Université de Lille

Thèse

pour obtenir le grade de :

Docteur de l'Université de Lille

dans la spécialité

« Mathématiques et leurs interactions »

par

Christopher Renaud Chan

Transition de phase liquide-gaz pour des processus ponctuels de Gibbs avec interaction saturée

Thèse soutenue le 1 juillet 2024 devant le jury composé de :

Mme	Sabine Jansen	Professeure Université de Munich	(Rapportrice)
M.	Yvan Velenik	Professeur Université de Genève	(Rapporteur)
M.	Thierry Gobron	Chargé de recherche Université de Lille	(Examinateur)
M.	Thomas Leblé	Chargé de recherche Université Paris-Cité	(Examinateur)
Mme	Mylène Maïda	Professeure Université de Lille	(Examinatrice)
Mme	Ellen Saada	Directrice de recherche Université Paris-Cité	(Examinatrice)
M.	David Dereudre	Professeur Université de Lille	(Directeur de Thèse)

Laboratoire Paul Painlevé
Bâtiment M2, Cité Scientifique, 59655 Villeneuve-d'Ascq
France

Remerciements

Je voudrais adresser mes remerciements en tout premier lieu à mon directeur de thèse, David Dereudre. Je voudrais te remercier pour ton soutien, tes conseils, qu'ils soient mathématiques, tennistiques ou autres... Je t'en suis extrêmement reconnaissant pour tout cela. Ces quatre années ont été pour moi une belle aventure scientifique et humaine, et c'est en grande partie grâce à toi. Tu es un véritable modèle et un mentor...

Je remercie chaleureusement Sabine Jansen et Yvan Velenik pour avoir accepté de rapporter cette thèse. Je suis d'autant plus touché, car vos travaux ont constitué une véritable source d'inspiration. Je tiens à exprimer toute ma gratitude envers Thomas Leblé, Mylène Maïda et Ellen Saada qui me font l'honneur d'être membres de ce jury. Je veux remercier particulièrement Thierry Gobron : sans toi et tes explications, la thèse n'aurait pas pu avancer, et c'était un plaisir de t'avoir dans mon jury.

Je voudrais aussi remercier les membres de l'équipe Proba-Stat pour toutes ces discussions passionnantes. J'ai une pensée particulière pour Serguei Dachian, qui a su me conseiller dans les enseignements et me guider dans les méandres de l'administration de Polytech, ainsi que Myriam Fradon pour toutes ces discussions sur le folklore régional du Poitou-Charentes. J'ai eu la chance et le plaisir de suivre le groupe de travail sur les processus ponctuels ; c'était pour moi un moment scientifique important, mais aussi de convivialité fort appréciable. Je veux donc remercier David Coupier, Rémi Bardenet, Mylène et David D. pour avoir animé ce GT.

Tout au long de cette thèse, j'ai eu le bonheur de nouer de fortes amitiés, et j'espère que l'on gardera contact malgré la distance. Tout d'abord, je veux remercier la "team Americano", Benoît Henry et Thibaut Vasseur, avec laquelle nous avons organisé les "cafés probas" et speedrunné les escape games. Ensuite, j'ai une pensée pour Raphaël Butez, mon partenaire de double, avec qui nous avons enfin réussi à battre le duo David/Barnabé. J'espère que ce n'est que le début d'une longue série de victoires. Barnabé, je voudrais te remercier d'être si pénible à jouer et pour tous ces retards. Plus sérieusement, tu as toujours apporté de la bonne humeur et de l'animation musicale en salle de convivialité, merci pour tout ça. Je n'oublie évidemment pas mes amis doctorants avec qui j'ai pu partager de très bons souvenirs et de beaux moments festifs : Keenan, Martin, Guillaume, Thomas, Iacopo, Nicola, Sophie, Julien, Jules, Justine, Imane, Etienne. I would also like to thank Fabio and Daniela, whom I met along the way. It is always a pleasure meeting you and discussing together.

Je souhaite aussi remercier les personnes qui m'ont transmis leurs savoirs et leur passion pour les sciences ; sans elles, je n'aurais pas pu faire cette thèse. En particulier, j'aimerais remercier Salah Belazreg, professeur de physique-chimie pendant mes années de lycée, pour ses encour-

agements, ses conseils et le temps qu'il a pris pour nous montrer les beautés de la science. C'était une chance d'avoir eu un enseignant aussi impliqué et exigeant.

J'ai une pensée pour mes amis Nicolas S, Nicolas R, Pierre, Niels, Victor, Alexis, Kévin, Loïc, Mathilde, Lucile, Antoine, Hariyanto (et je m'excuse pour ceux que j'ai oubliés). Nous avons passé des moments inoubliables et je suis sûr que d'autres aventures nous attendent.

Enfin, j'aimerais finir mes remerciements par les personnes qui me sont les plus chères au monde, ma famille. Untuk Mama dan ce Nova, terima kasih untuk dukungan dan kesabaran kalian terhadapku. Aku tau semua yang kalian bikin untuk diriku, semoga aku bisa membalas kasih di masa depan. Pour mon père, l'expression de nos sentiments respectifs n'a jamais été notre point fort, mais j'aimerais que tu saches que je t'aime ; tu as été un bon père, et je sais que tu as fait tout ce que tu pouvais pour moi dans les circonstances qui ont été les nôtres. Je te dédie ce travail, car il est le fruit de la graine de curiosité que tu as plantée en moi lors des multiples "expérimentations" et bricolages dans le jardin que nous avons faits ensemble.

Summary

In this thesis, we investigate the phenomenon of liquid-gas phase transition for Gibbs point processes with saturated interaction. In finite volume, the unnormalised density of the Gibbs measure with respect to a Poisson point process with activity z is given by the Boltzmann factor $e^{-\beta H}$, where β is the inverse temperature and H is the Hamiltonian that encodes the interaction between particles. The infinite volume Gibbs point processes are defined as solutions to the Dobrushin-Lanford-Ruelle equations, which describe the equilibrium of the system. Saturated interactions represent a class of models where the energy cost of adding a particle in areas of high particle density is constant. Specifically, we examine the question of uniqueness of infinite volume Gibbs point processes and study the discontinuities of the density with respect to z and β.

Within this context, we prove the non-uniqueness of the infinite volume Gibbs measures for special values of activity, provided the temperature is sufficiently low. Moreover, we demonstrate the non-differentiability of the pressure at these critical points. Our main tool in this proof is an adaptation of the Pirogov-Sinaï-Zahradník theory for continuous systems, that is particularly suited for saturated interactions. The saturation property allows for easy computation of energy in large areas with high particle density. One of the main assumption needed in this general result is that the Hamiltonian satisfies a form of Peierls condition, indicating a surplus of energy in areas of low density.

Subsequently, we apply our phase transition result for saturated interactions to two models. Firstly, we study the Quermass interaction, where the Hamiltonian is defined as a linear combination of the $d+1$ Minkowski functionals of the halo of a configuration, which is the union of closed balls of centered at the position of the particles with random radii. This family of models is a natural extension of the Widom-Rowlinson model, which only considered the volume of the halo. More specifically, we consider a linear combination of the volume, surface measure and Euler-Poincaré characteristic of the halo. For a family of such linear combination with bounded random radii and for β sufficiently large, we demonstrate the existence of a critical activity z_β^c at which the liquid-gas phase transition occurs. The value of z_β^c is not explicit, but we show that it is asymptotically equivalent to β at infinity.

Finally we investigate the family of diluted pairwise interactions, which is an approximation of the usual pairwise interaction. For an integrable pair potential ϕ, the Hamiltonian is given by $H(\omega) = \iint \phi(|x-y|)\mathbb{1}_{L_R(\omega)}(x)\mathbb{1}_{L_R(\omega)}(y)dxdy$ where $L_R(\omega)$ is the halo with radius R. For integrable pair potentials ϕ that are sufficiently repulsive in short range, we prove the liquid-gas phase transition of the diluted pairwise interaction for β sufficiently large. We demonstrate that

the interaction satisfy the Peierls condition using either a local or global analysis of the energy depending on the value of R. Again, we provide an equivalent of z_β^c as β tends to infinity.

Résumé

Dans cette thèse, nous étudions le phénomène de transition de phase liquide-gaz pour des processus ponctuels de Gibbs avec interaction saturée. En volume fini, la densité non normalisée de la mesure de Gibbs par rapport à un processus ponctuel de Poisson avec une activité z est donnée par le facteur de Boltzmann $e^{-\beta H}$, où β est la température inverse et H est l'hamiltonien qui encode l'interaction entre les particules. Les interactions saturées représentent une classe de modèles où le coût énergétique d'ajout d'une particule dans des zones de haute densité de particules est constant. Les processus ponctuels de Gibbs en volume infini sont définis comme solutions des équations de Dobrushin-Lanford-Ruelle, qui décrivent l'équilibre du système. Nous examinons la question de l'unicité des processus de Gibbs en volume infini et étudions les discontinuités de la fonction de densité par rapport à z et β.

Dans ce contexte, nous prouvons la non-unicité des mesures de Gibbs en volume infini pour des valeurs spécifiques d'activité, à condition que la température soit suffisamment basse. De plus, nous démontrons la non-différentiabilité de la pression en ces points critiques. Notre principal outil dans cette preuve est une adaptation de la théorie de Pirogov-Sinaï-Zahradník pour les systèmes continus, particulièrement adaptée aux interactions saturées. La propriété de saturation permet de calculer facilement l'énergie dans de grandes zones avec une forte densité de particules. L'une des principales hypothèses nécessaires dans ce résultat général est que l'hamiltonien satisfait des conditions de Peierls, indiquant un surplus d'énergie dans les zones de faible densité.

Par la suite nous appliquons notre résultat de transition de phase pour les interactions saturées à deux modèles. En tout premier lieu, nous étudions l'interaction Quermass, où l'hamiltonien est défini comme une combinaison linéaire des $d+1$ fonctionnelles de Minkowski sur le halo d'une configuration, qui est l'union de boules fermées centrées sur les positions des particules avec des rayons aléatoires. Cette famille de modèles est une extension naturelle du modèle de Widom-Rowlinson, qui ne considère que le volume du halo. Plus précisément, nous considérons une combinaison linéaire du volume, de la mesure de surface et de la caractéristique d'Euler-Poincaré de du halo. Pour une famille de telles combinaisons linéaires avec des rayons aléatoires bornés et pour β suffisamment grand, nous démontrons l'existence d'une activité critique z_β^c à laquelle la transition de phase liquide-gaz se produit. La valeur de z_β^c n'est pas explicite mais nous montrons qu'elle est proche de β à l'infini.

Enfin nous étudions la famille des interactions par paire diluées, qui est une approximation de l'interaction par paire habituelle. Pour un potentiel d'interaction intégrable ϕ, l'hamiltonien est donné par $H(\omega) = \iint \phi(|x-y|)\mathbb{1}_{L_R(\omega)}(x)\mathbb{1}_{L_R(\omega)}(y)dxdy$, où $L_R(\omega)$ est le halo avec un rayon R. Pour les potentiels d'interaction intégrables ϕ suffisamment répulsifs à courte portée, nous

démontrons la transition de phase liquide-gaz de l'interaction par paire diluée pour β grand. Nous démontrons que cette interaction satisfait les conditions de Peierls en analysant localement ou globalement l'énergie en fonction de la valeur de R. De nouveau, nous donnons un équivalent de z^c_β lorsque β tend vers l'infini.

Contents

Introduction

Les processus ponctuels sont des outils probabilistes et statistiques naturels pour décrire les données spatiales qui apparaissent dans de nombreux domaines tels que l'écologie, les télécommunications, l'astronomie, la science des données et la physique des systèmes de particules. Le processus ponctuel le plus populaire et le plus étudié est le processus ponctuel de Poisson [26], où les points n'interagissent pas et sont distribués indépendamment dans l'espace. Cependant, il est souvent plus intéressant d'étudier des points en interaction, notamment pour modéliser le comportement des particules physiques, pour faire de l'intégration numérique avec des méthodes de Monte-Carlo [1], ou encore pour sélectionner des données d'entraînement suffisamment différentes en machine learning [24]. Les interactions peuvent être de natures différentes : attractives, répulsives, ou les deux mais à des échelles différentes.

Les processus ponctuels de Gibbs ont été introduits dans les années 1960 et permettent de modéliser l'interaction entre les points, constituant ainsi le cadre théorique naturel de la physique statistique. Un ouvrage de référence est [31] pour les fondements rigoureux de la théorie. L'objet principal d'étude concerne les phénomènes de transition de phase, qui se caractérisent par un changement abrupt et discontinu d'une quantité macroscopique décrivant le système. Historiquement, les modèles gibbsiens ont permis de démontrer des changements d'état dans des systèmes de particules sur réseau avec spin. Notamment, le modèle d'Ising, modélisant les métaux ferromagnétiques, a permis d'étudier la magnétisation spontanée du métal. Ce phénomène apparaît lorsque la température est inférieure à un seuil critique appelé la température de Curie et s'explique par l'alignement des moments magnétiques des atomes. À haute température, cet effet disparaît car l'agitation thermique est trop forte et compense la tendance à l'alignement des spins.

Dans cette thèse, nous nous intéressons aux transitions liquide-gaz, qui se caractérisent par la coexistence de deux phases fluides de densités distinctes, sans cristallisation. L'état le moins dense correspond à l'état gazeux, tandis que l'autre correspond à l'état liquide. Dans ce contexte, les particules ne sont plus disposées sur un réseau, ce qui s'explique par l'absence de cristallisation. Cependant nous ne cherchons pas à démontrer la prédominance d'un des spins dans le système, même si cela constitue une question intéressante et qui plus est a été grandement étudiée dans de précédents travaux. Pour en citer quelques uns, nous avons les transitions de phase pour le modèle de Potts continu [18] et pour le modèle de Widom-Rowlinson à couleurs multiples non symétriques [2]. Sans considération sur les spins, nous souhaitons montrer la discontinuité de la densité de particule en utilisant des arguments géométriques et l'agencement des particules.

Une autre manière de voir la transition de phase vient du principe variationnel. Un système physique à l'équilibre minimise l'énergie libre, qui est la différence entre l'énergie du système et l'entropie. Ainsi, dans le problème de minimisation de l'énergie libre, il y a souvent un compromis entre la maximisation de l'entropie et la minimisation de l'énergie. Avec un jeu de paramètres convenable, on peut obtenir deux minimiseurs : un état avec une grande entropie, correspondant à l'état gazeux, et un état de moindre énergie, correspondant à l'état liquide.

Un processus ponctuel sur $\mathbb{R}^d$ est un variable aléatoire à valeur dans l'espace des configurations, où une configuration ω est un sous-ensemble localement fini de $\mathbb{R}^d$ (i.e. pour tout $\Delta \subset \mathbb{R}^d$ borné $\#\Delta \cap \omega < +\infty$). La loi d'un processus ponctuel de Gibbs dans un volume fini $\Delta \subset \mathbb{R}^d$, noté $P_\Delta^{z,\beta}$ est donnée par

$$P_\Delta^{z,\beta}(d\omega) = \frac{1}{Z_\Delta^{z,\beta}} e^{-\beta H(\omega)} \Pi_\Delta^z(d\omega)$$

où β est la température inverse, z est l'activité, H est l'hamiltonien, Π_Δ^z est la loi du processus de Poisson d'intensité z sur Δ et $Z_\Delta^{z,\beta}$ est la constante de normalisation aussi appelée fonction de partition. Cependant, comme nous voulons étudier les quantités macroscopiques et que nous souhaitons éviter les effets de bord, nous nous intéressons au processus ponctuel en volume infini. On définit alors les processus ponctuels de Gibbs sur $\mathbb{R}^d$ comme étant les solutions des équations de Dobrushin-Lanford-Ruelle (DLR) qui caractérisent l'état du système à l'équilibre thermodynamique. Ces équations donnent les lois conditionnelles des configurations dans un volume $\Delta \subset \mathbb{R}^d$ fini sachant la configuration η à l'extérieur de Δ, et dont l'expression est donnée par

$$P^{z,\beta}(d\omega_\Delta | \eta) = \frac{1}{Z_\Delta^{z,\beta}(\eta)} e^{-\beta H_\Delta(\omega_\Delta \cup \eta)} \Pi_\Delta^z(d\omega),$$

où ω_Δ est la restriction de la configuration au volume Δ et H_Δ est l'énergie locale d'une configuration et est définie par $H_\Delta(\omega) := H(\omega) - H(\omega_{\Delta^c})$. L'existence et l'unicité des solutions aux équations DLR ne sont pas toujours assurées. De plus, la non-unicité des mesures de Gibbs en volume infini est en fait liée au phénomène de transition de phase que nous allons étudier.

Jusqu'à présent, il existe peu de résultat démontrant la non-unicité des processus ponctuels de Gibbs à un seul type de particule (i.e. sans considération pour les spins). Le premier résultat de transition de phase liquide-gaz a été démontré pour le modèle de Widom-Rowlinson par Ruelle [30], Chayes, Chayes et Kotecký [3] et Giacomin, Lebowtiz, Maes [19]. L'hamiltonien pour ce modèle est

$$H(\omega) = \mathcal{V}(L_R(\omega)), \quad \text{où} \quad L_R(\omega) = \bigcup_{x \in \omega} B(x, R)$$

où $R > 0$, $L_R(\omega)$ est le halo des particules et $\mathcal{V}$ est la mesure de Lebesgue en dimension d. Ils démontrent l'existence d'une activité critique z_c telle que pour $\beta = z$ et $z > z_c$ on a une transition de phase liquide-gaz pour ce modèle. Les deux preuves utilisent les symétries du

modèle de Widom-Rowlinson à deux couleurs. Plus récemment, Dereudre et Houdebert [9] ont complété le diagrame de phase en démontrant l'unicité de la mesure de Gibbs en volume infini lorsque $z \neq \beta$ et hors d'un voisinage proche du point critique (z_c, z_c). Dans la littérature nous avons un deuxième résultat de transition liquide-gaz de Lebowitz, Mazel et Presutti [27] pour une interaction de Kac dont l'hamiltonien est donné par

$$H_\gamma(\omega) = -\frac{1}{2!} \sum_{\{x_1,x_2\}\subset\omega} \gamma^{2d} \mathcal{V}(\bigcap_{i=1}^{2} B(x_i, \gamma^{-1} R_d)) + \frac{1}{4!} \sum_{\{x_1,x_2,x_3,x_4\}\subset\omega} \mathcal{V}(\bigcap_{i=1}^{4} B(x_i, \gamma^{-1} R_d))$$

où $\gamma > 0$ est le paramètre de champ moyen. Dans leur preuve, ils font une adaptation au cadre continu de la théorie de Pirogov, Sinaï et Zaharadnik (PSZ). De la même manière, Pulvirenti et Tsagkarogiannis [33] ont montré que la transition de phase persiste lorsqu'on rajoute l'interaction hardcore au hamiltonien H_γ.

Initialement, la théorie PSZ a été developpée pour démontrer des transitions de phases dans le cas réseau et à basse température. C'est une généralisation de l'argument de Peierls à des modèles où les particules sur réseau interagissent à portée finie. A basse température le comportement du modèle peut être vu comme une perturbation du système à la température zéro absolu. A cette température, lorsque $\beta = +\infty$, le système adopte les configurations de plus basse énergie. Si on n'a pas unicité de ces configurations minimisant l'énergie, on retrouve à basse température des zones où les spins sont alignés, correspondant à l'un des minimiseurs de l'énergie. Ailleurs, les spins étant mélangés, on a un surplus d'énergie. Si ce surcoût énergétique est de l'ordre du volume des zones de mélange, c'est-à-dire que l'interaction vérifie les conditions de Peierls, alors on montre avec la théorie PSZ que la taille des zones d'inhomogénéité est très petite et qu'on a un spin dominant dans le système. Nous allons adapter la philosophie de cette méthode dans le cadre continu en nous appuyant sur les propriétés des interactions saturées. De fait, elle sera différente de l'adaptation faite par Lebowitz, Mazel et Presutti.

Dans cette thèse, nous nous intéressons à la classe des interactions saturées. Ces modèles présentent la bonne propriété que lorsqu'on considère une configuration suffisamment dense et homogène, dans le sens qu'on n'a pas de trop grand espace vide de particule, le coût énergétique pour ajouter un point est constant. On donne une définition plus détaillée d'interaction saturée dans la Section 1.3 du Chapitre 1. Cette classe d'interaction est relativement grande et contient des exemples intéressants. Un premier exemple est le modèle Quermass, dont l'hamiltonien est une combinaison linéaire des $d + 1$ fonctionnelles de Minkowski sur le halo des particules. En dimension 2, l'hamiltonien du Quermass est donné par

$$H(\omega) = a\mathcal{V}(L_R(\omega)) + b\mathcal{S}(L_R(\omega)) + c\,\chi(L_R(\omega)),$$

où $a, b, c \in \mathbb{R}$, $\mathcal{V}$ est la mesure de Lebesgue en dimension 2, $\mathcal{S}$ est la mesure du périmètre et χ est la caractéristique d'Euler-Poincaré. Cette interaction est une généralisation de l'interaction volumique du modèle Widom-Rowlinson. C'est un modèle qui a été introduit et étudié par Kendall, Van Lieshout et Baddeley [23]. Ils ont montré que le processus en volume infini existe lorsque que le rayon des boules est aléatoire mais borné. Ce résultat a été étendu par Dereudre en considérant des rayons aléatoires mais non bornés [5]. Ce modèle Quermass a la bonne propriété d'être saturé. En effet, soit $\Delta \subset \mathbb{R}^d$ de volume fini et ω une configuration telle que $\Delta \oplus B(0, R) \subset L_R(\omega)$

(où $\oplus$ est la somme de Minkowski) alors le coût énergétique de rajouter un point en $x \in \Delta$, $h(x,\omega) := H(\omega \cup \{x\}) - H(\omega) = 0$. Nous avons un deuxième exemple qui est la classe des interactions par paire diluées dont l'hamiltonien est

$$H(\omega) := \iint_{L_R(\omega)^2} \phi(|x-y|)dxdy$$

où ϕ est un potentiel d'interaction intégrable. Pour les mêmes raisons que le modèle Quermass, cette interaction est saturée. De plus, elle a un intérêt tout particulier car elle constitue une manière d'approximer les interactions par paire. En effet, on peut observer la chose suivante

$$\lim_{R\to 0} \frac{H(\omega)}{\mathcal{V}(B(0,R))^2} = \phi(0)\#\omega + 2\sum_{\{x,y\}\subset\omega} \phi(|x-y|).$$

Ainsi, obtenir des résultats de transition de phase liquide-gaz pour toute portée de dilution R serait une bonne nouvelle dans le but de démontrer une transition de phase pour des interactions par paire, comme pour l'interaction de Lennard-Jones dont la transition de phase a été observée par simulation [21] mais jamais démontrée rigoureusement.

Dans le Chapitre 1, nous définissons les processus de Gibbs marqués, qui constituent un cadre plus général permettant d'étudier des systèmes à plusieurs types de particules. Nous discutons du phénomène de transition de phase liquide-gaz. Notamment, nous établissons le lien entre le changement d'état et la régularité de la pression, et nous mettons en évidence le rôle des conditions aux bords. Ensuite, nous définissons les interactions saturées et en exposons les propriétés. Pour $\delta > 0$, nous définissons E_0 une fonction mesurable de l'espace des configurations dans $\mathbb{R} \cup \{+\infty\}$ tel que

$$H(\omega) = \sum_{i\in\mathbb{Z}^d} E_0(\tau_{-i\delta}(\omega))$$

où $\tau_{-i\delta}$ est la translation de vecteur $-i\delta$. La fonction E_0 correspond à l'énergie présente dans la case $T_0 = \left[-\frac{\delta}{2}, \frac{\delta}{2}\right[^d$. Une fois ce découpage case par case effectué, on dit que E_0 est saturée s'il existe $L > 0$ et une fonction $\overline{E}$ de $\mathbb{N}$ dans $\mathbb{R} \cup \{+\infty\}$ telle que pour toute configuration ω ayant un point dans chaque case à une distance L de T_0, on a $E_0(\omega) = \overline{E}(\#\omega \cap T_0)$. Ainsi, l'énergie d'une case saturée ne dépend plus de la position des particules mais uniquement de leur nombre dans celle-ci. Dans le cadre des interactions à portée finie, nous démontrons le lemme suivant, qui fait le lien avec l'intuition initiale de la propriété de saturation.

Lemme. *Si E_0 est une fonction locale sur $T_0 \oplus B(0,R)$ avec $R > 0$, alors il existe $A \in \mathbb{R}$ et $B \in \mathbb{R} \cup \{+\infty\}$ tel que*

$$\overline{E}(k) = (Ak + B)\mathbb{1}_{k\geq 1}.$$

Pour finir, nous présentons des résultats de simulations pour différentes interactions saturées qui sont obtenus en utilisant l'algorithme de Metropolis-Hastings. Nous illustrons les transitions

de phase pour les modèles Widom-Rowlinson et Quermass, ainsi que l'absence de ce phénomène pour l'interaction de Strauss à K-voisins, mettant en évidence que la saturation seule n'est pas suffisante.

Dans le Chapitre 2, nous introduisons les outils et objets nécessaires à la mise en oeuvre de la théorie PSZ. Notamment, nous définissons les contours, notés γ, qui représentent les zones connexes d'inhomogénéité créées par une configuration. Nous démontrons dans ce chapitre le résultat de transition de phase liquide-gaz suivant.

Théorème. *Soit H une interaction saturée, telle que E_0 ait de bonnes propriétés. On sait que pour tout $k \in \mathbb{N}$, $\overline{E}(k) = (Ak + B)\mathbb{1}_{k\geq 1}$ et on suppose que $A \geq 0$ et $-A \leq B < +\infty$. On suppose aussi que l'interaction vérifie les conditions de Peierls, i.e. l'interaction exhibe un surplus d'énergie dans les contours qui est de l'ordre du volume du contour (voir détails dans le Théorème 11). Pour tout $\beta > 0$, on pose z_β^- et z_β^+ de la façon suivante*

$$z_\beta^- := \frac{e^{\beta A}}{\delta^d} \ln\left(1 + e^{\beta B - 2}\right), \quad z_\beta^+ := \frac{e^{\beta A}}{\delta^d} \ln\left(1 + e^{\beta B + 2}\right)$$

et $U_\beta := (z_\beta^-, z_\beta^+)$. Alors il existe $\beta_c > 0$ tel que pour $\beta \geq \beta_c$ il existe $z_\beta^c \in U_\beta$ pour lequel on a une transition liquide-gaz. Plus précisément, on montre que la pression ψ vérifie

$$\frac{\partial \psi}{\partial z^+}(\beta, z_c) > \frac{\partial \psi}{\partial z^-}(\beta, z_c),$$

et on a deux mesures de Gibbs $P^+, P^- \in \mathcal{G}(H, z_\beta^c, \beta)$ telles que

$$\rho(P^+) := E_{P^+}(N_{[0,1]^d}) = z + z\frac{\partial \psi}{\partial z^+}(z_\beta^c, \beta) \quad \textit{and} \quad \rho(P^-) := E_{P^-}(N_{[0,1]^d}) = z + z\frac{\partial \psi}{\partial z^-}(z_\beta^c, \beta).$$

Les conditions de Peierls étant souvent très difficiles à vérifier, nous donnons une méthode simple permettant de les vérifier à l'aide de dominos. Les dominos sont constitués de cases adjacentes dont l'une est vide de point et pas l'autre. Nous appliquons cette méthode au modèle de Widom-Rowlinson à rayon aléatoire borné et démontrons un nouveau résultat de transition de phase liquide-gaz pour ce modèle. Le reste du chapitre est dédié à la preuve du théorème présenté ici.

Dans le chapitre 3, nous présentons deux résultats de transition de phase liquide-gaz. Ces résultats sont des conséquences du théorème démontré dans le chapitre 2. Dans cette partie l'objet sera de démontrer que pour les interactions considérées on vérifie effectivement les bonnes propriétés de saturation ainsi que les conditions de Peierls.

Dans un premier temps, nous étudions la transition de phase pour le modèle Quermass à rayon aléatoire mais borné. C'est un processus ponctuel de Gibbs marqué dans $\mathbb{R}^d$ avec des marques correspondant au rayon des boules constituant le halo et à valeur dans $[R_0, R_1]$ (avec $0 < R_0 \leq R_1$). L'hamiltonien qui va être considéré est

$$H(\omega) = \begin{cases} \mathcal{V}(L(\omega)) + \theta_1 S(L(\omega)) - \theta_2 \chi(L(\omega)) & \text{si } (d = 2) \\ \mathcal{V}(L(\omega)) + \theta_1 S(L(\omega)) & \text{si } (d \geq 3) \end{cases},$$

où $L(\omega) = \bigcup_{(x,R)\in\omega} B(x, R)$ et $\theta_1 \in \mathbb{R}$ et $\theta_2 \geq 0$. Nous montrons qu'il existe deux paramètres critiques $\theta_1^* = \frac{R_0}{d}$ et $\theta_2^*(\theta_1)$ (voir (3.4)) de telle sorte que pour $\theta_1 > -\theta_1^*$ et $0 \leq \theta_2 < \theta_2^*(\theta_1)$ cet hamiltonien vérifie les conditions de Peierls. En dimension $d \geq 3$, on prend toujours $\theta_2 = 0$. Ainsi nous obtenons le résultat suivant.

Théorème. *Soit $\theta_1 > -\theta_1^*$ et $0 \leq \theta_2 < \theta_2^*(\theta_1)$ ($\theta_2 = 0$ si $d \geq 3$). Alors il existe $\beta_c(\theta_1, \theta_2) > 0$ tel que pour tout $\beta > \beta_c(\theta_1, \theta_2)$ il existe $z_\beta^c > 0$ pour lequel on a une transition liquide-gaz. De plus, on sait qu'il existe $c > 0$ tel que $|z_\beta^c - \beta| = O(e^{-c\beta})$.*

Enfin, nous nous intéressons aux interactions par paire diluées avec un potentiel d'interaction ϕ. Comme expliqué précédemment, cette famille constitue une approximation de l'interaction par paire usuelle et est paramétrée par la portée de dilution R. Dans la section 3.2, nous démontrons ce dernier résultat, qui, nous l'espérons, sera notre porte d'entrée pour étudier les transitions de phase d'interaction par paire.

Théorème. *Soit $R > 0$ et $\phi \in L^1(\mathbb{R}^d)$ qui est radial, de portée finie R_1 et tel que $R_1 \geq R_0 := \sup\{r \in \mathbb{R}_+, \phi(r) > 0\} > 0$. De plus, on suppose que ϕ vérifie*

$$C_d \int_{B(0,R)} \phi^+ dx \geq \left[\left(\frac{R_0}{R}\right)^d - 1\right] \int_{B(0,R_0)\setminus B(0,R)} \phi^+ dx + \int_{\mathbb{R}^d} \phi^- dx$$

où $C_d > 0$ est une constante dépendant uniquement de la dimension. Alors il existe $\beta_c > 0$ tel que pour $\beta > \beta_c$ il existe $z_\beta^c > 0$ pour lequel on a une transition liquide gaz. De plus, on obtient qu'il existe $c > 0$ tel que

$$\left| z_\beta^c - \beta \int_{\mathbb{R}^d} \phi dx \right| = O(e^{-c\beta}).$$

Pour ϕ non-intégrable et positive à l'origine, comme par exemple le potentiel de Lennard-Jones, pour tout $R > 0$, nous pouvons tronquer le potentiel d'interaction pour obtenir ϕ_R, qui satisfait les hypothèses du théorème ci-dessus. Dans les perspectives de cette thèse, nous discutons des difficultés qui devront être surmontées afin de démontrer la transition de phase pour les interactions par paire en utilisant les interactions par paire diluées.

Chapter 1

Gibbs point process and saturated interaction

In this chapter, we give a general definition of a Gibbs marked point process. Initially, we define it in finite volume where the un-normalised density with respect to the Poisson point process is given by the Boltzmann factor $e^{-\beta H}$, with β representing the inverse temperature and H the energy functional, also called the Hamiltonian. Subsequently, we define the infinite volume Gibbs point process as solution to the DLR equations, these relationships reflects the behaviour of a system at equilibrium. In section 1.2, we discuss the conditions necessary for the existence of the infinite Gibbs measure and explore the question of its uniqueness. We showcase the relationship between solutions to the DLR equations with the minimisers of the free energy using the variational principle. In some context, the non-uniqueness of Gibbs measures is related to the phenomenon of Liquid-Gas phase transition. We elucidate the connection between this occurrence and the non-differentiability of the pressure. In section 1.3, we provide a definition of a saturated interaction which forms the framework within which we investigate the occurrence of the Liquid-Gas phase transition.

1.1 Gibbs marked point process

We denote by $\mathcal{B}_b(\mathbb{R}^d)$ the set of bounded Borel sets of $\mathbb{R}^d$ with positive Lebesgue measure. For any sets A and B in $\mathcal{B}_b(\mathbb{R}^d)$, $A \oplus B$ stands for the Minkowski sum of these sets. We denote by S a polish space that describe the spin state of particles. We denote by E the state space of a single marked point defined as $\mathbb{R}^d \times S$. For any $(x, s) \in \mathbf{E}$, the first coordinate x is for the location of the point and the second coordinate s is the mark representing the spin of a particle. For any set $\Delta \in \mathcal{B}_b(\mathbb{R}^d)$, E_Δ is the local state space $\Delta \times S$. A configuration of marked points ω is a locally finite set in E; i.e. $N_\Delta(\omega) := \#(\omega \cap E_\Delta)$ is finite for any $\Delta \in \mathcal{B}_b(\mathbb{R}^d)$. We denote by Ω the set of all marked point configurations and by Ω_f its restriction to finite configurations. For any $\omega \in \Omega$, its projection in $\Delta \subset \mathbb{R}^d$ is defined by $\omega_\Delta := \omega \cap E_\Delta$. We equip the state space Ω with the σ-algebra $\mathcal{F}$ generated by the counting functions on E, N_Δ for $\Delta \subset \mathbb{R}^d$ bounded. We consider the reference measure $\lambda \otimes P_S$ on $\mathbb{R}^d \times S$ where λ is the Lebesgue measure on $\mathbb{R}^d$ and

P_S a probability measure on the spin space S of the particle.

1.1.1 Hamiltonian and local energy

Gibbs point process arises naturally from statistical mechanics and physical models. It describes the physical system at equilibrium. In order to introduce Gibbs point processes we need to introduce the Hamiltonian. This functional describes the interaction between the particles and can encode the attraction or repulsion between particles at different scales.

Definition 1. The Hamiltonian or energy is a measurable function $H : \Omega_f \to \mathbb{R} \cup \{+\infty\}$ such that :

- $H(\emptyset) < +\infty$ (non-degenerate)
- $\exists A \geq 0, \forall \omega \in \Omega_f, H(\omega) \geq -AN(\omega)$ (stability)
- If $H(\omega) = +\infty \implies \forall x \in E, H(\omega \cup \{x\}) = +\infty$ (hereditary).

The stability condition means that the energy is at least linear, this assumption is natural as it is satisfied by most physical models. The hereditary condition means that the set of forbidden configurations is stable by addition of a new particle in the system. Finally the non-degeneracy condition is necessary otherwise by heredity the energy is equal to infinity everywhere.

Morphological interaction

We can define an Hamiltonian that depends on the morphology of a geometric object derived from the position of the particles. For example several interesting models are based upon a germ grain structure, also called the halo of the particle. We define the halo of a configuration $\omega \in \Omega$ by

$$L(\omega) = \bigcup_{(x,R)\in\omega} B(x, R)$$

where $R \in S \subset R_+$ and is the mark which corresponds to the size of each particle. The first interesting example is the Area interaction, also known as the Widom-Rowlinson model, that was introduced to model micro-emulsions or colloids with inter-penetrable particles [34] and the Hamiltonian is given by

$$H(\omega) = \mathcal{V}(L(\omega)) \tag{1.1}$$

where $\mathcal{V}$ is the Lebesgue measure on $\mathbb{R}^d$ and thus gives the volume of the halo. This model is popular as it is one the few models where phase transition has been proved [3, 9, 30], we will discuss this in more detail in Section 1.2.3. If we replace the Volume functional by a linear combination of Minkowski functionals we get the Quermass interaction like introduced in [23]. We are particularly interested in the following Hamiltonian,

$$H(\omega) = \mathcal{V}(L(\omega)) + \theta_1 S(L(\omega)) + \theta_2 \chi(L(\omega)) \tag{1.2}$$

where $S(L(\omega))$ is the $d-1$-dimensional Haussdorf measure of the boundary $\partial L(\omega)$ and $\chi(L(\omega)$ is the Euler-Poincaré characteristic of the halo. The stability assumption is not always guaranteed in this case, especially the addition of the Euler-Poincaré characteristic can yield un-stability of the Hamiltonian in higher dimension [23].

Pairwise interaction

The pairwise interaction is the most common type of interaction, as it is the most natural one to consider. For example, it includes the gravitational interaction for particles with mass and the Coulomb interaction for charged particles. In this context, the Hamiltonian is given by

$$H(\omega) = \sum_{\{x,y\}\subset\omega} \phi(|x-y|) \tag{1.3}$$

where ϕ is called the pair potential and is a measurable function $\phi : \mathbb{R}_+ \to \mathbb{R}\cup\{+\infty\}$. Under these assumptions the Hamiltonian is non-degenerate and hereditary but the stability assumption is not always guaranteed.

We have the important class of Riesz gas models, which is currently an active domain of research. These models describes the physics of charged particles. The pair potential is given by $\phi(r) = r^{-s}$ when $s \neq 0$ and $\phi(r) = -\ln(r)$ when $s = 0$. The parameter s encodes the weights of the interactions at long range. Usually we call the case $s > d$ the short range regime and $s \leq d$ the long range regime. The case where $s = d-2$ is the Coulomb interaction. The log-gas which is the case when $s = 0$ is not stable unless we add the interaction between the particles and a background that is used to balance the electric charges of the overall system.

Another example of pairwise interaction is the Lennard-Jones pair potential given by $\phi(r) = ar^{-12} + br^{-6}$ with $a > 0$ and $b \in \mathbb{R}$. This Hamiltonian describe the behaviour of noble gas fluids. The stability of this interaction is not direct when b is negative, but we can show that for $d \leq 5$ the stability is proved using Proposition 3.2.8 in [31].

The Strauss potential is another class of pair potential, given by $\phi(r) = a\mathbb{1}_{r\leq R}$ with $R > 0$ and $a \in \mathbb{R}\cup\{+\infty\}$. The special case where $a = +\infty$ is the hardcore interaction, or non penetrable spheres. Whenever $a \geq 0$ the potential is positive and therefore it is stable. On the contrary if $a < 0$ the interaction is no longer stable. Indeed, for any configuration $\omega \in \Omega_f$ such that all the particles are inside the ball $B(0, R/2)$ we have $H(\omega) = a/2 N(\omega)(N(\omega)-1)$.

Approximation of pairwise interaction

There are several ways to approximate the pairwise interaction. One way to do so is to restrict the interaction to a finite number of nearest neighbours. Let $n \in \mathbb{N}$, given a configuration $\omega \in \Omega_f$, a point $y \in \omega$ is the n-th neighbour of $x \in \omega$ if $\#B(x, |x-y|)\cap\omega = n$ and we denote by $y = v_n(x,\omega)$. In case there are several points at the same distance to x we can list these points using the lexicographical order on the cartesian coordinate. Given a pair potential ϕ the K-nearest neighbour Hamiltonian is given by

$$H_K(\omega) = \begin{cases} \sum_{x\in\omega} \sum_{m=1}^{\min(K,N(\omega)-1)} \phi(|x - v_m(x,\omega)|) & \text{if } N(\omega)\geq 2 \\ 0 & \text{otherwise} \end{cases}. \tag{1.4}$$

In this setting it is easier to obtain the stability of the Hamiltonian. For instance, let us consider a lower bounded pair potential : i.e. $\exists A \geq 0, \phi \geq -A$. Therefore we obtain by direct minoration

$$H_K(\omega) \geq -KAN(\omega).$$

Furthermore it is clear that as K tends to ∞ we obtain the classic pairwise interaction.

Another way to approximate the pairwise interaction is the mean of the pair potential over the volume of each particle, where the particle is a ball of radius $R > 0$. Given a pair potential ϕ the Hamiltonian is given by

$$H_m(\omega) = \sum_{\{x,y\}\subset\omega} \iint_{B(x,R)\times B(y,R)} \phi(|u-v|)du\,dv. \tag{1.5}$$

If we do a rescaling by dividing the potential by $\mathcal{V}(B(0,R))^2$ and take R to 0 we obtain the classical pairwise potential. In reality, this type of interaction is a pairwise interaction with a pair potential

$$\psi(x,y) = \iint_{B(x,R)\times B(y,R)} \phi(|u-v|)dudv.$$

This approach is used to smooth out the pair potential when it present any singularity. However it is often more interesting to regularize a potential with singularity using a C^∞ mollifier on the unit disk, such approach has been fruitful in demonstrating a law of large number for the max of the 2D Coulomb-gas potential [25].

Proposition 1. *Let ϕ a pair potential such that the pairwise interaction H is stable, then H_m is also stable.*

Proof. Let $A \geq 0$ the stability constant for H. For $\omega = \{x_1, \dots, x_n\} \in \Omega_f$, we construct $(X_i)_{1\leq i\leq n}$ independent random variable where X_i is a uniform random variable on $B(x_i, R)$. Therefore the Hamiltonian H_m can be written as

$$\begin{aligned} H_m(\omega) &= \lambda(B(0,R)) \sum_{\substack{i,j \\ i\neq j}} \mathbb{E}\left(\phi(|X_i - X_j|)\right) \\ &= \lambda(B(0,R))\,\mathbb{E}\left(\sum_{\substack{i,j \\ i\neq j}} \phi(|X_i - X_j|)\right) \\ &= \lambda(B(0,R))\mathbb{E}\left(H(\{X_1,\dots,X_n\})\right) \geq -A\lambda(B(0,R))N(\omega). \end{aligned}$$

□

Finally, another interesting way to approximate the pairwise interaction is the class of diluted pairwise interaction. The idea is similar to the previous example but instead of interacting between balls we look at the average interaction on the halo of particles. Therefore the Hamiltonian is given by

$$H(\omega) = \iint_{L_R(\omega)^2} \phi(|u-v|)dudv. \tag{1.6}$$

If we do the same rescaling as in the previous example, we obtain the following

$$\lim_{R\to 0} \frac{H(\omega)}{\mathcal{V}(B(0,R))^2} = N(\omega)\phi(0) + 2\sum_{\{x,y\}\subset\omega} \phi(|x-y|).$$

The term $N(\omega)\phi(0)$ is the energy coming from the self interaction of the particles. This can be interpreted as a modification of the activity z of the underlying Poisson point process. With this definition it is necessary that ϕ is integrable at the origin, otherwise it is not interesting since the energy will always be equal to $+\infty$ when we have at least one point.

Proposition 2. *Let ϕ be an integrable pair potential in $\mathbb{R}^d$ then H is bi-stable, i.e. there is $C > 0$ such that*

$$|H(\omega)| \leq CN(\omega).$$

Proof. Since ϕ is $L^1(\mathbb{R}^d)$, for any configuration $\omega \in \Omega_f$ we have

$$\begin{aligned}
|H(\omega)| &\leq \iint_{L_R(\omega)^2} |\phi(|y-x|)|dydx \\
&\leq \int_{L_R(\omega)} \int_{\mathbb{R}^d} |\phi|dydx \\
&\leq \mathcal{V}(L_R(\omega)) \int_{\mathbb{R}^d} |\phi|dy \\
&\leq \mathcal{V}(B(0,R))N(\omega) \int_{\mathbb{R}^d} |\phi|dy.
\end{aligned}$$

□

Definition 2. The Hamiltonian H is said to be stationary, if for all vector $u \in \mathbb{R}^d$ and any finite configuration $\omega \in \Omega_f$,

$$H(\tau_u(\omega)) = H(\omega)$$

where τ_u is the translation by vector u.

All the Hamiltonian presented previously are stationary. It is a way to consider a system of particles that is free from any external potential field like for example the gravitational field. Therefore, we are only considering the internal interaction between the particles in the system. Another quantity of interest is the local energy of a configuration. It is a quantity of a particular interest, as it is useful to give a description of the system through equilibrium equations known as the Dobrushin-Lanford-Ruelle (DLR) equations.

Definition 3. Let H be a stable, non-degenerate and hereditary Hamiltonian and $\Delta \in \mathcal{B}_b(\mathbb{R}^d)$ the local energy on Δ of a configuration $\omega \in \Omega_f$ is given by

$$H_\Delta(\omega) := H(\omega) - H(\omega_{\Delta^c})$$

with the convention $\infty - \infty = 0$.

Since the Hamiltonian is hereditary, the local energy is never equal to $-\infty$. This quantity can be interpreted as the energy cost given a configuration outside of Δ to transport particles from infinity to ω_Δ.

Definition 4. An Hamiltonian has a finite range $R \geq 0$, if for any $\Delta \in \mathcal{B}_b(\mathbb{R}^d)$ the local energy H_Δ is a local functional on $\Delta \oplus B(0, R)$; i.e. for any finite configuration $\omega \in \Omega_f$

$$H_\Delta(\omega) = H_\Delta(\omega_{\Delta \oplus B(0,R)}).$$

This definition is the richest as we will see down the road. Indeed there is another way to define the finite range interaction through some additivity property : for any configurations $\omega_1, \omega_2 \in \Omega_f$ such that $d_2(\omega_1, \omega_2) > R$ then $H(\omega_1 \cup \omega_2) = H(\omega_1) + H(\omega_2)$. It might seems to be a natural definition as this additivity of the Hamiltonian implies that the configurations ω_1 and ω_2 are non interacting and independent one from another. But these two definitions are not equivalent. The first definition under the assumption that $H(\emptyset) = 0$ implies the later. Indeed, when $\omega_1 \subset \Delta$ and $\omega_2 \in (\Delta \oplus B(0, R))^c$ we have

$$H_\Delta(\omega_1 \cup \omega_2) = H_\Delta(\omega_1) = H(\omega_1) - H(\emptyset).$$

And by definition we have

$$H_\Delta(\omega_1 \cup \omega_2) = H(\omega_1 \cup \omega_2) - H(\omega_2).$$

In summary, we obtain that

$$H(\omega_1 \cup \omega_2) = H(\omega_1) + H(\omega_2).$$

On the contrary, the additivity definition does not imply the proper definition. We consider the following interaction

$$H(\omega) = N_{cc}(L_R(\omega))$$

where N_{cc} counts the number of connected components. This interaction satisfies the additivity definition whenever the distance between two configurations is greater than $2R$, but according to the proper definition the range of this interaction is infinite. Indeed, for a configuration $\omega = \{x_0, \dots, x_{2n+3}\} \subset \mathbb{R}^d$ such that $x_i = (iR, 0, \dots, 0)$ for $i \leq n$, $x_{n+1} = (nR, R, 0, \dots, 0)$, $x_{n+2} = (nR, 2R, 0, \dots, 0)$ and $x_j = ((2n+3-j)R, 3R, 0, \dots, 0)$ for $j \geq n+3$, for any $r < (n-1)R$ we have $H_{[0,3R]^d}(\omega) = 0$ whereas $H_{[0,3R]^d}(\omega_{[0,3R]^d \oplus B(0,r)}) = 1$.

Among the examples of Hamiltonian previously cited some further condition is needed. For example, in the case of the morphological interaction of the Quermass type. The only assumption needed to obtain a finite range interaction is that the spin state is of the form $S = [R_0, R_1]$ where $0 \leq R_0 \leq R_1$. For the pairwise interaction and all the associated approximation, if the support of the pair potential is bounded then the overall interaction has a finite range. More generally, in this work, we consider only finite range interactions.

1.1.2 Finite Volume Gibbs point process and DLR equations

In order to study the state of a physical system at a microscopic scale we define the Gibbs point process in a finite volume. The law of the point process describes the state of the system of interacting particles at equilibrium. Before that we introduce the Poisson point process which is the most natural way to produce independent points in a marked space $\mathbb{R}^d \times S$.

Definition 5. Let μ be a sigma-finite measure on $\mathbb{R}^d \times S$. A Poisson point process with intensity μ is a point process Γ that verifies the following properties.

- For any bounded subset E in $\mathbb{R}^d \times S$, the random variable $N_E(\Gamma)$ is distributed following a Poisson distribution with parameter $\mu(E)$.
- For any finite sequence of disjoint bounded subsets in $\mathbb{R}^d \times S$, $E_1, \dots, E_n$, the random variables $N_{E_1}(\Gamma), \dots, N_{E_n}(\Gamma)$ are independent.

The distribution of the Poisson point process with intensity μ is denoted by Π^μ. When the intensity is $\mu = z\lambda^d \otimes P_S$ it is the stationary Poisson point process with intensity $z > 0$ and we denote its distribution with Π^z. For any measurable set $\Delta \subset \mathbb{R}^d$, we denote by Π^z_Δ the distribution of the Poisson point process with intensity $z\lambda^z_\Delta \otimes P_S$ which is also the restriction of the Poisson point process to Δ. The Poisson point process is the model for ideal gases where the particles are not interacting. Now we introduce the finite volume Gibbs point process to develop models of systems of interacting particles at equilibrium.

Definition 6. Let $\Delta \in \mathcal{B}_b(\mathbb{R}^d)$, the finite volume Gibbs measure on Δ with activity $z > 0$, inverse temperature $\beta \geq 0$ and Hamiltonian H is the distribution

$$P^{z,\beta}_\Delta = \frac{1}{Z^{z,\beta}_\Delta} e^{-\beta H} \Pi^z_\Delta, \tag{1.7}$$

where $Z^{z,\beta}_\Delta$, called the partition function, is the normalisation constant $\int e^{-\beta H} \Pi^z_\Delta$. A finite Gibbs point process (GPP) on Δ is a point process with distribution $P^{z,\beta}_\Delta$.

The distribution $P^{z,\beta}_\Delta$ is well defined since the partition function is positive and finite. Since the Hamiltonian is non-degenerate we have that

$$Z^{z,\beta}_\Delta \geq e^{-\beta H(\emptyset)} \Pi^z_\Delta(\emptyset) > 0.$$

On the other hand, the stability of the Hamiltonian yields

$$Z^{z,\beta}_\Delta \leq \int e^{\beta A N(\omega)} \Pi^z_\Delta(d\omega) = \exp(-z\lambda(\Delta)(e^{\beta A} - 1)).$$

The Gibbs measure verify DLR equations, that are due to Dobrushin Lanford and Ruelle, which gives the local distribution of a GPP inside a bounded window Λ given the position of particles in Λ^c.

Proposition 3 (DLR equations)**.** *For any $\Lambda \subset \Delta$ such that $\lambda(\Lambda) > 0$, for any bounded measurable function $f : \Omega \mapsto \mathbb{R}_+$ we have*

$$\int f(\omega) P_\Delta^{z,\beta}(\omega) = \iint \frac{1}{Z_\Lambda^{z,\beta}(\omega_{\Lambda^c})} f(\omega'_\Lambda \cup \omega_{\Lambda^c}) e^{-\beta H_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c})} \Pi_\Lambda^z(d\omega'_\Lambda) P_\Delta^{z,\beta}(d\omega) \tag{1.8}$$

where $Z_\Lambda^{z,\beta}(\omega_{\Lambda^c})$ is the normalisation constant $\int e^{-\beta H_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c})} \Pi_\Lambda^z(d\omega'_\Lambda)$. Equivalently, the local conditional marginal distribution of a GPP is given for $P_\Delta^{z,\beta}$-a.s all ω_{Λ^c} by

$$P_\Delta^{z,\beta}(\omega'_\Lambda | \omega_{\Lambda^c}) = \frac{1}{Z_\Lambda^{z,\beta}(\omega_{\Lambda^c})} e^{-\beta H_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c})} \Pi_\Lambda^z(d\omega'_\Lambda). \tag{1.9}$$

Proof. By definition of the local energy H_Λ and the independence of Poisson point process in disjoint areas we have

$$\begin{aligned} P_\Delta^{z,\beta}(d\omega) &= \frac{1}{Z_\Delta^{z,\beta}} e^{-\beta H(\omega)} \Pi_\Delta^z(d\omega) \\ &= \frac{1}{Z_\Delta^{z,\beta}} e^{-\beta H_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c})} e^{-\beta H(\omega_{\Lambda^c})} \Pi_\Lambda^z(d\omega'_\Lambda) \Pi_{\Delta\setminus\Lambda}^z(d\omega_{\Lambda^c}). \end{aligned}$$

Therefore $P(\omega'_\Lambda | \omega_{\Lambda^c})$ is absolutely continuous with respect to Π_Λ^z and the conditional density is given by

$$P(\omega'_\Lambda | \omega_{\Lambda^c}) = \frac{1}{Z_\Lambda^{z,\beta}(\omega_{\Lambda^c})} e^{-\beta H_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c})} \Pi_\Lambda^z(d\omega'_\Lambda).$$

□

The DLR equations can be seen as equilibrium equations of the system. It gives a way to locally resample the GPP that preserves the overall distribution.

1.1.3 Infinite Volume Gibbs point process

In order to study a system of particles in the bulk we need the infinite volume GPP which corresponds to the case where "$\Lambda = \mathbb{R}^d$". The previous Definition 6 fails as the Hamiltonian for an infinite configuration is meaningless. One way to construct the infinite volume is by considering a sequence of stationarised finite volume GPP on $\Delta_n = [-n, n]^d$. The limit of such sequence, if it exists is what we call the thermodynamic limit. We would want that the infinite volume GPP verify the same equilibrium equations. Therefore we define the infinite process using the DLR equations.

Definition 7. A probability measure P on Ω is a Gibbs measure for the Hamiltonian H, activity $z > 0$ and inverse temperature $\beta \geq 0$ if P is stationary in space (i.e. invariant for any translation) and if for any $\Delta \in \mathcal{B}_b(\mathbb{R}^d)$ and any bounded measurable function $f : \Omega \to \mathbb{R}$,

$$\int f(\omega) P(d\omega) = \int \int \frac{1}{Z_\Delta(\omega_{\Delta^c})} f(\omega'_\Delta \cup \omega_{\Delta^c}) e^{-\beta H_\Delta(\omega'_\Delta \cup \omega_{\Delta^c})} \Pi_\Delta^z(d\omega'_\Delta) P(d\omega) \tag{1.10}$$

where $Z_\Delta(\omega_{\Delta^c})$ is the partition function given the outer configuration ω_{Δ^c}

$$Z_\Delta(\omega_{\Delta^c}) = \int e^{-\beta H_\Delta(\omega'_\Delta \cup \omega_{\Delta^c})} \Pi^z_\Delta(d\omega').$$

We denote by $\mathcal{G}(H, z, \beta)$ the set of all Gibbs measures.

This definition brings up natural questions which are the existence of such probability measure and if it exists whether if it is unique. In the setting of finite range Hamiltonian, the following theorem proves the existence of the infinite GPP.

Theorem 1 (Theorem 1 [4])**.** *Let H be a stationary and finite range Hamiltonian, then for any $z > 0$ and $\beta \geq 0$, $\mathcal{G}(H, z, \beta)$ is a non-empty set.*

For infinite-range interactions, significant results by Ruelle [32] demonstrate the existence of the infinite volume GPP for pairwise interactions when the pair potential is both regular and superstable. Recent work has extended these results by establishing the existence of the infinite volume GPP for larger class of pairwise interaction by relaxing the superstable assumption [8]. For infinite-range morphological interactions, such as the Quermass interaction with unbounded radii (i.e. when $S = \mathbb{R}_+$), the existence of the infinite GPP has been proved under certain assumptions on P_S in [5]. The proof of Theorem 1 can be found in this introductory course on Gibbs point process [4], and for such it will not be done here. Nonetheless, we will discuss about the tools needed in this proof as it will be useful down the line.

Definition 8. A function $f : \Omega \to \mathbb{R}$ is a local tame function if there is a bounded set $\Delta \in \mathcal{B}_b(\mathbb{R}^d)$ and a constant $A \geq 0$ such that $f(\omega) = f(\omega_\Delta)$ and $|f(\omega)| \leq A N_\Delta(\omega)$.

We define the local convergence topology as it is the setting we use to prove the existence of accumulation point for the sequence of stationarised finite volume GPP on Δ_n.

Definition 9. The local convergence topology on the space of probability measures on Ω is the smallest topology such that for any local tamed function f the application $: P \to \int f dP$ is continuous. We denote by $\tau_{\mathcal{L}}$ this topology.

In this setting of local convergence topology, we use the specific entropy to prove the tightness of a collection of probability measures. For a stationary probability P on Ω the specific entropy is given by

$$I_\xi(P) = \lim_{n \to +\infty} \frac{1}{\lambda(\Delta_n)} I(P_{|\Delta_n} | \Pi^\xi_{\Delta_n}) \tag{1.11}$$

where $I(P_{|\Delta_n} | \Pi^\xi_{\Delta_n})$ is the relative entropy of $P_{|\Delta_n}$, the marginal distribution of P on Δ_n, with respect to $\Pi^\xi_{\Delta_n}$. For P and Q two probability measures, the relative entropy or the Kullback-Leibler divergence of P with respect to Q is defined as

$$I(P|Q) = \begin{cases} \int f \ln(f) dQ & \text{if } P << Q, \text{with } f = \frac{dP}{dQ} \\ +\infty & \text{otherwise} \end{cases}.$$

The limit always exists and is a consequence of a sub-additivity property of the relative entropy, for more detail see Chapter 15 in [14]. The specific entropy verify the following properties, the proof of which is also available in Chapter 15 of [14].

Proposition 4. *The specific entropy verify the following properties.*

- *For any stationary probability measure P on Ω,*

$$I_\xi(P) = \sup_{\Delta \in \mathbb{B}_b(\mathbb{R}^d)} \frac{1}{\lambda(\Delta)} I(P_\Delta | \Pi^\xi_\Delta).$$

  Therefore, the specific entropy does not depend on the sequence $(\Delta_n)_{n\in\mathbb{N}}$.

- *The specific entropy I_ξ is affine.*

Furthermore, we have the following fundamental proposition which essentially gives our tightness tool.

Proposition 5 (Proposition 2.6, Georgii-Zessin [17])**.** *For any $\xi > 0$ and $K \geq 0$, the set*

$$\{P \in \mathcal{P}, I_\xi(P) \leq K\}$$

is sequentially compact for the local convergence topology, where $\mathcal{P}$ is the space of stationary probability measures on Ω with finite intensity.

The finite volume Gibbs measure is not a stationary probability measure therefore it is not suited in the local convergence topology. For $\Delta_n = [-n, n]^d$ we construct the empirical field, denoted by $\overline{P}^{z,\beta}_{\Delta_n}$, associated to the finite volume Gibbs measure on Δ_n. The empirical field is the stationarised version of the finite volume GPP on Δ_n and it is defined as the probability measure on Ω such that for any test function f

$$\int f(\omega)\overline{P}^{z,\beta}_{\Delta_n}(d(\omega)) = \frac{1}{\lambda(\Delta_n)} \int_{\Delta_n} \int_\Omega f(\tau_u(\omega)) \widehat{P}^{z,\beta}_{\Delta_n}(d\omega)du \quad \text{where } \widehat{P}^{z,\beta}_{\Delta_n} = \otimes_{i\in\mathbb{Z}^d} P^{z,\beta}_{\tau_{2ni}(\Delta_n)}. \tag{1.12}$$

Any accumulation point of the sequence $\left(\overline{P}^{z,\beta}_{\Delta_n}\right)_{n\geq 1}$ is going to be stationary by construction.

Proposition 6. *The sequence $\left(\overline{P}^{z,\beta}_{\Delta_n}\right)_{n\geq 1}$ is tight for the local convergence topology if there exists $\xi > 0$ such that*

$$\sup_{n\geq 1} \frac{1}{\lambda(\Delta_n)} I(P^{z,\beta}_{\Delta_n} | \Pi^\xi_{\Delta_n}) < +\infty. \tag{1.13}$$

Proof. We know that the specific entropy is affine, therefore we obtain that

$$I_\xi(\overline{P}^{z,\beta}_{\Delta_n}) = \frac{1}{\lambda(\Delta_n)} \int_{\Delta_n} I_\xi(\widehat{P}^{z,\beta}_{\Delta_n} \circ \tau_{-u})du.$$

Since the specific entropy does not depend on the sequence of bounded areas, for each $\widehat{P}^{z,\beta}_{\Delta_n} \circ \tau_{-u}$ we can choose the sequence $\left(\tau_{-u}(\Delta_{(2k-1)n})\right)_{k\geq 1}$ and thus

$$\begin{aligned} I_\xi(\widehat{P}^{z,\beta}_{\Delta_n} \circ \tau_{-u}) &= \lim_{k\to+\infty} \frac{1}{(2k-1)^d \lambda(\Delta_n)} I(\widehat{P}^{z,\beta}_{\Delta_n | \Delta_{(2k-1)n}} | \Pi^\xi_{\Delta_{(2k-1)n}}) \\ &= \lim_{k\to+\infty} \frac{1}{(2k-1)^d \lambda(\Delta_n)} (2k-1)^d I(\widehat{P}^{z,\beta}_{\Delta_n} | \Pi^{z,\beta}_{\Delta_n}) \\ &= \frac{1}{\lambda(\Delta_n)} I(\widehat{P}^{z,\beta}_{\Delta_n} | \Pi^{z,\beta}_{\Delta_n}). \end{aligned}$$

As a consequence we have

$$I_\xi(\overline{P}^{z,\beta}_{\Delta_n}) = \frac{1}{\lambda(\Delta_n)} I(\widehat{P}^{z,\beta}_{\Delta_n} | \Pi^{z,\beta}_{\Delta_n}).$$

Therefore if (1.13) is verified Proposition 5 ensures that the empirical field is tight for the local convergence topology. □

The tools presented here will be useful in proving the existence of an accumulation point for a sequence of stationarised finite volume GPP or a sequence of infinite volume GPP. Nonetheless, we need to prove that such limit verify DLR equations for the interaction we want to study.

1.2 Liquid-Gas Phase Transition

The uniqueness of infinite volume GPP is an old and difficult question in statistical physics. It is related to the question of coexistence of different state of the system for the same couple of parameters (z, β). The parameters z and β play an important role in the description of the system via some measurable quantities, for example the density of particles. By direct computation on the microscopic system, i.e. on a bounded window, we have that the intensity of the finite volume GPP is a regular function with respect to z and β.

Proposition 7. *The function* $z \mapsto {}^{E_{P^{z,\beta}_\Delta}(N_\Delta)}\!/_{\lambda(\Delta)}$ *is continuous and differentiable with derivative* $z \mapsto {}^{\mathrm{Var}_{P^{z,\beta}_\Delta}(N_\Delta)}\!/_{z\lambda(\Delta)}$ *on* $(0, +\infty)$.

For the infinite volume GPP, the regularity of the intensity of the point process is not always guaranteed. The detection of the points of discontinuity of the intensity is a way to find the precise parameters at which different states coexist. As each state can be represented by an infinite GPP, we have therefore a strategy to find parameters for which we have non-uniqueness of the infinite volume Gibbs measure.

Definition 10. We say that a liquid-gas phase transition occurs for an interaction given by H and parameters $z > 0$ and $\beta \geq 0$ if there is $P, Q \in \mathcal{G}(H, z, \beta)$ such that

$$\rho(P) := E_P(N_{[0,1]^d}) > \rho(Q). \tag{1.14}$$

We also call this phenomenon a first order phase transition since the non-uniqueness of Gibbs measure is linked with the discontinuity of the intensity. The reason we call this a liquid-gas phase transition it is due to the fact that we only measure the jump of density between pure phases that corresponds to the liquid state or the gas state. The state we consider still have fluid property as we do not have crystallisation and symmetry breaking. Indeed, unlike a solid state the particles do not arrange themselves to form a lattice and the Gibbs measure obtained are still invariant by translation and rotation (i.e $P = P \circ r$ and $P = P \circ \tau_u$ for $r \in SO(\mathbb{R}^d)$ and u a vector in $\mathbb{R}^z$).

1.2.1 Variational principle

There is another point of view where the infinite volume Gibbs measure can be seen as the probability measure on Ω that solves the variational principle of statistical physics. Which state that a

Gibbs measure should minimise the free energy, that is the sum of the energy density and the specific entropy. The minimum of the free energy is equal to minus the pressure, where the pressure is defined as the following limit

$$\psi^{z,\beta} := \lim_{n\to+\infty} \frac{1}{\lambda(\Delta_n)} \ln(Z^{z,\beta}_{\Delta_n}). \tag{1.15}$$

In physics, we usually find that the pressure is defined via the same limit multiplied by $1/\beta$ in order to obtain a quantity whose unit is homogeneous to the energy per unit volume. This limit always exists since one can prove that the sequence $\left(\ln(Z^{z,\beta}_{\Delta_n})\right)_{n\geq 1}$ is sub-additive and we conclude using Fekete's lemma. Furthermore the stability and the non-degeneracy hypothesis on the Hamiltonian assures us that this limit is finite. For more detail, the proof of the existence of the limit can be found in the proof of Lemma 1 in [6]. The energy density for P a stationary probability measure on Ω is given by

$$e(P) := \lim_{n\to+\infty} \frac{1}{\lambda(\Delta_n)} \int H(\omega_{\Delta_n}) P(d\omega). \tag{1.16}$$

The existence of this limit is not guaranteed and we need to make the assumption that this limit exists for any stationary probability measure on Ω. Furthermore, we need the following technical assumption on the boundary effects of the Hamiltonian. We assume that for any infinite Gibbs measure P

$$\lim_{n\to+\infty} \frac{1}{\lambda(\Delta_n)} \int \partial H_{\Delta_n}(\omega) P(d\omega) = 0 \tag{1.17}$$

where $\partial H_{\Delta_n}(\omega) = H_{\Delta_n}(\omega) - H(\omega_{\Delta_n})$.

Theorem 2 (Variational Principle, Theorem 1 [6])**.** *For H a stationary and finite range Hamiltonian, $z > 0$ and $\beta \geq 0$. We assume that* (1.16) *is verified for any stationary probability measure on Ω and that the boundary effect condition* (1.17) *is verified for any Gibbs measure. Then for any stationary probability measure P on Ω with finite intensity*

$$F(P) := I_z(P) + \beta e(P) \geq -\psi^{z,\beta}, \tag{1.18}$$

with equality if and only if $P \in \mathcal{G}(H, z, \beta)$.

The variational principle is generally expected, but apart from the finite-range interactions, there are only a few examples where it has been demonstrated. For instance this result has been proved for superstable pairwise interactions [15, 16] and for the Delaunay interaction [7]. With the variational principle in mind, we can explain the heuristic behind the behaviour of different infinite volume GPP. Indeed, when a phase transition occurs, we have a competition between the entropy and the energy that cannot be solved simultaneously. For some special parameters z and β where the infinite volume Gibbs measure is non-unique, one Gibbs measure can minimise the energy density at the expense of the entropy of the system and another Gibbs measure will do the opposite. Therefore if an Hamiltonian has configurations that minimise the energy and if a slight modification of these configurations increases sharply the quantity of energy, this interaction is promising for exhibiting phase transition.

1.2.2 Regularity of the pressure

As we have seen previously the pressure is a quantity that is important in the description of the state. Before going further, let us recall some properties. For a bounded set $\Delta \in \mathcal{B}_b(\mathbb{R}^d)$ and $z > 0$ and $\beta \geq 0$ we call the finite volume pressure

$$\psi_\Delta^{z,\beta} := \frac{1}{\lambda(\Delta)} \ln Z_\Delta^{z,\beta}.$$

We can re-parametrize the model. Instead of the activity z, we consider the chemical potential $\mu \in \mathbb{R}$, where the relationship between μ and z is given $z = e^{-\beta\mu}$. The finite volume pressure is a convex function with respect to μ and β. Indeed, the partition function on Δ can be written as

$$Z_\Delta^{\mu,\beta} = e^{-(e^{-\beta\mu}-1)\lambda(\Delta)} E_{\Pi_\Delta^1}(e^{-\beta(H+\mu N_\Delta)}).$$

Therefore the convexity of the finite volume pressure is a consequence of the Hölder inequality. The finite volume pressure can be seen as a cumulant generating function. Therefore when $\Delta \to \mathbb{R}^d$ we have the following proposition.

Proposition 8. *The pressure $\psi^{\mu,\beta}$ is a convex function with respect to (μ, β).*

In general, we cannot assume that the pressure is smooth. On the contrary, the existence of non-regularities in the pressure is of interest to physicists, as it indicates a change in the system's behavior. In our case, we examine the non-differentiability of the pressure, which is referred to as a first-order phase transition. Furthermore, we can observe that via a direct computation of the derivative of the finite volume pressure with respect to z we have

$$z\frac{\partial \psi_\Delta^{z,\beta}}{\partial z} = -z + E_{P_\Delta^{z,\beta}}\left(\frac{N_\Delta}{\lambda(\Delta)}\right). \tag{1.19}$$

Under the assumption of a stationary Hamiltonian we have that for the empirical field $\overline{P}_{\Delta_n}^{z,\beta}$

$$\rho(\overline{P}_{\Delta_n}^{z,\beta}) = z + z\frac{\partial \psi_{\Delta_n}^{z,\beta}}{\partial z}. \tag{1.20}$$

If we make the assumption that the pressure $\psi^{z,\beta}$ is differentiable, we know that due to the convexity property that

$$\lim_{n\to+\infty} \frac{\partial \psi_{\Delta_n}^{z,\beta}}{\partial z} = \frac{\partial \psi^{z,\beta}}{\partial z}.$$

Therefore, we have for any Gibbs measure $P \in \mathcal{G}(H, z, \beta)$ obtained by thermodynamic limit

$$\rho(P^{z,\beta}) = z + z\frac{\partial \psi^{z,\beta}}{\partial z}. \tag{1.21}$$

We see here the relationship between the regularity of the pressure and liquid-gas phase transition. Indeed, if the pressure is non-differentiable for some parameter (z, β) we can hope to build two

sequences of stationary probability measures whose thermodynamic limits $P, Q \in \mathcal{G}(H, z, \beta)$ verify

$$\rho(P) = z + z\frac{\partial \psi^{z,\beta}}{\partial z^+} \quad \text{and} \quad \rho(Q) = z + z\frac{\partial \psi^{z,\beta}}{\partial z^-}.$$

One way to create such sequence is to build GPP with different boundary conditions. Up until now we have only considered the free boundary condition. Let us consider the setting for general boundary conditions. Let $R > 0$ and for any subset $\Delta \in \mathcal{B}_b(\mathbb{R}^d)$ we define the boundary of Δ the set $\partial\Delta := (\Delta^c \oplus B(0, R)) \cap \Delta$. An event A is said to be a boundary event on Δ if it is a local event on $\partial\Delta$, more precisely if it belongs to $\mathcal{F}_{\partial\Delta}$ which is the σ-algebra generated by counting functions N_B for any Borel sets $B \subset \partial\Delta$.

Definition 11. Let $\Delta \in \mathbb{B}_b(\mathbb{R}^d)$ and any boundary event $A \in \mathcal{F}_{\partial\Delta}$ such that $\Pi^z_\Delta(A) > 0$. The finite volume Gibbs measure on Δ with activity $z > 0$, inverse temperature $\beta \geq 0$, Hamiltonian H and boundary event A is the distribution

$$P^{z,\beta,A}_\Delta = \frac{1}{Z^{z,\beta,A}_\Delta} e^{-\beta H} \mathbb{1}_A \Pi^z_\Delta \tag{1.22}$$

where $Z^{z,\beta,A}_\Delta = \int_A e^{-\beta H} \Pi^z_\Delta$, is the normalisation constant.

We can define the empirical field $\overline{P}^{z,\beta,A}_\Delta$ as in (1.12) but with boundary condition A. With it we can define two distinct sequences of stationary probability measure on Ω using two different sequences of boundary event. In the following proposition, we show the existence of thermodynamic limit.

Proposition 9. *Let $(A_n)_{n\geq 1}$ be a sequence of boundary events on Δ_n and H be a stationary and finite range Hamiltonian, then for any $z > 0$ and $\beta \geq 0$ the sequence $(\overline{P}^{z,\beta,A_n}_{\Delta_n})_{n\geq 1}$ has an accumulation point that verify DLR equations.*

The proof of this proposition is similar to the proof of Theorem 1. For the moment we consider $\delta > 0$ and two boundary conditions, $A_n = \{\omega \in \Omega, N_{\partial\Delta_n}(\omega) = 0\}$ and $B_n = \{\omega \in \Omega, \forall i \in \delta\mathbb{Z}^d \cap \partial\Delta_n, N_{[-\delta/2,\delta/2]^d \oplus i}(\omega) = 1\}$. We denote by $P^{z,\beta,A}$ and $P^{z,\beta,B}$ the thermodynamic limit that is given by Proposition 9. In the case of unicity of the infinite Gibbs measure, we observe that the effect of the boundary condition vanishes as the volume increases. The behaviour in the bulk of the system tends to be more and more decorrelated from the behaviour at the boundary, which is something we might expect in the setting of finite range Hamiltonian. However if these different boundary conditions still have a strong influence on the behaviour in the bulk, it is a good setting for a phase transition phenomenon. Classically the way the boundary can influence the bulk is when there are percolation phenomenon involved as in those cases the correlation functions does not decay exponentially with the distance. A good choice of boundary condition is also important and for example A_n and B_n are prime examples. Indeed, if we prove that the bulk behaves in the same way as the boundary, via percolation arguments or other, we have for A_n a very sparse distribution of particles, thus $\rho(P^{z,\beta,A}) << 1$ and for B_n mostly packed configuration, thus $\rho(P^{z,\beta,B}) \geq 1/\delta^d$. Therefore, we will observe a liquid-gas phase transition since both $P^{z,\beta,A}$ and $P^{z,\beta,B}$ are infinite Gibbs measures.

1.2.3 Known results

In the setting of continuous point process, there are several results of phase transition. Among these we have an abundance of phase transition based on the existence of a dominant spin, where the considered system is a mixture of different particles, represented by their spin. The Hamiltonian for such systems takes into account the interaction between particles of the same type and the interaction between two different particles. If we consider only systems consisting of a single type of particles, thus a Hamiltonian that rely only on the self arrangement, the geometry and the density of particles, we have only a few known results. The first one being, the phase transition of the Area interaction.

Theorem 3 (Widom-Rowlinson [34] , Ruelle [30] , Chayes-Chayes-Kotecký [3], Giacomin-Lebowitz-Maes [19])**.** *For the Area interaction, there exists a critical activity z_c such that for $z > z_c$ and $\beta = z$ the system exhibits liquid-gas phase transitions.*

The underlying argument for the proof of phase transition of the Area interaction is the link between this model and the two colour Widom-Rowlinson model. For any $\Delta \in \mathbb{B}_b(\mathbb{R}^d)$, the distribution for the two colour Widom-Rowlinson model on Δ is given by

$$P_{\Delta}^{z_A,z_B}(d\omega^A, d\omega^B) = \frac{1}{Z_{\Delta}^{z_A,z_B}} \mathbb{1}_{d(\omega^A,\omega^B)\geq R} \Pi_{\Delta}^{z_A}(d\omega^A)\Pi_{\Delta}^{z_B}(d\omega^B)$$

where $R > 0$ is the distance of the hardcore interaction between different type of particles, $z_A > 0$ and $z_B > 0$ are the respective activity of particle A and B and $Z_{\Delta}^{z_A,z_B}$ is the normalisation constant. The marginal distributions of ω^A and ω^B are given by

$$P_{\Delta,A}^{z_A,z_B}(d\omega^A) = \frac{1}{Z_{\Delta}^{z_A,z_B}} e^{-z_B \mathcal{V}(L_R(\omega^A))} \Pi_{\Delta}^{z_A}(d\omega^A).$$

The marginal distributions are Gibbs measure on Δ for the Area interaction and z_B (in the case of particles A) plays the role of the inverse temperature β. This symmetry on z_A and z_B translates into a symmetry of z and β in the behaviour of the Area interaction model and this is what we call the duality for this model. The first proof of the phase transition for the Area interaction by Ruelle [30] relies on the two colour Widom-Rowlinson model and especially the symmetry for both particles when $z_A = z_B$. More specifically, the infinite volume distributions for each system of particle A or B are Gibbs measures for the Area interaction with $\beta = z$. The key element of this proof involve adapting the classical Peierls argument from lattice systems to continuous systems, as well as imposing boundary conditions such as the absence of particles B on the boundary. Ruelle proves that in a cell the probability that there is at least one particle B is smaller than the probability that there is at least one particle A.
The second proof by Chayes, Chayes and Kotecký [3] relies on the Fortuin-Kasteleyn representation for the two colour Widom-Rowlinson model. In this representation, the two color model with parameter $z = z_A = z_B$ can be seen a as realisation of the random cluster model with activity z and then we assign each connected component of $L_R(\omega)$ with a colour with probability $1/2$. The random cluster model is a Gibbs point process with $H(\omega) = -N_{cc}(L_R(\omega))$ and $\beta = \ln 2$,

whose finite volume distribution for $\Delta \subset \mathbb{R}^d$ is given by

$$P_{\Delta}^{z,\beta}(d\omega) = \frac{1}{Z_{\Delta}^{z,\beta}} 2^{N_{cc}(L_R(\omega))} \Pi_{\Delta}^{z}(d\omega).$$

We can show that for each colour the coloured point process stochastically dominates the Poisson point process with activity z. We know that there is a percolation threshold $z_c^p(R)$ such that for $z > z_c^p(R)$ the Boolean Poisson model with radius R percolates and thus the coloured point process also percolates by stochastic domination. Furthermore we know that the unbounded connected component is unique and thus in the two colour Widom-Rowlinson model only one type of particle is associated to this unique infinite volume connected component. Consequently, when we fix the boundary condition such as no particle B can be on the boundary we have fixed that it is necessarily particles A that has the infinite volume connected component.
The phase diagram for the Area interaction has been further explored in [9]. For $R > 0$ the radius of the balls around each particles, Houdebert and Dereudre proved the existence of an increasing Lipschitz function : $\beta \to \tilde{z}_c^a(\beta, R)$. For $\beta = 0$ we have $\tilde{z}_c^a(0, R) = z_c^p(R)$, where $z_c^p(R)$ is the percolation threshold of the Poisson Boolean model with balls of radius R. Moreover, there exists $\beta_R \in (0, +\infty)$ such that for $\beta \geq \beta_R$, $\tilde{z}_c^a(\beta, R) = \beta$. The quantity $\tilde{z}_c^a(\beta, R)$ is the percolation threshold for any Gibbs measure for the Area interaction, inverse temperature $\beta \geq 0$ and radius $R > 0$.

Theorem 4 (Dereudre-Houdebert [9])**.** *For the Area interaction with radius $R > 0$ and for all $\beta \geq 0$ and $z < \tilde{z}_c^a(\beta, R)$ there is an unique infinite volume Gibbs measure. Moreover, by duality, for every $z \geq 0$ and $\beta < \tilde{z}_c^a(z, R)$ we have uniqueness as well.*

The proof of this results uses something similar to the OSSS inequality. This inequality is obtained using the theory of randomised tree algorithms introduced by Duminil-Copin, Raoufi and Tassion in [10, 11, 12]. With Theorem 3 and 4, we have mostly completed the phase diagram as we can see in figure 1.1. The only remaining part that is yet to be explored is a small area around the critical point. It is conjectured that for the area interaction we have unicity of the Gibbs measure everywhere except when $\beta = z$ and $z > z_c$, where (z_c, z_c) is the critical point. The techniques used to prove phase transition or not for the Gibbs point process with Area interaction relies heavily on the symmetry of the two color Widom-Rowlinson model. The limit of such approaches are that it is not possible to extend phase transition results when the radii of the balls constituting the halo are stochastic.

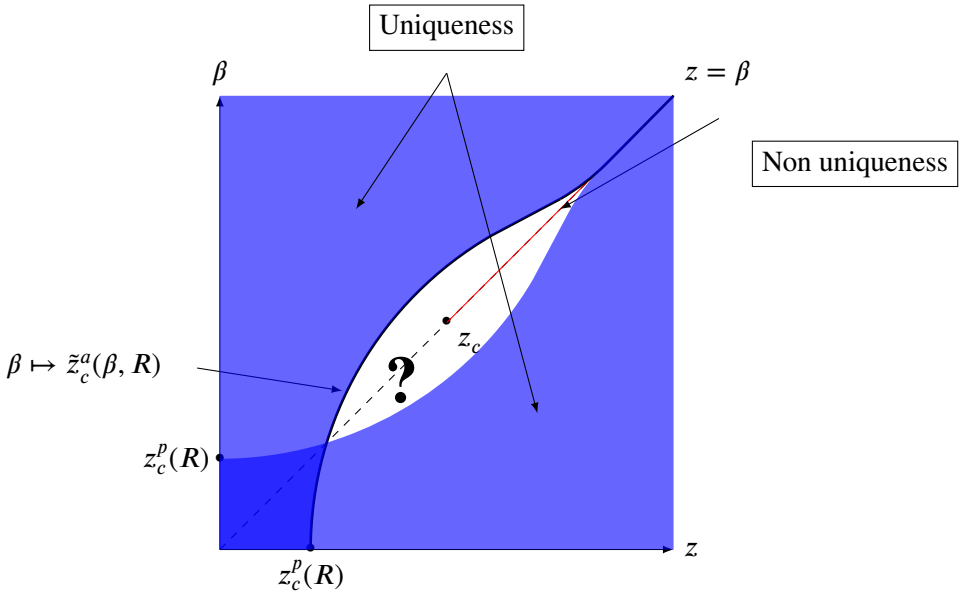


Figure 1.1: Uniqueness/non-uniqueness regimes for the area-interaction measures with parameters z, β. picture from [9]

Another important phase transition result is due to Lebowitz, Mazel and Presutti for the following Hamiltonian

$$H_\gamma(\omega) = -\frac{1}{2!} \sum_{\{x_1,x_2\}\subset\omega} \gamma^{2d} \mathcal{V}(\bigcap_{i=1}^{2} B(x_i, \gamma^{-1} R_d)) + \frac{1}{4!} \sum_{\{x_1,x_2,x_3,x_4\}\subset} \mathcal{V}(\bigcap_{i=1}^{4} B(x_i, \gamma^{-1} R_d)). \quad (1.23)$$

It is the first time that the Pirogov-Sinaï-Zahradník (PSZ) theory has been adapted in the continuous setting to prove a phase transition result.

Theorem 5 (Lebowitz-Mazel-Presutti[27]). *There exists $\beta_c \in (0, +\infty)$ such that for any $\beta > \beta_c$, there exists functions $\gamma_0(\beta)$ and $z(\gamma, \beta)$ such that for $0 < \gamma < \gamma_0(\beta)$ the interaction given by H_γ exhibit a liquid-gas phase transition at $(z(\beta, \gamma), \beta)$.*

In the precise statement of Theorem 5, we have the existence of two distinct Gibbs measures $P^+_{\gamma,\beta}, P^-_{\gamma,\beta} \in \mathcal{G}(H_\gamma, z(\beta, \gamma), \beta)$ such that $\rho(P^+_{\gamma,\beta}) > \rho(P^-_{\gamma,\beta})$. Furthermore, the limit for the quantities $z(\gamma, \beta)$ and $\rho(P^\pm_{\gamma,\beta})$ exists when $\gamma \to 0$.

The PSZ theory was originally developed for lattice systems. It has been successful in demonstrating phase transitions by analysing the system's behaviour at low temperatures as a perturbation of the system at absolute zero. When at zero temperature, the lattice system is arranged in configurations of lowest energy, which we call ground states, and these minimisers of the Hamiltonian might not be unique. When the ground state is not unique we expect that at low temperature the system would arrange itself in a way where one of the ground state is dominant. In the continuous setting, as we have seen with the variational principle, Theorem 2, the Gibbs states are the minimisers of the free energy. In this case, γ, which is the inverse range of the

interaction, acts as the temperature. When at absolute zero, $\gamma = 0$, the interaction becomes a mean field interaction and the mean field free energy has multiple minimisers. The density of the ground states verifies

$$-\rho + \frac{1}{3!}\rho^3 + \frac{1}{\beta}\ln(s) = \lambda. \tag{1.24}$$

Therefore, when β is large enough (1.24) has 3 roots, two corresponding to the minimiser of the free energy. When we choose $\lambda = \lambda(\beta)$ both local minima are global and thus the mean field interaction has two ground states. Finally, by using PSZ theory for $\gamma << 1$ we can find, via pertubative methods, two Gibbs measure for H_γ with particle densities close to the densities of the grounds states of the mean field interaction.

In the same spirit, Pulvirenti and Tsagkarogiannis, in [33], proved there are parameters for which liquid-gas phase transition occurs when the interaction is the one presented by Lebowitz, Mazel and Presutti with the hardcore interaction added to it, so for

$$H_{PT}(\omega) = H_\gamma(\omega) + \sum_{\{x,y\}\subset\omega} \phi(|x-y|)$$

where H_γ is defined as in (1.23) and for $R > 0$

$$\forall u \geq 0, \quad \phi(u) = \begin{cases} +\infty & \text{if } u \leq R \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 6 (Pulvirenti-Tsagkarogiannis [33])**.** *Consider the interaction given by H_{PT} in dimensions $d \geq 2$. For such model there are $R_0, \beta_{c,R}, \beta_{0,R}$ and for any $0 < R < R_0$ and $\beta \in (\beta_{c,R}, \beta_{0,R})$ there is $\gamma_{\beta,R} > 0$ so that for any $\gamma \leq \gamma_{\beta,R}$ there is $\lambda_{\beta,\gamma,R}$ such that : there are two distinct infinite volume Gibbs measures $\mu^{\pm}_{\beta,\gamma,R}$ with chemical potentials $\lambda_{\beta,\gamma,R}$ and inverse temperature β and two different densities, $0 < \rho_{\beta,\gamma,R,-} < \rho_{\beta,\gamma,R,+}$.*

If we consider models of systems with different type of particles, i.e. with different spin, we have an abundance of phase transition results based on the existence of a dominant spin in the mix. For instance, we can cite the phase transition of the continuous Potts models [18] or of the non symmetrical multiple colour Widom-Rowlinson model [2]. But if we consider liquid-gas phase transition, the only known results are the ones that we have presented in this section. Our goal in this thesis is to expand our knowledge on liquid-gas phase transition and prove the existence of this type of phenomenon to the class of saturated interaction.

1.3 Saturated Interaction

In this work, we are interested in a particular set of interactions called saturated interaction. The particularity of such interaction is that the cost of adding a single point whenever the configuration is sufficiently dense and homogeneous is tractable. More specifically there is $c \in \mathbb{R}$ such that for any $x \in \mathbb{R}^d$ and a dense and homogeneous configuration $\omega \in \Omega$

$$h(x,\omega) = H(\omega \cup \{x\}) - H(\omega) = c.$$

At this stage, we need to give to a proper definition of dense enough and homogeneous configuration. In the setting we will present, we are going to encompass a larger set of interaction than just the type of saturated interaction we just presented.

1.3.1 Coarse graining and saturation

The beginning of the approach consists in doing a coarse graining. We pave $\mathbb{R}^d$ with tiles of length $\delta > 0$. For any integers $i \in \mathbb{Z}^d$, the i-th tile is denoted by $T_i = \tau_{i\delta}([-\delta/2, \delta/2])$ where $\tau_{i\delta}$ is the translation by vector $i\delta$. For any $\Lambda \subset \mathbb{Z}^d$ we denote by $\widehat{\Lambda} = \bigcup_{i\in\Lambda} T_i$.

Definition 12. Let H be a Hamiltonian, we say that we can do a coarse graining if there exists a measurable function $E_0 : \Omega_f \to \mathbb{R} \cup \{+\infty\}$ such that for any $\omega \in \Omega_f$

$$H(\omega) = \sum_{i\in\mathbb{Z}^d} E_i(\omega) \tag{1.25}$$

where $E_i = E_0 \circ \tau_{-i\delta}$. E_i is the energy assigned to the tile T_i.

In order to simplify the notations, for any $\Lambda \subset \mathbb{Z}^d$, the total assigned energy on $\widehat{\Lambda}$ is denoted by

$$E_\Lambda(\omega) = \sum_{i\in\Lambda} E_i(\omega). \tag{1.26}$$

It can be noted that such function E_0 might not be unique. Indeed, we consider a Hamiltonian where we take the average of the pairwise potential ϕ over the volume of each particle, given by

$$H_m(\omega) = \sum_{\{x,y\}\subset\omega} \int_{B(x,r)} \int_{B(y,r)} \phi(|u-v|)dudv.$$

At first, we can choose to have E_0 such as

$$E_0(\omega) = \frac{1}{2} \sum_{x\in\omega_{T_0}} \sum_{y\in\omega\setminus\{x\}} \int_{B(x,r)} \int_{B(y,r)} \phi(|u-v|)dudv.$$

Indeed by doing so, when we sum over all integers we obtain

$$\begin{aligned}\sum_{i\in\mathbb{Z}^d} E_i(\omega) &= \sum_{i\in\mathbb{Z}^d} \frac{1}{2} \sum_{x\in\omega_{T_i}} \sum_{y\in\omega\setminus\{x\}} \int_{B(x,r)} \int_{B(y,r)} \phi(|u-v|)dudv \\ &= \frac{1}{2} \sum_{x\in\omega} \sum_{y\in\omega\setminus\{x\}} \int_{B(x,r)} \int_{B(y,r)} \phi(|u-v|)dudv = H_m(\omega).\end{aligned}$$

It is also possible to consider the following assignment of the energy

$$\widehat{E}_0(\omega) = \int_{T_0} \int_{\mathbb{R}^d} \left(N_{B(u,R)}(\omega) N_{B(v,R)}(\omega) - N_{B(u,R)\cap B(v,R)}(\omega)\right) \phi(|u-v|)dudv.$$

By summing over all integers we have

$$\sum_{i\in\mathbb{Z}^d} \widehat{E}_i(\omega) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(N_{B(u,R)}(\omega)N_{B(v,R)}(\omega) - N_{B(u,R)\cap B(v,R)}(\omega)\right)\phi(|u-v|)dudv.$$

We can observe that for any $u, v \in \mathbb{R}^d$

$$\begin{aligned}\sum_{x\in\omega}\sum_{y\in\omega\setminus\{x\}} \mathbb{1}_{B(u,R)}(x)\mathbb{1}_{B(v,R)}(y) &= \sum_{x\in\omega} \mathbb{1}_{B(u,R)}(x)N_{B(v,R)}(\omega\setminus\{x\}) \\ &= \sum_{x\in\omega} \mathbb{1}_{B(u,R)}(x)\left(N_{B(v,R)}(\omega) - \mathbb{1}_{B(v,R)}(x)\right) \\ &= N_{B(u,R)}(\omega)N_{B(v,R)}(\omega) - N_{B(u,R)\cap B(v,R)}(\omega).\end{aligned}$$

As a result of the previous relation and by exchanging the role of the pair (x, y) and (u, v) in left part of the equation we obtain that

$$\begin{aligned}\sum_{i\in\mathbb{Z}^d} \widehat{E}_i(\omega) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sum_{x\in\omega}\sum_{y\in\omega\setminus\{x\}} \mathbb{1}_{B(x,R)}(u)\mathbb{1}_{B(y,R)}(v)\phi(|u-v|)dudv \\ &= \sum_{x\in\omega}\sum_{y\in\omega\setminus\{x\}}\int_{B(x,r)}\int_{B(y,r)} \phi(|u-v|)dudv = H_m(\omega).\end{aligned}$$

Interestingly most properties of the Hamiltonian does not transfer naturally to E_0. For example, the stability or the finite range property of H does not necessarily imply the equivalent property on E_0. However the non-degeneracy of the Hamiltonian H implies that $E_0(\emptyset) = 0$. In fact, under the assumption that we can do a coarse graining on H then $E_0(\emptyset) = 0 \iff H$ is non-degenerate.

Definition 13. For an Hamiltonian H, let E_0 be an assigned energy to the tile T_0. We say that E_0 has a finite range $R > 0$ if for any configuration $\omega \in \Omega$, $E_0(\omega) = E_0(\omega_{T_0\oplus B(0,R)})$. Furthermore it is said to be stable if there exists $c \geq 0$ such that for all configuration $E_0(\omega) \geq -c(1 + N_{T_0\oplus B(0,R)}(\omega))$.

Lemma 7. *If E_0 has a finite range $R > 0$ and is stable then the original Hamiltonian H has a finite range and is stable.*

Proof. We consider $R > 0$ and $c \geq 0$ respectively the range and the stability constant of E_0. For any $\Delta \subset \mathbb{R}^d$, there exists $\Lambda_\Delta \subset \mathbb{Z}^d$ the smallest subset such that $\Delta \subset \widehat{\Lambda}_\Delta$. We denote by $\partial_{ext}\Lambda_\Delta$ the outside border area of Λ_Δ, more specifically $\partial_{ext}\Lambda_\Delta = \{i \in \Lambda_\Delta^c, \delta d(i, \Lambda_\Delta) \leq R + \delta\}$. For any configuration $\omega \in \Omega_f$ we have

$$H_\Delta(\omega) = E_{\Lambda_\Delta\cup\partial_{ext}\Lambda_\Delta}(\omega) - E_{\Lambda_\Delta\cup\partial_{ext}\Lambda_\Delta}(\omega_{\Delta^c}) + E_{\mathbb{Z}^d\setminus\partial_{ext}\Lambda_\Delta}(\omega) - E_{\mathbb{Z}^d\setminus\partial_{ext}\Lambda_\Delta}(\omega_{\Delta^c}).$$

Since E_0 has a finite range $R > 0$ and by construction we know that $\widehat{\Lambda}_\Delta^c \subset \Delta^c$ therefore

$$E_{\mathbb{Z}^d\setminus\partial_{ext}\Lambda_\Delta}(\omega) - E_{\mathbb{Z}^d\setminus\partial_{ext}\Lambda_\Delta}(\omega_{\Delta^c}) = E_{\mathbb{Z}^d\setminus\partial_{ext}\Lambda_\Delta}(\omega_{\widehat{\Lambda}_\Delta^c}) - E_{\mathbb{Z}^d\setminus\partial_{ext}\Lambda_\Delta}(\omega_{\widehat{\Lambda}_\Delta^c\cap\Delta^c}) = 0.$$

Consequently

$$\begin{aligned} H_\Delta(\omega) &= E_{\Lambda_\Delta \cup \partial_{ext}\Lambda_\Delta}(\omega) - E_{\Lambda_\Delta \cup \partial_{ext}\Lambda_\Delta}(\omega_{\Delta^c}) \\ &= E_{\Lambda_\Delta \cup \partial_{ext}\Lambda_\Delta}(\omega_{\widehat{\Lambda}_\Delta \cup \partial\widehat{\Lambda}_\Delta}) - E_{\Lambda_\Delta \cup \partial_{ext}\Lambda_\Delta}(\omega_{\widehat{\Lambda}_\Delta \cup \partial\widehat{\Lambda}_\Delta \cap \Delta^c}) \\ &= H_\Delta\left(\omega_{\Delta \oplus B\left(0, 2(R, \sqrt{d}\delta)\right)}\right). \end{aligned}$$

Therefore the interaction has a range of at most $2(R+\sqrt{d}\delta)$. Now we prove that H is necessarily stable. For any $\omega \in \Omega_f$ we define the set $\Lambda(\omega) = \{i \in \mathbb{Z}^d, \omega_{T_i \oplus B(0,R)} \neq \emptyset\}$. By construction we know that

$$\widehat{\Lambda}(\omega) \subset L_{R+\sqrt{d}\delta}(\omega), \quad \text{where} \quad L_{R+\sqrt{d}\delta}(\omega) = \bigcup_{x \in \omega} B(x, R + \sqrt{d}\delta),$$

and thus

$$\#\Lambda(\omega) \leq \frac{\lambda(B(0, R + \sqrt{d}\delta))}{\delta^d} N(\omega).$$

Therefore the energy for the configuration ω verify

$$\begin{aligned} H(\omega) &= \sum_{i \in \Lambda(\omega)} E_i(\omega) \\ &\geq -c \sum_{i \in \Lambda(\omega)} 1 + N_{T_i \oplus B(0,R)}(\omega) \\ &\geq -c \left(\#\Lambda(\omega) + \sum_{x \in \omega} \sum_{i \in \Lambda(x)} 1 \right) \\ &\geq -2c\,\lambda(B(0, R/\delta + \sqrt{d}))N(\omega). \end{aligned}$$

In conclusion, H is stable with the stability constant $A = 2c\,\lambda(B(0, R/\delta + \sqrt{d}))$. □

Since we don't have unicity in the way we do the coarse graining, it is important to find one such representation that is easy to compute whenever the configuration is dense enough and homogeneous. We define two sets of locally homogeneous configurations around the tile T_i for any $i \in \mathbb{Z}^d$,

$$\Omega^1_{i,L,\delta} = \{\omega \in \Omega, \omega_{T_j} \neq \emptyset, \forall j \in \mathbb{Z}^d, \delta\|i - j\| \leq L\},$$
$$\Omega^0_{i,L,\delta} = \{\omega \in \Omega, \omega_{T_j} = \emptyset, \forall j \in \mathbb{Z}^d, \delta\|i - j\| \leq L\}.$$

Moreover, for $\# \in \{0, 1\}$ and $\Lambda \subset \mathbb{Z}^d$ we denote by $\Omega^\#_{\Lambda,L,\delta} = \bigcap_{i \in \Lambda} \Omega^\#_{i,L,\delta}$ and $\Omega_{\Lambda,L,\delta} = \Omega^0_{\Lambda,L,\delta} \cup \Omega^1_{\Lambda,L,\delta}$. If $\Lambda = \mathbb{Z}^d$ we simply denote by $\Omega_{\mathbb{Z}^d,L,\delta} = \Omega_{L,\delta}$.

Definition 14. An interaction is said to be saturated if there exists $L > 0$ and $\delta > 0$, a coarse graining $E_0 : \Omega_f \to \mathbb{R} \cup \{+\infty\}$ and a measurable function $\overline{E} : \mathbb{N} \to \mathbb{R} \cup \{+\infty\}$ such that

$$\forall \omega \in \Omega^0_{0,L,\delta} \cup \Omega^1_{0,L,\delta}, \quad E_0(\omega) = \overline{E}(N_{T_0}(\omega)). \tag{1.27}$$

To simplify the notations, for any $\Lambda \subset \mathbb{Z}^d$, the total saturated energy on $\widehat{\Lambda}$ is denoted by

$$\overline{E}_\Lambda(\omega) = \sum_{i \in \Lambda} \overline{E}(N_{T_i}(\omega)). \tag{1.28}$$

Lemma 8. *For any saturated interaction with distance of saturation $L > 0$ and E_0 with a finite range $R > 0$, there is $A \in \mathbb{R}$ and $B \in \mathbb{R} \cup \{+\infty\}$ such that*

$$\overline{E}(k) = (Ak + B)\mathbb{1}_{k \geq 1}.$$

Proof. We fix $M = \max\{L, R\}$ and $k \in \mathbb{N}^*$ and we consider a configuration

$$\omega \in \{N_{T_0}(\omega) = k\} \bigcap_{\substack{i \in \left[-\frac{4M}{\delta}, \frac{4M}{\delta}\right]^d \\ i \neq 0}} \{N_{T_i}(\omega) = 1\}.$$

This way the tiles T_i where $i \in \left[-\frac{2M}{\delta}, \frac{2M}{\delta}\right]$ are saturated. Now we consider a point $x \in T_0$ really close to the border with an adjacent tile T_j and $\epsilon \in \mathbb{R}^d$ such that $x + \epsilon \in T_j$ and any point in the configuration $\omega + \epsilon$ still belong to their initial tile. Now if we compute the local energy of the point x in the configuration ω, by using finite range argument of E_0 and the saturation property, we have

$$h(x, \omega) := H(\omega \cup \{x\}) - H(\omega) = \overline{E}(k+1) - \overline{E}(k).$$

Similarly, the local energy of $x + \epsilon$ in the configuration $\omega + \epsilon$ we have

$$h(x + \epsilon, \omega + \epsilon) = \overline{E}(2) - \overline{E}(1).$$

Moreover, by stationarity property of the Hamiltonian we have that

$$h(x, \omega) = h(x + \epsilon, \omega + \epsilon).$$

Therefore, we have that

$$\overline{E}(k+1) - \overline{E}(k) = \overline{E}(2) - \overline{E}(1).$$

As a consequence, $\overline{E}$ is linear starting from 1, i.e there is $A \in \mathbb{R}$ and $B \in \mathbb{R} \cup \{+\infty\}$ such that

$$\overline{E}(k) = \begin{cases} Ak + B & \text{if } k \geq 1 \\ \overline{E}(0) & \text{otherwise} \end{cases}.$$

Furthermore, we know that $\overline{E}(0) = E_0(\emptyset) = 0$ and thus we have the desired expression. □

In general, we want that the configurations in $\Omega^1_{0,L,\delta}$ to be allowed by the system. If from the beginning we have $\overline{E}(1) = +\infty$, then any dense configuration that saturates an area are forbidden by the model and the only saturation observable is the saturation by large empty spaces. This is something we would like to avoid and therefore we would like to assume that at least $B < +\infty$, in order to allow these dense configurations.

1.3.2 Examples of saturated interaction

The class of saturated interaction is surprisingly quite large and many even unsuspected Hamiltonian falls into this category. A first example of such interaction that verify this saturation property is the K-nearest neighbour Strauss interaction, where the nearest neighbour interaction is defined in (1.4). Let $a \in \mathbb{R}$ and $R > 0$ the Hamiltonian is given by

$$H_K(\omega) = \begin{cases} \sum_{x\in\omega} \sum_{m=1}^{\min(K,N(\omega)-1)} a\mathbb{1}_{[0,R]}(|x - v_m(x,\omega)|) & \text{if } N(\omega) \geq 2 \\ 0 & \text{otherwise} \end{cases}. \tag{1.29}$$

In this situation, the natural choice for E_0 is obviously for any configuration $\omega \in \Omega_f$

$$E_0(\omega) = \sum_{x\in\omega_{T_0}} \sum_{m=1}^{\min(K,N(\omega)-1)} a\mathbb{1}_{[0,R]}(|x - v_m(x,\omega)|).$$

Then we will fix $L = R$ and δ small enough such that for any $x \in T_0, \#\{T_i, T_i \subset B(x,R)\} \geq K+1$. In this case whenever $\omega \in \Omega^1_{0,L,\delta}$ we have

$$E_0(\omega) = aKN(\omega_{T_0}) = \overline{E}(N(\omega_{T_0}))$$

and when $\omega_{T_0} = \emptyset$ and also for $\omega \in \Omega^0_{0,L,\delta}$ we have

$$E_0(\omega) = 0 = \overline{E}(0).$$

Other approximation of the pairwise interaction falls under this category, among them is the diluted pairwise interaction. We will treat this case in Section 3.2 as it seems to be a promising avenue in proving liquid-gas phase transition for pairwise interaction.
Another example of saturated interaction is the Area interaction, if we consider the marked point process where the mark corresponds to the radii of the balls forming the halo. If there is $R_1 \geq R_0 > 0$ such that the spin state $S = [R_0, R_1]$ and P_S any probability measure on S. The Hamiltonian is given by

$$H(\omega) = \mathcal{V}(L(\omega)), \quad \text{where} \quad L(\omega) = \bigcup_{\{x,R\}\in\omega} B(x,R).$$

The natural way to do the coarse graining is therefore

$$E_0(\omega) = \mathcal{V}(L(\omega) \cap T_0)$$

We fix the lengths δ and L such that $\sqrt{d}\delta < R_0$ and $L \geq 2R_1 + \delta$. Under these condition we know that as long as $N(\omega_{T_0}) \geq 1$ and therefore also for $\omega \in \Omega^1_{0,L,\delta}$ we have

$$E_0(\omega) = \delta^d = \overline{E}(N(\omega_{T_0})).$$

If $\omega \in \Omega^0_{0,L,\delta}$, we have that $L(\omega) \cap T_0 = \emptyset$ and thus

$$E_0(\omega) = 0 = \overline{E}(0).$$

In fact, the Quermass interaction under these assumption on the spin state is saturated in the same way even though it might not be as clear especially for the Euler-Poincaré characteristic. We will expand further about this in Section 3.1.

Unfortunately, most pairwise interaction are not saturated or at least we have not a coarse graining that exhibit this property. However, there is an exception to this statement which is the pairwise hardcore interaction who is saturated. We recall that for $R > 0$ the hardcore pair potential is given by

$$\phi(x) = +\infty \mathbb{1}_{x \leq R}.$$

The coarse graining is the usual for such pairwise interaction i.e.

$$E_0(\omega) = \frac{1}{2} \sum_{x \in \omega_{T_0}} \sum_{y \in \omega \setminus \{x\}} \phi(|x - y|).$$

If $R/2\sqrt{d} > \delta > 0$ and $L > 2(R + \delta)$ then for any configuration $\omega \in \Omega_{0,L,\delta}$

$$E_0(\omega) = \overline{E}(N_{T_0}(\omega))$$

where $\overline{E}(0) = 0$ and $\overline{E}(k) = +\infty$ for any $k \geq 1$. The pairwise hardcore interaction is a prime example of saturated interaction that does not allow the dense configurations.

1.3.3 Phase transition heuristic in saturated interaction setting

Let H be the Hamiltonian of a saturated interaction with E_0 that has a finite range $R > 0$ and a saturation range $L > 0$. We assume that for some $A, B \in \mathbb{R}$ we have $\overline{E}(k) = (Ak + B)\mathbb{1}_{k \geq 1}$. We consider two distinct distributions $P = \delta_\emptyset$ and $Q = \otimes_{i \in \mathbb{Z}^d} Q_i$ where

$$Q_i(d\omega_{T_i}) = \frac{1}{Z_1} e^{-\beta \overline{E}(N(\omega_{T_i}))} \mathbb{1}_{\omega \neq \emptyset} \Pi^z_{T_i}(d\omega_{T_i}). \tag{1.30}$$

More precisely we consider the empirical field of Q that is given by

$$\overline{Q} = \frac{1}{\delta^d} \int_{T_0} Q \circ \tau_{-u} du.$$

The entropy with respect to $\Pi^z_{\widehat{\Lambda}}$ of P and $\overline{Q}$ is given by

$$\begin{aligned} I(P_{|\widehat{\Lambda}} | \Pi^z_{\widehat{\Lambda}}) &= -|\Lambda| \ln(Z_0) \\ I(\overline{Q}_{|\widehat{\Lambda}} | \Pi^z_{\widehat{\Lambda}}) &= -|\Lambda| \ln(Z_1) - \beta \sum_{i \in \Lambda} \int_\Omega \overline{E}(N(\omega_{T_i})) Q_i(d\omega) \\ &= -|\Lambda| \ln(Z_1) - \beta |\Lambda| E_{Q_0}\left(\overline{E}(N_{T_0})\right) \end{aligned}$$

where

$$Z_0 = \int \mathbb{1}_{\omega=\emptyset} \Pi^z_{T_0}(d\omega) = e^{-z\delta^d}$$
$$Z_1 = \int e^{-\beta \overline{E}(N(\omega_{T_0}))} \mathbb{1}_{\omega\neq\emptyset} \Pi^z_{T_0}(d\omega) = e^{-(z\delta^d+\beta B)} \left(\exp(z\delta^d e^{-\beta A}) - 1\right).$$

Furthermore, if we compute the mean energy per unit volume we have for P

$$\begin{aligned} e(P) &= \lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|\delta^d} \int H(\omega_{\hat{\Lambda}}) P(d\omega) \\ &= \lim_{\Lambda\to\mathbb{Z}^d} \frac{H(\emptyset)}{|\Lambda|\delta^d} = 0. \end{aligned}$$

We define the boundary $B(\Lambda) = \{i \in \Lambda^c, \delta d_2(i,\Lambda) \leq R\} \cup \{i \in \Lambda, \delta d_2(i,\Lambda^c) \leq L\}$ and thus for $\overline{Q}$ we have

$$\begin{aligned} e(\overline{Q}) &= \lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|\delta^d} \int H(\omega_{\hat{\Lambda}}) \overline{Q}(d\omega) \\ &= \lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|\delta^d} \int \left(E_{B(\Lambda)}(\omega_{\hat{\Lambda}}) + \overline{E}_{\Lambda\setminus B(\Lambda)}(\omega_{\hat{\Lambda}})\right) \overline{Q}(d\omega) \\ &= \lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|\delta^d} \int E_{B(\Lambda)}(\omega_{\hat{\Lambda}}) \overline{Q}(d\omega) + \frac{|\Lambda\setminus B(\Lambda)|}{|\Lambda|\delta^d} E_{Q_0}\left(\overline{E}(N_{T_0})\right). \end{aligned}$$

Under the assumption that

$$\lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|\delta^d} \int E_{B(\Lambda)}(\omega_{\hat{\Lambda}}) \overline{Q}(d\omega) = 0,$$

we have

$$e(\overline{Q}) = \frac{1}{\delta^d} E_{Q_0}\left(\overline{E}(N_{T_0})\right).$$

As a result the free energy of $\overline{Q}$ and P verify

$$F(P) = -\frac{1}{\delta^d}\ln(Z_0) \quad \text{and} \quad F(\overline{Q}) = -\frac{1}{\delta^d}\ln(Z_1).$$

Therefore if we can find for the absolute zero temperature, $\beta = +\infty$, an activity $z_\infty > 0$ such that $Z_0 = Z_1$ and if we assume that P and $\overline{Q}$ are the minimisers of free energy, by the variational principle these two measures we constructed are two Gibbs measures for the Hamiltonian H. The system needs to be somehow attractive if we hope to have $\overline{Q}$ as a minimiser of the free energy. Heuristically, P and $\overline{Q}$ should be the minimisers at absolute zero if the system verify a Peierls condition i.e. we have a energy surplus per unit volume in mixing areas, where the state of each tiles are non homogeneous. Due to this energy surplus a probability measure that put weights on events where there are mixing areas cannot be a Gibbs measure for H at $\beta = +\infty$ and $z = z_\infty$. At low strictly positive temperature, we expect to find two Gibbs measures whose behaviour is close to the two ideal point processes. We use the PSZ theory to find these Gibbs measures with different intensities and we can interpret them as a small perturbation of the ideal states at low temperature.

1.3.4 Simulations

In order to gain some intuition on the behaviour of particles and to understand the occurrence of the liquid-gas phase transition, we conducted simulations for models with saturated interactions. Through this numerical study, we discovered that the saturation property alone is not sufficient for the phase transition to occur, prompting further investigation into which additional properties are needed. We begin by presenting the numerical results obtained for the Widom-Rowlinson model, for which we have theoretical results (see Theorem 3). Subsequently, we discuss the results of simulations for the Quermass interaction with both positive and negative contributions of the surface measure. Finally, we present the results of simulations for the K-nearest neighbour Strauss interaction, considering both repulsive and attractive cases.

Let us consider a finite box $\Lambda_F = [0, F]^d$ with F large enough as such we can consider that the finite Gibbs measure on Λ_F is an approximation the infinite volume Gibbs measure. As a result, for any tame local function f, with $f(\omega) = f(\omega_\Delta)$ and $\Delta \subset \Lambda_F$ we have

$$E_{P^{z,\beta}}(f) \approx E_{P^{z,\beta}_{\Lambda_F}}(f), \tag{1.31}$$

and particularly

$$\rho(P^{z,\beta}) = E_{P^{z,\beta}}(N_{[0,1]^d}) \approx \frac{1}{\lambda(\Lambda_F)} E_{P^{z,\beta}_{\Lambda_F}}(N_{\Lambda_F}). \tag{1.32}$$

The algorithm used to do the simulations is the Birth-death Metropolis-Hastings algorithm presented in [29] Section 7.1.2. We define the local energy of a point $x \in \mathbb{R}^d$ with in a configuration $\omega \in \Omega_f$

$$h(x, \omega) := H(\omega \cup \{x\}) - H(\omega).$$

For a configuration ω in $\Delta \subset \mathbb{R}^d$, we define $p(\omega)$ the probability for proposing a birth, $q_b(\omega, \cdot)$ the density function of a random variable on Δ, used to propose the site of the new point, and $q_d(\omega, \cdot)$ the discrete density on ω, used to propose the point to be erased from the configuration. The birth acceptance probability is given by

$$\alpha_b(\omega, x) = \min\{1, r(\omega, x)\},$$

with

$$r(\omega, x) = z e^{-\beta h(x,\omega)} \frac{(1 - p(\omega \cup \{x\}))q_d(\omega \cup \{x\}, x)}{p(\omega)q_b(\omega, x)}$$

The death acceptance probability is given by

$$\alpha_d(\omega, x) = \min\{1, r(\omega \setminus \{x\}, x)^{-1}\}.$$

Algorithm 1 Birth-death Metropolis-Hastings algorithm

Given $Y_n = \omega \subset \Delta$, generate Y_{n+1} as follows

1. draw $U_m^{(1)} \sim \text{Uniform}([0,1])$ and $U_m^{(2)} \sim \text{Uniform}([0,1])$;
2. if $U_m^{(1)} \leq p(\omega)$ then generate $x \sim q_b(\omega, \cdot)$ and set
$$Y_{m+1} = \begin{cases} \omega \cup \{x\} & \text{if } U_m^{(2)} \leq r(\omega, x) \\ \omega & \text{otherwise} \end{cases};$$
3. if $U_m^{(1)} > p(\omega)$, generate $x \sim q_d(\omega, \cdot)$ and set
$$Y_{m+1} = \begin{cases} \omega \setminus \{x\} & \text{if } U_m^{(2)} \leq r(\omega \setminus \{x\}, x)^{-1} \\ \omega & \text{otherwise} \end{cases},$$
and if $\omega = \emptyset$ then $Y_{m+1} = \emptyset$.

In practice, we will choose $p(\omega) = \frac{1}{2}$, $q_b(\omega, \cdot) = \frac{1}{\lambda(\Delta)} \mathbb{1}_\Delta(\cdot)$ and $q_d(\omega, \cdot) = \frac{1}{N(\omega)} \mathbb{1}_\omega(\cdot)$. By non heredity property of the Hamiltonian H, we know that $e^{\beta h(x, \omega \setminus \{x\})} \neq 0$ and therefore by Proposition 7.12 and Proposition 7.13 from [29], we know that the Markov Chain generated by Algorithm 1 is reversible, irreducible and aperiodic. More importantly, according to Proposition 7.4 from [29] there is a unique invariant distribution which is $P_\Delta^{z,\beta}$ and regardless of the initial distribution and for any $f \in \mathcal{L}^1(P_\Delta^{z,\beta})$ the ergodic averages verifies

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(Y_k) \overset{a.s.}{=} E_{P_\Delta^{z,\beta}}(f)$$

and especially for $f = N_\Delta$. As a result, we have that

$$\frac{1}{n} \sum_{k=0}^{n-1} N_{\Lambda_F}(Y_k) \approx \rho(P^{z,\beta}). \tag{1.33}$$

This gives an easy way to approximate the intensity of Gibbs measures for any set of parameters z, β provided that we are able to compute efficiently the Hamiltonian. The Quermass interaction is one of those Hamiltonian where it is quite difficult to compute. Let us recall that in dimension $d = 2$, the Quermass Hamiltonian, for $R > 0$ is given by

$$H(\omega) = \left(\mathcal{V} + \theta_1 \mathcal{S} + \theta_2 \chi\right)(L_R(\omega)), \quad \text{where } L_R(\omega) = \bigcup_{x \in \omega} B(x, R),$$

with $\mathcal{V}$ the Lebesgue measure in $\mathbb{R}^2$, $\mathcal{S}$ the 1-dimensional Haussdorf measure of the boundary and χ the Euler-Poincarré characteristic. The computation of such energy function is a difficult task and to do so we implemented an algorithm based on the following proposition.

Lemma 9 (Møller-Helisova [28]). *For any configuration $\omega \in \Omega_f$, we have the following inclusion-exclusion formula*

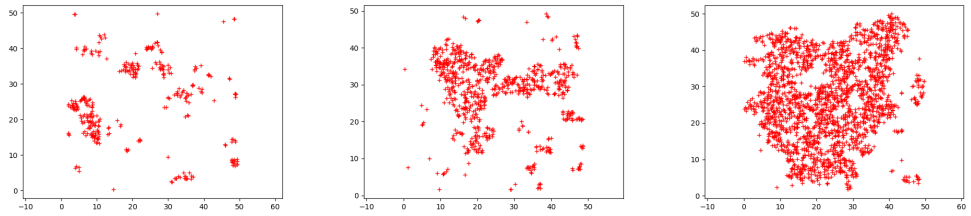
$$\mathcal{V}(L(\omega)) = \sum_{x \in I_1(\omega)} \mathcal{V}(B_x) - \sum_{\{x,y\} \in I_2(\omega)} \mathcal{V}(B_x \cap B_y) + \sum_{\{x,y,z\} \in I_3(\omega)} \mathcal{V}(B_x \cap B_y \cap B_z) \tag{1.34}$$

and

$$S(L(\omega)) = \sum_{x \in I_1(\omega)} S(B_x) - \sum_{\{x,y\} \in I_2(\omega)} S(B_x \cap B_y) + \sum_{\{x,y,z\} \in I_3(\omega)} S(B_x \cap B_y \cap B_z) \tag{1.35}$$

where $B_x = B(x, R)$, $I_1(\omega)$, $I_2(\omega)$ and $I_3(\omega)$ are respectively 1-cliques, 2-cliques and 3-cliques of the Delaunay triangulation, $\mathcal{D}(\omega)$.

Since we don't have any easy way to compute the Euler-Poincarré characteristic, in the simulation we will consider only the case where $\theta_2 = 0$. In general, we do our simulations in the box Λ_F with $F = 50$ and with 2 million iterations of the Birth-Death Metropolis-Hasting step and the ergodic average is computed by taking 400 values in the last 20 thousand steps of the algorithm taken with a spacing of 50 steps between them. This spacing between the taken values is heuristically done to get decorrelations between the values taken into account in the ergodic average and we take the last values in order to skip the burn-in period. Theoretically we should have better rate of convergence for the ergodic average. First for the Widom-Rowlinson model, $\theta_1 = \theta_2 = 0$, and with $R = 1$, in this situation, we know that a liquid-gas phase transition occurs when $z = \beta$ for z large enough by Theorem 3.



(a) $z \approx 1.68$ (b) $z \approx 1.7$ (c) $z \approx 1.72$

Figure 1.2: Widom-Rowlinson model for $\beta = 1.7$

We can clearly see that we have three distinct behaviours. We can observe in figure 1.2a that when $z \approx 1.68$ and $\beta = 1.7$ the particles are bunched up forming small islets. When we have a slightly larger activity, $z \approx 1.70$ as in figure 1.2b, we observe a large cluster that takes up a chunk of the box and we still have small islets. Finally, for z even larger $z \approx 1.72$, we observe a single large cluster that takes up a big portion of the box as in figure 1.2c.

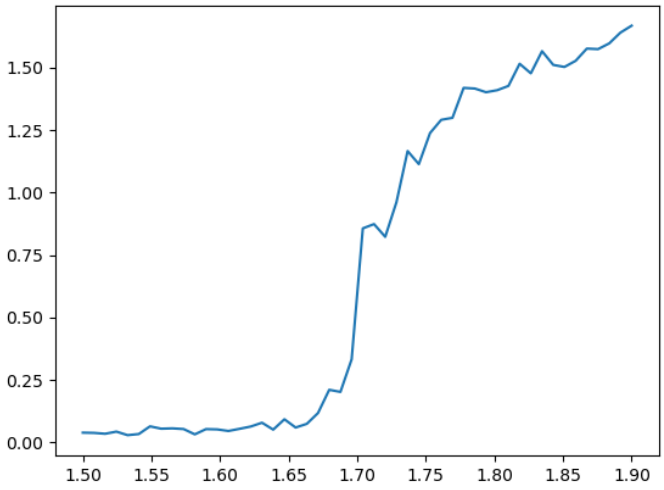
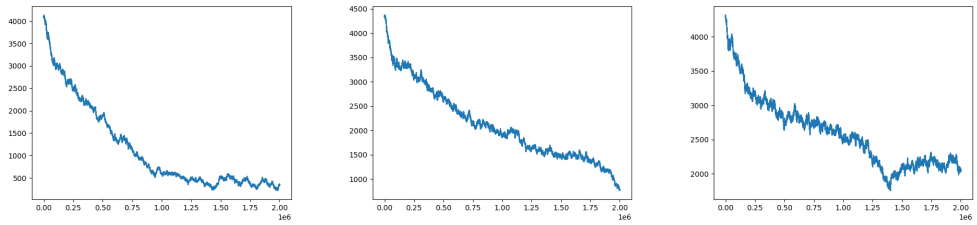


Figure 1.3: Intensity graph for the Widom-Rowlinson model with $\beta = 1.7$

The intensity presented here is given by the ergodic averages of the number of particles. If we look at the graph of intensity with respect to the activity z, we can clearly see a sharp increase of the density of particles at around $z = 1.7$ which is consistent with the theoretical results.



(a) $z \approx 1.68$ (b) $z \approx 1.70$ (c) $z \approx 1.72$

Figure 1.4: Widom-Rowlinson model for $\beta = 1.7$

Furthermore, if we look at the graph of evolution of the number of particles, used to control the convergence of the Markov chain. We can observe that outside the phase transition area the number of particles is quite stable at some point as in figures 1.4a and figure 1.4c. On the contrary, around the liquid-gas phase transition area the number of particles struggles to stabilize even for a big number of iteration of the Markov chain. Finally, if we plot the same intensity graph for $\beta \in \{1.7, 1.8, 1.9, 2, 2.1, 2.2\}$ and take the critical activity when the growth rate is the greatest, we obtain the following phase diagram that is quite close to the expected results of Theorem 3. The blue dotted line is the theoretical result and the red points are the critical values of the activity detected by our method.

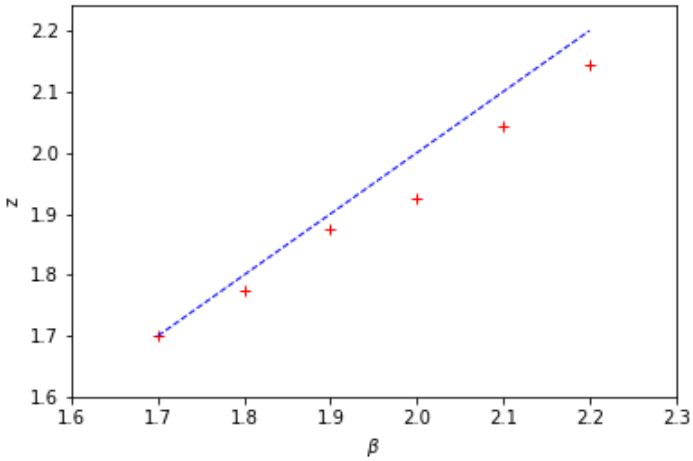
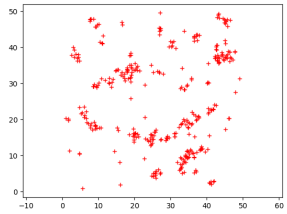
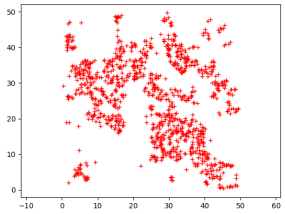


Figure 1.5: Widom-Rowlinson phase diagram

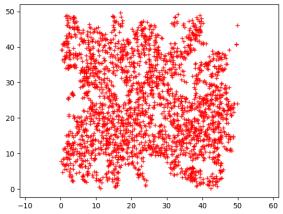
To investigate further the link between saturation and phase transition we have done simulations for the Quermass interaction for two sets of parameters $(\theta_1, \theta_2) = (0.05, 0)$ and $(\theta_1, \theta_2) = (-0.05, 0)$. By doing so we test the effect of the perimeter on the behaviour of the particles.



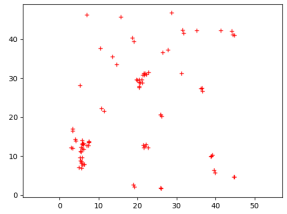
(a) $z \approx 2.86$, $\theta_1 = 0.05$, $\theta_2 = 0$



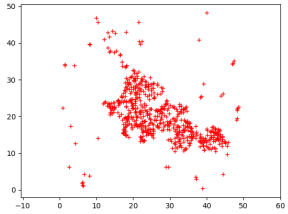
(b) $z \approx 2.98$, $\theta_1 = 0.05$, $\theta_2 = 0$



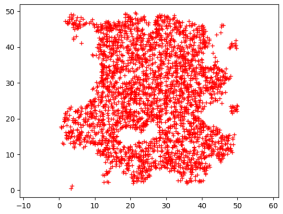
(c) $z \approx 3.07$, $\theta_1 = 0.05$, $\theta_2 = 0$



(d) $z \approx 0.92$, $\theta_1 = -0.05$, $\theta_2 = 0$



(e) $z \approx 0.96$, $\theta_1 = -0.05$, $\theta_2 = 0$

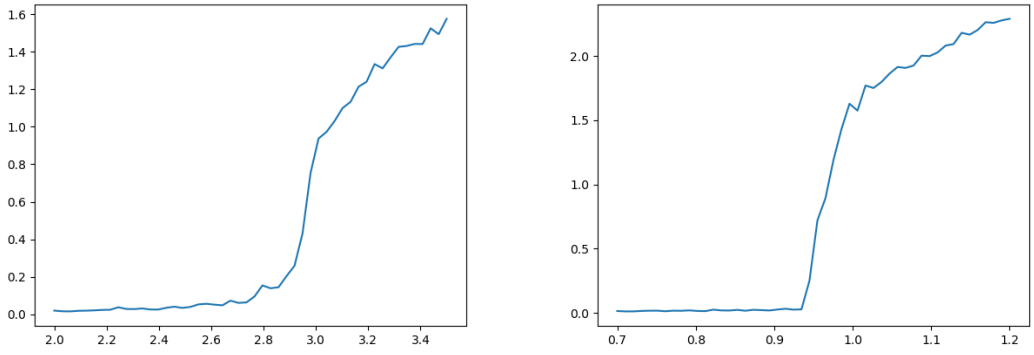


(f) $z \approx 0.99$, $\theta_1 = -0.05$, $\theta_2 = 0$

Figure 1.6: Quermass model for $\beta = 1.7$

We can clearly see that we obtain the same type of configurations as for the Widom-Rowlinson model, at low activity we have the particles bunched up in islets, figures 1.6a and 1.6d, and with

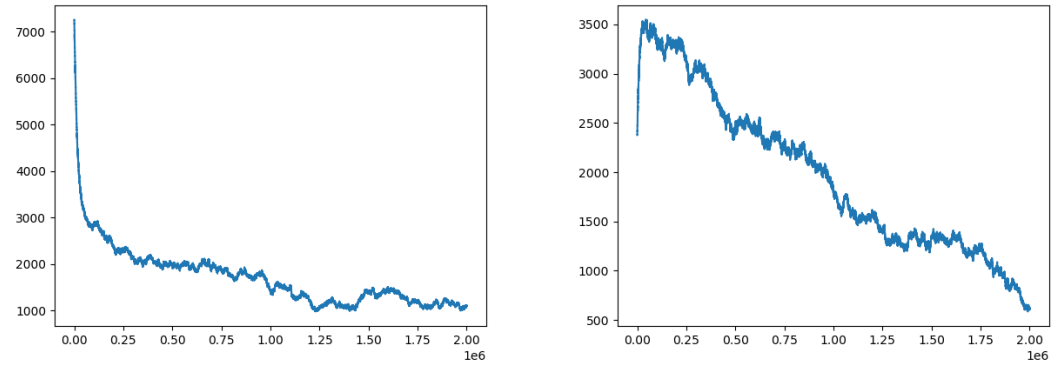
higher activity there appear a large cluster as in figures 1.6b and 1.6e and eventually for high activity this cluster takes up all the box as in figures 1.6c and 1.6f.



(a) $\theta_1 = 0.05, \theta_2 = 0$ (b) $\theta_1 = -0.05, \theta_2 = 0$

Figure 1.7: Intensity graph for the Quermass model for $\beta = 1.7$

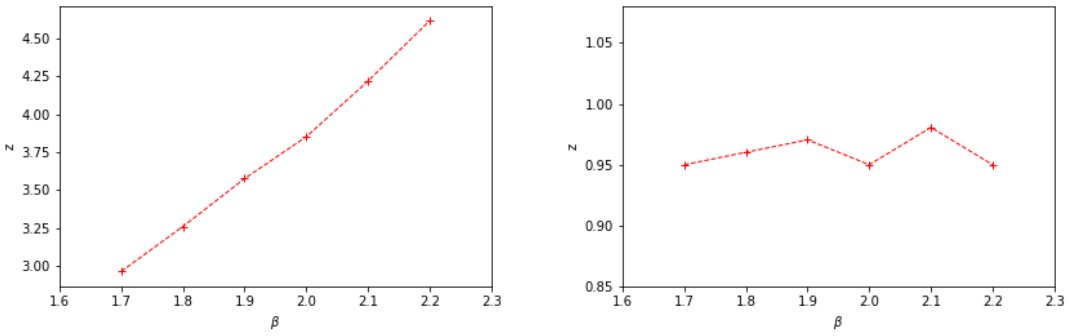
We can see that for both values of θ_1 we observe a sharp increase of the density of particles at different values of z. In the case of $\theta_1 = 0.05$ it occurs at $z \approx 2.98$ and for $\theta_1 = -0.05$ it is at $z \approx 0.96$. One hypothesis for why phase transition appears faster when θ_1 is negative, is probably because the interaction is slightly more repulsive. Indeed, the energy cost of adding a particle in the position $x \in \mathbb{R}^d$ to a configuration ω is lower in comparison to a positive θ_1. Therefore it is easier for the Markov chain to add this particle even if it is far away for the halo $L_R(\omega)$. As a consequence, it is easier to have the halo covering the entirety of the box Λ_F and once this happen the Quermass works in its saturated state as now the cost of adding is 0. In this saturated state it becomes easier to add points all over Λ_F and it is why we obtain the liquid state of matter.



(a) $z \approx 2.98$, $\theta_1 = 0.05$, $\theta_2 = 0$ (b) $z \approx 0.96$, $\theta_1 = -0.05$, $\theta_2 = 0$

Figure 1.8: Quermass model for $\beta = 1.7$

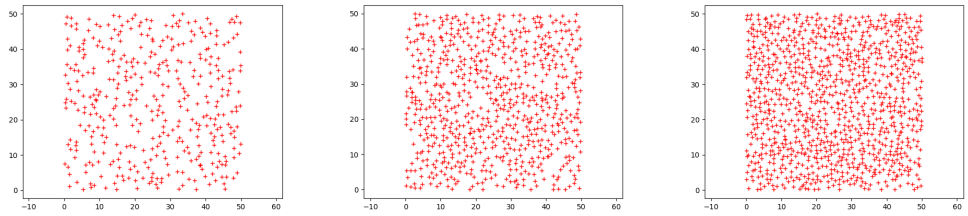
We choose the potential critical activity as being the point where we have the highest growth rate of the the density of particles in the graph 1.7a and 1.7b. This is further corroborated when we look into the evolution of the number of points with respect to the number of iteration of the Markov chain, we can observe that the number of points did not stabilize which gives another indication that a phase transition is happening at the identified critical activity. Thus we obtain the following phase diagram for both set of parameters.



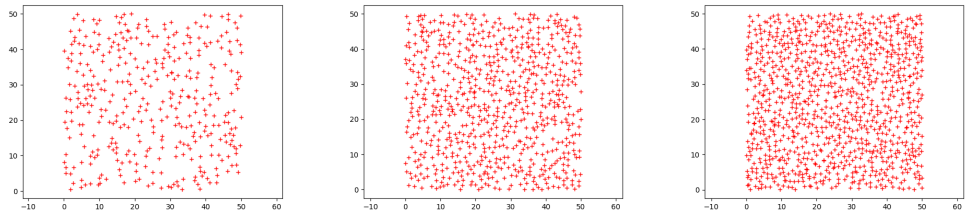
(a) $\theta_1 = 0.05$, $\theta_2 = 0$ (b) $\theta_1 = -0.05$, $\theta_2 = 0$

Figure 1.9: Quermass model phase diagram

Theses results seems to show that saturation seems to be a nice property to obtain a liquid-gas phase transition. In order to do further exploration, we also did some simulations for the K-nearest neighbour Strauss interaction, where the Hamiltonian is given in (1.29). As we have seen previously it is a saturated interaction. We fix $A = 1$ and with this parametrisation the interaction is repulsive. We obtain the following results via simulations with $K = 2$ and $K = 3$.



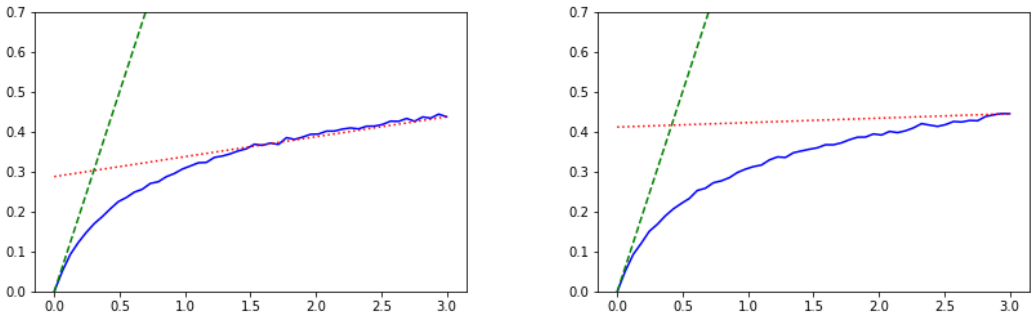
(a) $z \approx 0.31$, $K = 2$, $A = 1$ (b) $z \approx 0.92$, $K = 2$, $A = 1$ (c) $z \approx 2.45$, $K = 2$, $A = 1$



(d) $z \approx 0.31$, $K = 3$, $A = 1$ (e) $z \approx 0.92$, $K = 3$, $A = 1$ (f) $z \approx 2.45$, $K = 3$, $A = 1$

Figure 1.10: repulsive K-nearest neighbour Strauss interaction for $\beta = 1.5$

The behaviour is completely different from the one we saw with the Quermass model. Indeed, for repulsive K-nearest neighbour Strauss interaction at low intensity the particles does not form small clusters and but instead are pretty well distributed in the box (see figures 1.10a, 1.10d). For greater values of activity we can observe more and more cluster particles near one another but we have very few large holes (see figures 1.10b, 1.10c, 1.10e, 1.10f).



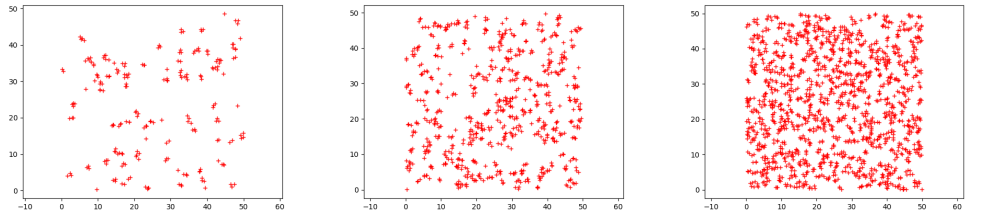
(a) $K = 2$, $A = 1$ (b) $K = 3$, $A = 1$

Figure 1.11: repulsive K-nearest neighbour Strauss interaction for $\beta = 1.5$

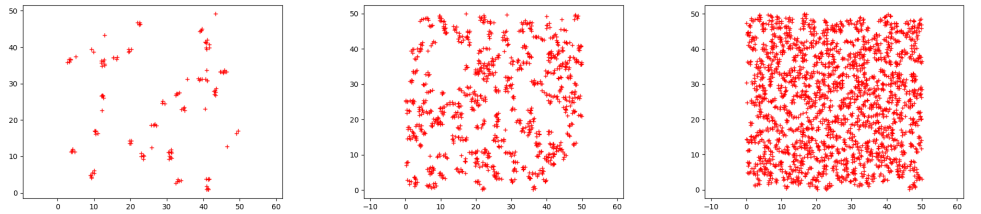
When we plot the density of particles with respect to the activity z, we can observe the saturated behaviour of this model. In figures 1.11a and 1.11b, the green dashed line is the graph of

$y = z$, that corresponds to the density of the Poisson point process with activity z, and the red dotted line the graph of $y = (z-3)e^{-\beta K A} + \rho(3)$, that corresponds to leading terms of the density of Q_i defined in (1.30) for a good choice of δ. In both figures, we can see that for small values of z the growth is linear like the Poisson point process. Then for larger values of activity the repulsive K-nearest neighbour Strauss interaction follows the behaviour of the fully saturated interaction Q_i which is more obvious in figure 1.11a. For the values of z in between, the model transition slowly from the Poisson behaviour to the fully saturated interaction behaviour. Also we do not observe any sharp increase of the intensity of the point process. This seems to indicate that no liquid-gas phase transition occurs for the repulsive K-nearest neighbour Strauss interaction.

From the previous simulations, we can conclude that attractiveness is needed and therefore we fix $A = -1$. We obtain the following results for the attractive K-nearest neighbour Strauss interaction with $K = 2$ and $K = 3$.



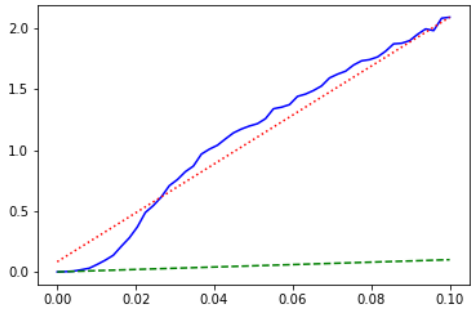
(a) $z \approx 0.014$, $K = 2$, $A = -1$ (b) $z \approx 0.02$, $K = 2$, $A = -1$ (c) $z \approx 0.029$, $K = 2$, $A = -1$



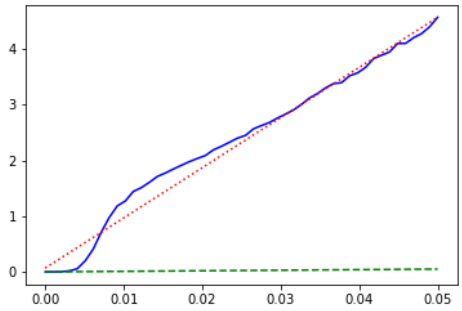
(d) $z \approx 0.005$, $K = 3$, $A = -1$ (e) $z \approx 0.007$, $K = 3$, $A = -1$ (f) $z \approx 0.009$, $K = 3$, $A = -1$

Figure 1.12: attractive K-nearest neighbour Strauss interaction for $\beta = 1.5$

For both values of K, we can observe the same typical configuration as for the Quermass and Widom-Rowlinson models for smaller values of z. Indeed, the particles form small islets as in figures 1.12a and 1.12d. However as the activity grows we do not observe the growth of a single cluster, like for the Quermass or Widom-Rowlinson models. Instead we observe more and bigger clusters (see figures 1.12b and 1.12e). At higher values of activity, we see that the particles covers the whole box again like the Quermass and Widom-Rowlinson models (see figures 1.12c and 1.12f).



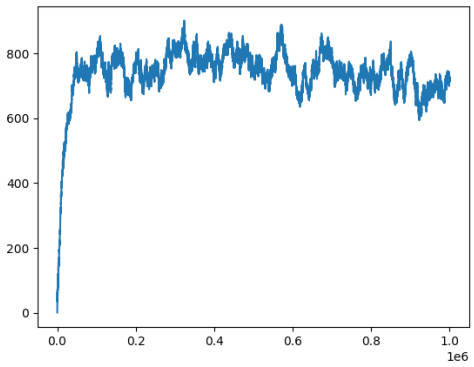
(a) $K = 2$, $A = -1$



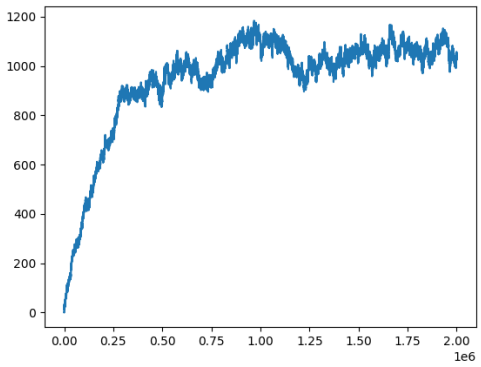
(b) $K = 3$, $A = -1$

Figure 1.13: attractive K-nearest neighbour Strauss interaction for $\beta = 1.5$

When we plot the intensity of the attractive K-nearest neighbour with respect to the activity, we can observe the saturated behaviour of the model just like the repulsive case. The green dashed line represents the density of the Poisson point process with activity z and the red dotted line corresponds to the leading terms of the density of Q_i. In both figures 1.13a and 1.13b, we can observe that the density of point of the attractive K-nearest neighbour Strauss interaction follows the density of the Poisson at small values of z and then it transitions to follow the density the saturated interaction Q_i. This transition is quite fast and is accompanied by a sharp increase of the intensity. It is even more visible for $K = 3$. This could be an indication for a possible liquid-gas phase transition. To verify this we looked the evolution of the number of points with respect to the number of iteration of the Markov chain at the potential critical activity.



(a) $z \approx 0.02$, $K = 2$, $A = -1$



(b) $z \approx 0.007$, $K = 3$, $A = -1$

Figure 1.14: attractive K-nearest neighbour Strauss interaction for $\beta = 1.5$

For both cases, we observe that the number of points in the systems has stabilised quickly and therefore the Markov chain has converged. This is something that indicates that there is no

phase transition as the system does not hesitate between two states with different density, which was the case for the Quermass and Widom-Rowlinson models. Even for larger values of β the attractive K-nearest neighbour Strauss interaction does not exhibit liquid-gas phase transition. Even though the growth rate is steeper for larger β, the graph of intensity remains smooth. Also the transition to the saturated behaviour happens at a lower value of activity.

From these simulations, we observed that saturation property alone is not sufficient to exhibit liquid gas phase transition phenomenon. Other properties seems to be needed in order to exhibit phase transition among those we have attractiveness of the interaction. The most important property that seems necessary is that the global energy of unsaturated tiles are greater than the corresponding energy these tiles are saturated, i.e. $E_\Lambda \geq \overline{E}_\Lambda$, and that this surplus of energy is non negligible with respect to the volume of mixing areas.

Chapter 2

Liquid-Gas Phase transition for Saturated interaction

In this chapter, we present the main results of our work concerning the liquid-gas phase transition for saturated interaction. Our proof is an adaptation of the PSZ theory in the continuous setting. This method has been very successful in demonstrating phase transition phenomenon in lattice systems. The core idea relies on a generalisation of Peierls argument, which has historically been employed to prove spontaneous magnetisation in the Ising model. Our adaptation differs from those made by Lebowitz, Mazel, and Presutti [27] or by Pulvirenti and Tsagkarogiannis [33], and is more suited to saturated interactions. This saturation property allows us to investigate the object while remaining entirely within continuous space. In contrast, in the aforementioned papers, the authors approximate the Gibbs point process using a lattice model and use this discretisation in order to prove the phase transition. For readers familiar with this technique, our proof follows the general idea presented in Chapter 7 of [13] with the required adaptation to our context. Before presenting the phase transition theorems, it is necessary to introduce the notion of contours, as it plays a crucial role in the statement of our results.

2.1 Contours

Let $\delta > 0$ be the length of the tiles $(T_i)_{i\in\mathbb{Z}^d}$. We consider the lattice $\mathbb{Z}^d$ underlying the tiles $(T_i)_{i\in\mathbb{Z}^d}$, where two sites $i, j \in \mathbb{Z}^d$ are connected if $\|i-j\|_\infty = 1$. We call the spin configuration the application

$$\begin{aligned}\sigma :&\Omega_f \times \mathbb{Z}^d \to \{0,1\}\\ &(\omega, i) \mapsto \begin{cases} 0 & \text{if } \omega_{T_i} = \emptyset \\ 1 & \text{otherwise} \end{cases}.\end{aligned}$$

In the following we use the notation # for either 0 or 1.

Definition 15. Let $L > 0$ and $\omega \in \Omega$, a site $i \in \mathbb{Z}^d$ is said to be #-correct if for all sites j such that $\delta\|i-j\| \leq 2L$, we have $\sigma(\omega, j) = \#$. A site i is non-correct when it fails to be #-correct

for any $\# \in \{0,1\}$. The set of all non-correct sites is denoted by $\overline{\Gamma}$. We can partition $\overline{\Gamma}$ into its maximum connected components that we denote by $\overline{\gamma}$ and we call it contour without types.

Since we are considering only finite configurations, the number of connected components is finite and for any $\overline{\gamma}$, the complementary set has a finite amount of maximum connected components that we denote by A and in particular we have only one unbounded connected component and we call it the exterior of $\overline{\gamma}$ that we denote by $ext(\overline{\gamma})$.

Definition 16. Let $\Lambda \subset \mathbb{Z}^d$ and $L > 0$, we define the exterior boundary $\partial_{ext}\Lambda$ and the interior boundary $\partial_{int}\Lambda$ of Λ as

$$\partial_{ext}\Lambda = \{j \in \Lambda^c, \delta d_2(j,\Lambda) \leq 2L\}$$
$$\partial_{int}\Lambda = \{i \in \Lambda, \delta d_2(i,\Lambda^c) \leq 2L + \delta\},$$

where d_2 is the Euclidean distance in $\mathbb{R}^d$.

Lemma 10. *Let $\omega \in \Omega_f$ be a finite configuration and $\overline{\gamma}$ any associated contour without type. Let A be a maximum connected component of $\overline{\gamma}^c$, then there is an unique $\# \in \{0,1\}$ such that for all $i \in \partial_{ext}A \cup \partial_{int}A, \sigma(\omega,i) = \#$. The value of the spin in the boundary is called the label of A and is denoted by* Label(A).

The proof of this lemma is classical and it corresponds to Lemma 7.23 in [13]. It relies on the fact that each set $\partial_{int}A$ and $\partial_{ext}A$ are connected and that the sites directly in contact with the contours are correct. Therefore there can be only one spin $\# \in \{0,1\}$ otherwise we would have two correct sites of opposite spin directly connected.

Definition 17. Let $\omega \in \Omega_f$, we call a contour γ the pair $(\overline{\gamma}, (\#_i)_{i\in\overline{\gamma}})$ where for all sites $i \in \overline{\gamma}$, $\sigma(\omega,i) = \#_i$. We denote by $\Gamma(\omega)$ the set of all contours that appear with the configuration ω.

Furthermore for a contour $\gamma = (\overline{\gamma}, (\#_j)_{j\in\overline{\gamma}})$ we call the type of γ the label of $ext(\overline{\gamma})$, $\text{Type}(\gamma) := \text{Label}(ext(\overline{\gamma}))$. And we call the interiors of a contour γ the sets

$$\text{Int}_{\#}\,\gamma = \bigcup_{\substack{A \neq ext(\overline{\gamma}) \\ \text{Label}(A)=\#}} A \quad \text{and} \quad \text{Int}\,\gamma = \text{Int}_0\,\gamma \cup \text{Int}_1\,\gamma.$$

Let $\omega \in \Omega_f$ be a finite configuration, a contour $\gamma \in \Gamma(\omega)$ is said to be external when for any other contour $\gamma' \in \Gamma(\omega)$, $\overline{\gamma} \subset ext(\overline{\gamma}')$. We denote by Γ_{ext} the subset of Γ comprised only of external contours.

Until now we have only considered collection of contours that can be achieved by a finite configuration of points. But classically in the Pirogov-Sinaï-Zahradník theory we need to introduce abstract collection of contours which are not achievable by any configuration. This is due to the cluster expansion development of the partition function using geometrically compatible collection of contours.

Definition 18. An abstract set of contours is a set of contours $\{\gamma_i = (\overline{\gamma}_i, (\#_j)_{j\in\overline{\gamma}_i}), i \in I \subset \mathbb{N}^*\}$ for which each contour γ_i is achievable for some configuration ω_i. We do not assume the global

achievability. We denote by Γ such set of contours. Moreover this set Γ is called geometrically compatible if for all $\{i, j\} \subset I$, $d_\infty(\gamma_i, \gamma_j) > 1$ and they all have the same type. Let $\Lambda \subset \mathbb{Z}^d$ we denote by $\mathcal{C}^\#(\Lambda)$ the collection of geometrically compatible sets of contours of the type # such that $d_\infty(\gamma_i, \Lambda^c) > 1$.

We allow the set $\Gamma = \{(\emptyset, \emptyset)\}$ to belong to the collection $\mathcal{C}^\#(\Lambda)$ for any Λ, this corresponds to the event where not a single contour appears in Λ. There are several interesting sub-collections of $\mathcal{C}^\#(\Lambda)$, one of them being the collection of sets such that all contours are external.

Definition 19. Let $\Lambda \subset \mathbb{Z}^d$ finite, we denote by $\mathcal{C}^\#_{ext}(\Lambda) \subset \mathcal{C}^\#(\Lambda)$ the sub-collection of sets Γ where any contours $\gamma \in \Gamma$ is external.

In a way, in the collection $\mathcal{C}^\#_{ext}(\Lambda)$ we are considering sets of contours where we have only one layer. In general, if we take a geometrically compatible abstract sets of contours Γ, a particular contour in this set can be encapsulated in the interior of another creating layers upon layers of contours. One method of exploration of the contours is by proceeding from the external layer and peel each layer to discover the other contours hidden under. Another sub-collection of $\mathcal{C}^\#(\Lambda)$ is the collection of sets such that for all contours the size of the interior is bounded.

Definition 20. A contour γ is of the class $k \in \mathbb{N}$ when $|\operatorname{Int}\gamma| = k$. Let $n \in \mathbb{N}$ and $\Lambda \subset \mathbb{Z}^d$ finite, we denote by $\mathcal{C}^\#_n(\Lambda) \subset \mathcal{C}^\#(\Lambda)$ the collection of contours Γ such that $\forall \gamma \in \Gamma$, γ is of the class $k \leq n$.

2.2 Liquid-Gas phase transition theorems

In this section, we start by giving a general theorem for Liquid-Gas phase transition. Subsequently, we investigate into the satisfaction of certain conditions, notably the Peierls condition 2.1. We will explore easier methods to verify this assumption and derive a more accessible theorem. Before proceeding, we need to define the different boundaries for any subset $\Lambda \subset \mathbb{Z}^d$,

$$\partial\Lambda = \{i \in \Lambda, \delta d_2(i, \Lambda^c) \leq L + \delta\}$$
$$\partial^-\Lambda = \{i \in \Lambda, \delta d_2(i, \Lambda^c) \leq L\}.$$

With these definitions in place, we are now ready to state our theorem on the liquid-gas phase transition for saturated interactions.

Theorem 11. *Let H be a saturated interaction, such that E_0 is stable with a finite range $R > 0$ and that there is $C \geq 0$ such that $E_0 \leq C(1 + N_{T_0 \oplus B(0,R)}(\omega)^2)$. By Lemma 8, we know that for any $k \in \mathbb{N}$, $\overline{E}(k) = (Ak + B)\mathbb{1}_{k \geq 1}$ and we assume that $A \geq 0$ and $-A \leq B < +\infty$. Furthermore, we suppose that the interaction verifies a Peierls-like condition, i.e. there is $\rho_0 > 0$ such that for any contours γ and any configuration ω that achieves this contour γ we have*

$$E_{\overline{\gamma} \setminus \partial^- \overline{\gamma}} - \overline{E}_{\overline{\gamma} \setminus \partial^- \overline{\gamma}}(\omega) \geq \rho_0 |\overline{\gamma}|. \tag{2.1}$$

In addition, we will assume that $\frac{\rho_0 l_0}{6} > \min\{A, A+B\}$*, where* l_0 *is the size of the smallest contour. For any* $\beta > 0$*, we fix* z_β^- *and* z_β^+ *as*

$$z_\beta^- := \frac{e^{\beta A}}{\delta^d} \ln\left(1 + e^{\beta B - 2}\right), \quad z_\beta^+ := \frac{e^{\beta A}}{\delta^d} \ln\left(1 + e^{\beta B + 2}\right) \tag{2.2}$$

and $U_\beta := (z_\beta^-, z_\beta^+)$*. Then there is* $\beta_c > 0$*, such that for* $\beta \geq \beta_c$ *there exists* $z_\beta^c \in U_\beta$ *for which a Liquid-Gas phase transition occurs. More specifically, we have*

$$\frac{\partial \psi}{\partial z^+}(\beta, z_c) > \frac{\partial \psi}{\partial z^-}(\beta, z_c), \tag{2.3}$$

and we can find two Gibbs measures $P^+, P^- \in \mathcal{G}(H, z_\beta^c, \beta)$ *such that*

$$\rho(P^+) = z + z\frac{\partial \psi}{\partial z^+}(z_\beta^c, \beta) \quad \textit{and} \quad \rho(P^-) = z + z\frac{\partial \psi}{\partial z^-}(z_\beta^c, \beta). \tag{2.4}$$

The stability, finite range property of E_0 and also the growth of E_0 to be at most equal to N^2 are all properties that can be easily verified for morphological interaction like the Quermass or the pairwise interaction and all the approximations we mentioned in Chapter 1 when the pair potential has a finite range and does not present any singularity. In this set of assumptions the hardest to verify is surely the Peierls condition (2.1).We present here an easy way to verify this hypothesis using dominoes which are pairs of adjacent empty and non empty tiles that we can find in any contours. According to the following lemma the number of such pair is of the same order as the volume of the contour.

Lemma 12. *There exists* $r_0 > 0$ *such that for any contour* γ*, the set of dominoes*

$$D(\gamma) := \{(i, j) \in \overline{\gamma}^2, \|i - j\|_\infty = 1, \#_i = 1, \#_j = 0\}$$

satisfies

$$|D(\gamma)| \geq r_0 |\overline{\gamma}|.$$

Proof. We start by choosing randomly in a contour γ a site k such that $\#_k = 1$. Since it is in a contour, it is non-correct, meaning that there is a site $j \in \gamma$, where $\#_j = 0$ and $\delta\|k - j\| \leq 2L$. We choose such j such as it is the closest to k. Forcibly we have a site i adjacent to j such that $\#_i = 1$ (at least in the direction of k). And we assign $S_1 = \{k\}$ and $D_1 = \{(i, j)\}$. We repeat the process to build S_{n+1} and D_{n+1} by choosing the points inside $\overline{\gamma} \setminus \bigcup_{k \in S_n} B(k, {}^{4L}/_{\delta})$. There is $p \in \mathbb{N}$, the number of step until the process stops because there is a finite number of sites with the spin equal to 1 in a contour. Since we have depleted the contour, we know that $D_p \subset D(\gamma)$ and that

$$\overline{\gamma}_1 := \{i \in \overline{\gamma}, \#_i = 1\} \subset \bigcup_{k \in S_p} B(k, {}^{4L}/_{\delta}) \cap \mathbb{Z}^d.$$

Furthermore, by non-correctness of sites with spin 0 in the contour, we have

$$\overline{\gamma}_0 := \{i \in \overline{\gamma}, \#_i = 0\} \subset \bigcup_{k \in S_p} B(k, {}^{6L}/_{\delta}) \cap \mathbb{Z}^d.$$

In summary we have

$$\overline{\gamma} \subset \bigcup_{k \in S_p} B(k, {}^{6L}/_{\delta}) \cap \mathbb{Z}^d.$$

Therefore the cardinals verify the following inequalities

$$|\overline{\gamma}| \leq |S_p||B(0, {}^{6L}/_{\delta}) \cap \mathbb{Z}^d|.$$

By construction we have $|S_p| = |D_p|$. Hence we obtain the desired inequality

$$|D(\gamma)| \geq |D_p| \geq r_0|\overline{\gamma}| \quad \text{where } r_0 = \frac{1}{|B(0, {}^{6L}/_{\delta}) \cap \mathbb{Z}^d|}.$$

□

Let us consider a saturated interaction such that there is $C > 0$ and for any $k \geq 1$, $\overline{E}(k) = C$, the energy of an non empty saturated tile is constant with respect to the number of points inside of it. Furthermore, we assume that for any configuration we have

$$\begin{aligned} E_0(\omega) &\geq 0 \quad \text{if} \quad N_{T_0}(\omega) = 0, \\ E_0(\omega) &\geq C \quad \text{if} \quad N_{T_0}(\omega) \geq 1, \end{aligned}$$

and there is $e_0 > 0$ such that for any contour γ and any domino $(i, j) \in D(\gamma)$ we have

$$E_j(\omega) \geq e_0. \tag{2.5}$$

This last condition (2.5) is a phenomenon we call the energy from vacuum. Under these hypotheses, it is easy to show that for any contour γ we have

$$E_{\overline{\gamma} \setminus \partial^- \overline{\gamma}} - \overline{E}_{\overline{\gamma} \setminus \partial^- \overline{\gamma}}(\omega) \geq |D(\gamma)|e_0.$$

Therefore, by applying Lemma 12, we obtain the desired Peierls-like with $\rho_0 = r_0 e_0$. As a result, we obtain a theorem with hypothesis that are easier to verify, as the Peierls condition is determined through a tile-by-tile investigation.

Theorem 13. *Let H be a saturated interaction, such that E_0 is stable with a finite range $R > 0$ and that there is $C \geq 0$ such that $E_0 \leq C(1 + N_{T_0 \oplus B(0,R)}(\omega)^2)$. We assume that for any $k \geq 1$, $\overline{E}(k) = \overline{E}(1) > 0$. Furthermore, we suppose that for any configuration $\omega \in \Omega_f$ we have*

$$E_0(\omega) \geq \overline{E}(N_{T_0}(\omega)), \tag{2.6}$$

and there is $e_0 > 0$ such that for any domino $(i, j) \in (\mathbb{Z}^d)^2$ (i.e. $\|i - j\|_\infty = 1$, $N_{T_i}(\omega) \geq 1$ and $N_{T_j}(\omega) = 0$) we have

$$E_j(\omega) \geq e_0. \tag{2.7}$$

Then there exists $\beta_c > 0$, such that for $\beta \geq \beta_c$ there exists $z_\beta^c > 0$ for which a Liquid-Gas phase transition occurs.

In this theorem, we simplify the Peierls-like condition into a simpler hypothesis on E_0. We handled the energy of a contour tile by tile and by doing so we proved the existence of this surplus of energy. This approach is certainly not optimal especially compared to a global approach but it can already yield interesting liquid-gas phase transition results not available before. For instance, let us consider the Area interaction with bounded random radii. It is a marked point process on $\mathbb{R}^d$ with the spin state $S = [R_0, R_1]$ where $R_1 > R_0 > 0$ and the Hamiltonian is given by

$$H(\omega) = \mathcal{V}(L(\omega)) \quad \text{where} \quad L(\omega) = \bigcup_{(x,R)\in\omega} B(x, R).$$

In this context, the arguments using the symmetry of the two color Widom-Rowlinson does not work any more, as for instance the one used by Ruelle [30], Chayes-Chayes-Kotecký [3] and Giacomin-Lebowitz-Maes [19]. With our approach, we can observe that when $2\sqrt{d}\delta \leq R_0$ and $L \geq 2R_1 + \delta$ this interaction is saturated with

$$E_i(\omega) = \mathcal{V}(L(\omega) \cap T_i), \quad \text{and} \quad \forall k \geq 1, \overline{E}(k) = \delta^d.$$

More interestingly we have by the size of the tile

$$E_0(\omega) \geq \overline{E}(N_{T_0}(\omega))$$

and for any domino (i, j), since for $(x, R) \in \omega_{T_i}$ we have $T_j \subset B(x, R)$ and thus $E_j(\omega) = \delta^d$. We can therefore apply Theorem 13 and obtain a new result.

Theorem 14 (Liquid-gas phase transition for the Area interaction with bounded random radii)**.** *Let R_0, R_1 be strictly positive and $R_0 < R_1$. Let us consider the marked point process of Area interaction model with random radii belonging to $[R_0, R_1] = S$ and any probability measure P_S. Then there exists $\beta_c > 0$ such that for $\beta > \beta_c$ there is $z^c_\beta > 0$ for which a Liquid-Gas phase transition occurs.*

In section 3.1, we improve on this result by considering a larger class of interactions and by deriving asymptotic behaviours for z^c_β when β tends to infinity. The rest of the chapter is dedicated to establishing Theorem 11 and introducing the tools needed for the proof.

2.3 Cluster Expansion

In this section, we provide a concise overview of cluster expansion, presenting the necessary tools for proving Theorem 11. For a more comprehensive understanding of this topic, we recommend consulting the detailed presentation by Velenik in Chapters 5 and 7 of [13], or to Chapter 7 of Jansen's course on Gibbsian point processes [22].

Let C be a collection of contours and for all $\Lambda \subset \mathbb{Z}^d$, $C(\Lambda)$ a sub-collection of contours γ such that $\overline{\gamma} \subset \Lambda$. For each contour γ in such collection we associate a weight w_γ, and we consider the weights to be invariant by translation. We set $l_0 = \min\{|\overline{\gamma}|, \gamma \in C\}$ and $\eta(\tau, l_0) = 2\exp(-\tau l_0/3)$. A set of contours $\Gamma = \{\gamma_1, \dots, \gamma_n\} \in C$, is said to be geometrically compatible if for all $i, j \in$

$\{1, \dots, n\}, i \neq j$ we have $d_\infty(\gamma_i, \gamma_j) \geq 1$. We define the polymer development associated to those weights as, for all $\Lambda \subset \mathbb{Z}^d$

$$\Phi(\Lambda) = \sum_{\substack{\Gamma \in \mathcal{C}(\Lambda) \\ \text{geometrically compatible}}} \prod_{\gamma \in \Gamma} w_\gamma.$$

A collection $C = \{\gamma_1, \cdots, \gamma_n\}$ is said to be decomposable if the support $\overline{C} = \bigcup_{\gamma \in C} \overline{\gamma}$ is not simply connected. A cluster, denoted by X, is a non-decomposable finite multiset of contours such that a same contour can appear multiple times and we define $\overline{X} := \bigcup_{\gamma \in X} \overline{\gamma}$. The cluster expansion for $\ln \Phi(\Lambda)$, if it converges, is given by

$$\ln \Phi(\Lambda) = \sum_{X \,:\, \overline{X} \subset \Lambda} \Psi(X)$$

where $\Psi(X) := \alpha(X) \prod_{\gamma \in X} w_\gamma$. We have this combinatorial term α whose expression is given by

$$\alpha(X) = \left\{ \prod_{\gamma \in \mathcal{C}(\Lambda)} \frac{1}{n_X(\gamma)!} \right\} \left\{ \sum_{\substack{G \subset G_n \\ \text{connected}}} \prod_{\{i,j\} \in G} \zeta(\gamma_i, \gamma_j) \right\}$$

where $n_X(\gamma)$ is the number of times a contour γ appears in the cluster X, $G_n = (V_n, E_n)$ is the undirected complete graph on $V_n = \{1, \cdots, n\}$ and ζ is defined as

$$\zeta(\gamma, \gamma') = \begin{cases} 0 & \text{if } \gamma, \gamma' \text{ are geometrically compatible} \\ -1 & \text{otherwise} \end{cases}.$$

According to Theorem 5.4 in [13] (see also section 7.4.1 in [13]) a sufficient condition for the convergence of the cluster expansion is

$$\sum_{\substack{\gamma \in \mathcal{C} \\ 0 \in \overline{\gamma}}} |w_\gamma| e^{3^d |\overline{\gamma}|} \leq 1. \tag{2.8}$$

Therefore, (2.8) holds whenever the weights w_γ decays exponentially fast with the volume of the contour. This leads naturally to the notion of τ-stable weights.

Definition 21. For $\tau > 0$, the weights $w_\gamma \in \mathcal{C}$ are τ-stable if

$$|w_\gamma| \leq e^{-\tau |\overline{\gamma}|}.$$

Under the assumption that the weights are τ-stable and if τ is large enough we have

$$\sum_{\substack{\gamma \in \mathcal{C} \\ 0 \in \overline{\gamma}}} e^{-\tau |\overline{\gamma}|} e^{3^d |\overline{\gamma}|} \leq 1$$

and therefore we have the convergence of the cluster expansion. In practice, we will take a larger value for τ such that the stronger assumption of Lemma 15 is verified. The following lemmas correspond to Lemma 7.30 and Lemma 7.31 in [13].

Lemma 15. *There exists $\tau_0 > 0$ such that when $\tau > \tau_0$*

$$\sum_{\gamma \in C : 0 \in \overline{\gamma}} |\overline{\gamma}|^{d/d-1} e^{-(\tau/2-1)|\overline{\gamma}|} e^{3^d |\overline{\gamma}|} \leq \eta(\tau, l_0) \leq 1. \tag{2.9}$$

Lemma 16. *Let us assume that the weights are τ-stable for $\tau > \tau_0$. Then for $L \geq l_0$*

$$\sum_{\substack{X : 0 \in \overline{X} \\ |\overline{X}| \geq L}} |\Psi(X)| \leq e^{-\frac{\tau L}{2}}. \tag{2.10}$$

Theorem 17. *Assume that, for all $\gamma \in C$, the weight w_γ is C^1 in a parameter $s \in (a, b)$ and that uniformly on (a, b),*

$$w_\gamma \leq e^{-\tau|\overline{\gamma}|}, \qquad \left| \frac{d w_\gamma}{d s} \right| \leq D |\overline{\gamma}|^{d/d-1} e^{-\tau|\overline{\gamma}|}, \tag{2.11}$$

where $D \geq 1$ is a constant. Then there exists $\tau_1 = \tau_1(D, d) < \infty$ such that the following holds. If $\tau > \tau_1$, we define the quantity g with the following absolutely convergent series,

$$g = \sum_{X : 0 \in \overline{X}} \frac{1}{|\overline{X}|} \Psi(X) \tag{2.12}$$

where the sum is over clusters X made of contours $\gamma \in C$ and $\overline{X} = \bigcup_{\gamma \in X} \overline{\gamma}$. Moreover,

$$|g| \leq \eta(\tau, l_0) \leq 1$$

and for all $\Lambda \subset \mathbb{Z}^d$ finite, g provides the volume contribution to $\log \Phi(\Lambda)$*, in the sense that*

$$\Phi(\Lambda) = \exp(g|\Lambda| + \Delta_\Lambda) \tag{2.13}$$

where Δ_Λ is a boundary term :

$$|\Delta_\Lambda| \leq \eta(\tau, l_0) |\partial_{ext} \Lambda|. \tag{2.14}$$

Finally, g and Δ_Λ are also C^1 in $s \in (a, b)$; its derivative equals

$$\frac{d g}{d s} = \sum_{X : 0 \in \overline{X}} \frac{1}{|\overline{X}|} \frac{d \Psi(X)}{d s} \tag{2.15}$$

and

$$\left| \frac{d g}{d s} \right| \leq D \eta(\tau, l_0), \qquad \left| \frac{d \Delta_\Lambda}{d s} \right| \leq D \eta(\tau, l_0) |\partial_{ext} \Lambda|. \tag{2.16}$$

This theorem is similar to Theorem 7.29 in [13], the only difference being is that the following statement is not included

$$\left| \frac{d \Delta_\Lambda}{d s} \right| \leq D \eta(\tau, l_0) |\partial_{ext} \Lambda|.$$

We provide a short proof of this claim. Following the computations in [13] we find

$$\frac{d \Delta_\Lambda}{d s} = \sum_{i \in \Lambda} \sum_{X : i \in \overline{X} \not\subset \Lambda} \frac{1}{|\overline{X}|} \frac{d \Psi}{d s}(X).$$

Whenever $i \in \overline{X} \not\subset \Lambda$ we know that $\overline{X} \cap \partial_{ext}\Lambda \neq \emptyset$ and since the weights are invariant by translation we have

$$\left|\frac{d\Delta_\Lambda}{ds}\right| \leq |\partial_{ext}\Lambda| \max_{j\in\partial_{ext}\Lambda} \sum_{X:j\in\overline{X}} \left|\frac{d\Psi}{ds}(X)\right| = |\partial_{ext}\Lambda| \sum_{X:0\in\overline{X}} \left|\frac{d\Psi}{ds}(X)\right|.$$

We can show that for any cluster X we have

$$\left|\frac{d\Psi}{ds}(X)\right| \leq |\overline{\Psi}(X)| \tag{2.17}$$

where

$$\overline{\Psi}(X) = \alpha(X)\prod_{\gamma\in X}\overline{w}_\gamma \quad \text{and} \quad \overline{w}_\gamma = D|\overline{\gamma}|^{d/d-1}e^{-(\tau-1)|\overline{\gamma}|}.$$

Following the same computation needed to prove (2.15) we can show that

$$\sum_{X:0\in\overline{X}} |\overline{\Psi}(X)| \leq \sum_{\gamma\in C:0\in\overline{\gamma}} \overline{w}_\gamma e^{3^d|\overline{\gamma}|}.$$

Therefore with Lemma 15 we have the desired control on the derivative of the boundary term.

2.4 Gibbs point processes with boundary conditions

2.4.1 Existence of infinite Gibbs measures with boundary conditions

Let H be a finite range and stationary Hamiltonian that verifies the saturation property (see Definition 14). Therefore there exists $\delta > 0$, $L > 0$ and two measurable functions E_0 and $\overline{E}$ corresponding to the coarse graining of the Hamiltonian and its behaviour when saturated by the tiles at distance L. Furthermore we assume that E_0 is stable with stability constant $C \geq 0$ and that E_0 has a finite range $R \in (0, L)$. We can remark that if E_0 is saturated with L it will be saturated with $L' > L$, therefore we can always choose $L > R$. Furthermore, we will consider that $\overline{E}(1) < +\infty$, in other words we make the assumption that the sufficiently dense configurations are allowed by the system. With theses assumptions on E_0 and $\overline{E}$ we build two distinct families of point processes using different boundary conditions. Let us recall the definition of these different boundaries of any subset $\Lambda \subset \mathbb{Z}^d$,

$$\partial\Lambda = \{i \in \Lambda, \delta d_2(i, \Lambda^c) \leq L + \delta\}$$
$$\partial^-\Lambda = \{i \in \Lambda, \delta d_2(i, \Lambda^c) \leq L\}.$$

Definition 22. For $\# \in \{0, 1\}$, $\Lambda \subset \mathbb{Z}^d$ and boundary condition $(\#)_\Lambda = \{\omega \in \Omega, \forall i \in \partial_{int}\Lambda, \sigma(\omega, i) = \#\}$, we define the distribution $P^\#_\Lambda$ such as

$$P^\#_\Lambda := \frac{1}{Z^\#_\Lambda} e^{-\beta(E_{\Lambda\setminus\partial\Lambda} + \overline{E}_{\partial\Lambda})} \mathbb{1}_{(\#)_\Lambda} \Pi^z_{\hat{\Lambda}} \tag{2.18}$$

where $Z^\#_\Lambda$ is the partition function and it is given by

$$Z^\#_\Lambda = \int e^{-\beta(E_{\Lambda\setminus\partial\Lambda} + \overline{E}_{\partial\Lambda})} \mathbb{1}_{(\#)_\Lambda} \Pi^z_{\hat{\Lambda}}. \tag{2.19}$$

This probability measure is well defined. For # = 0 we have

$$Z_\Lambda^{(0)} \geq \Pi_{\widehat{\Lambda}}^z(\emptyset) = e^{-z\delta^d|\Lambda|} > 0. \tag{2.20}$$

When # = 1 and with $A = \{\omega \in \Omega, \forall i \in \Lambda, N_{T_i}(\omega) = 1\}$ we have

$$Z_\Lambda^{(1)} \geq e^{-\beta\overline{E}(1)|\Lambda|}\Pi_{\widehat{\Lambda}}^z(A) = (z\delta^d)^{|\Lambda|}e^{-(z\delta^d+\beta\overline{E}(1))|\Lambda|} > 0. \tag{2.21}$$

Furthermore, using the same arguments as in Lemma 7 we can show that there is $c \geq 0$ such that for any configuration $\omega \in \Omega$

$$E_{\Lambda\setminus\partial\Lambda}(\omega) + \overline{E}_{\partial\Lambda}(\omega) \geq -cN(\omega) \tag{2.22}$$

and therefore,

$$Z_\Lambda^\# \leq \int_\Omega e^{\beta c N(\omega)}\Pi_{\widehat{\Lambda}}^z(d\omega) = \exp(z\delta^d|\Lambda|(e^{\beta c} - 1)) < +\infty.$$

Now that the point process is well defined, we prove in the following proposition that $P_\Lambda^\#$ verifies the same finite volume DLR equations with Hamiltonian H when Δ is in the bulk of the box $\widehat{\Lambda}$.

Proposition 10. *Let $\Lambda \subset \mathbb{Z}^d$ and $\Delta \subset \Lambda\widehat{\setminus\partial_{int}}\Lambda$ such that $\lambda(\Delta) > 0$. Then for $P_\Lambda^\#$-a.s. all ω_{Δ^c}*

$$P_\Lambda^\#(d\eta_\Delta|\omega_{\Delta^c}) = \frac{1}{Z_\Delta(\omega_{\Delta^c})}e^{-\beta H_\Delta(\eta_\Delta\cup\omega_{\Delta^c})}\Pi_\Delta^z(d\eta), \tag{2.23}$$

where $Z_\Delta(\omega_{\Delta^c})$ is the normalisation constant given by $Z_\Delta(\omega_{\Delta^c}) = \int e^{-\beta H_\Delta(\eta_\Delta\cup\omega_{\Delta^c})}\Pi_\Delta^z(d\eta)$.

Proof. We denote by $\Lambda_\Delta = \{i \in \Lambda, T_i \oplus B(0,R) \cap \Delta \neq \emptyset\}$ where $R > 0$ is the range of E_0. By definition of H_Δ we have

$$H_\Delta(\omega) = \sum_{i\in\Lambda_\Delta} E_i(\omega) - E_i(\omega_{\Delta^c}).$$

And for $i \in \Lambda \setminus \Lambda_\Delta$ by finite range property of E_0 we have

$$E_i(\omega) = E_i(\omega_{\Delta^c}).$$

Therefore we have

$$\begin{aligned} P_\Lambda^\#(d\omega) &= \frac{1}{Z_\Lambda^\#}e^{-\beta(\overline{E}_{\partial\Lambda}+E_{\Lambda\setminus\partial\Lambda})}\mathbb{1}_{(\#)_\Lambda}(\omega)\Pi_{\widehat{\Lambda}}^z(d\omega) \\ &= \frac{1}{Z_\Lambda^\#}e^{-\beta H_\Delta(\eta_\Delta\cup\omega_{\Delta^c})}e^{-\beta(\overline{E}_{\partial\Lambda}+E_{\Lambda\setminus(\partial\Lambda\cup\Lambda_\Delta)}-E_{\Lambda_\Delta})(\omega_{\Delta^c})}\mathbb{1}_{(\#)_\Lambda}(\omega_{\Delta^c})\Pi_\Delta^z(d\eta)\Pi_{\widehat{\Lambda}\setminus\Delta}^z(d\omega). \end{aligned}$$

Therefore the unnormalised conditional density of $P_\Lambda^\#(d\eta_\Delta|\omega_{\Delta^c})$ with respect to Π_Δ^z is $\eta \mapsto e^{-\beta H_\Delta(\eta_\Delta\cup\omega_{\Delta^c})}$ and with the proposed normalisation we obtain the desired DLR equations for $\Delta \subset \Lambda\widehat{\setminus\partial_{int}}\Lambda$. □

Now that we have proved some finite volume DLR equations with the same Hamiltonian H, we demonstrate in the following proposition that using $P^{\#}_{\Lambda}$ we can build an infinite volume Gibbs measure.

Proposition 11. *The empirical field* $\left(\overline{P}^{\#}_{\Lambda_n}\right)_{n\in\mathbb{N}}$ *has an accumulation point* $P^{\#}$ *that is a Gibbs measure for the initial Hamiltonian* H.

Proof. Using inequality (2.22) we obtain the following upper bound on the relative entropy

$$\begin{aligned} I(P^{\#}_{\Lambda}|\Pi^{\xi}_{\widehat{\Lambda}}) &= E_{P^{\#}_{\Lambda}}\left(-\beta(E_{\Lambda\setminus\partial\Lambda}+\overline{E}_{\partial\Lambda})+\ln\frac{z}{\xi}N_{\widehat{\Lambda}}\right)+(\xi-z)\delta^d|\Lambda|-\ln Z^{\#}_{\Lambda}\\ &\leq E_{P^{\#}_{\Lambda}}\left((\beta c+\ln\frac{z}{\xi})N_{\widehat{\Lambda}}\right)+(\xi-z)\delta^d|\Lambda|-\ln Z^{\#}_{\Lambda}. \end{aligned}$$

Therefore when we fix $\xi = ze^{\beta c}$ and using either (2.20) or (2.21) we have

$$\begin{aligned} &\frac{1}{\lambda(\widehat{\Lambda})}I(P^{(0)}_{\Lambda}|\Pi^{ze^{\beta c}}_{\widehat{\Lambda}}) \leq ze^{\beta c}\\ &\frac{1}{\lambda(\widehat{\Lambda})}I(P^{(1)}_{\Lambda}|\Pi^{ze^{\beta c}}_{\widehat{\Lambda}}) \leq ze^{\beta c}+\frac{\beta}{\delta^d}\overline{E}(1)-\frac{\ln(z\delta^d)}{\delta^d}. \end{aligned}$$

Either way, by Proposition 6, the sequence $\left(\overline{P}^{\#}_{\Lambda_n}\right)_{n\in\mathbb{N}}$ is tight for the local convergence topology and therefore there is $P^{\#}$ a probability measure on Ω that is an accumulation point for the sequence. Now we consider $\Delta\subset\mathbb{R}^d$ and f a bounded local function. Let us define a function f_Δ as

$$f_\Delta : \omega \mapsto \int f(\eta_\Delta\cup\omega_{\Delta^c})\frac{e^{-\beta H_\Delta(\eta_\Delta\cup\omega_{\Delta^c})}}{Z_\Delta(\omega_{\Delta^c})}\Pi^z_\Delta(d\eta)$$

where $Z_\Delta(\omega_{\Delta^c})$ is the normalisation constant given by $Z_\Delta(\omega_{\Delta^c}) = \int e^{-\beta H_\Delta(\eta_\Delta\cup\omega_{\Delta^c})}\Pi^z_\Delta(d\eta)$. Since E_0 has a finite range then H is finite range by Lemma 7. Therefore f_Δ is a bounded local function and we have

$$\begin{aligned} \int f_\Delta(\omega)\overline{P}^{\#}_{\Lambda_n}(d\omega) &= \frac{1}{\delta^d|\Lambda_n|}\int_{\widehat{\Lambda}_n}\int f_\Delta(\tau_u(\omega))P^{\#}_{\Lambda_n}(d\omega)du\\ &= \frac{1}{\delta^d|\Lambda_n|}\int_{\widehat{\Lambda}_n}\iint f(\eta_\Delta\cup\tau_u(\omega)_{\Delta^c})\frac{e^{-\beta H_\Delta(\eta_\delta\cup\tau_u(\omega)_{\Delta^c})}}{Z_\Delta(\tau_u(\omega)_{\Delta^c})}\Pi^z_\Delta(d\eta)P^{\#}_{\Lambda_n}(d\omega)du\\ &= \frac{1}{\delta^d|\Lambda_n|}\int_{\widehat{\Lambda}_n}\iint f(\tau_u(\eta_{\tau_{-u}(\Delta)}\cup\omega_{\tau_{-u}(\Delta)^c}))\frac{e^{-\beta H_{\tau_{-u}(\Delta)}(\eta_{\tau_{-u}(\Delta)}\cup\omega_{\tau_{-u}(\Delta)^c})}}{Z_{\tau_{-u}(\Delta)}(\omega_{\tau_{-u}(\Delta)^c})}\Pi^z_{\tau_{-u}(\Delta)}(d\eta)P^{\#}_{\Lambda_n}(d\omega)du. \end{aligned}$$

We define the subset $\widehat{\Lambda}^*_n = \{u\in\widehat{\Lambda}_n, \tau_{-u}(\Delta)\subset\Lambda_n\widehat{\setminus\partial_{int}}\Lambda_n\}$, when $u\in\widehat{\Lambda}^*_n$ we know that $P^{\#}_{\Lambda_n}$ verify DLR equations on $\tau_{-u}(\Delta)$ by Proposition 10 and therefore

$$\iint f(\tau_u(\eta_{\tau_{-u}(\Delta)}\cup\omega_{\tau_{-u}(\Delta)^c}))\frac{e^{-\beta H_{\tau_{-u}(\Delta)}(\eta_{\tau_{-u}(\Delta)}\cup\omega_{\tau_{-u}(\Delta)^c})}}{Z_{\tau_{-u}(\Delta)}(\omega_{\tau_{-u}(\Delta)^c})}\Pi^z_{\tau_{-u}(\Delta)}(d\eta)P^{\#}_{\Lambda_n}(d\omega) = \int f(\tau_u(\omega))P^{\#}_{\Lambda_n}(d\omega).$$

Now we need to deal with the boundary terms, i.e.

$$B_1 = \int_{\widehat{\Lambda}_n \setminus \widehat{\Lambda}_n^*} \iint f(\tau_u(\eta_{\tau_{-u}(\Delta)} \cup \omega_{\tau_{-u}(\Delta)^c})) \frac{e^{-\beta H_{\tau_{-u}(\Delta)}(\eta_{\tau_{-u}(\Delta)} \cup \omega_{\tau_{-u}(\Delta)^c})}}{Z_{\tau_{-u}(\Delta)}(\omega_{\tau_{-u}(\Delta)^c})} \Pi^z_{\tau_{-u}(\Delta)}(d\eta) P^{\#}_{\Lambda_n}(d\omega) du$$

$$B_2 = \int_{\widehat{\Lambda}_n \setminus \widehat{\Lambda}_n^*} \int f(\tau_u(\omega)) P^{\#}_{\Lambda_n}(d\omega),$$

since f is bounded we have

$$|B_1| + |B_2| \leq 2M\lambda\left(\widehat{\Lambda}_n \setminus \widehat{\Lambda}_n^*\right).$$

We can observe that $\lambda(\widehat{\Lambda}_n^*)$ is equivalent to $\lambda(\widehat{\Lambda}_n)$ and thus, for some sub-sequence n_k, we have

$$\begin{aligned}
\int f_\Delta(\omega) P^{\#}(d\omega) &= \lim_{k \to +\infty} \frac{1}{\delta^d |\Lambda_{n_k}|} \int_{\Lambda_{n_k}} f_\Delta(\omega) \overline{P}^{\#}_{\Lambda_{n_k}}(d\omega) \\
&= \lim_{k \to +\infty} \frac{1}{\delta^d |\Lambda_{n_k}|} \int_{\Lambda_{n_k}} \int f(\tau_u(\omega)) P^{\#}_{\Lambda_{n_k}}(d\omega) \\
&= \lim_{k \to +\infty} \int f(\omega) \overline{P}_{\Lambda^{\#}_{n_k}}(d\omega) \\
&= \int f(\omega) P^{\#}(d\omega),
\end{aligned}$$

that is the DLR equations for any subset $\Delta \subset \mathbb{R}^d$ and any bounded local function f. □

2.4.2 Polymer development

Now we have proved that we can obtain an infinite Gibbs measure for the saturated interaction H via $P^{\#}_{\Lambda}$, we would want to compare the partition function $Z^{\#}_{\Lambda}$ to the partition function $Z^{|\Lambda|}_{\#}$ which corresponds to the normalisation constant of the system with each tiles saturated i.e.

$$Q^{\#}_{\Lambda} = \frac{1}{Z^{|\Lambda|}_{\#}} e^{-\beta \overline{E}_\Lambda} \mathbb{1}_{\{\sigma(\omega,i)=\#, \forall i \in \Lambda\}} \Pi^z_{\widehat{\Lambda}}. \tag{2.24}$$

The constants Z_0 and Z_1 are equal to

$$Z_0 = e^{-z\delta^d} \quad \text{and} \quad Z_1 = \sum_{k=1}^{+\infty} \frac{(z\delta^d)^k}{k!} e^{-\beta \overline{E}(k)} e^{-z\delta^d}.$$

With the assumption that E_0 is stable and that $\overline{E}(1) < +\infty$ we get that $0 < Z_1 < +\infty$.

Proposition 12 (Polymer development)**.** *For any $\Lambda \subset \mathbb{Z}^d$ finite and any $\# \in \{0,1\}$ we have*

$$\Phi^{\#}_{\Lambda} := Z^{-|\Lambda|}_{\#} Z^{\#}_{\Lambda} = \sum_{\Gamma \in \mathcal{C}^{\#}(\Lambda)} \prod_{\gamma \in \Gamma} w^{\#}_{\gamma}$$

where $w^{\#}_{\gamma} = Z^{-|\overline{\gamma}|}_{\#} I_{\gamma} \frac{Z^{\#^*}_{\mathrm{Int}_{\#^*}\gamma}}{Z^{\#}_{\mathrm{Int}_{\#^*}\gamma}}$, it is called the weight of the contour γ, with $\#^* := 1 - \#$ and

$$I_{\gamma} := \int e^{-\beta(E_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega) + \overline{E}_{\partial^-\overline{\gamma}})} \mathbb{1}_{\left(\forall i \in \overline{\gamma}, \sigma(\omega,i) = \sigma_i\right)} \Pi^{z}_{\widehat{\gamma}}(d\omega).$$

Proof. We follow a similar development done in Chapter 7 in [13] with an adaptation to the setting of our model where the main difference is that the states of sites are random and have to be integrated under the Poisson measure. We can decompose the partition function $Z^{\#}_{\Lambda}$ according to the external contours $\mathcal{C}^{\#}_{ext}(\Lambda)$ and we have

$$Z^{\#}_{\Lambda} = \sum_{\Gamma \in \mathcal{C}^{\#}_{ext}(\Lambda)} \int e^{-\beta(E_{\Lambda\setminus\partial\Lambda} + \overline{E}_{\partial\Lambda})} \mathbb{1}_{(\#)_{\Lambda}} \mathbb{1}_{\{\Gamma_{ext}(\omega) = \Gamma\}} \Pi^{z}_{\widehat{\Lambda}}(d\omega).$$

For any $\Gamma \in \mathcal{C}^{\#}_{ext}(\Lambda)$ we can do a partition of Λ in the following way

$$\Lambda = \Lambda_{ext} \bigcup_{\gamma \in \Gamma} \left(\gamma \cup \mathrm{Int}_0\, \gamma \cup \mathrm{Int}_1\, \gamma\right)$$

where $\Lambda_{ext} = \bigcap_{\gamma \in \Gamma} ext(\overline{\gamma}) \cap \Lambda$. By the way the contours are built, we are assured that for any $\gamma \in \Gamma$, the sites in $\partial^-\overline{\gamma}$ or $\partial(\overline{\gamma}^c)$ are saturated. Therefore for any configuration $\omega \in (\#)_{\Lambda} \cap \{\Gamma_{ext}(\omega) = \Gamma\}$ we have

$$E_{\overline{\gamma}}(\omega) = \overline{E}_{\partial^-\overline{\gamma}}(\omega) + E_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega) \tag{2.25}$$

$$E_{\mathrm{Int}_{\#}\gamma}(\omega) = \overline{E}_{\partial\,\mathrm{Int}_{\#}\gamma}(\omega) + E_{\mathrm{Int}_{\#}\gamma\setminus\partial\,\mathrm{Int}_{\#}\gamma}(\omega). \tag{2.26}$$

Furthermore, since E_0 has a finite range $R < L$ and using the right hand term of equations (2.25) and (2.26), we have that the energy of the tiles in the contour γ and it's interior depends only on the configuration inside these areas. Concerning the energy of the tiles in Λ_{ext}, theses sites are either saturated or part of the boundary $\partial\Lambda$ and therefore

$$E_{\Lambda_{ext}}(\omega) = \overline{E}_{\Lambda_{ext}}(\omega). \tag{2.27}$$

As a consequence, using the independence of Poisson point process in disjoint areas we obtain

$$Z^{\#}_{\Lambda} = \sum_{\Gamma \in \mathcal{C}^{\#}_{ext}(\Lambda)} Z^{|\Lambda_{ext}|}_{\#} \prod_{\gamma \in \Gamma} I_{\gamma} Z^{\#}_{\mathrm{Int}_{\#}\gamma} Z^{\#^*}_{\mathrm{Int}_{\#^*}\gamma}$$

with the following convention $Z^{\#}_{\emptyset} = 1$. Therefore we have

$$\begin{aligned}
\Phi^{\#}_{\Lambda} &= Z^{-|\Lambda|}_{\#} Z^{\#}_{\Lambda} \\
&= \sum_{\Gamma \in \mathcal{C}^{\#}_{ext}(\Lambda)} \prod_{\gamma \in \Gamma} Z^{-|\overline{\gamma}|}_{\#} I_{\gamma} \frac{Z^{\#^*}_{\mathrm{Int}_{\#^*}\gamma}}{Z^{\#}_{\mathrm{Int}_{\#^*}\gamma}} \Phi^{\#}_{\mathrm{Int}_{\#^*}\gamma} \Phi^{\#}_{\mathrm{Int}_{\#}\gamma} \\
&= \sum_{\Gamma \in \mathcal{C}^{\#}_{ext}(\Lambda)} \prod_{\gamma \in \Gamma} w^{\#}_{\gamma} \Phi^{\#}_{\mathrm{Int}_{\#^*}\gamma} \Phi^{\#}_{\mathrm{Int}_{\#}\gamma}
\end{aligned}$$

We can iterate the same computations for $\Phi^{\#}_{\mathrm{Int}_{\#}\gamma}$ and $\Phi^{\#}_{\mathrm{Int}_{\#^*}\gamma}$ until we deplete the interior of the contours. It is a process of discovering the configuration by peeling layers after layers of contours. By doing this we obtain that

$$\Phi^{\#}_{\Lambda} = \sum_{\Gamma \in \mathcal{C}^{\#}(\Lambda)} \prod_{\gamma \in \Gamma} w^{\#}_{\gamma}.$$

□

We can remark that the saturation is needed to prove that the partition function has a polymer development but a weaker condition for this exist. All we need is something we call energy screening by the contours, i.e. for any contours γ

$$E_{\overline{\gamma}}(\omega) = E_{\overline{\gamma}}(\omega_{\widehat{\overline{\gamma}}}) \quad \text{and} \quad E_A(\omega) = E_A(\omega_{\widehat{A}})$$

where A is any maximum connected component of $\overline{\gamma}^c$.

Before proceeding further, we need the following lemma where we use the Peierls condition to obtain an upper bound on I_γ and its derivative with respect to z.

Lemma 18. *Let H be a saturated interaction, such that E_0 is stable and has a finite range. We assume that for any contours γ and any configuration ω that achieve the contour γ there exists $\rho_0 > 0$ such that*

$$E_{\overline{\gamma}\setminus\partial^-\overline{\gamma}} - \overline{E}_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega) \geq \rho_0 |\overline{\gamma}|. \tag{2.28}$$

Then we have

$$I_\gamma \leq Z_0^{|\overline{\gamma}_0|} Z_1^{|\overline{\gamma}_1|} e^{-\beta\rho_0|\overline{\gamma}|} \tag{2.29}$$

$$\left|\frac{\partial I_\gamma}{\partial z}\right| \leq \left(1 + \frac{1}{z\delta^d} E_{Q_0^{(1)}}(N_{T_0})\right) |\overline{\gamma}| \delta^d Z_0^{|\overline{\gamma}_0|} Z_1^{|\overline{\gamma}_1|} e^{-\beta\rho_0|\overline{\gamma}|} \tag{2.30}$$

where $Q_0^{(1)}$ is the probability measure given in (2.24)*.*

Proof. For the proof of (2.29) we simply inject (2.28) into I_γ and have

$$\begin{aligned} I_\gamma &\leq e^{-\beta\rho_0|\overline{\gamma}|} \int e^{-\beta\overline{E}_{\overline{\gamma}}} \mathbb{1}_{(\forall i \in \overline{\gamma}, \sigma(\omega,i)=\sigma_i)} \Pi^z_{\widehat{\overline{\gamma}}}(d\omega) \\ &\leq Z_0^{|\overline{\gamma}_0|} Z_1^{|\overline{\gamma}_1|} e^{-\beta\rho_0|\overline{\gamma}|}. \end{aligned}$$

For the second inequality (2.30), we need to observe that

$$\frac{\partial I_\gamma}{\partial z} = -\delta^d |\overline{\gamma}| I_\gamma + \frac{1}{z} \int N_{\widehat{\overline{\gamma}}}(\omega) e^{-\beta(E_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega) + \overline{E}_{\partial^-\overline{\gamma}})} \mathbb{1}_{(\forall i \in \overline{\gamma}, \sigma(\omega,i)=\sigma_i)} \Pi^z_{\widehat{\overline{\gamma}}}(d\omega).$$

We need to control the mean number of particles in the contours and for that we start by obviously injecting (2.28) and obtain

$$\begin{aligned}
\int N_{\widehat{\overline{\gamma}}}(\omega) e^{-\beta(E_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega)+\overline{E}_{\partial^-\overline{\gamma}})} \mathbb{1}_{(\forall i\in\overline{\gamma},\sigma(\omega,i)=\sigma_i)} \Pi^z_{\widehat{\overline{\gamma}}}(d\omega) & \\
&\leq e^{-\beta\rho_0|\overline{\gamma}|} \int N_{\widehat{\overline{\gamma}}}(\omega) e^{-\beta\overline{E}_{\overline{\gamma}}} \mathbb{1}_{(\forall i\in\overline{\gamma},\sigma(\omega,i)=\sigma_i)} \Pi^z_{\widehat{\overline{\gamma}}}(d\omega) \\
&\leq e^{-\beta\rho_0|\overline{\gamma}|} \sum_{i\in\overline{\gamma}_1} \int N_{T_i}(\omega) e^{-\beta\overline{E}_{\overline{\gamma}}} \mathbb{1}_{(\forall i\in\overline{\gamma},\sigma(\omega,i)=\sigma_i)} \Pi^z_{\widehat{\overline{\gamma}}}(d\omega) \\
&\leq e^{-\beta\rho_0|\overline{\gamma}|} Z_0^{|\overline{\gamma}_0|} Z_1^{|\overline{\gamma}_1|} \sum_{i\in\overline{\gamma}_1} \frac{1}{Z_1} \int N_{T_i}(\omega) e^{-\beta\overline{E}_i} \mathbb{1}_{(\sigma(\omega,i)=\sigma_i)} \Pi^z_{T_i}(d\omega) \\
&\leq |\overline{\gamma}_1| E_{Q_0^{(1)}}(N_{T_0}) Z_0^{|\overline{\gamma}_0|} Z_1^{|\overline{\gamma}_1|} e^{-\beta\rho_0|\overline{\gamma}|}.
\end{aligned}$$

Using this previous upper bound and (2.29) we obtain (2.30). □

At this stage of the proof, we almost have the τ-stability of the weights $w^{\#}_{\gamma}$, and find an upper bound for $\left|\frac{\partial w^{\#}_{\gamma}}{\partial z}\right|$ similar to (2.11) of Theorem 17. However, we are faced with the issue of bounding the ratio of partition functions, $\frac{Z^{\#^*}_{\mathrm{Int}_{\#^*}\gamma}}{Z^{\#}_{\mathrm{Int}_{\#^*}\gamma}}$, that appears in the weights and to achieve this we introduce the truncated weights.

2.5 Truncated weights and pressures

From the weights of each contour obtained in Proposition 12 we truncate them using an arbitrary cut-off function $\kappa : \mathbb{R} \to [0,1]$ that satisfies the following properties : $\kappa(s) = 1$ if $s \leq \frac{\beta\rho_0}{8}$, $\kappa(s) = 0$ if $s \geq \frac{\beta\rho_0}{4}$ where ρ_0 is a strictly positive constant and κ is C^1. Therefore such cut-off function κ satisfies $\|\kappa'\| = \sup_{\mathbb{R}} |\kappa'(s)| < +\infty$. The construction of truncated quantities is inductive. We start by defining the truncated pressure, for $n = 0$ as

$$\psi^{\#}_0 := \frac{\ln Z_{\#}}{\delta^d}. \tag{2.31}$$

The truncated weights for contours of class 0, $\gamma \in C^{\#}_0(\mathbb{R}^d)$ is defined as

$$\widehat{w}^{\#}_{\gamma} = w^{\#}_{\gamma} = Z^{-|\overline{\gamma}|}_{\#} I_{\gamma}. \tag{2.32}$$

Now we suppose that the truncated weights are well defined for contours γ of class $k \leq n$. We define $\widehat{\Phi}^{\#}_n$ with the following polymer development

$$\widehat{\Phi}^{\#}_n(\Lambda) := \sum_{\Gamma\in C^{\#}_n(\Lambda)} \prod_{\gamma\in\Gamma} \widehat{w}^{\#}_{\gamma}.$$

We can define the truncated partition function as

$$\widehat{Z}_n^{\#}(\Lambda) := Z_{\#}^{|\Lambda|}\widehat{\Phi}_n^{\#}(\Lambda).$$

With $\Lambda_k = [-k/2, k/2]^d \cap \mathbb{Z}^d$, the truncated pressure at rank n is given by

$$\begin{aligned}\widehat{\psi}_n^{\#} &:= \lim_{k\to+\infty} \frac{1}{\delta^d|\Lambda_k|}\ln(\widehat{Z}_n^{\#}(\Lambda_k)) \\ &= \widehat{\psi}_0^{\#} + \lim_{k\to+\infty} \frac{1}{\delta^d|\Lambda_k|}\ln(\widehat{\Phi}_n^{\#}(\Lambda_k)).\end{aligned}$$

In order to prove that the limit exists we observe that for $\Lambda, \Lambda' \subset \mathbb{Z}^d$ that are finite and disjoint we have

$$\begin{aligned}\widehat{\Phi}_n^{\#}(\Lambda)\widehat{\Phi}_n^{\#}(\Lambda') &= \left(\sum_{\Gamma\in\mathcal{C}_n^{\#}(\Lambda)}\prod_{\gamma\in\Gamma}\widehat{w}_\gamma^{\#}\right)\left(\sum_{\Gamma'\in\mathcal{C}_n^{\#}(\Lambda')}\prod_{\gamma'\in\Gamma'}\widehat{w}_{\gamma'}^{\#}\right) \\ &= \sum_{\Gamma\in\mathcal{C}_n^{\#}(\Lambda)}\sum_{\Gamma'\in\mathcal{C}_n^{\#}(\Lambda')}\prod_{\gamma\in\Gamma\cup\Gamma'}\widehat{w}_\gamma^{\#}.\end{aligned}$$

The contours in Γ and Γ' cannot be the same because their support exist in different volumes and since $\Gamma\cup\Gamma' \in \mathcal{C}_n^{\#}(\Lambda\cup\Lambda')$ therefore $|\mathcal{C}_n^{\#}(\Lambda\cup\Lambda')| \geq |\mathcal{C}_n^{\#}(\Lambda)| + |\mathcal{C}_n^{\#}(\Lambda')|$. The missing elements are eventually the contours that spans between Λ and Λ'. Consequently we have the following inequality

$$\widehat{\Phi}_n^{\#}(\Lambda_1)\widehat{\Phi}_n^{\#}(\Lambda_2) \leq \widehat{\Phi}_n^{\#}(\Lambda_1\cup\Lambda_2).$$

Furthermore, in every summation we have the empty contour $\gamma = (\emptyset, \emptyset)$ which by definition has its weight equal to 1. As a consequence for all $\Lambda \subset \mathbb{Z}^d$, $\widehat{\Phi}_n^{\#}(\Lambda) \geq 1$ and for $p = \lfloor k/m \rfloor$

$$\begin{aligned}\frac{1}{|\Lambda_k|\delta^d}\ln\widehat{\Phi}_n^{\#}(\Lambda_k) &\geq \frac{p^d}{|\Lambda_k|\delta^d}\ln\widehat{\Phi}_n^{\#}(\Lambda_m) \\ \liminf_{k\to\infty}\frac{1}{|\Lambda_k|\delta^d}\ln\widehat{\Phi}_n^{\#}(\Lambda_k) &\geq \frac{1}{|\Lambda_m|\delta^d}\ln\widehat{\Phi}_n^{\#}(\Lambda_m) \\ \liminf_{k\to\infty}\frac{1}{|\Lambda_k|\delta^d}\ln\widehat{\Phi}_n^{\#}(\Lambda_k) &\geq \limsup_{m\to\infty}\frac{1}{|\Lambda_m|\delta^d}\ln\widehat{\Phi}_n^{\#}(\Lambda_m).\end{aligned}$$

This ensures the existence of the limit and we denote it by $f_n^{\#}$. Furthermore, in every summation we have the empty contour $\gamma = (\emptyset, \emptyset)$ which by definition has its weight equal to 1 and $\widehat{\Phi}_n^{\#}(\Lambda) \leq \Phi_\Lambda^{\#}$ by construction. Hence we have that $\widehat{\psi}_n^{\#} \in [\widehat{\psi}_0^{\#}, \psi]$. In addition, we can easily see that the sequence of truncated pressure $(\widehat{\psi}_n^{\#})_{n\in\mathbb{N}}$ is increasing.

Definition 23. Given the truncated weight of contours of class $k \leq n$ (and therefore the truncated pressure $\widehat{\psi}_n^{\#}$), the truncated weight of a contour γ of class $n+1$ is defined by

$$\widehat{w}_\gamma^{\#} = Z_{\#}^{-|\overline{\gamma}|} I_\gamma\, \kappa\left((\widehat{\psi}_n^{\#^*} - \widehat{\psi}_n^{\#})\delta^d |\operatorname{Int}_{\#^*}\gamma|^{\frac{1}{d}}\right)\frac{Z_{\operatorname{Int}_{\#^*}\gamma}^{\#^*}}{Z_{\operatorname{Int}_{\#^*}\gamma}^{\#}}.$$

Furthermore, we define the following quantities of interest, $\widehat{\psi}_n := \max(\widehat{\psi}_n^0, \widehat{\psi}_n^1)$ and $a_n^\# := \widehat{\psi}_n - \widehat{\psi}_n^\#$. By definition, we have $a_n^\# \geq 0$ and for all contour γ of class $n+1$ we have the following implication

$$a_n^\# \delta^d (n+1)^{\frac{1}{d}} \leq \frac{\beta\rho_0}{8} \implies \widehat{w}_\gamma^\# = w_\gamma^\#.$$

Since the truncated weights are defined for any class, we can define the following polymer development for any boundary condition (#) and any $\Lambda \subset \mathbb{Z}^d$

$$\widehat{\Phi}^\#(\Lambda) := \sum_{\Gamma \in \mathcal{C}^\#(\Lambda)} \prod_{\gamma \in \Gamma} \widehat{w}_\gamma^\#$$

and alongside it the truncated partition function

$$\widehat{Z}_\Lambda^\# := Z_\#^{|\Lambda|} \widehat{\Phi}^\#(\Lambda) = e^{(\widehat{\psi}_0^\# + f^\#)\delta^d |\Lambda| + \Delta_\Lambda^\#}. \tag{2.33}$$

Definition 24. The truncated pressure associated to the boundary # is

$$\widehat{\psi}^\# := \lim_{n\to+\infty} \frac{1}{\delta^d |\Lambda_n|} \ln \widehat{Z}_{\Lambda_n}^\# = \widehat{\psi}_0^\# + f^\# \quad \text{where} \quad f\# = \lim_{n\to+\infty} \frac{1}{\delta^d |\Lambda_n|} \ln \widehat{\Phi}_{\Lambda_n}^\#.$$

The truncated pressure $\widehat{\psi}$ is defined as

$$\widehat{\psi} := \max(\widehat{\psi}^{(0)}, \widehat{\psi}^{(1)}).$$

We can also obtain $\widehat{\psi}^\#$ as the limit of the sequence of truncated pressure $(\widehat{\psi}_n^\#)_{n\in\mathbb{N}}$ and we define $a^\#$ as,

$$a^\# := \widehat{\psi} - \widehat{\psi}^\# = \lim_{n\to\infty} a_n^\#.$$

The following result is the key proposition in the proof Theorem 11, as we show that the truncated weights verify the inequalities (2.11) needed to use Theorem 17. Later, we show that for some value of z the truncated weights and actual weights coincide.

Proposition 13. *Let H be a saturated interaction, such that E_0 is stable with a finite range $R > 0$. When saturated we have $\overline{E} \geq 0$ and $\overline{E}(1) < +\infty$. We assume that it verifies a Peierls-like condition, i.e. there is $\rho_0 > 0$ such that for any contours γ and any configuration that achieves this contour γ we have*

$$E_{\overline{\gamma}\setminus\partial^-\overline{\gamma}} - \overline{E}_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega) \geq \rho_0 |\overline{\gamma}|. \tag{2.34}$$

Under these assumptions, for any $\beta > 0$ there are z_β^- and z_β^+ such that for $z \in U_\beta := (z_\beta^-, z_\beta^+)$ we have

$$e^{-2} = \frac{Z_0}{Z_1}(z_\beta^-) \leq \frac{Z_0}{Z_1}(z) \leq \frac{Z_0}{Z_1}(z_\beta^+) = e^2.$$

Let $\tau := \frac{1}{2}\beta\rho_0 - 8$. Then there exist $D \geq 1$ and $0 < \beta_0 < \infty$ such that for all $\beta > \beta_0$ there exist $C_1 > 0$ and $C_2 > 0$ where the following statements hold for any # and $n \geq 0$.

1. *(Bounds on the truncated weights) For all $k \leq n$, the truncated weights of each contour γ of class k is τ-stable uniformly on U_β :*

$$\widehat{w}_\gamma^\# \leq e^{-\tau|\overline{\gamma}|} \tag{2.35}$$

and

$$a_n^\# \delta^d |\operatorname{Int}\gamma|^{\frac{1}{d}} \leq \frac{\beta\rho_0}{16} \;\Rightarrow\; \widehat{w}_\gamma^\# = w_\gamma^\#. \tag{2.36}$$

Moreover, $z \mapsto \widehat{w}_\gamma^\#$ is C^1 and uniformly on U_β it verifies

$$\left|\frac{\partial \widehat{w}_\gamma^\#}{\partial z}\right| \leq D|\overline{\gamma}|^{\frac{d}{d-1}} e^{-\tau|\overline{\gamma}|}. \tag{2.37}$$

2. *(Bounds on the partition functions) For $\Lambda \subset \mathbb{Z}^d$ such that $|\Lambda| \leq n+1$, uniformly on U_β, we have*

$$Z_\Lambda^\# \leq e^{\widehat{\psi}_n \delta^d |\Lambda| + 2|\partial_{ext}\Lambda|}, \tag{2.38}$$

$$\left|\frac{\partial Z_\Lambda^\#}{\partial z}\right| \leq \left(C_1|\Lambda| + C_2|\partial_{ext}\Lambda|\right) e^{\widehat{\psi}_n \delta^d |\Lambda| + 2|\partial_{ext}\Lambda|}. \tag{2.39}$$

Proof. Before starting this proof by induction, we need to fix the some constants and in particular the quantity β which has to be sufficiently large. Recall that $\eta(\tau, l_0) := 2e^{-\frac{\tau l_0}{3}}$, where l_0 is is the size of the smallest contour. We set $D := (3 + 2\|\kappa'\|)C_1 + 2C_2$ where $C_1(\beta) := \sup_{U_\beta}(e+2)\delta^d - \frac{\ln(z\delta^d)}{z} + \frac{\beta}{z}\overline{E}(1)$ and $C_2(\beta) := 1/z_\beta^-$ and we choose $\beta \geq 1$ sufficiently large such that

$$\tau > \tau_0(d) \text{ where } \tau_0 \text{ is defined as in Lemma 15.} \tag{2.40}$$

$$D\eta(\tau, l_0) \leq 1 \tag{2.41}$$

$$\forall k \in \mathbb{N},\; 2k^{1/d} \exp\left(-\frac{\tau k^{d-1/d}}{2}\right) \leq \frac{\beta\rho_0}{16} \tag{2.42}$$

$$\forall x > 0,\; \delta^{-d} \exp\left(-\max\{(\beta\rho_0/16\delta^d x)^{d-1}, l_0\}\frac{\tau}{2}\right) \leq \frac{x}{2}. \tag{2.43}$$

Let us prove the proposition for $n = 0$. Let γ be a contour of class 0, for $z \in U_\beta$ and using (2.29) of Lemma 18 we have

$$\begin{aligned}\widehat{w}_\gamma^\# &\leq \left(\frac{Z_{\#^*}}{Z_\#}\right)^{|\overline{\gamma}_{\#^*}|} e^{-\beta\rho_0|\overline{\gamma}|} \\ &\leq e^{2|\overline{\gamma}|} e^{-\beta\rho_0|\overline{\gamma}|} = e^{-\tau|\overline{\gamma}|}.\end{aligned}$$

When computing the derivative of the truncated weights with respect to z we have

$$\frac{\partial \widehat{w}_\gamma^\#}{\partial z} = \frac{\partial w_\gamma^\#}{\partial z} = \left(\frac{\partial I_\gamma}{\partial z} - \frac{|\overline{\gamma}|}{Z_\#}\frac{\partial Z_\#}{\partial z} I_\gamma\right) Z_\#^{-|\overline{\gamma}|}.$$

Furthermore, we know that

$$\frac{1}{Z_\#}\frac{\partial Z_\#}{\partial z} = -\delta^d + \frac{1}{z}E_{Q_0^\#}(N_{T_0}), \tag{2.44}$$

and therefore

$$\frac{\partial \widehat{w}_\omega^\#}{\partial z} = \left(\frac{\partial I_\gamma}{\partial z} + |\overline{\gamma}|\delta^d\left(1 - \frac{1}{z\delta^d}E_{Q_0^\#}(N_{T_0})\right)I_\gamma\right) Z_\#^{-|\overline{\gamma}|}.$$

Therefore using inequalities (2.29) and (2.30) of Lemma 18 we have

$$\left|\frac{\partial \widehat{w}_\gamma^\#}{\partial z}\right| \leq \left(2 + \frac{2}{z\delta^d}E_{Q_0^{(1)}}(N_{T_0})\right)\delta^d|\overline{\gamma}|\frac{Z_{\#^*}}{Z_\#}^{|\overline{\gamma}_{\#^*}|}e^{-\beta\rho_0|\overline{\gamma}|} \tag{2.45}$$

$$\leq \left(2 + \frac{2}{z\delta^d}E_{Q_0^{(1)}}(N_{T_0})\right)\delta^d|\overline{\gamma}|e^{-(\beta\rho_0-2)|\overline{\gamma}|} \tag{2.46}$$

Since $\overline{E} \geq 0$, and using the Donsker-Varadhan entropic inequality we have that

$$\begin{aligned} E_{Q_0^{(1)}}(N_{T_0}) &\leq I(Z_0^{(1)}|\Pi_{T_0}^z) + \ln E_{\Pi_{T_0}^z}(e^{N_{T_0}}) \\ &\leq \int -\beta\overline{E}dQ_0^{(1)} - \ln Z_1 + (e-1)z\delta^d \\ &\leq -\ln Z_1 + (e-1)z\delta^d. \end{aligned}$$

Furthermore, we get that

$$Z_1 \geq e^{-\beta\overline{E}(1)}\Pi_{T_0}^z(N_{T_0}=1) = z\delta^d e^{-z\delta^d}e^{-\beta\overline{E}(1)}.$$

Thus, we have

$$E_{Q_0^{(1)}}(N_{T_0}) \leq -\ln(z\delta^d) + z\delta^d + \beta\overline{E}(1) + (e-1)z\delta^d. \tag{2.47}$$

When we introduce (2.47) into (2.46) we get

$$\left|\frac{\partial \widehat{w}_\gamma^\#}{\partial z}\right| \leq 2\left(e + 1 - \frac{\ln(z\delta^d)}{z\delta^d} + \frac{\beta\overline{E}(1)}{z\delta^d}\right)\delta^d|\overline{\gamma}|e^{-\tau|\overline{\gamma}|} < D|\overline{\gamma}|^{d/d-1}e^{-\tau|\overline{\gamma}|}. \tag{2.48}$$

When $|\Lambda| = 1$, we simply have the following inequality

$$Z_\Lambda^\# = Z_\# = e^{\widehat{\psi}_0^\#\delta^d} \leq e^{\widehat{\psi}_0\delta^d|\Lambda|+2|\partial_{ext}\Lambda|}.$$

Finally concerning the derivative of the partition function with respect to z using relation (2.44) and (2.47) we obtain

$$\begin{aligned} \left|\frac{\partial Z_\Lambda^\#}{\partial z}\right| = \left|\frac{\partial Z_\#}{\partial z}\right| &\leq \left((e+1)\delta^d - \frac{\ln(z\delta^d)}{z} + \frac{\beta}{z}\overline{E}(1)\right)Z_\# \\ &\leq C_1(\beta)|\Lambda|e^{\beta\delta^d\widehat{\psi}_0+2|\partial_{ext}\Lambda|}. \end{aligned}$$

Now the initialisation is done, we assume that the statements have been proved up to n and we have to prove that they hold also for $n+1$. Before proceeding properly into the induction we need the following lemma.

Lemma 19. *For $n \geq 1$, if for all contours of class at most n the truncated weights are τ-stable then for any $k \leq n$*

$$|\widehat{\psi}_n^{\#} - \widehat{\psi}_k^{\#}| \leq \frac{1}{\delta^d} e^{-\frac{\tau}{2} k^{d-1/d}} \quad \textit{and} \quad |\widehat{\psi}_n - \widehat{\psi}_k| \leq \frac{1}{\delta^d} e^{-\frac{\tau}{2} k^{d-1/d}}. \tag{2.49}$$

Proof. We have $|\widehat{\psi}_n^{\#} - \widehat{\psi}_k^{\#}| = |f_n^{\#} - f_k^{\#}|$. Since all contour of class at most n are τ-stable we know that the cluster expansion for $f_k^{\#}$ converges for $k \leq n$. We can notice that the clusters X that contributes to $f_n^{\#} - f_k^{\#}$ must have at least one contour γ of class greater than k. Therefore according to the isoperimetric inequality we have $|\overline{\gamma}| \geq k^{d-1/d}$ which in turn implies that $|\overline{X}| \geq k^{d-1/d}$. Thus by Lemma 16 we obtain

$$|\widehat{\psi}_n^{\#} - \widehat{\psi}_k^{\#}| \leq \frac{1}{\delta^d} \sum_{\substack{X : 0 \in \overline{X} \\ |\overline{X}| \geq k^{d-1/d}}} |\widehat{\Psi}^{\#}(X)| \leq \frac{1}{\delta^d} e^{-\frac{\tau}{2} k^{d-1/d}}. \tag{2.50}$$

Now we want to have the same kind of estimate for $|\widehat{\psi}_n - \widehat{\psi}_k|$. If $\widehat{\psi}_n = \widehat{\psi}_n^{\#}$ and $\widehat{\psi}_k = \widehat{\psi}_k^{\#}$, we get the same estimate since the difference is the same. In the case where $\widehat{\psi}_n = \widehat{\psi}_n^{\#}$ and $\widehat{\psi}_k = \widehat{\psi}_k^{\#^*}$, where $\# \neq \#^*$, we have on one side

$$\widehat{\psi}_n - \widehat{\psi}_k = \widehat{\psi}_n^{\#} - \widehat{\psi}_k^{\#} + \widehat{\psi}_k^{\#} - \widehat{\psi}_k^{\#^*} \leq \widehat{\psi}_n^{\#} - \widehat{\psi}_k^{\#}$$

since by definition $\widehat{\psi}_k^{\#} - \widehat{\psi}_k^{\#^*} = \widehat{\psi}_k^{\#} - \widehat{\psi}_k \leq 0$. On the other side, we have

$$\widehat{\psi}_n - \widehat{\psi}_k = \widehat{\psi}_n^{\#} - \widehat{\psi}_n^{\#^*} + \widehat{\psi}_n^{\#^*} - \widehat{\psi}_k^{\#^*} \geq 0$$

since $(\widehat{\psi}_i^{\#^*})_{i \in \mathbb{N}}$ is increasing and $\widehat{\psi}_n^{\#} - \widehat{\psi}_n^{\#^*} \geq 0$ by definition. In any case, we have

$$|\widehat{\psi}_n - \widehat{\psi}_k| \leq \frac{1}{\delta^d} e^{-\frac{\tau}{2} k^{d-1/d}}.$$

□

We move on the proof that (2.38) holds if $|\Lambda| = n + 2$. Note that any contour that appears inside of Λ is at most of class $k \leq n$. We say that a contour γ is stable if

$$a_n^{\#} \delta^d |\operatorname{Int} \gamma|^{1/d} \leq \frac{\beta \rho_0}{16}.$$

This property is hereditary, in the sense that for all contours γ' that can appear inside $\operatorname{Int} \gamma$ are stable as well. Since, we know that all contours of class at most n are τ-stable we can apply Lemma 19 and by (2.42) we have for any contour γ of class $k \leq n$

$$\begin{aligned} a_k^{\#} \delta^d |\operatorname{Int} \gamma|^{1/d} &= a_n^{\#} \delta^d |\operatorname{Int} \gamma|^{1/d} + (a_k^{\#} - a_n^{\#}) \delta^d |\operatorname{Int} \gamma|^{1/d} \\ &\leq a_n^{\#} \delta^d |\operatorname{Int} \gamma|^{1/d} + 2k^{1/d} e^{-\tau k^{d-1/d}/2} \\ &\leq a_n^{\#} \delta^d |\operatorname{Int} \gamma|^{1/d} + \frac{\beta \rho_0}{16}. \end{aligned}$$

Therefore, when the contours are stable it implies that $a_k^{\#}\delta^d |\operatorname{Int}\gamma|^{1/d} \leq \beta\rho_0/8$ and thus $\widehat{w}_\gamma^{\#} = w_\gamma^{\#}$. In contrast, we would call contours that doesn't satisfy this condition unstable. The stability of a contour depends on the parameter z as it affects the value of $a_n^{\#}$. Thus we have two cases to consider. The first case is $a_n^{\#} = 0$. Consequently all contours are stable, therefore according to Theorem 17 we have

$$\begin{aligned} Z_\Lambda^{\#} = \widehat{Z}_\Lambda^{\#} &= e^{\widehat{\psi}_n^{\#}\delta^d|\Lambda|+\Delta_\Lambda} \\ &\leq e^{\widehat{\psi}_n^{\#}\delta^d|\Lambda|+|\partial_{ext}\Lambda|} \\ &\leq e^{\widehat{\psi}_{n+1}^{\#}\delta^d|\Lambda|+2|\partial_{ext}\Lambda|} \end{aligned}$$

since $(\widehat{\psi}_i)_{i\in\mathbb{N}}$ is increasing and therefore (2.38) is proved. Now let us consider $a_n^{\#} > 0$, in this case some contours must be unstable. Therefore we can partition the configurations that generate among the external contours those that are unstable

$$Z_\Lambda^{\#} = \sum_{\substack{\Gamma\in C_{ext}^{\#}(\Lambda) \\ \text{unstable}}} \int e^{-\beta(E_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega)+\overline{E}_{\partial^-\overline{\gamma}})} \mathbb{1}_{(\#)_\Lambda} \mathbb{1}_{\{\Gamma\subset\Gamma_{ext}(\omega)\}} \Pi_{\widehat{\Lambda}}^{z}(d\omega).$$

Similar to what we did in the proof for the polymer development of the partition function, we can write each integral as a product of integrals with respect to Poisson point process distribution on different domains due to the saturation property of the Hamiltonian. The only difference is that we consider for the moment only unstable contours and so inside of Λ_{ext} we have to account for stable contours. Furthermore, these stable contours cannot encircle any external unstable contour due to the hereditary property of stable contours.

$$Z_\Lambda^{\#} = \sum_{\substack{\Gamma\in C_{ext}^{\#}(\Lambda) \\ \text{unstable}}} Z_{\Lambda_{ext},stable}^{\#} \prod_{\gamma\in\Gamma} I_\gamma Z_{\operatorname{Int}_0\gamma}^{(0)} Z_{\operatorname{Int}_1\gamma}^{(1)},$$

where $Z_{\Lambda_{ext},stable}^{\#}$ denotes the partition function restricted to configurations for which all contours are stable and by construction they are of class at most n. Since all those contours are of class at most n they are also τ-stable therefore they can be studied using a convergent cluster expansion according to Theorem 17 and thus

$$\begin{aligned} Z_{\Lambda_{ext},stable}^{\#} &= Z_{\#}^{|\Lambda_{ext}|}\widehat{\Phi}_{n,stable}^{\#}(\Lambda_{ext}) \\ &\leq Z_{\#}^{|\Lambda_{ext}|} e^{f_{n,stable}^{\#}\delta^d|\Lambda_{ext}|+|\partial_{ext}\Lambda_{ext}|} \\ &\leq e^{(\widehat{\psi}_0^{\#}+f_{n,stable}^{\#})\delta^d|\Lambda_{ext}|+|\partial_{ext}\Lambda_{ext}|} \end{aligned}$$

where $f_{n,stable}^{\#} = \lim_{k\to\infty} \frac{1}{|\Lambda_k|\delta^d} \ln \widehat{\Phi}_{n,stable}^{\#}(\Lambda_k)$. According to the induction hypothesis we have that

$$\begin{aligned} Z_{\operatorname{Int}_0\gamma}^{(0)} Z_{\operatorname{Int}_1\gamma}^{(1)} &\leq e^{\widehat{\psi}_n\delta^d|\operatorname{Int}\gamma|} e^{2(|\partial_{ext}\operatorname{Int}_1\gamma|+|\partial_{ext}\operatorname{Int}_0\gamma|)} \\ &\leq e^{\widehat{\psi}_n\delta^d|\operatorname{Int}\gamma|} e^{2|\overline{\gamma}|}. \end{aligned}$$

For any $\Gamma \in C^{\#}_{ext}(\Lambda)$ we have $|\partial_{ext}\Lambda_{ext}| \leq |\partial_{ext}\Lambda| + \sum_{\gamma\in\Gamma}|\overline{\gamma}|$, thus we get that

$$\begin{aligned} Z^{\#}_{\Lambda} &\leq \sum_{\substack{\Gamma\in C^{\#}_{ext}(\Lambda)\\ \text{unstable}}} e^{(\widehat{\psi}^{\#}_0 + f^{\#}_{n,stable})\delta^d|\Lambda_{ext}|} e^{|\partial_{ext}\Lambda| + \sum_{\gamma\in\Gamma}|\overline{\gamma}|} \prod_{\gamma\in\Gamma} I_\gamma e^{\widehat{\psi}_n\delta^d|\operatorname{Int}\gamma|} e^{2|\overline{\gamma}|} \\ &\leq e^{\widehat{\psi}_n\delta^d|\Lambda| + |\partial_{ext}\Lambda|} \sum_{\substack{\Gamma\in C^{\#}_{ext}(\Lambda)\\ \text{unstable}}} e^{-(\widehat{\psi}_n - \widehat{\psi}^{\#}_{n,stable})\delta^d|\Lambda_{ext}|} \prod_{\gamma\in\Gamma} I_\gamma e^{(3-\widehat{\psi}_n\delta^d)|\overline{\gamma}|} \end{aligned}$$

where we define $\widehat{\psi}^{\#}_{n,stable} := \widehat{\psi}^{\#}_0 + f^{\#}_{n,stable}$. Furthermore when we use the following inequalities $\widehat{\psi}_n \geq \widehat{\psi}^{\#}_n \geq \widehat{\psi}^{\#}_0 = \frac{\ln(Z_{\#})}{\delta^d}$ we get that $I_\gamma e^{-\widehat{\psi}_n\delta^d|\overline{\gamma}|} \leq e^{-\tau|\overline{\gamma}|}$. Therefore we have

$$Z^{\#}_{\Lambda} \leq e^{\widehat{\psi}_n\delta^d|\Lambda| + |\partial_{ext}\Lambda|} \sum_{\substack{\Gamma\in C^{\#}_{ext}(\Lambda)\\ \text{unstable}}} e^{-(\widehat{\psi}_n - \widehat{\psi}^{\#}_{n,stable})\delta^d|\Lambda_{ext}|} \prod_{\gamma\in\Gamma} e^{-(\tau-3)|\overline{\gamma}|}.$$

It remains to prove that the sum is bounded by $e^{|\partial_{ext}\Lambda|}$. First we note that $\widehat{\psi}_n - \widehat{\psi}^{\#}_{n,stable} = a^{\#}_n + f^{\#}_n - f^{\#}_{n,stable}$. By construction, the clusters that appear in the cluster expansion of $f^{\#}_n - f^{\#}_{n,stable}$ contain at least one unstable contour γ and therefore such contour verify the following

$$|\overline{\gamma}| \geq |\operatorname{Int}\gamma|^{d-1/d} \geq \left(\frac{\beta\rho_0}{16a^{\#}_n\delta^d}\right)^{(d-1)}$$

and by Lemma 16 and (2.43)

$$|f^{\#}_n - f^{\#}_{n,stable}| \leq \delta^{-d}\exp\left(-\max\{(\beta\rho_0/16a^{\#}_n\delta^d)^{d-1}, l_0\}\frac{\tau}{2}\right) \leq \frac{a^{\#}_n}{2}. \tag{2.51}$$

In the end, we obtain

$$\widehat{\psi}_n - \widehat{\psi}^{\#}_{n,\text{stable}} \geq \frac{a^{\#}_n}{2}. \tag{2.52}$$

Now let us define new weights w^*_γ as follow

$$w^*_\gamma = \begin{cases} e^{-(\tau-5)|\overline{\gamma}|} & \text{if } \gamma \text{ is unstable} \\ 0 & \text{otherwise.} \end{cases}$$

We denote by Φ^* the associated polymer development and have

$$g^* = \lim_{k\to\infty} \frac{1}{\delta^d|\Lambda_k|} \ln\Phi^*(\Lambda_k).$$

For sufficiently large β we can assure by Theorem 17 that it is a convergent cluster expansion. Since all contours that contribute to g^* are all unstable, we obtain an inequality similar to (2.51)

$$|g^*| \leq \delta^{-d}\exp\left(-\max\{(\beta\rho_0/16a^{\#}_n\delta^d)^{d-1}, l_0\}\frac{\tau}{2}\right) \leq \frac{a^{\#}_n}{2}. \tag{2.53}$$

Therefore with (2.52) and (2.53) we have $\widehat{\psi}_n - \widehat{\psi}^{\#}_n \geq g^*$ and thus

$$\sum_{\substack{\Gamma\in C^{\#}_{ext}(\Lambda)\\ \text{unstable}}} e^{-(\widehat{\psi}_n-\widehat{\psi}^{\#}_{n,stable})\delta^d|\Lambda_{ext}|}\prod_{\gamma\in\Gamma} e^{-(\tau-3)|\overline{\gamma}|} \leq \sum_{\substack{\Gamma\in C^{\#}_{ext}(\Lambda)\\ \text{unstable}}} e^{-g^*\delta^d|\Lambda_{ext}|}\prod_{\gamma\in\Gamma} e^{-(\tau-3)|\overline{\gamma}|}$$
$$\leq e^{-g^*\delta^d|\Lambda|}\sum_{\substack{\Gamma\in C^{\#}_{ext}(\Lambda)\\ \text{unstable}}}\prod_{\gamma\in\Gamma} e^{-(\tau-3)|\overline{\gamma}|}e^{g^*\delta^d(|\overline{\gamma}|+|\operatorname{Int}\gamma|)}.$$

By (2.53) we know that $\delta^d g^* \leq 1$, and again with Theorem 17 we know that $\Phi^*_{\operatorname{Int}\gamma} \geq e^{g^*\delta^d|\operatorname{Int}\gamma|-|\overline{\gamma}|}$ and so

$$\sum_{\substack{\Gamma\in C^{\#}_{ext}(\Lambda)\\ \text{unstable}}} e^{-(\widehat{\psi}_n-\widehat{\psi}^{\#}_{n,stable})\delta^d|\Lambda_{ext}|}\prod_{\gamma\in\Gamma} e^{-(\tau-3)|\overline{\gamma}|} \leq e^{-g^*\delta^d|\Lambda|}\sum_{\substack{\Gamma\in C^{\#}_{ext}(\Lambda)\\ \text{unstable}}}\prod_{\gamma\in\Gamma} e^{-(\tau-5)|\overline{\gamma}|}\Phi^*_{\operatorname{Int}\gamma}$$
$$= e^{-g^*\delta^d|\Lambda|}\Phi^*_\Lambda$$
$$\leq e^{|\partial_{ext}\Lambda|}.$$

In summary, we have

$$Z^{\#}_\Lambda \leq e^{\widehat{\psi}_n\delta^d|\Lambda|+2|\partial_{ext}\Lambda|} < e^{\widehat{\psi}_{n+1}\delta^d|\Lambda|+2|\partial_{ext}\Lambda|} \tag{2.54}$$

which is exactly (2.38) in the case where $|\Lambda| = n+2$. If $|\Lambda| \leq n+1$ it is sufficient to notice that $(\widehat{\psi}^{\#}_n)_{n\in\mathbb{N}}$ is increasing. Let us prove that (2.39) holds. We start with the case $|\Lambda| = n+2$ and by a similar argument it is true for any smaller Λ. By a direct computation we have

$$\frac{\partial Z^{\#}_\Lambda}{\partial z} = -|\Lambda|\delta^d Z^{\#}_\Lambda + \frac{1}{z}\int N_{\widehat{\Lambda}}(\omega)e^{-\beta(E_{\Lambda\setminus\partial\Lambda}(\omega)+\overline{E}_{\partial\Lambda})}\mathbb{1}_{(\#)_\Lambda}\Pi^z_{\widehat{\Lambda}}(d\omega)$$
$$= \left(-|\Lambda|\delta^d + \frac{1}{z}E_{P^{\#}_\Lambda}(N_{\widehat{\Lambda}})\right)Z^{\#}_\Lambda. \tag{2.55}$$

Furthermore, we know by our hypothesis (2.28) and $\overline{E} \geq 0$ that for any configuration $\omega \in \Omega_f$ we have $E_{\Lambda\setminus\partial\Lambda}(\omega) + \overline{E}_{\partial\Lambda}(\omega) \geq \overline{E}_\Lambda(\omega) \geq 0$. By Donsker-Varadhan entropic inequality we have the following

$$E_{P^{\#}_\Lambda}(N_{\widehat{\Lambda}}) \leq I(P^{\#}_\Lambda|\Pi^z_{\widehat{\Lambda}}) + \ln E_{\Pi^z_{\widehat{\Lambda}}}(e^{N_{\widehat{\Lambda}}})$$
$$\leq \int -\beta(E_{\Lambda\setminus\partial\Lambda}(\omega)+\overline{E}_{\partial\Lambda})dP^{\#}_\Lambda - \ln Z^{\#}_\Lambda + (e-1)z\delta^d|\Lambda|$$
$$\leq -\ln Z^{\#}_\Lambda + (e-1)z\delta^d|\Lambda|. \tag{2.56}$$

Furthermore, we know that the contours which appear in $|\Lambda|$ are at most of the class n. Therefore we know that their truncated weights are τ-stable and by Theorem 17

$$Z^{\#}_\Lambda \geq \widehat{Z}^{\#}_\Lambda \geq e^{\widehat{\psi}^{\#}_n\delta^d|\Lambda|-|\partial_{ext}\Lambda|}. \tag{2.57}$$

From inequalities (2.57) and (2.56) and by using the fact that $(\widehat{\psi}^{\#}_n)_{n\in\mathbb{N}}$ is increasing we obtain

$$E_{P^{\#}_\Lambda}(N_{\widehat{\Lambda}}) \leq \left((e-1)z - \widehat{\psi}^{\#}_0\right)\delta^d|\Lambda| + |\partial_{ext}\Lambda|.$$

And since

$$\widehat{\psi}_0^{(0)} = -z \quad \text{and} \quad \widehat{\psi}_0^{(1)} \geq -z + \frac{\ln(z\delta^d)}{\delta^d} - \frac{\beta}{\delta^d}\overline{E}(1),$$

we have

$$E_{P_\Lambda^\#}(N_{\widehat{\Lambda}}) \leq \Big((e+1)z\delta^d - \ln(z\delta^d) + \beta\overline{E}(1)\Big)\,|\Lambda| + |\partial_{ext}\Lambda|.$$

When we introduce the last inequality into (2.55) we get that

$$\begin{aligned}\left|\frac{\partial Z_\Lambda^\#}{\partial z}\right| &\leq \left[\left((e+2)\delta^d - \frac{\ln(z\delta^d)}{z} + \frac{\beta}{z}\overline{E}(1)\right)|\Lambda| + \frac{1}{z}|\partial_{ext}\Lambda|\right] Z_\Lambda^\# \\ &\leq \big(C_1(\beta)|\Lambda| + C_2(\beta)|\partial_{ext}\Lambda|\big)\, Z_\Lambda^\#.\end{aligned}$$

Therefore using (2.54) we obtain

$$\left|\frac{\partial Z_\Lambda^\#}{\partial z}\right| \leq \big(C_1(\beta)|\Lambda| + C_2(\beta)|\partial_{ext}\Lambda|\big)\, e^{\widehat{\psi}_{n+1}\delta^d|\Lambda|+2|\partial_{ext}\Lambda|}$$

and (2.39) is proved when $|\Lambda| = n+2$.

Now let us prove (2.35) which is the τ-stability of truncated weights for contours of class $n+1$. We consider a contour γ of class $n+1$. First of all, we can observe that $\widehat{\omega}_\gamma^\# = 0$ whenever $(\widehat{\psi}_n^{\#^*} - \widehat{\psi}_n^\#)\delta^d |\operatorname{Int}_{\#^*}\gamma|^{1/d} > \rho_0/4$. So we can assume that

$$(\widehat{\psi}_n^{\#^*} - \widehat{\psi}_n^\#)\delta^d |\operatorname{Int}_{\#^*}\gamma|^{1/d} \leq \frac{\beta\rho_0}{4}. \tag{2.58}$$

Since $|\operatorname{Int}\gamma| = n+1$ we can apply the induction hypothesis on the partition functions that appears in the truncated weights particularly we can use (2.38) and have

$$Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma} \leq e^{\widehat{\psi}_n\delta^d|\operatorname{Int}_{\#^*}\gamma|+2|\partial_{ext}\operatorname{Int}_{\#^*}\gamma|}. \tag{2.59}$$

By combining the previous inequalities (2.59) and (2.57) we have

$$\begin{aligned}\frac{Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma}}{Z^\#_{\operatorname{Int}_{\#^*}\gamma}} &\leq e^{(\widehat{\psi}_n - \widehat{\psi}_n^\#)\delta^d|\operatorname{Int}_{\#^*}\gamma|+3|\partial_{ext}\operatorname{Int}_{\#^*}\gamma|} \\ &\leq e^{(\widehat{\psi}_n - \widehat{\psi}_n^\#)\delta^d|\operatorname{Int}_{\#^*}\gamma|^{1/d}|\operatorname{Int}_{\#^*}\gamma|^{(d-1)/d}+3|\partial_{ext}\operatorname{Int}_{\#^*}\gamma|}.\end{aligned}$$

Furthermore, applying hypothesis (2.58), the isoperimetric inequality and $|\partial_{ext}\operatorname{Int}_{\#^*}\gamma| \leq |\overline{\gamma}|$ we have

$$\begin{aligned}\frac{Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma}}{Z^\#_{\operatorname{Int}_{\#^*}\gamma}} &\leq e^{\beta\rho_0/4|\operatorname{Int}_{\#^*}\gamma|^{(d-1)/d}+3|\partial_{ext}\operatorname{Int}_{\#^*}\gamma|} \\ &\leq e^{(\frac{1}{4}\beta\rho_0+3)|\partial_{ext}\operatorname{Int}_{\#^*}\gamma|} \leq e^{(\frac{1}{4}\beta\rho_0+3)|\overline{\gamma}|}.\end{aligned} \tag{2.60}$$

Therefore for $z \in U_\beta$ and according to (2.29) of Lemma 18 and the previous bound on the ratio of partition functions we have

$$\begin{aligned}\widehat{\omega}_\gamma^{\#} &\leq e^{-(\beta\rho_0-2)|\overline{\gamma}|} e^{(\frac{1}{4}\beta\rho_0+3)|\overline{\gamma}|} \\ &\leq e^{-(\frac{3}{4}\beta\rho_0-5)|\overline{\gamma}|} \leq e^{-\tau|\overline{\gamma}|}\end{aligned}$$

and as a consequence the weight $\widehat{\omega}_\gamma^{\#}$ is τ-stable.

Let us prove that (2.37) holds for a contour γ of class $n+1$. Similar to the proof of (2.35) we consider only the case when $(\widehat{\psi}_n^{\#^*} - \widehat{\psi}_n^{\#})\delta^d |\operatorname{Int}_{\#^*}\gamma|^{1/d} \leq \beta\rho_0/4$. By a direct computation

$$\frac{\partial \widehat{\omega}_\gamma^{\#}}{\partial z} = \left(-\frac{|\overline{\gamma}|}{Z_{\#}}\frac{\partial Z_{\#}}{\partial z} I_\gamma + \frac{\partial I_\gamma}{\partial z}\right)\left(Z_{\#}^{-|\overline{\gamma}|}\kappa \frac{Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma}}{Z^{\#}_{\operatorname{Int}_{\#^*}\gamma}}\right) + \frac{\partial}{\partial z}\left(\kappa \frac{Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma}}{Z^{\#}_{\operatorname{Int}_{\#^*}\gamma}}\right) Z_{\#}^{-|\overline{\gamma}|} I_\gamma.$$

The first term of the derivative can be bounded in a similar way as in (2.48) since $z \in U_\beta$ and using the inequalities in Lemma 18 and inequality (2.60) that controls the ratio of partition function we have

$$\left|-\frac{|\overline{\gamma}|}{Z_{\#}}\frac{\partial Z_{\#}}{\partial z} I_\gamma + \frac{\partial I_\gamma}{\partial z}\right|\left(Z_{\#}^{-|\overline{\gamma}|}\kappa \frac{Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma}}{Z^{\#}_{\operatorname{Int}_{\#^*}\gamma}}\right) \leq C_1(\beta)|\overline{\gamma}|e^{-\tau|\overline{\gamma}|}. \tag{2.61}$$

The second term that appears will yield the following bound

$$\frac{\partial \kappa}{\partial z} Z_{\#}^{-|\overline{\gamma}|} I_\gamma \frac{Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma}}{Z^{\#}_{\operatorname{Int}_{\#^*}\gamma}} \leq \left|\frac{\partial \kappa}{\partial z}\right| e^{-\tau|\overline{\gamma}|}$$

and regarding the derivative of κ with respect to z we have

$$\left|\frac{\partial \kappa}{\partial z}\right| \leq \left(\left|\frac{\partial \widehat{\psi}_n^{\#}}{\partial z}\right| + \left|\frac{\partial \widehat{\psi}_n^{\#^*}}{\partial z}\right|\right)\delta^d |\operatorname{Int}_{\#^*}\gamma|^{\frac{1}{d}} \|\kappa'\|.$$

Since all the contours appearing in $\widehat{\Phi}_n^{\#}$ are at most of class n, by the induction hypothesis, all these weights are τ-stable and satisfies the bound (2.37) on the derivatives. Therefore using Theorem 17 we have

$$\left|\frac{\partial f_n^{\#}}{\partial z}\right| \leq \frac{D\eta(\tau, l_0)}{\delta^d} \leq 1. \tag{2.62}$$

Using equation (2.44) and inequalities (2.47) and (2.62) we have

$$\begin{aligned}\frac{\partial \widehat{\psi}_n^{\#}}{\partial z} &= \frac{1}{\delta^d Z_{\#}}\frac{\partial Z_{\#}}{\partial z} + \frac{\partial f_n^{\#}}{\partial z} \\ &= \frac{1}{\delta^d}\left(\frac{1}{z}E_{Q_0^{\#}}(N_{T_0}) - \delta^d\right) + \frac{\partial f_n^{\#}}{\partial z} \\ \left|\frac{\partial \widehat{\psi}_n^{\#}}{\partial z}\right| &\leq \left(\frac{-\ln(z\delta^d)}{z\delta^z} + \frac{\beta\overline{E}(1)}{z\delta^d} + e + 1\right) + \left|\frac{\partial f_n^{\#}}{\partial z}\right| \leq \frac{1}{\delta^d}C_1(\beta).\end{aligned}$$

As a consequence we obtain that

$$\left|\frac{\partial \kappa}{\partial z}\right| \leq 2C_1(\beta)|\operatorname{Int}_{\#^*}\gamma|^{\frac{1}{d}}\|\kappa'\|.$$

When we introduce the isoperimetric inequality we have

$$\left|\frac{\partial \kappa}{\partial z}\right| \leq 2C_1\|\kappa'\||\partial_{ext}\operatorname{Int}_{\#^*}\gamma|^{\frac{1}{d-1}} < 2C_1\|\kappa'\||\overline{\gamma}|^{d/d-1}. \tag{2.63}$$

Lastly, for the derivative of the ratio of partition functions we have

$$\left|\frac{\partial}{\partial z}\frac{Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma}}{Z^{\#}_{\operatorname{Int}_{\#^*}\gamma}}\right| \leq \left|\frac{\partial Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma}/\partial z}{Z^{\#}_{\operatorname{Int}_{\#^*}\gamma}}\right| + \left|\frac{\partial Z^{\#}_{\operatorname{Int}_{\#^*}\gamma}/\partial z}{Z^{\#}_{\operatorname{Int}_{\#^*}\gamma}}\right| \frac{Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma}}{Z^{\#}_{\operatorname{Int}_{\#^*}\gamma}}.$$

We will proceed in the same way, and consider only the case where (2.58) is verified. Since $|\operatorname{Int}_{\#^*}\gamma| \leq n+1$ we can use (2.39) and (2.57) to obtain

$$\begin{aligned}\left|\frac{\partial Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma}/\partial z}{Z^{\#}_{\operatorname{Int}_{\#^*}\gamma}}\right| &\leq \left(C_1|\operatorname{Int}_{\#^*}\gamma| + C_2|\partial_{ext}\operatorname{Int}_{\#^*}\gamma|\right) e^{(\widehat{\psi}_n - \widehat{\psi}^{\#}_n)\delta^d|\operatorname{Int}_{\#^*}\gamma|+3|\partial_{ext}\operatorname{Int}_{\#^*}\gamma|} \\ &\leq \left(C_1|\operatorname{Int}_{\#^*}\gamma| + C_2|\partial_{ext}\operatorname{Int}_{\#^*}\gamma|\right) e^{\beta\rho_0/4|\operatorname{Int}_{\#^*}\gamma|^{(d-1)/d}+3|\partial_{ext}\operatorname{Int}_{\#^*}\gamma|}\end{aligned}$$

and this for any $\#^* \in \{0,1\}$. Furthermore, by introducing the isoperimetric inequality and $|\partial_{ext}\operatorname{Int}_{\#^*}\gamma| \leq |\overline{\gamma}|$ we have

$$\left|\frac{\partial Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma}/\partial z}{Z^{\#}_{\operatorname{Int}_{\#^*}\gamma}}\right| \leq \left(C_1|\overline{\gamma}|^{d/d-1} + C_2|\overline{\gamma}|\right) e^{(1/4\beta\rho_0+3)|\overline{\gamma}|}. \tag{2.64}$$

As a result, by combining (2.64) and (2.60) we have

$$\left|\frac{\partial}{\partial z}\frac{Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma}}{Z^{\#}_{\operatorname{Int}_{\#^*}\gamma}}\right| \leq 2\left(C_1 + C_2\right)|\overline{\gamma}|^{d/(d-1)}e^{(1/2\beta\rho_0+6)|\overline{\gamma}|}.$$

Therefore when we introduce inequality (2.29) of Lemme 18 we obtain

$$\kappa Z^{-|\overline{\gamma}|}_{\#} I_{\gamma}\left|\frac{\partial}{\partial z}\frac{Z^{\#^*}_{\operatorname{Int}_{\#^*}\gamma}}{Z^{\#}_{\operatorname{Int}_{\#^*}\gamma}}\right| \leq 2\left(C_1 + C_2\right)|\overline{\gamma}|^{d/(d-1)}e^{-\tau|\overline{\gamma}|}. \tag{2.65}$$

When we combine the inequalities (2.61), (2.63) and (2.65) we get the desired upper bound,

$$\left|\frac{\partial \widehat{w}^{\#}_{\gamma}}{\partial z}\right| \leq \left((3+2\|\kappa'\|)C_1 + 2C_2\right)|\overline{\gamma}|^{d/(d-1)}e^{-\tau|\overline{\gamma}|}.$$

To finish the proof let us show that (2.36) holds at the order $n+1$. Up to this point, we have proved that the truncated weights of class at most $n+1$ are τ-stable. Therefore we can apply

Lemma 19. Let γ of class $n+1$ if we have $a^{\#}_{n+1}\delta^d |\operatorname{Int}\gamma|^{1/d} \leq \frac{\beta\rho_0}{16}$ then by definition of truncated weights we would have $\widehat{w}^{\#}_{\gamma} = w^{\#}_{\gamma}$. Now we consider a contour γ of class $k \leq n$, according to Lemma 19 and (2.42) we have

$$\begin{aligned} a^{\#}_k\delta^d |\operatorname{Int}\gamma|^{1/d} &= a^{\#}_{n+1}\delta^d |\operatorname{Int}\gamma|^{1/d} + (a^{\#}_k - a^{\#}_{n+1})\delta^d |\operatorname{Int}\gamma|^{1/d} \\ &\leq a^{\#}_{n+1}\delta^d |\operatorname{Int}\gamma|^{1/d} + 2k^{1/d}e^{-\tau k^{d-1/d}/2} \\ &\leq a^{\#}_{n+1}\delta^d |\operatorname{Int}\gamma|^{1/d} + \frac{\beta\rho_0}{16}. \end{aligned}$$

Therefore if $a^{\#}_{n+1}\delta^d |\operatorname{Int}\gamma|^{1/d} \leq \beta\rho_0/16$ it implies that $a^{\#}_k\delta^d |\operatorname{Int}\gamma|^{1/d} \leq \beta\rho_0/8$ which in turn would imply $\widehat{w}^{\#}_{\gamma} = w^{\#}_{\gamma}$ by definition of the truncated weights. □

The consequence of the key proposition is that the truncated weights verify the condition of Theorem 17 and therefore we can write the truncated pressures $\widehat{\psi}^{\#}$ using a cluster expansion. In the following lemma, we prove under some assumption that the pressure of the system does not depend on the boundary condition # we impose.

Lemma 20. *Let H be a saturated interaction such that E_0 is stable with has a finite range $R > 0$, and that there is $C > 0$ such that $E_0 \leq C(1 + N_{T_0 \oplus B(0,R)}(\omega)^2)$. In addition, we assume that the truncated weights associated to the model are τ-stable, with $\tau = \frac{1}{2}\beta\rho_0 - 8$ and $\rho_0 > 0$, then the pressure is independent from the boundary conditions (#). More specifically, for β large enough we have*

$$\psi = \widehat{\psi} = \max\{\widehat{\psi}^{(1)}, \widehat{\psi}^{(0)}\}. \tag{2.66}$$

Proof. Before we begin, we recall that the pressure for each boundary condition is given by

$$\psi^{\#} := \lim_{n\to+\infty} \frac{\ln Z^{\#}_{\Lambda_n}}{\delta^d |\Lambda_n|}.$$

We denote by $p = \left\lceil \frac{L}{\delta} \right\rceil$, $P_n = \Lambda_n \setminus \Lambda_{n-p}$ and $B_n = \Lambda_{n-p} \setminus \Lambda_{n-2p}$. Since E has a finite range $R > 0$ with $R < L$, where L is the saturation distance, we have

$$Z_{\Lambda_n} = \int e^{-\beta E_{\Lambda_{n+p}}(\omega)} \Pi^z_{\widehat{\Lambda}_n}(d\omega).$$

Furthermore we denote by F_n the event

$$F_n = \bigcap_{i\in P_n\cup B_n} \{N_{T_i}(\omega) = 1\} \cap \bigcap_{j\in B_{n-2p}\cup P_{n-2p}} \{N_{T_j}(\omega) \geq 1\}.$$

For any $\omega \in F_n$ we have saturation for the tiles in B_n and P_{n-2p} and with it independence between the configurations in the bulk Λ_{n-2p} and in the boundary $B_n \cup P_n$. As a consequence we obtain the following

$$\begin{aligned} Z_{\Lambda_n} &\geq \int e^{-\beta(E_{\Lambda_{n+p}\setminus\Lambda_{n-p}} + \overline{E}_{B_n})} e^{-\beta(\overline{E}_{P_{n-2p}} + E_{\Lambda_{n-3p}})} \mathbb{1}_{F_n}(\omega) \Pi^z_{\widehat{\Lambda}_n}(d\omega) \\ &\geq Z^{(1)}_{\Lambda_{n-2p}} \int e^{-\beta(E_{\Lambda_{n+p}\setminus\Lambda_{n-p}} + \overline{E}_{B_n})} \mathbb{1}_{F_n}(\omega) \Pi^z_{\widehat{B_n\cup P_n}}(d\omega). \end{aligned}$$

Since $\omega \in F_n$ we know that for any $i \in \Lambda_{n+p} \setminus \Lambda_{n-p}$ that

$$\begin{aligned} N_{T_i \oplus B(0,R)}(\omega) &\leq \#\{j \in \mathbb{Z}^d, T_i \cap T_j + \oplus B(0,R) \neq \emptyset\} \\ &\leq \frac{\lambda(B(0, R + \sqrt{d}\delta))}{\delta^d}. \end{aligned}$$

As a result we have

$$\begin{aligned} E_{\Lambda_{n+p} \setminus \Lambda_{n-p}} &\leq \sum_{i \in \Lambda_{n+p} \setminus \Lambda_{n-p}} C(1 + N_{T_i \oplus B(0,R)}(\omega)) \\ &\leq C \left(1 + \frac{\lambda(B(0, R + \sqrt{d}\delta))}{\delta^d}\right) |\Lambda_{n+p} \setminus \Lambda_{n-p}|. \end{aligned}$$

Furthermore we have for $\omega \in F_n$

$$\overline{E}_{B_n}(\omega) = \overline{E}(1)|B_n|.$$

Consequently, there is a constant $c > 0$ such that

$$\int e^{-\beta(E_{\Lambda_{n+p} \setminus \Lambda_{n-p}} + \overline{E}_{B_n})} \mathbb{1}_{F_n}(\omega) \Pi^z_{\widehat{B_n \cup P_n}}(d\omega) \geq (z\delta^d e^{-(z\delta^d + \beta c)})^{|\Lambda_{n+p} \setminus \Lambda_{n-2p}|}.$$

By piecing everything together we obtain

$$\frac{\ln Z_{\Lambda_n}}{\delta^d |\Lambda_n|} \geq \frac{|\Lambda_{n+p} \setminus \Lambda_{n-2p}|}{|\Lambda_n|} \frac{\ln(z\delta^d e^{-(z\delta^d + \beta c)})}{\delta^d} + \frac{|\Lambda_{n-2p}|}{|\Lambda_n|} \frac{\ln Z^{(1)}_{\Lambda_{n-2p}}}{\delta^d |\Lambda_{n-2p}|}$$

and therefore we have $\psi \geq \psi^{(1)}$. Now let us consider the event E_n defined as

$$E_n = \bigcap_{i \in B_n \cup P_n} \{N_{T_i}(\omega) = 1\} \cap \bigcap_{j \in \Lambda_{n-2p} \setminus \Lambda_{n-5p}} \{N_{T_j}(\omega) = 0\}.$$

For any configuration $\omega \in E_n$ the tiles in P_{n-3p} are saturated by the empty space and we retrieve the empty boundary condition on Λ_{n-3p}. Therefore we have

$$\begin{aligned} Z^{(1)}_{\Lambda_n} &\geq \int e^{-\beta(\overline{E}_{P_n} + E_{\Lambda_{n-p} \setminus \Lambda_{n-3p}})} e^{-\beta E_{\Lambda_{n-4p}}} \mathbb{1}_{E_n}(\omega) \Pi^z_{\widehat{\Lambda}_n}(d\omega) \\ &\geq Z^{(0)}_{\Lambda_{n-3p}} \int e^{-\beta(\overline{E}_{P_n} + E_{\Lambda_{n-p} \setminus \Lambda_{n-3p}})} \mathbb{1}_{E_n}(\omega) \Pi^z_{\Lambda_n \setminus \Lambda_{n-3p}}(d\omega). \end{aligned}$$

With similar argument we can show that there exists $c > 0$ such that

$$\int e^{-\beta(\overline{E}_{P_n} + E_{\Lambda_{n-p} \setminus \Lambda_{n-3p}})} \mathbb{1}_{E_n}(\omega) \Pi^z_{\Lambda_n \setminus \Lambda_{n-3p}}(d\omega) \geq (z\delta^d e^{-(z\delta^d + \beta c)})^{|\Lambda_n \setminus \Lambda_{n-3p}|}.$$

Therefore we have $\psi^{(1)} \geq \psi^{(0)}$. Finally, let us recall that $(0)_{\Lambda_n} = \bigcap_{i \in P_n \cup B_n} \{N_{T_i}(\omega) = 0\}$ and therefore

$$\begin{aligned} Z^{(0)}_{\Lambda_n} &= \int e^{-\beta E_{\Lambda_{n-p}}} \mathbb{1}_{(0)_{\Lambda_n}} \Pi^z_{\widehat{\Lambda}_n}(d\omega) \\ &= e^{-z\delta^d |P_n \cup B_n|} \int e^{-\beta H} \Pi^z_{\widehat{\Lambda}_{n-2p}}(d\omega) \\ &= e^{-z\delta^d |P_n \cup B_n|} Z_{\Lambda_{n-p}}. \end{aligned}$$

As a result, we have $\psi^{(0)} = \psi$ and therefore

$$\psi = \psi^{(1)} = \psi^{(0)}. \tag{2.67}$$

Since the truncated weights are τ-stable, by Lemma 19 for any k and $n \geq k$ we have

$$|\widehat{\psi}^{\#}_n - \widehat{\psi}^{\#}_k| \leq \frac{1}{\delta^d} e^{-\frac{\tau}{2} k^{d-1/d}} \quad \text{and} \quad |\widehat{\psi}_n - \widehat{\psi}_k| \leq \frac{1}{\delta^d} e^{-\frac{\tau}{2} k^{d-1/d}}. \tag{2.68}$$

By taking the limit $n \to +\infty$ we obtain

$$|\widehat{\psi}^{\#} - \widehat{\psi}^{\#}_k| \leq \frac{1}{\delta^d} e^{-\frac{\tau}{2} k^{d-1/d}} \quad \text{and} \quad |\widehat{\psi} - \widehat{\psi}_k| \leq \frac{1}{\delta^d} e^{-\frac{\tau}{2} k^{d-1/d}}. \tag{2.69}$$

Therefore, for any contours γ of class k we have

$$\begin{aligned} a^{\#}_k \delta^d |\operatorname{Int} \gamma|^{1/d} &= a^{\#} \delta^d |\operatorname{Int} \gamma|^{1/d} + (a^{\#}_k - a^{\#}) \delta^d |\operatorname{Int} \gamma|^{1/d} \\ &\leq a^{\#} \delta^d |\operatorname{Int} \gamma|^{1/d} + 2k^{\frac{1}{d}} e^{-\tau k^{1/d}/2}. \end{aligned}$$

Thus for β large enough such that $2k^{\frac{1}{d}} e^{-\tau k^{1/d}/2} \leq \frac{\beta \rho_0}{16}$, we obtain

$$a^{\#}_k \delta^d |\operatorname{Int} \gamma|^{1/d} \leq a^{\#} \delta^d |\operatorname{Int} \gamma|^{1/d} + \frac{\beta \rho_0}{16}.$$

Consequently, if a contour γ verify $a^{\#} \delta^d |\operatorname{Int} \gamma|^{\frac{1}{d}} \leq \frac{\beta \rho_0}{16}$, then $\widehat{w}^{\#}_\gamma = w^{\#}_\gamma$ by definition of the truncated contours. Therefore when $a^{\#} = 0$ all the truncated weights are equal to the actual weight of the model and thus for all $\Lambda \subset \mathbb{Z}^d$ we have $\widehat{Z}^{\#}_\Lambda = Z^{\#}_\Lambda$ and thus $\widehat{\psi}^{\#} = \psi^{\#}$. Then with (2.67) we have

$$\psi = \widehat{\psi} = \max\{\widehat{\psi}^{(1)}, \widehat{\psi}^{(0)}\}.$$

□

As a consequence of the previous Lemma 20 we can extract properties of the pressure from the study of the truncated pressures. The truncated pressure is a way to extend the actual pressure of the model using cluster expansion.

2.6 Proofs of Liquid-Gas phase transition theorems

In this section we give the proof of Theorem 11. Since $\overline{E}(k) = (Ak + B)\mathbb{1}_{k\geq 1}$ with $A \geq 0$ and $-A \leq B < +\infty$ we have that $\overline{E} \geq 0$ and $\overline{E}(1) < +\infty$. Therefore, we verify the assumptions of Proposition 13. Thus we have that the truncated weights verify the hypothesis of Theorem 17, provided that β is large enough, and therefore we have

$$\widehat{\Phi}^{\#}(\Lambda) = e^{f^{\#}\delta^d|\Lambda| + \Delta^{\#}_{\Lambda}}$$

where $f^{\#}$ and $\Delta^{\#}_{\Lambda}$ are C^1 in U_β. Furthermore, according to Theorem 17 we also have

$$|f^{\#}| \leq \eta(\tau, l_0), \qquad |\Delta^{\#}_{\Lambda}| \leq \eta(\tau, l_0)|\partial_{ext}\Lambda| \tag{2.70}$$

$$\left|\frac{\partial f^{\#}}{\partial z}\right| \leq D\eta(\tau, l_0), \qquad \left|\frac{\partial \Delta^{\#}_{\Lambda}}{\partial z}\right| \leq D\eta(\tau, l_0)|\partial_{ext}\Lambda| \tag{2.71}$$

where $\tau = \frac{\beta\rho_0 l_0}{2} - 8$. For $z \in U_\beta$ let us denote by G the gap between the two truncated pressures

$$G(z) := \widehat{\psi}^{(1)} - \widehat{\psi}^{(0)} = \frac{1}{\delta^d}\ln\left(\frac{Z_1}{Z_0}\right) + f^{(1)} - f^{(0)} = \frac{1}{\delta^d}\ln S(z) + f^{(1)} - f^{(0)},$$

where

$$S(z) = \frac{Z_1}{Z_0} = \sum_{k=1}^{+\infty}\frac{(z\delta^d)^k}{k!}e^{-\beta(Ak+B)} = e^{-\beta B}\left(\exp(z\delta^d e^{-\beta A}) - 1\right).$$

By definition of z^-_β and z^+_β in (2.2) and by using (2.70), we have for sufficiently large β

$$G(z^-_\beta) \leq -\frac{2}{\delta^d} + 2\eta(\tau, l_0) < 0$$
$$G(z^+_\beta) \geq \frac{2}{\delta^d} - 2\eta(\tau, l_0) > 0.$$

By direct computation and using (2.71) we have

$$\begin{aligned}
\frac{\partial G}{\partial z}(z) &= \frac{e^{-\beta(A+B)}\exp(z\delta^d e^{-\beta A})}{e^{-\beta B}(\exp(z\delta^d e^{-\beta A}) - 1)} + \frac{\partial f^{(1)}}{\partial z} - \frac{\partial f^{(0)}}{\partial z} \\
&\geq \frac{e^{-\beta A}}{1 - \exp(-z\delta^d e^{-\beta A})} - 2D\eta(\tau, l_0) \\
&\geq \frac{e^{-\beta A}}{1 - \exp(-z^+_\beta\delta^d e^{-\beta A})} - 2D\eta(\tau, l_0) \\
&\geq e^{-\beta(A+B)-2} + e^{-\beta A} - 4De^{-\frac{\beta\rho_0 l_0}{6} + \frac{8}{3}}.
\end{aligned}$$

Since we assumed that either $A < \frac{\rho_0 l_0}{6}$ or $A + B < \frac{\rho_0 l_0}{6}$, for sufficiently large β and any $z \in U_\beta$ we have

$$\frac{\partial G}{\partial z}(z) > 0.$$

This ensures the existence of an unique $z_\beta^c \in U_\beta$ such that

$$\widehat{\psi} = \begin{cases} \widehat{\psi}^{(0)} & \text{when } z \in (z_\beta^-, z_\beta^c] \\ \widehat{\psi}^{(1)} & \text{when } z \in [z_\beta^c, z_\beta^+) \end{cases}$$

and also for all $z \in U_\beta$ (and thus also for $z = z_\beta^c$) we have

$$\frac{\partial \widehat{\psi}^{(1)}}{\partial z}(z) > \frac{\partial \widehat{\psi}^{(0)}}{\partial z}(z). \tag{2.72}$$

Therefore by Lemma 20 and (2.72) we have

$$\frac{\partial \psi}{\partial z^+}(\beta, z_c) > \frac{\partial \psi}{\partial z^-}(\beta, z_c).$$

We denote $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$ for $n \in \mathbb{N}^*$, we know by direct computation that

$$\frac{\partial \ln Z^\#_{\Lambda_n}}{\partial z} = -\delta^d |\Lambda_n| + \frac{1}{z} E_{P^\#_{\Lambda_n}}(N_{\widehat{\Lambda}_n}). \tag{2.73}$$

For β large enough and $z = z_\beta^c$ we know that $a^{(0)} = a^{(1)} = 0$ and therefore for any $\Lambda \subset \mathbb{Z}^d$ we have $Z^\#_\Lambda = \widehat{Z}^\#_\Lambda$ and consequently

$$\frac{\partial \ln Z^\#_{\Lambda_n}}{\partial z} = \frac{\partial \ln \widehat{Z}^\#_{\Lambda_n}}{\partial z} = \frac{\partial \widehat{\psi}^\#}{\partial z} \delta^d |\Lambda_n| + \frac{\partial \Delta^\#_{\Lambda_n}}{\partial z}. \tag{2.74}$$

Therefore by combining (2.73) and (2.74) we obtain

$$\frac{E_{P^\#_{\Lambda_n}}(N_{\widehat{\Lambda}_n})}{\delta^d |\Lambda_n|} = z + z \frac{\partial \widehat{\psi}^\#}{\partial z} + \frac{z}{\delta^d |\Lambda_n|} \frac{\partial \Delta^\#_{\Lambda_n}}{\partial z}.$$

According to Theorem 17 we know that

$$\left| \frac{\partial \Delta^\#_\Lambda}{\partial z} \right| \leq D\eta(\tau, l_0) |\partial_{ext} \Lambda| \implies \frac{1}{|\Lambda_n|} \frac{\partial \Delta^\#_{\Lambda_n}}{\partial z} \underset{n \to \infty}{\to} 0.$$

Furthermore we know by construction that

$$E_{P^\#_{\Lambda_n}}(N_{\widehat{\Lambda}_n}) = E_{\overline{P}^\#_{\Lambda_n}}(N_{\widehat{\Lambda}_n})$$

and since the empirical field is stationary we have

$$\frac{E_{\overline{P}^\#_{\Lambda_n}}(N_{\widehat{\Lambda}_n})}{\delta^d |\Lambda_n|} = E_{\overline{P}^\#_{\Lambda_n}}(N_{[0,1]^d}).$$

In summary, we obtain that

$$E_{\overline{P}^{\#}_{\Lambda_n}}(N_{[0,1]^d}) = z + z\frac{\partial \widehat{\psi}^{\#}}{\partial z} + \frac{z}{\delta^d |\Lambda_n|}\frac{\partial \Delta^{\#}_{\Lambda_n}}{\partial z}. \tag{2.75}$$

According to Proposition 11 both empirical fields $(\overline{P}^{\#}_{\Lambda_n})_{n\in\mathbb{N}}$ exhibit at least one accumulation point $P^{\#}$ that is a Gibbs measure. Therefore by local convergence and by taking the limit for the correct sub-sequence in (2.75) we have

$$\rho(P^{\#}) = z + z\beta\frac{\partial \psi^{\#}}{\partial z}(z^c_\beta)$$

and therefore using (2.72) we have that $\rho(P^{(1)}) > \rho(P^{(0)})$.

Under the assumption of Theorem 11, we have proved the existence of a Liquid-Gas phase transition at low temperature since β needs to be large enough. We have demonstrated the existence of the critical activity z^c_β at which this phenomenon occurs, although its determination is theoretical and, in general, we cannot expect to find an explicit expression for this quantity. The only case where this value is known is for the Area interaction with deterministic radii, where it is obtained via other methods. In chapter 3, concerning the Quermass interaction and the diluted pairwise interaction, although we lack an explicit expression, we are able to elucidate the asymptotic behaviour of z^c_β as β tends to infinity.

Chapter 3

Application to Quermass and diluted pairwise interaction

In this chapter, we present two results of liquid-gas phase transition. The first result concerns the Quermass interaction and the second result is about the diluted pairwise interaction. The proof of the phase transition theorems exposed in this chapter is an application of the results obtained in Chapter 2. For each type of interaction we are giving conditions for saturation and for them to verify Peierls condition. We are also going to investigate the behaviour of the critical activity z_β^c at which the phase transition happens as β tends to infinity. Lastly, we will present the perspectives on the work presented in this thesis.

3.1 Quermass interaction

The Quermass interaction in $\mathbb{R}^d$ are morphological interactions whose Hamiltonian is given by a linear combination of the $d+1$ Minkowski functionals on the halo of a configuration. It is a generalisation of the Area interaction and in fact by Hadwiger's characterisation Theorem [20] we know that it encompasses any functional F on finite union of convex compact spaces, continuous for the Haussdorff metric, invariant under isometric transformations and additive (i.e. $F(A\cup B) = F(A)+F(B)-F(A\cap B)$). This model was introduced by Kendall, Van Lieshout, and Baddeley [23], and for bounded random radii, they have proved that the infinite volume process exists under some conditions on the Minkowski functionals to ensure the stability of the interaction. This result of existence has been extended by Dereudre in dimension 2 for unbounded random radii with some assumptions on the tail of the distribution [5]. The Gibbs point process associated to the Quermass model is a marked point process on $\mathbb{R}^d$ and $S = [R_0, R_1]$ (with $R_1 \geq R_0 > 0$). In this case, for the Hamiltonian, we consider only linear combination of the volume $\mathcal{V}$, the surface measure $\mathcal{S}$ and the Euler-Poincaré characteristic χ (in dimension $d = 2$). This restriction is due to statistical physics considerations since we need the stability of the energy.

Definition 25. Let $\theta_1 \in \mathbb{R}$ and $\theta_2 \geq 0$. The energy of a finite configuration $\omega \in \Omega_f$ is given by

$$H(\omega) = \begin{cases} \mathcal{V}(L(\omega)) + \theta_1 S(L(\omega)) - \theta_2 \chi(L(\omega)) & \text{if } (d = 2) \\ \mathcal{V}(L(\omega)) + \theta_1 S(L(\omega)) & \text{if } (d \geq 3) \end{cases},$$

where

$$L(\omega) = \bigcup_{(x,R)\in\omega} B(x, R),$$

$\mathcal{V}(L(\omega))$ is the volume of $L(\omega)$ defined as the Lebesgue measure of $L(\omega)$, $S(L(\omega))$ is the surface of $L(\omega)$ defined as the $d-1$-dimensional Hausdorff measure of the boundary $\partial L(\omega)$ and $\chi(L(\omega))$ is the Euler-Poincaré characteristic of $L(\omega)$ defined as the difference between the number of connected components and the number of holes in $L(\omega)$ (in dimension $d = 2$).

The energy is parametrized with two parameters θ_1 and θ_2. We discuss below why we impose θ_2 to be non negative. With this choice of parameters the energy is stable which means that there exists a constant $C \geq 0$ such that for any finite configuration $\omega \in \Omega_f$,

$$H(\omega) \geq -CN(\omega).$$

The volume and the surface are clearly stable since the radii are uniformly bounded. The Euler-Poincaré characteristic is more delicate to study. In dimension 2, it is shown by Kendall et al. [23] that for the union of N closed balls, the number of holes is bounded above by $2N - 5$, and the number of connected components is clearly bounded by N. Therefore the Euler-Poincaré characteristic is stable for any parameter $\theta_2 \in \mathbb{R}$. In higher dimension $d \geq 3$, for some configurations, the maximum number of holes is of order N^2 and thus the Euler-Poincaré characteristic is not stable if $\theta_2 < 0$. More generally, the stability of this statistic is not obvious even if θ_2 is strictly positive. Therefore the existence of the infinite volume Gibbs point process is not well established. It is for this reason that we impose $\theta_2 = 0$ in the case $d \geq 3$.

Since the radii are uniformly bounded, we can show that the interaction has a finite range that is equal to $2R_1$. Indeed, using additivity of the Minkowski functionals we have for any configuration $\omega \in \Omega$ and any subset $\Delta \in \mathbb{B}_b(\mathbb{R}^d)$

$$\begin{aligned} F(L(\omega)) &:= (\mathcal{V} + \theta_1 S - \theta_2 \chi)(L(\omega)) \\ &= F(L(\omega_\Delta)) + F(L(\omega_{\Delta^c})) - F(L(\omega_\Delta) \cap L(\omega_{\Delta^c})) \\ &= F(L(\omega_\Delta)) + F(L(\omega_{\Delta^c})) - F(L(\omega_\Delta) \cap L(\omega_{\Delta \oplus B(0,2R_1)})). \end{aligned}$$

As a consequence we have

$$\begin{aligned} H_\Delta(\omega) &= F(L(\omega)) - F(L(\omega_{\Delta^c})) \\ &= F(L(\omega_\Delta)) - F(L(\omega_\Delta) \cap L(\omega_{\Delta \oplus B(0,2R_1)})) \\ &= F(L(\omega_{\Delta \oplus B(0,2R_1)})) - F(L(\omega_{\Delta \oplus B(0,2R_1)\setminus\Delta})) = H_\Delta(\omega_{\Delta \oplus B(0,2R_1)}). \end{aligned}$$

In summary, the Quermass interaction with bounded random radii and parameters $\theta_1 \in \mathbb{R}$ and $\theta_2 \geq 0$ is stable and has a finite range. Therefore, the existence of the infinite Gibbs point process associated to this interaction is a consequence of Theorem 1. Concerning the non-uniqueness of such point process, it involves one or two critical parameters θ_1^* and $\theta_2^*(\theta_1)$, depending on the dimension, with $\theta_1^* = R_0 \frac{\mathcal{V}(B(0,1))}{\mathcal{S}(B(0,1))} = \frac{R_0}{d}$ and for $\theta_1 > -\theta_1^*$ we have an expression of $\theta_2^*(\theta_1)$ which is given further down the line in equation (3.4).

Theorem 21. *Let θ_1, θ_2 be two parameters such that $\theta_1 > -\theta_1^*$ and $0 \leq \theta_2 < \theta_2^*(\theta_1)$ (recall that $\theta_2 = 0$ if $d \geq 3$). Then there exists $\beta_c(\theta_1, \theta_2) > 0$ such that for all $\beta > \beta_c(\theta_1, \theta_2)$, there exists $z_\beta^c > 0$ for which a liquid-gas phase transition occurs. Furthermore, we know that $|z_\beta^c - \beta|$ tends to zero exponentially fast when β tends to infinity.*

In the rest of this section, we will demonstrate the phase transition via the saturated interaction setting. So first, we will explicit the coarse graining and quickly prove some of the basic property of E_0 and $\overline{E}$. Then we prove that this interaction verify the Peierls-like condition. Finally, we will finish by studying the asymptotic behaviour of the critical activity z_β^c.

3.1.1 Coarse graining for the Quermass interaction

We call a facet F any non-empty intersection of closed tiles $(\overline{T}_i)_{i\in\mathbb{Z}}$. Clearly the dimension of a facet can be any integer between 0 and d. We denote by $\mathcal{F}$ the set of all facets and $\mathcal{F}_0 := \{F \in \mathcal{F}, F \cap T_0 \neq \emptyset\}$ the set of facets that touches the tile T_0.

Proposition 14. *The coarse graining for the Quermass interaction is given by*

$$\forall \omega \in \Omega_f, \quad E_0(\omega) = \mathcal{V}(L(\omega) \cap T_0) + \theta_1 S_0(L(\omega)) - \theta_2 \chi_0(L(\omega)), \tag{3.1}$$

where

$$S_0(A) = \sum_{k=d-1}^{d} \sum_{\substack{F\in\mathcal{F}_0 \\ \dim(F)=k}} (-1)^{d-k} \mathcal{S}(A \cap F)$$

$$\chi_0(A) = \sum_{k=0}^{d} \sum_{\substack{F\in\mathcal{F}_0 \\ \dim(F)=k}} (-1)^{d-k} \chi(A \cap F).$$

Furthermore, we have that E_0 has a finite range R_1 and there is $C > 0$ such that for any $\omega \in \Omega_f$,

$$|E_0(\omega)| \leq C(1 + N_{T_0 \oplus B(0,R_1)}(\omega)).$$

Proof. As a consequence of the additivity of the Minkowski functionals we have that

$$\sum_{i\in\mathbb{Z}^d} E_i(\omega) = H(\omega).$$

This ensures that E_0 is a proper coarse graining of the Hamiltonian of the Quermass interaction. It is clear that $E_0(\omega) = E_0(\omega_{T_0 \oplus B(0,R_1)})$ since for any marked point $x, r \in \omega_{(T_0 \oplus B(0,R_1))^c}$ and any

facet $F \in \mathcal{F}_0$ we have $B(x, r) \cap F = \emptyset$. Therefore E_0 has a range of R_1. Finally, we prove that E_0 is at most linear. First of all the volume contribution is bounded by the volume of the tile. Now for S_0 we have that for any facet $F \in \mathcal{F}_0$ of dimension d or $d-1$ and any configuration ω

$$\mathcal{S}(L(\omega) \cap F) \leq N_{T_0 \oplus B(0,R_1)}(\omega) \mathcal{S}(B(0, R_1))$$

and therefore there is $C_S > 0$ such that

$$|S_0(L(\omega))| \leq C_S N_{T_0 \oplus B(0,R_1)}(\omega)$$

Now we consider that we are in dimension 2 and we look into the Euler-Poincaré characteristic. If the facet F is of dimension 0 we have that $\chi(L(\omega) \cap F) \in \{0, 1\}$. Then if F is of dimension 1, then $L(\omega) \cap F$ has no hole then $\chi(L(\omega) \cap F) = N_{cc}(L(\omega) \cap F) \leq N_{T_0 \oplus B(0,R1)}(\omega)$. Finally, if $F = T_0$ Kendall et al. [23] has proved that the number of holes is at most linear. As a consequence, there is $C_\chi > 0$ such that

$$|\chi_0(L(\omega))| \leq C_\chi (1 + N_{T_0 \oplus B(0,R_1)}(\omega)).$$

□

Furthermore, we can observe that $S_i(\overline{T}_i) = 0$ and $\chi_i(\overline{T}_i) = 0$. This is due to the fact that for a $d-1$-dimensional facet F, we have $\mathcal{S}(F) = 2\lambda^{(d-1)}(F)$ where $\lambda^{(d-1)}$ is the $(d-1)$ Lebesgue measure. Therefore, for any configuration $\omega \in \Omega_f$ such that $T_i \subset L(\omega)$ we have $E_i(\omega) = \mathcal{V}(T_i) = \delta^d$. In the following proposition, we demonstrate that the Quermass interaction is saturated.

Proposition 15. *For $\delta \leq \frac{R_0}{\sqrt{d}}$ and $L > 2R_1 + \sqrt{d}\delta$ the Quermass interaction is saturated. More precisely, we have*

$$\overline{E}(k) := \delta^d \mathbb{1}_{k \geq 1} \tag{3.2}$$

such that

$$\forall \omega \in \Omega_{0,L,\delta}, \quad E_0(\omega) = \overline{E}(N_{T_0}(\omega)). \tag{3.3}$$

Proof. First, we begin with the case where $\omega \in \Omega^1_{0,L,\delta}$. We know that there is $(x, R) \in \omega_{T_0}$ and by our choice on δ we have that $T_0 \subset B(x, R)$. Therefore, we have $E_0(\omega) = \delta^d = \overline{E}(N_{T_0}(\omega))$.

Now we consider a configuration $\omega \in \Omega^0_{0,L,\delta}$. With our choice of L we know that for any $(x, R) \in \omega$ we have $B(x, R) \cap T_0 = \emptyset$. Therefore, we have $E_0(\omega) = 0 = \overline{E}(0)$. □

We have demonstrated through Proposition 14 and Proposition 15 that under some good choice of δ and L the Quermass interaction we are studying verifies most of the assumptions of Theorem 11 with the exception of the Peierls-like condition.

3.1.2 Peierls condition

In order to demonstrate that the Quermass interaction verify the Peierls condition, we will use a mix of the tile-by-tile approach and a global approach. Our goal, is to prove that in any contour we have enough tiles that are entirely covered by the halo in order to compensate any missing energy due to the surface or Euler-Poincaré characteristic contributions to the energy. A first way to do so is by using the dominoes. Therefore if we have $\delta < \frac{R_0}{2\sqrt{d}}$, in each contour γ and each configuration ω that achieves this contour the number of covered tiles is greater than $D(\gamma)$. In turn, by Lemma 12 we have that

$$D(\gamma) \geq r_0 |\overline{\gamma}|, \quad \text{where} \quad r_0 = \frac{1}{|B(0, {}^{6L}/_{\delta}) \cap \mathbb{Z}^d|}.$$

Another way to find such tiles in the contour is by counting the tiles covered by the halo which are close to the boundary of the halo. Indeed those tiles are guaranteed to be empty otherwise the boundary would be further away.

Lemma 22. *Let $R_0 \geq 2\delta\sqrt{d} > 0$, and let us define θ_1^δ as*

$$\theta_1^\delta := \inf_{\substack{\omega \in \Omega_f \\ \gamma : S_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(L(\omega))>0}} \left\{ \frac{V_{\omega,\gamma,\delta}}{S_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(L(\omega))} \right\}$$

and

$$V_{\omega,\gamma,\delta} := \max\left\{ \mathcal{V}(T_I), I \subset \overline{\gamma}\setminus\partial^-\overline{\gamma}, \forall i \in I, T_i \subset \partial L(\omega) \oplus B(0, R_0) \cap L(\omega) \right\}.$$

We have $\theta_1^\delta > 0$ and $\theta_1^\delta \underset{\delta\to 0}{\to} \theta_1^*$ where $\theta_1^* = R_0 \frac{\mathcal{V}(B(0,1))}{S(B(0,1))}$.

Proof. For any finite configuration $\omega \in \Omega_f$ and γ a contour that is created by this configuration such that $S_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(L(\omega)) > 0$, we have the following inequalities

$$V^+_{\omega,\gamma,\delta} \geq V_{\omega,\gamma,\delta} \geq V^-_{\omega,\gamma,\delta}$$

where

$$\begin{aligned} V^+_{\omega,\gamma,\delta} &= \mathcal{V}(\partial L(\omega) \oplus B(0, R_0) \cap L(\omega) \cap \widehat{\gamma\setminus\partial^-\gamma}) \\ V^-_{\omega,\gamma,\delta} &= \mathcal{V}((\partial L(\omega) \oplus B(0, R_0 - \delta))\setminus(\partial L(\omega) \oplus B(0, \delta)) \cap L(\omega) \cap \widehat{\gamma\partial^-\gamma}). \end{aligned}$$

The boundary of the halo $L(\omega)$ inside $\overline{\gamma}\setminus\partial^-\overline{\gamma}$, appearing in the computation of $S_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(L(\omega))$, is the union of spherical caps built via some marked points $(x_1, r_1), \dots, (x_m, r_m) \in \omega$. We denote by $\alpha_i \in [0, 1]$ the ratio of the surface of the ith spherical cap with respect to the total surface of its sphere. Therefore, by a simple geometrical argument

$$\begin{aligned} \frac{V^-_{\omega,\gamma,\delta}}{S_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(L(\omega))} &\geq \frac{\mathcal{V}(B(0,1))}{S(B(0,1))} \left(\frac{\sum_{i=1}^m \alpha_i (r_i^d - (r_i - R_0)^d)}{\sum_{i=1}^m \alpha_i r_i^{d-1}} - \epsilon_{\omega,\gamma}(\delta) \right) \\ &\geq \frac{\mathcal{V}(B(0,1))}{S(B(0,1))} (R_0 - \epsilon_{\omega,\gamma}(\delta)) \end{aligned}$$

where

$$\epsilon_{\omega,\gamma}(\delta) = \frac{\sum_{i=1}^{m} \alpha_i (r_i^d - (r_i - \delta)^d + (r_i - R_o + \delta)^d - (r_i - R_0)^d)}{\sum_{i=1}^{m} \alpha_i r_i^{d-1}}.$$

Therefore for any contours γ and any configuration $\omega \in \Omega_f$ that achieves this contour we have

$$\liminf_{\delta \to 0} \frac{V_{\omega,\gamma,\delta}}{S_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(L(\omega))} \geq \frac{R_0 \mathcal{V}(B(0,1))}{S(B(0,1))}.$$

Therefore, we obtain that

$$\liminf_{\delta \to 0} \theta_1^\delta \geq \theta_1^*.$$

Note that for $\omega = \{(0, R_0)\}$ we have

$$\frac{V^+_{\omega,\gamma,\delta}}{S_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(L(\omega))} = \frac{R_0 \mathcal{V}(B(0,1))}{S(B(0,1))}.$$

And thus,

$$\frac{R_0 \mathcal{V}(B(0,1))}{S(B(0,1))} \geq \inf_{\substack{\omega \in \Omega_f \\ \gamma : S_{\overline{\gamma}}(L(\omega)) > 0}} \left\{ \frac{V^+_{\omega,\gamma,\delta}}{S_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(L(\omega))} \right\}.$$

Which in turn implies that

$$\theta_1^* \geq \limsup_{\delta \to 0} \theta_1^\delta.$$

□

In the following lemma we compare the contribution of the Euler-Poincaré characteristic with the volume of the core of the contour. Indeed, the only tiles where the energy contribution of the surface measure or the Euler-Poincaré characteristic is non equal to 0 is for tiles in $\overline{\gamma}\setminus\partial^-\overline{\gamma}$.

Lemma 23. *Let $\delta \leq R_0/2\sqrt{d}$, $L \geq 2R_1 + \sqrt{d}\delta$ and $d = 2$. For any contour γ and any configuration ω that achieves this contour we have*

$$\chi_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega) \leq \frac{|\overline{\gamma}|\delta^d}{R_0^d \mathcal{V}(B(0,1))}.$$

Proof. By definition of the contours and with the conditions on δ and L we have

$$\chi_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(L(\omega)) = \chi_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(L(\omega_{\widehat{\gamma}})) \leq N_{cc}(L(\omega_{\widehat{\gamma}})).$$

Now the aim is to find for each connected components C of $L(\omega_{\widehat{\gamma}})$ a single point $(x, R) \in \omega_{\widehat{\gamma}}$ such that the ball $B(x, R) \subset C \cap \widehat{\gamma}$. Note that a ball $B(x, R) \subset C$ is not necessarily included inside the contour. If there exists such a ball not included in the contour, then there is $A \subset \overline{\gamma}^c$ a connected component such that $B(x, R) \cap \widehat{A} \neq \emptyset$. It gives the information that the site $i \in \mathbb{Z}^d$ such that

$x \in T_i$ is included inside $\partial_{ext}A$ and by Lemma 10 for all $j \in \partial_{ext}A$ we have $\sigma(j, \omega_{\omega_{\widehat{\gamma}}}) = 1$. Therefore all balls that are in the tiles corresponding to $\partial_{ext}A$ belong to the same connected component of the halo. We choose a site $j \in \partial_{ext}A$ such that $d_2(j, A) = \lceil \frac{2R_1}{\delta} \rceil$ and so there exists $(y, R') \in \omega_{T_j}$ such that $B(y, R') \subset \widehat{\gamma}$. Therefore we can replace the original representative of the connected component with one that is more suitable.

With this procedure we have now built, for each connected components C of $L(\omega_{\widehat{\gamma}})$, a single point $(x, R) \in \omega_{\widehat{\gamma}}$ such that the ball $B(x, R) \subset C \cap \widehat{\gamma}$. We define $I(\omega_{\widehat{\gamma}})$ as the set of all these points which represent the connected components of $L(\omega_{\widehat{\gamma}})$. By construction for any $(x, R) \neq (y, R') \in I(\omega_{\widehat{\gamma}})$, $B(x, R) \cap B(y, R') = \emptyset$ and therefore we have

$$N_{cc}(\omega_{\widehat{\gamma}}) = |I(\omega_{\widehat{\gamma}})| \leq \frac{|\overline{\gamma}|\delta^d}{\mathcal{V}(B(0, R_0))}.$$

□

For the following, we fix $L = 2R_1 + \sqrt{d}\delta$, therefore the constant r_0 in Lemma 12 has the following expression

$$r_0(\delta) = \frac{1}{\left| B\left(0, \frac{12R_1}{\delta} + 6\sqrt{d}\right) \cap \mathbb{Z}^d \right|}.$$

Let us consider $\theta_1 > -\theta_1^*$, we define $\theta_2^*(\theta_1)$ and $\theta_2^\delta(\theta_1)$ as

$$\theta_2^*(\theta_1) = \begin{cases} r_0\left(\frac{R_0}{2\sqrt{d}}\right)\mathcal{V}(B(0, R_0)) & \text{when } \theta_1 \geq 0 \\ \sup_{\delta \in \left]0, \frac{R_0}{2\sqrt{d}}\right[: \theta_1 > -\theta_1^\delta} \left\{ r_0(\delta)\mathcal{V}(B(0, R_0))(1 + \frac{\theta_1}{\theta_1^\delta}) \right\} & \text{when } \theta_1 < 0 \end{cases} \tag{3.4}$$

$$\theta_2^\delta(\theta_1) = \begin{cases} r_0\left(\frac{R_0}{2\sqrt{d}}\right)\mathcal{V}(B(0, R_0)) & \text{when } \theta_1 \geq 0 \\ \left\{ r_0(\delta)\mathcal{V}(B(0, R_0))(1 + \frac{\theta_1}{\theta_1^\delta}) \right\} & \text{when } \theta_1 < 0 \end{cases}.$$

Now let us prove that the Quermass for a large class of parameters verify the Peierls conditions.

Proposition 16. *Let θ_1, θ_2 be two parameters such that $\theta_1 > -\theta_1^*$ and $0 \leq \theta_2 < \theta_2^*(\theta_1)$ (recall that $\theta_2 = 0$ if $d \geq 3$). Then there is $\rho_0 > 0$ such that for any contours γ and any configuration ω that achieves this contours we have*

$$E_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega) - \overline{E}_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega) \geq \rho_0|\overline{\gamma}| \tag{3.5}$$

Proof. We detail the proof of Proposition 16 in dimension 2. In higher dimension, the proof works in the same manner and it is even easier since we assume that $\theta_2 = 0$. We know by

Lemma 22 that for sufficiently small δ we have $\theta_1 \geq -\theta_1^\delta > -\theta_1^*$ and $\theta_2 \leq \theta_2^\delta(\theta_1) < \theta_2^*(\theta_1)$. In the case where $\theta_1 < 0$ we need to consider a threshold t such that

$$\frac{\theta_2}{\theta_1^\delta + \theta_1} \frac{\delta^d}{\mathcal{V}(B(0,R_0))} < t < \frac{1}{\theta_1}\left(\frac{\theta_2 \delta^d}{\mathcal{V}(B(0,R_0))} - r_0(\delta)\delta^d\right).$$

We define the quantity ρ_0 as such

$$\rho_0 = \begin{cases} r_0(\delta)\delta^d - \frac{\theta_2\delta^d}{\mathcal{V}(B(0,R_0))} & \text{when } \theta_1 \geq 0 \\ \min\left\{(\theta_1^\delta + \theta_1)t - \frac{\theta_2\delta^d}{\mathcal{V}(B(0,R_0))},\ r_0(\delta)\delta^d + \theta_1 t - \frac{\theta_2\delta^d}{\mathcal{V}(B(0,R_0))}\right\} & \text{when } \theta_1 < 0 \end{cases}.$$

With the conditions on δ and t, it guarantees that $\rho_0 > 0$. We are going to start with the case when $\theta_1 < 0$. First we define for any contours γ the core of the contour with the spin equal to 1 as $C_1(\gamma) = \{i \in \overline{\gamma} \backslash \partial^- \overline{\gamma}, \sigma_i = 1\}$. With the condition $R_0 > 2\delta\sqrt{d}$ we know that we have a non negligible amount of empty tiles that are completely covered by the halo. Therefore using Lemmas 12 and 22 we have the following two lower bounds on the energy of the contour

$$\begin{aligned} E_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(\omega) &\geq \begin{cases} |C_1(\gamma)|\delta^d + r_0|\overline{\gamma}|\delta^d + \theta_1 S_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(L(\omega)) - \theta_2 \chi_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(L(\omega)) \\ |C_1(\gamma)|\delta^d + (\theta_1^\delta + \theta_1) S_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(L(\omega)) - \theta_2 \chi_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(L(\omega)) \end{cases} \\ &\geq \begin{cases} |C_1(\gamma)|\delta^d + r_0|\overline{\gamma}|\delta^d + \theta_1 S_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(L(\omega)) - \theta_2 \frac{|\overline{\gamma}|\delta^d}{\mathcal{V}(B(0,R_0))} \\ |C_1(\gamma)|\delta^d + (\theta_1^\delta + \theta_1) S_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(L(\omega)) - \theta_2 \frac{|\overline{\gamma}|\delta^d}{\mathcal{V}(B(0,R_0))} \end{cases} \quad \text{by Lemma 23.} \end{aligned}$$

Depending on the value of the surface inside the contour, one lower bound will be more preferable than the other. Since $\theta_1^\delta + \theta_1 > 0$ and given the threshold t that verifies our assumption we have

$$E_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(\omega) \geq \begin{cases} |C_1(\gamma)|\delta^d + \left(r_0\delta^d + \theta_1 t - \frac{\theta_2\delta^d}{\mathcal{V}(B(0,R_0))}\right)|\overline{\gamma}| & \text{if } S_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(L(\omega)) \leq t|\overline{\gamma}| \\ |C_1(\gamma)|\delta^d + (\theta_1^\delta + \theta_1)t|\overline{\gamma}| - \theta_2 \frac{|\overline{\gamma}|\delta^d}{\mathcal{V}(B(0,R_0))} & \text{if } S_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(L(\omega)) > t|\overline{\gamma}| \end{cases}.$$

In either cases, we have the desired lower boundary on the energy of a contour and since $|C_1(\gamma)|\delta^d = \overline{E}_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(\omega)$ we obtain

$$E_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(\omega) - \overline{E}_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(\omega) \geq \rho_0|\overline{\gamma}|.$$

Let us turn to the second case where $\theta_1 \geq 0$. It is even easier since we can simply drop the contribution of the surface in the energy and therefore we have

$$E_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(\omega) \geq |C_1(\gamma)|\delta^d + \left(r_0\delta^d - \frac{\theta_2\delta^d}{\mathcal{V}(B(0,R_0))}\right)|\overline{\gamma}| = |C_1(\gamma)|\delta^d + \rho_0|\overline{\gamma}|.$$

□

3.1.3 Proof of Theorem 21

We know by Proposition 14 and Proposition 16, for theses values of θ_1 and θ_2, the Quermass interaction verify the assumption of of Theorem 11. Therefore we have already the proof of the liquid-gas phase transition. All that is left is to find out the asymptotic behaviour of z_β^c and for that we need to go back to some details of the proof of Theorem 11. First of all, recall that for the Quermass interaction we have the following

$$\frac{Z_1}{Z_0} = S(z) = e^{-\beta\delta^d}(e^{z\delta^d} - 1).$$

We are going to fix $a(\beta) = \min\{2, e^{-\beta c}\}$ with $0 < c < \frac{\rho_0 l_0}{6}$ and

$$\widehat{U}_\beta = \left(\frac{\ln(1 + e^{\beta\delta^d - a(\beta)})}{\delta^d}, \frac{\ln(1 + e^{\beta\delta^d + a(\beta)})}{\delta^d} \right).$$

As such, we have for $z \in \widehat{U}_\beta$,

$$e^{-a} \leq \frac{Z_1}{Z_0} \leq e^a$$

and more generally,

$$e^{-2} \leq \frac{Z_1}{Z_0} \leq e^2.$$

Therefore, we have $\widehat{U}_\beta \subset U_\beta$. The question is whether or not z_β^c belongs to $\widehat{U}_\beta$. For the Quermass interaction the gap between the truncated pressures is given by

$$\begin{aligned} G(z) &:= \widehat{\psi}^{(1)} - \widehat{\psi}^{(0)} \\ &= \frac{1}{\delta^d} \ln\left(\frac{Z_1}{Z_0}\right) + f^{(1)} - f^{(0)} \\ &= (z - \beta) + \frac{\ln(1 - e^{-z\delta^d})}{\delta^d} + f^{(1)} - f^{(0)}. \end{aligned}$$

For $\hat{z}_\beta^- = \frac{\ln(1+e^{\beta\delta^d - a})}{\delta^d}$ and $\hat{z}_\beta^+ = \frac{\ln(1+e^{\beta\delta^d + a})}{\delta^d}$, which are the boundaries of $\widehat{U}_\beta$, we have for sufficiently large β

$$\begin{aligned} G(\hat{z}_\beta^-) &= -\frac{a(\beta)}{\delta^d} + f^{(1)} - f^{(0)} \leq -\frac{a}{\delta^d} + 2\eta(\tau, l_0) < 0 \\ G(\hat{z}_\beta^+) &= \frac{a(\beta)}{\delta^d} + f^{(1)} - f^{(0)} \geq \frac{a}{\delta^d} - 2\eta(\tau, l_0) > 0. \end{aligned}$$

Therefore, we know that $z_\beta^c \in \widehat{U}_\beta$. Furthermore, we can observe that

$$\begin{aligned} \hat{z}_\beta^- - \beta &= -\frac{a}{\delta^d} + \frac{1}{\delta^d} \ln(1 + e^{-\beta\delta^d + a}) = -\frac{a}{\delta^d} + o(a(\beta)) \\ \hat{z}_\beta^+ - \beta &= \frac{a}{\delta^d} + \frac{1}{\delta^d} \ln(1 + e^{-\beta\delta^d - a}) = \frac{a}{\delta^d} + o(a(\beta)). \end{aligned}$$

Therefore we have that $|z_\beta^c - \beta| = O(e^{-c\beta})$ with $c \in (0, \frac{\rho_0 l_0}{6})$. This proves the missing claim that the difference $|z_\beta^c - \beta|$ decays exponentially fast.

3.2 Diluted pairwise interaction

The diluted pairwise interaction is a class of interaction that approximates the classical pairwise interaction. The idea behind is that we average the pair potential on the halo of the particles. In this study, we consider the case of radial finite range and integrable pair potential. The Gibbs point process associated to this type of interaction is a point process on $\mathbb{R}^d$ and the Hamiltonian is given in the following definition.

Definition 26. Let ϕ be a radial pair potential such that $\phi \in L^1(\mathbb{R}^d)$ and has a compact support (there is R_1 such that for $r > R_1, \phi(r) = 0$). The Hamiltonian for the diluted pairwise interaction is given by

$$\forall \omega \in \Omega_f, \quad H(\omega) = \iint_{L_R(\omega)^2} \phi(|x-y|)dxdy \tag{3.6}$$

where

$$L_R(\omega) = \bigcup_{x \in \omega} B(x, R).$$

We have proved in Proposition 2 that the Hamiltonian is bi-stable because ϕ is integrable. Furthermore, since ϕ has a range of R_1 it is clear that H is a finite range interaction with a range of $R_1 + 2R$. As a consequence of Theorem 1, the infinite volume Gibbs measures for the diluted pairwise interaction exists. We call the scale of the dilution the radius R and define R_0 as

$$R_0 := \sup\{r \in \mathbb{R}_+, \phi(r) > 0\}.$$

The quantity R_0 is the range inside which the potential is repulsive. In this study, we will compare R to R_0 as it affects the way we prove the phase transition. For any $x, z \in \mathbb{R}^d$ with $x \neq z$ we denote $B_{sec}(x, z)$ the hyperspherical sector with polar angle $\pi/3$, radius R, x as the centre and directed toward z.

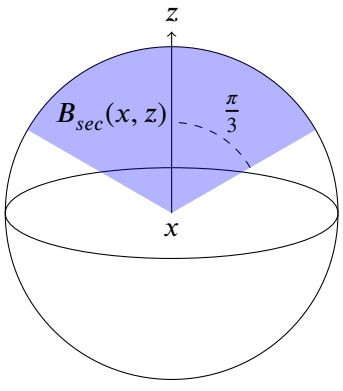


Figure 3.1: $B_{sec}(x, z)$

We define the constant C_d, dependant on the dimension d, which corresponds to the ratio of the volume of $B_{sec}(x, z)$ and the volume of $B(x, R)$ and it has the following expression

$$C_d = \frac{\int_0^{\frac{\pi}{3}} \sin(\theta)^{d-2} d\theta}{\int_0^{\pi} \sin(\theta)^{d-2} d\theta}. \tag{3.7}$$

In this section, we prove two liquid-gas phase transitions for the diluted pairwise interaction depending on the scale of dilution. This first theorem proves the phenomenon when we have a large dilution of the pair interaction (i.e $R > R_0$).

Theorem 24. *Let $\phi \in L^1(\mathbb{R}^d)$ be a radial pair potential with $+\infty > R_1 \geq R_0 > 0$. We assume that $R > R_0$ and that ϕ verifies*

$$C_d \int_{B(0,R_0)} \phi^+ dx > \int_{\mathbb{R}^d} \phi^- dx. \tag{3.8}$$

Then there is $\beta_c > 0$ such that for $\beta > \beta_c$, there is $z_\beta^c > 0$ for which a liquid-gas phase transition occurs. Furthermore, we know that there is $c > 0$ such that

$$\left| z_\beta^c - \beta \int_{\mathbb{R}^d} \phi dy \right| = O(e^{-c\beta}). \tag{3.9}$$

The following theorem demonstrates the liquid-gas phase transition phenomenon when we have a small dilution (i.e $R < R_0$).

Theorem 25. *Let $\phi \in L^1(\mathbb{R}^d)$ be a radial pair potential such that there are $+\infty > R_1 \geq R_0 > 0$. We assume that $R < R_0$ and that ϕ verifies*

$$C_d \int_{B(0,R)} \phi^+ dx \geq \left[\left(\frac{R_0}{R} \right)^d - 1 \right] \int_{B(0,R_0)\setminus B(0,R)} \phi^+ dx + \int_{\mathbb{R}^d} \phi^- dx \tag{3.10}$$

Then there is $\beta_c > 0$ such that for $\beta > \beta_c$, there is $z_\beta^c > 0$ for which a liquid-gas phase transition occurs. Furthermore, we know that there is $c > 0$ such that

$$\left| z_\beta^c - \beta \int_{\mathbb{R}^d} \phi dy \right| = O(e^{-c\beta}). \tag{3.11}$$

In reality, both cases are similar since the assumption (3.8) and (3.10) are the same. Indeed, we can observe that when $R > R_0$ both assumptions coincides. But we chose to present it separately since the proof in each cases differs. More specifically, the proofs of the Peierls condition are completely different, one rely on a tile-by-tile approach when the other is done globally. Furthermore, we can observe that for a pair potential repulsive and non integrable at the origin we can always truncate near the origin such that condition (3.10) is verified. Therefore we have the following corollary.

Corollary 25.1. *Let ϕ be a pair potential non integrable and positive at the origin with $+\infty > R_1 \geq R_0 > 0$. For any $R > 0$ there is $\epsilon \leq \min\{R_0, R\}$ for which we can build a truncated pair potential*

$$\phi_\epsilon(r) := \begin{cases} \phi(\epsilon) & \text{if } r \leq \epsilon \\ \phi(r) & \text{otherwise} \end{cases},$$

such that the diluted pairwise interaction for ϕ_ϵ exhibits a liquid-gas phase transition phenomenon for $\beta > \beta_c(\epsilon) > 0$ and a critical activity $z^c_\beta > 0$. Furthermore, we know that there is $c > 0$ such that

$$\left| z^c_\beta - \beta \int_{\mathbb{R}^d} \phi_\epsilon dy \right| = O(e^{-c\beta}). \tag{3.12}$$

This last corollary opens a new path to study phase transition of pairwise interaction with a strong short range repulsion, like for example the Lennard-Jones potential. In the rest of this section, we demonstrate the basic properties of the coarse graining and also the saturation of this interaction. Then we prove that it satisfy Peierls condition in both ways. Finally, we study the asymptotic behaviour of the critical activity.

3.2.1 Coarse graining for the diluted pairwise interaction

In the following, we consider radial pair potentials $\phi \in L^1(\mathbb{R}^d)$ such that there are $R_1 \geq R_0 > 0$ for which $\forall r \geq R_0, \phi(r) \leq 0$ and $\forall r \geq R_1, \phi(r) = 0$. In the beginning, we consider no further assumption on R other than $R > 0$. The coarse graining for the diluted pairwise interaction we consider is

$$\forall \omega \in \Omega_f, \quad E_0(\omega) = \int\limits_{L_R(\omega) \cap T_0} \int\limits_{L_R(\omega)} \phi(|x-y|) dx dy. \tag{3.13}$$

This choice of decomposition is natural and clearly we have that for any $\omega \in \Omega_f$

$$H(\omega) = \sum_{i \in \mathbb{Z}^d} E_i(\omega).$$

We demonstrate in the following proposition that this interaction is saturated .

Proposition 17. *The coarse graining E_0 has a finite range R_1 and is bounded,*

$$|E_0(\omega)| \leq \delta^d \int_{\mathbb{R}^d} |\phi| dx.$$

Furthermore, for $\delta \leq \frac{R}{\sqrt{d}}$ and $L > R_1 + 2\sqrt{d}R$ the diluted pairwise interaction is saturated. More precisely, we have

$$\overline{E}(k) := \delta^d C_\phi \mathbb{1}_{k \geq 1} \quad \text{where} \quad C_\phi := \int_{\mathbb{R}^d} \phi dx \tag{3.14}$$

such that

$$\forall \omega \in \Omega_{0,L,\delta}, \quad E_0(\omega) = \overline{E}(N_{T_0}(\omega)). \tag{3.15}$$

Proof. We know that ϕ is integrable. Therefore for any $\omega \in \Omega_f$ we have

$$\begin{aligned}|E_0(\omega)| &\leq \int\limits_{L_R(\omega)\cap T_0} \int\limits_{L_R(\omega)} |\phi| dx \\ &\leq \mathcal{V}(L_R(\omega)\cap T_0) \int_{\mathbb{R}^d} |\phi| dx \\ &\leq \delta^d \int_{\mathbb{R}^d} |\phi| dx.\end{aligned}$$

Furthermore, we have that E_0 has a range R_1. Indeed, by direct computation for any configuration $\omega \in \Omega_f$ we have the following

$$\begin{aligned}E_0(\omega) &= \int\limits_{L_R(\omega)\cap T_0} \int\limits_{L_R(\omega)\cap B(x,R_1)} \phi(|y-x|) dy dx \\ &= \int\limits_{L_R(\omega_{T_0\oplus B(0,R)})\cap T_0} \int\limits_{L_R(\omega_{T_0\oplus B(0,R_1)})} \phi(|y-x|) dy dx = E_0(\omega_{T_0\oplus B(0,R_1)}).\end{aligned}$$

With this choice of δ, we have that for any $x \in T_0$, $T_0 \subset B(x,R)$ and with this choice of L we are assured that for $\omega \in \Omega^1_{0,L,\delta}$ $B(x,R_1) \subset L_R(\omega)$. Consequently, for such configuration we have

$$\begin{aligned}E_0(\omega) &= \int_{T_0} \int_{B(x,R_1)} \phi(|y-x|) dy dx \\ &= \delta^d C_\phi = \overline{E}(N_{T_0}(\omega)).\end{aligned}$$

On the other hand, for configurations $\omega \in \Omega^0_{0,L,\delta}$, we have that $T_0 \oplus B(0,R_1) \cap L_R(\omega) = \emptyset$ and thus $E_0(\omega) = 0 = \overline{E}(0)$. □

We have demonstrated through the previous proposition that for δ small enough, L large enough and ϕ such that $C_\phi > 0$ that E_0 and $\overline{E}$ verify most of the assumptions of Theorem 11 or Theorem 13. In the following parts we are going to prove that under some assumption on ϕ, the interaction verifies the Peierls conditions, which we will do using either the dominoes or a global approach.

3.2.2 Peierls condition a tile-by-tile approach

In this part, we do the proof of the existence of the liquid gas phase transition phenomenon of Theorem 24. Our goal is to prove that with condition (3.8) and $R > R_0$ we verify the Peierls condition using the dominoes.

Proposition 18. *Let us have $R > R_0$, $L > R_1 + 2\sqrt{d}\delta$ and we fix δ small enough such that $\delta \leq \frac{R-R_0}{\sqrt{d}}$. We assume that*

$$e_0 := C_d \int_{B(0,R_0)} \phi^+ dx - \int_{\mathbb{R}^d} \phi^- dx > 0.$$

Then for any configuration $\omega \in \Omega_f$ we have

$$E_0(\omega) \geq \begin{cases} \delta^d C_\phi & \text{if } N_{T_0}(\omega) \geq 1 \\ \mathcal{V}(L_R(\omega) \cap T_0)e_0 & \text{otherwise} \end{cases}.$$

Proof. Let us start by the easy case, which is when $N_{T_0}(\omega) \geq 1$. Since $R > R_0$ and $\delta \leq \frac{R-R_0}{\sqrt{d}}$, we know that for any $x \in T_0$, $B(x, R_0) \subset L_R(\omega)$. Therefore we recover at least the positive part of the pair potential and we obtain

$$\int_{L_R(\omega)-x} \phi(|y|)dy \geq C_\phi.$$

This in turn leads to having

$$E_0(\omega) \geq \delta^d C_\phi.$$

Now let us consider the case when $N_{T_0}(\omega) = 0$ and that $L_R(\omega) \cap T_0 \neq \emptyset$ (otherwise we clearly have $E_0(\omega) = 0$). For any $x \in L_R(\omega) \cap T_0$ and $z \in \omega$ the closest point in the configuration to x, we know that at least we have $B_{sec}(x, z)$ included in the halo. Therefore, we directly obtain the following lower bound

$$\int_{L_R(\omega)-x} \phi dy \geq \int_{B_{sec}(x,z)} \phi^+ dy - \int_{\mathbb{R}^d} \phi^- dy.$$

Since the potential is radial and that $R > R_0$ we have that

$$\int_{B_{sec}(x,z)} \phi^+ dy = C_d \int_{B(0,R)} \phi^+ dy = C_d \int_{B(0,R_0)} \phi^+ dy.$$

As a consequence we obtain that

$$\begin{aligned} E_0(\omega) &= \int_{L_R(\omega)\cap T_0} \int_{L_R(\omega)-x} \phi(|y|)dydx \\ &\geq \int_{L_R(\omega)\cap T_0} e_0 dx = \mathcal{V}(L_R(\omega) \cap T_0)e_0. \end{aligned}$$

□

In Proposition 18 we have proved that under the assumptions of Theorem 24 (which is the case of large dilution, $R > R_0$) the diluted pairwise interaction verify the Peierls condition via the dominoes method. Therefore, the occurrence of a liquid-gas phase transition is a consequence of Theorem 13. In order to finish the proof of Theorem 24 all we need to demonstrate is the asymptotic behaviour of the critical activity, which will be done in Section 3.2.4.

3.2.3 Peierls condition a global approach

In this part, we do the proof of the existence of the liquid gas phase transition phenomenon of Theorem 25. We prove that with condition (3.10) and $R < R_0$ we verify the Peierls condition. We will be comparing the energy in the contours with the energy as if everything behaves like it is saturated. Before anything else we start with the following geometrical lemma.

Lemma 26. *For $0 < \epsilon < R \leq R_0$, we define θ_ϵ as*

$$\theta_\epsilon := \inf_{\substack{\omega \in \Omega_f \\ \gamma : V_{\omega,\gamma,\epsilon} > 0}} \left\{ \frac{V_{\omega,\gamma,\epsilon}}{V_{\omega,\gamma,R_0} - V_{\omega,\gamma,\epsilon}} \right\}$$

where

$$V_{\omega,\gamma,r} = \mathcal{V}(\partial L_R(\omega) \oplus B(0,r) \cap L_R(\omega) \cap \widehat{\gamma}).$$

Then we have

$$\theta_\epsilon = \frac{\epsilon^d}{R_0^d - \epsilon^d}.$$

Proof. For any contour γ and any configuration ω that achieves this contour, we observe that

$$\begin{aligned} \partial L_R(\omega) \oplus B(0,r) \cap L_R(\omega) \cap \widehat{\gamma} &= L_R(\omega)^c \oplus B(0,r) \cap \widehat{\gamma} \quad \text{if } r \leq R, \\ \partial L_R(\omega) \oplus B(0,r) \cap L_R(\omega) \cap \widehat{\gamma} &\subset L_R(\omega)^c \oplus B(0,r) \cap \widehat{\gamma} \quad \text{if } r > R. \end{aligned}$$

We can approximate $L_R(\omega)^c$ using union of open balls, so there is $\left((x_i, r_i)\right)_{i \in \mathbb{N}}$ and $K_n = \bigcup_{i=1}^n \mathring{B}(x_i, r_i)$ such that $K_n \subset L_R(\omega)^c$ and $L_R(\omega)^c \backslash K_n$ is decreasing and converges to $\emptyset$. For a fixed n, we have

$$\begin{aligned} \mathcal{V}\left(K_n \oplus B(0,R_0) \cap \widehat{\gamma}\right) &= \int_0^{R_0} H_{d-1}\left(\partial(K_n \oplus B(0,r)) \cap \widehat{\gamma}\right) dr \\ &= S_d \int_0^{R_0} \sum_{i=1}^n \alpha_{i,\gamma}(r)(r_i + r)^{d-1} dr \end{aligned}$$

where H_{d-1} is the $(d-1)$-Haussdorff measure, S_d the surface of the unit ball and $\alpha_{i,\gamma}(r)$ is the proportion of the surface of $\mathring{B}(x_i, r_i + r)$ that appears in $\partial(K_n \oplus B(0,r)) \cap \widehat{\gamma}$,

$$\alpha_{i,\gamma}(r) = \frac{H_{d-1}\left(\partial \mathring{B}(x_i, r_i + r) \cap \partial(K_n \oplus B(0,r)) \cap \widehat{\gamma}\right)}{H_{d-1}\left(\partial \mathring{B}(x_i, r_i + r)\right)}.$$

Furthermore, $\alpha_{i,\gamma}$ is decreasing. Indeed, let $r' > r$ for $z \in \partial \mathring{B}(x_i, r_i + r')$ that appears in $\partial(K_n \oplus B(0,r')) \cap \widehat{\gamma}$ then $y = {}^{r}/_{r'}(z - x_i) + x_i \in \partial \mathring{B}(x_i, r_i + r)$ would appear in $\partial(K_n \oplus$

$B(0,r)) \cap \widehat{\gamma}$, and this even though $\widehat{\gamma}$ is not convex because the contours are thick enough such that $d_2\left(\partial(K_n \oplus B(0,R)) \cap \widehat{\gamma}, \partial\gamma\right) > 2R$. As such, we have

$$\begin{aligned}
\mathcal{V}\left(K_n \oplus B(0,R_0) \cap \widehat{\gamma}\right) &= S_d \left(\frac{R_0}{\epsilon}\right)^d \int_0^{\epsilon} \sum_{i=1}^{n} \alpha_{i,\gamma}\left(\frac{R_0}{\epsilon} r\right) \left(\frac{\epsilon}{R_0} r_i + r\right)^{d-1} dr \\
&\leq S_d \left(\frac{R_0}{\epsilon}\right)^d \int_0^{\epsilon} \sum_{i=1}^{n} \alpha_{i,\gamma}(r) \left(r_i + r\right)^{d-1} dr \\
&\leq \left(\frac{R_0}{\epsilon}\right)^d \mathcal{V}\left(K_n \oplus B(0,\epsilon) \cap \widehat{\gamma}\right).
\end{aligned}$$

Since $\mathcal{V}$ is continuous and for $r \in \{\epsilon, R_0\}$, $K_n \oplus B(0,r) \cap \widehat{\gamma}$ converges to $L_R(\omega)^c \oplus B(0,r) \cap \widehat{\gamma}$ with the Hausdorff metric, we can take the limit as n tends to infinity on both sides of the inequality and we get

$$\begin{aligned}
V_{\omega,\gamma,R_0} &\leq \mathcal{V}\left(L_R(\omega)^c \oplus B(0,R_0) \cap \widehat{\gamma}\right) \\
&\leq \left(\frac{R_0}{\epsilon}\right)^d \mathcal{V}\left(L_R(\omega)^c \oplus B(0,\epsilon) \cap \widehat{\gamma}\right) \\
&\leq \left(\frac{R_0}{\epsilon}\right)^d V_{\omega,\gamma,\epsilon}.
\end{aligned}$$

Using the previous inequality, we have

$$\frac{V_{\omega,\gamma,\epsilon}}{V_{\omega,\gamma,R_0} - V_{\omega,\gamma,\epsilon}} \geq \frac{\epsilon^d}{R_0^d - \epsilon^d}.$$

We have equality for $L_R(\omega) = \mathbb{R}^d \backslash \{0\}$ but ω is not a configuration as we would have an infinity of point near the origin. But it can be obtained as a limit of configurations and therefore

$$\theta_\epsilon = \frac{\epsilon^d}{R_0^d - \epsilon^d}.$$

□

Proposition 19. *For $R < R_0$, let ϕ be a radial and integrable pair potential such that*

$$C_d \int_{B(0,R)} \phi^+ dy > \left[\left(\frac{R_0}{R}\right)^d - 1\right] \int_{B(0,R_0)\backslash B(0,R)} \phi^+ dy + \int_{\mathbb{R}^d} \phi^- dy. \tag{3.16}$$

Then for ϵ close enough to R and $\delta \leq \frac{R-\epsilon}{2\sqrt{d}}$ there is $v_0 > 0$ such that for any contour γ and any configuration ω that achieves this contour

$$E_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(\omega) - \overline{E}_{\overline{\gamma}\backslash\partial^-\overline{\gamma}}(\omega) \geq C_\phi v_0 |\overline{\gamma}|. \tag{3.17}$$

Proof. For any $r > 0$ and any configuration $\omega \in \Omega$ we denote by $L_R^{-r}(\omega) := L_R(\omega) \backslash \partial L_R(\omega) \oplus B(0,r)$. Since $L > R_1 \geq R_0$, we know that $L_R(\omega) \backslash L_R^{-R_0}(\omega) \subset \bigcup_{\gamma \in \Gamma(\omega)} \widehat{\gamma \backslash \partial^- \gamma}$. Now we consider a contour γ and a configuration ω that creates this contour. For $x \in L_R^{-R_0}(\omega)$, we know by construction that $B(x, R_0) \subset L_R(\omega)$ and thus completely recover the positive part of ϕ. As such we have that

$$\int_{L_R(\omega)-x} \phi dy \geq \int_{\mathbb{R}^d} \phi dy = C_\phi. \tag{3.18}$$

Now for $x \in L_R^{-\epsilon}(\omega) \backslash L_R^{-R_0}(\omega)$, by construction we know that $B(x, \epsilon) \subset L_R(\omega)$ and thus obtain the following inequality

$$\begin{aligned} \int_{L_R(\omega)-x} \phi dy &\geq \int_{B(0,\epsilon)} \phi^+ dy - \int_{\mathbb{R}^d} \phi^- dy \\ &\geq C_\phi - \int_{B(0,R_0)\backslash B(0,\epsilon)} \phi^+ dy. \end{aligned} \tag{3.19}$$

At last for $x \in L_R(\omega) \backslash L_R^{-\epsilon}(\omega)$, we know that there is $z \in \omega$ the closest to x and that $B_{sec}(x, z) \subset L_R(\omega)$. Since ϕ is radial we have

$$\begin{aligned} \int_{L_R(\omega)-x} \phi dy &\geq \int_{B_{sec}(0,z-x)} \phi^+ dy - \int_{\mathbb{R}^d} \phi^- dy \\ &\geq C_d \int_{B(0,R)} \phi^+ dy - \int_{\mathbb{R}^d} \phi^- dy. \end{aligned} \tag{3.20}$$

By combining inequalities (3.18), (3.19), (3.20), we obtain that

$$\begin{aligned} E_{\overline{\gamma} \backslash \partial^- \overline{\gamma}}(\omega) \geq C_\phi \mathcal{V}(L_R^{-\epsilon}(\omega) \cap \widehat{\gamma}) + \left(C_d \int_{B(0,R)} \phi^+ dy - \int_{\mathbb{R}^d} \phi^- dy \right) V_{\omega,\gamma,\epsilon} \\ - \left(V_{\omega,\gamma,R_0} - V_{\omega,\gamma,\epsilon} \right) \int_{B(0,R_0)\backslash B(0,\epsilon)} \phi^+ dy \end{aligned}$$

$$\begin{aligned} E_{\overline{\gamma} \backslash \partial^- \overline{\gamma}}(\omega) \geq C_\phi \mathcal{V}(L_R^{-\epsilon}(\omega) \cap \widehat{\gamma}) + \theta_\epsilon \left(C_d \int_{B(0,R)} \phi^+ dy - \int_{\mathbb{R}^d} \phi^- dy - \frac{1}{\theta_\epsilon} \int_{B(0,R_0)\backslash B(0,\epsilon)} \phi^+ dy \right) \\ \times \left(V_{\omega,\gamma,R_0} - V_{\omega,\gamma,\epsilon} \right). \end{aligned}$$

By Lemma 26, we know that for $\frac{1}{\theta_\epsilon} \to (\frac{R_0}{R})^d - 1$ as ϵ tends to R. Thus by assumption (3.16) and for ϵ close enough to R we have that

$$C_d \int_{B(0,R)} \phi^+ dy - \int_{\mathbb{R}^d} \phi^- dy - \frac{1}{\theta_\epsilon} \int_{B(0,R_0)\backslash B(0,\epsilon)} \phi^+ dy \geq 0.$$

Thus we get

$$E_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega) \geq \mathcal{V}(L_R^{-\epsilon}(\omega)\cap\widehat{\gamma})C_\phi.$$

Therefore if we compute the energy difference between the actual energy of the configuration in the contour and the energy as if the contour acts in a saturated manner we have

$$E_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega) - \overline{E}_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega) \geq C_\phi\left(\mathcal{V}(L_R^{-\epsilon}(\omega)\cap\widehat{\gamma}) - \delta^d C_1(\gamma)\right)$$

where $C_1(\gamma)$ is the sites in the core of the contour with the spin equal to 1, $C_1(\gamma) := \{i \in \overline{\gamma}\setminus\partial^-\overline{\gamma}, \sigma_i = 1\}$. The difference between the volumes is at least bounded from below by the volume of empty tiles covered by the halo of radius $R-\epsilon$. Since $\delta \leq \frac{R-\epsilon}{2\sqrt{d}}$, we know that for dominoes (i,j) the presence of a point in T_i assures that $T_j \subset L_{R-\epsilon}(\omega)$ even though T_j is void of point. Therefore using Lemma 12 we know that there is $r_0 > 0$ and $v_0 = r_0\delta^d$ such that

$$E_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega) - \overline{E}_{\overline{\gamma}\setminus\partial^-\overline{\gamma}}(\omega) \geq C_\phi v_0|\overline{\gamma}|.$$

□

Similarly to the tile-by-tile approach, in Proposition 19 we have proved that under the assumptions of Theorem 25 (which is the case of small dilution, $R < R_0$) the diluted pairwise interaction verifies the Peierls condition this time through a global approach. Therefore, the occurrence of a liquid-gas phase transition is a consequence of Theorem 11. In order to finish the proof of Theorem 25 all we need to demonstrate is the asymptotic behaviour of the critical activity which will be done in Section 3.2.4.

3.2.4 Asymptotic behaviour of the critical activity

In this part, we finish the proof of the liquid gas phase transition for the diluted pairwise interaction. In the following proposition we demonstrate the asymptotic behaviour of the critical activity when β tends to infinity.

Proposition 20. *Under the assumptions of Theorem 24 or Theorem 25, we have that $|z_\beta^c - \beta C_\phi|$ decays exponentially when β tends to infinity.*

Proof. We know by Proposition 17 and Proposition 18 (or Proposition 19) that the diluted pairwise interaction exhibit a phase transition for β sufficiently large and $z_\beta^c \in U_\beta$. Furthermore, we have that

$$\frac{Z_1}{Z_0} = S(z) = e^{-\beta C\phi\delta^d}(e^{z\delta^d} - 1).$$

We fix $a(\beta) = \min\{2, e^{-\beta c}\}$ with $0 < c < \frac{\rho_0 l_0}{6}$ and

$$\widehat{U}_\beta = \left(\frac{\ln(1+e^{\beta C_\phi\delta^d - a(\beta)})}{\delta^d}, \frac{\ln(1+e^{\beta C_\phi\delta^d + a(\beta)})}{\delta^d}\right).$$

As such, we have for $z \in \widehat{U}_\beta$,

$$e^{-a} \leq \frac{Z_1}{Z_0} \leq e^a$$

and thus $\widehat{U}_\beta \subset U_\beta$. Furthermore, the difference of truncated pressures is expressed as

$$G(z) = (z - C_\phi \beta) + \frac{\ln(1 - e^{-z\delta^d})}{\delta^d} + f^{(1)} - f^{(0)}.$$

For $\hat{z}_\beta^- = \frac{\ln(1+e^{\beta C_\phi \delta^d - a(\beta)})}{\delta^d}$, $\hat{z}_\beta^+ = \frac{\ln(1+e^{\beta C_\phi \delta^d + a(\beta)})}{\delta^d}$ and β sufficiently large we have

$$\begin{aligned} G(\hat{z}_\beta^-) &= -\frac{a(\beta)}{\delta^d} + f^{(1)} - f^{(0)} \leq -\frac{a(\beta)}{\delta^d} + 2\eta(\tau, l_0) < 0 \\ G(\hat{z}_\beta^+) &= \frac{a(\beta)}{\delta^d} + f^{(1)} - f^{(0)} \geq \frac{a(\beta)}{\delta^d} - 2\eta(\tau, l_0) > 0. \end{aligned}$$

Therefore, we know that $z_\beta^c \in \widehat{U}_\beta$ and we can observe that

$$\begin{aligned} \hat{z}_\beta^- - C_\phi \beta &= -\frac{a(\beta)}{\delta^d} + \frac{1}{\delta^d} \ln(1 + e^{-\beta C_\phi \delta^d + a(\beta)}) = -\frac{a(\beta)}{\delta^d} + o(a(\beta)) \\ \hat{z}_\beta^+ - C_\phi \beta &= \frac{a(\beta)}{\delta^d} + \frac{1}{\delta^d} \ln(1 + e^{-\beta C_\phi \delta^d - a(\beta)}) = \frac{a(\beta)}{\delta^d} + o(a(\beta)). \end{aligned}$$

As a consequence, we have that $|z_\beta^c - C_\phi \beta| = O(e^{-c\beta})$ when β tends to infinity with $c \in (0, \frac{\rho_0 l_0}{6})$. □

3.3 Perspectives

In this section, we present possible avenues for extending the results presented in this thesis and a conjecture on the liquid-gas phase transition for a pairwise interaction with strong short range repulsion.

Extension of Theorem 11

In the setting of saturated interaction, one of the assumption needed in Theorem 11 is that $\overline{E}(k) = (Ak + B)\mathbb{1}_{k \geq 1}$ with $A \geq 0$ and $A + B \geq 0$. With this condition, we have $\overline{E} \geq 0$. This is an assumption needed in Proposition 13 to have a nice upper bound on $E_{P_\Lambda^\#}(N_{\widehat{\Lambda}})$. We believe we can relax these assumptions considerably and still obtain a good upper bound on the mean number of particles. Generally using the entropic inequality we have the following

$$\begin{aligned} E_{P_\Lambda^\#}(n_{\widehat{\Lambda}}) &\leq I(P_\Lambda^\# | \Pi_{\widehat{\Lambda}^\xi}) + \ln E_{\Pi_{\widehat{\Lambda}^\xi}}(e^{N_{\widehat{\Lambda}}}) \\ &\leq \int \left[-\beta(E_{\Lambda \setminus \partial\Lambda}(\omega) + \overline{E}_{\partial\Lambda}) + N_{\widehat{\Lambda}} \ln \frac{z}{\xi} \right] d P_\Lambda^\# - \ln Z_\Lambda^\# + (e\xi - z)|\Lambda|\delta^d. \end{aligned}$$

Furthermore, since we verify the Peierls condition we have that $E_{\Lambda\setminus\partial\Lambda}(\omega) + \overline{E}_{\partial\Lambda} \geq \overline{E}_{\Lambda}$ and thus get

$$\begin{aligned}
E_{P_\Lambda^\#}(N_{\widehat{\Lambda}}) &\leq \int \left[-\beta \overline{E}_\Lambda + N_{\widehat{\Lambda}} \ln \frac{z}{\xi}\right] dP_\Lambda^\# - \ln Z_\Lambda^\# + (e\xi - z)|\Lambda|\delta^d \\
&\leq \sum_{i\in\Lambda} \int \left[-\beta(AN_{T_i} + B)\mathbb{1}_{N_{T_i}\geq 1}\right] dP_\Lambda^\# + \ln \frac{z}{\xi} \int N_{\widehat{\Lambda}} dP_\Lambda^\# - \ln Z_\Lambda^\# + (e\xi - z)|\Lambda|\delta^d \\
&\leq -\beta B \sum_{i\in\Lambda} \int \mathbb{1}_{N_{T_i}\geq 1} dP_\Lambda^\# + \int \left[-\beta A + \ln \frac{z}{\xi}\right] N_{\widehat{\Lambda}} dP_\Lambda^\# - \ln Z_\Lambda^\# + (e\xi - z)|\Lambda|\delta^d
\end{aligned}$$

If we choose $\xi = ze^{-\beta A}$ we can simplify and obtain

$$E_{P_\Lambda^\#}(N_{\widehat{\Lambda}}) \leq \begin{cases} -\beta B|\Lambda| - \ln Z_\Lambda^\# + (e^{-\beta A+1} - 1)z|\Lambda|\delta^d & \text{if } B < 0 \\ -\ln Z_\Lambda^\# + (e^{-\beta A+1} - 1)z|\Lambda|\delta^d & \text{if } B \geq 0 \end{cases}$$

We can see in these inequalities that if $A < 0$ then the leading term of the upper bound is $e^{-\beta A}$ as β tends to $+\infty$. At the moment it seems unclear how this affects the proof of Proposition 13 but we are confident that we can at least have a better lower bound on A.

More Liquid-Gas phase transition for the Quermass model

In our study of the Quermass model we focused on the contribution of the surface measure $\mathcal{S}$ and the Euler-Poincaré characteristic (in dimension 2), but we could have considered any other Minkowski functionals added to the Lebesgue measure $\mathcal{V}$. We believe that the volume needs to have a positive contribution to the Hamiltonian in order to have the Peierls conditions. Any other functional can be added as long as it does not negate the effect of the volume in the energy of the contours. In general the Quermass Hamiltonian is given by

$$H(\omega) = \sum_{k=0}^{d} \theta_{d-k} M_k^d(L(\omega))$$

where $\theta_k \in \mathbb{R}$ and M_k^d is the k-th Minkowski functional and we have $M_d^d = \mathcal{V}$, $M_{d-1}^d = \mathcal{S}$ and $M_0^d = \chi$. There might be some restrictions on θ_k, and this is due to stability constraints which is not always verified (i.e χ when $d \geq 3$ [23]). In order to generalize our results on the Quermass, one need to be able to compare M_k^d, the k-th Minkowsi functional for $0 \leq k \leq d-1$, with $\mathcal{V} = M_d^d$ or the volume of the contours. More specifically, we need to be able to prove that there is $c, C \in \mathbb{R}$ such that for any contours γ and any configuration ω that achieves this contour we have

$$c\mathcal{V}(L(\omega) \cap \widehat{\gamma}) \leq M_{k,\gamma}^d(L(\omega)) \quad \text{and/or} \quad M_{k,\gamma}^d(L(\omega)) \leq C\mathcal{V}(L(\omega) \cap \widehat{\gamma})$$

where $M_{k,\gamma}^d$ is the contribution of γ to the k-th Minkowski functional obtained via coarse graining. Then using our approach, we can tune the parameters θ_k in order to verify the Peierls condition.

Phase transition of pairwise interaction

The phase transition of pairwise interaction has long been conjectured. For instance, it has been observed via simulation that the Lennard-Jones interaction exhibits a liquid-gas phase transition [21]. In this thesis, we have considered approaching this question with the diluted pairwise interaction. As we have seen with Corollary 25.1, if we consider a finite range pair potential ϕ that is non integrable and positive at the origin for any $R > 0$ we have a truncation ϕ_R such that the diluted pair interaction with ϕ_R exhibit a phase transition. Recall that the Hamiltonian in this case is given by

$$H(\omega) = \iint_{L_R(\omega)^2} \phi_R(|x-y|)dxdy.$$

However, if we take the limit of this Hamiltonian as R goes to 0 we get

$$\lim_{R\to 0} \frac{H(\omega)}{\mathcal{V}(B(0,R))^2} = N(\omega)\phi(0) + 2\sum_{\{x,y\}\in\omega} \phi(|x-y|)$$

which is not the pairwise interaction we hoped for and since $\phi(0) = +\infty$ the system with this limiting Hamiltonian does not produce a nice physical behaviour. Therefore what we really need to consider is the corrected diluted pairwise interaction where the Hamiltonian is given by

$$H_R(\omega) = \frac{1}{R^{2d}b_d^2} \iint_{L_R(\omega)^2} \phi_R(|x-y|)dxdy - N(\omega)\phi_R(0)$$

where $b_d = \mathcal{V}(B(0,1))$. Therefore we have naturally

$$E_0(\omega) = \frac{1}{R^{2d}b_d^2} \int_{L_R(\omega)\cap T_0} \int_{L_R(\omega)} \phi_R(|x-y|)dxdy - N_{T_0}(\omega)\phi_R(0)$$

and with a good choice of $\delta > 0$ and $L > 0$ this interaction is saturated with

$$\overline{E}(k) = \left(\frac{\delta^d C_{\phi_R}}{R^{2d}b_d^2} - \phi_R(0)k\right)\mathbb{1}_{k\geq 1} \quad \text{where} \quad C_{\phi_R} = \int_{\mathbb{R}^d} \phi_R dx.$$

We can show that this interaction verify the Peierls condition. Indeed, we can observe that for any contour γ we have

$$E_{\gamma\setminus\partial^-\gamma} - \overline{E}_{\gamma\setminus\partial^-\gamma} = \frac{1}{R^{2d}b_d^2}\left(\int_{L_R(\omega)\cap\hat{\gamma}} \int_{L_R(\omega)} \phi_R(|x-y|)dxdy - C_{\phi_R}|C_1(\gamma)|\right)$$

where $|C_1(\gamma)|$ counts the number of non-empty tiles in the core of the contour. Using a similar approach as in Proposition 19, we can show that $\rho_0 = C_{\phi_R} r_0 \delta^d$. The only problem here is that we cannot apply Theorem 24, because in this case $A = -\phi_R(0) < 0$. It is why we need to improve

on the proof of Proposition 13, in order to prove phase transition for corrected diluted pairwise interaction.

Furthermore, there is a second difficulty with this approach. If we consider that we have a liquid-gas phase transition for the corrected diluted pairwise interaction, then for each $R > 0$ there is $\beta_c(R) > 0$ and for any $\beta \geq \beta_c(R)$ there is $z^c_\beta(R) > 0$ and the phase transition happens for (β, z^c_β). The problem is that we do not know how $\beta_c(R)$ behaves as R tends to 0. It is totally possible that $\beta_c \to +\infty$ and therefore we cannot conclude on the phase transition for a finite β. The same can be said for $z^c_\beta(R)$ if we admit that $\beta_c(R)$ is bounded. We need to guarantee that for $\beta \geq \sup\{\beta_c(R), R \in (0, R_0)\}$ the critical activity $z^c_\beta(R)$ is bounded.

Bibliography

[1] R. Bardenet and A. Hardy. *Monte Carlo with Determinantal Point Processes*. 2019.

[2] J. Bricmont, K. Kuroda, and J.L. Lebowitz. "The structure of Gibbs states and phase coexistence for nonsymmetric continuum Widom-Rowlinson models." In: *Z. Wahrsch. Verw. Gebiete* 67.2 (1984), pp. 121–138.

[3] J. T. Chayes, L. Chayes, and R. Kotecký. "The analysis of the Widom-Rowlinson model by stochastic geometric methods." In: *Comm. Math. Phys.* 172.3 (1995), pp. 551–569.

[4] D. Dereudre. "Introduction to the theory of Gibbs point processes." In: *Stochastic geometry*. Vol. 2237. Lecture Notes in Math. Springer, Cham, 2019, pp. 181–229.

[5] D. Dereudre. "The existence of quermass-interaction processes for nonlocally stable interaction and nonbounded convex grains." In: *Advances in Applied Probability* 41.3 (2009), pp. 664–681.

[6] D. Dereudre. "Variational principle for Gibbs point processes with finite range interaction." In: *Electron. Commun. Probab.* 21 (2016), Paper No. 10, 11.

[7] D. Dereudre and H.-O Georgii. "Variational characterisation of Gibbs measures with Delaunay triangle interaction." In: *Electron. J. Probab.* 14 (2009), no. 85, 2438–2462.

[8] D. Dereudre and T. Vasseur. "Existence of Gibbs point processes with stable infinite range interaction." In: *J. Appl. Probab.* 57.3 (2020), pp. 775–791.

[9] P. Dereudre D.and Houdebert. "Sharp phase transition for the continuum Widom-Rowlinson model." In: *Ann. Inst. Henri Poincaré Probab. Stat.* 57.1 (2021), pp. 387–407.

[10] H. Duminil-Copin, A. Raoufi, and V. Tassion. "Exponential decay of connection probabilities for subcritical Voronoi percolation in $$\mathbb {R}^d$$." In: *Probability Theory and Related Fields* 173.1 (Feb. 2019), pp. 479–490.

[11] H. Duminil-Copin, A. Raoufi, and V. Tassion. "Sharp phase transition for the random-cluster and Potts models via decision trees." In: *Annals of Mathematics* 189.1 (2019), pp. 75–99.

[12] H. Duminil-Copin, A. Raoufi, and V. Tassion. "Subcritical phase of d-dimensional Poisson–Boolean percolation and its vacant set." en. In: *Annales Henri Lebesgue* 3 (2020), pp. 677–700.

[13] S. Friedli and Y. Velenik. *Statistical mechanics of lattice systems*. A concrete mathematical introduction. Cambridge University Press, Cambridge, 2018, pp. xix+622.

[14] H.-O Georgii. *Gibbs measures and phase transitions*. Second. Vol. 9. De Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, 2011, pp. xiv+545.

[15] H.-O Georgii. "Large deviations and the equivalence of ensembles for Gibbsian particle systems with superstable interaction." In: *Probab. Theory Related Fields* 99.2 (1994), pp. 171–195.

[16] H.-O Georgii. "The equivalence of ensembles for classical systems of particles." In: *J. Statist. Phys.* 80.5-6 (1995), pp. 1341–1378.

[17] H.-O Georgii and H. Zessin. "Large deviations and the maximum entropy principle for marked point random fields." In: *Probab. Theory Related Fields* 96.2 (1993), pp. 177–204.

[18] H.-O. Georgii and O. Häggström. "Phase transition in continuum Potts models." In: *Communications in Mathematical Physics* 181.2 (1996), pp. 507–528.

[19] G. Giacomin, J. L. Lebowitz, and C. Maes. "Agreement percolation and phase coexistence in some Gibbs systems." English (US). In: *Journal of Statistical Physics* 80.5-6 (Sept. 1995), pp. 1379–1403.

[20] H. Hadwiger. *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957, pp. xiii+312.

[21] J.-P Hansen and L. Verlet. "Phase Transitions of the Lennard-Jones System." In: *Phys. Rev.* 184 (1 Aug. 1969), pp. 151–161.

[22] S. Jansen. *Lecture notes in Gibbsian point processes*. 2017.

[23] W. S. Kendall, M. N. M. van Lieshout, and A. J. Baddeley. "Quermass-interaction processes: conditions for stability." In: *Adv. in Appl. Probab.* 31.2 (1999), pp. 315–342.

[24] Alex Kulesza and Ben Taskar. "Determinantal Point Processes for Machine Learning." In: *Foundations and Trends® in Machine Learning* 5.2–3 (2012), pp. 123–286.

[25] G Lambert, T. Leblé, and O. Zeitouni. *Law of large numbers for the maximum of the two-dimensional Coulomb gas potential*. 2024.

[26] G. Last and M. Penrose. *Lectures on the Poisson Process*. Institute of Mathematical Statistics Textbooks. Cambridge University Press, 2017.

[27] J. L. Lebowitz, A. Mazel, and E. Presutti. "Liquid-vapor phase transitions for systems with finite-range interactions." In: *J. Statist. Phys.* 94.5-6 (1999), pp. 955–1025.

[28] J. Møller and K. Helisová. "Power diagrams and interaction processes for unions of discs." In: *Adv. in Appl. Probab.* 40.2 (2008), pp. 321–347.

[29] J. Møller and R. P. Waagepetersen. *Statistical inference and simulation for spatial point processes*. Vol. 100. Monographs on Statistics and Applied Probability. Chapman & Hall/CRC, Boca Raton, FL, 2004, pp. xvi+300.

[30] D. Ruelle. "Existence of a Phase Transition in a Continuous Classical System." In: *Phys. Rev. Lett.* 27 (16 Oct. 1971), pp. 1040–1041.

[31] D. Ruelle. *Statistical mechanics: Rigorous results*. W. A. Benjamin, Inc., New York-Amsterdam, 1969, pp. xi+219.

[32] D. Ruelle. "Superstable interactions in classical statistical mechanics." In: *Comm. Math. Phys.* 18 (1970), pp. 127–159.

[33] D. Tsagkarogiannis and E. Pulvirenti. "Finite Volume Corrections and Decay of Correlations in the Canonical Ensemble." In: *Journal of Statistical Physics* (Feb. 2015).

[34] B. Widom and J.S. Rowlinson. "New Model for the Study of Liquid–Vapor Phase Transitions." In: *Journal of Chemical Physics* 52 (1970), pp. 1670–1684.