Université de Lille

Doctoral School **MADIS-631**

University Department **Laboratoire Paul Painlevé**

Thesis defended by **Jules Candau-Tilh**

Defended on **October 11, 2024**

In order to become Doctor from Université de Lille

Academic Field **Mathematics**

Theoretical and numerical analysis of perturbed isoperimetric problems

Thesis supervised by Michael Goldman
Benoît Merlet

Committee members

Committee President	Simon Masnou	Professor at Université Claude Bernard Lyon 1
Referees	Guillaume Carlier	Professor at Université Paris Dauphine
	Annalisa Massaccesi	HDR Associate Professor at University of Padova
Examiners	Maxime Laborde	Associate Professor at Université Paris Cité
	Vincent Millot	Professor at Université Paris-Est Créteil Val-de-Marne
	Juliette Venel	Associate Professor at Université Polytechnique Hauts-de-France
Supervisors	Michael Goldman	HDR Junior Researcher at CNRS
	Benoît Merlet	Professor at Université de Lille

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École doctorale **MADIS-631**

Unité de recherche **Laboratoire Paul Painlevé**

Thèse présentée par **Jules CANDAU-TILH**

Soutenue le **11 octobre 2024**

En vue de l'obtention du grade de docteur de l'Université de Lille

Discipline **Mathématiques**

Etude théorique et numérique de problèmes isopérimetriques perturbés

Thèse dirigée par Michael GOLDMAN
Benoît MERLET

Composition du jury

Président du jury	Simon MASNOU	professeur à l'Université Claude Bernard Lyon 1
Rapporteurs	Guillaume CARLIER	professeur à l'Université Paris Dauphine
	Annalisa MASSACCESI	MCF HDR à l'University of Padova
Examinateurs	Maxime LABORDE	MCF à l'Université Paris Cité
	Vincent MILLOT	professeur à l'Université Paris-Est Créteil Val-de-Marne
	Juliette VENEL	MCF à l'Université Polytechnique Hauts-de-France
Directeurs de thèse	Michael GOLDMAN	chargé de recherche HDR au CNRS
	Benoît MERLET	professeur à l'Université de Lille

Theoretical and numerical analysis of perturbed isoperimetric problems

Abstract

In this thesis, we focus on perturbed isoperimetric problems. These problems involve the minimisation of an energy composed of a perimeter term that promotes mass aggregation, countered by a perturbation term favouring disaggregation.

We begin by presenting the concepts used as well as past and current research conducted on the isoperimetric problem and its variants.

In Chapter 1, we study a problem where the perimeter interacts with a non-local term called an exterior transport term, defined using optimal transport theory. We demonstrate the existence of solutions to this problem and, in regimes where the perimeter dominates, we prove that the minimisers are balls.

Chapter 2 is dedicated to the exterior transport term. In a general framework, we show that the variational problem defining it has solutions and a dual formulation. Using stronger assumptions, we finally show that this term is maximised only by balls.

In Chapter 3, we present a numerical study in dimension 2 of the problem from Chapter 1. We approximate the minimisers of the energy considered via a gradient descent algorithm. The numerical results lead us to conjecture the existence of a critical mass above which the minimisers are no longer balls, but elongated shapes with two axes of symmetry.

Chapter 4 focuses on a perturbed isoperimetric problem where the perimeter and perturbation terms are not explicit. We exhibit a general set of assumptions under which a relaxed version of the problem admits minimisers. Under stronger hypotheses, we then investigate whether these minimisers have density estimates.

Keywords: Calculus of Variations, perturbed isoperimetric problem, Wasserstein distance, generalized minimisers, Sinkhorn algorithm

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Etude théorique et numérique de problèmes isopérimetriques perturbés

Résumé

Nous étudions dans cette thèse des problèmes isopérimétriques perturbés. Ces problèmes consistent en la minimisation d'une énergie formée d'un terme de périmètre qui favorise l'agrégation de masse, auquel s'oppose un terme perturbatif favorisant la désagrégation.

Nous commençons par présenter les concepts utilisés ainsi que la recherche passée et actuelle effectuée sur le problème isopérimétrique et ses variantes.

Nous étudions dans le chapitre 1 un problème où le périmètre interagit avec un terme non-local dit de transport extérieur, défini à l'aide de la théorie du transport optimal. Nous montrons l'existence de solutions à ce problème et, dans les régimes où le périmètre domine, nous prouvons que les minimiseurs sont les boules.

Le chapitre 2 est consacré au terme de transport extérieur. Dans un cadre général, nous montrons que le problème le définissant admet des solutions et une formulation duale. A l'aide d'hypothèses plus fortes, nous montrons que ce terme est uniquement maximisé par les boules.

Dans le chapitre 3, nous présentons les travaux numériques effectués en dimension 2 sur le problème du premier chapitre. Nous approchons les minimiseurs de l'énergie considérée via une descente de gradient. Les résultats numériques nous amènent à conjecturer l'existence d'une masse critique à partir de laquelle les minimiseurs ne sont plus des boules, mais des formes allongées à deux axes de symétrie.

Le chapitre 4 porte sur un problème isopérimétrique perturbé où les termes de périmètre et de perturbation ne sont pas explicites. Pour des hypothèses assez générales, nous montrons que ce problème admet des minimiseurs en un sens faible. Nous montrons ensuite sous des hypothèses plus fortes que ces minimiseurs, appelés minimiseurs généralisés, possèdent des estimées de densité.

Mots clés : Calcul des Variations, problème isopérimétrique perturbé, distance de Wassertein, minimiseurs généralisés, algorithme de Sinkhorn

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Laboratoire Paul Painlevé
Cité scientifique – 59650 Villeneuve-d'Ascq – France

This thesis has been prepared at the following research units.

Laboratoire Paul Painlevé
Cité scientifique
59650 Villeneuve-d'Ascq
France

Centre INRIA de l'Université de Lille
Parc scientifique de la Haute Borne
59650 Villeneuve-d'Ascq
France

Remerciements

Je tiens tout d'abord à remercier chaleureusement mes directeurs de thèse Benoît et Michael. Le temps que vous m'avez consacré, la patience dont vous avez fait preuve et la confiance que vous avez eu en moi m'ont permis de produire un travail dont je suis maintenant fier. Votre exigeance et votre persévérance face aux problèmes mathématiques les plus subtils ont su grandement m'inspirer dans mes recherches.

Je suis également profondement reconnaissant envers Guillaume Carlier et Annalisa Massaccesi d'avoir rapporté ma thèse. Votre relecture et vos conseils m'ont été précieux pour aboutir à la version finale du manuscrit. Je remercie aussi très sincèrement Maxime Laborde, Simon Masnou, Vincent Millot et Juliette Venel d'avoir accepté de faire partie de mon jury et d'assister à ma soutenance.

Un grand merci à Marc pour son enthousiasme vis-à-vis de toutes les questions numériques que nous nous sommes posées. Ses nombreuses intuitions ont permis à notre collaboration d'être fructueuse et toujours passionnante. Merci également à Maxime Laborde et à Andrea Natale pour leur aide et leurs précieux conseils.

Je suis très reconnaissant envers Luigi Ambrosio, Antonin Chambolle et Guillaume Carlier, qui m'ont fait découvrir les domaines du calcul des variations et du transport optimal. Je n'ai jamais regretté un seul instant de m'être plongé dans ces questions.

Je remercie vivement les équipes RAPDSODI et PARADYSE de l'INRIA. Je garde des souvenirs précieux de l'ambiance et de la vie de l'équipe tant à l'intérieur qu'à l'extérieur des bureaux. Puissent vos cuves ne jamais se corroder.

Mille mercis également à mes compagnons de travail doctorants, post-doctorants ou encore stagiaires. Je pense également à tous les amis du CalVa que j'avais grand plaisir à recroiser à chaque évènement. Sachez que la précarité de votre statut professionnel a été proportionnelle au plaisir que j'ai pris à travailler à vos côtés.

J'exprime toute ma gratitude aux amis que je me suis fait au cours de ma vie lilloise : sans vous, je n'aurais imaginé que mon acclimatation se passe aussi bien. Mes pensées vont également à mes amis parisiens, de l'Amicale Fromagère, de l'ENS et mes anciens colocataires. Aller à Paris les week-ends pour vous voir fut toujours un immense plaisir, que vous disposiez d'un canapé-lit ou non.

Merci du fond du coeur à mes parents et à mon frère. L'amour et la confiance que vous me témoignez m'accompagnent toujours et me poussent à donner le meilleur de moi-même chaque jour.

Enfin et de manière générale, que toutes celles et ceux qui m'ont écouté me plaindre au cours de ce périple sachent qu'ils ont mon entière gratitude.

On ne reçoit pas la sagesse, il faut la découvrir soi-même, après un trajet que personne ne peut faire pour nous, ne peut nous épargner.

Marcel Proust.

Contents

Foreword

This manuscript presents the results obtained during the thesis conducted at the Paul Painlevé laboratory and the INRIA center of the University of Lille, under the supervision of Michael Goldman and Benoît Merlet. This document contains a general introduction, which presents the mathematical context in which the subject is situated and provides an overview of the contributions. We conclude the introduction by outlining some perspectives and possible future works based on this manuscript. Four chapters then follow, each corresponding to papers that are either published, submitted for publication or to be shortly submitted.

List of publications and preprints

- Chapter 1: written with Michael Goldman and published online in ESAIM : Control, Optimisation and Calculus of Variations under the title *Existence and stability results for an isoperimetric problem with a non-local interaction of Wasserstein type*. Available at

  https://www.esaim-cocv.org/articles/cocv/abs/2022/01/cocv210167/cocv210167.html

- Chapter 2: written with Michael Goldman and Benoît Merlet, to appear in Calculus of Variations and Partial Differential Equations (preprint: https://arxiv.org/abs/2309.02806).

- Chapter 3: to be submitted shortly.

- Chapter 4: submitted (prepint: https://arxiv.org/abs/2406.16379).

General introduction

Outline of the current chapter

In this manuscript, we investigate several questions related to a geometric variational problem involving a competition between the perimeter and a functional defined using optimal transport theory. This thesis therefore lies at the interface between geometric measure theory, calculus of variations, elliptic PDEs and optimal transportation.

Subject and objectives

Let us start with a brief overview of the work conducted in this manuscript. The central point of our research is to investigate the following nonlocal isoperimetric problem: given $\lambda, \alpha > 0$, $p \geq 1$ and $d \geq 2$ we consider

$$\inf_{|E|=\omega_d} \Big\{ \mathcal{E}(E) = \mathrm{Per}(E) + \lambda (\mathcal{W}_p^p(E))^{\alpha} \Big\}, \tag{0.0.1}$$

where ω_d denotes the volume of the unit ball of $\mathbb{R}^d$, Per is the Caccioppoli (or distributional) perimeter and $\mathcal{W}_p$ is a functional that we call the exterior transport functional. It is defined as follows: given $E \subset \mathbb{R}^d$ with finite Lebesgue measure (or volume) $|E|$ we set

$$\mathcal{W}_p(E) = \inf_F \Big\{ W_p(E, F) : |F \cap E| = 0 \Big\}, \tag{0.0.2}$$

where W_p is the p-Wasserstein distance and E, F are respectively identified with the measures $\chi_E dx, \chi_F dx$. By convention we set $W_p(\mu, \nu) = +\infty$ if μ, ν are nonnegative measures with different

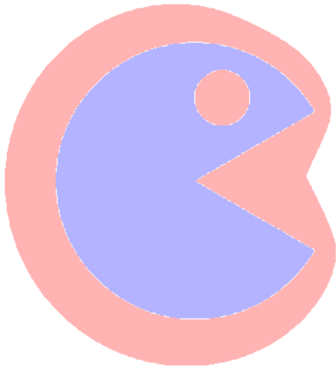

Figure 1: A set (in blue) and its minimiser for the exterior transport (in red).

total masses. See Figure 1 for an illustration of Problem (0.0.2).

It is worth mentioning that Problem (0.0.1) belongs to the family of perturbed isoperimetric problems, which may be generically defined as:

$$\inf\Big\{\mathcal{E}(E) = \mathrm{Per}(E) + V(E) : |E| = m\Big\},$$

where $m > 0$ and V is called the perturbative term. Typically, V represents a repulsive energy that decreases when a set is divided into smaller parts which are sent far away from each other. Additionally, V is often nonlocal; that is, altering a set E within a small open ball $B_r(x)$ results in an energy variation that depends not only on $E \cap B_r(x)$ but also on $E \setminus B_r(x)$. The archetypal form of V is that of an interaction term of the form

$$V(E) = \int_{E\times E} K(x-y)\,dx\,dy,$$

for some kernel $K : \mathbb{R}^d \to \mathbb{R}_+$.

Throughout the manuscript, we investigate four questions related to Problem (0.0.1), each one corresponding to a chapter. From now on, all considered subsets of $\mathbb{R}^d$ are assumed to be Lebesgue measurable. We also identify any two sets $E, E' \subset \mathbb{R}^d$ such that $|E\Delta E'| = 0$, where Δ is the symmetrical difference operator.

In Chapter 1, we establish that Problem (0.0.1) admits minimisers for any $m > 0$. We also obtain that any minimiser admits a finite number of connected components which are uniformly bounded and have a regular boundary. Eventually, when α and λ are such that the perimeter is the dominant term, we prove that the minimisers are balls.

In Chapter 2, we drop the perimeter term and focus on a generalization of the exterior transport term $\mathcal{W}_p$. Let $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ be a continuous cost function and let $\mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d)$ denote the set of nonnegative Radon measures over $\mathbb{R}^d \times \mathbb{R}^d$. Given $E, F \subset \mathbb{R}^d$ of finite volume, we introduce

$$\mathcal{T}_c(E,F) = \inf\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} c(x,y)d\pi(x,y) : \pi \in \mathcal{M}_+(\mathbb{R}^d\times\mathbb{R}^d),\, \pi_x = \chi_E dx,\, \pi_y = \chi_F dx\right\},$$

where π_x and π_y respectively denote the first and second marginal of $\pi \in \mathcal{M}_+(\mathbb{R}^d\times\mathbb{R}^d)$. We then define the associated exterior transport cost by

$$\Upsilon(E) = \inf_F\left\{\mathcal{T}_c(E,F) : |F\cap E| = 0\right\}. \tag{0.0.3}$$

Eventually, we consider the *maximization* problem

$$\sup\left\{\Upsilon(E) : |E| = m\right\}. \tag{0.0.4}$$

For very general cost functions c, we show that this problem admits maximisers for any $m > 0$. In the case where $c(x,y) = k(|y-x|)$ for some continuous, increasing and coercive function $k : \mathbb{R}_+ \to \mathbb{R}_+$ we show that balls are its unique maximisers. We thus confirm that the perimeter and the non-local functional $\mathcal{W}_p$ are competing terms in Problem (0.0.1).

In Chapter 3 we come back to the minimisation problem of the first chapter, which is by scaling equivalent to

$$\inf_{|E|=m}\left\{\mathrm{Per}(E) + (\mathcal{W}_p^p(E))^\alpha\right\}, \tag{0.0.5}$$

for some $m > 0$. We conduct a mostly numerical study of this problem in the case where $d = 2, p = 2$ and $\alpha = 1$. More precisely, we first restrict ourselves to the case of radially symmetric sets and study the behaviour of minimisers of (0.0.5) as m varies from 0 to ∞. Next, we consider the general case where we do not make any symmetry assumption. For the purposes of the numerical investigation, we replace the perimeter term with its Modica-Mortola approximation and the exterior transport cost with its entropic regularization. The resulting energy is then discretised over a Cartesian grid, and we employ an alternating minimisation method for the optimisation process. In particular, we compute the exterior transport term using the Sinkhorn algorithm.

In Chapter 4, we address more general functionals of the form

$$\mathcal{E}(E) = P(E) + V(E), \tag{0.0.6}$$

where P is an aggregative term generalizing the perimeter (e.g. the anisotropic or fractional perimeter) and V is a competing repulsive term. In a lot of cases, the disaggregation induced by V prevents this classical isoperimetric problem from having solutions. However, the corresponding generalised problem

$$\inf_{(E^i)_i}\left\{\sum_i \mathcal{E}(E^i) = \sum_i\left[P(E^i) + V(E^i)\right] : E^i \subset \mathbb{R}^d\ \forall i \geq 1,\ \sum_i |E^i| = m\right\} \tag{0.0.7}$$

may still admit minimisers, known as *generalised minimisers*. Our goal is to give general conditions on P and V guaranteeing that the infima in the classical and generalised problems coincide, and that the generalised problem admits minimisers. In the case where stronger assumptions on Per and V hold, we also show that generalised minimisers have (measure-theoretic) density estimates. We finally illustrate our results by giving some examples of perturbed isoperimetric problems with energy terms satisfying our assumptions.

Historical and mathematical context

Historically, the isoperimetric problem holds a central place in the field of calculus of variations. In $\mathbb{R}^2$, one of its possible formulation is to determine the shape of the largest possible area that can be enclosed by a closed curve of a given length. One of the earliest references to such a problem can be found in the legend of Queen Dido of Carthage. According to myth, after fleeing Tyre from her murderous brother, Dido arrived in North Africa, seeking land to establish a new city. The local ruler agreed to grant her as much land as she could encircle with a single oxhide. Demonstrating remarkable ingenuity, Dido cut the oxhide into thin strips and laid them out end-to-end, forming a large loop encircling an entire hill. By doing so, she maximised the enclosed area and founded the city of Carthage.

The isoperimetric problem was more rigorously analysed by ancient Greek mathematicians, who defined the perimeter as the length of the boundary of regular shapes of the plane, such as circles or polygons. Zenodorus, in the 2nd century BCE, is credited with some of the earliest known work on the problem, demonstrating that among all shapes with the same perimeter, the circle encloses the maximum area. This result laid the groundwork for future studies in calculus of variations and geometric analysis. However, it is not much later that significant progress was made on the isoperimetric problem. In 1838, Steiner used a symmetrisation process to show that in $\mathbb{R}^2$, if a solution exists, then it is a disk [75].

It is important to note that advances in understanding the isoperimetric problem are closely tied to progress in defining a sufficiently general notion of perimeter. In fact, defining which subsets of the ambient space are admissible in the isoperimetric problem is fundamental. Italian mathematician Renato Caccioppoli first introduced the notion of set of finite perimeter using the notion of finite variation in the sense of Tonelli in the 1920s [16]. His definition allowed for the extension of the classical definition of perimeter to more irregular sets, including those with fractal-like boundaries. In the 1950s, Caccioppoli improved on his previous results and introduced the concept of measure-theoretic boundary [14, 15]. Building on Caccioppoli's work, Ennio De Giorgi further developed the theory by introducing the notion of distributional perimeter [34, 33]:

$$\mathrm{Per}(E) = \sup_{\phi} \left\{ \int_{\mathbb{R}^d} \chi_E \mathrm{div}\phi \, dx : \phi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d), \|\phi\|_{L^\infty(\mathbb{R}^d)} \leq 1 \right\}. \tag{0.0.8}$$

A major reason why rigorous results on a problem known for several millennia were delayed until the 19th century is the issue of existence. Historically, mathematicians often overlooked whether a variational problem actually admitted a solution, leading to numerous false proofs in well-known problems.

Establishing the existence of solutions remained a major challenge, even in low dimensions. In the early 1900s, while working on the Dirichlet problem, D. Hilbert exhibited a general principle for constructing proofs of the existence of minimisers for a given functional [52]. Given a space X and $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$, this principle can be stated as follows:

1. Check that $\inf\{\mathcal{F}(x),\ x \in X\} = I > -\infty$, so that there exists a minimising sequence $(x_n)_{n\geq 1}$ such that $\lim_n \mathcal{F}(x_n) = I$;

2. Establish a compactness result on X so that (up to extracting a subsequence) there exists $x \in X$ such that $\lim_n x_n = x$;

3. Establish that $\mathcal{F}$ is lower semi-continuous on X, so that $\mathcal{F}(x) \leq \liminf_n \mathcal{F}(x_n)$, which eventually implies that $\mathcal{F}(x) = I$.

As an illustration let us apply this method, now called the Direct Method in the Calculus of Variations, to the relative isoperimetric problem. Given an open bounded set of finite perimeter A such that $|A| > \omega_d$, we follow [60, Proposition 12.30]: consider

$$\inf\{\mathrm{Per}(E,A) : E \in X\}, \quad \text{where} \quad X = \{E \subset A, |E| = \omega_d\} \tag{0.0.9}$$

and $\mathrm{Per}(E,A)$ is the perimeter of $E \subset \mathbb{R}^d$ relatively to A, that is

$$\mathrm{Per}(E,A) = \sup_{\phi} \left\{ \int \chi_E \mathrm{div}\phi dx : \phi \in C_c^1(A,\mathbb{R}^d), \|\phi\|_{L^\infty}(\mathbb{R}^d) \leq 1 \right\}.$$

Let us define for $t \in \mathbb{R}$ the set $E_t = A \cap \{x = (x_1, x_2, \ldots, x_d) : x_1 < t\}$. By continuity of the Lebesgue measure, there exists $t \in \mathbb{R}$ such that $|E_t| = m$, so that X is not empty and the first point of the Direct Method is verified. We now consider a minimising sequence $(E_n)_n$ for (0.0.9). By compactness of uniformly bounded sets of finite perimeter, up to extraction there exists $E \subset A$ of finite perimeter such that $E_n \to E$ as $n \to \infty$ (in the L^1 sense, that is $|E_n \Delta E| \to 0$ as $n \to \infty$). We additionally have $|E| = \omega_d$, which proves the second point of the Direct Method. The final requirement of the method follows from the lower semi-continuity of the relative perimeter with respect to the L^1 convergence, which is a direct consequence of its definition. In conclusion, we have proved that (0.0.9) admits a minimiser.

In many other shape optimisation problems, however, obtaining compactness in the space of candidate minimisers often proves more challenging. For instance, removing the constraint that A is bounded in the relative isoperimetric problem undermines the Direct Method as we applied it. Indeed, even though it is classical that $\sup_n \mathrm{Per}(E_n, A) < \infty$ implies that (up to extraction) the sequence $(E_n)_n$ is L^1_{loc}-converging to a set $E \subset A$, the local nature of this convergence provides no information about the volume of E. Moreover, when the perimeter term competes with a repulsive energy that acts over a longer or even infinite range, constraining the problem to a compact domain of $\mathbb{R}^d$ usually becomes unrealistic (see Figure 2).

Nevertheless, these perturbed isoperimetric problems, where the energy functional is the sum of a perimeter term and a typically non-local repulsive term, occupy an important place in

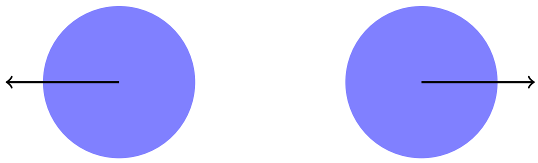

Figure 2: Two charged droplets repelling each other because of electrostatic forces.

mathematical physics. One of the first to formulate this type of variational problem was the physicist George Gamow in the 1930s to model the interaction between nucleons (i.e. protons and neutrons) inside the atomic nucleus [47]. For more details on the historical context of this well-known model see [26] and the references therein. For $\beta \in (0,d)$ and $m > 0$, it can be formulated as follows:

$$\inf_{|E|=m} \left\{ \mathrm{Per}(E) + \mathcal{G}_\beta(E) = \mathrm{Per}(E) + \int_{E\times E} \frac{dxdy}{|x-y|^{d-\beta}} \right\}. \tag{0.0.10}$$

Here, the perimeter is used to model a short-range attractive force such as surface tension competing against a high-range and repulsive functional encapsulated by $\mathcal{G}_\beta$. Let us mention that if we set $d = 3$ and $\beta = 2$, we recover the original physical case, where Per is called the surface term (which is analogous to the surface-tension term in liquids) and $\mathcal{G}_\beta$ accounts for the Coulombic repulsion between protons. From a mathematical standpoint this model raises several questions: do minimiser exist for all values of $m \in (0,\infty)$? When minimisers exist, are they balls, as in the non-perturbed isoperimetric problem? If they exist but are not balls, do they still retain some regularity properties? Let us stress that the non-local nature of the perturbation term introduces an additional level of complexity compared to the classical problem.

In the beginning of the 2010s, H. Knüpfer and C. B. Muratov answered to several of those questions in [56] and [54]. Their findings largely contributed to the resurgence of interest in isoperimetric problems among the mathematics community in recent years, and were reinforced in [7]. Eventually, the following result was obtained in [41]:

Theorem 0.0.1. *For every $d \geq 2$ and $\beta \in (0,d)$, there exists $m_0 > 0$ such that if $0 < m < m_0$, the ball of volume m is the unique minimiser (up to translation) of* (0.0.10).

Obtaining similar conclusions in the context of more general perturbed isoperimetric problems is currently an active field of research in isoperimetry theory. Given two generic perimeter and perturbative functionals P and V, one may study Problem (0.0.6) and attempt to find hypotheses on P and V under which existence and uniqueness of solutions can be established. Even in cases where the problem lacks solutions, applying the concentration-compactness principle may still yield valuable results such as the existence of solutions for a weaker, more general version of (0.0.6). This principle was developed by Pierre-Louis Lions in the 1980s to address minimisation problems posed in a space Ω that had solutions if Ω was compact but not necessarily when Ω was not compact [58]. In the context of shape optimisation, its heuristic can be summarised as follows (see also [43, 66, 67, 55] for additional discussions on the concentration-compactness principle in isoperimetric problems). Given a sequence of sets $(E_n)_n$ such that $|E_n| = m > 0$ for any $n \geq 1$, up to extraction one of the three situations occurs:

- *(compactness)* up to a translation, the sequence of measures $(\chi_{E_n} dx)_n$ is tight;
- *(vanishing)* for any $R > 0$, $\lim_n \sup_{y\in\mathbb{R}^d} |E_n \cap B_R(y)| = 0$;
- *(dichotomy)* there exists $\rho_n^1, \rho_n^2 \in L^1(\mathbb{R}^d, \mathbb{R}_+)$ concentrating almost all of the mass of E_n and such that $\mathrm{dist}(\mathrm{supp}\,\rho_n^1, \mathrm{supp}\,\rho_n^2) \to \infty$ as $n \to \infty$ (see [58] for the more precise definition).

The interest of such a classification is that in isoperimetric problems the vanishing case can often be excluded using the isoperimetric inequality. As a consequence, lack of existence in isoperimetric problems which are invariant by translation can only come from a splitting of the mass, where a minimising sequence separates into two or more non-vanishing components moving away from each other. In this regard, it is of interest to study (0.0.7): indeed, let us assume that we are in the case where the infima in the generalised and the classical problems coincide. We may then view a minimiser $(E^i)_i$ of (0.0.6) as the limit as $n \to \infty$ of the fleeing components $(E_n^i)_{i,n}$ components of a minimising sequence (E_n) for (0.0.7). We further investigate this topic in the first half of the fourth chapter of the manuscript.

As previously mentioned, ancient Greek mathematicians had already understood that studying (0.0.6) in the class of regular sets was significantly easier and allowed to say much more about eventual minimisers. The converse approach, i.e. establishing regularity properties of minimisers of variational problems, is also central in Calculus of Variations. In particular, establishing that a minimiser E of (0.0.6) (or a component of a minimiser $(E^i)_i$ of (0.0.7)) admits density estimates, i.e. that there exist $c_0, r_0 > 0$ such that for any $r \le r_0$

$$\forall\, x \in E,\, |E \cap B_r(x)| \ge c_0 r^d \quad \text{and} \quad \forall\, x \in E^c,\, |E^c \cap B_r(x)| \ge c_0 r^d$$

is now a frequently used concept in shape optimisation problems. It is common to resort to it to obtain the boundedness of minimisers [24], their connectedness [42] or the regularity of their boundary [49, 35, 65]. We discuss the regularity theory of minimisers of (0.0.6) and (0.0.7) in more details in the latter half of Chapter 4.

Let us now turn to the more specific case of $\mathcal{E} = \mathrm{Per} + \mathcal{W}$, where the perturbation term is defined using the Wasserstein distance. This model was originally introduced by M. A. Peletier and M. Röger in [70] in order to study the formation of cell membranes. Let us emphasise that after a rescaling, their variational problem can be expressed in terms of sets of finite perimeter as follows: given $\varepsilon > 0$, consider

$$\inf_{U_\varepsilon, V_\varepsilon} \left\{ \overline{\mathcal{F}_\varepsilon}(U_\varepsilon, V_\varepsilon) = \mathrm{Per}(U_\varepsilon) + \varepsilon^{-4} W_1(U_\varepsilon, V_\varepsilon) : |U_\varepsilon \cap V_\varepsilon| = 0,\ |U_\varepsilon| = |V_\varepsilon| = \varepsilon m \right\}, \tag{0.0.11}$$

where we recall that W_1 denotes the 1-Wasserstein distance. Most of the difficulty about solving this problem comes from the lack of an explicit formulation for the optimal transportation plan between two sets of $\mathbb{R}^d$. However, in the case $d = 2$ the geometric properties of the 1-Wasserstein distance can be exploited to ease the analysis. Within this framework, the authors of [70] were able to obtain a surprising compactness result on (0.0.11): if a sequence $(U_\varepsilon, V_\varepsilon)$ is such that the renormalised energy

$$\overline{\mathcal{G}_\varepsilon}(U_\varepsilon, V_\varepsilon) = \frac{\overline{\mathcal{F}_\varepsilon}(U_\varepsilon, V_\varepsilon) - 2m}{\varepsilon^2}$$

is bounded as $\varepsilon \to 0$, then the sequence $(\varepsilon^{-1} U_\varepsilon)$ has a support of "thickness" 2ε and converges in a measure-theoretic sense to a collection of closed curves of $\mathbb{R}^2$.

After the work of Peletier and Röger, Buttazzo, Carlier and Laborde introduced in [13] the following generalization of (0.0.11): given $\alpha, \lambda > 0$, $d \geq 2$, $p \geq 1$ and a domain $\Omega \subset \mathbb{R}^d$, they considered

$$\inf_{E,F} \Big\{ \mathrm{Per}(E) + \lambda (W_p^p(E,F))^\alpha : E, F \subset \Omega, |E \cap F| = 0, |E| = |F| = \omega_d \Big\}. \tag{0.0.12}$$

which is equivalent to Problem (0.0.1). Existence of solutions to (0.0.12) was shown in [13] for Ω being either $\mathbb{R}^2$ or a compact subset of $\mathbb{R}^d$. In the case $\Omega = \mathbb{R}^d$, it was shown in [77] that minimisers existed under the additional assumptions that $\alpha = 1$, $1/p + 2/d > 1$ and for λ small enough. We were able to obtain existence of solutions to (0.0.12) without these additional hypotheses in [19], which corresponds to Chapter 1 of this manuscript.

Let us fix $\alpha = 1$ and compare (0.0.12) to (0.0.10) where the perturbation term is given by the Riesz kernel and denoted by $\mathcal{G}_\beta$. By using the Riesz rearrangement inequality one obtains that among sets of same volume, balls are *maximisers* of $\mathcal{G}_\beta$ (see e.g. [57, Section 11.15]). Consequently, one may see (0.0.10) as a competition between two opposite forces, so that its eventual minimisers may exhibit various geometries as λ varies in $[0, \infty)$. In this context, it is thus natural to wonder whether balls also are maximisers of $\mathcal{W}_p$. The corresponding problem, defined for general costs in (0.0.4), is further investigated in the second chapter of the manuscript. The analysis proves to be more involved than that of the Riesz kernel, as one cannot directly apply a Riesz rearrangement to a maximiser of (0.0.4).

Context and state of the art regarding the numerical methods used in Chapter 3.

In [13], the authors also introduce a numerical method to compute $\mathcal{W}_p(E)$ and the set F minimising (0.0.2). Let us provide some context on the tools they used. Their approach should be linked to the growing interest in computational mathematics for efficient solvers of various optimal transport problems (such as the unbalanced optimal transport problem described in [25]). Indeed, even the simple problem of computing $W_p(\mu, \nu)$ given $\Omega \subset \mathbb{R}^d$ compact and $\mu, \nu \in \mathcal{P}(\Omega)$ can become extremely time-consuming as the number of points used to sample μ and ν increases. M. Cuturi showed in 2013 in [31] that considering an entropic relaxation of the Wasserstein distance could yield significant computational gains when approximating solutions of the optimal transport problem. Given $\gamma > 0$, its continuous formulation can be defined as

$$\inf \left\{ \int_{\Omega^2} |x-y|^p d\pi + \gamma \int_{\Omega^2} \pi(\log \pi - 1) dx dy : \pi \in \mathcal{M}_+(\Omega^2), \pi \ll dx dy, \pi_x = \mu, \pi_y = \nu \right\} \tag{0.0.13}$$

As a convex problem, (0.0.13) admits a dual formulation whose maximisers can be approximated by an alternate maximization process known as the Sinkhorn algorithm [31, 71, 32]. Similarly, to compute a numerical approximation of $\mathcal{W}_p(E)$, we apply a variant of the Sinkhorn algorithm adapted to the dual problem associated with (0.0.2).

To fully address (0.0.12), we also replace the perimeter by a functional that is more suitable to our numerical experiments. One of the standards methods to do so is the Modica-Mortola approximation of the perimeter functional below, which was shown to Γ-converge to the perimeter (up to a multiplication by 1/6) by Modica and Mortola in [62]. Given $\varepsilon > 0$, the set E is replaced by a function $u \in H^1(\Omega)$ and $\mathrm{Per}(E)$ is replaced by a term similar to the Ginzburg-Landau energy:

$$\mathcal{F}_\varepsilon(u) = \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon}\int_\Omega u^2(1-u)^2. \tag{0.0.14}$$

In this framework, the characteristic function of E is approximated with a smooth function u that transitions between 0 and 1 over a region of size $\varepsilon \ll 1$. The Modica-Mortola approximation thus belongs to the family of phase-field methods, which are popular tools in the study of interface dynamics [11], fracture problems [4] or multiphase flows [53]. Let us momentarily drop the exterior transport term in (0.0.12) and consider

$$\inf_u \left\{ \mathcal{F}_\varepsilon(u),\, u : \Omega \to [0,1],\, \int_\Omega u = \omega_d \right\}.$$

Computing the first variation of this last equation then yields an Allen-Cahn equation with a forcing term $\mu \in \mathbb{R}$:

$$\partial_t u = \varepsilon \Delta u - \frac{1}{\varepsilon} u(1-u)(1-2u) + \mu, \tag{0.0.15}$$

One may see $\mu \in \mathbb{R}$ as a Lagrangian multiplier associated with the conversation of the mass of u. In the case were $\mu = 0$, this equation was first derived by Allen and Cahn in the 1970s to describe the motion of boundaries in crystalline solids [2]. Together with its higher order counterpart the Cahn-Hilliard equation [18], it is now widespread in numerical analysis to model interface formation. A simple way to obtain approximate solutions of (0.0.15) is to proceed to a Lie splitting, where one alternatively solves

$$\partial_t u = \varepsilon \Delta u, \quad \text{and } \partial_t u = -\frac{1}{\varepsilon} u(1-u)(1-2u) + \mu.$$

We give more details on the implementation of this scheme when considered energy is the sum of $\mathcal{F}_\varepsilon$ and the entropic relaxation Υ_γ of the functional Υ in the third chapter of this thesis.

Main contributions

Let us now describe more precisely the results obtained and the methods employed in each chapter of the manuscript.

Chapter 1

This chapter corresponds to the article [19] published in *ESAIM : Control, Optimisation and Calculus of Variations* in collaboration with M. Goldman. For $d \geq 2$, $p \geq 1$, $\lambda > 0$ and $\alpha > 0$, the variational problem we consider is (0.0.1). Our first main result concerns the existence and regularity of solutions to this problem

Theorem 0.0.2. *For every $d \geq 2$, $p \geq 1$, $\alpha > 0$ and $\lambda > 0$, problem* (0.0.1) *has minimisers. Moreover, there exists $C = C(d,p,\alpha) > 0$ such that if $E = \cup_{i=1}^{I} E^i$ is such a minimiser with E^i the connected components of E, then*

$$\sum_{i=1}^{I} \mathrm{diam}(E^i) \leq C(1+\lambda)^{\frac{(d-1)(1+p)}{1+\alpha p}} \quad \text{and} \quad \inf_i \mathrm{diam}(E^i) \geq \frac{1}{C}(1+\lambda)^{-\frac{1+p}{1+\alpha p}}.$$

As a consequence $I \leq C(1+\lambda)^{\frac{d(1+p)}{1+\alpha p}}$ (in particular E has finitely many connected components).

This result was first obtained in the case $d = 2$ in [13] and then extended to the case $d \geq 3$ in [77] but under the assumption that λ is small together with some restrictions on α. As explained in the previous section, the idea of the proof, by now well-established in the context of geometrical analysis, is to follow a concentration-compactness type argument. We first show that thanks to the isoperimetric inequality, lack of compactness for minimising sequences can only come from splitting of the mass. This leads to the existence of so-called generalised minimisers. We then show that these generalised minimisers are actually $\Lambda-$minimisers of the perimeter (see [60, Section 21] for a definition) and therefore have uniform density bounds. As a direct consequence we obtain that they are made of a finite number of uniformly bounded connected components $(E^i)_{i=1}^{I}$. Additionally, for any $1 \leq 1 \leq I$, the boundary ∂E^i of E^i can be identified (up to a set of null $\mathcal{H}^{d-1}$ measure) with its reduced boundary $\partial^* E^i$, and $\partial^* E^i$ is a $C^{1,\gamma}$ set for any $0 < \gamma < 1/2$ (see e.g. [60, Chapter 15] for a definition of the reduced boundary). At this point the proof of the existence is concluded as in [13] using the fact that the non-local energy $\mathcal{W}_p^p$ is additive for sets which are sufficiently far apart.

The second main result we obtained was that if λ is small enough, then (0.0.1) is uniquely minimised by balls.

Theorem 0.0.3. *For every $d \geq 2$, $p \geq 1$ and $\alpha > 0$, there exists $\lambda_0 > 0$ such that for every $\lambda \leq \lambda_0$, balls are the only minimisers of* (0.0.1).

Let us first point out that if we consider the volume as the relevant parameter and replace (0.0.1) by (0.0.5) then by scaling we obtain that balls are the unique minimisers for small m if $\alpha\left(1+\frac{p}{d}\right)+\frac{1}{d} > 1$ while balls are the unique minimisers for large m if $\alpha\left(1+\frac{p}{d}\right)+\frac{1}{d} < 1$. Again, this result is neither surprising by its statement nor by the strategy to prove it. Indeed, following the pioneering work of Cicalese and Leonardi which gave in [28] an alternative proof of the quantitative isoperimetric inequality, it has been understood that such stability results may be obtained by combining the regularity theory for $\Lambda-$minimisers of the perimeter together with a (usually delicate) Taylor expansion of the energy around the ball. This second part of the proof is often referred to as a Fuglede type argument, see [45]. The main difficulty here is that our non-local energy depends in a very implicit way on the competitor. Moreover the underlying PDE is the Monge-Ampère equation, which is non-linear, making it very difficult to use standard tools from shape optimisation such as shape derivatives. We go around this difficulty by plugging the Kantorovich potentials corresponding to the ball into the dual formulation of the optimal transport problem.

More precisely, the starting point of the proof of Theorem 0.0.3 is to establish that for λ small enough, any minimiser E_λ of (0.0.1) is a nearly spherical set. Using the quantitative isoperimetric inequality (see e.g. [46]) and comparing the energies of E_λ and of B_1, up to translation we have

$$|E_\lambda \Delta B_1|^2 \leq C\left(\mathrm{Per}(E) - \mathrm{Per}(B_1)\right) \leq C\lambda\left(\left[\mathcal{W}_p^p(B_1)\right]^\alpha - \left[\mathcal{W}_p^p(E_\lambda)\right]^\alpha\right) \leq C\lambda\left[\mathcal{W}_p^p(B_1)\right]^\alpha. \tag{0.0.16}$$

Therefore, $E_\lambda \to B_1$ in L^1 to B_1 as $\lambda \to 0$. Using the properties of Λ-minimisers of the perimeter, we show in Chapter 1 that E_λ is actually smooth and that the convergence holds in $\mathcal{C}^{1,\gamma}$ for any $\gamma < 1/2$. Then, using the classical result from [3] for nearly-spherical sets yields

$$\int_{\partial B_1} f^2 \leq C(\mathrm{Per}(E_\lambda) - \mathrm{Per}(B_1)),$$

where $f : \partial B_1 \to \mathbb{R}$ is the function parametrizing the graph of E_λ:

$$\partial E_\lambda = \{(1 + f(x))x : x \in \partial B_1\}.$$

The heart of the proof of Theorem 0.0.3 is then to establish the following Taylor expansion of the energy of E_λ around the ball:

$$\left[\mathcal{W}_p^p(B_1)\right]^\alpha - \left[\mathcal{W}_p^p(E)\right]^\alpha \leq C\int_{\partial B_1} f^2,$$

for some constant $C = C(d) > 0$. Using this equation together with (0.0.16), we have

$$\int_{\partial B_1} f^2 \leq C(\mathrm{Per}(E_\lambda) - \mathrm{Per}(B_1)) \leq C\lambda\left(\left[\mathcal{W}_p^p(B_1)\right]^\alpha - \left[\mathcal{W}_p^p(E_\lambda)\right]^\alpha\right) \leq C\lambda\int_{\partial B_1} f^2.$$

From this chain of inequalities we can conclude that for $\lambda > 0$ small enough $f = 0$, so that E_λ must be B_1.

Chapter 2

The second chapter of the manuscript corresponds to the submitted article [20], written in collaboration with M. Goldman and B. Merlet. We study the optimisation problems associated with functionals which favour dispersion and are based on some Wasserstein energies. These functionals correspond to the non-local term of the energy studied in [19, 13, 70, 77, 64]. More precisely, we drop the perimeter term in the energy functional of Chapter 1 (which was defined as $\mathcal{E} = \mathrm{Per} + \lambda\mathcal{W}_p$) and focus on the perturbation term $\mathcal{W}_p$ and its corresponding *maximization* problem defined by (0.0.4). In this problem, $\mathcal{W}_p$ is replaced by a more general functional Υ which allows us to consider other cost functions than $c(x,y) = |x-y|^p$. The main goal of the chapter is to investigate the existence of maximisers for this problem and to characterise these latter.

If we apply the direct method of the Calculus of Variations, we obtain that, up to extraction, any maximising sequence E_n converges weakly to some function $u_\infty \in L^1(\mathbb{R}^d, [0,1])$. However, there is no guarantee at this point that u_∞ is a characteristic function or has mass m. Our strategy is to replace the functional Υ by a functional Υ_{fun} defined on $L^1(\mathbb{R}^d, [0,1])$. We then characterise the maximisers of the relaxed problem and are able to show that the supremum in (0.0.4) is actually reached. This relaxation approach is not new: it was successfully applied to several variational problems in the last few years (see for instance [27, 8, 69, 12]).

Next, we relax problem (0.0.4) as follows. Given $f \in L^1(\mathbb{R}^d, [0,1])$, the set of admissible exterior transport plans is defined as

$$\Pi_f = \left\{\gamma \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d) : \gamma_x = f,\, \gamma_y \leq 1 - f\right\}.$$

We then define the primal problem

$$\Upsilon_{\text{fun}}(f) = \inf\left\{\int c\, d\gamma : \gamma \in \Pi_f\right\}.$$

Under mild assumptions on c, we prove that there holds $\Upsilon_{\text{fun}}(\chi_E) = \Upsilon(E)$. Given $m > 0$, our maximisation problem is now

$$\mathcal{E}_{\text{fun}}(m) = \sup\left\{\Upsilon_{\text{fun}}(f) : f \in L^1(\mathbb{R}^d, [0,1]),\ \int f = m\right\}.$$

In what follows, $\mathcal{E}_{\text{fun}}(m)$ is called the exterior energy.

The first important result of this chapter is that maximisers of $\mathcal{E}(m)$ exist whenever c is of the form $c(x,y) = k(y-x)$ for some $k : \mathbb{R}^d \to \mathbb{R}_+$ which satisfies

(H1) $k \in C(\mathbb{R}^d, \mathbb{R}_+)$, $k(0) = 0$ and $k(x) \to \infty$ as $|x| \to \infty$,

(H2) $\forall x \neq 0$,
$$\limsup_{r\to 0} \frac{1}{r^d}\left|B_r(x) \cap \{y \in \mathbb{R}^d,\, k(y) < k(x)\}\right| > 0,$$

(H3) $\forall\, \sigma \in \mathbb{S}^{d-1}$, $r \mapsto k(r\sigma)$ is increasing on $\mathbb{R}_+$.

Observe that all the costs of the form $k(z) = |z|^p$ with $0 < p < \infty$ satisfy the above hypotheses.

Theorem 0.0.4. *Assume that $c(x,y) = k(y-x)$ for $x, y \in \mathbb{R}^d$ with k satisfying (H1), (H2) & (H3). Then, for any $m > 0$ the supremum in $\mathcal{E}_{\text{fun}}(m)$ is a maximum. Moreover, there exists $R_* = R_*(m)$ such that (up to translation) any maximiser is supported in the ball $\overline{B}_{R_*}$.*

Once the existence of maximisers for $\mathcal{E}_{\text{fun}}(m)$ is established, we use a concept known as the bathtub principle to show that maximisers are characteristic functions of sets. This finally allows us to show that (0.0.4) admits solutions.

Corollary 0.0.5. Assume that c satisfies the hypotheses of Theorem 0.0.4. Then, (0.0.4) admits a maximiser and $\mathcal{E}_{\text{fun}}(m) = \mathcal{E}(m)$ for any $m > 0$.

As a second main result, we establish that if k is furthermore radially symmetric then $\mathcal{E}_{\text{fun}}(m)$ and $\mathcal{E}(m)$ are uniquely maximised by balls of volume m.

Theorem 0.0.6. *Assume that $c(x,y) = k(|y-x|)$ for some $k \in C(\mathbb{R}_+, \mathbb{R}_+)$ (strictly) increasing and such that $k(0) = 0$ and $k(x) \to \infty$ as $x \to \infty$. Then, for any $m > 0$, the maximisers of $\mathcal{E}_{\text{fun}}(m)$ (and consequently those of $\mathcal{E}(m)$) are the balls of volume m.*

We point out that cost functions satisfying the hypotheses of Theorem 0.0.6 also satisfy hypotheses (H1), (H2) & (H3).

The proofs of Theorems 0.0.4 and 0.0.6 all strongly rely on the properties of the dual problem

$$\Upsilon^*_{\text{fun}}(f) := \sup\left\{\int f\varphi dx + \int (1-f)\psi dy : (\varphi,\psi) \in \Phi\right\},$$

where

$$\Phi := \left\{(\varphi,\psi) \in C_b(\mathbb{R}^d) \times C_b(\mathbb{R}^d),\ \psi \le 0,\ \varphi(x) + \psi(y) \le c(x,y)\ \forall\,(x,y) \in \mathbb{R}^d \times \mathbb{R}^d\right\}.$$

We establish Theorem 0.0.4 using the direct method of Calculus of Variations. The main difficulty is to establish compactness of maximising sequences. If we refer to the concentration-compactness principle [58], we have to prove that given a maximising sequence f_n, no mass escapes at infinity. To do so we establish two crucial results. The first one is that $m \mapsto \mathcal{E}(m)/m$ is increasing. This implies that $m \mapsto \mathcal{E}_{\text{fun}}(m)$ is strictly superadditive, i.e. that for $m > m' > 0$,

$$\mathcal{E}_{\text{fun}}(m) + \mathcal{E}_{\text{fun}}(m - m') < \mathcal{E}_{\text{fun}}(m). \tag{0.0.17}$$

Notice that this is the counterpart of the strict subadditivity inequality (also called binding inequality) which is known to provide compactness in minimisation problems, see e.g. [58, 43, 44]. Using the dual formulation Υ^*_{fun} of Υ_{fun}, we obtain the second crucial result for Theorem 0.0.4: a monotonicity principle on the sum of marginals of maximisers γ of $\Upsilon_{\text{fun}}(f)$. Combining this and (0.0.17), we prove that if f is almost maximising then most of its mass must remain in a bounded region. This gives tightness of maximising sequences for $\mathcal{E}_{\text{fun}}(m)$.

To prove Corollary 0.0.5, we consider a maximiser f of $\mathcal{E}_{\text{fun}}$ provided by Theorem 0.0.4 and a pair of potentials (φ,ψ) optimal for the dual problem $\Upsilon^*_{\text{fun}}(f)$. Using the definition of Υ^*_{fun} we see that f is a maximiser of

$$\sup\left\{\int \widetilde{f}(\varphi - \psi) : 0 \le \widetilde{f} \le 1,\ \int \widetilde{f} = m\right\}.$$

By the bathtub principle, $f = \chi_{\{\varphi-\psi>t\}} + \theta$ for some $t \in \mathbb{R}$ and some $\theta \in L^1(\mathbb{R}^d, [0,1])$ supported in $\{\varphi - \psi = t\}$. Then for any measurable subset $G \subset \{\varphi - \psi = t\}$ with $|G| = \int \theta$, the characteristic function of $E := \{\varphi - \psi > t\} \cup G$ is also a maximiser for $\Upsilon_{\text{fun}}(f)$. Exploiting uniqueness and a saturation result on the marginals of the optimal exterior transport plan for E, we obtain that there exists $F \subset \mathbb{R}^d$ such that any minimiser γ of $\Upsilon_{\text{fun}}(\chi_E)$ satisfies $\gamma_y = \chi_F$. This finally implies that E maximises (0.0.4).

Regarding Theorem 0.0.6 we may assume that $m = \omega_d$. We begin the proof by showing that the double supremum problem

$$\sup_{\int f = \omega_d} \left\{ \sup_{(\psi^c, \psi) \in \Phi} \left\{ \int f(\psi^c - \psi) + \int \psi \right\} \right\}, \tag{0.0.18}$$

coincides with $\mathcal{E}_{\text{fun}}(m)$ and admits a solution (f, ψ^c, ψ), where ψ^c is the c-transform of ψ, i.e. for $x \in \mathbb{R}^d$

$$\psi^c(x) = \inf_{y \in \mathbb{R}^d} \Big\{ k(|y - x|) - \psi(y) \Big\}. \tag{0.0.19}$$

To show that balls are maximisers of $\mathcal{E}_{\text{fun}}(m)$, we then establish that each term in (0.0.18) is improved by replacing f by χ_{B_1} and ψ by its symmetric increasing rearrangement ψ_* (i.e. ψ_* is the unique radially symmetric function whose sublevel sets are the same as the ones of ψ). This is quite an involved computation, which requires an extensive use of the Hardy-Littlewood and Brunn-Minkowski inequalities. The uniqueness part of the proof of Theorem 0.0.6 is then established by using the characterisation of the objects for which equality holds in the Hardy-Littlewood and the Brunn-Minkowski inequalities.

Chapter 3

The goal of this chapter is to numerically investigate (0.0.1) in the case $\alpha = 1$ and $p = d = 2$. Recall that in Chapter 1 we establish two main results regarding this problem: Theorem 0.0.2 on the existence and density estimates for its minimisers and Theorem 0.0.3 on the fact its minimisers are balls if λ is small enough (or if m is small enough if one considers the equivalent problem (0.0.5)).

More precisely, in Chapter 3 we provide answers or conjectures related to the following questions:

- how does the geometry of minimisers of (0.0.1) evolves as m varies in $[0, \infty)$? In particular, is radial symmetry preserved ?
- Among radially symmetric minimisers, can we observe a transition from a ball to an annulus as m increases?
- Is the transition sharp, or does an intermediate regime exist between the cases $m \ll 1$ and $m \gg 1$?

To address these questions, we perform two types of numerical explorations. The first one concerns Problem (0.0.2) restricted to radially symmetric and connect sets:

$$\inf_E \Big\{ \text{Per}(E) + \lambda \mathcal{W}_2^2(E) : E \text{ radially symmetric and connected}, |E| = \pi \Big\}. \tag{0.0.20}$$

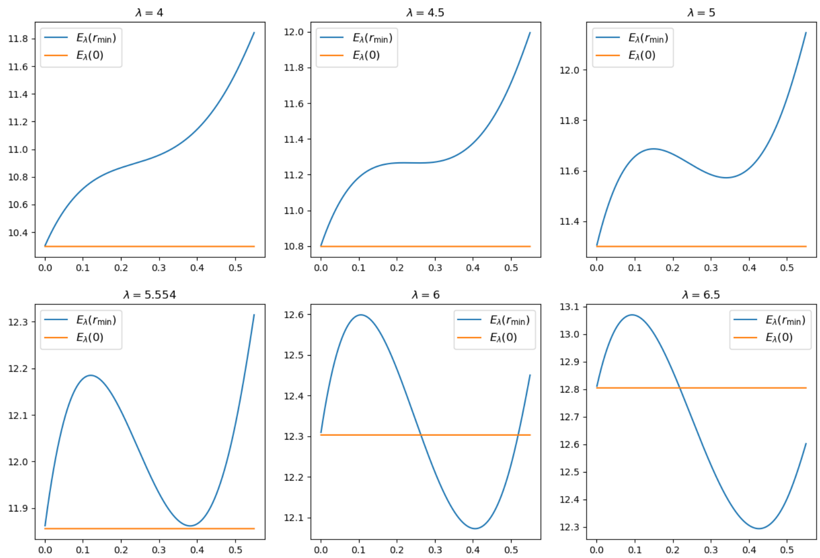

Figure 3: Energies of annuli of inner radius r_m (blue) and of the unit disk (orange).

The method we use for the computation of the energy of radially and connected sets is described later in this section.

The other exploration is conducted in the general case with non symmetry assumption. To implement a suitable algorithm in the non-radially symmetric case, it is convenient to replace the perimeter and Wasserstein functionals in (0.0.1) by smoother, approximate functionals. We then use a gradient descent algorithm to generate a sequence of shapes minimising the approximate energy.

Regarding Problem (0.0.20), let us denote by A_r the annulus of inner radius r and mass π, and by $A_{r,r'}$ a generic annulus of inner radius r and outer radius r'. We numerically observe the following, which we state as a conjecture:

Conjecture 0.0.7. There exist $0 < \lambda_1 < \lambda_2$ (where $\lambda_1 \approx 4.95$ and $\lambda_2 \approx 5.55$) such that:

- for $0 \leq \lambda \leq \lambda_1$, the unit disk B_1 is the unique local and global minimiser of (0.0.20),
- for $\lambda > \lambda_1$, (0.0.20) has two local minimisers : B_1 and an annulus A_{r_λ} where $r_\lambda > 0$,
- for $\lambda_1 < \lambda < \lambda_2$, the unit disk B_1 is the only global minimiser of (0.0.20),
- for $\lambda = \lambda_2$, (0.0.20) has two global minimisers: B_1 and a annulus A_{r_λ} where $r_\lambda > 0$,
- for $\lambda > \lambda_2$, the unique global minimiser of (0.0.20) is an annulus A_{r_λ} where $r_\lambda > 0$.

These guesses follow from the graphs presented in Figure 3 where the function $r_m \mapsto \mathcal{E}_\lambda(r_m) = \mathrm{Per}(A_{r_m}) + \lambda W_2^2(Ar_m)$ is represented. Each of the graphs corresponds to a different value of λ. To ease the comparison, we also give the graph of the constant function equals to the energy of the unit disk.

Let us now explain briefly how we compute the energy of a annulus A_{r_m} for some $r_m > 0$. Its outer radius r_M is fixed by the mass constraint, so that computing its perimeter presents can be

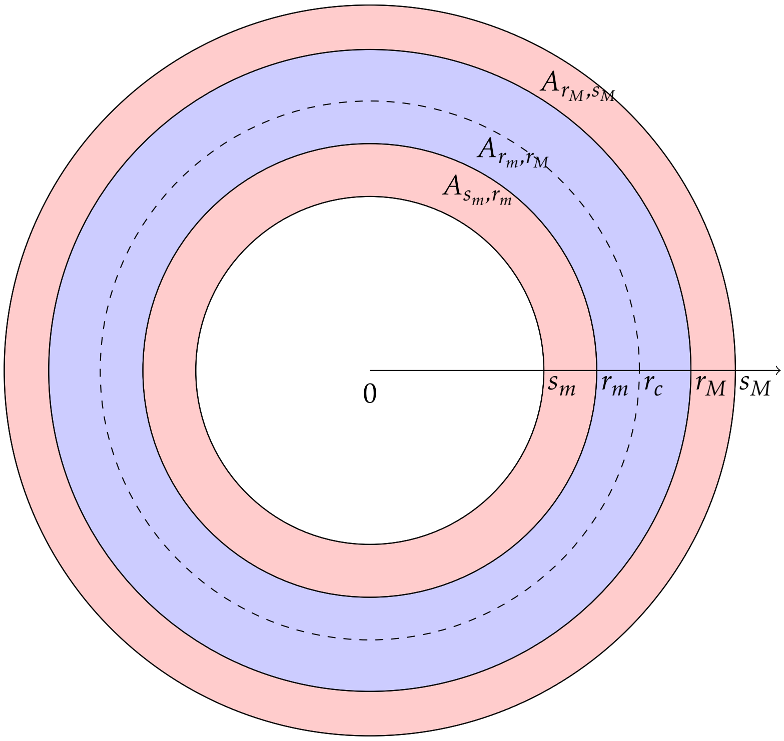

Figure 4: The annulus A_{r_m,r_M} and its corresponding minimiser $A_{s_m,r_m} \cup A_{r_M,s_M}$.

done explicitly. As for its exterior transport, we show in Chapter 3 that the optimal exterior set F corresponding to $\mathcal{W}_2^2(A_{r_m})$ is the reunion of one interior ring A_{s_m,r_m} and one exterior ring adjacent to A_r (see Figure 4). Additionally, the optimal exterior map between A_{r_m} and F is the c-cyclically monotone one. Therefore, the only parameter needed to determine $\mathcal{W}_2^2(A_{r_m})$ is the splitting radius r_c for which sending A_{r_m,r_c} to A_{s_m,r_m} and A_{r_c,r_M} to A_{r_M,s_M} is optimal in terms of energy.

We solve this problem by developing the expression of the transport cost $\tau(r_c)$ of sending A_{r_m,r_c} to A_{s_m,r_m} and A_{r_c,r_M} to A_{r_M,s_M} using the c-cyclically monotone, optimal transport map. We observe numerically that $r_c \mapsto \tau(r_c)$ is a convex and $\mathcal{C}^1$ function on $[r_m, r_M]$. We are thus able to solve the equation $\tau'(r_c) = 0$ using a variant of the secant method. This eventually allows us to compute $\mathcal{W}_2^2(A_{r_m})$.

Next, let us comment the results provided by the numerical simulations of the non-radially symmetric case. Recall that we obtain them by implementing a gradient descent on the energy $\mathcal{E}$ and solving the corresponding evolution equation by Lie splitting.

In a first experiment, whose result is depicted in Figure 5 below, we confirm the local minimality of unit disks for any $\lambda \geq 0$. We refer the reader to Chapter 3 for more experiments.

Let us now comment the experiment which gave us the most surprising results. We observe that when starting with initial data that exhibits highly non-radial symmetry, the final data does not converge to a radially symmetric shape. Instead, the initial shapes tend to evolve into thin, elongated forms with two axes of symmetry (see Figure 6 below). This leads us to consider the conjecture that thin and elongated shapes are the preferred minimisers as $\lambda \to \infty$, and not annuli. Refer to the last section of this introduction for a discussion on potential developments

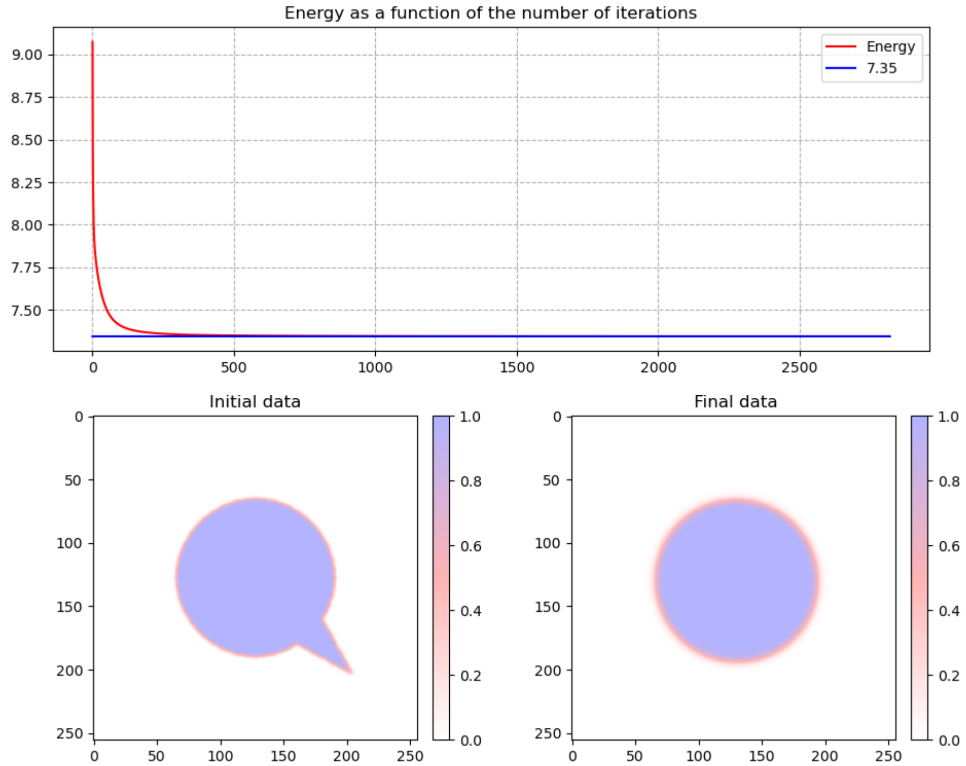

Figure 5: Starting with a perturbed ball ($\lambda = 1$).

on this matter.

Chapter 4

The last chapter of the manuscript focuses on general perturbed isoperimetric problems. Recall that an archetype of those kind of variational problems is Gamow's liquid drop model for the atomic nucleus [47], which correspond to (0.0.10). As explained before, the goal of this variational problem is to model an attractive, short-range force inducing surface tension (the "perimeter" term) that competes with a repulsive term V acting at a greater distance (the "perturbation" term, which is often nonlocal). This competition plays a pivotal role in the wide range of geometries the perturbed isoperimetric problem can describe (see for instance [55, Figure 1]) and both the physics and mathematics communities have explored numerous variants of this problem.

In this chapter we study the version of this problem defined in (0.0.6), where the energy is the sum of a nonnegative, perimeter-like term P and a perturbation term V. For instance, P may be the s-perimeter or anisotropic perimeter, while V can be the repulsive term given in (0.0.10) or the exterior transport cost $\mathcal{W}_p$. We also study the generalised problem (0.0.7) and discuss its similarities and differences with Problem (0.0.6).

Let us additionally mention that beyond the issue of existence of solutions, the question of their regularity is a significant aspect of the study of isoperimetric problems. A first step towards establishing regularity properties of minimisers is often to prove that they have density estimates. Recall that a set E admits interior (resp. exterior) density estimates when there exists

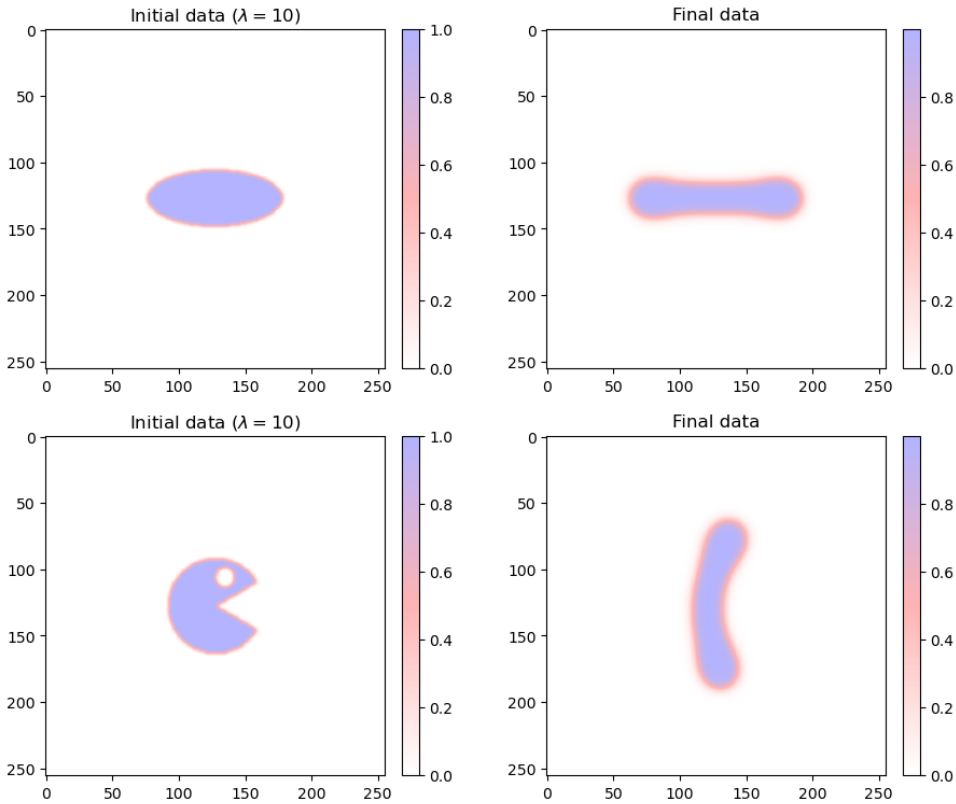

Figure 6: Starting with an ellipse and with a pacman ($\lambda = 10$).

$c_0, r_0 > 0$ such that for any $0 < r \le r_0$ and $x \in E$ (resp. E^c),

$$|E \cap B_r(x)| \ge c_0\, r^d \text{ (resp. } |E^c \cap B_r(x)| \ge c_0\, r^d\text{).}$$

Our goal is to exhibit in the case $V \neq 0$ general assumptions under which:

– (0.0.6) and (0.0.7) coincide,

– (0.0.7) admits solutions,

– solutions of (0.0.7) have density estimates.

Let us denote by $(e_i)_{i=1}^d$ the canonical basis of $\mathbb{R}^d$. Our first result is that (0.0.6) and (0.0.7) coincide under the following set of assumptions (S1).

- *Energy of small balls*: $\mathcal{E}(B_r) \to 0$ as $r \to 0$ and $\mathcal{E}(\emptyset) = 0$.
- *Convergence at infinity*: For any set E with $|E| < \infty$, $\mathcal{E}(E \cap B_R) \to \mathcal{E}(E)$ as $R \to \infty$.
- *Vanishing range of action*: If E, F are bounded, then $\mathcal{E}(E \cup (F + Le_1)) \to \mathcal{E}(E) + \mathcal{E}(F)$ as $L \to \infty$.

Proposition 0.0.8. *Assume that $\mathcal{E}$ satisfies (S1). Then the infima in* (0.0.6) *and* (0.0.7) *coincide.*

We then show that (0.0.7) admits minimisers. To prove this result we rely on the functionals $E \mapsto P(E, U)$ and $E \mapsto V(E, U)$, which are defined relatively to a Lebesgue measurable set U. By convention, we write $P(E, \mathbb{R}^d) = P(E)$ and $V(E, \mathbb{R}^d) = V(E)$. For the sake of conciseness we do not list the complete set of assumptions (S2) here (see the introduction of Chapter 2), but we will highlight some of the key points after the statement of the theorem.

Theorem 0.0.9. *Assume that the relative functionals of P and V satisfy (S2) and that* (0.0.6) *and* (0.0.7) *coincide. Then,* (0.0.7) *admits a solution.*

In the framework of the concentration-compactness principle, the lower semi-continuity of $P(\cdot, U)$ and $V(\cdot, U)$ for the L^1_{loc}-convergence is required. Additionally, one needs hypotheses guaranteeing that a minimising sequence $(E_n)_n$ for (0.0.6) has uniformly bounded perimeter (e.g. $V \geq 0$ is sufficient, but more general energies are included), and that sequence of uniformly bounded perimeter are compact for the L^1_{loc} topology. Lastly, an important technical tool is the relative isoperimetric inequality, i.e. the existence of $r_0 > 0$ and $f_1 : \mathbb{R}_+ \to \mathbb{R}_+$ increasing with $f_1(0) = 0$, $m \mapsto f_1(m)/m$ nonincreasing and $\lim_{m\to 0} f_1(m)/m = \infty$ such that for $r \leq r_0$, $x \in \mathbb{R}^d$ and $E \subset \mathbb{R}^d$:

$$\min\Big(f_1(|E \cap B_r(x)|), f_1(|B_r(x) \setminus E|)\Big) \leq P(E, B_r(x)).$$

This last inequality is fundamental because it allows to exhibit a candidate minimiser $(E^i)_i$ using the following method (which we describe formally). Given a minimising sequence $(E_n)_n$ for (0.0.7), we keep track of all the components $(E_n^i)_{i,n}$ which may "flee at infinity" and then define a limit set $E^i = \lim E_n^i$ for the L^1_{loc} topology. Repeating the process over $i \geq 1$ yields the candidate $(E^i)_i$, and using the relative isoperimetric we prove that $\sum_i |E^i| = m$. Let us stress the fact that this whole process is a slight generalization of the method employed in Chapter 1 to show that (0.0.7) admits a solution in the case where $P = \mathrm{Per}$ and $V = \mathcal{W}_p^p$.

In the second part of the chapter, we show that ρ-minimisers of the perimeter (see [60, Section 21] for the related concept of (Λ, r_0)-minimisers of the perimeter) have interior and exterior density estimates under the set of hypotheses (S3) (we refer to Chapter 4 for the precise list of the hypotheses).

Definition 0.0.10. Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing. We say that $E \subset \mathbb{R}^d$ is a ρ-minimiser of the perimeter (or simply a ρ-minimiser) if there exists $r_2 > 0$ such that for any $r \leq r_2$, $x \in \mathbb{R}^d$ and $E' \subset \mathbb{R}^d$ with $E \Delta E' \subset B_r(x)$ we have

$$P(E) \leq P(E') + \rho(r). \tag{0.0.21}$$

The function ρ is called the error function for E.

Theorem 0.0.11. *Let $E \subset \mathbb{R}^d$ be a ρ-minimiser of the perimeter for some error function ρ. If (S3) holds, then there exists $C_0, r_4 > 0$ such that for $r \leq r_4$,*

$$|E \cap B_r(x)| \geq C_0 r^d \text{ for every } x \in E^{(1)} \tag{0.0.22}$$

and

$$|B_r(x) \setminus E| \geq C_0 r^d \text{ for every } x \in E^{(0)}, \tag{0.0.23}$$

where for $t \in [0,1]$, $E^{(t)}$ denotes the points of density t of E.

The method employed to establish this kind of theorems is now well understood. Since the publication of De Giorgi's seminal papers on the classical isoperimetric problem in the 1950s, various strategies have been developed to address isoperimetric problems where the considered perimeter is anisotropic or nonlocal, or with different perturbation terms. However, most of these proofs revolve around the same idea: once the problem of relaxing the mass constraint is dealt with, one tests the minimality of E against $E \setminus B_r$ (resp. $E \cup B_r$) for r small enough, so that one obtains

$$\mathcal{E}(E) \leq \mathcal{E}(E \setminus B_r) \quad (\text{resp. } \mathcal{E}(E) \leq \mathcal{E}(E \cup B_r)). \tag{0.0.24}$$

Then, if P and V are regular enough one may prove that $P(E)$ and $P(E \setminus B_r)$ (resp. $P(E \cup B_r)$) and $V(E)$ and $V(E \setminus B_r)$ (resp. $V(E \cup B_r)$) coincide up to a small error term. One may thus apply the relative isoperimetric inequality to E (resp. E^c), inject it in (0.0.24) and integrate the resulting inequality to obtain interior (resp. exterior) density estimates (see [60, Remark 15.16]).

At the end of the second part of this chapter, we establish the connection between generalised minimisers and ρ-minimisers. We prove that generalised minimisers of (0.0.7) are ρ-minimisers of the perimeter for some ρ in two different situations: a case where P and V admit volume-fixing variations and a case where P and V have a scaling property.

Proposition 0.0.12. *Assume that the relative functionals of P and V satisfy (S4) and either both admit a scaling or both admit volume-fixing variations. Then every component of a generalised minimiser of* (0.0.7) *is a ρ-minimiser of the perimeter for an error function ρ.*

The purpose of the volume-fixing variations (or scaling) property is to allow us to compare the energy E^i with the energy of a set E' such that $E^i \Delta E'$ is localised in a small ball of $\mathbb{R}^d$. Eventually, we obtain that E^i is a ρ-minimiser of the perimeter.

Perspectives

To conclude this introductory chapter, we present several interesting questions that emerged during the course of this thesis, but remain unanswered or partially answered.

Recall that in Chapter 1 we studied Problem (0.0.1) and proved in Theorem 0.0.3 that there exists $\lambda_0 > 0$ such that for $\lambda \leq \lambda_0$, the unique minimisers of (0.0.1) are balls. One interpretation of this result is that when λ is small the perimeter is the dominant term and the problem can be seen as a small perturbation of the isoperimetric problem. One may thus wonder: what happens when λ is large? In this case, the exterior transport cost becomes the dominant term.

In the regime $\lambda \gg 1$, the quantitative isoperimetric inequality cannot be used anymore to show that minimisers are close to balls in $C^{1,\gamma}$ norm for any $\gamma < 1/2$. Thus, Flugede's results on quasi-spherical sets are no longer applicable. Let us set $\alpha = 1$, so that the regime $\lambda \gg 1$ in (0.0.1) is equivalent to the regime $m \gg 1$ in (0.0.5). Note that considering

$$\inf\{(\mathcal{W}_p^p(E))^\alpha : |E| = m\} \tag{0.0.25}$$

to gain insight on the behaviour of solutions of (0.0.1) in the regime of large masses yields an ill-posed problem. Indeed, let us consider in the case $p = d = 2$ a set E_N made N balls of radius

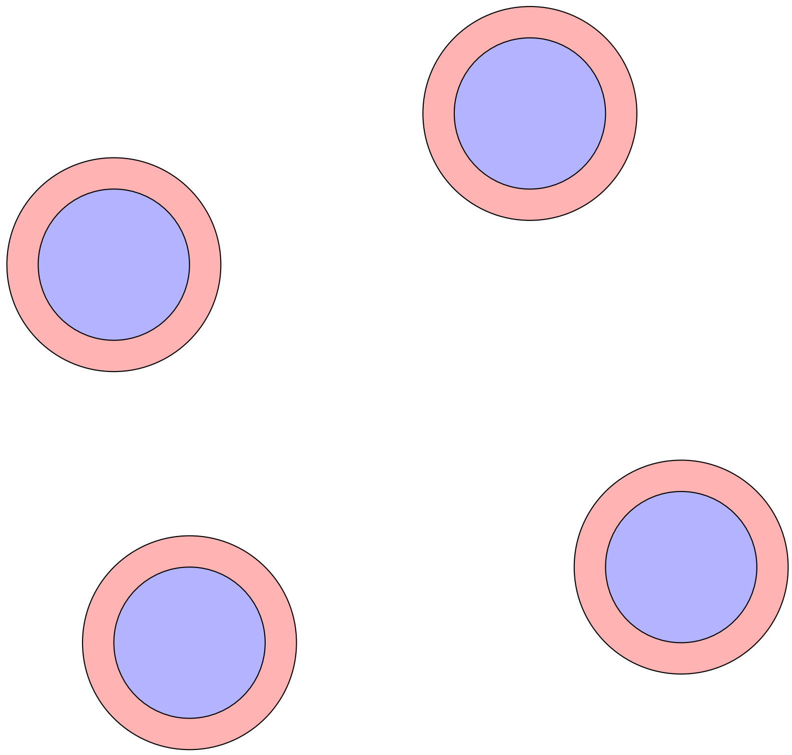

Figure 7: A set E_N made of N balls (in blue) and its corresponding minimiser (in red).

r_N (see Figure 7), so that $N\pi r_N^2 = m$. Sending each ball to the annulus of same mass surrounding it moves mass by at most r_N, so that $\mathcal{W}_2^2(E_N) \le r_N^2 m = m^3/N\pi$. Letting $N \to \infty$ we obtain that the infimum in (0.0.25) is 0, so that this problem cannot be used as a reference in the asymptotic development $\lambda = +\infty$.

We now take the perimeter back into consideration and compute some asymptotic estimates for potential minimisers of $\mathcal{E}$ in the regime $m \gg 1$. First notice that regarding the set E_N made of N ball of radius r_N, we have $\mathrm{Per}(E_N) = N2\pi r$. Additionally, thanks to the computations of Chapter 3 we know the exact value of $\alpha_2 = \mathcal{W}_2^2(B_1)$. By scaling, if the balls defining E_N are sufficiently far apart, we have

$$\mathcal{W}_2^2(E_N) = N\mathcal{W}_2^2(B_r) = Nr^4\mathcal{W}_2^2(B_1) = Nr^4\alpha_2.$$

Using the mass constraint $N\pi r_N^2 = m$ to eliminate r_N, we obtain regarding the total energy:

$$\mathcal{E}(E_n) = 2\sqrt{N\pi m} + \frac{\alpha_2 m^2}{N\pi^2}.$$

optimising this value in N yields that $N = \alpha^{2/3}\pi^{-5/3}m$ (assuming that $N \in \mathbb{R}_+$ for the sake of this estimate). Thus the total energy is of the form

$$\mathcal{E}(E_n) = \left(\frac{2\alpha_2^{1/3}}{\pi^{1/3}} + \frac{1}{\pi^{1/3}\alpha_2^{1/3}}\right) m \sim 2.05\, m. \tag{0.0.26}$$

Following the heuristics of [70, Section 2], we can exhibit another type of minimisers favoured in the regime $m \gg 1$. First notice that formally, $\mathcal{W}_p$ prefers elongated and thin shapes with most of its mass close to its boundary.

Let us explore the case of a rectangle E of length $L > 0$ and of width $\ell = m/L$. As a candidate for the minimisation problem defined by $\mathcal{W}_2(E)$, we consider the set F made of two rectangles F_1 and F_2 of length L and width $\ell/2$, placed on each side of E. A possible transport between E and F is to split E in two lengthwise and send each sub-rectangle of E to F_1 and F_2, so that each point of E travels a distance of $\ell/2$ (see Figure 8). Thus $\mathcal{W}_2^2(E) \sim m\ell^2/4$. Additionally, we have $\mathrm{Per}(E) = 2L + 2\ell = 2L + 2m/L$, so that eliminating ℓ yields

$$\mathcal{E}(E) \sim 2L + \frac{2m}{L} + \frac{m^3}{4L^2}.$$

Let us substitute $L = 2^{-2/3}m$ in this expression to obtain an expansion of $\mathcal{E}$ of order 1 in m . This yields

$$\mathcal{E}(E) \sim 2^{5/3} + m\left(2^{1/3} + 2^{-2/3}\right) \sim 3.17 + 1.89m. \tag{0.0.27}$$

For large values of m, the rectangle E is a long thin stripe and the constant term $2^{5/3}$ can be viewed as a boundary effect of the extremities of the stripe.

We may try to eliminate this contribution by folding the rectangle E back on itself: more precisely, we consider now an annulus A of inner radius $R > 0$ and width ℓ. Assuming that $\ell \ll R$, we have $|A| \sim 2\pi R\ell = m$ and $\mathrm{Per}(A) \sim 4\pi R$. As $R \to \infty$, the annulus A will locally resemble a rectangle, so that we can use a similar transport method on it to ease the computations. We cut the annulus along its mean radius, and have each point travel a distance approximately equals to $\ell/2$ to leave A and join the inner (our outer) annulus adjacent to A. Therefore,

$$\mathcal{E}(A) \sim 4\pi R + \frac{\ell^2 m}{4} = 4\pi R + \frac{m^3}{16\pi^2 R^2} \tag{0.0.28}$$

optimising this energy in R yields that $R = m/(2\cdot 4^{1/3}\pi)$. We then recover the expansion of the rectangle, minus the constant term $2^{5/3}$:

$$\mathcal{E}(A) \sim \left(2^{1/3} + 2^{-2/3}\right)m \sim 1.89\, m.$$

Let us however point out that this expansion corresponds to the energy of an annulus of infinite radius. In practice, we expect that the curvature of an annulus of inner radius $R > 0$ contributes to its exterior transport energy as a higher order penalizing term. Additionally, notice that while we underestimate the energy of the annulus in (0.0.28), we overestimate the energy of the rectangle in (0.0.27). This is a consequence of our choice of exterior transport map: the optimal exterior transport map would actually send some mass across the two shorter sides of the rectangle, and not only across its longer sides. We also believe that smoothing the two shorter sides of the rectangle and turning them into two half disk could further lower, see the final data of Figure 6 for a possible representation of these sausage-like minimisers.

These considerations lead us to formulate the following conjecture:

Conjecture 0.0.13. There exists $m_0 > 0$ such that the minimisers of Problem (0.0.5) stated with

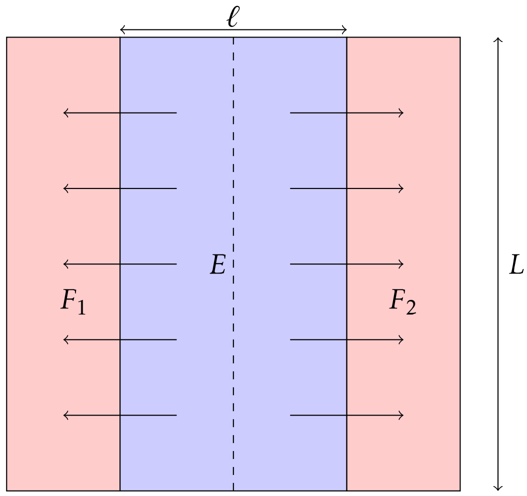

Figure 8: A possible exterior transport map for the rectangle E.

$\alpha = 1$ and $p = d = 2$ are balls for $m \leq m_0$ and stadium-like shapes (thin, elongated and with two axes of symmetry) for $m > m_0$.

Let us additionally mention that in the general case, the preferred geometry of the minimisers should also depend on the values of $p \geq 1$ and $d \geq 2$.

As a closing remark on this topic, let us stress the fact that the analysis conducted in [70] to obtain a Γ-convergence result relied on the geometrical properties of the 1-Wasserstein distance. More precisely, as in this case the transport cost is the Euclidean distance, it is possible to show that the optimal transport takes place along segments called transport rays (see e.g. [73, Section 3]) which cannot cross each other. This phenomenon allows for explicit computations in the case $p = 1$ but does not occur when $p > 1$, which heavily complicates the investigation.

Let us now point out some conjectures linked to the numerical computations conducted in Chapter 3. When we replaced problem in (0.0.1) by

$$\inf_{|E|=m} \left\{ \mathrm{Per}(E) + \mathcal{W}_2^2(E) : E \text{ radially symmetric and connected} \right\},$$

we excluded non-connected radial structures, i.e. the reunion of several disjoint annuli, from our study. Usually, to obtain information on the connectedness of minimisers, a first step is to establish that $m \mapsto e(m)/m$ is decreasing, where $e(m)$ is the value attained by the infimum in (0.0.1): this would indicate a preference in the problem for concentration of mass. Unfortunately, our simulations suggest that this is not the case and that $e(m)/m$ is increasing on a non-empty interval. Therefore, proving that radially symmetric minimisers are connected is still an open problem. It is also worth recalling that as of now, we do not believe that minimisers of (0.0.1) are radially symmetric for large values of m.

Furthermore, it should be noted that the computations of Chapter 3 on the radially symmetric case rely on several conjectures on the external transport functional for the annuli. Recall that given an annulus A_{r_m} of mass m and inner radius r_m and outer radius r_M the exterior transport is determined by the critical radius r_c separating the parts of A_{r_m} that are sent inwards and outwards. Notice that the largest possible interior ring for the exterior transport of A_{r_m} is the ball B_{r_m}, which yields the constraint $r_c \leq \sqrt{2}r_m$. Thus if we define $\hat{r}_M = \min(r_M, \sqrt{2}r_m)$, we have

$$\mathcal{W}_2^2(A_{r_m}) = \inf\{\tau(r_c) : r_c \in [r_m, \hat{r}_M]\},$$

where $\tau(r_c)$ is the cost of sending A_{r_m} to $F = A_{s_m,r_m} \cup A_{r_M,s_M}$, and the radii s_m and s_M are fixed by conservation of the mass. We have yet to theoretically confirm our numerical observation that τ is strictly convex on $[r_m, \hat{r}_M]$ and that $\tau'(r_m) < 0$. The simulations also suggested that $r_m \mapsto \mathcal{W}_2^2(A_{r_m})$ was smooth on $\mathbb{R}_+ \setminus r_m^1$, where r_m^1 is the supremum of the inner radii r_m such that the exterior transport of A_{r_m} completely fills B_{r_m}: this conjecture could also be investigated.

Lastly, it is worth mentioning that the generalised problem considered in Chapter 4 and defined in (0.0.6), where P and V not given by explicit formulas but are assumed to satisfy a list of properties, can be even further generalised. Indeed, we believe that many results obtained in Chapter 4 could be extended to the case of clusters of $\mathbb{R}^d$. Let $J \in \mathbf{N} \cup \{\infty\}$ be fixed. We say that $\mathbb{E} = \left(E_j\right)_{j=1}^J$ is a J-cluster in $\mathbb{R}^d$ if each E_j is a Borel set of $\mathbb{R}^d$ and $|E_{j_1} \cap E_{j_2}| = 0$ if $j_1 \neq j_2$. We then set $\mathbf{m}(\mathbb{E}) = (|E_j|)_{j=1}^J$ and

$$P(\mathbb{E}) = \frac{1}{2}\sum_{j=1}^{J} P(E_j) + \frac{1}{2}P\left(\bigcup_{j=1}^{J} E_j\right), \tag{0.0.29}$$

where P denotes a non-explicit perimeter functional. We assume that this perimeter term competes against a non-explicit perturbative functional of the form

$$V(\mathbb{E}) = V\left(E_1, E_2, \ldots, E_J\right),$$

so that the total energy of a cluster is given by

$$\mathcal{E}(\mathbb{E}) = P(\mathbb{E}) + V(\mathbb{E}).$$

Additionally, we say that $(\mathbb{E}^i)_{i\geq 1}$ is a generalised cluster in $\mathbb{R}^d$ if for any $i \geq 1$, $\mathbb{E}^i$ is a J-cluster in $\mathbb{R}^d$. We then define

$$\mathbf{m}\left((\mathbb{E}^i)_{i\geq 1}\right) = \left(\sum_{i\geq 1}|E_1^i|, \ldots, \sum_{i\geq 1}|E_J^i|\right),\ P\left((\mathbb{E}^i)_{i\geq 1}\right) = \sum_{i\geq 1} P(\mathbb{E}^i) \text{ and } V\left((\mathbb{E}^i)_{i\geq 1}\right) = \sum_{i\geq 1} V(\mathbb{E}^i).$$

The energy of a generalised cluster is

$$\widetilde{\mathcal{E}}\left((\mathbb{E}^i)_{i\geq 1}\right) = P\left((\mathbb{E}^i)_{i\geq 1}\right) + V\left((\mathbb{E}^i)_{i\geq 1}\right).$$

Let $\mathbf{m} = (m_1, \ldots, m_J)$ with $m_j > 0$ for any $1 \leq j \leq J$ be fixed. We consider the two following problems

$$\inf_{\mathbb{E}}\left\{\mathcal{E}(\mathbb{E}) : \mathbb{E}\ J-\text{cluster},\ \mathbf{m}(\mathbb{E}) = \mathbf{m}\right\} \tag{0.0.30}$$

and

$$\inf_{(\mathbb{E}^i)_i} \left\{ \widetilde{\mathcal{E}}\Big((\mathbb{E}^i)_{i\geq 1}\Big) : (\mathbb{E}^i)_{i\geq 1} \text{ generalised } J-\text{cluster, } \mathbf{m}\Big((\mathbb{E}^i)_{i\geq 1}\Big) = \mathbf{m} \right\}. \tag{0.0.31}$$

Refer to [74] for a study of a three-phase cluster where the energy functional is derived from Nakazawa and Ohta's density functional theory for block copolymers. See also [30] for an investigation of isoperimetric clusters for the fractional perimeter and for $V = 0$, where existence of minimisers is shown as well as partial regularity results.

In the very simple case of no interaction between phases in the perturbative term, where

$$V(\mathbb{E}) = \sum_{j=1}^{J} V(E_j),$$

Propositions 0.0.8 and 0.0.9 extend to finite clusters without much modification of the proofs. We believe that they may be extended to the case of infinite clusters as well. In the case $V = 0$, these two propositions were obtained for $J = \infty$ in [66].

When V cannot be decomposed as a sum of independent terms, our guess is that additional hypotheses are needed to establish Propositions 0.0.8 and 0.0.9. Additionally, establishing a version of Theorem 0.0.11 on density estimates for clusters or generalised clusters may be quite involved. One of the issues is the relaxation of the mass constraint in (0.0.30) or (0.0.31), even in the case where P and V admits a scaling: indeed, rescaling a given phase may cause it to overlap on the other phases. The classical strategy is thus to establish what is called a volume-fixing lemma: this approach was famously applied by Almgren in [3] to the case $V = 0$ to obtain existence and regularity of J-minimising clusters solving (0.0.30) (see also [60, Section 29]). Again, difficulties arise when a given phase E^j for $1 \leq j \leq J$ is totally surrounded by other phases: if one considers a local perturbation near the boundary of E^j to achieve an arbitrary (but suitably small) change in the volume of E^j, the effect of this perturbation on the energy of the surrounding phases has to be accounted for. In particular, we expect further complications when V is not comparable to the perimeter term, i.e. when it is not of the form

$$V(\mathbb{E}) = \sum_{i\neq j} V_{i,j}(E_i, E_j).$$

Chapter 1

Existence and stability results for an isoperimetric problem with a non-local interaction of Wasserstein type

Abstract. The aim of this chapter is to prove the existence of minimisers for a variational problem involving the minimisation under volume constraint of the sum of the perimeter and a non-local energy of Wasserstein type. This extends previous partial results to the full range of parameters. We also show that in the regime where the perimeter is dominant, the energy is uniquely minimised by balls.

Keywords and phrases. Optimal transport, isoperimetric problem, existence of minimisers.

2020 Mathematics Subject Classification. 49Q22, 49Q20, 49J35.

Outline of the current chapter

1.1 Introduction

In this chapter we consider a variational problem first proposed in [70] as a model describing the formation of bi-layers cellular membranes. Our first main result is the proof of the existence of minimisers in every space dimension and for every value of the parameters in the model.

This extends previous results obtained in [13, 77] to which we refer for further motivation of the problem. Our second main result is a proof of the minimality of the ball in the regime where the perimeter is dominant. To be more concrete, denoting by W_p the Wasserstein distance for $p \geq 1$ (see [76]) and identifying a set $E \subset \mathbb{R}^d$ with the restriction of the Lebesgue measure to E, we introduce the non-local energy

$$\mathcal{W}_p(E) = \inf_{|F \cap E|=0} W_p(E,F). \tag{1.1.1}$$

As already noticed in [13], this may be viewed as a projection problem for the Wasserstein distance (see [36]). We then consider for $\lambda, \alpha > 0$ the variational problem

$$\inf_{|E|=\omega_d} P(E) + \lambda \left[\mathcal{W}_p^p(E)\right]^\alpha, \tag{1.1.2}$$

where ω_d is the volume of the unit ball and $P(E)$ denotes the Caccioppoli perimeter of E, see [60]. Let us point out that probably the two most interesting cases are $\alpha = 1$ and $\alpha = \frac{1}{p}$. Our first main result is the following (we use the letter C to denote a generic constant whose value can change from line to line):

Theorem 1.1.1. *For every $d \geq 2$, $p \geq 1$, $\alpha > 0$ and $\lambda > 0$, problem* (1.1.2) *has minimisers. Moreover, there exists $C = C(d,p,\alpha) > 0$ such that if $E = \cup_{i=1}^I E^i$ is such a minimiser with E^i the connected components of E, then*

$$\sum_{i=1}^I \operatorname{diam}(E^i) \leq C(1+\lambda)^{\frac{(d-1)(1+p)}{1+\alpha p}} \quad \textit{and} \quad \inf_i \operatorname{diam}(E^i) \geq \frac{1}{C}(1+\lambda)^{-\frac{1+p}{1+\alpha p}}.$$

As a consequence $I \leq C(1+\lambda)^{\frac{d(1+p)}{1+\alpha p}}$ (in particular E has finitely many connected components).

Notice that we can actually say much more about the regularity of the minimisers, see Remark 1.3.6. This result was first obtained in the case $d = 2$ in [13] and then extended to the case $d \geq 3$ in [77] but under the assumption that λ is small together with some restrictions on α. The idea of the proof, which is by now well-established in the context of geometrical variational problems (see e.g. [49, 55, 43, 67]), is to follow a concentration-compactness type argument. We first show that thanks to the isoperimetric inequality, lack of compactness for minimising sequences can only come from splitting of the mass. This leads to the existence of so-called generalised minimisers (see Proposition 1.3.1). Then, we show following [13], that these generalised minimisers are actually Λ−minimisers of the perimeter (see [60]) and therefore have uniform density bounds. As a direct consequence, we obtain that they are made of a finite number of uniformly bounded connected components. At this point the proof of the existence is concluded as in [13] using the fact that the non-local energy $\mathcal{W}_p^p$ is additive for sets which are sufficiently far apart.

Our second main result is that if λ is small enough then (1.1.2) is uniquely minimised by balls.

Theorem 1.1.2. *For every $d \geq 2$, $p \geq 1$ and $\alpha > 0$, there exists $\lambda_0 > 0$ such that for every $\lambda \leq \lambda_0$, balls are the only minimisers of* (1.1.2).

Remark 1.1.3. Let us point out that if we considered the volume as the relevant parameter and replaced (1.1.2) by

$$\min_{|E|=m} P(E) + \left[\mathcal{W}_p^p(E)\right]^\alpha,$$

then by scaling (see [77]) we would obtain that balls are the unique minimisers for small m if $\alpha\left(1+\frac{p}{d}\right)+\frac{1}{d} > 1$ (which is essentially the case for which [77] obtained the existence of minimisers) while balls are the unique minimisers for large m if $\alpha\left(1+\frac{p}{d}\right)+\frac{1}{d} < 1$.

Again, this result is neither surprising by its statement nor by the strategy to prove it. Indeed, following the pioneering work of Cicalese and Leonardi which gave in [28] an alternative proof of the quantitative isoperimetric inequality, it has been understood that such stability results may be obtained by combining the regularity theory for $\Lambda-$minimisers of the perimeter together with a (usually delicate) Taylor expansion of the energy around the ball. This second part of the proof is often referred to as a Fuglede type argument, see [45]. Let us cite [54, 1, 41, 21, 63] as a few examples where this strategy has been carried out. The main difficulty here is that our non-local energy depends in a very implicit way on the competitor. Moreover, as opposed to [1, 63, 50], the underlying PDE is non-linear (namely the Monge-Ampère equation) making it very difficult to use standard tools from shape optimisation such as shape derivatives. We go around this difficulty by plugging in the dual formulation of optimal transport the Kantorovich potentials corresponding to the ball. This yields the estimate

Proposition 1.1.4. *There exists $C = C(d,p,\alpha) > 0$ such that for every set E such that $|E| = |B_1|$ and*

$$\partial E = \{(1+f(x))x \ : \ x \in \partial B_1\}$$

for some $f : \partial B_1 \mapsto \mathbb{R}$ with $\|f\|_\infty \leq 2^{1/d} - 1$,

$$\left[\mathcal{W}_p^p(B_1)\right]^\alpha - \left[\mathcal{W}_p^p(E)\right]^\alpha \leq C \int_{\partial B_1} f^2.$$

Related results in the literature. In the footsteps of [54] there has been an intense research activity around isoperimetric problems with non-local interactions. Probably the simplest and most studied one is the Gamow liquid-drop model where the non-local part of the energy is given by a Riesz type interaction energy. For this model, it has been shown that generalised minimisers exist and are balls for small volume (see [54, 41, 21, 67] and the review paper [26]). However, as opposed to our setting, it has been proven for the liquid-drop model that under some restrictions on the parameters, classical minimisers do not exist for large volumes (see [54, 44]). This is due to the long-range nature of the interactions induced by the Riesz kernel (in comparison with Proposition 1.2.2). Indeed, for compactly supported kernels it is shown in [72] that minimisers exist for all volumes (see also [68]).

1.2 The non-local energy

In this section we gather a few useful results about the energy $\mathcal{W}_p$ defined in (1.1.1). Most of these results were obtained in the case of bounded sets in [13, 77] but often with quite different proofs. We start with the well-posedness of (1.1.1) (see [77, Lemma 4.3] for comparison).

Proposition 1.2.1. *There exists $C = C(d,p) > 0$ such that for every set $E \subset \mathbb{R}^d$,*

$$\mathcal{W}_p(E) \le C|E|^{\frac{1}{p}+\frac{1}{d}}. \tag{1.2.1}$$

Moreover, if $|E| < \infty$, the minimisation problem (1.1.1) *is attained by a unique minimiser F and if π is an optimal transport plan*[1]*for $W_p(E,F)$, we have the estimate*

$$|x-y| \le C|E|^{\frac{1}{d}} \qquad \text{for } \pi - a.e. \ (x,y). \tag{1.2.2}$$

Proof. We may assume without loss of generality that $|E| < \infty$ otherwise there is nothing to prove. By scaling we can further assume that $|E| = 1$. In order to prove (1.2.1), we will construct a partition $(E_i)_{i\ge1}$ of E such that each E_i can be transported with a well-controlled cost. To this aim, consider a partition of $\mathbb{R}^d$ into cubes $(Q_i)_{i\ge1}$ of sidelength $\ell = 2^{1/d}$. Since $|E| = 1$, if we define $E_i = E \cap Q_i$ we have $|E_i| \le |Q_i|/2$ for every i. Therefore we can find a set $F_i \subset Q_i$ such that $|E \cap F_i| = 0$ and $|F_i| = |E_i|$. If T_i is the optimal transport map (in fact any transport map would work) from E_i to F_i we have

$$\sup_{E_i} |T_i - x| \le C.$$

Finally, consider $F = \cup_i F_i$ and T the map whose restriction to each Q_i is T_i. The map T is a transport map from E to F and

$$\mathcal{W}_p(E) \le W_p(E,F) \le \left(\sum_{i\ge1} \int_{E_i} |T_i - x|^p \right)^{\frac{1}{p}} \le C.$$

This proves (1.2.1).

Existence and uniqueness of a minimiser F for (1.1.1) follows from [36, Proposition 5.2] (which is stated for $p = 2$, but generalises easily to any $p \ge 1$) with $f = \chi_{E^c}$ and $\Omega = \mathbb{R}^d$. Moreover, as a consequence of [36, Proposition 5.2] we have

$$\widetilde{\mathcal{W}_p}(E) = \inf_{\mu} \Big\{ W_p(E,\mu) \ : \mu \text{ absolutely continuous, and } 0 \le \mu \le \chi_{E^c} \Big\} = \mathcal{W}_p(E). \tag{1.2.3}$$

and χ_F is also the unique minimiser of $\widetilde{\mathcal{W}_p}(E)$. Let π be an optimal transport plan for $W_p(E,F)$ and let us show (1.2.2). For this we adapt the proof of [77, Lemma 4.3] to the case of plans instead of maps. Letting

$$\Gamma = \{(x,y) \in \operatorname{spt}\pi \ : \ |x-y| \ge C\}$$

[1] for $p > 1$ we know from [76, Theorem 2.44] that π is unique and is induced by a map but for $p = 1$, since we do not assume finite moments for E it does not follow from [76, Theorem 2.50].

let us show that for C large enough, $\pi(\Gamma)=0$. Assume that it is not the case and let R be such that $|B_R|=3$. Then, there exists $x\in\mathbb{R}^d$ such that $m=\pi(\Gamma\cap(B_R(x)\times\mathbb{R}^d))>0$. Without loss of generality we may assume that $x=0$. Let $\pi_{\text{bad}}=\chi_{\Gamma\cap(B_R\times\mathbb{R}^d)}\pi$. Since $|B_R|-|E|-|F|\ge 1\ge m>0$, there exists $\widetilde{\mu}\le\chi_{B_R}(1-\chi_E-\chi_F)$ with $\widetilde{\mu}(\mathbb{R}^d)=\pi_{\text{bad}}(\mathbb{R}^d\times\mathbb{R}^d)$. Finally let θ be the first marginal of π_{bad} and set

$$\tilde{\pi}=\pi-\pi_{\text{bad}}+\frac{1}{m}\theta\otimes\tilde{\mu}.$$

It is readily checked that the first marginal of $\tilde{\pi}$ is χ_E and that its second marginal μ satisfies $\mu\le\chi_{E^c}$. We thus have on the one hand by definition of Γ

$$W_p^p(E,F)\ge\int_{(\Gamma\cap(B_R\times\mathbb{R}^d))^c}|x-y|^p\,d\pi+mC^p.$$

On the other hand, by minimality of F for $\widetilde{W_p}(E)$,

$$W_p^p(E,F)\le W_p^p(E,\mu)\le\int_{(\Gamma\cap(B_R\times\mathbb{R}^d))^c}|x-y|^p\,d\pi+m2^pR^p.$$

This implies $C<2R$ and concludes the proof that $\pi(\Gamma)=0$ if C is large enough.

□

We now turn to the super-additivity and lower semi-continuity of $\mathcal{W}_p$ (compare to [77, Lemmma 4.4]).

Proposition 1.2.2. *We have:*

(i) If E and E' are disjoint sets then

$$\mathcal{W}_p^p(E\cup E')\ge\mathcal{W}_p^p(E)+\mathcal{W}_p^p(E'). \tag{1.2.4}$$

As a consequence, if $E\subset E'$ then $\mathcal{W}_p(E)\le\mathcal{W}_p(E')$;

(ii) There exists $C>0$ such that if

$$d(E,E')\ge C\max(|E|^{\frac{1}{d}},|E'|^{\frac{1}{d}}),$$

then

$$\mathcal{W}_p^p(E\cup E')=\mathcal{W}_p^p(E)+\mathcal{W}_p^p(E');$$

(iii) If E_n converges in L^1_{loc} to E then

$$\mathcal{W}_p(E)\le\liminf_n\mathcal{W}_p(E_n). \tag{1.2.5}$$

Proof. To prove (i), let F be the $\mathcal{W}_p$-minimiser for $E\cup E'$, and π be an optimal transport plan from $E\cup E'$ to F. Let μ_E be the second marginal of $\chi_{E\times\mathbb{R}^d}\pi$ and $\mu_{E'}$ be the second marginal of $\chi_{E'\times\mathbb{R}^d}\pi$. By definition μ_E is $\widetilde{\mathcal{W}_p}$-admissible (recall (1.2.3)) for E. Moreover, $\chi_{E\times\mathbb{R}^d}\pi$ is an optimal transport plan between E and μ_E. The corresponding statement also holds for E' instead of E. Therefore, appealing once more to (1.2.3),

$$\mathcal{W}_p^p(E)+\mathcal{W}_p^p(E') \le W_p^p(E,\mu_E)+W_p^p(E',\mu_{E'}) = \int_{(E\cup E')\times\mathbb{R}^d} |x-y|^p\, d\pi = \mathcal{W}_p^p(E\cup E').$$

Property (ii) is a direct consequence of (1.2.2). Indeed, if F and F' are the $\mathcal{W}_p$-minimisers for E and E', by (1.2.2), $|(E\cup F)\cap(E'\cup F')| = 0$ so that $F\cup F'$ is admissible for $E\cup E'$ which gives $\mathcal{W}_p^p(E\cup E') \le \mathcal{W}_p^p(E)+\mathcal{W}_p^p(E')$.

We finally prove (iii), and consider a sequence $(E_n)_{n\ge1}$ that is $L^1_{\rm loc}$ converging to E. For every $R>0$ set $E_{R,n} = E_n\cap B_R$ so that $E_{R,n}$ converges in L^1 to $E_R = E\cap B_R$. Using the continuity of W_p with respect to weak convergence, (1.2.2) and (1.2.3) it is not hard to check that $\mathcal{W}_p$ is lower semi-continuous with respect to L^1 convergence (in Lemma 1.2.4 below we will actually prove a much stronger result). Since by (1.2.4), $\mathcal{W}_p(E_{R,n}) \le \mathcal{W}_p(E_n)$ we have

$$\mathcal{W}_p(E_R) \le \liminf_{n\to\infty} \mathcal{W}_p(E_{R,n}) \le \liminf_{n\to\infty} \mathcal{W}_p(E_n).$$

Since E_R converges in L^1 to E as $R\to\infty$, using once more the lower semi-continuity of $\mathcal{W}_p$ for this convergence we conclude the proof. □

Remark 1.2.3. Let us point out that for every set E with $|E|<\infty$, since $E\cap B_R$ converges in L^1 to E as $R\to\infty$, we have by lower semi-continuity and $\mathcal{W}_p(E\cap B_R) \le \mathcal{W}_p(E)$ that $\lim_{R\to\infty}\mathcal{W}_p(E\cap B_R) = \mathcal{W}_p(E)$.

We then prove that $\mathcal{W}_p^p$ is Lipschitz continuous with respect to L^1 convergence. This is a crucial ingredient in order to obtain the $\Lambda-$minimality property of generalised minimisers. See [13, Lemma 4.5] or [77, Theorem 5.4] for comparison.

Lemma 1.2.4. *There exists a constant $C = C(d,p) > 0$ such that for any Lebesgue sets E, E'*

$$|\mathcal{W}_p^p(E)-\mathcal{W}_p^p(E')| \le C(|E|^{\frac{p}{d}}+|E'|^{\frac{p}{d}})|E\Delta E'|. \tag{1.2.6}$$

Moreover, there exists $C = C(d,p,\alpha) > 0$ such that for every family of sets $(E^i)_{i\ge1}$ and $((E')^i)_{i\ge1}$,

$$\left|\left[\sum_i \mathcal{W}_p^p(E^i)\right]^\alpha - \left[\sum_i \mathcal{W}_p^p((E')^i)\right]^\alpha\right| \le CM_{\mathcal{W}}\left|\sum_i \mathcal{W}_p^p(E^i) - \sum_i \mathcal{W}_p^p((E')^i)\right|. \tag{1.2.7}$$

where

$$M_{\mathcal{W}} = \max\left(\left(\sum_i \mathcal{W}_p^p(E^i)\right)^{\alpha-1}, \left(\sum_i \mathcal{W}_p^p((E')^i)\right)^{\alpha-1}\right).$$

Proof. We start with the proof of (1.2.6). Thanks to Remark 1.2.3 we may assume that E and E' are bounded sets. By symmetry of the roles of E and E', it is sufficient to show that

$$\mathcal{W}_p^p(E')-\mathcal{W}_p^p(E) \le C|E'|^{\frac{p}{d}}|E'\setminus E|. \tag{1.2.8}$$

By scaling we may assume that $|E'| = 1$. Let F with $|E \cap F| = 0$ be such that $\mathcal{W}^p(E) = W_p^p(E, F)$. Let T_E be an optimal transport map from E to F (which exists by [76, Theorem 2.44 & Theorem 2.50] since E and F are bounded), and denote $T_F = T_E^{-1}$ which is an optimal transport map from F to E. We define $\widetilde{F} = F \setminus E'$, set $F^- = T_E(E') \cap \widetilde{F}$ and decompose E' as

$$E' = (E' \cap T_F(\widetilde{F})) \cup (E' \setminus T_F(\widetilde{F}))$$

so that $T_E(E' \cap T_F(\widetilde{F})) = F^-$. Our goal is now to construct a set $F^+ \subset (E' \cup F^-)^c$ and a map T^+ from $E' \setminus T_F(\widetilde{F})$ to F^+ with controlled transport cost. We proceed as in the proof of (1.2.1) and consider a partition of $\mathbb{R}^d$ into cubes $(Q_i)_{i \geq 1}$ of sidelength $\ell = 3^{1/d}$. We thus have for every $i \geq 1$,

$$|Q_i| - |E' \cap Q_i| - |F^- \cap Q_i| \geq |E' \setminus T_F(\widetilde{F})|.$$

Therefore, for any $i \geq 1$, there exists $F_i \subset Q_i \cap (E' \cup F^-)^c$ such that $|F_i| = |(E' \setminus T_F(\widetilde{F})) \cap Q_i|$ and an optimal transport map T_i from $(E' \setminus T_F(\widetilde{F})) \cap Q_i$ to F_i. We set $F^+ = \cup_{i \geq 1} F_i$ and define T^+ from $E' \setminus T_F(\widetilde{F})$ to F^+ by setting its restriction on any Q_i to be T_i. By construction,

$$\sup_{E' \setminus T_F(\widetilde{F})} |T^+ - x| \leq C.$$

We can now set $T = T_E$ on $E' \cap T_F(\widetilde{F})$ and $T = T^+$ on $E' \setminus T_F(\widetilde{F})$ and obtain

$$\begin{aligned}
\mathcal{W}_p^p(E') - \mathcal{W}_p^p(E) &\leq \int_{E' \cap T_F(\widetilde{F})} |T_E - x|^p + \int_{E' \setminus T_F(\widetilde{F})} |T^+ - x|^p - \mathcal{W}_p^p(E) \\
&\leq \int_{E' \setminus T_F(\widetilde{F})} |T^+ - x|^p \\
&\leq C|E' \setminus T_F(\widetilde{F})|.
\end{aligned}$$

We finally observe that

$$\begin{aligned}
|E' \setminus T_F(\widetilde{F})| &\leq |E' \setminus E| + |E \setminus T_F(\widetilde{F})| \\
&= |E' \setminus E| + |E| - |F \setminus E'| \\
&\leq |E' \setminus E| + |E' \cap F| \\
&\leq 2|E' \setminus E|.
\end{aligned}$$

This proves (1.2.8).

We now turn to (1.2.7). For this we simply use the fact that there exists $C = C(\alpha) > 0$ such that for every $a > 0$ and $b > 0$

$$|a^\alpha - b^\alpha| \leq C \max(a^{\alpha-1}, b^{\alpha-1})|a - b|.$$

□

From (1.2.7), we see that in order to obtain a good Lipschitz bound for $E \mapsto \left[\mathcal{W}_p^p(E)\right]^\alpha$ when $\alpha < 1$ (recall that we are particularly interested in the case $\alpha = \frac{1}{p} \leq 1$), we will need a control from below on the transport term. This is obtained through the following interpolation result between the perimeter and $\mathcal{W}_p$. This result may be seen as a particular case of the more general estimate obtained in [23]. However since in this setting the proof is very elementary we decided to keep it.

Proposition 1.2.5. *There exists a constant $C = C(d) > 0$ such that for every family of sets $(E^i)_{i\geq 1}$ we have*

$$\left(\sum_i \mathcal{W}_p^p(E^i)\right)^{\frac{1}{p}} \left(\sum_i P(E^i)\right) \geq C\left(\sum_i |E^i|\right)^{1+\frac{1}{p}} \tag{1.2.9}$$

Proof. Since by Hölder inequality, $\mathcal{W}_1(E) \leq \mathcal{W}_p(E)|E|^{1-\frac{1}{p}}$, using Hölder inequality once more for the sum we see that it is enough to prove (1.2.9) for $p = 1$. Let E be a set of finite perimeter and volume. We will first show that there exists $C = C(d) > 0$ such that

$$\mathcal{W}_1(E) \geq Cr(|E| - CrP(E)). \tag{1.2.10}$$

Take η a standard mollifier, rescale it by setting $\eta_r(x) = r^{-d}\eta(x/r)$ and consider $\phi_r = \eta_r * \chi_E$. Using Young's inequality, we have

$$|\nabla\phi_r|_\infty \leq |\chi_E|_\infty |\nabla\eta_r|_1 \leq Cr^{-1}.$$

Therefore, by Kantorovich duality for W_1, we obtain using $F \subset E^c$,

$$\mathcal{W}_1(E) = W_1(E,F) = \sup_{|\nabla\psi|\leq 1} \int \psi(\chi_E - \chi_F) \geq C\int r\phi_r(\chi_E - \chi_F) \geq Cr\int \phi_r(\chi_E - \chi_{E^c}).$$

Since $\int \phi_r = |E|$,

$$\int \phi_r\chi_E = |E| - \int \phi_r(1-\chi_E) = |E| - \int \phi_r\chi_{E^c},$$

so that

$$\mathcal{W}_1(E) \geq Cr\left(|E| - 2\int \phi_r\chi_{E^c}\right).$$

We now re-express the term $\int \phi_r\chi_{E^c}$ in order to bound it by the perimeter of E :

$$\begin{aligned}
\int \phi_r\chi_{E^c} &= \iint \eta_r(y-x)\chi_E(x)\chi_{E^c}(y)\,dxdy \\
&= \frac{1}{2}\iint \eta_r(x-y)|\chi_E(x) - \chi_E(y)|\,dxdy \\
&= \frac{1}{2}\iint \eta_r(z)|\chi_E(x) - \chi_E(x+z)|\,dxdz \\
&\leq CP(E)\int |z|\eta_r(z)\,dz \\
&\leq CrP(E).
\end{aligned}$$

This proves (1.2.10).

Let now $(E^i)_{i\geq 1}$ be a family of sets and let us show (1.2.9). We may assume that $\sum_i P(E^i) + |E^i| <$

∞ since otherwise there is nothing to prove. Summing (1.2.10) over $i \geq 1$ yields

$$\sum_{i\geq 1} \mathcal{W}_1(E^i) \geq Cr\left(\sum_{i\geq 1} |E^i| - Cr\sum_{i\geq 1} P(E^i)\right).$$

We conclude the proof by taking

$$r = \varepsilon \frac{\sum_{i\geq 1} |E^i|}{\sum_{i\geq 1} P(E^i)},$$

with ε chosen small enough so that $\varepsilon C \leq 1/2$. □

1.3 Existence of minimisers

In this section we prove Theorem 1.1.1. As already explained in the introduction, we will first prove the existence of generalised minimisers and then prove that they are Λ–minimisers of the perimeter to obtain a bound on their diameter which readily implies the existence of minimisers in a classical sense.

1.3.1 Existence of generalised minimisers

We start with some notation. For a set E we define the energy (we keep the dependence in p and α implicit)

$$\mathcal{E}_\lambda(E) = P(E) + \lambda\left[\mathcal{W}_p^p(E)\right]^\alpha.$$

We call a family $\widetilde{E} = (E^i)_{i\geq 1}$ a generalised set and define the generalised energy as

$$\widetilde{\mathcal{E}}_\lambda(\widetilde{E}) = \sum_i P(E^i) + \lambda\left[\sum_i \mathcal{W}_p^p(E^i)\right]^\alpha. \tag{1.3.1}$$

We say that $\widetilde{E}$ is a generalised minimiser if $\sum_i |E^i| = \omega_d$ and

$$\widetilde{\mathcal{E}}_\lambda(\widetilde{E}) = \inf\left\{\widetilde{\mathcal{E}}_\lambda(\widetilde{E}') \ : \ \sum_i |(E')^i| = \omega_d\right\}.$$

Proposition 1.3.1. *For every $d \geq 2$, $p \geq 1$, $\alpha > 0$ and $\lambda > 0$, there exists generalised minimisers and*

$$\inf\{\mathcal{E}_\lambda(E) \ : |E| = \omega_d\} = \inf\left\{\widetilde{\mathcal{E}}_\lambda(\widetilde{E}) \ : \ \sum_i |E^i| = \omega_d\right\}. \tag{1.3.2}$$

Proof. We start by pointing out that using Proposition 1.2.2 and a simple rescaling argument (see for instance [77]), it is not hard to modify a generalised minimising sequence into a classical minimising sequence so that (1.3.2) holds.

By (1.3.2), in order to prove the existence of a generalised minimiser we can consider a classical minimising sequence $(E_n)_{n\ge 1}$ such that

$$\lim_{n\to\infty} \mathcal{E}_\lambda(E_n) = \inf\left\{\widetilde{\mathcal{E}}_\lambda(\widetilde{E}) \ : \ \sum_i |E^i| = \omega_d\right\}.$$

We now follow relatively closely the proof of [49, Theorem 4.9]. We first notice that using the unit ball B_1 as competitor, we may assume that $\mathcal{E}_\lambda(E_n) \le \mathcal{E}_\lambda(B_1) \le C(1+\lambda)$. For every $n \ge 1$, let $Q_{n,i}$ be a partition of $\mathbb{R}^d$ into disjoint cubes of side-length 2 and such that

$$m_{n,i} = |E_n \cap Q_{n,i}|$$

is a decreasing sequence in i. By the relative isoperimetric inequality we have

$$\sum_i m_{n,i}^{\frac{d-1}{d}} \le C \sum_i P(E_n, Q_{n,i}) = CP(E_n) \le C(1+\lambda).$$

Since $\sum_i m_{n,i} = \omega_d$, we have for every $I \ge 1$, and every $i \ge I$, $m_{n,i} \le m_{n,I} \le \omega_d/I$ and thus

$$\sum_{i\ge I} m_{n,i} = \sum_{i\ge I} m_{n,i}^{\frac{d-1}{d}} m_{n,i}^{\frac{1}{d}} \le C I^{-\frac{1}{d}} \sum_{i\ge I} m_{n,i}^{\frac{d-1}{d}} \le C(1+\lambda) I^{-\frac{1}{d}}.$$

This proves uniform tightness of $m_{n,i}$ and thus up to extraction we may assume that for every i, $m_{n,i} \to m_i$ with $\sum_i m_i = \omega_d$. Let now $z_{n,i} \in Q_{n,i}$. Up to a further extraction we may assume that for every i, j, $|z_{n,i} - z_{n,j}| \to c_{ij} \in [0,\infty]$ and $E_n - z_{n,i} \to E^i$ in $L^1_{loc}(\mathbb{R}^d)$. We now introduce an equivalence class by saying that $i \sim j$ if $c_{ij} < \infty$ and denote by $[i]$ the equivalence class of i. Notice that if $i \sim j$, E^i and E^j coincide up to a translation. For every equivalence class $[i]$ let $m_{[i]} = \sum_{j\in[i]} m_j$ so that

$$\sum_{[i]} m_{[i]} = \sum_i m_i = \omega_d.$$

By the L^1_{loc} convergence of $E_n - z_{n,i}$ to E^i and the definition of the equivalence relation, we have for every $j \in [i]$, $|E^j| = m_{[i]}$. Up to a relabeling we may now assume that there is a unique element E^i in each equivalence class. We have thus constructed a generalised set $\widetilde{E} = (E^i)_{i\ge 1}$ such that $\sum_i |E^i| = \omega_d$. We are left with the proof of

$$\widetilde{\mathcal{E}}_\lambda(\widetilde{E}) \le \liminf_{n\to\infty} \mathcal{E}_\lambda(E_n) = \inf\left\{\widetilde{\mathcal{E}}_\lambda(\widetilde{E}) \ : \ \sum_i |E^i| = \omega_d\right\}. \tag{1.3.3}$$

To this aim let $I \in \mathbf{N}$. And let $z_{n,1}, \cdots, z_{n,I}$ be as before such that $E_n - z_{n,i}$ converges to E^i and $|z_{n,i} - z_{n,j}| \to \infty$ as $n \to \infty$ if $i \ne j$. For every $R > 0$, if n is large enough, $\min_{i\ne j} |z_{n,i} - z_{n,j}| \ge 4R$. Therefore, the co-area formula yields

$$|E_n| \ge \sum_{i=1}^I \left|E_n \cap \Big(B(z_{n,i}, 2R) \setminus B(z_{n,i}, R)\Big)\right| = \int_R^{2R} \sum_{i=1}^I \mathcal{H}^{d-1}(E_n \cap \partial B(z_{n,i}, t))\, dt.$$

By the mean value theorem, we can thus find $R_n \in (R, 2R)$ such that

$$\sum_{i=1}^{I} \mathcal{H}^{d-1}(\partial B_{R_n}(z_{n,i}) \cap E_n) \le \frac{C}{R}.$$

We now define $E^{i,R_n} = (B_{R_n}(z_{n,i}) \cap E_n) - z_{n,i}$ so that on the one hand,

$$\sum_{i=1}^{I} P(E^{i,R_n}) \le P(E_n) + \frac{C}{R} \tag{1.3.4}$$

and on the other hand by (1.2.4),

$$\sum_{i=1}^{I} \mathcal{W}_p^p(E^{i,R_n}) \le \mathcal{W}_p^p\left(\cup_{i=1}^{I} B_{R_n}(z_{n,i}) \cap E_n\right) \le \mathcal{W}_p^p(E_n).$$

From the bound (1.3.4), we conclude that up to extraction, E^{i,R_n} converges in L^1 to a set $E^{i,R}$ as $n \to \infty$. Moreover, from the L^1_{loc} convergence of $E_n - z_{n,i}$ to E^i it is not hard to see that also $E^{i,R}$ converges to E^i in L^1 as $R \to \infty$. We thus conclude that by lower semi-continuity of the perimeter and (1.2.5) that

$$\sum_{i=1}^{I} P(E^{i,R}) + \lambda\left(\sum_{i=1}^{I} \mathcal{W}_p^p(E^{i,R})\right)^{\alpha} \le \liminf_{n\to\infty}\left(P(E_n) + \lambda\left[\mathcal{W}_p^p(E_n)\right]^{\alpha}\right) + \frac{C}{R}.$$

Letting then $R \to \infty$ and finally $I \to \infty$ we conclude the proof of (1.3.3). □

Before proceeding further let us study the scaling of the energy.

Proposition 1.3.2. *For every fixed $d \ge 2$, $p \ge 1$ and $\alpha > 0$, there exists $C = C(d, p, \alpha) > 0$ such that for every $\lambda > 0$,*

$$\frac{1}{C}(1+\lambda)^{\frac{1}{1+\alpha p}} \le \inf_{|E|=\omega_d} \mathcal{E}_\lambda(E) \le C(1+\lambda)^{\frac{1}{1+\alpha p}}. \tag{1.3.5}$$

Moreover, if $\widetilde{E} = (E^i)_{i\ge 1}$ is a generalised minimiser, then

$$\frac{1}{C}(1+\lambda)^{\frac{1}{1+\alpha p}} \le \sum_i P(E^i) \le C(1+\lambda)^{\frac{1}{1+\alpha p}} \tag{1.3.6}$$

and

$$\frac{1}{C}(1+\lambda)^{-\frac{p}{1+\alpha p}} \le \sum_i \mathcal{W}_p^p(E^i) \le C(1+\lambda)^{-\frac{p}{1+\alpha p}}. \tag{1.3.7}$$

Proof. Let us first consider the case $\lambda \le 1$. Using the ball B_1 as competitor and the isoperimetric inequality we have for every generalised minimiser $\widetilde{E}$,

$$\begin{aligned} P(B_1) + \lambda\left[\sum_i \mathcal{W}_p^p(E^i)\right]^{\alpha} &\le \sum_i P(E^i) + \lambda\left[\sum_i \mathcal{W}_p^p(E^i)\right]^{\alpha} \\ &\le P(B_1) + \lambda\left[\mathcal{W}_p^p(B_1)\right]^{\alpha}. \end{aligned}$$

From this combined with the isoperimetric inequality we obtain (1.3.5) and (1.3.6) together with $\sum_i \mathcal{W}_p^p(E^i) \leq \mathcal{W}_p^p(B_1)$. To obtain the first inequality in (1.3.7) we combine (1.2.9) with $\sum_i P(E^i) \leq C$.

Let now $\lambda \geq 1$. We consider the competitor made of N balls E^i of radius r so that the constraint $\sum_i |E^i| = \omega_d$ translates into $Nr^d = 1$. The energy of such a competitor is such that

$$\widetilde{\mathcal{E}}_\lambda(\widetilde{E}) \leq C\left(r^{-1} + \lambda r^{p\alpha}\right).$$

minimising in r by choosing $r = \lambda^{-\frac{1}{1+\alpha p}}$ (which is admissible since the corresponding N is large) gives the upper bounds in (1.3.5), (1.3.6) and (1.3.7). Using (1.2.9) we see that the upper bound in (1.3.6) gives the lower bound in (1.3.7) and vice-versa. These lower bounds then also imply the lower bound in (1.3.5). □

1.3.2 Quasi-minimality properties of generalised minimisers

As in many similar variational problems, in order to prove a quasi-minimality property, it will be convenient to relax the volume constraint. To this aim, for $\Lambda > 0$ and $\widetilde{E}$ a generalised set we introduce the penalised energy

$$\widetilde{\mathcal{E}}_{\lambda,\Lambda}(\widetilde{E}) = \sum_i P(E^i) + \lambda\left[\sum_i \mathcal{W}_p^p(E^i)\right]^\alpha + \Lambda\left|\sum_i |E_i| - \omega_d\right|.$$

We start by proving that if Λ is large enough, then every generalised minimiser is also an unconstrained minimiser of $\widetilde{\mathcal{E}}_{\lambda,\Lambda}$.

Proposition 1.3.3. *There exists $C = C(d,p,\alpha) > 0$ such that for every $\lambda > 0$, if $\Lambda \geq C(1+\lambda)^{\frac{1}{1+\alpha p}}$ then every generalised minimiser of* (1.3.1) *is also a minimiser of $\widetilde{\mathcal{E}}_{\lambda,\Lambda}$.*

Proof. Let C_0 to be fixed below and assume that $\Lambda \geq C_0(1+\lambda)^{\frac{1}{1+\alpha p}}$. By (1.3.5), if there exists $\widetilde{E}$ such that

$$\widetilde{\mathcal{E}}_{\lambda,\Lambda}(\widetilde{E}) < \inf\left\{\widetilde{\mathcal{E}}_\lambda(\widetilde{E}') \ : \ \sum_i |(E')^i| = \omega_d\right\}$$

we must have

$$\Lambda\left|\sum_i |E^i| - \omega_d\right| \leq C(1+\lambda)^{\frac{1}{1+\alpha p}}. \tag{1.3.8}$$

and $\sum_i |E^i| \neq \omega_d$. Let

$$t = \omega_d^{\frac{1}{d}}\left(\sum_i |E^i|\right)^{-\frac{1}{d}}$$

so that $t\widetilde{E} = (tE^i)_{i\geq 1}$ satisfies $\sum_i |tE^i| = \omega_d$. From (1.3.8), we see that $t = 1+\varepsilon$ with $|\varepsilon| \leq C\Lambda^{-1}(1+\lambda)^{\frac{1}{1+\alpha p}}$. By hypothesis we have

$$\widetilde{\mathcal{E}}_{\lambda,\Lambda}(\widetilde{E}) < \widetilde{\mathcal{E}}_\lambda(t\widetilde{E}) = (1+\varepsilon)^{d-1}\sum_i P(E^i) + \lambda(1+\varepsilon)^{(d+p)\alpha}\left[\sum_i \mathcal{W}_p^p(E^i)\right]^\alpha.$$

By Taylor expansion we have for $\Lambda \geq C(1+\lambda)^{\frac{1}{1+\alpha p}}$,

$$\Lambda\varepsilon < C\varepsilon\left(\sum_i P(E^i) + \lambda\left[\sum_i \mathcal{W}_p^p(E^i)\right]^\alpha\right) \leq C\varepsilon(1+\lambda)^{\frac{1}{1+\alpha p}}.$$

This gives the bound $\Lambda \leq C(1+\lambda)^{\frac{1}{1+\alpha p}}$ which yields the conclusion provided $C_0 > C$. □

Combining Lemma 1.2.4 together with Proposition 1.3.3 we may now prove that generalised minimisers are Λ–minimisers of the perimeter. We point out that a related quasi-minimality property was derived in [13, Theorem 4.6].

Proposition 1.3.4. *There exists $C = C(d,p,\alpha) > 0$ such that if $\Lambda \geq C(1+\lambda)^{\frac{1+p}{1+\alpha p}}$, every generalised minimiser $\widetilde{E} = (E^i)_{i\geq 1}$ of $\widetilde{\mathcal{E}}_\lambda$ is a Λ–minimiser of the perimeter in the sense that for every $i \geq 1$ and every set $E \subset \mathbb{R}^d$,*

$$P(E^i) \leq P(E) + \Lambda|E^i \Delta E|. \tag{1.3.9}$$

Proof. Let $\Lambda_0 = C(1+\lambda)^{\frac{1}{1+\alpha p}}$ be such that Proposition 1.3.3 applies and let $\widetilde{E} = (E^i)_{i\geq 1}$ be a generalised minimiser of $\widetilde{\mathcal{E}}_\lambda$. Without loss of generality, let us prove (1.3.9) for E^1. Using as competitor $E \times (E^i)_{i\geq 2}$ for $\widetilde{\mathcal{E}}_{\lambda,\Lambda_0}$ we find after simplification that

$$P(E^1) \leq P(E) + \lambda\left(\left[\mathcal{W}_p^p(E) + \sum_{i\geq 2}\mathcal{W}_p^p(E^i)\right]^\alpha - \left[\sum_i \mathcal{W}_p^p(E^i)\right]^\alpha\right) + \Lambda_0|E^1\Delta E|. \tag{1.3.10}$$

Notice that we can now assume that

$$\mathcal{W}_p^p(E) \geq \mathcal{W}_p^p(E^1) \tag{1.3.11}$$

since otherwise we can already conclude that (1.3.9) holds. Moreover, (1.3.6) implies in particular that $P(E^1) \leq C(1+\lambda)^{\frac{1}{1+\alpha p}}$ so that we can assume that

$$|E^1\Delta E| \leq C\Lambda^{-1}(1+\lambda)^{\frac{1}{1+\alpha p}} \leq C(1+\lambda)^{-\frac{p}{1+\alpha p}},$$

which in particular yields $|E^1| \leq C$. From (1.2.6), this implies that we can work under the assumption

$$\mathcal{W}_p^p(E) \leq \mathcal{W}_p^p(E^1) + C(1+\lambda)^{-\frac{p}{1+\alpha p}} \overset{(1.3.7)}{\leq} C(1+\lambda)^{-\frac{p}{1+\alpha p}}. \tag{1.3.12}$$

Combining (1.3.10), (1.2.7) and (1.2.6) we find

$$
\begin{aligned}
P(E^1) &\le P(E) + \lambda \max\left(\left(\sum_i \mathcal{W}_p^p(E^i)\right)^{\alpha-1}, \left(\mathcal{W}_p^p(E) + \sum_{i\ge 2} \mathcal{W}_p^p(E^i)\right)^{\alpha-1}\right)|E^1\Delta E| \\
&\quad + \Lambda_0|E^1\Delta E| \\
&\overset{(1.3.11)\&(1.3.12)\&(1.3.7)}{\le} P(E) + C\left(\lambda(1+\lambda)^{-\frac{p(\alpha-1)}{1+\alpha p}} + (1+\lambda)^{\frac{1}{1+\alpha p}}\right)|E^1\Delta E| \\
&\le P(E) + C(1+\lambda)^{\frac{1+p}{1+\alpha p}}|E^1\Delta E|.
\end{aligned}
$$

This proves (1.3.9). □

As a direct corollary we obtain uniform density estimates for generalised minimisers (see [60, Theorem 21.11]).

Proposition 1.3.5. *There exists $C = C(d,p,\alpha) > 0$ such that if $r < C(1+\lambda)^{-\frac{1+p}{1+\alpha p}}$, every generalised minimiser $\widetilde{E} = (E^i)_{i\ge 1}$ of $\widetilde{\mathcal{E}}_\lambda$ satisfies for every i and every $x \in \partial E^i$ (here ∂E^i denotes the measure-theoretic boundary of E^i)*

$$|E^i \cap B(x,r)| \ge \frac{\omega_d}{4^d} r^d. \tag{1.3.13}$$

As a consequence, up to relabeling, we have $\widetilde{E} = (E^i)_{i=1}^I$ where for every i, E^i are compact connected sets such that $\mathcal{H}^{d-1}(\partial E^i) = P(E^i)$. Moreover, there is a constant $C = C(d,p,\alpha) > 0$ such that

$$\sum_{i=1}^I \operatorname{diam}(E^i) \le C(1+\lambda)^{\frac{(d-1)(1+p)}{1+\alpha p}} \quad \text{and} \quad \min_i \operatorname{diam}(E^i) \ge \frac{1}{C}(1+\lambda)^{-\frac{1+p}{1+\alpha p}}. \tag{1.3.14}$$

As a consequence $I \le C(1+\lambda)^{\frac{d(1+p)}{1+\alpha p}}$.

Remark 1.3.6. Let us notice that the regularity theory for $\Lambda-$minimisers of the perimeter gives us actually much more. Denote by $\partial^* E$ the reduced boundary of E (see [60]) and $\Sigma(E) = \partial E \backslash \partial^* E$. Then if E is a $\Lambda-$minimiser of the perimeter, $\partial^* E$ is $C^{1,\gamma}$ for every $\gamma < 1/2$ and $\Sigma(E)$ is empty if $d \le 7$, an at most finite union of points if $d = 8$ and satisfies $\mathcal{H}^s(\Sigma(E)) = 0$ for every $s > d-8$ if $d \ge 9$. For classical or generalised minimisers of our energy we expect higher regularity to hold but this goes beyond the scope of this chapter.

Proof of Proposition 1.3.5. By Proposition 1.3.4, there exists $\Lambda = C(1+\lambda)^{\frac{1+p}{1+\alpha p}}$ such that every generalised minimiser $\widetilde{E} = (E^i)_{i\ge 1}$ is a $\Lambda-$minimiser of the perimeter. By [60, Theorem 21.11], (1.3.13) holds as long as $\Lambda r < 1$. This proves the first part of the claim. We can further make the identification

$$E^i = \{x \in \mathbb{R}^d \ : \ \liminf_{r\to 0} |E^i \cap B(x,r)| > 0\}$$

so that thanks to (1.3.13), E^i are closed sets with $\mathcal{H}^{d-1}(\partial E^i) = P(E^i)$. By (1.2.4) we may further assume that each E^i is connected. Fix now r such that $\Lambda r = 1/2$. By Vitali's covering Lemma, for every i let $x_1, \cdots, x_{N_i} \in E^i$ be such that $E^i \subset \cup_{j=1}^{N_i} B(x_j, r)$ and $B(x_j, r/5)$ are pairwise disjoint.

Using (1.3.13) we have $N_i \le Cr^{-d}|E^i|$. Since $\mathrm{diam}(E^i) \le CrN_i$ we have

$$\sum_i \mathrm{diam}(E^i) \le Cr^{-(d-1)} \le C(1+\lambda)^{\frac{(d-1)(1+p)}{1+\alpha p}}.$$

This proves the first part of (1.3.14). The second part follows from $\mathrm{diam}(E^i) \ge C|E^i|^{1/d} \ge Cr$ which is a direct consequence of (1.3.13). □

1.3.3 Proof of Theorem 1.1.1

We may now conclude the proof of Theorem 1.1.1 and show the existence of (classical) minimisers for (1.1.2).

Proof of Theorem 1.1.1. For every fixed $d \ge 2$, $p \ge 1$, $\alpha > 0$ and $\lambda > 0$, Proposition 1.3.1 gives the existence of a generalised minimiser $\widetilde{E} = (E^i)_{i\ge1}$. We thus have by (1.3.2),

$$\sum_i P(E^i) + \lambda \left[\sum_i \mathcal{W}_p^p(E^i) \right]^\alpha = \inf_{|E|=\omega_d} \mathcal{E}_\lambda(E).$$

Thanks to Proposition 1.3.5, if $R = C(1+\lambda)^{\frac{(d-1)(1+p)}{1+\alpha p}}$ with $C > 0$ large enough, then $\widetilde{E} = (E^i)_{i=1}^I$ with $I \le R^{\frac{d}{d-1}}$ and for every $i \le I$, E_i is a connected compact set with $\sum_{i=1}^I \mathrm{diam}(E^i) \le \frac{1}{2}R$. Let $(e_1, \cdots, e_d)$ be the canonical basis of $\mathbb{R}^d$ and define the set

$$E = \cup_{i=1}^I (E^i + Rie_1).$$

By Proposition 1.2.2, if C is large enough, $\mathcal{W}_p^p(E) = \sum_i \mathcal{W}_p^p(E^i)$. Since E^i are pairwise disjoint we also have $P(E) = \sum_i P(E^i)$ (and $|E| = \sum_i |E^i| = \omega_d$) so that

$$\mathcal{E}_\lambda(E) = \sum_i P(E^i) + \lambda \left[\sum_i \mathcal{W}_p^p(E^i) \right]^\alpha \overset{(1.3.2)}{=} \inf_{|E|=\omega_d} \mathcal{E}_\lambda(E).$$

Therefore E is a minimiser of (1.1.2) and the proof is complete. □

1.4 Minimality of the ball when the perimeter is dominant

We now turn to Theorem 1.1.2 and prove that for small λ the unique minimisers of (1.1.2) are balls. We first show that for λ small enough, up to a translation, every minimiser of (1.1.2) is a small $C^{1,\gamma}$ perturbation of the ball B_1.

Proposition 1.4.1. *For every $d \ge 2$, $p \ge 1$, $\alpha > 0$, $\gamma \in (0, 1/2)$ and $\varepsilon > 0$, there exists $\lambda_0 = \lambda_0(d, p, \alpha, \gamma, \varepsilon)$ such that for every $\lambda \le \lambda_0$, up to translation, every minimiser E of* (1.1.2) *is nearly spherical in the sense that its barycentre is in* 0 *and there exists $f : \partial B_1 \mapsto \mathbb{R}$ with $\|f\|_{C^{1,\gamma}} \le \varepsilon$ such that*

$$\partial E = \{(1 + f(x))x \;:\; x \in \partial B_1\}.$$

Proof. The proof is quite classical and mostly rests on the (uniform in λ) Λ−minimising property of E. Let E_λ be a sequence of minimisers of (1.1.2). For fixed $\gamma \in (0,1/2)$ we aim at proving that up to translation E_λ converges in $C^{1,\gamma}$ to B_1. We start by noting that using B_1 as a competitor together with the quantitative isoperimetric inequality we have up to translation,

$$|E_\lambda \Delta B_1|^2 \le C\,(P(E)-P(B_1)) \le C\lambda\left(\left[\mathcal{W}_p^p(B_1)\right]^\alpha - \left[\mathcal{W}_p^p(E_\lambda)\right]^\alpha\right) \le C\lambda\left[\mathcal{W}_p^p(B_1)\right]^\alpha. \tag{1.4.1}$$

Therefore E_λ converges in L^1 to B_1. It is now a classical fact that if a sequence of Λ−minimisers converges in L^1 to a smooth set then the whole sequence is actually smooth (with the notation of Remark 1.3.6, $\Sigma(E_\lambda) = \emptyset$) and the convergence holds in $C^{1,\gamma}$ (see e.g. [28, Lemma 3.6]). As a consequence the barycentre of E_λ also converges to 0 and the proof is concluded. □

We now recall that for nearly spherical sets, it was shown in [45] that there exists $C = C(d) > 0$ such that

$$\int_{\partial B_1} f^2 \le C\,(P(E)-P(B_1)). \tag{1.4.2}$$

Moreover, Proposition 1.1.4 states that for such sets we also have

$$\left[\mathcal{W}_p^p(B_1)\right]^\alpha - \left[\mathcal{W}_p^p(E)\right]^\alpha \le C\int_{\partial B_1} f^2. \tag{1.4.3}$$

Postponing the proof of (1.4.3) to the next section we may conclude the proof of Theorem 1.1.2.

Proof of Theorem 1.1.2. By Proposition 1.4.1, if λ is small enough then every minimiser E of (1.1.2) is nearly spherical. Arguing as in (1.4.1) we have

$$\int_{\partial B_1} f^2 \overset{(1.4.2)}{\le} C\,(P(E)-P(B_1)) \le C\lambda\left(\left[\mathcal{W}_p^p(B_1)\right]^\alpha - \left[\mathcal{W}_p^p(E)\right]^\alpha\right) \overset{(1.4.3)}{\le} C\lambda\int_{\partial B_1} f^2,$$

which implies that if λ is small enough, $f = 0$ and thus $E = B_1$. □

1.4.1 Proof of Proposition 1.1.4

We start with a few simple facts about $\mathcal{W}_p(B_1)$. We let $A = B_{2^{1/d}} \backslash B_1$ be the annulus of volume ω_d around B_1. With a slight abuse of notation, we will write $\phi(x) = \phi(|x|)$ if ϕ is a radially symmetrical function.

Lemma 1.4.2. *We have the following properties:*

(i) The minimiser F of (1.1.1) *for B_1 is A;*

(ii) The map

$$T(x) = \left(1+|x|^d\right)^{\frac{1}{d}}\frac{x}{|x|} \tag{1.4.4}$$

is an optimal transport map (the unique one if $p > 1$) between B_1 and F;

(iii) the corresponding Kantorovich potentials (ϕ,ψ) are radially symmetric and $r \mapsto \psi(r)$ is increasing. Finally, (ϕ,ψ) are locally Lipschitz continuous.

Proof. We start with (i). By Proposition 1.2.1, let F be the unique minimiser of (1.1.1) for B_1, so that $\mathcal{W}_p(B_1) = W_p(B_1, F)$. If R is any rotation of $\mathbb{R}^d$, since $W_p(R(B_1), R(F)) = W_p(B_1, F)$, we see that $R(F)$ is also a minimiser of (1.1.1) for B_1. By uniqueness we have $F = R(F)$ and thus F is radially symmetric. Let us now prove that $F = A$. We denote by T an optimal transport map from B_1 to F. For $y \in F$ let $x \in B_1$ be such that $T(x) = y$. Applying [36, Lemma 5.1.] with $f = \chi_{B_1^c}$ yields that $\chi_F = \chi_{B_1^c}$ on $B(x, |y-x|)$. By the radial symmetry of F, we obtain that $B_{|y|} \setminus B_1$ is included in F. Since this is true for every $y \in F$ we conclude that $F = A$.

Regarding (ii), we note that T defined in (1.4.4) is the unique radially symmetric map (in the sense that $T(x) = f(|x|)\frac{x}{|x|}$) which solves $\det \nabla T = 1$ and $f(0) = 1$. Let us argue that T is $c-$cyclically monotone for the cost $c(x,y) = |x-y|^p$ and thus an optimal transport map between any bounded radially symmetric set E and $T(E)$ (see [76, Definition 2.33 & Remark 2.39]). This follows from the fact that $f(r) = (1+r^d)^{1/d}$ is monotone on $\mathbb{R}^+$ and thus also $c-$cyclically monotone on $\mathbb{R}^+$ (as these two notions coincide for convex costs in dimension one) so that for every $x_1, \cdots, x_I$, using the convention $x_0 = x_I$

$$\begin{aligned}\sum_{i=1}^{I} |T(x_i) - x_i|^p &= \sum_{i=1}^{I} |f(|x_i|) - |x_i||^p \\ &\le \sum_{i=1}^{I} |f(|x_{i-1}|) - |x_i||^p \\ &\le \sum_{i=1}^{I} |f(|x_{i-1}|)\frac{x_{i-1}}{|x_{i-1}|} - x_i|^p \\ &= \sum_{i=1}^{I} |T(x_{i-1}) - x_i|^p.\end{aligned}$$

Notice that the inverse map $T^{-1} : B_1^c \mapsto \mathbb{R}^d$ is given by

$$T^{-1}(y) = \left(|y|^d - 1\right)^{\frac{1}{d}} \frac{y}{|y|}.$$

As for (iii), we argue a bit differently for $p > 1$ and $p = 1$. Let us start with the easier case $p = 1$. Denoting $\phi(x) = -|x|$ we have that ϕ is $1-$Lipschitz, radially symmetric and decreasing (and thus $\psi = -\phi$ is radially symmetric and increasing) and satisfies for $x \in \mathbb{R}^d$

$$\phi(x) - \phi(T(x)) = |T(x)| - |x| = |T(x) - x| \tag{1.4.5}$$

so that $(\phi, -\phi)$ is indeed a couple of Kantorovich potentials. As a side note, it is easily seen from (1.4.5) that on the one hand, up to a constant ϕ is the unique Kantorovich potential and on the other hand that every optimal transport map must be radially symmetric (there is however no uniqueness of the optimal transport map). Note also that the validity of (1.4.5) gives an alternative proof of the optimality of T when $p = 1$.

For $p > 1$, we first argue that ϕ is radially symmetric and decreasing. For this we use that by

[73, Theorem 1.17], if we let $h(z) = |z|^p$, then the unique Kantorovich potential ϕ is given by

$$\nabla\phi(x) = \nabla h(x - T(x)) = -p\left((1+|x|^d)^{\frac{1}{d}} - |x|\right)^{p-1} \frac{x}{|x|} = \phi'(|x|)\frac{x}{|x|}$$

with $\phi' \le 0$. Now since ϕ and ψ are c−conjugate, we have

$$\psi(y) = \inf_x [|x-y|^p - \phi(x)] \tag{1.4.6}$$

from which we deduce that also ψ is radially symmetric. Arguing exactly as for ϕ but with T replaced by T^{-1} we see that ψ is increasing on B_1^c. In order to conclude that ψ is in fact increasing on $\mathbb{R}^d$ we will prove that for $y \in B_1$,

$$\psi(y) = |y|^p - \phi(0) \tag{1.4.7}$$

or in other words that (1.4.6) is attained at $x = 0$. We first point out that (1.4.7) holds for $|y| = 1$ since $T^{-1}(y) = 0$ and thus by definition of Kantorovich potentials

$$\phi(0) + \phi(y) = |y|^p.$$

We also observe that since ϕ is decreasing, for every $y \in \mathbb{R}^d$ the optimal x in (1.4.6) must satisfy $|x| \le |y|$ (and $x = |x|y/|y|$). Fix now $y \in B_1$ and let x be such that

$$\psi(y) = |y-x|^p - \phi(x).$$

Let $\bar{y} = y/|y| \in \partial B_1$. Using x as a competitor in (1.4.6) for $\bar{y}$ we have

$$\psi(\bar{y}) = 1 - \phi(0) \le (1-|x|)^p - \phi(x).$$

Using now 0 as competitor in (1.4.6) for y we also have

$$(|y| - |x|)^p - \phi(x) \le |y|^p - \phi(0)$$

so that

$$1 - (1-|x|)^p \le \phi(0) - \phi(x) \le |y|^p - (|y|-|x|)^p.$$

However the function $t \to t^p - (t-|x|)^p$ is increasing in $[|x|, \infty)$ so that we reach a contradiction unless $x = 0$.

To conclude, the local Lipschitz continuity of (ϕ, ψ) is standard, see [76, Proposition 2.43]. □

In order to prove (1.4.3) we will need the following simple result.

Lemma 1.4.3. *Let ψ be a radially symmetric and increasing function and let $E \subset B_{2^{1/d}}$ with $|E| = \omega_d$. Then*

$$\inf_F \left\{ \int_F \psi \; : \; |F \cap E| = 0 \textit{ and } |F| = \omega_d \right\} = \int_{B_{2^{1/d}} \setminus E} \psi. \tag{1.4.7}$$

Proof. We first show that for any $r > 0$,

$$\min_{|E|=|B_r|} \int_E \psi = \int_{B_r} \psi. \tag{1.4.8}$$

For E with $|E| = |B_r|$, we write

$$\int_E \psi - \int_{B_r} \psi = \int_{E\backslash B_r} \psi - \int_{B_r\backslash E} \psi.$$

Since ψ is radially increasing we have

$$\inf_{E\backslash B_r} \psi \geq \psi(r) \geq \sup_{B_r\backslash E} \psi.$$

Using $|E\backslash B_r| = |B_r\backslash E|$, we find

$$\int_E \psi - \int_{B_r} \psi \geq 0$$

and thus (1.4.8) holds.
Now if $E \subset B_{2^{1/d}}$ with $|E| = \omega_d$, for every set F with $|F \cap E| = 0$ and $|F| = |E| = \omega_d$, we have $|E \cup F| = |B_{2^{1/d}}|$ and thus

$$\int_F \psi = \int_{F\cup E} \psi - \int_E \psi \overset{(1.4.8)}{\geq} \int_{B_{2^{1/d}}} \psi - \int_E \psi = \int_{B_{2^{1/d}}\backslash E} \psi,$$

which is the desired conclusion. □

We may now prove Proposition 1.1.4.

Proof of Proposition 1.1.4. We may assume that $\mathcal{W}_p(B_1) \geq \mathcal{W}_p(E)$ since otherwise there is nothing to prove. Using (1.2.7) we see that it is enough to prove the estimate for $\alpha = 1$, that is

$$\mathcal{W}_p^p(B_1) - \mathcal{W}_p^p(E) \leq C \int_{\partial B_1} f^2. \tag{1.4.9}$$

Let (ϕ, ψ) be the Kantorovich potentials associated with $W_p(B_1, A)$ and recall that by Lemma 1.4.2, ψ is radially symmetric and increasing. By hypothesis, $E \subset B_{2^{1/d}}$. For every admissible competitor F for $\mathcal{W}_p(E)$ we have by duality

$$W_p^p(E,F) \geq \int_E \phi + \int_F \psi.$$

Taking the infimum over F we get

$$\mathcal{W}_p^p(E) \geq \int_E \phi + \inf_F \left\{ \int_F \psi \;:\; |F \cap E| = 0 \text{ and } |F| = \omega_d \right\} \overset{(1.4.3)}{\geq} \int_E \phi + \int_{B_{2^{1/d}}\backslash E} \psi.$$

Therefore,

$$
\begin{aligned}
\mathcal{W}_p^p(B_1) - \mathcal{W}_p^p(E) &\leq \int_{B_1} \phi + \int_A \psi - \int_E \phi - \int_{B_{2^{1/d}}\setminus E} \psi \\
&= \int_{B_1} (\phi - \psi) - \int_E (\phi - \psi) \\
&= \int_{B_1\setminus E} (\phi - \psi) - \int_{E\setminus B_1} (\phi - \psi).
\end{aligned}
$$

We may now argue as in [54, Proposition 6.2]. We let $c = \phi(1) - \psi(1)$ and use that ϕ and ψ are Lipschitz continuous in a neighbourhood of ∂B_1 to infer

$$
\begin{aligned}
\int_{B_1\setminus E} (\phi - \psi) - \int_{E\setminus B_1} (\phi - \psi) &= \int_{B_1\setminus E} [(\phi - \psi) - c] - \int_{E\setminus B_1} [(\phi - \psi) - c] \\
&\leq C \int_{B_1 \Delta E} |1 - |x|| \\
&\leq C \int_{\partial B_1} \int_0^{f(x)} t\, dt\, d\mathcal{H}^{d-1}(x) \\
&\leq C \int_{\partial B_1} f^2.
\end{aligned}
$$

This concludes the proof of (1.4.9). □

Chapter 2

An exterior optimal transport problem

Abstract. This chapter deals with a variant of the optimal transportation problem. Given $f \in L^1(\mathbb{R}^d, [0,1])$ and a cost function $c \in C(\mathbb{R}^d \times \mathbb{R}^d)$ of the form $c(x,y) = k(y-x)$, we minimise $\int c\, d\gamma$ among transport plans γ whose first marginal is f and whose second marginal is not prescribed but constrained to be smaller than $1-f$. Denoting by $\Upsilon(f)$ the infimum of this problem, we then consider the maximisation problem $\sup\{\Upsilon(f) : \int f = m\}$ where $m > 0$ is given. We prove that maximisers exist under general assumptions on k, and that for k radial, increasing and coercive these maximisers are the characteristic functions of the balls of volume m.

Keywords and phrases. Optimal transport, dual problem, existence of maximisers.

2020 Mathematics Subject Classification. 49Q22, 49Q20, 49J35.

Outline of the current chapter

2.1 Introduction

In this chapter, we study the optimisation problems associated with functionals which favour dispersion and are based on some Wasserstein energies. These functionals correspond to the non-local term of the energy studied in [19, 13, 70, 77, 64]. Our main result is that for a very large class of radial costs, balls are the unique volume-constrained maximisers of these functionals. This confirms that they enter in strong competition with their perimeter for which balls are volume-constrained minimisers.

We denote by $\mathcal{M}_+(\mathbb{R}^d)$ the set of positive Radon measures on $\mathbb{R}^d$. Given a cost function c and $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^d)$, we let $\mathcal{T}_c(\mu,\nu)$ be the c-transport cost between μ and ν (see Section 2.2 for the exact definition of $\mathcal{T}_c$). Given a measurable set $E \subset \mathbb{R}^d$ with finite volume, we consider the optimisation problem

$$\Upsilon_{\text{set}}(E) := \inf\Big\{\mathcal{T}_c(E,F) : F \subset \mathbb{R}^d \text{ Lebesgue measurable}, |F| = |E|, |F \cap E| = 0\Big\} \tag{2.1.1}$$

where we identify E with the restriction of the Lebesgue measure on E. Given $m > 0$, we introduce the maximisation problem

$$\mathcal{E}_{\text{set}}(m) := \sup_{|E|=m} \Upsilon_{\text{set}}(E). \tag{2.1.2}$$

The main goal of the chapter is to investigate the existence of maximisers for this problem and to characterise these latter.

If we apply the direct method of the Calculus of Variations, we obtain that, up to extraction, any maximising sequence E_n converges weakly to some function $u_\infty \in L^1(\mathbb{R}^d,[0,1])$. However, there is no guarantee at this point that u_∞ is a characteristic function or has mass m. Our strategy is to extend the functional Υ_{set} as a functional Υ defined on $L^1(\mathbb{R}^d,[0,1])$. Applying the bathtub principle (see Proposition 2.4.11) to a maximiser of the relaxed problem, we show that the supremum in (2.1.2) is actually reached (see Corollary 2.1.2). This relaxation approach is not new: it was successfully applied to several variational problems in the last few years (see for instance [27, 8, 69, 12]).

Given $f \in L^1(\mathbb{R}^d, [0,1])$, the set of admissible exterior transport plans is defined as

$$\Pi_f := \Big\{\gamma \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d) : \gamma_x = f,\, \gamma_y \le 1 - f\Big\}.$$

Here, the measures $f\,dx$ and $(1-f)\,dy$ are identified with their respective densities and γ_x and γ_y denote respectively the first and second marginals of γ. We then define the primal problem

$$\Upsilon(f) := \inf\left\{\int c\,d\gamma : \gamma \in \Pi_f\right\}.$$

We have $\Upsilon(\chi_E) = \Upsilon_{\text{set}}(E)$ under mild assumptions on c (see Theorem 2.4.4). Given $m > 0$, our maximisation problem is now to compute the exterior transport energy $\mathcal{E}(m)$, which is defined as

$$\mathcal{E}(m) := \sup\left\{\Upsilon(f) : f \in L^1(\mathbb{R}^d, [0,1]),\ \int f\,dx = m\right\}. \tag{2.1.3}$$

By abuse of notation and when no confusion is possible, we refer to the variational problems by the values they attain (*e.g.* we write $\Upsilon_{\text{set}}(E)$ for (2.1.1)).

2.1.1 Main results

The first important result of this chapter is that maximisers of $\mathcal{E}(m)$ exist whenever c is of the form $c(x,y) = k(y-x)$ for some $k : \mathbb{R}^d \to \mathbb{R}_+$ and satisfies

(H1) $k \in C(\mathbb{R}^d, \mathbb{R}_+)$, $k(0) = 0$ and $k(x) \to \infty$ as $|x| \to \infty$,

(H2) $\forall x \neq 0$,

$$\limsup_{r\to 0} \frac{1}{r^d}\left|B_r(x) \cap \{y \in \mathbb{R}^d,\ k(y) < k(x)\}\right| > 0,$$

(H3) $\forall\, \sigma \in \mathbb{S}^{d-1}$, $r \mapsto k(r\sigma)$ is increasing on $\mathbb{R}_+$.

Notice that k is not assumed to be strictly convex, so that our results hold in cases where the existence of an optimal transport map is not guaranteed. Also observe that all the costs of the form $k(z) = |z|^p$ with $0 < p < \infty$ satisfy the above hypotheses. However, radial symmetry is not required and the costs $k(z) = |z|^p\, h(z/|z|)$ with h positive and Lipschitz continuous on $\mathbb{S}^{d-1}$ are also admissible.

Theorem 2.1.1. *Assume that $c(x,y) = k(y-x)$ for $x, y \in \mathbb{R}^d$ with k satisfying (H1), (H2) and (H3). Then, for any $m > 0$ the supremum in $\mathcal{E}(m)$ is attained. Moreover, there exists $R_* = R_*(m)$ such that (up to translation) any maximiser is supported in the ball $\overline{B}_{R_*}$.*

Once the existence of maximisers for $\mathcal{E}(m)$ is established, the bathtub principle (see Proposition 2.4.11) and a saturation result (see Theorem 2.4.4) imply that (2.1.2) admits solutions.

Corollary 2.1.2. Assume that c satisfies the hypotheses of Theorem 2.1.1. Then, (2.1.2) admits a maximiser and $\mathcal{E}_{\text{set}}(m) = \mathcal{E}(m)$ for any $m > 0$.

As a second main result, we establish that if k is furthermore radially symmetric then $\mathcal{E}(m)$ and $\mathcal{E}_{\text{set}}(m)$ are uniquely maximised by balls of volume m.

Theorem 2.1.3. *Assume that $c(x,y) = k(|y-x|)$ for some $k \in C(\mathbb{R}_+, \mathbb{R}_+)$ increasing and such that $k(0) = 0$ and $k(x) \to \infty$ as $x \to \infty$. Then, for any $m > 0$, the maximisers of $\mathcal{E}(m)$ (and consequently those of $\mathcal{E}_{\text{set}}(m)$) are the balls of volume m. Moreover the minimiser of $\Upsilon_{\text{set}}(B_1)$ is the annulus $B_{2^{1/d}} \setminus B_1$.*

We point out that cost functions satisfying the hypotheses of Theorem 2.1.3 also satisfy hypotheses (H1), (H2) and (H3). Let us briefly sketch the proofs of these three results. They all strongly rely on the properties of the dual problem

$$\Upsilon^*(f) := \sup\left\{\int (f\varphi + (1-f)\psi)\,dx : (\varphi, \psi) \in \Phi\right\},$$

where

$$\Phi := \Big\{(\varphi,\psi) \in C_b(\mathbb{R}^d)\times C_b(\mathbb{R}^d),\ \psi \le 0,\ \varphi(x)+\psi(y) \le c(x,y)\ \forall\,(x,y)\in\mathbb{R}^d\times\mathbb{R}^d\Big\}.$$

We establish Theorem 2.1.1 using the direct method of Calculus of Variations. The main difficulty is to establish compactness of maximising sequences. If we refer to the concentration-compactness principle [58], we have to prove that given a maximising sequence f_n, no mass escapes at infinity. To do so we establish two crucial results. The first one is that $m \mapsto \mathcal{E}(m)/m$ is increasing (see Proposition 2.4.7). This implies that $m \mapsto \mathcal{E}(m)$ is strictly superadditive, *i.e.* that for $m > m' > 0$,

$$\mathcal{E}(m') + \mathcal{E}(m-m') < \mathcal{E}(m). \tag{2.1.4}$$

Notice that this is the counterpart of the strict subbadditivity inequality (also called binding inequality) which is known to provide compactness in minimisation problems, see *e.g.* [58, 43, 44]. Using the dual formulation Υ^* of Υ, we obtain the second crucial result for Theorem 2.1.3: a monotonicity principle on the sum of marginals of minimisers γ of $\Upsilon(f)$ (see Corollary 2.4.6). This is the most delicate part of the proof. Combining this and (2.1.4), we prove that if f is almost maximising then most of its mass must remain in a bounded region (see Proposition 2.4.9). This gives tightness of maximising sequences for $\mathcal{E}(m)$.

To prove Corollary 2.1.2, we consider a maximiser f of $\mathcal{E}(m)$ provided by Theorem 2.1.1 and a pair of potentials (φ,ψ) optimal for the dual problem $\Upsilon^*(f)$. Using the definition of Υ^* we see that f is a maximiser of

$$\sup\left\{\int \widetilde{f}(\varphi-\psi) : 0 \le \widetilde{f} \le 1,\ \int \widetilde{f} = m\right\}.$$

By the bathtub principle, $f = \chi_{\{\varphi-\psi>t\}} + \theta$ for some $t \in \mathbb{R}$ and some $\theta \in L^1(\mathbb{R}^d,[0,1])$ supported in $\{\varphi-\psi=t\}$. Then for any measurable subset $G \subset \{\varphi-\psi=t\}$ with $|G| = \int\theta$, the characteristic function of $E := \{\varphi-\psi>t\}\cup G$ is also a maximiser for $\mathcal{E}(m)$. By Theorem 2.4.4 and Corollary 2.4.5 applied to E, there exists $F \subset \mathbb{R}^d$ such that any minimiser γ of $\Upsilon(\chi_E)$ satisfies $\gamma_y = \chi_F$. This finally implies that E maximises (2.1.2).

Regarding Theorem 2.1.3, as explained in Section 2.5, we may assume without loss of generality that $m = \omega_d$, the volume of the unit ball. Combining Theorem 2.1.1 and Lemma 2.4.10 yields that

$$\sup_{\int f=\omega_d}\left\{\sup_{(\psi^c,\psi)\in\Phi}\left\{\int f(\psi^c-\psi) + \int\psi\right\}\right\}, \tag{2.1.5}$$

coincides with $\mathcal{E}(m)$ and admits a solution (f,ψ^c,ψ), where ψ^c is the c-transform of ψ (see Definition 2.2.2). To show that balls are maximisers of $\mathcal{E}(m)$, we establish that each term in (2.1.5) is improved by replacing f by χ_{B_1} and ψ by its symmetric increasing rearrangement ψ_* (see Definition 2.5.1). As $\int\psi = \int\psi_*$, the third term in (2.1.5) does not change under rearrangement. Regarding the second term, combining the Hardy-Littlewood inequality (see [57, Theorem 3.4]) and the bathtub principle yields (recall that $\psi \le 0$)

$$-\int f\psi \le -\int f^*\psi_* \le -\int \chi_{B_1}\psi_*,$$

where f^* is the symmetric decreasing rearrangement of f (see Definition 2.5.1). The study of the first term $\int f\psi^c$ is more involved. Indeed it requires to understand the interactions between the operations of c–transform and symmetrization. To the best of our knowledge, this type of questions have not been addressed so far. Using the Brunn-Minkowski inequality, we obtain the following crucial comparison:

$$(\psi^c)^* \le (\psi_*)^c.$$

Combining this inequality with the Hardy-Littlewood inequality yields

$$\int f\psi^c \le \int f^*(\psi^c)^* \le \int f^*(\psi_*)^c. \tag{2.1.6}$$

Additionally, as $(\psi_*)^c$ is non-increasing, χ_{B_1} is a maximiser of

$$\sup\left\{\int \widetilde{f}(\psi_*)^c : 0 \le \widetilde{f} \le 1, \int \widetilde{f} = \omega_d\right\}, \tag{2.1.7}$$

so that $\int f^*(\psi_*)^c \le \int(\psi_*)^c\chi_{B_1}$. Lastly, by (2.1.6), $\int f\psi^c \le \int \chi_{B_1}(\psi_*)^c$. This eventually proves that unit balls maximise $\mathcal{E}(\omega_d)$.

As for uniqueness, the key property to establish is that $(\psi_*)^c$ is decreasing on B_1 (see Lemma 2.5.3). Indeed, by [57, Theorem 3.4], this implies that χ_{B_1} is the unique maximiser of (2.1.7). Combining this with the fact that the inequalities in (2.1.6) are now equalities, we obtain that $f^* = \chi_{B_1}$, so that $f = \chi_E$ for some $E \subset \mathbb{R}^d$. Using the equality case of the Brunn-Minkowski inequality, we then show that (up to a translation) $f = \chi_{B_1}$, concluding the proof.

2.1.2 Motivation

In [13], the following variational problem was introduced:

$$\inf_{|E|=\omega_d}\left\{P(E) + \alpha\Upsilon_p(E)\right\}, \tag{2.1.8}$$

where $\alpha > 0$ and where Υ_p is the functional Υ defined in (2.1.1) with the cost $c(x,y) = |x-y|^p$. Such a variational problem may be used to model the formation of bi-layer biological membranes (see [70, 59]). Existence of minimisers were obtained in the series of work [13, 77, 64, 19].

Notice that (2.1.8) is an isoperimetric problem with a non-local term Υ_p. One of the best-known examples of this type of problem is Gamow's liquid drop model for the atomic nucleus. Since the beginning of the 2010s (see [26] for an historical perspective), this model has received a lot of attention from the mathematical community, and several versions of it have been studied, see for instance [54, 50, 55, 51]. In this framework, the perimeter term represents the local attractive forces while the repulsive non-local term is given by the Riesz potential

$$V_\beta(E) := \int_E \int_E \frac{dxdy}{|x-y|^{d-\beta}},$$

where $\beta \in (0,d)$. A consequence of Riesz's rearrangement inequality is that balls are the volume constrained maximisers of V_β. This illustrates the competition between the perimeter and the Riesz potential. It is thus natural to investigate whether similar properties hold for (2.1.8). In our case, the proof is much more involved since the rearrangement argument does not seem to work well for the primal problem. We consider instead the dual problem Υ^* and study the (fortunately favourable) interplay between rearrangement and c–transforms.

As a closing remark, we point out that the functional Υ is a particular case of the optimal partial transport problem studied in [40, 35].

2.1.3 Organization of the chapter

The chapter is structured as follows. In Section 2.2, we introduce the notation and review standard facts related to optimal transport in complete separable metric spaces. In Section 2.3, we obtain preliminary results on the functional Υ defined in compact spaces. In Section 2.4, we establish Theorem 2.1.1. Eventually, in Section 2.5, we prove Theorem 2.1.3.

2.2 Notation and preliminary results

2.2.1 Notation

Let (X, d_X) be a Polish space endowed with a positive Radon measure λ.

Given a function $f : X \to \mathbb{R}$, we decompose it as:

$$f = f_+ + f_- \quad \text{with} \quad f_+ := \max(0, f) := 0 \vee f \quad \text{and} \quad f_- := \min(0, f) := 0 \wedge f.$$

Let us stress that f_- is non-positive, contrary to the classical decomposition of a function into its positive and negative parts.

We endow $\mathcal{M}_+(X)$ with the topology of weak-$*$ convergence, that is the topology induced by duality with $C_b(X)$. The convergence of a sequence $\mu_n \in \mathcal{M}_+(X)$ to $\mu \in \mathcal{M}_+$ is written: $\mu_n \overset{*}{\rightharpoonup} \mu$ as $n \to \infty$.

Given a measure $\mu \in \mathcal{M}_+(X)$ and a set $A \subset X$, the restriction of μ to A is the measure $\mu \llcorner A$ defined as $\mu \llcorner A(B) := \mu(B \cap A)$ for every Borel set B of X. The support of μ, denoted by $\operatorname{supp} \mu$, is the closed set defined by

$$\operatorname{supp} \mu := \{x \in X : \mu(A) > 0 \text{ for all open set A containing } x\}.$$

Given $f \in L^1(X, \lambda)$ the support of f is defined as the support of the measure $f d\lambda$ and denoted by $\operatorname{supp} f$. We identify the measure $f d\lambda$ with its density f and write $f_n \overset{*}{\rightharpoonup} f$ as $n \to \infty$ to signify that $\int f_n \xi$ converges to $\int f \xi$ for every $\xi \in C_b(X)$.

Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R})$, we denote by $\mathrm{Leb}(f)$ the set of its Lebesgue points.

Given $x \in \mathbb{R}^d$ and $r > 0$, $B_r(x)$ denotes the open ball of radius r centred at x, and B_r denotes the open ball of radius r centred at 0. The closed ball of radius r centred at x is denoted by $\overline{B}_r(x)$. The volume of the unit ball in $\mathbb{R}^d$ is denoted by ω_d.

Given two sets A, B of $\mathbb{R}^d$, we define their sum $A + B := \{a + b,\, a \in A,\, b \in B\}$. The gap between A and B is $d(A,B) := \inf\{|a - b|,\, a \in A,\, b \in B\}$.

2.2.2 Optimal transport theory

In this subsection, we recall some results regarding standard optimal transport theory. Most of the material presented here comes from [73, Chapter 1].

Let (X, d_X) be a complete separable metric space (*i.e.* a Polish space) and let $c : X \times X \to \mathbb{R}$ be measurable. Given $\mu, \nu \in \mathcal{M}_+(X)$ such that $\mu(X) = \nu(X)$, the Kantorovitch problem with marginals μ and ν and cost c is

$$\mathcal{T}_c(\mu,\nu) := \inf\left\{ \int c\, d\gamma : \gamma \in \Pi(\mu,\nu) \right\}, \tag{2.2.1}$$

where $\Pi(\mu,\nu)$ is the set of transport plans between μ and ν, *i.e.*

$$\Pi(\mu,\nu) := \left\{ \gamma \in \mathcal{M}_+(X \times X) : \gamma_x = \mu,\, \gamma_y = \nu \right\}.$$

Problem (2.2.1) admits a dual formulation given by

$$\mathcal{T}_c^*(\mu,\nu) := \sup\left\{ \int \varphi\, d\mu + \int \psi\, d\nu : \varphi, \psi \in C_b(X),\, \varphi \oplus \psi \le c \right\}, \tag{2.2.2}$$

where the function $\varphi \oplus \psi$ is defined on $X \times X$ by $(\varphi \oplus \psi)(x,y) := \varphi(x) + \psi(y)$.

Theorem 2.2.1 (Theorem 1.7 of [73])**.** *Let $c : X \times X \to \mathbb{R}$ be lower semi-continuous and bounded from below and let $\mu, \nu \in \mathcal{M}_+(X)$ with $\mu(X) = \nu(X)$. Then* (2.2.1) *admits a solution and*

$$\mathcal{T}_c(\mu,\nu) = \mathcal{T}_c^*(\mu,\nu).$$

Using the notion of c-transform of a function, the maxima of (2.2.2) can be further characterised.

Definition 2.2.2. Given a function $\xi : X \to \mathbb{R} \cup \{+\infty\}$, we define its c-transform (or c-conjugate) $\xi^c : X \to \mathbb{R} \cup \{-\infty\}$ by

$$\xi^c(y) := \inf_{x \in X} \{c(x,y) - \xi(x)\}.$$

Denoting $\bar{c}(y,x) := c(x,y)$, the $\bar{c}$-transform of $\zeta : X \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\zeta^{\bar{c}}(x) := \inf_{y \in X} \{\bar{c}(y,x) - \zeta(y)\}.$$

A function $\psi : X \to \mathbb{R} \cup \{-\infty\}$ is said to be $\bar{c}$-concave if there exists $\xi : X \to \mathbb{R} \cup \{+\infty\}$ such that $\psi = \xi^c$ (the definition of c-concavity is analogous).

Definition 2.2.3. Let (X, d_X) be a metric space and $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ be increasing and such that $\omega(0) = 0$. A function $\varphi : X \to \mathbb{R}$ is ω-continuous if for all $x, x' \in X$,

$$|\varphi(x) - \varphi(x')| \leq \omega(d_X(x, x')).$$

Similarly, we say that $c : X \times X \to \mathbb{R}$ is ω-continuous if for all $x, x', y, y' \in X$,

$$|c(x, y) - c(x', y')| \leq \omega(d_X(x, x') + d_X(y, y')).$$

Proposition 2.2.4. *Let $\varphi, \psi : X \to \mathbb{R}$ be fixed and assume that φ^c and $\psi^{\bar{c}}$ take real values. The following statements hold:*

(i) If c is ω-continuous, then φ^c is also ω-continuous,

(ii) $\varphi^{c\bar{c}} \geq \varphi$, and $\varphi^{c\bar{c}} = \varphi$ if and only if φ is c-concave,

(iii) φ^c is the largest function ψ compatible with the constraint $\varphi \oplus \psi \leq c$ and $\psi^{\bar{c}}$ is the largest function φ compatible with the constraint $\varphi \oplus \psi \leq c$.

Remark that if X is compact and φ, ψ and c are bounded then φ^c and $\psi^{\bar{c}}$ take real values. Moreover, if c is continuous, say ω-continuous, the proposition states that φ^c and $\psi^{\bar{c}}$ are ω-continuous. This yields the following existence result for (2.2.2).

Theorem 2.2.5 (Proposition 1.11 of [73])**.** *Let X be a compact metric space and $c : X \times X \to \mathbb{R}$ be continuous. Then there exists a solution (φ, ψ) to (2.2.2), where φ is c-concave and $\psi = \varphi^c$. In particular,*

$$\mathcal{T}_c^*(\mu, \nu) = \max\left\{ \int \varphi \, d\mu + \int \varphi^c \, d\nu : \varphi \text{ } c\text{-concave} \right\}.$$

A pair of functions maximising (2.2.2) is called a pair of Kantorovitch potentials.

2.3 Study of the exterior transport functional in compact metric spaces

Let (X, d_X) be a compact metric space and let $c : X \times X \to \mathbb{R}$ be a continuous cost function. We endow (X, d_X) with a measure $\lambda \in \mathcal{M}_+(X)$ such that $\lambda(X) > 0$ and denote by $L^1(X)$ the set of $\mathbb{R}$-valued functions integrable with respect to λ. Given $f \in L^1(X)$, we define the set of admissible transport plans

$$\Pi_f := \left\{ \gamma \in \mathcal{M}_+(X \times X) : \gamma_x = f, \, \gamma_y \leq 1 - f \right\}$$

and the primal problem

$$\Upsilon(f) := \inf\left\{ \int c \, d\gamma : \gamma \in \Pi_f \right\}. \tag{2.3.1}$$

Notice that Π_f is empty whenever f does not satisfy $0 \le f \le 1$ or when $\int f\, d\lambda > \lambda(X)/2$. In the other cases, there exists $g \in L^1(X)$ such that $g \ge 0$, $f + g \le 1$ and $\int g\, d\lambda = \int f\, d\lambda$. Thus,

$$\gamma := \frac{1}{\int f\, d\lambda}(f\, d\lambda) \otimes (g\, d\lambda) \in \Pi_f,$$

and Π_f is not empty. We now fix $0 < m \le \lambda(X)/2$ and define

$$L^1_m := \left\{ f \in L^1(X, [0,1]) : \int f \le m \right\}.$$

Given $f \in L^1_m$ and $\varphi, \psi \in C(X)$, we set

$$K_f(\varphi, \psi) := \int (f\varphi + (1-f)\psi)\, d\lambda \tag{2.3.2}$$

and define the dual problem

$$\Upsilon^*(f) := \sup\left\{ K_f(\varphi,\psi) : (\varphi,\psi) \in \Phi \right\}, \tag{2.3.3}$$

where

$$\Phi := \{(\varphi,\psi) \in C(X) \times C(X),\ \psi \le 0,\ \varphi \oplus \psi \le c\}.$$

For the remainder of the section we fix $f \in L^1_m$. As in the classical theory of optimal transport, a simple application of the direct method of Calculus of Variations shows that (2.3.1) admits a minimiser.

Proposition 2.3.1. *Assume that X is a compact metric space and that $c \in C(X \times X, \mathbb{R})$. Then, the infimum in* (2.3.1) *is a minimum.*

Remark 2.3.2. If we let $f \in L^1_m$, by Proposition 2.3.1, there exists $\gamma \in \mathcal{M}_+(X \times X)$ optimal for $\Upsilon(f)$. Notice that γ solves the classical optimal transport problem from f towards $g := \gamma_y$ defined by (2.2.1). Moreover, we have the identity $\Upsilon(f) = \mathcal{T}_c(f,g)$.

Let us now show that $\Upsilon^*(f) = \Upsilon(f)$ and that (2.3.3) admits a maximising pair (φ,ψ). We first establish that we can reduce the set of competitors for (2.3.3). To simplify the notation we denote by φ^c_- the function $(\varphi^c)_- := \varphi^c \wedge 0$.

Lemma 2.3.3. *Assume that X is a compact metric space and that $c \in C(X \times X, \mathbb{R})$. Then, there holds*

$$\Upsilon^*(f) = \sup\{K_f(\psi^{\bar{c}}, \psi) : \psi = \varphi^c_- \text{ for some } \varphi \in \Phi'\}, \tag{2.3.4}$$

where

$$\Phi' := \left\{ \varphi \in C(X),\ \varphi = (\varphi^c_-)^{\bar{c}},\ \max \varphi^c \ge 0 \right\}. \tag{2.3.5}$$

Proof. Step 1. We can replace ψ by φ^c_- and assume that $\max \varphi^c \ge 0$.

Let $(\varphi,\psi) \in \Phi$. By Proposition 2.2.4 (iii), $\psi \le \varphi^c$, so that $\psi \le \varphi^c \wedge 0 = \varphi^c_-$. As $1 - f \ge 0$, $K_f(\varphi, \varphi^c_-) \ge K_f(\varphi,\psi)$. Therefore, we can restrict the maximisation to the pairs (φ, φ^c_-) in the supremum (2.3.3). Now, if $\max \varphi^c = -t < 0$ we set $\widetilde{\varphi} := \varphi - t$ so that $\widetilde{\varphi}^c = \varphi^c + t$. Consequently, $\max \widetilde{\varphi}^c = 0$ and in particular, $\widetilde{\varphi}^c_- = \widetilde{\varphi}^c$ so that $(\widetilde{\varphi}, \widetilde{\varphi}^c) \in \Phi$. We then compute

$$\begin{aligned} K_f(\widetilde{\varphi}, \widetilde{\varphi}^c) &\geq \int f(\widetilde{\varphi} - \widetilde{\varphi}^c)\, d\lambda + \int \widetilde{\varphi}^c \, d\lambda \\ &= \int f(\varphi - \varphi^c)\, d\lambda + \int \varphi^c \, d\lambda + t(\lambda(X) - 2m) \\ &= K_f(\varphi, \varphi^c_-) + t(\lambda(X) - 2m). \end{aligned}$$

As $2m \leq \lambda(X)$ we obtain $K_f(\widetilde{\varphi}, \widetilde{\varphi}^c_-) \geq K_f(\varphi, \varphi^c_-)$. Hence

$$\Upsilon^*(f) = \sup\Big\{K_f(\varphi, \varphi^c_-) : \varphi \in C(X), \max \varphi^c \geq 0\Big\}.$$

Step 2. There holds $\varphi = (\varphi^c_-)^{\bar{c}}$.

Let us introduce the mapping $P : C(X) \to C(X)$ defined by $P(\varphi) := (\varphi^c_-)^{\bar{c}}$. For $\varphi \in C(X)$, $\varphi^c \geq \varphi^c_-$, so that $P(\varphi) = (\varphi^c_-)^{\bar{c}} \geq \varphi^{c\bar{c}}$. By Proposition 2.2.4 (i), $\varphi^{\bar{c}c} \geq \varphi$, hence

$$P(\varphi) \geq \varphi. \tag{2.3.6}$$

By Proposition 2.2.4 (ii) again there holds $P(\varphi)^c = (\varphi^c_-)^{\bar{c}c} \geq \varphi^c_-$. Taking the negative part yields

$$P(\varphi)^c_- \geq \varphi^c_-. \tag{2.3.7}$$

We deduce from (2.3.6) and (2.3.7) that

$$K_f(P(\varphi), P(\varphi)^c_-) \geq K_f(\varphi, \varphi^c_-).$$

Now, we observe that if $\max \varphi^c \geq 0$ we also have $\max \varphi^c_- = 0$ and, by (2.3.7), $\max P(\varphi)^c_- = 0$ which implies that $\max P(\varphi)^c \geq 0$. Hence,

$$\Upsilon^*(f) = \sup\Big\{K_f(\widetilde{\varphi}, \widetilde{\varphi}^c_-) : \widetilde{\varphi} \in C(X), \max \widetilde{\varphi}^c \geq 0, \widetilde{\varphi} = P(\varphi) \text{ for some } \varphi \in C(X)\Big\}. \tag{2.3.8}$$

To conclude, we show that $P(P(\varphi)) = P(\varphi)$ for any $\varphi \in C(X)$. By (2.3.6), $P(P(\varphi)) \geq P(\varphi)$. Taking the $\bar{c}$-transform in (2.3.7) yields $P(P(\varphi)) \leq P(\varphi)$ and we have indeed $P(P(\varphi)) = P(\varphi)$. Hence we have $\tilde{\varphi} \in \Phi'$ in (2.3.8) and we get

$$\Upsilon^*(f) = \sup\Big\{K_f(\tilde{\varphi}, \tilde{\varphi}^c_-) : \tilde{\varphi} \in \Phi'\Big\}. \tag{2.3.9}$$

Finally, by definition $\tilde{\varphi} = (\tilde{\varphi}^c_-)^{\bar{c}}$ for $\tilde{\varphi} \in \Phi'$ and (2.3.4) follows from (2.3.9) by letting $\psi := \tilde{\varphi}^c_-$. □

We can now establish that the supremum in (2.3.4) is reached.

Proposition 2.3.4. *Assume that X is a compact metric space and that $c \in C(X \times X, \mathbb{R})$. Then, the set Φ' is compact in $(C(X), \|\cdot\|_\infty)$ and the suprema in (2.3.4) and (2.3.3) are attained.*

Proof. Let us show that Φ' is compact. Let φ_n be a sequence in Φ'. The function c is ω-continuous for some modulus of continuity $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$, so that by Proposition 2.2.4 (ii) for every $n \geq 0$, $\varphi_n^c := (\varphi_n)^c$ and $\varphi_{n-}^c := ((\varphi_n)^c)_-$ are ω-continuous. By definition of Φ', $\varphi_n = (\varphi_{n-}^c)^{\bar{c}}$, so that φ_n is also ω-continuous for every $n \geq 0$. Let us show that the sequences φ_n and φ_{n-}^c are uniformly bounded in $(C(X), \|\cdot\|_\infty)$. We observe that for every $n \geq 0$, $\max \varphi_n^c \geq 0$. Denoting by x_n a point of X such that $\varphi_{n-}^c(x_n) = 0$, by ω-continuity we have for $x \in X$ and $n \geq 0$,

$$-\omega(\operatorname{diam}(X)) \leq -\omega(d_X(x, x_n)) \leq \varphi_{n-}^c(x) - \varphi_{n-}^c(x_n) = \varphi_{n-}^c(x) \leq 0.$$

Thus the sequence φ_{n-}^c is uniformly bounded in $(C(X), \|\cdot\|_\infty)$. By definition of the c-transform

$$\min_{X \times X} c - \max_X \varphi_{n-}^c \leq (\varphi_{n-}^c)^{\bar{c}} \leq \max_{X \times X} c - \min_X \varphi_{n-}^c.$$

Hence the sequence φ_n is also uniformly bounded. By Arzelá-Ascoli's theorem, there exists a pair $(\varphi, \psi) \in C(X) \times C(X)$ such that, up to extraction of a subsequences, $(\varphi_n, \varphi_{n-}^c)$ converges uniformly to (φ, ψ).

Let us show that $\varphi \in \Phi'$. By Proposition 2.2.4 (iii) and by uniform convergence $\varphi_n^c \to \varphi^c$ as $n \to \infty$ so that

$$\varphi_{n-}^c \to \varphi_-^c \quad \text{uniformly as } n \to \infty, \tag{2.3.10}$$

which yields $\psi = \varphi_-^c$. From (2.3.10) and the uniform continuity of c, we deduce that

$$(\varphi_{n-}^c)^{\bar{c}} = \varphi_n \to (\varphi_-^c)^{\bar{c}} \quad \text{uniformly as } n \to \infty.$$

Since $\varphi_n \to \varphi$ as $n \to \infty$, we obtain $\varphi = (\varphi_-^c)^{\bar{c}}$. Lastly, by uniform convergence, the fact that $\max \varphi_n^c \geq 0$ for all $n \geq 0$ implies that $\max \varphi^c \geq 0$, so that $\varphi \in \Phi'$. This shows that Φ' is a compact subset of $(C(X), \|\cdot\|_\infty)$.

Let now ψ_n be a maximising sequence for (2.3.4). For all $n \geq 0$, there exists $\varphi_n \in \Phi'$ such that $\psi_n = \varphi_{n-}^c$. By compactness of Φ', $\varphi_n \to \varphi$ as $n \to \infty$ for some $\varphi \in \Phi'$. Setting $\psi = \varphi_-^c$, we have $\psi_n \to \psi$ and $\psi_n^{\bar{c}} \to \psi^{\bar{c}}$ as $n \to \infty$. The functional K_f being continuous with respect to uniform convergence, we obtain

$$K_f(\psi^{\bar{c}}, \psi) = \lim K_f(\psi_n^{\bar{c}}, \psi_n) = \Upsilon^*(f).$$

This proves that ψ is a maximiser for (2.3.4) and by Lemma 2.3.3, ψ also maximises (2.3.3). □

We are now ready to prove that there is no duality gap between (2.3.1) and (2.3.3). The proof is an adaptation of [73, Section 1.6.3].

Proposition 2.3.5. *Assume that X is a compact metric space and that $c \in C(X \times X, \mathbb{R})$. Then,*

$$\Upsilon^*(f) = \Upsilon(f).$$

Proof. *Step 1. Definition of H and first properties.*

For $p \in C(X \times X)$, we define

$$H(p) := -\sup\left\{\int (f\varphi + (1-f)\psi)\, d\lambda : (\varphi,\psi) \in \Phi_p\right\}$$

where

$$\Phi_p := \{(\varphi,\psi) \in C(X) \times C(X),\ \psi \le 0,\ \varphi \oplus \psi \le c - p\}.$$

We first observe that $c - p$ is continuous and bounded from below. Thus, by applying Proposition 2.3.4 with $c - p$ in place of c, we see that the above supremum is a maximum.

Let us now show that H is convex. Let $p_0, p_1 \in C(X \times X)$ and $\theta \in [0,1]$ and let us set $p := (1-\theta)p_0 + \theta p_1$. We denote by (φ_0, ψ_0) and (φ_1, ψ_1) two maximising pairs associated with p_0 and p_1 and set $\varphi := (1-\theta)\varphi_0 + \theta\varphi_1$, $\psi := (1-\theta)\psi_0 + \theta\psi_1$. We see that (φ,ψ) is an admissible pair ($\psi \le 0$ and $\varphi \oplus \psi \le c - p$), so that

$$H(p) \le -\int (f\varphi + (1-f)\psi)\, d\lambda = (1-\theta)H(p_0) + \theta H(p_1).$$

This proves that H is convex.

Next, we establish that H is lower semi-continuous in $(C(X \times X), \|\cdot\|_\infty)$. Let p_n and p be elements of $C(X \times X)$ such that $p_n \to p$ uniformly as $n \to \infty$. The sequence $c - p_n$ is uniformly equi-continuous. Therefore, proceeding as in the proof of Proposition 2.3.4, there exists a sequence of uniformly bounded and equi-continuous admissible pairs (φ_n, ψ_n) such that

$$H(p_n) = -\int (f\varphi_n + (1-f)\psi_n)\, d\lambda \qquad \text{for every } n \ge 0.$$

We first extract a subsequence $p_{n'}$ such that $\lim_{n'} H(p_{n'}) = \liminf_n H(p_n)$. By Arzelà-Ascoli's theorem, there exists $(\varphi,\psi) \in C(X) \times C(X)$ such that $\varphi_{n'} \to \varphi$ and $\psi_{n'} \to \psi$ uniformly as $n' \to \infty$. By pointwise convergence, $\psi \le 0$ and $\varphi \oplus \psi \le c - p$. Passing to the limit yields

$$H(p) \le -\int (f\varphi + (1-f)\psi)\, d\lambda = -\lim_{n'} \int (f\varphi_{n'} + (1-f)\psi_{n'})\, d\lambda = \liminf_n H(p_n).$$

Thus H is lower semi-continuous.

Step 2. Absence of duality gap.

Since H is convex and lower semi-continuous on the Banach space $(C(X \times X), \|\cdot\|_\infty)$, we have $H(0) = H^{**}(0)$. Here, for a Banach space $\mathcal{X}$ and a function $F : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$, F^* denotes the Legendre transform of F defined on the topological dual $\mathcal{X}^*$ of $\mathcal{X}$ by

$$F^*(x^*) := \sup\{x^*(x) - F(x) : x \in \mathcal{X}\}.$$

In particular,

$$\Upsilon^*(f) = -H(0) = -H^{**}(0) = \inf\{H^*(\gamma) : \gamma \in \mathcal{M}(X \times X)\}. \tag{2.3.11}$$

We now compute H^*. Let $\gamma \in \mathcal{M}(X \times X)$. By definition,

$$H^*(\gamma) = \sup_{p \in C(X\times X)} \left\{ \int p \, d\gamma + \sup_{(\varphi,\psi)\in\Phi_p} \left\{ \int (f\varphi + (1-f)\psi) \, d\lambda \right\} \right\}.$$

Let us first assume that there exists $q \in C(X \times X, \mathbb{R}_+)$ such that $t := -\int q \, d\gamma > 0$. We set $\varphi = \min c$, $\psi = 0$ and $p_n := -nq$ for $n \geq 1$. We obtain

$$H^*(\gamma) \geq nt - m \min c \ \to \infty \qquad \text{as } n \to \infty.$$

Thus, when computing $H^*(\gamma)$, we may assume that $\gamma \geq 0$. We rewrite $H^*(\gamma)$ as

$$\begin{aligned} H^*(\gamma) = \int c \, d\gamma + \sup_{p\in C(X\times X)} \sup_{(\varphi,\psi)\in\Phi_p} \bigg\{ \int (p - c + \varphi \oplus \psi) \, d\gamma \\ + \int \varphi \, d(f\lambda - \gamma_x) + \int \psi \, d((1-f)\lambda - \gamma_y)) \bigg\}. \end{aligned} \tag{2.3.12}$$

Let us set

$$G(\gamma) := \sup \left\{ \int \varphi \, d(f\lambda - \gamma_x) + \int \psi \, d((1-f)\lambda - \gamma_y) : (\varphi, \psi) \in C(X) \times C(X), \ \psi \leq 0 \right\}.$$

On the one hand, given $(\varphi, \psi) \in \Phi_p$ and $\gamma \geq 0$,

$$\int (p - c + \varphi \oplus \psi) \, d\gamma \leq 0.$$

Therefore, $H^*(\gamma) \leq \int c \, d\gamma + G(\gamma)$. On the other hand, given (φ, ψ) admissible for $G(\gamma)$, setting $p = c - \varphi \oplus \psi$ yields the converse inequality thanks to (2.3.12). Hence

$$H^*(\gamma) = \int c \, d\gamma + G(\gamma). \tag{2.3.13}$$

Given $\gamma \in \mathcal{M}_+(X \times X)$, we have $G(\gamma) = 0$ if $\gamma \in \Pi_f$ and $G(\gamma) = +\infty$ otherwise. Combining this with (2.3.13), we obtain that for $\gamma \in \mathcal{M}(X \times X)$,

$$H^*(\gamma) = \begin{cases} \int c \, d\gamma & \text{if } \gamma \in \Pi_f, \\ +\infty & \text{in the other cases.} \end{cases}$$

Taking the infimum with respect to $\gamma \in \mathcal{M}(X \times X)$ and recalling (2.3.11), we get

$$\Upsilon^*(f) = \inf\{H^*(\gamma) : \gamma \in \mathcal{M}(X \times X)\} = \inf \left\{ \int c \, d\gamma : \gamma \in \Pi_f \right\} = \Upsilon(f),$$

which concludes the proof. □

Remark 2.3.6. There is still no duality gap between (2.3.1) and (2.3.3) if we only assume c to be lower semi-continuous. This result can be obtained by approximating c pointwise from below by a non-decreasing sequence of continuous functions.

In the remainder of the section, we focus on the properties of the potentials $(\psi^{\bar{c}}, \psi)$ maximising (2.3.4). We first show that the sign of $\psi^{\bar{c}c}$ enforces constraints on the local values of the marginals of any plan γ optimal for $\Upsilon(f)$.

Proposition 2.3.7. *Assume that X is a compact metric space and that $c \in C(X \times X, \mathbb{R})$. Let $(\psi^{\bar{c}}, \psi)$ be a maximiser of* (2.3.4) *and let γ be a minimiser of* (2.3.1)*. We set $g := \gamma_y$. Then, γ is a minimiser of* (2.2.1) *with $(\mu, \nu) = (f, g)$ and $(\psi^{\bar{c}}, \psi^{\bar{c}c})$ is a pair of Kantorovitch potentials realising the maximum in* (2.2.2)*. Moreover, up to λ-negligible sets,*

$$f + g \equiv 1 \quad \text{on } \{\psi^{\bar{c}c} < 0\} \qquad \text{and} \qquad g \equiv 0 \quad \text{on } \{\psi^{\bar{c}c} > 0\}. \tag{2.3.14}$$

Proof. By Remark 2.3.2, γ realises the minimum in (2.2.1) and $\Upsilon(f) = \mathcal{T}_c(f, g)$. As there is no duality gap in (2.3.1) nor in (2.2.1), $\Upsilon^*(f) = \mathcal{T}_c^*(f, g)$. Additionally, $(\psi^{\bar{c}}, \psi^{\bar{c}c})$ is admissible for $\mathcal{T}_c^*(f, g)$ and $\psi = (\psi^{\bar{c}c})_-$. Thus

$$K_f(\psi^{\bar{c}}, \psi) = \int f \psi^{\bar{c}} \, d\lambda + \int (1 - f)(\psi^{\bar{c}c})_- \, d\lambda = \mathcal{T}_c^*(f, g) \geq \int f \psi^{\bar{c}} \, d\lambda + \int g \psi^{\bar{c}c} \, d\lambda. \tag{2.3.15}$$

Hence

$$\int (1 - f - g)(\psi^{\bar{c}c})_- \, d\lambda \geq \int g (\psi^{\bar{c}c})_+ \, d\lambda.$$

Since $(1 - f - g)(\psi^{\bar{c}c})_- \leq 0$ and $g(\psi^{\bar{c}c})_+ \geq 0$, the integrands must vanish λ-almost everywhere: we deduce (2.3.14). Additionally, the inequality in (2.3.15) is an equality. Consequently, $(\psi^{\bar{c}}, \psi^{\bar{c}c})$ is a pair of Kantorovitch potentials for (2.2.2). □

To end this section, we establish a comparison principle on the potentials maximising (2.3.4). We say that a set $\Psi \subset C(X)$ admits a minimal (respectively maximal) element for the relation $\leq$ if there exists $\psi_0 \in \Psi$ such that for any $\psi \in \Psi$, $\psi_0 \leq \psi$ (respectively $\psi_0 \geq \psi$).

Proposition 2.3.8. *Assume that X is a compact metric space and that $c \in C(X \times X, \mathbb{R})$. Let $f \in L^1_m$ and let us define*

$$\Psi_f := \Big\{\psi,\, \psi = \varphi^c_- \text{ for some } \varphi \in \Phi' \text{ and } K_f(\psi^{\bar{c}}, \psi) = \Upsilon^*(f)\Big\},$$

where K_f is defined in (2.3.2) *and Φ' in Lemma 2.3.3.*
Then:

(i) Ψ_f admits a maximal element for the relation $\leq$, denoted by ψ_f in the sequel,

(ii) For $f_1, f_2 \in L^1_m$, there holds $f_1 \leq f_2 \implies \psi_{f_1} \geq \psi_{f_2}$.

Proof. Step 1. Sufficient condition and preliminary claim.

Notice that Ψ_f is not empty by Proposition 2.3.4. To obtain (i), we prove that the set (which is not empty since Ψ_f is not empty)

$$\Phi_f := \left\{\varphi \in \Phi',\, K_f(\varphi, \varphi^c_-) = \Upsilon^*(f)\right\}$$

admits a minimal element φ_f and then $\psi_f := (\varphi^c_f)_-$ is the desired maximal element of Ψ_f. Let us make a preliminary observation.

Claim. Let $f_1, f_2 \in L^1_m$ with $f_1 \leq f_2$ and set $\varphi_i \in \Phi_{f_i}$ for $i \in \{1,2\}$. Then $\varphi_\wedge := \varphi_1 \wedge \varphi_2 \in \Phi_{f_1}$.

Let us first prove that $\varphi_\wedge \in \Phi'$. In the sequel we write $\varphi^c_i := (\varphi_i)^c$ and $\varphi^c_{i\,-} := ((\varphi_i)^c)_-$ for $i \in \{1,2,\wedge\}$. We observe that $\varphi_\wedge \in C(X)$. By definition of the c-transform, we obtain

$$\varphi^c_\wedge = (\varphi_1 \wedge \varphi_2)^c \geq \varphi^c_1 \vee \varphi^c_2 \tag{2.3.16}$$

and

$$(\varphi_1 \vee \varphi_2)^c = \varphi^c_1 \wedge \varphi^c_2. \tag{2.3.17}$$

Since for $i \in \{1,2\}$, $\max \varphi^c_i \geq 0$ we have by (2.3.16) that $\max \varphi^c_\wedge \geq 0$.

We now prove that $(\varphi^c_{\wedge-})^{\bar{c}} = \varphi_\wedge$. We observe that $\varphi^c_{\wedge-} \leq \varphi^c_\wedge$ which implies $(\varphi^c_{\wedge-})^{\bar{c}} \geq \varphi^{c\bar{c}}_\wedge$. By Proposition 2.2.4 (ii), $\varphi^{c\bar{c}}_\wedge \geq \varphi_\wedge$ so that $(\varphi^c_{\wedge-})^{\bar{c}} \geq \varphi_\wedge$. Conversely, taking the negative part of (2.3.16), we have $\varphi^c_{\wedge-} \geq (\varphi^c_1 \vee \varphi^c_2)_- = \varphi^c_{1-} \vee \varphi^c_{2-}$. Taking the $\bar{c}$-transform and using (2.3.17) (with $\bar{c}$ instead of c) yields

$$(\varphi^c_{\wedge-})^{\bar{c}} \leq (\varphi^c_{1-} \vee \varphi^c_{2-})^{\bar{c}} = (\varphi^c_{1-})^{\bar{c}} \wedge (\varphi^c_{2-})^{\bar{c}} = \varphi_1 \wedge \varphi_2 = \varphi_\wedge.$$

Hence $(\varphi^c_{\wedge-})^{\bar{c}} = \varphi_\wedge$ and $\varphi_\wedge \in \Phi'$.

We now show that the pair $(\varphi_\wedge, \varphi^c_{\wedge-})$ maximises $\Upsilon^*(f_1)$. We set

$$\Delta_K := K_{f_1}(\varphi_\wedge, \varphi^c_{\wedge-}) - K_{f_1}(\varphi_1, \varphi^c_{1-}) = \int f_1(\varphi_\wedge - \varphi_1) + \int (1-f_1)(\varphi^c_{\wedge-} - \varphi^c_{1-}).$$

By optimality of φ_1, $\Delta_K \leq 0$. Let us prove the converse inequality. Substituting $f_1 = f_2 + f_1 - f_2$ in the definition of Δ_K, we obtain

$$\Delta_K = \int f_2(\varphi_\wedge - \varphi_1) + \int (1-f_2)(\varphi^c_{\wedge-} - \varphi^c_{1-}) + \int (f_2 - f_1)(\varphi_1 - \varphi_\wedge + \varphi^c_{\wedge-} - \varphi^c_{1-}).$$

We have $f_2 - f_1 \geq 0$ and $\varphi_1 - \varphi_\wedge \geq 0$. Additionally, $\varphi^c_\wedge \geq \varphi^c_1$, so that $\varphi^c_{\wedge-} \geq \varphi^c_{1-}$. Thus the last integral in Δ_K is non-negative. Adding and subtracting $f_2\varphi_2$ in the first integral yields

$$\Delta_K \geq \int f_2\varphi_2 + \int f_2(\varphi_\wedge - \varphi_1 - \varphi_2) + \int (1-f_2)(\varphi^c_{\wedge-} - \varphi^c_{1-}). \tag{2.3.18}$$

Let us set $\varphi_\vee := \varphi_1 \vee \varphi_2$. By optimality of φ_2, we have $K_{f_2}(\varphi_2, \varphi^c_{2-}) \geq K_{f_2}(\varphi_\vee, \varphi^c_{\vee-})$, which rewrites as

$$\int f_2\varphi_2 \geq \int f_2\varphi_\vee + \int (1-f_2)(\varphi^c_{\vee-} - \varphi^c_{2-}).$$

Injecting this inequality in the first term of the right-hand side of (2.3.18) yields

$$\Delta_K \geq \int f_2(\varphi_\wedge + \varphi_\vee - \varphi_1 - \varphi_2) + \int (1-f_2)(\varphi^c_{\vee -} + \varphi^c_{\wedge -} - \varphi^c_{1-} - \varphi^c_{2-}). \tag{2.3.19}$$

The integrand in the first integral of (2.3.19) vanishes. Regarding the second term, using (2.3.17) and (2.3.16) we obtain

$$\varphi^c_{\vee -} + \varphi^c_{\wedge -} \geq \varphi^c_{1-} \wedge \varphi^c_{2-} + \varphi^c_{1-} \vee \varphi^c_{2-} = \varphi^c_{1-} + \varphi^c_{2-}.$$

Hence the integrand in the second integral is non-negative. We conclude that $\Delta_K \geq 0$ and finally that $\Delta_K = 0$ so that the claim is proved.

Step 2. Construction of the minimal element of Φ_f.

By Lemma 2.3.4, Φ' is compact. As $\varphi \mapsto K_f(\varphi, \varphi^c_-)$ is continuous for the norm of uniform convergence, Φ_f is compact as well. Let $(\varphi_j)_{j\geq 0}$ be a dense subset of Φ_f. For $x \in X$ and $j \geq 0$, we define $\widetilde{\varphi}_j$ and φ_f by

$$\widetilde{\varphi}_j(x) := \min(\varphi_0(x), \ldots, \varphi_j(x)) \qquad \text{and} \qquad \varphi_f(x) := \inf\{\varphi(x),\, \varphi \in \Phi_f\}.$$

Using our preliminary claim with $f_1 = f_2 = f$ recursively, we obtain that for any $j \geq 0$, $\widetilde{\varphi}_j \in \Phi_f$. As Φ_f is compact and $\widetilde{\varphi}_j \to \varphi_f$ pointwise, we obtain that $\widetilde{\varphi}_j \to \varphi_f$ uniformly and $\varphi_f \in \Phi_f$, so that φ_f is the desired minimal element of Φ_f.

Step 3. Conclusion.

Taking $\psi_f := \varphi^c_{f\,-}$ proves (i). Let $f_1 \leq f_2$ as given in the statement of (ii). By the previous step, there exist φ_1, φ_2 respective minimal elements for Φ_{f_1} and Φ_{f_2} such that $\psi_1 := \varphi^c_{1-}$ and $\psi_2 := \varphi^c_{2-}$ are respective maximal elements for Ψ_{f_1} and Ψ_{f_2}. By the preliminary claim, $\varphi_1 \wedge \varphi_2 \in \Phi_{f_1}$ and by minimality of φ_1 we have $\varphi_1 \leq \varphi_1 \wedge \varphi_2$, so that $\varphi_2 \geq \varphi_1$. Hence $\psi_2 \leq \psi_1$. □

2.4 The case of translation invariant costs in Euclidean spaces

We now assume that $X = \mathbb{R}^d$, that λ is the Lebesgue measure and that $c(x,y) = k(y-x)$, with $k : \mathbb{R}^d \to \mathbb{R}_+$. We recall the following hypotheses on k.

(H1) $k \in C(\mathbb{R}^d, \mathbb{R}_+)$, $k(0) = 0$ and $k(x) \to \infty$ as $|x| \to \infty$,

(H2) $\forall x \neq 0$,

$$\limsup_{r\to 0} \frac{1}{r^d} \left| B_r(x) \cap \{y \in \mathbb{R}^d,\, k(y) < k(x)\} \right| > 0,$$

(H3) $\forall\, \sigma \in \mathbb{S}^{d-1}$, $r \mapsto k(r\sigma)$ is increasing on $\mathbb{R}_+$.

Notice that under hypotheses (H1) and (H2), there holds $k(x) > 0$ for $x \neq 0$.

The primal problem is now defined as

$$\Upsilon(f) := \inf\left\{ \int c\, d\gamma : \gamma \in \Pi_f \right\}, \tag{2.4.1}$$

where

$$\Pi_f := \Big\{\gamma \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d) : \gamma_x = f,\, \gamma_y \le 1 - f\Big\}.$$

The goal of this section is to prove that for every $m > 0$ the energy

$$\mathcal{E}(m) := \sup\left\{\Upsilon(f) : f \in L^1(\mathbb{R}^d, [0,1]),\, \int f = m\right\} \tag{2.4.2}$$

admits a maximiser.

2.4.1 First properties of the exterior transport functional and saturation theorem

In this subsection, we collect some properties of the functional Υ defined in $\mathbb{R}^d$ and establish a saturation property (Theorem 2.4.4), namely that if γ is a minimiser for $\Upsilon(f)$ then $\gamma_y(x) \in \{f(x), 1 - f(x)\}$ for almost every $x \in \mathbb{R}^d$.

We start by proving that minimisers of (2.4.1) exist. The proof of this result is similar to the proof of [19, Proposition 2.1], but with weaker assumptions on the cost c and in the context of functions taking values in $[0,1]$ rather than in $\{0,1\}$.

Proposition 2.4.1. *Assume that k satisfies (H1). Then, for any $m > 0$ and $f \in L^1_m$, the infimum in* (2.4.1) *is attained. Additionally, given any minimiser γ of* (2.4.1) *we have $\Upsilon(f) = \mathcal{T}_c(f,g)$, where $g := \gamma_y$.*

Lastly, there exists $R = R(m)$ non-decreasing in m such that for any $f \in L^1_m$,

$$\Upsilon(f) = \min\left\{\int c\, d\gamma : \gamma \in \Pi_f,\, \forall\, (x,y) \in \operatorname{supp}\gamma,\, |x-y| \le R\right\} \tag{2.4.3}$$

and for any minimiser γ of (2.4.1)*, there holds $|x-y| \le R$ on* $\operatorname{supp}\gamma$*.*

Proof. The strategy of the proof is to first establish (2.4.3) with an infimum in place of the minimum. Then we use this property to derive compactness for (2.4.1).

Step 1. Restricting the set of competitors for (2.4.1)*.*

We let $\gamma \in \Pi_f$ and set $g := \gamma_y$. We want to build a competitor $\widetilde{\gamma}$ for $\Upsilon(f)$ such that for some $R > 0$, $|x-y| \le R$ for every $(x,y) \in \operatorname{supp}\widetilde{\gamma}$. For $R > 0$, we define

$$\Gamma_R := \Big\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d,\, |x-y| \ge R\Big\}.$$

We consider a standard partition of $\mathbb{R}^d$ into cubes $(Q_i)_{i\ge 0}$ with side-length $\rho_1(m) := (3m)^{1/d}$. We define

$$I := \{i \ge 0,\, m_i := \gamma(\Gamma_R \cap (Q_i \times \mathbb{R}^d)) > 0\},$$

and for $i \in I$, we set

$$\gamma_{\text{bad},i} := \chi_{\Gamma_R \cap (Q_i \times \mathbb{R}^d)}\gamma.$$

As $|Q_i| - \int f - \int g \ge m \ge m_i$, there exists a positive measure $\mu_i \ll \chi_{Q_i}\lambda$ such that

$$\mu_i \le \chi_{Q_i}(1 - f - g) \qquad \text{and} \qquad \mu_i(\mathbb{R}^d) = \gamma_{\text{bad},i}(\mathbb{R}^d \times \mathbb{R}^d).$$

Denoting by θ_i the first marginal of $\gamma_{\text{bad},i}$ we set

$$\widetilde{\gamma_i} := \chi_{Q_i\times\mathbb{R}^d}\gamma - \gamma_{\text{bad},i} + \frac{1}{m_i}\theta_i \otimes \mu_i \ \geq 0.$$

For $i \in I^c$, we simply define $\widetilde{\gamma_i} := \chi_{(Q_i\times\mathbb{R}^d)}\gamma$. As a consequence, $\widetilde{\gamma} := \sum_{i\geq 0}\widetilde{\gamma_i}$ is a transport plan whose first marginal is f and second marginal $\widetilde{g}$ verifies $\widetilde{g} \leq g \leq 1-f$. By construction, for $R > \sqrt{d}\rho_1(m)$, we have $\widetilde{\gamma}(\Gamma_R) = 0$.

Let us now compare the transportation cost of γ and $\widetilde{\gamma}$. We compute:

$$\begin{aligned}\int c\,d\widetilde{\gamma} - \int c\,d\gamma &= \sum_{i\in I}\int_{Q_i\times\mathbb{R}^d} c\,d\left(\frac{\theta_i\otimes\mu_i}{m_i} - \gamma_{\text{bad},i}\right)\\ &\leq \left(\sum_{i\in I} m_i\right)\left(\max_{z\in\overline{Q}_{\rho_1(m)}} k(z) - \inf_{|z|\geq R} k(z)\right).\end{aligned}$$

Let us set $\overline{Q}_{\rho_1(m)} := [0;\rho_1(m)]^d$ and then $M := \max\{k(z), z\in\overline{Q}_{\rho_1(m)}\}$. By (H1), there exists $R > \sqrt{d}\rho_1(m)$ such that if $|z| > R$, then $k(z) > M$. With this choice of R we have $\int c\,d\widetilde{\gamma} \leq \int c\,d\gamma$. Lastly, whenever $\gamma(\Gamma_R) > 0$,

$$\int c\,d\widetilde{\gamma} < \int c\,d\gamma. \tag{2.4.4}$$

Step 2 : Lower semi-continuity of the transportation cost.

This step is classical. To prove that $\gamma \mapsto \int c\,d\gamma$ is lower semi-continuous with respect to weak convergence, we proceed by approximation. Let us assume that $\gamma_n \overset{*}{\rightharpoonup} \gamma$ as $n\to\infty$. For $j\geq 0$, we define $c_j := c\wedge j$. The sequence c_j is non-decreasing and converges pointwise to c. For every $j \geq 0$, $c_j \in C_b(\mathbb{R}^d\times\mathbb{R}^d)$, so that

$$\int c_j\,d\gamma = \lim_n \int c_j\,d\gamma_n \leq \liminf_n \int c\,d\gamma_n.$$

By the monotone convergence theorem,

$$\int c\,d\gamma = \lim_j \int c_j\,d\gamma \leq \liminf_n \int c\,d\gamma_n,$$

which concludes the second step of the proof.

Step 3. $\Upsilon(f)$ admits a minimiser.

Let γ_n be a minimising sequence for (2.4.1). Let us show that the sequence γ_n is tight. By the first step, we can assume that there exists $R = R(m)$ such that for any $n \geq 0$ there holds $|x-y| \leq R$ on $\operatorname{supp}\gamma_n$. Now, because $\int f \leq m < \infty$, there exists $R' = R'(m) > 0$ such that $\int_{\mathbb{R}^d\setminus B_{R'}} f \leq \varepsilon$. Hence,

$$\gamma_n\left(\mathbb{R}^d\times\mathbb{R}^d \setminus (B_{R'}\times B_{R+R'})\right) = \gamma_n\left(B_{R'}\times(\mathbb{R}^d\setminus B_{R+R'})\right) + \gamma_n\left((\mathbb{R}^d\setminus B_{R'})\times\mathbb{R}^d\right) \leq 0 + \varepsilon m$$

which proves that the sequence γ_n is tight. Together with the second step, this shows that (2.4.1) admits a minimiser. Moreover, by (2.4.4) for any minimiser γ of (2.4.1) there holds $|x-y| \leq R$ on $\operatorname{supp}\gamma$. Lastly, setting $g := \gamma_y$ the identity $\Upsilon(f) = \mathcal{T}_c(f,g)$ is immediate. □

We now establish some basic properties of the functional Υ. The results here are similar to [19, Proposition 2.2 & Lemma 2.4].

Proposition 2.4.2. *Assume that k satisfies (H1). Given $m > 0$ and $f_1, f_2 \in L^1_m$ we have:*

(i) If $f_1 + f_2 \le 1$, then

$$\Upsilon(f_1 + f_2) \ge \Upsilon(f_1) + \Upsilon(f_2).$$

As a consequence, if $f_1 \le f_2$, then $\Upsilon(f_1) \le \Upsilon(f_2)$.

(ii) There exists $R = R(m)$ such that if $d(\operatorname{supp} f_1, \operatorname{supp} f_2) \ge R$, then

$$\Upsilon(f_1 + f_2) = \Upsilon(f_1) + \Upsilon(f_2).$$

(iii) There exists $C = C(m) > 0$ such that

$$|\Upsilon(f_1) - \Upsilon(f_2)| \le C\|f_1 - f_2\|_{L^1}.$$

(iv) Let $f, f_n \in L^1(\mathbb{R}^d, [0,1])$ be such that the sequence f_n is tight and $f_n \overset{*}{\rightharpoonup} f$. Then $\Upsilon(f_n) \to \Upsilon(f)$.

Proof. Step 1. Proof of (i)&(ii).

To prove (i), we consider a transport plan γ optimal for $\Upsilon(f_1 + f_2)$ whose existence is guaranteed by Proposition 2.4.1. We would like to extract from γ two plans γ^1 and γ^2 admissible for $\Upsilon(f_1)$ and $\Upsilon(f_2)$ respectively. Using the convention $0/0 = 0$, we define γ^1 and γ^2 through

$$d\gamma^1(x,y) := \frac{f_1(x)}{(f_1 + f_2)(x)}\, d\gamma(x,y) \qquad \text{and} \qquad d\gamma^2(x,y) := \frac{f_2(x)}{(f_1 + f_2)(x)}\, d\gamma(x,y).$$

By construction, $\gamma^1_x = f_1$ and $\gamma^2_x = f_2$. We also have $\gamma^1 \le \gamma$, so that

$$\gamma^1_y \le \gamma_y \le 1 - (f_1 + f_2) \le 1 - f_1.$$

Likewise, $\gamma^2_y \le 1 - f_2$. Therefore, γ^1 and γ^2 are admissible for $\Upsilon(f_1)$ and $\Upsilon(f_2)$ respectively. Moreover

$$\Upsilon(f_1) + \Upsilon(f_2) \le \int c\, d\gamma^1 + \int c\, d\gamma^2 = \int c\, d\gamma = \Upsilon(f_1 + f_2),$$

which is the desired conclusion.

To prove (ii), we consider transport plans γ^1 and γ^2 which are optimal for $\Upsilon(f_1)$ and $\Upsilon(f_2)$ respectively. We define $g_1 := \gamma^1_y$ and $g_2 := \gamma^2_y$. If we set $\gamma := \gamma^1 + \gamma^2$, we have $\gamma_x = f_1 + f_2$ and $\gamma_y = g_1 + g_2$. Moreover, by Proposition 2.4.1, if $d(\operatorname{supp} f_1, \operatorname{supp} f_2) \ge R$ for $R = R(m)$ large enough, then the supports of g_1 and g_2 are also disjoint. Consequently, $g_1 + g_2 \le 1 - (f_1 + f_2)$, so that γ is admissible for $\Upsilon(f_1 + f_2)$ and we have the desired converse inequality

$$\Upsilon(f_1 + f_2) \le \int c\, d(\gamma^1 + \gamma^2) \le \Upsilon(f_1) + \Upsilon(f_2).$$

Step 2. Proof of (iii).

Exchanging the roles of f_1 and f_2, it is enough to prove the estimate

$$\Upsilon(f_2) - \Upsilon(f_1) \le C\|f_2 - f_1\|_{L^1}. \tag{2.4.5}$$

Let γ^1 be a minimiser of $\Upsilon(f_1)$ and let us set $g_1 := \gamma^1_x$. In the next substeps, we build from γ^1 an exterior transport plan γ^2 for f_2 with controlled cost.

Step 2.a. Transporting most of $f_1 \wedge f_2$.

Using the convention $0/0 = 0$, we define a plan γ' by

$$d\gamma'(x,y) := \frac{(f_1 \wedge f_2)(x)}{f_1(x)} d\gamma_1(x,y).$$

We set $g' := \gamma'_y$. Notice that $\gamma' \le \gamma^1$, which implies $g' \le g_1$. Additionally, $\gamma'_x = f_1 \wedge f_2$, so that

$$\gamma^1(\mathbb{R}^d \times \mathbb{R}^d) - \gamma'(\mathbb{R}^d \times \mathbb{R}^d) = \int (f_1 - f_1 \wedge f_2) = \int (f_1 - f_2)_+. \tag{2.4.6}$$

Heuristically, γ' corresponds to sending through γ^1 as much mass from f_2 as possible. However, we have to remove some of this mass because the constraint $g' \le 1 - f_2$ might not hold true everywhere. Let

$$u := (f_2 + g' - 1)_+$$

and define γ'' as

$$d\gamma''(x,y) := \frac{g'(y) - u(y)}{g'(y)} d\gamma'(x,y).$$

We set $f'' := \gamma''_x$ and $g'' := \gamma''_y$. By construction, $g'' = g' - u$ so $g'' \le 1 - f_2$ as desired. Since $\gamma'' \le \gamma'$, we also have $f'' \le f_1 \wedge f_2 \le f_2$. Now since $g' \le g_1 \le 1 - f_1$ we have $u \le (f_2 - f_1)_+$ from which we infer

$$\gamma'(\mathbb{R}^d \times \mathbb{R}^d) - \gamma''(\mathbb{R}^d \times \mathbb{R}^d) = \int (g' - g'') = \int u \le \int (f_2 - f_1)_+.$$

Summing this and (2.4.6) yields

$$\gamma^1(\mathbb{R}^d \times \mathbb{R}^d) - \gamma''(\mathbb{R}^d \times \mathbb{R}^d) \le \|f_2 - f_1\|_{L^1}. \tag{2.4.7}$$

Eventually since $\gamma'' \le \gamma' \le \gamma^1$ and $c \ge 0$ we have

$$\int c\, d\gamma'' - \int c\, d\gamma^1 \le 0. \tag{2.4.8}$$

Step 2.b. Final construction.

We are now ready to build an admissible transport plan γ^2 for $\Upsilon(f_2)$. Noticing that $f_2 - f'' \ge 0$ we write $f_2 = f'' + (f_2 - f'')$. By (2.4.7) we have

$$\int (f_2 - f'') = \int (f_2 - f_1) + \int (f_1 - f'') \le 2\|f_2 - f_1\|_{L^1}. \tag{2.4.9}$$

Arguing as in the proof of Proposition 2.4.1, we can find a function $0 \leq g''' \leq 1 - f_2 - g''$ with $\int g''' \leq \int (f_2 - f'')$ and a transport plan γ''' between $f_2 - f''$ and g''' such that for some $C = C(m) > 0$,

$$\int c\, d\gamma''' \leq C \int (f_2 - f'') \overset{(2.4.9)}{\leq} C\|f_2 - f_1\|_{L^1}. \tag{2.4.10}$$

Finally we define $\gamma^2 := \gamma'' + \gamma'''$ which is admissible for $\Upsilon(f_2)$ by construction. Summing (2.4.8) and (2.4.10), we get

$$\Upsilon(f_2) \leq \int c\, d\gamma^2 \leq \int c\, d\gamma_1 + C\|f_2 - f_1\|_{L^1} = \Upsilon(f_1) + C\|f_2 - f_1\|_{L^1}.$$

This proves (2.4.5) and thus point (iii).

Step 3. Proof of (iv).

Let f_n and f be as in the statement of the proposition. By weak convergence, we have $f_n, f \in L^1_m$ for some $m > 0$. Using the Lipschitz continuity of Υ with respect to L^1 convergence, we may assume without loss of generality that f_n (and thus also f) are supported in $\overline{B}_{R_0}$ for some $R_0 > 0$. Applying Proposition 2.4.1 we get that minimisers of $\Upsilon(f_n)$ and $\Upsilon(f)$ are supported in $\overline{B}_R \times \overline{B}_R$ for some $R > R_0 > 0$. We may thus restrict these problems to the compact set $\overline{B}_R$. Using Proposition 2.3.5 we have $\Upsilon(f_n) = \Upsilon^*(f_n)$ and it is thus enough to prove the continuity of Υ^* with respect to the weak-$*$ topology.

By Proposition 2.3.4, for every $n \geq 0$ there exists a pair of potentials (φ_n, ψ_n) maximising $\Upsilon^*(f_n)$. Since for every n, φ_n belongs to Φ' (where Φ' is defined by (2.3.5)) and since this set is compact by Proposition 2.3.4 we have that a subsequence $\varphi_{n'}$ of φ_n converges in $C(\overline{B}_R)$ to some $\varphi \in \Phi'$. Arguing as in the proof of Proposition 2.3.4 we see that $\psi_{n'}$ also converges to ψ with (φ, ψ) admissible for $\Upsilon^*(f)$. By weak-strong convergence we then have

$$\limsup \Upsilon^*(f_{n'}) = \limsup \int f_{n'}\varphi_{n'} + (1 - f_{n'})\psi_{n'} = \int f\varphi + (1 - f)\psi \leq \Upsilon^*(f).$$

Similarly, if (φ, ψ) are optimal potentials for $\Upsilon^*(f)$, they are admissible for $\Upsilon^*(f_n)$ and thus

$$\liminf \Upsilon^*(f_n) \geq \liminf \int f_n\varphi + (1 - f_n)\psi = \int f\varphi + (1 - f)\psi = \Upsilon^*(f).$$

We then have $\lim \Upsilon^*(f_{n'}) = \Upsilon^*(f)$ and by uniqueness of the limit we see that the extraction was not necessary. This establishes (iv) and ends the proof of the proposition. □

The next lemma and theorem state very important saturation properties satisfied by the optimal exterior transport plan. These results extend [36, Lemma 5.1 & Proposition 5.2] to more general costs c.

Lemma 2.4.3. *Assume that k satisfies (H1) and (H2). For $f \in L^1_m$ let γ be optimal for $\Upsilon(f)$. Then for every $(x_0, y_0) \in \operatorname{supp} \gamma$ there holds $f + \gamma_y \equiv 1$ almost everywhere on the saturation set*

$$S(x_0, y_0) := \{y \in \mathbb{R}^d,\ k(y - x_0) < k(y_0 - x_0)\}.$$

Proof. In the proof we set $g := \gamma_y$ and $h := f + g$. Let $(x_0, y_0) \in \operatorname{supp}\gamma$ and assume without loss of generality that $x_0 = 0$. We suppose by contradiction that there exists $\varepsilon > 0$ such that the set

$$S_\varepsilon := \{h < 1\} \cap \{y \in \mathbb{R}^d,\, k(y) < k(y_0) - \varepsilon\}$$

has positive Lebesgue measure. Notice that by (H1), $k(x) \to \infty$ as $|x| \to \infty$ so that S_ε is bounded. Therefore,

$$m_\varepsilon := \int_{S_\varepsilon} (1-h) \ \in (0,\infty).$$

We now exhibit an exterior transport plan $\tilde\gamma$ whose transportation cost is strictly smaller than the one of γ. Given $r > 0$, we define the measure $\gamma^0 := \gamma \llcorner (B_r \times B_r(y_0))$. As $(0, y_0) \in \operatorname{supp}\gamma$, for every $r > 0$,

$$0 < \gamma^0(\mathbb{R}^d \times \mathbb{R}^d) \le \int_{B_r} f \ \le |B_r|. \tag{2.4.11}$$

Thus, by the last inequality in (2.4.11) there exists $r_\varepsilon > 0$ such that for every $r \in (0, r_\varepsilon]$,

$$\gamma^0(\mathbb{R}^d \times \mathbb{R}^d) = \alpha m_\varepsilon$$

for some $0 < \alpha \le 1$. Let us fix $r \in (0, r_\varepsilon]$. We define a competitor $\widetilde\gamma$ for $\Upsilon(f)$ by setting $\widetilde\gamma := \gamma - \gamma^0 + \eta$, where

$$\eta := \gamma^0_x \otimes \frac{1-h}{m_\varepsilon} \chi_{S_\varepsilon}.$$

By construction, $\widetilde\gamma_x = \gamma_x = f$. We also have

$$f + \widetilde\gamma_y \le f + g + \alpha(1-h)\chi_{S_\varepsilon} \le h + \alpha(1-h) = 1 - (1-h)(1-\alpha) \le 1,$$

so that $\widetilde\gamma$ is admissible for $\Upsilon(f)$. We compute

$$\int c\, d\widetilde\gamma - \int c\, d\gamma = \int c\, d\eta - \int c\, d\gamma^0 \le \alpha m_\varepsilon \left(\max_{\overline{B}_r \times \overline{S}_\varepsilon} c(x,y) - \min_{\overline{B}_r \times \overline{B}_r(y_0)} c(x,y) \right).$$

By continuity of c there exists $r_\varepsilon > 0$ such that for $0 < r \le r_\varepsilon$,

$$\max_{\overline{B}_r \times \overline{S}_\varepsilon} c(x,y) \le k(y_0) - \varepsilon/2 \qquad \text{and} \qquad \min_{\overline{B}_r \times \overline{B}_r(y_0)} c(x,y) \ge k(y_0) - \varepsilon/4.$$

Thus for $0 < r \le r_\varepsilon$,

$$\int c\, d\widetilde\gamma - \int c\, d\gamma \le -\alpha\varepsilon m_\varepsilon/4 < 0,$$

which contradicts the fact that γ is a minimiser for $\Upsilon(f)$. □

Theorem 2.4.4. *Assume that k satisfies (H1) and (H2). For $f \in L^1_m$, let $\gamma \in \Pi_f$ be a minimiser of* (2.4.1) *and set $g := \gamma_y$. Then, defining*

$$E := \{x : \exists\, y \ne x \text{ such that } (x,y) \in \operatorname{supp}\gamma \text{ or } (y,x) \in \operatorname{supp}\gamma\},$$

the set E is Lebesgue measurable and we have the identity $g = (1-f)\chi_E + f\chi_{E^c}$.

Proof. Step 1. A preliminary claim.

We first prove the following. Let $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^d)$ be such that $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$, and let $\gamma \in \Pi(\mu,\nu)$. Assume that μ and ν are two nonnegatives Radon measure on $\mathbb{R}^d$. If we define the set

$$\mathcal{A}(\gamma) := \{x : \exists\, y \neq x \text{ such that } (x,y) \in \operatorname{supp}\gamma\},$$

then $\mu \leq \nu$ on $\mathcal{A}(\gamma)^c$.

To prove the claim, let us first show that $\mathcal{A}(\gamma)$ is Lebesgue measurable. We define

$$\mathcal{D}(\gamma) := \operatorname{supp}\gamma \setminus \{(x,x) : x \in \mathbb{R}^d\},$$

which is a Borel set of $\mathbb{R}^d \times \mathbb{R}^d$. If we denote by $p_x : X \times X \to X$ the canonical projection on the first variable, we have

$$p_x(\mathcal{D}(\gamma)) = \{x \in \mathbb{R}^d : \exists y \neq x,\, (x,y) \in \operatorname{supp}\gamma\} = \mathcal{A}(\gamma).$$

Thus $\mathcal{A}(\gamma)$ is the continuous image of a Borel set. By [39, Proposition 2.2.13], $\mathcal{A}(\gamma)$ is Lebesgue measurable, as well as μ-measurable and ν-measurable.

We now show that $\mu \leq \nu$ on $\mathcal{A}(\gamma)^c$. Let $\phi \in C_c(\mathbb{R}^d, \mathbb{R}_+)$. By definition of $\mathcal{A}(\gamma)$, if $(x,y) \in \operatorname{supp}\gamma$ and $x \in \mathcal{A}(\gamma)^c$ then $x = y$. Therefore

$$\begin{aligned}\int_{\mathcal{A}(\gamma)^c} \phi\, d\mu = \int \phi(x)\chi_{\mathcal{A}(\gamma)^c}(x)\, d\gamma(x,y) &= \int \phi(y)\chi_{\mathcal{A}(\gamma)^c}(y)\chi_{\mathcal{A}(\gamma)^c}(x)\, d\gamma(x,y)\\ &\leq \int \phi(y)\chi_{\mathcal{A}(\gamma)^c}(y)\, d\gamma(x,y) = \int_{\mathcal{A}(\gamma)^c} \phi\, d\nu,\end{aligned}$$

As μ (resp. ν) is a Radon measure, $C_c(\mathbb{R}^d)$ is dense in $L^1(\mu)$ (resp. $L^1(\nu)$). We thus conclude that $\mu \leq \nu$, which proves the claim.

Step 2. Construction of E.

We now consider an optimal exterior transport plan γ for $\Upsilon(f)$ and set $g := \gamma_y$, $h := f + g$. By Proposition 2.4.1, γ is an optimal transport plan from f to g. Let $\overline{\gamma}$ be the image of γ through the map $(x,y) \mapsto (y,x)$ and define

$$E := \mathcal{A}(\gamma) \cup \mathcal{A}(\overline{\gamma}).$$

We have $E^c = \mathcal{A}(\gamma)^c \cap \mathcal{A}(\overline{\gamma})^c$ and by the first step there holds $f \leq g$ and $g \leq f$ almost everywhere on E^c. Hence,

$$g\chi_{E^c} = f\chi_{E^c}.$$

To conclude the proof, we have to show that $g \equiv 1 - f$ on E or equivalently that up to Lebesgue negligible sets $\mathcal{A}(\gamma)$ and $\mathcal{A}(\overline{\gamma})$ are included in $\{g = 1 - f\}$.

On the one hand, if $x_0 \in \mathcal{A}(\gamma)$ is a Lebesgue point of both f and g, there exists $y_0 \neq x_0$ such that $(x_0, y_0) \in \operatorname{supp}\gamma$. By Lemma 2.4.3, denoting

$$S(x_0, y_0) := \{y \in \mathbb{R}^d, k(y - x_0) < k(y_0 - x_0)\},$$

we have $g = 1 - f$ almost everywhere on $S(x_0, y_0)$. Notice that $S(x_0, y_0)$ is an open set and that $x_0 \in S(x_0, y_0)$ (since for $x \neq 0$, $k(x) > 0 = k(0)$). Hence $g(x_0) = 1 - f(x_0)$ and $\mathcal{A}(\gamma) \subset \{g = 1 - f\}$ up to a set of Lebesgue measure zero.

On the other hand, if $y_0 \in \mathcal{A}(\overline{\gamma})$ there exists $x_0 \neq y_0$ such that $(x_0, y_0) \in \operatorname{supp}\gamma$. Let us assume by contradiction that $g(y_0) < 1 - f(y_0)$. Without loss of generality, we can assume that y_0 is a point of Lebesgue density one of $\{g < 1 - f\}$. Then

$$\lim_{r\to 0} \frac{1}{r^d} \Big|\{g = 1 - f\} \cap B(y_0, r)\Big| = 0.$$

Thus by Lemma 2.4.3,

$$\lim_{r\to 0} \frac{1}{r^d} \left|S(x_0, y_0) \cap B(y_0, r)\right| = 0,$$

which contradicts (H2) as $y_0 \neq x_0$. Hence $g(y_0) = 1 - f(y_0)$ and $\mathcal{A}(\overline{\gamma}) \subset \{g = 1 - f\}$. This concludes the proof of the theorem. □

An important corollary is the uniqueness of the second marginal of minimisers of (2.4.1).

Corollary 2.4.5. Assume that k satisfies (H1) and (H2). Let $f \in L^1(\mathbb{R}^d)$. Then all minimisers γ of $\Upsilon(f)$ have the same second marginal γ_y.

Proof. We let γ, γ' be two minimisers of (2.4.1) and define $\widetilde{\gamma} := (\gamma + \gamma')/2$ which also minimises (2.4.1). We denote $g := \gamma_y$, $g' := \gamma'_y$ and $\widetilde{g} := \widetilde{\gamma}_y$ and introduce the set

$$F := \{x \in \mathbb{R}^d : g(x), g'(x), \widetilde{g}(x) \in \{f(x), 1 - f(x)\}\}.$$

Assuming by contradiction that $g(x) \neq g'(x)$ for some $x \in F$, we have $1/2 = \widetilde{g}(x) \in \{f(x), 1 - f(x)\}$ so that $f(x) = 1/2$ and $g(x) = g'(x) = 1/2$, which is absurd. Hence $g = g'$ on F and since F is of full measure by Theorem 2.4.4, the proof is complete. □

2.4.2 Preliminary results for the existence of a maximiser of the exterior transport energy

We now gather results which, combined with Theorem 2.4.4, allow us to prove existence of a maximiser for both (2.1.2) and (2.1.3).

We first establish a corollary of Theorem 2.4.4 regarding the monotonicity of the sum of the marginals of solutions to (2.4.1).

Corollary 2.4.6. Assume that k satisfies (H1) and (H2). Let $m > 0$, let $f_1, f_2 \in L^1_m$ be such that $f_1 \leq f_2$ and let γ^1, γ^2 be respective minimisers of $\Upsilon(f_1)$ and $\Upsilon(f_2)$. Then setting $g_1 := \gamma^1_y$ and $g_2 := \gamma^2_y$, we have $f_1 + g_1 \leq f_2 + g_2$.

Proof. Let $f_1, f_2 \in L^1_m$ be such that $f_1 \le f_2$. In the first three steps of the proof, we additionally assume that they are compactly supported. This condition is relaxed in the fourth and final step.

By Proposition 2.4.1, we can assume that the ambient space is a compact ball $\overline{B}_R$. Let γ^1, γ^2 be minimisers for $\Upsilon(f_1)$ and $\Upsilon(f_2)$ respectively. For $i \in \{1,2\}$ we define $g_i := \gamma^i_y$, $h_i := f_i + g_i$ and set

$$F := \{h_1 > h_2\}.$$

We shall prove that $|F| = 0$. By Theorem 2.4.4, there exists $E_1, E_2 \subset \overline{B}_R$ such that

$$h_1 = \chi_{E_1} + 2f_1\chi_{E_1^c} \qquad \text{and} \qquad h_2 = \chi_{E_2} + 2f_2\chi_{E_2^c}.$$

Since $h_2 \ge 0$, $h_1 \le 1$ and $h_2 \ge f_2 \ge f_1$ we have

$$h_1 > 0, \qquad h_2 = 2f_2 < 1 \qquad \text{and} \qquad f_1 < 1 \quad \text{on } F. \tag{2.4.12}$$

Step 1. $|E_1^c \cap F| = 0$.

By definition of E_1 we have $h_1 = 2f_1$ on E_1^c and by (2.4.12) we have $h_2 = 2f_2$ on F and since $f_1 \le f_2$ we get $h_1 \le h_2$ on $E_1^c \cap F$. This contradicts the definition of F, hence $E_1^c \cap F = \emptyset$ and in particular $|E_1^c \cap F| = 0$. Notice that as a consequence $h_1 = 1$ on F.

Step 2. Intermediate claim.

Let ψ_1 be the maximal potential for $\Upsilon^*(f_1)$ given by Proposition 2.3.8. We define

$$G := \{\psi_1^{\bar{c}c} = 0\} \cap E_1 \cap F$$

and claim that $|G| = 0$. Let us assume by contradiction that $|G| > 0$. First notice that on F,

$$f_1 + g_1 = h_1 > h_2 = 2f_2,$$

so that

$$g_1 > 2f_2 - f_1 \ge f_2 \ge f_1.$$

Thus

$$G \subset E_1 \cap F \subset \{g_1 > f_1\}. \tag{2.4.13}$$

Now recall that by Theorem 2.4.4,

$$E_1 = \{x : \exists y \ne x \text{ such that } (x,y) \in \operatorname{supp}\gamma^1 \text{ or } (y,x) \in \operatorname{supp}\gamma^1\}.$$

Together with (2.4.13) we obtain that for almost every $y_0 \in G$ there exists $x_0 \ne y_0$ with $(x_0, y_0) \in \operatorname{supp}\gamma^1$. Without loss of generality, we assume that y_0 is a point of positive density of G and we set

$$S(x_0,y_0) := \Big\{y \in \mathbb{R}^d \,:\, k(y - x_0) < k(y_0 - x_0)\Big\}.$$

By (H2), we have $|G \cap S(x_0,y_0)| > 0$. Let now $\widetilde{y} \in G \cap S(x_0,y_0)$. By Proposition 2.3.7, $(\psi_1^{\bar{c}}, \psi_1^{\bar{c}c})$

forms a pair of Kantorovich potentials for the optimal transport from f_1 to g_1. Thus

$$\psi_1^{\bar{c}}(x_0) + \psi_1^{\bar{c}c}(y_0) = k(y_0 - x_0) \qquad \text{and} \qquad \psi_1^{\bar{c}}(x_0) + \psi_1^{\bar{c}c}(\widetilde{y}) \leq k(\widetilde{y} - x_0).$$

However, $y_0, \widetilde{y} \in G$, so that $\psi_1^{\bar{c}c}(y_0) = \psi_1^{\bar{c}c}(\widetilde{y}) = 0$, hence

$$\psi_1^{\bar{c}}(x_0) = k(y_0 - x_0) \qquad \text{and} \qquad \psi_1^{\bar{c}}(x_0) \leq k(\widetilde{y} - x_0).$$

Eventually, as $\widetilde{y} \in S(x_0, y_0)$, we conclude that

$$\psi_1^{\bar{c}}(x_0) \leq k(\widetilde{y} - x_0) < k(y_0 - x_0) = \psi_1^{\bar{c}}(x_0),$$

obtaining a contradiction. Thus $|G| = 0$, which is the claim.

Step 3. $|E_1 \cap F| = 0$.

By Proposition 2.3.7,

$$\{\psi_1^{\bar{c}c} > 0\} \subset \{g_1 = 0\} \quad \text{so that} \quad \{g_1 > 0\} \subset \{\psi_1^{\bar{c}c} \leq 0\}.$$

We observe that $g_1 = 1 - f_1$ on $E_1 \cap F$. By (2.4.12), $E_1 \cap F \subset \{g_1 > 0\}$ and by the previous step, $E_1 \cap F \subset \{\psi_1^{\bar{c}c} \neq 0\}$, hence $\psi_1^{\bar{c}c} < 0$ almost everywhere on $E_1 \cap F$.

Let ψ_2 be the maximal potential for $\Upsilon^*(f_2)$ given by Proposition 2.3.8. As $f_1 \leq f_2$, we have $\psi_1 \geq \psi_2$ so that $\psi_1^{\bar{c}c} \geq \psi_2^{\bar{c}c}$. Thus

$$\psi_2^{\bar{c}c} < 0 \text{ on } E_1 \cap F.$$

By Proposition 2.3.7 we deduce that

$$h_2 = g_2 + f_2 = 1 \text{ on } E_1 \cap F.$$

But since $h_2 < 1$ on F we get that $|E_1 \cap F| = 0$ and with the first step we conclude that $|F| = 0$.

Step 4. Extension to the non-compact case.

Let $f_1, f_2 \in L^1_m$ be such that $f_1 \leq f_2$. For $i \in \{1,2\}$, we set $f_{i,R} = f_i \chi_{B_R}$, consider γ_R^i an optimal exterior transport plan for $\Upsilon(f_{i,R})$ and set $g_{i,R} := (\gamma_R^i)_y$. Applying the previous steps to $f_{1,R}$ and $f_{2,R}$, we obtain

$$f_{1,R} + g_{1,R} \leq f_{2,R} + g_{2,R}. \tag{2.4.14}$$

For $i \in \{1,2\}$, $f_{i,R}$ L^1-converges to f_i as $R \to \infty$. By Proposition 2.4.2 (iii), $\Upsilon(f_{i,R}) \to \Upsilon(f_i)$ as $R \to \infty$. Additionally, γ_R^i admits a subsequence converging weakly-$*$ to some $\widetilde{\gamma}^i$ admissible for $\Upsilon(f_i)$. By lower semi-continuity of $\gamma \mapsto \int c\, d\gamma$ with respect to weak-$*$ convergence, we get

$$\int c\, d\widetilde{\gamma}^i \leq \liminf_R \int c\, d\gamma_R^i = \liminf_R \Upsilon(f_{i,R}) = \Upsilon(f_i).$$

Hence $\widetilde{\gamma}^i$ is optimal for $\Upsilon(f_i)$, so that by Corollary 2.4.5, $\widetilde{\gamma}^i_y = g_i$. Finally, as $\gamma_R^i \overset{*}{\rightharpoonup} \widetilde{\gamma}^i$ as $R \to \infty$, $g_{i,R}$ converges in duality with $C_b(\mathbb{R}^d)$ to g_i as $R \to \infty$. Multiplying (2.4.14) by $\phi \in C_c(\mathbb{R}^d, \mathbb{R}_+)$, integrating and passing to the limit we obtain that for any $\phi \in C_c(\mathbb{R}^d, \mathbb{R}_+)$,

$$\int (f_1 + g_1)\phi \leq \int (f_2 + g_2)\phi.$$

Hence $f_1 + g_1 \leq f_2 + g_2$ which completes the proof. $\square$

We now prove that $\mathcal{E}$ is strictly superadditive.

Proposition 2.4.7. *Assume that k satisfies (H1) and (H3). Let $m \in (0,\infty)$ and define $e(m) := \mathcal{E}(m)/m$. Then, e is increasing on $(0,\infty)$. As a consequence, given $0 < m' < m$,*

$$\mathcal{E}(m') + \mathcal{E}(m - m') < \mathcal{E}(m).$$

Proof. Let $M > m > 0$. We have to establish that $\mathcal{E}(m) < (m/M)\mathcal{E}(M)$.

Step 1. $\mathcal{E}(m) \leq (m/M)\mathcal{E}(M)$.
For $R > 0$ we set

$$\Gamma_R := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| > R\}.$$

Let $0 \leq \varepsilon < \mathcal{E}(m)/2$ and $f \in L^1_m$ of mass exactly m and such that $\Upsilon(f) \geq \mathcal{E}(m) - \varepsilon$. We denote $\lambda := (M/m)^{1/d} > 1$ and we set

$$f_\lambda(x) := f(x/\lambda) \qquad \text{for } x \in \mathbb{R}^d,$$

so that $\int f_\lambda = M$. Let γ^λ be an optimal transport plan for $\Upsilon(f_\lambda)$. We define a Radon measure γ by

$$\int \xi(x,y)\, d\gamma(x,y) := \frac{m}{M} \int \xi(x/\lambda, y/\lambda)\, d\gamma^\lambda(x,y) \qquad \text{for } \xi \in C_c(\mathbb{R}^d \times \mathbb{R}^d).$$

Observe that γ is admissible for $\Upsilon(f)$. By Proposition 2.4.1, there exists $R_\lambda = R_\lambda(M)$ such that $\gamma^\lambda(\Gamma_{R_\lambda}) = 0$. Setting $R := R_\lambda/\lambda$, we then have

$$\gamma(\Gamma_R) = \frac{m}{M}\gamma^\lambda(\Gamma_{R_\lambda}) = 0. \tag{2.4.15}$$

Let us define

$$\kappa(r) := \min\{k(z) - k(z/\lambda) : r \leq |z| \leq R_\lambda\}.$$

As $\lambda > 1$ we have by (H3) that $\kappa(r) > 0$ for $0 < r < R_\lambda$. Additionally, $k(z/\lambda) \leq k(z)$ for any $z \in \mathbb{R}^d$. Consequently, for any $0 < r < R_\lambda$,

$$\begin{aligned}
\Upsilon(f) &\leq \int k(y-x)\, d\gamma(x,y) \\
&= \frac{m}{M} \int k\left(\frac{y-x}{\lambda}\right) d\gamma^\lambda(x,y) \\
&= \frac{m}{M} \int \left[k\left(\frac{y-x}{\lambda}\right) - k(y-x)\right] d\gamma^\lambda(x,y) + \frac{m}{M} \int k(y-x)\, d\gamma^\lambda(x,y) \\
&\leq \frac{m}{M} \int_{\Gamma_r} \left[k\left(\frac{y-x}{\lambda}\right) - k(y-x)\right] d\gamma^\lambda(x,y) + \frac{m}{M}\mathcal{E}(M).
\end{aligned}$$

In the integral over Γ_r, the term in brackets is smaller than $-\kappa(r)$. Hence, for every $0 < r < R_\lambda$

$$\mathcal{E}(m) - \varepsilon \le \Upsilon(f) \le \frac{m}{M}\mathcal{E}(M) - \frac{m}{M}\kappa(r)\gamma^\lambda(\Gamma_r). \tag{2.4.16}$$

At this point we can send ε to 0 and deduce that $\mathcal{E}(m) \le (m/M)\mathcal{E}(M)$. However we need to establish a strict inequality. For this we prove in the next step that there exist $r^*, \delta > 0$ not depending on ε or f such that $\gamma^\lambda(\Gamma_{r^*}) \ge \delta$.

Step 2. Conclusion.
For $r \ge 0$, we set

$$\overline{k}(r) := \max\{k(z) : |z| \le r\}.$$

This function is increasing, continuous and there holds $\overline{k}(0) = 0$. Notice that using a ball of mass m as a candidate for the energy $\mathcal{E}(m)$, we see that $\mathcal{E}(m) > 0$ for any $m > 0$. Let us fix $0 < r^* < R$ such that

$$m\overline{k}(r^*) \le \mathcal{E}(m)/4. \tag{2.4.17}$$

By (2.4.15) we have $|x-y| \le R$ for $(x,y) \in \operatorname{supp}\gamma$ and by definition $|x-y| \le r^*$ for $(x,y) \notin \Gamma_{r^*}$. We deduce

$$\frac{\mathcal{E}(m)}{2} < \Upsilon(f) \le \int_{\Gamma_{r^*}} c\, d\gamma + \int_{\Gamma_{r^*}^c} c\, d\gamma \le \gamma(\Gamma_{r^*})\overline{k}(R) + (m - \gamma(\Gamma_{r^*}))\,\overline{k}(r^*)$$
$$\overset{(2.4.17)}{\le} \gamma(\Gamma_{r^*})\overline{k}(R) + (m - \gamma(\Gamma_{r^*}))\,\frac{\mathcal{E}(m)}{4m}.$$

This implies

$$\gamma(\Gamma_{r^*})\left(\overline{k}(R) - \frac{\mathcal{E}(m)}{4m}\right) > \frac{\mathcal{E}(m)}{4}.$$

Thus $4m\overline{k}(R) > \mathcal{E}(m)$ and

$$\frac{m}{M}\gamma^\lambda(\Gamma_{\lambda r^*}) = \gamma(\Gamma_{r^*}) \ge \frac{m\mathcal{E}(m)}{4m\overline{k}(R) - \mathcal{E}(m)} =: m^* > 0.$$

Plugging this in (2.4.16) with $r = \lambda r^* < R_\lambda$ we obtain

$$\mathcal{E}(m) - \varepsilon \le \frac{m}{M}\mathcal{E}(M) - m^*\kappa(\lambda r^*).$$

Since $\varepsilon \in [0, \mathcal{E}(m)/2]$ is arbitrary and $m^*\kappa(r^*) > 0$, this proves the proposition. □

We close this subsection with a lemma establishing that if a function f nearly maximises $\mathcal{E}(m)$ for some $m > 0$ then there exists a cube which is at least half filled by f.

Lemma 2.4.8. *Let $m > 0$. There exists a non-decreasing function $r_0 : m \mapsto r_0(m)$ such that for $m > 0$ and $f \in L^1_m$ with $\Upsilon(f) \ge \mathcal{E}(m)/2$, there exists a cube Q_0 of side-length $r_0(m)$ such that:*

$$\int_{Q_0} f \ge \frac{|Q_0|}{2}.$$

Proof. Let $r_0 > 0$ to be fixed later and assume by contradiction that there exists a partition $\mathcal{Q}$ of $\mathbb{R}^d$ in cubes with side-length r_0 such that for every $Q \in \mathcal{Q}$,

$$\int_Q f < \frac{|Q|}{2}.$$

The strategy to get a contradiction from this hypothesis is to build an exterior transport plan for f with too small transport cost. Let $Q \in \mathcal{Q}$. Since $\int_Q (1-f) \geq \int_Q f$ there exists a function $g_Q \geq 0$ supported in Q such that $\int g_Q = \int_Q f$ and $f\chi_Q + g_Q \leq 1$. We then set

$$\gamma_Q := f\chi_Q \otimes \frac{g_Q}{\int_Q f} \qquad \text{and} \qquad \gamma := \sum_{Q\in\mathcal{Q}} \gamma_Q.$$

Notice that γ is a valid competitor for $\Upsilon(f)$. Next for $R > 0$, we define

$$\overline{k}(R) := \max\{k(x), |x| \leq R\}.$$

We compute:

$$0 < \frac{\mathcal{E}(m)}{2} \leq \Upsilon(f) \leq \sum_{Q\in\mathcal{Q}} \int_{Q\times Q} k(y-x)\, d\gamma_Q$$
$$\leq \overline{k}\left(\sqrt{d}r_0\right) \sum_{Q\in\mathcal{Q}} \int_Q f = \overline{k}\left(\sqrt{d}r_0\right) \int f \leq \overline{k}\left(\sqrt{d}r_0\right) m. \tag{2.4.18}$$

Remarking that $\overline{k} : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous at 0, increasing and with $\overline{k}(0) = 0$, we set

$$r_0 := \max\left\{ r > 0 : \overline{k}\left(\sqrt{d}r\right) \leq \frac{\mathcal{E}(m)}{4m} \right\} > 0$$

and obtain a contradiction with (2.4.18). This concludes the proof. □

2.4.3 Existence of a maximiser for the exterior transport energy

In the following subsection, we assume that (H1), (H2) and (H3) hold and prove the existence of maximisers for (2.1.3).

We only have to prove that maximising sequences for $\mathcal{E}(m)$ are tight. However our result is more precise. We obtain that if f nearly maximises $\mathcal{E}(m)$ then almost all its mass concentrates in a closed ball with radius $R_* = R_*(m)$. In the limit, maximisers are supported in such balls.

Proposition 2.4.9. *Let $m > 0$. There exist $R_* = R_*(m) > 0$, $\varepsilon_0 = \varepsilon_0(m) > 0$ non-decreasing in m with the following property. Let $0 < \varepsilon \leq \varepsilon_0$ and let $f \in L^1_m$ such that $\int f = m$ and $\Upsilon(f) \geq \mathcal{E}(m) - \varepsilon$, then up to a translation there holds*

$$\int_{\mathbb{R}^d \setminus B_{R_*}} f \leq \frac{2m}{\mathcal{E}(m)} \varepsilon.$$

Proof. *Outline of the proof.* (Step 1) We start by using Lemma 2.4.8 to get a collection $\mathcal{Q}_0$ of cubes Q of side-length $r_0 = r_0(m)$ such that $\int_Q f + g \geq |Q|/2$. We denote $\Omega_0 := \cup \mathcal{Q}_0$. We also consider the set Ω obtained by thickening Ω_0 by adding the cubes closer than some distance $R = R(m)$. The real R is chosen so that no mass of $f\chi_{\Omega_0}$ is sent outside Ω by any optimal exterior transport plan of f.

(Step 2) We build an exterior transport plan for f whose cost is very close to $\Upsilon(f\chi_\Omega)$.

(Step 3) Next, we show that Ω concentrates almost all the mass of f. Using the strict superadditivity of $m \mapsto \mathcal{E}(m)$ and the previous step, we deduce that $m_\Omega := \int f\chi_\Omega$ is close to m.

(Step 4) Eventually, we show that the distance between cubes in $\mathcal{Q}_0$ is uniformly bounded. As the cardinal of $\mathcal{Q}_0$ is also bounded, we conclude that the diameter of Ω is bounded by a distance only depending on m.

Step 1. Construction of a collection of cubes on which $\int_Q (f+g) \geq |Q|/2$.

Let $m > 0$ and f as in the statement of the proposition and assume that $\Upsilon(f) \geq \mathcal{E}(m)/2$ so that

$$\varepsilon := \mathcal{E}(m) - \Upsilon(f) \leq \mathcal{E}(m)/2. \tag{2.4.19}$$

Let γ be a minimiser for $\Upsilon(f)$ and let us set $g := \gamma_y$. Let r_0 and Q_0 be given by Lemma 2.4.8. We denote by $\hat{\mathcal{Q}}$ the regular partition of $\mathbb{R}^d$ into cubes of side-length r_0 such that $Q_0 \in \hat{\mathcal{Q}}$. For $j \geq 0$ to be fixed later, we set $r_j := 2^{-j} r_0$. Considering the partition $\mathcal{Q}$ of $\mathbb{R}^d$ into cubes of side-length r_j obtained by refining $\hat{\mathcal{Q}}$, we define $\mathcal{Q}_0$ as the subset formed by the elements $Q \in \mathcal{Q}$ such that

$$\int_Q (f+g) \geq \frac{|Q|}{4}.$$

We remark that $\mathcal{Q}_0$ is not empty since

$$\int_Q (f+g) \geq \int_Q f \geq \frac{|Q|}{2}$$

for at least one of the 2^j sub-cubes of Q_0 in the partition $\mathcal{Q}$.

Let us define $\Omega_0 := \cup \mathcal{Q}_0$. By Proposition 2.4.1, there exists $R = R(m)$ such that $|x-y| \leq R$ on $\operatorname{supp}\gamma$. We denote by $\mathcal{Q}_R$ the collection of cubes $Q \in \mathcal{Q}$ such that $d(Q, \Omega_0) \leq R$, and by Ω their union. By construction, there holds $\gamma(\Omega_0 \times \Omega^c) = 0$. We now define

$$f_\Omega := f\chi_\Omega, \qquad \text{and} \qquad m_\Omega := \int f_\Omega,$$

and we let γ_Ω be an optimal exterior transport plan for f_Ω, that is $\gamma \in \Pi_{f_\Omega}$ with $\int c\, d\gamma_\Omega = \Upsilon(f_\Omega)$. We then set $g_\Omega := (\gamma_\Omega)_y$.

By Proposition 2.4.1 again, we have (since $m_\Omega \leq m$) that

$$\gamma_\Omega(\Omega_0 \times \Omega^c) = 0. \tag{2.4.20}$$

Step 2. Building a transport plan for f whose cost is close to $\Upsilon(f_\Omega)$.

In this step we modify γ_Ω to build an exterior transport plan γ for f with a cost close to

$\Upsilon(f_\Omega)$. More precisely, we require that for some constant $C = C(r_j) > 0$ with $C(r_j) \to 0$ as $r_j \to 0$,

$$\int c\,d\gamma - \int c\,d\gamma_\Omega \le C(m - m_\Omega).$$

The proof is a refinement of the proof of the Lipschitz continuity of Υ, see Proposition 2.4.2 (iii). In the following we define successively the plans γ^0, γ^1, γ^2, γ^3 which satisfy in particular

$$\operatorname{supp}\gamma^0 \subset \overline{\Omega}\times\overline{\Omega}, \qquad \operatorname{supp}\gamma^1 \subset \overline{\Omega}\times\overline{\Omega^c}, \qquad \operatorname{supp}\gamma^2 \subset \overline{\Omega\setminus\Omega_0}\times\overline{\Omega\setminus\Omega_0}, \qquad \operatorname{supp}\gamma^3 \subset \overline{\Omega^c}\times\overline{\Omega^c}.$$

First we set $\gamma^0 := \gamma_\Omega \llcorner \Omega\times\Omega$ and denote $f^0 := \gamma^0_x$, $g^0 := \gamma^0_y$. We build the three remaining plans in the following substeps. These constructions will satisfy

$$(\gamma^1+\gamma^2)_x = f_\Omega - \gamma^0_x = f\chi_\Omega - f^0 \qquad \text{and} \qquad \gamma^3_x = f - f_\Omega = f\chi_{\Omega^c}.$$

We will set eventually $\widehat{\gamma} := \gamma^0+\gamma^1+\gamma^2+\gamma^3$ which will be an admissible transport plan for f. The difficulty is to preserve the constraint $f + \widehat{\gamma}_y \le 1$ while controlling the cost.

Step 2.a. Construction of γ^1.

Let us denote $\gamma^1_\Omega := \gamma_\Omega \llcorner \Omega\times\Omega^c$, $f^1_\Omega := (\gamma^1_\Omega)_x$ and $g^1_\Omega := (\gamma^1_\Omega)_y = \chi_{\Omega^c} g_\Omega$. We can not rule out the possibility that $f + g_\Omega > 1$ in some part of Ω^c so that we cannot set $\gamma^1 = \gamma^1_\Omega$. However, we will transport as much as possible mass through γ^1_Ω. Let us define

$$u := (f + g_\Omega - 1)_+,$$

which corresponds to the excess mass transported through γ^1_Ω. Using the convention $0/0 = 0$, we define γ^1 by

$$d\gamma^1(x,y) := \frac{g_\Omega(y) - u(y)}{g_\Omega(y)}\,d\gamma^1_\Omega(x,y).$$

At this point, we have

$$\int c\,d(\gamma^0+\gamma^1) \le \int c\,d\gamma_\Omega = \Upsilon(f_\Omega). \tag{2.4.21}$$

Moreover setting $f^1 := \gamma^1_x$ and $g^1 := \gamma^1_y$, there holds $\operatorname{supp} g^1 \subset \overline{\Omega^c}$. Notice that since $f_\Omega \le f$, by Corollary 2.4.6 we have $f_\Omega + g_\Omega \le f + g$, so that $g_\Omega \le f + g$ in Ω^c which implies $g^1 \le f + g$. Thus

$$\int_Q (f + g^1) \le \int_Q (2f + g) < \frac{|Q|}{2}, \qquad \text{for every } Q \in \mathcal{Q}\setminus\mathcal{Q}_R, \tag{2.4.22}$$

where we used the definition of $\mathcal{Q}_0$ and the fact that $[\mathcal{Q}\setminus\mathcal{Q}_R]\cap\mathcal{Q}_0 = \emptyset$.

Let us compute for later use the mass from Ω that still requires to be transported. By construction

$$\int_\Omega (f - f^0 - f^1) = \int d(\gamma^1_\Omega - \gamma^1) = \int \frac{u(y)}{g_\Omega(y)}\,d\gamma^1_\Omega(x,y) = \int_{\Omega^c} (f + g_\Omega - 1)_+ \le \int_{\Omega^c} f. \tag{2.4.23}$$

Step 2.b. Construction of γ^2.

We now define

$$\gamma_\Omega^2 := \gamma_\Omega^1 - \gamma^1 = \gamma_\Omega - \gamma^0 - \gamma^1.$$

Notice that by (2.4.20), $\gamma_\Omega^1(\Omega_0 \times \Omega^c) = 0$, so that

$$\operatorname{supp} \gamma_\Omega^2 \subset \overline{\Omega \setminus \Omega_0} \times \overline{\Omega^c}.$$

In particular, $f^2 := f_\Omega - f^0 - f^1$ is supported in $\overline{\Omega \setminus \Omega_0}$. Let $Q \in \mathcal{Q}_R \setminus \mathcal{Q}_0$. Since $g^0 \le g_\Omega$ and $f = f_\Omega$ on Q, using Corollary 2.4.6 again we see that

$$\int_Q (f + g^0) \le \int_Q (f_\Omega + g_\Omega) \le \int_Q (f + g) \le \frac{|Q|}{4}.$$

Therefore for such Q there exists a function $g_Q^2 : Q \to \mathbb{R}_+$ such that $f + g^0 + g_Q^2 \le 1$ and $\int g_Q^2 = \int_Q f^2$. Defining

$$\gamma_Q^2 := \frac{1}{\int_Q f^2}\left[\chi_Q f^2\right] \otimes g_Q^2 \quad \text{for } Q \in \mathcal{Q}_R \setminus \mathcal{Q}_0 \quad \text{and then} \quad \gamma^2 := \sum_{Q \in \mathcal{Q}_R \setminus \mathcal{Q}_0} \gamma_Q^2,$$

we have $\gamma_x^2 = f^2$ and $g^2 := \gamma_y^2 = \sum_Q g_Q^2$. Hence $f + g^0 + g^2 \le 1$ and

$$\int c \, d\gamma^2 \le \left(\int f^2 \right) \overline{k}\left(\sqrt{d} r_j\right),$$

where as in the proof of Proposition 2.4.7 we denote $\overline{k}(r) := \max\{k(x) : |x| \le r\}$.

By construction $f^2 = f_\Omega - f^0 - f^1$, so by (2.4.23) there holds $\int f^2 \le m - m_\Omega$ which leads to the cost estimate

$$\int c \, d\gamma^2 \le (m - m_\Omega) \overline{k}\left(\sqrt{d} r_j\right). \tag{2.4.24}$$

Step 2.c. Construction of γ^3.

We still have to transport the mass corresponding to $\chi_{\Omega^c} f$. For every $Q \in \mathcal{Q} \setminus \mathcal{Q}_R$ we have $\int_Q f \le \int_Q (f + g) \le |Q|/4$, therefore, in view of (2.4.22), there exists a function $g_Q^3 : Q \to \mathbb{R}_+$ such that $\int g_Q^3 = \int_Q f$ and $f + g^1 + g_Q^3 \le 1$. As in the previous step, we define

$$\gamma_Q^3 := \frac{1}{\int_Q f}\left[\chi_Q f\right] \otimes g_Q^3 \qquad \text{and} \qquad \gamma^3 := \sum_{Q \in \mathcal{Q} \setminus \mathcal{Q}_R} \gamma_Q^3.$$

By construction, $(\gamma^3)_x = \chi_{\Omega^c} f$ and denoting $g^3 := (\gamma^3)_y$, we have $\operatorname{supp} g^3 \subset \overline{\Omega^c}$ as well as $f + g^1 + g^3 \le 1$. Moreover

$$\int c \, d\gamma^3 \le \left(\int_{\Omega^c} f \right) \overline{k}\left(\sqrt{d} r_j\right) = (m - m_\Omega) \overline{k}\left(\sqrt{d} r_j\right). \tag{2.4.25}$$

Step 2.d. Conclusion : definition and properties of $\widehat{\gamma}$.

Eventually, we set $\widehat{\gamma} := \gamma^0 + \gamma^1 + \gamma^2 + \gamma^3$ and $\widehat{g} := \widehat{\gamma}_y$. There holds $\widehat{\gamma}_x = f^0 + f^1 + f^2 + f^3 = f$ and $f + \widehat{g} \le 1$ so that $\widehat{\gamma}$ is an admissible exterior transport plan for f. Besides, collecting the estimates (2.4.21),(2.4.24)&(2.4.25) we get

$$\Upsilon(f) \leq \int c\, d\widehat{\gamma} \leq \Upsilon(f_\Omega) + 2(m - m_\Omega)\overline{k}\left(\sqrt{d}\, r_j\right). \tag{2.4.26}$$

Step 3. We show that $m - m_\Omega \leq C(m)\varepsilon$ (recall the definition (2.4.19) *of ε).*

As $\Upsilon(f_\Omega) \leq \mathcal{E}(m_\Omega)$ and $\mathcal{E}(m) - \varepsilon = \Upsilon(f)$, (2.4.26) yields

$$\mathcal{E}(m) - \varepsilon \leq \mathcal{E}(m_\Omega) + 2\overline{k}\left(\sqrt{d}\, r_j\right)(m - m_\Omega).$$

Additionally by Proposition 2.4.7, $\mathcal{E}(m_\Omega) \leq \frac{m_\Omega}{m}\mathcal{E}(m)$. Hence

$$\left(\frac{\mathcal{E}(m)}{m} - 2\overline{k}\left(\sqrt{d}\, r_j\right)\right)(m - m_\Omega) \leq \varepsilon.$$

By continuity of k, $\overline{k}(\sqrt{d}\, r_j) \to 0$ as $r_j \to 0$. Recalling that $r_j = 2^{-j} r_0$, we fix $j \geq 0$ as the first integer such that $\overline{k}(\sqrt{d}\, r_j) \leq \mathcal{E}(m)/4m$ (notice that j does not depend on ε). Therefore

$$m - m_\Omega \leq \frac{2m\varepsilon}{\mathcal{E}(m)}. \tag{2.4.27}$$

This yields

$$\int_{\mathbb{R}^d \setminus \Omega} f = \int f - \int f_\Omega = m - m_\Omega \leq \frac{2m\varepsilon}{\mathcal{E}(m)}. \tag{2.4.28}$$

For future use, let us also notice that injecting (2.4.27) into (2.4.26) we obtain

$$\mathcal{E}(m) - \varepsilon \leq \Upsilon(f) \leq \Upsilon(f_\Omega) + \varepsilon. \tag{2.4.29}$$

Step 4 : Bounding the diameter of Ω.

We finally prove that Ω is uniformly bounded which would conclude the proof. For Q_-, $Q_+ \in \mathcal{Q}_0$, we write $Q_- \sim Q_+$ if there exists a finite chain

$$Q_- = Q_0, Q_1, \ldots, Q_n = Q_+ \tag{2.4.30}$$

such that $Q_i \in \mathcal{Q}_0$ and $d(Q_{i-1}, Q_i) \leq 4R + \sqrt{d}\, r_j$ for $1 \leq i \leq n$. This defines an equivalence relation. Let us show that there exists only one equivalence class. We assume by contradiction that there exist at least two equivalence classes, and we let $\mathcal{C}^1$ be one of these classes and $\mathcal{C}^2$ be the union of the remaining classes. For $i \in \{1,2\}$, we then define Ω^i to be the union of the cubes Q such that $d(Q, \mathcal{C}^i) \leq R$. By construction, $d(\Omega_1, \Omega_2) > 2R$. Recalling that Ω is the union of the cubes Q such that $d(Q, \Omega_0) \leq R$, we have $\Omega^1 \cup \Omega^2 = \Omega$.

For $i \in \{1,2\}$, we set $f_\Omega^i := f_\Omega \chi_{\Omega^i}$ and $m_\Omega^i = \int f_\Omega^i$. We have $m_\Omega^1 + m_\Omega^2 = m_\Omega \leq m$ and $m_\Omega^1, m_\Omega^2 \geq 2^{-jd}|Q_0|/4 = 2^{-jd-2}|Q_0|$. Additionally, by Proposition 2.4.2 (ii),

$$\Upsilon(f_\Omega) = \Upsilon(f_\Omega^1) + \Upsilon(f_\Omega^2) \leq \mathcal{E}(m_\Omega^1) + \mathcal{E}(m_\Omega^2).$$

Injecting this inequality into (2.4.29) yields

$$\mathcal{E}(m) - \varepsilon \leq \Upsilon(f) \leq \Upsilon(f_\Omega) + \varepsilon \leq \mathcal{E}(m^1_\Omega) + \mathcal{E}(m^2_\Omega) + \varepsilon.$$

Recalling that $e(m) = \mathcal{E}(m)/m$, this rewrites as

$$me(m) \leq m^1_\Omega e(m^1_\Omega) + m^2_\Omega e(m^2_\Omega) + 2\varepsilon. \tag{2.4.31}$$

As $m^1_\Omega + m^2_\Omega \leq m$ and for $i \in \{1,2\}$, $m^i_\Omega \geq 2^{-jd-2}|Q_0|$, we have $m^i_\Omega \leq m - 2^{-jd-2}|Q_0|$. Recall that by Proposition 2.4.7, e is increasing, so that $e(m^i_\Omega) \leq e(m - 2^{-jd} m_0)$. Hence

$$m^1_\Omega e(m^1_\Omega) + m^2_\Omega e(m^2_\Omega) \leq me\left(m - 2^{-jd} m_0\right).$$

With (2.4.31), we obtain

$$me(m) \leq me\left(m - 2^{-jd} m_0\right) + 2\varepsilon,$$

which is absurd for ε small enough because e is increasing. It follows that for $\varepsilon > 0$ small enough the relation $\sim$ has a single class. Recall that for all $Q \in \mathcal{Q}_0$, $\int_Q (f+g) \geq 2^{-jd-2}|Q_0|$. Thus the maximal length of a chain in (2.4.30) without any repetition is bounded by $N := \lfloor 2^{jd+3} m/|Q_0| \rfloor$. Therefore, the diameter of Ω is bounded by $(4R + 2\sqrt{d} r_j)(N+1)$ with r_j and N only depending on m, the dimension d and the cost c. Together with (2.4.28) this proves the proposition. □

We can now apply the direct method of Calculus of Variations to establish the existence of a maximiser for (2.1.3).

Proof of Theorem 2.1.1. Let f_n be a maximising sequence for (2.1.3) and let $R_* = R_*(m)$ be given by Proposition 2.4.9 so that up to translation,

$$\int_{\mathbb{R}^d \setminus B_{R_*}} f_n \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, f_n is a tight sequence of $\mathcal{M}_+(\mathbb{R}^d)$ and up to extraction of a subsequence it converges weakly-$*$ to f where f is admissible for (2.1.3). By Proposition 2.4.2 (iv),

$$\Upsilon(f) = \lim \Upsilon(f_n) = \mathcal{E}(m),$$

so that f is a maximiser for $\mathcal{E}(m)$.

Let now f be any maximiser of $\mathcal{E}(m)$. Applying Proposition 2.4.9 to f we have that up to a translation $\operatorname{supp} f \subset \overline{B}_{R_*}$. This concludes the proof. □

Let us show that when f is compactly supported there exist Kantorovitch potentials for the problem (2.4.1) (this is the situation of interest as we have just established that the maximisers of $\mathcal{E}(m)$ are compactly supported in $\mathbb{R}^d$).

Lemma 2.4.10. *Let $m > 0$ and assume that $f \in L^1_m$ is compactly supported. Let $R = R(m)$ be given by Proposition 2.4.1 such that all maximisers γ of $\Upsilon(f)$ are supported in $X := \operatorname{supp} f + \overline{B}_R$. Then, there* *exists a pair* $(\varphi,\psi)\in C_c(\mathbb{R}^d)\times C_c(\mathbb{R}^d)$ *optimal for* $\Upsilon^*(f)$*. Additionally,* $\varphi=\psi^c$*,* $\psi=\varphi^c_-$ *and both* φ *and* ψ *are compactly supported in* X*.*

Proof. Let us introduce $\tilde c := c_{|X\times X}$ which is a continuous cost function on the compact set X. By Proposition 2.3.4, there exists $\widetilde\psi\in C(X)$ with $\widetilde\psi=(\widetilde\psi^{\tilde c\tilde c})_-$ such that $\Upsilon^*(f)=K_f(\widetilde\psi^{\tilde c},\widetilde\psi)$. By Proposition 2.3.7,

$$\{\widetilde\psi^{\tilde c\tilde c}<0\}\subset\{f+g=1\}\subset X.$$

Combining this with $\widetilde\psi=(\widetilde\psi^{\tilde c\tilde c})_-$ and $f+g=0$ on ∂X , we get $\widetilde\psi=0$ on ∂X. We extend the potentials on $\mathbb{R}^d$ by setting

$$\psi:=\begin{cases}\widetilde\psi & \text{in} \quad X,\\ 0 & \text{in} \quad X^c,\end{cases}\qquad \text{and for } x\in\mathbb{R}^d,\quad \varphi(x):=\psi^c(x)=\inf\{c(x,y)-\psi(y): y\in\mathbb{R}^d\}.$$

We now show that the pair (φ,ψ) satisfies the conclusion of the lemma.

Observe that ψ is continuous and supported in X and that $\psi\le 0$. Hence $\varphi\ge 0$. Moreover, for $x\in\mathbb{R}^d\setminus\operatorname{supp}\psi$,

$$\varphi(x)\le c(x,x)-\psi(x)=0,$$

so that φ is also supported in X.

Next, for $x\in X$,

$$\varphi(x)=\min\left(\min_{y\in\mathbb{R}^d\setminus X}c(x,y),\ \min_{y\in X}\{c(x,y)-\widetilde\psi(y)\}\right).$$

Let $x\in X$. For $y\in\mathbb{R}^d\setminus X$, there exists $\widetilde y$ in the intersection of the segment $[x,y]$ with ∂X. By continuity, $\widetilde\psi(\widetilde y)=0=\widetilde\psi(y)$ and moreover by (H3), $c(x,\widetilde y)\le c(x,y)$ so that $c(x,\widetilde y)-\widetilde\psi(\tilde y)\le c(x,y)$. We deduce that for $x\in X$ the above formula simplifies as

$$\varphi(x)=\min_{y\in X}\{c(x,y)-\widetilde\psi(y)\}=\widetilde\psi^{\tilde c}(x).$$

This proves that φ is continuous and that $K_f(\varphi,\psi)=K_f(\widetilde\psi^{\tilde c},\widetilde\psi)=\Upsilon^*(f)$. Moreover, using the same argument as above, we have $\varphi^c(y)=0$ for $y\notin X$. For $y\in X$,

$$\varphi^c(y)=\min\left(\min_{x\in\mathbb{R}^d\setminus X}c(x,y),\ \min_{x\in X}\{c(x,y)-\widetilde\psi^{\tilde c}(x)\}\right)=\min_{x\in X}\{c(x,y)-\widetilde\psi^{\tilde c}(x)\}=\widetilde\psi^{\tilde c\tilde c}(y).$$

We deduce $\varphi^c_-=0=\psi$ in X^c, and $\varphi^c_-=(\psi^{\tilde c\tilde c})_-=\widetilde\psi$ in X. Thus $\varphi^c_-=\psi$ everywhere. This ends the proof of the lemma. □

Let us now recall a variant of the bathtub principle, see [57, Theorem 1.14].

Proposition 2.4.11. *Let* $\xi:\mathbb{R}^d\to\mathbb{R}_+$ *be measurable and such that for all* $t\ge 0$, $|\{\xi>t\}|<\infty$*. Given* $m>0$*, let*

$$t:=\inf\{s\ge 0, |\{\xi>s\}|\le m\}.$$

Then, the maximisers of

$$\sup_{\widetilde{f}}\left\{\int \widetilde{f}\xi \,:\, \widetilde{f}\in L^1(\mathbb{R}^d),\, 0\le \widetilde{f}\le 1,\, \int \widetilde{f}=m\right\}$$

are the functions $f := \chi_{\{\xi>t\}}+\theta$, *where* $\theta\in L^1(\mathbb{R}^d,[0,1])$ *is supported in* $\{\xi=t\}$ *and satisfies*

$$\int \theta = m-|\{\xi>t\}|.$$

We are now ready to establish Corollary 2.1.2.

Proof of Corollary 2.1.2.

By Theorem 2.1.1, the optimisation problem (2.1.3) admits a compactly supported solution f. Let $(\varphi,\psi)\in C_c(\mathbb{R}^d)\times C_c(\mathbb{R}^d)$ be an optimal pair for $\Upsilon^*(f)$ provided by Lemma 2.4.10, so that

$$\Upsilon^*(f)=\int f(\varphi-\psi)+\int\psi.$$

We see that f is a maximiser of:

$$\sup\left\{\int \widetilde{f}(\varphi-\psi) \,:\, \widetilde{f}\in L^1(\mathbb{R}^d),\, 0\le \widetilde{f}\le 1,\, \int \widetilde{f}=m\right\}.$$

Let us set $\xi:=\varphi-\psi\ge 0$. By Proposition 2.4.11 there exists $t\ge 0$ and $\theta\in L^1(\mathbb{R}^d,[0,1])$ supported in $\{\xi=t\}$ such that $f=\chi_{\{\xi>t\}}+\theta$. Notice in particular that since $\theta\in[0,1]$, we have

$$|\{\xi=t\}|\ge\int\theta=m-|\{\xi>t\}|.$$

and there exist measurable subsets $G\subset\{\xi=t\}$ with $|G|=m-|\{\xi>t\}|$. For any such set, setting

$$\bar{f}:=\chi_{\{\xi>t\}}+\chi_G,$$

we have $\Upsilon^*(\bar{f})=\Upsilon^*(f)$ and $\bar{f}$ is also a maximiser of (2.1.3). Since $\bar{f}$ is a characteristic function, by Theorem 2.4.4 and Corollary 2.4.5, there exists $F\subset\mathbb{R}^d$ such that any minimiser γ of $\Upsilon(\bar{f})$ satisfies $\gamma_y=\chi_F$. Setting $E:=\{\xi>t\}\cup G$, we deduce that

$$\Upsilon_{\mathrm{set}}(E)=\Upsilon^*(\bar{f})=\Upsilon(\bar{f})=\Upsilon(f)$$

so that $\mathcal{E}(m)=\mathcal{E}_{\mathrm{set}}(m)$, which concludes the proof. □

2.5 Maximisers of the exterior transport energy are characteristic functions of balls

In this section we prove Theorem 2.1.3. We assume that $c(x,y)=k(|y-x|)$ with $k\in C(\mathbb{R}_+,\mathbb{R}_+)$ increasing and coercive and with $k(0)=0$. In particular, we have now $c=\bar{c}$, so that the operations of c-transform and $\bar{c}$-transform coincide. Also notice that by Theorem 2.4.4, if $f = \chi_E$ for some Lebesgue measurable set E then $\Upsilon_{\text{set}}(E) = \Upsilon(\chi_E)$. By abuse of notation, we write $\Upsilon(E)$ for $\Upsilon(\chi_E)$. Since the class of costs that we consider is invariant by scaling we assume without loss of generality that $m = \omega_d$.

We now recall the definition of symmetric rearrangement of functions with constant sign (see [57, Chapter 3] for more details on symmetric rearrangements).

Definition 2.5.1.

(i) Given a measurable set $A \subset \mathbb{R}^d$, we define the symmetric rearrangement of A as the open ball A^* centred at the origin and of volume $|A|$.

(ii) Let $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ be measurable and such that for every $t \geq 0$, $|\{\varphi > t\}| < \infty$. Its symmetric decreasing rearrangement is defined by

$$\varphi^*(x) := \int_{\mathbb{R}_+} \chi_{\{\varphi > t\}^*}(x)\, dt.$$

(iii) Let $\psi : \mathbb{R}^d \to \mathbb{R}_-$ be measurable and such that for every $t \leq 0$, $|\{\psi < t\}| < \infty$. Its symmetric increasing rearrangement is defined by

$$\psi_*(x) := -(-\psi)^*(x) = -\int_{\mathbb{R}_-} \chi_{\{\psi < t\}^*}(x)\, dt.$$

The following lemma recalls some basic properties of the symmetric increasing rearrangement ψ_* of a non-positive function ψ. All these properties but the continuity of ψ_* follow immediately from the definition. The fact that continuity is preserved by symmetric rearrangement is well-known but we have no reference for this at hand. We provide a short proof for the reader's convenience.

Lemma 2.5.2. *Let $\psi : \mathbb{R}^d \to \mathbb{R}_-$ be as in Definition 2.5.1. Then, ψ_* is non-positive, radial, non-decreasing, and for any $t \leq 0$, $\{\psi_* < t\} = \{\psi < t\}^*$. Besides, if ψ is supported in a compact set of diameter bounded by $2R > 0$ then ψ_* is supported in $\overline{B}_R$. If moreover ψ is continuous then ψ_* is also continuous.*

Proof of the last point. Let $\psi \in C_c(\mathbb{R}^d, \mathbb{R}_-)$. First, as the strict sublevels sets $\{\psi_* < t\}$ are the open balls $\{\psi < t\}^*$, ψ_* is upper semi-continuous (note that this is true even when ψ is not continuous).

Let us now establish that ψ_* is lower semi-continuous, *i.e.* that for any $t \leq 0$, $\{\psi_* \leq t\}$ is closed. We first notice that $\{\psi_* \leq 0\} = \mathbb{R}^d$ is closed. Given $t < 0$, let $t_n < 0$ be a decreasing sequence converging to t. Observe that if for some $n \geq 0$, $\{\psi < t_n\} = \emptyset$, then $\{\psi_* \leq t\} = \emptyset$ is closed. Next, we assume that for every $n \geq 0$,

$$\{\psi < t_n\} \neq \emptyset. \tag{2.5.1}$$

We denote by R_n the radius of the ball $\{\psi_* < t_n\}$. Notice that the sequence R_n is non-increasing and bounded by 0, so that R_n converges to some $R \geq 0$.

Let us show that the sequence R_n is decreasing. By contradiction, we assume that $R_n = R_{n+1}$ for some $n \geq 0$. Then $\{\psi < t_n\}^* = \{\psi < t_{n+1}\}^*$ and $|\{t_{n+1} \leq \psi < t_n\}| = 0$. Using (2.5.1) and the fact that ψ is compactly supported, there exists x such that $\psi(x) < t_{n+1}$ and y such that $\psi(y) > t_n$. Thus by continuity of ψ there exists z such that $\psi(z) = (t_{n+1} + t_n)/2$. By continuity of ψ again, there exists $\eta > 0$ such that $B_\eta(z) \subset \{t_{n+1} < \psi < t_n\}$, contradicting the fact that $|\{t_{n+1} \leq \psi < t_n\}| = 0$. As a conclusion, the sequence R_n is decreasing and

$$\{\psi_* \leq t\} = \bigcap_{n\geq 0} \{\psi_* < t_n\} = \bigcap_{n\geq 0} B_{R_n} = \overline{B}_R.$$

Hence ψ_* is lower semi-continuous and therefore continuous. □

To prove Theorem 2.1.3, we need a last lemma characterising optimal potentials of $\Upsilon^*(\chi_{B_1})$. Along the way we will prove that the set F minimising $\Upsilon_{\text{set}}(E)$ (recall (2.1.1)) is the annulus $A = B_{2^{1/d}} \backslash B_1$.

Lemma 2.5.3. *Let (ψ^c, ψ) be a pair of optimal potentials for $\Upsilon^*(\chi_{B_1})$ such that ψ is radially symmetric and non-decreasing. Then ψ^c is radially symmetric and non-increasing. Besides, ψ^c is radially decreasing on B_1. Finally, if γ is a minimiser of $\Upsilon(\chi_{B_1})$ then $\gamma_y = \chi_A$.*

Proof. Combining the facts that k is continuous, that $k(r) \to \infty$ as $r \to \infty$ and that ψ is bounded by Lemma 2.4.10, we see that for any $x \in \mathbb{R}^d$,

$$\psi^c(x) = \min\{k(|y-x|) - \psi(y) : y \in \mathbb{R}^d\}.$$

As ψ is radially symmetric non-decreasing and k is increasing, we easily see that

$$\psi^c(x) = \min\{k(|y-x|) - \psi(y) : y, \exists \lambda \geq 1, y = \lambda x\}, \tag{2.5.2}$$

which in turn implies that ψ^c is radially symmetric.

From now on, for radial functions $\zeta : \mathbb{R}^d \to \mathbb{R}$, we make the abuse of notation $\zeta(r) = \zeta(r\sigma)$ for $r \geq 0$ where σ is some fixed element of $\mathbb{S}^{d-1}$. With this convention (2.5.2) reads

$$\psi^c(r) = \min_{s\geq r} k(s-r) - \psi(s). \tag{2.5.3}$$

Let us prove that ψ^c is non-increasing. Let $0 \leq r_1 \leq r_2$. By (2.5.3), there exists $r \geq r_1$ such that

$$\psi^c(r_1) = k(r - r_1) - \psi(r). \tag{2.5.4}$$

If $r \leq r_2$, we use $\psi^c(r_2) \leq k(0) - \psi(r_2) = -\psi(r_2)$ and deduce from (2.5.4) and the fact that ψ is non-decreasing that

$$\psi^c(r_2) - \psi^c(r_1) \leq \psi(r) - \psi(r_2) - k(r - r_1) \leq 0.$$

If $r > r_2$, we use $\psi^c(r_2) \leq k(r - r_2) - \psi(r)$ to get

$$\psi^c(r_2) - \psi^c(r_1) \leq k(r - r_2) - k(r - r_1) \leq 0,$$

because $r_1 \le r_2 < r$ and k is increasing. In both cases $\psi^c(r_2) - \psi^c(r_1) \le 0$. Hence ψ^c is non-increasing on $\mathbb{R}^d$.

We now prove that ψ^c is decreasing on B_1. Let $0 < r_1 < r_2 < 1$. Given γ a minimiser for $\Upsilon(B_1)$, there exists $y \in \mathbb{R}^d \setminus B_1$ such that $(r_1\sigma, y) \in \operatorname{supp}\gamma$. By Proposition 2.3.7, γ is an optimal transport plan between $f := \chi_{B_1}$ and $g := \gamma_y$ and (ψ^c, ψ^{cc}) is a pair of Kantorovitch potentials for the transport between f and g. Therefore,

$$\psi^c(r_1) + \psi^{cc}(|y|) = k(|y - r_1\sigma|). \tag{2.5.5}$$

Let us prove by contradiction that $y \in [1, +\infty)\sigma$. Assume it is not and let $y' := |y|\sigma$. Recalling that $|y| \ge 1 > r_1$, we have $|y' - r_1\sigma| = |y| - r_1 < |y - r_1\sigma|$ and since k is increasing we deduce

$$k(|y' - r_1\sigma|) < k(|y - r_1\sigma|).$$

Then, by definition of ψ^{cc} and taking into account that it is radially symmetric we get

$$\psi^c(r_1) + \psi^{cc}(|y|) \le k(|y' - r_1\sigma|) < k(|y - r_1\sigma|)$$

which contradicts (2.5.5). Therefore, $y = r\sigma$ for some $r \ge 1$. By definition of the c-transform,

$$\psi^c(r_2) + \psi^{cc}(r) \le k(r - r_2). \tag{2.5.6}$$

Subtracting (2.5.5) to (2.5.6), we obtain

$$\psi^c(r_2) - \psi^c(r_1) \le k(r - r_2) - k(r - r_1) < 0,$$

where we used $r_1 < r_2 < 1 \le r$. This shows that ψ^c is decreasing on B_1.

Finally we notice that as a consequence of the above discussion, the plan γ is radial. Combining this with Lemma 2.4.3 proves that $g = \chi_A$. □

We are now ready to prove Theorem 2.1.3.

Proof of Theorem 2.1.3.

Part I : Unit balls are maximisers of $\mathcal{E}(\omega_d)$.

By Theorem 2.1.1, there exists a compactly supported maximiser f for (2.1.3) with $m = \omega_d$. By Lemma 2.4.10, there exists an optimal pair $(\psi^c, \psi) \in C_c(\mathbb{R}^d) \times C_c(\mathbb{R}^d)$ for problem $\Upsilon^*(f)$ such that $\psi = (\psi^{cc})_-$.

Step 1. We build a radially symmetric maximiser for (2.1.3).

Let ψ_* be the symmetric increasing rearrangement of ψ. By Lemma 2.5.2, as $\psi \in C_c(\mathbb{R}^d)$, we also have $\psi_* \in C_c(\mathbb{R}^d)$. We denote by ψ_*^c the function $(\psi_*)^c$. By definition, $\psi_*^c \oplus \psi_* \le c$. Proceeding as in the proof of Lemma 2.4.10, we obtain $\psi_*^c \in C_c(\mathbb{R}^d)$. Thus (ψ_*^c, ψ_*) is admissible for $\Upsilon^*(B_1)$.

Notice that (f,ψ) solves the double supremum problem (recall the definition (2.3.2) of K_f)

$$\sup_f \sup_{\psi\in C_c(\mathbb{R}^d)} \left\{ K_f(\psi^c,\psi) : 0\le f\le 1, \int f = \omega_d,\ \psi\le 0 \right\}.$$

Hence

$$\mathcal{E}(\omega_d) = K_f(\psi^c,\psi) = \int f(\psi^c-\psi) + \int \psi \ge \Upsilon^*(B_1) \ge K_{\chi_{B_1}}(\psi_*^c,\psi_*) = \int_{B_1}(\psi_*^c-\psi_*) + \int \psi_*.$$

In the remainder of this step, we establish the converse inequality

$$K_f(\psi^c,\psi) \le K_{\chi_{B_1}}(\psi_*^c,\psi_*), \tag{2.5.7}$$

so that B_1 is a maximiser of $\mathcal{E}(\omega_d)$ and the first part of Theorem 2.1.3 is proved. Notice that (2.5.7) also implies that (ψ_*^c,ψ_*) is a pair of optimal potentials for $\Upsilon^*(B_1)$. To establish (2.5.7), we first notice that by construction $\int\psi = \int\psi^*$ so that we only need to prove

$$\int f(\psi^c-\psi) \le \int_{B_1}(\psi_*^c-\psi_*). \tag{2.5.8}$$

In Step 2 below we establish the inequality

$$(\psi^c)^* \le (\psi_*)^c = \psi_*^c, \tag{2.5.9}$$

where $(\psi^c)^*$ denotes the symmetric decreasing rearrangement of ψ^c. Admitting that (2.5.9) holds we deduce (2.5.8) as follows. Since f is non-negative and compactly supported we have by the Hardy-Littlewood inequality (see [57, Theorem 3.4])

$$-\int f\psi \le -\int f^*\psi_* \qquad \text{and} \qquad \int f\psi^c \le \int f^*(\psi^c)^* \overset{(2.5.9)}{\le} \int f^*\psi_*^c. \tag{2.5.10}$$

Using that $-\psi_*$ and ψ_*^c are radially symmetric and non-increasing, we may appeal to Proposition 2.4.11 and conclude that separately,

$$-\int f\psi \le -\int \chi_{B_1}\psi_* \qquad \text{and} \qquad \int f\psi^c \le \int \chi_{B_1}\psi_*^c. \tag{2.5.11}$$

Summing these inequalities gives (2.5.8) and thus (2.5.7). This proves that χ_{B_1} is a maximiser for $\mathcal{E}(\omega_d)$ and then that B_1 is a maximiser for $\mathcal{E}_{\text{set}}(\omega_d)$.

Step 2. Proof of (2.5.9).

As ψ_*^c and $(\psi^c)^*$ are both continuous radially symmetric functions, to prove (2.5.9) it is sufficient to establish that for any $t>0$, $\{(\psi^c)^*>t\}\subset\{\psi_*^c>t\}$, *i.e.* that

$$|\{(\psi^c)^*>t\}| = |\{\psi^c>t\}| \le |\{\psi_*^c>t\}|. \tag{2.5.12}$$

Recall that as $\psi\in C_c(\mathbb{R}^d)$ and $k\in C(\mathbb{R}_+,\mathbb{R}_+)$ with $k(x)\to\infty$ as $x\to\infty$, for any $x\in\mathbb{R}^d$ the function

$k(|y-x|)-\psi(y)$ admits a minimum on $\mathbb{R}^d$. Thus for any $x\in\mathbb{R}^d$ the infimum defining $\psi^c(x)$ (see Definition 2.2.2) is reached. Recalling that k is also radially symmetric and increasing, we obtain

$$\begin{aligned}\{\psi^c>t\}&=\{x\in\mathbb{R}^d:\min\{k(|y-x|)-\psi(y):y\in\mathbb{R}^d\}>t\}\\&=\{x\in\mathbb{R}^d:-\psi>t-k(r)\text{ on }\overline{B}_r(x)\ \forall r\geq 0\}\\&=\bigcap_{r\geq 0}\{x\in\mathbb{R}^d:-\psi>t-k(r)\text{ on }\overline{B}_r(x)\}\\&=\bigcap_{r\geq 0}\{-\psi>t-k(r)\}_r,\end{aligned}$$

where for $\Omega\subset\mathbb{R}^d$ and $r\geq 0$, Ω_r is defined as $\Omega_r:=\{x\in\Omega:d(x,\mathbb{R}^d\setminus\Omega)>r\}$. In particular,

$$|\{\psi^c>t\}|\leq\inf_{r\geq 0}|\{-\psi>t-k(r)\}_r|. \tag{2.5.13}$$

We observe that $\{-\psi>t-k(r)\}$ is an open set for any $t>0$ and $r\geq 0$. We also notice that (2.5.13) holds for all $\psi\in C_c(\mathbb{R}^d)$. In particular, it holds for ψ_*. Moreover, as ψ_* is radially non-decreasing by construction, the sets $\{-\psi_*>t-k(r)\}_r$ are open balls centred at the origin and we have in fact

$$|\{\psi_*^c>t\}|=\inf_{r\geq 0}|\{-\psi_*>t-k(r)\}_r|.$$

Let us now prove the following claim.

Claim. Let $s>0$ and $V>0$.

(i) If $V>\omega_d s^d$ then, among open sets $\Omega\subset\mathbb{R}^d$ of volume V, $|\Omega_s|$ is maximal if and only if Ω is a ball.

(ii) If $V\leq\omega_d s^d$ then $|\Omega_s|=0$ for any set of volume V.

Let $V>0$ and $s>0$ and let $\Omega\subset\mathbb{R}^d$ be an open set. We assume without loss of generality that $|\Omega|=V$ and $|\Omega_s|>0$. Notice that we always have $\Omega_s+B_s\subset\Omega$ (but the converse inclusion may fail). By the Brunn-Minkowski inequality (see for instance [48]) applied to Ω_s and B_s, there holds

$$V^{1/d}=|\Omega|^{1/d}\geq|\Omega_s+B_s|^{1/d}\geq|\Omega_s|^{1/d}+|B_s|^{1/d}. \tag{2.5.14}$$

If Ω_s is a ball, then Ω is a ball of volume V, $\Omega=\Omega_s+B_s$ and we have equality in (2.5.14). Conversely if we have equality in (2.5.14), by the equality case of the Brunn-Minkowski inequality and the fact that $s>0$, Ω_s is a ball and $|\Omega|=|\Omega_s+B_s|$, so that Ω is a ball. This proves the first part of the claim.

Regarding the second part, we assume that $|\Omega|\leq\omega_d s^d$ and (by contradiction) that $|\Omega_s|>0$. The above reasoning applies and we have $\Omega=\Omega_s+B_s$ so that $|\Omega_s|>0$ implies $|\Omega|>|B_s|=\omega_d s^d$ and we get a contradiction. This proves the claim.

By definition, $\{-\psi>t-k(r)\}$ and $\{-\psi_*>t-k(r)\}$ have the same volume. As a consequence of the claim, for any $t>0$ and $r>0$,

$$|\{-\psi>t-k(r)\}_r|\leq|\{-\psi_*>t-k(r)\}_r|. \tag{2.5.15}$$

Notice that the previous inequality is an equality if $r = 0$, as $\Omega_0 = \Omega$ for any open set Ω. Taking the infimum on $r \geq 0$ yields

$$|\{\psi^c > t\}| \leq \inf_{r\geq 0} |\{-\psi > t - k(r)\}_r| \leq \inf_{r\geq 0} |\{-\psi_* > t - k(r)\}_r| = |\{\psi_*^c > t\}|. \tag{2.5.16}$$

This proves (2.5.12) which in turn implies (2.5.9).

Part II : Unit balls are the unique maximisers of $\mathcal{E}(\omega_d)$.

Step 1. Proof of $f = \chi_{\{\psi^c > \psi_*^c(1)\}}$ (exploiting the equality case in the bathtub principle).

We now show that any maximiser f is of the form $\chi_{\{\psi^c > \psi_*^c(1)\}}$. By Lemma 2.5.3, ψ_*^c is radially decreasing on B_1 and non-increasing on $\mathbb{R}^d$. Thus χ_{B_1} is the only function maximising

$$\sup_{\widetilde{f}} \left\{ \int \widetilde{f} \psi_*^c : 0 \leq \widetilde{f} \leq 1, \int \widetilde{f} = \omega_d \right\}.$$

As $\Upsilon^*(\chi_{B_1}) = \mathcal{E}(\omega_d)$, the inequalities in (2.5.10) and (2.5.11) are in fact equalities (and (2.5.9) is also an equality in B_1). Namely, there hold

$$(\psi^c)^* = \psi_*^c \quad \text{in} \quad B_1, \qquad -\int f\psi = -\int f^*\psi_* \qquad \text{and} \qquad \int f\psi^c = \int f^* \psi_*^c.$$

This leads to

$$\int f\psi^c = \int f^*(\psi^c)^* = \int f^*\psi_*^c = \int_{B_1} \psi_*^c,$$

and f is a maximiser of

$$\sup_{\widetilde{f}} \left\{ \int \widetilde{f} \psi^c : 0 \leq \widetilde{f} \leq 1, \int \widetilde{f} = \omega_d \right\}.$$

Let us now prove that $|\{\psi^c > \psi_*^c(1)\}| = \omega_d$ (which with Proposition 2.4.11 yields $f = \chi_{\{\psi^c > \psi_*^c(1)\}}$). Since $(\psi^c)^* = \psi_*^c$ in B_1, and ψ_*^c is decreasing in B_1 by Lemma 2.5.3, there holds for $t \geq \psi_*^c(1)$,

$$|\{\psi^c > t\}| = |\{(\psi^c)^* > t\}| = |\{\psi_*^c > t\}|. \tag{2.5.17}$$

Using this for $t = \psi_*^c(1)$ we get $|\{\psi^c > \psi_*^c(1)\}| = \omega_d$ and we conclude with Proposition 2.4.11 that $f = \chi_{\{\psi^c > \psi_*^c(1)\}}$.

Step 2. We prove that $\{\psi^c > t\}$ is a ball for $t > \psi_*^c(1)$ (exploiting the equality case in the Brunn-Minkowski inequality).

Step 2.a.

We fix $t > \psi_*^c(1)$. Combining (2.5.17) and (2.5.16), we get that

$$|\{\psi^c > t\}| = \inf_{r\geq 0} |\{-\psi > t - k(r)\}_r| = \inf_{r\geq 0} |\{-\psi_* > t - k(r)\}_r| = |\{\psi_*^c > t\}|. \tag{2.5.18}$$

The following claim is established in Step 2.b below.

Claim. There exists $r_* = r_*(t) > 0$ such that

$$|\{\psi_*^c > t\}| = \inf_{r\geq 0} |\{-\psi_* > t - k(r)\}_r| = |\{-\psi_* > t - k(r_*)\}_{r_*}|.$$

Provisionally assuming the claim let us prove that $\{\psi^c > t\}$ is a ball.

We assume without loss of generality that $|\{\psi_*^c > t\}| > 0$ (otherwise $|\{\psi^c > t\}| \leq |\{\psi_*^c > t\}| = 0$ by (2.5.15) and the open set $\{\psi^c > t\}$ is empty). Next, the claim, (2.5.18) and (2.5.15) yield that $r_*(t)$ also minimises $\inf_{r\geq 0} |\{-\psi > t - k(r)\}_r|$. Thus by (2.5.18), $\{-\psi_* > t - k(r_*(t))\}_{r_*(t)}$ is a ball of positive volume. As $r_*(t) > 0$, by the equality case of the claim of Part I, Step 2, the set $\{-\psi > t - k(r_*(t))\}_{r_*(t)}$ is also a ball. As $\{\psi^c > t\} \subset \{-\psi > t - k(r_*(t))\}_{r_*(t)}$, by (2.5.18) the inclusion is actually an equality. Hence $\{\psi^c > t\}$ is a ball.

Step 2.b. Proof of the claim.

We first show that there exist $0 < r_t < R_t < \infty$ such that

$$\inf_{r\geq 0} |\{-\psi_* > t - k(r)\}_r| = \inf_{r_t \leq r \leq R_t} |\{-\psi_* > t - k(r)\}_r|. \tag{2.5.19}$$

We start with the upper bound on r. By (H1) and (H3), there exists R_t such that $k(R_t) = t + 1$. Hence, if $r > R_t$, $\{-\psi_* > t - k(r)\} = \mathbb{R}^d$. We can thus only consider the radii $r \leq R_t$.

We now prove the lower bound on r. Recall that $t > \psi_*^c(1)$ and that ψ_*^c is decreasing in B_1. Therefore there exists $R_*(t) < 1$ such that

$$\{\psi_*^c > t\} = B_{R_*(t)}.$$

We set $r_t := \frac{1-R_*(t)}{2}$ and claim that (2.5.19) holds for this value. To ease notation, let us set for $r > 0$

$$S_r := \{-\psi_* > t - k(r)\}_r = \{x \in \mathbb{R}^d : -\psi_* > t - k(r) \text{ on } \overline{B}_r(x)\}.$$

We also define $\overline{R} := \frac{1+R_*(t)}{2}$. In order to prove (2.5.19) it is enough to show that

$$\{\psi_*^c > t\} = \cap_{r\geq r_t} S_r. \tag{2.5.20}$$

Recalling that the sets S_r are centred balls and that $B_{R_*(t)} \subset B_{\overline{R}}$, we have

$$\{\psi_*^c > t\} = \cap_{r\geq 0}(S_r \cap B_{\overline{R}}).$$

We now claim that

$$\cap_{r\geq r_t}(S_r \cap B_{\overline{R}}) \subset \cap_{r<r_t}(S_r \cap B_{\overline{R}}),$$

which is equivalent to

$$\cup_{r<r_t}(S_r^c \cap B_{\overline{R}}) \subset \cup_{r\geq r_t}(S_r^c \cap B_{\overline{R}}). \tag{2.5.21}$$

To prove this let $x \in S_r^c \cap B_{\overline{R}}$ for some $r < r_t$. By definition of S_r^c,

$$\min_{y\in \overline{B}_r(x)} k(r) - \psi_*(y) \leq t.$$

In particular since k is increasing, there exists $y \in \overline{B}_r(x)$ such that

$$k(|x-y|) - \psi^*(y) \le t.$$

As $x \in B_{\overline{R}} \subset B_1$, and (ψ_*^c, ψ_*^{cc}) are Kantorovitch potentials for the external transport minimising $\Upsilon(B_1)$ (see Proposition 2.3.7) there exists $z \in B_1^c$ such that $\psi_*^{cc}(z) = \psi_*(z)$ (by (2.3.14)) and

$$\psi_*^c(x) = k(|x-z|) - \psi_*(z) = \min_y k(|x-y|) - \psi_*(y) \le t.$$

Since $z \in B_1^c$ and $x \in B_{\overline{R}}$ we have

$$r' = |z-x| \ge 1 - \overline{R} = \frac{1 - R_*(t)}{2} = r_t$$

and thus

$$\min_{z \in \overline{B}_{r'}(x)} k(r') - \psi_*(z) \le t$$

so that $x \in S_{r'}$. This shows (2.5.21) which implies

$$\{\psi_*^c > t\} = \cap_{r \ge r_t} (S_r \cap B_{\overline{R}}).$$

Eventually, we must have $S_r \subset B_{\overline{R}}$ for some $r \ge r_t$ (otherwise $\{\psi_*^c > t\} = B_{\overline{R}}$ which is absurd). This concludes the proof of (2.5.20) and thus of (2.5.19).

Next, setting

$$L(r) := |\{-\psi_* > t - k(r)\}_r|,$$

we still have to establish that the infimum of L over $[r_t, R_t]$ is reached. For this we establish that L is lower semi-continuous (together with (2.5.19) this will conclude the proof of the existence of $r_*(t) > 0$ minimising L over $\mathbb{R}_+$). We start by noticing that, $r \mapsto |\{-\psi_* > r\}|$ is lower semi-continuous on $\mathbb{R}_+$. Let us denote by $\rho_t(r)$ the radius of the ball $\{-\psi_* > t - k(r)\}$. As k is continuous, the function $r \mapsto \rho_t(r)$ is also lower semi-continuous. Finally, as $L(r) = \omega_d[(\rho_t(r) - r)_+]^d$, L is lower semi-continuous as well. This ends the proof of the claim.

Step 3. Conclusion.

Let now t_n be a decreasing sequence converging to $\psi_*^c(1)$. We have

$$\{\psi^c > \psi_*^c(1)\} = \bigcup_{n \ge 0} \{\psi^c > t_n\} \quad \text{and} \quad \{\psi_*^c > \psi_*^c(1)\} = \bigcup_{n \ge 0} \{\psi_*^c > t_n\} = B_1. \tag{2.5.22}$$

By (2.5.18), for every $n \ge 0$, $\{\psi^c > t_n\} = B_{r_n}(z_n)$, where r_n is the radius of $\{\psi_*^c > t_n\}$ and $z_n \in \mathbb{R}^d$. Since t_n is decreasing the sequence $B_{r_n}(z_n)$ is non-decreasing. Moreover, by (2.5.22) $r_n \to 1$ as $n \to \infty$. Hence there exists $z \in \mathbb{R}^d$ such that $\chi_{B_{r_n}} \to \chi_{B_1}(z)$ monotonically in $L^1(\mathbb{R}^d)$ as $n \to \infty$. Eventually (2.5.22) implies that $\{\psi^c > \psi_*^c(1)\} = B_1(z)$. Consequently, $f = \chi_{\{\psi^c > \psi_*^c(1)\}} = \chi_{B_1(z)}$. This concludes the proof of the fact that balls are the unique maximisers to (2.1.3). □

Chapter 3

Numerical explorations on an isoperimetric problem with a non-local interaction of Wasserstein type

Abstract. We present a numerical study of a non-local isoperimetric problem in $\mathbb{R}^2$ where the perturbation term is defined using the 2-Wasserstein distance. We first investigate the problem restricted to radially symmetric and connected sets. Then, we present numerical simulations obtained by regularizing and discretising the isoperimetric problem and implementing a gradient descent algorithm. Our results suggest that the minimisers are no longer radially symmetric for large masses. Finally, we detail the implementation of our algorithm, which consists in solving an Allen-Cahn equation by splitting and alternate minimization. The optimal transport term is computed using a variant of the Sinkhorn algorithm.

Keywords and phrases. Nonlocal isoperimetric problems, Wasserstein distance, Allen-Cahn equation, Sinkhorn algorithm, gradient descent, alternate splitting.

2020 Mathematics Subject Classification. 49J45, 90C25, 65K10.

Outline of the current chapter

3.1 Introduction

In this article, we present some numerical considerations on the non-local isoperimetric problem studied in [19]. Let us denote by W_p the Wasserstein distance for $p \geq 1$ and identify a set $E \subset \mathbb{R}^d$ of finite Lebesgue measure with the restriction $\chi_E dx$ of the Lebesgue measure to E. The non-local energy we consider is the exterior transport functional

$$\mathcal{W}_p(E) = \inf_{|F \cap E| = 0} W_p(E, F),$$

where by convention $W_p(\mu, \nu) = +\infty$ if μ, ν are nonnegative measures with different total masses. Given $\lambda > 0$, we then introduce the variational problem

$$\inf\left\{P(E) + \lambda \mathcal{W}_p^p(E) : |E| = \omega_d\right\} \tag{3.1.1}$$

where ω_d is the volume of the unit ball and $P(E)$ denotes the perimeter of E. Let us recall the main results of [19] regarding (3.1.1). The first one states that solutions do exist and have a bounded number of connected components, all of which being bounded as well.

Theorem 3.1.1. *For every $d \geq 2$, $p \geq 1$ and $\lambda > 0$, problem* (3.1.1) *has minimisers and every minimiser is $\mathcal{H}^{d-1}$-equivalent to a representative set whose boundary is $C^{1,\alpha}$-regular for any $\alpha \in (0, 1/2)$. Moreover, there exists $C = C(d, p) > 0$ such that if $E = \cup_{i=1}^I E^i$ is such a minimiser with E^i the connected components of E, then*

$$\sum_{i=1}^{I} \mathrm{diam}(E^i) \leq C(1+\lambda)^{d-1} \quad \textit{and} \quad \inf_i \mathrm{diam}(E^i) \geq C(1+\lambda)^{-1}.$$

As a consequence $I \leq C(1+\lambda)^d$.

The second theorem of [19] establishes that in the regime where the perimeter is the dominant term, balls are the only minimisers of (3.1.1).

Theorem 3.1.2. *For every $d \geq 2$, $p \geq 1$, there exists $\lambda_0 > 0$ such that for every $\lambda \leq \lambda_0$, balls are the only minimisers of* (3.1.1).

Notice that by scaling (3.1.1) is equivalent to

$$\inf\left\{P(E) + \mathcal{W}_p^p(E) : |E| = m\right\} \quad \text{with} \quad m = \lambda^{\frac{d}{p+1}}.$$

In particular, the regime $\lambda \ll 1$ where the perimeter is dominant corresponds to $m \ll 1$, and the case $\lambda \gg 1$ corresponds to $m \gg 1$. When $\lambda \gg 1$ the exterior transport functional, which strongly penalises balls, becomes the dominant term. Indeed, we proved in [20, Theorem 1.3] (which corresponds to Theorem 2.1.3 in the second chapter of this manuscript) that the ball was the maximiser of Υ among sets of fixed mass. We thus expect that other geometries emerge in the case $\lambda \gg 1$. To be more specific, we expect that the exterior transport functional favours sets which are thin and with as much mass close to its boundary as possible, so that structures such as strips and annuli may be favoured as λ increases.

In the rest of the article, we fix $d = p = 2$. We are interested in answering the following questions:

- How does the geometry of minimisers of (3.1.1) evolve as m varies in $[0, \infty)$?
- If we restrict the study to Problem (3.1.2) defined just below, can we numerically observe the minimiser evolving from a ball to other shapes as m increases?
- Is the transition sharp, or does an intermediate regime exist between the cases $m \ll 1$ and $m \gg 1$?

We perform two types of numerical explorations to address these questions: one in the radially symmetric case, where we compute the energies of the ball and annuli, and another one in the general case for comparison. The radially symmetric version of (3.1.1) we study is defined as follows:

$$\inf_E\left\{P(E) + \lambda \mathcal{W}_2^2(E) : E \text{ radially symmetric and connected, } |E| = \pi\right\}. \tag{3.1.2}$$

In this radial case, the simulations are quite accurate and they lead us to formulate the following conjecture:

Conjecture 3.1.3. There exist $0 < \lambda_1 < \lambda_2$ (where $\lambda_1 \approx 4.95$ and $\lambda_2 \approx 5.55$) such that:

- for $0 \leq \lambda \leq \lambda_1$, the unit disk B_1 is the unique local and global minimiser of (3.1.2),
- for $\lambda > \lambda_1$, (3.1.2) has two local minimisers : B_1 and an annulus A_{r_λ} where $r_\lambda > 0$,
- for $\lambda_1 < \lambda < \lambda_2$, the unit disk B_1 is the only global minimiser of (3.1.2),
- for $\lambda = \lambda_2$, (3.1.2) has two global minimisers: B_1 and a annulus $A_{r'_\lambda}$ where $r'_\lambda > 0$,
- for $\lambda > \lambda_2$, the unique global minimiser of (3.1.2) is an annulus $A_{r''_\lambda}$ where $r''_\lambda > 0$.

Let us next describe our study of the model without any a priori symmetry, which corresponds to the minimisation problem:

$$\inf\Big\{P(E)+\lambda \mathcal{W}_2^2(E) : |E|=\pi\Big\}. \tag{3.1.3}$$

To carry out the numerical simulations, we replace the perimeter and Wasserstein functionals in (3.1.3) by more suitable, smoothed functionals. We use a phase-field method, where the characteristic function χ_E of a set $E\subset\mathbb{R}^2$ is replaced with a function $u:\mathbb{R}^2\to[0,1]$ such that $\int u=\pi$.

In this context, we approximate the perimeter term by the classical Modica-Mortola functional (see [62]), defined for $\varepsilon>0$ and $u\in H^1(\mathbb{R}^2)$ as follows:

$$\mathcal{F}_\varepsilon(u)=3\varepsilon\int_{\mathbb{R}^2}|\nabla u|^2+\frac{3}{\varepsilon}\int_{\mathbb{R}^2}u^2(1-u)^2. \tag{3.1.4}$$

We refer the reader to [9, Section 7] for more details on the Γ-convergence of this functional to the perimeter as $\varepsilon\to 0$. We would also like to point out the versatility of such phase-field methods, which are used in the study of interface dynamics [11], fracture problems [4] or multiphase flows [53].

The exterior transport term is defined for a function $u:\mathbb{R}^2\to[0,1]$ by

$$\Upsilon(u)=\inf\Big\{W_2^2(u,v) : v \text{ Lebesgue},\ 0\le v\le 1-u\Big\}, \tag{3.1.5}$$

where we identify u and the measure udx when no confusion can arise. For numerical purposes, we consider its entropic relaxation Υ_γ. It is given for $\gamma>0$ by

$$\Upsilon_\gamma(u)=\inf_{\Pi}\Bigg\{\int_{\mathbb{R}^2\times\mathbb{R}^2}|x-y|^2d\Pi(x,y)+\gamma\mathcal{H}(\pi) : \Pi\in\mathcal{M}(\mathbb{R}^2\times\mathbb{R}^2), \\ \Pi_x=udx,\ 0\le\Pi_y\le(1-u)dy\Bigg\}, \tag{3.1.6}$$

where Π_x and Π_y are the first and second marginals of $\Pi\in\mathcal{M}(\mathbb{R}^2\times\mathbb{R}^2)$, and where the (negative) entropy $\mathcal{H}$ is defined by

$$\mathcal{H}(\Pi)=\begin{cases}\int_{\mathbb{R}^2\times\mathbb{R}^2}\Pi(\log(\Pi)-1)\mathrm{d}x\mathrm{d}y & \text{if } \Pi\ll dx\otimes dy \text{ and } \Pi\ge 0,\\ +\infty & \text{otherwise.}\end{cases}$$

This relaxation is classical in computational optimal transport and is used to compute accurate approximate solutions to various optimal transportation problems (see e.g. [31] and [32, Section 4] for more details). It is also worth pointing out that the algorithm we implement to compute Υ_γ is based on the numerical method described in [13], which was itself based on [71].

Eventually, the problem that we are interested in solving is what we call the Modica-Mortola-entropic approximation of (3.1.3): given $\lambda, \varepsilon, \gamma > 0$, we consider

$$\inf_u \left\{ \mathcal{E}(u) = 3\varepsilon \int_\Omega |\nabla u|^2 + \frac{3}{\varepsilon} \int_\Omega u^2(1-u)^2 + \lambda \Upsilon_\gamma(u) : u : \Omega \to [0,1], \int_{\mathbb{R}^2} u dx = \pi \right\}. \tag{3.1.7}$$

Formally, λ represents the "physical" parameter that models the contribution of the exterior transport term to the total energy, while ε and γ are two small approximation parameters.

The ambient space Ω we consider is the torus $\mathbb{R}^2/L\mathbf{Z}^2$ defined for $L > 0$. We discretise Ω using a regular grid and implement a finite difference scheme to compute each energy term. For the optimization, we use an alternate minimisation algorithm and treat the optimal transport term using a variant of the Sinkhorn algorithm.

The chapter is organised as follows. In Section 3.2, we present the theoretical results and conjectures associated to the study of the radially symmetric problem (3.1.2). In Section 3.3 we consider the approximated version (3.1.7) of Problem (3.1.3). We present the numerical simulations we conducted to study (3.1.7) and compare them to the previous conjectures. In Section 3.4, we give the proofs and computations on which is based our investigation of Problem (3.1.2). In Section 3.5, we detail the transition of the original problem (3.1.3) to its approximation (3.1.7). We also derive the evolution equation we use for the implementation of the gradient descent algorithm. In Section 3.6, we detail the space discretisation of (3.1.7) and the implementation of the gradient descent in this setting. Eventually, in Section 3.7 we detail the derivation of the Sinkhorn algorithm we use to solve the exterior optimal transport problem (3.1.6)

3.2 Results in the two-dimensional radially symmetric case

We now present some theoretical considerations on Problem (3.1.2). First notice that due to the non-local nature of optimal transport, obtaining a closed form expression for $\mathcal{W}_2^2(E)$ given a generic shape $E \subset \mathbb{R}^2$ is in general impossible. However, restricting the study to radially symmetric, connected sets allows for more explicit computations.

Given $0 \leq r_m$, we denote by A_{r_m,r_M} the centered open annulus of mass π with inner radius r_m and outer radius $r_M = \sqrt{1 + r_m^2}$. As the mass is fixed, when no confusion can arise we simply write $A_{r_m,r_M} = A_{r_m}$. Notice that with our notation, the case $r_m = 0$ corresponds to the unit disk $A_{r_m} = B_1$.

Given $0 \leq r_m$, for the perimeter we always have

$$P(A_{r_m}) = 2\pi\left(r_m + \sqrt{1 + r_m^2}\right)$$

(see its graph in Figure 3.1). The main difficulty is to compute

$$\mathcal{W}_2^2(A_{r_m}) = \inf\left\{W_2^2(A_{r_m}, F) : F \subset \mathbb{R}^2, |F \cap A_{r_m}| = 0\right\}. \tag{3.2.1}$$

Given $r_c \in [r_m, r_M]$, we denote by $\tau(r_c)$ the cost of sending A_{r_m} to the reunion of the inner

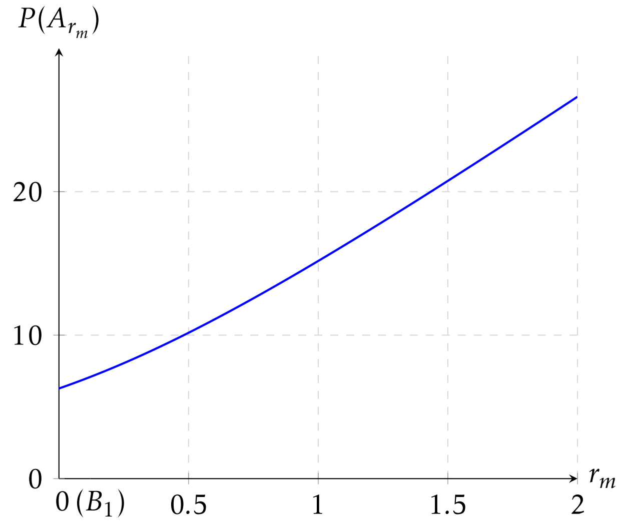

Figure 3.1: Perimeter $P(A_{r_m})$ of the annulus of inner radius r_m.

and outer annuli A_{s_m,r_m} and A_{r_M,s_M} as follows: A_{r_m,r_c} is sent to A_{s_m,r_m} and A_{r_c,r_M} to A_{r_M,s_M}, using the c-cyclically monotonous and unique optimal transport map T_{r_c} (which is defined in (3.4.1)). See Figure 3.2 below for an illustration. Also notice that as A_{r_m,r_c} can at most be of mass B_{r_m}, we necessarily have $r_c \leq \sqrt{2}r_m$. Eventually, τ_{r_c} is well defined for $r_m \leq r_c \leq \hat{r}_M$, where $\hat{r}_M = \min(\sqrt{2}r_m, r_M)$.

The following proposition allows us to reduce the study $\mathcal{W}_2^2(A_{r_m})$ to a single variable optimization problem involving $\tau(r_c)$. Its proof is postponed to Section 3.4.1.

Proposition 3.2.1. *Problem* (3.2.1) *is uniquely minimised by the reunion F of two annuli adjacent to* A_{r_m}*, that is*

$$F = A_{s_m,r_m} \cup A_{r_M,s_M},$$

where $0 \leq s_m \leq r_m$ *and* $r_M \leq s_M$*. Additionally, determining F is equivalent to finding the unique optimal splitting radius* $\overline{r}_c \in [r_m, \hat{r}_M]$ *which minimises the cost of sending* A_{r_m,r_c} *to* A_{s_m,r_m} *and* A_{r_c,r_M} *to* A_{r_M,s_M} *using the map* T_{r_c}*. Therefore:*

$$W_2^2(A_{r_m}) = \inf\Big\{\tau(r_c) : r_c \in [r_m, \hat{r}_M]\Big\} = \tau(\overline{r}_c), \tag{3.2.2}$$

The previous proposition gives existence and uniqueness of F and of the optimal transport map $T_{\overline{r}_c}$, but $\overline{r}_c$ is still to be determined explicitely. For this matter, we exhibit an equation satisfied by $\overline{r}_c$ which is used to compute a numerical approximation of r_c through a secant method. By conservation of the mass, we can then deduce s_m and s_M.

The computation of τ and τ' is carried out in Section 3.4.2, where we in particular establish that τ is $\mathcal{C}^1$ on $[r_m, \hat{r}_M]$. We numerically observe that

Assumption 3.2.2. The function τ is strictly convex on $[r_m, \hat{r}_M]$ and $\tau'(r_m) < 0$.

This observation leads us to use the following method to compute $\overline{r}_c$: we first check if $\tau'(\hat{r}_M) \leq 0$. In this case the minimum is reached at $\hat{r}_M$. Otherwise, we solve $\tau'(r_c) = 0$ for

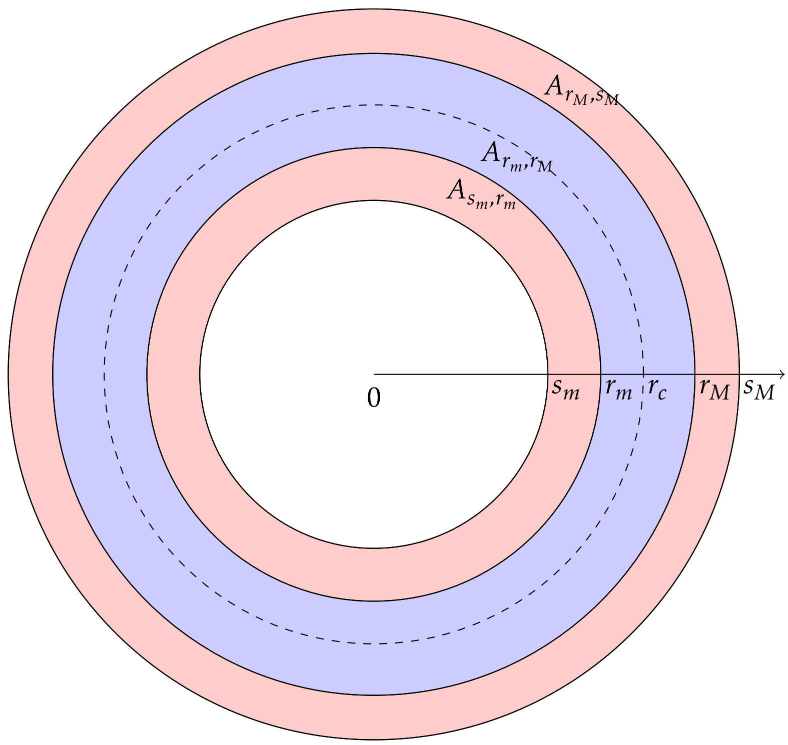

Figure 3.2: The annulus A_{r_m,r_M} and its corresponding minimiser $A_{s_m,r_m} \cup A_{r_M,s_M}$.

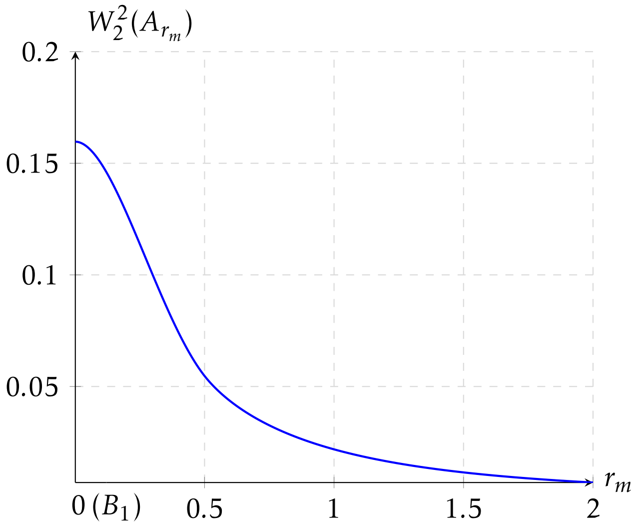

Figure 3.3: Exterior transport energy $W_2^2(A_{r_m})$ of the annulus inner radius r_m.

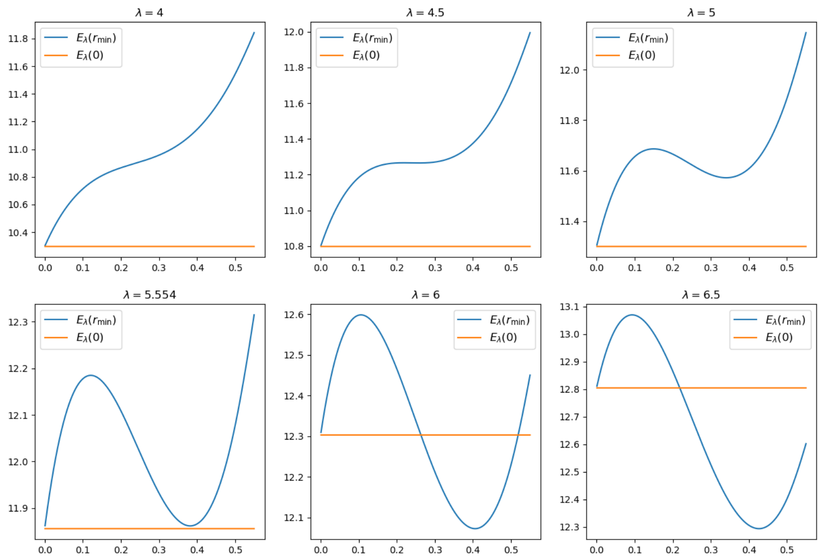

Figure 3.4: Energies of annuli of inner radius r_m (blue) and of the unit disk (orange).

$r_c \in (r_m, \hat{r}_M)$ using a variant of the secant method (see Section 3.4.3 for the detailed explanation). We thus are able to compute $W_2^2(A_{r_m})$ using (3.2.2), and we give its graph is given in Figure 3.3.

Eventually, we can compute the energy

$$\mathcal{E}(A_{r_m}) = 2\pi\left(r_m + \sqrt{1+r_m^2}\right) + \lambda W_2^2(A_{r_m})$$

for different values of $\lambda \geq 0$. Our computations are summarised in Figure 3.4. We plotted the function $r_m \mapsto \mathcal{E}(A_{r_m})$ for increasing values of $\lambda \geq 0$. Recall that the case $r_m = 0$ corresponds to the unit disk B_1. We added the constant function equals to $\mathcal{E}(B_1)$ on each graph to ease the comparison of the annulus and the disk. Let us point out the fact that these numerical results are correct up to computer accuracy, and that they lead us to make Conjecture 3.1.3 on the behaviour of the minimisers of (3.1.2).

3.3 Numerical experiments in the non radially symmetric case

We now present the results of the numerical experiments conducted in order to study the non radially symmetric problem (3.1.3). While the detailed implementation of the algorithm is written down in Section 3.5, let us now briefly summarise it.

3.3.1 Breakdown of the algorithm

Given $L > 0$, we work on the torus $\Omega = \mathbb{R}^2/L\mathbf{Z}^2$ that we discretise in a regular grid of $N \times N$ squares of side-length $h = L/N$. To solve

$$\inf\left\{\mathcal{E}(u) = \mathcal{F}_\varepsilon(u) + \lambda\Upsilon_\gamma(u),\, u : \Omega \to [0,1],\, \int_\Omega u = \pi\right\},$$

Parameter	Description
L	$\Omega = \mathbb{R}^2/L\mathbf{Z}^2 \subset \mathbb{R}^2$ is the ambient space.
N	Ω is discretised in $N \times N$ squares.
h	$h = L/N$ is the size of the griud.
ε	Modica-Mortola parameter, $\varepsilon = 2h$.
γ	Entropic parameter.
λ	The energy is $\mathcal{F}_\varepsilon + \lambda\Upsilon_\gamma$.
δt	Time step.
tol	Relative tolerance.

Table 3.1: Parameters of the fully discretised problem.

we implement a gradient descent algorithm of time step δt. In the next lines, we consider the continuous problem rather than the discretised one to lighten notation. Let us denote by W the double well function $u \mapsto u^2(1-u)^2/2$. Formally computing the first variation of $\mathcal{F}_\varepsilon(u) + \lambda\Upsilon_\gamma(u)$, we obtain the following evolution equation for the L^2-gradient flow of $\mathcal{E}$:

$$\partial_t u = 3\varepsilon\Delta u - \frac{3}{\varepsilon}W'(u) + \xi_u + \mu. \tag{3.3.1}$$

The function ξ_u corresponds to the first variation of Υ_γ, whose detailed computation in the discretised setting is given in Section 3.7. The real μ is a Lagrange multiplier associated with the preservation of mass. Notice that Equation (3.3.1) can be seen as an Allen-Cahn equation with a forcing term $\xi_u + \mu$. To solve it, we proceed to a Lie splitting and alternatively solve on a time interval of length δt:

$$\partial_t u = 3\varepsilon\Delta u \quad \text{and} \quad \partial_t u = -\frac{3}{\varepsilon}W'(u) + \lambda\xi_u + \mu. \tag{3.3.2}$$

We stop the process when the relative variation of the energy from one iteration to another is inferior to some fixed tolerance threshold. We summarise the notation in Table 3.1.

3.3.2 Presentation of the numerical simulations

Comparison of the splitting algorithm with the radial computations

In this experiment, our goal is to confirm that the limit shapes generated by the splitting algorithm possess energy levels similar to those of the shapes computed in the radially symmetric case. To obtain the set of curves of Figure 3.5, we set $L = 4$ and use the splitting algorithm for λ taking sampled values in $[3.5, 7.5]$ and N successively equal to 128, 192 and 256. We initialize the splitting algorithm with a figure U_0 that is already close to be the theoretical optimal figure (i.e. the ball or an optimal annulus) computed in the radially symmetric case. Let us point out that this experiment relies on the fact that radially symmetric shapes are stable for our splitting algorithm, so that we are guaranteed to have limit shapes that are also radially symmetric. Our goal here is not to compare radial shapes with the thin and elongated sets observed in subsequent experiments.

Also notice that as N increases, ε decreases so that we fix decreasing values of γ to have

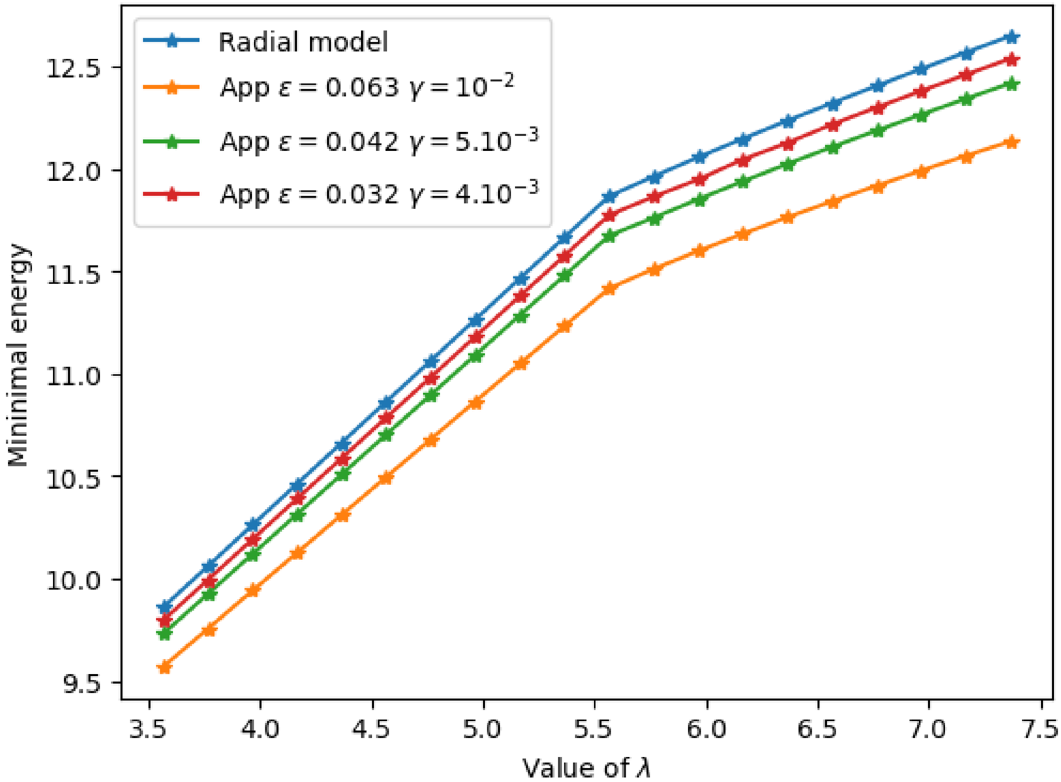

Figure 3.5: Comparison of the 2D model and the radial case.

$\gamma \ll \varepsilon$. We fix $\delta t = \varepsilon/8$ and tol $= 10^{-8}$ in each experiment.

We observe on Figure 3.5 that the alternate splitting algorithm yields energy values that tends to the ones obtained in the radially symmetric case. Notice that the derivative of the minimal energy with respect to λ has a jump at the critical value $\lambda = \lambda_2 \simeq 5.554$. This corresponds to the transition from the optimality of the ball to the one of the annulus in the radially symmetric case.

Stability of the ball and of the annulus

In this experiment, we set $L = 4$, $N = 256$, $\gamma = 10^{-2}$, tol $= 10^{-8}$ and $\delta t = \varepsilon/10$ (recall that $\varepsilon = 2h = 2L/N$). In the first simulation, which corresponds to Figure 3.6, we set $\lambda = 1$ and we do observe that a non radially symmetric shape close to the ball evolves to the ball. In the second one, which corresponds to Figure 3.7, we set $\lambda = 10$ and we start with an annulus. In the limit, we obtain a final annulus close to the optimal one predicted by the radial computations.

Some experiments in the critical regime.

In this experiment, we set $L = 4$, $N = 256$, $\gamma = 10^{-2}$, tol $= 10^{-8}$, $\delta t = \varepsilon/10$ and $\lambda = \lambda_2 \simeq 5.554$. The results we obtained are depicted in Figure 3.8. They tend to confirm the existence of a critical value λ_2 of λ for which both the disk and some annulus A_{r_m} are global minimisers (compare Figure 3.8 and Figure 3.9).

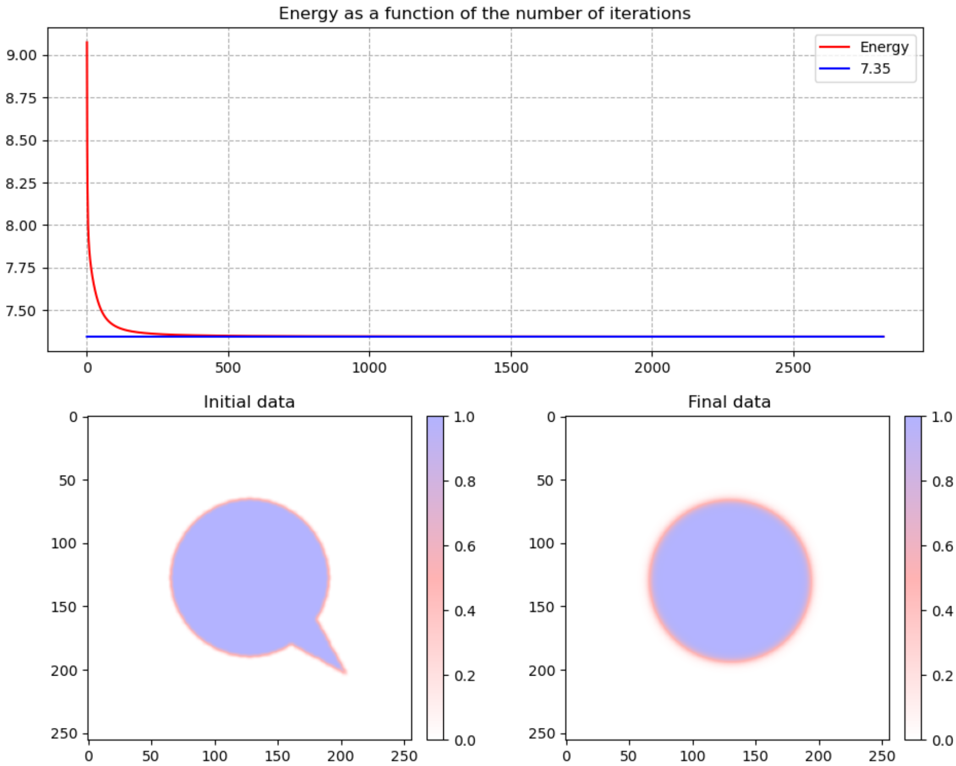

Figure 3.6: Starting with a perturbed ball ($\lambda = 1$).

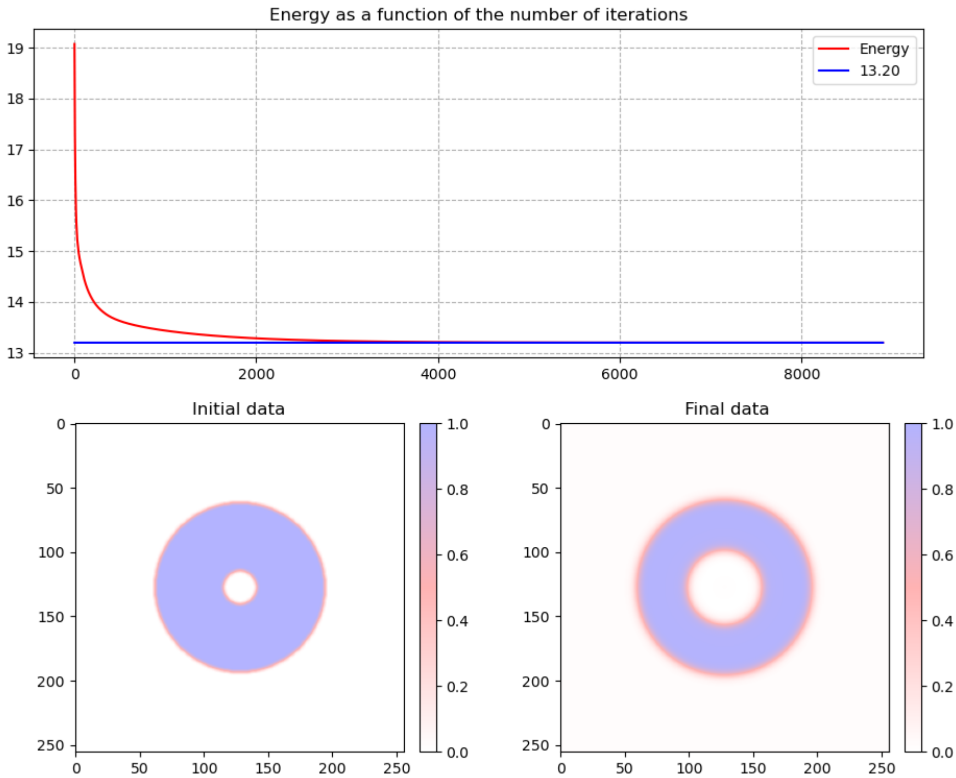

Figure 3.7: Starting with a thick annulus ($\lambda = 10$).

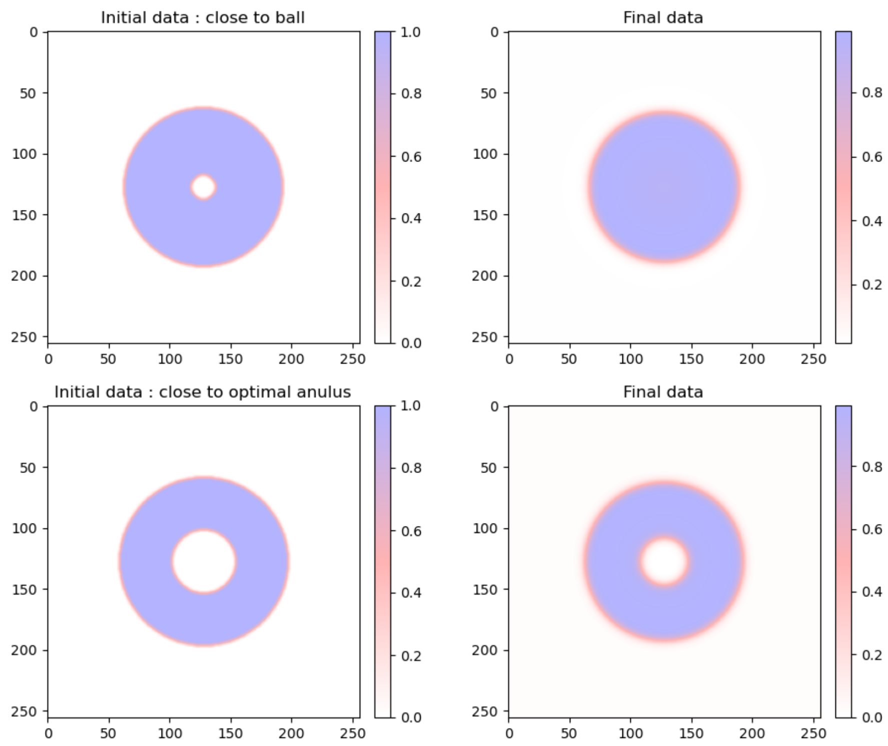

Figure 3.8: Behaviour in the critical regime $\lambda = \lambda_2 \simeq 5.554$.

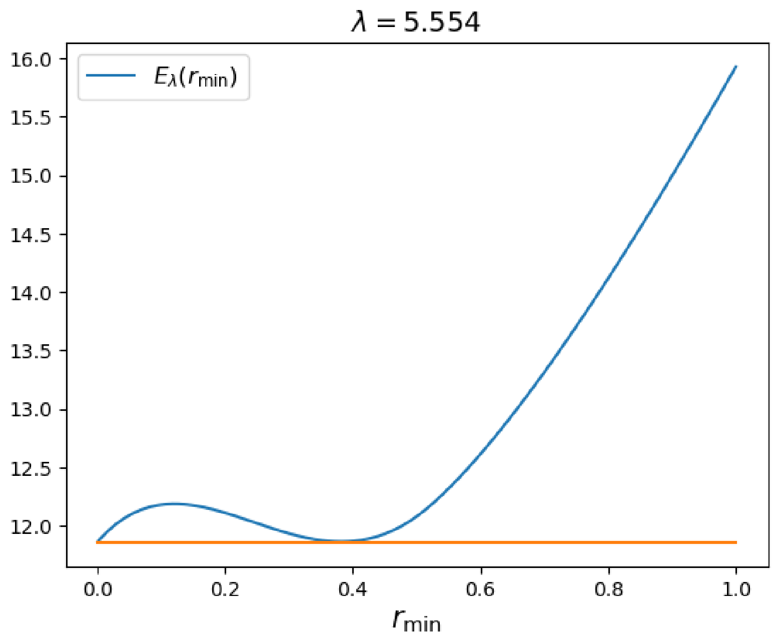

Figure 3.9: Energy of A_{r_m} when r_m varies in the case $\lambda = \lambda_2 \simeq 5.554$.

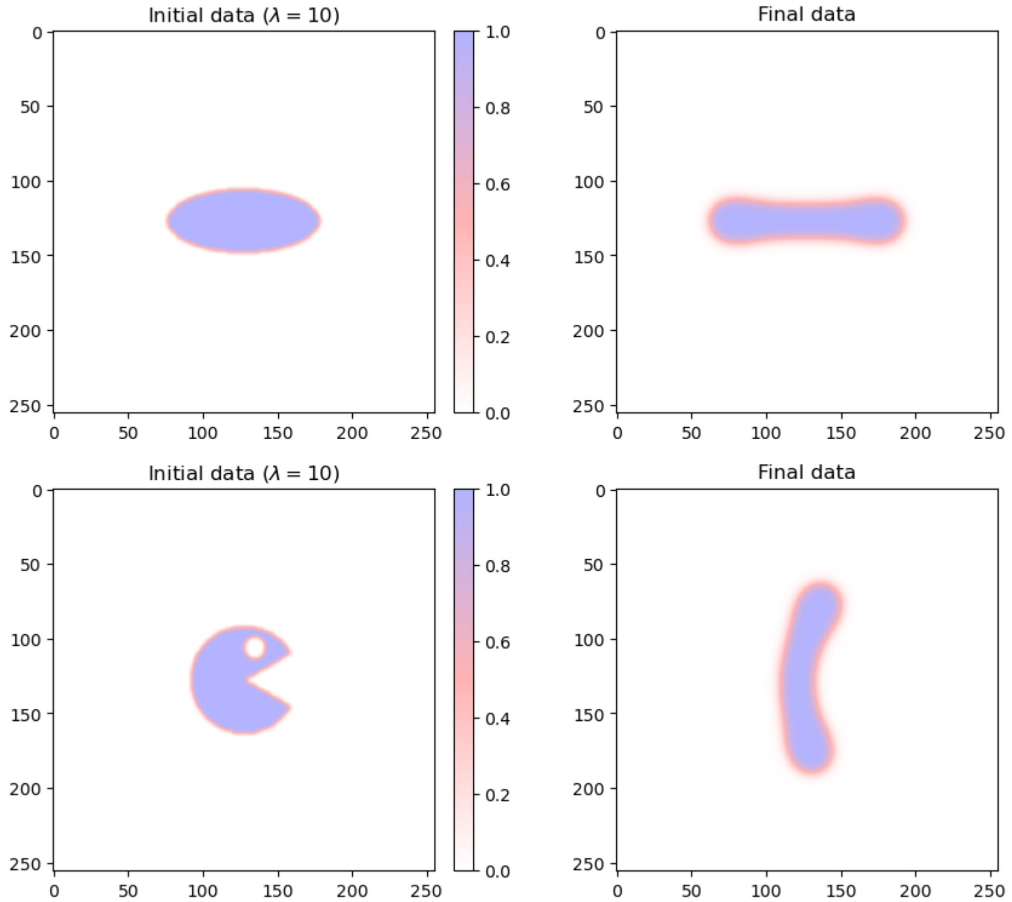

Figure 3.10: Starting with an ellipse ($\lambda = 10$).

Starting away from radially symmetric sets.

When we start with initial data that is highly non radially symmetric, the algorithm struggles to converge to a precisely defined set. Most of the mass uniformly diffuses into the background, while a small set in the centre evolves into a disk. To counter this phenomenon, we use a slightly different version of our algorithm, where the mass constraint is more carefully enforced. See Section 3.6.3 for more details.

In the experiments conducted with this new method, we set $L = 8$, $N = 256$, $\gamma = 10^{-2}$, $\delta t = \varepsilon/10$, tol $= 10^{-8}$ and $\lambda = 10$. The results we obtained are depicted in Figure 3.10. We observe that the limit shape is not at all radially symmetric set. Instead, it is elongated, thin and seems to have two axes of symmetry. This leads us to consider the conjecture that thin and elongated shapes are the preferred minimisers as $\lambda \to \infty$, and not annuli.

3.4 Computation and proofs in the two dimensional radially symmetric case

3.4.1 The exterior transport cost functional for a single annulus

Proof of Proposition 3.2.1. By [19, Proposition 2.1] Problem (3.2.1) admits a unique minimiser F, and the unique optimal transport plan Π between A_{r_m} and F is induced by a map T. By proceeding as in the proof of [19, Lemma 4.2 (i)] we obtain that F is radially symmetric, and eventually that F is the reunion of the two annuli adjacent to A_{r_m} (one of the annuli being possibly empty). Regarding the transport map T, it is sufficient to follow the proof of [19, Lemma 4.2 (ii)]. We introduce

$$T_{r_c}(x) = f_{r_c}(|x|)\frac{x}{|x|}, \quad \text{where} \quad f_{r_c}(x) = \begin{cases} f_-(x) = \sqrt{s^2 - (r_c - r_m)^2} & \text{if} \quad s \le r_c, \\ f_+(x) = \sqrt{s^2 + (1 + r_m^2 - r_c^2)} & \text{if} \quad s \ge r_c. \end{cases} \tag{3.4.1}$$

Notice that T_{r_c} is the unique radially symmetric map (in the sense that $T(x) = f(|x|)\frac{x}{|x|}$) which solves $\det\nabla T = 1$ and sends A_{r_m} to F. More precisely, T_{r_c} sends A_{r_m,r_c} to A_{s_m,r_m} and A_{r_c,r_M} to A_{r_M,s_M}. As the map

$$g : \mathbb{R}_+ \to \mathbb{R}_+, \, r \mapsto \sqrt{1 + r^2}$$

is monotone on $\mathbb{R}_+$, we have that the set $S = \{(x, T(x)) : x \in A_{r_m}\}$ is c-cyclically monotone for the cost $c(x,y) = |x-y|^2$ (see [5, Definition 1.7] for a definition of c-cyclical monotonicity). Therefore, if we define the transport plan $\Pi_{r_c} = (\mathrm{Id}, T)_{\#}\chi_{A_{r_m}}dx$, we have that $\mathrm{supp}(\Pi_{r_c}) = S$ is c-cyclically monotone. Thus by [5, Theorem 1.13], π_{r_c} is optimal between A_{r_m} and $F = T(A_{r_m})$. Consequently by uniqueness of the optimal transport plan, we have that $\Pi = \Pi_{r_c}$ and $T = T_{r_c}$. □

3.4.2 The splitting radius for the exterior transport of a single annulus

Recall that $0 \le r_m$ is fixed, that $\hat{r}_M = \min(r_M, \sqrt{2}r_m)$ and that given $r_c \in [r_m, \hat{r}_M]$, we write $\tau(r_c)$ to denote the transport cost of A_{r_m} to $A_{s_m,r_m} \cup A_{r_M,s_M}$ by sending A_{r_m,r_c} to A_{s_m,r_m} and A_{r_c,r_M} to A_{r_M,s_M} (see Figure 3.2). Also recall that as a consequence of Proposition 3.2.1,

$$W_2^2(A_{r_m}) = \min\{\tau(r_c) : r_c \in [r_m, \hat{r}_M]\} = \tau(\overline{r}_c),$$

i.e. that the exterior transport problem for A_{r_m} is solved by finding the optimal splitting radius $\overline{r}_c$.

Let us now explain how we obtain an equation on $\overline{r}_c$. We split $\tau(r_c)$ into $\tau_-(r_c)$ and $\tau_+(r_c)$, which denote the cost of sending A_{r_m,r_c} (resp. A_{r_c,r_M}) to A_{s_m,r_m} (resp. A_{r_M,s_M}).By (3.4.1), we have

$$\begin{aligned}\tau(r_c) &= \int_{A_{r_m}} |T_{r_c}(x) - x|^2 dx = \int_{A_{r_m,r_c}} |T_{r_c}(x) - x|^2 dx + \int_{A_{r_c,r_M}} |T_{r_c}(x) - x|^2 dx \\ &= \int_{r_m}^{r_c} 2\pi s\Big((s^2 - r_c^2 + r_m^2)^{1/2} - s\Big)^2 ds + \int_{r_c}^{r_M} 2\pi s\Big((s^2 + 1 + r_m^2 - r_c^2)^{1/2} - s\Big)^2 ds.\end{aligned}$$

After integration and denoting by $\delta^2 = r_c^2 - r_m^2$, we have

$$\tau_-(r_c) = \frac{2\pi\delta^2}{4}\ln\left(\frac{r_m + r_c}{r_m + \sqrt{2r_m^2 - r_c^2}}\right) + \frac{2\pi\delta^8}{16}\left(\frac{1}{(r_m + r)^4} - \frac{1}{(r_m + \sqrt{2r_m^2 - r_c^2})^4}\right),$$

$$\tau_+(r_c) = \frac{2\pi(1+\delta^2)^2}{4}\ln\left(\frac{\sqrt{1+r_m^2} + \sqrt{2+2r_m^2 - r_c^2}}{\sqrt{1+r_m^2} + r_c}\right) + \frac{2\pi(1+\delta^2)^4}{16}\left(\frac{1}{(\sqrt{1+r_m^2} + \sqrt{2+2r_m^2 - r_c^2})^4} - \frac{1}{(\sqrt{1+r_m^2} + r_c)^4}\right).$$

By Proposition 3.2.1, solving (3.2.1) amounts to finding the unique r_c which minimises $\tau(r_c)$. First, notice that τ_- is of class C^∞ on $[r_m, \sqrt{2}r_m)$ and of class C^1 on $[r_m, \sqrt{2}r_m]$. The function τ_+ is of class C^∞ on $[r_m, \hat{r}_M]$. Hence τ is of class C^1 on $[r_m, \hat{r}_M]$ and on this interval we compute

$$\tau_-{}'(r_c) = 2\pi r_m r_c\left(2r_m - r_c - \sqrt{2r_m^2 - r_c^2}\right) + 2\pi r_c\delta^2\ln\left(\frac{r_m + r_c}{r_m + \sqrt{2r_m^2 - r_c^2}}\right),$$

$$\tau_+{}'(r_c) = \tau_{+,1}{}'(r_c) + \tau_{+,2}{}'(r_c) + \tau_{+,3}{}'(r_c) + \tau_{+,4}{}'(r_c),$$

where

$$\tau_{+,1}{}'(r_c) = \frac{\pi(1+\delta^2)}{2}\sqrt{1+r_m^2}\left(\frac{r_c}{\sqrt{2+2r_m^2 - r_c^2}} - 1\right),$$

$$\tau_{+,2}{}'(r_c) = -2\pi(1+\delta^2)r_m\ln\left(\frac{\sqrt{1+r_m^2} + \sqrt{2+2r_m^2 - r_c^2}}{\sqrt{1+r_m^2} + r_c}\right),$$

$$\tau_{+,3}{}'(r_c) = \frac{\pi(1+\delta^2)^3}{2}\left(\frac{r_m^2 + 1 + r_c^2 + 2r_c\sqrt{1+r_m^2}}{(\sqrt{1+r_m^2} + r_c)^5}\right),$$

$$\tau_{+,4}{}'(r_c) = -\frac{\pi(1+\delta^2)^3}{2}\left(\frac{3r_m^2 + 3 - r_c^2}{\sqrt{2r_m^2 + 2 - r_c^2}} + 2\sqrt{1+r_m^2}\right)\frac{r_c}{(\sqrt{1+r_m^2} + \sqrt{2r_m^2 + 2 - r_c^2})^5}.$$

Recall that by Assumption 3.2.2, τ is strictly convex on $[r_m, \hat{r}_M]$ and $\tau'(r_m) < 0$. We then search for the minimum of τ as follows: we first check if $\tau'(\hat{r}_M) \leq 0$. In this case the minimum is reached at $\hat{r}_M$. Otherwise, we solve $\tau'(r_c) = 0$ for $r_c \in (r_m, \hat{r}_M)$. For this, we implement the modified secant method described below.

3.4.3 A modified secant method to compute the splitting radius

Recall that τ_+ is smooth on $[r_m, \hat{r}_M]$ and that τ_- is of class C^1 on $[r_m, \hat{r}_M]$ but not twice differentiable at $\sqrt{2}r_m$. In particular, we have the following asymptotics for τ':

$$\tau'(r_c) = a - b\sqrt{2r_m^2 - r_c^2} + O(2r_m^2 - r_c^2) \quad \text{for} \quad 0 < \sqrt{2}r_m - r_c \ll 1, \tag{3.4.2}$$

for some $a \in \mathbb{R}$, $b > 0$ depending on r_m. We observe that if $\overline{r}_c$ is close to $\sqrt{2}r_m$, the derivative of τ' is very large, so that the performance of the classical secant method deteriorates. Namely, we approximate τ' by some functions $f(r_c) = a - b\sqrt{2r_m^2 - r_c^2}$ instead of affine functions $f(r_c) = a + br_c$.

Given $a \in \mathbb{R}$ and $b > 0$, we define on $[r_m, \hat{r}_M]$ the function $\eta(a, b, r) = a - b\sqrt{2r_m^2 - r^2}$. In the case where $\tau(\hat{r}_M) > 0$, we initialise the method by setting

$$r[0] = \frac{r_m + r_M}{2} + \frac{r_M - rm}{10}, \quad r[1] = \frac{r_m + r_M}{2} - \frac{r_M - r_m}{10}.$$

Then for $n \geq 1$, we define $a_n \in \mathbb{R}$ and $b_n > 0$ to be such that

$$\eta(a_n, b_n, r[n-1]) = \tau'(r[n-1]) \quad \text{and} \quad \eta(a_n, b_n, r[n]) = \tau'(r[n]). \tag{3.4.3}$$

Notice that by strict convexity of τ, we have $\tau'(r[n-1]) \neq \tau'(r[n])$. We then define $r[n+1]$ to be the solution of $\eta(a_n, b_n, r) = 0$. Solving the system (3.4.3), we finally obtain that

$$r[n+1] = \left(2r_m^2 - \frac{\left(\tau'(r[n])\sqrt{2r_m^2 - r[n-1]^2} - \tau'(r[n-1])\sqrt{2r_m^2 - r[n]^2}\right)^2}{\left(\tau'(r[n]) - \tau'(r[n-1])\right)^2} \right)^{1/2}. \tag{3.4.4}$$

We stop the iterations and consider that the method has provided an acceptable approximation for r_c when $|\tau'(r[n+1])| \leq \alpha |\tau'(r[1])|$ with $\alpha = 10^{-12}$. This secant method thus grants us a way of computing an approximate value of $\mathcal{W}_2^2(A_{r_m})$ given $r_m \geq 0$.

3.4.4 Asymptotic expansions of the exterior transport cost

Thanks to the numerical simulations, we also notice that there exists r_m^1 such that for $r_m \leq r_m^1$, the entire disk inside A_{r_m} is filled by $A_{s_m, r_m} = B_{r_m}$. In this case, $r_c = \sqrt{2}r_m$, and an expansion at order 5 with respect to $r_m \ll 1$ yields

$$\frac{W_2^2(A_{r_m})}{2\pi} = \frac{\ln(1+\sqrt{2}) + 4 - 3\sqrt{2}}{4} - \frac{\ln(1+\sqrt{2})}{2} r_m^2 + \frac{4\sqrt{2}}{3} r_m^3 + \frac{\ln(1+\sqrt{2}) - 2\sqrt{2}}{2} r_m^4 + \frac{2\sqrt{2}}{15} r_m^5 + O(r_m^6).$$

In the case $r_m \to \infty$, following [70, Appendix B] we expect that r_c admits an a priori expansion of the form

$$\frac{r_c}{r_m} = 1 + \frac{1}{4r_m^2} + \frac{\theta}{16r_m^4} + O\left(\frac{1}{r_m^6}\right),$$

for some $\theta \in \mathbb{R}$ which we compute by optimizing the quantity $\tau(r_c)$ for fixed $r_m \gg 1$ (we omit the details of this particular optimization). We finally obtain that as $r_m \to \infty$ we have

$$r_c(r_m) = r_m + \frac{1}{4r_m} - \frac{3}{32r_m^3} + O\left(\frac{1}{r_m^5}\right) \quad \text{and} \quad \frac{W_2^2(A_{r_m})}{2\pi} = \frac{1}{32r_m^2} - \frac{1}{64r_m^4} + \frac{67}{6144r_m^6} + O\left(\frac{1}{r_m^8}\right).$$

3.5 From the theoretical non-radial problem to the numerical implementation

Let us explain how we build approximate solutions of (3.1.7). Recall that the ambient space Ω is the torus $\mathbb{R}^2/L\mathbf{Z}^2$, where $L > 0$ is such that $|\Omega| \geq 2\pi$. Given $\lambda > 0$ we thus aim at solving

$$\inf\left\{P(E) + \lambda\Upsilon(E) \ : E \subset \Omega, |E| = \omega_2\right\}.$$

To obtain algorithms that are efficient in terms of computational cost, we substitute each term of the previous problem by an approximate counterpart which is easier to discretise and evaluate.

3.5.1 Approximate problem and duality

For the perimeter term we use the Modica-Mortola approximation and define $W : \mathbb{R} \to \mathbb{R}$ by

$$W(s) = \frac{1}{2}s^2(1-s)^2 \quad \text{and set} \quad C_W = \int_0^1 \sqrt{2W(s)}\,ds = \frac{1}{6}.$$

Given $\varepsilon > 0$ and $u \in H^1(\Omega)$ we then set

$$\mathcal{F}_\varepsilon(u) = \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(u)dx.$$

We denote by $BV(\mathbb{R}^2)$ the set of functions with finite total variation and recall that $\mathcal{F}_\varepsilon$ Γ-converges in $L^1(\Omega)$ (see e.g. [9, Section 7.2] for more details on the Γ-convergence of $\mathcal{F}_\varepsilon$) as $\varepsilon \to 0$ to the functional $\mathcal{F}_0$, where

$$\mathcal{F}_0(u) = \begin{cases} C_W \int_\Omega |\nabla u| \text{ if } u \in \mathrm{BV}(\mathbb{R}^2, \{0,1\}), \\ \infty \quad \text{otherwise.} \end{cases}$$

As for Υ, we follow the usual approach in computational optimal transport and replace it by its entropic approximation (see e.g [22]). The (negative) entropy $\mathcal{H}$ is defined for $\Pi \in \mathcal{M}(\Omega \times \Omega)$ by

$$\mathcal{H}(\Pi) = \begin{cases} \int_{\Omega\times\Omega} \Pi(\log(\Pi) - 1)\mathrm{d}x\mathrm{d}y & \text{if } \Pi \ll dxdy \text{ and } \Pi \geq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

and by convention we set $0 \log 0 = 0$. The entropy of a measure $\mu \in \mathcal{M}(\Omega)$ is defined similarly.

For $x, y \in \mathbb{R}^2$, let us set $c(x,y) = |x-y|^p$. Given $\gamma > 0$ and $u : \mathbb{R}^2 \to [0,1]$ with $\int u = \pi$, we define

$$\Upsilon_\gamma(u) = \inf_{\Pi \in \mathcal{M}(\Omega\times\Omega)} \left\{ \int_{\Omega\times\Omega} c\,d\pi + \gamma\mathcal{H}(\Pi) : \Pi_x = u\,dx,\ 0 \leq \Pi_y \leq (1-u)dy \right\}. \tag{3.5.1}$$

A simple application of the Direct Method in the Calculus of Variations allows us to prove the following existence result for (3.5.1) (which in fact holds for any $p \geq 1$ and $d \geq 2$)

Proposition 3.5.1. *Let $u : \Omega \to [0,1]$ be such that $\int u = \pi$. Assume that there exists at least one exterior transport plan Π such that $\mathcal{H}(\Pi) < \infty$. Then, the infimum in* (3.5.1) *is attained by a unique minimiser $\pi_0 \in \mathcal{M}_+(\Omega \times \Omega)$.*

As a linear optimization problem under convex constraints, which are given by a continuous linear equality and a continuous linear inequality, (3.5.1) admits a dual problem. By the Fenchel-Rockafellar theorem, we have:

Proposition 3.5.2. *Recall that $\Omega \subset \mathbb{R}^2$ is compact. Given $u : \Omega \to [0,1]$ such that $\int u = \pi$, we have the following dual formulation for $\Upsilon_\gamma(u)$:*

$$\begin{gathered}\Upsilon_\gamma(u) = \sup_{\varphi,\psi\in\Phi} \left\{ \int_\Omega u\varphi dx + \int_\Omega (1-u)\psi dy - \gamma \int_{\Omega\times\Omega} e^{\frac{-c+\varphi+\psi}{\gamma}} u(x)\mathrm{d}x(1-u(y))\mathrm{d}y \right\}, \\ \Phi = \Big\{ (\varphi,\psi) \in C(\Omega)\times C(\Omega),\ \psi \leq 0 \Big\}.\end{gathered} \tag{3.5.2}$$

Eventually, we are interested in solving the Modica-Mortola-entropic approximation of (3.1.2): given $\varepsilon, \lambda, \gamma > 0$,

$$\inf_u \left\{ \mathcal{E}(u) = 6\mathcal{F}_\varepsilon(u) + \lambda\Upsilon_\gamma(u) : 0 \leq u \leq 1,\ \int u dx = \pi \right\}. \tag{3.5.3}$$

3.5.2 Subdifferential of the approximated energy

We compute approximate solutions to (3.5.3) by implementing a gradient descent algorithm. Again, to lighten notation we describe the optimization of the continuous functional $\mathcal{E}(u)$ instead of its discretised version. In this regard, we have to exhibit functions belonging the subdifferential of $\mathcal{E}$, which is defined as follows:

$$\partial\mathcal{E}(u) = \left\{ q : \Omega \to \mathbb{R},\ \liminf_{\delta\to 0} \frac{\mathcal{E}(u+\delta(v-u)) - \mathcal{E}(u)}{\delta} \geq q \text{ for any } v:\ \Omega \to [0,1],\ \int v = \pi \right\}.$$

Let us start with the subdifferential of $\mathcal{F}_\varepsilon$. It turns out that we have a stronger result because its first variation is well-known in the literature. Given a functional $\mathcal{G} : L^1(\Omega) \to \mathbb{R}$, we define its first variation in our setting by

$$\delta_u\mathcal{G}(v) = \lim_{\delta\to 0} \frac{\mathcal{G}((1-\delta)u+\delta v) - \mathcal{G}(u)}{\delta} = \lim_{\delta\to 0} \frac{\mathcal{G}(u+\delta(v-u)) - \mathcal{G}(u)}{\delta},.$$

where $u, v : \Omega \to [0,1]$ such that $\int u = \int v = \pi$. For the Modica-Mortola functional, given $u \in H^2(\Omega)$ we have:

$$\delta_u\mathcal{F}_\varepsilon(v) = -\varepsilon \int_\Omega \Delta u(v-u)dx + \frac{1}{\varepsilon}\int_\Omega W'(u)(v-u)dx, \tag{3.5.4}$$

with $W'(u) = u(1-u)(1-2u)$.

We now turn to the subdifferential of $\Upsilon_\gamma(u)$. We assume that there exists a couple of potentials (φ_u, ψ_u) maximizing (3.5.2). Using them as candidates for $\Upsilon_\gamma(u+\delta(v-u))$ and setting

$w = v - u$ yield

$$\Upsilon_\gamma(u+\delta w) \geq \int_\Omega \Big[u+\delta w\Big]\varphi_u dx + \int_\Omega \Big[1-(u+\delta w)\Big]\psi_u dy$$
$$-\gamma \int_{\Omega\times\Omega} \exp\Big(\gamma^{-1}\Big(-c(x,y)+\varphi_u(x)+\psi_u(y)\Big)\Big)\Big[u(x)+\delta w(x)\Big]\Big[1-\Big(u(y)+\delta w(y)\Big)\Big]dxdy.$$

By rearranging the terms and relabelling the integration variables, we recognise that

$$\Upsilon_\gamma(u+\delta w) \geq \Upsilon_\gamma(u) + \delta\int_\Omega (\varphi_u - \psi_u) w dx$$
$$-\gamma\,\delta \int_{\Omega\times\Omega} \exp\Big(\gamma^{-1}(-c(x,y)+\varphi_u(x)+\psi_u(y))\Big)[w(x)(1-u(y)) - w(y)u(x))]dxdy + O(\delta^2).$$

We finally obtain that

$$\liminf_{\delta\to 0} \frac{\Upsilon(u+\delta w) - \Upsilon_\gamma(u)}{\delta} \geq \int_\Omega (\varphi_u - \psi_u) w dx - \gamma \int_\Omega G w dx \tag{3.5.5}$$

where the function G is defined for $x \in \mathbb{R}^2$ by

$$G(x) = \int_\Omega \exp\Big(\frac{-c(x,y)}{\gamma}\Big)\Big[\exp\Big(\frac{\varphi_u(x)+\psi_u(y)}{\gamma}\Big)(1-u(y)) - \exp\Big(\frac{\varphi_u(y)+\psi_u(x)}{\gamma}\Big)u(y)\Big]dy.$$

Combining (3.5.4) and (3.5.5), we finally have that the function

$$-6\varepsilon\Delta u + \frac{6}{\varepsilon}W'(u) + \lambda(\varphi_u - \psi_u) - \lambda\gamma G \tag{3.5.6}$$

belongs to the subdifferential of $\mathcal{E}$. In the numerical implementation of our algorithms, we always have that $\gamma \ll \lambda, \varepsilon$, so that we will omit the last term in (3.5.6). Consequently, using (3.5.6) to implement a gradient descent algorithm for $\mathcal{E}$ yields the following evolution equation:

$$\partial_t u = 6\varepsilon\Delta u - \frac{6}{\varepsilon}u(1-u)(1-2u) - \lambda(\varphi_u - \psi_u) + \mu, \tag{3.5.7}$$

where $\mu \in \mathbb{R}$ is the Lagrange multiplier associated with the preservation of the mass of u. Setting $\xi_u = \varphi_u - \psi_u$, we notice that (3.5.7) corresponds to an Allen-Cahn equation with an additional term $-\lambda\xi_u + \mu$. Following the notation of [11, Chapter 2], given $t > 0$ we denote by

$S(t)$ the flow of (3.5.7),

$e^{t\Delta}$ the flow of the diffusion equation $\partial_t u = \varepsilon\Delta u$,

$Y(t)$ the flow of the reaction equation $\partial_t u = -\frac{1}{\varepsilon}u(1-u)(1-2u) - \lambda\xi_u + \mu$,

and we approximate $S(t)$ by using the Lie splitting $L(t) = Y(t)e^{t\Delta}$.

3.6 Space discretisation and implementation of the Lie splitting

Let us now describe how we discretise the domain $\Omega = \mathbb{R}^2/L\mathbf{Z}^2$ and solve each term of the reaction-diffusion equation described just above. In this section, we assume that the term ξ_u related to the exterior transport is known. Its computation is based on the Sinkhorn algorithm which is detailed in the next and last section.

3.6.1 Solving the diffusion equation by Fourier transform

We first have to solve the classical heat equation on Ω

$$\partial_t u = 6\varepsilon\Delta u.$$

In a discretised context, we implement a finite difference scheme and use the discrete Fourier transform, which we now comment. Consider the regular partition of Ω into N^2 cubes and identify each cube by a double index (i_1, i_2) with $0 \leq i_1, i_2 \leq N-1$. We specify u to be a function $\overline{u}$ constant on each square of Ω and by abuse of notation identify $\overline{u}$ with the $N\times N$ square matrix of the values it takes. In this context, we compute the discrete partial derivatives of $\overline{u}$ over the x and y axis in the forward fashion, i.e.

$$D_1\,\overline{u}(i_1,i_2) = h^{-1}(\overline{u}(i_1+1,i_2) - \overline{u}(i_1,i_2)) \quad \text{and} \quad D_2\,\overline{u}(i_1,i_2) = h^{-1}(\overline{u}(i_1,i_2+1) - \overline{u}(i_1,i_2)),$$

where the boundary conditions are periodic: $N-1+1=0$ and $N-1+1=0$. The equation verified by $\overline{u}$ is then

$$\partial_t\overline{u}(i_1,i_2,t) = \varepsilon\Big[D_1^2\overline{u}(i_1,i_2,t) + D_2^2\overline{u}(i_1,i_2,t)\Big]. \tag{3.6.1}$$

where D^2 corresponds to the forward derivative iterated with the backward one. We can then write $\overline{u}$ as the linear sum of its Fourier coefficients as follows

$$\overline{u}(i_1,i_2,t) = \sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1}\mathcal{F}(\overline{u})(k_1,k_2,t)e^{-2\pi i\left(\frac{i_1k_1}{N}+\frac{i_2k_2}{N}\right)},$$

Injecting this expression into (3.6.1), expanding the derivatives and factorizing twice by the half-angle yields

$$\partial_t\mathcal{F}(\overline{u})(k_1,k_2,t) = -4\varepsilon\left(\frac{\sin^2(\pi k_1h^{-1}) + \sin^2(\pi k_2h^{-1})}{h^2}\right)\mathcal{F}(\overline{u})(k_1,k_2,t)$$

Solving this equation in $t > 0$ we obtain

$$\mathcal{F}(\overline{u})(t) = e^{\varepsilon C_1 t}\mathcal{F}(\overline{u})(0) \quad \text{where} \quad C_1(k_1,k_2) = -4\left(\frac{\sin^2(\pi k_1h^{-1}) + \sin^2(\pi k_2h^{-1})}{h^2}\right)$$

Inverting the Fourier transform, we finally obtain the following expression for the discretised flow e^{tD} of $e^{t\Delta}$:

$$e^{tD}(\overline{u}_0)(t) = \mathcal{F}^{-1}\left(e^{\varepsilon C_1 t}\mathcal{F}(\overline{u}_0)\right),$$

which we will discretise in time accordingly during the implementation of the algorithm.

3.6.2 Solving the reaction equation by the explicit Euler method

We now have to solve the following non-linear differential equation

$$\partial_t u = -\frac{6}{\varepsilon}u(1-u)(1-2u) - \lambda\xi_u + \mu. \tag{3.6.2}$$

The discretisation and computation of ξ_u is done using a modified Sinkhorn algorithm described in the following section. Now assuming that ξ_u is given, a simple way of approximately solving (3.6.2) is by the use of an explicit Euler scheme. Given an initial condition u_0, we define u at time $t > 0$ as

$$u(t) = u_0 - t\left(\frac{1}{\varepsilon}u_0(1-u_0)(1-2u_0) + \xi_{u_0} + \mu\right), \tag{3.6.3}$$

In the discrete setting, the implementation simply consists in applying (3.6.3) on each cube of the discretisation of Ω.

3.6.3 Slight modification of the algorithm for non radially symmetric cases

In the first version of the splitting algorithm, our handling of the mass constraint simply amounts to having a step where the missing mass is uniformly distributed over the grid. However, in some situations this process can lead to situations of over-diffusion, where most of the mass of the initial data trivially fades into the background. To circumvent this difficulty, we introduce a second algorithm with an alternative handling of the mass constraint. This corresponds to implementing the second model of the conservative Allen-Cahn equation in [11, Section 3.5] (see also [10] on this exact topic).

For the sake of the explanation we assume that $\gamma = 0$. We consider the function $\Phi : [0,1] \to [0,1]$ such that

$$\Phi(s) = 6\int_0^s \sqrt{2W(t)}dt = 6\int_0^s (t-t^2)dt = (3-2s)s^2,$$

and a slightly modified version of Problem (3.5.3):

$$\inf_u \left\{ \overline{\mathcal{E}}(u) = 3\varepsilon\int_\Omega |\nabla u|^2 dx + \frac{1}{\varepsilon}\int_\Omega 6\,W(u)dx + \lambda\Upsilon(\Phi(u)) : 0 \le u \le 1,\ \int_\Omega \Phi(u) = m \right\}. \tag{3.6.4}$$

Let us denote by u_ε a solution of (3.6.4). Notice that $\Phi(u) = u$ if $u = \chi_E$ for some set $E \subset \Omega$. Additionally, if u_ε is an approximation of a characteristic function χ_E as a consequence of Modica and Mortola's Γ-convergence result, so is $\Phi(u)$. Consequently, the energies $\overline{\mathcal{E}}(u)$ and $\overline{\mathcal{E}}(u)$ should coincide as $\varepsilon \to 0$ and $u_\varepsilon \to \chi_E$ for some set $E \subset \Omega$.

Replacing u by $\Phi(u)$ in the exterior transport term and in the mass constraint yields the following Allen-Cahn equation when computing the first variation of $\overline{\mathcal{E}}$:

$$\frac{\partial_t u}{6} = \varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) - \lambda \sqrt{2W(u)}\xi_u + \sqrt{2W(u)}\mu, \tag{3.6.5}$$

where $\sqrt{2W(u)}\mu = u(1-u)\mu$ is the Lagrange multiplier associated with the mass constraint. The main purpose of this new phase-field equation is the non-uniformity of the Lagrange multiplier, which allows for a more precise handling of potential mass loss. We also observe that $\Phi(u) = u$ if $u = \chi_E$ for some set $E \subset \Omega$. Consequently, the alternative and classic algorithm coincide as $\varepsilon \to 0$ and the solutions of the Allen-Cahn equation converge to characteristic functions of sets.

3.7 A Sinkhorn-like algorithm for the entropic exterior transport

In this section, we provide a detailed exposition of the computation of the discretised version of Υ. In particular, we explain how the exterior transport term ξ_u appearing in the reaction step of the Allen-Cahn equation (3.5.7) is computed. Given $\Omega = [0, L]^2$ as in the previous section, recall that Υ was defined on the set of Borel functions $u : \Omega \to [0, 1]$ such that $\int u = \pi$ as

$$\Upsilon(u) = \inf_{\Pi \in \mathcal{M}_+(\Omega\times\Omega)} \left\{ \int_{\Omega\times\Omega} |x-y|^2 d\Pi : \Pi_x = u dx,\ \Pi_y \leq (1-u) dy \right\}, \tag{3.7.1}$$

with Π_x and Π_y being the first and second marginals of Π. Given $N \geq 1$, we denote by

$$\mathcal{Q} = \{Q_i\}_{i=1}^{N^2}$$

the collection of N^2 regularly distributed squares partitioning Ω, and set $h = L/N$. In this context, u is a piecewise constant function of the form

$$u = \sum_{i=1}^{N^2} u_i \mathbf{1}_{Q_i} \quad \text{with} \quad u_i \in [0,1] \quad \text{and} \quad \sum_{i=1}^{N^2} u_i h^2 = \pi.$$

3.7.1 Entropic regularization of the discrete exterior transport problem

We now set $n = m = N^2$ and in the rest of the article, we write $\mathbb{R}_+^n$ and $\mathbb{R}_+^m$ to denote respectively the initial and target vector spaces of our exterior optimal transport problem. Let us define $\mathbf{M}_+^{n,m}$ as the convex set of matrices of size $n \times m$ with nonnegative coefficients. Given $1 \leq i \leq n$ and $1 \leq j \leq m$, we denote

$$C = (C_{i,j})_{i,j} = (h^2 |i-j|^2)_{i,j}$$

the cost matrix associated with the 2-Wasserstein distance. We now define $\overline{u}$ to be the vector $(u_1, u_2, \ldots, u_n) \in \mathbb{R}_+^n$ of the values taken by u. Therefore (3.7.1) rewrites in its discretised form:

$$\inf \left\{ \langle C, P \rangle \ : P \in \mathbf{M}_+^{n,m},\ P\mathbb{1}^m = \overline{u},\ {}^tP\mathbb{1}^n \leq \mathbb{1}^m - \overline{u} \right\}.$$

Here, $\langle\cdot,\cdot\rangle$ is the canonical inner product of matrices and $\mathbb{1}^n = (1,\dots,1) \in \mathbb{R}^n$. To solve this problem numerically, we consider its usual entropic regularization (see [32, Chapter 4]) and consider instead for $\varepsilon > 0$

$$\inf\left\{\langle C,P\rangle + \varepsilon H(P) \ : P \in \mathbf{M}_+^{n,m},\ P\mathbb{1}^m = \overline{u},\ {}^tP\mathbb{1}^n \leq \mathbb{1}^m - \overline{u}\right\}, \tag{3.7.2}$$

where the discrete entropy H is defined as

$$H(P) = \sum_{i,j} P_{i,j}(\log(P_{i,j}) - 1).$$

To quantify the difference between $P \in \mathbf{M}_+^{n,m}$ and a reference kernel $K \in \mathbf{M}_+^{n,m}$ (see [32, Remark 4.2]), we introduce the discrete Kullback–Leibler divergence:

$$KL(P|K) = \sum_{i,j} P_{i,j} \log\left(\frac{P_{i,j}}{K_{i,j}}\right) - P_{i,j} + K_{i,j}.$$

We now set for $1 \leq i \leq n$ and $1 \leq j \leq m$

$$\overline{K} = (\overline{K}_{i,j})_{i,j} = u_i\, e^{-\frac{C_{i,j}}{\varepsilon}}(1 - u_j).$$

Notice that by definition, $KL(P|K) = +\infty$ if $K_{i,j} = 0$ and $P_{i,j} > 0$ for some couple (i,j). Thus when explicitly minimising (3.7.2) we can simply set $P_{i,j} = 0$ if $K_{i,j} = 0$ (i.e. if $u_i = 0$ or $u_j = 1$). In what follows, we assume that $u_i > 0$ and $1 - u_j > 0$ for any $1 \leq i \leq n$ and $1 \leq j \leq m$. By direct computation, given $P \in \mathbf{M}_+^{n,m}$ we notice that

$$\varepsilon KL(P|\overline{K}) - \varepsilon \sum_{i,j} \overline{K}_{i,j} = \langle \overline{C}, P\rangle + \varepsilon H(P),$$

where the modified cost $\overline{C} \in \mathbf{M}_+^{n,m}$ is defined as

$$\overline{C}_{i,j} = C_{i,j} - \varepsilon \log(u_i(1 - u_j)).$$

Solving (3.7.2) with $\overline{C}$ thus amounts to minimising $KL(P|\overline{K})$ with the same constraints on P.

3.7.2 Dual formulation of the discrete exterior transport problem

As a constrained convex minimisation problem, (3.7.2) admits a dual formulation. We denote by $\mathbb{R}_-^m$ the vectors of $\mathbb{R}^m$ with nonpositive coefficients. Let us introduce the dual variables $f, g \in \mathbb{R}^n \times \mathbb{R}_-^m$ used to encode the constraints. As a consequence of the convexity and the lower semi-continuity of the functional and the constraint, there is no duality gap (see [20, Proposition 3.5] for a proof in a Polish space and with $\gamma = 0$), so that the primal and dual problems coincide.

We then compute

$$\inf\Big\{\langle\overline{C},P\rangle+\varepsilon H(P)\ :P\in\mathbf{M}_+^{n,m},\ P\mathbb{1}^m=\overline{u},\ {}^tP\mathbb{1}^n\le\mathbb{1}^m-\overline{u}\Big\}$$

$$=\inf_{P\in\mathbf{M}_+^{n,m}}\sup_{f,g\in\mathbb{R}^n\times\mathbb{R}^m_-}\Big(\langle\overline{C},P\rangle+\varepsilon H(P)-\langle P\mathbb{1}^m-\overline{u},f\rangle-\langle{}^tP\mathbb{1}^n-\mathbb{1}^m+\overline{u},g\rangle\Big)$$

$$=\sup_{f,g\in\mathbb{R}^n\times\mathbb{R}^m_-}\inf_{P\in\mathbf{M}_+^{n,m}}\Big(\langle\overline{C},P\rangle+\varepsilon H(P)-\langle P\mathbb{1}^m,f\rangle-\langle{}^tP\mathbb{1}^n,g\rangle\Big)+\langle\overline{u},f\rangle+\langle\mathbb{1}^m-\overline{u},g\rangle.$$

Let us mention that the variables f,g in the discretised setting respectively correspond to the approximate Kantorovitch potentials φ and ψ of the continuous setting. By convexity of the functions involved,

$$\inf_{P\in\mathbf{M}_+^{n,m}}\Big(\langle\overline{C},P\rangle+\varepsilon H(P)-\langle P\mathbb{1}^m,f\rangle-\langle{}^tP\mathbb{1}^n,g\rangle\Big)$$

admits a minimiser characterised by

$$\nabla_P\Big(\langle\overline{C},P\rangle+\varepsilon H(P)-\langle P\mathbb{1}^m,f\rangle-\langle{}^tP\mathbb{1}^n,g\rangle\Big)=0.$$

Solving this equation we obtain the following expression for P:

$$P_{i,j}=e^{-\frac{\overline{C}_{ij}-(f_i+g_j)}{\varepsilon}}.$$

Injecting it in the last line of (3.7.2) we obtain a second duality formula:

$$\inf\Big\{\langle\overline{C},P\rangle+\varepsilon H(P)\ :P\in\mathbf{M}_+^{n,m},\ P\mathbb{1}^m=\overline{u},\ {}^tP\mathbb{1}^n\le\mathbb{1}^m-\overline{u}\Big\}$$

$$=\sup_{f,g\in\mathbb{R}^n\times\mathbb{R}^m_-}\Big(\langle f,\overline{u}\rangle+\langle g,\mathbb{1}^m-\overline{u}\rangle-\varepsilon\sum_{i,j}e^{-\frac{\overline{C}_{ij}-(f_i+g_j)}{\varepsilon}}\Big).\tag{3.7.3}$$

The Sinkhorn algorithm consists in iteratively building approximate solutions of the dual problem in (3.7.3). To do so, we first solve (3.7.3) for f considering g is fixed, and then compute g fixing f to be the previously computed value. Precisely, for g fixed we consider

$$\sup_{f\in\mathbb{R}^n}\Big(\langle f,\overline{u}\rangle+\langle g,\mathbb{1}^m-\overline{u}\rangle-\varepsilon\sum_{i,j}e^{-\frac{\overline{C}_{ij}-(f_i+g_j)}{\varepsilon}}\Big).$$

Computing the criticality condition and using the definition of $\overline{C}$ yield that for $1\le i\le n$

$$u_i-e^{\frac{f_i}{\varepsilon}}u_i\sum_j e^{\frac{g_j-C_{ij}}{\varepsilon}}(1-u_j)=0,$$

that is (as we assumed $u_i > 0$)

$$f_i = -\varepsilon \log\left(\sum_j e^{\frac{g_j - C_{ij}}{\varepsilon}}(1-u_j)\right).$$

Next, assuming that f is given we solve

$$\sup_{g\in\mathbb{R}^m_-}\left(\langle f,\overline{u}\rangle + \langle g, \mathbb{1}^m - \overline{u}\rangle - \varepsilon\sum_{i,j} e^{-\frac{\overline{C}_{ij}-(f_i+g_j)}{\varepsilon}}\right).$$

and obtain that for $1 \le j \le m$ the maximiser exists and is given by

$$g_j = \min\left(0, -\varepsilon\log\left(\sum_i u_i e^{\frac{f_i - C_{ij}}{\varepsilon}}\right)\right).$$

We can now initialise the Sinkhorn algorithm by setting $f^{(0)} = g^{(0)} = 0$ and then compute for $k \geq 0$:

$$\begin{aligned} f_i^{(k+1)} &= -\varepsilon\log\left(\sum_j e^{\frac{g_j^{(k)} - C_{i,j}}{\varepsilon}}(1-u_j)\right),\\ g_j^{(k+1)} &= \min\left(0, -\varepsilon\log\left(\sum_i u_i e^{\frac{f_i^{(k+1)} - C_{i,j}}{\varepsilon}}\right)\right). \end{aligned}$$

This formulation of the algorithm is often referred to as Sinkhorn *in the log domain* (see [32, Remark 4.23]). Usually one instead works with the variables $p_i = \exp(f_i/\varepsilon)$ and $q_j = \exp(g_j/\varepsilon)$. Defining $K = (K_{i,j})_{i,j} = \exp(-C_{i,j}/\varepsilon)_{i,j}$, the alternate minimisation then rewrites

$$\begin{aligned} p^{(k+1)} &= \frac{\mathbb{1}^n}{K((\mathbb{1}^m - \overline{u})\odot q^{(k)})},\\ q^{(k+1)} &= \min\left(\mathbb{1}^m, \frac{\mathbb{1}^m}{{}^tK(\overline{u}\odot p^{(k+1)})}\right). \end{aligned}$$

The division is to be understood as element-wise and $\odot$ denotes the element-wise multiplication. The usual multiplication of a vector X of $\mathbb{R}^n$ by a matrix M of $\mathbb{R}^n\times\mathbb{R}^n$ is simply written MX.

Remark 3.7.1. Recall that the problem is set in a squared domain $\Omega = [0,L]^2$ partitioned into N^2 squares of $\mathbb{R}^2$ and that $h = L/N$. In this context, it is possible to significantly speed-up the computation done in the Sinkhorn algorithm by avoiding computing matrices of size $n^2 = N^4$ at each iteration. Let $i, j \in \{1,2,\ldots,N^2\}$ denote the center of the i-th (resp j-th) cube of the partition. We introduce an x-axis and a y-axis in Ω by writing

$$i = (i_1, i_2) \quad \text{and} \quad j = (j_1, j_2) \quad \text{with} \quad i_1, j_1, i_2, j_2 \in \{0, \ldots, N-1\}$$

so that

$$C(i,j) = (|(i_1 - j_1)h|^2 + |(i_2 - j_2)h|^2).$$

Defining the matrices of $\mathbf{M}_+^{N,N}$

$$K_1 = (K^1_{i_1,j_1}) = e^{-\frac{1}{\varepsilon}((i_1-j_1)h)^2} \quad \text{and} \quad K_2 = (K^2_{i_2,j_2}) = e^{-\frac{1}{\varepsilon}((i_2-j_2)h)^2},$$

and seeing $\overline{u}$, p and q as $N \times N$ matrices, the products Kp and $K^T q$ respectively amount to $K^1 p K^2$ and $K^1 q K^2$ (since K^1 and K^2 are symmetric). We denote by $\mathbb{1}^{N\times N}$ the square matrix of size $N \times N$ filled with ones. In the end, the algorithm rewrites

$$\begin{aligned} p^{(k+1)} &= \frac{\mathbb{1}^{N\times N}}{K_1((\mathbb{1}^{N\times N} - \overline{u}) \odot q^{(k)})K_2}, \\ q^{(k+1)} &= \min\left(\mathbb{1}^{N\times N}, \frac{\mathbb{1}^{N\times N}}{K_1(\overline{u} \odot p^{(k+1)})K_2}\right). \end{aligned}$$

Chapter 4

A concentration-compactness principle for perturbed isoperimetric problems with general assumptions

Abstract. Derived from the concentration-compactness principle, the concept of generalised minimiser can be used to define generalised solutions of variational problems which may have components “infinitely" distant from each other. In this chapter and under mild assumptions we establish existence and density estimates of generalised minimisers of perturbed isoperimetric problems. Our hypotheses encapsulate a wide class of functionals including the classical, anisotropic and fractional perimeter. The perturbation term may for instance take the form of a potential, a translation invariant kernel or a nonlocal term involving the Wasserstein distance.

Keywords and phrases. Nonlocal isoperimetric problems, concentration-compactness principle, generalised minimisers.

2020 Mathematics Subject Classification. 49J45, 49Q10, 49Q20.

Outline of the current chapter

4.1 Introduction

One of the first perturbed isoperimetric problems was formulated by George Gamow in the 1930s for investigating the stability of the atomic nucleus [47]. Given $d \geq 2$, $m > 0$ and $\alpha \in (0,d)$ a possible formulation of this variational problem is

$$\inf_{|E|=m} \left\{ \mathrm{Per}(E) + \int_{E\times E} \frac{dxdy}{|x-y|^{d-\alpha}} \right\},$$

where Per is the Caccioppoli perimeter and $|E|$ is the Lebesgue measure of E. The goal of this variational problem is to model an attractive, short-range force inducing surface tension (the "perimeter" term) that competes with a repulsive term V acting at a greater distance (the "perturbation" term, which is often nonlocal). This competition plays a pivotal role in the wide range of geometries the perturbed isoperimetric problem can describe (see for instance [55, Figure 1]) and both the physics and mathematics communities have explored numerous variants of this problem. In this chapter we study a generalised version of this problem:

$$e(m) = \inf_{|E|=m} \left\{ \mathcal{E}(E) = P(E) + V(E) \right\}, \tag{4.1.1}$$

where P is a non-explicit, nonnegative perimeter term and V a perturbation term.

Mathematically speaking, a primary concern in tackling this optimisation problem is establishing the existence of solutions. The most challenging task often is to exhibit convergent minimising sequences. From a concentration-compactness principle standpoint [58], lack of compactness in isoperimetric problems can occur when a minimising sequence admits components fleeing infinitely far apart. However, in such scenarios it may still be possible to show that some relaxed versions of the problem admit minimisers. It is within this framework that we introduce the generalised minimisation problem:

$$e_{\mathrm{gen}}(m) = \inf \left\{ \mathcal{E}_{\mathrm{gen}}((E^i)_{i\geq 1}) = \sum_{i\geq 1} \mathcal{E}(E^i) : \sum_{i\geq 1} |E^i| = m \right\}. \tag{4.1.2}$$

In the context of metric measure spaces and with $V = 0$, research towards finding minimal assumptions guaranteeing existence of solutions to (4.1.2) was carried out in [66], where existence of generalised isoperimetric clusters was also shown. See also [6] for a characterization of minimising sequences for the isoperimetric problem on noncompact $RCD(K,N)$ spaces.

In addition to the matter of their existence, the question of the regularity of minimisers constitutes a significant aspect of the study of isoperimetric problems. It is indeed well-known in shape optimisation theory that studying variational problems in the class of sets whose boundary has some regularity allows for much easier computations and characterization of the solutions. A first step towards establishing regularity properties of minimisers is often to prove that, when they exist, they have density estimates. We say that a set E admits interior (resp. exterior) density estimates when there exists $c, r' > 0$ such that for any $0 < r \leq r'$ and $x \in E$ (resp. E^c),

$$|E \cap B_r(x)| \geq c\, r^d \text{ (resp. } |E^c \cap B_r(x)| \geq c\, r^d \text{)}.$$

It can also be useful to show density estimates for sets that are close (for a given topology) to minimisers of a given isoperimetric problem. Indeed, it is then often possible to modify a bit those sets to obtain actual minimisers of the considered problem. See for instance results obtained in [37] in the context of Riemannian manifolds regarding the regularity of volume-constrained local minimisers of anisotropic surface energies.

Our present goal is to exhibit in the case $V \neq 0$ general assumptions under which:

- (4.1.1) and (4.1.2) coincide,
- (4.1.2) admits solutions,
- solutions of (4.1.2) have density estimates.

4.1.1 Main results

Let us denote by $(e_i)_{i=1}^d$ the canonical basis of $\mathbb{R}^d$ and specify that all the sets considered in the chapter are assumed to be at least (Lebesgue) measurable. We start by proving in Section 4.2 that (4.1.1) and (4.1.2) coincide under the following set of assumptions (S1).

(H1) *Energy of small balls*: $\mathcal{E}(B_r) \to 0$ as $r \to 0$ and $\mathcal{E}(\emptyset) = 0$.

(H2) *Convergence at infinity*: For any set E with $|E| < \infty$, $\mathcal{E}(E \cap B_R) \to \mathcal{E}(E)$ as $R \to \infty$.

(H3) *Vanishing range of action*: If E, F are bounded sets, then $\mathcal{E}(E \cup (F + Le_1)) \to \mathcal{E}(E) + \mathcal{E}(F)$ as $L \to \infty$.

Proposition 4.1.1. *Assume that $\mathcal{E}$ satisfies (S1). Then* (4.1.1) = (4.1.2).

Let us comment a bit on (S1). We use (H1) to compensate for any potential mass deficit when we modify a set E to construct a generalised minimiser $(E^i)_i$. However, there are alternative methods to ensure that the mass constraint is satisfied when solving (4.1.1) or (4.1.2) (see e.g. Remark 4.2.1). (H3) states that bounded sets do not interact when infinitely far apart from each other, so that they may be seen as components of a generalised minimiser.

We then show that (4.1.2) admits minimisers. To prove this result we introduce the functionals $E \mapsto P(E, U)$ and $E \mapsto V(E, U)$, which are defined relatively to a Lebesgue measurable set U. By convention, we write $P(E, \mathbb{R}^d) = P(E)$ and $V(E, \mathbb{R}^d) = V(E)$. The set of assumptions (S2) we require to establish that (4.1.2) has solutions is as follows (we use the letter $\mathcal{F}$ to denote P or V in hypotheses applying to both terms):

(H4) *Relative isoperimetric inequality*: There exists $r_0 > 0$ and $f_1 : \mathbb{R}_+ \to \mathbb{R}_+$ increasing with $f_1(0) = 0$, $m \mapsto f_1(m)/m$ nonincreasing and $\lim_{m \to 0} f_1(m)/m = \infty$ such that for $r \leq r_0$, $x \in \mathbb{R}^d$ and $E \subset \mathbb{R}^d$:

$$\min\Big(f_1(|E \cap B_r(x)|), f_1(|B_r(x) \setminus E|)\Big) \leq P(E, B_r(x)).$$

(H5) *Periodicity*: There exists $0 < r_1 \le 2r_0/\sqrt{d}$ such that $\mathcal{F}(E + r_1, U + r_1 e_k) = \mathcal{F}(E, U)$ for any $1 \le k \le d$ and $E, U \subset \mathbb{R}^d$.

(H6) *Perimeter set operations*: Given $E \subset \mathbb{R}^d$, if $U \subset U'$ are open, then $P(E,U) \le P(E,U')$, and if $(U_i)_{i=1}^I$ are open disjoint sets, then $\sum_{i=1}^I P(E,U_i) \le P(E, \cup_{i=1}^I U_i)$.

(H7) *Bounded perimeter*: If $(E_n)_n$ is such that $\sup_n \mathcal{E}(E_n) < \infty$ and $\sup_n |E_n| < \infty$, then $\sup_n P(E_n) < \infty$.

(H8) *Compactness*: If $(E_n)_n$ satisfies $\sup_n P(E_n, U) < \infty$, then up to extraction there exists $E \subset \mathbb{R}^d$ such that $E_n \cap U \to E$ in L^1_{loc} as $n \to \infty$.

(H9) *Lower semicontinuity*: Given $E \subset \mathbb{R}^d$ and a bounded open set U, if $E_n \to E$ in L^1_{loc} as $n \to \infty$ with $\sup_n |E_n| < \infty$, then $\mathcal{F}(E,U) \le \liminf_n \mathcal{F}(E_n, U)$.

(H10) *Beppo-Levi*: If $(U_n)_{n\ge 0}$ is a nondecreasing sequence of open sets exhausting $\mathbb{R}^d$, then for any $E \subset \mathbb{R}^d$, we have $\mathcal{F}(E, U_n) \to \mathcal{F}(E)$ as $n \to \infty$.

(H11) *Weak superadditivity*: For any $m > 0$ there exists $\eta_1, \eta_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with η_1 continuous, $\eta_1(0) = 0$ and $\eta_2(r) \to 0$ as $r \to \infty$ such that the following holds: for any $E \subset \mathbb{R}^d$ with $|E| \le m$ and any finite family of balls $(B^i)_{i=1}^I$ of radius $R > 0$ such that $\min_{i \ne j} \mathrm{dist}(B^i, B^j) \ge 5R$,

$$\sum_{i=1}^I V(E, B^i) \le V(E) + \eta_1\left(\left|E \setminus \cup_{i=1}^I B_i\right|\right) + \eta_2(R).$$

Theorem 4.1.2. *Assume that the relative functionals of P and V satisfy (S2) and that* (4.1.1) *and* (4.1.2) *coincide. Then,* (4.1.2) *admits a solution.*

These assumptions deserve some comments. In terms of the concentration-compactness principle, the relative isoperimetric inequality (H4) allows us to exclude the "vanishing" case. (H5) is a weakened form of the invariance by translation. (H7) is trivial when the perturbation term is nonnegative. Let us point out that in [66] the authors establish existence of isoperimetric clusters in homogeneous metric spaces and with $V = 0$. In particular, their results imply existence of isoperimetric sets in $\mathbb{R}^d$. A comparison between (S2) and their set of hypotheses reveals that they are essentially identical. Indeed when $V = 0$, hypotheses (H7) and (H11) are superfluous and (S2) is analogous to the hypotheses of [66, Section 2 & Theorem 3.3]. Points (i) and (ii) of [66, Theorem 3.3] are obtained in our case through the partition of $\mathbb{R}^d$ into cubes.

In the first part of Section 4.3, we show that ρ-minimisers of the perimeter (see [60, Section 21] for the related concept of (Λ, r_0)-minimisers of the perimeter) have interior and exterior density estimates under the set of hypotheses (S3).

Definition 4.1.3. Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing. We say that $E \subset \mathbb{R}^d$ is a ρ-minimiser of the perimeter (or simply a ρ-minimiser) if there exists $r_2 > 0$ such that for any $r \le r_2$, $x \in \mathbb{R}^d$ and $E' \subset \mathbb{R}^d$ with $E \Delta E' \subset B_r(x)$ we have

$$P(E) \le P(E') + \rho(r). \tag{4.1.3}$$

The function ρ is called the error function for E.

The set of assumptions (S3) is made of (H4) (relative isoperimetric inequality) and (H6) (set operations) as well as three new hypotheses:

(H12) *Local comparisons*: For $E \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$ and a.e. $r > 0$,

$$P(E) = P(E, B_r(x)) + P(E, B_r(x)^c).$$

Additionally, for some $C > 0$:

$$P(E \setminus B_r(x)) \le P(E, B_r(x)^c) + CP(B_r(x), E),$$
$$P(E \cup B_r(x)) \le P(B_r(x), E^c) + P(E) - CP(E^c, B_r(x)).$$

(H13) *Integral inequality*: There exists $f_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and r_2 such that for any $E \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$ and $0 < r \le r_2$

$$\frac{1}{r}\int_0^r P(B_s(x), E)ds \le f_2(r, |E \cap B_r(x)|).$$

(H14) *Density scale factor*: Let f_1 and f_2 be given by (H4) and (H13) respectively and define

$$f_3(r,m) = 2^d \left(\frac{f_2(r,m) + \rho(r)}{f_1(m)}\right) \quad \text{for} \quad r, m > 0.$$

Then, there exist $r_3, \varepsilon_1 > 0$ such that

$$f_3(r,m) \le 1 \ \text{ for every } \ r \le r_3 \ \text{ and every } \ \frac{\varepsilon_1}{2^d} r^d < m \le \varepsilon_1 r^d.$$

Theorem 4.1.4. *Let $E \subset \mathbb{R}^d$ be a ρ-minimiser of the perimeter for some error function ρ. If (S3) holds, then there exists $C_0, r_4 > 0$ such that for $r \le r_4$,*

$$|E \cap B_r(x)| \ge C_0 r^d \text{ for every } x \in E^{(1)} \tag{4.1.4}$$

and

$$|B_r(x) \setminus E| \ge C_0 r^d \text{ for every } x \in E^{(0)}, \tag{4.1.5}$$

where for $t \in [0,1]$, $E^{(t)}$ denotes the points of density t of E.

Let us provide some context on (S3) and Theorem 4.1.4. Density estimates for ρ-minimisers of the perimeter are often an important tool in the study of isoperimetric problems. Indeed, it can be used to show that said ρ-minimisers have bounded connected components. Additionally, it is usually a crucial first step in the study of the spherical excess of minimisers (see [60, Section 22 & 26]), a central concept of the regularity theory for minimisers of isoperimetric problems. For illustrations of this concept, one can refer to [50] for the anisotropic perimeter and to [17] for the fractional perimeter. Additionally, when the density estimates are independent of the considered ρ-minimiser, it may be possible to exhibit minimisers for the classical problem

(4.1.1) provided additional assumptions on the perturbation term V (such as finite or rapidly decreasing range of action). See [72] for an illustration of this approach with P the classical perimeter and V a nonlocal kernel. Refer also to [24] for an example where P is the fractional perimeter and V is the integral of a periodic function and is not necessarily positive, or to [19] for the case of $P = \mathrm{Per}$ and V is defined using the Wasserstein distance.

The methodology employed to establish this kind of theorems is now well understood. Since the publication of De Giorgi's seminal papers on the classical isoperimetric problem in the 1950s, various strategies been developed to address isoperimetric problems where the considered perimeter is anisotropic or nonlocal, or with different perturbation terms. However, most of these proofs revolve around the same idea: apply the relative isoperimetric inequality to E (resp. E^c) and integrate this inequality to obtain interior (resp. exterior) density estimates (see [60, Remark 15.16]). Consequently, we have aimed at formulating streamlined hypotheses to encompass this shared framework, and also to simplify the process of establishing density estimates in future research on isoperimetric problems. In our framework, we need (H12) to deduce local results from the ρ-minimality of a set E, which is a priori a global property. (H13) is used together with the relative isoperimetric inequality to allow us to compare perimeters and Lebesgue measures. Finally, (H14) ensures that the error function ρ of a ρ-minimiser E is a perturbation of higher order of $P(E)$.

Remark 4.1.5. The conditions on ρ specified in (H14) are mild enough that ρ-minimisers of the anisotropic perimeter P_ϕ (resp. of the fractional perimeter P_s) have density estimates in the following two cases:

- $\rho(r) = Cr^{d-1+\alpha}$ (resp. $\rho(r) = Cr^{d-s+\alpha}$) with $\alpha \in (0,1)$ and any $C > 0$,
- $\rho(r) = Cr^{d-1}$ (resp. $\rho(r) = Cr^{d-s}$) and C small enough.

Theorem 4.1.4 is thus in accordance with [50, Proposition 3.1] and [29, Theorem 5.7].

In the second part of Section 4.3, we establish the connection between generalised minimisers and ρ-minimisers. We prove that generalised minimisers of (4.1.2) are ρ-minimisers of the perimeter for some ρ in two different situations: a case where P admits volume-fixing variations and a case where both P and V have a scaling property.

Definition 4.1.6. Let $E \subset \mathbb{R}^d$ be such that $P(E) + V(E) < \infty$. We say that E admits volume fixing variations if there exist $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ nondecreasing and $r_5, \varepsilon_2 > 0$ with the following properties : if $E' \subset \mathbb{R}^d$ is such that $E \Delta E' \subset B_{r_5}(x)$ for some $x \in \mathbb{R}^d$, then

1. for any ε such that $|\varepsilon| < \varepsilon_2$, there exist $F \subset \mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$ such that $B_{r_5}(x)$ and $B_{r_5}(x_0)$ are disjoint and
$$F \Delta E \subset B_{r_5}(x_0), \quad |F| - |E| = \varepsilon, \quad \mathcal{E}(F) \le \mathcal{E}(E) + g_1(|\varepsilon|).$$
2. If $F, F' \subset \mathbb{R}^d$ are such that $E \Delta E' = F \Delta F' \subset B_r(x)$ for $r \le r_5$ and $E \Delta F = E' \Delta F' \subset B_{r_5}(x_0)$ with $B_{r_5}(x)$ and $B_{r_5}(x_0)$ disjoint, then
$$P(F') - P(F) \le P(E') - P(E) + g_2(r). \tag{4.1.6}$$

We now introduce the following set of hypotheses, denoted (S4):

(H15) *Scaling*: If E minimises (4.1.1), then there exists $\alpha, \beta \in \mathbb{R}$ and $t_0 > 0$ such that

$$P(tE) \le t^\alpha P(E) \quad \text{and} \quad V(tE) \le t^\beta V(E) \quad \text{for any } t \text{ such that } |t-1| \le t_0. \tag{4.1.7}$$

Additionally there exists $\delta \in [0,1]$, $\gamma \ge 0$ and $C_1 > 0$ (if $\delta = 0$ we require $0 < C_1 < 1$) such that for any $E \subset \mathbb{R}^d$,

$$V(E) \ge -C_1 |E|^\delta P(E)^{1-\delta}. \tag{4.1.8}$$

(H16) *Volume-fixing variations*: If E solves (4.1.1), then E admits volume-fixing variations.

(H17) *Local perturbation control*: There exists $v : \mathbb{R}_+ \to \mathbb{R}_+$ nondecreasing and $r_6 > 0$ such that for $r \le r_6$, if $E, E' \subset \mathbb{R}^d$ satisfy $E \Delta E' \subset B_r(x)$ for some $x \in \mathbb{R}^d$, then

$$\left| V(E) - V(E') \right| \le v(r).$$

Proposition 4.1.7. *Assume that the relative functionals of P and V satisfy (S4) with either (H15) or (H16). Then every component of a generalised minimiser of* (4.1.2) *is a ρ-minimiser of the perimeter for an error function ρ. The error function ρ is defined by selecting the function equivalent to $r \mapsto Cr^c$ with the smallest possible $c > 0$ among*

- *$r \mapsto Cr^d$ and v if (H15) holds (C depending on the constants appearing in (H15)),*
- *g_1, g_2 and v if (H16) holds.*

Allow us to comment on (S4). We rely on the classical idea that if for some $i \ge 1$, $E^i \subset \mathbb{R}^d$ is the component of a generalised minimiser of (4.1.2), then it is a minimiser of (4.1.1) with the constraint $m = |E^i|$. We then need to use either (H15) or (H16) to relax the mass constraint in order to be able to compare E^i with a set E' such that $E^i \Delta E' \subset B_r(x)$ for some $x \in \mathbb{R}^d$ and r small enough. The scaling hypothesis (H15) is a well-known method (see e.g [65, Proposition 4.6] or [19, Proposition 3.3]), but when V is not necessarily positive, we lose the fact that the boundedness of $\mathcal{E}$ implies boundedness of P and V so that additional hypotheses are needed to control the growth of V. The first point of Definition 4.1.6 appearing in (H16) is inspired by the classical “volume-fixing variations" lemma (or “Almgren's lemma” (see [60, Lemma 17.21]). Let us also point out that the second point of Definition 4.1.6 is to account for perimeters with nonlocal properties, as one may take $g_2 = 0$ if P is the classical or anisotropic perimeter (see the definitions below). Eventually using (H17) to deal with local perturbations of V, we obtain that E verifies (4.1.3) for some error function ρ.

4.1.2 Application to three perturbed isoperimetric problems

Let us present some examples from the literature of perimeter and perturbation terms satisfying the sets of hypotheses (S1) to (S4), or only (S1) and (S2) in the case of the considered Dirichlet energy. In Section 4.4 we provide a proof of this statement for three different perturbed isoperimetric problems. Additionally, we briefly comment on the other examples mentioned below.

Regarding the perimeter, we consider its anisotropic and anisotropic nonlocal versions. For $E, U \subset \mathbb{R}^d$ we set

$$P_\phi(E,U) = \int_{(\partial^* E)\cap U} \phi(\nu_E(x))\, d\mathcal{H}^{d-1}(x),$$
$$P_K(E,U) = \int_{(E\cap U)\times E^c} K(x-y) dx dy.$$

The value $P_\phi(E)$ is well-defined if E is of finite Caccioppoli perimeter, and then $\partial^* E$ denotes the reduced boundary of E. The anisotropy ϕ is a nonnegative, one-homogeneous, convex and coercive functional. In particular, there exists $0 < C'_\phi \leq C_\phi$ such that for $x \in \mathbb{R}^d$,

$$C'_\phi |x| \leq \phi(x) \leq C_\phi |x|.$$

If $\phi = |\cdot|$ we recover the classical perimeter, which we denote by Per in this chapter. Regarding the nonlocal perimeter, we require that there exist $C'_K, C_K > 0$ and $s \in (0,1)$ such that for $x \in \mathbb{R}^d$,

$$C'_K\, |x|^{-(d+s)} \leq K(x) \leq C_K\, |x|^{-(d+s)}.$$

We additionally require that $K \in W^{1,1}_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ and that for $x \in \mathbb{R}^d$,

$$|\nabla K(x)| \leq |x|^{-(d+s+1)}.$$

Let us point out that if $U \neq \mathbb{R}^d$, the definition of the relative nonlocal perimeter differs from the one found in the literature (see e.g. [24, Section 2]). When $K = |\cdot|^{-d-s}$, we recover the fractional perimeter and simply write $P_K = P_s$.

The perturbation terms encompassed by our hypotheses can be split into several categories.

The first and perhaps most studied in the literature is the Riesz-type kernel: given $E, U \subset \mathbb{R}^d$, we consider

$$V_G(E,U) = \int_{(E\cap U)\times(E\cap U)} G(x-y)\, dx dy$$

where $G : \mathbb{R}^d \to \mathbb{R}_+$ is continuous on $\mathbb{S}^{d-1}$ and such that there exists $\beta \in (0,d)$ such that for any $t \geq 0$ and $x \in \mathbb{R}^d$

$$G(tx) \leq t^{-\beta} G(x).$$

We refer to [56, 55, 7] for seminal examples where $P = \text{Per}$ and $G(x) = |x|^{-\beta}$ and to [41] for an example where $P = P_s$ and G is explicit as well. See also [42] for a study of the anisotropic case and [65] for a recent development in the case $P = \text{Per}$ and with general kernels.

Remark 4.1.8. Proceeding as in [65], one can obtain that generalised minimisers to $P + V_G$ exist and have density estimates for $P = \text{Per}$, P_ϕ or P_s and with G, nonnegative, symmetric with respect to the origin, vanishing at infinity and such that

$$G(tx) \leq tG(x) \quad \text{for } x \in \mathbb{R}^d \text{ and } t \geq 1.$$

This case is not encompassed in our setting, because with these assumptions, neither the scaling hypothesis (H15) nor the volume-fixing hypothesis (H16) hold. However, the mass constraint can still be dealt with using the fact that for the considered perimeters

$$P(E \cap B_R) \leq P(E) \quad \text{ for any } E \subset \mathbb{R}^d \text{ and } R > 0.$$

This classical result is a consequence of the monotonicity of the perimeter regarding intersection with convex sets, which holds for the classical, anisotropic and fractional perimeter, but not for the generalised nonlocal perimeter P_K.

A second family of perturbation terms appears in the prescribed curvature problem. We consider

$$V_T(E, U) = -\int_{E \cap U} T(x)\, dx,$$

and assume that T is L-periodic and Lipschitz continuous. Refer for instance to [49] for the case $P = \text{Per}$ and to [24] for the case $P = P_s$.

Remark 4.1.9. If one only wants to establish that (S1) and (S2) are verified for $P + V_T$, weaker hypotheses on T can be considered. We use the Lipschitz continuity assumption to establish that the volume-fixing hypothesis (H16) holds (see Section 4.2).

Perturbation terms involving optimal transport are studied in [19]. Given $p \in [1, \infty)$ and denoting by $W_p(E, F)$ the p-Wasserstein distance between $E, F \subset \mathbb{R}^d$, one can set for $U \subset \mathbb{R}^d$

$$V_{\mathcal{W}}(E, U) = \inf_{|F \cap E \cap U| = 0} W_p(E \cap U, F)^p.$$

Eventually, we can consider a Dirichlet energy as in [35] and show that it satisfies (S1) and (S2). Given $E \subset \mathbb{R}^d$, we define the Sobolev-like space

$$\hat{H}_0^1(E) = \left\{ u \in H^1(\mathbb{R}^d) : u = 0 \ \text{ a.e. on } E^c \right\}$$

which is a Hilbert space as it is closed in $H^1(\mathbb{R}^d)$. For $p \in (d, \infty)$ and $h \in L^p(\mathbb{R}^d)$, the Dirichlet (or torsion) energy of E is then

$$V_{\text{Dir}}(E) = \min_u \left\{ \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2\, dx - \int_{\mathbb{R}^d} uh\, dx : u \in \hat{H}_0^1(E) \right\}.$$

and given $U \subset \mathbb{R}^d$, we set $V_{\text{Dir}}(E, U) = V_{\text{Dir}}(E \cap U)$.

Remark 4.1.10. It is actually possible to show that minimisers of $P_\phi + V_{\text{Dir}}$ or of $P_K + V_{\text{Dir}}$ admit interior and exterior density estimates. However, because we focus on the set $\hat{H}_0^1(E)$ instead of the Sobolev space $H_0^1(E)$ (we have to introduce $\hat{H}_0^1(E)$ because if E is not open, there may exist a set E' such that $|E\Delta E'| = 0$ but $H^1_0(E') \neq H^1_0(E)$), we were not able to prove (S3) and (S4) exactly as they stand. One can proceed as in [35, Theorem 1.1] and first prove that E admits exterior density estimates, so that it can be correctly be identified with then open set $E^{(1)}$, and then establish interior density estimates for $E^{(1)}$.

4.1.3 Notation and organisation of the chapter

All the constants of the chapter depend on d, the functions $(f_i)_i, (g_j)_j, h, \eta, \rho, v$ and the parameters r, ε used in the hypotheses, where $i = 1,2,3$ and $j = 1,2$. We denote them with the same letter C when differentiating the constants from one another is not relevant. We write $C = C(E, m)$ to specify an additional dependency on a set E or a parameter m. In some statements we write $A \ll B$ to indicate that there exists a constant $\varepsilon > 0$ such that if $A \leq \varepsilon B$ then the conclusion of the statement holds.

In Section 4.2, we prove that the infima of (4.1.1) and (4.1.2) coincide, and that (4.1.2) admits solutions. In Section 4.3, we first establish that ρ-minimisers of (4.1.2) have interior and exterior density estimates. We then discuss two cases where generalised minimisers of (4.1.2) are ρ-minimisers of the perimeter as well. In Section 4, we study three examples of perturbed isoperimetric problems.

4.2 Existence of generalised minimisers

We start off by establishing Proposition 4.1.1, i.e. that (4.1.1) and (4.1.2) coincide.

Proof of Proposition 4.1.1. Given a set E with $|E| = m$, we define the generalised set $(E^i)_i = (E, \emptyset, \ldots, \emptyset)$ and have $\mathcal{E}_{\text{gen}}((E^i)_i) = \mathcal{E}(E)$. Hence $e_{\text{gen}}(m) \leq e(m)$.

Conversely if we let $\varepsilon > 0$, there exists $(E^i)_i$ admissible for (4.1.2) such that

$$\mathcal{E}_{\text{gen}}((E^i)_i) \leq e_{\text{gen}}(m) + \varepsilon.$$

Let us show that there exists a set E admissible for (4.1.1) such that

$$\mathcal{E}(E) \leq e_{\text{gen}}(m) + 5\varepsilon.$$

By (H1) small balls have vanishing energy: there exists $\delta = \delta(\varepsilon)$ such that if B_r is a centred ball of radius $r > 0$, then

$$|B_r| \leq 2\delta \implies \mathcal{E}(B_r) \leq \varepsilon. \tag{4.2.1}$$

As $(E^i)_i$ is of finite energy and mass, there exists an integer $I = I(\varepsilon, \delta)$ large enough that

$$\sum_{i=1}^{I} \mathcal{E}(E^i) \leq \sum_{i\geq 1} \mathcal{E}(E^i) + \varepsilon \leq e_{\text{gen}}(m) + 2\varepsilon \quad \text{and} \quad \sum_{i\geq I+1} |E^i| \leq \delta.$$

Combining this with the convergence at infinity (H2) of $\mathcal{E}$, there exists $R = R(\varepsilon, \delta, I)$ large enough that

$$\sum_{i=1}^{I}\mathcal{E}(E^i\cap B_R)\le\sum_{i=1}^{I}\mathcal{E}(E^i)+\varepsilon\le e_{\text{gen}}(m)+3\varepsilon\quad\text{and}\quad\sum_{i=1}^{I}|E^i\cap B_R^c|\le\delta.\tag{4.2.2}$$

Let B_r be the centred ball with volume

$$|B_r|=\sum_{i=1}^{I}|E^i\cap B_R^c|+\sum_{i\ge I+1}|E^i|\le 2\delta.$$

Given $L>0$, we define the set

$$E_L=\left[\bigcup_{i=1}^{I}\Big((E^i\cap B_R)+iLe_1\Big)\right]\bigcup\Big[B_r+(I+1)Le_1\Big].$$

By construction, for L large enough $|E_L|=m$. Using recursively (H3) on the vanishing range of action of $\mathcal{E}$ then yields for that for L large enough

$$\mathcal{E}(E_L)\le\sum_{i=1}^{I}\mathcal{E}(E^i\cap B_R)+\mathcal{E}(B_r)+\varepsilon\le e_{\text{gen}}(m)+5\varepsilon$$

where we used (4.2.1) and (4.2.2) in the last inequality. Thus

$$e(m)\le e_{\text{gen}}+5\varepsilon,$$

and as $\varepsilon>0$ is arbitrary the proof is complete. □

Remark 4.2.1. The vanishing energy of small balls (H1) does not hold for perturbation terms V which are α-homogeneous with $\alpha<0$. However, this hypothesis can be replaced by the assumption that P and V are homogeneous for some reals α,β and that $V\ge 0$. Proceeding as in the proof of [19, Proposition 3.3], one can then show that there exists $\Lambda=\Lambda(m)\ge 0$ such that

$$e(m)=\inf_{E}\left\{\mathcal{E}(E)+\Lambda\Big||E|-m\Big|\right\},$$

and

$$e_{\text{gen}}(m)=\inf_{(E^i)_i}\left\{\mathcal{E}_{\text{gen}}((E)_i^i)+\Lambda\Big|\sum_i|E^i|-m\Big|\right\}.$$

We can subsequently reproduce the proof of Proposition 4.1.1 without introducing a small ball to compensate mass deficit.

Under the set of hypotheses $(S2)$, we can prove Theorem 4.1.2, i.e. that generalised minimisers of (4.1.2) exist. Recall that given $U\subset\mathbb{R}^d$, the localised versions of P and V are denoted by $P(\cdot,U)$ and $V(\cdot,U)$.

Proof of Theorem 4.1.2.

We follow the direct method in the Calculus of Variations. First, we use a classical minimising sequence to build a generalised set, and then establish lower semi-continuity results to prove that this generalised set is a generalised minimiser.

Step 1. Construction of a generalised set.

Let $(E_n)_n$ be a minimising sequence for (4.1.1). As assumed in the statement of Theorem 4.1.2, $(E_n)_n$ is also a minimising sequence for (4.1.2). Let $r_0, r_1 > 0$ be as in assumptions (H4) and (H5) on the relative isoperimetric inequality and the periodicity of P and V. Notice that B_{r_0} contains the centred cube of side-length r_1. We consider a partition $(Q_n^i)_{i,n}$ of $\mathbb{R}^d$ into cubes of side-length r_1 and we set

$$m_n^i = |E_n \cap Q_n^i| \quad \text{and} \quad M_n^i = |E_n \cap B_n^i|,$$

where B_n^i is the ball of radius r_0 with the same centred as Q_n^i. Rearranging the sequence we assume that for every $n \geq 0$, $i \mapsto M_n^i$ is nonincreasing.

Step 1.1 Let us now show that the series $\sum_i M_n^i$ is uniformly summable with respect to $n \geq 0$. Notice that there exists $C = C(r_1/r_0)$ such that for every $n \geq 0$

$$\sum_{i\geq 1} \chi_{B_n^i} \leq C, \quad \text{so that} \quad \sum_{i\geq 1} M_n^i \leq Cm.$$

Thus as M_n^i is nonincreasing in i, for every $I \geq 1$ and $i \geq I$ we have

$$M_n^i \leq M_n^I \leq Cm/I. \tag{4.2.3}$$

Let $\varepsilon > 0$. Recall that the function f_1 involved in (H4) is such that there exists $\delta = \delta(\varepsilon)$ such that $m \leq \varepsilon f_1(m)$ for any $m \leq \delta$. By (4.2.3) there exists $I = I(\delta)$ such that for any $i \geq I$ we have $M_n^i \leq \delta$. Up to reducing δ we assume without loss of generality that $|M_n^i| \leq |B_{r_0}|/2$. Then, by (H4) we have

$$\sum_{i\geq I} M_n^i \leq \varepsilon \sum_{i\geq I} f_1(M_n^i) \leq \varepsilon \sum_{i\geq I} P(E_n, B_n^i). \tag{4.2.4}$$

Given $n \geq 0$, we split the covering $(B_n^i)_i$ of $\mathbb{R}^d$ into N families $\mathcal{B}_n^1, \dots, \mathcal{B}_n^N$ such that $|B_n^i \cap B_n^j| = 0$ if $B_n^i, B_n^j \in \mathcal{B}_n^k$ for some $1 \leq k \leq N$ and $i \neq j$. Notice that $N = N(r_1/r_0)$ is uniformly bounded in $n \in \mathbf{N}$. By (H6) we may write

$$\begin{aligned} \sum_{i\geq I} P(E_n, B_n^i) &= \sum_{k=1}^{N} \sum_{B_n^i \in \mathcal{B}_n^k,\, i\geq I} P(E_n, B_n^i) \\ &\leq \sum_{k=1}^{N} P\Big(E_n, \bigcup_{B_n^i \in \mathcal{B}_n^k} B_n^i\Big) \leq \sum_{k=1}^{N} P(E_n) \leq NP(E_n). \end{aligned} \tag{4.2.5}$$

By (H7) $\sup_n \mathcal{E}(E_n) < \infty$ implies that $\sup_n P(E_n) < \infty$. Thus plugging (4.2.5) into (4.2.4) yields

$$\sum_{i\geq I} M_n^i \leq \varepsilon N P(E_n) \leq \varepsilon C'$$

for some constant $C' > 0$. This proves that the series $\sum_i M_i^n$ is uniformly summable with respect to $n \geq 0$. As for any $i \geq 1$ and $n \geq 0$, we have $m_n^i \leq M_n^i$, the series $\sum_i m_n^i$ is uniformly summable in $n \geq 0$ as well.

Step 1.2. By the previous substep, there exists a sequence $(m_i)_{i\geq 1}$, such that up to extraction $m_n^i \to m^i$ as $n \to \infty$ for every $i \geq 1$. Besides, $m_i \geq 0$ for every i and by uniform summability,

$$\sum_i m^i = m.$$

We now build a generalised set of total mass m. Let x_n^i be the centred of Q_n^i. Up to further extraction, we assume that for every $i, j \geq 1$, $|x_n^i - x_n^j| \to d_{i,j} \in [0, \infty]$ as $n \to \infty$. Recall that r_1 is chosen so that (H5) holds, so that $\sup_n P(E_n - x_n^i) = \sup_n P(E_n) < \infty$. Thus by the compactness assumption (H8), for every $i \geq 1$, there exists E^i such that $E_n - x_n^i \to E^i$ in L^1_{loc}.

We now define an equivalence class in the set $\{1, 2, \dots\}$ by setting

$$i \sim j \quad \text{if} \quad d_{i,j} < \infty.$$

Notice that if $i \sim j$, then E^i and E^j coincide up to a translation. We denote by $\mathcal{C}$ the set of all equivalence classes. For every equivalence class $c \in \mathcal{C}$ let $m_c = \sum_{i\in c} m_i$ so that

$$\sum_{c\in\mathcal{C}} m_c = \sum_{i\geq 1} m_i = m. \tag{4.2.6}$$

Step 1.3. Let us fix $c \in \mathcal{C}$ and let us establish that $|E^i| = m_c$ for every $i \in c$. Given $\ell \geq 1$, by definition, there exists R_ℓ such that for all $n \geq 0$

$$\bigcup_{1\leq j\leq \ell,\, j\in c} Q_n^j \subset B_{R_\ell}(x_n^i).$$

Thus

$$\begin{aligned}\sum_{1\leq j\leq \ell,\, j\in c} m_n^j = \sum_{1\leq j\leq \ell,\, j\in c} |E_n \cap Q_n^j| &= \left| E_n \bigcap \Big(\bigcup_{1\leq j\leq \ell,\, j\in c} Q_n^j \Big) \right| \\ &\leq |E_n \cap B_{R_\ell}(x_n^i)| = |(E_n - x_n^i) \cap B_{R_\ell}|.\end{aligned}$$

As $E_n - x_n^i \to E^i$ in L^1_{loc}, taking $n \to \infty$ yields

$$\sum_{1\leq j\leq \ell,\, j\in c} m^j \leq |E^i \cap B_{R_\ell}| \leq |E^i|.$$

Letting $\ell \to \infty$ we finally obtain

$$m_c \leq |E^i|. \tag{4.2.7}$$

Let us prove the converse inequality. For this, thanks to (4.2.6) and (4.2.7) it is sufficient to establish the inequality

$$\sum_{c\in\mathcal{C}} |E^{i_c}| \leq m, \tag{4.2.8}$$

where for each c we select one $i_c \in c$, for instance $i_c = \min\{j : j \in c\}$. Let us fix $N \geq 1$ and define $\mathcal{C}_N = \{c \in \mathcal{C} : i_c \leq N\}$, which is a finite subset of $\mathcal{C}$. Given $R > 0$, by definition of the equivalence relation for n large enough $|B_R(x_n^{i_c}) \cap B_R(x_n^{i_{c'}})| = 0$ for $c, c' \in \mathcal{C}_N$ with $c \neq c'$. Hence

$$m = |E_n| \geq \left| E_n \bigcap \bigcup_{c\in\mathcal{C}_N} B_R(x_n^{i_c}) \right| = \sum_{c\in\mathcal{C}_N} |E_n \cap B_R(x_n^{i_c})| = \sum_{c\in\mathcal{C}_N} |(E_n - x_n^{i_c}) \cap B_R|.$$

Passing to the limit in $n \to \infty$ yields

$$m \geq \sum_{c\in\mathcal{C}_N} |E^{i_c} \cap B_R|.$$

Eventually, letting $R \to \infty$ and then $N \to \infty$ proves (4.2.8).

Consequently for $c \in \mathcal{C}$ and $i \in c$ we have $|E^i| = m_c$. Relabelling, we write $\{E^{i_c} : c \in \mathcal{C}\} = \{\widetilde{E}^1, \widetilde{E}^2, \widetilde{E}^3, \dots\}$ so that $(\widetilde{E}^i)_i$ is admissible for (4.1.2). Given $i \geq 1$, we also denote $\tilde{x}_n^i = x_n^j$ where $j \geq 1$ is such that $E^j = \widetilde{E}^i$.

Step 2 : Lower semi-continuity of the energy.
We are left with the proof of

$$\mathcal{E}_{\text{gen}}((\widetilde{E}^i)_i) \leq \liminf_{n\to\infty} \mathcal{E}(E_n).$$

Keeping the notation of the previous step, we let $I \geq 1$ and consider the family $\tilde{x}_n^1, \cdots, \tilde{x}_n^I$. Note that if we let $R > 0$, for n large enough $\min_{i\neq j} |\tilde{x}_n^i - \tilde{x}_n^j| \geq 5R$.

We start with the perimeter term. Using the periodicity assumption (H5) and then the set operations property (H6), we have

$$\sum_{i=1}^{I} P(E_n - \tilde{x}_n^i, B_R) = \sum_{i=1}^{I} P(E_n, B_R(\tilde{x}_n^i)) \leq P\Big(E_n, \bigcup_{i=1}^{I} B_R(\tilde{x}_n^i)\Big) \leq P(E_n).$$

Recall that $E_n - \tilde{x}_n^i \to \widetilde{E}^i$ in L^1_{loc} as $n \to \infty$. Using the lower semicontinuity and Beppo-Levi assumptions (H9)&(H10) in that order, letting $n \to \infty$ and then $R \to \infty$ we obtain

$$\sum_{i=1}^{I} P(\widetilde{E}^i) \leq P(E_n).$$

Thus sending $I \to \infty$ yields

$$\sum_{i\geq 1} P(\widetilde{E}^i) \leq \liminf_{n} P(E_n). \tag{4.2.9}$$

Let us turn to the perturbation term. Using the functions η_1, η_2 of the weak superadditivity assumption (H11), we write

$$\begin{aligned}\sum_{i=1}^{I} V(E_n - \tilde{x}_n^i, B_R) &= \sum_{i=1}^{I} V(E_n, B_R(\tilde{x}_n^i)) \\ &\leq V(E_n) + \eta_1 \left(\left| E_n \setminus \bigcup_{i=1}^{I} B_R(\tilde{x}_n^i) \right| \right) + \eta_2 \left(\min_{i \neq j} |\tilde{x}_n^i - \tilde{x}_n^j| - 2R \right).\end{aligned}$$

Notice that

$$\left| E_n \setminus \bigcup_{i=1}^{I} B_R(\tilde{x}_n^i) \right| = |E_n| - \sum_{i=1}^{I} |(E_n - \tilde{x}_n^i) \cap B_R|.$$

Recall that $\eta_2(r) \to 0$ as $r \to \infty$ and that η_1 is continuous. Letting $n \to \infty$ in the previous inequality and using (H9) yields

$$\sum_{i=1}^{I} V(\widetilde{E}^i, B_R) \leq \liminf_n \sum_{i=1}^{I} V(E_n - \tilde{x}_n^i, B_R) \leq \liminf_n V(E_n) + \eta_1 \left(m - \sum_{i=1}^{I} |\widetilde{E}^i \cap B_R| \right).$$

Notice that by (4.2.6), letting $R \to \infty$ and then $I \to \infty$ we have

$$m - \sum_{i=1}^{I} |\widetilde{E}^i \cap B_R| \to 0.$$

Therefore, using that $\eta_1(t) \to 0$ as $t \to 0$ and letting $R \to \infty$ and then $I \to \infty$ we obtain from (H10) that

$$\sum_{i=1}^{\infty} V(\widetilde{E}^i) \leq \liminf_n V(E_n).$$

Combining this inequality with (4.2.9) yields

$$\mathcal{E}_{\text{gen}}((\widetilde{E}^i)_i) = \sum_{i=1}^{\infty} \left[P(\widetilde{E}^i) + V(\widetilde{E}^i) \right] \leq \liminf_n \left[V(E_n) + P(E_n) \right] = e_{\text{gen}}(m).$$

This proves that $(\widetilde{E}^i)_i$ is a generalised minimiser of (4.1.2). □

4.3 Characterization of approximate minimisers of perturbed isoperimetric problems

4.3.1 Establishing Density estimates

In this subsection, we establish Theorem 4.1.4, i.e. that ρ-minimisers admit density estimates under the set of hypotheses (S3).

Proof of Theorem 4.1.4. Take E as in the statement of Theorem 4.1.4, $\varepsilon_1 > 0$ provided by (H14) and for $x \in \mathbb{R}^d$, $r > 0$, set $m(r) = |E \cap B_r(x)|$. We will show that there exists $r_0 > 0$ (depending on the functions and parameters r, ε appearing in the hypotheses) such that:

$$\text{if for some } r \le r_0,\ \frac{m(r)}{r^d} \le \varepsilon_1,\ \text{then } \frac{m(r/2)}{(r/2)^d} \le \varepsilon_1, \tag{4.3.1}$$

and

$$\text{if for some } r \le r_0,\ \frac{m(r)}{r^d} \ge \varepsilon_1,\ \text{then } \frac{m(r/2)}{(r/2)^d} \ge \varepsilon_1. \tag{4.3.2}$$

Combining (4.3.1) and (4.3.2) and the definitions of $E^{(1)}$ and $E^{(0)}$ then yields (4.1.4) and (4.1.5).

We start by proving (4.3.1): assume that $m(r) \le \varepsilon_1 r^d$ for some $r > 0$ to be fixed later. Translation invariance does not necessarily hold, but up to a change of coordinates we may assume that $x = 0$. Notice that $t \mapsto P(B_t, E)$ can not be strictly greater than its mean value over $[r/2, r]$ for any $t \in [r/2, r]$. Hence there exists $t \in [r/2, r]$ such that

$$\frac{r}{2} P(B_t, E) \le \int_{r/2}^{r} P(B_s, E) ds \le \int_0^r P(B_s, E) ds \le r f_2(r, m(r)), \tag{4.3.3}$$

where the last inequality comes from (H13). Up to multiplying f_2 by a constant, we omit the factor 1/2 in what follows. Now, applying Definition 4.1.3 with $F = E \setminus B_t$ yields

$$P(E) \le P(E \setminus B_t) + \rho(t).$$

By applying (H12) to $P(E \setminus B_t)$ and plugging it into the previous inequality we have

$$P(E, B_t) \le C P(B_t, E) + \rho(t).$$

Then, using the monotonicity (H6) of $U \mapsto P(E, U)$ and the one of ρ,

$$P(E, B_{r/2}) \le C P(B_t, E) + \rho(r).$$

Together with (4.3.3), we obtain (again replacing $C f_2$ by f_2)

$$P(E, B_{r/2}) \le \overline{f}_2(r, m(r)), \tag{4.3.4}$$

where

$$\overline{f}_2(r) = f_2(r, m(r)) + \rho(r).$$

Recall that $m(r/2) \le m(r) \le \varepsilon_1 r^d$. Without loss of generality, we may assume that $\varepsilon_1 \le \omega_d / 2^{d+1}$, which implies $m(r/2) \le |B_{r/2}|/2$, so that in particular $m(r/2) \le |B_{r/2} \setminus E|$. Combining the relative isoperimetric inequality (H4) and (4.3.4) yields

$$f_1(m(r/2)) \le P(E, B_{r/2}) \le \overline{f}_2(r, m(r))$$

so that

$$m(r/2) \le \overline{f}_2(r, m(r)) \frac{m(r/2)}{f_1(m(r/2))} \le \frac{\overline{f}_2(r, m(r))}{f_1(m(r))} m(r),$$

where we used the fact that $m \mapsto f_1(m)/m$ is nonincreasing in the last inequality. By hypothesis, $m(r) \le \varepsilon_1 r^d$, and recalling that $f_3 = 2^d(f_2+\rho)/f_1 = 2^d\overline{f}_2/f_1$ we obtain

$$\frac{m(r/2)}{(r/2)^d} \le f_3(r, m(r))\frac{m(r)}{r^d} \le f_3(r, m(r))\varepsilon_1. \tag{4.3.5}$$

By contradiction, assume that $m(r/2) > \varepsilon_1(r/2)^d$. Then $m(r) \ge m(r/2) > \varepsilon_1(r/2)^d$. Thus by (H14), we have $f_3(r, m(r)) \le 1$ and by (4.3.5), $m(r/2) \le \varepsilon_1(r/2)^d$, which is absurd. Hence $m(r/2) \le \varepsilon_1(r/2)^d$, proving (4.3.1).

To establish (4.3.2), we define $m^c(r) = |E^c \cap B_r|$ and assume that $m^c(r) \le \varepsilon_1 r^d$ for some $r > 0$. Again, applying the mean value theorem to $r \mapsto P(B_r, E^c)$ yields the existence of $t \in [r/2, r]$ such that

$$P(B_t, E^c) \le f_2(r, m^c(r)).$$

Next, comparing the ρ-minimiser E with $F = E \cup B_t$ yields

$$P(E) \le P(E \cup B_t) + \rho(t).$$

Using (H12) to bound the local variations of the perimeter, we obtain

$$CP(E^c, B_t) \le P(B_t, E^c) + \rho(t).$$

The proof of (4.3.2) is then exactly as the one of (4.3.1) with E replaced by E^c. □

4.3.2 Generalised minimisers as approximate minimisers of the perimeter

In this subsection, we prove Proposition 4.1.7, which describes two cases where generalised minimisers of (4.1.2) are also ρ-minimisers of the perimeter for some function ρ.

Proof of Proposition 4.1.7.

We first observe that given $i \ge 1$, if E^i is a component of a generalised minimiser of mass $|E^i| = m_i$, then it is a minimiser of (4.1.1) with the mass constraint $m = m_i$. We now show that E^i is a ρ-minimiser of the perimeter and split the proof on whether hypothesis (H15) or (H16) holds. To ease the notation, we write $E = E^i$ and $m = m_i$.

Case 1 : Scaling.

We first assume that P and V admit the scaling property given by (H15). Let us establish that for some $\Lambda \gg 1$, E is a minimiser of

$$\inf_{E'}\Big\{\mathcal{E}_\Lambda(E') = \mathcal{E}(E') + \Lambda\Big|m - |E'|\Big|\Big\}. \tag{4.3.6}$$

By contradiction, let us assume that there exists $\Lambda_n \to \infty$ and $(E_n)_{n\in\mathbf{N}}$ such that

$$\mathcal{E}_{\Lambda_n}(E_n) < \mathcal{E}(E). \tag{4.3.7}$$

We first notice that we must have $|E_n| \ne m$.

Step 1: Boundedness of P and V. Let us show that

$$\sup_n P(E_n) < \infty \quad \text{and} \quad \sup_n |V(E_n)| < \infty. \tag{4.3.8}$$

By (4.1.8) from (H15), for every $n \in \mathbf{N}$ we have

$$V(E_n) \geq -C_1 |E_n|^{\delta} P(E_n)^{1-\delta}. \tag{4.3.9}$$

If $\delta = 0$, then $C_1 < 1$ and by (4.3.7)

$$P(E_n)(1 - C_1) \leq P(E_n) + V(E_n) = \mathcal{E}(E_n) \leq \mathcal{E}_{\Lambda_n}(E_n) < \mathcal{E}(E) \leq \mathcal{E}(B_{\ell(m)}),$$

where $B_{\ell(m)}$ is the centred ball of volume m. Hence (4.3.8) holds.

If $\delta \in (0,1]$, applying Young's inequality to (4.3.9) yields that for every $n \in \mathbf{N}$

$$V(E_n) \geq -C_1(\delta |E_n| + (1-\delta) P(E_n)), \tag{4.3.10}$$

so that

$$(1 - C_1(1-\delta)) P(E_n) \leq \mathcal{E}(E_n) + \delta C_1 |E_n|.$$

Thus for $\Lambda_n \geq \delta C_1$, by the triangle inequality

$$(1 - C_1(1-\delta)) P(E_n) \leq \mathcal{E}(E_n) + \Lambda_n \big| m - |E_n| \big| + \delta C_1 m \leq \mathcal{E}_{\Lambda_n}(E_n) + \delta C_1 m.$$

Therefore by (4.3.7) $\sup_n P(E_n) < \infty$. By (4.3.9), up to relabelling C_1 we have

$$V(E_n) \geq -C_1 |E_n|^{\delta}. \tag{4.3.11}$$

Therefore to prove (4.3.8) it is sufficient to establish that $\sup_n |E_n| < \infty$. By (4.3.10) and replacing C_1 by $\max(1, C_1)$

$$|E_n|(\Lambda_n - \delta C_1) \leq V(E_n) + C_1(1-\delta) P(E_n) + \Lambda_n |E_n| \leq C_1 \mathcal{E}_{\Lambda_n}(E_n) + C_1 \Lambda_n m,$$

so that dividing by Λ_n yields

$$|E_n|(1 - \delta C_1 \Lambda_n^{-1}) \leq C_1 \Lambda_n^{-1} \mathcal{E}_{\Lambda_n}(E_n) + C_1 m.$$

Hence $\sup_n |E_n| < \infty$ and (4.3.8) holds.

Step 2: Showing that E minimises $\mathcal{E}_\Lambda$. We are now ready to prove (4.3.6). We set $t_n^d = m|E_n|^{-1}$ so that $|t_n E_n| = m$ and write $t_n = 1 + \varepsilon_n$ where $\varepsilon_n \in (-1, +\infty)$. Combining (4.3.7) and (4.3.8) one has

$$\Lambda_n \big| |E_n| - m \big| \leq \mathcal{E}(E) - P(E_n) - V(E_n) \leq \mathcal{E}(B_{\ell m}) - V(E_n),$$

so that

$$\sup_n \Lambda_n ||E_n| - m| \le \sup_n [\mathcal{E}(B_{\ell m}) - V(E_n)] = \mathcal{E}(B_{\ell(m)}) - \inf_n V(E_n) < \infty$$

As $\Lambda_n \to \infty$ as $n \to \infty$, the previous inequality implies that $|E_n| \to m$ and $\varepsilon_n \to 0$ as $n \to \infty$. Now using the scaling part of (H15), by definition of E we have

$$\mathcal{E}_{\Lambda_n}(E_n) < \mathcal{E}(E) \le \mathcal{E}(t_n E_n) \le t_n^{\alpha} P(E_n) + t_n^{\beta} V(E_n) = (1+\varepsilon_n)^{\alpha} P(E_n) + (1+\varepsilon_n)^{\beta} V(E_n).$$

Therefore, a Taylor expansion yields that for some $C_3 = C_3(\alpha, \beta, \delta, \gamma, m) > 0$

$$\Lambda_n \Big| m - |E_n| \Big| \le |\varepsilon_n| \mathcal{E}(E_n) \le |\varepsilon_n| C_3 \mathcal{E}(B_{\ell(m)}) = |\varepsilon_n| C_3. \tag{4.3.12}$$

Finally, notice that by definition of ε_n,

$$\Big| m - |E_n| \Big| = m|1 - t_n^{-d}| = m|1 - (1+\varepsilon_n)^{-d}| = m\varepsilon_n + O(\varepsilon_n^2).$$

Injecting this last equation into (4.3.12), we obtain

$$\Lambda_n m |\varepsilon_n| \le |\varepsilon_n| C_3$$

so that $\Lambda_n \le C_3$, contradicting the fact that $\Lambda_n \to n$ as $n \to \infty$. We thus have that E minimises (4.3.6) for $\Lambda \gg 1$.

Step 3 : Conclusion. We finally consider E' with $E' \Delta E \subset B_r(x)$ for some $x \in \mathbb{R}^d$ and $r > 0$. Let us consider $\Lambda \gg 1$ such that E minimises (4.3.6). We have

$$P(E) \le P(E') + \big[V(E') - V(E)\big] + \Lambda \Big| |E'| - m \Big| \le P(E') + \big[V(E') - V(E)\big] + \Lambda \omega_d r^d.$$

Using the local control (H17) on V, for $r \ll 1$ we have

$$P(E) \le P(E') + v(r) + \Lambda \omega_d r^d.$$

Therefore E is a quasi-minimiser of the perimeter. Its error function ρ is v or $r \mapsto \Lambda \omega_d r^d$, whichever is equivalent to $r \mapsto C r^c$ with the smallest possible $c > 0$.

Case 2 : Local variation.
We now assume that (H16) holds. Given $x \in \mathbb{R}^d$ and $0 < r \le r_5/2$, we consider E' such that $E' \Delta E \subset B_r(x)$. We notice that $\Big| |E'| - |E| \Big| \le \omega_d r_5^d$ so that for r_5 small enough we may write $|E'| = |E| + \varepsilon$ with $|\varepsilon| \le \varepsilon_2$. By Definition 4.1.6 (1), there exist $x_0 \in \mathbb{R}^d$ and $F \subset \mathbb{R}^d$ such that $|F| = |E'| - 2\varepsilon = |E| - \varepsilon'$ and $E \Delta F \subset B_{r_5}(x_0)$ with $|B_{r_5}(x) \cap B_{r_5}(x_0)| = 0$ and

$$\mathcal{E}(F) \le \mathcal{E}(E) + g_1(\omega_d r^d). \tag{4.3.13}$$

We then define

$$F' = \Big(F \cap B_r(x_0)\Big) \cup \Big(E \setminus (B_r(x) \cup B_r(x_0))\Big) \cup \Big(E' \cap B_r(x)\Big)$$

and observe that $|F'| = |E|$, $F\Delta F' = E\Delta E' \subset B_r(x)$ and $E\Delta F = E'\Delta F' \subset B_r(x_0)$. By Definition 4.1.6 (2),

$$P(F') - P(F) \leq P(E') - P(E) + g_2(r),$$

so that by minimality of E and the locality assumption of V (H17):

$$\mathcal{E}(E) - \mathcal{E}(F) \leq \mathcal{E}(F') - \mathcal{E}(F) \leq P(E') - P(E) + g_2(r) + v_3(r).$$

Injecting this inequality into (4.3.13) yields

$$P(E) \leq P(E') + g_1(\omega_d r^d) + g_2(r) + v_3(r),$$

so that E is a quasi-minimiser of the perimeter. Its error function ρ is g_1, g_2 or v, whichever is equivalent to $r \mapsto Cr^c$ with the smallest possible $c > 0$.

□

4.4 Application to three perturbed isoperimetric problems

In this section, we consider three perturbed isoperimetric problems and investigate whether they satisfy the sets of hypotheses (S1) to (S4). We let the reader check that the hypotheses not mentioned in the proofs are indeed verified.

4.4.1 An anisotropic liquid drop model

Let ϕ and G be as in Section 4.1.2. We consider for $E \subset \mathbb{R}^d$ and U open,

$$P_\phi(E,U) = \int_{(\partial^* E)\cap U} \phi(\nu_E(x))d\mathcal{H}^{d-1}(x) \quad \text{and} \quad V_G(E,U) = \int_{(E\cap U)\times(E\cap U)} G(x-y)dxdy.$$

Proposition 4.4.1. *The perimeter term P_ϕ and perturbation term V_G satisfy (S1) to (S4).*

Proof. Regarding $(S2)$, the isoperimetric inequality (H4) and compactness property (H8) for P_ϕ are implied by the fact that $C'_\phi \operatorname{Per} \leq P_\phi \leq C_\phi \operatorname{Per}$. We can thus take $f_1(m) = C_1 m^{(d-1)/d}$. As for the weak superadditivity (H11) of V_G, let us denote $V_G(E) = L_G(E,E)$ where $L_G(E,F) = \int_{E\times F} G(x-y)dxdy$ and given some $x^i \in \mathbb{R}^d$ we set $B_i = B_R(x^i)$. We compute

$$L_G(E\cap\cup_i B_i, E\cap\cup_i B_i) = \sum_{i=1}^{I} L_G(E\cap B_i, E\cap B_i) + \sum_{i\neq j} L_G(E\cap B_i, E\cap B_j). \tag{4.4.1}$$

Thus

$$V_G(E) \geq V_G(E, \cup_i B_i) = \sum_{i=1}^{I} V_G(E, B_i) + \sum_{i\neq j} L_G(E\cap B_i, E\cap B_j)$$

and we conclude that (H11) holds with $\eta_1 = \eta_2 = 0$.

We now prove (S4) and finish with (S3). The scaling hypothesis (H15) is verified by P_ϕ and V_G by hypothesis on ϕ and G. Regarding (H17), first notice that by hypothesis on G

$$G(x) \le \frac{C}{|x|^\beta} \quad \text{for any } x \in \mathbb{R}^d, \text{ where } C = \sup\left\{G(x) : x \in \mathbb{S}^{d-1}\right\}. \tag{4.4.2}$$

Now given $E, E' \subset \mathbb{R}^d$ such that $E\Delta E' \subset B_r(x)$ for some $x \in \mathbb{R}^d$ and $r > 0$, we compute

$$\begin{aligned} V_G(E) - V_G(E') &= L_G(E,E) - L_G(E',E') \\ &= L_G(E \setminus E', E) + L_G(E \cap E', E) - L_G(E' \setminus E, E') - L_G(E' \cap E, E') \\ &\le L_G(E \setminus E', E) + L_G(E \cap E', E \setminus E'). \end{aligned}$$

Thus by symmetry of the roles of E and E', and defining $E_\Delta = E\Delta E'$ and $E_\cup = E \cup E'$,

$$\left|V_G(E) - V_G(E')\right| \le L_G(E \cap E', E\Delta E') + L_G(E\Delta E', E \cup E') \le C \int_{E_\Delta \times E_\cup} \frac{dxdy}{|x-y|^\beta},$$

where the last inequality is a consequence of (4.4.2). By (4.4.2), there also exists $R = R(G) > 0$ such that $G \le 1$ outside of B_R. As $\beta \in (0,d)$, (4.4.2) implies that G is integrable on B_R. We define for $R > 0$the set $S_R = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d, |x-y| < R\}$ and we have

$$\begin{aligned} \int_{E_\Delta \times E_\cup} \frac{dxdy}{|x-y|^\beta} &\le \int_{(E_\Delta \times E_\cup)\cap S_R} G(x-y)dydx + \int_{(E_\Delta \times E_\cup)\cap S_R^c} G(x-y)dydx \\ &\le \int_{E\Delta E'} \int_{B_R} G(z)dz + |E\Delta E'||E \cup E'| = C(G,m)|E\Delta E'|, \end{aligned}$$

so that (H17) holds with $v(r) = Cr^d$. Thus by Proposition 4.1.7, generalised minimisers of $P_\phi + V_G$ are quasi-minimisers of the perimeter with error function $\rho(r) = Cr^d$.

As for $(S3)$, the fact that P_ϕ satisfies (H12) on local comparisons is classical and a consequence of [60, Theorem 16.3]. Thanks to the fact that $P_\phi \le C_\phi \mathrm{Per}$, the integral inequality (H13) is verified with $f_2(r,m) = C_3 m/r$. Finally, regarding the density scale factor assumption (H14), recall that

$$f_1(m) = C_1 m^{\frac{d-1}{d}}.$$

Therefore, up to replacing f_3 by Cf_3 we have

$$f_3(r,m) = \frac{f_2(r,m) + \rho(r)}{f_1(m)} = \frac{m^{\frac{1}{d}}}{r} + r\left(\frac{m}{r^d}\right)^{\frac{1-d}{d}} = \left(\frac{m}{r^d}\right)^{\frac{1}{d}} + r\left(\frac{m}{r^d}\right)^{\frac{1-d}{d}}.$$

Taking for instance $r_3 = 2^{-d}$ and $\varepsilon_1 = 2^{-d}$, we obtain that if $r \le r_3$ and m is such that $\varepsilon_1 2^{-d} \le mr^{-d} \le \varepsilon_1$, then $f_3(r,m) \le 1$. Therefore, (H14) holds. □

Remark 4.4.2. Another possible nonlocal kernel is of the form

$$V_K(E,U) = -P_K(E,U) = -\int_{E\cap U \times E^c} K(x-y)\,dxdy,$$

so that the corresponding isoperimetric problem $P - P_K$ may be seen as the difference between a (local or nonlocal) perimeter and a nonlocal perimeter. Notice that if $K \in L^1(\mathbb{R}^d)$ we may write

$$V_K(E,U) = \int_{(E\cap U)\times E} K(x-y)\,dxdy - |E\cap U|\|K\|_{L^1(\mathbb{R}^d)},$$

so that the analysis of this case is exactly as in the case $V = V_G$ studied above. However in the recent works [38, 51, 61] the considered problem is

$$\omega_{d-1}\mathrm{Per}(\cdot) - (1-s)P_s(\cdot) \quad \text{or} \quad (1-t)P_t(\cdot) - (1-s)P_s(\cdot)$$

with $0 < s < t < 1$, so that the assumption $V_K \in L^1(\mathbb{R}^d)$ cannot be used. While various sets of hypotheses on K are used in the aforementioned articles, proving that (S1) and (S2) hold is similar to the case $V = V_G$. To obtain density estimates however, the known approaches revolve around showing that there exists $0 < s_0 < 1$ such that for $E\Delta E' \subset B_r(x)$,

$$\left|V_K(E) - V_K(E')\right| \le C(K)|E\Delta E'|^{1-s_0}\mathrm{Per}(E\Delta E')^{s_0}.$$

The dependency on the perimeter of $E\Delta E'$ appearing in the bound on the local variations of V then prevents one from establishing (H17). Thus (S4) does not hold even though assumption (H15) is verified. This problem in turn prevents us from establishing (H14), so that (S3) does not hold even though (H12) and (H13) of (S3) are verified. Usually, one has to first show density estimates for the perimeter before establishing volume density estimates, which is outside the framework of this chapter.

4.4.2 A prescribed nonlocal curvature problem

Let K and T be as in Section 4.1.2 and fix $s \in (0,1)$. In particular, recall that T is L-periodic for some $L > 0$ and Lipschitz continuous. Given $E, U \subset \mathbb{R}^d$ we consider

$$P_K(E) = \int_{(E\cap U)\times E^c} K(x-y)dxdy \quad \text{and} \quad V_T(E,U) = -\int_{E\cap U} T(x)\,dx.$$

Proposition 4.4.3. *The perimeter term P_K and perturbation term V_T satisfy (S1) to (S4).*

Proof. Regarding (S2), recall that by hypothesis $C_K' P_s \le P_K \le C_K P_s$. It is thus enough to prove that the relative isoperimetric inequality holds for P_s. We first observe that that for any $E \subset \mathbb{R}^d$, $r > 0$ and $x \in \mathbb{R}^d$, writing by abuse of notation $B_r = B_r(x)$

$$2P(E,B_r) \ge 2\int_{(E\cap B_r)\times(E^c\cap B_r)} \frac{dxdy}{|x-y|^{d+s}} = \int_{B_r\times B_r} \frac{|\chi_E(x)-\chi_E(y)|^2}{|x-y|^{d+s}}. \tag{4.4.3}$$

Then if $|E\cap B_r| \le |B_r|/2$, we proceed exactly as in the proof of [38, Lemma 2.5]: by a Poincaré-type inequality for fractional Sobolev spaces, we obtain that for any $r_0 > 0$, there exists $C = C(r_0,d,s)$ such that for any $r \le r_0$,

$$P(E,B_r) \ge C|E\cap B_r|^{\frac{d-s}{d}}.$$

Conversely, if $|E^c \cap B_r| \le |B_r|/2$, we can proceed as in the preceding case because the roles of E and E^c are symmetrical in (4.4.3). Eventually, we have that the relative isoperimetric inequality (H4) holds for

$$f_1(m) = C_1 m^{\frac{d-s}{d}}.$$

The compactness property (H8) also holds because of the embedding theorems for fractional Sobolev spaces. The periodicity assumption (H5) holds for V_T because T is L-periodic and the constraint $L = r_1 \le 2r_0/\sqrt{d}$ is verified by setting $r_0 = \sqrt{d}L/2$. Let us prove (H7) on the boundedness of the perimeter by contraposition. We have $V_T(E) \ge -\|T\|_\infty |E|$, so that if there exists $(E_n)_n$ with $\sup_n P_K(E_n) = \infty$ and $\sup_n |E_n| = C < \infty$, then

$$\mathcal{E}(E_n) = P_K(E_n) + V_T(E_n) \ge P_K(E_n) - \|T\|_\infty |E_n|.$$

Hence $\sup_n \mathcal{E}(E_n) = \infty$ and (H7) is proved.

Regarding (S4), as P_K and V_T have no scaling property, we have to establish that (H16) on volume-fixing variations holds. Regarding Definition 4.1.6 (1), we proceed as in [24, Lemma 3.1]. Given a minimiser E of (4.1.1), we have that $|E| < \infty$ and $P_K(E) < \infty$. We consider $E' \subset \mathbb{R}^d$ such that $E \Delta E' \subset B_r(x)$ for $r \le r_5$ and $x \in \mathbb{R}^d$. Let us show that there exists $x_0 \ne x \in \mathbb{R}^d$ such that

$$|E \cap B_{r_5}(x_0)| > 0, \quad |E^c \cap B_{r_5}(x_0)| > 0 \quad \text{and} \quad |B_{r_5}(x) \cap B_{r_5}(x_0)| = 0. \tag{4.4.4}$$

Up to reducing r_5, we may assume that $\omega_d r_5^d \le |E|/4$. Thus there exists $\theta \in \mathbb{S}^{d-1}$ such that $x_\theta = x + 2r_5\theta \in E$. Additionally, $|B_{r_5}(y) \cap B_{r_5}(x)| = 0$. Finally, there also exists $r_0 \ge 2r_5$ such that for $x_0 = x + r_0\theta$, we have $x_0 \in E$ and $|B_{r_5}(x_0) \cap E^c| \ne 0$. Thus (4.4.4) holds.

Hence by the relative isoperimetric inequality we have

$$\operatorname{Per}(E, B(x_0, r_5)) = \sup\left\{ \int_E \operatorname{div} S(x)\, dx : S \in C_c^1(B(x_0, r_5), \mathbb{R}^d),\ \|S\|_\infty \le 1 \right\} > 0.$$

Hence there exists $S \in C_c^1(B(x_0, r_5), \mathbb{R}^d)$ such that $M = \int_E \operatorname{div} S(x)\, dx > 0$.

Let us now define for $t \in (-1, 1)$ the maps $\Phi_t(x) = x + tS(x)$. By a changing of variables, we compute

$$P_K(\Phi_t(E)) = \int_{E \times E^c} K(\Phi_t(x) - \Phi_t(y))\,(1 + t\operatorname{div} S(x) + t\operatorname{div} S(y) + o(t))\, dx dy. \tag{4.4.5}$$

By hypothesis on K we may write that for any $x, y \in \mathbb{R}^d$ and $t \in (-1, 1)$

$$K(\Phi_t(x) - \Phi_t(y)) - K(x - y) = t \int_0^1 \nabla K\left[x - y + ut(S(x) - S(y))\right] \cdot (S(x) - S(y)) du.$$

Combining the regularity of S with the fact that $|\nabla K| \le |\cdot|^{-(d+s+1)}$ yields that for some $C = C(S, K)$

$$\int_{E \times E^c} |K(\Phi_t(x) - \Phi_t(y)) - K(x - y)| dx dy \le C|t| \int_{E \times E^c} \frac{dx dy}{|x - y|^{d+s}} \le C|t| P_K(E).$$

where we used the fact that $C_K'|\cdot|^{-(d+s)} \le K(\cdot)$ in the last inequality. Reinjecting this into (4.4.5) we obtain

$$(1 - C|t|)P_K(E) \le P_K(\Phi_t(E)) \le (1 + C|t|)P_K(E) \tag{4.4.6}$$

for some $C = C(S,K)$. We proceed similarly for the perturbative term, writing

$$V_T(\Phi_t(E)) = -\int_{\Phi_t(E)} T(x)dx = -\int_E T(x + S(x))(1 + t\mathrm{div}S(x) + o(t))dx.$$

Using the Lipschitz continuity of T and (4.4.6) we obtain that for some $C = C(S,K)$:

$$\mathcal{E}(\Phi_t(E)) = \mathcal{E}(E) + Ct + o(|t|).$$

Finally, notice that

$$|\Phi_t(E)| = \int_E (1 + t\,\mathrm{div}\,S(x) + o(t))\,dx = |E| + Mt + o(t).$$

Thus for $|\varepsilon| \ll 1$ we can find $t(\varepsilon) = \varepsilon/M + o(|\varepsilon|)$ such that $F = \Phi_{t(\varepsilon)}(E)$ satisfies $|F| = |E| + \varepsilon$, $E\Delta F \subset B_{r_5}(x_0)$ and

$$\mathcal{E}(F) \le \mathcal{E}(E) + C|\varepsilon|,$$

so that Definition 4.1.6 (1) holds for $g_1(|\varepsilon|) = C|\varepsilon|$.

Regarding Definition 4.1.6 (2), let us consider $E, E', F, F' \subset \mathbb{R}^d$ and $r_5 > 0$ such that $E\Delta E' = F\Delta F' \subset B_r(x)$ for some $r \le r_5$ and $E\Delta F = E'\Delta F' \subset B_{r_5}(x_0)$ with $B_{r_5}(x)$ and $B_{r_5}(x_0)$ disjoint. Recall that we want an estimate of the form

$$\Delta P_K = P_K(E) - P_K(E') - (P_K(F) - P_K(F')) \le g_2(r), \tag{4.4.7}$$

for some nondecreasing function $g_2 : \mathbb{R}_+ \to \mathbb{R}_+$. Up to a change of coordinates we assume that $x = 0$ and given $A, B \subset \mathbb{R}^d$ we define

$$L(A,B) = \int_{A\times B} K(x-y)dxdy.$$

We first decompose E over $B_r, B_{r_5}(x_0)$ and $H = B_r^c \cap B_{r_0}^c$ and obtain

$$L(E,E^c) = L(E\cap B_r, E^c) + L(E\cap B_{r_5}(x_0), E^c) + L(E\cap H, E^c\cap B_r) + L(E\cap H, E^c\cap B_r^c).$$

We write the same formula for $L(E', E'^c)$, and thus obtain for $\Delta P_{E,F} = P_K(E) - P_K(F)$ that

$$\begin{aligned}\Delta P_{E,F} =& L(E\cap B_r, B_{r_5}(x_0)\cap E^c) - L(E\cap B_r, B_{r_5}(x_0)\cap F^c)\\ &+ L(E\cap B_{r_5}(x_0), E^c) - L(F\cap B_{r_5}(x_0), F^c)\\ &+ L(E\cap H, E^c\cap B_r^c) - L(E\cap H, F^c\cap B_r^c).\end{aligned}$$

The corresponding expression for $\Delta P_{E',F'} = P_K(E') - P_K(F')$ is

$$\begin{aligned}\Delta P_{E',F'} =& L(E'\cap B_r, E'^c \cap B_{r_5}(x_0)) - L(E'\cap B_r, F'^c\cap B_{r_5}(x_0))\\ &+ L(E'\cap B_{r_5}(x_0), E'^c) - L(F'\cap B_{r_5}(x_0), F'^c)\\ &+ L(E'\cap H, E'^c\cap B_r^c) - L(E'\cap H, F'^c \cap B_r^c).\end{aligned}$$

Notice that $E\cap H = E'\cap H$, that $E^c \cap B_r^c = E'^c\cap B_r^c$ and that $F^c\cap B_r^c = F'^c \cap B_r^c$. Hence the last lines of $\Delta P_{E,F}$ and $\Delta P_{E',F'}$ will cancel each other out in the difference $\Delta P_K = \Delta P_{E,F} - \Delta P_{E',F'}$. Therefore

$$\begin{aligned}\Delta P_K =& L(E\cap B_r, E^c\cap B_{r_5}(x_0)) - L(E\cap B_r, F^c \cap B_{r_5}(x_0))\\ &+ L(E'\cap B_r, F'^c\cap B_{r_5}(x_0)) - L(E'\cap B_r, E'^c\cap B_{r_5}(x_0))\\ &+ L(E\cap B_{r_5}(x_0), E^c\cap B_r) - L(E\cap B_{r_5}(x_0), E'^c \cap B_r)\\ &+ L(F\cap B_{r_5}(x_0), E'^c\cap B_r) - L(F\cap B_{r_5}(x_0), E^c\cap B_r).\end{aligned}$$

Notice that the right-hand side of the previous equality is a sum of terms of the form $L_K(A,B)$, with either $A\subset B_r$, and $B\subset B_{r_5}(x_0)$ or vice versa. In both cases, there exists $C = C(K, r_5) > 0$ such that $\inf\{|x-y| : (x,y)\in A\times B\} \geq C$. Also recalling that $K(\cdot) \leq C_K|\cdot|^{-(d+s)}$, up to relabelling C_K we have

$$L_K(A,B) \leq C \iint_{B_r\times B_{r_5}(x_0)} \frac{dz dy}{|z-y|^{d+s}} \leq C|B_r||B_{r_5}(x_0)| = Cr^d \tag{4.4.8}$$

Therefore $\Delta P_K \leq Cr^d$, so that (4.4.7) holds with $g_2(r) = Cr^d$.

Finally, (H17) on the local Lipschitz continuity of V is verified with $v(r) = \|T\|_\infty \omega_d r^d$. Thus by Proposition 4.1.7, there exists $C = C(g_1, g_2, v)$ such that for $r \ll 1$, generalised minimisers of (4.1.2) are ρ-minimisers for $\rho(r) = Cr^d$.

We conclude with $(S3)$. Regarding the integral inequality (H13), let $E\subset \mathbb{R}^d$ and $x\in\mathbb{R}^d$. Up to a translation, we assume that $x = 0$. Now given $z \in B_u$, we write

$$\int_{B_u^c} \frac{dy}{|z-y|^s} \leq \int_{B_u^c}\frac{dy}{(|y|-|z|)^s} \leq C\int_{u-|z|}^\infty \frac{v^{d-1}dv}{v^{d+s}} = \frac{C}{(u-|z|)^s}.$$

Hence

$$P_K(B_u, E) = \int_{(E\cap B_u)\times B_u^c} \frac{dz dy}{|z-y|^s} \leq C\int_{E\cap B_u}\frac{dz}{s(u-|z|)^s} = C\int_0^u \frac{\mathcal{H}^{d-1}(E\cap B_v)}{(u-v)^s}\,dv.$$

Hence by Fubini-Tonelli

$$\begin{aligned}\frac{1}{r}\int_0^r P_K(B_u,E)\,du &= \frac{C}{r}\int_0^r\int_v^r \frac{\mathcal{H}^{d-1}(E\cap B_v)}{(u-v)^s}\,du dv\\ &\leq \frac{Cr^{1-s}}{r}\int_0^r \mathcal{H}^{d-1}(E\cap B_v)\,dv = \frac{C}{r^s}|E\cap B_r|,\end{aligned}$$

and one can take $f_2(r,m) = C_3 r^{-s} m$. We finish by establishing (H14) on the density scale factor.

Recall that

$$f_1(m) = C_1 m^{\frac{d-s}{d}}$$

so that up to a multiplicative constant

$$f_3(r,m) = \frac{f_2(r,m)+\rho(r)}{f_1(m)} = \left(\frac{m}{r^d}\right)^{\frac{s}{d}} + r\left(\frac{m}{r^d}\right)^{\frac{s-d}{d}}.$$

We then conclude as in the proof of Proposition 4.4.1 : for some $r_3, \varepsilon_1 > 0$,

$$f_3(r,m) \leq 1 \quad \text{for every} \quad r \leq r_3 \quad \text{and} \quad \frac{\varepsilon_1}{2^d} < \frac{m}{r^d} \leq \varepsilon_1.$$

□

Remark 4.4.4. Notice that excepted (H14) on the density scale factor and (H15) on the scaling of P and V, all the hypotheses in (S1) to (S4) can be checked separately in P and V. Also notice that in Propositions 4.4.1 on $P_\phi + V_T$ and 4.4.3 on $P_K + V_G$ we respectively checked that P_ϕ and P_K satisfied (S1) to (S4). Thus if one wants to check that (S1) to (S4) are satisfied for $P_K + V_G$ or $P_\phi + V_T$, only (H14) and either (H15) or (H16) have to be checked.

For (H15) to hold with $P_K + V_G$, one has to add a scaling hypothesis on K (which is verified if P_K is the fractional perimeter P_s). If one instead wants to show that (H16) holds, hypotheses on G must be added: as for P_K, assuming that $G \in W^{1,1}_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ and that G and ∇G are controled by sufficently integrable functions allows to conclude. Establishing that (H14) holds is identical to the case $P_\phi + V_G$. Indeed, having $f_1(m) = Cm^{(d-s)/d}$ or $f_1(m) = Cm^{(d-1)/d}$ does not change the proof of (H14), because the perturbation introduced by V_G is of the form Cm^d and $d > \max(d-1, d-s)$.

We finish with $P_\phi + V_T$. On the one hand, to show that (H16) holds we have to add the hypothesis that $\phi \in C^1(\mathbb{S}^{d-1})$. Indeed, it implies that P_ϕ admits a first variation (see [60, Exercise 20.7]) so that the first point of (4.1.6) holds. The second point of (4.1.6) holds with $g_2 = 0$ because of the locality of P_ϕ. On the other hand, for (H15) to hold one has to add a scaling hypothesis on T. Lastly, one can show that (H14) holds by proceeding as in the previous paragraph on $P_K + V_G$.

Remark 4.4.5. We also mentioned in the introduction that for $p \geq 1$ and $E, U \subset \mathbb{R}^d$ the Wasserstein functional

$$V_{\mathcal{W}}(E,U) = \inf_{|F \cap E \cap U| = 0} W_p(E \cap U, F)^p$$

could be considered as a perturbative term. Let us briefly explain why $P_\phi + V_{\mathcal{W}}$ satisfy (S1) to (S4). The hypotheses of (S1) and (S2) are verified for $V_{\mathcal{W}}$ as a consequence of [19, Proposition 2.2]. Additionally, $V_{\mathcal{W}}(tE) = t^\beta V_{\mathcal{W}}(E)$ with $\beta = p + d$, so that (H15) is verified. Lastly by [19, Lemma 2.4] for $E, E' \subset \mathbb{R}^d$:

$$|V_{\mathcal{W}}(E) - V_{\mathcal{W}}(E')| \leq C(|E|^{\frac{p}{d}} + |E|^{\frac{p}{d}})|E \Delta E'|, \tag{4.4.9}$$

so that (H17) holds with $v(r) = Cr^d$. We establish that (H14) holds as in the previous remark.

Lastly, if one wants to study $P_K + V_{\mathcal{W}}$ in the case where K admits no scaling, one will need to show that $V_{\mathcal{W}}$ admits volume-fixing variations. This last fact remains an open question for now, although we believe it may hold without additionnal assumption on $\mathcal{W}$.

4.4.3 A model of a nonlocal perimeter interacting with Dirichlet eigenvalues

In this example, the perimeter is P_ϕ and the perturbation term is V_{Dir}, i.e. for $E, U \subset \mathbb{R}^d$

$$V_{\mathrm{Dir}}(E,U) = V_{\mathrm{Dir}}(E\cap U) = \min_u \left\{ \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2\,dx - \int_{\mathbb{R}^d} uh\,dx : u \in \hat{H}^1_0(E\cap U)\right\},$$

where $h \in L^p(\mathbb{R}^d)$ for some $p > d$. Before proving that V_{Dir} satisfies (S1) and (S2), let us recall some of its properties. Given E with $|E| < \infty$, $V_{\mathrm{Dir}(E)}$ admits an unique minimiser w_E, which is bounded (see [35, Section 2]):

$$\|w_E\|_{L^\infty} \le C(p)\|h\|_{L^p}|E|^{2/d-1/p}. \tag{4.4.10}$$

It also satisfies

$$\int_{\mathbb{R}^d} \nabla w_E \cdot \nabla\varphi\,dx = \int_{\mathbb{R}^d} h\varphi\,dx \tag{4.4.11}$$

for any $\varphi \in \hat{H}^1_0(E)$, so that

$$V_{\mathrm{Dir}}(E) = -\frac{1}{2}\int_{\mathbb{R}^d} hw_E\,dx = -\frac{1}{2}\int_E hw_E\,dx. \tag{4.4.12}$$

Proposition 4.4.6. *The perimeter term P_ϕ and perturbation term V_{Dir} satisfy (S1) and (S2).*

Proof. Regarding (S1), by combining (4.4.12) and the Hölder inequality, for $r > 0$ we have

$$|V_{\mathrm{Dir}}(B_r)| \le \int_{B_r} hw_{B_r} \le \|h\|_{L^p(B_r)}\|w_{B_r}\|_{L^{p'}(E)},$$

where $p' = p/(p-1)$. Thus by (4.4.10)

$$|V_{\mathrm{Dir}}(B_r)| \le C\|h\|^2_{L^p}|B_r|^{1/p'+2/d-1/p},$$

so that $V_{\mathrm{Dir}}(B_r) \to 0$ as $r \to 0$, proving (H1). Next, notice that if $A, B \subset \mathbb{R}^d$ have finite volume and are such that $\mathrm{dist}(A,B) > 0$, then $V_{\mathrm{Dir}}(A\cup B) = V_{\mathrm{Dir}}(A) + V_{\mathrm{Dir}}(B)$. Thus (H3) holds.

We now move to (S2). Regarding (H7), given $E \subset \mathbb{R}^d$ we proceed as for (H1) and we notice that

$$V_{\mathrm{Dir}}(E) \ge -C\|h\|^2_{L^p}|E|^{1/p'+2/d-1/p}.$$

Therefore, given $(E_n)_n$ with $\sup_n |E_n| < \infty$, $\sup P_\phi(E_n) = \infty$ implies $\sup_n \mathcal{E}(E_n) = \infty$, proving (H7) by contraposition. Assumption (H9) follows from [35, Remark 2.3]. We conclude by proving the weak superaddivity assumption (H11). Given $m > 0$, we fix E with $|E| \le m$ and $I \ge 1$ disjoints balls$B_R(x^1),\dots,B_R(x^I)$ such that $\min_{i\neq j}\mathrm{dist}(B^i,B^j) \ge 5R$. We first notice that

$$\sum_{i=1}^I V_{\mathrm{Dir}}(E\cap B_R(x^i)) = V(E\cap\mathcal{B}), \quad \text{where} \quad \mathcal{B} = \bigcup_{i=1}^I B_R(x^i). \tag{4.4.13}$$

Let us denote $B^i = B_{R/2}(x^i)$ for $1 \le i \le I$. There exists a mollifier $\psi \in C_c^\infty(\mathbb{R}^d, [0,1])$ satisfying

$$\psi = 1 \ \text{ on } \ \mathcal{B} = \bigcup_{i=1}^{I} B^i, \ \psi = 0 \ \text{ on } \ \bigcap_{i=1}^{I} B_R^c(x_i) \ \text{ and } \ \|\nabla\psi\|_{L^\infty} \le \frac{C}{R}.$$

Noticing that $\psi w_E \in \hat{H}_0^1(E \cap \mathcal{B})$, we can use it as a candidate for the minimisation in $V_{\mathrm{Dir}}(E \cap \mathcal{B})$. We obtain

$$V_{\mathrm{Dir}}(E \cap \mathcal{B}) - V_{\mathrm{Dir}}(E) \le \frac{1}{2}\int_{\mathbb{R}^d} |\nabla(\psi w_E)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^d} h w_E (1 - 2\psi) dx.$$

Notice that $|\nabla(\psi w_E)|^2 = \nabla w_E \cdot \nabla(\psi^2 w_E) + w_E |\nabla\psi|^2$. Applying (4.4.11) with $\varphi = \psi^2 w_E$ yields

$$\begin{aligned} V_{\mathrm{Dir}}(E \cap \mathcal{B}) - V_{\mathrm{Dir}}(E) &\le \frac{1}{2}\int_{\mathbb{R}^d} w_E^2 |\nabla\psi|^2 dx + \frac{1}{2}\int_{\mathbb{R}^d} w_E h \psi^2 dx + \frac{1}{2}\int_{\mathbb{R}^d} w_E h (1 - 2\psi) dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} w_E^2 |\nabla\psi|^2 dx + \frac{1}{2}\int_{\mathbb{R}^d} w_E h (1 - \psi)^2 dx \\ &\le \frac{C}{R^2} + \sum_{i=1}^{I} \int_{(B^i)^c} |w_E h| dx. \end{aligned}$$

By the Hölder inequality we estimate that for every $1 \le i \le I$,

$$\int_{(B^i)^c} |w_E h| \le \left(\int_{E \cap (B^i)^c} |w_E|^{p'} \right)^{1/p'} \|h\|_{L^p((B^i)^c)} \le \|w_E\|_{L^\infty}^{p'} |E|^{1/p'} \|h\|_{L^p((B^i)^c)}.$$

Recall that for $1 \le i \le I$, $B^i = B_{R/2}(x^i)$. Thus if we denote

$$\eta_2(R) = \frac{C}{R^2} + C \sum_{i=1}^{I} \|h\|_{L^p((B^i)^c)},$$

we have that $\eta_2(R) \to 0$ as $R \to \infty$. Thus (H11) holds, as

$$\sum_{i=1}^{I} V_{\mathrm{Dir}}(E \cap B^i) = V_{\mathrm{Dir}}(E \cap \mathcal{B}) \le V_{\mathrm{Dir}}(E) + \eta_2(R).$$

□

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Theoretical and numerical analysis of perturbed isoperimetric problems

Abstract

In this thesis, we focus on perturbed isoperimetric problems. These problems involve the minimisation of an energy composed of a perimeter term that promotes mass aggregation, countered by a perturbation term favouring disaggregation.

We begin by presenting the concepts used as well as past and current research conducted on the isoperimetric problem and its variants.

In Chapter 1, we study a problem where the perimeter interacts with a non-local term called an exterior transport term, defined using optimal transport theory. We demonstrate the existence of solutions to this problem and, in regimes where the perimeter dominates, we prove that the minimisers are balls.

Chapter 2 is dedicated to the exterior transport term. In a general framework, we show that the variational problem defining it has solutions and a dual formulation. Using stronger assumptions, we finally show that this term is maximised only by balls.

In Chapter 3, we present a numerical study in dimension 2 of the problem from Chapter 1. We approximate the minimisers of the energy considered via a gradient descent algorithm. The numerical results lead us to conjecture the existence of a critical mass above which the minimisers are no longer balls, but elongated shapes with two axes of symmetry.

Chapter 4 focuses on a perturbed isoperimetric problem where the perimeter and perturbation terms are not explicit. We exhibit a general set of assumptions under which a relaxed version of the problem admits minimisers. Under stronger hypotheses, we then investigate whether these minimisers have density estimates.

Keywords: Calculus of Variations, perturbed isoperimetric problem, Wasserstein distance, generalized minimisers, Sinkhorn algorithm

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Etude théorique et numérique de problèmes isopérimetriques perturbés

Résumé

Nous étudions dans cette thèse des problèmes isopérimétriques perturbés. Ces problèmes consistent en la minimisation d'une énergie formée d'un terme de périmètre qui favorise l'agrégation de masse, auquel s'oppose un terme perturbatif favorisant la désagrégation.

Nous commençons par présenter les concepts utilisés ainsi que la recherche passée et actuelle effectuée sur le problème isopérimétrique et ses variantes.

Nous étudions dans le chapitre 1 un problème où le périmètre interagit avec un terme non-local dit de transport extérieur, défini à l'aide de la théorie du transport optimal. Nous montrons l'existence de solutions à ce problème et, dans les régimes où le périmètre domine, nous prouvons que les minimiseurs sont les boules.

Le chapitre 2 est consacré au terme de transport extérieur. Dans un cadre général, nous montrons que le problème le définissant admet des solutions et une formulation duale. A l'aide d'hypothèses plus fortes, nous montrons que ce terme est uniquement maximisé par les boules.

Dans le chapitre 3, nous présentons les travaux numériques effectués en dimension 2 sur le problème du premier chapitre. Nous approchons les minimiseurs de l'énergie considérée via une descente de gradient. Les résultats numériques nous amènent à conjecturer l'existence d'une masse critique à partir de laquelle les minimiseurs ne sont plus des boules, mais des formes allongées à deux axes de symétrie.

Le chapitre 4 porte sur un problème isopérimétrique perturbé où les termes de périmètre et de perturbation ne sont pas explicites. Pour des hypothèses assez générales, nous montrons que ce problème admet des minimiseurs en un sens faible. Nous montrons ensuite sous des hypothèses plus fortes que ces minimiseurs, appelés minimiseurs généralisés, possèdent des estimées de densité.

Mots clés : Calcul des Variations, problème isopérimétrique perturbé, distance de Wassertein, minimiseurs généralisés, algorithme de Sinkhorn

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Laboratoire Paul Painlevé
Cité scientifique – 59650 Villeneuve-d'Ascq – France