UNIVERSITÉ DE LILLE

THÈSE

pour obtenir le grade de :

DOCTEUR DE L'UNIVERSITÉ DE LILLE

dans la spécialité

« MATHÉMATIQUES »

par

Huan DAI

Analytic and asymptotic properties of solutions of non-homogeneous pantograph functional differential equations

Propriétés analytiques et asymptotiques de solutions d'équations différentielles fonctionnelles de pantographe non homogènes

Thèse soutenue le 3 mai 2025 devant le jury composé de :

Mme	HUI LIANG	Professeur, Harbin Institute of Technology, China	Président (Examinateur)
M.	JEAN-PIERRE RAMIS	Professeur émérite, Université de Toulouse, France	(Rapporteur)
M.	ALBERTO LASTRA	Professeur, Universidad de Alcalá, Espagne	(Rapporteur)
M.	GUOTING CHEN	Professeur, Great Bay University, China	(Examinateur)
M.	CHANGGUI ZHANG	Professeur, Université de Lille, France	(Directeur de Thèse)
M.	WENBIN LI	Associate Professor, Harbin Institute of Technology, China	(Co-Directeur de Thèse)

LABORATOIRE PAUL PAINLEVÉ, L'ÉCOLE DOCTORALE EG MADIS
THÈSE EN COTUTELLE ENTRE L'UNIVERSITÉ DE LILLE ET L'INSTITUT DE TECHNOLOGIE DE HARBIN(SHENZHEN)

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Chapitre 1

Introduction

1.1 Background and objective

In addressing numerous scientific problems, it is essential to characterize phenomena based on their rates of change, which are often represented by derivatives. Consequently, the study of differential equations and the properties of their solutions is pivotal for solving many practical issues and holds significant research importance and application value. A current research hotspot in differential equation theory is functional differential equations. These equations describe changes in phenomena that depend not only on the current state but also on states over past or future periods. Functional differential equations have important practical applications in fields such as physics, information technology, chemistry, engineering, economics, and biomathematics. Due to their widespread application in scientific research, studying the stability theory and asymptotic behavior of their solutions has become particularly important.

The pantograph equation

$$y'(x) = ay(qx) + by(x) \tag{1.1.1}$$

serves as an ideal mathematical model for describing the wave motion problem of the overhead supply line in an electrified railway system, where a and b are non-zero real numbers, $y(x)$ is a real-valued function. Equation (1.1.1) is classified as a functional differential equation because substituting $y = e^t$, $q = e^c$, and $y(x) = z(t)$ transforms it into the equation $e^{-t}z'(t) = a(t + c) + bz(t)$. If $0 < q < 1$, then $c = \ln(q) < 0$, and the equation becomes a delayed functional differential equation. Conversely, if $q > 1$, then $c = \ln(q) > 0$, and the equation becomes an advanced functional differential equation.

According to [46], the term “pantograph” originates from the works of Ockendon, Tayler and Iserles [70, 44]. This equation has significant applications in various fields, such as numerical analysis [66], nonlinear dynamical systems [29], among others. In 1971, Kato and Mcleod studied the existence, analytic and asymptotic properties of solutions to Equation (1.1.1) [54]. They examined the effect of the parameter q on the solution : if $0 < q < 1$, then Equation (1.1.1) with $y(0) = 1$ is "well-posed", meaning there is a unique analytic solution in the neighborhood of $x = 0$; if $q > 1$, then there is no analytic solution. Therefore, in this thesis, we consider the case $0 < q < 1$.

The pantograph equation also belongs to the class of differential q-difference equations. Unlike ordinary differential equations, the solutions at infinity of Equation (1.1.1) are infinitely dimensional due to the influence of the parameter q. This complexity makes it challenging to analyze the asymptotic properties of the solution at zero. To address this problem, Kato and Mcleod first proposed establishing a connection formula between the solutions at zero and infinity. They then used this connection formula to derive the asymptotic properties of the solution at zero. This approach is also an important idea that will be utilized in this thesis.

Some authors have also studied the non-homogeneous form of the equation corresponding to the pantograph equation. For example, Lim [60] investigated the asymptotic bounds of solutions to the equation

$$y'(x) = ay(qx) + by(x) + g(x), \tag{1.1.2}$$

by assuming g is a function defined on $[0, \infty)$ and $g(x) = O(x^{\alpha})$ as $x \to \infty$.

Currently, most research on functional differential equations focuses on the properties of solutions in the real plane. Analyzing the properties of solutions in the complex plane is a complicated and challenging process, and the related theories are not yet fully developed. As a result, this remains a significant research hotspot in the field of differential equations. Inspired by previous work, this thesis primarily investigates the analytical and asymptotic properties of solutions of the non-homogeneous pantograph functional differential Equation (1.1.2) in the complex plane.

Consider x on the Riemann surface of the logarithm. Let $x = -\frac{t}{b}$ $(b \neq 0)$ and define $u(t) = y(x) = y(-\frac{t}{b})$. Then, the derivative of $u(t)$ with respect to t is given by $u'(t) = -\frac{1}{b}y'(-\frac{t}{b})$. Substituting $x = -\frac{t}{b}$ into Equation (1.1.2), we obtain :

$$u'(t) = \alpha u(qt) - u(t) + f(t),$$

where $\alpha = -\frac{a}{b} \neq 0$ and $f(t) = -\frac{1}{b}g(-\frac{t}{b})$. Thus, we focus on analyzing the following equation :

$$y'(x) = \alpha y(qx) - y(x) + f(x). \tag{1.1.3}$$

We assume that the non-homogeneous term f is a rational function. Then, we only need to consider f in one of the three forms :

(i) polynomials;

(ii) fractions with a singularity at zero : $f(x) = \frac{1}{x^m}$, $m > 0$;

(iii) fractions with a non-zero singularity : $f(x) = \frac{1}{(x+c)^m}$, $m > 0, c \in \mathbb{C}^*$.

We will discuss each of these cases in Chapters 2, 4, and 5, respectively.

When f is a fraction with singularity at zero, we mainly study the asymptotic behaviors of solutions of Equation (1.1.3). Following Kato and Mcleod's ideas on the connection formula mentioned earlier, we first study the connection formula between the solutions of Equation (1.1.3) at zero and infinity. Since the theory of elliptic functions is closely related to the analytic theory of linear functional q-difference equations [20, 97, 82, 30, 78], we use the Jacobi function to represent the solution of (1.1.3). Then the asymptotic form of the solutions at zero is analyzed according to the connection formula. When f is a fraction with singularity at non-zero constant, the solutions may have an infinite number of singularities due to the action of q, These singularities are like $-c, -cq, -cq^2, \cdots$, or $-c, -\frac{c}{q}, -\frac{c}{q^2}, \cdots$. Therefore, different from the previous situation, the focus of the study changes to the study of the analytical properties of the solution.

The homogeneous equation corresponding to Equation (1.1.3) is

$$y'(x) = \alpha y(qx) - y(x). \tag{1.1.4}$$

Zhang [98] proved that on the complex plane, a unique power series solution of Equation (1.1.4) satisfying the initial condition $y(0) = 1$ can be represented by a linear combination of all elements of the fundamental solution system at infinity. This relation formula is called the connection formula between solutions at zero and infinity, and it plays an important role in determining the asymptotic form of the solution with the initial condition. The difference between the two solutions of the non-homogeneous Equation (1.1.3) is a solution of the homogeneous Equation (1.1.4). Therefore, the connection formula between the solutions of the non-homogeneous pantograph equation can be derived using the connection formula obtained by Zhang, and this simple observation will play a key role in this study.

In this thesis, a theoretical study is conducted on the properties of solutions to non-homogeneous pantograph equations. The main results are as follows :

(1) A connection formula between the solutions at zero and infinity for non-homogeneous functional differential equations is established in the complex plane.

(2) By utilizing the properties of q-periodic function to analyze the asymptotic properties of solutions, the asymptotic form of the solution with the initial condition is described more precisely.

(3) Index theorems of the operators corresponding to the differential q-difference equations are established.

(4) The correlation theory of the properties of solutions for a class of non-homogeneous functional differential equations in the complex plane is perfected.

1.2 Current research of the pantograph equation

The study of the homogeneous pantograph equation dates back to 1971 when Ockendon et al. [70] established a mathematical model for the motion of the pantograph head on the electric supply system of British railway trams, i.e., Equation (1.1.1), and investigated the dynamical properties of the tram power supply system. In the same year, Fox et al. [36] classified the parameter ranges in Equation (1.1.1) and obtained solutions expressed in series or integral forms, along with their asymptotic properties. Kato et al. [54] studied the existence and uniqueness of solutions and discussed the asymptotic properties of solutions under various parameter values. Carr et al. [25] extended the work of Kato et al. [54] by supplementing the case where parameter b is purely imaginary, analyzing the asymptotic properties of solutions when $\operatorname{Re} b = 0$. Liu [63] investigated numerical methods for Equation (1.1.4), utilizing perturbation techniques and Kuruklis's results [56] to analyze the discretized form of the equation and explore the asymptotic behavior of numerical solutions. Zhang [98] studied solutions of the equation in the complex domain at the origin and infinity, deriving connection formulas between these solutions and analyzing the asymptotic form of the origin solution at infinity.

It is worth mentioning that discussions on the non-homogeneous pantograph equation began as early as 1924, before the term "pantograph" was proposed. Flamant [35] studied the existence and uniqueness of the solution of equation

$$f'(x) = a(x)f(\frac{x}{\gamma}) + b(x)$$

through the successive approximations, and also discussed the behavior of the solution when the non-homogeneous term $b(x)$ has singularities at the origin or other points. When $a(x)$ and $b(x)$ are holomorphic at the origin, and $b(x)$ has a simple pole at the origin, the behavior of the solution $f(x)$ near the origin is similar to $f(x) = \phi(x) + \psi(x)\log x$, where $\phi(x)$ and $\psi(x)$ are holomorphic functions.

The pantograph equation also belongs to a class of linear functional q-difference equations. In the study of functional q-difference equations, distinct or even opposite results often arise for $q > 1$ and $0 < q < 1$ [21, 26, 96]. Equation (1.1.1) exhibits similar phenomena. Kato et al. [54] separately discussed the properties of solutions in the real domain for these ranges of q. They proved that when $0 < q < 1$, there exists a unique analytic solution near $x = 0$ satisfying boundary conditions, with all solutions having specific asymptotic properties. Conversely, when $q > 1$, Equation (1.1.1) lacks analytic solutions, and only one solution matches a prescribed asymptotic form.

The analytic theory of linear functional q-difference equations is closely linked to elliptic function theory [20, 91, 72, 1]. Birkhoff [20] generalized the Riemann problem to ordinary differential equations with irregular singularities, linear difference equations, and q-difference equations. By prescribing singularities and characteristic constants, he described solution properties near singularities via asymptotic expansions and matrix analysis. Di et al. [91] studied power series solutions of differential q-difference equations, including hypergeometric and basic hypergeometric functions. Adams [1] classified series solutions and analytic conditions for n-th order linear differential q-difference equations based on root distributions of characteristic equations : (i) all roots are finite and non-zero ; (ii) not all roots are finite and non-zero. Ohyama et al. [71] derived explicit Euler-type formulas for analytic solution bundles of arbitrary linear algebraic q-difference equations, relating to “intermediate singularities.”

When $b = 0$, Equation (1.1.1) reduces to :

$$y'(x) = ay(qx) \tag{1.2.1}$$

Despite its resemblance to ordinary differential equations, this equation exhibits distinct behaviors. Morris et al. [69] showed that solutions of Equation (1.1.1) with $a = -1$ are entire functions and proved unbounded oscillations as $t \to \infty$ via the Phragmén-Lindelöf principle. Iserles [44] established that $\limsup_{t\to\infty} \|y(t)\| = \infty$ holds regardless of a and q. Kato et al. [54] further discussed the complex asymptotic behavior of solutions in this case.

For nonhomogeneous pantograph equations, Lim [60] assumed $g(x) = x^\alpha$ and introduced $\kappa = \ln|b/a|/\ln q$, classifying asymptotic bounds based on b : (i) $y(x) = O(e^{bx})$ $(x \to \infty)$ for $b > 0$; (ii) for $b < 0$, $y(x) = O(x^\kappa)$ if $\alpha < \kappa$, $O(x^\kappa \ln x)$ if $\alpha = \kappa$, and $O(x^\alpha)$ if $\alpha > \kappa$. Aldosari et al. [6] employed Maclaurin series expansions to solve equations with exponential nonhomogeneous terms, deriving closed-form series solutions with explicit convergence proofs. Saadeh et al. [81] proposed the Residual Power Series Method (RPSM) to construct analytic approximations for pantograph systems, validated via numerical examples. Aljoufi [8] transformed nonhomogeneous equations with polynomial terms into standard homogeneous forms, deriving explicit solution formulas.

Regarding series solutions, Ramis [76] introduced Gevrey asymptotic expansions and multisummability for divergent series. Balser [16] outlined advances in formal solutions for complex differential equations. Ismail et al. [48] derived convergent asymptotic expansions for q-functions using integral representations. Andrews et al. [14] discussed q-series relations and connections to elliptic functions. Chen et al. [27] applied Borel-Laplace summation to holomorphic solutions of nonlinear PDEs.

Liu et al. [61] studied analytic and numerical solutions for multi-delay pantograph equations, proposing h-methods with stability conditions. Li et al. [59] analyzed Runge-Kutta (RK) stability for multi-delay equations. Anakira et al. [13] enhanced RPSM with Laplace transforms and Padé approximants for multi-delay equations. Singh et al. [86] applied fractional differential transform methods to solve fractional pantograph equations.

Fractional and higher-order pantograph equations were explored by Bhalekar et al. [18], deriving special functions and their properties. Jafari et al. [52] implemented Legendre pseudospectral methods for delay differential equations. Alzabut et al. [12] studied asymptotic stability of nonlinear fractional pantograph equations using Krasnoselskii's fixed-point theorem. Almalahi et al. [9] established existence and stability for implicit fractional pantograph equations. Yang et al. [95] proposed Jacobi spectral Galerkin methods for nonlinear fractional pantograph equations.

For variable coefficients in Equation (1.1.1), Alrebdi et al. [10] combined Maclaurin series and Adomian decomposition methods. Guglielmi et al. [40] analyzed one-leg θ-methods for stability on quasi-geometric grids. Wang et al. [92] developed shifted Chebyshev polynomial algorithms for variable-coefficient equations. Dix [32] studied asymptotic behavior via Lyapunov functionals.

Following the establishment of Equation (1.1.1), its generalizations have been extensively studied [39, 31, 3, 57, 41, 83, 64, 42, 15, 85, 24, 90, 33, 2, 79, 53]. Buhmann et al. [22] analyzed stability of neutral differential equations with variable delays :

$$\begin{aligned} &y'(t) = Ay(t) + By(qt) + Cy'(pt), \\ &y(0) = y_0, \end{aligned} \tag{1.2.2}$$

where A, B, C are matrices and $p, q \in (0, 1)$. Stability conditions for discretized models align with the original equation under step-size constraints. Iserles [45] termed Equation (1.2.2) the "generalized pantograph equation," deriving asymptotic stability via Dirichlet series : $\lim_{t\to\infty} y(t) = 0$ if $\mathrm{Re}(\lambda(A)) < 0$ and $\|A^{-1}B\| < 1$. Zhao et al. [100] proved L-stable RK methods preserve asymptotic stability.

For the neutral equation :

$$\begin{aligned} &y'(t) = ay(t) + \sum_{i=1}^{l} b_i y(q_i t) + \sum_{i=1}^{l} c_i y'(p_i t), \\ &y(0) = y_0, \end{aligned} \tag{1.2.3}$$

Iserles et al. [47] established existence, uniqueness, and asymptotic behavior. Liu [62] analyzed θ-methods for numerical stability. When $p_i = q_i$, Buhmann et al. [23] studied RK stability, while Koto [55] introduced P_∞-stability concepts.

Wang [93] proposed $G_q(\bar{q})$-stable geometric grid methods for :

$$y'(t) = f(t, y(t), y(\lambda t), y'(\lambda t)). \tag{1.2.4}$$

Shammakh et al. [84] investigated Hilfer fractional pantograph equations, while Aly et al. [11] applied fixed-point theorems to ψ-Hilfer equations. Yang et al. [94] implemented Jacobi spectral methods for nonlinear fractional pantograph equations.

Existing research predominantly focuses on real-domain asymptotic bounds [99, 38, 94, 80, 68, 87] and numerical methods [17, 19, 50, 51, 7, 88, 49, 73, 4, 43, 89, 5]. In contrast, complex-domain analytic solutions remain underexplored. This study focuses on the analyticity and asymptotic behavior of solutions to the nonhomogeneous pantograph equation (1.1.2) in the complex domain, aiming to advance understanding in this field.

1.3 Preliminary knowledge

For the convenience of analysis, we introduce notations common to all chapters.

Let $\widetilde{\mathbb{C}}^*$ be the Riemann surface of the complex logarithm, and let log be the complex logarithmic function on the Riemann surface $\widetilde{\mathbb{C}}^*$, i.e., $x^a = e^{a \log x}$ for all $a \in \mathbb{C}$ and $x \in \widetilde{\mathbb{C}}^*$.

For $c \in \mathbb{C}$, we denote by $\widetilde{\mathbb{C} \setminus \{-c\}}$ the Riemann surface of the complex logarithmic function $\log(x + c)$. Let a, b be two real numbers such that $a < b$, and let $S_n^c(a, b)$ be the open sector on $\widetilde{\mathbb{C} \setminus \{-c\}}$:

$$S_n^c(a, b) = \{x \in \widetilde{\mathbb{C} \setminus \{-c\}} | a < \arg(q^n x + c) < b\},$$

especially,

$$\begin{aligned} S(a, b) &\triangleq S_0^0(a, b) = \{x \in \widetilde{\mathbb{C}^*} | a < \arg(x) < b\}, \\ S_1(a, b) &\triangleq S_0^1(a, b) = \big\{x \in \widetilde{\mathbb{C} \setminus \{-1\}} \big| a < \arg(x + 1) < b\big\}, \\ S_n^1(a, b) &= \big\{x \in \widetilde{\mathbb{C} \setminus \{-1\}} \big| a < \arg(q^n x + 1) < b\big\}. \end{aligned} \tag{1.3.1}$$

We denote the Jacobi-θ function as

$$\theta(q, x) = \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n,$$

simplyfied by $\theta(x)$. For any $n \in \mathbb{Z}$, we have

$$\theta(q^n x) = q^{-\frac{n(n-1)}{2}} x^{-n} \theta(x). \tag{1.3.2}$$

For any $\alpha \in \mathbb{C}$ and $n \geq 1$, let $(\alpha;q)_n$ and $(\alpha)_n$ be the product of the following fomulas :

$$(\alpha;q)_n = \prod_{j=0}^{n-1}(1-\alpha q^j),\ (\alpha)_n = \prod_{j=0}^{n-1}(\alpha+j),$$

and $(\alpha;q)_0 = (\alpha)_0 = 1$. It follows that $(\alpha;q)_n$ and $(\alpha)_n$ can be extended to $(\alpha;q)_\infty$ and $(\alpha)_\infty$ as $n \to \infty$. Furthermore,

$$(\alpha q^{-n};q)_n = (-1)^n(\frac{q}{\alpha};q)_n(\frac{\alpha}{q})^n q^{-\frac{n(n-1)}{2}}. \tag{1.3.3}$$

Some equations that will be used (see [14]) :

$$\begin{aligned} \sum_{n\geq 0}\frac{1}{(q;q)_n}x^n &= \frac{1}{(x;q)_\infty}, \qquad |x|<1, |q|<1, \\ \sum_{n\geq 0}\frac{(-1)^n q^{\frac{n(n-1)}{2}}x^n}{(q;q)_n} &= (x;q)_\infty, \qquad |q|<1. \end{aligned} \tag{1.3.4}$$

As in [98], we define

$$F_0(\alpha;q,x) = \sum_{n\geq 0}\frac{(-1)^n(\alpha;q)_n}{n!}x^n, \tag{1.3.5}$$

then $F_0(\alpha;q,x)$ is the unique solution of Equation (1.1.4) with initial condition $y(0) = 1$. In case of no ambiguity, it is abbreviated as F_0.

1.4 Main contents of this study

Based on the above analysis, this thesis primarily investigates the existence, analyticity, and asymptotic forms of solutions to the non-homogeneous pantograph functional differential Equation (1.1.3) in the complex plane. First, we study the existence and uniqueness of solutions to Equation (1.1.3) with non-homogeneous terms under specific assumptions. Next, we explore solutions expressed as series at zero and infinity, respectively. We estimate the growth trend of these series and analyze the solutions' analyticity based on the series' radius of convergence. Furthermore, we establish the relationship between the integral solution and the series solution, which facilitates the analysis of the connection formula between solutions at zero and infinity. Finally, we study the asymptotic behavior of the solution with given initial conditions using the connection formula. The thesis is structured as follows :

In Chapter 2, we focus on the existence and uniqueness of the Laurent series solution when the non-homogeneous term is expressed as a Laurent series expanded at zero and infinity. We also examine the existence and uniqueness of the solution when the non-homogeneous term is a polynomial. When the non-homogeneous term is represented as a Laurent series expanded at zero, we seek solutions in the form of a Laurent series at zero. Conversely, when the non-homogeneous term is given as a Laurent series expanded at infinity, we look for solutions expressed as a Laurent series at infinity. For analytic analysis, we assume the non-homogeneous term to be an analytic function. If this term is a rational function, it can be considered as a combination of three cases. In the scenario where the non-homogeneous term is a polynomial, we study the existence and uniqueness of the polynomial solution and explore the relationship between solutions at zero and infinity. The other two cases are discussed in Chapters 4 and 5.

Chapter 3 mainly studies the index theory of Equation (1.1.3). We analyze the index theorem of the operator corresponding to Equation (1.1.3) in the complex plane, based on the index theory of differential q-difference equations, and examine how the non-homogeneous term affects the properties of the solution. We distinguish between two cases : $0 < q < 1$ and $q > 1$. For $0 < q < 1$, which aligns with the range of q studied in this thesis, we discuss the operator form and the corresponding index theorem when the operator acts on series at zero and infinity. The solutions are used to verify the theoretical results for $0 < q < 1$. For $q > 1$, this represents the opposite situation to the solutions studied in this thesis. It can be regarded as the perfection of the operator index theory when the equation does not limit the range of q.

In Chapter 4, we mainly consider the second case : when the non-homogeneous term is a fraction with a singularity at zero. We study the asymptotic properties of the analytic solutions. The solutions expressed in terms of the series at zero and infinity are found, respectively, and the analytical properties of the solutions are analyzed. By applying the Laplace transform to the equation, we obtain two Laplace integral solutions. We then explore the relationship between these integral functions by examining the equation satisfied by the Laplace integrals and the solutions of the corresponding homogeneous equation. The relationship between the integral solution and the series solution is analyzed by using perturbations of the equation. This analysis allows us to study the connection formula between solutions at zero and infinity, ultimately leading to an understanding of the asymptotic form of the solution given an initial condition.

In Chapter 5, we delve into the third case, where the non-homogeneous term is a fraction with a singularity at a nonzero constant, we study the analytic and asymptotic properties of solutions. We identify solutions expressed as series expansions at zero and infinity, as well as divergent series solutions with an infinite number of singularities at nonzero constants. Using the equation's perturbation, solutions in terms of the integral-sum function are studied, and the analytical properties of solutions are analyzed. Additionally, we utilize the Borel summation method to compute the integral-sum function corresponding to these divergent series solutions, confirming that it aligns with the solution obtained through perturbation of the equation. This consistency supports the robustness of our approach. The uniqueness of the solution with the initial condition is studied, and the relationship between the series at zero and the integral-sum function is analyzed by using the uniqueness of the solution with the initial condition.

Chapitre 2

Laurent series solution at zero and infinity

2.1 Introduction

In this chapter, we begin by examining the Laurent series solutions of the equation at both zero and infinity. These solutions can be expressed in terms of the series expansions around these points, respectively. The following lemma establishes that the solutions at zero can be represented as a combination of power series and logarithmic terms.

Lemma 2.1. *For the formal power series $u(x)$ and $v(x)$, if $y(x) = u(x) \cdot \log x + v(x)$ is a solution of Equation* (1.1.3)*, then it has the following form :*

$$y(x) = u_0 \log x \cdot F_0(\alpha; q, x) + v(x),$$

where u_0 is an arbitrary constant and $v(x)$ is a solution of equation

$$v'(x) = \alpha v(qx) - v(x) + \widetilde{f}(x)$$

with

$$\widetilde{f}(x) = \alpha(\ln q)u_0 F_0(\alpha; q, qx) - \frac{u_0 F_0(\alpha; q, x)}{x} + f(x).$$

Proof. Assuming that $y(x) = u(x) \cdot \log x + v(x)$ with $u(x) = \sum_{n\geq 0} u_n x^n$, $v(x) = \sum_{n\geq 0} v_n x^n$ and substituting them into (1.1.3), we have

$$\begin{aligned} &\log x \left[u'(x) - \alpha u(qx) + u(x)\right] = 0, \\ &v'(x) = \alpha v(qx) - v(x) + \widetilde{f}(x), \end{aligned} \tag{2.1.1}$$

where $\widetilde{f}(x) = \alpha(\ln q)u(qx) - u(x) \cdot \frac{1}{x} + f(x)$. The first equation of (2.1.1) is equivalent to (1.1.4), then

$$u_n = \frac{(-1)^n(\alpha;q)_n u_0}{n!},$$

where u_0 is an arbitrary constant. □

2.2 Laurent series solution at zero

Let $\Omega = D(O,R)$ be an open disk centered at zero with radius R on the complex plane. If $f(x) = x^\gamma\phi(x)$ ($\gamma \in \mathbb{C}$) with $\phi(x)$ is an analytic function on Ω and $\phi(0) \neq 0$, then one can write

$$f(x) = \sum_{n\geq 0} \phi_n x^{\gamma+n}.$$

We assume that $y = \sum_{n\geq 0} a_n x^{\gamma+n}$ ($a_0 \neq 0$) is a solution of Equation (1.1.3) and substitute it into the equation, then

$$\sum_{n\geq 0}(n+\gamma)a_n x^{n-1} + \sum_{n\geq 1}(1-\alpha q^{n+\gamma-1})a_{n-1}x^{n-1} = \sum_{n\geq 1}\phi_{n-1}x^{n-1}.$$

Comparing the coefficients of x^{n-1}, we get

$$\begin{aligned} &\gamma a_0 = 0, && n = 0, \\ &a_n = \frac{(-1)(1-\alpha q^{n+\gamma+1})a_{n-1} + \phi_{n-1}}{n+\gamma}, && n \geq 1. \end{aligned} \tag{2.2.1}$$

Equation (2.2.1) holds only if $\gamma = 0$. Consequently, we can categorize the values of γ into three distinct cases :

(i) $\gamma = 0$,
(ii) $\gamma \in \mathbb{C}^* \setminus \mathbb{Z}_{<0}$,
(iii) $\gamma \in \mathbb{Z}_{<0}$.

We will discuss each of these cases in the subsequent sections.

Remark 2.1. *If $\gamma \in \mathbb{Z}_{>0}$, then it falls under case (ii). From Equation (2.2.1), we observe that there are no non-zero solutions of the form $y = \sum_{n\geq 0} a_n x^n$. Therefore, we assume that a solution of Equation (1.1.3) can be expressed as $y(x) = x^{\gamma+c}\sum_{n\geq 0} a_n x^n$, where c is an undetermined constant. Substituting this expression into the equation yields :*

$$\sum_{n\geq 0}(n+\gamma+c)a_n x^{n+c-1} = \sum_{n\geq 0}(\alpha q^{n+\gamma+c}-1)a_n x^{n+c} + \sum_{n\geq 0}\phi_n x^n. \tag{2.2.2}$$

Comparing the coefficients of the constant term in (2.2.2), we find that $c = 1$. The solution $y(x) = x^{\gamma+1}\sum_{n\geq 0} a_n x^n$ can be written as $\sum_{n\geq 0} b_n x^n$, where $b_0 = b_1 = \cdots = b_\gamma = 0$, and $b_{n+\gamma+1} = a_n$ for $n \geq 0$. Similarly, $f = \sum_{n\geq 0}\phi_n x^{\gamma+n}$ is equivalent to $\sum_{n\geq 0}\psi_n x^n$, where $\psi_0 = \psi_1 = \cdots = \psi_{\gamma-1} = 0$, and $\psi_{n+\gamma} = \phi_n$ for $n \geq 0$. This leads us to conclude that the situation is analogous to the case where $\gamma = 0$.

2.2.1 Parameter $\gamma = 0$ in the non-homogeneous term

In this section, we will explore the existence and uniqueness of solutions for Equation (1.1.3). To analyze the analyticity of the solution, let Ω be a connected domain. Let f be an analytic function on Ω, denoted as $f \in \mathcal{A}(\Omega;\mathbb{C})$. Consider the n-th derivative of (1.1.3) :

$$y^{(n+1)}(x) = \alpha q^n y^{(n)}(qx) - y^{(n)}(x) + f^{(n)}(x), \quad n \in \mathbb{N}. \tag{2.2.3}$$

When $\alpha \notin q^{\mathbb{Z}_{\leq 0}}$, we introduce the series

$$F(\alpha;q,c,f,x) = c\cdot\sum_{n\geq 0}\frac{(-1)^n(\alpha;q)_n}{n!}x^n + \sum_{n\geq 1}\sum_{k=1}^{n}\frac{(-1)^{n-k}(\alpha;q)_n f^{(k-1)}(0)}{n!(\alpha;q)_k}x^n,$$

where $c \in \mathbb{C}$ is an arbitrary constant. Then, we have the following theorem.

Theorem 2.1. *Let $f \in \mathcal{A}(\Omega;\mathbb{C})$ and assume that $\alpha \notin q^{\mathbb{Z}_{\leq 0}}$. Given a constant $c_0 \in \mathbb{C}^*$, then the function $F(\alpha;q,c_0,f,x)$ is a unique analytic solution of Equation* (1.1.3) *with* $y(0) = c_0$ *on* Ω.

Proof. Since $f \in \mathcal{A}(\Omega;\mathbb{C})$, the function $f(x)$ can be expanded as a Taylor series on Ω :

$$f(x) = \sum_{n\geq 0}\frac{f^{(n)}(0)}{n!}x^n,$$

and $|f^{(j)}(0)| \leq Mj!(R')^{-j}$ ($j \in \mathbb{N}$) for $0 < R' < R$, where M is a positive constant. Let $x = 0$, Equation (2.2.3) becomes

$$y^{(n+1)}(0) = \alpha q^n y^{(n)}(0) - y^{(n)}(0) + f^{(n)}(0), \quad n \in \mathbb{N}.$$

We have

$$F = \sum_{n\geq 0}\frac{y^{(n)}(0)}{n!}x^n$$

with

$$y^{(n)}(0) = (-1)^n(\alpha;q)_n y(0) + \sum_{k=1}^{n}(-1)^{n-k}\frac{(\alpha;q)_n}{(\alpha;q)_k}f^{(k-1)}(0).$$

The radius of convergence for the series $\sum_{n\geq 0} \frac{(-1)^n(\alpha;q)_n y(0)}{n!} x^n$ is infinity. Then, we will calculate the radius of convergence of the series

$$\sum_{n\geq 1}\sum_{k=1}^{n} \frac{(-1)^{n-k}(\alpha;q)_n}{n!(\alpha;q)_k} f^{(k-1)}(0)x^n.$$

Let

$$u_n = \sum_{k=1}^{n} \frac{(-1)^{n-k}(\alpha;q)_n}{n!(\alpha;q)_k} f^{(k-1)}(0).$$

From $\lim_{n\to\infty} |(\alpha;q)_n| = \bar{M}$ ($\bar{M}$ is a positive constant), we have $M_2 \leq |(\alpha;q)_i| \leq M_1$ ($i \in \mathbb{N}$), where M_1, M_2 are positive constants. Therefore,

$$|u_n| = \Big|\sum_{k=1}^{n} \frac{(-1)^{n-k}(\alpha;q)_n}{n!(\alpha;q)_k} f^{(k-1)}(0)\Big| \leq \sum_{k=1}^{n} \frac{M_1 M(k-1)!}{M_2 n!(R')^{k-1}}$$

for sufficiently large n, we get

$$\frac{1}{n!}\sum_{k=1}^{n} \frac{M_2 M(k-1)!}{M_1(R')^{k-1}} \leq \frac{1}{n!}\sum_{k=1}^{n} \frac{M_2 M(n-1)!}{M_1(R')^{n-1}} \leq \frac{M_2 M}{M_1(R')^{n-1}}.$$

Since

$$\sup\left\{\frac{M_2 M}{M_1(R')^{n-1}}(R')^n\right\} = \frac{M_2 M R}{M_1},$$

the radius of convergence for the series $\sum_{n\geq 0} \frac{M_2 M}{M_1(R')^{n-1}} x^n$ is R'. Therefore, the radius of convergence of the series

$$\sum_{n\geq 1}\sum_{k=1}^{n} (-1)^{n-k} \frac{(\alpha;q)_n}{n!(\alpha;q)_k} f^{(k-1)}(0)x^n$$

is at least R'. Since R' is any constant satisfying $R' < R$, the radius of convergence of F is R, and for $x \in \Omega = D(O,R)$, the function F is a unique analytic solution of Equation (1.1.3) with $y(0) = c_0$. $\square$

Remark 2.2. *The parameter $\alpha \in q^{\mathbb{Z}_{\leq 0}}$ does not affect the result ; we just need to replace $\frac{(\alpha;q)_n}{(\alpha;q)_k}$ with $(\alpha q^{k+1};q)_{n-k}$.*

Remark 2.3. *If $\alpha \in q^{\mathbb{Z}_{\leq 0}}$, then there is a constant $k_0 \in \mathbb{N}$, such that $\alpha = q^{-k_0}$. Then we have the following properties :*

(1) $(\alpha;q)_n = 0$ for $n \geq k_0 + 1$;

(2) $(\alpha q^{k+1};q)_{n-k} = (1-\alpha q^k)\cdots(1-\alpha q^{n-1}) = 0$ *for* k,n *satisfying* $k \le k_0 \le n-1$, *and* $(\alpha q^{k+1};q)_{n-k} = \frac{(\alpha;q)_n}{(\alpha;q)_k} = 1$ *for* $k=n$.

The solution $F(\alpha;q,c_0,f,x)$ *can be written as the following form :*

$$c_0 \cdot \sum_{n=0}^{k_0} \frac{(-1)^n(\alpha;q)_n}{n!}x^n + \sum_{n=2}^{k_0}\sum_{k=1}^{n-1} \frac{(-1)^{n-k}(\alpha;q)_n f^{(k-1)}(0)}{n!(\alpha;q)_k}x^n$$
$$+ \sum_{n\ge k_0+2}\sum_{k=k_0+1}^{n-1} \frac{(-1)^{n-k}(\alpha;q)_n f^{(k-1)}(0)}{n!(\alpha;q)_k}x^n + \sum_{n\ge 1}\frac{f^{(n-1)}(0)}{n!}x^n.$$

If $k_0 < 2$, *then* $\sum_{n=2}^{k_0}\sum_{k=1}^{n-1} \frac{(-1)^{n-k}(\alpha;q)_n f^{(k-1)}(0)}{n!(\alpha;q)_k}x^n$ *becomes* 0.

From the above results, we obtain that, if $f \in \mathcal{A}(\Omega,\mathbb{C})$, then the radius of convergence of the Taylor series solution that satisfies the Equation (1.1.3) expanded at zero is R. In other words, for $f \in \mathcal{A}(\Omega,\mathbb{C})$, if y is a solution of Equation (1.1.3), then $y \in \mathcal{A}(\Omega,\mathbb{C})$.

Corollary 2.1. *Let* $E = \mathcal{A}(\Omega,\mathbb{C})$ *be the set of all analytic functions on* Ω, *and let* $L : E \to E$ *be an operator satisfying* $L(y) = y'(x) - \alpha y(qx) + y(x)$ $(\alpha \neq 0, 0 < q < 1)$, *then* L *is surjective but not injective. In other words, for any* $f \in E$, *there are an infinite number of solutions* $y \in E$, *such that* $L(y) = f$.

Proof. From Theorem 2.1 and its following Remark, one can obtain that the operator L is surjective. Since

$$y = c \cdot \sum_{n\ge 0} \frac{(-1)^n(\alpha;q)_n}{n!}x^n$$

is a solution of $L(y) = 0$ and c can be any constant, the operator L is not injective. □

Corollary 2.2. *Let* $\hat{E} = \mathcal{A}(\mathbb{C},\mathbb{C})$ *be the set of all entire functions, and let* $\hat{L} : \hat{E} \to \hat{E}$ *be the operator satisfying* $\hat{L}(y) = y'(x) - \alpha y(qx) + y(x)$ $(\alpha \neq 0, 0 < q < 1)$. *Then* $\hat{L}$ *is surjective but not injective. In other words, for any* $f \in \hat{E}$, *there are an infinite number of solutions* $y \in \hat{E}$, *such that* $\hat{L}(y) = f$.

Proof. Letting $R \to +\infty$ prove the conclusion, where R is as in the above. □

According to [67], we can calculate the index of the operator L, the definition is as follows.

Definition 2.1. *Let* E *and* F *be two vector spaces. The linear operator* $u : E \to F$ *has an index if the dimensions of the Spaces* $Ker(u)$ *and* $Coker(u)$ *are finite. The index of the operator* u *is defined as :*

$$\chi(u) = \dim Ker(u) - \dim Coker(u).$$

Remark 2.4. *Since L is surjective, we have* $\dim Coker(L) = 0$. *From*

$$Ker(L) = \Big\{c \cdot \sum_{n\geq 0} \frac{(-1)^n(\alpha;q)_n}{n!}x^n : c \in \mathbb{C}\Big\},$$

we obtain that $\dim Ker(L) = 1$. *Therefore,*

$$\chi(L) = \dim Ker(L) - \dim Coker(L) = 1,$$

and the index of $\hat{L}$ is also 1.

2.2.2 Parameter $\gamma \in \mathbb{C}^* \setminus \mathbb{Z}_{<0}$ in the non-homogeneous term

To demonstrate the convergence of the series solution derived later, we first introduce an estimate of the gamma function, as outlined in the following lemma.

Lemma 2.2. *Let $a, b \in \mathbb{C}$. Then*

$$\frac{\Gamma(a+n)}{\Gamma(b+n)} \sim n^{a-b}$$

as $n \to +\infty$.

Proof. From the Stirling formula of the gamma function :

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\Big(\frac{z}{e}\Big)^z\Big(1 + O\big(\frac{1}{z}\big)\Big),$$

we have

$$\begin{aligned}
\frac{\Gamma(a+n)}{\Gamma(b+n)} \sim & \sqrt{\frac{b+n}{a+n}}\Big(\frac{a+n}{e}\Big)^{a+n}\Big/\Big(\frac{b+n}{e}\Big)^{b+n} \\
\sim & \exp\{b - a + (a+n)\log(a+n) - (b+n)\log(b+n)\} \\
= & e^{b-a}\exp\Big\{a\log(a+n) - b\log(b+n) + n\log\Big(\frac{a+n}{b+n}\Big)\Big\} \\
= & e^{b-a}\exp\Big\{a\log(1+\frac{a}{n}) - b\log(1+\frac{b}{n}) + (a-b)\log n + \log\Big(1+\frac{a-b}{n+b}\Big)^n\Big\}
\end{aligned}$$

as $n \to +\infty$. Both $\log(1+\frac{a}{n})$ and $\log(1+\frac{b}{n})$ tend to 0 as $n \to +\infty$, and

$$\begin{aligned}
\lim_{n\to+\infty}\log\Big(1+\frac{a-b}{n+b}\Big)^n = & \lim_{n\to+\infty}\log\Big(1+\frac{a-b}{n+b}\Big)^{\frac{n+b}{a-b}\cdot\frac{a-b}{1+b/n}} \\
= & \lim_{n\to+\infty}\log e^{\frac{a-b}{1+b/n}} \\
= & a - b.
\end{aligned}$$

Then $\frac{\Gamma(a+n)}{\Gamma(b+n)} \sim n^{a-b}$ as $n \to +\infty$. $\square$

If $\gamma \in \mathbb{C}^* \backslash \mathbb{Z}_{<0}$ and $\alpha \notin q^{-\gamma+\mathbb{Z}_{<0}}$, we assume that $y(x) = x^{\gamma+1}\sum_{n\geq 0} a_n x^n$ and substitute it into (1.1.3), where $f(x) = x^\gamma \sum_{n\geq 0}\phi_n x^n$ ($\gamma \notin \mathbb{Z}_{<0}, \phi_0 \neq 0$), then

$$\sum_{n\geq 0}(\gamma+n+1)a_n x^n = \sum_{n\geq 1}(\alpha q^{\gamma+n}-1)a_{n-1}x^n + \sum_{n\geq 0}\phi_n x^n.$$

Then

$$\begin{aligned}
&(\gamma+1)a_0 = \phi_0, && n=0,\\
&a_n = \frac{(\alpha q^{\gamma+n}-1)a_{n-1}+\phi_n}{\gamma+n+1}, && n\geq 1.
\end{aligned}$$

One can get that

$$\begin{aligned}
a_n =& \frac{(-1)^n(\alpha q^{1+\gamma};q)_n}{(\gamma+1)_{n+1}}\phi_0 + \sum_{k=1}^{n}\frac{(-1)^{n-k}(\gamma+2)_{k-1}(\alpha q^{1+\gamma};q)_n}{(\gamma+2)_n(\alpha q^{1+\gamma};q)_k}\phi_k\\
=&\sum_{k=0}^{n}\frac{(-1)^{n-k}(\gamma+1)_k(\alpha q^{1+\gamma};q)_n}{(\gamma+1)_{n+1}(\alpha q^{1+\gamma};q)_k}\phi_k.
\end{aligned}$$

If $\alpha \in q^{-\gamma+\mathbb{Z}_{<0}}$, we only need to replace $\frac{(\alpha q^{1+\gamma};q)_n}{(\alpha q^{1+\gamma};q)_k}$ with $(\alpha q^{\gamma+k+1};q)_{n-k}$. Define

$$\widetilde{F}_1(\alpha;q,\gamma,\phi,x) = \sum_{n\geq 0}a_n x^n = \sum_{n\geq 0}\sum_{k=0}^{n}\frac{(-1)^k(\gamma+1)_k(\alpha q^{1+\gamma};q)_n}{(\gamma+1)_{n+1}(\alpha q^{1+\gamma};q)_k}\phi_k(-x)^n,$$

we have the following theorem.

Theorem 2.2. *Let $f(x) = x^\gamma\phi(x)$ ($\gamma \in \mathbb{C}^* \setminus \mathbb{Z}_{<0}, \phi(0) \neq 0$) and $\phi(x) = \sum_{n\geq 0}\phi_n x^n \in \mathcal{A}(\Omega,\mathbb{C})$. Then $F_1(\alpha;q,\gamma,\phi,x) = x^{\gamma+1}\widetilde{F}_1(\alpha;q,\gamma,\phi,x)$ is an analytic solution of Equation (1.1.3) on $\{x \in \widetilde{\mathbb{C}}^* : |x| < R\}$, and $\widetilde{F}_1(\alpha;q,\gamma,\phi,x) \in \mathcal{A}(\Omega,\mathbb{C})$.*

Proof. We only need to prove that $\widetilde{F}_1(\alpha;q,\gamma,\phi,x) \in \mathcal{A}(\Omega,\mathbb{C})$. Since $\phi(x) \in \mathcal{A}(\Omega,\mathbb{C})$, we have $|\phi_k| \leq M(R')^{-k}$ ($k \in \mathbb{N}$) for any $0 < R' < R$, where M is a positve constant. Let $M_2 \leq |(\alpha q^{1+\gamma};q)_i| \leq M_1$ ($i \in \mathbb{N}$), where M_1, M_2 are positve constants, we have

$$\Big|\sum_{k=0}^{n}\frac{(-1)^k(\gamma+1)_{k-1}(\alpha q^{1+\gamma};q)_n}{(\gamma+1)_n(\alpha q^{1+\gamma};q)_k}\phi_k\Big| \leq \sum_{k=0}^{n}\frac{M_1 M(|\gamma|+1)_{k-1}}{M_2|(\gamma+1)_n|(R')^k}.$$

If

$$\frac{(|\gamma|+1)_k(R')^k}{(|\gamma|+1)_{k-1}(R')^{k+1}} = \frac{|\gamma|+k}{R'} \geq 1,$$

then $k \geq R' - |\gamma|$. Therefore, for $k \geq R' - |\gamma|$, the function $\frac{(|\gamma|+1)_{k-1}}{(R')^k}$ is increasing with respect to k. Then

$$\frac{(|\gamma|+1)_{n-1}}{(R')^n} = \max\left\{\frac{(|\gamma|+1)_{k-1}}{(R')^k}\right\}$$

for sufficinetly large n. From Lemma 2.2, we get

$$\begin{aligned}\sum_{k=0}^{n}\frac{M_1M(|\gamma|+1)_{k-1}}{M_2|(\gamma+1)_n|(R')^k} \leq & \frac{M_1M}{M_2}\sum_{k=0}^{n}\frac{(|\gamma|+1)_{n-1}}{|(\gamma+1)_n|(R')^n} \\ \leq & \frac{M_1M}{M_2}\sum_{k=0}^{n}\frac{\Gamma(|\gamma|+n+1)}{|\Gamma(\gamma+n+1)|(R')^n} \\ \leq & C(n+1)n^{|\gamma|-\gamma}(R')^{-n}\end{aligned}$$

as $n \to +\infty$. Then the radius of convergence of $\widetilde{F}_1(\alpha; q, \gamma, \phi, x)$ is R. $\square$

From (2.1), we have

$$b_n = (-1)^n\frac{(\alpha;q)_n b_0}{n!} + \sum_{k=1}^{n}\frac{(-1)^{n-k}(\alpha;q)_n(k-1)!}{(\alpha;q)_k n!}\psi_{k-1}.$$

Together with $b_0 = 0$ and $\psi_0 = \psi_1 = \cdots = \psi_{\gamma-1} = 0$, we obtain that the above expression for b_n is equivalent to

$$\sum_{k=\gamma+1}^{n}\frac{(-1)^{n-k}(\alpha;q)_n(k-1)!}{(\alpha;q)_k n!}\psi_{k-1} = \sum_{k=0}^{n-\gamma-1}\frac{(-1)^{n-k-\gamma-1}(\alpha;q)_n(k+\gamma)!}{(\alpha;q)_{k+\gamma+1}n!}\psi_{k+\gamma}.$$

Then $\sum_{n\geq 0} b_n x^n = \sum_{n\geq 0}\sum_{k=0}^{n}\frac{(-1)^{n-k}(\alpha;q)_{n+\gamma+1}(k+\gamma)!}{(\alpha;q)_{k+\gamma+1}(n+\gamma+1)!}\psi_{k+\gamma}x^{n+\gamma+1}$. Since $\psi_{k+\gamma} = \phi_k$, the function

$$\hat{F}(\alpha;q,\gamma,\phi,x) = x^{\gamma+1}\sum_{n\geq 0}\sum_{k=0}^{n}\frac{(-1)^{n-k}(\alpha;q)_{n+\gamma+1}(k+\gamma)!}{(\alpha;q)_{k+\gamma+1}(n+\gamma+1)!}\phi_k x^n$$

is an analytic solution of Equation (1.1.3) on Ω ($\phi(x)$ is an analytic function on Ω).

Remark 2.5. *For $\gamma \in \mathbb{Z}_{>0}$, the functions $F_1(\alpha; q, \gamma, \phi, x)$ and $\hat{F}(\alpha; q, \gamma, \phi, x)$ have the same expression.*

For the operator $L : E \to E$ satisfying $L(y) = y'(x) - \alpha y(qx) + y(x)$ ($\alpha \neq 0, 0 < q < 1$), let $L(y) = f$ ($f = x^\gamma\phi$), it has a solution of the form $y = x^{\gamma+1}u$. Substituting $y = x^{\gamma+1}u$ into $L(y)$, we get

$$L(x^{\gamma+1}u) = (\gamma+1)x^\gamma u(x) + x^{\gamma+1}u'(x) - \alpha q^{\gamma+1}x^{\gamma+1}u(qx) + x^{\gamma+1}u(x).$$

Then $L(y) = f$ is equivalent to

$$\widetilde{L}(u) = u'(x) - \alpha q^{\gamma+1} u(qx) + (1 + \frac{\gamma+1}{x})u(x),$$

where $\widetilde{L}(u) = x^{-(\gamma+1)} L(x^{\gamma+1} u)$.

Corollary 2.3. *Let $E = \mathcal{A}(\Omega, \mathbb{C})$ be the set of all analytic functions on Ω, let $\widetilde{L} : E \to E$ be the operator such that $\widetilde{L}(u) = u'(x) - \alpha q^{\gamma+1} u(qx) + (1 + \frac{\gamma+1}{x})u(x)$ $(\alpha \neq 0, 0 < q < 1, \gamma \notin \mathbb{Z}_{\leq 0})$, then $\widetilde{L}$ is a surjective but not injective. In other words, for any $\phi \in E$, there exist an infinite number of solutions $u \in E$, such that $\widetilde{L}(u) = \phi$.*

Since $\widetilde{L}$ is a surjective, we have $\dim Coker(\widetilde{L}) = 0$. From

$$Ker(\widetilde{L}) = \left\{ c\widetilde{F}_1(\alpha; q, \gamma, f, x) : c \in \mathbb{C} \right\},$$

we get $\dim Ker(\widetilde{L}) = 1$. From the definition 2.1, one can obatin that $\chi(\widetilde{L}) = 1$.

2.2.3 Parameter $\gamma \in \mathbb{Z}_{<0}$ in the non-homogeneous term

With regard to parameters of non-homogeneous terms, we note that Theorem 2.2 holds under the condition $\gamma \notin \mathbb{Z}_{\leq 0}$, and the case of $\gamma = 0$ has already been discussed. Therefore, we next consider the case of $\gamma \in \mathbb{Z}_{<0}$. To facilitate the analysis, we first discuss the case where $\gamma = -1$.

2.2.3.1 Parameter $\gamma = -1$

For $\alpha \notin q^{\mathbb{Z}_{\leq 0}}$, define

$$\begin{aligned} F_{11} &= \sum_{n\geq 1} \sum_{k=1}^{n} \frac{(-1)^{n-1}(\alpha;q)_n [1 + \alpha q^{k-1}(k \ln q - 1)]}{n! k (1 - \alpha q^{k-1})} x^n, \\ F_{12} &= \sum_{n\geq 1} \sum_{k=1}^{n} \frac{(-1)^{n-k}(\alpha;q)_n (k-1)!}{(\alpha;q)_k n!} \phi_k x^n. \end{aligned} \tag{2.2.4}$$

If $\alpha \in q^{\mathbb{Z}_{\leq 0}}$, we only need to replace $\frac{(\alpha;q)_n}{1-\alpha q^{k-1}}$ in F_{11} with

$$\begin{array}{ll} (1-\alpha q)(1-\alpha q^2)\cdots(1-\alpha q^{n-1}) & k = 1, \\ (1-\alpha)\cdots(1-\alpha q^{k-2})(1-\alpha q^k)\cdots(1-\alpha q^{n-1}) & k = 2, 3, \cdots, n-1, \end{array}$$

and replace $\frac{(\alpha;q)_n}{(\alpha;q)_k}$ in F_{12} with $(\alpha q^k; q)_{n-k}$. We get the following theorem.

Theorem 2.3. *Let $f(x) = x^{-1}\phi(x)$, where $\phi(x) = \sum_{n\geq 0}\phi_n x^n \in \mathcal{A}(\Omega,\mathbb{C})$ ($\phi(0) \neq 0$). Given $v_0 \in \mathbb{C}$, the function $\bar{F}(\alpha; q, v_0, \phi, x) = \phi_0 \log x \cdot F_0 + v_0 F_0 + \phi_0 F_{11} + F_{12}$ defined on $\widetilde{\mathbb{C}}^*$ is a unique analytic solution of Equation* (1.1.3) *that satisfies the initial asymptotic property $y(x) = \phi_0 \log x + v_0 + o(1)$ ($x \to 0$). The function $\bar{F}$ is analytic on $\widetilde{\Omega} = \{x \in \widetilde{\mathbb{C}}^* : |x| < R\}$.*

Proof. We suppose the solution is of the form $y(x) = u(x)\cdot \log x + v(x)$. From Lemma 2.1, the series

$$u(x) = \sum_{n\geq 0} \frac{(-1)^n(\alpha;q)_n u_0}{n!} x^n$$

is a solution of homogeneous Equation (1.1.4). For $f = \sum_{n\geq 0}\phi_n x^{n-1}$, substituting $v(x) = \sum_{n\geq 0} v_n x^n$ into the second equation of (2.1.1), we have

$$\begin{aligned}\sum_{n\geq 1} n v_n x^{n-1} = &\sum_{n\geq 1}(\alpha q^{n-1} - 1)v_{n-1}x^{n-1} + \alpha \ln q \sum_{n\geq 1} u_{n-1}q^{n-1}x^{n-1} \\ &- \sum_{n\geq 0} u_n x^{n-1} + \sum_{n\geq 0}\phi_n x^{n-1}.\end{aligned}$$

When $n = 0$, the above equation becoms $u_0 = \phi_0$. When $n \geq 1$, conparing the coefficients of x^{n-1}, one can get that

$$\begin{aligned} v_n &= \frac{(-1)(1-\alpha q^{n-1})v_{n-1}}{n} + \frac{\alpha \ln q \cdot u_{n-1}q^{n-1} - u_n}{n} + \frac{\phi_n}{n} \\ &= \frac{(-1)(1-\alpha q^{n-1})v_{n-1}}{n} + \frac{(-1)^{n-1}(\alpha;q)_n \phi_0[1 + \alpha q^{n-1}(n\ln q - 1)]}{nn!(1-\alpha q^{n-1})} + \frac{\phi_n}{n} \\ &= \cdots \\ &= \frac{(-1)^n(\alpha;q)_n v_0}{n!} + \phi_0 \sum_{k=1}^{n} \frac{(-1)^{n-1}(\alpha;q)_n[1+\alpha q^{k-1}(k\ln q - 1)]}{n!k(1-\alpha q^{k-1})} \\ &\quad + \sum_{k=1}^{n} \frac{(-1)^{n-k}(\alpha;q)_n (k-1)!}{(\alpha;q)_k n!}\phi_k. \end{aligned}$$

The proof is completed. $\square$

It can be easily calculated that the radius of convergence of series F_0 and F_{11} are infinite. If the radius of convergence of $\phi(x)$ is R, then the radius of convergence of series F_{12} is also R.

2.2.3.2 Parameter $\gamma \in \mathbb{Z}_{<0} \setminus \{-1\}$

When $\gamma \in \mathbb{Z}_{<0}$ and $\gamma \neq -1$, we have a similar result to the case $\gamma = -1$.

Theorem 2.4. *Let $f(x) = x^\gamma \phi(x)$, where $\gamma = -k_0$ and k_0 is a positive integer that is not less than 2, $\phi(x) = \sum_{n\geq 0} \phi_n x^n \in \mathcal{A}(\Omega, \mathbb{C})$ $(\phi(0) \neq 0)$. Then*

$$\begin{aligned} &F_{k_0}(\alpha; q, \phi, x) \\ &\quad = u_0 \log x \sum_{n\geq 0} \frac{(-1)^n (\alpha; q)_n}{n!} x^n + \sum_{n=0}^{k_0-2} \sum_{k=0}^{n} \frac{(-1)^{n-k} (\alpha q^{k-k_0+1}; q)_{n-k}}{(k-k_0+1)_{n-k+1}} \phi_k x^n + \sum_{n\geq 0} w_n x^n \end{aligned}$$

is a solution of Equation (1.1.3) *if and only if $u_0 \neq 0$. The function $F_{k_0}(\alpha; q, \phi, x)$ is analytic on $\widetilde{\Omega} = \{x \in \widetilde{\mathbb{C}}^* : |x| < R\}$, where*

$$u_0 = \sum_{k=0}^{k_0-1} \frac{(-1)^{k_0-k-1} (\alpha q^{k-k_0+1}; q)_{k_0-k-1}}{(k-k_0+1)_{k_0-k-1}} \phi_k.$$

The expression for w_n is shown as (2.2.7).

Proof. For $\gamma = -k_0$, we assume that $y(x) = u(x) \cdot \log x + v(x)$, where

$$u(x) = \sum_{n\geq 0} u_n x^n, \quad v(x) = \frac{P(x)}{x^{k_0-1}} + w(x),$$

the function $w(x)$ is a power series, and $P(x)$ is a polynomial with the highest degree k_0-2. Write $w(x) = \sum_{n\geq 0} w_n x^n$ and $P(x) = \sum_{n=0}^{k_0-2} p_n x^n$. Substituting them into Equation (1.1.3), it has the same form as Equation (2.1.1), where $\hat{f}(x) = \alpha \ln q \cdot u(qx) - u(x) \cdot \frac{1}{x} + f(x)$ $(f(x) = \sum_{n\geq 0} \phi_n x^{n-k_0})$. the function $u(x) = u_0 \sum_{n\geq 0} \frac{(-1)^n (\alpha; q)_n}{n!} x^n$. Equation

$$v'(x) = \alpha v(qx) - v(x) + \hat{f}(x)$$

becomes

$$\frac{(-k_0+1)P(x)}{x^{k_0}} + \frac{P'(x)}{x^{k_0-1}} + w'(x) = \frac{\alpha P(qx)}{q^{k_0-1} x^{k_0-1}} + \alpha w(qx) - \frac{P(x)}{x^{k_0-1}} - w(x) + \hat{f}(x).$$

Then

$$(-k_0+1)P(x) + xP'(x) + x^{k_0} w'(x) = \alpha q^{-k_0+1} x P(qx) + \alpha x^{k_0} w(qx) - xP(x) - x^{k_0} w(x) + x^{k_0} \hat{f}(x).$$

Since $P(x) = \sum_{n=0}^{k_0-2} p_n x^n$ is a polynomial with the highest degree k_0-2, we have $p_{k_0-2} \neq 0$. Divide the above equation into two parts, one containing $x^0, x^1, \cdots, x^{k_0-1}$, and the other terms forming the other part. The above equation is equivalent to

$$(-k_0+1) \sum_{n=0}^{k_0-2} p_n x^n + \sum_{n=0}^{k_0-2} n p_n x^n = \sum_{n=1}^{k_0-1} (\alpha q^{n-k_0} - 1) p_{n-1} x^n - u_0 x^{k_0-1} + \sum_{n=0}^{k_0-1} \phi_n x^n \quad (2.2.5)$$

and

$$\sum_{n\geq 1} nw_n x^{n+k_0-1} = \sum_{n\geq 1}(\alpha q^{n-1}-1)w_{n-1}x^{n+k_0-1} + \alpha\ln q\sum_{n\geq 1}u_{n-1}q^{n-1}x^{n+k_0-1} - \sum_{n\geq 1}u_n x^{n+k_0-1} + \sum_{n\geq 1}\phi_{n+k_0-1}x^{n+k_0-1}. \tag{2.2.6}$$

For $n = 0, 1, \cdots, k_0 - 2$, comparing the coefficients of x^n in (2.2.5), we get

$$(-k_0+1)p_0 = b_0, \qquad n = 0,$$
$$p_n = \frac{(\alpha q^{n-k_0}-1)p_{n-1}+\phi_n}{n-k_0+1}, \qquad n = 1, 2, \cdots, k_0-2.$$

Comparing the coefficients of x^{k_0-1} in (2.2.5), we get

$$(\alpha q^{-1}-1)p_{k_0-2} - u_0 + \phi_{k_0-1} = 0.$$

Then

$$p_n = \sum_{k=0}^{n}(-1)^{n-k}\frac{(\alpha q^{k-k_0+1};q)_{n-k}}{(k-k_0+1)_{n-k+1}}\phi_k, \qquad n = 0, 1, \cdots, k_0-2,$$

and

$$u_0 = (\alpha q^{-1}-1)p_{k_0-2} + \phi_{k_0-1} = \sum_{k=0}^{k_0-1}\frac{(-1)^{k_0-k-1}(\alpha q^{k-k_0+1};q)_{k_0-k-1}}{(k-k_0+1)_{k_0-k-1}}\phi_k.$$

Therefore,

$$P(x) = \sum_{n=0}^{k_0-2}\sum_{k=0}^{n}(-1)^{n-k}\frac{(\alpha q^{k-k_0+1};q)_{n-k}}{(k-k_0+1)_{n-k+1}}b_k x^n.$$

Comparing the coefficients of x^i ($i \geq k_0$) in (2.2.6), we have

$$w_1 = (\alpha-1)w_0 + \alpha(\ln q)u_0 - u_1 + \phi_{k_0},$$
$$w_n = \frac{1}{n}\left[(\alpha q^{n-1}-1)w_{n-1} + \alpha(\ln q)q^{n-1}u_{n-1} - u_n + \phi_{n+k_0-1}\right].$$

Then

$$w_n = \frac{(-1)^n(\alpha;q)_n}{n!}w_0 + \frac{(-1)^{n-1}(\alpha q;q)_{n-1}}{n!}\phi_{k_0} + \sum_{k=1}^{n-1}\frac{(-1)^{n-k-1}k!(\alpha q;q)_{n-1}}{n!(\alpha q;q)_k}\phi_{k+k_0} + \sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}k!(\alpha q;q)_{n-1}}{n!(\alpha q;q)_k}[\alpha(\ln q)q^k u_k - u_{k+1}].$$

Substituting the expression of $u_n : \sum_{n\geq 0} \frac{(-1)^n(\alpha;q)_n}{n!}$, we obtain that

$$\begin{aligned} w_n =& \frac{(-1)^n(\alpha;q)_n}{n!} w_0 + \frac{(-1)^{n-1}(\alpha q;q)_{n-1}}{n!}\phi_{k_0} + \sum_{k=1}^{n-1} \frac{(-1)^{n-k-1}k!(\alpha q;q)_{n-1}}{n!(\alpha q;q)_k}\phi_{k+k_0} \\ &+ \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}k!(\alpha q;q)_{n-1}}{n!(\alpha q;q)_k}\Big[\frac{(-1)^k\alpha(\ln q)(\alpha;q)_k u_0}{k!} - \frac{(-1)^{k+1}(\alpha;q)_{k+1}u_0}{(k+1)!}\Big] \\ =& \frac{(-1)^n(\alpha;q)_n}{n!}(w_0+u_0) + \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}k!(\alpha q;q)_{n-1}}{n!(\alpha q;q)_k}\phi_{k+k_0} \\ &+ u_0 \sum_{k=1}^{n} \frac{(-1)^{n-1}(\alpha q;q)_{n-1}[\alpha q^{k-1}(k\ln q - 1)+1]}{n!}. \end{aligned} \tag{2.2.7}$$

Similar to the previous proof, the radius of convergence of the series in the function $F_{k_0}(\alpha;q,\phi,x)$ is the same as that of the series $\phi(x)$. □

Remark 2.6. *Under the same conditions as the theorem 2.4, if*

$$u_0 = \sum_{k=0}^{k_0-1} \frac{(-1)^{k_0-k-1}(\alpha q^{k-k_0+1};q)_{k_0-k-1}}{(k-k_0+1)_{k_0-k-1}}\phi_k = 0,$$

then for any constant w_0, the function

$$\begin{aligned} \hat{F}_{k_0}(\alpha;q,\phi,x) =& w_0 \sum_{n\geq 0} \frac{(-1)^n(\alpha;q)_n}{n!}x^n + \sum_{n=0}^{k_0-2}\sum_{k=0}^{n}(-1)^{n-k}\frac{(\alpha q^{k-k_0+1};q)_{n-k}}{(k-k_0+1)_{n-k+1}}\phi_k x^n \\ &+ \sum_{n\geq 1}\sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}k!(\alpha q;q)_{n-1}}{n!(\alpha q;q)_k}\phi_{k+k_0}x^n \end{aligned}$$

is the analytic solution of Equation (1.1.3) *on* $\widetilde{\Omega}$.

Corollary 2.4. *Let $\phi(x) = \sum_{n\geq 0} a_n x^n$ be the formal series, and $E_0 = \{x^\gamma\phi(x) \big| \gamma \in \mathbb{Z}_{<0}\}$. Define $H_0 = \log xE_0 + E_0$. If $L : H_0 \to E_0$ is the operator such that $L(y) = y'(x) - \alpha y(qx) + y(x)$ ($\alpha \neq 0, 0 < q < 1$), then L is surjective but not injective. In other words, for any $f \in E_0$, there are an infinite number of $y \in H_0$, such that $L(y) = f$.*

2.3 Laurent series solution at infinity

In order to study the series solution at infinity, we assume that the non-homogeneous term $f(x)$ can be represented by a series expanded at infinity, that is : $f = x^\delta\psi(x)$, where

$\psi(x) = \sum_{n\geq 0} \psi_n x^{-n}$ is a series expanded at infinity, and $\psi(\infty) \neq 0$. We first consider the case $\delta = 0$. In this case, Equation (1.1.3) becomes

$$y'(x) = \alpha y(qx) - y(x) + \sum_{n\geq 0} \psi_n x^{-n}. \tag{2.3.1}$$

For α, given

$$\lambda_m = \frac{-\log\alpha + 2m\pi\mathrm{i}}{\ln q},$$

where $-\log\alpha$ is the principal value of $\arg(q^{\lambda_m})$, i.e., $\alpha q^{\lambda_m} - 1 = 0$.

2.3.1 Parameter $\alpha \notin q^{\mathbb{Z}_{\geq 0}}$

When $\alpha \notin q^{\mathbb{Z}_{\geq 0}}$. Supposing $y = x^\lambda \sum_{n\geq 0} c_n x^{-n} + \sum_{n\geq 0} a_n x^{-n}$ and substituting it into (2.3.1), we obtain that

$$\begin{aligned}&\sum_{n\geq 1}(\lambda - n + 1)c_{n-1}x^{\lambda-n} + \sum_{n\geq 2}(-n+1)a_{n-1}x^{-n}\\ &= \sum_{n\geq 0}(\alpha q^{\lambda-n} - 1)c_n x^{\lambda-n} + \sum_{n\geq 0}(\alpha q^{-n} - 1)a_n x^{-n} + \sum_{n\geq 0}\psi_n x^{-n}.\end{aligned}$$

Comparing the coefficients of $x^{\lambda-n}$ on both sides of the equation, we obtain

$$\begin{aligned}&(\alpha q^\lambda - 1)c_0 = 0,\\ &c_n = \frac{(-\lambda + n - 1)}{1 - \alpha q^{\lambda-n}}c_{n-1}, \quad n \geq 1.\end{aligned}$$

Then

$$\lambda = \lambda_m = \frac{-\log\alpha + 2m\pi\mathrm{i}}{\ln q}, \quad m \in \mathbb{Z}.$$

From the expression of α and λ_m shown in the Introduction, one can obtain that,

$$c_n = \frac{-\lambda + n - 1}{1 - \alpha q^{-n}}c_{n-1} = \frac{(-\lambda_m)_n}{(q^{-1}; q^{-1})_n}c_0$$

for $n \geq 1$, where c_0 is an arbitrary constant.

Define $a_{-1} = 0$. Comparing the coefficients of x^{-n}, we have

$$a_n = \frac{(n-1)a_{n-1} + \psi_n}{1 - \alpha q^{-n}}, \quad n \geq 0.$$

Then

$$a_n = \sum_{k=0}^{n} \frac{(k)_{n-k}}{(\alpha q^{-k}; q^{-1})_{n-k+1}}\psi_k.$$

We obtain the conclusion as follows.

Lemma 2.3. *If $\alpha \notin q^{\mathbb{Z}_{\geq 0}}$, and $f = \psi(x)$ is the series expanded at infinity ($\psi(x) = \sum_{n\geq 0}\psi_n x^{-n}$, $\psi(\infty) \neq 0$), then*

$$G(\alpha; q, \lambda_m, c_0, f, x) = c_0 x^{\lambda_m} \sum_{n\geq 0} \frac{(-\lambda_m)_n}{(q^{-1}; q^{-1})_n} x^{-n} + \sum_{n\geq 0}\sum_{k=0}^{n} \frac{(k)_{n-k}}{(\alpha q^{-k}; q^{-1})_{n-k+1}} \psi_k x^{-n}$$

is a solution of Equation (1.1.3)*, where c_0 is an arbitrary constant. The series*

$$c_0 x^{\lambda_m} \sum_{n\geq 0} \frac{(-\lambda_m)_n}{(q^{-1}; q^{-1})_n} x^{-n}$$

is analytic on $\widetilde{\mathbb{C}}^*$. The radius of convergence of the series $\sum_{n\geq 0}\sum_{k=0}^{n} \frac{(k)_{n-k}}{(\alpha q^{-k}; q^{-1})_{n-k+1}} \psi_k x^{-n}$ is the same as that of $\psi(x)$.

Proof. Since

$$\lim_{n\to\infty} \left| \frac{(-\lambda_m)_{n+1}}{(q^{-1}; q^{-1})_{n+1}} \cdot \frac{(q^{-1}; q^{-1})_n}{(-\lambda_m)_n} \right| = \lim_{n\to\infty} \left| \frac{-\lambda_m + n}{1 - q^{-n-1}} \right| = 0,$$

the radius of convergence of the series $x^{\lambda_m} \sum_{n\geq 0} \frac{(-\lambda_m)_n}{(q^{-1};q^{-1})_n} c_0 x^{-n}$ is infinity. Then we prove that the radius of convergence of the series

$$\sum_{n\geq 0}\sum_{k=0}^{n} \frac{(k)_{n-k}}{(\alpha q^{-k}; q^{-1})_{n-k+1}} \psi_k x^{-n}$$

is the same as that of the series $\psi(x)$. Since $\psi(x)$ is analytic at infinity, there is a constant $R > 0$, such that $\psi(x) = \sum_{n\geq 0}\psi_n x^{-n}$, and $|b_j| \leq M(R')^{-j}$ $(j \in \mathbb{N})$ for any $0 < R' < R$. One can obtain that the radius of convergence of $G(\alpha; q, \lambda_m, f, x)$ is also $\frac{1}{R}$. $\square$

From the lemma above, we derive the corresponding result for any δ.

Theorem 2.5. *If $\alpha \notin q^{-\delta+\mathbb{Z}_{\geq 0}}$ and $f = x^{\delta}\psi(x)$ ($\psi(\infty) \neq 0$) is a series expanded at infinity, then*

$$\hat{G}(\alpha; q, \lambda_m, c_0, f, x) = c_0 x^{\lambda_m} \sum_{n\geq 0} \frac{(-\lambda_m)_n}{(q^{-1}; q^{-1})_n} x^{-n} + \sum_{n\geq 0}\sum_{k=0}^{n} \frac{(-\delta + k)_{n-k}}{(\alpha q^{\delta-k}; q^{-1})_{n-k+1}} \psi_k x^{\delta-n}$$

is a solution of Equation (1.1.3)*, where c_0 is an arbitrary constant. The series*

$$c_0 x^{\lambda_m} \sum_{n\geq 0} \frac{(-\lambda_m)_n}{(q^{-1}; q^{-1})_n} x^{-n}$$

is analytic on $\widetilde{\mathbb{C}}^*$, *the radius of convergence of the series*

$$\sum_{n\geq 0}\sum_{k=0}^{n}\frac{(-\delta+k)_{n-k}}{(\alpha q^{\delta-k};q^{-1})_{n-k+1}}\psi_k x^{\delta-n}$$

is the same as that of $\psi(x)$.

In the following, we look for solutions expressed by power series. Introduce the following series :

$$G_\delta(\alpha;q,\psi,x)=x^\delta\sum_{n\geq 0}\sum_{k=0}^{n}\frac{(-1)^n(\alpha q^\delta;q^{-1})_k(-\delta+k)_{n-k}\psi_k}{(\alpha q^\delta;q^{-1})_{n+1}}x^{-n},$$

where

$$\begin{aligned}&(-\delta+k)_{n-k}=(-\delta+n-1)(-\delta+n-2)\cdots(-\delta+k), && k=1,2,\cdots,n-1,\\ &(-\delta+k)_{n-k}=1, && k=n.\end{aligned}$$

Theorem 2.6. *If* $\alpha\notin q^{-\delta+\mathbb{Z}_{>0}}$, *and the non-homogeneous term* $f=x^\delta\psi(x)$ $(\psi(\infty)\neq 0)$ *with* $\psi(x)$ *is the series expanded at infinity, then* $G_\delta(\alpha;q,\psi,x)$ *is a solution of Equation* (1.1.3), *and the radius of convergence of the power series* $G_\delta(\alpha;q,\psi,x)$ *is the same as that of* $\psi(x)$.

Proof. Since $\psi(x)$ is analytic at infinity, we have $\psi(x)=\sum_{n\geq 0}\psi_n x^{-n}$, and there exists a positive constant R, such that $|\psi_k|\leq M(R')^{-k}$ $(k\in\mathbb{N})$ for constant $M>0$ and any $0<R'<R$. Letting $y=x^\delta\sum_{n\geq 0}a_n x^{-n}$ and putting it into (1.1.3), we have

$$(\delta-n)\sum_{n\geq 0}a_n x^{\delta-1-n}=\alpha\sum_{n\geq 0}a_n(qx)^{\delta-n}-\sum_{n\geq 0}a_n x^{\delta-n}+\sum_{n\geq 0}\psi_n x^{\delta-n}.$$

Comparing the coefficients of x^δ, we get $(\alpha q^\delta-1)a_0+\psi_0=0$. Comparing the coefficients of $x^{\delta-n-1}$, we obtain

$$(\alpha q^{\delta-n}-1)a_n=-(-\delta+n-1)a_{n-1}-\psi_n$$

for $n\geq 1$. Then

$$\begin{aligned}a_n&=\frac{-\delta+n-1}{1-\alpha q^{\delta-n}}a_{n-1}+\frac{1}{1-\alpha q^{\delta-n}}\psi_n\\ &=\frac{(-1)\alpha^{-1}q^{n-\delta}(-\delta+n-1)}{1-\alpha^{-1}q^{1-\delta}q^{n-1}}a_{n-1}+\frac{(-1)\alpha^{-1}q^{n-\delta}}{1-\alpha^{-1}q^{1-\delta}q^{n-1}}\psi_n.\end{aligned}$$

According to the recurrence formula, the expression of a_n is

$$\frac{(-1)^n\alpha^{-n}q^{-n\delta+\frac{n(n+1)}{2}}(-\delta)_n a_0}{(\alpha^{-1}q^{1-\delta};q)_n}+\sum_{k=1}^{n}\frac{(-1)^n\alpha^{-n}q^{-n\delta+\frac{n(n+1)}{2}}(\alpha^{-1}q^{1-\delta};q)_{k-1}(-\delta+k)_{n-k}\psi_k}{(-1)^{k-1}\alpha^{-(k-1)}q^{-(k-1)\delta+\frac{k(k-1)}{2}}(\alpha^{-1}q^{1-\delta};q)_n}.$$

The radius of convergence of the second series in the right-hand side of the above equation is calculated in the following remark, the radius of convergence of the series $G_\delta(\alpha; q, f, x)$ is R, it is analytic on $|x| \geq \frac{1}{R}$. $\square$

Remark 2.7. *The radius of convergence of the series*

$$\sum_{n\geq 0}\sum_{k=1}^{n} \frac{(-1)^n \alpha^{-n} q^{-n\delta+n(n+1)/2}(\alpha^{-1}q^{1-\delta};q)_{k-1}(-\delta+k)_{n-k}}{(-1)^{k-1}\alpha^{-(k-1)}q^{-(k-1)\delta+k(k-1)/2}(\alpha^{-1}q^{1-\delta};q)_n}\psi_k x^n$$

is R. Therefore, the radius of convergence of $G_\delta(\alpha; q, f, x)$ is also R.

Proof. It is easily to get that

$$\begin{aligned}&\Big|\sum_{k=1}^{n} \frac{(-1)^n \alpha^{-n} q^{-n\delta+n(n+1)/2}(\alpha^{-1}q^{1-\delta};q)_{k-1}(-\delta+k)_{n-k}}{(-1)^{k-1}\alpha^{-(k-1)}q^{-(k-1)\delta+k(k-1)/2}(\alpha^{-1}q^{1-\delta};q)_n}\psi_k\Big|\\ =&\Big|\sum_{k=0}^{n}(-1)^{n-k+1}\alpha^{k-n-1}q^{(k-n-1)\delta+(n-k+1)(n+k)/2}\frac{(\alpha^{-1}q^{1-\delta};q)_{k-1}(-\delta+k)_{n-k}}{(\alpha^{-1}q^{1-\delta};q)_n}\psi_k\Big|.\end{aligned}$$

From the beginning of the proof of Theorem 2.6, it is obtained that $|\psi_k| \leq M(R')^{-k}$ $(k \in \mathbb{N})$ for constant $M > 0$ and any $0 < R' < R$. In addition, $\alpha = q^\mu$ and μ can be expressed as $\mathrm{Re}(\mu) + \mathrm{i}\mathrm{Im}(\mu)$. Therefore, the above equation is equivalent to

$$\begin{aligned}&\Big|\sum_{k=0}^{n}(-1)^{n-k+1}q^{(k-n-1)(\delta+\tau)+\mathrm{i}(k-n-1)\mathrm{Im}\mu-k(k-1)/2}q^{n(n+1)/2}\frac{(\alpha^{-1}q^{1-\delta};q)_{k-1}(-\delta+k)_{n-k}\psi_k}{(\alpha^{-1}q^{1-\delta};q)_n}\Big|\\ \leq&\sum_{k=0}^{n}\frac{M_1M|(-\delta+k)_{n-k}|q^{(k-n-1)(|\delta|+\tau)+n(n+1)/2}}{M_2q^{k(k-1)/2}(R')^k}.\end{aligned}$$

Denote the above series by $\sum_{k=0}^{n} u_k$. Since

$$\begin{aligned}\Big|\frac{u_{k+1}}{u_k}\Big| =&\frac{|(-\delta+k+1)_{n-k-1}|\,q^{(k-n)(|\delta|+\tau)+n(n+1)/2}}{q^{k(k+1)/2}(R')^{k+1}}\cdot\frac{q^{k(k-1)/2}(R')^k}{|(-\delta+k)_{n-k}|\,q^{(k-n-1)(|\delta|+\tau)+n(n+1)/2}}\\ =&\frac{q^{|\delta|+\tau-k}}{|-\delta+k|R'},\end{aligned}$$

we have $|\frac{u_{k+1}}{u_k}| \geq 1$ for sufficiently large k. Thus, for sufficiently large n,

$$\Big|\sum_{k=1}^{n} \frac{(-1)^n \alpha^{-n} q^{-n\delta+n(n+1)/2}(\alpha^{-1}q^{1-\delta};q)_{k-1}(-\delta+k)_{n-k}}{(-1)^{k-1}\alpha^{-(k-1)}q^{-(k-1)\delta+k(k-1)/2}(\alpha^{-1}q^{1-\delta};q)_n}\psi_k\Big| \leq Cnq^{n-|\delta|-\tau}(R')^{-n}.$$

The radius of convergence of the series $\sum_{n\geq 1} Cnq^{n-\sigma-\tau}(R')^{-n}x^n$ is $R'/q > R'$. Then the radius of convergence of $G_\delta(\alpha; q, f, x)$ is at least R' for any $0 < R' < R$, i.e., the radius of convergence equals R, and the proof is completed. $\square$

2.3.2 Parameter $\alpha \in q^{\mathbb{Z}_{\geq 0}}$

From Lemma 2.1, if $y = \log x \cdot u(x) + v(x)$ is a solution of Equation (2.3.1), then it is equivalent to Equation (2.1.1) with $f = \sum_{n\geq 0} b_n x^{-n}$. For $u(x) = x^{\lambda_m} w(x)$ with

$$\lambda_m = \frac{-\log\alpha + 2m\pi \mathrm{i}}{\ln q},\ w(x) = \sum_{n\geq 0} c_n x^{-n},$$

the first equation in Equation (2.1.1) becomes

$$\sum_{n\geq 1}(\lambda_m - n + 1)c_{n-1}x^{\lambda_m - n} = \sum_{n\geq 0}(\alpha q^{\lambda_m - n} - 1)c_n x^{\lambda_m - n}.$$

Then

$$\begin{array}{ll} (\alpha q^{\lambda_m} - 1)c_0 = 0, & n = 0, \\ c_n = \frac{-\lambda_m + n - 1}{1 - q^{-n}} c_{n-1}, & n \geq 1. \end{array}$$

We get

$$c_n = \frac{(-\lambda_m)_n}{(q^{-1};q^{-1})_n} c_0,$$

where c_0 is an undetermined constant. Therefore,

$$u(x) = x^{\lambda_m} \sum_{n\geq 0} \frac{(-\lambda_m)_n}{(q^{-1};q^{-1})_n} c_0 x^{-n}.$$

Supposing $v(x) = \sum_{n\geq 0} a_n x^{-n}$ and putting it into the second equation in (2.1.1), we have

$$\begin{aligned} \sum_{n\geq 2}(-n+1)a_{n-1}x^{-n} = & \sum_{n\geq 0}(\alpha q^{-n} - 1)a_n x^{-n} + \alpha(\ln q) \sum_{n\geq -\lambda_m} \frac{(-\lambda_m)_{n-\lambda_m}}{(q^{-1};q^{-1})_{n-\lambda_m}} c_0 q^{\lambda_m - n} x^{-n} \\ & - \sum_{n\geq -\lambda_m + 1} \frac{(-\lambda_m)_{n+\lambda_m - 1}}{(q^{-1};q^{-1})_{n+\lambda_m - 1}} c_0 x^{-n} + \sum_{n\geq 0} \psi_n x^{-n}. \end{aligned}$$

To ensure that the above equation holds, we need to compare the coefficients of each term. Consequently, the parameter λ_m must be an integer that is less than or equal to zero.

When $\lambda_m = 0$, the above equation becomes

$$\sum_{n\geq 2}(-n+1)a_{n-1}x^{-n} = \sum_{n\geq 0}(\alpha q^{-n} - 1)a_n x^{-n} + \alpha c_0 \ln q - \frac{c_0}{x} + \sum_{n\geq 0} \psi_n x^{-n},$$

then

$$\begin{array}{ll} (\alpha - 1)\cdot a_0 + \alpha c_0 \ln q + \psi_0 = 0, & n = 0, \\ (\alpha q^{-1} - 1)a_1 - c_0 + \psi_1 = 0, & n = 1, \\ a_n = \frac{(n-1)a_{n-1} + \psi_n}{1 - \alpha q^{-n}}, & n \geq 2. \end{array}$$

For $\alpha = q^{-\lambda_m} = 1$, we have

$$
\begin{aligned}
&0 \cdot a_0 + c_0 \ln q + \psi_0 = 0, \\
&a_1 = \frac{\psi_1 - c_0}{1 - \alpha q^{-1}}, \\
&a_n = \frac{(1-q^{-1})(n-1)!}{(q^{-1};q^{-1})_n} a_1 + \sum_{k=2}^{n} \frac{(k)_{n-k}\psi_k}{(q^{-k};q^{-1})_{n-k+1}}, \quad n \geq 2.
\end{aligned}
\tag{2.3.2}
$$

When λ_m is an integer less than or equal to -1 ($\alpha = q, q^2, \cdots$), it is similar to the steps above. Therefore, if $\alpha \in q^{\mathbb{Z}_{\geq 0}}$, then we obtain a solution of Equation (2.3.1).

If $\alpha = 1, q, q^2, \cdots$, then the corresponding term of

$$
\frac{\psi_0}{1-\alpha}, \frac{\psi_1}{(1-\alpha q^{-1})x}, \frac{\psi_1 + (1-\alpha q^{-1})\psi_2}{(1-\alpha q^{-1})(1-\alpha q^{-2})x^{-2}}, \cdots
$$

of the series

$$
\sum_{n\geq 0}\sum_{k=0}^{n} \frac{(k)_{n-k}}{(\alpha q^{-k};q^{-1})_{n-k+1}} \psi_k x^{-n}
$$

in $G(\alpha; q, \lambda_m, f, x)$ will be meaningless. For example, when $\alpha = 1$, the constant term does not make sense. But if we add the logarithmic term (suppose $y = \log x \cdot u(x) + v(x)$) to the other equation (the first equation in (2.3.2)), we can make the equation true by determining the value of the undetermined constant c_0. In this case, for the first equation in the system, we compare the coefficients of x^{-n}, and the term multiplied by a_0 is 0. When $\alpha = q$, the term containing x^{-1} becomes meaningless. The specific steps are omitted here, but we can imagine that for the second equation in the system, comparing the coefficients of x^{-n}, the term multiplied by a_1 is 0, and to make the equation hold, we get another fixed value of c_0. The calculation details are shown in the proof below.

Theorem 2.7. *If $\alpha = q^{k_0}$ and $f = \psi(x)$ is a convergent series expanded at infinity ($\psi(\infty) \neq 0$), then*

$$
\begin{aligned}
G_{k_0}(\alpha; q, f, x) = &-\frac{1}{\ln q}\Big(\sum_{k=1}^{k_0-1} \frac{(k)_{k_0-k}}{(q;q)_{k_0-k}}\psi_k + \psi_{k_0}\Big) \log x \sum_{n\geq 0} \frac{(k_0)_n x^{-n-k_0}}{(q^{-1};q^{-1})_n} + \frac{\psi_0}{1-q^{k_0}} \\
&+ \sum_{n=1}^{k_0-1}\sum_{k=1}^{n} \frac{(k)_{n-k}}{(q^{k_0-k};q^{-1})_{n-k+1}} b_k x^{-n} + \sum_{n\geq k_0} \frac{(k_0)_{n-k_0}}{(q^{-1};q^{-1})_{n-k_0}} a_{k_0} x^{-n} \\
&+ \sum_{n\geq k_0+1}\sum_{k=k_0+1}^{n} \frac{\left[q^{k_0-k}(k_0)_{k-k_0}\ln q + (q^{k_0-k}-1)(k_0)_{k-k_0-1}\right](k)_{n-k}c_0}{(q^{-1};q^{-1})_{n-k_0}(q^{k_0-k};q^{-1})_{n-k+1}} x^{-n} \\
&+ \sum_{n\geq k_0+1}\sum_{k=k_0+1}^{n} \frac{(k)_{n-k}\psi_k}{(q^{k_0-k};q^{-1})_{n-k+1}} x^{-n}
\end{aligned}
$$

is an analytic solution of Equation (1.1.3) *on the Riemann surface of the logrithm, where* a_{k_0} *is an arbitrary constant,*

$$c_0 = -\frac{1}{\ln q}\Big(\sum_{k=1}^{k_0-1} \frac{(k)_{k_0-k}}{(q;q)_{k_0-k}}\psi_k + \psi_{k_0}\Big),$$

and ψ_k $(k \in \mathbb{N})$ *are coefficients of* x^k*. For* $\psi(x) = \sum_{k\geq 0} b_k x^k$*, the radius of convergence of* $G_{k_0}(\alpha; q, f, x)$ *is the same as that of* $\psi(x)$*. When* k_0 *is equal to* 0 *or* 1*, the series*

$$\sum_{k=1}^{k_0-1} \frac{(k)_{k_0-k}}{(q;q)_{k_0-k}}\psi_k, \quad \sum_{n=1}^{k_0-1}\sum_{k=1}^{n} \frac{(k)_{n-k}}{(q^{k_0-k};q^{-1})_{n-k+1}}\psi_k x^{-n}$$

are equal to 0*. When* $k_0 = 0$*,* $\frac{b_0}{1-q^{k_0}} = 0$.

Proof. Specific calculations for the case of $\alpha = 1$, $\alpha = q$ and $\alpha = q^2$ $(k_0 = 0, 1, 2)$ are in the next section, which can be expressed by the corresponding solution for the case of $k_0 \geq 3$. When $\alpha = q^{k_0}$ $(k_0 \geq 3)$, Equation (2.3.1) becomes

$$\begin{aligned} &\log x \Big[u'(x) - q^{k_0}u(qx) + u(x)\Big] = 0, \\ &v'(x) = q^{k_0}v(qx) - v(x) + q^{k_0}(\ln q)u(qx) - u(x)\cdot\frac{1}{x} + f(x). \end{aligned} \tag{2.3.3}$$

Putting $u(x) = x^{\lambda_m}\sum_{n\geq 0} c_n x^{-n}$ $(\lambda_m = \frac{2m\pi \mathrm{i}}{\ln q} - k_0)$ into the first equation in (2.3.3), we have

$$\sum_{n\geq 1}(\lambda_m - n + 1)c_{n-1}x^{\lambda_m - n} = \sum_{n\geq 0}(q^{\lambda_m - n + k_0} - 1)c_n x^{\lambda_m - n}.$$

Then

$$\begin{aligned} &(q^{\lambda_m + k_0} - 1)c_0 = 0, \quad n = 0, \\ &c_n = \tfrac{-\lambda_m + n - 1}{1 - q^{-n}}c_{n-1}, \quad n \geq 1. \end{aligned}$$

We get

$$c_n = \frac{(-\lambda_m)_n}{(q^{-1};q^{-1})_n}c_0,$$

where c_0 is a constant to be determined. Therefore,

$$u(x) = x^{\lambda_m}\sum_{n\geq 0}\frac{(-\lambda_m)_n}{(q^{-1};q^{-1})_n}c_0 x^{-n}.$$

Supposing $v(x) = \sum_{n\geq 0} a_n x^{-n}$ and substituting it into the second equation of the above eqution, we obtain

$$\begin{aligned}\sum_{n\geq 2}(-n+1)a_{n-1}x^{-n} =&\sum_{n\geq 0}(q^{-n+k_0}-1)a_n x^{-n} + (\ln q)x^{\lambda_m}\sum_{n\geq 0}\frac{(-\lambda_m)_n}{(q^{-1};q^{-1})_n}c_0 q^{-n}x^{-n}\\ &- x^{\lambda_m}\sum_{n\geq 1}\frac{(-\lambda_m)_{n-1}}{(q^{-1};q^{-1})_{n-1}}c_0 x^{-n} + \sum_{n\geq 0}\psi_n x^{-n}.\end{aligned}$$

To compare the coefficients and make the above equation hold, $\lambda_m + k_0$ should be an integer, so $\lambda_m = \lambda_0 = -k_0$. We have $c_n = \frac{(k_0)_n}{(q^{-1};q^{-1})_n}c_0$ $(n\geq 1)$ and $u(x) = x^{-k_0}\sum_{n\geq 0}\frac{(k_0)_n}{(q^{-1};q^{-1})_n}c_0 x^{-n}$. The equation is equivalent to

$$\begin{aligned}\sum_{n\geq 2}(-n+1)a_{n-1}x^{-n} =&\sum_{n\geq 0}(q^{-n+k_0}-1)a_n x^{-n} + (\ln q)\sum_{n\geq k_0}\frac{(k_0)_{n-k_0}}{(q^{-1};q^{-1})_{n-k_0}}c_0 q^{-n+k_0}x^{-n}\\ &- \sum_{n\geq k_0+1}\frac{(k_0)_{n-k_0-1}}{(q^{-1};q^{-1})_{n-k_0-1}}c_0 x^{-n} + \sum_{n\geq 0}\psi_n x^{-n}.\end{aligned}$$

Then

$$\begin{aligned}
&(q^{k_0}-1)a_0 + \psi_0 = 0, && n = 0,\\
&(q^{k_0-1}-1)a_1 + \psi_1 = 0, && n = 1,\\
&a_n = \frac{1}{1-q^{k_0-n}}\left[(n-1)a_{n-1} + \psi_n\right], && 2\leq n\leq k_0-1,\\
&(1-k_0)a_{k_0-1} = 0\cdot a_{k_0} + c_0\ln q + \psi_{k_0}, && n = k_0,\\
&a_n = \frac{1}{1-q^{k_0-n}}\Big[(n-1)a_{n-1} + \frac{q^{-n+k_0}(k_0)_{n-k_0}\ln q c_0}{(q^{-1};q^{-1})_{n-k_0}} - \frac{(k_0)_{n-k_0-1}c_0}{(q^{-1};q^{-1})_{n-k_0-1}} + \psi_n\Big], && n\geq k_0+1.
\end{aligned}$$

We have

$$
\begin{aligned}
a_0 &= \frac{\psi_0}{1-q^{k_0}},\\
a_1 &= \frac{\psi_1}{1-q^{k_0-1}},\\
a_n &= \frac{(n-1)!}{(q^{k_0-1};q^{-1})_n}\psi_1 + \sum_{k=2}^{n}\frac{(k)_{n-k}}{(q^{k_0-k};q^{-1})_{n-k+1}}\psi_k, \qquad 2\le n\le k_0-1,\\
a_{k_0-1} &= \frac{c_0\ln q+\psi_{k_0}}{1-k_0},\\
a_n &= \frac{(k_0)_{n-k_0}}{(q^{-1};q^{-1})_{n-k_0}}a_{k_0} + \sum_{k=k_0+1}^{n}\frac{\left[q^{k_0-k}(k_0)_{k-k_0}\ln q+(q^{k_0-k}-1)(k_0)_{k-k_0-1}\right](k)_{n-k}c_0}{(q^{-1};q^{-1})_{k-k_0}(q^{k_0-k};q^{-1})_{n-k+1}}\\
&\quad + \sum_{k=k_0+1}^{n}\frac{(k)_{n-k}\psi_k}{(q^{k_0-k};q^{-1})_{n-k+1}}, \qquad n\ge k_0+1,
\end{aligned}
$$

where a_{k_0} is an arbitrary constant. From the third and fourth equations in the system above, we obtain that, when $n=k_0-1$,

$$
\begin{aligned}
c_0 =& \frac{1}{\ln q}\Big[\frac{(1-k_0)(k_0-2)!}{(q;q)_{k_0-1}}\psi_1 + \sum_{k=2}^{k_0-1}\frac{(1-k_0)(k)_{k_0-k-1}}{(q;q)_{k_0-k}}\psi_k - \psi_{k_0}\Big]\\
=& -\frac{1}{\ln q}\Big(\sum_{k=1}^{k_0-1}\frac{(k)_{k_0-k}}{(q;q)_{k_0-k}}\psi_k + \psi_{k_0}\Big).
\end{aligned}
$$

The proof is completed. □

2.3.3 Specific calculations for the special cases of α

This section mainly shows the computational details about some special case of $\alpha=1$, $\alpha=q$ and $\alpha=q^2$ in the proof of Theorem 2.7.

Remark 2.8. *If $\alpha=1$, $f=\psi(x)$ is analytic at infinity ($\psi(\infty)\neq 0$), then*

$$
G_0(\alpha;q,f,x) = -\frac{b_0}{\ln q}\log x + a_0 + \frac{b_0}{\ln q}\sum_{n\ge1}\frac{(n-1)!}{(q^{-1};q^{-1})_n}x^{-n} + \sum_{n\ge1}\sum_{k=1}^{n}\frac{(k)_{n-k}b_k}{(q^{-k};q^{-1})_{n-k+1}}x^{-n}
$$

is an analytic solution of (1.1.3) on Riemann surface of the logarithm, where b_k ($k\in\mathbb{N}$) are coefficients of x^k for $\psi(x)=\sum_{k\ge0}b_kx^k$. The radius of convergence of $G_0(\alpha;q,f,x)$ is the same as that of $\psi(x)$.

Proof. When $\alpha = 1$, suppose $y(x) = \log x \cdot u(x) + v(x)$, Equation (4.2.1) becomes

$$\begin{aligned} &\log x \left[u'(x) - u(qx) + u(x)\right] = 0, \\ &v'(x) = v(qx) - v(x) + \ln q \cdot u(qx) - u(x) \cdot \frac{1}{x} + f(x). \end{aligned} \tag{2.3.4}$$

Suppose $u(x) = x^{\lambda_m} w(x)$ ($\lambda_m = \frac{2m\pi \mathrm{i}}{\ln q}, w(x) = \sum_{n\geq 0} c_n x^{-n}$) and substitute it into the first equation of (2.3.4), we have

$$\sum_{n\geq 1} (\lambda_m - n + 1) c_{n-1} x^{\lambda_m - n} = \sum_{n\geq 0} (q^{\lambda_m - n} - 1) c_n x^{\lambda_m - n}.$$

Then

$$\begin{aligned} &(q^{\lambda_m} - 1)c_0 = 0, && n = 0, \\ &c_n = \frac{-\lambda_m + n - 1}{1 - q^{-n}} c_{n-1}, && n \geq 1. \end{aligned}$$

We get

$$c_n = \frac{(-\lambda_m)_n}{(q^{-1}; q^{-1})_n} c_0,$$

where c_0 is a constant to be determined. Then

$$u(x) = x^{\lambda_m} \sum_{n\geq 0} \frac{(-\lambda_m)_n}{(q^{-1}; q^{-1})_n} c_0 x^{-n}.$$

Supposing $v(x) = \sum_{n\geq 0} a_n x^{-n}$ and substituting it into the second equation of (2.3.4), we have

$$\begin{aligned} \sum_{n\geq 2} (-n+1) a_{n-1} x^{-n} =& \sum_{n\geq 0} (q^{-n} - 1) a_n x^{-n} + \ln q \cdot x^{\lambda_m} \sum_{n\geq 0} \frac{(-\lambda_m)_n}{(q^{-1}; q^{-1})_n} c_0 q^{-n} x^{-n} \\ & - x^{\lambda_m} \sum_{n\geq 1} \frac{(-\lambda_m)_{n-1}}{(q^{-1}; q^{-1})_{n-1}} c_0 x^{-n} + \sum_{n\geq 0} b_n x^{-n}. \end{aligned} \tag{2.3.5}$$

To compare the coefficients and make Equation (2.3.5) true, λ_m should be an integer, so $\lambda_m = \lambda_0 = 0$. We have $c_n = 0$ ($n \geq 1$) and $u(x) = c_0$. Equation (2.3.5) becomes

$$\sum_{n\geq 2} (-n+1) a_{n-1} x^{-n} = \sum_{n\geq 0} (q^{-n} - 1) a_n x^{-n} + c_0 \ln q - \frac{c_0}{x} + \sum_{n\geq 0} b_n x^{-n}.$$

Then

$$\begin{aligned} &0 \cdot a_0 + c_0 \ln q + b_0 = 0, && n = 0, \\ &(q^{-1} - 1) a_1 - c_0 + b_1 = 0, && n = 1, \\ &a_n = \frac{(n-1) a_{n-1} + b_n}{1 - q^{-n}}, && n \geq 2. \end{aligned}$$

Consequently,

$$
\begin{aligned}
c_0 &= -\frac{b_0}{\ln q},\\
a_1 &= \frac{b_1 - c_0}{1-q^{-1}} = \frac{1}{1-q^{-1}}\left(b_1 + \frac{b_0}{\ln q}\right),\\
a_n &= \frac{(1-q^{-1})(n-1)!}{(q^{-1};q^{-1})_n} a_1 + \sum_{k=2}^{n} \frac{(k)_{n-k} b_k}{(q^{-k};q^{-1})_{n-k+1}}, \quad n \geq 2,\\
&= \frac{b_0}{\ln q} \cdot \frac{(n-1)!}{(q^{-1};q^{-1})_n} + \sum_{k=1}^{n} \frac{(k)_{n-k} b_k}{(q^{-k};q^{-1})_{n-k+1}},
\end{aligned}
$$

where a_0 is an arbitrary constant. □

Remark 2.9. *If $\alpha = q$, $f = \psi(x)$ is analytic at infinity ($\psi(\infty) \neq 0$), then*

$$
\begin{aligned}
G_1(\alpha;q,f,x) = &-\frac{b_1}{\ln q} \log x \sum_{n\geq 0} \frac{n! x^{-n-1}}{(q^{-1};q^{-1})_n} + \frac{b_0}{1-q} + \sum_{n\geq 1} \frac{(n-1)!}{(q^{-1};q^{-1})_n} a_1 x^{-n}\\
&+ \sum_{n\geq 2}\sum_{k=2}^{n} \left(\frac{\left[q^{1-k}(k-1)! \ln q + (q^{1-k}-1)(k-2)!\right](k)_{n-k} c_0}{(q^{-1};q^{-1})_{n-1}(q^{1-k};q^{-1})_{n-k+1}} + \frac{(k)_{n-k} b_k}{(q^{1-k};q^{-1})_{n-k+1}} \right) x^{-n}
\end{aligned}
$$

is an analytic solution of (1.1.3) on the Riemann surface of the logarithm, the radius of convergence of $G_1(\alpha;q,f,x)$ is the same as that of $\psi(x)$.

Proof. When $\alpha = q$, Equation (4.2.1) becomes

$$
\begin{aligned}
&\log x \left[u'(x) - qu(qx) + u(x)\right] = 0,\\
&v'(x) = qv(qx) - v(x) + q \ln q \cdot u(qx) - u(x) \cdot \frac{1}{x} + f(x).
\end{aligned}
\tag{2.3.6}
$$

Substituting $u(x) = x^{\lambda} \sum_{n\geq 0} c_n x^{-n}$ into the first equation of (2.3.6), we have

$$
\sum_{n\geq 1} (\lambda - n + 1) c_{n-1} x^{\lambda-n} = \sum_{n\geq 0} (q^{\lambda-n+1} - 1) c_n x^{\lambda-n}.
$$

Then

$$
\begin{aligned}
&(q^{\lambda+1} - 1) c_0 = 0, && n = 0,\\
&c_n = \frac{-\lambda + n - 1}{1 - q^{-n}} c_{n-1}, && n \geq 1.
\end{aligned}
$$

For $n \geq 0$, we have

$$\lambda = \lambda_m - 1 = \frac{2m\pi \mathrm{i}}{\ln q} - 1,$$
$$c_n = \frac{(-\lambda_m + 1)_n}{(q^{-1}; q^{-1})_n} c_0,$$

where c_0 is a constant to be determined. Then

$$u(x) = x^{\lambda_m - 1} \sum_{n\geq 0} \frac{(-\lambda_m + 1)_n}{(q^{-1}; q^{-1})_n} c_0 x^{-n}.$$

Supposing $v(x) = \sum_{n\geq 0} a_n x^{-n}$ and substituting it into the second equation of (2.3.6), we have

$$\begin{aligned}\sum_{n\geq 2}(-n+1)a_{n-1}x^{-n} = &\sum_{n\geq 0}(q^{-n+1} - 1)a_n x^{-n} + \ln q \cdot x^{\lambda_m - 1} \sum_{n\geq 0} \frac{(-\lambda_m + 1)_n}{(q^{-1}; q^{-1})_n} c_0 q^{-n} x^{-n} \\ &- x^{\lambda_m - 1} \sum_{n\geq 1} \frac{(-\lambda_m + 1)_{n-1}}{(q^{-1}; q^{-1})_{n-1}} c_0 x^{-n} + \sum_{n\geq 0} b_n x^{-n}.\end{aligned}$$

To compare the coefficients and make the above equation true, λ_m should be an integer, so $\lambda_m = 0$. We have $c_n = \frac{n!}{(q^{-1};q^{-1})_n} c_0$ $(n \geq 1)$ and $u(x) = x^{-1} \sum_{n\geq 0} \frac{n!}{(q^{-1};q^{-1})_n} c_0 x^{-n}$. The above equation becomes

$$\begin{aligned}\sum_{n\geq 2}(-n+1)a_{n-1}x^{-n} = &\sum_{n\geq 0}(q^{-n+1} - 1)a_n x^{-n} + \ln q \cdot \sum_{n\geq 1} \frac{(n-1)!}{(q^{-1}; q^{-1})_{n-1}} c_0 q^{-n+1} x^{-n} \\ &- \sum_{n\geq 2} \frac{(n-2)!}{(q^{-1}; q^{-1})_{n-2}} c_0 x^{-n} + \sum_{n\geq 0} b_n x^{-n}.\end{aligned}$$

Then

$$\begin{aligned}&(q-1)a_0 + b_0 = 0, && n = 0,\\ &0 \cdot a_1 + c_0 \ln q + b_1 = 0, && n = 1,\\ &a_n = \frac{1}{1 - q^{1-n}} \left[(n-1)a_{n-1} + \frac{(n-1)! \ln q}{(q^{-1}; q^{-1})_{n-1}} c_0 q^{-n+1} - \frac{(n-2)!}{(q^{-1}; q^{-1})_{n-2}} c_0 + b_n\right], && n \geq 2.\end{aligned}$$

Consequently,

$$
\begin{aligned}
c_0 &= -\frac{b_1}{\ln q},\\
a_0 &= \frac{b_0}{1-q},\\
a_n &= \frac{(n-1)!}{(q^{-1};q^{-1})_n}a_1 + \sum_{k=2}^{n}\frac{\left[q^{1-k}(k-1)!\ln q + (q^{1-k}-1)(k-2)!\right](k)_{n-k}c_0}{(q^{-1};q^{-1})_{n-1}(q^{1-k};q^{-1})_{n-k+1}}\\
&\quad + \sum_{k=2}^{n}\frac{(k)_{n-k}b_k}{(q^{1-k};q^{-1})_{n-k+1}}, \quad n \geq 2,
\end{aligned}
$$

where a_1 is an arbitrary constant. □

Remark 2.10. *If $\alpha = q^2$, $f = \psi(x)$ is analytic at infinity ($\psi(\infty) \neq 0$), then*

$$
\begin{aligned}
&G_2(\alpha;q,f,x)\\
&= -\frac{1}{\ln q}\left(\frac{b_1}{1-q}+b_2\right)\log x \sum_{n\geq 0}\frac{(2)_n x^{-n-2}}{(q^{-1};q^{-1})_n} + \frac{b_0}{1-q^2} + \frac{b_1}{1-q}x^{-1} + \sum_{n\geq 2}\frac{(2)_{n-2}}{(q^{-1};q^{-1})_{n-2}}a_2 x^{-n}\\
&\quad + \sum_{n\geq 3}\sum_{k=3}^{n}\left(\frac{\left[q^{2-k}(2)_{k-2}\ln q + (q^{2-k}-1)(2)_{k-3}\right](k)_{n-k}c_0}{(q^{-1};q^{-1})_{n-2}(q^{2-k};q^{-1})_{n-k+1}} + \frac{(k)_{n-k}b_k}{(q^{2-k};q^{-1})_{n-k+1}}\right)x^{-n}
\end{aligned}
$$

is an analytic solution of (1.1.3) *on Riemann surface of the logarithm, the radius of convergence of $G_2(\alpha;q,f,x)$ is the same as that of $\psi(x)$.*

Proof. When $\alpha = q^2$, Equation (4.2.1) becomes

$$
\begin{aligned}
&\log x\left[u'(x) - q^2u(qx) + u(x)\right] = 0,\\
&v'(x) = q^2v(qx) - v(x) + q^2\ln q\cdot u(qx) - u(x)\cdot\frac{1}{x} + f(x).
\end{aligned}
\tag{2.3.7}
$$

Substituting $u(x) = x^{\lambda}\sum_{n\geq 0}c_nx^{-n}$ into the first equation of (2.3.7), we have

$$
\sum_{n\geq 1}(\lambda-n+1)c_{n-1}x^{\lambda-n} = \sum_{n\geq 0}(q^{\lambda-n+2}-1)c_nx^{\lambda-n}.
$$

Then

$$
\begin{aligned}
(q^{\lambda+2}-1)c_0 &= 0, & n&=0,\\
c_n &= \frac{-\lambda+n-1}{1-q^{-n}}c_{n-1}, & n&\geq 1.
\end{aligned}
$$

We get

$$\lambda = \lambda_m - 2 = \frac{2m\pi \mathrm{i}}{\ln q} - 2,$$
$$c_n = \frac{(-\lambda_m + 2)_n}{(q^{-1}; q^{-1})_n} c_0,$$

where c_0 is a constant to be determined. Then

$$u(x) = x^{\lambda_m - 2} \sum_{n \geq 0} \frac{(-\lambda_m + 2)_n}{(q^{-1}; q^{-1})_n} c_0 x^{-n}.$$

Supposing $v(x) = \sum_{n \geq 0} a_n x^{-n}$ and substituting it into the second equation of (2.3.7), we have

$$\begin{aligned} \sum_{n \geq 2} (-n+1) a_{n-1} x^{-n} =& \sum_{n \geq 0} (q^{-n+2} - 1) a_n x^{-n} + c_0 \ln q \cdot x^{\lambda_m - 2} \sum_{n \geq 0} \frac{(-\lambda_m + 2)_n}{(q^{-1}; q^{-1})_n} (qx)^{-n} \\ & - x^{\lambda_m - 2} \sum_{n \geq 1} \frac{(-\lambda_m + 2)_{n-1}}{(q^{-1}; q^{-1})_{n-1}} c_0 x^{-n} + \sum_{n \geq 0} b_n x^{-n}. \end{aligned} \tag{2.3.8}$$

To compare the coefficients and make Equation (2.3.8) true, λ_m should be an integer, so $\lambda_m = 0$. We have $c_n = \frac{(2)_n}{(q^{-1}; q^{-1})_n} c_0$ $(n \geq 1)$ and $u(x) = x^{-2} \sum_{n \geq 0} \frac{(2)_n}{(q^{-1}; q^{-1})_n} c_0 x^{-n}$. Equation (2.3.8) becomes

$$\begin{aligned} \sum_{n \geq 2} (-n+1) a_{n-1} x^{-n} =& \sum_{n \geq 0} (q^{-n+2} - 1) a_n x^{-n} + \ln q \cdot \sum_{n \geq 2} \frac{(2)_{n-2}}{(q^{-1}; q^{-1})_{n-2}} c_0 q^{-n+2} x^{-n} \\ & - \sum_{n \geq 3} \frac{(2)_{n-3}}{(q^{-1}; q^{-1})_{n-3}} c_0 x^{-n} + \sum_{n \geq 0} b_n x^{-n}. \end{aligned}$$

Then

$$\begin{aligned} &(q^2 - 1) a_0 + b_0 = 0, && n = 0, \\ &(q - 1) a_1 + b_1 = 0, && n = 1, \\ &- a_1 = 0 \cdot a_2 + c_0 \ln q + b_2, && n = 2, \\ &a_n = \frac{1}{1 - q^{2-n}} \left[(n-1) a_{n-1} + \frac{(2)_{n-2} \ln q}{(q^{-1}; q^{-1})_{n-2}} q^{-n+2} c_0 - \frac{(2)_{n-3}}{(q^{-1}; q^{-1})_{n-3}} c_0 + b_n \right], && n \geq 3. \end{aligned}$$

Consequently,

$$\begin{aligned}
a_0 &= \frac{b_0}{1-q^2},\\
a_1 &= \frac{b_1}{1-q},\\
c_0 &= -\frac{a_1+b_2}{\ln q},\\
a_n &= \frac{(2)_{n-2}}{(q^{-1};q^{-1})_{n-2}}a_2 + \sum_{k=3}^{n}\frac{\left[q^{2-k}(2)_{k-2}\ln q + (q^{2-k}-1)(2)_{k-3}\right](k)_{n-k}c_0}{(q^{-1};q^{-1})_{n-2}(q^{2-k};q^{-1})_{n-k+1}}\\
&\quad + \sum_{k=3}^{n}\frac{(k)_{n-k}b_k}{(q^{2-k};q^{-1})_{n-k+1}}, \quad n \geq 3,
\end{aligned}$$

where a_2 is an arbitrary constant. □

2.4 Properties of solutions when the non-homogeneous term is a polynomial

If f is a polynomial of order m, then it can be written as

$$f = e_0 + e_1 x + \cdots + e_m x^m, \quad e_0, e_1, \cdots, e_m \in \mathbb{C},$$

we have the following theorem.

Theorem 2.8. *Let P be a polynomial, and let $\mathbb{C}[x]$ be the set of all polynomials. Let $L : \mathbb{C}[x] \to \mathbb{C}[x]$ be the operator satisfying $L(P) = P' - \alpha P(qx) + P$. For $\alpha = q^{\mu}$, if $\mu \notin \mathbb{Z}_{\leq 0} \cup \frac{2\pi \mathrm{i}}{\ln q}\mathbb{Z}$, then L is a bijection. Accordingly, the polynomial solution $y(x) = \sum_{n=0}^{m} l_n x^n$ satisfies $L(y) = f$, where m is the order of the polynomial $f(x)$, and*

$$l_n = \sum_{k=0}^{m-n}\frac{(-1)^k (n+k)!(\alpha;q)_n e_{n+k}}{n!(\alpha;q)_{n+k+1}}.$$

Proof. For $y(x) \in \mathbb{C}[x]$, we have $L(y) = y'(x) - \alpha y(qx) + y(x) = f(x)$. Since $f(x)$ is a polynomial of order m, the polynomial solution $y(x)$ is also of order m. Assuming that $y(x) = l_0 + l_1 x + \cdots + l_m x^m$ and substituting it into the Equation (1.1.3), we have

$$\begin{aligned}
&l_1 + 2l_2 x + \cdots + m l_m x^{m-1} - \alpha l_0 - \alpha l_1 q x - \cdots - \alpha l_m q^m x^m + l_0 + l_1 x + \cdots\\
&+ l_m x^m = e_0 + e_1 x + \cdots + e_m x^m.
\end{aligned}$$

Comparing the coefficients on both sides of the equation, we obtain

$$\begin{aligned}
e_m &= (1-\alpha q^m)l_m,\\
e_{m-1} &= (1-\alpha q^{m-1})l_{m-1} + ml_m,\\
e_{m-2} &= (1-\alpha q^{m-2}) + (m-1)l_{m-1},\\
&\vdots\\
e_0 &= (1-\alpha)l_0 + l_1.
\end{aligned} \tag{2.4.1}$$

If $1-\alpha, 1-\alpha q, \cdots, 1-\alpha q^m \neq 0$, then Equation (2.4.1) has a unique solution. Therefore, if $\mu \notin \mathbb{Z}_{\leq 0} \cup \frac{2\pi \mathrm{i}}{\ln q}\mathbb{Z}$ for $\alpha = q^\mu$, then L is a bijection. From Equation (2.4.1), we get

$$\begin{aligned}
l_m &= \frac{e_m}{1-\alpha q^m},\\
l_{m-1} &= \frac{e_{m-1}}{1-\alpha q^{m-1}} - \frac{me_m}{(1-\alpha q^m)(1-\alpha q^{m-1})},\\
l_{m-2} &= \frac{e_{m-2}}{1-\alpha q^{m-2}} - \frac{(m-1)e_{m-1}}{(1-\alpha q^{m-1})(1-\alpha q^{m-2})} + \frac{m(m-1)e_m}{(1-\alpha q^m)(1-\alpha q^{m-1})(1-\alpha q^{m-2})},\\
&\cdots\\
l_{m-i} &= \sum_{k=0}^{i}(-1)^k \frac{(m-i+k)!(\alpha;q)_{m-i}e_{m-i+k}}{(m-i)!(\alpha;q)_{m-i+k+1}}.
\end{aligned}$$

Then $y(x) = \sum_{m\geq i} l_{m-i}x^{m-i}$ is a polynomial solution of Equation (2.4.1). Replacing $m-i$ with n, we get $y(x) = \sum_{n=0}^{m} l_n x^n$, where

$$l_n = \sum_{k=0}^{m-n} \frac{(-1)^k (n+k)!(\alpha;q)_n e_{n+k}}{n!(\alpha;q)_{n+k+1}},$$

and m is the order of the polynomial $f(x)$. □

2.4.1 Solutions at zero and infinity

When $y(x)$ is a polynomial of order m, by Theorem 2.1, the corresponding solution can be written as

$$F_p(\alpha;q,x) = \sum_{n=0}^{m} \frac{(-1)^n(\alpha;q)_n y(0)}{n!}x^n + \sum_{n=1}^{m}\sum_{k=1}^{n} \frac{(-1)^{n-k}(\alpha;q)_n f^{(k-1)}(0)}{n!(\alpha;q)_k}x^n. \tag{2.4.2}$$

From Theorem 2.6, a polynomial solution of order m at infinity has the following form :

$$\begin{aligned}
G_p(\alpha;q,x) &= x^m G_p(-m;q,\frac{1}{x})\\
&= \sum_{n=0}^{m} \frac{(-1)^n \alpha^{-n} q^{-nm+n(n+1)/2}(-m)_n y(0)}{(\alpha^{-1}q^{1-m};q)_n}x^{m-n} + \sum_{n=1}^{m} \frac{(-1)^n g_{-m,k,n}}{(\alpha^{-1}q^{1-m};q)_n}x^{m-n},
\end{aligned}$$

where $y(0)=\frac{e_m}{1-\alpha q^m}$, and

$$g_{-m,k,n}=\sum_{k=1}^{n}(-1)^{k-1}\alpha^{k-n-1}q^{(k-n-1)m+(n-k+1)(n+k)/2}(\alpha^{-1}q^{1-m};q)_{k-1}(-m+k)_{n-k}e_{m-k}.$$

2.4.2 Connection formula between solutions at zero and infinity

From Theorem 2.8, the constant term $y(0)$ in the solution $F_p(\alpha;q,x)$ can be determined, we proceed to a conclusion.

Theorem 2.9. *If y is a polynomial solution of order m, then $F_p(\alpha;q,x)=G_p(\alpha;q,x)$, they can be expressed by the polynomial in Theorem 2.8.*

Proof. Since $f=e_0+e_1x+\cdots+e_mx^m$, we have $f^{(k-1)}(0)=(k-1)!e_{k-1}$. From Theorem 2.8, we get

$$y(0)=l_0=\sum_{k=0}^{m}\frac{(-1)^k k!e_k}{(\alpha;q)_{k+1}}.$$

Then Equation (2.4.2) is equivalent to

$$\begin{aligned}F_p(\alpha;q,x)&=\sum_{n=0}^{m}\sum_{k=0}^{m}\frac{(-1)^{n-k}k!(\alpha;q)_n e_k}{n!(\alpha;q)_{k+1}}x^n+\sum_{n=0}^{m}\sum_{k=1}^{n}\frac{(-1)^{n-k}(k-1)!(\alpha;q)_n e_{k-1}}{n!(\alpha;q)_k}x^n\\&=\sum_{n=0}^{m}\sum_{k=0}^{m}\frac{(-1)^{n-k}k!(\alpha;q)_n e_k}{n!(\alpha;q)_{k+1}}x^n-\sum_{n=0}^{m}\sum_{k=0}^{n-1}\frac{(-1)^{n-k}k!(\alpha;q)_n e_k}{n!(\alpha;q)_{k+1}}x^n\\&=\sum_{n=0}^{m}\sum_{k=n}^{m}\frac{(-1)^{n-k}k!(\alpha;q)_n e_k}{n!(\alpha;q)_{k+1}}x^n.\end{aligned}$$

Replacing k with $n+k$, one can get that

$$\sum_{n=0}^{m}\sum_{k=n}^{m}\frac{(-1)^{n-k}k!(\alpha;q)_n e_k}{n!(\alpha;q)_{k+1}}x^n=\sum_{n=0}^{m}\sum_{k=0}^{m-n}\frac{(-1)^{k}(n+k)!(\alpha;q)_n e_{n+k}}{n!(\alpha;q)_{n+k+1}}x^n,$$

the right-hand side of the above equation has the same form as the solution shown in Theorem 2.8.

Peplacing n with $m-n$ and replacing k with $m-k$, we have

$$
\begin{aligned}
G_p(\alpha;q,x) &= \sum_{n=0}^{m}\Big[\frac{(-1)^n\alpha^{-n}q^{-nm+n(n+1)/2}(-m)_n a_0}{(\alpha^{-1}q^{1-m};q)_n} + \frac{(-1)^n g_{-m,k,n}}{(\alpha^{-1}q^{1-m};q)_n}\Big]x^{m-n} \\
&= \sum_{n=0}^{m}\Big[\frac{(-1)^{m-n}m!e_m}{(1-\alpha q^n)\cdots(1-\alpha q^m)} + \frac{(-1)^n g_{-m,k,n}}{(\alpha^{-1}q^{1-m};q)_n}\Big]x^{n} \\
&= \sum_{n=0}^{m}\Big[\frac{(-1)^{m-n}m!e_m}{(1-\alpha q^n)\cdots(1-\alpha q^m)} + \sum_{k=1}^{m-n}\frac{(-m+k)_{m-n-k}e_{m-k}}{(1-\alpha q^{m-k})\cdots(1-\alpha q^n)}\Big]x^{n} \\
&= \sum_{n=0}^{m}\Big[\frac{(-1)^{m-n}m!e_m}{(1-\alpha q^n)\cdots(1-\alpha q^m)} + \sum_{k=n}^{m-1}\frac{(-1)^{k-n}k!(\alpha;q)_n e_k}{n!(\alpha;q)_{k+1}}\Big]x^{n} \\
&= \sum_{n=0}^{m}\sum_{k=n}^{m}\frac{(-1)^{k-n}k!(\alpha;q)_n e_k}{n!(\alpha;q)_{k+1}}x^n,
\end{aligned}
$$

then $G_p(\alpha;q,x) = F_p(\alpha;q,x)$. □

2.5 Summary of this chapter

In this chapter, we study the properties of solutions at zero and infinity, including the existence, uniqueness, and analytic properties of the Laurent series solution. The convergence radius of the series part of the solution at zero and infinity is calculated respectively, which is the same as that of the non-homogeneous term. For the Laurent series solutions at zero, the parameters in the non-homogeneous term are discussed in three cases. The Taylor series, Laurent series, and the solution with the logarithmic term are obtained, respectively. For the Laurent series solutions at infinity, parameter α is discussed in two cases. The solution in general case and the solution with logarithmic terms in special case are obtained. When the non-homogeneous term is a polynomial, there is a unique polynomial solution that satisfies the equation, and its order is equal to the order of the non-homogeneous term.

Chapitre 3

Index theorem for differential q-difference equation

3.1 Introduction

We have introduced the index concept of operators in Definition 2.1. Next, we will derive the index calculation method of the operator corresponding to the equation studied in this chapter. We follow the steps in [65].

Consider the operator that the space $\mathbb{C}[[x]]$ to itself :

$$L = \frac{\mathrm{d}}{\mathrm{d}x} - \alpha\sigma_q + 1,$$

where $0 < q < 1$, $\mathbb{C}[[x]]$ is the set of formal series, and σ_q is the operator that maps the formal series $\sum_{n\geq 0} a_n x^n$ to the formal series $\sum_{n\geq 0} a_n q^n x^n$. Equation (1.1.3) can be written as $Ly = f$.

From the definitions, it is easy to see that the operators x and $\frac{\mathrm{d}}{\mathrm{d}x}$ that map space $\mathbb{C}[[x]]$ to itself have indexes of -1 and 1, repectively. Then the index of the operator $x\frac{\mathrm{d}}{\mathrm{d}x}$ on $\mathbb{C}[[x]] \to \mathbb{C}[[x]]$ is 0. For the sake of discussion, we first study the operators $L_1 = x\frac{\mathrm{d}}{\mathrm{d}x} - \alpha x\sigma_q + x$ from the space $\mathbb{C}[[x]]$ to itself.

We introduce some definitions and results of reference [65] as follows.

Definition 3.1. *For a formal series $g(x) = \sum_{n\geq 0} a_n x^n$, let $s, s' \in \mathbb{R}$ and $A > 0$. If for any $n \in \mathbb{N}$, there exists $C > 0$, such that*

$$|a_n| < C|q|^{-\frac{sn(n+1)}{2}}(n!)^{s'}A^n,$$

we call that g is q-Gevrey of s-order and Gevrey of s'-order. Denoted as $g(x) \in \mathbb{C}[[x]]_{q,s,s',A}$.

Define

$$\mathbb{C}[[x]]_{q,s,s'} = \bigcup_{A>0} \mathbb{C}[[x]]_{q,s,s',A},$$
$$\mathbb{C}[[x]]_{(q,s,s')} = \bigcap_{A>0} \mathbb{C}[[x]]_{q,s,s',A}.$$

Lemma 3.1. *Let $u : E \to F$ and $v : F \to G$ be two operators with indexes. Then $v \circ u : E \to G$ is an operator with index, whose index is equal to : $\chi(v \circ u) = \chi(v) + \chi(u)$.*

Inspired by the method outlined in [65], we review the study of operators in the context of Banach spaces, particularly focusing on the calculation of indexes.

Let $s, s' \in \mathbb{R}$, $\rho > 0$ and $\mu \in \mathbb{R}$. We define the mapping :

$$\begin{aligned} \psi_{s,s',\rho,\mu} :\ \mathbb{C}[[x]] &\longrightarrow \mathbb{C}[[x]] \\ \sum_{n\geq 0} a_n x^n &\longmapsto \sum_{n\geq 0} b_n x^n \end{aligned}$$

with $b_n = |q|^{-\frac{sn(n+1)}{2}} (n!)^{s'} \rho^{-n}(1+n)^{-\mu} a_n$.

Let $l^1_{s,s',\rho,\mu} = \psi_{s,s',\rho,\mu}\left(l^1(\mathbb{C})\right)$. For $b = \{b_n\} \in l^1_{s,s',\rho,\mu}$, define

$$\|b\|_{s,s',\rho,\mu} = \sum_{n=0}^{\infty} \frac{|b_n|}{|q|^{-\frac{sn(n+1)}{2}} (n!)^{s'} \rho^{-n}(1+n)^{-\mu}} < \infty,$$

which is a norm of the space $l^1_{s,s',\rho,\mu}$. Then $l^1_{s,s',\rho,\mu}$ is a Banach space, and we identify the sequence space $\mathbb{C}^{\mathbb{N}}$with the space of formal series.

Lemma 3.2. $x\frac{\mathrm{d}}{\mathrm{d}x} : l^1_{s,s',\rho,\mu} \longrightarrow l^1_{s,s',\rho,\mu-1}$.

Proof. Since $\{a_n\} \in l^1_{s,s',\rho,\mu}$, we get

$$\|a_n\|_{s,s',\rho,\mu} = \sum_{n=0}^{\infty} \frac{|a_n|}{|q|^{-\frac{sn(n+1)}{2}} (n!)^{s'} \rho^{-n}(1+n)^{-\mu}} < \infty.$$

For $\sum_{n\geq 0} b_n x^n = x\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n\geq 0} a_n x^n\right) = \sum_{n\geq 0} n a_n x^n$, we have $b_n = n a_n$, and

$$\begin{aligned} \|b_n\|_{s,s',\rho,\mu} &= \sum_{n=0}^{\infty} \frac{|n a_n|}{|q|^{-\frac{sn(n+1)}{2}} (n!)^{s'} \rho^{-n}(1+n)^{-(\mu-1)}} \\ &= \sum_{n=0}^{\infty} \frac{|a_n|}{|q|^{-\frac{sn(n+1)}{2}} (n!)^{s'} \rho^{-n}(1+n)^{-\mu}} \cdot \frac{n}{1+n} \\ &< \sum_{n=0}^{\infty} \frac{|a_n|}{|q|^{-\frac{sn(n+1)}{2}} (n!)^{s'} \rho^{-n}(1+n)^{-\mu}} \\ &< \infty. \end{aligned}$$

Then $\{b_n\} \in l^1_{s,s',\rho,\mu-1}$, the proof is completed. $\square$

Lemma 3.3. $x : l^1_{s,s',\rho,\mu} \longrightarrow l^1_{s,s',\rho|q|^{-s},\mu+s'}$.

Lemma 3.4. $x\sigma_q : l^1_{s,s',\rho,\mu} \longrightarrow l^1_{s,s',\rho|q|^{-(1+s)},\mu+s'}$.

The proof is similar to the proof in lemma 3.2.

3.2 The operator for a formal series expanded at 0

In this section, we will calculate the index of the operator acting on the series expanded at the origin, following a similar step to the reference [65]. According to the above lemma, we get the following conclusion about the operator L_1.

Proposition 3.1. *Let $s > 0$ and $s' \in \mathbb{R}$ or $s = 0$ and $s' \geq -1$. The operator*

$$L_1 = x\frac{\mathrm{d}}{\mathrm{d}x} - \alpha x\sigma_q + x$$

maps space $l^1_{s,s',\rho,\mu}$ to space $l^1_{s,s',\rho,\mu-1}$. In other words, the operator L_1 is a $(\rho, 1, \mu, -1)$-operator.

Proof. From Lemma 3.2, Lemma 3.3 and Lemma 3.4, we have $x\frac{\mathrm{d}}{\mathrm{d}x} : l^1_{s,s',\rho,\mu} \to l^1_{s,s',\rho,\mu-1}$, $-\alpha x\sigma_q : l^1_{s,s',\rho,\mu} \to l^1_{s,s',\rho|q|^{-(1+s)},\mu+s'}$ and $x : l^1_{s,s',\rho,\mu} \to l^1_{s,s',\rho|q|^{-s},\mu+s'}$. If $s > 0$ and $s' \in \mathbb{R}$ or $s = 0$ and $s' \geq -1$, the union of sets $l^1_{s,s',\rho,\mu-1}$, $l^1_{s,s',\rho|q|^{-(1+s)},\mu+s'}$ and $l^1_{s,s',\rho|q|^{-s},\mu+s'}$ is $l^1_{s,s',\rho,\mu-1}$, then the operator $L_1 : l^1_{s,s',\rho,\mu} \to l^1_{s,s',\rho,\mu-1}$. $\square$

Remark 3.1. *The operator L_1 is $(\rho, 1, \mu, -1)$-operator can be obtained by Definition 4.3.6 in reference [65], the main contents of the definition are :*

Let $s, s', \rho, r, \mu, \nu \in \mathbb{R}$, where $\rho, r > 0$. Let L be an endomorphism of the space $\mathbb{C}[[x]]_{q,s,s'}$ ($\mathbb{C}[[x]]_{(q,s,s')}$). Then L is called (ρ, r, μ, ν)-operator. If for suitable $0 < \rho < \rho_0$ ($\rho > \rho_0$), there is a continuous linear mapping $L_{s,s',\rho,r,\mu,\nu} : l^1_{s,s',\rho,\mu} \to l^1_{s,s',\rho r,\mu+\nu}$ makes the following diagrams commutative :

$$\begin{array}{ccc}
l^1_{s,s',\rho,\mu} & \xrightarrow{L_{s,s',\rho,r,\mu,\nu}} & l^1_{s,s',\rho r,\mu+\nu} \\
\downarrow & & \downarrow \\
\mathbb{C}[[x]]_{q,s,s'} & \xrightarrow{L} & \mathbb{C}[[x]]_{q,s,s'}
\end{array}$$

Proposition 3.2. *For $s > 0$ and $s' \in \mathbb{R}$ or $s = 0$ and $s' \geq -1$. Let $\mu \in \mathbb{R}$ and let $L_1 : \mathbb{C}[[x]]_{q,s,s'} \to \mathbb{C}[[x]]_{q,s,s'}$ ($\mathbb{C}[[x]]_{(q,s,s')}$) be $(\rho, 1, \mu, -1)$-operator. For $0 < \rho < \rho_0$ (resp. $\rho > \rho_0$) and suitable ρ_0, we assume that every operator $L_{s,s',\rho,1,\mu,-1}$ has an index, and the index is independent of the value of ρ. Then L_1 has an index and $\chi(L_1) = \chi\left(L_{s,s',\rho,1,\mu,-1}\right)$.*

Next we will show that if $s > 0$ and $s' \in \mathbb{R}$ or $s = 0$ and $s' > -1$, then the study of the operator $L_1 = x\frac{\mathrm{d}}{\mathrm{d}x} - \alpha x\sigma_q + x$ is reduced to the study of the operator $L_2 = x\frac{\mathrm{d}}{\mathrm{d}x}$.

Proposition 3.3. *Let $s > 0$ and $s' \in \mathbb{R}$ or $s = 0$ and $s' > -1$. If $L_{s,s',\rho,1,\mu,-1}$ has an index and is independent of the value of ρ for $0 < \rho < \rho_0$ ($\rho > \rho_0$) and suitable ρ_0, then the operator L_1 on $\mathbb{C}[[x]]_{q,s,s'}$ ($\mathbb{C}[[x]]_{(q,s,s')}$) has an index and $\chi(L_1) = \chi(L_2)$.*

Proof. It is obviously that $L_1 = L_2 + L_3 + L_4$, where $L_3 = -\alpha x\sigma_q$ is $(\rho, |q|^{-(1+s)}, \mu, s')$-operator, and $L_4 = x$ is $(\rho, |q|^{-s}, \mu, s')$-operator. If $s > -1$ and $s' \in \mathbb{R}$ or $s = -1$ and $s' \geq -1$, then $L_2 + L_3$ is $(\rho, 1, \mu, -1)$-operator, and $\chi(L_2 + L_3) = \chi(L_2)$. If $s > 0$ and $s' \in \mathbb{R}$ or $s = 0$ and $s' > -1$, then $L_2 + L_3 + L_4$ is $(\rho, 1, \mu, -1)$-operator, and $\chi(L_2 + L_3 + L_4) = \chi(L_2 + L_3) = \chi(L_2)$. Regarding the intersection of the values of s and s', i.e. $s > 0$ and $s' \in \mathbb{R}$ or $s = 0$ and $s' > -1$, we have $L_2 + L_3 + L_4 = L_1$, then $\chi(L_1) = \chi(L_2)$. □

Proposition 3.4. *Let $s = 0$ and $s' = -1$. For suitable ρ_0, if $L_{s,s',\rho,1,\mu,-1}$ has an index, and the index is independent of the value of ρ for $0 < \rho < \rho_0$ ($\rho > \rho_0$), then L_1 on $\mathbb{C}[[x]]_{q,s,s'}$ ($\mathbb{C}[[x]]_{(q,s,s')}$) is an index operator, with an index $\chi(L_1) = \chi(L_2 + L_4)$.*

Proof. The proof is similar to that of the Proposition 3.3. □

After the above steps, we reduce the operator L_1 (L_2) to the operator of the ordinary differential equation. J. Ramis [74, 77] studied the situation :

If $s > 0$ and $s' \in \mathbb{R}$ or $s = 0$ and $s' \geq -1$, then

$$\chi\left(T_L, \mathbb{C}[[x]]_{q,s,s'}\right) = -\inf\{k|\exists i, \exists j, (i,j,k) \in N(s,s')\},$$
$$\chi\left(T_L, \mathbb{C}[[x]]_{(q,s,s')}\right) = -\sup\{k|\exists i, \exists j, (i,j,k) \in N(s,s')\}.$$

For

$$T_L = \sum_{i\in\Im}(\sum_{k\in K_i} \alpha_{i,j_{(i,k)},k} q^{\frac{sk(k+1)}{2}} x^k)(x\frac{\mathrm{d}}{\mathrm{d}x})^i$$

and $\Im = \{i = 0, \cdots, I|\exists j, \exists k, (i,j,k) \in N(s,s')\}$, $K_i = \{k \geq 0|\exists j, (i,j,k) \in N(s,s')\}$, where

$$N(s,s') = \{(i,j,k) \in M(s)|ks' - i = v_{s,s'}(L)\},$$
$$M(s) = \{(i,j,k)|\alpha_{i,j,k} \neq 0, j + ks = p_s^0(L)\},$$

with $p_s^0(L) = \inf_{\alpha_{i,j,k}\neq 0}(j + ks)$ and $v_{s,s'}(L) = \inf_{\alpha_{i,j,k}\neq 0, j+ks=p_s^0(L)}(ks' - i)$.

When $i = 1, k = 0$ and $\alpha_{1,0,0} = 1$, the operator L_2 is equivalent to T_L, we have

$$\chi\left(L_2, \mathbb{C}[[x]]_{q,s,s'}\right) = \chi\left(L_2, \mathbb{C}[[x]]_{(q,s,s')}\right) = 0,$$

for $s > 0$ and $s' \in \mathbb{R}$ or $s = 0$ and $s' > -1$.

The operator $L_2 + L_4$ can regarded as T_L if $s = 0$, $\alpha_{1,0,0} = 1$ and $\alpha_{0,0,1} = 1$. Therefore, for $s = 0$ and $s' = -1$, we get

$$\chi\left(L_2 + L_4, \mathbb{C}[[x]]_{q,s,s'}\right) = 0, \quad \chi\left(L_2 + L_4, \mathbb{C}[[x]]_{(q,s,s')}\right) = -1.$$

Lemma 3.5. *For the operator $L_1 = x\frac{\mathrm{d}}{\mathrm{d}x} - \alpha x \sigma_q + x$.*

(i) If $s > 0$ and $s' \in \mathbb{R}$ or if $s = 0$ and $s' > -1$, then

$$\chi\left(L_1, \mathbb{C}[[x]]_{q,s,s'}\right) = \chi\left(L_1, \mathbb{C}[[x]]_{(q,s,s')}\right) = 0.$$

(ii) If $s = 0$ and $s' = -1$, then

$$\chi\left(L_1, \mathbb{C}[[x]]_{q,s,s'}\right) = 0, \quad \chi\left(L_1, \mathbb{C}[[x]]_{(q,s,s')}\right) = -1.$$

Proof. From Proposition 3.3 and Propostion 3.4, if $s > 0$ and $s' \in \mathbb{R}$ or $s = 0$ and $s' > -1$, then

$$\chi(L_1, \mathbb{C}[[x]]_{q,s,s'}) = \chi(L_2, \mathbb{C}[[x]]_{q,s,s'}) = 0,$$
$$\chi(L_1, \mathbb{C}[[x]]_{(q,s,s')}) = \chi(L_2, \mathbb{C}[[x]]_{(q,s,s')}) = 0.$$

If $s = 0$ and $s' = -1$, then

$$\chi(L_1, \mathbb{C}[[x]]_{q,s,s'}) = \chi(L_2 + L_4, \mathbb{C}[[x]]_{q,s,s'}) = 0,$$
$$\chi(L_1, \mathbb{C}[[x]]_{(q,s,s')}) = \chi(L_2 + L_4, \mathbb{C}[[x]]_{(q,s,s')}) = -1.$$

The proof is completed. □

Theorem 3.1. *For the operator $L = \frac{\mathrm{d}}{\mathrm{d}x} - \alpha \sigma_q + 1$.*

(i) If $s < 0$ and $s' \in \mathbb{R}$ or $s = 0$ and $s' < -1$, then

$$\chi\left(L, \mathbb{C}[[x]]_{q,s,s'}\right) = \chi\left(L, \mathbb{C}[[x]]_{(q,s,s')}\right) = 0.$$

(ii) If $s = 0$ and $s' = -1$, then

$$\chi\left(L, \mathbb{C}[[x]]_{q,s,s'}\right) = 1, \quad \chi\left(L, \mathbb{C}[[x]]_{(q,s,s')}\right) = 0.$$

Proof. The operator L_1 can be seen as $\hat{L} = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0} \alpha_{i,j,k} x^k (x\frac{\mathrm{d}}{\mathrm{d}x})^i \sigma_q^j$, where $\alpha_{1,0,0} = 1$, $\alpha_{0,1,1} = -\alpha$ and $\alpha_{0,0,1} = 1$. Let

$$\widetilde{L} = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0} \alpha_{i,j,k} x^k (\frac{\mathrm{d}}{\mathrm{d}x})^i \sigma_q^j.$$

Since
$$\chi\left(\hat{L}, \mathbb{C}[[x]]_{q,s,s'}\right) = -\inf\{k | \exists i, \exists j, (i,j,k) \in N(s,s')\},$$
we have
$$\chi\left(\widetilde{L}, \mathbb{C}[[x]]_{q,s,s'}\right) = -\inf\{k-i | \exists i, \exists j, (i,j,k) \in N(s,s')\},$$
by replacing k with $k-i$, where

$$N(s,s') = \{(i,j,k) \in M(s) | (k-i)s' - i = v_{s,s'}(L)\},$$
$$M(s) = \{(i,j,k) | \alpha_{i,j,k} \neq 0, j + (k-i)s = p_s^0(L)\},$$

with $p_s^0(L) = \inf_{\alpha_{i,j,k}\neq 0}(j+(k-i)s)$ and $v_{s,s'}(L) = \inf_{\alpha_{i,j,k}\neq 0, j+ks=p_s^0(L)}((k-i)s'-i)$. The operator L can be written as $\hat{L}$ ($\alpha_{1,0,0}=1$, $\alpha_{0,1,0}=-\alpha$, and $\alpha_{0,0,0}=1$).

(i) If $s<0$ and $s' \in \mathbb{R}$ or $s=0$ and $s'<-1$, then $\inf j+(k-i)s = \inf\{-s,0\} = 0$ and $M(s) = (0,0,0)$ or $M(s) = \{(1,0,0),(0,0,0)\}$, $N(s,s') = (0,0,0)$, therefore,

$$\chi\left(L, \mathbb{C}[[x]]_{q,s,s'}\right) = \chi\left(L, \mathbb{C}[[x]]_{(q,s,s')}\right) = 0.$$

(ii) If $s=0$ and $s'=-1$, then $\inf\{j+(k-i)s\} = 0$, $M(s) = \{(1,0,0),(0,0,0)\}$, $\inf\{(k-i)s'-i = \inf(-s'-1)\} = 0$ and $N(s,s') = \{(1,0,0),(0,0,0)\}$, therefore,

$$\chi\left(L, \mathbb{C}[[x]]_{q,s,s'}\right) = 1,\ \chi\left(L, \mathbb{C}[[x]]_{(q,s,s')}\right) = 0.$$

The proof is completed. □

Remark 3.2. *Let $\mathbb{C}\{x\}$ be the space of series whose radius of convergence is not zero. When $s,s'=0$, the space $\mathbb{C}[[x]]_{q,s,s'}$ becomes $\mathbb{C}\{x\}$. The result of Theorem 3.1 is the same as that of Remark 2.4.*

According to the index theorem, we can classify the solutions of the equations. First of all, we introduce some definitions and results in [65] (corresponding to this case).

Definition 3.2. *We say that $s \in \mathbb{R}$ is an exceptional value for L_1, if it exists two triples $(i_1, j_1, k_1 - i_1)$ and $(i_2, j_2, k_2 - i_2)$ in $M(s)$ such that $k_1 - i_1 \neq k_2 - i_2$.*

Definition 3.3. *Let $s \in \mathbb{R}$. The real s' is an exceptional value to s for the operator L_1, if there are two triples $(i_1, j_1, k_1 - i_1)$ and $(i_2, j_2, k_2 - i_2)$ in $N(s,s')$ such that $k_1 - i_1 \neq k_2 - i_2$.*

Definition 3.4. *We denote $\mathbb{L}_{q,s,s'}$ (resp. $\mathbb{L}_{(q,s,s')}$) the algebra of operators of this form with $a_{i,j}(x) \in \mathbb{C}[[x]]_{q,s,s'}$ ($\mathbb{C}[[x]](q,s,s')$). If $a_{i,j}(x) \in \mathbb{C}[[x]]$, the algebra is denoted $\hat{\mathbb{L}}$. Finally, if the coefficients are convergent series, the corresponding notation is $\mathbb{L}$.*

Lemma 3.6. *Let $L \in \mathbb{L}$. Let $s_1 > s_2 > \cdots > s_l$ be the possible exceptional values ≥ 0 for L. For each of them, s_p, let $s'_{p,1} > s_{p,2} > \cdots > s_{p,l_p}$ be the possible exceptional values relative to s_p (if $s_p = 0$ we limit ourselves to positive or zero s'-values). Let $\hat{f} \in \mathbb{C}[[x]]$ and $g \in \mathbb{C}\{x\}$ such that $L(\hat{f}) = g$. Then :*

— *Either $\hat{f} \in \mathbb{C}\{x\}$,*

— *Either there exists a unique real $s' > 0$ such that $\hat{f} \in \mathbb{C}[[x]]_{q,0,s'}$ and $\hat{f} \notin \mathbb{C}[[x]]_{(q,0,s')}$: we say that $\hat{f}$ is Gevrey of exact order s',*

— *Either there exists a unique real $s > 0$ and a unique real $s' \in \mathbb{R}$ such that $\hat{f} \in \mathbb{C}[[x]]_{q,s,s'}$ and $\hat{f} \notin \mathbb{C}[[x]]_{(q,s,s')}$: we say that $\hat{f}$ is q-Gevrey of exact order s and Gevrey of exact order s'.*

From these results in [65] and Theorem 3.1, no positive exceptional value s and s' for the operator $L_1 = x\frac{\mathrm{d}}{\mathrm{d}x} - \alpha x\sigma_q + x$ and $L = \frac{\mathrm{d}}{\mathrm{d}x} - \alpha\sigma_q + 1$. Then we have the following results.

Lemma 3.7. *For $L_1 = x\frac{\mathrm{d}}{\mathrm{d}x} - \alpha x\sigma_q + x \in \mathbb{L}$, if $\hat{f} \in \mathbb{C}[[x]]$ and $g \in \mathbb{C}\{x\}$ such that $L_1(\hat{f}) = g$, then $\hat{f} \in \mathbb{C}\{x\}$.*

Theorem 3.2. *For $L = \frac{\mathrm{d}}{\mathrm{d}x} - \alpha\sigma_q + 1 \in \mathbb{L}$, if $\hat{f} \in \mathbb{C}[[x]]$ and $g \in \mathbb{C}\{x\}$ such that $L(\hat{f}) = g$, then $\hat{f} \in \mathbb{C}\{x\}$.*

The result in the above theorem is consistent with that of Corollary 2.1.

3.3 The operator for a formal series expanded at ∞

For the operator for a formal series expanded at ∞, we tranform it into an operator for a formal series expanded at 0, so that we can analyze it as described in the previous subsection.

For $\hat{f}(x) = \sum_{n\geq 0} a_n x^n$ a formal series expanded at 0, we have $L(\hat{f}(x)) = (\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\sigma_q + 1)(\hat{f})$. Let $t = \frac{1}{x}$, then $\hat{f}(t) = \sum_{n\geq 0} a_n t^n$ is a formal series expanded at ∞. We have

$$L(\hat{f}(t)) = (\frac{\mathrm{d}}{\mathrm{d}t} - \alpha\sigma_q + 1)(\hat{f}) = (-x^2\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\sigma_q^{-1} + 1)(\hat{f}),$$

i.e.,

$$L_\infty = \left[-x^2(\frac{\mathrm{d}}{\mathrm{d}x})\sigma_q - \alpha + \sigma_q\right]\sigma_q^{-1}$$

is the operator for a formal series $\hat{f} \in \mathbb{C}[[x^{-1}]]$, then we study the index theorem by the process similar to above.

Denote $\widetilde{L} = -x(x\frac{\mathrm{d}}{\mathrm{d}x})\sigma_q - \alpha + \sigma_q$, then $L_\infty = \widetilde{L} \circ \sigma_q^{-1}$. From Lemma 3.1, we get $\chi(L_\infty) = \chi(\widetilde{L}) + \chi(\sigma_q^{-1})$. The operator σ_q^{-1} is an isomorphism of $l^1_{s,s',\rho,\mu} \to l^1_{s,s',\rho|q|,\mu}$ (resp. an automorphisme of $\mathbb{C}[[x]]_{q,s,s'}$ and $\mathbb{C}[[x]]_{(q,s,s')}$). Therefore, it has zero index in these spaces. Then $\chi(L_\infty) = \chi(\widetilde{L})$.

Let $s > -1$ and $s' \in \mathbb{R}$ or $s = -1$ and $s' \geq 1$ in the following discussion.

Definition 3.5. *We denote $\mathbb{L}_{q,s,s',-1}$ (resp. $\mathbb{L}_{(q,s,s',-1)}$) the algebra of operators of this form with $a_{i,j}(x) \in \mathbb{C}[[x^{-1}]]_{q,s,s'}$ (resp. $\mathbb{C}[[x^{-1}]](q,s,s')$). If $a_{i,j}(x) \in \mathbb{C}[[x^{-1}]]$, the algebra is denoted $\hat{\mathbb{L}}_{-1}$. Finally, if the coefficients are convergent series, the corresponding notation is $\mathbb{L}_{-1}$.*

Proposition 3.5. *The operator*

$$\widetilde{L} = -x(x\frac{\mathrm{d}}{\mathrm{d}x})\sigma_q - \alpha + \sigma_q$$

operates from $l^1_{s,s',\rho,\mu}$ to $l^1_{s,s',\rho|q|^{-p^0_s(\widetilde{L})},\mu+v_{s,s'}(\widetilde{L})}$ with

$$p^0_s(\widetilde{L}) = \inf\{s+1, 0, 1\}, \quad v_{s,s'}(\widetilde{L}) = \inf\{s'-1, 0\}.$$

In other words, $\widetilde{L}$ is a $(\rho, |q|^{-p^0_s(\widetilde{L})}, \mu, v_{s,s'}(\widetilde{L}))$-operator.

Proof. From Proposition 4.3.5 in [65] : Let $i, j, k \in \mathbb{N}$. The operator $x^k(x\frac{\mathrm{d}}{\mathrm{d}x})^i\sigma_q^j$ operators from $l^1_{s,s',\rho,\mu}$ to $l^1_{s,s',\rho|q|^{-(j+ks)},\mu-i+ks'}$. Then

the operator $-x(x\frac{\mathrm{d}}{\mathrm{d}x})\sigma_q : l^1_{s,s',\rho,\mu} \to l^1_{s,s',\rho|q|^{-(1+s)},\mu-1+s'}$,

the operator $\sigma_q : l^1_{s,s',\rho,\mu} \to l^1_{s,s',\rho|q|^{-1},\mu}$, and the operator $-\alpha : l^1_{s,s',\rho,\mu} \to l^1_{s,s',\rho,\mu}$.

The union of $l^1_{s,s',\rho|q|^{-(1+s)},\mu-1+s'}$, $l^1_{s,s',\rho|q|^{-1},\mu}$ and $l^1_{s,s',\rho,\mu}$ is $l^1_{s,s',\rho|q|^{-p^0_s(\widetilde{L})},\mu+v_{s,s'}(\widetilde{L})}$, then the operator $\widetilde{L} : l^1_{s,s',\rho,\mu} \to l^1_{s,s',\rho|q|^{-p^0_s(\widetilde{L})},\mu+v_{s,s'}(\widetilde{L})}$. □

The discussion will be divided into three cases.

(i) If $s > -1, s' \in \mathbb{R}$, then $p^0_s(\widetilde{L}) = 0$, $M(s) = N(s,s') = \{(0,0,0)\}$. The study of $\widetilde{L}$ reduces to the study of $\hat{L} = -\alpha$.

(ii) If $s = -1, s' > 1$, then $M(s) = \{(0,0,0),(1,1,1)\}$ and $N(s,s') = \{(0,0,0)\}$, which is the same case as (i).

(iii) If $s = -1, s' = 1$, $M(s) = N(s,s') = \{(0,0,0),(1,1,1)\}$. In addition, $s = -1$ is an exceptional value for $\widetilde{L}$, and $s' = 1$ is an exceptional value relatively to $s = -1$ for $\widetilde{L}$. The study of $\widetilde{L}$ reduces to the study of $\bar{L} = -x^2(\frac{\mathrm{d}}{\mathrm{d}x})\sigma_q - \alpha$.

Then, we reduce the operator $\bar{L}$ to an opertator of ordinary differential equation. We define

$$\eta(x) = \sum_{n\geq 0} q^{\frac{n(n+1)}{2}} x^n, \quad \eta^*(x) = \sum_{n\geq 0} q^{-\frac{n(n+1)}{2}} x^n,$$

and denote the Hadamard product of the formal series by

$$(\sum_{n\geq 0} a_n x^n)\square(\sum_{n\geq 0} b_n x^n) = (\sum_{n\geq 0} a_n b_n x^n).$$

Define the new operator $\tilde{\tilde{L}}(f) = \eta\square\bar{L}(\eta^*\square f)$. Then $\tilde{\tilde{L}} = -q^{-1}x^2(\frac{\mathrm{d}}{\mathrm{d}x}) - \alpha$.

Since the mapping

$$\begin{array}{rll} \eta: & l^1_{0,1,\rho,\mu} & \longrightarrow l^1_{0,1,\rho,\mu} \\ & f & \longmapsto \eta^*\square f \end{array}$$

and

$$\begin{array}{rll} \eta^{-1}: & l^1_{0,1,\rho,\mu} & \longrightarrow l^1_{0,1,\rho,\mu} \\ & f & \longmapsto \eta\square f \end{array}$$

are isomorphisms, we have $\chi(\eta) = \chi(\eta^{-1}) = 0$. For $\tilde{\tilde{L}}(f) = \eta^{-1}\circ\bar{L}\circ\eta$, we obtain the following lemma.

Lemma 3.8. *The operator $\bar{L}: l^1_{0,1,\rho,\mu} \longrightarrow l^1_{0,1,\rho,\mu}$ is indexed if and only if*

$$\tilde{\tilde{L}}: l^1_{0,1,\rho,\mu} \longrightarrow l^1_{0,1,\rho,\mu}$$

is indexed. In this case, $\chi(\bar{L}) = \chi(\tilde{\tilde{L}})$ (similar statement holds in the cases $\mathbb{C}[[x]]_{q,0,1}$ and $\mathbb{C}[[x]]_{(q,0,1)}$).

Theorem 3.3. *For the operator $\widetilde{L} = -x(x\frac{\mathrm{d}}{\mathrm{d}x})\sigma_q - \alpha + \sigma_q$.*

(i) If $s > -1$ and $s' \in \mathbb{R}$ or $s = -1$ and $s' > 1$, then

$$\chi\left(\widetilde{L}, \mathbb{C}[[x]]_{q,s,s'}\right) = \chi\left(\widetilde{L}, \mathbb{C}[[x]]_{(q,s,s')}\right) = 0.$$

(ii) If $s = -1$ and $s' = 1$, then

$$\chi\left(\widetilde{L}, \mathbb{C}[[x]]_{q,s,s'}\right) = 0, \quad \chi\left(\widetilde{L}, \mathbb{C}[[x]]_{(q,s,s')}\right) = -1.$$

Proof. By using the theory of index for operator in ordinary differential equations, caculating the index of $\tilde{\tilde{L}}$ directly, we can easily get the result. $\square$

No positive exceptional value s and s' for the operator $\widetilde{L}$. Then, we have the following results.

Lemma 3.9. *For $\widetilde{L} = -x(x\frac{\mathrm{d}}{\mathrm{d}x})\sigma_q - \alpha + \sigma_q \in \mathbb{L}$, if $\hat{f} \in \mathbb{C}[[x]]$ and $g \in \mathbb{C}\{x\}$ such that $\widetilde{L}(\hat{f}) = g$, then $\hat{f} \in \mathbb{C}\{x\}$.*

Theorem 3.4. *For $L = \frac{\mathrm{d}}{\mathrm{d}x} - \alpha\sigma_q + 1 \in \mathbb{L}_{-1}$, if $\hat{f} \in \mathbb{C}[[x^{-1}]]$ and $g \in \mathbb{C}\{x^{-1}\}$ such that $\widetilde{L}(\hat{f}) = g$, then $\hat{f} \in \mathbb{C}\{x^{-1}\}$.*

3.4 The operator with $|q| > 1$

There are no positive exceptional values s and s' for the operator L as $0 < |q| < 1$. In this section, we consider the case $|q| > 1$.

Let $p = \frac{1}{q}$. Thus, $\sigma_q = \sigma_p^{-1}$, and the operator $L = \frac{\mathrm{d}}{\mathrm{d}x} - \alpha\sigma_q + 1$ becomes

$$L = \Big[\frac{\mathrm{d}}{\mathrm{d}x}\sigma_p - \alpha + \sigma_p\Big]\sigma_p^{-1},$$

with $0 < |p| < 1$. Denote $\widetilde{L} = \frac{\mathrm{d}}{\mathrm{d}x}\sigma_p - \alpha + \sigma_p$. It's similar to the case $0 < |q| < 1$ of the two sections above, by replacing q with p. The operator σ_p^{-1} is an isomorphism of $l^1_{s,s',\rho,\mu} \to l^1_{s,s',\rho|p|,\mu}$ (resp. an automorphisme of $\mathbb{C}[[x]]_{p,s,s'}$ and $\mathbb{C}[[x]]_{(p,s,s')}$), therefore has zero index in these spaces. Then $\chi(L) = \chi(\widetilde{L})$. Let $s < 1$ and $s' \in \mathbb{R}$ or $s = 1$ and $s' \leq -1$.

Proposition 3.6. *The operator $\widetilde{L} = \frac{\mathrm{d}}{\mathrm{d}x}\sigma_p - \alpha + \sigma_p$ operates from $l^1_{s,s',\rho,\mu}$ to $l^1_{s,s',\rho|p|^{-p_s^0(\widetilde{L})},\mu+v_{s,s'}(\widetilde{L})}$ with*

$$p_s^0(\widetilde{L}) = \inf\{1-s, 0\}, \quad v_{s,s'}(\widetilde{L}) = \inf\{-s'-1, 0\}.$$

In other words, $\widetilde{L}$ is a $(\rho, |p|^{-p_s^0(\widetilde{L})}, \mu, v_{s,s'}(\widetilde{L}))$-operator.

Since $\widetilde{L} = \frac{\mathrm{d}}{\mathrm{d}x}\sigma_p - \alpha + \sigma_p$ can be seen as

$$\hat{L} = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}\alpha_{i,j,k}x^k(x\frac{\mathrm{d}}{\mathrm{d}x})^i\sigma_q^j$$

with $\alpha_{1,1,0} = 1$, $\alpha_{0,0,0} = -\alpha$, and $\alpha_{0,1,0} = 1$, the discussion will be devided into three cases.

(i) If $s < 1, s' \in \mathbb{R}$, then $p_s^0(\widetilde{L}) = 0$, $M(s) = N(s,s') = \{(0,0,0)\}$. The study of $\widetilde{L}$ reduces to the study of $L = -\alpha$.

(ii) If $s = 1, s' < -1$, then $M(s) = \{(1,1,0),(0,0,0)\}$ and $N(s,s') = \{(0,0,0)\}$, the same case as (i).

(iii) If $s = 1, s' = -1$, $M(s) = N(s,s') = \{(1,1,0),(0,0,0)\}$. In addition, $s = 1$ is an exceptional value for $\widetilde{L}$, and $s' = -1$ is an exceptional value relatively to $s = -1$ for $\widetilde{L}$. The study of $\widetilde{L}$ reduces to the study of $L = \frac{\mathrm{d}}{\mathrm{d}x}\sigma_p - \alpha$.

Lemma 3.10. *For the operator $\widetilde{L} = \frac{\mathrm{d}}{\mathrm{d}x}\sigma_p - \alpha + \sigma_p$.*

(i) If $s < 1$ and $s' \in \mathbb{R}$ or $s = 1$ and $s' < -1$, then

$$\chi\left(\widetilde{L}, \mathbb{C}[[x]]_{p,s,s'}\right) = \chi\left(\widetilde{L}, \mathbb{C}[[x]]_{(p,s,s')}\right) = 0.$$

(ii) If $s = 1$ and $s' = -1$, then

$$\chi\left(\widetilde{L}, \mathbb{C}[[x]]_{p,s,s'}\right) = 1, \quad \chi\left(\widetilde{L}, \mathbb{C}[[x]]_{(p,s,s')}\right) = 0.$$

Theorem 3.5. *For the operator $L=\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\sigma_q+1\in\mathbb{L}$ with $|q|>1$.*

(i) If $s<1$ and $s'\in\mathbb{R}$ or $s=1$ and $s'<-1$, then

$$\chi\left(L,\mathbb{C}[[x]]_{q,-s,s'}\right)=\chi\left(L,\mathbb{C}[[x]]_{(q,-s,s')}\right)=0.$$

(ii) If $s=1$ and $s'=-1$, then

$$\chi\left(L,\mathbb{C}[[x]]_{q,-s,s'}\right)=1,\quad \chi\left(L,\mathbb{C}[[x]]_{(q,-s,s')}\right)=0.$$

Theorem 3.6. *Let $L=\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\sigma_q+1\in\mathbb{L}$ with $|q|>1$, $\hat{f}\in\mathbb{C}[[x]]$ and $g\in\mathbb{C}\{x\}$ such that $L(\hat{f})=g$. Then :*

- *Either $\hat{f}\in\mathbb{C}\{x\}$,*
- *Either $\hat{f}\in\mathbb{C}[[x]]_{q,-1,-1}$ and $\hat{f}\notin\mathbb{C}[[x]]_{(q,-1,-1)}$: we say that $\hat{f}$ is q-Gevrey of exact order -1 and Gevrey of exact order -1.*

Proof. Since $s=1$ is the positive exceptional value for L in $\mathbb{C}[[x]]_{q,-s,s'}$ $(\mathbb{C}[[x]]_{(q,-s,s')})$, and $s'=-1$ is an exceptional value relatively to $s=-1$ for L, we can easily get the result from Lemma 3.6. □

Remark 3.3. *For the operator $L=\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\sigma_q+1\in\mathbb{L}$ $(|q|>1)$ for a formal series expansions at ∞, it becomes $L_\infty=-x^2\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\sigma_p+1$ $(0<|p|<1)$.*

(i) If $s>0$ and $s'\in\mathbb{R}$ or $s=0$ and $s'>1$, then

$$\chi\left(L,\mathbb{C}[[x]]_{q,-s,s'}\right)=\chi\left(L,\mathbb{C}[[x]]_{(q,-s,s')}\right)=0.$$

(ii) If $s=0$ and $s'=1$, then

$$\chi\left(L,\mathbb{C}[[x]]_{q,-s,s'}\right)=0,\quad \chi\left(L,\mathbb{C}[[x]]_{(q,-s,s')}\right)=-1.$$

We get that $s=0$ is the exceptional value for L_∞, and $s'=1$ is an exceptional value relatively to $s=0$ for L_∞.

Proposition 3.7. *Let $L=\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\sigma_q+1\in\mathbb{L}_{-1}$ with $|q|>1$, $\hat{f}\in\mathbb{C}[[x^{-1}]]$ and $g\in\mathbb{C}\{x^{-1}\}$ such that $L(\hat{f})=g$. Then :*

- *Either $\hat{f}\in\mathbb{C}\{x^{-1}\}$,*
- *Either $\hat{f}\in\mathbb{C}[[x^{-1}]]_{q,0,1}$ and $\hat{f}\notin\mathbb{C}[[x^{-1}]]_{(q,0,1)}$: we say that $\hat{f}$ is q-Gevrey of exact order 0 and Gevrey of exact order 1.*

3.5 Summary of this chapter

In this chapter, we study index theorem of the non-homogeneous pantograph equation. It is divided into two cases : $0 < q < 1$ and $q > 1$. The index and the exceptional values of the operators of formal power series at the origin and infinity are calculated, and the growth rates of the solutions of formal series are classified according to the existence of exceptional value. When $0 < q < 1$, there is no positive exceptional value, and then the influence of the non-homogeneous term on the solution satisfying the equation is obtained : if the non-homogeneous term is a series with a non-zero radius of convergence at the origin (infinity), then the corresponding solution of the formal series is also a series with a non-zero radius of convergence at the origin (infinity). This is consistent with the conclusion obtained later when the non-homogeneous term is analyzed in a more specific form.

Chapitre 4

Properties of solutions when the non-homogeneous term has a singularity at zero

4.1 Introduction

When the non-homogeneous term is a fraction with a singularity at zero, there is a unique polynomial transformation such that equation (1.1.3) with $f(x) = \frac{1}{x^m}$ is transformed into an equation (1.1.3) with $f(x) = \frac{1}{x}$ (see proposition below). Therefore, we will study the simplified non-homogeneous functional differential equation

$$y'(x) = \alpha y(qx) - y(x) + \frac{1}{x}, \tag{4.1.1}$$

where α is a non-zero complex number.

Proposition 4.1. *Let m be an integer with $m > 1$. If $\alpha \notin \{q, q^2, \cdots, q^{m-1}\}$, then there is a unique transformation of the form $y(x) = \sum_{j=1}^{m-1} a_j x^{-j} + dz(x)$, where $a_j, d \in \mathbb{C}^*$, such that the equation $y'(x) = \alpha y(qx) - y(x) + \frac{1}{x^m}$ is converted to $z'(x) = \alpha z(qx) - z(x) + \frac{1}{x}$.*

Proof. By direct computation, one gets that, for $j = 1, 2, \cdots, m-1$,

$$a_j = -(\alpha q^{-m+1}; q)_{m-j-1} \frac{(j-1)!}{(m-1)!}, \quad d = \frac{(\alpha q^{-m+1}; q)_{m-1}}{(m-1)!}.$$

The proof is completed. □

Remark 4.1. *If $\alpha = q^k$ for some $k \in \{1, 2, \cdots, m-1\}$, then there is a transformation of the form $y(x) = \sum_{j=k}^{m-1} a_j x^{-j} + z(x)$, such that equation $y'(x) = \alpha y(qx) - y(x) + \frac{1}{x^m}$ can be converted to the homogeneous equation $z'(x) = \alpha z(qx) - z(x)$.*

In order to prove the uniqueness of the solution of a non-homogeneous equation, we first prove the uniqueness of the solution of the corresponding homogeneous equation satisfying the initial value conditions, as follows.

Lemma 4.1. *Let $y(x)$ be a solution of Equation* (1.1.4) *analytic on* $(0, a)$ $(a > 0)$. *Then $y(x) = 0$ for all $x \in (0, a)$ if and only if $\lim_{x \to 0} y(x) = 0$.*

Proof. For $x \in (0, a)$, let $z(x) = y(x)e^x$. Then

$$z'(x) = (\alpha y(qx) - y(x))e^x + y(x)e^x = \alpha e^x y(qx) = \alpha e^{(1-q)x} z(qx).$$

If $0 < x_0 < x < x_1 < a$, then

$$z(x) = z(x_0) + \alpha \int_{x_0}^{x} e^{(1-q)t} z(qt) \mathrm{d}t.$$

Therefore, by letting $x_0 \to 0$, we have

$$z(x) = \alpha \int_0^x e^{(1-q)t} z(qt) \mathrm{d}t.$$

That means,

$$|z(x)| \leq |\alpha| x_1 e^{(1-q)x_1} \max_{0<t\leq qx_1} |z(t)|$$

holds for all $x \in (0, x_1]$. Therefore,

$$|y(x)| \leq |\alpha| x_1 e^{-x} e^{(1-q)x_1} \max_{0<t\leq qx_1} |e^t y(t)| \leq C x_1 \max_{0<t\leq qx_1} |y(t)|,$$

where $C = |\alpha| e^{x_1}$. Then

$$\max_{x \in (0, x_1]} |y(x)| \leq C x_1 \max_{0<t\leq qx_1} |y(t)|.$$

By iterating, we get

$$\max_{x \in (0, x_1]} |y(x)| \leq C^n q^{n(n-1)/2} x_1^n \max_{0<t\leq q^n x_1} |y(t)|.$$

One can obtain that $\max_{0<x\leq x_1} |y(x)| = 0$ as $n \to +\infty$. $\square$

4.2 Solutions expressed in terms of power series

Notice that Equation (1.1.3) with $f(x) = \frac{1}{x^m}$ can be tranformed into Equation (1.1.3) with $f(x) = \frac{1}{x}$. Therefore, in this section, we will study the solution of Equation (4.1.1) at zero and infinity.

We now study solutions of equation (4.1.1) by means of power series at zero. As usual, we denote by $H_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$, the n-th harmonic number.

Lemma 4.2. *Let $F_0(\alpha; q, x)$ be as in (1.3.5), and let u, v be two convergent power series. The function*

$$y(x) = u(x) \log x + v(x)$$

is a solution of equation (4.1.1) iff the two functions $u(x)$ and $v(x)$ satisfy the following equations :

$$\begin{aligned} &u'(x) = \alpha u(qx) - u(x), \quad u(0) = 1, \\ &v'(x) = \alpha v(qx) - v(x) + \alpha \ln q \cdot u(qx) - \tfrac{u(x)}{x} + \tfrac{1}{x}. \end{aligned} \tag{4.2.1}$$

Hence, $u(x) = F_0(\alpha; q, x)$.

Proof. From Lemma 2.1, we obtain that

$$\begin{aligned} &\log x [u'(x) - \alpha u(qx) + u(x)] = 0, \\ &v'(x) = \alpha v(qx) - v(x) + \alpha \ln q \cdot u(qx) - \tfrac{u(x)}{x} + \tfrac{1}{x}, \end{aligned} \tag{4.2.2}$$

The first equation of (4.2.2) is equivalent to Equation (1.1.4), hence $u(x) = u_0 \, F_0(\alpha; q, x)$, where $u_0 = u(0)$. Furthermore, the second equation of (4.2.2) admits a power series solution v only when $u(0) = 1$. So, we find the system given in (4.2.1). □

Theorem 4.1. *Let $\alpha \notin q^{\mathbb{Z}_{\leq 0}}$. For any constant c_0, we define*

$$F(\alpha; q, c_0, x) = (c_0 + \log x) F_0(\alpha; q, x) + F_1(\alpha; q, x), \tag{4.2.3}$$

where $F_0(\alpha; q, x)$ is as in (1.3.5) and where

$$F_1(\alpha; q, x) = \sum_{n \geq 1} \big(H_n + \sum_{k=0}^{n-1} \frac{\alpha q^k \ln q}{1 - \alpha q^k}\big) \frac{(-1)^{n-1} (\alpha; q)_n}{n!} x^n.$$

Then the function $F(\alpha; q, c_0, x)$ is a unique analytic solution on $\widetilde{\mathbb{C}^*}$ of equation (4.1.1) with the initial asymptotic condition $y(x) = \log x + c_0 + o(1)$, as $x \to 0$ along a certain direction.

Proof. Since $F_0(\alpha; q, x)$ is an analytic solution of Equation (1.1.4), substituting $F(\alpha; q, c_0, x)$ into Equation (4.1.1), we see that the first equation in (4.2.1) is satisfied. By taking $v(x) = F_1(\alpha; q, x)$, the second equation in (4.2.1) is satisfied.

Since the sequence $|1 + \alpha q^{k-1}(k \ln q - 1)]|$ is convergent as $k \to \infty$, then there exists a constant C_1 such that $|1 + \alpha q^{k-1}(k \ln q - 1)]| \leq C_1$, for all $k \geq 1$. Because $\alpha \notin q^{-\mathbb{N}}$, there exists a constant C_2 such that $|1 - \alpha q^{k-1}| \geq C_2$ for all $k \geq 1$. Therefore, for $n \geq 1$,

$$|\sum_{k=1}^{n} \frac{(-1)^{n-1}(\alpha; q)_n[1 + \alpha q^{k-1}(k \ln q - 1)]}{n!k(1 - \alpha q^{k-1})}| \leq \frac{C_1|(\alpha; q)_n|}{C_2(n-1)!}.$$

Hence, the radius of convergence of $F_1(\alpha; q, x)$ is ∞.

Furthermore, it is trivial that $F(\alpha; q, c_0, x) = \log x + c_0 + o(1)$ as $x \to 0$ along any direction. From the analysis above, the function $F(\alpha; q, c_0, x)$ is a solution of Equation (4.1.1) as described in the theorem.

For uniqueness, we can easily see that using a suitable change of variable, the difference of two solutions with the given asymptotic initial condition satisfies the conditions of Lemma 4.1. □

Next, we look for a power series solution at ∞.

Theorem 4.2. *If $\alpha \notin q^{\mathbb{Z}_{>0}}$, then the following power series*

$$G(\alpha; q, x) = x^{-1} \sum_{n \geq 0} \frac{(-1)^{n+1} \alpha^{-(n+1)} q^{(n+1)(n+2)/2} n!}{(\alpha^{-1}q; q)_{n+1}} x^{-n} \tag{4.2.4}$$

represents the unique analytic solution of Equation (4.1.1) *in $\mathbb{C}^* \cup \{\infty\}$ which vanishes at infinity.*

Proof. We suppose $y = \sum_{n \geq 0} a_n x^{-n-1}$ and substitute it into Equation (4.1.1), then

$$-\sum_{n \geq 0}(n+1)a_n x^{-n-2} = \sum_{n \geq 0}(\alpha q^{-n-1} - 1)a_n x^{-n-1} + x^{-1}.$$

Comparing the coefficients of x^{-1}, one obtains $a_0 = \frac{1}{1 - \alpha q^{-1}}$ (for $\alpha \neq q$). By comparing the coefficients of $x^{-2}, x^{-3}, \cdots$, we have $(\alpha q^{-n-1} - 1)a_n = -n a_{n-1}$ for $n \geq 1$. Therefore,

$$a_n = \frac{n}{1 - \alpha q^{-n-1}} a_{n-1} = \cdots = \frac{(-1)^n \alpha^{-n} q^{n(n+3)/2} n!}{(\alpha^{-1}q^2; q)_n} a_0$$

for $\alpha \notin q^{\mathbb{Z}_{>0}}$. Thus, the expression for a_n is unique. Since

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)\, q^{n+2}}{|\alpha - q^n|} = 0\,,$$

the power series (in $\frac{1}{x}$) $G(\alpha; q, x)$ defined by (4.2.4) is the unique analytic solution of Equation (4.1.1) in Riemann sphere $\mathbb{C}^* \cup \{\infty\}$, such that $\lim\limits_{x \to \infty} G(\alpha; q, x) = 0$. □

Remark 4.2. *When $\alpha \neq 1$, any analytic solution at infinity is null there. In this case, $G(\alpha; q, x)$ is the unique analytic solution at infinity.*

Proof. This comes from the fact that $\lim_{x\to\infty}(y'(x) - \frac{1}{x}) = 0$, which implies that $(\alpha - 1)\lim_{x\to\infty} y(x) = 0$. $\square$

The connection formula between $F(\alpha; q, c_0, x)$ and $G(\alpha; q, x)$ will be given in Section 4.5. In formula (4.2.4), one can notice that, for any $\alpha \notin q^{\mathbb{Z}_{>0}}$,

$$\Big|\frac{\alpha^{-(n+1)}}{(\alpha^{-1}q; q)_{n+1}}\Big| = \frac{1}{|(\alpha - q)(\alpha - q^2)\cdots(\alpha - q^{n+1})|} \leq \frac{1}{\epsilon^{n+1}}, \tag{4.2.5}$$

where $\epsilon = \min_{k\in\mathbb{Z}_{>0}} |\alpha - q^k|$. Furthermore, we have the following Remark.

Remark 4.3. *The function $F(\alpha; q, c_0, x)$ of Theorem 4.1 is analytic for $\alpha \notin q^{\mathbb{Z}_{\leq 0}}$, while the function $G(\alpha; q, x)$ of Theorem 4.2 is analytic for $\alpha \notin q^{\mathbb{Z}_{>0}}$.*

Indeed, by using the uniform convergence theorem, one can see that $F_0(\alpha; q, x)$ and $F_1(\alpha; q, x)$ are respectively analytic for $(\alpha, x) \in \mathbb{C} \times \mathbb{C}$ and for all $(\alpha, x) \in \mathbb{C} \times \mathbb{C}$ such that $\alpha q^k \neq 1$ for any $k \in \mathbb{Z}_{\geq 0}$. This implies the analyticity of $F(\alpha; q, c_0, x)$ for $(\alpha, x) \in \big(\mathbb{C} \setminus q^{\mathbb{Z}_{\leq 0}}\big) \times \mathbb{C}$. In the same way, we get the analyticity of $G(\alpha; q, x)$, by using (4.2.5).

4.3 Two Laplace integrals and relationship between them

In this section, we will look for solutions of (4.1.1) in the form of Laplace integral. For the convergence of Laplace integral, we refer to [67, p. 216, Definition A.2.3], whose main content is : Let $f \in C^2([0, +\infty))$. Suppose there are $A, B > 0$ such that $|f(x)| \leq Ae^{Bx}$, then the function $\xi \mapsto \int_0^{+\infty} f(x)e^{-x\xi}\mathrm{d}x$ converges for $\mathrm{Re}(\xi) > B$. It is denoted Lf, Laplace transform of f, which holomorphic for $\mathrm{Re}(\xi) > B$. We first give an elementary lemma as follows.

Lemma 4.3. *Assume that $d \in \mathbb{R}$, $\epsilon > 0$, and h is an analytic function in the open sector $\{t \in \widetilde{\mathbb{C}}^* \big| |\arg(t) - d| < \epsilon\}$, verifying $h(t) = O(t)$ for $t \to 0$, and $h(t) = O(e^{\lambda|t|})$ for $t \to \infty$ $(\lambda \geq 0)$. An integral of the form*

$$y(x) = \int_0^{\infty e^{id}} h(t)e^{-tx}\frac{\mathrm{d}t}{t}$$

is a solution of Equation (4.1.1) *in the domain*

$$S^d_{\epsilon,\lambda} = \bigcup_{\delta\in(d-\epsilon,d+\epsilon)} \{x \in \widetilde{\mathbb{C}}^* \big| \mathrm{Re}(xe^{i\delta}) > \lambda\},$$

if and only if the function $h(t)$ satisfies

$$\alpha h(t) + (qt-1)h(qt) = -qt. \tag{4.3.1}$$

Proof. One has $\dfrac{1}{x} = \displaystyle\int_0^{\infty e^{id}} te^{-tx}\frac{\mathrm{d}t}{t}$ for $\mathrm{Re}(xe^{\mathrm{i}d}) > 0$, and $y(x)$ can be extended into an analytic function in $S^d_{\epsilon,\lambda}$ under the given conditions. Thus, we can obtain the result by substituting $y(x)$ in Equation (4.1.1). □

4.3.1 Two types of Laplace integral solutions

Suppose $\phi(t) = \sum_{n\geq 1}\phi_n t^n$ and substisute it into Equation (4.3.1). Comparing the coefficients of t^n, we obtain

$$\phi_1 = \frac{1}{1-\alpha q^{-1}}, \text{ and for } n \geq 2, \phi_n = \frac{1}{1-\alpha q^{-n}}\phi_{n-1}. \tag{4.3.2}$$

Next, we use it to get an integral solution of Equation (4.1.1), as shown in the following lemma.

Lemma 4.4. *Let $\alpha \notin q^{\mathbb{Z}_{>0}}$, $d \in [0, 2\pi]$, and*

$$\phi(t) = \sum_{n\geq 0}\phi_{n+1}t^{n+1} = \sum_{n\geq 0}\frac{t^{n+1}}{(\alpha q^{-n-1};q)_{n+1}}. \tag{4.3.3}$$

Then the integral $L_1^{[d]}(\alpha;q,x) = \displaystyle\int_0^{\infty e^{id}}\phi(t)e^{-tx}\frac{\mathrm{d}t}{t}$ is an analytic solution of Equation (4.1.1) *for x in the Riemann surface $\widetilde{\mathbb{C}}^*$ such that* $\arg(x) \in (-d-\frac{\pi}{2}, -d+\frac{\pi}{2})$.

Consequently, glueing all the functions $L_1^{[d]}(\alpha;q,x)$ ($d \in [0,2\pi]$) gives rise to an analytic solution of Equation (4.1.1) *in $\mathbb{C}^* \cup \{\infty\}$, which will be denoted by $L_1(\alpha;q,x)$.*

Proof. First, from (4.3.2) and (1.3.3), we have $\phi_n = O(\alpha^{-n}q^{n(n+1)/2})$, we get that $\phi(t)$ is an entire function satisfying (4.3.1) and $\phi(0) = 0$. According to Lemma 4.3, it suffices to prove that $\phi(t)$ has at most exponential growth at ∞.

From [77, p. 59, Proposition 2.1], we obtain that

$$\left|\frac{\phi(t)}{t}\right| \leq Ce^{-\frac{(\log|t/\alpha\sqrt{q}|)^2}{2\ln q}}, \quad \text{as } |t| \to +\infty.$$

Therefore, as $t \to \infty$ in $\mathbb{C}$, $\phi(t) = O(e^{\lambda|t|})$ for any $\lambda > 0$. Hence, the integral $L_1^{[d]}(\alpha;q,x)$ is well-defined in any direction $d \in [0, 2\pi]$. This is to say, there is no singular direction. Thus, one obtains an analytic function on $\mathbb{C}^* \cup \{\infty\}$. □

Note that the proof above for Lemma 4.4 is based on a basic idea of the Borel-Laplace summation theory (see [67, p. 216, Proposition A.2.3]).

Theorem 4.3. *If $\alpha \notin q^{\mathbb{Z}_{>0}}$, then the relation*

$$G(\alpha; q, x) = L_1(\alpha; q, x) \tag{4.3.4}$$

holds for $x \in \mathbb{C}^* \cup \{\infty\}$.

Proof. By the definition in (4.2.4), one can get that

$$G(\alpha; q, x) = \sum_{n \geq 0} \frac{n!}{(\alpha q^{-n-1}; q)_{n+1}} x^{-n-1}.$$

Given any direction d and x such that $\mathrm{Re}(xe^{id}) > 0$, by using dominated convergence theorem, we have

$$L_1^{[d]}(\alpha; q, x) = \int_0^{\infty e^{id}} e^{-tx} \phi(t) \frac{\mathrm{d}t}{t} = \sum_{n \geq 0} \frac{n!}{(\alpha q^{-n-1}; q)_{n+1}} x^{-n-1}.$$

Then $G(\alpha; q, x) = L_1^{[d]}(\alpha; q, x)$. By the analytic continuation process, relation (4.3.4) holds for all $x \in \mathbb{C}^* \cup \{\infty\}$. □

In the following, we will introduce another kind of integral solution.

Lemma 4.5. *Let $|\alpha| < 1$, $d \in (0, 2\pi)$, and*

$$\psi(t) = (\alpha; q)_\infty \sum_{n \geq 0} \frac{\alpha^n \, t}{(q; q)_n (q^n - t)}. \tag{4.3.5}$$

The integral

$$L_2^{[d]}(\alpha; q, x) = \int_0^{\infty e^{id}} \psi(t) e^{-tx} \frac{\mathrm{d}t}{t}$$

is an analytic solution of Equation (4.1.1) for x in the Riemann surface $\widetilde{\mathbb{C}}^*$ such that $\arg(x) \in (-d - \frac{\pi}{2}, -d + \frac{\pi}{2})$.

Consequently, glueing all the functions $L_2^{[d]}(\alpha; q, x)$ ($d \in (0, 2\pi)$) gives rise to an analytic solution of Equation (4.1.1) for $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$, which will be denoted by $L_2(\alpha; q, x)$.

Proof. To see that $L_2^{[d]}(\alpha; q, x)$ is a solution of (4.1.1), we first prove that $\psi(t)$ is a solution of (4.3.1) (Lemma 4.3). Letting $g(t) = \psi(t)/t$, we only need to prove that $g(t)$ is a solution of

$$(1 - t) g(t) - \frac{\alpha}{q} g(\frac{t}{q}) = 1.$$

It can be proved by direct computation with the help of the following identity :

$$(\alpha; q)_\infty \sum_{n \geq 0} \frac{\alpha^n}{(q; q)_n} = 1.$$

Then $\psi(t)$ is a solution of Equation (4.3.1).

Next, we consider the growth of $\psi(t)$ at infinity to prove the convergence of the integral. Since

$$\begin{aligned} |\psi(t)| \leq & |(\alpha;q)_\infty| \sum_{n\geq 0} \frac{|\alpha|^n(|q^n| + |t-q^n|)}{(q;q)_n|q^n-t|} \\ = & |(\alpha;q)_\infty| \sum_{n\geq 0} \frac{|\alpha|^n q^n}{(q;q)_n|q^n-t|} + |(\alpha;q)_\infty| \sum_{n\geq 0} \frac{|\alpha|^n}{(q;q)_n}, \end{aligned} \tag{4.3.6}$$

the function $\psi(t)$ is bounded as $t \to \infty$ in any direction $d \in (0, 2\pi)$. The result follows by applying Lemma 4.3 ($\lambda = 0, d = \epsilon = \pi$). □

The following theorem describes the Stokes phenomenon of $L_2(\alpha; q, x)$.

Theorem 4.4. *Let* $|\alpha| < 1$ *and* $\arg(x) \in (-\frac{5\pi}{2}, -\frac{3\pi}{2})$*, the following formula holds :*

$$L_2(\alpha; q, xe^{2\pi i}) - L_2(\alpha; q, x) = 2\pi i(\alpha;q)_\infty \sum_{n\geq 0} \frac{\alpha^n e^{-q^n x}}{(q;q)_n}, \tag{4.3.7}$$

the right-hand side of (4.3.7) *is a solution of the homogenous Equation* (1.1.4).

Proof. Let L be any smooth and anti-clockwise curve whose interior contains the set $\{1, q, q^2 \cdots\}$. By the residue theorem,

$$\begin{aligned} L_2(\alpha; q, xe^{2\pi i}) - L_2(\alpha; q, x) &= \int_L (\alpha;q)_\infty \sum_{n\geq 0} \frac{\alpha^n}{(q;q)_n(t-q^n)} e^{-tx} \mathrm{d}t \\ &= 2\pi i(\alpha;q)_\infty \sum_{n\geq 0} \frac{\alpha^n e^{-q^n x}}{(q;q)_n}, \end{aligned}$$

which is (4.3.7). Since $|\alpha| < 1$, the series in (4.3.7) is convergent for $\mathrm{Re}(x) > 0$, which is satisfied when $\arg(x) \in (-\frac{5\pi}{2}, -\frac{3\pi}{2})$. By direct calculation, one can prove that the right-hand side of (4.3.7) is a solution of Equation (1.1.4). □

4.3.2 Relationship between the two integral solutions

We have established that $L_1(\alpha; q, x)$ and $L_2(\alpha; q, x)$ are solutions of Equation (4.1.1). Then $L_1(\alpha; q, x) - L_2(\alpha; q, x)$ is a solution of the homogeneous Equation (1.1.4). We now study the link formula between $L_1(\alpha; q, x)$ and $L_2(\alpha; q, x)$, by giving an explicit expression of $L_1(\alpha; q, x) - L_2(\alpha; q, x)$. For doing this, we should study the relation between the functions ϕ and ψ.

Lemma 4.6. *Let $\psi(t)$ be as in Lemma 4.5. For $|t| > 1$, the following equation holds :*

$$\psi(t) = -\sum_{n\geq 0}(\alpha; q)_n\big(\frac{1}{t}\big)^n.$$

Proof. By using $\frac{1}{1-\frac{q^n}{t}} = \sum_{m\geq 0}(\frac{q^n}{t})^m$ for every fixed n, Equation (4.3.5) becomes

$$\psi(t) = -(\alpha; q)_\infty \sum_{n\geq 0, m\geq 0} \frac{(\alpha q^m)^n}{(q;q)_n}\Big(\frac{1}{t}\Big)^m. \tag{4.3.8}$$

For every fixed m, by applying Equation (1.3.4), we obtain

$$(\alpha; q)_\infty \sum_{n\geq 0} \frac{(\alpha q^m)^n}{(q;q)_n} = \frac{(\alpha;q)_\infty}{(\alpha q^m;q)_\infty} = (\alpha; q)_m. \tag{4.3.9}$$

Putting (4.3.9) into (4.3.8) allows us to complete the proof. □

The functions ϕ and ψ are solutions of (4.3.1), and ϕ is an entire function while ψ has simple poles on the set $q^{\mathbb{N}}$. The homogeneous equation of (4.3.1) has special solutions, for example,

$$H(t) = \frac{\theta(-\frac{q}{\alpha}t)}{(\frac{q}{\alpha};q)_\infty(\frac{1}{t};q)_\infty}, \tag{4.3.10}$$

who has the same poles as ψ. We will obtain the relationship between $\phi(t), \psi(t)$, and $H(t)$, as shown in Lemma 4.7. From (1.3.3), we have

$$\phi(t) = \sum_{n\geq 0} \frac{t^{n+1}}{(\alpha q^{-n-1};q)_{n+1}} = \sum_{n\geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(\frac{q}{\alpha};q)_{n+1}}\Big(-\frac{qt}{\alpha}\Big)^{n+1}.$$

According to the definition of basic hypergeometric series in [37, p. 4, formula (1.2.22)] :

$${}_r\phi_s(a_1, a_2, \cdots, a_r; b_1, b_2, \cdots, b_s; q, z) = \sum_{n\geq 0} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\Big[(-1)^n q^{\frac{n(n-1)}{2}}\Big]^{1+s-r} z^n,$$

the right-hand side of the series in (4.3.8) can be viewed as a basic hypergeometic series, written as ${}_1\phi_1(q; \frac{q}{\alpha}; q, \frac{qt}{\alpha}) - 1$. The connection formula for ${}_2\phi_1$ is proposed in [37, p. 117, formula (4.3.2)], and the following relationship can also be regarded as the connection formula for ${}_1\phi_1$.

Lemma 4.7. *Let $\phi(t)$ and $\psi(t)$ be as in (4.3.3) and (4.3.5). For $|t| > 1$, the following equation holds :*

$$\phi(t) = H(t) + \psi(t). \tag{4.3.11}$$

Proof. Let $c = -\frac{q}{\alpha}$, Equation (4.3.10) can be rewritten as $H(t) = \frac{\theta(ct)}{(-c;q)_\infty(\frac{1}{t};q)_\infty}$. From formula (1.3.4), we have, for $|t| > 1$,

$$\frac{1}{(\frac{1}{t};q)_\infty} = \sum_{k\geq 0} \frac{1}{(q;q)_k} \big(\frac{1}{t}\big)^k.$$

Multiplying with the series $\theta(ct) = \sum_{l\in\mathbb{Z}} q^{\frac{l(l-1)}{2}}(ct)^l$, we obtain

$$H(t) = \frac{1}{(-c;q)_\infty} \sum_{k\geq 0} \frac{1}{(q;q)_k} \big(\frac{1}{t}\big)^k \sum_{l\in\mathbb{Z}} q^{\frac{l(l-1)}{2}}(ct)^l = H^+(t) + H^-(t),$$

where

$$H^+(t) = \sum_{k\geq 0}\sum_{l>k} \frac{c^l q^{\frac{l(l-1)}{2}}}{(-c;q)_\infty(q;q)_k} t^{l-k}, \quad H^-(t) = \sum_{k\geq 0}\sum_{l\leq k} \frac{c^l q^{\frac{l(l-1)}{2}}}{(-c;q)_\infty(q;q)_k} \big(\frac{1}{t}\big)^{k-l}.$$

Letting $m = l - k$ and replacing l with $k+m$, it follows that

$$H^+(t) = \sum_{m>0,k\geq 0} \frac{(cq^m)^k q^{\frac{k(k-1)}{2}} q^{\frac{m(m-1)}{2}}}{(-c;q)_\infty(q;q)_k}(ct)^m.$$

By applying Equation (1.3.4), for fixed m, we have

$$\sum_{k\geq 0}\frac{(cq^m)^k q^{\frac{k(k-1)}{2}}}{(q;q)_k} = (-cq^m;q)_\infty.$$

Therefore,

$$H^+(t) = \sum_{m>0} \frac{(-cq^m;q)_\infty}{(-c;q)_\infty} q^{\frac{m(m-1)}{2}}(ct)^m = \sum_{m>0} \frac{q^{\frac{m(m-1)}{2}}}{(\frac{q}{\alpha};q)_m}\big(-\frac{q}{\alpha}t\big)^m = \phi(t).$$

Next, letting $n = k - l$ and replacing l with $k - n$, we have

$$H^-(t) = \sum_{n\geq 0,k\geq 0} \frac{c^{k-n} q^{\frac{(k-n)(k-n-1)}{2}}}{(-c;q)_\infty(q;q)_k}\big(\frac{1}{t}\big)^n.$$

Notice that $\frac{(k-n)(k-n-1)}{2} = \frac{k(k-1)}{2} + \frac{n(n+1)}{2} - kn$. By summing on k and using Equation (1.3.4), we obtain

$$H^-(t) = \sum_{n\geq 0} c_n \big(\frac{1}{t}\big)^n,$$

where

$$c_n = \frac{(-\frac{c}{q^n};q)_\infty}{(-c;q)_\infty} c^{-n} q^{\frac{n(n+1)}{2}} = (-\frac{c}{q^n};q)_n c^{-n} q^{\frac{n(n+1)}{2}}.$$

By using formula (1.3.3), we have $c_n = (\alpha;q)_n$. From Lemma 4.6, we can obtain that $H^-(t) = -\psi(t)$, which completes the proof. □

Remark 4.4. *We note that Equation* (4.3.11) *is still valid for all* $t \in \mathbb{C}^* \backslash q^{\mathbb{N}}$. *This property will be used in the sequel.*

We obtain the link formula between the two integral solutions as follows.

Theorem 4.5. *Let* $|\alpha| < 1$ *and* $\alpha \notin q^{\mathbb{Z}_{>0}}$. *For any* x *verifying* $\arg(x) \in \left(-\frac{5\pi}{2}, \frac{\pi}{2}\right)$, *there exists a direction* $d \in (0, 2\pi)$ *such that the following relation holds :*

$$L_1(\alpha;q,x) = L_2(\alpha;q,x) + \int_0^{\infty e^{id}} e^{-tx} H(t) \frac{\mathrm{d}t}{t}, \tag{4.3.12}$$

where $H(t)$ *is shown in* (4.3.10).

Proof. For any given such x, one can find a direction $d \in (0, 2\pi)$ such that $\arg(x) \in \left(-d - \frac{\pi}{2}, -d + \frac{\pi}{2}\right)$, by using Lemma 4.7 and Remark 4.4, the proof is completed. □

4.3.3 Another approach to derive the relationship : elliptic functions

The relationship between the L_1 and L_2 integral solutions (4.3.12) can also be derived from related knowledge of elliptic functions.

Let $\phi(t)$ and $\psi(t)$ be defined as in (4.3.3) and (4.3.5), they are both solutions of Equation (4.3.1), so their difference : $I(t) = \phi(t) - \psi(t)$ is a solution of the homogeneous equation

$$(1-t)h(t) = \alpha h(\frac{t}{q}). \tag{4.3.13}$$

We will study the form of the meromorphic solution for $I(t)$ in the following.

$I(t)$ satisfies two conditions : (1) The set of sigularities for $I(t)$ is $\{q^n; n \in \mathbb{N}\} \cup \{0\}$; (2) For $n \in \mathbb{N}$, $Res[I(t), q^n] = \frac{(\alpha;q)_\infty \alpha^n q^n}{(q;q)_n}$. We will find the meromorphic solution of Equation (4.3.13) to detemine $I(t)$. Equation (4.3.13) has the solution

$$H_0(t) = C_0 (qt;q)_\infty \frac{\theta(\lambda t)}{\theta(\mu t)},$$

where $\frac{\mu}{\lambda} = \alpha$ and C_0 is an arbitrary constant. Let

$$H_1(t) = C_0 (qt;q)_\infty \frac{\theta(-\frac{q}{\alpha} t)}{\theta(-qt)}$$

by choosing $\lambda = -\frac{q}{\alpha}$ and $\mu = -q$. In the following, we will prove that $I(t)$ is equivalent to $H_1(t)$ as the constant C_0 determined.

For convenience, we introduce some notations from [58]. Let ω_1, ω_2 be two complex numbers linearly independent over the real numbers, which means that there is no relation $a\omega_1 + b\omega_2 = 0$ (with $a, b \in \mathbb{R}$ not both 0). Define $L[\omega_1, \omega_2]$ as the lattice generated by ω_1, ω_2, we obtained a fundamental period-parallelogram $L[\omega_1, \omega_2] = m\omega_1 + n\omega_2$ $(m, n \in \mathbb{Z})$, which is the lattice looks like the intersection of two families of parallel lines. For $\alpha \in \mathbb{C}$, we call the set consisting of all points $\alpha + t_1\omega_1 + t_2\omega_2$ $(0 \leq t_i < 1)$ a fundamental parallelgram for the lattice. From [58], we have the following two lemmas.

Lemma 4.8. *An elliptic function $f(z)$, which is the entire (without poles), must be constant.*

Proof. If $f(z)$ has no poles in a fundamental parallelogram, then it is analytic and consequently bounded in the parallelogram. There is a positive constant K, such that $|f(z)| < K$ for any z in the parallelogram. From the period properties of $f(z)$, we have $|f(z)| < K$ for any $z \in \mathbb{C}$. Then $f(z)$ is a constant from Liouville's theorem : Let $f(z)$ be analytic for all values of z and let $|f(z)| < K$ for all values of z, then $f(z)$ is a constant. See P105 of [14] for more details. $\square$

Lemma 4.9. *Let P be a fundamental parallelogram, and assume that the elliptic function of f has no zero and pole on its boundary. Let $\{a_i\}$ be the singular points (zeros and poles) of f inside P, and let f have order m_i at a_i. Then $\sum m_i = 0$.*

Lemma 4.10. *The functions with simple poles at $q^{\mathbb{Z}_{\geq 0}}$ that satisfy Equation (4.3.13) are all in the form of*

$$H_1(t) = C_0(qt;q)_\infty \frac{\theta(-\frac{q}{\alpha}t)}{\theta(-qt)}.$$

Proof. We only need to prove that the meromorphic solutions of Equation (4.3.13) differ by at most a constant multiple. If $H_2(t)$ is also a meromorphic solution of (4.3.13), then function $H_3(t) = \frac{H_2(t)}{H_1(t)}$ satisfies :

$$(i)\ H_3(qt) = H_3(t); \quad (ii)\ H_3(te^{2\pi \mathrm{i}}) = H_3(t),$$

which equivalent to :

$$(i)\ H_3(z+1) = H_3(z); \quad (ii)\ H_3(z + \frac{2\pi \mathrm{i}}{\ln q}) = H_3(z)$$

for $q^z = t$. Therefore, $H_3(t)$ is an elliptic function with periods $1, \mathrm{i}\tau$. Let P_0 be the fundamental parallelogram defined as the set consisting of all points : $\alpha + t_1 + t_2\mathrm{i}\tau$ $(0 \leq t_i < 1)$.

$H_2(t)$ is analytic in $\mathbb{C}^* \setminus q^{\mathbb{Z}_{\geq 0}}$, then $H_3(t)$ has as most simple poles $q^{\mathbb{Z}_{\geq 0}}$ in P_0. From lemma 4.8, $H_3(t)$ is analytic in P_0. From lemma 4.9, we have $H_3(t) = C$ (C is a constant) and then $H_2(t) = CH_1(t)$. □

Remark 4.5. *The function $H_1(t)$ with simple poles $q^{\mathbb{Z}_{\geq 0}}$ is the form of the meromorphic solution for $I(t)$, and C_0 is determined by the residue of $H_1(t)$.*

Then we determine the value of C_0, From the Jacobi triple product formula

$$\theta(x) = (q;q)_\infty(-x;q)_\infty(-\frac{q}{x};q)_\infty \triangleq (q,-x,-\frac{q}{x};q)_\infty, \tag{4.3.14}$$

we have

$$\begin{aligned}
I(t) =& C_0(qt;q)_\infty \cdot \frac{\theta(-\frac{q}{\alpha}t)}{\theta(-qt)} \\
=& C_0(qt;q)_\infty \cdot \frac{(q;q)_\infty(\frac{qt}{\alpha};q)_\infty(\frac{\alpha}{t};q)_\infty}{(q;q)_\infty(qt;q)_\infty(\frac{1}{t};q)_\infty} \\
=& \frac{C_0}{(q;q)_\infty} \cdot \frac{\theta(-\frac{\alpha}{t})}{(\frac{1}{t};q)_\infty} \\
=& \frac{C_0}{(q;q)_\infty} \cdot \frac{\theta(-\frac{\alpha}{t})}{(1-\frac{q^n}{t})(\frac{1}{t};q)_n(\frac{q^{n+1}}{t};q)_\infty}.
\end{aligned}$$

From Remark 1.3.2, we have

$$Res[I(t),q^n] = \frac{C_0}{(q;q)_\infty^2} \cdot \frac{\theta(-\frac{\alpha}{q^n})q^n}{(q^{-n};q)_n} = C_0\frac{\theta(-\alpha)}{(q;q)_\infty^2} \cdot \frac{\alpha^n q^n}{(q;q)_n}.$$

From the condition $Res[I(t),q^n] = \frac{(\alpha;q)_\infty \alpha^n q^n}{(q;q)_n}$, we obtain that $C_0 = \frac{(q;q)_\infty}{(\frac{q}{\alpha};q)_\infty}$. Therefore,

$$I(t) = \frac{\theta(-\frac{\alpha}{t})}{(\frac{q}{\alpha};q)_\infty(\frac{1}{t};q)_\infty}.$$

Theorem 4.6. *Let $|\alpha| < 1$ and $\arg x \in (-\frac{5\pi}{2}, \frac{\pi}{2})$. Then $I(t)$ is a meromorphic solution of Equation* (4.3.13)*, and the integral $I(\alpha;q,x) = \int_0^\infty e^{-tx} I(t)\frac{\mathrm{d}t}{t}$ is a solution of homogeneous equation $y'(x) = \alpha y(qx) - y(x)$. Futhermore, the following relation holds :*

$$L_0(\alpha;q,x) - y^*(\alpha;q,x) = \int_0^\infty e^{-tx}\frac{\theta(-\frac{\alpha}{t})}{(\frac{q}{\alpha};q)_\infty(\frac{1}{t};q)_\infty}\frac{\mathrm{d}t}{t},$$

for $\arg x \in (-\frac{5\pi}{2}, \frac{\pi}{2})$.

Proof. From

$$\begin{aligned}(1-t)I(t) &= \frac{(1-t)\theta(-\frac{\alpha}{t})}{(\frac{q}{\alpha};q)_\infty(\frac{1}{t};q)_\infty}\\ &= \frac{(-t)}{(\frac{q}{\alpha};q)_\infty(\frac{q}{t};q)_\infty}\sum_{n\in\mathbb{Z}}(-1)^n\alpha^n q^{n(n-1)/2}t^{-n}\\ &= \frac{\alpha}{(\frac{q}{\alpha};q)_\infty(\frac{q}{t};q)_\infty}\sum_{n\in\mathbb{Z}}(-1)^{n-1}(\alpha q)^{n-1}q^{(n-1)(n-2)/2}t^{-(n-1)}\\ &= \alpha I(\frac{t}{q}),\end{aligned}$$

we have $I(t)$ is a solution of Equation (4.3.13). Since $d \in (0, 2\pi)$, we have t satisfies that $t \notin q^{\mathbb{N}} \cup \{0\}$ and

$$I^{[d]}(\alpha;q,x) = \int_0^{\infty e^{id}} e^{-tx} I(t)\frac{\mathrm{d}t}{t}$$

is a solution of $y'(x) = \alpha y(qx) - y(x)$. Together with the analytic continuation process, it follows that $I(\alpha;q,x)$ converges for $\arg x \in (-\frac{5\pi}{2}, \frac{\pi}{2})$. Since

$$L_1(\alpha;q,x) - L_2(\alpha;q,x) = \int_0^{\infty} e^{-tx}\left(\phi(t) - \psi(t)\right)\frac{\mathrm{d}t}{t} = \int_0^{\infty} e^{-tx} I(t)\frac{\mathrm{d}t}{t},$$

the proof is completed. □

4.4 Expression of integral solutions in terms of series

This section focuses on exploring the relationship between series and integral solutions, which is structured into the following aspects. By applying perturbations of the equation, we derive an alternative expression for $L_2(\alpha;q,x)$, which is represented in terms of series expansions at zero. Finally, we establish the link formula between $F(\alpha;q,c_0,x)$ and $L_2(\alpha;q,x)$ by using Theorem 4.3.

4.4.1 Perturbations of the parameter α

The idea of perturbations of the equation gives another representation of $L_2(\alpha;q,x)$. In the following, we consider α as a parameter close to zero.

If $\alpha = 0$, then Equation (4.1.1) becomes

$$y'(x) = -y(x) + \frac{1}{x}, \tag{4.4.1}$$

which has the divergent series solution $\sum_{n\geq 0} n!(\frac{1}{x})^{n+1}$. From the Definition 1.3.1.2 in [67] : We place ourselves in the plane of $\tau = \frac{1}{x}$. $\sum \frac{a_n}{\tau^n}$ is Borel-summable in $U = \{\mathrm{Re}(\tau) > 0\}$, of sum f, if :

$$\exists C > 0,\ \forall n \in \mathbb{N},\ \forall \tau \in U, \quad \big| f(\tau) - \sum_{k<n} \frac{a_k}{\tau^k} \big| \leq \frac{C^n \cdot n!}{|\tau|^n}.$$

We obtain that, given any $d \in (0, 2\pi)$, the corresponding Borel-sum in the direction d is

$$E^{[d]}(x) = \int_0^{\infty e^{\mathrm{i}d}} e^{-xt} \frac{\mathrm{d}t}{1-t},$$

for $\mathrm{Re}(xe^{\mathrm{i}d}) > 0$, *i.e.*, $\arg(x) \in (-d - \frac{\pi}{2}, -d + \frac{\pi}{2})$. By the analytic continuation process, it yields a function $E(x)$, which is a solution of (4.4.1) for $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$ on $\widetilde{\mathbb{C}}^*$, for more details, see Ramis [75, p. 183, Definition 3.1] and Malgrange [67, p. 217, Proposition A.2.4].

Since $L_2(\alpha; q, x)$ is a solution of (4.1.1) with $\alpha \neq 0$ and $E(x)$ is a solution of (4.1.1) with $\alpha = 0$, Equation (4.1.1) can be seen as the perturbation of Equation (4.4.1). We expect to expand $L_2(\alpha; q, x)$ as a perturbation series related to $E(x)$.

Proposition 4.2. *Let $\alpha \neq 0$ and $|\alpha| < 1$. For $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$ on $\widetilde{\mathbb{C}}^*$, the following relation holds*

$$L_2(\alpha; q, x) = (\alpha; q)_\infty \sum_{n\geq 0} \frac{\alpha^n}{(q;q)_n} E(q^n x). \tag{4.4.2}$$

Consequently, $L_2(\alpha; q, x) \to E(x)$ as $\alpha \to 0$.

Proof. For any given x with $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$, one can find a direction $d \in (0, 2\pi)$ such that $\arg(x) \in \big(-d - \frac{\pi}{2}, -d + \frac{\pi}{2}\big)$. By the definition of $L_2(\alpha; q, x)$, we have

$$L_2^{[d]}(\alpha; q, x) = (\alpha; q)_\infty \int_0^{\infty e^{id}} \sum_{n\geq 0} \frac{\alpha^n\, t}{(q;q)_n (q^n - t)} e^{-tx} \frac{\mathrm{d}t}{t},$$

where the series under the integral verifies (4.3.6). Since

$$\int_0^{\infty e^{id}} \frac{t}{q^n - t} e^{-tx} \frac{\mathrm{d}t}{t} = E^{[d]}(q^n x),$$

we can apply the dominated convergence theorem to obtain (4.4.2). □

Let $\alpha \neq 0$ and close to 0. Note that a function in the form $y(x) = \sum_{n\geq 0} y_n(x)\alpha^n$ is a solution of Equation (4.1.1) iff the functions $y_n(x)$ satisfy the following system of differential equations :

$$\begin{aligned} &y_0'(x) + y_0(x) = \tfrac{1}{x}, \\ &y_{n+1}'(x) + y_{n+1}(x) = y_n(qx), \quad n \geq 0. \end{aligned} \tag{4.4.3}$$

By using equations (1.3.4) and (4.4.2), we obtain the following corollary, which gives a particular solution to the above system.

Corollary 4.1. *For* $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$*, a particular solution of system* (4.4.3) *is given by the following*

$$y_n(x) = \sum_{k=0}^{n} \frac{(-1)^k q^{k(k-1)/2}}{(q;q)_k (q;q)_{n-k}} E(q^{n-k}x), \text{ for } n \geq 0.$$

4.4.2 Relationship between $L_2(\alpha; q, x)$ and $F(\alpha; q, c_0, x)$

To derive the link formula between $L_2(\alpha; q, x)$ and $F(\alpha; q, c_0, x)$, we will express $L_2(\alpha; q, x)$ in terms of power series at 0. This is done by establishing the relationship between $E(x)$ and the series solution at 0 of Equation (4.4.1), because formula (4.4.2) indicates that $L_2(\alpha; q, x)$ can be expressed in terms of $E(x)$, and $E(x)$ is a solution of (4.4.1).

Lemma 4.11. *Let* $\gamma = \Gamma'(1)$ *be the Euler's constant* ([*28*, p.185, *formula* (10.8.1)]) *and*

$$w_\gamma(x) = \gamma e^{-x} + e^{-x}\log(e^{i\pi}x) + \sum_{n\geq 1} \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-k)(n-k)!\,k!} x^n. \tag{4.4.4}$$

Then, for $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$*, we have* $E(x) = w_\gamma(x)$.

Proof. Choose $d = \pi$ for $E^{[d]}(x)$. Given x such that $\arg(x) \in (-\frac{3\pi}{2}, -\frac{\pi}{2})$, we have $\mathrm{Re}(x) < 0$, and

$$E^{[\pi]}(x) = \int_0^{\infty e^{i\pi}} \frac{e^{-tx}}{1-t} \mathrm{d}t = -\int_0^{+\infty} \frac{e^{tx}}{1+t} \mathrm{d}t.$$

From [28, p. 276, formula (14.1.10)], we have the exponential integral

$$E_1(z) := \int_1^{\infty} \frac{e^{-zt}}{t} \mathrm{d}t = -\gamma - \mathrm{Ln}(z) - \sum_{k\geq 1} \frac{(-1)^k z^k}{k!k},$$

where $\mathrm{Re}(z) > 0$. If $z = e^{i\pi}x$, then we have

$$E^{[\pi]}(x) = -e^{-x}E_1(e^{i\pi}x) = \gamma e^{-x} + e^{-x}\log(e^{i\pi}x) + e^{-x} \sum_{k\geq 1} \frac{x^k}{k!k}.$$

One completes the proof by direct computation and analytic continuation. □

As usual, we denote by $H_n = \sum_{k=1}^{n} \frac{1}{k}$, the n-th harmonic number.

Remark 4.6. *The series in* (4.4.4) *has another expression as follows*

$$w_\gamma(x) = \gamma e^{-x} + e^{-x}\log(e^{i\pi}x) + \sum_{n\geq 1} \frac{(-1)^{n-1} H_n}{n!} x^n. \tag{4.4.5}$$

By utilizing formula (4.4.2) and Lemma 4.11, we have the following corollary.

Corollary 4.2. *Let $|\alpha| < 1$ and $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$. The function $L_2(\alpha; q, x)$ can be expressed in terms of $w_\gamma(x)$ as follows :*

$$L_2(\alpha; q, x) = (\alpha; q)_\infty \sum_{m\geq 0} \frac{\alpha^m}{(q;q)_m} w_\gamma(q^m x). \tag{4.4.6}$$

The following lemma presents a relation that will be used to study the relationship between $L_2(\alpha; q, x)$ and solutions at 0 of Equation (4.1.1).

Lemma 4.12. *For any $|x| < 1$, the following equation holds*

$$\sum_{m\geq 0} \frac{m x^m}{(q;q)_m} = \frac{1}{(x;q)_\infty} \sum_{k\geq 0} \frac{x q^k}{1 - x q^k}. \tag{4.4.7}$$

Proof. For any $|x| < 1$, by using (1.3.4), we have

$$\Big[\frac{x}{(x;q)_\infty}\Big]' = \Big[\sum_{m\geq 0} \frac{x^{m+1}}{(q;q)_m}\Big]' = \sum_{m\geq 0} \frac{(m+1)x^m}{(q;q)_m}.$$

Then

$$\begin{aligned}\sum_{m\geq 0} \frac{m x^m}{(q;q)_m} &= \Big[\frac{x}{(x;q)_\infty}\Big]' - \sum_{m\geq 0} \frac{x^m}{(q;q)_m} \\ &= \frac{(x;q)_\infty - x[(x;q)_\infty]'}{(x;q)_\infty^2} - \sum_{m\geq 0} \frac{x^m}{(q;q)_m} = \frac{-x[(x;q)_\infty]'}{(x;q)_\infty^2}.\end{aligned}$$

Since $[(x;q)_\infty]' = (x;q)_\infty \sum_{k\geq 0} \frac{-q^k}{1 - x q^k}$, Equation (4.4.7) can be easily proved. □

We now examine the relationship between $L_2(\alpha; q, x)$ and $F(\alpha; q, c_0, x)$, where the function $F(\alpha; q, c_0, x)$ is defined in Theorem 4.1. For a special value of c_0, we have the following result.

Theorem 4.7. *Let $\alpha \notin q^{\mathbb{Z}_{\leq 0}}$ and $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$. The following relation holds :*

$$L_2(\alpha; q, x) = F(\alpha; q, \gamma_0, x), \tag{4.4.8}$$

where

$$\gamma_0 = \sum_{k\geq 0} \frac{\alpha q^k \ln q}{1 - \alpha q^k} + \gamma + i\pi. \tag{4.4.9}$$

Proof. According to Remark 4.3, the function $F(\alpha; q, \gamma_0, x)$ is analytic for $\alpha \notin q^{\mathbb{Z}_{\leq 0}}$. Thus, we need only to establish the validity of (4.4.8) for $|\alpha| < 1$.

First, we have (see [98, p. 7, Proposition 2.1])

$$F_0(\alpha; q, x) = (\alpha; q)_\infty \sum_{m\geq 0} \frac{\alpha^m e^{-q^m x}}{(q;q)_m} = \sum_{m\geq 0} \frac{(-1)^m(\alpha;q)_m}{m!} x^m,$$

Then, from (4.4.5) and (4.4.6), we obtain

$$L_2(\alpha; q, x) = (\log(e^{i\pi}x) + \gamma)F_0(\alpha; q, x) + A_1 + A_2,$$

where

$$A_1 = (\alpha; q)_\infty \sum_{m\geq 0} \frac{\alpha^m}{(q;q)_m} (\log(q^m))e^{-q^m x}$$

and

$$A_2 = (\alpha; q)_\infty \sum_{m\geq 0} \frac{\alpha^m}{(q;q)_m} \sum_{n\geq 1}\sum_{k=1}^{n} \frac{(-1)^{n-1}}{n!k} q^{mn} x^n.$$

By using the exponential series and inversion of the summation order of convergent power series, we get

$$\begin{aligned} A_1 &= (\alpha; q)_\infty (\ln q) \sum_{m\geq 0} \frac{m\alpha^m}{(q;q)_m} \sum_{k\geq 0} \frac{(-q^m x)^k}{k!} \\ &= (\alpha; q)_\infty (\ln q) \sum_{k\geq 0} \frac{(-x)^k}{k!} \sum_{m\geq 0} \frac{m(\alpha q^k)^m}{(q;q)_m}. \end{aligned}$$

By applying (4.3.9) and (4.12), we further have

$$\begin{aligned} A_1 &= (\alpha; q)_\infty (\ln q) \sum_{k\geq 0} \frac{(-x)^k}{k!(\alpha q^k; q)_\infty} \sum_{m\geq 0} \frac{\alpha q^{k+m}}{1-\alpha q^{k+m}} \\ &= \sum_{n\geq 0}\sum_{k\geq n} \frac{\alpha q^k \ln q}{1-\alpha q^k} \frac{(-1)^n(\alpha;q)_n}{n!} x^n. \end{aligned}$$

Together with (1.3.4), we get

$$A_2 = (\alpha; q)_\infty \sum_{n\geq 1}\sum_{k=1}^{n} \Big(\sum_{m\geq 0} \frac{\alpha^m q^{mn}}{(q;q)_m}\Big) \frac{(-1)^{n-1}}{n!} \frac{x^n}{k} = \sum_{n\geq 1} \frac{(-1)^{n-1} H_n}{n!} (\alpha; q)_n x^n.$$

Hence,

$$A_1 + A_2 = F_1(\alpha; q, x) + \sum_{k\geq 0} \frac{\alpha q^k \ln q}{1-\alpha q^k} F_0(\alpha; q, x).$$

Thus, the proof is completed. $\square$

4.5 Connection formula and asymptotic behavior

We recall that $F(\alpha;q,c_0,x)$ and $G(\alpha;q,x)$ are as in (4.2.3) and (4.2.4). In this section, we first introduce some lemmas about q-periodic functions. Then, we present the connection formula between $F(\alpha;q,c_0,x)$ and $G(\alpha;q,x)$ for a special case ($c_0=\gamma_0$). The asymptotic behaviors of $F(\alpha;q,\gamma_0,x)$ are obtained using the connection formula. Finally, we draw conclusions about the connection formula and the asymptotic behaviors at ∞ of solutions around zero for the general case.

4.5.1 Two families of q-periodic functions

Let μ be a fixed complex number such that $\alpha=q^{\mu}$ and $-\frac{\pi}{|\ln q|}<\mathrm{Im}(\mu)\leq\frac{\pi}{|\ln q|}$. There are infinity numbers of $\mu_l=\mu+i\kappa l$ ($l\in\mathbb{Z}$ and $\kappa=-\frac{2\pi}{\ln q}$), such that $q^{\mu_l}=\alpha$. All values of μ_l form a set, which we call $\Lambda_\alpha=\{\mu_l\in\mathbb{C}:q^{\mu_l}=\alpha\}$.

Lemma 4.13. *Let $\alpha\notin q^{\mathbb{Z}}$. For $\arg(x)\in(-\frac{5\pi}{2},\frac{\pi}{2})$, the function*

$$g_n(\alpha;q,x)=\sum_{\mu_l\in\Lambda_\alpha}\frac{\Gamma(n+\mu_l)x^{-\mu_l}}{1-e^{2\pi i\mu_l}},\ n\geq 0 \tag{4.5.1}$$

is an analytic solution of the equation $y(x)=\alpha y(qx)$.

Proof. For $\mu_l\in\Lambda_\alpha$, we have

$$|x^{-\mu_l}|=|e^{-(\mathrm{Re}(\mu_l)+i\mathrm{Im}(\mu_l))(\log|x|+i\arg(x))}|=e^{\mathrm{Im}(\mu_l)\arg(x)}|x|^{-\mathrm{Re}(\mu)}. \tag{4.5.2}$$

Therefore,

$$|1-e^{2\pi i\mu_l}|\geq\big|1-|e^{2\pi i\mu_l}|\big|=|1-e^{-2\pi\mathrm{Im}\mu}e^{-2\pi\kappa l}|. \tag{4.5.3}$$

Since $\alpha\notin q^{\mathbb{Z}}$, we have $\mu_l\notin\mathbb{Z}\oplus i\kappa\mathbb{Z}$. By using Stirling's formula for the Gamma function (see [14, p. 21, Corollary 1.4.4]), it yields that

$$\Gamma(n+\mu_l)=\sqrt{2\pi}|\mathrm{Im}(\mu_l)|^{n+\mathrm{Re}(\mu_l)-\frac{1}{2}}e^{-\frac{\pi|\mathrm{Im}(\mu_l)|}{2}}\left[1+O\Big(\frac{1}{|\mathrm{Im}(\mu_l)|}\Big)\right] \tag{4.5.4}$$

as $|\mathrm{Im}(\mu_l)|\to+\infty$.

(i) If $\mathrm{Im}(\mu_l)>0$ ($l>0$), then for $\mathrm{Im}(\mu_l)=\mathrm{Im}(\mu)+\kappa l$, we have

$$\Gamma(n+\mu_l)=O\big(l^{n+\mathrm{Re}(\mu)-\frac{1}{2}}e^{-\frac{\pi\kappa l}{2}}\big),\quad\text{as } l\to+\infty.$$

Together with equations (4.5.2) and (4.5.3), we obtain that

$$\Big|\sum_{\mathrm{Im}(\mu_l)>0}\frac{\Gamma(n+\mu_l)x^{-\mu_l}}{1-e^{2\pi i\mu_l}}\Big|\leq Ce^{\mathrm{Im}(\mu)\arg(x)}\sum_{l\in\mathbb{Z}_{>0}}\frac{l^{n+\mathrm{Re}(\mu)-\frac{1}{2}}e^{\kappa l(\arg(x)-\frac{\pi}{2})}}{1-e^{-2\pi\mathrm{Im}(\mu)-2\pi\kappa l}}|x|^{-\mathrm{Re}(\mu)}. \tag{4.5.5}$$

The series of the right-hand side of (4.5.5) is convergent if the ratio of two consecutive terms tends to a limit smaller than 1, that is to say :

$$\lim_{l\to+\infty}\Big|\big(1+\frac{1}{l}\big)^{n+\mathrm{Re}(\mu)-\frac{1}{2}}\frac{(1-e^{-2\pi\mathrm{Im}\mu-2\pi\kappa l})e^{\kappa(\arg(x)-\frac{\pi}{2})}}{1-e^{-2\pi\mathrm{Im}\mu-2\pi\kappa(l+1)}}\Big|=e^{\kappa(\arg(x)-\frac{\pi}{2})}<1,$$

i.e., $\arg(x)<\frac{\pi}{2}$.

(ii) If $\mathrm{Im}(\mu_l)<0$ $(l<0)$, then Equation (4.5.4) becomes

$$\Gamma(n+\mu_l)=O\big((-\mathrm{Im}(\mu_l))^{n+\mathrm{Re}(\mu)-\frac{1}{2}}e^{\frac{\pi\mathrm{Im}(\mu_l)}{2}}\big),\ \text{when}\ \ \mathrm{Im}(\mu_l)\to-\infty.$$

By taking $m=-l$, we have

$$\Gamma(n-\mu_m)=O\big((-m)^{n+\mathrm{Re}(\mu)-\frac{1}{2}}e^{-\frac{\pi\kappa m}{2}}\big),\ \text{as}\ \ m\to+\infty.$$

Equation (4.5.2) becomes $\big|x^{-\mu_l}\big|=e^{(\mathrm{Im}(\mu)-\kappa m)\arg(x)}|x|^{-\mathrm{Re}(\mu)}$. Therefore, we have

$$\Big|\sum_{\mathrm{Im}(\mu_l)<0}\frac{\Gamma(n+\mu_l)x^{-\mu_l}}{1-e^{2\pi i\mu_l}}\Big|\le Ce^{\mathrm{Im}(\mu)\arg(x)}\sum_{m\in\mathbb{Z}_{>0}}\frac{m^{n+\mathrm{Re}(\mu)-\frac{1}{2}}e^{-\kappa m(\arg(x)+\frac{\pi}{2})}}{e^{-2\pi\mathrm{Im}(\mu)+2\pi\kappa m}-1}|x|^{-\mathrm{Re}(\mu)}. \tag{4.5.6}$$

The series of the right-hand side of (4.5.6) is convergent if

$$\lim_{m\to+\infty}\Big|\big(1+\frac{1}{m}\big)^{n+\mathrm{Re}(\mu)-\frac{1}{2}}\frac{(1-e^{-2\pi\mathrm{Im}\mu+2\pi\kappa m})e^{-\kappa(\arg(x)+\frac{\pi}{2})}}{1-e^{-2\pi\mathrm{Im}\mu+2\pi\kappa(m+1)}}\Big|=e^{-\kappa(\arg(x)+\frac{5\pi}{2})}<1,$$

i.e., $\arg(x)>-\frac{5\pi}{2}$.

To sum up, the Laurent series in (4.5.1) is convergent if the series of the right-hand side of (4.5.5) and (4.5.6) are convergent. Therefore, the function $g_n(\alpha;q,x)$ is well-defined and analytic for $\arg(x)\in(-\frac{5\pi}{2},\frac{\pi}{2})$. Finally, it is obvious that $g_n(\alpha;q,x)$ satisfies $\alpha y(qx)=y(x)$ by direct computation. □

Remark 4.7. *For* $\arg(x)\in(-\frac{\pi}{2},\frac{\pi}{2})$*, let*

$$h_n(\alpha;q,x)=g_n(\alpha;q,x)-g_n(\alpha;q,xe^{-2\pi i}).$$

Then, we have $h_n(\alpha;q,x)=\sum_{\mu_l\in\Lambda_\alpha}\Gamma(n+\mu_l)x^{-\mu_l}$.

The next lemma gives estimates of the functions $g_n(\alpha;q,x)$ and $h_n(\alpha;q,x)$.

Lemma 4.14. *For any fixed n, the functions $\hat{g}_n(\alpha;q,x)=x^\mu g_n(\alpha;q,x)$ and $\hat{h}_n(\alpha;q,x)=x^\mu h_n(\alpha;q,x)$ are q-periodic functions and bounded as $x\to\infty$ in any direction $d\in(-\frac{5\pi}{2},\frac{\pi}{2})$ and $d\in(-\frac{\pi}{2},\frac{\pi}{2})$, respectively.*

Proof. Both $g_n(\alpha;q,x)$ and $h_n(\alpha;q,x)$ are solutions of equation $y(x)=\alpha y(qx)$. It is easy to verify that $\hat{g}_n$ and $\hat{h}_n$ are q-periodic functions, i.e., satisfying $\hat{y}(qx)=\hat{y}(x)$. We only need to prove that $\hat{g}_n$ satisfies $|\hat{g}_n(\alpha;q,x)|\le C$ as $x\to\infty$ in any direction $\arg(x)\in(-\frac{5\pi}{2},\frac{\pi}{2})$. For $\hat{h}_n$, we change the direction to $\arg(x)\in(-\frac{\pi}{2},\frac{\pi}{2})$.

For any $d\in(-\frac{5\pi}{2},\frac{\pi}{2})$ and $x\in[\frac{1}{q}e^{id},\frac{1}{q^2}e^{id}]$, we have $\hat{y}(x)=\hat{y}(qx)$, where $qx\in[e^{id},\frac{1}{q}e^{id}]$. Therefore, using the continuity and taking $C=\max\limits_{x\in[e^{id},\frac{1}{q}e^{id}]}|\hat{y}(x)|$, we have $|\hat{y}(x)|\le C$ for all $x=te^{id}$ with $t\ge 1$. $\square$

4.5.2 The connection formula for the critical value $c_0=\gamma_0$

In this section, we will present the connection formula between $F(\alpha;q,\gamma_0,x)$ and $G(\alpha;q,x)$. We first introduce a lemma that will be used later.

Define $e(q;x)=e^{-\frac{\log^2\frac{x}{\sqrt{q}}}{2\ln q}}$. One can get that both $e(q;x)$ and $\theta(q;x)$ satisfy equation $xy(qx)=y(x)$. Let $q^*=e^{-2\pi\kappa}$ and $x^*=x^{-i\kappa}$. From [14, p. 498, (10.4.2)] and [98, p. 12, (4.2)], we have

$$\theta(q;e^{i\pi}x)=\sqrt{\kappa}\,e(q;e^{i\pi}x)\theta(q^*;e^{i\pi}x^*), \tag{4.5.7}$$

and from [98, p. 12, Lemma 4.1], we have

$$\frac{\theta(q;-q^\mu x)}{\theta(q;-x)}=q^{-\mu(\mu-1)/2}(e^{i\pi})^{-\mu}\frac{\theta(q^*;-e^{2\pi i\mu}x^*)}{\theta(q^*;-x^*)}. \tag{4.5.8}$$

Define $(q^\mu,q^{1-\mu};q)_\infty=(q^\mu;q)_\infty(q^{1-\mu};q)_\infty$. We have the following lemma.

Lemma 4.15. *Let $\mu\in\mathbb{C}\setminus\mathbb{Z}$. For $\arg(t)\in(0,2\pi)$, the following relation holds :*

$$\frac{\theta(q;-\frac{q^\mu}{t})}{(\frac{1}{t};q)_\infty}=\frac{\kappa(q^\mu,q^{1-\mu};q)_\infty}{i(q;q)_\infty}\sum_{n\ge 0}\sum_{l\in\mathbb{Z}}\frac{(-1)^n q^{n(n+1)/2}t^{n+\mu+i\kappa l}}{(q;q)_n(1-e^{2\pi i(\mu+i\kappa l)})}. \tag{4.5.9}$$

Proof. Let $x=\frac{1}{t}$. Since $\arg(t)\in(0,2\pi)$, we have $\arg(x)\in(-2\pi,0)$. Then

$$|x^*|=e^{\kappa\arg(x)}\in(e^{-2\kappa\pi},1)=(q^*,1).$$

From [14, p. 502, (10.5.3)] : for $|q|<1$ and $|ba^{-1}|<1$,

$$\sum_{l\in\mathbb{Z}}\frac{(a;q)_l}{(b;q)_l}x^l=\frac{(ax;q)_\infty(q/ax;q)_\infty(q;q)_\infty(b/a;q)_\infty}{(x;q)_\infty(b/ax;q)_\infty(b;q)_\infty(q/a;q)_\infty}.$$

Letting $q=q^*$, $x=x^*$, $a=e^{2\pi i\mu}$ and $b=q^*e^{2\pi i\mu}$ yields that

$$\frac{\theta(q^*;-e^{2\pi i\mu}x^*)}{\theta(q^*;-x^*)}=\frac{\theta(q^*;-e^{2\pi i\mu})}{(q^*,q^*)^3_\infty}\sum_{l\in\mathbb{Z}}\frac{x^{*l}}{1-e^{2\pi i\mu}q^{*l}}. \tag{4.5.10}$$

From (4.5.8) and (4.5.10), we have

$$\frac{\theta(q;-q^{\mu}x)}{\theta(q;-x)}=q^{-\mu(\mu-1)/2}(e^{i\pi})^{-\mu}\frac{\theta(q^*;-e^{2\pi i\mu})}{(q^*,q^*)_{\infty}^3}\sum_{l\in\mathbb{Z}}\frac{x^{*l}}{1-e^{2\pi i\mu}q^{*l}}.$$

By using (4.5.7) to $\theta(q^*;-e^{2\pi i\mu})$ and $(q^*;q^*)_{\infty}=\frac{q^{1/24}}{\sqrt{\kappa}}e^{\kappa\pi/12}(q;q)_{\infty}$, it yields that

$$\begin{aligned}\frac{\theta(q;-q^{\mu}x)}{(x;q)_{\infty}}=&\frac{\kappa(q^{\mu},q^{1-\mu};q)_{\infty}}{i(q;q)_{\infty}}(\frac{q}{x};q)_{\infty}\sum_{l\in\mathbb{Z}}\frac{x^{-(\mu+i\kappa l)}}{1-e^{2\pi i(\mu+i\kappa l)}}\\ =&\frac{\kappa(q^{\mu},q^{1-\mu};q)_{\infty}}{i(q;q)_{\infty}}\sum_{n\geq 0}\frac{(-1)^nq^{n(n+1)/2}x^{-n}}{(q;q)_n}\sum_{l\in\mathbb{Z}}\frac{x^{-(\mu+i\kappa l)}}{1-e^{2\pi i(\mu+i\kappa l)}},\end{aligned}$$

the series in the right-hand side is normally convergent on any compact of $\{x|\arg(x)\in(-2\pi,0)\}$ ($|x^*|<1$). The proof is completed by replacing x with $\frac{1}{t}$. □

Theorem 4.8. *Let* $\alpha\notin q^{\mathbb{Z}}$ *and* γ_0 *be as in* (4.4.9)*. The following relation holds for any* $x\in\widetilde{\mathbb{C}^*}$ *with* $\arg(x)\in(-\frac{5\pi}{2},\frac{\pi}{2})$ *:*

$$F(\alpha;q,\gamma_0,x)=G(\alpha;q,x)+\frac{i\kappa(\alpha;q)_{\infty}}{(q;q)_{\infty}}\sum_{n\geq 0}\frac{(-1)^nq^{n(n+1)/2}}{(q;q)_n}g_n(\alpha;q,x)x^{-n}.$$

Proof. By Jacobi triple product formula (4.3.14), we obtain that $\theta(-\frac{qt}{\alpha})=\theta(-\frac{\alpha}{t})$. From equations (4.3.4), (4.3.12) and (4.4.8), letting $d=\pi$, we have

$$F(\alpha;q,\gamma_0,x)=G(\alpha;q,x)-\int_0^{\infty e^{i\pi}}\frac{e^{-tx}\,\theta\left(-\frac{\alpha}{t}\right)}{\left(\frac{q}{\alpha};q\right)_{\infty}\left(\frac{1}{t};q\right)_{\infty}}\frac{\mathrm{d}t}{t},$$

for $\arg(x)\in(-\frac{3\pi}{2},\frac{\pi}{2})$.

Assuming that $|\alpha|<1$, then we obtain $\mathrm{Re}(\mu)>0$. From $\alpha\notin q^{\mathbb{Z}}$, we have $\mu+i\kappa l\notin\mathbb{Z}$. By using Equation (4.5.9), Lebesgue's dominated convergence Theorem and

$$\int_0^{\infty}e^{-t}t^{n+\mu+i\kappa l}\frac{\mathrm{d}t}{t}=\Gamma(n+\mu+i\kappa l),$$

we obtain the equation given in the theorem, where $g_n(\alpha;q,x)$ is shown in Lemma 4.13. By applying Remark 4.3 and analytic continuation, the equation in the theorem holds for any value of α and $\arg(x)\in(-\frac{5\pi}{2},\frac{\pi}{2})$. □

Then, we obtain the asymptotic form of solution at zero for the critical value $c_0=\gamma_0$.

Theorem 4.9. *Let $\alpha \notin q^{\mathbb{Z}}$, and let γ_0 be as in* (4.4.9).

(*i*) *If $|\alpha| < q$, then*

$$F(\alpha; q, \gamma_0, x) = \frac{1}{1-\alpha q^{-1}} x^{-1} + o(x^{-1})$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(*ii*) *If $|\alpha| > q$, then*

$$F(\alpha; q, \gamma_0, x) = \frac{i\kappa(\alpha; q)_\infty}{(q;q)_\infty} \hat{g}_0(\alpha; q, x) x^{-\mu} + o(x^{-\mu})$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, *where $\hat{g}_0(\alpha; q, x)$ is a bounded q-periodic function shown in Lemma 4.14 with $n = 0$.*

(*iii*) *If $|\alpha| = q$, then*

$$F(\alpha; q, \gamma_0, x) = \frac{1}{1-\alpha q^{-1}} x^{-1} + \frac{i\kappa(\alpha; q)_\infty}{(q;q)_\infty} \hat{g}_0(\alpha; q, x) x^{-\mu} + o(x^{-1})$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Proof. In order to study the asymptotic behavior of $F(\alpha; q, c_0, x)$, we distinguish three cases :

(i) If $|\alpha| < q$, then $-\mathrm{Re}(\mu) = -\ln|\alpha|/\ln q < -1$. From Theorem 4.8, we have

$$\begin{aligned} F(\alpha; q, \gamma_0, x) &= x^{-1}\big[\tfrac{1}{1-\alpha q^{-1}} + \sum_{n\geq 1} \frac{\alpha^{-(n+1)} q^{(n+1)(n+2)/2} n!}{(\alpha^{-1}q;q)_{n+1}} x^{-n} + O(x^{-\mathrm{Re}(\mu)+1})\big] \\ &= \tfrac{1}{1-\alpha q^{-1}} x^{-1}\big[1 + O(x^{-1}) + O(x^{-\mathrm{Re}(\mu)+1})\big] \end{aligned}$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(ii) If $|\alpha| > q$, then $\mathrm{Re}(\mu) - 1 < 0$. From Lemma 4.14 and Theorem 4.8, we get

$$F(\alpha; q, \gamma_0, x) = \frac{i\kappa(\alpha; q)_\infty}{(q;q)_\infty} g_0(\alpha; q, x) + x^{-\mu}\big[O(x^{-1}) + O(x^{\mathrm{Re}(\mu)-1})\big]$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(iii) If $|\alpha| = q$, then we assume that $\mu = 1 + \frac{ia}{\ln q}$ ($a \notin 2\pi\mathbb{Z}$). From Theorem 4.8, we have

$$F(\alpha; q, \gamma_0, x) = \frac{1}{1-\alpha q^{-1}} x^{-1}\big[1 + O(x^{-1})\big] + \frac{i\kappa(\alpha; q)_\infty}{(q;q)_\infty} \hat{g}_0(\alpha; q, x) x^{-\mu}\big[1 + O(x^{-1})\big]$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. $\square$

4.5.3 Concluding results for general c_0

From the above analysis, we draw conclusions about the connection formula and asymptotic behaviors in the general case.

Corollary 4.3. *Let $\alpha \notin q^{\mathbb{Z}}$. The following relation holds for any $c_0 \in \mathbb{C}^*$ and $x \in \widetilde{\mathbb{C}^*}$ with* $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$F(\alpha; q, c_0, x) = G(\alpha; q, x) + \frac{i\kappa(\alpha; q)_\infty}{(q; q)_\infty} \sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(q; q)_n} [g_n(\alpha; q, x) + \frac{c_0 - \gamma_0}{2\pi i} h_n(\alpha; q, x)] x^{-n}.$$

Proof. The proof is completed, by using Theorem 4.8,

$$F(\alpha; q, c_0, x) = F(\alpha; q, \gamma_0, x) + (c_0 - \gamma_0) F_0(\alpha; q, x),$$

and the connection formula

$$F_0(\alpha; q, x) = \frac{\kappa(\alpha; q)_\infty}{2\pi(q; q)_\infty} \sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(q; q)_n} h_n(\alpha; q, x) x^{-n}$$

for $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ in [98, p. 6, Theorem 1.2]. □

Therefore, we obtain the asymptotic form at ∞ of solutions around zero.

Theorem 4.10. *Let $\alpha \notin q^{\mathbb{Z}}$.*

(i) If $|\alpha| < q$, then

$$F(\alpha; q, c_0, x) = \frac{x^{-1}}{1 - \alpha q^{-1}} + o(x^{-1})$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(ii) If $|\alpha| > q$, then

$$F(\alpha; q, c_0, x) = \frac{i\kappa(\alpha; q)_\infty}{(q; q)_\infty} \big(\hat{g}_0(\alpha; q, x) + \frac{c_0 - \gamma_0}{2\pi i} \hat{h}_0(\alpha; q, x)\big) x^{-\mu} + o(x^{-\mu})$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, *where $\hat{g}_0(\alpha; q, x)$ and $\hat{h}_0(\alpha; q, x)$ are bounded q-period functions in any direction* $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, *as shown in Lemma 4.14 ($n = 0$).*

(iii) If $|\alpha| = q$, then

$$F(\alpha; q, c_0, x) = \frac{x^{-1}}{1 - \alpha q^{-1}} + \frac{i\kappa(\alpha; q)_\infty}{(q; q)_\infty} \big(\hat{g}_0(\alpha; q, x) + \frac{c_0 - \gamma_0}{2\pi i} \hat{h}_0(\alpha; q, x)\big) x^{-\mu} + o(x^{-1})$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Proof. Similar to the proof of Theorem 4.9, it can be easily obtained by using the above corollary. □

As mentioned in the Introduction, Lim obtained the asymptotic boundaries for all solutions of Equation (1.1.2). We now compare the asymptotic form obtained in this chapter with the results in [60]. In this chapter, the non-homogeneous term $g(x) = \frac{1}{x}$ has a singularity at 0, which is defined on $(0, \infty)$, does not satisfy the condition in [60] that g is defined on $[0, \infty)$, but it satisfies other conditions such as $g = O(x^{-1})$ and $g' = O(x^{-2})$ (equivalent to the case where $\alpha = -1$ in [60]). Recall Lim's results in [60, Theorem 1] (let's make $\alpha = -1$ for comparison) :

Let $b < 0$. Assume that g' exists. Let $g(x) = O(x^{-1})$ and $g'(x) = O(x^{-2})$. Then :

(i) If $\frac{\ln|b/a|}{\ln q} > -1$, every solution of (1.1.2) is $O(x^{\frac{\ln|b/a|}{\ln q}})$ as $x \to \infty$.

(ii) If $\frac{\ln|b/a|}{\ln q} = -1$, every solution of $O(x^{\frac{\ln|b/a|}{\ln q}} \ln x)$ as $x \to \infty$.

(iii) If $\frac{\ln|b/a|}{\ln q} < -1$, every solution of (1.1.2) is $O(x^{-1})$ as $x \to \infty$.

From the tranform shown in the Introduction, we know that $\alpha = -\frac{a}{b}$. Therefore, the above case (i) is equivalent to : if $|\alpha| > q$, every solution of (1.1.2) is $O(x^{-\frac{\ln|\alpha|}{\ln q}})$ as $x \to \infty$, which is consistent with the case (ii) in Theorem 4.10 (because $\frac{\ln|\alpha|}{\ln q} = \mathrm{Re}(\mu)$ and every solution is $O(x^{-\mathrm{Re}(\mu)})$ as $x \to \infty$). The above case (iii) is equivalent to : if $|\alpha| < q$, every solution of (1.1.2) is $O(x^{-1})$ as $x \to \infty$, which is also consistent with the case (i) in Theorem 4.10.

4.6 Summary of this chapter

This chapter studies the asymptotic properties of the analytic solutions of the corresponding equation at the origin and infinity when the non-homogeneous term has a singularity at zero. Using the Laplace transform, we obtain two types of Laplace integral solutions and establish the relationship formula between them. Using the idea of equation perturbation, we derive the relationship between the series solutions and the Laplace integral solutions. With the relationship between the two Laplace integral solutions as a bridge, we establish the connection formula between the solution at zero and the solution at infinity based on the earlier relationship formula, thereby obtaining the asymptotic form of the solution at zero. Based on the conclusion about the asymptotic form, we conclude that for small $|\alpha|$, Equation (4.1.1) has a form similar to ordinary differential equations. On the other hand, for large $|\alpha|$, the q-difference operator plays a more important role than the differential operator.

Chapitre 5

Properties of solutions when the non-homogeneous term has a singularity at non-zero constant

5.1 Introduction

If $y(x)$ is an analytic solution of

$$y'(x) = \alpha y(qx) - y(x) + \frac{1}{c+x}, \tag{5.1.1}$$

then, by taking $(m-1)$-th derivative of the equation, one obtains that $z(x) = \frac{y^{(m-1)}(x)}{(m-1)!}$ is a solution of $z'(x) = \alpha' z(qx) - z(x) + \frac{1}{(c+x)^m}$ with $\alpha' = \alpha q^{m-1}$. Hence, we only need to consider equation (5.1.1). We mainly investigate (5.1.1) in a special case where $c = 1$,

$$y'(x) = \alpha y(qx) - y(x) + \frac{1}{1+x}. \tag{5.1.2}$$

For the case of general c, we divide the discussion into two cases based on its value.

(i) If $\mathrm{Re}(c) > 0$. We can get similar results to the solutions of (5.1.2), except that the expression and asymptotic behaviors of solutions are both related to c.

(ii) If $\mathrm{Re}(c) < 0$. From the previous analysis and change of variables, without loss of generality, we only need to study the following equation

$$y'(x) = \alpha y(qx) - y(x) + \frac{1}{1-x}, \tag{5.1.3}$$

the asymptotic behaviors will be discussed at the end for general c. In the last section of this chapter, we will discuss the properties of solutions to the Equation (5.1.1) when c is a general value. We first introduce the formulas and lemmas that will be used later.

According to reference [14, P490 Corollary 10.2.2 (c)-(d)], we have

$$\sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}} x^k = (x;q)_N, \quad \sum_{k=0}^{\infty} \begin{bmatrix} N+k-1 \\ k \end{bmatrix}_q x^k = \frac{1}{(x;q)_N}, \tag{5.1.4}$$

where $\begin{bmatrix} N \\ k \end{bmatrix}_q = \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$ is called the q-binomial coefficient.

Lemma 5.1. *Let $k \in \mathbb{N}$ and integer $m \in [0,k]$. The following equation*

$$\frac{(\alpha q^{-1}; q^{-1})_m}{(\alpha q^{-1}; q^{-1})_{k+1}} = \sum_{n\geq 0} \frac{(\alpha q^{-(k+1)})^n (q^{k-m+1}; q)_n}{(q;q)_n} \tag{5.1.5}$$

holds for any $|\alpha| < q^{k+1}$.

Proof. We will prove the equality holds by induction. If $k=0$, then $m=0$, equality (5.1.5) becomes $\frac{1}{1-\alpha q^{-1}} = \sum_{n\geq 0} (\alpha q^{-1})^n$ for $|\alpha| < q$, which is obviously true. If equality (5.1.5) holds for k, then for $k+1$, the left-hand side of (5.1.5) is equivalent to $\frac{(\alpha q^{-1};q^{-1})_m}{(\alpha q^{-1};q^{-1})_{k+1}} \cdot \frac{1}{1-\alpha q^{-(k+2)}}$, which can be viewed as the product of two series

$$\sum_{n\geq 0} \frac{(\alpha q^{-(k+1)})^n (q^{k-m+1};q)_n}{(q;q)_n} \cdot \sum_{n\geq 0} (\alpha q^{-(k+2)})^n = \sum_{n\geq 0} (\alpha q^{-(k+2)})^n \sum_{i=0}^{n} \frac{q^i (q^{k-m+1};q)_i}{(q;q)_i}.$$

Letting $b=1$ and $b' = k-m+1$ in [34, p. 379 (10.97)], we have

$$\sum_{i=0}^{n} \frac{q^i (q^{k-m+1};q)_i}{(q;q)_i} = \frac{(q^{k-m+2};q)_n}{(q;q)_n},$$

it can also be proved by induction. Therefore, Equation (5.1.5) holds for $k+1$. □

5.2 Solutions expressed by power series at zero and infinity

In this section, we obtain the solutions of Equation (5.1.2), which can be expressed by power series at 0 and ∞, and study the analytic continuation properties of these solutions.

5.2.1 Power series solutions at zero

By letting $\frac{1}{1+x} = \sum_{n\geq 0} (-x)^n$ ($|x| < 1$) and assuming $y = \sum_{n\geq 0} a_n x^n$, we have

$$\sum_{n\geq 1} n a_n x^{n-1} = \sum_{n\geq 1} (\alpha q^{n-1} - 1) a_{n-1} x^{n-1} + \sum_{n\geq 1} (-x)^{n-1}.$$

Compraing the coefficients of x^{n-1}, one get $a_n = -\frac{1-\alpha q^{n-1}}{n}\, a_{n-1} + \frac{(-1)^{n-1}}{n}$ $(n \geq 1)$. Define

$$F_1(\alpha; q, x) = \sum_{n\geq 1}\sum_{k=1}^{n} \frac{(-1)^{n-1}(k-1)!\,(\alpha q^k; q)_{n-k}}{n!}\, x^n. \tag{5.2.1}$$

Let $D(0,1)$ be an open disk centered at 0 and with radius 1. One can find the following result.

Theorem 5.1. *Let F_0 and F_1 be defined as in* (1.3.5) *and* (5.2.1). *Given a constant $c_0 \in \mathbb{C}$, the function*

$$F(\alpha; q, c_0, x) = c_0 F_0(\alpha; q, x) + F_1(\alpha; q, x) \tag{5.2.2}$$

is the unique series solution analytic in $D(0,1)$ of Equation (5.1.2) *with $y(0) = c_0$. Furthermore, the series F_0 is an entire function, the radius of convergence of F_1 is* 1.

Proof. It is obvious that F_0 is an entire function. Since $(\alpha q^k; q)_{n-k}$ is bounded, the radius of convergence of F_1 is at least 1. We suppose that the radius is R, then $F_1(\alpha; q, qx)$ is analytic on $|x| < \frac{R}{q}$. Then $F_1'(\alpha; q, x) - \alpha F_1(\alpha; q, qx) + F_1(\alpha; q, x)$ is analytic on $|x| < R$. If $R > 1$, then $y'(x) - \alpha y(qx) + y(x) = \frac{1}{1+x}$ would have no singularity for any $|x| < R$, which contradicts the fact that $\frac{1}{1+x}$ has a singularity at $x = -1$. Hence $R = 1$. According to [98] we obtain that $F(\alpha; q, c_0, x)$ is a unique solution of (5.1.2) with $y(0) = c_0$ on $D(0,1)$. □

Remark 5.1. *Notice that $\frac{1}{1+x} = \sum_{n\geq 0}(-x)^n \in \mathbb{C}\{x\}$. According to Corollary 3.2, we obtain that : solutions of Equation* (5.1.3) *satisfy $y \in \mathbb{C}\{x\}$. From Theorem 5.1, we have $F(\alpha; q, c_0, x) \in \mathbb{C}\{x\}$, it is consistent with the result in Corollary 3.2.*

5.2.2 Power series solutions at infinity

Next, we look for solutions of Equation (5.1.2), which can be expressed by power series at infinity. Let $\frac{1}{1+x} = \frac{1}{x}\cdot\frac{1}{1+\frac{1}{x}} = \sum_{n\geq 0}(-1)^n x^{-n-1}$ ($|x| > 1$). By assuming $y = \sum_{n\geq 0} a_n x^{-n-1}$ and substituting it into Equation (5.1.2), we have $\sum_{n\geq 1}(-n)a_{n-1}x^{-n-1} = \sum_{n\geq 0}(\alpha q^{-n-1} - 1)a_n x^{-n-1} + \sum_{n\geq 0}(-1)^n x^{-n-1}$. Comparing the coefficients of x^{-n-1} in the above equation, we get the expression for all a_n $(n \geq 0)$.

Theorem 5.2. *Let $\alpha \notin q^{\mathbb{Z}_{>0}}$. Then the function*

$$G(\alpha; q, x) = \sum_{n\geq 0}\sum_{k=0}^{n} \frac{(-1)^k n!(\alpha q^{-1}; q^{-1})_k}{k!(\alpha q^{-1}; q^{-1})_{n+1}} x^{-n-1} \tag{5.2.3}$$

is the unique analytic solution of (5.1.2) *in $|x| > q$ vanishing at infinity. Furthermore, the following relation holds :*

$$\lim_{x\to\infty} x \cdot G(\alpha; q, x) = \frac{1}{1-\alpha q^{-1}}.$$

Proof. From $G(\alpha;q,x)=\sum_{n\geq 0}\frac{n!}{(\alpha q^{-1};q^{-1})_{n+1}}\sum_{k=0}^{n}u_k x^{-n-1}$, where $u_k=\frac{(-1)^k(\alpha q^{-1};q^{-1})_k}{k!}$. We have $\left|\frac{u_k}{u_{k-1}}\right|=\frac{|1-\alpha q^{-k}|}{k}>1$ for sufficient large k and n. Then

$$|G(\alpha;q,x)|\leq C\sum_{n\geq 0}\frac{n+1}{|1-\alpha q^{-n-1}|}|x|^{-n-1}.$$

The above series is convergent for $|x|>q$, then G is analytic on $|x|>q$.

For $\lim_{x\to\infty}x\cdot G(\alpha;q,x)=\frac{1}{1-\alpha q^{-1}}$, it can be proved by direct calculation. □

Remark 5.2. *By using the uniform convergence theorem, we can see that $F(\alpha;q,c_0,x)$ and $G(\alpha;q,x)$ are respectively analytic for $(\alpha,x)\in\mathbb{C}\times D(0,1)$ and for all $(\alpha,x)\in\mathbb{C}\times\{|x|>q\}$ such that $\alpha q^{-k}\neq 1$ for any $k\in\mathbb{Z}_{>0}$.*

Remark 5.3. *The series solution $G(\alpha;q,x)\in\mathbb{C}\{x^{-1}\}$, it is consistent with the result in Corollary* 3.4.

5.2.3 Analytic continuation of solutions at zero and infinity

The following two theorems show the analytic continuation properties of the solutions represented by power series at zero and infinity, respectively.

Theorem 5.3. *The function F_1 given by* (5.2.1) *satisfies Equation* (5.1.2) *with $y(0)=0$, which can be analytically extended to $\mathbb{C}\setminus(-\infty,-1]$, and with the following property : Given any integer $n\geq 0$, there exists a function $y_n(x)$ analytic in $|x|<q^{-n-1}$ such that*

$$F_1(\alpha;q,x)=y_n(x)+\sum_{k=0}^{n}A_k(x)\log(1+q^k x),\tag{5.2.4}$$

where

$$A_k(x)=\alpha^k\sum_{j=0}^{k}\frac{(-1)^j q^{\frac{j(j-1)}{2}}}{(q;q)_j(q;q)_{k-j}}e^{-(1+q^k x)q^{-j}}.$$

Proof. We prove formula (5.2.4) by induction on n. Let $x\notin(-\infty,-1]$. By Theorem 5.1 and taking $F_1=u(x)e^{-x}$ into the equation, we get $u'(x)e^{-x}-\alpha u(qx)e^{-qx}=\frac{1}{1+x}$. Then $u(x)=\int_0^x\frac{e^t}{1+t}\mathrm{d}t+\alpha\int_0^x e^t F_1(\alpha;q,qt)\mathrm{d}t$. The function defined by the power series solution $F_1(\alpha;q,x)$ satisfies

$$F_1(\alpha;q,x)=e^{-x}\int_0^x\frac{e^t}{1+t}\mathrm{d}t+\alpha e^{-x}\int_0^x e^t F_1(\alpha;q,qt)\mathrm{d}t.\tag{5.2.5}$$

Notice that $e^t = e^{-1} \cdot e^{1+t} = e^{-1} \cdot \sum_{m\geq 0} \frac{(1+t)^m}{m!}$, we have

$$\int_0^x \frac{e^t}{1+t}\mathrm{d}t = -e^{-1}\log(1+x) + g_0(x), \tag{5.2.6}$$

where $g_0(x)$ is an entire function. Hence, formula (5.2.4) is true for $n = 0$, where

$$y_0(x) = g_0(x)e^{-x} + \alpha e^{-x}\int_0^x e^t F_1(\alpha; q, qt)\mathrm{d}t$$

is an analytic function in $|x| < q^{-1}$.

Supposing that assertion (5.2.4) is true for some n and substituting (5.2.4) and (5.2.6) into the right-hand side of (5.2.5), we have

$$\begin{aligned} F_1(\alpha; q, x) =& e^{-(1+x)}\log(1+x) + e^{-x}g_0(x) + \alpha e^{-x}\int_0^x e^t y_n(qt)\mathrm{d}t \\ &+ \alpha e^{-x}\int_0^x e^t \sum_{k=0}^n A_k(qt)\log(1+q^{k+1}t)\mathrm{d}t. \end{aligned}$$

In the following, we will take the terms that contain $\log(1+q^k x)$ out of the last integral of the above equation. Since

$$\begin{aligned} &\int_0^x e^t \sum_{k=0}^n A_k(qt)\log(1+q^{k+1}t)\mathrm{d}t \\ =& \int_0^x e^t \sum_{k=0}^n \alpha^k \sum_{j=0}^k \frac{(-1)^j q^{j(j-1)/2} e^{-(1+q^{k+1}t)q^{-j}}}{(q;q)_j (q;q)_{k-j}} \log(1+q^{k+1}t)\mathrm{d}t \\ =& \sum_{k=1}^{n+1} \alpha^{k-1} \sum_{j=0}^{k-1} \frac{(-1)^j q^{j(j-1)/2}}{(q;q)_j (q;q)_{k-j-1}} \int_0^x e^{-q^{-j}+(1-q^{k-j})t}\log(1+q^k t)\mathrm{d}t, \end{aligned}$$

where the last integral $\int_0^x e^{-q^{-j}+(1-q^{k-j})t}\log(1+q^k t)\mathrm{d}t$ is equal to

$$\frac{e^{-q^{-j}+(1-q^{k-j})x}}{1-q^{k-j}}\log(1+q^k x) + \int_0^x \frac{q^k e^{-q^{-j}+(1-q^{k-j})t}}{(1-q^{k-j})(1+q^k t)}\mathrm{d}t,$$

and $\int_0^x \frac{q^k e^{-q^{-j}+(1-q^{k-j})t}}{(1-q^{k-j})(1+q^k t)}\mathrm{d}t$ can be written as $\frac{e^{-q^{-k}}}{1-q^{k-j}}\log(1+q^k x) + g_k(x)$, ($g_k(x)$ is an entire function), one obtains that $F_1(\alpha; q, x) =$

$$\begin{aligned} &e^{-(1+x)}\log(1+x) + e^{-x}g_0(x) + \alpha e^{-x}\int_0^x e^t y_n(qt)\mathrm{d}t + \sum_{k=1}^{n+1} A_{k-1}(x)g_k(x) \\ &+ \sum_{k=1}^{n+1} \alpha^k \sum_{j=0}^{k-1} \frac{(-1)^j q^{\frac{j(j-1)}{2}}\left[e^{(1+q^k x)q^{-j}} - e^{-(1+q^k x)q^{-k}}\right]}{(q;q)_j (q;q)_{k-j}} \log(1+q^k x). \end{aligned} \tag{5.2.7}$$

From the first equation of (5.1.4), it follows that

$$\sum_{j=0}^{k-1} \frac{(-1)^j q^{\frac{j(j-1)}{2}}}{(q;q)_j (q;q)_{k-j}} = \frac{(-1)^{k-1} q^{\frac{k(k-1)}{2}}}{(q;q)_k}.$$

Substituting it into (5.2.7), we have $y(x) = y_{n+1}(x) + \sum_{k=0}^{n+1} A_k(x) \log(1 - q^k x)$, where $y_{n+1}(x)$ is an analytic function on $|x| < q^{-n-2}$. □

Theorem 5.4. *Let $G(\alpha; q, x)$ be as in* (5.2.3)*, then it can be analytically extended to $\mathbb{C}^* \setminus \{-q, -q^2, -q^3, \cdots\}$. In other words, there is a function $g_n(x)$ analytic in neighborhood of $x = -q^n$ $(n \geq 1)$ and polynomial $P_n(x)$ such that*

$$G(\alpha; q, x) = P_n(x) + g_n(x), \tag{5.2.8}$$

where

$$P_n(x) = (q;q)_{n-1} \Big(\frac{q}{\alpha}\Big)^n \sum_{k=1}^{n} \frac{(-1)^k (k-1)! q^{\frac{k(k-1)}{2}}}{(q;q)_{k-1} (q;q)_{n-k}} \Big(\frac{1}{x+q^n}\Big)^k.$$

Proof. We will prove (5.2.8) by induction. The equation

$$y'(x) = \alpha y(qx) - y(x) + \frac{1}{1+x}$$

can be written as $y(x) = \frac{1}{\alpha}[y'(\frac{x}{q}) + y(\frac{x}{q}) - \frac{q}{x+q}]$, then

$$G(\alpha; q, x) = \frac{1}{\alpha}\Big[G'\big(\alpha; q, \frac{x}{q}\big) + G\big(\alpha; q, \frac{x}{q}\big) - \frac{q}{x+q}\Big], \tag{5.2.9}$$

which is analytic on $|x| > q$. Then $G(\alpha; q, x) = -\frac{q}{\alpha} \cdot \frac{1}{x+q} + g_1(x)$ with $g_1(x)$ analytic near $x = -q$. Hence, the assertion holds for $n = 1$. Suppose, Equation (5.2.8) holds for some $n \geq 1$, i.e., $G(\alpha; q, x) = P_n(x) + g_n(x)$ as above, $g_n(x)$ is an analytic function near $x = -q^n$. Substituting formula (5.2.8) into (5.2.9), we obtain $G(\alpha; q, x) = \frac{1}{\alpha}[P_n'(\frac{x}{q}) + P_n(\frac{x}{q})] + g_{n+1}(x)$, where $g_{n+1}(x) = \frac{1}{\alpha}[g_n'(\frac{x}{q}) + g_n(\frac{x}{q}) - \frac{q}{x+q}]$ is an analytic function near $x = -q^{n+1}$. We only need to prove that $\frac{1}{\alpha}[P_n'(\frac{x}{q}) + P_n(\frac{x}{q})] = P_{n+1}(x)$. From the expression of $P_n(x)$, it yields that

$$\begin{aligned}\frac{1}{\alpha}\big[P_n'(\frac{x}{q}) + P_n(\frac{x}{q})\big] =& \Big(\frac{q}{\alpha}\Big)^{n+1} \Big[\sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k k! q^{\frac{k(k+1)}{2}} \frac{1}{(x+q^{n+1})^{k+1}} \\ &+ \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^{k+1} (k-1)! q^{\frac{k(k-1)}{2}} q^{k-1} \frac{1}{(x+q^{n+1})^k}\Big],\end{aligned}$$

where the notation $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$ is shown in Section 5.1. By replacing k in the last series of the above equation with $k+1$, and combining the sum of k on the right of the equation from 1 to $n-1$, the right-hand side of the above formula becomes

$$\big(\frac{q}{\alpha}\big)^{n+1}\sum_{k=1}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_q(-1)^k k! q^{\frac{k(k+1)}{2}}\frac{1}{(x+q^{n+1})^{k+1}}+\big(\frac{q}{\alpha}\big)^{n+1}\frac{(-1)^n n! q^{\frac{n(n+1)}{2}}}{(x+q^{n+1})^{n+1}}-\big(\frac{q}{\alpha}\big)^{n+1}\frac{1}{x+q^{n+1}}.$$

By calculation, the above equation is equal to

$$\big(\frac{q}{\alpha}\big)^{n+1}\sum_{k=1}^{n+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q(-1)^k (k-1)! q^{\frac{k(k-1)}{2}}\frac{1}{(x+q^{n+1})^{k}}.$$

Then $\frac{1}{\alpha}\big[P_n'(\frac{x}{q})+P_n(\frac{x}{q})\big]=P_{n+1}(x)$, which accomplishes the proof. $\square$

5.3 Two approaches to obtain the integral-sum function

In this section, we propose two ways to obtain a solution which is expressed by the sum of integral. The first way : we apply the idea of equations' perturbation to obtain the integral-sum solution. The second way : we look for the formal series solution, and calculate its corresponding Borel-sum by using the method of Borel summation for divergent series. The Borel-sum solution has the same form as that of the integral-sum solution.

5.3.1 First approach by perturbation

Let α be a parameter that is close to 0. If $\alpha=0$, then Equation (5.1.2) becomes

$$y'(x)=-y(x)+\frac{1}{1+x}, \tag{5.3.1}$$

it has the same form as Euler equation $y'(x)=-y(x)+\frac{1}{x}$. Let $S(a,b)$ be as in (1.3.1). Notice that Euler has a formal series solution $\sum_{n\geq 0}n!x^{-n-1}$, whose Borel-sum $\hat{E}(x)$ is an analytic function defined on $S(-\frac{\pi}{2},\frac{5\pi}{2})$, and $\hat{E}(x)$ consists of the analytical continuation of all functions $\int_0^{\infty e^{id}}\frac{e^{-xt}}{1-t}\mathrm{d}t$ with integral directions $d\in(-\pi,\pi)$. Then Equation (5.3.1) has a divergent series solution $\sum_{n\geq 0}n!(1+x)^{-n-1}$. one can get that the divergent series is Borel-summable (see [67, p. 175 Definition 1.3.1.2]). Let $S_1(a,b)$ be as in (1.3.1). Similar to the Borel-summation for the Euler series, Equation (5.3.1) has a Borel-sum solution $\hat{E}(x+1)$, which is defined on $S_1(-\frac{\pi}{2},\frac{5\pi}{2})$.

If we view Equation (5.1.2) as the perturbation of Equation (5.3.1) and expanded the solution of (5.1.2) into $\sum_{n\geq 0}y_n(x)\alpha^n$, then

$$y_{n+1}'(x)+y_{n+1}(x)=y_n(qx),\ n\geq 0. \tag{5.3.2}$$

Lemma 5.2. *Let* $S_n^1(a,b)$ *be as in* (1.3.1). *There is a family of funtions* $\{I_n(x)\}_{n\geq 0}$ *defined on* $S_n^1(-\frac{\pi}{2},\frac{5\pi}{2})$ *that satisfy the system* (5.3.2).

Proof. Let $S_1(a,b)$ be as in (1.3.1). Choose $y_0(x)=\int_0^{\infty e^{-\mathrm{i}\pi}}\frac{e^{-(1+x)t}}{1-t}\mathrm{d}t$ as a solution of (5.3.1) which is defined on $S_1(\frac{\pi}{2},\frac{3\pi}{2})$. Let $y_{n+1}=e^{-x}z$, we get $z=\int_{\ell_x}e^{t_n}y_n(qt_n)\mathrm{d}t_n$, where $\ell_x=\left\{t+\mathrm{i}\cdot\mathrm{Im}(x)\middle|t\in(-\infty,\mathrm{Re}(x)]\right\}$. Therefore, $y_{n+1}=e^{-x}z=\int_{\ell_x}e^{-x+t_n}y_n(qt_n)\mathrm{d}t_n$.

For $n=0$:

$$y_1(x)=\int_{\ell_x}\int_0^{\infty e^{-\mathrm{i}\pi}}e^{-x+t_0}\cdot\frac{e^{-(qt_0+1)t}}{1-t}\mathrm{d}t\mathrm{d}t_0=\int_0^{\infty e^{-\mathrm{i}\pi}}\frac{e^{-(qx+1)t}}{(1-t)(1-qt)}\mathrm{d}t.$$

For $n=1$:

$$y_2(x)=\int_{\ell_x}\int_0^{\infty e^{-\mathrm{i}\pi}}e^{-x+t_1}\cdot\frac{e^{-(q^2t_1+1)t}}{(1-t)(1-qt)}\mathrm{d}t\mathrm{d}t_1=\int_0^{\infty e^{-\mathrm{i}\pi}}\frac{e^{-(q^2x+1)t}}{(1-t)(1-qt)(1-q^2t)}\mathrm{d}t.$$

By induction, we have $y_n(x)=\int_0^{\infty e^{-\mathrm{i}\pi}}\frac{e^{-(q^nx+1)t}}{(1-t)(1-qt)\cdots(1-q^nt)}\mathrm{d}t$ for $n\geq 0$.

Let $S_n^1(a,b)$ be as in (1.3.1). Each function y_n is well-defined on the domain $S_n^1(\frac{\pi}{2},\frac{3\pi}{2})$. Replacing the integral path $[0,\infty e^{-\mathrm{i}\pi})$ by $[0,\infty e^{\mathrm{i}d})$, where $d\in(-2\pi,0)$, we obtain

$$I_n^{[d]}(x)=\int_0^{\infty e^{\mathrm{i}d}}\frac{e^{-(q^nx+1)t}}{(t;q)_{n+1}}\mathrm{d}t. \tag{5.3.3}$$

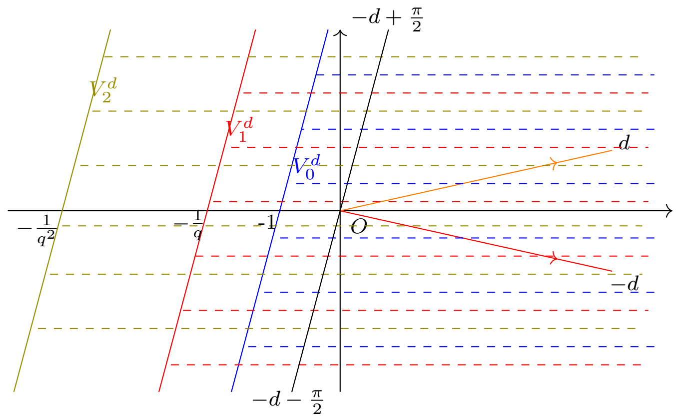

FIGURE 5.1 – Domain of analyticity of $I_n^{[d]}(x)$

Let

$$V_n^d=S_n^1(-d-\frac{\pi}{2},-d+\frac{\pi}{2}), \tag{5.3.4}$$

we will prove the analyticity of the function $I_n^{[d]}$ in any compact subset of V_n^d. By writing $t = re^{\mathrm{i}d}$, one can get that, in any compact subset V_n^d,

$$\Big|\frac{e^{-(q^n x+1)t}}{(t;q)_{n+1}}\Big| \le \frac{e^{\mathrm{Re}\{-(q^n x+1)t\}}}{[\sin(d)]^n} = \frac{e^{-[\mathrm{Re}(q^n x+1)\cos d - \mathrm{Im}(q^n x+1)\sin d]r}}{[\sin(d)]^n} \le \frac{e^{-\epsilon_n r}}{[\sin(d)]^n},$$

where $\epsilon_n = \min_{x\in K}\big\{\mathrm{Re}(q^n x+1)\cos(d) - \mathrm{Im}(q^n x+1)\sin(d)\big\} > 0$. Since $\epsilon_n \to \cos(d)$ as $n\to\infty$, we have $\epsilon_n \ge \epsilon' > 0$. Then $\big|I_n^{[d]}(x)\big| \le \int_0^{+\infty} \frac{e^{-\epsilon' r}}{[\sin(d)]^n}\mathrm{d}r = \frac{1}{[\sin(d)]^n\epsilon'}$. Then $I_n^{[d]}(x)$ is analytic on V_n^d for every fixed $d \in (-2\pi, 0)$. By analytic continuation, gluing all functions $I_n^{[d]}(x)$ yields a function defined on the domain $S_n^1(-\frac{\pi}{2}, \frac{5\pi}{2})$, denoted by $I_n(x)$. $\square$

Theorem 5.5. *Let* $|\alpha| < 1$ *and let* $S_1(-\pi,\pi)$ *be as in* (1.3.1). *There are two analytic functions* T_1 *and* T_2 *on* $S_1(-\pi,\pi)$*, such that :*

(i) They are both solutions of Equation (5.1.2).

(ii) $T_2(\alpha;q,x) - T_1(\alpha;q,x) = (\eta_2-\eta_1)F_0(\alpha;q,x)$*, where* F_0 *is given in* (1.3.5)*, and where* η_1 *and* η_2 *are two constants defined as follows :*

$$\eta_1 = \sum_{k\ge 0}\alpha^k \int_0^{\infty e^{\mathrm{i}d_1}} \frac{e^{-t}}{(t;q)_{k+1}}\mathrm{d}t, \quad \eta_2 = \sum_{k\ge 0}\alpha^k \int_0^{\infty e^{\mathrm{i}d_2}} \frac{e^{-t}}{(t;q)_{k+1}}\mathrm{d}t, \tag{5.3.5}$$

with $d_1 \in (0,\frac{\pi}{2})$ *and* $d_2 \in (-\frac{\pi}{2}, 0)$.

(iii) For $x \in S_1(-\frac{\pi}{2},\frac{\pi}{2})$,

$$T_2(\alpha;q,x) - T_1(\alpha;q,x) = 2\pi i\sum_{n\ge 0}\alpha^n \sum_{j=0}^{n} \frac{(-1)^j q^{\frac{j(j-1)}{2}}}{(q;q)_j(q;q)_{n-j}} e^{-(q^n x+1)q^{-j}}. \tag{5.3.6}$$

Proof. From the perturbation of equation, $\sum_{n\ge0}\alpha^n I_n^{[d]}(x)$ satisfies Equation (5.1.2), then we study the analycity of $\sum_{n\ge0}\alpha^n I_n^{[d]}(x)$. Let V_n^d be as in (5.3.4). If $d\in(-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2})$, one can see from Figure 5.1 that $V_0^d \subset V_1^d \subset \cdots \subset V_n^d \subset \cdots$ and $\bigcap_{n\ge0} V_n^d = V_0^d$. But if $d \in (\frac{\pi}{2},\frac{3\pi}{2})$, then $\bigcap_{n\ge0} V_n^d$ will be empty, and we can not prove the analyticity of $\sum_{n\ge0}\alpha^n I_n^{[d]}(x)$ for all n. Therefore, for sufficiently small $\epsilon > 0$, we choose $d_1 = \frac{\pi}{2} - \epsilon \in (0,\frac{\pi}{2})$ and $d_2 = -\frac{\pi}{2}+\epsilon \in (-\frac{\pi}{2},0)$. Let $S_1(a,b)$ and $S_n^1(a,b)$ be as in (1.3.1). Define $T_1(\alpha;q,x)$ and $T_2(\alpha;q,x)$ as two functions analytic for $x\in S_1(-\pi+\epsilon,\epsilon)$ and $x \in S_1(-\epsilon,\pi-\epsilon)$ respectively, which can be seen as the analytic continuation of $\sum_{n\ge0}|\alpha|^n I_n^{[d_1]}$ and $\sum_{n\ge0}|\alpha|^n I_n^{[d_2]}$.

From the previous definition, we first consider the analyticity of T_1 on $S_1(-\pi+\epsilon,\epsilon)$, so we only need to prove the analyticity of $\sum_{n\ge0}|\alpha|^n I_n^{[d_1]}$ (the function T_2 admits the same result). For $d_1\in(0,\frac{\pi}{2})$, we have $V_0^{d_1} \subset V_1^{d_1} \subset \cdots \subset V_n^{d_1} \subset \cdots$. From the proof of Lemma 5.2, we have $\big|\sum_{n\ge0}\alpha^n I_n^{[d_1]}(x)\big| \le \sum_{n\ge0}\big(\frac{|\alpha|}{\cos(\epsilon)}\big)^n\frac{1}{\epsilon'}$. For $|\alpha|<1$, there exists a sufficiently small $\epsilon > 0$, such that $|\alpha| < \cos(\epsilon)$, the previous series is normally convergent in K. Therefore, the function T_1 is defined and analytic on $S_1(-\pi+\epsilon, \epsilon)$. Similarly, the function T_2 is defined and analytic on $S_1(-\epsilon, \pi-\epsilon)$.

Notice that T_1 and T_2 are well-defined at $x=0$. Let η_1 and η_2 be as in (5.3.5). From Theorem 5.1, we obtain that T_1 and T_2 can be analytically extended to $\mathbb{C}\setminus(-\infty,-1]$, such that $T_1(\alpha;q,0)=\eta_1$ and $T_2(\alpha;q,0)=\eta_2$, respectively.

Let L be an anti-clockwise smooth curve whose interior contains $\left\{q^{-j}\right\}$ for $0\leq j\leq n$. Since

$$T_2(\alpha;q,x)-T_1(\alpha;q,x)=\sum_{n\geq 0}\alpha^n\int_L \frac{e^{-(q^n x+1)t}}{(t;q)_j(1-tq^j)(tq^{j+1};q)_{n-j}}\mathrm{d}t,$$

Equation (5.3.6) can be proved by the Residue theorem. □

Remark 5.4. *We will only consider the function T_1 in the following, the function T_2 also admits the result.*

5.3.2 Another approach by Borel summation

This section will propose another way to obtain the integral-sum function, i.e., by applying the summation method of Borel-Laplace to the formal series solution of Equation (5.1.2). We first look for the formal series solution with singularities. Since the non-homogeneous term $\frac{1}{1+x}$ has a singularity at $x=-1$ and the action of opertator $y(qx)$ in (5.1.2), we obtain that the solution may contain an infinite number of singularities like $-q^{-1}, -q^{-2}, \cdots$. Assume that the solution of Equation (5.1.2) contains the terms $\frac{1}{1+q^n x}$ ($n\in\mathbb{N}$). Together with the action of operator $y'(x)$ in Equation (5.1.2), the terms $\frac{1}{1+q^n x}$ become $\frac{1}{(1+q^n x)^{m+1}}$ ($m,n\in\mathbb{N}$). Therefore, we assume that the formal series

$$\hat{y}(x)=\sum_{n\geq 0}\Big(\sum_{m\geq 0}\frac{A_{n,m}}{(1+q^n x)^{m+1}}\Big).$$

We have the following result.

Proposition 5.1. *Equation* (5.1.2) *has a formal series solution* $\hat{y}(x)=\sum_{n\geq 0}\hat{y}_n(x)\alpha^n$, *where*

$$\hat{y}_n=\sum_{m\geq 0}\frac{m!(q^{m+1};q)_n}{(1+q^n x)^{m+1}(q;q)_n}$$

for every fixed n, the Borel-sum of $\hat{y}_n(x)$ is $I_n^{[d]}(x)$, which has the same form as that of the integral-sum solution shown in Theorem 5.5.

Proof. To simply notations, in the following we shall denotevthe successive $\sum_{n\geq 0}\sum_{m\geq 0}$ by $\sum_{n\geq 0, m\geq 0}$. By putting $\hat{y}(x)$ into Equation (5.1.2), we get

$$-\sum_{n\geq 0,m\geq 0}\frac{(m+1)q^n A_{n,m}}{(1+q^n x)^{m+2}} = \alpha\sum_{n\geq 0,m\geq 0}\frac{A_{n,m}}{(1+q^{n+1}x)^{m+1}} - \sum_{n\geq 0,m\geq 0}\frac{A_{n,m}}{(1+q^n x)^{m+1}} + \frac{1}{1+x}.$$

Then

$$\begin{aligned} A_{0,0} &= 1, & A_{0,m} &= mA_{0,m-1} = m!, \\ A_{n,0} &= \alpha A_{n-1,0} = \alpha^n, & A_{n,m} &= \alpha A_{n-1,m} + mq^n A_{n,m-1}. \end{aligned}$$

Therefore, $A_{n,m} = m!\alpha^n \frac{(q^{m+1};q)_n}{(q;q)_n}$, and $\hat{y}(x)$ is a formal sries solution of (5.1.2).

For every fixed n, we will derive the borel-sum of $\hat{y}_n(x)$. From the second equation of (5.1.4), we have

$$\sum_{m\geq 0}\frac{(q^{m+1};q)_n}{(q;q)_n}t^m = \sum_{m\geq 0}\begin{bmatrix} m+n \\ m \end{bmatrix}_q t^m = \frac{1}{(t;q)_{n+1}}.$$

Then for every fixed n, the Borel-sum of $\hat{y}_n(x)$ with respect to $x_0 = q^n x + 1$ is equal to

$$\int_0^{\infty e^{\mathrm{i}d}} e^{-x_0 t}\sum_{m\geq 0}\frac{(q^{m+1};q)_n}{(q;q)_n}t^m \mathrm{d}t = \int_0^{\infty e^{\mathrm{i}d}}\frac{e^{-(q^n x+1)t}}{(t;q)_{n+1}}\mathrm{d}t = I_n^{[d]}(x),$$

where $I_n^{[d]}(x)$ is defined as in (5.3.3). □

5.4 Asymptotic behavior of the series solution with initial condition

In this section, we first study the asymptotic behaviors of the integral-sum function $T_1(\alpha;q,x)$ (or $T_2(\alpha;q,x)$) as $x\to\infty$. Then, we establish the relationship between the power series $F(\alpha;q,c_0,x)$ and the integral-sum function $T(\alpha;q,x)$. Finally, we investigate the asymptotic behaviors of $F(\alpha;q,c_0,x)$ at infinity according to the asymptotic properties and relation formulas obtained above.

5.4.1 Asymptotic behaviors of integral-sum functions

Before analyzing the asymptotic behaviors of T_1 and T_2, we first introduce the following lemma, which shows the asymptotic expansion of the function I_n for every fixed n (see reference [16, P. 70] for the definition).

Lemma 5.3. *Let $S_1(a,b)$ be as in* (1.3.1). *Consider the family of analytic functions $\{I_n(x)\}_{n\geq 0}$ mentioned in Section 5.3.1. Then for every relatively-compact subsector $S \Subset S_1(-\pi,\pi)$, there exists $C_S > 0$, such that the following Gevrey asymptotic expansion*

$$I_n(x) = \sum_{m=0}^{M} \frac{m!(q^{m+1};q)_n}{(q;q)_n} \cdot \frac{1}{(q^n x+1)^{m+1}} + r_{n,M}(x) \tag{5.4.1}$$

holds for any $n, M \in \mathbb{N}$ and any $x \in S$, where

$$|r_{n,M}(x)| \leq \frac{(-q^{M+1};q)_n}{(C_S)^{M+2}(q;q)_n} \cdot (M+1)! \cdot \frac{1}{|q^n x+1|^{M+2}}.$$

Proof. We will follow the similar steps to the proof of [16, p. 79, Theorem 22]. Let S be any relatively-compact subsector $S_1(-\pi,\pi)$. Then there exist $0 < \epsilon < \pi$, such that $S \Subset S_1(-\pi+\epsilon, \pi-\epsilon)$. Let $d_1 = \frac{\pi}{2} - \frac{\epsilon}{2}$, $d_2 = -\frac{\pi}{2} + \frac{\epsilon}{2}$, and let

$$\begin{aligned} D_1 =& \big\{x \in \widetilde{\mathbb{C} \setminus \{-1\}} \big| \arg(x+1) \in [-d_1 - \frac{\pi}{2} + \frac{\epsilon}{2}, -d_1 + \frac{\pi}{2} - \frac{\epsilon}{2}]\big\} \\ =& \big\{x \in \widetilde{\mathbb{C} \setminus \{-1\}} \big| \arg(x+1) \in [-\pi+\epsilon, 0]\big\}, \\ D_2 =& \big\{x \in \widetilde{\mathbb{C} \setminus \{-1\}} \big| \arg(x+1) \in [-d_2 - \frac{\pi}{2} + \frac{\epsilon}{2}, -d_2 + \frac{\pi}{2} - \frac{\epsilon}{2}]\big\} \\ =& \big\{x \in \widetilde{\mathbb{C} \setminus \{-1\}} \big| \arg(x+1) \in [0, \pi-\epsilon]\big\}. \end{aligned}$$

Since $S_1(-\pi+\epsilon, \pi-\epsilon) \subset D_1 \cup D_2$, we have $S \subset D_1 \cup D_2$. Let $X = q^n x + 1$, we only need to prove that $I_n^{[d_1]}(x) = \int_0^{\infty e^{\mathrm{i}d_1}} \frac{e^{-Xt}}{(t;q)_{n+1}} \mathrm{d}t$ has the same asymptotic expansion as (5.4.1), the other directions follow the same process. From the second equation of (5.1.4), one can obtain that

$$\frac{1}{(t;q)_{n+1}} = \sum_{m=0}^{M} \frac{(q^{m+1};q)_n}{(q;q)_n} t^m + \hat{r}_{n,M}(t),$$

where the function

$$\hat{r}_{n,M}(t) = \frac{1}{(t;q)_{n+1}} - \sum_{m=0}^{M} \frac{(q^{m+1};q)_n}{(q;q)_n} t^m. \tag{5.4.2}$$

Then we will simplify $\hat{r}_{n,M}(t)$. From the first equation of (5.1.4), we have

$$\sum_{m=0}^{M} \frac{(q^{m+1};q)_n}{(q;q)_n} t^m = \sum_{m=0}^{M} \sum_{k=0}^{n} \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q;q)_k (q;q)_{n-k}} (tq^k)^m = \sum_{k=0}^{n} \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q;q)_k (q;q)_{n-k}} \cdot \frac{1-(tq^k)^{M+1}}{1-tq^k}.$$

Substituting the above equation into (5.4.2) and using

$$\frac{1}{(t;q)_{n+1}} = \sum_{k=0}^{n} \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q;q)_k (q;q)_{n-k}} \cdot \frac{1}{1-tq^k},$$

we have

$$\hat{r}_{n,M}(t) = \frac{1}{(t;q)_{n+1}} - \sum_{k=0}^{n} \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q;q)_k (q;q)_{n-k}} \cdot \frac{1-(tq^k)^{M+1}}{1-tq^k} = \sum_{k=0}^{n} \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q;q)_k (q;q)_{n-k}} \cdot \frac{(tq^k)^{M+1}}{1-tq^k}.$$

Then, for $d_1 = \frac{\pi}{2} - \frac{\epsilon}{2}$, we have $\mathrm{Re}(t) > 0$, and $|\hat{r}_{n,M}(t)| \leq \frac{(-q^{M+1};q)_n}{(q;q)_n} |t|^{M+1}$.

From $\int_0^{\infty e^{\mathrm{i}d_1}} t^m e^{-Xt} \mathrm{d}t = \frac{m!}{X^{m+1}}$, we have

$$\int_0^{\infty e^{\mathrm{i}d_1}} \sum_{m=0}^{M} \frac{(q^{m+1};q)_n}{(q;q)_n} t^m e^{-Xt} \mathrm{d}t = \sum_{m=0}^{M} \frac{m!(q^{m+1};q)_n}{X^{m+1}(q;q)_n}.$$

Define $r_{n,M}(x) = \int_0^{\infty e^{\mathrm{i}d_1}} \hat{r}_{n,M}(t) e^{Xt} \mathrm{d}t$. Then $r_{n,M}(x) = \int_0^{\infty e^{\mathrm{i}[d_1+\arg(X)]}} \frac{\hat{r}_{n,M}(\frac{t}{X})}{X} e^t \mathrm{d}t$ by the change of variables. By letting $t = se^{\mathrm{i}[d_1+\arg(X)]}$ ($s \in \mathbb{R}$) and using the estimation of $\hat{r}_{n,M}(t)$, we have

$$|r_{n,M}(x)| \leq \frac{(-q^{M+1};q)_n}{(q;q)_n} \cdot \frac{1}{|X|^{M+2}} \int_0^{+\infty} s^{M+1} e^{\cos(d_1+\arg(X))s} \mathrm{d}s.$$

For any $x \in D_1$, the integral in the above equation satisfies

$$\int_0^{+\infty} s^{M+1} e^{\cos(d_1+\arg(X))s} \mathrm{d}s \leq \int_0^{+\infty} s^{M+1} e^{-\kappa s} \mathrm{d}s = \frac{(M+1)!}{\kappa^{M+2}},$$

where κ (dependent on D_1) is a postive constant equal to $\inf\left\{-\cos(d_1+\arg(X))\right\} = \sin(\frac{\epsilon}{2})$. The direction d_2 follows the same process. Therefore, for every $S \Subset S_1(-\pi,\pi)$, there is a positive constant $C_S = \sin(\frac{\epsilon}{2})$ (dependent on S), such that the function $I_n(x)$ has the Gevrey asymptotic expansion shown in (5.4.1), and

$$|r_{n,M}(x)| \leq \frac{(M+1)!(-q^{M+1};q)_n}{(C_S)^{M+2}(q;q)_n} \cdot \frac{1}{|X|^{M+2}}, \quad \text{for any} \quad x \in S.$$

□

Lemma 5.4. *For any relatively-compact subsector S of $\mathbb{C} \setminus [0,-\infty)$, there is a positive constant C_S, such that for any couple of integers M and $m \in [0,M]$, the function $\frac{t^{m+1}}{(1+t)^{m+1}}$ has the following expansion :*

$$\frac{t^{m+1}}{(1+t)^{m+1}} = \sum_{k=m}^{M} \frac{(-1)^{k+m} k!}{m!(k-m)!} \cdot t^{k+1} + \nu_{m,M}(t), \tag{5.4.3}$$

where

$$|\nu_{m,M}(t)| \leq \frac{(1+C_S)^{M+1}}{(C_S)^{m+1}} \cdot |t|^{M+2}$$

for any $t \in S$.

Proof. Let S be any relatively-compact subsector of $\mathbb{C} \setminus [0, -\infty)$. There is a sufficiently small $\epsilon > 0$, so that $S \Subset \{t \in \mathbb{C}^* \big| \arg(t) \in (-\pi + \epsilon, \pi - \epsilon)\}$. Assume that $t \in S$. Let $\rho = |t| \sin(\frac{\epsilon}{2})$ and $L_\rho = \{z \in \mathbb{C} \big| z = t + \rho e^{i\theta}, \theta \in [0, 2\pi]\}$ rotating counterwise centered at t.

For any fixed n and $m \in [0, M]$, we have

$$\frac{t^{m+1}}{(1+t)^{m+1}} = \frac{(-1)^m t^{m+1}}{m!} \cdot \frac{\mathrm{d}^m}{\mathrm{d}t^m}\Big(\frac{1}{1+t}\Big). \tag{5.4.4}$$

For $t \in S$, one can get that $\frac{1}{1+t} = \sum_{k=0}^{M}(-t)^k + \frac{(-t)^{M+1}}{1+t}$. Substituting it into (5.4.4), we have

$$\frac{t^{m+1}}{(1+t)^{m+1}} = \sum_{k=m}^{M} \frac{(-1)^{k+m} k!}{m!(k-m)!} t^{k+1} + \nu_{m,M}(t),$$

where

$$\nu_{m,M}(t) = \frac{(-1)^m t^{m+1}}{m!} \cdot \frac{\mathrm{d}^m}{\mathrm{d}t^m}\Big(\frac{(-t)^{M+1}}{1+t}\Big) = \frac{(-1)^m t^{m+1}}{2\pi \mathrm{i}} \int_{L_\rho} \frac{(-z)^{M+1}}{1+z} \cdot \frac{1}{(z-t)^{m+1}} \mathrm{d}z.$$

For $z \in L_\rho \subset \{z \in \mathbb{C} \big| \arg(z) \in (-\pi + \frac{\epsilon}{2}, \pi - \frac{\epsilon}{2})\}$, we have $|1+z| \geq \sin\frac{\epsilon}{2}$ and $|z| \leq |t|[1 + \sin(\frac{\epsilon}{2})]$. Then

$$|\nu_{m,M}(t)| \leq \frac{|t|^{m+1}}{2\pi \sin\frac{\epsilon}{2}} \int_0^{2\pi} \frac{|t|^{M+1}[1+\sin(\frac{\epsilon}{2})]^{M+1}}{(|t|\sin\frac{\epsilon}{2})^m} \mathrm{d}\theta = \frac{[1+\sin(\frac{\epsilon}{2})]^{M+1}}{(\sin\frac{\epsilon}{2})^{m+1}} |t|^{M+2}.$$

The proof is completed. □

Then we obtain the asymptotic behavior of T_1 (or T_2).

Theorem 5.6. *Let $|\alpha| < q^{M+2}$ and let $S(-\pi, \pi)$ be as in (1.3.1). For every relatively-compact subsector $S \Subset S(-\pi, \pi)$, there exists a constant $D_{S,M} > 0$, so that for every non-negative integer M and $x \in S$,*

$$T_1(\alpha; q, x) = \sum_{k=0}^{M} \sum_{m=0}^{k} \frac{(-1)^k k! (\alpha q^{-1}; q^{-1})_m}{m! (\alpha q^{-1}; q^{-1})_{k+1}} \cdot \frac{1}{x^{k+1}} + R_{S,M}(x), \tag{5.4.5}$$

where $|R_{S,M}(x)| \leq D_{S,M} \cdot \dfrac{1}{|x|^{M+2}}$.

Proof. From (1.3.1), we have $S(-\pi, \pi) \subset S_1(-\pi, \pi)$. Letting $t = \frac{1}{q^n x}$ in (5.4.3), we obtain that for any relatively-compact subsector S of $S(-\pi, \pi)$, there is a constant $C_S = \sin(\frac{\epsilon}{2}) > 0$, such that

$$\frac{1}{(q^n x + 1)^{m+1}} = \sum_{k=m}^{M} \frac{(-1)^{k+m} k! q^{-n(k+1)}}{m!(k-m)!} \cdot \frac{1}{x^{k+1}} + \nu_{n,m,M}(x), \tag{5.4.6}$$

where

$$|\nu_{n,m,M}(x)| \leq \frac{(1+C_S)^{M+1}q^{-n(M+2)}}{(C_S)^{m+1}} \cdot \frac{1}{|x|^{M+2}} \tag{5.4.7}$$

for any n and any $x \in S$.

Similar to Lemma 5.4, for remainder $r_{n,M}(x)$ in Lemma 5.3, it can be estimated as

$$|r_{n,M}(x)| \leq \frac{\left(1+\frac{1}{C_S}\right)^{M+1}(M+1)!(-q^{M+1};q)_n q^{-n(M+2)}}{(C_S)^{M+3}(q;q)_n} \cdot \frac{1}{|x|^{M+2}} \tag{5.4.8}$$

for any fixed n and any $x \in S$.

Substituting (5.4.6) and (5.4.8) into (5.4.1), we have

$$\begin{aligned} T_1(\alpha;q,x) &= \sum_{n\geq 0} \alpha^n I_n(x) \\ &= \sum_{n\geq 0}\sum_{m=0}^{M}\sum_{k=m}^{M} \frac{(-1)^{k+m}(\alpha q^{-(k+1)})^n (q^{m+1};q)_n k!}{(q;q)_n (k-m)!} \cdot \frac{1}{x^{k+1}} + R_M(x), \end{aligned} \tag{5.4.9}$$

where

$$R_M(x) = \sum_{n\geq 0} \alpha^n r_{n,M}(x) + \sum_{n\geq 0} \alpha^n \sum_{m=0}^{M} \nu_{n,m,M}(x).$$

Since (5.4.7) shows $\left|x^{M+2}\sum_{n\geq 0}\alpha^n\sum_{m=0}^{M}\nu_{n,m,M}(x)\right| \leq \sum_{n\geq 0}\sum_{m=0}^{M}\frac{(1+C_S)^{M+1}|\alpha q^{-(M+2)}|^n}{(C_S)^{m+1}}$, the last series of the above formula converges as $|\alpha| < q^{M+2}$. Together with (5.4.8), we have

$$\left|x^{M+2}R_M(x)\right| \leq$$

$$\sum_{n\geq 0} \frac{\left(1+\frac{1}{C_S}\right)^{M+1}(M+1)!(-q^{M+1};q)_n|\alpha q^{-(M+2)}|^n}{(C_S)^{M+3}(q;q)_n} + \sum_{n\geq 0}\sum_{m=0}^{M} \frac{(1+C_S)^{M+1}|\alpha q^{-(M+2)}|^n}{(C_S)^{m+1}}$$

for any $x \in S$. Denote the right-hand side of the above equation by $D_{S,M}$. We replace m by $k-m$ in (5.4.9) and simplify the coefficients of $\frac{1}{x^{k+1}}$ ($m \in \mathbb{N}$), the proof is completed by using Equation (5.1.5). □

Remark 5.5. *The functions T_1 and T_2 have asymptotic behavior as shown in formula* (5.4.5)*, whose first M terms are the same as the first M terms of the expression of the function G, but formula* (5.4.5) *holds when both the parameter α and the coefficient $D_{S,M}$ are related to M. Therefore, formula* (5.4.5) *does not belong to Gevrey asymptotic expansion, which does not contradict the uniqueness of the Gevrey asymptotic expansion.*

5.4.2 The asymptotic behaviors of series solutions

To obtain the asymptotic behavior of the power series solution with the initial condition, we first study the relationship between the power series F and the integral-sum function T_1 (or T_2). Let η_1, η_2 be as in (5.3.5), we have $T_1(\alpha; q, 0) = \eta_1$ and $T_2(\alpha; q, 0) = \eta_2$. From Theorem 5.1, the function $F(\alpha; q, c_0, x)$ satisfies Equation (5.1.2) with $y(0) = c_0$. We obtain

$$F(\alpha; q, c_0, x) = T_1(\alpha; q, x) + (c_0 - \eta_1)F_0(x) \tag{5.4.10}$$

for any $x \in S(-\pi, \pi)$. Similarly, $F(\alpha; q, c_0, x) = T_2(\alpha; q, x) + (c_0 - \eta_2)F_0(x)$.

Let $\mu \in \mathbb{C}$ be a fixed number such that $\alpha = q^\mu$. There are infinity numbers of $\mu_l = \mu + i\kappa l$ ($l \in \mathbb{Z}$ and $\kappa = -\frac{2\pi}{\ln q}$), such that $q^{\mu_l} = \alpha$. In the following, we introduce a q-periodic function representing the relationship between F and T_1 (or T_2).

Lemma 5.5. *For every fixed n, the function*

$$g_n(\mu; q, x) = \sum_{l\in\mathbb{Z}} \Gamma(n + \mu + i\kappa l)x^{-i\kappa l},\ n \geq 0 \tag{5.4.11}$$

of the solution of equation : $y(qx) = y(x)$. Furthermore, g_n is called a q-periodic function and bounded as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Proof. Let x_0 be the parameter in $[e^{id}, \frac{1}{q}e^{id}]$ with $d \in (-\frac{\pi}{2}, \frac{\pi}{2})$. For every solution of $y(qx) = y(x)$, we have $y(x_0) = y(\frac{1}{q}x_0) = \cdots$. By using the continuity and letting $C = \max_{x\in[e^{id},\frac{1}{q}e^{id}]} |g_n(\mu; q, x)|$, we obtain that $|g_n(\mu; q, x)| \leq C$ for $x = te^{id}$ with $t \geq 1$. □

Theorem 5.7. *Let $\alpha \notin q^{\mathbb{Z}_{\leq 0}}$. The relation*

$$F(\alpha; q, c_0, x) = T_1(\alpha, q, x) + C_\alpha x^{-\mu} \sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2} g_n(\mu; q, x)}{(q; q)_n} x^{-n} \tag{5.4.12}$$

holds for $x \in S(-\frac{\pi}{2}, \frac{\pi}{2})$, where

$$C_\alpha = \frac{\kappa(c_0 - \eta_1)(\alpha; q)_\infty}{2\pi(q; q)_\infty}. \tag{5.4.13}$$

Proof. One can complete the proof by using Equation (5.4.10), and the connection formula in [98, p. 6, formula (1.7)] :

$$F_0 = \frac{\kappa(\alpha; q)_\infty}{2\pi(q; q)_\infty} \sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(q; q)_n} \sum_{l\in\mathbb{Z}} (\mu + i\kappa l)_n \Gamma(\mu + i\kappa l) x^{-\mu - i\kappa l} (\frac{1}{x})^n$$

for $\alpha \notin q^{\mathbb{Z}_{\leq 0}}$ and $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Together with (5.4.10), the proof is completed. □

Remark 5.6. *Equation* (5.4.12) *still holds after replacing* T_1 *with* T_2*, where*

$$C_\alpha = \frac{\kappa(c_0 - \eta_2)(\alpha; q)_\infty}{2\pi(q; q)_\infty}.$$

From Theorem 5.6 and Theorem 5.7, we obtain the asymptotic behavior of $F(\alpha; q, c_0, x)$ as follows.

Corollary 5.1. *Assume that* $|\alpha| < q^2$*. Then*

(i) $F(\alpha; q, c_0, x) = \frac{q}{q-\alpha} \cdot x^{-1} + o(x^{-1})$ *as* $x \to \infty$ *in any relatively-compact subsector of* $S(-\frac{\pi}{2}, \frac{\pi}{2})$*.*

(ii) $F(\alpha; q, c_0, x) = (\alpha; q)_\infty e^{-x} + o(e^{-x})$ *as* $x \to \infty$ *in any relatively-compact subsector of* $S(\frac{\pi}{2}, \pi) \cup S(\pi, \frac{3\pi}{2})$*.*

Proof. (i) For x in the right-half plane $\mathbb{C}^+$. From $q^\mu = \alpha$, we have

$$|x^{-\mu}| = \left|e^{-\mathrm{Re}(\mu)\log|x| + \mathrm{Im}(\mu)\arg x + i[\mathrm{Re}(\mu)\arg x - \mathrm{Im}(\mu)\log|x|]}\right| = e^{\mathrm{Im}(\mu)\arg x}|x|^{-\mathrm{Re}(\mu)}.$$

(1) If $c_0 \neq \eta_1, \eta_2$, then we need to compare the values of $-\mathrm{Re}(\mu)$ and -1. Since $|\alpha| < q^2$, we have $-\mathrm{Re}(\mu) = -\ln|\alpha|/\ln q < -1$. From (5.4.12) and Theorem 5.6 ($M = 0$), we obtain that the function F has the same asymptotic behavior as T_1 and T_2 for $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore, $F(\alpha; q, c_0, x) = x^{-1}\left[\frac{1}{1-\alpha q^{-1}} + O(x^{-\mathrm{Re}(\mu)+1})\right]$ as $x \to \infty$ in any relaticely-compact subsector of $\mathbb{C}^+$. (2) If $c_0 = \eta_1$ or $c_0 = \eta_2$, then $C_\alpha = 0$. The fucntion F still has the same asymptotic behavior as T_1 and T_2.

(ii) For $x \in \mathbb{C}^-$, according to [98, p. 8, Theorem 2.1], we obtain that $F_0(\alpha; q, x) = (\alpha; q)_\infty e^{-x} + o(e^{-x})$ as $x \to \infty$. One can obtain the result from the analytic region of T_1 (or T_2). $\square$

Consider Equation (5.1.3) : $y'(x) = \alpha y(qx) - y(x) + \frac{1}{1-x}$. Similar to the analysis in Section 5.2.1 and 5.2.2, the series solutions at 0 and ∞ of Equation (5.1.3) have a similar form as the solutions F and G in Theorem 5.1 and 5.2. Similar to the function $T(\alpha; q, x)$, an integral-sum function is obtained, which is an analytic solution of (5.1.3) for $\arg(x-1) \in (0, 2\pi)$. Finally, we obtain the asymptotic behavior of the series solution of Equation (5.1.3) with initial condition (denoted by $\hat{F}$). Assume that $|\alpha| < q^2$. Then, for $x \in \mathbb{C}^+$, we have $\hat{F}(\alpha; q, c_0, x) = \frac{q}{\alpha - q} \cdot x^{-1} + o(x^{-1})$ as $x \to \infty$ in any relatively-compact subsector of $S(-\frac{\pi}{2}, 0) \cup S(0, \frac{\pi}{2})$. For $x \in \mathbb{C}^-$ we have $\hat{F}(\alpha; q, c_0, x) = (\alpha; q)_\infty e^{-x} + o(e^{-x})$ as $x \to \infty$ in any relatively-compact subsector of $S(\frac{\pi}{2}, \frac{3\pi}{2})$.

5.5 The properties of solutions to equation $y'(x) = \alpha y(qx) - y(x) + \frac{1}{c+x}$

In this section, we consider the properties of solutions of Equation (5.1.1) : $y'(x) = \alpha y(qx) - y(x) + \frac{1}{c+x}$, where $c \in \mathbb{C}^*$. The non-homogeneous term $\frac{1}{c+x}$ can be expanded at 0 and ∞ as follows : $\sum_{n\geq 0}(-1)^n c^{-n-1} x^n$ (for $|x| < |c|$) and $\sum_{n\geq 0}(-c)^n x^{-n-1}$ (for $|x| > |c|$). Similar to the analysis process in Section 5.2, the series solutions at 0 and ∞ of equation (5.1.1) have similar forms as the solutions $F(\alpha; q, c_0, x)$ in Theorem 5.1 and $G(\alpha; q, x)$ in Theorem 5.2, except that the expression and their radius of convergence at 0 and ∞ are all related to c. Define

$$\hat{F}_1(\alpha; q, x, c) = \sum_{n\geq 1}\sum_{k=1}^{n} \frac{(-1)^{n-1}(k-1)!\,(\alpha q^k; q)_{n-k}}{n!}\, x^n \tag{5.5.1}$$

and

$$\hat{G}(\alpha; q, c, x) = \sum_{n\geq 0}\sum_{k=0}^{n} \frac{(-c)^k n!\,(\alpha q^{-1}; q^{-1})_k}{k!\,(\alpha q^{-1}; q^{-1})_{n+1}}\, x^{-n-1}. \tag{5.5.2}$$

Let $D(0, R)$ be an open disk centered at 0 and with radius R, one get the following results.

Corollary 5.2. *Let F_0 amd F_1 be as in* (1.3.5) *and* (5.5.1)*. Given a constant $c_0 \in \mathbb{C}$, the function $F(\alpha; q, c_0, c, x) = c_0 F_0(\alpha; q, x) + \hat{F}_1(\alpha; q, c, x)$ is the unique series solution analytic in $D(0, |c|)$ of Equation* (5.1.1) *with $y(0) = c_0$. Furthermore, the series F_0 is an entire function, the radius of convergence of $\hat{F}_1$ is $|c|$.*

Corollary 5.3. *Let $\alpha \notin q^{\mathbb{Z}_{>0}}$, and let $\hat{G}(\alpha; q, c, x)$ be as in* (5.5.2)*. Then the function $\hat{G}(\alpha; q, c, x)$ is the unique analytic solution of (5.1.1) in $|x| > q|c|$ vanishes at infinity. Furthermore, the following relation holds : $\lim_{x\to\infty} x \cdot \hat{G}(\alpha; q, c, x) = \frac{1}{1-\alpha q^{-1}}$.*

From the analysis in the previous section, one can obtain that the asymptotic form of $\hat{F}(\alpha; q, c_0, c, x)$ is the same as that of $F(\alpha; q, c_0, x)$.

Corollary 5.4. *Assume that $|\alpha| < q^2$. Then*

(i) $\hat{F}(\alpha; q, c_0, c, x) = \frac{q}{q-\alpha} \cdot x^{-1} + o(x^{-1})$ as $x \to \infty$ in any relatively-compact subector of $S(-\frac{\pi}{2}, \frac{\pi}{2})$.

(ii) $\hat{F}(\alpha; q, c_0, c, x) = (\alpha; q)_\infty e^{-x} + o(e^{-x})$ as $x \to \infty$ in any relatively-compact subector of $S(\frac{\pi}{2}, \pi) \cup S(\pi, \frac{3\pi}{2})$.

5.6 Summary of this chapter

In this chapter, we study the analytic and asymptotic properties of the solution of the equation when the non-homogeneous term has a singularity at $x = 1$. According to the series form of the non-homogeneous term expanded at zero and infinity, respectively, the power series solutions at zero and infinity are calculated. By using the method of equation's perturbation, the solutions in the form of integral-sum function are obtained. By using the Borel summation method of divergent series solutions, the solutions of divergent series with singularities at 1, q^{-1}, q^{-2}, $\cdots$ are summed to obtain the corresponding Borel-sum function solution, which is the same form as the integral-sum function, and can be regarded as another derivation method of the solution of the integral form. The relationship between the solution at zero and the integral-sum function is established, and then the analytic form of the solution at zero is obtained. Finally, the results are extended to the case where the non-homogeneous term has a singularity at $x = c$.

Conclusions

In this thesis, we study the existence, uniqueness, analytic, and asymptotic properties of solutions of non-homogeneous pantograph functional differential equations over the complex number field by using the theory of analytic functions, Laplace integral, perturbation of equations, connection formulas of solutions, etc. The main contents and results of the study are as follows :

(1) The existence and uniqueness of solutions in the Laurent series at zero and infinity of the equations are studied. For the Laurent series solutions at zero, the non-homogeneous term is the product of x^γ and a series at zero. The value of γ can be the following three cases : zero, a negative integer, and the other cases, which are discussed separately. When γ equals to zero, it can be classified as the case of a Taylor series, and a solution in the form of a Taylor series is obtained. When γ is a negative integer, the solution of the equation contains the term $\log x$. In the other cases, the solution is the product of $x^{\gamma+1}$ and a series at zero. They all have the same radius of convergence as the series of the non-homogeneous term. The solution in the Laurent series at infinity is obtained by taking the form of the non-homogeneous term as the Laurent series at infinity. For the sake of analyticity, we assume that the non-homogeneous terms are analytic functions. If the non-homogeneous term is a rational function, then it can be viewed as a combination of the following three conditions : (i) the non-homogeneous term is a polynomial ; (ii) the non-homogeneous term is a fraction with a singularity at the origin ; (iii) the non-homogeneous term is a fraction with singularities at non-zero constants. When the non-homogeneous term is a polynomial, the existence and uniqueness of the solution at zero and infinity are studied, and the connection formula between the solutions at zero and infinity is obtained.

(2) The index theorem corresponding to the non-homogeneous pantograph equation is studied. The non-homogeneous pantograph equations studied in this thesis belong to a class of differential q-difference equations. According to index theorems of differential q-difference equations in the literature, the corresponding conclusions of non-homogeneous pantograph equations are discussed. By calculating the index and exceptional values of the operator corresponding to the equation, the influence of the non-homogeneous term on solutions

is obtained. The value of q is divided into two cases : $0 < q < 1$ and $q > 1$. Since the pantograph equation studied in this thesis considers the case of $0 < q < 1$, we use the index theorem corresponding to $0 < q < 1$ to analyze the influence of non-homogeneous terms on the solutions. For an operator acting on the series expanded at zero, if the formal power series solution at zero under the action of the operator yields a power series with a (non-zero) radius of convergence at zero, the solution is a series with a (non-zero) radius of convergence ; For an operator acting on a series expanded at infinity, if the formal power series solution at infinity under the action of the operator yields a power series with a (non-zero) radius of convergence at infinity, the solution is a series with a (non-zero) radius of convergence at infinity.

(3) The asymptotic properties of the analytic solutions of the equation are studied when the non-homogeneous term has a singularity at zero. We obtain the solution at zero and infinity in terms of power series based on the hypothesis that the solution is expressed by the power series expanded at zero and infinity. By calculating the convergent radius of the series, the analytic property is analyzed, and the uniqueness of the analytic solution is proved. According to the Laplace transform, assuming that the solution has the form of a Laplace integral, it is obtained that the function under the Laplace integral satisfies a class of q-difference equations. Solving the q-difference equation gives two solutions, and their corresponding Laplace integral is the solution of the non-homogeneous pantograph equation. The relationship between the two solutions of the q-difference equation is established using the special solution of the corresponding homogeneous q-difference equation, Then, the relationship between their corresponding Laplace product solutions is obtained. Using the equation's perturbation, the relationship between the series solution at zero and one of the Laplace integral solutions is studied. The relationship between the power series solution at infinity and another Laplace integral solution is obtained from the integral representation of the gamma function. According to the relation formulas obtained above, the connection formula between the solution at zero and infinity is established. The asymptotic form of the solutions with initial conditions is studied by using the connection formula.

(4) The analytical and asymptotic properties of the solutions are studied when the non-homogeneous term has a singularity at a nonzero constant c. We first consider the case where the non-homogeneous term has a singularity at 1. By expanding the non-homogeneous term into a series at zero and infinity, assuming that the solution of the equation has the form of a power series at infinity, the power series solution at zero and infinity and their radius of convergence are calculated. Using the idea of the equation's perturbation, a solution in the form of an integral-sum function is obtained, and the analytic property is analyzed. Due to the differential and q-difference operators, the equation has a divergent series solution with singularities at 1, q^{-1}, q^{-2}, $\cdots$, etc. By the Borel summation method of the divergent

series, the corresponding Borel-sum solution is obtained, which is the same as the solution of the previous integral-sum form and can be regarded as another method of deriving an integral-sum function solution. The relationship between the power series solution at zero and the integral-sum function is established. Then, the asymptotic form of the solution at zero is obtained from this relationship.

The follow-up research of this thesis can be carried out from the following aspects :

(1) Parameter $q > 1$: The equation studied in this thesis considers the case of $0 < q < 1$. When $q > 1$, the obtained result may not be valid or opposite. For example, for the analytic proof of solutions, the series converges for $0 < q < 1$, but if $q > 1$ the series will be divergent. The method of Borel summation for divergent series might be used. In addition, it is difficult to study the asymptotic form of the solution.

(2) Combine the three cases that rational functions have : the combination of the non-homogeneous term is a polynomial, the fraction with a singularity at zero and the fraction with a singularity at a non-zero constant, what effect it has on the analyticity of the solution, the region of convergence, and the connection formula.

(3) In this thesis, we study the connection formulas between solutions at zero and infinity in three cases where the non-homogeneous term is a polynomial, a fraction with a singularity at zero, and a fraction with a singularity at a non-zero constant. For non-homogeneous terms with more general forms, for example, the connection formula between the solution of a Laurent series at zero and infinity, as well as the connection formula for the solution of a non-homogeneous functional differential equation, can be studied in further detail.

Acknowledgments.

I sincerely thank the Joint PhD Program between Université de Lille and Harbin Institute of Technology, Shenzhen (HIT Shenzhen). My gratitude extends to the Paul Painlevé Laboratory and l'école doctorale EG MADIS at Université de Lille, and to HIT Shenzhen for their essential support.

My deepest appreciation goes to my supervisors for their expert guidance : Prof. Changgui Zhang for his patient direction and critical insights, Prof. Guoting Chen for his invaluable advice, and Assoc. Prof. Wenbin Li for his consistent support throughout this research.

I also thank my Thesis Examination Committee members for their time and constructive feedback.

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Abstract.

Functional differential equations are a research hotspot in differential equations, describing changes related to current, past, and future states. The pantograph equation, also known as the differential q-difference equation, models wave motion in electrified railway systems. Due to its wide applications, studying the properties of its solutions is crucial. However, research on solutions in the complex plane is lacking, and the non-homogeneous term and q-difference operator complicate asymptotic analysis.

This paper investigates the existence, uniqueness, analyticity, and asymptotic forms of solutions for non-homogeneous pantograph equations in the complex plane. We analyze Laurent series solutions at zero and infinity, and explore the influence of rational functions on these solutions. The operator index theorem is applied to classify solution growth and verify theoretical consistency.

When the non-homogeneous term has a singularity at zero, we study series solutions at zero and infinity, using Laplace transforms and Jacobi functions to establish relationships between integral and power series solutions. For singularities at nonzero constants, we analyze polynomial-transformed equations, deriving asymptotic forms and verifying solution uniqueness through initial value conditions.

Overall, this research provides a comprehensive analysis of non-homogeneous pantograph equations in the complex plane, addressing key challenges and advancing theoretical understanding.

Keywords. Functional differential equation ; Laplace transformation ; Pantograph equation ; Connection fomula ; Asymptotic behaviors ; Laurent series.

Résumé.

Les équations différentielles fonctionnelles constituent un sujet de recherche majeur en analyse différentielle, décrivant les changements liés aux états actuels, passés et futurs. L'équation du pantographe, également appelée équation différentielle à q-différence, modélise le mouvement des ondes dans les systèmes ferroviaires électrifiés. En raison de ses nombreuses applications, l'étude des propriétés de ses solutions revêt une importance cruciale. Cependant, les recherches sur les solutions dans le plan complexe restent limitées, et le terme non homogène ainsi que l'opérateur à q-différence complexifient l'analyse asymptotique.

Cet article examine l'existence, l'unicité, l'analyticité et les formes asymptotiques des solutions des équations du pantographe non homogènes dans le plan complexe. Nous analysons les solutions sous forme de séries de Laurent en zéro et à l'infini, tout en explorant l'influence des fonctions rationnelles sur ces solutions. Le théorème de l'indice des opérateurs est appliqué pour classifier la croissance des solutions et vérifier la cohérence théorique.

Lorsque le terme non homogène présente une singularité en zéro, nous étudions les solutions en série en zéro et à l'infini, en utilisant les transformées de Laplace et les fonctions de Jacobi pour établir des liens entre les solutions intégrales et les solutions en série entière. Pour les singularités en des constantes non nulles, nous analysons des équations transformées par des polynômes, en dérivant des formes asymptotiques et en vérifiant l'unicité des solutions via des conditions initiales.

Globalement, cette recherche offre une analyse complète des équations du pantographe non homogènes dans le plan complexe, abordant des défis clés et faisant progresser la compréhension théorique.

Mots clés. Équation différentielle fonctionnelle ; Transformation de Laplace ; Équation du pantographe ; Formule de connexion ; Comportement asymptotique ; Séries de Laurent.