$ by

$$\mathcal{T}f(z) = \sum_{\{n\geq 3; T(n)\geq 3\}} c_n z^{T(n)} \quad \text{for every } z \in \mathbf{D}.$$

Since $T(0) = 0$, $T(1) = 2$ and $T(2) = 1$, we prefer to consider $\mathcal{T}$ as acting on $\mathcal{X}$ instead of $\mathcal{B}^2$ in order to avoid the fixed point 0 and the trivial cycle $(1, 2)$. The operator $\mathcal{T}$ is then well defined and is continuous on the space $\mathcal{X}$, and this will allow us to study its dynamical properties.

Proposition 2.2.7 ([36, Section 2])**.** The operator $\mathcal{T}$ is well defined and continuous on $\mathcal{X}$.

Proof. For every $f \in \mathcal{X}$ with $f(z) = \sum_{n=3}^{\infty} c_n z^n$ we have

$$\mathcal{T}f(z) = \sum_{\{n\geq 3; T(n)\geq 3\}} c_n z^{T(n)} = \sum_{k=3}^{\infty} \left(\sum_{\{j\geq 3; T(j)=k\}} c_j \right) z^k$$

and by the Cauchy-Schwarz inequality

$$\begin{aligned}
\|\mathcal{T}f\|^2 &= \sum_{k=3}^{\infty} \frac{\pi}{k+1} \left| \sum_{\{j\geq 3; T(j)=k\}} c_j \right|^2 \\
&= \sum_{k=3}^{\infty} \frac{\pi}{k+1} \left| \sum_{\{j\geq 3; T(j)=k\}} \frac{c_j}{\sqrt{j+1}} \sqrt{j+1} \right|^2 \\
&\leq \sum_{k=3}^{\infty} \frac{\pi}{k+1} \left(\sum_{\{j\geq 3; T(j)=k\}} \frac{|c_j|^2}{j+1} \right) \left(\sum_{\{j\geq 3; T(j)=k\}} j+1 \right) \\
&\leq \left(\sup_{k\geq 3} \frac{1}{k+1} \sum_{\{j\geq 3; T(j)=k\}} j+1 \right) \sum_{k=3}^{\infty} \sum_{\{j\geq 3; T(j)=k\}} \frac{\pi\,|c_j|^2}{j+1},
\end{aligned}$$

where

$$\begin{aligned}
\sup_{k\geq 3} \frac{1}{k+1} \sum_{\{j\geq 3; T(j)=k\}} j+1 &= \max_{r\in\{0,1,2\}} \left(\sup_{m\geq 1} \frac{1}{3m+r+1} \sum_{T(j)=3m+r} j+1 \right) \\
&= \max \left(\sup_{m\geq 1} \frac{6m+1}{3m+1}, \sup_{m\geq 1} \frac{6m+3}{3m+2}, \sup_{m\geq 1} \frac{6m+5+2m+2}{3m+3} \right) \\
&= \max \left(\frac{6}{3}, \frac{6}{3}, \frac{8}{3} \right) \\
&= \frac{8}{3}.
\end{aligned}$$

Thus $\|Tf\|^2 \leq 8/3 \cdot \|f\|^2$ for every $f \in \mathcal{X}$, hence T defines a continuous operator acting on $\mathcal{X}$. □

The main result obtained by Neklyudov concerning the dynamical properties of $\mathcal{T}$ can be stated as follows:

Theorem 2.2.8 ([36, Theorem 2.2])**.** If T has no non-trivial cycle, then $\mathcal{T}$ is hypercyclic.

Proof. It suffices to show that $\mathcal{T}$ satisfies the assumptions of Theorem 2.1.28. Consider $X_0 = Y_0 = \{\sum_{k=3}^{d} c_k z^k; d \geq 3, c_3, \ldots, c_d \in \mathbf{C}\}$. Fix $k \geq 3$. If T has no non-trivial cycle, the orbit of k under the action of T reaches 1 or diverges toward infinity. If the orbit reaches 1 then there exists $N \geq 0$ such that $T^n(k) \in \{1,2\}$ for every $n \geq N$. Hence $\mathcal{T}^n z^k = z^{T^n(k)} = 0$ in $\mathcal{X}$ for every $n \geq N$. If the orbit of k under the action of T diverges toward infinity then

$$\|\mathcal{T}^n z^k\|^2 = \|z^{T^n(k)}\|^2 = \frac{\pi}{T^n(k)+1} \xrightarrow[n\to+\infty]{} 0.$$

So we have proved that $\mathcal{T}^n z^k \to 0$ as $n \to +\infty$ for every $k \geq 3$. Thus $\mathcal{T}^n x \to 0$ as $n \to +\infty$ for every $x \in X_0$.

Consider the map $S\colon Y_0 \to Y_0$ defined by $Sz^k = z^{2k}$ for every $k \geq 3$. Then for every $k \geq 3$

$$\|S^n z^k\|^2 = \|z^{2^n k}\|^2 = \frac{\pi}{2^n k + 1} \xrightarrow[n\to+\infty]{} 0.$$

Thus $S^n y \to 0$ as $n \to +\infty$ for every $y \in Y_0$. Eventually $\mathcal{T}Sz^k = \mathcal{T}z^{2k} = z^k$ for every $k \geq 3$, so $\mathcal{T}Sy = y$ for every $y \in Y_0$. Hence $\mathcal{T}$ satisfies the Hypercyclicity Criterion, which gives that $\mathcal{T}$ is hypercyclic by Theorem 2.1.28. □

The assumption that T has no non-trivial cycle comes from the fact that the orbit of $k \geq 1$ under the action of T can admit three different behaviors. It can either reach the trivial cycle, which means that k satisfies the Collatz Conjecture, or it can reach a non-trivial cycle, or eventually it can diverge toward infinity. Thus by assuming that there is no such non-trivial cycle, Neklyudov has showed that $\mathcal{T}$ is hypercyclic thanks to the Hypercyclicity Criterion.

We already have presented in section 2.1.3 the role played by eigenvectors and eigenvalues in linear dynamics. In order to study the dynamics of $\mathcal{T}$, Neklyudov explicits some families of eigenvectors which, under some assumptions, span dense subspaces.

Proposition 2.2.9 ([36, Theorem 2.3])**.** For every $\mu \in \mathbf{C}$ such that $0 \leq |\mu| < \sqrt{2}$, the functions

$$h_m(\mu,\cdot)\colon z \mapsto \sum_{n=0}^{\infty} \mu^n (z^{(6m+4)2^n} - z^{(2m+1)2^n}) \quad \text{and} \quad h_0(\mu,\cdot)\colon z \mapsto \sum_{n=0}^{\infty} \mu^n z^{2^{n+2}}$$

belong to $\mathcal{X}$ and are, for every $m \geq 1$, eigenvectors for $\mathcal{T}$ associated to the eigenvalue μ.

Proof. For every $m \geq 1$ the sets $\{(6m+4)2^n; n \geq 0\}$ and $\{(2m+1)2^n; n \geq 0\}$ are disjoint sets. Thus for every $\mu \in \mathbf{C}$ and every $m \geq 1$

$$\|h_m(\mu,\cdot)\|^2 = \sum_{n=0}^{\infty} \frac{\pi\,|\mu|^{2n}}{(6m+4)2^n + 1} + \sum_{n=0}^{\infty} \frac{\pi\,|\mu|^{2n}}{(2m+1)2^n + 1} < +\infty$$

and

$$\|h_0(\mu,\cdot)\|^2 = \sum_{n=0}^{\infty} \frac{\pi\,|\mu|^{2n}}{2^{n+2} + 1} < +\infty$$

since $|\mu|^2 < 2$. Moreover for every $m \geq 1$ and every $z \in \mathbf{D}$

$$\begin{aligned}
\mathcal{T}h_m(\mu,\cdot)(z) &= \sum_{n=0}^{\infty} \mu^n \left(z^{T((6m+4)2^n)} - z^{T((2m+1)2^n)}\right) \\
&= \sum_{n=1}^{\infty} \mu^n \left(z^{(6m+4)2^{n-1}} - z^{(2m+1)2^{n-1}}\right) + z^{3m+2} - z^{3m+2} \\
&= \mu \sum_{n=0}^{\infty} \mu^n \left(z^{(6m+4)2^n} - z^{(2m+1)2^n}\right) \\
&= \mu h_m(\mu, z)
\end{aligned}$$

and

$$\mathcal{T}h_0(\mu,\cdot)(z) = \sum_{n=0}^{\infty} \mu^n z^{T(2^{n+2})} = \sum_{n=1}^{\infty} \mu^n z^{2^{n+1}} + z^2 = \mu \sum_{n=0}^{\infty} \mu^n z^{2^{n+2}} = \mu h_0(\mu, z).$$

□

We start by generalizing the setting of Neklyudov, and we consider weighted Bergman spaces.

Definition 2.2.10. We denote by $\mathcal{B}^2_\omega$, where $\omega\colon \mathbf{Z}_+ \to (0,+\infty)$ is a positive weight, the space

$$\left\{ f\colon z \mapsto \sum_{n=0}^{\infty} c_n z^n \in \mathrm{Hol}(\mathbf{D}); \|f\|^2_\omega = \sum_{n=0}^{\infty} \frac{|c_n|^2}{\omega(n)} < +\infty \right\},$$

which is called a *weighted Bergman space.* We also denote by $\mathcal{X}_\omega$ the quotient space $\mathcal{B}^2_\omega/\mathrm{span}[1,z,z^2]$ and still by $\mathcal{T}$ the operator acting on $\mathcal{X}_\omega$ defined for every $f \in \mathcal{X}_\omega$ with $f(z) = \sum_{n=3}^{\infty} c_n z^n$ by

$$\mathcal{T}f(z) = \sum_{\{n\geq 3; T(n)\geq 3\}} c_n z^{T(n)} \quad \text{for every } z \in \mathbf{D}.$$

We denote by ω_0 the weight such that $\omega_0(n) = (n+1)/\pi$ for every $n \geq 0$. This gives $\mathcal{X}_{\omega_0} = \mathcal{X}$; it will allow us to observe that our results generalize Neklyudov's ones.

Our first result, proved in the third chapter, characterizes the continuity of the operator $\mathcal{T}$ acting on the quotient of the weighted Bergman space $\mathcal{X}_\omega$.

Proposition A. The operator $\mathcal{T}$ is well defined and continuous on $\mathcal{X}_\omega$ if and only if the sequence

$$\left(\sum_{\{j\geq 3; T(j)=k\}} \frac{\omega(j)}{\omega(k)} \right)_{k\geq 3}$$

is bounded, that is to say if and only if the three sequences $(\omega(6m)/\omega(3m))_{m\geq 1}$, $(\omega(6m+2)/\omega(3m+1))_{m\geq 1}$ and $((\omega(6m+4)+\omega(2m+1))/\omega(3m+2))_{m\geq 1}$ are bounded. In this case we have

$$\begin{aligned} \|\mathcal{T}\|^2_\omega &= \sup_{k\geq 3} \sum_{\{j\geq 3; T(j)=k\}} \frac{\omega(j)}{\omega(k)} \\ &= \max\left\{ \sup_{m\geq 1} \frac{\omega(6m)}{\omega(3m)}, \sup_{m\geq 1} \frac{\omega(6m+2)}{\omega(3m+1)}, \sup_{m\geq 1} \frac{\omega(6m+4)+\omega(2m+1)}{\omega(3m+2)} \right\} \end{aligned}$$

and for every $n \geq 0$

$$\|\mathcal{T}^n\|^2_\omega = \sup_{k\geq 3} \sum_{\{j\geq 3; T^n(j)=k\}} \frac{\omega(j)}{\omega(k)}.$$

We will need in our work the following conditions implying that the eigenvectors $h_m(\mu,\cdot)$ explicited by Neklyudov ([36]) span a dense subspace of $\mathcal{X}_\omega$. This is proved in the third chapter and stated as follows:

Theorem B. If ω is bounded from below, then $h_m(\mu, \cdot)$ belongs to $\mathcal{X}_\omega$ for every $m \geq 0$ and every $\mu \in \mathbf{D}$. Moreover

$$\text{span}[h_m(\mu, \cdot); m \geq 0, \mu \in \mathbf{D}]$$

is dense in $\mathcal{X}_\omega$. This is in particular the case if $\omega = \omega_0$.

Using the same arguments, we can more precisely show that if $h_m(\mu, \cdot)$, $m \geq 0$, belongs to $\mathcal{X}_\omega$ for every $\mu \in A$ where A is a centered open disk, then the subspace

$$\text{span}[h_m(\mu, \cdot); m \geq 0, \mu \in \Delta]$$

is dense in $\mathcal{X}_\omega$ as soon as $\Delta \subset A$ has an accumulation point in A. We will also show in a similar way that the adjoint operator $\mathcal{T}^*$ has no eigenvalue.

Theorem C. If ω is bounded from below then $\sigma_p(\mathcal{T}^*) = \emptyset$. This is in particular the case if $\omega = \omega_0$.

Since the eigenvectors of $\mathcal{T}$ span a dense subspace of $\mathcal{X}_\omega$, we can generalize Theorem 2.2.8. Indeed we will have the tools to show, under a weak assumption on the weight ω, that $\mathcal{T}$ is hypercyclic thanks to the Hypercyclicity Criterion. Contrary to Theorem 2.2.8, note that we do not need any additional condition on the cycles of the Collatz map T.

Theorem D. If ω is bounded from below and if $\omega(k2^n) \to +\infty$ as $n \to +\infty$ for every $k \geq 3$ then $\mathcal{T}$ is hypercyclic on $\mathcal{X}_\omega$. This is in particular the case if $\omega = \omega_0$.

We give in the third chapter a condition on the weight ω which ensures that the operator $\mathcal{T}$ satisfies the Godefroy-Shapiro Criterion.

Theorem E. If ω is bounded from below and if there exists $\rho > 1$ such that for every $k \geq 3$ the sequence $(\rho^n/\omega(k2^n))_{n \geq 0}$ is bounded, then $\mathcal{T}$ satisfies the Godefroy-Shapiro Criterion. This is in particular the case if $\omega = \omega_0$.

If we make an additional assumption on the weight ω, the eigenvectors $h_m(\mu, \cdot)$ actually form a perfectly spanning set of eigenvectors associated to unimodular eigenvalues. This will allow us to prove that the operator $\mathcal{T}$ is ergodic with respect to a measure of full support.

Theorem F. If ω is bounded from below and if $\sum_{n=0}^{\infty} 1/\omega(k2^n) < +\infty$ for every $k \geq 3$, then $\mathcal{T}$ is ergodic with respect to a gaussian measure of full support in $\mathcal{X}_\omega$. This is in particular the case if $\omega = \omega_0$.

Since an ergodic operator, with respect to a measure of full support, is in particular frequently hypercyclic, we will obtain under the same assumption on the weight that $\mathcal{T}$ is frequently hypercyclic.

Theorem G. If ω is bounded from below and if $\sum_{n=0}^{\infty} 1/\omega(k2^n) < +\infty$ for every $k \geq 3$, then $\mathcal{T}$ is frequently hypercyclic on $\mathcal{X}_\omega$.

Eventually under the same assumption on the weight ω, the Hypercyclicity Criterion and the fact that eigenvectors $h_m(\mu, \cdot)$ associated to eigenvalues $\mu \in \{e^{i\alpha\pi}; \alpha \in \mathbf{Q}\}$ span a dense subspace in $\mathcal{X}_\omega$ will allow us to prove that the operator $\mathcal{T}$ is chaotic. Thus we give a positive answer to the question asked by Neklyudov ([36, Introduction]).

Theorem H. If ω is bounded from below and if $\sum_{n=0}^{\infty} 1/\omega(k2^n) < +\infty$ then $\mathcal{T}$ is chaotic on $\mathcal{X}_\omega$. This is in particular the case if $\omega = \omega_0$.

2.3 Invariant Subspace Problem and Bishop operators

We start this section by presenting some results concerning the Invariant Subspace Problem. This problem motivates the study of the dynamics of Bishop operators, which are the subject of the fourth chapter of this manuscript.

Let X be a complex, separable, infinite-dimensional Banach space.

Definition 2.3.1. Let $T \in \mathcal{B}(X)$ be an operator acting on X and let F be a closed subspace of X. We say that F is *non-trivial* if $F \neq \{0\}$ and if $F \neq X$. We say that F is an *invariant subspace for* T if $T(F) \subset F$. We say that F is a *hyperinvariant subspace for* T if F is invariant for every operator commuting with T.

The Invariant Subspace Problem is stated as follows.

Conjecture 2.3.2. Let T be a bounded linear operator acting on X. Does there exist a non-trivial closed subspace in X which is invariant for T?

We refer to the two books [40] of Radjavi and Rosenthal and [17] of Chalendar and Partington for overviews of this invariant subspace problem.

This famous problem in functional analysis has been open for more than a half century, despite very substantial (positive or negative) results obtained for numerous classes of operators or spaces.

Von Neumann and also Aronzajn and Smith ([2]) proved that a compact operator acting on a Banach space always admits a non-trivial invariant subspace. Then this result was generalized by Lomonosov ([33]) who showed that if an operator commutes with a non-zero compact operator, then it admits a non-trivial hyperinvariant subspace.

Nevertheless Enflo, and then Read, obtained counter-examples to the Invariant Subspace Problem in the case of Banach spaces. First Enflo showed in [21] the existence of a Banach space X and of an operator $T \in \mathcal{B}(X)$ that admits no non-trivial invariant subspace. Read also contructed in [41] on operator acting on a certain Banach space without any non-trivial invariant subspace, and then gave ([42]) an example of an operator which has no non-trivial invariant subspace on the Banach space of summable sequences ℓ^1. This was the first counter-example on a classical Banach space. Ten years later he constructed ([43]) quasinilpotent operators, that is to say such that the spectral radius $\lim_{n\to+\infty} \|T^n\|^{1/n}$ is zero, acting on $\ell^1(\mathbf{N})$ without any non-trivial invariant subspace.

The problem is still widely open in the reflexive case, and in particular in the case of Hilbert spaces. One of the most spectacular results in the Hilbertian case is the Brown-Chevreau-Pearcy's Theorem ([14]), which states that a contraction on a complex, separable, infinite-dimensional Hilbert space $\mathcal{H}$ whose spectrum contains the unit circle admits a non-trivial invariant subspace.

In this thesis, we will be interested in the Bishop operators. This family of operators was suggested as possible counter-examples of the Invariant Subspace Problem on the Hilbert space $L^2([0,1])$.

Definition 2.3.3 ([19])**.** For every $\alpha \in [0,1]$, we call *Bishop operator* the operator T_α acting on $L^2([0,1])$ and defined for every $f \in L^2([0,1])$ by

$$T_\alpha f(x) = xf(\{x+\alpha\}) \quad \text{almost everywhere on } [0,1],$$

where $\{\cdot\}$ denotes the fractional part of a real number.

We begin by observing that if α is a rationnal number r/q, T_α^q commutes with its adjoint operator and thus necessarily admits a non-trivial invariant subspace.

Definition 2.3.4 ([17, Definition 1.2.15])**.** An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *normal* if $TT^* = T^*T$.

We can prove by using the spectral theorem and a Borel functional calculus that a normal operator admits a non-trivial invariant subspace.

Theorem 2.3.5 ([17, Theorem 3.4.1])**.** An normal operator $T \in \mathcal{B}(\mathcal{H})$, that is not a multiple of the identity on $\mathcal{H}$, admits a non-trivial hyperinvariant subspace.

Proposition 2.3.6 ([17, Section 5.4])**.** Let $\alpha \in [0,1]$ be a rational number. The operator T_α admits a non-trivial hyperinvariant subspace.

Proof. Consider $\alpha = r/q$ where r and q are coprime. So for every $f \in L^2([0,1])$ and for almost every $x \in [0,1]$

$$T_\alpha^q f(x) = x\{x+r/q\}\dots\{x+(q-1)r/q\}f(\{x+qr/q\}) = x\{x+1/q\}\dots\{x+(q-1)/q\}f(x).$$

Then T_α^q is a multiplication operator M_ω by ω on $L^2([0,1])$, where $\omega\colon [0,1] \to [0,1]$ is defined by $\omega(x) = x\{x+1/q\}\dots\{x+(q-1)/q\}$. Thus the operator T_α^q is normal and is not a multiple of the identity on $L^2([0,1])$, hence it admits a non-trivial hyperinvariant subspace by Theorem 2.3.5. However a hyperinvariant subspace for T_α^q appears to be a hyperinvariant subspace for T_α since an operator commuting with T_α commutes in particular with T_α^q. So T_α admits a non-trivial hyperinvariant subspace. □

Thus we restrict the study of Bishop operators to the case where the parameter α is irrationnal. Davie proved ([19]) that if α is not a Liouville number, then T_α admits a non-trivial hyperinvariant subspace. We recall that a real number α is said to be a *Liouville number* if it can be well approached by rational numbers, in the following sense: there exists a sequence $(r_n/q_n)_{n\geq 1}$ of rational such that $|\alpha - r_n/q_n| < 1/q_n^n$ for every $n \geq 1$. It can be interesting to note at this point that we only know that T_α is not a counter-example to the Invariant Subspace Problem when α is a rational number or, when α is a non-Liouville number: these two classes of parameters are of a very different nature.

Some other authors tried to enlarge this set of parameters α such that T_α admits a non-trivial hyperinvariant subspace by trying to explicit some Liouville numbers α such that T_α has a non-trivial (hyper)invariant subspace. To do so, they were interested in the way an irrational number is approched by rational numbers.

Let us introduce the notions of continued fractions and of convergents, which are useful to find rational numbers approaching an irrational number. We refer to the book of Bugeaud [15] for further information.

Definition 2.3.7 ([15, Section 1.2]). Let $n \geq 0$ and let $(a_k)_{0\leq k\leq n}$ be integers such that $a_0 \in \mathbf{Z}$ and $a_k \geq 1$ for every $k \in \{1,\dots,n\}$. We denote by $[a_0; a_1, \dots, a_n]$ and we call *finite continued fraction* the rational

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_n}}}.$$

We apply an algorithm to any irrational number α in order to associate to it a sequence of finite continued fractions, which will be called convergents of α.

Definition 2.3.8 ([15, Definition 1.2]). Let $x \in [0,1]$ be an irrational number. Let $a_0 \in \mathbf{Z}$ and let $\xi_0 > 1$ such that $x = a_0 + 1/\xi_0$. Let $(a_k)_{k\geq 1}$ be a sequence of positive integers and $(\xi_k)_{k\geq 1}$ be a sequence of real numbers in $(1, +\infty)$ such that $\xi_k = a_{k+1} + 1/\xi_{k+1}$ for every $k \geq 0$. We define the sequences $(r_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$ in the following way: r_n and q_n are coprime and $r_n/q_n = [a_0; a_1, \dots, a_n]$ for every $n \geq 0$. The rational numbers $(r_n/q_n)_{n\geq 1}$ are called the *convergents of* x.

The following theorem claims that convergents satisfy some recurrence relations, which will be particularly useful to estimate the error done when we approach an irrational number by its convergents.

Theorem 2.3.9 ([15, Theorem 1.3]). Let $a_0 \in \mathbf{Z}_+$, and let three sequences of integers $(a_n)_{n\geq 1}$, $(r_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$ be such that $a_n \geq 1$, r_n and q_n are coprime, and $r_n/q_n = [a_0; a_1, \dots, a_n]$ for every $n \geq 1$. If we note $r_{-1} = 1$, $r_0 = a_0$, $q_{-1} = 0$ and $q_0 = 1$, then $r_n = a_n r_{n-1} + r_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$ for every $n \geq 1$.

Proof. Let us proceed by recursion. If $n = 1$ we have $r_1/q_1 = [a_0; a_1] = a_0 + 1/a_1 = (a_1 a_0 + 1)/a_1$, so $r_1 = a_1 a_0 + 1 = a_1 r_0 + r_{-1}$ and $q_1 = a_1 = a_1 q_0 + q_{-1}$ since $a_1 a_0 + 1$ and a_1 are coprime.

Suppose now that the property is true until a rank $n \geq 1$ for every sequence $(b_k)_{k\geq 1}$ of integers where $b_0 \in \mathbf{Z}$ and $b_k \geq 1$ for every $k \geq 1$. Remark that for every $k \in \{2, \dots, n+1\}$

$$\frac{r_k}{q_k} = [a_0; a_1, \dots, a_k] = a_0 + \frac{1}{[a_1; a_2, \dots, a_k]}.$$

Let $r'_1/q'_1, \dots, r'_n/q'_n$ be rational numbers such that $r'_j/q'_j = [a_1; a_2, \dots, a_{j+1}]$ for every $j \in \{1, \dots, n\}$ with $r'_{-1} = 1$, $r'_0 = a_1$, $q'_{-1} = 0$ and $q'_0 = 1$. Hence for every $k \in \{2, \dots, n+1\}$

$$\frac{r_k}{q_k} = a_0 + \frac{1}{[a_1; a_2, \dots, a_k]} = a_0 + \frac{q'_{k-1}}{r'_{k-1}} = \frac{a_0 r'_{k-1} + q'_{k-1}}{r'_{k-1}},$$

so $r_k = a_0 r'_{k-1} + q'_{k-1}$ and $q_k = r'_{k-1}$ since $a_0 r'_{k-1} + q'_{k-1}$ and r'_{k-1} are coprime. By assumption we have $r'_n = a_{n+1} r'_{n-1} + r'_{n-2}$ and $q'_n = a_{n+1} q'_{n-1} + q'_{n-2}$, hence

$$\begin{aligned} r_{n+1} &= a_0 r'_n + q'_n \\ &= a_0(a_{n+1} r'_{n-1} + r'_{n-2}) + a_{n+1} q'_{n-1} + q'_{n-2} \\ &= a_{n+1}(a_0 r'_{n-1} + q'_{n-1}) + a_0 r'_{n-2} + q'_{n-2} \\ &= a_{n+1} r_n + r_{n-1}. \end{aligned}$$

and

$$q_{n+1} = r'_n = a_{n+1}r'_{n-1} + r'_{n-2} = a_{n+1}q_n + q_{n-1}.$$

□

We can estimate the error done when we approach a real number by its convergents thanks to the relations they satisfy.

Theorem 2.3.10 ([15, Corollary 1.4])**.** Let x be an irrational number and let $(r_n/q_n)_{n\geq 1} = ([a_0; a_1, \ldots, a_n])_{n\geq 1}$ be its convergents. For every $n \geq 1$

$$\frac{1}{q_n(q_n + q_{n+1})} < \left| x - \frac{r_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

Proof. Let $(\xi_n)_{n\geq 0}$ be the sequence of elements in $(1, +\infty)$ such that $x = a_0 + 1/\xi_0$ and $\xi_n = a_{n+1} + 1/\xi_{n+1}$ for every $n \geq 0$. Then $x = [a_0; a_1, \ldots, a_n, \xi_n]$ for every $n \geq 1$. We will show that for every $n \geq 1$

$$x = [a_0; a_1, \ldots, a_n, \xi_n] = \frac{\xi_n r_n + r_{n-1}}{\xi_n q_n + q_{n-1}}.$$

First we have

$$x = [a_0; a_1, \xi_1] = a_0 + \frac{1}{a_1 + 1/\xi_1} = \frac{a_0(a_1 + 1/\xi_1) + 1}{a_1 + 1/\xi_1} = \frac{\xi_1(a_1 a_0 + 1) + a_0}{\xi_1 a_1 + 1} = \frac{\xi_1 r_1 + r_0}{\xi_1 q_1 + q_0},$$

and for every $n \geq 1$

$$\begin{aligned}
\frac{\xi_{n+1} r_{n+1} + r_n}{\xi_{n+1} q_{n+1} + q_n} &= \frac{\xi_{n+1}(a_{n+1} r_n + r_{n-1}) + r_n}{\xi_{n+1}(a_{n+1} q_n + q_{n-1}) + q_n} \\
&= \frac{(a_{n+1} + 1/\xi_{n+1}) r_n + r_{n-1}}{(a_{n+1} + 1/\xi_{n+1}) q_n + q_{n-1}} \\
&= \frac{\xi_n r_n + r_{n-1}}{\xi_n q_n + q_{n-1}} \\
&= \ldots \\
&= \frac{\xi_1 r_1 + r_0}{\xi_1 q_1 + q_0} \\
&= x.
\end{aligned}$$

Hence

$$\begin{aligned}
q_n \left(x - \frac{r_n}{q_n} \right) &= q_n \frac{\xi_n r_n + r_{n-1}}{\xi_n q_n + q_{n-1}} - r_n \\
&= \frac{\xi_n r_n q_n + r_{n-1} q_n - \xi_n r_n q_n - r_n q_{n-1}}{\xi_n q_n + q_{n-1}} \\
&= \frac{r_{n-1} q_n - r_n q_{n-1}}{\xi_n q_n + q_{n-1}}.
\end{aligned}$$

However for every $n \geq 1$

$$\begin{aligned} r_{n-1}q_n - r_nq_{n-1} &= r_{n-1}(a_nq_{n-1} + q_{n-2}) - (a_nr_{n-1} + r_{n-2})q_{n-1} \\ &= -(r_{n-2}q_{n-1} - r_{n-1}q_{n-2}) \\ &= \dots \\ &= (-1)^n(r_{-1}q_0 - r_0q_{-1}) \\ &= (-1)^n, \end{aligned}$$

hence

$$\left| x - \frac{r_n}{q_n} \right| = \frac{1}{q_n(\xi_nq_n + q_{n-1})}.$$

Moreover $\xi_n = a_{n+1} + 1/\xi_{n+1} \in (a_{n+1}, a_{n+1} + 1)$ so

$$\left| x - \frac{r_n}{q_n} \right| > \frac{1}{q_n((a_{n+1} + 1)q_n + q_{n-1})} = \frac{1}{q_n(q_n + q_{n+1})}$$

and

$$\left| x - \frac{r_n}{q_n} \right| < \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} = \frac{1}{q_nq_{n+1}}.$$

□

Flattot ([22]) was the first to succeed in expliciting some Liouville numbers α such that T_α admits a non-trivial hyperinvariant subspace.

Theorem 2.3.11 ([22, Theorem 5.3])**.** Let $\alpha \in [0, 1]$ be an irrational number. Suppose that there exist rational numbers $(r_n/q_n)_{n\geq 0}$ such that $1 \leq q_n \leq n \log(n)^{-3}$ and such that $|\alpha - r_n/q_n| \leq 1/q_n^2$ for every $n \geq 0$. Moreover if $|\alpha - r_n/q_n| \geq e^{-q_n^{1/3}}$ when n is sufficiently large enough, then T_α admits a non-trivial hyperinvariant subspace.

More recently, Chamizo, Gallardo-Guttiérrez, Monsalve-López and Ubis ([18]) generalized the result obtained by Flattot and obtained a condition on the growth of the denominators of the convergents of an irrational number α that implies the existence of a non-trivial hyperinvariant subspace.

Theorem 2.3.12 ([18, Theorem 3.7])**.** Let $\alpha \in [0, 1]$ be an irrational number and let $(r_n/q_n)_{n\geq 0}$ be the sequence of its convergents. If $\log(q_{n+1}) = O(q_n/\log(q_n)^3)$ then T_α admits a non-trivial hyperinvariant subspace.

According to the approach taken by Davie who showed that T_α admits a non-trivial hyperinvariant subspace as soon as α is not a Liouville number, the assumptions of the generalizations above require a control of the growth of the denominators of the convergents of the irrational number α.

These Bishop operators can be more generally seen as weighted translation operators.

Definition 2.3.13 ([22, Introduction])**.** Let $\phi \in L^\infty([0, 1])$ be a bounded function and let $\alpha \in [0, 1]$. We respectively call *multiplication operator* and *translation operator* the operators M_ϕ and U_α defined on $L^2([0,1])$ by

$$M_\phi f = \phi f \quad \text{and} \quad U_\alpha f = f(\{\cdot + \alpha\}) \quad \text{for every } f \in L^2([0,1]).$$

We then call *weighted translation operator* the operator $T_{\phi,\alpha}$ defined for every $f \in L^2([0,1])$ by

$$T_{\phi,\alpha} f(x) = M_\phi U_\alpha f(x) = \phi(x) f(\{x + \alpha\}) \quad \text{almost everywhere on } [0,1].$$

We will now be interested in the dynamical properties of Bishop operators, and more generally of weighted translation operators. This study is motivated by the Invariant Subspace Problem: indeed the closure of the subspace spanned by an orbit under the action of an operator T is an invariant subspace. Therefore an operator $T \in \mathcal{B}(X)$ does not admit any non-trivial invariant subspace if and only if the subspace spanned by the orbit $\{T^n x; n \geq 0\}$ is dense in X for every $x \in X \backslash \{0\}$. That means that an operator is a counter-example to the Invariant Subspace Problem if and only if every non-zero vector in X is cyclic for this operator.

To study the dynamical properties of Bishop operators, we can consider them acting on $L^p([0,1])$ with $1 < p < +\infty$. We will denote by m the Lebesgue measure on $[0,1]$.

We first remark that T_α is a contraction, that is to say that $\|T_\alpha\| \leq 1$, for every $\alpha \in [0,1]$. So every orbit under the action of T_α is bounded, which prevents T_α from being hypercyclic.

Proposition I. For every $\alpha \in [0,1]$, the Bishop operators T_α is not hypercyclic.

Since $\|T_{\phi,\alpha}\| = \|\phi\|_\infty$, the same argument shows that a weighted translation operator $T_{\phi,\alpha}$ is not hypercyclic if $\|\phi\|_\infty \leq 1$.

Proposition J. Let $\alpha \in [0,1]$ and let $\phi \in L^\infty([0,1])$. If $\|\phi\|_\infty \leq 1$ or if $m(\{\phi = 0\}) > 0$, then $T_{\phi,\alpha}$ is not hypercyclic.

In the general case, we will show in the fourth chapter that weighted translation operators cannot satisfy the Hypercyclicity Criterion, thanks to its equivalence with the Gethner-Shapiro Criterion.

Theorem K. For every $\alpha \in [0,1]$ and every $\phi \in L^\infty([0,1])$, the operator $T_{\phi,\alpha}$ does not satisfy the Hypercyclicity Criterion.

The fact that the weight appearing in the definition of Bishop operators $\phi(x) = x$ is almost everywhere postive prevents Bishop operators from being supercyclic. To show such a result, we will need the Positive Supercyclicity Theorem, which is stated as follows.

Theorem 2.3.14 (Positive Supercyclicity Theorem, [4, Corollary 3.4])**.** Let $T \in \mathcal{B}(X)$ be an operator such that $\sigma_p(T^*) = \emptyset$. A vector $x \in X$ is supercyclic for T if and only if $\{aT^n x; n \geq 0, a > 0\}$ is dense in X.

Davie and Flattot showed that a Bishop operator has no eigenvalue. Adjustments of their proof allow us to show that the adjoint operator T_α^* has no eigenvalue as well. We denote by p' the real number such that $1/p + 1/p' = 1$.

Theorem 2.3.15 ([19, Theorem 2])**.** For every $\alpha \in [0,1]$, the operators $T_\alpha \in \mathcal{B}(L^p([0,1]))$ and $T_\alpha^* \in \mathcal{B}(L^{p'}([0,1]))$ have no eigenvalue.

We will now be able to prove that a Bishop operator cannot be supercyclic.

Theorem L. For every $\alpha \in [0,1]$, the Bishop operator T_α is not supercyclic.

This approach can be generalized to the case of weighted translation operators. The fact that $T^*_{\phi,\alpha}$ has no eigenvalue will be proved in a similar way, under the additional assumption that ϕ is an increasing convex function.

Theorem 2.3.16 ([22, Theorem 2.1])**.** Let $\alpha \in [0,1]$ and let $\phi \in L^\infty([0,1])$ be an increasing convex function such that $\phi(0) = 0$. The weighted translation operators $T_{\phi,\alpha} \in \mathcal{B}(L^p([0,1]))$ and $T^*_{\phi,\alpha} \in \mathcal{B}(L^{p'}([0,1]))$ have no eigenvalue.

Since the adjoint operator $T^*_{\phi,\alpha}$ has no eigenvalue and since the weight ϕ is almost everywhere positive, we are able to prove that weighted translation operators are not supercyclic either.

Theorem M. Let $\alpha \in [0,1]$ and let $\phi \in L^\infty([0,1])$ be an increasing convex function such that $\phi(0) = 0$. The weighted translation operator $T_{\phi,\alpha}$ is not supercyclic.

In the general case of weighted translation operators, we will show that they cannot satisfy the Supercyclicity Criterion. To do so we will need a Gethner-Shapiro-type Supercyclicity Criterion. By adjusting the arguments of the known proof of the equivalence between the Hypercyclicity Criterion and the Gethner-Shapiro Criterion, we will prove that the Supercyclicity Criterion is equivalent to such a criterion.

Theorem N. An operator $T \in \mathcal{B}(X)$ satisfies the assumptions of the Supercyclicity Criterion if and only if it satisfies the following assumptions: there exist X_0 and Y_0 two dense subsets of X, an increasing sequence $(n_k)_{k\geq 0}$ of integers, a sequence $(\lambda_{n_k})_{k\geq 0}$ of non-zero complex numbers and a map $S\colon Y_0 \to Y_0$ such that:

(i) $\lambda_{n_k} T^{n_k} x \to 0$ as $k \to +\infty$ for every $x \in X_0$;

(ii) $\lambda_{n_k}^{-1} S^{n_k} y \to 0$ as $k \to +\infty$ for every $y \in Y_0$;

(iii) $TSy = y$ for every $y \in Y_0$.

It will then be possible to show that no weighted translation operator can satisfy the Supercyclicity Criterion.

Theorem O. For every $\alpha \in [0,1]$ and every $\phi \in L^\infty([0,1])$ the weighted translation operator $T_{\phi,\alpha}$ does not satisfy the Supercyclicity Criterion.

We are now interested in the study of the cyclicity of Bishop operators T_α. It has already been studied by Chalendar and Partington in [16] and characterized when α is rational. We now present these results.

Definition 2.3.17 ([17, Definition 5.4.1])**.** Let $\alpha = r/q$ be a rational such that r and q are coprime and $0 < r < q$. We denote by $\Delta(f, r/q)$ the function defined for almost every $t \in [0,1]$ by

$$\begin{vmatrix} f(t) & T_{r/q} f(t) & \dots & T_{r/q}^{q-1} f(t) \\ f(\{t + r/q\}) & T_{r/q} f(\{t+r/q\}) & \dots & T_{r/q}^{q-1} f(\{t + r/q\}) \\ \vdots & \vdots & & \vdots \\ f(\{t + (q-1)r/q\}) & T_{r/q} f(\{t + (q-1)r/q\}) & \dots & T_{r/q}^{q-1} f(\{t + (q-1)r/q\}) \end{vmatrix}.$$

The characterization obtained by Chalendar and Partington claims that a function is cyclic for T_α if and only if $\Delta(f,\alpha)$ almost never vanishes.

We recall that m denotes here the Lebesgue measure on $[0,1]$.

Theorem 2.3.18 ([17, Theorem 5.4.4])**.** Let $\alpha = r/q$ be a rational such that r and q are coprime and $0 < r < q$. A function $f \in L^p([0,1])$ is cyclic for $T_{r/q}$ if and only if the function $\Delta(f, r/q)$ satisfies $m(\{t \in [0,1]; \Delta(f, r/q)(t) = 0\}) = 0$.

Using this theorem, we will be able to show in the last chapter of this thesis that all Bishop operators T_α, with α rational, are cyclic and have common cyclic vectors.

Theorem P. Any holomorphic function f on a open neighborhood of $[0,1]$ such that $f(0) \neq 0$ is cyclic for T_α for every $\alpha \in (0,1) \cap \mathbf{Q}$.

In particular, for every $\alpha \in (0,1) \cap \mathbf{Q}$ the Bishop operator T_α is cyclic and the constant function $\mathbf{1}$ is a common cyclic vector for these operators.

Our aim is now to prove the cyclicity of Bishop operators for some irrational parameters α. To do so we will use the Kuratwoski-Ulam Theorem, whose proof requires an introduction of the Baire property. We say that a set satisfies it if it is an open set, up to a meager set.

Definition 2.3.19 ([29, Definition 8.21])**.** Let $\mathcal{X}$ be a Polish space. We say that a subset $A \subset \mathcal{X}$ *satisfies the Baire property* if there exists an open set U in $\mathcal{X}$ such that $A \Delta U = (A \backslash U) \cup (U \backslash A)$ is a meager set in $\mathcal{X}$.

To show the Kuratowski-Ulam Theorem, remark that the complement of a set with the Baire property also has the Baire property, and that a co-meager set necessarily satisfies the Baire property.

Proposition 2.3.20 ([29, Section 8.F])**.** Let $\mathcal{X}$ be a Polish space and let A be a subset of $\mathcal{X}$. The subset A satisfies the Baire property if and only if $\mathcal{X} \backslash A$ satisfies it. If A is a co-meager set in $\mathcal{X}$, then A satisfies the Baire property.

Proof. Suppose that A satisfies the Baire property. Then there exists an open set U in $\mathcal{X}$ such that $M = A \Delta U$ is a meager set in $\mathcal{X}$. Thus $A = U \Delta M$ and

$$(\mathcal{X} \backslash \overline{U}) \Delta (\partial U \Delta M) = \mathcal{X} \backslash (\overline{U} \Delta (\partial U \Delta M)) = \mathcal{X} \backslash ((\overline{U} \Delta \partial U) \Delta M) = \mathcal{X} \backslash (U \Delta M) = \mathcal{X} \backslash A.$$

However $\mathcal{X} \backslash \overline{U}$ is an open set in $\mathcal{X}$ and $\partial U \Delta M$ is a meager set in $\mathcal{X}$ since $\partial U \Delta M \subset \partial U \cup M$ where ∂U is a meager set in $\mathcal{X}$ because $\mathcal{X} \backslash \partial U$ is a dense open set. So $\mathcal{X} \backslash A$ satisfies the Baire property.

Moreover if A is a co-meager set in $\mathcal{X}$ then $A \Delta \mathcal{X} = \mathcal{X} \backslash A$ where $\mathcal{X}$ is an open set in $\mathcal{X}$ and $\mathcal{X} \backslash A$ is a meager set in $\mathcal{X}$. So A satisfies the Baire property. □

We have now the necessary tools to show the Kuratowski-Ulam Theorem.

Theorem 2.3.21 (Kuratowski-Ulam, [29, Theorem 8.41])**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be two Polish spaces and let A be a subset of $\mathcal{X} \times \mathcal{Y}$ satisfying the Baire property. The subset A is a co-meager set in $\mathcal{X} \times \mathcal{Y}$ if and only if

$$\{x \in \mathcal{X}; \{y \in \mathcal{Y}; (x,y) \in A\} \text{ is co-meager in } \mathcal{Y}\} \text{ is co-meager in } \mathcal{X}$$

and if and only if

$$\{y \in \mathcal{Y}; \{x \in \mathcal{X}; (x, y) \in A\} \text{ is co-meager in } \mathcal{X}\} \text{ is co-meager in } \mathcal{Y}.$$

Proof. Since $\mathcal{X}$ and $\mathcal{Y}$ play the same role, let us show the first equivalence. For every $(x, y) \in \mathcal{X} \times \mathcal{Y}$ we consider the continuous maps $\varphi_x \colon \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ and $\phi_y \colon \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ defined by $\varphi_x(z) = (x, z)$ for every $z \in \mathcal{Y}$ and by $\phi_y(z) = (z, y)$ for every $z \in \mathcal{X}$. Hence for every susbset $B \subset \mathcal{X} \times \mathcal{Y}$ and every $(x, y) \in \mathcal{X} \times \mathcal{Y}$ we have $\{z \in \mathcal{Y}; (x, z) \in B\} = \varphi_x^{-1}(B)$ and $\{z \in \mathcal{X}; (z, y) \in B\} = \phi_y^{-1}(B)$.

Suppose that A is a co-meager set in $\mathcal{X} \times \mathcal{Y}$. By definition there exists a sequence $(A_n)_{n \geq 0}$ of dense open sets in $\mathcal{X} \times \mathcal{Y}$ such that $\cap_{n \geq 0} A_n \subset A$. For every $x \in \mathcal{X}$, by Baire Category Theorem, the G_δ-set $\varphi_x^{-1}(\cap_{n \geq 0} A_n) = \cap_{n \geq 0} \varphi_x^{-1}(A_n)$ is dense in $\mathcal{Y}$ if and only if $\varphi_x^{-1}(A_n)$ is dense in $\mathcal{Y}$ for every $n \geq 0$ since φ_x is continuous. So

$$\left\{ x \in \mathcal{X}; \varphi_x^{-1} \left(\bigcap_{n \geq 0} A_n \right) \text{ is dense} \right\} = \bigcap_{n \geq 0} \{x \in \mathcal{X}; \varphi_x^{-1}(A_n) \text{ is dense}\}$$

and

$$\bigcap_{n \geq 0} \{x \in \mathcal{X}; \varphi_x^{-1}(A_n) \text{ is dense}\} \subset \left\{ x \in \mathcal{X}; \varphi_x^{-1}(A) \text{ is co-meager} \right\}$$

because φ_x is continuous for every $x \in \mathcal{X}$ and

$$\bigcap_{n \geq 0} \varphi_x^{-1}(A_n) = \varphi_x^{-1} \left(\bigcap_{n \geq 0} A_n \right) \subset \varphi_x^{-1}(A).$$

It suffices then to prove that $\{x \in \mathcal{X}; \varphi_x^{-1}(A_n) \text{ is dense}\}$ is a co-meager set in $\mathcal{X}$ for every $n \geq 0$. Fix $n \geq 0$. Since $\mathcal{Y}$ is separable, consider a basis $(V_p)_{p \geq 0}$ of non-empty open sets in $\mathcal{Y}$ and consider $U_p = \{x \in \mathcal{X}; \exists y \in V_p, (x, y) \in A_n\}$. First U_p is an open set in $\mathcal{X}$ since ϕ_y is continuous for every $y \in \mathcal{Y}$ and

$$U_p = \bigcup_{y \in V_p} \phi_y^{-1}(A_n).$$

Besides U_p is dense in $\mathcal{X}$. Indeed let U be a non-empty open set in $\mathcal{X}$. The set $U \times V_p$ is a non-empty open set in $\mathcal{X} \times \mathcal{Y}$, so by density of A_n there exists $(x, y) \in A_n \cap (U \times V_p)$. So $x \in U_p \cap U$, hence U_p is dense in $\mathcal{X}$. Let us show the inclusion

$$\bigcap_{p \geq 0} U_p \subset \{x \in \mathcal{X}; \varphi_x^{-1}(A_n) \text{ is dense}\}.$$

Let $x \in \cap_{p \geq 0} U_p$ and $p \geq 0$. There exists $y \in V_p$ such that $(x, y) \in A_n$ since $x \in U_p$. So $y \in V_p \cap \varphi_x^{-1}(A_n)$, hence $\varphi_x^{-1}(A_n) \cap V_p \neq \emptyset$ for every $p \geq 0$. Therefore $\varphi_x^{-1}(A_n)$ is dense for every $n \geq 0$, which implies that $\{x \in \mathcal{X}; \varphi_x^{-1}(A) \text{ is co-meager}\}$ is a co-meager set in $\mathcal{X}$.

Conversely suppose that $\{x \in \mathcal{X}; \varphi_x^{-1}(A) \text{ is co-meager}\}$ is a co-meager set in $\mathcal{X}$. Suppose by contradiction that A is not a co-meager set in $\mathcal{X} \times \mathcal{Y}$ and note $A' = (\mathcal{X} \times \mathcal{Y}) \backslash A$. Then A' is not a meager set and A' satisfies the Baire property since A does by Proposition 2.3.20. So there exist an open set B in $\mathcal{X} \times \mathcal{Y}$ and a meager set in $\mathcal{X} \times \mathcal{Y}$ such that $A' = B\Delta M$. The open set B cannot be a meager set since $A' \subset B \cup M$, which would make A' a meager set in $\mathcal{X} \times \mathcal{Y}$. The spaces $\mathcal{X}$ and $\mathcal{Y}$ being separable, there exist a non-empty open set U in $\mathcal{X}$ and a non-empty open set V in $\mathcal{Y}$ such that $U \times V \subset B$. By assumption $C_1 = \{x \in \mathcal{X}; \varphi_x^{-1}(A') \text{ is meager}\}$ is a co-meager set in $\mathcal{X}$. Besides $M' = (\mathcal{X} \times \mathcal{Y})\backslash M$ is a co-meager set in $\mathcal{X} \times \mathcal{Y}$, so it satisfies the Baire property. Hence by the first implication already proved, the set $\{x \in \mathcal{X}; \varphi_x^{-1}(M') \text{ is co-meager}\}$ is a co-meager set in $\mathcal{X}$, which implies that $C_2 = \{x \in \mathcal{X}; \varphi_x^{-1}(M) \text{ is meager}\}$ is a co-meager set in $\mathcal{X}$. So $C_1 \cap C_2$ is a co-meager set in $\mathcal{X} \times \mathcal{Y}$. It is in particular dense in $\mathcal{X} \times \mathcal{Y}$ and intersects the non-empty open set U. Hence there exists $x \in U$ such that $\varphi_x^{-1}(A')$ and $\varphi_x^{-1}(M)$ are meager sets in $\mathcal{Y}$. However $A' = B\Delta M$ so $\varphi_x^{-1}(A') = \varphi_x^{-1}(B)\Delta\varphi_x^{-1}(M)$. Besides $V \subset \varphi_x^{-1}(B)$ since $U \times V \subset B$, hence $V\backslash\varphi_x^{-1}(M) \subset \varphi_x^{-1}(B)\backslash\varphi_x^{-1}(M) \subset \varphi_x^{-1}(A')$. Therefore $V \subset \varphi_x^{-1}(A') \cup \varphi_x^{-1}(M)$, which implies that V is a meager set in $\mathcal{Y}$. It contradicts the fact that V is a non-empty open set in $\mathcal{Y}$, which implies that A is a co-meager set in $\mathcal{X} \times \mathcal{Y}$. □

The Kuratowski-Ulam Theorem allows us to show that there exist irrational numbers α such that T_α is cyclic, which was unknown until now. More precisely, the following theorem is proved in the fourth chapter.

Theorem Q. The set $\{\alpha \in [0,1]; T_\alpha \text{ is cyclic}\}$ is a co-meager set in $[0,1]$.

This is proved by showing that the set

$$\{(\alpha, f) \in [0,1] \times L^p([0,1]); f \text{ is cyclic for } T_\alpha\}$$

is a co-meager subset of $[0,1] \times L^p([0,1])$ and by applying the Kuratowski-Ulam theorem.

Theorem Q allows us to show the existence of irrational parameters α such that T_α is cyclic, but does not provide explicit examples. A second goal of the fourth chapter is to obtain such examples of irrational parameters α.

By using the estimates of the error done by approaching an irrational number by its convergents, we show that T_α is cyclic as soon as the sequence of denominators of the convergents of α admits infinitely many sufficiently large gaps. More presicely, in this case if f is cyclic for every Bishop operator T_β with $\beta \in \mathbf{Q}$ then f is cyclic for T_α.

Theorem R. Let $f \in L^p([0,1])$ be a common cyclic function to all Bishop operators T_α where $\alpha \in (0,1)$ is rational. There exists a function $\psi_f \colon \mathbf{N} \to \mathbf{R}_+$ satisfying the following property: if $(r_n/q_n)_{n\geq 0}$ are the convergents of an irrational number $\alpha \in [0,1]$ and if for every $n \geq 0$ there exists $n_0 \geq n$ such that $q_{n_0+1} > \psi_f(q_{n_0})$, then f is cyclic for T_α.

The assumption above could require a strong growth of the denominators of the convergents, while Theorems 2.3.11 and 2.3.12 assume that their growth is controlled. This can be explained by the fact that cyclicity and admitting a non-trivial invariant subspace can be seen as opposed properties: an operator T has no non-trivial invariant subspace if and only if every non-zero vector is cyclic for T.

Our aim is now to generalize Theorem R to the case of weighted translation operators $T_{\phi,\alpha}$. To do so, we first search for a characterization of a cyclic function for $T_{\phi,\alpha}$, where α is rational, which is similar to the characterization from Chalendar and Partington for Bishop operators.

Definition 2.3.22. Let $\phi \in L^p([0,1])$ and let $\alpha = r/q$ be a rational number such that r and q are coprime and $0 < r < q$. The function $\Delta_\phi(f, r/q)$ is defined almost everywhere on $[0,1]$ by

$$\begin{vmatrix} f(t) & T_{\phi,\alpha}f(t) & \dots & T_{\phi,\alpha}^{q-1}f(t) \\ f(\{t+r/q\}) & T_{\phi,\alpha}f(\{t+r/q\}) & \dots & T_{\phi,\alpha}^{q-1}f(\{t+r/q\}) \\ \vdots & \vdots & & \vdots \\ f(\{t+(q-1)r/q\}) & T_{\phi,\alpha}f(\{t+(q-1)r/q\}) & \dots & T_{\phi,\alpha}^{q-1}f(\{t+(q-1)r/q\}) \end{vmatrix}.$$

We obtain a generalization of the characterization of Chalendar and Partington, stated as follows.

Theorem S. Let $\phi \in L^\infty([0,1])$ and let $\alpha = r/q$ be a rational number such that r and q are coprime and $0 < r < q$. A function $f \in L^p([0,1])$ is cyclic for $T_{\phi,\alpha}$ if and only if the function $\Delta_\phi(f, r/q)$ satisfies $m(\{t \in [0,1]; \Delta_\phi(f, r/q)(t) = 0\}) = 0$.

With this characterization at hands, and with additional assumptions on the weight ϕ, we observe that there are common cyclic functions for all operators $T_{\phi,\alpha}$, where α is rational.

Theorem T. Let $\phi \in \mathcal{C}([0,1], \mathbf{R})$ be an increasing function satisfying $\phi(0) = 0$ and which is holomorphic on an open neighborhood of $[0,1]$. Any holomorphic function f on an open neighborhood of $[0,1]$ such that $f(0) \neq 0$ is cyclic for $T_{\phi,\alpha}$ for every $\alpha \in (0,1) \cap \mathbf{Q}$.

In particular for every $\alpha \in (0,1) \cap \mathbf{Q}$ the Bishop operator $T_{\phi,\alpha}$ is cyclic and the constant function $\mathbf{1}$ is a common cyclic vector for these operators.

Under the same assumptions on the weight ϕ and if the sequence of denominators of the convergents of an irrational number α admits infinitely many sufficiently large gaps, we show that $T_{\phi,\alpha}$ is cyclic.

Theorem U. Let $\phi \in \mathcal{C}([0,1], \mathbf{R})$ be an increasing function satisfying $\phi(0) = 0$ and which is holomorphic on an open neighborhood of $[0,1]$. Let $f \in L^p([0,1])$ be a common cyclic vector for $T_{\phi,\alpha}$ for every $\alpha \in (0,1) \cap \mathbf{Q}$. There exists a function $\psi_{\phi,f} \colon \mathbf{N} \to \mathbf{R}_+$ with the following property: if $(r_n/q_n)_{n \geq 0}$ are the convergents of an irrational number α in $[0,1]$ and if for every $n \geq 0$ there exists $n_0 \geq n$ such that $q_{n_0+1} > \psi_{\phi,f}(q_{n_0})$, then f is cyclic for $T_{\phi,\alpha}$.

In Theorem R (or Theorem U), the function ψ_f (or $\psi_{\phi,f}$) is not explicit. Take the case where $\alpha = 1/q$ and $f = \mathbf{1}$. In the approach we followed in order to explicit the function ψ_f, we needed at some point to estimate from below on $[0, 1/q)$ the modulus of the determinant

$$\Delta(\mathbf{1}, 1/q)(x) = \begin{vmatrix} 1 & x & \dots & x\{x+1/q\}\dots\{x+(q-2)/q\} \\ 1 & \{x+1/q\} & \dots & \{x+1/q\}\{x+2/q\}\dots\{x+(q-1)/q\} \\ \vdots & \vdots & & \vdots \\ 1 & \{x+(q-1)/q\} & \dots & \{x+(q-1)/q\}x\dots\{x+(q-3)/q\} \end{vmatrix}.$$

Computer simulations seem to show that this determinant does not vanish and is strictly monotonic on $[0, 1/q)$. Nonetheless we were not able to prove it, which prevented us from expliciting a function ψ_f.

Chapter 3

Collatz Conjecture and linear dynamics

The following manuscript [1] has been submitted to the Proceedings of the American Mathematical Society in February 2023. The proof of the Lemma 3.2.8 has been added for the sake of completeness.

Abstract. In this paper, we study the dynamics of an operator $\mathcal{T}$ naturally associated to the so-called *Collatz map*, which maps an integer $n \geq 0$ to $n/2$ if n is even and $3n+1$ if n is odd. This operator $\mathcal{T}$ is defined on certain weighted Bergman spaces $\mathcal{B}^2_\omega$ of analytic functions on the unit disk. Building on previous work of Neklyudov, we show that $\mathcal{T}$ is hypercyclic on $\mathcal{B}^2_\omega$, independently of whether the Collatz Conjecture holds true or not. Under some assumptions on the weight ω, we show that $\mathcal{T}$ is actually ergodic with respect to a Gaussian measure with full support, and thus frequently hypercyclic and chaotic.

3.1 Introduction

Our aim in this paper is to investigate from the point of view of linear dynamics the properties of an operator naturally associated to the so-called Collatz map. This map $T_0\colon \mathbf{Z}_+ \to \mathbf{Z}_+$ is defined as

$$T_0(n) = \begin{cases} n/2 & \text{if } n \text{ is even;} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases}$$

The dynamics of this simple-looking map are still very mysterious, and the famous Collatz Conjecture (called also the $3n+1$-Conjecture, or the Syracuse Conjecture) states that the orbit of any integer $k \geq 1$ under the action of T_0 eventually reaches the point 1.

We will use in this paper the following modified Collatz map $T\colon \mathbf{Z}_+ \to \mathbf{Z}_+$, defined as

$$T(n) = \begin{cases} n/2 & \text{if } n \text{ is even;} \\ (3n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

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whose dynamics are the same as those of T_0 as far as the Collatz Conjecture is concerned.

We will here give some of the etablished results about the Collatz Conjecture. We refer the reader to the survey [31] written by Lagarias for an overview of this problem.

The conjecture had been empirically verified for every $k \leq 20 \times 10^9$ and is now known to be true for every $k \leq 20 \times 2^{58} \approx 5.764 \times 10^{18}$ ([37]).

Since $T(1) = 2$ and $T(2) = 1$, $(1, 2)$ is called the trivial cycle of T and Eliahou showed in [20] that a non-trivial cycle of T must have a length greater than 10 439 860 591.

A possible approach to the Collatz Conjecture is to study the proportion of positive integers k having a T-orbit that reaches 1. Krasikov and Lagarias proved for instance in [30] that if X is sufficiently large, the number of integers $1 \leq k \leq X$ whose T-orbit reaches 1 is at least $X^{0.84}$.

It is also of interest to investigate the time it takes a T-orbit to reach 1. Applegate and Lagarias proved in [1] that infinitely many positive integers k have a T-orbit that reaches 1 in at least $6.143 \log(k)$ steps.

Moreover, Tao showed in [44] that "almost all" T-orbits attain almost bounded values. That is to say, if we fix a function $f\colon \mathbf{N} \to \mathbf{R}$ such that $f(k) \to +\infty$ as $k \to +\infty$, the infinimum of the T-orbit of k is less than $f(k)$ for almost every integer $k \geq 1$, in the sense of logarithmic density.

It is possible to associate to T in a natural way bounded operators on some Hilbert spaces of analytic functions on the unit disk $\mathbf{D}$, and to link the dynamics of T to those of these operators. An approach of this kind was first proposed by Berg and Meinardus in [9]. Consider the operator $\mathcal{F}$ defined on the space $Hol(\mathbf{D})$ of holomorphic functions on the unit disk $\mathbf{D}$ in the following way: for every $f \in Hol(\mathbf{D})$ with $f(z) = \sum_{n=0}^{\infty} c_n z^n$,

$$\mathcal{F}f(z) = \sum_{n=0}^{\infty} c_{T(n)} z^n, \quad z \in \mathbf{D}.$$

Berg and Meinardus expressed in [9] the Collatz Conjecture in terms of functional equations and obtained that it is equivalent to the fact that 1 is an eigenvalue of $\mathcal{F}$ of multiplicity 2.

Then Neklyudov took in [36] a different perspective, and considered the adjoint $\mathcal{T}$, for the Hardy space $\mathcal{H}^2(\mathbf{D})$, of the operator $\mathcal{F}$: for every $f \in \mathcal{H}^2(\mathbf{D})$ with $f(z) = \sum_{n=0}^{\infty} c_n z^n$,

$$\mathcal{T}f(z) = \sum_{n=0}^{\infty} c_n z^{T(n)}, \quad z \in \mathbf{D}.$$

This operator $\mathcal{T}$ is considered in [36] as acting on the Bergman space

$$\mathcal{B}^2 = \left\{ f\colon z \mapsto \sum_{n=0}^{\infty} c_n z^n \in Hol(\mathbf{D}); \|f\|^2 = \sum_{n=0}^{\infty} \frac{\pi \, |c_n|^2}{n+1} < +\infty \right\}$$

on $\mathbf{D}$. Since $T(0) = 0$, $T(1) = 2$ and $T(2) = 1$, the vector space $\mathcal{H}_0 = \operatorname{span}[1, z, z^2]$ is invariant under the action of $\mathcal{T}$, and $\mathcal{T}$ induces a bounded operator on the quotient space $\mathcal{X} = \mathcal{B}^2/\mathcal{H}_0$, which we still denote by $\mathcal{T}$. Considering this quotient space allows to avoid the trivial cycle $(1, 2)$ and the fixed point 0 of the Collatz map, and simplifies the study of the dynamics of $\mathcal{T}$. Neklyudov proved in [36] several results pertaining to the behavior of the iterates of $\mathcal{T}$, in connection with the Collatz Conjecture. For instance, he showed that if T has no non-trivial cycle, then $\mathcal{T}$ is hypercyclic, i.e. admits a vector with dense orbit [36, Theorem 2.2]. He also undertook a study of the periodic points of $\mathcal{T}$ and showed that each hypothetical cycle of T and each hypothetical diverging T-orbit can be associated to a fixed point of $\mathcal{T}$. He also constructed fixed points of $\mathcal{T}$ independently of any cycle or diverging orbit of T.

Neklyudov asked in [36] about the existence of a criterion allowing to distinguish these fixed points and also whether $\mathcal{T}$ is chaotic (i.e. hypercyclic and admitting a dense set of periodic points).

In this work, we continue this study of operators on Hilbert spaces of analytic functions associated to the Collatz map and give an affirmative answer to this question of Neklyudov. We consider the operator $\mathcal{T}$ above as acting on weighted Bergman spaces of holomorphic functions on $\mathbf{D}$ of the form

$$\mathcal{B}^2_\omega = \left\{ f\colon z \mapsto \sum_{n=0}^{\infty} c_n z^n \in Hol(\mathbf{D}); \|f\|^2_\omega = \sum_{n=0}^{\infty} \frac{|c_n|^2}{\omega(n)} < +\infty \right\}$$

where $\omega\colon \mathbf{Z}_+ \to (0, +\infty)$ is a positive weight. Just as above, it will be more natural to study the action of $\mathcal{T}$ on the quotient space $\mathcal{X}_\omega = \mathcal{B}^2_\omega/\mathcal{H}_0$. Let $\omega_0(n) = (n+1)/\pi$ for every $n \geq 0$. Then $\mathcal{B}^2_{\omega_0}$ is the classical Bergman space $\mathcal{B}^2$. We write $\mathcal{X}_{\omega_0} = \mathcal{X}$.

Whenever the weight ω is such that $\mathcal{T}$ defines a bounded operator on $\mathcal{X}_\omega$, it is of interest to study the dynamics of $\mathcal{T}$. The main properties that we will consider are those of hypercyclicity (existence of a dense orbit, or, equivalently, topological transitivity), chaos, frequent hypercyclicity and ergodicity with respect to an invariant measure with full support. We give a brief overview of these notions in Section 3 of the present paper, and refer the reader to one of the works [4] and [26] for more on linear dynamics.

Our first main result states that under some mild assumptions on the weight ω, the operator $\mathcal{T}$ is hypercyclic.

Theorem 3.1.1. If ω is bounded from below and if $\omega(k2^n) \to +\infty$ as $n \to +\infty$ for every $k \geq 3$, then $\mathcal{T}$ is hypercyclic.

Theorem 3.1.1 applies in particular to the weight ω_0, i.e. to the case where $\mathcal{T}$ acts on the Bergman space, and shows that the result [36, Theorem 2.2] of Neklyudov is in fact not conditional to the Collatz map having no non-trivial cycle.

Our second main result shows, under slightly stronger assumptions on ω, that $\mathcal{T}$ acting on $\mathcal{X}_\omega$ enjoys some strong dynamical properties:

Theorem 3.1.2. If ω is bounded from below and if $\sum_{n=0}^{\infty} 1/\omega(k2^n) < +\infty$ for every $k \geq 3$, then $\mathcal{T}$ is chaotic, frequently hypercyclic and ergodic with respect to a Gaussian invariant measure with full support.

In particular, $\mathcal{T}$ acting on the Bergman space $\mathcal{X}$ is chaotic. This provides an affirmative answer to Neklyudov's question.

The paper is organized as follows. We first provide in Section 2 conditions on the weight ω ensuring that $\mathcal{T}$ acts as a bounded operator on $\mathcal{X}_\omega$ (Proposition 3.2.1); building on a description given in [36] of some eigenvectors of $\mathcal{T}$, we show (still under some mild conditions on ω) that these eigenvectors span a dense subspace of $\mathcal{X}_\omega$ (Theorem 3.2.4) and that the adjoint $\mathcal{T}^*$ of $\mathcal{T}$ on $\mathcal{X}_\omega$ has no eigenvalue (Theorem 3.2.7). After recalling in Section 3 some background in linear dynamics, we prove Theorems 3.1.1 and 3.1.2 above. The proofs of these two results rely heavily on the properties of the eigenvectorfields of $\mathcal{T}$ presented in Section 2, and in particular on Theorem 3.2.4. Section 4 collects some additional results as well as some open questions.

3.2 Boundedness of $\mathcal{T}$ and properties of its eigenvectors

Let ω be a positive weight on $\mathbf{Z}_+$. We first consider the operator $\mathcal{T}$ acting on the space $\mathbf{C}[\xi]/\mathrm{span}[1,\xi,\xi^2]$, defined by

$$\mathcal{T}\sum_{n=3}^{d} c_n\xi^n = \sum_{3\leq n\leq d, T(n)\geq 3} c_n\xi^{T(n)} \quad \text{for every } \sum_{n=3}^{d} c_n\xi^n \in \mathbf{C}[\xi]/\mathrm{span}[1,\xi,\xi^2],$$

and give conditions on ω implying that $\mathcal{T}$ can be extended to a bounded operator on $\mathcal{X}_\omega$.

Proposition 3.2.1. The operator $\mathcal{T}$, defined for every $f \in \mathcal{X}_\omega$ with $f(z) = \sum_{n=3}^{\infty} c_n z^n$ by

$$\mathcal{T}\sum_{n=3}^{\infty} c_n z^n = \sum_{T(j)\geq 3} c_j z^{T(j)}, \quad z \in \mathbf{D},$$

is a bounded operator acting on $\mathcal{X}_\omega$ if and only if the three sequences $(\omega(6m)/\omega(3m))_{m\geq 1}$, $(\omega(6m+2)/\omega(3m+1))_{m\geq 1}$ and $((\omega(6m+4)+\omega(2m+1))/\omega(3m+2))_{m\geq 1}$ are bounded. In this case

$$\|\mathcal{T}\|_\omega^2 = \max\left\{\sup_{m\geq 1}\frac{\omega(6m)}{\omega(3m)}, \sup_{m\geq 1}\frac{\omega(6m+2)}{\omega(3m+1)}, \sup_{m\geq 1}\frac{\omega(6m+4)+\omega(2m+1)}{\omega(3m+2)}\right\}.$$

Moreover for every $n \geq 0$

$$\|\mathcal{T}^n\|_\omega^2 = \sup_{k\geq 3}\sum_{T^n(j)=k}\frac{\omega(j)}{\omega(k)}.$$

Taking $\omega = \omega_0$ we obtain in particular that $\mathcal{T}$ is bounded on $\mathcal{X}$ and that $\|\mathcal{T}\|^2 = 8/3$.

Proof. Suppose that the three sequences above are bounded, which means that the maximum

$$\max\left\{\sup_{m\geq 1}\frac{\omega(6m)}{\omega(3m)}, \sup_{m\geq 1}\frac{\omega(6m+2)}{\omega(3m+1)}, \sup_{m\geq 1}\frac{\omega(6m+4)+\omega(2m+1)}{\omega(3m+2)}\right\} = \sup_{k\geq 3}\sum_{T(j)=k}\frac{\omega(j)}{\omega(k)}$$

is finite. Let $n \geq 0$ and $f \in \mathcal{X}_\omega$ with $f(z) = \sum_{k=3}^{\infty} c_k z^k$. We have

$$\mathcal{T}^n f(z) = \sum_{T^n(j)\geq 3} c_j z^{T^n(j)} = \sum_{k=3}^{\infty}\sum_{T^n(j)=k} c_j z^k, \quad z \in \mathbf{D},$$

and

$$\|\mathcal{T}^n f\|_\omega^2 = \sum_{k=3}^{\infty} \frac{1}{\omega(k)} \left| \sum_{T^n(j)=k} c_j \right|^2 = \sum_{k=3}^{\infty} \frac{1}{\omega(k)} \left| \sum_{T^n(j)=k} \sqrt{\omega(j)} \frac{c_j}{\sqrt{\omega(j)}} \right|^2 .$$

By the Cauchy-Schwarz's inequality

$$\|\mathcal{T}^n f\|_\omega^2 \leq \sum_{k=3}^{\infty} \frac{1}{\omega(k)} \sum_{T^n(j)=k} \omega(j) \sum_{T^n(j)=k} \frac{|c_j|^2}{\omega(j)} \leq \left(\sup_{k\geq 3} \sum_{T^n(j)=k} \frac{\omega(j)}{\omega(k)} \right) \|f\|_\omega^2,$$

which gives that $\mathcal{T}$ is bounded if the sequence $(\sum_{T(j)=k} \omega(j)/\omega(k))_{k\geq 3}$ is bounded and that $\|\mathcal{T}^n\|_\omega^2 \leq \sup_{k\geq 3} \sum_{T^n(j)=k} \omega(j)/\omega(k)$.

Conversely, let $n \geq 0$ and suppose that $\mathcal{T}$ is a bounded operator on $\mathcal{X}_\omega$. Let $(k_p)_{p\geq 1}$ such that $\sum_{T^n(j)=k_p} \omega(j)/\omega(k_p) \to \sup_{k\geq 3} \sum_{T^n(j)=k} \omega(j)/\omega(k)$ as $p \to +\infty$. Consider the function $f_p \in \mathcal{X}_\omega$ with $f_p(z) = \sum_{T^n(j)=k_p} \omega(j) z^j$ whose norm satisfies $\|f_p\|_\omega^2 = \sum_{T^n(j)=k_p} |\omega(j)|^2 / \omega(j) = \sum_{T^n(j)=k_p} \omega(j)$ and

$$\|\mathcal{T}^n f_p\|_\omega^2 = \frac{1}{\omega(k_p)} \left| \sum_{T^n(j)=k_p} \omega(j) \right|^2 = \left(\sum_{T^n(j)=k_p} \frac{\omega(j)}{\omega(k_p)} \right) \|f_p\|_\omega^2.$$

So we have $\|\mathcal{T}^n f_p\|_\omega^2 / \|f_p\|_\omega^2 \to \sup_{k\geq 3} \sum_{T^n(j)=k} \omega(j)/\omega(k)$ as $p \to +\infty$ and it follows that $\sup_{k\geq 3} \sum_{T^n(j)=k} \omega(j)/\omega(k) \leq \|\mathcal{T}^n\|_\omega^2 < +\infty$ for every $n \geq 0$, which concludes the proof. □

For instance, it follows from Proposition 3.2.1 that if ω has polynomial growth, then $\mathcal{T}$ is a bounded operator on $\mathcal{X}_\omega$.

Eigenvectors play a very important role in linear dynamics. We will thus be interested in the eigenvalues and the eigenvectors of $\mathcal{T}$, and will provide some condition on the weight ω ensuring that the eigenvectors of $\mathcal{T}$ span a dense subspace of $\mathcal{X}_\omega$.

Definition 3.2.2. For every $\mu \in \mathbf{C}$ and every $m \geq 1$, consider the analytic functions on $\mathbf{D}$

$$h_m(\mu, \cdot) \colon z \mapsto \sum_{n=0}^{\infty} \mu^n \left(z^{(6m+4)2^n} - z^{(2m+1)2^n} \right) \quad \text{and} \quad h_0(\mu, \cdot) \colon z \mapsto \sum_{n=0}^{\infty} \mu^n z^{2^{n+2}}.$$

These functions have been introduced by Neklyudov in [36, Theorem 2.3] and are the only eigenvectors that will be needed in order to obtain results concerning the spanning of dense subspaces.

Proposition 3.2.3. For every $m \geq 0$ and every $\mu \in \mathbf{C}$, the vector $h_m(\mu, \cdot)$ belongs to $\mathcal{X}_\omega$ if and only if the series $\sum_{n\geq 0} |\mu|^{2n} / \omega((2m+1)2^n)$ and $\sum_{n\geq 0} |\mu|^{2n} / \omega((6m+4)2^n)$ converge. In this case $\mathcal{T} h_m(\mu, \cdot) = \mu h_m(\mu, \cdot)$.

In particular if ω is bounded from below then $h_m(\mu, \cdot)$ belongs to $\mathcal{X}_\omega$ for every $m \geq 0$ and every $\mu \in \mathbf{D}$. Moreover if $\omega = \omega_0$ then $h_m(\mu, \cdot)$ belongs to $\mathcal{X}$ for every $m \geq 0$ and every $\mu \in \mathbf{C}$ such that $|\mu| < \sqrt{2}$.

Proof. Let $\mu \in \mathbf{C}$ and $m \geq 1$. The function $h_m(\mu, \cdot)$ belongs to $\mathcal{X}_\omega$ if and only if $\sum_{n=0}^{\infty} |\mu|^{2n} / \omega((6m+4)2^n) + \sum_{n=0}^{\infty} |\mu|^{2n} / \omega((2m+1)2^n) < +\infty$. And $h_0(\mu, \cdot)$ belongs to $\mathcal{X}_\omega$ if and only if $\sum_{n=0}^{\infty} |\mu|^{2n} / \omega(2^{n+2}) < +\infty$.

Moreover for every $m \geq 1$ and every $z \in \mathbf{D}$ we have

$$\begin{aligned}
\mathcal{T} h_m(\mu, \cdot)(z) &= \sum_{n=0}^{\infty} \mu^n \left(z^{T((6m+4)2^n)} - z^{T((2m+1)2^n)} \right) \\
&= \sum_{n=1}^{\infty} \mu^n \left(z^{(6m+4)2^{n-1}} - z^{(2m+1)2^{n-1}} \right) + z^{3m+2} - z^{3m+2} \\
&= \mu h_m(\mu, z)
\end{aligned}$$

and

$$\mathcal{T} h_0(\mu, \cdot)(z) = \sum_{n=0}^{\infty} \mu^n z^{T(2^{n+2})} = \sum_{n=1}^{\infty} \mu^n z^{2^{n+1}} + z^2 = \mu h_0(\mu, z),$$

which shows that the functions $h_m(\mu, \cdot)$, $m \geq 0$, are eigenvectors of $\mathcal{T}$ associated to the eigenvalue μ as soon as they belong to $\mathcal{X}_\omega$. $\square$

Our aim is now to show that if the weight ω is bounded from below, the eigenvectors $h_m(\mu, \cdot)$, $m \geq 0$, $\mu \in \mathbf{D}$, span a dense subspace of $\mathcal{X}_\omega$.

Theorem 3.2.4. If ω is bounded from below, then $\operatorname{span}[h_m(\mu, \cdot); m \geq 0, \mu \in \mathbf{D}]$ is dense in $\mathcal{X}_\omega$.

In particular if $\omega = \omega_0$ then $\{h_m(\mu, \cdot); m \geq 0, \mu \in \mathbf{D}\}$ spans a dense subspace of $\mathcal{X}$.

Since T admits a particular behavior on $\{2^i; i \geq 0\}$, we will denote by $\mathcal{P}_0$ the set $\{2^i; i \geq 0\}$ and by $\mathcal{P}_2$ the set $\{2^i; i \geq 2\}$. The proof of Theorem 3.2.4 relies on the following lemma.

Lemma 3.2.5. Let $f \in \mathcal{X}_\omega$, with $f(z) = \sum_{k=3}^{\infty} c_k z^k$, such that for every $n \geq 0$ and every $m \geq 1$

$$\frac{c_{(6m+4)2^n}}{\omega((6m+4)2^n)} = \frac{c_{(2m+1)2^n}}{\omega((2m+1)2^n)} \quad \text{and} \quad c_{2^{n+2}} = 0.$$

Then for every $k \geq 3$ such that $k \notin \mathcal{P}_2$, there exist integer sequences $(m_n)_{n \geq 1}$, $(p_n)_{n \geq 1}$ and $(j_n)_{n \geq 1}$ depending on k such that:

(i) $c_k = (\omega((2m_1+1)2^{p_1}) / \omega((6m_n+4)2^{p_n})) c_{(6m_n+4)2^{p_n}}$ for every $n \geq 1$;

(ii) $m_n \geq 1$ for every $n \geq 1$;

(iii) $3m_n + 2 = T^{j_n}(k)$ for every $n \geq 1$;

(iv) $j_{n+1} > j_n$ if and only if $3m_n + 2 \notin \mathcal{P}_2$ for every $n \geq 1$;

(v) $(j_n)_{n \geq 1}$ and $((6m_n+4)2^{p_n})_{n \geq 1}$ are either both strictly increasing or both stationary.

Proof. Since $k \geq 3$ and $k \notin \mathcal{P}_2$, there exist $m_1 \geq 1$ and $p_1 \geq 0$ such that $k = (2m_1+1)2^{p_1}$. Then

$$c_k = c_{(2m_1+1)2^{p_1}} = \frac{\omega((2m_1+1)2^{p_1})}{\omega((6m_1+4)2^{p_1})} c_{(6m_1+4)2^{p_1}}$$

and

$$3m_1 + 2 = T(2m_1 + 1) = T^{p_1+1}((2m_1 + 1)2^{p_1}) = T^{p_1+1}(k).$$

We set then $j_1 = p_1 + 1$.

Suppose now that $(m_i)_{1\leq i\leq n}$, $(p_i)_{1\leq i\leq n}$ and $(j_i)_{1\leq i\leq n}$ have been defined so as to satisfy the following properties:

(i) $c_k = (\omega((2m_1 + 1)2^{p_1})/\omega((6m_i + 4)2^{p_i}))c_{(6m_i+4)2^{p_i}}$ for every $1 \leq i \leq n$;

(ii) $m_i \geq 1$ for every $1 \leq i \leq n$;

(iii) $3m_i + 2 = T^{j_i}(k)$ for every $1 \leq i \leq n$;

(iv) $j_{i+1} > j_i$ if and only if $3m_i + 2 \notin \mathcal{P}_2$ for every $1 \leq i \leq n - 1$.

Let us construct m_{n+1}, p_{n+1} and j_{n+1}.

If $3m_n + 2 \in \mathcal{P}_2$, then we set $m_{n+1} = m_n$, $p_{n+1} = p_n$ and $j_{n+1} = j_n$. Otherwise there exist $m_{n+1} \geq 1$ and $q_{n+1} \geq 0$ such that $3m_n + 2 = (2m_{n+1} + 1)2^{q_{n+1}}$. So

$$(6m_n + 4)2^{p_n} = (3m_n + 2)2^{p_n+1} = (2m_{n+1} + 1)2^{q_{n+1}+p_n+1}.$$

We set then $p_{n+1} = q_{n+1} + p_n + 1 > p_n$. Moreover

$$\begin{aligned} c_k &= \frac{\omega((2m_1 + 1)2^{p_1})}{\omega((2m_{n+1} + 1)2^{p_{n+1}})} c_{(2m_{n+1}+1)2^{p_{n+1}}} \\ &= \frac{\omega((2m_1 + 1)2^{p_1})}{\omega((2m_{n+1} + 1)2^{p_{n+1}})} \frac{\omega((2m_{n+1} + 1)2^{p_{n+1}})}{\omega((6m_{n+1} + 4)2^{p_{n+1}})} c_{(6m_{n+1}+4)2^{p_{n+1}}} \\ &= \frac{\omega((2m_1 + 1)2^{p_1})}{\omega((6m_{n+1} + 4)2^{p_{n+1}})} c_{(6m_{n+1}+4)2^{p_{n+1}}}. \end{aligned}$$

Remark that

$$3m_{n+1}+2 = T^{p_{n+1}+1}((2m_{n+1}+1)2^{p_{n+1}}) = T^{p_{n+1}+1}((6m_n+4)2^{p_n}) = T^{p_{n+1}}((3m_n+2)2^{p_n}).$$

So $3m_{n+1}+2 = T^{p_{n+1}-p_n}(3m_n+2) = T^{p_{n+1}-p_n+j_n}(k)$ and we set $j_{n+1} = p_{n+1}-p_n+j_n > j_n$. We also have $(6m_{n+1} + 4)2^{p_{n+1}} > (2m_{n+1} + 1)2^{p_{n+1}} = (6m_n + 4)2^{p_n}$.

Defined this way, the sequences $(m_n)_{n\geq 1}$, $(p_n)_{n\geq 1}$ and $(j_n)_{n\geq 1}$ satisfy the first four properties and it remains to prove the last one. On the one hand, if there exists $n_0 \geq 1$ such that $3m_{n_0}+2 \in \mathcal{P}_2$, then the sequences $(j_n)_{n\geq 1}$ and $((6m_n+4)2^{p_n})_{n\geq 1}$ are stationary by construction. On the other hand if $3m_n + 2 \notin \mathcal{P}_2$ for every $n \geq 1$, then the sequences $(j_n)_{n\geq 1}$ and $((6m_n + 4)2^{p_n})_{n\geq 1}$ are strictly increasing by construction, which proves that the fifth property is satisfied. □

Proof of Theorem 3.2.4. By Proposition 3.2.3, $h_m(\mu, \cdot)$ belongs to $\mathcal{X}_\omega$ for every $m \geq 0$ and every $\mu \in \mathbf{D}$ since ω is bounded from below. Let $f \in \text{span}[h_m(\mu, \cdot); m \geq 0, \mu \in \mathbf{D}]^\perp$ in $\mathcal{X}_\omega$ with $f(z) = \sum_{k=3}^{\infty} c_k z^k$. Our aim is to show that $f = 0$. For every $\mu \in \mathbf{D}$ and every $m \geq 1$

$$\varphi_m(\mu) \colon = \langle h_m(\mu, \cdot), f\rangle = \sum_{n=0}^{\infty} \left(\frac{\overline{c_{(6m+4)2^n}}}{\omega((6m + 4)2^n)} - \frac{\overline{c_{(2m+1)2^n}}}{\omega((2m + 1)2^n)} \right) \mu^n = 0,$$

$$\varphi_0(\mu)\colon = \langle h_0(\mu,\cdot), f\rangle = \sum_{n=0}^{\infty} \frac{\overline{c_{2^{n+2}}}}{\omega(2^{n+2})}\mu^n = 0.$$

For every $k \geq 3$ the radius of convergence of the power series $\sum_{n\geq 0}(\overline{c_{k2^n}}/\omega(k2^n))z^n$ is greater than 1. Indeed by the Cauchy-Schwarz's inequality

$$\left(\sum_{n=0}^{\infty}\left|\frac{\overline{c_{k2^n}}}{\omega(k2^n)}z^n\right|\right)^2 \leq \sum_{n=0}^{\infty}\frac{|c_{k2^n}|^2}{\omega(k2^n)}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{\omega(k2^n)} < +\infty$$

for every $z \in \mathbf{D}$ because f belongs to $\mathcal{X}_\omega$ and ω is supposed to be bounded from below. Thus the functions φ_m and φ_0 are holomorphic and they identically vanish on $\mathbf{D}$ for every $m \geq 1$. So for every $m \geq 1$

$$\frac{c_{(6m+4)2^n}}{\omega((6m+4)2^n)} = \frac{c_{(2m+1)2^n}}{\omega((2m+1)2^n)} \quad \text{and} \quad c_{2^{n+2}} = 0 \quad \text{for every } n \geq 0.$$

Claim 3.2.6. Let ω be a positive weight on $\mathbf{Z}_+$ which is bounded from below, and let $f \in \mathcal{X}_\omega$ with $f(z) = \sum_{k=3}^{\infty} c_k z^k$. Suppose that for every $m \geq 1$

$$\frac{c_{(6m+4)2^n}}{\omega((6m+4)2^n)} = \frac{c_{(2m+1)2^n}}{\omega((2m+1)2^n)} \quad \text{and} \quad c_{2^{n+2}} = 0 \quad \text{for every } n \geq 0,$$

Then $c_k = 0$ for every $k \geq 3$.

Proof. Fix $k \geq 3$. If $k \in \mathcal{P}_2$ then $c_k = 0$. Otherwise consider the sequences $(m_n)_{n\geq 1}$, $(p_n)_{n\geq 1}$ and $(j_n)_{n\geq 1}$ associated to k given by Lemma 3.2.5.

On the one hand suppose that $\{T^n(k); n \geq 0\} \cap \mathcal{P}_0 = \emptyset$, then in particular $3m_n + 2 \notin \mathcal{P}_2$ for every $n \geq 1$ by (iii). So by (iv), $(j_n)_{n\geq 1}$ is strictly increasing and then so is $((6m_n+4)2^{p_n})_{n\geq 1}$ by (v). Therefore by (i)

$$\|f\|_\omega^2 = \sum_{j=3}^{\infty}\frac{|c_j|^2}{\omega(j)} \geq \sum_{n=1}^{\infty}\frac{\left|c_{(6m_n+4)2^{p_n}}\right|^2}{\omega((6m_n+4)2^{p_n})} = \sum_{n=1}^{\infty}\frac{\omega((6m_n+4)2^{p_n})}{\omega((2m_1+1)2^{p_1})^2}\left|c_k\right|^2.$$

Since ω is bounded from below, the previous series can converge only in the case where $c_k = 0$.

On the other hand suppose that $\{T^n(k); n \geq 0\} \cap \mathcal{P}_0 \neq \emptyset$, then $(j_n)_{n\geq 1}$ is stationary. Indeed suppose that $(j_n)_{n\geq 1}$ is not stationary, that is to say strictly increasing by (v). There exists $N \geq 1$ such that $T^N(k) \in \mathcal{P}_0$, so $T^n(k) \in \mathcal{P}_0$ for every $n \geq N$ and $T^{j_N}(k) \in \mathcal{P}_0$ since $j_N \geq N$. Then $T^{j_N}(k) = 3m_N + 2 \in \mathcal{P}_2$ by (ii) and (iii), which implies that $j_{N+1} = j_N$. This is impossible, so $(j_n)_{n\geq 1}$ is indeed stationary. Thus there exists $N \geq 1$ such that $3m_N + 2 \in \mathcal{P}_2$ by (iv), which implies by (i) that

$$c_k = \frac{\omega((2m_1+1)2^{p_1})}{\omega((6m_N+4)2^{p_N})}c_{(6m_N+4)2^{p_N}} = 0$$

because $(6m_N+4)2^{p_N} = (3m_N+2)2^{p_N+1} \in \mathcal{P}_2$.

We have shown that $c_k = 0$ for every $k \geq 3$. □

Thus Claim 3.2.6 gives that $f = 0$, which concludes the proof of Theorem 3.2.4. $\square$

We conclude this section by proving that the adjoint $\mathcal{T}^*$ of $\mathcal{T}$ acting on $\mathcal{X}_\omega$ has no eigenvalue. We will denote by $\sigma_p(\mathcal{T}^*)$ the set of its eigenvalues.

Theorem 3.2.7. If ω is bounded from below then $\sigma_p(\mathcal{T}^*) = \emptyset$.
In particular if $\omega = \omega_0$ then $\sigma_p(\mathcal{T}^*) = \emptyset$.

The proof proceeds along the same lines as the proof of Theorem 3.2.4. We first need the following Lemma, whose proof is similar to the proof of Lemma 3.2.5.

Lemma 3.2.8. Let $\mu \in \mathbf{C}\backslash\{0\}$ and let $f \in \mathcal{X}_\omega$ with $f(z) = \sum_{k=3}^{\infty} c_k z^k$. Suppose that for every $m \geq 1$ and every $n \geq 0$

$$\frac{\mu^{-1} c_{(3m+2)2^n}}{\omega((3m+2)2^n)} = \frac{c_{(2m+1)2^n}}{\omega((2m+1)2^n)} \quad \text{and} \quad c_{2^{n+2}} = 0.$$

Then for every $k \geq 3$ such that $k \notin \mathcal{P}_2$, there exist integer sequences $(m_n)_{n\geq 1}$, $(p_n)_{n\geq 1}$ and $(j_n)_{n\geq 1}$ depending on k such that:

(i) $c_k = (\mu^{-n}\omega((2m_1+1)2^{p_1})/\omega((3m_n+2)2^{p_n}))c_{(3m_n+2)2^{p_n}}$ for every $n \geq 1$;

(ii) $m_n \geq 1$ for every $n \geq 1$;

(iii) $3m_n + 2 = T^{j_n}(k)$ for every $n \geq 1$;

(iv) $j_{n+1} > j_n$ if and only if $3m_n + 2 \notin \mathcal{P}_2$ for every $n \geq 1$;

(v) $(j_n)_{n\geq 1}$ and $((3m_n+2)2^{p_n})_{n\geq 1}$ are either both strictly increasing or both stationary.

Proof. Since $k \geq 3$ and $k \notin \mathcal{P}_2$, there exist $m_1 \geq 1$ and $p_1 \geq 0$ such that $k = (2m_1+1)2^{p_1}$. Then

$$c_k = c_{(2m_1+1)2^{p_1}} = \frac{\mu^{-1}\omega((2m_1+1)2^{p_1})}{\omega((3m_1+2)2^{p_1})} c_{(3m_1+2)2^{p_1}}.$$

and

$$3m_1 + 2 = T(2m_1+1) = T^{p_1+1}((2m_1+1)2^{p_1}) = T^{p_1+1}(k).$$

We set then $j_1 = p_1 + 1$.

Suppose now that $(m_i)_{1\leq i\leq n}$, $(p_i)_{1\leq i\leq n}$ and $(j_i)_{1\leq i\leq n}$ have been defined so as to satisfy the following properties:

(i) $c_k = (\mu^{-i}\omega((2m_1+1)2^{p_1})/\omega((3m_i+2)2^{p_i}))c_{(3m_i+2)2^{p_i}}$ for every $1 \leq i \leq n$;

(ii) $m_i \geq 1$ for every $1 \leq i \leq n$;

(iii) $3m_i + 2 = T^{j_i}(k)$ for every $1 \leq i \leq n$;

(iv) $j_{i+1} > j_i$ if and only if $3m_i + 2 \notin \mathcal{P}_2$ for every $1 \leq i \leq n-1$.

Let us construct m_{n+1}, p_{n+1} and j_{n+1}.

If $3m_n + 2 \in \mathcal{P}_2$, then we set $m_{n+1} = m_n$, $p_{n+1} = p_n$ and $j_{n+1} = j_n$. Otherwise there exist $m_{n+1} \geq 1$ and $q_{n+1} \geq 0$ such that $(3m_n+2)2^{p_n} = (2m_{n+1}+1)2^{q_{n+1}+p_n}$. We set then

$p_{n+1} = q_{n+1} + p_n \geq p_n$. Moreover

$$\begin{aligned} c_k &= \frac{\mu^{-n}\omega((2m_1+1)2^{p_1})}{\omega((2m_{n+1}+1)2^{p_{n+1}})} c_{(2m_{n+1}+1)2^{p_{n+1}}} \\ &= \frac{\mu^{-n-1}\omega((2m_1+1)2^{p_1})}{\omega((2m_{n+1}+1)2^{p_{n+1}})} \frac{\omega((2m_{n+1}+1)2^{p_{n+1}})}{\omega((3m_{n+1}+2)2^{p_{n+1}})} c_{(3m_{n+1}+2)2^{p_{n+1}}} \\ &= \frac{\mu^{-(n+1)}\omega((2m_1+1)2^{p_1})}{\omega((3m_{n+1}+2)2^{p_{n+1}})} c_{(3m_{n+1}+2)2^{p_{n+1}}}. \end{aligned}$$

Remark that

$$3m_{n+1}+2 = T^{p_{n+1}+1}((2m_{n+1}+1)2^{p_{n+1}}) = T^{p_{n+1}+1}((3m_n+2)2^{p_n}) = T^{p_{n+1}+1-p_n+j_n}(k),$$

then we set $j_{n+1} = p_{n+1} + 1 - p_n + j_n > j_n$. We also have $(3m_{n+1}+2)2^{p_{n+1}} > (2m_{n+1}+1)2^{p_{n+1}} = (3m_n+2)2^{p_n}$.

Thus the sequences $(m_n)_{n\geq 1}$, $(p_n)_{n\geq 1}$ and $(j_n)_{n\geq 1}$ satisfy the properties (i), (ii), (iii) and (iv). We prove now that they also satisfy (v). Suppose that there exists $n_0 \geq 1$ such that $3m_{n_0} + 2 \in \mathcal{P}_2$, then the sequences $(j_n)_{n\geq 1}$ and $((3m_n+2)2^{p_n})_{n\geq 1}$ are stationary. Suppose now that $3m_n + 2 \notin \mathcal{P}_2$ for every $n \geq 1$, then the sequences $(j_n)_{n\geq 1}$ and $((3m_n+2)2^{p_n})_{n\geq 1}$ are strictly increasing, which concludes the proof. □

Proof of Theorem 3.2.7. For every $f, g \in \mathcal{X}_\omega$ with $f(z) = \sum_{n=3}^{\infty} c_n(f)z^n$ and with $g(z) = \sum_{n=3}^{\infty} c_n(g)z^n$, we have

$$\langle \mathcal{T}f \mid g\rangle = \sum_{n=3}^{\infty} \frac{c_n(\mathcal{T}f)\overline{c_n(g)}}{\omega(n)} = \sum_{n=3}^{\infty} \frac{\sum_{T(j)=n} c_j(f)\overline{c_n(g)}}{\omega(n)} = \sum_{j\geq 3, T(j)\geq 3} \frac{c_j(f)\overline{c_{T(j)}(g)}}{\omega(T(j))},$$

so

$$\langle \mathcal{T}f \mid g\rangle = \frac{c_3(f)\overline{c_5(g)}}{\omega(5)} + \sum_{k=5}^{\infty} \frac{c_k(f)\overline{c_{T(k)}(g)}}{\omega(T(k))}.$$

Then $\mathcal{T}^*$ is defined in the following way:

$$\mathcal{T}^* \sum_{n=3}^{\infty} c_n z^n = \frac{\omega(3)}{\omega(5)} c_5 z^3 + \sum_{k=5}^{\infty} \frac{\omega(k)}{\omega(T(k))} c_{T(k)} z^k \quad \text{for every } f\colon z \mapsto \sum_{n=3}^{\infty} c_n z^n \in \mathcal{X}_\omega.$$

Let $\mu \in \mathbf{C}$, let $f \in \mathcal{X}_\omega$ with $f(z) = \sum_{n=3}^{\infty} c_n z^n$ and suppose that $\mathcal{T}^* f = \mu f$. We have

$$\begin{cases} (\omega(3)/\omega(5))c_5 = \mu c_3 \\ (\omega(k)/\omega(T(k)))c_{T(k)} = \mu c_k, \quad k \geq 5 \\ 0 = \mu c_4, \end{cases}$$

which is equivalent to

$$\begin{cases} (\omega(2m)/\omega(m))c_m = \mu c_{2m}, & m \geq 3 \\ (\omega(2m+1)/\omega(3m+2))c_{3m+2} = \mu c_{2m+1}, & m \geq 1 \\ 0 = \mu c_4. & \end{cases}$$

Firstly if $\mu = 0$ then $c_m = 0$ for every $m \geq 3$. So 0 does not belong to $\sigma_p(\mathcal{T}^*)$.

We suppose now that $\mu \neq 0$. Then we have $\mathcal{T}^* f = \mu f$ if and only if

$$\begin{cases} c_{m2^n} = \mu^{-n}(\omega(m2^n)/\omega(m))c_m, & m \geq 3, n \geq 0 \\ \mu^{-1}c_{(3m+2)2^n}/\omega((3m+2)2^n) = c_{(2m+1)2^n}/\omega((2m+1)2^n), & m \geq 1, n \geq 0 \\ c_{2^{n+2}} = 0, & n \geq 0 \end{cases}$$

because for every $m \geq 1$ and every $n \geq 0$

$$c_{(2m+1)2^n} = \frac{\mu^{-1}\omega((2m+1)2^n)}{\omega((2m+1)2^{n-1})}c_{(2m+1)2^{n-1}} = \ldots = \frac{\mu^{-n}\omega((2m+1)2^n)}{\omega(2m+1)}c_{2m+1},$$

so

$$c_{(2m+1)2^n} = \frac{\mu^{-n-1}\omega((2m+1)2^n)}{\omega(3m+2)}c_{3m+2} = \ldots = \frac{\mu^{-n-1+n}\omega((2m+1)2^n)}{\omega((3m+2)2^n)}c_{(3m+2)2^n}.$$

Fix $k \geq 3$. If $k \in \mathcal{P}_2$ then $c_k = 0$. Suppose now that $k \notin \mathcal{P}_2$. If $|\mu| \leq 1$, we can remark that

$$\|f\|_\omega^2 = \sum_{j=3}^{\infty} \frac{|c_j|^2}{\omega(j)} \geq \sum_{n=0}^{\infty} \frac{|c_{k2^n}|^2}{\omega(k2^n)} = \sum_{n=0}^{\infty} \frac{|\mu|^{-2n}\,\omega(k2^n)}{\omega(k)^2}\,|c_k|^2\,.$$

Since ω is bounded from below, the previous series can converge only in the case where $c_k = 0$ and it follows that $f = 0$. Thus μ does not belong to $\sigma_p(\mathcal{T}^*)$. If $|\mu| > 1$, consider the sequences $(m_n)_{n\geq 1}$, $(p_n)_{n\geq 1}$ and $(j_n)_{n\geq 1}$ associated to k given by Lemma 3.2.8.

On the one hand suppose that $\{T^n(k); n \geq 0\} \cap \mathcal{P}_0 = \emptyset$, then in particular $3m_n + 2 \notin \mathcal{P}_2$ for every $n \geq 1$ by (iii). So by (iv), $(j_n)_{n\geq 1}$ is strictly increasing and then so is $((3m_n+2)2^{p_n})_{n\geq 1}$ by (v). Therefore by (i)

$$\|f\|_\omega^2 = \sum_{j=3}^{\infty} \frac{|c_j|^2}{\omega(j)} \geq \sum_{n=1}^{\infty} \frac{\left|c_{(3m_n+2)2^{p_n}}\right|^2}{\omega((3m_n+2)2^{p_n})} = \sum_{n=1}^{\infty} \frac{|\mu|^{2n}\,\omega((3m_n+2)2^{p_n})}{\omega((2m_1+1)2^{p_1})^2}\,|c_k|^2\,.$$

Since ω is bounded from below, the previous series converges only in the case where $c_k = 0$.

On the other hand suppose that $\{T^n(k); n \geq 0\} \cap \mathcal{P}_0 \neq \emptyset$, we will follow the lines of the proof of Theorem 3.2.4 and claim that $(j_n)_{n\geq 1}$ is stationary. Indeed suppose that $(j_n)_{n\geq 1}$ is not stationary, that is to say is strictly increasing by (v). There exists $N \geq 1$ such that $T^N(k) \in \mathcal{P}_0$, so $T^n(k) \in \mathcal{P}_0$ for every $n \geq N$ and $T^{j_N}(k) \in \mathcal{P}_0$ since $j_N \geq N$. Then $T^{j_N}(k) = 3m_N + 2 \in \mathcal{P}_2$ by (ii) and (iii), which implies that $j_{N+1} = j_N$. This is impossible. So $(j_n)_{n\geq 1}$ is stationary and there exists $N \geq 1$ such that $3m_N + 2 \in \mathcal{P}_2$ by (iv). This implies by (i) that

$$c_k = \frac{\mu^{-N}\omega((2m_1+1)2^{p_1})}{\omega((3m_N+2)2^{p_N})}c_{(3m_N+2)2^{p_N}} = 0$$

because $(3m_N+2)2^{p_N} \in \mathcal{P}_2$. We have shown that $c_k = 0$ for every $k \geq 3$. Thus $f = 0$ and so μ does not belong to $\sigma_p(\mathcal{T}^*)$ either. □

3.3 Linear dynamics of the operator $\mathcal{T}$

Our aim is now to understand the links between the dynamics of the Collatz map and those of the operator $\mathcal{T}$. We begin by recalling briefly the definition of some important properties in linear dynamics (hypercyclicity, chaos, frequent hypercyclicity and ergodicity), before investigating whether the operator $\mathcal{T}$ acting on $\mathcal{X}_\omega$ satisfies each of these properties.

3.3.1 Hypercyclicity

A much studied dynamical property is called *hypercyclicity*. We refer the reader to the books [4] and [26] for an in-depth study of this notion. Let X be a separable infinite-dimensional complex Banach (or Fréchet) space. We will denote by $\mathcal{B}(X)$ the set of bounded (or continuous) linear operators on X.

Definition 3.3.1 ([26, Definition 2.15])**.** An operator $\mathcal{A} \in \mathcal{B}(X)$ is said to be *hypercyclic* if there exists $x \in X$ such that its orbit $\{\mathcal{A}^n x; n \geq 0\}$ under $\mathcal{A}$ is dense in X. In this case x is called a hypercyclic vector for $\mathcal{A}$.

Recall that a G_δ-set in X is a countable intersection of open sets in X. The following theorem shows that either the hypercyclic vectors of an operator $\mathcal{A} \in \mathcal{B}(X)$ forms a dense G_δ-set in X or $\mathcal{A}$ has no hypercyclic vector.

Theorem 3.3.2 (Birkhoff's transitivity theorem, [26, Theorems 1.16 and 2.19])**.** An operator $\mathcal{A} \in \mathcal{B}(X)$ is hypercyclic if and only if for every non-empty open sets U and V in X, there exists $n \geq 0$ such that $\mathcal{A}^n(U) \cap V \neq \emptyset$. In this case, the hypercyclic vectors for $\mathcal{A}$ form a dense G_δ-set in X.

Important examples of hypercyclic operators can be found in the literature. On the Fréchet space $Hol(\mathbf{C})$ of entire functions endowed with the family of seminorms $\|f\|_{n,\infty} = \sup\{|f(z)|\,; |z| \leq n\}$ for $n \geq 1$, the Birkhoff's operator $T^{(a)} \colon f \mapsto f(\cdot + a)$ is hypercyclic if and only if $a \neq 0$. The differentiation operator D acting on $Hol(\mathbf{C})$ is also hypercyclic. In the Banach space setting, the simplest examples of hypercyclic operators are given by the Rolewicz's operators λB for $|\lambda| > 1$, where B is the backward shift on $\ell^2(\mathbf{N})$ defined by $B(x_n)_{n\geq 1} = (x_{n+1})_{n\geq 1}$.

The following criterion provides a practical mean to prove the hypercyclicity of an operator.

Theorem 3.3.3 (Hypercyclicity Criterion, [26, Theorem 3.12])**.** Let $\mathcal{A} \in \mathcal{B}(X)$. Suppose that there exist dense subsets X_0 and Y_0 of X, an increasing sequence $(n_k)_{k\geq 1}$ of integers and a sequence $(S_{n_k} \colon Y_0 \to X)_{k\geq 1}$ of maps such that:

(i) $\mathcal{A}^{n_k} x \to 0$ as $k \to +\infty$ for every $x \in X_0$;

(ii) $S_{n_k} y \to 0$ as $k \to +\infty$ for every $y \in Y_0$;

(iii) $\mathcal{A}^{n_k} S_{n_k} y \to y$ as $k \to +\infty$ for every $y \in Y_0$.

Then the operator $\mathcal{A}$ is hypercyclic.

In the case where $\omega = \omega_0$, and provided that the map T has no non-trivial cycle, it is shown in [36, Theorem 2.2] that $\mathcal{T}$ satisfies the Hypercyclicity Criterion, and is thus hypercyclic. Our first theorem generalizes this result by showing that $\mathcal{T}$ is hypercyclic on $\mathcal{X}_\omega$ under a rather mild hypothesis on the weight ω, independently of any assumption on the existence of cycles for the Collatz map.

Theorem 3.3.4. If ω is bounded from below and if $\omega(k2^n) \to +\infty$ as $n \to +\infty$ for every $k \geq 3$, then $\mathcal{T}$ is hypercyclic.

In particular if $\omega = \omega_0$ then $\mathcal{T}$ is hypercyclic on $\mathcal{X}$.

Proof. It suffices to prove that $\mathcal{T}$ satisfies the Hypercyclicity Criterion (Theorem 3.3.3). Consider the subspace $X_0 = \mathrm{span}[h_m(\mu, \cdot); \mu \in \mathbf{D}, m \geq 0]$, which is dense in $\mathcal{X}_\omega$ by Theorem 3.2.4. We have $\mathcal{T}^n h_m(\mu, \cdot) = \mu^n h_m(\mu, \cdot) \to 0$ as $n \to +\infty$ for every $\mu \in \mathbf{D}$ and every $m \geq 0$. Thus this is true for any linear combination in X_0. Consider now the dense subspace $Y_0 = \mathrm{span}[z^k; k \geq 3]$ of $\mathcal{X}_\omega$ and the map $S\colon Y_0 \to Y_0$ defined by $Sz^k = z^{2k}$ for every $k \geq 3$. Then $\|S^n z^k\|_\omega^2 = \|z^{k2^n}\|_\omega^2 = 1/\omega(k2^n) \to 0$ as $n \to +\infty$ and $\mathcal{T} S z^k = \mathcal{T} z^{2k} = z^{T(2k)} = z^k$ for every $k \geq 3$. Thus this is still true for any linear combination in Y_0. Then $\mathcal{T}$ satisfies the Hypercyclicity Criterion, so $\mathcal{T}$ is hypercyclic. □

Actually, the role played by eigenvectors in the hypercyclicity of an operator appears explicitly in the following criterion.

Theorem 3.3.5 (Godefroy-Shapiro Criterion, [26, Theorem 3.1])**.** Let $\mathcal{A} \in \mathcal{B}(X)$. Suppose that the subspaces

$$X_0 = \mathrm{span}[\ker(\mathcal{A} - \mu); |\mu| < 1] \quad \text{and} \quad Y_0 = \mathrm{span}[\ker(\mathcal{A} - \mu); |\mu| > 1]$$

are dense in X. Then $\mathcal{A}$ is hypercyclic.

This criteria is satisfied as soon as there are enough complex numbers μ such that $h_m(\mu, \cdot)$ belongs to $\mathcal{X}_\omega$ for every $m \geq 0$.

Theorem 3.3.6. If ω is bounded from below and if there exists $\rho > 1$ such that for every $k \geq 3$ the sequence $(\rho^n/\omega(k2^n))_{n \geq 0}$ is bounded, then $\mathcal{T}$ satisfies the Godefroy-Shapiro Criterion.

In particular if $\omega = \omega_0$, then taking $\rho = 2$ gives us that $\mathcal{T}$ satisfies the Godefroy-Shapiro Criterion.

Proof. By Proposition 3.2.3, $h_m(\mu, \cdot)$ belongs to $\mathcal{X}_\omega$ for every $m \geq 0$ and every $\mu \in \mathbf{C}$ such that $|\mu| < \sqrt{\rho}$. By Theorem 3.2.4, it suffices to show that the subspace $Y_0 = \mathrm{span}[h_m(\mu, \cdot); m \geq 0, 1 < |\mu| < \sqrt{\rho}]$ is dense in $\mathcal{X}_\omega$. This can be proved by following the steps of the proof of Theorem 3.2.4 and considering the holomorphic functions φ_m and φ_0 on $\{\mu \in \mathbf{C}; |\mu| < \sqrt{\rho}\}$ instead of $\mathbf{D}$. □

3.3.2 Frequent hypercyclicity and ergodicity

The notion of frequent hypercyclicity is a reinforcement of that of hypercyclicity. It quantifies the frequency with which the orbit of a vector visits a non-empty open set. We refer the reader to [3] for more on this notion.

Definition 3.3.7 ([26, Definition 9.2])**.** An operator $\mathcal{A} \in \mathcal{B}(X)$ is said to be *frequently hypercyclic* if there exists a vector $x \in X$ such that for every non-empty open set U in X

$$\liminf_{N \to +\infty} \frac{\mathrm{card}\{0 \leq n \leq N; \mathcal{A}^n x \in U\}}{N+1} > 0.$$

In this case x is called a frequently hypercyclic vector for $\mathcal{A}$.

The Hypercyclicity Criterion admits a frequently hypercyclic version, which is the following theorem.

Theorem 3.3.8 (Frequent Hypercyclicity Criterion, [26, Theorem 9.9])**.** Let $\mathcal{A} \in \mathcal{B}(X)$. Suppose that there exist a dense set X_0 in X and a map $S\colon X_0 \to X_0$ such that for every $x \in X_0$:

1. $\sum_{n \geq 0} \mathcal{A}^n x$ converges unconditionally;
2. $\sum_{n \geq 0} S^n x$ converges unconditionally;
3. $\mathcal{A}Sx = x$.

Then $\mathcal{A}$ is frequently hypercyclic.

The three hypercyclic operators presented before are actually frequently hypercyclic. The operators λB, D and $T^{(a)}$ are frequently hypercyclic respectively on $\ell^2(\mathbf{N})$ and $Hol(\mathbf{C})$ if $|\lambda| > 1$ and if $a \neq 0$.

In order to prove that $\mathcal{T}$ is frequently hypercyclic, we will actually show that $\mathcal{T}$ is ergodic with respect to a Gaussian measure of full support.

Let H be a complex Hilbert space, let $\mathcal{B}$ be the σ-algebra of Borel subsets of H and let m be a probability measure on $(H, \mathcal{B})$.

Definition 3.3.9 ([3, Definition 3.9])**.** A transformation $\mathcal{A}\colon (H, \mathcal{B}, m) \to (H, \mathcal{B}, m)$ in $\mathcal{B}(H)$ is said to *preserve the measure* m if $m(\mathcal{A}^{-1}(B)) = m(B)$ for every $B \in \mathcal{B}$. A measure-preserving transformation $\mathcal{A}\colon (H, \mathcal{B}, m) \to (H, \mathcal{B}, m)$ is said to be *ergodic* if for every $B \in \mathcal{B}$, $\mathcal{A}^{-1}(B) = B$ implies that $m(B) \in \{0, 1\}$.

The eigenvectors of an operator play an important role in the study of its dynamics. The fact that the eigenvectors of $\mathcal{T}$ span a dense subspace of $\mathcal{X}_\omega$ allowed us to show that $\mathcal{T}$ is hypercyclic under some assumptions on the weight ω. In order to show that $\mathcal{T}$ is ergodic with respect to a Gaussian measure with full support, we will rely on the properties of the eigenvectors of $\mathcal{T}$ associated to unimodular eigenvalues. We first recall a few relevant definitions.

Definition 3.3.10 ([3, Definition 3.1])**.** An operator $\mathcal{A} \in \mathcal{B}(H)$ is said to have a *perfectly spanning set of eigenvectors associated to unimodular eigenvalues* if there exists a continuous probability measure σ on the unit circle $\mathbf{T}$ such that the eigenvectors of $\mathcal{A}$ associated to eigenvalues $\mu \in A$ span a dense subspace in H for every σ-measurable subset A of $\mathbf{T}$ satisfying $\sigma(A) = 1$.

Definition 3.3.11 ([3, Definitions 3.13 and 3.14])**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. A measurable function $f\colon (\Omega, \mathcal{F}, P) \to \mathbf{C}$ is said to have *complex symmetric Gaussian distribution* if $\Re(f)$ and $\Im(f)$ have independent centered Gaussian distribution and the same variance.

A probability measure m on $(H, \mathcal{B})$ is said to be a *Gaussian measure* if the function $y \mapsto \langle y, x\rangle$ has complex symmetric Gaussian distribution for every $x \in H$.

A measure m on $(H, \mathcal{B})$ is said to have a *full support* if $m(U) > 0$ for every non-empty open set $U \in \mathcal{B}$.

Theorem 3.3.12 ([3, Theorem 3.22])**.** If an operator $\mathcal{A} \in \mathcal{B}(H)$ admits a perfectly spanning set of eigenvectors associated to unimodular eigenvalues, then there exists a Gaussian invariant measure with full support m on H such that $\mathcal{A}\colon (H, \mathcal{B}, m) \to (H, \mathcal{B}, m)$ is ergodic.

We are now ready to prove the ergodicity of $\mathcal{T}$.

Theorem 3.3.13. If ω is bounded from below and if $\sum_{n=0}^{\infty} 1/\omega(k2^n) < +\infty$ for every $k \geq 3$, then $\mathcal{T}$ is ergodic with respect to a Gaussian invariant measure with full support on $\mathcal{X}_\omega$.

This is in particular the case when $\omega = \omega_0$.

Proof. By Theorem 3.3.12, it suffices to prove that $\mathcal{T}$ admits a perfectly spanning set of eigenvectors associated to unimodular eigenvalues. Since $\sum_{n=0}^{\infty} 1/\omega(k2^n) < +\infty$ for every $k \geq 3$, the function $h_m(\mu, \cdot)$ belongs to $\mathcal{X}_\omega$ for every $m \geq 0$ and every $\mu \in \mathbf{C}$ such that $|\mu| \leq 1$ by Proposition 3.2.3. We will prove that for every Borel set $A \subset \mathbf{T}$ such that $\sigma(A) = 1$, where σ is the Lebesgue measure on the unit circle, the subspace $\operatorname{span}[h_m(\mu, \cdot); m \geq 0, \mu \in A]$ is dense in $\mathcal{X}_\omega$.

Claim 3.3.14. Let ω be a weight on $\mathbf{Z}_+$ which is bounded from below and such that $\sum_{n=0}^{\infty} 1/\omega(k2^n) < +\infty$ for every $k \geq 3$. Then the subspace $\operatorname{span}[h_m(\mu, \cdot); m \geq 0, \mu \in D]$ is dense in $\mathcal{X}_\omega$ for every dense subset D of $\mathbf{T}$.

Proof. Let $f \in \operatorname{span}[h_m(\mu, \cdot); m \geq 0, \mu \in D]^\perp$ in $\mathcal{X}_\omega$ with $f(z) = \sum_{k=3}^{\infty} c_k z^k$. Our aim is to prove that $f = 0$. For every $\mu \in D$ and every $m \geq 1$

$$\varphi_m(\mu)\colon = \langle h_m(\mu, \cdot), f\rangle = \sum_{n=0}^{\infty} \left(\frac{\overline{c_{(6m+4)2^n}}}{\omega((6m+4)2^n)} - \frac{\overline{c_{(2m+1)2^n}}}{\omega((2m+1)2^n)} \right) \mu^n = 0$$

and

$$\varphi_0(\mu)\colon = \langle h_0(\mu, \cdot), f\rangle = \sum_{n=0}^{\infty} \frac{\overline{c_{2^{n+2}}}}{\omega(2^{n+2})} \mu^n = 0.$$

For every $k \geq 3$ the power series $\sum_{n=0}^{\infty} (\overline{c_{k2^n}}/\omega(k2^n)) z^n$ is uniformly convergent on the closed disk $\overline{\mathbf{D}}$. Indeed for every $n \geq 0$ and every $z \in \overline{\mathbf{D}}$

$$\left| \frac{\overline{c_{k2^n}}}{\omega(k2^n)} z^n \right| \leq \frac{|c_{k2^n}|}{\sqrt{\omega(k2^n)}} \frac{1}{\sqrt{\omega(k2^n)}}$$

and by the Cauchy-Schwarz's inequality

$$\left(\sum_{n=0}^{\infty} \frac{|c_{k2^n}|}{\sqrt{\omega(k2^n)}} \frac{1}{\sqrt{\omega(k2^n)}}\right)^2 \leq \sum_{n=0}^{\infty} \frac{|c_{k2^n}|^2}{\omega(k2^n)} \sum_{n=0}^{\infty} \frac{1}{\omega(k2^n)} < +\infty$$

since f belongs to $\mathcal{X}_\omega$ and by assumption on ω. Then φ_m and φ_0 are holomorphic on $\mathbf{D}$, continuous on $\overline{\mathbf{D}}$ and they vanish on D for every $m \geq 1$. Since D is dense in the unit circle $\mathbf{T}$, the functions φ_m and φ_0 vanish on $\mathbf{T}$ by continuity. By the maximum modulus principle, φ_m and φ_0 identically vanish on $\mathbf{D}$ for every $m \geq 1$ and

$$\frac{c_{(6m+4)2^n}}{\omega((6m+4)2^n)} = \frac{c_{(2m+1)2^n}}{\omega((2m+1)2^n)} \quad \text{and} \quad c_{2^{n+2}} = 0 \quad \text{for every } n \geq 0.$$

It follows then from Claim 3.2.6 that $f = 0$, which gives that $\text{span}[h_m(\mu, \cdot); m \geq 0, \mu \in D]$ is dense in $\mathcal{X}_\omega$. □

Thus since a Borel set $A \subset \mathbf{T}$ satisfying $\sigma(A) = 1$ is dense in $\mathbf{T}$, Claim 3.3.14 gives that $\mathcal{T}$ admits a perfectly spanning set of eigenvectors associated to unimodular eigenvalues, which concludes the proof. □

We now will link the ergodicity of an operator to its frequent hypercyclicity. Let $\mathcal{A} \in \mathcal{B}(H)$ be ergodic with respect to an invariant measure with full support. First it follows from the Birkhoff's transitivity theorem that $\mathcal{A}$ is hypercyclic. Then it follows from Birkhoff's pointwise ergodic theorem that:

Theorem 3.3.15 ([4, Proposition 6.23])**.** If an operator $\mathcal{A} \in \mathbf{B}(H)$ is ergodic with respect to a probability measure m on H with full support, then $\mathcal{A}$ is frequently hypercyclic. Moreover the frequently hypercyclic vectors of $\mathcal{A}$ form a set of full measure for m.

We deduce from Theorems 3.3.13 and 3.3.15 the following result:

Theorem 3.3.16. If ω is bounded from below and if $\sum_{n=0}^{\infty} 1/\omega(k2^n) < +\infty$ for every $k \geq 3$, then $\mathcal{T}$ is frequently hypercyclic on $\mathcal{X}_\omega$.

In particular if $\omega = \omega_0$, $\mathcal{T}$ is frequently hypercyclic on $\mathcal{X}$.

We finish this section by proving that, under the assumptions of Theorem 3.3.13, $\mathcal{T}$ is chaotic.

Definition 3.3.17 ([26, Definition 2.29])**.** An operator $\mathcal{A} \in \mathcal{B}(X)$ is said to be *chaotic* if $\mathcal{A}$ is hypercyclic and has a dense set of periodic points.

Theorem 3.3.18. If ω is bounded from below and if $\sum_{n=0}^{\infty} 1/\omega(k2^n) < +\infty$ for every $k \geq 3$, then $\mathcal{T}$ is chaotic on $\mathcal{X}_\omega$.

In particular if $\omega = \omega_0$, then $\mathcal{T}$ is chaotic on $\mathcal{X}$.

Proof. Since $\sum_{n=0}^{\infty} 1/\omega(k2^n) < +\infty$ for every $k \geq 3$, the function $h_m(\mu, \cdot)$ belongs to $\mathcal{X}_\omega$ for every $m \geq 0$ and every $\mu \in \mathbf{C}$ such that $|\mu| \leq 1$ by Proposition 3.2.3. By Theorem 3.3.4, $\mathcal{T}$ is hypercyclic and it suffices to show that set of its perdiodic points $\text{Per}(\mathcal{T})$ is dense in $\mathcal{X}_\omega$. It follows from [26, Proposition 2.33] that $\text{Per}(\mathcal{T}) = \text{span}[h_m(\mu, \cdot); m \geq 0, \mu \in \{e^{\alpha i \pi}; \alpha \in \mathbf{Q}\}]$, and its density is given by Claim 3.3.14. □

3.4 Open questions

We finish this paper by presenting a few open questions connected to the results we have presented.

Question 3.4.1. In the case where $\omega = \omega_0$, it follows from Propositions 3.2.1 and 3.2.3 that the spectrum $\sigma(\mathcal{T})$ satisfies

$$\{z \in \mathbf{C}; |z| < \sqrt{2}\} \subset \sigma_p(\mathcal{T}) \subset \sigma(\mathcal{T}) \subset \{z \in \mathbf{C}; |z| \leq \sqrt{8/3}\}.$$

What is the exact value of the spectral radius $\rho(\mathcal{T})$ of $\mathcal{T}$? Is it true that $\rho(\mathcal{T}) = \sqrt{2}$?

In order to answer this question, one could perhaps build on the following proposition, which provides an explicit expression for the norm of the iterates $\mathcal{T}^n$ of $\mathcal{T}$:

Proposition 3.4.2. If $\omega = \omega_0$, then for every $n \geq 1$

$$\|\mathcal{T}^n\|^2 = \max_{0 \leq r \leq 3^n - 1} \sum_{a\xi + b \in P_{n,r}^{(n)} \subset \mathbf{Q}[\xi]} \frac{a}{3^n}$$

where $P_{0,r}^{(n)} = \{3^n\xi + r\}$ and $P_{k+1,r}^{(n)} = \{2P; P \in P_{k,r}^{(n)}\} \cup \{(2P-1)/3; P \in P_{k,r}^{(n)}, P(0) \equiv 2 \mod 3\}$ for every $k \geq 0$ and $0 \leq r \leq 3^n - 1$.

Proof. According to Proposition 3.2.1

$$\|\mathcal{T}^n\|^2 = \max_{0 \leq r \leq 3^n - 1} \left\{ \sup_{m \geq \delta_r} \sum_{T^n(j) = 3^n m + r} \frac{j+1}{3^n m + r + 1} \right\}$$

where $\delta_r = 1$ if $r \leq 2$ and $\delta_r = 0$ otherwise. Since $T(j) = k$ if and only if $j = 2k$ or $j = (2k-1)/3$ if $k \equiv 2 \mod 3$, the sets $P_{n,r}^{(n)}$ are such that $T^n(j) = 3^n m + r$ if and only if $j \in \{P(m); P \in P_{n,r}^{(n)}\}$ for every $0 \leq r \leq 3^n - 1$ and every $m \geq \delta_r$. Then

$$\|\mathcal{T}^n\|^2 = \max_{0 \leq r \leq 3^n - 1} \left\{ \sup_{m \geq \delta_r} \sum_{a\xi + b \in P_{n,r}^{(n)}} \frac{am + b + 1}{3^n m + r + 1} \right\}.$$

Fix $0 \leq r \leq 3^n - 1$. If $\alpha, \beta, \gamma, \delta > 0$, one can remark that a sequence $((\alpha m + \beta)/(\gamma m + \delta))_{m \geq 0}$ is strictly increasing or constant if and only if $\alpha\delta \geq \beta\gamma$. We will prove by recursion that $a(r+1) \geq 3^n(b+1)$ for every $a\xi + b \in P_{n,r}^{(n)}$. Indeed this is firstly true for $P_{0,r}^{(n)} = \{3^n\xi + r\}$. Besides if $P_{k,r}^{(n)}$ satisfies it, let $P \in P_{k+1,r}^{(n)}$. Either $P = 2a\xi + 2b$ with $a\xi + b \in P_{k,r}^{(n)}$, or $P = 2a\xi/3 + (2b-1)/3$ with $a\xi + b \in P_{k,r}^{(n)}$ and $b \equiv 2 \mod 3$. On the one hand we would have $2a(r+1) \geq 3^n(2b+2) > 3^n(2b+1)$, and on the other hand we would have $2a(r+1)/3 \geq 3^n(2b+2)/3 \geq 3^n((2b-1)/3+1)$. Then this is true for $P_{k+1,r}^{(n)}$, which proves that $P_{n,r}^{(n)}$ satisfies it. So for every $a\xi + b \in P_{n,r}^{(n)}$

$$\sup_{m \geq \delta_r} \frac{am + b + 1}{3^n m + r + 1} = \lim_{m \to +\infty} \frac{am + b + 1}{3^n m + r + 1} = \frac{a}{3^n},$$

which concludes the proof. $\square$

Question 3.4.3. If $\liminf_{n\to+\infty} \omega(n) = 0$, could $\mathcal{T}$ be bounded, hypercyclic or ergodic ? Could $\mathcal{T}^*$ have any eigenvalue ?

The operator $\mathcal{T}$ can actually be seen as acting on a space of sequences $(c_n)_{n\geq 3}$ instead of on a space of holomorphic functions $f\colon z \mapsto \sum_{n=3}^{\infty} c_n z^n$.

Question 3.4.4. What would remain from these results if we considered $\mathcal{T}$ as acting on the space of complex sequences $\mathbf{C}^{\mathbf{N}}$?

Chapter 4

Invariant Subspace Problem and Bishop operators

The following article [1] has been published by Journal of Mathematical Analysis and Application in March 2023. The proof of Theorem 4.3.10, which includes the Lemma 4.3.9, has been added for the sake of completeness.

Abstract. In this paper, we study the so-called Bishop operators T_α on $L^p([0,1])$, with $\alpha \in (0,1)$ and $1 < p < +\infty$, from the point of view of linear dynamics. We show that they are never hypercyclic nor supercyclic, and investigate extensions of these results to the case of weighted translation operators. We then investigate the cyclicity of the Bishop operators T_α. Building on results by Chalendar and Partington in the case where α is rational, we show that T_α is cyclic for a dense G_δ-set of irrational α's, discuss cyclic functions and provide conditions in terms of convergents of $\alpha \in \mathbf{R}\backslash\mathbf{Q}$ implying that certain functions are cyclic.

4.1 Introduction

One of the most famous open problems in modern operator theory is the Invariant Subspace Problem in the Hilbertian setting, which can be stated as follows:

Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space. For every bounded linear operator T acting on $\mathcal{H}$, does there exist a non-trivial closed invariant subspace for T?

Bishop suggested in the fifties a family of operators as counterexamples to the Invariant Subspace Problem on the complex Hilbert space $L^2([0,1])$, defined for every $\alpha \in [0,1]$ and every $f \in L^2([0,1])$ by

$$T_\alpha f(x) = xf(\{x+\alpha\}) \quad \text{a.e. on } [0,1],$$

where $\{\cdot\}$ denotes the fractional part of a real number. These Bishop operators can be seen as weighted translation operators defined for every $\alpha \in [0,1]$, every $\phi \in L^\infty([0,1])$

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and every $f \in L^2([0,1])$ by

$$T_{\phi,\alpha} f(x) = \phi(x) f(\{x+\alpha\}) \quad \text{a.e. on } [0,1],$$

that is to say the composition of the multiplication operator M_ϕ and of the translation operator U_α defined on $L^2([0,1])$ by

$$M_\phi f = \phi f \quad \text{ and } \quad U_\alpha f = f(\{\cdot + \alpha\}) \quad \text{for every } f \in L^2([0,1]).$$

Even if these two types of operators are separately very well understood, the behavior of the Bishop operators or of the weighted translation operators are still rather mysterious in the general case of irrational numbers α.

In 1965, Parrott first studied in [38] these weighted translation operators and succeeded to compute the spectrum of the Bishop operators, proving that

$$\sigma(T_\alpha) = \left\{z \in \mathbf{C}; |z| \leq e^{-1}\right\} \quad \text{for every irrational number } \alpha \in [0,1].$$

By means of a functional calculus approach, Davie proved in [19] that whenever α is a non-Liouville number in $[0,1]$, the operator T_α admits a non-trivial hyperinvariant subspace, that is to say a non-trivial subspace which is invariant by every operator which commutes with T_α. Recall that an irrational number α is a Liouville number if there exists a sequence $(p_n/q_n)_{n\geq 0}$ of rational numbers such that

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^n} \quad \text{for every } n \geq 0.$$

These results have been extended for instance by MacDonald in [34] to the weighted translation operators $T_{\phi,\alpha}$. However the extensions only involve new weights ϕ while the parameter α still must be a non-Liouville number.

The first one who was able to pass this barrier of Liouville numbers was Flattot in 2008, who proved in [22] that T_α admits non-trivial hyperinvariant subspaces for *some* Liouville numbers α.

The set of such parameters was recently further extended in [18] by Chamizo, Gallardo-Gutiérrez, Monsalve-López and Ubis to include more Liouville numbers. However the existence of non-trivial closed invariant subspaces for T_α for *every* irrational number α in $[0,1]$ is still a tantalizing open problem.

We recall that an operator T does not admit any non-trivial closed invariant subspace if and only if the subspace generated by the T-orbit $\{T^n x; n \geq 0\}$ is dense in $\mathcal{H}$ for every $x \in \mathcal{H}\backslash\{0\}$, that is to say if every $x \in \mathcal{H}\backslash\{0\}$ is cyclic for T.

It is thus an interesting question to investigate whether the operators T_α are cyclic, that is to say admit a vector whose orbit under the action of T_α spans a dense subspace of $L^2([0,1])$. Surprisingly enough, this natural question seems to have been considered only in the case where $\alpha \in \mathbf{Q}$, by Chalendar, Partington and Pozzi, see [16] and [17]. It is the main goal of this paper to study cyclicity of T_α for $\alpha \in \mathbf{R}\backslash\mathbf{Q}$. We also consider the related hypercyclicity (existence of a dense orbit) and supercyclicity (existence of a projective dense orbit) notions. Since the Bishop operators and the weighted translation operators are well-defined on the spaces $L^p([0,1])$ with $1 < p < +\infty$, we will carry out our study in this setting.

The paper is organized as follows. Section 2 deals with hypercyclicity properties of Bishop and weighted translation operators. After recalling the main definitions, we observe (Proposition 4.2.7) that the operators T_α cannot be hypercyclic, and that the weighted translation operators do not satisfy the so-called Hypercyclicity Criterion (Theorem 4.2.9). In Section 3, we will study supercyclicity properties of our operators. We show (Proposition 4.3.14) that weighted translation operators are not supercyclic for a large class of weights, and that no weighted translation operator satisfies the Supercyclicity Criterion (Theorem 4.3.15). The proofs of these results rely on the Positive Supercyclicity Criterion from [32] as well as on an observation of the point spectrum of the adjoints of these operators (Theorem 4.3.13). In Section 4 we study cyclicity of the operators T_α acting on $L^p([0,1])$. Building on results from [17] which characterize cyclic functions for T_α when $\alpha \in \mathbf{Q}$, and using Baire Category arguments, we show (Theorem 4.4.9) that T_α is cyclic for a co-meager set of parameters $\alpha \in [0,1]$. We then explicit a set of irrational numbers $\alpha \in [0,1]$ such that T_α is cyclic, using the continued fraction expansion of α. More precisely, for a certain class of functions $f \in L^p([0,1])$, we show that f is cyclic for T_α as soon as the sequence $(q_n)_{n\geq 0}$ of denominators of the convergents of $\alpha \in \mathbf{R}\backslash\mathbf{Q}$ has infinitely many sufficiently large gaps (Theorem 4.4.16). We then generalize the results from [17] to the case of weighted translation operators with α rational, and generalize our cyclicity results for irrational α to this setting (Theorems 4.4.24, 4.4.27 and 4.4.28). The final Section 5 presents some further remarks and open questions.

4.2 Hypercyclicity

The first well known dynamical property we study is called *hypercyclicity*. We refer the reader to the books [4] and [26] for an in-depth study of this notion, as well as further topics in linear dynamics. Let X be a separable infinite-dimensional complex Banach (or Fréchet) space. We will denote by $\mathcal{B}(X)$ the set of linear and continuous operators on X.

Definition 4.2.1 ([13, Introduction])**.** An operator $T \in \mathcal{B}(X)$ is said to be *hypercyclic* if there exists $x \in X$ such that the T-orbit $\{T^n x; n \geq 0\}$ is dense in X. In this case x is called a hypercyclic vector for T.

This notion is linked to a problem similar to the Invariant Subspace Problem called the Invariant Subset Problem. Indeed, an operator $T \in \mathcal{B}(X)$ does not admit any non-trivial invariant closed subset if and only if for every $x \in X\backslash\{0\}$ the T-orbit $\{T^n x; n \geq 0\}$ is dense in X, that is to say, if and only if every $x \in X\backslash\{0\}$ is hypercyclic for T.

The following theorem shows that the set of hypercyclic vectors of an operator $T \in \mathcal{B}(X)$ is either empty or a dense G_δ-set in X. We recall that a G_δ-set in X is a countable intersection of open sets in X.

Theorem 4.2.2 (Birkhoff's transitivity theorem, [12, §62])**.** An operator $T \in \mathcal{B}(X)$ is hypercyclic if and only if for every pair (U, V) of non-empty open sets in X, there exists $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$. In this case, the set of hypercyclic vectors for T is a dense G_δ-set in X.

Hypercyclicity being a well studied notion, we give here some important examples of hypercyclic operators.

Example 4.2.3 ([26, Examples 2.20, 2.21 and 2.22])**.** Let us consider the Hilbert space $\ell^2(\mathbf{N})$ and the Fréchet space $H(\mathbf{C})$ of entire functions endowed with the family of semi-norms $\|\cdot\|_{n,\infty}\colon f \in H(\mathbf{C}) \mapsto \sup_{|z|\leq n} |f(z)|$, $n \geq 1$. The operators λB on $\ell^2(\mathbf{N})$, D on $H(\mathbf{C})$ and $T^{(a)}$ on $H(\mathbf{C})$, defined for every $(x_1, x_2, \dots) \in \ell^2(\mathbf{N})$ and every $f \in H(\mathbf{C})$ by

$$\lambda B(x_1, x_2, \dots) = (\lambda x_2, \lambda x_3, \dots), \quad D(f) = f' \quad \text{and} \quad T^{(a)} f = f(\cdot + a)$$

and respectively called Rolewicz's operator, MacLane's operator and Birkhoff's operator, are hypercyclic whenever $|\lambda| > 1$ and $a \neq 0$.

A useful tool to prove the hypercyclicity of an operator is the following criterion.

Theorem 4.2.4 (Hypercyclicity Criterion, [11, Theorem 2.3])**.** Let $T \in \mathcal{B}(X)$. Suppose that there exist dense subsets X_0 and Y_0 of X, an increasing sequence $(n_k)_{k\geq 1}$ of integers and a sequence $(S_{n_k}\colon Y_0 \to X)_{k\geq 1}$ of maps such that:

(i) $T^{n_k} x \to 0$ as $k \to +\infty$ for every $x \in X_0$;

(ii) $S_{n_k} y \to 0$ as $k \to +\infty$ for every $y \in Y_0$;

(iii) $T^{n_k} S_{n_k} y \to y$ as $k \to +\infty$ for every $y \in Y_0$.

Then the operator T is hypercyclic.

Bès and Peris actually proved in [11, Theorem 2.3] that an operator T on X satisfies the Hypercyclicity Criterion if and only if the operator $T \oplus T\colon (x, y) \in X \oplus X \mapsto (Tx, Ty)$ is hypercyclic. It was an open question for a long time to give examples of hypercyclic operators not satisfying the Hypercyclicity Criterion.

It has been proved by Peris in [39, Theorem 2.3] that the Hypercyclicity Criterion as stated in Theorem 4.2.4 is actually equivalent to a similar criterion involving a partial inverse map S and its iterates $(S^{n_k})_{k\geq 0}$ instead of a sequence of maps $(S_{n_k})_{k\geq 0}$.

Theorem 4.2.5 (Gethner-Shapiro Criterion, [23, Theorem 2.2] and [26, Theorem 3.10])**.** Let $T \in \mathcal{B}(X)$. Suppose that there exist dense subsets X_0 and Y_0 of X, an increasing sequence $(n_k)_{k\geq 1}$ of integers and a map $S\colon Y_0 \to Y_0$ such that:

(i) $T^{n_k} x \to 0$ as $k \to +\infty$ for every $x \in X_0$;

(ii) $S^{n_k} y \to 0$ as $k \to +\infty$ for every $y \in Y_0$;

(iii) $TSy = y$ for every $y \in Y_0$.

Then the operator T is hypercyclic.

These two criteria are in fact equivalent, although the Hypercyclicity Criterion looks more general than the Gethner-Shapiro Criterion.

Theorem 4.2.6 ([39, Theorem 2.3])**.** An operator $T \in \mathcal{B}(X)$ satisfies the Hypercyclicity Criterion if and only if it satisfies the Gethner-Shapiro Criterion.

4.2.1 Hypercyclicity of Bishop operators

First of all, it is useful to compute the iterates of T_α. For every $\alpha \in [0, 1]$, every $f \in L^p([0, 1])$ and every $n \geq 0$

$$T_\alpha^n f(x) = x\{x + \alpha\} \dots \{x + (n-1)\alpha\} f(\{x + n\alpha\}) \quad \text{a.e. on } [0, 1].$$

We will be interested here in the hypercyclicity of the Bishop operators, and later on in the hypercyclicity of the weighted translation operators. Bishop operators are easily seen not to be hypercyclic.

Proposition 4.2.7. For every $\alpha \in [0,1]$, the Bishop operator T_α is not hypercyclic.

Proof. For every $f \in L^p([0,1])$,

$$\|T_\alpha f\|_p^p = \int_0^1 x^p \, |f(\{x+\alpha\})|^p \, dx \le \int_0^1 |f(y)|^p \, dy \le \|f\|_p^p.$$

Thus $\|T_\alpha\| \le 1$, so every T_α-orbit is bounded in $L^p([0,1])$ and T_α is not hypercyclic. □

4.2.2 Hypercyclicity of weighted translation operators

Since $\|T_{\phi,\alpha}\| = \|\phi\|_\infty$ for every $\phi \in L^\infty([0,1])$ and every $\alpha \in [0,1]$, $T_{\phi,\alpha}$ cannot be hypercyclic if $\|\phi\|_\infty \le 1$. We also observe that $T_{\phi,\alpha}$ cannot be hypercyclic in the case where $m(\{\phi = 0\}) > 0$ if m denotes the Lebesgue measure on $[0,1]$.

Proposition 4.2.8. Let $\phi \in L^\infty([0,1])$ and let $\alpha \in [0,1]$. If $\|\phi\|_\infty \le 1$ or if $m(\{\phi = 0\}) > 0$, then the operator $T_{\phi,\alpha}$ is not hypercyclic.

Proof. Suppose that $m(\{\phi = 0\}) > 0$. Since for every $f \in L^p([0,1])$ and every $n \ge 0$, $T_{\phi,\alpha}^n f(x) = \phi(x)\phi(\{x+\alpha\})\ldots\phi(\{x+(n-1)\alpha\})f(\{x+n\alpha\})$ a.e. on $[0,1]$, then $\{\phi = 0\} \subset \{T_{\phi,\alpha}^n f = 0\}$. Hence the orbit $\{T_{\phi,\alpha}^n f; n \ge 0\}$ cannot be dense in $L^p([0,1])$ since for every $n \ge 0$

$$\|T_{\phi,\alpha}^n f - \mathbf{1}_{\{\phi=0\}}\|_p^p \ge \int_{\{\phi=0\}} 1 dx \ge m(\{\phi = 0\}) > 0.$$

□

In the next section, we will extend the set of weights ϕ such that $T_{\phi,\alpha}$ is not hypercyclic, but we don't know if $T_{\phi,\alpha}$ can ever be hypercyclic. However we will prove that it cannot satisfy the Gethner-Shapiro Criterion, and hence cannot satisfy the Hypercyclicity Criterion either.

Theorem 4.2.9. Let $\phi \in L^\infty([0,1])$ and let $\alpha \in [0,1]$. The operator $T_{\phi,\alpha}$ cannot satisfy the Hypercyclicity Criterion.

Proof. Suppose that $T_{\phi,\alpha}$ satisfies the Hypercyclicity Criterion. Then by Theorem 4.2.6, it satisfies the Gethner-Shapiro Criterion. Suppose also that $m(\{\phi = 0\}) = 0$. One can remark that for every $f, h \in L^p([0,1])$, we have $T_{\phi,\alpha} f = h$ if and only if $f(x) = h(\{x-\alpha\})/\phi(\{x-\alpha\})$ a.e. on $[0,1]$.

Let X_0 and Y_0 be the two dense sets given by the Gethner-Shapiro Criterion, and let $S\colon Y_0 \to Y_0$ be the associated inverse map necessarily given by $Sh(x) = h(\{x-\alpha\})/\phi(\{x-\alpha\})$ a.e. on $[0,1]$ for every $h \in Y_0$. Since X_0 and Y_0 are dense subsets of $L^p([0,1])$, there exist $f \in X_0$ and $g \in Y_0$ such that $m(\{|f| > 1\}) \ge 3/4$ and $m(\{|g| > 1\}) \ge 3/4$. Otherwise one would have for every $f \in X_0$

$$\|f - \mathbf{2}\|_p^p \ge \int_{\{|f| \le 1\}} |2 - |f(x)||^p \, dx \ge m(\{|f| \le 1\}) \ge \frac{1}{4},$$

contradicting the density of X_0 in $L^p([0,1])$ (a similar argument holds for $g \in Y_0$). Then by the Cauchy-Schwarz inequality, we have for every $k \geq 0$

$$\begin{aligned}
\left(\int_0^1 |f(\{x+n_k\alpha\})g(x)|^{p/2}\,dx\right)^2 &\leq \int_0^1 |\phi(x)\ldots\phi(\{x+(n_k-1)\alpha\})f(\{x+n_k\alpha\})|^p\,dx \\
&\quad \times \int_0^1 \frac{|g(x)|^p}{|\phi(x)\ldots\phi(\{x+(n_k-1)\alpha\})|^p}dx \\
&\leq \|T_{\phi,\alpha}^{n_k}f\|_p^p \int_0^1 \frac{|g(\{y-n_k\alpha\})|^p}{|\phi(\{y-n_k\alpha\})\ldots\phi(\{y-\alpha\})|^p}dy \\
&\leq \|T_{\phi,\alpha}^{n_k}f\|_p^p \|S^{n_k}g\|_p^p \xrightarrow[k\to+\infty]{} 0.
\end{aligned}$$

Hence there exists a subsequence $(m_k)_{k\geq 0}$ of $(n_k)_{k\geq 0}$ such that $f(\{x+m_k\alpha\})g(x) \to 0$ as $k \to +\infty$ a.e. on $[0,1]$. We consider now the sets $\omega = \{|f| > 1\}$, $\Omega = \{|g| > 1\}$ and $\Omega' = \Omega \cap \{x \in [0,1]; f(\{x+m_k\alpha\})g(x) \to 0 \text{ as } k \to +\infty\}$. We have $m(\Omega') = m(\Omega) \geq 3/4$. Since

$$\Omega' \subset \bigcup_{K\geq 0}\bigcap_{k\geq K}\{x \in [0,1]; \{x+m_k\alpha\} \notin \omega\}$$

and since the sequence $(\cap_{k\geq K}\{x \in [0,1]; \{x+m_k\alpha\} \notin \omega\})_{K\geq 0}$ is increasing, it follows that the sequence $(m(\cap_{k\geq K}\{x \in [0,1]; \{x+m_k\alpha\} \notin \omega\}))_{K\geq 0}$ admits a limit satisfying

$$m(\Omega') \leq \lim_{K\to+\infty} m\left(\bigcap_{k\geq K}\{x \in [0,1]; \{x+m_k\alpha\} \notin \omega\}\right) \leq \liminf_{K\to+\infty} m(R_\alpha^{-m_K}([0,1]\backslash\omega))$$

where R_α is the translation $R_\alpha\colon x \in [0,1] \mapsto \{x+\alpha\}$ which preserves the Lebesgue measure m. So

$$\frac{3}{4} \leq m(\Omega') \leq \liminf_{K\to+\infty} m(R_\alpha^{-m_k}([0,1]\backslash\omega)) \leq \liminf_{K\to+\infty} m([0,1]\backslash\omega) \leq \frac{1}{4},$$

which is a contradiction. □

4.3 Supercyclicity

We now consider a less restrictive dynamical property, called *supercyclicity*, which has been much studied in the literature.

Definition 4.3.1 ([28, Definition §4])**.** An operator $T \in \mathcal{B}(X)$ is said to be *supercyclic* if there exists $x \in X$ such that the subset $\{\lambda T^n x; \lambda \in \mathbf{C}, n \geq 0\}$ is dense in X. In this case x is called a supercyclic vector for T.

The Birkhoff's Theorem admits a supercyclic version, which runs at follows.

Theorem 4.3.2 ([4, Theorem 1.12])**.** An operator $T \in \mathcal{B}(X)$ is supercyclic if and only if for every pair (U,V) of non-empty open sets in X, there exist $n \geq 0$ and $\lambda \in \mathbf{C}$ such that $\lambda T^n(U) \cap V \neq \emptyset$. In this case, the set of supercyclic vectors for T is a dense G_δ-set in X.

The so-called Supercyclicity Criterion is the most useful tool to prove that an operator is supercyclic. It is patterned after the Hypercyclicity Criterion.

Theorem 4.3.3 (Supercyclicity Criterion, [10, Lemma 3.1])**.** Let $T \in \mathcal{B}(X)$. Suppose that there exist dense subsets X_0 and Y_0 of X, an increasing sequence $(n_k)_{k\geq 1}$ of integers, a sequence $(\lambda_{n_k})_{k\geq 1}$ of non-zero complex numbers and a sequence $(S_{n_k}\colon Y_0 \to X)_{k\geq 1}$ of maps such that:

(i) $\lambda_{n_k} T^{n_k} x \to 0$ as $k \to +\infty$ for every $x \in X_0$;

(ii) $\lambda_{n_k}^{-1} S_{n_k} y \to 0$ as $k \to +\infty$ for every $y \in Y_0$;

(iii) $T^{n_k} S_{n_k} y \to y$ as $k \to +\infty$ for every $y \in Y_0$.

Then the operator T is supercyclic.

We are looking for a supercyclic version of Theorem 4.2.9. To do so, we will prove that the Supercyclicity Criterion is equivalent to a Gethner-Shapiro-type Supercyclicity Criterion. For this we will need the notion of *universality* which is a generalization of hypercyclicity applied to a family of operators $(T_n)_{n\geq 0}$ instead of the iterates of a unique operator T.

Definition 4.3.4 ([25, Definition 1])**.** Let Y be a metric space. A sequence $(T_n\colon X \to Y)_{n\geq 0}$ of continuous maps is said to be *universal* if there exists $x \in X$ such that its orbit $\{T_n x; n \geq 0\}$ under $(T_n)_{n\geq 0}$ is dense in Y. In this case x is called a universal vector for $(T_n)_{n\geq 0}$.

Theorem 4.3.5 (Universality Criterion, [25, Theorem 1])**.** Let Y be a separable metric space and $(T_n\colon X \to Y)_{n\geq 0}$ a family of continuous maps. The set of universal vectors for $(T_n)_{n\geq 0}$ is dense in X if and only if for every non-empty open sets U and V in X and Y, there exists $n \geq 0$ such that $T_n(U) \cap V \neq \emptyset$. In this case the set of universal vectors for $(T_n)_{n\geq 0}$ is a dense G_δ-set in X.

With some adjustments in the proof of the equivalence between the Hypercyclicity Criterion and the Gethner-Shapiro Criterion, given in [26, Theorem 3.22], one can prove that the Supercyclicity Criterion is equivalent to the following Gethner-Shapiro-type Supercyclicity Criterion.

Theorem 4.3.6. An operator $T \in \mathcal{B}(X)$ satisfies the Supercyclicity Criterion if and only if the following holds true: there exist dense subsets X_0 and Y_0 of X, an increasing sequence $(n_k)_{k\geq 1}$ of integers, a sequence $(\lambda_{n_k})_{k\geq 1}$ of non-zero complex numbers and a map $S\colon Y_0 \to Y_0$ such that:

(i) $\lambda_{n_k} T^{n_k} x \to 0$ as $k \to +\infty$ for every $x \in X_0$;

(ii) $\lambda_{n_k}^{-1} S^{n_k} y \to 0$ as $k \to +\infty$ for every $y \in Y_0$;

(iii) $TSy = y$ for every $y \in Y_0$.

Proof. We only have to prove that the Supercyclicity Criterion implies the criterion given by Theorem 4.3.6. So let us suppose that T satisfies the Supercyclicity Criterion. Since T has dense range the Mittag-Leffler's Theorem, as stated in [26, Theorem 3.21] with $X_n = X$ and $f_n = T\colon X \to X$ for every $n \in \mathbf{N}$, gives that the subspace

$$Y = \big\{x \in X;\ \text{there exists } (x_n)_{n\in\mathbf{N}} \in X^{\mathbf{N}}, x = x_1 \text{ and } Tx_{n+1} = x_n \text{ for every } n \in \mathbf{N}\big\}$$

is dense in X. Consider $X^{\mathbf{N}}$ endowed with the product topology, and its closed subspace

$$\mathcal{X} = \left\{(x_n)_{n\in\mathbf{N}} \in X^{\mathbf{N}}; Tx_{n+1} = x_n \text{ for every } n \in \mathbf{N}\right\},$$

which is a separable Fréchet space endowed with the induced topology. We consider the operator $\mathcal{T}\colon \mathcal{X} \to \mathcal{X}$ defined by $\mathcal{T}(x_1, x_2, \ldots) = (Tx_1, Tx_2, \ldots)$ for every $(x_1, x_2, \ldots) \in \mathcal{X}$ and its inverse $\mathcal{B}\colon \mathcal{X} \to \mathcal{X}$ defined by $\mathcal{B}(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ for every $(x_1, x_2, \ldots) \in \mathcal{X}$. Let us now prove that the family $(\lambda_{n_k}\mathcal{T}^{n_k})_{k\geq 1}$ is topologically transitive on $\mathcal{X}$, that is to say that for every pair $(\mathcal{U}_0, \mathcal{V}_0)$ of non-empty open sets in $\mathcal{X}$, there exists $k \geq 1$ such that $\lambda_{n_k}\mathcal{T}^{n_k}(\mathcal{U}_0) \cap \mathcal{V}_0 \neq \emptyset$. Given $\mathcal{U}_0$ and $\mathcal{V}_0$, there exist $N \geq 1$ and non-empty open sets $U_1, \ldots, U_N, V_1, \ldots, V_N$ of X such that

$$\mathcal{U} = \{(x_n)_{n\in\mathbf{N}} \in \mathcal{X}; x_j \in U_j \text{ for every } j \in \{1, \ldots, N\}\} \subset \mathcal{U}_0$$

and

$$\mathcal{V} = \{(y_n)_{n\in\mathbf{N}} \in \mathcal{X}; y_j \in V_j \text{ for every } j \in \{1, \ldots, N\}\} \subset \mathcal{V}_0.$$

Let $x = (x_n)_{n\in\mathbf{N}} \in \mathcal{U}$ and $y = (y_n)_{n\in\mathbf{N}} \in \mathcal{V}$. Since $T^j x_N = x_{N-j} \in U_{N-j}$ and $T^j y_N = y_{N-j} \in V_{N-j}$ whenever $j \in \{1, \ldots, N\}$, there exist non-empty open neighborhoods $U'_N \subset U_N$ and $V'_N \subset V_N$ of x_N and y_N such that $T^j(U'_N) \subset U_{N-j}$ and $T^j(V'_N) \subset V_{N-j}$ whenever $j \in \{1, \ldots, N\}$. Since T satisfies the Supercyclicity Criterion, there exists $k \geq 1$ such that $\lambda_{n_k}T^{n_k}(U'_N) \cap V'_N \neq \emptyset$, and hence there exists a non-empty open set $U''_N \subset U'_N$ such that $\lambda_{n_k}T^{n_k}(U''_N) \subset V'_N$. Let $u_N \in Y \cap U''_N$, which exists by density of Y. Since $u_N \in Y$, there exists $(u_n)_{n\geq N}$ such that $Tu_{n+1} = u_n$ whenever $n \geq N$. Moreover we consider $u_j = T^{N-j}u_N \in U_j$ whenever $j \in \{1, \ldots, N-1\}$ since $u_N \in U''_N \subset U'_N$. So $u = (u_n)_{n\in\mathbf{N}} \in \mathcal{U}$ and besides $\lambda_{n_k}\mathcal{T}^{n_k}u \in \mathcal{V}$. Indeed $\lambda_{n_k}\mathcal{T}^{n_k}u = (\lambda_{n_k}T^{n_k}u_1, \lambda_{n_k}T^{n_k}u_2, \ldots)$ and $\lambda_{n_k}T^{n_k}u_j = \lambda_{n_k}T^{n_k}T^{N-j}u_N = T^{N-j}(\lambda_{n_k}T^{n_k}u_N) \in T^{N-j}(V'_N) \subset V_j$ whenever $j \in \{1, \ldots, N\}$ because $\lambda_{n_k}T^{n_k}u_N \in \lambda_{n_k}T^{n_k}(U''_N) \subset V'_N$. Thus $\lambda_{n_k}\mathcal{T}^{n_k}u \in \lambda_{n_k}\mathcal{T}^{n_k}(\mathcal{U}) \cap \mathcal{V}$, which is hence non-empty.

Let us prove now that there exist a dense subset Y'_0 of X, a map $S\colon Y'_0 \to Y'_0$ and a subsequence $(m_k)_{k\geq 1}$ of $(n_k)_{k\geq 1}$ such that $TSy = y$ and $\lambda_{m_k}^{-1}S^{m_k}y \to 0$ as $k \to +\infty$ whenever $y \in Y'_0$. Since $(\lambda_{n_k}\mathcal{T}^{n_k})_{k\geq 1}$ is topologically transitive, so is $(\lambda_{n_k}^{-1}\mathcal{B}^{n_k})_{k\geq 1}$ and the sets

$$\{x \in \mathcal{X}; \{\lambda_{n_k}\mathcal{T}^{n_k}x; k \geq 1\} \text{ is dense in } \mathcal{X}\} \text{ and } \left\{x \in \mathcal{X}; \left\{\lambda_{n_k}^{-1}\mathcal{B}^{n_k}x; k \geq 1\right\} \text{ is dense in } \mathcal{X}\right\}$$

are dense G_δ-subsets of $\mathcal{X}$ by Theorem 4.3.5. So there exists $y = (y_n)_{n\in\mathbf{N}} \in \mathcal{X}$ such that the sets

$$\{\lambda_{n_k}\mathcal{T}^{n_k}y; k \geq 1\} = \{(\lambda_{n_k}T^{n_k}y_1, \lambda_{n_k}T^{n_k}y_2, \ldots); k \geq 1\}$$

and

$$\left\{\lambda_{n_k}^{-1}\mathcal{B}^{n_k}y; k \geq 1\right\} = \left\{(\lambda_{n_k}^{-1}y_{1+n_k}, \lambda_{n_k}^{-1}y_{2+n_k}, \ldots); k \geq 1\right\}$$

are dense in $\mathcal{X}$. We now consider the subset $Y'_0 = \{\lambda y_n; \lambda \in \mathbf{C}\backslash\{0\} \text{ and } n \in \mathbf{N}\}$ of X, which is dense in X by density of the projection onto the first coordinate of the dense subset $\{\lambda_{n_k}^{-1}\mathcal{B}^{n_k}y; k \geq 1\}$ in $\mathcal{X}$. Let us consider the map $S\colon Y'_0 \to Y'_0$ defined by $S(\lambda y_n) = \lambda y_{n+1}$ for every $\lambda y_n \in Y'_0$. It is well defined because if $\lambda y_n = \mu y_m$ with $n < m$, then $y_n = (\mu/\lambda)y_m$

and $T^{m-n}y_m = y_n = (\mu/\lambda)y_m$ since $y \in \mathcal{X}$. Hence the subset

$$\{\lambda_{n_k}T^{n_k}y_m; k \geq 1\} \subset \mathrm{span}[T^k y_m; k \in \{0, \ldots, m-n-1\}]$$

could not be dense in X, which would contradict the density of the projection onto the m-th coordinate of the dense subset $\{\lambda_{n_k}\mathcal{T}^{n_k}y; k \geq 1\}$ of $\mathcal{X}$.

Moreover $TS\lambda y_n = \lambda T y_{n+1} = \lambda y_n$ whenever $\lambda y_n \in Y_0'$ because $y \in \mathcal{X}$. Since the subset $\{\lambda_{n_k}^{-1}\mathcal{B}^{n_k}y; k \geq 1\} = \{(\lambda_{n_k}^{-1}y_{1+n_k}, \lambda_{n_k}^{-1}y_{2+n_k}, \ldots)\}$ is dense in $\mathcal{X}$, there exists a subsequence $(m_k)_{k\geq 1}$ of $(n_k)_{k\geq 1}$ such that $\lambda_{m_k}^{-1}\mathcal{B}^{m_k}y = (\lambda_{m_k}^{-1}y_{1+m_k}, \lambda_{m_k}^{-1}y_{2+m_k}, \ldots) \to 0$ as $k \to +\infty$ and hence $\lambda_{m_k}^{-1}S^{m_k}\lambda y_n = \lambda\lambda_{m_k}^{-1}y_{n+m_k} \to 0$ as $k \to +\infty$ whenever $\lambda y_n \in Y_0'$. Eventually since T satisfies the Supercyclicity Criterion, we consider the dense subset $X_0' = X_0$ of X which has the property that $\lambda_{n_k}T^{n_k}x \to 0$ as $k \to +\infty$ whenever $x \in X_0'$. We have thus shown that T satisfies the following three properties:

(i) $\lambda_{m_k}T^{m_k}x \to 0$ as $k \to +\infty$ for every $x \in X_0'$;

(ii) $\lambda_{m_k}^{-1}S^{m_k}y \to 0$ as $k \to +\infty$ for every $y \in Y_0'$;

(iii) $TSy = y$ for every $y \in Y_0'$,

and this terminates the proof. □

4.3.1 Supercyclicity of Bishop operators

In order to prove that the Bishop operator cannot be supercyclic we will need the so-called *Positive Supercyclicity Theorem*, proved by León-Saavedra and Müller in [32, Corollary 4], see also [4, Corollary 3.4]. We will denote by $\sigma_p(T)$ the set of eigenvalues of an operator $T \in \mathcal{B}(X)$.

Theorem 4.3.7 (Positive Supercyclicity Theorem, [4, Corollary 3.4])**.** *Let $T \in \mathcal{B}(X)$. If $\sigma_p(T^*) = \emptyset$, then $x \in X$ is supercyclic for T if and only if the set $\{aT^n x; a \in (0, +\infty), n \geq 0\}$ is dense in X.*

We will also need the following straightforward lemma.

Lemma 4.3.8. *If a subset A is dense in $L^p([0,1])$, then the set of the Lebesgue measures $\{m(\{\Re(f) > 0\}); f \in A\}$ is dense in $[0,1]$.*

Davie proved in [19, Theorem 2] that the Bishop operator T_α has no eigenvalue whatever the value of $\alpha \in [0,1]$. The same ideas are recalled and generalized by Flattot in [22, Theorem 2.1]. In fact, the approach proposed by Flattot in [22] allows us to prove that the adjoint of the Bishop operator $T_\alpha^* \in \mathcal{B}(L^{p'}([0,1]))$, defined for every $f \in L^{p'}([0,1])$ by

$$T_\alpha^* f(x) = \{x - \alpha\} f(\{x - \alpha\}) \quad \text{a.e. on } [0,1]$$

with $1/p + 1/p' = 1$, has no eigenvalue. To do so, we will need the following lemma.

Lemma 4.3.9 ([22, Lemma 2.2])**.** *If $f, g\colon [a,b] \to \mathbf{R}$ are increasing, convex, non-negative functions, then the product fg also satisfies these properties.*

Moreover if $f(a) = 0$ then $m(\{x \in [a,b]; |1 - f(x)| > 1/2\}) \geq (b-a)/3$.

Proof. It suffices to show that fg is convex. Let $t \in [0,1]$ and let $x, y \in [a,b]$. We know that $0 \leq f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$ and $0 \leq g(tx+(1-t)y) \leq tg(x)+(1-t)g(y)$, thus

$$\begin{aligned} 0 &\leq fg(tx+(1-t)y) \\ &\leq (tf(x)+(1-t)f(y))(tg(x)+(1-t)g(y)) \\ &\leq t^2 fg(x) + t(1-t)(f(x)g(y)+f(y)g(x)) + (1-t)^2 fg(y). \end{aligned}$$

It suffices now to show that

$$t^2 fg(x) + t(1-t)(f(x)g(y)+f(y)g(x)) + (1-t)^2 fg(y) \leq tfg(x) + (1-t)fg(y),$$

which is equivalent to

$$0 \leq t(1-t)(fg(x)+fg(y)-f(x)g(y)-f(y)g(x)).$$

However $fg(x)+fg(y)-f(x)g(y)-f(y)g(x) = (f(y)-f(x))(g(y)-g(x)) > 0$ since f and g are increasing. So fg is convex.

Suppose now that $f(a) = 0$.

If $f(b) \leq 1/2$ then $[a,b] \subset \{|1-f| \geq 1/2\}$, which gives that $m(\{x \in [a,b]; |1-f(x)| \geq 1/2\}) = b-a \geq (b-a)/3$.

If $f(b) > 1/2$, there exists $t_1 \in (a,b)$ such that $f(t_1) = 1/2$. On the one hand if $t_1 - a \geq (b-a)/3$ then the inclusion $[a,t_1] \subset \{|1-f| \geq 1/2\}$ gives that $m(\{|1-f(x)| \geq 1/2\}) \geq t_1 - a \geq (b-a)/3$. On the other hand if $t_1 - a < (b-a)/3$, then $f(b) > 3/2$. Indeed by convexity

$$\frac{f(b)-f(t_1)}{b-t_1} \geq \frac{f(t_1)-f(a)}{t_1-a},$$

which is equivalent to

$$f(b) \geq \frac{1}{2} + \frac{b-t_1}{2(t_1-a)}.$$

Thus $f(b) > 3/2$ since $t_1 - a < (b-a)/3$ and $b - t_1 > 2(b-a)/3$. So there exists $t_2 \in (t_1, b)$ such that $f(t_2) = 3/2$. Then $\{|1-f| \geq 1/2\} = [a,t_1] \cup [t_2,b]$ and $m(\{x \in [a,b]; |1-f(x)| \geq 1/2\}) = t_1 - a + b - t_2$. We now show that $t_2 - t_1 \leq 2(b-a)/3$. By convexity

$$\frac{f(t_2)-f(t_1)}{t_2-t_1} \geq \frac{f(t_1)-f(a)}{t_1-a},$$

which is equivalent to $t_2 - t_1 \leq 2(t_1 - a)$. Thus $t_2 - t_1 \leq 2(b-a)/3$ since $t_1 - a < (b-a)/3$. □

Let us now prove that the adjoint of the Bishop operator T_α^* has no eigenvalue.

Theorem 4.3.10. For every $\alpha \in [0,1]$, the adjoint operator $T_\alpha^* \in \mathcal{B}(L^{p'}([0,1]))$ has no eigenvalue.

Proof. First, let us suppose that $\alpha = r/q$ is a rational number where r and q are coprime.

Let $\lambda \in \mathbf{C}$ and let $f \in L^{p'}([0,1])$ satisfying $T_\alpha^* f = \lambda f$. One can compute that a.e. on $[0,1]$

$$\begin{aligned} T_\alpha^{q*} f(x) &= \{x - r/q\}\{x - 2r/q\} \ldots \{x - qr/q\} f(\{x - qr/q\}) \\ &= \{x + (q-1)r/q\}\{x + (q-2)r/q\} \ldots x f(x). \end{aligned}$$

Since the map $k \in \{0, \ldots, q-1\} \mapsto \{kr/q\}$ is one-to-one from the set $\{0, \ldots, q-1\}$ onto the set $\{0, 1/q, \ldots, (q-1)/q\}$, then $T_\alpha^{q*} f(x) = x\{x + 1/q\} \ldots \{x + (q-1)/q\} f(x)$. If w is the function defined by $w(x) = x\{x + 1/q\} \ldots \{x + (q-1)/q\}$ for every $x \in [0,1]$, the function f satisfies $T_\alpha^{q*} f = wf = \lambda^q f$. One can remark that $\{f \neq 0\} \subset \{w = \lambda^q\}$, implying that $m(\{f \neq 0\}) \leq m(\{w = \lambda^q\})$. However w is $1/q$-periodic and since $w(x) = x(x + 1/q) \ldots (x + (q-1)/q)$ for every $x \in [0, 1/q)$, it is strictly increasing on $[0, 1/q)$. So the set $\{w = \lambda^q\}$ is finite and its Lebesgue measure satisfies $m(\{w = \lambda^q\}) = 0$. Then $f = 0$, that is to say that the point spectrum of the adjoint operator is empty.

Now suppose that $\alpha \in [0,1]$ is irrational. The proof follows the lines of the proof of Theorem 2.1 in [22]. By the Dirichlet's approximation Theorem, there exists a sequence $(p_n/q_n)_{n \geq 1}$ of rational numbers satisfying $q_n \to +\infty$ as $n \to +\infty$ such that we have $|\alpha - p_n/q_n| \leq 1/q_n^2$ for every $n \geq 1$.

Since T_α^* is injective, let $\lambda \in \mathbf{C} \backslash \{0\}$ and let $f \in L^{p'}([0,1])$ be a function satisfying $T_\alpha^* f = \lambda f$. Then for $n \geq 1$, we have $T_\alpha^{q_n *} f = \lambda^{q_n} f$, so $T_\alpha^{q_n *} f(\{x + q_n \alpha\})/\lambda^{n_k} = f(\{x + q_n \alpha\})$ a.e. on $[0,1]$, that is to say we have $\{x + (q_n - 1)\alpha\} \ldots \{x + \alpha\} x f(x)/\lambda^{n_k} = f(\{x + q_n \alpha\})$. Hence $f(\{x + q_n \alpha\}) = F_n(x) f(x)$ a.e. on $[0,1]$, where F_n is the function defined by $F_n(x) = \{x + (q_n - 1)\alpha\} \ldots \{x + \alpha\} x/\lambda^{q_n}$ for every $x \in [0,1]$. Then we have $f(x) - f(\{x + q_n \alpha\}) = (1 - F_n(x)) f(x)$ a.e. on $[0,1]$, and thus $|f(x)| = |f(x) - f(\{x + q_n \alpha\})| / |1 - F_n(x)|$.

As in the proof of Theorem 2.1 in [22], $(f(\{\cdot + q_n \alpha\}))_{n \geq 1}$ tends to f in measure and one can construct a sequence $(n_k)_{k \geq 1}$ of positive integers such that

$$m\left(\bigcap_{k=1}^{\infty} \left\{ x \in [0,1]; |f(x) - f(\{x + q_{n_k} \alpha\})| < \frac{1}{2k} \right\} \right) \geq \frac{5}{6}.$$

We will note

$$A = \bigcap_{k=1}^{\infty} \left\{ x \in [0,1]; |f(x) - f(\{x + q_{n_k} \alpha\})| < \frac{1}{2k} \right\}$$

and fix $k \geq 1$. Let $\sigma \colon \{1, \ldots, q_{n_k} - 1\} \to \{1, \ldots, q_{n_k} - 1\}$ be the unique permutation of the set $\{1, \ldots, q_{n_k} - 1\}$ which satisfies $0 < 1 - \{\sigma(1)\alpha\} < \ldots < 1 - \{\sigma(q_{n_k} - 1)\alpha\}$. Then one can remark that $|F_{n_k}| : x \mapsto x\{x + \alpha\} \ldots \{x + (q_{n_k} - 1)\alpha\} / |\lambda|^{n_k}$ has a continuous extension which satisfies the conditions of Lemma 4.3.9 on each interval $[1 - \{\sigma(j)\alpha\}, 1 - \{\sigma(j+1)\alpha\}]$ if $j \in \{1, \ldots, q_{n_k} - 2\}$ and on the intervals $[0, 1 - \{\sigma(1)\alpha\}]$ and $[1 - \{\sigma(q_{n_k} - 1)\alpha\}, 1]$ denoted by $I_0, \ldots, I_{q_{n_k} - 1}$.

Indeed $F_{n_k}(0) = F_{n_k}(1 - \{\sigma(j)\alpha\}) = 0$ and the map $x \mapsto \{x + \sigma(j)\alpha\} = \{x + \{\sigma(j)\alpha\}\}$ is an increasing, non-negative, affine function on both intervals $[0, 1 - \{j\alpha\})$ and $[1 - \{j\alpha\}, 1)$ for every $j \in \{1, \ldots, q_{n_k} - 1\}$. Thus by Lemma 4.3.9, for every $i \in \{0, \ldots, q_{n_k} - 1\}$ $m\left(\{x \in I_i; |1 - |F_{n_k}(x)|| > 1/2\}\right) \geq \text{length}(I_i)/3$ and then we have the inequality $m\left(\{x \in [0,1]; |1 - |F_{n_k}(x)|| > 1/2\}\right) \geq 1/3$ by summing. Since $|1 - F_{n_k}| \geq |1 - |F_{n_k}||$, we have $m(\{x \in [0,1]; |1 - F_{n_k}(x)| > 1/2\}) \geq 1/3$.

If we set $B_k = \{x \in [0,1]; |1 - F_{n_k}(x)| > 1/2\}$ and $B = \cap_{n=1}^{\infty} \cup_{k \geq n} B_k$ then $m(B) =$

$\lim_{n\to+\infty} m(\cup_{k\geq n} B_k) \geq 1/3$ and we obtain the inequality

$$m(A\cap B) = 1 - m([0,1]\backslash(A\cup B)) \geq 1 - \frac{1}{6} - \frac{2}{3} \geq \frac{1}{6}.$$

Moreover for every $x \in A\cap B$ and every $n \geq 1$, there exists $k \geq n$ such that $x \in B_k$ and $|f(x) - f(\{x + q_{n_k}\alpha\})| < 1/2k$, thus $|f(x)|/2 \leq |f(x)|\,|1 - F_{n_k}(x)| < 1/2k$, which implies that $|f(x)| \leq 1/k$. So $f(x) = 0$, and thus we have $m(\{x \in [0,1]; f(x) = 0\}) \geq m(A\cap B) \geq 1/6$. Since $T_\alpha^* f = \lambda f$, then for almost every x in $[0,1]$, $f(x) = 0$ implies that $f(\{x+\alpha\}) = 0$. Then the set $\{x \in [0,1]; f(x) = 0\}$ is α-invariant, implying that

$$1 = m\left(\bigcup_{k=1}^{\infty} R_\alpha^k(\{x \in [0,1]; f(x) = 0\})\right) \leq m(\{x \in [0,1]; f(x) = 0\})$$

because $m(\{x \in [0,1]; f(x) = 0\}) \geq 1/6$ and R_α is an ergodic transformation. So $f = 0$, that is to say, the point spectrum of the adjoint operator is empty. □

We can now deduce from the Positive Supercyclicity Theorem that T_α is not supercyclic.

Theorem 4.3.11. For every $\alpha \in [0,1]$, the Bishop operator T_α is not supercyclic.

Proof. Let $f \in L^p([0,1])$. By Theorem 4.3.7, f is supercyclic for T_α if and only if the set $\{aT_\alpha^n f; a > 0, n \geq 0\}$ is dense in $L^p([0,1])$. Then by Lemma 4.3.8, the set of Lebesgue measures $\{m(\{\Re(aT_\alpha^n f) > 0\}); a > 0, n \geq 0\}$ is dense in $[0,1]$. Since $aT_\alpha^n f(x) = ax\{x+\alpha\}\dots\{x+(n-1)\alpha\}f(\{x+n\alpha\})$ a.e. on $[0,1]$, for every $a > 0$ and every $n \geq 1$, we have $\Re(aT_\alpha^n f)(x) > 0$ if and only if $ax\{x+\alpha\}\dots\{x+(n-1)\alpha\}\Re(f)(\{x+n\alpha\}) > 0$. So

$$\begin{aligned} m(\{\Re(aT_\alpha^n f) > 0\}) &= m(\{\Re(f)(\{\cdot + n\alpha\}) > 0\}) \\ &= m(R_\alpha^{-n}(\{\Re(f) > 0\})) \\ &= m(\{\Re(f) > 0\}) \end{aligned}$$

since R_α preserves the measure m. Then the set of the Lebesgue measures, which is equal to the singleton $\{m(\{\Re(f) > 0\})\}$, is not dense in $[0,1]$. Hence f is not supercyclic for T_α. □

4.3.2 Supercyclicity of weighted translation operators

Let $\phi \in L^\infty([0,1])$, one can remark already that $T_{\phi,\alpha}$ cannot be supercyclic if the measure $m(\{\phi = 0\})$ is positive.

Proposition 4.3.12. Let ϕ be a function in $L^\infty([0,1])$ satisfying $m(\{\phi = 0\}) > 0$ and let $\alpha \in [0,1]$. The operator $T_{\phi,\alpha}$ is not supercyclic.

Proof. For every $\lambda \in \mathbf{C}$, every $n \geq 0$ and every $f \in L^p([0,1])$, we have the inequality $0 < m(\{\phi = 0\}) \leq m(\{\lambda T_{\phi,\alpha}^n f = 0\})$, so the set $\{\lambda T_{\phi,\alpha}^n f; \lambda \in \mathbf{C}, n \geq 0\}$ cannot be dense in $L^p([0,1])$. □

In order to generalize the result of Theorem 4.3.11 about the non-supercyclicity to the weighted translation operators $T_{\phi,\alpha}$, one would need the adjoint operator $T_{\phi,\alpha}^*$, defined on

$L^{p'}([0,1])$ by $T^*_{\phi,\alpha}f(x) = \phi(\{x-\alpha\})f(\{x-\alpha\})$ for every $f \in L^{p'}([0,1])$ and a.e. on $[0,1]$, to have no eigenvalue whenever $\alpha \in [0,1]$.

The argument of Flattot in [22, Theorem 2.1] proving that weighted translation operators $T_{\phi,\alpha}$ have no eigenvalue can be adjusted to show the following result. We omit the proof because it follows the proof of Theorem 4.3.10.

Theorem 4.3.13. Let $\phi \in L^\infty([0,1])$ be an increasing convex function such that $\phi(0) = 0$. For every $\alpha \in [0,1]$, the operators $T_{\phi,\alpha} \in \mathcal{B}(L^p([0,1]))$ and $T^*_{\phi,\alpha} \in \mathcal{B}(L^{p'}([0,1]))$ have no eigenvalue.

As in the case of Bishop operators (Theorem 4.3.11), the non-negativity of the weight gives the following result.

Proposition 4.3.14. Let $\phi \in L^\infty([0,1])$ be an increasing convex function such that $\phi(0) = 0$. For every $\alpha \in [0,1]$, the weighted translation operator $T_{\phi,\alpha}$ cannot be supercyclic.

Proof. Let $f \in L^p([0,1])$. Since $\sigma_p(T^*_{\phi,\alpha}) = \emptyset$, by Theorem 4.3.7, f is supercyclic for $T_{\phi,\alpha}$ if and only if the set $\{aT^n_{\phi,\alpha}f; a > 0, n \geq 0\}$ is dense in $L^p([0,1])$. Then by Lemma 4.3.8, the set $\{m(\{\Re(aT^n_{\phi,\alpha}f) > 0\}); a > 0, n \geq 0\}$ is dense in $[0,1]$. Since $aT^n_{\phi,\alpha}f(x) = \phi(x)\phi(\{x+\alpha\})\ldots\phi(\{x+(n-1)\alpha\})f(\{x+n\alpha\})$ a.e. on $[0,1]$, for every $a > 0$ and every $n \geq 0$, then $\Re(aT^n_{\phi,\alpha}f)(x) > 0$ if and only if $\phi(x)\phi(\{x+\alpha\})\ldots\phi(\{x+(n-1)\alpha\})\Re(f)(\{x+n\alpha\}) > 0$. So

$$m(\{\Re(aT^n_{\phi,\alpha}f) > 0\}) = m(R_\alpha^{-n}(\{\Re(f) > 0\})) = m(\{\Re(f) > 0\})$$

since ϕ is positive on $(0,1]$ and R_α preserves the measure m. Then the set of the Lebesgue measures, which is equal to the singleton $\{m(\{\Re(f)(x) > 0\})\}$, is not dense in $[0,1]$, and f is not supercyclic for $T_{\phi,\alpha}$. ☐

Using Theorem 4.3.6, we can prove that the Supercyclicity Criterion cannot be satisfied by any weighted translation operator. The proof proceeds in the same way as in the hypercyclic case.

Theorem 4.3.15. For every $\phi \in L^\infty([0,1])$ and every $\alpha \in [0,1]$, the operator $T_{\phi,\alpha}$ does not satisfy the Supercyclicity Criterion.

Proof. Suppose that $T_{\phi,\alpha}$ satisfies the Supercyclicity Criterion. Then by Theorem 4.3.6, it satisfies the Gethner-Shapiro-type Supercyclicity Criterion with data X_0, Y_0 and S. Suppose also that $m(\{\phi = 0\}) = 0$. Since X_0 and Y_0 are dense subsets of $L^p([0,1])$, let $f \in X_0$ and $g \in Y_0$ be functions such that $m(\{|f| > 1\}) \geq 3/4$ and $m(\{|g| > 1\}) \geq 3/4$. Then by the Cauchy-Schwarz inequality, for every $k \geq 0$

$$\begin{aligned}\left(\int_0^1 |f(\{x+n_k\alpha\})g(x)|^{p/2}\,dx\right)^2 &\leq \int_0^1 |\lambda_{n_k}\phi(x)\ldots\phi(\{x+(n_k-1)\alpha\})f(\{x+n_k\alpha\})|^p\,dx \\ &\quad\times \int_0^1 \frac{|g(x)|^p}{|\lambda_{n_k}\phi(x)\ldots\phi(\{x+(n_k-1)\alpha\})|^p}dx \\ &\leq \|\lambda_{n_k}T^{n_k}_{\phi,\alpha}f\|_p^p \int_0^1 \frac{|g(\{y-n_k\alpha\})|^p}{|\lambda_{n_k}\phi(\{y-n_k\alpha\})\ldots\phi(\{y-\alpha\})|^p}dy \\ &\leq \|\lambda_{n_k}T^{n_k}_{\phi,\alpha}f\|_p^p \left\|\lambda_{n_k}^{-1}S^{n_k}g\right\|_p^p \xrightarrow[k\to+\infty]{} 0\end{aligned}$$

with the map $S\colon Y_0 \to Y_0$ defined by $Sg(x) = g(\{x-\alpha\})/\phi(\{x-\alpha\})$ a.e. on $[0,1]$ for every $g \in Y_0$. As in the hypercyclic case if $(m_k)_{k\geq 0}$ is a subsequence of $(n_k)_{k\geq 0}$ such that $f(\{x+m_k\alpha\}g(x) \to 0$ as $k \to +\infty$ for every x in a full measure set A in $[0,1]$, then the sets $\omega = \{|f| > 1\}$ and $\Omega' = \{|g| > 1\} \cap A$ satisfy

$$\frac{3}{4} \leq m(\Omega') \leq m\left(\bigcup_{K\geq 0}\bigcap_{k\geq K}\{x \in [0,1]; \{x+m_k\alpha\} \notin \omega\}\right) \leq m([0,1]\backslash\omega) \leq \frac{1}{4},$$

which is impossible. □

4.4 Cyclicity

Our aim is now to investigate the cyclicity properties of the Bishop and weighted translation operators. Cyclicity is the less restrictive dynamical property of operators which we consider in this paper. We will denote by $\mathbf{C}[\xi]$ the set of complex polynomials.

Definition 4.4.1 ([24, Introduction]). An operator $T \in \mathcal{B}(X)$ is said to be *cyclic* if there exists $x \in X$ such that the linear subspace $\text{span}[T^n x; n \geq 0] = \{P(T)x; P \in \mathbf{C}[\xi]\}$ is dense in X. In this case x is called a cyclic vector for T.

As in the case of hypercyclicity and supercyclicity, there exists a well known necessary and sufficient condition for the cyclicity of an operator in the case where T^* has no eigenvalue, proved by Baire Category arguments.

Proposition 4.4.2. Let $T \in \mathcal{B}(X)$ such that $\sigma_p(T^*) = \emptyset$. The following assertions are equivalent:

(i) T is cyclic;

(ii) For every pair (U,V) of non-empty open sets in X, there exists $P \in \mathbf{C}[\xi]$ such that $P(T)(U) \cap V \neq \emptyset$.

In this case, the set of cyclic vectors for T is a dense G_δ-set in X.

Proof. Suppose that the operator T is cyclic. Let x be a cyclic vector for T and let U and V be two non-empty open sets in X. There exists $P \in \mathcal{C}[\xi]$ such that $P(T)x \in U$. Since $\sigma_p(T^*) = \emptyset$, the operator $P(T)$ has dense range. Hence one can find an open set W of X such that $P(T)(W) \subset V$. Once again by cyclicity there exists $Q \in \mathcal{C}[\xi]$ such that $Q(T)x \in W$ and then $P(T)Q(T)x = Q(T)P(T)x \in Q(T)(U) \cap V \neq \emptyset$.

Conversely, let us suppose that the condition (ii) holds and let $(U_k)_{k\geq 1}$ be a basis of non-empty open sets in X. A vector $x \in X$ is cyclic for T if and only if for every $k \geq 1$, there exists $P \in \mathbf{C}[\xi]$ such that $P(T)x \in U_k$, that is to say if and only if $x \in \bigcap_{k\geq 1}\bigcup_{P\in\mathbf{C}[\xi]} P(T)^{-1}(U_k)$. The union $\cup_{P\in\mathbf{C}[\xi]}P(T)^{-1}(U_k)$ is a dense open set in X for every $k \geq 1$. Indeed for every non-empty open set V in X there exists $P \in \mathbf{C}[\xi]$ such that $P(T)(V) \cap U_k \neq \emptyset$, that is to say that $P(T)^{-1}(U_k) \cap V \neq \emptyset$. So $\cup_{P\in\mathbf{C}[\xi]}P(T)^{-1}(U_k) \cap V \neq \emptyset$. By the Baire Category Theorem, the set of cyclic vectors is a dense G_δ-set in X and thus a non-empty set. □

4.4.1 Cyclicity of Bishop operators in the rational case

The cyclicity of the Bishop operator T_α has already been studied in the case $\alpha \in \mathbf{Q}$ by Chalendar and Partington in [17, Section 5.4] and generalized to multivariable Bishop operators by Chalendar, Partington and Pozzi in [16, Section 4].

Definition 4.4.3 ([17, Definition 5.4.1])**.** Let $f \in L^p([0,1])$ and let $\alpha = r/q$ be a rational number such that $0 < r < q$ and r and q are coprime. We define $\Delta(f, r/q)$ a.e. on $[0,1]$ by

$$\begin{vmatrix} f(t) & T_{r/q}f(t) & \cdots & T_{r/q}^{q-1}f(t) \\ f\left(\{t+r/q\}\right) & T_{r/q}f\left(\{t+r/q\}\right) & \cdots & T_{r/q}^{q-1}f\left(\{t+r/q\}\right) \\ \vdots & \vdots & & \vdots \\ f\left(\{t+(q-1)r/q\}\right) & T_{r/q}f\left(\{t+(q-1)r/q\}\right) & \cdots & T_{r/q}^{q-1}f\left(\{t+(q-1)r/q\}\right) \end{vmatrix}.$$

This function is used in [17, Theorem 5.4.4] to give a necessary and sufficient condition for a function $f \in L^p([0,1])$ to be cyclic for T_α, where $\alpha \in (0,1) \cap \mathbf{Q}$.

Theorem 4.4.4 ([17, Theorem 5.4.4])**.** Let $\alpha = r/q$ be a rational number such that $0 < r < q$ and r and q are coprime. A function $f \in L^p([0,1])$ is cyclic for $T_{r/q}$ if and only if the function $\Delta(f, r/q)$ satisfies $m(\{t \in [0,1]; \Delta(f, r/q)(t) = 0\}) = 0$.

One can deduce from the previous theorem a set of common cyclic vectors for the family of operators T_α, $\alpha \in (0,1) \cap \mathbf{Q}$.

Theorem 4.4.5. Any holomorphic function f on a open neighborhood of $[0,1]$ such that $f(0) \neq 0$ is cyclic for T_α for every $\alpha \in (0,1) \cap \mathbf{Q}$.

Proof. Let $\alpha = r/q$ be a rational number such that $0 < r < q$ and r and q are coprime. One can remark that the function $|\Delta(f, r/q)|$ is a $1/q$-periodic function by permutation of the rows. Besides the function $\Delta(f, r/q)$ is holomorphic on $(0, 1/q)$ and right-continuous at 0 since $\Delta(f, r/q)(t)$ is equal for every $t \in [0, 1/q)$ to the determinant

$$\left| \left(\left(t + \left\{\tfrac{ir}{q}\right\}\right) \cdots \left(t + \left\{\tfrac{(i+j-1)r}{q}\right\}\right) f\left(t + \left\{\tfrac{(i+j)r}{q}\right\}\right) \right)_{0 \le i,j \le q-1} \right|.$$

Suppose that f is not cyclic for $T_{r/q}$. Thus $m(\{t \in [0,1]; \Delta(f, r/q)(t) = 0\}) > 0$ by Theorem 4.4.4. Since $|\Delta(f, r/q)|$ is $1/q$-periodic, $m(\{t \in [0, 1/q); |\Delta(f, r/q)(t)| = 0\}) > 0$. Hence $\Delta(f, r/q)(t) = 0$ for every $t \in [0, 1/q)$ because $\Delta(f, r/q)$ is holomorphic on $(0, 1/q)$ and right-continuous at 0. However $\Delta(f, r/q)(0)$ is equal to the determinant

$$\begin{vmatrix} f(0) & 0 & \cdots & \cdots & 0 \\ f(\{r/q\}) & \{r/q\}f(\{2r/q\}) & \cdots & \cdots & \{r/q\}\dots\{(q-1)r/q\}f(0) \\ f(\{2r/q\}) & \{2r/q\}f(\{3r/q\}) & & \iddots & 0 \\ \vdots & \vdots & \iddots & \iddots & \vdots \\ f(\{(q-1)r/q\}) & \{(q-1)r/q\}f(0) & 0 & \cdots & 0 \end{vmatrix}$$

which is in turn equal to

$$(-1)^{(q-2)(q-1)/2} f(0)^q \prod_{i=0}^{q-2} \{(q-1)r/q\} \cdots \{(q-1-i)r/q\} \neq 0.$$

This contradiction shows that f is cyclic for T_α. □

Remark 4.4.6. In particular, for every $\alpha \in (0,1) \cap \mathbf{Q}$ the Bishop operator T_α is cyclic and the constant function $\mathbf{1}$ is a common cyclic vector for these operators.

Definition 4.4.7. For every subset A of $[0,1]$, we define the set

$$\mathrm{Cycl}_A = \bigcap_{\alpha \in A} \{f \in L^p([0,1]); f \text{ is cyclic for } T_\alpha\}.$$

of common cyclic vectors for all operators T_α, $\alpha \in A$.

In particular $\mathrm{Cycl}_{\mathbf{Q}\cap(0,1)}$ contains any holomorphic function f on an open neighborhood of $[0,1]$ such that $f(0) \neq 0$.

4.4.2 Cyclicity of Bishop operators in the irrational case

We are now going to use the cyclicity of the Bishop operator T_α for all rational numbers $\alpha \in (0,1)$ to deduce the cyclicity of T_α for some irrational numbers $\alpha \in (0,1)$. To do so we will first need a notion of large sets in Baire spaces, which are called *co-meager sets*. We refer the reader to [29, Section 8.A] for more on this notion.

Definition 4.4.8 ([29, Section 8.A])**.** Let X be a Polish space, that is to say a separable completely metrizable topological space. A subset A of X is said to be:

(i) a co-meager set in X if it contains a dense G_δ-set in X;

(ii) a meager set in X if the complement $X \backslash A$ is a co-meager set in X.

(iii) satisfying the Baire property if there exists an open set U in X such that $A \Delta U = A \backslash U \cup U \backslash A$ is a meager set in X.

Our aim is now to prove the following theorem:

Theorem 4.4.9. The set $\{\alpha \in [0,1]; T_\alpha \text{ is cyclic}\}$ is a co-meager set in $[0,1]$.

To do so, we recall the Kuratowski-Ulam Theorem.

Theorem 4.4.10 (Kuratowski-Ulam, [29, Theorem 8.41])**.** Let X and Y be two Polish spaces and let A be a subset of $X \times Y$ satisfying the Baire property. Then the following assertions are equivalent:

(i) A is a co-meager set in $X \times Y$;

(ii) $\{x \in X; \{y \in Y; (x,y) \in A\} \text{ is a co-meager set in } Y\}$ is a co-meager set in X;

(iii) $\{y \in Y; \{x \in X; (x,y) \in A\} \text{ is a co-meager set in } X\}$ is a co-meager set in Y.

One can remark that a co-meager set satisfies the Baire property, the chosen open set being the full space. Then we deduce from the Kuratowski-Ulam's Theorem the following corollary.

Corollary 4.4.11. Let X and Y be two Polish spaces and let A be a co-meager set in $X \times Y$. Then the sets

$$\{x \in X; \{y \in Y; (x,y) \in A\} \text{ is a co-meager set in } Y\}$$

and

$$\{y \in Y; \{x \in X; (x,y) \in A\} \text{ is a co-meager set in } X\}$$

are co-meager sets in X and Y respectively.

To prove Theorem 4.4.9, our first strategy is to directly apply Corollary 4.4.11 to the set

$$A = \{(\alpha, f) \in [0,1] \times L^p([0,1]); f \text{ is cyclic for } T_\alpha\},$$

which does not give any information about the irrationals α such that T_α is cyclic. A second more explicit strategy would be to find a G_δ-set in $\{\alpha \in [0,1]; T_\alpha \text{ is cyclic}\}$ (see Remark 4.4.17).

We will now show that A is a dense G_δ-set in $[0,1] \times L^p([0,1])$. To do so, we will need the following lemma.

Lemma 4.4.12. For every $Q \in \mathbf{C}[\xi]$, the map

$$\begin{array}{rccc} \phi_Q\colon & ([0,1],|\cdot|) \times (L^p([0,1]), \|\cdot\|_p) & \to & (L^p([0,1]), \|\cdot\|_p) \\ & (\alpha, f) & \mapsto & Q(T_\alpha)f \end{array}$$

is continuous.

Proof. For any $M > 0$ we define $\mathcal{B}_M(L^p([0,1]))$ to be the set of the operators T of $L^p([0,1])$ such that $\|T\| \leq M$. We denote by SOT the *Strong Operator Topology* on $L^p([0,1])$, i.e. the topology generated by the seminorms $T \mapsto \|Tf\|_p$ for every $f \in L^p([0,1])$. Then one can write

$$\phi_Q = \varphi_{3,Q} \circ ((\varphi_{2,Q} \circ \varphi_1) \oplus id)$$

where

$$\begin{array}{rccc} id\colon & (L^p([0,1]), \|\cdot\|_p) & \to & (L^p([0,1]), \|\cdot\|_p) \\ & f & \mapsto & f, \end{array}$$

$$\begin{array}{rccc} \varphi_1\colon & ([0,1],|\cdot|) & \to & (\mathcal{B}_1(L^p([0,1])), SOT) \\ & \alpha & \mapsto & T_\alpha, \end{array}$$

$$\begin{array}{rccc} \varphi_{2,Q}\colon & (\mathcal{B}_1(L^p([0,1])), SOT) & \to & \big(\mathcal{B}_{\|Q\|_1}(L^p([0,1])), SOT\big) \\ & T & \mapsto & Q(T), \end{array}$$

$$\begin{array}{rccc} \varphi_{3,Q}\colon & (\mathcal{B}_{\|Q\|_1}(L^p([0,1])) \times SOT) \times (L^p([0,1]), \|\cdot\|_p) & \to & (L^p([0,1]), \|\cdot\|_p) \\ & (T,f) & \mapsto & Tf. \end{array}$$

The continuity of the maps $\varphi_{2,Q}$ and $\varphi_{3,Q}$ being easily proved, we only need to prove that φ_1 is a continuous map. Let $\alpha \in [0,1]$, and let $(\alpha_n)_{n \geq 1}$ be a sequence of elements of $[0,1]$ such that $\alpha_n \to \alpha$ as $n \to +\infty$. Given $f \in L^p([0,1])$, we are going to show that $\|T_{\alpha_n} f - T_\alpha f\|_p \to 0$ as $n \to +\infty$.

Let $\varepsilon > 0$. By density there exists $g \in \mathcal{C}([0,1])$ such that $\|f-g\|_p < \varepsilon$. For every $n \geq 1$,

$$\|T_{\alpha_n} f - T_\alpha f\|_p \leq \|T_{\alpha_n} f - T_{\alpha_n} g\|_p + \|T_{\alpha_n} g - T_\alpha g\|_p + \|T_\alpha g - T_\alpha f\|_p.$$

On the one hand, we have for every $\beta \in [0,1]$ that

$$\|T_\beta f - T_\beta g\|_p^p = \int_0^1 |tf(\{t+\beta\}) - tg(\{t+\beta\})|^p \, dt \leq \int_0^1 |f(s) - g(s)|^p \, ds \leq \|f-g\|_p^p,$$

so that $\|T_{\alpha_n} f - T_{\alpha_n} g\|_p + \|T_\alpha g - T_\alpha f\|_p \leq 2\|f-g\|_p < 2\varepsilon$. On the other hand

$$\begin{aligned}\|T_{\alpha_n} g - T_\alpha g\|_p^p &= \int_0^1 |tg(\{t+\alpha_n\}) - tg(\{t+\alpha\})|^p \, dt \\ &\leq \int_0^1 |g(\{t+\alpha_n\}) - g(\{t+\alpha\})|^p \, dt \\ &\leq \int_0^1 |g(y) - g(\{y+\alpha-\alpha_n\})|^p \, dy \\ &\leq \int_0^{1-\{\alpha-\alpha_n\}} |g(y) - g(y+\{\alpha-\alpha_n\})|^p \, dy \\ &\qquad + \int_{1-\{\alpha-\alpha_n\}}^1 |g(y) - g(y+\{\alpha-\alpha_n\}-1)|^p \, dy.\end{aligned}$$

Since g is uniformly continuous on $[0,1]$, there exists $\delta > 0$ such that $|g(x)-g(y)| < \varepsilon$ whenever $x, y \in [0,1]$ satisfy $|x-y| < \delta$. Moreover since $\alpha_n \to \alpha$ as $n \to +\infty$, there exists $n_0 \geq 1$ such that $|\alpha - \alpha_n| < \min(\delta, \varepsilon^p/(2\|g\|_\infty)^p, 1)$ for every $n \geq n_0$. Fix $n \geq n_0$.

First, we suppose that $0 \leq \alpha - \alpha_n$. Then $\{\alpha-\alpha_n\} = \alpha - \alpha_n = |\alpha - \alpha_n|$ and

$$\begin{aligned}\|T_{\alpha_n} g - T_\alpha g\|_p^p &\leq (1 - \{\alpha-\alpha_n\})\varepsilon^p + 2^p\|g\|_\infty^p \{\alpha-\alpha_n\} \\ &\leq (1 - |\alpha-\alpha_n|)\varepsilon^p + 2^p\|g\|_\infty^p \, |\alpha-\alpha_n| \\ &\leq 2\varepsilon^p.\end{aligned}$$

Then, we suppose that $0 > \alpha - \alpha_n$. So $\{\alpha-\alpha_n\} = \alpha - \alpha_n + 1 = 1 - |\alpha-\alpha_n|$ and

$$\begin{aligned}\|T_{\alpha_n} g - T_\alpha g\|_p^p &\leq (1 - \{\alpha-\alpha_n\})2^p\|g\|_\infty^p + \{\alpha-\alpha_n\}\varepsilon^p \\ &\leq |\alpha-\alpha_n| \, 2^p\|g\|_\infty^p + (1 - |\alpha-\alpha_n|)\varepsilon^p \\ &\leq 2\varepsilon^p.\end{aligned}$$

Eventually we get that $\|T_{\alpha_n} f - T_\alpha f\|_p \leq (2 + 2^{1/p})\varepsilon$ for every $n \geq n_0$, and thus φ_1 is continuous at α. □

We can now show Theorem 4.4.9.

Proof of Theorem 4.4.9. First, we prove that the set previously defined as $A = \{(\alpha, f) \in [0,1] \times L^p([0,1]); f \text{ is cyclic for } T_\alpha\}$ is a G_δ-set in $[0,1] \times L^p([0,1])$. Let $(U_n)_{n\geq 1}$ be a basis of non-empty open sets in $L^p([0,1])$. For every $(\alpha, f) \in [0,1] \times L^p([0,1])$, f is cyclic for T_α

if and only if for every $n \geq 1$, there exists $Q \in \mathbf{C}[\xi]$ such that $Q(T_\alpha)f \in U_n$. Thus

$$A = \bigcap_{n\geq 1} \bigcup_{Q\in\mathbf{C}[\xi]} \phi_Q^{-1}(U_n)$$

where the map ϕ_Q is defined by $\phi_Q(\alpha, f) = Q(T_\alpha)f$ for every $(\alpha, f) \in [0,1] \times L^p([0,1])$ and is continuous by Lemma 4.4.12. So A is a G_δ-set in $[0,1] \times L^p([0,1])$.

We next prove that the set A is dense in $[0,1] \times L^p([0,1])$. Let $g \in L^p([0,1])$, let $\beta \in [0,1]$, and $\varepsilon > 0$. There exists $\alpha \in (0,1) \cap \mathbf{Q}$ such that $|\beta - \alpha| < \varepsilon$. Since T_α is cyclic by Theorem 4.4.5 and since $\sigma_p(T_\alpha^*) = \emptyset$ by Theorem 4.3.10, the set of cyclic vectors for T_α is a dense G_δ-set by Proposition 4.4.2. Thus there exists $f \in L^p([0,1])$ cyclic for T_α such that $\|g - f\|_p < \varepsilon$. So $(\alpha, f) \in A$ and satisfies $|\beta - \alpha| < \varepsilon$ and $\|g - f\|_p < \varepsilon$. Hence A is a dense G_δ-set in $[0,1] \times L^p([0,1])$.

Then the set

$$B = \{\alpha \in [0,1]; \{f \in L^p([0,1]); f \text{ is cyclic for } T_\alpha\} \text{ is a co-meager set in } L^p([0,1])\}$$

is a co-meager set in $[0,1]$ by Corollary 4.4.11. Moreover, $B \subset \{\alpha \in [0,1]; T_\alpha \text{ is cyclic}\}$, and Theorem 4.4.9 follows. □

We want now to explicit a set of irrational numbers α in $[0,1]$ such that T_α is a cyclic operator. More precisely, we will give a sufficient condition on α expressed in terms of rational approximations implying that T_α is cyclic. Some results on approximations of irrationals by rational are recalled here and can be found in [15].

Definition 4.4.13 ([15, Section 1.2])**.** Let $n \geq 0$ and let $(a_k)_{0\leq k\leq n}$ be such that $a_0 \in \mathbf{Z}$ and $a_k \in \mathbf{N}$ for every $k \in \{1, \ldots, n\}$. The rational number

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_n}}}$$

is called a *finite continued fraction* and is written $[a_0; a_1, \ldots, a_n]$.

Let us now recall the continued fraction expansion of an irrational number.

Definition 4.4.14 ([15, Definition 1.2])**.** Let $x \in [0,1]\backslash\mathbf{Q}$. Let $a_0 \in \mathbf{Z}$ and $\xi_0 \in (0,1)$ such that $x = a_0 + \xi_0$. Let $(a_k)_{k\geq 1}$ be a sequence of positive integers and $(\xi_k)_{k\geq 1}$ be a sequence of elements in $(0,1)$ such that $1/\xi_k = a_{k+1} + \xi_{k+1}$ for every $k \geq 0$.

The rational numbers of the sequence $(p_n/q_n)_{n\geq 0} = ([a_0; a_1, \ldots, a_n])_{n\geq 0}$ are called the *convergents* of x.

The following result shows that these convergents give an approximation by rational numbers of an irrational number, which turns out to be optimal.

Proposition 4.4.15 ([15, Theorem 1.3, Corollary 1.4 and Theorem 1.4])**.** Let x be an irrational number and $(p_n/q_n)_{n\geq 0}$ its convergents. If $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$ and $q_0 = 1$, then $p_n = a_n p_{n-1} + p_{n-2}, q_n = a_n q_{n-1} + q_{n-2}$ and p_n and q_n are coprime for every $n \geq 1$.

Moreover the convergents $(p_n/q_n)_{n\geq 0}$ converge to x and satisfy

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} \qquad \text{for every } n \geq 0.$$

We are now able to prove the following theorem, which gives a sufficient condition on the convergents of an irrational number α in $[0,1]$ implying the cyclicity of T_α. We recall that the set $\text{Cycl}_{\mathbf{Q}\cap(0,1)}$ has been defined in Definition 4.4.7 as the set of common cyclic vectors for all T_α, $\alpha \in \mathbf{Q} \cap (0,1)$.

Theorem 4.4.16. Let $f \in \text{Cycl}_{\mathbf{Q}\cap(0,1)}$. There exists a function $\psi_f \colon \mathbf{N} \to \mathbf{R}_+$ with the following property: if $(p_n/q_n)_{n\geq 0}$ are the convergents of an irrational number α in $[0,1]$ and if for every $n \geq 0$ there exists $n_0 \geq n$ such that $q_{n_0+1} > \psi_f(q_{n_0})$, then f is cyclic for T_α.

Proof. Let $(g_j)_{j\geq 1}$ be a dense set of functions in $L^p([0,1])$. Let $q \geq 2$ be an integer. We consider the sets

$$R_q = \left\{ \frac{r}{q} \,; 0 < r < q \text{ and } gcd(r,q) = 1 \right\} \subset [0,1] \text{ and } G_q = \{g_1, \ldots, g_q\} \subset L^p([0,1]).$$

Since $T_{r/q}$ is cyclic for every $r/q \in R_q$, for every $j \in \{1, \ldots, q\}$ there exists $Q_{r/q,j} \in \mathbf{C}[\xi]$ such that $\|Q_{r/q,j}(T_{r/q})f - g_j\|_p < 1/2^q$. Then the finite set

$$\mathcal{P}_q = \left\{ Q_{r/q,j} ; (r/q, j) \in R_q \times \{1, \ldots, q\} \right\}$$

is such that for every $j \in \{1, \ldots, q\}$ and every $r/q \in R_q$, there exists $Q \in \mathcal{P}_q$ such that $\|Q(T_{r/q})f - g_j\|_p < 1/2^q$. Since the maps $\phi_{Q,f} \colon \alpha \mapsto Q(T_\alpha)f$ are continuous for every polynomial Q in $\mathcal{P}_q$ by Lemma 4.4.12, there exists $\delta(q) > 0$ such that for every $Q \in \mathcal{P}_q$ and every $r/q \in R_q$, $\|Q(T_\beta)f - Q(T_{r/q})f\|_p < 1/2^q$ whenever $\beta \in [0,1]$ satisfies $|\beta - r/q| < \delta(q)$.

Let us consider the function $\psi_f \colon \mathbf{N} \to \mathbf{R}_+$ defined by $\psi_f(q) = 1/(q\delta(q))$ and an irrational number $\alpha \in (0,1)$ whose convergents $(p_n/q_n)_{n\geq 0}$ are such that for every $n \geq 0$, there exists $n_0 \geq n$ such that $q_{n_0+1} > \psi_f(q_{n_0})$. Since $(q_n)_{n\geq 0}$ is an increasing sequence of positive integers, $q_n \geq n$ for every $n \geq 0$. Fix $n \geq 0$ and $n_0 \geq n$ such that $q_{n_0+1} > \psi_f(q_{n_0})$. Then $g_n \in G_{q_n} \subset G_{q_{n_0}}$ and $p_{n_0}/q_{n_0} \in R_{q_{n_0}}$, so there exists $Q_{n_0} \in \mathcal{P}_{q_{n_0}}$ such that $\|Q_{n_0}(T_{p_{n_0}/q_{n_0}})f - g_n\|_p < 2^{-q_{n_0}}$. Since $|\alpha - p_{n_0}/q_{n_0}| < 1/(q_{n_0}q_{n_0+1}) < \delta(q_{n_0})$, we have $\|Q_{n_0}(T_\alpha)f - Q_{n_0}(T_{p_{n_0}/q_{n_0}})f\|_p < 2^{-q_{n_0}}$. So

$$\begin{aligned} \|Q_{n_0}(T_\alpha)f - g_n\|_p &\leq \|Q_{n_0}(T_\alpha)f - Q_{n_0}(T_{p_{n_0}/q_{n_0}})f\|_p + \|Q_{n_0}(T_{p_{n_0}/q_{n_0}})f - g_n\|_p \\ &< 2^{-q_{n_0}} + 2^{-q_{n_0}} \leq 2^{-(q_{n_0}-1)} \leq 2^{-(n_0-1)} \leq 2^{-(n-1)}. \end{aligned}$$

Hence f is cyclic for T_α. $\square$

Remark 4.4.17. As in Theorem 4.4.9, Theorem 4.4.16 implies that the set of parameters α such that T_α is cyclic is a co-meager set in $[0,1]$. Indeed, if $q_k(\alpha)$ denotes the denominator of the k-th convergent of α, then for every $f \in \mathrm{Cycl}_{\mathbf{Q}\cap(0,1)}$ the set

$$\bigcap_{n\geq 0}\bigcup_{n_0\geq n}\{\alpha \in \mathbf{R}\backslash\mathbf{Q}\cap(0,1); q_{n_0+1}(\alpha) > \psi_f(q_{n_0}(\alpha))\}$$

is a G_δ-set in $[0,1]$.

Remark 4.4.18. The proof of the previous theorem does not require the function f to be cyclic for T_α for *every* rational number α, but only for $\alpha \in \mathbf{Q}\cap(0,1)\backslash F$, where F is a finite set.

We now move over to the study of cyclicity properties of weighted translation operators.

4.4.3 Cyclicity of weighted translation operators

Let ϕ be a function in $L^\infty([0,1])$. Again, we observe that $T_{\phi,\alpha}$ cannot be cyclic in the case where $m(\{\phi=0\})>0$.

Proposition 4.4.19. Let $\phi \in L^\infty([0,1])$ satisfying $m(\{\phi=0\})>0$. For every $\alpha\in[0,1]$, the operator $T_{\phi,\alpha}$ is not cyclic.

Proof. Let $A=\{\phi=0\}$. Then for every $f\in L^p([0,1])$ and every $P=\sum_{0\leq k\leq d} a_k\xi^k \in \mathbf{C}[\xi]$

$$P(T_{\phi,\alpha})f(x) = \sum_{k=0}^{d} a_k\phi(x)\dots\phi(\{x+(k-1)\alpha\})f(\{x+k\alpha\}) = a_0 f(x) \quad \text{a.e. on } A.$$

Suppose that f is cyclic for $T_{\phi,\alpha}$. Then the set $\{P(T_{\phi,\alpha})f_{|A}; P\in\mathbf{C}[\xi]\} = \mathrm{span}[f_{|A}]$ is dense in $L^p(A)$, which is impossible. ☐

Our aim is now to extend the result of Chalendar and Partington characterizing cyclic functions for rational Bishop operators to the case of weighted translation operators $T_{\phi,\alpha}$, with $\alpha\in\mathbf{Q}$ and an increasing function $\phi\in\mathcal{C}([0,1],\mathbf{R})$.

Definition 4.4.20. Let $\phi\in L^\infty([0,1])$, let $f\in L^p([0,1])$ and let $\alpha=r/q$ be a rational number such that $0<p<q$ and that r and q are coprime. We define $\Delta_\phi(f,r/q)(t)$ a.e. on $[0,1]$ by

$$\begin{vmatrix} f(t) & T_{\phi,r/q}f(t) & \cdots & T^{q-1}_{\phi,r/q}f(t) \\ f(\{t+r/q\}) & T_{\phi,r/q}f(\{t+r/q\}) & \cdots & T^{q-1}_{\phi,r/q}f(\{t+r/q\}) \\ \vdots & \vdots & & \vdots \\ f(\{t+(q-1)r/q\}) & T_{\phi,r/q}f(\{t+(q-1)r/q\}) & \cdots & T^{q-1}_{\phi,r/q}f(\{t+(q-1)r/q\}) \end{vmatrix}.$$

We will prove the following theorem:

Theorem 4.4.21. Let $\alpha=r/q$ be a rational number in $(0,1)$ such that $0<r<q$ and r and q are coprime. Let $\phi\in\mathcal{C}([0,1],\mathbf{R})$ be an increasing function. A function $f\in L^p([0,1])$ is cyclic for $T_{\phi,\alpha}$ if and only if the function $\Delta_\phi(f,r/q)$ satisfies $m(\{t\in[0,1];\Delta_\phi(f,r/q)(t)=0\})=0$.

The proof of Theorem 4.4.21 follows the lines of the proof of Theorem 5.4.4 in [17]. The first step in the proof of Theorem 4.4.21 is the following lemma:

Lemma 4.4.22. Let $\phi \in \mathcal{C}([0,1], \mathbf{R})$ be an increasing function and let $\alpha = r/q$ be a rational number such that $0 < r < q$ and that r and q are coprime. Let $f \in L^p([0,1])$, let $n \geq 1$ and let $h \in L^\infty([0,1])$ vanishing on the set

$$\begin{aligned}\Omega_{n,f,\phi} =& \{t \in [0,1]; |\Delta_\phi(f, r/q)(t)| < 1/n\} \cup \\ & \bigcup_{0 \leq i,j \leq q-1} \{t \in [0,1]; \left|T^j_{\phi,r/q} f(\{t + ir/q\})\right| > n\}.\end{aligned}$$

There exist $1/q$-periodic functions $h_0, \ldots, h_{q-1} \in L^\infty([0,1])$ such that $h = \sum_{j=0}^{q-1} h_j T^j_{\phi,r/q} f$.

Proof. The proof of Lemma 4.4.22 is patterned after the proof of Lemma 5.4.2 in [17]. A decomposition of h of the form $h = h_0 f + \ldots + h_{q-1} f$ is by $1/q$-periodicity equivalent to

$$\left\{\begin{array}{rcccccc} h(t) & = & h_0(t)f(t) & + & \ldots & + & h_{q-1}(t)T^{q-1}_{\phi,r/q}f(t) \\ h(t+1/q) & = & h_0(t)f(t+1/q) & + & \ldots & + & h_{q-1}(t)T^{q-1}_{\phi,r/q}f(t) \\ & \vdots & & & & & \\ h(t+(q-1)/q) & = & h_0(t)f(t+(q-1)/q) & + & \ldots & + & h_{q-1}(t)T^{q-1}_{\phi,r/q}f(t+(q-1)/q) \end{array}\right.$$

a.e. on $[0, 1/q)$, that is to say to equivalent to the system

$$\begin{pmatrix} h(t) \\ \vdots \\ h(t+(q-1)/q) \end{pmatrix} = \begin{pmatrix} f(t) & \ldots & T^{q-1}_{\phi,r/q}f(t) \\ \vdots & & \vdots \\ f(t+(q-1)/q) & \ldots & T^{q-1}_{\phi,r/q}f(t+(q-1)/q) \end{pmatrix} \begin{pmatrix} h_0(t) \\ \vdots \\ h_{q-1}(t) \end{pmatrix}$$

whose determinant is $\pm\Delta_\phi(f, r/q)(t)$ by permutation of the rows since r and q are coprime. On the one hand there exist such solutions if $t \in [0,1] \backslash \Omega_{n,f,\phi}$ because $|\Delta_\phi(f, r/q)(t)| \geq 1/n$. On the other hand it suffices to set $h_0(t) = \ldots = h_{q-1}(t) = 0$ whenever $t \in \Omega_{n,f,\phi}$. Such functions $h_0, \ldots, h_{q-1}$ are then bounded by the definition of the set $\Omega_{n,f,\phi}$. □

We will also need the following density lemma, which is an analogue of Lemma 5.4.3 in [17].

Lemma 4.4.23. Let $\phi \in \mathcal{C}([0,1], \mathbf{R})$ be an increasing function and let F be a function in $L^p([0,1/q])$ such that $m(\{t \in [0,1/q]; F(t) = 0\}) = 0$. The set $\{Q(w)F; Q \in \mathbf{C}[\xi]\}$ is dense in $L^p([0,1/q])$ where w is the function defined on $[0,1/q)$ by $w(x) = \phi(x)\phi(x + 1/q)\ldots\phi(x + (q-1)/q)$.

Proof. Suppose that $1/p + 1/p' = 1$ and that G is a function in $L^{p'}([0,1/q])$ such that

$$\int_0^{1/q} Q(w(t))F(t)\overline{G(t)}dt = 0 \quad \text{for every } Q \in \mathbf{C}[\xi].$$

Since ϕ is continuous on $[0,1]$, the function w admits a continuous extension on $[0,1/q]$, denoted by w_0. Then

$$\int_0^{1/q} Q(w_0(t))F(t)\overline{G(t)}dt = 0 \quad \text{for every } Q \in \mathbf{C}[\xi].$$

Since $\phi \in \mathcal{C}([0,1],\mathbf{R})$ is an increasing function, the algebra $\{Q(w_0); Q \in \mathbf{C}[\xi]\}$ in $\mathcal{C}([0,1/q])$ separates points, contains the constant functions and is closed under complex conjugation. Then this algebra is dense in $(\mathcal{C}([0,1/q]), \|\cdot\|_\infty)$ by the Stone-Weierstrass's Theorem. Therefore

$$\int_0^{1/q} f(t)F(t)\overline{G(t)}dt = 0 \quad \text{for every } f \in \mathcal{C}([0,1/q]).$$

Thus the function $F\overline{G}$ in $L^1([0,1/q])$ is the constant function equal to zero and so is the function G in $L^{p'}([0,1/q])$. □

With these tools at hand, we can now prove Theorem 4.4.21.

Proof of Theorem 4.4.21. Assume that f satisfies $m(\{t \in [0,1]; \Delta_\phi(f,r/q)(t) = 0\}) = 0$. The union

$$A = \bigcup_{n\in\mathbf{N}} \{h \in L^\infty([0,1]); h = 0 \text{ on } \Omega_{n,f,\phi}\}$$

is dense in $L^p([0,1])$ since $m(\cap_{n\in\mathbf{N}}\Omega_{n,f,\phi}) = 0$. Fix $\varepsilon > 0$ and $h = \sum_{j=0}^{q-1} h_j T_{\phi,\alpha}^j f \in A$. In order to prove that there exists $Q \in \mathbf{C}[\xi]$ such that $\|Q(T_{\phi,\alpha})f - h\|_p < \varepsilon$, one can remark that

$$Q(T_{\phi,\alpha})f = \sum_{k=0}^{d} a_k T_{\phi,\alpha}^k f = \sum_{j=0}^{q-1}\sum_{n=0}^{m} a_{nq+j} T_{\phi,\alpha}^{nq+j} f = \sum_{j=0}^{q-1}\sum_{n=0}^{m} a_{nq+j} w^n T_{\phi,\alpha}^j f = \sum_{j=0}^{q-1} Q_j(w) T_{\phi,\alpha}^j f$$

for every $Q = \sum_{k=0}^{d} a_k\xi^k \in \mathbf{C}[\xi]$ since $T_{\phi,\alpha}^q f = wf$ and by $1/q$-periodicity of w. Therefore, in order to prove that there exists $Q \in \mathbf{C}[\xi]$ such that $\|Q(T_{\phi,\alpha})f - \sum_{j=0}^{q-1} h_j T_{\phi,\alpha}^j f\|_p < \varepsilon$, it suffices to show that for every $j \in \{0,\ldots,q-1\}$, there exists $Q_j \in \mathbf{C}[\xi]$ such that $\|Q_j(w)(T_{\phi,\alpha}) - h_j T_{\phi,\alpha}^j f\|_p < \varepsilon/q$ and then the polynomial Q defined by $Q = \sum_{j=0}^{q-1} Q_j(\xi^q)\xi^j$ will satisfy $\|Q(T_{\phi,\alpha})f - h\|_p < \varepsilon$. Let $j \in \{0,\ldots,q-1\}$ and consider the function

$$F_j \colon t \in [0,1/q] \mapsto \left|T_{\phi,\alpha}^j f(t)\right| + \left|T_{\phi,\alpha}^j f(\{t+1/q\})\right| + \ldots + \left|T_{\phi,\alpha}^j f(\{t+(q-1)/q\})\right|.$$

For every $t \in [0,1/q]$, if $F_j(t) = 0$ then $\Delta_\phi(f,r/q)(t) = 0$ because its j-th column would be zero. So $m(\{t \in [0,1/q]; F_j(t) = 0\}) = 0$ since $m(\{t \in [0,1]; \Delta_\phi(f,r/q)(t) = 0\}) = 0$. By Lemma 4.4.23 there exists $Q_j \in \mathbf{C}[\xi]$ such that $\|Q_j(w)F_j - h_jF_j\|_{[0,1/q],p} < \varepsilon/q$. However since $p>1$ and by $1/q$-periodicity of $Q_j(w)-h_j$

$$\begin{aligned}\|(Q_j(w)-h_j)F_j\|^p_{[0,1/q],p} &= \int_0^{1/q} |Q_j(w(t))-h_j(t)|^p \left(\sum_{k=0}^{q-1}\left|T^j_{\phi,\alpha}f(t+k/q)\right|\right)^p dt\\ &\geq \sum_{k=0}^{q-1}\int_0^{1/q} |Q_j(w(t))-h_j(t)|^p \left|T^j_{\phi,\alpha}f(t+k/q))\right|^p dt\\ &\geq \int_0^1 \left|(Q_j(w(t))-h_j(t))T^j_{\phi,\alpha}f(t)\right|^p dt\\ &\geq \|Q_j(w)T^j_{\phi,\alpha}f-h_jT^j_{\phi,\alpha}f\|^p_p.\end{aligned}$$

Then $\|Q(T_{\phi,\alpha})f-h\|_p \leq \sum_{j=0}^{q-1}\|Q_j(w)T^j_{\phi,\alpha}f-h_jT^j_{\phi,\alpha}f\|_p < \sum_{j=0}^{q-1}\varepsilon/q \leq \varepsilon$, so f is cyclic for $T_{\phi,\alpha}$.

Conversely, let us now assume that the set $A=\{t\in[0,1];\Delta_\phi(f,r/q)(t)=0\}$ satisfies $m(A)>0$. Then there exist functions $a_0,\ldots,a_{q-1}$ on $[0,1]$, not all zero, such that

$$\begin{pmatrix} f(t) & \ldots & f(\{t+(q-1)r/q\})\\ T_{\phi,\alpha}f(t) & \ldots & T_{\phi,\alpha}f(\{t+(q-1)r/q\})\\ \vdots & & \vdots\\ T^{q-1}_{\phi,\alpha}f(t) & \ldots & T^{q-1}_{\phi,\alpha}f(\{t+(q-1)r/q\})\end{pmatrix}\begin{pmatrix}a_0(t)\\ a_1(t)\\ \vdots\\ a_{q-1}(t)\end{pmatrix}=\begin{pmatrix}0\\0\\ \vdots\\0\end{pmatrix}$$

a.e. on A, that is to say $\sum_{j=0}^{q-1}a_j(t)T^i_{\phi,\alpha}f(\{t+jr/q\})=0$ for every $i\in\{0,\ldots,q-1\}$. If $\varphi=Q(T_{\phi,\alpha})f$ with $Q=\sum_{k=0}^{d}b_k\xi^k\in\mathbf{C}[\xi]$, then

$$\begin{aligned}\sum_{j=0}^{q-1}a_j(t)\varphi(\{t+jr/q\}) &= \sum_{k=0}^{d}b_k\sum_{j=0}^{q-1}a_j(t)T^k_{\phi,\alpha}f(\{t+jr/q\})\\ &= \sum_{n=0}^{m}\sum_{i=0}^{q-1}b_{nq+i}\sum_{j=0}^{q-1}a_j(t)T^{nq+i}_{\phi,\alpha}f(\{t+jr/q\})\\ &= \sum_{n=0}^{m}\sum_{i=0}^{q-1}b_{nq+i}w^n(t)\sum_{j=0}^{q-1}a_j(t)T^{i}_{\phi,\alpha}f(\{t+jr/q\})\\ &= 0\end{aligned}$$

a.e. on A since $T^{nq+i}_{\phi,\alpha}f=w^nT^i_{\phi,\alpha}f$ and by $1/q$-periodicity of w. Therefore $\sum_{j=0}^{q-1}a_j(t)\varphi(\{t+jr/q\})=0$ whenever φ is a function in the closed subspace generated by $\{T^n_{\phi,\alpha}f;n\geq 0\}$. So the subspace $\{Q(T_{\phi,\alpha})f;Q\in\mathbf{C}[\xi]\}$ is not dense in $L^p([0,1])$, that is to say, f is not cyclic for $T_{\phi,\alpha}$. □

Hence if ϕ is a holomorphic function, Theorem 4.4.5 still holds for the weighted translation operators.

Theorem 4.4.24. Let $\phi\in\mathcal{C}([0,1],\mathbf{R})$ be an increasing function satisfying $\phi(0)=0$ and which is holomorphic on an open neighborhood of $[0,1]$. Any holomorphic function f on an open neighborhood of $[0,1]$ such that $f(0)\neq 0$ is cyclic for $T_{\phi,\alpha}$ for every $\alpha\in(0,1)\cap\mathbf{Q}$.

Proof. Let $\alpha = r/q$ be a rational number such that $0 < r < q$ and r and q are coprime. It suffices to show that $m(\{t \in [0,1]; \Delta_\phi(f,r/q)(t) = 0\}) = 0$ by Theorem 4.4.21. The function $|\Delta_\phi(f,r/q)|$ is $1/q$-periodic on $[0,1]$ and $\Delta_\phi(f,r/q)$ is holomorphic on $(0,1/q)$, since ϕ is holomorphic, and right-continuous at 0. Indeed $\Delta_\phi(f,r/q)(t)$ is equal to the determinant

$$\left|\left(\phi\left(t+\left\{\tfrac{ir}{q}\right\}\right)\dots\phi\left(t+\left\{\tfrac{(i+j-1)r}{q}\right\}\right)f\left(t+\left\{\tfrac{(i+j)r}{q}\right\}\right)\right)_{0\le i,j\le q-1}\right|$$

whenever $t \in [0,1/q)$. Then as in the case of the Bishop operators, $\Delta_\phi(f,r/q)(t) = 0$ for every $t \in [0,1/q)$ if f is not cyclic for $T_{\phi,\alpha}$. However $\Delta_\phi(f,r/q)(0)$ is equal to the determinant

$$\begin{vmatrix} f(0) & 0 & \dots & \dots & 0 \\ f(\{r/q\}) & \phi(\{r/q\})f(\{2r/q\}) & \dots & \dots & \phi(\{r/q\})\dots\phi(\{(q-1)r/q\})f(0) \\ f(\{2r/q\}) & \phi(\{2r/q\})f(\{3r/q\}) & & \iddots & 0 \\ \vdots & \vdots & \iddots & \iddots & \vdots \\ f(\{(q-1)r/q\}) & \phi(\{(q-1)r/q\})f(0) & 0 & \dots & 0 \end{vmatrix}$$

which is in turn equal to

$$(-1)^{(q-2)(q-1)/2} f(0)^q \prod_{i=0}^{q-2} \phi(\{(q-1)r/q\})\dots\phi(\{(q-1-i)r/q\}) \neq 0.$$

So f is cyclic for $T_{\phi,\alpha}$. □

Definition 4.4.25. Let $\phi \in L^\infty([0,1])$. For every subset A of $[0,1]$, we define the set

$$\mathrm{Cycl}_A^{(\phi)} = \bigcap_{\alpha\in A} \{f \in L^p([0,1]); f \text{ is cyclic for } T_{\phi,\alpha}\}$$

of common cyclic vectors for all operators $T_{\phi,\alpha}$, $\alpha \in A$.

In particular $\mathrm{Cycl}^{(\phi)}_{\mathbf{Q}\cap(0,1)}$ contains any holomorphic function f on an open neighborhood of $[0,1]$ such that $f(0) \neq 0$ if ϕ is an increasing function on $[0,1]$ that is holomorphic on an open neighborhood of $[0,1]$ satisfying $\phi(0) = 0$.

It follows that Theorem 4.4.9 can be generalized to the weighted translation operators.

Lemma 4.4.26. Let $\phi \in L^\infty([0,1])$. For every $Q \in \mathbf{C}[\xi]$, the map

$$\begin{array}{rrcl} \varphi_Q\colon & ([0,1],|\cdot|) \times (L^p([0,1]),\|\cdot\|_p) & \to & (L^p([0,1]),\|\cdot\|_p) \\ & (\alpha,f) & \mapsto & Q(T_{\phi,\alpha})f \end{array}$$

is continuous.

Proof. Let $Q = \sum_{k=0}^d a_k\xi^k \in \mathbf{C}[\xi]$. As in the case of the Bishop operators, one can write

$$\varphi_Q = \varphi_{3,Q} \circ ((\varphi_{2,Q} \circ \varphi_1) \oplus id)$$

where

$$\begin{array}{rrcl} id\colon & (L^p([0,1]),\|\cdot\|_p) & \to & (L^p([0,1]),\|\cdot\|_p) \\ & f & \mapsto & f, \end{array}$$

$$\begin{array}{rrcl} \varphi_1\colon & ([0,1],|\cdot|) & \to & (\mathcal{B}_{\|\phi\|_\infty}(L^p([0,1])),SOT) \\ & \alpha & \mapsto & T_{\phi,\alpha}, \end{array}$$

$$\begin{array}{rrcl} \varphi_{2,Q}\colon & (\mathcal{B}_{\|\phi\|_\infty}(L^p([0,1])),SOT) & \to & (\mathcal{B}_{M_Q}(L^p([0,1])),SOT) \\ & T & \mapsto & Q(T), \end{array}$$

$$\begin{array}{rrcl} \varphi_{3,Q}\colon & (\mathcal{B}_{M_Q}(L^p([0,1])),SOT)\times(L^p([0,1]),\|\cdot\|_p) & \to & (L^p([0,1]),\|\cdot\|_p) \\ & (T,f) & \mapsto & Tf \end{array}$$

with $M_Q=\sum_{k=0}^d |a_k|\,\|\phi\|_\infty^k$. We will only prove the continuity of φ_1 since the other are easily proved. Let $\alpha\in[0,1]$, and let $(\alpha_n)_{n\geq 1}$ be a sequence of elements of $[0,1]$ such that $\alpha_n\to\alpha$ as $n\to+\infty$. Given $f\in L^p([0,1])$, we are going to show that $\|T_{\phi,\alpha_n}f-T_{\phi,\alpha}f\|_p\to 0$ as $n\to+\infty$. Let $\varepsilon>0$. By density there exists $g\in\mathcal{C}([0,1])$ such that $\|f-g\|_p<\varepsilon$. For every $n\geq 1$, $\|T_{\phi,\alpha_n}f-T_{\phi,\alpha}f\|_p\leq\|T_{\phi,\alpha_n}f-T_{\phi,\alpha_n}g\|_p+\|T_{\phi,\alpha_n}g-T_{\phi,\alpha}g\|_p+\|T_{\phi,\alpha}g-T_{\phi,\alpha}f\|_p$. On the one hand for every $\beta\in[0,1]$

$$\|T_{\phi,\beta}f-T_{\phi,\beta}g\|_p^p=\int_0^1|\phi(t)f(\{t+\beta\})-\phi(t)g(\{t+\beta\})|^p\,dt\leq\|\phi\|_\infty^p\|f-g\|_p^p,$$

so that $\|T_{\phi,\alpha_n}f-T_{\phi,\alpha_n}g\|_p+\|T_{\phi,\alpha}g-T_{\phi,\alpha}f\|_p\leq 2\|\phi\|_\infty\|f-g\|_p$. On the other hand as in the proof of Lemma 4.4.12

$$\begin{aligned}\|T_{\phi,\alpha_n}g-T_{\phi,\alpha}g\|_p^p&=\int_0^1|\phi(t)g(\{t+\alpha_n\})-\phi(t)g(\{t+\alpha\})|^p\,dt\\&\leq\|\phi\|_\infty^p\int_0^1|g(\{t+\alpha_n\})-g(\{t+\alpha\})|^p\,dt\\&\leq 2\|\phi\|_\infty^p\varepsilon^p\end{aligned}$$

whenever n is sufficiently large since g is uniformly continuous and since $\alpha_n\to\alpha$ as $n\to+\infty$. Eventually if n is sufficiently large, we get that $\|T_{\phi,\alpha_n}f-T_{\phi,\alpha}f\|_p\leq(2+2^{1/p})\|\phi\|_\infty\varepsilon$ and thus φ_1 is continuous at α. □

Theorem 4.4.27. Let $\phi\in\mathcal{C}([0,1],\mathbf{R})$ be an increasing convex function satisfying $\phi(0)=0$ and which is holomorphic on an open neighborhood of $[0,1]$. The set of parameters $\{\alpha\in[0,1];T_{\phi,\alpha}\text{ is cyclic}\}$ is a co-meager set in $[0,1]$.

Proof. Let us set $A=\{(\alpha,f)\in[0,1]\times L^p([0,1]);f\text{ is cyclic for }T_{\phi,\alpha}\}$. We will prove that this is a dense G_δ-set in $[0,1]\times L^p([0,1])$. Let $(U_n)_{n\geq 1}$ be a basis of non-empty open sets in $L^p([0,1])$. For every $(\alpha,f)\in[0,1]\times L^p([0,1])$ f is cyclic for $T_{\phi,\alpha}$ if and only if for every $n\geq 1$, there exists $Q\in\mathbf{C}[\xi]$ such that $Q(T_{\phi,\alpha})f\in U_n$. Thus

$$A=\bigcap_{n\geq 1}\bigcup_{Q\in\mathbf{C}[\xi]}\varphi_Q^{-1}(U_n)$$

where the map φ_Q is defined on $[0,1]\times L^p([0,1])$ by $\varphi_Q(\alpha,f)=Q(T_{\phi,\alpha})f$ and is continuous by Lemma 4.4.26 since ϕ is bounded on $[0,1]$. So A is a G_δ-set in $[0,1] \times L^p([0,1])$. Let now $\varepsilon > 0$ and $(\beta, g) \in [0,1] \times L^p([0,1])$. There exists $\alpha \in (0,1) \cap \mathbf{Q}$ such that $|\beta - \alpha| < \varepsilon$. Since $T_{\phi,\alpha}$ is cyclic by Theorem 4.4.24 and since $\sigma_p(T^*_{\phi,\alpha}) = \emptyset$ by Theorem 4.3.13, the set of cyclic vectors for $T_{\phi,\alpha}$ is a dense G_δ-set in $L^p([0,1])$. Then there exists a cyclic vector f for $T_{\phi,\alpha}$ such that $\|g - f\|_p < \varepsilon$. So A is a dense G_δ-set in $[0,1] \times L^p([0,1])$.

By the Kuratowski-Ulam's Theorem, the set

$$B = \{\alpha \in [0,1]; \{f \in L^p([0,1]); f \text{ is cyclic for } T_{\phi,\alpha}\} \text{ is co-meager set in } L^p([0,1])\}$$

is a co-meager set in $[0,1]$ and $B \subset \{\alpha \in [0,1]; T_{\phi,\alpha} \text{ is cyclic}\}$. □

The function $\varphi_{P,f} \colon \alpha \mapsto P(T_{\phi,\alpha})f$ still being continuous, one can extend Theorem 4.4.16 for a weighted translation operator $T_{\phi,\alpha}$ whenever α is an irrational number in $[0,1]$ sufficiently well approached by its convergents.

Theorem 4.4.28. Let $\phi \in \mathcal{C}([0,1], \mathbf{R})$ be an increasing function satisfying $\phi(0) = 0$ and which is holomorphic on an open neighborhood of $[0,1]$. Let $f \in \mathrm{Cycl}^{(\phi)}_{\mathbf{Q}\cap(0,1)}$ be a common cyclic vector for $T_{\phi,\alpha}$ for every $\alpha \in (0,1) \cap \mathbf{Q}$. There exists a function $\psi_{\phi,f} \colon \mathbf{N} \to \mathbf{R}_+$ with the following property: if $(p_n/q_n)_{n\geq 0}$ are the convergents of an irrational number α in $[0,1]$ and if for every $n \geq 0$ there exists $n_0 \geq n$ such that $q_{n_0+1} > \psi_{\phi,f}(q_{n_0})$, then f is cyclic for $T_{\phi,\alpha}$.

Proof. Since the weighted translation operator $T_{\phi,\alpha}$ is cyclic whenever $\alpha \in (0,1) \cap \mathbf{Q}$ and since the map $\varphi_{Q,f} \colon \alpha \mapsto Q(T_{\phi,\alpha})f$ is continuous whenever $Q \in \mathbf{C}[\xi]$, the result follows from the proof of Theorem 4.4.16. □

Remark 4.4.29. Since we don't assume in Theorem 4.4.28 that ϕ is a convex function, the operator $T^*_{\phi,\alpha}$ may have eigenvalues according to Theorem 4.3.13. Then we don't know if the set of cyclic functions for $T_{\phi,\alpha}$ is a dense set in $L^p([0,1])$ whenever $T_{\phi,\alpha}$ is cyclic.

4.5 Open questions

We present in this last section some further comments and open questions.

To begin with, although we know that the weighted translation operators $T_{\phi,\alpha}$ do not satisfy the Hypercyclicity Criterion, we are unable to prove that they are not hypercyclic.

Question 4.5.1. For every $\phi \in L^\infty([0,1])$ and every $\alpha \in [0,1]$, is it true that $T_{\phi,\alpha}$ is not hypercyclic on $L^p([0,1])$?

A negative answer would lead to a natural example of a hypercyclic but not weakly mixing operator on a Banach space.

As in the hypercyclic case, we do not know if the weighted translation operators $T_{\phi,\alpha}$ are not supercyclic in general, even if we do know that they cannot satisfy the Supercyclicity Criterion.

Question 4.5.2. For every $\phi \in L^\infty([0,1])$ and every $\alpha \in [0,1]$, is it true that $T_{\phi,\alpha}$ is not supercyclic on $L^p([0,1])$?

Our results on cyclicity of operators T_α for $\alpha \in \mathbf{R}\backslash\mathbf{Q}$ give rise to many questions.

Question 4.5.3. Given $f \in L^p([0,1])\backslash\{\mathbf{0}\}$, does there exist a finite set $F \subset \mathbf{Q}$ such that f is cyclic for T_α for every $\alpha \in \mathbf{Q}\backslash F$?

Thanks to Theorem 4.4.16, this set could help to explicit a set of common cyclic vectors for T_α for a large set of $\alpha \in [0,1]$.

It would also be interesting to take a closer look at the function ψ appearing in the statement of Theorem 4.4.16.

Question 4.5.4. Can the function ψ appearing in the statement of Theorem 4.4.16 be made explicit, for instance in the case where $f = \mathbf{1}$?

We know that $f = \mathbf{1}$ is a common cyclic vector for T_α whenever $\alpha \in (0,1) \cap \mathbf{Q}$. Let $\varepsilon > 0$ and let $\alpha = r/q$ be a rational number in $(0,1)$. In order to obtain an explicit form for ψ, we would need for every $g \in L^\infty([0,1])$ vanishing on a set $\Omega_{n,f}$ to explicit a polynomial $Q \in \mathbf{C}[\xi]$ such that $\|Q(T_\alpha)\mathbf{1} - g\|_p < \varepsilon$. To do so, writing $g = g_0 T_\alpha^0 f + \ldots + g_{q-1} T_\alpha^{q-1} f$, we would have to take a look at polynomials $Q_j \in \mathbf{C}[\xi]$ such that $\|Q_j - g_j\|_\infty < \varepsilon$ whenever $j \in \{0, \ldots, q-1\}$. In the study of such polynomials, an obstacle is the study of the determinant

$$\Delta(\mathbf{1}, r/q)(t) = \begin{vmatrix} 1 & t & \ldots & t\{t+r/q\}\ldots\{t+(q-2)r/q\} \\ 1 & \{t+r/q\} & \ldots & \{t+r/q\}\ldots\{t+(q-1)r/q\} \\ \vdots & \vdots & & \vdots \\ 1 & \{t+(q-1)r/q\} & \ldots & \{t+(q-1)r/q\}t\ldots\{t+(q-3)r/q\} \end{vmatrix}.$$

This determinant seems to never vanish on $[0,1/q)$ and to be a monotone function on $[0,1/q)$. Proving such properties should lead to estimates allowing to obtain an explicit form of the function ψ.

Getting back to the study of invariant subspaces, we recall that in [18], the authors have proved that T_α admits a non-trivial closed hyperinvariant subspace in $L^p([0,1])$ as soon as the convergents $(p_n/q_n)_{n\geq 0}$ of α satisfy

$$\log(q_{n+1}) \underset{n\to+\infty}{=} O\left(\frac{q_n}{\log(q_n)^3}\right).$$

This condition gives a bound on the growth of the sequence $(q_n)_{n\geq 0}$ of denominators of the convergents of α. On the other hand, the results we proved in Theorem 4.4.16 state that T_α will be cyclic as soon as the sequence $(q_n)_{n\geq 0}$ has infinitely many sufficiently large gaps. This is not surprising. Indeed cyclicity and admitting non-trivial closed invariant subspaces can be seen as opposed properties of an operator T since T does not admit such a subspace if and only if every non-zero vector is cyclic for T. Nevertheless, an optimization of the function in the statement of Theorem 4.4.16 ψ could possibly lead to a positive answer to the following question.

Question 4.5.5. Is it possible to explicit irrational numbers α such that T_α is cyclic and admits a non-trivial (hyper)invariant subspace?

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