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Operads in 2-categories and models of structure interchange

Opérades dans les 2-catégories et modèles de lois d'échange de structures

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Notice to readers

This thesis was intended to contain a fourth chapter, which we were unable to include due to time constraints. This chapter, which will be detailed in future works, is meant to provide further insights on A_∞-structures from a cellular point of view. While the constructions in the first chapter, notably regarding generalized versions of Yoneda's Lemma, Kan extensions, and Day's convolution product, have intrinsic interest in our opinion, they were originally intended to serve as the theoretical background for the aforementioned missing chapter.

Due tu the same time constraints, this version still contains some inaccuracies and is intended to be refined and corrected.

It should also be noted that the notation we employ for ends and coends is reversed compared to modern references (notably [**20**]) on the subject. This difference in notation was not intentional but rather intuitive due to the exponential notation, so that it seemed more natural to us to write the category on which the end depends as an exponent.

Introduction

The main purpose of this thesis is to provide an operadic way of understanding operadic actions that interchange. The main motivation for this study is the construction of combinatorial models of E_n-operads based on models of structure interchanges governed by operads.

Interchange of structures occur when we form the category of $\mathcal{P}$-algebras in a category of $\mathcal{Q}$-algebras, for operads $\mathcal{P}$ and $\mathcal{Q}$. Indeed, we can identify the objects of this category with objects equipped with compatible actions of the operads $\mathcal{P}$ and $\mathcal{Q}$ and the interchange of structure gives precisely the shape of this compatibility relation. Boardman and Vogt introduced the tensor product of (set theoretical or topological) operads in order to represent this category as a category of algebras over an operad $\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$ (see for instance [**5**]). They provided, as a follow-up, an approach to understand the compatibility relations based on structure interchanges in the context of operads.

Recall that a topological operad is said to be an E_n-operad if it is homotopy equivalent to the operad of little n-cubes $\mathcal{C}^n$, which model operations acting on n-fold loop spaces (see again [**5, 4**] and [**24**]). The construction of the Boardman-Vogt tensor product was originally motivated by applications to the study of iterated loop spaces, and hence, is related to the study of E_n-operads. The idea is to get information about the structure of an n-fold loop space $Y = \Omega^n X$ in terms of its n distinct compatible structures of 1-fold loop space, and hence, to get information about structures governed by an E_n-operad in terms of n compatible structures governed by an E_1-operad. The Boardman-Vogt tensor product can be used to model the structure interchange that governs the compatibility relation between such iterated structures. But we face difficulties to obtain effective information from the Boardman-Vogt tensor product. The operad of associative algebras $\mathcal{A}s$ provides a set-theoretic (discrete) model of E_1-operad, but set-theoretic operads do not carry enough homotopical information to retrieve E_n-operads by using the tensor product operation. In fact, according to [**6**], the Eckmann-Hilton argument yields an isomorphism of operads $\mathcal{A}s \otimes_{\mathrm{BV}} \mathcal{A}s \cong Com$, where Com represents the operad of commutative algebras. The topological operads, on the other hand, lack of explicit description by generators and relations. The homotopy type of $\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$, for topological operads $\mathcal{P}$ and $\mathcal{Q}$, is therefore hard to determine and few examples are known. The most powerful result on the tensor product asserts that the tensor product operad $\mathcal{C} \otimes_{\mathrm{BV}} \mathcal{D}$ of an E_n-operad $\mathcal{C}$ and an E_m-operad $\mathcal{D}$ is an E_{n+m}-operad as soon as the operads $\mathcal{C}$ and $\mathcal{D}$ are both cofibrant (see [**9**]).

To address these issues, we propose to work in the context of categorical operads, and to take inspiration from operads $\mathcal{M}^n$ that describe the structure of n-fold monoidal categories. The operads $\mathcal{M}^n$ were introduced by Balteanu, Fiedorowitch, Schwänzle and Vogt for the study of n-fold loop spaces from a categorical point of view [**2**]. The realization of the nerve of the operad $\mathcal{M}^n$ is homotopy equivalent to the operad of the little n-cubes. Hence, the operads $\mathcal{M}^n$ can be understood as categorical analogues of E_n-operads.

In fact, operads defined in the monoidal 2-category of categories $(\mathrm{CAT}, \times)$ enjoy several properties which make them convenient for the study of algebraic structures which interchange. Indeed, the 2-category CAT is particular among the other 2-categories in that each 2-category is naturally equipped with a category of morphisms between each pair of objects, so that it makes sense to consider algebras over non symmetric CAT-operads in any monoidal 2-category and over symmetric CAT-operads in any symmetric monoidal 2-category.

In this thesis, we formalize a notion of presentation by generators and relations for operads defined in the category of small categories **cat**, in terms of operadic polygraphs. We show that the operads $\mathcal{M}^n$, in particular, admit such a presentation. This notion of presentation gives explicit conditions for objects to have the structure of an algebra over a categorical operad. Then we will equip the category $\mathrm{OP}_{\mathbf{cat}}$ with the model structure transported from the canonical model structure on **cat**. In this model category, we characterize the cofibrant operads as the categorical operads whose underlying operad of objects forms a free operad. We construct a tensor product both at the level of operads and at the level of polygraphic presentations in a compatible way, so that we obtain an explicit presentation of the tensor product of operads in terms of the presentation of each factors. We obtain isomorphisms $\mathcal{M}^p \otimes_{\mathrm{BV}} \mathcal{M}^q \cong \mathcal{M}^{p+q}$ and a homotopy invariance of the tensor product without cofibrancy hypothesis. We therefore obtain an explicit presentation of a cofibrant resolution $\mathcal{M}^n_\infty$ of the operads $\mathcal{M}^n$ from an explicit cofibrant resolution $\mathcal{M}^1_\infty$ of the operad $\mathcal{M}^1$, given by a categorical counterpart of the Stasheff operad of associahedra. We finally provide an explicit $\mathcal{M}^n_\infty$-algebra structure on n-fold loop spaces by constructing a morphism of operads $\mathcal{M}^n_\infty \to \Pi\mathcal{C}^n$, where $\Pi\mathcal{C}^n$ refers to the categorical operad obtained from the operad $\mathcal{C}^n$ by using the fundamental groupoid functor $\Pi : \mathrm{TOP} \to \mathbf{cat}$.

We give a more detailed outline of these ideas and results in the next paragraphs.

Iterated monoids and iterated loop spaces

To any monoidal 2-category $(\Lambda, \otimes_\Lambda)$ we can associate a 2-category of monoids $\mathrm{MON}_{(\Lambda,\otimes_\Lambda)}$ in Λ. Let $(\Lambda, \otimes_1)$ be a monoidal 2-category and $\otimes_2 : \Lambda \times \Lambda \to \Lambda$ be a 2-functor. The 2-category of monoidal 2-categories is monoidal (with respect to the cartesian product), so that $(\Lambda, \otimes_1) \times (\Lambda, \otimes_1)$ inherits the structure of a monoidal 2-category. Hence we can assume that the 2-functor $\otimes_2$ is lax monoidal with respect to this structure. In this case, it induces a 2-functor between their 2-categories of monoids

$$\otimes_2 : \mathrm{MON}_{(\Lambda,\otimes_1)} \times \mathrm{MON}_{(\Lambda,\otimes_1)} \to \mathrm{MON}_{(\Lambda,\otimes_1)},$$

providing $(\mathrm{MON}_{(\Lambda,\otimes_1)}, \otimes_2)$ with the structure of a monoidal 2-category. We then define the 2-category of 2-fold monoids in Λ as $\mathrm{MON}^2_{(\Lambda,\otimes_1,\otimes_2)} = \mathrm{MON}_{(\mathrm{MON}_{(\Lambda,\otimes_1)},\otimes_2)}$. We can apply this construction inductively, to get the 2-categories of n-fold monoids

$$\mathrm{MON}^n_{(\Lambda,\otimes_1,\ldots,\otimes_n)} = \mathrm{MON}_{(\mathrm{MON}^{n-1}_{(\Lambda,\otimes_1,\ldots,\otimes_{n-1})},\otimes_n)}$$

If the monoidal 2-category structure $(\Lambda, \otimes_\Lambda)$ on Λ is symmetric, then its 2-category of monoids inherits a monoidal structure as well, so that we can also define the

2-category of n-fold monoids $\mathrm{Mon}^n_{(\Lambda,\otimes)}$ in Λ by

$$\mathrm{Mon}^n_{(\Lambda,\otimes)} = \mathrm{Mon}_{(\mathrm{Mon}^{n-1}_{(\Lambda,\otimes)},\otimes)}.$$

We have an explicit description of n-fold monoids. The case of n-fold monoids in the monoidal 2-category of small categories can be described as follows. The 2-category of n-fold monoidal categories $\mathrm{Mon}^n_{(\mathbf{Cat},\times)}$ has for objects the small categories $\mathcal{C}$ equipped with n strictly associative and unital monoidal products

$$\otimes_1, \dots, \otimes_n : \mathcal{C}\times\mathcal{C} \to \mathcal{C},$$

and natural transformations $\otimes_i^j$ such that

$$\begin{tikzcd}
(\mathcal{C}\times\mathcal{C})\times(\mathcal{C}\times\mathcal{C}) \arrow[rr, "(2\ \ 3)"] \arrow[d, "\otimes_j\times\otimes_j"'] & & (\mathcal{C}\times\mathcal{C})\times(\mathcal{C}\times\mathcal{C}) \arrow[d, "\otimes_i\times\otimes_i"] \\
\mathcal{C}\times\mathcal{C} \arrow[dr, "\otimes_i"'] \arrow[rr, phantom, "\overset{\Longrightarrow}{\otimes_i^j}"] & & \mathcal{C}\times\mathcal{C} \arrow[dl, "\otimes_j"] \\
& \mathcal{C} &
\end{tikzcd},$$

satisfying coherence diagrams:

- $\Box_i^j, \Box_j^i$ for $i<j$ relative to the compatibility of $\otimes_i^j$ with the associativity of $\otimes_i$ and $\otimes_j$,
- $\hexagon_{1\leq i<j<k\leq n}$ which ensure that $\otimes_j^k$ is a lax monoidal 2-morphism with respect to $\otimes_i$.

The structure of an n-fold monoidal small category is considered by Balteanu, Fiedorowitch, Schwänzle and Vogt in [**2**]. One of the main outcome of their work asserts that the geometric realization of the nerve $\mid \mathcal{N}\mathcal{C} \mid$ of an n-fold monoidal category $\mathcal{C} \in \mathrm{Mon}^n_{(\mathbf{Cat},\times)}$ is weakly equivalent to an n-fold loop space up to group completion.

Categorical operads and their algebras

We review in depth the definition of categorical operads in this thesis. We let $\mathfrak{S}$ denote the category which has the natural numbers $n \in \mathbb{N}$ as objects and the symmetric groups Σ_n, regarded as groups of automorphisms of the objects n, as morphisms. We define the 2-category of symmetric sequences in a monoidal 2-category Λ as the 2-category $\Lambda^{\mathfrak{S}^{op}}$ of 2-functors $\mathfrak{S}^{op} \to \Lambda$ We equip the category of symmetric sequences with a composition product $\circ$, which gives to $\left(\Lambda^{\mathfrak{S}^{op}}, \circ\right)$ the structure of a monoidal 2-category. We then define the 2-category $\mathrm{Op}_{\mathrm{Cat}}$ of symmetric operads defined in the symmetric monoidal 2-category $(\mathrm{Cat}, \times)$ as the category of monoids in the monoidal 2-category $(\mathrm{Cat}^{\mathfrak{S}^{op}}, \circ)$.

The operads $\mathcal{M}^n$ of iterated monoidal categories give examples of symmetric operads in Cat, so that $\mathcal{M}^n\mathrm{Alg}{\cong}\mathrm{Mon}^n_{(\mathrm{Cat},\times)}$. We introduce a suitable notion of presentation of operads by operadic polygraphs in order to revisit the construction of these operads. In the thesis, we explain a general definition of this notion of operadic polygraph presentation in the context of operads in n-categories. In the case of a small categorical operad $\mathcal{P}$, the idea is to take a presentation of the operad of objects of $\mathcal{P}$ as a set-theoretic operad, and then to take morphisms that generate $\mathcal{P}$ with respect to both operadic compositions and categorical compositions, together with generating relations involving both directions of compositions as well. We give an idea of this construction in the next subsection.

Operadic polygraphs. Recall we have a free/forgetfull adjunction for $\textsc{Set}$-theoretical operads

$$\mathcal{U} : \textsc{Op}_{\textsc{Set}} \rightleftharpoons \textsc{Set}^{\mathfrak{S}^{op}} : \mathcal{T}^{(0)}.$$

We have different manners of forgetting some structure on categorical operads: we can first forget about the categorical structure and the operadic structure and get a symmetric sequence of graphs, then forget about the 1-cells of a graph and get a symmetric sequence in $\textsc{Set}$. Since the objects of an operad in Cat form an operad in $\textsc{Set}$, we can also forget about the morphisms of the operad, and then forget about the operad structure. We call $\textsc{Op}^{+}_{\textsc{Set}}$ the result of the following pull back:

$$\begin{tikzcd}
\textsc{Op}_{\mathbf{Cat}} \arrow[dr, dashed, "\mathcal{W}^{(1)}"] \arrow[drr, bend left, "\mathcal{Y}^{(1)}"] \arrow[ddr, bend right, "\Pi^{(0)}"'] & & \\
& \textsc{Op}^{+}_{\textsc{Set}} \arrow[r, dashed, "Y^{(0)}_{+}"] \arrow[d, dashed, "\Pi^{(0)}_{+}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Gph^{\mathfrak{S}} \arrow[d, "\Pi^{(0)}"] \\
& \textsc{Op}_{\textsc{Set}} \arrow[r, "\mathcal{Y}^{(0)}"] & \textsc{Set}^{\mathfrak{S}}.
\end{tikzcd}$$

The induced morphism $\mathcal{W}^{(1)} : \textsc{Op}_{\mathbf{Cat}} \to \textsc{Op}^{+}_{\textsc{Set}}$ has a left adjoint we call $\mathcal{T}^{(1)} : \textsc{Op}^{+}_{\textsc{Set}} \to \textsc{Op}_{\mathbf{Cat}}$. The category $\textsc{Op}^{+}_{\textsc{Set}}$ can be described as the the category whose objects are symmetric sequences of graphs whose underlying sequence of 0-cells has the structure of an operad. For $\mathcal{P} \in \textsc{Op}_{\textsc{Set}^{+}}$ we write $\mathcal{P}_0 = \mathcal{V}^{(0)}_{+}(\mathcal{P})$ for the operad of 0-cells, $\mathcal{P}_1$ for the symmetric sequence of 1-cells, and we adopt the notation $\mathcal{P} = \mathcal{P}_1 \rightrightarrows \mathcal{P}_0$. An operadic polygraph $\mathcal{E}$ is the data of

- $\mathcal{E}^{(1)} \in \textsc{Op}^{+}_{\textsc{Set}}$,
- $\mathcal{E}^{(0)} \in \textsc{Set}^{\mathfrak{S}^{op}}$,

such that $\mathcal{E}^{(1)}_0 \cong \mathcal{T}^{(0)}(E_0)$. Thus, an operadic polygraph is just a graph of generating objects and morphisms

$$\mathcal{E} = \left(E_1 \rightrightarrows \mathcal{T}^{(0)}(E_0)\right),$$

where we wrote E_1 for $\mathcal{E}^{(1)}_1$ and E_0 for $\mathcal{E}^{(0)}_0$. Call $\mathcal{Q}\textsc{Op}_{\mathbf{Cat}}$ the category of operadic polygraphs, we have a pair of adjoints functors extending the free/forgetful adjunction

$$\mathcal{T} : \mathcal{Q}\textsc{Op}_{\mathbf{Cat}} \rightleftarrows \textsc{Op}_{\mathbf{Cat}} : \mathcal{W}.$$

We need to define how to endow operadic polygraphs with relations in a compatible way, so that we can talk about polygraphic presentations of categorical operads. Let $\mathcal{E}$ be an operadic polygraph. We define a system of compatible relations $\mathcal{R} = (R_1, R_0)$ on $\mathcal{E}$ as the data of

- a symmetric sequence $R_0 \in \textsc{Set}^{\mathfrak{S}^{op}}$, together with morphisms
$$R_0 \underset{r_2}{\overset{r_1}{\rightrightarrows}} \mathcal{T}(\mathcal{E})_0 \ ,$$
- a symmetric sequence $R_1 \in \textsc{Set}^{\mathfrak{S}^{op}}$, together with morphisms
$$R_1 \underset{s_2}{\overset{s_1}{\rightrightarrows}} \mathcal{T}(\mathcal{E})_1 \ ,$$

such that $\pi_0 s$ and $\pi_0 t$ both equalize s_0 and s_1, where $\pi_0 = \mathrm{coeq}(r_1, r_2)$ and $\pi_1 = \mathrm{coeq}(s_1, s_2)$, so that we have induced source and target morphisms

$$\begin{tikzcd}
R_0 \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & \mathcal{T}(\mathcal{E})_0 \arrow[r, dashed, "\pi_0"] & \mathcal{T}(\mathcal{E}:\mathcal{R})_0 \\
R_1 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & \mathcal{T}(\mathcal{E})_1 \arrow[r, dashed, "\pi_1"] \arrow[u, shift left, "s"] \arrow[u, shift right, "t"'] & \mathcal{T}(\mathcal{E}:\mathcal{R})_1 \arrow[u, dashed, shift left] \arrow[u, dashed, shift right]
\end{tikzcd} .$$

Hence we have a well defined functor

$$\mathcal{T}(-:-) : \mathcal{Q}^{\mathrm{REL}}\mathrm{OP}_{\mathbf{Cat}} \to \mathrm{OP}_{\mathbf{Cat}},$$

where $\mathcal{Q}^{\mathrm{REL}}\mathrm{OP}_{\mathbf{Cat}}$ is the category whose objects are operadic polygraphs equipped with compatible relations.

DEFINITION. A polygraphic presentation of an operad $\mathcal{P}$ is an isomorphism

$$\mathcal{T}(\mathcal{E}:\mathcal{R}) \xrightarrow{\cong} \mathcal{P}.$$

The homotopy theory of categorical operads. In what follows, we say that an operad $\mathcal{P}$ is free on its objects if there exists a symmetric sequence E and an isomorphism of operads $\mathcal{T}^{(0)}(E) \cong \mathcal{P}_0$, where $\mathcal{P}_0$ is the operad of objects of $\mathcal{P}$. We rely on the following result to do homotopy theory for operads.

THEOREM (Corollary II.3.2.5 and Proposition II.3.2.15). *There exists a model category structure on* $\mathrm{OP}_{\mathrm{CAT}}$ *such that*

- *The weak equivalences are aritywise equivalences of categories*
- *Fibrations are aritywise isofibrations*
- *An operad is cofibrant if and only if it is free on its objects.*

The second main result of the work of Balteanu, Fiedorowitch, Schwänzle and Vogt in [**2**] is that the operad of n-fold monoidal categories provides a model of E_n-operad in the sense that we have a weak equivalence of topological operads

$$|\,\mathcal{N}\mathcal{M}^n\,| \xrightarrow{\sim} \mathcal{C}_n,$$

where $\mathcal{C}_n$ denotes the operad of little n-cubes. Note however that the operads $\mathcal{M}^n$ are not free on their objects because of the associativity and unit relations. Hence they are not cofibrant. We will therefore aim to provide a construction of a cofibrant resolution of these operads.

Note that we did not considered the 2-categorical structure of CAT yet. In order to have some notions of compatible actions of operads, it can be convenient to define algebras over operads in CAT in a monoidal 2-category which is not CAT.

Algebras over categorical operads. We now examine the definition of the 2-category of algebras over a symmetric categorical operad. We generally work in the setting of a symmetric monoidal 2-category $(\Lambda, \otimes_\Lambda)$. [1]

[1] In this thesis, we also study the case of non symmetric operads, as well as their algebras in a monoidal 2-category $(\Lambda, \otimes_\Lambda)$. It is worth noting that we do *not* require the monoidale structure on Λ to be symmetric. See 1.4.8

For an object A of Λ, we define the category $\mathrm{End}_\Lambda^A(r) = \Lambda(A^{\otimes_\Lambda^r}, A)$ for $r \geq 0$, with the obvious action of the symmetric group. This collection of categories form a symmetric operad End_Λ^A in Cat. The composition is given by the composite

$$\Lambda\left(A^{\otimes_\Lambda^r}, A\right) \times \prod_{i=1}^r \Lambda\left(A^{\otimes_\Lambda^{n_i}}, A\right) \to \Lambda\left(A^{\otimes_\Lambda^r}, A\right) \times \Lambda\left(\bigotimes_{i=1}^r{}_\Lambda A^{\otimes_\Lambda^{n_i}}, A\right) \to \Lambda\left(A^{\otimes_\Lambda^n}, A\right),$$

where the first functor is the tensor product functor $\otimes_\Lambda$ induced in the category of morphisms of Λ and where the second functor is given by the composition of morphisms in Λ.

Definition (Definition 1.4.9). Let $\mathcal{P}$ be an operad in Cat and Λ be a symmetric monoidal 2-category.

- A $\mathcal{P}$-algebra is an object X of Λ equipped with a morphism of Cat-operads
$$\psi_X : \mathcal{P} \to \mathrm{End}_X^\Lambda.$$
In particular, each $p \in \mathcal{P}(r)$ yields a morphism in Λ from $X^{\otimes_\Lambda^r}$ to X, which we may also denote by p or $p_X : X^{\otimes_\Lambda^r} \to X$.
- Let X and Y be $\mathcal{P}$-algebras. We let $\mathcal{P}\text{-}\mathrm{Alg}_\Lambda(X,Y)$ be the category whose objects are lax morphisms of $\mathcal{P}$-algebras from X to Y, and whose set of morphisms between lax morphisms of $\mathcal{P}$-algebras F, G is given by $\mathcal{P}\text{-}\mathrm{Alg}_\Lambda(X,Y)(F,G)$, where
  - a lax morphism of $\mathcal{P}$-algebras from X to Y is a pair $(F, \otimes_F^\bullet)$, where $F : X \to Y$ is a morphism in Λ and $\otimes_F^\bullet$ is a 2-morphism in $[\mathbb{A}^{op}, \mathrm{Cat}]$:
$$\begin{tikzcd} & \Lambda\left(X^{\otimes_\Lambda^\bullet}, X\right) \arrow[dr, "\Lambda\left(X^{\otimes_\Lambda^\bullet}, F\right)"] & \\ \mathcal{P} \arrow[ur, "\psi_X"] \arrow[dr, "\psi_Y"'] & \Uparrow \otimes_F^\bullet & \Lambda\left(X^{\otimes_\Lambda^\bullet}, Y\right) \\ & \Lambda\left(Y^{\otimes_\Lambda^\bullet}, Y\right) \arrow[ur, "\Lambda\left(F^{\otimes_\Lambda^\bullet}, Y\right)"'] & \end{tikzcd}$$
which fulfils natural commutativity constraints (see Definition 1.4.9).
  - For $F, G \in \mathcal{P}\text{-}\mathrm{Alg}_\Lambda(X,Y)$, a 2-morphism of $\mathcal{P}$-algebras is a morphism $\alpha : F \to G$ in $\Lambda(X,Y)$ such that the following diagram commutes for all $p \in \mathcal{P}(r)$:
$$\begin{tikzcd} p_Y(F,\dots,F) \arrow[r, "\otimes_F^p"] \arrow[d, "p_Y(\alpha,\dots,\alpha)"'] & F(p_X,\dots,p_X) \arrow[d, "\alpha(p_X,\dots,p_X)"] \\ p_Y(G,\dots,G) \arrow[r, "\otimes_G^p"'] & G(p_X,\dots,p_X). \end{tikzcd}$$

We obtain a 2-category $\mathcal{P}\text{-}\mathrm{Alg}_\Lambda$ whose objects are given by $\mathcal{P}$-algebras in Λ, and whose category of morphisms between $\mathcal{P}$-algebras X and Y is given by $\mathcal{P}\text{-}\mathrm{Alg}_\Lambda(X,Y)$.

We have the following observation.

Proposition. *The 2-category $\mathcal{P}\text{-}\mathrm{Alg}_\Lambda$ is symmetric monoidal. Moreover,* Alg *yields a 3-functor*
$$\mathrm{Alg} : \mathrm{Op}_{\mathrm{Cat}}^{op} \times \mathrm{Mon}_{\mathrm{Cat}_2}^{\mathfrak{S}} \to \mathrm{Mon}_{\mathrm{Cat}_2}^{\mathfrak{S}},$$

where we use the notation $\mathrm{Mon}^{\mathfrak{S}}_{\mathrm{Cat}_2}$ *for the* 3*-category of symmetric monoidal* 2*-categories.*

The category of iterated monoids, defined in the first section of this introduction, can now be described as the 2-category of algebras over the Cat-operads $\mathcal{M}^n$, so that we have an isomorphism of 2-categories

$$\mathcal{M}^n\text{-}\mathrm{Alg}_{(\Lambda,\otimes_\Lambda)} \cong \mathrm{Mon}^n_{(\Lambda,\otimes_\Lambda)}.$$

We can also form the following definition.

Definition. The 2-category of $\mathcal{Q}$-algebras in the category of $\mathcal{P}$-algebras in Λ is the category $\mathcal{Q}\text{-}\mathrm{Alg}_{\mathcal{P}\text{-}\mathrm{Alg}_\Lambda}$ obtained by the composite

$$\begin{tikzcd}
\mathrm{Mon}^{\mathfrak{S}}_{\mathrm{Cat}_2} \arrow[r, "{(\mathcal{P},\mathcal{Q})}"] & \mathrm{Op}^{op}_{\mathrm{Cat}} \times \mathrm{Op}^{op}_{\mathrm{Cat}} \times \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{Cat}_2} \arrow[r, "\mathrm{Op}^{op}_{\mathrm{Cat}} \times \mathrm{Alg}"] \arrow[d, dotted, "\otimes_{\mathrm{BV}} \times \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{Cat}_2} ?"'] & \mathrm{Op}^{op}_{\mathrm{Cat}} \times \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{Cat}_2} \arrow[d, "\mathrm{Alg}"] \\
& \mathrm{Op}^{op}_{\mathrm{Cat}} \times \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{Cat}_2} \arrow[r, "\mathrm{Alg}"] & \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{Cat}_2}
\end{tikzcd}.$$

Note that, in the case of the operads $\mathcal{M}^n$, we have the isomorphism

$$\mathcal{M}^q\text{-}\mathrm{Alg}_{\mathcal{M}^p\text{-}\mathrm{Alg}_{(\Lambda,\otimes_\Lambda)}} \cong \mathcal{M}^{p+q}\text{-}\mathrm{Alg}_{(\Lambda,\otimes_\Lambda)}.$$

The (lax) Boardman-Vogt tensor product of operads and homotopy iterated monoids

The idea of the Boardman-Vogt tensor product is to form a tensor product of operads in order to fill the square of the diagram of the previous definition.

The construction of the (lax) Boardman-Vogt tensor product. Let $\mathcal{P},\mathcal{Q} : \mathfrak{S} \to \mathrm{Set}$ be symmetric sequences. The isomorphism of sets $\{1,\dots,n\} \times \{1,\dots,m\} \cong \{1,\dots,nm\}$ induces group morphisms $\Sigma_n \times \Sigma_m \to \Sigma_{nm}$. Hence, we have a functor

$$\times : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$$

which gives to $\mathfrak{S}$ the structure of a monoidal category. We define the matrix product of $\mathcal{P},\mathcal{Q}$ by a Day convolution operation:

$$\begin{tikzcd}
\mathfrak{S} \times \mathfrak{S} \arrow[r, "\mathcal{P}\times\mathcal{Q}"] \arrow[d, "\times"'] \arrow[dr, phantom, "\Downarrow"] & Cat \\
\mathfrak{S} \arrow[ur, dashed, "\mathcal{P}\square\mathcal{Q}"'] &
\end{tikzcd}$$

so that we have $\mathcal{P}\square\mathcal{Q} = Lan_\times(\mathcal{P}\times\mathcal{Q})$. Hence the data of a morphism of symmetric sequence $\mathcal{P}\square\mathcal{Q} \to \mathcal{R}$ is equivalent to the data of a morphism $\mathcal{P}(n)\times\mathcal{Q}(m) \to \mathcal{R}(nm)$ for each n,m in $\mathbb{N}$. Note that the square product induces a monoidal structure on the category of symmetric sequences in Cat. Moreover, the isomorphisms $\mathcal{P}(n) \times \mathcal{Q}(m) \xrightarrow{\cong} \mathcal{Q}(m) \times \mathcal{P}(n)$ induces a morphism

$$\mathfrak{s}_{\mathcal{P},\mathcal{Q}} : \mathcal{P}\square\mathcal{Q} \xrightarrow{\cong} \mathcal{Q}\square\mathcal{P}.$$

We also consider the morphism $\iota_{\mathcal{P},\mathcal{Q}} : \mathcal{P}\square\mathcal{Q} \to \mathcal{P}\circ\mathcal{Q}$ induced by the composite

$$\mathcal{P}(n) \times \mathcal{Q}(m) \xrightarrow{\mathtt{id}\times\Delta^n} \mathcal{P}(n) \times \prod_{i=1}^{r} \mathcal{Q}(m) \hookrightarrow \mathcal{P}\circ\mathcal{Q}(nm).$$

The usual Boardman-Vogt tensor product $\mathcal{P}\otimes_{\mathrm{BV}}\mathcal{Q}$ is identified with the quotient of the coproduct operad $\mathcal{P}\vee\mathcal{Q}$ by the operadic ideal generated by $\mathcal{P}\square\mathcal{Q}$, with projections σ,τ on the free operad given by

$$\sigma : \mathcal{P}\square\mathcal{Q} \xrightarrow{\iota_{\mathcal{P},\mathcal{Q}}} \mathcal{P}\circ\mathcal{Q} \hookrightarrow \mathcal{P}\vee\mathcal{Q}$$
$$\tau : \mathcal{P}\square\mathcal{Q} \xrightarrow{\mathfrak{s}_{\mathcal{P},\mathcal{Q}}} \mathcal{Q}\square\mathcal{P} \xrightarrow{\iota_{\mathcal{Q},\mathcal{P}}} \mathcal{Q}\circ\mathcal{P} \hookrightarrow \mathcal{P}\vee\mathcal{Q}.$$

so that the operad $\mathcal{P}\otimes_{\mathrm{BV}}\mathcal{Q}$ is universal amoung the operads equalizing σ and τ:

$$\mathcal{P}\square\mathcal{Q} \underset{\tau}{\overset{\sigma}{\rightrightarrows}} \mathcal{P}\bigvee\mathcal{Q} \dashrightarrow \mathcal{P}\otimes_{\mathrm{BV}}\mathcal{Q}\ .$$

The main interest of this construction is that the category of $\mathcal{P}\otimes_{\mathrm{BV}}\mathcal{Q}$-algebras is equivalent to the category of $\mathcal{P}$-algebras in the category of $\mathcal{Q}$-algebras, so that this tensor product is notably suitable for the study of interchanging structures in general and iterated loop spaces in particular.

In a sense, the idea of this thesis is to consider a lax version Boardman-Vogt tensor product in the context of categorical operads. The idea precisely consists in generalizing the construction of the operads $\mathcal{M}^n$ and adding a morphism between $\sigma(p,q)$ and $\tau(p,q)$ for each $p\in\mathcal{P}$ and $q\in\mathcal{Q}$, instead of making these operations equal. This construction will enables us to have a strong additivity property. We make the generators of this tensor product explicit on polygraphic presentations to give the ideas behind the construction.

Definition. Let $(\mathcal{E},\mathcal{R})$ and $(\mathcal{F},\mathcal{S})$ be operadic polygraphs together with compatible relations. We define their tensor product $(\mathcal{E},\mathcal{R})\otimes_{\mathrm{BV}}(\mathcal{F},\mathcal{S})$ as an operadic polygraph with relations $(\mathcal{E}\sqcup\mathcal{F}\sqcup\mathcal{G},\mathcal{R}\sqcup\mathcal{S}\sqcup\mathcal{U})$ with:

- $G_0 = U_0 = \emptyset$
- $G_1 = E_0\square F_0 \underset{\tau}{\overset{\sigma}{\rightrightarrows}} \mathcal{F}(E_0\sqcup F_0)$
- $U_1 = R_0\square F_0 \sqcup E_0\square S_0 \sqcup E_1\square S_0 \sqcup R_0\square F_1$.

This construction can also be made on the categories of categorical operads $\mathrm{Op}_{\mathrm{Cat}}$, without reference to a presentation. The operad $\mathcal{P}\otimes_{\mathrm{BV}}\mathcal{Q}$ is constructed as the coproduct operad $\mathcal{P}\vee\mathcal{Q}$, to which we add a sequence $\mathcal{P}\square\mathcal{Q}$ of generating morphisms with source and target given by σ and τ, and subject to coherence relations involving operadic composition of the objects of $\mathcal{P}$ and $\mathcal{Q}$ and morphisms in $\mathcal{P}$ and $\mathcal{Q}$. In particular, the objects of $\mathcal{P}\otimes_{\mathrm{BV}}\mathcal{Q}$ is the coproduct of the object operads of $\mathcal{P}$ and $\mathcal{Q}$. This construction is compatible with the polygraphic presentations, in the sense that we get the following proposition.

Proposition. *We have an isomorphism* $\mathcal{T}\left((\mathcal{E},\mathcal{R})\otimes_{\mathrm{BV}}(\mathcal{F},\mathcal{S})\right)\cong\mathcal{T}(\mathcal{E},\mathcal{R})\otimes_{\mathrm{BV}}\mathcal{T}(\mathcal{F},\mathcal{S})$.

We then have the following main result.

Theorem. *The lax Boardman-Vogt tensor product extends to a 2-functor which provides the 2-category of categorical operads with a monoidal structure*

$$\otimes_{\mathrm{BV}} : \mathrm{Op}_{\mathbf{Cat}}\times\mathrm{Op}_{\mathbf{Cat}} \longrightarrow \mathrm{Op}_{\mathbf{Cat}}.$$

Moreover, the algebra 2-functor Alg *gives to the 2-category of symmetric monoidal 2-categories the structure of a left module over the monoidal 2-category* $(\mathrm{Op}_{\mathrm{Cat}},\otimes_{\mathrm{BV}})$, *in the sense that the square*

$$\begin{array}{ccc} \mathrm{Op}^{op}_{\mathrm{Cat}} \times \mathrm{Op}^{op}_{\mathrm{Cat}} \times \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{Cat}_2} & \xrightarrow{\mathrm{Op}^{op}_{\mathrm{Cat}} \times \mathrm{Alg}} & \mathrm{Op}^{op}_{\mathrm{Cat}} \times \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{Cat}_2} \\ {\scriptstyle \otimes^{op}_{\mathrm{BV}} \times \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{Cat}_2}} \downarrow & & \downarrow {\scriptstyle \mathrm{Alg}} \\ \mathrm{Op}^{op}_{\mathrm{Cat}} \times \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{Cat}_2} & \xrightarrow{\mathrm{Alg}} & \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{Cat}_2} \end{array}$$

commutes up to a canonical isomorphism. The tensor product also preserves acyclic fibrations, and $\mathcal{P}\otimes_{\mathrm{BV}}\mathcal{Q}$ is a cofibrant operad if and only if both $\mathcal{P}$ and $\mathcal{Q}$ are cofibrant.

This result admits the following corollary.

Corollary. *If we have cofibrant resolutions $\mathcal{P}^\infty \xrightarrow{\sim} \mathcal{P}$ and $\mathcal{Q}^\infty \xrightarrow{\sim} \mathcal{Q}$ of operads $\mathcal{P}$ and $\mathcal{Q}$, then the tensor product*

$$\mathcal{P}^\infty \otimes_{\mathrm{BV}} \mathcal{Q}^\infty \overset{\sim}{\twoheadrightarrow} \mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$$

is a cofibrant resolution of $\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$.

Note that the tensor product is not symmetric. However, we have an isomorphism

$$\mathcal{P}^{op} \otimes_{\mathrm{BV}} \mathcal{Q}^{op} \cong (\mathcal{Q} \otimes_{\mathrm{BV}} \mathcal{P})^{op}.$$

We say that this tensor product is lax because the $\mathcal{Q}$-algebra structure on a $\mathcal{P}\otimes_{\mathrm{BV}}\mathcal{Q}$-algebra yields lax morphisms of $\mathcal{P}$-algebras. The monoidal structure induced on the 2-category of operads in Cat is associative up isomorphism.

The application to the definition of cofibrant resolutions of the operads $\mathcal{M}^n$. In the case of the operads $\mathcal{M}$, we have an isomorphism $\mathcal{M}^n \otimes_{\mathrm{BV}} \mathcal{M}^m \cong \mathcal{M}^{n+m}$. The idea of this thesis is therefore to use the strong additivity property of the lax Boardman-Vogt tensor product in order to produce a cofibrant resolution $\mathcal{M}^n_\infty$ of the operad $\mathcal{M}^n$ from a cofibrant resolution $\mathcal{M}^1_\infty$ of the operad $\mathcal{M}^1$. We with a description of the construction of this operad $\mathcal{M}^1_\infty$.

We define the operad $\mathcal{M}^1_\infty$ by the following presentation.

We first define a non symmetric operadic polygraph $\mathcal{E}^{\cong}$ aritywise by

- $E^{\cong}{}_0(n) = \{\otimes_n\}$,
- $E^{\cong}{}_1(n) = \coprod_{\substack{r+s=n+1\\ r,s\geq 0}} \{\otimes^i_{r,s}, {\otimes^i_{r,s}}^{-1}\}_{1\leq i\leq r}$,

  with $s,t : E^{\cong}_1(n) \to \mathcal{T}(E^{\cong}_0)(n)$ given by
  - $s(\otimes^i_{r,s}) = \otimes_r \circ_i \otimes_s$,
  - $s({\otimes^i_{r,s}}^{-1}) = \otimes_n$,
  - $t(\otimes^i_{r,s}) = \otimes_n$.
  - $t({\otimes^i_{r,s}}^{-1}) = \otimes_r \circ_i \otimes_s$,

We then consider the system of relations $\mathcal{R}^{\cong}$ for the operadic polygraph $\mathcal{E}$ that we form as follows

- $R^{\cong}{}_0 = \emptyset$
- $$R^{\cong}_1(n) = \{\eta^T\}_{(T\in\mathcal{T}\mathcal{E}(n), \rho T=3)} \coprod \coprod_{\substack{r+s=n+1\\ r,s\geq 0}} \{{\otimes^i_{r,s}}^{-1}\otimes^i_{r,s} = 1_{\otimes_r\circ_i\otimes_s}\}_{1\leq i\leq r} \coprod \coprod_{\substack{r+s=n+1\\ r,s\geq 0}} \{\otimes^i_{r,s}{\otimes^i_{r,s}}^{-1} = 1_{\otimes_n}\}_{1\leq i\leq r},$$

where η_T implements the relations corresponding to the comutativity of the diagrams

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T/\{e\} \arrow[d] \\ T/\{f\} \arrow[r] & T/\{e,f\} = \otimes_n \end{tikzcd}$$

for each $T \in \mathcal{T}E(n)$ with 3 vertices.

We eventually set $\mathcal{M}^1_\infty = \mathcal{T}(\mathcal{E}^{\cong} : \mathcal{R}^{\cong})$. We get the following first statement.

THEOREM. *The operad $\mathcal{M}^1_\infty$ is cofibrant. We have an acyclic fibration*

$$\mathcal{M}^1_\infty \overset{\sim}{\twoheadrightarrow} \mathcal{M}^1$$

providing a cofibrant resolution of the operad $\mathcal{M}^1$.

In chapter II, we actually define a cofibrant resolution $\mathcal{A}s_\infty$ of $\mathcal{M}^1$ by applying a general construction of a cofibrant resolution $\mathcal{C}(\mathcal{P})$ for categorical operads. For $\mathcal{P} = \mathcal{M}^1$, the result of this construction $\mathcal{A}s_\infty = \mathcal{C}(\mathcal{M}^1)$ is identified with the operad $\mathcal{M}^1_\infty$ given by the above presentation.

We then use our result on the Boardman Vogt tensor product to get the following main result.

THEOREM (Corollary III.3.1.1). *The operad $\mathcal{M}^n_\infty := (\mathcal{M}^1_\infty)^{\otimes^n_{\mathrm{BV}}}$ is cofibrant, and provides a cofibrant resolution of the operad $\mathcal{M}^n$ through the following acyclic fibration*

$$\mathcal{M}^n_\infty \overset{\sim}{\twoheadrightarrow} \mathcal{M}^n.$$

Moreover, we have the following operadic presentation of $\mathcal{M}^n_\infty$:

- $E^n_0 = \coprod_{i=1}^{n} E_0$
- $E^n_1 = \coprod_{i=1}^{n} E_1 \sqcup \coprod_{1\leq i<j\leq n} E_0 \square E_0$
- $R^n_1 = \coprod_{i=1}^{n} R_1 \sqcup \coprod_{1\leq i<j\leq n} E_1 \square E_0 \sqcup \coprod_{1\leq i<j\leq n} E_0 \square E_1 \sqcup \coprod_{1\leq i<j<k\leq n} E_0 \square E_0 \square E_0.$

Hence its objects are freely generated by n operations in each arity and its morphisms are edge contractions between vertices of the same color together with interchange morphisms between each pair of operation, subjected to coherence relations.

We can define a homotopy n-fold monoid in a symmetric monoidal 2-category $(\Lambda, \otimes_\Lambda)$ is the data of

- an object X of Λ
- for each $1 \leq i \leq n$ and $r \in \mathbb{N}$, a morphism in Λ

$$\otimes^r_i : X^{\otimes^r_\Lambda} \to X,$$

– for each $1 \leq i \leq r$ and $p,q,r \in \mathbb{N}$, a 2-morphism $\otimes_i^{p,q,r}$ in Λ such that

$$\begin{tikzcd}
X^{\otimes_\Lambda^{p+q+r}} \arrow[dd, "\otimes_i^{p+q+r}"'] & X^{\otimes_\Lambda^p} \otimes_\Lambda X^{\otimes_\Lambda^q} \otimes_\Lambda X^{\otimes_\Lambda^r} \arrow[l] \arrow[d, "X^{\otimes_\Lambda^p} \otimes_\Lambda \otimes_i^q \otimes_\Lambda X^{\otimes_\Lambda^r}"] \\
 & X^{\otimes_\Lambda^p} \otimes_\Lambda X \otimes_\Lambda X^{\otimes_\Lambda^r} \arrow[d] \arrow[l, phantom, "\overset{\Longleftarrow}{\otimes_i^{p,q,r}}"] \\
X & X^{\otimes_\Lambda^{p+r+1}} \arrow[l, "\otimes_i^{p+r+1}"]
\end{tikzcd}$$

which fulfils coherence constraints expressed by the commutativity of a familly of squares for each $1 \leq i \leq n$,

– for each $1 \leq i < j \leq n$ and $r,s \in \mathbb{N}$, an interchange 2-morphism $\square_{i,r}^{j,s}$ in Λ, such that

$$\begin{tikzcd}
 & \left(X^{\otimes_\Lambda^r}\right)^{\otimes_\Lambda^s} \arrow[r, "\otimes_i^r \otimes_\Lambda^s"] & (X)^{\otimes_\Lambda^s} \arrow[d, "\otimes_j^s"] \\
\left(X^{\otimes_\Lambda^s}\right)^{\otimes_\Lambda^r} \arrow[ur, "\cong"] \arrow[r, "\otimes_j^s \otimes_\Lambda^r"'] & (X)^{\otimes_\Lambda^r} \arrow[r, "\otimes_i^r"'] \arrow[ur, phantom, "\square_{i,r}^{j,s}"] & X
\end{tikzcd}$$

and which fulfils coherence constraints expressed by the commutativity of

– a family of pentagons $\left\{\pentagon_{i,j}^{r,l,s,p}, 1 \leq l \leq r\right\}_{1 \leq i < j \leq n}^{r,s,p \geq 0}$ where the pentagon $\pentagon_{i,j}^{r,l,s,p}$ ensures that the 1-morphism $\otimes_j^p$ is compatible with the 2-morphism $\otimes_{l,i}^{r,s}$ if $i < j$ and ensures that the 2-morphism $\otimes_{l,i}^{r,p}$ is compatible with the 1-morphism $\otimes_j^s$ if $j < i$,

– a family of hexagons $\left\{\hexagon_{i,j,k}^{r,s,p}, 1 \leq i < j < k \leq n\right\}_{r,s,p \geq 0}$, where the hexagon $\hexagon_{i,j,k}^{r,s,p}$ ensures the compatibility of the 2-morphism $\square_{j,s}^{k,p}$ with respect to the 1-morphism $\otimes_i^r$.

We can describe an $\mathcal{M}_\infty^n$-algebra structure on the n-fold loop space $\Omega^n X$ of a topological space X, in the monoidal 2-category $(\mathrm{Top}, \times)$, where the 2-morphisms are the homotopy classes of the homotopies. For this purpose, we first observe that the n-cube I^n has the structure of an $\mathcal{M}_\infty^n$-algebra in the monoidal 2-category $(\mathrm{Top}, \sqcup)$, which induces an $\mathcal{M}_\infty^n$-algebra structure in $(\mathrm{Top}, \sqcup)$ on the n-sphere S^n via the isomorphism $S^n \cong I^n/\partial I^n$.

Let $\otimes_i^r : \bigsqcup\limits_{k=1}^{r} I^n \to I^n$ be the map which sends the j-th little n-cube to I^n by inclusion in contracting its i-th coordinate by $1/r$. Explicitly, we set

$$\otimes_i^r(t^{(j)}) = \left(t_1^{(j)}, \ldots, \frac{t_i^{(j)} + j - 1}{r}, \ldots, t_n^{(j)}\right)$$

where $t^{(j)} = \left(t_1^{(j)}, \ldots, t_r^{(j)}\right)$. Note that the interchange laws hold strictly. We also can define homotopies $\otimes_i^r \circ_l \otimes_i^s \to \otimes_i^{r+s-1}$ which fullfil the coherence constraints, so that $I^n \in \mathcal{M}_\infty^n\text{-}\mathrm{Alg}_{(\mathrm{Top}, \sqcup)}$. Moreover, the maps $\otimes_i^r$ together with the homotopies, induce maps and homotopies on the quotient $I^n/\partial I^n \cong S^n$, so that S^n has an induced structure of an $\mathcal{M}_\infty^n$-algebra in $(Top, \sqcup)$.

Let X be a topological space. We construct an $\mathcal{M}^n_\infty$-algebra structure on $\Omega^n X = [S^n, X]$ by the composite

$$[S^n, X]^r \cong \left[\bigsqcup_r S^n, X\right] \xrightarrow{[(\otimes_i^r)^*, X]} [S^n, X]\,,$$

where $(\otimes_i^r)^*$ is an extension of $\otimes_i^r$, such that:

$$\begin{tikzcd} \bigsqcup_r S^n \arrow[r, "(\gamma_1,\ldots,\gamma_r)"] \arrow[d, "\otimes_i^r"'] & X \\ S^n \arrow[ur, dashed, "(\otimes_i^r)^*(\gamma_1,\ldots,\gamma_r)"'] & \end{tikzcd}$$

We therefore obtain a 2-functor

$$\Omega^n : \textsc{Top} \to \mathcal{M}^n_\infty\text{-}\textsc{Alg}_{(\textsc{Top},\times)}.$$

Logical background and size issues

Questions related to the size of categories play a significant role in this thesis. Indeed, the operads $\mathcal{M}^n$ of iterated monoidal categories are defined in the *category* of small categories, so that a significant part of our work is situated within this framework. However, in order to establish the constructions defined in this thesis in a *formal* way, it seemed appropriate to place ourselves in a 2-categorical framework, with particular attention given to the 2-categories $\textsc{Cat}$ and $\textsc{Cat}^{\mathbb{N}}$ of categories and sequences of categories. By category, we mean, unless otherwise specified, what is usually referred to as a locally small category. The appendix of this thesis is dedicated to establish a logical framework enabling precise reasoning, particularly with regard to questions related to size. The first chapter of this thesis may be seen as a detailed application of the framework introduced in the appendix, focusing in the case where $n = 1, 2$.

CHAPTER I

Categorical algebra in a 2-category

This chapter aims to lay down the foundational framework employed in this thesis. To be specific, we focus on the context of 2-categories, as the main objects of this thesis consist in operads defined in the 2-category CAT of categories.

In a first section, we provide a comprehensive review of general constructions of category theory in the framework of 2-category theory, and delve deeper into the exploration of those categorical concepts We inspect the 2-category of categories, examining its structure and the properties that render it an appropriate fundamental object amidst all other objects in the 3-category CAT_2 of 2-categories. We review the customary definitions involving 2-categories, such as 2-adjunctions and monoidality, and we recall the different notions of Kan extensions in a 2-category.

In a second section, we study the internal structures of a given 2-category. We begin with a digression on the structure of 2-categories Λ equipped with a distinguished object $*_\Lambda$, *a unit.* This unit enables us to associate a category to any object of Λ, the category of its points. We use this distinguished object to unravel the underlying structure of objects in Λ, serving as a starting point for subsequent internal reasoning we will implement in this chapter. In this context, it becomes pertinent to examine the nature of the 2-functor $\Lambda(*_\Lambda, -) : \Lambda \to \mathrm{CAT}$, which maps the objects of Λ to categories. Indeed, whether this 2-functor is locally faithful or not will provide information on whether the structures defined internally and externally in Λ coincide. Notably, when Λ is 2-category CAT, this 2-functor is an equivalence. Then we introduce various notions of limits and colimits within a 2-category Λ, both from internal and external perspectives. We revisit the definitions of Kan extensions and provide internal versions of them. We show that internal Kan extensions pointwise coincide with external Kan extensions in a sense that we make precise.

In a third subsection we extensively explore the concepts of monoidality, examining their intricacies and fundamental aspects. We examine the definition of monoidal 2-categories. Then we study a notion of monoid internal to a monoidal 2-category, and we examine the definition of iterated monoid structures in iterated monoidal 2-categories. Finally, we extend day convolution product to the context of monoids internal to a monoidal 2-category which is also cartesian closed.

In a fourth section, we examine the case where the 2-category Λ is a cartesian closed monoidal 2-category equipped with a distinguished object $\mathcal{S}$. We require some assumptions on the structure of $\mathcal{S}$, such as internal completeness and cocompleteness, so that this object will provide an analogue of the category of sets. We use the object $\mathcal{S}$ to provide each object of Λ with a structure analogous to a category, which we require to satisfy coherence conditions. Those conditions will ensure that the 2-category structure of Λ is compatible with the internal category structure of its objects. Finally, we define internal ends and coends in Λ and use them to provide an explicit description of Kan extensions and day convolution products. We conclude this section by defining free cocomplete completion of objects in Λ with respect to $\mathcal{S}$. We also provide an analogue of the Yoneda embedding and prove an equivalent of the Yoneda lemma.

1. Basic 2-category theory

The purpose of this section is to review some basic categorical definitions, mainly to fix the background of the constructions of this thesis. To be specific, in the subsequent sections of this chapter, we will generalize usual categorical constructions within the framework of a 2-category satisfying some properties that make this 2-category looks like the 2-category of categories. Therefore, we recall some facts involving sets, categories, 2-categories, and the way these concepts are entangled[1].

In this thesis, the concept of equality is only defined for elements of a given set. The categories and the 2-categories we consider are, a priori, large[2]. For this reason, we no not consider strict functors between large categories, neither do we consider strict 2-functors or strict 2-natural transformations between 2-categories. Note that this perspective aligns with expectations that have been extensively observed in the literature (see for instance [**13, 22**]). We propose in Appendix A a logical framework, on which we can found our constructions and work out set theoretic difficulties that generally occur in such contexts. We refer to this appendix for further precision on the background of the definitions of this section[3].

1.1. Conventions. We start with a review of conventions regarding sets, categories, 2-categories and 3-categories (referring to Appendix A for the details of the definitions). We make a particular emphasis on the basic case of sets in order to give a hint of our interpretation of sets within this hierarchy of categorical structures. In what follows, we generally let $\mathrm{S{\scriptsize ET}}$ denote the category of sets, we let $\mathrm{C{\scriptsize AT}}$ denote the 2-category of categories with the categories of functors as categories of morphisms, we let $\mathrm{C{\scriptsize AT}}_2$ denote the 3-category of 2-categories with the 2-categories of 2-functors as 2-categories of morphisms, and so on. We also use the convention $\mathrm{C{\scriptsize AT}}_0 = \mathrm{S{\scriptsize ET}}$, $\mathrm{C{\scriptsize AT}}_1 = \mathrm{C{\scriptsize AT}}$.

NOTATION 1.1.1.

- Let X be a set.
  - If x is an element of X, then we write $x \in X$.
  - If $x, y \in X$, then we may write $X(x, y)$ for the truth value of the relation $x = y$.
- Let $\mathcal{C}$ be a category.

---

[1] We have $\mathrm{S{\scriptsize ET}} \in \mathrm{C{\scriptsize AT}}, \mathrm{C{\scriptsize AT}} \in \mathrm{C{\scriptsize AT}}_2, \mathrm{C{\scriptsize AT}}_2 \in \mathrm{C{\scriptsize AT}}_3$, and so on. Most of the constructions made in $\mathrm{C{\scriptsize AT}}$ are based on a similar construction at the level immediately lower of sets. In the same way, we also need to examine the structure of $\mathrm{C{\scriptsize AT}}_2$, so that categorical constructions can be defined into a global, functorial framework.

[2] See Appendix A. for a definition of small objects.

[3] In general, we adopt standard mathematical notation in the main text of the thesis. Note however that these notations may occasionally differ from the conventions adopted in the appendix. For instance, we adopt the usual notation '$\in$' for the belonging relation of objects in the main text, while in the appendix we prefer to use the notation ':' in order to stress a logical interpretation of this relation. In the same way, in the appendix, we use the notation '$\Leftrightarrow$' and '$\Rightarrow$' as logical constructors for definitions that we explain more informally in the account of this section.

  - If X is an object of $\mathcal{C}$, then we write $X \in \mathcal{C}$.
  - If $X, Y \in \mathcal{C}$, then we write $\mathcal{C}(X, Y)$ for the set of morphisms from X to Y. We also write $f : X \to Y$ for $f \in \mathcal{C}(X, Y)$.
- Let Λ be a 2-category.
  - If $\mathcal{C}$ is an object of Λ, then we write $\mathcal{C} \in \Lambda$.
  - If $\mathcal{C}, \mathcal{D} \in \Lambda$, then we write $\Lambda(\mathcal{C}, \mathcal{D})$ for the category of morphisms from $\mathcal{C}$ to $\mathcal{D}$. We also write $F : \mathcal{C} \to \mathcal{D}$ for $F \in \Lambda(\mathcal{C}, \mathcal{D})$.
- Let $\mathcal{T}$ be a 3-category.
  - If Λ is an object of $\mathcal{T}$, then we write $\Lambda \in \mathcal{T}$.
  - If $\Lambda, \Gamma \in \mathcal{T}$, then we write $\mathcal{T}(\Lambda, \Gamma)$ for the 2-category of morphisms from Λ to Γ. We also write $\mathcal{F} : \Lambda \to \Gamma$ for $\mathcal{F} \in \mathcal{T}(\Lambda, \Gamma)$.

NOTATION 1.1.2.

- Let $\mathcal{C} \in \mathrm{CAT}$. We write
$$\mathcal{C}(\_, \_) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{SET}$$
for the functor given on objects X, Y of $\mathcal{C}$ by the set of morphisms $\mathcal{C}(X, Y)$ from X to Y.[4]
- Let $\Lambda \in \mathrm{CAT}_2$. We write
$$\Lambda(\_, \_) : \Lambda^{op} \times \Lambda \to \mathrm{CAT}$$
for the 2-functor given on objects $\mathcal{C}, \mathcal{D}$ of Λ by the category of morphisms $\Lambda(\mathcal{C}, \mathcal{D})$ from $\mathcal{C}$ to $\mathcal{D}$.[5]

1.2. Categorical constructions.

Oppositization. We review and unravel the definition of oppositization functors in the setting of categories, 2-categories, 3-categories.

DEFINITION 1.2.1. Let $\mathcal{C} \in \mathrm{CAT}$. We let $\mathcal{C}^{op} \in \mathrm{CAT}$ be the opposite category of $\mathcal{C}$, defined as follows:

- The objects $X \in \mathcal{C}^{op}$ correspond to objects $X \in \mathcal{C}$.
- For $X, Y \in \mathcal{C}$, we set $\mathcal{C}^{op}(X, Y) = \mathcal{C}(Y, X)$.

DEFINITION 1.2.2. Let $\Lambda \in \mathrm{CAT}_2$.

- We let $\Lambda^{op} \in \mathrm{CAT}_2$ be the opposite 2-category of Λ, defined as follows:
  - The objects $\mathcal{C} \in \Lambda^{op}$ correspond to objects $\mathcal{C} \in \Lambda$.
  - For $\mathcal{C}, \mathcal{D} \in \Lambda$, we set $\Lambda^{op}(\mathcal{C}, \mathcal{D}) = \Lambda(\mathcal{D}, \mathcal{C})$.
- We let $\Lambda^{op_2} \in \mathrm{CAT}_2$ be the 2-opposite 2-category of Λ, defined as follows:
  - The objects $\mathcal{C} \in \Lambda^{op_2}$ correspond to objects $\mathcal{C} \in \Lambda$.
  - For $\mathcal{C}, \mathcal{D} \in \Lambda$, we set $\Lambda^{op_2}(\mathcal{C}, \mathcal{D}) = \Lambda(\mathcal{C}, \mathcal{D})^{op}$.

DEFINITION 1.2.3. We let $\_^{op} : \mathrm{CAT}^{op_2} \to \mathrm{CAT}$ be the oppositization 2-functor, defined as follows:

- To the object $\mathcal{C} \in \mathrm{CAT}^{op_2}$, corresponding to $\mathcal{C} \in \mathrm{CAT}$, we associate $\mathcal{C}^{op} \in \mathrm{CAT}$.

---

[4]This functor may also be obtained from the definition of a category (cf. Appendix A). The functor $\mathcal{C}(\_, \_)$ yields maps $\mathcal{C}(Y_1, X_1) \times \mathcal{C}(X_2, Y_2) \to \mathrm{SET}(\mathcal{C}(X_1, X_2), \mathcal{C}(Y_1, Y_2))$ natural in the objects X_1, X_2, Y_1, Y_2 of $\mathcal{C}$. This map can be regarded as the composition of morphims in $\mathcal{C}$, simultaneously on both sides. The associativity of this simultaneous composition operation is ensured by the naturality constraints that $\mathcal{C}(\_, \_)$ must satisfy to be a functor.

[5]This functor may again be obtained from the definition of a 2-category.

- For $\mathcal{C}, \mathcal{D} \in \mathrm{CAT}^{op_2}$, we define the functor $\_^{op}_{\mathcal{C},\mathcal{D}} : \mathrm{CAT}^{op_2}(\mathcal{C}, \mathcal{D}) = \mathrm{CAT}(\mathcal{C}, \mathcal{D})^{op} \to \mathrm{CAT}(\mathcal{C}^{op}, \mathcal{D}^{op})$ by the following correspondence:
  - To $F \in \mathrm{CAT}(\mathcal{C}, \mathcal{D})^{op_2}$, we associate the functor $F^{op} \in \mathrm{CAT}(\mathcal{C}^{op}, \mathcal{D}^{op})$ using that every object $X \in \mathcal{C}^{op}$, corresponding to $X \in \mathcal{C}$, can be mapped to $FX \in \mathcal{D}$ giving $F^{op}X = FX \in \mathcal{D}^{op}$, and for every pair $X, Y \in \mathcal{C}^{op}$, we can form $F^{op}(X, Y) = F(Y, X) : \mathcal{C}^{op}(X, Y) \to \mathcal{D}^{op}(FX, FY)$.
  - For $F, G \in \mathrm{CAT}(\mathcal{C}, \mathcal{D})^{op_2}$, we define the map $\mathrm{CAT}(\mathcal{C}, \mathcal{D})^{op_2}(F, G) \to \mathrm{CAT}(\mathcal{C}^{op}, \mathcal{D}^{op})(F^{op}, G^{op})$ through

$$\begin{aligned}\mathrm{CAT}(\mathcal{C}, \mathcal{D})^{op_2}(F, G) &= \mathrm{CAT}(\mathcal{C}, \mathcal{D})(G, F) \\ &\cong \int^{X \in \mathcal{C}} \mathcal{D}(GX, FX) \\ &\cong \int^{X \in \mathcal{C}^{op}} \mathcal{D}^{op}(F^{op}X, G^{op}X) \\ &\cong \mathrm{CAT}(\mathcal{C}^{op}, \mathcal{D}^{op})(F^{op}, G^{op}).\end{aligned}$$

Definition 1.2.4. Let $\mathcal{U} \in \mathrm{CAT}_3$.

- We let $\mathcal{U}^{op} \in \mathrm{CAT}_3$ be the opposite 3-category of $\mathcal{U}$, defined as follows:
  - The objects $\Lambda \in \mathcal{U}^{op}$ correspond to objects $\Lambda \in \mathcal{U}$.
  - For $\Lambda, \Gamma \in \mathcal{U}^{op}$, we set $\mathcal{U}^{op}(\Lambda, \Gamma) = \mathcal{U}(\Gamma, \Lambda)$.
- We let $\mathcal{U}^{op_2} \in \mathrm{CAT}_3$ be the 2-opposite 3-category of $\mathcal{U}$, defined as follows:
  - The objects $\Lambda \in \mathcal{U}^{op_2}$ correspond to objects $\Lambda \in \mathcal{U}$,
  - For $\Lambda, \Gamma \in \mathcal{U}^{op_2}$, we set $\mathcal{U}^{op_2}(\Lambda, \Gamma) = \mathcal{U}(\Lambda, \Gamma)^{op}$.
- We let $\mathcal{U}^{op_3} \in \mathrm{CAT}_3$ be the 3-opposite 3-category of $\mathcal{U}$, defined as follows:
  - The objects $\Lambda \in \mathcal{U}^{op_3}$ correspond to objects $\Lambda \in \mathcal{U}$,
  - For $\Lambda, \Gamma \in \mathcal{U}^{op3}$, we set $\mathcal{U}^{op_3}(\Lambda, \Gamma) = \mathcal{U}(\Lambda, \Gamma)^{op_2}$.

Definition 1.2.5. We let $\_^{op} : \mathrm{CAT}_2^{op_2} \to \mathrm{CAT}_2$ be the oppositization 3-functor on 2-categories, defined as follows:

- To the object $\Lambda \in \mathrm{CAT}_2^{op_2}$, corresponding to $\Lambda \in \mathrm{CAT}_2$, we associate $\Lambda^{op} \in \mathrm{CAT}_2$.
- For $\Lambda, \Gamma \in \mathrm{CAT}_2^{op_2}$, we define the 2-functor $\_^{op_2}_{\Lambda,\Gamma} : \mathrm{CAT}_2^{op_2}(\Lambda, \Gamma) = \mathrm{CAT}_2(\Lambda, \Gamma)^{op} \to \mathrm{CAT}_2(\Lambda^{op}, \Gamma^{op})$ by the following correspondence:
  - To $F : \Lambda \to \Gamma$, we associate the 2-functor $F^{op_2} : \Lambda^{op} \to \Gamma^{op}$ using that every object $\mathcal{C} \in \Lambda^{op}$, corresponding to $\mathcal{C} \in \Lambda$, can be mapped to $\mathcal{FC} \in \Gamma$, giving $F^{op}\mathcal{C} = F\mathcal{C} \in \Gamma$, and for every pair $\mathcal{C}, \mathcal{D} \in \Lambda^{op}$, we can form $F^{op}_{\mathcal{C},\mathcal{D}} = F_{\mathcal{D},\mathcal{C}} : \Lambda(\mathcal{D}, \mathcal{C}) \to \Gamma(F\mathcal{D}, F\mathcal{C})$.
  - For $F, G : \Lambda \to \Gamma$, we define $\mathrm{CAT}_2(\Lambda, \Gamma)^{op}(F, G) = \mathrm{CAT}_2(\Lambda, \Gamma)(G, F) \to \mathrm{CAT}_2(\Lambda^{op}, \Gamma^{op})(F^{op}, G^{op})$ through

$$\begin{aligned}\mathrm{CAT}_2(\Lambda, \Gamma)(G, F) &\xrightarrow{\cong} \int^{\mathcal{C} \in \Lambda} \Gamma(G\mathcal{C}, F\mathcal{C}) \xrightarrow{\cong} \int^{\mathcal{C} \in \Lambda^{op}} \Gamma^{op}(F^{op}\mathcal{C}, G^{op}\mathcal{C}) \\ &\xrightarrow{\cong} \mathrm{CAT}_2(\Lambda^{op}, \Gamma^{op})(F^{op}, G^{op}).\end{aligned}$$

Definition 1.2.6. We let $\_^{op_2} : \mathrm{CAT}_2^{op_3} \to \mathrm{CAT}_2$ be the 2-oppositization 3-functor on 2-categories, defined as follows:

– To the objects $\Lambda \in \mathrm{CAT}_2^{op_3}$, corresponding to objects $\Lambda \in \mathrm{CAT}_2$, we associate $\Lambda^{op_2} \in \mathrm{CAT}_2$
– For $\Lambda, \Gamma \in \mathrm{CAT}_2^{op_3}$, we define the 2-functor $\_\,^{op_2}_{\Lambda,\Gamma} : \mathrm{CAT}_2^{op_3}(\Lambda, \Gamma) = \mathrm{CAT}_2(\Lambda, \Gamma)^{op_2} \to \mathrm{CAT}_2(\Lambda^{op_2}, \Gamma^{op_2})$ by the following correspondence:
  – To $F : \Lambda \to \Gamma$, we associate the 2-functor $F^{op_2} : \Lambda^{op_2} \to \Gamma^{op_2}$ using that every object $\mathcal{C} \in \Lambda$ can be mapped to $F\mathcal{C} \in \Gamma$, and for every pair $\mathcal{C}, \mathcal{D} \in \Lambda$, we can form $F^{op_2}(\mathcal{C}, \mathcal{D}) : \Lambda^{op_2}(\mathcal{C}, \mathcal{D}) \to \Gamma^{op_2}(F\mathcal{C}, F\mathcal{D})$ by taking

$$\Lambda^{op_2}(\mathcal{C}, \mathcal{D}) = \Lambda(\mathcal{C}, \mathcal{D})^{op} \xrightarrow{F(\mathcal{C},\mathcal{D})^{op}} \Gamma(F\mathcal{C}, F\mathcal{D})^{op} = \Gamma^{op_2}(F\mathcal{C}, F\mathcal{D}).$$

– For $F, G : \Lambda \to \Gamma$, we define $\mathrm{CAT}_2^{op_3}(\Lambda, \Gamma)(F, G) = \mathrm{CAT}_2(\Lambda, \Gamma)(F, G)^{op}$
$\to \mathrm{CAT}_2(\Lambda^{op_2}, \Gamma^{op_2})(F^{op_2}, G^{op_2})$

through

$$\mathrm{CAT}_2(\Lambda, \Gamma)(F, G)^{op} \xrightarrow{\cong} \int^{\mathcal{C} \in \Lambda^{op}} \Gamma(F\mathcal{C}, G\mathcal{C})^{op} \xrightarrow{\cong} \int^{\mathcal{C} \in \Lambda^{op_2}} \Gamma^{op_2}(F^{op_2}\mathcal{C}, G^{op_2}\mathcal{C})$$
$$\xrightarrow{\cong} \mathrm{CAT}_2(\Lambda^{op_2}, \Gamma^{op_2})(F^{op_2}, G^{op_2}).$$

REMARK 1.2.7. More generally, we have an r-oppositization $n+1$-functor

$$\_\,^{op_r} : \mathrm{CAT}_n^{op_{r+1}} \to \mathrm{CAT}_n$$

whose construction is made precise in A.2.6.2.

Terminal objects. Initial objects. We review the definition of terminal and initial objects. We rely on the construction of the one-point set, which we explain in detail in Definition A.6.0.7, using the formalism developed in the appendix, and of the one-object n-category, which we can obtain by a straightforward inductive construction from the definition of the one-point set. We just make explicit the result of these constructions in the next Definition. Recall that we adopt the notation $\mathbb{B}$ for the boolean set and $\top$ for the element true of $\mathbb{B}$ (see Appendix A).

DEFINITION 1.2.8. The one-point set, which we will denote by $*_0$, is the set such that $*_0(x, y) = \top$, for all $x, y \in *_0$[6]. The one-object n-category, which we will denote by $*_n$, is so that $*_n(x, y) = *_{n-1}$ for all $x, y \in *_n$[7].

REMARK 1.2.9. We have $X \in \mathrm{SET} \Rightarrow \mathrm{SET}(X, *_0) \cong *_0$ and $\mathcal{C} \in \mathrm{CAT}_n \Rightarrow \mathrm{CAT}_n(\mathcal{C}, *_n) \cong *_n$.

DEFINITION 1.2.10. Let $\dot{*}_n : *_{n+1} \to \mathrm{CAT}_n$ be defined as follows:

– To $x \in *_{n+1}$ we associate $\dot{*}_n x = *_n \in \mathrm{CAT}_n$.
– For $x, y \in *_{n+1}$, we take $\dot{*}_n(x, y) = 1_{*_n} : *_n = *_{n+1}(x, y) \to \mathrm{CAT}_n(*_n, *_n) \cong *_n$.

DEFINITION 1.2.11. For $\mathcal{C} \in \mathrm{CAT}_n$, we define $\overline{*}_{n-1} : \mathcal{C}^{op} \to *_n$ as follows:

– To $X \in \mathcal{C}^{op}$, we associate $\overline{*}_{n-1} X = *_{n-1} \in *_n$.
– For $X, Y \in \mathcal{C}^{op}$, we take $\overline{*}_{n-1}(X, Y) := \overline{*}_{n-2} : \mathcal{C}^{op}(X, Y) \to *_n(*_{n-1}, *_{n-1}) \cong *_{n-1}$.

[6]Thus, all elements x and y of $*_0$ are equal in $*_0$ by definition. It is convenient to write $\top \in *_0$ for the unique element of $*_0$.

[7]The definition implies that all objects are equal in $*_n$, as in the case $n = 0$. We may also write $*_{n-1} \in *_n$ for the unique object of $*_n$.

Definition 1.2.12. Let $\mathcal{C}$ be a category. We say that an object $*_{\mathcal{C}} \in \mathcal{C}$ is terminal if it is equipped with a canonical isomorphism

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[rr, bend left, "\mathcal{C}(_,*_{\mathcal{C}})"] \arrow[dr, "\overline{*}_0"'] & \cong & \textsc{Set} \\
& *_1. \arrow[ur, "*_0"'] &
\end{tikzcd}$$

In other words, an object is terminal in $\mathcal{C}$ if it represents the composite $\mathcal{C}^{op} \xrightarrow{\overline{*}_0} *_1 \xrightarrow{*_0} \textsc{Set}$.

Example 1.2.13. The one-point set $*_0 \in \textsc{Set}$ is terminal.

Definition 1.2.14. Let Λ be a 2-category. An object $*_\Lambda \in \Lambda$ is terminal if it is equipped with a canonical isomorphism

$$\begin{tikzcd}
\Lambda^{op} \arrow[rr, bend left, "\Lambda(_,*_{\Lambda})"] \arrow[dr, "\overline{*}_1"'] & \cong & \textsc{Cat} \\
& *_2. \arrow[ur, "*_1"'] &
\end{tikzcd}$$

Example 1.2.15. The one-object category $*_1 \in \textsc{Cat}$ is terminal.

The definition of an initial object is dual to the definition of a terminal object.

Definition 1.2.16. Let $\mathcal{C}$ be a category. We say that an object $\emptyset_{\mathcal{C}} \in \mathcal{C}$ is initial if it is equipped with a canonical isomorphism

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "\mathcal{C}(\emptyset_{\mathcal{C}},_)"] \arrow[dr, "\overline{*}_0"'] & \cong & \textsc{Set} \\
& *_1. \arrow[ur, "*_0"'] &
\end{tikzcd}$$

In other words, an object is initial in $\mathcal{C}$ if it corepresents the composite $\mathcal{C} \xrightarrow{\overline{*}_0} *_1 \xrightarrow{*_0} \textsc{Set}$.

Definition 1.2.17. Let Λ be a 2-category. An object $\emptyset_\Lambda \in \Lambda$ is initial if it is equipped with a canonical isomorphism

$$\begin{tikzcd}
\Lambda \arrow[rr, bend left, "\Lambda(\emptyset_{\Lambda},_)"] \arrow[dr, "\overline{*}_1"'] & \cong & \textsc{Cat} \\
& *_2. \arrow[ur, "*_1"'] &
\end{tikzcd}$$

Cartesian products. We now review the definition of cartesian products. We make explicit a construction of cartesian products for sets in Definition A.6.0.16, using the formalism developed in the appendix. We can still extend this construction to n-categories, using a straightforward inductive procedure. We just state the result of these constructions in the case of sets, categories and 2-categories, in the next definition.

Definition 1.2.18.

- For $X, Y \in \text{Set}$, we let $X \times Y \in \text{Set}$ denote the cartesian product of X and Y, formed as a set equipped with projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$.
- For $\mathcal{C}, \mathcal{D} \in \text{Cat}$, we let $\mathcal{C} \times \mathcal{D} \in \text{Cat}$ denote the cartesian product of $\mathcal{C}$ and $\mathcal{D}$, formed as a category equipped with projection functors $\pi_{\mathcal{C}} : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\pi_{\mathcal{D}} : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$.
- For $\Lambda, \Gamma \in \text{Cat2}$, we let $\Lambda \times \Gamma \in \text{Cat}_2$ denote the cartesian product of Λ and Γ, formed as a 2-category equipped with projection 2-functors $\pi_\Lambda : \Lambda \times \Gamma \to \Lambda$ and $\pi_\Gamma : \Lambda \times \Gamma \to \Gamma$.

Example 1.2.19.

- We have $\text{Set} \in \text{Cat}$, and hence the cartesian product $\text{Set} \times \text{Set}$ is defined in Cat. The cartesian product of sets extends to a functor
$$\times : \text{Set} \times \text{Set} \to \text{Set}.$$
- We have $\text{Cat} \in \text{Cat}_2$, and hence we can form the cartesian product $\text{Cat} \times \text{Cat}$ in Cat_2. The cartesian product of categories extends to a 2-functor
$$\times : \text{Cat} \times \text{Cat} \to \text{Cat}.$$

Definition 1.2.20. We say that a category $\mathcal{C} \in \text{Cat}$ is cartesian if it has a terminal object and if there exists a functor
$$\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$
and an isomorphism
$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{C} \times \mathcal{C} \arrow[r, "1\times\times"] \arrow[d, "\tau\Delta"'] & \mathcal{C}^{op} \times \mathcal{C} \arrow[dd, "\mathcal{C}(-,-)"] \\
\mathcal{C}^{op} \times \mathcal{C} \times \mathcal{C}^{op} \times \mathcal{C} \arrow[d, "\mathcal{C}(-,-)\times\mathcal{C}(-,-)"'] \arrow[r, phantom, "\cong \;\; \pi^*"] & \\
\text{Set} \times \text{Set} \arrow[r, "\times"'] & \text{Set},
\end{tikzcd}$$
where $\tau\Delta$ is the composite of the functor induced by the diagonal Δ
$$\mathcal{C}^{op} \times \mathcal{C} \times \mathcal{C} \xrightarrow{\Delta\times 1\times 1} \mathcal{C}^{op} \times \mathcal{C}^{op} \times \mathcal{C} \times \mathcal{C}$$
with the functor induced by the permutation τ of factors in the cartesian product
$$\mathcal{C}^{op} \times \mathcal{C}^{op} \times \mathcal{C} \times \mathcal{C} \xrightarrow{1\times\tau\times 1} \mathcal{C}^{op} \times \mathcal{C} \times \mathcal{C}^{op} \times \mathcal{C}.$$
Note that in this case, the terminal object $*_{\mathcal{C}}$ of $\mathcal{C}$ is such that $X \in \mathcal{C} \Rightarrow * \times X \cong X$.

Example 1.2.21. The cartesian product functor on sets $\times : \text{Set} \times \text{Set} \to \text{Set}$ gives to $(\text{Set}, \times, *_0)$ the structure of a cartesian category. The canonical isomorphism π^* such that
$$\begin{tikzcd}
\text{Set}^{op} \times \text{Set} \times \text{Set} \arrow[r, "1\times\times"] \arrow[d, "\tau\Delta"'] & \text{Set}^{op} \times \text{Set} \arrow[dd, "\text{Set}(-,-)"] \\
\text{Set}^{op} \times \text{Set} \times \text{Set}^{op} \times \text{Set} \arrow[d, "\text{Set}(-,-)\times\text{Set}(-,-)"'] \arrow[r, phantom, "\cong \;\; \pi^*"] & \\
\text{Set} \times \text{Set} \arrow[r, "\times"'] & \text{Set}
\end{tikzcd}$$

can be given, componentwise, for $Z, X, Y \in \text{Set}$, by the canonical bijection $\pi^*_{Z,X,Y} : \text{Set}(Z, X \times Y) \to \text{Set}(Z, X) \times \text{Set}(Z, Y)$.

Definition 1.2.22. We say that a 2-category $\Lambda \in \text{Cat}_2$ is cartesian if it has a terminal object $*_\Lambda \in \Lambda$ and if there exists a 2-functor

$$\times : \Lambda \times \Lambda \to \Lambda$$

and a canonical isomorphism

$$\begin{tikzcd}
\Lambda^{op} \times \Lambda \times \Lambda \arrow[r, "1\times\times"] \arrow[d, "\tau\Delta"'] & \Lambda^{op} \times \Lambda \arrow[dd, "\Lambda(-,-)"] \\
\Lambda^{op} \times \Lambda \times \Lambda^{op} \times \Lambda \arrow[d, "\Lambda(-,-)\times\Lambda(-,-)"'] \arrow[r, phantom, "\underset{\pi^*}{\cong}"] & \\
\text{Cat} \times \text{Cat} \arrow[r, "\times"'] & \text{Cat},
\end{tikzcd}$$

where $\tau\Delta$ refers to the composite of the functor induced by the diagonal Δ

$$\Lambda^{op} \times \Lambda \times \Lambda \xrightarrow{\Delta\times 1\times 1} \Lambda^{op} \times \Lambda^{op} \times \Lambda \times \Lambda$$

with the functor induced by the permutation τ of factors in the cartesian product

$$\Lambda^{op} \times \Lambda^{op} \times \Lambda \times \Lambda \xrightarrow{1\times\tau\times 1} \Lambda^{op} \times \Lambda \times \Lambda^{op} \times \Lambda.$$

Note that if $*_\Lambda \in \Lambda$ is a terminal object, we have a canonical isomorphism the terminal object $*_\Lambda$ of Λ is such that $\mathcal{C} \in \Lambda \Rightarrow *_\Lambda \times \mathcal{C} \cong \mathcal{C}$.

Example 1.2.23. The cartesian product 2-functor $\times : \text{Cat} \times \text{Cat} \to \text{Cat}$ of Example 1.2.19 gives to $(\text{Cat}, \times, *_1)$ the structure of a cartesian 2-category.

Internal morphisms.

Definition 1.2.24. We say that a category $\mathcal{C}$ is cartesian closed if it is cartesian and if it is equipped with a functor

$$[-,-]_\mathcal{C} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C},$$

referred to as the internal hom of $\mathcal{C}$, together with an isomorphism

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{C}^{op} \times \mathcal{C} \arrow[r, "\times\times 1"] \arrow[d, "1\times[-,-]_\mathcal{C}"'] \arrow[dr, phantom, "\cong"] & \mathcal{C}^{op} \times \mathcal{C} \arrow[d, "\mathcal{C}(-,-)"] \\
\mathcal{C}^{op} \times \mathcal{C} \arrow[r, "\mathcal{C}(-,-)"'] & \text{Set}.
\end{tikzcd}$$

In this case, the canonical isomorphism given in $X \in \mathcal{C}$ by $*_\mathcal{C} \times X \cong X$ induces an isomorphism

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{C} \arrow[r, "\mathcal{C}(-,-)"] \arrow[d, "{[-,-]_\mathcal{C}}"'] \arrow[dr, phantom, "\cong"] & \text{Set} \\
\mathcal{C}. \arrow[ur, "\mathcal{C}(*_\mathcal{C},-)"'] &
\end{tikzcd}$$

If $\mathcal{C}$ is cartesian closed, then we also have the following isomorphisms

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{C}^{op} \times \mathcal{C} \arrow[r, "\times\times 1"] \arrow[d, "1\times[-,-]_\mathcal{C}"'] \arrow[dr, phantom, "\cong"] & \mathcal{C}^{op} \times \mathcal{C} \arrow[d, "{[-,-]_\mathcal{C}}"] \\
\mathcal{C}^{op} \times \mathcal{C} \arrow[r, "{[-,-]_\mathcal{C}}"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{C} \arrow[r, "\mathcal{C}(-,-)"] \arrow[d, "{[-,-]_\mathcal{C}}"'] \arrow[dr, phantom, "\cong"] & \text{Set} \\
\mathcal{C}. \arrow[ur, "\mathcal{C}(*_\mathcal{C},-)"'] &
\end{tikzcd}$$

EXAMPLE 1.2.25. The functor

$$\mathrm{SET}(\_,\_) : \mathrm{SET}^{op} \times \mathrm{SET} \to \mathrm{SET}$$

gives a cartesian closed category structure to the category of sets. We also write $[\_,\_]$ for the functor $\mathrm{SET}(\_,\_)$.

DEFINITION 1.2.26. Let Λ be a 2-category. We say that Λ is closed if there exists a 2-functor

$$[\_,\_]_\Lambda : \Lambda^{op} \times \Lambda \to \Lambda,$$

which we call the internal hom of Λ.

DEFINITION 1.2.27. We say that a 2-category Λ is cartesian closed if it is cartesian and if it is equipped with an internal hom 2-functor $[\_,\_]_\Lambda$, together with an isomorphism

$$\begin{tikzcd}
\Lambda^{op} \times \Lambda^{op} \times \Lambda \arrow[r, "\times\times 1"] \arrow[d, "1\times[_,_]_\Lambda"'] \arrow[dr, phantom, "\cong"] & \Lambda^{op} \times \Lambda \arrow[d, "\Lambda(_,_)"] \\
\Lambda^{op} \times \Lambda \arrow[r, "\Lambda(_,_)"'] & \mathrm{CAT}.
\end{tikzcd}$$

In this case, the isomorphism given in $\mathcal{C} \in \Lambda \Rightarrow$ by $*_\Lambda \times \mathcal{C} \cong \mathcal{C}$ induces an isomorphism

$$\begin{tikzcd}
\Lambda^{op} \times \Lambda \arrow[r, "\Lambda(_,_)"] \arrow[d, "{[_,_]_\Lambda}"'] & \mathrm{CAT} \arrow[dl, phantom, "\cong"] \\
\Lambda. \arrow[ur, "\Lambda(*_\Lambda,_)"'] &
\end{tikzcd}$$

If Λ is cartesian closed, then Λ also is internally cartesian closed in the sense that we have canonical isomorphisms

$$\begin{tikzcd}
\Lambda^{op} \times \Lambda^{op} \times \Lambda \arrow[r, "\times\times 1"] \arrow[d, "1\times[_,_]_\Lambda"'] \arrow[dr, phantom, "\cong"] & \Lambda^{op} \times \Lambda \arrow[d, "{[_,_]_\Lambda}"] \\
\Lambda^{op} \times \Lambda \arrow[r, "{[_,_]_\Lambda}"'] & \Lambda
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^{op} \times \Lambda \arrow[r, "\Lambda(_,_)"] \arrow[d, "{[_,_]_\Lambda}"'] & \mathrm{CAT} \arrow[dl, phantom, "\cong"] \\
\Lambda. \arrow[ur, "\Lambda(*_\Lambda,_)"'] &
\end{tikzcd}$$

EXAMPLE 1.2.28. The structural 2-functor of CAT

$$\mathrm{CAT}(\_,\_) : \mathrm{CAT}^{op} \times \mathrm{CAT} \to \mathrm{CAT}$$

gives the structure of a cartesian closed 2-category to the 2-category of categories. We also write $[\_,\_]$ for the 2-functor $\mathrm{CAT}(\_,\_)$.

REMARK 1.2.29. Let $\mathcal{C} \in \mathrm{CAT}$ and $X \in \mathcal{C}$. The cartesian closed structure of CAT justifies the existence of the functor $\mathcal{C}(\_, X) : \mathcal{C}^{op} \to \mathrm{SET}$. More precisely, the isomorphism given by the cartesian closed structure of CAT induces an isomorphism

$$[\mathcal{C}^{op} \times \mathcal{C}, \mathrm{SET}] \xrightarrow{\cong} [\mathcal{C}, [\mathcal{C}^{op}, \mathrm{SET}]]$$

which takes the functor $\mathcal{C}(\_,\_) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{SET}$ to the Yoneda embedding $h_\mathcal{C} : \mathcal{C}^{op} \hookrightarrow [\mathcal{C}^{op}, \mathrm{SET}]$. The Yoneda Lemma can be stated as follows.

LEMMA 1.2.30. *Let $\mathcal{C} \in \mathrm{CAT}$. Write η for the morphism corresponding to* $1 \in \mathrm{CAT}\left([\mathcal{C}^{op}, \mathrm{SET}], [\mathcal{C}^{op}, \mathrm{SET}]\right)$ *under the isomorphism*

$$[[\mathcal{C}^{op}, \mathrm{SET}], [\mathcal{C}^{op}, \mathrm{SET}]] \cong [\mathcal{C}^{op} \times [\mathcal{C}^{op}, \mathrm{SET}], \mathrm{SET}]\,.$$

We have a canonical isomorphism

$$\begin{tikzcd}
\mathcal{C}^{op} \times [\mathcal{C}^{op}, \mathrm{SET}] \arrow[r, "h_{\mathcal{C}}^{op} \times 1"] \arrow[dr, "\eta"'] & {[\mathcal{C}^{op}, \mathrm{SET}] \times [\mathcal{C}^{op}, \mathrm{SET}]} \arrow[d, "{[\mathcal{C}^{op}, \mathrm{SET}](_,_)}"] \arrow[dl, phantom, "\cong"] \\
 & \mathrm{SET}.
\end{tikzcd}$$

As a consequence, the Yoneda embedding $h_{\mathcal{C}} : \mathcal{C} \hookrightarrow [\mathcal{C}^{op}, \mathrm{SET}]$ is fully faithful.

CONSEQUENCE 1.2.31. *For any category $\mathcal{C}$, it makes sense to define objects by universal property, which amounts to finding an object representing a given functor $\mathcal{C} \to \mathrm{SET}$ or $\mathcal{C}^{op} \to \mathrm{SET}$. Indeed, by fully faithfulness of the Yoneda embedding, two objects representing the same functor are canonically isomorphic in $\mathcal{C}$.*

DEFINITION 1.2.32. Suppose that P is a property regarding functors. Let $\Lambda_1, \Lambda_2 \in \mathrm{CAT}_2$ and $F : \Lambda_1 \to \Lambda_2$ be a 2-functor. We say that F satisfies the property P locally if for every pair of objects $\mathcal{C}, \mathcal{D} \in \Lambda_1$, the functor $F_{\mathcal{C},\mathcal{D}} : \Lambda_1(\mathcal{C}, \mathcal{D}) \to \Lambda_2(F\mathcal{C}, F\mathcal{D})$ satisfies P.

EXAMPLE 1.2.33. A 2-functor $F : \Lambda_1 \to \Lambda_2$ is locally fully faithful if for every pair of objects $\mathcal{C}, \mathcal{D} \in \Lambda_1$, the functor $F_{\mathcal{C},\mathcal{D}} : \Lambda_1(\mathcal{C}, \mathcal{D}) \to \Lambda_2(F\mathcal{C}, F\mathcal{D})$ is fully faithful.

Adjunctions between 2-categories.

DEFINITION 1.2.34. Let $\Lambda_1, \Lambda_2 \in \mathrm{CAT}_2$. An adjunction between Λ_1 and Λ_2, which we write

$$\begin{tikzcd}
F : \Lambda_1 \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \Lambda_2 : G, \arrow[l, bend left]
\end{tikzcd}$$

is a pair of 2-functors $F : \Lambda_1 \to \Lambda_2$, $G : \Lambda_2 \to \Lambda_1$, equipped with an equivalence between the 2-functors

$$\Lambda_1(\_, G\_) : \Lambda_1^{op} \times \Lambda_2 \xrightarrow{\Lambda_1^{op} \times G} \Lambda_1^{op} \times \Lambda_1 \xrightarrow{\Lambda_1(\_,\_)} \mathrm{CAT}$$

and

$$\Lambda_2(F\_, \_) : \Lambda_1^{op} \times \Lambda_2 \xrightarrow{F^{op} \times \Lambda_2} \Lambda_2^{op} \times \Lambda_2 \xrightarrow{\Lambda_2(\_,\_)} \mathrm{CAT},$$

that is, equivalences of categories

$$\Lambda_2(F\mathcal{C}, \mathcal{D}) \xrightarrow{\cong} \Lambda_1(\mathcal{C}, G\mathcal{D})$$

which are 2-natural in the objects $\mathcal{C}$ of Λ_1 and $\mathcal{D}$ of Λ_2. We also say that F is left adjoint to G or that G is right adjoint to F, or that (F, G) form a pair of adjoint 2-functors.

Cartesian, cocartesian and bicartesian 2-categories. In Definition 1.2.22, we give a definition of a cartesian 2-category Λ in terms of a globally defined cartesian product 2-functor $\times : \Lambda \times \Lambda \to \Lambda$. In the next definition, we make explicit the definition of the cartesian product of a pair of objects in a 2-category, without assuming the existence of such a 2-functor defined in the whole 2-category.

DEFINITION 1.2.35. Let Λ be a 2-category and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be objects of Λ. A cartesian product of $\mathcal{C}_1$ and $\mathcal{C}_2$ in Λ is a universal pair $(\mathcal{C}_1 \times \mathcal{C}_2, (\pi_1, \pi_2))$, where $\mathcal{C}_1 \times \mathcal{C}_2 \in \Lambda$ and for $i = 1, 2$, $\pi_i : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_i$ is a morphism in Λ.

Precisely, $(\mathcal{C}_1 \times \mathcal{C}_2, (\pi_1, \pi_2))$ represents the cartesian product of $\mathcal{C}_1$ and $\mathcal{C}_2$ in Λ if any other pair $(\mathcal{D}, (p_1, p_2))$, with $\mathcal{D} \in \Lambda$ and $p_i : \mathcal{D} \to \mathcal{C}_i$ for $i = 1, 2$, yields a pair

$(p,(\rho_1^p,\rho_2^p))$ where $p:\mathcal{D}\to\mathcal{C}_1\times\mathcal{C}_2$ is a morphism in Λ and ρ_1^p,ρ_2^p are 2-isomorphisms in Λ:

$$\begin{tikzcd}
\mathcal{D} \arrow[dr, dashed, "p"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, phantom, shift left=3ex, "\overset{\rho_2^p}{\Rightarrow}\cong"] \arrow[dr, phantom, shift right=3ex, "\cong\Downarrow\rho_1^p"'] & & \\
& \mathcal{C}_1\times\mathcal{C}_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathcal{C}_2\ , \\
& \mathcal{C}_1 &
\end{tikzcd}$$

such that for any pair of morphisms $q_1:\mathcal{D}\to\mathcal{C}_1, q_2:\mathcal{D}\to\mathcal{C}_2$ and any pair of 2-morphisms $f_1\in\Lambda(\mathcal{D},\mathcal{C}_1)(p_1,q_1), f_2\in\Lambda(\mathcal{D},\mathcal{C}_2)(p_2,q_2)$, there exists a unique 2-morphism $f\in\Lambda(\mathcal{D},\mathcal{C}_1\times\mathcal{C}_2)(p,q)$ such that $f_1\rho_1^p=\rho_1^q f$ and $f_2\rho_2^p=\rho_2^q f$:

$$\begin{tikzcd}
& \mathcal{D} \arrow[ddl, "p_1"'] \arrow[dd, dashed, "p"] \arrow[ddr, "p_2"] \arrow[drrr, no head] \arrow[dd, phantom, shift right=4ex, "\overset{\rho_1^p}{\Leftarrow}"] \arrow[dd, phantom, shift left=4ex, "\overset{\rho_2^p}{\Rightarrow}"] & & & & \\
& & & & \mathcal{D} \arrow[ddl, "q_1"'] \arrow[dd, "q"] \arrow[ddr, "q_2"] \arrow[dd, phantom, shift right=4ex, "\underset{\rho_1^q}{\Leftarrow}"] \arrow[dd, phantom, shift left=4ex, "\underset{\rho_2^q}{\Rightarrow}"] & \\
\mathcal{C}_1 \arrow[drrr, no head] & \mathcal{C}_1\times\mathcal{C}_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[drrr, no head] \arrow[urrr, phantom, "\overset{f_1}{\Rightarrow}\ \overset{f}{\Rightarrow}\ \overset{f_2}{\Rightarrow}"] & \mathcal{C}_2 \arrow[drrr, no head] & & & \\
& & & \mathcal{C}_1 & \mathcal{C}_1\times\mathcal{C}_2 \arrow[l, "\pi_2"] \arrow[r, "\pi_1"'] & \mathcal{C}_2.
\end{tikzcd}$$

Equivalently, $(\mathcal{C}_1\times\mathcal{C}_2,(\pi_1,\pi_2))$ represents the cartesian product of $\mathcal{C}_1$ and $\mathcal{C}_2$ if $\mathcal{C}_1\times\mathcal{C}_2$ represents the 2-functor $\Lambda(\_,\mathcal{C}_1)\times\Lambda(\_,\mathcal{C}_2)$ through (π_1,π_2), in the sense that (π_1,π_2) yields an equivalence

$$\Lambda(\_,\mathcal{C}_1\times\mathcal{C}_2)\xrightarrow{\cong}\Lambda(\_,\mathcal{C}_1)\times\Lambda(\_,\mathcal{C}_2).$$

REMARK 1.2.36. Thus, a 2-category Λ is cartesian if it has a terminal object $*_\Lambda$ and the cartesian product of any pair of objects exists (the universal property ensures that the cartesian product defines a 2-functor in this case).

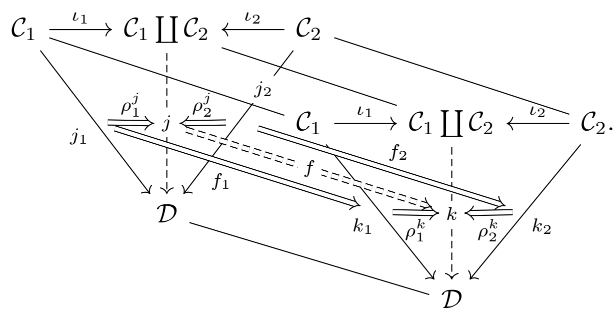
DEFINITION 1.2.37. Let Λ be a 2-category and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be objects of Λ. A cartesian coproduct of $\mathcal{C}_1$ and $\mathcal{C}_2$ in Λ is a universal pair $(\mathcal{C}_1\coprod\mathcal{C}_2,(\iota_1,\iota_2))$, where $\mathcal{C}_1\times\mathcal{C}_2\in\Lambda$ and for $i=1,2$, $\iota_i:\mathcal{C}_i\to\mathcal{C}_1\times\mathcal{C}_2$ is a morphism in Λ.

Precisely, $(\mathcal{C}_1\times\mathcal{C}_2,(\iota_1,\iota_2))$ represents the cartesian coproduct of $\mathcal{C}_1$ and $\mathcal{C}_2$ in Λ if any other pair $(\mathcal{D},(j_1,j_2))$, with $\mathcal{D}\in\Lambda$ and $j_i:\mathcal{D}\to\mathcal{C}_i$ for $i=1,2$, yields a pair $(j,(\rho_1^j,\rho_2^j))$ where $j:\mathcal{C}_1\coprod\mathcal{C}_2\to\mathcal{D}$ is a morphism in Λ and ρ_1^j,ρ_2^j are 2-isomorphisms in Λ:

$$\begin{tikzcd}
& \mathcal{C}_2 \arrow[d, "\iota_2"'] \arrow[ddr, bend left, "j_2"] & \\
\mathcal{C}_1 \arrow[r, "\iota_1"] \arrow[drr, bend right, "j_1"'] & \mathcal{C}_1\coprod\mathcal{C}_2 \arrow[dr, dashed, "j"] \arrow[dr, phantom, shift left=3ex, "\cong\overset{\rho_2^j}{\Leftarrow}"] \arrow[dr, phantom, shift right=3ex, "\cong\Uparrow\rho_1^j"'] & \\
& & \mathcal{D},
\end{tikzcd}$$

such that for any pair of morphisms $k_1:\mathcal{C}_1\to\mathcal{D}, k_2:\mathcal{C}_2\to\mathcal{D}$ and any pair of 2-morphisms $f_1\in\Lambda(\mathcal{C}_1,\mathcal{D})(j_1,k_1), f_2\in\Lambda(\mathcal{C}_2,\mathcal{D})(j_2,k_2)$, there exists a unique

2-morphism $f \in \Lambda(\mathcal{C}_1 \coprod \mathcal{C}_2, \mathcal{D})(j,k)$ such that $f\rho_1^j = \rho_1^k f_1$ and $f\rho_2^j = \rho_2^k f_2$:



Equivalently, $(\mathcal{C}_1 \times \mathcal{C}_2, (\pi_1, \pi_2))$ represents the cartesian coproduct of $\mathcal{C}_1$ and $\mathcal{C}_2$ if $\mathcal{C}_1 \coprod \mathcal{C}_2$ represents the 2-functor $\Lambda(\mathcal{C}_1, \_) \times \Lambda(\mathcal{C}_2, \_)$ through (ι_1, ι_2), in the sense that (ι_1, ι_2) yields an equivalence

$$\Lambda(\mathcal{C}_1 \coprod \mathcal{C}_2, \_) \xrightarrow{\cong} \Lambda(\mathcal{C}_1, \_) \times \Lambda(\mathcal{C}_2, \_).$$

Definition 1.2.38. We say that the 2-category Λ is cocartesian if it has an initial object and if the coproduct of any pair of objects exists.

Definition 1.2.39. Let Λ be a 2-category which is both cartesian and cocartesian. The morphisms obtained from

$$\mathcal{C}_1 \times \mathcal{D} \to \mathcal{C}_1 \to \mathcal{C}_1 \coprod \mathcal{C}_2$$

and

$$\mathcal{C}_1 \times \mathcal{D} \to \mathcal{C}_1 \to \mathcal{C}_2 \coprod \mathcal{C}_2,$$

together with the projections onto $\mathcal{D}$, yield a distribution morphism

$$(\mathcal{C}_1 \times \mathcal{D}) \coprod (\mathcal{C}_2 \times \mathcal{D}) \to (\mathcal{C}_1 \coprod \mathcal{C}_2) \times \mathcal{D}$$

which is natural in the objects $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_2$ of Λ. We say that Λ is bicartesian if the distribution morphism is an isomorphism, as well as the initial morphism $\emptyset_\Lambda \to \emptyset_\Lambda \times \mathcal{D}$ when we form a cartesian product with the initial object of Λ.

1.3. The 2-category of small categories. We revisit the definition of the notion of a small category in this section. Recall that for a set S, we denote by $\mathrm{I}S$ the discrete category associated to S[8].

Definition 1.3.1. A small category is a triple $(\mathcal{C}, \mathcal{C}_0, \epsilon_\mathcal{C})$, where

- $\mathcal{C}$ is a category,
- $\mathcal{C}_0$ is a set,
- $\epsilon_\mathcal{C} : \mathrm{I}\mathcal{C}_0 \to \mathcal{C}$ is a functor yielding an equivalence on objects[9]:

$$x \in \mathcal{C}_0 \Leftrightarrow x \in \mathcal{C}^{10}.$$

We also say that $\mathcal{C}$ is a small category, and we write $\mathcal{C} \in \mathbf{Cat}$. For objects $x, y \in \mathcal{C}$, we write $x = y$ if the corresponding elements in the set $\mathcal{C}_0$ are equal.

[8] See A.8.2.1.

[9] A 0-equivalence in the sense of Definition 2.5.1.

[10] The predicates $\_ \in \mathcal{C}_0$ and $\_ \in \mathcal{C}$ are equivalent, i.e. they are equal in the set of predicates - cf Appendix A, Definition 5.0.11.

REMARK 1.3.2. Let $\mathcal{C}$ be a small category. The functor $\epsilon_{\mathcal{C}}$ yields morphisms

$$\epsilon_{\mathcal{C}}(x, y) : \mathrm{I}\mathcal{C}_0(x, y) \to \mathcal{C}(x, y),$$

which are natural in $x, y \in \mathcal{C}$. Recall that $\mathrm{I}\mathcal{C}_0(x, y)$ is non empty if and only if $x = y$ in $\mathcal{C}_0$, in which case we have $\mathrm{I}\mathcal{C}_0(x, y) = *$. Therefore, any elements $x, y \in \mathcal{C}_0$ such that $x = y$ yield a map $\epsilon_{\mathcal{C}}(x, y) : * \to \mathcal{C}(x, y)$, such that $\epsilon_{\mathcal{C}}(x, x) \in \mathcal{C}(x, x)$ is the identity of x in $\mathcal{C}$. By naturality of $\epsilon_{\mathcal{C}}$, the diagram

$$\begin{tikzcd}
\mathrm{I}\mathcal{C}_0(x, y) \times \mathrm{I}\mathcal{C}_0(y, x) \arrow[r, "\epsilon_{\mathcal{C}}(x,y)\times\epsilon_{\mathcal{C}}(y,x)"] \arrow[d] & \mathcal{C}(x, y) \times \mathcal{C}(y, x) \arrow[d] \\
\mathrm{I}\mathcal{C}_0(x, x) \arrow[r, "\epsilon_{\mathcal{C}}(x,x)=\iota_x"'] & \mathcal{C}(x, x)
\end{tikzcd}$$

commutes, ensuring the coherence of this definition.

DEFINITION 1.3.3. Let $(\mathcal{C}, \mathcal{C}_0), (\mathcal{D}, \mathcal{D}_0) \in \mathbf{Cat}$. A functor F from $(\mathcal{C}, \mathcal{C}_0)$ to $(\mathcal{D}, \mathcal{D}_0)$ is the data of

- a map $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$
- a functor $F : \mathcal{C} \to \mathcal{D}$
- such that the diagram

$$\begin{tikzcd}
\mathrm{I}\mathcal{C}_0 \arrow[r, "\mathrm{I}F_0"] \arrow[d, "\epsilon_{\mathcal{C}}"'] & \mathrm{I}\mathcal{D}_0 \arrow[d, "\epsilon_{\mathcal{D}}"] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D}
\end{tikzcd}$$

  strictly commutes, in the sense that for each element $x \in \mathcal{C}_0$, the elements of $\mathcal{D}_0$ corresponding to the objects $F\epsilon_{\mathcal{C}}x$ and $\epsilon_{\mathcal{D}}F_0x$ in $\mathcal{D}$ are equal.

If (F, F_0) is a functor from $(\mathcal{C}, \mathcal{C}_0)$ to $(\mathcal{D}, \mathcal{D}_0)$, then we write $F : \mathcal{C} \to \mathcal{D}$ or $F \in \mathbf{Cat}(\mathcal{C}, \mathcal{D})$.

DEFINITION 1.3.4. Let $\mathbf{Cat}$ be the 2-category such that

- $\mathcal{C} \in \mathbf{Cat}$ is given by Definition 1.3.1,
- $\mathbf{Cat}(\mathcal{C}, \mathcal{D}) \in \mathrm{CAT}$ is such that $F \in \mathbf{Cat}(\mathcal{C}, \mathcal{D})$ if F is as defined in Definition 1.3.3, and for $F, G \in \mathbf{Cat}(\mathcal{C}, \mathcal{D})$, the set $\mathbf{Cat}(\mathcal{C}, \mathcal{D})(F, G)$ is given by the end
$$\mathbf{Cat}(\mathcal{C}, \mathcal{D})(F, G) = \int^{x\in\mathcal{C}} \mathcal{D}(Fx, Gx)$$
  of the following composite in CAT
$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{F^{op}\times G} \mathcal{D}^{op} \times \mathcal{D} \xrightarrow{\mathcal{D}(-,-)} \mathrm{SET}.$$

PROPOSITION 1.3.5. *For any pair of small categories $\mathcal{C}, \mathcal{D} \in \mathbf{Cat}$, the category $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ of functors from $\mathcal{C}$ to $\mathcal{D}$ admits a small category structure $[\mathcal{C}, \mathcal{D}]_{\mathbf{Cat}} \in \mathbf{Cat}$. We can form a 2-functor*

$$[-, -]_{\mathbf{Cat}} : \mathbf{Cat}^{op} \times \mathbf{Cat} \to \mathbf{Cat},$$

which gives to the 2-category of small categories $\mathbf{Cat}$ the structure of a cartesian closed 2-category. In particular, the 2-category of small categories $\mathbf{Cat}$ is monoidal, so that $(\mathbf{Cat}, \times) \in \mathrm{MON}_{\mathrm{CAT}_2}$.

Proof. Let $\mathcal{C}, \mathcal{D} \in \mathbf{Cat}$ and $F, G \in \mathbf{Cat}(\mathcal{C}, \mathcal{D})$. Recall that F, respectively G, is equipped with a map $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$, respectively $G_0 : \mathcal{C}_0 \to \mathcal{D}_0$. By definition, the maps F_0 and G_0 satisfy $F_0 = G_0$ if and only if $F_0 x = G_0 x$ in $\mathcal{D}_0$ for all $x \in \mathcal{C}_0$. Suppose that $(F, F_0), (G, G_0)$ is such that $F_0 = G_0$. The functor $\epsilon_{\mathcal{D}}$ yields natural maps

$$\mathcal{D}_0(F_0 y, G_0 y) \to \mathcal{D}(Fy, Gy).$$

By Remark 1.3.2, we obtain natural isomorphisms $\mathcal{D}(Fx, Fy) \xrightarrow{\cong} \mathcal{D}(Fx, Gy)$ for all $x, y \in \mathcal{C}$. Let us write

$$F(\_,\_), G(\_,\_) : \mathcal{C}(\_,\_) \to \mathcal{D}(F\_, G\_)$$

for the natural transformations corresponding to $F(\_,\_)$ and $G(\_,\_)$ under this identification, as displayed on the following diagram

(1)
$$\begin{tikzcd}
\mathcal{C}(x,y) \arrow[r, "G(x,y)"] \arrow[d, "F(x,y)"'] & \mathcal{D}(Gx, Gy) \arrow[d, "\cong"] \\
\mathcal{D}(Fx, Fy) \arrow[r, "\cong"'] & \mathcal{D}(Fx, Gy).
\end{tikzcd}$$

Define $F = G$ as $(F = G) := [\mathcal{C}^{op} \times \mathcal{D}, \mathrm{Set}](\mathcal{C}(\_,\_), \mathcal{D}(F\_, G\_))(F(\_,\_), G(\_,\_))$. We have

$$\begin{aligned}
(F = G) &\Leftrightarrow \int^{x,y\in\mathcal{C}} \mathrm{Set}\left(\mathcal{C}(x,y), \mathcal{D}(Fx,Gy)\right)\left(F(x,y), G(x,y)\right) \\
&\Leftrightarrow \int^{x,y\in\mathcal{C}} \int^{h\in\mathcal{C}(x,y)} \mathcal{D}(Fx,Gy)(F(x,y)h, G(x,y)h) \\
&\Leftrightarrow \prod_{x,y\in\mathcal{C}_0} \prod_{h\in\mathcal{C}(x,y)} (F(x,y)h = G(x,y)h).
\end{aligned}$$

To sum up, we have $F = G$ if and only if for all objects x, y of $\mathcal{C}$ and each morphism $h : x \to y$ in $\mathcal{C}$, the morphisms $Fh : Fx \to Fy$ and $Gh : Gx \to Gy$ are equal as morphisms in $\mathcal{D}$ from Fx to Gy. Note that we used the equality $F_0 = G_0$.

Let $[\mathcal{C}, \mathcal{D}]_0$ be the set such that $F \in [\mathcal{C}, \mathcal{D}]_0 \Leftrightarrow F \in \mathbf{Cat}(\mathcal{C}, \mathcal{D})$, with $[\mathcal{C}, \mathcal{D}]_0(F, G) := (F_0 = G_0) \times (F = G)$ for $F, G \in [\mathcal{C}, \mathcal{D}]_0$. The functor $\epsilon_{\mathcal{D}}$ yields a map

$$\mathrm{I}\mathcal{D}_0(F_0 x, G_0 x) \to \mathcal{D}(Fx, Gx),$$

whose naturality in $x \in \mathcal{C}_0$ follows from the commutativity of the above diagram (1), itself given by $F = G$. We obtain a map

$$\mathrm{I}\left((F_0 = G_0) \times (F = G)\right) \to \int^{x\in\mathcal{C}} \mathcal{D}(F_0 x, G_0 x),$$

or equivalently,

$$\mathrm{I}[\mathcal{C}, \mathcal{D}]_0(F, G) \to \mathbf{Cat}(\mathcal{C}, \mathcal{D})(F, G)$$

which is natural in $F, G \in \mathbf{Cat}(\mathcal{C}, \mathcal{D})$. Those natural maps yield a functor

$$\epsilon_{[\mathcal{C},\mathcal{D}]_{\mathbf{Cat}}} : \mathrm{I}[\mathcal{C}, \mathcal{D}]_0 \to \mathrm{Cat}(\mathcal{C}, \mathcal{D})$$

which is identical on objects, and hence a 0-equivalence. The set $[\mathcal{C}, \mathcal{D}]_0$, together with the functor $\epsilon_{[\mathcal{C},\mathcal{D}]_{\mathbf{Cat}}}$, gives to the category $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ the structure of a small category which we call $[\mathcal{C}, \mathcal{D}]_{\mathbf{Cat}}$. We obtain an internal hom 2-functor

$$[\_,\_]_{\mathbf{Cat}} : \mathbf{Cat}^{op} \times \mathbf{Cat} \to \mathbf{Cat}.$$

The cartesian structure closed structure of **Cat** is directly inherited from the cartesian closed structure of SET and of CAT. $\square$

DEFINITION 1.3.6. We let **cat** be the category of small categories, defined as follows:

- $\mathcal{C} \in \mathbf{cat} \Leftrightarrow \mathcal{C} \in \mathbf{Cat}$,
- $\mathcal{C}, \mathcal{D} \in \mathbf{cat} \Rightarrow \mathbf{cat}(\mathcal{C}, \mathcal{D}) := [\mathcal{C}, \mathcal{D}]_0$, where the set $[\mathcal{C}, \mathcal{D}]_0$ is given in the proof of Proposition 1.3.5

The category **cat** thus obtained inherits a cartesian closed structure from **Cat**.

REMARK 1.3.7. We have a 2-functor $\mathbf{Icat} \to \mathbf{Cat}$ given by

- $\mathcal{C} \in \mathbf{cat} \Leftrightarrow \mathcal{C} \in \mathbf{Cat}$,
- for $\mathcal{C}, \mathcal{D} \in \mathbf{cat}$, $F \in \mathbf{cat}(\mathcal{C}, \mathcal{D}) \Leftrightarrow F \in \mathbf{Cat}(\mathcal{C}, \mathcal{D})$,
- and for $F, G \in \mathbf{cat}(\mathcal{C}, \mathcal{D})$, $\mathbf{Icat}(\mathcal{C}, \mathcal{D})(F, G) \to \mathbf{Cat}(\mathcal{C}, \mathcal{D})(F, G)$ is induced by $x, y \in \mathcal{C} \Rightarrow (\mathrm{I}\mathcal{D}_0(Fx, Gx) \to \mathcal{D}(Fx, Gx))$.

The 2-functor thus obtained is a 1-equivalence by construction. According to Definition 7.0.1, the 2-category **Cat** of small categories is 1-small.

2. Categorical constructions within a 2-category

We extend some categorical notions, usually defined in the 2-category of categories, to the objects of a given 2-category Λ. In particular, we define limits, colimits, and Kan extension within a 2-category in such a way that we recover the usual notions of limits, colimits and Kan extensions of functors when this 2-category is the 2-category of categories. We first observe that there is a forgetful 2-functor $\Lambda(*_\Lambda, \_) : \Lambda \to \mathrm{CAT}$ from Λ to the 2-category CAT as soon as Λ is equipped with a distinguished object $*_\Lambda$. We get in this context that every object of Λ has an underlying structure of a category.

2.1. The forgetful 2-functor.

DEFINITION 2.1.1. Let Λ be a 2-category with a distinguished object $*_\Lambda \in \Lambda$. The structural 2-functor $\Lambda(\_, \_)$ of Λ yields[11] a 2-functor

$$\Lambda(*_\Lambda, \_) : \Lambda \to \mathrm{CAT}.$$

If $\mathcal{C} \in \Lambda$, then we write $\mathcal{C}^* \in \mathrm{CAT}$ for the image of $\mathcal{C}$ under $\Lambda(*_\Lambda, \_)$. We say that $\mathcal{C}^*$ is the category of objects of $\mathcal{C}$.

REMARK 2.1.2. The 2-functor $\mathrm{CAT}(*, \_)$ induces an equivalence of 2-categories $\mathrm{CAT} \xrightarrow{\cong} \mathrm{CAT}$. One way to see this is to notice, first, that the one-object category $*$ is a unit for the cartesian product. The 2-category CAT is cartesian closed with respect to its own structural 2-functor $\mathrm{CAT}(\_, \_)$, hence the result.

OBSERVATION 2.1.3. *Let $\Lambda \in \mathrm{CAT}_2$ and $*_\Lambda \in \Lambda$. If the forgetful 2-functor $\Lambda(*_\Lambda, \_)$ is an equivalence of 2-categories, then Λ is equivalent to* CAT. *For each pair of objects $\mathcal{C}, \mathcal{D} \in \Lambda$, the 2-functor $\Lambda(*_\Lambda, \_)$ yields a functor which we write*

$$\Lambda^*(\mathcal{C}, \mathcal{D}) : \Lambda(\mathcal{C}, \mathcal{D}) \to [\mathcal{C}^*, \mathcal{D}^*]\,.$$

---

[11] Note that we use the cartesian closed structure of CAT_2.

– *Suppose that for all $\mathcal{C}, \mathcal{D} \in \Lambda$, the functor $\Lambda^*(\mathcal{C}, \mathcal{D})$ is an equivalence of categories, so that the forgetful 2-functor $\Lambda(*_\Lambda, \_)$ is a local equivalence. In particular, we have an explicit description of the morphisms in Λ: the data of a morphism $F : \mathcal{C} \to \mathcal{D}$ in Λ is equivalent to the data of*
  – *a morphism $FX : *_\Lambda \to \mathcal{D}$ for all $X : *_\Lambda \to \mathcal{C}$,*
  – *for all $X, Y : *_\Lambda \to \mathcal{C}$, and $f : X \Rightarrow Y$ a 2-morphism $Ff : FX \Rightarrow FY$:*

$$\begin{tikzcd} *_\Lambda \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "f\Downarrow"] & \mathcal{C} \end{tikzcd} \qquad \Rightarrow \qquad \begin{tikzcd} *_\Lambda \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "Ff\Downarrow"] & \mathcal{C} \end{tikzcd}$$

  which satisfies naturality conditions.

  We also obtain an explicit description of the 2-morphisms by their value on the generalized objects $*_\Lambda \to \mathcal{C}$.

– *Suppose that for all $\mathcal{C}, \mathcal{D} \in \Lambda$, the functor $\Lambda^*(\mathcal{C}, \mathcal{D})$ is fully faithful, so that the forgetful 2-functor $\Lambda(*_\Lambda, \_)$ is locally fully faithful. In this case, for $F, G : \mathcal{C} \to \mathcal{D}$, the morphism*

$$\Lambda(\mathcal{C}, \mathcal{D})(F, G) \to [\mathcal{C}^*, \mathcal{D}^*]\,(F^*, G^*)$$

  is an isomorphism, so that the characterization of natural transformations in terms of an end yields the isomorphism

$$\Lambda(\mathcal{C}, \mathcal{D})(F, G) \xrightarrow{\cong} \int^{X \in \mathcal{C}^*} \mathcal{D}^*(FX, GX).$$

  As a consequence the 2-morphisms of such a 2-category are determined by their values on points, in the sense that the data of a 2-morphism $\eta : F \Rightarrow G$ in Λ

$$\begin{tikzcd} *_\Lambda \arrow[r, "X"] & \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \end{tikzcd}$$

  is equivalent to the data of natural morphisms $\eta_X \in \mathcal{D}^*(FX, GX)$ in $X : *_\Lambda \to \mathcal{C}$

$$\begin{tikzcd} *_\Lambda \arrow[r, bend left, "FX"] \arrow[r, bend right, "GX"'] \arrow[r, phantom, "\Downarrow\eta_X"] & \mathcal{D}. \end{tikzcd}$$

– *Suppose that for all $\mathcal{C}, \mathcal{D} \in \Lambda$, the functor $\Lambda^*(\mathcal{C}, \mathcal{D})$ is faithful, so that the forgetful 2-functor $\Lambda(*_\Lambda, \_)$ is locally faithful. This means that for $F, G : \mathcal{C} \to \mathcal{D}$ and $\eta, \mu \in \Lambda(\mathcal{C}, \mathcal{D})(F, G)$, we have $\eta = \mu$ as soon as $X \in \mathcal{C} \Rightarrow \eta_X = \mu_X$. In particular, the commutativity of diagrams in Λ can be shown pointwise.*

Definition 2.1.4. We say that a 2-category Λ is $*_\Lambda$-primary if the 2-functor $\Lambda(*_\Lambda, \_)$ is locally fully faithful. In this case, the commutativity of (3-dimensional) diagrams can be shown pointwise and 2-morphisms can be defined by their value on the objects of their domain.

Example 2.1.5. We will see in the next section that the 2-category of $\mathcal{V}$-enriched categories is $*_\mathcal{V}$-primary.

Remark 2.1.6. Let Λ be a closed monoidal 2-category with unit $*_\Lambda \in \Lambda$. For all objects $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_1, \mathcal{D}_2 \in \Lambda$, the 2-functor $\Lambda(*_\Lambda, \_)$ yields a functor

$$\Lambda([\mathcal{C}_1, \mathcal{D}_1]_\Lambda, [\mathcal{C}_2, \mathcal{D}_2]_\Lambda) \to [\Lambda(\mathcal{C}_1, \mathcal{D}_1), \Lambda(\mathcal{C}_2, \mathcal{D}_2)]\,,$$

which is fully faithful if and only if Λ is $*_\Lambda$-primary. In this case, the set of 2-morphisms between morphisms $L, K : [\mathcal{C}_1, \mathcal{D}_1]_\Lambda \to [\mathcal{C}_2, \mathcal{D}_2]_\Lambda$ in Λ is such that

$$\Lambda([\mathcal{C}_1, \mathcal{D}_1]_\Lambda, [\mathcal{C}_2, \mathcal{D}_2]_\Lambda)(L, K) \xrightarrow{\cong} \int^{F:\mathcal{C}_1 \to \mathcal{D}_1} \Lambda(\mathcal{C}_2, \mathcal{D}_2)(LF, KF).$$

2.2. Adjunctions.

DEFINITION 2.2.1. Let $\Lambda \in \text{CAT}_2$, $\mathcal{C}, \mathcal{D} \in \Lambda$, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. We say that F and G form a pair of adjoint morphisms if there exists 2-morphisms $\eta : 1_\mathcal{C} \Rightarrow GF, \epsilon : FG \Rightarrow 1_\mathcal{D}$ in Λ, such that the following diagrams commute

$$\begin{tikzcd} F \arrow[r, "1_F \circ \eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon \circ 1_F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta \circ 1_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "1_G \circ \epsilon"] \\ & G. \end{tikzcd}$$

The triangle on the left is written in the category $\Lambda(\mathcal{C}, \mathcal{D})$, whereas the one on the right is written in the category $\Lambda(\mathcal{D}, \mathcal{C})$. We also say that (F, G) is a pair of adjoint morphisms.

REMARK 2.2.2. Let $\Lambda_1, \Lambda_2 \in \text{CAT}_2$. Any 2-functor $\Lambda_1 \to \Lambda_2$ takes a pair of adjoint morphisms in Λ_1 to a pair of adjoint morphisms in Λ_2.

The following lemma can be easily deduced from the last remark and the characterization of adjoint functors in CAT.

LEMMA 2.2.3. *Let Λ be a 2-category and $*_\Lambda$ be an object of Λ. Each pair of adjoint morphisms*

$$F : \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{D} : G,$$

yields isomorphisms

$$\mathcal{D}^*(FX, Y) \cong \mathcal{C}^*(X, GY)$$

which are natural in $X : *_\Lambda \to \mathcal{C}$ and $Y : *_\Lambda \to \mathcal{D}$.

PROPOSITION 2.2.4. *Let $\Lambda \in \text{CAT}_2$. If Λ is $*_\Lambda$-primary, then a pair of morphisms form an adjunction in Λ*

$$F : \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{D} : G,$$

if and only if there is natural isomorphisms in $X : *_\Lambda \to \mathcal{C}, Y : *_\Lambda \to \mathcal{D}$

$$\mathcal{D}^*(FX, Y) \cong \mathcal{C}^*(X, GY).$$

PROOF. Suppose that there is an isomorphism

$$\mathcal{D}^*(FX, Y) \cong \mathcal{C}^*(X, GY)$$

natural in $X : *_\Lambda \to \mathcal{C}$ and $Y : *_\Lambda \to \mathcal{D}$. The morphism

$$\eta_Z : * \xrightarrow{1_{FZ}} \mathcal{D}^*(FZ, FZ) \xrightarrow{\cong} \mathcal{C}^*(Z, GFZ)$$

is natural in $Z : *_\Lambda \to \mathcal{C}$, and hence induces a morphism

$$\eta : * \to \int_{Z:*_\Lambda \to \mathcal{C}} \mathcal{C}^*(Z, GFZ) \cong \Lambda(\mathcal{C}, \mathcal{C})(1_\mathcal{C}, GF).$$

Similarly, we obtain a 2-morphism $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ from the natural morphisms

$$* \xrightarrow{1_{GZ}} \mathcal{C}^*(GZ, GZ) \xrightarrow{\cong} \mathcal{D}^*(FGZ, Z).$$

in $Z : *_\Lambda \to \mathcal{D}$. Consequently, the data of a 2-morphism $\eta : 1_{\mathcal{C}} \Rightarrow GF$, respectively $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ in Λ is equivalent to the data of morphisms $\phi_{X,Y} : \mathcal{D}^*(FX, Y) \to \mathcal{C}^*(X, GY)$, respectively $\psi_{X,Y} : \mathcal{C}^*(FX, Y) \to \mathcal{D}^*(X, GY)$, natural in $X : *_\Lambda \to \mathcal{C}$ and $Y : *_\Lambda \to \mathcal{D}$. The morphism thus obtained $\phi_{X,Y}$, respectively $\psi_{X,Y}$, satisfies $\psi_{X,Y}\phi_{X,Y} = 1_{\mathcal{D}^*(FX,Y)}$, respectively $\phi_{X,Y}\psi_{X,Y} = 1_{\mathcal{C}^*(X,GY)}$, if and only if the following triangle on the left hand side, respectively on the right hand side,

$$\begin{tikzcd} F \arrow[r, "1_F \circ \eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon \circ 1_F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta \circ 1_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "1_G \circ \epsilon"] \\ & G \end{tikzcd}$$

commutes in the category $\Lambda(\mathcal{C}, \mathcal{D})$, respectively in the category $\Lambda(\mathcal{D}, \mathcal{C})$. □

2.3. Limits and colimits.

We define pointwise and global limits of morphisms in a 2-category Λ. For this purpose, we require Λ to be equipped with a closed monoidal structure $(\Lambda, \times_\Lambda, [\_, \_], *_\Lambda)$ and an augmentation $1_\Lambda \Rightarrow *_\Lambda$, so that each $\mathcal{C} \in \Lambda$ yields a morphism $\mathcal{C} \to *_\Lambda$ in a natural way. If Λ is equipped with a cartesian closed structure, then the unit $*_\Lambda$ is terminal in Λ. Consequently, we obtain an augmentation given in each object $\mathcal{C}$ of Λ by the unique morphism $\mathcal{C} \to *_\Lambda$.

It is convenient to assume that the unit of Λ is send to the terminal object of Cat by the forgetful 2-functor $\Lambda(*_\Lambda, \_)$, especially when dealing with pointwise limits and colimits. Indeed, we will observe that pointwise limits and colimits are shaped on set-theoritical limits and colimits. We therefore assume that this assumption holds. Let us also notice that the closed monoidal structure on Λ is not required to define pointwise limits and colimits.

Definition 2.3.1. Let $\mathcal{S} \in \Lambda$. The unit object $*_\Lambda$ of Λ induces a functor

$$\mathcal{S}^* \xrightarrow{\cong} \Lambda(*_\Lambda, \mathcal{S}) \times * \to \Lambda(*_\Lambda, \mathcal{S}) \times \Lambda(I, *_\Lambda) \to \Lambda(I, \mathcal{S})$$

natural in each object $I \in \Lambda$, which we call the constant functor. For any $X : *_\Lambda \to \mathcal{S}$, we write $\overline{X} : I \to \mathcal{S}$ for the image of X under the constant functor.

Definition 2.3.2. Let $I, \mathcal{S} \in \Lambda$ and $F : I \to \mathcal{S}$. We define the limit of F as a universal pair $(\lim F, \rho)$, where $\lim F : *_\Lambda \to \mathcal{S}$ and $\rho \in \Lambda(I, \mathcal{S})(\overline{\lim F}, F)$. Explicitly, ρ consists in the data for each $i \in I$ of a 2-morphism ρ_i in Λ such that

$$\begin{tikzcd} I \arrow[r, "F"] & S \\ *_\Lambda \arrow[u, "i"] \arrow[ur, "\lim F"'] \arrow[u, phantom, shift left=4ex, "\Leftarrow \rho_i"] \end{tikzcd}$$

and such that the following equality holds for all $i, j \in I$ and $f : i \to j$.

$$\begin{tikzcd} I \arrow[r, "F"] & S \\ *_\Lambda \arrow[u, bend left, "j"] \arrow[u, bend right, "i"'] \arrow[u, phantom, "\Leftarrow f"] \arrow[ur, "\lim F"'] \arrow[u, phantom, shift left=4ex, "\Leftarrow \rho_i"] \end{tikzcd} \qquad = \qquad \begin{tikzcd} I \arrow[r, "F"] & S \\ *_\Lambda \arrow[u, "j"] \arrow[ur, "\lim F"'] \arrow[u, phantom, shift left=4ex, "\Leftarrow \rho_j"] \end{tikzcd}$$

REMARK 2.3.3. Let $X : *_\Lambda \to \mathcal{S}$. Write $\mathcal{S}^*(X, F) : I^* \to \textsc{Set}$ for the functor

$$\begin{array}{rccc} \mathcal{S}^*(X, F) : & I^* & \longrightarrow & \textsc{Set} \\ & i & \mapsto & \mathcal{S}^*(X, Fi). \end{array}$$

The pair $(\mathcal{S}^*(X, \lim F), \mathcal{S}^*(X, \rho))$, where $\mathcal{S}^*(X, \rho) \in [I^*, \textsc{Set}]\left(\overline{\mathcal{S}^*(X, \lim F)}, \mathcal{S}^*(X, F)\right)$, is the limit of the functor $\mathcal{S}^*(X, F)$. In particular, we have an isomorphism for all $X \in \mathcal{S}^*$

$$\mathcal{S}^*(X, \lim F) \cong \lim \mathcal{S}^*(X, F).$$

DEFINITION 2.3.4. Let $I, \mathcal{S} \in \Lambda$ and $F : I \to \mathcal{S}$. We define the colimit of F as a universal pair $(\mathrm{colim} F, \iota)$, where $\mathrm{colim} F : *_\Lambda \to \mathcal{S}$ and $\iota \in \Lambda(I, \mathcal{S})(F, \overline{\mathrm{colim} F})$. Explicitly, ι consists in the data for each $i \in I$ of a 2-morphism ι_i in Λ such that

$$\begin{tikzcd} I \arrow[r, "F"] \arrow[dr, phantom, "\Downarrow \iota_i"] & S \\ *_\Lambda \arrow[u, "i"] \arrow[ur, "\mathrm{colim} F"'] & \end{tikzcd}$$

and such that the following equality holds for all $i, j \in I$ and $f : i \to j$.

$$\begin{tikzcd} I \arrow[r, "F"] \arrow[dr, phantom, "\Downarrow \iota_i"] & S \\ *_\Lambda \arrow[u, bend left, "i"] \arrow[u, bend right, "j"'] \arrow[u, phantom, "\Rightarrow f"] \arrow[ur, "\mathrm{colim} F"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} I \arrow[r, "F"] \arrow[dr, phantom, "\Downarrow \iota_j"] & S \\ *_\Lambda \arrow[u, "j"] \arrow[ur, "\mathrm{colim} F"'] & \end{tikzcd}$$

REMARK 2.3.5. Let $X : *_\Lambda \to \mathcal{S}$. The morphism $F : T \to \mathcal{S}$ yields a functor

$$\begin{array}{rccc} \mathcal{S}^*(F, X) : & I^{*op} & \longrightarrow & \textsc{Set} \\ & i & \mapsto & \mathcal{S}^*(Fi, X). \end{array}$$

The pair $(\mathcal{S}^*(\mathrm{colim} F, X), \mathcal{S}^*(\iota, X))$, where $\mathcal{S}^*(\iota, X) \in [I^{*op}, \textsc{Set}](\overline{\mathcal{S}^*(\mathrm{colim} F, X)}, \mathcal{S}^*(F, X))$, is the limit of the functor $\mathcal{S}^*(F, X)$. In particular, we have an isomorphism for all $X \in \mathcal{S}^*$

$$\mathcal{S}^*(\mathrm{colim} F, X) \cong \lim \mathcal{S}^*(F, X).$$

REMARK 2.3.6. This definition of limits and colimits of morphisms in a 2-category coincides with the usual definition of limits and colimits of functors, when regarded as morphisms in the 2-category $\textsc{Cat}$ of categories.

DEFINITION 2.3.7. Let $\mathcal{S} \in \Lambda$ and $*_\mathcal{S} : *_\Lambda \to \mathcal{S}$. We say that $*_\mathcal{S}$ is terminal in $\mathcal{S}$ if each $X : *_\Lambda \to \mathcal{S}$ yields an isomorphism $\mathcal{S}^*(X, *_\mathcal{S}) \cong *$.

DEFINITION 2.3.8. Let $\mathcal{S} \in \Lambda$ and $\emptyset_\mathcal{S} : *_\Lambda \to \mathcal{S}$. We say that $\emptyset_\mathcal{S}$ is initial in $\mathcal{S}$ if each $X : *_\Lambda \to \mathcal{S}$ yields an isomorphism $\mathcal{S}^*(\emptyset_\mathcal{S}, X) \cong *$.

DEFINITION 2.3.9. We say that an object $\mathcal{S}$ of Λ is locally I-complete if any morphism $F : I \to \mathcal{S}$ admits a limit. Dually, we say that $\mathcal{S}$ is locally I-cocomplete if any morphism $F : I \to \mathcal{S}$ admits a colimit.

EXAMPLE 2.3.10. The object $\mathcal{S}$ of Λ has a terminal object $*_\mathcal{S}$ if and only if the unique morphism $\emptyset_\Lambda \to \mathcal{S}$ has a limit. Consequently, $\mathcal{S}$ is locally $\emptyset_\Lambda$-complete if and only if it has an terminal object.

EXAMPLE 2.3.11. The object $\mathcal{S}$ of Λ has an initial object $\emptyset_\mathcal{S}$ if and only if the unique morphism $\emptyset_\Lambda \to \mathcal{S}$ has a colimit. Consequently, $\mathcal{S}$ is locally $\emptyset$-cocomplete if and only if it has an initial object.

PROPOSITION 2.3.12. *Let $\mathcal{S}$ be an object of Λ.*

– *If $\mathcal{S}$ is locally I-complete, then each morphism $F : I \to \mathcal{S}$ in Λ yields a morphism $\lim F : *_\Lambda \to \mathcal{S}$ in a natural way. We can form a functor*

$$\lim : \Lambda(I, \mathcal{S}) \to \mathcal{S}^*$$

which sends F to $\lim F : *_\Lambda \to \mathcal{S}$. Moreover, the functor $\lim$ is right adjoint to the constant functor.

Conversely, suppose that the constant functor $\mathcal{S}^* \to \Lambda(I, S)$ has a right adjoint $\lim : \Lambda(I, \mathcal{S}) \to \mathcal{S}^*$. For $F : I \to \mathcal{S}$, the pair $(\lim F, \rho)$ is the limit of F, where $\rho \in \Lambda(I, \mathcal{S})(\overline{lF}, F)$ is given by the counit of the adjunction thus obtained.

– *If $\mathcal{S}$ is locally I-cocomplete, then each morphism $F : I \to \mathcal{S}$ in Λ yields a morphism colim$F : * \to \mathcal{S}$ in a natural way. We can form a functor*

$$colim : \Lambda(I, \mathcal{S}) \to \mathcal{S}^*$$

which sends F to colim$F : *_\Lambda \to \mathcal{S}$. Moreover, the functor colim is left adjoint to the constant functor.

Conversely, suppose that the constant functor $\mathcal{S}^* \to \Lambda(I, S)$ has a left adjoint colim : $\Lambda(I, \mathcal{S}) \to \mathcal{S}^*$. For $F : I \to \mathcal{S}$, the pair $(colimF, \iota)$ is the colimit of F, where $\iota \in \Lambda(I, \mathcal{S})(F, \overline{colimF})$ is given by the unit of the adjunction thus obtained.

PROOF. Let $F, G : I \to \mathcal{S}$ and $\alpha \in \Lambda(I, \mathcal{S})(F, G)$. Write $\alpha_i : Fi \to Gi$ for the composite

$$\begin{tikzcd} *_\Lambda \arrow[r, "i"] & I \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{S} \end{tikzcd} .$$

The pair $\left(\lim F, \left\{\alpha_i \rho_i^F\right\}_i\right)$ satisfies the universal property of $\lim G$, and hence yields a morphism $\lim F \to \lim G$ by universality of $\left(\lim G, \rho^G\right)$. Moreover, the pair $\left(\mathrm{colim} G, \left\{\iota_i^G \alpha_i\right\}_i\right)$ satisfies the properties of $\mathrm{colim} F$, and hence yields a morphism $\mathrm{colim} F \to \mathrm{colim} G$. The converse statements are immediate from the definition. □

DEFINITION 2.3.13. Let $\mathcal{S}$ be an object of Λ and $X, Y : *_\Lambda \to \mathcal{S}$. A cartesian product of X and Y in $\mathcal{S}$ is a universal pair $(X \times_\mathcal{S} Y, (\pi_X, \pi_Y))$ with

– $X \times_\mathcal{S} Y : *_\Lambda \to \mathcal{S}$,
– $\pi_X : X \times_\mathcal{S} Y \to X$,
– $\pi_Y : X \times_\mathcal{S} Y \to Y$.

DEFINITION 2.3.14. Let $\mathcal{S}$ be an object of Λ and $X, Y : *_\Lambda \to \mathcal{S}$. A coproduct of X and Y in $\mathcal{S}$ is a universal pair $(X \coprod_\mathcal{S}, Y, (\iota_X, \iota_Y))$ with

– $X \coprod_\mathcal{S} Y : *_\Lambda \to \mathcal{S}$
– $\iota_X : X \to X \coprod_\mathcal{S} Y$
– $\iota_Y : Y \to X \coprod_\mathcal{S} Y$.

DEFINITION 2.3.15. We say that an object $\mathcal{S}$ of Λ is cartesian if it has a terminal object $*_\mathcal{S} : *_\Lambda \to \mathcal{S}$ and if the cartesian product of any $X, Y : *_\Lambda \to \mathcal{S}$ exists.

DEFINITION 2.3.16. We say that an object $\mathcal{S}$ of Λ is cocartesian if it has an initial object $\emptyset_\mathcal{S} : *_\Lambda \to \mathcal{S}$ and if the cartesian coproduct of any $X, Y : *_\Lambda \to \mathcal{S}$ exists.

REMARK 2.3.17. For $X, Y : *_\Lambda \to \mathcal{S}$, the product $X \times_{\mathcal{S}} Y : *_\Lambda \to \mathcal{S}$ can also be defined as the object of $\mathcal{S}^*$ representing the functor

$$\mathcal{S}^*(\_, X) \times \mathcal{S}^*(\_, Y) : \mathcal{S}^{*op} \to \mathrm{SET},$$

given on $Z : *_\Lambda \to \mathcal{S}$ by $\mathcal{S}^*(Z, X) \times \mathcal{S}^*(Z, Y)$. Consequently, the product of X and Y in $\mathcal{S}$ is such that

$$\mathcal{S}^*(Z, X \times_{\mathcal{S}} Y) \xrightarrow{\cong} \mathcal{S}^*(Z, X) \times \mathcal{S}^*(Z, Y)$$

for all $Z : * \to \mathcal{S}$.

REMARK 2.3.18. For $X, Y : *_\Lambda \to \mathcal{S}$, the coproduct $X \coprod_{\mathcal{S}} Y : *_\Lambda \to \mathcal{S}$ may also be defined as the object of $\mathcal{S}^*$ co-representing the functor

$$\mathcal{S}^*(X, \_) \times \mathcal{S}^*(Y, \_) : \mathcal{S}^* \to \mathrm{SET},$$

given on $Z : *_\Lambda \to \mathcal{S}$ by $\mathcal{S}^*(X, Z) \times \mathcal{S}^*(Y, Z)$. Consequently, the coproduct of X and Y in $\mathcal{S}$ is such that

$$\mathcal{S}^*(X \coprod_{\mathcal{S}} Y, Z) \xrightarrow{\cong} \mathcal{S}^*(X, Z) \times \mathcal{S}^*(Y, Z)$$

for all $Z : * \to \mathcal{S}$.

REMARK 2.3.19. Let Λ be a bicartesian closed 2-category. We obtain a sequence of natural isomorphisms in each object $\mathcal{C}$ of Λ:

$$\begin{aligned}
\Lambda\left(\mathcal{C}, [*_\Lambda \coprod *_\Lambda, \mathcal{S}]_\Lambda\right) &\cong \Lambda\left((*_\Lambda \coprod *_\Lambda) \times \mathcal{C}, \mathcal{S}\right) \\
&\cong \Lambda\left((*_\Lambda \times \mathcal{C}) \coprod (*_\Lambda \times \mathcal{C}), \mathcal{S}\right) \\
&\cong \Lambda(\mathcal{C} \coprod \mathcal{C}, \mathcal{S}) \\
&\cong \Lambda(\mathcal{C}, \mathcal{S}) \times \Lambda(\mathcal{C}, \mathcal{S}) \\
&\cong \Lambda(\mathcal{C}, \mathcal{S} \times \mathcal{S}).
\end{aligned}$$

As a consequence, we have an isomorphism $[*_\Lambda \coprod *_\Lambda, \mathcal{S}]_\Lambda \cong \mathcal{S} \times \mathcal{S}$ in Λ. In particular, the data of $X, Y : *_\Lambda \to \mathcal{S}$ corresponds to the data of a morphism

$$(X, Y) : *_\Lambda \coprod *_\Lambda \to \mathcal{S},$$

whose limit is given by the cartesian product of X and Y in $\mathcal{S}$, and whose colimit is given by the cartesian coproduct of X and Y in $\mathcal{S}$. Therefore, the cartesian product of all $X, Y : *_\Lambda \to \mathcal{S}$ exists in $\mathcal{S}$ if and only if $\mathcal{S}$ is $(*_\Lambda \coprod *_\Lambda)-$complete. In this case, the limit yields a functor

$$\Lambda(*_\Lambda, \mathcal{S} \times_\Lambda \mathcal{S}) \to \mathcal{S}^*.$$

It would be convenient to obtain a monoidal structure on $\mathcal{S}$ internally in Λ, such that the tensor product is given pointwise by the catesian product. This motivates the subsequent notion of internal limits.

In order to manipulate the notions of limits and colimits internally in Λ, we define internal completeness and cocompleteness. For this purpose, we assume that Λ is a closed 2-category.

DEFINITION 2.3.20. Let $I, \mathcal{S} \in \Lambda$. We let the constant morphism $c : \mathcal{S} \to [I, \mathcal{S}]_\Lambda$ be defined by the composite

$$\mathcal{S} \xrightarrow{\cong} [*_\Lambda, S]_\Lambda \to [I, *_\Lambda]_\Lambda \times [*_\Lambda, \mathcal{S}]_\Lambda \to [I, \mathcal{S}]_\Lambda$$

DEFINITION 2.3.21. Let $\mathcal{S}$ be an object of Λ. We say that $\mathcal{S}$ is internally I-complete if the constant morphism has a right adjoint, denoted by $\lim_\Lambda$

$$c : \quad \mathcal{S} \underset{\perp}{\rightleftarrows} [I, \mathcal{S}]_\Lambda \quad : \lim_\Lambda .$$

Dually, we say that $\mathcal{S}$ is internally I-cocomplete if the constant morphism has a left adjoint, denoted by $\operatorname{colim}_\Lambda$

$$\operatorname{colim}_\Lambda : \quad [I, \mathcal{S}]_\Lambda \underset{\perp}{\rightleftarrows} \mathcal{S} \quad : c.$$

DEFINITION 2.3.22. Suppose that $\mathcal{S}$ has an initial object and is $(*_\Lambda \coprod *_\Lambda)$-complete. The internal cartesian product provides $(\mathcal{S}, \times_\mathcal{S})$ with the structure of a monoid in Λ with respect to the cartesian product. We say that $\mathcal{S}$ is a cartesian monoid in Λ.

REMARK 2.3.23. $\mathcal{S}$ is internally $(*_\Lambda \coprod *_\Lambda)$-complete if and only if it is $\coprod\limits_n *_\Lambda$-complete for each strictly positive natural number n. Consequently, $\mathcal{S}$ has the structure of a cartesian monoid if and only if it is $\coprod\limits_n *_\Lambda$-complete for each natural number n.

REMARK 2.3.24. Suppose that S has a terminal object $*_\mathcal{S} : *_\Lambda \to \mathcal{S}$. We can regard $*_\mathcal{S} : *_\Lambda \to \mathcal{S}$ as a morphism and consider its colimit, which exists and is given by $*_\mathcal{S} : *_\Lambda \to \mathcal{S}$. Suppose that $\mathcal{S}$ is internally $(*_\Lambda \coprod *_\Lambda)$-cocomplete. The colimit of $(*_\mathcal{S}, *_\mathcal{S}) : *_\Lambda \coprod *_\Lambda \to \mathcal{S}$ yields an element $*_\mathcal{S} \coprod *_\mathcal{S} : *_\Lambda \to \mathcal{S}$ of $\mathcal{S}^*$. Let $\coprod\limits_n *_\mathcal{S} : *_\Lambda \to \mathcal{S}$ be inductively defined by

- $1_\mathcal{S} = \coprod\limits_1 *_\mathcal{S} = *_\mathcal{S}$
- $(n+1)_\mathcal{S} = \coprod\limits_n *_\mathcal{S} = \operatorname{colim}\left(\left(\coprod\limits_n *_\mathcal{S}, *_\mathcal{S}\right) : *_\Lambda \coprod *_\Lambda \to \mathcal{S}\right) = \coprod\limits_n *_\mathcal{S} \coprod *_\mathcal{S}.$

Suppose that $\mathcal{S}$ also has an initial object $\emptyset_\mathcal{S} : *_\Lambda \to \mathcal{S}$ and write $0_\mathcal{S} := \emptyset_\mathcal{S}$. We obtain elements $n_\mathcal{S} : *_\Lambda \to \mathcal{S}$ for each natural number, which we can add by using the internal coproduct of $\mathcal{S}$, so that $n_\mathcal{S} + m_\mathcal{S} \cong (n+m)_\mathcal{S}$.

REMARK 2.3.25. A category $\mathcal{C}$ is cartesian in the sense of Definition 1.2.35 if and only if it is a cartesian monoid in $(\mathrm{CAT}, \times, *)$.

DEFINITION 2.3.26. We say that $\mathcal{S}$ is internally complete in Λ if for each object I of Λ, the constant morphism has a right adjoint.

DEFINITION 2.3.27. We say that $\mathcal{S}$ is internally cocomplete in Λ if for each object I of Λ, the constant morphism has a left adjoint.

PROPOSITION 2.3.28. *Suppose that Λ is $*_\Lambda$-primary. The object $\mathcal{S}$ is internally complete, respectively cocomplete, if and only if for each object I of Λ, any morphism $I \to \mathcal{S}$ has a limit, respectively a colimit, in $\mathcal{S}$.*

PROOF. For any object I of Λ, we have $\Lambda([I, \mathcal{S}]_\Lambda, \mathcal{S}) \cong [\Lambda(I, \mathcal{S}), \mathcal{S}^*]$ by Remark 2.1.6. The result follows from Proposition 2.3.12. $\square$

EXAMPLE 2.3.29. The category of sets is internally complete and cocomplete in CAT.

REMARK 2.3.30. We require the existence of arbitrary limits, no matter how large. It is worth notifying that Freyd's theorem on large limits does not apply to our framework. Indeed, the 'collection' of all morphisms on a category can not be given the structure of a set,[12] mainly because equality between objects of an arbitrary category is not defined.

Nevertheless, limits defined over small categories often hold more significance than limits defined over large categories. For instance, the cartesian product is defined as a small limit. Conversely, the limit and the colimit of the identity functor SET $\to$ SET is respectively given by the one element set and the empty set.

The fact that limits and colimits are better suited for functors from a small category can be understood as follows. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ be a functor. Both the limit and the colimit of F produce a single object of $\mathcal{D}$ from all objects and morphisms in $\mathcal{C}$. In this way, any object of $\mathcal{C}$ collapses into an element of the limit or colimit object of F, reducing the dimension by one level.

2.4. Kan extensions.

In this subsection, we fix an object $\mathcal{S}$ of Λ.

DEFINITION 2.4.1. Let $\mathcal{C}_1, \mathcal{C}_2 \in \Lambda$ and $\mu : \mathcal{C}_2 \to \mathcal{C}_1$. A local left Kan extension of a morphism $F : \mathcal{C}_2 \to \mathcal{S}$ along μ is a universal pair $(Lan_\mu F, \alpha)$, where $Lan_\mu F : \mathcal{C}_1 \to \mathcal{S}$ and where $\alpha \in \Lambda(\mathcal{C}_2, \mathcal{S})(F, (Lan_\mu F)\mu)$ is a 2-morphism in Λ, such that

$$\begin{tikzcd} \mathcal{C}_2 \arrow[r, "F"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\Swarrow\alpha"] & \mathcal{S} \\ \mathcal{C}_1 \arrow[ur, dashed, "Lan_\mu F"'] & \end{tikzcd}.$$

Hence for any other pair (L, β) such that $L : \mathcal{C}_1 \to \mathcal{S}$ and $\beta \in \Lambda(\mathcal{C}_2, \mathcal{S})(F, (Lan_\mu F)\mu)$, there exists a unique $\lambda \in \Lambda(\mathcal{C}_1, \mathcal{S})(Lan_\mu F, L)$ such that the following diagram commutes in the category $\Lambda(\mathcal{C}_1, \mathcal{S})$.

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & Lan_\mu F\mu \arrow[d, "\lambda \circ 1_\mu"] \\ & L\mu \end{tikzcd}$$

REMARK 2.4.2. The pair $(Lan_\mu F, \alpha)$ can be equivalently defined as the unique pair such that α induces an isomorphism

$$\Lambda(\mathcal{C}_1, \mathcal{S})(Lan_\mu F, L) \cong \Lambda(\mathcal{C}_2, \mathcal{S})(F, L\mu).$$

REMARK 2.4.3. Let $\mathcal{C}_1, \mathcal{C}_2 \in \Lambda$, the 2-functor $\Lambda(\_, \mathcal{S}) : \Lambda^{op} \to$ CAT induces a functor

$$\Lambda(\mathcal{C}_2, \mathcal{C}_1) \to \text{CAT}\left(\Lambda(\mathcal{C}_1, \mathcal{S}), \Lambda(\mathcal{C}_2, \mathcal{S})\right),$$

which is given in $\mu : \mathcal{C}_2 \to \mathcal{C}_1$ by the functor $\Lambda(\mu, \mathcal{S}) : \Lambda(\mathcal{C}_1, \mathcal{S}) \to \Lambda(\mathcal{C}_2, \mathcal{S})$ obtained by precomposition with μ. We often omit the dependance on $\mathcal{S}$ on this expression and write μ_* instead of $\Lambda(\mu, \mathcal{S})$ for simplicity.

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[12]We actually can. However, the elements of the resulting set will correspond to equivalence classes of the objects of the category of arrows. See Appendix A for more details.

DEFINITION 2.4.4. Let $\mathcal{C}_1, \mathcal{C}_2 \in \Lambda$ and $\mu : \mathcal{C}_2 \to \mathcal{C}_1$. We say that $Lan_\mu : \Lambda(\mathcal{C}_1, \mathcal{S}) \to \Lambda(\mathcal{C}_2, \mathcal{S})$ is the global $\mathcal{S}$-valued left Kan extension along μ with values in $\mathcal{S}$ if Lan_μ is left adjoint to μ_* in CAT, so that

$$\begin{tikzcd} Lan_\mu : \Lambda(\mathcal{C}_2, \mathcal{S}) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \Lambda(\mathcal{C}_1, \mathcal{S}) : \mu_*. \arrow[l, bend left] \end{tikzcd}$$

PROPOSITION 2.4.5. *Let* $\mathcal{C}_1, \mathcal{C}_2 \in \Lambda$ *and* $\mu : \mathcal{C}_2 \to \mathcal{C}_1$.

- *Suppose that* $Lan_\mu : \Lambda(\mathcal{C}_1, \mathcal{S}) \to \Lambda(\mathcal{C}_2, \mathcal{S})$ *is the global left Kan extension along* μ *with values in* $\mathcal{S}$. *Write* $\eta \in [\Lambda(\mathcal{C}_1, \mathcal{S}), \Lambda(\mathcal{C}_1, \mathcal{S})]\,(1, \mu_* Lan_\mu)$ *for the counit of the adjunction thus defined. The pair* $(Lan_\mu F, \eta_F)$ *is the local left Kan extension of* F *along* μ.
- *Suppose that for all* $F : \mathcal{C}_2 \to \mathcal{S}$, *the pair* $(Lan_\mu F, \alpha_F)$ *is the local left Kan extension of* F *along* μ. *Then we can form a functor*

$$Lan_\mu : \Lambda(\mathcal{C}_2, \mathcal{S}) \to \Lambda(\mathcal{C}_1, \mathcal{S})$$

given on $F \in \Lambda(\mathcal{C}_2, \mathcal{S})$ *by* $Lan_\mu F : \mathcal{C}_1 \to \mathcal{S}$ *which is left adjoint to* μ_*.

PROOF. Let $F, G : \mathcal{C}_2 \to \mathcal{S}$ and define

$$Lan_\mu(F,G) : \Lambda(\mathcal{C}_2, \mathcal{S})(F,G) \to \Lambda(\mathcal{C}_1, \mathcal{S})(Lan_\mu F, Lan_\mu G)$$

as follows. Let $\lambda \in \Lambda(\mathcal{C}_2, \mathcal{S})(F,G)$, we obtain a pair $(Lan_\mu G, \alpha_G \lambda)$ with $Lan_\mu G : \mathcal{C}_1 \to \mathcal{S}$ and $\alpha_G \lambda : \Lambda(\mathcal{C}_2, \mathcal{S})(F, Lan_\mu G \mu)$. By universality of the pair $(Lan_\mu F, \alpha_F)$, there is a unique 2-morphism $Lan_\mu(F,G)\lambda : Lan_\mu F \to Lan_\mu G$ such that the following diagram commutes in $\Lambda(\mathcal{C}_2, \mathcal{S})$.

$$\begin{tikzcd} F \arrow[r, "\lambda"] \arrow[drr, "\alpha_F"'] & G \arrow[r, "\alpha_G"] & Lan_\mu G\mu \\ & & Lan_\mu F\mu \arrow[u, "Lan_\mu(F,G)\lambda\mu"'] \end{tikzcd}$$

□

PROPOSITION 2.4.6. *Let* $\mathcal{C}_1, \mathcal{C}_2 \in \Lambda$ *and suppose that for all* $\mu : \mathcal{C}_2 \to \mathcal{C}_1$, $Lan_\mu : \Lambda(\mathcal{C}_2, \mathcal{S}) \to \Lambda(\mathcal{C}_1, \mathcal{S})$ *is the global left Kan extension along* μ. *We can form a functor*

$$Lan : \Lambda(\mathcal{C}_2, \mathcal{C}_1) \to [\Lambda(\mathcal{C}_2, \mathcal{S}), \Lambda(\mathcal{C}_1, \mathcal{S})]\,,$$

given on $\mu \in \Lambda(\mathcal{C}_2, \mathcal{C}_1)$ *by* $Lan_\mu : \Lambda(\mathcal{C}_2, \mathcal{S}) \to \Lambda(\mathcal{C}_1, \mathcal{S})$.

PROOF. Let $\mu, \nu \in \Lambda(\mathcal{C}_2, \mathcal{C}_1)$ and $\phi \in \Lambda(\mathcal{C}_2, \mathcal{C}_1)(\mu, \nu)$. Let

$$Lan_\phi \in [\Lambda(\mathcal{C}_2, \mathcal{S}), \Lambda(\mathcal{C}_1, \mathcal{S})]\,(Lan_\nu, Lan_\mu)$$

be defined in $F : \mathcal{C}_2 \to \mathcal{S}$ as follows. The canonical morphism $F \to Lan_\nu F\nu$ induces a morphism $F \to Lan_\nu F\mu$ by precomposition with $\phi : \mu \to \nu$. By universality of the pair $(Lan_\mu F, \alpha_\mu)$, there exists a unique morphism $Lan_\phi F : Lan_\mu F \to Lan_\nu F$ such that the following diagram commutes in $\Lambda(\mathcal{C}_2, \mathcal{S})$.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & Lan_\nu F\nu \arrow[r] & Lan_\nu F\mu \\ Lan_\mu F\mu \arrow[urr, dashed, "Lan_\phi F\mu"'] & & \end{tikzcd}$$

We show that $Lan_\phi F$ is natural in F. Let $G : \mathcal{C}_2 \to \mathcal{S}$ and $\lambda \in \Lambda(\mathcal{C}_2, \mathcal{S})(F, G)$. The naturality condition can be expressed by the commutativity of the diagram

$$\begin{tikzcd}
Lan_\nu F \arrow[r, "Lan_\nu \lambda"] \arrow[d, "Lan_\phi F"'] & Lan_\nu G \arrow[d, "Lan_\phi G"] \\
Lan_\mu F \arrow[r, "Lan_\mu \lambda"'] & Lan_\mu G.
\end{tikzcd}$$

Both of the resulting composites induce a map $F \to Lan_\mu G$. The commitativity of the diagram follows from the universality of $Lan_\nu F$.

$$\begin{tikzcd}
F \arrow[rr, "\lambda"] \arrow[dr, "\alpha_\nu"] \arrow[dd, "\alpha_\mu"'] & & G \arrow[d, "\alpha_\nu"] \arrow[ddr, "\alpha'_\mu"] & \\
& Lan_\nu F\nu \arrow[r, "Lan_\nu \lambda\nu"] \arrow[dd, "Lan_\phi F\nu"] \arrow[ddrr, dashed] & Lan_\nu G\nu \arrow[ddr, "Lan_\phi G\nu"] & \\
Lan_\mu F\mu \arrow[dr, "(Lan_\mu F)\circ\phi"'] & & & Lan_\mu G\mu \arrow[d, "(Lan_\mu G)\circ\phi"] \\
& Lan_\mu F\nu \arrow[rr, "Lan_\mu \lambda\nu"'] & & Lan_\mu G\nu
\end{tikzcd}$$

We obtain a natural transformation $Lan_\phi : Lan_\nu \to Lan_\mu$ for each ϕ. Let $\phi_1 \in \Lambda(\mathcal{C}_2, \mathcal{C}_1)(\nu_1, \nu_2)$ and $\phi_2 \in \Lambda(\mathcal{C}_2, \mathcal{C}_1)(\nu_2, \nu_3)$, we obtain a canonical isomorphism $Lan_{\phi_2\phi_1} \cong Lan_{\phi_2} Lan_{\phi_1}$ immediately from the universality of the constructions, hence the result. $\square$

In what follows, we suppose that Λ is equipped with a closed monoidal structure $(\Lambda, \otimes_\Lambda, \mathbf{1}_\Lambda)$. The internal hom 2-functor induces a 2-functor

$$[-, \mathcal{S}]_\Lambda : \quad \Lambda^{op} \quad \to \quad \Lambda.$$

Definition 2.4.7. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be objects of Λ and $\mu : \mathcal{C}_2 \to \mathcal{C}_1$. We say that $Lan_\mu^{\mathcal{S}} : [\mathcal{C}_2, \mathcal{S}]_\Lambda \to [\mathcal{C}_1, \mathcal{S}]_\Lambda$ is the internal left Kan extension along μ with values in $\mathcal{S}$, if it is left adjoint to $[\mu, \mathcal{S}]_\Lambda$, so that we have the following adjunction in Λ

$$Lan_\mu^{\mathcal{S}} : [\mathcal{C}_2, \mathcal{S}]_\Lambda \rightleftarrows [\mathcal{C}_1, \mathcal{S}]_\Lambda : [\mu, \mathcal{S}]_\Lambda.$$

Remark 2.4.8. We may still consider ordinary left Kan extensions in the sense of Definition 2.4.4. To make the distinction precise, we refer to the functor $Lan_\mu : \Lambda(\mathcal{C}_2, \mathcal{S}) \to \Lambda(\mathcal{C}_1, \mathcal{S})$ as the external left Kan extension along μ.

Definition 2.4.9. We say that the internal left Kan extension is globally defined for $\mathcal{S}$-valued morphisms if it exists along any morphism $\mu : \mathcal{C}_2 \to \mathcal{C}_1$ in Λ and extends to a functor

$$Lan^{\mathcal{S}} : \Lambda(\mathcal{C}_2, \mathcal{C}_1) \to \Lambda\left([\mathcal{C}_2, \mathcal{S}]_\Lambda, [\mathcal{C}_1, \mathcal{S}]_\Lambda\right).$$

We say that the internal left Kan extension is internally defined if the internal left Kan extension is globally defined and if there is a morphism

$$\mathbf{Lan}^{\mathcal{S}} : [\mathcal{C}_2, \mathcal{C}_1]_\Lambda \to \left[[\mathcal{C}_2, \mathcal{S}]_\Lambda, [\mathcal{C}_1, \mathcal{S}]_\Lambda\right]_\Lambda$$

in Λ whose image under the forgetful 2-functor $\Lambda(1_\Lambda, \_)$ is given by $Lan^{\mathcal{S}}$.

Proposition 2.4.10. *Let $\mathcal{C}_1, \mathcal{C}_2 \in \Lambda$, $\mu : \mathcal{C}_2 \to \mathcal{C}_1$ and $F : \mathcal{C}_2 \to \mathcal{S}$. Suppose that the internal left Kan extension along μ with values in $\mathcal{S}$ exists. The composite*

$$Lan_\mu^{\mathcal{S}} F : *_\Lambda \xrightarrow{F} [\mathcal{C}_2, \mathcal{S}]_\Lambda \xrightarrow{Lan_\mu^{\mathcal{S}}} [\mathcal{C}_1, \mathcal{S}]_\Lambda$$

corresponds to the left Kan extension of F along μ under the isomorphism of categories $\Lambda(\mathcal{C}_1,\mathcal{S}) \cong [\mathcal{C}_1,\mathcal{S}]^*_\Lambda$.

PROOF. We deduce from Lemma 2.2.3 that $Lan^\Lambda_\mu F$ satisfies the universal property of $Lan_\mu F$, hence $Lan_\mu F \cong Lan^\Lambda_\mu F$. □

COROLLARY 2.4.11. *If the internal left Kan extension is globally defined for $\mathcal{S}$-valued morphisms, then the external left Kan extension exists, so that the following diagram*

$$\begin{tikzcd}
\Lambda(\mathcal{C}_2,\mathcal{C}_1) \arrow[r, "Lan^{\mathcal{S}}"] \arrow[d, "Lan^{\mathcal{S}}"'] & {[\Lambda(\mathcal{C}_2,\mathcal{S}),\Lambda(\mathcal{C}_1,\mathcal{S})]} \\
\Lambda([\mathcal{C}_2,\mathcal{S}]_\Lambda,[\mathcal{C}_1,\mathcal{S}]_\Lambda) \arrow[ur, "([\mathcal{C}_1,\mathcal{S}]_\Lambda,[\mathcal{C}_2,\mathcal{S}]_\Lambda)^*"'] &
\end{tikzcd}$$

commutes.

PROPOSITION 2.4.12. *If the 2-category Λ is $*_\Lambda$-primary, then the internal left Kan extension is globally defined for $\mathcal{S}$-valued morphisms if and only if the internal left Kan extension exists along any morphism $\mu : \mathcal{C}_2 \to \mathcal{C}_1$.*

PROOF. Suppose that the internal left Kan extension with values in $\mathcal{S}$ exists along any morphism of Λ, and let $\mu,\nu : \mathcal{C}_2 \to \mathcal{C}_1$ be morphisms in Λ. Since Λ is $*_\Lambda$-primary, we have the following isomorphisms

$$\begin{aligned}
\Lambda\left([\mathcal{C}_2,\mathcal{S}]_\Lambda,[\mathcal{C}_1,\mathcal{S}]_\Lambda\right)\left(Lan^{\mathcal{S}}_\mu, Lan^{\mathcal{S}}_\nu\right) &\cong \int_{F\in[\mathcal{C}_2,\mathcal{S}]^*_\Lambda} [\mathcal{C}_1,\mathcal{S}]^*_\Lambda(Lan^{\mathcal{S}}_\mu F, Lan^{\mathcal{S}}_\nu F) \\
&\cong \int_{F:\mathcal{C}_2\to\mathcal{S}} \Lambda(\mathcal{C}_1,\mathcal{S})(Lan_\mu F, Lan_\nu F) \\
&\cong \left[\Lambda(\mathcal{C}_2,\mathcal{S}),\Lambda(\mathcal{C}_1,\mathcal{S})\right](Lan_\mu, Lan_\nu).
\end{aligned}$$

By Proposition 2.4.6, $\mu \mapsto Lan^{\mathcal{S}}_\mu$ extends to a functor, hence the result. □

COROLLARY 2.4.13. *If the 2-category Λ is $*_\Lambda$-primary, then the internal left Kan extension is internally defined if and only if the internal left Kan extension is globally defined, if and only if the internal left Kan extension exists along any morphism.*

In what follows, we suppose that Λ is an $*_\Lambda$-primary 2-category. We state some properties of global Kan extension functor

$$Lan^{\mathcal{S}} : \Lambda(\mathcal{C}_2,\mathcal{C}_1) \longrightarrow \Lambda([\mathcal{C}_2,\mathcal{S}]_\Lambda,[\mathcal{C}_1,\mathcal{S}]_\Lambda).$$

We let Λ^{Lan} be the 2-category whose objects are the objects $\mathcal{S}$ of Λ such that the internal left Kan extension is globally defined for $\mathcal{S}$-valued morphisms. For any objects $\mathcal{C}_1$ and $\mathcal{C}_2$ of Λ, we write $[\mathcal{C}_2,-]_\Lambda,[\mathcal{C}_1,-]_\Lambda : \Lambda^{Lan} \to \Lambda$ for the 2-functors induced by $[\mathcal{C}_1,\_]_\Lambda,[\mathcal{C}_2,\_]_\Lambda : \Lambda \to \Lambda$ by restriction.

LEMMA 2.4.14. *Let $\mu : \mathcal{C}_2 \to \mathcal{C}_1$. The functor $Lan^{\mathcal{S}}_\mu$ is natural in $\mathcal{S}$, in the sense that the functors $Lan^{\mathcal{S}}_\mu$ yield a colax 2-natural transformation*

$$Lan^\bullet_\mu : [\mathcal{C}_2,-]_\Lambda \to [\mathcal{C}_1,-]_\Lambda$$

Proof. We let the colax 2-natural transormation $Lan_\mu^\bullet$ be defined as follows. Let $\mathcal{S}_2, \mathcal{S}_1$ be such that the internal left Kan extension if globally defined and let $\Phi : \mathcal{S}_2 \to \mathcal{S}_1$. Let Lan_μ^Φ be the 2-morphism in Λ such that

$$\begin{tikzcd}
{[\mathcal{C}_2,\mathcal{S}_2]_\Lambda} \arrow[r, "{[\mathcal{C}_2,\Phi]_\Lambda}"] \arrow[d, "Lan_\mu^{\mathcal{S}_2}"'] & {[\mathcal{C}_2,\mathcal{S}_1]_\Lambda} \arrow[d, "Lan_\mu^{\mathcal{S}_1}"] \arrow[dl, Rightarrow, shorten=12pt, "Lan_\mu^\Phi"'] \\
{[\mathcal{C}_1,\mathcal{S}_2]_\Lambda} \arrow[r, "{[\mathcal{C}_1,\Phi]_\Lambda}"'] & {[\mathcal{C}_1,\mathcal{S}_1]_\Lambda,}
\end{tikzcd}$$

and defined by using the natural isomorphism

$$\begin{aligned}
&\Lambda\left([\mathcal{C}_2,\mathcal{S}_2]_\Lambda,[\mathcal{C}_1,\mathcal{S}_1]_\Lambda\right)\left(Lan_\mu^{\mathcal{S}_1}\circ[\mathcal{C}_2,\Phi]_\Lambda,[\mathcal{C}_1,\Phi]_\Lambda\circ Lan_\mu^{\mathcal{S}_2}\right)\\
&\qquad\cong\left[\Lambda\left(\mathcal{C}_2,\mathcal{S}\right),\Lambda\left(\mathcal{C}_1,\mathcal{S}_1\right)\right]\left(Lan_\mu^{\mathcal{S}}\Phi,\Phi Lan_\mu^{\mathcal{S}}\right)\\
&\qquad\cong\int^{F:\mathcal{C}_2\to\mathcal{S}_2}\Lambda(\mathcal{C}_1,\mathcal{S}_1)\left(Lan_\mu^{\mathcal{S}_1}\Phi F,\Phi Lan_\mu^{\mathcal{S}_2}F\right).
\end{aligned}$$

Let $F : \mathcal{C}_2 \to \mathcal{S}_2$, the canonical morphism $F \to Lan_\mu^{\mathcal{S}_2} F\mu$ in $\Lambda(\mathcal{C}_2,\mathcal{S}_2)$ yields a morphism $\Phi F \to \Phi Lan_\mu^{\mathcal{S}_2} F\mu$ in $\Lambda(\mathcal{C}_2,\mathcal{S}_1)$ as displayed in the following diagram.

$$\begin{tikzcd}
\mathcal{C}_2 \arrow[r, "F", "Lan_\mu^{\mathcal{S}_1}\Phi F"'] \arrow[d, "\mu"'] & \mathcal{S}_2 \arrow[r, "\Phi"] \arrow[d, phantom, "\Downarrow"] & \mathcal{S}_1 \\
\mathcal{C}_1 \arrow[urr, dashed, bend left] \arrow[urr, dashed, bend right, "Lan_\mu^\Phi F", "\Phi Lan_\mu^{\mathcal{S}_2} F"'] & & .
\end{tikzcd}$$

By universality of $Lan_\mu^{\mathcal{S}_1}\Phi F$, we obtain $Lan_\mu^\Phi F : Lan_\mu^{\mathcal{S}_1}\Phi F \to \Phi Lan_\mu^{\mathcal{S}_2} F$ as the unique morphism in $\Lambda(\mathcal{C}_1,\mathcal{S}_1)$ such that the following diagram commutes

$$\begin{tikzcd}
\Phi F \arrow[r, "\alpha_{\Phi F}"] \arrow[dr, "\Phi\alpha_F"'] & Lan_\mu \Phi F\mu \arrow[d, "Lan_\mu^\Phi F\mu"] \\
& \Phi Lan_\mu F\mu.
\end{tikzcd}$$

Let $\mathcal{S}_3 \xrightarrow{\Phi_2} \mathcal{S}_2 \xrightarrow{\Phi_2} \mathcal{S}_1$. The 2-morphism resulting from composition of the left hand side diagram is equal to the 2-morphism displayed on the right hand side diagram by the universality of the definition.

$$\begin{tikzcd}
{[\mathcal{C}_2,\mathcal{S}_3]_\Lambda} \arrow[r, "{[\mathcal{C}_2,\Phi_2]}"] \arrow[d, "Lan_\mu^{\mathcal{S}_3}"'] & {[\mathcal{C}_2,\mathcal{S}_2]_\Lambda} \arrow[r, "{[\mathcal{C}_2,\Phi_1]}"] \arrow[d, "Lan_\mu^{\mathcal{S}_2}"'] \arrow[dl, Rightarrow, shorten=12pt, "Lan_\mu^{\Phi_2}"'] & {[\mathcal{C}_2,\mathcal{S}_2]_\Lambda} \arrow[d, "Lan_\mu^{\mathcal{S}_1}"] \arrow[dl, Rightarrow, shorten=12pt, "Lan_\mu^{\Phi_1}"'] & \overset{\circ}{\longmapsto} & {[\mathcal{C}_2,\mathcal{S}_3]_\Lambda} \arrow[r, "{[\mathcal{C}_2,\Phi_2\Phi_2]}"] \arrow[d, "Lan_\mu^{\mathcal{S}_3}"'] & {[\mathcal{C}_2,\mathcal{S}_1]_\Lambda} \arrow[d, "Lan_\mu^{\mathcal{S}_1}"] \arrow[dl, Rightarrow, shorten=12pt, "Lan_\mu^{\Phi_2\Phi_1}"'] \\
{[\mathcal{C}_1,\mathcal{S}_3]_\Lambda} \arrow[r, "{[\mathcal{C}_1,\Phi_1]}"'] & {[\mathcal{C}_1,\mathcal{S}_1]_\Lambda} \arrow[r, "{[\mathcal{C}_1,\Phi_1]}"'] & {[\mathcal{C}_1,\mathcal{S}_1]_\Lambda} & & {[\mathcal{C}_1,\mathcal{S}_3]_\Lambda} \arrow[r, "{[\mathcal{C}_1,\Phi_2\Phi_1]}"'] & {[\mathcal{C}_1,\mathcal{S}_1]_\Lambda.}
\end{tikzcd}$$

Let $\Phi_1, \Phi_2 : \mathcal{S}_2 \to \mathcal{S}_1$ and $\chi : \Phi_1 \to \Phi_2$. The equality

$$Lan_\mu^{\Phi_2}\circ\left(Lan_\mu^{\mathcal{S}_1}.[\mathcal{C}_2,\chi]_\Lambda\right)=\left([\mathcal{C}_1,\chi]_\Lambda.Lan_\mu^{\mathcal{S}_2}\right)\circ Lan_\mu^{\Phi_1}$$

holds[13] by universality, hence the result. $\square$

Lemma 2.4.15. *The $\mathcal{S}$-valued global left Kan extension is compatible with composition of morphisms in the following sense. Let $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ be objects of Λ*

[13]In the set $\Lambda\left([\mathcal{C}_2,\mathcal{S}_2]_\Lambda,[\mathcal{C}_1,\mathcal{S}_1]_\Lambda\right)\left(Lan_\mu^{\mathcal{S}_1}\circ[\mathcal{C}_2,\Phi_1]_\Lambda,[\mathcal{C}_1,\Phi_2]_\Lambda\circ Lan_\mu^{\mathcal{S}_2}\right)$.

and let μ_1, μ_2 be morphisms in Λ such that $\mathcal{C}_3 \overset{\mu_2}{\to} \mathcal{C}_2 \overset{\mu_1}{\to} \mathcal{C}_1$. We have a canonical isomorphism

$$Lan^{\mathcal{S}}_{\mu_1\mu_2} \cong Lan^{\mathcal{S}}_{\mu_1} Lan^{\mathcal{S}}_{\mu_2}.$$

Proof. Let $F : \mathcal{C}_3 \to \mathcal{S}$. Write $\alpha_{\mu_1\mu_2} : F \to Lan^{\mathcal{S}}_{\mu_1\mu_2}F\mu_1\mu_2$ for the left Kan extension of F along $\mu_1\mu_2$, $\alpha_{\mu_2} : F \to Lan^{\mathcal{S}}_{\mu_2}F\mu_2$ for the left Kan extension of F along μ_2, and $\alpha^{\mu_2}_{\mu_1} : Lan^{\mathcal{S}}_{\mu_2}F \to Lan^{\mathcal{S}}_{\mu_1}Lan^{\mathcal{S}}_{\mu_2}F\mu_1$ for the left Kan extension of $Lan^{\mathcal{S}}_{\mu_2}F$ along μ_1.

$$\begin{tikzcd}
\mathcal{C}_3 \arrow[rr, "F"] \arrow[d, "\mu_2"'] \arrow[r, phantom, "\alpha_{\mu_1\mu_2}\Downarrow \; \alpha_{\mu_2}\Downarrow"] & & \mathcal{S} \\
\mathcal{C}_2 \arrow[urr, dashed, "Lan^{\mathcal{S}}_{\mu_2}F"'] \arrow[d, "\mu_1"'] \arrow[r, phantom, "\alpha^{\mu_2}_{\mu_1}\Downarrow"] & & \\
\mathcal{C}_1 \arrow[uurr, dashed, "Lan^{\mathcal{S}}_{\mu_1\mu_2}F"] \arrow[uurr, dashed, bend right, "Lan^{\mathcal{S}}_{\mu_1}Lan^{\mathcal{S}}_{\mu_2}F"'] & &
\end{tikzcd}$$

- By universality of $Lan^{\mathcal{S}}_{\mu_2\mu_1}F$, the composite morphism

$$F \xrightarrow{\alpha_{\mu_2}} Lan^{\mathcal{S}}_{\mu_2}F\mu_2 \xrightarrow{\alpha^{\mu_2}_{\mu_1}\mu_2} Lan^{\mathcal{S}}_{\mu_1}Lan^{\mathcal{S}}_{\mu_2}F\mu_1\mu_2$$

  yields a canonical morphism $\beta_{\mu_1\mu_2} : Lan^{\mathcal{S}}_{\mu_1\mu_2}F \to Lan^{\mathcal{S}}_{\mu_1}Lan^{\mathcal{S}}_{\mu_2}F$.
- By universality of $Lan^{\mathcal{S}}_{\mu_2}F$, the morphism $\alpha_{\mu_1\mu_2} : F \to Lan^{\mathcal{S}}_{\mu_1\mu_2}F\mu_1\mu_2$ yields a morphism $\beta_{\mu_2} : Lan^{\mathcal{S}}_{\mu_2}F \to Lan^{\mathcal{S}}_{\mu_1\mu_2}F\mu_1$.
- By universality of $Lan^{\mathcal{S}}_{\mu_1}$ in $Lan_{\mu_2}F$, the morphism β_{μ_2} yields a morphism $\beta^{\mu_2}_{\mu_1} : Lan^{\mathcal{S}}_{\mu_1}Lan^{\mathcal{S}}_{\mu_2}F \to Lan^{\mathcal{S}}_{\mu_1\mu_2}F$.

We obtain a canonical isomorphism $Lan^{\mathcal{S}}_{\mu_1}Lan^{\mathcal{S}}_{\mu_2}F \cong Lan^{\mathcal{S}}_{\mu_1\mu_2}F$ from the canonicity. The result holds internally when Λ is an $*_\Lambda$-primary 2-category. □

Proposition 2.4.16. *Let $\mathcal{S}$ be an object of Λ such that the internal Kan extensions is globally defined for $\mathcal{S}$-valued morphisms in Λ. The internal Kan extension yields a 2-functor*

$$[\_, \mathcal{S}]^{\#}_{\Lambda} : \Lambda \to \Lambda$$

which is natural in the objects $\mathcal{S}$ of Λ such that $\mathcal{S}$-valued internal Kan extensions exist.

Proof. Let $\mathcal{C}$ be an object of Λ, we set $[\mathcal{C}, \mathcal{S}]^{\#}_{\Lambda} = [\mathcal{C}, \mathcal{S}]_{\Lambda}$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be objects of Λ. The global Kan extension functor

$$Lan^{\mathcal{S}} : \Lambda(\mathcal{C}_2, \mathcal{C}_1) \to \Lambda\left([\mathcal{C}_2, \mathcal{S}]_{\Lambda}, [\mathcal{C}_1, \mathcal{S}]_{\Lambda}\right).$$

is natural in the objects $\mathcal{C}_1, \mathcal{C}_1$ of Λ by Lemma 2.4.15, and hence defines a 2-functor $\Lambda \to \Lambda$. □

3. Monoidal structures in 2-categories

We study in depth the definition of monoidal structures in 2-categories. In a preliminary step, we examine the definition of monoidal 2-categories. Then we study the definition of monoids in monoidal 2-categories, and hence of internal monoidal structures on objects of a monoidal 2-category. We rely on this construction to give a definition of a 2-category of 2-fold monoids, and more generally, of iterated monoids in 2-categories equipped with iterated monoidal structures. To complete the account of this section, we explain the definition of generalized Day convolutions for this notion of monoid internal to a monoidal 2-category.

3.1. Monoidal 2-categories. We make explicit the definition of a monoidal 2-category in this subsection. But, before that, we give the following definition of a notion of an $\mathbb{N}$-unbiased monoidal 2-category, which is more general, and has the advantage of making explicit higher operations hidden in the structure of a monoidal 2-category.

DEFINITION 3.1.1. An $\mathbb{N}$-unbiased monoidal 2-category is a 2-category Λ equipped with:

- For each natural number n, a 2-functor $\otimes_\Lambda^n$, which we call the tensor product[14]:
$$\otimes_\Lambda^n : \Lambda^n \to \Lambda$$
- For each natural number r, and all $n_1, \ldots, n_r$, a 2-natural transformation $\otimes_\Lambda^{r,n_\bullet}$, which we call the associator:
$$\begin{tikzcd}
\prod_{i=1}^r \Lambda^{n_i} \arrow[r, "\cong"] \arrow[d, "\prod_{i=1}^r \otimes_\Lambda^{n_i}"'] \arrow[dr, phantom, "\overset{\otimes_\Lambda^{r,n_\bullet}}{\Longrightarrow}"] & \Lambda^n \arrow[d, "\otimes_\Lambda^n"] \\
\Lambda^r \arrow[r, "\otimes_\Lambda^r"'] & \Lambda
\end{tikzcd}$$
where $n = n_1 + \ldots n_r$.
- For each natural number p, for all natural numbers $r_1, \ldots, r_p$, and for all natural numbers $n_i^1, \ldots, n_i^{r_i}$ for $i = 1, \ldots, p$, a modification $\otimes_\Lambda^{p,r_\bullet,n_\bullet^\bullet}$, called the 2-associator of Λ, between the 2-natural transformations resulting from the composition of the following cubical diagram:
$$\begin{tikzcd}
\prod_{i=1}^p \prod_{j=1}^{r_i} \Lambda^{n_i^j} \arrow[rr, "\cong"] \arrow[d, "\prod_{i=1}^p \prod_{j=1}^{r_i} \otimes_\Lambda^{n_i^j}"'] \arrow[ddr, "\cong"] \arrow[drr, phantom, "\overset{\prod_{i=1}^p \otimes_\Lambda^{r_i,n_i^\bullet}}{\Longrightarrow}"] & & \prod_{i=1}^p \Lambda^{n_i} \arrow[d, "\prod_{i=1}^p \otimes_\Lambda^{n_i}"] \arrow[ddr, "\cong"] & \\
\prod_{i=1}^p \Lambda^{r_i} \arrow[rr, "\prod_{i=1}^p \otimes_\Lambda^{r_i}"] \arrow[ddr, "\cong"'] \arrow[dr, phantom, "\cong"] & & \Lambda^p \arrow[ddr, "\otimes_\Lambda^p"] \arrow[dr, phantom, "\otimes_\Lambda^{p,n_\bullet}"] \arrow[dl, phantom, "\otimes_\Lambda^{p,r_\bullet}"] & \\
& \prod_{k=1}^r \Lambda^{n^k} \arrow[rr, "\cong"] \arrow[d, "\prod_{k=1}^r \otimes_\Lambda^{n^k}"'] \arrow[drr, phantom, "\overset{\otimes_\Lambda^{r,n^\bullet}}{\Longrightarrow}"] & & \Lambda^n \arrow[d, "\otimes_\Lambda^n"] \\
& \Lambda^r \arrow[rr, "\otimes_\Lambda^r"'] & & \Lambda
\end{tikzcd}$$
where:
  - $r = r_1 + \cdots + r_p$.
  - For $i = 1, \ldots, p$, $n_i = n_i^1 + \cdots + n_i^{r_i}$.
  - $n = n_1 + \cdots + n_p$.
  - For $k = 1, \ldots, r$, n^k corresponds to n_i^j under the isomorphism
$$\coprod_{i=1}^p \{1, \ldots, r_i\} \cong \{1, \ldots, r\}.$$

[14]Note that we obtain a unit by taking $n = 0$.

  - The 2-natural transformation on the top square is obtained from the 2-associator $\otimes^{p,r_\bullet,n_\bullet^\bullet}_{\mathrm{C{\small AT}}_2}$ of the monoidal 3-category $(\mathrm{C{\small AT}}_2,\times)$ in the object corresponding to Λ under the diagonal.
  - The 2-natural transformation on the far left square results from the naturality of the associator $\otimes^{p,r_\bullet}_{\mathrm{C{\small AT}}_2}$ of the monoidal 3-category $(\mathrm{C{\small AT}}_2,\times)$ with respect to $\otimes^{n_1^1}_\Lambda, \cdots \otimes^{n_p^{r_p}}_\Lambda$.

  The domain and the codomain of this modification are made explicit on Figure 1.
- The last condition involves a fourth-dimensional diagram made of composites of the 2-functors, 2-natural transformations, and modifications defined above. We state this condition by using the definition of equality between two modifications ψ, ϕ between 2-natural transformations. The modifications ψ and ψ are equal if and only if for each object $\mathcal{C}$ of their domain 2-category Γ, the 2-morphisms obtained in the codomain category $\psi\mathcal{C}, \phi\mathcal{C}$ are equal. Consequently, a fourth-dimensional diagram made of compositions of modifications in the 3-category of 2-categories commutes if and only if the 3-dimensional diagram obtained in the codomain 2-category by precomposition with the 2-functor $\mathcal{C} : * \to \Gamma$ commutes. We require that for all natural number s, for all $(p_i)_{i=1}^s$, for all $\coprod_{i=1}^s \left(r_i^j\right)_{j=1}^{p_i}$, and for all $\coprod_{i=1}^s \coprod_{j=1}^{p_i} (n_{i,j,k})_{k=1}^{r_j^i}$, for all object $\mathcal{C}_\bullet \to \prod_{i=1}^s \prod_{j=1}^{p_i} \prod_{k=1}^{r_j^i} \Lambda^{n_{i,j,k}}$, make commute this 3-dimensional cubical diagram.

If Λ is an $\mathbb{N}$-unbiased monoidal 2-category, then we write $\Lambda \in \mathrm{M{\small ON}}^{\mathbb{N}}_{\mathrm{C{\small AT}}_2}$.

We say that an $\mathbb{N}$-unbiased monoidal 2-category Λ is strongly monoidal if its associator and 2-associator are isomorphisms.

Definition 3.1.2. Let Λ and Γ be $\mathbb{N}$-unbiased monoidal 2-categories. A monoidal 2-functor $\Lambda \to \Gamma$ is a 2-functor $F : \Lambda \to \Gamma$ equipped with:

- For each natural number n, a 2-natural transformation $\otimes_F^n$ called the product of F:

$$\begin{tikzcd}
\Lambda^n \arrow[r, "F^n"] \arrow[d, "\otimes^n_\Lambda"'] \arrow[dr, phantom, "\xRightarrow{\otimes^n_F}"] & \Gamma^n \arrow[d, "\otimes^n_\Gamma"] \\
\Lambda \arrow[r, "F"'] & \Gamma
\end{tikzcd}$$

- For each natural number r, and for all natural numbers $n_1, \dots, n_r$, a modification $\otimes_F^{r,n_\bullet}$, called the associator of F, between the 2-natural transformations displayed on the following cubical diagram:

$$\begin{tikzcd}
 & & \Lambda^n \arrow[r, "F^n"] \arrow[dd, "\otimes^n_\Lambda"] \arrow[r, phantom, shift right=6, "\xRightarrow{\otimes^n_F}"] & \Gamma^n \arrow[dd, "\otimes^n_\Gamma"] \\
\prod_{i=1}^r \Lambda^{n_i} \arrow[r, "\prod_{i=1}^r F^{n_i}"] \arrow[urr, "\cong"] \arrow[dd, "\prod_{i=1}^r \otimes^{n_i}_\Lambda"'] \arrow[dr, phantom, "\xRightarrow{\prod_{i=1}^r \otimes^{n_i}_F}"] \arrow[urr, phantom, shift right=4, "\cong"] & \prod_{i=1}^r \Gamma^{n_i} \arrow[urr, "\cong"] \arrow[dd, "\prod_{i=1}^r \otimes^{n_i}_\Gamma"] \arrow[urr, phantom, shift right=6, "\xRightarrow{\otimes^{r,n_\bullet}_\Gamma}"] \arrow[r, phantom, shift left=2, "\xRightarrow{\otimes^{r,n_\bullet}_\Lambda}"] & & \\
 & & \Lambda \arrow[r, "F"] & \Gamma. \\
\Lambda^r \arrow[r, "F^r"'] \arrow[urr, "\otimes^r_\Lambda"] \arrow[urr, phantom, shift right=4, "\xRightarrow{\otimes^r_F}"] & \Gamma^r \arrow[urr, "\otimes^r_\Gamma"'] & &
\end{tikzcd}$$

The domain and the codomain of this modification are made explicit on Figure 2 by using an hemispherical decomposition of this cube.
– such that

We write $F \in \mathrm{Mon}^{\mathbb{N}}_{\mathrm{Cat}_2}(\Lambda, \Gamma)$. We say that a monoidal functor is strongly monoidal if its product and associator are isomorphisms.

EXAMPLE 3.1.3. Let $\Lambda \in \mathrm{Mon}^{\mathbb{N}}_{\mathrm{Cat}_2}$. The oppositization 3-functor[15] $\_^{op}$: $\mathrm{Cat}_2^{op_2} \to \mathrm{Cat}_2$ yields a 2-functor

$$\mathrm{Cat}_2(\Lambda^n, \Lambda)^{op} \to \mathrm{Cat}_2((\Lambda^{op})^n, \Lambda^{op}),$$

and hence gives to $(\Lambda^{op}, \otimes_\Lambda^{op})$ the structure of an $\mathbb{N}$-unbiased monoidal 2-category whose associator is given in the opposite direction. The direction of the 2-associator is not changed. We say that a 2-category equipped with an $\mathbb{N}$-unbiased monoidal structure is $\mathbb{N}$-unbiased *op*-monoidal if its associator is given in the opposite direction. In this way, the opposite 2-category Λ^{op} of any monoidal 2-category Λ inherits an $\mathbb{N}$-unbiased *op*-monoidal 2-category structure. In particular, Λ^{op} is equipped with an $\mathbb{N}$-unbiased monoidal structure as soon as the associator of Λ is an isomorphism.

EXAMPLE 3.1.4. Let $\Lambda \in \mathrm{Mon}^{\mathbb{N}}_{\mathrm{Cat}_2}$. The 2-oppositization 3-functor[16] $\_^{op_2}$: $\mathrm{Cat}_2^{op_3} \to \mathrm{Cat}_2$ yields a 2-functor

$$\mathrm{Cat}_2(\Lambda^n, \Lambda)^{op_2} \to \mathrm{Cat}_2((\Lambda^{op_2})^n, \Lambda^{op_2}),$$

and hence sends the tensor product of Λ to a tensor product $(\otimes_\Lambda^n)^{op_2} : (\Lambda^{op_2})^n \to \Lambda^{op_2}$. While the direction of the associator is not changed, the direction of the 2-associator is reversed. We say that a 2-category equipped with an $\mathbb{N}$-unbiased monoidal structure is $\mathbb{N}$-unbiased op_2-monoidal if its 2-associator is given in the opposite direction. In this way, the 2-opposite 2-category Λ^{op_2} of any monoidal 2-category Λ inherits an $\mathbb{N}$-unbiased op_2-monoidal 2-category structure. In particular, Λ^{op_2} is equipped with an $\mathbb{N}$-unbiased monoidal structure as soon as the 2-associator of Λ is an isomorphism.

DEFINITION 3.1.5. Let $\Lambda, \Gamma \in \mathrm{Mon}^{\mathbb{N}}_{\mathrm{Cat}_2}$. We obtain a 2-category $\mathrm{Mon}^{\mathbb{N}}_{\mathrm{Cat}_2}(\Lambda, \Gamma)$, which gives to $\mathrm{Mon}^{\mathbb{N}}_{\mathrm{Cat}_2}$ the structure of a 3-category.

REMARK 3.1.6. Monoidal 2-categories admit a smaller presentation. The unbiased point of view consists in providing an explicit description of each of the monoidal laws on a 2-category, which makes it often easier to work with. In fact, we will see that the unbiased definition corresponds to the barycentric subdivision of the associahedra. We give the smaller definition of a monoidal 2-category.

DEFINITION 3.1.7. A monoidal 2-category $(\Lambda, \otimes_\Lambda, 1_\Lambda)$ is a 2-category Λ, equipped with

– a unit object $1_\Lambda \in \Lambda$,
– a 2-functor $\otimes_\Lambda : \Lambda \times \Lambda \to \Lambda$ called the tensor product,

---

[15]See Definition 1.2.5
[16]See Definition 1.2.6.

– a 2-natural transformation α_Λ called the associator:

$$
\begin{tikzcd}
(\Lambda\times\Lambda)\times\Lambda \arrow[r, "\alpha_\Lambda"] \arrow[dd, "\otimes_\Lambda\times\Lambda"'] & \Lambda\times(\Lambda\times\Lambda) \arrow[dr, "\Lambda\times\otimes_\Lambda"] & \\
 & \overset{\alpha_\Lambda}{\Rightarrow} & \Lambda\times\Lambda \arrow[d, "\otimes_\Lambda"] \\
\Lambda\times\Lambda \arrow[rr, "\otimes_\Lambda"'] & & \Lambda
\end{tikzcd}
$$

– a 2-natural transformation l_Λ and a 2-natural transformation r_Λ, respectively called left and right unitors:

$$
\begin{tikzcd}
\ast\times\Lambda \arrow[r, "\cong"] \arrow[d, "1_\Lambda\times\Lambda"'] \arrow[dr, phantom, "l_\Lambda\Downarrow"] & \Lambda \\
\Lambda\times\Lambda \arrow[ur, "\otimes_\Lambda"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda\times\ast \arrow[r, "\cong"] \arrow[d, "\Lambda\times 1_\Lambda"'] \arrow[dr, phantom, "r_\Lambda\Downarrow"] & \Lambda \\
\Lambda\times\Lambda \arrow[ur, "\otimes_\Lambda"'] &
\end{tikzcd}
$$

– a modification, called the 2-unitor, providing an isomorphism in the 2-category of 2-functors $[\Lambda^2, \Lambda]$:

$$
\begin{tikzcd}
(-\otimes_\Lambda 1_\Lambda)\otimes_\Lambda - \arrow[r, "\alpha^\Lambda(-{,}1_\Lambda{,}-)"] \arrow[d, "r^\Lambda(-)\otimes_\Lambda -"'] \arrow[dr, phantom, "\cong"] & -\otimes_\Lambda(1_\Lambda\otimes_\Lambda -) \arrow[dl, "-\otimes_\Lambda l^\Lambda(-)"] \\
-\otimes_\Lambda - &
\end{tikzcd}
$$

– a modification α_2^Λ, called the 2-associator, whose domain and codomain are the given by the 2-natural transformations resulting from the composition of the diagrams displayed in Figure 3 and 4, and which can we written as follows in the 2-category of 2-functors $[\Lambda^4, \Lambda]$:

$$
\begin{tikzcd}
 & ((-\otimes_\Lambda -)\otimes_\Lambda -)\otimes_\Lambda - \arrow[dl, "\alpha^\Lambda(-\otimes_\Lambda -{,}-{,}-)"'] \arrow[dr, "\alpha^\Lambda(-{,}-{,}-)\otimes_\Lambda -"] & \\
(-\otimes_\Lambda -)\otimes_\Lambda(-\otimes_\Lambda -) \arrow[d, "\alpha^\Lambda(-{,}-{,}-\otimes_\Lambda -)"'] & \overset{\alpha_2^\Lambda}{\Leftarrow} & (-\otimes_\Lambda(-\otimes_\Lambda -))\otimes_\Lambda - \arrow[d, "\alpha^\Lambda(-{,}-\otimes_\Lambda -{,}-)"] \\
-\otimes_\Lambda(-\otimes_\Lambda(-\otimes_\Lambda -)) & & -\otimes_\Lambda((-\otimes_\Lambda -)\otimes_\Lambda -) \arrow[ll, "-\otimes_\Lambda\alpha^\Lambda -{,}-{,}-"]
\end{tikzcd}
$$

– a 3-associator and which fulfill the coherence constraints expressed by the commutativity of diagrams given by the associahedra K_4

If Λ is a monoidal 2-category, then we write $\Lambda \in \mathrm{Mon}_{\mathrm{Cat}_2}$. If the associator and the 2-associator of Λ are isomorphisms, then we say that Λ is a strongly monoidal 2-category.

Example 3.1.8. Recall from Definition 1.2.22 that a 2-category Λ is cartesian if it has a terminal object $\ast_\Lambda$ and a cartesian product 2-functor

$$\times : \Lambda\times\Lambda \to \Lambda.$$

We easily check that the cartesian product is associative up to a canonical isomorphism. We accordingly obtain a monoidal 2-category structure on Λ, which we call the cartesian monoidal structure, with the terminal object of Λ as unit.

EXAMPLE 3.1.9. As expected, the 2-category of categories $(\mathrm{CAT}, \times, *)$ is (cartesian) monoidal.

REMARK 3.1.10. The definition of the 3-category $\mathrm{MON}_{\mathrm{CAT}_2}$ of monoidal 2-categories involves the use of the cartesian product of 2-categories, and hence, the monoidal structure of CAT_2 given by the cartesian product. The 3-category $(\mathrm{CAT}_2, \times)$ is monoidal in $(\mathrm{CAT}_3, \times)$, which is itself monoidal in $(\mathrm{CAT}_4, \times)$, and so on. In this way, the 3-category $\mathrm{MON}_{\mathrm{CAT}_2}$ should be regarded as the 3-category of monoids in the monoidal 4-category $(\mathrm{CAT}_3, \times)$. More generally, the framework that enables the definition of monoidal structures is provided by an infinite sequence of entangled monoids $(\mathrm{CAT}_n, \times) \in \mathrm{MON}_{\mathrm{CAT}_{n+1}}$.

OBSERVATION 3.1.11. *There is a notion of monoidality for* 3*-categories as well. When the monoidal structure is given by the cartesian product, the universality of the product ensures that the coherence conditions hold.*

The 3*-category* $\mathrm{MON}_{\mathrm{CAT}_2}$ *is monoidal. We will extensively use this monoidal structure in what follows. Let* $(\Lambda_1, \otimes_{\Lambda_1}, 1_{\Lambda_1})$ *and* $(\Lambda_2, \otimes_{\Lambda_2}, 1_{\Lambda_2})$ *be monoidal* 2*-categories. The cartesian product* $\Lambda := \Lambda_1 \times \Lambda_2$ *inherits a monoidal* 2*-category structure. Explicitly, we set* $1_\Lambda = 1_{\Lambda_1} \times 1_{\Lambda_2}$. *We let* $\otimes_\Lambda$ *be defined by the composite*

$$\otimes_\Lambda : (\Lambda_1 \times \Lambda_2) \times (\Lambda_1 \times \Lambda_2) \cong (\Lambda_1 \times \Lambda_1) \times (\Lambda_2 \times \Lambda_2) \overset{\otimes_{\Lambda_1} \times \otimes_{\Lambda_2}}{\to} \Lambda_1 \times \Lambda_2.$$

We take for α^Λ *the natural transformation defined on the objects* $\mathcal{C}, \mathcal{D}, \mathcal{E}$ *of* $\Lambda_1 \times \Lambda_2$ *by* $\alpha^\Lambda(\mathcal{C}, \mathcal{D}, \mathcal{E}) = \alpha^{\Lambda_1}(\mathcal{C}_1, \mathcal{D}_1, \mathcal{E}_1) \times \alpha^{\Lambda_2}(\mathcal{C}_2, \mathcal{D}_2, \mathcal{E}_2)$, *so that*

$$\begin{array}{c}
(\mathcal{C} \otimes_\Lambda \mathcal{D}) \otimes_\Lambda \mathcal{E} = ((\mathcal{C}_1 \otimes_{\Lambda_1} \mathcal{D}_1) \otimes_{\Lambda_1} \mathcal{E}_1, (\mathcal{C}_2 \otimes_{\Lambda_2} \mathcal{D}_2) \otimes_{\Lambda_2} \mathcal{E}_2) \\
\Big\downarrow{\scriptstyle \alpha^{\Lambda_1}(\mathcal{C}_1, \mathcal{D}_1, \mathcal{E}_1) \times \alpha^{\Lambda_2}(\mathcal{C}_2, \mathcal{D}_2, \mathcal{E}_2)} \\
\mathcal{C} \otimes_\Lambda (\mathcal{D} \otimes_\Lambda \mathcal{E}) = (\mathcal{C}_1 \otimes_{\Lambda_1} (\mathcal{D}_1 \otimes_{\Lambda_1} \mathcal{E}_1), \mathcal{C}_2 \otimes_{\Lambda_2} (\mathcal{D}_2 \otimes_{\Lambda_2} \mathcal{E}_2)).
\end{array}$$

We wrote $\mathcal{B}_i \in \Lambda_i$ *for the projection on* Λ_i *of any object* $\mathcal{B} \in \Lambda$. *The left and right unitors* l^Λ, r^Λ *are given by the composite*

$$l_\Lambda : 1_\Lambda \otimes_\Lambda \_ \xrightarrow{\cong} (1_{\Lambda_1} \times 1_{\Lambda_2}) \otimes_\Lambda \_ \xrightarrow{\cong} (1_{\Lambda_1} \otimes_{\Lambda_1} \_) \times (1_{\Lambda_2} \otimes_{\Lambda_2} \_) \xrightarrow{l_{\Lambda_1} \times l_{\Lambda_2}} \Lambda$$

and

$$r_\Lambda : - \otimes_\Lambda 1_\Lambda \xrightarrow{\cong} - \otimes_\Lambda (1_{\Lambda_1} \times 1_{\Lambda_2}) \xrightarrow{\cong} (- \otimes_{\Lambda_1} 1_{\Lambda_1}) \times (- \otimes_{\Lambda_2} 1_{\Lambda_2}) \xrightarrow{r_{\Lambda_1} \times r_{\Lambda_2}} \Lambda.$$

We obtain a monoidal 2*-category* $(\Lambda, \otimes_\Lambda, 1_\Lambda)$, *which moreover represents the cartesian product of* Λ_1 *and* Λ_2 *in the* 3*-category of monoidal* 2*-categories. Consequently,* $(\mathrm{MON}_{\mathrm{CAT}_2}, \times, *)$ *is a cartesian monoidal* 3*-category.*

Let $(\Lambda, \otimes_\Lambda, 1_\Lambda)$ be a monoidal 2-category. The 3-category of 2-categories is cartesian closed, so that the tensor product 2-functor $\otimes_\Lambda : \Lambda \times \Lambda \to \Lambda$ may be regarded as 2-functor $\Lambda \to [\Lambda, \Lambda]_{\mathrm{CAT}_2}$. Therefore, each object $\mathcal{C}$ of Λ yields to a 2-functor

$$\mathcal{C} \otimes_\Lambda \_ \ : \Lambda \to \Lambda.$$

DEFINITION 3.1.12. Let $(\Lambda, \otimes_\Lambda, 1_\Lambda)$ be a monoidal 2-category. We say that Λ is closed respect to $\otimes_\Lambda$ if is equipped with an internal hom 2-functor

$$[-, -]_\Lambda : \Lambda^{op} \times \Lambda \to \Lambda$$

such that each object $\mathcal{C}$ of Λ yields an adjunction

$$\begin{tikzcd} \mathcal{C} \otimes_\Lambda - \ : \Lambda \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \Lambda : [\mathcal{C}, -]_\Lambda . \arrow[l, bend left] \end{tikzcd}$$

Definition 3.1.13. Let Γ, Λ be closed monoidal 2-categories. We say that a 2-functor $F : \Gamma \to \Gamma$ is closed if it is equipped with a 2-natural transformation $[-,-]_F$:

$$\begin{tikzcd} \Gamma^{op} \times \Gamma \arrow[r, "F^{op} \times F"] \arrow[d, "{[-,-]_\Gamma}"'] & \Lambda^{op} \times \Lambda \arrow[d, "{[-,-]_\Lambda}"] \\ \Gamma \arrow[r, "F"'] \arrow[ur, phantom, "\Nearrow {[-,-]_F}"] & \Psi . \end{tikzcd}$$

The isomorphism

$$[\Gamma^{op} \times \Gamma, \Lambda] \left(F [-,-]_\Gamma , [F-, F-]_\Lambda\right) \xrightarrow{\cong} \int^{\Gamma \times \Gamma^{op}} \Lambda \left(F [-,-]_\Gamma , [F-, F-]_\Lambda\right)$$

then yields morphisms $[X,Y]_F : F[X,Y]_\Gamma \to [FX, FY]_\Psi$ in Ψ which are natural in $X, Y : * \to \Gamma$.

Example 3.1.14. A cartesian 2-category Λ is closed for its cartesian monoidal structure if and only if it is cartesian closed in the sense of Definition 1.2.27.

We have the following extra observation

Proposition 3.1.15. *Let Λ be a cartesian closed 2-category. If Λ is also cocartesian, then it is necessarily bicartesian. We say that Λ is bicartesian closed.*

Proof. Let $\Lambda \in \mathrm{Cat}_2$ be cartesian closed and cocartesian. Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}, \mathcal{E}$ be objects of Λ. We have the following natural isomorphisms

$$\begin{aligned} \Lambda \left((\mathcal{C}_1 \coprod \mathcal{C}_2) \times \mathcal{D}, \mathcal{E} \right) &\cong \Lambda \left(\mathcal{C}_1 \coprod \mathcal{C}_2, [\mathcal{D}, \mathcal{E}]_\Lambda \right) \\ &\cong \Lambda \left(\mathcal{C}_1, [\mathcal{D}, \mathcal{E}]_\Lambda \right) \times \Lambda \left(\mathcal{C}_2, [\mathcal{D}, \mathcal{E}]_\Lambda \right) \\ &\cong \Lambda \left(\mathcal{C}_1 \times \mathcal{D}, \mathcal{E} \right) \times \left(\mathcal{C}_2 \times \mathcal{D}, \mathcal{E} \right) \\ &\cong \Lambda \left((\mathcal{C}_1 \times \mathcal{D}) \coprod (\mathcal{C}_2 \times \mathcal{D}), \mathcal{E} \right) \end{aligned}$$

□

3.2. Monoids in a monoidal 2-category. We can now give the definition of monoidal objects in an ambient monoidal 2-category.

Definition 3.2.1. Let $(\Lambda, \otimes_\Lambda, 1_\Lambda)$ be a monoidal 2-category. We let the 2-category of monoids in Λ be defined as the 2-category of monoidal 2-functors

$$\mathrm{Mon}_{(\Lambda, \otimes_\Lambda)} := \mathrm{Mon}_{\mathrm{Cat}_2} \left(*, (\Lambda, \otimes_\Lambda)\right).$$

Explicitly, a monoid $(\mathcal{C}, \otimes_\mathcal{C}, 1_\mathcal{C})$ in Λ consists in the data of

- an object $\mathcal{C}$ of Λ,
- a morphism $1_\mathcal{C} : 1_\Lambda \to \mathcal{C}$ in Λ called the unit of $\mathcal{C}$,
- a morphism $\otimes_\mathcal{C} : \mathcal{C} \otimes_\Lambda \mathcal{C} \to \mathcal{C}$ in Λ called the tensor product of $\mathcal{C}$,

– a 2-morphism $\alpha^{\mathcal{C}}$ called the associator:

$$\begin{tikzcd}
(\mathcal{C} \otimes_\Lambda \mathcal{C}) \otimes_\Lambda \mathcal{C} \arrow[r, "\alpha_\Lambda"] \arrow[dd, "\otimes_{\mathcal{C}} \otimes_\Lambda \mathcal{C}"'] & \mathcal{C} \otimes_\Lambda (\mathcal{C} \otimes_\Lambda \mathcal{C}) \arrow[dr, "\mathcal{C} \otimes_\Lambda \otimes_{\mathcal{C}}"] & \\
 & \arrow[phantom, "\overset{\alpha_{\mathcal{C}}}{\Nearrow}"] & \mathcal{C} \otimes_\Lambda \mathcal{C} \arrow[d, "\otimes_{\mathcal{C}}"] \\
\mathcal{C} \otimes_\Lambda \mathcal{C} \arrow[rr, "\otimes_{\mathcal{C}}"'] & & \mathcal{C}
\end{tikzcd}$$

– a 2-morphism $l_{\mathcal{C}}$ and a 2-morphism $r_{\mathcal{C}}$, respectively called left and right unitors:

$$\begin{tikzcd}
1_\Lambda \otimes_\Lambda \mathcal{C} \arrow[r, "\cong"] \arrow[d, "1_{\mathcal{C}} \otimes_\Lambda \mathcal{C}"'] \arrow[r, phantom, shift right=3, "l_{\mathcal{C}} \Downarrow"] & \mathcal{C} \\
\mathcal{C} \otimes_\Lambda \mathcal{C} \arrow[ur, "\otimes_{\mathcal{C}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \otimes_\Lambda 1_\Lambda \arrow[r, "\cong"] \arrow[d, "\mathcal{C} \otimes_\Lambda 1_{\mathcal{C}}"'] \arrow[r, phantom, shift right=3, "r_{\mathcal{C}} \Downarrow"] & \mathcal{C} \\
\mathcal{C} \otimes_\Lambda \mathcal{C} \arrow[ur, "\otimes_{\mathcal{C}}"'] &
\end{tikzcd}$$

and which fulfils the coherence constraints expressed by the commutativity of the diagrams of Figure 5 and 6.

Remark 3.2.2. We can also define monoidal 3-categories and monoidal 3-functors between them, so that the 2-category of monoidal 2-categories in the sense of Definition 3.1.1 satisfies

$$\textsc{Mon}_{\textsc{Cat}_3}(*, \textsc{Cat}_2) \cong \textsc{Cat}_3(*, \textsc{Mon}_{\textsc{Cat}_2}) \cong \textsc{Mon}_{\textsc{Cat}_2}.$$

For any monoidal 2-category $(\Lambda, \otimes_\Lambda)$, the 2-category of monoids in Λ satisfies

$$\begin{aligned}
\textsc{Mon}_{(\Lambda, \otimes_\Lambda)} &\cong \textsc{Mon}_{\textsc{Cat}_3}(*, \textsc{Cat}_2)(*, (\Lambda, \otimes_\Lambda)) \cong \textsc{Cat}_3(*, \textsc{Mon}_{\textsc{Cat}_2})(*, (\Lambda, \otimes_\Lambda)) \\
&\cong \textsc{Mon}_{\textsc{Cat}_2}(*, (\Lambda, \otimes_\Lambda)) \cong \textsc{Cat}_2(*, \textsc{Mon}_{(\Lambda, \otimes_\Lambda)}).
\end{aligned}$$

Example 3.2.3. The cartesian product gives to the category of small categories **cat** the structure of a monoid in the monoidal 2-category of categories $(\textsc{Cat}, \times)$.

Definition 3.2.4. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}), (\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}}) \in \textsc{Mon}_{(\Lambda, \otimes_\Lambda)}$. By definition, we have

$$\textsc{Mon}_{(\Lambda, \otimes)}(\mathcal{C}, \mathcal{D}) = \textsc{Mon}_{\textsc{Cat}_2}(*, (\Lambda, \otimes_\Lambda))(\mathcal{C}, \mathcal{D}).$$

A monoidal morphisms from $\mathcal{C}$ to $\mathcal{D}$ in Λ is an element of this category. Explicitly, a monoidal morphism from $\mathcal{C}$ to $\mathcal{D}$ is a morphism $F : \mathcal{C} \to \mathcal{D}$ in Λ equipped with a 2-morphisms $\otimes_F$, called the tensor product, and a 2-morphism 1_F, called the unit:

$$\begin{tikzcd}
\mathcal{C} \otimes_\Lambda \mathcal{C} \arrow[r, "\otimes_{\mathcal{C}}"] \arrow[d, "F \otimes_\Lambda F"'] \arrow[dr, phantom, "\overset{\otimes_F}{\Nearrow}"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D} \otimes_\Lambda \mathcal{D} \arrow[r, "\otimes_{\mathcal{D}}"] & \mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
1_\Lambda \arrow[r, "1_{\mathcal{C}}"] \arrow[dr, "1_{\mathcal{D}}"'] & \mathcal{C} \arrow[d, "F"] \arrow[d, phantom, shift right=3, "\Nearrow 1_F"] \\
 & \mathcal{D}
\end{tikzcd}$$

such that

– the following diagram commutes in the category $\Lambda(\mathcal{C}^{\otimes^3_\Lambda}, \mathcal{C})$:

$$\begin{tikzcd}
(F \otimes_{\mathcal{D}} F) \otimes_{\mathcal{D}} F \arrow[rr, "\alpha^{\mathcal{D}}_{F_\,,F_\,,F_}"] \arrow[d, "\otimes_F \otimes_{\mathcal{D}} _"'] & & F \otimes_{\mathcal{D}} (F \otimes_{\mathcal{D}} F) \arrow[d, "_ \otimes_{\mathcal{D}} \otimes_F"] \\
F(_ \otimes_{\mathcal{C}} _) \otimes_{\mathcal{D}} F \arrow[d, "\otimes_F(_ \otimes_{\mathcal{C}} _\,,_)"'] & & _ \otimes_{\mathcal{D}} F(_ \otimes_{\mathcal{C}} _) \arrow[d, "\otimes_F(_\,,_ \otimes_{\mathcal{C}} _)"] \\
F((_ \otimes_{\mathcal{C}} _) \otimes_{\mathcal{C}} _) \arrow[rr, "F\alpha^{\mathcal{C}}_{_\,,_\,,_}"'] & & F(_ \otimes_{\mathcal{C}} (_ \otimes_{\mathcal{C}} _))
\end{tikzcd}$$

– the following diagrams commute in the category $\Lambda(\mathcal{C}, \mathcal{D})$:

$$\begin{tikzcd}
1_{\mathcal{D}} \otimes_{\mathcal{D}} F \arrow[r, "1_F \otimes_{\mathcal{D}} F"] \arrow[d, "l_{\mathcal{D}}"'] & F1_{\mathcal{C}} \otimes_{\mathcal{D}} F \arrow[d, "1_{\mathcal{C}} \otimes_F _"] \\
F & F(1_{\mathcal{C}} \otimes_{\mathcal{C}} _) \arrow[l, "Fl_{\mathcal{C}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F \otimes_{\mathcal{D}} 1_{\mathcal{D}} \arrow[r, "F \otimes_{\mathcal{D}} 1_F"] \arrow[d, "r_{\mathcal{D}}"'] & F \otimes_{\mathcal{D}} F1_{\mathcal{D}} \arrow[d, "_ \otimes_F 1_{\mathcal{D}}"] \\
F \arrow[r, "Fr_{\mathcal{C}}"'] & F(_ \otimes_{\mathcal{C}} 1_{\mathcal{C}})
\end{tikzcd}$$

The diagrams above involve some identifications between isomorphic objects in Λ. The accurate diagram for associativity can be found in Figure 8.

Definition 3.2.5. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}), (\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be monoids in $(\Lambda, \otimes_\Lambda)$ and let $F, G \in \mathrm{Mon}_{(\Lambda, \otimes_\Lambda)}(\mathcal{C}, \mathcal{D})$. By definition, the set of monoidal 2-morphisms from F to G is given by

$$\mathrm{Mon}_{(\Lambda, \otimes_\Lambda)}(\mathcal{C}, \mathcal{D})(F, G) = \mathrm{Mon}_{\mathrm{Cat}_2}(*, (\Lambda, \otimes_\Lambda))(\mathcal{C}, \mathcal{D})(F, G).$$

Explicitly, a monoidal 2-morphism $\alpha : F \Rightarrow G$ is a 2-morphism in Λ such that following diagrams commute:

Associativity

$$\begin{tikzcd}
\mathcal{C} \otimes_\Lambda \mathcal{C} \arrow[r, "\otimes_{\mathcal{C}}"] \arrow[d, "F \otimes F"'] \arrow[dr, phantom, "\overset{\otimes_F}{\nearrow}"] & \mathcal{C} \arrow[d, bend right=40, "F"'] \arrow[d, bend left=40, "G"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\
\mathcal{D} \otimes_\Lambda \mathcal{D} \arrow[r, "\otimes_{\mathcal{D}}"'] & \mathcal{D}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C} \otimes_\Lambda \mathcal{C} \arrow[r, "\otimes_{\mathcal{C}}"] \arrow[d, bend right=40, "F \otimes F"'] \arrow[d, bend left=40, "G \otimes G"] \arrow[d, phantom, "\overset{\alpha \otimes \alpha}{\Rightarrow}"] \arrow[dr, phantom, "\underset{\otimes_G}{\nearrow}"] & \mathcal{C} \arrow[d, "G"] \\
\mathcal{D} \otimes_\Lambda \mathcal{D} \arrow[r, "\otimes_{\mathcal{D}}"'] & \mathcal{D}
\end{tikzcd}$$

Unitality

$$\begin{tikzcd}
1_\Lambda \arrow[r, "1_{\mathcal{C}}"] \arrow[dr, "1_{\mathcal{D}}"'] \arrow[r, phantom, shift right=3, "\overset{1_F}{\nearrow}"] & \mathcal{C} \arrow[d, bend right=40, "F"'] \arrow[d, bend left=40, "G"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\
& \mathcal{D}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
1_\Lambda \arrow[r, "1_{\mathcal{C}}"] \arrow[dr, "1_{\mathcal{D}}"'] \arrow[r, phantom, shift right=3, "\overset{1_G}{\nearrow}"] & \mathcal{C} \arrow[d, "G"] \\
& \mathcal{D}
\end{tikzcd}$$

Remark 3.2.6. Recall from Example 3.1.11 that the 3-category of monoidal 2-categories is cartesian, so that $\mathrm{Mon}(\mathrm{Cat}_2, \times)$ is monoidal. Let $(\Lambda, \otimes_\Lambda)$ and $(\Gamma, \otimes_\Gamma)$ be monoidal 2-categories. The cartesian product 2-category $\Lambda \times \Gamma$ inherits a monoidal 2-category structure $(\Lambda \times \Gamma, \otimes_{\Lambda \times \Lambda})$. We have an isomorphism of 2-categories

$$\mathrm{Mon}_{(\Lambda \times \Gamma, \otimes_{\Lambda \times \Gamma})} \cong \mathrm{Mon}_{(\Lambda, \otimes_\Lambda)} \times \mathrm{Mon}_{(\Gamma, \otimes_\Gamma)}.$$

EXAMPLE 3.2.7. The usual 2-category of monoidal categories can be obtained as the 2-category of monoids in the monoidal 2-category of categories.

EXAMPLE 3.2.8. The unit object $1_\Lambda : * \to \Lambda$ of Λ has the structure of a monoid in $(\Lambda, \otimes_\Lambda)$.

If Λ has a terminal object, then it also has the structure of a monoid in Λ. The corresponding object in the 2-category of monoids in Λ is terminal in $\mathrm{Mon}_{(\Lambda,\otimes_\Lambda)}$.

DEFINITION 3.2.9. Let $\mathcal{C}$ be a monoid in Λ. Let $\mathrm{Mon}_{(\Lambda,\otimes_\Lambda)}(\mathcal{C}, \otimes_\mathcal{C})$ be the category of monoids internal to $\mathcal{C}$, given by

$$\mathrm{Mon}_{(\Lambda,\otimes_\Lambda)}(\mathcal{C}, \otimes_\mathcal{C}) := \mathrm{Mon}_{(\Lambda,\otimes_\Lambda)}(*_\Lambda, (\mathcal{C}, \otimes_\mathcal{C})).$$

Thus, a monoid internal to $\mathcal{C}$ is, by definition, a monoidal morphism $X : *_\Lambda \to \mathcal{C}$, and explicitly consists in

- a morphism $X : *_\Lambda \to \mathcal{C}$ in Λ
- a 2-morphism 1_X, called the unit:

$$\begin{tikzcd} *_\Lambda \arrow[r] \arrow[dr, "1_\mathcal{C}"'] & *_\Lambda \arrow[d, "X"] \arrow[dl, phantom, "\Leftarrow 1_X"] \\ & \mathcal{C} \end{tikzcd}$$

- a 2-morphism $\otimes_X$, called the tensor product:

$$\begin{tikzcd} *_\Lambda \arrow[r, "X \otimes_\Lambda X"] \arrow[dr, "X"'] & \mathcal{C} \otimes_\Lambda \mathcal{C} \arrow[d, "\otimes_\mathcal{C}"] \arrow[dl, phantom, "\Leftarrow \otimes_X"] \\ & \mathcal{C} \end{tikzcd}$$

such that:

- The following diagrams commute in the category $\Lambda(*_\Lambda, \mathcal{C})$:

$$\begin{tikzcd} 1_\mathcal{C} \otimes_\mathcal{C} X \arrow[r, "1_X \otimes_\mathcal{C} X"] \arrow[d, "l_\mathcal{C}"'] & X \otimes_\mathcal{C} X \arrow[d, "\otimes_X"] \\ X & X \arrow[l, "X"] \end{tikzcd} \qquad \begin{tikzcd} X \otimes_\mathcal{C} 1_\mathcal{C} \arrow[r, "X \otimes_\mathcal{C} 1_X"] \arrow[d, "r_\mathcal{C}"'] & X \otimes_\mathcal{C} X \arrow[d, "\otimes_X"] \\ X \arrow[r, "X"'] & X \end{tikzcd}$$

- The diagram of Figure 9 commutes. The 3-dimensional diagram may be written internal to the object $\mathcal{C}$ of Λ, so that we retrieve the usual pentagon condition in the category $\Lambda(*, \mathcal{C})$:

$$\begin{tikzcd} (X \otimes_\mathcal{C} X) \otimes_\mathcal{C} X \arrow[r, "\alpha_\mathcal{C}"] \arrow[dd, "\otimes_X \otimes_\mathcal{C} X"'] & X \otimes_\mathcal{C} (X \otimes_\mathcal{C} X) \arrow[dr, "X \otimes_\mathcal{C} \otimes_X"] & \\ & & X \otimes_\mathcal{C} X \arrow[d, "\otimes_X"] \\ X \otimes_\mathcal{C} X \arrow[rr, "\otimes_X"'] & & X. \end{tikzcd}$$

Let X, Y be monoids internal to $\mathcal{C}$. By definition, a monoidal morphism $F : X \to Y$ is a monoidal 2-morphism in Λ:

$$\begin{tikzcd} *_\Lambda \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\Downarrow F"] & \mathcal{C}. \end{tikzcd}$$

Explicitly, a monoidal morphism $F : X \to Y$ is a morphism in $\Lambda(*_\Lambda, \mathcal{C})$, which satisfies the coherence conditions given by the commutativity of the following diagrams:

Associativity

$$
\begin{tikzcd}
*_\Lambda \otimes_\Lambda *_\Lambda \arrow[r, "\otimes_{*_\Lambda}"] \arrow[d, "X\otimes X"'] \arrow[dr, phantom, "\Nearrow \otimes_X"] & *_\Lambda \arrow[d, bend right, "X"'] \arrow[d, bend left, "Y"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\
\mathcal{C} \otimes_\Lambda \mathcal{C} \arrow[r, "\otimes_\mathcal{C}"'] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
*_\Lambda \otimes_\Lambda *_\Lambda \arrow[r, "\otimes_{*_\Lambda}"] \arrow[d, bend right, "X\otimes X"'] \arrow[d, bend left, "Y\otimes Y"] \arrow[d, phantom, "\overset{\alpha\otimes\alpha}{\Rightarrow}"] \arrow[dr, phantom, "\Nearrow \otimes_Y"] & *_\Lambda \arrow[d, "Y"] \\
\mathcal{C} \otimes_\Lambda \mathcal{C} \arrow[r, "\otimes_\mathcal{C}"'] & \mathcal{C}
\end{tikzcd}
$$

Unitality

$$
\begin{tikzcd}
1_\Lambda \arrow[r, "1_{*_\Lambda}"] \arrow[dr, "1_\mathcal{C}"'] \arrow[dr, phantom, "\Nearrow 1_X"] & *_\Lambda \arrow[d, bend right, "X"'] \arrow[d, bend left, "Y"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\
& \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
1_\Lambda \arrow[r, "1_{*_\Lambda}"] \arrow[dr, "1_\mathcal{C}"'] \arrow[dr, phantom, "\Nearrow 1_Y"] & *_\Lambda \arrow[d, "Y"] \\
& \mathcal{C}
\end{tikzcd}
$$

We can write those diagrams internally to $\mathcal{C}$, in the category $\Lambda(*_\Lambda, \mathcal{C})$. We obtain:

$$
\begin{tikzcd}
X \otimes_\mathcal{C} X \arrow[r, "F\otimes_\mathcal{C} F"] \arrow[d, "\otimes_X"'] & Y \otimes_\mathcal{C} Y \arrow[d, "\otimes_Y"] \\
X \arrow[r, "F"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
1_\mathcal{C} \arrow[r, "1_X"] \arrow[dr, "1_Y"'] & X \arrow[d, "F"] \\
& Y.
\end{tikzcd}
$$

Example 3.2.10. A monoid in the monoidal 2-category of categories $(\textsc{Cat}, \times, *)$ is a monoidal category. With this definition, a monoid internal to $\mathcal{C}$ corresponds to the usual notion of a monoid in a monoidal category.

Definition 3.2.11. Recall from Example 3.2.3 that the cartesian product gives to the category of small categories **cat** the structure of a monoidal category. Let

$$\mathbf{moncat} := \textsc{mon}_{(\textsc{Cat},\times)}(\mathbf{cat}, \times).$$

According to Definition 3.2.9, an object of **moncat** is a small category equipped with a monoidal structure which is strictly associative and strictly unital. We say that **moncat** is the category of strictly monoidal categories.

Definition 3.2.12. Recall from Proposition 1.3.5 that the 2-category of small categories **Cat** is monoidal, so that $(\mathbf{Cat}, \times) \in \textsc{Mon}_{\textsc{Cat}_2}$. Let $\mathbf{MonCat} := \textsc{Mon}_{\textsc{Cat}_2}(\mathbf{Cat}, \times)$ be the 2-category of monoids in **Cat**.

Let $\mathcal{C}, \mathcal{D} \in \mathbf{MonCat}$. We say that a morphism $F : \mathcal{C} \to \mathcal{D}$ in **MonCat** is strictly unital if the diagram

$$
\begin{tikzcd}
1_\Lambda \arrow[r, "1_\mathcal{C}"] \arrow[dr, "1_\mathcal{D}"'] \arrow[dr, phantom, "\Nearrow 1_F"] & \mathcal{C} \arrow[d, "F"] \\
& \mathcal{D}
\end{tikzcd}
$$

commutes strictly in **Cat**, so that $F1_\mathcal{C} = 1_\mathcal{D}$ in the set $\mathcal{D}_0$. If $\mathcal{C}, \mathcal{D} \in \mathbf{MonCat}$, then we let $\mathbf{MonCat}^\bullet(\mathcal{C}, \mathcal{D})$ be the full subcategory of $\mathbf{Mon}_\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ whose objects are strictly unital monoidal morphisms. We set $\mathcal{C} \in \mathbf{MonCat}^\bullet \Leftrightarrow \mathcal{C} \in \mathbf{MonCat}$, and obtain a 2-category whose objects are monoidal small categories, whose morphisms are given by monoidal morphisms which are strictly unital, and whose 2-morphisms are monoidal 2-morphisms.

Definition 3.2.13. Let **Moncat** $\in$ Cat be defined as follows.

– We set $\mathcal{C} \in \mathbf{Moncat} \Leftrightarrow \mathcal{C} \in \mathbf{moncat}$.
– Let $\mathcal{C}, \mathcal{D} \in \mathbf{moncat}$, we let $\mathbf{Moncat}(\mathcal{C}, \mathcal{D}) = \mathbf{MonCat}^{\bullet}(\mathcal{C}, \mathcal{D})_0$, where $\mathbf{MonCat}^{\bullet}(\mathcal{C}, \mathcal{D})_0$ is the underlying set of the small category $\mathbf{MonCat}^{\bullet}(\mathcal{C}, \mathcal{D})$.

Consequently, the objects of **Moncat** are strict monoidal categories in the sense of Definition 3.2.11. If $\mathcal{C}$ and $\mathcal{D}$ are strictly monoidal categories, then a morphism $F \in \mathbf{Moncat}(\mathcal{C}, \mathcal{D})$ is a monoidal morphism $F \in \mathrm{Mon}_{(\mathbf{Cat}, \times)}(\mathcal{C}, \mathcal{D})$ in the sense of Definition 3.2.4, with the additional condition of being strictly unital.

Proposition 3.2.14. *Let $(\Lambda, \otimes_\Lambda), (\Gamma, \otimes_\Gamma) \in \mathrm{Mon}_{\mathrm{Cat}_2}$. Any morphism $F \in \mathrm{Mon}_{\mathrm{Cat}_2}((\Lambda, \otimes_\Lambda), (\Gamma, \otimes_\Gamma))$ yields a 2-functor*

$$\mathrm{Mon}_F : \mathrm{Mon}_{(\Lambda, \otimes_\Lambda)} \to \mathrm{Mon}_{(\Gamma, \otimes_\Gamma)}.$$

The monoidal 2-category structure on the terminal 2-category $*$ yields a 2-functor $\mathrm{Mon}_{\mathrm{Cat}_2}(*, \_\,) : \mathrm{Mon}_{\mathrm{Cat}_2} \to \mathrm{Cat}_2$. For any monoidal 2-category Λ, the 2-category $\mathrm{Mon}_{\mathrm{Cat}_2}(*, \Lambda)$ corresponds, by definition, to the 2-category of monoids in Λ. Moreover, for any monoidal 2-functor $F \in \mathrm{Mon}_{\mathrm{Cat}_2}((\Lambda, \otimes_\Lambda), (\Gamma, \otimes_\Gamma))$, the 2-functor $\mathrm{Mon}_{\mathrm{Cat}_2}(*, F)$ also corresponds to the 2-functor Mon_F. Consequently, we write

$$\mathrm{Mon} : \mathrm{Mon}_{\mathrm{Cat}_2} \to \mathrm{Cat}_2$$

for the 3-functor $\mathrm{Mon}_{\mathrm{Cat}_2}(*, \_\,)$. Moreover, we deduce from Remark 3.2.6 that the 3-functor Mon *is monoidal.*

Proof. This proposition follows from straightforward verifications. $\square$

Remark 3.2.15. Suppose Λ is a monoidal 2-category and let $\mathcal{C}$ and $\mathcal{D}$ be monoids in Λ. If $F : \mathcal{C} \to \mathcal{D}$ is a monoidal morphism in Λ, then F takes monoids internal to $\mathcal{C}$ to monoids internal to $\mathcal{D}$. Therefore, F induces a functor between the categories of monoids

$$F : \mathrm{Mon}_{(\mathcal{C}, \otimes_\mathcal{C})} \to \mathrm{Mon}_{(\mathcal{D}, \otimes_\mathcal{D})}.$$

Let $(\Lambda, \otimes_\Lambda, 1_\Lambda)$ be a monoidal 2-category. We assume that Λ is cartesian, so that $(\Lambda, \times, *_\Lambda)$ has the structure of a monoidal 2-category too. We show that the cartesian monoidal structure on Λ is necessarily compatible with its former monoidal structure. In particular, any monoidal 2-category which is also cartesian naturally possesses a 2-monoidal 2-category structure.

Lemma 3.2.16. *The cartesian product is compatible with any monoidal structure $(\Lambda, \otimes_\Lambda)$ on a 2-category Λ, in the sense that the 2-functor*

$$\times : (\Lambda \times \Lambda, \otimes_{\Lambda \times \Lambda}) \to (\Lambda, \otimes_\Lambda).$$

is naturally equipped with a monoidal structure.

Proof. We show there is a natural transformation

$$\begin{tikzcd}
(\Lambda \times \Lambda) \times (\Lambda \times \Lambda) \arrow[r, "(-\times -)\times(-\times -)"] \arrow[d, "\otimes_{\Lambda\times\Lambda}"'] \arrow[dr, phantom, "\swarrow \mu"] & \Lambda \times \Lambda \arrow[d, "\otimes_\Lambda"] \\
\Lambda \times \Lambda \arrow[r, "-\times -"'] & \Lambda
\end{tikzcd}.$$

Let $(\mathcal{C}_1, \mathcal{D}_1), (\mathcal{C}_2, \mathcal{D}_2) \in (\Lambda \times \Lambda) \times (\Lambda \times \Lambda)$. The canonical projections

– $p_1 : \mathcal{C}_1 \times \mathcal{D}_1 \to \mathcal{C}_1, p_2 : \mathcal{C}_2 \times \mathcal{D}_2 \to \mathcal{C}_2$,
– $q_1 : \mathcal{C}_1 \times \mathcal{D}_1 \to \mathcal{D}_1, q_2 : \mathcal{C}_2 \times \mathcal{D}_2 \to \mathcal{D}_2$

yield morphisms

– $p_1 \otimes_\Lambda p_2 : (\mathcal{C}_1 \times \mathcal{D}_1) \otimes_\Lambda (\mathcal{C}_2 \times \mathcal{D}_2) \to \mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2$,
– $q_1 \otimes_\Lambda q_2 : (\mathcal{C}_1 \times \mathcal{D}_1) \otimes_\Lambda (\mathcal{C}_2 \times \mathcal{D}_2) \to \mathcal{D}_1 \otimes_\Lambda \mathcal{D}_2$.

We obtain interchange morphisms

$$\mu_{\mathcal{C}_1,\mathcal{D}_1,\mathcal{C}_2,\mathcal{D}_2} : (\mathcal{C}_1 \times \mathcal{D}_1) \otimes_\Lambda (\mathcal{C}_2 \times \mathcal{D}_2) \to \mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2 \times \mathcal{D}_1 \otimes_\Lambda \mathcal{D}_2$$

which are natural in the objects $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_1, \mathcal{D}_2$ of Λ. The interchange thus obtained readily satisfies the unit and associativity constraints due to the universality of the cartesian product. □

Corollary 3.2.17. *Let $(\Lambda, \otimes_\Lambda, 1_\Lambda)$ be a monoidal 2-category. The 2-category $\mathrm{Mon}_{(\Lambda,\otimes_\Lambda,1_\Lambda)}$ is cartesian as soon as Λ is. The cartesian, product $X \times Y$ of monoids in Λ is represented by the image of (X, Y) by the 2-functor*

$$\times : \mathrm{Mon}_{(\Lambda,\otimes_\Lambda,1_\Lambda)} \times \mathrm{Mon}_{(\Lambda,\otimes_\Lambda,1_\Lambda)} \to \mathrm{Mon}_{(\Lambda,\otimes_\Lambda,1_\Lambda)}$$

induced by the cartesian product 2-functor on Λ.

Example 3.2.18. Let $(\Lambda, \otimes_\Lambda)$ be a symmetric monoidal 2-category. The symmetric structure provides the tensor product $\otimes_\Lambda$, the associator and the 2-associator of Λ with the structure of a monoidal 2-functor with respect to its own monoidal structure. In particular, the tensor product yields a 2-functor

$$\otimes : \mathrm{Mon}_{(\Lambda,\otimes_\Lambda)} \times \mathrm{Mon}_{(\Lambda,\otimes_\Lambda)} \cong \mathrm{Mon}_{(\Lambda\times\Lambda,\otimes_{\Lambda\times\Lambda})} \to \mathrm{Mon}_{(\Lambda,\otimes_\Lambda)}.$$

We obtain a monoidal 2-category structure on $\mathrm{Mon}_{(\Lambda,\otimes_\Lambda)}$, and hence, a 2-category $\mathrm{Mon}_{(\mathrm{Mon}_{(\Lambda,\otimes_\Lambda)},\otimes_\Lambda)}$. Note that the monoidal 2-category $\big(\mathrm{Mon}_{(\Lambda,\otimes_\Lambda)}, \otimes_\Lambda\big)$ is also symmetric, so that we can iterate this construction.

Definition 3.2.19. Let $(\Lambda, \otimes_1)$ be a monoidal 2-category and suppose that it is equipped with an other monoidal 2-category structure $(\Lambda, \otimes_2)$ such that both the unit, the tensor product, the associator and the 2-associator are monoidal morphisms. By Proposition **??**, the 3-functor Mon is monoidal, and hence takes monoids to monoids. We obtain a monoidal 2-category structure $(\mathrm{Mon}_{(\Lambda,\otimes_1)}, \otimes_2)$ on the 2-category $\mathrm{Mon}_{(\Lambda,\otimes_1)}$. We let the 2-category of 2-fold monoids in Λ be defined as

$$\mathrm{Mon}^2_{(\Lambda,\otimes_1,\otimes_2)} := \mathrm{Mon}_{(\mathrm{Mon}_{(\Lambda,\otimes_1)},\otimes_2)}.$$

We recursively obtain a 2-category of p-fold monoid in Λ as soon as Λ is equipped with p monoidal structures such that the r-th monoidal structure on Λ has the structure of a morphism in $\mathrm{Mon}^{r-1}_{(\Lambda,\otimes_1,\ldots,\otimes_{r-1})}$ for all $r = 1, \ldots, n$. In this case, we obtain a 2-category

$$\mathrm{Mon}^n_{(\Lambda,\otimes_1,\ldots,\otimes_n)} = \mathrm{Mon}_{(\mathrm{Mon}^{n-1}_{(\Lambda,\otimes_1,\ldots,\otimes_{n-1})},\otimes_n)}.$$

Remark 3.2.20. The compatibility of the monoidal structures are ordered because monoidal morphisms have a direction. Hence the 2-categories $\mathrm{Mon}^2_{(\Lambda,\otimes_j,\otimes_i)}$ does not have the same meaning as $\mathrm{Mon}^2_{(\Lambda,\otimes_i,\otimes_j)}$.

DEFINITION 3.2.21. Suppose that $(\Lambda, \otimes_\Lambda)$ is a symmetric monoidal 2-category. We deduce from Example 3.2.18 that there is a well defined 2-category of n-fold monoids in Λ defined by

$$\mathrm{Mon}^n_{(\Lambda,\otimes)} := \mathrm{Mon}^n_{(\Lambda,\otimes,\dots,\otimes)}$$

for all $n \in \mathbb{N}$. It follows from the symmetry of the cartesian product and by Lemma 3.2.16 that when Λ is complete we always can define the 2-category of cartesian n-fold monoids

$$\mathrm{Mon}^n_{(\Lambda,\times)} = \mathrm{Mon}^n_{(\Lambda,\times,\dots,\times)}.$$

PROPOSITION 3.2.22. *Let $(\Lambda, \otimes_1, \dots, \otimes_n, 1_\Lambda)$ be an n-fold monoidal 2-category. We have the following characterization of the 2-category $\mathrm{Mon}^n_{(\Lambda,\otimes_1,\dots,\otimes_n)}$.*

- *An n-monoid in Λ consists in the data of*
  - *an object $\mathcal{C}$ of Λ,*
  - *morphisms $\mu_1, \dots, \mu_n$ in Λ, such that for $1 \le i \le n$,*

$$\mu_i : \mathcal{C} \otimes_i \mathcal{C} \to \mathcal{C}$$

  - *2-morphisms $\square_i^j$ in Λ for all $1 \le i < j \le n$, such that*

$$\begin{tikzcd}
 & (\mathcal{C} \otimes_j \mathcal{C}) \otimes_i (\mathcal{C} \otimes_j \mathcal{C}) \arrow[dl, "\mu_j \otimes_i \mu_j"'] \arrow[dr, "\eta_i^j"] & \\
\mathcal{C} \otimes_i \mathcal{C} \arrow[d, "\mu_i"'] \arrow[rr, phantom, "\xRightarrow{\square_i^j}"] & & (\mathcal{C} \otimes_i \mathcal{C}) \otimes_j (\mathcal{C} \otimes_i \mathcal{C}) \arrow[d, "\mu_i \otimes_j \mu_i"] \\
\mathcal{C} & & \mathcal{C} \otimes_j \mathcal{C} \arrow[ll, "\mu_j"]
\end{tikzcd},$$

  *and such that the diagrams of Figure **??** and **??** commute.*
- *An n-fold monoidal morphism $F : (\mathcal{C}, \mu_1, \dots, \mu_n) \to (\mathcal{D}, \mu_1, \dots, \mu_n)$ consists in the data of*
  - *a morphism $F : \mathcal{C} \to \mathcal{D}$ in Λ*
  - *for all $1 \le i \le n$, a 2-morphism τ_i in Λ, such that*

$$\begin{tikzcd}
\mathcal{C} \otimes_i \mathcal{C} \arrow[r, "\mu_i"] \arrow[d, "F \otimes_i F"'] \arrow[dr, phantom, "\overset{\tau_i}{\nearrow}"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D} \otimes_i \mathcal{D} \arrow[r, "\nu_i"'] & \mathcal{D}
\end{tikzcd}$$

  *and such that the faces of the 3-dimensional diagrams of Figure 10 and **??** are equal.*

EXAMPLE 3.2.23. Consider the monoidal 2-category $(\mathrm{Cat}, \times)$ of categories with respect to the cartesian product. Recall that monoids in $(\mathrm{Cat}, \times)$ are usual monoidal categories, so that strict monoids in $(\mathrm{Cat}, \times)$ are monoidal categories which are strictly associative and unital. The 2-category $\mathrm{Mon}^n_{(\mathrm{Cat},\times)}$ of n-fold monoidal categories has for objects the categories $\mathcal{C}$ equipped with

- a unit object $1_\mathcal{C} \in \mathcal{C}$
- for each $1 \le i \le n$, a functor

$$\otimes_i : \mathcal{C} \times \mathcal{C} \to C$$

  which gives to $(\mathcal{C}, \otimes_i, 1_\mathcal{C})$ the structure of a strict monoidal category

– for each $1 \leq i < j \leq n$, a natural transformation η_i^j, such that

$$\begin{tikzcd}
(\mathcal{C}\times\mathcal{C})\times(\mathcal{C}\times\mathcal{C}) \arrow[rr, "(2\ \ 3)"] \arrow[d, "\otimes_j\times\otimes_j"'] & & (\mathcal{C}\times\mathcal{C})\times(\mathcal{C}\times\mathcal{C}) \arrow[d, "\otimes_i\times\otimes_i"] \\
\mathcal{C}\times\mathcal{C} \arrow[dr, "\otimes_i"'] \arrow[rr, phantom, "\overset{}{\underset{\eta_i^j}{\Rightarrow}}"] & & \mathcal{C}\times\mathcal{C} \arrow[dl, "\otimes_j"] \\
& \mathcal{C} &
\end{tikzcd},$$

and which satisfies coherence diagrams:

– $\square_i^j, \square_j^i$ for $i < j$ relative to the compatibility of η_i^j with the associativity of $\otimes_i$ and $\otimes_j$,

– $\pentagon_{1\leq i<j<k\leq n}$ which ensure that η_j^k is a lax monoidal 2-morphism with respect to $\otimes_i$.

3.3. Day convolution. Let $\mathcal{C}$ be a monoidal category. By Day's convolution, the category of presheaves $[\mathcal{C}^{op}, \textsc{Set}]$ inherits a monoidal structure from $\mathcal{C}$ such that the Yoneda embedding $y_{\mathcal{C}} : \mathcal{C} \hookrightarrow [\mathcal{C}^{op}, \textsc{Set}]$ has the structure of a lax monoidal functor. The pair $([\mathcal{C}^{op}, \textsc{Set}], y_{\mathcal{C}})$ is universal among those consisting in a cocomplete monoidal category $\mathcal{D}$ together with a lax monoidal functor $\mathcal{C} \to \mathcal{D}$, so that the category of presheaves of $\mathcal{C}$ is the free cocomplete completion of $\mathcal{C}$ in the 2-category of monoidal categories. This result generalizes to the framework of $\mathcal{V}$-enriched categories. If $\mathcal{C}$ is a $\mathcal{V}$-enriched monoidal category, then the $\mathcal{V}$-enriched category of $\mathcal{V}$-enriched functors $[\mathcal{C}^{op}, \mathcal{V}]_{\mathcal{V}}$ inherits the structure of a $\mathcal{V}$-enriched monoidal category from $\mathcal{C}$, such that the Yoneda enriched embedding is the free cocompletion of $\mathcal{C}$ in the 2-category of monoidal $\mathcal{V}$-enriched categories. As previously observed in this section, monoids can be defined internally into any monoidal 2-category, encompassing the notions of a monoidal category and of a $\mathcal{V}$-enriched monoidal category as monoids internal to $(\textsc{Cat}, \times)$ or $(\textsc{Cat}_{\mathcal{V}}, \times_{\mathcal{V}})$.

The purpose of this subsection is to make precise the conditions under which one can generalize the Day convolution product for monoids internal to some closed monoidal 2-category Λ, notably for a monoidal structure that is not necessarily the one for which the 2-category Λ is closed. In the next section, we will formalize the presheaf construction within a 2-category. To each object $\mathcal{C}$, we will associate a cocomplete object $\hat{C}$, together with a fully faithful morphism $y_{\mathcal{C}} : \mathcal{C} \hookrightarrow \hat{\mathcal{C}}$, such that the pair $(\hat{C}, y_{\mathcal{C}})$ is the free cocompletion of $\mathcal{C}$ in Λ. For any monoid $\mathcal{C}$ in Λ, we will use the results of this subsection to give to $\hat{\mathcal{C}}$ the structure of a monoid in Λ, such that the fully faithful morphism $\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$ has the structure of a lax monoidal morphism. As a result, we will obtain that the pair $(\hat{C}, y_{\mathcal{C}})$ is free cocompletion of $\mathcal{C}$ in the 2-category of monoids in Λ.

In the next chapter, we will give to the cartesian closed 2-category $\textsc{Cat}^{\mathbb{N}}$ the structure of a monoidal 2-category $(\textsc{Cat}^{\mathbb{N}}, \circ)$, and will define $\textsc{Cat}$-operads as monoids in $(\textsc{Cat}^{\mathbb{N}}, \circ)$. Subsequently, the presheaf object of any operad will be given the structure of an operad by using the results of this subsection.[17]

[17] In Chapter 4, we make the construction of presheaf operads explicit for a cofibrant model Υ of the associative operad in $\textsc{Cat}$. We will use our results on generalized Day convolution to construct a topological realization of operads internal to $\hat{\Upsilon}$ from the associahedra.

Let $(\Lambda, \otimes_\Lambda, 1_\Lambda)$ be a monoidal 2-category. Suppose that Λ is closed, with the internal hom 2-functor given by

$$[-,-]_\Lambda : \Lambda^{op} \times \Lambda \to \Lambda.$$

We assume that $[-,-]_\Lambda$ has the structure of a strongly unital[18] lax monoidal morphism[19].

Definition 3.3.1. The lax monoidal structure of $[-,-]_\Lambda$ yields the following morphism in Λ

$$\overline{\otimes}_\Lambda : [\mathcal{C}_1, \mathcal{D}_1]_\Lambda \otimes_\Lambda [\mathcal{C}_2, \mathcal{D}_2]_\Lambda \to [\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2, \mathcal{D}_1 \otimes_\Lambda \mathcal{D}_2]_\Lambda,$$

which is natural in the objects $\mathcal{C}_1, \mathcal{D}_1, \mathcal{C}_2, \mathcal{D}_2$ of Λ. We call this morphism the external product.

Lemma 3.3.2. *Let $\mathcal{S}$ be a monoid in $(\Lambda, \otimes_\Lambda, 1_\Lambda)$. The 2-functor*

$$[-, \mathcal{S}]_\Lambda : \Lambda^{op} \to \Lambda$$

is lax monoidal.

Proof. The external product and the monoidal structure of $\mathcal{S}$ induce a morphism

$$[\mathcal{C}_1, \mathcal{S}]_\Lambda \otimes_\Lambda [\mathcal{C}_2, \mathcal{S}]_\Lambda \xrightarrow{\overline{\otimes}_\Lambda} [\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2, \mathcal{S} \otimes_\Lambda \mathcal{S}]_\Lambda \xrightarrow{[\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2, \otimes_\mathcal{S}]_\Lambda} [\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2, \mathcal{S}]_\Lambda$$

natural in $\mathcal{C}_1$ and $\mathcal{C}_2$, which gives the composition product of $[\_, \mathcal{S}]_\Lambda$. We let the unit be defined by the composite

$$1_\Lambda \xrightarrow{\cong} [1_\Lambda, 1_\Lambda] \xrightarrow{[1_\Lambda, 1_\mathcal{S}]} [1_\Lambda, \mathcal{S}]_\Lambda,$$

where $1_\mathcal{S} : 1_\Lambda \to \mathcal{S}$ is the unit for the monoidal structure of $\mathcal{S}$. The unit of $[\_, \mathcal{S}]_\Lambda$ inherits the structure of a unit for the composition product from the unit of $\mathcal{S}$. The composition product satisfies the associativity conditions, which also are inherited from the associative structure of the monoidal product of $\mathcal{S}$. Let us define the associator

$$\begin{tikzcd}
([\mathcal{C}_1, \mathcal{S}]_\Lambda \otimes_\Lambda [\mathcal{C}_2, \mathcal{S}]_\Lambda) \otimes_\Lambda [\mathcal{C}_3, \mathcal{S}]_\Lambda \arrow[r, "\alpha^\Lambda"] \arrow[d, "\overline{\otimes}_\Lambda \otimes_\Lambda [\mathcal{C}_3, \mathcal{S}]_\Lambda"'] & [\mathcal{C}_1, \mathcal{S}]_\Lambda \otimes_\Lambda ([\mathcal{C}_2, \mathcal{S}]_\Lambda \otimes_\Lambda [\mathcal{C}_3, \mathcal{S}]_\Lambda) \arrow[d, "{[\mathcal{C}_1, \mathcal{S}]_\Lambda \otimes_\Lambda \overline{\otimes}_\Lambda}"] \\
([\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2, \mathcal{S} \otimes_\Lambda \mathcal{S}]_\Lambda) \otimes_\Lambda [\mathcal{C}_3, \mathcal{S}]_\Lambda \arrow[d, "{[\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2, \otimes_\mathcal{S}]_\Lambda \otimes_\Lambda [\mathcal{C}_3, \mathcal{S}]_\Lambda}"'] & [\mathcal{C}_1, \mathcal{S}]_\Lambda \otimes_\Lambda ([\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3, \mathcal{S} \otimes_\Lambda \mathcal{S}]_\Lambda) \arrow[d, "{[\mathcal{C}_1, \mathcal{S}]_\Lambda \otimes_\Lambda [\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3, \otimes_\mathcal{S}]_\Lambda}"] \\
([\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2, \mathcal{S}]_\Lambda) \otimes_\Lambda [\mathcal{C}_3, \mathcal{S}]_\Lambda \arrow[d, "\overline{\otimes}_\Lambda"'] \arrow[r, phantom, "\overset{}{\underset{\alpha}{\Longrightarrow}}"] & [\mathcal{C}_1, \mathcal{S}]_\Lambda \otimes_\Lambda ([\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3, \mathcal{S}]_\Lambda) \arrow[d, "\overline{\otimes}_\Lambda"] \\
[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, \mathcal{S} \otimes_\Lambda \mathcal{S}]_\Lambda \arrow[d, "{[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, \otimes_\mathcal{S}]_\Lambda}"'] & [\mathcal{C}_1 \otimes_\Lambda (\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3), \mathcal{S} \otimes_\Lambda \mathcal{S}]_\Lambda \arrow[d, "{[\mathcal{C}_1 \otimes_\Lambda (\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3), \otimes_\mathcal{S}]_\Lambda}"] \\
[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, \mathcal{S}]_\Lambda & [\mathcal{C}_1 \otimes_\Lambda (\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3), \mathcal{S}]_\Lambda \arrow[l, "{[\alpha^\Lambda, \mathcal{S}]}"]
\end{tikzcd}$$

[18] In the sense that we have a canonical isomorphism $[1_\Lambda, 1_\Lambda]_\Lambda \cong 1_\Lambda$ in Λ.

[19] Note that the internal hom is always lax monoidal with respect to the cartesian product.

First, by naturality of $\overline{\otimes}_\Lambda$ with respect to $\otimes_{\mathcal{S}} : \mathcal{S} \otimes_\Lambda \mathcal{S} \to \mathcal{S}$, this diagram is equivalent to

$$\begin{tikzcd}
([\mathcal{C}_1,\mathcal{S}]_\Lambda \otimes_\Lambda [\mathcal{C}_2,\mathcal{S}]_\Lambda) \otimes_\Lambda [\mathcal{C}_3,\mathcal{S}]_\Lambda \arrow[r, "\alpha^\Lambda"] \arrow[d, "\overline{\otimes}_\Lambda \otimes_\Lambda [\mathcal{C}_1,\mathcal{S}]_\Lambda"'] & [\mathcal{C}_1,\mathcal{S}]_\Lambda \otimes_\Lambda ([\mathcal{C}_2,\mathcal{S}]_\Lambda \otimes_\Lambda [\mathcal{C}_3,\mathcal{S}]_\Lambda) \arrow[d, "{[\mathcal{C}_1,\mathcal{S}]_\Lambda \otimes_\Lambda \overline{\otimes}_\Lambda}"] \\
([\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2, \mathcal{S} \otimes_\Lambda \mathcal{S}]_\Lambda) \otimes_\Lambda [\mathcal{C}_3,\mathcal{S}]_\Lambda \arrow[d, "\overline{\otimes}_\Lambda"'] & [\mathcal{C}_1,\mathcal{S}]_\Lambda \otimes_\Lambda ([\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3, \mathcal{S} \otimes_\Lambda \mathcal{S}]_\Lambda) \arrow[d, "\overline{\otimes}_\Lambda"] \\
[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, (\mathcal{S} \otimes_\Lambda \mathcal{S}) \otimes_\Lambda \mathcal{S}]_\Lambda \arrow[d, "{[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, \otimes_{\mathcal{S}} \otimes_\Lambda \mathcal{S}]_\Lambda}"'] & [\mathcal{C}_1 \otimes_\Lambda (\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3), \mathcal{S} \otimes_\Lambda (\mathcal{S} \otimes_\Lambda \mathcal{S})]_\Lambda \arrow[d, "{[\mathcal{C}_1 \otimes_\Lambda (\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3), \mathcal{S} \otimes_\Lambda \otimes_{\mathcal{S}}]_\Lambda}"] \\
[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, \mathcal{S} \otimes_\Lambda \mathcal{S}]_\Lambda \arrow[d, "{[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, \otimes_{\mathcal{S}}]_\Lambda}"'] & [\mathcal{C}_1 \otimes_\Lambda (\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3), \mathcal{S} \otimes_\Lambda \mathcal{S}]_\Lambda \arrow[d, "{[\mathcal{C}_1 \otimes_\Lambda (\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3), \otimes_{\mathcal{S}}]_\Lambda}"] \\
[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, \mathcal{S}]_\Lambda & [\mathcal{C}_1 \otimes_\Lambda (\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3), \mathcal{S}]_\Lambda \arrow[l, "{[\alpha^\Lambda, \mathcal{S}]}"]
\end{tikzcd},$$

and therefore, to

$$\begin{tikzcd}
 & & [\mathcal{C}_1,\mathcal{S}]_\Lambda \otimes_\Lambda ([\mathcal{C}_2,\mathcal{S}]_\Lambda \otimes_\Lambda [\mathcal{C}_3,\mathcal{S}]_\Lambda) \arrow[dd, "\overline{\otimes_\Lambda} \circ ({[\mathcal{C}_1,\mathcal{S}]_\Lambda} \otimes_\Lambda \overline{\otimes_\Lambda})"] \\
([\mathcal{C}_1,\mathcal{S}]_\Lambda \otimes_\Lambda [\mathcal{C}_2,\mathcal{S}]_\Lambda) \otimes_\Lambda [\mathcal{C}_3,\mathcal{S}]_\Lambda \arrow[urr, "\alpha^\Lambda"] \arrow[dd, "\overline{\otimes_\Lambda} \circ (\overline{\otimes_\Lambda} \otimes_\Lambda {[\mathcal{C}_3,\mathcal{S}]_\Lambda})"'] & & \\
 & & [\mathcal{C}_1 \otimes_\Lambda (\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3), \mathcal{S} \otimes_\Lambda (\mathcal{S} \otimes_\Lambda \mathcal{S})]_\Lambda \arrow[ddll, "{[\alpha_\Lambda, \mathcal{S} \otimes_\Lambda (\mathcal{S} \otimes_\Lambda \mathcal{S})]_\Lambda}"'] \arrow[dd, "{[\mathcal{C}_1 \otimes_\Lambda (\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3), \otimes_{\mathcal{S}} \circ (\mathcal{S} \otimes_\Lambda \otimes_{\mathcal{S}})]_\Lambda}"] \\
[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, (\mathcal{S} \otimes_\Lambda \mathcal{S}) \otimes_\Lambda \mathcal{S}]_\Lambda \arrow[urr, Rightarrow] \arrow[d, "{[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, \alpha^\Lambda]_\Lambda}"] \arrow[dd, bend right=60, "{[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, \otimes_{\mathcal{S}} \circ (\otimes_{\mathcal{S}} \otimes_\Lambda \mathcal{S})]_\Lambda}"'] & & \\
[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, \mathcal{S} \otimes_\Lambda (\mathcal{S} \otimes_\Lambda \mathcal{S})]_\Lambda \arrow[d, "{[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, \otimes_{\mathcal{S}} \circ (\mathcal{S} \otimes_\Lambda \otimes_{\mathcal{S}})]_\Lambda}"] \arrow[l, phantom, "\Uparrow {[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, \alpha_{\mathcal{S}}]_\Lambda}"] & \cong & [\mathcal{C}_1 \otimes_\Lambda (\mathcal{C}_2 \otimes_\Lambda \mathcal{C}_3), \mathcal{S})]_\Lambda \arrow[dll, "{[\alpha^\Lambda, \mathcal{S}]_\Lambda}"] \\
[(\mathcal{C}_1 \otimes_\Lambda \mathcal{C}_2) \otimes_\Lambda \mathcal{C}_3, \mathcal{S}]_\Lambda & &
\end{tikzcd}$$

The isomorphism on the square is provided by the naturality of $[\alpha^\Lambda, \_]_\Lambda$ with respect to $\otimes_{\mathcal{S}} \circ (\mathcal{S} \otimes_\Lambda \otimes_{\mathcal{S}})$. The 2-morphism on the pentagone on the top is provided by the lax monoidal structure of the tensor product of Λ, and the 2-morphism on the triangle is provided by the associator of $\mathcal{S}$. The coherence conditions can be deduced in the same way from the conditions fullfiled by the associator of $\mathcal{S}$, the associators of Λ, and the monoidal structure on the tensor product of Λ.

$\square$

Proposition 3.3.3. *Let $\mathcal{S}$ be a monoid in $(\Lambda, \otimes_\Lambda)$. Suppose that $\mathcal{S}$ is internally cocomplete in Λ, so that $\mathcal{S}$-valued left Kan extensions exist along any morphism of Λ. Let $(\mathcal{C}, \otimes_{\mathcal{C}})$ be a colax monoid in Λ. The image of $\mathcal{C}$ in Λ through the morphism*

$$[\_, \mathcal{S}]_\Lambda : \Lambda^{op} \to \Lambda$$

inherits a lax monoidal structure from $\mathcal{C}$. The tensor product is given by the composite

$$\otimes_{[\mathcal{C},\mathcal{S}]_\Lambda} : [\mathcal{C}, \mathcal{S}] \otimes_\Lambda [\mathcal{C}, \mathcal{S}]_\Lambda \xrightarrow{\overline{\otimes}_\Lambda} [\mathcal{C} \otimes_\Lambda \mathcal{C}, \mathcal{S} \otimes_\Lambda \mathcal{S}]_\Lambda \xrightarrow{[1, \otimes_{\mathcal{S}}]_\Lambda} [\mathcal{C} \otimes_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda \xrightarrow{Lan^{\mathcal{S}}_{\otimes_{\mathcal{C}}}} [\mathcal{C}, \mathcal{S}]_\Lambda \ .$$

The unit $\overline{1_{\mathcal{C}}} : 1_\Lambda \to [\mathcal{C}, \mathcal{S}]_\Lambda$ is given by the global left Kan extension of the unit of $\mathcal{S}$ along the unit of $\mathcal{C}$, hence by the composite

$$\overline{1_C} : 1_\Lambda \xrightarrow{1_{\mathcal{C}}} \mathcal{C} \xrightarrow{\cong} [1_\Lambda, \mathcal{C}]_\Lambda \xrightarrow{Lan^{\mathcal{S}}} [[1_\Lambda, \mathcal{S}]_\Lambda, [\mathcal{C}, \mathcal{S}]_\Lambda]_\Lambda \xrightarrow{[1_{\mathcal{S}}, 1]_\Lambda} [1_\Lambda, [\mathcal{C}, \mathcal{S}]_\Lambda]_\Lambda \xrightarrow{\cong} [\mathcal{C}, \mathcal{S}]_\Lambda.$$

Proof. The cohererence conditions can be more easily proven by using the unbiased framework. We let the tensor product be defined in arity $n \in \mathbb{N}$ by the following composite.

$$\otimes^n_{[\mathcal{C},\mathcal{S}]_\Lambda} : [\mathcal{C}, \mathcal{S}]_\Lambda^{\otimes^n_\Lambda} \xrightarrow{\bar{\otimes}^n_\Lambda} [\mathcal{C}^{\otimes^n_\Lambda}, \mathcal{S}^{\otimes^n_\Lambda}]_\Lambda \xrightarrow{[\mathcal{C}^{\otimes^n_\Lambda}, \otimes^n_{\mathcal{S}}]_\Lambda} [\mathcal{C}^{\otimes^n_\Lambda}, \mathcal{S}]_\Lambda \xrightarrow{Lan^{\mathcal{S}}_{\otimes^n_{\mathcal{C}}}} [\mathcal{C}, \mathcal{S}]_{\mathcal{S}}.$$

Note that we obtain the unit by taking $n = 0$. Let $r \in \mathbb{N}$ and $n_1, \ldots, n_r \in \mathbb{N}$. Let us show that there is a natural transformation $\overline{\alpha_{\mathcal{C}}}$ such that

$$\begin{tikzcd}
\bigotimes\limits_{i=1}^{n}{}_\Lambda [\mathcal{C}, \mathcal{S}]_\Lambda \arrow[r, "\alpha^\Lambda"] \arrow[ddd, "\bar{\otimes}^n(\mathcal{C},\mathcal{S})"'] & \bigotimes\limits_{i=1}^{r}{}_\Lambda [\mathcal{C}, \mathcal{S}]^{\otimes^{n_i}_\Lambda} \arrow[r, "\bigotimes\limits_{i=1}^{r}{}_\Lambda \bar{\otimes}^{n_i}(\mathcal{C},\mathcal{S})"] & \bigotimes\limits_{i=1}^{r}{}_\Lambda \left[\mathcal{C}^{\otimes^{n_i}_\Lambda}, \mathcal{S}^{\otimes^{n_i}_\Lambda}\right]_\Lambda \arrow[dr, "\bigotimes\limits_{i=1}^{r}{}_\Lambda \left[\mathcal{C}^{\otimes^{n_i}_\Lambda}, \otimes^{n_i}_{\mathcal{S}}\right]_\Lambda"] & \\
 & & & \bigotimes\limits_{i=1}^{r}{}_\Lambda \left[\mathcal{C}^{\otimes^{n_i}_\Lambda}, \mathcal{S}\right]_\Lambda \arrow[d, "\bigotimes\limits_{i=1}^{r}{}_\Lambda Lan^{\mathcal{S}}_{\otimes^{n_i}_{\mathcal{C}}}"] \\
 & & \arrow[l, phantom, "\Leftarrow \overline{\alpha_{\mathcal{C}}}"] & \bigotimes\limits_{i=1}^{r}{}_\Lambda [\mathcal{C}, \mathcal{S}]_\Lambda \arrow[d, "\bar{\otimes}^r(\mathcal{C},\mathcal{S})"] \\
\left[\mathcal{C}^{\otimes^n_\Lambda}, \mathcal{S}^{\otimes^n_\Lambda}\right]_\Lambda \arrow[dd, "Lan^{\mathcal{S}^{\otimes^n_\Lambda}}_{\otimes^n_{\mathcal{C}}}"'] & & & \left[\bigotimes\limits_{i=1}^{r}{}_\Lambda \mathcal{C}, \bigotimes\limits_{i=1}^{r}{}_\Lambda \mathcal{S}\right]_\Lambda \arrow[d, "\left[\bigotimes\limits_{i=1}^{r}{}_\Lambda \mathcal{C}, \otimes^r_{\mathcal{S}}\right]_\Lambda"] \\
 & & & \left[\bigotimes\limits_{i=1}^{r}{}_\Lambda \mathcal{C}, \mathcal{S}\right]_\Lambda \arrow[d, "Lan^{\mathcal{S}}_{\otimes^r_{\mathcal{C}}}"] \\
\left[\mathcal{C}, \mathcal{S}^{\otimes^n_\Lambda}\right]_\Lambda \arrow[rrr, "{[\mathcal{C}, \otimes^n_{\mathcal{S}}]_\Lambda}"'] & & & {[\mathcal{C}, \mathcal{S}]_\Lambda} .
\end{tikzcd}$$

For this purpose, we use the properties of Kan extensions to exchange step by step the morphisms involved in the diagram above, so that we can use the associators of $\mathcal{C}$ and $\mathcal{S}$. We obtain a chain of 2-morphisms which arrange into the diagram of Figure 11. $\square$

4. Entangled enrichment on a 2-category

In the constructions of this section, we require some additional structures on our 2-category Λ, which will enables us to endow the objects of Λ with a structure that is close to the structure of a category. For this purpose, we will need an object $\mathcal{S}$ of Λ, which will play a role analogous to the category of sets. We will assume that each object $\mathcal{C}$ of Λ is equipped with a morphism $\mathcal{C}(\_,\_) : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ in Λ. In particular, each pair of objects $X, Y : *_\Lambda \to \mathcal{C}$ of $\mathcal{C}$ yields an object $\mathcal{C}(X,Y) : * \to \mathcal{S}$ of $\mathcal{S}$, which may be regarded as the object of morphisms in $\mathcal{C}$ from X to Y.

It is well known that the Yoneda lemma expresses, in particular, the compatibility between the internal structure of objects and the global structure of $\mathrm{C{\scriptsize AT}}$, notably with regards to the closed structure of $\mathrm{C{\scriptsize AT}}$. In this section, we aim to provide a conceptual explanation of the interplays between the internal structure on the objects of Λ and the global structure of Λ. We obtain an analogue of the Yoneda lemma. In particular, we will be able to define objects internally in the objects of Λ by universal property.

4.1. Opposite objects in a 2-category.

Definition 4.1.1. An oppositization on a 2-category is the data of a 2-functor

$$(\_)^{op} : \Lambda^{op_2} \to \Lambda$$

such that:

- It is idempotent in the sense that the diagram commutes up to canonical isomorphism

$$\begin{tikzcd} \Lambda^{op_2} \arrow[r, "(-)^{op}"] \arrow[dr, "\mathtt{id}"'] & \Lambda \cong (\Lambda^{op_2})^{op_2} \arrow[d, "((-)^{op})^{op_2}"] \\ & \Lambda^{op_2}. \end{tikzcd}$$

  In particular, for object $\mathcal{C}$ of Λ, we have $(\mathcal{C}^{op})^{op} \cong \mathcal{C}$.
- It is compatible with the 2-functor $\Lambda(*_\Lambda, \_) : \Lambda \to \mathrm{C{\scriptsize AT}}$ in that there is a canonical isomorphism

$$\begin{tikzcd} \Lambda \arrow[r, "(-)^{op}"] \arrow[d, "\Lambda(*_\Lambda,-)"'] \arrow[dr, phantom, "\cong"] & \Lambda^{op_2} \arrow[d, "\Lambda(*_\Lambda,-)^{op_2}"] \\ \mathrm{C{\scriptsize AT}} \arrow[r, "(-)^{op}"'] & \mathrm{C{\scriptsize AT}}^{op_2} \end{tikzcd} .$$

Remark 4.1.2. The compatibility of oppositization with the 2-functor $\Lambda(*_\Lambda, \_)$ ensures that for object $\mathcal{C}$ of Λ, we have an isomorphism

$$\Lambda(*_\Lambda, \mathcal{C}^{op}) \cong \Lambda(*_\Lambda, \mathcal{C})^{op},$$

which may be understood as 'the underlying category of $\mathcal{C}^{op}$ has the same objects than $\mathcal{C}$ and morphisms in the opposite direction.

Definition 4.1.3. We say that an oppositization on a closed monoidal 2-category $(\Lambda, \otimes_\Lambda, [\_,\_]_\Lambda)$ is

– *monoidal* if the oppositization 2-functor is strongly monoidal with respect to the monoidal structure of Λ. In particular, we have isomorphisms $(\mathcal{C} \otimes_\Lambda \mathcal{D})^{op} \cong \mathcal{C}^{op} \otimes_\Lambda \mathcal{D}^{op}$ natural in the objects $\mathcal{C}, \mathcal{D}$ of Λ.
– *closed* if the oppositization 2-functor is closed in the sense of Definition 3.1.13. In this case, each pair of objects $\mathcal{C}, D$ of Λ yields a morphism $[\mathcal{C}, \mathcal{D}]_\Lambda^{op} \to [\mathcal{C}^{op}, \mathcal{D}^{op}]_\Lambda$.
– *closed monoidal* if it is both closed and monoidal.

Example 4.1.4. The oppositization 2-functor $\mathrm{C{\scriptsize AT}}op_2 \to \mathrm{C{\scriptsize AT}}$ given in Definition 1.2.3 yields a closed monoidal oppositization on $(\mathrm{C{\scriptsize AT}}, \times, [\,\_\,,\,\_\,])$.

Definition 4.1.5. Let $(\Lambda, \times_\Lambda, [\_, \_]_\Lambda, *_\Lambda)$ be a closed monoidal 2-category equipped with an oppositization. Let $(\mathcal{S}, \times_\mathcal{S}, *_\mathcal{S})$ be a monoid in Λ. We say that the monoidal structure on $\mathcal{S}$ is closed, or that $\mathcal{S}$ is a closed monoidal object of Λ, if $\mathcal{S}$ is equipped with a monoidal morphism

$$[\_, \_]_\mathcal{S} : \mathcal{S}^{op} \times_\Lambda \mathcal{S} \to \mathcal{S}$$

together with an isomorphism

$$\begin{tikzcd}
\mathcal{S}^{op} \times_\Lambda \mathcal{S}^{op} \times_\Lambda \mathcal{S} \arrow[r, "- \times_\mathcal{S}^{op} - \times_\Lambda 1"] \arrow[d, "1 \times_\Lambda [_,_]_\mathcal{S}"'] \arrow[dr, phantom, "\cong"] & \mathcal{S}^{op} \times_\Lambda \mathcal{S} \arrow[d, "[_,_]_\mathcal{S}"] \\
\mathcal{S}^{op} \times_\Lambda \mathcal{S} \arrow[r, "[_,_]_\mathcal{S}"'] & \mathcal{S},
\end{tikzcd}$$

such that the following diagram commutes in $\mathcal{S}^*$ for all $X, Y, Z, T : *_\Lambda \to \mathcal{S}$.

$$\begin{tikzcd}
{[(X \times_\mathcal{S} Y) \times_\mathcal{S} Z, T]_\mathcal{S}} \arrow[r] \arrow[d] & {[X \times_\mathcal{S} (Y \times_\mathcal{S} Z), T]_\mathcal{S}} \arrow[d] \\
{[Z, [[X \times_\mathcal{S} Y]_\mathcal{S}, T]_\mathcal{S}]_\mathcal{S}} \arrow[dr] & {[Y \times_\mathcal{S} Z, [X, T]_\mathcal{S}]_\mathcal{S}} \arrow[d] \\
 & {[Z, [Y, [X, T]_\mathcal{S}]_\mathcal{S}]_\mathcal{S}}
\end{tikzcd}$$

In this section, Λ is a closed monoidal 2-category equipped with an oppositization. We also assume that Λ is equipped with an object $(\mathcal{S}, \times_\mathcal{S}, [\_, \_]_\mathcal{S}, *_\mathcal{S})$ which is closed monoidal.

4.2. Ends and coends. The standard definition of ends and coends involves using the set of morphisms between objects within a category, which render their generalization to an arbitrary 2-category intricate. Before we introduce the suitable framework for defining internal ends and coends, we first define ends and coends externally by using the forgetful 2-functor $\Lambda \to \mathrm{C{\scriptsize AT}}$. We also introduce the notion of an end and a coend relatively to a morphism $\mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ in Λ, which we will use in the subsequent section to define entangled enrichments. We will be able to define ends and coends internally in a 2-category as soon as it is provided with an entangled enrichment by using the relative ends and coends.

Definition 4.2.1. Let $\mathcal{C}$ and $\mathcal{S}$ be objects of Λ and let

$$F : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$$

be a morphism in Λ. An end of F is a universal pair $\left(\int^{\mathcal{C}} F, \pi\right)$, where

$$\int^{\mathcal{C}} F : *_\Lambda \to \mathcal{S}$$

is a morphism in Λ, and where π consists in the data of a 2-morphism π_X for each $X : *_\Lambda \to \mathcal{C}$:

$$\begin{tikzcd} & \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Uparrow \pi_X"] & \\ *_\Lambda \arrow[ur, "{(X,X)}"] \arrow[rr, "\int^{\mathcal{C}} F"'] & {} & \mathcal{S} \end{tikzcd},$$

such that for all $X, Y : *_\Lambda \to \mathcal{C}$ and all morphism $f : X \to Y$ in $\mathcal{C}^*$, the induced 2-morphisms

$$\begin{tikzcd} & \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Uparrow \pi_X"] & \\ *_\Lambda \arrow[ur, bend left=60, "{(X,Y)}"] \arrow[ur, "{(X,X)}"'] \arrow[ur, phantom, bend left=30, "\Leftarrow (X,f)"] \arrow[rr, "\int^{\mathcal{C}} F"'] & {} & \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Uparrow \pi_Y"] & \\ *_\Lambda \arrow[ur, bend left=60, "{(X,Y)}"] \arrow[ur, "{(Y,Y)}"'] \arrow[ur, phantom, bend left=30, "\Leftarrow (f^{op},Y)"] \arrow[rr, "\int^{\mathcal{C}} F"'] & {} & \mathcal{S} \end{tikzcd}$$

satisfies $F(X, f)\pi_X = F(f^{op}, Y)\pi_Y$.

Remark 4.2.2. The morphism $F : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ in Λ yields a functor

$$F^* : \mathcal{C}^{*op} \times \mathcal{C}^* \to \mathcal{S}^*,$$

whose end $\int^{\mathcal{C}} F^* \in \mathcal{S}^*$ precisely is the end $\int^{\mathcal{C}} F : *_\Lambda \to \mathcal{S}$ of F thus defined.

Definition 4.2.3. We say that the end is globally defined is there is a morphism

$$\int^{\mathcal{C}} : [\mathcal{C}^{op} \times \mathcal{C}, \mathcal{S}]_{\mathcal{S}} \to \mathcal{S}$$

which corresponds to the usual end under the forgetful 2-functor $\Lambda \to$ Cat.

Remark 4.2.4. Let $F : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ be a morphism in Λ. Each $X : *_\Lambda \to \mathcal{S}$ yields a functor

$$\mathcal{S}^*(X, F(-,-)) : (\mathcal{C}^*)^{op} \times \mathcal{C}^* \to \text{Set}.$$

Its end $\int^{\mathcal{C}} \mathcal{S}^*(X, F(-,-)) \in$ Set is such that the canonical morphism

$$\int^{\mathcal{C}} \mathcal{S}^*(X, F(-,-)) \xrightarrow{\cong} \mathcal{S}^*\left(X, \int^{\mathcal{C}} F(\_,\_)\right)$$

is an isomorphism.

Definition 4.2.5. Let $\mathcal{C}$ and $\mathcal{S}$ be objects of Λ and let

$$F : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$$

be a morphism in Λ. A coend of F is a universal pair $\left(\int_{\mathcal{C}} F, \iota\right)$, where

$$\int_{\mathcal{C}} F : *_\Lambda \to \mathcal{S}$$

is a morphism in Λ, and where ι consists in the data of a 2-morphism ι_X for each $X : *_\Lambda \to \mathcal{C}$:

$$\begin{tikzcd} & \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow \iota_X"] & \\ *_\Lambda \arrow[ur, "{(X,X)}"] \arrow[rr, "\int_{\mathcal{C}} F"'] & {} & \mathcal{S} \end{tikzcd} ,$$

such that for all $X, Y : *_\Lambda \to \mathcal{C}$ and all morphism $f : X \to Y$ in $\mathcal{C}^*$, the induced 2-morphisms

$$\begin{tikzcd} & \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow \iota_X"] & \\ *_\Lambda \arrow[ur, bend left=60, "{(Y,X)}"] \arrow[ur, bend left=20, "{(X,X)}"'] \arrow[ur, phantom, "\Rightarrow (f^{op},X)"] \arrow[rr, "\int_{\mathcal{C}} F"'] & {} & \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow \iota_Y"] & \\ *_\Lambda \arrow[ur, bend left=60, "{(Y,X)}"] \arrow[ur, bend left=20, "{(Y,Y)}"'] \arrow[ur, phantom, "\Rightarrow (Y,f)"] \arrow[rr, "\int_{\mathcal{C}} F"'] & {} & \mathcal{S} \end{tikzcd}$$

satisfy $\iota_X F(f^{op}, X) = \iota_Y F(Y, f)$.

REMARK 4.2.6. The morphism $F : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ in Λ yields a functor

$$F^* : \mathcal{C}^{*op} \times \mathcal{C}^* \to \mathcal{S}^*,$$

whose coend $\int_{\mathcal{C}} F^* : *_\Lambda \to \mathcal{S}$ precisely is the coend $\int_{\mathcal{C}} F : *_\Lambda \to \mathcal{S}$ of F thus defined.

DEFINITION 4.2.7. We say that the coend is globally defined is there is a morphism

$$\int_{\mathcal{C}} : [\mathcal{C}^{op} \times \mathcal{C}, \mathcal{S}]_{\mathcal{S}} \to \mathcal{S}$$

which corresponds to the usual coend under the forgetful 2-functor $\Lambda \to \mathrm{CAT}$.

REMARK 4.2.8. Let $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}$ be a morphism in Λ. Each $X : *_\Lambda \to \mathcal{S}$ yields a functor

$$\mathcal{S}^*(F(-,-), X) : \mathcal{C}^* \times (\mathcal{C}^*)^{op} \to \mathrm{SET}.$$

Its end $\int^{\mathcal{C}} \mathcal{S}^*(F(-,-), X)$ is such that the canonical morphism

$$\int^{\mathcal{C}} \mathcal{S}^*(F(-,-), X) \xrightarrow{\cong} \mathcal{S}^* \left(\int_{\mathcal{C}} F(\_,\_), X \right)$$

is an isomorphism.

DEFINITION 4.2.9. Let $F : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$. The morphism F, together with the closed monoidal structure on $\mathcal{S}$, yields a morphism

$$F(\_,\_) \times_{\mathcal{S}} \_ : \mathcal{S} \to [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda.$$

We say that a morphism

$$\int^F : [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda \to \mathcal{S}$$

is an F-relative end if $\int^F$ is right adjoint to $F(\_,\_) \times_{\mathcal{S}} \_$.

DEFINITION 4.2.10. Let $F : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$. The morphism F, together with the closed monoidal structure on $\mathcal{S}$, yields a morphism

$$[F(\_,\_),\_]_{\mathcal{S}} : \mathcal{S} \to [\mathcal{C} \times_\Lambda \mathcal{C}^{op}, \mathcal{S}]_\Lambda.$$

We say that a morphism

$$\int_F : [\mathcal{C} \times_\Lambda \mathcal{C}^{op}, \mathcal{S}]_\Lambda \to \mathcal{S}$$

is an F-relative coend if $\int_F$ is left adjoint to $[F(\_,\_),\_]_{\mathcal{S}}$.

PROPOSITION 4.2.11. *Suppose that the 2-category Λ is $*_\Lambda$-primary. Let $\mathcal{C}$ be an object of Λ, and let $F : \mathcal{C}^{op} \times_{\mathcal{S}} \mathcal{C} \to \mathcal{S}$. The closed structure on $\mathcal{S}$ yields a morphism*

$$[F(\_,\_),\_]_{\mathcal{S}} : [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda \to [(\mathcal{C}^{op} \times_\Lambda \mathcal{C})^{op} \times_\Lambda (\mathcal{C}^{op} \times_\Lambda \mathcal{C}), \mathcal{S}]_\Lambda.$$

The F-relative end factors through the composite

$$\int^F : [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda \xrightarrow{[F(\_,\_),\_]_{\mathcal{S}}} [(\mathcal{C}^{op} \times_\Lambda \mathcal{C})^{op} \times_\Lambda (\mathcal{C}^{op} \times_\Lambda \mathcal{C}), \mathcal{S}]_\Lambda \xrightarrow{\int^{\mathcal{C}^{op} \times_\Lambda \mathcal{C}}} \mathcal{S}$$

so that the F-relative end of any morphism $G(\_,\_) : \mathcal{C}^{op} \times_{\mathcal{S}} \mathcal{C} \to \mathcal{S}$ satisfies

$$\int^F G(\_,\_) \xrightarrow{\cong} \int^{\mathcal{C}^{op} \times_\Lambda \mathcal{C}} [F(\_,\_), G(\_,\_)]_{\mathcal{S}}.$$

PROOF. We use the $*_\Lambda$-primary structure of Λ to reason pointwise. Let $Z : *_\Lambda \to \mathcal{S}$ and $G : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$. We have the following natural isomorphisms

$$\begin{aligned}
\mathcal{S}^*\left(Z, \int^{\mathcal{C}^{op} \times_\Lambda \mathcal{C}} [F(_,_), G(_,_)]_{\mathcal{S}}\right) &\xrightarrow{\cong} \int^{X,Y \in \mathcal{C}^{op} \times_\Lambda \mathcal{C}} \mathcal{S}^*\left(Z, [F(X,Y), G(X,Y)]_{\mathcal{S}}\right) \\
&\xrightarrow{\cong} \int^{X,Y \in \mathcal{C}^{op} \times_\Lambda \mathcal{C}} \mathcal{S}^*\left(Z \times_{\mathcal{S}} F(X,Y), G(X,Y)\right) \\
&\xrightarrow{\cong} [\mathcal{C}^{op} \times \mathcal{C}, \mathcal{S}]^*_\Lambda \left(Z \times_{\mathcal{S}} F(_,_), G(_,_)\right) \\
&\xrightarrow{\cong} \mathcal{S}^*\left(Z, \int^F G(_,_)\right),
\end{aligned}$$

hence the result. $\square$

PROPOSITION 4.2.12. *Suppose that the 2-category Λ is $*_\Lambda$-primary. Let $\mathcal{C}$ be an object of Λ, and let $F : \mathcal{C}^{op} \times_{\mathcal{S}} \mathcal{C} \to \mathcal{S}$. The monoidal structure on $\mathcal{S}$ yields a morphism*

$$\_ \times_{\mathcal{S}} F(\_,\_) : [\mathcal{C} \times_\Lambda \mathcal{C}^{op}, \mathcal{S}]_\Lambda \to [\mathcal{C} \times_\Lambda \mathcal{C}^{op} \times_\Lambda (\mathcal{C} \times_\Lambda \mathcal{C}^{op})^{op}, \mathcal{S}]_\Lambda.$$

The F-relative coend factors through the composite

$$\int_F : [\mathcal{C} \times_\Lambda \mathcal{C}^{op}, \mathcal{S}]_\Lambda \xrightarrow{\_ \times_{\mathcal{S}} F(\_,\_)} [\mathcal{C} \times_\Lambda \mathcal{C}^{op} \times_\Lambda (\mathcal{C} \times_\Lambda \mathcal{C}^{op})^{op}, \mathcal{S}]_\Lambda \xrightarrow{\int_{\mathcal{C} \times_\Lambda \mathcal{C}^{op}}} \mathcal{S}$$

so that the F-relative coend of any morphism $G(\_,\_) : \mathcal{C} \times_{\mathcal{S}} \mathcal{C}^{op} \to \mathcal{S}$ satisfies

$$\int_F G(\_,\_) \xrightarrow{\cong} \int_{\mathcal{C} \times_\Lambda \mathcal{C}^{op}} G(\_,\_) \times_{\mathcal{S}} F(\_,\_).$$

PROOF. Again, we use the $*_\Lambda$-primary structure of Λ to reason pointwise. Let $Z : *_\Lambda \to \mathcal{S}$ and $G : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$. We have the following natural isomorphisms

$$\begin{aligned}
\mathcal{S}^*\left(\int_{\mathcal{C}\times_\Lambda\mathcal{C}^{op}} G(_,_) \times_{\mathcal{S}} F(_,_)_{\mathcal{S}}, Z\right) &\xrightarrow{\cong} \int^{X,Y\in\mathcal{C}\times_\Lambda\mathcal{C}^{op}} \mathcal{S}^*\left(G(X,Y)\times_{\mathcal{S}} F(X,Y)_{\mathcal{S}}, Z\right)\\
&\xrightarrow{\cong} \int^{X,Y\in\mathcal{C}\times_\Lambda\mathcal{C}^{op}} \mathcal{S}^*(G(X,Y),[F(X,Y),Z]_{\mathcal{S}})\\
&\xrightarrow{\cong} [\mathcal{C}\times_\Lambda\mathcal{C}^{op},\mathcal{S}]^*\left(G(_,_),[F(_,_),Z]_{\mathcal{S}}\right)\\
&\xrightarrow{\cong} \mathcal{S}^*(\int_F G(_,_),Z),
\end{aligned}$$

hence the result. □

REMARK 4.2.13. As a consequence of Proposition 4.2.11 and 4.2.12, relative ends and coends can be made internally in Λ. We obtain a morphism in Λ

$$\int^{\bullet} : [\mathcal{C}^{op}\times_\Lambda\mathcal{C},\mathcal{S}]_\Lambda^{op} \to [[\mathcal{C}^{op}\times_\Lambda\mathcal{C},\mathcal{S}]_\Lambda,\mathcal{S}]_\Lambda\,,$$

which maps F to $\int^F$. Dually, we obtain

$$\int_{\bullet} : [(\mathcal{C}^{op}\times_\Lambda\mathcal{C})^{op},\mathcal{S}]_\Lambda^{op} \to [[\mathcal{C}^{op}\times_\Lambda\mathcal{C},\mathcal{S}]_\Lambda,\mathcal{S}]_\Lambda\,,$$

which maps F to $\int_F$.

4.3. Entangled enrichment. Let Λ be a cartesian[20] closed 2-category equipped with an oppositization and let $\mathcal{S}$ be an object of Λ. In this subsection, we assume that the 2-category Λ is $*_\Lambda$-primary. We define a 2-category $\Lambda_{/\mathcal{S}}$ together with a closed monoidal structure, and use it to efficiently describe the coherence conditions that should satisfy the internal morphism $\mathcal{S}$-objects in Λ. For this purpose, we assume that the oppositization 2-functor is closed monoidal. We also make the following assumptions on $\mathcal{S}$:

– $\mathcal{S}$ is internally bicomplete in Λ
– $\mathcal{S}$ is equipped with the structure of a closed monoid $(\mathcal{S},\times_{\mathcal{S}},[\_,\_]_{\mathcal{S}},*_{\mathcal{S}})$.

DEFINITION 4.3.1. Let $\Lambda_{/\mathcal{S}}$ be the 2-category defined as follows:

– An object of $\Lambda_{/\mathcal{S}}$ is a pair $(\mathcal{C},F)$, which we also write $F^{\mathcal{C}}$, where
  – $\mathcal{C}$ is an object of Λ
  – F is a morphism $F : \mathcal{C}^{op}\times_\Lambda\mathcal{C}\to\mathcal{S}$ in Λ

– For each pair of objects $F^{\mathcal{C}}, G^{\mathcal{D}}$ of $\Lambda_{/\mathcal{S}}$, we let $\Lambda_{/\mathcal{S}}(F^{\mathcal{C}}, G^{\mathcal{D}})$ be the category such that
  – An object of $\Lambda_{/\mathcal{S}}(F^{\mathcal{C}}, G^{\mathcal{D}})$ is a triple (ψ_1,ψ_2,α), where
    * $\psi_1,\psi_2 : \mathcal{C}\to\mathcal{D}$ are morphisms in Λ
    * α is a morphism $\alpha : F\to G(\psi_1^{op},\psi_2)$ in $\Lambda(\mathcal{C}^{op}\times_\Lambda\mathcal{C}, S)$

[20]We assume that the monoidal structure on Λ is cartesian for the sake of simplicity. Though we will not use the property of the cartesian product, we will sometimes need to exchange factors. We suspect this exchange of factors to comes from the global structure of oppositization rather than from symmetry. In particular, the subsequent constructions may generalize to any biclosed monoidal structure on Λ. But in practise, the monoidal structure for which the constructions of this subsection are relevant is given by the cartesian product.

  - For each pair of objects $(\psi_1, \psi_2, \alpha), (\phi_1, \phi_2, \beta)$ of $\Lambda_{/\mathcal{S}}(F^{\mathcal{C}}, G^{\mathcal{D}})$, a morphism $\nu : (\psi, \alpha) \to (\phi, \beta)$ is a pair (ν_1, ν_2), where $\nu_1 : \psi_1 \to \phi_1$ and $\nu_2 : \psi_2 \to \phi_2$ are morphisms in $\Lambda(\mathcal{C}, \mathcal{D})$ such that the following equality holds:

$$\begin{tikzcd}
\mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[dd, bend right=50, "\psi_1^{op}\times_\Lambda\phi_2"'] \arrow[dd, bend left=50, "\psi_1^{op}\times_\Lambda\psi_2"] \arrow[dd, phantom, "\overset{1\times_\Lambda\nu_2}{\Longleftarrow}"] \arrow[dr, "F"] & & & \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[dd, bend right=50, "\psi_1^{op}\times_\Lambda\phi_2"'] \arrow[dd, bend left=50, "\phi_1^{op}\times_\Lambda\phi_2"] \arrow[dd, phantom, "\overset{\nu_1^{op}\times_\Lambda 1}{\Longleftarrow}"] \arrow[dr, "F"] & \\
 & \mathcal{S} \arrow[l, phantom, "\Downarrow\alpha"] & = & & \mathcal{S} \arrow[l, phantom, "\Downarrow\beta"] \\
\mathcal{D}^{op} \times_\Lambda \mathcal{D} \arrow[ur, "G"'] & & & \mathcal{D}^{op} \times_\Lambda \mathcal{D} \arrow[ur, "G"'] &
\end{tikzcd}$$

We write $\Pi : \Lambda_{/\mathcal{S}} \to \Lambda$ for the evident projection 2-functor.

Definition 4.3.2. A section of Π is a 2-functor $\mathrm{M} : \Lambda \to \Lambda_{/\mathcal{S}}$ such that

  - the composite $\Lambda \xrightarrow{\mathrm{M}} \Lambda_{/\mathcal{S}} \xrightarrow{\Pi} \Lambda$ is the identity of Λ
  - the image of $\mathcal{S}$ by M is given by the internal hom of $[\_, \_]_{\mathcal{S}} : \mathcal{S}^{op} \times_\Lambda \mathcal{S} \to \mathcal{S}$ of $\mathcal{S}$.

We say that a section satisfies a property regarding 2-functors if its underlying 2-functor does.

Remark 4.3.3. We introduce the following notations in order to make the manipulation of $\Lambda_{/\mathcal{S}}$ easier.

  - We write $\Lambda_{/\mathcal{S}} = \coprod\limits_{\mathcal{C}\in\Lambda} \Lambda(\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S})$.
  - For any pair of objects $F^{\mathcal{C}}, G^{\mathcal{D}}$ of $\Lambda_{/\mathcal{S}}$, we write

$$\Lambda_{/\mathcal{S}}\left(F^{\mathcal{C}}, G^{\mathcal{D}}\right) = \coprod_{\psi_1,\psi_2\in\Lambda(\mathcal{C},\mathcal{D})} \Lambda(\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S})(F, G \circ (\psi_1^{op}, \psi_2)),$$

  Note that the data of a morphism in $\Lambda_{/\mathcal{S}}$ from $F^{\mathcal{C}}$ to $G^{\mathcal{D}}$ effectively corresponds to the data of morphisms ψ_1, ψ_2 in Λ from $\mathcal{C}$ to $\mathcal{D}$, together with a morphism α in $\Lambda(\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S})$ from F to $G(\psi_1, \psi_2)$.
  - Let $(\psi_1, \psi_2, \alpha), (\phi_1, \phi_2, \beta) \in \Lambda_{/\mathcal{S}}(F^{\mathcal{C}}, G^{\mathcal{D}})$. We write

$$\coprod_{\substack{\nu_1\in\Lambda(\mathcal{C},\mathcal{D})(\psi_1,\phi_1)\\ \nu_2\in\Lambda(\mathcal{C},\mathcal{D})(\psi_2,\phi_2)}} \Lambda(\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S})\left((F, G(\psi_1^{op}, \phi_2))(G.(1, \nu_2) \circ \alpha, G.(\nu_1^{op}, 1) \circ \beta\right)$$

  for the set $\Lambda_{/\mathcal{S}}\left(F^{\mathcal{C}}, G^{\mathcal{D}}\right)((\psi, \alpha), (\phi, \beta))$. In the same way, the data of a morphism in $\Lambda_{/\mathcal{S}}\left(F^{\mathcal{C}}, G^{\mathcal{D}}\right)$ from (ψ, α) to (ϕ, β) effectively corresponds to the data of morphisms ν_i in $\Lambda(\mathcal{C}, \mathcal{D})$ from ψ_i to ϕ_i, $i = 1, 2$, such that $G.(1, \nu_2) \circ \alpha = G.(\nu_1^{op}, 1) \circ \beta$. Following our conventions, each factor of the coproduct below is a proposition, or equivalently a truth value, which is true if and only if the equality $(G.\eta) \circ \alpha = \beta$ holds[21].

Proposition 4.3.4. *The 2-category $\Lambda_{/\mathcal{S}}$ inherits a monoidal structure from the monoidal structure of Λ and the monoidal structure on $\mathcal{S}$. Moreover, the object of*

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[21] Recall from Appendix A that a truth value has an element if and only if this truth value is 'true'. Consequently, $(G.(1, \nu_2) \circ \alpha = G.(\nu_1^{op}, 1) \circ \beta)$ has an element if and only if the corresponding equality holds, and the data of an element of $(G.(1, \nu_2) \circ \alpha = G.(\nu_1^{op}, 1) \circ \beta)$ corresponds to the data of a proof of this equality.

$\Lambda_{/\mathcal{S}}$ corresponding to the internal hom $[\_,\_]_{\mathcal{S}} : \mathcal{S}^{op} \times_\Lambda \mathcal{S} \to \mathcal{S}$ of $\mathcal{S}$ has a monoidal structure in $\Lambda_{/\mathcal{S}}$.

Proof. We let the tensor product

$$\times_{/\mathcal{S}} : \coprod_{\mathcal{C}\in\Lambda} \Lambda(\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}) \times \coprod_{\mathcal{D}\in\Lambda} \Lambda(\mathcal{D}^{op} \times_\Lambda \mathcal{D}, \mathcal{S}) \to \coprod_{\mathcal{E}\in\Lambda} \Lambda(\mathcal{E}^{op} \times_\Lambda \mathcal{E}, \mathcal{S})$$

be induced by the tensor product $\times_\Lambda : \Lambda \times_\Lambda \Lambda \to \Lambda$, together with the induced functor

$$\Lambda(\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}) \times \Lambda(\mathcal{D}^{op} \times_\Lambda \mathcal{D}, \mathcal{S}) \xrightarrow{\times_\Lambda} \Lambda((\mathcal{C}^{op} \times_\Lambda \mathcal{C}) \times_\Lambda (\mathcal{D}^{op} \times_\Lambda \mathcal{D}), \mathcal{S} \times_\Lambda \mathcal{S})$$
$$\xrightarrow{\Lambda(\cong, \times_{\mathcal{S}})} \Lambda((\mathcal{C} \times_\Lambda \mathcal{D})^{op} \times_\Lambda (\mathcal{C} \times_\Lambda \mathcal{D}), \mathcal{S}).$$

We readily obtain a monoidal 2-category structure on $\Lambda_{/\mathcal{S}}$ whose unit $1_{/\mathcal{S}} : *_\Lambda \to \mathcal{S}$ is given by the object of $\Lambda_{/\mathcal{S}}$ corresponding to the unit of $\mathcal{S}$. The lax monoidal structure of $[\_,\_]_{\mathcal{S}}$ yields a 2-morphism in Λ

$$\begin{tikzcd}
(\mathcal{S} \times_\Lambda \mathcal{S})^{op} \times_\Lambda (\mathcal{S} \times_\Lambda \mathcal{S}) \arrow[r, "\cong"] \arrow[d, dashed, "\times_{\mathcal{S}}^{op} \times_\Lambda \times_{\mathcal{S}}"'] & (\mathcal{S}^{op} \times_\Lambda \mathcal{S}) \times_\Lambda (\mathcal{S}^{op} \times_\Lambda \mathcal{S}) \arrow[r, "{[_,_]_{\mathcal{S}} \times_\Lambda [_,_]_{\mathcal{S}}}"] \arrow[d, phantom, "\Leftarrow"] & \mathcal{S} \times \mathcal{S} \arrow[r, "\times_{\mathcal{S}}"] & \mathcal{S} \\
\mathcal{S}^{op} \times \mathcal{S} \arrow[rrru, "{[_,_]_{\mathcal{S}}}"'] & & &
\end{tikzcd}$$

from which we deduce a monoidal structure on $[\_,\_]_{\mathcal{S}} : \mathcal{S}^{op} \times_\Lambda \mathcal{S} \to \mathcal{S}$. $\square$

Proposition 4.3.5. *The 2-category $\Lambda_{/\mathcal{S}}$ is equipped with an internal hom 2-functor*

$$[\![\_,\_]\!]_{\Lambda/\mathcal{S}} : \Lambda_{/\mathcal{S}}^{op} \times \Lambda_{/\mathcal{S}} \to \Lambda_{/\mathcal{S}}$$

which gives to $\Lambda_{/\mathcal{S}}$ the structure of a closed monoidal 2-category.

Let $F : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ and $G : \mathcal{D}^{op} \times_\Lambda \mathcal{D} \to \mathcal{S}$ be regarded as objects of $\Lambda_{/\mathcal{S}}$. The internal hom yields an object

$$[\![F, G]\!]_{/\mathcal{S}} : [\mathcal{C}, \mathcal{D}]_\Lambda^{op} \times_\Lambda [\mathcal{C}, \mathcal{D}]_\Lambda \to \mathcal{S}.$$

whose value on $\psi_1, \psi_2 \in [\mathcal{C}, D]_\Lambda$ is given by

$$[\![F, G]\!]_{/\mathcal{S}}(\psi_1, \psi_2) \xrightarrow{\cong} \int^F G \circ (\psi_1, \psi_2) \xrightarrow{\cong} \int^{\mathcal{C}^{op}\times_\Lambda \mathcal{C}} [F(\_,\_), G(\psi_1 \_, \psi_2 \_)]_{\mathcal{S}}.$$

Proof. By using the notation introduced in Remark 4.3.3, the 2-category $\Lambda_{/\mathcal{S}}^{op}$ can be written as $\Lambda_{/\mathcal{S}}^{op} = \coprod_{\mathcal{C}\in\Lambda^{op}} \Lambda(\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S})^{op}$. Let $\mathcal{C}$ and $\mathcal{D}$ be objects of Λ and define natural morphism

$$[\![\_,\_]\!]_{/\mathcal{S}} : [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda^{op} \times_\Lambda [\mathcal{D}^{op} \times_\Lambda \mathcal{D}, \mathcal{S}]_\Lambda \to \big[[\mathcal{C}, \mathcal{D}]_\Lambda^{op} \times_\Lambda [\mathcal{C}, \mathcal{D}]_\Lambda, \mathcal{S}\big]_\Lambda$$

as follows. The closed monoidal structure of Λ yields a morphism

$$[\mathcal{C}, \_]_\Lambda : [\mathcal{D}^{op} \times_\Lambda \mathcal{D}, \mathcal{S}]_\Lambda \to \big[[\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{D}^{op} \times_\Lambda \mathcal{D}]_\Lambda, [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda\big]_\Lambda.$$

which is natural in the objects $\mathcal{C}$ and $\mathcal{D}$ of Λ. The lax monoidal structure of the internal hom of Λ yields a morphism

$$\big[[\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{D}^{op} \times_\Lambda \mathcal{D}]_\Lambda, [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda\big]_\Lambda \to \big[[\mathcal{C}, \mathcal{D}]_\Lambda^{op} \times_\Lambda [\mathcal{C}, \mathcal{D}]_\Lambda, [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda\big]_\Lambda.$$

We write $\chi : [\mathcal{D}^{op} \times_\Lambda \mathcal{D}, \mathcal{S}]_\Lambda \to \left[[\mathcal{C},\mathcal{D}]_\Lambda^{op} \times_\Lambda [\mathcal{C},\mathcal{D}]_\Lambda, [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda\right]_\Lambda$ for the morphism obtained by composition of the two morphisms just defined. Recall from Remark 4.2.13 that the relative end yields a morphism

$$\int^\bullet : [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda^{op} \to \left[[\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda, \mathcal{S}\right]_\Lambda .$$

We let $[\![-,-]\!]_{/\mathcal{S}}$ be defined by the composite

$$\begin{aligned}&[\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda^{op} \times_\Lambda [\mathcal{D}^{op} \times_\Lambda \mathcal{D}, \mathcal{S}]_\Lambda \\ &\qquad \xrightarrow{\int^\bullet \times_\Lambda \chi} \left[[\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda, \mathcal{S}\right]_\Lambda \times_\Lambda \left[[\mathcal{C},\mathcal{D}]_\Lambda^{op} \times_\Lambda [\mathcal{C},\mathcal{D}]_\Lambda, [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda\right]_\Lambda \\ &\qquad \to \left[[\mathcal{C},\mathcal{D}]_\Lambda^{op} \times_\Lambda [\mathcal{C},\mathcal{D}]_\Lambda, \mathcal{S}\right]_\Lambda\end{aligned}$$

where the last morphism is the internal composition in Λ obtained from its closed monoidal structure. We obtain a 2-functor

$$\Lambda_{/\mathcal{S}}^{op} \times \Lambda_{/\mathcal{S}} \xrightarrow{\cong} \coprod_{(\mathcal{C},\mathcal{D}):\Lambda^{op}\times\Lambda} \Lambda(\mathcal{C},\mathcal{S})^{op} \times \Lambda(\mathcal{D},\mathcal{S}) \xrightarrow{\coprod_{[-,-]_\Lambda} [\![-,-]\!]^*_{/\mathcal{S}}} \Lambda([\mathcal{C},\mathcal{D}]_\Lambda, \mathcal{S}) \to \Lambda_{/\mathcal{S}}$$

from the internal hom $[-,-]_\Lambda$ of Λ and the morphism $[\![-,-]\!]_{/\mathcal{S}}$ defined above, where $[\![-,-]\!]^*_{/\mathcal{S}}$ is the functor obtained from $[\![-,-]\!]_{/\mathcal{S}}$ by the forgetful 2-functor $\Lambda \to \textsc{Cat}$.

By construction, the internal hom thus defined gives to $\Lambda_{/\mathcal{S}}$ the structure of a closed monoidal 2-category. We provide the explicit calculation for better clarity.

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be objects of Λ, $F : \mathcal{C} \to \mathcal{S}, G : \mathcal{D} \to \mathcal{S}$ and $H : \mathcal{E} \to \mathcal{S}$ be morphisms in Λ, regarded as objects of $\Lambda_{/\mathcal{S}}$. First write

$$\Lambda_{/\mathcal{S}}(F \times_{/\mathcal{S}} G, H) = \coprod_{\psi_1,\psi_2 \in \Lambda(\mathcal{C}\times_\Lambda\mathcal{D},\mathcal{E})} \Lambda((\mathcal{C}\times_\Lambda\mathcal{D})^{op} \times_\Lambda (\mathcal{C}\times_\Lambda\mathcal{D}), \mathcal{S})(F \times_{/\mathcal{S}} G, H \circ (\psi_1^{op}, \psi_2)).$$

On the other hand, we have

$$\Lambda_{/\mathcal{S}}(F, [\![G,H]\!]_{\Lambda/\mathcal{S}}) = \coprod_{\psi_1,\psi_2 \in \Lambda(\mathcal{C},[\mathcal{D},\mathcal{E}]_\Lambda)} \Lambda(\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S})(F, [\![G,H]\!]_{/\mathcal{S}}(\psi_1^{op}, \psi_2)).$$

Let $\psi_1, \psi_2 \in \Lambda(\mathcal{C}, [\mathcal{D},\mathcal{E}]_\Lambda)$. By definition, we have

$$\begin{aligned}\Lambda(\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S})(F, [\![G,H]\!]_{/\mathcal{S}}(\psi_1^{op}, \psi_2)) &\xrightarrow{\cong} [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]^*_\Lambda \left(F, \int^G H \circ (\psi_1, \psi_2)\right) \\ &\cong [(\mathcal{C} \times_\Lambda \mathcal{D})^{op} \times_\Lambda (\mathcal{C} \times_\Lambda \mathcal{D}), \mathcal{S}]^*_\Lambda \,(F \times_{/\mathcal{S}} G, H \circ (\psi_1^{op}, \psi_2)).\end{aligned}$$

The closed monoidal structure on Λ yields an isomorphism $\Lambda(\mathcal{C}\times_\Lambda\mathcal{D},\mathcal{E}) \xrightarrow{\cong} \Lambda(\mathcal{C},[\mathcal{D},\mathcal{E}]_\Lambda)$, so that we obtain

$$\coprod_{\psi \in \Lambda(\mathcal{D}\times_\Lambda\mathcal{C},\mathcal{E})} \Lambda(\mathcal{D}\times_\Lambda\mathcal{C},\mathcal{S})(G \times_{/\mathcal{S}} F, H\psi) \xrightarrow{\cong} \coprod_{\psi^\# \in \Lambda(\mathcal{C},[\mathcal{D},\mathcal{E}]_\Lambda)} \Lambda(\mathcal{C},[\mathcal{D},\mathcal{S}]_\Lambda)(F, [\![G,H]\!]_{/\mathcal{S}}\psi^\#),$$

hence the result. □

Definition 4.3.6. An entangled $\mathcal{S}$-enrichment on Λ is a strongly closed monoidal section of Π

$$\textsc{Map}_\mathcal{S} : \Lambda \to \Lambda_{/\mathcal{S}}.$$

Notation 4.3.7.

– For each object $\mathcal{C}$ of Λ, we write $\mathcal{C}(\_,\_) : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ for $\mathrm{Map}_{\mathcal{S}}\, \mathcal{C}$
– For each pair of objects $\mathcal{C}, \mathcal{D}$ of Λ and each morphism $F : \mathcal{C} \to \mathcal{D}$, we write

$$F(\_,\_) : \mathcal{C}(\_,\_) \to \mathcal{D}(F\_, F\_)$$

for the morphism $\mathrm{Map}_{\mathcal{S}}\, F$ in $\Lambda_{/\mathcal{S}}$. This morphism can be displayed by the following diagram in Λ:

$$\begin{tikzcd} \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[r, "\mathcal{C}(-,-)"] \arrow[d, "F^{op}\times_\Lambda F"'] & S \arrow[dl, phantom, "\Swarrow F(\_,\_)"] \\ \mathcal{D}^{op} \times_\Lambda \mathcal{D} \arrow[ur, bend right, "\mathcal{D}(-,-)"'] & \end{tikzcd}$$

Remark 4.3.8. Note that in particular, for all $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \Lambda$, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, we have

– a natural isomorphism

$$\begin{tikzcd} \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[r, "\mathcal{C}(-,-)"] \arrow[d, "\tau\cong"'] & \mathcal{S} \arrow[dl, phantom, "\cong\Downarrow"] \\ \mathcal{C} \times_\Lambda \mathcal{C}^{op} \arrow[ur, bend right, "\mathcal{C}^{op}(-,-)"'] & \end{tikzcd}$$

– a natural isomorphism

$$\begin{tikzcd} (\mathcal{C} \times_\Lambda \mathcal{D})^{op} \times_\Lambda (\mathcal{C} \times_\Lambda \mathcal{D}) \arrow[r, "\mathcal{C}\times_\Lambda\mathcal{D}(-,-)"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\cong\Downarrow"] & \mathcal{S} \\ (\mathcal{C}^{op} \times_\Lambda \mathcal{C}) \times_\Lambda (\mathcal{D}^{op} \times_\Lambda \mathcal{D}) \arrow[r, "\mathcal{C}(\_,\_)\times_\Lambda\mathcal{D}(\_,\_)"'] & \mathcal{S} \times_\Lambda \mathcal{S} \arrow[u, "\times_{\mathcal{S}}"'] \end{tikzcd}$$

Moreover,
– the morphism

$$*_\Lambda(-,-) : *_\Lambda^{op} \times_\Lambda *_\Lambda \to \mathcal{S}$$

corresponds to $*_{\mathcal{S}} : *_\Lambda \to \mathcal{S}$ under the isomorphism $*_\Lambda^{op} \times_\Lambda *_\Lambda \cong *_\Lambda$,
– $G(F\_, F\_) \circ F(\_,\_) = GF(\_,\_)$, so that

$$\begin{tikzcd} \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[d, "F^{op}\times_\Lambda F"'] \arrow[dr, "\mathcal{C}(-,-)"] \arrow[dr, phantom, "\Swarrow F(\_,\_)"', shift right=3] & \\ \mathcal{D}^{op} \times_\Lambda \mathcal{D} \arrow[r, "\mathcal{D}(-,-)"] \arrow[d, "G^{op}\times_\Lambda G"'] & S \\ \mathcal{E}^{op} \times_\Lambda \mathcal{E} \arrow[ur, "\mathcal{E}(-,-)"'] \arrow[ur, phantom, "\Swarrow G(\_,\_)", shift left=3] & \end{tikzcd} = \begin{tikzcd} \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[dd, "GF^{op}\times_\Lambda GF"'] \arrow[dr, "\mathcal{C}(-,-)"] & \\ \arrow[r, phantom, "\Swarrow GF(\_,\_)"] & S \\ \mathcal{E}^{op} \times_\Lambda \mathcal{E} \arrow[ur, "\mathcal{E}(-,-)"'] & \end{tikzcd}$$

– the 2-morphisms obtained by composition of the diagrams

$$\begin{tikzcd} \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[rr, "\mathcal{C}(-,-)"] \arrow[dr, bend right, "F^{op}\times_\Lambda G"'] \arrow[dr, bend left, "F^{op}\times_\Lambda F"] \arrow[dr, phantom, "\Swarrow id\times_\Lambda\eta"] & \arrow[d, phantom, "\Downarrow F(\_,\_)"] & \mathcal{S} \\ & \mathcal{D}^{op} \times_\Lambda \mathcal{D} \arrow[ur, "\mathcal{D}(-,-)"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[rr, "\mathcal{C}(-,-)"] \arrow[dr, bend right, "F^{op}\times_\Lambda G"'] \arrow[dr, bend left, "G^{op}\times_\Lambda G"] \arrow[dr, phantom, "\Swarrow \eta^{op}\times_\Lambda id"] & \arrow[d, phantom, "\Downarrow G(\_,\_)"] & \mathcal{S} \\ & \mathcal{D}^{op} \times_\Lambda \mathcal{D}. \arrow[ur, "\mathcal{D}(-,-)"'] & \end{tikzcd}$$

are equal.

REMARK 4.3.9. If $(\Lambda, \mathcal{S})$ is an entangled 2-category, then we have a forgetful 2-functor

$$\Lambda \to \mathrm{CAT}_{\mathcal{S}^*}.$$

REMARK 4.3.10. Let $\mathcal{C} \in \Lambda$. For $X, Y \in \mathcal{C}$, the object $\mathcal{C}(X, Y) \in \mathcal{S}$ may be regarded as the object of morphisms in $\mathcal{C}$ from X to Y. The morphism $\mathcal{C}(\_,\_) : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ induces a 2-morphism in Λ

$$\mathcal{C}^{op}(\_,\_) \times_{\mathcal{S}} \mathcal{C}(\_,\_) \to \mathcal{S}(\mathcal{C}(\_,\_), \mathcal{C}(\_,\_)).$$

In particular, we obtain natural morphisms in $\mathcal{S}$

$$\mathcal{C}(X_1, Y_1) \times_{\mathcal{S}} \mathcal{C}(X_2, Y_2) \to \mathcal{S}(\mathcal{C}(Y_1, Y_2), \mathcal{C}(X_1, X_2)),$$

which we can regard as composition of morphisms on the left and on the right.

DEFINITION 4.3.11. Let $\mathcal{C} \in \Lambda$. Any $X : *_\Lambda \to \mathcal{C}$ yields a 2-morphism

$$\begin{tikzcd} *_\Lambda^{op} \times *_\Lambda \arrow[r, "*_\Lambda(-,-)"] \arrow[d, "X^{op}\times X"'] & \mathcal{S} \arrow[dl, phantom, "\swarrow \alpha_X"] \\ \mathcal{C}^{op} \times \mathcal{C} \arrow[ur, bend right, "\mathcal{C}(-,-)"'] & \end{tikzcd},$$

hence a morphism $\imath_X : *_{\mathcal{S}} \Rightarrow \mathcal{C}(X, X)$ in $\mathcal{S}^*$, which we call the identity of X.

REMARK 4.3.12. In the previous section, we observed that each object $\mathcal{C}$ of Λ has an underlying category $\mathcal{C}^*$, so that for $X, Y : * \to \mathcal{C}$, the set $\mathcal{C}^*(X, Y)$ can be regarded as the set of morphisms from X to Y in $\mathcal{C}$. The entangled $\mathcal{S}$-enrichment on Λ yields an object $\mathcal{C}(X, Y) \in \mathcal{S}^*$. Consequently, the set $\mathcal{S}^*(*_{\mathcal{S}}, \mathcal{C}(X, Y))$ may also represent a set of morphisms from X to Y in $\mathcal{C}$.

Let $f : X \to Y$ be morphism in $\mathcal{C}^*$. The structural morphism $\mathcal{C}(\_,\_)$ of $\mathcal{C}$ yields a morphism $\mathcal{C}(-, X) \xrightarrow{\mathcal{C}(-,f)} \mathcal{C}(-, Y)$ in the category $\Lambda(\mathcal{C}^{op}, \mathcal{S})$, and hence a map

$$\mathcal{C}^*(X, Y) \to \Lambda(\mathcal{C}^{op}, \mathcal{S})(\mathcal{C}(-, X), \mathcal{C}(-, Y)).$$

The precomposition of 2-morphisms with $X : *_\Lambda \to \mathcal{C}^{op}$ yields a morphism

$$\Lambda(\mathcal{C}^{op}, \mathcal{S})(\mathcal{C}(-, X), \mathcal{C}(-, Y)) \to \Lambda(*_\Lambda, \mathcal{S})(\mathcal{C}(X, X), \mathcal{C}(X, Y)).$$

We obtain a morphism

$$\Lambda(*_\Lambda, \mathcal{S})(\mathcal{C}(X, X), \mathcal{C}(X, Y)) \to \mathcal{S}^*(*_{\mathcal{S}}, \mathcal{C}(X, Y))$$

by precomposition in the category $\mathcal{S}^*$ with the internal identity $\imath_X : *_* \to \mathcal{C}(X, X)$ of X, hence a comparison morphism

$$\mathcal{C}^*(X, Y) \to \mathcal{S}^*(*_{\mathcal{S}}, \mathcal{C}(X, Y)).$$

Note that the identity of X in $\mathcal{C}^*$ is send to the internal unit of $\mathcal{C}$.

REMARK 4.3.13. The compatibility of the 2-functor $\mathrm{MAP}_{\mathcal{S}}$ with respect to the closed structures of Λ and $\Lambda_{/\mathcal{S}}$ can be expressed as follows. Let $\mathcal{C}$ and $\mathcal{D}$ be objects of Λ. The 2-functor $\mathrm{MAP}_{\mathcal{S}}$ sends the object $[\mathcal{C}, \mathcal{D}]_\Lambda$ of Λ to an object $[\mathcal{C}, \mathcal{D}](\_,\_) : [\mathcal{C}, \mathcal{D}]_\Lambda^{op} \times_\Lambda [\mathcal{C}, \mathcal{D}]_\Lambda \to \mathcal{S}$ of $\Lambda_{/\mathcal{S}}$.

On the other hand, the object $\mathcal{C}$ of Λ is send to the object $\mathcal{C}(\_,\_) : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ of $\Lambda_{/\mathcal{S}}$, and the object $\mathcal{D}$ of Λ is send to the object $\mathcal{D}(\_,\_) : \mathcal{D}^{op} \times_\Lambda \mathcal{D} \to \mathcal{S}$. The closed monoidal structure on $\Lambda_{/\mathcal{S}}$ yields an object

$$[\![\mathcal{C}(\_,\_), \mathcal{D}(\_,\_)]\!]_{/\mathcal{S}} : [\mathcal{C}, \mathcal{D}]_\Lambda^{op} \times_\Lambda [\mathcal{C}, \mathcal{D}]_\Lambda \to \mathcal{S}$$

of $\Lambda_{/\mathcal{S}}$, such that

$$\begin{tikzcd} \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[r, "\mathcal{C}(\_,\_)"] \arrow[d, dashed] & \mathcal{S} \arrow[dl, phantom, "\Leftarrow"] \\ \mathcal{D}^{op} \times_\Lambda \mathcal{D} \arrow[ur, "\mathcal{D}(\_,\_)"'] & \end{tikzcd} \iff \begin{tikzcd} *_\Lambda \arrow[r, "*_\mathcal{S}"] \arrow[d, dashed] & \mathcal{S} \arrow[dl, phantom, "\Leftarrow"] \\ {[\mathcal{C},\mathcal{D}]_\Lambda^{op} \times_\Lambda [\mathcal{C},\mathcal{D}]_\Lambda} \arrow[ur, "{[\![\mathcal{C}(\_,\_),\mathcal{D}(\_,\_)]\!]_{/\mathcal{S}}}"'] & \end{tikzcd}.$$

The closed structure of $\mathrm{MAP}_\mathcal{S}$ yields an isomorphism $[\mathcal{C},\mathcal{D}]_\Lambda(\_,\_) \cong [\![\mathcal{C}(\_,\_),\mathcal{D}(\_,\_)]\!]_{/\mathcal{S}}$. Let $F, G : \mathcal{C} \to \mathcal{D}$ be morphisms in Λ. By definition, we have

$$[\![\mathcal{C}(\_,\_),\mathcal{D}(\_,\_)]\!]_{/\mathcal{S}}(F,G) = \int^{\mathcal{C}(\_,\_)} \mathcal{D}(F\_,G\_),$$

and hence $[\mathcal{C},\mathcal{D}]_\Lambda(F,G)$ is given by the same formula.

REMARK 4.3.14. Let $\mathcal{C},\mathcal{D}$ be objects of Λ and $F, G : \mathcal{C} \to \mathcal{D}$ be morphisms. The 2-functor $\mathrm{MAP}_\mathcal{S}$ yields a map

$$\Lambda(\mathcal{C},\mathcal{D})(F,G) \to \Lambda(\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S})(\mathcal{C}(\_,\_),\mathcal{D}(F\_,G\_)).$$

By definition of relative ends, we obtain an isomorphism

$$\Lambda(\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S})\left(\mathcal{C}(\_,\_),\mathcal{D}(F\_,G\_)\right) \xrightarrow{\cong} \mathcal{S}^*\left(*_\mathcal{S}, \int^{\mathcal{C}(\_,\_)} \mathcal{D}(F\_,G\_)\right)$$

PROPOSITION 4.3.15. *The 2-functor* $\mathrm{MAP}_\mathcal{S}$ *is locally fully faithful.*

PROOF. Let $\mathcal{C}$ and $\mathcal{D}$ be objects of Λ and $F, G : \mathcal{C} \to \mathcal{D}$ be morphisms. The 2-functor $\mathrm{MAP}_\mathcal{S}$ sends an element η of $\Lambda(\mathcal{C},\mathcal{D})(F,G)$ to an element (η,η) of $\Lambda_{/\mathcal{S}}$, which is such that the composite of the left hand side diagram is equal to the composite of the right hand side diagram:

$$\begin{tikzcd} \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[rr, "\mathcal{C}(-,-)"] \arrow[dr, bend left, "F^{op}\times_\Lambda F"] \arrow[dr, bend right, "F^{op}\times_\Lambda G"'] \arrow[dr, phantom, "\Leftarrow id\times_\Lambda \eta"] & \arrow[d, phantom, "\Downarrow F(\_,\_)"] & \mathcal{S} \\ & \mathcal{D}^{op} \times_\Lambda \mathcal{D} \arrow[ur, "\mathcal{D}(-,-)"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}^{op} \times_\Lambda \mathcal{C} \arrow[rr, "\mathcal{C}(-,-)"] \arrow[dr, bend left, "G^{op}\times_\Lambda G"] \arrow[dr, bend right, "F^{op}\times_\Lambda G"'] \arrow[dr, phantom, "\Leftarrow \eta^{op}\times_\Lambda id"] & \arrow[d, phantom, "\Downarrow G(\_,\_)"] & \mathcal{S} \\ & \mathcal{D}^{op} \times_\Lambda \mathcal{D}. \arrow[ur, "\mathcal{D}(-,-)"'] & \end{tikzcd}.$$

A morphism from $F(\_,\_)$ to $G(\_,\_)$ in $\Lambda_{/\mathcal{S}}$ consists in the data of elements α_1, α_2 of $\Lambda(\mathcal{C},\mathcal{D})(F,G)$ satisfying

$$\mathcal{D}(F,\nu_1) \circ F(\_,\_) = \mathcal{D}(\nu_2, G) \circ G(\_,\_)$$

in the set

$$\Lambda(\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S})(\mathcal{C}(\_,\_),\mathcal{D}(F\_,G\_)).$$

Let $X : *_\Lambda \to \mathcal{C}$, by precomposition with $\imath_X : *_\mathcal{S} \to \mathcal{C}(X,X)$, we obtain $\nu_1^X \circ F(X,X)(\epsilon_X) = G(X,X)(\epsilon_X) \circ \nu_1^X$ in the set $\mathcal{D}^*(FX,GX)$. We have $F(X,X)(\epsilon_X) = \epsilon_{FX}$, and hence $\nu_1^X \circ F(X,X)(\epsilon_X) = \nu_1^X$. We obtain the equality $\nu_1^X = \nu_2^X$ in the set $\mathcal{D}(FX,GX)$. Recall that on an $*_\Lambda$-primary category, the forgetful 2-functor $\Lambda(*_\Lambda,\_)$ is in particular locally faithful. Consequently, for ν_1, ν_2 in $\Lambda(\mathcal{C},\mathcal{D})(F,G)$, we have

$$\Lambda(\mathcal{C},\mathcal{D})(F,G)(\nu_1,\nu_2) = \int^{X\in\mathcal{C}} \mathcal{D}^*(FX,GX)(\nu_1^X,\nu_2^X).$$

Hence $\nu_1 = \nu_2$ and $\mathrm{MAP}_\mathcal{S}$ is locally fully faithful. $\square$

REMARK 4.3.16. The statement of Proposition 4.3.15 may be seen as a more general version of the Yoneda Lemma. We will see that the Yoneda lemma directly follows from this proposition.

COROLLARY 4.3.17. *For any object $\mathcal{C}$ of Λ and for any objects X, Y of $\mathcal{C}$, we obtain an isomorphism*

$$\mathcal{C}^*(X,Y) \xrightarrow{\cong} \mathcal{S}^*(*_{\mathcal{S}}, \mathcal{C}(X,Y)).$$

COROLLARY 4.3.18. *Let $\mathcal{C}$ be an object of Λ and $F : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$. We have the following isomorphisms in $\mathcal{S}^*$:*

$$\int^{\mathcal{C}} F \;\cong \int^{\mathcal{C}(-,-)} F,$$

and dually,

$$\int_{\mathcal{C}} F \;\cong \int_{\mathcal{C}^{op}(-,-)} F.$$

In the next subsection, we will define ends and coends internally so that they pointwise correspond to the ends and coends of Definition 4.2.1 and 4.2.5. Those isomorphisms impose the definition of internal ends and coends.

COROLLARY 4.3.19. *For any objects $\mathcal{C}$ and $\mathcal{D}$ of Λ, and for any morphisms $F, G : \mathcal{C} \to \mathcal{D}$, we have the following expression of the internal object of morphisms from F to G:*

$$[\mathcal{C}, \mathcal{D}]_\Lambda(F,G) \xrightarrow{\cong} \int^{X : *_\Lambda \to \mathcal{C}} \mathcal{D}(FX, GX).$$

4.4. Back to internal constructions. In this subsection, $(\Lambda, \times_\Lambda, [-,-]_\Lambda, *_\Lambda)$ is a closed monoidal 2-category equipped with an entangled $\mathcal{S}$-enrichement. In particular, $\mathcal{S}$ is bicomplete, and $(\mathcal{S}, \times_{\mathcal{S}}, [-,-]_{\mathcal{S}}, *_{\mathcal{S}})$ has the structure of a closed monoid in Λ.

We use the results of the previous subsection to provide a characterization of the internal constructions defined in the previous section in terms of their values on objects[22]. In particular, we provide explicit formulas for Kan extensions.

We study the case where Λ is equipped with another monoidal structure which is compatible with its cartesian closed structure and provide an explicit formula for the Day convolution product.

DEFINITION 4.4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be objects of Λ and $F : \mathcal{C} \to \mathcal{D}$ be a morphism. We say that F is fully faithful if its image $F(-,-) : \mathcal{C}(-,-) \to \mathcal{D}(F(-), F(-))$ under $\mathrm{MAP}_{\mathcal{S}}$ is an isomorphism.

Adjunctions.

DEFINITION 4.4.2. An internal adjunction between objects $\mathcal{C}, \mathcal{D}$ of Λ consists in the data of a morphism $F : \mathcal{C} \to \mathcal{D}$, a morphism $G : \mathcal{D} \to \mathcal{C}$, and an isomorphism

$$\mathcal{C}(-, G-) \xrightarrow{\cong} \mathcal{D}(F-, -)$$

in the category $\Lambda(\mathcal{C}^{op} \times_\Lambda \mathcal{D}, \mathcal{S})$.

[22]If $\mathcal{C}$ is an object of Λ, then we say that X is an object of $\mathcal{C}$ if X is a morphism $*_\Lambda \to \mathcal{C}$ in Λ.

PROPOSITION 4.4.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be objects of Λ. An internal adjunction between $\mathcal{C}$ and $\mathcal{D}$ yields an adjunction in the sense of Definition 2.2.1. If the 2-category Λ is $*_\Lambda$-primary, then a pair of functors form an internal adjunction if and only if they form an adjunction in the sense of Definition 2.2.1.*

PROOF. This is a straightforward consequence of the following isomorphism

$$[\mathcal{C}^{op} \times_\Lambda \mathcal{D}, \mathcal{S}]_\Lambda \left(\mathcal{C}(\_, G\_), \mathcal{D}(F\_, \_)\right) \xrightarrow{\cong} \int^{X,Y \in \mathcal{C}^{op} \times_\Lambda \mathcal{D}} \mathcal{S}\left(\mathcal{C}(X, GY), \mathcal{D}(FX, Y)\right).$$

Indeed, we also have

$$[\mathcal{C}, \mathcal{C}]_\Lambda (1_\mathcal{C}, GF) \xrightarrow{\cong} \int^{X \in \mathcal{C}} \mathcal{C}(X, GFX).$$

We use the forgetful 2-functor $\Lambda \to \textsc{Cat}$ to obtain the result. $\square$

Tensored objects.

DEFINITION 4.4.4. We say that an object $\mathcal{C}$ of Λ is tensored over $\mathcal{S}$ if it equipped with a morphism

$$\times_\mathcal{C} : \mathcal{S} \times_\Lambda \mathcal{C} \to \mathcal{C},$$

in Λ such that for each object $X : *_\Lambda \to \mathcal{C}$, the morphism $\_ \times_\mathcal{C} X : \mathcal{S} \to \mathcal{C}$ is left adjoint to $\mathcal{C}(X, \_) : \mathcal{C} \to \mathcal{S}$. In particular, for each objects X, Y of $\mathcal{C}$ and each object Z of $\mathcal{S}$, the resulting adjunction yields natural isomorphisms

$$\mathcal{C}(X \times_\mathcal{C}^\mathcal{S} Z, Y) \cong \mathcal{S}(Z, \mathcal{C}(X, Y)).$$

EXAMPLE 4.4.5. The object $\mathcal{S}$ is tensored over itself.

Ends and coends. Let $\mathcal{D}$ be an object of Λ such that $\mathcal{D}$ is tensored over $\mathcal{S}$. By definition, $\mathcal{D}$ is equipped with a morphism $\_ \times_\mathcal{D} \_ : \mathcal{S} \times_\Lambda \mathcal{D} \to \mathcal{D}$ such that for each object Z of $\mathcal{D}$, the induced morphism $Z \times_\mathcal{D} \_ : \mathcal{S} \to \mathcal{D}$ which is left adjoint to $\mathcal{D}(Z, \_)$. Each object $\mathcal{C}$ of Λ yields a morphism

$$\mathcal{C}(\_, \_) \times_\mathcal{D} \_ : \mathcal{D} \to [\mathcal{C}^{op} \times \mathcal{C}, \mathcal{D}]_\Lambda,$$

obtained by the composite

$$\mathcal{D} \xrightarrow{\mathcal{C}(\_,\_)} \mathcal{D} \times_\Lambda [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{S}]_\Lambda \to [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{D} \times_\Lambda \mathcal{S}]_\Lambda \xrightarrow{[\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \_ \times_\mathcal{D} \_]_\Lambda} [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{D}]_\Lambda.$$

DEFINITION 4.4.6. For any object $\mathcal{D}$ of Λ which is tensored over $\mathcal{S}$, and for any object $\mathcal{C}$ of Λ, we let the $\mathcal{D}$-valued end morphism

$$\int^\mathcal{C} : [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{D}]_\Lambda \to \mathcal{D}$$

be defined as the right adjoint of

$$\mathcal{C}(\_, \_) \times_\mathcal{D} \_ : \mathcal{D} \to [\mathcal{C}^{op} \times_\Lambda \mathcal{C}, \mathcal{D}]_\Lambda.$$

Note that the $\mathcal{S}$-valued end morphism corresponds to the $\mathcal{C}(\_, \_)$-relative end.

DEFINITION 4.4.7. For any object $\mathcal{C}$ of Λ, we let the coend morphism

$$\int_\mathcal{C} : [\mathcal{C} \times_\Lambda \mathcal{C}^{op}, \mathcal{S}]_\Lambda \to \mathcal{S}$$

be defined as the left adjoint of

$$[\mathcal{C}(\_, \_), \_]_\mathcal{S} : \mathcal{S} \to [\mathcal{C} \times_\Lambda \mathcal{C}^{op}, \mathcal{S}]_\Lambda.$$

Note that the coend morphisms thus defined corresponds to the $\mathcal{C}(\_,\_)$-relative coend.

REMARK 4.4.8. The forgetful 2-functor $\Lambda \to \mathrm{CAT}$ sends the internal end to the end defined in 4.2.1, and the internal coend to the coend of Definition 4.2.5

REMARK 4.4.9. Let $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}$. By Proposition 4.2.11, we obtain

$$\int^{\mathcal{C}} F \xrightarrow{\cong} \int^{\mathcal{C}^{op} \times_\Lambda C} [\mathcal{C}(\_,\_), F]_{\mathcal{S}},$$

and by Proposition 4.2.12, we obtain

$$\int_{\mathcal{C}} F \xrightarrow{\cong} \int_{\mathcal{C}^{op} \times_\Lambda \mathcal{C}} \mathcal{C}^{op}(\_,\_) \times_{\mathcal{S}} F.$$

Kan extensions.

PROPOSITION 4.4.10. *The internal left Kan extension with values in $\mathcal{S}$ is internally defined, in the sense that we have a morphism*

$$\mathbf{Lan}^{\mathcal{S}} : [\mathcal{C}_2, \mathcal{C}_1]^{op}_\Lambda \to [[\mathcal{C}_2, \mathcal{S}]_\Lambda, [\mathcal{C}_1, \mathcal{S}]_\Lambda]_\Lambda\,,$$

whose value $\mathbf{Lan}^{\mathcal{S}}_\mu$ on a morphism $\mu : \mathcal{C}_2 \to \mathcal{C}_1$ is the internal left adjoint of $[\mu, \mathcal{S}]_\Lambda$. For any morphism $F : \mathcal{C}_2 \to \mathcal{S}$, we obtain the following expression of $\mathbf{Lan}^{\mathcal{S}}_\mu F$:

$$\mathbf{Lan}^{\mathcal{S}}_\mu F = \int_{X : *_\Lambda \to \mathcal{C}_2} \mathcal{C}_1\,(\mu X, -) \times_{\mathcal{S}} FX.$$

Where the coend is taken over the morphism

$$\mathcal{C}_1\,(\mu\_, \_) \times_{\mathcal{S}} F(-) : \mathcal{C}_2^{op} \times \mathcal{C}_2 \to [\mathcal{C}_1, \mathcal{S}]_\Lambda.$$

PROOF. We have the following sequence of isomorphisms which are natural in each variable

$$\begin{aligned}
[\mathcal{C}_1, \mathcal{S}]_\Lambda \left(\int^{X : *_\Lambda \to \mathcal{C}_2} \mathcal{C}_1\,(\mu X, -) \times_{\mathcal{S}} FX, G \right) &\cong \int_{X : *_\Lambda \to \mathcal{C}_2} [\mathcal{D}, \mathcal{S}]_\Lambda\,(\mathcal{C}_1\,(\mu X, -) \times_{\mathcal{S}} FX, G) \\
&\cong \int_{X : *_\Lambda \to \mathcal{C}_2} \mathcal{S}\,(FX, [\mathcal{C}_1, \mathcal{S}]_\Lambda\,(\mathcal{D}\,(\mu X, -)\,, G)) \\
&\cong \int_{X : *_\Lambda \to \mathcal{C}_2} \mathcal{S}\,(FX, G\mu X) \cong [\mathcal{C}_2, \mathcal{S}]_\Lambda\,(F, G\mu).
\end{aligned}$$

We obtain

$$[\mathcal{C}_1, \mathcal{S}]_\Lambda \left(\mathbf{Lan}^{\mathcal{S}}_\mu F, G \right) \cong [\mathcal{C}_2, \mathcal{S}]_\Lambda\,(F, G\mu)\,,$$

hence the result. □

4.5. Presheaf objects.

PROPOSITION 4.5.1. *For each object $\mathcal{C}$ of Λ, the object $[\mathcal{C}^{op}, \mathcal{S}]_\Lambda$ is tensored over $\mathcal{S}$. The action of $\mathcal{S}$ over $[\mathcal{C}^{op}, \mathcal{S}]_\Lambda$*

$$\times^{\mathcal{S}}_{\mathcal{C}} : [\mathcal{C}^{op}, \mathcal{S}]_\Lambda \times_\Lambda \mathcal{S} \to [\mathcal{C}^{op}, \mathcal{S}]_\Lambda$$

is defined by the composite

$$\begin{tikzcd}
{[\mathcal{C}^{op},\mathcal{S}]_\Lambda \times_\Lambda \mathcal{S}} \arrow[rr, "\times^{\mathcal{S}}_{\mathcal{C}}"] \arrow[d, "\cong"'] & & {[\mathcal{C}^{op},\mathcal{S}]_\Lambda} \\
{[\mathcal{C}^{op},\mathcal{S}]_\Lambda \times_\Lambda [*_\Lambda,\mathcal{S}]_\Lambda} \arrow[r] & {[\mathcal{C}^{op}\times_\Lambda *_\Lambda, \mathcal{S}\times_\Lambda \mathcal{S}]_\Lambda.} \arrow[ur, "{[\mathcal{C}^{op},\times_{\mathcal{S}}]_\Lambda}"'] &
\end{tikzcd}$$

Proof. Let $Z : *_\Lambda \to \mathcal{S}$ and $F, G : *_\Lambda \to [\mathcal{C}^{op},\mathcal{S}]_\Lambda$, we have natural isomorphisms

$$[\mathcal{C}^{op},\mathcal{S}]_\Lambda\left(Z \times^{\mathcal{S}}_{\mathcal{C}} F, G\right) \cong \int^{X:*_\Lambda\to\mathcal{C}} \mathcal{S}\left(Z \times_{\mathcal{S}} FX, GX\right) \cong \int^{X:*_\Lambda\to\mathcal{C}} \mathcal{S}\left(Z, \mathcal{S}(FX,GX)\right)$$
$$\cong \mathcal{S}\left(Z, \int^{X:*_\Lambda\to\mathcal{C}} \mathcal{S}(FX,GX)\right) \cong \mathcal{S}\left(Z, [\mathcal{C}^{op},\mathcal{S}]_\Lambda(F,G)\right).$$

□

The Yoneda embedding. Let Λ be equipped with an entangled $\mathcal{S}$-enrichement. Recall that we assumed Λ to be equipped with a closed monoidal structure. For each object $\mathcal{C}$ of Λ, the data of a morphism

$$\mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$$

is therefore equivalent to the data of a morphism $\mathcal{C} \to [\mathcal{C}^{op},\mathcal{S}]_\Lambda$. In particular, the morphism $\mathcal{C}(-,-) : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ corresponds to a morphism $y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{op},\mathcal{S}]_\Lambda$. If $X : *_\Lambda \to \mathcal{C}$, then we also write $\mathcal{C}(-,X) : \mathcal{C}^{op} \to \mathcal{S}$ for $y_{\mathcal{C}}X : *_\Lambda \to [\mathcal{C}^{op},\mathcal{S}]_\Lambda$.

Definition 4.5.2. Let $\mathcal{C}$ be an object of Λ. Define its object of presheaves as the object $[\mathcal{C}^{op},\mathcal{S}]_\Lambda$ of Λ. We call the morphism

$$y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{op},\mathcal{S}]_\Lambda$$

the Yoneda embedding.

Proposition 4.5.3. *Let $\mathcal{C}$ be an object of Λ, let X be an object of $\mathcal{C}$, and let $F : \mathcal{C}^{op} \to \mathcal{S}$. We have natural isomorphisms*

$$\int^{Y:*_\Lambda\to\mathcal{C}} \mathcal{S}(\mathcal{C}(Y,\_), FY) \xrightarrow{\cong} F$$

and

$$F \xrightarrow{\cong} \int_{X:*_\Lambda\to\mathcal{C}} \mathcal{C}(\_,X) \times_{\mathcal{S}} FX$$

in $\mathcal{S}^*$, which is natural in X and F. In particular, we obtain the following isomorphisms in $\mathcal{S}^*$ for each object X of $\mathcal{C}$, and each morphism $F : \mathcal{C}^{op} \to \mathcal{S}$:

$$[\mathcal{C}^{op},\mathcal{S}]_\Lambda(\mathcal{C}(\_,X), F) \xrightarrow{\cong} FX.$$

Proof. This is an immediate consequence of Proposition 4.3.15 and 4.3.19 by using the closed monoidal structure on $\mathcal{S}$ and the definition of ends and coends. First observe that any morphism $F : \mathcal{C}^{op} \to \mathcal{S}$ yields a morphism in $\Lambda(\mathcal{C}^{op} \times \mathcal{C}, \mathcal{S})$:

$$F(\_,\_) : \mathcal{C}^{op}(\_,\_) \to \mathcal{S}(F\_, F\_)$$

which by closed monoidal structure on $\mathcal{S}$, and by definition of the coend, corresponds to

$$F^\# : \int_{X:*_\Lambda\to\mathcal{C}} \mathcal{C}(\_,X) \times_{\mathcal{S}} FX \to F\_\,,$$

which corresponds to

$$F_\# : F_\_ \to \int^{Y:*_\Lambda \to \mathcal{C}} \mathcal{S}(\mathcal{C}(Y,\_), FY).$$

We show that both $F^\#$ and $F_\#$ are isomorphisms. By Proposition 4.3.15, we have

$$\begin{aligned}
[\mathcal{C}^{op}, \mathcal{S}]_\Lambda(F,G) &\xrightarrow{\cong} [\mathcal{C} \times \mathcal{C}^{op}, \mathcal{S}]_\Lambda \left(\mathcal{C}^{op}(_,_), \mathcal{S}(F_, G_)\right) \\
&\xrightarrow{\cong} \int^{X,Y \in \mathcal{C}} \mathcal{S}\left(\mathcal{C}(Y,X), \mathcal{S}(FX, GY)\right) \\
&\xrightarrow{\cong} \int^{X,Y \in \mathcal{C}} \mathcal{S}\left(\mathcal{C}(Y,X) \times_{\mathcal{S}} FX, GY\right) \\
&\xrightarrow{\cong} \int^{Y \in \mathcal{C}} \mathcal{S}\left(\int_{X \in \mathcal{C}} \mathcal{C}(Y,X) \times_{\mathcal{S}} FX, GY\right) \\
&\xrightarrow{\cong} [\mathcal{C}^{op}, \mathcal{S}]_\Lambda \left(\int_{X \in \mathcal{C}} \mathcal{C}(_,X) \times_{\mathcal{S}} FX, G\right),
\end{aligned}$$

and hence, F represents the coend $\int_{X:*_\Lambda \to \mathcal{C}} \mathcal{C}(\_,X) \times_{\mathcal{S}} FX$. On the other hand, we have

$$\begin{aligned}
[\mathcal{C}^{op}, \mathcal{S}]_\Lambda \left(G, \int^{Y:*_\Lambda \to \mathcal{C}} \mathcal{S}(\mathcal{C}(Y,_), FY)\right) &\xrightarrow{\cong} \int^{X:*_\Lambda \to \mathcal{C}} \mathcal{S}\left(GX, \int^{Y:*_\Lambda \to \mathcal{C}} \mathcal{S}\left(\mathcal{C}(Y,X), FY\right)\right) \\
&\xrightarrow{\cong} \int^{X:*_\Lambda \to \mathcal{C}} \int^{Y:*_\Lambda \to \mathcal{C}} \mathcal{S}\left(GX, \mathcal{S}\left(\mathcal{C}(Y,X), FY\right)\right) \\
&\xrightarrow{\cong} \int^{X,Y:*_\Lambda \to \mathcal{C}} \mathcal{S}\left(GX \times_{\mathcal{S}} \mathcal{C}(Y,X), FY\right) \\
&\xrightarrow{\cong} \int^{X,Y:*_\Lambda \to \mathcal{C}} \mathcal{S}\left(\mathcal{C}(Y,X), \mathcal{S}\left(GX, FY\right)\right) \\
&\xrightarrow{\cong} [\mathcal{C} \times \mathcal{C}^{op}, \mathcal{S}]_\Lambda \left(\mathcal{C}^{op}(_,_), \mathcal{S}(G_, F_)\right) \\
&\xrightarrow{\cong} [\mathcal{C}^{op}, \mathcal{S}]_\Lambda \left(G, F\right).
\end{aligned}$$

□

REMARK 4.5.4. We obtain the Yoneda lemma by taking $\Lambda = \text{CAT}$ and $\mathcal{S} = \text{SET}$.

PROPOSITION 4.5.5. *The Yoneda embedding is fully faithful. In particular, for each object* $\mathcal{C}$ *of* Λ *and each pair of objects* X, Y *of* $\mathcal{C}$*, the canonical morphism*

$$\mathcal{C}(X,Y) \xrightarrow{\cong} [\mathcal{C}^{op}, \mathcal{S}]_\Lambda \left(\mathcal{C}(-,X), \mathcal{C}(-,Y)\right)$$

is an isomorphism.

CONSEQUENCE 4.5.6. *The previous definitions of objects by universal property is well defined. Indeed, two objects satisfying the same universal property are isomorphic by Proposition 4.5.5.*

Completeness.

PROPOSITION 4.5.7. *Let* $\mathcal{C}$ *be an object of* Λ*. The object* $[\mathcal{C}^{op}, \mathcal{S}]_\Lambda$ *is complete and cocomplete. Let* I *be an object of* Λ*. The biclosed structure of* Λ *yields an* *isomorphism* $[\mathrm{I}, [\mathcal{C}^{op}, \mathcal{S}]_\Lambda]_\Lambda \xrightarrow{\cong} [\mathcal{C}^{op}, [\mathrm{I}, \mathcal{S}]_\Lambda]_\Lambda$. *The colimit, respectively the limit morphism, is given by the composite*

$$[\mathrm{I}, [\mathcal{C}^{op}, \mathcal{S}]_\Lambda]_\Lambda \xrightarrow{\cong} [\mathcal{C}^{op}, [\mathrm{I}, \mathcal{S}]_\Lambda]_\Lambda \to [\mathcal{C}^{op}, \mathcal{S}]_\Lambda,$$

where the morphism on the right hand side is obtained from the colimit, respectively from the limit morphism $[\mathrm{I}, \mathcal{S}]_\mathcal{S} \to \mathcal{S}$ *of* $\mathcal{S}$.

Proof. Let $G : \mathcal{C}^{op} \to \mathcal{S}$ and $F : \mathrm{I} \to [\mathcal{C}^{op}, \mathcal{S}]_\Lambda$. We also write $F : \mathcal{C}^{op} \to [\mathrm{I}, \mathcal{S}]_\Lambda$ for the corresponding element under the isomorphism $[\mathrm{I}, [\mathcal{C}^{op}, \mathcal{S}]_\Lambda]_\Lambda \xrightarrow{\cong} [\mathcal{C}^{op}, [\mathrm{I}, \mathcal{S}]_\Lambda]_\Lambda$. We show that the colimit of F is represented by object $\operatorname{colim} F : \mathcal{C}^{op} \to \mathcal{S}$ defined above. We have the following isomorphisms in Λ

$$\begin{aligned}
[\mathrm{I}, [\mathcal{C}^{op}, \mathcal{S}]_\Lambda]_\Lambda \left(F, \overline{G}\right) &\xrightarrow{\cong} [\mathcal{C}^{op}, [\mathrm{I}, \mathcal{S}]_\Lambda]_\Lambda \left(F, \overline{G}\right) \\
&\xrightarrow{\cong} \int^{X \in \mathcal{C}} [\mathrm{I}, \mathcal{S}]_\Lambda (FX, GX) \\
&\xrightarrow{\cong} \int^{X \in \mathcal{C}} \mathcal{S}(\operatorname{colim} FX, GX) \\
&\xrightarrow{\cong} [\mathcal{C}^{op}, \mathcal{S}]_\Lambda (\operatorname{colim} F, \overline{G}),
\end{aligned}$$

hence the result. □

Proposition 4.5.8. *For any object* $\mathcal{C}$ *of* Λ, *the Yoneda embedding* $\mathcal{C} \to [\mathcal{C}^{op}, \mathcal{S}]_\Lambda$ *preserves colimits and is universal among such morphisms in the sense that for each pair* (D, F) *where* $\mathcal{D}$ *is a cocomplete object of* Λ *and* $F : \mathcal{C} \to \mathcal{D}$ *is a morphism in* Λ, *there is an essentially unique morphism* $\overline{F} \in [\mathcal{C}^{op}, \mathcal{S}]_\Lambda$ *which preserves colimits such that*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, hook] \arrow[dr, phantom, "\cong"] & \mathcal{D} \\
{[\mathcal{C}^{op}, \mathcal{S}]_\Lambda.} \arrow[ur, dashed, "\overline{F}"'] &
\end{tikzcd}$$

Proof. We use the fact that $\mathcal{D}$ is cocomplete. In particular, $\mathcal{D}$ is tensored over $\mathcal{S}$. We write $\cdot : \mathcal{S} \times_\Lambda \mathcal{D} \to \mathcal{D}$ for the action of $\mathcal{S}$ on $\mathcal{D}$ thus obtained. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism in Λ such that F preserves colimits. Let $\overline{F} : [\mathcal{C}^{op}, \mathcal{D}]_\Lambda \to \mathcal{D}$ be defined as the left Kan extension of F along the Yoneda embedding, so that

$$\bar{F} = \int_{X \in \mathcal{C}} [\mathcal{C}^{op}, \mathcal{S}]_\Lambda (\mathcal{C}(\_, X), \_) \cdot FX \xrightarrow{\cong} \int_{X \in \mathcal{C}} ev_{X\,\_} \cdot FX.$$

By Proposition 4.5.3, the restriction of $\bar{F}$ on $\mathcal{C}$ is canonically isomorphic to F. The morphism $\bar{F} : [\mathcal{C}^{op}, \mathcal{D}]_\Lambda \to \mathcal{D}$ preserves colimits by construction. □

Remark 4.5.9. In particular, the object $\mathcal{S}$ of Λ is the free cocomplete completion of $*_\Lambda$.

Monoidality. We deduce an expression of the Day convolution product in terms of coends from the expression of Kan extensions. We assume that Λ is equipped with an additional monoidal structure $(\Lambda, \otimes_\Lambda, 1_\Lambda)$ which is compatible with its former closed monoidal structure[23], so that in particular, the internal hom of Λ is lax monoidal with respect to $(\Lambda, \otimes_\Lambda, 1_\Lambda)$. We also assume that this monoidal structure

[23]Recall that a monoidal structure on a complete 2-category always has the structure of a lax monoidal 2-functor with respect to the cartesian product.

is compatible with the oppositization. We suppose that the object $\mathcal{S}$ is also equipped with another monoidal structure $(\mathcal{S},\otimes_{\mathcal{S}})$ in $(\Lambda,\otimes_\Lambda)$, which is compatible with its cartesian closed structure. In particular, $(\mathcal{S},\times_{\mathcal{S}},\otimes_{\mathcal{S}})$ has the structure of a 2-fold monoid in the 2-monoidal 2-category $(\Lambda,\times_\Lambda,\otimes_\Lambda)$. [24].

Proposition 4.5.10. *The 2-category $\Lambda_{/\mathcal{S}}$ inherits a monoidal structure from $(\Lambda,\otimes_\Lambda)$, which is compatible with its closed monoidal structure defined in Proposition 4.3.5.*

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be objects of Λ. The compatibility of $\otimes_\Lambda$ with respect to $\times_\Lambda$ yields an interchange morphism

$$\eta : (\mathcal{C}^{op}\otimes_\Lambda \mathcal{D}^{op})\times_\Lambda (\mathcal{C}\otimes_\Lambda \mathcal{D}) \to (\mathcal{C}^{op}\times_\Lambda \mathcal{C})\otimes_\Lambda (\mathcal{D}^{op}\times_\Lambda \mathcal{D}).$$

The compatibility of $\otimes_\Lambda$ with respect to oppositization yields an isomorphism $\mathcal{C}^{op}\otimes_\Lambda \mathcal{D}^{op}\cong(\mathcal{C}\otimes_\Lambda \mathcal{D})^{op}$. Let

$$\otimes_{/\mathcal{S}} : \Lambda(\mathcal{C}^{op}\times_\Lambda \mathcal{C},\mathcal{S})\times\Lambda(\mathcal{D}^{op}\times_\Lambda \mathcal{D},\mathcal{S})\to\Lambda((\mathcal{C}\otimes_\Lambda \mathcal{D})^{op}\times_\Lambda(\mathcal{C}\otimes_\Lambda \mathcal{D}),\mathcal{S})$$

be defined by the composite

$$\begin{aligned}\Lambda(\mathcal{C}^{op}\times_\Lambda \mathcal{C},\mathcal{S})\times\Lambda(\mathcal{D}^{op}\times_\Lambda \mathcal{D},\mathcal{S}) &\xrightarrow{\otimes_\Lambda}\Lambda((\mathcal{C}^{op}\times_\Lambda \mathcal{C})\otimes_\Lambda(\mathcal{D}^{op}\times_\Lambda \mathcal{D}),\mathcal{S})\\ &\xrightarrow{\Lambda(\eta,\mathcal{S})}\Lambda\left((\mathcal{C}^{op}\otimes_\Lambda \mathcal{D}^{op})\times_\Lambda(\mathcal{C}\otimes_\Lambda \mathcal{D}),\mathcal{S}\right)\\ &\xrightarrow{\cong}\Lambda\left((\mathcal{C}\otimes_\Lambda \mathcal{D})^{op}\times_\Lambda(\mathcal{C}\otimes_\Lambda \mathcal{D}),\mathcal{S}\right).\end{aligned}$$

The functor $\otimes_{\mathcal{S}}$ thus obtained, together with the 2-functor $\otimes_\Lambda$, yield a 2-functor

$$\coprod_{\mathcal{C},\mathcal{D}\in\Lambda}\Lambda(\mathcal{C}^{op}\times_\Lambda \mathcal{C},\mathcal{S})\times\Lambda(\mathcal{D}^{op}\times_\Lambda \mathcal{D},\mathcal{S})\xrightarrow[\otimes_\Lambda]{\coprod\otimes_{/\mathcal{S}}}\Lambda\left((\mathcal{C}\otimes_\Lambda \mathcal{D})^{op}\times_\Lambda(\mathcal{C}\otimes_\Lambda \mathcal{D}),\mathcal{S}\right)$$

and hence a 2-functor

$$\otimes_{\Lambda_{/\mathcal{S}}} : \Lambda_{/\mathcal{S}}\times\Lambda_{/\mathcal{S}}\to\Lambda_{/\mathcal{S}}.$$

The unit $1_{\mathcal{S}} : 1_\Lambda\to\mathcal{S}$ of $(\mathcal{S},\otimes_{\mathcal{S}})$ yields a morphism

$$1_{\Lambda/\mathcal{S}} : 1_\Lambda^{op}\times_\Lambda 1_\Lambda \xrightarrow{1_{\mathcal{S}}^{op}\times_\Lambda 1_{\mathcal{S}}}\mathcal{S}^{op}\times_\Lambda\mathcal{S}\xrightarrow{[-,-]_{\mathcal{S}}}\mathcal{S}.$$

We obtain a monoidal structure $(\Lambda_{/\mathcal{S}},\otimes_{\Lambda/\mathcal{S}},1_{\Lambda/\mathcal{S}})$ on $\Lambda_{/\mathcal{S}}$. The compatibility of $\otimes_\Lambda$ with respect to $\times_\Lambda$, together with the compatibility of $\otimes_{\mathcal{S}}$ with respect to $\times_{\mathcal{S}}$, yield an exchange morphism between $\otimes_{\Lambda/\mathcal{S}}$ and $\times_{\Lambda/\mathcal{S}}$ from which we deduce the compatibility of $\otimes_{\Lambda/\mathcal{S}}$ with respect to $\times_{\Lambda/\mathcal{S}}$. □

Definition 4.5.11. We say that the entangled $\mathcal{S}$-enrichement on Λ is lax monoidal if the structural 2-functor $\mathrm{Map}_{\mathcal{S}} : \Lambda\to\Lambda_{/\mathcal{S}}$ has the structure of a lax monoidal morphism $(\Lambda,\otimes_\Lambda)\to(\Lambda_{/\mathcal{S}},\otimes_{\Lambda_{/\mathcal{S}}})$ which is compatible with its former strongly closed monoidal structure.

Proposition 4.5.12. *Let $(\mathcal{C},\otimes_{\mathcal{C}})$ be a monoid in $(\Lambda,\otimes_\Lambda)$. The presheaf object $[\mathcal{C}^{op},\mathcal{S}]_\Lambda$ of $\mathcal{C}$ inherits a monoidal structure such that the embedding*

$$y_{\mathcal{C}} : \mathcal{C}\to[\mathcal{C}^{op},\mathcal{S}]_\Lambda$$

[24]Note that the cartesian product distributes with itself, so that we can restrict to the case where the additional monoidal structure on Λ also is given by the cartesian product. In this case, we can both consider a 2-fold monoidal structure on $\mathcal{S}$ in $(\Lambda,\times_\Lambda)$ given by 2 distinct laws, or its 2-fold monoidal structure $(\mathcal{S},\times_{\mathcal{S}})$ given by the cartesian product.

is a monoidal morphism. Moreover, the pair $([\mathcal{C}^{op}, \mathcal{S}]_\Lambda, y_{\mathcal{C}})$ *is universal among such pairs, in the sense that for any cocomplete monoid* $\mathcal{D}$ *equipped with a monoidal morphism* $\mathcal{C} \to \mathcal{D}$*, there is an essentially unique cocomplete monoidal morphism* $\overline{F} : [\mathcal{C}^{op}, \mathcal{S}]_\Lambda \to \mathcal{D}$ *such that*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, hook] & \mathcal{D} \\ {[\mathcal{C}^{op}, \mathcal{S}]_\Lambda.} \arrow[ur, dashed, "\overline{F}"'] \arrow[u, phantom, shift left=8, "\cong"] & \end{tikzcd}$$

For each natural number n*, the tensor product thus obtained*

$$\otimes^n_{[\mathcal{C}^{op},\mathcal{S}]_\Lambda} : [\mathcal{C}^{op}, \mathcal{S}]_\Lambda^{\otimes^n_\Lambda} \to [\mathcal{C}^{op}, \mathcal{S}]_\Lambda$$

satisfies

$$\otimes^n_{[\mathcal{C}^{op},\mathcal{S}]_\Lambda} F_\bullet \cong \int^{X_\bullet \in \mathcal{C}^{\otimes^n_\Lambda}} \mathcal{C}(\otimes^n_{\mathcal{C}} X_\bullet, \_) \times_{\mathcal{S}} \otimes^n_{\mathcal{S}} [\overline{\otimes}_\Lambda F_\bullet](X_\bullet)$$

for all $F_\bullet : *_\Lambda \to [\mathcal{C}^{op} \to \mathcal{S}]_\Lambda^{\otimes^n_\Lambda}$.

Remark 4.5.13. Given that the monoidal structure on Λ is not necessarily cartesian, the objects of $[\mathcal{C}^{op}, \mathcal{S}]_\Lambda^{\otimes^n_\Lambda}$ do not restrict to the elements of the form $F_1 \otimes_\Lambda \cdots \otimes_\Lambda F_n$ for some $F_1, \ldots, F_n : \mathcal{C}^{op} \to \mathcal{S}$. Let $F_\bullet = F_1 \otimes_\Lambda \cdots \otimes_\Lambda F_n : *_\Lambda \to [\mathcal{C}^{op} \to \mathcal{S}]_\Lambda^{\otimes^n_\Lambda}$ be obtained from the tensor product in Λ of the objects $F_1, \ldots, F_n : \mathcal{C}^{op} \to \mathcal{S}$ and let $X_\bullet = X_1 \otimes_\Lambda \cdots \otimes_\Lambda X_n : *_\Lambda \to \mathcal{C}^{\otimes^n_\Lambda}$ be obtained from $X_1, \ldots X_n : *_\Lambda \to \mathcal{C}$. We have

$$\otimes^n_{\mathcal{S}} [\overline{\otimes}_\Lambda F_\bullet](X_\bullet) = F_1 X_1 \otimes_{\mathcal{S}} \cdots \otimes_{\mathcal{S}} F_n X_n.$$

Proof. The formula is a straightforward consequence of the definition of the Day convolution product by using Proposition 4.4.10. The monoidal structure of the embedding is consequence of the aforementioned formula and Proposition 4.5.3. $\square$

4.6. Example: enriched categories. Let $(\mathcal{V}, \otimes_{\mathcal{V}}, *_{\mathcal{V}})$ be a bicomplete closed symmetric monoidal category. If $X \in \mathcal{V}$, then we write $x \in X$ for $x \in \mathcal{V}(*_{\mathcal{V}}, X)$.

Definition 4.6.1. We say that $\mathcal{C}$ is a $\mathcal{V}$-enriched category, and we write $\mathcal{C} \in \mathrm{Cat}_{\mathcal{V}}$, if $\mathcal{C}$ is equipped with

- objects $X \in \mathcal{C}$ [25]
- for each pair of objects $X, Y \in \mathcal{C}$, an object $\mathcal{C}(X, Y) \in \mathcal{V}$ of morphisms from X to Y,
- for each object $X \in \mathcal{C}$, an element $1_X \in \mathcal{V}(*_{\mathcal{V}}, \mathcal{C}(X, X))$ called the identity of X

[25] In fact, a predicate $\_ \in \mathcal{C}$, so that $X \in \mathcal{C}$ is a proposition for all symbol X -cf Appendix A.

- for $X, Y, Z \in \mathcal{C}$, an object $\circ \in \mathcal{V}(\mathcal{C}(Y,Z) \otimes_{\mathcal{V}} \mathcal{C}(X,Y)$[26], called the composition morphism[27],
- such that:
  - for $X, Y \in \mathcal{C}$ and $f \in \mathcal{C}(X,Y)$, we have $f = f \circ \imath_X$,
  - for $X, Y \in \mathcal{C}$ and $f \in \mathcal{C}(X,Y)$, we have $\imath_Y \circ f = f$,
  - for $X, Y, Z, T \in \mathcal{C}$, and $f \in \mathcal{C}(X,Y)$, $g \in \mathcal{C}(Y,Z)$, $h \in \mathcal{C}(Y,T)$, we have $(hg)f = h(gf)$.

Definition 4.6.2. We let $\mathbb{V} \in \mathrm{Cat}_{\mathcal{V}}$ denote the obvious $\mathcal{V}$-enriched category, associated to $\mathcal{V}$, that we formally define as follows.

- The objects $X \in \mathbb{V}$ correspond to objects $X \in \mathcal{V}$.
- For $X, Y \in \mathbb{V}$, we take $\mathbb{V}(X,Y) = [X,Y]_{\mathcal{V}} \in \mathcal{V}$.
- For $X \in \mathbb{V}$, we let $\imath_X \in \mathcal{V}(*_{\mathcal{V}}, [X,X]_{\mathcal{V}})$ be defined by the element corresponding to the identity $\imath^{\mathcal{V}}_X \in \mathcal{V}(X,X)$ of X in the category $\mathcal{V}$ under the isomorphism $\mathcal{V}(X,X) \cong \mathcal{V}(*_{\mathcal{V}}, [X,X]_{\mathcal{V}})$..
- For $X, Y, Z \in \mathbb{V}$, we let the composition $\circ \in \mathcal{V}([X,Y]_{\mathcal{V}} \otimes_{\mathcal{V}} [Y,Z]_{\mathcal{V}}, [X,Z]_{\mathcal{V}})$ be defined by the element resulting from the composite

$$\begin{aligned} *_{\mathcal{V}} &\xrightarrow{\eta_{Y,Z}} \mathcal{V}(Y \otimes_{\mathcal{V}} [Y,Z]_{\mathcal{V}}, Z) \\ &\xrightarrow{\mathcal{V}(\eta_{X,Y} \otimes_{\mathcal{V}} [Y,Z]_{\mathcal{V}}, Z)} \mathcal{V}((X \otimes_{\mathcal{V}} [X,Y]_{\mathcal{V}}) \otimes_{\mathcal{V}} [Y,Z]_{\mathcal{V}}, Z) \\ &\xrightarrow{\cong} \mathcal{V}(X \otimes_{\mathcal{V}} ([X,Y]_{\mathcal{V}} \otimes_{\mathcal{V}} [Y,Z]_{\mathcal{V}}), Z) \\ &\xrightarrow{\cong} \mathcal{V}([X,Y]_{\mathcal{V}} \otimes_{\mathcal{V}} [Y,Z]_{\mathcal{V}}, [X,Z]_{\mathcal{V}}). \end{aligned}$$

  We deduce the unit and associativity constraints from the definition and the naturality of the morphisms involved.

Definition 4.6.3. For $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_{\mathcal{V}}$, we let $\mathrm{Cat}_{\mathcal{V}}(\mathcal{C}, \mathcal{D}) \in \mathrm{Cat}$ denote the category such that:

- an object $F \in \mathrm{Cat}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ consists in a mapping, which to any $X \in \mathcal{C}$ associates an object $FX \in \mathcal{D}$, together with $F(\_,\_) \in \int^{\mathcal{C}^{*op} \times \mathcal{C}^*} \mathcal{V}(\mathcal{C}(\_,\_), \mathcal{D}(F\_, F\_))$,
- and for $F, G \in \mathrm{Cat}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})$, we take $\mathrm{Cat}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})(F,G) = \int^{X \in \mathcal{C}} \mathcal{V}(*_{\mathcal{V}}, \mathcal{D}(FX, GX)) \in \mathrm{Set}$.

Definition 4.6.4. We let $\mathrm{Cat}_{\mathcal{V}} \in \mathrm{Cat}_2$ be the 2-category such that:

- the objects $X \in \mathrm{Cat}_{\mathcal{V}}$ consist of the $\mathcal{V}$-enriched categories in the sense of Definition 4.6.1,
- and to $X, Y \in \mathrm{Cat}_{\mathcal{V}}$, we associate the category $\mathrm{Cat}_{\mathbb{V}}(X,Y) \in \mathrm{Cat}$ given by Definition 4.6.3.

---

[26] $\mathcal{C}(Y,Z) \otimes_{\mathcal{V}} \mathcal{C}(X,Y) \in \mathcal{V}$ denotes the image of $(\mathcal{C}(Y,Z), \mathcal{C}(X,Y)) \in \mathcal{V} \times \mathcal{V}$ under the functor $\otimes_{\mathcal{V}} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$.

[27] For $f \in \mathcal{C}(X,Y), g \in \mathcal{C}(Y,Z)$, we write $g \circ f \in \mathcal{C}(X,Z)$ for the element resulting from the composite

$$\begin{tikzcd} * \arrow[r, "{(f,g)}"] \arrow[rrdd, dashed, "g \circ f"'] & \mathcal{V}(*_{\mathcal{V}}, \mathcal{C}(X,Y)) \times \mathcal{V}(*_{\mathcal{V}}, \mathcal{C}(Y,Z)) \arrow[r, "\cong"] & \mathcal{V} \times \mathcal{V}((*_{\mathcal{V}}, *_{\mathcal{V}}), (\mathcal{C}(X,Y), \mathcal{C}(Y,Z))) \arrow[d, "\otimes_{\mathcal{V}}"] \\ & & \mathcal{V}(*_{\mathcal{V}} \otimes_{\mathcal{V}} *_{\mathcal{V}}, \mathcal{C}(X,Y) \otimes_{\mathcal{V}} \mathcal{C}(Y,Z)) \arrow[d, "\mathcal{V}(\epsilon_{\mathcal{V}}, \circ)"] \\ & & \mathcal{V}(*_{\mathcal{V}}, \mathcal{C}(X,Z)). \end{tikzcd}$$

Remark 4.6.5. The forgetful 2-functor $\mathrm{CAT}_{\mathbb{V}}(*_{\mathcal{V}}, \_)$ takes the $\mathbb{V}$-enriched category $\mathbb{V}$ to the category $\mathcal{V}$.

Definition 4.6.6. For $\mathcal{C}, \mathcal{D} \in \mathrm{CAT}_{\mathcal{V}}$, we let $\mathcal{C} \times_{\mathcal{V}} \mathcal{D} \in \mathrm{CAT}_{\mathcal{V}}$ denote the $\mathcal{V}$-enriched category such that:

– the objects $X \in \mathcal{C} \times_{\mathcal{V}} \mathcal{D}$ consist of pairs $(\pi_{\mathcal{C}} X \in \mathcal{C}, \pi_{\mathcal{D}} X \in \mathcal{D})$,
– and for $X, Y \in \mathcal{C} \times_{\mathcal{V}} \mathcal{D}$, we take $\mathcal{C} \times_{\mathcal{V}} \mathcal{D}(X, Y) = \mathcal{C}(\pi_{\mathcal{C}} X, \pi_{\mathcal{C}} Y) \otimes_{\mathcal{V}} \mathcal{D}(\pi_{\mathcal{D}} X, \pi_{\mathcal{D}} Y)$.

Definition 4.6.7. To $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_1, \mathcal{D}_2 \in \mathrm{CAT}_{\mathcal{V}}$, we associate a functor

$$\times_{\mathcal{V}} : \mathrm{CAT}_{\mathcal{V}}(\mathcal{C}_1, \mathcal{C}_2) \times \mathrm{CAT}_{\mathcal{V}}(\mathcal{D}_1, \mathcal{D}_2) \to \mathrm{CAT}_{\mathcal{V}}(\mathcal{C}_1 \times_{\mathcal{V}} \mathcal{D}_1, \mathcal{C}_2 \times_{\mathcal{V}} \mathcal{D}_2)$$

that we define as follows.

For $F_{\mathcal{C}} \in \mathrm{CAT}_{\mathcal{V}}(\mathcal{C}_1, \mathcal{C}_2), F_{\mathcal{D}} \in \mathrm{CAT}_{\mathcal{V}}(\mathcal{D}_1, \mathcal{D}_2)$, we define

$$F_{\mathcal{C}} \times_{\mathcal{V}} F_{\mathcal{D}} \in \mathrm{CAT}_{\mathcal{V}}(\mathcal{C}_1 \times_{\mathcal{V}} \mathcal{D}_1, \mathcal{C}_2 \times_{\mathcal{V}} \mathcal{D}_2)$$

on objects by

$$F_{\mathcal{C}} \times_{\mathcal{V}} F_{\mathcal{D}}\ X = (F_{\mathcal{C}} \pi_{\mathcal{C}_1} X, F_{\mathcal{D}} \pi_{\mathcal{D}_1} X) \in \mathcal{C}_2 \times_{\mathcal{V}} \mathcal{D}_2,$$

and for $X, Y \in \mathcal{C}_1 \times_{\mathcal{V}} \mathcal{D}_1$, we let

$$F_{\mathcal{C}} \times_{\mathcal{V}} F_{\mathcal{D}}(X, Y) : \mathcal{C}_1 \times_{\mathcal{V}} \mathcal{D}_1(X, Y) \to \mathcal{C}_2 \times_{\mathcal{V}} \mathcal{D}_2(F_{\mathcal{C}} \times_{\mathcal{V}} F_{\mathcal{D}}\ X, F_{\mathcal{C}} \times_{\mathcal{V}} F_{\mathcal{D}}\ Y)$$

be the morphism in $\mathcal{V}$ obtained as the composite

$$\begin{aligned}
\mathcal{C}_1 \times_{\mathcal{V}} \mathcal{D}_1(X, Y) &\cong \mathcal{C}_1(\mathcal{C}_0, Y_{\mathcal{C}}) \otimes_{\mathcal{V}} \mathcal{D}_1(\mathcal{D}_0, Y_{\mathcal{D}}) \\
&\xrightarrow{F_{\mathcal{C}}(\mathcal{C}_0, Y_{\mathcal{C}}) \otimes_{\mathcal{V}} F_{\mathcal{D}}(\mathcal{D}_0, Y_{\mathcal{D}})} \mathcal{C}_2(F_{\mathcal{C}}\mathcal{C}_0, F_{\mathcal{C}} Y_{\mathcal{C}}) \otimes_{\mathcal{V}} \mathcal{D}_2(F_{\mathcal{D}}\mathcal{D}_0, F_{\mathcal{D}} Y_{\mathcal{D}}) \\
&\xrightarrow{\cong} \mathcal{C}_2 \times_{\mathcal{V}} \mathcal{D}_2(F_{\mathcal{C}} \times_{\mathcal{V}} F_{\mathcal{D}}\ X, F_{\mathcal{C}} \times_{\mathcal{V}} F_{\mathcal{D}}\ Y)
\end{aligned}$$

Proposition 4.6.8. *We let*

$$\times_{\mathbb{V}} : \mathrm{CAT}_{\mathbb{V}} \times \mathrm{CAT}_{\mathbb{V}} \to \mathrm{CAT}_{\mathbb{V}}$$

denote the 2*-functor given by Definition 4.6.6 on objects and by Definition 4.6.7 on morphisms.*

Proposition 4.6.9. *The functor*

$$\mathrm{CAT}_{\mathbb{V}}([\mathcal{C}_1, \mathcal{D}_1]_{\mathbb{V}}, [\mathcal{C}_2, \mathcal{D}_2]_{\mathbb{V}}) \longrightarrow [\mathrm{CAT}_{\mathbb{V}}(\mathcal{C}_1, \mathcal{D}_1), \mathrm{CAT}_{\mathcal{V}}(\mathcal{C}_2, \mathcal{D}_2)]$$

induced by $\mathrm{CAT}_{\mathbb{V}}(*_{\mathcal{V}}, \_)$ *is an equivalence of categories, natural in* $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_1, \mathcal{D}_2 \in \mathrm{CAT}_{\mathbb{V}}$*. Consequently, the forgetful* 2*-functor* $\mathrm{CAT}_{\mathbb{V}}(*_{\mathcal{V}}, \_) : \mathrm{CAT}_{\mathbb{V}} \to \mathrm{CAT}$ *is a local equivalence. In particular, the* 2*-category* $\mathrm{CAT}_{\mathbb{V}}$ *is* $*_{\mathcal{V}}$*-primary.*

Definition 4.6.10. For $\mathcal{C} \in \mathrm{CAT}_{\mathcal{V}}$, we let $\mathcal{C}^{op} \in \mathrm{CAT}_{\mathcal{V}}$ denote the $\mathcal{V}$-enriched category such that:

– the objects $X \in \mathcal{C}^{op}$ correspond to objects $X \in \mathcal{C}$,
– and for $X, Y \in \mathcal{C}^{op}$, we take $\mathcal{C}^{op}(X, Y) = \mathcal{C}(Y, X) \in \mathcal{V}$.

Remark 4.6.11. Each $\mathcal{C} \in \mathrm{CAT}_{\mathbb{V}}$ yields a morphism $\mathcal{C}(\_, \_) \in \mathrm{CAT}_{\mathbb{V}}(\mathcal{C}^{op} \times_{\mathbb{V}} \mathcal{C}, \mathbb{V})$.

Definition 4.6.12. Let $\mathcal{C}, \mathcal{D} \in \mathrm{CAT}_{\mathcal{V}}$. We let $[\mathcal{C}, \mathcal{D}]_{\mathcal{V}} \in \mathrm{CAT}_{\mathcal{V}}$ be the $\mathcal{V}$-enriched category such that

– an $F \in [\mathcal{C},\mathcal{D}]_{\mathcal{V}}$ consists in a mapping, which to any $X \in \mathcal{C}$ associates an object $FX \in \mathcal{D}$, together with $F(\_,\_) \in \int^{\mathcal{C}^{op}\times\mathcal{C}} \mathbb{V}(\mathcal{C}(\_,\_),\mathcal{D}(F\_,F\_))$,
– and for $F,G \in [\mathcal{C},\mathcal{D}]_{\mathcal{V}}$, we take $[\mathcal{C},\mathcal{D}]_{\mathcal{V}}(F,G) = \int^{\mathcal{C}} \mathcal{D}(F\_,G\_) \in \mathcal{V}$.

Proposition 4.6.13. *Let $\mathcal{B},\mathcal{C},\mathcal{D} \in \mathrm{Cat}_{\mathbb{V}}$. We have a natural isomorphism*

$$\mathrm{Cat}_{\mathbb{V}}(\mathcal{B},[\mathcal{C},\mathcal{D}]_{\mathbb{V}}) \cong \mathrm{Cat}_{\mathbb{V}}(\mathcal{B}\times_{\mathbb{V}}\mathcal{C},\mathcal{D}).$$

Moreover, we can form a 2-functor

$$[\_,\_]_{\mathbb{V}} : \mathrm{Cat}_{\mathbb{V}}^{op} \times \mathrm{Cat}_{\mathbb{V}} \to \mathrm{Cat}_{\mathbb{V}},$$

which gives a closed monoidal structure to the 2-category of $\mathcal{V}$-enriched categories.

Remark 4.6.14. The 2-category $\mathrm{Cat}_{\mathbb{V}}$ is naturally equipped with an entangled $\mathbb{V}$-enrichement, and the forgetful functor of Remark 4.3.9 is an equivalence.

Remark 4.6.15. Let $\mathcal{C}_1,\mathcal{C}_2,\mathcal{D}_1,\mathcal{D}_2 \in \mathrm{Cat}_{\mathbb{V}}$. We can form a morphism in $\mathrm{Cat}_{\mathbb{V}}$

$$\times_{\mathbb{V}} : [\mathcal{C}_1,\mathcal{D}_1]_{\mathbb{V}} \times_{\mathbb{V}} [\mathcal{C}_2,\mathcal{D}_2]_{\mathbb{V}} \to [\mathcal{C}_1\times_{\mathbb{V}}\mathcal{C}_2,\mathcal{D}_1\times_{\mathbb{V}}\mathcal{D}_2]_{\mathbb{V}}$$

as follows.

Let $(F_1,F_2) \in [\mathcal{C}_1,\mathcal{D}_1]_{\mathbb{V}} \times_{\mathbb{V}} [\mathcal{C}_2,\mathcal{D}_2]_{\mathbb{V}}$. We let $F_1 \times_{\mathbb{V}} F_2 \in [\mathcal{C}_1\times_{\mathbb{V}}\mathcal{C}_2,\mathcal{D}_1\times_{\mathbb{V}}\mathcal{D}_2]_{\mathbb{V}}$ be defined as follows.

– For $(X_1,X_2) \in \mathcal{C}_1\times_{\mathbb{V}}\mathcal{C}_2$, we set $F_1 \times_{\mathbb{V}} F_2\,(X_1,X_2) = (F_1X_1,F_2X_2) \in \mathcal{D}_1\times_{\mathbb{V}}\mathcal{D}_2$.
– For $(X_1,X_2),(Y_1,Y_2) \in \mathcal{C}_1\times_{\mathbb{V}}\mathcal{C}_2$, we let $F_1 \times_{\mathbb{V}} F_2\ \left((X_1,X_2),(Y_1,Y_2)\right)$ be defined by the composite

$$\begin{aligned}
\mathcal{C}_1\times_{\mathbb{V}}\mathcal{C}_2\left((X_1,X_2),(Y_1,Y_2)\right) &\xrightarrow{\cong} \mathcal{C}_1(X_1,Y_1)\otimes_{\mathbb{V}}\mathcal{C}_2(X_2,Y_2)\\
&\xrightarrow{F_1(X_1,Y_1)\otimes_{\mathbb{V}}F_2(X_2,Y_2)} \mathcal{D}_1(F_1X_1,F_1Y_1)\otimes_{\mathbb{V}}\mathcal{D}_2(F_2X_2,F_2Y_2)\\
&\xrightarrow{\cong} \mathcal{D}_1\times_{\mathbb{V}}\mathcal{D}_2\left((F_1X_1,F_2X_2),(F_1Y_1,F_2Y_2)\right).
\end{aligned}$$

Let $(F_1,F_2),(G_1,G_2) \in [\mathcal{C}_1,\mathcal{D}_1]_{\mathbb{V}} \times_{\mathbb{V}} [\mathcal{C}_2,\mathcal{D}_2]_{\mathbb{V}}$. The composite of the following morphisms in $\mathcal{V}$

$$\begin{aligned}
[\mathcal{C}_1,\mathcal{D}_1]_{\mathbb{V}} \times_{\mathbb{V}} [\mathcal{C}_2,\mathcal{D}_2]_{\mathbb{V}}\left((F_1,F_2),(G_1,G_2)\right) &\xrightarrow{\cong} [\mathcal{C}_1,\mathcal{D}_1]_{\mathbb{V}}(F_1,G_1)\otimes_{\mathbb{V}}[\mathcal{C}_2,\mathcal{D}_2]_{\mathbb{V}}(F_2,G_2)\\
&\xrightarrow{\cong} \int^{X\in\mathcal{C}_1}\mathcal{D}_1(F_1X,G_1X)\otimes_{\mathbb{V}}\int^{Y\in\mathcal{C}_2}\mathcal{D}_2(F_2Y,G_2Y)\\
&\xrightarrow{\pi_X\otimes_{\mathbb{V}}\pi_Y} \int^{(X,Y)\in\mathcal{C}_1\times_{\mathbb{V}}\mathcal{C}_2}\mathcal{D}_1(F_1X,G_1X)\otimes_{\mathbb{V}}\mathcal{D}_2(F_2Y,G_2Y)\\
&\xrightarrow{\cong} \int^{(X,Y)\in\mathcal{C}_1\times_{\mathbb{V}}\mathcal{C}_2}\mathcal{D}_1\times_{\mathbb{V}}\mathcal{D}_2\left(F_1\times_{\mathbb{V}}F_2(X,Y),G_1\times_{\mathbb{V}}G_2(X,Y)\right)
\end{aligned}$$

gives to $\times_{\mathbb{V}}$ the structure of a morphism in $\mathcal{V}$.

Observation 4.6.16. *In fact, the structure of $\mathrm{Cat}_{\mathbb{V}}$ is not fully exploited when regarded as a 2-category. To illustrate this point, we informally describe an object denoted as $\mathrm{Cat}^2_{\mathbb{V}}$, which appears to offer a more natural framework for enriched category theory. Say that $\Lambda \in \mathrm{Cat}^2_{\mathbb{V}}$ if Λ has objects $\mathcal{C} \in \Lambda$, and if each pair of objects $\mathcal{C},\mathcal{D} \in \Lambda$ yields a $\mathcal{V}$-enriched category $\Lambda(\mathcal{C},\mathcal{D}) \in \mathrm{Cat}_{\mathbb{V}}$. For $\Lambda,\Gamma \in \mathrm{Cat}^2_{\mathbb{V}}$, we let $\mathrm{Cat}^2_{\mathbb{V}}(\Lambda,\Gamma) \in \mathrm{Cat}^2_{\mathbb{V}}$ be such that:*

- *an object* $F \in \mathrm{C{\scriptstyle AT}}^2_{\mathbb{V}}(\Lambda, \Gamma)$ *consists in a mapping, which to any* $\mathcal{C} \in \Lambda$ *associates* $F\mathcal{C} \in \Gamma$*, together with* $F(\_,\_) \in \int^{\Lambda^{op}\times\Lambda} \mathrm{C{\scriptstyle AT}}_{\mathbb{V}}(\Lambda(\_,\_), \Gamma(F\_, F\_))$,
- *and for* $F, G \in \mathrm{C{\scriptstyle AT}}^2_{\mathbb{V}}(\Lambda, \Gamma)$ *we take* $\mathrm{C{\scriptstyle AT}}^2_{\mathbb{V}}(\Lambda, \Gamma)(F, G) = \int^{\Lambda} \Gamma(F\_, G\_) \in \mathrm{C{\scriptstyle AT}}_{\mathbb{V}}$.

For $\Lambda, \Gamma \in \mathrm{C{\scriptstyle AT}}^2_{\mathbb{V}}$*, we let* $\Lambda \times_{\mathbb{V}} \Gamma \in \mathrm{C{\scriptstyle AT}}^2_{\mathbb{V}}$ *be such that:*

- *an object* $\mathcal{C} \in \Lambda \times \Gamma$ *is given by a pair* $\mathcal{C}_\Lambda \in \Lambda$*,* $\mathcal{C}_\Gamma \in \Gamma)$*,*
- *and for* $\mathcal{C}, \mathcal{D} \in \Lambda \times_{\mathbb{V}} \Gamma$*, we take* $\Lambda \times_{\mathbb{V}} \Gamma = \Lambda(\mathcal{C}_\Lambda, \mathcal{D}_\Lambda) \times_{\mathbb{V}} \Gamma(\mathcal{C}_\Gamma, \mathcal{D}_\Gamma) \in \mathrm{C{\scriptstyle AT}}_{\mathbb{V}}$.

Then we let $\mathrm{C{\scriptstyle AT}}^1_{\mathbb{V}} \in \mathrm{C{\scriptstyle AT}}^2_{\mathbb{V}}$ *be such that:*

- *the objects* $\mathcal{C} \in \mathrm{C{\scriptstyle AT}}^1_{\mathbb{V}}$ *correspond to objects* $\mathcal{C} \in \mathrm{C{\scriptstyle AT}}_{\mathbb{V}}$*,*
- *and for* $\mathcal{C}, \mathcal{D} \in \mathrm{C{\scriptstyle AT}}^1_{\mathbb{V}}$*, we take* $\mathrm{C{\scriptstyle AT}}^1_{\mathbb{V}}(\mathcal{C}, \mathcal{D}) = [\mathcal{C}, \mathcal{D}]_{\mathbb{V}} \in \mathrm{C{\scriptstyle AT}}_{\mathbb{V}}$.

By Definition 4.6.6 and Remark 4.6.15, we obtain a morphism

$$\times_{\mathbb{V}} \in \mathrm{C{\scriptstyle AT}}^1_{\mathbb{V}} \times_{\mathbb{V}} \mathrm{C{\scriptstyle AT}}^1_{\mathbb{V}} \to \mathrm{C{\scriptstyle AT}}^1_{\mathbb{V}}$$

in $\mathrm{C{\scriptstyle AT}}^2_{\mathbb{V}}$*, which provides* $\mathrm{C{\scriptstyle AT}}^1_{\mathbb{V}}$ *with the structure of a monoid internal to* $(\mathrm{C{\scriptstyle AT}}^2_{\mathbb{V}}, \times_{\mathbb{V}})$*. Note that we obtain a hierarchic system* $\mathbb{V} \in \mathrm{C{\scriptstyle AT}}^1_{\mathbb{V}}, \mathrm{C{\scriptstyle AT}}^1_{\mathbb{V}} \in \mathrm{C{\scriptstyle AT}}^2_{\mathbb{V}}, \ldots$*, which behaves like* $\mathrm{S{\scriptstyle ET}} \in \mathrm{C{\scriptstyle AT}}, \mathrm{C{\scriptstyle AT}} \in \mathrm{C{\scriptstyle AT}}_2, \ldots$.

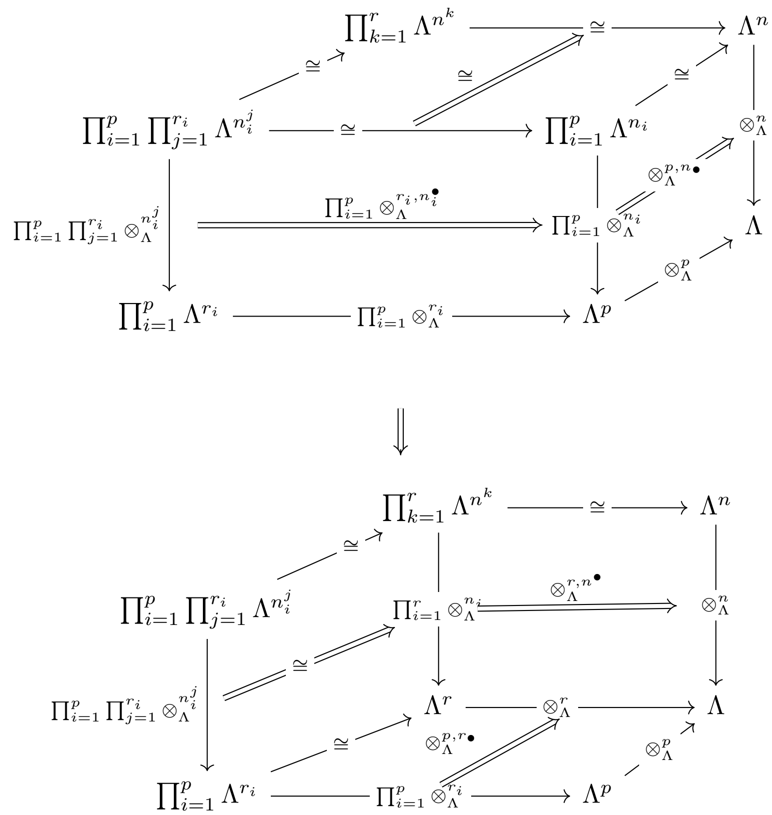


FIGURE 1. The domain and codomain of the 3-associator for a monoidal 2-category Λ.

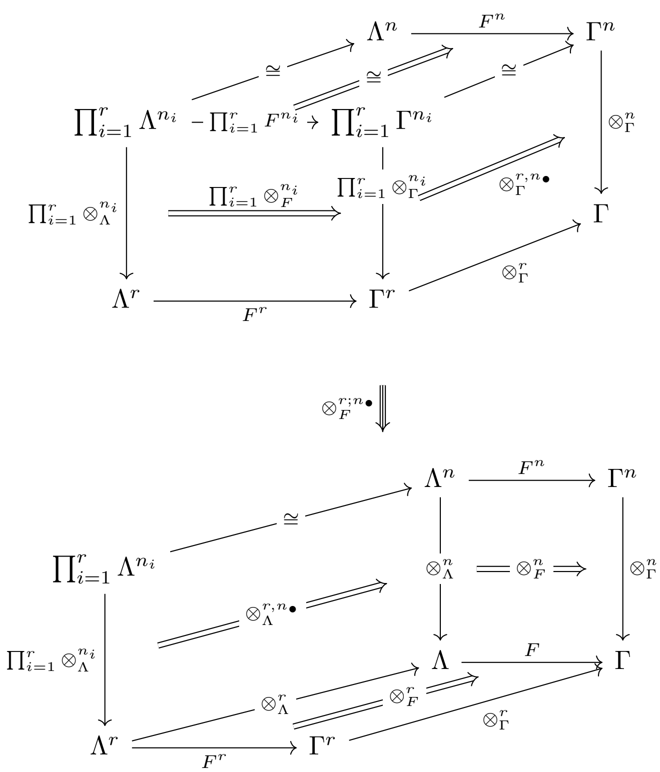


FIGURE 2. Hemispherical decomposition providing the domain and codomain of the associator attached with a monoidal 2-functor $F : \Lambda \to \Gamma$. The isomorphism on the top square of the upper diagram is obtained by using the monoidal 3-category structure of $(\mathrm{CAT}_2, \times)$.

$$\begin{tikzcd}
& ((\Lambda\times\Lambda)\times\Lambda)\times\Lambda \arrow[dl, "\alpha^{2\mathrm{Cat}}(\Lambda,\Lambda,\Lambda)\times\Lambda"'] \arrow[rr, "(\otimes_\Lambda\times\Lambda)\times\Lambda"] & & (\Lambda\times\Lambda)\times\Lambda \arrow[d, "\otimes_\Lambda\times\Lambda"] & \\
(\Lambda\times(\Lambda\times\Lambda))\times\Lambda \arrow[r, "(\Lambda\times\otimes_\Lambda)\times\Lambda"] \arrow[d, "\alpha^{2\mathrm{Cat}}(\Lambda,\Lambda\times\Lambda,\Lambda)"'] & (\Lambda\times\Lambda)\times\Lambda \arrow[rr, "\otimes_\Lambda\times\Lambda"] \arrow[d, "\alpha^{\mathrm{Cat}_2}(\Lambda,\Lambda,\Lambda)"'] \arrow[ur, phantom, "\cong\ \Rightarrow\alpha^\Lambda"] & & \Lambda\times\Lambda \arrow[dr, "\otimes_\Lambda"] & \\
\Lambda\times((\Lambda\times\Lambda)\times\Lambda) \arrow[r, "\Lambda\times(\otimes_\Lambda\times\Lambda)"] \arrow[d, "\Lambda\times\alpha^{2\mathrm{Cat}}(\Lambda,\Lambda,\Lambda)"'] \arrow[dr, phantom, "\cong"] & \Lambda\times(\Lambda\times\Lambda) \arrow[dr, "\Lambda\times\otimes_\Lambda"] \arrow[rr, phantom, "\Rightarrow\alpha^\Lambda\ \cong"] & & & \Lambda \\
\Lambda\times(\Lambda\times(\Lambda\times\Lambda)) \arrow[r, "\Lambda\times(\Lambda\times\otimes_\Lambda)"'] & \Lambda\times(\Lambda\times\Lambda) \arrow[r, "\Lambda\times\otimes_\Lambda"'] \arrow[u, phantom, "\Rightarrow\alpha^\Lambda"] & \Lambda\times\Lambda \arrow[urr, "\otimes_\Lambda"] & &
\end{tikzcd}$$

FIGURE 3. Associativity 1. Here, α^{Cat_2} refers to the associator of $(\mathrm{Cat}_2,\times)$. The isomorphism on the square in the middle of the diagram is provided by naturality of α^{Cat_2}.

$$\begin{tikzcd}
& ((\Lambda\times\Lambda)\times\Lambda)\times\Lambda \arrow[dl, "\alpha^{2\mathrm{Cat}}(\Lambda,\Lambda,\Lambda)\times\Lambda"'] \arrow[dd, "\alpha^{2\mathrm{Cat}}(\Lambda\times\Lambda,\Lambda,\Lambda)"] \arrow[rr, "(\otimes_\Lambda\times\Lambda)\times\Lambda"] & & (\Lambda\times\Lambda)\times\Lambda \arrow[dd, "\alpha^{\mathrm{Cat}_2}(\Lambda,\Lambda,\Lambda)"'] \arrow[ddrrr, "\otimes_\Lambda\times\Lambda"] & & & & \\
(\Lambda\times(\Lambda\times\Lambda))\times\Lambda \arrow[dd, "\alpha^{2\mathrm{Cat}}(\Lambda,\Lambda\times\Lambda,\Lambda)"'] \arrow[dr, phantom, "\cong\ \Leftarrow"] & & & & & & & \\
& (\Lambda\times\Lambda)\times(\Lambda\times\Lambda) \arrow[rr, "\otimes_\Lambda\times(\Lambda\times\Lambda)"] \arrow[dr, "(\Lambda\times\Lambda)\times\otimes_\Lambda"'] \arrow[dd, "\alpha^{2\mathrm{Cat}}(\Lambda,\Lambda,\Lambda\times\Lambda)"] & & \Lambda\times(\Lambda\times\Lambda) \arrow[dr, "\Lambda\times\otimes_\Lambda"] \arrow[rrr, phantom, "\Rightarrow\alpha^\Lambda\ \cong"] & & & \Lambda\times\Lambda \arrow[ddr, "\otimes_\Lambda"] & \\
\Lambda\times((\Lambda\times\Lambda)\times\Lambda) \arrow[dr, "\Lambda\times\alpha^{2\mathrm{Cat}}(\Lambda,\Lambda,\Lambda)"'] & & (\Lambda\times\Lambda)\times\Lambda \arrow[rr, "\otimes_\Lambda\times\Lambda"] \arrow[dd, "\alpha^{\mathrm{Cat}_2}(\Lambda,\Lambda,\Lambda)"] & & \Lambda\times\Lambda \arrow[drrr, "\otimes_\Lambda"] & & & \\
& \Lambda\times(\Lambda\times(\Lambda\times\Lambda)) \arrow[dr, "\Lambda\times(\Lambda\times\otimes_\Lambda)"'] & & & & & & \Lambda \\
& & \Lambda\times(\Lambda\times\Lambda) \arrow[drrr, "\Lambda\times\otimes_\Lambda"] \arrow[urrr, phantom, "\cong"] & & & & & \\
& & & & & \Lambda\times\Lambda \arrow[uurr, "\otimes_\Lambda"] \arrow[ur, phantom, "\Uparrow\alpha^\Lambda"] & &
\end{tikzcd}$$

FIGURE 4. Associativity 2. This diagram is made of 3 pentagons and 3 squares. There is one square on the top, one square on the middle and one on the bottom. We obtain the isomorphism on the middle square by functoriality of the tensor product of Λ. The isomorphisms on the squares of the top and the bottom are given by naturality of the associator α^{Cat_2} provided by the monoidal structure of Cat_2.

$$
\begin{tikzcd}
& ((\,\_ \otimes_{\mathcal{C}} \_\,) \otimes_{\mathcal{C}} \_\,) \otimes_{\mathcal{C}} \_ \arrow[dl, "\alpha^{\mathcal{C}}(\,\_ \otimes_{\mathcal{C}} \_\,,\,\_\,,\,\_\,)"'] \arrow[dr, "\alpha^{\mathcal{C}}_{(\,\_\,,\,\_\,,\,\_\,)} \otimes_{\mathcal{C}} \_"] & \\
(\,\_ \otimes_{\mathcal{C}} \_\,) \otimes_{\mathcal{C}} (\,\_ \otimes_{\mathcal{C}} \_\,) \arrow[d, "\alpha^{\mathcal{C}}_{(\,\_\,,\,\_\,,\,\_ \otimes_{\mathcal{C}} \_\,)}"'] & & (\,\_ \otimes_{\mathcal{C}} (\,\_ \otimes_{\mathcal{C}} \_\,)) \otimes_{\mathcal{C}} \_ \arrow[d, "\alpha^{\mathcal{C}}(\,\_\,,\,\_ \otimes_{\mathcal{C}} \_\,,\,\_\,)"] \\
\_ \otimes_{\mathcal{C}} (\,\_ \otimes_{\mathcal{C}} (\,\_ \otimes_{\mathcal{C}} \_\,)) & & \_ \otimes_{\mathcal{C}} ((\,\_ \otimes_{\mathcal{C}} \_\,) \otimes_{\mathcal{C}} \_\,) \arrow[ll, "\_ \otimes_{\mathcal{C}} \alpha^{\mathcal{C}}(\,\_\,,\,\_\,,\,\_\,)"]
\end{tikzcd}
$$

FIGURE 5. Coherence constraint for the associator. Here, we see the vertices of the pentagone as objects in the category of morphisms $\Lambda(\mathcal{C}^{\otimes^4_\Lambda}, \mathcal{C})$, where $\mathcal{C}^{\otimes^4_\Lambda}$ can be written without ambiguity since all braidings are equivalent up to a unique isomorphism. The corresponding 3-dimensional coherence diagram has a hemispherical decomposition into 2-dimensional faces that are required to be equal. Those hemispheres respectively correspond to Figure 3 and Figure 4.

$$
\begin{tikzcd}
(- \otimes_{\mathcal{C}} 1_{\mathcal{C}}) \otimes_{\mathcal{C}} - \arrow[r, "\alpha^{\mathcal{C}}(-,1_{\mathcal{C}},-)"] \arrow[d, "r^{\mathcal{C}}(-) \otimes_{\mathcal{C}} -"'] & - \otimes_{\mathcal{C}} (1_{\mathcal{C}} \otimes_{\mathcal{C}} -) \arrow[dl, "- \otimes_{\mathcal{C}} l^{\mathcal{C}}(-)"] \\
- \otimes_{\mathcal{C}} - &
\end{tikzcd}
$$

FIGURE 6. Coherence constraint for the left and right unitors. In the same way, this diagram lies in the category of morphisms $\Lambda(\mathcal{C}^{\otimes^2_\Lambda}, \mathcal{C})$..

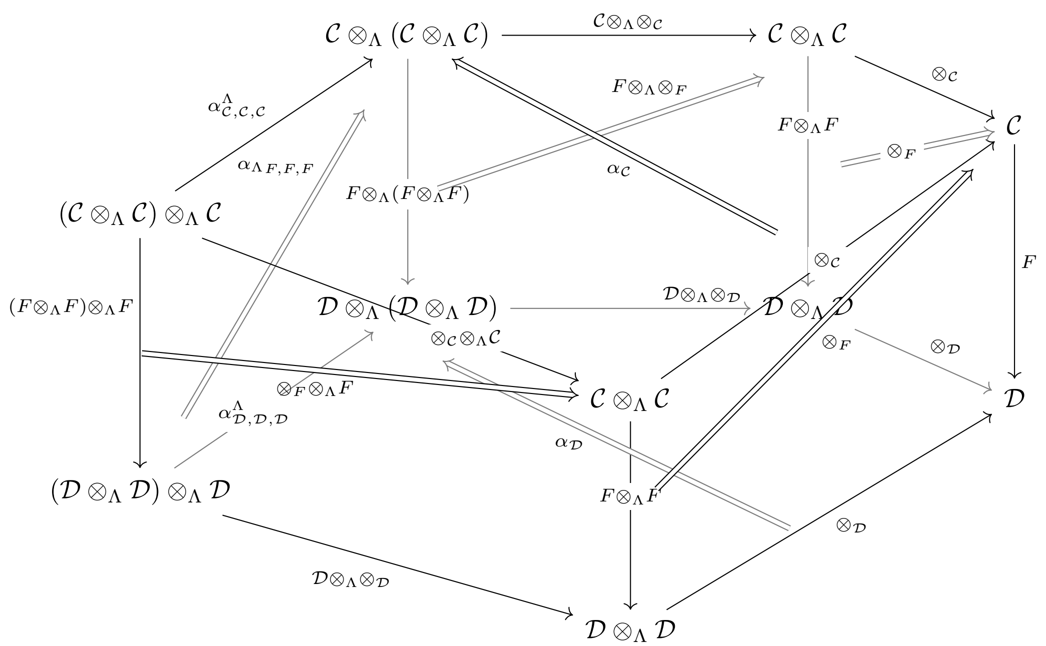


FIGURE 7. Associativity constraint for a monoidal morphisms $F : \mathcal{C} \to \mathcal{D}$ in $\mathrm{MON}_{\Lambda,\otimes_\Lambda}$

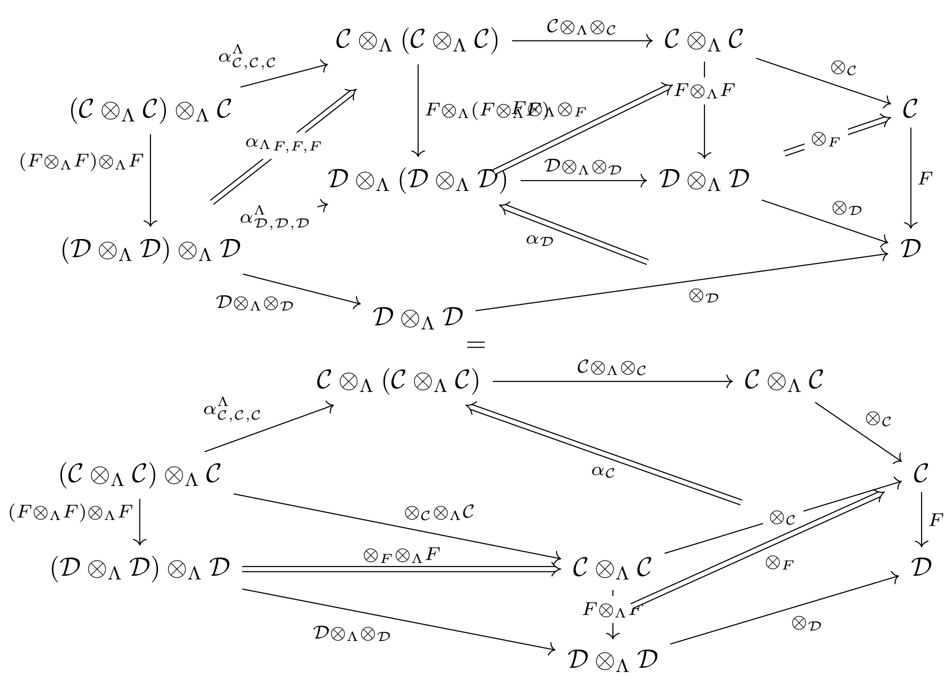


FIGURE 8. Hemispherical decomposition of the diagram below.

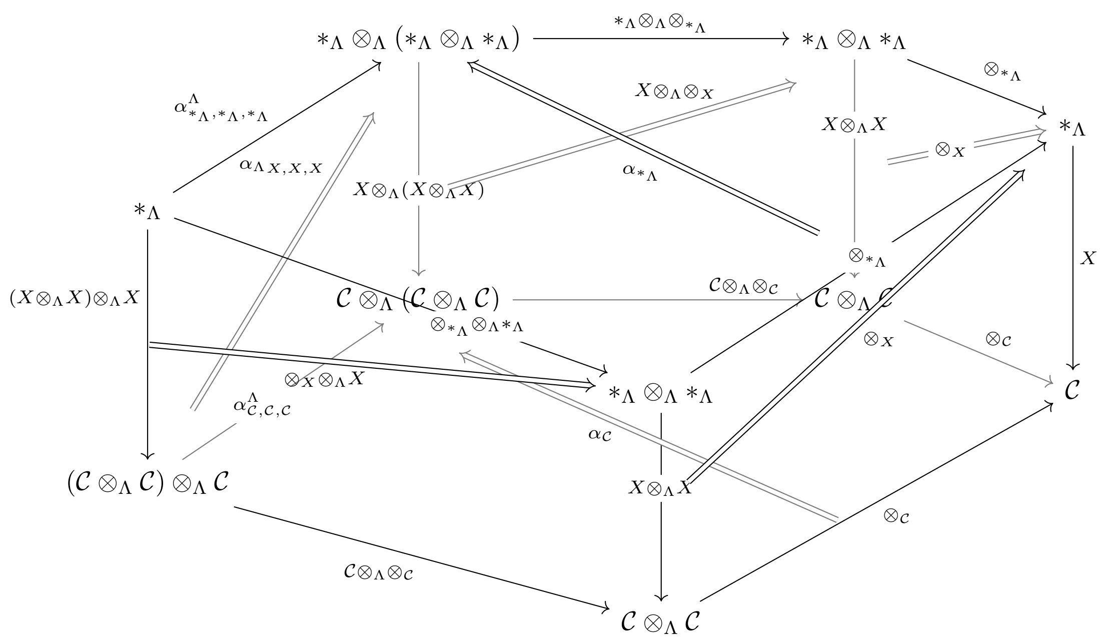


FIGURE 9. Associativity constraint for a monoid X in $\mathcal{C}$, where $\mathcal{C}$ is a monoid in $(\Lambda, \otimes_\Lambda)$

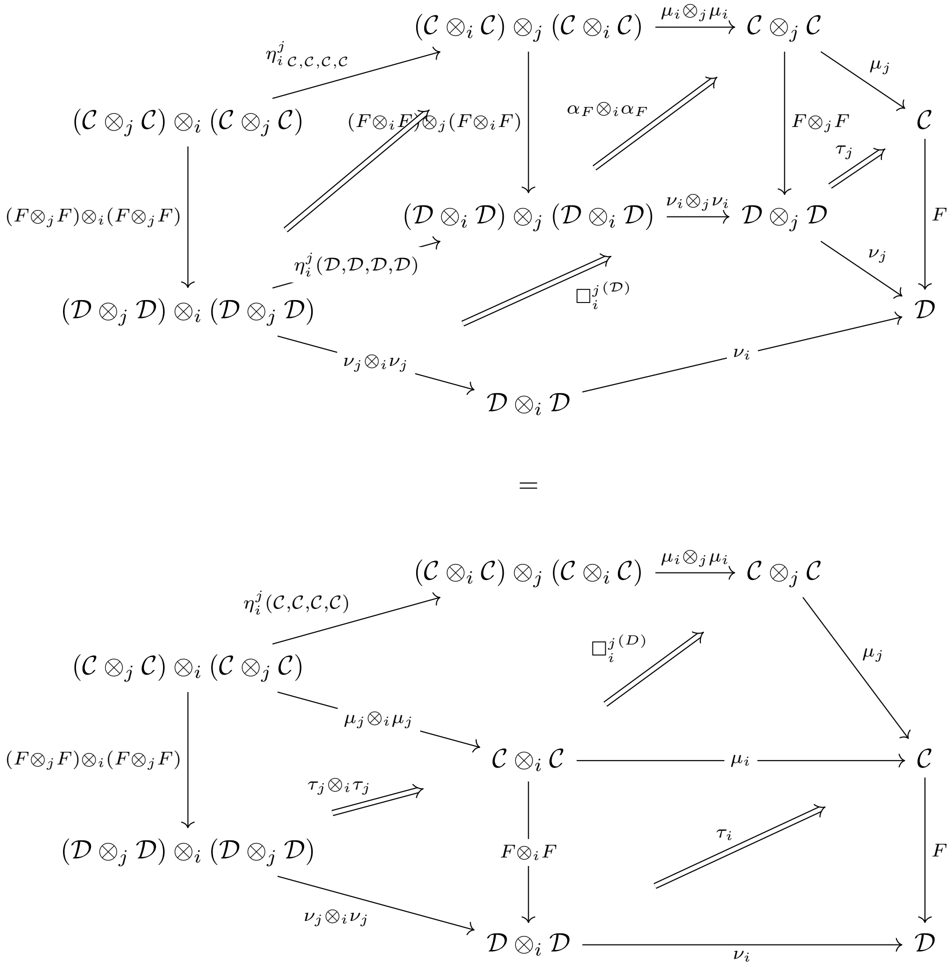


FIGURE 10. Coherence constraint for n-fold lax monoidal morphisms. While the 2-morphisms τ_j ensure that F is a morphism of monoids $(\mathcal{C}, \mu_i) \to (\mathcal{D}, \nu_i)$, this 3 dimensional diagram ensure that the 2-morphisms τ_j are lax monoidal 2-morphisms with respect to μ_i, ν_i for all $i < j$.

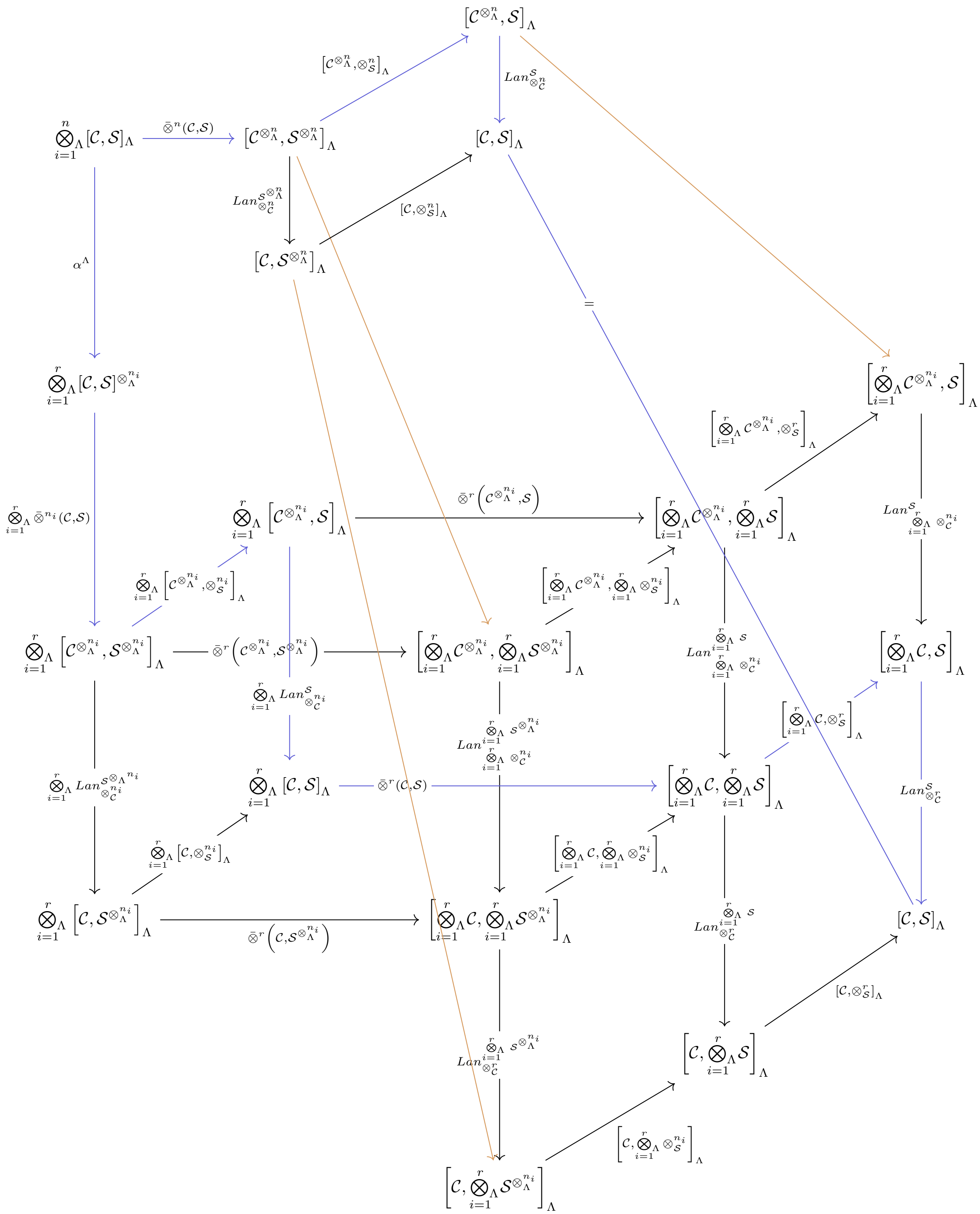


FIGURE 11. Construction of the associator $\otimes^{r,n_\bullet}_{[\mathcal{C},\mathcal{S}_\Lambda]}$. The former diagram displaying the domain and the codomain of the associator is the one obtained from the purple arrows, forgetting about the others arrows.

CHAPTER II

Operads

The concept of an operad was initially introduced by Michael Boardman and Rainer Vogt [**5**], and Peter May [**24**], for the study of (iterated) loop spaces. These works also initiated general applications of operads for the study of structures up to homotopy, including higher associative structures. The idea of an operad is to govern collections of operations of varying arities and their compositions, and to use this approach to manage the coherence issues that occur in the description of structures up to homotopy. In fact, Max Kelly simultaneously perceived the need for such a structure in the context of categorical coherence [**14**].

The topological little n-cubes operads, denoted by $\mathcal{C}_n$, were defined in order to govern the information associated to the structure of n-fold loop spaces. The key assertion is that a connected space X is weakly homotopy equivalent to an n-fold loop space as soon as it is equipped with the structure of an algebra over $\mathcal{C}_n$. To those operads, we can more precisely associate a delooping machine which produces a sequence of spaces $\mathcal{B}^p X$ for $p = 0, \dots, n$ such that $\mathcal{B}^0 X = X$ and $\Omega B^p X \sim B^{p-1} X$ for $p > 0$. In homotopy theory, in order to perform delooping constructions, we may deal with spaces equipped with actions of operads that are only weakly homotopy equivalent to $\mathcal{C}_n$. We refer to such an operad as an E_n-operad.

For $n = 1$ and $n = \infty$, we have also delooping machines associated to discrete operads, and hence for algebra structures on spaces governed by discrete collections of operations. Indeed, we can form a delooping $\mathcal{B}X$ out of a monoid structure on a space X, and iterate this construction infinitely many times as soon as the monoidal structure on X is commutative. In fact, we can establish a purely algebraic definition of structures underlying iterated loop spaces for all $n = 1, 2, \dots, \infty$ by passing to categories, and considering the nerve to get the connection with topological spaces. The notion of a monoid extends to the notion of a monoidal category, and the notion of a commutative monoid extends to the notion of a symmetric monoidal category. Balteanu, Fiedorowicz, Schwnzle, and Vogt defined an operad $\mathcal{M}^n$ for any n, governing n-fold monoidal category structures, which are intermediate between monoidal categories and symmetric monoidal categories (see [**2**]). They proved that to any n-fold monoidal category $\mathcal{C}$, one can associate a space $\mathcal{B}^n\mathcal{C}$ together with a map $|\mathcal{N}\mathcal{C}| \to \Omega^n B^n \mathcal{C}$, which is a weak homotopy equivalence when $|\mathcal{N}\mathcal{C}|$ is connected, and a group completion in general. In the particular case $n = 2$, we have similar results using an operad governing braided monoidal categories instead of $\mathcal{M}_2$. Moreover, Balteanu, Fiedorowicz, Schwnzle, and Vogt showed that the realization of the nerve of $\mathcal{M}^n$ is weakly homotopy equivalent to the operad of the little n-cubes, so that those operads can be understood as categorical analogues of E_n-operads.

The purpose of this chapter is to revisit the general definitions of the theory of operads and to go deeper in the study of categorical operads, with motivating examples given by the operads of iterated monoidal categories.

In a first section, we recall some basic notions about operads defined in monoidal categories and their algebras. We give a new definition of the composition product of collections, on which the structure of an operad is shaped, with the aim of providing a more general framework for the definition of operads. The idea is to investigate the distinct roles played by two monoidal structures in the composition product of sequences. The first law yields a closed monoidal structure, while the second law depends on the structure of the objects under consideration. We also give a natural generalization of the definition of the composition structure of operads for operads defined in 2-categories. We extend day convolution product to the context of algebras over a categorical operad $\mathcal{P}$ in a monoidal 2-category Λ, which is equipped with the structure described in the first chapter such that the object $\mathcal{S}$ collecting the internal morphisms has the structure of a $\mathcal{P}$-algebra. In this way, the presheaf object of any $\mathcal{P}$-algebra inherits a $\mathcal{P}$-algebra structure. The operads $\mathcal{M}^n$ are defined in the category of small categories. Nevertheless, it will be convenient to use operads defined in the 2-category of all categories and to regard $\mathcal{M}^n$ as such. Indeed, just like set operads may act on any monoidal category because of the definition of a category, categorical operads naturally act on any monoidal 2-category. In the case of $\mathcal{M}^n$, we will be able to retrieve the 2-categories of n-fold monoids in a symmetric monoidal 2-category as a 2-category of algebras over $\mathcal{M}^n$. In addition, the application of some constructions of the first chapter will lead to consider, in the fourth chapter, operads defined in categories which are not small but defined in categories of presheaves. To complete our study, we recall the n-fold delooping construction described in [**2**] and briefly investigate the properties a 2-category should have to make this delooping possible, in an attempt to realize n-fold monoids in a 2-category as topological n-fold loop spaces.

In a second section, after having recalled the way the category of operads inherits a model category structure from the monoidal category over which it is defined, we equip the category Op_{Cat} of small categorical operads with the model structure transported from the canonical model structure on Cat. We identify cofibrant operads as those operads whose objects form a free operad. We provide a functorial cofibrant resolution of operads. We observe that the resolution ${\mathcal{M}_1}^{\infty}$ of $\mathcal{M}_1$ is given aritywise by the codiscrete groupoid on the set of trees of this arity. But for $\mathcal{M}^n$ with $n > 1$, the functorial resolution becomes very big, and is therefore hardly exploitable.

In the third section, we formalize a definition of presentations by generators and relations adapted to small categorical operads. The interplays between the aritywise structure of a category and the global structure of an operad make the structure of categorical operads intricate, so that the usual presentations of operads in terms of generators and relations cannot fit the cases we are investigating. Indeed, the operads $\mathcal{M}^n$ have generating morphisms between operadic composites of its generating objects. To handle such generation schemes, we take inspiration from polygraphs, which provide convenient generators for ω-categories. By employing this approach, we will be able to generate categorical operads in both the operadic direction and the categorical direction. We will also impose relations on operadic polygraphs to obtain a polygraphic notion of presentations of operads. As an application, we provide the operads $\mathcal{M}^n$ with a polygraphic presentation. We also exhibit a polygraphic presentation of the operad ${\mathcal{M}_1}^{\infty}$, thanks to which it identifies with the face poset of the unital associahedra (see[**26**]).

A notion of operadic polygraphs previously arose in the context of higher rewritting for operads defined in vector spaces. To be specific, in [**23**], Philippe Malbos and Isaac Ren introduced operadic polygraphs in a way that is actually constructed in the reverse direction to ours. Precisely, the objects arising from their construction are resolutions of linear operads by ω-categories internal to linear operads, where ours aim to provide presentations of operads internal to categories. However, it seems that the free omega category monad and the free operad monad distribute over each other, so that ω-categories internal to operads should identify with operads defined in ω-categories. It should also be mentioned that what we call a polygraphic presentation of operads differs from their definition in that we actually need our generating cells to be subjected to relations in each dimension, while they inductively add some cells of higher dimensions to relax the relations involving the operations of linear operads in a homotopical coherent way.

1. Generalities on categorical operads and their algebras

We define operads and their algebras in a 2-categorical context. We both consider symmetric and non symmetric operads with values in the 2-category of categories. We do not explicit the case of operads with values in a monoidal 2-category which is not CAT. However, we make all of the constructions formal, so that more general cases should be easily deducible from the one established in this section. In particular, one can easily obtain the definition of non symmetric operads with values in a monoidal 2-category $(\Lambda, \otimes_\Lambda)$, provided that Λ is also equipped with a closed monoidal 2-category structure $(\Lambda, \times_\Lambda, [-,-]_\Lambda)$, possibly different, but compatible with the first monoidal 2-category structure. On the other hand, symmetric operads can be defined in any symmetric monoidal 2-category $(\Lambda, \otimes_\Lambda)$ equipped with a compatible closed monoidal structure $(\Lambda, \times_\Lambda, [-,-]_\Lambda)$. Notably, a symmetric closed monoidal 2-category yields a suitable framework for symmetric operads as soon as the the internal hom is monoidal.
We also introduce the categorical operads $\mathcal{M}^n$ of n-fold monoidal categories, and describe their algebras in any symmetric monoidal 2-category.

1.1. The unbiased structures on which operads are shaped. In Chapter I, we explained the definition of $\mathbb{N}$-unbiaised monoidal 2-categories with the purpose of unravelling fine structures underlying associativity relations in monoidal 2-categories. This definition of $\mathbb{N}$-unbiaised monoidal structures for 2-categories is a 2-categorical counterpart of a classical definition of $\mathbb{N}$-unbiaised monoidal structures for categories (see [**18**]).

The purpose of this subsection is to review the definition of unbiased structures that give the shape of the composition schemes of categorical operads, in both the non symmetric and the symmetric case. This subsection is expository, and is mostly intended to fix the background of our constructions and some conventions.

The unbiased structure of natural numbers. Let $\mathbb{N}$ be the set of natural numbers. The addition of natural numbers gives to $(\mathbb{N}, +, 0)$ the structure of a monoid in the monoidal category $(\text{SET}, \times)$. In fact, the structure of a monoid is precisely shaped on the structure of the natural numbers. To be explicit, recall that a monoidal structure on an object is defined as the data of a unit object, a binary operation (the tensor product), together with unit and associativity isomorphisms, which satisfy coherence constraints ensuring that any different orders between iterations of the tensor product yield equivalent results, so that the n-th iteration of the tensor product makes sense. In the unbiased setting, we define a monoidal structure as the data of an operation $\otimes^n$ of arity n for all $n \in \mathbb{N}$, which model an n-fold tensor product. These operations are subject to compatibility isomorphisms as well, so that the composite $\otimes^r \circ (\otimes^{n_1}, \dots, \otimes^{n_r})$ is equivalent to $\otimes^{n_1+\cdots+n_r}$. Recall simply that the structure of an unbiased monoidal category, which we obtain by this approach, is equivalent to the usual notion of a monoidal category (see [**18**]). In what follows, we use the expression '$\mathbb{N}$-unbiased monoidal category' to refer to a monoidal category equipped with the equivalent unbiased structure. Hence, an $\mathbb{N}$-unbiased monoidal category comes equipped with an operation $\otimes^n$ of arity n for each $n \in \mathbb{N}$.

In the appendix, we observe that the structure of the set of natural numbers is precisely shaped on the principle of iteration, which underlies this correspondence between the usual and unbiased monoidal structures. Recall that each natural number n corresponds to the n-th iteration of the map $s : \mathbb{N} \to \mathbb{N}$ applied to the element 0, so that we can write $n = s^n(0)$. Note that this notation makes sense, again by virtue of the associativity of the composition of maps. The unbiased definition of the monoidal structure inherent to the set of natural numbers precisely gives the way composites of operations should arrange for unbiased monoidal structures, and may be formalized as follows. Let $r \in \mathbb{N}$. We have an operation

$$+^r : \mathbb{N}^r \to \mathbb{N},$$

which to $n_1, \dots, n_r$ associates the sum $+^r(n_1, \dots, n_r) = n_1 + \cdots + n_r$. Since each n_i satisfies $n_i = +^{n_i}(1, \dots, 1)$, we obtain the relation

$$+^r(+^{n_1}(1, \dots, 1), \dots, +^{n_r}(1, \dots, 1)) = +^{+^r(+^{n_1}(1,\dots,1),\dots,+^{n_r}(1,\dots,1))}(1, \dots, 1).$$

In the context of operads, we will interpret the above equality of natural numbers as an equality of operations $+^r(+^{n_1}, \dots, +^{n_r}) = +^{+^r(n_1,\dots,n_r)}$.

For the moment, we just form an $\mathbb{N}$-sequence

$$[\mathbb{N}^\bullet, \mathbb{N}] : \mathbb{N} \to \mathrm{S\small{ET}},$$

to collect such operations $+^r$. Basically, this $\mathbb{N}$-sequence associates to r the set of functions $\mathbb{N}^r \to \mathbb{N}$. Let $*^{\mathbb{N}} : \mathbb{N} \to \mathrm{S\small{ET}}$ be the terminal $\mathbb{N}$-sequence, so that $*^n = *$. The collection of operations $+^r$ can now be expressed as a morphism in the category $[\mathbb{N}, \mathrm{S\small{ET}}]$ such that

$$+^\bullet : *^{\mathbb{N}} \to [\mathbb{N}^\bullet, \mathbb{N}].$$

We will see that the terminal sequence $*^{\mathbb{N}}$ defines an $\mathbb{N}$-operad, whose algebras precisely correspond to $\mathbb{N}$-unbiased monoidal objects. In this setting, we can interpret the set of natural numbers $\mathbb{N}$ as the free $\mathbb{N}$-unbiased monoidal object generated by the terminal set $*$ in the $\mathbb{N}$-unbiased monoidal category of sets.

The unbiased structure of symmetric groups. We adapt the previous construction to define a collection that models unbiased structures attached to symmetric monoidal structures. We will generally use the phrase '$\mathfrak{S}$-unbiased structures' to refer to these unbiased structures, where the notation $\mathfrak{S}$ refers to the symmetric groupoid, on which we shape these structures, and of which we review the definition first.

Let $n \in \mathbb{N}$. In what follows, we generally write $|n| = \{1, \ldots, n\}$ for the set with n elements, and we adopt the notation Σ_n for the symmetric group on n letters, regarded as the group of automorphisms of this set $|n|$. Then we define the symmetric groupoid $\mathfrak{S}$ as the category which has the natural numbers n as objects and the elements of these automorphism groups as morphisms. Note that $\mathfrak{S}$ forms a groupoid. Nevertheless, in what follows, we generally regard $\mathfrak{S}$ as an object of the 2-category Cat. We may also write $\mathfrak{S}$ for the 2-category $\mathrm{I}\mathfrak{S}$, which we obtain by adding identity 2-morphisms to $\mathfrak{S}$. We then call 'symmetric sequences of categories' the objects of the 2-category such that $\mathrm{Cat}^{\mathfrak{S}} = [\mathfrak{S}, \mathrm{Cat}]$.

The symmetric groupoid naturally arises in the description of the different ways of permuting coordinates. To see this, we can first notice that the set $|n| : * \to \mathrm{Set}$ with n elements can be defined as the n-th iterated coproduct of the terminal set, so that $|n| \cong * \sqcup \cdots \sqcup *$. The n distinct elements of $|n|$ are given by the structural morphisms of the coproduct $\iota_i : * \to * \sqcup \cdots \sqcup *$ for $i = 1, \ldots, n$. By the universal property of the coproduct, the data of a morphism $\sigma : |n| \to |n|$ in the category of sets is equivalent to the data of a morphism $\sigma_i : * \to * \sqcup \cdots \sqcup *$ for each $i = 1, \ldots, n$, and hence to an element $\iota_{\sigma_i} : * \to |n|$. In this formalism, we obtain $\Sigma_n \cong \mathrm{Set}^{\cong}(|n|, |n|)$, so that each $\sigma \in \Sigma_r$ corresponds to the automorphism of $|n|$ given by the family $\left\{\iota_{\sigma(i)}\right\}_{i=1,\ldots,r}$.

Let $\mathcal{C}$ be a category and observe that for each $n \in \mathbb{N}$, the n-th iterated cartesian product of $\mathcal{C}$ satisfies

$$\mathcal{C}^n = \underbrace{\mathcal{C} \times \cdots \times \mathcal{C}}_{n} \cong \prod_{i=1}^{n} \mathcal{C} \cong [* \sqcup \cdots \sqcup *, \mathcal{C}],$$

where the terminal object appears n times. Hence, each category induces a 2-functor

$$\mathcal{C}^{\bullet} : \mathfrak{S}^{op} \to \mathrm{Cat},$$

given by $r \mapsto \mathcal{C}^r = \underbrace{\mathcal{C} \times \cdots \times \mathcal{C}}_{r}$ on objects, and that we determine as follows on morphisms. Let $\sigma \in \Sigma_r$. Then σ induces a functor

$$[* \sqcup \cdots \sqcup *, \mathcal{C}] \xrightarrow{[\sigma,\mathcal{C}]} [* \sqcup \cdots \sqcup *, \mathcal{C}]$$

by pre-composition, which is an isomorphism. In particular, for $r \in \mathbb{N}$, we obtain a 2-functor

$$\mathfrak{S}^{\bullet} : \mathfrak{S}^{op} \to \mathrm{Cat},$$

which to $r : * \to \mathfrak{S}^{op}$ associates the cartesian product category $\mathfrak{S}^r$, and which to a permutation $\sigma \in \Sigma_r$ associates a functor

$$\sigma^* : \mathfrak{S}^r \xrightarrow{\cong} \mathfrak{S}^r.$$

The functor σ^* is given

- by $\sigma^*(n_1, \ldots, n_r) = (n_{\sigma^{-1}(1)}, \ldots, n_{\sigma^{-1}(r)})$ on objects,
- by $\sigma^*(\tau_1, \ldots, \tau_r) = (\tau_{\sigma^{-1}(1)}, \ldots, \tau_{\sigma^{-1}(r)})$ on morphisms.

In order to fully describe how the different compositions of permutations arrange, observe that the 2-functor $\mathfrak{S}^{\bullet}$ induces a 2-functor

$$[\mathfrak{S}^{\bullet}, \mathfrak{S}] : \mathfrak{S} \to \mathrm{Cat},$$

such that $[\mathfrak{S}^\bullet,\mathfrak{S}](r) = [\mathfrak{S}^r,\mathfrak{S}]$, and $[\mathfrak{S}^\bullet,\mathfrak{S}](r,r)(\sigma) = [\mathfrak{S}^r,\mathfrak{S}] \xrightarrow{[\sigma^*,\mathfrak{S}]} [\mathfrak{S}^r,\mathfrak{S}]$. For our purpose, we also consider the terminal symmetric sequence

$$*^{\mathfrak{S}} : \mathfrak{S} \to \mathrm{CAT}$$

given by $*(r) = *$.

The multiplicative structure of permutations can now be expressed in terms of a morphism

$$+^\bullet : *^{\mathfrak{S}} \to [\mathfrak{S}^\bullet,\mathfrak{S}]$$

in the 2-category of symmetric sequences of categories $\mathrm{CAT}^{\mathfrak{S}} = [\mathfrak{S},\mathrm{CAT}]$. To be explicit, for $r : * \to \mathfrak{S}$, we define

$$+^r : \mathfrak{S}^r \to \mathfrak{S},$$

by $+^r(n_1,\dots,n_r) = n_1 + \cdots + n_r$ on objects and by

$$\prod_{i=1}^r \mathrm{SET}^{\cong}(|n_i|,|n_i|) \xrightarrow{\amalg} \mathrm{SET}^{\cong}\left(\coprod_{i=1}^r |n_i|, \coprod_{i=1}^r |n_i|\right) \xrightarrow{\cong} \mathrm{SET}^{\cong}(|n_1+\cdots+n_r|,|n_1+\cdots+n_r|)$$

on morphisms. To give $+^\bullet$ the structure of a 2-natural transformation, we define, for $\sigma \in \Sigma_r$, a natural isomorphism $+^\sigma$ such that

$$\begin{tikzcd} * \arrow[r, "+^r"] \arrow[d, "\sigma"'] & {[\mathfrak{S}^r,\mathfrak{S}]} \arrow[d, "\sigma_*"] \arrow[dl, phantom, "\overset{+^\sigma}{\Swarrow}\cong"] \\ * \arrow[r, "+^r"'] & {[\mathfrak{S}^r,\mathfrak{S}].} \end{tikzcd}$$

For this purpose, let $(n_1,\dots,n_r) : * \to \mathfrak{S}^r$. Then we define

$$+^\sigma(n_1,\dots,n_r) : +^r((n_{\sigma^{-1}(1)},\dots,n_{\sigma^{-1}(r)}) \to +^r(n_1,\dots,n_r)$$

by the isomorphism

$$|n_{\sigma^{-1}(1)}+\cdots+n_{\sigma^{-1}(r)}| \cong |n_{\sigma^{-1}(1)}|\sqcup\cdots\sqcup|n_{\sigma^{-1}(r)}| \xrightarrow{\sigma} |n_1|\sqcup\cdots\sqcup|n_r| \cong |n_1+\cdots+n_r|.$$

For all morphism $(\tau_1,\dots,\tau_r) \in \mathfrak{S}^r((n_1,\dots,n_r),(n_1,\dots,n_r))$, the diagram

$$\begin{tikzcd} {|n_{\sigma^{-1}(1)}| \sqcup \cdots \sqcup |n_{\sigma^{-1}(r)}|} \arrow[r, "+^\sigma(n_1,\dots,n_r)"] \arrow[d, "\tau_{\sigma^{-1}(1)}\sqcup\cdots\sqcup\tau_{\sigma^{-1}(r)}"'] & {|n_1| \sqcup \cdots \sqcup |n_r|} \arrow[d, "\tau_1\sqcup\cdots\sqcup\tau_r"] \\ {|n_{\sigma^{-1}(1)}| \sqcup \cdots \sqcup |n_{\sigma^{-1}(r)}|} \arrow[r, "+^\sigma(n_1,\dots,n_r)"'] & {|n_1| \sqcup \cdots \sqcup |n_r|} \end{tikzcd}$$

commutes. Here we write $\tau_1 \sqcup \cdots \sqcup \tau_r$ instead of $+^r(\tau_1,\dots,\tau_r)$ for clarity. It follows that $+^\sigma$ defines a natural transformation $\sigma_* +^r \Rightarrow +^r$, so that we have defined the 2-functor $+^\bullet : *^{\mathfrak{S}} \to [\mathfrak{S}^\bullet,\mathfrak{S}]$ governing the structure of symmetries.

We will define $\mathfrak{S}$-operads as objects of the 2-category $\mathrm{CAT}^{\mathfrak{S}}$ equipped with an extra structure called composition. We will see that the terminal symmetric sequence $*^{\mathfrak{S}}$ defines a $\mathfrak{S}$-operad, whose algebras precisely correspond to $\mathfrak{S}$-unbiased monoidal objects. We can then interpret the symmetric groupoid $\mathfrak{S}$ as the free $\mathfrak{S}$-unbiased monoidal object generated by the terminal category $*$ in the $\mathfrak{S}$-unbiased monoidal category of categories.

Conventions. In what follows, we let $\mathbb{A}$ denote either $\mathbb{N}$ or $\mathfrak{S}$, regarded either as a category or as a 2-category depending on the context. Most of the time, we will treat simultaneously the cases of monoidal structures ($\mathbb{A} = \mathbb{N}$) and of symmetric monoidal structures ($\mathbb{A} = \mathfrak{S}$) using the 2-category $\mathbb{A}$. In both cases, the objects of $\mathbb{A}$ are given by the set of natural numbers and there exists a morphism between n and m if and only if n and m are equal.

When dealing with symmetric monoidal structures, we often need to use the morphisms of $\mathbb{A}$. Hence, we may write $\sigma : * \to \mathbb{A}(n, n)$ for constructions in the general $\mathbb{A}$-unbiased monoidal framework, so that when $\mathbb{A}$ is $\mathfrak{S}$, the morphism σ corresponds to a permutation, whereas when $\mathbb{A}$ is $\mathbb{N}$, the morphism σ is always the identity of n.

1.2. Miscellaneous structures on 2-categories. In this subsection, we revisit the definition of structures on 2-categories that we need for the definition of operads.

Tensored 2-categories.

Definition 1.2.1. Let Λ be an $\mathbb{A}$-unbiased monoidal 2-category. Let Γ be a 2-category. We say that Γ is tensored over Λ if it is equipped with a 2-functor

$$\cdot : \Lambda \times \Gamma \to \Gamma,$$

which satisfies natural unit and associativity relations with respect to the internal monoidal structure of the category Λ. In the case where the 2-category Γ is also $\mathbb{A}$-unbiased monoidal, we say that the tensoring is monoidal if this 2-functor forms an $\mathbb{A}$-unbiased monoidal morphism.

We say that Γ is enriched over Λ is it is equipped with a 2-functor

$$\Gamma^{\Lambda}(-, -) : \Gamma^{op} \times \Gamma \to \Lambda,$$

with unit morphisms $1_{\Gamma} : \mathbb{1}_{\Lambda} \to \Gamma^{\Lambda}(\mathcal{P}, \mathcal{P})$ and composition morphisms $\circ : \Gamma^{\Lambda}(\mathcal{R}, \mathcal{Q}) \otimes \Gamma^{\Lambda}(\mathcal{P}, \mathcal{R}) \to \Gamma^{\Lambda}(\mathcal{P}, \mathcal{Q})$, which again satisfy natural unit and associativity relations.

We say that Γ is tensored and enriched over Λ if it is both tensored over Λ and enriched over Λ and, for each object $\mathcal{C}$ of Λ and each pair of objects $\mathcal{P}, \mathcal{Q}$ of Γ, we have an equivalence of categories

$$\Gamma(\mathcal{C} \cdot \mathcal{P}, \mathcal{Q}) \cong \Lambda(\mathcal{C}, \Gamma^{\Lambda}(\mathcal{P}, \mathcal{Q})).$$

We also assume in this context that the unit $1_{\Gamma} : \mathbb{1}_{\Lambda} \to \Gamma^{\Lambda}(\mathcal{P}, \mathcal{P})$ corresponds to the unit isomorphism $\mathbb{1}_{\Lambda} \cdot \mathcal{P} \cong \mathcal{P}$ of the tensor structure under the equivalence of categories $\Lambda(\mathbb{1}_{\Lambda}, \Gamma^{\Lambda}(\mathcal{P}, \mathcal{Q})) \cong \Gamma(\mathbb{1}_{\Lambda} \cdot \mathcal{P}, \mathcal{Q})$, while the composition morphism $\circ : \Gamma^{\Lambda}(\mathcal{R}, \mathcal{Q}) \otimes \Gamma^{\Lambda}(\mathcal{P}, \mathcal{R}) \to \Gamma^{\Lambda}(\mathcal{P}, \mathcal{Q})$ corresponds to the composite of adjunction augmentations $\Gamma^{\Lambda}(\mathcal{P}, \mathcal{Q}) \cdot (\Gamma^{\Lambda}(\mathcal{P}, \mathcal{R}) \cdot \mathcal{P}) \to \Gamma^{\Lambda}(\mathcal{R}, \mathcal{Q}) \cdot \mathcal{R} \to \mathcal{Q}$ under the equivalences of categories

$$\begin{aligned}\Lambda(\Gamma^{\Lambda}(\mathcal{R}, \mathcal{Q}) \otimes \Gamma^{\Lambda}(\mathcal{P}, \mathcal{R}), \Gamma^{\Lambda}(\mathcal{P}, \mathcal{Q})) &\cong \Gamma((\Gamma^{\Lambda}(\mathcal{R}, \mathcal{Q}) \otimes \Gamma^{\Lambda}(\mathcal{P}, \mathcal{R})) \cdot \mathcal{P}, \mathcal{Q}) \\ &\cong \Gamma(\Gamma^{\Lambda}(\mathcal{R}, \mathcal{Q}) \cdot (\Gamma^{\Lambda}(\mathcal{P}, \mathcal{R}) \cdot \mathcal{P}), \mathcal{Q}).\end{aligned}$$

Note also that every 2-category is canonically enriched over Cat. The following proposition provides a sufficient condition for the existence of a tensoring associated to this enriched structure.

EXAMPLE 1.2.2. Every closed monoidal 2-category is tensored and enriched over itself, which is monoidal if and only if the closed structure is.

PROPOSITION 1.2.3. *Let* Γ *be a cartesian* 2*-category such that for each category* $\mathcal{C}$*, any constant* 2*-functor* $\mathbb{I}\mathcal{C} \to \Gamma$ *has a colimit in* Γ*. Then* Γ *is tensored and enriched over* CAT*.e*

PROOF. We define a 2-functor $\coprod *_\Gamma : \mathrm{CAT} \to \Gamma$ as follows. Let $\mathcal{C}$ be a category and consider the constant 2-functor

$$*_\Gamma^{\mathcal{C}} : \mathbb{I}\mathcal{C} \to * \xrightarrow{*_\Gamma} \Gamma.$$

We define $\coprod\limits_{\mathcal{C}} *_\Gamma := \coprod *_\Gamma(\mathcal{C}) : * \to \Gamma$ as the colimit of $*_\Gamma^{\mathcal{C}}$, which extends to a 2-functor $\coprod *_\Gamma : \mathrm{CAT} \to \Gamma$. Define an action of CAT on Γ as the composite

$$\cdot : \mathrm{CAT} \times \Gamma \xrightarrow{\coprod *_\Gamma} \Gamma \times \Gamma \xrightarrow{\times} \Gamma.$$

Note that we have $\mathcal{C} \cdot \mathcal{P} \cong \operatorname{colim} \coprod_{\mathcal{C}} \mathcal{P}$, where $\coprod_{\mathcal{C}} \mathcal{P}$ is the constant 2-functor

$$\mathcal{P}^{\mathcal{C}} : \mathbb{I}\mathcal{C} \xrightarrow{*} * \xrightarrow{\mathcal{P}} \Gamma,$$

The colimit of this 2-functor is an object of Γ, which we write

$$\coprod_{\mathcal{C}} \mathcal{P} : * \to \Gamma.$$

This construction extends to a 2-functor

$$\cdot : \mathrm{CAT} \times \Gamma \to \Gamma.$$

Let $\mathcal{C}$ be a category and $\mathcal{P}, \mathcal{Q}$ be objects of Γ. We have the following equivalences of categories

$$\Gamma(\mathcal{C} \cdot \mathcal{P}, \mathcal{Q}) \cong \Gamma\left(\coprod_{\mathcal{C}} \mathcal{P}, \mathcal{Q}\right) \cong \prod_{\mathcal{C}} \Gamma(\mathcal{P}, \mathcal{Q}) \cong \mathrm{CAT}(\mathcal{C}, \Gamma(\mathcal{P}, \mathcal{Q})).$$

The unit and associativity relations of this tensoring $\cdot : \mathrm{CAT} \times \Gamma \to \Gamma$ follows from universal properties. The correspondence between the unit and the composition morphisms in Γ and the universal morphisms attached to this tensoring follows from similar arguments. □

EXAMPLE 1.2.4. Suppose that a 2-category Γ is tensored over a 2-category Λ, then we have a 2-functor

$$\cdot : \Lambda \times [\mathbb{A}^{op}, \Gamma] \to [\mathbb{A}^{op}, \Lambda \times \Gamma] \xrightarrow{[\mathbb{A}^{op}, \times]} [\mathbb{A}^{op}, \Gamma],$$

so that the 2-category of 2-functors $[\mathbb{A}^{op}, \Gamma]$ is tensored over Λ[1].

---

[1]This observation is valid for any 2-category $\mathbb{A}$, not restricted to $\mathbb{A} = \mathbb{N}, \mathfrak{S}$.

The integral product.

DEFINITION 1.2.5. Let Γ be a bicomplete 2-category and Λ be a 2-category such that Γ is tensored over Λ. We define the integral product over $\mathbb{A}^2$ as the 2-functor

$$\dagger : [\mathbb{A}, \Lambda] \times [\mathbb{A}^{op}, \Gamma] \to \Gamma$$

obtained by the composite

$$[\mathbb{A}, \Lambda] \times [\mathbb{A}^{op}, \Gamma] \xrightarrow{\times} [\mathbb{A} \times \mathbb{A}^{op}, \Lambda \times \Gamma] \xrightarrow{[\mathbb{A}\times\mathbb{A}^{op}, \cdot]} [\mathbb{A} \times \mathbb{A}^{op}, \Gamma] \xrightarrow{\int_{\mathbb{A}}} \Gamma.$$

PROPOSITION 1.2.6. *Let Γ be a bicomplete 2-category and Λ a 2-category such that Γ is tensored and enriched over Λ. Then for each $\mathcal{C} : \mathbb{A}^{op} \to \Gamma$, the 2-functor induced by the integral product*

$$- \dagger \mathcal{C} : [\mathbb{A}, \Lambda] \to \Gamma$$

has a right adjoint $\Gamma^{\mathbb{A}}_{\Lambda}(\mathcal{C}, -) : \Gamma \to [\mathbb{A}, \Lambda]$, which extends to a 2-functor

$$\Gamma^{\mathbb{A}}_{\Lambda}(-, -) : [\mathbb{A}^{op}, \Gamma]^{op} \times \Gamma \to [\mathbb{A}, \Lambda].$$

Hence for $\mathcal{P} : \mathbb{A} \to \Lambda$, $\mathcal{C} : \mathbb{A}^{op} \to \Gamma$ and $X : * \to \Gamma$, we have a natural isomorphism

$$[\mathbb{A}, \Lambda]\left(\mathcal{P}, \Gamma^{\mathbb{A}}_{\Lambda}(\mathcal{C}, X)\right) \cong \Gamma\left(\mathcal{P} \dagger \mathcal{C}, X\right).$$

PROOF. Recall that the oppositization 3-functor takes the 2-category of 2-functors $[\mathbb{A}^{op}, \Gamma]^{op}$ to the 2-category $[\mathbb{A}, \Gamma^{op}]$. We let $\Gamma^{\mathbb{A}}_{\Lambda}$ be the 2-functor defined as the composite

$$\Gamma^{\mathbb{A}}_{\Lambda}(-, -) : [\mathbb{A}^{op}, \Gamma]^{op} \times \Gamma \to [\mathbb{A}, \Gamma^{op}] \times \Gamma \xrightarrow{\times} [\mathbb{A}, \Gamma^{op} \times \Gamma] \xrightarrow{[\mathbb{A}, \Gamma_{\Lambda}(-,-)]} [\mathbb{A}, \Lambda].$$

Let $\mathcal{C} : \mathbb{A}^{op} \to \Gamma$, $X : * \to \Gamma$ and $n : * \to \mathbb{A}$. We explicitly have $\Gamma^{\mathbb{A}}_{\Lambda}(\mathcal{C}, X)(n) = \Gamma^{\Lambda}(\mathcal{C}(n), X)$. Let $\mathcal{P} : \mathbb{A} \to \Lambda$, $\mathcal{C} : \mathbb{A}^{op} \to \Gamma$ and $X : * \to \Gamma$. The folowing natural isomorphisms

$$\begin{aligned}
\Gamma(\mathcal{P} \dagger \mathcal{C}, X) &\cong \Gamma\left(\int_{n:*\to\mathbb{A}} \mathcal{P}(n) \cdot \mathcal{C}(n), X\right) \\
&\cong \int^{n:*\to\mathbb{A}} \Gamma\left(\mathcal{P}(n) \cdot \mathcal{C}(n), X\right) \\
&\cong \int^{n:*\to\mathbb{A}} \Gamma\left(\mathcal{P}(n), \Gamma_{\Lambda}\left(\mathcal{C}(n), X\right)\right) \\
&\cong \int^{n:*\to\mathbb{A}} \Gamma\left(\mathcal{P}(n), \Gamma^{\mathbb{A}}_{\Lambda}\left(\mathcal{C}, X\right)(n)\right) \\
&\cong [\mathbb{A}, \Lambda]\left(\mathcal{P}, \Gamma^{\mathbb{A}}_{\Lambda}(\mathcal{C}, X)\right)
\end{aligned}$$

show that for each $\mathcal{C} : \mathbb{A}^{op} \to \Gamma$, the 2-functor $\Gamma^{\mathbb{A}}_{\Lambda}(\mathcal{C}, -)$ is right adjoint to $- \dagger \mathcal{C}$. $\square$

---

[2]The construction of this definition makes sense for any CAT-small 2-category $\mathbb{A}$, not restricted to $\mathbb{A} = \mathbb{N}, \mathfrak{S}$.

Exponentiation.

Definition 1.2.7. Let Λ be a 2-category. We define its exponentiation $\Lambda^\bullet$ as the 2-functor

$$\Lambda^\bullet : \mathbb{A}^{op} \to \mathrm{Cat}_2,$$

which to n associates the 2-category $\Lambda^n = \underbrace{\Lambda \times \cdots \times \Lambda}_{n}$, and for $\mathbb{A} = \mathfrak{S}$, associates to any permutation $\sigma \in \Sigma_n$ the isomorphism of 2-categories

$$\Lambda^n \cong [* \sqcup \cdots \sqcup *, \Lambda] \xrightarrow{[\sigma,\Lambda]} [* \sqcup \cdots \sqcup *, \Lambda] \cong \Lambda^n.$$

Remark 1.2.8. Let Λ and Γ be 2-categories. The cartesian product $\Lambda^\bullet \times \Gamma^\bullet$ in $[\mathbb{A}^{op}, \mathrm{Cat}_2]$ is naturally equivalent to $(\Lambda \times \Gamma)^\bullet$.

Proposition 1.2.9. *The 2-functor $\mathcal{U}^{\mathbb{A}}$ which forgets about $\mathbb{A}$-unbiased monoidal structures has a left adjoint*

$$\begin{tikzcd} \mathcal{F}^{\mathbb{A}} : \mathrm{Cat}_2 \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathrm{Mon}^{\mathbb{A}}\mathrm{Cat}_2 : \mathcal{U}^{\mathbb{A}}, \arrow[l, bend left] \end{tikzcd}$$

which associates, to every 2-category Λ, a free $\mathbb{A}$-unbiased monoidal 2-category $\mathcal{F}^{\mathbb{A}}\Lambda$. This object $\mathcal{F}^{\mathbb{A}}\Lambda$ can be expressed in terms of the following coend

$$\mathcal{F}^{\mathbb{A}}\Lambda \cong \int^{n:*\to\mathbb{A}} *^{\mathbb{A}} \times \Lambda^\bullet.$$

Proof. Let Λ be a 2-category and Γ be an $\mathbb{A}$-unbiased monoidal 2-category. Let $\psi : \Lambda \to \mathcal{U}^{\mathbb{A}}\Gamma$ be a 2-functor and write $\bar{\psi} : \int_{\mathbb{A}} *^{\mathbb{A}} \times \Lambda^\bullet \to \Gamma$ for the image of ψ under the composite

$$[\Lambda, \Gamma] \xrightarrow{\int_{\mathbb{A}} *^{\mathbb{A}} \times (-)^\bullet} \left[\int_{\mathbb{A}} *^{\mathbb{A}} \times \Lambda^\bullet, \int_{\mathbb{A}} *^{\mathbb{A}} \times \Gamma^\bullet\right] \xrightarrow{\left[\int_{\mathbb{A}} *^{\mathbb{A}} \times \Lambda^\bullet, \otimes_\Gamma^{\mathbb{A}}\right]} \left[\int_{\mathbb{A}} *^{\mathbb{A}} \times \Lambda^\bullet, \Gamma\right].$$

We equip $\bar{\psi}$ with the structure of an $\mathbb{A}$-unbiased monoidal 2-functor extending ψ, in the sense that the diagram

$$\begin{tikzcd} \Lambda \arrow[r, "\psi"] \arrow[d, hook] \arrow[dr, phantom, "\cong"] & \Gamma \\ \mathcal{F}^{\mathbb{A}}\Lambda \arrow[ur, "\bar{\psi}"'] & \end{tikzcd}$$

commutes up to an equivalence. Let $n : * \to \mathbb{A}$. The component $\bar{\psi}_n : \Lambda^n \to \Gamma$ of $\bar{\psi}$ in n is defined by the composite

$$\Lambda^n \xrightarrow{\psi^n} \Gamma^n \xrightarrow{\otimes_\Gamma^n} \Gamma.$$

Let $\sigma : * \to \mathbb{A}(n,n)$. The natural isomorphism $\bar{\psi}_\sigma : \otimes_\Gamma^n \bar{\psi}_n \overset{\cong}{\Rightarrow} \bar{\psi}_n$ is given by the composite

$$\begin{tikzcd} \Lambda^n \arrow[r, "\psi^n"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\cong"] & \Gamma^n \arrow[r, "\otimes_\Gamma^n"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\otimes_\Gamma^{\sigma}\ \cong"] & \Gamma \\ \Lambda^n \arrow[r, "\psi^n"'] & \Gamma^n \arrow[ur, "\otimes_\Gamma^n"'] & \end{tikzcd}$$

The composite $\Lambda \to \mathcal{F}^{\mathbb{A}}\Lambda \xrightarrow{\bar{\psi}} \Gamma$ precisely gives ψ. The data of an isomorphism

$$\begin{tikzcd}
\mathcal{F}^{\mathbb{A}}\mathcal{F}^{\mathbb{A}}\Lambda \arrow[r, "\mathcal{F}^{\mathbb{A}}\bar{\psi}"] \arrow[d, "\eta_\Lambda"'] \arrow[dr, phantom, "\overset{\alpha}{\cong}"] & \mathcal{F}^{\mathbb{A}}\Gamma \arrow[d, "\otimes^{\mathbb{A}}_{\Gamma}"] \\
\mathcal{F}^{\mathbb{A}}\Lambda \arrow[r, "\bar{\psi}"'] & \Gamma
\end{tikzcd}$$

corresponds to the data of an isomorphism $\alpha_r^{n_1,\dots,n_r}$ for each $r : * \to \mathbb{A}$ and $n_1, \dots, n_r \to \mathbb{A}$

$$\begin{tikzcd}
\prod\limits_{i=1}^{r} \Lambda^{n_i} \arrow[r, "\prod\limits_{i=1}^{r} \psi^{n_i}"] \arrow[d, "\cong"'] & \prod\limits_{i=1}^{r} \Gamma^{n_i} \arrow[r, "\prod\limits_{i=1}^{r} \otimes^{n_i}_{\Gamma}"] \arrow[d, phantom, "\overset{\cong}{\alpha_r^{n_1,\dots,n_r}}"] & \Gamma^r \arrow[d, "\otimes^r_\Gamma"] \\
\Lambda^n \arrow[rr, "\bar{\psi}_n = \otimes^n_\Gamma \psi^n"'] & & \Gamma
\end{tikzcd}$$

where $n = n_1 + \cdots + n_r$, and an isomorphism $\alpha_\sigma^{\sigma_1,\dots,\sigma_r} : \sigma(\sigma_1, \dots, \sigma_r).\alpha_r^{n_1,\dots,n_r} \xrightarrow{\cong} \alpha_r^{n_1,\dots,n_r}$ for all $\sigma : * \to \mathbb{A}(r,r)$ and $\sigma_i : * \to \mathbb{A}(n_i, n_i), i = 1, \dots, r$. Let $\alpha_r^{n_1,\dots,n_r}$ be defined by the isomorphism $\otimes^r_\Gamma \circ \prod\limits_{i=1}^{r} \otimes^{n_i}_\Gamma \cong \otimes^n_\Gamma$. The existence of the required isomorphism $\alpha_\sigma^{\sigma_1,\dots,\sigma_r}$ follows from the commutativity of the following diagram

$$\begin{tikzcd}
 & & & \prod\limits_{i=1}^{r} \Lambda^{n_i} \arrow[rr, "\prod\limits_{i=1}^{r} \psi^{n_i}"] & & \prod\limits_{i=1}^{r} \Gamma^{n_i} \arrow[rr, "\prod\limits_{i=1}^{r} \otimes^{n_i}_\Gamma"] & & \Gamma^r \arrow[d, "\otimes^r_\Gamma"] \\
\prod\limits_{i=1}^{r} \Lambda^{n_i} \arrow[rr, "\prod\limits_{i=1}^{r} \psi^{n_i}"] \arrow[urrr, "(\sigma;\sigma_1,\dots,\sigma_r)"] \arrow[d, "\cong"'] & & \prod\limits_{i=1}^{r} \Gamma^{n_i} \arrow[rr, "\prod\limits_{i=1}^{r} \otimes^{n_i}_\Gamma"] \arrow[urrr, "(\sigma;\sigma_1,\dots,\sigma_r)"] \arrow[urr, phantom, "\overset{\sigma.\prod\limits_{I=1}^{r} \otimes^{\sigma_i}_\Gamma}{\Longrightarrow}"] & & \Gamma^r \arrow[urrr, "\sigma"] \arrow[d, "\otimes^r_\Gamma"] \arrow[urr, phantom, "\Downarrow \otimes^\sigma_\Gamma"] & \Lambda^n \arrow[rr, "\bar{\psi}_n = \otimes^n_\Gamma \psi^n"] & & \Gamma. \\
\Lambda^n \arrow[rrrr, "\bar{\psi}_n = \otimes^n_\Gamma \psi^n"'] \arrow[urrrrr, dotted, "\sigma(\sigma_1,\dots,\sigma_r)"] & & & & \Gamma \arrow[urrr, "id"'] \arrow[ur, phantom, "\overset{\otimes^{\sigma(\sigma_1,\dots,\sigma_r)}_\Gamma}{\Longrightarrow}"] & & &
\end{tikzcd}$$

This shows that $\bar{\psi} : \mathcal{F}^{\mathbb{A}}\Lambda \to \Gamma$ defines an $\mathbb{A}$-unbiased monoidal morphism extending ψ. Conversely, let $\phi : \mathcal{F}^{\mathbb{A}}\Lambda \to \Gamma$ be an $\mathbb{A}$-unbiased monoidal morphism. Let $n : * \to \mathbb{A}$ and let ϕ_n be defined by the composite

$$\phi_n : \Lambda^n \xrightarrow{\iota_n} \int_{\mathbb{A}} *^{\mathbb{A}} \times \Lambda^{\bullet} \xrightarrow{\phi} \Gamma.$$

We show that ϕ is determined by ϕ_1. Let $n : * \to \mathbb{A}$. Since ϕ is $\mathbb{A}$-unbiased monoidal, we have an isomorphism

$$\begin{tikzcd}
\left(\int_{\mathbb{A}} *^{\mathbb{A}} \times \Lambda^{\bullet}\right)^r \arrow[r] \arrow[d, "\phi^r"'] \arrow[dr, phantom, "\cong"] & \int_{\mathbb{A}} *^{\mathbb{A}} \times \Lambda^{\bullet} \arrow[d, "\phi"] \\
\Gamma^r \arrow[r, "\otimes^r_\Gamma"'] & \Gamma,
\end{tikzcd}$$

so that in particular, we have isomorphisms

$$\begin{tikzcd}
\Lambda^r \arrow[rr, "\iota_{(1,\ldots,1)}"] \arrow[dr, "{\iota_1}^r"] \arrow[ddr, "{\phi_1}^r"'] \arrow[drr, phantom, "\cong"] & & \int_{(n_1,\ldots,n_r):*\mathbb{A}^r} * \times \prod_{I=1}^{r} \Lambda^{n_i} \arrow[d] \arrow[dl, "\cong"] \\
 & \left(\int_{\mathbb{A}} *^{\mathbb{A}} \times \Lambda^{\bullet}\right)^r \arrow[r] \arrow[d, "\phi^r"'] \arrow[ur, "\cong"] \arrow[dr, phantom, "\cong"] & \int_{n:*\mathbb{A}} * \times \Lambda^{n_1+\ldots+n_r} \arrow[d, "\int_{n:*\mathbb{A}} \phi_{n_1+\ldots+n_r}"] \\
 & \Gamma^r \arrow[r, "\otimes_{\Gamma}^{r}"'] & \Gamma
\end{tikzcd}$$

The composite $\left(\int_{n:*\to\mathbb{A}} \phi_{n_1+\cdots+n_r}\right) \mu\iota_{(1,\ldots,1)}$ precisely gives ϕ_r, so that $\phi_r \cong {\phi_1}^r$. $\square$

Remark 1.2.10. For $\mathbb{A} = \mathbb{N}$, we obtain that the free monoidal 2-category generated by a 2-category Λ satisfies

$$\mathcal{F}^{\mathbb{A}}\Lambda \cong \coprod_{n\in\mathbb{N}} \Lambda^n.$$

In particular, the 2-category $\mathbb{N}$ is freely generated by the terminal 2-category $*$.

Remark 1.2.11. For $\mathbb{A} = \mathfrak{S}$, we first construct a 2-category Λ^n/Σ_n for each $n : * \to \mathfrak{S}$ as the 2-category whose objects are the objects of Λ^n, and whose category of morphisms $\Lambda^n/\Sigma_n(\mathcal{C}, \mathcal{D})$ for objects $\mathcal{C}$ and $\mathcal{D}$ is defined as follows. The data of an object of $\Lambda^n/\Sigma_n(\mathcal{C}, \mathcal{D})$ is equivalent to the data of an object of $\coprod_{\sigma:*\to\Sigma_n} \Lambda^n(\sigma_*\mathcal{C}, \mathcal{D})$. Let $F : * \to \Lambda^n(\sigma_{F*}\mathcal{C}, \mathcal{D})$ and $G : * \to \Lambda^n(\sigma_{G*}\mathcal{C}, \mathcal{D})$. We let the set of morphisms from F to G be defined as

$$\Lambda^n/\Sigma_n(\mathcal{C}, \mathcal{D})(F, G) = \Lambda^n(\sigma_F.\mathcal{C}, \mathcal{D})(F, {\sigma_F}_*^{-1}\sigma_{G*}G),$$

which can be graphically described as the following diagram in Λ^n

$$\begin{tikzcd}
\sigma_F.\mathcal{C} \arrow[rr, "F"] \arrow[d, "{\sigma_F}_*^{-1}"'] & & \mathcal{D} \\
\mathcal{C} \arrow[r, "\sigma_{G*}"'] & \sigma_G.\mathcal{C}. \arrow[ur, "G"'] &
\end{tikzcd}$$

In particular, if for $i = 1, \ldots, n$, we have a morphism $f_i : \mathcal{C}_i \to \mathcal{D}_i$ in Λ, then for all $\sigma \in \Sigma_n$, both $f = (f_1, \ldots, f_n) : (\mathcal{C}_1, \ldots, \mathcal{C}_n) \to (\mathcal{D}_1, \ldots, \mathcal{D}_n)$ and $\sigma.f = (f_{\sigma(1)}, \ldots, f_{\sigma(n)}) : (\mathcal{C}_{\sigma(1)}, \ldots, \mathcal{C}_{\sigma(n)}) \to (\mathcal{D}_1, \ldots, \mathcal{D}_n)$ are objects of $\Lambda^n/\Sigma_n\left((\mathcal{C}_1, \ldots, \mathcal{C}_n), (\mathcal{D}_1, \ldots, \mathcal{D}_n)\right)$. Moreover, the identity of $\sigma.F$ in Λ^n provides an isomorphism between $\sigma.f$ and f in Λ^n/Σ_n, whose inverse is given by the identity of f in Λ^n. We give to $\Lambda^n/\Sigma_n(\mathcal{C}, \mathcal{D})$ the structure of a category as follows. Let $F : * \to \Lambda^n(\sigma_{F*}\mathcal{C}, \mathcal{D})$, $G : * \to \Lambda^n(\sigma_{G*}\mathcal{C}, \mathcal{D})$ and $H : * \to \Lambda^n(\sigma_{H*}\mathcal{C}, \mathcal{D})$. We define a composition morphism by the composite

$$\begin{aligned}
&\Lambda^n(\sigma_F.\mathcal{C}, \mathcal{D})(F, {\sigma_F}_*^{-1}\sigma_{G*}G) \times \Lambda^n(\sigma_G.\mathcal{C}, \mathcal{D})(G, {\sigma_G}_*^{-1}\sigma_{H*}H) \\
&\quad \to \quad \Lambda^n(\sigma_F.\mathcal{C}, \mathcal{D})(F, {\sigma_F}_*^{-1}\sigma_{G*}G) \times \Lambda^n(\sigma_F.\mathcal{C}, \mathcal{D})({\sigma_F}_*^{-1}\sigma_{G*}G, {\sigma_F}_*^{-1}\sigma_{H*}H) \\
&\quad \to \quad \Lambda^n(\sigma_F.\mathcal{C}, \mathcal{D})(F, {\sigma_F}_*^{-1}\sigma_{H*}H),
\end{aligned}$$

where the last arrow is given by the composition of morphisms in the category $\Lambda^n(\sigma_F.\mathcal{C},\mathcal{D})$. The composition in $\Lambda^n/\Sigma_n(\mathcal{C},\mathcal{D})$ therefore corresponds to the following composition scheme in Λ^n

$$\begin{tikzcd}
\sigma_F.\mathcal{C} \arrow[d, "{\sigma_F}_*^{-1}"'] \arrow[rrd, "F"] & & \\
\mathcal{C} \arrow[r, "\sigma_{G*}"] \arrow[d, "\sigma_{H*}"'] & \sigma_G.\mathcal{C} \arrow[r, "G"] \arrow[u, phantom, "\Leftarrow"] \arrow[d, phantom, "\Rightarrow"] & \mathcal{D}. \\
\sigma_H.\mathcal{C} \arrow[rru, "H"'] & &
\end{tikzcd}$$

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be objects of Λ^n/Σ_n. We define the composition functor

$$\Lambda^n/\Sigma_n(\mathcal{C},\mathcal{D}) \times \Lambda^n/\Sigma_n(\mathcal{D},\mathcal{E}) \to \Lambda^n/\Sigma_n(\mathcal{C},\mathcal{E})$$

in the components $\sigma, \tau \in \Sigma_n$ by the composite

$$\Lambda^n(\sigma_*\mathcal{C},\mathcal{D})\times\Lambda^n(\tau_*\mathcal{D},\mathcal{E}) \to \Lambda^n(\tau_*\sigma_*C,\tau_*\mathcal{D})\times\Lambda^n(\tau_*\mathcal{D},\mathcal{E}) \to \Lambda^n(\tau_*\sigma_*\mathcal{C},\mathcal{E}) \hookrightarrow \Lambda^n/\Sigma_n(\mathcal{C},\mathcal{E}).$$

We obtain the following description of underlying 2-category of the free symmetric monoidal 2-category generated by Λ:

$$\mathcal{F}^{\mathbb{A}}\Lambda \cong \coprod_{n\in\mathbb{N}} \Lambda^n/\Sigma_n.$$

EXAMPLE 1.2.12. Let $\mathfrak{S}^{(2)}$ be the free monoidal 2-category generated by the terminal 2-category $*$, so that $\mathfrak{S}^{(2)} \cong \coprod\limits_{n\in\mathbb{N}} *^n/\Sigma_n$. Hence, the data of an object of $\mathfrak{S}^{(2)}$ is equivalent to the data of a natural numbers. For each natural number n, the category of morphisms $\mathfrak{S}^{(2)}_n := \mathfrak{S}^{(2)}(n,n)$ can be described as follows. The data of an object of $\mathfrak{S}^{(2)}_n$ is equivalent to the data of a permutation $\sigma \in \Sigma_n$, corresponding to the unique object of the category $*^n(\sigma.*^n,*^n)$. Let $\sigma,\tau \in \Sigma_n$. The set of morphisms $\mathfrak{S}^{(2)}_n(\sigma,\tau)$ has precisely one element, which is given by $*^n(\sigma.*^n,*^n)(*^n,\sigma^{-1}\tau.*^n) \cong *$.

REMARK 1.2.13. If we use the 2-category $\mathfrak{S}^{(2)}$ instead of $\mathfrak{S}$ to model symmetric monoidal 2-categories, then an $\mathfrak{S}^{(2)}$-unbiased monoidal structure on a 2-category Λ consists in the data of

- for each $n \in \mathbb{N}$, a functor

  $$\otimes^n_\Lambda : \Lambda^n \to \Lambda,$$

  which satisfies unit and associativity conditions,
- for each $\sigma : * \to \Sigma_n$, a natural transformation $\otimes^\sigma_\Lambda$ such that

  $$\begin{tikzcd}
  \Lambda^n \arrow[rr, "\sigma_*"] \arrow[dr, "\otimes^n_\Lambda"'] & \arrow[d, phantom, "\overset{\otimes^\sigma_\Lambda}{\underset{\cong}{\Rightarrow}}"] & \Lambda^n \arrow[dl, "\otimes^n_\Lambda"] \\
  & \Lambda, &
  \end{tikzcd}$$

  with natural isomorphisms $\otimes^\sigma_\Lambda \otimes^\tau_\Lambda \cong \otimes^{\sigma\tau}_\Lambda$ for each $\sigma,\tau : * \to \sigma_n$,

– for each $\sigma, \tau : * \to \Sigma_n$, an invertible modification $\otimes_\Lambda^{\tau,\sigma}$ such that

$$
\begin{tikzcd}
\Lambda^n \arrow[rr, bend left=40, "\tau_*"] \arrow[rr, "\sigma_*"'] \arrow[rr, phantom, bend left=20, "\cong"] \arrow[dr, "\otimes^n_\Lambda"'] & \arrow[d, phantom, "\overset{\otimes^\sigma_\Lambda}{\Rightarrow}\,\cong"] & \Lambda^n \arrow[dl, "\otimes^n_\Lambda"] & \arrow[r, "\otimes^{\sigma,\tau}_\Lambda", "\cong"'] & {} & \Lambda^n \arrow[rr, "\tau_*"] \arrow[dr, "\otimes^n_\Lambda"'] & \arrow[d, phantom, "\overset{\otimes^\tau_\Lambda}{\Rightarrow}\,\cong"] & \Lambda^n \arrow[dl, "\otimes^n_\Lambda"] \\
& \Lambda & & & & & \Lambda, &
\end{tikzcd}
$$

and which is compatible with composition.

In fact, the compatibility relations impose the modification $\otimes_\Lambda^{\sigma,\tau}$ to be given by the following composite

$$
\begin{tikzcd}
\Lambda^n \arrow[rr, bend left=40, "\tau_*"] \arrow[rr, "\sigma_*"'] \arrow[rr, phantom, bend left=20, "\cong"] \arrow[dr, "\otimes^n_\Lambda"'] & \arrow[d, phantom, "\overset{\otimes^\sigma_\Lambda}{\Rightarrow}"] & \Lambda^n \arrow[dl, "\otimes^n_\Lambda"] \\
& \Lambda &
\end{tikzcd}
\cong
\begin{tikzcd}
\Lambda^n \arrow[rr, bend left=40, "id"] \arrow[rr, phantom, bend left=20, "\cong"] \arrow[r, "\sigma_*"] \arrow[ddrr, "\otimes^n_\Lambda"'] & \Lambda^n \arrow[r, "\sigma_*^{-1}"] \arrow[d, phantom, "\overset{\otimes^{\sigma\,-1}_\Lambda \otimes^\sigma_\Lambda}{\Rightarrow}"] & \Lambda^n \arrow[r, "\tau_*"] \arrow[dd, "\otimes^n_\Lambda"] \arrow[dr, phantom, "\overset{\otimes^\tau_\Lambda}{\Rightarrow}"] & \Lambda^n \arrow[ddl, "\otimes^n_\Lambda"] \\
& {} & & \\
& & \Lambda, &
\end{tikzcd}
\cong
\begin{tikzcd}
\Lambda^n \arrow[rr, "\tau_*"] \arrow[dr, "\otimes^n_\Lambda"'] & \arrow[d, phantom, "\overset{\otimes^\tau_\Lambda}{\Rightarrow}\,\cong"] & \Lambda^n \arrow[dl, "\otimes^n_\Lambda"] \\
& \Lambda &
\end{tikzcd}
$$

so that a $\mathfrak{S}$-unbiased monoidal 2-category structure is equivalent to an $\mathfrak{S}^{(2)}$-unbiased monoidal 2-category structure on Λ.

Remark 1.2.14. The adjunction of Proposition 1.2.9 induces a monad $\mathcal{U}^{\mathbb{A}}\mathcal{F}^{\mathbb{A}}$ which we also write $\mathcal{F}^{\mathbb{A}}$ for simplicity

$$\mathcal{F}^{\mathbb{A}} : \mathrm{Mon}^{\mathbb{A}}\mathrm{Cat}_2 \to \mathrm{Mon}^{\mathbb{A}}\mathrm{Cat}_2.$$

A 2-category is an $\mathbb{A}$-unbiased monoidal 2-category if and only if it is an algebra over $\mathcal{F}^{\mathbb{A}}$. Indeed, the structural morphism given by the $\mathcal{F}^{\mathbb{A}}$-algebra structure on a 2-category Λ provides a morphism

$$\otimes^{\mathbb{A}}_\Lambda : \mathcal{F}^{\mathbb{A}}\Lambda \to \Lambda,$$

which is $\mathbb{A}$-unbiased monoidal. In particular, we obtain a 2-functor $\otimes^n_\Lambda : \Lambda^n \to \Lambda$ for each $n \in \mathbb{N}$, which is invariant under the action of the symmetric group when $\mathbb{A} = \mathfrak{S}$.

Definition 1.2.15. Let $n : * \to \mathbb{A}$ and consider the n-diagonal 2-functor $\Delta_n : \Lambda \to \Lambda^n$. Let $\sigma : * \to \mathbb{A}(n,n)$, and write $\Delta_\sigma : \sigma_*\Delta_n \overset{\cong}{\Rightarrow} \Delta_n$ for the natural isomorphism induced by σ. The collection $\{\Delta_n, \Delta_\sigma\}_{n,\sigma}$ induce a morphism into the end

$$\Delta : \Lambda \to \int^{\mathbb{A}} *^{\mathbb{A}} \times \Lambda^{\bullet}.$$

The composite

$$\otimes^{\mathbb{A}}_\Lambda \Delta_n : \Lambda \xrightarrow{\Delta} \int^{\mathbb{A}} *^{\mathbb{A}} \times \Lambda^{\bullet} \xrightarrow{\pi_n} \Lambda^n \xrightarrow{\iota_n} \int_{\mathbb{A}} *^{\mathbb{A}} \times \Lambda^{\bullet} \xrightarrow{\otimes^{\mathbb{A}}_\Lambda} \Lambda$$

induces a 2-functor $\otimes^{\mathbb{A}}_\Lambda \Delta : \mathbb{A}^{op} \to [\Lambda, \Lambda]$.

DEFINITION 1.2.16. Let Λ be an $\mathbb{A}$-unbiased monoidal 2-category. We define the exponentiation as the 2-functor

$$-^{\bullet} : \Lambda \to [\mathbb{A}^{op}, \Lambda]$$

that corresponds to the 2-functor $\otimes_{\Lambda}^{\mathbb{A}}\Delta$ under the equivalence $[\mathbb{A}^{op}, [\Lambda, \Lambda]] \cong [\Lambda, [\mathbb{A}^{op}, \Lambda]]$. Let $\mathcal{C}$ be an object of Λ. The object $\mathcal{C}^{\bullet} : \mathbb{A}^{op} \to \Lambda$ is defined on $n : * \to \mathbb{A}^{op}$ by $\mathcal{C}^{\otimes_{\Lambda}^{n}}$, and when $\mathbb{A}$ is the symmetric groupoid $\mathfrak{S}$, the right action of Σ_n on $\mathcal{C}^{\otimes_{\Lambda}^{n}}$ is yielded by the permutation of the coordinates on $\Delta_n\mathcal{C} = (\mathcal{C}, \dots, \mathcal{C}) : * \to \Lambda^n$.

PROPOSITION 1.2.17. *Suppose that Λ is an $\mathbb{A}$-unbiased monoidal 2-category which is bicomplete, and recall that the 2-category $[\mathbb{A}^{op}, \Lambda]$ inherits an $\mathbb{A}$-unbiased monoidal structure from the $\mathbb{A}$-unbiased monoidal structures of $\mathbb{A}^{op}$ and Λ.*

DEFINITION 1.2.18. Let Λ and Γ be $\mathbb{A}$-unbiased monoidal 2-categories such that Γ is tensored over Λ. We define the composition product as the composite

$$\circ : [\mathbb{A}, \Lambda] \times \Gamma \xrightarrow{[\mathbb{A},\Lambda]\times -^{\bullet}} [\mathbb{A}, \Lambda] \times [\mathbb{A}^{op}, \Gamma] \xrightarrow{\dagger} \Gamma,$$

so that for $\mathcal{P} : \mathbb{A} \to \Lambda$ and $\mathcal{C} : * \to \Gamma$, we have $\mathcal{P} \circ \mathcal{C} = \mathcal{P} \dagger \mathcal{C}^{\bullet} \cong \int_{\mathbb{A}} \mathcal{P}(n) \cdot \mathcal{C}^{\otimes_{\Gamma}^{n}}$. If Γ is tensored and enriched over Λ, then we also have an equivalence

$$\Gamma\left(\mathcal{P} \circ \mathcal{C}, \mathcal{D}\right) \cong [\mathbb{A}, \Lambda]\left(\mathcal{P}, \Gamma_{\Lambda}^{\mathbb{A}}(\mathcal{C}^{\bullet}, \mathcal{C})\right).$$

1.3. Sequences of categories and categorical operads. We can now give the definition of operads in categories, relying on the constructions of the previous subsection.

Sequences of categories. For simplicity, we write $\mathbb{N}$ for the discrete 2-category $I^2\mathbb{N}$ whose objects are given by the natural numbers. The monoid structure of the set of natural numbers extends to a monoidal 2-category structure on $\mathbb{N}$ seen as a 2-category.

DEFINITION 1.3.1. We define the 2-category $\mathrm{CAT}^{\mathbb{N}}$ of categorical sequences as the 2-category of 2-functors $\mathrm{CAT}_2(\mathbb{N}, \mathrm{CAT})$. Explicitly, the 2-category $\mathrm{CAT}^{\mathbb{N}}$ is such that

- An object $\mathcal{C}$ of $\mathrm{CAT}^{\mathbb{N}}$ consists in the data of a category $\mathcal{C}(n)$ for each $n \in \mathbb{N}$,
- If $\mathcal{C}$ and $\mathcal{D}$ are objects of $\mathrm{CAT}^{\mathbb{N}}$, then the category of morphisms $\mathrm{CAT}^{\mathbb{N}}(\mathcal{C}, \mathcal{D})$ is defined as follows:
    - The data of an object $F : \mathcal{C} \to \mathcal{D}$ consists in the data of a functor $F(n) : \mathcal{C}(n) \to \mathcal{D}(n)$ for each $n \in \mathbb{N}$,
    - If $F, G : \mathcal{C} \to \mathcal{D}$, then a morphism η from F to G is given by a natural transformation $\eta(n) : F(n) \to G(n)$ for each $n \in \mathbb{N}$.

The 2-category $\mathrm{CAT}^{\mathbb{N}}$ is cartesian closed. Indeed, the cartesian product is given by the pointwise cartesian product. In the same way, if $\mathcal{C}$ and $\mathcal{D}$ are sequences of categories, we define the sequence of categories $[\mathcal{C}, \mathcal{D}]^{\mathbb{N}}$ by the category of functors $[\mathcal{C}, \mathcal{D}]_{\mathbb{N}}(n) = [\mathcal{C}(n), \mathcal{D}(n)]$ for $n \in \mathbb{N}$. We obtain a 2-functor

$$[-,-]^{\mathbb{N}} : \mathrm{CAT}^{\mathbb{N}\ op} \times \mathrm{CAT}^{\mathbb{N}} \to \mathrm{CAT}^{\mathbb{N}}$$

such that for each sequence of categories $\mathcal{C}$, the induced 2-functor

$$[\mathcal{C}, -]^{\mathbb{N}} : \mathrm{CAT}^{\mathbb{N}} \to \mathrm{CAT}^{\mathbb{N}}$$

is right adjoint to the 2-functor

$$\mathcal{C} \times - : \textsc{Cat}^{\mathbb{N}} \to \textsc{Cat}^{\mathbb{N}}.$$

Observe that a morphism $F : [\mathcal{C}_1, \mathcal{D}_1]^{\mathbb{N}} \to [\mathcal{C}_2, \mathcal{D}_2]^{\mathbb{N}}$ in $\textsc{Cat}^{\mathbb{N}}$ is a sequence of functors $F(n)$ between the categories $[\mathcal{C}_1(n), \mathcal{D}_1(n)]$ and $[\mathcal{C}_2(n), \mathcal{D}_2(n)]$, whereas a functor $\textsc{Cat}^{\mathbb{N}}(\mathcal{C}_1, \mathcal{D}_1) \to \textsc{Cat}^{\mathbb{N}}(\mathcal{C}_2, \mathcal{D}_2)$ takes a sequence of functors to another sequence of functors, but is not necessarily itself a sequence of functors. Therefore, the category of functors from $\textsc{Cat}^{\mathbb{N}}(\mathcal{C}_1, \mathcal{D}_1)$ to $\textsc{Cat}^{\mathbb{N}}(\mathcal{C}_2, \mathcal{D}_2)$ carries less structure than the category of morphisms from $[\mathcal{C}_1, \mathcal{D}_1]_{\mathbb{N}}$ to $[\mathcal{C}_2, \mathcal{D}_2]_{\mathbb{N}}$ in $\textsc{Cat}^{\mathbb{N}}$. However, the following proposition allows us to identify the functors that can we written as sequences of functors with morphisms of $\textsc{Cat}^{\mathbb{N}}$.

Proposition 1.3.2. *The 2-functor of objects of sequences of categories*

$$\textsc{Cat}^{\mathbb{N}}(*^{\mathbb{N}}, -) = \textsc{Cat}^{\mathbb{N}} \to \textsc{Cat}$$

is injective, locally injective, and locally fully faithful. In particular, for each sequences of categories $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_1, \mathcal{D}_2$, the functor

$$\textsc{Cat}^{\mathbb{N}}\left([\mathcal{C}_1, \mathcal{D}_1]^{\mathbb{N}}, [\mathcal{C}_1, \mathcal{D}_1]^{\mathbb{N}}\right) \to \left[\textsc{Cat}^{\mathbb{N}}(\mathcal{C}_1, \mathcal{D}_1), \textsc{Cat}^{\mathbb{N}}(\mathcal{C}_2, \mathcal{D}_2)\right]$$

is fully faithful.

Proof. Let $\mathcal{C}$ be a sequence of categories. Since $\textsc{Cat}^{\mathbb{N}}$ is a 2-category of 2-functors, we have

$$\mathcal{C}^* := \textsc{Cat}^{\mathbb{N}}(*^{\mathbb{N}}, \mathcal{C}) \cong \int^{n \in \mathbb{N}} \textsc{Cat}(*, \mathcal{C}(n)) = \prod_{n \in \mathbb{N}} \mathcal{C}(n),$$

so that the 2-functor $\textsc{Cat}^{\mathbb{N}} \to \textsc{Cat}$ is injective on the objects. Moreover, if $\mathcal{D}$ is another sequence of categories, we have

$$\textsc{Cat}^{\mathbb{N}}(\mathcal{C}, \mathcal{D}) \cong \prod_{n \in \mathbb{N}} [\mathcal{C}(n), \mathcal{D}(n)]$$

and

$$[\mathcal{C}^*, \mathcal{D}^*] = \left[\prod_{n \in \mathbb{N}} \mathcal{C}(n), \prod_{n \in \mathbb{N}} \mathcal{D}(n)\right],$$

so that the functor $\textsc{Cat}^{\mathbb{N}}(*^{\mathbb{N}}, -)(\mathcal{C}, \mathcal{D}) : \textsc{Cat}^{\mathbb{N}}(\mathcal{C}, \mathcal{D}) \to [\mathcal{C}^*, \mathcal{D}^*]$ induced by the 2-functor of objects corresponds to the natural morphism

$$\prod_{n \in \mathbb{N}} [\mathcal{C}(n), \mathcal{D}(n)] \to \left[\prod_{n \in \mathbb{N}} \mathcal{C}(n), \prod_{n \in \mathbb{N}} \mathcal{D}(n)\right],$$

which is injective on the objects. Let $F, G : * \to \textsc{Cat}^{\mathbb{N}}(\mathcal{C}, \mathcal{D})$. The morphism $\textsc{Cat}^{\mathbb{N}}(*^{\mathbb{N}}, -)(\mathcal{C}, \mathcal{D})(F, G) : \textsc{Cat}^{\mathbb{N}}(\mathcal{C}, \mathcal{D})(F, G) \to [\mathcal{C}^*, \mathcal{D}^*]\,(F^*, G^*)$ factors through the following isomorphisms of sets

$$
\begin{aligned}
\mathrm{C{\scriptstyle AT}}^{\mathbb{N}}(\mathcal{C},\mathcal{D})(F,G) &\cong \prod_{n\in\mathbb{N}} \left[\mathcal{C}(n),\mathcal{D}(n)\right](F(n),G(n)) \\
&\cong \prod_{n\in\mathbb{N}} \int^{X_n:*\to\mathcal{C}(n)} \mathcal{D}(n)(F(n)X_n, G(n)X_n) \\
&\cong \int^{\prod\limits_{n\in\mathbb{N}} X_n:*\to \prod\limits_{n\in\mathbb{N}}\mathcal{C}(n)} \prod_{n\in\mathbb{N}} \mathcal{D}(n)\left(\left(\prod_{n\in\mathbb{N}} F(n)\right)X, \left(\prod_{n\in\mathbb{N}} G(n)\right)X\right) \\
&\cong \int^{X:*\to\mathcal{C}^*} \mathcal{D}^*\left(F^*X, G^*X\right) \cong \left[\mathcal{C}^*,\mathcal{D}^*\right](F^*,G^*)
\end{aligned}
$$

which shows that $\mathrm{C{\scriptstyle AT}}^{\mathbb{N}}(*^{\mathbb{N}},-)$ is locally fully faithful.

□

Definition 1.3.3. We define the oppositization 2-functor

$$-^{op} : \left(\mathrm{C{\scriptstyle AT}}^{\mathbb{N}}\right)^{co} \to \mathrm{C{\scriptstyle AT}}^{\mathbb{N}}$$

as follows. Let $\mathcal{C}$ be a sequence of categories and define its opposite sequence $\mathcal{C}^{op}$ in $n \in \mathbb{N}$ by $\mathcal{C}^{op}(n) = \mathcal{C}(n)^{op}$. Let $\mathcal{D}$ be a sequence of categories. Observe that any sequence of functors from $\mathcal{C}$ to $\mathcal{D}$ naturally induces a functor from $\mathcal{C}^{op}$ to $\mathcal{D}^{op}$, whereas any $\eta : * \to \mathrm{C{\scriptstyle AT}}^{\mathbb{N}}(\mathcal{C},\mathcal{D})(F,G)$ induces a 2-morphism $\eta^{op} : * \to \mathrm{C{\scriptstyle AT}}^{\mathbb{N}}(\mathcal{C}^{op},\mathcal{D}^{op})(G^{op},F^{op})$. We obtain a functor

$$\mathrm{C{\scriptstyle AT}}^{\mathbb{N}}(\mathcal{C},\mathcal{D})^{op} \to \mathrm{C{\scriptstyle AT}}^{\mathbb{N}}(\mathcal{C}^{op},\mathcal{D}^{op}).$$

Definition 1.3.4. Let $\mathrm{S{\scriptstyle ET}}^{\mathbb{N}} : * \to \mathrm{C{\scriptstyle AT}}^{\mathbb{N}}$ be the constant sequence of categories defined on $n \in \mathbb{N}$ by $\mathrm{S{\scriptstyle ET}}^{\mathbb{N}}(n) = \mathrm{S{\scriptstyle ET}}$.

The object $\mathrm{S{\scriptstyle ET}}^{\mathbb{N}}$ of $\mathrm{C{\scriptstyle AT}}^{\mathbb{N}}$ is bicomplete cartesian closed, and as a consequence, is suitable for collecting internal morphisms in $\mathrm{C{\scriptstyle AT}}^{\mathbb{N}}$.

We define internal morphisms in $\mathrm{C{\scriptstyle AT}}^{\mathbb{N}}$ with values in $\mathrm{S{\scriptstyle ET}}^{\mathbb{N}}$. Let $\mathcal{C} : * \to \mathrm{C{\scriptstyle AT}}^{\mathbb{N}}$ be a sequence of categories. We define a morphism in $\mathrm{C{\scriptstyle AT}}^{\mathbb{N}}$

$$\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{S{\scriptstyle ET}}^{\mathbb{N}}.$$

We let $\mathcal{C}(-,-)$ be the sequence of functors defined on $n \in \mathbb{N}$ by the functor

$$\mathcal{C}(n)(-,-) : \mathcal{C}^{op}(n) \times \mathcal{C}(n) \to \mathrm{S{\scriptstyle ET}},$$

which, to objects X, Y of $\mathcal{C}(n)$, associates the set $\mathcal{C}(n)(X,Y)$ of morphisms from X to Y in the category $\mathcal{C}(n)$.

Let $X : *^{\mathbb{N}} \to \mathcal{C}$ and let the internal unit $1_X \in \mathcal{C}(X,X)$ of X be defined by the sequence of functions given in arity $n \in \mathbb{N}$ by

$$1_X(n) = 1_{X(n)} : * \to \mathcal{C}(n)(X(n),X(n)).$$

Let $X, Y, Z : * \to \mathcal{C}$. We define a composition morphism in $\mathrm{S{\scriptstyle ET}}^{\mathbb{N}}$

$$\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$$

given in arity $n \in \mathbb{N}$ by the composition of morphisms in $\mathcal{C}(n)$. The verification of the unit and associativity axioms is immediate from the associativity and unit axioms of a category.

LEMMA 1.3.5. *For each pair of objects $\mathcal{C}, \mathcal{D}$ and morphisms $F, G : \mathcal{C} \to \mathcal{D}$ of* $\mathrm{CAT}^{\mathbb{N}}$*, the canonical morphism*

$$[\mathcal{C}, \mathcal{D}]^{\mathbb{N}}(F, G) \to \int^{X:*^{\mathbb{N}} \to \mathcal{C}} \mathcal{D}(FX, GX)$$

is an isomorphism in $\mathrm{SET}^{\mathbb{N}}$*, so that the internal* $\mathrm{SET}^{\mathbb{N}}$*-category structure of objects of* $\mathrm{CAT}^{\mathbb{N}}$ *is regular.*

PROOF. The sequence of sets $[\mathcal{C}, \mathcal{D}]^{\mathbb{N}}(F, G)$ is defined on $n \in \mathbb{N}$ by

$$\begin{aligned} [\mathcal{C}, \mathcal{D}]^{\mathbb{N}}(F, G)(n) &= [\mathcal{C}, \mathcal{D}](n)(F(n), G(n)) \\ &= [\mathcal{C}(n), \mathcal{D}(n)]\,(F(n), G(n)) \\ &\cong \int^{X_n:*\to\mathcal{C}(n)} \mathcal{D}(n)(F(n)X, G(n)X) \\ &\cong \left(\int^{X:*^{\mathbb{N}} \to \mathcal{C}} \mathcal{D}(FX, GX)\right)(n). \end{aligned}$$

□

COROLLARY 1.3.6. *Let $\mathcal{C} : *^{\mathbb{N}} \to \mathrm{CAT}^{\mathbb{N}}$, $X \in \mathcal{C}$ and $F : \mathcal{C}^{op} \to \mathrm{SET}^{\mathbb{N}}$. We have an isomorphism in $\mathcal{S}$*

$$[\mathcal{C}^{op}, \mathrm{SET}^{\mathbb{N}}]^{\mathbb{N}}(\mathcal{C}(-, X), F) \cong FX.$$

As a consequence, the objects defined by universal property internally in $\mathrm{SET}^{\mathbb{N}}$ *are unique up to a canonical isomorphism.*

LEMMA 1.3.7. *For each $\mathcal{C} : * \to \mathrm{CAT}^{\mathbb{N}}$ and each pair of objects $X, Y : * \to \mathcal{C}$, we have an isomorphism*

$$\mathrm{CAT}^{\mathbb{N}}(*, \mathcal{C})(X, Y) \cong \mathrm{CAT}^{\mathbb{N}}(*, \mathrm{SET}^{\mathbb{N}})(*, \mathcal{C}(X, Y)),$$

so that the internal $\mathrm{SET}^{\mathbb{N}}$*-category structure of objects of* $\mathrm{CAT}^{\mathbb{N}}$ *is compatible with their external category structure.*

PROOF. On the one hand, we have

$$\begin{aligned} \mathrm{CAT}^{\mathbb{N}}(*, \mathcal{C})(X, Y) &\cong \int^{n \in \mathbb{N}} \mathrm{CAT}(*, \mathcal{C}(n))(X(n), Y(n)) \\ &\cong \prod_{n \in \mathbb{N}} \mathcal{C}(n)(X(n), Y(n)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \mathrm{CAT}^{\mathbb{N}}(*, \mathrm{SET}^{\mathbb{N}})(*, \mathcal{C}(X, Y)) &\cong \int^{n \in \mathbb{N}} \mathrm{CAT}(*, \mathrm{SET})(*, \mathcal{C}(X, Y)(n)) \\ &\cong \prod_{n \in \mathbb{N}} \mathrm{SET}(*, \mathcal{C}(n)(X(n), Y(n))) \\ &\cong \prod_{n \in \mathbb{N}} \mathcal{C}(n)(X(n), Y(n)) \end{aligned}$$

□

Categorical operads. Recall that the 2-category $[\mathbb{N}, \textsc{Cat}]$ inherits a monoidal structure $([\mathbb{N}, \textsc{Cat}], +_{Day})$ from the monoidal 2-category structure of $(\mathbb{N}, +)$ by Day's convolution product for 2-categories. Let $\mathcal{P} : \mathbb{N} \to \textsc{Cat}$. The cartesian closed structure of $\textsc{Cat}_2$ allows us to see $\mathcal{P}$ as a 2-functor $\mathcal{P} : * \to [\mathbb{N}, \textsc{Cat}]$.

Remark 1.3.8. The addition of natural numbers gives to $(\mathbb{N}, +, 0)$ the structure of a monoid in the monoidal category $(\textsc{Set}, \times, *)$. Moreover, $(\mathbb{N}, +, 0)$ is the free monoid generated by the terminal set $*$. As a consequence, we have an equivalence of 2-categories

$$[\mathbb{N}, \textsc{Cat}] \cong \textsc{MonCat}_2\left((\mathbb{N}, +), ([\mathbb{N}, \textsc{Cat}], +_{Day}\right).$$

Under this isomorphism, any categorical sequence $\mathcal{P}$ corresponds to a monoidal 2-functor

$$\mathcal{P}^{\otimes} : (\mathbb{N}, +) \to (\textsc{Cat}^{\mathbb{N}}, +_{Day}).$$

Definition 1.3.9. We define the power sequence 2-functor $-^{\otimes} : \textsc{Cat}^{\mathbb{N}} \to [\mathbb{N}, \textsc{Cat}^{\mathbb{N}}]$ as the 2-functor obtained by the composite

$$\textsc{Cat}^{\mathbb{N}} \cong \textsc{MonCat}_2\left((\mathbb{N}, +), (\textsc{Cat}^{\mathbb{N}}, +_{Day})\right) \to \left[\mathbb{N}, \textsc{Cat}^{\mathbb{N}}\right],$$

where the last arrow forgets about the monoidal structure of the 2-functors. Let $\mathcal{P}$ be a categorical sequence and $r \in \mathbb{N}$. The sequence of categories $\mathcal{P}^{\otimes^r}$ is explicitly given by

$$\mathcal{P}^{\otimes^r} = \int_{(n_1, \dots, n_r) \in \mathbb{N}^r} \prod_{i=1}^{r} \mathcal{P}(n_i) \times \mathbb{N}(-, n_1 + \cdots + n_r),$$

so that the sequence $\mathcal{P}^{\otimes^r}$ is given in arity $n \in \mathbb{N}$ by the category

$$\mathcal{P}^{\otimes^r}(n) = \coprod_{n_1 + \cdots + n_r = n} \prod_{i=1}^{r} \mathcal{P}(n_i)$$

Definition 1.3.10. We define the composition of categorical sequences as the 2-functor

$$\circ : \textsc{Cat}^{\mathbb{N}} \times \textsc{Cat}^{\mathbb{N}} \to \textsc{Cat}^{\mathbb{N}}$$

obtained by the composite

$$\begin{tikzcd}
\textsc{Cat}^{\mathbb{N}} \times \textsc{Cat}^{\mathbb{N}} \arrow[r, "id \times -^{\otimes}"] \arrow[ddddrr, "\circ"'] & {[\mathbb{N}, \textsc{Cat}] \times \left[\mathbb{N}, \textsc{Cat}^{\mathbb{N}}\right]} \arrow[r] & {[\mathbb{N} \times \mathbb{N}, \textsc{Cat} \times [\mathbb{N}, \textsc{Cat}]]} \arrow[d] \\
 & & {[\mathbb{N} \times \mathbb{N}, [\mathbb{N}, \textsc{Cat} \times \textsc{Cat}]]} \arrow[d, "{[\mathbb{N} \times (\mathbb{N} \cong \mathbb{N}^{op}), [\mathbb{N}, \times]]}"] \\
 & & {[\mathbb{N} \times \mathbb{N}^{op}, [\mathbb{N}, \textsc{Cat}]]} \arrow[d, "\int_{\mathbb{N}}"] \\
 & & \textsc{Cat}^{\mathbb{N}}.
\end{tikzcd}$$

Let $\mathcal{P}$ and $\mathcal{Q}$ be categorical sequences. We obtain

$$\mathcal{P} \circ \mathcal{Q} = \int_{\mathbb{N}} \mathcal{P} \times \mathcal{Q}^{\otimes}.$$

LEMMA 1.3.11. *For any categorical sequences $\mathcal{Q}, \mathcal{R} : \mathbb{N} \to \mathrm{CAT}$, we have an isomorphism*

$$(\mathcal{Q} \circ \mathcal{R})^{\otimes} \cong \mathcal{Q}^{\otimes} \circ \mathcal{R}.$$

PROOF. Let $r \in \mathbb{N}$. We first observe that

$$\mathcal{Q}^{\otimes^r} \circ \mathcal{R} = \int_{n \in \mathbb{N}} \mathcal{Q}^{\otimes^r}(n) \times \mathcal{R}^{\otimes^n}.$$

We also have

$$\begin{aligned}
(\mathcal{Q} \circ \mathcal{R})^{\otimes^r} &= \int_{m_1,\ldots,m_r} \left(\prod_{i=1}^{r} (\mathcal{Q} \circ \mathcal{R})(m_i) \right) \times \mathbb{N}(-, m_1 + \cdots + m_r) \\
&\cong \int_{m_1,\ldots,m_r} \left(\prod_{i=1}^{r} \int_{n_i \in \mathbb{N}} \mathcal{Q}(n_i) \times \mathcal{R}^{\otimes^{n_i}}(m_i) \right) \times \mathbb{N}(-, m_1 + \cdots + m_r) \\
&\cong \int_{m_1,\ldots,m_r} \int_{n_1,\ldots,n_r} \mathcal{Q}^{\otimes^r}(n_1 + \cdots + n_r) \times \mathcal{R}^{\otimes^{n_1 + \cdots + n_r}}(m_1 + \cdots + m_r) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{N}(-, m_1 + \cdots + m_r) \\
&\cong \int_{n \in \mathbb{N}} \mathcal{Q}^{\otimes^r}(n) \times \mathcal{R}^{\otimes^n},
\end{aligned}$$

and hence, we get the result of the lemma. □

LEMMA 1.3.12. *The composition product of categorical sequences is associative.*

PROOF. Let $\mathcal{P}, \mathcal{Q}, \mathcal{R} : \mathbb{N} \to \mathrm{CAT}$. We obtain the result by the following isomorphisms:

$$\begin{aligned}
(\mathcal{P} \circ \mathcal{Q}) \circ \mathcal{R} &= \int_{n \in \mathbb{N}} (\mathcal{P} \circ \mathcal{Q})(n) \times \mathcal{R}^{\otimes^n} \\
&= \int_{n \in \mathbb{N}} \int_{r \in \mathbb{N}} \mathcal{P}(r) \times \mathcal{Q}^{\otimes^r}(n) \times \mathcal{R}^{\otimes^n} \\
\mathcal{P} \circ (\mathcal{Q} \circ \mathcal{R}) &= \int_{r \in \mathbb{N}} \mathcal{P}(r) \times (\mathcal{Q} \circ \mathcal{R})^{\otimes^r} \\
&\cong \int_{r \in \mathbb{N}} \mathcal{P}(r) \times \mathcal{Q}^{\otimes^r} \circ \mathcal{R} \\
&= \int_{r \in \mathbb{N}} \mathcal{P}(r) \times \int_{n \in \mathbb{N}} \mathcal{Q}^{\otimes^r}(n) \times \mathcal{R}^{\otimes^n} \\
&\cong \int_{n \in \mathbb{N}} \int_{r \in \mathbb{N}} \mathcal{P}(r) \times \mathcal{Q}^{\otimes^r}(n) \times \mathcal{R}^{\otimes^n}.
\end{aligned}$$

□

DEFINITION 1.3.13. Let $\mathrm{I} : * \to \mathrm{CAT}^{\mathbb{N}}$ be the sequence defined on $n \in \mathbb{N}$ by

$$\mathrm{I}(n) = \begin{cases} * & \text{if } n = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

We refer to I as the unit categorical sequence.

LEMMA 1.3.14. *Let $\mathcal{P}$ be a categorical sequence. The unit categorical sequence is a unit for the composition product of categorical sequences.*

PROOF. First observe that $\mathrm{I}^{\otimes^n}(n) \cong *$ and $\mathrm{I}^{\otimes^n}(m) = \emptyset$ for $m \neq n$. Hence

$$\mathcal{P} \circ \mathrm{I} = \int_{n \in \mathbb{N}} \mathcal{P}(n) \times \mathrm{I}^{\otimes^n} \cong \mathcal{P}.$$

We also immediately obtain

$$\mathrm{I} \circ \mathcal{P} = \int_{n \in \mathbb{N}} \mathrm{I}(n) \times \mathcal{P}^{\otimes^n} \cong \mathrm{I}(1) \times \mathcal{P}^{\otimes^1} \cong \mathcal{P}.$$

□

PROPOSITION 1.3.15. *The composition product of categorical sequences gives to* $(\mathrm{CAT}^{\mathbb{N}}, \circ, \mathrm{I})$ *the structure of a monoidal* 2*-category.*

DEFINITION 1.3.16. We define the 2-category of categorical operads as the 2-category of monoids

$$\mathrm{OP}^{\mathbb{N}}_{\mathrm{CAT}} := \mathrm{MON}_{(\mathrm{CAT}_2, \times)}\left(\mathrm{CAT}^{\mathbb{N}}, \circ, \mathrm{I}\right).$$

EXAMPLE 1.3.17. The 2-category $\mathrm{OP}^{\mathbb{N}}_{\mathrm{CAT}}$ has a terminal object, denoted by $\mathcal{A}s$ or $*^{\mathbb{N}}$ depending on the context. Its underlying sequence is given in each arity $n \in \mathbb{N}$ by the terminal category. We also refer to the terminal operad as the associative operad.

EXAMPLE 1.3.18. The cartesian product of sets gives to the categorical sequence $\mathrm{SET}^{\mathbb{N}}$ the structure of a categorical operad.

DEFINITION 1.3.19. Let $\mathcal{P}$ be a categorical operad. We define the category of operads internal to $\mathcal{P}$ as the category of monoids internal to $\mathcal{P}$, so that

$$\mathcal{P}\text{-}\mathrm{OP} := \mathrm{MON}_{(\mathrm{CAT}_2, \times)}\left(\mathrm{CAT}^{\mathbb{N}}, \circ, \mathrm{I}\right)(*, \mathcal{P}).$$

Explicitly, an operad X internal to $\mathcal{P}$ consists in the data of

- a sequence of objects $\{X(r)\}_{r \in \mathbb{N}}$, where $X(r)$ is an object of $\mathcal{P}(r)$ for each $r \in \mathbb{N}$,
- a morphism $\epsilon \to X(1)$ in $\mathcal{P}(1)$, where ϵ is the unit object of $\mathcal{P}$,
- for each $r \in \mathbb{N}$ and each collection $n_1, \ldots, n_r \in \mathbb{N}$, a morphism in $\mathcal{P}(n_1 + \cdots + n_r)$

  $$\mu_{\mathcal{P}}\left(X(r), X(n_1), \ldots, X(n_r)\right) \to X(n_1 + \cdots + n_r),$$

  where $\mu_{\mathcal{P}}$ denotes the product operation of the operad $\mathcal{P}$,

so that the unit and associativity axioms of internal monoids hold.

EXAMPLE 1.3.20. An operad internal to the operad $\mathrm{SET}^{\mathbb{N}}$ is a sequence of sets $X : \mathbb{N} \to \mathrm{SET}$ equipped with a distinguished element $1_X : * \to X(1)$ and maps

$$X(r) \times \prod_{i=1}^{r} X(n_i) \to X(n),$$

satisfying the usual associativity and unit axioms.

DEFINITION 1.3.21. We define the 2-category $\mathrm{CAT}^{\mathfrak{S}}$ of symmetric categorical sequences as the 2-category of 2-functors $\mathrm{CAT}_2(\mathfrak{S}^{op}, \mathrm{CAT})$.

1.4. Algebras over categorical operads. The algebras over a CAT-operad naturally lie in a monoidal 2-category, just like the algebras over a SET-operad lie in a monoidal category. We associate a CAT-operad to any object of a monoidal 2-category, namely, its endomorphism operad, and define an algebra over a CAT-operad $\mathcal{P}$ in a monoidal 2-category Λ as an object X of Λ equipped with a morphism of CAT-operad from $\mathcal{P}$ to the endomorphism operad of X. We obtain a 2-category of $\mathcal{P}$-algebras in Λ for each CAT-operad $\mathcal{P}$. We define algebras over symmetric categorical operads in a symmetric monoidal 2-category in the same way. We also obtain a 2-category of algebras over a symmetric categorical operad in a symmetric monoidal 2-category. We show that we obtain a 2-functor ALG, which to a categorical operad $\mathcal{P}$, respectively a symmetric categorical operad $\mathcal{P}$, and a monoidal 2-category Λ, respectively a symmetric monoidal 2-category Λ, associates the 2-category of algebras over $\mathcal{P}$ in Λ. In the case where the monoidal 2-category Λ is symmetric, we show in this subsection that the 2-category of algebras over a (symmetric or non symmetric) operad in Λ inherits a symmetric monoidal 2-category structure. Hence, we will be able to define the 2-category of $\mathcal{P}$-algebras *in* the 2-category of $\mathcal{Q}$-algebras in Λ, for each (symmetric or non symmetric) categorical operads $\mathcal{P}$ and $\mathcal{Q}$.

Algebras over categorical operads in a monoidal 2*-category.* We define the 2-category of algebras over a categorical operad in a monoidal 2-category. We investigate the conditions under which the 2-category of $\mathcal{P}$-algebras in a monoidal 2-category can be equipped with a monoidal 2-category structure. Let $(\Lambda, \otimes_\Lambda, 1_\Lambda)$ be a monoidal 2-category and $X : *_\Lambda \to \Lambda$ be an object of Λ.

DEFINITION 1.4.1. The tensor product 2-functor of Λ induces a 2-functor

$$X^{\otimes_\Lambda^\bullet} : \mathbb{N} \to \Lambda,$$

which to $n \in \mathbb{N}$ associates the image $X^{\otimes_\Lambda^n}$ of $(X, \dots, X) : * \to \Lambda^n$ under the n-th tensor product $\otimes_\Lambda^n$ of Λ. We refer to $X^{\otimes_\Lambda^\bullet}$ as the power sequence of X.

REMARK 1.4.2. The power sequence $X^{\otimes_\Lambda^\bullet}$ yields a monoidal 2-functor

$$X^{\otimes_\Lambda^\bullet} : * \to [\mathbb{N}, \Lambda]\,.$$

We have, as a consequence, a 2-natural transformation in the 2-category $[\mathbb{N}, \mathrm{CAT}_2]$

$$\begin{tikzcd} *^{\mathbb{N}} \arrow[r, "{X^{\otimes_\Lambda^\bullet}}^{\mathbb{N}}"] \arrow[d] & {[\mathbb{N},\Lambda]^{\mathbb{N}}} \arrow[d, "\otimes_{Day}"] \arrow[dl, Rightarrow, "\cong"] \\ * \arrow[r, "X^{\otimes_\Lambda^\bullet}"'] & {[\mathbb{N},\Lambda]} \end{tikzcd}$$

which is an isomorphism. We accordingly have an isomorphism for each $r \in \mathbb{N}$

$$\begin{tikzcd} *^{r} \arrow[r, "{X^{\otimes_\Lambda^\bullet}}^{r}"] \arrow[d] & {[\mathbb{N},\Lambda]^{r}} \arrow[d, "\otimes_{Day}^r"] \arrow[dl, Rightarrow, "\cong"] \\ * \arrow[r, "X^{\otimes_\Lambda^\bullet}"'] & {[\mathbb{N},\Lambda]\,.} \end{tikzcd}$$

The composite $\otimes_{Day}^r {X^{\otimes_\Lambda^\bullet}}^r : \mathbb{N} \to \Lambda$ is given by

$$\otimes_{Day}^r {X^{\otimes_\Lambda^\bullet}}^r \cong \int_{(n_1,\dots,n_r):*\to\mathbb{N}^r} \mathbb{N}(n_1 + \cdots + n_r, -) \times \bigotimes_{i=1}^{r}{}_{\Lambda} X^{\otimes_\Lambda^{n_i}},$$

so that the isomorphism of sequences $\otimes^r_{Day} X^{\otimes^\bullet_\Lambda} \cong X^{\otimes^\bullet_\Lambda}$ is given in arity n by

$$\int_{(n_1,\dots,n_r):*\to\mathbb{N}^r} \mathbb{N}(n_1+\cdots+n_r,n)\times \bigotimes_{i=1}^{r}{}_{\Lambda} X^{\otimes^{n_i}_\Lambda} \cong \coprod_{n_1+\cdots+n_r=n} \bigotimes_{i=1}^{r}{}_{\Lambda} X^{\otimes^{n_i}_\Lambda} \cong X^{\otimes^n_\Lambda}.$$

DEFINITION 1.4.3. Let X be an object of Λ. We let $\mathrm{END}^\Lambda_X : \mathbb{N}^{op} \to \mathrm{CAT}$ be the sequence of categories obtained by the composite

$$\begin{tikzcd} \mathbb{N}^{op} \arrow[r, dashed, "\mathrm{END}^\Lambda_X"] \arrow[d, "(X^{\otimes^\bullet_\Lambda},X)"'] & \mathrm{CAT}. \\ \Lambda^{op}\times\Lambda \arrow[ur, "\Lambda(-,-)"'] & \end{tikzcd}$$

We may also write $\Lambda(X^\bullet, X)$ for this sequence.

DEFINITION 1.4.4. Let X be an object of Λ. The sequence END^Λ_X corresponds to a 2-functor $\mathrm{END}^\Lambda_X : * \to [\mathbb{N}^{op}, \mathrm{CAT}]$. We let

$$\mathrm{END}^{\Lambda}_X{}^{\bullet} : \mathbb{N} \to [\mathbb{N}^{op}, \mathrm{CAT}]\,.$$

be the unique monoidal 2-functor extending END^Λ_X for the monoidal structure given on $[\mathbb{N}^{op}, \mathrm{CAT}]$ by Day's convolution, and by using the isomorphism $\mathbb{N} \cong \mathcal{F}^{\mathbb{N}}*$. Let $r \in \mathbb{N}$. The sequence $\mathrm{END}^{\Lambda}_X{}^{r}$ is therefore given by the coend

$$\mathrm{END}^{\Lambda}_X{}^{r} = \int_{(n_1,\dots,n_r):*\to\mathbb{N}^r} \mathbb{N}(n_1+\cdots+n_r,-)\times\prod_{i=1}^{r}\Lambda\left(X^{\otimes^{n_i}_\Lambda}, X\right),$$

so that for each $n \in \mathbb{N}$ we obtain

$$\mathrm{END}^{\Lambda}_X{}^{r}(n) = \coprod_{n_1+\cdots+n_r=n}\prod_{i=1}^{r}\Lambda\left(X^{\otimes^{n_i}_\Lambda}, X\right).$$

REMARK 1.4.5. For each $r \in \mathbb{N}$ and $n_1, \dots, n_r \in \mathbb{N}$, the tensor product of Λ yields a functor

$$\otimes^r_\Lambda : \prod_{i=1}^{r}\Lambda\left(X^{\otimes^{n_i}_\Lambda}, X\right) \to \Lambda\left(\bigotimes_{i=1}^{r} X^{\otimes^{n_i}_\Lambda}, X^{\otimes^r_\Lambda}\right).$$

DEFINITION 1.4.6. We let the unit of X be the morphism of categorical sequences $\mathrm{I}_X : \mathrm{I} \to \mathrm{END}^\Lambda_X$ obtained from the identity $1_X : * \to \Lambda(X,X)$.

DEFINITION 1.4.7. We let

$$\mu : \int_{r:*\to\mathbb{N}} \mathrm{END}^\Lambda_X(r)\times\left(\mathrm{END}^\Lambda_X\right)^r \to \mathrm{END}^\Lambda_X$$

be the morphism of categorical sequences that we form as follows. Let $r \in \mathbb{N}$. We first obtain a morphism from

$$\begin{aligned} \mathrm{END}^{\Lambda}_X{}^{r} &= \int_{(n_1,\dots,n_r):*\to\mathbb{N}^r} \mathbb{N}(n_1+\cdots+n_r,-)\times\prod_{i=1}^{r}\Lambda\left(X^{\otimes^{n_i}_\Lambda}, X\right) \\ &\xrightarrow{\int_{(n_i)_i:*\to\mathbb{N}^r}\mathbb{N}(\Sigma n_i,-)\times\otimes^r_\Lambda} \int_{(n_1,\dots,n_r):*\to\mathbb{N}^r} \mathbb{N}(n_1+\cdots+n_r,-)\times\Lambda\left(\bigotimes_{i=1}^{r}{}_{\Lambda} X^{\otimes^{n_i}_\Lambda},\ \bigotimes_{i=1}^{r}{}_{\Lambda} X\right) \\ &\to \int_{(n_1,\dots,n_r):*\to\mathbb{N}^r} \mathbb{N}(n_1+\cdots+n_r,-)\times\Lambda\left(X^{\otimes^{n_1+\cdots+n_r}_\Lambda}, X^{\otimes^r_\Lambda}\right). \end{aligned}$$

We define μ as the composite

$$
\begin{aligned}
&\int_{r:*\to\mathbb{N}} \mathrm{End}_X^\Lambda(r) \times \left(\mathrm{End}_X^\Lambda\right)^r \\
\to &\int_{r:*\to\mathbb{N}} \Lambda\left(X^{\otimes_\Lambda^r}, X\right) \times \int_{(n_1,\dots,n_r):*\to\mathbb{N}^r} \mathbb{N}(n_1+\cdots+n_r,-) \times \Lambda\left(X^{\otimes_\Lambda^{n_1+\cdots+n_r}}, X^{\otimes_\Lambda^r}\right) \\
\cong &\int_{r:*\to\mathbb{N}} \int_{(n_1,\dots,n_r):*\to\mathbb{N}^r} \mathbb{N}(n_1+\cdots+n_r,-) \times \Lambda\left(X^{\otimes_\Lambda^r}, X\right) \times \Lambda\left(X^{\otimes_\Lambda^{n_1+\cdots+n_r}}, X^{\otimes_\Lambda^r}\right) \\
\to &\int_{r:*\to\mathbb{N}} \int_{(n_1,\dots,n_r):*\to\mathbb{N}^r} \mathbb{N}(n_1+\cdots+n_r,-) \times \Lambda\left(X^{\otimes_\Lambda^{n_1+\cdots+n_r}}, X\right) \cong \mathrm{End}_X^\Lambda,
\end{aligned}
$$

where the last morphism is induced by the composition of morphisms in Λ. The associativity and unit isomorphisms provided by the associative and unitary structure of the composition of morphisms in Λ yield isomorphisms which give to End_X^Λ the structure of a monoid for the composition of sequences.

Definition 1.4.8. For each object X of Λ, we define the endomorphism operad of X as the categorical sequence End_X^Λ, equipped with the unit and composition morphisms given in Definition 1.4.6 and Definition 1.4.7.

Definition 1.4.9. Let $\mathcal{P}$ be an operad in Cat and Λ be a monoidal 2-category.

- A $\mathcal{P}$-algebra is an object X of Λ equipped with a morphism of Cat-operads
$$\psi_X : \mathcal{P} \to \mathrm{End}_X^\Lambda.$$
In particular, each $p \in \mathcal{P}(r)$ yields a morphism in Λ from $X^{\otimes_\Lambda^r}$ to X, which we may also denote by p or $p_X : X^{\otimes_\Lambda^r} \to X$.
- Let X and Y be $\mathcal{P}$-algebras. We let $\mathcal{P}\text{-}\mathrm{Alg}_\Lambda(X,Y)$ be the category whose objects are lax morphisms of $\mathcal{P}$-algebras from X to Y, and whose set of morphisms between lax morphisms of $\mathcal{P}$-algebras F, G is given by $\mathcal{P}\text{-}\mathrm{Alg}_\Lambda(X,Y)(F,G)$, where
  - a lax morphism of $\mathcal{P}$-algebras from X to Y is a pair $(F, \otimes_F^\bullet)$, where $F : X \to Y$ is a morphism in Λ and $\otimes_F^\bullet$ is a 2-morphism in $[\mathbb{A}^{op}, \mathrm{Cat}]$:
$$
\begin{tikzcd}
 & \Lambda\left(X^{\otimes_\Lambda^\bullet}, X\right) \arrow[dr, "\Lambda\left(X^{\otimes_\Lambda^\bullet}, F\right)"] & \\
\mathcal{P} \arrow[ur, "\psi_X"] \arrow[dr, "\psi_Y"'] & \Uparrow \otimes_F^\bullet & \Lambda\left(X^{\otimes_\Lambda^\bullet}, Y\right) \\
 & \Lambda\left(Y^{\otimes_\Lambda^\bullet}, Y\right) \arrow[ur, "\Lambda\left(F^{\otimes_\Lambda^\bullet}, Y\right)"'] &
\end{tikzcd}
$$
which fulfils the constraint expressed by the commutativity of the diagram **??**.
  - For $F, G \in \mathcal{P}\text{-}\mathrm{Alg}_\Lambda(X,Y)$, a 2-morphism of $\mathcal{P}$-algebras is a morphism $\alpha : F \to G$ in $\Lambda(X,Y)$ such that the following diagram commutes for

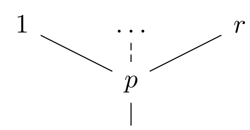
all $p \in \mathcal{P}(r)$:

$$\begin{tikzcd} p_Y(F,\dots,F) \arrow[r, "\otimes_F^p"] \arrow[d, "p_Y(\alpha,\dots,\alpha)"'] & F(p_X,\dots,p_X) \arrow[d, "\alpha(p_X,\dots,p_X)"] \\ p_Y(G,\dots,G) \arrow[r, "\otimes_G^p"'] & G(p_X,\dots,p_X). \end{tikzcd}$$

We obtain a 2-category $\mathcal{P}$-$\mathrm{A{\scriptsize LG}}_\Lambda$ whose objects are given by $\mathcal{P}$-algebras in Λ, and whose category of morphisms between $\mathcal{P}$-algebras X and Y is given by $\mathcal{P}$-$\mathrm{A{\scriptsize LG}}_\Lambda$.

NOTATION 1.4.10. We may use the following notation, which are convenient when working in the framework of operads and algebras over operads.

– If $\mathcal{P}$ be an operad in $(\Lambda, \otimes_\Lambda)$ and $p \in \mathcal{P}(r)$, we write

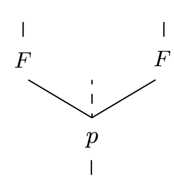


and may omit the numbering. If X is an object of Λ equipped with a $\mathcal{P}$-algebra structure, then each $p \in \mathcal{P}(r)$ yields a morphism in Λ from $X^{\otimes_\Lambda^r}$ to X. We use the same notation for this morphism.

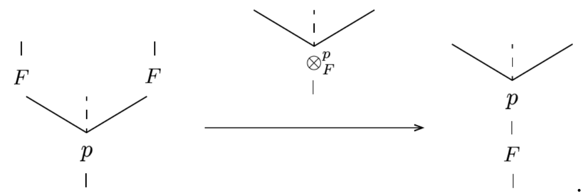
– If $f \in \mathcal{P}(r)(p_1, p_2)$ is a morphism between the operations $p_1, p_2 \in \mathcal{P}(r)$, we write

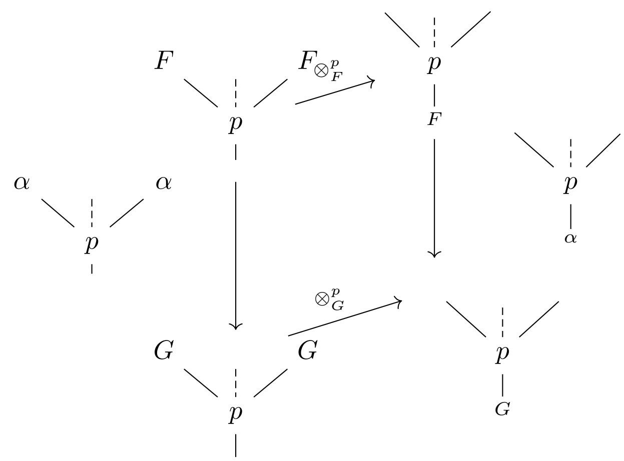
for the morphism f in the category $\mathcal{P}(r)$. If X is an object of Λ equipped with a $\mathcal{P}$-algebra structure, then f yields a morphism in the category $\Lambda(X^{\otimes_\Lambda^r}, X)$ from p_1 to p_2. We use the same notation for this morphism. Note that if Λ is $\mathrm{C{\scriptsize AT}}$, then p_1 and p_2 are functors, and f is a natural transformation.

– Suppose that X and Y are equipped with a $\mathcal{P}$-algebra structure in Λ. Let $p \in \mathcal{P}(r)$ and $F : X \to Y$ be a morphism in Λ. The morphism F yields a morphism $F^{\otimes_\Lambda^r} : X^{\otimes_\Lambda^r} \to Y^{\otimes_\Lambda^r}$ in Λ. We write



for the composite morphism $X^{\otimes_\Lambda^r} \xrightarrow{F^{\otimes_\Lambda^r}} Y^{\otimes_\Lambda^r} \xrightarrow{p} Y$ in Λ. More generally, we extend the usual notation of operadic composites to morphisms in Λ involving iterated tensor products. We give the following characterization of For instance, we obtain the following for $\otimes^p_F$:





REMARK 1.4.11. Let X, Y be $\mathcal{P}$-algebras in Λ. Let $F : * \to \mathcal{P}\text{-}\mathrm{ALG}_\Lambda(X, Y)$ be a lax morphism of $\mathcal{P}$-algebras. For each $r \in \mathbb{N}$ and each operation $p \in \mathcal{P}(r)$, the natural transformation $\otimes^p_F$ yields a morphism in the category of objects of Y

$$p(Fx_1, \ldots, Fx_r) \to Fp(x_1, \ldots, x_r),$$

which is natural in $x_1, \ldots, x_r : *_\Lambda \to X$. In particular, we obtain the notion of a lax monoidal functor when $\mathcal{P}$ is the associative operad and Λ is the 2-category CAT.

PROPOSITION 1.4.12. *Let X be an object of Λ and $\mathcal{P}$ be an operad in* CAT. *The data of a morphism of categorical operads*

$$\mathcal{P} \to \mathrm{END}^\Lambda_X$$

is equivalent to the data of a morphism in Λ

$$\int_{\mathbb{N}} \mathcal{P} \times X^{\otimes_\Lambda \bullet} \to X$$

such that the following diagram commutes up to a 2-isomorphism in Λ:

$$\begin{tikzcd}
& \int_{\mathbb{N}} \mathcal{P} \times \left(\int_{\mathbb{N}} \mathcal{P} \times X^{\otimes_\Lambda \bullet}\right)^{\otimes_\Lambda \bullet} \arrow[dl, "\int_{\mathbb{N}} \mathcal{P} \times \psi_X^{\otimes_\Lambda \bullet}"'] \arrow[dr, "\cong"] & \\
\int_{\mathbb{N}} \mathcal{P} \times X^{\otimes_\Lambda \bullet} \arrow[d, "\psi_X"'] & \cong & \int_{r \in \mathbb{N}} \int_{(n_1, \ldots, n_r) \in \mathbb{N}^r} \mathcal{P}(r) \times \mathcal{P}^{\otimes^r} \times X^{\otimes_\Lambda \bullet} \arrow[d, "\int \mu_{\mathcal{P}} \times X^{\otimes_\Lambda}"] \\
X & & \int_{\mathbb{N}} \mathcal{P} \times X^{\otimes_\Lambda \bullet} \arrow[ll, "\psi_X"]
\end{tikzcd}$$

PROPOSITION 1.4.13. *Let $X, Y : * \to \mathcal{P}\text{-}\mathrm{ALG}_{(\Lambda,\otimes_1)}$. Suppose that Λ is equipped with another monoidal structure $(\Lambda,\otimes_2)$ which is compatible with $(\Lambda,\otimes_1)$, so that $(\Lambda,\otimes_1,\otimes_2,1_\Lambda)$ has the structure of a 2-monoidal 2-category. Then the tensor product $\otimes_2$ yields a tensor product on the 2-category of $\mathcal{P}$-algebras in $(\Lambda,\otimes_1)$*

$$\otimes_2 : \mathcal{P}\text{-}\mathrm{ALG}_{(\Lambda,\otimes_1)} \times \mathcal{P}\text{-}\mathrm{ALG}_{(\Lambda,\otimes_1)} \to \mathcal{P}\text{-}\mathrm{ALG}_{(\Lambda,\otimes_1)}.$$

More generally, the monoidal structure $(\Lambda,\otimes_2)$ on the monoidal 2-category $(\Lambda,\otimes_1)$ yields a monoidal structure $\big(\mathcal{P}\text{-}\mathrm{ALG}_{(\Lambda,\otimes_1)},\otimes_2\big)$ on the 2-category of $\mathcal{P}$-algebras.

PROOF. Let X and Y be $\mathcal{P}$-algebras in $(\Lambda,\otimes_1)$. The monoidal structure of the 2-functor $\otimes_2$ yields morphisms in Λ for each $n \in \mathbb{N}$:

$$\otimes_2^1 : (X \otimes_2 Y)^{\otimes_1^n} \to X^{\otimes_1^n} \otimes_2 Y^{\otimes_1^n}$$

which are natural in X, Y, and which satisfy the coherence conditions of monoidal morphisms. We equip $X \otimes_2 Y$ with a $\mathcal{P}$-algebra structure as follows. The sequence of morphisms given in arity $r \in \mathbb{N}$ by the composite

$$\begin{tikzcd}
\mathcal{P}(r) \arrow[r, "\psi_X(r)\times\psi_Y(r)"] \arrow[d, dashed, "\psi_{X\otimes_2 Y}(r)"'] & \Lambda(X^{\otimes_1^n}, X) \times \Lambda(Y^{\otimes_1^n}, Y) \arrow[d, "\otimes_2"] \\
\Lambda\big((X\otimes_2 Y)^{\otimes_1^n}, X\otimes_2 Y\big) & \Lambda\big(X^{\otimes_1^n}\otimes_2 Y^{\otimes_1^n}, X\otimes_2 Y\big) \arrow[l, "\Lambda(\otimes_2^n, X\otimes_2 Y)"]
\end{tikzcd}$$

yields a morphism of CAT-operads, and hence a $\mathcal{P}$-algebra structure on $X \otimes_2 Y$. The verification of associativity and unit constraints is easily deducible from the constraints on $\otimes_2$, for instance by using the $\mathbb{N}$-unbiased monoidal structure attached to $(\mathrm{MON}_{(\Lambda,\otimes_1)},\otimes_2)$. □

REMARK 1.4.14. It is important to notice that we used the universal property of the cartesian product to produce the morphisms $\mathcal{P}(n) \to \Lambda(X^{\otimes_\Lambda^n}, X) \times \Lambda(Y^{\otimes_\Lambda^n}, Y)$ from the structural morphisms given by the $\mathcal{P}$-algebra structure on X and on Y. As a consequence, this construction can a priori not be adapted in the enriched setting unless the monoidal structure is cartesian.

DEFINITION 1.4.15. Let $\mathcal{P}$ and $\mathcal{Q}$ be CAT-operads and let $(\Lambda,\otimes_1,\otimes_2)$ be a 2-monoidal 2-category. We define the 2-category of $\mathcal{Q}$-algebras in the 2-category of $\mathcal{P}$-algebras in $(\Lambda,\otimes_1,\otimes_2)$ as the 2-category $\mathcal{Q}\text{-}\mathrm{ALG}_{(\mathcal{P}\text{-}\mathrm{ALG}(\Lambda,\otimes_1),\otimes_2)}$. If the 2-fold monoidal structure on Λ is given by a symmetric monoidal structure $(\Lambda,\otimes)$, then we just write $\mathcal{Q}\text{-}\mathrm{ALG}_{\mathcal{P}\text{-}\mathrm{ALG}_\Lambda}$ for the 2-category of $\mathcal{Q}$-algebras in the 2-category of $\mathcal{P}$-algebras in $(\Lambda,\otimes)$.

PROPOSITION 1.4.16. *We have a 2-functor*

$$\mathrm{ALG} : \mathrm{OP}_{\mathrm{CAT}}^{op} \times \Pi\mathrm{MON}_{\mathrm{CAT}_2} \to \Pi\mathrm{CAT}_2,$$

which, to an operad $\mathcal{P}$ and a monoidal category $(\Lambda,\otimes)$, associates the 2-category of $\mathcal{P}$-algebras in $(\Lambda,\otimes)$ as defined in Definition 1.4.9. If $(\Lambda,\otimes)$ is a symmetric monoidal 2-category, then the 2-category $\mathcal{P}\text{-}\mathrm{ALG}_\Lambda$ inherits a symmetric monoidal structure. In this case, ALG *extends to a 2-functor*

$$\mathrm{ALG} : \mathrm{OP}_{\mathrm{CAT}}^{op} \times \Pi\mathrm{MON}_{\mathrm{CAT}_2}^{\mathfrak{S}} \to \Pi\mathrm{MON}_{\mathrm{CAT}_2}^{\mathfrak{S}}.$$

PROOF. Let $\mathcal{P},\mathcal{Q} \in \mathrm{OP}_{\mathrm{CAT}}$ and $(\Lambda,\otimes_\Lambda),(\Gamma,\otimes_\Gamma) \in \mathrm{MON}_{\mathrm{CAT}_2}$. We define a functor

$$\mathrm{OP}_{\mathrm{CAT}}(\mathcal{Q},\mathcal{P}) \times \Pi\mathrm{MON}_{\mathrm{CAT}_2}(\Lambda,\Gamma) \to \Pi\mathrm{CAT}_2(\mathcal{P}\text{-}\mathrm{ALG}_\Lambda, \mathcal{Q}\text{-}\mathrm{ALG}_\Gamma)$$

as follows.

- Let $\alpha \in \mathrm{O\scriptstyle P}_{\mathrm{C\scriptstyle AT}}(\mathcal{Q},\mathcal{P})$ and $\chi \in \mathrm{M\scriptstyle ON}_{\mathrm{C\scriptstyle AT}_2}(\Lambda,\Gamma)$. The 2-functor
$$\mathcal{P}\text{-}\mathrm{A\scriptstyle LG}_\Lambda \to \mathcal{Q}\text{-}\mathrm{A\scriptstyle LG}_\Gamma$$
is defined as follows.
  - let $(\mathcal{C},\psi) \in \mathcal{P}\text{-}\mathrm{A\scriptstyle LG}_\Lambda$ such that $\mathcal{C} \in \Lambda$ and $\psi \in \mathrm{O\scriptstyle P}_{\mathrm{C\scriptstyle AT}}(\mathcal{P},\Lambda(\mathcal{C}^\bullet,\mathcal{C}))$, then $\chi\mathcal{C} \in \Gamma$ inherits a $\mathcal{Q}$-algebra structure by the composite
$$\mathcal{Q} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\psi} \Lambda(\mathcal{C}^\bullet,\mathcal{C}) \xrightarrow{\chi(\mathcal{C}^\bullet,\mathcal{C})} \Gamma(\chi(\mathcal{C}^\bullet),\chi\mathcal{C}) \xrightarrow{\Gamma(\eta_\chi,1_\mathcal{C})} \Gamma((\chi\mathcal{C})^\bullet,\chi\mathcal{C}),$$
where $\eta_\chi : (\chi\mathcal{C})^\bullet \to \chi(\mathcal{C}^\bullet)$ is given by the monoidal structure of χ.

□

Algebras over symmetric categorical operads in a symmetric monoidal 2-category. We define the 2-category of algebras over a symmetric categorical operad in a symmetric monoidal 2-category. We show that the 2-category of $\mathcal{P}$-algebras in a symmetric monoidal 2-category can be equipped with a symmetric monoidal 2-category structure. Let $(\Lambda,\otimes_\Lambda,1_\Lambda)$ be a symmetric monoidal 2-category and $X : *_\Lambda \to \Lambda$ be an object of Λ.

Definition 1.4.17. For each $n \in \mathbb{N}$ and for each object X of Λ the symmetric group Σ_n acts on the left on $X^{\otimes_\Lambda^n}$ by permutation of the factors, so that each object X of Λ induces a 2-functor
$$X^{\otimes_\Lambda^\bullet} : \mathfrak{S} \to \Lambda.$$
We refer to $X^{\otimes_\Lambda^\bullet}$ as the power symmetric sequence of X.

Remark 1.4.18. The power symmetric sequence $X^{\otimes_\Lambda^\bullet}$ defines a symmetric monoidal 2-functor, so that

- for $r : * \to \mathfrak{S}$ and $n_1,\dots,n_r * \to \mathfrak{S}$, we have
$$X^{\otimes_\Lambda^{n_1+\cdots+n_r}} \cong \bigotimes_{i=1}^{r}{}_\Lambda X^{\otimes_\Lambda^{n_i}},$$
- for each $\sigma : * \to \Sigma_r$ and $\sigma_i : * \to \Sigma_{n_i}$, we have a natural isomorphism
$$\begin{tikzcd} X^{\otimes_\Lambda^{n_1+\cdots+n_r}} \arrow[r, "X^{\otimes_\Lambda^{\sigma(\sigma_1,\dots,\sigma_r)}}"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\cong"] & X^{\otimes_\Lambda^{n_{\sigma(1)}+\cdots+n_{\sigma(r)}}} \arrow[d, "\cong"] \\ \bigotimes_{i=1}^{r}{}_\Lambda X^{\otimes_\Lambda^{n_i}} \arrow[r, "\sigma.\ \bigotimes_{i=1}^{r}{}_\Lambda X^{\otimes_\Lambda^{\sigma_i}}"'] & \bigotimes_{i=1}^{r}{}_\Lambda X^{\otimes_\Lambda^{n_{\sigma_i}}}. \end{tikzcd}$$

Definition 1.4.19. We define a symmetric sequence of categories $\mathrm{E\scriptstyle ND}_X^\Lambda : \mathfrak{S}^{op} \to \mathrm{C\scriptstyle AT}$ by the composite
$$\begin{tikzcd} \mathfrak{S}^{op} \arrow[r, dashed, "\mathrm{E\scriptstyle ND}_X^\Lambda"] \arrow[d, "(X^{\otimes_\Lambda^\bullet}, X)"'] & \mathrm{C\scriptstyle AT}. \\ \Lambda^{op} \times \Lambda \arrow[ur, "\Lambda(-,-)"'] & \end{tikzcd}$$

Definition 1.4.20. We call unit of X the morphism of symmetric sequences $\epsilon : \mathrm{I} \to \mathrm{E\scriptstyle ND}_X^\Lambda$ induced by the identity $1_X : * \to \Lambda(X,X)$.

Definition 1.4.21. We let

$$\int_{r:*\to\mathfrak{S}} \mathrm{End}^{\Lambda}_{X}(r)\times\left(\mathrm{End}^{\Lambda}_{X}\right)^{r}\to\mathrm{End}^{\Lambda}_{X}$$

be the morphism of symmetric sequences that we form as follows. Let $r\in\mathbb{N}$. The tensor product 2-functor $\otimes_\Lambda$ also induces functors on the categories of morphisms in Λ. In particular, for each $r\in\mathbb{N}$ and $n_1,\dots,n_r\in\mathbb{N}$, we have a functor

$$\prod_{i=1}^{r}\Lambda\left(X^{\otimes^{n_i}_{\Lambda}},X\right)\to\Lambda\left(\bigotimes_{i=1}^{r}X^{\otimes^{n_i}_{\Lambda}},X^{\otimes^{r}_{\Lambda}}\right).$$

We form the composite

$$\Lambda\left(X^{\otimes^{r}_{\Lambda}},X\right)\times\prod_{i=1}^{r}\Lambda\left(X^{\otimes^{n_i}_{\Lambda}},X\right)\to\Lambda\left(X^{\otimes^{r}_{\Lambda}},X\right)\times\Lambda\left(\bigotimes_{i=1}^{r}X^{\otimes^{n_i}_{\Lambda}},X^{\otimes^{r}_{\Lambda}}\right)\to\Lambda\left(X^{\otimes^{n}_{\Lambda}},X\right),$$

where the last arrow is obtained from the 2-category structure of Λ by composition of the morphisms, to get the requested composition functor at the level of our coend. The compatibility of this composition functor with the action of Σ_r comes from the symmetric monoidal structure of Λ. We use, as in the non-symmetric case, the natural isomorphism provided by the associativity of the composition of morphisms in Λ to get an associativity isomorphism for this composition functor, which again satisfies the usual coherence conditions of associativity isomorphisms. The morphism ϵ similarly satisfies the required conditions for being a unit with respect to the operadic composition product in $\mathrm{End}^{\Lambda}_{X}$.

Definition 1.4.22. For each object X of Λ, we define the endomorphism symmetric operad of X as the categorical symmetric sequence $\mathrm{End}^{\Lambda}_{X}$, equipped with the unit and composition morphisms given in Definition 1.4.20 and Definition 1.4.21.

Definition 1.4.23. Let $\mathcal{P}$ be a symmetric operad in Cat. We define the 2-category $\mathcal{P}$-Alg$_\Lambda$ of $\mathcal{P}$-algebras, lax $\mathcal{P}$-algebras 1-morphisms and lax $\mathcal{P}$-algebras 2-morphisms by the following.

- A $\mathcal{P}$-algebra is an object X of Λ equipped with a morphism of symmetric Cat-operads

$$\psi_X:\mathcal{P}\to\mathrm{End}^{\Lambda}_{X}.$$

- If X and Y are $\mathcal{P}$-algebras, then the category $\mathcal{P}$-Alg$_\Lambda(X,Y)$ is such that
  - its objects are the morphisms $F:X\to Y$ in Λ equipped with a natural transformation $\eta(r)$

$$\begin{tikzcd} & \Lambda\left(X^{\otimes^{r}_{\Lambda}},X\right) \arrow[dr, "\Lambda\left(X^{\otimes^{r}_{\Lambda}},F\right)"] \arrow[dd, phantom, "\Uparrow\eta(r)"] & \\ \mathcal{P}(r) \arrow[ur, "\psi_X"] \arrow[dr, "\psi_Y"'] & & \Lambda\left(X^{\otimes^{r}_{\Lambda}},Y\right), \\ & \Lambda\left(Y^{\otimes^{r}_{\Lambda}},Y\right) \arrow[ur, "\Lambda\left(F^{\otimes^{r}_{\Lambda}},Y\right)"'] & \end{tikzcd}$$

    for each $r\in\mathbb{N}$, which is compatible with the action of Σ_r and such that the diagram of Figure **??** commutes,

– the set of 2-morphisms from F to G, for $F, G : * \to \mathcal{P}\text{-}\mathrm{ALG}_\Lambda(X,Y)$ lax morphisms of $\mathcal{P}$-algebras in Λ, consists of the 2-morphisms $\alpha : F \Rightarrow G$ in Λ, such that the corresponding diagram commutes.

REMARK 1.4.24. Suppose that Λ has a distinguished object 1_Λ, let X, Y be $\mathcal{P}$-algebras in Λ and let $F : * \to \mathcal{P}\text{-}\mathrm{ALG}_\Lambda(X,Y)$ be a lax morphism of $\mathcal{P}$-algebras. For each $r \in \mathbb{N}$, the natural transformation $\eta(r)$ part of the $\mathcal{P}$-algebra morphism structure of F provides morphisms in the category of objects of Y for each $x_1, \ldots, x_r : *_\Lambda \to X$ for each operation $p : * \to \mathcal{P}(r)$, such that $\eta(r)$ corresponds to

$$\eta(r)_{x_1,\ldots,x_r} : p(F(x_1), \ldots, F(x_r)) \to F(p(x_1, \ldots, x_r)).$$

The compatibility with the action of Σ_r ensures that given a permutation $\sigma \in \Sigma_r$, the diagram

$$\begin{tikzcd} \sigma.p(F(x_1), \ldots, F(x_r)) \arrow[r, "\eta(r)(\sigma.p)_{x_1,\ldots,x_r}"] \arrow[d, "\cong"'] & F(\sigma.p(x_1, \ldots, x_r)) \arrow[d, "\cong"] \\ p\left(F(x_{\sigma(1)}), \ldots, F(x_{\sigma(r)})\right) \arrow[r, "\eta(r)(p)_{x_{\sigma(1)},\ldots,x_{\sigma(r)}}"'] & F\left(p(x_{\sigma(1)}, \ldots, x_{\sigma(r)})\right) \end{tikzcd}$$

commutes.

PROPOSITION 1.4.25. *Let $X, Y : * \to \mathcal{P}\text{-}\mathrm{ALG}_{(\Lambda,\otimes)}$. The tensor product $\otimes$ extends to $\mathcal{P}$-algebras*

$$\otimes : \mathcal{P}\text{-}\mathrm{ALG}_{(\Lambda,\otimes)} \times \mathcal{P}\text{-}\mathrm{ALG}_{(\Lambda,\otimes)} \to \mathcal{P}\text{-}\mathrm{ALG}_{(\Lambda,\otimes)},$$

so that $\left(\mathcal{P}\text{-}\mathrm{ALG}_{(\Lambda,\otimes)}, \otimes\right)$ has the structure of a monoidal 2-category.

PROOF. Let X and Y be $\mathcal{P}$-algebras in $(\Lambda, \otimes)$. We define a $\mathcal{P}$-algebra structure on $X \otimes Y$ as follows. Let $r \in \mathbb{N}$. We let $\psi_{X\otimes Y}$ be the morphism of symmetric CAT-operads defined in arity r by the composite

$$\begin{tikzcd} \mathcal{P}(r) \arrow[r, "\psi_X(r)\times\psi_Y(r)"] \arrow[drr, dashed, "\psi_{X\otimes Y}(r)"'] & \Lambda(X^{\otimes^n_\Lambda}, X) \times \Lambda(Y^{\otimes^n_\Lambda}, Y) \arrow[r, "\otimes_2"] & \Lambda\left(X^{\otimes^n_\Lambda} \otimes Y^{\otimes^n_\Lambda}, X \otimes Y\right) \arrow[d, "\cong"] \\ & & \Lambda\left((X \otimes Y)^{\otimes^n_\Lambda}, X \otimes Y\right), \end{tikzcd}$$

where the vertical isomorphism is given by the commutativity of $\otimes$. $\square$

DEFINITION 1.4.26. Let $\mathcal{P}$ and $\mathcal{Q}$ be symmetric CAT-operads and $(\Lambda, \otimes)$ be a symmetric monoidal 2-category. We define the 2-category of $\mathcal{Q}$-algebras in the 2-category of $\mathcal{P}$-algebras in $(\Lambda, \otimes)$ as the 2-category $\mathcal{Q}\text{-}\mathrm{ALG}_{(\mathcal{P}\text{-}\mathrm{ALG}(\Lambda,\otimes),\otimes)}$.

PROPOSITION 1.4.27. *Let $(\Lambda, \otimes)$ be a symmetric monoidal 2-category. The 2-category $\mathcal{P}\text{-}\mathrm{ALG}_\Lambda$ is symmetric monoidal. Moreover,* ALG *defines a 2-functor*

$$\mathrm{ALG} : {\mathrm{OP}^{\mathfrak{S}}_{\mathrm{CAT}}}^{op} \times \Pi\mathrm{MON}^{\mathfrak{S}}_{\mathrm{CAT}_2} \to \Pi\mathrm{MON}^{\mathfrak{S}}_{\mathrm{CAT}_2},$$

which to a symmetric CAT*-operad $\mathcal{P}$ and a symmetric monoidal 2-category Λ associates the 2-category $\mathcal{P}\text{-}\mathrm{ALG}_\Lambda$ of $\mathcal{P}$-algebras in Λ.*

Generalized Day convolution product for algebras over operads. Let $(\Lambda, \otimes)$ be a bicomplete monoidal 2-category which is cartesian closed, and suppose it is equipped with internal morphisms with values in the bicomplete object $\mathcal{S}_\Lambda$ of Λ. Let $\mathcal{P}$ be a categorical operad such that $\mathcal{S}_\Lambda$ is equipped with a $\mathcal{P}$-algebra structure. We generalise Day's convolution product in the context of $\mathcal{P}$-algebras in Λ, so that when $\mathcal{P}$ is the associative categorical operad, we precisely obtain Day's convolution product as defined in section 1. Precisely, recall that we have a presheaf 2-functor

$$[-, \mathcal{S}_\Lambda]_\Lambda : \Lambda_{\mathcal{S}}^{op} \to \Lambda.$$

(Here $\Lambda_{\mathcal{S}}$ is the full sub-2-category of $\mathcal{S}_\Lambda$-small objects in Λ, and $\Lambda_{\mathcal{S}}^{op}$ has the same objects than $\Lambda_{\mathcal{S}}$, with categories of morphisms between objects $\mathcal{C}$ and $\mathcal{D}$ of Λ defined by $\Lambda_{\mathcal{S}}^{op}(\mathcal{C}, \mathcal{D}) = \Lambda_S(\mathcal{D}, C)^{op}$.)

Let $\mathcal{C}$ be an $\mathcal{S}_\Lambda$-small object of Λ which is equipped with a $\mathcal{P}$-algebra structure. We show that the object $[\mathcal{C}^{op}, \mathcal{S}_\Lambda]_\Lambda$ inherits a $\mathcal{P}$-algebra structure.

Proposition 1.4.28. *The presheaf 2-functor $[-, \mathcal{S}]_\Lambda$ restricts to $\mathcal{P}$-algebras, hence it induces a 2-functor*

$$[-, \mathcal{S}]_\Lambda : \mathcal{P}\text{-}\mathrm{ALG}_{\Lambda_{\mathcal{S}}}{}^{op} \to \mathcal{P}\text{-}\mathrm{ALG}_\Lambda.$$

Digression on multidimensional algebra and commutativity. The main benefit of working in a symmetric monoidal framework for defining algebras over operads lies in the possibility of iterating the algebra process. However, it is important to note that symmetric monoidal objects inherit significantly less structure than n-fold monoidal objects, and hence make the information more difficult to manage.

Indeed, let $\mathcal{P}$ and $\mathcal{Q}$ be non symmetric categorical operads and let $(\Lambda, \otimes_1, \otimes_2)$ be a 2-monoidal 2-category. The 2-category $\mathcal{Q}\text{-}\mathrm{ALG}_{(\mathcal{P}\text{-}\mathrm{ALG}(\Lambda,\otimes_1),\otimes_2)}$ has for objects the objects of Λ which are equipped with an action of $\mathcal{P}$ with respect to $\otimes_1$, together with a compatible action of $\mathcal{Q}$ with respect to $\otimes_2$. The distinction between the monoidal structures on which these operads act imposes structural constraints, resulting in a better understanding on the combinatorial aspects involving the symmetries induced by the interchange between $\mathcal{P}$ and $\mathcal{Q}$. In fact, the two monoidal laws may be seen as distinct directions in a 2-dimensional algebraic framework, in a way analogous to the vertical and horizontal concatenation of topological homotopies. With this paradigm, working in a symmetric monoidal framework amounts to study an infinite dimensional structure through a projection on a single dimensional space.

In this section, we noticed that non-symmetric operads are shaped on associativity, while symmetric operads are shaped on commutativity.

In the next section, we introduce generalized operads, which can be shaped on any given structure, in an attempt to model particular composition schemes in a more accurate way. For instance, we will be able to define operads $\mathrm{OP}_{\mathrm{CAT}}^{(n)}$ shaped on n-fold associativity, which we will call n-fold operads. To such an operad, we will associate a 2-category of algebras in any n-fold monoidal 2-category.

We claim that n-fold operads efficiently govern structures which can be decomposed into n compatible actions of usual (symmetric or non-symmetric) operads. To see this, we briefly examine the structure corresponding to compatible actions of some operads $\mathcal{P}$ and $\mathcal{Q}$. Recall from Proposition 1.4.13 that the 2-category of $\mathcal{P}$-algebras in a monoidal 2-category $(\Lambda, \otimes_1)$ inherits a monoidal structure whenever

Λ is equipped with another monoidal structure whose tensor product $\otimes_2$ defines a monoidal morphism with respect to $(\Lambda, \otimes_1)$. Consequently, given another operad $\mathcal{Q}$, it makes sense to consider the 2-category of $\mathcal{Q}$-algebras in the monoidal 2-category $(\mathcal{P}\text{-}\mathrm{A}\textsc{lg}_{(\Lambda,\otimes_1)}, \otimes_2)$, whose objects are $\mathcal{P}$-algebras in $(\Lambda, \otimes_1)$ equipped with the structure of a $\mathcal{Q}$-algebra with respect to $\otimes_2$, such that the structural morphisms given by the action of $\mathcal{Q}$ are morphisms of $\mathcal{P}$-algebras. Its seems natural to ask whether the 2-category $\mathcal{Q}\text{-}\mathrm{A}\textsc{lg}_{(\mathcal{P}\text{-}\mathrm{A}\textsc{lg}(\Lambda,\otimes_1),\otimes_2)}$ can be represented as the 2-category of algebras over an operad $\mathcal{P} \otimes_{\otimes\ \mathrm{BV}} \mathcal{Q}$. Necessarily, algebras over such an operad have to be defined in a 2-fold monoidal 2-category, so that we can require $\mathcal{P} \otimes_{\otimes\ \mathrm{BV}} \mathcal{Q}$ to be such that

$$\mathcal{P} \otimes_{\otimes\ \mathrm{BV}} \mathcal{Q}\text{-}\mathrm{A}\textsc{lg}_{(\Lambda,\otimes_1,\otimes_2)} \cong \mathcal{Q}\text{-}\mathrm{A}\textsc{lg}_{(\mathcal{P}\text{-}\mathrm{A}\textsc{lg}(\Lambda,\otimes_1),\otimes_2)}.$$

This observation motivates the definition of generalized operads, and specifically, of operads shaped on n-fold associativity. In this thesis, we mostly treat the case where the operads under consideration are symmetric. In Chapter III, we will give an explicit description of the tensor product of symmetric operads. For any symmetric monoidal 2-category $(\Lambda, \otimes)$, we will obtain an isomorphism

$$\mathcal{P} \otimes_{\otimes\ \mathrm{BV}} \mathcal{Q}\text{-}\mathrm{A}\textsc{lg}_{(\Lambda,\otimes)} \cong \mathcal{Q}\text{-}\mathrm{A}\textsc{lg}_{(\mathcal{P}\text{-}\mathrm{A}\textsc{lg}(\Lambda,\otimes),\otimes)}.$$

Recall that associative structures can be encoded via both symmetric and non symmetric operad. In practice, working with the non-symmetric associative operad is much more natural and straightforward. Indeed, the symmetric version of the associative operad requires managing unnecessary additional relations involving the action of symmetric groups. In contrast, it could seems that the use of symmetric operads is necessary to describe structures governing compatible actions of operads.

Indeed, suppose that $\mathcal{P}$ and $\mathcal{Q}$ are non-symmetric operads, let $(\Lambda, \otimes)$ be a symmetric monoidal 2-category, and let $A \in \mathcal{Q}\text{-}\mathrm{A}\textsc{lg}_{(\mathcal{P}\text{-}\mathrm{A}\textsc{lg}(\Lambda,\otimes),\otimes)}$. By definition, for each $q \in \mathcal{Q}(n)$, the object $A \in \mathcal{P}\text{-}\mathrm{A}\textsc{lg}_{(\Lambda,\otimes)}$ is equipped with a lax morphism of $\mathcal{P}$-algebras

$$\psi_q : A^{\otimes^r} \to A$$

in Λ. In particular, for each $p \in \mathcal{P}(r)$, A is equipped with a 2-morphism ψ_q^p in Λ such that

$$\begin{tikzcd}
(A^{\otimes^n})^{\otimes^r} \arrow[r, "\tau"] \arrow[dd, "\psi_q^{\otimes^r}"'] \arrow[from=1-1, to=3-2, phantom, "\Rightarrow \psi_q^p"] & (A^{\otimes^r})^{\otimes^n} \arrow[dr, "\psi^{p\otimes^n}"] & \\
 & & A^{\otimes^n} \arrow[dl, "\psi_q"] \\
A^{\otimes^r} \arrow[r, "\psi^p"'] & A. &
\end{tikzcd}$$

Hence, if we try to encapsulate the resulting structure on $A \in \Lambda$ by a single operad $\mathcal{P} \otimes_{\otimes\ \mathrm{BV}} \mathcal{Q}$, this operad would in particular be equipped with a morphism $p \square q$ encoding the aforementioned exchange involving the 2-morphism ψ_q^p. Note that the morphism τ involved in the exchange diagram is an isomorphism in Λ which realises a permutation of the coordinates in the product $A^{\otimes^{nr}}$. It follows that the action of the symmetric group is needed to encode the source and the target of the operation

$$p \square q \in \mathcal{P} \otimes_{\otimes\ \mathrm{BV}} \mathcal{Q}\left(p(q, \dots, q), \tau.q(p, \dots, p)\right).$$

Consequently, the operad $\mathcal{P} \otimes_{\otimes \text{ BV}} \mathcal{Q}$ can not be defined as a non symmetric operad, resulting in the well known combinatorial intricacies regarding the exchange of operadic structures, notably with regard to the structure of iterated loop spaces.

For a better understanding of operadic actions which interchange, we briefly state the general scheme of n-fold operads and their relation with interchange. Although the construction of n-fold operads and their algebras in n-fold monoidal 2-categories can be deduced from the next section as a particular case, we do not provide a proof of what follows for the moment, neither construct the tensor product of an n-fold operad with an m-fold operad. We simply state propositions whose proof and details will be given in future work. Therefore, the following can be viewed as conjectural, as well as a remark of interest for the understanding of iterated structures.

In the case of n-fold operads, the algebra 2-functor analogous to the one constructed in Proposition 1.4.16 will have the following expression

$$\textsc{Alg} : {\textsc{Op}^{(n)}_{\textsc{Cat}}}^{op} \times \Pi\textsc{Mon}^{n}_{\textsc{Cat}_2} \to \Pi\textsc{Cat}_2.$$

By similar arguments as in Proposition 1.4.13, given an n-fold operad $\mathcal{P}$ and an $(n+m)$-monoidal 2-category $(\Lambda, \otimes_1, \ldots, \otimes_{n+m})$, the 2-category $\mathcal{P}\text{-}\textsc{Alg}_{(\Lambda,\otimes_1,\ldots,\otimes_n)}$ will inherits the structure of an m-fold monoidal 2-category from Λ, with products given by $\otimes_{n+1}, \ldots, \otimes_{n+m}$. Consequently, any m-fold operad $\mathcal{Q}$ will yield a 2-category of algebras internal to the 2-category of $\mathcal{P}$-algebras in Λ:

$$\mathcal{Q}\text{-}\textsc{Alg}_{(\mathcal{P}\text{-}\textsc{Alg}_{(\Lambda,\otimes_1,\ldots,\otimes_n)},\otimes_{n+1},\ldots,\otimes_{n+m})},$$

whose objects may be seen as objects of Λ equipped with an action of $\mathcal{P}$ and an action of $\mathcal{Q}$ which are compatible with each other. We expect a graded tensor product

$$\otimes_{\otimes \text{BV}} : \textsc{Op}^{(n)}_{\textsc{Cat}} \times \textsc{Op}^{(m)}_{\textsc{Cat}} \to \textsc{Op}^{(n+m)}_{\textsc{Cat}}$$

such that for any n-fold operad $\mathcal{P}$ and any m-fold operad $\mathcal{Q}$, the $(n+m)$-fold operad $\mathcal{P} \otimes_{\otimes \text{ BV}} \mathcal{Q}$ thus obtained satisfies

$$\mathcal{P} \otimes_{\otimes \text{ BV}} \mathcal{Q}\text{-}\textsc{Alg}_{(\Lambda,\otimes_1,\ldots,\otimes_{n+m})} \cong \mathcal{Q}\text{-}\textsc{Alg}_{(\mathcal{P}\text{-}\textsc{Alg}_{(\Lambda,\otimes_1,\ldots,\otimes_n)},\otimes_{n+1},\ldots,\otimes_{n+m})},$$

and more generally, such that the following diagram commutes up to isomorphism.

$$\begin{tikzcd}
{\textsc{Op}^{(m)}_{\textsc{Cat}}}^{op} \times {\textsc{Op}^{(n)}_{\textsc{Cat}}}^{op} \times \Pi\mathcal{M}^{n+m}\text{-}\textsc{Alg}_{(\textsc{Cat}_2,\times)} \arrow[r, "\cong"] \arrow[d, "\otimes_{\otimes \text{BV}}"'] \arrow[dr, phantom, "\cong"] & {\textsc{Op}^{(m)}_{\textsc{Cat}}}^{op} \times {\textsc{Op}^{(n)}_{\textsc{Cat}}}^{op} \times \Pi\mathcal{M}^{n}\text{-}\textsc{Alg}_{(\mathcal{M}^m\text{-}\textsc{Alg}_{(\textsc{Cat}_2,\times)},\times)} \arrow[d, "1\times\textsc{Alg}"] \\
{\textsc{Op}^{(n+m)}_{\textsc{Cat}}}^{op} \times \Pi\mathcal{M}^{n+m}\text{-}\textsc{Alg}_{(\textsc{Cat}_2,\times)} \arrow[dr, "\textsc{Alg}"'] & {\textsc{Op}^{(m)}_{\textsc{Cat}}}^{op} \times \Pi\mathcal{M}^{m}\text{-}\textsc{Alg}_{(\textsc{Cat}_2,\times)} \arrow[d, "\textsc{Alg}"] \\
& \Pi\textsc{Cat}_2
\end{tikzcd}$$

Note that if the horizontal isomorphism on the top of the diagram has first been observed in Chapter I as a direct consequence of the definition of iterated monoids, it will also comes straightforward from the isomorphism of symmetric categorical operads $\mathcal{M}^{n+m} \cong \mathcal{M}^n \otimes \mathcal{M}^m$ which will be stated after having defined the tensor product in Chapter III.

2. Generalized operads

The purpose of this section is to formalize the definition of a generalized notion of operad within the framework of a monoidal category and with a structure shaped on a categorical operad. The operadic shape provides a way of extending the notion of a non-symmetric operad (where we take associative composition shapes) and the notion of a symmetric operad (where we take symmetric and associative composition shapes).

We still work in a 2-categorical framework. We precisely define, for Υ a small (symmetric) categorical operad and Λ a monoidal 2-category, a 2-category of Υ-operads internal in Λ. For this purpose, we have to assume that Λ is bicomplete close, with internal morphisms taking values in a bicomplete closed object $\mathcal{S}_\Lambda$, which has the structure of an Υ-algebra in Λ.

To make our account shorter, we only describe the case where Υ is a symmetric categorical operad in detail. We just have to forget about symmetric structures to get the non-symmetric case of our constructions.

2.1. Definition and examples. Let Υ be a symmetric categorical operad and let $(\Lambda, \otimes_\Lambda, \times_\Lambda, [-,-]_\Lambda, *_\Lambda)$ be a bicomplete closed and symmetric monoidal 2-category equipped with internal morphisms in $\mathcal{S}_\Lambda$. Suppose that $\mathcal{S}_\Lambda$ has the structure of an Υ-algebra in Λ with unit given by the terminal object $*_\mathcal{S} : *_\Lambda \to \mathcal{S}$. We provide a description of free Υ-algebras in the monoidal 2-category $(\Lambda, \otimes)$ and use the free Υ-algebra $\mathcal{F}^\Upsilon(*_\Lambda)$ generated by the terminal object $*_\Lambda$ of Λ to define a category $\mathrm{OP}_\Lambda^\Upsilon$ of generalized Υ-operads in Λ. For this purpose, we define a composition product $\circ$ and equip the object $[\mathcal{F}^\Upsilon(*_\Lambda)^{op}, \mathcal{S}]_\Lambda$ with the structure of a monoid in the monoidal 2-category $(\Lambda, \otimes)$. We will use the condition of the following definition to define a unit for this composition product on $[\mathcal{F}^\Upsilon(*_\Lambda)^{op}, \mathcal{S}]_\Lambda$.

DEFINITION 2.1.1. For a small categorical operad Υ and an Υ-algebra $\mathcal{S}$ in a monoidal 2-category $(\Lambda, \otimes)$, we say that an object $s : *_\Lambda \to \mathcal{S}$ defines a 0-object for the Υ-algebra structure if the following property holds: "for each collection $s_1, \ldots, s_r : *_\Lambda \to \mathcal{S}^{\otimes_\Lambda^r}$, if there is some i such that $s_i \cong s$, then we have $\mu_T^\mathcal{S}(s_1, \ldots, s_r) \cong s$ for all $T : * \to \Upsilon(r)$.

PROPOSITION 2.1.2. *The obvious forgetful* 2*-functor* $\mathcal{U}^\Upsilon$*, from the* 2*-category of* Υ*-algebras in* Λ *to the* 2*-category* Λ*, has a left adjoint* $\mathcal{F}^\Upsilon$*, which associates to every object of* Λ *a free* Υ*-algebra in* Λ*. Hence, we have an adjunction of* 2*-categories*

$$\mathcal{F}^\Upsilon : \Lambda \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \Upsilon\text{-}\mathrm{ALG}_{(\Lambda,\otimes)} : \mathcal{U}^\Upsilon.$$

Moreover, this adjunction restricts to $\mathcal{S}$*-small objects and commutes with the Yoneda embedding, so that the diagram*

$$\begin{tikzcd}
\Lambda_\mathcal{S} \arrow[r, bend left, "\mathcal{F}^\Upsilon"] \arrow[r, phantom, "\perp"] \arrow[d, hook] & \Upsilon\text{-}\mathrm{ALG}_{(\Lambda_\mathcal{S},\otimes)} \arrow[l, "\mathcal{U}^\Upsilon"] \arrow[d, hook] \\
\Lambda \arrow[r, "\mathcal{F}^\Upsilon"] \arrow[r, phantom, "\perp"] & \Upsilon\text{-}\mathrm{ALG}_{(\Lambda,\otimes)} \arrow[l, bend left, "\mathcal{U}^\Upsilon"]
\end{tikzcd}$$

commutes.

PROOF. Let $E : * \to \Lambda$. We construct an object $\mathcal{F}^\Upsilon E : * \to \Lambda$ and an Υ-algebra structure on $\mathcal{F}^\Upsilon E$. We define

$$\mathcal{F}^\Upsilon E = \int_{r:*\to\mathfrak{S}} \Upsilon(r) \times E^{\otimes_\Lambda^r}.$$

We equip $\mathcal{F}^\Upsilon E$ with the structure of an Υ-algebra in $(\Lambda, \otimes)$. For this purpose, we let

$$\mu : \int_{\mathfrak{S}} \Upsilon \times (\mathcal{F}^\Upsilon E)^{\otimes_\Lambda} \to \mathcal{F}^\Upsilon E$$

be defined by the composite

$$\int_{\mathfrak{S}} \Upsilon \times (\mathcal{F}^\Upsilon E)^{\otimes_\Lambda} \cong \int_{\mathfrak{S}} \Upsilon \times \left(\int_{\mathfrak{S}} \Upsilon \times E^{\otimes_\Lambda} \right)^{\otimes_\Lambda} \xrightarrow{\cong}$$

$\square$

Now we assume that the initial object $\emptyset_{\mathcal{S}}$ of $\mathcal{S}$ is a 0-object for the Υ-algebra structure of $\mathcal{S}$. Recall that the terminal object $*_\Lambda$ of Λ satisfies $*_\Lambda = I(*_{\mathcal{S}})$, so that it is $\mathcal{S}$-small. Let Υ_Λ be the free Υ-algebra in $(\Lambda, \otimes_\Lambda)$ generated by the terminal object of Λ, so that $\Upsilon_\Lambda = \mathcal{F}^\Upsilon(*_\Lambda)$. By generalized Day's convolution product for algebras over an operad, the presheaf object $[\Upsilon_\Lambda^{op}, \mathcal{S}]_\Lambda$ inherits the structure of an Υ-algebra in $(\Lambda, \otimes_\Lambda)$. We recall the expression of this structure. Let $T : * \to \Upsilon(r)$. The Υ-algebra structure of $[\Upsilon_\Lambda^{op}, \mathcal{S}]_\Lambda$ provides a morphism

$$\mu_T : [\Upsilon_\Lambda^{op}, \mathcal{S}]_\Lambda^{\otimes_\Lambda^r} \to [\Upsilon_\Lambda^{op}, \mathcal{S}]_\Lambda.$$

in Λ. Let $\bar{\mathcal{P}} : * \to [\Upsilon_\Lambda^{op}, \mathcal{S}]_\Lambda^{\otimes_\Lambda^r}$. Then $\mu_T \bar{\mathcal{P}} : \Upsilon_\Lambda^{op} \to \mathcal{S}$ is given on $V : * \to \Upsilon_\Lambda^{op}$ by

$$\mu_T \bar{\mathcal{P}}_V \cong \int_{\bar{T}:*\to\Upsilon_\Lambda^{\otimes_\Lambda^r}} \mu_T^{\mathcal{S}} \bar{\mathcal{P}} \bar{T} \times_{\mathcal{S}} \Upsilon_\Lambda \left(V, \mu_T^\Upsilon \bar{T} \right).$$

Recall that if we have $T_i : * \to \Upsilon_\Lambda$ for $i = 1, \ldots, r$, then we can form the composite

$$\begin{tikzcd} * \arrow[r, bend left, "\Upsilon_\Lambda^r"] \arrow[r, bend right, "*_\Lambda^r"'] \arrow[r, phantom, "(T_i)\ \Uparrow"] & \Lambda^r \arrow[r, "\otimes_\Lambda"] & \Lambda \end{tikzcd}$$

to define an object $\bar{T} : *_\Lambda \to \Upsilon^{\otimes_\Lambda^r}$. In what follows, we also use the notation $T_1 \otimes_\Lambda \cdots \otimes_\Lambda T_r$ for this object $\bar{T} : *_\Lambda \to \Upsilon^{\otimes_\Lambda^r}$. Suppose that $\bar{\mathcal{P}} : * \to [\Upsilon_\Lambda^{op}, \mathcal{S}]_\Lambda^{\otimes_\Lambda^r}$ is also induced by the data of $\mathcal{P}_i : * \to [\Upsilon_\Lambda^{op}, \mathcal{S}]$ for $i = 1, \ldots, r$, then the summand indexed by $\bar{T}$ in the coend is given by

$$\mu_T^{\mathcal{S}} \bar{\mathcal{P}} \bar{T} \times_{\mathcal{S}} \Upsilon_\Lambda \left(V, \mu_T^\Upsilon \bar{T} \right) \cong \mu_T^{\mathcal{S}} \left(\mathcal{P}_1 T_1 \otimes_\Lambda \cdots \otimes_\Lambda \mathcal{P}_r T_r \right) \times_{\mathcal{S}} \Upsilon_\Lambda \left(V, \mu_T^\Upsilon \left(T_1 \otimes_\Lambda \cdots \otimes_\Lambda T_r \right) \right).$$

We have an equivalence of categories

$$\Lambda(\Upsilon_\Lambda^{op}, \mathcal{S}) \cong \Upsilon\text{-}Alg_\Lambda(\Upsilon_\Lambda, [\Upsilon_\Lambda^{op}, \mathcal{S}]_\Lambda])$$

induced by the adjunction. Let $\mathcal{P} : \Upsilon_\Lambda^{op} \to \mathcal{S}$ be a morphism in Λ. We write

$$\mathcal{P}^\bullet : \Upsilon_\Lambda \to [\Upsilon_\Lambda^{op}, \mathcal{S}]_\Lambda$$

for the morphism of Υ-algebras in Λ corresponding to $\mathcal{P}$ under this equivalence.

LEMMA 2.1.3. *The morphism $\mathcal{P}^\bullet$ has the following description.*

– *Let $T : * \to \Upsilon_\Lambda$. Then $\mathcal{P}^T : *_\Lambda \to [\Upsilon_\Lambda^{op}, \mathcal{S}]_\Lambda$ is given by*

$$\mathcal{P}^T \cong \int_{T_1,\dots,T_r : *_\Lambda \to \Upsilon^r} \mu_T^{\mathcal{S}}(\mathcal{P}_{T_1},\dots,\mathcal{P}_{T_r}) \times_{\mathcal{S}} \Upsilon_\Lambda(-, \mu_\Upsilon(T; T_1,\dots,T_r)).$$

– *Let $T, W : *_\Lambda \to \Upsilon_\Lambda$. The corresponding morphism*

$$\mathcal{P}^{T,W} : \Upsilon_\Lambda(T,W) \to [\Upsilon_\Lambda^{op}, \mathcal{S}]_\Lambda(\mathcal{P}^T, \mathcal{P}^W)$$

in $\mathcal{S}^*$ associates to each $\phi : *_S \to \Upsilon_\Lambda(T,W)$ the element $\mathcal{P}^{T,W}\phi : *_{\mathcal{S}} \to [\Upsilon_\Lambda^{op}, \mathcal{S}]_\Lambda(\mathcal{P}^T, \mathcal{P}^W)$ of $\mathcal{S}$, whose term

$$\mathcal{P}^{T,W}\phi_V : *_{\mathcal{S}} \to \mathcal{S}\left(\mathcal{P}_V^T, \mathcal{P}_V^W\right),$$

for $V : *_\Lambda \to \Upsilon_\Lambda^{op}$, is given by

$$\mathcal{P}^{T,W}\phi_V = \int_{T_1,\dots,T_r : *_\Lambda \to \Upsilon^r} \mu_T^{\mathcal{S}}(\mathcal{P}_{T_1},\dots,\mathcal{P}_{T_r}) \times_{\mathcal{S}} \Upsilon_\Lambda(V, \mu_\Upsilon(\phi; T_1,\dots,T_r)).$$

Definition 2.1.4. We define the composition product morphism in Λ as the composite

$$\begin{tikzcd}
{[\Upsilon^{op}, \mathcal{S}]_\Lambda \times [\Upsilon^{op}, \mathcal{S}]_\Lambda} \arrow[r] \arrow[ddrr, dashed, "\circ"] & {[\Upsilon^{op}, \mathcal{S}]_\Lambda \times [\Upsilon, [\Upsilon^{op}, \mathcal{S}]_\Lambda]_\Lambda} \arrow[r] & {[\Upsilon^{op} \times \Upsilon, \mathcal{S} \times [\Upsilon^{op}, \mathcal{S}]_\Lambda]_\Lambda} \arrow[d, "{[\Upsilon^{op}\times\Upsilon, \times_{\mathcal{S}}]_\Lambda}"] \\
& & {[\Upsilon^{op} \times \Upsilon, [\Upsilon^{op}, \mathcal{S}]_\Lambda]_\Lambda} \arrow[d, "\int_\Upsilon"] \\
& & {[\Upsilon^{op}, \mathcal{S}]_\Lambda.}
\end{tikzcd}$$

The composition product is given for $\mathcal{P}, \mathcal{Q} : *_\Lambda \to [\Upsilon^{op}, \mathcal{S}]_\Lambda$ by

$$\mathcal{P} \circ \mathcal{Q} = \int_\Upsilon \mathcal{P} \times_{\mathcal{S}} \mathcal{Q}^\bullet.$$

Definition 2.1.5. The Υ-algebra structure on Υ_Λ induces a functor $\psi(0) : \Upsilon(0) \cong * \to \Lambda(*_\Lambda, \Upsilon_\Lambda)$, and hence, a morphism $1_\Upsilon : *_\Lambda \to \Upsilon_\Lambda$ in Λ. We define

$$I^\Upsilon : *_\Lambda \to [\Upsilon_\Lambda^{op}, \mathcal{S}]_\Lambda$$

by $I^\Upsilon 1_\Upsilon = *_{\mathcal{S}}$ and $I^\Upsilon T = \emptyset_{\mathcal{S}}$ for $T : *_\Lambda \to \Upsilon$ such that $T \neq 1_\Upsilon$.

Remark 2.1.6. Note that we can say that two objects $T, V : *_\Lambda \to \Upsilon_\Lambda$ are distinct because Υ_Λ is $\mathcal{S}$-small.

Lemma 2.1.7. *The object I^Υ defines a unit for the composition product.*

Proof. Let $\mathcal{P} : \Upsilon_\Lambda^{op} \to \mathcal{S}$ and $V : *_\Lambda \to \Upsilon_\Lambda$. We have

$$\begin{aligned}
\mathcal{P} \circ {I^\Upsilon}_V &\cong \int_{T : *_\Lambda \to \Upsilon_\Lambda} \mathcal{P}_T \times_{\mathcal{S}} {I^\Upsilon}_V^T \\
&\cong \int_{T : *_\Lambda \to \Upsilon_\Lambda} \mathcal{P}_T \times_{\mathcal{S}} \int_{T_1,\dots,T_r : *_\Lambda \to \Upsilon_\Lambda} \mu_T^{\mathcal{S}}(I_{T_1}^\Upsilon, \dots, I_{T_r}^\Upsilon) \times_{\mathcal{S}} \Upsilon_\Lambda\left(V, \mu_T^{\Upsilon_\Lambda}(T_1,\dots,T_r)\right) \\
&\cong \int_{T : *_\Lambda \to \Upsilon_\Lambda} \mathcal{P}_T \times_{\mathcal{S}} \mu_T^{\mathcal{S}}(*_{\mathcal{S}}, \dots, *_{\mathcal{S}}) \times_{\mathcal{S}} \Upsilon_\Lambda\left(V, \mu_T^{\Upsilon_\Lambda}(1_\Upsilon, \dots, 1_\Upsilon)\right) \\
&\cong \int_{T : *_\Lambda \to \Upsilon_\Lambda} \mathcal{P}_T \times_{\mathcal{S}} \Upsilon_\Lambda(V, T) \\
&\cong \ \mathcal{P}_V,
\end{aligned}$$

where the last isomorphism follows from Lemma **??**. On the other hand, we have

$$I^{\Upsilon} \circ \mathcal{P} \cong \int_{T:*_\Lambda \to \Upsilon_\Lambda} I_T \times_{\mathcal{S}} \mathcal{P}^T \cong \mathcal{P}^{1_\Upsilon} \cong \mathcal{P}.$$

□

PROPOSITION 2.1.8. *The presheaf object* $[\Upsilon, \mathcal{S}]_\Lambda : *_\Lambda \to \Lambda$*, with the composition product* $\circ$ *and unit* I^{Υ} *defines a monoid in the* 2*-category* $(\Lambda, \times)$.

DEFINITION 2.1.9. We define the category of Υ-operads in Λ as the category

$$\mathrm{OP}^{\Upsilon}_{\Lambda} = \mathrm{MON}_{(\Lambda, \otimes_\Lambda)}\left([\Upsilon, \mathcal{S}]_\Lambda, \circ, I^{\Upsilon}\right).$$

2.2. Algebras over generalized operads. In order to provide a meaning to generalized operads, we proceed to define the objects on which a generalized operad can act. Just like generalized Υ-operads in Λ are defined in a framework internal to Υ-algebra structures, an algebra over an Υ-operad will naturally lies in an object which is equipped with an Υ-algebra structure. We associate an Υ-operad to any point of an Υ-algebra in Λ, and we call this operad the endomorphism operad of this point.

DEFINITION 2.2.1. Let $\mathcal{C} : * \to \Upsilon\text{-}\mathrm{ALG}_{(\Lambda, \otimes)}$ and $X : *_\Lambda \to \mathcal{C}$. We define a morphism

$$\mathrm{END}^{\mathcal{C}}_X : \Upsilon^{op}_\Lambda \to \mathcal{S}$$

as follows. Let $T : *_\Lambda \to \Upsilon_\Lambda$. We let $\mathrm{END}^{\mathcal{C}}_X T : *_\Lambda \to \mathcal{S}$ be defined by

$$\mathrm{END}^{\mathcal{C}}_X T = \mathcal{C}(\mu^{\mathcal{C}}_T(X, \dots, X), X).$$

In the first chapter, we observed that monoids can be defined within any monoidal 2-category. Given a monoidal 2-category and a monoid, we can also define monoids internal to this monoid. In the same way, a monoidal 2-category precisely is a monoid within the monoidal 3-category of 2-categories, with product given by the cartesian product. Accordingly, monoidal structures lie in an entangled system of monoids internal to them others. Formally, the definition of monoidal structures involves a hierarchic system $\{\mathbb{T}_n\}_n$ such that each $\mathbb{T}_n : \mathbb{T}_{n+1}$ has the structure of a monoidal object in $\mathbb{T}_{n+1}$, so that $(\mathbb{T}_n, \otimes_{\mathbb{T}_n}) : \mathrm{MON}_{(\mathbb{T}_{n+1}, \otimes_{\mathbb{T}_{n+1}})}$. In the previous section, we introduced the notion of symmetric and non symmetric operads and observed that the 2-category $\mathrm{MON}_{(\mathbb{T}, \otimes_{\mathbb{T}})}$ of monoids in a monoidal 2-category $(\mathbb{T}, \otimes_{\mathbb{T}})$ could be obtained as the 2-category of algebras in $\mathbb{T}$ over the terminal non-symmetric operad.

The purpose of this section is to generalise the notion of an operad by using the aforementioned observations on monoidal structures and on the structure of operads. For this purpose, we consider a hierarchic system $\{\mathbb{T}_n\}_n$ as defined in Appendix A, so that each $\mathbb{T}_n$ is an object of $\mathbb{T}_{n+1}$. We assume each $\mathbb{T}_n$ to be complete and cocomplete in $\mathbb{T}_{n+1}$ and write $*_{n-1} \in \mathbb{T}_n$ for the terminal object of $\mathbb{T}_n$. We show how the notion of an operad arises from the notion of a monad through exponentiation and free objects.

All of the constructions established in this section rely on the entangled nature of the notion of an algebra over an monad. In particular, we extensively use the connexion between the levels provided by the hierarchic type system $\mathbb{T}$. We first place at level $n+1$ and suppose that there is a monad in $\mathbb{T}_{n+1}$ such that $\mathbb{T}_n$ has the structure of an algebra over this monad.

2.3. Monads. We first introduce some basic facts on monads and their algebras and show some technical results that we will use in the next section to set the framework of generalized operads.

2.3.1. *Generalities on monads and their algebras.*

Definition 2.3.1. Let $\Lambda : \mathbb{T}_{n+1}$. Any $\mathcal{C} : \Lambda$ yields an object $\Lambda(\mathcal{C}, \mathcal{C}) : \mathbb{T}_n$ which we call $\Omega\Lambda_{\mathcal{C}}$. Suppose that each morphism in Λ is equipped with an inverse in a coherent way. In this case, we obtain a morphism

$$\Omega\Lambda : \Lambda \to \mathbb{T}_n.$$

The following is a straightforward consequence of the definition of the objects and morphisms of $\mathbb{T}_n$.

Proposition 2.3.2. *The composition of morphisms in Λ yields a monoidal structure on $\Omega\Lambda_{\mathcal{C}}$ in $(\mathbb{T}_n, \times)$ whose unit is given by the identity morphism of $\mathcal{C}$.*

Definition 2.3.3. A monad in $\Lambda : \mathbb{T}_{n+1}$ on $\mathcal{C} : \Lambda$ is a monoid in $\Omega\Lambda_{\mathcal{C}}$. Explicitly, a monad in Λ consists in

- an object $\mathcal{C} : \Lambda$
- a morphism $\mathbb{P} : \mathcal{C} \to \mathcal{C}$ in Λ
- a 2-morphism $\epsilon : I_{\mathcal{C}} \Rightarrow \mathbb{P}$, hence, an element $\epsilon : \Lambda(\mathcal{C}, \mathcal{C})(I_{\mathcal{C}}, \mathbb{P})$
- a 2-morphism $\mu : FF \Rightarrow F$, hence, an element $\mu : \Lambda(\mathcal{C}, \mathcal{C})(\mathbb{P}\mathbb{P}, \mathbb{P})$
- satisfying associativity and unit conditions.

Algebras over a monad. The notion of an algebra over such a monad requires a distinguished object $\star_\Lambda : \Lambda$ so that each object $\mathcal{C} : \Lambda$ has internal objects $X : \mathcal{C}$ given by the morphisms $X : \star_\Lambda \to \mathcal{C}$ in Λ. We can then define an algebra over a monad $\mathbb{P} : \mathcal{C} \to \mathcal{C}$ in Λ as an object $X : \mathcal{C}$ equipped with a morphism $\mathbb{P}X \to X$ in $\mathcal{C}^{\star_\Lambda}$, and a unit $*_{\mathcal{C}} \to X$ provided that $\mathcal{C}$ also has a distinguished object. However, we need stronger notions of completeness are required so that the algebras over $\mathbb{P}$ can be given the structure of an object of Λ. For this reason, we only consider monads internal to $\mathbb{T}_{n+1}$. We will later extend the notion of an algebra over a monad internally in some object by using the notion of morphism of algebras.

Definition 2.3.4. An algebra over a monad $\mathbb{P} : \mathbb{T}_{n+1} \to \mathbb{T}_{n+1}$ is an object $\mathcal{C} : \mathbb{T}_{n+1}$ equipped with

- a morphism $\otimes_{\mathcal{C}} : \mathbb{P}\mathcal{C} \to \mathcal{C}$ in $\mathbb{T}_{n+1}$, called the composition
- a morphism $\epsilon_{\mathcal{C}} : *_n \to \mathcal{C}$, called the unit
- satisfying coherence conditions.

Example 2.3.5. The monad structure on $\mathbb{P} : \mathbb{T}_{n+1} \to \mathbb{T}_{n+1}$ yields a $\mathbb{P}$-algebra structure on $\mathbb{P}\Lambda$ for any $\Lambda : \mathbb{T}_{n+1}$. We say that $\mathbb{P}\Lambda$ is the free $\mathbb{P}$-algebra generated by Λ.

Definition 2.3.6. Suppose that $\mathbb{P} : \mathbb{T}_{n+1} \to \mathbb{T}_{n+1}$ is a monad and let $\mathcal{C}$ and $\mathcal{D}$ be objects of $\mathbb{T}_{n+1}$ equipped with the structure of an algebra over $\mathbb{P}$. Let $\mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}}(\mathcal{C}, \mathcal{D}) : \mathbb{T}_{n+1}$ be such that

- an element $f : \mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}}(\mathcal{C}, \mathcal{D})$ is a morphism $f : \mathcal{C} \to \mathcal{D}$ in $\mathbb{T}_{n+1}$ equipped with
- for $f, g : \mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}}(\mathcal{C}, \mathcal{D})$, we let $\mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}}(\mathcal{C}, \mathcal{D})(f, g) : \mathbb{T}_n$ be such that

DEFINITION 2.3.7. Let $\mathbb{P} : \mathbb{T}_{n+1} \to \mathbb{T}_{n+1}$ be a monad. We let $\mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}} : \mathbb{T}_{n+2}$ be the type of $\mathbb{P}$-algebras in $\mathbb{T}_{n+1}$, whose objects are $\mathbb{P}$-algebras in $\mathbb{T}_{n+1}$, and whose morphisms $\mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}}(\mathcal{C}, \mathcal{D}) : \mathbb{T}_{n+1}$ between $\mathbb{P}$-algebras $\mathcal{C}, \mathcal{D}$ is the element of type $\mathbb{T}_n$ given in Definition 2.3.6.

DEFINITION 2.3.8. Let $\mathbb{P} : \mathbb{T}_{n+1} \to \mathbb{T}_{n+1}$ be a monad and let $\Lambda : \mathbb{T}_{n+1}$ be equipped with the structure of a $\mathbb{P}$-algebra. We let the type $\mathbb{P}\text{-}\mathrm{ALG}_\Lambda : \mathbb{T}_{n+1}$ of $\mathbb{P}$-algebras in Λ be defined as

$$\mathbb{P}\text{-}\mathrm{ALG}_\Lambda := \mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}n+1}(*_n, \Lambda).$$

More generally, $\mathbb{P}$-algebras can be defined internally in any $\mathbb{P}$-algebra through the structural morphism of $\mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}}$. We obtain a functor

$$\mathbb{P}\text{-}\mathrm{ALG}_{\_} : \mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}} \to \mathbb{T}_{n+1}$$

which is precisely given by the functor of points

$$\mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}}(*_n, \_) : \mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}} \to \mathbb{T}_{n+1}.$$

DEFINITION 2.3.9. We let $\mathcal{U}_\mathbb{P} : \mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}} \to \mathbb{T}_{n+1}$ be the functor which maps a $\mathcal{F}$-algebra to its underlying object in $\mathbb{T}_{n+1}$.

2.3.2. *Monads and adjunctions.*

PROPOSITION 2.3.10. *Let $\mathcal{C}, \mathcal{D} : \Lambda$ and suppose that there is an adjunction in Λ*

$$\mathcal{F} : \mathcal{C} \leftrightharpoons \mathcal{D} : \mathcal{G}.$$

The unit and the counit, together with the coherence relations, yield the structure of a monad on the composite morphism $\mathbb{P} := \mathcal{G}\mathcal{F} : \mathcal{C} \to \mathcal{C}$. The composition is given by

$$\mathbb{P}\mathbb{P} = \mathcal{G}\mathcal{F}\mathcal{G}\mathcal{F} \xrightarrow{\mathcal{G}\epsilon\mathcal{F}} \mathcal{G}\mathcal{F} = \mathbb{P}$$

The unit of the adjunction $\eta : I_\mathcal{C} \Rightarrow \mathcal{G}\mathcal{F}$ yields a unit for this composition. In the case where Λ is a well pointed object[3]*, we can directly deal with the equivalence in $\mathbb{T}_n(\mathcal{C}^{op} \times \mathcal{D}, \mathbb{T}_{n-1})$*

$$\mathcal{D}(\mathcal{F}\_, \_) \cong \mathcal{C}(\_, \mathcal{G}\_).$$

We can state a reciprocal as soon as the monad has a well defined object of algebras.

PROPOSITION 2.3.11. *Let $\mathbb{P} : \mathbb{T}_{n+1} \to \mathbb{T}_{n+1}$ be a monad and recall from Example 2.3.5 that the morphism $\mathbb{P}$ factors through its object of algebras. The forgetful functor*

$$\mathcal{U}_\mathbb{P} : \mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}} \to \mathbb{T}_{n+1}$$

yields an adjunction

$$\mathbb{P} : \mathbb{T}_{n+1} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}} : \mathcal{U}_\mathbb{P}.$$

Moreover, the monad induced by this adjunction precisely corresponds to the former monad $\mathbb{P}$.

---

[3]Recall that $\Lambda : \mathbb{T}_{n+1}$ is well pointed if it is equipped with an object $*_\Lambda : \Lambda$ such that the canonical morphism $\Lambda(*_\Lambda, \_) : \Lambda \to \mathbb{T}_n$ is locally fully faithful.

2.3.3. *Closed morphisms.*

DEFINITION 2.3.12. Let Γ, Ψ be closed objects of $\mathbb{T}_{n+1}$. We say that a morphism $F : \Gamma \to \Psi$ is closed if it is equipped with a 2-morphism $[\_,\_]_F$ in $\mathbb{T}_{n+1}$:

$$\begin{tikzcd} \Gamma^{op} \times \Gamma \arrow[r, "F^{op}\times F"] \arrow[d, "{[\_,\_]_\Gamma}"'] & \Psi^{op} \times \Psi \arrow[d, "{[\_,\_]_\Psi}"] \\ \Gamma \arrow[r, "F"'] \arrow[ur, phantom, "\Nearrow {[\_,\_]_F}"] & \Psi. \end{tikzcd}$$

The isomorphism

$$[\Gamma^{op} \times \Gamma, \Psi]_{\mathbb{T}_{n+1}} \left(F\,[\_,\_]_\Gamma\,, [F\_, F\_]_\Psi\right) \xrightarrow{\cong} \int^{\Gamma\times\Gamma^{op}} \Psi\left(F\,[\_,\_]_\Gamma\,, [F\_, F\_]_\Psi\right)$$

then yields morphisms $[X,Y]_F : F\,[X,Y]_\Gamma \to [FX, FY]_\Psi$ in Ψ which are natural in $X, Y : *_{n+1} \to \Gamma$.

LEMMA 2.3.13. *The free $\mathbb{P}$-algebra monad is closed.*

2.3.4. *Generalized Day's convolution product.* Let $\mathcal{F}_\mathbb{P} : \mathbb{T}_{n+1} \to \mathbb{T}_{n+1}$ be a monad and suppose that $\mathbb{T}_n$ has he structure of a $\mathbb{P}$-algebra. Recall that any $\Gamma : \mathbb{T}_{n+1}$ yields a presheaf object $\mathbb{T}_{n+1}(\Gamma^{op}, \mathbb{T}_n) : \mathbb{T}_{n+1}$. We use the closed structure of the free $\mathbb{P}$-algebra monad to give to $[\Gamma^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}}$ the structure of a $\mathbb{P}$-algebra.

The closed structure of $\mathcal{F}_\mathbb{P}$ yields a morphism

$$\mathcal{F}_\mathbb{P}\,[\Gamma^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}} \to [\mathcal{F}_\mathbb{P}\Gamma^{op}, \mathcal{F}_\mathbb{P}\mathbb{T}_n]_{\mathbb{T}_{n+1}}\,.$$

The $\mathbb{P}$-algebra structure on $\mathbb{T}_n$ yields a morphism

$$\mathcal{F}_\mathbb{P}\,[\Gamma^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}} \to [\mathcal{F}_\mathbb{P}\Gamma^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}}$$

by composition with the structural morphism of $\mathbb{P}$-algebras $\otimes^{\mathbb{P}}_{\mathbb{T}_n} : \mathcal{F}_\mathbb{P}\mathbb{T}_n \to \mathbb{T}_n$.

PROPOSITION 2.3.14. *Suppose that $\Gamma : \mathbb{T}_{n+1}$ has the structure of a $\mathbb{P}$-algebra. The left Kan extension along the opposite of the structural morphism $\otimes^{\mathbb{P}}_{\Gamma} : \mathcal{F}_\mathbb{P}\Gamma \to \Gamma$ yields a $\mathbb{P}$-algebra structure on the presheaf object $[\Gamma^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}}$. The structural morphism is explicitly given by the composite*

$$\mathcal{F}_\mathbb{P}\,[\Gamma^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}} \to [\mathcal{F}_\mathbb{P}\Gamma^{op}, \mathcal{F}_\mathbb{P}\mathbb{T}_n]_{\mathbb{T}_{n+1}} \xrightarrow{\left[\mathcal{F}_\mathbb{P}\Gamma^{op}, \otimes^{\mathbb{P}\mathbb{T}_n}\right]} [\mathcal{F}_\mathbb{P}\Gamma^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}} \xrightarrow{Lan_{\otimes^{\mathbb{P}}_{\Gamma}}} [\Gamma^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}}\,.$$

REMARK 2.3.15.

PROPOSITION 2.3.16. *The presheaf functor therefore restricts to $\mathbb{P}$-algebras, and hence yields a functor*

$$[\_^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}} : \mathbb{P}\text{-}\mathrm{ALG}^{op}_{\mathbb{T}_{n+1}} \to \mathbb{P}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}}.$$

The $\mathbb{P}$-algebra structure thus obtained on the presheaf object of a $\mathbb{P}$-algebra in $\mathbb{T}_{n+1}$ is universal among those providing the Yoneda embedding

$$\Gamma \to [\Gamma^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}}$$

with the structure of a morphism of $\mathbb{P}$-algebras.

2.4. Operads and monads. In this section, we place at level $n+1$ and consider a monad

$$\mathcal{F}_{\mathbb{P}_{n+1}} : \mathbb{T}_{n+1} \to \mathbb{T}_{n+1}$$

We assume that $\mathbb{T}_n : \mathbb{T}_{n+1}$ has the structure of a $\mathbb{P}_{n+1}$-algebra in $\mathbb{T}_{n+1}$.

2.4.1. *Exponentiation and free algebras.*

Definition 2.4.1. We let $\mathbb{P}_n : \mathbb{T}_{n+1}$ be the free $\mathbb{P}_{n+1}$-algebra in $\mathbb{T}_{n+1}$ generated by the terminal object $*_n$, equivalently given by the image of $*_n$ under $\mathcal{F}_{\mathbb{P}_{n+1}}$, so that

$$\mathbb{P}_n : *_{n+1} \xrightarrow{*_n} \mathbb{T}_{n+1} \xrightarrow{\mathcal{F}_{\mathbb{P}_{n+1}}} \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_{\mathbb{T}_{n+1}}.$$

Definition 2.4.2. Let $\Gamma : \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_{\mathbb{T}_{n+1}}$. We let the $\mathbb{P}_n$-exponentiation be the morphism in $\mathbb{T}_{n+1}$

$$(\_)^{\otimes^{\mathbb{P}_n}} : \Gamma \to \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_{\mathbb{T}_{n+1}}(\mathbb{P}_n, \Gamma)$$

obtained by using the adjunction of Definition 2.3.11. The exponentiation is explicitly given by the following composite in $\mathbb{T}_{n+1}$

$$\Gamma \xrightarrow{\cong} \mathbb{T}_{n+1}(*_n, \Gamma) \xrightarrow{\cong} \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_{\mathbb{T}_{n+1}}(\mathbb{P}_n, \Gamma).$$

Let $p : \mathbb{P}_n$ and $X : \Gamma$. We write $X^{\otimes^p} : \Gamma$ for the corresponding object of Γ.

Remark 2.4.3. Let Γ be a $\mathbb{P}_{n+1}$-algebra in $\mathbb{T}_{n+1}$ and let $\otimes_\Gamma^{\mathbb{P}_{n+1}} : \mathcal{F}_{\mathbb{P}_{n+1}}\Gamma \to \Gamma$ be the morphism in $\mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_{\mathbb{T}_{n+1}}$ provided by this $\mathbb{P}_{n+1}$-algebra structure. The functor $\mathcal{F}_{\mathbb{T}_{n+1}} : \mathbb{T}_{n+1} \to \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_{\mathbb{T}_{n+1}}$ yields a functor at the level of morphisms

$$\mathbb{T}_{n+1}(*_n, \Gamma) \to \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_{\mathbb{T}_{n+1}}(\mathcal{F}_{\mathbb{P}_{n+1}}*_n, \mathcal{F}_{\mathbb{P}_{n+1}}\Gamma).$$

We therefore obtain a morphism

$$\Gamma \to \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_{\mathbb{T}_{n+1}}(\mathbb{P}_n, \Gamma)$$

which precisely corresponds to the exponentiation given in Definition 2.4.2. The $\mathbb{P}_n$- exponentiation of an object X of Γ is therefore given by

$$X^{\otimes^{\mathbb{P}_n}} \cong \otimes_\Gamma^{\mathbb{P}_{n+1}} \mathcal{F}_{\mathbb{P}_{n+1}} X.$$

Example 2.4.4. The $\mathbb{P}_{n+1}$-algebra structure on $\mathbb{T}_n$ yields the exponentiation

$$(\_)^{\otimes^{\mathbb{P}_n}} : \mathbb{T}_n \to \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_{\mathbb{T}_{n+1}}(\mathbb{P}_n, \mathbb{T}_n).$$

Proposition 2.4.5. *Let Γ be a $\mathbb{P}_{n+1}$-algebra in $\mathbb{T}_{n+1}$ and let $X : \Gamma$. The left Kan extension of the unique morphism $* : \mathbb{P}_n \to *_n$ in $\mathbb{T}_{n+1}$ yields a morphism*

$$\mathcal{F}^\Gamma_{\mathbb{P}_{n+1}} : \Gamma \xrightarrow{\_^{\otimes^{\mathbb{P}_n}}} \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_{\mathbb{T}_{n+1}}(\mathbb{P}_n, \Gamma) \xrightarrow{Lan_*} \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_{\mathbb{T}_{n+1}}(*_n, \Gamma) = \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_\Gamma$$

which is left adjoint to the forgetful functor

$$\mathcal{U}^\Gamma_{\mathbb{P}_{n+1}} : \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_\Gamma \to \Gamma.$$

For any object $X : \Gamma$, we therefore obtain a $\mathbb{P}_n$-algebra structure on $F^\Gamma_{\mathbb{P}_{n+1}} X$, which satisfies

$$\mathcal{F}^\Gamma_{\mathbb{P}_{n+1}} X \cong \int_{\mathbb{P}_n} * \times X^{\otimes^{\mathbb{P}_n}}.$$

We say that $\mathcal{F}^\Gamma_{\mathbb{P}_{n+1}} X$ is the free $\mathbb{P}_{n+1}$-algebra generated by X in Γ.

Proof. The universal property of the left Kan extension yields isomorphisms

$$\begin{aligned} \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_\Gamma(Lan_* X^{\otimes^{\mathbb{P}_n}}, Y) &\cong \mathbb{P}_{n+1}\text{-}\mathrm{A}\textsc{lg}_{\mathbb{T}_{n+1}}(\mathbb{P}_n, \Gamma)\left(X^{\otimes^{\mathbb{P}_n}}, \overline{Y}\right) \\ &\cong \Gamma(X, Y) \end{aligned}$$

which are natural in $X : \Gamma$ and $Y : \mathbb{P}_{n+1}\text{-}\mathrm{ALG}_\Gamma$. The formula $\mathcal{F}^\Gamma_{\mathbb{P}_{n+1}} X \cong \int_{\mathbb{P}_n} * \times X^{\otimes^{\mathbb{P}_n}}$ is a direct consequence of the expression of left Kan extensions in terms of coends. □

2.4.2. *On $\mathbb{P}_n$-sequences.* We only treat the case of $\mathbb{P}_n$-sequences with values in the canonical object $\mathbb{T}_n$ but also have the more general notion of $\mathbb{P}_n$-sequences in a $\mathbb{P}_{n+1}$-algebra Γ. In order to have the same results on $\mathbb{P}_n$-sequences with values in a $\mathbb{P}_{n+1}$-algebra Γ, this object must have similar notions of completeness that $\mathbb{T}_n$. The case of sequences with values in some Γ can be obtained by using a different hierarchic system of types with Γ instead of $\mathbb{T}_n$.

Definition 2.4.6. We let the type of $\mathbb{P}_n$-sequences in $\mathbb{T}_n$ be the type of presheaves $[\mathbb{P}_n^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}}$. According to Proposition 2.3.14, the generalized Day convolution product yields a $\mathbb{P}_{n+1}$-algebra structure on the type of $\mathbb{P}_n$-sequences in $\mathbb{T}_n$.

Remark 2.4.7. The $\mathbb{P}_{n+1}$-algebra structure on $\mathbb{P}_n$-sequences obtained by Day's convolution yields the exponentiation of $\mathbb{P}_n$-sequences

$$(\_)^{\otimes^{\mathbb{P}_n}} : [\mathbb{P}_n^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}} \to \mathbb{P}_{n+1}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}} \left(\mathbb{P}_n, [\mathbb{P}_n^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}}\right).$$

Let $\mathcal{P} : \mathbb{P}_n^{op} \to \mathbb{T}_n$. The exponentiation of $\mathcal{P}$ is the morphism

$$\mathcal{P}^{\otimes^{\mathbb{P}_{n+1}}} : \mathbb{P}_n \to [\mathbb{P}_n^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}}$$

given in $p : \mathbb{P}_n$ by the presheaf

$$\mathcal{P}^{\otimes^p} \cong \int_{\bar{q}:\mathbb{P}_n^{\otimes^p_{\mathbb{T}_{n+1}}}} \mathbb{P}_n(\_, \otimes^p_{\mathbb{P}_n} \bar{q}) \times \otimes^p_{\mathbb{T}_n} \mathcal{P}^{\otimes^p_{\mathbb{T}_{n+1}}} \bar{q} \quad : \; \mathbb{P}_n^{op} \to \mathbb{T}_n.$$

Definition 2.4.8. Let Γ be a $\mathbb{P}_{n+1}$-algebra in $\mathbb{T}_{n+1}$. Each object X of Γ yields a $\mathbb{P}_n$-sequence

$$\Gamma\left(X^{\otimes^{\mathbb{P}_n}}, X\right) : \mathbb{P}_n^{op} \to \mathbb{T}_n$$

obtained by the composite

$$\mathbb{P}_n^{op} \xrightarrow{\cong} \mathbb{P}_n^{op} \times *_n \xrightarrow{\left(X^{\otimes^{\mathbb{P}_n}}\right)^{op} \times X} \Gamma^{op} \times \Gamma \xrightarrow{\Gamma(\_,\_)} \mathbb{T}_n.$$

Definition 2.4.9. Let $*_{\mathbb{P}_n} : \mathbb{P}_n^{op} \to \mathbb{T}_n$ be the constant $\mathbb{P}_n$-sequence given by the terminal object $*_{n-1} : \mathbb{T}_n$, obtained by the composite

$$\mathbb{P}_n^{op} \to *_n \xrightarrow{*_{n-1}} \mathbb{T}_n.$$

Note that $*_{\mathbb{P}_n}$ effectively is terminal in $[\mathbb{P}_n^{op}, \mathbb{T}_n]_{\mathbb{T}_{n+1}}$.

2.4.3. *On $\mathbb{P}_n$-operads.*

Operads as monoids.

Definition 2.4.10. Let $\mathbb{I} : \mathbb{P}_n^{op} \to \mathbb{T}_n$ be the $\mathbb{P}_n$-sequence obtained as the left Kan extension of the unit of $\mathbb{T}_n$ for its $\mathbb{P}_{n+1}$-algebra structure along the unit of $\mathbb{P}_n$ for its $\mathbb{P}_{n+1}$-algebra structure. We say that $\mathbb{I}$ is the unit $\mathbb{P}_n$-sequence.

Definition 2.4.11. Let $\circ : [\mathbb{P}_n^{op}, \mathbb{T}_n] \times [\mathbb{P}_n^{op}, \mathbb{T}_n] \to [\mathbb{P}_n^{op}, \mathbb{T}_n]$ be defined as the composite

$$[\mathbb{P}_n^{op}, \mathbb{T}_n] \times [\mathbb{P}_n^{op}, \mathbb{T}_n] \xrightarrow{\_^{\otimes^{\mathbb{P}_n}}} [\mathbb{P}_n^{op}, \mathbb{T}_n] \times [\mathbb{P}_n, [\mathbb{P}_n^{op}, \mathbb{T}_n]] \to [\mathbb{P}_n^{op} \times \mathbb{P}_n, [\mathbb{P}_n^{op}, \mathbb{T}_n]] \xrightarrow{\int_{\mathbb{P}_n}} [\mathbb{P}_n^{op}, \mathbb{T}_n].$$

We say that $\circ$ is the substitution product of $\mathbb{P}_n$-sequences. Let $\mathcal{P}, \mathcal{Q} : \mathbb{P}_n^{op} \to \mathbb{T}_n$, we explicitly have

$$\mathcal{P} \circ \mathcal{Q} = \int_{\mathbb{P}_n} \mathcal{P} \times \mathcal{Q}^{\otimes^{\mathbb{P}_n}}.$$

REMARK 2.4.12. The $\mathbb{P}_n$-sequence $\mathcal{P} \circ \mathcal{Q}$ can be explicitly given in some $q : \mathbb{P}_n$ by

$$\mathcal{P} \circ \mathcal{Q}(q) \cong \int_{p:\mathbb{P}_n} \int_{\bar{p}:\mathbb{P}_n^{\otimes^p}} \mathbb{P}_n(q, \otimes_{\mathbb{P}_n}^p \bar{p}) \times \mathcal{P}(p) \times \otimes_{\mathbb{T}_n}^p \mathcal{Q}^{\otimes^p} \bar{p}$$

PROPOSITION 2.4.13. *The substitution product of $\mathbb{P}_n$-sequences in $\mathbb{T}_n$ yields a monoidal structure on $\mathbb{P}_n$-sequences whose unit is given by the unit $\mathbb{P}_n$-sequence $\mathbb{I}$ defined in 2.4.10.*

DEFINITION 2.4.14. We let $\mathbb{P}_n\text{-}\mathrm{OP}_{\mathbb{T}_n}$ be the element of type $\mathbb{T}_{n+1}$ defined as

$$\mathbb{P}_n\text{-}\mathrm{OP}_{\mathbb{T}_n} := \mathrm{MON}_{(\mathbb{T}_{n+1},\times)}\left(\left[\mathbb{P}_n^{op}, \mathbb{T}_n\right], \circ, \mathbb{I}\right)$$

EXAMPLE 2.4.15. The terminal sequence $*_{\mathbb{P}_n} : \mathbb{P}_n^{op} \to \mathbb{T}_n$ trivially has a $\mathbb{P}_n$-operad structure.

PROPOSITION 2.4.16. *Let Γ be a $\mathbb{P}_{n+1}$-algebra in $\mathbb{T}_{n+1}$ and let X be an object of Γ. The endomorphism sequence*

$$\Gamma\left(X^{\otimes^{\mathbb{P}_n}}, X\right) : \mathbb{P}_n^{op} \to \mathbb{T}_n$$

defined in 2.4.8 can be given the structure of a $\mathbb{P}_n$-operad in $\mathbb{T}_n$.

PROOF. We use the closed structure of the exponentiation and the composition of morphisms in Γ and obtain the operadic composition. The identity of X yields a unit for this product. The associativity is ensured by the associativity of the composition of morphisms in Γ and the associativity of the exponentiation (itself ensured by its universal property). $\square$

Operads as algebras over a monad. The notion of an operad could be more accurately stated without involving the notion of a monoid. For this purpose, we need to endow the object of $\mathbb{P}_n$-sequences in $\mathbb{T}_n$ with the structure of an operad in $\mathbb{P}_{n+2}$. In this way, it will be possible to define an operad as an operad internal to this operad.

2.4.4. *Algebras over $\mathbb{P}_n$-operads.* Let Γ be a $\mathbb{P}_{n+1}$-algebra in $\mathbb{T}_{n+1}$ and let $\mathcal{P}$ be a $\mathbb{P}_n$-operad in $\mathbb{T}_n$.

DEFINITION 2.4.17. A $\mathcal{P}$-algebra in Γ is an object $X : \Gamma$ equipped with a morphism of operads $\otimes_X^{\mathcal{P}} : \mathbb{P}_n\text{-}\mathrm{OP}_{\mathbb{T}_n}\left(\mathcal{P}, \Gamma\left(X^{\otimes^{\mathbb{P}_n}}, X\right)\right)$. We write $X : \mathcal{P}\text{-}\mathrm{ALG}_\Gamma$.

DEFINITION 2.4.18. Let $X, Y : \Gamma$ and suppose that X and Y are equipped with the structure of a $\mathcal{P}$-algebra. We say that a morphism $F : X \to Y$ in Γ is a morphism of $\mathcal{P}$-algebras and we write $F : \mathcal{P}\text{-}\mathrm{ALG}_\Gamma(X, Y)$ if it is equipped with a natural transformation $\otimes_F^{\mathcal{P}}$ such that

$$\begin{tikzcd}
 & \Gamma\left(X^{\otimes_{\mathbb{P}_n}}, X\right) \arrow[dr, "\Gamma\left(X^{\otimes_{\mathbb{P}_n}}, F\right)"] & \\
\mathcal{P} \arrow[ur, "\otimes_X^{\mathcal{P}}"] \arrow[dr, "\otimes_Y^{\mathcal{P}}"'] & \Uparrow \otimes_F^{\mathcal{P}} & \Gamma\left(X^{\otimes_{\mathbb{P}_n}}, Y\right) \\
 & \Gamma\left(Y^{\otimes_{\mathbb{P}_n}}, Y\right) \arrow[ur, "\Gamma\left(F^{\otimes_{\mathbb{P}_n}}, Y\right)"'] &
\end{tikzcd}$$

and which satisfy the coherence constraints expressed by

If F, G are morphisms of $\mathcal{P}$-algebras in Γ, we let $\mathcal{P}\text{-}\mathrm{A}{\scriptstyle\mathrm{LG}}_\Gamma(X,Y)(F,G) : \mathbb{T}_{n-1}$ be given by

We obtain an element $\mathcal{P}\text{-}\mathrm{A}{\scriptstyle\mathrm{LG}}_\Gamma : \mathbb{T}_{n+1}$.

The following proposition establishes a link between the algebras internal to a $\mathbb{P}_{n+1}$-algebra Γ and the algebras over the terminal $\mathbb{P}_n$-operad.

Proposition 2.4.19. *The algebras in Γ over the terminal operad $*_{\mathbb{P}_n}$ precisely correspond to $\mathbb{P}_n$-algebras in Γ, so that we have an isomorphism*

$$*_{\mathbb{P}_n}\text{-}\mathrm{A}{\scriptstyle\mathrm{LG}}_\Gamma \cong \mathbb{P}_n\text{-}\mathrm{A}{\scriptstyle\mathrm{LG}}_\Gamma = \mathbb{P}_{n+1}\text{-}\mathrm{A}{\scriptstyle\mathrm{LG}}(*_n, \Gamma).$$

Proof. First recall that a $\mathbb{P}_{n+1}$-algebra internal to Γ is an object $X : \Gamma$ equipped with a morphism

$$\begin{tikzcd}
\mathbb{P}_n \arrow[r, "\mathcal{F}_{\mathbb{P}_{n+1}} X"] \arrow[d, "*"'] \arrow[dr, phantom, "\Leftarrow \otimes_X^{\mathbb{P}_{n+1}}"] & \mathcal{F}_{\mathbb{P}_{n+1}} \Gamma \arrow[d, "\otimes_\Gamma^{\mathbb{P}_{n+1}}"] \\
*_n \arrow[r, "X"'] & \Gamma
\end{tikzcd}$$

such that $\otimes_X^{\mathbb{P}_{n+1}}$ has the structure of a morphism of $\mathbb{P}_{n+1}$-algebras. We have isomorphisms

$$\begin{aligned}
\mathbb{T}_{n+1}\left(\mathbb{P}_n, \Gamma\right)\left(\otimes_\Gamma^{\mathbb{P}_{n+1}} \mathcal{F}_{\mathbb{P}_{n+1}} X, \overline{X}\right) &\cong \int^{p:\mathbb{P}_n} \Gamma\left(X^{\otimes_\Gamma^p}, X\right) \\
&\cong \int^{p:\mathbb{P}_n^{op}} \mathbb{T}_{n+1}\left(*, \Gamma\left(X^{\otimes_\Gamma^p}, X\right)\right) \\
&\cong \left[\mathbb{P}_n^{op}, \mathbb{T}_n\right]\left(*_{\mathbb{P}_n}, \Gamma\left(X^{\otimes_\Gamma^p}, X\right)\right)
\end{aligned}$$

A morphism is a morphism of $\mathbb{P}_{n+1}$-algebras if and only if the corresponding morphism $*_{\mathbb{P}_n} \to \Gamma\left(X^{\otimes_\Gamma^p}, X\right)$ under the aforementioned equivalence, is a morphism of $\mathbb{P}_n$-operads. We therefore obtain an equivalence

$$\mathbb{P}_{n+1}\text{-}\mathrm{A}{\scriptstyle\mathrm{LG}}_{\mathbb{T}_{n+1}}\left(\mathbb{P}_n, \Gamma\right)\left(\otimes_\Gamma^{\mathbb{P}_{n+1}} \mathcal{F}_{\mathbb{P}_{n+1}} X, \overline{X}\right) \cong \mathbb{P}_n\text{-}\mathrm{O}{\scriptstyle\mathrm{P}}_{\mathbb{T}_n}\left(*_{\mathbb{P}_n}, \Gamma\left(X^{\otimes_\Gamma^p}, X\right)\right),$$

hence the result. $\square$

Corollary 2.4.20. *We have an equivalence*

$$\mathbb{P}_n\text{-}\mathrm{A}{\scriptstyle\mathrm{LG}}_{\mathbb{T}_n} \cong \mathbb{P}_{n+1}\text{-}\mathrm{A}{\scriptstyle\mathrm{LG}}_{\mathbb{T}_{n+1}}(*_n, \mathbb{T}_n)$$

Free $\mathcal{P}$-algebras. The proof of Proposition 2.4.19 suggests a candidate for an explicite expression of the free $\mathbb{P}_n$-algebra generated by an object of Γ. We actually have the following general result on free $\mathcal{P}$-algebras for an operad $\mathcal{P}$.

PROPOSITION 2.4.21. *Let $\Gamma : \mathbb{P}_{n+1}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}}$ and suppose that Γ is tensored over $\mathbb{T}_n$. There is an isomorphism*

$$[\mathbb{P}_n^{op}, \mathbb{T}_n]\left(\mathcal{P}, \Gamma\left(X^{\otimes^{\mathbb{P}_n}}, X\right)\right) \xrightarrow{\cong} \Gamma\left(\int_{\mathbb{P}_n} \mathcal{P} \times X^{\otimes_{\mathbb{P}_n}}, X\right).$$

Moreover, a morphism $\mathcal{P} \to \Gamma\left(X^{\otimes^{\mathbb{P}_n}}, X\right)$ of $\mathbb{P}_n$-sequences is a morphism of operads if and only if the corresponding morphism $\int_{\mathbb{P}_n} \mathcal{P} \times X^{\otimes_{\mathbb{P}_n}} \to X$ in Γ yields a $\mathcal{P}$-algebra structure on X.

PROOF. The result follows from the isomorphisms

$$\begin{aligned}
[\mathbb{P}_n^{op}, \mathbb{T}_n]\left(\mathcal{P}, \Gamma\left(X^{\otimes^{\mathbb{P}_n}}, X\right)\right) &\xrightarrow{\cong} \int^{p:\mathbb{P}_n} \mathbb{T}_n\left(\mathcal{P}(p), \Gamma\left(X^{\otimes^p}, X\right)\right) \\
&\xrightarrow{\cong} \int^{p:\mathbb{P}_n} \Gamma\left(\mathcal{P}(p) \cdot X^{\otimes^p}, X\right) \\
&\xrightarrow{\cong} \Gamma\left(\int_{p:\mathbb{P}_n} \mathcal{P}(p) \cdot X^{\otimes^p}, X\right).
\end{aligned}$$

□

We obtain the following corollary, which generalizes Proposition 2.4.5.

COROLLARY 2.4.22. *The forgetful functor $\mathcal{U}_{\mathcal{P}} : \mathcal{P}\text{-}\mathrm{ALG}_\Gamma \to \Gamma$ has a left adjoint*

$$\mathcal{F}_{\mathcal{P}} : \Gamma \to \mathcal{P}\text{-}\mathrm{ALG}_\Gamma.$$

The underlying object of the image of $X : \Gamma$ by $\mathcal{F}_{\mathcal{P}}$ is given by

$$\mathcal{F}_{\mathcal{P}} X = \int_{p:\mathbb{P}_n} \mathcal{P}(p) \times X^{\otimes^p_\Gamma}.$$

2.4.5. *Free $\mathbb{P}_n$-operads.* The characterization of operads in terms of monoids makes the construction of free operads difficult to manage. In this section, we use the free operad generated by the terminal sequence to provide an easier description of the free operad generated by a $\mathbb{P}_n$-sequence. For this purpose, we assume that the free $\mathbb{P}_{n+1}$-algebra monad

$$\mathcal{F}_{\mathbb{P}_{n+1}} : \mathbb{T}_{n+1} \to \mathbb{T}_{n+1}$$

follows the same scheme that the one we established in the previous section. In particular, we assume $\mathbb{T}_{n+1}$ to have the structure of a $\mathbb{P}_{n+2}$-algebra in $\mathbb{T}_{n+2}$. According to the microcosm principle, we precisely assume that the hierarchic type system in which we we work satisfies

$$\mathbb{T}_n : \mathbb{P}_{n+1}\text{-}\mathrm{ALG}_{\mathbb{T}_{n+1}}.$$

In the same way, the element $\mathbb{P}_{n+1} : \mathbb{T}_{n+2}$ refers to the free $\mathbb{P}_{n+2}$-algebra generated by the terminal object $*_{n+2}$ in $\mathbb{T}_{n+2}$. We obtain similar notions of $\mathbb{P}_{n+1}$-operad and algebras for each level. We still assume that the $\mathbb{P}_{n+1}$-algebra structure on $\mathbb{T}_n$ in $\mathbb{T}_{n+1}$ is compatible with its cartesian monoidal structure.

In particular, we obtain from Proposition 2.4.5 that the object $\mathbb{P} : \mathbb{T}_{n+1}$ satisfies

$$\mathbb{P}_n \cong \int_{p:\mathbb{P}_{n+1}} *_n \times *_n^{\otimes^p} \cong \int_{p:\mathbb{P}_n+1} *_n.$$

LEMMA 2.4.23. *Let $\Lambda, \Gamma : \mathbb{T}_{n+1}$. The constant presheaves defined in 2.1.6 $\overline{\Lambda}, \overline{\Gamma} : \mathbb{P}_{n+1}^{op} \to \mathbb{T}_{n+1}$ satisfy*

$$\left[\mathbb{P}_{n+1}^{op}, \mathbb{T}_{n+1}\right](\overline{\Lambda}, \overline{\Gamma}) \xrightarrow{\cong} \int^{\mathbb{P}_{n+1}} \mathbb{T}_{n+1}(\Lambda, \Gamma) \xrightarrow{\cong} \left[\mathbb{P}_n^{op}, \mathbb{T}_{n+1}^{op}(\Lambda, \Gamma)\right].$$

PROOF. $\mathbb{P}_n \xrightarrow{\cong} \Pi \mathbb{P}_{n+1}$. □

We obtain the following as a direct consequence.

LEMMA 2.4.24. *Let $\Gamma : \mathbb{T}_{n+1}$ be regarded as a constant presheaf $\overline{\Gamma} : \mathbb{P}_n^{op} \to \mathbb{T}_{n+1}$. The end $\int^{p:\mathbb{P}_{n+1}} *_n \times \Gamma$, where $\bar{*}_n$ is seen as a constant presheaf as well, satisfies*

$$\int^{p:\mathbb{P}_{n+1}} *_n \times \Gamma \xrightarrow{\cong} [\mathbb{P}_n^{op}, \Gamma].$$

PROOF. We proceed inductively as follows. We have

$$- \ X : \int^{p:\mathbb{P}_{n+1}} \Gamma \Leftrightarrow \left(p : \mathbb{P}_{n+1} \Rightarrow X_p : \Gamma; p, q : \mathbb{P}_{n+1} \Rightarrow X_{p,q} : \int^{\mathbb{P}(p,q)} \mathbb{T}_n(X_p, X_q)\right)$$
$$\Leftrightarrow \quad M : [\mathbb{P}_n^{op}, \mathbb{T}_n]$$
$$- \ X, Y : \int^{p:\mathbb{P}_{n+1}} \Gamma \Rightarrow \left(\left(\int^{p:\mathbb{P}_{n+1}} \Gamma\right)(X, Y) \cong \int\right)$$

We also have

$$\int^{p:\mathbb{P}_{n+1}} \Gamma \xrightarrow{\cong} \left[\mathbb{P}_{n+1}^{op} \times \mathbb{P}_{n+1}, \mathbb{T}_{n+1}\right](\mathbb{P}_{n+1}(\_, \_), \Gamma)$$

□

PROPOSITION 2.4.25. *The $\mathbb{P}_{n+1}$-algebra structure on $\mathbb{T}_n$ gives to the associated constant $\mathbb{P}_{n+1}$-sequence $\overline{\mathbb{T}_n} : \mathbb{P}_{n+1}^{op} \to \mathbb{T}_{n+1}$ the structure of an operad.*

PROOF. The exponentiation of $\overline{\mathbb{T}_n}$ satisfies

$$\overline{\mathbb{T}_n}^{\otimes^p} \cong \int_{\bar{q}:\mathbb{P}_{n+1}{}^{\otimes^p}} \mathbb{P}_{n+1}(\_, p \cdot \bar{q}) \cdot \otimes^p_{\mathbb{T}_{n+1}} \mathbb{T}_n.$$

The $\mathbb{P}_{n+1}$-algebra structure on $\mathbb{T}_n$ yields a morphism

$$\otimes^{\mathbb{P}_{n+1}}_{\mathbb{T}_n} : \int_{p:\mathbb{P}_{n+1}} \otimes^p_{\mathbb{T}_{n+1}} \mathbb{T}_n \to \mathbb{T}_n.$$

The projection $\int_{\bar{q}:\mathbb{P}_{n+1}{}^{\otimes^p}} \mathbb{P}_{n+1}(\_, p.\bar{q}) \to *$ in each $p : \mathbb{P}_{n+1}$, together with the $\mathbb{P}_{n+1} \otimes \mathbb{M}$ - algebra structure on $\mathbb{T}_n$ then yields a product

$$\begin{aligned}
\overline{\mathbb{T}_n} \circ \overline{\mathbb{T}_n} &\xrightarrow{\cong} \int_{p:\mathbb{P}_{n+1}} \int_{\bar{q}:\mathbb{P}_{n+1}{}^{\otimes^p}} \mathbb{P}_{n+1}(_, p.\bar{q}) \cdot \mathbb{T}_n \times \otimes^p_{\mathbb{T}_{n+1}} \mathbb{T}_n \\
&\to \int_{p:\mathbb{P}_{n+1}} \mathbb{T}_n \times \otimes^p_{\mathbb{T}_{n+1}} \mathbb{T}_n \to \mathbb{T}_n \times \mathbb{T}_n \xrightarrow{\times_{\mathbb{T}_n}} \mathbb{T}_n \\
&\xrightarrow{\cong} \overline{\mathbb{T}_n}.
\end{aligned}$$

□

PROPOSITION 2.4.26. *The data of a morphism* $*_{\mathbb{P}_{n+1}} \to \overline{\mathbb{T}_n}$ *in* $\left[\mathbb{P}_{n+1}^{op}, \mathbb{T}_{n+1}\right]$ *is equivalent to the data of a morphism* $\mathbb{P}_n^{op} \to \mathbb{T}_n$. *This equivalence extends to an equivalence*

$$[\mathbb{P}_n^{op}, \mathbb{T}_n] \cong \left[\mathbb{P}_{n+1}^{op}, \mathbb{T}_{n+1}\right]\left(*_{\mathbb{P}_{n+1}}, \overline{\mathbb{T}_n}\right).$$

PROOF. We have isomorphisms

$$\left[\mathbb{P}_{n+1}^{op}, \mathbb{T}_{n+1}\right]\left(*_{\mathbb{P}_{n+1}}, \overline{\mathbb{T}_n}\right) \xrightarrow{\cong} \int^{p:\mathbb{P}_{n+1}} \mathbb{T}_{n+1}\left(*_n, \mathbb{T}_n\right) \xrightarrow{\cong} \int^{p:\mathbb{P}_{n+1}} \mathbb{T}_n.$$

The results follows from Lemma 2.4.24. □

DEFINITION 2.4.27. We let $\mathcal{F}_n^{\mathbb{P}} : \mathbb{P}_n\text{-}\mathrm{OP}_{\mathbb{T}_n}$ be the free operad generated with the terminal $\mathbb{P}_n$ - sequence. We say that $\mathcal{F}_n^{\mathbb{P}}$ is the operad of $\mathbb{P}$-trees in $\mathbb{T}_n$.

PROPOSITION 2.4.28. *There is an isomorphism*

$$\mathbb{P}_{n+1}\text{-}\mathrm{OP}_{\mathbb{T}_{n+1}}\left(\overline{*}_n, \overline{\mathbb{T}}_n\right) \xrightarrow{\cong} \mathbb{P}_n\text{-}\mathrm{OP}_{\mathbb{T}_n}.$$

COROLLARY 2.4.29. *The free operad functor factors as*

$$\begin{aligned}\mathbb{T}_{n+1}\left(\mathbb{P}_n^{op}, \mathbb{T}_n\right) &\xrightarrow{\cong} \left[\mathbb{P}n+1^{op}, \mathbb{T}_{n+1}\right]\left(\overline{*}_n, \overline{\mathbb{T}}_n\right) \xrightarrow{\cong} \mathbb{P}_{n+1}\text{-}\mathrm{OP} - \mathbb{T}_{n+1}\left(\mathbb{F}_{\mathbb{P}_n}, \overline{\mathbb{T}}_n\right)\\ &\xrightarrow{Lan_{(\mathbb{F}_{\mathbb{P}_n} \twoheadrightarrow *_n)}} \mathbb{P}_{n+1}\text{-}\mathrm{OP}_{\mathbb{T}_{n+1}}\left(\overline{*}_n, \overline{\mathbb{T}}_n\right) \xrightarrow{\cong} \mathbb{P}_n\text{-}\mathrm{OP}_{\mathbb{T}_n}.\end{aligned}$$

EXAMPLE 2.4.30. The free monoid monad yields usual non symmetric operads, which may be regarded as monoids internal to the monoidal category of $\mathbb{N}$-sequences. The free $\mathbb{N}$ - operad generated by the terminal sequence precisely gives the operad of non symmetric trees.

REMARK 2.4.31. More generally, for any $\mathbb{P}_{n+1} \otimes_{\mathrm{BV}} \mathbb{M}^{\times}$ - algebra Γ, we let $\mathbb{F}_n^{\Gamma}$ be the operad of $\mathbb{P}_n$-trees in Γ, freely generated by the terminal sequence in Γ. As usual, we only treat the (non restrictive) case where Γ is $\mathbb{T}_n$.

DEFINITION 2.4.32. We also let $\mathbb{F}_n : \mathbb{T}_{n+1}$ be the left Kan extension of $\mathcal{F}_n^{\mathbb{P}}$ along the terminal morphism,

$$\begin{tikzcd} \mathbb{P}_{n+1}^{op} \arrow[r, "\mathcal{F}_n^{\mathbb{P}}"] \arrow[d, "*"'] & \mathbb{T}_{n+1} \\ *_n \arrow[ur, dashed, "Lan_* \mathcal{F}_n^{\mathbb{P}}"'] & \end{tikzcd}$$

which is hence explicitly given by

$$\mathbb{F}_n \cong \int_{p:\mathbb{P}_{n+1}} \mathcal{F}_n^{\mathbb{P}}(p).$$

Any element $T : \mathbb{F}_n$ hence corresponds to an element $T : \mathcal{F}_n^{\mathbb{P}}(p)$ for some $p_T : \mathbb{P}_{n+1}$. We refer to $p_T : \mathbb{P}_{n+1}$ as the arity of T.

The following is straightforward.

PROPOSITION 2.4.33. *The mapping* $T : \mathbb{F}_n \mapsto p_T : \mathbb{P}_{n+1}$ *extends to a functor*

$$p : \mathbb{F}_n \to \mathbb{P}_n.$$

Definition 2.4.34. Each $p : \mathbb{P}_n$ yields a morphism $\mathcal{F}_{\mathbb{P}_n} \twoheadrightarrow *_n$ which corresponds to the composition in the terminal operad $*_{\mathbb{P}_n}$. The coend of this morphism of operads yields a morphism

$$\int_{p:\mathbb{P}_{n+1}} \mathcal{F}_{\mathbb{P}_n} \cong \mathbb{F}_{\mathbb{P}_n} \to \int_{p:\mathbb{P}_{n+1}} *_n \cong \mathbb{P}_n$$

in $\mathbb{T}_{n+1}$, which corresponds to the arity defined in 2.4.33

Example 2.4.35. The category of $\mathfrak{S}$-trees $\mathbb{F}_{\mathfrak{S}} \cong \int_{n:\mathfrak{S}} \mathcal{F}_{\mathfrak{S}}(*)$ is equipped with a morphism

$$\mathbb{F}_{\mathfrak{S}} \to \mathfrak{S}$$

which to a symmetric tree associates its arity. The arity of a tree is given by operadic composition.

Definition 2.4.36. Each $p : \mathbb{P}_n$ yields an operation $\mathrm{I}_p : \mathcal{F}_{\mathbb{P}_n}(p)$ which we call the p - corolla.

Remark 2.4.37. Let $p : \mathbb{P}_n$ and $\overline{q} : \mathbb{P}_n^{\otimes^p}$. The composition in the free operad $\mathcal{F}_{\mathbb{P}_n}$ yields an operation $I_p \circ I_{\overline{q}} : \mathcal{F}_{\mathbb{P}_n}(p \circ \overline{q})$.

Example 2.4.38. Symmetric and non symmetric corollas.

Definition 2.4.39. Let $M : \mathbb{P}_n^{op} \to \mathbb{T}_n$ be a $\mathbb{P}_n$-sequence in $\mathbb{T}_n$, which we regard as a morphism $M : \overline{*}_n \to \overline{\mathbb{T}}_n$ in $\left[\mathbb{P}_{n+1}^{op}, \mathbb{T}_{n+1}\right]$. The structure of an operad on $\overline{\mathbb{T}}_n$ yields a morphism of operads

$$M_{\mathbb{P}}^{\mathcal{F}} : \mathcal{F}_n^{\mathbb{P}} \to \overline{\mathbb{T}}_n.$$

Each $T : \mathbb{F}_n(p)$ for $p : \mathbb{P}_n$ yields an object which we just write $M^T : \mathbb{T}_n$.

Remark 2.4.40. Let $p : \mathbb{P}_n$, $\overline{q} : \mathbb{P}_n^{\otimes^p}$ and write $T_p^{\overline{q}}$ for the operation of $\mathbb{F}_{\mathbb{P}_n}$ resulting from the composition of the associated corollas. The morphism

$$\mathbb{F}_{\mathbb{P}_n} M(p \circ \overline{q}) : \mathcal{F}_{\mathbb{P}_n}(p \circ \overline{q}) \to \mathbb{T}_n$$

sends $T_p^{\overline{q}}$ to $M(p) \times \bigotimes^p M\ \overline{q}$.

Proposition 2.4.41. *The forgetful functor* $\mathcal{U}_{\mathbb{T}_n}^{\mathbb{P}_n\text{-}\mathrm{Op}} : \mathbb{P}_n\text{-}\mathrm{Op}_{\mathbb{T}_n} \to [\mathbb{P}_n^{op}, \mathbb{T}_n]$ *has a left adjoint*

$$\mathcal{F}_{\mathbb{T}_n}^{\mathbb{P}_n\text{-}\mathrm{Op}} : [\mathbb{P}_n^{op}, \mathbb{T}_n] \to \mathbb{P}_n\text{-}\mathrm{Op}_{\mathbb{T}_n}$$

which is explicitly given by

$$\mathcal{F}_{\mathbb{T}_n}^{\mathbb{P}_n\text{-}\mathrm{Op}} M \cong \int_{T:\mathbb{F}_n} \mathbb{P}_{n+1}(\_, \mu T) \times M^T$$

Proof. Left Kan extension of the exponentiation of M along the terminal morphism of sequences. □

2.5. Examples.

2.5.1. $\mathbb{M}^2$- *operads.* Let $\mathcal{P} : \mathbb{M}^{2^{op}} \to \mathbb{T}_n$ be an $\mathbb{M}^2$-sequence. The substitution product yields a sequence

$$\mathcal{P} \circ \mathcal{P} \cong \int_{T:\mathbb{M}^2} \mathcal{P}_T \times \mathcal{P}^{\otimes^T}.$$

Suppose that $\mathcal{P}$ has the structure of an $\mathbb{M}^2$-operad in $\mathbb{T}_n$, then any $T : \mathbb{M}^2$ yields a composition

$$\mu_T : \mathcal{P}_T \circ \mathcal{P}^{\otimes^T} \to \mathcal{P}.$$

Let $T : \mathbb{M}^2$ be the operation given by . The $\mathbb{M}^2$-sequence $\mathcal{P}^{\otimes^T}$ satisfies

$$P^{\otimes^T} \xrightarrow{\cong} \int_{T_1,T_2,T_3,T_4} \mathbb{M}^2(\_,(T_1 \circ T_2) \bullet (T_3 \circ T_4)) \times (P_{T_1} \otimes_2 P_{T_2}) \otimes_1 (P_{T_3} \otimes_2 P_{T_4})$$

3. Small categorical operads

Let $\mathbb{A}$ denotes either $\mathbb{N}$ or $\mathfrak{S}$. Recall from Definition A.7.0.1 that an object $\mathcal{P}$ of $\mathrm{CAT}^{\mathbb{A}}$ is said to be $\mathrm{SET}^{\mathbb{A}}$-small, or just small, if there exists an object X of $\mathrm{SET}^{\mathbb{A}}$ such that $\mathcal{P} \Leftrightarrow \mathcal{P}_0$ and which satisfy some coherence conditions. Also recall that we can define a category $\mathbf{Cat}^{\mathbb{A}}$ whose objects are small objects in $\mathrm{CAT}^{\mathbb{A}}$, and whose set of morphisms is given by strictly associative functors, with equality defined pointwise. Let $\mathcal{P}$ and $\mathcal{Q}$ be small categorical operads and let $X_{\mathcal{P}}$ and $X_{\mathcal{Q}}$ be objects of $\mathrm{SET}^{\mathbb{A}}$ such that $p \in \mathcal{P} \Leftrightarrow p \in X_{\mathcal{P}}$ and $q \in \mathcal{Q} \Leftrightarrow q \in X_{\mathcal{Q}}$. The set of morphisms $\mathbf{Cat}^{\mathbb{A}}(\mathcal{P},\mathcal{Q})$ is then defined as

- $F : * \to \mathbf{Cat}^{\mathbb{A}}(\mathcal{P},\mathcal{Q}) \Leftrightarrow F : * \to \mathrm{CAT}^{\mathbb{A}}(\mathcal{P},\mathcal{Q}) \times (F \text{ is strict})$, and
- for $F, G : * \to \mathbf{Cat}^{\mathbb{A}}(\mathcal{P},\mathcal{Q})$, the truth value $F = G$ is defined for $\mathbb{A} = \mathbb{N}$ by

$$\mathbf{Cat}^{\mathbb{N}}(\mathcal{P},\mathcal{Q})(F,G) = \prod_{n\in\mathbb{N}} \int^{x:*\to X_{\mathcal{P}}} X_{\mathcal{Q}}(Fx,Gx)(n),$$

an for $\mathbb{A} = \mathfrak{S}$ by

$$\mathbf{Cat}^{\mathfrak{S}}(\mathcal{P},\mathcal{Q})(F,G) = \int^{n:*\to\mathfrak{S}} \int^{x(n):*\to X_{\mathcal{P}}(n)} X_{\mathcal{Q}}(n)(F(n)x(n),G(n)x(n)).$$

Remark 3.0.1. We can also give to $\mathbf{Cat}^{\mathbb{N}}$ the structure of an object of $\mathrm{CAT}^{\mathbb{N}}$ as follows.

- $\mathcal{P} : \mathbf{Cat}^{\mathbb{N}} \Leftrightarrow (\mathcal{P} : \mathrm{CAT}^{\mathbb{N}}) \times (\mathcal{P} \text{ is small })$
- if $\mathcal{P},\mathcal{Q} : \mathbf{Cat}^{\mathbb{N}}$ we define $\mathbf{Cat}^{\mathbb{N}}(\mathcal{P},\mathcal{Q}) : *^{\mathbb{N}} \to \mathrm{SET}^{\mathbb{N}}$ by
  - $F : \mathbf{Cat}^{\mathbb{N}}(\mathcal{P},\mathcal{Q}) \Leftrightarrow F : \mathrm{CAT}^{\mathbb{N}}(\mathcal{P},\mathcal{Q})$
  - for $F, G : Cat^{\mathbb{N}}(\mathcal{P},\mathcal{Q})$, the truth value $(F = G) : \mathbb{B}^{\mathbb{A}}$ is defined by

$$\mathbf{Cat}^{\mathbb{N}}(\mathcal{P},\mathcal{Q})(F,G) = \int^{x:*\to X_{\mathcal{P}}} X_{\mathcal{Q}}(Fx,Gx).$$

We will however mostly use the category structure of $\mathbf{Cat}^{\mathbb{N}}$ in this section, so that we can make use of model category theory.

In this section, the word operad denotes small categorical operads, and the word symmetric operad denotes small symmetric categorical operads.

3.1. The operads $\mathcal{M}_n$ and their algebras. We recall the definition of the categorical operads $\mathcal{M}^n$ governing iterated monoidal categories introduced in [**2**], and the way they are linked with n-fold loop spaces.

Construction and main results on the operads $\mathcal{M}^n$. We briefly recall the construction of the operads $\mathcal{M}^n$ introduced in [**2**]. We do not give its explicit description for the moment, which can be found in the same article, Definition 3.1. We will instead provide a description of $\mathcal{M}^n$ in terms of generators and relations after having introduced polygraphic presentations of operads. We also state the main theorem of the article [**2**], thanks to which the operads $\mathcal{M}^n$ can be seen an E_n-operads in the category of small categories.

Recall from Definition A. 7.0.1 that a category $\mathcal{C}$ is said to be small if there exists a set $\mathcal{C}_0$ such that the data of an object of $\mathcal{C}$ is equivalent to the data of an element of $\mathcal{C}_0$. With this formalism, equality between objects of $\mathcal{C}$ makes sense: we say that objects are equal if the corresponding elements are equal in the set $\mathcal{C}_0$. We also say that two objects are distinct if they are not equal. Recall that we have a small category $\textsc{Mon}^n(\mathbf{Cat}, \times, *)$ whose objects are small categories equipped with a strictly unital and stricly associative n-fold monoidal structure in the monoidal category $(\mathbf{Cat}, \times, *)$, and whose morphisms are strictly associative and strictly unital lax monoidal functors.

The forgetful functor $\textsc{Mon}^n(\mathbf{Cat}, \times, *) \to \mathbf{Cat}$ has a left adjoint

$$\mathcal{F}_n : Cat \to \textsc{Mon}^n(\mathbf{Cat}, \times, *).$$

Let $\mathcal{C}$ be a small category. The category $\mathcal{F}_n\mathcal{C}$ has for objects are all finite expressions generated by the objects of $\mathcal{C}$ using associative operations $\otimes_1, \dots, \otimes_n$. Define the length of an expression as the number of objects of $\mathcal{C}$ involved in this expression and let $\mathcal{F}_n\mathcal{C}(r)$ be the full subcategory of $\mathcal{F}_n\mathcal{C}$ whose objects are expressions of length r. In particular, we have a map $\mathcal{F}_n\mathcal{C}(r)_0 \to \mathcal{C}_0^r$ which to an expression of length r associates the objects of $\mathcal{C}$ involved in this expression. The category $\mathcal{F}_n\mathcal{C}(r)$ admits the decomposition

$$\mathcal{F}_n\mathcal{C} \cong \coprod_{r \in \mathbb{N}} \mathcal{F}_n\mathcal{C}(r).$$

Let $\bar{\mathcal{C}_0^r} \subset \mathcal{C}_0^r$ be the subset of $\mathcal{C}_0^r$ such that its elements are all distinct and $\bar{\mathcal{F}_n}\mathcal{C}(r)$ be the full subcategory of $\mathcal{F}_n\mathcal{C}(r)$ whose objects are send to $\bar{\mathcal{C}_0^r}$. The symmetric group Σ_r acts freely on $\bar{\mathcal{F}_n}\mathcal{C}_0(r)$ by permutation of the coordinates in $\mathcal{C}_0^r$, and this action extends to the morphisms of $\bar{\mathcal{F}_n}\mathcal{C}(r)$ in a compatible way, so that we obtain a functor $\bar{\mathcal{F}_n}\mathcal{C} : \mathfrak{S}^{op} \to \mathbf{Cat}$. Let E_r be the set with r elements and let the symmetric sequence $\mathcal{M}^n : \mathfrak{S}^{op} \to \mathbf{Cat}$ be defined in arity r by $\mathcal{M}^n(r) = \bar{\mathcal{F}_n}E_r(r)$, where we obtain the symmetric action from the observation that we have an isomorphism $\bar{\mathcal{F}_n}E_r(r) \cong \bar{\mathcal{F}_n}\mathbb{N}(r)$ for each arity r. The coproduct

$$\coprod_{r \in \mathbb{N}} \bar{\mathcal{F}_n}\mathbb{N}(r)$$

carries the structure of an n-fold monoidal category, providing an n-fold monoidal category structure on the coproduct $\coprod_{r \in \mathbb{N}} \mathcal{M}^n(r)$. It should also be noted that the category $\mathcal{F}_n\mathcal{C}(r)$ is isomorphic to $\mathcal{M}^n(r) \underset{\Sigma_r}{\times} \mathcal{C}^r$ for each r, so that the decomposition of $\mathcal{F}_n\mathcal{C}$ reduces to

$$\mathcal{F}_n\mathcal{C} \cong \coprod_{r \in \mathbb{N}} \mathcal{M}^n(r) \underset{\Sigma_r}{\times} \mathcal{C}^r.$$

REMARK 3.1.1. When $\mathcal{C}$ is the terminal category, we obtain the following description of the free n-fold monoidal category generated by the terminal category

$$\mathcal{F}_n(*) \cong \coprod_{r \in \mathbb{N}} \mathcal{M}^n(r)/\Sigma_r.$$

It is also worth notifying that if the symmetric group action is necessary to define the morphisms involving the interchange of the different monoidal structures, the action of Σ_r on $\mathcal{M}^n(r)$ is free on objects.

REMARK 3.1.2. Suppose that $\mathcal{C}$ has the structure of an n-fold monoidal category, the counit of the adjunction provides an evaluation morphism $\mathcal{F}_n\mathcal{C} \to \mathcal{C}$. It follows that we get a functor for each r

$$\mathcal{M}^n(r) \underset{\Sigma_r}{\times} \mathcal{C}^r \to \mathcal{C}.$$

In the particular case where $\mathcal{C}$ is the coproduct over $r \in \mathbb{N}$ of the categories $\mathcal{M}^n(r)$, we obtain a morphism of symmetric categorical sequences

$$\int_{r \in \mathbb{N}} \mathcal{M}^n(r) \times \mathcal{M}^{n\otimes^r} \to \mathcal{M}^n,$$

providing the symmetric sequence of categories $\mathcal{M}^n$ with the structure of a categorical operad.

REMARK 3.1.3. The symmetric categorical operad $\mathcal{M}_1$ is defined in arity r by $\mathcal{M}_1(r) = \Sigma_r$, with free right action of Σ_r given by translation, so that it precisely corresponds to the symmetric associative operad. It will also be convenient to consider the non symmetric version of $\mathcal{M}_1$, which is given in each arity by the terminal category.

The following coherence theorem is one of the main results of [**2**], where a complete description of the morphisms in $\mathcal{M}^n$ is also given.

THEOREM 3.1.4. *(see* [**2**, Theorem 3.6]*) The operad $\mathcal{M}^n$ consists of posets.*

Recall that both the nerve and geometric realization functors preserve products, so that they define monoidal functors. It follows that the geometric realization of the nerve of any categorical operad yields a topological operad. The main interest of the operads $\mathcal{M}^n$ will be given by the following theorem. Because of this result, we refer to any categorical operad which is equivalent to the operad $\mathcal{M}^n$ as an E_n-operad.

THEOREM 3.1.5. *(see* [**2**, Theorem 3.14]*) The operad $|\mathcal{N}\mathcal{M}^n|$ is weakly equivalent to the little n-cubes operad $\mathcal{C}_n$.*

On $\mathcal{M}^n$-algebras and n-fold monoids. We describe the algebras over the operads $\mathcal{M}^n$. For this purpose, it will be convenient to see the operads $\mathcal{M}^n$ as operads defined in the 2-category of all categories, so that we will be able to define $\mathcal{M}^n$-algebras in any symmetric monoidal 2-category. In particular, we recover the 2-category of n-fold monoids in a monoidal 2-category as the 2-category of algebras over $\mathcal{M}^n$ in this monoidal 2-category.

PROPOSITION 3.1.6. *Iterated monoids as defined in Chapter II can now be described as the* 2*-category of algebras over the* CAT*-operads $\mathcal{M}^n$, so that we have an isomorphism of* 2*-categories*

$$\mathcal{M}^n\text{-}\mathrm{ALG}_{(\Lambda,\otimes_\Lambda)} \cong \mathrm{MON}^n_{(\Lambda,\otimes_\Lambda)}$$

On n-fold loop spaces and n-fold delooping.

THEOREM 3.1.7. *(see [**2**, Theorem 2.2]) If $\mathcal{C} \in \mathrm{MON}^n_{(\mathrm{CAT},\times)}$ is an n-fold monoidal category, then the geometric realization of its nerve $\mid \mathcal{NC} \mid$ is an n-fold loop space up to group completion.*

REMARK 3.1.8. The proof of Theorem 3.1.7 consists in iterating the delooping construction and constructing an n-simplicial space $\mathcal{B}^n X$ and a group completion $\Omega^n \mid \mathcal{B}^n\mathcal{C} \mid \rightarrow \mathcal{N} \mid \mathcal{C} \mid$. The properties of the monoidal 2-category $(\mathrm{CAT}, \times)$ that make this n-fold delooping possible can be adapted to a monoidal 2-category $(\Lambda, \otimes_\Lambda)$ so that we can obtain a similar result. For this purpose, we need the category Λ to be equipped with a pseudomonoidal 2-functor into the 2-category of topological spaces with cartesian product (which is given by the geometric realization of the nerve for CAT). We also need to be able to produce a strict functor $(\Delta^{op})^n \rightarrow \Lambda$ from a lax one in such a way that the pseudomonoidal functor into TOP takes the lax functor and its strictification to objectwise equivalent lax functors.

Let $X \in \mathrm{PSMON}^n_{(\Lambda,\otimes_\Lambda)}$. We construct a lax 2-functor

$$\mathcal{B}^n X : (\Delta^{op})^n \longrightarrow \Lambda.$$

We set $\mathcal{B}^n X_{r_1,\ldots,r_n} = X^{\otimes_\Lambda^{r_1\ldots r_n}}$ for a collection of objects $(r_1,\ldots,r_n)$. We first define the value of $\mathcal{B}^n X$ on the collections of morphisms $(f_1,\ldots,f_n)$ in $(\Delta^{op})^n$ that have only one non identity morphism f_i in coordinate $1 \leq i \leq n$. We proceed in the same way than the usual delooping, of the monoid $X^{\otimes_\Lambda^{r_1\ldots r_{i-1}}}$, with respect to the monoidal product $\otimes_i$, raised to power of $r_{i+1}\ldots r_n$. Then we decompose any morphism $(f_1,\ldots,f_n)$ in $(\Delta^{op})^n$ as the composite

$$(f_1,\ldots,f_n) = (id,\ldots,id,f_n) \circ \cdots \circ (f_1,id,\ldots,id)$$

and we take

$$B^n X(f_1,\ldots,f_n) = B^n X(id,\ldots,id,f_n) \circ \cdots \circ B^n X(f_1,id,\ldots,id).$$

The 2-morphisms η_i^j are 2-morphisms in Λ and provide $\mathcal{B}^n X$ with the structure of a lax monoidal 2-functor. The coherence constraints of η_i^j ensure that $\mathcal{B}^n X$ also satisfies the coherence constraints for lax monoidal 2-functors. Now suppose that

– the lax 2-functor $\mathcal{B}^n X$ can be 'strictified' to a 2-functor $\hat{\mathcal{B}}^n X$,
– the 2-category Λ is equipped with a pseudomonoidal 2-functor

$$\mid - \mid^\Lambda : (\Lambda, \otimes_\Lambda) \rightarrow (\mathrm{TOP}, \times),$$

– for all $r_1,\ldots,r_n$ we have a homotopy equivalence $\mid \hat{\mathcal{B}}^n X \mid^\Lambda_{r_1,\ldots,r_n} \sim \mid \mathcal{B}^n X \mid^\Lambda_{r_1,\ldots n r_n}$ in *Top*.

Then $\mid \hat{\mathcal{B}}^n X \mid^\Lambda$ is a special Δ^n-space. It follows that the map

$$\Omega^n \mid\mid \hat{\mathcal{B}}^n X \mid^\Lambda \mid \longrightarrow \mid X \mid^\Lambda$$

is a group completion. Hence we get the result of Theorem 3.1.7 in the case of the monoidal 2-category $(\mathrm{CAT}, \times)$ equipped with Street's rectification of lax functors, and the pseudomonoidal functor realization of the nerve $\mid \mathcal{N} \mid : \mathrm{CAT} \rightarrow Top$.

3.2. Model category structure on small categorical operads. We define a model category structure on the category of small categorical operads and small symmetric categorical operads. We provide a simple characterisation of cofibrant operads for this model structure. Finally, we make explicit a cofibrant replacement functor and apply our results to exhibit a cofibrant model of the operad $\mathcal{M}_1$ governing monoidal categories.

Transferred model category structure. We briefly recall the conditions under which the category of operads defined in a symmetric monoidal category inherits a model category structure from a model category structure on its based category. We observe that the symmetric monoidal category $(\mathbf{Cat}, \times, *)$ satisfies the required properties. We apply the results of [**11**] and [**3**] to define a model category structure on the category of small categorical operads and small symmetric categorical operads.

Recall that the category **Cat** is bicomplete closed and is equipped with a model structure such that

– the weak equivalences are the equivalences of categories,
– the fibrations are the isofibrations,
– the cofibrations are the injections on the objects.

We refer to this model category structure as the canonical model structure on **Cat**. Note that this model category structure is cofibrantly generated (see for instance [**28**]) and that every category is both fibrant and cofibrant. Recall that the category $\mathbf{Cat}^{\mathbb{N}}$ inherits a model structure from the model structure on **Cat** such that the weak equivalences, fibrations and cofibrations are defined degreewise. In particular, every object of $\mathbf{Cat}^{\mathbb{N}}$ is both fibrant and cofibrant. The general statement of [**11**] implies that the right transferred model structure along the adjunction

$$\mathfrak{S} \times - : Cat^{\mathbb{N}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Cat}^{\mathfrak{S}} : U$$

is well defined, so that we have the following proposition.

Proposition 3.2.1. *There is a model category structure on the category* $\mathbf{Cat}^{\mathfrak{S}}$ *such that*

$$\mathfrak{S} \times - : Cat^{\mathbb{N}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Cat}^{\mathfrak{S}} : U$$

is a Quillen adjunction, and where the weak equivalences are the aritywise equivalences of categories and the fibrations are the aritywise isofibrations.

Definition 3.2.2. Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category equipped with a model structure. We say that this model structure is monoidal if the following conditions are satisfied.

Unit axiom The unit is cofibrant.

Pushout-product axiom If $\iota_1 : X_1 \to Y_1$ and $\iota_2 : X_2 \to Y_2$ are cofibrations, then the natural morphism

$$X_1 \otimes Y_2 \coprod_{X_1 \otimes X_2} X_2 \otimes Y_1 \to Y_1 \otimes Y_2$$

is a cofibration, which is acyclic if either ι_1 or ι_2 is acyclic.

Note that the unit axiom is trivially satisfied for the canonical model structure on **Cat** since all the objects of are cofibrant. The pushout-product axiom also holds ([**28**],Theorem 5.1), so that we obtain the following proposition.

PROPOSITION 3.2.3. *The canonical model structure on the symmetric monoidal category* $(\mathbf{Cat}, \times, *)$ *is monoidal.*

We recall the following theorem.

THEOREM 3.2.4. *(see* [**3**]*) Let* $\mathcal{E}$ *be a cartesian closed model category such that*

– $\mathcal{E}$ *is cofibrantly generated and the terminal object of* $\mathcal{E}$ *is cofibrant,*
– E *has a symmetric monoidal fibrant replacement functor.*

Then the right transferred model structure on $\mathrm{Op}_{\mathcal{E}}^{\mathfrak{S}}$ *along the adjunction*

$$\mathcal{F} : \mathcal{E}^{\mathfrak{S}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Op}_{\mathcal{E}}^{\mathfrak{S}} : \mathcal{U}$$

exists, so that there is a cofibrantly generated model category structure on the category $\mathrm{Op}_{\mathcal{E}}^{\mathfrak{S}}$ *of operads in* $\mathcal{E}$*.*

COROLLARY 3.2.5. *There exists a cofibrantly generated model category structure on both the categories* $\mathrm{Op}_{\mathbf{Cat}}^{\mathfrak{S}}$ *and* $\mathrm{Op}_{\mathbf{Cat}}$*, such that all the adjunctions involved in the commutative square*

$$\begin{tikzcd}
\mathbf{Cat}^{\mathbb{N}} \arrow[rr, bend left, "\mathcal{F}"] \arrow[rr, phantom, "\perp"] \arrow[d, bend right, "\mathfrak{S}\times -"'] \arrow[d, phantom, "\dashv"] & & \mathrm{Op}_{\mathbf{Cat}} \arrow[ll, bend left, "\mathcal{U}"] \arrow[d, bend right, "\mathfrak{S}\times -"'] \arrow[d, phantom, "\dashv"] \\
\mathbf{Cat}^{\mathfrak{S}} \arrow[rr, bend left, "\mathcal{F}"] \arrow[rr, phantom, "\perp"] \arrow[u, bend right, "U"'] & & \mathrm{Op}_{\mathbf{Cat}}^{\mathfrak{S}} \arrow[ll, bend left, "\mathcal{U}"] \arrow[u, bend right, "U"']
\end{tikzcd}$$

are Quillen adjunctions, and such that both in $\mathrm{Op}_{\mathbf{Cat}}$ *and* $\mathrm{Op}_{\mathbf{Cat}}^{\mathfrak{S}}$*,*

– *the weak equivalences are the aritywise equivalences of categories*
– *the fibrations are the aritywise isofibrations.*

REMARK 3.2.6. Quillen adjunctions are stable under composition. We write

$$\mathcal{F}^{\mathfrak{S}} : Cat^{\mathbb{N}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Op}_{\mathbf{Cat}}^{\mathfrak{S}} : \mathcal{U}^{\mathfrak{S}}$$

for the Quillen adjunction obtain by the diagonal composition of the adjunctions in the commutative square.

REMARK 3.2.7. Recall that an equivalence of categories $F : \mathcal{C} \to \mathcal{D}$ yields a weak equivalence of topological spaces $|\mathcal{N}f| : |\mathcal{N}\mathcal{C}| \xrightarrow{\sim} |\mathcal{N}\mathcal{D}|$ via the geometric realisation of the nerve. Moreover, an isofibration $f : \mathcal{C} \to \mathcal{D}$ induces a fibration of topological spaces $|\mathcal{N}f| : |\mathcal{N}\mathcal{C}| \xrightarrow{\sim} |\mathcal{N}\mathcal{D}|$. Hence, the weak equivalences we consider are stronger than the weak equivalences given by Thomason's model structure on **Cat**, and there are more cofibrant objects in the model structure we use. It follows any cofibrant resolution of a categorical operad $\mathcal{P}$ will provide a weak equivalence of topological operads, which will however not necessarily be a cofibrant resolution of $|\mathcal{N}\mathcal{P}|$.

Cofibrant objects. We construct cofibrant resolution functors for the model category structures we defined on the categories of categorical operads and of symmetric categorical operads. We also characterize cofibrant objects.

Definition 3.2.8. We say that the symmetric group acts freely on the objects of symmetric sequence $E : \mathfrak{S} \to \mathbf{Cat}$ if there exists a sequence $A : \mathbb{N} \to \mathrm{SET}$ and an isomorphism of symmetric sequences $E_0 \cong \mathfrak{S} \times A$.

Definition 3.2.9. Let $E : \mathfrak{S} \to \mathbf{Cat}$ be a symmetric sequence. We define a symmetric sequence CE as follows. We let $CE_0 = \mathfrak{S} \times UE_0$. If $r \in \mathbb{N}$ and $(\sigma, x), (\tau, y) : * \to \Sigma_r \times E_0(r)$, then we define

$$CE(r)((\sigma, x), (\tau, y)) = E(r)(\sigma.x, \tau.y).$$

Let $E, F : \mathfrak{S} \to \mathbf{Cat}$. We define a morphism $C_{E,F}$ in $\mathrm{SET}^{\mathfrak{S}}$

$$C_{E,F} : \mathbf{Cat}^{\mathfrak{S}}(E, F) \to \mathbf{Cat}^{\mathfrak{S}}(CE, CF)$$

as follows. Let $f : * \to \mathbf{Cat}^{\mathfrak{S}}(E, F)$. Then f induces a morphism $C_{E,F}f : * \to \mathbf{Cat}^{\mathfrak{S}}(CE, CF)$. It is given on the objects by $\mathfrak{S} \times f_0 : \mathfrak{S} \times E_0 \to \mathfrak{S} \times F_0$. Let $r \in \mathbb{N}$ and let $(\sigma, x), (\tau, y) : * \to \Sigma_r \times E_0(r)$. The morphism induced by f on the morphisms is given componentwise by $f_{x,y} : E(r)(\sigma.x, \tau.y) \to F(r)(f\sigma.x, f\tau.y) \cong F(r)(\sigma.fx, \tau.fy)$.

Hence C defines a functor $C : \mathbf{Cat}^{\mathfrak{S}} \to \mathbf{Cat}^{\mathfrak{S}}$. The unit of the adjunction between categorical sequences and symmetric categorical sequences induces an acyclic fibration

$$\pi_E : CE \overset{\sim}{\twoheadrightarrow} E.$$

Proposition 3.2.10. *A symmetric sequence in* **Cat** *is cofibrant if and only if the symmetric group acts freely on its objects.*

Proof. We show that for any symmetric sequence of categories E such that $E_0 \cong \mathfrak{S} \times A$ for some $A \in \mathrm{SET}^{\mathbb{N}}$, the symmetric sequence E is cofibrant. Let $\pi : F \overset{\sim}{\twoheadrightarrow} G$ be an acyclic fibration. By definition, π defines an equivalence of categories for each $r \in \mathbb{N}$, so that for each pair of operations p, q of $F_0(r)$, the functor $\pi(r)$ induces a natural isomorphism

$$\pi(r)(p, q) : F(r)(p, q) \cong G(r)(\pi q, \pi r)$$

which is compatible with the symmetric group action. More precisely, π induces a 2-morphism α_π in $\mathbf{Cat}^{\mathfrak{S}}$

$$\begin{tikzcd} F^{op} \times F \arrow[r, "F(-,-)"] \arrow[d, "\pi^{op}\times\pi"'] & \mathrm{SET}^{\mathfrak{S}} \\ G^{op} \times G \arrow[ur, "G(-,-)"'] \arrow[u, phantom, shift left=6, "\overset{\alpha_\pi}{\Downarrow}\cong"] & \end{tikzcd}$$

which is an isomorphism. Let $f : E \to G$. We define a morphism $\bar{f} : E \to F$ such that the diagram

$$\begin{tikzcd} & F \arrow[d, two heads, "\sim\pi"] \\ E \arrow[ur, dashed, "\bar{f}"] \arrow[r, "f"'] & G \end{tikzcd}$$

commutes. Since $\pi(r)$ is an isofibration, it is in particular surjective on the objects. Hence we obtain a map $g(r) : A(r) \to F_0(r)$ such that $\pi(r)g(r)x = f(r)x$ for each $x \in A(r)$. We let $\bar{f}$ be defined on the objects by the morphism of symmetric sequences induced by the collection $g(r)$. We define a 2-morphism $\alpha_{\bar{f}}$ in $\mathrm{Set}^{\mathfrak{S}}$ by the composite

$$\begin{tikzcd}
 & F^{op}\times F \arrow[d, "\pi^{op}\times\pi"'] \arrow[dr, "F(-,-)"] & \\
*^{\mathbb{N}} \arrow[ur, "(\bar{f}x,\bar{f}x)"] \arrow[r, "(fx,fx)"] \arrow[dr, "(x,x)"'] & G^{op}\times G \arrow[r, "G(-,-)"] \arrow[ur, phantom, "\overset{\alpha_\pi^{-1}}{\Rightarrow}\ \cong"] & \mathrm{Set}^{\mathfrak{S}}, \\
 & E^{op}\times E \arrow[u, "f^{op}\times f"] \arrow[ur, "E(-,-)"'] \arrow[u, phantom, shift left=6, "\overset{\alpha_f}{\Rightarrow}"] &
\end{tikzcd}$$

so that we obtain a 2-morphism in $\mathrm{Cat}^{\mathfrak{S}}$

$$\alpha_{\bar{f}}x : E(x,x) \Rightarrow F(\bar{f}x, \bar{f}x),$$

such that the induced morphism of symmetric categorical sequences $\bar{f} : E \to F$ lifts f.

Let E be a cofibrant symmetric sequence and let $\lambda : E \to CE$ be a lifting in the diagram

$$\begin{tikzcd}
 & CE \arrow[d, two heads, "\sim"] \\
E \arrow[ur, dashed, "\lambda"] \arrow[r, "id"'] & E.
\end{tikzcd}$$

In particular, λ provides a morphism of symmetric sequences

$$\lambda : E \to \Sigma \times UE.$$

Let $A : *^{\mathbb{N}} \to \mathrm{Set}^{\mathbb{N}}$ be defined by the equalizer

$$\begin{tikzcd}
A \arrow[r, dashed, "i"] & UE_0 \arrow[r, shift left, "\epsilon_{UE_0}"] \arrow[r, shift right, "U\lambda"'] & U\mathfrak{S}\times UE_0.
\end{tikzcd}$$

Write $\hat{i} : \mathfrak{S} \times A \to E$ for the morphism of symmetric sequences induced by i. Let $r \in \mathbb{N}$ and $x \in E_0(r)$. Let $\sigma_x \in \Sigma r$ and $z_x \in E_0(r)$ be such that $\lambda x = (\sigma_x, z_x)$. Since $\pi\lambda = id$ we have $\sigma_x.z_x = x$, and since λ is a morphism of symmetric sequences we have $\lambda x = \sigma_x.\lambda z_x = (\sigma_x, z_x)$, so that $\lambda x = \sigma_x.\epsilon z_x$. Hence $\lambda z_x = \epsilon z_x$, and there is some $a_x \in A(r)$ such that $ia_x = z_x$. We obtain a morphism $h : E_0 \to \Sigma \times A$ which to x associates (σ_x, a_x). Indeed, if $x \in E_0(r)$ and $\sigma \in \Sigma_r$, then $\sigma.\lambda x = \sigma.(\sigma_x, z_x) = (\sigma.\sigma_x, z_x)$. On the other hand, $\lambda\sigma.x = (\sigma_{\sigma_x.x}, z_{\sigma.x})$. Note that h is such that $hia = a$ for each $a \in A(r)$. Since λ is a morphism of symmetric sequences, we obtain $z_x = z_{\sigma.x}$ and $\sigma.\sigma_x = \sigma_{\sigma.x}$. We have for $x \in E_0(r)$, $\hat{i}hx = \hat{i}(\sigma_x, a_x) = \sigma_x ia_x = \sigma_x z_x = x$. Conversely, for $(\sigma, a) \in \Sigma_r \times A(r)$, we have $h\hat{i}(\sigma, a) = h(\sigma.ia) = \sigma.hia = (\sigma, a)$. This proves that we have an isomorphism of symmetric sequences $E_0 \cong \mathfrak{S} \times A$. $\square$

Corollary 3.2.11. *The category* $\mathbf{Cat}^{\mathfrak{S}}$ *of symmetric categorical sequences is equipped with a functorial cofibrant resolution* $C : Cat^{\mathfrak{S}} \to \mathbf{Cat}^{\mathfrak{S}}$.

Definition 3.2.12. We say that a categorical operad $\mathcal{P}$ is free on its object if there exists a sequence $E : \mathbb{N} \to \mathrm{Set}$ and an isomorphism of operads $\mathcal{P}_0 \cong \mathcal{F}(E)$.

DEFINITION 3.2.13. Let $\mathcal{P}$ be an operad. We define a sequence of categories $\mathcal{C}(\mathcal{P})$ as follows:

– We let the sequence of objects of $\mathcal{C}(\mathcal{P})$ be freely generated by the objects of $\mathcal{P}$, so that $\mathcal{C}(\mathcal{P})_0 = \mathcal{F}(\mathcal{U}\mathcal{P}_0)$.
– Let $r \in \mathbb{N}$ and $p, q : * \to \mathcal{C}(\mathcal{P})_0(r)$. We define the set of morphisms in $\mathcal{C}(\mathcal{P})(r)$ from p to q by
$$\mathcal{C}(P)(r)(p,q) = \mathcal{P}(r)\left(\mu(p), \mu(q)\right).$$

PROPOSITION 3.2.14. *The categorical sequence $\mathcal{C}(\mathcal{P})$ is naturally equipped with the structure of an operad in* **Cat**. *Moreover, the operadic composition of $\mathcal{P}$ induces an acyclic fibration of categorical operads.*
$$\pi : \mathcal{C}(\mathcal{P}) \overset{\sim}{\twoheadrightarrow} \mathcal{P}.$$

PROOF. Let $r, n_1, \dots, n_r \in \mathbb{N}$. We define the composition functor
$$\mathcal{C}(\mathcal{P})(r) \times \prod_{i=1}^{r} \mathcal{C}(\mathcal{P})(n_i) \to \mathcal{C}(\mathcal{P})(n_1 + \cdots + n_r)$$
on the objects by the operadic composition of the free operad $\mathcal{C}(\mathcal{P}_0) = \mathcal{F}\mathcal{U}\mathcal{P}_0$. Let $p, q : * \to \mathcal{C}(\mathcal{P})_0(r)$ and $p_i, q_i : * \to \mathcal{C}(\mathcal{P})_0(n_i)$. Let $n = n_1 + \cdots + n_r$. We define the operadic composition for morphisms in $\mathcal{C}(\mathcal{P})$ as the map
$$\mathcal{C}(\mathcal{P})(r)(p,q) \times \prod_{i=1}^{r} \mathcal{C}(\mathcal{P})(n_i)(p_i, q_i) \to \mathcal{C}(\mathcal{P})(n)(p(p_1, \dots, p_r), q(q_1, \dots, q_r))$$
given by the composition of morphisms in $\mathcal{P}$
$$\mathcal{P}(r)(\mu p, \mu q) \times \prod_{i=1}^{r} \mathcal{P}(n_i)(\mu p_i, \mu q_i) \to \mathcal{P}(n)\left(\mu\left[\mu p; (\mu p_1, \dots, \mu p_r)\right], \mu\left[\mu q; (\mu q_1, \dots, \mu q_r)\right]\right),$$
using that the associativity of the composition provides an equality between the composite operations
$$\mu(\mu p; (\mu p_1, \dots, \mu p_r)) = \mu(p; (p_1, \dots, p_r)).$$
From the definition of $\mathcal{C}(\mathcal{P})$, it is straightforward to see that the counit $\mathcal{F}\mathcal{U}\mathcal{P}_0 \to \mathcal{P}_0$ of the free-forgetful adjunction
$$\mathcal{F} : \mathrm{SET}^{\mathbb{N}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{OP}_{\mathrm{SET}} : \mathcal{U}.$$
induces an acyclic fibration of operads $\pi : \mathcal{C}(\mathcal{P}) \overset{\sim}{\twoheadrightarrow} \mathcal{P}$. $\square$

PROPOSITION 3.2.15. *An operad is cofibrant if and only if it is free on its objects.*

PROOF. Let $\mathcal{P}$ be a categorical operad which is free on its objects. Let $\pi : \mathcal{Q} \overset{\sim}{\twoheadrightarrow} \mathcal{R}$ be an acyclic fibration and $f : \mathcal{P} \to \mathcal{R}$ be a morphism of operads. We construct a morphism of operads $\bar{f} : \mathcal{P} \to \mathcal{Q}$ which lifts f, so that the following diagram commutes
$$\begin{tikzcd} & \mathcal{Q} \arrow[d, two heads, "\pi", "\sim"'] \\ \mathcal{P} \arrow[r, "f"'] \arrow[ur, dashed, "\bar{f}"] & \mathcal{R}. \end{tikzcd}$$

Let $E : *^{\mathbb{N}} \to Sets^{\mathbb{N}}$ and let $\phi : \mathcal{F}(E) \xrightarrow{\cong} \mathcal{P}_0$ be an isomorphism of operads. We first construct a morphism of operads $\bar{f}_0 : \mathcal{P}_0 \to \mathcal{Q}_0$ lifting f_0 as follows. The morphism of operads $f_0 : \mathcal{P}_0 \to \mathcal{R}_0$ is induced by a morphism of sequences $\hat{f} : E \to \mathcal{U}\mathcal{R}_0$. Let $r \in \mathbb{N}$ and let x be an element of $E(r)$. The image of x under $\hat{f}(r)$ is an element $\hat{f}(r)x$ of the underlying set of $\mathcal{R}_0(r)$. Since $\pi(r)$ is an equivalence of categories, there is some element z of the underlying set of $\mathcal{Q}_0(r)$ and an isomorphism $\phi : \pi(r)z \xrightarrow{\cong} \hat{f}(r)x$. Since $\pi(r)$ is an isofibration, there exists an element y_x of $\mathcal{Q}(r)$ and an isomorphism $\psi : z \xrightarrow{\cong} y_x$ such that $\pi(r)y_x = \hat{f}(r)x$ and $\pi(r)(\psi) = \phi$. We let the morphism of operads $\bar{f} : \mathcal{P}_0 \to \mathcal{Q}_0$ be induced by the morphism of sequences given in arity $r \in \mathbb{N}$ by the map $E(r) \to \mathcal{Q}_0(r)$ which to x associates y_x.

We lift f at the level of the morphisms. For this purpose, we define a 2-morphism

$$\bar{f}_1 : \mathcal{P}(-,-) \Rightarrow \mathcal{Q}(\bar{f}-, \bar{f}-)$$

between the lax morphisms of $\textsc{Cat}$ operads

$$\mathcal{P}(-,-) : * \to \mathcal{P}^{op} \times \mathcal{P} \to \textsc{Set}^{\mathbb{N}}$$

and

$$\mathcal{R}(f-, f-) : * \to \mathcal{P}^{op} \times \mathcal{P}^{op} \xrightarrow{f^{op} \times f} \mathcal{R}^{op} \times \mathcal{R} \xrightarrow{\mathcal{R}(-,-)} \textsc{Set}^{\mathbb{N}}.$$

Since π is an equivalence, it defines a 2-isomorphism in $\textsc{Op}_{\mathbf{Cat}}$ such that

$$\begin{tikzcd}
\mathcal{Q}^{op} \times \mathcal{Q} \arrow[r, "\mathcal{Q}(-,-)"] \arrow[d, "\pi^{op} \times \pi"'] \arrow[dr, phantom, "\overset{\cong}{\Swarrow}"] & \textsc{Set}^{\mathbb{N}} \\
\mathcal{R}^{op} \times \mathcal{R}. \arrow[ur, "\pi(_,_)\ \mathcal{R}(-,-)"'] &
\end{tikzcd}$$

In particular, the diagram

$$\begin{tikzcd}
 & \mathcal{Q}^{op} \times \mathcal{Q} \arrow[d, "\pi^{op} \times \pi"'] \arrow[dr, "\mathcal{Q}(-,-)"] \arrow[dr, phantom, "\pi(_,_)\ \overset{\cong}{\Swarrow}"'] & \\
 & \mathcal{R}^{op} \times \mathcal{R} \arrow[r, "\mathcal{R}(-,-)"] & \textsc{Set}^{\mathfrak{S}} \\
* \arrow[uur, "\bar{f}^{op} \times \bar{f}"] \arrow[ur, dashed] \arrow[r] & \mathcal{P}^{op} \times \mathcal{P} \arrow[u, "f^{op} \times f"] \arrow[ur, "\mathcal{P}(-,-)"'] \arrow[ur, phantom, "\overset{f(_,_)}{\Nwarrow}"] &
\end{tikzcd}$$

shows that $\pi(\_,\_)$ induces an isomorphism

$$\mathcal{Q}(\bar{f}-, \bar{f}-) \overset{\cong}{\Rrightarrow} \mathcal{R}(\pi\bar{f}-, \pi\bar{f}-) : * \to \textsc{Set}$$

and we let $\bar{f}_1$ be defined by the composite

$$\mathcal{P}(-,-) \overset{f_1}{\Rrightarrow} \mathcal{R}(f-, f-) \overset{\cong}{\Rrightarrow} \mathcal{R}(\pi\bar{f}-, \pi\bar{f}-) \overset{\cong}{\Rrightarrow} \mathcal{Q}(\bar{f}-, \bar{f}-).$$

Hence we obtain a morphism of operads $\bar{f} : \mathcal{P} \to \mathcal{Q}$ such that $\pi\bar{f} = f$ and the operad $\mathcal{P}$ is cofibrant.

Now we show that any cofibrant operad is free on its objects. Let $\mathcal{P}$ be a cofibrant operad. Let $\lambda : \mathcal{P} \to \mathcal{C}(\mathcal{P})$ be a lifting of the identity morphism of $\mathcal{P}$:

$$\begin{tikzcd} & \mathcal{C}(\mathcal{P}) \arrow[d, two heads, "\mu"] \\ \mathcal{P} \arrow[ur, dashed, "\lambda"] \arrow[r, "\mathrm{id}"'] & \mathcal{P}. \end{tikzcd}$$

Also write λ for the morphism of operads induced by λ on objects

$$\lambda : \mathcal{P}_0 \to \mathcal{F}\mathcal{U}\mathcal{P}_0.$$

Write $\epsilon : \mathcal{U}\mathcal{P}_0 \to \mathcal{U}\mathcal{F}\mathcal{U}\mathcal{P}_0$ for the counit of the free-forgetful adjunction

$$\mathcal{F} : \mathrm{S{\scriptstyle ET}}^{\mathbb{N}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{O{\scriptstyle P}}_{\mathrm{S{\scriptstyle ET}}} : \mathcal{U}.$$

applied to the object $\mathcal{U}\mathcal{P}_0$ of $\mathrm{S{\scriptstyle ET}}^{\mathbb{N}}$. Consider the equalizer in $\mathrm{S{\scriptstyle ET}}^{\mathbb{N}}$

$$\begin{tikzcd} E \arrow[r, dashed, "i"] & \mathcal{U}\mathcal{P}_0 \arrow[r, shift left, "\epsilon"] \arrow[r, shift right, "\mathcal{U}\lambda"'] & \mathcal{U}\mathcal{F}\mathcal{U}\mathcal{P}_0 \end{tikzcd} ,$$

we can think about E as the subsequence of the underlying sequence of the operad $\mathcal{P}_0$ of objects of $\mathcal{P}$ made of operations which can not be written as a non trivial operadic composite. Let $\hat{i} : \mathcal{F}(E) \to \mathcal{P}_0$ be the morphism of operads induced by i. We will show $\hat{i}$ is an isomorphism.

We choose a canonical representation of each element in the free operad so that for each element $\bar{x} \in \mathcal{F}\mathcal{U}\mathcal{P}_0(n)$, there is a unique tree T together with elements $x_\nu \in \mathcal{P}_0(n_\nu)$ of $\mathcal{P}_0$ for each $\nu \in V(T)$ such that $\bar{x} = \prod_{\nu \in V(T)} x_\nu$. Let T be a tree and $\bar{x}_\nu \in \mathcal{F}\mathcal{U}\mathcal{P}_0$ be an operation of the free operad for $\nu \in V(T)$. We write $c\left\{\bar{x}_\nu\right\}_{\nu \in V(T)} \in \mathcal{F}\mathcal{U}\mathcal{P}_0(n)$ for the resulting composite operation in the free operad. In particular, if for each $\nu \in V(T)$ we have $x_\nu \in \mathcal{P}_0(n_\nu)$, then the composition in the free operad of the operations $\epsilon x_\nu \in \mathcal{F}\mathcal{U}\mathcal{P}_0(n_\nu)$ along the tree T is such that

$$c\left\{\epsilon x_\nu\right\}_{\nu \in V(T)} = \prod_{\nu \in V(T)} x_\nu.$$

Let $x \in \mathcal{P}_0(n)$. Let T be a tree and $x_\nu \in \mathcal{P}_0(n_\nu)$, for $\nu \in V(T)$, be such that we have $\lambda x = \prod_{\nu \in V(T)} x_\nu$. Since $\mu\lambda = id$ we have

$$x = \mu\left(\prod_{\nu \in V(T)} x_\nu\right).$$

Since λ is a morphism of operads we have

$$c\left\{\lambda x_\nu\right\}_{\nu \in V(T)} = \lambda\mu\left(\prod_{\nu \in V(T)} x_\nu\right) = \lambda x.$$

Hence we obtain

$$c\left\{\epsilon x_\nu\right\}_{\nu \in V(T)} = \prod_{\nu \in V(T)} x_\nu = \lambda x = c\left\{\lambda x_\nu\right\}_{\nu \in V(T)},$$

so that for all $\nu \in V(T)$ we have $\epsilon x_\nu = \lambda x_\nu$. By the universal property of the morphism $i : E \to \mathcal{U}\mathcal{P}_0$, there exists a unique $e_\nu \in E(n_\nu)$ such that $ie_\nu = x_\nu$. We take

$$\phi x = \prod_{\nu \in V(T)} e_\nu \in \mathcal{F}E(n).$$

We show that the map $x \mapsto \phi x$ defines a morphism of operads $\phi : \mathcal{P}_0 \to \mathcal{F}E$. Let $x \in \mathcal{P}_0(r)$ and $x_i \in \mathcal{P}_0(n_i)$ for $i = 1, \dots, r$. The equality $\phi x(x_1, \dots, x_r) = \phi x \times \prod_{i=1}^{r} \phi x_i$ holds since λ is a morphism of operads, which is inverse to $\hat{i}$. $\square$

Proposition 3.2.16. *For each operad $\mathcal{P}$, the operad $\mathcal{C}(\mathcal{P})$ is cofibrant, and $\mathcal{C}$ extends to a functor*

$$\mathcal{C} : \mathrm{Op}_{\mathbf{Cat}} \to \mathrm{Op}_{\mathbf{Cat}}.$$

The operadic composition in $\mathcal{P}$ induces a morphism of operads $\mathcal{C}(\mathcal{P}) \overset{\sim}{\twoheadrightarrow} \mathcal{P}$ which is an acyclic fibration. Therefore, $\mathcal{C}$ provides a functorial cofibrant replacement for categorical operads.

Proof. Any morphism of operads $\mathcal{P} \to \mathcal{Q}$ extends functorially to a morphism of operads $\mathcal{C}(\mathcal{P}) \to \mathcal{C}(\mathcal{Q})$, in a way which is compatible with the replacement morphisms of $\mathcal{P}$ and $\mathcal{Q}$. $\square$

Definition 3.2.17. We say that a symmetric categorical operad is $\mathfrak{S}$-cofibrant if it is cofibrant as a symmetric categorical sequence.

Definition 3.2.18. We say that a symmetric operad $\mathcal{P}$ is free on its objects if there exists a sequence of sets $E : *^{\mathbb{N}} \to \mathrm{Set}^{\mathbb{N}}$ and an isomorphism of symmetric operads $\mathcal{F}^{\mathfrak{S}}(E) \cong \mathcal{P}_0$.

Definition 3.2.19. Let $\mathcal{P}$ be a symmetric operad. We define a symmetric sequence $\mathcal{C}(\mathcal{P})$ in **Cat** as follows:

- We let the symmetric sequence of objects of $\mathcal{C}(\mathcal{P})$ be freely generated by the objects of $\mathcal{P}$, so that $\mathcal{C}(\mathcal{P})_0 = \mathcal{F}^{\mathfrak{S}}(\mathcal{U}^{\mathfrak{S}}\mathcal{P}_0)$.
- Let $r \in \mathbb{N}$ and $p, q : * \to \mathcal{C}(\mathcal{P})_0(r)$. We define the set of morphisms in $\mathcal{C}(\mathcal{P})(r)$ from p to q by

$$\mathcal{C}(P)(r)(p, q) = \mathcal{P}(r)\left(\mu(p), \mu(q)\right),$$

  and we equip this set with the action of the symmetric group induced by the action of Σ_r on $\mathcal{P}(r)$.

Proposition 3.2.20. *The symmetric sequence $\mathcal{C}(\mathcal{P})$ is naturally equipped with the structure of an operad in* **Cat**. *Moreover, the operadic composition of $\mathcal{P}$ induces an acyclic fibration of symmetric categorical operads.*

$$\pi : \mathcal{C}(\mathcal{P}) \overset{\sim}{\twoheadrightarrow} \mathcal{P}.$$

Proof. Let $r, n_1, \dots, n_r \in \mathbb{N}$. We define the composition functor

$$\int^{r:*\to\mathfrak{S}} \mathcal{C}(\mathcal{P})(r) \times \prod_{i=1}^{r} \mathcal{C}(\mathcal{P})(n_i) \to \mathcal{C}(\mathcal{P})(n_1 + \cdots + n_r)$$

by the operadic composition of the free operad $\mathcal{C}(\mathcal{P}_0) = \mathcal{F}^{\mathfrak{S}}\mathcal{U}^{\mathfrak{S}}\mathcal{P}_0$ on objects, and we proceed as follows for morphisms.

Let $p, q : * \to \mathcal{C}(\mathcal{P})_0(r)$ and $p_i, q_i : * \to \mathcal{C}(\mathcal{P})_0(n_i)$. Let $n = n_1 + \cdots + n_r$. We define the operadic composition for morphisms in $\mathcal{C}(\mathcal{P})$ as the map

$$\mathcal{C}(\mathcal{P})(r)(p,q) \times \prod_{i=1}^{r} \mathcal{C}(\mathcal{P})(n_i)(p_i,q_i) \to \mathcal{C}(\mathcal{P})(n)(p(p_1,\dots,p_r), q(q_1,\dots,q_r))$$

given by the composition of morphisms in $\mathcal{P}$

$$\mathcal{P}(r)(\mu p,\mu q)\times\prod_{i=1}^{r}\mathcal{P}(n_i)(\mu p_i,\mu q_i) \to \mathcal{P}(n)\left(\mu\left[\mu p;(\mu p_1,\dots,\mu p_r)\right],\mu\left[\mu q;(\mu q_1,\dots,\mu q_r)\right]\right),$$

The associativity of the composition provides an equality between the composite operations

$$\mu(\mu p;(\mu p_1,\dots,\mu p_r)) = \mu(p;(p_1,\dots,p_r)).$$

From the definition of $\mathcal{C}(\mathcal{P})$, it is straightforward to see that the counit $\mathcal{F}^{\mathfrak{S}}\mathcal{U}^{\mathfrak{S}}\mathcal{P}_0 \to \mathcal{P}_0$ of the free-forgetful adjunction

$$\mathcal{F}^{\mathfrak{S}} : \mathrm{SET}^{\mathbb{N}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{OP}^{\mathfrak{S}}_{\mathrm{SET}} : \mathcal{U}^{\mathfrak{S}}.$$

induces an acyclic fibration of symmetric operads $\pi : \mathcal{C}(\mathcal{P}) \overset{\sim}{\twoheadrightarrow} \mathcal{P}$. $\square$

PROPOSITION 3.2.21. *A symmetric categorical operad is cofibrant if and only if it is free on its objects.*

PROOF. Let $\mathcal{P}$ be a symmetric categorical operad which is free on its objects. Let $E : *^{\mathbb{N}} \to Sets^{\mathbb{N}}$ and let $\phi : \mathcal{F}^{\mathfrak{S}}(E) \xrightarrow{\cong} \mathcal{P}_0$ be an isomorphism of operads. Let $\pi : \mathcal{Q} \overset{\sim}{\twoheadrightarrow} \mathcal{R}$ be an acyclic fibration and $f : \mathcal{P} \to \mathcal{R}$ be a morphism of symmetric operads. We have $\mathcal{F}^{\mathfrak{S}}E \cong \mathfrak{S} \times \mathcal{F}E$, so that we obtain a morphism of operads on objects $\bar{f}$ such that the diagram

$$\begin{tikzcd} & U\mathcal{Q}_0 \arrow[d, two heads, "\pi"] \\ \mathcal{F}E \arrow[r, "f"'] \arrow[ur, dashed, "\bar{f}"] & U\mathcal{R}_0 \end{tikzcd}$$

commutes. By adjunction we obtain a morphism of symmetric operads $\bar{f}^{\mathfrak{S}} : \mathfrak{S} \times \mathcal{F}E \to \mathcal{Q}_0$ which lifts f. Since any acyclic fibration induces isomorphisms at the level of morphisms, we can lift f at the level of morphisms as well. Hence any symmetric categorical operad which is free on objects is cofibrant.

Conversely, suppose that $\mathcal{P}$ is a symmetric cofibrant operad. In particular, both the operad $U\mathcal{P}$ and the symmetric sequence $\mathcal{U}\mathcal{P}$ are cofibrant. Let $E : \mathbb{N} \to \mathrm{SET}$ be a sequence such that $\mathcal{F}E \cong U\mathcal{P}_0$ in $\mathrm{OP}_{\mathbf{Cat}}$ and let $G : \mathfrak{S} \to \mathrm{SET}$ be $\mathcal{U}\mathcal{P}_0 \cong \Sigma \times G$. We have an isomorphism $U(\Sigma \times G) \cong \mathcal{U}\mathcal{F}E$, so that for each $r \in \mathbb{N}$, we have an isomorphism $\mathcal{F}G(r) \cong \Sigma_r \times E(r)$. $\square$

EXAMPLE 3.2.22. The operads $\mathcal{M}^n$ are $\mathfrak{S}$-cofibrant but not cofibrant. Indeed, their objects are subject to associativity and unit relations.

PROPOSITION 3.2.23. *For each symmetric operad $\mathcal{P}$, the symmetric operad $\mathcal{C}(\mathcal{P})$ is cofibrant, and $\mathcal{C}$ extends to a functor*

$$\mathcal{C} : \mathrm{OP}^{\mathfrak{S}}_{\mathbf{Cat}} \to \mathrm{OP}^{\mathfrak{S}}_{\mathbf{Cat}}.$$

The operadic composition in $\mathcal{P}$ induces a morphism of operads $\mathcal{C}(\mathcal{P}) \xrightarrow{\sim} \mathcal{P}$ which is an acyclic fibration. Therefore, $\mathcal{C}$ provides a functorial cofibrant replacement for symmetric categorical operads.

Proof. Any morphism of operads $\mathcal{P} \to \mathcal{Q}$ extends functorially to a morphism of operads $\mathcal{C}(\mathcal{P}) \to \mathcal{C}(\mathcal{Q})$ in a way that is compatible with the replacement morphisms of $\mathcal{P}$ and $\mathcal{Q}$. $\square$

Remark 3.2.24. Let $\mathcal{P}$ be a symmetric categorical operad and suppose that $\mathcal{P}$ is $\mathfrak{S}$-cofibrant, so that its sequence of objects $\mathcal{P}_0$ is $\mathfrak{S}$-free. Let $\mathcal{C}^{\mathbb{N}}\mathcal{P} : * \to \mathrm{Op}^{\mathfrak{S}}_{\mathbf{Cat}}$ be such that

- $\mathcal{C}^{\mathbb{N}}(\mathcal{P})_0 = \mathcal{F}\mathcal{U}\mathcal{P}_0$
- For $p, q : * \to \mathcal{C}^{\mathbb{N}}(\mathcal{P})_0(r)$ we define $\mathcal{C}^{\mathbb{N}}(\mathcal{P})(p,q) = \mathcal{P}(\mu p, \mu q)$.

The canonical cofibrant resolution of $\mathcal{P}$ factors as

$$\mathcal{C}\mathcal{P} \xrightarrow{\sim} \mathcal{C}^{\mathbb{N}}\mathcal{P} \xrightarrow{\sim} \mathcal{P}$$

There is a natural acyclic fibration $\mathcal{C}^{\mathbb{N}}\mathcal{P} \xrightarrow{\sim} \mathcal{P}$. The symmetric group acts freely on the objects of $\mathcal{C}^{\mathbb{N}}(\mathcal{P})$, so that $\mathcal{C}^{\mathbb{N}}(\mathcal{P})$ provides a smaller cofibrant resolution of $\mathcal{P}$.

Application to the monoidal category operad $\mathcal{M}_1$. Recall that the operad $\mathcal{M}_1$ governing monoidal categories can be defined both as a symmetric operad and as a non symmetric operad. We write $\mathcal{A}s$ for the monoidal category operad and $\mathcal{A}s^{\mathfrak{S}}$ for the monoidal category symmetric operad. We apply the results of the previous subsection to construct a cofibrant model of both $\mathcal{A}s$ and $\mathcal{A}s^{\mathfrak{S}}$.

Definition 3.2.25. Let $\mathcal{A}s^{\mathfrak{S}}_{\infty} \in \mathrm{Op}^{\mathfrak{S}}_{\mathbf{Cat}}$ be the operad defined as $\mathcal{A}s^{\mathfrak{S}}_{\infty} = \mathcal{C}^{\mathbb{N}}(\mathcal{M}_1)$. The isofibration $\mathcal{A}s^{\mathfrak{S}}_{\infty} \xrightarrow{\sim} \mathcal{A}s^{\mathfrak{S}}$ provides a cofibrant resolution of $\mathcal{A}s^{\mathfrak{S}}$.

The operad $\mathcal{A}s^{\mathfrak{S}}_{\infty}$ is freely generated on objects by the symmetric sequence given in arity $r \in \mathbb{N}$ by the symmetric group Σ_r, with a symmetric action given by translation. There is a morphism between two operations of each arity if and only if their operadic composite in $\mathcal{A}s^{\mathfrak{S}}$ correspond to the same permutation. Hence, the operad $\mathcal{A}s^{\mathfrak{S}}_{\infty}$ is a poset. It is convenient to consider this operad as a non symmetric operad in order to have an easier description of a cofibrant model.

Definition 3.2.26. Let $\mathcal{A}s_{\infty} \in \mathrm{Op}_{\mathbf{Cat}}$ be the operad defined as $\mathcal{A}s_{\infty} = \mathcal{C}(\mathcal{A}s)$. The isofibration $\mathcal{A}s_{\infty} \xrightarrow{\sim} \mathcal{A}s$ provides a cofibrant resolution of $\mathcal{A}s$. The operad $\mathcal{A}s_{\infty}$ is freely generated by one operation in each arity on its objects, with a unique morphism between each pair of operations.

Definition 3.2.27. Define the category of $\mathcal{A}s_{\infty}$ monoidal small categories as the category $\mathcal{A}s_{\infty}$-*Alg*. An $\mathcal{A}s_{\infty}$ monoidal small category consisits in the data of a small category $\mathcal{C}$ equipped with

- for each $n \in \mathbb{N}$, a functor $\mu_n : \mathcal{C}^n \to \mathcal{C}$
- for each $r \in \mathbb{N}$ and $n_1, \ldots, n_r \in \mathbb{N}$, a natural transformation

$$\begin{tikzcd} \prod_{i=1}^r \mathcal{C}^{n_i} \arrow[r, "\prod_{i=1}^r \mu_{n_i}"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\overset{\cong}{\swarrow}"] & \mathcal{C}^r \arrow[d, "\mu_r"] \\ \mathcal{C}^n \arrow[r, "\mu_n"'] & \mathcal{C} \end{tikzcd}$$

which is an isomorphism, and such that any two ways of producing a natural transformation between functors involving compositions of the functors $(\mu_p)_p$ are equal.

REMARK 3.2.28. Let us insist that in Definition 3.2.27 the natural isomorphisms ensuring associativity are required to satisfy a condition on all possible composites of the generating functors. Consequently, it may be difficult to verify such a condition. In the next chapter, we will provide a presentation of $\mathcal{A}s_\infty$ in terms of generators and relations, from which we will extract more accurate coherence conditions. In this way, the commutativity of specific diagrams will be enough to deduce the commutativity of all the required diagrams.

REMARK 3.2.29. In [**10**], Fiedorowicz, Gubkin and Vogt defined a categorical operad k providing a categorical analogue of Stasheff's associahedra. The operad k is a contractible poset whose objects are freely generated by an operation in each arity, and whose order is given by Tamari's. They defined an $\mathcal{A}_\infty$-monoidal category as a small category equipped with the structure of an algebra over k. They proved that an $\mathcal{A}_\infty$-monoidal category is equivalent to a monoidal category as soon as all the natural transformations involved are isomorphisms. Such an $\mathcal{A}_\infty$-monoidal category corresponds to an algebra over the operad $\mathcal{A}s_\infty$ that we define. However, they only considered A_∞-monoidal categories with a strict unit, whereas an algebra over our operad $\mathcal{A}s_\infty$ is required to satisfy unit axioms only up to isomorphism, so that the operad $\mathcal{A}s_\infty$ has infinitely many operations in each arity. We will relax the isomorphism conditions and provide a lax form of our operad $\mathcal{A}s_\infty$ after having defined polygraphic presentations of operads. This lax version precisely corresponds to the operad k, except for the fact that again, we consider weaker unit conditions. It also is worth notifying that this lax or directed version of the operad $\mathcal{A}s_\infty$ can not be equivalent to the operad $\mathcal{A}s$ governing monoidal categories in the model structure that we use. Indeed, the equivalences we defined involve the strong notion of an equivalences of categories. The directed version of $\mathcal{A}s_\infty$ will instead be topologically equivalent to $\mathcal{A}s$ after passing to nerve and geometric realization.

Recall that any small categorical operad can also be seen as an operad in the 2-category of all categories $(\mathrm{CAT}, \times, *)$. From now we will see the operad $\mathcal{A}s_\infty$ as such. In this way, we will be able to define $\mathcal{A}s_\infty$-algebras in any monoidal 2-category, even if its monoidal structure is not symmetric. For the moment, we only give a formal definition of this weakened version of monoids in a monoidal 2-category. We will provide an explicit description of them after having introduced suitable generators for operads in **Cat**.

DEFINITION 3.2.30. Let $(\Lambda, \otimes_\Lambda, 1_\Lambda)$ be a monoidal 2-category. We define an $\mathcal{A}s_\infty$ monoid in Λ as an algebra over the CAT-operad $\mathcal{A}s_\infty$. More precisely, we define the 2-category of $\mathcal{A}s_\infty$-algebras in Λ as the 2-category $\mathcal{A}s_\infty\text{-}Alg_{(\Lambda,\otimes_\Lambda,1_\Lambda)}$.

REMARK 3.2.31. Note that we also obtain cofibrant models of the operads $\mathcal{M}^n$ for $n > 1$ from the canonical cofibrant resolution constructed in Remark 3.2.24. However, the objects of $\mathcal{C}^{\mathbb{N}}(\mathcal{M}^n)$ are freely generated by the underlying symmetric sequence of the objects of $\mathcal{M}^n$, but still lack of an explicit description. We will fix this issue in two steps. One of them consists in introducing suitable generators for operads defined in the category of small categories. Next, we will construct a tensor product for **Cat**-operads and their generators. Observing that this tensor product behaves well with respect to the model structure we considered in $\mathrm{Op}_{\mathbf{Cat}}$, we will deduce an explicit cofibrant model of $\mathcal{M}^n$ from the description of the operad $\mathcal{A}s_\infty$ in terms of generators of relations.

3.3. Operadic polygraphs and presentations of small categorical operads. We want to construct categorical operads whose generating morphisms can have their source and target expressed as operadic composition of generating objects rather than generating objects only. For this purpose, we define a kind of generators that we will call operadic polygraphs by analogy with polygraphs for ω-categories. In this way, an operad generated by an operadic polygraph will be aritywise a category generated by operadic compositions of the generating morphisms. The interaction between the operadic and the categorical structure will induce some relations on the morphisms, so that in each arity we do not have a free category.

Operads generated by polygraphs. We introduce operadic polygraphs and use them as convenient generators for small symmetric categorical operads. Let Gph_n be the category of n-graphs (see for instance [**18**] or [**25**]), which we equip with its cartesian monoidal category structure. We write $\mathbf{Cat}_n$ for the category of small n-categories, which we also equip with its cartesian monoidal category structure. A small n-category may be defined as an n-graph equipped with additional structure allowing compositions of p-morphisms for $p = 0, \ldots, n$. In particular, we have an adjunction

$$\begin{tikzcd} Gph_n \arrow[r, bend left, "F^{(n)}"] \arrow[r, phantom, "\perp"] & \mathbf{Cat}_n \arrow[l, bend left, "U^{(n)}"] \end{tikzcd}$$

for each $n \in \mathbb{N}$, which is also monoidal, so that it induces an adjunction between the resulting categories of symmetric operads

$$\begin{tikzcd} \mathrm{Op}_{Gph_n} \arrow[r, bend left, "\mathcal{F}^{(n)}"] \arrow[r, phantom, "\perp"] & \mathrm{Op}_{\mathbf{Cat}_n} \arrow[l, bend left, "\mathcal{U}^{(n)}"] \end{tikzcd} .$$

Again, we refer to [**18**] or [**25**] for further details on higher small categories, and focus on the lower cases $n = 0$ and $n = 1$ for the purpose of this thesis. However, it is worth observing that the constructions we establish in this subsection may be generalized to higher values of n (see [**23**] for the $\mathbb{K}$-linear case). We set $Gph_{-1} = \mathbf{Cat}_{-1} = *$ for convenience. Note that $Gph_0 = \mathbf{Cat}_0 = \mathrm{Set}$, and that $\mathbf{Cat}_1 = Cat$.

In general, we adopt the notation $\mathcal{C}_r$ for the set of r-cells of an n-graph, respectively, of a small n-category. In the case of a categorical n-operad $\mathcal{P}$, the sequence $\mathcal{P}_r$, formed by the sets of r-cells in $\mathcal{P}$, has the structure of an operad in the category of sets.

Let C be a small n-category and $x, y \in C_{p-1}$ be p-1-cells of C. We write $C_p(x, y)$ for the subset of the set of p-cells of C obtained by the pull back

$$\begin{tikzcd} C_p(x,y) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "{(x,y)}"] \\ C_p \arrow[r, "s\times t"'] & C_{p-1}\times C_{p-1}. \end{tikzcd}$$

Write C_{p-1} for the discrete category associated to the set C_{p-1}, we obtain a functor $C_p(-,-) : C_{p-1} \times C_{p-1} \to \mathrm{SET}$ which to (x,y) associates the set $C_p(x,y)$.

Each n-graph has an underlying n-1-graph when forgetting about the n-cells. In the same way, each n-category has an underlying n-1-category. The induced forgetful functors are both monoidal, hence induces forgetful functors between their resulting categories of operads. We can summarize the compatibility of these functors as the commutativity of the following cube of adjunctions

$$\begin{tikzcd}[column sep=small] & & Gph^{\mathfrak{S}}_{n-1} \arrow[r, shift left, "T^{(n-1)}"] \arrow[d, shift right, "F^{(n-1)\mathfrak{S}}"'] & \mathrm{OP}_{Gph_{n-1}} \arrow[l, shift left, "V^{(n-1)}"] \arrow[d, shift right, "\mathcal{F}^{(n-1)}"'] \\ & & \mathbf{Cat}^{\mathfrak{S}}_{n-1} \arrow[r, shift left, "\mathcal{T}^{(n-1)}"] \arrow[u, shift right, "U^{(n-1)\mathfrak{S}}"'] & \mathrm{OP}_{\mathbf{Cat}_{n-1}} \arrow[l, shift left, "\mathcal{V}^{(n-1)}"] \arrow[u, shift right, "\mathcal{U}^{(n-1)}"'] \\ Gph^{\mathfrak{S}}_{n} \arrow[r, shift left, "T^{(n)}"] \arrow[d, shift right, "F^{(n)\mathfrak{S}}"'] \arrow[uurr, dotted, "\Pi^{(n)}"] & \mathrm{OP}_{Gph_{n}} \arrow[l, shift left, "V^{(n)}"] \arrow[d, shift right, "\mathcal{F}^{(n)}"'] \arrow[uurr, dotted] & & \\ \mathbf{Cat}^{\mathfrak{S}}_{n} \arrow[r, shift left, "\mathcal{T}^{(n)}"] \arrow[u, shift right, "U^{(n)\mathfrak{S}}"'] \arrow[uurr, dotted] & \mathrm{OP}_{\mathbf{Cat}_{n}}, \arrow[l, shift left, "\mathcal{V}^{(n)}"] \arrow[u, shift right, "\mathcal{U}^{(n)}"'] \arrow[uurr, dotted] & & \end{tikzcd}$$

where the dotted arrows forget about the n-cells. Let

$$\mathcal{O}^{(n)} : \; \begin{tikzcd} Gph^{\mathfrak{S}}_n \arrow[r, shift left, bend left] \arrow[r, phantom, "\perp"] & \mathrm{OP}_{\mathbf{Cat}_n} \arrow[l, shift left, bend left] \end{tikzcd} \; : \mathcal{Y}^{(n)}$$

be the functors that result from the diagonal adjunction in the above square. Let $0 \le p, q \le n$. Recall that each small n-category C yields a category $\Pi^{(n)}_{p,q}C$ whose underlying graph is given by $P_q \rightrightarrows P_p$, so that $\Pi^{(n)}_{p,q}$ defines both a functor $\mathbf{Cat}_n \to \mathbf{Cat}$ and a functor $Gph_n \to Gph$. These functors are compatible with the free forgetful functors, so that the following diagram commutes

$$\begin{tikzcd} Gph_n \arrow[r, "\Pi^{(n)}_{p,q}"] \arrow[d, shift right, "F^{(n)}"'] & Gph \arrow[d, shift right, "F^{(1)}"'] \\ \mathbf{Cat}_n \arrow[r, "\Pi^{(n)}_{p,q}"] \arrow[u, shift right, "U^{(n)}"'] & \mathbf{Cat} \arrow[u, shift right, "U^{(1)}"'] \end{tikzcd} .$$

Moreover, the functors $\Pi^{(n)}_{p,q}$ are strongly monoidal, and hence induce functors on the resulting categories of operads. We therefore also have commutative cubes

$$\begin{tikzcd}
 & & Gph_1^{\mathfrak{S}} \arrow[r, shift left, "T^{(1)}"] \arrow[d, shift right, "F^{(1)\mathfrak{S}}"'] & \mathrm{Op}_{Gph_1} \arrow[l, shift left, "V^{(1)}"] \arrow[d, shift right, "\mathcal{F}^{(1)}"'] \\
 & & \mathbf{Cat}_1^{\mathfrak{S}} \arrow[r, shift left, "\mathcal{T}^{(1)}"] \arrow[u, shift right, "U^{(1)\mathfrak{S}}"'] & \mathrm{Op}_{\mathbf{Cat}_1} \arrow[l, shift left, "\mathcal{V}^{(1)}"] \arrow[u, shift right, "\mathcal{U}^{(1)}"'] \\
Gph_n^{\mathfrak{S}} \arrow[r, shift left, "T^{(n)}"] \arrow[d, shift right, "F^{(n)\mathfrak{S}}"'] \arrow[uurr, dotted, "\Pi^{(n)}_{p,q}"] & \mathrm{Op}_{Gph_n} \arrow[l, shift left, "V^{(n)}"] \arrow[d, shift right, "\mathcal{F}^{(n)}"'] \arrow[uurr, dotted] & & \\
\mathbf{Cat}_n^{\mathfrak{S}} \arrow[r, shift left, "\mathcal{T}^{(n)}"] \arrow[u, shift right, "U^{(n)\mathfrak{S}}"'] \arrow[uurr, dotted] & \mathrm{Op}_{\mathbf{Cat}_n}. \arrow[l, shift left, "\mathcal{V}^{(n)}"] \arrow[u, shift right, "\mathcal{U}^{(n)}"'] \arrow[uurr, dotted] & &
\end{tikzcd}$$

LEMMA 3.3.1. *The category of $n+1$-graphs Gph_{n+1} can be obtained from the category of n-graphs by adding $n+1$-cells, in the sense that the following diagram is a pull back*

$$\begin{tikzcd}
Gph_{n+1} \arrow[r, dashed, "\Pi^{(n+1)}_{n,n+1}"] \arrow[d, dashed, "\Pi^{(n+1)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Gph \arrow[d, "\Pi^{(1)}_0"] \\
Gph_n \arrow[r, "\Pi^{(n)}_n"'] & \mathrm{SET}
\end{tikzcd} .$$

DEFINITION 3.3.2. Let $\mathrm{Op}^+_{\mathbf{Cat}_n}$ be the category defined as the result of the pull back

$$\begin{tikzcd}
\mathrm{Op}^+_{\mathbf{Cat}_n} \arrow[r, dashed, "\mathcal{Y}^{(n)}_+"] \arrow[d, dashed, "\Pi^{(n)}_+"'] \arrow[dr, phantom, "\lrcorner", very near start] & Gph^{\mathfrak{S}}_{n+1} \arrow[d, "\Pi^{(n)}"] \\
\mathrm{Op}_{\mathbf{Cat}_n} \arrow[r, "\mathcal{Y}^{(n)}"] & Gph^{\mathfrak{S}}_n .
\end{tikzcd}$$

An object $\mathcal{P}$ of $\mathrm{Op}^+_{\mathbf{Cat}_n}$ corresponds to the data of an operad $\mathcal{P}_{\leq n} = \Pi^{(n)}_+\mathcal{P}$ in $\mathbf{Cat}_n$ and of a symmetric sequence of $n+1$-graphs $P = \mathcal{Y}^{(n)}_+\mathcal{P}$ such that the underlying sequence of n-graphs $P_{\leq n} = \Pi^{(n)}P$ of P is identified with the underlying n-graph $\mathcal{Y}^{(n)}\mathcal{P}_{\leq n}$ of $\mathcal{P}$. Let $\mathcal{P}$ and $\mathcal{Q}$ be objects of $\mathrm{Op}^+_{\mathbf{Cat}_n}$. The set $\mathrm{Op}^+_{\mathbf{Cat}_n}(\mathcal{P},\mathcal{Q})$ is obtained by the pull back

$$\begin{tikzcd}
\mathrm{Op}^+_{\mathbf{Cat}_n}(\mathcal{P},\mathcal{Q}) \arrow[r, dashed, "\mathcal{Y}^{(n)}_+"] \arrow[d, dashed, "\Pi^{(n)}_+"'] \arrow[dr, phantom, "\lrcorner", very near start] & Gph^{\mathfrak{S}}_{n+1}(P,Q) \arrow[d, "\Pi^{(n)}"] \\
\mathrm{Op}_{\mathbf{Cat}_n}(\mathcal{P}_{\leq n},\mathcal{Q}_{\leq n}) \arrow[r, "\mathcal{Y}^{(n)}"] & Gph^{\mathfrak{S}}_n(P_{\leq n},Q_{\leq n}).
\end{tikzcd}$$

LEMMA 3.3.3. *The data of an object of $\mathrm{Op}^+_{\mathbf{Cat}_n}$ consists in the data of an operad $\mathcal{P}$ in $\mathbf{Cat}_n$ together with a symmetric sequence P_{n+1} of additional $n+1$-cells and morphisms $s,t : P_{n+1} \to \mathcal{P}_n$ such that the following diagrams commute*

$$\begin{tikzcd}
P_{n+1} \arrow[r, "s"] \arrow[d, "t"'] & \mathcal{P}_n \arrow[d, "s_n"] \\
\mathcal{P}_n \arrow[r, "s_n"'] & \mathcal{P}_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
P_{n+1} \arrow[r, "s"] \arrow[d, "t"'] & \mathcal{P}_n \arrow[d, "t_n"] \\
\mathcal{P}_n \arrow[r, "t_n"'] & \mathcal{P}_{n-1}.
\end{tikzcd}$$

Formally, the category $\mathrm{OP}^{+}_{\mathbf{Cat}_n}$ *exhibits the fiber product* $\mathrm{OP}_{\mathbf{Cat}_n} \underset{\mathrm{SET}^{\mathfrak{S}}}{\times} Gph^{\mathfrak{S}}$*, so that the following diagram is a pull back*

$$\begin{tikzcd}
\mathrm{OP}^{+}_{\mathbf{Cat}_n} \arrow[r, dashed, "\Pi^{(n)}_{1,0}\mathcal{Y}^{(n)}_{+}"] \arrow[d, dashed, "\Pi^{(n)}_{+}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Gph^{\mathfrak{S}} \arrow[d, "\Pi^{(1)}_{1}"] \\
\mathrm{OP}_{\mathbf{Cat}_n} \arrow[r, "(\Pi\mathcal{Y})^{(n)}_{n}"'] & \mathrm{SET}^{\mathfrak{S}}.
\end{tikzcd}$$

Let $\mathcal{P}, \mathcal{Q} : * \to \mathrm{OP}^{+}_{\mathbf{Cat}_n}$*. The set of morphisms* $\mathrm{OP}^{+}_{\mathbf{Cat}_n}(\mathcal{P}, \mathcal{Q})$ *is therefore obtained by the following pull back*

$$\begin{tikzcd}
\mathrm{OP}^{+}_{\mathbf{Cat}_n}(\mathcal{P}, \mathcal{Q}) \arrow[r, dashed, "\Pi^{(n)}_{1,0}\mathcal{Y}^{(n)}_{+}"] \arrow[d, dashed, "\Pi^{(n)}_{+}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Gph^{\mathfrak{S}}(P_{n,n+1}, Q_{n,n+1}) \arrow[d, "\Pi^{(1)}_{1}"] \\
\mathrm{OP}_{\mathbf{Cat}_n}(\mathcal{P}_{\leq n}, \mathcal{Q}_{\leq n}) \arrow[r, "(\Pi\mathcal{Y})^{(n)}_{n}"'] & \mathrm{SET}^{\mathfrak{S}}(P_n, Q_n).
\end{tikzcd}$$

It follows that the data of a morphism $f = \mathcal{P} \to \mathcal{Q}$ *in* $\mathrm{OP}^{+}_{\mathbf{Cat}_n}$ *is equivalent to the data a morphism of operads* $f_{\bullet} : \mathcal{P}_{\leq n} \to \mathcal{Q}_{\leq n}$ *together with a morphism of symmetric sequences* $f_{n+1} : P_{n+1} \to Q_{n+1}$ *such that the following diagrams commute*

$$\begin{tikzcd}
P_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "s"'] & Q_{n+1} \arrow[d, "s"] \\
\mathcal{P}_n \arrow[r, "f_n"'] & \mathcal{Q}_n
\end{tikzcd}
\qquad
\begin{tikzcd}
P_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "t"'] & Q_{n+1} \arrow[d, "f_n"] \\
\mathcal{P}_n \arrow[r, "t"'] & \mathcal{Q}_n.
\end{tikzcd}$$

PROOF. We deduce the following sequence of isomorphisms from Lemma 3.3.1 and by associativity of the fiber product

$$\mathrm{OP}^{+}_{\mathbf{Cat}_n} \cong \mathrm{OP}_{\mathbf{Cat}_n} \underset{Gph^{\mathfrak{S}}_{n}}{\times} Gph^{\mathfrak{S}}_{n+1} \cong \mathrm{OP}_{\mathbf{Cat}_n} \underset{Gph^{\mathfrak{S}}_{n}}{\times} \left(Gph^{\mathfrak{S}}_{n} \underset{\mathrm{SET}^{\mathfrak{S}}}{\times} Gph^{\mathfrak{S}} \right) \cong \mathrm{OP}_{\mathbf{Cat}_n} \underset{\mathrm{SET}^{\mathfrak{S}}}{\times} Gph^{\mathfrak{S}}.$$

$\square$

DEFINITION 3.3.4. Let $\mathcal{T}^{(n+1)}_{+} : \mathrm{OP}^{+}_{\mathbf{Cat}_n} \to \mathrm{OP}_{\mathbf{Cat}_{n+1}}$ be the functor such that, for each object $\mathcal{P}$ of $\mathrm{OP}^{+}_{\mathbf{Cat}_n}$, we have

$$\mathrm{OP}_{\mathbf{Cat}_{n+1}}\left(\mathcal{T}^{(n+1)}_{+}\mathcal{P}, \mathcal{Q}\right) \cong \mathrm{OP}_{\mathbf{Cat}_n}(\mathcal{P}_{\leq n}, \mathcal{Q}_{\leq n}) \underset{\mathrm{SET}^{\mathfrak{S}}(P_n, Q_n)}{\times} Gph^{\mathfrak{S}}(P_{n,n+1}, Q_{n,n+1}).$$

The operad $\mathcal{T}^{(n+1)}_{+}\mathcal{P}$ has the same underlying n-categorical operad as $\mathcal{P}$ and has all operadic and categorical compositions of elements of P_{n+1} for its $n+1$-cells.

Proposition 3.3.5. *Let* $\mathcal{W}^{(n+1)} : \mathrm{Op}_{\mathbf{Cat}_{n+1}} \to \mathrm{Op}^+_{\mathbf{Cat}_n}$ *be the functor induced by the functors* $\mathcal{Y}^{(n+1)}$ *and* $\Pi^{(n)}$*, so that*

$$\begin{tikzcd}
\mathrm{Op}_{\mathbf{Cat}_{n+1}} \arrow[dr, dashed, "\mathcal{W}^{(n+1)}"] \arrow[drr, bend left, "\mathcal{Y}^{(n+1)}"] \arrow[ddr, bend right, "\Pi^{(n)}"'] & & \\
& \mathrm{Op}^+_{\mathbf{Cat}_n} \arrow[r, dashed, "Y_+^{(n)}"] \arrow[d, dashed, "\Pi_+^{(n)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Gph^{\mathfrak{S}}_{n+1} \arrow[d, "\Pi^{(n)}"] \\
& \mathrm{Op}_{\mathbf{Cat}_n} \arrow[r, "\mathcal{Y}^{(n)}"] & Gph^{\mathfrak{S}}_n .
\end{tikzcd}$$

The functor $\mathcal{T}_+^{(n+1)}$ *is left adjoint to* $\mathcal{W}^{(n+1)}$*, so that we have the following adjunction*

$$\mathcal{T}_+^{(n+1)} : \ \mathrm{Op}^+_{\mathbf{Cat}_n} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Op}_{\mathbf{Cat}_{n+1}} \ : \mathcal{W}^{(n+1)}.$$

Proof. We have the following isomorphisms for each object $\mathcal{P}$ of $\mathrm{Op}^+_{\mathbf{Cat}_n}$ and each object $\mathcal{Q}$ of $\mathrm{Op}_{\mathbf{Cat}_n}$:

$$\begin{aligned}
\mathrm{Op}^+_{\mathbf{Cat}_n}\left(\mathcal{P}, \mathcal{W}^{(n+1)}\mathcal{Q}\right) &\cong \mathrm{Op}_{\mathbf{Cat}_n}\left(\mathcal{P}_{\leq n}, \mathcal{Q}_{\leq n}\right) \underset{\mathrm{Set}^{\mathfrak{S}}(P_n, Q_n)}{\times} Gph^{\mathfrak{S}}(P_{n,n+1}, Q_{n,n+1}) \\
&\cong \mathrm{Op}_{\mathbf{Cat}_{n+1}}\left(\mathcal{T}^{(n+1)}\mathcal{P}, \mathcal{Q}\right).
\end{aligned}$$

The first isomorphism is a consequence of Lemma 3.3.3, and the second isomorphism holds by definition of $\mathcal{T}_+^{(n+1)}\mathcal{P}$. ☐

Remark 3.3.6. For $n = -1$, we obtain $\mathrm{Op}^+_{\mathbf{Cat}_{-1}} \cong \mathrm{Set}^{\mathfrak{S}}$, and the previous constructions fit into the following diagram

$$\begin{tikzcd}
\mathrm{Op}_{\mathrm{Set}} \arrow[dr, dashed, bend left, "\mathcal{W}^{(0)}"] \arrow[drr, bend left, "\mathcal{Y}^{(0)}"] \arrow[ddr, bend right] & & \\
& \mathrm{Set}^{\mathfrak{S}} \arrow[ul, bend left, "\mathcal{T}^{(0)}"] \arrow[r, dashed, "id"] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Set}^{\mathfrak{S}} \arrow[d] \\
& *^{\mathfrak{S}} \arrow[r] & *^{\mathfrak{S}} .
\end{tikzcd}$$

We obtain the usual adjunction between symmetric sequences of sets and set-theoretical operads

$$\mathcal{T}^{(0)} : \ \mathrm{Set}^{\mathfrak{S}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Op}_{\mathrm{Set}} \ : \mathcal{W}^{(0)}.$$

Remark 3.3.7. For $n = 0$, we have the following diagram

$$\begin{tikzcd}
\mathrm{Op}_{\mathbf{Cat}} \arrow[dr, dashed, "\mathcal{W}^{(1)}"] \arrow[drr, bend left, "\mathcal{Y}^{(1)}"] \arrow[ddr, bend right, "\Pi^{(0)}"'] & & \\
& \mathrm{Op}^+_{\mathrm{Set}} \arrow[r, dashed, "Y_+^{(0)}"] \arrow[d, dashed, "\Pi_+^{(0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Gph^{\mathfrak{S}} \arrow[d, "\Pi^{(0)}"] \\
& \mathrm{Op}_{\mathrm{Set}} \arrow[r, "\mathcal{Y}^{(0)}"] & \mathrm{Set}^{\mathfrak{S}} .
\end{tikzcd}$$

In what follows, we sometimes omit the exponents and simply write $\mathcal{T}_+$ for $\mathcal{T}_+^{(1)}$ and $\mathcal{W}$ for $\mathcal{W}_+^{(1)}$. We obtain an adjunction

$$\mathcal{T}_+ : \ \mathrm{Op}^+_{\mathrm{Set}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Op}_{\mathbf{Cat}} \ : \mathcal{W}.$$

An object $\mathcal{P}$ of the category $\mathrm{Op}^+_{\mathrm{Set}}$ consists in the data of a set-theoretical operad $\mathcal{P}_0$ together with a symmetric sequence P_1 equipped with morphisms

$$s, t : P_1 \to \mathcal{P}_0.$$

Such an object $\mathcal{P}$ can equivalently be defined by the data of a set-operad $\mathcal{P}_0$ of objects, and for all $r \in \mathbb{N}$ and $x, y : * \to \mathcal{P}_0(r)$, of a set of 1-cells $P_1(r)(x, y)$ equipped with an action of Σ_r such that each σ acts as a morphism

$$\sigma : P_1(r)(x, y) \to P_1(r)(\sigma.x, \sigma.y).$$

An operadic n-polygraph $\mathcal{P}$ can also be inductively characterized as the data of

- an operad $\mathcal{P}_0 : * \to \mathrm{Op}_{\mathrm{Set}}$,
- for all $r \in \mathbb{N}$ and $x, y : * \to \mathcal{P}_0(r)$, an operadic n-1-polygraph $\mathcal{P}(r)(x, y)$
- a lax morphism

For $\mathcal{P}, \mathcal{Q} : * \to \mathrm{Op}^+_{\mathrm{Set}}$, the set of morphisms $\mathrm{Op}^+_{\mathrm{Set}}(\mathcal{P}, \mathcal{Q})$ has for elements the morphisms of operads $f_0 : \mathcal{P}_0 \to \mathcal{Q}_0$ equipped for each $r \in \mathbb{N}$ and each $x, y : * \to \mathcal{P}_0(r)$ with a morphism $f(r)(x, y) : P_1(r)(x, y) \to \mathcal{Q}_1(r)(f_0 x, f_0 y)$ which is invariant under the action of Σ_r.

Each object $\mathcal{P}$ of $\mathrm{Op}^+_{\mathrm{Set}}$ yields a categorical operad $\mathcal{T}_+\mathcal{P}$, whose operad of objects $\mathcal{T}_+\mathcal{P}_0 \cong \mathcal{P}_0$ is given by the set-operad $\mathcal{P}_0$, and whose morphisms are freely generated by the elements of the symmetric sequence P_1, both operadically and categorically.

Definition 3.3.8. For $n \in \mathbb{N}$, we let

$$\Pi, \mathcal{T} : \prod_{p=0}^{n} \mathrm{Op}^+_{\mathbf{Cat}_{p-1}} \to \prod_{p=0}^{n-1} \mathrm{Op}_{\mathbf{Cat}_p}$$

be the functors whose projections $\mathcal{T}_p, \Pi_p$, for $p = 0, \ldots, n-1$, are given by the composites

$$\mathcal{T}_p : \prod_{p=0}^{n} \mathrm{Op}^+_{\mathbf{Cat}_{p-1}} \xrightarrow{\pi_p} \mathrm{Op}^+_{\mathbf{Cat}_{p-1}} \xrightarrow{\mathcal{T}_+^{(p)}} \mathrm{Op}_{\mathbf{Cat}_p}$$

and

$$\Pi_p : \prod_{p=0}^{n} \mathrm{Op}^+_{\mathbf{Cat}_{p-1}} \xrightarrow{\pi_{p+1}} \mathrm{Op}^+_{\mathbf{Cat}_p} \xrightarrow{\Pi_+^{(p)}} \mathrm{Op}_{\mathbf{Cat}_p}.$$

We define the category $\mathcal{Q}\mathrm{Op}_{\mathbf{Cat}_n}$ of operadic n-polygraphs as the equalizer

$$\mathcal{Q}\mathrm{Op}_{\mathbf{Cat}_n} \dashrightarrow \prod_{p=0}^{n} \mathrm{Op}^+_{\mathbf{Cat}_{p-1}} \underset{\Pi}{\overset{\mathcal{T}}{\rightrightarrows}} \prod_{p=0}^{n-1} \mathrm{Op}_{\mathbf{Cat}_p}.$$

Hence, an operadic n-polygraph $\mathcal{E}$ consists in the data, for each $0 \leq p \leq n$, of

- an object $\mathcal{E}_{(p)} : * \to \mathrm{Op}^+_{\mathbf{Cat}_{p-1}}$
- and an isomorphism of p-categorical operads $\mathcal{T}_+^{(p)}\mathcal{E}_{(p)} \cong \Pi_+^{(p)}\mathcal{E}_{(p+1)}$.

The projections $\prod_{p=0}^{n} \mathrm{OP}^{+}_{\mathbf{Cat}_{p-1}} \to \prod_{p=0}^{r} \mathrm{OP}^{+}_{\mathbf{Cat}_{p-1}}$ and $\prod_{p=0}^{n-1} \mathrm{OP}_{\mathbf{Cat}_p} \to \prod_{p=0}^{r-1} \mathrm{OP}_{\mathbf{Cat}_p}$ for $r < n$ yield a truncation morphism $\mathcal{Q}\mathrm{OP}_{\mathbf{Cat}_n} \to \mathcal{Q}\mathrm{OP}_{\mathbf{Cat}_r}$. If $\mathcal{E}$ is an operadic n-polygraph, then we write $\mathcal{E}_{\leq r}$ for the operadic r-polygraph obtained from $\mathcal{E}$ by truncation.

Definition 3.3.9. Let $n \in \mathbb{N}$ and $\mathcal{T}^{(n)} : \mathcal{Q}\mathrm{OP}_{\mathbf{Cat}_n} \to \mathrm{OP}_{\mathbf{Cat}_n}$ be the functor defined as the composite

$$\mathcal{Q}\mathrm{OP}_{\mathbf{Cat}_n} \to \prod_{p=0}^{n} \mathrm{OP}^{+}_{\mathbf{Cat}_{p-1}} \xrightarrow{\pi_n} \mathrm{OP}^{+}_{\mathbf{Cat}_{n-1}} \xrightarrow{\mathcal{T}^{(n)}_{+}} \mathrm{OP}_{\mathbf{Cat}_n}.$$

We say that an n-categorical operad $\mathcal{P}$ is quasi free if there exists an operadic n-polygraph $\mathcal{E}$ and an isomorphism $\mathcal{P} \cong \mathcal{T}^{(n)}\mathcal{E}$.

Proposition 3.3.10. *The data of an operadic n-polygraph $\mathcal{E}$ is equivalent to the data of*

- *a symmetric sequence of sets E_p for $p = 0, \ldots, n$,*
- *together with source and target morphisms*

$$s_p, t_p : E_p \rightrightarrows \mathcal{T}^{(p-1)}(\mathcal{E}_{<p})_{p-1}$$

for $p = 1, \ldots, n$, such that $s_{p-1}s_p = s_{p-1}t_p$ and $t_{p-1}s_p = t_{p-1}t_p$.

We may directly make the source and target of the generating cells explicit and use the following equivalent description of an operadic n-polygraph, which consists in the data of

- *a symmetric sequence $E_0 : * \to \mathrm{SET}^{\mathfrak{S}}$,*
- *for all $1 \leq p \leq n$ and $0 \leq i < p$, for all $x_i, y_i : * \to \mathcal{T}^{(i)}(\mathcal{E}_{\leq i})(x_0, y_0) \ldots (x_{i-1}, y_{i-1})$, a symmetric sequence of generating p-cells*

$$E_p(x_0, y_0) \ldots (x_{p-1}, y_{p-1}) : * \to \mathrm{SET}^{\mathfrak{S}}.$$

We obtain symmetric sequences E_p for all $p = 0, \ldots, n$ by setting

$$E_p = \coprod_{\substack{x_0, y_0 : * \to \mathcal{T}^{(0)}(E_0) \\ \ldots \\ x_p, y_p : * \to \mathcal{T}^{(p-1)}(\mathcal{E}_{\leq p-1})(x_0, y_0) \ldots (x_{p-1}, y_{p-1})}} E_p(x_0, y_0) \ldots (x_{p-1}, y_{p-1})$$

In both cases, we write $\mathcal{E} = (E_0, \ldots, E_n)$ for the operadic n-polygraph with generating n-cells given by E_n.

Let $\mathcal{E}$ and $\mathcal{F}$ be operadic n-polygraphs. The data of a morphism $f : \mathcal{E} \to \mathcal{F}$ is equivalent to the data of morphisms of sequences $f_p : E_p \to F_p$ for $p = 0, \ldots, n$ such that the diagrams

$$\begin{tikzcd} E_p \arrow[r, "f_p"] \arrow[d, "s_p"'] & F_p \arrow[d, "s_p"] \\ \mathcal{T}^{(p-1)}(\mathcal{E}_{<p})_{p-1} \arrow[r, "\mathcal{T}^{(p-1)}(f_{<p})_{p-1}"'] & \mathcal{T}^{(p-1)}(\mathcal{F}_{<p})_{p-1} \end{tikzcd} \qquad \begin{tikzcd} E_p \arrow[r, "f_p"] \arrow[d, "s_p"'] & F_p \arrow[d, "s_p"] \\ \mathcal{T}^{(p-1)}(\mathcal{E}_{<p})_{p-1} \arrow[r, "\mathcal{T}^{(p-1)}(f_{<p})_{p-1}"'] & \mathcal{T}^{(p-1)}(\mathcal{F}_{<p})_{p-1} \end{tikzcd}$$

commute for all $p = 1, \ldots, n$. Equivalently, the data of a morphism $f : \mathcal{E} \to \mathcal{F}$ is equivalent to the data of

- *a morphism of symmetric sequences $f_0 : E_0 \to F_0$*

– *for* $1 \leq p \leq n$, $0 \leq i < p$, *and* $x_i, y_i : * \to \mathcal{T}^{(i)}(\mathcal{E}_{\leq i})(x_0, y_0) \ldots (x_{i-1}, y_{i-1})$, *a morphism of symmetric sequences*

$$f_p(x_0, y_0) \ldots (x_{p-1}, y_{p-1}) : E_p(x_0, y_0) \ldots (x_{p-1}, y_{p-1}) \to F_p(f_0 x_0, f_0 y_0) \ldots (f_{p-1} x_{p-1}, f_{p-1} y_{p-1}).$$

Proof. We proceed by induction on $n \in \mathbb{N}$. Since an operadic 0-polygraph is a symmetric sequence of sets, the case $n = 0$ is trivial. Suppose that the category of operadic p-polygraphs admits the following presentation for $p = 0, \ldots, n$ and let $\mathcal{E}$ be an operadic $n+1$-polygraph. The category of $n+1$-operadic polygraphs equalizes the diagram

$$\mathcal{Q}\mathrm{Op}_{\mathbf{Cat}_{n+1}} \dashrightarrow \mathcal{Q}\mathrm{Op}_{\mathbf{Cat}_n} \times \mathrm{Op}^+_{\mathbf{Cat}_n} \underset{\Pi_+^{(n)} \pi_2}{\overset{\mathcal{T}^{(n)} \pi(-,-)}{\rightrightarrows}} \mathrm{Op}_{\mathbf{Cat}_n} \ ,$$

so that $\mathcal{E}$ is determined by the data of the operadic n-polygraph $\mathcal{E}_{\leq n}$ together with an object $\mathcal{E}^+$ of $\mathrm{Op}^+_{\mathbf{Cat}_n}$ which has the same underlying n-categorical operad than $\mathcal{T}^{(n)}(\mathcal{E}_{\leq n})$, so that $\mathcal{E}$ is determined by the data of $\mathcal{E}_{\leq n}$ and $\mathcal{E}^+_{n+1} \rightrightarrows \mathcal{E}^+_n$. In particular, we obtain an isomorphism at the level of n-cells

$$\mathcal{E}^+_n \cong \mathcal{T}^{(n)}(\mathcal{E}_{\leq n})_n,$$

so that $\mathcal{E}$ is determined by the data of symmetric sequences $E_0, \ldots, E_n$ with the source and target morphisms characterizing $\mathcal{E}$, and by the symmetric sequence $\mathcal{E}^+_{n+1}$, together with source and target morphisms $\mathcal{E}^+_{n+1} \rightrightarrows \mathcal{T}^{(n)}(\mathcal{E}_{\leq n})_n$ that satisfy the required compatibility conditions because $\mathcal{E}^+$ has the structure of an n-operad. The characterization of the morphisms of operadic n-polygraphs can be obtained in the same way by induction on n by using the expression of $\mathcal{Q}\mathrm{Op}_{\mathbf{Cat}_{n+1}}(\mathcal{E}, \mathcal{F})$ as the equalizer

$$\mathcal{Q}\mathrm{Op}_{\mathbf{Cat}_n}(\mathcal{E}_{\leq n}, \mathcal{F}_{\leq n}) \times \mathrm{Op}^+_{\mathbf{Cat}_n}(\mathcal{E}^+, \mathcal{F}^+) \underset{\Pi_+^{(n)} \pi_2}{\overset{\mathcal{T}^{(n)} \pi(-,-)}{\rightrightarrows}} \mathrm{Op}_{\mathbf{Cat}_n}(\Pi_+^{(n)} \mathcal{E}^+, \Pi_+^{(n)} \mathcal{F}^+) \ .$$

□

Proposition 3.3.11. *For all* $n \in \mathbb{N}$, *the functor* $\mathcal{T}^{(n)}$ *has a right adjoint* $\mathcal{Q}^{(n)}$, *such that*

$$\mathcal{T}^{(n)} : \ \mathcal{Q}\mathrm{Op}_{\mathbf{Cat}_n} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Op}_{\mathbf{Cat}_n} \ : \mathcal{Q}^{(n)}.$$

Proof. We proceed by induction on $n \in \mathbb{N}$. For this purpose, we first make explicit the cases $n = 0$ and $n = 1$. For $n = 0$, observe that we have an isomorphism $\mathcal{Q}\mathrm{Op}_{\mathrm{Set}} \cong \mathrm{Set}^{\mathfrak{S}}$. The functor $\mathcal{T}^{(0)}$ corresponds to the free operad functor, whose right adjoint is given by the usual forgetful functor

$$\mathcal{T}^{(0)} : \ \mathrm{Set}^{\mathfrak{S}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Op}_{\mathrm{Set}} \ : \mathcal{Q}^{(0)}.$$

Let $\mathcal{P}$ be a categorical operad. We define $\mathcal{Q}^{(1)}_0 : \mathrm{Op}_{\mathbf{Cat}} \to Sets^{\mathfrak{S}}$ as the composite

$$\mathcal{Q}^{(1)}_0 : \mathrm{Op}_{\mathbf{Cat}} \xrightarrow{\Pi^{(0)}} \mathrm{Op}_{\mathrm{Set}} \xrightarrow{\mathcal{Q}^{(0)}} \mathrm{Set}^{\mathfrak{S}}.$$

We also define a functor $\mathcal{Q}^{(1)}_1 : \mathrm{Op}_{\mathbf{Cat}} \to \mathrm{Op}^+_{\mathrm{Set}}$ as follows. First recall that $\mathrm{Op}^+_{\mathrm{Set}} \cong \mathrm{Op}_{\mathrm{Set}} \underset{\mathrm{Set}^{\mathfrak{S}}}{\times} Gph^{\mathfrak{S}}$. We define an object $\mathcal{Q}^{(1)}_1 \mathcal{P}$ of $\mathrm{Op}^+_{\mathrm{Set}}$ for each categorical operad $\mathcal{P}$ by its components $\mathcal{Q}_1^{(1)}\mathcal{P} \cong (\bar{\mathcal{P}}, \mathcal{P}^+)$. We let the underlying set-theoretical operad $\bar{\mathcal{P}}$ of $\mathcal{Q}_1^{(1)}\mathcal{P}$ be given by the composite

$$\bar{\mathcal{P}} : * \xrightarrow{\mathcal{P}} \mathrm{O}\mathrm{P}_{\mathbf{Cat}} \xrightarrow{\Pi^{(0)}} \mathrm{O}\mathrm{P}_{\mathrm{SET}} \xrightarrow{\mathcal{Q}^{(0)}} \mathrm{SET}^{\mathfrak{S}} \xrightarrow{\mathcal{T}^{(0)}} \mathrm{O}\mathrm{P}_{\mathrm{SET}},$$

so that $\bar{\mathcal{P}}$ corresponds to the operad freely generated by the 0-cells P_0 of $\mathcal{P}$. We define the symmetric sequence of graphs $\mathcal{P}^+$ as follows. Let ${\mathcal{P}^+}_0 = \mathcal{Q}^{(0)}\bar{\mathcal{P}}$ and let ${\mathcal{P}^+}_1$ be the colimit of the composite

$$\bar{\mathcal{P}} \times \bar{\mathcal{P}} \xrightarrow{\mu\times\mu} P_0 \times P_0 \xrightarrow{P_1(-,-)} \mathrm{SET}^{\mathfrak{S}},$$

where μ is given by composition of the unit $\mathcal{T}^{(0)}\mathcal{Q}^{(0)}\mathcal{P}_0 \to \mathcal{P}_0$ with $\mathcal{Q}^{(0)}$. Hence for each $r \in \mathbb{N}$ we can write

$$ {\mathcal{P}^+}_1(r) = \coprod_{x_1,x_2 \in {\mathcal{P}^+}_0(r)} P_1(r)(\mu x_1, \mu x_2).$$

For all $x_1, x_2 \in {\mathcal{P}^+}_0(r)$ and $i = 0, 1$, we have a morphism $\pi_i^{(x_1,x_2)}(r)$ obtained as the composite

$$\pi_i^{(x_1,x_2)}(r) : P_1(r)(\mu x_1, \mu x_2) \to * \xrightarrow{(x_1,x_2)} {\mathcal{P}^+}_0(r) \times {\mathcal{P}^+}_0(r) \xrightarrow{\pi_i} {\mathcal{P}^+}_0(r),$$

which induces morphisms on the colimit $\pi(\_\,,\_)(r), \pi_2(r) : {\mathcal{P}^+}_1(r) \to {\mathcal{P}^+}_0(r)$. We give to $\mathcal{P}^+$ the structure of a symmetric sequence of graphs by setting $s(r) = \pi(\_\,,\_)(r)$ and $t(r) = \pi_2(r)$. Since the symmetric sequence of 0-cells of $\bar{\mathcal{P}}$ coincides with the 0-cells of $\mathcal{P}^+$, the pair $(\bar{\mathcal{P}}, \mathcal{P}^+)$ defines an object $\mathcal{Q}_1^{(1)}\mathcal{P}$ of $\mathrm{O}\mathrm{P}^+_{\mathrm{SET}}$. Observe that we have isomorphisms

$$\Pi_+^{(0)}\mathcal{Q}_1^{(1)}\mathcal{P} \cong \bar{P} \cong \mathcal{T}^{(0)}\mathcal{Q}^{(0)}\Pi^{(0)}\mathcal{P} \cong \mathcal{T}_+^{(0)}\mathcal{Q}_0^{(1)}\mathcal{P},$$

so that the pair of functors $(\mathcal{Q}_0^{(1)}, \mathcal{Q}_1^{(1)})$ induce a functor $\mathcal{Q}^{(1)} : \mathrm{O}\mathrm{P}_{\mathbf{Cat}} \to \mathcal{Q}\mathrm{O}\mathrm{P}_{\mathbf{Cat}}$. Let $\mathcal{P} = (\mathcal{P}^{(0)}, \mathcal{P}^{(1)})$ be an operadic polygraph, where $\mathcal{P}^{(i)}$ is an object of $\mathrm{O}\mathrm{P}^+_{\mathbf{Cat}_{i-1}}$. We set $\mathcal{T}^{(1)}\mathcal{P} = \mathcal{T}_+^{(1)}\mathcal{P}^{(1)}$. We accordingly get a functor $\mathcal{T}^{(1)} : \mathcal{Q}\mathrm{O}\mathrm{P}_{\mathbf{Cat}} \to \mathrm{O}\mathrm{P}_{\mathbf{Cat}}$. We show that $\mathcal{T}^{(1)}$ and $\mathcal{Q}^{(1)}$ are adjoint to each other. On the one hand, we have natural isomorphisms

$$\begin{aligned}
\mathrm{O}\mathrm{P}_{\mathbf{Cat}}(\mathcal{T}^{(1)}\mathcal{P}, \mathcal{R}) &\cong \mathrm{O}\mathrm{P}^+_{\mathrm{SET}}(\mathcal{P}^{(1)}, \mathcal{W}^{(1)}\mathcal{R}) \\
&\cong \mathrm{O}\mathrm{P}_{\mathrm{SET}}({\mathcal{P}^{(1)}}_0, \mathcal{R}_0) \underset{\mathrm{SET}^{\mathfrak{S}}({P^{(1)}}_0, R_0)}{\times} Gph^{\mathfrak{S}}(P^{(1)}, R) \\
&\cong \mathrm{O}\mathrm{P}_{\mathrm{SET}}(\mathcal{T}^{(0)}\mathcal{P}^{(0)}, \mathcal{R}_0) \underset{\mathrm{SET}^{\mathfrak{S}}(\mathcal{Q}^{(0)}\mathcal{T}^{(0)}\mathcal{P}^{(0)}, R_0)}{\times} Gph^{\mathfrak{S}}(P^{(1)}, R) \\
&\cong \mathrm{SET}^{\mathfrak{S}}(\mathcal{P}^{(0)}, R_0) \underset{\mathrm{SET}^{\mathfrak{S}}(\mathcal{Q}^{(0)}\mathcal{T}^{(0)}P_0, R_0)}{\times} Gph^{\mathfrak{S}}(P^{(1)}, R).
\end{aligned}$$

On the other hand, we have a natural isomorphism

$$\mathcal{Q}\mathrm{O}\mathrm{P}_{\mathbf{Cat}}(\mathcal{P}, \mathcal{Q}^{(1)}\mathcal{R}) \cong \mathrm{O}\mathrm{P}^+_{\mathrm{SET}}(\mathcal{P}^{(1)}, \mathcal{Q}^{(1)}\mathcal{R}_{(1)}) \underset{\mathrm{O}\mathrm{P}_{\mathrm{SET}}(\mathcal{T}^{(0)}\mathcal{P}^{(0)}, \mathcal{T}^{(0)}R_0)}{\times} \mathrm{SET}^{\mathfrak{S}}(\mathcal{P}^{(0)}, R_0)$$

Recall that the set of morphisms $\mathrm{O}\mathrm{P}^+_{\mathrm{SET}}(\mathcal{P}^{(1)}, \mathcal{Q}^{(1)}\mathcal{R}_{(1)})$ is isomorphic to

$$\mathrm{O}\mathrm{P}_{\mathrm{SET}}(\mathcal{T}^{(0)}\mathcal{P}^{(0)}, \mathcal{T}^{(0)}R_0) \underset{\mathrm{SET}^{\mathfrak{S}}(\mathcal{Q}^{(0)}\mathcal{T}^{(0)}\mathcal{P}^{(0)}, \mathcal{Q}^{(0)}\mathcal{T}^{(0)}R_0)}{\times} Gph^{\mathfrak{S}}(P, R^+ \rightrightarrows \bar{R}),$$

where P is the underlying graph of $\mathcal{P}$ and where $R^+ \rightrightarrows \bar{R}$ is the underlying graph of $\mathcal{Q}^{(1)}\mathcal{R}$. As a consequence, we obtain the isomorphism

$$\mathcal{Q}\mathrm{OP}_{\mathbf{Cat}}(\mathcal{P},\mathcal{Q}^{(1)}\mathcal{R}) \cong \ Gph^{\mathfrak{S}}(P, R^+ \rightrightarrows \bar{R}) \underset{\mathrm{SET}^{\mathfrak{S}}(\mathcal{Q}^{(0)}\mathcal{T}^{(0)}\mathcal{P}^{(0)},\mathcal{Q}^{(0)}\mathcal{T}^{(0)}R_0)}{\times} \mathrm{SET}^{\mathfrak{S}}(\mathcal{P}^{(0)}, R_0).$$

Let $\phi : * \to \mathcal{Q}\mathrm{OP}_{\mathbf{Cat}}(\mathcal{P},\mathcal{Q}^{(1)}\mathcal{R})$ with its component $\phi^{(1)} : * \to Gph^{\mathfrak{S}}(P, R^+ \rightrightarrows \bar{R})$. The graph morphism $\phi^{(1)}$ is given on the 0-cells by $\phi^{(1)}_0 = \mathcal{T}^{(0)}\phi^{(0)}$ where $\phi^{(0)} : \mathcal{P}^{(0)} \to R_0$. Since $\phi^{(1)}$ is a graph morphism, its value on some $f \in \mathcal{P}^{(1)}$ is given by $(sf, tf, \phi_1 f) \in R^+1$ for a unique $\phi_1 : \mathcal{P}^{(1)} \to R_1$, so that we have

$$\begin{aligned}\mathcal{Q}\mathrm{OP}_{\mathbf{Cat}}(\mathcal{P},\mathcal{Q}^{(1)}\mathcal{R}) &\cong \ Gph^{\mathfrak{S}}(P^{(1)}, R) \underset{\mathrm{SET}^{\mathfrak{S}}(\mathcal{Q}^{(0)}\mathcal{T}^{(0)}\mathcal{P}^{(0)},\mathcal{Q}^{(0)}\mathcal{T}^{(0)}R_0)}{\times} \mathrm{SET}^{\mathfrak{S}}(\mathcal{P}^{(0)}, R_0) \\ &\cong \ \mathrm{OP}_{\mathbf{Cat}}(\mathcal{T}^{(1)}\mathcal{P},\mathcal{R}).\end{aligned}$$

This shows that $\mathcal{T}^{(1)}$ is left adjoint to $\mathcal{Q}^{(1)}$. Now suppose that we have an adjunction

$$\mathcal{T}^{(p)} : \ \mathcal{Q}\mathrm{OP}_{\mathbf{Cat}_p} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{OP}_{\mathbf{Cat}_p} \ : \mathcal{Q}^{(p)}$$

for each $p \leq n-1$, such that for each operadic p-polygraph $\mathcal{E}$, the p-categorical operad $\mathcal{T}^{(p)}\mathcal{E}$ is given by $\mathcal{T}^{(p)}\mathcal{E} \cong \mathcal{T}^{(p)}_+\mathcal{E}^{(p)}$. We show that we have an adjunction

$$\mathcal{T}^{(n)} : \ \mathcal{Q}\mathrm{OP}_{\mathbf{Cat}_n} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{OP}_{\mathbf{Cat}_n} \ : \mathcal{Q}^{(n)}.$$

Let $\mathcal{P}$ be an n-categorical operad. We define an operadic n-polygraph $\mathcal{Q}^{(n)}\mathcal{P}$ by its underlying objects $\mathcal{Q}^{(n)}_p\mathcal{P}$ of $\mathrm{OP}^+_{\mathbf{Cat}_{p-1}}$ for $p = 0, \dots, n$. We have an operadic n-1-polygraph $\mathcal{Q}^{(n-1)}\mathcal{P}_{<n}$. We write $\mathcal{Q}^{(n-1)}_p\mathcal{P}_{<n}$ for its underlying objects of $\mathrm{OP}^+_{\mathbf{Cat}_{p-1}}$ for $p = 0, \dots, n-1$. We set $\mathcal{Q}^{(n)}_p\mathcal{P} = \mathcal{Q}^{(n-1)}_p\mathcal{P}_{<n}$ for $p = 0, \dots, n-1$. We form the n-1-categorical operad

$$\bar{\mathcal{P}} = \mathcal{T}^{(n-1)}\mathcal{Q}^{(n-1)}\mathcal{P}_{<n}$$

and we write $\mu : \bar{P}_{n-1} \to P_{n-1}$ for the restriction of the counit morphism

$$\mathcal{T}^{(n-1)}\mathcal{Q}^{(n-1)}\mathcal{P}_{<n} \to \mathcal{P}_{<n}$$

on the $n-1$-cells. We consider a symmetric sequence of graphs $\mathcal{P}^+$ given in arity r by the following graph

$$\coprod_{x,y \in \bar{P}_{n-1}(r)} P_n(r)(\mu x, \mu y) \rightrightarrows \bar{P}_{n-1}(r),$$

where the source and target morphisms are given by the canonical projections. The pair $\mathcal{Q}^{(n)}_n\mathcal{P} := (\bar{\mathcal{P}}, \mathcal{P}^+)$ defines an object of $\mathrm{OP}^+_{\mathbf{Cat}_{n-1}}$. We have isomorphisms of n-1-categorical operads

$$\Pi^{(n-1)}_+\mathcal{Q}^{(n)}_n\mathcal{P} \cong \bar{\mathcal{P}} \cong \mathcal{T}^{(n-1)}\mathcal{Q}^{(n-1)}\mathcal{P}_{<n} \cong \mathcal{T}^{(n-1)}_+\mathcal{Q}^{(n-1)}_{n-1}\mathcal{P}_{<n} \cong \mathcal{T}^{(n-1)}_+\mathcal{Q}^{(n)}_{n-1}\mathcal{P},$$

from which we deduce that the objects $\mathcal{Q}^{(n)}_0\mathcal{P}, \dots, \mathcal{Q}^{(n)}_n\mathcal{P}$ yield an operadic n-polygraph $\mathcal{Q}^{(n)}\mathcal{P}$. We obtain a functor $\mathcal{Q}^{(n)} : \mathrm{OP}_{\mathbf{Cat}_n} \to \mathcal{Q}\mathrm{OP}_{\mathbf{Cat}_n}$, which we show is right adjoint to $\mathcal{T}^{(n)}$. We have natural isomorphisms

$$\begin{aligned}
\mathrm{Op}_{\mathbf{Cat}_n}(\mathcal{T}^{(n)}\mathcal{E},\mathcal{P}) &\cong \mathrm{Op}_{\mathbf{Cat}_n}(\mathcal{T}_+^{(n)}\mathcal{E}^{(n)},\mathcal{P})\\
&\cong \mathrm{Op}^+_{\mathbf{Cat}_{n-1}}(\mathcal{E}^{(n)},\mathcal{W}^{(n)}\mathcal{P})\\
&\cong \mathrm{Op}_{\mathbf{Cat}_{n-1}}(\Pi_+^{(n-1)}\mathcal{E}^{(n)},\mathcal{P}_{<n}) \underset{\mathrm{Set}^{\mathfrak{S}}\left(\mathcal{E}^{(n)}_{n-1},P_{n-1}\right)}{\times} Gph^{\mathfrak{S}}\left(\mathcal{E}^{(n)}_n \rightrightarrows \mathcal{E}^{(n)}_{n-1}, P_n \rightrightarrows P_{n-1}\right)\\
&\cong \mathrm{Op}_{\mathbf{Cat}_{n-1}}(\mathcal{T}_+^{(n-1)}\mathcal{E}^{(n-1)},\mathcal{P}_{<n}) \underset{\mathrm{Set}^{\mathfrak{S}}\left(\mathcal{E}^{(n)}_{n-1},P_{n-1}\right)}{\times} Gph^{\mathfrak{S}}\left(\mathcal{E}^{(n)}_n \rightrightarrows \mathcal{E}^{(n)}_{n-1}, P_n \rightrightarrows P_{n-1}\right)\\
&\cong \mathcal{Q}\mathrm{Op}_{\mathbf{Cat}_{n-1}}(\mathcal{E}_{<n},\mathcal{Q}^{(n-1)}\mathcal{P}_{<n}) \underset{\mathrm{Set}^{\mathfrak{S}}\left(\mathcal{E}^{(n)}_{n-1},P_{n-1}\right)}{\times} Gph^{\mathfrak{S}}\left(\mathcal{E}^{(n)}_n \rightrightarrows \mathcal{E}^{(n)}_{n-1}, P_n \rightrightarrows P_{n-1}\right).
\end{aligned}$$

We also have natural isomorphisms

$$\mathcal{Q}\mathrm{Op}_{\mathbf{Cat}_n}(\mathcal{E},\mathcal{Q}^{(n)}\mathcal{P}) \cong \mathcal{Q}\mathrm{Op}_{\mathbf{Cat}_{n-1}}(\mathcal{E}_{<n},\mathcal{Q}^{(n-1)}\mathcal{P}_{<n}) \underset{\mathrm{Op}_{\mathrm{Set}}\left(\mathcal{E}^{(n)}_{n-1},\mathcal{Q}^{(n)}\mathcal{P}_{n-1}\right)}{\times} \mathrm{Op}^+_{\mathrm{Set}}\left(\mathcal{E}^{(n)}_n \rightrightarrows \mathcal{E}^{(n)}_{n-1}, \mathcal{Q}^{(n)}\mathcal{P}_{n-1,n}\right),$$

where

$$\mathcal{Q}^{(n)}\mathcal{P}_{n-1,n}(r) = \coprod_{x,y\in\mathcal{Q}^{(n)}\mathcal{P}_{n-1}(r)} \mathcal{P}_n(r)(\mu x,\mu y) \rightrightarrows \mathcal{Q}^{(n)}\mathcal{P}_{n-1}(r).$$

We have

$$\begin{aligned}
\mathrm{Op}^+_{\mathrm{Set}}\left(\mathcal{E}^{(n)}_n \rightrightarrows \mathcal{E}^{(n)}_{n-1}, \mathcal{Q}^{(n)}\mathcal{P}_{n-1,n}\right) &\cong \mathrm{Op}^+_{\mathrm{Set}}\left(\mathcal{E}^{(n)}_n \rightrightarrows \mathcal{E}^{(n)}_{n-1}, \coprod_{x,y\in\mathcal{Q}^{(n)}\mathcal{P}_{n-1}} \mathcal{P}_n(\mu x,\mu y) \rightrightarrows \mathcal{Q}^{(n)}\mathcal{P}_{n-1}\right)\\
\cong \mathrm{Op}_{\mathrm{Set}}\left(\mathcal{E}^{(n)}_{n-1},\mathcal{Q}^{(n)}\mathcal{P}_{n-1}\right) &\underset{\mathrm{Set}^{\mathfrak{S}}\left(\mathcal{E}^{(n)}_{n-1},\mathcal{Q}^{(n)}\mathcal{P}_{n-1}\right)}{\times} Gph^{\mathfrak{S}}\left(\mathcal{E}^{(n)}_n \rightrightarrows \mathcal{E}^{(n)}_{n-1}, \mathcal{Q}^{(n)}\mathcal{P}_n \rightrightarrows \mathcal{Q}^{(n)}\mathcal{P}_{n-1}\right).
\end{aligned}$$

Hence we obtain that the category $\mathcal{Q}\mathrm{Op}_{\mathbf{Cat}_n}(\mathcal{E},\mathcal{Q}^{(n)}\mathcal{P})$ is isomorphic to

$$\mathcal{Q}\mathrm{Op}_{\mathbf{Cat}_{n-1}}(\mathcal{E}_{<n},\mathcal{Q}^{(n-1)}\mathcal{P}_{<n}) \underset{\mathrm{Set}^{\mathfrak{S}}\left(\mathcal{E}^{(n)}_{n-1},\mathcal{Q}^{(n)}\mathcal{P}_{n-1}\right)}{\times} Gph^{\mathfrak{S}}\left(\mathcal{E}^{(n)}_n \rightrightarrows \mathcal{E}^{(n)}_{n-1}, \mathcal{Q}^{(n)}\mathcal{P}_n \rightrightarrows \mathcal{Q}^{(n)}\mathcal{P}_{n-1}\right).$$

Given an element of this fiber product, its component given by the graph morphism is freely generated by the morphism of sequences $(\mathcal{E}_{<n})_{n-1} \to (\mathcal{Q}^{(n-1)}\mathcal{P}_{<n})_{n-1}$. □

Remark 3.3.12. – An operadic 0-polygraph is the data of a generating symmetric sequence and $\mathcal{T}^{(0)}$ is the free operad functor

$$\mathcal{T} : \mathcal{Q}_0\mathrm{Op} = \mathrm{Set}^{\mathfrak{S}} \to \mathrm{Op}_{\mathrm{Set}}.$$

– An operadic 1-polygraph is the data of sequences of generating objects E_0 and generating morphisms E_1, where the source and the target of each generating morphism is a free composite of the generating objects. Write $= \mathcal{Q}_1 Op$ for the category of operadic (1)-polygraphs, the functor

$$\mathcal{T} : \mathcal{Q}\mathrm{Op} \to \mathrm{Op}_{\mathbf{Cat}}$$

takes an operadic polygraph (E_0, E_1) to an operad whose objects are freely generated by E_0 and whose morphisms are categorical composites of operadic compositions of the generating morphisms and objects

Polygraphic presentation of operads. We want to add some relations on polygraphic generators for operads, so that we have a notion of polygraphic presentation. We make precise the conditions under which we can subject the generating n-cells to relations, in a way that we can produce an operad presented by polygraphic generators and relations.

Let $\mathcal{E}$ be an operadic polygraph with generating n-cells E_n. Say $\mathcal{R} = (R_n, (p_n, q_n))_n$, where p_n, q_n are morphisms $R_n \to \mathcal{T}(\mathcal{E})_n$, define compatible relations on $\mathcal{E}$ whenever $\pi_{n-1}s_n$ and $\pi_{n-1}t_n$ both equalize p_n and q_n, where $\pi_{n-1} = coeq(p_{n-1}, q_{n-1})$

$$\begin{tikzcd} R_{n-1} \arrow[r, shift left, "p_{n-1}"] \arrow[r, shift right, "q_{n-1}"'] & \mathcal{T}(\mathcal{E})_{n-1} \arrow[r, dashed, "\pi_{n-1}"] & \mathcal{T}(\mathcal{E}:\mathcal{R})_{n-1} \\ R_n \arrow[r, shift left, "p_n"] \arrow[r, shift right, "q_n"'] & \mathcal{T}(\mathcal{E})_n \arrow[u, shift left, "s_n"] \arrow[u, shift right, "t_n"'] \arrow[r, dashed, "\pi_n"] & \mathcal{T}(\mathcal{E}:\mathcal{R})_n, \arrow[u, dashed, shift left] \arrow[u, dashed, shift right] \end{tikzcd}$$

so that we have induced source and target morphisms

$$\mathcal{T}(\mathcal{E}:\mathcal{R})_n \overset{s_n}{\underset{t_n}{\rightrightarrows}} \mathcal{T}(\mathcal{E}:\mathcal{R})_{n-1}.$$

Proposition 3.3.13. *If $\mathcal{R}$ defines compatible relations on an operadic polygraph $\mathcal{E}$, then $\mathcal{T}(\mathcal{E}:\mathcal{R})(r)$ has an induced structure of n-category and we obtain a $\mathbf{Cat}_n$-valued operad $\mathcal{T}(\mathcal{E}:\mathcal{R})$.*

Definition 3.3.14. Let $\mathcal{P}$ be an operad in $\mathbf{Cat}_n$. A polygraphic presentation of $\mathcal{P}$ is the data of an operadic polygraph $\mathcal{E}$ equipped with compatible relations $\mathcal{R}$ and an isomorphism $\mathcal{T}(\mathcal{E}:\mathcal{R}) \overset{\cong}{\to} \mathcal{P}$.

Definition 3.3.15. We define the category $\mathcal{Q}^{\text{REL}}Op$ whose objects are operadic polygraphs equipped with compatible relations, and whose morphisms $(\mathcal{E},\mathcal{R}) \to (\mathcal{G},\mathcal{S})$ are pairs

$$(f,g) : (\mathcal{E},\mathcal{R}) \longrightarrow (\mathcal{G},\mathcal{S})$$

where

- $f : \mathcal{E} \to \mathcal{G}$ is a morphism of operadic polygraphs
- $g = (g_n)_n$ consists of morphisms $g_n : R_n \to S_n$ such that the diagrams

$$\begin{tikzcd} R_n \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "g_n"'] & \mathcal{T}(\mathcal{E})_n \arrow[d, "\mathcal{T}(f)_n"] \\ S_n \arrow[r, shift left] \arrow[r, shift right] & \mathcal{T}(\mathcal{G})_n \end{tikzcd}$$

commute.

Proposition 3.3.16. *We have a well defined functor $\mathcal{T} : \mathcal{Q}_n^{\text{REL}}\text{OP} \to \text{OP}_{\mathbf{Cat}_n}$.*

Proof. Let $(\mathcal{E}, R) \xrightarrow{(f,g)} (\mathcal{G}, S)$ be a morphism in $\mathcal{Q}_1^{\text{REL}}Op$. Recall that a morphism

$$\mathcal{T}(\mathcal{E}:R) \longrightarrow \mathcal{T}(\mathcal{G}:S)$$

is uniquely determined by a morphism $\mathcal{E} \to \mathcal{T}(\mathcal{G} : S)$ that equalizes both R_0 and R_1. The following diagrams

$$\begin{tikzcd} R_0 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "g_0"'] & \mathcal{T}(E_0) \arrow[r, dashed] \arrow[d, "\mathcal{T}(f_0)"'] & \mathcal{T}(\mathcal{E} : R)_0 \arrow[d, dashed, "\mathcal{T}(f:g)_0"'] \\ S_0 \arrow[r, shift left] \arrow[r, shift right] & \mathcal{T}(G_0) \arrow[r, dashed] & \mathcal{T}(\mathcal{G} : S)_0 \end{tikzcd} \qquad \begin{tikzcd} R_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "g_1"'] & \mathcal{T}(\mathcal{E})_1 \arrow[r, dashed] \arrow[d, "\mathcal{T}(f_1)"'] & \mathcal{T}(\mathcal{E} : R)_1 \arrow[d, dashed, "\mathcal{T}(f:g)_1"'] \\ S_1 \arrow[r, shift left] \arrow[r, shift right] & \mathcal{T}(\mathcal{G})_1 \arrow[r, dashed] & \mathcal{T}(\mathcal{G} : S)_1 \end{tikzcd}$$

show we have a well defined induced (functorial) morphism

$$\mathcal{T}(f : g) : \mathcal{T}(\mathcal{E} : R) \longrightarrow \mathcal{T}(\mathcal{G} : S).$$

□

PROPOSITION 3.3.17. *The functor $\mathcal{T}$ has a right adjoint $\mathcal{W}$:*

$$\mathcal{W} : \begin{tikzcd} \mathrm{O}\mathrm{P}_{\mathbf{Cat}} \arrow[r, bend left, ""{name=U}] & \mathcal{Q}^{\mathrm{REL}}\mathrm{O}\mathrm{P} \arrow[l, bend left, dashed, ""{name=D}] \arrow[from=U, to=D, phantom, "\top"] \end{tikzcd} : \mathcal{T}.$$

PROOF. By definition a morphism $\mathcal{T}(\mathcal{E}, R) \to \mathcal{Q}$ is equivalent to the data of a morphism $\mathcal{T}(\mathcal{E}) \to \mathcal{Q}$ such that the induced morphisms equalize the diagrams

$$R_0 \rightrightarrows \mathcal{T}(\mathcal{E})_0 \longrightarrow \mathcal{Q}_0$$

$$R_1 \rightrightarrows \mathcal{T}(\mathcal{E})_1 \longrightarrow \mathcal{Q}_1.$$

Thus it is equivalent to the data of a morphism $\mathcal{E} \to V(\mathcal{Q})$ together with a morphism of relations

$$\begin{tikzcd} R_0 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{T}(\mathcal{E})_0 \arrow[d] \\ \mathcal{T}(\mathcal{Q}_0) \arrow[r, shift left, "id"] \arrow[r, shift right, "id"'] & \mathcal{T}(V(\mathcal{Q})_0) \end{tikzcd} \qquad \begin{tikzcd} R_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{T}(\mathcal{E})_1 \arrow[d] \\ \mathcal{T}(V(\mathcal{Q}))_1 \arrow[r, shift left, "id"] \arrow[r, shift right, "id"'] & \mathcal{T}(V(\mathcal{Q}))_1. \end{tikzcd}$$

Let

$$S = \left(\mathcal{T}(\mathcal{Q}_0) \underset{id}{\overset{id}{\rightrightarrows}} \mathcal{T}(V(\mathcal{Q})_0), \mathcal{T}(V(\mathcal{Q}))_1 \underset{id}{\overset{id}{\rightrightarrows}} \mathcal{T}(V(\mathcal{Q}))_1 \right).$$

Then S is a system of compatible relations for $\mathcal{Q}$.

This shows that $W : \mathcal{Q} \mapsto (V(\mathcal{Q}), S)$ defines a functor which is right adjoint to $\mathcal{T}$. □

3.4. Applications.

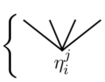
EXAMPLE 3.4.1 (Polygraphic presentation of the operads $\mathcal{M}^n$). Let $\mathcal{M}_n$ be the operad in **Cat** from [**2**]. Let $(\mathcal{E}^n, \mathcal{R}^n) \in \mathcal{Q}_1^{\mathrm{REL}}\mathrm{O}\mathrm{P}$ be defined by

- generating objects ${E^n}_0$ concentrated in arity 2 with

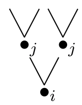
$$ {E^n}_0(2) = \Sigma_2 . \left\{ \quad \right\}_{1 \leq i \leq n}, $$

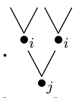
- generating morphisms ${E^n}_1$ concentrated in arity 4, with

$$ {E^n}_1(4) = \Sigma_4 . \left\{ \quad \right\}_{1 \leq i < j \leq n}, $$

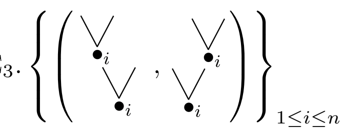


where $s, t : {E^n}_1 \rightrightarrows {E^n}_0$ are defined by

– $s : \eta_i^j \mapsto$ 

– $t : \eta_i^j \mapsto (2\,3).$  ,

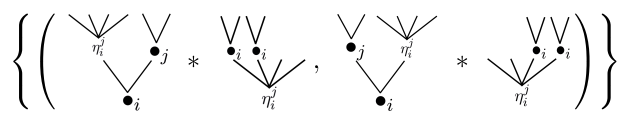
– the relations on the objects ${R^n}_0$ are concentrated in arity 3 and ensure the associativity of the binary operations, so that

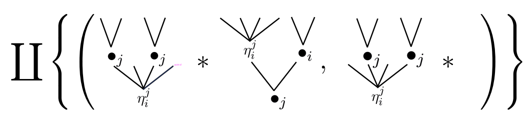
$${R^n}_0(3) = \Sigma_3.$$  ,

– the relations on the morphisms ${R^n}_1$ encode compatibility with the interchange morphisms and separate as

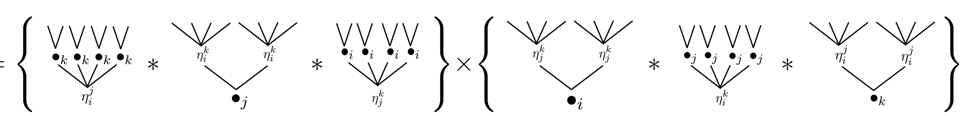
$${R^n}_1 = \coprod_{1\leq i<j\leq n} \Sigma_6.{R^n}_1^{(i,j)}(6) \coprod \coprod_{1\leq i<j<k\leq n} \Sigma_8.{R^n}_1^{(i,j,k)}(8),$$

with

$${R^n}_1^{(i,j)}(6) =$$ 



and

$${R^n}_1^{(i,j,k)}(8) =$$  ,

where the trees labelled by generating operations represent the free composition of these and $*$ denotes the free composition of the morphisms on the free operad. We wrote $f * g$ for the composition $g \circ f$. We also identified an object operation with the identity morphism it induces on the free operad (whose morphisms are freely generated).

Proposition 3.4.2. *The operads $\mathcal{M}_n$ admit a polygraphic presentation*

$$\mathcal{M}_n \cong \mathcal{F}(\mathcal{E}^n : R^n).$$

Definition 3.4.3. Define a non symmetric operadic polygraph $\mathcal{E}$ aritywise by

– $E_0(n) = \{\mu_n\}$,
– $E_1(n) = \coprod_{\substack{r+s=n+1\\ r,s\geq 0}} \{\alpha^i_{r,s}\}_{1\leq i\leq r}$,

with $s, t : E_1(n) \to \mathcal{T}(E_0)(n)$ given by
– $s(\alpha^i_{r,s}) = \mu_r \circ_i \mu_s$,
– $t(\alpha^i_{r,s}) = \mu_n$.

REMARK 3.4.4. We can describe the non symmetric operad generated by $\mathcal{E}$ as follows. Its operad of objects is generated by a single operation in each arity, and hence precisely corresponds to the operad of trees. With this identification, its morphisms are freely generated by edge contractions. In what follows, we introduce a system of compatible relations for $\mathcal{E}$ which ensures that edges can be contracted in any order.

NOTATION 3.4.5. Given a tree T corresponding to an object of any arity of the operad $\mathcal{T}E$, we write $\rho T :\in \mathbb{N}$ for the number of vertices of T. For instance, each generator $\mu_n \in E_0(n)$ corresponds the n-corolla Y_n, which satisfies $\rho Y_n = 1$.

DEFINITION 3.4.6. Define a system of relations $\mathcal{R}$ for the operadic polygraph $\mathcal{E}$ as follows

- $R_0 = \emptyset$
- $R_1(n) = \left\{\eta^T\right\}_{(T\in\mathcal{T}\mathcal{E}(n),\rho T=3)}$, where η_T implements the relations corresponding to the commutativity of the diagrams

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T/\{e\} \arrow[d] \\ T/\{f\} \arrow[r] & T/\{e,f\} = \mu_n \end{tikzcd}$$

  for each $T \in \mathcal{T}E(n)$ such that $\rho T = 3$.

PROPOSITION 3.4.7. *The operad ${\mathcal{M}_1}^\infty := \mathcal{T}(\mathcal{E} : \mathcal{R})$ is a poset whose objects are given by trees, and where there is a morphism $T \to S$ if and only if S can be obtained from T by contracting internal edges. Hence this poset corresponds to the face poset of the unital associahedra (see* [**26**]*). As a consequence, the operad ${\mathcal{M}_1}^\infty$ is contractible. Moreover, it corresponds to the operad κ defined by* [**10**] *(up to unit).*

REMARK 3.4.8. Recall that the weak equivalences associated with the model structure we defined on $\mathrm{Op}_{\mathbf{Cat}}$ are given by equivalences of categories. Hence, despite the fact that we have an equivalence of topological operads $|{\mathcal{M}_1}^\infty| \xrightarrow{\sim} |\mathcal{M}_1|$, the operad $\mathcal{M}^1_\infty$ can not be equivalent to $\mathcal{M}_1$ with respect to with model structure.

DEFINITION 3.4.9. Define a non symmetric operadic polygraph $\mathcal{E}^{\cong}$ aritywise by

- ${E^{\cong}}_0(n) = \{\mu_n\}$,
- ${E^{\cong}}_1(n) = \coprod\limits_{\substack{r+s=n+1\\ r,s\geq 0}} \{\alpha^i_{r,s}, {\alpha^i_{r,s}}^{-1}\}_{1\leq i\leq r}$, with $s,t : E^{\cong}_1(n) \to \mathcal{T}(E^{\cong}_0)(n)$ given by
  - $s(\alpha^i_{r,s}) = \mu_r \circ_i \mu_s$,
  - $s({\alpha^i_{r,s}}^{-1}) = \mu_n$,
  - $t(\alpha^i_{r,s}) = \mu_n$.
  - $t({\alpha^i_{r,s}}^{-1}) = \mu_r \circ_i \mu_s$,

REMARK 3.4.10. The operad of objects of $\mathcal{T}\mathcal{E}^{\cong}$ also corresponds to the operad of trees. Hence, we see the generating morphisms as edge contractions as well. The relations that we will define on $\mathcal{E}^{\cong}$ will both ensure that these edge contractions are isomorphisms, and that edges can be contracted in any order.

DEFINITION 3.4.11. Define a non symmetric operadic polygraph $\mathcal{E}^{\cong}$ aritywise by

- ${E^{\cong}}_0(n) = \{\mu_n\}$,
- ${E^{\cong}}_1(n) = \coprod_{\substack{r+s=n+1\\ r,s\geq 0}} \{\alpha^i_{r,s}, {\alpha^i_{r,s}}^{-1}\}_{1\leq i\leq r}$, with $s,t : E^{\cong}_1(n) \to \mathcal{T}(E^{\cong}_0)(n)$ given by
  - $s(\alpha^i_{r,s}) = \mu_r \circ_i \mu_s$,
  - $s({\alpha^i_{r,s}}^{-1}) = \mu_n$,
  - $t(\alpha^i_{r,s}) = \mu_n$.
  - $t({\alpha^i_{r,s}}^{-1}) = \mu_r \circ_i \mu_s$,

DEFINITION 3.4.12. We define a system of relations $\mathcal{R}^{\cong}$ for the operadic polygraph $\mathcal{E}$ as follows

- ${R^{\cong}}_0 = \emptyset$
- $R^{\cong}_1(n) = \{\eta^T\}_{(T\in\mathcal{T}\mathcal{E}(n),\rho T=3)} \coprod \coprod_{\substack{r+s=n+1\\ r,s\geq 0}} \{{\alpha^i_{r,s}}^{-1}\alpha^i_{r,s} = 1_{\mu_r\circ_i\mu_s}\}_{1\leq i\leq r}$
$$\coprod \coprod_{\substack{r+s=n+1\\ r,s\geq 0}} \{\alpha^i_{r,s}{\alpha^i_{r,s}}^{-1} = 1_{\mu_n}\}_{1\leq i\leq r},$$
  where η_T implements the relations corresponding to the commutativity of the diagrams
$$\begin{tikzcd} T \arrow[r] \arrow[d] & T/\{e\} \arrow[d] \\ T/\{f\} \arrow[r] & T/\{e,f\} = \mu_n \end{tikzcd}$$
  for each $T \in \mathcal{T}E(n)$ such that $\rho T = 3$.

PROPOSITION 3.4.13. *This operad is still aritywise a poset. Moreover, there is an isofibration*
$$\mathcal{M}^1_\infty \twoheadrightarrow {\mathcal{M}^1_\infty}^{\cong}$$
which induces an equivalence of topological operads
$$|\, \mathcal{N}\mathcal{M}^1_\infty \,| \xrightarrow{\sim} |\, \mathcal{N}{\mathcal{M}^1_\infty}^{\cong} \,| \,.$$
presentation of the operad defined in 3.2.26

REMARK 3.4.14. Observe that the categories $\mathcal{M}^1_\infty(r)$ identifies with the posets of planar trees with r leaves, with a free symmetric action and a morphism $S \to T$ whenever T is obtained from S by contracting an edge. It is shown in [**10**] that $\mathcal{M}^1_\infty(r)$ is precisely the face poset of the associahedron $\mathcal{K}(r)$, so that we have an equivalence of topological operads $|\, \mathcal{N}\mathcal{M}^1_\infty \,| \xrightarrow{\sim} \mathcal{K}$.

PROPOSITION 3.4.15. *The projection ${\mathcal{M}^1_\infty}^{\cong} \twoheadrightarrow \mathcal{M}^1$ is an acyclic fibration, so that we have the following cofibrant resolution of the operad $\mathcal{M}^1$:*
$${\mathcal{M}^1_\infty}^{\cong} \overset{\sim}{\twoheadrightarrow} \mathcal{M}^1.$$

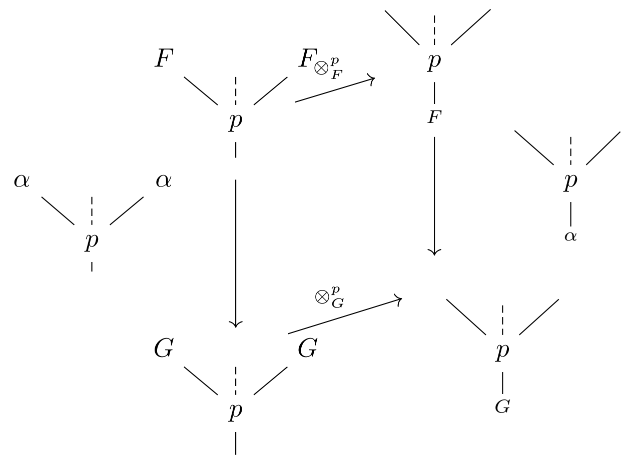


FIGURE 1. Coherence constraint for a 2-morphism of $\mathcal{P}$-algebras $\alpha : F \to G$, where $F, G : X \to Y$ are lax morphisms of $\mathcal{P}$-algebras.

CHAPTER III

Tensor product of operads in Cat and interchange

In [**15**], Lawvere introduced the tensor product of algebraic theories as a way to define a single algebraic theory describing the structure of compatible combinations of algebraic theories. Building upon Lawvere's work, Boardman and Vogt applied this construction to the framework of topological and set-theoretical operads. For set-theoretical or topological operads $\mathcal{P}$ and $\mathcal{Q}$, the Boardman-Vogt tensor product $\mathcal{P} \otimes_{BV} \mathcal{Q}$ is defined by a universal property, yielding a representation of the category of $\mathcal{Q}$-algebras in the category of $\mathcal{P}$-algebras as a category of algebras over $\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$, hence providing a better understanding on objects on which P and Q act in a compatible way. In what follows, we merely use the expression "tensor product" for this operation on operads. (We will actually use the same expression for generalizations of this tensor product construction that we formulate in the context of categorical operads.) In [**9**], Fiedorowicz and Vogt, motivated by a previous additivity result established by Dunn, proved that the tensor product $E_m \otimes_{BV} E_n$ is an E_{m+n}-operad as soon as E_m is a cofibrant model of E_m operad and E_n is a cofibrant model of E_n operad. Jacob Lurie's work in [**21**] further explored the tensor product of $(\infty, 1)$-operads, specifically on the E_n-operads obtained from the topological operads $\mathcal{C}_n$ via the homotopy coherent nerve construction. Lurie constructed a map of simplicial sets $E_n \times E_m \to E_{n+m}$ which exhibits E_{n+m} as the tensor product of E_n and E_m, providing an $(\infty, 1)$-operadic analogue of Dunn's additivity. To obtain a straightforward description of the tensor product, such as through generators and relations, the topological framework is inadequate due to the absence of explicit generators. Additionally, the set-theoretical framework encounters difficulties with the Eckmann-Hilton argument, which can be interpreted as the existence of an isomorphism $As \otimes_{BV} As \cong Com$, contradicting the expected additivity property for E_1 operads. In the $(\infty, 1)$-categorical framework, the tensor product $\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$ of two $(\infty, 1)$-operads $\mathcal{P}$ and $\mathcal{Q}$ lacks an explicit description since its existence relies on a fibrant replacement of an $(\infty, 1)$-preoperad obtained from the Cartesian product of $\mathcal{P}$ and $\mathcal{Q}$. This section aims to provide a clear description of the tensor product in a categorical framework. To achieve this, we take inspiration from the structure of the categorical operads $\mathcal{M}_n$.

In a first subsection, we provide a concise review of the tensor product's construction in the discrete setting. Additionally, we revisit the construction of the box product of symmetric sequences, which gives the scheme of interchange relations.

In the second subsection, our focus shifts towards the construction of the tensor product. The main idea behind this construction is to relax the interchange condition in the categorical setting. The tensor product that we define endows the 2-category of categorical operads with the structure of a monoidal

2-category which is not commutative. We establish that the tensor product of operads satisfies the desired universal property, leading to the equivalence $\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}\text{-}\mathrm{ALG}_{(\Lambda,\otimes)} \cong \mathcal{Q}\mathrm{ALG}_{(\mathcal{P}-\mathrm{ALG}_{(\Lambda,\otimes)},\otimes)}$. Additionally, we lift the tensor product to polygraphic presentations of operads, enabling us to explicitly describe the tensor product in terms of the presentations of its individual factors. We get, as a particular case of this description, the additivity relation $\mathcal{M}_n \otimes_{\mathrm{BV}} \mathcal{M}_m \cong \mathcal{M}_{n+m}$. Moreover, we investigate the behavior of the tensor product within the model structures studied in the previous section of the thesis. We observe that the tensor product of two operads is cofibrant if and only if each factor is cofibrant. Furthermore, we prove that the tensor product maps cofibrant resolutions of operads to a cofibrant resolution of their tensor product.

In the third subsection, we elaborate on the applications of the tensor product to the operads $\mathcal{M}_n$. As a main result, we provide an explicit presentation of cofibrant models $\mathcal{M}_n^\infty$ of $\mathcal{M}_n$ using the presentation of $\mathcal{M}_1^\infty$ described earlier and the relation $\mathcal{M}_1^{\otimes_{BV}^n} \cong \mathcal{M}_n$. Then, we focus on studying the algebras over the $\mathcal{M}_n^\infty$ operads within the 2-category $(\mathrm{TOP}, \times)$. We describe an $\mathcal{M}_n^\infty$-algebra structure on the n-cubes I^n in $(\mathrm{TOP}, \sqcup)$, enabling us to equip each n-fold loop space with an $\mathcal{M}_n^\infty$-algebra structure. However, we note that the $\mathcal{M}_n^\infty$-algebra structure on n-fold loop spaces factors through a morphism of operads $\mathcal{M}_n^\infty \to \Pi\mathcal{C}_n$, where $\Pi\mathcal{C}_n$ is the fundamental groupoid operad of $\mathcal{C}_n$. Consequently, the structure of an $\mathcal{M}_n^\infty$-algebra on a topological space does not provide significant information on that space for $n > 2$. Nevertheless, the case $n = 2$ remains of interest, and we describe an explicit non-trivial $\mathcal{M}_2^\infty$-algebra structure on the fundamental groupoid of any 2-fold loop space. We obtain, as a consequence, that a topological space has the homotopy 1-type of a 2-fold loop space up to group completion if and only if its fundamental groupoid has the structure of an $\mathcal{M}_2^\infty$-algebra.

Recall that that the categorical operad $\mathcal{M}_1^\infty$ is identified with the face poset of the associahedra. Considering that the operads $\mathcal{M}_n^\infty$ are posets too, we aim to investigate the possibility of realizing the nerve of the operads $\mathcal{M}_n^\infty$ as cellular spaces, so that the morphisms in $\mathcal{M}_n^\infty$ correspond to inclusions of boundaries. Due to the inhomogeneity of the cell dimensions in relation to the interchange morphisms, the cell structure we expect will require an unconventional way of gluing along boundaries. In fact, these gluing operations will involve partial gluing along separate dimensions, and these partial gluings themselves will be subjected to relations that involve interchange. This approach will allow us to efficiently describe the symmetries present in the operads $\mathcal{M}_n$, arising from the interchangeable role of each operation. While it may not be possible for these cellular spaces to have the structure of an operad like the associahedra due to dimension arguments, we conjecture that they have the structure of what we will call an n-fold operad. Notably, this structure may not require aritywise actions of the symmetric group. With this terminology, 1-fold operads should correspond to non symmetric operads and can have algebras over any monoidal category, whereas the algebras over n-fold operads naturally fit in the framework of n-fold monoidal categories. Developing these tools will involve to extend usual constructions to an n-fold monoidal framework.

1. The Boardman-Vogt tensor product of operads

The Boardman-Vogt tensor product was originally defined for symmetric operads defined in the closed monoidal category of topological spaces or in the monoidal category of sets, with monoidal product both given by the cartesian product. As noticed in Remark II.1.4.14 in the framework of operads defined in the 2-category Cat, the property of the cartesian product, and the compatibility of the product with respect to itself due to its commutativity, will permit us to endow the category of $\mathcal{P}$-algebras in $(\mathrm{Top}, \times)$ with the structure of a monoidal category, for any set-theoretical or topological operad $\mathcal{P}$.

In this section, $\mathcal{C}$ denotes a cartesian closed category, which in practise is either Top of Set. The word operads will refer to operads defined in $\mathcal{C}$. We first recall the definition of actions of operads which interchange, and the resulting Eckmann-Hilton argument.

1.1. Interchange and Eckmann-Hilton argument, revisited.

Definition 1.1.1. Let $\mathcal{V}$ be a cartesian closed category and recall that the 2-category $\mathrm{Cat}_{\mathcal{V}}$ of $\mathcal{V}$-enriched categories is symmetric monoidal. We say that an object X of $\mathrm{Cat}_{\mathcal{V}}$ is a $\mathcal{V}$-enriched n-monoidal category if it has the structure of an internal n-pseudomonoid in $\mathrm{Cat}_{\mathcal{V}}$.

Definition 1.1.2. Let $\mathcal{B}$ be an operad and $(\mathcal{D}, \otimes)$ be a topologically enriched monoidal category. Let X be an object of $\mathcal{D}$, and let $\mathrm{End}_X^{\mathcal{D}}$ be the topological operad defined as follows. For $r \in \mathbb{N}$, we let $\mathrm{End}_X^{\mathcal{D}}(r) = \mathcal{D}(X^{\otimes^r}, X) \in \mathrm{Top}$. The operad structure is induced by the following composite for each $r \in \mathbb{N}$ and $n_1, ..., n_r \in \mathbb{N}$

$$\mathcal{D}(X^{\otimes^r}, X)\times\prod_{i=1}^{r}\mathcal{D}(X^{\otimes^{n_i}}, X) \xrightarrow{1\times\otimes^r} \mathcal{D}(X^{\otimes^r}, X)\times\mathcal{D}(\prod_{i=1}^{r} X^{\otimes^{n_i}}, X^{\otimes^r}) \xrightarrow{\circ} \mathcal{D}(X^{\otimes^n}, X),$$

where $n = n_1 + ... + n_r$. We may also write $\mathrm{End}_X^{\mathcal{D}} = \mathcal{D}(X^{\otimes^\bullet}, X)$.

Let $\mathcal{B}\text{-}\mathrm{Alg}_{\mathcal{D}} \in \mathrm{Cat}$ be the topologically enriched category of $\mathcal{B}$-algebras in $\mathcal{D}$, defined as follows.

label=- A $\mathcal{B}$-algebra in $\mathcal{D}$ is a pair (X, ψ) where $X \in \mathcal{D}$ and $\psi : \mathcal{B} \to \mathrm{End}_X^{\mathcal{D}}$ is a morphism of operads.

label=- For $X, Y \in \mathcal{B}\text{-}\mathrm{Alg}_{\mathcal{D}}$, we let the space of morphisms $\mathcal{B}\text{-}\mathrm{Alg}_{\mathcal{D}}(X, Y)$ be the subspace of $\mathcal{D}(X, Y)$ whose elements are the morphisms $F \in \mathcal{D}(X, Y)$ such that for each $r \in \mathbb{N}$, the diagram

$$\begin{tikzcd}
\mathcal{B}(r) \arrow[r, "\psi^X(r)"] \arrow[d, "\psi^Y(r)"'] & \mathcal{D}(X^{\otimes^r}, X) \arrow[d, "\mathcal{D}(X^{\otimes^r}, F)"] \\
\mathcal{D}(Y^{\otimes^r}, Y) \arrow[r, "\mathcal{D}(F^{\otimes^r}, X)"'] & \mathcal{D}(X^{\otimes^r}, Y)
\end{tikzcd}$$

commutes in Top.

Proposition 1.1.3. *Let $\mathcal{B}$ be an operad and $(\mathcal{D}, \otimes_1)$ be a topologicaly enriched monoidal category. Suppose that $\mathcal{D}$ is equipped with an other monoidal category structure $(\mathcal{D}, \otimes_2)$ such that $(\mathcal{D}, \otimes_1, \otimes_2)$ is a 2-monoidal topologically enriched category. Then the topologically enriched category of $\mathcal{B}$-algebras in $(\mathcal{D}, \otimes_1)$ inherits a* *monoidal structure from $\otimes_2$. More generally, if $(\mathcal{D}, \otimes_1, ..., \otimes_n)$ is an n-fold topologically enriched monoidal category, then the topologically enriched monoidal category of $\mathcal{B}$-algebras in $(\mathcal{D}, \otimes_1)$ inherits the structure of an $(n-1)$-fold topologically enriched monoidal category.*

PROOF. We construct the tensor product of the underlying topologically enriched monoidal structure

$$\otimes_2 : \mathcal{B}\text{-}\mathrm{ALG}_{\mathcal{D}} \times \mathcal{B}\text{-}\mathrm{ALG}_{\mathcal{D}} \to \mathcal{B}\text{-}\mathrm{ALG}_{\mathcal{D}}$$

by

label=- $(X, \psi_X), (Y, \psi_Y) \in \mathcal{B}\text{-}\mathrm{ALG}_{\mathcal{D}} \Rightarrow (X \otimes_2 Y, \psi_X \otimes_2 \psi_Y) \in \mathcal{B}\text{-}\mathrm{ALG}_{\mathcal{D}}$, where

$$\psi_X \otimes_2 \psi_Y : \mathcal{B} \to \mathrm{END}^{\mathcal{D}}_{X \otimes_2 Y}$$

is defined by the composite

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\psi_X \times \psi_Y"] \arrow[drr, dashed, "\psi_{X \otimes_2 Y}"'] & \mathcal{D}(X^{\otimes_1^{\bullet}}, X) \times \mathcal{D}(Y^{\otimes_1^{\bullet}}, Y) \arrow[r, "\otimes_2"] & \mathcal{D}(X^{\otimes_1^{\bullet}} \otimes_2 Y^{\otimes_1^{\bullet}}, X \otimes_2 Y), \arrow[d, "\mathcal{D}(\tau, X \otimes_2 Y)"] \\
 & & \mathcal{D}((X \otimes_2 Y)^{\otimes_1^{\bullet}}, X \otimes_2 Y)
\end{tikzcd}$$

where τ refers to the interchange morphism between $\otimes_1$ and $\otimes_2$.

label=- Let $(X_1, \psi^1_X), (X_2, \psi^2_X), (Y_1, \psi^1_Y), (Y_2, \psi^2_Y) \in \mathcal{B}\text{-}\mathrm{ALG}_{\mathcal{D}}$, and define the map $\otimes_2$ on the space of morphisms

$$\mathcal{B}\text{-}\mathrm{ALG}_{\mathcal{D}}(X_1, X_2) \times \mathcal{B}\text{-}\mathrm{ALG}_{\mathcal{D}}(Y_1, Y_2) \to \mathcal{B}\text{-}\mathrm{ALG}_{\mathcal{D}}(X_1 \otimes_2 Y_1, X_2 \otimes_2 Y_2)$$

by restriction of the map induced by $\otimes_2$ on the morphisms on $\mathcal{D}$

$$\mathcal{D}(X_1, X_2) \times \mathcal{D}(Y_1, Y_2) \to \mathcal{D}(X_1 \otimes_2 Y_1, X_2 \otimes_2 Y_2).$$

For this purpose, we need to show that for each $F_X \in \mathcal{B}\text{-}\mathrm{ALG}_{\mathcal{D}}(X_1, X_2)$ and $F_Y \in \mathcal{B}\text{-}\mathrm{ALG}_{\mathcal{D}}(Y_1, Y_2)$, the tensor product $F_X \otimes_2 F_Y \in \mathcal{D}(X_1 \otimes_2 Y_1, X_2 \otimes_2 Y_2)$ of the underlying morphisms F_X and F_Y in $\mathcal{D}$ is a morphism of $\mathcal{B}$-algebras. The diagram expressing the compatibility of the morphism $F_X \otimes_2 F_Y$ with respect to the $\mathcal{B}$-algebra structures on $X_1 \otimes Y_1$ and $X_2 \otimes_2 Y_2$ in

$(\mathcal{D}, \otimes_1)$ admits the following decomposition into commutative subsquares

$$\begin{tikzcd}
 & \mathcal{B}(r) \arrow[dl, "\psi_X^2 \times \psi_Y^2"'] \arrow[dr, "\psi_X^1 \times \psi_Y^1"] & \\
D(X_2^{\otimes \overset{\bullet}{1}}, X_2) \times \mathcal{D}(Y_2^{\otimes \overset{\bullet}{1}}, Y_2) \arrow[dr, "D(F_X^{\otimes \overset{\bullet}{1}}, X_2) \times \mathcal{D}(F_Y^{\otimes \overset{\bullet}{1}}, Y_2)"] \arrow[dd, "\otimes_2"'] & & D(X_1^{\otimes \overset{\bullet}{1}}, X_1) \times \mathcal{D}(Y_1^{\otimes \overset{\bullet}{1}}, Y_1) \arrow[dl, "D(X_1^{\otimes \overset{\bullet}{1}}, F_X) \times \mathcal{D}(Y_1^{\otimes \overset{\bullet}{1}}, F_Y)"'] \arrow[dd, "\otimes_2"] \\
 & D(X_1^{\otimes \overset{\bullet}{1}}, X_2) \times \mathcal{D}(Y_1^{\otimes \overset{\bullet}{1}}, Y_2) \arrow[dd, "\otimes_2"] & \\
\mathcal{D}(X_2^{\otimes \overset{\bullet}{1}} \otimes_2 Y_2^{\otimes \overset{\bullet}{1}}, X_2 \otimes_2 Y_2) \arrow[dr, "\mathcal{D}(F_X^{\otimes \overset{\bullet}{1}} \otimes_2 F_Y^{\otimes \overset{\bullet}{1}}, X_2 \otimes_2 Y_2)"] \arrow[dd, "\mathcal{D}(\tau, X_2 \otimes_2 Y_2)"'] & & \mathcal{D}(X_1^{\otimes \overset{\bullet}{1}} \otimes_2 Y_1^{\otimes \overset{\bullet}{1}}, X_1 \otimes_2 Y_1) \arrow[dl, "\mathcal{D}(X_1^{\otimes \overset{\bullet}{1}} \otimes_2 Y_1^{\otimes \overset{\bullet}{1}}, F_X \otimes_2 F_Y)"'] \arrow[dd, "\mathcal{D}(\tau, X_1 \otimes_2 Y_1)"] \\
 & \mathcal{D}(X_1^{\otimes \overset{\bullet}{1}} \otimes_2 Y_1^{\otimes \overset{\bullet}{1}}, X_2 \otimes_2 Y_2) \arrow[dd, "\mathcal{D}(\tau, X_2 \otimes_2 Y_2)"] & \\
\mathcal{D}((X_2 \otimes_2 Y_2)^{\otimes \overset{\bullet}{1}}, X_2 \otimes_2 Y_2) \arrow[dr, "\mathcal{D}((F_X \otimes_2 F_Y)^{\otimes \overset{\bullet}{1}}, X_1 \otimes_2 Y_1)"'] & & \mathcal{D}((X_1 \otimes_2 Y_1)^{\otimes \overset{\bullet}{1}}, X_1 \otimes_2 Y_1) \arrow[dl, "\mathcal{D}((X_1 \otimes_2 Y_1)^{\otimes \overset{\bullet}{1}}, F_X \otimes_2 F_Y)"] \\
 & \mathcal{D}((X_1 \otimes_2 Y_1)^{\otimes \overset{\bullet}{1}}, X_2 \otimes_2 Y_2) &
\end{tikzcd}$$

and hence commutes, which shows that $F_X \otimes_2 F_Y$ is a morphism of $\mathcal{B}$-algebras.

$\square$

Corollary 1.1.4. *Let $(\mathcal{D}, \otimes)$ be a topologically enriched symmetric monoidal category and $\mathcal{B}$ be an operad. The topologically enriched category $\mathcal{B}$-$\mathrm{ALG}_{\mathcal{D}}$ of $\mathcal{B}$ algebras in $\mathcal{D}$ is symmetric monoidal. Notably, the topologically enriched monoidal category of $\mathcal{B}$-algebras in* $(\mathrm{TOP}, \times)$ *is symmetric monoidal.*

Remark 1.1.5. In the topologically enriched category TOP, a diagram commutes if and only if it commutes pointwise. As a consequence, if $\mathcal{B}$ is an operad and $X, Y \in \mathcal{B}$-$\mathrm{ALG}_{\mathrm{TOP}}$, a continuous map $F : X \to Y$ is a morphism of $\mathcal{B}$-algebras if and only if for each $r \in \mathbb{N}$ and $p \in \mathcal{B}(r)$, the following diagram commutes

$$\begin{tikzcd}
X^r \arrow[r, "\psi^X p"] \arrow[d, "F^r"'] & X \arrow[d, "F"] \\
Y^r \arrow[r, "\psi^Y p"'] & Y
\end{tikzcd}.$$

Definition 1.1.6. Let $\mathcal{A}$ and $\mathcal{B}$ be operads and $(\mathcal{D}, \otimes_1, \otimes_2)$ be a topologically enriched 2-monoidal category. We let $(\mathcal{A}\text{-}\mathrm{Alg}_{\mathcal{B}\text{-}\mathrm{Alg}_{(\mathcal{D},\otimes_1)}}, \otimes_2)$ be the topologically enriched category of $\mathcal{A}$-algebras in $\mathcal{B}$-algebras. If $\mathcal{D}$ is symmetric with $\otimes_1 = \otimes_2$, we write $\mathcal{A}\text{-}\mathrm{Alg}_{\mathcal{B}\text{-}\mathrm{Alg}_{\mathcal{D}}}$. In this case, we have an equivalence of categories $\mathcal{A}\text{-}\mathrm{Alg}_{\mathcal{B}\text{-}\mathrm{Alg}_{\mathcal{D}}} \cong \mathcal{B}\text{-}\mathrm{Alg}_{\mathcal{A}\text{-}\mathrm{Alg}_{\mathcal{D}}}$. When $\mathcal{D}$ is Top, we write $\mathcal{A}\text{-}\mathrm{Alg}_{\mathcal{B}\text{-}\mathrm{Alg}}$.

Example 1.1.7. Let $\mathcal{A}$ and $\mathcal{B}$ be operads and let X be a space equipped with both an $\mathcal{A}$-algebra and a $\mathcal{B}$-algebra structure. Then $X \in \mathcal{A}\text{-}\mathrm{Alg}_{\mathcal{B}\text{-}\mathrm{Alg}_{\mathcal{D}}}$ if and only if for each $r, s \in \mathbb{N}$ and $p \in \mathcal{A}(r), q \in \mathcal{B}(s)$, the diagram

$$\begin{tikzcd}
(X^r)^s \arrow[r, "\cong"] \arrow[d, "p^s"'] & (X^s)^r \arrow[r, "q^r"] & X^r \arrow[d, "p"] \\
X^s \arrow[rr, "q"'] & & X
\end{tikzcd}$$

commutes. We also say that the actions of $\mathcal{A}$ and $\mathcal{B}$ on X interchange.

Example 1.1.8. Let $\mathcal{C}_n$ be the topological of little n-cubes, and recall that for each $1 \leq i \leq n$, there is a morphism of topological operads

$$\alpha_i : \mathcal{C}_1 \to \mathcal{C}_n$$

explicitly given in arity $r \in \mathbb{N}$ by

$$(c_1, ..., c_r) \in \mathcal{C}_1(r) \mapsto (\iota_i c_1, ..., \iota_i c_r) \in \mathcal{C}_n(r)$$

where $c_k \subset [0,1]$ and $\iota_i c_k = [0,1]^{i-1} \times c_k \times [0,1]^{n-i} \subset [0,1]^n$ for $k = 1, ..., r$. Then for any $\mathcal{C}_n$-space X and each $i \neq j$, the actions of $\mathcal{C}_1$ on X induced by α_i, α_j interchange.

Lemma 1.1.9 (Eckmann-Hilton's argument). *Let $(\Lambda, \times_1, \times_2, \mathbb{1}_\Lambda) \in Mon_{\mathrm{Cat}_2}$ be a 2-monoidal 2-category. Suppose that $\mathcal{C} \in \Lambda$ has the structure of a 2-fold monoid in Λ, so that $(\mathcal{C}, \otimes_1, \otimes_2, \mathbb{1}_\mathcal{C}) \in Mon^2_{(\Lambda, \times_1, \times_2)}$. Let $(X, \mu_1, \mu_2, 1_X) \in Mon^2_{(C, \otimes_1, \otimes_2)}$ be a 2-fold monoid internal to $\mathcal{C}$ such that the monoidal structures $(\mathcal{C}, \mu_1)$ and $(\mathcal{C}, \mu_2)$ share the same unit ϵ. Then for each pair of generalized objects x, y of X, the following equalities hold in the set $\Lambda(\mathbb{1}_\Lambda, \mathcal{C})(\mathbb{1}_\mathcal{C}, X)$*

- $\mu_1(x \otimes_1 y) = \mu_2(x \otimes_2 y)$
- $\mu_1(x \otimes_1 y) = \mu_1(y \otimes_1 x)$
- $\mu_2(x \otimes_2 y) = \mu_2(y \otimes_2 x)$.

Proof. First observe that the objects under consideration arrange as follow within Cat_2

$$\begin{tikzcd}
* \arrow[r, bend left=50, "\mathcal{C}"] \arrow[r, bend right=50, "\mathbb{1}_\Lambda"'] \arrow[r, phantom, "\mathbb{1}_\mathcal{C}\Uparrow\ \overset{y}{\underset{x}{\rightrightarrows}}\ \Uparrow X"] & \Lambda.
\end{tikzcd}$$

We obtain the following diagram from the properties of the cartesian product and the 2-fold monoidal structure on Λ

$$\begin{tikzcd}
* \arrow[rr, bend left=50, "((\mathcal{C},\mathbb{1}_\Lambda),(\mathbb{1}_\Lambda,\mathcal{C}))"] \arrow[rr, bend right=50, "(\mathbb{1}_\Lambda,\mathbb{1}_\Lambda),(\mathbb{1}_\Lambda,\mathbb{1}_\Lambda)"'] \arrow[rr, phantom, "\Uparrow (\mathbb{1}_\mathcal{C},\mathbb{1}_\Lambda),(\mathbb{1}_\Lambda,\mathbb{1}_\mathcal{C},) \xrightarrow{(x,1_X),(1_X,y)} \Uparrow (X,\mathbb{1}_\Lambda),(\mathbb{1}_\Lambda,X)"] & & (\Lambda\times\Lambda)\times(\Lambda\times\Lambda) \arrow[dl, "\times_1\times\times_1"'] \arrow[dr, "\tau"] & \\
& \Lambda\times\Lambda \arrow[d, "\times_2"'] \arrow[rr, phantom, "\Uparrow\beta"] & & (\Lambda\times\Lambda)\times(\Lambda\times\Lambda) \arrow[d, "\times_2\times\times_2"] \\
& \Lambda & & \Lambda\times\Lambda. \arrow[ll, "\times_1"]
\end{tikzcd}$$

By composition with $(\Lambda\times\Lambda)\times(\Lambda\times\Lambda) \xrightarrow{\times_1\times\times_1} \Lambda\times\Lambda \xrightarrow{\times_2}$, we obtain

$$\begin{tikzcd}
* \arrow[rr, bend left=50, "((\mathcal{C}\times_1\mathbb{1}_\Lambda)\times_2(\mathbb{1}_\Lambda\times_1\mathcal{C}))"] \arrow[rr, bend right=50, "(\mathbb{1}_\Lambda\times_1\mathbb{1}_\Lambda)\times_2(\mathbb{1}_\Lambda\times_1\mathbb{1}_\Lambda)"'] \arrow[rr, phantom, "\Uparrow (\mathbb{1}_\mathcal{C}\times_1\mathbb{1}_\Lambda)\times_2(\mathbb{1}_\Lambda\times_1\mathbb{1}_\mathcal{C},) \xRightarrow{(x\times_1 1_X)\times_2(1_X\times_1 y)} \Uparrow (X\times_1\mathbb{1}_\Lambda)\times_2(\mathbb{1}_\Lambda\times_1 X)"] & & \Lambda,
\end{tikzcd}$$

and by composition with $(\Lambda\times\Lambda)\times(\Lambda\times\Lambda) \xrightarrow{\cong} (\Lambda\times\Lambda)\times(\Lambda\times\Lambda) \xrightarrow{\times_2\times\times_2} \Lambda\times\Lambda \xrightarrow{\times_1} \Lambda$, we obtain

$$\begin{tikzcd}
* \arrow[rr, bend left=50, "((\mathcal{C}\times_2\mathbb{1}_\Lambda)\times_1(\mathbb{1}_\Lambda\times_2\mathcal{C}))"] \arrow[rr, bend right=50, "(\mathbb{1}_\Lambda\times_2\mathbb{1}_\Lambda)\times_1(\mathbb{1}_\Lambda\times_2\mathbb{1}_\Lambda)"'] \arrow[rr, phantom, "\Uparrow (\mathbb{1}_\mathcal{C}\times_2\mathbb{1}_\Lambda)\times_1(\mathbb{1}_\Lambda\times_2\mathbb{1}_\mathcal{C},) \xrightarrow{(x\times_2 1_X)\times_1(1_X\times_2 y)} \Uparrow (X\times_2\mathbb{1}_\Lambda)\times_1(\mathbb{1}_\Lambda\times_2 X)"] & & \Lambda.
\end{tikzcd}$$

The transformation β provides a transformation between those diagrams. For better clarity, let us step down a level and place internally within Λ. Under the identifications

- $(\mathbb{1}_\Lambda \times_1 \mathbb{1}_\Lambda) \times_2 (\mathbb{1}_\Lambda \times_1 \mathbb{1}_\Lambda) \cong \mathbb{1}_\Lambda$,
- $(\mathbb{1}_\Lambda \times_2 \mathbb{1}_\Lambda) \times_1 (\mathbb{1}_\Lambda \times_2 \mathbb{1}_\Lambda) \cong \mathbb{1}_\Lambda$,
- $((\mathcal{C} \times_1 \mathbb{1}_\Lambda) \times_2 (\mathbb{1}_\Lambda \times_1 \mathcal{C})) \cong \mathcal{C} \times_2 \mathcal{C}$,
- $((\mathcal{C} \times_2 \mathbb{1}_\Lambda) \times_1 (\mathbb{1}_\Lambda \times_2 \mathcal{C})) \cong \mathcal{C} \times_1 \mathcal{C}$,

we obtain

$$\begin{tikzcd}
 & \mathcal{C}\times_2\mathcal{C} \arrow[rr, bend left, "\beta"] \arrow[dr, "\otimes_2"] \arrow[d, phantom, "\Downarrow\mu_2"] & & \mathcal{C}\times_1\mathcal{C} \arrow[dl, "\otimes_1"'] \arrow[ll, phantom, "\Leftarrow\eta"] \arrow[d, phantom, "\Downarrow\mu_1"] & \mathbf{1}_\Lambda \arrow[l, bend right, "\mathbb{1}_\mathcal{C}\times_1\mathbb{1}_\mathcal{C}"'] \arrow[l, bend left, "X\times_1 X"] \arrow[l, phantom, "\Downarrow x\times_1 y"] \arrow[dll, bend left, "x\times_1 y"] \\
\mathbf{1}_\Lambda \arrow[ur, bend left, "\mathbb{1}_\mathcal{C}\times_2\mathbb{1}_\mathcal{C}"] \arrow[ur, bend right, "X\times_2 X"'] \arrow[ur, phantom, "\Downarrow x\times_2 y"] \arrow[rr, bend right, "X"'] & & \mathcal{C} & &
\end{tikzcd}$$

where η is the interchange morphism obtained from the 2-fold monoidal structure of $\mathcal{C}$. The resulting composite can be displayed internally in $\mathcal{C}$ as

$$\begin{tikzcd} \mathbb{1}_{\mathcal{C}} \arrow[r, "x\otimes_1 y"] & X \otimes_1 X \arrow[rr, "\eta"] \arrow[dr, "\mu_1"'] & \arrow[d, phantom, "\xLeftarrow{\lambda}"] & X \otimes_2 X \arrow[dl, "\mu_2"] & \mathbb{1}_{\mathcal{C}} \arrow[l, "x\otimes_2 y"'] \\ & & X, & & \end{tikzcd}$$

where λ is the interchange morphism obtained from the 2-monoidal structure of X. As a consequence, λ provides an element of type $p_\lambda \in \Lambda(\mathbb{1}_\Lambda, \mathcal{C})(\mathbb{1}_{\mathcal{C}}, X)(\mu_2 x \otimes_2 y, x \otimes_1 Y)$, with $\Lambda(\mathbb{1}_\Lambda, \mathcal{C})(\mathbb{1}_{\mathcal{C}}, X)(\mu_2\, x \otimes_2 y, \mu_1\, x \otimes_1 y) \in \mathbb{B}$, so that p_λ is a proof of the equality $\mu_1\, x \otimes_1 y = \mu_2 x \otimes_2 y$. The equalities $\mu_1\, x \otimes_1 y = \mu_1 y \otimes_1 x$ and $\mu_2\, x \otimes_2 y = \mu_2 y \otimes_2 x$ can be proven similarly. For this purpose, the interchange may be applied to $(\mathbb{1}_\Lambda, \mathcal{C}), (\mathcal{C}, \mathbb{1}_\Lambda)$. $\square$

REMARK 1.1.10. Most of the time, the 2-category Λ in which one applies Eckmman-Hilton's argument is symmetric monoidal, with monoidal product given by cartesian product. In fact, Λ is often the 2-category of categories, with internal 2-fold monoid given by a category $\mathcal{C}$, which also is often symmetric, with monoidal product given by the cartesian product. Indeed, the framework usually employed for the study of iterated monoids stands in the category of topological spaces or in the category of sets. The interest in stating Eckmann-Hilton's argument with such a level of generality [1] is in our opinion, at least two fold. First, it may be convenient to work in the 2-category of topologically enriched categories, so that we can consider 2-fold monoids internal to a topologically enriched category of algebras over an operad. Furthermore, the intricacies regarding the combinatorial aspects of iterated monoids appear with greater clarity as the amount of structure increases, as noticed in the previous chapter[2]. This generalized expression of Eckmann-Hilton's argument can be interpreted as follow. Let X be a 2-fold monoid internal to a 2-fold monoid, itself internal to a 2-fold monoidal 2-category. If $\mu_1 : X \otimes_1 X \to X$ and $\mu_2 : X \otimes_2 X \to X$ send each pair of generalized objects $x, y : \mathbb{1}_{\mathcal{C}} \to X$ of X to the same element, their global comparison as monoidal structures does not make sense for that all, in that their domain differ. Still, when both Λ and $\mathcal{C}$ are symmetric, we retrieve the customary statement of Eckmann-Hilton's argument.

1.2. Pairing of operads and matrix monoidal product of symmetric sequences. We recall the definition of a pairing of operads, due to May, which will be useful to express the universal property of the tensor product. For this purpose, we first introduce a way of arranging permutations on blocks, which will be useful both for the description of a pairing of operads and for the subsequent constructions in this subsection. We recall a symmetric monoidal structure on the category of symmetric sequences described in [**8**], called the matrix monoidal product, and use it to reformulate the universal property of the tensor product in categorical terms.

[1] This framework is however, still inaccurate. The right approach for considering iterated monoids is to place within an infinite sequence of monoids internal to each others, so that $\Lambda_n \in Mon^2\Lambda_{n+1}$ for all n. This paradigm is implicitly used in the statement of Eckmann-Hilton's argument. Indeed $(\mathrm{CAT}_2, \times)$, is a symmetric monoid, in particular a 2-fold monoid, internal to $(\mathrm{CAT}_3, \times)$, and so on. For more details see Appendix A.

[2] See II.1.4

In the next section, we will extend the matrix monoidal product to the 2-category of symmetric sequences of categories and will extensively use it to describe the generators and relations involved in the construction of the tensor product.

DEFINITION 1.2.1. Let $n, m \in \mathbb{N}$. We let $\sigma_n^m : |n| \times |m| \cong |nm|$ be the isomorphism consisting in the concatenation of m lines of n elements, which formally sends $(i,j) \in |n| \times |m|$ to $(j-1)n + i \in |nm|$. The isomorphisms $\sigma_n^m : [n] \times [m] \cong [nm]$ induce a group morphism $\mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{nm}$ for each $n, m \in \mathbb{N}$. Those group morphisms induce a functor

$$\square : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S},$$

which gives to the category $\mathfrak{S}$ the structure of a symmetric monoidal category whose unit is given by $1 \in \mathfrak{S}$.

DEFINITION 1.2.2. Let $n, m \in \mathbb{N}$. We let τ_n^m be the permutation in Σ_{nm}, which 'exchanges rows and columns ', defined as the composite

$$\tau_n^m : |nm| \xrightarrow{\sigma_n^{m\,-1}} |n| \times |m| \xrightarrow{\cong} |m| \times |n| \xrightarrow{\sigma_m^n} |nm|.$$

DEFINITION 1.2.3. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be operads. A pairing $\mu : (\mathcal{A}, \mathcal{B}) \to \mathcal{C}$ is the data for each $r, s \in \mathbb{N}$, of a morphism of symmetric sequences $\mu : \mathcal{A}(r) \times \mathcal{B}(s) \to \mathcal{C}(rs)$ such that

(a) for each $\sigma_r \in \Sigma_r$ and $\sigma_s \in \Sigma_s$, for each $a \in \mathcal{A}(r)$ and $b \in \mathcal{B}(s)$ we have

$$\mu(\sigma_r.a, \sigma_s.b) = \sigma_r \square \sigma_s.\mu(a,b),$$

(b) for each $n_1, ..., n_r \in \mathbb{N}$ and each $m_1, ..., m_s \in \mathbb{N}$, for all $a \in \mathcal{A}(r), a_i \in \mathcal{A}(n_i)$ and $b \in \mathcal{B}(s), b_j \in \mathcal{B}(m_j)$,

$$\mu(a(a_1, ..., a_r), b(b_1, ..., b_r)) = \tau.\mu(a,b)\left(\mu(a_i, b_j)_{i,j}\right),$$

where the order on the family $\left(\mu(a_i, b_j)\right)_{i,j}$ is given by the isomorphism σ_n^m of Definition III.1.2.2, and where τ is the permutation obtained by the following sequence of operadic compositions in the operad of permutations

$$\begin{tikzcd}
{|r|(|n_1|, ..., |n_r|) \circ (|s|(|m_1|, ..., |m_s|)^n)} \arrow[r, dashed] \arrow[d] & {|rs|(|n_i m_j|_{i,j})} \\
{|r|\left(|n_1|\left(|s|(|m_1|, ..., |m_s|\right)^{n_1}, ..., |n_r|\left(|s|(|m_1|, ..., |m_s|\right)^{n_r}\right).} \arrow[ur] &
\end{tikzcd}$$

PROPOSITION 1.2.4. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be operads and $\mu : (\mathcal{A}, \mathcal{B}) \to \mathcal{C}$ be a pairing of operads. We obtain morphisms of operads $\mu_{\mathcal{A}} : \mathcal{A} \to \mathcal{C}$ and $\mu_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}$. Moreover, those morphisms satisfy the exchange law in the sense that for all $a \in \mathcal{A}(r)$ and $b \in \mathcal{B}(s)$, we have*

$$\mu_{\mathcal{A}}(a)(\{\mu_{\mathcal{B}}(b)\}_{i=1}^r) = \sigma_r^s.\mu_{\mathcal{B}}(b)(\{\mu_{\mathcal{A}}(a)\}_{j=1}^s).$$

PROOF. Let $\mu_{\mathcal{A}} : \mathcal{A} \to \mathcal{C}$ be given in arity $r \in \mathbb{N}$ by the composite

$$\mu_{\mathcal{A}}(r) : \mathcal{A}(r) \xrightarrow{\cong} \mathcal{A}(r) \times 1_{\mathcal{C}} \xrightarrow{1_{\mathcal{A}} \times \epsilon_{\mathcal{B}}} \mathcal{A}(r) \times \mathcal{B}(1) \xrightarrow{\mu} \mathcal{C}(r),$$

by condition 1 on μ, the morphism $\mu_{\mathcal{A}}$ yields a morphism of symmetric sequence, and by condition 2, the morphism $\mu_{\mathcal{A}}$ is compatible with the operadic composition. Consequently, we obtain a morphism of operads $\mu_{\mathcal{A}} : \mathcal{A} \to \mathcal{C}$. In the same way, we obtain a morphism of operads $\mu_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}$. Write $1_{\mathcal{A}} \in \mathcal{A}(1)$ and $1_{\mathcal{B}} \in \mathcal{B}(1)$ for the units of $\mathcal{A}$ and $\mathcal{B}$. For $a \in \mathcal{A}(r)$ and $b \in \mathcal{B}(r)$, the second condition ensures that

$$\mu_{\mathcal{A}}(a)(\{\mu_{\mathcal{B}}(b)\}_{i=1}^{r}) = \mu(a, 1_{\mathcal{B}})(\{\mu(1_{\mathcal{A}}, b)\}_{i=1}^{r}) = \mu(a, b)$$

and

$$\sigma_r^s.\mu_{\mathcal{B}}(b)(\{\mu_{\mathcal{A}}(a)\}_{j=1}^{s}) = \sigma_r^s.\mu(1_{\mathcal{A}}, b)(\{\mu((a, 1_{\mathcal{B}})\}_{j=1}^{s}) = \sigma_r^s.\sigma_s^r.\mu(a, b) = \mu(a, b),$$

which proves the second assertion of the proposition. $\square$

Definition 1.2.5. The symmetric monoidal structure on $\mathfrak{S}$ given in Definition III.1.2.2 induces a symmetric monoidal structure on the category of symmetric sequences in $\mathcal{C}$ by Day's convolution. We write

$$\square : \mathcal{C}^{\mathfrak{S}} \times \mathcal{C}^{\mathfrak{S}} \to \mathcal{C}^{\mathfrak{S}}$$

for the tensor product functor thus obtained. For $A, B \in \mathcal{C}^{\mathfrak{S}}$, the symmetric sequence $A \square B \in \mathcal{C}^{\mathfrak{S}}$ is given by the left Kan extension

$$\begin{tikzcd} \mathfrak{S} \times \mathfrak{S} \arrow[r, "A \times B"] \arrow[d, "\square"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{C} \\ \mathfrak{S}. \arrow[ur, dashed, "A \square B"'] & \end{tikzcd}$$

Hence, we obtain the following formula in arity $r \in \mathbb{N}$

$$\begin{aligned} A \square B(r) &\cong \int_{n,m \in \mathfrak{S} \times \mathfrak{S}} \mathfrak{S}(n \times m, r) \times A(n) \times B(m) \\ &\cong \coprod_{nm = r} \mathfrak{S}_r \underset{\mathfrak{S}_n \times \mathfrak{S}_m}{\times} A(n) \times B(m). \end{aligned}$$

Remark 1.2.6. By construction, the data of a morphism of symmetric sequence $A \square B \to R$ is equivalent to the data for each n, m in $\mathbb{N}$ of a morphism $A(n) \times B(m) \to R(nm)$ which is compatible with the action of the symmetric group.

1.3. The Boardman-Vogt tensor product. We recall the customary definition of the Boardman-Vogt tensor product of topological operads and provide an equivalent characterization in terms of coequalizers, which will be more suitable for our purpose. We refer to [**7**] and [**6**] for details on the existence of the tensor product.

Definition 1.3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be operads. The Boardman-Vogt tensor product of $\mathcal{A}$ and $\mathcal{B}$ is a universal pair $(\mathcal{A} \otimes \mathcal{B}, \mu : (\mathcal{A}, \mathcal{B}) \to \mathcal{A} \otimes \mathcal{B})$, where $\mathcal{A} \otimes \mathcal{B}$ is an operad and where μ is a pairing of operads. Precisely, a pair $(\mathcal{A} \otimes \mathcal{B}, \mu : (\mathcal{A}, \mathcal{B}) \to \mathcal{A} \otimes \mathcal{B})$ represents the Boardman-Vogt tensor product of $\mathcal{A}$ and $\mathcal{B}$ if for any operad $\mathcal{C}$ equipped with a pairing $\nu : (\mathcal{A}, \mathcal{B}) \to \mathcal{C}$, there is a unique morphism of operads $\bar{\nu} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} (\mathcal{A}, \mathcal{B}) \arrow[r, "\mu"] \arrow[dr, "\nu"'] & \mathcal{A} \otimes \mathcal{B} \arrow[d, "\bar{\nu}"] \\ & \mathcal{C}. \end{tikzcd}$$

Definition 1.3.2. Let $\mathcal{A}$ and $\mathcal{B}$ be symmetric sequences. The sequence of isomorphisms given in $n, m \in \mathbb{N}$ by $\mathcal{P}(n) \times \mathcal{Q}(m) \xrightarrow{\cong} \mathcal{Q}(m) \times \mathcal{P}(n)$ induces an isomorphism of symmetric sequences

$$\mathfrak{s}_{\mathcal{P},\mathcal{Q}} : \mathcal{P}\square\mathcal{Q} \xrightarrow{\cong} \mathcal{Q}\square\mathcal{P}$$

which maps $p\square q \in \mathcal{P}\square\mathcal{Q}(nm)$ to $\tau_n^m.q\square p \in \mathcal{Q}\square\mathcal{P}(nm)$ for $p \in \mathcal{P}(n)$ and $q \in \mathcal{Q}(m)$. The sequence of morphisms given in $n, m \in \mathbb{N}$ by the composite

$$\mathcal{P}(n) \times \mathcal{Q}(m) \xrightarrow{\mathtt{id}\times\Delta^n} \mathcal{P}(n) \times \prod_{i=1}^{n} \mathcal{Q}(m) \hookrightarrow \mathcal{P} \circ \mathcal{Q}(nm) \hookrightarrow \mathcal{P} \vee \mathcal{Q}(nm),$$

where the morphism Δ^n is induced by the identity in each factor, yield a morphism $\iota_{\mathcal{P},\mathcal{Q}} : \mathcal{P}\square\mathcal{Q} \to \mathcal{P} \vee \mathcal{Q}$. We also let $\iota^\tau_{\mathcal{P},\mathcal{Q}}$ be the morphism obtained from the following composite:

$$\iota^\tau_{\mathcal{P},\mathcal{Q}} : \mathcal{P}\square\mathcal{Q} \xrightarrow{\mathfrak{s}_{\mathcal{P},\mathcal{Q}}} \mathcal{Q}\square\mathcal{P} \xrightarrow{\iota_{\mathcal{Q},\mathcal{P}}} \mathcal{Q} \vee \mathcal{P} \cong \mathcal{P} \vee \mathcal{Q}.$$

Notation 1.3.3. Let $\mathcal{R}$ be a symmetric sequence and $\mathcal{P}, \mathcal{Q}$ be operads in $\mathcal{C}$. We say that a diagram

$$\mathcal{R} \rightrightarrows \mathcal{P} \longrightarrow \mathcal{Q}$$

is a coequalizer if the following induced diagram is a coequalizer in the category of operads

$$\mathcal{T}\mathcal{R} \rightrightarrows \mathcal{P} \longrightarrow \mathcal{Q}.$$

Proposition 1.3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be operads. The data of a pairing of operads $\mu : (\mathcal{A}, \mathcal{B}) \to \mathcal{C}$ is equivalent to the data of a morphism $\mathcal{A} \vee \mathcal{B} \xrightarrow{\nu} \mathcal{C}$ equalizing the diagram*

$$\mathcal{A}\square\mathcal{B} \underset{\iota^\tau_{\mathcal{A},\mathcal{B}}}{\overset{\iota_{\mathcal{A},\mathcal{B}}}{\rightrightarrows}} \mathcal{A} \vee \mathcal{B} \xrightarrow{\ \nu\ } \mathcal{C}\ .$$

Moreover, this diagram is a coequalizer if and only if the pairing $\mu : (\mathcal{A}, \mathcal{B}) \to \mathcal{C}$ is universal.

Proof. By Proposition III.1.2.4, a pairing of operads $(\mathcal{A}, \mathcal{B}) \to \mathcal{C}$ yields morphisms of operads $\mathcal{A} \to \mathcal{C}$ and $\mathcal{B} \to \mathcal{C}$ which satisfy the exchange law. Equivalently, the induced morphism of operads $\mathcal{A} \vee \mathcal{B} \to \mathcal{C}$ equalizes $\iota_{\mathcal{A},\mathcal{B}}$ and $\iota^\tau_{\mathcal{A},\mathcal{B}}$. Conversely, let $\nu : \mathcal{A} \vee \mathcal{B} \to \mathcal{C}$ be a morphism of operads such that $\nu\iota_{\mathcal{A},\mathcal{B}} = \nu\iota^\tau_{\mathcal{A},\mathcal{B}}$. The family of morphisms $\mu(r, s) : \mathcal{A}(r) \times \mathcal{B}(s) \to \mathcal{C}(rs)$ given in $a \in \mathcal{A}(r), b \in \mathcal{B}(s)$ by $\nu(a(b, ..., b))$ provides a pairing of operads. The equivalence between the universality of the pairing and the universal property of equalizers is straightforward. $\square$

Corollary 1.3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be operads and let $\mathcal{D}$ be a topologically enriched monoidal category. The category of $\mathcal{B}$-algebras in $\mathcal{A}$-$\mathrm{Alg}_{\mathcal{D}}$ is equivalent to the category of $\mathcal{A} \otimes \mathcal{B}$-algebras in $\mathcal{D}$.*

Proposition 1.3.6. *The Boardman-Vogt tensor product yields a functor*

$$\otimes_{BV} : \mathrm{Op} \times \mathrm{Op} \to \mathrm{Op},$$

which gives to the category of operads the structure of a symmetric monoidal category.

REMARK 1.3.7. The main interest of the Boardman-Vogt tensor product lies in that the category of $\mathcal{P}\otimes_{BV}\mathcal{Q}$-algebras is equivalent to the category of $\mathcal{P}$-algebras in the category of $\mathcal{Q}$-algebras, which makes it notably suited for the study of iterated loop spaces. However, this tensor product may be too rigid to capture the information on higher commutativity. Notably, in the discrete case, Eckmann-Hilton's argument states that we have an isomorphism $As\otimes_{BV} As\cong Com$. Under specific cofibrancy assumptions, the subsequent additivity theorem was established in [**9**].

THEOREM 1.3.8. *Let $\mathcal{A}$ be an E_n-operad and $\mathcal{B}$ an E_m-operad. If both of them are cofibrant, then the tensor product $\mathcal{A}\otimes_{BV}\mathcal{B}$ is a E_{n+m}-operad.*

REMARK 1.3.9. The resulting operad is not necessarily cofibrant.

2. Lax tensor product of categorical operads

The idea of the tensor product that we define in this section is taken from the construction of the operads $\mathcal{M}^n$. Instead of requiring a strict interchange between each pair of operations, we add a morphism encoding the compatibility between those operations.

2.1. Preliminary remarks on the interchange relation.

NOTATION 2.1.1. We write $\bullet_r\in\mathcal{M}_2(r)$ for the arity r-th operation of $\mathcal{M}_2$ corresponding to the first monoidal law, and we write let $\circ_s\in\mathcal{M}_2(s)$ be the operation of arity s corresponding to the second one.

OBSERVATION 2.1.2. *We can write the interchange morphism in the category $\mathcal{M}_2(4)$ as follows:*

$$\bullet(\circ,\circ)\to\sigma_2^2\cdot\circ(\bullet,\bullet),$$

where $\tau_2^2\in\Sigma_4$ is the permutation exchanging rows and columns, as defined in III.1.2.2. Note that τ_2^2 is the transposition $\tau_2^2=(2\,3)$. More generally, for each pair of natural numbers n,m, the operadic composition of $\bullet_n$ with $\{\circ_m\}_{i=1}^n$ yields an operation $\bullet_n(\circ_m,\dots,\circ_m)\in\mathcal{M}_2(nm)$, which is minimal in the sense that there is no non identical morphism into it in $\mathcal{M}_2(nm)$. Conversely, the operadic composition of $\circ_m$ with $\{\bullet_n\}_{i=1}^m$ yield an operation $\circ_m(\bullet_n,\dots,\bullet_n)\in\mathcal{M}_2(nm)$, which is maximal in the sense that there is no non identical morphism from it in $\mathcal{M}_2(nm)$. The interchange morphism yields chains of morphisms from the minimal operation $\bullet_n(\circ_m,\dots,\circ_m)$ to the maximal operation $\tau_n^m\cdot\circ_m(\bullet_n,\dots,\bullet_n)$, which are all equal in the poset $\mathcal{M}_2(nm)$, so that we have a unique morphism:

$$\bullet_n(\circ_m,\dots,\circ_m)\to\tau_n^m\cdot\circ_m(\bullet_n,\dots,\bullet_n).$$

DEFINITION 2.1.3. For $n\in\mathbb{N}^*$. We let $\mathbb{M}_n$ be the free n-fold monoidal category generated by the terminal category $*$. Recall from Remark II.3.1.1 that $\mathbb{M}_n$ satisfies

$$\mathbb{M}_n\cong\coprod_{r\in\mathbb{N}}\mathcal{M}_n(r)/\Sigma_r.$$

Note that in particular, $\mathbb{M}_1\cong\mathbb{N}$ as monoidal categories.

LEMMA 2.1.4. *The sum of permutations $+:\mathfrak{S}\times\mathfrak{S}\to\mathfrak{S}$ yields an $\mathcal{M}_2$-algebra structure on the category $\mathfrak{S}$.*

PROOF. The sum of permutation is strictly unital and strictly associative. Let the interchange morphism β:

$$\begin{tikzcd}
(\mathfrak{S}\times\mathfrak{S})\times(\mathfrak{S}\times\mathfrak{S}) \arrow[r, "\tau"] \arrow[d, "+\times+"'] & (\mathfrak{S}\times\mathfrak{S})\times(\mathfrak{S}\times\mathfrak{S}) \arrow[d, "+\times+"] \\
\mathfrak{S}\times\mathfrak{S} \arrow[dr, "+"'] \arrow[r, phantom, "\Nearrow\beta"] & \mathfrak{S}\times\mathfrak{S} \arrow[d, "+"] \\
& \mathfrak{S}
\end{tikzcd}$$

be defined in $n_1, n_2, m_1, m_2 \in \mathfrak{S}$ by the permutation $(n_1+n_2)+(m_1+m_2) \xrightarrow{\cong} (n_1+m_1)+(n_2+m_2)$ corresponding to the evident isomorphism of sets

$$(|n_1| \coprod |n_2|) \coprod (|m_1| \coprod |m_2|) \xrightarrow{\cong} (|n_1| \coprod |m_1|) \coprod (|n_2| \coprod |m_2|).$$

The verification of the unit and associativity constraints is straightforward. $\square$

DEFINITION 2.1.5. Let $\pi_{\mathfrak{S}} : \mathbb{M}_2 \to \mathfrak{S}$ be the 2-fold monoidal functor induced by the functor $1 : * \to \mathfrak{S}$. Each object of the summand of $\mathbb{M}_2$ indexed by $r \in \mathbb{N}$ is send to $r \in \mathfrak{S}$ by $\pi_{\mathfrak{S}}$. Notably, the morphism corresponding to the binary interchange law $\bullet(\circ,\circ) \to \sigma_2^2 \cdot \circ(\bullet,\bullet)$ in $\mathcal{M}_2(4)/\Sigma_4$ is send to the permutation $\tau_2^2 = (2\,3)$ of Σ_4.

DEFINITION 2.1.6. Let $\boxbar : \mathbb{M}_1 \times \mathbb{M}_1 \to \mathbb{M}_2$ and $\boxminus : \mathbb{M}_1 \times \mathbb{M}_1 \to \mathbb{M}_2$ be defined in $(n,m) \in \mathbb{M}_1 \times \mathbb{M}_1$ by the following elements of $\mathbb{M}_2$:

– $n \boxbar m = \bullet_n(\circ_m, \dots, \circ_m)$,
– $n \boxminus m = \tau_n^m \cdot \circ_m(\bullet_n, \dots, \bullet_n)$.

2.2. Construction of the tensor product.

DEFINITION 2.2.1. Let Λ be a 2-category and $\iota_1, \iota_2 : \mathcal{R} \to \mathcal{P}$ be morphisms in Λ. A lax coequalizer of (ι_1, ι_2) in Λ is a universal triple $(\mathcal{P}/\mathcal{R}, \pi_{\mathcal{R}}, \tau)$, where

– $\mathcal{P}/\mathcal{R}$ is an object of Λ,
– $\pi_{\mathcal{R}} : \mathcal{P} \to \mathcal{P}/\mathcal{R}$ is a morphism in Λ,
– $\tau : \pi\iota_1 \Rightarrow \pi\iota_2$ is a 2-morphism in Λ.

Precisely, we say that the triple $(\mathcal{P}/\mathcal{R}, \pi_{\mathcal{R}}, \tau)$ is universal if for any other triple $(\mathcal{Q}, \pi, \sigma)$, there exists a unique pair $(\overline{\pi}, \beta)$:

$$\begin{tikzcd}
\mathcal{R} \arrow[r, shift left, "\iota_1"] \arrow[r, shift right, "\iota_2"'] & \mathcal{P} \arrow[r, "\pi_{\mathcal{R}}"] \arrow[dr, "\pi"'] & \mathcal{P}/\mathcal{R} \arrow[d, dashed, "\overline{\pi}"] \arrow[dl, phantom, "\Swarrow\beta"] \\
& & \mathcal{Q}
\end{tikzcd}$$

such that the following diagram commutes in Λ:

$$\begin{tikzcd}
\mathcal{R} \arrow[r, bend left, "\pi_{\mathcal{R}}\iota_1"] \arrow[r, "\pi_{\mathcal{R}}\iota_2"'] \arrow[r, phantom, shift left=2, "\Downarrow\tau"] \arrow[dr, "\pi\iota_1"] \arrow[dr, bend right, "\pi\iota_2"'] \arrow[dr, phantom, shift right=2, "\Downarrow\alpha"] & \mathcal{P}/\mathcal{R} \arrow[d, dashed, "\overline{\pi}"] \arrow[dl, phantom, "\Swarrow\beta"] \\
& \mathcal{Q}
\end{tikzcd}$$

LEMMA 2.2.2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be small symmetric categorical operads, regarded as objects of the category* $\mathrm{Op}^{\mathfrak{S}}_{(\mathbf{cat},\times)}$. *Let* $\mathcal{P}\square\mathcal{Q} : \mathfrak{S}^{op} \to \mathbf{cat}$ *be the symmetric sequence as defined in III.1.2.5 and recall that it comes equipped with with morphisms* ι, ι_τ : $\mathcal{P}\square\mathcal{Q} \to \mathcal{P} \vee \mathcal{Q}$. *The following diagrams strictly commute in* $[\mathfrak{S}^{op}, \mathbf{cat}]$*:*

DEFINITION 2.2.3. Let $\mathcal{P}$ and $\mathcal{Q}$ be small symmetric categorical operads, regarded as objects of the 2-category $\mathrm{Op}^{\mathfrak{S}}_{\mathbf{Cat}}$. The Boardman-Vogt tensor product of $\mathcal{P}$ and $\mathcal{Q}$ is a universal triple $(\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}, \pi, \tau)$, where

- $\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$ is a symmetric categorical operad
- $\pi : \mathcal{P} \vee \mathcal{Q} \to \mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$ is a morphism in $\mathrm{Op}^{\mathfrak{S}}_{\mathbf{Cat}}$
- $\tau : \pi\iota \Rightarrow \pi\iota^\tau$ is a 2-morphism in $\mathbf{Cat}^{\mathfrak{S}}$

and which satisfies the following:

- the triple $(\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}, \pi, \tau)$ is a lax coequalizer for (ι, ι^τ) in the 2-category $\mathbf{Cat}^{\mathfrak{S}}$
-

DEFINITION 2.2.4. Let $(\mathcal{E}, \mathcal{R})$ and $(\mathcal{F}, \mathcal{S})$ be some operadic polygraphs together with compatible relations. We define their tensor product $(\mathcal{E}, \mathcal{R}) \otimes_{\mathrm{BV}} (\mathcal{F}, \mathcal{S})$ as an operadic polygraph with relations $(\mathcal{E} \sqcup \mathcal{F} \sqcup \mathcal{G}, \mathcal{R} \sqcup \mathcal{S} \sqcup \mathcal{U})$ with:

- $G_0 = U_0 = \emptyset$
- $G_1 = E_0 \square F_0 \underset{\tau}{\overset{\sigma}{\rightrightarrows}} \mathcal{F}(E_0 \sqcup F_0)$
- $U_1 = R_0 \square F_0 \sqcup E_0 \square S_0 \sqcup E_1 \square S_0 \sqcup R_0 \square F_1$.

This construction also can be made on the categories of categorical operads Op_{Cat}. The operad $\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$ is constructed as the coproduct operad $\mathcal{P} \vee \mathcal{Q}$, to which we add a sequence $\mathcal{P}\square\mathcal{Q}$ of generating morphism with source and target also defined by σ and τ, and subjected to coherence relations involving operadic composition of the objects of $\mathcal{P}$ and $\mathcal{Q}$ and morphisms in $\mathcal{P}$ and $\mathcal{Q}$. In particular, the objects of $\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$ is the coproduct of the objects operads of $\mathcal{P}$ and $\mathcal{Q}$. This construction is compatible with the polygraphic presentations, in the sense that we get the following proposition.

PROPOSITION 2.2.5. $\mathcal{T}((\mathcal{E}, \mathcal{R}) \otimes_{\mathrm{BV}} (\mathcal{F}, \mathcal{S})) \cong \mathcal{T}(\mathcal{E}, \mathcal{R}) \otimes_{\mathrm{BV}} \mathcal{T}(\mathcal{F}, \mathcal{S})$.

THEOREM 2.2.6. *There is a monoidal structure on the* 2*-category of categorical operads whose tensor product is given by the lax Boardman-Vogt tensor product, which extends to a* 2*-functor*

$$\otimes_{\mathrm{BV}} : \mathrm{Op}_{Cat} \times \mathrm{Op}_{Cat} \longrightarrow \mathrm{Op}_{Cat}.$$

Moreover, the algebra 2*-functor* ALG *gives to the* 2*-category of symmetric monoidal* 2*-categories the structure of a left module over the monoidal* 2*-category* $(\mathrm{Op}^{op}_{\mathrm{CAT}}, \otimes_{\mathrm{BV}})$*, in the sense that the square*

$$\begin{tikzcd}
\mathrm{Op}^{op}_{\mathrm{CAT}} \times \mathrm{Op}^{op}_{\mathrm{CAT}} \times \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{CAT}_2} \arrow[r, "id \times \mathrm{Alg}"] \arrow[d, "\otimes^{op}_{\mathrm{BV}} \times id"'] & \mathrm{Op}^{op}_{\mathrm{CAT}} \times \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{CAT}_2} \arrow[d, "\mathrm{Alg}"] \\
\mathrm{Op}^{op}_{\mathrm{CAT}} \times \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{CAT}_2} \arrow[r, "\mathrm{Alg}"] & \mathrm{Mon}^{\mathfrak{S}}_{\mathrm{CAT}_2}
\end{tikzcd}$$

commutes up to isomorphism. The tensor product also preserves acyclic fibrations, and $\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$ *is a cofibrant operad if and only if both* $\mathcal{P}$ *and* $\mathcal{Q}$ *are cofibrant.*

Corollary 2.2.7. $\mathcal{M}^n \otimes_{\mathrm{BV}} \mathcal{M}^m \cong \mathcal{M}^{n+m}$.

Corollary 2.2.8. *If we have cofibrant resolutions $\mathcal{P}_\infty \xrightarrow{\sim} \mathcal{P}$ and $\mathcal{Q}_\infty \xrightarrow{\sim} \mathcal{Q}$ of operads $\mathcal{P}$ and $\mathcal{Q}$, then the tensor product*

$$\mathcal{P}_\infty \otimes_{\mathrm{BV}} \mathcal{Q}_\infty \xrightarrow{\sim} \mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$$

is a cofibrant resolution of $\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q}$.

Remark 2.2.9. The term "lax" of the lax tensor product $\mathcal{P} \otimes_{BV} \mathcal{Q}$ only refers to the interchange between $\mathcal{P}$ and $\mathcal{Q}$, in that an algebra over $\mathcal{P} \otimes_{BV} \mathcal{Q}$ is equipped with regular structures of a $\mathcal{P}$-algebra and of a $\mathcal{Q}$-algebra, which are "lax" compatible.

2.3. Additivity and further properties.

Remark 2.3.1. The monoidal structure on categorical operads induced by the tensor product fails to be symmetric. Indeed, any morphism of sequences

$$\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{Q} \xrightarrow{\tau} \mathcal{Q} \otimes_{\mathrm{BV}} \mathcal{P}$$

mapping the morphism $p\square q$ to $q\square p$ also should map the object $p(q, ..., q)$ to $q(p, ..., p)$, thus it can not be a morphism of operads. However, we have an isomorphism of operads

$$\mathcal{P}^{op} \otimes_{\mathrm{BV}} \mathcal{Q}^{op} \cong (\mathcal{Q} \otimes_{\mathrm{BV}} \mathcal{P})^{op}.$$

3. Application: cofibrant models of the operads $\mathcal{M}_n$ and n-fold loop spaces

3.1. Description of cofibrant models of $\mathcal{M}_n$.

Corollary 3.1.1. *The operad $\mathcal{M}^n_\infty := (\mathcal{M}^1_\infty)^{\otimes^n_{\mathrm{BV}}}$ is cofibrant, and provides a cofibrant resolution of the operad $\mathcal{M}^n$, so that we have an acyclic fibration*

$$\mathcal{M}^n_\infty \xrightarrow{\sim} \mathcal{M}^n.$$

Moreover, we have the following operadic presentation of $\mathcal{M}^n_\infty$:

- $E^n_0 = \coprod_{i=1}^{n} E_0$
- $E^n_1 = \coprod_{i=1}^{n} E_1 \sqcup \coprod_{1\leq i<j\leq n} E_0\square E_0$
- $R^n_1 = \coprod_{i=1}^{n} R_1 \sqcup \coprod_{1\leq i<j\leq n} E_1\square E_0 \sqcup \coprod_{1\leq i<j\leq n} E_0\square E_1 \sqcup \coprod_{1\leq i<j<k\leq n} E_0\square E_0\square E_0.$

Hence its objects are freely generated by n operations in each arity and its morphisms are edge contractions between vertices of the same color together with interchange morphisms between each pair of operation, subjected to coherence relations.

Definition 3.1.2. An n-fold monoid up to homotopy in a symmetric monoidal 2-category $(\Lambda, \otimes_\Lambda)$ is the data of

- an object X of Λ
- for each $1 \leq i \leq n$ and $r \in \mathbb{N}$, a 1-morphism in Λ

$$\mu^r_i : X^{\otimes^r_\Lambda} \to X,$$

– for each $r, s \in \mathbb{N}$, $1 \leq k \leq r$ and $1 \leq i \leq r$, a 2-morphism $\alpha^{r,s}_{k,i}$ in Λ such that

$$\begin{tikzcd}
X^{\otimes_\Lambda^{r+s-1}} \arrow[r, "\cong"] \arrow[dd, "\mu_i^{r+s-1}"'] & X^{\otimes_\Lambda^{k-1}} \otimes_\Lambda X^{\otimes_\Lambda^{s}} \otimes_\Lambda X^{\otimes_\Lambda^{r-k}} \arrow[d, "\mathtt{id}\otimes_\Lambda \mu_i^s \otimes_\Lambda \mathtt{id}"] \\
\arrow[r, phantom, "\overset{\Longleftarrow}{\alpha^{r,s}_{i,k}}"] & X^{\otimes_\Lambda^{k-1}} \otimes_\Lambda X \otimes_\Lambda X^{\otimes_\Lambda^{r-k}} \arrow[d, "\cong"] \\
X & X^{\otimes_\Lambda^{r}} \arrow[l, "\mu_i^r"]
\end{tikzcd}$$

and which fulfill coherence constraints expressed by the commutativity of a familly of squares for each $1 \leq i \leq n$,

– for each $1 \leq i < j \leq n$ and $r, s \in \mathbb{N}$, an interchange 2-morphism $\square^{j,s}_{i,r}$ in Λ, such that

$$\begin{tikzcd}
& \left(X^{\otimes_\Lambda^{r}}\right)^{\otimes_\Lambda^{s}} \arrow[r, "\mu_i^{r\otimes_\Lambda^{s}}"] & (X)^{\otimes_\Lambda^{s}} \arrow[d, "\mu_j^s"] \\
\left(X^{\otimes_\Lambda^{s}}\right)^{\otimes_\Lambda^{r}} \arrow[ur, "\cong"] \arrow[r, "\mu_j^{s\otimes_\Lambda^{r}}"'] & (X)^{\otimes_\Lambda^{r}} \arrow[r, "\mu_i^r"'] \arrow[ur, phantom, "\Uparrow \square^{j,s}_{i,r}"] & X
\end{tikzcd}$$

and which fullfil coherence constraints expressed by the commutativity of

– a family of pentagons $\left\{\pentagon^{r,l,s,p}_{i,j}, 1 \leq l \leq r\right\}^{r,s,p\geq 0}_{1\leq i<j\leq n}$ where the pentagone $\pentagon^{r,l,s,p}_{i,j}$ ensures that the 1-morphism μ_j^p is compatible with the 2-morphism $\alpha^{r,s}_{l,i}$ if $i < j$ and ensures that the 2-morphism $\alpha^{r,p}_{l,i}$ is compatible with the 1-morphism μ^s_j if $j < i$,

– a family of hexagons $\left\{\hexagon^{r,s,p}_{i,j,k}, 1 \leq i < j < k \leq n\right\}_{r,s,p\geq 0}$, where the hexagon $\hexagon^{r,s,p}_{i,j,k}$ ensures the compatibility of the 2-morphism $\square^{k,p}_{j,s}$ with respect to the 1-morphism μ_i^r.

3.2. Link with n-fold loop spaces. We describe an $\mathcal{M}^n_\infty$-algebra structure on the n-fold loop space $\Omega^n X$ of a topological space X, in the monoidal 2-category $(\textsc{Top}, \times)$, where the 2-morphisms are the homotopy classes of the homotopies. For this purpose, we first observe that the n-cube I^n has the structure of an $\mathcal{M}^n_\infty$-algebra in the monoidal 2-category $(\textsc{Top}, \sqcup)$, which induces an $\mathcal{M}^n_\infty$-algebra structure in $(\textsc{Top}, \sqcup)$ on the n-sphere S^n via the isomorphism $S^n \cong I^n/\partial I^n$.

Let $\mu_i^r : \bigsqcup\limits_{k=1}^{r} I^n \to I^n$ be the map which sends the j-th little n-cube to I^n by inclusion in contracting its i-th coordinate by $1/r$. Explicitely, we set

$$\mu_i^r(t^{(j)}) = \left(t_1^{(j)}, ..., \frac{t_i^{(j)} + j - 1}{r}, ..., t_n^{(j)}\right)$$

where $t^{(j)} = \left(t_1^{(j)}, ..., t_r^{(j)}\right)$. Note that the interchange laws hold strictly. We also can define homotopies $\mu_i^r \circ_l \mu_i^s \to \mu_i^{r+s-1}$ which fullfil the coherence constraints, so that $I^n \in \mathcal{M}^n_\infty\text{-}\textsc{Alg}_{(\textsc{Top},\sqcup)}$. Moreover, the maps μ_i^r together with the homotopies, induce maps and homotopies on the quotient $I^n/\partial I^n \cong S^n$, so that S^n has an induced structure of an $\mathcal{M}^n_\infty$-algebra in $(Top, \sqcup)$.

Let X be a topological space. We construct an $\mathcal{M}^n_\infty$-algebra structure on $\Omega^n X = [S^n, X]$ by the composite

$$[S^n, X]^r \cong \left[\bigsqcup_r S^n, X\right] \xrightarrow{[(\mu_i^r)^*, X]} [S^n, X]\,,$$

where $(\mu_i^r)^*$ is an extension of μ_i^r, such that:

$$\begin{tikzcd}
\bigsqcup_r S^n \arrow[r, "(\gamma_1,\ldots,\gamma_r)"] \arrow[d, "\mu_i^r"'] & X \\
S^n \arrow[ur, dashed, "(\mu_i^r)^*(\gamma_1,\ldots,\gamma_r)"'] &
\end{tikzcd}$$

Hence we have

$$\Omega^n : \textsc{Top} \to \mathcal{M}^n_\infty\text{-}\textsc{Alg}_{(\textsc{Top},\times)}.$$

APPENDIX A

Hierarchical categorization of mathematical structures

Since Zermelo Frankel's formalization of mathematics through set theory, mathematical knowledge has progressively diverged from its set-theoretical foundations (see for instance the introduction of [**19**]). Apart from the increasingly recognized inadequacy of ZFC theory in formalizing problems arising from modern research - notably with regards to higher algebra - the paradoxes inherent in this theory yield strong restrictions on the nature of mathematical objects [**29**]. Notably, it seems that we still lack a well defined framework for the treatment of the so-called big sets, or proper classes, leading to sort of a confusion on the very definition of a category. In this thesis, we extensively work in a framework where the objects under consideration are not small. It is indeed necessary to work in the 2-category of all categories $\mathrm{C{\small AT}}$ for our purpose, as well as in the 2-category $\mathrm{C{\small AT}}^{\mathbb{N}}$ of sequences of categories, so that we can introduce the notion of a $\mathrm{C{\small AT}}$-operad. The constant operad $\overline{\mathrm{S{\small ET}}}$, given in each arity by the category of sets, is one of the fundamental exemple of such a $\mathrm{C{\small AT}}$-operad structure. The former aim of this appendix is to formalize the notion of a small object. For this purpose, we propose novel logical foundations that enable precise definitions of the objects used within this thesis.

The framework that we sketch meets the expectations raised by the scientific community in the last decades. Far from being exotic in practice, the language we present should be regarded instead as a natural formalization of what has been repeatedly observed in modern research, notably in attempts to lay the foundation of mathematics from a categorical perspective [**22, 17, 1**]. It should nonetheless be noted that the underlying logic of what we aim to propose as a foundation significantly differs from the view widely adopted since the advent of modern logic. In what follows, we aim to outline the main issues identified by modern research as well as the solutions we propose. We will also highlight the specificities related to the method we employ when establishing the logical framework of our approach.

Although the main ideas are present in this appendix, the treatment of the few details missing for a rigorous presentation will be the subject of a dedicated article, which is currently in preparation.

Logical foundations of mathematics. The foundation of mathematics is usually approached through formal systems. This method allows truths to be inferred from axioms by using a specified set of rules. For this purpose, one usually deals with primitive notions, the meaning of which is understood by means of the axioms that they satisfy. The underlying logic of formal systems used to prove statements about the theory always reduces to Boolean logic with implication, conjunction and disjunction[1]. Our framework is close to dependant type theory and homotopy type theory and aims to reconcile different perspectives. In type theory and homotopy type theory, there are multiple notions of equality (judgmental, definitional, equality type, equivalence of types) and implication (entailment, implication in the deductive system, and implication internal to the theory). The logical framework we define contains its own metalanguage and lacks undefined basic objects that satisfy axioms (or rules). Equality is explicitly defined within our framework. We also have a syntactical equality, which, similar to type theory, is dynamic. For instance, defining an element X of n-type $A : \mathbb{T}_n$ by $X := Y$ where $Y : A$ amounts to 'add'a literal equality $X = Y$ to the language $\mathbb{L} : \mathbb{T}_0$. This process is made formal by considering quotient of sets (see 6.0.34). While in homotopy type theory, n-types are defined as particular kinds of types, we proceed in the reverse direction. Indeed, we define an infinity type as the data of an n-type for each n equipped with additional structure enabling to see an infinity type as a limit of its underlying n-types. We make this construction precise in 8.

While the question of size is usually treated in the literature as a hierarchy of universes, so that the collection of all sets forms a class, we offer a fundamentally different approach in this appendix. In particular, we avoid the typical ambiguities due to such hierarchies, so that each object under consideration has a well defined unique type.

In particular, in 6.0.1, we *define* a set as the data of its elements, together with the data of a truth value between each couple of elements, satisfying a naturality condition, from which we will deduce transitivity. With this paradigm, we say that two elements of a set are equal if the corresponding truth value is *true*. While reflexivity comes as a necessary consequence of the structure of the category of sets, the symmetry of the equality relation thus defined requires to be stated as an extra condition. The fundamental construction of the category of sets will be treated in detail in subsection 6.

We propose to define truth, truth values, sets, categories and so on, as a mutual data through a linear hierarchy of containers $\mathbb{T}_n : \mathbb{T}_{n+1}$, which can be interpreted either as n-dimensional spaces, n-types, n-categories or n-toposes. Recall that a topos has an underlying cartesian closed category $\mathcal{C}$, equipped with a subobject classifier Ω with a distinguished element $1 : * \to \Omega$, where $*$ is the terminal object of the topos. The subobject classifier is usually interpreted as the object of truth values internal to the topos, with truth value 'true'given by the distinguished element 1. From our point of view, this data corresponds to the first 3 levels of a hierarchy starting with $1 : \Omega$, $\Omega : \mathcal{C}$, $\mathcal{C} : \textsc{Cat}$, and continuing with $\textsc{Cat}_n : \textsc{Cat}_{n+1}$.

We address the usual coherence issues regarding the higher morphisms of higher categories through naturality conditions. The interest is that the framework we propose is particularly suited for inductive reasoning.

Chapter I may be seen as a partial application of the theory developed in this appendix at levels below 2. We precisely start from the usual hierarchy of categories $\textsc{Cat}_n : \textsc{Cat}_{n+1}$ and proceed to internalize categorical constructions, leading

[1]Suppose for instance that we are given judgments $J_1, \ldots, J_r, J$ about a theory. One usually emploies the notation $\frac{J_1,\ldots,J_r}{J}$ for the implication $\prod_{i=1}^r J_i \Rightarrow J$.

to another hierarchy starting from[2] $S : \Lambda$, with $\Lambda : \mathrm{CAT}_2$, and going back to $\mathrm{CAT}_2 : \mathrm{CAT}_3$, and so on.

1. Introduction

1.1. The general shape of an $\mathbb{N}$-language. The language that we define is shaped on the set of natural numbers $\mathbb{N}$, which encodes the linear form of the process of comprehension and iteration. We define an $\mathbb{N}$-language as the data of a family of types $\mathbb{T}_n$ for each n equipped with an entangled structure which makes mathematical reasoning possible in this language. In particular, each type $\mathbb{T}_n$ must be given the structure of an element of type $\mathbb{T}_{n+1}$, so that $\mathbb{T}_n$ is both a container for the elements of type $\mathbb{T}_n$ and an element of type $\mathbb{T}_{n+1}$. We define the notion of a proposition internally to this system, as well as the more general notion of a predicate, which may be regarded as a functional proposition on a given domain. We will define the property of being of type $\mathbb{T}_n$ as a predicate $\_ : \mathbb{T}_n$ depending on a domain -usually the familiar alphabet- corresponding to the letters or words we use when writing. To any element X of the domain will thereby correspond a proposition $X : \mathbb{T}_n$ whose meaning may be interpreted as 'X is an element of type $\mathbb{T}_n$' , and which is therefore assigned with a truth value -usually, but not necessarily, true or false. In order to avoid classical paradoxes, it will be appropriate to remove the word corresponding to the literal expression of the predicate from its domain of definition. For instance, the domain of the predicate $\_ : \mathbb{T}_n$ will be required not to contain the element '$\mathbb{T}_n$'of the language, so that the expression $\mathbb{T}_n : \mathbb{T}_n$ always is meaningless. The former fixed expressions of the language consist of the elements $\mathbb{T}_n$ of type $\mathbb{T}_{n+1}$. Given that our language is dynamic, in the same way that the language of type theory is, me may want to fix, or equivalently, define, some elements $X : \mathbb{T}_n$. It is convenient to remove the literal expression of fixed elements from the domain of predicates, so that such elements can not be taken as variables involved in the definition of predicates. For instance, we will define the set of natural numbers $\mathbb{N}$ as an element of type $\mathbb{T}_0$ in 6.0.27, providing a proof of the proposition $\mathbb{N} : \mathbb{T}_0$. The literal expression $\mathbb{N} : \mathbb{L}$ will thereby be forbidden in the domain of any predicates, so that it will not be possible to define another element $\mathbb{N}$ of some type after '$\mathbb{N}$' has been assigned the value 'set of natural numbers'.

On n-dimensional spaces and n-types. Each element of type $\mathbb{T}_n$ should be regarded as an n-dimensional space. We proceed inductively as follows to formalize this idea. Suppose that X is an element of type $\mathbb{T}_n$, so that we are given a proof of the proposition $X : \mathbb{T}_n$. Such a proof consists, in particular, to the data of a predicate $\_ : X$ which gives the conditions for some x to be an element of type X. We may regard the elements $x : X$ as the points of X. Any pair of elements $x, y : X$, formally given by a proof of the proposition $(x : X) \times (y : Y)$ obtained by conjunction, yields an element $X(x, y)$ of type $\mathbb{T}_{n-1}$, which we regard as the n-1-dimensional space of morphisms, or paths, from x to y in X. Let us give an example in the case where X is an element of type $\mathbb{T}_2$. Let $x, y : X$, we obtain a 1-dimensional space $X(x, y) : \mathbb{T}_1$. The $\mathbb{T}_1$-type structure on $X(x, y)$ yields a predicate $\_ : X(x, y)$. Let $f, g : X(x, y)$, we obtain a 0-dimensional space $X(x, y)(f, g) : \mathbb{T}_0$, on which we consider a point $\eta : X(x, y)(f, g)$. The way x, y, f, g and η arrange can

[2]By using the notation employed in Chapter 1.

be displayed as follows

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & y \end{tikzcd} \qquad\qquad f \xrightarrow{\eta} g \qquad\qquad \eta.$$

The diagram on the left hand side is written within the 2-dimensional space X. While the elements x, y of X correspond to points, the morphisms $f, g : X(x, y)$ correspond to path between them, and the 2-morphism η may be regarded as the 2-dimensional disk which provides a path between the paths f and g. The diagram on the middle is written in the 1-dimensional space $X(x, y)$ of morphisms between x and y in X. Here, the morphisms f and g are regarded as elements, or points of the space $X(x, y)$, and the 2-morphism η hence corresponds to a morphism, or path, between f and g. The diagram on the right hand side is written on the 0-dimensional space $X(x, y)(f, g)$, on which we just have the point η.

In this regard, any object that we consider within the system $\mathbb{T}$ corresponds to an element of some space, itself corresponding to an element of $\mathbb{T}_n$ for some n. From now on, we use the diagram notation just introduced. In particular, for any elements x, y of an element X of type $\mathbb{T}_n$, $f : x \to y$ equivalently means $f : X(x, y)$.

1.2. Alpha. In order to enable reasoning entirely internal to such a system, notably for the definition of the system itself, it is necessary to have an infinite graduation in both the increasing and the decreasing direction, so that the sequence of types $\mathbb{T}_n$ neither starts nor ends. It is however desirable to have this sequence constant until a certain level, so that we can have a meaningful internal notion of truth values, as well as a meaningful notion of element and unicity. In this way, we will see that elements of a given type are indistinguishable below level -2 . Note that, due to the $\mathbb{T}_{n+1}$-type structure on each $\mathbb{T}_n$, the minimal language could not be 'empty', but would rather be 'constant'.

We will fix the hierarchic system of types $\mathbb{T}$ constant until $n = -2$, so that $\mathbb{T}_{-1}$ is the first level where there is an element which differs from the one necessarily present, given by the type at the immediately lower level $\mathbb{T}_{-2}$. We say that the system splits at level -1. We will write $\mathbb{B}_{\mathbb{T}} := \mathbb{T}_{\text{-}1}$ for the first non constant type, and will call truth values the elements of type $\mathbb{B}_{\mathbb{T}}$. We will also write $\top = \mathbb{T}_{\text{-}2}$ and $\alpha = \mathbb{T}_{\text{-}3}$, so that α is the unique element of type $\top$. By construction, $\top$ is an element of type $\mathbb{B}_{\mathbb{T}}$, which we call true, and there is, at least, one other element of type $\mathbb{B}_{\mathbb{T}}$. By construction again, $\top$ is the unique element of type $\mathbb{B}_{\mathbb{T}}$ which has an element, or proof. Note that it is necessary for the system $\mathbb{T}$ to split so that the element ' true ' as defined acquires meaning as a truth value. Conversely, the concept of ' truth ' necessarily exists as soon as the system splits.

Necessarily, the element $\alpha : \top$ has a unique element, which itself has a unique element, and so on. For each n, we will define an element $*_n : \mathbb{T}_{n+1}$ from the primary element α, so that $\mathbb{T}_n \cong *_n$ at levels below -2. The element $*_n$ should be regarded as a reflection of α at level n, and is closely related to the concept of element. Notably, the data of a proof of the proposition $X : \mathbb{T}_n$ corresponds to the data of a morphism $X : *_n \to \mathbb{T}_n$ between the elements $*_n, \mathbb{T}_n$ of type $\mathbb{T}_{n+1}$, and hence, provides a way to shift dimension. In parallel, if $X : \mathbb{T}_n$, the data of an element x of X, that is, the data of a proof of the proposition $x : X$, will be equivalent to the data of a morphism $x : *_{n-1} \to X$ in $\mathbb{T}_n$.

1.3. Omega. In topology and homotopy theory, the objects we mainly deal with have elements, paths between each pair of elements, paths between pairs of paths and so on. Consequently, the structure of these objects can not be fully encapsulated by the structure of an element of some type $\mathbb{T}_n$ for some finite n. In general, we can nonetheless obtain sufficient information on a topological space from its sequence of homotopy groups. In a similar fashion, we will define a predicate $\_ : \mathbb{T}_\omega$ by using the hierarchic system of types $\mathbb{T}$, so that any $X : \mathbb{T}_\omega$ will in particular be equipped with an element $X_n : \mathbb{T}_n$ for each n. Notably, the sequence of elements $*_n : \mathbb{T}_{n+1}$ yields an element $*_\omega : \mathbb{T}_\omega$.

We extend the notion on morphisms and elements previously defined on $\mathbb{T}_n$ to $\mathbb{T}_\omega$. For this purpose, we will first associate a predicate $\_ : X$ to each $X : \mathbb{T}_\omega$ and call x an element of X as soon as x satisfies $x : X$. Given $x, y : X$, we will define an element $X(x, y) : \mathbb{T}_\omega$, whose elements will be called ω-morphisms in X from x to y. Then, we show that each pair of elements $X, Y : \mathbb{T}_\omega$ yield an element $\mathbb{T}_\omega(X, Y) : \mathbb{T}_\omega$. We show that the predicates $\_ : X$ and $\_ : \mathbb{T}_\omega(*_\omega, X)$ are equivalent, and this equivalence of predicates extends to an equivalence $X \cong \mathbb{T}_\omega(*_\omega, X)$ in $\mathbb{T}_\omega$. The data of an element of X is therefore equivalent to the data of an ω-morphism $*_\omega \to X$, and for each $x, y : X$, we obtain an equivalence $X(x, y) \cong \mathbb{T}_\omega(*_\omega, X)(x, y)$ in $\mathbb{T}_\omega$.

The hierarchy provided by the system $\mathbb{T}$ ensures that any mathematical construction involving objects, morphisms and higher morphisms is made internal to some other object. Thanks to this perspective, the framework that is used is always made accurate, and any diagram acquires a precise meaning. In contrast, we may encounter a difficulty with $\mathbb{T}_\omega$, which is itself not contained, and hence cannot be seen as a proper object. Indeed, while $\mathbb{T}_\omega$ does have elements and morphisms between each pair of elements X, Y, respectively given by the predicate $\_ : \mathbb{T}_\omega$, and by the element $\mathbb{T}_\omega(X, Y) : \mathbb{T}_\omega$, there is no object $\mathbb{T}_\omega$ that encapsulates this entire structure.

We aim to address this issue by proceeding as follows. We let $\mathbb{T}_{\omega-1} : \mathbb{T}_\omega$ be given in level n by the element $\mathbb{T}_{n-1}$ of $\mathbb{T}_n$. We obtain an equivalence between the predicates $\_ : \mathbb{T}_{\omega-1}$ and $\_ : \mathbb{T}_\omega$. We also obtain an equivalence $\mathbb{T}_{\omega-1}(X, Y) \cong \mathbb{T}_\omega(X, Y)$ in $\mathbb{T}_\omega$ for each $X, Y : \mathbb{T}_\omega$. We therefore obtain a container $\mathbb{T}_{\omega-1}$, which we may just write $\mathbb{T}_\omega$, and which we can regard as contained in itself by shifting levels. We then provide each $\mathbb{T}_n$ with the structure of an element of type $\mathbb{T}_\omega$, so that all the constructions previously made levelwise can be gathered in $\mathbb{T}_\omega$.

1.4. Hierarchic type systems. This subsection is dedicated to the definition of a hierarchic type system $\mathbb{T}$.

DEFINITION 1.4.1. We use the notations given in 1.4.2 and the definitions of 1.4.3. A hierarchic type system $\mathbb{T}$ is the data of

- A language[3] $\mathbb{L} : \mathbb{T}_0$
- A formal[4] definition

[3]That is, an element $\mathbb{L}$ of type $\mathbb{T}_0$, or equivalently, a proof of the proposition $(\mathbb{L} : \mathbb{T}_0)$, which by definition corresponds to the data of an element $\alpha_{\mathbb{L}} : (\mathbb{L} : \mathbb{T}_0)$. Hence, in the basic case, $\mathbb{L}$ is a set of symbols in which we write.

[4]The symbol n which we use in this definition does not have a meaning a priori and is therefore exchangeable with any other symbol. Notably, the data of such an element $def_n^{\mathbb{T}}$ yields an element $def_{n+1}^{\mathbb{T}}$. For instance, while $def_n^{\mathbb{T}}$ consists, in particular, in the data of a proof of

$$\begin{aligned} def_n^{\mathbb{T}} : (\ & (\,(X : \mathbb{T}_n) : \mathbb{B}\,), (\,(X : \mathbb{T}_n) \Rightarrow (\,(\_ : X) : \mathrm{P}_{\mathrm{RED}_{\mathbb{L}}}\,)\,), \\ & (\,(X : \mathbb{T}_n) \Rightarrow (\,X(\_,\_) : \mathbb{T}_n(X^{op} \times X, \mathbb{T}_{n-1})\,)\,), \\ & (\mathbb{T}_n : \mathbb{T}_{n+1})\ \ ) \end{aligned}$$

together with an adjunction in $\mathbb{T}_{n+1}$

$$\Pi_{n-1} : \mathbb{T}_n \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{I}_{n+1}\mathbb{T}_{n-1} : \mathrm{I}_n,$$

such that [5]

- Each $\mathbb{T}_n$ is internally complete and cocomplete in $\mathbb{T}_{n+1}$.
- The internal hom

$$[\_,\_]_n^{\mathbb{T}} := \mathbb{T}_n(\_,\_) : \mathbb{T}_n^{op} \times \mathbb{T}_n \to \mathbb{T}_n$$

  provided by the proof of the proposition $(\mathbb{T}_n : \mathbb{T}_{n+1})$, together with the monoidal structure $(\mathbb{T}_n, \times) : \mathrm{M{\small ON}}(\mathbb{T}_{n+1}, \times)$ obtained from the cartesian product in $\mathbb{T}_n$, give to $(\mathbb{T}_n, \times, [\_,\_]_n^{\mathbb{T}})$ the structure of a closed monoid in $(\mathbb{T}_{n+1}, \times, [\_,\_]_{n+1}^{\mathbb{T}})$.

NOTATION 1.4.2. Let $\mathbb{T}$ be a hierarchic type system and $X, Y : \mathbb{T}_n$. We write

- α for $\mathbb{T}_{-3}$,
- $\top$ for $\mathbb{T}_{-2}$,
- $\mathbb{B}$ for $\mathbb{T}_{-1}$,
- $(x, y : X)$ for $(x : X) \times (y : X)$,
- $f : X \to Y$ for $f : \mathbb{T}_n(X, Y)$.

If $n = -1$, we may also write

- $f : X \Rightarrow Y$ for $f : X \to Y$, and
- $X \Leftrightarrow Y := \mathbb{B}(\mathcal{X}, Y) \times \mathbb{B}(Y, X)$.

We therefore have equalities in $\mathbb{L}$ between the denoted element and its notation, so that for instance, the (litteral) expressions $f : X \to Y$ and $f : \mathbb{T}_n(X, Y)$ are equal in $\mathbb{L}$.

DEFINITION 1.4.3. Let $\mathbb{T}$ be a hierarchic type system.

- For $X, Y : \mathbb{T}_n$, the n-type of adjunctions $\mathbb{T}_n(X, Y)^{\leftrightarrows} : \mathbb{T}_n$ is given in 2.7.1.
- We say that an element $X : \mathbb{T}_n$ is internally complete in $\mathbb{T}_n$ if it has all limits in the sense of Definition 2.9.2.
- We say that an element $X : \mathbb{T}_n$ is internally cocomplete in $\mathbb{T}_n$ if it has all colimits in the sense of Definition 2.9.2.
- We say that an element $X : \mathbb{T}_n$ is internally complete and cocomplete in $\mathbb{T}_n$ if it is both internally complete in $\mathbb{T}_n$ and internally cocomplete in $\mathbb{T}_n$.
- The cartesian product in an internally complete object $X : \mathbb{T}_n$ is defined in **??** and yields a symmetric monoidal structure on X, which is made precise in **??**.
- The definition of a closed monoidal structure on an object $X : \mathbb{T}_n$ is given on **??**.
- For $X : \mathbb{T}_n$, the opposite element $X^{op} : \mathbb{T}_n$ is defined in 2.6.1

the proposition $(\mathbb{T}_n : \mathbb{T}_{n+1})$, we obtain a proof of the proposition $(\mathbb{T}_{n+1} : \mathbb{T}_{n+1+1})$ as well by substitution.

[5]If P is a property depending on $X : \mathbb{T}_n$, so that, for instance, P yields $(X : \mathbb{T}_n) \Rightarrow (PX : \mathbb{B})$, we say that X is such that PX if X comes equipped with a proof $\alpha_X : PX$. We make this notion precise in **??**.

EXAMPLE 1.4.4. Let us define a constant hierarchic type system $\mathbb{c}$ as follows. We use the usual alphabet as a language, so that $(x : \mathbb{L}) = \top$.

– We let $x : \mathbb{c}_n \Leftrightarrow x = \mathbb{c}_{n-1}$, where the equality is taken in $\mathbb{L}$,
– and $\mathbb{c}_n(\mathbb{c}_{n-1}, \mathbb{c}_{n-1}) := \mathbb{c}_{n-1}$.

Each $\mathbb{c}_n$ has therefore a unique element $\mathbb{c}_{n-1}$.

1.5. Formal definition of a hierarchic type system. Formally, a hierarchic type system consists in a sequence of formal symbols, or assertions, and whose meaning will be given by the definition of the system $\mathbb{T}$.We present some of these assertions and provide an explanation for each one.

The sequence

(a) $p_X : ((X : \mathbb{T}_n) : \mathbb{B})$
(b) $p_n : (\mathbb{T}_{n-1} : \mathbb{T}_n)$
(c) $p_x^X : \mathbb{B}(X : \mathbb{T}_n, (x : X) : \mathbb{B})$
(d) $p_{\mathcal{C}}^{op} : \mathbb{B}(\mathcal{C} : \mathbb{T}_n, \mathcal{C}^{op} : \mathbb{T}_n)$
(e) $p_{\mathcal{C},X}^{op} : \mathbb{B}(\mathcal{C} : \mathbb{T}_n, \mathbb{B}(X : \mathcal{C}, X : \mathcal{C}^{op}))$
(f) $p_{\mathcal{C},X}^{op'} : \mathbb{B}(\mathcal{C} : \mathbb{T}_n, \mathbb{B}(X : \mathcal{C}^{op}, X : \mathcal{C}))$
(g) $p_{\mathcal{C}(\_,\_)} : \mathbb{B}(\mathcal{C} : \mathbb{T}_n, \mathcal{C}(\_,\_) : \mathbb{T}_n(\mathcal{C}, \mathbb{T}_n(\mathcal{C}^{op}, \mathbb{T}_{n-1})))$
(h) $p_{\mathcal{C},\mathcal{D},X,F} : \mathbb{B}(\mathcal{C} : \mathbb{T}_n, \mathbb{B}(\mathcal{D} : \mathbb{T}_n^{op}, \mathbb{B}(X : \mathcal{C}, \mathbb{B}(F : \mathbb{T}_n(\mathcal{C}, \mathcal{D}), FX : \mathcal{D}))))$
(i) $p_{X\times Y} : \mathbb{B}(X, \mathbb{B}(Y : \mathbb{T}_n, X \times Y : \mathbb{T}_n))$
(j) $p_{X\times Y}^{z,X} : \mathbb{B}(X : \mathbb{T}_n, \mathbb{B}(Y : \mathbb{T}_n, \mathbb{B}(z : X \times Y, \pi_X z : X)))$
(k) $p_{X\times Y}^{z,Y} : \mathbb{B}(X : \mathbb{T}_n, \mathbb{B}(Y : \mathbb{T}_n, \mathbb{B}(z : X \times Y, \pi_Y z : Y)))$
(l) $p_{X\times Y}^{x,y} : \mathbb{B}(X : \mathbb{T}_n, \mathbb{B}(Y : \mathbb{T}_n, \mathbb{B}(x : X, \mathbb{B}(y : Y, (x, y) : X \times Y))))$

therefore corresponds to the following.

(a) For any symbol Y, the assertion $Y : \mathbb{T}_n$ should be understood as "Y is an element of type $\mathbb{T}_n$". In the case where $n = -1$, the assertion $\tau : \mathbb{B}$ should be understood as "τ is a truth value". Define a proposition as an assertion to which a truth value is assigned, so that $(X : \mathbb{T}_n) : \mathbb{B}$ becomes an assertion which states that $X : \mathbb{T}_n$ is a proposition. Necessarily, the assertion $((X : \mathbb{T}_n) : \mathbb{B})$ becomes a proposition. To see this, take $X : \mathbb{T}_n$ for X and $n = -1$. Let P be a proposition, and define a proof p of P as an element of type P, so that p is a proof of P means $p : P$. Say that P is true if it is equipped with a proof $p : P$. Then, our first axiom states that there is a proof p_X of the proposition asserting that the assertion $(X : \mathbb{T}_n)$ is a proposition. We will define this proposition in the next subsection.
(b) The second assertion requires a given proof of the proposition $\mathbb{T}_{n-1} : \mathbb{T}_n$, so that for each n, "$\mathbb{T}_{n-1}$ is an element of type $\mathbb{T}_n$" is true. In particular, the propositions $\top : \mathbb{B}$ and $\alpha : \top$ must be true. Hence, there is a truth value $\top$ which is equipped with a proof α that $\top$ is true if we see this truth value as a proposition. We refer to $\top$ as "true", and we will show that a proposition is true in the sense that there is a proof of this proposition, if and only if it is true in the sense that the truth value assigned to this proposition is given by $\top$.
(c) The meaning of this assertion depends on assertions 7 and 8, and must be understood as an implication, which states that when X is an element of type $\mathbb{T}_n$, then we have a proposition $x : X$. We will interpret $x : X$ as "x is an element of type X".

(d) The meaning of this assertions depends on assertions 4, 5, 7 and 8. By assertion 4 we have $\mathbb{B}^{op} : \mathbb{T}_0$. Given propositions $P : \mathbb{B}$ and $Q : \mathbb{B}$, by assertion 5 there is a proposition $Q : \mathbb{B}^{op}$, and by assertion 8, there will be a proposition $\mathbb{B}(P, Q)$, which we will interpret as "if the proposition P is true, then the proposition Q is true", or just "if P then Q". We write $P \Rightarrow Q$ for the proposition $\mathbb{B}(P, Q)$. This justifies the assertion that if $Q : \mathbb{B}$ then $Q : \mathbb{B}^{op}$ as well as $\mathbb{B} : \mathbb{T}_0 \Rightarrow \mathbb{B}^{op} : \mathbb{T}_0$.

(e) The forth assertion states that for any element $\mathcal{C}$ of type $\mathbb{T}_n$, we have an element $\mathcal{C}^{op}$ of type $\mathbb{T}_n$. It will correspond to the opposite category of a category in the case where $n = 1$, or the the opposite poset of a poset in the case where $n = 0$.

(f) This assertion states that the opposite element $\mathcal{C}^{op}$ of an element $\mathcal{C}$ of type $\mathbb{T}_n$ has X as an element whenever X is an element of type $\mathcal{C}$.

(g) Symmetrically, we need to be able to produce an object of type $\mathcal{C}$ for any object of type $\mathcal{C}^{op}$. In fact, after having defined equivalence of truth values, we will require $X : \mathcal{C} \Leftrightarrow X : \mathcal{C}^{op}$.

(h) This assertions states that whenever $\mathcal{C}$ is an element of type $\mathbb{T}_n$, it comes equipped with an element $\mathcal{C}(\_, \_)$ of type $\mathbb{T}_n(\mathcal{C}, \mathbb{T}_n(\mathcal{C}^{op}, \mathbb{T}_{n-1}))$, which will correspond to the Yoneda embedding and provide a type of morphisms between each pair of objects of $\mathcal{C}$. In the case where $\mathcal{C} = \mathbb{B}$, we will obtain a truth value of morphisms between each pair of truth values, which we will refer to implication.

(i) First observe that $\mathbb{T}_n(\mathcal{C}, \mathcal{D})$ is not define. We used a notation for the sake of clarity, which depends on the next assertion and is given as follows. For $\mathcal{C} : \mathbb{T}_n$, we have $\mathcal{C}(\_, \_) : \mathbb{T}_n(\mathcal{C}^{op}, \mathbb{T}_n(\mathcal{C}^{op}, \mathbb{T}_{n-1}))$. By axiom 7, if $X : \mathcal{C}$, we obtain $\mathcal{C}(\_, \_)X : \mathbb{T}_n(\mathcal{C}^{op}, \mathbb{T}_n)$. We let $\mathcal{C}(X, \_) := \mathcal{C}(\_, \_)X$. Now for $Y : \mathcal{C}^{op}$, we obtain $\mathcal{C}(X, \_)Y : \mathbb{T}_n$, and we write $\mathcal{C}(X, Y) := \mathcal{C}(X, \_)Y$. Note that this notation makes $\mathbb{B}(\tau_1, \tau_2)$ into a proposition when $\tau_1 : \mathbb{B}$ and $\tau_2 : \mathbb{B}$. Now, since $\mathbb{T}_n : \mathbb{T}_{n+1}$ is true by axiom 2, we obtain $\mathbb{T}_n(\_, \_) : \mathbb{T}_{n+1}(\mathbb{T}_n, \mathbb{T}_{n+1}(\mathbb{T}_n^{op}, \mathbb{T}_n))$. Hence, for $\mathcal{C} : \mathbb{T}_n$ and $\mathcal{D} : \mathbb{T}_n^{op}$, we obtain $\mathbb{T}_n(\mathcal{C}, \mathcal{D}) : \mathbb{T}_n$. It follows that $F : \mathbb{T}_n(\mathcal{C}, \mathcal{D})$ is a proposition by axiom 1.

(j) One of the most essential notions remaining to define is the conjunction of truth values. In fact, the conjunction of truth values not only is a particular instance of cartesian product, but also states the essence of its definition. For this reason, we will use the symbol $\times$ to denote the conjunction of truth values. It is also worth notifying that it is closely related to enumeration. For instance, we want to be able to consider several elements of type $\mathbb{T}_n$, so that $X : \mathbb{T}_n, Y : \mathbb{T}_n$ becomes a proposition asserting that $X : \mathbb{T}_n$ and $Y : \mathbb{T}_n$. Assertion 7 states that given an element X of type $\mathbb{T}_n$, then whenever Y is an element of type $\mathbb{T}_n$, there is a proposition $(X, Y : \mathbb{T}_n)$. Of course, this statement is purely syntactical, so that it makes sense to require this proposition to be true, which is the statement of the next axiom.

(k) Now that any propositions $X : \mathbb{T}_n$ and $Y : \mathbb{T}_n$ yield a proposition $X, Y : \mathbb{T}_n$, our eight assertion ensures that $X, Y : \mathbb{T}_n$ is judge as true whenever $X : \mathbb{T}_n$ and $Y : \mathbb{T}_n$ is. It will be convenient to obtain the converse statement, so that $X, Y : \mathbb{T}_n$ provide a proof of both $X : \mathbb{T}_n$ and $Y : \mathbb{T}_n$. For this purpose, we need to have access either to conjunction of truth values or to equivalence of them, which requires conjunction.

2. Terminology

2.1. The n-th dimensional point. We first give the full definition of the n-th point $*_n : \mathbb{T}_{n+1}$. We also provide a definition of $*_n$ in terms of generators and relations and show that those definitions agree. In both cases, we proceed by induction on n and let $*_n : \mathbb{T}_{n+1}$ be such that $*_n = \mathbb{T}_n$ for $n \leq -2$.

We use a particular notation for the first values $*_{-3}$ and $*_{-2}$ to highlight the particular meaning attached to those objects. Recall that we write $\mathbb{B} : \mathbb{T}_0$ for the element $\mathbb{T}_{-1} : \mathbb{T}_0$, which we call the element of truth values.

We use the symbol $\top : \mathbb{B}$ for the truth value $*_{-2}$, which we call true. We also use the symbol $\alpha : \top$ for the element $*_{-3} : \top$. We will see that α is the unique element of type true after having defined the notion of unicity.

Definition 2.1.1. We let $*_n : \mathbb{T}_{n+1}$ be such that

– $\beta : *_n \Leftrightarrow \top \Leftrightarrow *_{-2}$
– $\beta, \gamma : *_n \Rightarrow *_n(\beta, \gamma) = *_{n-1}$.

Definition 2.1.2. We let $*'_n : \mathbb{T}_{n+1}$ be such that $*'_{n-1} : *'_n$.

Proposition 2.1.3. *The data of an element of type $*_n$ is equivalent to the data of an element of type $*'_n$ for each n. Consequently, we obtain an equivalence between $*_n$ and $*'_n$ in $\mathbb{T}_{n+1}$.*

Proof. We proceed by induction on n. The definition of $*_n$ and $*'_n$ agree for $n \leq -2$. ☐

Proposition 2.1.4. *Let $X : \mathbb{T}_n$. The data of an element of type X is equivalent to the data of a morphism $*_n \to X$ in $\mathbb{T}_{n+1}$. More generally, there is an isomorphism $\mathbb{T}_{n+1}(*_n, X) \cong X$ in $\mathbb{T}_{n+1}$.*

Proof. This result is a straightforward consequence of the equivalent definition of $*_n$ in terms of its generators given in 2.1.2. We also have equivalences

$$\begin{aligned} \_ : *_n \to X &\Leftrightarrow - \_ : *_n \Rightarrow \_ : X, \\ &\quad\; - *_n(\_,\_) \to X(\_,\_) \\ &\Leftrightarrow - \top \Rightarrow \_ : X \\ &\quad\; - \overline{*}_{n-1} \to X(\_,\_) \end{aligned}$$

The data of an element of type $\mathbb{T}_{n+1}(*_n, X)$ is therefore equivalent to the data of an element $x : X$, together with the identity $\imath_x : X(x,x)$. The result follows from the isomorphisms

$$\mathbb{T}_{n+1}(*_n, X)(\_,\_) \xrightarrow{\cong} \int^{-:*_n} X(\_,\_) \xrightarrow{\cong} \int^{\top} X(\_,\_) \xrightarrow{\cong} X(\_,\_).$$

☐

Proposition 2.1.5. *For each $X : \mathbb{T}_{n+1}$, there is a unique morphism $X \to *_n$. Precisely, any $X : \mathbb{T}_{n+1}$ yields an isomorphism $\mathbb{T}_{n+1}(X, *_n) \cong *_n$ in $\mathbb{T}_{n+1}$.*

Definition 2.1.6. Let $\Lambda : \mathbb{T}_{n+1}$. The isomorphism $\mathbb{T}_{n+1}(\Lambda, *_n) \cong *_n$ yields a functor

$$\overline{(\_)} : \mathbb{T}_n \to \mathbb{T}_{n+1}(\Lambda^{op}, \mathbb{T}_n)$$

which is obtained by the composite

$$\mathbb{T}_n \xrightarrow{\cong} \mathbb{T}_n \times *_n \xrightarrow{\cong} \mathbb{T}_{n+1}(*_n, \mathbb{T}_n) \times \mathbb{T}_{n+1}(\Lambda^{op}, *_n) \to \mathbb{T}_{n+1}(\Lambda^{op}, \mathbb{T}_n),$$

where the last morphism is the composition of morphisms in $\mathbb{T}_{n+1}$. We say that the presheaf thus obtained

$$\overline{Z} : \Lambda^{op} \xrightarrow{*} *_n \xrightarrow{Z} \mathbb{T}_n$$

is the constant presheaf equals to Z. Note that $\overline{Z}$ effectively sends any element of type Λ to the element Z of type $\mathbb{T}_n$.

Terminal and initial objects.

Definition 2.1.7. Let $X : \mathbb{T}_n$ and $*_X : X$. We say that $*_X$ is terminal in X is there is an isomorphism $X(\_, *_X) \cong *_{n-1}$.

Remark 2.1.8. If an element $*_X : X$ is terminal in X, then by Yoneda's Lemma $*_X$ is unique among the elements of X satisfying this property.

Example 2.1.9. The element $*_n : \mathbb{T}_{n+1}$ satisfies $\mathbb{T}_{n+1}(\_, *_n) \cong *_n$ and is therefore terminal in $\mathbb{T}_{n+1}$.

2.2. Composition and units.

Definition 2.2.1. Let $X : \mathbb{T}_n$ and recall that $x : X$ equivalently corresponds to a morphism $x : *_{n-1} \to X$ in $\mathbb{T}_n$. We therefore obtain a morphism

$$1_x : *_{n-2} \to X(x, x)$$

in $\mathbb{T}_{n-1}$, which we call the unit of x.

Definition 2.2.2. Let $X : \mathbb{T}_n$ and $x, y, z : X$. We obtain the following morphisms from the $\mathbb{T}_n$-type structure on X:

- $X(x, \_)_{y,z} : X(y, z) \to \mathbb{T}_{n-1}(X(x, y), X(x, z))$,
- $X(\_, y)_{x,z} : X(x, y) \to \mathbb{T}_{n-1}(X(y, z), X(x, z))$.

We write $X(x, \_) : X(x, y) \to \mathbb{T}_{n-1}(X(y, z), X(x, z))$ for the morphism obtained from $X(x, \_)$ by using the cartesian closed structure of $\mathbb{T}_{n-1}$. By naturality of the structural morphism of X, the morphisms $X(x, \_)$ and $X(\_, y)$ are equivalent. We obtain equivalent morphisms

$$X(x, f), X(f, y) : X(y, z) \to X(x, z)$$

which are natural in $f : X(x, y)$. We write $gf : X(x, z)$ or $x \xrightarrow{f} y \xrightarrow{g} z$ for the image of $g : X(y, z)$ under those equivalent morphisms.

Proposition 2.2.3. *Let $X : \mathbb{T}_n$ and $x, y : X$. For any morphism $f : x \to y$ in X, we have $f1_x \cong f$ and $1_y f \cong f$ in $X(x, y)$.*

2.3. Properties.

Definition 2.3.1. A property P on an element $\Gamma : \mathbb{T}_n$ is a morphism $P : \Gamma \to \mathbb{T}_{n-1}$. We also say that P is a property regarding X. We say that an element $X : \Gamma$ satisfies P if there is some $p : PX$.

Let P_n be a property regarding morphisms in $\mathbb{T}_n$.

Definition 2.3.2. Let $\Gamma, \Lambda : \mathbb{T}_n$. We say that $F : \Gamma \to \Lambda$ satisfies P_{n-1} locally if for each pair of elements $X, Y : \Gamma$, the morphism $F(X,Y) : \Gamma(X,Y) \to \Lambda(FX, FY)$ induced by F in $\mathbb{T}_{n-1}$ satisfies P_{n-1}.

Let $\phi, \psi : \Gamma \to \mathbb{T}_{n-1}$. We say that $G : \mathbb{T}_n(\Gamma, \mathbb{T}_{n-1})(\phi, \psi)$ satisfies P_{n-1} if GX satisfies P_{n-1} for all $X : \Gamma$. With this terminology, $F : \Gamma \to \Lambda$ satisfies P_{n-1} locally if and only if $F(\_,\_) : \mathbb{T}_n(\Gamma^{op} \times \Gamma, \mathbb{T}_{n-1})(\Gamma(\_,\_), \Lambda(F\_, F\_))$ satisfies P_{n-1}.

Definition 2.3.3. Let $P : \Gamma \to \mathbb{T}_n$ be a property on $\Gamma : \mathbb{T}_{n+1}$ and let $X : \Gamma$. We let $\Gamma_P : \mathbb{T}_{n+1}$ be the element obtained by the directed pull back

$$\begin{tikzcd} \Gamma_P \arrow[r, dashed] \arrow[d, dashed, "\iota"'] & * \arrow[d, "*_{n-1}"] \arrow[dl, Rightarrow] \\ \Gamma \arrow[r, "P"'] & \mathbb{T}_n. \end{tikzcd}$$

The data of an element $(X, p_X) : \Gamma_P$ is therefore equivalent to the data of an element $X : \Gamma$ together with a proof $p_X : PX$ that X satisfies P. We say that Γ_P is the subtype of the elements of type Γ which satisfy P.

Definition 2.3.4. Let $P : \Gamma \to \mathbb{T}_n$ be a property on $\Gamma : \mathbb{T}_{n+1}$ and let $X : \Gamma_P$, so that X is an element of Γ which satisfies P. We say that X is unique such that P if X yields an isomorphism $\Gamma_P \cong *_n$.

Example 2.3.5. For any $X : \mathbb{T}_n$, there is a unique morphism $X \to *_{n-1}$.

Definition 2.3.6. Let $P : \Gamma \to \mathbb{T}_n$ be a property on $\Gamma : \mathbb{T}_{n+1}$. We say that there is no element satisfying P if $\Gamma_P \cong \emptyset_n$.

Example 2.3.7. There is no element satisfying the property $\mathbb{T}_n(\_, \emptyset) : \mathbb{T}_n^{op} \to \mathbb{T}_n$. In other worlds, $\mathbb{T}_n(X, \emptyset_{n-1}) \cong \emptyset_{n-1}$ for any $X : \mathbb{T}_n$.

2.4. Injectivity and surjectivity.

Definition 2.4.1. Let $p : E \to B$ be a morphism in $\mathbb{T}_m$. We say that it is 0-surjective if for each $b : * \to B$ there is some $e : * \to E$ and an isomorphism $pe \cong b$ in B. We say that p is $n+1$-surjective if for all $x, y : * \to B$, the morphism $p(x,y) : E(x,y) \to B(px, py)$ in $\mathbb{T}_{n-1}$ is n-surjective.

Remark 2.4.2. Any morphism in $\mathbb{T}_n$ is m-surjective for $m > n+1$.

2.5. Equivalences.

Definition 2.5.1. A morphism $F : \mathcal{C} \to \mathcal{D}$ in $\mathbb{T}_n$ is

- a 0-equivalence if it yields an equivalence of predicates $\_ : \mathcal{C} \Leftrightarrow \_ : \mathcal{D}$,
- an r-equivalence if
  - it induces an equivalence of predicates $\_ : \mathcal{C} \Leftrightarrow \_ : \mathcal{D}$
  - it is a local r-1 - equivalence.

Definition 2.5.2. We say that a morphism $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if it is an $n+1$-equivalence.

Example 2.5.3. A 0-equivalence of sets is a surjective function. An equivalence of sets $f : X \to Y$ is a surjection such that f induces an equivalence $X(x,y) \Leftrightarrow Y(fx, fy)$. Hence an equivalence of sets is an isomorphism of sets.

Example 2.5.4. An equivalence of categories $F : \mathcal{C} \to \mathcal{D}$ in the sense of Definition 2.5.2 is an equivalence of categories is the usual sense.

2.6. Oppositization.

Definition 2.6.1. Let $X : \mathbb{T}_n$. We let $X^{op} : \mathbb{T}_n$ be such that

– $\_ : X^{op} = \_ : X$
– $X^{op}(\_,\_) : \mathbb{T}_n\left((X^{op})^{op} \times X^{op}, \mathbb{T}_{n-1}\right)$ is obtained by the composite

$$(X^{op})^{op} \times X^{op} \xrightarrow{\cong} X \times X^{op} \xrightarrow{\tau} X^{op} \times X \xrightarrow{X(\_,\_)} \mathbb{T}_{n-1},$$

where τ is the morphism that exchanges the factors in the cartesian product, and where the isomorphism $(X^{op})^{op} \cong X$ is tautological on the predicates, and is given on the morphisms by using the involutive structure of τ.

Definition 2.6.2. More generally, we have an r-oppositization morphism in $\mathbb{T}_{n+1}$

$$\_^{op_r} : \mathbb{T}_n^{op_{r+1}} \to \mathbb{T}_n,$$

which we define inductively by the following.

– $\_ : \mathbb{T}_n^{op_{r+1}} \Leftrightarrow \_ : \mathbb{T}_n \Rightarrow \_^{op_r} : \mathbb{T}_n$, where $X : \mathbb{T}_n \Rightarrow X^{op_r} : \mathbb{T}_n$ is defined by
  – $\_ : X^{op_r} = \_ : X$
  – $X^{op_r}(\_,\_) : (X^{op_r})^{op} \times X^{op_r} \to \mathbb{T}_n$ is given by the morphism $X(\_,\_)^{op_{r-1}}$ obtained by the the image of $X(\_,\_)$ through the composite

$$\begin{aligned}
&\mathbb{T}_n^{op_{r+1}}(X^{op} \times X, \mathbb{T}_{n-1}) = \mathbb{T}_n(X^{op} \times X, \mathbb{T}_{n-1})^{op_r} \\
&\xrightarrow{_^{op_r}(X^{op} \times X, \mathbb{T}_{n-1}^{op_r})} \mathbb{T}_n((X^{op})^{op_r} \times X^{op_r}, \mathbb{T}_{n-1}^{op_r}) \\
&\xrightarrow{\mathbb{T}_n((X^{op})^{op_r} \times X, _^{op_{r-1}})} \mathbb{T}_n((X^{op})^{op_r} \times X^{op_r}, \mathbb{T}_{n-1}) \\
&\xrightarrow{\cong} \mathbb{T}_n((X^{op_r})^{op} \times X^{op_r}, \mathbb{T}_{n-1}),
\end{aligned}$$

The last isomorphism is induced by $(X^{op_r})^{op} \xrightarrow{\cong} (X^{op})^{op_r}$, which is identical both on the objects and on the morphisms. To see this, let $x, y : X$. On the one hand, we have

$$(X^{op_r})^{op}(x,y) = X^{op_r}(y,x) = X(y,x)^{op_{r-1}},$$

and on the other hand,

$$(X^{op})^{op_r}(x,y) = (X^{op}(x,y))^{op_{r-1}} = X(y,x)^{op_{r-1}}.$$

– We let $\_^{op_r}(\_,\_) : \mathbb{T}_n^{op_{r+1}}(\_,\_) \to \mathbb{T}_n(\_^{op_r}, \_^{op_r})$ be the morphism in $\mathbb{T}_{n+1}((\mathbb{T}_n^{op_r})^{op} \times \mathbb{T}_n^{op_r}, \mathbb{T}_n)$ given in $X, Y : \mathbb{T}_n$ by the morphism

$$\mathbb{T}_n^{op_{r+1}}(X,Y) = \mathbb{T}_n(X,Y)^{op_r} \to \mathbb{T}_n(X^{op_r}, Y^{op_r}),$$

defined as follows.

– For $F : \mathbb{T}_n(X,Y)$, we let $F^{op_r} : X^{op_r} \to Y^{op_r}$ be given by F on the objects, and on the morphism by $F^{op_r}(\_,\_)$, obtained by the composite

$$X^{op_r}(\_,\_) = X(\_,\_)^{op_{r-1}} \xrightarrow{F(\_,\_)^{op_{r-1}}} Y(F\_,F\_)^{op_{r-1}} = Y^{op_r}(F\_,F\_).$$

– Let $F, G : \mathbb{T}_n(X,Y)$, the isomorphisms

$$\begin{aligned}
\mathbb{T}_n(X^{op_r}, Y^{op_r})(F^{op_r}, G^{op_r}) &\xrightarrow{\cong} \int^{X^{op_r}} Y^{op_r}(F^{op_r}_, G^{op_r}_) \\
&\xrightarrow{\cong} \int^{X^{op_r}} Y(F_, G_)^{op_{r-1}} \\
&\xrightarrow{\cong} \left(\int^{X} Y(F_, G_)\right)^{op_{r-1}} \\
&\xrightarrow{\cong} \mathbb{T}_n(X,Y)(F,G)^{op_{r-1}} \\
&\xrightarrow{\cong} \mathbb{T}_n(X,Y)^{op_r}(F,G)
\end{aligned}$$

give $\_^{op_r}$ on the morphisms.

2.7. Adjunctions.

Definition 2.7.1. Let $X, Y : \mathbb{T}_n$ and $F : X \to Y$. We say that $G : Y \to X$ is right adjoint to F if it is equipped with an equivalence

$$\eta : X(F\_,\_) \cong Y(\_,G\_).$$

We also say that F is left adjoint of G, or that (F,G) form a pair of adjoint morphisms in $\mathbb{T}_n$. We write $(F,G,\eta) : \mathbb{T}_n(X,Y)^{\leftrightarrows}$, or just

$$\begin{tikzcd} F : X \arrow[r, bend left] \arrow[r, phantom, "\perp"] & Y : G \arrow[l, bend left] \end{tikzcd}.$$

Let $(F_1, G_1, \eta_1), (F_2, G_2, \eta_2) : \mathbb{T}_n(X,Y)^{\leftrightarrows}$. We let $\mathbb{T}_n(X,Y)^{\leftrightarrows}((F_1,G_1,\eta_1),(F_2,G_2,\eta_2)) : \mathbb{T}_{n-1}$ be defined as

$$\coprod_{(\mu_F,\mu_G)} \mathbb{T}_n(X^{op} \times Y, \mathbb{T}_{n-1})\left(\mathbb{T}_n(F_1\_,\_), \mathbb{T}_n(\_,G_2\_)\right)\left(\eta_2 \mathbb{T}_n(\mu_F,\_), \mathbb{T}_n(\_,\mu_G)\eta_1\right)^{\simeq},$$

where the corproduct is taken over $(\mu_F,\mu_G) : \mathbb{T}_n(X,Y)(F_2,F_1) \times \mathbb{T}_n(Y,X)(G_1,G_2)$. We therefore obtain an n-type $\mathbb{T}_n(X,Y)^{\leftrightarrows}$ of adjunctions in $\mathbb{T}_n$ from X to Y. The (n-1)-type of morphisms between two adjunctions (F_1,G_1,η_1) and (F_1,G_2,η_2) has for objects the pairs (μ_F,μ_G) of 2-morphisms $\mu_F : F_2 \Rightarrow F_1$, $\mu_G : G_1 \Rightarrow G_2$ in $\mathbb{T}_n$ equipped with an equivalence

$$\begin{tikzcd}
\mathbb{T}_n(F_1_,_) \arrow[r, "\eta_1", "\cong"'] \arrow[d, "\mathbb{T}_n(\mu_F,_)"'] \arrow[dr, phantom, "\cong"] & \mathbb{T}_n(_,G_1_) \arrow[d, "\mathbb{T}_n(_,\mu_G)"] \\
\mathbb{T}_n(F_2_,_) \arrow[r, "\eta_2", "\cong"'] & \mathbb{T}_n(_,G_2_).
\end{tikzcd}$$

in the n-type $\mathbb{T}_n(X^{op} \times Y, \mathbb{T}_{n-1})$.

Definition 2.7.2. Let $X : \mathbb{T}_n$. We let

$$\_ \times X(\_,\_) : \mathbb{T}_{n-1} \to \mathbb{T}_n(X^{op} \times X, \mathbb{T}_{n-1})$$

be the morphism in $\mathbb{T}_n$ obtained by the composite

$$\begin{aligned}\mathbb{T}_{n-1} \xrightarrow{\cong} \mathbb{T}_{n-1} \times *_n \xrightarrow{X(\_,\_)} & \mathbb{T}_{n-1} \times \mathbb{T}_n(X^{op} \times X, \mathbb{T}_{n-1}) \\ \xrightarrow{\cong} & \mathbb{T}_n(*_{n-1}, \mathbb{T}_{n-1}) \times \mathbb{T}_n(X^{op} \times X, \mathbb{T}_{n-1}) \\ \xrightarrow{\times} & \mathbb{T}_n(X^{op} \times X, \mathbb{T}_{n-1} \times \mathbb{T}_{n-1}) \\ \xrightarrow{\mathbb{T}_n(X^{op} \times X, \times)} & \mathbb{T}_n(X^{op} \times X, \mathbb{T}_{n-1}).\end{aligned}$$

DEFINITION 2.7.3. Let $X : \mathbb{T}_n$. We let

$$\mathbb{T}_{n-1}(X(\_,\_),\_) : \mathbb{T}_{n-1} \to \mathbb{T}_n(X \times X^{op}, \mathbb{T}_{n-1})$$

be the morphism in $\mathbb{T}_n$ obtained by the composite

$$\mathbb{T}_{n-1} \xrightarrow{\mathbb{T}_{n-1}(\_,\_)} \mathbb{T}_n(\mathbb{T}_{n-1}^{op}, \mathbb{T}_{n-1}) \xrightarrow{\mathbb{T}_{n-1}(X(\_,\_)^{op}, \mathbb{T}_{n-1})} \mathbb{T}_{n-1}(X \times X^{op}, \mathbb{T}_{n-1}).$$

DEFINITION 2.7.4. Let $X : \mathbb{T}_n$. We let the end morphism

$$\int^{\_:X} : \mathbb{T}_n(X^{op} \times X, \mathbb{T}_{n-1}) \to \mathbb{T}_{n-1}$$

be the right adjoint of $\_ \times X(\_,\_)$, so that we have the following adjunction in $\mathbb{T}_n$:

$$\_ \times X(\_,\_) : \mathbb{T}_{n-1} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbb{T}_n(X^{op} \times X, \mathbb{T}_{n-1}) : \textstyle\int^{\_:X}$$

OBSERVATION 2.7.5. *The definition of $X : \mathbb{T}_n$ involves a morphism $X(\_,\_) : X^{op} \times X \to \mathbb{T}_{n-1}$ in $\mathbb{T}_n$, whose definition itself involves $X(\_,\_)$. We let $X(\_,\_)$ be defined inductively as follows. By definition, the data of a morphism $X(\_,\_) : X^{op} \times X \to \mathbb{T}_{n-1}$ consists in:*

- $x, y : X \Rightarrow X(x, y) : \mathbb{T}_{n-1}$
- $X_{((\_,\_),(\_,\_))} : X^{op}(\_,\_) \times X(\_,\_) \to \mathbb{T}_{n-1}(X(\_,\_), X(\_,\_))$ *in* $\mathbb{T}_n\left((X^{op} \times X)^{op} \times X^{op} \times X, \mathbb{T}_{n-1}\right)$*, and hence,*

$$X_{((\_,\_),(\_,\_))} : \int^{x_1,x_2,y_1,y_2} \mathbb{T}_{n-1}\left(X(x_2,x_1) \times X(y_1,y_2), \mathbb{T}_{n-1}(X(x_1,y_1), X(x_2,y_2))\right).$$

We definition of the structural morphism $X(\_,\_)$ of any element $X : \mathbb{T}_n$, as well as the more general definition of naturality, follows inductively.

DEFINITION 2.7.6. Let $X : \mathbb{T}_n$. We let the coend morphism

$$\int_{\_:X} : \mathbb{T}_n(X \times X^{op}, \mathbb{T}_{n-1}) \to \mathbb{T}_{n-1}$$

be the left adjoint of $\mathbb{T}_n(X(\_,\_),\_)$, so that we have the following adjunction in $\mathbb{T}_n$:

$$\textstyle\int_{\_:X} : \mathbb{T}_n(X \times X^{op}, \mathbb{T}_{n-1}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbb{T}_{n-1} : \mathbb{T}_n(X(\_,\_),\_).\ .$$

2.8. Terminal object.

DEFINITION 2.8.1. Let $\Lambda : \mathbb{T}_n$. A terminal object in Λ is an object $*_\Lambda : \Lambda$ equipped with an equivalence

$$\Lambda(\_, *_\Lambda) \cong \overline{*_{n-2}}.$$

2.9. Limits and colimits.

DEFINITION 2.9.1. Let $X : \mathbb{T}_n$ and let $*_X : X \to *$ be the unique morphism. If $Y : \mathbb{T}_n$, we write $\varrho : Y \to \mathbb{T}_n(X, Y)$ for the morphism in $\mathbb{T}_n$ obtained by the composite

$$\varrho : Y \cong \mathbb{T}_n(*, Y) \xrightarrow{*_X \times 1} \mathbb{T}_n(X, *) \times \mathbb{T}_n(*, Y) \xrightarrow{\circ} \mathbb{T}_n(X, Y).$$

We refer to ϱ as the constant morphism.

DEFINITION 2.9.2. For any elements X and Y of type $\mathbb{T}_n$, we define, the limit morphism, respectively the colimit morphism, as the right adjoint, respectively as the left adjoint, to the constant morphism in $\mathbb{T}_n$:

$$\begin{tikzcd} \mathbb{T}_n(X, Y) \arrow[r, bend left, "\mathrm{colim}"] \arrow[r, bend right, "\mathrm{lim}"'] & Y. \arrow[l, "\underset{\perp}{\overset{\perp}{\varrho}}"'] \end{tikzcd}$$

3. The hierarchic system of categories

DEFINITION 3.0.1. We let $\_ : \mathbb{T}_n : \mathrm{PRED}_{\mathbb{L}}$ be defined on X by the proposition $X : \mathbb{T}_n$, whose truth value is given by

$$(X : \mathbb{T}_n) \Leftrightarrow ((\_ : X) : \mathrm{PRED}, X(\_, \_) : \mathbb{T}_n(X^{op} \times X, \mathbb{T}_{n-1}))\,.$$

Let $X, Y : \mathbb{T}_n$, we let $\mathbb{T}_n(X, Y) : \mathbb{T}_n$ be defined by

- $F : \mathbb{T}_n(X, Y) \Leftrightarrow \big(\_ : X \Rightarrow F\_ : Y, F(\_, \_) : \int^{x,y} \mathbb{T}_{n-1}(X(x, y), Y(Fx, Fy))\big)$
- $F, G : \mathbb{T}_n(X, Y) \Rightarrow \mathbb{T}_n(X, Y)(F, G) = \int^x Y(Fx, Gx)$.

The notation $\int^z H(z, z)$ is made precise in 2.7.4 for any $Y : \mathbb{T}_n$ and $H : Z^{op} \times Z \to \mathbb{T}_{n-1}$, and encodes the concept of naturality. We say that $z : Z \Rightarrow \eta_z : H(z, z)$ is natural if it yields an element $\eta : \int^z H(z, z)$. Conversely, we assume that any $\eta : \int^z H(z, z)$ yields natural maps $z : Z \Rightarrow \eta_z : H(z, z)$.

REMARK 3.0.2. This definition involves a $\mathbb{T}_n$-type structure on $\mathbb{T}_{n-1}$ for all n. We first assume that $\mathbb{T}_{n-1} : \mathbb{T}_n$ for all n and introduce the notion of naturality. We will then be able to prove $\mathbb{T}_{n-1} : \mathbb{T}_n$.

OBSERVATION 3.0.3. *Let $X : \mathbb{T}_n$ and let $H : X^{op} \times X \to \mathbb{T}_{n-1}$. By definition, H yields an element $H(x, y) : \mathbb{T}_{n-1}$ for each $x, y : X$, as well as an element of the end*

$$\int^{x_1, y_1, x_2, y_2} \mathbb{T}_{n-1}\left(X^{op}(x_1, x_2) \times X(y_1, y_2), \mathbb{T}_{n-1}(H(x_1, y_1), H(x_2, y_2))\right).$$

For $f_x : x_2 \to x_1$ and $f_y : y_1 \to y_2$, we write $H(f_x, f_y) : H(x_1, y_1) \to H(x_2, y_2)$ for the morphism obtained from H. In particular, for $x, y : X$ we obtain morphisms

- $H(1_x, \_) : X(x, y) \to \mathbb{T}_{n-1}(H(x, x), H(x, y))$*, and*
- $H(\_, 1_y) : X(x, y) \to \mathbb{T}_{n-1}(H(y, y), H(x, y))$.

Suppose that we are given some $\eta_x : H(x, x)$ for all $x : X$, we obtain the following morphisms for $x, y : X$:

- $H(1_x, \_)\eta_x : X(x, y) \to H(x, y)$,
- $H(\_, 1_y)\eta_y : X(x, y) \to H(x, y)$,

and hence

(2) $$H(x,y)\left(H(1_x,\_)\eta_x, H(\_,1_y)\eta_y\right) : X(x,y)^{op} \times X(x,y) \to \mathbb{T}_{n-2}.$$

Suppose that for all $x, y : X$ *and* $f : X(x,y)$*, we are given a 2-morphism* $\eta_f^{x,y}$ *in* $\mathbb{T}_{n-1}$*:*

$$\begin{tikzcd}
* \arrow[r, "\eta_x"] \arrow[d, "\eta_y"'] \arrow[dr, phantom, "\Swarrow \eta_f^{x,y}"] & H(x,x) \arrow[d, "H(1_x,f)"] \\
H(y,y) \arrow[r, "H(f,1_y)"'] & H(x,y)
\end{tikzcd}$$

so that $x, y : X \Rightarrow \Big(f : X(x,y) \Rightarrow \eta_f^{x,y} : H(x,y)\left(H(x,f)\eta_x, H(f,y)\eta_y\right)\Big)$*. Let* $X' := \mathbb{T}_n$ *be defined by* $X' = X^{op} \times X$ *and let* $K : X'^{op} \times X' \to \mathbb{T}_{n-1}$ *be defined by the composite*

$$(X^{op} \times X)^{op} \times (X^{op} \times X) \xrightarrow{X(\_,\_)^{op} \times H} \mathbb{T}_{n-1}^{op} \times \mathbb{T}_{n-1} \xrightarrow{\mathbb{T}_{n-1}(\_,\_)} \mathbb{T}_{n-1}.$$

Both $H(1_x,\_)\eta_x$ *and* $H(\_,1_y)\eta_y$ *yield an element of* $K(z,z)$ *for all* $z : X^{op} \times X$ *given by* $z = (x,y)$*. We write* $\mu_z : K(z,z)$ *for* $H(1_x,\_)\eta_x$ *and* $\mu^z : K(z,z)$ *for* $H(\_,1_y)\eta_y$*. Let* $f : Z(z_1,z_2)$ *be given by* $f_x : X^{op}(x_1,x_2)$ *and* $f_y : X(y_1,y_2)$ *with* $z_1 = (x_1,y_1)$ *and* $z_2 = (x_2,y_2)$ *in* $X^{op} \times X$*. We can observe the following.*

– *By definition, the data of a 2-morphism* β_{f_x,f_y} *in* $\mathbb{T}_{n-1}$ *such that*

$$\begin{tikzcd}
* \arrow[r, "H(1_{x_1},_)\eta_{x_1}"] \arrow[d, "H(1_{x_2},_)\eta_{x_2}"'] \arrow[dr, phantom, "\Swarrow \beta_{f_x,f_y}"] & \mathbb{T}_{n-1}\left(X(x_1,y_1), H(x_1,y_1)\right) \arrow[d, "\mathbb{T}_{n-1}(X(x_1,y_1),H(f_x,f_y))"] \\
\mathbb{T}_{n-1}\left(X(x_2,y_2), H(x_2,y_2)\right) \arrow[r, "\mathbb{T}_{n-1}(X(f_x,f_y),H(x_2,y_2))"'] & \mathbb{T}_{n-1}\left(X(x_1,y_1), H(x_2,y_2)\right)
\end{tikzcd}$$

corresponds to the data of an element

$$\beta_{f_x,f_y} : \int^{g:X(x_1,y_1)} H(x_2,y_2)\left(H(f_x,f_y)H(1_{x_1},g)\eta_{x_1}, H(1_{x_2},f_y g f_x)\eta_{x_2}\right)$$

and hence to morphisms $\beta^g_{f_x,f_y} : H(f_x,f_y)H(1_{x_1},g)\eta_x \to H(1_{x_2},f_y g f_x)\eta_{x_2}$ *in* $H(x_2,y_2)$ *which are natural in* $g : X(x_1,y_1)$*. For* $g : X(x_1,y_1)$*, we let* $\beta^g_{f_1,f_2}$ *be the morphism in* $H(x_2,y_2)$ *resulting from the composite of the diagram*

$$\begin{tikzcd}
* \arrow[r, "\eta_{x_1}"] \arrow[dd, "\eta_{x_2}"'] & H(x_1,x_1) \arrow[rr, "H(x_1,g)"] \arrow[d, "H(f_x,x_1)"] \arrow[dl, phantom, "\Swarrow \eta_{f_x}^{x_1,x_2}"] \arrow[dr, phantom, "\cong \Downarrow H(f_x,f_y g)"] & & H(x_1,y_1) \arrow[dd, "H(f_x,f_y)"] \\
& H(x_2,x_1) \arrow[r, "H(x_2,g)"] \arrow[d, phantom, "\Downarrow \cong"] & H(x_2,y_1) \arrow[dr, "H(x_2,f_y)"] & \\
H(x_2,x_2) \arrow[ur, "H(x_2,f_x)"] \arrow[rrr, "H(x_2,f_y g f_x)"'] & & & H(x_2,y_2).
\end{tikzcd}$$

The mapping $g \mapsto \beta^g_{f_x,f_y}$ *thus obtained inherits naturality in* $g : X(x_1,y_1)$ *from naturality of the mappings*

– $g \mapsto \Big(H(f_x,f_y)H(x_1,g) \overset{\cong}{\Rightarrow} H(x_2,f_y)H(x_2,g)H(f_x,x_1)\Big)$ *and*

– $g \mapsto \Big(H(x_2,f_y)H(x_2,g)H(x_2,f_x) \overset{\cong}{\Rightarrow} H(x_2,f_y g f_x)\Big)$.

We therefore obtain a 2*-morphism* $\beta_f : K(z_1, z_2)\,(K(z_1, f)\mu_{z_1}, K(f, z_2)\mu_{z_2})$ *for all* $f : Z(z_1, z_2)$*.*

– *The data of a morphism* γ_{f_x,f_y} *in* $\mathbb{T}_{n-1}$ *such that*

$$\begin{tikzcd}
* \arrow[r, "H(_,1_{y_1})\eta_{y_1}"] \arrow[d, "H(_,1_{y_2})\eta_{y_2}"'] \arrow[dr, phantom, "\overset{\gamma_{f_x,f_y}}{\swarrow}"] & \mathbb{T}_{n-1}\left(X(x_1,y_1), H(x_1,y_1)\right) \arrow[d, "\mathbb{T}_{n-1}(X(x_1,y_1),H(f_x,f_y))"] \\
\mathbb{T}_{n-1}\left(X(x_2,y_2), H(x_2,y_2)\right) \arrow[r, "\mathbb{T}_{n-1}(X(f_x,f_y),H(x_2,y_2))"'] & \mathbb{T}_{n-1}\left(X(x_1,y_1), H(x_2,y_2)\right)
\end{tikzcd}$$

corresponds to the data of an element

$$\gamma_{f_x,f_y} : \int^{g:X(x_1,y_1)} H(x_2,y_2)\left(H(f_x,f_y)H(g,y_1)\eta_{y_1}, H(f_y g f_x, y_2)\eta_{y_2}\right),$$

and hence, to morphisms $\gamma^g_{f_x,f_y} : H(f_x,f_y)H(g,y_1)\eta_{y_1} \to H(f_y g f_x, y_2)\eta_{y_2}$ *in* $H(x_2,y_2)$ *which are natural in* $g : X(x_1,x_2)$*. For* $g : X(x_1,y_1)$*, we let* $\gamma^g_{f_x,f_y}$ *be the morphism in* $H(x_2,y_2)$ *resulting from the composite of the diagram*

$$\begin{tikzcd}
* \arrow[r, "\eta_{y_1}"] \arrow[dd, "\eta_{y_2}"'] & H(y_1,y_1) \arrow[rr, "H(g,y_1)"] \arrow[d, "H(y_1,f_y)"] \arrow[dl, phantom, "\overset{\eta^{y_1,y_2}_{f_y}}{\swarrow}"] & & H(x_1,y_1) \arrow[dd, "H(f_x,f_y)"] \\
& H(y_1,y_2) \arrow[r, "H(g,y_2)"] & H(x_1,y_2) \arrow[dr, "H(f_x,y_2)"] \arrow[u, phantom, "\cong\Downarrow H(f_x g,f_y)"] & \\
H(y_2,y_2) \arrow[ur, "H(f_y,y_2)"] \arrow[rrr, "H(f_y g f_x,y_2)"'] & & \arrow[u, phantom, "\Downarrow\cong"] & H(x_2,y_2).
\end{tikzcd}$$

The mapping $g : X(x_1,y_1) \Rightarrow \gamma^g_{f_1,f_2}$ *thus obtained inherits naturality in* $g : X(x_1,y_1)$ *from naturality of the mappings*

– $g \mapsto \Big(H(f_x,f_y)H(g,y_1) \overset{\cong}{\Rrightarrow} H(f_x,y_2)H(g,y_2)H(y_1,f_y)\Big)$*, and*
– $g \mapsto \Big(H(f_x,y_2)H(g,y_2)H(f_y,y_2) \overset{\cong}{\Rrightarrow} H(f_x g f_y, y_2)\Big)$*.*

We therefore obtain a 2*-morphism* $\gamma_f : K(z_1,z_2)\,(K(z_1,f)\mu^{z_1}, K(f,z_2)\mu^{z_2})$ *for all* $f : K(z_1,z_2)$*.*

Definition 3.0.4. Let $X : \mathbb{T}_n$ and $H : X^{op} \times X \to \mathbb{T}_{n-1}$. We say that a mapping $(x : X \Rightarrow \eta_x : H(x,x))$ is natural in $x : X$, and we write $\eta : \int^{x:X} H(x,x)$, if the following holds.

– For $x, y : X$, we are given

$$\eta^{x,y} : \mathbb{T}_{n-1}(X(x,y), H(x,y))\,(H(\_,x)\eta_x, H(\_,y)\eta_y)\,,$$

and hence, a mapping $f : X(x,y) \Rightarrow \eta^{x,y}_f : H(x,y)\,(H(x,f)\eta_x, H(f,y)\eta_y)$ which is natural in $f : X(x,y)$, where $\eta^{x,y}_f$ is seen as an element of $L^{x,y}(f,f)$ with $L^{x,y} : X(x,y)^{op} \times X(x,y) \to \mathbb{T}_{n-2}$ is made explicit in (2).
– for

In particular, the end of any $H : Y^{op} \times Y \to \mathbb{T}_{n-1}$ will satisfy $\int^Y H \cong \mathbb{T}_n(Y^{op} \times Y, \mathbb{T}_{n-1})(Y(\_,\_), H)$.

EXPLANATION 1. Let $X : \mathbb{T}_n$. By definition, X is equipped with a morphism $X(\_,\_) : X^{op} \times X \to \mathbb{T}_{n-1}$ in $\mathbb{T}_n$. We sketch a characterisation of $X(\_,\_)$ by using the definition of $\_ : \mathbb{T}_n(X^{op} \times X, \mathbb{T}_{n-1})$.

- $x, y : X \Rightarrow X(x,y) : \mathbb{T}_{n-1}$
- $X(\_\_,\_\_) : \int^{X^{op}\times X} \mathbb{T}_{n-1}(X^{op}(\_,\_) \times X(\_,\_), \mathbb{T}_{n-1}(X(\_,\_), X(\_,\_)))$.

The expression of $X(\_\_,\_\_)$ as an end can not be taken as a definition in itself, as it presupposes that $X(\_,\_)$ already is a morphism. It should instead be understood as a necessary condition that X satisfy so that $X : \mathbb{T}_n$.

The idea is to give to $x, y : X \Rightarrow X(x,y) : \mathbb{T}_{n-1}$ the structure of a morphism inductively. For this purpose, $X(x,y)$ must satisfy naturality conditions encapsulated by $X(\_\_,\_\_)$. Recall that for $x, y : X$, the element $X(x,y)$ of $\mathbb{T}_{n-1}$ must be regarded as $\mathbb{T}_{n-1}$-type of morphisms in X from x to y. Providing $x, y \mapsto X(x,y)$ with the structure of a morphism $X^{op} \times X \to \mathbb{T}_{n-1}$ will then amount to define a composition for morphisms in X.

REMARK 3.0.5. Defining an element X of type $\mathbb{T}_n$ amonts to provide a proof that $X : \mathbb{T}_n$ is true. We may write 'let $X : \mathbb{T}_n$ be such that ... ' to define an element X of type $\mathbb{T}_n$, with the corresponding proof given instead of the dots.

4. The poset of truth values

We unravel the definition of CAT_n for $n = -1$ to obtain the definition of the poset $\mathbb{B} := \mathrm{CAT}_{-1} : \mathbb{T}_0$ of truth values. According to Definition 3.0.1, the poset $\mathbb{B}$ satisfies

- $\nu : \mathbb{B} \Leftrightarrow (((\beta : \nu) : \mathbb{B}) \times (\beta_1, \beta_2 : \nu \Rightarrow \nu(\beta_1, \beta_2) : \top))$,
- $\nu, \tau : \mathbb{B} \Rightarrow \mathbb{B}(\nu, \tau) : \mathbb{B}$,
- $\nu : \mathbb{B} \Rightarrow \mathbb{B}(\nu, \nu)$,
- $\nu, \tau, \mu : \mathbb{B} \Rightarrow \mathbb{B}(\nu, \tau) \times \mathbb{B}(\tau, \mu) \Rightarrow \mathbb{B}(\nu, \mu)$.

DEFINITION 4.0.1. Let $\top : \mathbb{B}$ be such that $(\beta : \top) \Leftrightarrow \top$. Necessarily, we have $(\beta_1, \beta_2 : \top \Rightarrow \top(\beta_1, \beta_2) = \alpha)$.

REMARK 4.0.2. Let $\nu : \mathbb{B}$. Recall that α is the unique element of type $\top$ in the sense that $\beta : \top \Rightarrow \top(\beta, \alpha) = \alpha$. For any $\beta_1, \beta_2 : \nu$, we must have $\nu(\beta_1, \beta_2) = \alpha$. Hence, the data of an element $\nu : \mathbb{B}$ is equivalent to the data of a proposition $(\beta : \nu) : \mathbb{B}$. In the same way, we have $\gamma : \mathbb{B}(\tau, \nu) \Leftrightarrow ((\beta : \tau) \Rightarrow (\gamma\beta : \nu))$. Also note that for $\nu : \mathbb{B}$, there is at most one element of type ν.

PROPOSITION 4.0.3. *Let* $\nu : \mathbb{B}$*, we have* $\mathbb{B}(\nu, \top) \Leftrightarrow \top$.

PROOF. Let $p : \nu \to \top$ in $\mathbb{B}$ be such that

- $\beta : \nu \Rightarrow p\beta := \alpha : \top$,
- $\beta, \gamma : \nu \Rightarrow p(\beta, \gamma) : \nu(\beta, \gamma) \to \top(\alpha, \alpha)$ is given by

$$\alpha : \top \Leftrightarrow \mathbb{B}(\top, \top) \Leftrightarrow (\nu(\beta, \gamma) \to \top(\alpha, \alpha)).$$

It follows that $p : \mathbb{B}(\nu, \top)$ is true, that is, $p : \top \Rightarrow \mathbb{B}(\nu, \top)$. In the same way, we obtain $\mathbb{B}(\nu, \top) \Rightarrow \top$, so that $\mathbb{B}(\nu, \top) \Leftrightarrow \top$. □

DEFINITION 4.0.4. We define $\bot : \mathbb{B}$ by $\beta : \bot \Leftrightarrow \bot$.

REMARK 4.0.5. The element $\bot : \mathbb{B}$ satisfies $\nu : \mathbb{B} \Rightarrow \mathbb{B}(\bot, \nu) = \top$. Indeed, suppose $\nu : \mathbb{B}$, the data of an element $\gamma : \mathbb{B}(\bot, \nu)$ is given by $(\beta : \bot \to \gamma\beta : \nu) \Leftrightarrow \mathbb{B}(\bot, \gamma\beta : \nu) \Rightarrow \top$.

5. The poset of propositions

We make the concept of a proposition formal and investigate the nature of the assertion 'Every proposition is either true or false'. Recall that we defined a proposition as an assertion to which a truth value is assigned. In fact, an assertion may depend on elements of a given domain, and is formulated in a certain language.

DEFINITION 5.0.1. A language $\mathbb{L}^{\mathfrak{a}}$ on an alphabet $\mathfrak{a} \in \text{SET}$ is a subset $\mathbb{L}^{\mathfrak{a}}$ of the underlying set of the free monoid generated by $\mathfrak{a}$.

DEFINITION 5.0.2. We let the set $\mathcal{A}^{\mathfrak{a}}_{\mathbb{L}}$ of assertions expressed on a language $\mathbb{L}^{\mathfrak{a}}$ be defined as the free associative monoid on the set $\mathbb{L}^{\mathfrak{a}}$.

DEFINITION 5.0.3. Let $\mathfrak{a} \in \text{SET}$ be an alphabet and $\mathbb{L}^{\mathfrak{a}}$ be a language on $\mathfrak{a}$. We define the poset $\mathbb{L}^{\mathfrak{a}}\text{PROP}$ of propositions on the language $\mathbb{L}^{\mathfrak{a}}$ as follows. We set

- $P : \mathbb{L}^{\mathfrak{a}}\text{PROP} \Leftrightarrow P : \mathcal{A}_{\mathbb{L}} \times \mathbb{B}$,
- $P, Q : \mathbb{L}^{\mathfrak{a}}\text{PROP} \Rightarrow \mathbb{L}^{\mathfrak{a}}\text{PROP}(P, Q) = \mathbb{B}((P), (Q))$.

Consequently, the data of a proposition consists in the data of an assertion expressed in a certain language, and a truth value. We write '$\_$' $: \mathbb{L}^{\mathfrak{a}}\text{PROP} \to \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}}$ and $(\_) : \mathbb{L}^{\mathfrak{a}}\text{PROP} \to \mathbb{B}$ for the induced projections. For any proposition P, we say that 'P' is the assertion of P and that (P) is its evaluation. We may define a proposition P by its value $(P) : \mathbb{B}$. In this case, we let the corresponding assertion be given by 'P'.

DEFINITION 5.0.4. Let P and Q be propositions. Write $(P \Rightarrow Q)$ for the truth value $\mathbb{L}^{\mathfrak{a}}\text{PROP}(P, Q)$. We obtain a proposition $P \Rightarrow Q$, whose truth value satisfies

$$(P \Rightarrow Q) \Leftrightarrow ((P) \Rightarrow (Q))\,.$$

The implication thus defined yields an internal hom on the poset of propositions

$$\Rightarrow : \mathbb{L}^{\mathfrak{a}}\text{PROP}^{op} \times \mathbb{L}^{\mathfrak{a}}\text{PROP} \to \mathbb{L}^{\mathfrak{a}}\text{PROP}.$$

DEFINITION 5.0.5. The set of propositions inherits conjunction and disjunction from the set of truth values $\mathbb{B}$. Let $I : \text{SET}$ and $P_\bullet : I \to \mathbb{L}^{\mathfrak{a}}\text{PROP}$. Let $\coprod_{i:I} P_i : \mathbb{B}$ for the proposition such that

$$\left(\alpha : \coprod_{i:I} P_i\right) \Leftrightarrow (i : I, p : P_i).$$

The proposition $\coprod_{i:I} P_i$, together with the natural projections, represents the colimit of $P : I \to \mathbb{L}^{\mathfrak{a}}\text{PROP}$.

In the same way, let $\prod_{i:I} P_i : \mathbb{B}$ be the proposition such that

$$\left(\alpha : \prod_{i:I} P_i\right) \Leftrightarrow (i : I \Rightarrow p_i : P_i).$$

The proposition $\prod_{i:I} P_i$, together with the natural projections, represents the limit of $P : I \to \mathbb{L}^{\mathfrak{a}}\text{PROP}$.

PROPOSITION 5.0.6. *We obtain a product from the conjunction of propositions*

$$\times : \mathbb{L}^{\mathfrak{a}}\text{PROP} \times \mathbb{L}^{\mathfrak{a}}\text{PROP} \to \text{PROP}$$

which gives to the poset $\mathbb{L}^{\mathfrak{a}}\text{PROP} \to \text{PROP}$ *a cartesian closed structure, so that for any propositions* P, Q *and* R*, we have*

$$(P \times Q) \Rightarrow R \;\Leftrightarrow\; P \Rightarrow (Q \Rightarrow R)$$

and

$$P \Rightarrow (Q \times R) \;\Leftrightarrow\; (P \Rightarrow Q) \times (P \Rightarrow R).$$

PROPOSITION 5.0.7. *We obtain a product from the disjunction of propositions*

$$\coprod : \mathbb{L}^{\mathfrak{a}}\text{PROP} \times \mathbb{L}^{\mathfrak{a}}\text{PROP} \to \text{PROP}$$

which gives to the poset $\mathbb{L}^{\mathfrak{a}}\text{PROP} \to \text{PROP}$ *a bicartesian closed structure, so that for any propositions* P, Q *and* R*, we have*

$$\left(P \coprod Q\right) \Rightarrow R \;\Leftrightarrow\; (P \Rightarrow R) \coprod (Q \Rightarrow R)$$

and

$$P \Rightarrow \left(Q \coprod R\right) \;\Leftrightarrow\; (P \Rightarrow Q) \Rightarrow R.$$

DEFINITION 5.0.8. Define the category of propositional domains $\mathbb{L}^{\mathfrak{a}}\text{DOM}$ on the language $\mathbb{L}^{\mathfrak{a}}$ as follows.

- $\mathbf{D} : \mathbb{L}^{\mathfrak{a}}\text{DOM} \Leftrightarrow (\mathcal{E}_{\mathbf{D}} : \text{SET}, I_{\mathbf{D}} : \mathcal{E}_{\mathbf{D}} \to \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}})$
- $\mathbf{D}_1, \mathbf{D}_2 : \mathbb{L}^{\mathfrak{a}}\text{DOM} \Rightarrow \mathbb{L}^{\mathfrak{a}}\text{DOM}(\mathbf{D}_1, \mathbf{D}_2) : \text{SET}$, where

$$\begin{aligned}
\text{- } f : \mathbb{L}^{\mathfrak{a}}\text{DOM}(\mathbf{D}_1, \mathbf{D}_2) \quad &\Leftrightarrow (\mathcal{E}_f : \text{SET}(\mathcal{E}_{\mathbf{D}_1}, \mathcal{E}_{\mathbf{D}_2}),\\
&\qquad I_f : \text{SET}(\mathcal{E}_{\mathbf{D}_1}, \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}})(I_{\mathbf{D}_2}\mathcal{E}_f, I_{\mathbf{D}_1})),\\
\text{- } f, g : \mathbb{L}^{\mathfrak{a}}\text{DOM}(\mathbf{D}_1, \mathbf{D}_2) &\Rightarrow \mathbb{L}^{\mathfrak{a}}\text{DOM}(\mathbf{D}_1, \mathbf{D}_2)(f, g)\\
&= \text{SET}(\mathcal{E}_{\mathbf{D}_1}, \mathcal{E}_{\mathbf{D}_2})(\mathcal{E}_f, \mathcal{E}_g).
\end{aligned}$$

The function I may be seen as an interpretation of the elements of the domain in terms of words or sentences. We may refer to $\mathcal{E}_{\mathbf{D}}$ as the set of subjects of $\mathbf{D}$.

EXAMPLE 5.0.9. The category of propositional domains on the language $\mathbb{L}^{\mathfrak{a}}$ has a terminal object which is given by the set $\mathcal{A}^{\mathfrak{a}}_{\mathbb{L}}$ together with the identical interpretation $\imath : \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}} \to \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}}$.

EXAMPLE 5.0.10. The projection '$\_$' $: \mathbb{L}^{\mathfrak{a}}\text{PROP} = \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}} \times \mathbb{B} \to \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}}$ gives to the set of propositions the structure of a propositional domain, such that the interpretation of a proposition is given by its underlying assertion. We write $\mathbb{L}^{\mathfrak{a}}\mathbf{D}_{\text{PROP}}$ for this propositional domain.

DEFINITION 5.0.11. Let $\mathbf{D} : \mathbb{L}^{\mathfrak{a}}\text{DOM}$. A predicate P on the domain $\mathbf{D}$, expressed in the language $\mathbb{L}^{\mathfrak{a}}$, is the data of

- an expression '$\_\ P$' $: \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}}$
- an evaluation $(\_\ P) : \mathcal{E}_{\mathbf{D}} \to \mathbb{B}$.

We obtain a set $\mathbb{L}^{\mathfrak{a}}\text{PRED}_{\mathbf{D}}$ whose elements are predicates on the propositional domain $\mathbf{D}$ by setting $\mathbb{L}^{\mathfrak{a}}\text{PRED}_{\mathbf{D}} = \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}} \times \text{SET}(\mathcal{E}_{\mathbf{D}}, \mathbb{B})$. We write the canonical projections as follows:

- '$\_\ \_$' $: \mathbb{L}^{\mathfrak{a}}\text{PRED}_{\mathbf{D}} \to \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}}$,
- $(\_\ \_) : \mathbb{L}^{\mathfrak{a}}\text{PRED}_{\mathbf{D}} \to \text{SET}(\mathcal{E}_{\mathbf{D}}, \mathbb{B})$,

so that '$\_\ \_$' takes a predicate P to its expression '$\_\ P$', and $(\_\ \_)$ takes a predicate P to its evaluation function $(\_\ P)$. We may define a predicate by its evaluation function $(\_\ P) : \mathcal{E}_{\mathbf{D}} \to \mathbb{B}$, in which case we let the expression of this predicate be given by '$\_\ P$'.

Definition 5.0.12. Let $\mathbf{D} : \mathbb{L}^{\mathfrak{a}}\text{Dom}$. We let the evaluation function

$$\mathbb{L}^{\mathfrak{a}}\text{Pred}_{\mathbf{D}} \times \mathcal{E}_{\mathbf{D}} \to \mathbb{B}$$

be defined as the projection $(\_\ \_) : \mathbb{L}^{\alpha}\text{Pred}_{\mathbf{D}} \to \text{Set}(\mathcal{E}_{\mathbf{D}}, \mathbb{B})$ by using the cartesian closed structure of Set. For a predicate $\_\ P$ and a subject $x : \mathcal{E}_{\mathbf{D}}$, the evaluation provides an element $(x\, P)$ of $\mathbb{B}$.

Definition 5.0.13. Let $\mathbf{D} : \mathbb{L}^{\mathfrak{a}}\text{Dom}$. We let the expression function

$$\mathbb{L}^{\mathfrak{a}}\text{Pred}_{\mathbf{D}} \times \mathcal{E}_{\mathbf{D}} \to \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}}$$

be defined as the composite

$$\mathbb{L}^{\mathfrak{a}}\text{Pred}_{\mathbf{D}} \times \mathcal{E}_{\mathbf{D}} \xrightarrow{`\_\ \_\text{'} \times I_{\mathbf{D}}} \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}} \times \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}} \xrightarrow{\coprod} \mathcal{A}^{\mathfrak{a}}_{\mathbb{L}},$$

where the last morphism is given by the free monoidal structure on $\mathcal{A}^{\mathfrak{a}}_{\mathbb{L}}$, which we can regard as concatenation. The expression function thus defined yields an element '$x\, P$' of the language $\mathcal{A}^{\mathfrak{a}}_{\mathbb{L}}$ for each proposition P and each subject x of the domain of P.

Observation 5.0.14. *For any propositional domain $\mathbf{D}$ on $\mathcal{A}^{\mathfrak{a}}_{\mathbb{L}}$, the evaluation and expression functions yield a map*

$$\mathbb{L}^{\mathfrak{a}}\text{Pred}_{\mathbf{D}} \times \mathcal{E}_{\mathbf{D}} \to \mathbb{L}^{\mathfrak{a}}\text{Prop}.$$

Given a predicate P and an element of the domain x, we obtain a proposition $x\, P$.

Example 5.0.15. By definition, any element X of type $\mathbb{T}_n$ is equipped with a predicate $\_ : X$ on the terminal domain. For any symbol x, the proposition $x : X$ thus obtained is true if and only if x is an element of X.

Example 5.0.16. Let us define a predicate $\_$ IsTrue on the propositional domain $\mathbb{L}^{\mathfrak{a}}\mathbf{D}_{\text{Prop}}$ of propositions defined in Example 5.0.10. For this purpose, we let the evaluation map $(\_\ \text{IsTrue}) : \mathbb{L}^{\mathfrak{a}}\text{Prop} \to \mathbb{B}$ be given by the canonical projection, so that for any proposition P,

$$(P\ \text{IsTrue}) = (P),$$

so that the proposition P IsTrue is true if and only if P is true.

Conversely, we obtain a predicate $\_$ IsFalse whose evaluation map is given by the composite

$$(\_\ \text{IsFalse}) : \mathbb{L}^{\mathfrak{a}}\text{Prop} \xrightarrow{\cong} \mathbb{L}^{\mathfrak{a}}\text{Prop}^{op} \xrightarrow{(\_)^{op}} \mathbb{B}^{op} \xrightarrow{\mathbb{B}(\_,\perp)} \mathbb{B}.$$

so that for any proposition P, the proposition P IsFalse is true if and only if the proposition P is false.

Definition 5.0.17.

Proposition 5.0.18. *For any proposition P, Let P : Prop. The proposition*

$$\text{IsTrueOrIsFalse}\ P := \text{IsTrue}\ P \coprod \text{IsFalse}\ P$$

obtained by disjonction of the propositions IsTrue *and* IsFalse, *is true.*

PROOF. Since we identified the proposition P with its evaluation, we have $P : \mathbb{B}$. Hence by definition P is either given by $\top$, in which case $\textsc{IsTrue}\, P \coprod \textsc{IsFalse}\, P \Leftrightarrow \mathbb{B}(P, \perp) \coprod \mathbb{B}(\top, P) \Leftrightarrow \mathbb{B}(\top, \perp) \coprod \mathbb{B}(\top, \top) \Leftrightarrow \perp \coprod \top \Leftrightarrow \top$, or P is given by $\perp$, in which case $\textsc{IsTrue}\, P \coprod \textsc{IsFalse}\, P \Leftrightarrow \mathbb{B}(P, \perp) \coprod \mathbb{B}(\top, P) \Leftrightarrow \mathbb{B}(\perp, \perp) \coprod \mathbb{B}(\top, \perp) \Leftrightarrow \top \coprod \perp \Leftrightarrow \top$. □

COROLLARY 5.0.19. *The proposition' Every proposition is either true or false', formally defined by the product* $\prod_{P:\textsc{Prop}} P\ \textsc{IsTrueOrFalse}$*, is true.*

PROOF. Proposition 5.0.18 yields a proof of the proposition $P\ \textsc{IsTrueOrFalse}$ for each proposition P, and hence a proof of the proposition $\prod_{P:\textsc{Prop}} P\ \textsc{IsTrueOrFalse}$. □

6. The category of sets

In this subsection, we focus on the category of sets, defined as 0-th level of the hierarchic type of categories. In [**19**], Tom Leinster, following Lawvere's Elementary Theory of the Category of Sets (see [**16**]), states the axioms of the theory of sets. To be precise, recall that Lawvere's Set Theory involves undefined terms, or primitive concepts, which are supposed to satisfy some axioms. These primitive concepts are sets, functions, composition and identity. We unravel the definitions of the last subsection in the case where $n = 0$, use them to propose a definition of the terms to which the axioms apply, and show that these axioms hold when applied to the terms thus defined.

DEFINITION 6.0.1. We let the category of sets be the element $\textsc{Set} := \textsc{Cat}_0^{\simeq} : \textsc{Cat}_1$. According to Definition 3.0.1, the data of an element $X : \textsc{Set}$ is equivalent to the data of

- a predicate $\_ : X$,
- a morphism $X(\_, \_) : X^{op} \times X \to \mathbb{B}$,
- together with an isomorphism $X \to X^{op}$, which sends any $x : X$ to the same literal value $x : X^{op}$.

We call a set any element of $\textsc{Set}$. If X is a set and $x : X$, we say that x is an element of X. If X and Y are sets, we define a function from X to Y as an element of the set $\textsc{Set}(X, Y)$, which is defined as follows.

- The data of an element $f : \textsc{Set}(X, Y)$ is given by the data for each element $x : X$ of an element $fx : Y$ such that $fx = fy$ in Y whenever $x = y$ in X.[6]
- Given elements $f, g : \textsc{Set}(X, Y)$, the truth value $f = g$ is defined as

$$\textsc{Set}(X, Y)(f, g) = \int^{x:X} Y(fx, gx) \cong \prod_{x:X} Y(fx, gx).$$

  For any element f of $\textsc{Set}(X, Y)$, we write $f : X \to Y$. Note that by definition, two functions $f, g : X \to Y$ are equal if for each $x : X$ we have $fx = gx$.

NOTATION 6.0.2. Let $X : \textsc{Set}$ and $x, y : X$ be elements of X. We let $(x = y) : \mathbb{B}$ be the truth value defined as $(x = y) := X(x, y)$. We say that elements x, y pf the set X are equal if this truth value is *true*.

---

[6] In the sense of Notation 6.0.2.

Notably, the set $\mathbb{L}$ whose elements are the words of the language, is equipped with an equality which we call *literal* equality, and which we recall is dynamic.

EXPLANATION 2. The data of a set $X : \text{SET}$ is therefore equivalent to the data of

- a truth value $(x : X) : \mathbb{B}$ for any $x : \mathbb{L}$ such that $(x : X) \Leftrightarrow (y : X)$ whenever x and y are literally equal in $\mathbb{L}$,
- for each pair of elements $x, y : X$, a truth value $X(x, y) : \mathbb{B}$, such that
  - $1_x : \top \Rightarrow X(x, x)$, which means that $x = x$ in X, from which we deduce the reflexivity of equality in X ;
  - $x, y : X \Rightarrow (X(x, y) \Leftrightarrow X(y, x))$, so that equality of elements is symmetric ;
  - $x, y, z : X \Rightarrow (X(x, y) \times X(y, z) \Rightarrow X(x, z))$, so that equality is transitive.

DEFINITION 6.0.3. Let X, Y, Z be sets, $f : X \to Y$ and $g : Y \to Z$. We let the composite of f and g be the function $gf : X \to Z$ defined as

- $x : X \stackrel{f}{\Rightarrow} fx : Y \stackrel{g}{\Rightarrow} gfx : Z$
- $x, y : X \Rightarrow gf(x, y) : X(x, y) \Rightarrow Z(gfx, gfy)$ where gf(x,y) is given by the composite

$$gf(x, y) : X(x, y) \stackrel{f(x,y)}{\Rightarrow} Y(fx, gx) \stackrel{g(fx,fy)}{\Rightarrow} Z(gfx, gfy).$$

DEFINITION 6.0.4. Let X be a set. We let the identity $1_X : X \to X$ be defined as follows.

- $x : X \Rightarrow 1_X x := x : X$, which is always true since $\nu : \mathbb{B} \Rightarrow \mathbb{B}(\nu, \nu) = \top$,
- $x, y : X \Rightarrow 1_X(x, y) : X(x, y) \Rightarrow X(1_X x, 1_X y)$, where $1_X(x, y)$ is the unique element α of type $\top \Leftrightarrow \mathbb{B}(X(x, y), X(x, y)) \Leftrightarrow \mathbb{B}(X(x, y), X(1_X x, 1_X y))$.

PROPOSITION 6.0.5. *For each sets X, Y, Z, T and each functions $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$, we have $(hg)f = h(gf)$. Moreover, for each sets X, Y, we have $1_Y f = f = f 1_X$.*

PROOF. Let $X, Y, Z, T : \text{SET}$ and $f : \text{SET}(X, Y), g : \text{SET}(Y, Z), h : \text{SET}(Z, T)$. On the one hand, $x : X \Rightarrow (hg)fx : T$ is defined as the composite

$$x : X \stackrel{f}{\Rightarrow} fx : Y \stackrel{hg}{\Rightarrow} (hg)(fx) : T,$$

hence by

$$x : X \stackrel{f}{\Rightarrow} fx : Y \stackrel{g}{\Rightarrow} g(fx) : Z \stackrel{h}{\Rightarrow} h(g(fx)) : T.$$

On the other hand, $x : X \Rightarrow h(gf)x : T$ is defined as the composite

$$x : X \stackrel{gf}{\Rightarrow} (gf)x : Y \stackrel{h}{\Rightarrow} h((gf)x) : T,$$

hence by

$$x : X \stackrel{f}{\Rightarrow} fx : Y \stackrel{g}{\Rightarrow} g(fx) : Z \stackrel{h}{\Rightarrow} h(g(fx)) : T.$$

It follows that $x : X \Rightarrow (hg)fx = h(gf)x$, so that $(hg)f = h(gf)$. Let $X, Y : \text{SET}$ and $f : X \to Y$. We have $x : X \Rightarrow 1_Y fx = 1_Y(fx) = fx$ and $x : X \Rightarrow f 1_X x = fx$, hence $1_Y f = f = f 1_X$. $\square$

Definition 6.0.6. We say that $X : \mathrm{SET}$ has a unique element if $x, y : X \Rightarrow X(x, y)$. If X is a set with a unique element, we therefore obtain $x, y : X \Rightarrow (x = y)$.

We say that $X : \mathrm{SET}$ has no element if $x : X \Leftrightarrow \perp$. Equivalently, $X : \mathrm{SET}$ has no element if $x : X \Rightarrow \perp$.

Definition 6.0.7. We let the empty set $\emptyset$ and the one point set $*$ be defined as follows.

We let $* : \mathrm{SET}$ be such that

- $x : * \Leftrightarrow \top$
- $x, y : * \Rightarrow *(x, y) = \top : \mathbb{B}$.

We let $\emptyset : \mathrm{SET}$ be such that

- $x : \emptyset \Leftrightarrow \perp$,
- we have $\mathbb{B}(x, y : \emptyset, \emptyset(x, y) : \mathbb{B}) \Leftrightarrow \mathbb{B}(\perp, \emptyset(x, y) : \mathbb{B}) \Leftrightarrow \top$.

Definition 6.0.8. Let $T : \mathrm{SET}$. We say that T is terminal if for each $X : \mathrm{SET}$, the set $\mathrm{SET}(X, T)$ has a unique element. We say that $I : \mathrm{SET}$ is initial if for each $X : \mathrm{SET}$, the set $\mathrm{SET}(I, X)$ has a unique element.

Proposition 6.0.9. *There exists a terminal set.*

Proof. We show that the set $*$ has a unique element. Let $X : \mathrm{SET}$ and $*_X : \mathrm{SET}(X, *)$ be such that

- $x : X \Rightarrow *_X x = x : *$
- let $x, y : X$, since
$$\mathbb{B}(X(x, y), *(x, y)) \Leftrightarrow \mathbb{B}(X(x, y), \top) \Leftrightarrow \top,$$
we let $x, y : X \Rightarrow *_X(x, y) : X(x, y) \Rightarrow *(x, y)$ be defined as $x, y : X \Rightarrow *_X(x, y) := \alpha : \top$.

Let $*'_X : \mathrm{SET}(X, *)$, we have
$$\mathrm{SET}(X, *)(*_X, *'_X) = \int^{x:X} *(*_X x, *'_X x) = \int^{x:X} \top = \top,$$
so that the symmetric equation shows that $*_X = *'_X$. Hence $\mathrm{SET}(X, *)$ has a unique element and $*$ is terminal. $\square$

Proposition 6.0.10. *The set $\emptyset$ is initial.*

Proof. Let X be a set, we have
$$\begin{aligned}
\emptyset_X : \mathrm{SET}(\emptyset, X) &\Leftrightarrow (x : \emptyset \Rightarrow \emptyset_X x : X) \times (x, y : \emptyset \Rightarrow \emptyset_X(x, y) : \mathbb{B}(\emptyset(x, y), X(\emptyset_X x, \emptyset_X y))) \\
&\Leftrightarrow \mathbb{B}(x : \emptyset, \emptyset_X x : X) \times \mathbb{B}(x, y : \emptyset, \mathbb{B}(\emptyset_X(x, y) : \mathbb{B}(\emptyset(x, y), X(\emptyset_X x, \emptyset_X y)))) \\
&\Leftrightarrow \mathbb{B}(\perp, \emptyset_X x : X) \times \mathbb{B}(\perp, \mathbb{B}(\emptyset_X(x, y) : \mathbb{B}(\emptyset(x, y), X(\emptyset_X x, \emptyset_X y)))) \\
&\Leftrightarrow \top \times \top \Leftrightarrow \top.
\end{aligned}$$
We also have
$$\begin{aligned}
\emptyset_X, \emptyset'_X : \mathrm{SET}(\emptyset, X) \Leftrightarrow \mathrm{SET}(\emptyset, X)(\emptyset_X, \emptyset'_X) \\
&= \int^{x:\emptyset} X(\emptyset_X x, \emptyset'_X x) \\
&= \int^{\perp} X(\emptyset_X x, \emptyset'_X x) \\
&= \perp.
\end{aligned}$$

Hence the set $\text{Set}(\emptyset, X)$ has a unique element and $\emptyset$ is initial. □

REMARK 6.0.11. For any set X, the data of a proof of the proposition $x : X$ is equivalent to the data of an element $\bar{x} : \text{Set}(*, X)$. Indeed, we have

$$\begin{aligned}\bar{x} : \text{Set}(*, X) &\Leftrightarrow (z : * \Rightarrow \bar{x}z : X) \times (z, t : * \Rightarrow \bar{x}(z,t) : *(z,t) \Rightarrow X(\bar{x}z, \bar{x}t)) \\ &\Leftrightarrow (\top \Rightarrow \bar{x}z : X) \times (\top \Rightarrow \bar{x}(z,t) : \top \Rightarrow X(\bar{x}z, \bar{x}t)) \\ &\Leftrightarrow (\top \Rightarrow x : X).\end{aligned}$$

REMARK 6.0.12. A set X has no element in the sense of Definition 6.0.6 if and only if the set of functions $\text{Set}(*, X)$ has no element.

PROPOSITION 6.0.13. *There exists a set with no element.*

PROOF. The set $\emptyset$ has no element by definition. □

PROPOSITION 6.0.14. *Let X and Y be sets and let $f, g : X \to Y$. If for all $x : X$ we have $fx = gx$, then $f = g$.*

PROOF. This axiom holds by definition of the equality of functions. □

DEFINITION 6.0.15. Let X and Y be sets. A product of X and Y is a set $X \times Y$ together with functions $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, such that for any set E, the functions π_X, π_Y induce an equivalence between the data of a function $f : E \to X \times Y$ and the data of functions $f_X : E \to X, f_Y : E \to Y$.

DEFINITION 6.0.16. Let $X, Y : \text{Set}$. We define $X \times Y : \text{Set}$ as follows.

- $z : X \times Y \Leftrightarrow (z^x : X) \times (z^y : Y)$,
- $z_1, z_2 : X \times Y \Rightarrow X \times Y(z_1, z_2) = X(z_1^x, z_2^x) \times Y(z_1^y, z_2^y)$

Let $\pi_X : X \times Y \to X$ be such that

- $z : X \times Y \Rightarrow z^X : X$
- $z_1, z_2 : X \Rightarrow \pi_X(z_1, z_2) : X(z_1^x, z_2^x) \times X(z_1^y, z_2^y) \Rightarrow X(z_1^x, z_2^x)$ is given by the structural implications of the conjunction in $\mathbb{B}$.

We define $\pi_Y : X \times Y \to Y$ similarly.

PROPOSITION 6.0.17. *Every pair of sets has a product.*

PROOF. Let $X, Y : \text{Set}$. We show that $X \times Y$, together with π_X and π_Y, defines a product for X and Y. Let E be a set and $f_X : E \to X, f_Y : E \to Y$. We show that there exists a unique $f : E \to X \times Y$ such that $\pi_X f = f_X$ and $\pi_Y f = f_Y$. Let $f : E \to X \times Y$ be defined as

- $e : E \Rightarrow f_X e : X, f_Y e : Y \Rightarrow fe : X \times Y$
- $e_1, e_2 : E \Rightarrow f(x, y) : E(e_1, e_2) \overset{(f_X(e_1,e_2), f_Y(e_1,e_2))}{\Rightarrow} X(f_X e_1, f_X e_2) \times Y(f_Y e_1, f_Y e_2) \Leftrightarrow X \times Y(fe_1, fe_2)$.

We obtain a function $f : E \to X \times Y$ such that $\pi_X f = f_X$ and $\pi_Y f = f_Y$. Suppose $g : E \to X \times Y$ is such that $\pi_X g = f_X$ and $\pi_Y g = f_Y$, then

$$\begin{aligned}\text{Set}(E, X \times Y)(f, g) &\cong \int^{e:E} X \times Y(fe, ge) \\ &\cong \int^{e:E} X(\pi_X fe, \pi_X ge) \times Y(\pi_Y fe, \pi_Y ge) \\ &\cong \text{Set}(E, X)(\pi_X f, \pi_X g) \times \text{Set}(E, Y)(\pi_Y f, \pi_Y g) \cong \top.\end{aligned}$$

Similarly, we obtain

$$\text{S}\textsc{et}(E, X \times Y)(g, f) \cong \text{S}\textsc{et}(E, X)(\pi_X g, \pi_X f) \times \text{S}\textsc{et}(E, Y)(\pi_Y g, \pi_Y f) \cong \top.$$

This shows that $X \times Y$ is a product for X and Y. $\square$

Definition 6.0.18. Let $X, Y : \text{S}\textsc{et}$. A function set for X and Y is a set $[X, Y]$ equipped with a function $ev : [X, Y] \times X \to Y$ such that for each set E equipped with a function $e : E \times X \to Y$, there exists a unique function $\bar{e} : E \to [X, Y]$ such that $ev(\bar{e} \times \imath_X) = e$.

Proposition 6.0.19. *For all sets X and Y, there exist a function set from X to Y.*

Proof. Let X and Y be some sets. Recall that the category of sets is cartesian closed. The counit provides a morphism of sets

$$ev : \text{S}\textsc{et}(X, Y) \times X \to Y,$$

which is defined by

- $(f : \text{S}\textsc{et}(X, Y), x : X) \Rightarrow ev\, f\, x := fx : Y$,
- $(f, g : \text{S}\textsc{et}(X, Y), x, y : X) \Rightarrow ev((f, x), (g, y)) : \text{S}\textsc{et}(X, Y)(f, g) \times X(x, y) \to Y(fx, gy)$, where $ev((f, x), (g, y))$ is given by the composite

$$\text{S}\textsc{et}(X, Y)(f, g) \times X(x, y) \xrightarrow{\pi_x \times g(x,y)} Y(fx, gx) \times Y(gx, gy) \xrightarrow{\circ} Y(fx, gy).$$

Let E be a set equipped with a function $e : E \times X \to Y$, and let $\bar{e} : E \to \text{S}\textsc{et}(X, Y)$ correspond to the adjoint of e provided by the cartesian closed structure of $\text{S}\textsc{et}$, which is such that for $p : E$ and $x : X$, we have $ev(\bar{e} \times \imath_X)(p, x) = ev\,(ep, x) = e(p)(x) = \bar{e}(p, x)$. The construction by adjunction of the function $\bar{e}$ ensures that it is unique among the functions satisfying $ev(\bar{e} \times \imath_X) = e$. Hence the set $\text{S}\textsc{et}(X, Y)$, together with the function ev, defines a function set from X to Y. $\square$

Definition 6.0.20. Let $X, Y : \text{S}\textsc{et}$, $f : X \to Y$ and $y : Y$. An inverse image of y under f is a set $f^{-1}y$ together with a morphism $\iota : f^{-1}y \to X$ such that $x : f^{-1}y \Rightarrow f\iota x = y$, and such that for each set E equipped with a function $j : E \to X$ satisfying $x : X \Rightarrow fjx = y$, there is a unique function $\bar{j} : E \to f^{-1}y$ such that $\iota j = \bar{j}$.

Proposition 6.0.21. *If $X, Y : \text{S}\textsc{et}$ and $f : X \to Y$, for each $y : Y$, there exists an inverse image of y under f.*

Proof. The element $y : Y$ induces a function $y : * \to Y$. Let $f^{-1}y : \text{S}\textsc{et}$ and $\iota : f^{-1}y \to X$ be defined as the pull back of the following diagram

$$\begin{tikzcd}
f^{-1}y \arrow[r, dashed] \arrow[d, dashed, "\iota"'] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}.$$

By construction, the composite $f\iota$ maps any element x of $f^{-1}y$ to y. The set $f^{-1}y$ and the morphism $\iota : f^{-1}y \to X$ have the following explicit description.

- $p : f^{-1}y \Leftrightarrow (x_p : X \times fx_p = y) \Rightarrow \iota p := x_p : X$,
- $p, q : f^{-1}y \Leftrightarrow f^{-1}y(p, q) := X(x_p, x_q)$, so that $\iota(p, q) := \imath_{X(x_p, x_q)} : X(x_p, x_q) \to X(x_p, x_q)$.

The pair $(f^{-1}y, \iota)$ satisfies the uniqueness property by the universal property of the pull back. $\square$

DEFINITION 6.0.22. We say that a function $f : X \to Y$ is injective if

$$(x, y : X) \Rightarrow (X(x, y) \Leftrightarrow Y(fx, fy)) .$$

DEFINITION 6.0.23. We say that a function $f : X \to Y$ is surjective if it is 0-surjective in the sense of Definition 2.4.1. Hence, f is surjective if

$$y : Y \Rightarrow \coprod_{x:X} Y(y, fx).$$

DEFINITION 6.0.24. A subset classifier is a set Ω together with a distinguished object $\top : \Omega$ such that for each sets U, X and each injective function $f : U \to X$, there exists a unique function $\chi : X \to \Omega$ such that $(U, \iota : U \to X)$ defines an inverse image of $\top$ under χ.

PROPOSITION 6.0.25. *There exists a subset classifier.*

PROOF. We show that the set of truth values $\mathbb{B}$: SET, together with the element true $\top : \mathbb{B}$, defines a subset classifier. Let U and X be sets and let $\iota : U \to X$ be an injection. We let $\chi : X \to \mathbb{B}$ be defined as follows.

– $x : X \Rightarrow \chi x = \coprod_{u:U} X(\iota u, x)$
– $x, y : X \Rightarrow \chi(x, y) : X(x, y) \to \mathbb{B}\left(\coprod_{u:U} X(\iota u, x), \coprod_{u:U} X(\iota u, y)\right)$, where $\chi(x, y)$ is induced by $u : U \Rightarrow \left(X(x, y) \times X(\iota u, x) \xrightarrow{\circ} X(\iota u, y) \hookrightarrow \coprod_{u:U} X(\iota u, y)\right)$.

We have $v : U \Rightarrow X(\iota v, \iota v) \xrightarrow{i_v} \coprod_{u:U} X(\iota u, \iota v)$, so that $v : U \Rightarrow \chi\iota v = \top$. Hence, we have $v : U \Rightarrow \iota v : X, \chi\iota v = \top$. Let $x : X$ be such that $\chi x = \top$. Since an implication $\top \Rightarrow \coprod_{u:U} X(\iota u, x)$ is equivalently given by an element $u_X : U$ and an implication $\top \Rightarrow X(\iota u_x, x)$, we obtain in particular $(x : X, \chi x = \top) \Rightarrow u_x : U$, hence, an equivalence

$$(x : X, \chi x = \top) \Leftrightarrow u_x : U.$$

Moreover, we have $(u, v : U \Rightarrow (U(u, v) \Leftrightarrow X(\iota u, \iota v))$, which shows that $U \cong \chi^{-1}\top$. Let $\chi' : X \to \mathbb{B}$ be such that $\iota : U \to X$ defines an inverse image of $\top$ under χ'. Then for each $x : X$, we have $\chi' x = \top$ if and only if there is some $u : U$ such that $x = \iota u$, if and only if $\chi x = \top$, so that $\chi = \chi'$. Hence $\chi : X \to \mathbb{B}$ is unique such that $\iota : U \to X$ defines an inverse image of $\top$ under χ. $\square$

DEFINITION 6.0.26. A natural number system is a set $\mathbb{N}$ together with

– an element $0 : \mathbb{N}$
– a function $s : \mathbb{N} \to \mathbb{N}$

such that for each set N equipped with an element $e : N$ and a function $r : N \to N$, there exists a unique function $\bar{r} : \mathbb{N} \to N$ such that $\bar{r}0 = e$ and $n : \mathbb{N} \Rightarrow \bar{r}sn = s\bar{r}n$.

DEFINITION 6.0.27. Let $\mathbb{N}$: SET be the set generated by $0 : \mathbb{N}$ and $n : \mathbb{N} \Rightarrow n + 1 : \mathbb{N}$. The predicate $\_ : \mathbb{N}$ is therefore defined as

$$(n : \mathbb{N}) \Leftrightarrow (n = 0) \sqcup ((m : \mathbb{N}), (n = m + 1)) ,$$

where the equality is taken literally in $\mathbb{L}$. The equality of natural numbers also comes straightforward from the equivalence $(n = m) \Leftrightarrow (n + 1 = m + 1)$ for each $n, m : \mathbb{N}$, together with $(0 = 0 + 1) \Leftrightarrow \perp$.

Note that we obtain a function $\_ + 1 : \mathbb{N} \to \mathbb{N}$. It should also be noted that our definition precisely corresponds to Peano's axioms of natural numbers stated in [**27**].

NOTATION 6.0.28. We set $1 =: 0+1$, $2 := 1+1$, $3 := 2+1, \ldots, 9 := 8+1$, $10 := 9+1$, and recover the usual expression of natural numbers by using decimal notation.

REMARK 6.0.29. We obtain a monoidal structure directly from the definition of the set of natural numbers, which will be discussed in detail in a forthcoming article currently in preparation. The product of this monoidal structure corresponds to the addition of natural numbers. We therefore obtain a function $+^p : \mathbb{N}^p \to \mathbb{N}$ for each $p : \mathbb{N}$ by using the structure of an algebra over the associative operad thus obtained on $\mathbb{N}$. It is worth noting that the operadic composite

$$+^p \circ (+^q, \ldots, +^q) : (\mathbb{N}^q)^p \to \mathbb{N}$$

sends the element $(1, \ldots, 1) : \mathbb{N}^{pq}$ to the usual product $p \times q : \mathbb{N}$ of p and q in $\mathbb{N}$, so that the commutativity of the product of natural numbers precisely corresponds to interchange. It is therefore possible to apply the tools of operad theory, especially regarding cofibrant resolutions, to deduce results on the factorization of integers into products of prime factors.

PROPOSITION 6.0.30. *There exists a natural number system.*

PROOF. We show that the set $\mathbb{N}$ of Definition 6.0.27 satisfies this universal property. Let $N : \text{SET}$, $e : N$ and $r : N \to N$. The assignment $\bar{r}0 := e, \bar{r}sn := s\bar{r}n$ yields a function $\bar{r} : \mathbb{N} \to R$, which is unique among these satisfying $\bar{r}0 = e$ and $n : \mathbb{N} \Rightarrow \bar{r}sn = s\bar{r}n$. $\square$

DEFINITION 6.0.31. A right inverse of a function $f : X \to Y$ is a function $g : Y \to X$ such that $gf = 1_X$.

REMARK 6.0.32. The choice axiom of set theory is equivalent to the requirement that every surjective function has a right inverse.

We do not provide a proof of this axiom in our approach. Consider a surjective function $f : X \to Y$ and try to construct a right inverse $g : Y \to X$. By definition, we have

$$y : Y \Rightarrow \coprod_{x:X} Y(fx, y),$$

so that each $y : Y$ provides an element $\alpha_y : \coprod_{x:X} Y(fx, y)$, that is, an element $x_y : X$ such that $y = fx$.

- At the level of the elements, we can define g as $y : Y \Rightarrow gx := x_y : X$.
- Suppose $y, z : Y$, we need to provide an implication $g(x, y) : Y(y, z) \Rightarrow X(x_y, x_z)$.

In fact, nothing ensures that the chosen antecedent x_y of y under f is the same than the one we chose for z. To illustrate this, consider the surjection $\pi : X \to X/R$ associated to the quotient of a set X by a relation R. Suppose that $q : X/R \to X$ is a right inverse for π. If q maps $x : X/R$ to its canonical antecedent $x : X$ by p, then q induces an implication $q(x, y) : X/R(x, y) \Rightarrow X(x, y)$ for each $x, y : X$ if and only if R is the trivial relation on X.

DEFINITION 6.0.33. A relation on a set X is a morphism $R : X \times X \to \mathbb{B}$. We say that R satisfies

- **reflexivity** if it is equipped with a 2-morphism $r : X(\_,\_) \to R(\_,\_)$:

$$\begin{tikzcd} X \times X \arrow[d, "1\times 1"'] \arrow[dr, "X(\_,\_)"] & \\ X \times X \arrow[r, "R(\_,\_)"'] \arrow[r, phantom, shift left=3, "\Leftarrow r"] & \mathbb{B}. \end{tikzcd}$$

- **symmetry** if it is equipped with an isomorphism

$$\begin{tikzcd} X \times X \arrow[d, "\tau"'] \arrow[dr, "R(\_,\_)"] & \\ X \times X \arrow[r, "R(\_,\_)"'] \arrow[r, phantom, shift left=3, "\cong"] & \mathbb{B}, \end{tikzcd}$$

where τ exchanges the factors in the product. We obtain

$$(x, y \in X \Rightarrow (R(x,y) \Leftrightarrow R(y,x)))\,.$$

- **transitivity** if it is equipped with a 2-morphism

$$\begin{tikzcd} X \times X \times X \arrow[r, "1\times\Delta\times 1"] \arrow[d, "(\pi_1,\pi_2)"'] & (X \times X) \times (X \times X) \arrow[r, "R\times R"] \arrow[d, phantom, "\Leftarrow"] & \mathbb{B} \times \mathbb{B} \arrow[d, "\times"] \\ X \times X \arrow[rr, "R"'] & & \mathbb{B} \end{tikzcd}$$

Where Δ is the diagonal morphism. We obtain

$$(x, y, z \in X \Rightarrow (R(x,y) \times R(y,z) \Rightarrow R(x,z)))\,.$$

DEFINITION 6.0.34. Let X be a set and $R : X \times X \to \mathbb{B}$ be a reflexive, symmetric and transitive relation on X. A quotient of X by R is a universal pair $(X/R, \pi)$, where X/R is a set, and $\pi : X \to X/R$ is a morphism in SET such that

$$x, y \in X \Rightarrow (R(x,y) \Rightarrow X/R(\pi x, \pi y)).$$

The following description of the quotient X/R is tautological.

PROPOSITION 6.0.35. *Let X be a set and R be a relation on X. Let X/R be the set such that*

- $\_ : X/R := \_ : X$,
- *the equality relation $X/R(\_,\_) : X/R \times X/R \to \mathbb{B}$ is obtained from R by using the transitivity and reflexivity properties. If $x, y \in X$, we have $X/R(x,y) = R(x,y)$.*

We have a canonical projection $\pi : X \to X/R$ which is identical on the objects, and which is given on the morphisms by $r(\_,\_) : X(\_,\_) \to R(\_,\_) \xrightarrow{\cong} X/R(\pi\_, \pi\_)$.

The pair $(X/R, \pi)$ represents the quotient of X by R.

7. Small objects

Definition 7.0.1. An r-small object in $\mathbb{T}_n$ is the data of
- an object $\mathcal{C} \in \mathbb{T}_n$
- an object $\mathcal{C}_r \in \mathbb{T}_r$
- an r-equivalence $\mathrm{I}^{n-r}\mathcal{C}_r \to \mathcal{C}$.

Remark 7.0.2. Any object of $\mathbb{T}_n$ is canonically equipped with n-small structure.

Definition 7.0.3. Let $\mathbb{T}_n^{(r)}$ be the element of type $\mathbb{T}_{r+1}$ whose elements are given by r-small objects. Let $(\mathcal{C}, \mathcal{C}_r), (\mathcal{D}, \mathcal{D}_r)$ be r-small objects. Let $\mathbb{T}_n^{(r)}(\mathcal{C}, \mathcal{D}) : * \to \mathbb{T}_r$ be defined as follows. The data of an element of $\mathbb{T}_n^{(r)}(\mathcal{C}, \mathcal{D})$ is given by the data of an element of $\mathbb{T}_n(\mathcal{C}, \mathcal{D})$. Let $F, G : X \to Y$ be such elements. We let $\mathbb{T}_n^{(r)}(X, Y)(F, G) : * \to \mathbb{T}_{r-1}$ be given by the end

$$\mathbb{T}_n^{(r)}(X, Y)(F, G) \cong \int^{x:*\to E_X} E_Y(Fx, Gx).$$

We have a forgetful morphism

$$\mathrm{I}^{n-r-1}\mathbb{T}_n^{(r)} \to \mathbb{T}_n,$$

which to any r-small object of type $\mathbb{T}_n$ associates its underlying object of type $\mathbb{T}_n$. Given r-small objects $(X, E_X), (Y, E_Y)$, the morphism

$$\mathrm{I}^{n-r-1}\mathbb{T}_n^{(r)}((X, E_X), (Y, E_Y)) \to \mathbb{T}_n(X, Y)$$

Example 7.0.4. An element $\mathcal{C}$ of type $\mathbb{T}_n$ is 0-small if there exists a set E such that the data of an element of $\mathcal{C}$ is equivalent to the data of an element of E. We obtain a category of 0-small objects of type $\mathbb{T}_n$, where two morphisms are equal if they are pointwise equal in the underlying set of the target object.

7.1. Size and limits. Recall that the notions of completeness and cocompleteness usually restrict to the existence of limits and colimits of functors from a *small* category. The impact of the size of sets on categorical constructions has been addressed in detail in [**29**]. In particular, the following theorem, due to Freyd (see [**12**]), is recalled and provides a compelling argument for this restriction. We recall the proof of Freyd's theorem as well, and show the reason why this theorem is not valid in our framework (as previously observed by Shulman in [**30**]).

Theorem 7.1.1. *If a category $\mathcal{C}$ has products indexed by the collection $Arr(\mathcal{C})$ of arrows in $\mathcal{C}$, then $\mathcal{C}$ is a preorder. In particular, any small complete category is a preorder, and no large category that is not a preorder can admit products indexed by proper classes.*

Proof. Suppose that we had two different arrows $f, g : X \to Y$, and form the product $\prod_{Arr_\mathcal{C}} Y$. Then f and g gives us $2^{|Arr_\mathcal{C}|}$ arrows $X \to \prod_{Arr_\mathcal{C}} Y$, but there is only $\mid Arr_\mathcal{C} \mid$ arrows in $\mathcal{C}$, which is a contradiction. $\square$

Here, the notion of a small category is equivalent to the one we defined, and the notion of a large category corresponds to our notion of a category with the property of not being small. In fact, this theorem can not be stated in the context of our theory, mainly because for us, all sets have the same 'size', and hence, we do not use the terminology of proper class, neither of the one of collections. In our point of view, sets can be distinguishable only up to isomorphism, and two morphisms in a category are therefore distinguishable only when the source and target are fixed. It is therefore meaningless to consider the cardinal of all the arrows in a category without taking the isomorphism classe of the objects. Try for instance to define a set of all sets SETS : Set, so that the data of an element of SETS corresponds to the data of an element of Set. Then each pair of sets E and F must be endowed with a truth value SETS(E, F). Since equality of sets does not make sense in our framework, a reasonable way of assigning a truth value to each pair of sets consists in setting

$$\mathrm{SETS}(E, F) := \Pi\mathrm{Set}^{\cong}(E, F),$$

in which case a set is said to be equal to another if and only if they are isomorphic. We therefore obtain the set of cardinal numbers, which we write Card, and which is such that the data of an element of Card is equivalent to the data of a set, and where two cardinals are equal if and only if the corresponding sets are isomorphic. In the same way, the collection of arrows of a category naturally carries the structure of a category, and can hence be given the structure of a set either when the category is small, or by taking isomorphism classes of objects.

Definition 7.1.2. Let $\mathcal{C}$: Cat be a category, and let $\mathrm{Arr}_{\mathcal{C}}$: Cat be defined as follows.

- The data of an element of $\mathrm{Arr}_{\mathcal{C}}$ corresponds to the data of
  - $X, Y : \mathcal{C}$
  - $f : \mathcal{C}(X, Y)$.
- Let $(X, Y, f), (X', Y', f') : \mathrm{Arr}_{\mathcal{C}}$. We let the set $\mathrm{Arr}_{\mathcal{C}}((X, Y, f), (X', Y', f'))$ be defined as
  - $g : \mathrm{Arr}_{\mathcal{C}}((X, Y, f), (X', Y', f'))$

    $\Leftrightarrow (g_X : \mathcal{C}(X, X'), g_Y : \mathcal{C}(Y, Y'), \alpha_g : \mathcal{C}(X, Y')(f' g_X, g_Y f)$,
  - $g, g' : \mathrm{Arr}_{\mathcal{C}}((X, Y, f), (X', Y', f'))$

$$\Leftrightarrow \mathrm{Arr}_{\mathcal{C}}((X, Y, f), (X', Y', f'))(g, g') := \mathcal{C}(X, X')(g_X, g'_X) \times \mathcal{C}(Y, Y')(g_Y, g'_Y).$$

Suppose that $(\mathcal{C}, E)$ is a small category, and let $Arr_{\mathcal{C}}$: Set be such that

- $g : Arr_{\mathcal{C}} \Leftrightarrow g : \mathrm{Arr}_{\mathcal{C}}$,
- $(X, Y, f), (X', Y', f') : Arr_{\mathcal{C}} \Leftrightarrow Arr_{\mathcal{C}}((X, Y, f), (X', Y', f')) : \mathbb{B}$, with

$$Arr_{\mathcal{C}}((X, Y, f), (X', Y', f')) := E(X, X') \times E(Y, Y') \times \mathcal{C}(X, Y')(f' \epsilon_{X,X'}, \epsilon_{Y,Y'} f).$$

When $\mathcal{C}$ is a category which is not necessarily small, we still can define a 'set of arrows' by considering $\Pi\mathrm{Arr}_{\mathcal{C}}$.

Let $\mathcal{C}$ be a category and suppose that there is some $X, Y : \mathcal{C}$ and $f, g : \mathcal{C}(X, Y)$ such that $\mathcal{C}(X, Y(f, g) = \perp$. Let $\mathrm{Arr}_{\mathcal{C}} \to \mathcal{C}$ be defined by the composite

$$\mathrm{Arr}_{\mathcal{C}} \to * \xrightarrow{Y} \mathcal{C},$$

and suppose that its limit $\prod_{\mathrm{Arr}_{\mathcal{C}}} Y : \mathcal{C}$ exists. By definition, we have

$$\mathcal{C}(X, \prod_{\mathrm{Arr}_{\mathcal{C}}} Y) \cong \prod_{\mathrm{Arr}_{\mathcal{C}}} \mathcal{C}(X, Y).$$

Moreover, the data of an element of $\prod_{\mathrm{Arr}_{\mathcal{C}}} \mathcal{C}(X, Y)$ is given by the data for each $h : \mathcal{C}(X_h, Y_h)$, of an element $\phi_h : \mathcal{C}(X, Y)$, such that

$$j : \mathrm{Arr}_{\mathcal{C}}(h : X_h \to Y_h, k : X_k \to Y_k) \Rightarrow \mathcal{C}(X, Y)(\phi_h, \phi_k).$$

In particular, suppose that $i : X_h \xrightarrow{\cong} X_k$ and $i' : Y_h \xrightarrow{\cong} Yk$, then we must have $\phi_h = \phi_{i'hi}$ in $\mathcal{C}(X,Y)$. Hence, distinct morphisms $f, g : X \to Y$ induce an inclusion

$$\Pi\textsc{Cat}\,(\textsc{Arr}_{\mathcal{C}}, \mathrm{I}\,\{f,g\}) \hookrightarrow \mathcal{C}(X, \prod_{\textsc{Arr}_{\mathcal{C}}}, Y),$$

which do not lead to a contradiction. However, suppose that a small category $\mathcal{C}$ has objects $X, Y : \mathcal{C}$ and distinct morphisms $f, g : \mathcal{C}(X,Y)$, and suppose that the discrete product $\prod_{Arr_{\mathcal{C}}} Y : \mathcal{C}$ obtained by the limit of the functor

$$\mathrm{I}Arr_{\mathcal{C}} \to * \xrightarrow{Y} \mathcal{C}$$

exists, then $f = g$ by the same argument than Theorem **??**. We obtain the following proposition as a direct consequence.

Proposition 7.1.3. *Any small category with small limits is a poset.*

Proposition 7.1.4. *Let $\mathcal{C}$ be a category such that for each pair of objects $X, Y : \mathcal{C}$, the set $\mathcal{C}(X,Y)$ has at most one element. Then there exists a poset P and an isomorphism $\mathcal{C} \cong \mathrm{I}P$. In particular, the category $\mathcal{C}$ is small.*

Proof. Let P be the poset whose elements are given by the elements of $\mathcal{C}$, and such that for each $X, Y : \mathcal{C}$, $P(X,Y) := \Pi\mathcal{C}(X,Y) : \mathbb{B}$. The categorical structure of $\mathcal{C}$ provides P with the structure of a poset. Moreover, it is immediate that we have an isomorphism $\mathrm{I}P \cong \mathcal{C}$. $\square$

8. The omega type of omega types

Definition 8.0.1. We say that X is an element of type $\mathbb{T}_\omega$ if for each $n \geq 0$, X comes equipped with

– an element $X_{\leq n}$ of type $\mathbb{T}_n$,
– and an adjunction in $\mathbb{T}_n$

$$\pi_n : X_{\leq n} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{I}X_{\leq n-1} : \iota_n \ .$$

Definition 8.0.2. Let X be an element of type $\mathbb{T}_\omega$. We say that x is an element of type X, and we write $x : * \to X$, if x comes equipped with

– for each $n \geq 0$, an element $x_{\leq n} : * \to X_{\leq n}$,
– for each $n \geq 0$, an adjunction internal to $X_{\leq n}$

$$\pi_n^x : x_{\leq n} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \iota_n x_{\leq n-1} : \iota_n^x \ .$$

Definition 8.0.3. Let X and Y be elements of type $\mathbb{T}_\omega$. For $n \geq 0$,

– let $\mathbb{T}_\omega(X,Y)_{\leq n} := \mathbb{T}_n(X_{\leq n}, Y_{\leq n}) : * \to \mathbb{T}_n$,
– let $\pi_n^* : \mathbb{T}_\omega(X,Y)_{\leq n} \to \mathrm{I}\mathbb{T}_\omega(X,Y)_{\leq n-1}$ be defined by the composite of

$$\mathbb{T}_n(X_{\leq n}, Y_{\leq n}) \xrightarrow{\pi_n^Y} \mathbb{T}_n(X_{\leq n}, \mathrm{I}Y_{\leq n-1}) \xrightarrow{\iota_{n-1}^X} \mathbb{T}_n(\mathrm{I}X_{\leq n-1}, \mathrm{I}Y_{\leq n-1})$$

and

$$\mathbb{T}_n(\mathrm{I}X_{\leq n-1}, \mathrm{I}Y_{\leq n-1}) \cong \mathrm{I}\mathbb{T}_{n-1}(\mathrm{I}X_{\leq n-1}, \mathrm{I}Y_{\leq n-1}) \cong \mathrm{I}\mathbb{T}_{n-1}(X_{\leq n-1}Y_{\leq n-1}).$$

The adjoint $\pi_n : \Pi\mathbb{T}_\omega(X,Y)_{\leq n} \to \mathbb{T}_\omega(X,Y)_{\leq n-1}$ of π_n^* is given by the composite

$$\begin{tikzcd}
\Pi\mathbb{T}_n(X_{\leq n}, Y_{\leq n}) \arrow[r, "\Pi\pi_n^Y"] \arrow[ddrr, dashed, "\pi_n"'] & \Pi\mathbb{T}_n(X_{\leq n}, \mathrm{I}Y_{\leq n-1}) \arrow[r, "\Pi\iota_{n-1}^X"] & \Pi\mathbb{T}_n(\mathrm{I}X_{\leq n-1}, \mathrm{I}Y_{\leq n-1}) \arrow[d, "\cong"] \\
& & \Pi\,\mathrm{I}\mathbb{T}_{n-1}(X_{\leq n-1}, Y_{\leq n-1}) \arrow[d, "\epsilon"] \\
& & \mathbb{T}_{n-1}(X_{\leq n-1}, Y_{\leq n-1})
\end{tikzcd}$$

– let $\iota_n : \mathrm{I}\mathbb{T}_\omega(X,Y)_{\leq n-1} \to \mathbb{T}_\omega(X,Y)_{\leq n}$ be defined by the composite of

$$\mathrm{I}\mathbb{T}_{n-1}(X_{\leq n-1}, Y_{\leq n-1}) \cong \mathrm{I}\mathbb{T}_{n-1}(\mathrm{I}X_{\leq n-1}, \mathrm{I}Y_{\leq n-1}) \cong \mathbb{T}_n(\mathrm{I}X_{\leq n-1}, \mathrm{I}Y_{\leq n-1})$$

and

$$\mathbb{T}_n(\mathrm{I}X_{\leq n-1}, \mathrm{I}Y_{\leq n-1}) \xrightarrow{\iota_n^Y} \mathbb{T}_n(\mathrm{I}X_{\leq n-1}, Y_{\leq n}) \xrightarrow{\pi_n^X} \mathbb{T}_n(X_{\leq n}, Y_{\leq n}),$$

–

This gives to $\mathbb{T}_\omega(X,Y)$ the structure of an element of type $\mathbb{T}_\omega$. We write $F : X \to Y$ for any element F of type $\mathbb{T}_\omega(X,Y)$.

Definition 8.0.4. More generally, let X be an element of type $\mathbb{T}_\omega$ and $x, y : * \to X$. We give to $X(x,y)$ the structure of an element of type $\mathbb{T}_\omega$, as follows. Let $n \leq 0$, and define

- $X(x,y)_{\leq n} := X_{\leq n+1}(x_{\leq n+1}, y_{\leq n+1}) : * \to \mathbb{T}_n$
- $\pi_n^{X(x,y)} : x_{\leq n} \to \iota y_{\leq n-1}$
- $\iota_n^{X(x,y)} : \iota x_{\leq n-1} \to y_{\leq n}$
- isos

Proposition 8.0.5. *Let $\mathbb{T}$ be the element of type $\mathbb{T}_\omega$ be defined as $\mathbb{T}_{\leq n} := \mathbb{T}_{n-1} : \mathbb{T}_n$ equipped with the adjunction*

$$\begin{tikzcd} \Pi_n : \mathbb{T}_{n-1} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathrm{I}\mathbb{T}_{n-2} : \mathrm{I}_n \arrow[l, bend left] \end{tikzcd}$$

that will be defined in Proposition 8.2.9. The sequence of elements $\mathbb{T}_{\leq n} := \mathbb{T}_{n-1} : \mathbb{T}_n$, together with the morphisms $\mathrm{I} : \mathrm{I}\mathbb{T}_{n-1} \to \mathbb{T}_n$ and $\Pi : \mathbb{T}_{n-1} \to \mathrm{I}\mathbb{T}_n$, give to $\mathbb{T}$ the structure of an element of type $\mathbb{T}_\omega$. Moreover,

- *the data of an element of type $\mathbb{T}_\omega$ is equivalent to the data of an element of type $\mathbb{T}$,*
- *for each elements X, Y of type $\mathbb{T}$, we have an isomorphism between the following elements of type $\mathbb{T}_\omega$*

$$\mathbb{T}(X,Y) \cong \mathbb{T}_\omega(X,Y).$$

Hence, T_ω may be seen as the internalization of $\mathbb{T}_\omega$.

Proof. Immediate. □

Definition 8.0.6. Let $\mathbb{T}_\mathbb{N} : * \to \mathbb{T}_0$ be defined as follows.

- $0 : * \to \mathbb{T}_\mathbb{N}$ and $n : * \to \mathbb{T}_\mathbb{N} \Rightarrow n+1 : * \to \mathbb{T}_\mathbb{N}$,
- $\mathbb{T}_\mathbb{N}(0,0) = \top, \mathbb{T}_\mathbb{N}(0,sn) = \bot, \mathbb{T}_\mathbb{N}(sn,0) = \bot, \mathbb{T}_\mathbb{N}(sn,sm) = \mathbb{T}_\mathbb{N}(n,m)$.

We refer to $\mathbb{T}_\mathbb{N}$ as the natural number object internal to $\mathbb{T}$.

PROPOSITION 8.0.7. $\mathbb{T} \cong \lim \mathbb{T}_n$.

PROPOSITION 8.0.8. *Let* $X : * \to \mathbb{T}_n$ *be an element of type* $\mathbb{T}_n$*. For* $0 \leq p \leq n$*, let* $X_{\leq n-p} := \Pi^p X : * \to \mathbb{T}_{n-p}$ *and let* $\pi_{n-p} : \Pi^p X \to \mathrm{I}\Pi^p X$*,* $\iota_{n-p} : \mathrm{I}\Pi^p X \to \Pi^p X$ *be deduced from the adjunction of Proposition 8.0.5. For* $p \geq n$*, let* $X_{\leq n+p} := \mathrm{I}^p X : * \to \mathbb{T}_{n+p}$ *and let* π_{n+p} :*,* ι_{n+p} :*. These data define an element* $\mathrm{I}^{\omega-n} X$ *of type* $\mathbb{T}_\omega$*. Let* X *and* Y *be elements of type* $\mathbb{T}_n$*, and let*

$$\mathrm{I}^{\omega-n}(X,Y) : \mathrm{I}^{\omega-n-1}\mathbb{T}_n(X,Y) \to \mathbb{T}(\mathrm{I}^{\omega-n}X, \mathrm{I}^{\omega-n}Y)$$

be defined as follows. [...]

We obtain a fully faithful morphism in $\mathbb{T}_\omega$

$$\mathrm{I}^{\omega-n} : \mathrm{I}^{\omega-n-1}\mathbb{T}_n \to \mathbb{T}_\omega.$$

PROPOSITION 8.0.9. *The data of an element of type* $\mathrm{I}^{\omega-n-1}\mathbb{T}_n$ *is equivalent to the data of an element of type* $\mathbb{T}_n$*.*

8.1. Postnikov decomposition.

PROPOSITION 8.1.1. *Let* $p : E \to B$ *be a morphism in* $\mathbb{T}_n$*. There exists* $E_n, ..., E_0 : \mathbb{T}_n$ *such that* p *factors as*

$$\begin{tikzcd} E \arrow[rrrr, "p"] \arrow[dr, "p_{n+1}"'] & & & & B \\ & E_n \arrow[r, "p_n"'] & \dots \arrow[r, "p_1"'] & E_0 \arrow[ur, "p_0"'] & \end{tikzcd},$$

where p_k *is* r*-surjective for* $r \neq k$*.*

PROOF. We proceed by induction on n. Let $p : E \to B$ be a morphism in $\mathbb{T}_0$ and define $E_0 : * \to \mathbb{T}_0$ such that $* \to E_0 \Leftrightarrow * \to E$, and for $x, y : * \to E_0$, $E_0(x,y) = B(px, py)$. Let $p_1 : E \to E_0$ be the identity on the objects, and for $x, y : * \to E$, let $p_1(x,y) = p(x,y) : E(x,y) \to E_0(x,y) = B(px,py)$. In particular, p_1 is 0-surjective. Also define $p_0 : E_0 \to B$ such that $p_0 x = px$, and for $x, y : * \to E_0$, $p_0(x,y) = id : E_0(x,y) = B(x,y) \to B(x,y)$. Note that p_0 is 1-surjective, and that $p_0 p_1 = p$. Now let $p : E \to B$ be a morphism in $\mathbb{T}_{n+1}$. For $x, y : * \to E$, p defines a morphism $p(x,y) : E(x,y) \to B(px,py)$ in $\mathbb{T}_n$, which by induction admits a factorisation

$$\begin{tikzcd} E(x,y) \arrow[rrrr, "p(x,y)"] \arrow[dr, "q_{n+1}(x,y)"'] & & & & B(px,py) \\ & F_n(x,y) \arrow[r, "q_n(x,y)"'] & \dots \arrow[r, "q_1(x,y)"'] & F_0(x,y) \arrow[ur, "q_0(x,y)"'] & \end{tikzcd}$$

Let $k = 0, ..., n+1$, and let $E_k : * \to \mathbb{T}_{n+1}$ be such that $* \to E_k \Leftrightarrow * \to E$. For $x, y : * \to E_k$, define $E_k(x,y) = F_{k-1}(x,y)$ if $k > 0$ and $E_0(x,y) = B(px,py)$. For $k = 1, ..., n+2$, we let $E_{n+2} := E$ and we define $p_k : E_k \to E_{k-1}$ as the identity on the objects, and for $x, y : * \to E$, we set $p_k(x,y) := q_{k-1}(x,y)$. In particular, p_k is 0-surjective. Moreover, since $p_k(x,y)$ is r-surjective for $r \neq k-1$ by induction hypothesis, it follows that p_k is r-surjective whenever $r \neq k$. Let $p_0 : E_0 \to B$ be defined on the objects by $x : * \to E_0 \Rightarrow p_0 x = px : * \to B$, and on the morphisms by the identity of $B(px,py)$ for each $x, y : * \to E$. Hence p_0 is r-surjective for each $r \neq 0$. Define $E_{n+2} := E$ and $E_{-1} := B$. We obtain a sequence of objects $E_{n+1}, ..., E_0$ of $\mathbb{T}_{n+1}$, together with morphisms $p_k : E_k \to E_{k-1}$ which are r-surjective for $r \neq k$, for each $k = n+2, ..., 0$. The composite $p_0 ... p_{n+2} : E \to B$ maps any object x of E to the object px of B. For each pair of objects x, y of E, we have

$$p_0 ... p_{n+2}(x, y) = p_0 ... p_{n+1}(x, y)\, \epsilon_{B(px,py)} = q_0(x, y) ... q_{n+1}(x, y) = p(x, y),$$

so that the sequence $p_{n+2}, ..., p_0$ factors p. $\square$

8.2. Dimensional shifts.

Definition 8.2.1. Let X be an element of type $\mathbb{T}_n$. We define an element $\mathrm{I}_n X$ of type $\mathbb{T}_{n+1}$ by induction as follows. We write I for I_n for clarity.

- The data of an element of type $\mathrm{I}X$ is equivalent to the data of an element of type X. We write $\mathrm{I}x$ for the element of type $\mathrm{I}X$ corresponding to the element x of type X.
- For each pair of elements $\mathrm{I}x, \mathrm{I}y$ of type $\mathrm{I}X$, we set $\mathrm{I}X(\mathrm{I}x, \mathrm{I}y) = \mathrm{I}X(x, y)$.

Note that for any element ρ of type $\mathbb{T}_n$ with $n < -2$, there is a unique element of type $\mathbb{T}_{n+1}$, so that $\mathrm{I}\rho : * \to \mathbb{T}_{n+1}$ is necessarily given by this element.

Let X and Y be elements of type $\mathbb{T}_n$. We define a morphism $\mathrm{I}_n(X, Y)$ in $\mathbb{T}_{n+2}$ inductively on n, such that

$$\mathrm{I}_n(X, Y) : \mathrm{I}_n \mathbb{T}_n(X, Y) \to \mathbb{T}_{n+1}(\mathrm{I}X, \mathrm{I}Y).$$

We just write I for clarity.

- Let F be an element of type $\mathbb{T}_n(X, Y)$. We define an element $\mathrm{I}F$ of type $\mathbb{T}_{n+1}(\mathrm{I}X, \mathrm{I}Y)$.
  - For $\mathrm{I}x : * \to \mathrm{I}X$ we set $\mathrm{I}F\mathrm{I}x := \mathrm{I}Fx : * \to \mathrm{I}Y$.
  - Let $\mathrm{I}x, \mathrm{I}y : * \to \mathrm{I}X$. We obtain a morphism by induction

    $$\mathrm{I}(F_{x,y}) : \mathrm{I}X(x, y) \to \mathrm{I}Y(Fx, Fy),$$

    hence a morphism

    $$\mathrm{I}F_{\mathrm{I}x,\mathrm{I}y} : \mathrm{I}X(\mathrm{I}x, \mathrm{I}y) \to \mathrm{I}Y(\mathrm{I}Fx, \mathrm{I}Fy) = \mathrm{I}Y(\mathrm{I}F\mathrm{I}x, \mathrm{I}F\mathrm{I}y).$$
- If F and G are elements of type $\mathbb{T}_n(X, Y)$, we define the morphism

  $$\mathrm{I}_{\mathrm{I}F,\mathrm{I}G} : \mathrm{I}\mathbb{T}_n(X, Y)(\mathrm{I}F, \mathrm{I}G) \to \mathbb{T}_{n+1}(\mathrm{I}X, \mathrm{I}Y)(\mathrm{I}F, \mathrm{I}G)$$

  as follows. On the one hand, we have

  $$\mathrm{I}\mathbb{T}_n(X, Y)(\mathrm{I}F, \mathrm{I}G) \cong \mathrm{I}\int^{x:*\to X} Y(Fx, Gx),$$

  which can be shown by induction to be equivalent to $\int^{\mathrm{I}x:*\to\mathrm{I}X} \mathrm{I}Y(\mathrm{I}F\mathrm{I}x, \mathrm{I}G\mathrm{I}x)$, and on the other hand, we have

  $$\mathbb{T}_{n+1}(\mathrm{I}X, \mathrm{I}Y)(\mathrm{I}F, \mathrm{I}G) \cong \int^{\mathrm{I}x:*\to\mathrm{I}X} \mathrm{I}Y(\mathrm{I}F\mathrm{I}x, \mathrm{I}G\mathrm{I}x),$$

  so that we let $\mathrm{I}_{\mathrm{I}F,\mathrm{I}G}$ be induced by the isomorphism

$$\mathrm{I}\int^{x:*\to X} Y(Fx, Gx) \cong \int^{\mathrm{I}x:*\to\mathrm{I}X} \mathrm{I}Y(\mathrm{I}F\mathrm{I}x, \mathrm{I}G\mathrm{I}x) \cong \int^{\mathrm{I}x:*\to\mathrm{I}X} \mathrm{I}Y(\mathrm{I}F\mathrm{I}x, \mathrm{I}G\mathrm{I}x).$$

PROPOSITION 8.2.2. *We obtain a morphism in* $\mathbb{T}_{n+2}$ *for each* $n \in \mathbb{N}$

$$\mathrm{I}_n : \mathrm{I}_{n+1}\mathbb{T}_n \to \mathbb{T}_{n+1}.$$

Moreover, I_n *is fully faithful, so that for each elements* X, Y *of type* $\mathbb{T}_n$*, we have an isomorphism*

$$\mathrm{I}_n(X,Y) : \mathrm{I}_{n+1}\mathbb{T}_n(X,Y) \xrightarrow{\cong} \mathbb{T}_{n+1}(\mathrm{I}_n X, \mathrm{I}_n Y).$$

PROOF. We proceed by induction on $n \in \mathbb{N}$. For $n < -2$, we have $\mathrm{I}\mathbb{T}_n = \mathrm{I}\mathbb{T}_{n+1}$, so that I_n is the identity morphism. Let $X, Y : * \to \mathbb{T}_n$. Then the data of an element $f : * \to \mathbb{T}_{n+1}(\mathrm{I}X, \mathrm{I}Y)$ consists in the data for each $x : * \to X$, of an element $fx : * \to Y$, and for each $x, y : * \to X$, of a morphism $f_{x,y} : \mathrm{I}_{n-1}X(x,y) \to \mathrm{I}_{n-1}Y(fx, fy)$, which by induction uniquely corresponds to an element of $\mathrm{I}_n\mathbb{T}_{n-1}(X(x,y), Y(fx,fy))$, which we also write $f_{x,y}$. It follows that the data of an element of type $\mathrm{I}_{n+1}\mathbb{T}_n(X,Y)$ is equivalent to the data of an element of type $\mathbb{T}_{n+1}(\mathrm{I}_n X, \mathrm{I}_n Y)$. Let $f, g : * \to \mathrm{I}_{n+1}\mathbb{T}_n(X,Y)$. Then

$$\mathrm{I}\mathbb{T}_n(X,Y)(f,g) \cong \mathrm{I}\mathbb{T}_n(\mathrm{I}X, \mathrm{I}Y)(f,g) \cong \int^{x:*\to\mathrm{I}X} \mathrm{I}_n Y(fx, gx) \cong \mathbb{T}_{n+1}(\mathrm{I}X, \mathrm{I}Y).$$

□

NOTATION 8.2.3. For each $p \geq 0$ and $n \in \mathbb{N}$ we write $\mathrm{I}^p : \mathrm{I}^p\mathbb{T}_n \to \mathbb{T}_{n+p}$ for the morphism in $\mathbb{T}_{n+p+1}$ defined by $\mathrm{I}^0 = Id$ and $\mathrm{I}^{p+1} = \mathrm{I}\,\mathrm{I}^p\mathbb{T}_n \xrightarrow{\mathrm{I}\,\mathrm{I}^p} \mathrm{I}\mathbb{T}_{n+p} \xrightarrow{\mathrm{I}} \mathbb{T}_{n+p+1}$.

EXAMPLE 8.2.4. Recall that we wrote $\alpha = \mathbb{T}_{-3}$, $\top = \mathbb{T}_{-2}$ and $\mathbb{B}_{\mathbb{T}} = \mathbb{T}_{-1}$. The element $\mathrm{I}\alpha$ of type $\mathbb{B}_{\mathbb{T}}$ has a unique element, so that we obtain $\mathrm{I}\alpha = \top$ in $\mathbb{B}_{\mathbb{T}}$. Moreover, the object $\mathrm{I}\top$ in $\mathbb{T}_0$ has a unique element. We write $* := \mathrm{I}\top$. More generally, we define the element $*_n$ of type $\mathbb{T}_n$ by $*_n := \mathrm{I}\,*_{n-1} = \mathrm{I}^{n+2}\alpha$, and will often just write $*$ for this element.

REMARK 8.2.5. The data of an element of type $\mathbb{T}_n$ is equivalent to the data of a morphism $* \to \mathbb{T}_n$ in $\mathbb{T}_{n+1}$. Hence, for each element X of type $\mathbb{T}_n$, we write $X : * \to \mathbb{T}_n$.

REMARK 8.2.6. Suppose that $\mathbb{B}_{\mathbb{T}}$ has an initial object $\perp$, and suppose that the element $\mathrm{I}\perp$ of type $\mathbb{T}_0$ has an element $x : * \to \mathrm{I}\perp$, then there is some β of type $\mathbb{B}_{\mathbb{T}}(\top, \perp)$ such that $x = \mathrm{I}\beta$. Hence $\mathrm{I}\perp$ has no element. We write $\mathrm{I}\perp = \emptyset$.

DEFINITION 8.2.7. Let $X : * \to \mathbb{T}_n$ with $n \geq 0$ and let $\Pi X : * \to \mathbb{T}_{n-1}$ be such that

- $* \to \Pi X \Leftrightarrow * \to X$,
- for $x, y : * \to X$, the element $\Pi X(x,y) : * \to \mathbb{T}_{n-2}$ is defined as $\Pi X(x,y) = \alpha$ if $n = 0$ and $\Pi X(x,y) = \Pi(X(x,y))$ else.

Let $X, Y : * \to \mathbb{T}_n$ and let

$$\Pi_{X,Y} : \mathbb{T}_n(X,Y) \to \mathrm{I}\mathbb{T}_{n-1}(\Pi X, \Pi Y)$$

be the morphism in $\mathbb{T}_{n+1}$ inductively obtained as follows.

- Let $F : X \to Y$. We let $\Pi F : \Pi X \to \Pi Y$ be such that $x : * \to X \Rightarrow Fx : * \to Y$, and for $x, y : * \to X$, we let the morphism in $\mathbb{T}_{n-2}$

$$\Pi F_{x,y} : \Pi X(x,y) \to \Pi Y(Fx, Fy)$$

be defined by the image of $F_{x,y} : * \to \mathbb{T}_{n-1}\left(X(x,y), Y(Fx,Fy)\right)$ by

$$\Pi_{X(x,y),Y(Fx,Fy)} : \mathbb{T}_{n-1}\left(X(x,y), Y(Fx,Fy)\right) \to \mathrm{I}\mathbb{T}_{n-2}\left(\Pi X(x,y), \Pi Y(Fx,Fy)\right)$$

if $n > 0$, and by the unique element of type $\mathbb{T}_{n-2}(\alpha,\alpha)$ else.

– Let $F, G : X \to Y$ and $x : * \to X$. By induction we have a morphism

$$\Pi_{*,Y(Fx,Gx)} : \mathbb{T}_{n-1}\left(*, Y(Fx,Gx)\right) \to I\mathbb{T}_{n-2}\left(\Pi *, \Pi Y(Fx,Gx)\right) \cong \mathrm{I}\Pi Y(\Pi Fx, \Pi Gx).$$

We let

$$\Pi_{X,Y}(F,G) : \mathbb{T}_n(X,Y)(F,G) \to \mathrm{I}\mathbb{T}_{n-1}(\Pi X, \Pi Y)(\Pi F, \Pi G)$$

be defined by the end

$$\int^{x:*\to X} \Pi_{*,Y(Fx,Gx)} : \int^{x:*\to X} Y(Fx,Gx) \to \int^{x:*\to X} \mathrm{I}\Pi Y(\Pi Fx, \Pi Gx)$$
$$\cong \mathrm{I}\int^{x:*\to X} \Pi Y(\Pi Fx, \Pi Gx).$$

Remark 8.2.8. If $X : * \to \mathbb{T}_0$ is a set, then $\Pi X = \bot \Leftrightarrow X \cong \emptyset$, and $\Pi X = \top$ whenever X has an element.

Proposition 8.2.9. *We obtain a morphism in $\mathbb{T}_{n+1}$ for each n such that*

$$\Pi_n : \mathbb{T}_n \to \mathrm{I}_n\mathbb{T}_{n-1}$$

which is left adjoint to $\mathrm{I}_{n+1} : \mathrm{I}_n\mathbb{T}_{n-1} \to \mathbb{T}_n$. Hence, for each element $\mathcal{C}$ of type $\mathbb{T}_n$ and each element X of type $\mathbb{T}_{n-1}$, we have an isomorphism

$$\mathbb{T}_n(\mathcal{C}, \mathrm{I}X) \cong \mathrm{I}\mathbb{T}_{n-1}(\Pi\mathcal{C}, X),$$

which equivalently corresponds to an isomorphism

$$\Pi\mathbb{T}_n(\mathcal{C}, \mathrm{I}X) \cong \mathbb{T}_{n-1}(\Pi\mathcal{C}, X).$$

In particular, the morphism Π_n corresponds under this isomorphism to

$$\Pi_n^* : \Pi_{n+1}\mathbb{T}_n \to \mathbb{T}_{n-1}.$$

Proof. We define a morphism $\eta : Id \Rightarrow \mathrm{I}\,\Pi$ by induction on n, so that $\eta : * \to \mathbb{T}_{n+1}(\mathbb{T}_n, \mathbb{T}_n)(Id, \mathrm{I}\,\Pi)$. Let $\mathcal{C}$ be an element of type $\mathbb{T}_n$ and let $\eta_{\mathcal{C}} : \mathcal{C} \to \mathrm{I}\,\Pi\mathcal{C}$ be such that

– $Y : * \to \mathcal{C} \Rightarrow \eta_{\mathcal{C}}Y = \mathrm{I}\,\Pi Y : * \to \mathrm{I}\,\Pi\mathcal{C}$,

– for $Y, Z : * \to \mathcal{C}$, we let

$$\eta_{\mathcal{C}}(Y,Z) : \mathcal{C}(Y,Z) \to \mathrm{I}\,\Pi\mathcal{C}(\mathrm{I}\,\Pi X, \mathrm{I}\,\Pi Y) \cong \mathrm{I}\,\Pi\left(\mathcal{C}(X,Y)\right)$$

be the morphism in bT_{n-1} obtained by induction hypothesis. For $n = 0$, let E be an element of type $\mathbb{T}_0$ and suppose E has at least an element. Hence $\Pi E = \top : * \to \mathbb{T}_{-1}$, and $\mathrm{I}\,\Pi E = *$, so that we define η_E as the unique morphism $E \to *$ in $\mathbb{T}_0$. If $E = \emptyset$, then $\Pi E = \bot$ and $\mathrm{I}\,\Pi E =$, so that we define as the identity $\eta_\emptyset : \emptyset \to \emptyset$.

We define a natural isomorphism $\epsilon_X : \Pi\,\mathrm{I}X \to X$ by induction. Let X be an element of type bT_{-1}. Then either $X = \top$ and $\mathrm{I}X = *$, so that $\Pi\,\mathrm{I}X = *$, or $X = \bot$ and $\mathrm{I}X = \emptyset$, so that $\Pi\,\mathrm{I}X = \bot$. Hence we let ϵ_X be the equivalence $\Pi\,\mathrm{I}X \Leftrightarrow X$. Let $X : * \to \mathbb{T}_n$. We define $\epsilon_X : \Pi\,\mathrm{I}X \to X$ by the identity on the objects, and for $x, y : * \to X$, by the morphism obtained by induction hypothesis $\epsilon_{X(x,y)} : \Pi\,\mathrm{I}X(x,y) \to X(x,y)$.

Let $X : * \to \mathbb{T}_{n-1}$. Then by ϵ_X we have $\mathrm{I}\,\Pi\,\mathrm{I}X \cong X$, and the composite

$$\mathrm{I}X \xrightarrow{\eta_{\mathrm{I}X}} \mathrm{I}\,\Pi\,\mathrm{I}X \xrightarrow{I\epsilon_X} IX$$

can easily be shown to be the identity. Let $\mathcal{C} : * \to \mathbb{T}_n$. The composite

$$\Pi C \xrightarrow{\Pi\eta_{\mathcal{C}}} \Pi\,\mathrm{I}\,\Pi\mathcal{C} \xrightarrow{\epsilon_{\Pi X}} \mathcal{C}$$

also is the identity, so that Π is left adjoint to I. $\square$

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