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par

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Homological algebra of Frobenius-twisted polynomial superfunctors

Algèbre homologique des superfoncteurs polynômiaux tordus à la Frobenius

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Abstract

The aim of this manuscript is to study the recently-introduced category $\boldsymbol{\mathcal{P}}$ of polynomial superfunctors in characteristic p. That is the $\mathbb{Z}/2\mathbb{Z}$-graded version of its older and better-known relative $\mathcal{P}$, introduced to help computing the cohomology of rational modules over a scheme. A particular interest lies within superfunctors that are twisted, i.e. obtained via precomposition by the Frobenius twist $\mathbf{I}_0^{(r)}$. Only few of their Ext-groups in $\boldsymbol{\mathcal{P}}$ had been computed so far. We were able to conjecture a general formula for them, surprisingly similar to the known one in $\mathcal{P}$, and managed to prove it true in many interesting cases. Depeding on situations, the results were pursued both in by direct approches and by more technical ones involving spectral sequences. Finally, the existence of an interesting cohomology class is pointed out as a possible waypoint towards the total comprehension of the subject.

Résumé

Le but de ce manuscrit est d'étudier la catégorie $\boldsymbol{\mathcal{P}}$ des superfoncteurs polynômiaux en caractéristique p. Celle-ci a été introduite récemment en tant que version $\mathbb{Z}/2\mathbb{Z}$-graduée de la plus ancienne $\mathcal{P}$, à son tour introduite comme instrument pour calculer la cohomologie des représentations rationnelles sur un schéma. Un intérêt particulier réside dans les superfoncteurs tordus, i.e. obtenus en précomposant par le twist de Frobenius $\mathbf{I}_0^{(r)}$. À la base, on ne connaît que quelqu'un des groupes Ext rélatifs à ces foncteurs. La contribution de cette thèse est de conjecturer une formule générale pour leur calcul, étonnamment similaire à celle connue pour $\mathcal{P}$, et d'en démontrer positivement plusieurs cas particuliers. Selon les divers cas, tels résultats ont été obtenus par d'approches soit directes soit plus techniques à l'aide de suites spectrales. Enfin, on met le doigt sur l'apparition d'une classe cohomologique spéciale, qui pourrait servir de passage vers la maîtrise totale du sujet.

Contents

Chapter 0

Introduction

Strict polynomial functors were introduced by Friedlander and Suslin in their paper [FS97] as a tool to prove the finite generation of $H^*(G, \Bbbk)$ where G is a finite group scheme over a field $\Bbbk$ of positive characteristic p. Informally, a *strict polynomial functor* is a functor from the category of vector spaces to itself, such that the maps between Hom-spaces given by this functor are polynomial. It is *homogeneous of degree* d if so are the latter maps. An example of d-homogeneous strict polynomial functor is given by the symmetric powers S^d, i.e. the functor defined by $V \mapsto S^d(V)$ on objects and by $\varphi \mapsto \varphi^d$ on morphisms. In particular there is the identity functor I. The category of strict polynomial functors with their natural transformations is noted $\mathcal{P}$. It is closely related to the category of rational GL_n-modules. There is indeed an exact functor $\mathcal{P} \to GL_n\text{-}\mathsf{mod}$, $F \mapsto F(\Bbbk^n)$ which induces an isomorphism

$$\mathrm{Ext}^*_{\mathcal{P}}(F, G) \xrightarrow{\simeq} \mathrm{Ext}^*_{GL_n}(F(\Bbbk^n), G(\Bbbk^n))$$

if n is big enough [FS97, Cor 3.13]. Hence, Ext-computations for rational modules can be replaced by Ext-computations in $\mathcal{P}$, which often happen to be easier. The authors take advantage of this fact in the very same paper, as they pass through the computation of $\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$ to prove the existence of a family of nonzero classes in $H^{2p^{r-1}}(GL_n, gl_n^{(r)})$. Here the $-^{(r)}$ index denotes, for $r \geq 1$, the r-th *Frobenius twist* of a module or of a polynomial functor[1]. In the literature, starting from [FS97], various computations of Ext between twisted functors have been made [FFSS99, Cha05, Tou13] until the most general one, which provides a graded natural isomorphism

$$\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r}) \tag{0.1}$$

where the right-hand side is graded by the total degree. Here $E_r := \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$ and G_{E_r} denotes the parametrised functor $G(E_r \otimes -)$ (cf. §2.3.3 or [Tou12] for details about parametrisation).

The Frobenius twist and its behaviour with respect to Ext are in fact the main topic of this manuscript. Recently, Axtell [Axt13] introduced a generalisation of the theory of strict polynomial functors in a $\mathbb{Z}/2\mathbb{Z}$-graded way. Throughout all this manuscript, the word *super* will always be short for "$\mathbb{Z}/2\mathbb{Z}$-graded". In this sense one speaks of super vector spaces, superalgebras etcetera, until one gets to define the main object of our work: the *strict polynomial superfunctors*. They can then be seen as an enhanced version of the objects of $\mathcal{P}$. Following Drupieski's spirit [Dru16], the "super" version of an object will be very often denoted by the same symbol in boldface. For example, the category of strict polynomial superfunctors will be noted by $\boldsymbol{\mathcal{P}}$. The Ext in this category are theirselves super vector spaces. The Frobenius twist functor $I^{(r)}$ admits

[1] Conventionally $F^{(0)} = F$, i.e. the 0-th twist is the identity. In the manuscript we will mostly ignore this extreme case since, as we are going to see in Chapter 2, some nice properties of the Frobenius twist do not extend to the $r = 0$ case.

a super-counterpart $\mathbf{I}^{(r)}$. It has the remarkable property of decomposing as the direct sum of two subfunctors $\mathbf{I}_0^{(r)}$ and $\mathbf{I}_1^{(r)}$, which are respectively concentrated in even and odd superdegrees. These subfunctors have in turn another notable property: one can precompose a strict polynomial functor (not superfunctor!) $F \in \mathcal{P}$ by $\mathbf{I}_0^{(r)}$ or $\mathbf{I}_1^{(r)}$ and obtain a strict polynomial superfunctor, denoted respectively $F_0^{(r)}$ and $F_1^{(r)}$. Some Ext-computations have been successfully performed in $\boldsymbol{\mathcal{P}}$ on this kind of objects (for example in [Dru16, DK22]) showing as many analogies as differences compared to the analogous computations in $\mathcal{P}$.

One of the main purposes of this PhD thesis is to study an analogue of the isomorphism (0.1) in $\boldsymbol{\mathcal{P}}$. Set $\boldsymbol{E_r} := \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(r)}, \mathbf{I}_0^{(r)})$. Our main conjecture is then the following.

Conjecture 0.1. *Let F, G be strict polynomial functors and let r be a positive integer. Then the superspace $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)})$ is purely even and there is a graded natural isomorphism*

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{\boldsymbol{E_r}})$$

where $G_{\boldsymbol{E_r}} = G(\boldsymbol{E_r} \otimes -)$ and the right-hand side is $\mathbb{Z}$-graded by the total degree.

We do not know if Conjecture 0.1 is true in general but we have very interesting partial results in this direction.:

- The first general assertion is true. Even more generally, for all $\ell, m \in \{0,1\}$ the superspace $\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}(F_\ell^{(r)}, G_m^{(r)})$ is concentrated in even superdegree (Theorem 4.1.3).
- Conjecture 0.1 is true if F or G are additive (Corollary 3.0.2).
- Conjecture 0.1 is true if F is projective or, dually, if G is injective (Theorem 4.2.5).
- In general, we have proved a nontrivial inclusion of a big direct summand of $\mathrm{Ext}^*_{\mathcal{P}}(F, G_{\boldsymbol{E_r}})$ into $\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)})$. Theorem 4.4.10 reads indeed:

  Theorem 0.2. *Let $r \geq 1$ and let F, G be homogeneous polynomial functors of degree d. The purely even space $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)})$ contains a copy of $\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)})$ as well as, for each $k \geq 1$, a copy of $\mathrm{Ext}^{<2p^r}_{\mathcal{P}}(F^{(r)}, G^{(r)})$ with cohomological degree shifted by $2dkp^r$.*

The key for this last computation is the construction of certain universal classes, which in some sense embody the "real" new content of the super theory in comparison to the *classical* (=ungraded) one. For a future perspective, in Problem 4.4.12 we conjecture the existence of a finer family of classes, which could give precious extra information and play a role in future improvements of the results.

Our results admit remarkable generalisations if we replace the Frobenius twists $\mathbf{I}_0^{(r)}, \mathbf{I}_1^{(r)}$ by a general additive superfunctor A. We managed indeed to prove a complete classification of the additive objects of $\boldsymbol{\mathcal{P}}$ (Theorem 2.3.5) which can be resumed in the following statement.

Theorem 0.3. *Let A be a homogeneous additive polynomial superfunctor. Then A has degree p^r for some $r \geq 0$ and it is isomorphic:*

- *to a sum of copies of $\boldsymbol{I}$ and $\boldsymbol{\Pi}$ if $r = 0$;*
- *to a sum of copies of $\boldsymbol{I}_0^{(r)}, \boldsymbol{I}_1^{(r)}, \boldsymbol{I}_0^{(r)} \circ \boldsymbol{\Pi}, \boldsymbol{I}_1^{(r)} \circ \boldsymbol{\Pi}$ if $r \geq 1$.*

Thanks to this fact, we can define more generally a superfunctor of the form $F \circ A$, where $F \in \mathcal{P}$ and A is additive, and investigate its homology. That leads to generalising Conjecture 0.1 in the following way.

Conjecture 0.4. *Let A, B be additive homogeneous superfunctors of degree strictly greater than 1 and let F, G be strict polynomial functors. There is a graded isomorphism, natural in all variables:*

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ B) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)})$$

where the right-hand side is graded by the total degree.

Note that the Ext appearing on the left-hand side of Conjecture 0.4 is a super vector space, while the Ext on the right-hand side is a priori only a vector space. In Section 2.3.4 we explain how to define a superdegree on the right-hand side in a natural way (using the super structure of $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A, B)$) and we prove that, if the isomorphism of the conjecture holds, then it must be an isomorphism of graded super vector spaces.

We will show that Conjecture 0.4 has positive response in the following cases:

- If F is also additive (Corollary 3.0.2);
- In degree zero: there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{P}}(F, G_{\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A,B)}) \simeq \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ B)$$

  proved in Theorem 2.3.33;
- If F is projective or, dually, if G is injective (Theorem 4.2.5).

We also prove at the end, in a similar (but weaker) way as in Theorem 0.2, that there is an actual inclusion of a big direct summand of $\mathrm{Ext}^*_{\mathcal{P}}(F, G_{\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)})$ into $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ B)$.

0.1 Contents of the manuscript

Chapter 1 is intended to introduce the reader to our objects of interest, the strict polynomial superfunctors. In first instance, we describe all our "super" framework. Notably we give the definition of super vector space and superlinear category. This will be the base to generalise classical polynomial functors [FS97] to their super counterparts. Once these are defined, we speak about their main basic properties and manipulations. In particular, we explain how the category $\boldsymbol{\mathcal{P}}$ admits a set of (co)generating projectives (injectives) which allow to perform Ext-computations in $\boldsymbol{\mathcal{P}}$.

In Chapter 2 we introduce the operation on (super)functors which will concern all our computations: the Frobenius twist. In characteristic p, such notion of twist of a vector space yields a strict polynomial functor $I^{(1)}$ of degree p [FS97, Pir03]. Iterating the construction, one gets the r-th twist $I^{(r)}$. In particular, given a functor F, one can then obtain by precomposition a new functor $F^{(r)} := F \circ I^{(r)}$. These constructions are replicable in the super context in more than one way. As we anticipated before, there are three super analogues of $I^{(r)}$, and by consequence at least three ways of twisting (cf. [Dru16]). In both contexts, the homological properties of a twisted (super)functor are different from the ones of F. For example, the functor $(-)^{(r)} : \mathcal{P} \to \mathcal{P}$, as well as its super analogue, is exact and fully faithful, but does not preserve projectives/injectives. Computation of Ext spaces between classical twisted functors has been performed by finding an explicit injective coresolution [Tro05, Tou12] or using the De Rham and Koszul complexes [FS97, Pir03]. Unfortunately, both ways are not quite direct to generalise. For example, the explicit injective coresolution found in [Tro05] is not easily replicable in the super context, as we explain in detail in Section 2.4. The latter is a short excursus about this very topic, and has the double goal of performing some known computations in a more optimal way and, as anticipated, showing why the super Ext are generally not computable in the same way as the classical ones.

A first fundamental result in Chapter 2 is the classification of all additive polynomial superfunctors. Generalising the classical result [Tou17] which states that the only additive polynomial functors are sums of copies of $I^{(r)}$, we prove (Theorem 2.3.5) that the only additive polynomial superfunctors are sums of copies of four different types if $r \geq 1$ (and two types if $r = 0$). Specifically, the ones of degree > 1 are all obtained by Frobenius twist superfunctors. We can then investigate, in greater generality, the homological algebra of a superfunctor of the form $F \circ A$ where A is additive.

In this direction, Chapters 3 and 4 contain the main computations of the thesis. They are meant to give an answer to Conjecture 0.4. We start by assuming that one of the two functors, for example F, is additive. By Touzé's classification of additive polynomial functors [Tou17, Prop. 3.5] we can furtherly assume that $F = I^{(r)}$. Write $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)^{(r)}$ for the Frobenius twist (degree-wise) of the space $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)$. In Chapter 3 we are able to build an explicit map

$$\Psi_{A,B} : \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)^{(r)} \to \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, G \circ B)$$

and prove that it is an isomorphism. In the formalism of parametrised functors, this says exactly that Conjecture 0.4 is true whenever at least one between F and G is additive (cf. Corollary 3.0.2).

In complete generality, things become more difficult. We focus on the case where F and G are homogeneous of the same degree d.[2] The approach developed in Chapter 4 consists in approaching $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ B)$ via the *twisting spectral sequence*:

$$\boldsymbol{II}_2^{s,t} := \mathrm{Ext}^s_{\mathcal{P}}(F, (G_{\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)})^t) \Rightarrow \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ B)\ .$$

The isomorphism of Conjecture 0.4 is then formally equivalent to the collapsing of this sequence. Such collapsing is unfortunately not easy to prove either. We then specialise into more explicit cases, starting from $A = B = \mathbf{I}_0^{(r)}$ which gives Conjecture 0.1. By means of a restriction morphism $res_0 : \boldsymbol{\mathcal{P}} \to \mathcal{P}$, which sends $F_0^{(r)} \mapsto F^{(r)}$ for all $F \in \mathcal{P}$, we find a morphism between our twisting spectral sequence and its classical counterpart, that is completely known [Tou13]. In this way we get to prove that Conjecture 0.4 holds in low degrees (Theorem 4.3.4). A nearly equivalent and equally suggestive form of this result is given just before (Theorem 4.3.3) and says that

$$res_0 : \mathrm{Ext}^n_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)}) \longrightarrow \mathrm{Ext}^n_{\mathcal{P}}(F^{(r)}, G^{(r)})$$

is an isomorphism for all $n < 2p^r$. However, this bound is small and independent of d, so not completely satisfying. In order to find more information, we pass to the case $A = \mathbf{I}_0^{(r)}, B = \mathbf{I}_1^{(r)}$. The interesting thing is that, in this case, there is no extension of degree less than dp^r and, additionally, we deduce the existence (not trivial at all) of a nonzero class $c_G \in \mathrm{Ext}^{dp^r}_{\boldsymbol{\mathcal{P}}}(G_0^{(r)}, G_1^{(r)})$. For $G = I$, this class corresponds to the generating class of $\mathrm{Ext}^{p^r}_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(r)}, \mathbf{I}_1^{(r)})$. We make use of this class to build a morphism between two twisting spectral sequences, in order to propagate as much as possible the information we already found. This results in the announced Theorem 0.2, ensuring an inclusion of a big summand into $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)})$. This inclusion is given by a formula which only involves res_0 and the product by a certain class $\varepsilon_G \in \mathrm{Ext}^{2dp^r}_{\boldsymbol{\mathcal{P}}}(G_0^{(r)}, G_0^{(r)})$ built from c_G. In particular, it is natural with respect to F and G. In conclusion, $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)})$ contains (at least) an infinite sum of copies of a subspace of $\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)})$, shifted in higher and higher cohomological degree. Despite being significantly weaker than Conjecture 0.4, this result points out a nice phenomenon of periodicity about the Ext in $\boldsymbol{\mathcal{P}}$, as well as a lower bound for its dimension in infinitely many degrees.

For general additive functors A and B, this last result can be generalised, provided that we know the dimension of $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)$ (which is easy, for example, when we know the decomposition of A and B). The space $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ B)$ will then contain several copies of $\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)})$ and its shiftings, with super-degrees encoded by the information about $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)$ (cf. Theorem 4.5.6).

[2] This is harmless to the computations, as explained in Convention 1.5.5 and in §1.5.1.

We leave space to an open question. Our universal class ε_G is supposed to somehow play the role of the known class $(e_r)^p \in \mathrm{Ext}^{2p^r}_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(r)}, \mathbf{I}_0^{(r)})$. The latter is the lowest-degree class that restricts to zero in $\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$ and is in fact the one that generates all $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(r)}, \mathbf{I}_0^{(r)})$ as a module over $\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$. For this reason, the class e_r embodies the truly original and mysterious ingredient of the super theory. It is then natural to wonder if, for example, ε_G admits a p-th root in general. We give a partial response in Problem 4.4.12, showing that this is the case if G is injective. The response for a generic G is not clear and would make an interesting point in the future development of the subject.

0.2 Conventions

Throughout all this manuscript, $\Bbbk$ is a field of characteristic $p \geq 3$ (in characteristic 2 there are no signs, hence the super theory coincides with the classical one). If not specified otherwise, all algebraic structures (vector spaces, algebras...) will be over $\Bbbk$, as well as all unadorned tensor products. If A is an algebra, $A - \mathsf{mod}$ denotes the category of finite-dimensional left modules over A. Finally, to be short, we will usually drop the word "strict" next to "polynomial (super)functor".

Chapter 1

Strict polynomial functors and superfunctors

1.1 Classical theory

Before introducing the super objects, we start by recalling the classical theory that we are going to generalise. For brevity, in this section some details are omitted: we will treat them more carefully in the super case. We start by setting some notation.

Notation 1.1.1. If V, W are vector spaces over a field $\Bbbk$, we will write:

- $\mathrm{Hom}(V, W)$ for the space of $\Bbbk$-linear morphisms from V to W.
- $V \otimes W$ for their tensor product over $\Bbbk$.
- $V^\vee := \mathrm{Hom}(V, \Bbbk)$ for the linear dual of V.
- If $E^* = \bigoplus_{i\in\mathbb{Z}} E^i$ is a graded vector space, the linear dual $(E^\vee)^*$ will always intended to be the restricted dual $\bigoplus_{i\in\mathbb{Z}}(E^i)^\vee$.

All unadorned Hom and $\otimes$ symbols will be assumed to be over $\Bbbk$. We will also say *linear* instead of $\Bbbk$-*linear* when no ambiguity about $\Bbbk$ is possible.

Notation 1.1.2. We denote by $\mathfrak{vec}$ the category of vector spaces and linear morphisms, and by $\mathcal{V}$ the full subcategory of $\mathfrak{vec}$ formed by finite-dimensional vector spaces.

They are an example of *linear* categories, i.e. enriched over $\mathfrak{vec}$.

Definition 1.1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two linear categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *linear* if so are the structure maps $\mathrm{Hom}_{\mathcal{C}}(c_1, c_2) \to \mathrm{Hom}_{\mathcal{D}}(Fc_1, Fc_2)$.

Example 1.1.4. For any linear category $\mathcal{C}$, the identity functor Id_C is linear. For all vector spaces W, the endofunctor $-\otimes W : \mathfrak{vec} \to \mathfrak{vec}$ is linear. On the other side, for example, the symmetric power functor $V \mapsto S^2(V)$ is not.

We recall that, for V a vector space, $S^n(V)$ and $\Gamma^n(V)$ are defined respectively as the coinvariants and invariants of $V^{\otimes n}$ under the action of the symmetric group Σ_n which permutes the tensors. They are respectively called n-th *symmetric* and *divided powers*. Summing over n, we obtain two algebras: the symmetric power algebra $S^*(V)$, equipped with concatenation of tensors, and the divided power algebra $\Gamma^*(V)$ equipped with the shuffle product[1]. We will sometimes use the following explicit presentation of the classical divided power algebra Γ^*V.

[1]We will give more details about these structures in the next section.

Proposition 1.1.5. *Let V be a vector space. $\Gamma^*(V)$ is generated as an algebra by the symbols $\gamma_n(v)$, as n runs in $\mathbb{N}$ and $v \in V$, subject to the relations:*

$$\gamma_n(\lambda v) = \lambda^n \gamma_n(v)$$
$$\gamma_n(v) \cdot \gamma_m(v) = \binom{m+n}{m} \gamma_{n+m}(v)$$
$$\gamma_n(v+w) = \sum_{i=0}^{n} \gamma_i(v) \cdot \gamma_{n-i}(w)$$
$$\gamma_n(\gamma_m(v)) = \frac{(mn)!}{(m)!^n n!} \gamma_{mn}(v) \,.$$

Proof. A possible reference is [Pir03, §1.5]. □

Remark 1.1.6. For all $d \geq 0$ and all vector spaces V, W the linear isomorphism

$$V^{\otimes d} \otimes W^{\otimes d} \longrightarrow (V \otimes W)^{\otimes d}$$
$$(v_1 \otimes ... \otimes v_d) \otimes (w_1 \otimes ... \otimes w_d) \longmapsto (v_1 \otimes w_1) \otimes ... \otimes (v_d \otimes w_d)$$

is Σ_d-equivariant if on the left-hand side we consider the diagonal permutation action of Σ_d. This implies that $\Gamma^d V \otimes \Gamma^d W$ is naturally a subspace of $\Gamma^d(V \otimes W)$. In particular, if A is an algebra, this induces a map

$$\Gamma^d A \otimes \Gamma^d A \longrightarrow \Gamma^d(A \otimes A) \longrightarrow \Gamma^d A$$

which equips $\Gamma^d A$ with the structure of algebra.

Definition 1.1.7. For $d \geq 0$, define $\Gamma^d \mathcal{V}$ as the category with the same objects as $\mathcal{V}$ and morphisms $\mathrm{Hom}_{\Gamma^d \mathcal{V}}(V, W) := \Gamma^d \mathrm{Hom}(V, W)$. One way to define composition in $\Gamma^d \mathcal{V}$ is to use the isomorphism

$$\Gamma^d \mathrm{Hom}(V, W) \simeq \mathrm{Hom}_{\Bbbk \Sigma_d}(V^{\otimes d}, W^{\otimes d})$$

(which is proved in a more general version in Lemma 1.3.16) and pass to the standard composition of equivariant maps.

Remark 1.1.8. $\Gamma^d \mathcal{V}$ is a linear category.

Definition 1.1.9. A d-homogeneous *strict polynomial functor* is a linear functor $F : \Gamma^d \mathcal{V} \to \mathcal{V}$.

The symbol $\mathcal{P}_d$ denotes the category of d-homogeneous strict polynomial functors with their natural transformations. The product $\prod_{d \geq 0} \mathcal{P}_d$ of all these categories is denoted $\mathcal{P}$.

Example 1.1.10. A strict polynomial functor of degree 0 is a constant functor. Strict polynomial functors of degree 1 are just linear functors from $\mathcal{V}$ on itself: for example $\mathrm{Hom}(V, -)$ and $V \otimes -$, where V is any fixed vector space of finite dimension. In particular there is the identity functor I. More generally $\otimes^d, S^d, \Gamma^d, \Lambda^d$ can be viewed as strict polynomial functors of degree d (the argument is the same as in Example 1.5.7).

Remark 1.1.11. Let $F \in \mathcal{P}_d$, $G \in \mathcal{P}_e$. Then $F \oplus G \in \mathcal{P}$ (non-homogeneous), $F \otimes G \in \mathcal{P}_{d+e}$, while $F \circ G \in \mathcal{P}_{de}$.

The divided and symmetric power functors play a quite important role in $\mathcal{P}$. For a finite-dimensional vector space V, let

$$\Gamma^{d,V} := \Gamma^d \mathrm{Hom}(V, -)$$
$$S^d_V := S^d(V \otimes -)$$

which are polynomial functors of degree d, since they are the composition of Γ^d (resp. S^d) with $\mathrm{Hom}(V, -)$ (resp. $V \otimes -$).

Theorem 1.1.12 (Yoneda lemma)**.** *There is a linear isomorphism, natural in $F \in \mathcal{P}_d$ and $V \in \Gamma^d\mathcal{V}$:*

$$\mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,V}, F) \simeq F(V)\ .$$

As a consequence, $\Gamma^{d,V}$ is a projective object of $\mathcal{P}_d$. Moreover, the associated natural transformation $\Gamma^{d,V} \otimes F(V) \to F$ given by $f \otimes v \longmapsto (Ff)(v)$, is surjective whenever $\dim(V) \geq d$ [Fri03, Prop. 3.11]. So, the collection of $\Gamma^{d,V}$ forms a set of projective generators for $\mathcal{P}_d$. We now end the section with a description of $\mathcal{P}_d$ as a category of left modules. Remember by Remark 1.1.6 that $\Gamma^d A$ is an algebra whenever A is.

Definition 1.1.13. For any $n, d \geq 1$, the *Schur algebra* is defined as $S(n,d) := \Gamma^d\mathrm{End}(\Bbbk^n)$.

If $F \in \mathcal{P}_d$, then $F(\Bbbk^n)$ is a left $S(n,d)$-module via

$$\begin{aligned} S(n,d) \otimes F(\Bbbk^n) &\to F(\Bbbk^n) \\ \varphi \otimes v &\longmapsto (F\varphi)(v)\ . \end{aligned}$$

In addition, for all $T \in \mathrm{Hom}_{\mathcal{P}}(F,G)$, $T_{\Bbbk^n}$ is a $S(n,d)$-equivariant map. Thus evaluation at $\Bbbk^n$ defines a functor

$$\mathcal{P}_d \longrightarrow S(n,d) - \mathsf{mod} \tag{1.1.1}$$

where the notation $A - \mathsf{mod}$ stands for the category of finite-dimensional left modules over an algebra A.

Theorem 1.1.14 ([Fri03, Thm 3.12])**.** *If $n \geq d$, the functor (1.1.1) gives rise to an equivalence of categories*

$$\mathcal{P}_d \simeq S(n,d) - \mathsf{mod}$$

with quasi-inverse provided by $M \mapsto \Gamma^{d,\Bbbk^n} \otimes_{S(n,d)} M$.

1.2 The super world

Definition 1.2.1. A *super vector space* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space, that is, a vector space V together with a decomposition $V = V_0 \oplus V_1$. Here V_0 is called the *even* part of V, while V_1 is called the *odd* part.

By definition, all elements of V decompose uniquely as the sum of an even and an odd one, i.e. of a vector belonging to V_0 and V_1 respectively. The overbarred symbol $\overline{v}$ stands for the $\mathbb{Z}/2\mathbb{Z}$-degree (or *super degree*) of v. Hence, $\overline{v} = 0$ if $v \in V_0$ and $\overline{v} = 1$ if $v \in V_1$.

Notation 1.2.2. In our vocabulary, *super* will always mean $\mathbb{Z}/2\mathbb{Z}$-graded. In contrast, the adjective *classical* will denote the ungraded version of an object or a result (ex: vector spaces are the classical version of super vector spaces).

Notation 1.2.3. We would be supposed to put the bar over 0 and 1 as well, to mark them as elements of $\mathbb{Z}/2\mathbb{Z}$. In order to lighten notation, we will drop it. Whenever they appear explicitly as super degrees of an element or as subscripts of V, they are to be intended modulo 2, as well as all operations involving them.

Convention 1.2.4. When making definitions on graded objects (in particular on super vector spaces) which involve explicit formulas, we will nearly always write them for homogeneous elements. In words this means that if a degree symbol, e.g. $\overline{v}$, appears in a definition, we are always assuming that v is homogeneous in order for the formula to make sense, and tacitly extending the definition by linearity.

Definition 1.2.5. The *superdimension* of a super vector space V is the pair of non-negative integers $\mathrm{sdim}(V) := (\dim V_0, \dim V_1)$.

If $V_1 = 0$, we say that V is *purely even*; if $V_0 = 0$, it is *purely odd.* If W is a vector space, we can see it as a purely even super vector space $W \oplus 0$. Abusing notation, we will still denote it by W. We can also see it as a purely odd super vector space $0 \oplus W$, in which case we will denote it ΠW. In particular, a super vector space of superdimension (n, m) identifies with $\Bbbk^{n|m} := \Bbbk^n \oplus \Pi\Bbbk^m$. In general, the letter Π denotes a parity change operation, as in the following definition.

Definition 1.2.6 (Parity change)**.** If $V = V_0 \oplus V_1$, then $\Pi V := V_1 \oplus V_0$ is the same vector space with interchanged homogeneous components. If $v \in V$ is a homogeneous vector, πv will stand for the same vector seen in ΠV. In particular (by construction) $\overline{\pi v} = \overline{v} + 1$.

The general notion of linear morphism is the same as in the ungraded case. The symbol Hom with no subscript will always stand for $\mathrm{Hom}_{\Bbbk}$, i.e. the space of linear morphisms between two super vector spaces.

Definition 1.2.7. A linear morphism $f : V \to W$ between two super vector spaces is called *even* if $f(V_i) \subset W_i$, *odd* if $f(V_i) \subset W_{i+1}$, $i = 0, 1$.

Informally speaking, an even morphism preserves the grading while an odd morphism inverses it. Since by definition every super vector space decomposes (uniquely) as the sum of its even and odd parts, it follows that any linear morphism $f : V \to W$ decomposes uniquely into the sum of an even and an odd morphism:

$$\mathrm{Hom}(V, W) = \mathrm{Hom}_0(V, W) \oplus \mathrm{Hom}_1(V, W) .$$

In particular, this gives $\mathrm{Hom}(V, W)$ itself the structure of a super vector space. We keep track of an identity, which is just a rephrasing of the definition of even/odd morphism: for all homogeneous $f \in \mathrm{Hom}(V, W)$ and any $v \in V$,

$$\overline{f(v)} = \overline{f} + \overline{v} .$$

Using this identity one can readily verify, for example, that if f, g are two homogeneous composable morphisms, then

$$\overline{f \circ g} = \overline{f} + \overline{g} .$$

We recall how classical operations on vector spaces generalise on two super vector spaces V, W.

- **Direct sum:** $V \oplus W$ is a super vector space, its even (resp. odd) part being $V_0 \oplus W_0$ (resp. $V_1 \oplus W_1$).
- **Linear dual:** $V^\vee = \mathrm{Hom}(V, \Bbbk)$ comes as a special case of $\mathrm{Hom}(V, W)$ above, with $\Bbbk$ considered as purely even. In particular, its super decomposition is $V^\vee = V_0^\vee \oplus V_1^\vee$. If $f : V \to W$ is a linear map, its transpose $f^\vee : W^\vee \to V^\vee$ is given by the formula

  $$< f^\vee\varphi, x >:= (-1)^{\overline{f}\,\overline{\varphi}} < \varphi, f(x) >$$

  which implies that $(f \circ g)^\vee = (-1)^{\overline{f}\,\overline{g}} g^\vee \circ f^\vee$.
- **Tensor product:** The super structure on $V \otimes W$ can be induced by the previous ones via the isomorphism $V \otimes W \simeq \mathrm{Hom}(V^\vee, W)$. Explicitly, this gives the following super decomposition:

  $$(V \otimes W)_0 = V_0 \otimes W_0 \;\oplus\; V_1 \otimes W_1$$
  $$(V \otimes W)_1 = V_0 \otimes W_1 \;\oplus\; V_1 \otimes W_0$$

Remark 1.2.8. Nearly all classical isomorphisms have their super version. For example there is an isomorphism, which is natural with respect to the super vector space V:

$$\begin{aligned} V &\xrightarrow{\simeq} (V^\vee)^\vee \\ v &\longmapsto ev_v \end{aligned} \tag{1.2.1}$$

where ev_v acts on a function $f \in V^\vee$ by $f \mapsto (-1)^{\overline{f}\,\overline{v}} f(v)$.

Definition 1.2.9. The *interchange* or *twisting* map of the tensor product of two super vector spaces V, W is defined as

$$\begin{aligned} V \otimes W &\longrightarrow W \otimes V \\ v \otimes w &\longmapsto (-1)^{\overline{v}\,\overline{w}}\, w \otimes v \ . \end{aligned} \tag{1.2.2}$$

Definition 1.2.10 (Tensor product of maps)**.** Let $f : V \to W$ and $g : V' \to W'$ be linear maps between super vector spaces. The tensor product map $f \otimes g : V \otimes V' \to W \otimes W'$ is defined by

$$(f \otimes g)(v \otimes v') = (-1)^{\overline{g}\,\overline{v}} f(v) \otimes g(v') \ .$$

This operation is associative, hence one can define recursively the tensor product of any number of maps.

Remark 1.2.11. Composition is compatible with tensor product of maps up to a Koszul sign. Using the definition, one checks indeed that

$$(f \otimes g) \circ (h \otimes l) = (-1)^{\overline{g}\,\overline{h}} (f \circ h) \otimes (g \circ l)$$

for any two pairs of composable morphisms (f, h) and (g, l).

1.3 Superalgebras

Definition 1.3.1. A *superalgebra* is the data $(A, u, \cdot)$ of a super vector space A with a unit $u : \mathbb{k} \to A$ and an associative, unital product $\cdot : A \otimes A \to A$ which is even linear as a map of super vector spaces.

All superalgebras appearing in the manuscript are associative and unital, so we drop these two adjectives from now on. Last condition means in words that $A_i \cdot A_j \subset A_{i+j}$ for $i, j \in \{0, 1\}$. Equivalently, for all homogeneous $a, b \in A$, $\overline{a \cdot b} = \overline{a} + \overline{b}$. When there is no risk of confusion, we will drop the symbol $\cdot$ of the product. A morphism of superalgebras is simply a morphism of the underlying algebras. Given two superalgebras A and B, their tensor product $A \otimes B$ is considered with the signed product

$$(a \otimes b)(c \otimes d) := (-1)^{\overline{b}\,\overline{c}}\, ac \otimes bd \ . \tag{1.3.1}$$

These conventions produce the following.

Lemma 1.3.2. *Let A, B be superalgebras. The twisting map (1.2.2) commutes with the product (1.3.1), hence defines an isomorphism of superalgebras.*

Proof. Call τ the twist map (1.2.2). Then

$$\begin{aligned} \tau(a \otimes b) \cdot \tau(c \otimes d) &= (-1)^{\overline{a}\,\overline{b} + \overline{c}\,\overline{d}} (b \otimes a)(d \otimes c) \\ &= (-1)^{\overline{a}\,\overline{b} + \overline{c}\,\overline{d} + \overline{a}\,\overline{d}} (bd \otimes ac) \end{aligned}$$

and on the other side

$$\begin{aligned}\tau((-1)^{\overline{b}\,\overline{c}}\,(ac\otimes bd)) &= (-1)^{\overline{b}\,\overline{c}+\overline{ac}\,\overline{bd}}\,(bd\otimes ac)\\ &= (-1)^{\overline{b}\,\overline{c}+(\overline{a}+\overline{c})(\overline{b}+\overline{d})}(bd\otimes ac)\\ &= (-1)^{\overline{a}\,\overline{b}+\overline{c}\,\overline{d}+\overline{a}\,\overline{d}}\,(bd\otimes ac)\end{aligned}$$

so the two images coincide. □

The map (1.2.2) will be our notion of twist by default. Consequently, in our glossary a superalgebra will be said *commutative* if multiplication commutes with (1.2.2), i.e. if $ab = (-1)^{\overline{a}\,\overline{b}}\,ba$ for all a, b homogeneous.

Definition 1.3.3. A *graded superalgebra* is a superalgebra A which additionally bears a $\mathbb{Z}$-grading that is preserved by the product. The $\mathbb{Z}$-degree of a $\mathbb{Z}$-homogeneous element $a \in A$ will be denoted by $|a|$. The condition on the product is then $|a{\cdot}b| = |a|+|b|$ for all a, b $\mathbb{Z}$-homogeneous.

Let A, B be graded superalgebras. There is more than one definition for the tensor product of A and B. The symbol $A \otimes B$ will denote the graded superalgebra with the same product (1.3.1), which then ignores the $\mathbb{Z}$-degrees. Alternatively, one can define the graded tensor product $A \otimes^g B$, which has still $A \otimes B$ as underlying graded super vector space but a product that takes account of both gradings:

$$(a\otimes b)\cdot(c\otimes d) := (-1)^{\overline{b}\,\overline{c}+|b||c|}\,ac\otimes bd\ .$$

The proof of Lemma 1.3.2 generalises without problems to the following:

Lemma 1.3.4. *There is a graded even isomorphism of graded superalgebras*

$$\begin{aligned}A\otimes^g B &\longrightarrow B\otimes^g A\\ a\otimes b &\longmapsto (-1)^{\overline{a}\,\overline{b}+|a||b|}b\otimes a\ .\end{aligned}\tag{1.3.2}$$

As above, a superalgebra will be said *graded-commutative* if it is graded and its product commutes with the graded twist (1.3.2). It will be just *commutative* if it is so as an ungraded superalgebra.

Definition 1.3.5. A *supercoalgebra* is the data (C, η, Δ) of a super vector space C with a counity $\eta : C \to \Bbbk$ and a coassociative counital map $\Delta : C \to C \otimes C$ (coproduct) which is even linear as a map of super vector spaces.

A supercoalgebra is *cocommutative* if the coproduct is compatible with the twist (1.2.2). A *graded supercoalgebra* is a supercoalgebra bearing a $\mathbb{Z}$-grading preserved by Δ. It is *graded cocommutative* if the coproduct is compatible with the graded twist (1.3.2). We provide some basic examples, which are the super version of well known (co)algebras.

Example 1.3.6. Let V be a finite-dimensional super vector space. The *tensor superalgebra* on V is $\boldsymbol{\mathcal{T}}^*(V) := \bigoplus_n V^{\otimes n}$ equipped with the standard *concatenation* product

$$(v_1\otimes ...\otimes v_n)\otimes(v_1'\otimes ...\otimes v_m')\longmapsto v_1\otimes ...\otimes v_n\otimes v_1'\otimes ...\otimes v_m'\ .$$

In fact, $\boldsymbol{\mathcal{T}}^*(V)$ is free in the category of superalgebras. If we take its restricted dual, we obtain an isomorphic vector space equipped with the *deconcatenation* coproduct

$$\Delta(v_1\otimes ...\otimes v_n) = \sum_{i=1}^{n}(v_1\otimes ...\otimes v_i)\otimes(v_{i+1}\otimes ...\otimes v_n)$$

which is the cofree conilpotent object in the category of supercoalgebras. Thanks to the (co)freeness, we can equip both with a structure of a superbialgebra. Define the *shuffle product* as the unique supercoalgebra morphism $\mu : \boldsymbol{\mathcal{T}}^*(V) \otimes \boldsymbol{\mathcal{T}}^*(V) \to \boldsymbol{\mathcal{T}}^*(V)$ described in degree 1 by $\mu(v \otimes 1) = \mu(1 \otimes v) = v$. Its dual, the *shuffle coproduct* (or *deshuffle*, as we will call it to be shorter) is the only superalgebra morphism $\Delta' : \boldsymbol{\mathcal{T}}^*(V) \to \boldsymbol{\mathcal{T}}^*(V) \otimes \boldsymbol{\mathcal{T}}^*(V)$ such that $\Delta'(v) = v \otimes 1 + 1 \otimes v$. By construction we have that

$$\boldsymbol{\mathcal{T}}_1^*(V) := (\boldsymbol{\mathcal{T}}^*(V), \text{concatenation, deshuffle})$$
$$\boldsymbol{\mathcal{T}}_2^*(V) := (\boldsymbol{\mathcal{T}}^*(V), \text{shuffle, deconcatenation})$$

provide two structures of superbialgebras on $\boldsymbol{\mathcal{T}}^*$ that are dual to each other, i.e.

$$(\boldsymbol{\mathcal{T}}_1^*(V^\vee))^\vee \simeq \boldsymbol{\mathcal{T}}_2^*(V)$$

and vice-versa. Moreover, $\boldsymbol{\mathcal{T}}_2^*(V)$ (resp. $\boldsymbol{\mathcal{T}}_1^*(V)$) is commutative (resp. cocommutative), while the other one is obviously not.

Remark 1.3.7. There is a general explicit description for the shuffle product, which justifies also its name. Let $\mathfrak{S}_{m,n} \subset \Sigma_{m+n}$ be the subset of (m,n)-*shuffles*, i.e. the permutations σ such that $\sigma(1) < ... < \sigma(n)$ and $\sigma(n+1) < ... < \sigma(n+m)$. The shuffle product is then given at each level $(\otimes^n) \otimes (\otimes^m) \to \otimes^{n+m}$ by the map $\sum_{\sigma \in \mathfrak{S}_{m,n}} (-) \cdot \sigma$. A similar description holds for the deshuffle.

Example 1.3.8. Consider the right action of the symmetric group Σ_n on $V^{\otimes n}$ determined by transpositions of the form $(i, i+1)$ in the following way:

$$(v_1 \otimes \cdots \otimes v_n) \cdot (i, i+1) := (-1)^{\overline{v_i}\,\overline{v_{i+1}}} (v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n)\,. \tag{1.3.3}$$

The *super divided power* $\mathbf{\Gamma}^n V$ is defined as the invariants of this action; the *super symmetric power* $\mathbf{S}^n V$ as its coinvariants. Set $\mathbf{\Gamma}^* V = \bigoplus_n \mathbf{\Gamma}^n V$ and $\mathbf{S}^* V = \bigoplus_n \mathbf{S}^n V$. They are both commutative superalgebras. The product (coproduct) on $\mathbf{S}^*$ is given by concatenation (deshuffle), while on $\mathbf{\Gamma}^*$ by shuffle (deconcatenation). In other words, $\mathbf{S}^*(V)$ is a quotient bialgebra of $\boldsymbol{\mathcal{T}}_1^*(V)$, while $\mathbf{\Gamma}^*(V)$ is a sub-bialgebra of $\boldsymbol{\mathcal{T}}_2^*(V)$.

Remark 1.3.9. Let $d \geq 0$ and let V, W be super vector spaces. Similarly to the classical case (Remark 1.1.6), the isomorphism $V^{\otimes d} \otimes W^{\otimes d} \to (V \otimes W)^{\otimes d}$ induced by the super twist (1.2.2) is Σ_d-equivariant, once the left-hand side is endowed with the diagonal action of Σ_d. Hence $\mathbf{\Gamma}^d V \otimes \mathbf{\Gamma}^d W$ is naturally a subspace of $\mathbf{\Gamma}^d(V \otimes W)$. In particular, if A is a superalgebra, a superalgebra structure is induced on $\mathbf{\Gamma}^d A$.

Example 1.3.10. Consider the action of Σ_n which modifies (1.3.3) by a minus sign, i.e. determined on transpositions $(i, i+1)$ by

$$(v_1 \otimes \cdots \otimes v_n) \cdot (i, i+1) := -(-1)^{\overline{v_i}\,\overline{v_{i+1}}} (v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n)\,.$$

The *super alternating power* $\mathbf{A}^n V$ is defined as the invariants of this action; the *super exterior power* $\mathbf{\Lambda}^n V$ as its coinvariants. Here V is considered as a $\mathbb{Z}$-graded super vector space concentrated in $\mathbb{Z}$-degree 1. Set $\mathbf{A}^* V = \bigoplus_n \mathbf{A}^n V$ and $\mathbf{\Lambda}^* V = \bigoplus_n \mathbf{\Lambda}^n V$. As above, $\mathbf{\Lambda}^*(V)$ inherits the structure of bialgebra from $\boldsymbol{\mathcal{T}}_1^*(V)$, while $\mathbf{A}^*(V)$ from $\boldsymbol{\mathcal{T}}_2^*(V)$.

Remark 1.3.11. All products and coproducts defined on $\mathbf{S}^*(V)$, $\mathbf{\Gamma}^*(V)$, $\mathbf{\Lambda}^*(V)$ and $\mathbf{A}^*(V)$ are manifestly natural with respect to V. Moreover, $\mathbf{S}^*(V)$ and $\mathbf{\Gamma}^*(V)$ are commutative and cocommutative, while $\mathbf{\Lambda}^*(V)$ and $\mathbf{A}^*(V)$ are graded- commutative and graded-cocommutative.

Remark 1.3.12. Let G be a group and M be a G-module. The linear dual $M^\vee$ is also a G-module via $(gf)(m) := f(g^{-1}m)$. In this sense, there is a natural isomorphism $(M^\vee)^G \simeq (M_G)^\vee$. In our case this means that, as graded super vector spaces,

$$\mathbf{\Gamma}^*(V^\vee) \simeq (\mathbf{S}^* V)^\vee, \quad \mathbf{A}^*(V^\vee) \simeq (\mathbf{\Lambda}^* V)^\vee$$

where we recall that $(\mathbf{S}^* V)^\vee$ and $(\mathbf{\Lambda}^* V)^\vee$ denote the *restricted* duals. We will see later that these are in fact natural isomorphisms of bialgebras.

Remark 1.3.13. The functor $\mathbf{S}^*$, resp. $\mathbf{\Lambda}^*$, is left adjoint to the forgetful functor from the category of commutative (resp. graded-commutative) superalgebras to the category of super vector spaces (resp. $\mathbb{Z}$-graded super vector spaces). In particular they respect coproducts. Since the coproduct in the category of (graded-)commutative superalgebras is given by $\otimes$, this means that there exist natural isomorphisms

$$\mathbf{S}^*(V \oplus W) \simeq \mathbf{S}^*(V) \otimes \mathbf{S}^*(W)$$

$$\mathbf{\Lambda}^*(V \oplus W) \simeq \mathbf{\Lambda}^*(V) \otimes \mathbf{\Lambda}^*(W)$$

of which we a give an explicit formula in the next proposition.

Proposition 1.3.14 (Exponential property)**.** *For all finite-dimensional super vector spaces V, W there are isomorphisms of graded superbialgebras*

$$\boldsymbol{S}^* V \otimes \boldsymbol{S}^* W \simeq \boldsymbol{S}^*(V \oplus W)$$
$$\mathbf{\Lambda}^* V \otimes \mathbf{\Lambda}^* W \simeq \mathbf{\Lambda}^*(V \oplus W)$$

induced respectively by concatenation product and shuffle coproduct. The dual operations induce natural isomorphisms of graded superbialgebras

$$\mathbf{\Gamma}^* V \otimes \mathbf{\Gamma}^* W \simeq \mathbf{\Gamma}^*(V \oplus W)$$
$$\boldsymbol{A}^* V \otimes \boldsymbol{A}^* W \simeq \boldsymbol{A}^*(V \oplus W).$$

Proof. We only treat the case of $\mathbf{S}^*$ and $\mathbf{\Lambda}^*$, the proof for the other two following entirely by the duality of Remark 1.3.12. Let us start by $\mathbf{S}^*$. Call μ the concatenation map $\mathbf{S}^*(V) \otimes \mathbf{S}^*(W) \to \mathbf{S}^*(V \oplus W)$. It is clearly surjective by the bilinear property of the tensor product, hence by dimension reasons it is a natural isomorphism of super vector spaces. We have just to verify that it is compatible with the product and coproduct of $\mathbf{S}^*$. Let $v, v' \in \mathbf{S}^*(V)$ and $w, w' \in \mathbf{S}^*(W)$. As defined in (1.3.1), the product $(v \otimes w) \cdot (v' \otimes w')$ in $\mathbf{S}^*(V) \otimes \mathbf{S}^*(W)$ equals $(-1)^{\overline{w}\,\overline{v'}}\, vv' \otimes ww'$, which is sent by μ to $(-1)^{\overline{w}\,\overline{v'}}\, vv'ww'$. But this is equal to the product of vw and $v'w'$ in $\mathbf{S}^*(V \oplus W)$, since interchanging w and v' brings a sign $(-1)^{\overline{w}\,\overline{v'}}$. Thus $\mu((v \otimes w) \cdot (v' \otimes w')) = \mu(v \otimes w)\mu(v' \otimes w')$. To prove that it respects the coproduct Δ' amounts to verifying the commutativity of the diagram

$$\begin{tikzcd}
\mathbf{S}^*(V) \otimes \mathbf{S}^*(W) \arrow[r, "\Delta' \otimes \Delta'"] \arrow[dd, "\mu"] & \mathbf{S}^*(V) \otimes \mathbf{S}^*(V) \otimes \mathbf{S}^*(W) \otimes \mathbf{S}^*(W) \arrow[d, "1 \otimes \tau \otimes 1"] \\
 & \mathbf{S}^*(V) \otimes \mathbf{S}^*(W) \otimes \mathbf{S}^*(V) \otimes \mathbf{S}^*(W) \arrow[d, "\mu \otimes \mu"] \\
\mathbf{S}^*(V \oplus W) \arrow[r, "\Delta'"] & \mathbf{S}^*(V \oplus W) \otimes \mathbf{S}^*(V \oplus W)
\end{tikzcd}$$

where τ is the twist map (1.2.2). Since Δ' is determined by the product and by its expression in degree 1 (see Example 1.3.6), it will suffice to prove the commutativity of the diagram for an element of the form $v \otimes 1 \in \mathbf{S}^1(V) \otimes \mathbf{S}^0(W)$. We have $\Delta'(\mu(v \otimes 1)) = v \otimes 1 + 1 \otimes v$. On the other hand, $\Delta' \otimes \Delta'$ takes it onto $(v \otimes 1 + 1 \otimes v) \otimes (1 \otimes 1) = v \otimes 1 \otimes 1 \otimes 1 \ + \ 1 \otimes v \otimes 1 \otimes 1$, then $1 \otimes \tau \otimes 1$ onto $v \otimes 1 \otimes 1 \otimes 1 \ + \ 1 \otimes 1 \otimes v \otimes 1$ and finally $\mu \otimes \mu$ onto $v \otimes 1 + 1 \otimes v$. The verification is thus completed for $\mathbf{S}^*$. Let now $u : \mathbf{\Lambda}^*(V \oplus W) \to \mathbf{\Lambda}^*(V) \otimes \mathbf{\Lambda}^*(W)$ be the only morphism of superalgebras that restricts in degree 1 to $v + w \mapsto v \otimes 1 \ + \ 1 \otimes w$. It respects the multiplication by definition, and a verification similar to the above shows that it respects the coproduct as well. It is only left to show that it is an isomorphism. Again, for dimensions reasons, it suffices to prove that it is a surjection. We will provide a preimage of a generic element $(v_1 \wedge ... \wedge v_n) \otimes (w_1 \wedge ... \wedge w_m) \in \mathbf{\Lambda}^n(V) \otimes \mathbf{\Lambda}^m(W)$. The case $n = m = 0$ is trivial. Otherwise

$$(v_1 \wedge ... \wedge v_n) \otimes (w_1 \wedge ... \wedge w_m) = \pm[(v_1 \wedge ... \wedge v_{n-1}) \otimes (w_1 \wedge ... \wedge w_{m-1})] \cdot [v_n \otimes w_m]$$

thus the existence of a preimage follows by induction and by the multiplicativity of u. $\square$

Applying the exponential isomorphisms on the decomposition $V = V_0 \oplus V_1$ and studying the actions on each homogeneous component, one finds $\mathbb{Z}$-graded isomorphisms of the underlying algebras

$$\mathbf{S}^*V \simeq S^*(V_0) \otimes \Lambda^*(V_1) \tag{1.3.4}$$

$$\mathbf{\Gamma}^*V \simeq \Gamma^*(V_0) \otimes \Lambda^*(V_1) \tag{1.3.5}$$

$$\mathbf{A}^*V \simeq \Lambda^*(V_0) \otimes^g \Gamma^*(V_1) \tag{1.3.6}$$

$$\mathbf{\Lambda}^*V \simeq \Lambda^*(V_0) \otimes^g S^*(V_1) \tag{1.3.7}$$

The first two become isomorphisms of superalgebras by placing V_i in super degree i on the right. The last two become isomorphisms of graded superalgebras by doing the same and placing V in $\mathbb{Z}$-degree 1. Note that the decomposition of V in even and odd part is not natural, because of the odd morphisms that switch them. Hence *a fortiori* the displayed isomorphism are not natural in V.

Remark 1.3.15. Let V be a super vector space. Then $S^*(V_0), \Gamma^*(V_0), \Lambda^*(V_1)$ (with V_i in $\mathbb{Z}$-degree i) are bicommutative superbialgebras, since they are sub-superbialgebras of $\mathbf{S}^*(V)$ or $\mathbf{\Gamma}^*(V)$. Similarly, $\Lambda^*(V_0), S^*(V_1), \Gamma^*(V_1)$ are graded-bicommutative because sub-superbialgebras of $\mathbf{\Lambda}^*(V)$ or $\mathbf{A}^*V$.

We give a useful interpretation of divided powers as a kind of equivariant mapping space. Remember the sign convention of Definition 1.2.10 for the tensor product of two linear maps.

Lemma 1.3.16. *Let V, W be super vector spaces and let the symmetric group Σ_n act on $\mathrm{Hom}(V^{\otimes n}, W^{\otimes n})$ by $(f \cdot \sigma)(-) := (f(- \cdot \sigma^{-1})) \cdot \sigma$. Then the map*

$$\begin{aligned} \mathrm{Hom}(V, W)^{\otimes n} &\longrightarrow \mathrm{Hom}(V^{\otimes n}, W^{\otimes n}) \\ f_1 \otimes ... \otimes f_n &\longmapsto f_1 \otimes ... \otimes f_n \end{aligned}$$

is a Σ_n-equivariant isomorphism. In particular, it restricts to an isomorphism

$$\mathbf{\Gamma}^n\mathrm{Hom}(V, W) \simeq \mathrm{Hom}_{\Bbbk\Sigma_n}(V^{\otimes n}, W^{\otimes n})\ .$$

Proof. Since the product $\otimes$ of maps is defined recursively, for the first assertion it suffices to treat the case $n = 2$. Let σ be the non-identity element of Σ_2. The equivariance amounts then to prove, for all $f, g \in \mathrm{Hom}(V, W)$, $v_1, v_2 \in V$, that:

$$(f \otimes g)((v_1 \otimes v_2) \cdot \sigma) \cdot \sigma = (-1)^{\overline{f}\,\overline{g}}\,(g \otimes f)(v_1 \otimes v_2)\ .$$

By definition of $\otimes$, the right-hand side equals

$$(-1)^{\overline{f}\,\overline{g} + \overline{f}\,\overline{v_1}}\, g(v_1) \otimes f(v_2)$$

while the left-hand side is equal to

$$\begin{aligned} ((-1)^{\overline{v_1}\,\overline{v_2}}\,(f \otimes g)(v_2 \otimes v_1)) \cdot \sigma &= ((-1)^{\overline{v_1}\,\overline{v_2} + \overline{g}\,\overline{v_2}}\, f(v_2) \otimes g(v_1)) \cdot \sigma \\ &= (-1)^{\overline{v_1}\,\overline{v_2} + \overline{g}\,\overline{v_2} + \overline{f(v_2)}\,\overline{g(v_1)}}\, g(v_1) \otimes f(v_2) \\ &= (-1)^{\overline{v_1}\,\overline{v_2} + \overline{g}\,\overline{v_2} + (\overline{f} + \overline{v_2})(\overline{g} + \overline{v_1})} g(v_1) \otimes f(v_2) \\ &= (-1)^{\overline{f}\,\overline{g} + \overline{f}\,\overline{v_1}}\, g(v_1) \otimes f(v_2) \end{aligned}$$

so the two expressions are equal as wanted. This proves that the map is equivariant. It is clearly injective and the source has the same dimension as the target, so it is an isomorphism. The second isomorphism follows from the first by taking invariants. □

Definition 1.3.17 (Supermodules)**.** Let A be a superalgebra. A left supermodule over A is a super vector space V together with a left action $A \otimes V \to V$ that is linear and even as a map of super vector spaces. The category of finite-dimensional left A-supermodules and equivariant morphisms (in the usual sense) will be noted by $A - \mathsf{smod}$.

As for superalgebras, last condition means in formulas that $\overline{a \cdot v} = \overline{a} + \overline{v}$ for all $a \in A, v \in V$. A morphism of left A-supermodules is a linear map $\psi : V \to W$ such that $\psi(a \cdot v) = (-1)^{\overline{\psi}\,\overline{a}}\, a \cdot \psi(v)$. The notion of right A-supermodule is defined similarly, except for morphisms: a morphism of right A-supermodules is simply a morphism of the underlying A-modules, i.e. there is no sign involved in the compatibility condition.

1.4 Superlinear categories

Now that we have introduced super objects, we want to generalise the notion of linear category. We have seen that the super structures propagate via elementary operations (Hom, tensors...), which means that our categories will often be enriched. We start by the larger and most important ones for us.

Notation 1.4.1. We denote by $\mathfrak{svec}$ the category of super vector spaces and linear morphisms, and by $\boldsymbol{\mathcal{V}}$ the full subcategory of $\mathfrak{svec}$ with finite-dimensional objects.

Definition 1.4.2. A category is *superlinear* if it is enriched over $\mathfrak{svec}$.

Any linear category is trivially superlinear. In particular, $\mathfrak{vec}$, $\mathcal{V}$ and the two categories appearing above are superlinear. In fact, $\mathcal{V}$ and $\boldsymbol{\mathcal{V}}$ are enriched over themselves.

Definition 1.4.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two superlinear categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *linear* (resp. *even linear*) if the structure maps $\mathrm{Hom}_{\mathcal{C}}(c_1, c_2) \to \mathrm{Hom}_{\mathcal{D}}(Fc_1, Fc_2)$ are linear (resp. even linear).

Example 1.4.4. An obvious example of even linear functor is $\mathrm{Id}_{\mathfrak{svec}}$. There is also the parity change functor $\boldsymbol{\Pi} : \mathfrak{svec} \to \mathfrak{svec}$ defined by $V \mapsto \Pi V$ and $f \mapsto (-1)^{\overline{f}} f$. Since $\boldsymbol{\Pi}$ is manifestly an involution, it defines an equivalence of categories. In particular there is a chain of even isomorphisms $\mathrm{Hom}(\Pi V, W) \simeq \Pi\mathrm{Hom}(V, W) \simeq \mathrm{Hom}(V, \Pi W)$.

Definition 1.4.5. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two even linear functors. A homogeneous natural transformation $T : F \to G$ is a collection of linear maps $T_V \in \mathrm{Hom}_{\mathcal{D}}(FV, GV)$ such that $\overline{T_V}$ is independent of V (and noted by $\overline{T}$) and, for all homogeneous morphisms $\varphi \in \mathrm{Hom}_{\mathcal{C}}(V, W)$,

$$G\varphi \circ T_V = (-1)^{\overline{T}\,\overline{\varphi}}\, T_W \circ F\varphi\ .$$

A natural transformation is by definition the sum of an even and an odd one.

Remark 1.4.6. By construction, if $T = T_0 + T_1$ is such a natural transformation, its even and odd part are the collections of maps (for $V \in \mathcal{C}$)

$$(T_0)_V = (T_V)_0$$
$$(T_1)_V = (T_V)_1.$$

Definition 1.4.7. Let $\mathcal{C}$ be a superlinear category. Then

- $\mathcal{C}_{ev}$ is the category with the same objects but just the even morphisms.

- $\mathcal{C}^-$ is the category with same objects and morphisms but the composition law $\circ^-$ changed by a sign:
$$\varphi \circ^- \psi := (-1)^{\overline{\varphi}\,\overline{\psi}}\, \varphi \circ \psi \;.$$
- $\mathcal{C}_0$ and $\mathcal{C}_1$ are the full subcategories generated by the purely even objects, resp. purely odd objects.

In particular $\mathcal{C}_0, \mathcal{C}_1$ are full subcategories of $\mathcal{C}_{ev}$. Moreover, there is an equivalence of categories $\mathcal{C} \simeq \mathcal{C}^-$ which is the identity on objects and sends a morphism f onto f^- defined by $f^-(v) := (-1)^{\overline{v}\,\overline{f}}\, f(v)$.

Example 1.4.8. There are equivalences of categories $\mathfrak{svec}_0 \simeq \mathfrak{vec} \simeq \mathfrak{svec}_1$ and $\boldsymbol{\mathcal{V}}_0 \simeq \mathcal{V} \simeq \boldsymbol{\mathcal{V}}_1$. Moreover, $\mathfrak{svec}_{ev} \simeq \mathfrak{svec}_0 \times \mathfrak{svec}_1$ and $\boldsymbol{\mathcal{V}}_{ev} \simeq \boldsymbol{\mathcal{V}}_0 \times \boldsymbol{\mathcal{V}}_1$.

1.4.1 The Yoneda Lemma

In this section we are going to state the fundamental result known as Yoneda Lemma, in the context of superlinear categories and in two slightly different formulations. Later, we will specialise all this to our specific category of functors.

Let $\mathcal{C}$ be a superlinear category and set $\mathcal{F} := \mathrm{Func}(\mathcal{C}, \mathfrak{svec})$ the category of even linear covariant functors $\mathcal{C} \to \mathfrak{svec}$. The following fact is nothing but a consequence of the definition of natural transformation.

Lemma 1.4.9. *$\mathcal{F}$ is a superlinear category.*

We can then state a superlinear version of the Yoneda lemma. For an object $C \in \mathcal{C}$, consider the representable functor $h^C := \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathcal{F}$.

Theorem 1.4.10 (Yoneda Lemma). *There is an even linear isomorphism, natural with respect to $F \in \mathcal{F}$ and to $C \in \mathcal{C}$*
$$\mathrm{Hom}_{\mathcal{F}}(h^C, F) \simeq F(C) \;.$$

Proof. There is a set map sending each natural transformation $T : h^C \to F$ to the element $T_C(\mathrm{Id}_C) \in F(C)$. In the opposite way, there is a set map sending an element $v \in F(C)$ to the natural transformation defined by $\varphi \mapsto (F\varphi)(v)$. It is easy to verify that these two maps are inverse to one another. They are also linear and even, since $\overline{T_C(\mathrm{Id}_C)} = \overline{T} + \overline{\mathrm{Id}_C} = \overline{T}$ because identities are always even. This proves the assertion. $\square$

It can be sometimes useful to reformulate this result on $\mathcal{F}_{ev}$. If $T = T_0 + T_1$ is the super decomposition of a natural transformation T, we are in the position to define the following functor:
$$\mathcal{F} \longrightarrow \mathcal{F}_{ev}$$
$$\begin{aligned} F &\longmapsto F \oplus (\mathbf{\Pi} \circ F) \\ T &\longmapsto \begin{pmatrix} T_0 & T_1 \\ T_1 & T_0 \end{pmatrix} \end{aligned}$$
This functor is left adjoint to the canonical inclusion $\mathcal{F}_{ev} \subset \mathcal{F}$. Indeed, for all $F, G \in \mathcal{F}$ there is an even isomorphism
$$\mathrm{Hom}_{\mathcal{F}}(F, G) \simeq \mathrm{Hom}_{\mathcal{F}_{ev}}(F \oplus \; \mathbf{\Pi} \circ F, \; G)$$
given by $T \mapsto T_0 \oplus T_1$, that is natural with respect to both variables. If we take $F = h^C$ and compose this isomorphism with the one provided by the Yoneda lemma, we obtain the following.

Corollary 1.4.11. *There is an even linear isomorphism*
$$\begin{aligned} \mathrm{Hom}_{\mathcal{F}_{ev}}(h^C \oplus \mathbf{\Pi} \circ h^C, G) &\longrightarrow G(C) \\ T^0 \oplus T^1 &\longmapsto (T^0_C + T^1_C)(\mathrm{Id}_C) \;. \end{aligned}$$

1.5 Strict polynomial superfunctors

We now introduce the main protagonist of our manuscript. The objects of interest will be the functors $F : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}$ such that the morphism-level map $F_{V,W} : \mathrm{Hom}(V,W) \to \mathrm{Hom}(FV,FW)$ is polynomial. In this fashion, superlinear functors appear as the special case where $F_{V,W}$ is homogeneous of degree 1. Let $\mathrm{Hom}_{pol}(A,B) = \mathbf{S}^*(A^\vee) \otimes B$ denote the super space of polynomial maps between two super spaces A and B. Then we are asking

$$\begin{aligned} F_{V,W} &\in \mathrm{Hom}_{pol}(\mathrm{Hom}(V,W), \mathrm{Hom}(FV,FW)) \\ &= \mathbf{S}^*(\mathrm{Hom}(V,W)^\vee) \otimes \mathrm{Hom}(FV,FW) \\ &\simeq (\mathbf{\Gamma}^*\mathrm{Hom}(V,W))^\vee \otimes \mathrm{Hom}(FV,FW) \\ &\simeq \mathrm{Hom}(\mathbf{\Gamma}^*\mathrm{Hom}(V,W),\ \mathrm{Hom}(FV,FW)) \end{aligned}$$

where the first isomorphism comes from Remark 1.3.12. We have then found a linear way to define a polynomial superfunctor, at the price of changing the source and hence the category.

Definition 1.5.1. For $n \geq 0$, let $\mathbf{\Gamma}^n\boldsymbol{\mathcal{V}}$ be the category having the same objects as $\boldsymbol{\mathcal{V}}$ but morphism spaces $\mathrm{Hom}_{\mathbf{\Gamma}^n\boldsymbol{\mathcal{V}}}(V,W) := \mathbf{\Gamma}^n\mathrm{Hom}(V,W)$. The composition law is defined using the linear isomorphism of Lemma 1.3.16

$$\mathbf{\Gamma}^n\mathrm{Hom}(V,W) \simeq \mathrm{Hom}_{\Bbbk\Sigma_n}(V^{\otimes n}, W^{\otimes n}) \tag{1.5.1}$$

and thus makes $\mathbf{\Gamma}^n\boldsymbol{\mathcal{V}}$ into a superlinear category.

Remark 1.5.2. Recall that $\mathbf{\Gamma}^0 V \simeq \Bbbk$ for all super vector spaces V. Hence, in the category $\mathbf{\Gamma}^0\boldsymbol{\mathcal{V}}$, between any two objects there is precisely one morphism along with its $\Bbbk$-multiples.

Remark 1.5.3. We want to point out a compatibility property of the composition law in $\mathbf{\Gamma}^n\boldsymbol{\mathcal{V}}$ (which here we will note $\circ$) with respect to the product $\cdot$ in $\mathbf{\Gamma}^*$. First, one can easily check that, if f, g are composable morphisms,

$$\gamma_n(f) \circ \gamma_n(g) = \gamma_n(f \circ g)\ .$$

But more in general

$$(\gamma_n(f) \cdot \gamma_m(g)) \circ \gamma_{n+m}(h) = \gamma_n(f \circ h) \cdot \gamma_m(g \circ h)\ .$$

Definition 1.5.4. A homogeneous *strict polynomial superfunctor* of degree n is a linear even functor $F : \mathbf{\Gamma}^n\boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}$. The category consisting of these objects and their natural transformations is noted $\boldsymbol{\mathcal{P}}_n$.

Set $\boldsymbol{\mathcal{P}} = \prod_n \boldsymbol{\mathcal{P}}_n$. This means that all strict polynomial superfunctors, i.e. elements of $\boldsymbol{\mathcal{P}}$, write uniquely as the sum of its homogeneous components.

Convention 1.5.5. Remark that, if F, G are homogeneous superfunctors, then $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F,G)$ can only be nonzero if they are of the same degree d, and in that case $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F,G) = \mathrm{Hom}_{\boldsymbol{\mathcal{P}}_d}(F,G)$. For that reason, we feel free to interchange the notation $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}$ and $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}_d}$ based on whether we want to emphasize the degree or not. The same convention applies to $\mathcal{P}$.

We summarise the standard operations that one can perform on polynomial superfunctors. Given $F \in \boldsymbol{\mathcal{P}}_n, G \in \boldsymbol{\mathcal{P}}_m$, we can make:

- **Direct sum:** $F \oplus G$ is given by $(F \oplus G)(V) = F(V) \oplus G(V)$ on objects and the obvious sum on morphisms.

- **Tensor product:** $F \otimes G \in \boldsymbol{\mathcal{P}}_{m+n}$ is given on objects by $(F \otimes G)(V) = F(V) \otimes G(V)$. To define it on morphisms, remark that the inclusion of groups $\Sigma_n \times \Sigma_m \subset \Sigma_{m+n}$ induces a canonical inclusion $\mathbf{\Gamma}^{n+m}V \subset \mathbf{\Gamma}^n V \otimes \mathbf{\Gamma}^m V$ for all super vector spaces V. Then the action on morphisms is given by the even linear composition

$$\begin{tikzcd}
\mathbf{\Gamma}^{n+m}\mathrm{Hom}(V,W) \arrow[r, hook] & \mathbf{\Gamma}^n\mathrm{Hom}(V,W) \otimes \mathbf{\Gamma}^m\mathrm{Hom}(V,W) \arrow[d, "F\otimes G"] \\
& \mathrm{Hom}(F(V),F(W)) \otimes \mathrm{Hom}(G(V),G(W)) \arrow[d, "\otimes"] \\
& \mathrm{Hom}(F(V)\otimes G(V), F(W)\otimes G(W))\,.
\end{tikzcd}$$

- **Composition:** $F \circ G \in \boldsymbol{\mathcal{P}}_{nm}$ is given by $(F\circ G)(V) = F(G(V))$ on objects. On morphisms, use the inclusion of groups $\Sigma_m^{\times n} \subset \Sigma_{nm}$. It induces an inclusion $\mathbf{\Gamma}^{nm}\mathrm{Hom}(V,W) \subset \mathbf{\Gamma}^n(\mathbf{\Gamma}^m\mathrm{Hom}(V,W))$ which is used similarly to above to build a morphism

$$\begin{tikzcd}
\mathbf{\Gamma}^{nm}\mathrm{Hom}(V,W) \arrow[r, hook] & \mathbf{\Gamma}^n(\mathbf{\Gamma}^m\mathrm{Hom}(V,W)) \arrow[d, "\mathbf{\Gamma}^n G"] \\
& \mathbf{\Gamma}^n\mathrm{Hom}(G(V),G(W)) \arrow[d, "F"] \\
& \mathrm{Hom}(F(G(V)),F(G(W)))\,.
\end{tikzcd}$$

Remark 1.5.6. $(F\otimes G)\circ H = (F\circ H)\otimes(G\circ H)$ for all superfunctors F, G, H.

Example 1.5.7. Superfunctors of degree 0 are just constant functors. For all super vector spaces V, examples of superfunctors of degree 1 are $\mathrm{Hom}(V,-)$ and $V\otimes -$. In particular, we have the identity[2] $\mathbf{I}$ and the parity change $\mathbf{\Pi}$ defined in Example 1.4.4. More generally, $\otimes^n, \mathbf{S}^n, \mathbf{\Gamma}^n, \mathbf{\Lambda}^n, \mathbf{A}^n$ are all superfunctors of degree n. To see it, note by F any of them. Any Σ_n-equivariant map $f : V^{\otimes n} \to W^{\otimes n}$ induces then a well-defined map $Ff : F(V) \to F(W)$. Composing with (1.5.1) we have a morphism

$$\mathbf{\Gamma}^n\mathrm{Hom}(V,W) \xrightarrow{\simeq} \mathrm{Hom}_{\Bbbk\Sigma_n}(V^{\otimes n}, W^{\otimes n}) \to \mathrm{Hom}(FV,FW)$$

which makes F into a n-homogeneous polynomial superfunctor.

Remark 1.5.8. Sometimes, to be concise we will use the notation $\mathbf{\Pi}^n$ to indicate the superfunctor $\mathbf{\Pi}\circ ... \circ\mathbf{\Pi}$ composed n times. This coincides with $\mathbf{\Pi}$ if n is odd, with $\mathbf{I}$ if n is even. For all $F_1, ..., F_n \in \boldsymbol{\mathcal{P}}$ there is the identity

$$(\mathbf{\Pi}\circ F_1)\otimes ... \otimes(\mathbf{\Pi}\circ F_n) = \mathbf{\Pi}^n \circ (F_1 \otimes ... \otimes F_n)\,.$$

1.5.1 Basic properties of $\boldsymbol{\mathcal{P}}$

We introduce the projective and injective objects of our category. First note that $\boldsymbol{\mathcal{P}}$ and its subcategories of homogeneous degree are not abelian: in fact not even $\mathfrak{svec}$ is, since there is no natural $\mathbb{Z}/2\mathbb{Z}$-grading for the (co)kernel of a non-homogeneous linear map. This obstruction is removed in $\boldsymbol{\mathcal{P}}_{ev}$ by the absence of odd morphisms. Hence, we say that an object $F \in \boldsymbol{\mathcal{P}}$ is *projective* (resp. *injective*) if the functor $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F,-) : \boldsymbol{\mathcal{P}}_{ev} \to \mathfrak{svec}_{ev}$ (resp. $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(-,F) : \boldsymbol{\mathcal{P}}_{ev} \to \mathfrak{svec}_{ev}$) is exact.

[2]We use a bold symbol to distinguish it from the classical identity $I \in \mathcal{P}$ and from other identity functors appearing hereafter, which will be noted Id.

Definition 1.5.9. For a non-negative integer d and a finite-dimensional super vector space V, define:

$$\begin{aligned}\mathbf{\Gamma}^{d,V} &:= \mathbf{\Gamma}^d \mathrm{Hom}(V,-)\\ \mathbf{S}^d_V &:= \mathbf{S}^d(V\otimes -)\ .\end{aligned}$$

They are both homogeneous polynomial superfunctors of degree d, as they are obtained by composing $\mathbf{\Gamma}^d$ resp. $\mathbf{S}^d$ with $\mathrm{Hom}(V,-)$ resp. $V\otimes -$. Note that $\mathbf{\Gamma}^{d,V} = \mathrm{Hom}_{\mathbf{\Gamma}^d\boldsymbol{\mathcal{V}}}(V,-)$ by definition. In particular, Theorem 1.4.10 specialises as follows.

Theorem 1.5.10 (Yoneda lemma in $\boldsymbol{\mathcal{P}}$)**.** *There is an isomorphism of super vector spaces*

$$\mathrm{Hom}_{\boldsymbol{\mathcal{P}}_d}(\mathbf{\Gamma}^{d,V}, F) \simeq F(V)$$

which is natural with respect to $V\in \mathbf{\Gamma}^d\boldsymbol{\mathcal{V}}$ and $F\in\boldsymbol{\mathcal{P}}_d$.

As a consequence, $\mathbf{\Gamma}^{d,V}$ is a projective object in the category $\boldsymbol{\mathcal{P}}_d$. The *Yoneda morphism*

$$\begin{aligned}\mathbf{\Gamma}^{d,V}\otimes F(V) &\longrightarrow F\\ f\otimes x &\longmapsto Ff(x)\end{aligned} \tag{1.5.2}$$

is surjective under mild hypotheses on the superdimension of V. We are actually going to give a stronger result, which generalises Theorem 1.1.14 to the super context. For the next definition, recall by Remark 1.3.9 that $\mathbf{\Gamma}^d A$ is a superalgebra whenever A is. Recall also the notation $\Bbbk^{m|n}$ for a super vector space of superdimension (m,n).

Definition 1.5.11. Let $m,n,d\geq 1$. The *Schur superalgebra* is defined as $S(m|n,d) := \mathbf{\Gamma}^d\mathrm{End}(\Bbbk^{m|n})$.

If $F\in\boldsymbol{\mathcal{P}}_d$, then $F(\Bbbk^{m|n})$ has the structure of a finite-dimensional left $S(m|n,d)$-supermodule via

$$\begin{aligned}S(m|n,d)\otimes F(\Bbbk^{m|n}) &\longrightarrow F(\Bbbk^{m|n})\\ \varphi\otimes v &\longmapsto F\varphi(v)\ .\end{aligned}$$

Moreover, if $T\in \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F,G)$, then $T_{\Bbbk^{m|n}}$ is a $S(m|n,d)$-equivariant map. So, evaluation at $\Bbbk^{m|n}$ defines a functor

$$\boldsymbol{\mathcal{P}}_d \longrightarrow S(m|n,d) - \mathsf{smod}\ . \tag{1.5.3}$$

Theorem 1.5.12 ([Axt13, Thm 4.2])**.** *Let m,n,d be positive integers such that $m,n\geq d$. Then (1.5.3) gives an equivalence of categories*

$$\boldsymbol{\mathcal{P}}_d \simeq S(m|n,d) - \mathsf{smod}$$

a quasi-inverse being provided by $M\longmapsto \mathbf{\Gamma}^{d,\Bbbk^{m|n}}\otimes_{S(m|n,d)} M$.

As anticipated, this gives the following important consequence:

Corollary 1.5.13. *If $m,n\geq d$, $\mathbf{\Gamma}^{d,\Bbbk^{m|n}}$ is a projective generator for $\boldsymbol{\mathcal{P}}_d$.*

Proof. The Yoneda morphism (1.5.2) coincides with the composite

$$\mathbf{\Gamma}^{d,\Bbbk^{m|n}}\otimes F(\Bbbk^{m|n}) \twoheadrightarrow \mathbf{\Gamma}^{d,\Bbbk^{m|n}}\otimes_{S(m|n,d)} F(\Bbbk^{m|n}) \longrightarrow F$$

and the right arrow is an isomorphism by Theorem 1.5.12. In particular, the composition is a surjection. □

1.5.2 Alternative versions of the Yoneda lemma

We now apply the results of Section 1.4 to rewrite the Yoneda lemma in $\boldsymbol{\mathcal{P}}_{ev}$:

Theorem 1.5.14 (Yoneda lemma (II)). *The isomorphism of Theorem 1.5.10 rewrites as an even linear isomorphism*

$$\mathrm{Hom}_{(\boldsymbol{\mathcal{P}}_d)_{ev}}(\boldsymbol{\Gamma}^{d,V} \oplus \boldsymbol{\Pi} \circ \boldsymbol{\Gamma}^{d,V},\ F) \simeq F(V)$$

where the even part, resp. odd part, of the left-hand side is the summand $\mathrm{Hom}_{(\boldsymbol{\mathcal{P}}_d)_{ev}}(\boldsymbol{\Gamma}^{d,V}, F)$, *resp.* $\mathrm{Hom}_{(\boldsymbol{\mathcal{P}}_d)_{ev}}(\boldsymbol{\Pi} \circ \boldsymbol{\Gamma}^{d,V}, F)$.

Proposition 1.5.15 ([Axt13, Prop A.1]). *If $m, n \geq d$, then $\boldsymbol{\Gamma}^{d,\Bbbk^{m|n}} \oplus \boldsymbol{\Pi} \circ \boldsymbol{\Gamma}^{d,\Bbbk^{m|n}}$ is a projective generator of $(\boldsymbol{\mathcal{P}}_d)_{ev}$.*

Besides making explicit the even and odd part of the isomorphism of the Yoneda lemma, this formulation is also going to be useful later for the definition of extensions in $\boldsymbol{\mathcal{P}}$.

Corollary 1.5.16. *The Yoneda morphism (1.5.2) rewrites as*

$$\begin{aligned} (\boldsymbol{\Gamma}^{d,V} \oplus \boldsymbol{\Pi} \circ \boldsymbol{\Gamma}^{d,V}) \otimes F(V) &\longrightarrow F \\ (f + \pi g) \otimes v &\longmapsto Ff(v_0) + Fg(v_1) \end{aligned} \tag{1.5.4}$$

where v_0, v_1 are the even and odd part of v.

We give a last, very useful version of the Yoneda lemma at the level of weights. We recall some basic concepts about representation theory, which the reader can find in detail for example in [Wat79] and [FS97, §2].

Definition 1.5.17. An *algebraic group scheme* over $\Bbbk$ - in the following, simply a *scheme* - is a representable covariant functor from the category of commutative $\Bbbk$-algebras to the category of groups. It is hence by definition of the form

$$G = \mathrm{Hom}_{\,\Bbbk-\mathrm{alg}}(A, -)$$

for some (unique) algebra A, which is said to *represent* G. A morphism of schemes $\rho : G \to H$ is a natural transformation between G and H, i.e. a collection of group homomorphisms $\rho(A) : G(A) \to H(A)$ natural with respect to A.

Definition 1.5.18. If G and H are schemes represented respectively by algebras A and B, the direct product $G \times H$ is defined by $(G \times H)(A') := G(A') \times H(A')$ and is represented by $A \otimes B$.

Let V be a vector space of dimension n. A classical example of scheme is the general linear group $GL(V)$, represented by $\Bbbk[x_{i,j}, \det(x_{i,j})^{-1}]_{1 \leq i,j \leq n}$. The extreme case $n = 1$ gives the multiplicative group $\mathbb{G}_m := GL_1$.

Let now G be any algebraic group scheme. A *rational representation* of G (or rational G-module) is the data of a finite-dimensional vector space M and a morphism of schemes $G \longrightarrow GL(M)$. This is equivalent to giving a collection of $\Bbbk$-linear actions

$$G(A) \otimes (M \otimes A) \to M \otimes A$$

which is natural with respect to A. A rational G-module is *simple* if it has no nontrivial G-submodules, *semisimple* if it decomposes as the direct sum of its simple submodules. There is the following well-known example.

Theorem 1.5.19. *All rational $\mathbb{G}_m^{\times n}$-modules are semisimple ($n \geq 1$).*

We can describe the nontrivial submodules of a rational $\mathbb{G}_m$-module M. In fact, all simple $\mathbb{G}_m$-submodules of M are of the form

$$M^r := \{m \otimes a \in M \otimes A \text{ such that } \lambda \cdot (m \otimes a) = m \otimes \lambda^r a\} .$$

for some $r \in \mathbb{Z}$. This submodule is rational if and only if $r \geq 0$. Therefore, Theorem 1.5.19 states that $M = \bigoplus_{r \geq 0} M^r$. Here r is called *weight* of the representation and M^r is the associated *weight space*. Generalising, the weights of $\mathbb{G}_m^{\times n}$ are *compositions*, i.e. n-uples $\lambda = (\lambda_1, ..., \lambda_n)$ of non-negative integers, and the weight spaces are of the form

$$M^\lambda = \{m \otimes a \in M \otimes A \;:\; \mathrm{diag}(a_1, ..., a_{n+m}) \cdot (m \otimes a) = m \otimes (a_1^{\lambda_1} \cdots a_n^{\lambda_n})a\}$$

and, again by Theorem 1.5.19, M decomposes into the direct sum $\bigoplus_\lambda M^\lambda$. Note that a weight space M^λ can possibly be the zero space.

Convention 1.5.20. We will say "λ is a weight of M" if the weight space M^λ is nonzero.

We specialise that to the context of polynomial superfunctors. Given $F \in \boldsymbol{\mathcal{P}}_d$ and a finite-dimensional super vector space V, the super vector space $F(V)$ is naturally endowed with a structure of $\mathbf{\Gamma}^d\mathrm{End}(V)$-supermodule. Equivalently, F induces a morphism of schemes $\mathrm{End}(V) \to \mathrm{End}(F(V))$, which restricts to $GL(V) \to GL(F(V))$. Set now $(n, m) = \mathrm{sdim}(V)$. Then there is an inclusion of schemes $\mathbb{G}_m^{\times n+m} \subset GL(V)$. Composing it with the previous morphism, we get a morphism of schemes $\mathbb{G}_m^{\times n+m} \to GL(F(V))$. Then $F(V)$ can be given the structure of a rational $\mathbb{G}_m^{\times n+m}$-module.

Remark 1.5.21. Let us make two little abuses of notation, first by identifying a diagonal matrix φ with $\gamma_d(\varphi)$ and secondly by hiding the dependence on an algebra A from the definition of weight space. Then, if $V \simeq \mathbb{k}^{n|m}$

$$F(V)^\lambda = \{x \in F(V) \mid F(\mathrm{diag}(a_1, ..., a_{n+m}))(x) = a_1^{\lambda_1} \cdots a_{n+m}^{\lambda_{n+m}}\, x\} . \tag{1.5.5}$$

Definition 1.5.22. A *composition of d in n parts* is a composition $(\lambda_1, ..., \lambda_n)$ such that $\lambda_1 + ... + \lambda_n = d$. The set of such elements is denoted by $\Lambda(n, d)$.

Lemma 1.5.23. *Let $F \in \mathcal{P}_d$ and suppose that $\lambda = (\lambda_1, ..., \lambda_{n+m})$ is a weight of $F(\mathbb{k}^{n|m})$. Then $\lambda_1 + ... + \lambda_{n+m} = d$.*

Proof. We have $\gamma_d(\mathrm{diag}(a, ..., a)) = a^d \gamma_d(\mathrm{Id}_V)$ by the relations of Γ^* (Proposition 1.1.5). Consequently

$$F(\gamma_d(\mathrm{diag}(a, ..., a)) = a^d\, \mathrm{Id}_{F(V)}$$

by linearity and functoriality of F. Hence, if $0 \neq x \in F(V)^\lambda$, by (1.5.5) we obtain the identity $a^{\lambda_1 + \dots + \lambda_{n+m}} x = a^d x$ arbitrarily on a, which gives the assertion. □

Example 1.5.24. The weights of the representation $\mathbb{k}^{n|m}$ are the 1-compositions $\varepsilon_i = (0, ..., 0, 1, 0, ..., 0)$ with 1 in position i, for $1 \leq i \leq n + m$. More generally, the weights of $\mathbf{S}^d(\mathbb{k}^{n|m})$ are the compositions of d in $n + m$ parts.

Definition 1.5.25. Let $\Lambda(n|m, d) := \Lambda(n + m, d)$ as a set. For a composition $\lambda \in \Lambda(n|m, d)$, set:

$$\mathbf{\Gamma}^\lambda := \mathbf{\Gamma}^{\lambda_1} \otimes ... \otimes \mathbf{\Gamma}^{\lambda_n} \otimes (\mathbf{\Gamma}^{\lambda_{n+1}} \circ \mathbf{\Pi}) \otimes ... \otimes (\mathbf{\Gamma}^{\lambda_{n+m}} \circ \mathbf{\Pi}) .$$

$$\mathbf{S}^\lambda := \mathbf{S}^{\lambda_1} \otimes ... \otimes \mathbf{S}^{\lambda_n} \otimes (\mathbf{S}^{\lambda_{n+1}} \circ \mathbf{\Pi}) \otimes ... \otimes (\mathbf{S}^{\lambda_{n+m}} \circ \mathbf{\Pi}) .$$

Given an ordered homogeneous basis $\{e_1, ..., e_{n+m}\}$ of $\mathbb{k}^{n|m}$, the one dimensional space $\mathbf{S}^\lambda(\mathbb{k}^{n|m}) = \mathbb{k}\, e_1^{\lambda_1} ... e_{n+m}^{\lambda_{n+m}}$ is exactly the λ-weight space of the representation of Example 1.5.24. The presence of $\mathbf{\Pi}$ in the definition of $\mathbf{\Gamma}^\lambda$ and $\mathbf{S}^\lambda$ is the reason why we put a bar in the notation

$\Lambda(n|m, d)$: one should remember to distinguish between the first n and the last m components of λ. Let $V = \Bbbk^{n|m}$. By the exponential property, $\mathbf{\Gamma}^{d,V}$ and $\mathbf{S}^d_V$ split then into direct sums

$$\mathbf{\Gamma}^{d,V} \simeq \bigoplus_{\lambda\in\Lambda(n|m,d)} \mathbf{\Gamma}^\lambda$$
$$\mathbf{S}^d_V \simeq \bigoplus_{\lambda\in\Lambda(n|m,d)} \mathbf{S}^\lambda \ .$$

We are in position to state the "weighted" version of the Yoneda lemma, on the model of [FS97, Cor. 2.12].

Theorem 1.5.26 (Weighted Yoneda lemma). *Let $V \simeq \Bbbk^{n|m}$ and $F \in \boldsymbol{\mathcal{P}}_d$. Consider the decomposition of the rational $\mathbb{G}_m^{\times n+m}$-module $F(V)$*

$$F(V) = \bigoplus_{\lambda\in\Lambda(n|m,d)} F(V)^\lambda \ .$$

Then, for all $\lambda \in \Lambda(n|m, d)$, the isomorphism of Theorem 1.5.10 restricts to an isomorphism

$$\mathrm{Hom}_{\boldsymbol{\mathcal{P}}_d}(\mathbf{\Gamma}^\lambda, F) \simeq F(V)^\lambda \ .$$

Proof. Consider a natural transformation $T : \mathbf{\Gamma}^{d,V} \to F$. Let $\{e_1, ..., e_{n+m}\}$ be an ordered homogeneous basis of $\Bbbk^{n|m}$ (i.e. such that the first n elements are even and the last m are odd). Then the element $\gamma_d(\mathrm{Id}_V) \in \mathbf{\Gamma}^d\mathrm{End}(V)$ corresponds via the exponential isomorphism (Proposition 1.3.14) to the element

$$\sum_{\mu\in\Lambda(n|m,d)} \gamma_{\mu_1}(e_1) \otimes ... \otimes \gamma_{\mu_{n+m}}(e_{n+m}) \in \bigoplus_{\mu\in\Lambda(n|m,d)} \mathbf{\Gamma}^\mu(V) \ .$$

As a consequence, the restricted transformation $T|_{\mathbf{\Gamma}^\lambda}$ is sent onto $T_V(\gamma_{\lambda_1}(e_1)\otimes...\otimes\gamma_{\lambda_{n+m}}(e_{n+m}))$ by the isomorphism of Theorem 1.5.10. Call this element x. Using the naturality of T, we have

$$\begin{aligned} F(\mathrm{diag}(a_1, ..., a_{n+m}))(x) &= T_V(F(\mathrm{diag}(a_1, ..., a_{n+m}))(\gamma_{\lambda_1}(e_1) \otimes ... \otimes \gamma_{\lambda_{n+m}}(e_{n+m}))) \\ &= T_V(\, a_1^{\lambda_1}...a_{n+m}^{\lambda_{n+m}}(\gamma_{\lambda_1}(e_1) \otimes ... \otimes \gamma_{\lambda_{n+m}}(e_{n+m}))) \\ &= a_1^{\lambda_1}...a_{n+m}^{\lambda_{n+m}}\ x \end{aligned}$$

thus the Yoneda isomorphism maps $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}_d}(\mathbf{\Gamma}^\lambda, F)$ into $F(V)^\lambda$, which proves the theorem. □

1.5.3 Kuhn duals

Definition 1.5.27. Let $F \in \boldsymbol{\mathcal{P}}_n$. The *Kuhn dual* of F is the superfunctor $F^\# \in \boldsymbol{\mathcal{P}}_n$ defined on objects by $F^\#(V) = F(V^\vee)^\vee$ and on morphisms by the composition

$$\begin{aligned} \mathbf{\Gamma}^n\mathrm{Hom}(V, W) \simeq \mathbf{\Gamma}^n\mathrm{Hom}(W^\vee, V^\vee) &\xrightarrow{F} \mathrm{Hom}(F(W^\vee), F(V^\vee)) \\ &\simeq \mathrm{Hom}(F(V^\vee)^\vee, F(W^\vee)^\vee) \ . \end{aligned}$$

The definition extends to $F \in \boldsymbol{\mathcal{P}}$ by applying the Kuhn dual to each homogeneous component of F.

Proposition 1.5.28. *Kuhn duality commutes with finite direct sums, tensor products and compositions. In particular, the functor $\otimes^n$ is self-dual.*

Example 1.5.29. Let V be a finite-dimensional super vector space. Using the binatural isomorphism

$$V^\vee \otimes W \simeq \mathrm{Hom}(V, W)$$

it is immediate to check that the two 1-homogeneous functors $V \otimes -$ and $\mathrm{Hom}(V, -)$ are the Kuhn duals of each other.

Proposition 1.5.30. *Kuhn duality is involutive. Furthermore, $(-)^\# : \boldsymbol{\mathcal{P}} \to (\boldsymbol{\mathcal{P}}^-)^{op}$ is an equivalence of categories.*

Proof. Note by $i_V : V \xrightarrow{\simeq} (V^\vee)^\vee$ the natural isomorphism (1.2.1). It gives rise to a natural isomorphism $i : \mathbf{I} \to \mathbf{I}^\#$. For all $F \in \boldsymbol{\mathcal{P}}$, the composite

$$F = \mathbf{I} \circ F \circ \mathbf{I} \xrightarrow{F \circ i} \mathbf{I} \circ F \circ \mathbf{I}^\# \xrightarrow{i \circ F} \mathbf{I}^\# \circ F \circ \mathbf{I}^\# = F^{\#\#}$$

shows that $F^{\#\#} \simeq F$. Moreover, given $T \in \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F, G)$, its dual transformation is $T^\#_V := (T_{V^\vee})^\vee$. By the definitions on linear duals, if $U \in \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(G, H)$, then $(U \circ T)^\# = (-1)^{\overline{T}\,\overline{U}} T^\# \circ U^\#$. That gives the stated category equivalence. ☐

In particular, if $F^\# \simeq G$ it follows automatically that $G^\# \simeq F$.

Proposition 1.5.31. *Let $V \in \boldsymbol{\mathcal{V}}$. Then $\mathbf{\Gamma}^{d,V}$ and $\boldsymbol{S}^d_V$ are the Kuhn duals of each other.*

Since Kuhn duality commutes with composition of functors, by Example 1.5.29 it will be enough to show that $\mathbf{\Gamma}^d$ and $\mathbf{S}^d$ are the Kuhn dual of each other. In the next section we are actually going to prove a stronger result (Proposition 1.5.39). As in many different contexts, the proof happens to be carried out much more easily by summing over d and using the bialgebra structure of these superfunctors. Meanwhile we give an immediate corollary.

Corollary 1.5.32. *If $m, n \geq d$, $\boldsymbol{S}^d_{\Bbbk^{m|n}}$ is an injective cogenerator for $\boldsymbol{\mathcal{P}}_d$.*

1.5.4 $\boldsymbol{\mathcal{P}}$-algebras

Definition 1.5.33. A $\boldsymbol{\mathcal{P}}$*-algebra* is a superfunctor $A \in \boldsymbol{\mathcal{P}}$ endowed with even natural maps $u : \Bbbk \to A$ and $m : A \otimes A \to A$ which satisfy the usual associativity and unitarity properties.

In other words, (u, m) make A into a $\boldsymbol{\mathcal{P}}$-algebra if and only if (u_V, m_V) make $A(V)$ into a superalgebra for all V. A is *graded* if there is a decomposition $A = \bigoplus_d A^d$ with respect to which m is homogeneous.

Remark 1.5.34. Any $\boldsymbol{\mathcal{P}}$-algebra can be seen as a graded $\boldsymbol{\mathcal{P}}$-algebra by its polynomial grading, i.e. $A^d \in \boldsymbol{\mathcal{P}}_d$ being its d-th degree part. Indeed, the restriction $m : A^d \otimes A^e \to A^f$ can only be nonzero if $f = d + e$.

We say that A is (graded-)commutative if each $A(V)$ is. A morphism of $\boldsymbol{\mathcal{P}}$-algebras is a natural map which commutes with u and m.

Definition 1.5.35. A $\boldsymbol{\mathcal{P}}$-coalgebra is a superfunctor $A \in \boldsymbol{\mathcal{P}}$ endowed with natural maps $\eta : A \to \Bbbk$ and $\Delta : A \to A \otimes A$ who satisfy the usual coassociativity and counitarity properties.

As above, a $\boldsymbol{\mathcal{P}}$-coalgebra C is graded if there is a decomposition $C = \bigoplus_d C^d$ with respect to which Δ is homogeneous. Let $A \otimes^g B$ be the graded tensor product introduced in Section 1.3. Then a *graded* $\boldsymbol{\mathcal{P}}$*-bialgebra* is both a $\boldsymbol{\mathcal{P}}$-algebra and $\boldsymbol{\mathcal{P}}$-coalgebra such that $\Delta : A \to A \otimes^g A$ is a morphism of $\boldsymbol{\mathcal{P}}$-algebras. If A is concentrated in even $\mathbb{Z}$-degrees, we speak simply of a $\boldsymbol{\mathcal{P}}$*-bialgebra.*

Example 1.5.36. Recalling the notation from Example 1.3.6, $\boldsymbol{\mathcal{T}}^*_1$ and $\boldsymbol{\mathcal{T}}^*_2$ are both $\boldsymbol{\mathcal{P}}$-bialgebras. In particular so are $\mathbf{\Gamma}^*$, $\mathbf{S}^*$, $\mathbf{\Lambda}^*$ and $\mathbf{A}^*$, since they are all sub-bialgebras or quotient bialgebras of one of them. Specifically, $\mathbf{\Gamma}^*$ and $\mathbf{S}^*$ are commutative and cocommutative, while $\mathbf{\Lambda}^*$ and $\mathbf{A}^*$ are graded-commutative and graded-cocommutative.

Proposition 1.5.37. *If A is a $\boldsymbol{\mathcal{P}}$-algebra (resp. coalgebra), then $A^{\#}$ is a $\boldsymbol{\mathcal{P}}$-coalgebra (resp. algebra). Moreover, A is commutative if and only if $A^{\#}$ is cocommutative.*

Proof. Follows from the fact that Kuhn dual is an equivalence of categories $\boldsymbol{\mathcal{P}} \to (\boldsymbol{\mathcal{P}}^{-})^{op}$ (Proposition 1.5.30) and that product and coproduct are even maps, hence no sign appears in the commutative diagrams involving them. □

Remark 1.5.38. $\boldsymbol{\mathcal{T}}_1^* \simeq (\boldsymbol{\mathcal{T}}_2^*)^{\#}$ as $\boldsymbol{\mathcal{P}}$-bialgebras and vice-versa.

We now show our main couple of Kuhn duals, namely $\mathbf{S}^*$ and $\mathbf{\Gamma}^*$.

Theorem 1.5.39. *$\mathbf{S}^* \simeq (\mathbf{\Gamma}^*)^{\#}$ as $\boldsymbol{\mathcal{P}}$-bialgebras.*

Proof. By Proposition 1.5.30, Kuhn dual takes natural injections onto surjections and vice-versa. Remember by the definitions that $\mathbf{\Gamma}^*$ is a sub-bialgebra of $\boldsymbol{\mathcal{T}}_2^*$, hence $(\mathbf{\Gamma}^*)^{\#}$ is a quotient bialgebra of $\boldsymbol{\mathcal{T}}_1^*$. Moreover, by Proposition 1.5.37, $(\mathbf{\Gamma}^*)^{\#}$ is commutative and cocommutative. It follows that the quotient map $\boldsymbol{\mathcal{T}}_1^* \twoheadrightarrow (\mathbf{\Gamma}^*)^{\#}$ factors through $\mathbf{S}^*$:

$$\begin{tikzcd} \boldsymbol{\mathcal{T}}_1^* \arrow[r, two heads] \arrow[d, two heads] & (\mathbf{\Gamma}^*)^{\#} \\ \mathbf{S}^* \arrow[ur, dashed, "\alpha"'] & \end{tikzcd}$$

and α is a surjective morphism of bialgebras, since the other two are. By Remark 1.3.12 the source and target of α have equal superdimension when evaluated on a super space V, thus α is an isomorphism. □

1.6 Homological algebra in $\boldsymbol{\mathcal{P}}$

Recall that $\boldsymbol{\mathcal{P}}$ is not abelian. The following definition is motivated by the discussion at the beginning of §1.5.1.

Definition 1.6.1. If F is a polynomial superfunctor, $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F,-)$ is defined as the right derived functor of $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F,-) : \boldsymbol{\mathcal{P}}_{ev} \to \mathfrak{svec}_{ev}$.

In particular $\mathrm{Ext}^0_{\boldsymbol{\mathcal{P}}}(F,-) = \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F,-)$. In the same spirit of Convention 1.5.5, for d-homogeneous superfunctors F, G we will sometimes write $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}_d}$ instead of the equivalent $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}$ just to emphasize the degree.

Example 1.6.2. Set for brevity $\mathrm{Ext}^{>0}_{\boldsymbol{\mathcal{P}}}(-,-) := \bigoplus_{i>0} \mathrm{Ext}^i_{\boldsymbol{\mathcal{P}}}(-,-)$. Then

$$\mathrm{Ext}^{>0}_{\boldsymbol{\mathcal{P}}}(\mathbf{\Gamma}^{d,V},-) = 0 = \mathrm{Ext}^{>0}_{\boldsymbol{\mathcal{P}}}(-,\mathbf{S}^d_V)$$

for all finite dimensional super vector space V. That comes by the injectivity of $\mathbf{S}^d_V$, resp. projectivity of $\mathbf{\Gamma}^{d,V}$.

Extensions in $\boldsymbol{\mathcal{P}}_{ev}$ are purely even spaces. To distinguish between "even" and "odd" extensions, we remember of the superdegree distinction made in Proposition 1.5.14 to have:

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F,G)_0 \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}_{ev}}(F,G)$$

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F,G)_1 \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}_{ev}}(F,\mathbf{\Pi}\circ G) \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}_{ev}}(\mathbf{\Pi}\circ F,G)\,.$$

Remark 1.6.3. By construction, postcomposition with $\mathbf{\Pi}$ comes out of the Ext:

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F,\mathbf{\Pi}\circ G) \simeq \Pi(\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F,G)) \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi}\circ F,G)$$

Lemma 1.6.4. *The space $\mathrm{Ext}^i_{\boldsymbol{\mathcal{P}}}(F,G)$ is finite-dimensional for all homogeneous superfunctors F, G and for all $i \geq 0$.*

Proof. Say $d := \deg F = \deg G$. Let P_* be a projective resolution of F. Since by definition strict polynomial superfunctors take finite-dimensional values, each P_i is a direct factor of a finite sum of terms like $\mathbf{\Gamma}^{d,V}$. This implies by Theorem 1.5.10 that $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(P_i, G)$ is finite-dimensional, whence in particular its subquotient $\mathrm{Ext}^i_{\boldsymbol{\mathcal{P}}}(F, G)$. $\square$

When computing Ext, it is sometimes useful to use *conjugation.*

Definition 1.6.5. If $F \in \boldsymbol{\mathcal{P}}$, we define its conjugate to be $F^{\Pi} := \mathbf{\Pi} \circ F \circ \mathbf{\Pi}$.

Remark 1.6.6. The operation $-^{\Pi} : \boldsymbol{\mathcal{P}}_{ev} \to \boldsymbol{\mathcal{P}}_{ev}$ is exact and preserves projectives. Hence for all $F, G \in \boldsymbol{\mathcal{P}}$ it induces an even isomorphism $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F, G) \xrightarrow{\simeq} \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ F \circ \mathbf{\Pi},\ \mathbf{\Pi} \circ G \circ \mathbf{\Pi})$. The image of an extension e via this map will be similarly denoted by e^{Π}. Since conjugation is involutive, $(e^{\Pi})^{\Pi} = e$.

1.6.1 Yoneda product

Let $F, G, H \in \boldsymbol{\mathcal{P}}$ and let P_*, resp. Q_*, be a projective resolution of F, resp. G. Identify an element of $\mathrm{Ext}^s_{\boldsymbol{\mathcal{P}}}(F, G)$ with a cocycle $P_s \to G$. By the lifting property of projective objects, such a cocycle gives rise to a chain map $P_{s+*} \to Q_*$, so in particular to a map $P_{s+t} \to Q_t$. Composing the latter with a cocycle $Q_t \to H$, we obtain a new cocycle $P_{s+t} \to H$. That yields a well-defined map

$$\mathrm{Ext}^s_{\boldsymbol{\mathcal{P}}}(G, H) \otimes \mathrm{Ext}^t_{\boldsymbol{\mathcal{P}}}(F, G) \xrightarrow{\cdot} \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(F, H) \tag{1.6.1}$$

linear and associative, called the *Yoneda product.* In particular $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F, F)$ is a superalgebra, called the *Yoneda superalgebra* associated to F.

Notation 1.6.7. If $\varphi : \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(G, H)$ is fixed, the map

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F, G) \xrightarrow{\varphi \cdot -} \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F, H)$$

will be sometimes denoted as $\mathrm{Ext}(F, \varphi)$ depending on convenience.

As in the classical case, there is a link between $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F, G)$ and the "extensions of G by F". Precisely, define an n-extension of G by F as an exact sequence $E : 0 \to G \to E_1 \to \ldots \to E_n \to F \to 0$ in which each morphism is homogeneous. Define the parity $|E|$ as the sum of their degrees modulo 2. Consider the equivalence relation $\sim$ such that $E \sim E'$ if there is a morphism of complexes $E \to E'$ that is the identity on F and G and whose components are all homogeneous. Such relation preserves the parity of extensions.

Definition 1.6.8. For $n \geq 1$ denote by $\mathrm{Yext}^n_{\boldsymbol{\mathcal{P}}}(F, G)$ the super vector space generated by the equivalence classes of n-extensions of G by F. The superdegree of an element E is given by the parity $|E|$. Set by convention $\mathrm{Yext}^0_{\boldsymbol{\mathcal{P}}}(F, G) := \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F, G)$.

Proposition 1.6.9 ([Dru16, Prop. 3.5.1])**.** *There is an even isomorphism of graded super vector spaces*

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F, G) \simeq \mathrm{Yext}^*_{\boldsymbol{\mathcal{P}}}(F, G)$$

under which the Yoneda product (1.6.1) corresponds to the concatenation of extensions.

1.6.2 Cross and cup products

Given superfunctors F_1, G_1, F_2, G_2 there is a graded natural map

$$\times : \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_1, G_1) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_2, G_2) \longrightarrow \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_1 \otimes F_2, G_1 \otimes G_2) \tag{1.6.2}$$

given by tensoring cocycles or equivalently (via Proposition 1.6.9) by tensoring extensions seen as cochain complexes. This operation is called *cross product* of extensions. It commutes with the supertwist in the same way as in the classical situation, as the following lemma shows.

Lemma 1.6.10. *Let $F_1, G_1, F_2, G_2 \in \boldsymbol{\mathcal{P}}$ and let T be the supertwist map (1.2.2). The following diagram*

$$\begin{tikzcd}
\mathrm{Ext}^s_{\boldsymbol{\mathcal{P}}}(F_1, G_1) \otimes \mathrm{Ext}^t_{\boldsymbol{\mathcal{P}}}(F_2, G_2) \arrow[r, "\times"] \arrow[d, "T"'] & \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(F_1 \otimes F_2, G_1 \otimes G_2) \arrow[d, "\mathrm{Ext}(T,T)"] \\
\mathrm{Ext}^t_{\boldsymbol{\mathcal{P}}}(F_2, G_2) \otimes \mathrm{Ext}^s_{\boldsymbol{\mathcal{P}}}(F_1, G_1) \arrow[r, "\times"] & \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(F_2 \otimes F_1, G_2 \otimes G_1)
\end{tikzcd}$$

commutes up to a sign $(-1)^{st}$.

Proof. Let $P_* \to F_1$ and $Q_* \to F_2$ be projective resolutions and take cocycles $f : P_s \to G_1$, $g : Q_t \to G_2$. The unique map $P_* \otimes Q_* \to Q_* \otimes P_*$ which lifts T at the level of resolutions is explicitly given by

$$(-1)^{ab}\, T : P_a \otimes Q_b \to Q_b \otimes P_a$$

for all $a, b \geq 0$. Therefore the proof amounts to verifying the identity

$$T \circ (f \otimes g) \circ T = (-1)^{\overline{f}\,\overline{g}}\, g \otimes f \ .$$

We do it by inspection on an element $p \otimes q \in P_s \otimes Q_t$:

$$\begin{aligned}
[T \circ (f \otimes g) \circ T]\, (p \otimes q) &= (-1)^{\overline{p}\,\overline{q}}\ [T \circ (f \otimes g)]\, (q \otimes p) \\
&= (-1)^{\overline{p}\,\overline{q} + \overline{g}\,\overline{q}}\ T(f(q) \otimes g(p)) \\
&= (-1)^{\overline{p}\,\overline{q} + \overline{g}\,\overline{q} + (\overline{f} + \overline{q})(\overline{g} + \overline{p})}\ g(p) \otimes f(q) \\
&= (-1)^{\overline{f}\,(\overline{g} + \overline{p})}\ g(p) \otimes f(q) \\
&= \left[(-1)^{\overline{f}\,\overline{g}}\, g \otimes f \right] (p \otimes q)
\end{aligned}$$

which gives the desired verification. □

The cross product behaves well with respect to the Yoneda product $\cdot$ just defined. This is the content of the following lemma, which is an adaptation of [Yon58, Prop. 1] to the super framework.

Lemma 1.6.11. *Let $e \in \mathrm{Ext}^n_{\boldsymbol{\mathcal{P}}}(G_1, H_1)$, $e' \in \mathrm{Ext}^{n'}_{\boldsymbol{\mathcal{P}}}(F_1, G_1)$, $f \in \mathrm{Ext}^m_{\boldsymbol{\mathcal{P}}}(G_2, H_2)$ and $f' \in \mathrm{Ext}^{m'}_{\boldsymbol{\mathcal{P}}}(F_2, G_2)$. Then:*

$$(e \cdot e') \times (f \cdot f') = (-1)^{n'm + \overline{e'}\,\overline{f}}\ (e \times f) \cdot (e' \times f')\ .$$

If A is a $\boldsymbol{\mathcal{P}}$-algebra, one can apply cross product and multiplication and obtain a map

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(C, A) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(C, A) \longrightarrow \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(C \otimes C, A) \tag{1.6.3}$$

and, similarly, if C is a $\boldsymbol{\mathcal{P}}$-coalgebra, do the same with comultiplication to have a map

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(C, A) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(C, A) \longrightarrow \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(C, A \otimes A)\ . \tag{1.6.4}$$

Finally, if both conditions on A and C are satisfied, there is a third notion of product

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(C, A) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(C, A) \longrightarrow \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(C, A) \tag{1.6.5}$$

which is associative and hence endows $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(C, A)$ with the structure of a graded superalgebra. When there is no risk of confusion, we will use the same symbol $\cup$ for (1.6.3), (1.6.4) and (1.6.5). The following proposition (which can be found in [Dru16, Lemma 3.4.1] and follows easily from Lemma 1.6.10) shows that the type of commutativity of (1.6.5) depends just on the commutativity of C and A, other than the cohomological sign. To be concise, set $\varepsilon(A) = 0$ if A is commutative, $\varepsilon(A) = 1$ if A is graded-commutative, and dually for the cocommutativity of C.

Proposition 1.6.12. *Let C be a $\boldsymbol{\mathcal{P}}$-coalgebra and A a $\boldsymbol{\mathcal{P}}$-algebra. Let $e \in \mathrm{Ext}^s_{\boldsymbol{\mathcal{P}}}(C^i, A^i)$, $f \in \mathrm{Ext}^t_{\boldsymbol{\mathcal{P}}}(C^j, A^j)$. Then*

$$e \cup f = (-1)^{st + \overline{e}\,\overline{f} + \varepsilon(C)\cdot ij + \varepsilon(A)\cdot ij}\ f \cup e\ .$$

Chapter 2

Additive superfunctors

2.1 The Frobenius twist

We start by recalling the classical version of the Frobenius twist, for which the reader can refer to [FS97]. Let V be a finite-dimensional vector space and $r \geq 1$ an integer fixed throughout. Let $\varphi_r : \Bbbk \to \Bbbk$ be the p^r-th power map.

Definition 2.1.1. The *Frobenius twist* of V is $V^{(r)} := \Bbbk \otimes_{\varphi_r} V$, i.e. the base change of V along φ_r. Its elements of the form $1 \otimes v$ are shortly denoted by $v^{(r)}$.

Equivalently, $V^{(r)}$ is the vector space generated by symbols $\{v^{(r)},\ v \in V\}$ subject to the relations

$$(v+w)^{(r)} = v^{(r)} + w^{(r)}$$
$$(\lambda v)^{(r)} = \lambda^{p^r} v^{(r)} .$$

If $\{e_1, ..., e_n\}$ is a basis of V, the map $e_i \mapsto e_i^{(r)}$ $(i = 1, ..., n)$ defines a non-natural isomorphism $V \simeq V^{(r)}$. Frobenius twist commutes with direct sums, duals and tensor products; in particular $\mathrm{Hom}(V,W)^{(r)} \simeq \mathrm{Hom}(V^{(r)}, W^{(r)})$. The latter isomorphism is given explicitly by the formula

$$\mathrm{Hom}(V,W)^{(r)} \xrightarrow{\simeq} \mathrm{Hom}(V^{(r)}, W^{(r)})$$
$$f^{(r)} \longmapsto f_r$$

where by definition $f_r(v^{(r)}) := (f(v))^{(r)}$.

Remark 2.1.2. The inclusion $V^{(r)} \hookrightarrow S^{p^r}(V)$, $v^{(r)} \mapsto v^{p^r}$ is linear.

This observation shows how to make the Frobenius twist into a strict polynomial functor. Namely, at the level of objects it is given by $V \mapsto V^{(r)}$, while the structure map on morphisms is given by the composition

$$\Gamma^{p^r}\mathrm{Hom}(V,W) \to \mathrm{Hom}(V,W)^{(r)} \simeq \mathrm{Hom}(V^{(r)}, W^{(r)})$$

where the first map is the Kuhn dual of the inclusion of Remark 2.1.2.

Definition 2.1.3 (Frobenius twist functor)**.** The polynomial functor just defined is denoted $I^{(r)}$. It is by construction homogeneous of degree p^r.

One of our protagonists is the super version of the Frobenius twist. If V is a super vector space, the definition of $V^{(r)}$ as an ungraded space is exactly the same as the classical one. If $V = V_0 \oplus V_1$ is the decomposition in even and odd parts, then $V_0^{(r)}$ and $V_1^{(r)}$ are respectively the even and odd parts of $V^{(r)}$. For all $V, W \in \boldsymbol{\mathcal{V}}$, the isomorphism $\mathrm{Hom}(V^{(r)}, W^{(r)}) \simeq \mathrm{Hom}(V,W)^{(r)}$ is even. The same formula as in the classical context gives a linear map $i : V^{(r)} \to \mathbf{S}^{p^r}(V)$. Therefore, as above, one can define a polynomial superfunctor which is given on objects by $V \mapsto V^{(r)}$ and on morphisms by the composition of even linear maps:

$$\mathbf{\Gamma}^{p^r}(\mathrm{Hom}(V,W)) \xrightarrow{i^\vee} \mathrm{Hom}(V,W)^{(r)} \simeq \mathrm{Hom}(V^{(r)}, W^{(r)}) \tag{2.1.1}$$

where $i^\vee$ is the linear dual of the map i.

Definition 2.1.4 (Frobenius twist superfunctor)**.** The polynomial superfunctor just defined is denoted $\mathbf{I}^{(r)}$ and is homogeneous of degree p^r.

The equation (2.1.1) hides a special feature. If $v = v_0 + v_1$ is the decomposition of v into even and odd components, then $v^{p^r} = (v_0)^{p^r} + (v_1)^{p^r}$ by the Newton formula. But since $\mathbf{S}^*(V_1) = \Lambda^*(V_1)$ (1.3.4), we have $(v_1)^{p^r} = 0$ and by consequence $v^{p^r} = (v_0)^{p^r}$. So the image of i is contained in the subspace $\mathbf{S}^{p^r}(V_0)$. Dually, this means that $i^\vee$ vanishes anywhere but on $\mathbf{\Gamma}^{p^r}(V_0)$, in particular its image is contained in $V_0^{(r)}$. This fact implies that (2.1.1) has image in the *even* subspace $\mathrm{Hom}_0(V^{(r)}, W^{(r)}) = \mathrm{Hom}(V_0^{(r)}, W_0^{(r)}) \oplus \mathrm{Hom}(V_1^{(r)}, W_1^{(r)})$. Thanks to that, the formulas

$$\mathbf{I}_0^{(r)}(V) = V_0^{(r)}, \qquad \mathbf{I}_1^{(r)}(V) = V_1^{(r)},$$

define actual polynomial sub-superfunctors of $\mathbf{I}^{(r)}$ such that $\mathbf{I}^{(r)} = \mathbf{I}_0^{(r)} \oplus \mathbf{I}_1^{(r)}$.

Lemma 2.1.5. *$\boldsymbol{I}^{(r)}, \boldsymbol{I}_0^{(r)}$ and $\boldsymbol{I}_1^{(r)}$ are self-dual.*

Proof. Straightforward, since the Frobenius twist commutes with linear dual. □

The following fact is an easy verification.

Lemma 2.1.6. *There are isomorphisms of polynomial superfunctors*

$$\mathbf{\Pi} \circ \boldsymbol{I}_0^{(r)} \simeq \boldsymbol{I}_1^{(r)} \circ \mathbf{\Pi}\,, \qquad \mathbf{\Pi} \circ \boldsymbol{I}_1^{(r)} \simeq \boldsymbol{I}_0^{(r)} \circ \mathbf{\Pi}\,.$$

2.2 Frobenius precomposition

In §1.5 we saw the general recipe to compose two polynomial superfunctors (which is formally the same for classical polynomial functors). Thanks to the special features of the Frobenius twist, we can now perform a hybrid kind of composition: it makes indeed sense to postcompose $\mathbf{I}_0^{(r)}$ or $\mathbf{I}_1^{(r)}$ by a *classical* polynomial functor. To see how, we first extend the definition of the p^r-th power map i of the previous subsection.

Notation 2.2.1. The p^r-th power map induces a superalgebra homomorphism

$$\mathbf{S}^*(V^{(r)}) \to \mathbf{S}^*(V)$$

that we denote i_*. It decomposes into the sum of maps $i_n : \mathbf{S}^n(V^{(r)}) \to \mathbf{S}^{np^r}(V)$, in particular $i_1 = i$. We have in particular a dual map $i_*^\vee : \mathbf{\Gamma}^*(V) \to \mathbf{\Gamma}^*(V^{(r)})$ which decomposes as the sum of $i_n^\vee : \mathbf{\Gamma}^{np^r}(V) \to \mathbf{\Gamma}^n(V^{(r)})$.

The assertions in the next lemma are proven in [Dru16, §2.7].

Lemma 2.2.2. *$i_*^\vee$ is an algebra homomorphism determined on generators by*

$$i_*^\vee(z) = \begin{cases} \gamma_{p^{e-r}}(v) & \text{if } z = \gamma_{p^e}(v) \text{ for some } e \geq r \text{ and } v \in V_0 \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, $i_*^\vee$ has image in the even subspace of $\mathbf{\Gamma}^*(V^{(r)})$. This implies in particular that $\mathbf{I}_0^{(r)}$ and $\mathbf{I}_1^{(r)}$ have image in the category $\boldsymbol{\mathcal{V}}_0$ and $\boldsymbol{\mathcal{V}}_1$ respectively (the full subcategories on purely even, resp. purely odd spaces). This enables us to make the following definition.

Definition 2.2.3. For $F \in \mathcal{P}_d$, the polynomial superfunctors $F_0^{(r)}$ and $F_1^{(r)}$ are defined in the following way:

$$F_0^{(r)} : \mathbf{\Gamma}^{dp^r}\boldsymbol{\mathcal{V}} \xrightarrow{i_{d,0}^\vee} \mathbf{\Gamma}^d\boldsymbol{\mathcal{V}}_0 \simeq \Gamma^d\mathcal{V} \xrightarrow{F} \mathcal{V} \simeq \boldsymbol{\mathcal{V}}_0 \subset \boldsymbol{\mathcal{V}} \tag{2.2.1}$$

$$F_1^{(r)} : \mathbf{\Gamma}^{dp^r}\boldsymbol{\mathcal{V}} \xrightarrow{i_{d,1}^\vee} \mathbf{\Gamma}^d\boldsymbol{\mathcal{V}}_1 \simeq \Gamma^d\mathcal{V} \xrightarrow{F} \mathcal{V} \simeq \boldsymbol{\mathcal{V}}_d \subset \boldsymbol{\mathcal{V}}\,. \tag{2.2.2}$$

where:

- $i_{d,\ell}^\vee : \mathbf{\Gamma}^{dp^r}\boldsymbol{\mathcal{V}} \to \mathbf{\Gamma}^d\boldsymbol{\mathcal{V}}$ is the functor defined, for $\ell \in \{0,1\}$, by $V \mapsto V_\ell^{(r)}$ on objects and by $i_d^\vee$ on Hom.
- $\boldsymbol{\mathcal{V}}_d := \boldsymbol{\mathcal{V}}_0$ if d is even and $\boldsymbol{\mathcal{V}}_1$ if d is odd. The reason for this distinction will be clear later on.

Observe that one can optionally compose (2.2.1) and (2.2.2) by $\mathbf{\Pi}$, which gives sense to the precomposition of F by respectively $\mathbf{I}_0^{(r)} \circ \mathbf{\Pi}$ and $\mathbf{I}_1^{(r)} \circ \mathbf{\Pi}$.

Remark 2.2.4. This definition is more strongly related to the general recipe of composition in §1.5 than it seems. If F is a *super*functor of degree n, we could form the composite

$$F^{(r)} : \mathbf{\Gamma}^{np^r}\boldsymbol{\mathcal{V}} \xrightarrow{i_n^\vee} \mathbf{\Gamma}^n\boldsymbol{\mathcal{V}} \xrightarrow{F} \boldsymbol{\mathcal{V}} \tag{2.2.3}$$

as above (but without the restrictions to $\mathbf{\Gamma}^n\boldsymbol{\mathcal{V}}_0$ or $\mathbf{\Gamma}^n\boldsymbol{\mathcal{V}}_1$). We claim that this coincides with the original definition of $F^{(r)} = F \circ \mathbf{I}^{(r)}$. There is indeed an evident commutative diagram

$$\begin{tikzcd}
\mathbf{S}^{np^r}(V) & \mathbf{S}^n(\mathbf{S}^{p^r}(V)) \arrow[l] \\
\mathbf{S}^n(V^{(r)}) \arrow[u, "i_n"] \arrow[ur, "\mathbf{S}^n(i)"']
\end{tikzcd}$$

where the top map is multiplication. Note that the latter coincides with the map induced by the inclusion $\Sigma_{p^r}^{\times n} \subset \Sigma_{p^r n}$ at the level of coinvariants. This gives by duality a commutative diagram

$$\begin{tikzcd}
\mathbf{\Gamma}^{np^r}(V) \arrow[r] \arrow[d, "i_n^\vee"'] & \mathbf{\Gamma}^n(\mathbf{\Gamma}^{p^r}(V)) \arrow[dl, "\mathbf{\Gamma}^n(i^\vee)"] \\
\mathbf{\Gamma}^n(V^{(r)})
\end{tikzcd}$$

where the above arrow is induced by the inclusion $\Sigma_{p^r}^{\times n} \subset \Sigma_{p^r n}$ at the level of invariants. But the right-down path on Hom spaces is exactly what appears in the composition defined in §1.5. Therefore, it coincides with (2.2.3) as stated.

Example 2.2.5. Apply the exponential isomorphisms (1.3.4)-(1.3.7) to a twisted space $V^{(r)} = V_0^{(r)} \oplus V_1^{(r)}$. In this case, since $r \geq 1$, such decomposition *is* natural with respect to V, as explained at the end of §2.1 (this is a rephrasing of the fact that there exist superfunctors $\mathbf{I}_0^{(r)}$ and $\mathbf{I}_1^{(r)}$). As a consequence, we have isomorphisms of $\boldsymbol{\mathcal{P}}$-algebras:

$$\begin{aligned}
\mathbf{S}^* \circ \mathbf{I}^{(r)} &\simeq S_0^{*(r)} \otimes \Lambda_1^{*(r)}, \\
\mathbf{\Gamma}^* \circ \mathbf{I}^{(r)} &\simeq \Gamma_0^{*(r)} \otimes \Lambda_1^{*(r)}, \\
\mathbf{\Lambda}^* \circ \mathbf{I}^{(r)} &\simeq \Lambda_0^{*(r)} \otimes^g S_1^{*(r)}, \\
\mathbf{A}^* \circ \mathbf{I}^{(r)} &\simeq \Lambda_0^{*(r)} \otimes^g \Gamma_1^{*(r)}.
\end{aligned}$$

Moreover, the following relations can be verified straightforwardly: if d is odd,

$$\begin{aligned}
S_1^{d(r)} &\simeq \mathbf{\Pi} \circ S_0^{d(r)} \circ \mathbf{\Pi}, \\
\Lambda_1^{d(r)} &\simeq \mathbf{\Pi} \circ \Lambda_0^{d(r)} \circ \mathbf{\Pi}, \\
\Gamma_1^{d(r)} &\simeq \mathbf{\Pi} \circ \Gamma_0^{d(r)} \circ \mathbf{\Pi},
\end{aligned} \tag{2.2.4}$$

while if d is even

$$\begin{aligned}
S_1^{d(r)} &\simeq S_0^{d(r)} \circ \mathbf{\Pi}, \\
\Lambda_1^{d(r)} &\simeq \Lambda_0^{d(r)} \circ \mathbf{\Pi}, \\
\Gamma_1^{d(r)} &\simeq \Gamma_0^{d(r)} \circ \mathbf{\Pi}.
\end{aligned} \tag{2.2.5}$$

In the notation of Remark 1.5.8, this can be rewritten more concisely as

$$S_0^{d(r)} \circ \mathbf{\Pi} \;\simeq\; \mathbf{\Pi}^d \circ S_1^{d(r)} \tag{2.2.6}$$

and similarly for Λ and Γ.

This interchangeability between twist and parity shifts is actually true for any classical functor.

Proposition 2.2.6. *Let $F \in \mathcal{P}_d$ and $\ell \in \{0,1\}$. Then there is a natural isomorphism*

$$F_\ell^{(r)} \circ \mathbf{\Pi} \simeq \mathbf{\Pi}^d \circ F_{\ell-1}^{(r)}$$

where $\mathbf{\Pi}^d = \mathbf{\Pi} \circ \ldots \circ \mathbf{\Pi}$ for d times, as in Remark 1.5.8.

Proof. By (2.2.4) and (2.2.5) this is the case if $F = S^d$. Thanks to Remarks 1.5.6 and 1.5.8 we deduce the validity of the statement for $F = S^\lambda$ with λ a composition. Hence, it is true for all injectives. Since the two sides are exact functors with respect to F, the result follows by taking injective coresolutions. □

2.2.1 Vanishing property

We saw that the Frobenius superfunctors all have a particular behaviour on morphisms, namely they vanish on a big part of them. It could be useful to understand if they keep this property when post-composed with a polynomial functor.

Definition 2.2.7. Let $V \in \boldsymbol{\mathcal{V}}$, $d \geq 0$ and recall the decomposition $\mathbf{\Gamma}^d V \simeq \bigoplus_{a=0}^d \Gamma^{d-a}(V_0) \otimes \Lambda^a(V_1)$. For $a \in \{0, ..., d\}$, an element belonging to the a-summand is said to be of *parity a*.

By definition, reducing this $\mathbb{Z}$-grading modulo 2 gives the $\mathbb{Z}/2\mathbb{Z}$-grading that was introduced in Section 1.3.

Proposition 2.2.8. *Let $F \in \mathcal{P}_d$ and $A \in \{\boldsymbol{I}_0^{(r)}, \boldsymbol{I}_1^{(r)}, \boldsymbol{I}_0^{(r)} \circ \mathbf{\Pi}, \boldsymbol{I}_1^{(r)} \circ \mathbf{\Pi}\}$. If $f \in \mathbf{\Gamma}^{dp^r}\mathrm{Hom}(V, W)$ has parity > 0, then $(F \circ A)(f) = 0$.*

Proof. For all the listed A, by definition $(F \circ A)(f) = F(i_d^\vee(f))$ where we recall that $i_d^\vee : \mathbf{\Gamma}^{dp^r}(\mathrm{Hom}(V,W)) \to \mathbf{\Gamma}^d(\mathrm{Hom}_0(V,W)^{(r)})$ is the dual of the p^r-th power map. If f has parity $0 < a \leq dp^r$, by (1.3.5) it can be written as a sum of terms of the form

$$\gamma_{s_1}(f_1) \cdot \ldots \cdot \gamma_{a_s}(s_{d-a}) \cdot g_1 \cdot \ldots \cdot g_a$$

with $g_1, ..., g_a \in \mathrm{Hom}_1(V,W)$. But Lemma 2.2.2 says that $i_*^\vee$ kills $g_1, ..., g_a$ and commutes with products. Hence $i^\vee(f) = 0$ and in particular $(F \circ A)(f) = 0$. □

2.2.2 Fully faithfulness

In this subsection we prove that the Frobenius precomposition functors are fully faithful. Set:

$$\Phi_r := - \circ \mathbf{I}_0^{(r)} : \mathcal{P} \longrightarrow \boldsymbol{\mathcal{P}}$$
$$\overline{\Phi_r} := - \circ \mathbf{I}_1^{(r)} : \mathcal{P} \longrightarrow \boldsymbol{\mathcal{P}}.$$

By Convention 1.5.5, we may restrict our attention to homogeneous functors.

Theorem 2.2.9 (Fully faithfulness of Φ_r). *Let $F \in \mathcal{P}_d$ and $\lambda \in \Lambda(n,d)$. There is a chain of even isomorphisms (where the left-hand side is considered as purely even)*

$$\mathrm{Hom}_{\mathcal{P}_d}(F, S^\lambda) \xrightarrow{\simeq} \mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(F_0^{(r)}, S_0^{\lambda(r)}) \xrightarrow{\simeq} \mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(F_0^{(r)}, \boldsymbol{S}^{p^r\lambda}) \qquad (2.2.7)$$

the first one being induced by Φ_r and the second by the inclusion $S_0^{\lambda(r)} \hookrightarrow \boldsymbol{S}^{p^r\lambda}$. Hence, Φ_r is fully faithful: for all $F, G \in \mathcal{P}$ it induces an isomorphism

$$\mathrm{Hom}_{\mathcal{P}_d}(F,G) \simeq \mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(F_0^{(r)}, G_0^{(r)})\,. \qquad (2.2.8)$$

Proof. We prove first the injectivity of (2.2.8). Let $T \in \mathrm{Hom}_{\mathcal{P}_d}(F,G)$ be such that $\Phi_r(T) = 0$: in other words, this means that T_V is zero whenever $V = W_0^{(r)}$ for some finite-dimensional super vector space W. It means in particular that $T_{V^{(r)}} = 0$ for all finite-dimensional vector spaces V. Consider now a (non-natural) isomorphism $\varphi : V \xrightarrow{\simeq} V^{(r)}$. Then $\gamma_d(\varphi) \in \mathrm{Hom}_{\mathbf{\Gamma}^d\boldsymbol{\mathcal{V}}}(V, V^{(r)})$ is an isomorphism, hence so are $F(\gamma_d(\varphi))$ and $G(\gamma_d(\varphi))$. By naturality of T, the diagram

$$\begin{tikzcd} F(V) \arrow[r, "T_V"] \arrow[d, "F(\gamma_d(\varphi))"'] & G(V) \arrow[d, "G(\gamma_d(\varphi))"] \\ F(V^{(r)}) \arrow[r, "T_{V^{(r)}}"'] & G(V^{(r)}) \end{tikzcd}$$

commutes. Since the vertical arrows are isomorphisms and the lower arrow is zero by hypothesis, by arbitrariness of V we deduce that $T = 0$. This proves that (2.2.8) is injective, i.e. Φ_r is faithful. In particular, the first map of the composition (2.2.7) is an inclusion.

The second map is induced by the p^r-th power map $S_0^{\lambda(r)} \hookrightarrow \mathbf{S}^{p^r\lambda}$. Since the latter is an inclusion, the induced map is an inclusion as well by left exactness of $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, -)$.

Therefore the whole composition is injective. By Theorem 1.5.26 we know that the first and last Hom space are of the same superdimension - namely $\mathrm{sdim}(F^\#(\Bbbk^n)^\lambda)$ - thus the composition is an isomorphism as stated. In particular, this proves that (2.2.8) is an isomorphism whenever G is injective. For a general G, take a coresolution $G \hookrightarrow J^*$ and form the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{P}}(F,G) \arrow[r, "\Phi_r"] \arrow[d] & \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)}) \arrow[d] \\ \mathrm{Hom}_{\mathcal{P}}(F,J^*) \arrow[r, "\Phi_r"] & \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, J_0^{*(r)})\,. \end{tikzcd}$$

By the previous part of the proof, the lower arrow is an isomorphism in each degree. Note that, even if the coresolution $J_0^{*(r)}$ is not injective, we have nevertheless $H^0\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, J_0^{*(r)}) \simeq \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)})$. Hence the upper arrow is an isomorphism as well. □

Theorem 2.2.10 (Fully faithfulness of $\overline{\Phi_r}$)**.** *Let $F \in \mathcal{P}_d$ and $\lambda \in \Lambda(n,d)$. There is a chain of isomorphisms*

$$\mathrm{Hom}_{\mathcal{P}_d}(F, S^\lambda) \xrightarrow{\simeq} \mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(F_1^{(r)}, S_1^{\lambda(r)}) \xrightarrow{\simeq} \mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(F_1^{(r)}, \mathbf{\Pi} \circ \boldsymbol{S}^{p^r\lambda} \circ \mathbf{\Pi}) \tag{2.2.9}$$

the first one being induced by $\overline{\Phi_r}$ and the second one having parity equal to $d+1$ mod 2. Hence, $\overline{\Phi_r}$ is fully faithful: for all $F, G \in \mathcal{P}$ it induces an isomorphism

$$\mathrm{Hom}_{\mathcal{P}_d}(F, G) \simeq \mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(F_1^{(r)}, G_1^{(r)})\ . \tag{2.2.10}$$

Proof. By a similar argument to that used in the proof of Theorem 2.2.9, one proves that (2.2.10) is injective, hence in particular that the first map in (2.2.9) is an inclusion. Now let us consider the second map of (2.2.9). By (2.2.6), applying conjugation to the even inclusion $S_0^{\lambda(r)} \hookrightarrow \mathbf{S}^{p^r\lambda}$ produces an even inclusion $\mathbf{\Pi} \circ S_0^{\lambda(r)} \circ \mathbf{\Pi} \hookrightarrow \mathbf{\Pi} \circ \mathbf{S}^{p^r\lambda} \circ \mathbf{\Pi}$. We want to rewrite the source. Using Remarks 1.5.6 and 1.5.8, as well as (2.2.6), we have

$$\begin{aligned}
\mathbf{\Pi} \circ S_0^{\lambda(r)} \circ \mathbf{\Pi} &= \mathbf{\Pi}^{n+1} \circ \left[(\mathbf{\Pi} \circ S_0^{\lambda_1(r)} \circ \mathbf{\Pi}) \otimes ... \otimes (\mathbf{\Pi} \circ S_0^{\lambda_n(r)} \circ \mathbf{\Pi})\right] \\
&\simeq \mathbf{\Pi}^{n+1} \circ \left[(\mathbf{\Pi}^{\lambda_1+1} S_1^{\lambda_1(r)}) \otimes ... \otimes (\mathbf{\Pi}^{\lambda_n+1} S_1^{\lambda_n(r)})\right] \\
&= \mathbf{\Pi}^{n+1} \circ \mathbf{\Pi}^{\sum_{i=1}^n(\lambda_i+1)} \circ S_1^{\lambda(r)} \\
&= \mathbf{\Pi}^{d+1} S_1^{\lambda(r)}
\end{aligned}$$

since $\sum_{i=1}^n(\lambda_i+1) = n+d$. This means that conjugation induces an inclusion $S_1^{\lambda(r)} \hookrightarrow \mathbf{\Pi} \circ \mathbf{S}^{p^r\lambda} \circ \mathbf{\Pi}$ of parity $d+1$, as asserted. Now, since the functor $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F_1^{(r)}, -)$ is left exact, the second map of the composition is an inclusion, hence the whole (2.2.10) is an inclusion. Remembering that $\mathbf{S}^* \circ \mathbf{\Pi} = \mathbf{S}^*_{\Pi\Bbbk}$, we have by the Yoneda lemma

$$\begin{aligned}
\mathrm{sdim}(\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F_1^{(r)}, \mathbf{\Pi} \circ \mathbf{S}^{p^r\lambda} \circ \mathbf{\Pi})) &= \mathrm{sdim}(\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ F_1^{(r)}, \mathbf{S}^{p^r\lambda}_{\Pi\Bbbk})) \\
&= \mathrm{sdim}(\Pi F^{\#}(\Bbbk^{n(r)})^{p^r\lambda}) \\
&= \mathrm{sdim}(\Pi F^{\#}(\Bbbk^n)^\lambda) \\
&= \mathrm{sdim}(\Pi \mathrm{Hom}_{\mathcal{P}}(F, S^\lambda))\ .
\end{aligned}$$

In particular the source and target of (2.2.10) have the same dimension as ungraded spaces. This is enough to deduce that the composite is an isomorphism. In particular so is $\overline{\Phi_r}$. This proves fully faithfulness when G is injective. The general case follows as in the proof of Theorem 2.2.9. □

As a corollary, we have an interesting vanishing lemma. We say that a composition $\mu \in \Lambda(m|n,d)$ is *purely even* is $\lambda_{m+1} = ... = \lambda_{m+n} = 0$ and that it is *purely odd* if $\lambda_1 = ... = \lambda_m = 0$.

Corollary 2.2.11. *Let $\mu \in \Lambda(m|n,d)$.*

- *If μ is not divisible by p^r or is not purely even, then* $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, \boldsymbol{S}^\mu) = 0$.
- *If μ is not divisible by p^r or is not purely odd, then* $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F_1^{(r)}, \boldsymbol{S}^\mu) = 0$.

Proof. By the Yoneda lemma, the super vector space $\mathrm{Hom}_{\mathcal{P}}(F, S^d_{\Bbbk^m})$ has the same superdimension as $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, \mathbf{S}^{dp^r}_{\Bbbk^{m|n}})$, namely equal to $\mathrm{sdim}(F^{\#}(\Bbbk^m))$. Decomposing both via the exponential property and computing the superdimensions, we deduce that

$$\sum_{\lambda\in\Lambda(m,d)} \mathrm{sdim}(\mathrm{Hom}_{\mathcal{P}}(F, S^\lambda)) = \sum_{\mu\in\Lambda(m|n,dp^r)} \mathrm{sdim}(\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, \mathbf{S}^\mu)) \ .$$

On the other hand, the isomorphism (2.2.7) implies in particular that

$$\sum_{\lambda\in\Lambda(m,d)} \mathrm{sdim}(\mathrm{Hom}_{\mathcal{P}}(F, S^\lambda)) = \sum_{\lambda\in\Lambda(m|0,dp^r)} \mathrm{sdim}(\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, \mathbf{S}^{p^r\lambda}))$$

which forces $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, \mathbf{S}^\mu)$ to vanish whenever μ is not a p^r-multiple of a composition in $\Lambda(m,d)$. This proves the first statement. The second one follows by an analogous reasoning from Theorem 2.2.10. ☐

2.2.3 An exponential formula on Ext

In this subsection we prove a super analogue of [FS97, Prop. 5.2]. It is a result that allows to compute the extensions between a tensor product of superfunctors and twisted symmetric powers. We first state a key lemma about the weights of a twisted functor.

Lemma 2.2.12. *Let $F \in \mathcal{P}_d$ and $m, n \geq 0$.*

- *The weights of $F_0^{(r)}(\Bbbk^{m|n}) = F(\Bbbk^{m(r)})$ are purely even and divisible by p^r.*
- *The weights of $F_1^{(r)}(\Bbbk^{m|n}) = F(\Bbbk^{n(r)})$ are purely odd and divisible by p^r.*

Proof. By Theorem 1.5.26 and Kuhn duality, $G(V)^\lambda \simeq \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(G^{\#}, \mathbf{S}^\lambda)$ for any $G \in \boldsymbol{\mathcal{P}}$ and any composition λ. The result follows then by Corollary 2.2.11. ☐

Theorem 2.2.13. *Let F, G be strict polynomial superfunctors of degree s, t respectively, such that $s + t = dp^r$. Then*

- $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F \otimes G, (S^d_V)^{(r)}_\ell) = 0$ *if s, t are not divisible by p^r.*
- *If $s = p^r s'$ and $t = p^r t'$ there is a isomorphism*

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F \otimes G, (S^d_V)^{(r)}_\ell) \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F, (S^{s'}_V)^{(r)}_\ell) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(G, (S^{t'}_V)^{(r)}_\ell)$$

induced by cup product, in particular natural in all variables.

Proof. By Proposition 2.2.6, it suffices to treat the case $\ell = 0$. We first prove the assertion for F, G projective, say $F = \mathbf{\Gamma}^\lambda$ and $G = \mathbf{\Gamma}^\mu$ for $\lambda \in \Lambda(m|n, s)$ and $\mu \in \Lambda(m'|n', t)$. In particular $F \otimes G$ is again projective, so that the Ext isomorphism reduces to a Hom isomorphism. Define now $(\lambda, \mu) \in \Lambda(m + m'|n + n', s + t)$ as the composition

$$(\lambda, \mu) := (\lambda_1, ..., \lambda_m, \mu_1, ..., \mu_{m'}, \lambda_{m+1}, ..., \lambda_{m+n}, \mu_{m'+1}, ..., \mu_{m'+m'}) \ .$$

Note that $F \otimes G \subset \mathbf{\Gamma}^{dp^r, \Bbbk^{m+m'|n+n'}}$ is the direct factor corresponding to (λ, μ). By the Weighted Yoneda lemma (Theorem 1.5.26) we have an isomorphism

$$\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F \otimes G, (S^d_V)^{(r)}_0) \simeq S^d(V \otimes \Bbbk^{m+m'(r)})^{(\lambda,\mu)} \ .$$

The action of $\mathbb{G}_m^{\times m+m'+n+n'}$ on $S^d(V \otimes \Bbbk^{m+m'(r)})$, has only weights divisible by p^r (Lemma 2.2.12). As a consequence, both s and t are forced to be multiples of p^r or the Hom space is zero. This proves the first point. Suppose then $\lambda = p^r\lambda'$, $\mu = p^r\mu'$. Again by the Weighted Yoneda lemma, there are isomorphisms fitting in a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{P}}(F, (S^s_V)^{(r)}_0) \otimes \mathrm{Hom}_{\mathcal{P}}(G, (S^t_V)^{(r)}_0) \arrow[r, "\simeq"] \arrow[d] & S^{\lambda'}(V \otimes \Bbbk^{m(r)}) \otimes S^{\mu'}(V \otimes \Bbbk^{m'(r)}) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathcal{P}}(F \otimes G, (S^d_V)^{(r)}_0) \arrow[r, "\simeq"] & S^d(V \otimes \Bbbk^{m+m'(r)})^{(\lambda,\mu)}
\end{tikzcd}$$

where the right map is the multiplication m and the left map is, by direct inspection, equal to $m \circ (-_1 \otimes -_2)$. But the latter is exactly the definition of cup product at the Hom level, hence the second point follows. If F, G are general superfunctors, take P_*, Q_* projective resolutions. Then $P_* \otimes Q_*$ is a projective resolution of $F \otimes G$. Since the cup product of extensions comes from tensoring cocycles, if we apply the previous computations to $\mathrm{Hom}_{\mathcal{P}}(P_* \otimes Q_*, (S^d_V)^{(r)}_0)$ the result follows. $\square$

Corollary 2.2.14. *Let $\ell, m \in \{0,1\}$ and $X = \Gamma$ or S. Cup product induces then a natural isomorphism*

$$\mathrm{Ext}^*_{\mathcal{P}}(X^{\lambda(r)}_\ell, (S^d_V)^{(r)}_m) \simeq \bigotimes_i \mathrm{Ext}^*_{\mathcal{P}}(X^{\lambda_i(r)}_\ell, (S^{\lambda_i}_V)^{(r)}_m) \ .$$

2.3 Additive superfunctors

In this section we describe a class of superfunctors containing Frobenius twists.

Definition 2.3.1. A homogeneous polynomial superfunctor F is *additive* if there is an isomorphism $F(V \oplus W) \simeq F(V) \oplus F(W)$ natural in V, W.

It is sometimes useful to make such isomorphism explicit. The next lemma shows that it is always possible. Let $i_V : V \hookrightarrow V \oplus W$ and $i_W : W \hookrightarrow V \oplus W$ denote the canonical inclusions.

Lemma 2.3.2. *A homogeneous polynomial superfunctor F is additive if and only if the natural transformation (a priori injective)*

$$F(V) \oplus F(W) \xrightarrow{F(i_V)+F(i_W)} F(V \oplus W)$$

is an isomorphism.

Proof. The "if" part is obvious. To prove the other one, suppose that F is additive and call $\Theta_{V,W}$ the isomorphism of Definition 2.3.1. Its naturality gives commutative diagrams

$$\begin{tikzcd}
F(V) \oplus F(W) \arrow[r, "\Theta_{V,W}"] & F(V \oplus W) \\
F(V) \arrow[u, "i_{F(V)}"] \arrow[r, "\Theta_{V,0}"] & F(V) \arrow[u, "F(i_V)"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(V) \oplus F(W) \arrow[r, "\Theta_{V,W}"] & F(V \oplus W) \\
F(W) \arrow[u, "i_{F(W)}"] \arrow[r, "\Theta_{0,W}"] & F(W) \arrow[u, "F(i_W)"']
\end{tikzcd}$$

which together yield a commutative triangle

$$\begin{tikzcd}
F(V) \oplus F(W) \arrow[r, "\Theta_{V,W}"] \arrow[dr, "\Theta_{V,0}+\Theta_{0,W}"'] & F(V \oplus W) \\
 & F(V) \oplus F(W) \, . \arrow[u, "F(i_V)+F(i_W)"']
\end{tikzcd}$$

By hypothesis, the two left-most arrows of this last diagram are isomorphisms. Hence $F(i_V) + F(i_W)$ is an isomorphism too, as we had to show. □

Example 2.3.3. $\mathbf{I}$ and $\mathbf{\Pi}$ are the easiest examples of additive superfunctors. One sees immediately that so are the Frobenius twists $\mathbf{I}_0^{(r)}, \mathbf{I}_1^{(r)}$ and $\mathbf{I}^{(r)}$. In the next section we will prove the fact (absolutely nontrivial) that any additive superfunctor is built from these ones.

Additive superfunctors have an important homological property, often referred to as Pirashvili's vanishing lemma, since it was first proved [Pir88] by Pirashvili in the context of non-strict functors.

Lemma 2.3.4. [Dru16, Fri03] *Let $\mathcal{A} := \mathcal{P}$ or $\boldsymbol{\mathcal{P}}$. Let $A, F, G \in \mathcal{A}$ be such that A is additive and F, G are reduced, i.e. $F(0) = G(0) = 0$. Then*

$$\mathrm{Ext}^*_{\mathcal{A}}(A, F \otimes G) = \mathrm{Ext}^*_{\mathcal{A}}(F \otimes G, A) = 0 \ .$$

2.3.1 Classification of additive superfunctors

We dedicate this section to a theorem that classifies all additive superfunctors as sums of indecomposable ones. The theorem is precise: we know completely such indecomposables as well as their occurrence in the decomposition.

Theorem 2.3.5. *Let A be an additive homogeneous superfunctor of degree d. Then $d = p^r$ for some $r \geq 0$ and*

1. *if $r = 0$, then A decomposes as a direct sum of copies of the superfunctors $\boldsymbol{I}$ and $\mathbf{\Pi}$;*
2. *if $r > 0$ then A decomposes as a direct sum of copies of the superfunctors $\boldsymbol{I}_0^{(r)}$, $\boldsymbol{I}_1^{(r)}$, $\boldsymbol{I}_0^{(r)} \circ \mathbf{\Pi}$ and $\boldsymbol{I}_1^{(r)} \circ \mathbf{\Pi}$.*

Moreover, the number of copies of $\boldsymbol{I}_0^{(r)}$ (resp. $\boldsymbol{I}_0^{(r)} \circ \mathbf{\Pi}$, $\boldsymbol{I}_1^{(r)} \circ \mathbf{\Pi}$, $\boldsymbol{I}_1^{(r)}$) in the decomposition is equal to the dimension of the vector space $A(\Bbbk)_0$ (resp. $A(\Bbbk)_1, A(\Pi\Bbbk)_0, A(\Pi\Bbbk)_1$, where $\Pi\Bbbk := \Bbbk^{0|1}$). For $r = 0$ the number of copies of $\boldsymbol{I}$ (resp. $\mathbf{\Pi}$) is equal to the dimension of $A(\Bbbk)_0$ (resp. $A(\Bbbk)_1$). In particular, the decomposition of A is unique up to isomorphism.

The proof of this result will pass through several steps. We follow the approach used in [Tou17] to classify the additive functors in $\mathcal{P}$. We start by pointing out a useful property of additive superfunctors. For the rest of the section, we keep the already introduced notation $\Pi\Bbbk = \Bbbk^{0|1}$.

Proposition 2.3.6. *Let $F, G \in \boldsymbol{\mathcal{P}}$ be superfunctors and suppose that F is additive (resp. that G is additive). Let $\varphi : F \to G$ be a natural transformation between them. Then φ is a monomorphism (resp. epimorphism) if and only if $\varphi_\Bbbk : F(\Bbbk) \to G(\Bbbk)$ and $\varphi_{\Pi\Bbbk} : F(\Pi\Bbbk) \to G(\Pi\Bbbk)$ are monomorphisms (resp. epimorphisms).*

Proof. One implication is obvious. For the other one, we only treat the case of a monomorphism, the other one being dual. We have then to check that $\varphi_V : F(V) \to G(V)$ is injective for any $V \in \boldsymbol{\mathcal{V}}$. Choose a homogenous basis to identify $V \simeq \Bbbk^{m|n}$. Then, by naturality of φ and additivity of F, we can factor φ_V by

$$F(V) \simeq F(\Bbbk^{m|n}) \simeq F(\Bbbk)^{\oplus m} \oplus F(\Pi\Bbbk)^{\oplus n} \hookrightarrow G(\Bbbk)^{\oplus m} \oplus G(\Pi\Bbbk)^{\oplus n} \hookrightarrow G(V)$$

where the middle arrow is a sum of $\varphi_\Bbbk$ and $\varphi_{\Pi\Bbbk}$, which are by hypothesis monomorphisms, and the final inclusion is given by functoriality on the inclusions $\Bbbk, \Pi\Bbbk \subset V$. The assertion follows. □

Corollary 2.3.7. *Let $F, G \in \boldsymbol{\mathcal{P}}$ be additive and let $\varphi : F \to G$ be a natural transformation. Then φ is an isomorphism if and only if $\varphi_{\Bbbk} : F(\Bbbk) \to G(\Bbbk)$ and $\varphi_{\Pi\Bbbk} : F(\Pi\Bbbk) \to G(\Pi\Bbbk)$ are.*

Set now $d \geq 1$. Let $\mathbf{Q}^d$ denote the superfunctor defined as the cokernel of the natural map induced by multiplication in $\mathbf{\Gamma}^d$

$$\bigoplus_{k=1}^{d-1} \mathbf{\Gamma}^k \otimes \mathbf{\Gamma}^{d-k} \longrightarrow \mathbf{\Gamma}^d . \tag{2.3.1}$$

Note that $\mathbf{Q}^1 = \mathbf{I}$, since in that case $\mathbf{\Gamma}^1 = \mathbf{I}$ and the direct sum is equal to zero.

Lemma 2.3.8. *1. $\mathbf{Q}^d$ is an additive superfunctor.*

2. $\mathbf{Q}^d = 0$ if d is not a power of p.

3. $\mathbf{Q}^{p^r}(\Bbbk) \simeq \Bbbk$ for all $r \geq 0$.

4. $\mathbf{Q}^d(\Pi\Bbbk) \simeq \Pi\Bbbk$ if $d = 1$ and zero otherwise.

Proof. We start by point (4): it follows from the fact that $\mathbf{Q}^1 = \mathbf{I}$ and $\mathbf{\Gamma}^d(\Pi\Bbbk) = \Lambda^d(\Bbbk) = 0$ if $d > 1$. Let us now prove point (1). By Remark 1.3.14 multiplication induces a natural isomorphism $\bigoplus_{k=0}^{d} \mathbf{\Gamma}^k(V) \otimes \mathbf{\Gamma}^{d-k}(W) \simeq \mathbf{\Gamma}^d(V \oplus W)$. Since (2.3.1) is also the multiplication map, the cokernel lets only survive the first and last direct summands, giving then a natural isomorphism $\mathbf{Q}^d(V \oplus W) \simeq \mathbf{Q}^d(V) \oplus \mathbf{Q}^d(W)$. To prove points (2) and (3), recall that $\mathbf{\Gamma}^i(\Bbbk) \simeq \Bbbk$ for any i, so that the composite

$$\Bbbk \simeq \mathbf{\Gamma}^k(\Bbbk) \otimes \mathbf{\Gamma}^{d-k}(\Bbbk) \xrightarrow{\text{mult}} \mathbf{\Gamma}^d(\Bbbk) \simeq \Bbbk$$

is equal to multiplication by $\binom{d}{k}$ by Proposition 1.1.5. Then $\mathbf{Q}^d(\Bbbk)$ is the quotient $\Bbbk/n_d\Bbbk$ where $n_d = \gcd_{1 \leq k \leq d} \binom{d}{k}$. It is known [Tou17, Lemma 3.14] that $n_d = 1$ if d is not a prime power, hence in this case $\mathbf{Q}^d(\Bbbk) = 0$. Using point (4) and additivity, $\mathbf{Q}^d = 0$. Otherwise, if d is a prime power, n_d is equal to p. That completes the proof. ☐

Let F be a superfunctor of degree d. Recall that $\mathbf{\Gamma}^{d,\Bbbk} = \mathbf{\Gamma}^d$ and $\mathbf{\Gamma}^{d,\Pi\Bbbk} \simeq \mathbf{\Gamma}^d \circ \mathbf{\Pi}$, so that the Yoneda maps (1.5.4) yield morphisms

$$\begin{aligned} (\mathbf{\Gamma}^d \oplus \mathbf{\Pi} \circ \mathbf{\Gamma}^d) \otimes F(\Bbbk) &\longrightarrow F , \\ (\mathbf{\Gamma}^d \circ \mathbf{\Pi} \oplus \mathbf{\Pi} \circ \mathbf{\Gamma}^d \circ \mathbf{\Pi}) \otimes F(\Pi\Bbbk) &\longrightarrow F , \end{aligned} \tag{2.3.2}$$

which can be decomposed into four morphisms

$$\begin{aligned} \mathbf{\Gamma}^d \otimes F(\Bbbk)_0 &\longrightarrow F , \\ \mathbf{\Pi} \circ \mathbf{\Gamma}^d \otimes F(\Bbbk)_1 &\longrightarrow F , \\ (\mathbf{\Gamma}^d \circ \mathbf{\Pi}) \otimes F(\Pi\Bbbk)_0 &\longrightarrow F , \\ (\mathbf{\Pi} \circ \mathbf{\Gamma}^d \circ \mathbf{\Pi}) \otimes F(\Pi\Bbbk)_1 &\longrightarrow F . \end{aligned} \tag{2.3.3}$$

Now evaluate the first two ones on $\Bbbk$ and the second two ones on $\Pi\Bbbk$. By the explicit formula in (1.5.4), the direct factor $\mathbf{\Gamma}^d(\Bbbk) \otimes F(\Bbbk)_0$ is mapped onto the even part $F(\Bbbk)_0 \subset F(\Bbbk)$, and similarly for the other ones:

$$\begin{aligned} \mathbf{\Gamma}^d(\Bbbk) \otimes F(\Bbbk)_0 &\longrightarrow F(\Bbbk)_0 , \\ \mathbf{\Pi} \circ \mathbf{\Gamma}^d(\Bbbk) \otimes F(\Bbbk)_1 &\longrightarrow F(\Bbbk)_1 , \\ \mathbf{\Gamma}^d(\Bbbk) \otimes F(\Pi\Bbbk)_0 &\longrightarrow F(\Pi\Bbbk)_0 , \\ \mathbf{\Pi} \circ \mathbf{\Gamma}^d(\Bbbk) \otimes F(\Pi\Bbbk)_1 &\longrightarrow F(\Pi\Bbbk)_1 . \end{aligned} \tag{2.3.4}$$

Since $\mathbf{\Gamma}^d(\Bbbk) \simeq \mathbf{\Gamma}^d\mathrm{End}(\Bbbk)$ is spanned by the identity, the morphisms (2.3.4) are immediately isomorphisms due to the explicit formula of the Yoneda maps (1.5.4).

Suppose from now that F is additive. Let us go back for a moment to the natural maps (2.3.3). Precompose them by the multiplication map (2.3.1), duly composed with $\mathbf{\Pi}$ where needed. The composite map is zero by Lemma 2.3.4, thus the morphisms (2.3.3) factor through natural maps

$$\begin{aligned} \mathbf{Q}^d \otimes F(\Bbbk)_0 &\longrightarrow F\,, \\ \mathbf{\Pi} \circ \mathbf{Q}^d \otimes F(\Bbbk)_1 &\longrightarrow F\,, \\ (\mathbf{Q}^d \circ \mathbf{\Pi}) \otimes F(\Pi\Bbbk)_0 &\longrightarrow F\,, \\ (\mathbf{\Pi} \circ \mathbf{Q}^d \circ \mathbf{\Pi}) \otimes F(\Pi\Bbbk)_1 &\longrightarrow F\,. \end{aligned} \tag{2.3.5}$$

As we did before, evaluate the first two ones on $\Bbbk$ and the second two ones on $\Pi\Bbbk$ to get

$$\begin{aligned} \mathbf{Q}^d(\Bbbk) \otimes F(\Bbbk)_0 &\longrightarrow F(\Bbbk)_0\,, \\ \mathbf{\Pi} \circ \mathbf{Q}^d(\Bbbk) \otimes F(\Bbbk)_1 &\longrightarrow F(\Bbbk)_1\,, \\ \mathbf{Q}^d(\Bbbk) \otimes F(\Pi\Bbbk)_0 &\longrightarrow F(\Pi\Bbbk)_0\,, \\ \mathbf{\Pi} \circ \mathbf{Q}^d(\Bbbk) \otimes F(\Pi\Bbbk)_1 &\longrightarrow F(\Pi\Bbbk)_1\,. \end{aligned} \tag{2.3.6}$$

The morphisms (2.3.6) are isomorphisms because they are the factorisation of the isomorphisms (2.3.4) through the quotient map $\mathbf{\Gamma}^d(\Bbbk) \to \mathbf{Q}^d(\Bbbk)$.

Proposition 2.3.9. *Let F be an additive superfunctor of degree $d > 1$. Then there is a natural isomorphism*

$$\begin{aligned} F \simeq\ & \boldsymbol{Q}^d \otimes F(\Bbbk)_0 \ \oplus\ (\mathbf{\Pi} \circ \boldsymbol{Q}^d) \otimes F(\Bbbk)_1 \\ &\oplus (\boldsymbol{Q}^d \circ \mathbf{\Pi}) \otimes F(\Pi\Bbbk)_0 \ \oplus\ (\mathbf{\Pi} \circ \boldsymbol{Q}^d \circ \mathbf{\Pi}) \otimes F(\Pi\Bbbk)_1\,. \end{aligned}$$

Proof. The candidate isomorphism is the sum of the four natural morphisms (2.3.5). Lemma 2.3.8 and isomorphisms (2.3.6) imply that it is an isomorphism when evaluated on $\Bbbk$ and $\Pi\Bbbk$, thus the assertion follows from Corollary 2.3.7. □

The analogous statement of Proposition 2.3.9 for functors of degree 1 requires a slight modification. We need the following:

Lemma 2.3.10. *Let F be a strict polynomial superfunctor of degree 1. Then there exists an odd isomorphism $F(\Bbbk) \simeq F(\Pi\Bbbk)$.*

Proof. By hypothesis F is defined on morphisms by an even linear map

$$F_{V,W} : \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(F(V), F(W))\,.$$

Let $\pi : \Bbbk \to \Pi\Bbbk$, $\pi' : \Pi\Bbbk \to \Bbbk$ be the parity change maps. Clearly $\pi' \circ \pi = \mathrm{Id}_{\Bbbk}$ and $\pi \circ \pi' = \mathrm{Id}_{\Pi\Bbbk}$. Since F is by definition even as a functor $\boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}$, by functoriality $F(\pi) : F(\Bbbk) \to F(\Pi\Bbbk)$ is an odd isomorphism with inverse $F(\pi')$. □

Proposition 2.3.11. *Let F be an additive superfunctor of degree 1. Then there is a natural isomorphism*

$$F \simeq \boldsymbol{I} \otimes F(\Bbbk)_0 \ \oplus\ \mathbf{\Pi} \otimes F(\Bbbk)_1\,.$$

Proof. The natural morphism is given by the sum of the first two ones in (2.3.5) (recall that $\mathbf{Q}^1 = \mathbf{I}$). It is clearly an isomorphism when evaluated on $\Bbbk$, and it is also when evaluated on $\Pi\Bbbk$ by Lemma 2.3.10. Thus it is an isomorphism by Corollary 2.3.7. □

We now handle the case of additive superfunctors of any degree. The only thing left is to identify $\mathbf{Q}^d$.

Proposition 2.3.12. *$\boldsymbol{Q}^d = 0$ if d is not a power of p. If $d = p^r$, then $\boldsymbol{Q}^d \simeq \boldsymbol{I}_0^{(r)}$.*

Proof. If d is not a power of p, Lemma 2.3.8(1) ensures that $\mathbf{Q}^d = 0$. Otherwise, if $d = p^r$, simply note that $\mathbf{I}_0^{(r)}$ vanishes on purely odd spaces and sends $\Bbbk$ onto itself (up to isomorphism). The statement follows then by applying Proposition 2.3.9 to $F = \mathbf{I}_0^{(r)}$. □

We can finally gather the results of the section to prove the classification theorem.

Proof of Theorem 2.3.5. Propositions 2.3.9 and 2.3.12 imply in particular that a nonzero additive superfunctor has degree p^r. If $r = 0$, it is isomorphic to a direct sum of copies of $\mathbf{I}$ and $\mathbf{\Pi}$ by Proposition 2.3.11; if $r > 0$, it is isomorphic to a direct sum of copies of $\mathbf{I}_0^{(r)}$, $\mathbf{\Pi} \circ \mathbf{I}_0^{(r)}$, $\mathbf{I}_0^{(r)} \circ \mathbf{\Pi}$ and $\mathbf{\Pi} \circ \mathbf{I}_0^{(r)} \circ \mathbf{\Pi}$ by Proposition 2.3.9. The statement follows then by the isomorphisms $\mathbf{\Pi} \circ \mathbf{I}_1^{(r)} \simeq \mathbf{I}_0^{(r)} \circ \mathbf{\Pi}$ (Lemma 2.1.6) and $\mathbf{\Pi} \circ \mathbf{\Pi} = \mathbf{I}$. □

Theorem 2.3.5 is a powerful tool: whenever one makes a statement about an additive superfunctor, it is enough to prove it for the *indecomposable* superfunctors $\{\mathbf{I}_0^{(r)}, \mathbf{I}_1^{(r)}, \mathbf{I}_0^{(r)} \circ \mathbf{\Pi}, \mathbf{I}_1^{(r)} \circ \mathbf{\Pi}\}$ and to show that its validity is preserved by direct sums. For example we have the following easy fact.

Lemma 2.3.13. *Additive superfunctors are self-dual. In particular, for all additive A, B we have $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A, B) \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(B, A)$.*

Proof. $\mathbf{I}$ and $\mathbf{\Pi}$ are manifestly self-dual, while $\mathbf{I}_0^{(r)}$ and $\mathbf{I}_1^{(r)}$ by Lemma 2.1. Moreover, Kuhn dual commutes with the direct sum of superfunctors. The assertion follows then from Theorem 2.3.5. □

For deeper applications, we introduce the precomposition by an additive superfunctor.

2.3.2 Precomposition by a general additive superfunctor

We are in condition to define precomposition by a general additive (homogeneous) superfunctor A of degree strictly greater than 1. To explain that, we start by a more general definition that we will use all along the manuscript. Consider the category $\mathcal{V}^*$ of finite dimensional $\mathbb{Z}$-graded vector spaces and morphisms which preserve the gradings. If $E^* \in \mathcal{V}^*$ and $F \in \mathcal{P}$, one can define $F(E^*) \in \mathcal{V}^*$ in the following way [Tou12, §2.5].

Definition 2.3.14. Let E^* be a finite-dimensional graded vector space and $F \in \mathcal{P}$. We define $F(E^*)$ this way:

- $F(E^*) := F(E)$ as an ungraded vector space;
- To put the grading, let the multiplicative group $\mathbb{G}_m$ act on E^* with weight i on E^i. Then by functoriality there is an action of $\mathbb{G}_m$ on $F(E^*)$ and the latter decomposes in its weight spaces by semisimplicity. We let then $F(E^*)^i$ be the i-th weight space with respect to this action.

If $f : E^* \to E'^*$ is a morphism in $\mathcal{V}^*$, then Ff sends each $F(E^*)^i$ on $F(E'^*)^i$, hence it is a morphism in $\mathcal{V}^*$ too. In this way F becomes a linear functor $\Gamma^* \mathcal{V}^* \to \mathcal{V}^*$.

Example 2.3.15. Fix $d = \deg F$ and $E^* = \bigoplus_{i=0}^n \Bbbk e_i$ with $\deg(e_i) = \varepsilon_i$. Then $F(E^*) = \bigoplus_{\lambda\in\Lambda(n,d)} F^\lambda$ as in Definition 1.5.5. Set the degree of a composition to be $|\lambda| := \sum_{i=0}^n \varepsilon_i\lambda_i$. Then the $\mathbb{Z}$-grading of Definition 2.3.14 on $F(E^*)$ is induced by the degree of compositions:

$$F(E^*)^i = \bigoplus_{|\lambda|=i} F^\lambda \ .$$

This is very intuitive if one thinks of the special case $F = S^d$.

A super vector space can be seen as a $\mathbb{Z}$-graded vector space with V_0 in degree 0, V_1 in degree 1 and zero elsewhere. This gives an inclusion of categories $\boldsymbol{\mathcal{V}}_{ev} \subset \mathcal{V}^*$, as well as a functor $\mathcal{V}^* \to \boldsymbol{\mathcal{V}}_{ev}$ reducing gradings modulo 2. In particular we can restrict this construction to make F into a linear functor $\mathbf{\Gamma}^d\boldsymbol{\mathcal{V}}_{ev} \to \boldsymbol{\mathcal{V}}_{ev}$.

Now let A be an additive superfunctor of degree > 1. By Theorem 2.3.5, its degree is p^r for some $r > 0$ and it can be written as a direct sum of copies of $\mathbf{I}_0^{(r)}, \mathbf{I}_1^{(r)}$ and their precomposition by $\mathbf{\Pi}$. From this and (2.1.1) we deduce that the image of A is contained in $\boldsymbol{\mathcal{V}}_{ev}$. Hence the following even linear composition makes sense:

$$F \circ A : \mathbf{\Gamma}^{dp^r}\boldsymbol{\mathcal{V}} \to \mathbf{\Gamma}^d\boldsymbol{\mathcal{V}}_{ev} \xrightarrow{F} \boldsymbol{\mathcal{V}}_{ev} \subset \boldsymbol{\mathcal{V}}$$

where the first map is given by A at the object level and by

$$\mathbf{\Gamma}^{dp^r}\mathrm{Hom}(V,W) \subset \mathbf{\Gamma}^d(\mathbf{\Gamma}^{p^r}\mathrm{Hom}(V,W)) \xrightarrow{\mathbf{\Gamma}^{p^r}A} \mathbf{\Gamma}^d\mathrm{Hom}(A(V),A(W))$$

at the morphism level. Note that, by Remark 2.2.4, this composition consists of a sum of copies of $i_d^\vee$. Summing up, we have the following.

Proposition 2.3.16. *Let A be an additive superfunctor of degree p^r, $r > 0$. Then A restricts to a superlinear functor $\mathbf{\Gamma}^{p^r}\boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}_{ev}$. In particular, precomposition by A yields an exact functor*

$$- \circ A : \mathcal{P}_d \to (\boldsymbol{\mathcal{P}}_{dp^r})_{ev} \ . \tag{2.3.7}$$

We call Φ_A this precomposition functor. Explicitly, at the level of objects, $F(A(V))$ is computed by considering $A(V)$ as an ungraded space. Let then $\pi : V \to V$ be the automorphism $v \to (-1)^{\overline{v}}\, v$. Then the $\mathbb{Z}/2\mathbb{Z}$-grading on $F(A(V))$ is defined by setting in superdegree i the $(-1)^i$-eigenspace of the action of $F(\gamma_d(A\pi)) : F(A(V)) \to F(A(V))$. At the morphisms level, one just applies (2.1.1) to each indecomposable factor of A (possibly composing by $\mathbf{\Pi}$). In particular, using Theorem 2.3.5 one can generalise Proposition 2.2.8.

Proposition 2.3.17. *Let $F \in \mathcal{P}_d$ and let $A \in \boldsymbol{\mathcal{P}}_{p^r}$ be an additive superfunctor. If a morphism $f \in \mathbf{\Gamma}^{dp^r}\mathrm{Hom}(V,W)$ has parity > 0 (according to Definition 2.2.7), then $(F \circ A)(f) = 0$.*

Such property gives an easier access to many proofs. In first instance, we show that, for a fixed $F \in \mathcal{P}_d$, postcomposition by F yields a functor from the full subcategory of additive superfunctors to $\boldsymbol{\mathcal{P}}$.

Lemma 2.3.18. *Let $F \in \mathcal{P}_d$. Let $A, B \in \boldsymbol{\mathcal{P}}_{p^r}$ be additive and $T : A \to B$ be an even natural transformation. Then the collection of maps*

$$(F_T)_V := F(\gamma_d(T_V)) : F(A(V)) \to F(B(V))$$

defines an even natural transformation $F_T : F \circ A \to F \circ B$. This construction is compatible with composition of transformations.

Proof. Compatibility with composition is explicit by the defining formula, so our goal is to prove the naturality. In words, we have to prove the identity

$$(F_T)_W \circ (F \circ A)(f) = (F \circ B)(f) \circ (F_T)_V$$

for all $V, W \in \boldsymbol{\mathcal{V}}$ and all $f \in \mathbf{\Gamma}^{dp^r}\mathrm{Hom}(V, W)$. By Proposition 2.3.17 we can assume that f has parity 0, i.e. $f \in \Gamma^{dp^r}\mathrm{Hom}_0(V, W)$. By Theorem 1.1.5, we can write f as a sum of terms of the form $\gamma_{a_1}(f_1) \cdot ... \cdot \gamma_{a_s}(f_s)$ for some $s \geq 0$ and $f_1, ..., f_s \in \mathrm{Hom}_0(V, W)$. By Lemma 2.2.2 we can further suppose that the a_i-s are all powers of p, i.e. we can rewrite

$$f = \gamma_{p^{b_1}}(f_1) \cdot ... \cdot \gamma_{p^{b_s}}(f_s)$$

for some $b_1, ..., b_s \geq r$. Since A acts like a sum of $i_d^\vee$, we have

$$(F \circ A)(f) = F(\gamma_{p^{b_1 - r}}(Af_1) \cdot ... \cdot \gamma_{p^{b_s - r}}(Af_s))$$

whence we obtain by functoriality

$$\begin{aligned}(F_T)_W \circ (F \circ A)(f) =& F(\gamma_d(T_W)) \circ F(\gamma_{p^{b_1-r}}(Af_1) \cdot ... \cdot \gamma_{p^{b_s-r}}(Af_s)) \\ & F(\gamma_d(T_W) \circ \left[\gamma_{p^{b_1-r}}(Af_1) \cdot ... \cdot \gamma_{p^{b_s-r}}(Af_s)\right]) .\end{aligned}$$

Since T is natural, $T_W \circ Ag = Bg \circ T_V$ for any morphism g. Therefore, using Remark 1.5.2, the last member of the equation is equal to

$$\begin{aligned}&F(\left[\gamma_{p^{b_1-r}}(Bf_1) \cdot ... \cdot \gamma_{p^{b_s-r}}(Bf_s)\right] \circ \gamma_d(T_V)) \\ &F(\gamma_{p^{b_1-r}}(Bf_1) \cdot ... \cdot \gamma_{p^{b_s-r}}(Bf_s)) \circ F(\gamma_d(T_V)) = (F \circ B)(f) \circ (F_T)_V\end{aligned}$$

which ends the proof.

□

An immediate and interesting application of Lemma 2.3.18 is the following generalisation of [Dru16, Lemma 2.7.1].

Lemma 2.3.19. *Let $A \subset B$ be an inclusion of additive superfunctors and let $F \in \mathcal{P}$. Then $F \circ A$ is a direct summand of $F \circ B$.*

Proof. Denote by $i : A \to B$ the natural even inclusion which we have by hypothesis. By Theorem 2.3.5, A is necessarily a direct summand of B, so there exists a natural transformation $p : B \to A$ such that $p \circ i = \mathrm{Id}_A$. By Lemma 2.3.18 there exist F_i, F_p such that $F \circ A \xrightarrow{F_i} F \circ B \xrightarrow{F_p} F \circ A$ is the identity. Hence $F \circ A$ is a direct summand of $F \circ B$. □

Proposition 2.3.20. *For all nonzero additive A, the functor Φ_A is faithful.*

Proof. By Theorem 2.2.9 and 2.2.10, Φ_A is fully faithful when A is $\mathbf{I}_0^{(r)}$, $\mathbf{I}_1^{(r)}$,$\mathbf{I}_0^{(r)} \circ \mathbf{\Pi}$ or $\mathbf{I}_1^{(r)} \circ \mathbf{\Pi}$ (since $(-) \circ \mathbf{\Pi}$ is an equivalence of categories). Otherwise, choose an additive indecomposable subfunctor $A' \subset A$, which exists in view of Theorem 2.3.5. By Lemma 2.3.19, $F \circ A'$ is a direct summand of $F \circ A$ for any $F \in \mathcal{P}$. Since A' is indecomposable, one can form the composition

$$\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ A) \to \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F \circ A', G \circ A') \to \mathrm{Hom}_{\mathcal{P}}(F, G)$$

where the first arrow is induced by the inclusion of $F \circ A'$ and $G \circ A'$ as direct summands, while the second one is the inverse of $\Phi_{A'}$ which exists by the first part of the proof. If we show that this arrow is a left inverse of Φ_A at the Hom level, we are done. Let then $T \in \mathrm{Hom}_{\mathcal{P}}(F, G)$, and write for short $T_A := \Phi_A(T) \in \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ A)$. Denote by $i : A' \to A$ the inclusion and $p : A \to A'$ the projection, so that $p \circ i = \mathrm{Id}_{A'}$. With this notation, what we have to prove is that $G_p \circ T_A \circ F_i = T_{A'}$. But this follows immediately by the identity $T_A \circ F_i = G_i \circ T_{A'}$ that we have by naturality of T. □

Remark 2.3.21. If A is decomposable, it is easy for Φ_A to be faithful but not fully faithful. For example $\mathrm{Hom}_{\mathcal{P}}(I,I) \simeq \Bbbk$ while $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\mathbf{I}^{(1)},\mathbf{I}^{(1)}) \simeq \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(1)},\mathbf{I}_0^{(1)}) \oplus \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_1^{(1)},\mathbf{I}_1^{(1)}) \simeq \Bbbk^2$.

Remark 2.3.22. If we change the source from $\mathcal{P}$ to $\boldsymbol{\mathcal{P}}$, the precomposition $\boldsymbol{\Phi_r} := (-)\circ \mathbf{I}_0^{(r)} : \boldsymbol{\mathcal{P}} \to \boldsymbol{\mathcal{P}}$ fails even to be faithful. A counterexample is provided by $F = \mathbf{I}_1^{(r)}$: indeed, $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F,F) \neq 0$ but $\boldsymbol{\Phi_r}(F) = 0$.

We end the section by generalising Theorem 2.2.13.

Corollary 2.3.23. *Let $A \in \boldsymbol{\mathcal{P}}_{p^r}$ additive and let $F \in \boldsymbol{\mathcal{P}}_s$, $G \in \boldsymbol{\mathcal{P}}_t$ such that $s+t = dp^r$. Then*

- $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F \otimes G, S^d_V \circ A) = 0$ *if s,t are not divisible by p^r.*
- *If $s = p^r s'$ and $t = p^r t'$ there is a natural isomorphism induced by cup product*

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F \otimes G, S^d_V \circ A) \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F, S^{s'}_V \circ A) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(G, S^{t'}_V \circ A)\,.$$

Proof. We know that the statement is true when $A = \mathbf{I}_0^{(r)}, \mathbf{I}_1^{(r)}$ (Theorem 2.2.13). It follows for the two other indecomposables by moving the $\mathbf{\Pi}$ from one argument to the other in the Ext. By Theorem 2.3.5, we can proceed inductively and write $A = A' \oplus A''$ with A', A'' additive superfunctors on which the corollary is true. As in the proof of Theorem 2.2.13, we can reduce to the case where F and G are projective, say $F = \mathbf{\Gamma}^{s,W}, G = \mathbf{\Gamma}^{t,Z}$, so that we only have a Hom isomorphism to prove. By the Weighted Yoneda lemma and additivity of A', A'' we have

$$\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F \otimes G, S^d_V \circ (A' \oplus A'')) \simeq [S^d_V(A'(W) \oplus A'(Z) \oplus A''(W) \oplus A''(Z))]^{(s,t)}$$

where (s,t) indicates the 2-composition of d. Since A' and A'' satisfy the statements by hypothesis, s and t must be multiple of p^r for this weight space to be nonzero. If this is the case, the weight space coincides with

$$S^{s'}_V(A'(W) \oplus A''(W)) \otimes S^{t'}_V(A'(Z) \oplus A''(Z))$$

which is isomorphic via multiplication on S^* to

$$\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F, S^{s'}_V \circ (A' \oplus A'')) \otimes \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(G, S^{t'}_V \circ (A' \oplus A''))$$

again by Yoneda lemma and additivity of A', A''. $\square$

Corollary 2.3.24. *Let $X = \Gamma$ or S. Cup product defines a natural isomorphism*

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(X^\lambda \circ A, S^d_V \circ B) \simeq \bigotimes_i \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(X^{\lambda_i} \circ A, S^{\lambda_i}_V \circ B)$$

for all additive superfunctors A and B.

2.3.3 The adjoints of Φ_A

We devote this section to the explicit computation of the adjoints of Φ_A. For our purposes, we focus mainly on the right adjoint.

Convention 2.3.25. Although we defined the precomposition functor Φ_A with target in the category $\boldsymbol{\mathcal{P}}_{ev}$, from now we will consider it as a functor $\mathcal{P} \to \boldsymbol{\mathcal{P}}$ by means of the inclusion $\boldsymbol{\mathcal{P}}_{ev} \subset \boldsymbol{\mathcal{P}}$. Remember that the adjoint of the such inclusion is $H \to H \oplus (\mathbf{\Pi} \circ H)$ (cf. the end of Section 1.4) which allows to retrieve the even and odd parts in the computations. This is similar to the discussion on Ext-spaces in $\boldsymbol{\mathcal{P}}$ in §1.6. For reasons that will be clear later, we make such slightly counter-intuitive but equivalent choice for practical reasons that will appear later (together with a *vademecum* in Section 2.3.4 to read the superdegrees).

In order to compute the adjoint, we are going to make use of Theorems 1.5.12 and 1.1.14 which realise $\boldsymbol{\mathcal{P}}$ and $\mathcal{P}$ as categories of (super)modules. To be short, just in this section we will call them *equivalence theorems*. Working in a category of (super)modules is very convenient when it comes to compute adjoints. So, from now on, let $p^r = \deg A$ and fix $n, m \geq dp^r$. By direct inspection, Φ_A corresponds via the equivalence theorems to the functor

$$S(n,d) - \mathsf{mod} \longrightarrow S(m|n, dp^r) - \mathsf{smod}$$

$$M \longmapsto \Gamma^{d,\Bbbk^n}(A(\Bbbk^{m|n})) \otimes_{S(n,d)} M$$

whose right adjoint is given by a well-known general formula

$$S(m|n, dp^r) - \mathsf{smod} \longrightarrow S(n,d) - \mathsf{mod}$$

$$N \longmapsto \mathrm{Hom}_{S(m|n,dp^r)}(\Gamma^{d,\Bbbk^n}(A(\Bbbk^{m|n})), N) \ .$$

Going back up again through the equivalence theorems, we arrive to an expression of the right adjoint of (2.3.7):

$$\begin{gathered}\boldsymbol{\mathcal{P}}_{dp^r} \longrightarrow \mathcal{P}_d \\ G \longmapsto \Gamma^{d,\Bbbk^n} \otimes_{S(n,d)} \mathrm{Hom}_{S(m|n,dp^r)}(\Gamma^{d,\Bbbk^n}(A(\Bbbk^{m|n})), G(\Bbbk^{n|m})) \ .\end{gathered} \tag{2.3.8}$$

This last formula is quite weighty, but we can relieve it. Indeed, by the proof of Theorem 1.1.14, for any functor $H \in \mathcal{P}_d$ with $d \leq n$ the canonical map

$$\Gamma^{d,\Bbbk^n} \otimes_{S(n,d)} H(\Bbbk^n) \longrightarrow H$$

is an isomorphism of functors. Moreover, by the same theorem, the evaluation functor is fully faithful. Thanks to these two facts we can rewrite

$$\begin{aligned}\Gamma^{d,\Bbbk^n} \otimes_{S(n,d)} &\mathrm{Hom}_{S(m|n,dp^r)}(\Gamma^{d,\Bbbk^n}(A(\Bbbk^{m|n})), G(\Bbbk^{n|m})) \\ &\simeq \Gamma^{d,\Bbbk^n} \otimes_{S(n,d)} \mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(\Gamma^{d,\Bbbk^n} \circ A,\ G) \\ &\simeq \mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(\Gamma^{d,-} \circ A,\ G)\end{aligned}$$

which gives the computation we wanted:

Proposition 2.3.26. *The right adjoint $\rho_A : \boldsymbol{\mathcal{P}}_{dp^r} \longrightarrow \mathcal{P}_d$ of the precomposition functor Φ_A is given by*

$$\rho_A(H) := \mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(\Gamma^{d,-} \circ A,\ H) \ .$$

As a corollary we obtain the left adjoint too.

Corollary 2.3.27. *The left adjoint of Φ_A is given by*

$$s_A(F) := \mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(S^d_- \circ A, F)^\# \ .$$

Proof. By the properties of Kuhn dual (§1.5.3), we have

$$\mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(F, G \circ A) \simeq \mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(G^\# \circ A^\#, F^\#)$$

and the latter, by Proposition 2.3.26, is isomorphic to

$$\mathrm{Hom}_{\mathcal{P}_d}(G^\#, \rho_{A^\#}(F^\#)) = \mathrm{Hom}_{\mathcal{P}_d}(G^\#, \mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(\Gamma^{d,-} \circ A^\#, F^\#)) \ .$$

Since $\Gamma^{d,V}$ and S^d_V are (for any V) the Kuhn dual of each other, this is naturally isomorphic to $\mathrm{Hom}_{\mathcal{P}_d}(\mathrm{Hom}_{\boldsymbol{\mathcal{P}}_{dp^r}}(S^d_- \circ A, F)^\#, G)$ as we had to show. □

Remark 2.3.28. One should recall that the Hom space in the formula of Proposition 2.3.26 is to be seen as a classical vector space by forgetting the $\mathbb{Z}/2\mathbb{Z}$-grading (remember that it is supposed to define a *classical* polynomial functor).

When A is additive, we are able to compute the explicit value of ρ_A on an important collection of superfunctors. For that, we need to introduce the concept of parametrisation.

Definition 2.3.29 (Parametrised functor)**.** Let E^* be a $\mathbb{Z}$-graded vector space, finite-dimensional in each degree, and let $F \in \mathcal{P}_d$.

1. If E^* is finite-dimensional, we define a graded functor $F_{E^*} \in \mathcal{P}_d$ by $F_{E^*}(V) := F(E^* \otimes V)$ with the grading introduced in Definition 2.3.14.

2. If E^* is infinite-dimensional, we define the *lower parametrisation* F_{E^*} by
$$F_{E^*} = \underset{E'^* \subset E^*}{colim}\, F_{E'^*} \ ,$$
where the colimit is taken over the poset of finite dimensional graded vector subspaces E'^* of E^*, ordered by inclusion.

3. The *upper parametrisation* of F by E^* is defined as
$$F^{E^*} := F_{(E^\vee)^*}$$
where $(E^\vee)^*$ denotes as usual the restricted dual of E^*.

The lower parametrisation will be our default one, so that we drop the adjective *lower*. It is clear that there is no ambiguity in the definition, since if E^* has finite dimension then the poset $E'^* \subset E^*$ has E^* as final object, so the colimit equals the first definition of F_{E^*}. Write $\mathcal{P}_d^*$ for the category of $\mathbb{Z}$-graded d-homogeneous strict polynomial functors. Then $F_{E^*}, F^{E^*} \in \mathcal{P}_d^*$ and they have the following properties.

Proposition 2.3.30. *1. Lower and upper parametrisations by E^* define exact functors $\mathcal{P}_d \to \mathcal{P}_d^*$. Moreover, both operations are natural with respect to E^*.*

2. Assume that E^* is degreewise finite-dimensional and zero in negative degrees. For all degrees i, let $E^{*\leq i}$ be the graded subspace of E^* which is equal to E^* in degrees less or equal to i and zero in degrees higher than i. Then the inclusion $E^{*\leq i} \hookrightarrow E^*$ induces a monomorphism $F_{E^{*\leq i}} \to F_{E^*}$ which is an isomorphism in degrees less or equal to i.

3. Make the same hypothesis on E^* as in point (2). Then there are bigraded isomorphisms, natural in F, G and E^*:
$$\mathrm{Ext}^*_{\mathcal{P}}(F^{E^*}, G) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E^*}) \ .$$

Proof. (1) follows from [Tou12, Lemma 2.8] and exactness of filtered colimits.

To prove (2), the graded monomorphism $\tau_i : F_{E^{*\leq i}} \hookrightarrow F_{E^*}$ is simply the one induced by functoriality by the canonical inclusion $E^{*\leq i} \hookrightarrow E^*$. Moreover, the hypothesis implies that each $E^{*\leq i}$ has finite total dimension. Since any finite-dimensional subspace of E^* is included in $E^{*\leq n}$ for some n, F_{E^*} is equal to the colimit of the chain of inclusions $F_{E^{*\leq 0}} \subset F_{E^{*\leq 1}} \subset \cdots \subset F_{E^{*\leq n}} \subset \ldots$, i.e. to the quotient of $\bigoplus_{d\geq 0} F_{E^{*\leq d}}$ obtained by identifying all components of the same degree. In particular, the components of degree less than i have all representants in $F_{E^{*\leq i}}$. Then the source and the target of τ_i have the same dimension in degrees less than i, making τ_i an isomorphism in that case.

Let us prove (3). By point (2) it is sufficient to provide bigraded isomorphisms
$$\mathrm{Ext}^*_{\mathcal{P}}(F^{E^{*\leq i}}, G) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_{*\leq i}})$$

for all $i \geq 0$. In other words, we may suppose that E^* is of finite total dimension and use the explicit description of point (1). Denote by $V^\vee = \mathrm{Hom}(V, \Bbbk)$ the dual of a (possibly graded) vector space. One has

$$\begin{aligned}(\Gamma^{d,V})^{E^*}(W) &= \Gamma^{d,V}((E^*)^\vee \otimes W) = \Gamma^d \mathrm{Hom}(V, (E^*)^\vee \otimes W) \\ &\simeq \Gamma^d \mathrm{Hom}(E^* \otimes V, W) = \Gamma^{d, E^* \otimes V}(W)\end{aligned}$$

giving an isomorphism $(\Gamma^{d,V})^{E^*} \simeq \Gamma^{d, E^* \otimes V}$ natural in V and E^*. In particular, in each degree, $(\Gamma^{d,V})^{E^*}$ is a projective object in $\mathcal{P}_d$. A double application of Yoneda lemma gives then for each G

$$\mathrm{Hom}_{\mathcal{P}}((\Gamma^{d,V})^{E^*}, G) \simeq G(E^* \otimes V) \simeq \mathrm{Hom}_{\mathcal{P}}(\Gamma^{d,V}, G_{E^*})$$

which respects the gradings by [FS97, Cor. 2.12]. By projectivity of $\Gamma^{d,V}$ and $(\Gamma^{d,V})^{E^*}$, the isomorphism on the Ext^* follows at once for $F = \Gamma^{d,V}$. For a general F, take a projective resolution $P_* \to F$. By exactness of the parametrisation and by the fact that each P_n is a sum of $\Gamma^{d,V}$, $(P_*)^{E^*}$ is a projective resolution of F^{E^*} degreewise. Then from the previous case we deduce

$$\mathrm{Ext}^*_{\mathcal{P}}(F^{E^*}, G) \simeq H^* \mathrm{Hom}_{\mathcal{P}}((P_*)^{E^*}, G) \simeq H^* \mathrm{Hom}_{\mathcal{P}}(P_*, G_{E^*}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E^*})$$

as we had to show. □

We will mostly use the lower parametrisation, also in force of Proposition 2.3.30. We will then suppress the adjective *lower* if it creates no confusion. In order to state many important results of the manuscript, starting by the following, we have to enlarge our definition of parametrised functor.

Definition 2.3.31. Let E^* be a $\mathbb{Z}$-graded *super* vector space, finite-dimensional in each degree, and let $F \in \mathcal{P}_d$. The parametrised functor F_{E^*} is defined via Definition 2.3.29 by forgetting the super-grading on E^*.

Lemma 2.3.32. *Let $A, B \in \boldsymbol{\mathcal{P}}_{p^r}$ be additive and $G \in \mathcal{P}_d$. Then there is an isomorphism of polynomial functors, natural in all variables:*

$$\rho_A(G \circ B) \simeq G_{\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A,B)} \ .$$

Proof. Consider the even natural transformation $\eta_{A,B}$ given by

$$\begin{aligned}\eta_{A,B}(V) : \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A,B) \otimes A(V) &\to B(V) \\ T \otimes v &\longmapsto T_V(v).\end{aligned}$$

and apply Lemma 2.3.18 to obtain an even natural transformation

$$G_\eta = G_{\eta_{A,B}} : G_{\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A,B)} \circ A \longrightarrow G \circ B \ .$$

Define then $\Theta_{A,B}$ to be the composite

$$\begin{tikzcd} G_{\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A,B)} \simeq \mathrm{Hom}_{\mathcal{P}}(\Gamma^{d,-}, G_{\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A,B)}) \arrow[r, "\circ A"] & \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\Gamma^{d,-} \circ A, G_{\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A,B)} \circ A) \arrow[d, "(G_\eta)_*"] \\ & \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\Gamma^{d,-} \circ A, G \circ B) \arrow[d, equal] \\ & \rho_A(G \circ B). \end{tikzcd}$$

Our goal is to prove that $\Theta_{A,B}$ is an isomorphism. Since $\rho_A(- \circ B)$ is left exact, it is enough to prove that for G injective. First we handle the case where A and B are indecomposable additives. Note that in that case $\eta_{A,B}$ is either an isomorphism or zero. We prove this claim examining case by case, starting from the pairs (A, B) such that $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A, B) = 0$. For these pairs $\eta_{A,B} = 0$, thus $\Theta_{A,B} = 0$. For $\Theta_{A,B}$ to be an isomorphism, we have then to prove that $\rho_A(G \circ B) = 0$. Let then $\ell \in \{0, 1\}$ and $m = \ell - 1$.

$A = \mathbf{I}_\ell^{(r)}, B = \mathbf{I}_m^{(r)}$: we know by [DK22, Thm 5.1.2] that $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\Gamma_\ell^{d(r)}, S_m^{d(r)}) = 0$. By a double application of Corollary 2.2.14 we can deduce that $\rho_A(S_V^d \circ B) = \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\Gamma^{d,V} \circ A, S_V^d \circ B) = 0$ for all V.

$A = \mathbf{I}_\ell^{(r)} \circ \mathbf{\Pi}, B = \mathbf{I}_m^{(r)} \circ \mathbf{\Pi}$: follows by the previous point and fully faithfulness of $- \circ \mathbf{\Pi}$.

$A = \mathbf{I}_\ell^{(r)}, B = \mathbf{\Pi} \circ \mathbf{I}_m^{(r)}$: by Lemma 2.1.6 and (2.2.4) (tensoring with a purely even space V changes nothing)

$$S_V^d \circ B = S_V^d \circ \mathbf{\Pi} \circ \mathbf{I}_m^{(r)} \simeq (S_V^d)_\ell^{(r)} \circ \mathbf{\Pi}$$

is isomorphic to $\mathbf{\Pi} \circ (S_V^d)_m^{(r)}$ or $(S_V^d)_m^{(r)}$ depending on the parity of d. So, up to a parity change, $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\Gamma^d \circ A, S_V^d \circ B) \simeq \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\Gamma_\ell^{d(r)}, (S_V^d)_m^{(r)})$ which is zero by the previous point. This means that $\rho(G \circ B) = 0$ whenever G is injective.

We pass to other pairs (A, B), for which $\eta_{A,B}$ is an isomorphism.

$A = B = \mathbf{I}_\ell^{(r)}$: then $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A, B) \simeq \Bbbk$ and $\Theta_{A,B} = \mathit{\Phi}_r$ or $\overline{\mathit{\Phi}_r}$, which are isomorphisms by Theorem 2.2.9 and 2.2.10.

$A = B = \mathbf{I}_\ell^{(r)} \circ \mathbf{\Pi}$: follows as above by fully faithfulness of $- \circ \mathbf{\Pi}$.

$A = \mathbf{I}_\ell^{(r)}, B = \mathbf{\Pi} \circ \mathbf{I}_\ell^{(r)}$: as in the previous point, $S_V^d \circ B \simeq (S_V^d)_m^{(r)}$ if d is even and $\mathbf{\Pi} \circ (S_V^d)_m^{(r)}$ if d is odd. In both cases, since $\rho_A(S_V^d \circ B)$ is viewed as un ungraded space, $\rho_A(S_V^d \circ B) \simeq \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}((\Gamma^{d,-})_\ell^{(r)}, (S_V^d)_\ell^{(r)})$ which is naturally isomorphic to S_V^d by the first point of the proof. Since $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A, B) \simeq \Pi\Bbbk$ and by definition $(S_V^d)_{\Pi\Bbbk} = S_V^d$, we conclude that $\Theta_{A,B}$ is an isomorphism for all G injective.

We now pass to the case where only A is indecomposable. By making induction on the number of indecomposable summands of B, there exists two additives B', B'' such that $B \simeq B' \oplus B''$ and such that $\Theta_{A,B'}, \Theta_{A,B''}$ are isomorphisms. We have then to show that $\Theta_{A,B' \oplus B''}$ is an isomorphism too. Since source and target are left exact with respect to G, we can restrict to the special case $G = S_V^d$. For the first case, $\Theta_{A,B \oplus B'}$ decomposes as

$$\begin{array}{c}
(S_V^d)_{\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A, B' \oplus B'')} \\
\Big\downarrow \simeq \\
\bigoplus_{a=0}^{d} (S_V^{d-a})_{\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A,B')} \otimes (S_V^a)_{\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A,B'')} \\
\Big\downarrow \simeq \\
\bigoplus_{a=0}^{d} \mathrm{Hom}_{\mathcal{P}}(\Gamma^{d-a,-}, (S_V^{d-a})_{\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A,B')}) \otimes \mathrm{Hom}_{\mathcal{P}}(\Gamma^{a,-}, (S_V^a)_{\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A,B'')}) \\
\Big\downarrow \sum (\eta_{A,B'} \circ \mathit{\Phi}_A) \otimes (\eta_{A,B''} \circ \mathit{\Phi}_A) \\
\bigoplus_{a=0}^{d} \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\Gamma^{d-a,-} \circ A, S_V^{d-a} \circ B') \otimes \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\Gamma^{a,-} \circ A, S_V^a \circ B'') \\
\Big\downarrow \cup \\
\bigoplus_{a=0}^{d} \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\Gamma^{d,-} \circ A, S_V^{d-a} \circ B' \otimes S_V^d \circ B'') \\
\Big\downarrow \simeq \\
\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\Gamma^{d,-} \circ A, S_V^d \circ (B' \oplus B'')) .
\end{array}$$

The cup product map is an isomorphism by Corollary 2.3.23 and the middle map is an isomorphism by the first part of the proof. Theorem 2.3.5 implies then that $\Theta_{A,B}$ is an isomorphism whenever A is indecomposable. A symmetric argument, using again Corollary 2.3.23 and the exponentiality of Γ^*, completes the proof that $\Theta_{A,B}$ is an isomorphism for all A, B additive. □

The isomorphism we just proved implies by definition the following one between Hom spaces. It is worth remarking that it is in fact the degree-zero case of an Ext-isomorphism that we will state in Chapter 4 (Conjecture 4.1.5).

Theorem 2.3.33. *Let A, B be additive superfunctors. There is an isomorphism, natural in F, G:*

$$\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ B) \simeq \mathrm{Hom}_{\mathcal{P}}(F, G_{\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A,B)}) \,.$$

2.3.4 How to read superdegrees in classical formulas

We end the section with a short discussion. One may protest because Definition 2.3.31 deliberately loses the information about the superdegrees of the parameter. By consequence, the one of Theorem 2.3.33 is itself just an isomorphism of ungraded spaces, since the right-hand side is *a priori* ungraded. This is harmless for the dimensional computations we are going to perform; it just may not be clear how to recover the superdegrees from a "classical" term like the right-hand side. There is, actually, a $\mathbb{Z}/2\mathbb{Z}$-grading on a space of the form $\mathrm{Hom}_{\mathcal{P}}(F, G_E)$ which does the job. Recall the parity automorphism $\pi_V : V \to V$, $v \mapsto (-1)^{\overline{v}}\, v$.

Definition 2.3.34 (Supergrading on $G(E)$)**.** Let $G \in \mathcal{P}_d$ and let E be a (possibly $\mathbb{Z}$-graded) *super* vector space. The supergrading on $G(E)$ is defined by declaring the even (resp. odd) subspace to be the $+1$ (resp. -1) eigenspace for the action of $G(\gamma_d(\pi)) : G(E) \to G(E)$.

It is important to remark that this definition is just formal and does not yield a polynomial superfunctor, an obstruction being made by the odd morphisms in $\boldsymbol{\Gamma}^d\boldsymbol{\mathcal{V}}$ (unless $E = V^{(r)}$ for some V, which is the definition of $G^{(r)}$ given at the beginning of Section 2.3.2). Hence this is not in conflict with the $G_E \in \mathcal{P}$ of Definition 2.3.31. Whenever we write G_E, we refer to the latter polynomial functor.

Remark 2.3.35. If E is purely even, then so is $G(E)$. If E is purely odd, then $G(E)$ is concentrated in degree $\deg G \mod 2$.

The Definition 2.3.34 induces the following one.

Definition 2.3.36. A natural transformation $T : F \to G_E$ is declared even (resp. odd) if, for all $V \in \mathcal{V}$, the map T_V has image in the even (resp. odd) subspace of $G(E^* \otimes V)$. In that case we baptise $\overline{T} = 0$ or 1 respectively.

Equivalently, $\overline{T} = i$ if and only if the diagram

$$\begin{tikzcd} F \arrow[r, "T"] \arrow[dr, "T"'] & G_E \arrow[d, "G\left(\gamma_d(\pi\otimes -)\right)"] \\ & G_E \end{tikzcd}$$

commutes up to a sign $(-1)^i$. If $F = \Gamma^d$ we retrieve the supergrading of Definition 2.3.34. The strong link between this "artificial" supergrading and the real one is the following.

Lemma 2.3.37. *Let E be a $\mathbb{Z}$-graded super vector space and $F, G \in \mathcal{P}_d$. Let $A \in \boldsymbol{\mathcal{P}}$ be additive. Then the graded natural map $\mathrm{Hom}_{\mathcal{P}}(F, G_E) \to \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F \circ A, G_E \circ A)$ given by precomposition is an even map of super vector spaces, as soon as the left-hand side is $\mathbb{Z}/2\mathbb{Z}$-graded according to Definition 2.3.36.*

Proof. Recall that $(T \circ A)_V = T_{AV} : F(AV) \to G(E \otimes AV)$ with AV considered without superdegrees. Set by $\overline{T} \in \{0, 1\}$ the super degree of Definition 2.3.34. Let $x \in F(AV)$. Remark that Definition 2.3.34 can be summed up in the implicit relation $F(A(\pi_V))(x) = (-1)^{\overline{x}}\, x$ for a homogeneous $x \in F(AV)$ (we suppress the γ_d everywhere for briefness). We then have to show that

$$\big[G(\gamma_d(\pi_{E\otimes AV})) \circ T_{AV}\big](x) = (-1)^{\overline{T}+\overline{x}}\, T_{AV}(x)$$

since that will imply $\overline{T \circ A} = \overline{T}$ as desired. Consider the following diagram:

$$\begin{tikzcd}
 & G(E\otimes AV) \arrow[d, "G(\pi_E\otimes 1)"] \arrow[dr, "G(\pi_{E\otimes AV})"] & \\
F(AV) \arrow[ur, "T_{AV}"] \arrow[r, "T_{AV}"] \arrow[dr, "F(A(\pi_V))"'] & G(E\otimes AV) \arrow[r, "G(1\otimes\pi_{AV})"] & G(E\otimes AV) \\
 & F(AV) \arrow[ur, "T_{AV}"'] &
\end{tikzcd}$$

The inferior triangle is commutative because T is a natural transformation. The up-left triangle commutes up to a sign $(-1)^{\overline{T}}$ by definition. The up-right is trivially commutative. Hence

$$\begin{aligned}\big[G(\gamma_d(\pi_{E\otimes AV})) \circ T_{AV}\big](x) &= (-1)^{\overline{T}}\,\big[T_{AV} \circ F(A(\pi_V))\big](x)\\ &= (-1)^{\overline{T}+\overline{x}}\, T_{AV}(x)\end{aligned}$$

which is what we needed. □

Thanks to this lemma, the map $\Theta_{A,B}(V)$ in the proof of Lemma 2.3.32 is an even map of super vector spaces. Hence we can update the statement by saying that the isomorphism $\rho_A(G \circ B) \simeq G_{\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(A,B)}$ respects the superdegrees.

There is an analogous supergrading for $\mathrm{Ext}^*_{\mathcal{P}}(F, G_E)$ that is induced by the one for Hom via passage to the derived functors. In particular:

- If E^* is purely even, then so is $\mathrm{Ext}^*_{\mathcal{P}}(F, G_E)$. This concerns the majority of cases we are interested in. We will see indeed that most of the time E^* will be a subspace of $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}^{(r)}, \mathbf{I}^{(r)})$.
- If E^* is purely odd, then $\mathrm{Ext}^*_{\mathcal{P}}(F, G_E)$ is concentrated in superdegree $\deg F \mod 2$. This is for example the case when E^* is (a subspace of) $\mathrm{Ext}^*(\mathbf{I}^{(r)}, \mathbf{I}^{(r)} \circ \mathbf{\Pi})$. Note that, when in presence of a Frobenius twist precomposed by $\mathbf{\Pi}$, one can workaround by using Proposition 2.2.6. This gives a result that is in accord with this last definition.

2.4 Exponential superfunctors and super Troesch complex

In this section (which is an excursus and may be skipped with no consequence for the comprehension of the next chapters) we provide a first good application of the classification of additive superfunctors. Namely, we study the basic properties of superfunctors equipped with an exponential structure. A good feature of such objects is that, when they form complexes via a compatible differential, the exponential structure passes to cohomology, which makes its computation much more accessible. In classical literature, such complexes happen to provide explicit resolutions of twisted functors. Indeed, a construction which was performed first [LSF94] in characteristic 2 and then generalised by Troesch [Tro05] in odd characteristic provides an injective coresolution of the twisted symmetric power algebra $S^{*(1)}$. This coresolution has turned out to be very handy for computations: for example, Touzé used it in [Tou12] to compute spaces $\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, S^{\mu(r)})$ for all polynomial functors F and all compositions μ.

Our main goal is to investigate the cohomology of the "superized" version of Troesch complex, which was introduced and studied by Drupieski and Kujawa [DK22]. Thanks to the theory of exponential superfunctors and our Theorem 2.3.5, we are able to perform a special case of their computations with nearly no effort. Namely, we prove that the cohomology of the super Troesch complex is isomorphic in a graded way to $\mathbf{S}^{*(1)}$. Unfortunately, unlike the classical case, this does not yield a coresolution of $S_0^{*(r)}$ as hoped, because many more pieces of cohomology appear in positive degrees. It turns out to be quite difficult to get rid of them and build an explicit coresolution. The direct computation of $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, S_0^{\mu(r)})$ seems then still inaccessible, which is the reason why in Chapter 4 we switch to different methods.

Although exponential functors can be treated in a more general context [Tou19], we will focus for our purposes only on polynomial superfunctors all along. In particular, one can retrieve all the definitions and statements for classical polynomial functors by restricting to the subcategory $\boldsymbol{\mathcal{V}}_0 \simeq \mathcal{V}$.

2.4.1 Exponential superfunctors

Definition 2.4.1. An *exponential superfunctor* is a triple (F, e, u) such that $F = \bigoplus_d F^d \in \boldsymbol{\mathcal{P}}$ is a graded polynomial superfunctor and e, u are natural isomorphisms

$$e_{V,W} : F(V \oplus W) \simeq F(V) \otimes F(W)$$
$$u : \Bbbk \simeq F(0)$$

that are associative and unital.

By the evident analogy with the property of "taking sums into products", we refer to $e_{V,W}$ as the *exponential isomorphism.*

Remark 2.4.2. If F, G are two exponential superfunctors, then so is their tensor product $F \otimes G$. If F is exponential and H is additive, then $F \circ H$ is exponential.

Remark 2.4.3. Let $F := \bigoplus_d F^d$ an exponential superfunctor. For reasons that will be clear later on, we call F^d the component of *weight* d of F, instead of "degree" d. The exponential isomorphism restricts then to

$$F^d(V \oplus W) \simeq \bigoplus_{a+b=d} F^a(V) \otimes F^b(W) \ . \tag{2.4.1}$$

Example 2.4.4. If A is a $\boldsymbol{\mathcal{P}}$-algebra[1] satisfying some additional properties (which we will speak about shortly), its grading as an exponential superfunctor is the same as the polynomial grading, i.e. $A = \bigoplus_d A^d$ with $A^d \in \boldsymbol{\mathcal{P}}_d$.

Definition 2.4.5. Let (F, e, u), (G, e', u') be two exponential superfunctors. An *exponential morphism* is a natural transformation $T \in \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F, G)$ which commutes with the exponential structure, i.e. such that the following diagrams commute:

$$\begin{tikzcd} F(V \oplus W) \arrow[r, "T_{V\oplus W}"] \arrow[d, "e_{V,W}"'] & G(V \oplus W) \arrow[d, "e'_{V,W}"] \\ F(V) \otimes F(W) \arrow[r, "T_V \otimes T_W"] & G(V) \otimes G(W) \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \arrow[r, "u"] \arrow[dr, "u'"'] & F(0) \arrow[d, "T_0"] \\ & G(0) \end{tikzcd}$$

We denote then by $\boldsymbol{\mathcal{P}}_{exp}$ the category formed by exponential superfunctors and exponential morphisms.

[1] Recall the definition in §1.5.4.

Let $\sigma_V : V \oplus V \to V$ be the sum map. One can easily turn an exponential superfunctor (F, e, u) into a $\boldsymbol{\mathcal{P}}$-algebra (F, μ, η) by setting for each V multiplication and unit

$$\mu_V : F(V) \otimes F(V) \xrightarrow{e_{V,V}} F(V \oplus V) \xrightarrow{F(\sigma_V)} F(V)$$
$$\eta_V : \Bbbk \xrightarrow{u} F(0) \xrightarrow{F(0)} F(V)\ .$$

The definitions imply that (F, μ, η) satisfies the following properties:

(I) $\eta_0 : \Bbbk \to F(0)$ is an isomorphism.

(II) For all V, W the following composition is an isomorphism (where i_V, i_W are the inclusions of V and W in $V \oplus W$):

$$F(V) \otimes F(W) \xrightarrow{F(i_V)\otimes F(i_W)} F(V \oplus W)^{\otimes 2} \xrightarrow{\mu_{V\oplus W}} F(V \oplus W)\ .$$

Vice-versa, as stated in the next proposition, a $\boldsymbol{\mathcal{P}}$-algebra with these properties is always built like that by an exponential superfunctor. This fact is a troubleless generalisation of [Tou19, Lemma 2.7].

Proposition 2.4.6. *Let $\boldsymbol{\mathcal{P}}_{alg}$ denote the category of $\boldsymbol{\mathcal{P}}$-algebras and let $\widetilde{\boldsymbol{\mathcal{P}}_{alg}}$ be its full subcategory whose objects satisfy (I)-(II). Then the map $(F, e, u) \longmapsto (F, \mu, \eta)$ defines an equivalence of categories $\boldsymbol{\mathcal{P}}_{exp} \xrightarrow{\simeq} \widetilde{\boldsymbol{\mathcal{P}}_{alg}}$.*

Example 2.4.7. In virtue of Proposition 1.3.14, examples of such $\boldsymbol{\mathcal{P}}$-algebras (and then of exponential superfunctors) are $\mathbf{S}^*, \mathbf{\Gamma}^*, \mathbf{\Lambda}^*, \mathbf{A}^*$.

A first indication of the utility of the exponential structure is the following.

Proposition 2.4.8. *Let $T : F \longrightarrow G$ be an exponential transformation. Then T is an isomorphism (monomorphism, epimorphism) if and only if $T_{\Bbbk} : F(\Bbbk) \to G(\Bbbk)$ and $T_{\Pi\Bbbk} : F(\Pi\Bbbk) \to G(\Pi\Bbbk)$ are.*

Proof. One sense is trivial. The other one follows by decomposing any super vector space V into the sum of its one-dimensional subspaces and using the hypothesis iteratively. ☐

Proposition 2.4.9. *Let F be an exponential superfunctor and F^d its component of weight d. Then*

1. *$F^0(0) \simeq \Bbbk$ and $F^d(0) = 0$ for all $d > 0$.*

2. *Let m be the minimum strictly positive weight such that $F^m \neq 0$. Then F^m is an additive functor.*

Proof. Since $F(0)$ is one-dimensional, there is exactly one component F^d such that $F^d(0)$ does not vanish and is isomorphic to $\Bbbk$. Since by (2.4.1) $F^{2d}(0) \simeq F^d(0) \otimes F^d(0) \simeq \Bbbk$, the only possibility is $d = 0$. For the second point, use again (2.4.1) for $d = m$. The hypothesis and the first point imply that, on the right side of the formula, the only nonzero terms are the ones corresponding to $(a, b) = (0, m)$ and $(m, 0)$, alias respectively $F^m(V)$ and $F^m(W)$. That means $F^m(V \oplus W) \simeq F^m(V) \oplus F^m(W)$ naturally in both variables, as it has to be for F^m to be additive. ☐

The natural framework where we are going to use exponential objects is the one of p-complexes, which are generalisation of cochain complexes and which we introduce in the next subsection.

2.4.2 p-complexes

We start in full generality, following the work of Troesch [Tro05]. Let $\mathcal{C}$ be a monoidal abelian category.

Definition 2.4.10. Let $p \geq 2$ a prime. A *p-complex* in $\mathcal{C}$ is the data of:

- a graded object $C^* := \bigoplus_{d \geq 0} C^d$ of $\mathcal{C}$,
- a family of morphisms $\delta = \{\delta_n \in \mathrm{Hom}_{\mathcal{C}}(C^n, C^{n+1})\}_n$ such that $\delta^p = 0$.

A map like δ is called *p-differential*. When $p = 2$ we retrieve the usual complexes.

Example 2.4.11. Let $q \leq p - 1$. A sequence of q isomorphisms $0 \to C_1 \xrightarrow{\simeq} C_2 \xrightarrow{\simeq} \ldots \xrightarrow{\simeq} C_q \xrightarrow{\simeq} C_{q+1} \to 0$ is a p-complex.

There are several ways to produce a complex out of a p-complex. For each $1 \leq s \leq p - 1$, the sequence

$$C^*_{[s]} := \ldots \xrightarrow{\delta^{p-s}} C^n \xrightarrow{\delta^s} C^{n+s} \xrightarrow{\delta^{p-s}} C^{n+p} \xrightarrow{\delta^s} \ldots$$

is indeed a complex, called the s-contraction of C^*.

Definition 2.4.12. Let C^* be a p-complex and $1 \leq s \leq p - 1$. The s-th cohomology group of C^* is defined as the cohomology of $C^*_{[s]}$, i.e.

$$H^i_{[s]}(C) = \frac{\mathrm{Ker}(\delta^s : C^i \to C^{i+s})}{\mathrm{Im}(\delta^{p-s} : C^{i-p+s} \to C^i)} .$$

We call *s-cocyles*, resp. *s-coboundaries*, the elements of the numerator, resp. denominator. We have then a family of $p - 1$ cohomology groups. In general, there is no reason for them to be equal, as the following basic example shows.

Example 2.4.13. Let $p = 3$ and $C^* : 0 \to C \xrightarrow{\simeq} C \to 0$ be a 3-complex consisting of two copies of C in degrees $0, 1$ and an isomorphism between them. Then $H^0_{[1]}(C) = 0$, while $H^0_{[2]}(C) \simeq C$.

Theorem 2.4.14 (Kapranov [Kap96])**.** *Let C^* be a p-complex and suppose that there is $s \in \{1, \ldots, p-1\}$ such that $H^*_{[s]}(C) = 0$. Then $H^*_{[t]}(C) = 0$ for all $t \in \{1, \ldots, p-1\}$.*

It makes then some sense to define a notion of acyclicity. A p-complex which satisfies the hypothesis of Kapranov is called *p-acyclic*. For example, a chain of $p - 1$ isomorphisms is p-acyclic.

In general, for integers $1 \leq s \leq t < p$, there are inclusions $\mathrm{Ker}(\delta^s) \subset \mathrm{Ker}(\delta^t)$ and $\mathrm{Im}(d^{p-t}) \subset \mathrm{Im}(d^{p-s})$. In particular, there are well-defined maps

$$H^*_{[s]}(C) \to H^*_{[t]}(C) . \tag{2.4.2}$$

Definition 2.4.15. A p-complex is called *normal* if (2.4.2) is an isomorphism for all $1 \leq s \leq t < p$.

Hence in a normal p-complex C^* all cohomologies are the same; we note them under the same notation $H^*(C)$. By Kapranov Theorem, a p-acyclic p-complex is normal.

Definition 2.4.16. Let (C^*, δ_C) and (D^*, δ_D) be two p-complexes. Their tensor product $C^* \otimes D^*$ is also a p-complex when equipped with the p-differential defined on $x \otimes y \in C^a \otimes D^t$ by

$$x \otimes y \longmapsto \delta_C(x) \otimes y \; + \; (-1)^a x \otimes \delta_D(y) .$$

Clearly, if x, y are 1-cocyles so is $x \otimes y$. If moreover one of them is a 1-coboundary, say $x = \delta_C^{p-1} x'$, then

$$\delta^{p-1}(x' \otimes y) = \sum_{k=0}^{p-1} \binom{p-1}{k} \delta_C^k x' \otimes \delta_D^{p-1-k} y = \delta_C^{p-1} x' \otimes y = x \otimes y$$

(since $\delta_D(y) = 0$), so $x \otimes y$ is also a 1-coboundary. The following map is thus well defined:

$$\begin{aligned} H^*_{[1]}(C) \otimes H^*_{[1]}(D) &\longrightarrow H^*_{[1]}(C \otimes D) \\ [x] \otimes [y] &\longmapsto [x \otimes y] . \end{aligned} \tag{2.4.3}$$

Unfortunately, no analogue for the Kunneth formula is known for general p-complexes, firstly because it is not clear how the different cohomologies should interact. Nevertheless, it has been proved that when C^* and D^* are normal there is no obstruction.

Theorem 2.4.17 ([DK22, Thm 3.2.1])**.** *Let C^*, D^* be normal p-complexes. Then $C^* \otimes D^*$ is also normal and (2.4.3) defines an isomorphism*

$$H^*(C) \otimes H^*(D) \simeq H^*(C \otimes D) .$$

We want now to generalise the notion of exponential superfunctor to the context of p-complexes, and see if similar results on its cohomology still hold.

Definition 2.4.18. A *p-differential graded exponential superfunctor* (from now, *p-dg-EF*) is a tuple (F, e, u, δ) such that:

i. (F, e, u) is an exponential superfunctor;

ii. (F, δ) is a p-complex;

iii. δ is compatible with the exponential structure, i.e. the following diagrams are commutative:

$$\begin{tikzcd} F(V \oplus W) \arrow[r, "\delta"] \arrow[d, "e_{V,W}"] & F(V \oplus W) \arrow[d, "e_{V,W}"] \\ F(V) \otimes F(W) \arrow[r, "\delta \otimes 1 + 1 \otimes \delta"] & F(V) \otimes F(W) \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \arrow[r, "u"] \arrow[dr, "u"'] & F(0) \arrow[d, "\delta"] \\ & F(0) \end{tikzcd}$$

where one should remember the sign convention of Definition 1.2.10 when looking at the map $\delta \otimes 1 + 1 \otimes \delta$.

If we see F as a $\boldsymbol{\mathcal{P}}$-algebra, condition (iii.) is equivalent to δ being a derivation. This condition ensures a nice structure-preserving property at the level of cohomology. For that we need to recall a standard but fundamental result, that the reader can find for example in [Mac12, Thm 10.1] (in a more general formulation than the one we need).

Theorem 2.4.19 (Kunneth formula)**.** *Let A^*, B^* be (usual) cochain complexes in the category of super $\Bbbk$-vector spaces. Then $H^*(A \otimes B) \simeq H^*(A) \otimes H^*(B)$.*

We say that a p-dg-EF is normal if its underlying p-complex is. As we saw, for objects with such property there is a well-defined notion of "cohomology". That is what the following result is about.

Theorem 2.4.20. *If C^* is a normal p-dg-EF, then $H^*(C)$ is an exponential superfunctor. In particular, its first positive cohomology group that does not vanish is an additive superfunctor.*

Proof. The commutative square in Definition 2.4.18 (iii.) yields an isomorphism $H^*(F(V \oplus W)) \simeq H^*(F(V) \otimes F(W))$ and Theorem 2.4.19 yields $H^*(F(V) \otimes F(W)) \simeq H^*(F(V)) \otimes H^*(F(W))$. This shows that $H^*(C)$ is exponential. The last assertion comes from Proposition 2.4.9. $\square$

Remark that $H^*(C)$ a double grading: the cohomological degree $H^i(C)$, which we simply call *degree*, and the weight. The weight-d component of $H^*(C)$ is $H^*(C^d)$. That is why we chose a different name for this latter one. In the next section we are going to apply this machinery to a specific p-complex.

2.4.3 The super Troesch p-complex

Let $\mathbf{B}^*$ be the polynomial superfunctor defined on every $V \in \boldsymbol{\mathcal{V}}$ by

$$\mathbf{B}^*(V) := \mathbf{S}^*(V)^{\otimes p} .$$

As a tensor product of exponential superfunctors, it is exponential. We denote its summands as $\mathbf{S}^\lambda := \mathbf{S}^{\lambda_0} \otimes ... \otimes \mathbf{S}^{\lambda_{p-1}}$.

Definition 2.4.21. Let $x \in \mathbf{S}^\lambda$. The *weight* of x is $\sum_{i=0}^{p-1} \lambda_i$, while its *degree* is $\sum_{i=0}^{p-1} i\lambda_i$.

One can define $\mathbf{B}^*$ in the following equivalent way. Let E^* be the graded even vector space with basis $\{e_1, ..., e_p\}$ such that $\deg(e_i) = i - 1$. Then $\mathbf{B}^*(V) \simeq \mathbf{S}^*(E^* \otimes V)$ as a graded super space. As explained when we introduced parametrisation in Section 2.3.3, $\mathbf{B}^* = \mathbf{S}^*_E$ decomposes into the sum of $\mathbf{S}^\lambda$ as λ runs through the p-uples of positive integers. This explains the definition of weight.

Consider now the linear map $\alpha : E^* \to E^*$ such that $\alpha(e_i) = e_{i+1}$ for $1 \leq i \leq p-1$ and $\alpha(e_p) = 0$. By a little abuse, use the same name for the natural transformation induced on the functor Id_E. Set

$$\partial_n : \mathbf{S}^n_E \xrightarrow{\Delta_{n-1,1}} \mathbf{S}^{n-1}_E \otimes \mathbf{I}_E \xrightarrow{\mathrm{Id}\otimes\alpha} \mathbf{S}^{n-1}_E \otimes \mathbf{I}_E \xrightarrow{\mu} \mathbf{S}^n_E$$

where $\Delta_{n-1,1}$ is the component of the coproduct Δ which has the indicated target. Then $\partial := \sum_n \partial_n : \mathbf{B}^* \to \mathbf{B}^*$ raises degrees by one and has p-th power zero. It hence endows $\mathbf{B}^*$ with the structure of a p-complex. For detailed proofs of that, see [DK22, §3.3].

Remark 2.4.22. The map ∂ preserves weights by construction. Therefore $\mathbf{B}^*$ decomposes as the direct sum of its homogeneous (with respect to the weight) components $\mathbf{B}^*_n$.

Remark 2.4.23. We are going to make the differential explicit. Let $\delta = (\delta_{i,j})_{i,j \geq 0}$ be the following map:

$$\mathbf{S}^i(V) \otimes \mathbf{S}^j(V) \longrightarrow \mathbf{S}^{i-1}(V) \otimes \mathbf{S}^{j+1}(V)$$

$$v_1 \cdots v_i \otimes w_1 \cdots w_j \longmapsto \sum_{k=1}^{i} (-1)^{\overline{v_k}\,(\overline{v_{k+1}} + ... + \overline{v_i})} v_1 \cdots \hat{v_k} \cdots v_i \otimes v_k w_1 \cdots w_j$$

where the hat notation $\hat{v}$ indicates that v has been removed. Now replace tensors by bars and let $v_0, ..., v_{p-1} \in \mathbf{S}^*(V)$ denote some symmetric tensors. The formula for ∂ is then given by

$$\partial(v_0 \mid ... \mid v_{p-1}) = \sum_{k=0}^{p-2} (v_1 \mid ... \mid \delta(v_k \mid v_{k+1}) \mid ... \mid v_{p-1}) .$$

Proposition 2.4.24 ([DK22, Thm 3.4.1])**.** $\boldsymbol{B}^*$ *is a normal p-complex.*

We can thus speak of *the* cohomology $H^*(\mathbf{B})$ and apply Theorem 2.4.20.

Proposition 2.4.25. $\boldsymbol{B}^*$ *is a p-dg-EF. As a consequence,* $H^*(\boldsymbol{B})$ *is an exponential superfunctor.*

We may thus start the machinery to compute $H^*(\mathbf{B})$. First, remark that $\mathbf{B}_1^* = 0 \to \mathbf{I} \to \ldots \to \mathbf{I} \to 0$ is a chain of $p-1$ isomorphisms, therefore p-acyclic. Thus, by Theorem 2.4.20 the first non-trivial cohomology group is additive. In force of Theorem 2.3.5, it has to be in weight at least p. Let us check $H^*(\mathbf{B}_p)$ to see if it is zero or not. Since it is in any case an additive superfunctor, we can completely retrieve it from its value on a one-dimensional super space. If V is a purely even space, then $\mathbf{B}^*(V)$ coincides with the classical Troesch p-complex and its cohomology is known [Tro05, Thm 3.1.2], namely $H^*(\mathbf{B}_p(\Bbbk)) \simeq \Bbbk^{(1)}$ if $* = 0$ and zero otherwise. On the other hand, $\mathbf{B}_p^*(\Pi\Bbbk)$ vanishes everywhere but on the term $\bigotimes^p(\Pi\Bbbk) \simeq \Pi\Bbbk$ which is in degree $\binom{p}{2}$. In particular, $H^*(\mathbf{B}_p(\Pi\Bbbk))$ is also isomorphic to $\Pi\Bbbk$ placed in degree $\binom{p}{2}$. Putting all together and using Theorem 2.3.5, we conclude the following:

Lemma 2.4.26.
- *$H^*(\boldsymbol{B}_n) = 0$ for $0 < n < p$.*
- *$H^*(\boldsymbol{B}_p) \simeq (\boldsymbol{I}_0^{(1)})_{[0]} \oplus (\boldsymbol{I}_1^{(1)})_{[\binom{p}{2}]}$.*

The higher cohomology groups are no longer additive, hence not that easy to compute directly. However, we will not need to do that. We are rather going to find them all at once thanks to the information of Lemma 2.4.26 and the superalgebra structure on $H^*(\mathbf{B})$. To lighten notation, we make a little abuse until the end of this section and consider $\mathbf{I}^{(1)}$ as a graded functor with $\mathbf{I}_0^{(1)}$ in degree 0 and $\mathbf{I}_1^{(1)}$ in degree $\binom{p}{2}$. Hence Lemma 2.4.26 says that there is an isomorphism of graded superfunctors $\mathbf{I}^{(1)} \simeq H^*(\mathbf{B}_p)$. This and the inclusion $\mathbf{B}_p^* \subset \mathbf{B}^*$ induce a graded morphism

$$\mathbf{I}^{(1)} \longrightarrow H^*(\mathbf{B}) \tag{2.4.4}$$

which can be extended, by the universal property of $\mathbf{S}^*$, to a unique morphism of $\boldsymbol{\mathcal{P}}$-algebras

$$\mathbf{S}^{*(1)} \longrightarrow H^*(\mathbf{B})\ . \tag{2.4.5}$$

Proposition 2.4.27. *The map (2.4.5) is a graded isomorphism.*

Corollary 2.4.28. *There is an isomorphism of graded polynomial superfunctors $H^*(\boldsymbol{B}) \simeq \boldsymbol{S}^{*(1)}$, where the grading on the right side is induced by considering $\boldsymbol{I}_0^{(1)}$ in degree 0 and $\boldsymbol{I}_1^{(1)}$ in degree $\binom{p}{2}$. In terms of weights, the isomorphism reads in the following way:*

$$H^*(\boldsymbol{B}_n) = 0 \ \text{ if } n \text{ is not divisible by } p,$$
$$H^*(\boldsymbol{B}_{pm}) \simeq \bigoplus_{0\le \ell \le m} (S_0^{m-\ell(1)} \otimes \Lambda_1^{\ell(1)})_{[\ell\binom{p}{2}]}\ .$$

Proof of Proposition 2.4.27. Since the exponential structures on source and target are induced by the product on $\mathbf{S}^*$, (2.4.5) is compatible with them by construction. We can then apply Proposition 2.4.8 and reduce ourselves to prove that it is an isomorphism when evaluated on $\Bbbk$ and $\Pi\Bbbk$. The first case is covered from the classical computation in [Tro05]. On the other hand, $\mathbf{B}_n(\Pi\Bbbk) = 0$ in weights $n > p$. This and Lemma 2.4.26 imply that $H^*(\mathbf{B}(\Pi\Bbbk)) \simeq \Bbbk_{[0]} \oplus (\Pi\Bbbk^{(1)})_{[\binom{p}{2}]}$ (the first $\Bbbk$ being in weight zero). This is equivalent to writing $H^*(\mathbf{B}(\Pi\Bbbk)) \simeq \Lambda^*((\Bbbk^{(1)})_{[\binom{p}{2}]})$ with $\Bbbk^{(1)}$ placed in odd $\mathbb{Z}/2\mathbb{Z}$-degree. But the latter is exactly equal to the source of (2.4.5) evaluated on $\Pi\Bbbk$. □

Remark 2.4.29. We did not need it for our purposes, but a simple explicit expression for the map (2.4.4) can be the following:

$$\mathbf{I}^{(1)} \longrightarrow H^*(\mathbf{B}_p)$$
$$v \longmapsto [v^p] \ \text{ if } \ \overline{v} = 0$$
$$v \longmapsto [v^{\otimes p}] \ \text{ if } \ \overline{v} = 1$$

It is immediate to verify that v^p and $v^{\otimes p}$ are indeed cocycles. We have to check that the map is linear. Take homogeneous vectors $v \in V,\ w \in W$. If $\overline{v} \neq \overline{w}$, the two are sent on distinct summands of $H^*(\mathbf{B}(V \oplus W))$ so the verification is trivial. If $\overline{v} = \overline{w} = 0$, linearity follows by the identity $(v+w)^p = v^p + w^p$ that holds in characteristic p. For the last case $\overline{v} = \overline{w} = 1$ we have to show that $[(v+w)^{\otimes p}] = [v^{\otimes p}] + [w^{\otimes p}]$, which is a little trickier. Consider the odd sub-vector space $W := \Bbbk v \oplus \Bbbk w$. The classes $[v^{\otimes p}], [w^{\otimes p}]$ and $[(v+w)^{\otimes p}]$ can be seen as elements of $H^{\binom{p}{2}}(\mathbf{B}_p(W))$. By Lemma 2.4.26 this has dimension 2, and by the exponential property $\{[v^{\otimes p}], [w^{\otimes p}]\}$ is a basis (one can obviously discard the extreme case $v = w \in V = W$, where the verification is trivial.). So there exist $a, b \in \Bbbk$, independent of the choice of V and W, such that

$$[(v + w)^{\otimes p}] = a[v^{\otimes p}] + b[w^{\otimes p}] \ .$$

As a consequence of their independence, one can take alternatively $v = 0$ and $w = 0$ to deduce $a = b = 1$, that ends our verification.

Chapter 3

Extensions between additive compositions

We get started with the main interest of this manuscript, which is making Ext computations in $\boldsymbol{\mathcal{P}}$. As declared at the beginning (Conjecture 0.4), our goal is to study extensions between *classical* polynomial functors precomposed by an additive superfunctor. In this chapter, we treat the special case where one of the two classical functors is additive too. Since all additive polynomial functors are direct sums of $I^{(r)}$ [Tou17, §3.1], we are then reduced to computing

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, G \circ B)$$

where A and B are additive homogeneous polynomial superfunctors of the same degree p^s, $s \geq 1$, and G is a polynomial functor of degree p^r. The hypothesis $s \geq 1$ is needed in order for the compositions to make sense, as we explained in Section 2.3.2. Moreover, in force of Theorem 2.3.5, we control all additive superfunctors by mean of the *indecomposable* ones, namely the Frobenius twists and their parity shifts. We can then start by studying extensions between Frobenius-twisted functors. The result we are going to prove at the end of the chapter is the following.

Theorem 3.0.1. *Let $G \in \mathcal{P}_{p^r}$ and let $A, B \in \boldsymbol{\mathcal{P}}_{p^s}$ ($s \geq 1$) be additive. There is a graded isomorphism, natural in G, A, B:*

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, G \circ B) \simeq \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A, B)^{(r)} .$$

Remembering the definition and properties of parametrisation (§2.3.3) this result leads to a positive answer to a special case of Conjecture 0.4.

Corollary 3.0.2. *Conjecture 0.4 is true if F or G is additive.*

Proof. Since Kuhn duality preserves additivity, it is enough to treat the case where F is additive. An additive polynomial functor is isomorphic to a sum of copies of $I^{(r)}$ [Tou17, Prop. 3.5], so we can suppose $F = I^{(r)}$. Recall that for any graded super vector space E^* there is an isomorphism $(I^{(r)})^{E^*} \simeq I^{(r)} \otimes E^{(r)\,*}$. Moreover, $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A, B)^{(r)}$ is finite-dimensional in each degree by Lemma 1.6.4. So by Proposition 2.3.30(3) we have

$$\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G_{\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}((I^{(r)})^{\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)}, G) \simeq \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A, B)^{(r)}$$

which, together with Theorem 3.0.1, proves the statement. □

3.1 The Yoneda superalgebra

We start by recollecting the general structure of the fundamental Ext spaces between Frobenius twists. Our reference for that is [Dru16]. Set:

$$\begin{aligned} E_r^* &:= \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)}), \\ \boldsymbol{E_r}^* &:= \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(r)}, \mathbf{I}_0^{(r)}) \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_1^{(r)}, \mathbf{I}_1^{(r)}), \\ \overline{\boldsymbol{E_r}}^* &:= \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(r)}, \mathbf{I}_1^{(r)}) \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_1^{(r)}, \mathbf{I}_0^{(r)}). \end{aligned}$$

To make notation more agile, we will drop the index *. Recall that the Yoneda product described in (1.6.1) makes $\boldsymbol{E_r}$ into a superalgebra and $\overline{\boldsymbol{E_r}}$ into a $\boldsymbol{E_r}$-supermodule. In the classical case, the analogous product makes E_r into an algebra.

Theorem 3.1.1 ([FS97, Cor. 4.8])**.**

$$E_r^s \simeq \begin{cases} \Bbbk & \text{if } \ s < 2p^r \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

As a commutative algebra, E_r is generated by elements $\{\epsilon_{1,r}, ..., \epsilon_{r,r}\}$, with $\deg(\epsilon_{i,r}) = 2p^{i-1}$, subject to relations $\epsilon_{1,r}^p = ... = \epsilon_{r,r}^p = 0$.

Theorem 3.1.2 ([Dru16, Cor. 4.5.5])**.**

$$\boldsymbol{E_r}^s \simeq \begin{cases} \Bbbk & \text{if } \ s \text{ is even} \\ 0 & \text{if } \ s \text{ is odd.} \end{cases}$$

$$\overline{\boldsymbol{E_r}}^s \simeq \begin{cases} \Bbbk & \text{if } \ s \geq p^r \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.1.3. Using the classification of additive superfunctors together with Remark 1.6.3, we can deduce from Theorem 3.1.2 all the Ext spaces between indecomposable additive superfunctors. In particular, for all additive superfunctors A, B, $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A, B)$ is a sum of copies of $\boldsymbol{E_r}, \overline{\boldsymbol{E_r}}, \boldsymbol{\Pi E_r}$ and $\boldsymbol{\Pi}\overline{\boldsymbol{E_r}}$.

The superalgebra structure of $\boldsymbol{E_r}$ is similar to the one of E_r, but with one striking difference.

Theorem 3.1.4 ([Dru16, Thm 4.7.1])**.** *We have the following descriptions:*

- *$\boldsymbol{E_r}$ is the commutative graded superalgebra generated by even elements $\{e_{1,r}, ..., e_{r,r}\}$, with $\deg(e_{i,r}) = 2p^{i-1}$, subject to relations $e_{1,r}^p = ... = e_{r-1,r}^p = 0$.*
- *$\overline{\boldsymbol{E_r}}$ is the free $\boldsymbol{E_r}$-supermodule on one even generator c_r of degree p^r.*

The first glaring difference between E_r and its super counterpart $\boldsymbol{E_r}$ is, at a merely dimensional level, the infiniteness of the latter one. Examining the algebra structures, we see that the reason lies in the presence of a non-nilpotent generator, namely $e_{r,r} \in \boldsymbol{E_r}^{2p^{r-1}}$, which is absent in E_r. This one class is hence responsible for the existence of all the nonzero classes in $\boldsymbol{E_r}$ of degree greater than $2p^r$.

From Theorem 3.1.4 we also deduce the following important remark, which makes another substantial difference between the classical theory and the super theory.

Remark 3.1.5. For all $r \geq 1$, the classical twist defines a morphism of algebras $E_r \to E_{r+1}$ which is injective (see for example [Tou12, Lemma 5.2]). This is no longer the case in the super context. Indeed, by Theorem 3.1.2, as graded super vector spaces all the $\boldsymbol{E_r}$ are the same, regardless of r. Therefore the twist map $\boldsymbol{E_r} \to \boldsymbol{E_{r+1}}$ cannot be injective, otherwise it would yield an isomorphism of superalgebras which would contradict Theorem 3.1.4. In equivalent but more explicit terms, the kernel of the twist map is generated as an ideal by the nonzero class $(e_{r,r})^p$. Such class is thus responsible for the actual qualitative difference between classical and super twist. The general computations investigated in Chapter 4 rely largely on this mysterious class.

We will also need the description of the more general spaces

$$\boldsymbol{E_{j,r}}^* := \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(r)}, S_0^{p^{r-j}(j)}) \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_1^{(r)}, S_1^{p^{r-j}(j)})$$

$$\overline{\boldsymbol{E_{j,r}}}^* := \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(r)}, S_1^{p^{r-j}(j)}) \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_1^{(r)}, S_0^{p^{r-j}(j)})$$

for $0 \leq j \leq r$ (the isomorphisms come from Proposition 2.2.6). Note that $\boldsymbol{E_{r,r}}^* = \boldsymbol{E_r}^*$ and $\overline{\boldsymbol{E_{r,r}}}^* = \overline{\boldsymbol{E_r}}^*$. As graded spaces we have [Dru16, Thm 4.5.1]:

$$\boldsymbol{E_{j,r}}^s \simeq \begin{cases} \Bbbk & \text{if} \quad s \equiv 0 \mod 2p^{r-j} \\ 0 & \text{otherwise} \end{cases}$$

$$\overline{\boldsymbol{E_{j,r}}}^s \simeq \begin{cases} \Bbbk & \text{if} \quad s \equiv p^r \mod 2p^{r-j} \text{ and } s \geq 2p^r \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3.1.6 ([Dru16, Prop. 4.6.4])**.** *For $0 \leq j \leq r$, the canonical morphism $\boldsymbol{I}_\ell^{(r)} \to S_\ell^{p^{r-j}(j)}$ induces surjective maps*

$$\boldsymbol{E_r} \longrightarrow \boldsymbol{E_{j,r}}$$
$$\overline{\boldsymbol{E_r}} \longrightarrow \overline{\boldsymbol{E_{j,r}}}$$

whose kernels are the right $\boldsymbol{E_r}$-supermodules generated respectively by the family $\{e_{1,r}, \ldots, e_{r-j,r}\}$ and $\{c_r \cdot e_{1,r},\ \ldots,\ c_r \cdot e_{r-j,r}\}$.

3.2 The Ext computation

We dedicate to the proof of Theorem 3.0.1 by first inspecting the cases where A, B are indecomposable. Throughout the rest of the chapter, $\ell \in \{0,1\}$ and $\overline{\ell} := \ell + 1 \mod 2$. We recall that $y \cdot x$ denotes the Yoneda product (1.6.1) of two extensions x, y. If $V = \bigoplus_{i \geq 0} V^i$ is a graded vector space, we also recall that $V^{(r)} := I^{(r)}(V)$ is graded by

$$(V^{(r)})^d = \begin{cases} (V^i)^{(r)} & \text{if } d = p^r i \\ 0 & \text{otherwise} \end{cases}$$

in accordance with Definition 2.3.14. Finally, we keep in mind the notation $e_{i,r}$ and c_r of Theorem 3.1.4 for the generators of $\boldsymbol{E_r}^*$ and $\overline{\boldsymbol{E_r}}^*$.

Convention 3.2.1. As explained in Convention 1.5.5, it is not restrictive to only consider Ext spaces between *homogeneous* (super)functors. In what follows, G will be always assumed to be homogeneous of degree p^r and A, B homogeneous of degree p^s ($s \geq 1$). This assumption being fixed throughout, we suppress the degrees from the notations $\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}$ and $\mathrm{Ext}_{\mathcal{P}}$ in order to make them lighter.

3.2.1 The case $A = B = \mathbf{I}_\ell^{(r)}$

Proposition 3.2.2. *There is a morphism of superalgebras determined on generators by*

$$\sigma : \boldsymbol{E_s}^{(r)} \to \boldsymbol{E_{r+s}}$$
$$e_{i,s}^{(r)} \longmapsto e_{r+i,r+s}$$

which is graded injective.

Proof. It is graded because $\deg(e_{i,s}^{(r)}) = p^r \deg(e_{i,s}) = 2p^{r+i-1} = \deg(e_{r+i,r+s})$. Moreover, if $i < s$ then $r+i < r+s$, hence $e_{r+i,r+s}^p = 0$. This means that σ respects the relations on the two superalgebras and is then well defined. To see that σ is injective, consider a nonzero element $e^{(r)} \in \boldsymbol{E_s}^{(r)}$. By Theorem 3.1.4, e is a sum of terms of the form

$$e_{1,s}^{d_1} \cdot \ldots \cdot e_{s,s}^{d_s}$$

with $d_1, ..., d_{s-1} < p$. Therefore by definition $\sigma(e^{(r)})$ is equal to the sum of the corresponding terms

$$e_{r+1,r+s}^{d_1} \cdot \ldots \cdot e_{r+s,r+s}^{d_s}$$

each one being nonzero because $d_1, ..., d_{s-1} < p$. So $\sigma(e^{(r)}) \neq 0$. □

Notation 3.2.3. For an extension $x \in \mathrm{Ext}^*_{\mathcal{P}}(F, G)$, denote by $x_\ell^{[s]}$ the extension in $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_\ell^{(s)}, G_\ell^{(s)})$ obtained by precomposing x with $\mathbf{I}_\ell^{(s)}$.

We define then a graded map, natural in G:

$$\Psi : \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \boldsymbol{E_s}^{(r)} \longrightarrow \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_\ell^{(r+s)}, G_\ell^{(s)})$$
$$x \otimes e^{(r)} \longmapsto x_\ell^{[s]} \cdot \sigma(e^{(r)}) \ .$$

Proposition 3.2.4. Ψ *is an isomorphism for all* G.

Proof. We carry out the proof in two steps.

- **Step 1**: assume G injective. It is enough to consider the case $G = S^\mu$, since all injective are direct factors of sums of such functors. Let us treat first the case $\mu = (p^r)$. Then $\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, S^{p^r})$ is only nonzero in degree 0, where it is one-dimensional and generated by the twisted p^r-th power map $\varphi : I^{(r)} \to S^{p^r}$. Denote $\varphi_s := \varphi \circ \mathbf{I}_\ell^{(s)} : \mathbf{I}_\ell^{(r+s)} \to S_\ell^{p^r(s)}$. Thus Ψ identifies to the composition

$$\boldsymbol{E_s}^{(r)} \xrightarrow{\sigma} \boldsymbol{E_{r+s}} \xrightarrow{\varphi_s \cdot -} E_{s,r+s} \ .$$

  By Theorem 3.1.6, the right map is a surjection and its kernel is generated as a super vector space by the classes $\{e_{1,r+s}, \ldots, e_{r,r+s}\}$. On the other side, σ is an injection on all the other generators of $\boldsymbol{E_{r+s}}$, so the composition is an isomorphism as desired. In the case where $\mu \neq (p^r)$, then S^μ and $S_\ell^{\mu(s)}$ are both tensor products of reduced functors. Hence the source and the target of Ψ are zero by Lemma 2.3.4, so that Ψ is trivially an isomorphism.

- **Step 2**: we make the proof for a general G by a spectral sequence argument. As a first step, we construct a lifting of our map Ψ on the level of chain (bi)complexes. To be more specific, we choose an injective coresolution J^* of F, a projective resolution P_* of $\mathbf{I}_\ell^{(r+s)}$, and we consider the bicomplexes (the bicomplex $B^{m,n}$ has trivial vertical differentials)

$$B^{m,n} = \mathrm{Hom}_{\mathcal{P}}(I^{(r)}, J^m) \otimes \mathrm{Ext}^n_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_\ell^{(s)}, \mathbf{I}_\ell^{(s)})^{(r)} \ ,$$
$$C^{m,n} = \mathrm{Hom}_{\mathcal{P}}(I^{(r)}, J^m) \otimes \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(P_n, \mathbf{I}_\ell^{(r+s)}) \ ,$$
$$D^{m,n} = \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(P_n, J_m \circ \mathbf{I}_\ell^{(r)}) \ .$$

The homology of the total complexes of these bicomplexes are given by

$$\begin{aligned} H^*(\mathrm{Tot}B) &= \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}^{(s)}_\ell, \mathbf{I}^{(s)}_\ell)^{(r)} \,, \\ H^*(\mathrm{Tot}C) &= \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \mathrm{Ext}_{\boldsymbol{\mathcal{P}}}(\mathbf{I}^{(r+s)}_\ell, \mathbf{I}^{(r+s)}_\ell) \,, \\ H^*(\mathrm{Tot}D) &= \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}^{(r+s)}_\ell, G^{(s)}_\ell) \,. \end{aligned}$$

We choose a basis of the graded space $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}^{(s)}_\ell, \mathbf{I}^{(s)}_\ell)$ and for each element b of this basis we choose a cocyle $z(b)$ representing $\sigma(b)$ in the complex $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(P_*, \mathbf{I}^{(r+s)}_\ell)$. This induces a morphism of bicomplexes

$$z : B^{m,n} \to C^{m,n}$$

such that $H^*(\mathrm{Tot}z) : H^*(\mathrm{Tot}B) \to H^*(\mathrm{Tot}C)$ is equal to $\mathrm{id}\otimes\sigma$. We also define a morphism of bicomplexes $\phi : C^{m,n} \to D^{m,n}$ as the composition

$$C^{m,n} \to \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\mathbf{I}^{(r+s)}_\ell, J^m \circ \mathbf{I}^{(s)}_\ell) \otimes \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(P_n, \mathbf{I}^{(r+s)}_\ell) \to D^{m,n}$$

where the first map is induced by precomposition by the functor $\mathbf{I}^{(s)}_\ell$ and the second one by the composition of morphisms in $\boldsymbol{\mathcal{P}}$. By construction, the map $H^*(\mathrm{Tot}\phi)$ sends $x \otimes e^{(r)}$ to $x^{[s]}_\ell \cdot e^{(r)}$. Thus $H^*(\mathrm{Tot}(\phi \circ z))$ coincides with Ψ.

Let us now consider the first quadrant spectral sequences associated to the bicomplexes $B^{m,n}$ and $D^{m,n}$:

$$\begin{aligned} E^{m,n}_1(B) &= \mathrm{Hom}_{\mathcal{P}}(I^{(r)}, J^m) \otimes \mathrm{Ext}^n_{\boldsymbol{\mathcal{P}}}(\mathbf{I}^{(s)}_\ell, \mathbf{I}^{(s)}_\ell)^{(r)} \Rightarrow H^{m+n}(\mathrm{Tot}B) \,. \\ E^{m,n}_1(D) &= \mathrm{Ext}^n_{\boldsymbol{\mathcal{P}}}(\mathbf{I}^{(r+s)}_\ell, J^m \circ \mathbf{I}^{(s)}_\ell) \Rightarrow H^{m+n}(\mathrm{Tot}D) \,. \end{aligned}$$

The morphism of bicomplexes $\phi \circ z$ induces a morphism of spectral sequences, which is equal to Ψ on the abutment. Thus to prove that Ψ is an isomorphism, it suffices to prove that $E^{m,n}_\infty(\phi \circ z)$ is an isomorphism. But by construction $E^{m,*}_1(\phi \circ z) = \Psi$ on each column of index m. Since J^m is injective, we deduce by Step 1 that $E^{m,*}_1(\phi\circ z)$ it is an isomorphism on the first pages. Therefore $E^{m,n}_\infty(\phi \circ z)$ is an isomorphism, as we had to show.

□

3.2.2 The case $A = \mathbf{I}^{(r)}_\ell, B = \mathbf{I}^{(r)}_{\overline{\ell}}$

The idea is to perform the same construction but for two twists of different parities. More precisely, we will construct a morphism analogous to Ψ but having $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}^{(r+s)}_\ell, G^{(s)}_{\overline{\ell}})$ as target. First we start by defining a counterpart of the morphism σ of Proposition 3.2.2. Recall that $\overline{\boldsymbol{E_s}}$ is the free $\boldsymbol{E_s}$-supermodule on the generator c_s (Proposition 3.1.4). Note also that σ endows $\boldsymbol{E_{r+s}}$, thus also $\overline{\boldsymbol{E_{r+s}}}$, with the structure of an $\boldsymbol{E_s}^{(r)}$-supermodule.

Proposition 3.2.5. *There is a morphism of graded $\boldsymbol{E_s}^{(r)}$-supermodules determined by*

$$\begin{aligned} \sigma' : \overline{\boldsymbol{E_s}}^{(r)} &\to \overline{\boldsymbol{E_{r+s}}} \\ c^{(r)}_s &\longmapsto c_{r+s} \end{aligned}$$

which is graded injective.

Proof. By construction, $\sigma'(c^{(r)}_s \cdot e^{(r)}) = c_{r+s} \cdot \sigma(e^{(r)})$ and this determines σ' completely. In particular σ' is injective, since σ is. Moreover $\deg(c^{(r)}_s) = p^r \deg(c_s) = p^{r+s} = \deg c_{r+s}$, which shows that σ' is graded. □

Define, as above, a graded map natural in G:

$$\begin{aligned}\Psi' : \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \overline{\boldsymbol{E_s}}^{(r)} &\longrightarrow \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_\ell^{(r+s)}, G_{\overline{\ell}}^{(s)}) \\ x \otimes e^{(r)} &\longmapsto x_{\overline{\ell}}^{[s]} \cdot \sigma'(e^{(r)})\end{aligned}$$

Proposition 3.2.6. *Ψ' is a graded isomorphism for all G.*

Proof. The proof is conceptually similar to the one of Proposition 3.2.4. When $G = S^\mu$ is an injective cogenerator, the assertion follows in the same way by Theorem 3.1.6 if $\mu = (p^r)$ and from Lemma 2.3.4 otherwise. For an arbitrary G, take the very same P_* and J^* and form bicomplexes

$$\begin{aligned} B^{m,n} &= \mathrm{Hom}_{\mathcal{P}}(I^{(r)}, J^m) \otimes \mathrm{Ext}^n_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_\ell^{(s)}, \mathbf{I}_{\overline{\ell}}^{(s)})^{(r)} , \\ C^{m,n} &= \mathrm{Hom}_{\mathcal{P}}(I^{(r)}, J^m) \otimes \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(P_n, \mathbf{I}_{\overline{\ell}}^{(r+s)}) , \\ D^{m,n} &= \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(P_n, J^m \circ \mathbf{I}_{\overline{\ell}}^{(r)}) . \end{aligned}$$

Take a basis of $\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_\ell^{(s)}, \mathbf{I}_{\overline{\ell}}^{(s)})$ and to any element b of this basis associate a cocycle $z'(b)$ representing $\sigma'(b)$ in $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(P_n, \mathbf{I}_{\overline{\ell}}^{(r+s)})$. It induces a morphism of bicomplexes $z' : B^{m,n} \to C^{m,n}$ such that $H^*(\mathrm{Tot} z') = \mathrm{id} \otimes \sigma'$. Then define a morphism φ' as in the previous proof, replacing ℓ by $\overline{\ell}$ everywhere in the composition. The map $H^*(\mathrm{Tot}\varphi')$ sends $x \otimes e^{(r)}$ to $x_{\overline{\ell}}^{[s]} \cdot e^{(r)}$, so $H^*(\mathrm{Tot}(\varphi' \circ z'))$ is equal to Ψ'. In the second step, the spectral sequence associated to these new $B^{*,*}$ and $D^{*,*}$ are exactly the source and the target of Ψ'. The rest of the proof is a word-for-word repetition. □

3.2.3 Parity shifted twists

Half work is done for indecomposable additives. We now have to treat the shifted ones, i.e. additives of the form $\mathbf{\Pi} \circ \mathbf{I}_\ell^{(s)}$.

Proposition 3.2.7. *Let π_s denote the generator of* $\mathrm{Hom}(\mathbf{\Pi} \circ \boldsymbol{I}_\ell^{(s)}, \boldsymbol{I}_\ell^{(s)})$. *Then there are injective morphisms of graded super vector spaces*

$$\tau : \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ \boldsymbol{I}_\ell^{(s)}, \boldsymbol{I}_\ell^{(s)})^{(r)} \to \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ \boldsymbol{I}_\ell^{(r+s)}, \boldsymbol{I}_\ell^{(r+s)})$$

$$\tau' : \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ \boldsymbol{I}_\ell^{(s)}, \boldsymbol{I}_{\overline{\ell}}^{(s)})^{(r)} \to \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ \boldsymbol{I}_\ell^{(r+s)}, \boldsymbol{I}_{\overline{\ell}}^{(r+s)})$$

which fit in diagrams

$$\begin{tikzcd} \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ \boldsymbol{I}_\ell^{(s)}, \boldsymbol{I}_\ell^{(s)})^{(r)} \arrow[r, "\tau"] & \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ \boldsymbol{I}_\ell^{(r+s)}, \boldsymbol{I}_\ell^{(r+s)}) \\ \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\boldsymbol{I}_\ell^{(s)}, \boldsymbol{I}_\ell^{(s)})^{(r)} \arrow[u, "(-)\cdot\pi_s^{(r)}"] \arrow[r, "\sigma"] & \mathrm{Ext}_{\boldsymbol{\mathcal{P}}}(\boldsymbol{I}_\ell^{(r+s)}, \boldsymbol{I}_\ell^{(r+s)}) \arrow[u, "(-)\cdot\pi_{r+s}"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ \boldsymbol{I}_\ell^{(s)}, \boldsymbol{I}_{\overline{\ell}}^{(s)})^{(r)} \arrow[r, "\tau'"] & \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ \boldsymbol{I}_\ell^{(r+s)}, \boldsymbol{I}_{\overline{\ell}}^{(r+s)}) \\ \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\boldsymbol{I}_\ell^{(s)}, \boldsymbol{I}_{\overline{\ell}}^{(s)})^{(r)} \arrow[u, "(-)\cdot\pi_s^{(r)}"] \arrow[r, "\sigma'"] & \mathrm{Ext}_{\boldsymbol{\mathcal{P}}}(\boldsymbol{I}_\ell^{(r+s)}, \boldsymbol{I}_{\overline{\ell}}^{(r+s)}) \arrow[u, "(-)\cdot\pi_{r+s}"'] \end{tikzcd}$$

Proof. By Remark 1.6.3, the vertical arrows in both diagrams are (odd) isomorphisms. Hence the existence of such injective τ and τ' is straightforward. □

We now have two sister maps of Ψ, Ψ', namely:

$$\Omega : \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ \mathbf{I}_\ell^{(s)}, \mathbf{I}_\ell^{(s)})^{(r)} \to \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ \mathbf{I}_\ell^{(r+s)}, G_\ell^{(s)})$$
$$x \otimes e^{(r)} \longmapsto x_\ell^{[s]} \cdot \tau(e^{(r)})$$

$$\Omega' : \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ \mathbf{I}_\ell^{(s)}, \mathbf{I}_{\overline{\ell}}^{(s)})^{(r)} \to \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{\Pi} \circ \mathbf{I}_\ell^{(r+s)}, G_{\overline{\ell}}^{(s)})$$
$$x \otimes e^{(r)} \longmapsto x_{\overline{\ell}}^{[s]} \cdot \tau'(e^{(r)})$$

Proposition 3.2.8. *Ω and Ω' are isomorphisms for all G.*

Proof. Since τ fits in the diagram of Proposition 3.2.7, by its definition Ω fits in a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \mathrm{Ext}^*(\mathbf{\Pi} \circ \mathbf{I}_\ell^{(s)}, \mathbf{I}_\ell^{(s)})^{(r)} \arrow[r, "\Omega"] & \mathrm{Ext}^*(\mathbf{\Pi} \circ \mathbf{I}_\ell^{(r+s)}, G_\ell^{(s)}) \\
\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_\ell^{(s)}, \mathbf{I}_\ell^{(s)})^{(r)} \arrow[r, "\Psi"] \arrow[u, "\mathrm{id}\otimes(-)\cdot\pi_s^{(r)}"] & \mathrm{Ext}_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_\ell^{(r+s)}, G_\ell^{(s)}) \arrow[u, "(-)\cdot\pi_{r+s}"']
\end{tikzcd}$$

but Ψ is an isomorphism by Proposition 3.2.4 and the vertical arrows are isomorphisms, then so is Ω. The proof for Ω' is analogous. □

3.2.4 General computation

We have computed extensions of the form $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, G \circ B)$ by inspecting the possibilities for A, B indecomposable. Now it is time to generalise our computations to arbitrary additive $A, B \in \boldsymbol{\mathcal{P}}$. For that, we will make use of a technical lemma.

Convention 3.2.9. If $\mathcal{A}, \mathcal{B}$ are additive categories and $H : \mathcal{A} \to \mathcal{B}$ is a functor, we say that H is *additive* if there exists a binatural isomorphism $H(A \oplus B) \simeq H(A) \oplus H(B)$. In other words, we generalise Definition 2.3.1. Since the content of Lemma 2.3.2 is entirely valid in this context, the definition is equivalent if moreover we require the isomorphism to be the one induced via H by the canonical inclusions $A, B \subset A \oplus B$. This allows us to make a little abuse in the following proof, where we identify $H(A \oplus B)$ with $H(A) \oplus H(B)$.

Lemma 3.2.10. *Let $\mathcal{A}, \mathcal{B}$ be additive categories and $H, L : \mathcal{A} \to \mathcal{B}$ two additive functors. Let $\mathcal{I}$ be a full subcategory of $\mathcal{A}$ such that every object of $\mathcal{A}$ can be written as a finite direct sum of copies of objects in $Ob(\mathcal{I})$. Denote by $i : \mathcal{I} \hookrightarrow \mathcal{A}$ the canonical inclusion. Then:*

1. *Every natural transformation $T : H \circ i \to L \circ i$ extends uniquely to a natural transformation $\widetilde{T} : H \to L$.*

2. *$\widetilde{T}$ is an isomorphism (monomorphism, epimorphism) if and only if T is.*

Proof. Let us first prove uniqueness. Suppose that there are two natural transformations $\widetilde{T}, \widetilde{T}'$ with the required properties. For an object A, choose an isomorphism $\varphi : A \simeq \bigoplus_{i \in I} A_i$ with $A_i \in Ob(\mathcal{I})$ and I a finite set. Since $\widetilde{T}$ and $\widetilde{T}'$ are both natural and lift T, in the following diagram

$$\begin{tikzcd} H(A) \arrow[r, "\widetilde{T}_A"] \arrow[d, "H\varphi"'] & L(A) \arrow[d, "L\varphi"] \\ \bigoplus_i H(A_i) \arrow[r, "\bigoplus_i T_{A_i}"] \arrow[d, "H\varphi^{-1}"'] & \bigoplus_i L(A_i) \arrow[d, "L\varphi^{-1}"] \\ H(A) \arrow[r, "\widetilde{T}'_A"] & L(A) \end{tikzcd}$$

both squares are commutative, hence so is the whole diagram. But the vertical composition are identities, so $\widetilde{T}_A = \widetilde{T}'_A$ for all A, which shows that $\widetilde{T} = \widetilde{T}'$.

We now prove the existence part. For an object A, consider the same data $\{(A_i)_{i\in I},\ \varphi\}$ as above and define:

$$\widetilde{T}_A : H(A) \xrightarrow[\simeq]{H\varphi} \bigoplus_i H(A_i) \xrightarrow{\bigoplus_i T_{A_i}} \bigoplus_i L(A_i) \xrightarrow[\simeq]{L\varphi^{-1}} L(A)\ .$$

Claim: if B is another object, $\psi : B \simeq \bigoplus_{j\in J} B_j$ a decomposition with $B_j \in Ob(\mathcal{I})$ and $f \in \mathrm{Hom}_{\mathcal{A}}(A,B)$, then the following diagram commutes:

$$\begin{tikzcd} H(A) \arrow[r, "H\varphi", "\simeq"'] \arrow[d, "Hf"'] & \bigoplus_i H(A_i) \arrow[r, "\bigoplus_i T_{A_i}"] & \bigoplus_i L(A_i) \arrow[r, "L\varphi^{-1}", "\simeq"'] & L(A) \arrow[d, "Lf"] \\ H(B) \arrow[r, "\simeq", "H\psi"'] & \bigoplus_j H(B_j) \arrow[r, "\bigoplus_j T_{B_j}"'] & \bigoplus_j L(B_j) \arrow[r, "\simeq", "L\psi^{-1}"'] & L(B) \end{tikzcd} \tag{3.2.1}$$

Once we have proved the claim, we can conclude that:

- $\widetilde{T}_A$ does not depend on the choice of the decomposition of A (by posing $B = A$ and $f = \mathrm{Id}_A$);
- $\widetilde{T}$ is a natural transformation;
- Statement (2) holds (by the explicit definition of $\widetilde{T}$).

We proceed then check the commutativity of (3.2.1). By the universal property of the direct sum, for all $i \in I$ and $j \in J$ there exist morphisms $f_{ij} : A_i \to B_j$ such that the matrices $\widetilde{Hf} = (Hf_{ij})$ and $\widetilde{Lf} = (Lf_{ij})$ provide arrows which fill the two outer squares of (3.2.1). We are then reduced to check that the resulting central square

$$\begin{tikzcd} \bigoplus_i H(A_i) \arrow[r, "\bigoplus_i T_{A_i}"] \arrow[d, "\widetilde{Hf}"'] & \bigoplus_i L(A_i) \arrow[d, "\widetilde{Lf}"] \\ \bigoplus_j H(B_j) \arrow[r, "\bigoplus_j T_{B_j}"'] & \bigoplus_j L(B_j) \end{tikzcd} \tag{3.2.2}$$

commutes. First, we have $\widetilde{Lf} \circ (\bigoplus_i T_{A_i}) = \Big(Lf_{ij} \circ T_{A_i}\Big)_{i,j}$. Now, since each f_{ij} is a morphism in the subcategory $\mathcal{I}$ and by hypothesis $T = \widetilde{T} \circ i$ is natural, the latter is equal to $\Big(T_{B_j} \circ Hf_{ij}\Big)_{i,j} = (\bigoplus_j T_{B_j}) \circ \widetilde{Hf}$, which proves commutativity of (3.2.2) and then of (3.2.1) as wanted. □

Let s be a positive integer, let $\mathcal{A}$ be the full subcategory of $\boldsymbol{\mathcal{P}}_{p^s}$ consisting of additive super-functors with finite dimensional values, and let $\mathcal{I}$ be the full subcategory of $\mathcal{A}$ on the objects $\{\mathbf{I}_0^{(s)}, \mathbf{I}_1^{(s)}, \mathbf{\Pi} \circ \mathbf{I}_0^{(s)}\, \mathbf{\Pi} \circ \mathbf{I}_1^{(s)}\}$. It follows from Theorem 2.3.5 that $\mathcal{A}$ and $\mathcal{I}$ satisfy the hypothesis of Lemma 3.2.10. We keep this notation for the proof of the following proposition.

Proposition 3.2.11. *Let A and B be two additive superfunctors of degree p^s $(s \geq 1)$. There is an injective morphism of graded super vector spaces*

$$\sigma_{A,B} : \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)^{(r)} \to \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, I^{(r)} \circ B)$$

natural with respect to A and B, determined by the following conditions:

1. *When $A = B = \boldsymbol{I}_\ell^{(s)}$, $\sigma_{A,B}$ equals the map σ of Proposition 3.2.2.*
2. *When $A = \boldsymbol{I}_\ell^{(s)}$ and $B = \boldsymbol{I}_{\bar\ell}^{(s)}$, $\sigma_{A,B}$ equals the map σ' of Proposition 3.2.5.*
3. *When $A = \boldsymbol{\Pi} \circ \boldsymbol{I}_\ell^{(s)}$ and $B = \boldsymbol{I}_\ell^{(s)}$, $\sigma_{A,B}$ equals the map τ of Proposition 3.2.7.*
4. *When $A = \boldsymbol{\Pi} \circ \boldsymbol{I}_\ell^{(s)}$ and $B = \boldsymbol{I}_{\bar\ell}^{(s)}$, $\sigma_{A,B}$ equals the map τ' of Proposition 3.2.7.*
5. *There is a commutative square*

$$\begin{array}{ccc}
\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)^{(r)} & \xrightarrow{\sigma_{A,B}} & \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, I^{(r)} \circ B) \\
\simeq \big\downarrow \circ\boldsymbol{\Pi} & & \simeq \big\downarrow \circ\boldsymbol{\Pi} \\
\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A \circ \boldsymbol{\Pi}, B \circ \boldsymbol{\Pi})^{(r)} & \xrightarrow{\sigma_{A\circ\boldsymbol{\Pi}, B\circ\boldsymbol{\Pi}}} & \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A \circ \boldsymbol{\Pi}, I^{(r)} \circ B \circ \boldsymbol{\Pi}).
\end{array}$$

Proof. Whenever A and B are as in (1)-(4) set

$$\sigma_{A\circ\boldsymbol{\Pi}, B\circ\boldsymbol{\Pi}}(e^{(r)}) := (\sigma_{A,B}(e^{(r)} \circ \boldsymbol{\Pi})) \circ \boldsymbol{\Pi}$$

which gives by construction commutativity of the diagram in (5). Hence, $\sigma_{A,B}$ is completely determined by (1)-(5) when A, B are in $\mathcal{I}$. We want to conclude by invoking Lemma 3.2.10 with $\mathcal{A}$ and $\mathcal{I}$ as above, $\mathcal{B}$ the category of $\mathbb{Z}$-graded vector spaces of finite dimension degree-wise, $H = \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,-)^{(r)}$ (resp. $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(-,B)^{(r)}$) and $L = \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, I^{(r)} \circ -)$ (resp. $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ -, I^{(r)} \circ B)$) for a fixed A (resp. B) in $\mathcal{A}$. The only thing to verify is that H and L are additive in either definition. For both this is automatic from biadditivity of $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(-,-)$, additivity of the twist and additivity of the operation $I^{(r)} \circ -$. □

Let now $x_B := x \circ B$ for an extension x. Define the graded map

$$\begin{aligned}
\Psi_{A,B} : \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)^{(r)} &\to \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, G \circ B) \\
x \otimes e^{(r)} &\longmapsto x_B \cdot \sigma_{A,B}(e^{(r)}) .
\end{aligned}$$

Theorem 3.0.1 can now be proved in its generality.

Theorem 3.2.12. *For all additive homogeneous superfunctors A, B of degree p^s and for all homogeneous strict polynomial functor G of degree p^r, $\Psi_{A,B}$ is an isomorphism.*

Proof. When A and B are of the forms in (1)-(4) of the previous proposition, $\Psi_{A,B}$ is an isomorphism by respectively Propositions 3.2.4, 3.2.6 and 3.2.8. Then condition (5) of Proposition 3.2.11 implies that it is an isomorphism for all $A, B \in \mathcal{I}$. To conclude, we want to invoke Lemma 3.2.10. Take the same categorical settings as the previous proof and, for a fixed A, set:

$$H = \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,-)^{(r)}$$

$$L = \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, G \circ -)$$

resp. for a fixed B:

$$H = \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, G) \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(-,B)^{(r)}$$

$$L = \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ -, G \circ B)\ .$$

We have to prove that all these functors are additive. The only non straightforward case is L in its first definition, i.e. $L = \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, G \circ -)$. Let then $C, C' \in \mathcal{P}$ be additive. Let us first consider the case $G = S_V^{p^r}$. Recall that

$$S_V^{p^r} \circ (C \oplus C') \simeq \bigoplus_{a+b=p^r} S_V^a \circ C \ \otimes\ S_V^b \circ C'.$$

Apply now $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, -)$ to the last direct sum. Since $I^{(r)} \circ A$ is additive, by Pirashvili's vanishing lemma (2.3.4) the only terms which survive are the ones corresponding to $a = 0$ and $b = 0$, that is,

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, S_V^{p^r} \circ (C \oplus C')) \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, S_V^{p^r} \circ C) \ \oplus\ \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, S_V^{p^r} \circ C')$$

which proves that L is additive for $G = S_V^{p^r}$. For an arbitrary G, take an injective coresolution $G \hookrightarrow J^*$ and form the spectral sequences

$$\begin{aligned} E_1^{s,t} &= \mathrm{Ext}^t_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, J^s \circ C\ \oplus J^s \circ C') \Rightarrow \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, G \circ C\ \oplus\ G \circ C') \\ F_1^{s,t} &= \mathrm{Ext}^t_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, J^s \circ (C \oplus C')) \Rightarrow \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, G \circ (C \oplus C')) \end{aligned}$$

The inclusions $C, C' \subset C \oplus C'$ induce a morphism

$$\psi : \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, G \circ C\ \oplus\ G \circ C') \to \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(I^{(r)} \circ A, G \circ (C \oplus C'))$$

as well as a morphism of spectral sequences $\varphi : E^{*,*} \to F^{*,*}$ which equals ψ on the abutment. But we know that φ is an isomorphism on the first page, since the J^s are injective. Then ψ is an isomorphism and the theorem is proved. $\square$

Chapter 4

Extensions between twisted functors

In the previous chapter (Corollary 3.0.2) we positively solved Conjecture 0.4 with the additional assumption that the first functor F is additive. Dropping this hypothesis forces us to change completely approach. The idea that we develop in this chapter is to approach $\mathrm{Ext}^*_{\mathcal{P}}(F \circ A, G \circ B)$ by means of a spectral sequence, which identifies to $\mathrm{Ext}^*_{\mathcal{P}}(F, G_{\mathrm{Ext}_{\mathcal{P}}(A,B)})$ at the second page. Our conjecture can thus be reformulated in terms of the collapsing of this spectral sequence (Conjecture 4.1.5). We mainly focus on the cases $A = B = \mathbf{I}_0^{(r)}$ and $A = \mathbf{I}_0^{(r)}, B = \mathbf{I}_1^{(r)}$, which are expected to tell the greatest part of the story . Since *a priori* we know nothing about the differentials of the spectral sequence, we start by comparing it to its classical counterpart which is completely understood [Tou13]. It is furthermore possible to compare it with itself, by means of morphisms induced by cup product with special classes (Proposition 4.3.16 and 4.4.1). In fact, deducing the existence and the properties of such classes is in some sense the real new content of the theory. At the end, we will be able to come out with a partial (but satisfying) response to our original question.

4.1 The twisting spectral sequence

As anticipated in the introduction, the graded isomorphism that we desire is meant to come from the study of a spectral sequence that we introduce in this section. Its construction is highly standard, as we see in the next proposition, but having found an explicit formula for the right adjoint of the twisting functor is fundamental to extract information from it. The reader can refer to [Wei94] for the generalities about spectral sequences.

Proposition 4.1.1. *Let $\mathcal{A}, \mathcal{B}$ be abelian $\Bbbk$-linear categories with enough projectives and injectives, $c_1, c_2 : \mathcal{A} \to \mathcal{B}$ two $\Bbbk$-linear functors, with c_1 exact. Denote by ρ_1 the right adjoint of c_1 and by $\boldsymbol{R}^*\rho_1$ its right derived functor. Then, for all $F, G \in \mathcal{A}$, there is a cohomological spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}}(F, \boldsymbol{R}^t\rho_1(c_2G)) \Rightarrow \mathrm{Ext}^{s+t}_{\mathcal{B}}(c_1F, c_2G) \quad . \tag{4.1.1}$$

Proof. Let P_* be a projective resolution of F and J^* an injective coresolution of c_2G. Consider the bicomplex $E_0^{s,t} := \mathrm{Hom}_{\mathcal{B}}(c_1P_s, J^t)$. Since c_1 is exact, the homology of its totalization computes $\mathrm{Ext}^*_{\mathcal{B}}(c_1F, c_2G)$, which is then the abutment of the associated spectral sequence. Now, by adjunction $\mathrm{Hom}_{\mathcal{B}}(c_1P_s, J^t) \simeq \mathrm{Hom}_{\mathcal{A}}(P_s, \rho_1J^t)$ so the first page is isomorphic to $\mathrm{Hom}_{\mathcal{A}}(P_s, \mathbf{R}^t\rho_1(c_2G))$. By consequence, the second page identifies to $\mathrm{Ext}^s_{\mathcal{A}}(F, \mathbf{R}^t\rho_1(c_2G))$. $\square$

It is then a first quadrant spectral sequence with the differential going right-downwards. Since the Ext-groups are by hypothesis $\Bbbk$-vector spaces, its collapsing is controlled by the following criterion.

Proposition 4.1.2. *Suppose that* $\mathrm{Ext}^i_{\mathcal{B}}(c_1F, c_2G)$ *is finite-dimensional for all* i*. Then the spectral sequence (4.1.1) collapses at the second page if and only if there is a graded isomorphism (not necessarily natural) with respect to total degree on the right side*

$$\mathrm{Ext}^*_{\mathcal{B}}(c_1F, c_2G) \simeq \mathrm{Ext}^*_{\mathcal{A}}(F, \boldsymbol{R}^*\rho_1(c_2G)) \quad .$$

When $B = \boldsymbol{\mathcal{P}}_d$ ($d \geq 0$), the criterion of Proposition 4.1.2 applies in the same way even if $\boldsymbol{\mathcal{P}}_d$ is $\mathbb{k}$-superlinear (and not abelian). That is because the Ext in $\boldsymbol{\mathcal{P}}_d$ are computed by passing to an abelian subcategory of $\boldsymbol{\mathcal{P}}_d$, as explained in §1.5.1. The same formal transition to $\boldsymbol{\mathcal{P}}_{ev}$ is made in order for the right derived functor $\mathbf{R}^*\rho_A$ to make sense (since in our computations we consider it as a functor $\boldsymbol{\mathcal{P}} \to \mathcal{P}$). Finally, the hypothesis of finiteness in Proposition 4.1.2 is ensured in $\boldsymbol{\mathcal{P}}_d$ by Lemma 1.6.4.

We can then apply this machinery to our context. Fix from now $r \geq 1$, $d \geq 0$. Set $\mathcal{A} = \mathcal{P}_d$, $\mathcal{B} = \boldsymbol{\mathcal{P}}_{dp^r}$ and take $F, G \in \mathcal{P}_d$. For additive superfunctors $A, B \in \boldsymbol{\mathcal{P}}_{p^r}$, set $c_1 = \Phi_A$, $c_2 = \Phi_B$ the precomposition functors defined in §2.3.2. We will drop the cryptic notation $\varphi_A(F)$ in favor of the explicit $F \circ A$. With these settings, the spectral sequence of Proposition 4.1.1 reads

$$\boldsymbol{II}^{s,t}_{A,B} := \mathrm{Ext}^s_{\mathcal{P}_d}(F, \mathbf{R}^t\rho_A(G \circ B)) \Rightarrow \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}_{dp^r}}(F \circ A, G \circ B) \quad . \tag{4.1.2}$$

The existence alone of such a spectral sequence implies a general but very interesting fact.

Theorem 4.1.3. *For all* $F, G \in \mathcal{P}_d$ *and* $\ell, m \in \{0, 1\}$ *we have the following:*

- $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F^{(r)}_\ell, G^{(r)}_m)$ *is a purely even superspace.*
- $\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}(F^{(r)}_\ell \circ \mathbf{\Pi}, G^{(r)}_m)$ *and* $\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}(F^{(r)}_\ell, G^{(r)}_m \circ \mathbf{\Pi})$ *are purely even (resp. odd) if* d *is even (resp. odd).*

Proof. By the discussion in Section 2.3.4, the Ext at the second page bears a superdegree that is exactly the one specified in the respective cases. In particular, the abutment is concentrated in the same respective superdegrees. □

The right derived functor appearing in the formula is easily computed thanks to Proposition 2.3.26.

Lemma 4.1.4. *Let* $A \in \boldsymbol{\mathcal{P}}_{p^r}$ *be additive and let* $H \in \boldsymbol{\mathcal{P}}_{dp^r}$*. Then* $\boldsymbol{R}^*\rho_A(H)$ *is the graded* d*-homogeneous classical polynomial functor defined by*

$$V \longmapsto \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}_{dp^r}}(\Gamma^{d,V} \circ A, H)$$

where the Ext *is seen as an ungraded space.*

We will prove later (Proposition 4.2.6) that, if G is a classical polynomial functor and B is an additive polynomial superfunctor, then $\mathbf{R}^*\rho_A(G \circ B) \simeq G_{\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)}$ naturally with respect to G. In view of this and of Prop 4.1.2, we can reformulate Conjecture 0.4 as follows.

Conjecture 4.1.5. *For all* $F, G \in \mathcal{P}_d$ *the spectral sequence (4.1.2) collapses at the second page. In particular, there is a graded isomorphism (a priori not natural)*

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}_{dp^r}}(F \circ A, G \circ B) \simeq \mathrm{Ext}^*_{\mathcal{P}_d}(F, G_{\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)}) \,.$$

Convention 4.1.6. As in Chapter 3, we suppress from now the polynomial degree from the notations $\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}$ and $\mathrm{Ext}_{\mathcal{P}}$ in order to be short. Nevertheless, for the reasons explained in Convention 1.5.5, our (super)functors will still be assumed to be homogeneous all along: F, G of degree d and A, B of degree p^r, $r \geq 1$.

4.2 The right derived functor of ρ_A

A crucial step to study the twisting spectral sequence is to compute the right derived functor $\mathbf{R}^*\rho_A$ in an explicit form. This will be equivalent to proving the validity of Conjecture 4.1.5 when G is injective. We start to do this by assuming, in addition, G to be injective. Consider the transformation, natural in A, B, V, W,

$$(\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)\otimes V\otimes W)^{\otimes d}\simeq(\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A\otimes W^\vee, B\otimes V))^{\otimes d}\to\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\Gamma^{d,W}\circ A, S^d_V\circ B)\quad(4.2.1)$$

induced by additivity of A and B and by cup product.

Lemma 4.2.1. *The cup product (4.2.1) factors through a map*

$$\eta_{A,B}(V,W): S^d(\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)\otimes V\otimes W)\longrightarrow\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\Gamma^{d,W}\circ A, S^d_V\circ B)\;.\quad(4.2.2)$$

Proof. Since the parameters V, W do not affect the cup product, the essential of the proof carries out the same if we suppose $V = W = \Bbbk$. It is also enough to treat the case $d = 2$. With these assumptions, let us start from the case where A, B are indecomposable. Taken $e\in\mathrm{Ext}^s_{\boldsymbol{\mathcal{P}}}(A,B)$ and $f\in\mathrm{Ext}^t_{\boldsymbol{\mathcal{P}}}(A,B)$, we have to prove that $e\cup f = f\cup e$ (with no sign: note that we are taking S^d - not $\mathbf{S}^d$ - and everything here lives in even superdegree). We treat case by case and to conclude we always use Proposition 1.6.12 (with $i = j = 1$). Remember that $S_0^{*(r)}$ and $\Gamma_0^{*(r)}$ are bicommutative superbialgebras, while $S_1^{*(r)}$ and $\Gamma_1^{*(r)}$ are graded-bicommutative (cf. Remark 1.3.15). First let us consider the cases where $\overline{e} = \overline{f} = 0$, which are the following ones.

- $A = B = \mathbf{I}_0^{(r)}$: in this case s, t are even (since $\boldsymbol{E_r}$ is concentrated in purely even cohomological degrees). Moreover, in the notation of Proposition 1.6.12, $\varepsilon(S_0^{*(r)}) = \varepsilon(\Gamma_0^{*(r)}) = 0$. Hence by the latter proposition we conclude $e\cup f = f\cup e$.
- $A = B = \mathbf{I}_1^{(r)}$: s, t are still even, but $\varepsilon(S_0^{*(r)}) = \varepsilon(\Gamma_0^{*(r)}) = 1$. Nonetheless, Proposition 1.6.12 gives again $e\cup f = f\cup e$.
- $A = \mathbf{I}_0^{(r)}, B = \mathbf{I}_1^{(r)}$: here $\varepsilon(\Gamma_0^{*(r)}) = 0$ and $\varepsilon(S_1^{*(r)}) = 1$. In contrast, s, t are odd because $\overline{\boldsymbol{E}}_{\boldsymbol{r}}$ is concentrated in odd cohomological degrees. Hence there are two signs -1 in the formula of Proposition 1.6.12 which cancel, giving $e\cup f = f\cup e$ as desired.
- $A = \mathbf{I}_1^{(r)}, B = \mathbf{I}_0^{(r)}$: symmetric to the previous one.

We invoke the same Proposition for the other cases where $\overline{e} = \overline{f} = 1$. Remark also that $\varepsilon(A\circ\mathbf{\Pi}) = \varepsilon(A)$ for any $\boldsymbol{\mathcal{P}}$-(co)algebra A.

- $A = \mathbf{I}_0^{(r)}, B = \mathbf{I}_1^{(r)}\circ\mathbf{\Pi}$: here $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)\simeq\mathbf{\Pi}\boldsymbol{E_r}$, hence s, t are even. Since $\varepsilon(\Gamma_0^{*(r)}) = 0$, $\varepsilon(S_1^{*(r)}\circ\mathbf{\Pi}) = 1$, we obtain $e\cup f = f\cup e$.
- $A = \mathbf{I}_0^{(r)}, B = \mathbf{I}_0^{(r)}\circ\mathbf{\Pi}$: same conclusion, since now $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)\simeq\mathbf{\Pi}\overline{\boldsymbol{E}}_{\boldsymbol{r}}$, hence s, t are odd and $\varepsilon(\Gamma_0^{*(r)}) = 0$, $\varepsilon(S_0^{*(r)}\circ\mathbf{\Pi}) = 0$.

These two cases are also symmetrical in A, B. Finally, the cases where both A, B are twists precomposed by $\mathbf{\Pi}$ follow from the even isomorphism $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F,G)\simeq\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F\circ\mathbf{\Pi}, G\circ\mathbf{\Pi})$. We conclude that the lemma is true whenever A and B are indecomposable. To deal with the general case, by an argument of induction and Kuhn duality we are reduced to study the cases where $A = \mathbf{I}_0^{(r)}$ and $B = \mathbf{I}_\ell^{(r)}\oplus\mathbf{I}_m^{(r)}$ for $\ell, m\in\{0,1\}$.

- If $\ell = m = 0$, then $S^*\circ B\simeq S_0^{*(r)}\otimes S_0^{*(r)}$ is commutative.
- If $\ell = m = 1$, then $S^*\circ B\simeq S_1^{*(r)}\otimes^g S_1^{*(r)}$ is graded-commutative.

In either case we can conclude again using Proposition 1.6.12. We are just left with the case $\ell = 0, m = 1$ (the other one being symmetric) i.e. $B = \mathbf{I}^{(r)}$. Remark that $S^* \circ B \simeq S_0^{(r)} \otimes S_1^{(r)}$ contains $S_0^{(r)}$ and $S_1^{(r)}$ as sub-superalgebras. Take now $e \in \mathrm{Ext}^s_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(r)}, \mathbf{I}^{(r)})$, $f \in \mathrm{Ext}^t_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(r)}, \mathbf{I}^{(r)})$. Since $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(r)}, \mathbf{I}^{(r)}) = \boldsymbol{E_r} \oplus \overline{\boldsymbol{E}}_{\boldsymbol{r}}$, there are in turn three possibilities:

- s, t are even: then e, f belong to $\boldsymbol{E_r}$, hence $e \cup f \in \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(\Gamma_0^{2(r)}, S_0^{2(r)})$. We then conclude that $e \cup f = f \cup e$ using Proposition 1.6.12 and the commutativity of $S_0^{*(r)}$.
- s, t are odd: then e, f belong to $\overline{\boldsymbol{E}}_{\boldsymbol{r}}$, hence $e \cup f \in \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(\Gamma_0^{2(r)}, S_1^{2(r)})$. We conclude using Proposition 1.6.12 and the graded-commutativity of $S_1^{*(r)}$.
- s is even and t is odd: the $e \in \boldsymbol{E_r}$ and $f \in \overline{\boldsymbol{E}}_{\boldsymbol{r}}$. In this case st is even and the following diagram commutes

$$\begin{tikzcd} \mathbf{I}_0^{(r)} \otimes \mathbf{I}_1^{(r)} \arrow[dd, "T"'] \arrow[dr, "m"] & \\ & S^2(\mathbf{I}_0^{(r)} \oplus \mathbf{I}_1^{(r)}) \\ \mathbf{I}_1^{(r)} \otimes \mathbf{I}_0^{(r)} \arrow[ur, "m"'] & \end{tikzcd}$$

whence we deduce that $e \cup f$ equals $f \cup e$ as an element of $\mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(\Gamma_0^{2(r)}, S^2(\mathbf{I}_0^{(r)} \oplus \mathbf{I}_1^{(r)}))$.

That completes the proof. □

Hence we have a natural transformation of polynomial functors

$$\eta_{A,B}(V, -) : (S^d_V)_{\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)} \longrightarrow \mathbf{R}^* \rho_A(S^d_V \circ B) \tag{4.2.3}$$

which we are going to show to be an isomorphism. The basic case where $V = \Bbbk$ and A, B are Frobenius twists is exhausted by the following result.

Theorem 4.2.2 ([DK22, Thm 5.1.2])**.** *For all $\ell, m \in \{0, 1\}$, the cup product map (4.2.2) yields a graded natural isomorphism*

$$S^d(\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\boldsymbol{I}_\ell^{(r)}, \boldsymbol{I}_m^{(r)})) \xrightarrow{\simeq} \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\Gamma_\ell^{d(r)}, S_m^{d(r)}) .$$

Starting from this result, we are able to prove that (4.2.2) - and therefore (4.2.3) - is an isomorphism for all A, B, V.

Proposition 4.2.3. *The map (4.2.3) is a graded natural isomorphism*

$$(S^d_V)_{\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)} \simeq \boldsymbol{R}^* \rho_A(S^d_V \circ B)$$

for all V and all A, B.

Proof. We prove in steps that $\eta_{A,B}(V, W)$ is an isomorphism for all V, W.

- **Step 1:** proof for $V = W = \Bbbk$. Call $\eta_{A,B} = \eta_{A,B}(\Bbbk, \Bbbk)$. If $A = \mathbf{I}_\ell^{(r)}$ and $B = \mathbf{I}_m^{(r)}$, $\ell, m \in \{0, 1\}$, then we know from Theorem 4.2.2 that $\eta_{A,B}$ is an isomorphism. If one between A, B or both are precomposed by $\boldsymbol{\Pi}$, the assertion follows from Proposition 2.2.6. Hence all cases where A, B are indecomposables are settled. To conclude, we have to show that $\eta_{A \oplus A', B}$ is an isomorphism whenever $\eta_{A,B}$ and $\eta_{A',B}$ are (and the same for $\eta_{A, B \oplus B'}$, whose proof is entirely symmetric). This follows from the fact that $\eta_{A\oplus A',B}$ equals the composition

$$\begin{array}{c}
S^d(\mathrm{Ext}^*_{\mathcal{P}}(A\oplus A', B)) \\
\big\downarrow\simeq \\
\bigoplus_{a=0}^d S^{d-a}(\mathrm{Ext}^*_{\mathcal{P}}(A,B))\otimes S^a(\mathrm{Ext}^*_{\mathcal{P}}(A',B)) \\
\big\downarrow \sum_a \eta_{A,B}\otimes\eta_{A',B} \\
\bigoplus_{a=0}^d \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^{d-a}\circ A, S^{d-a}\circ B)\otimes \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^a\circ A', S^a\circ B) \\
(*)\big\downarrow\simeq \\
\bigoplus_{a=0}^d \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^{d-a}\circ A \,\otimes\, \Gamma^a\circ A', S^d\circ B) \\
\big\downarrow\simeq \\
\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^d\circ(A\oplus A'), S^d\circ B)
\end{array}$$

where $(*)$ is induced by cup product and is an isomorphism by Corollary 2.3.23.

- **Step 2:** proof for $W=\Bbbk$. Then $\eta_{A,B}(V,\Bbbk)$ factors through the composition

$$\begin{array}{ccc}
S^d(\mathrm{Ext}^*_{\mathcal{P}}(A,B)\otimes V) & \simeq & \bigoplus_{\lambda\in\Lambda(\dim V,d)} S^\lambda(\mathrm{Ext}^*_{\mathcal{P}}(A,B)) \\
 & & \big\downarrow \sum_\lambda(\eta_{A,B})^{\otimes\dim V} \\
 & & \bigoplus_{\lambda\in\Lambda(\dim V,d)} \bigotimes_i \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^{\lambda_i}\circ A, S^{\lambda_i}\circ B) \\
 & & \big\downarrow\simeq \\
 & & \bigoplus_{\lambda\in\Lambda(\dim V,d)} \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^d\circ A, S^\lambda\circ B) \\
 & & \big\downarrow\simeq \\
 & & \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^d\circ A, S^d_V\circ B)
\end{array}$$

  where the first vertical arrow is an isomorphism by Step 1 and the second one by Corollary 2.3.23 again.

- **Step 3:** we assert that, for all composition λ of length n, there is a graded isomorphism $S^\lambda(\mathrm{Ext}^*_{\mathcal{P}}(A,B)\otimes V)\simeq \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\lambda\circ A, S^d_V\circ B)$ induced by $\eta_{A,B}(V,\Bbbk)^{\otimes n}$ and cup product. Indeed, Step 2 provides such isomorphism for $\lambda=(d)$. Generalisation to any composition comes from Corollary 2.3.24.

- **Step 4:** conclusion. Analogously to Step 2, $\eta_{A,B}(V,W)$ decomposes as the sum of the restrictions

$$\eta_{A,B}(V,\Bbbk)^{\otimes\dim W}:\ S^\lambda(\mathrm{Ext}^*_{\mathcal{P}}(A,B)\otimes V)\longrightarrow \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^\lambda\circ A, S^d_V\circ B)$$

  for λ ranging through $\Lambda(\dim W,d)$. Each one is an isomorphism in force of Step 3. It follows that so is $\eta_{A,B}(V,W)$.

This concludes the proof. □

For later use, we keep track of the following special case treated explicitly in the proof.

Corollary 4.2.4. *For any finite composition λ, cup product induces a natural isomorphism*

$$S^\lambda(\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)\otimes V)\simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\Gamma^\lambda\circ A, S^d_V\circ B)$$

and in particular

$$(\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)\otimes V)^{\otimes d}\simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\otimes^d\circ A, S^d_V\circ B)\ . \tag{4.2.4}$$

In particular, Proposition 4.2.3 says that Conjecture 4.1.5 is true whenever F is projective and G is injective. Using this very same computation, we will now be able to extend the result to any polynomial functor F. Explicitly, we want to compute $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F\circ A, S^d_V\circ B)$. Denote by t the grading on the parametrised functor $(S^d_V)_{\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)} = S^d_{V\otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)}$. Thanks to Proposition 4.2.3, the spectral sequence (4.1.2) for $G=S^d_V$ reads at the second page:

$$\mathrm{Ext}^s_{\mathcal{P}}(F,(S^d_{V\otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)})^t)\Rightarrow \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(F\circ A, S^d_V\circ B)\ .$$

Since $S^d_{V\otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)}$ is injective in each degree, the sequence is concentrated in a row and hence collapses at the second page. This proves Conjecture 4.1.5 when $G=S^d_V$. We prove an even stronger assertion in the following theorem.

Theorem 4.2.5. *There is a graded isomorphism, natural in all variables*

$$F^\#(\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)\otimes V)\simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F\circ A, S^d_V\circ B) \tag{4.2.5}$$

which, in the case $F=\Gamma^{d,W}$, coincides with the cup product (4.2.3).

Proof. Since the spectral sequence is concentrated in a single row, the edge homomorphisms induce a natural isomorphism

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F\circ A, S^d_V\circ B)\simeq \mathrm{Hom}_{\mathcal{P}}(F,\mathbf{R}^*\rho_A(S^d_V))\ .$$

If we call again $\eta_{A,B}(V):=\eta_{A,B}(V,-)$ the mentioned cup product map, the isomorphism (4.2.5) is then induced by

$$\mathrm{Hom}_{\mathcal{P}}(F,\eta_{A,B}(V)):\mathrm{Hom}_{\mathcal{P}}(F,S^d_{V\otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)})\to \mathrm{Hom}_{\mathcal{P}}(F,\mathbf{R}^*\rho_A(S^d_V))$$

and by Yoneda lemma

$$\mathrm{Hom}_{\mathcal{P}}(F,S^d_{V\otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)})\simeq F^\#(\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)\otimes V)\ .$$

Finally, set $F=\Gamma^{d,W}$ in all these morphisms. Since Yoneda lemma is natural in all variables, the map $\mathrm{Hom}_{\mathcal{P}}(\Gamma^{d,W},\eta_{A,B}(V))$ identifies with $\eta_{A,B}(V,W)\,:\,S^d(W\otimes V\otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B))\longrightarrow \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\Gamma^{d,W}\circ A, S^d_V\circ B)$, as stated. □

Our next step is using (4.2.5) to identify $\mathbf{R}^*\rho_A(G\circ B)$ for all $G\in\mathcal{P}_d$.

Proposition 4.2.6. *There is a natural graded isomorphism of polynomial functors*

$$\boldsymbol{R}^*\rho_A(G\circ B)\simeq G_{\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)}\ .$$

Proof. Dualise and use (4.2.5) to get

$$\begin{aligned}\mathbf{R}^*\rho_A(G\circ B)\,(V) = \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\Gamma^{d,V}\circ A, G\circ B) &\simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(G^\#\circ B, S^d_V\circ A)\\ &\simeq \mathrm{Hom}_{\mathcal{P}}(G^\#, S^d_{\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(B,A)\otimes V})\simeq G(\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(B,A)\otimes V)\end{aligned}$$

natural in G and V. Since $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(B,A)\simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)$ by Lemma 2.3.13, the formula follows. □

We make a very short detour to point out a remark that will be important later. Call by m_{S_A} the multiplication map $(W \otimes A)^{\otimes d} \to S^d_W \circ A$ and define the morphism

$$\begin{tikzcd} \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(S^d_W \circ A, S^d_V \circ B) \arrow[r, "-\cdot m_{S_A}"] & \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}((W \otimes A)^{\otimes d}, S^d_V \circ B) \arrow[d, "\simeq"] \\ & (W^\vee \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B) \otimes V)^{\otimes d} \end{tikzcd}$$

where the vertical isomorphism is the inverse of (4.2.4). By Proposition 1.6.12, this is Σ_d-invariant and thus factors into a morphism

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(S^d_W \circ A, S^d_V \circ B) \longrightarrow \Gamma^d(W^\vee \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B) \otimes V)\ . \tag{4.2.6}$$

Lemma 4.2.7. *(4.2.6) is an isomorphism and coincides with (4.2.5) in the special case* $F = S^d_W$.

Proof. Naturality of the isomorphism (4.2.5) applied with respect to the multiplication map $m_{S_A} : (W \otimes A)^{\otimes d} \to S^d_W \circ A$ gives a commutative diagram

$$\begin{tikzcd} \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(S^d_W \circ A, S^d_V \circ B) \arrow[r, "-\cdot m_{S_A}"] & \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}((W \otimes A)^{\otimes d}, S^d_V \circ B) \arrow[d, "\simeq"] \\ \Gamma^d(W^\vee \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B) \otimes V) \arrow[u, "\simeq"] \arrow[r, hook] & (W^\vee \otimes \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B) \otimes V)^{\otimes d} \end{tikzcd}$$

where the two vertical arrows are (tautologically) the ones given by (4.2.5) in the respective cases $F = S^d_W$ and $F = (W \otimes \mathbf{I})^{\otimes d}$. The right one coincides with the cup product map (4.2.4): this follows from Theorem 4.2.5, since $\otimes^d$ is a direct summand of $\Gamma^{d,\Bbbk^d}$. We then conclude that, by construction, the left one is exactly (4.2.6). □

4.3 The isomorphism in low degrees

We keep in mind the notation introduced at the beginning of Chapter 3:

$$\begin{aligned} E_r^* &:= \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)}),\\ \boldsymbol{E_r}^* &:= \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(r)}, \mathbf{I}_0^{(r)}) \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_1^{(r)}, \mathbf{I}_1^{(r)}),\\ \overline{\boldsymbol{E_r}}^* &:= \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_0^{(r)}, \mathbf{I}_1^{(r)}) \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}_1^{(r)}, \mathbf{I}_0^{(r)}) \end{aligned}$$

where again we will drop the symbol $*$ if no explicit reference to the grading is needed. Since r is fixed at this point, the notation $e_{i,r}$ for the generators of $\boldsymbol{E_r}$ (cf. Theorem 3.1.4) will be shortened into e_i.

4.3.1 First case: same parity twists

In this section we prove a weak version of Conjecture 4.1.5. In fact, we will prove it in low degrees but, in exchange, we gain naturality of the isomorphism. We specialise for the moment to the case $A = B = \mathbf{I}_0^{(r)}$, which by conjugation yields the parallel result for $A = B = \mathbf{I}_1^{(r)}$. Applying Proposition 4.2.6, the spectral sequence (4.1.2) becomes:

$$\boldsymbol{II}^{s,t} := \boldsymbol{II}^{s,t}_{\mathbf{I}_0^{(r)},\mathbf{I}_0^{(r)}} = \mathrm{Ext}^s_{\mathcal{P}}(F, (G_{\boldsymbol{E_r}})^t) \Rightarrow \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)}) \quad . \tag{4.3.1}$$

Our work in this section is based on the comparison of our spectral sequence with the classical one of same type. More precisely, this sequence is given for all $F, G \in \mathcal{P}$ by Proposition 4.1.1 applied to the classical setting:

$$II^{s,t} := \mathrm{Ext}^s_{\mathcal{P}}(F, (G_{E_r})^t) \Rightarrow \mathrm{Ext}^{s+t}_{\mathcal{P}}(F^{(r)}, G^{(r)})\ . \tag{4.3.2}$$

It is known that this sequence collapses at the second page [Tou13, Cor. 5], but there is a stronger result that can be found for example in [Cha15, Cor. 3.7]:

Theorem 4.3.1. *There is a graded natural isomorphism* $\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r})$.

In order to compare $\boldsymbol{II}^{*,*}$ and $II^{*,*}$ we will make use of a restriction morphism. Set $i_0 : \mathcal{V} \to \boldsymbol{\mathcal{V}}$, $i_0(V) = V \oplus 0$ and $u : \boldsymbol{\mathcal{V}} \to \mathcal{V}$ the functor which forgets the $\mathbb{Z}/2\mathbb{Z}$-grading.

Definition 4.3.2. For a polynomial superfunctor F, define $res_0 F := u \circ F \circ i_0$. This operation defines an exact functor $\boldsymbol{\mathcal{P}}_{ev} \longrightarrow \mathcal{P}$.

In words, restricting a superfunctor means just to evaluate it on purely even spaces and consider the result as an ungraded space. The theorem we are going to prove can be stated in two nearly equivalent fashions. The first one gives an immediate comparison between the super and classical twists, while the second one is really related to the Conjecture 4.1.5. Both proofs will be given at the end of the section.

Theorem 4.3.3. *For all $n < 2p^r$, the restriction morphism*

$$res_0 : \mathrm{Ext}^n_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)}) \longrightarrow \mathrm{Ext}^n_{\mathcal{P}}(F^{(r)}, G^{(r)})$$

is an isomorphism.

Theorem 4.3.4. *For all $n \le 2p^r + 1$, the spectral sequence (4.3.1) induces an isomorphism*

$$\mathrm{Ext}^n_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)}) \simeq \bigoplus_{s+t=n} \mathrm{Ext}^s_{\mathcal{P}}(F, (G_{\boldsymbol{E_r}})^t)$$

which is natural in F, G if $n < 2p^r$.

Let us start by some simple properties of res_0. Firstly, it maps $\boldsymbol{\mathcal{P}}_d$ onto $\mathcal{P}_d$. Moreover, it is immediate from the definition that the restricted functor $res_0 : \boldsymbol{\mathcal{P}}_{ev} \to \mathcal{P}$ is exact.

Lemma 4.3.5.
1. *For all $G \in \boldsymbol{\mathcal{P}}$, $res_0(G_0^{(r)}) = (res_0 G)^{(r)}$.*
2. *For all $F \in \mathcal{P}$, $res_0(F_0^{(r)}) = F^{(r)}$.*
3. *For all $G, H \in \boldsymbol{\mathcal{P}}$, $res_0(G \otimes H) \simeq res_0(G) \otimes res_0(H)$.*

Proof. For (1), evaluate on $V \in \mathcal{V}$ to have $(res_0(G_0^{(r)}))\,(V) = G_0^{(r)}(V \oplus 0) = G(V^{(r)} \oplus 0) = (res_0 G)(V^{(r)}) = (res_0 G)^{(r)}(V)$. The verification for (2) is analogous, while (3) is immediate from the definition of tensor product of polynomial superfunctors. □

Since it is an exact functor, res_0 induces for all $F, G \in \boldsymbol{\mathcal{P}}$ a morphism $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F, G) \to \mathrm{Ext}^*_{\mathcal{P}}(res_0 F, res_0 G)$. In the following proposition we make res_0 into a morphism between our two spectral sequences. To make no confusion, note by π_r the particular restriction $res_0 : \boldsymbol{E_r} \to E_r$. Remark that, as a map of graded spaces, it is the projection on the first $2p^r - 1$ degrees.

Proposition 4.3.6. *There exists a morphism of spectral sequences $\boldsymbol{II}^{*,*} \to II^{*,*}$ that identifies*

- *with $G(\pi_r)_* : \mathrm{Ext}^*_{\mathcal{P}}(F, G_{\boldsymbol{E_r}}) \to \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r})$ on the second pages,*
- *with res_0 on the abutments.*

Proof. We have to manipulate the explicit construction of our spectral sequences, namely the one made in general in the proof of Proposition 4.1.1 and its classical counterpart. Let $J^* \hookleftarrow G_0^{(r)}$ be an injective coresolution and $P_* \twoheadrightarrow F$ be a projective resolution. Then the spectral sequence

$\boldsymbol{II}^{*,*}$ is induced by the bicomplex[1] $\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}((P_s)_0^{(r)}, J^t)$. We proceed to define our restriction morphism of spectral sequences. Take K^* an injective coresolution of $G^{(r)}$ in $\mathcal{P}$. Since $res_0(J^*)$ is a (not injective) coresolution of $G^{(r)}$, there exists a morphism of complexes $res_0(J^*) \to K^*$ lifting the identity. Define the restriction morphism at the page zero by the composite

$$\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}((P_*)_0^{(r)}, J^*) \xrightarrow{res_0} \mathrm{Hom}_{\mathcal{P}}(P_*^{(r)}, res_0(J^*)) \to \mathrm{Hom}_{\mathcal{P}}(P_*^{(r)}, K^*)\ . \tag{4.3.3}$$

Note that the right-most bicomplex gives rise to $II^{*,*}$. The morphism (4.3.3) identifies to the restriction of extensions between the abutments

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)}) \to \mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)})$$

and between the first pages

$$\mathrm{Ext}^t_{\boldsymbol{\mathcal{P}}}((P_s)_0^{(r)}, G_0^{(r)}) \to \mathrm{Ext}^t_{\mathcal{P}}((P_s)^{(r)}, G^{(r)})\ . \tag{4.3.4}$$

Now by (4.2.5) there is an isomorphism

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}((P_s)_0^{(r)}, G_0^{(r)}) \simeq \mathrm{Hom}_{\mathcal{P}}(P_s, G_{\boldsymbol{E_r}}) \tag{4.3.5}$$

by which we compute the second page. In the classical case, there is an analogous isomorphism

$$\mathrm{Ext}^*_{\mathcal{P}}(P_s^{(r)}, G^{(r)}) \simeq \mathrm{Hom}_{\mathcal{P}}(P_s, G_{E_r})\ . \tag{4.3.6}$$

Hence, to conclude we have to verify the commutativity of the following square:

$$\begin{tikzcd}
\mathrm{Ext}^t_{\boldsymbol{\mathcal{P}}}((P_s)_0^{(r)}, G_0^{(r)}) \arrow[r, "(4.3.4)"] \arrow[d, "(4.3.5)"'] & \mathrm{Ext}^t_{\mathcal{P}}((P_s)^{(r)}, G^{(r)}) \arrow[d, "(4.3.6)"] \\
\mathrm{Hom}_{\mathcal{P}}(P_s, (G_{\boldsymbol{E_r}})^t) \arrow[r, "G(\pi_r)_*"'] & \mathrm{Hom}_{\mathcal{P}}(P_s, (G_{E_r})^t)\ .
\end{tikzcd}$$

First note that, for any polynomial functor H and any natural transformation $T : G \to H$, the diagram

$$\begin{tikzcd}
G(\boldsymbol{E_r} \otimes -) \arrow[r, "G(\pi_r)"] \arrow[d, "T_{\boldsymbol{E_r}}"'] & G(E_r \otimes -) \arrow[d, "T_{E_r}"] \\
H(\boldsymbol{E_r} \otimes -) \arrow[r, "H(\pi_r)"'] & H(E_r \otimes -)
\end{tikzcd}$$

commutes (by definition of naturality). This implies that, if G is not injective, one can argue by taking an injective coresolution, since all sides of the diagram are exact with respect to G. We then assume G injective. Let us then suppose $P_s := \Gamma^{d,W}$ and $G := S^d_V$. In this case, by Theorem 4.2.5, the isomorphism $S^d(W \otimes V \otimes \boldsymbol{E_r}) \simeq \mathrm{Hom}_{\mathcal{P}}(\Gamma^{d,W}, (S^d_V)_{\boldsymbol{E_r}}) \simeq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}((\Gamma^{d,W})_0^{(r)}, (S^d_V)_0^{(r)})$, as well as its classical analogue, is provided explicitly by the cup product $e_1 \cdots e_d \longmapsto e_1 \cup ... \cup e_d$. Moreover, Lemma 4.3.5(3) implies that res_0 commutes with the cup product. Hence (4.3.4) identifies to the morphism $S^d_V(\boldsymbol{E_r}) \to S^d_V(E_r)$ given by $e_1 \cdots e_d \mapsto res_0(e_1) \cdots res_0(e_d)$, which is by definition $S^d_V(\pi_r)$. □

The restriction morphism gives us a good tool to compare the two sequences. Since $II^{*,*}$ collapses at the second page, we have that $G(\pi_r)_* \circ \delta = 0$ for any differential δ at any page

[1] For convenience, we do not pass immediately to the right adjoint ρ as we did in the proof of Proposition 4.1.1.

of $\boldsymbol{II}^{*,*}$. What we want to study is then the kernel of $G(\pi_r)_*$. Let us make some definitions. Consider the set of *unbounded* compositions of the integer d:

$$\Lambda(\infty, d) = \{\lambda = (\lambda_i)_{i\in\mathbb{N}} \mid \lambda_i \geq 0, \sum_{i\geq 0} \lambda_i = d\}$$

and define the weight of λ to be $|\lambda| := \sum_{i\geq 0} 2i\lambda_i$ (this makes sense because λ can only have finitely many nonzero components). In this sense, there is a graded decomposition

$$G_{\boldsymbol{E_r}} = \bigoplus_{\lambda\in\Lambda(\infty,d)} G^\lambda$$

such that $(G_{\boldsymbol{E_r}})^t = \bigoplus_{|\lambda|=t} G^\lambda$.

Remark 4.3.7. Since $|\lambda|$ is always even, $(G_{\boldsymbol{E_r}})^t = 0$ if t is odd. In particular, the odd rows in $\boldsymbol{II}^{*,*}$ are all zero.

Call λ *n-bounded* if it zero in all components λ_i for $i \geq n$. Then

$$G(\pi_r)|_{G^\lambda} = \begin{cases} \text{isomorphism on the image} & \text{if } \lambda \text{ is } p^r\text{-bounded} \\ 0 & \text{otherwise} \end{cases}$$

thus the same holds for the push forward $G(\pi_r)_*$. The property of being bounded is partially controlled by the weight:

Lemma 4.3.8. *If $|\lambda| < 2n$, then λ is n-bounded.*

Proof. The non-n-bounded composition with minimal weight is

$$(d-1, 0, \dots, 0, 1, 0, \dots)$$

with 1 in place n. Its weight is $2n$, which proves the assertion. □

Corollary 4.3.9. *If $t < 2p^r$, the restriction of $G(\pi_r) : (G_{\boldsymbol{E_r}})^t \to (G_{E_r})^t$ is an isomorphism.*

Proof. $(G_{\boldsymbol{E_r}})^t = \bigoplus_{|\lambda|=t} G^\lambda$, so the hypothesis implies that all the compositions λ in the sum have weight $< 2p^r$. By Lemma 4.3.8 they are all p^r-bounded, hence the restriction of res_0 is an isomorphism on its image. But the image is the sum of the G^λ with λ p^r-bounded of weight t, which coincides with $(G_{E_r})^t$. □

Proposition 4.3.10. *The elements lying in the strip $\{\boldsymbol{II}^{*,t}, t < 2p^r\}$ survive.*

Proof. $\boldsymbol{II}^{s,t} = \mathrm{Ext}^s_{\boldsymbol{\mathcal{P}}}(F, (G_{\boldsymbol{E_r}})^t)$ is sent by $G(\pi_r)_*$ onto $II^{s,t} = \mathrm{Ext}^s_{\boldsymbol{\mathcal{P}}}(F, (G_{E_r})^t)$. Recall that the differentials in the latter sequence are all zero in consequence of [Tou13, Cor. 5]. Therefore, if ∂ is any differential landing somewhere in the strip $T_r := \{t < 2p^r\} \subset \boldsymbol{II}^{*,*}$, then $G(\pi_r)_* \circ \partial = 0$. In force of Corollary 4.3.9, $G(\pi_r)_*$ is an isomorphism on T_r, which forces $\partial = 0$. In particular (recall that our differentials go right-downwards) nonzero differentials can neither land nor depart from T_r. This means that all the elements in the strip survive. □

Proof of Theorem 4.3.3. It follows by Corollary 4.3.9 that the morphism

$$G(\pi_r)_* : \mathrm{Ext}^*_{\mathcal{P}}(F, (G_{\boldsymbol{E_r}})^t) \to \mathrm{Ext}^*_{\mathcal{P}}(F, (G_{E_r})^t)$$

is an isomorphism for all $t < 2p^r$. By Proposition 4.3.6, it comes from a spectral sequence morphism that identifies to $res_0 : \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)}) \to \mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)})$ between the abutments. Hence, the latter is also an isomorphism in degrees strictly less than $2p^r$, as stated. □

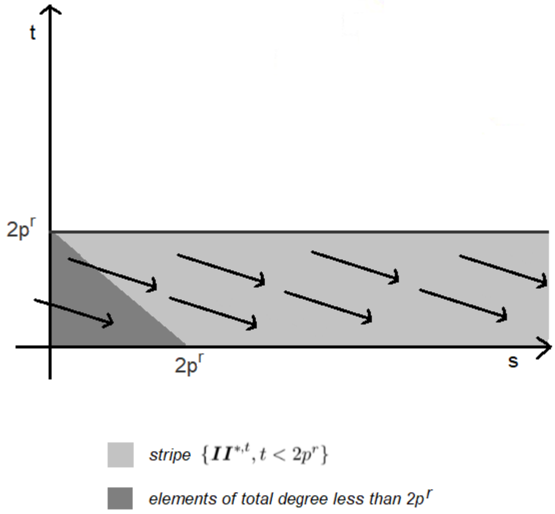

Figure 4.1: The "critic" strip of $\boldsymbol{II}^{*,*}$.

We are now ready to prove that $\boldsymbol{II}^{*,*}$ collapses in low total degrees, as announced in Theorem 4.3.4 at the beginning of the section. Formally, this might be proved as a direct consequence of Proposition 4.3.10, but that would not assure any naturality. In order to gain this extra information, we carry out a slightly less direct proof. One may find help in Figure 4.1.

Proof of Theorem 4.3.4. In degrees strictly less than $2p^r$ (considering the total degree on the two rightmost spaces) we have a chain of natural isomorphisms

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{\boldsymbol{E_r}})$$

the first one coming from Theorem 4.3.3, the second one from Theorem 4.3.1 and the last one being $G(\pi_r)_*^{-1}$. Since they are all natural, this concludes the proof for degrees strictly less than $2p^r$. In addition, looking at the spectral sequence $\boldsymbol{II}^{*,*}$, we observe that all differentials departing from the line $t = 2p^r$ fall into the critical strip $\{\boldsymbol{II}^{*,t}, t < 2p^r\}$, thus vanish by Proposition 4.3.10. In particular, the terms $\boldsymbol{II}^{0,2p^r}$ and $\boldsymbol{II}^{1,2p^r}$ survive, since by geometric reasons they can receive no nontrivial differential either. Moreover, $\boldsymbol{II}^{0,2p^r+1} = 0$ because $\boldsymbol{II}^{*,*}$ vanishes in odd strips. This forces the sequence to collapse in degrees $2p^r$ and $2p^r + 1$ as well. $\square$

4.3.2 Second case: different parity twists

We now handle the case where A and B are Frobenius twists of different parity. By conjugation we can reduce to the case $A = \mathbf{I}_0^{(r)}, B = \mathbf{I}_1^{(r)}$ and consider the special case of the spectral sequence (4.1.2):

$$\overline{\boldsymbol{II}}^{s,t} := \boldsymbol{II}^{s,t}_{\mathbf{I}_0^{(r)},\mathbf{I}_1^{(r)}} = \mathrm{Ext}^s_{\mathcal{P}}(F, (G_{\overline{\boldsymbol{E}}_{\boldsymbol{r}}})^t) \Rightarrow \mathrm{Ext}^{s+t}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_1^{(r)}) \tag{4.3.7}$$

about which we are going to prove the following statement.

Theorem 4.3.11. *For all $n \le (d+2)p^r+1$, the spectral sequence (4.3.7) induces an isomorphism*

$$\mathrm{Ext}^n_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_1^{(r)}) \simeq \bigoplus_{s+t=n} \mathrm{Ext}^s_{\mathcal{P}}(F, (G_{\overline{\boldsymbol{E}}_{\boldsymbol{r}}})^t)$$

which is natural in F, G for $n < (d+2)p^r$.

We recall (Theorem 3.1.4) that $\overline{\boldsymbol{E}_{\boldsymbol{r}}}^*$ is isomorphic to the free $\boldsymbol{E}_{\boldsymbol{r}}{}^*$-supermodule with one even generator c_r placed in degree p^r. In particular, as a super vector space

$$\overline{\boldsymbol{E}_{\boldsymbol{r}}}^s \simeq \begin{cases} \Bbbk & s \ge p^r \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

The idea is to replicate such sort of shifting isomorphism on the spectral sequence $\overline{\boldsymbol{II}}^{*,*}$. We start with an easy fact.

Lemma 4.3.12. *Let $G \in \mathcal{P}_d$. Then the functor $G_{\overline{\boldsymbol{E}}_{\boldsymbol{r}}}$ is only nonzero in degrees greater or equal than dp^r, and $(G_{\overline{\boldsymbol{E}}_{\boldsymbol{r}}})^{dp^r} = G$.*

Proof. Let λ be a composition in $\Lambda(\infty, d)$ and $G^\lambda \subset G_{\overline{\boldsymbol{E}}_{\boldsymbol{r}}}$ the corresponding subfunctor. By the explicit structure of $\overline{\boldsymbol{E}}_{\boldsymbol{r}}$, we see that G^λ is in degree

$$\sum_{k\ge 0}(p^r+2k)\lambda_k = dp^r + \sum_k 2k\lambda_k$$

which is in particular at least dp^r. The last statement comes from the fact that the only composition of weight dp^r is $\lambda = (d, 0, 0, ...)$ and $G^{(d,0,0,...)} = G$. □

Corollary 4.3.13. *Let $F, G \in \mathcal{P}_d$. Then:*

1. $\mathrm{Ext}^n_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_1^{(r)}) = 0$ *for all $n < dp^r$.*
2. *There is an isomorphism*

$$\mathrm{Ext}^{dp^r}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_1^{(r)}) \simeq \mathrm{Hom}_{\mathcal{P}}(F, G) \tag{4.3.8}$$

   natural in F, G.

3. *For all G there exists a unique nonzero class in $\mathrm{Ext}^{dp^r}_{\boldsymbol{\mathcal{P}}}(G_0^{(r)}, G_1^{(r)})$ corresponding to the identity Id_G via (4.3.8).*

Proof. Lemma 4.3.12 implies that the lower horizontal strip $t < dp^r$ of the spectral sequence $\overline{\boldsymbol{II}}^{*,*}$ is identically zero, which implies *a fortiori* its collapsing in total degrees $< dp^r$, hence the first point. As a consequence, $\overline{\boldsymbol{II}}^{0,dp^r} = \mathrm{Hom}_{\mathcal{P}}(F, (G_{\overline{\boldsymbol{E}}_{\boldsymbol{r}}})^{dp^r}) = \mathrm{Hom}_{\mathcal{P}}(F, G)$ is the only nonzero term of total degree dp^r and survives. By these reasons we have the desired natural isomorphism $\mathrm{Ext}^{dp^r}_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_1^{(r)}) \simeq \mathrm{Hom}_{\mathcal{P}}(F, G)$. The last point follows immediately. □

We give a name to this nonzero class, which will play a remarkable role.

Definition 4.3.14. For all G, we denote by c_G the class introduced in point (3) of Corollary 4.3.13.

Lemma 4.3.15. *1. $c_{G\oplus H} = c_G + c_H$ for all polynomial functors G, H.*

2. For all natural transformations $T \in \mathrm{Hom}_{\mathcal{P}}(G, H)$ the following identity holds:

$$T_1^{(r)} \cdot c_G = c_H \cdot T_0^{(r)}\,.$$

3. *The class* $c_{S_V^d} \in \mathrm{Ext}_{\boldsymbol{\mathcal{P}}}^{dp^r}((S_V^d)_0^{(r)}, (S_V^d)_1^{(r)})$ *satisfies the following relation:*

$$c_{S_V^d} \cdot m_{S_0} = m_{S_1} \cdot (1_V \otimes c_r)^{\times d}$$

where m_{S_ℓ} *denotes the multiplication map* $(V \otimes \boldsymbol{I}_\ell^{(r)})^{\otimes d} \to (S_V^d)_\ell^{(r)}$.

4. c_I *is equal to* c_r *(the generating class of* $\overline{\boldsymbol{E}}_{\boldsymbol{r}}$ *as an* $\boldsymbol{E}_{\boldsymbol{r}}$*-supermodule).*

Proof. 1. Since the isomorphism (4.3.8) is natural in either variable, the inclusion $\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}^{dp^r}(G,G) \oplus \mathrm{Ext}_{\boldsymbol{\mathcal{P}}}^{dp^r}(H,H) \hookrightarrow \mathrm{Ext}_{\boldsymbol{\mathcal{P}}}^{dp^r}(G \oplus H, G \oplus H)$ corresponds via (4.3.8) to the inclusion $\mathrm{Hom}_{\mathcal{P}}(G,G) \oplus \mathrm{Hom}_{\mathcal{P}}(H,H) \hookrightarrow \mathrm{Hom}_{\mathcal{P}}(G \oplus H, G \oplus H)$. The latter sends $\mathrm{Id}_G + \mathrm{Id}_H$ onto $\mathrm{Id}_{G \oplus H}$. Hence, by definition the sum of c_G and c_H must equal $c_{G \oplus H}$.

2. Using again naturality of (4.3.8), there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}^{dp^r}(G_0^{(r)}, G_1^{(r)}) \arrow[d, "T_1^{(r)} \cdot -"'] & \mathrm{Hom}_{\mathcal{P}}(G,G) \arrow[l, "\simeq"'] \arrow[d, "T_*"] \\
\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}^{dp^r}(G_0^{(r)}, H_1^{(r)}) & \mathrm{Hom}_{\mathcal{P}}(G,H) \arrow[l, "\simeq"'] \\
\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}^{dp^r}(H_0^{(r)}, H_1^{(r)}) \arrow[u, "- \cdot T_0^{(r)}"] & \mathrm{Hom}_{\mathcal{P}}(H,H) \arrow[l, "\simeq"] \arrow[u, "T^*"']
\end{tikzcd}$$

and the desired identity comes by following the paths of Id_G and Id_H.

3. By the proof of Lemma 4.2.7, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}^*((S_V^d)_0^{(r)}, (S_V^d)_1^{(r)}) \arrow[r, "- \cdot m_{S_0}"] & \mathrm{Ext}_{\boldsymbol{\mathcal{P}}}^*((\mathbf{I}_0^{(r)} \otimes V)^{\otimes d}, (S_V^d)_1^{(r)}) \\
\Gamma^d(\mathrm{End}(V) \otimes \overline{\boldsymbol{E}}_{\boldsymbol{r}}) \arrow[u, "\simeq"] \arrow[r, hook] & (\mathrm{End}(V) \otimes \overline{\boldsymbol{E}}_{\boldsymbol{r}})^{\otimes d} \arrow[u, "\simeq"]
\end{tikzcd}$$

where the right vertical arrow is given by cup product. Restrict this diagram in degree dp^r. Then by Lemma 4.3.12 $(\Gamma^d(\mathrm{End}(V) \otimes \overline{\boldsymbol{E}}_{\boldsymbol{r}}))^{dp^r} = \Gamma^d(\mathrm{End}(V) \otimes \Bbbk c_r)$ generated by $\gamma_d(1_V \otimes c_r)$. In fact, in the isomorphism

$$\mathrm{Hom}_{\mathcal{P}}(S_V^d, S_V^d) \simeq (\Gamma^d(\mathrm{End}(V) \otimes \overline{\boldsymbol{E}}_{\boldsymbol{r}}))^{dp^r}$$

the identity of S_V^d corresponds exactly to $\gamma_d(1_V \otimes c_r)$. Thus, by definition, $c_{S_V^d}$ is the image of $\gamma_d(1_V \otimes c_r)$ via the left vertical map. It follows that $\gamma_d(1_V \otimes c_r)$ goes onto $c_{S_V^d} \cdot m_{S_0}$ by the left-most path of the diagram. The other path sends it by definition onto $m_{S_1} \cdot (1_V \otimes c_r)^{\times d}$, which proves the statement.

□

We use this class to construct a morphism between the spectral sequence (4.3.7) and the already treated one (4.3.1). It is visualised in Figure 4.2.

Proposition 4.3.16. *There is a morphism of spectral sequences* $\varphi^{*,*} : \boldsymbol{II}^{*,*} \to \overline{\boldsymbol{II}}^{*,*}$ *of bidegree* $(0, dp^r)$ *(with* $d = \deg F = \deg G$*) that identifies*

- *with* $G(c_r \cdot -)_* : \mathrm{Ext}_{\mathcal{P}}^*(F, G_{\boldsymbol{E}_{\boldsymbol{r}}}) \to \mathrm{Ext}_{\mathcal{P}}^*(F, G_{\overline{\boldsymbol{E}}_{\boldsymbol{r}}})$ *on the second pages,*
- *with Yoneda product by* c_G *on the abutments.*

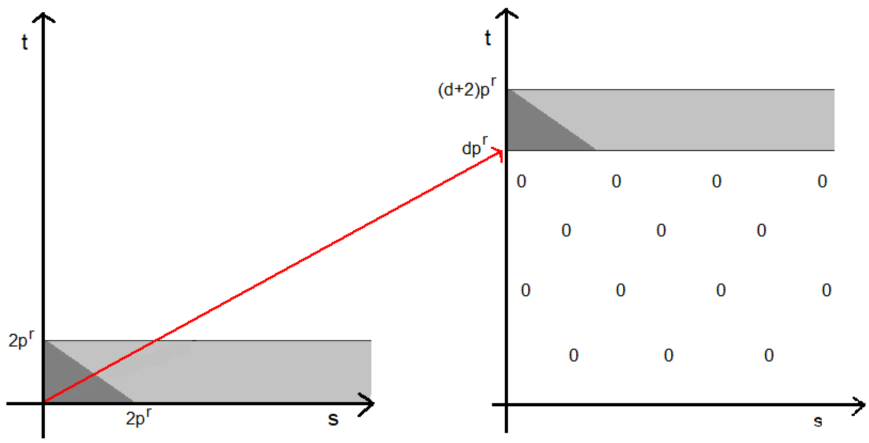

Figure 4.2: The morphism $\varphi^{*,*}$ of Proposition 4.3.16.

Proof. Similarly to the proof of Proposition 4.3.5, take

$$\begin{array}{ll} P_* \twoheadrightarrow F & \text{projective resolution} \\ G_0^{(r)} \hookrightarrow J^* & \text{injective coresolution} \\ G_1^{(r)} \hookrightarrow K^* & \text{injective coresolution} \end{array}$$

and form the bicomplexes

$$A^{s,t} := \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}((P_s)_0^{(r)}, J^t),$$
$$\overline{A}^{s,t} := \mathrm{Hom}_{\boldsymbol{\mathcal{P}}}((P_s)_0^{(r)}, K^t).$$

They give rise respectively to $\boldsymbol{II}^{*,*}$ and $\overline{\boldsymbol{II}}^{*,*}$. Identify c_G with a homotopy class of a morphism $J^* \to K^*[dp^r]$ and define a morphism of bicomplexes $A^{*,*} \to \overline{A}^{*,*+dp^r}$ by push forward. This identifies with $c_G \cdot (-)$ between the abutments and between the first pages $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}((P_s)_0^{(r)}, G_0^{(r)}) \to \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}((P_s)_0^{(r)}, G_1^{(r)})$. The latter identifies via (4.2.5) to a morphism

$$\mathrm{Hom}(P_s, G_{\boldsymbol{E_r}}) \to \mathrm{Hom}(P_s, G_{\overline{\boldsymbol{E}}_{\boldsymbol{r}}}) \tag{4.3.9}$$

that we must show to be the push-forward by $G(- \cdot c_r)$. In first instance, we can suppose $P_s := \Gamma^{d,W}$ and G injective, since otherwise one can take an injective coresolution of G and conclude with the help of Lemma 4.3.15 (2). Furthermore, by Lemma 4.3.15 (1), the case $G = S^d_V$ is sufficient. Our goal is then to show the commutativity of the following diagram

$$\begin{tikzcd} \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}((\Gamma^{d,W})_0^{(r)}, (S^d_V)_0^{(r)}) \arrow[r, "c_{S^d_V}\cdot(-)"] & \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}((\Gamma^{d,W})_0^{(r)}, (S^d_V)_1^{(r)}) \\ S^d(V \otimes W \otimes \boldsymbol{E_r}) \arrow[u, "\simeq"] \arrow[r, "S^d_{V\otimes W}(c_r \cdot -)"'] & S^d(V \otimes W \otimes \overline{\boldsymbol{E}}_{\boldsymbol{r}}) \arrow[u, "\simeq"'] \end{tikzcd} \tag{4.3.10}$$

where the upper vertical arrows are (4.2.5) composed with the Yoneda lemma, i.e. cup product (by Theorem 4.2.5). To do that, we split it into a diagram

$$\begin{tikzcd}
\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}((\Gamma^{d,W})^{(r)}_0, (S^d_V)^{(r)}_0) \arrow[r, "c_{S^d_V}\cdot(-)"] & \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}((\Gamma^{d,W})^{(r)}_0, (S^d_V)^{(r)}_1) \\
\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}((\Gamma^{d,W})^{(r)}_0, (\mathbf{I}^{(r)}_0 \otimes V)^{\otimes d}) \arrow[u, "m_{S_0}\cdot -"] \arrow[r, "(1_V\otimes c_r)^{\times d}\cdot(-)"] & \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}((\Gamma^{d,W})^{(r)}_0, (\mathbf{I}^{(r)}_1 \otimes V)^{\otimes d})) \arrow[u, "m_{S_1}\cdot -"'] \\
(V\otimes W\otimes \boldsymbol{E_r})^{\otimes d} \arrow[u, "\simeq"] \arrow[r, "(1_V\otimes 1_W \otimes (c_r\cdot -))^{\otimes d}"] \arrow[d, two heads] & (V\otimes W\otimes \overline{\boldsymbol{E}}_{\boldsymbol{r}})^{\otimes d} \arrow[u, "\simeq"'] \arrow[d, two heads] \\
S^d(V\otimes W\otimes \boldsymbol{E_r}) \arrow[r, "S^d_{V\otimes W}(c_r\cdot -)"'] \arrow[uuu, bend left=60, "\simeq"] & S^d(V\otimes W\otimes \overline{\boldsymbol{E}}_{\boldsymbol{r}}) \arrow[uuu, bend right=60, "\simeq"']
\end{tikzcd}$$

where the middle vertical arrows are given by cup product and the curved arrows are exactly the vertical ones of (4.3.10). In particular, the outer perimeter coincides indeed with (4.3.10). First, remark that the outer semi-spheres are commutative by definition of the two cup products. The upper square is commutative by Lemma 4.3.15 (3). Commutativity of the middle square comes from the identity

$$(c_r \cdot -_1) \times \ldots \times (c_r \cdot -_d) = c_r^{\times d} \cdot (-_1 \times \ldots \times -_d)$$

which holds by Lemma 1.6.11 (no sign appears because all extensions in $\boldsymbol{E_r}$ are of even cohomological degree). The lower square is trivially commutative. All this shows that (4.3.10) is commutative, which concludes the proof. □

Proof of Theorem 4.3.11. Since $G(- \cdot c_r)$ is an isomorphism of degree dp^r and since the first $dp^r - 1$ lines of $\overline{\boldsymbol{II}}^{*,*}$ are zero by Lemma 4.3.13, it follows that $\varphi^{*,*}$ yields an isomorphism of bidegree $(0, dp^r)$ on the second pages, hence a natural isomorphism on the abutments

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F^{(r)}_0, G^{(r)}_0) \simeq \mathrm{Ext}^{*+dp^r}_{\boldsymbol{\mathcal{P}}}(F^{(r)}_0, G^{(r)}_1)\ . \tag{4.3.11}$$

The result follows then by this and by Theorem 4.3.4. □

4.4 A weaker version in any degree

We now make some sort of iteration process out of Proposition 4.3.16. First, remembering Remark 1.6.6, consider the conjugate extension $c^{\Pi}_G \in \mathrm{Ext}^{dp^r}_{\boldsymbol{\mathcal{P}}}(G^{(r)}_1, G^{(r)}_0)$. Via conjugation, one can repeat the procedure of the proposition to construct a morphism of spectral sequences $\overline{\boldsymbol{II}}^{*,*} \to \boldsymbol{II}^{*,*}$, again of bidegree $(0, dp^r)$, which identifies to $G(c^{\Pi}_r \cdot -)$ at the level of second pages and to $c^{\Pi}_G \cdot -$ on the abutments. In particular, by composition one obtains an endomorphism of $\boldsymbol{II}^{*,*}$ of bidegree $(0, 2dp^r)$.

Proposition 4.4.1. *There is a morphism of spectral sequences $\psi^{*,*} : \boldsymbol{II}^{*,*} \to \boldsymbol{II}^{*,*}$ of bidegree $(0, 2dp^r)$ which identifies*

- *with $G((e_r)^p \cdot -) : \mathrm{Ext}^*_{\mathcal{P}}(F, G_{\boldsymbol{E_r}}) \to \mathrm{Ext}^*_{\mathcal{P}}(F, G_{\boldsymbol{E_r}})$ on the second pages,*
- *with Yoneda product by $c^{\Pi}_G \cdot c_G$ on the abutments.*

Proof. The only thing left to prove is the identification on the second pages. That follows by the relations on $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\mathbf{I}^{(r)}, \mathbf{I}^{(r)})$ [Dru16, Thm. 4.7.1] which say that $c^{\Pi}_r \cdot c_r = (e_r)^p$ is nonzero and generates $\boldsymbol{E_r}^{2p^r}$. □

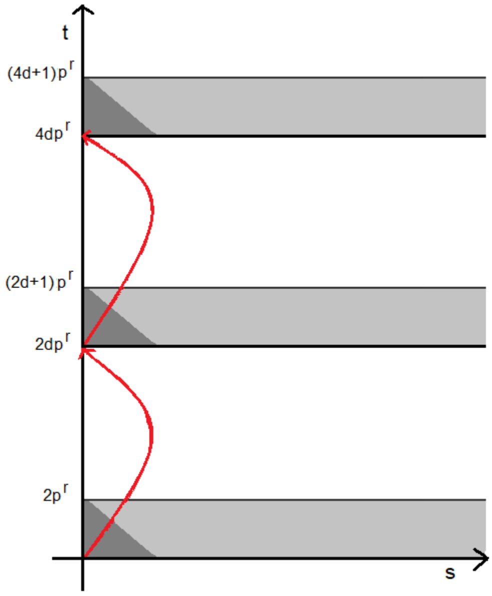

Figure 4.3: The endomorphism of Proposition 4.4.1 and its iterations.

Unfortunately $G(-\cdot(e_r)^p)$, and consequently $\psi^{*,*}$, is not an isomorphism. This is because, while $c_r\cdot - : \boldsymbol{E_r} \to \overline{\boldsymbol{E}}_{\boldsymbol{r}}$ is an isomorphism, $c_r^{\Pi}\cdot - : \overline{\boldsymbol{E}}_{\boldsymbol{r}} \to \boldsymbol{E_r}$ is not. Its image includes just classes of degree at least $2p^r$. In particular, Yoneda product with $(e_r)^p$ is not an isomorphism of $\boldsymbol{E_r}$ on itself. Such loss of information can only get worse as d gets bigger. For example, $(S^d_{\boldsymbol{E_r}})^0$ is equal to one copy of S^d - corresponding to the composition $(d, 0, \dots)$ - while $(S^d_{\boldsymbol{E_r}})^{2dp^r}$ contains at least a copy of S^d and a copy of $S^{d-1}\otimes S^1$ - corresponding respectively to compositions $(0, \dots, d, 0, \dots)$ with d in p^r-th position and $(d-1, 0, \dots, 1, \dots)$ with 1 in dp^r-th position. Anyway, the good news is that $\psi^{*,*}$ is still injective. We want then to identify the its image.

Definition 4.4.2. Let $\lambda \in \Lambda(\infty, d)$ and m an integer. The associated m-shifted composition is defined as

$$(\lambda_{[m]})_i := \begin{cases} \lambda_{i-m} & \text{if } i \geq m \\ 0 & \text{otherwise} \end{cases}$$

It is immediate to check that $\lambda_{[m]} \in \Lambda(\infty, d)$ and that $|\lambda_{[m]}| = |\lambda| + 2dm$.

Proposition 4.4.3. *The morphism $\psi^{*,*}$ of Proposition 4.4.1 is an isomorphism on its image, which coincides (as a vector space) with*

$$Im(\psi) = \bigoplus_{\lambda \in \Lambda(\infty, d)} \mathrm{Ext}^*_{\mathcal{P}}(F, G^{\lambda_{[p^r]}}).$$

Proof. Since $-\cdot(e_r)^p : \boldsymbol{E_r} \to \boldsymbol{E_r}$ is an isomorphism on its image, so is $G(-\cdot\ (e_r)^p)$. It is now immediate to see that the latter sends G^λ onto $G^{\lambda_{[p^r]}}$. □

Corollary 4.4.4. *All the elements of the form*

$$\mathrm{Ext}^s_{\mathcal{P}}(F, G^{\lambda_{[p^r]}}) \quad \text{with } s + |\lambda| \leq 2p^r + 1$$

survive in $\boldsymbol{II}^{*,*}$.

Proof. The map ψ sends the critical strip $T_r := \{\boldsymbol{II}^{*,t}, t < 2p^r\}$ on the strip $\{2dp^r \le t < 2(d+1)p^r\}$, more precisely on the subspace

$$A := \{\mathrm{Ext}^s_{\mathcal{P}}(F, G^{\lambda_{[p^r]}}) :\ |\lambda| < 2p^r,\ s \ge 0\}\ .$$

If a differential ∂ departs from A, then it can be precomposed by ψ and $\partial \circ \psi = 0$, since the differentials in T_r are zero. We conclude that no nonzero differential departs from A. Now, let A' be the subspace of A containing only the elements of total degree $< 2(d+1)p^r$. Explicitly, this means that

$$s + |\lambda_{[p^r]}| < 2(d+1)p^r$$

which, since $|\lambda_{[p^r]}| = |\lambda| + 2dp^r$, means that

$$A' = \{\mathrm{Ext}^s_{\mathcal{P}}(F, G^{\lambda_{[p^r]}}) \quad : \quad s + |\lambda| < 2p^r\}\ .$$

For geometric reasons, differentials landing on A' must come from the second quadrant, thus are trivially zero. Resuming, we have proved that no differential departs from or lands on A'. This shows the surviving of all elements of the asserted form, for $s + |\lambda| < 2p^r$. Now consider

$$A'' = \{\mathrm{Ext}^s_{\mathcal{P}}(F, G^{\lambda_{[p^r]}}) \quad : \quad s + |\lambda| = 2p^r \text{ or } 2p^r + 1\}\ .$$

As in the proof of Theorem 4.3.4, they receive no nontrivial differential for geometrical reasons, thus they survive as well. □

By applying ψ *ad libitum* (as depicted in Figure 4.3) one can make induction on the previous result and get the following.

Proposition 4.4.5. *For all $k \ge 1$, the elements of the form*

$$\mathrm{Ext}^s_{\mathcal{P}}(F, G^{\lambda_{[kp^r]}}) \quad \textit{with } s + |\lambda| \le 2p^r + 1$$

survive in $\boldsymbol{II}^{*,*}$.

Visually, what we have proved to survive is the lowest strip $\{t < 2p^r\}$ and infinite copies of the leftmost flag $\{s + t \le 2p^r + 1\}$. Namely, there is one copy for each $k \ge 1$, each one living in the strip $\{2dkp^r \le t \le 2(kd+1)p^r + 1\}$. Now we want to gather such family of surviving elements to state a weak version of Conjecture 4.1.5. We do that in the following (quite heavy) proposition, which we next translate in a more readable and suggestive theorem.

Proposition 4.4.6. *Let $F, G \in \mathcal{P}_d$. For a composition $\lambda \in \Lambda(\infty, d)$, denote by G^λ the correspondent weight space of $G_{\boldsymbol{E_r}}$. Then there is an inclusion*

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)}) \supseteq \bigoplus_{0 \le t \le 2p^r+1} \mathrm{Ext}^*_{\mathcal{P}}(F, (G_{\boldsymbol{E_r}})^t) \oplus \bigoplus_{\substack{k \ge 1 \\ s+|\lambda| \le 2p^r+1}} \mathrm{Ext}^s_{\mathcal{P}}(F, G^{\lambda_{[kp^r]}})\ .$$

Proof. The first summand comes from Theorem 4.3.4 and the second ones are the surviving elements of Proposition 4.4.5. □

Notation 4.4.7 (Cohomological shifting). Let V^* be a $\mathbb{Z}$-graded vector space and n an integer. The n-shifted space $(V_{[n]})^*$ is defined by $(V_{[n]})^i := V^{i-n}$.

Notation 4.4.8. Set for briefness $\mathrm{Ext}^{<2p^r}_{\mathcal{P}}(-,-) := \bigoplus_{0 \le s < 2p^r} \mathrm{Ext}^s_{\mathcal{P}}(-,-)$.

Definition 4.4.9. For all $G \in \mathcal{P}$, set $\varepsilon_G := c^{\Pi}_G \cdot c_G \in \mathrm{Ext}^{2dp^r}(G_0^{(r)}, G_0^{(r)})$ where c_G is the class of Definition 4.3.14.

Theorem 4.4.10. *Let $F, G \in \mathcal{P}_d$. The map $\sum_{k\geq 0}(\varepsilon_G)^k \cdot res_0^{-1}$ induces an even graded natural inclusion*

$$\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \oplus \bigoplus_{k\geq 1} \left(\mathrm{Ext}^{<2p^r}_{\mathcal{P}}(F^{(r)}, G^{(r)})\right)^*_{[2dkp^r]} \subseteq \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)})$$

where the left-hand side is placed in superdegree 0*.*

Proof. Let us look at Proposition 4.4.6 and pick only the summands of the right-hand side such that, respectively, $t < 2p^r$ and $s + |\lambda| < 2p^r$. By Theorem 4.3.1, the first one is naturally isomorphic to $\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)})$ and by Theorem 4.3.3 it is included in $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)})$ via res_0^{-1}. The second terms are just shiftings of this one via $\psi^{*,*}$, which is a natural injection in force of Proposition 4.4.3. By Proposition 4.4.1, $\psi^{*,*}$ identifies here with $-\cdot\varepsilon_G$. Therefore, each k-summand is isomorphic to a copy of $\mathrm{Ext}^{<2p^r}_{\mathcal{P}}(F^{(r)}, G^{(r)})$ shifted by $2kdp^r$, and is included in $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)})$ via res_0^{-1} composed with $-\cdot(\varepsilon_G)^k$. The result follows. ☐

With an application of (4.3.11) we immediately get the analogous result for twists of different parity.

Theorem 4.4.11. *Let $F, G \in \mathcal{P}_d$. Then there is an even graded natural inclusion*

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_1^{(r)}) \supseteq \left(\mathrm{Ext}_{\mathcal{P}}(F^{(r)}, G^{(r)})\right)^*_{[-dp^r]} \oplus \left(\bigoplus_{k\geq 1} \mathrm{Ext}^{<2p^r}_{\mathcal{P}}(F^{(r)}, G^{(r)})\right)^*_{[(2k-1)dp^r]}$$

where the right-hand side is in superdegree 0 *and the bracket* $[-]$ *denotes a shifting of cohomological degree.*

Problem 4.4.12. One might wonder if the extension $\varepsilon_G := c_G^{\Pi} \cdot c_G$ writes always as a p-th power. This is the case for $G = I$, as pointed out at the beginning of the section. We can show that the answer is positive for any G injective. Let us consider first the case $G = \otimes^d$. Indeed, thanks to Theorem 4.2.5

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\otimes_0^{d(r)}, \otimes_1^{d(r)}) \simeq \mathrm{Hom}_{\mathcal{P}}(\otimes^d, \otimes^d_{\boldsymbol{E_r}}) \simeq \overline{\boldsymbol{E}_{\boldsymbol{r}}}^{\otimes d} \otimes \Bbbk\Sigma_d$$

which implies in particular that $\mathrm{Ext}^{dp^r}_{\boldsymbol{\mathcal{P}}}(\otimes_0^{d(r)}, \otimes_1^{d(r)})$ is generated by the set $\{\sigma_*(c_r^{\times d}),\ \sigma \in \Sigma_d\}$. Therefore, $c_{\otimes^d}$ is by construction a multiple of $c_r^{\times d}$. Applying Lemma 4.3.15 (2)-(3) on the multiplication map $\otimes^d \to S^d$, one sees that in fact $c_{\otimes^d} = c_r^{\times d}$. Inverting cross product and Yoneda product with Lemma 1.6.11, one concludes that $e_{\otimes^d} := c^{\Pi}_{\otimes^d} \cdot c_{\otimes^d} = (c_r^{\Pi})^{\times d} \cdot c_r^{\times d} = (c_r^{\Pi} \cdot c_r)^{\times d} = ((e_r)^p)^{\times d} = (e_r^{\times d})^p$. So $e_{\otimes^d}$ admits a p-th root. Let us now prove the case $G = S^d_V$. Since adding a parameter is harmless, we suppose $V = \Bbbk$ to lighten notations. From the previous discussion and Lemma 4.3.15, we can deduce an implicit relation $e_{S^d} \cdot m_{S_0} = m_{S_0} \cdot (e_r^{\times d})^p$. Moreover, the functor $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(-_0^{(r)}, \otimes_0^{d(r)})$ is exact as a consequence of Theorem 4.2.5, hence we have an exact sequence

$$0 \to \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(S_0^{d(r)}, \otimes_0^{d(r)}) \xrightarrow{-\cdot m_{S_0}} \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\otimes_0^{d(r)}, \otimes_0^{d(r)}) \to \bigoplus_{\sigma\in\Sigma_d} \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(\otimes_0^{d(r)}, \otimes_0^{d(r)})$$

where the second map is $\sum\limits_{\sigma\in\Sigma_d} \mathrm{Ext}^*(\sigma - \mathrm{Id}, \otimes_0^{d(r)})$. Since $e_r^{\times d}$ is manifestely in its kernel, by exactness there exists $e'_r \in \mathrm{Ext}^{2dp^{r-1}}_{\boldsymbol{\mathcal{P}}}(S_0^{d(r)}, \otimes_0^{d(r)})$ such that $e_r^{\times d} = e'_r \cdot m_{S_0}$. In particular, by the previous relation

$$e_{S^d} \cdot m_{S_0} = m_{S_0} \cdot (e_r^{\times d})^p = m_{S_0} \cdot (e'_r \cdot m_{S_0})^p = (m_{S_0} \cdot e'_r)^p \cdot m_{S_0}$$

where the last equality is simply associativity of the Yoneda product. Note now that $m_{S_0} \cdot e'_r$ is an element of $\mathrm{Ext}^{2dp^{r-1}}_{\boldsymbol{\mathcal{P}}}(S_0^{d(r)}, S_0^{d(r)})$. By exactness of the short sequence, that implies $e_{S^d} =$

$(m_{S_0} \cdot e'_r)^p$ i.e. we have found a p-th root of e_{S^d}. In particular, since the class ε_G is additive with respect to G, we can conclude that ε_G is a p-th power whenever G is injective. On the other hand, we do not know if the same holds for non-injective G. The general existence of such $2dp^{r-1}$-extension would lead to a consistent improvement of Theorem 4.4.10. In a slightly different and more ambitious perspective, it might also provide a "universal class" similar to the ones used by Chałupnik [Cha15] in order to construct by hand an isomorphism $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{\boldsymbol{E_r}})$.

4.5 Generalisation to all additives

The goal now is to generalise the results we have found to the case of general A and B. Investigating the case of Frobenius twists, we have found isomorphisms in bounded degrees, with a bound that unfortunately depends on the parity of the two twists. So, to compactify the statements, we start by introducing the following definition.

Definition 4.5.1. Let A, B be indecomposable additive superfunctors of degree p^r. Define the quantity $\varepsilon_{A,B}$ according to the following table:

	$\mathbf{I}_0^{(r)}$	$\mathbf{I}_1^{(r)}$	$\mathbf{I}_0^{(r)} \circ \mathbf{\Pi}$	$\mathbf{I}_1^{(r)} \circ \mathbf{\Pi}$
$\mathbf{I}_0^{(r)}$	0	p^r	p^r	0
$\mathbf{I}_1^{(r)}$	p^r	0	0	p^r
$\mathbf{I}_0^{(r)} \circ \mathbf{\Pi}$	p^r	0	0	p^r
$\mathbf{I}_1^{(r)} \circ \mathbf{\Pi}$	0	p^r	p^r	0

The value of ε on any pair of additives is determined by the additional axiom:

$$\varepsilon_{A \oplus A', B} = \varepsilon_{B, A \oplus A'} = \min(\varepsilon_{A,B}, \varepsilon_{A',B}) \,.$$

Lemma 4.5.2. *$\varepsilon_{A,B}$ is the lowest degree where $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A, B)$ is nonzero.*

Proof. For the indecomposables, this is known (Theorem 3.1.2). For a sum of indecomposables, the assertion follows by the biadditivity of Ext and the last axiom of the definition. □

Corollary 4.5.3. *If $\varepsilon_{A,B} = p^r$, then $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A, B)$ is isomorphic to a sum of copies of $\overline{\boldsymbol{E}}_{\boldsymbol{r}}$ and $\mathbf{\Pi}\overline{\boldsymbol{E}}_{\boldsymbol{r}}$.*

We can now state the main result. Recall the general spectral sequence $\boldsymbol{II}^{*,*}_{A,B}$ introduced in (4.1.2).

Theorem 4.5.4. *Let $F, G \in \mathcal{P}_d$ and let $A, B \in \boldsymbol{\mathcal{P}}_{p^r}$ additive. Then:*

1. $\mathrm{Ext}^n_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ B) = 0$ *for all $n < d \cdot \varepsilon_{A,B}$.*
2. $\mathrm{Ext}^{d \cdot \varepsilon_{A,B}}_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ B)$ *contains a sum of copies of* $\mathrm{Hom}_{\mathcal{P}}(F, G)$*. In particular,* $\mathrm{Ext}^{d \cdot \varepsilon_{A,B}}_{\boldsymbol{\mathcal{P}}}(G \circ A, G \circ B) \neq 0$.
3. *If $\varepsilon_{A,B} = p^r$, then for all n there is an isomorphism*

$$\begin{aligned}\mathrm{Ext}^{n-dp^r}_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ (\mathbf{\Pi} \circ B \circ \mathbf{\Pi})) &\simeq \mathrm{Ext}^n_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ B) \\ &\simeq \mathrm{Ext}^{n-dp^r}_{\boldsymbol{\mathcal{P}}}(F \circ (\mathbf{\Pi} \circ A \circ \mathbf{\Pi}), G \circ B) \,.\end{aligned}$$

Proof. 1. In force of Lemma 4.5.2, $G_{\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}(A,B)}$ is zero in degrees strictly less than $d \cdot \varepsilon_{A,B}$. By consequence, the second page of $\boldsymbol{II}^{*,*}_{A,B}$ is all zero in the lowest $d \cdot \varepsilon_{A,B} - 1$ rows, so the abutment vanishes in degrees strictly less than $d \cdot \varepsilon_{A,B}$, as stated.

2. Continuing of the proof of the first point, we see that in $\boldsymbol{II}^{*,*}_{A,B}$ there is only one nonzero element of total degree $d \cdot \varepsilon_{A,B}$, namely

$$\boldsymbol{II}^{0,d\cdot\varepsilon_{A,B}}_{A,B} \simeq \mathrm{Hom}_{\mathcal{P}}(F, (G_{\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}(A,B)})^{d\cdot\varepsilon_{A,B}})$$

which survives for trivial geometric reasons. But $(G_{\mathrm{Ext}_{\boldsymbol{\mathcal{P}}}(A,B)})^{d\cdot\varepsilon_{A,B}}$ contains a sum of copies of G, so in particular $\boldsymbol{II}^{0,d\cdot\varepsilon_{A,B}}_{A,B}$ contains a sum of copies of $\mathrm{Hom}_{\mathcal{P}}(F,G)$.

3. We only prove the first isomorphism, the other one being a consequence of it by means of Kuhn duality. By hypothesis and Corollary 4.5.3, $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)$ is a sum of copies of $\overline{\boldsymbol{E}}_{\boldsymbol{r}}$ and $\mathbf{\Pi}\overline{\boldsymbol{E}}_{\boldsymbol{r}}$. Since $\varepsilon_{A,\mathbf{\Pi}\circ B\circ\mathbf{\Pi}} = 0$, we deduce that $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A, \mathbf{\Pi}\circ B\circ\mathbf{\Pi})$ is a sum of *the same number* of copies of, respectively, $\boldsymbol{E}_{\boldsymbol{r}}$ and $\mathbf{\Pi}\boldsymbol{E}_{\boldsymbol{r}}$. Therefore, there is a linear map $f_{A,B} : \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,\mathbf{\Pi}\circ B\circ\mathbf{\Pi}) \to \mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)$, namely a diagonal matrix consisting of Yoneda product by c_r or πc_r, where c_r is the generator of $\overline{\boldsymbol{E}}_{\boldsymbol{r}}{}^{p^r}$. It is by construction a map of degree p^r which is an isomorphism on its image. Now, consider by point (2) a nonzero class $c_{B,G} \in \mathrm{Ext}^{dp^r}_{\boldsymbol{\mathcal{P}}}(G\circ(\mathbf{\Pi}\circ B\circ\mathbf{\Pi}),\, G\circ B)$ corresponding to a sum of identities. Repeating the process of Proposition 4.3.16, one can construct a morphism of spectral sequences $\boldsymbol{II}^{*,*}_{A,\,\mathbf{\Pi}\circ B\circ\mathbf{\Pi}} \to \boldsymbol{II}^{*,*}_{A,B}$ of bidegree $(0, dp^r)$, which identifies to the pushforward by $G(f_{A,B})$ between the second pages and to $(-)\cdot c_{B,G}$ between the abutments. Now, $G(f_{A,B})$ is an isomorphism on its image as well. In force of this discussion and of point (1), we get on the abutments the desired isomorphism

$$\mathrm{Ext}^n_{\boldsymbol{\mathcal{P}}}(F\circ A, G\circ B) \simeq \mathrm{Ext}^{n-dp^r}_{\boldsymbol{\mathcal{P}}}(F\circ A, G\circ(\mathbf{\Pi}\circ B\circ\mathbf{\Pi}))\ .$$

□

To state the general form of the inclusions of Theorems 4.4.10 and 4.4.11, we need a little more notation. If A,B are indecomposable, then $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)$ is concentrated in either even or odd superdegree. Set $\delta_{A,B}$ to be 0 in the first case and 1 in the second case.

Proposition 4.5.5. *Let $F,G \in \mathcal{P}_d$ and A,B be indecomposable additives (Frobenius twists and their parity shifts). If $\varepsilon_{A,B} = 0$, there is a natural inclusion*

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F\circ A, G\circ B) \supseteq \mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \oplus \bigoplus_{k\geq 1}\left(\mathrm{Ext}^{<2p^r}_{\mathcal{P}}(F^{(r)}, G^{(r)})\right)^*_{[2dkp^r]}$$

with the right-hand side in superdegree $d\cdot\delta_{A,B} \mod 2$. If $\varepsilon_{A,B} = p^r$, there is a natural inclusion

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F\circ A, G\circ B) \supseteq \left(\mathrm{Ext}_{\mathcal{P}}(F^{(r)}, G^{(r)})\right)^*_{[-dp^r]} \oplus \bigoplus_{k\geq 1}\left(\mathrm{Ext}^{<2p^r}_{\mathcal{P}}(F^{(r)}, G^{(r)})\right)^*_{[(2k-1)dp^r]}$$

again with the right-hand side in superdegree $d\cdot\delta_{A,B} \mod 2$.

Proof. By Proposition 2.2.6, the case $A = B = \mathbf{I}^{(r)}_0$ is sufficient for the first part and the case $A = \mathbf{I}^{(r)}_0, B = \mathbf{I}^{(r)}_1$ for the second one, with $\delta_{A,B}$ taking account of the superdegrees as explained. In these two cases, we retrieve the content of Theorems 4.4.10 and 4.4.11. □

As a consequence of Theorem 2.3.5, for any additive pair A,B the superspace $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B)$ is a sum of copies of $\boldsymbol{E}_{\boldsymbol{r}}, \overline{\boldsymbol{E}}_{\boldsymbol{r}}, \mathbf{\Pi}\boldsymbol{E}_{\boldsymbol{r}}$ and $\mathbf{\Pi}\overline{\boldsymbol{E}}_{\boldsymbol{r}}$. Knowing how many copies of each there are, we can apply Proposition 4.5.5 multiple times to get the following.

Theorem 4.5.6. *Let $F,G \in \mathcal{P}_d$ and let $A,B \in \boldsymbol{\mathcal{P}}_{p^r}$ be additive. Let n,m,n',m' be positive integers such that*

$$\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(A,B) = \boldsymbol{E}_{\boldsymbol{r}}{}^{\oplus n} \oplus \mathbf{\Pi}\boldsymbol{E}_{\boldsymbol{r}}{}^{\oplus n'} \oplus \overline{\boldsymbol{E}}_{\boldsymbol{r}}{}^{\oplus m} \oplus \mathbf{\Pi}\overline{\boldsymbol{E}}_{\boldsymbol{r}}{}^{\oplus m'}$$

and define the graded super vector space

$$\Omega^* := \mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \oplus \bigoplus_{k\geq 1} \left(\mathrm{Ext}^{<2p^r}_{\mathcal{P}}(F^{(r)}, G^{(r)})\right)^*_{[2dkp^r]} .$$

Then $\mathrm{Ext}^*_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ B)$ *contains, in a natural way with respect to* F *and* G*:*

- n *copies of* Ω^* *in superdegree zero;*
- n' *copies of* Ω^* *in superdegree* $d \mod 2$*;*
- m *copies of* $(\Omega_{[-dp^r]})^*$ *in superdegree zero;*
- m' *copies of* $(\Omega_{[-dp^r]})^*$ *in superdegree* $d \mod 2$.

Proof. By Lemma 2.3.19, for all A, A' additives, $F \circ (A \oplus A')$ contains $F \circ A$ and $F \circ A'$ as direct summands. The statement follows then by induction on n, m, n', m' with Proposition 4.5.5 as base case for all of them. □

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