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Mericel AYAMOU

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Mericel AYAMOU

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Acronyms

- a.e. Almost Everywhere
- PDEs Partial Differential Equations
- ODEs Ordinary Differential Equations
- TDS Time Delay Systems
- CLF Control Lyapunov Function
- FT Finite-time
- AS Asymptotically Stable/Stability
- LMIs Linear Matrix Inequalities

Notation

Some notations and preliminary definitions used throughout the thesis are as follows:

- $\mathbb{R}$ denotes the set of real numbers;
- $|z| = \sqrt{z^\top z}$ is Euclidean norm of $z = (z_i)_{i=1}^n = (z_1, \dots, z_n)^\top \in \mathbb{R}^n$;
- we write $P \succ 0$ if the symmetric matrix $P = P^\top \in \mathbb{R}^{n\times n}$ is positive definite ;
- $\lambda_{min}(Q)$ and $\lambda_{max}(Q)$ represent the minimal and maximal eigenvalue of a matrix $Q = Q^\top$;
- $\mathrm{diag}\{v\}$ denotes the diagonal matrix with the components of the vector $v \in \mathbb{R}^n$ on the main diagonal;
- $\mathcal{R}_e(A)$ represent the real parts of eigenvalues of the matrix A;
- $\|x\|_P = \sqrt{x^\top P x}$ is the weighted Euclidean norm with $P \succ 0$;
- given a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$, $\|A\|_P = \sup_{x\neq 0} \frac{\|Ax\|_P}{\|x\|_P}$;
- the sign function is defined as $\mathrm{sign}(x) := \begin{cases} 1, & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x = 0, \\ -1, & \text{if} \quad x < 0; \end{cases}$
- $\langle \cdot, \cdot \rangle$ is a scalar product on $L^2(0,1)$ such that $\forall f, g \in L^2(0,1)$, $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$, $H^k(0,1)$ is Sobolev space having k square integrable weak derivatives ;
- $f \in H^1_{loc}([0,+\infty))$ if for all $a \in (0,+\infty)$, the restriction of f to $(0,a)$ is in $H^1(0,a)$;
- ℓ denotes the set of all real sequences $\{w_i\}_{i=0}^\infty$, $w_i \in \mathbb{R}$; a column vector $w_\infty := (w_0, w_1, ...)^\top$ with an infinite number of elements can be associated with such a sequence, so we can write $w_\infty \in \ell$;
- the Hilbert space ℓ^2 of quadratically summable sequences, the Banach space ℓ^∞ of uniformly bounded sequences and all other $\ell^p, p \geq 1$ spaces are subsets of ℓ ;
- we denote by $\|Q\|_{\ell^p}$ the operator norm for a linear bounded operator $Q : \ell^p \to \ell^p$ and $\|Q\|_{\ell^p}$ is defined as follows $\|Q\|_{\ell^p} = \sup_{w_\infty \in \ell^p \setminus \{\mathbf{0}\}} \frac{\|Qw_\infty\|_{\ell^p}}{\|w_\infty\|_{\ell^p}}$;
- A function $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is said to be a class-$\mathcal{K}$ function if it is continuous, zero at zero, and strictly increasing;

- A class-$\mathcal{K}$ function $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is said to be a class-$\mathcal{K}_\infty$ function if it is unbounded with its argument;
- A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class-$\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$, and $\beta(r, \cdot)$ is decreasing and $\lim\limits_{t \to +\infty} \beta(r, t) = 0$ for each fixed $r \in \mathbb{R}_+$;
- A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *Generalized class-$\mathcal{KL}$ function* ($\mathcal{GKL}-$function) if *i*) the mapping $r \mapsto \beta(r, 0)$ is a class-$\mathcal{K}$ function; *ii*) for each fixed $r \geq 0$ the mapping $t \mapsto \beta(r, t)$ is continuous, decreasing to zero and $\exists \tilde{T}(r) \in [0, +\infty)$ such that $\beta(r, t) = 0$ for all $t \geq \tilde{T}(r)$;
- Let us denote by $\partial_t f(t, x)$ or $f_t(t, x)$ (resp. $\partial_x f(t, x)$ or $f_x(t, x)$) the partial derivative of a function f with respect to the time (resp. space) variable t (resp. x);
- Let us also denote by $\partial_{xx} f(t, x)$ or $f_{xx}(t, x)$ the second partial derivative of a function f with respect to the space variable x;

Résumé long

Cette thèse se concentre sur la conception de contrôleurs nonlinéaires précisément homogènes en dimension finie pour les phénomènes physiques modélisés par les EDPs paraboliques tels que les procédés thermiques et les écoulements. Pour la stabilisation des systèmes de contrôle en dimension infinie étudiés, comme méthode de stabilisation nous avons opté pour la décomposition modale et comme outils de contrôle, nous avons considéré l'utilisation de contrôles homogènes, qui constituent un intermédiaire entre le contrôle linéaire et le contrôle modes glissants. L'approche par décomposition modale est utilisée comme méthode clé pour stabiliser les équations aux dérivées partielles paraboliques. Cette technique consiste à séparer l'équation aux dérivées partielles paraboliques en deux composantes distinctes : la première composante est constituée d'une partie contrôlée de dimension finie qui capture les modes instables, tandis que la seconde composante est une partie résiduelle de dimension infinie qui englobe les modes stables. En isolant les modes instables, n'importe quelle technique de conception de contrôle pour les systèmes en dimension finie peut être utilisée pour concevoir des stratégies de contrôle pour ce sous-système. Dans notre cas, pour la stabilisation du sous-système en dimension finie nous avons utilisé des contrôles homogènes qui permettent d'avoir des propriétés de convergence en temps finie ou en temps fixe pour le sous système en dimension finie comparativement au contrôle linéaire qui ne peut garantir que la convergence exponentielle . Également, nous avons aussi opté pour le contrôle homogène car il permet d'avoir une meilleure convergence avec un faible débordement du signal du contrôle et moins de pic du système durant la phase initiale de stabilisation comparativement au contrôle linéaire qui en voulant garantir une meilleure convergence entraîne un large débordement du signal du contrôle et un large pic du système durant la phase initial de stabilisation. Cette approche de stabilisation pour les systèmes en dimension infinie est avantageuse car elle permet de concevoir des contrôle en dimension finie et donc par conséquent de concevoir des gains de contrôles en dimension finie, de résoudre des inégalités matricielle d'ordre réduits malgré qu'il s'agisse d'un problème de stabilisation de système en dimension infinie. Grâce à cette approche (décomposition modale + conception de contrôle homogène), nous avons obtenu des résultats significatifs, en particulier en ce qui concerne la stabilisation des équations aux dérivées partielles paraboliques, qu'elles soient linéaires ou non linéaires. Plus précisément, cette thèse présente une étude approfondie sur la conception de contrôleurs homogènes au bord pour la stabilisation des équations aux dérivées partielles (EDP) modélisant les phénomènes d'écoulement, en se concentrant plus particulièrement sur l'équation de Burgers 1D avec viscosité.

Dans un premier temps, nous nous sommes concentrés sur la stabilisation des équations de réaction-diffusion en employant la décomposition modale et en utilisant des contrôleurs homogènes. En stabilisant la partie instable avec le contrôle homogène et puisque la partie résiduelle comporte des modes stables déjà, le système en boucle fermée est juste un système en cascade. L'analyse de la stabilité de tout le système (comprenant à la fois le sous-système instable et la partie résiduelle stable) peut être étudiée via la stabilité de Lyapunov et l'attractivité asymptotique de l'origine. Bien vrai que cette approche permet de conclure la stabilité asymptotique globale du système en dimension infinie, elle ne permet pas d'une part de conclure sur la décroissance de la norme de l'état mais aussi d'autre part est inapplicable dans le cas des équations aux dérivées partielles non-linéaires comme l'équation de Burgers qui constitue l'objectif principale de nôtre thèse. Il s'est donc avéré nécessaire de construire une fonction de Lyapunov pour prouver la stabilité asymptotique des équations de réaction diffusion avec contrôle homogène au bord et d'espérer qu'on peut s'inspirer de cette fonction de Lyapunov pour garantir la stabilité locale des équations de Burgers avec contrôle homogène au bord. Dans les travaux existants employant la décomposition modale pour la stabilisation des équations de réaction-diffusion, principalement les contrôleurs linéaires sont conçus et la fonction de Lyapunov quadratique est utilisée pour conclure sur la stabilité exponentielle du système. Cette fonction de Lyapunov quadratique ne nous permet pas de conclure la stabilité du système en boucle fermée avec le contrôle homogène due à la nature non linéaire du contrôle homogène. La contribution majeure de la thèse a été de construire une fonction de Lyapunov non-quadratique pour garantir la stabilité asymptotique globale du système en boule fermée avec le contrôle homogène. Dans un second temps, nous avons proposé la construction de ce contrôle homogène pour la stabilisation de l'équation de Burgers en utilisant toujours la décomposition modale. Puisque pour les systèmes en dimension infinie la stabilité du linéarisé d'un système non-linéaire ne suffit pas pour conclure la stabilité locale du système non-linéaire, nous nous sommes inspiré de notre travail fait dans le cadre de la stabilisation des équations de réaction-diffusion avec contrôle homogène au bord pour construire une fonction de Lyapunov garantissant la stabilité asymptotique de l'équation de Burgers avec le contrôle homogène au bord. Ainsi on a justifié qu'on peut résoudre les problèmes du contrôle actif d'écoulements turbulents principalement la stabilisation des équations de Burgers en dimension en utilisant des contrôleurs homogènes. Après avoir conçu des contrôleurs homogènes au bord pour la stabilisation des équations de réaction-diffusion et de Burgers et prouvé la stabilité asymptotique des systèmes en boucle fermée via la construction de fonction de Lyapunov, il est crucial d'un point de vue mathématique d'étudier le caractère bien posé des systèmes en boucle fermée c'est à dire vérifier si le système en boucle fermée admet de solution et si la solution est unique. La plupart des résultats existants sur l'étude du caractère bien posé des systèmes en dimension infinie repose sur la propriété de Lipschitz mais dans nôtre cas, le contrôle homogène considéré n'est pas Lipschitzienne et ainsi ces méthodes classiques d'étude du caractère bien posé sont inapplicables .Nous avons donc développées dans cette thèse des méthodes spécifiques permettant d'étudier le caractère bien posée des systèmes en dimension infinie comportant des termes non-Lipschitziennes nous permettant de justifier que l'équation de réaction-diffusion avec le contrôle homogène au bord en boucle fermée admet une solution unique et que l'équation de Burgers avec contrôle homogène au bord en boucle fermé admet de solution. Également dans cette thèse nous avons conçus des prédicteurs homogènes au bord pour la stabilisation des équations de réaction-diffusion avec un retard à l'entrée. L'analyse de stabilité du système en boucle fermée a été faite via construction d'une fonction de Lyapunov. Enfin la question naturelle qui s'est posée est de savoir pourquoi utiliser les contrôleurs homogènes pour stabiliser les systèmes en dimension infinie alors que les contrôleurs linéaires stabilisent déjà les systèmes en dimension infinie tout en garantissant une stabilité exponentielle et un caractère bien posée du système en boucle fermée avec des algorithmes de contrôles plus faciles à implémenter lors des expérimentations. Pour répondre à cette question, nous avons présenté des avantages de l'utilisation des contrôleurs homogènes pour la stabilisation des systèmes en dimension infinie étudiées dans cette thèse. Dans chacun des cas suivants: stabilisation de l'équation de réaction-diffusion et de l'équation de Burgers , prédicteur homogène pour l'équation de réaction diffusion avec retard à l'entrée, nous avons présentés des simulations numériques prouvant que le contrôleur homogène permet d'avoir une meilleure convergence avec moins de débordement du signal de contrôle et peu de pic de la solution du système en boucle fermée durant la phase initiale de stabilisation comparativement au contrôleur linéaire qui en voulant garantir une meilleure convergence en utilisant un grand gain entraîne un large débordement du signal du contrôle et aussi un grand pic de la solution du système en boucle fermée. Pour finir, dans cette thèse nous avons également conçus des contrôles homogènes pour des modèles réduits qui peuvent être utilisés pour modéliser les écoulements. En effet, dans le domaine du contrôle des écoulements, deux approches sont généralement utilisées pour la conception de contrôle: la première consiste à concevoir le contrôleur pour les systèmes en dimension infinie comme expliquée plus haut et la seconde approche consiste à utiliser des méthodes de réductions tels que la méthode de Galerkin, la méthode des différences finies ou la méthodes des éléments finis pour obtenir des modèles réduits et ensuite concevoir le contrôleur sur ces modèles réduits; l'objectif de cette dernière approche est de capturer une large partie de la dynamique du système en dimension infinie à travers le modèle réduit, de concevoir le contrôle en utilisant le modèle réduit et d'implémenter ce contrôleur directement sur le système en dimension infinie espérant que ça puisse accomplir les objectifs désirés par exemple assurer la stabilisation de ce système en dimension infinie. Puisque les systèmes d'équations en dimension infinie modélisant les écoulements sont non-linéaires, les modèles réduits obtenus en utilisant les techniques de réduction sont des équations différentielles ordinaires non-linéaires. Nous avons conçus des contrôleurs homogènes pour des systèmes quasi-linéaires assurant la stabilité locale en temps finis de ces systèmes. Par ailleurs, l'avantage de méthode de conception de contrôle stabilisant les systèmes quasi-linéaire en temps fini comparé aux autres méthodes qui existent et garantissant aussi la convergence en temps fini tels que le contrôle optimale, l'approche par la fonction de contrôlabilité, contrôle par fonction de Lyapunov ou aussi l'approche par fonction de Lyapunov implicite est que nôtre méthode de conception de contrôle temps-fini est basée sur la résolution simples inégalités matricielles.

En résumé, cette thèse offre de nouvelles contributions au domaine de la théorie du contrôle des équations aux dérivées partielles en intégrant des stratégies de conception de contrôle homogènes via la décomposition modale. On s'est intéressé à la stabilisation des équations de réaction-diffusion avec un contrôle homogène au bord où l'analyse de stabilité a été faite avec une construction de fonction de Lyapunov non-quadratique. Cette construction de fonction de Lyapunov nous a permis de développer ce même type de contrôle homogène au bord pour la stabilisation de l'équation de Burgers et aussi pour l'équation de réaction diffusion avec retard à l'entrée. Les travaux présentés dans cette thèse constituent la toute première investigation de conception de contrôleurs homogènes au bord pour la stabilisation des équations aux dérivées partielles linéaires et non-linéaires.

Chapter 1

Introduction

1.1 Context and motivation: modeling and control using infinite dimensional systems

The modeling of various phenomena in science and complex processes in engineering is typically carried out using finite-dimensional systems—namely, ordinary differential equations (ODEs)—and infinite-dimensional systems, such as time-delay systems (TDS) and partial differential equations (PDEs). ODEs are particularly effective for modeling systems where interactions can be analyzed through a single independent variable. Examples include population growth, pendulum motion, and the dynamics of electrical circuits, among many others. In contrast, PDEs capture interactions across space and time and are essential for describing diffusion and propagation phenomena like heat transfer, fluid flow, and wave propagation. Overall PDEs are essential for modeling several complex physical systems, including networks (hydraulic [1], gas pipe-line [2], road traffic [3], multi-agent systems [4]), chemical and biological processes, thermal systems [5], flow control ([6], [7], [8], [9], [10]) among others. The aforementioned system and phenomena are typically described by PDEs such as parabolic PDEs [11], Hyperbolic PDEs [1], the viscous Burgers equation [12], [13], Korteweg-de Vries equation (KdV) [14], [15], [16], [17], [18], [19], [20], Kuramoto–Sivashinsky equation [21], [22], etc.

While those PDEs provide significant advantages for modeling complex and large-scale systems, they are generally more challenging to handle than ODEs, specially when it comes for control-oriented purposes. Fortunately, over several decades, the control of infinite-dimensional systems has consolidated as an active research field that encompasses various aspects of mathematical analysis, control theory, and the foundational principles of dynamics.

For those systems, there are two ways of modeling an actuation, dependent of the physical configuration: *interior control* or *boundary control*. Interior control (or in-domain control) is directly integrated into the dynamics of the PDE model and operates on a subset of the domain, whereas boundary control acts at the boundaries of the domain. This distinction means that interior control can influence the behavior of the system throughout its entire volume, allowing for localized adjustments to specific areas. In contrast, boundary control allows for the application of control directly at the boundaries, which is more suitable in many practical cases. It can be more energy-efficient than interior control. For instance, in thermal systems, regulating temperature at the boundaries can minimize energy consumption compared to heating or cooling when actuating through the entire volume of the system. For flow systems, adjusting the boundary velocities is practically easier to implement than adjusting conditions within the flow field, or changing the speed of inflow at an inlet boundary is often more straightforward than trying to manipulate flow properties throughout the entire volume. Thermal and flow systems are indeed clear examples of complex systems that are modeled by PDEs having several control objectives, and therefore motivate the development of advanced boundary control strategies. We briefly elaborate on both of them in the sequel.

Control of Thermal systems

Control of thermal systems is critical for example in applications such as HVAC (heating, ventilation, and air conditioning) systems, where the thermal regulation is crucial for maintaining comfortable indoor environments while maximizing energy efficiency. In solar thermal systems, managing the temperature of heat transfer fluids is key to optimizing energy extraction and storage. In many manufacturing processes, thermal control improves the final mechanical properties (such as hardness, toughness, and resistance), as well as the electrical or optical properties of the product. Maintaining a stable temperature is crucial in many industrial processes to prevent hazardous situations. For instance, in chemical reactions, excessive heat can lead to dangerous reactions or even explosions. Moreover, many processes are optimized for specific temperature ranges; therefore, stabilizing temperature can enhance efficiency, reduce waste, and lower energy consumption in heating, cooling, or chemical reactions. In laboratories and data centers, temperature stability is vital for preserving the integrity of sensitive materials, equipment, and stored information. Consequently, effective temperature control not only ensures safety and quality but also contributes to operational efficiency and reliability across various applications.

A thermal process can be mathematically modeled by relating temperature, time, and spatial coordinates. The modeling can be done using the heat equation with a source term, often referred to as reaction term, that may induce instability. Hence, the stabilization problem for the heat equation involves controlling temperature distributions to maintain or steer them to a desired state, especially when the system is subjected to disturbances or uncertainties.

Consider, for instance, a metal bar of length L that can be heated along its length.

The temperature distribution is governed by the following equation:

$$\frac{\partial T}{\partial t}(t,x) = \nu \frac{\partial^2 T}{\partial x^2}(t,x) + Q(t,x), \quad (t,x) \in (0,+\infty) \times (0,L),$$

where $T(t,x)$ represents the temperature at position x and time t, $Q(t,x)$ represents internal or external heat sources within the bar, and ν is the thermal diffusivity. The initial condition is given by

$$T(0,x) = T_0(x), \quad x \in [0,L], \tag{1.1}$$

where T_0 is a known function representing the initial temperature distribution. We assume that the temperature at the left end of the metal bar (at position $x = 0$) is maintained at a constant value (e.g., zero degree Celsius), and the temperature at the right end (at position $x = L$) is controlled by an external heat/cooling source modeled by a function $U(t)$ as follows:

$$T(t,0) = 0, \quad T(t,L) = U(t).$$

The control $U(t)$ can be used to maintain or adjust the temperature according to de-

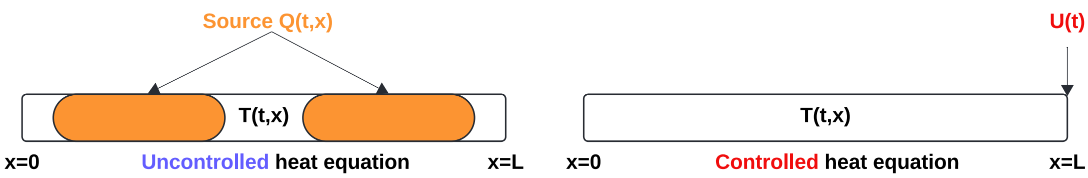

Figure 1.1: Uncontrolled and controlled temperature distribution along a metal bar of length L influenced by reactive sources.

sired performance criteria, allowing for effective thermal management. Additionally, $U(t)$ influences how heat propagates through the entire system and enables the system to respond to changing external conditions or demands, such as maintaining a specific temperature in thermal processing applications. Figure 1.2 shows a numerical example of the boundary stabilization of a 1D Reaction-Diffusion PDE.

Flow control

Flow control is an interdisciplinary field that integrates fluid mechanics ([24], [25], [26]) and control theory ([27]). The primary objectives of flow control are to manage fluid dynamics in order to prevent the adverse effects associated with flow separation, thereby enhancing performance and efficiency across various engineering domains [27]. In the aeronautics sector, flow control significantly enhances aircraft performance by reducing

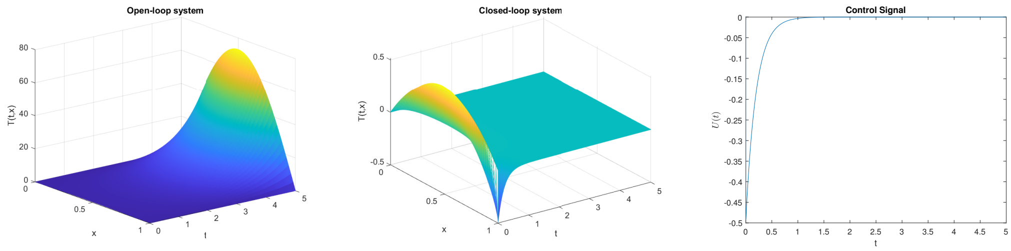

Figure 1.2: Left: Uncontrolled reaction-diffusion equation with a source term $Q(t,x) = 1.1\pi^2 T(t,x)$. Middle: controlled reaction-diffusion equation with a linear boundary controller (designed using e.g., the Modal decomposition approach [23] (as we will detail in Section 1.2.2.2). Right: Time-evolution of the linear boundary control.

drag. In the transportation industry, flow separation generates drag forces; the higher the drag, the greater the fuel consumption, which subsequently leads to increased carbon dioxide (CO2) emissions. Therefore, the challenge lies in controlling the flow to minimize separation, primarily by reducing the recirculation zone. In marine engineering, flow control improves the hydrodynamics of ships, thereby increasing their energy efficiency. Furthermore, in the energy sector, it optimizes the operation of wind turbines.

Fluid flow control can be categorized into two main types: passive and active control. Passive control ([28], [29], [30]) typically involves modifications to the surface design, such as wall-mounted devices, optimized shapes, vortex generators, streamwise ribs, or riblets. This approach does not require auxiliary power but has limitations in adaptability to changing experimental conditions, resulting in a lack of robustness. In contrast, active control ([31]) includes strategies such as vibrating flaps, acoustic excitation, or suction/blowing, which involve the continuous adjustment of variables that influence the flow based on measurements from the flow field (feedback). In recent years, several contributions addressing various active flow control strategies in synergy well-established methods of analysis within fluid mechanics [32], [33], [34],[35], [36].

The physical phenomena behind turbulent flows, separation phenomena are modeled by nonlinear PDEs, namely the Navier-Stokes equations or Burgers equation model flows (understood as a reduced-order model of the Navier-Stokes equation). We recall that the $2D$ or $3D$ Navier-Stokes equations are given as follows [37]:

$$\frac{\partial \rho V}{\partial t} - \Delta V + (V \cdot \nabla)V = -\nabla p + \nabla \cdot \tau + \rho g,$$

where ρ is the density of the fluid, p is the pressure, V is a vector representing flow velocities, τ is the stress tensor, and g is the vector representing gravitational forces. This equation is often accompanied by the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0,$$

which represents the conservation of mass. The Navier-Stokes equations are nonlinear partial differential equations that are very difficult and complex to study [1], especially in the context of Automatic Control. Given the complexity of these equations, they are often too intricate to be used for the design of real-time controllers without incurring extremely high computational costs. Consequently, they are frequently approximated to reduce the complexity of the system. The Burgers equation [12], [13], is an example of a reduced-order model for the Navier-Stokes, which shares similar nonlinearity with the Navier-Stokes equations, and captures essential dynamics often observed in turbulent flows. Therefore, it can be utilized to approximate fluid flows in certain cases and under some assumptions. These scenarios include one-dimensional flow problems, cases where external forces like gravity or pressure gradients are negligible, and conditions in which viscosity plays a critical role in influencing fluid behavior—such as in low Reynolds number flows. It is frequently employed as a benchmark equation in both theoretical studies and numerical simulations to gain a deeper understanding of more complex fluid behavior. The one-dimensional viscous Burgers equation is expressed as:

$$\partial_t v(t,x) = \nu \partial_{xx} v(t,x) - v(t,x)\partial_x v(t,x), \quad (t,x) \in (0,+\infty) \times (0,L),$$

where $v(t,x)$ denotes the velocity of the fluid at position x and time t, and ν represents the kinematic viscosity. This equation effectively models the dynamics of viscous fluid flow, enabling the study of shock formation and wave propagation in a simpler context compared to the Navier-Stokes equations. One can assume a no-slip condition at the left boundary $x = 0$, indicating that the fluid is at rest in contact with this boundary and at the right boundary $x = L$, the fluid flow can be modified by a time-varying function $U(t)$, represented by the following boundary conditions:

$$v(t,0) = 0, \quad v(t,L) = U(t).$$

The initial condition for the system is given by:

$$v(0,x) = v_0(x), \quad x \in [0,L],$$

where $v_0(x)$ represents the initial velocity distribution along the spatial domain $(0,L)$. Boundary stabilization of the viscous Burgers equation is an essential topic in control theory and mathematical physics, with numerous applications in both active and passive flow control, particularly in the context of laminar flow stabilization, aerodynamic drag reduction, and turbulence control [10], [27], [39].

1.1.1 Reduced order techniques

Model reduction is essential for various applications, particularly in control design, simulation, and real-time modeling, where full-scale simulations of PDEs may be compu-

[1] To date, the existence of a global-in-time classical solution of these equations in three dimensions remains an open ("millennium") problem in mathematics [37], [38].

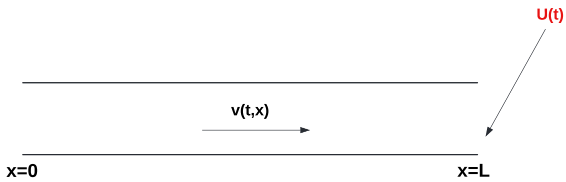

Figure 1.3: Flow

tationally expensive or infeasible. The model reduction techniques aim to derive low-dimensional systems that accurately represent the dominant dynamics of PDEs governing fluid flows or thermal systems. Several methodologies for approximating PDEs are available, employing techniques such as Galerkin's method, finite difference approximations, Proper Orthogonal Decomposition (POD), and the Finite Element Method (FEM) [40], [41], [42], [43], [44]. The Finite Differences method relies on spatial discretization using a grid and approximates derivatives, typically employing first- or second-order approximations. POD is a statistical and empirical technique used to derive a reduced set of basis functions from the solutions of the PDE over time. This method involves analyzing snapshots of the solution and identifying modes that capture the most energy of the system. The FEM approximates solutions by using a discretized mesh, which simplifies the process compared to determining an exact solution for the continuous system. The Galerkin method involves projecting a PDE onto a finite-dimensional set of basis functions, ensuring that the residual of the PDE is orthogonal to the subspace defined by these basis functions. The model reduction techniques are extensively used in fluid dynamics to analyze and control turbulent flows efficiently. In thermal systems, reduced-order models can facilitate the design of more efficient heating and cooling systems by accurately predicting temperature distributions while minimizing computational load. However, developing models that accurately reflect the relevant dynamics of a system can be quite challenging. Nevertheless, these models—despite their limitations—can still be beneficial for control applications.

1.2 Overview of PDE control

1.2.1 Early and late lumping approaches

After a model reduction via spatial discretization, the resulting system of ODEs is used in control design. The wide body of synthesis methods available for finite-dimensional systems can be used to design a controller in order to achieve some goals, for example, flow reattachment, set-point tracking (i.e., reaching a prescribed state of the flow system), and maintaining a stable temperature defined by the user. This procedure is referred to as the early lumping approach [45], [46]. The early-lumping approach, also known as indirect controller design, still plays an important role in engineering practice, which certainly comes from the fact that its application does not require a deep theoretical background. In contrast, the late lumping approach [47] aims at tackling directly the control design for systems described by PDEs, without model reduction [39], [48]. The main obstacle in the late lumping approach for fluid flow control system design is the infinite-dimensional nature of the Navier-Stokes equations, which prohibits their direct use for the synthesis of practically implementable (low-order) feedback controllers. There is quite a large literature on active flow control based on the early lumping approach. What mainly differentiates in the approaches used for is the design of a control. One can cite for example center manifold approach [49], optimal closed-loop approach [42], machine learning approach [50].

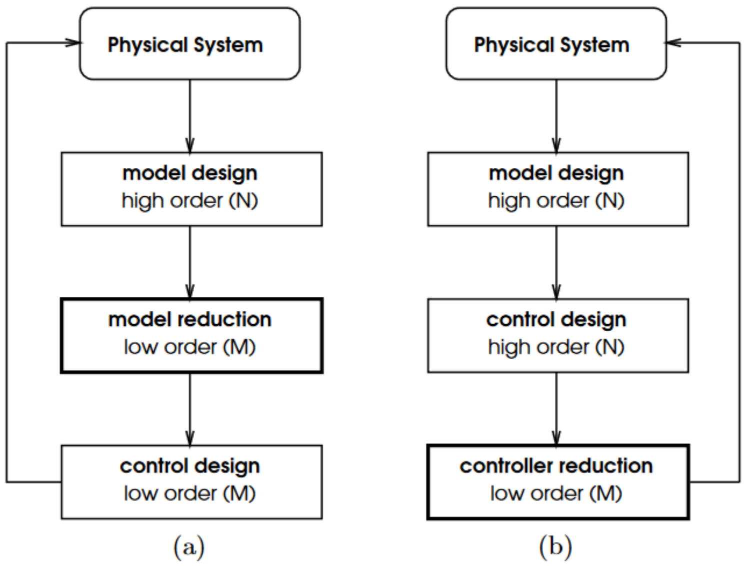

Figure 1.4: (a) Early lumping approach. (b) Late lumping approach.

1.2.2 Methods for PDE control design

As discussed in previous sections, both linear and nonlinear partial differential equations (PDEs) are central to modeling complex systems that involve diffusion and flow phenomena. Controlling these systems—especially those governed by parabolic PDEs—has been a longstanding focus of research [51], [52]. Several methods have been developed for stabilizing both linear and nonlinear parabolic PDEs, including: 1) The backstepping method; 2) The modal decomposition approach; 3) The control Lyapunov function method.

Given the numerous advantages of boundary control over interior control–such as broader applicability, and easier implementation in practical settings–it is worth recalling that most boundary stabilization strategies lead to the design of either finite-dimensional or infinite-dimensional controllers. Among these, finite-dimensional controllers are generally preferred for experimental and real-world applications due to their simpler implementation. This practicality stems from their reliance on reduced-order models, which capture the dominant system dynamics while simplifying the control design.

The feasibility of designing finite-dimensional controllers depends on the spectral properties of the underlying differential operator—typically the Laplacian. Its spectrum can be decomposed into a finite-dimensional part, which includes slow (and possibly unstable) eigenmodes, and an infinite-dimensional residual, comprising fast-decaying stable modes. By focusing on the finite-dimensional component, one can design controllers that achieve stabilization while reducing complexity.

Finite-dimensional controllers can be further categorized into linear and nonlinear types. While linear finite-dimensional controllers have been extensively studied in the literature, research on nonlinear finite-dimensional controllers is relatively limited, despite their potential to address more complex nonlinear behaviors in systems.

1.2.2.1 Backstepping method

The backstepping method is typically used for linear PDEs [53]. It involves transforming the original system into another system of the same type, referred to as the target system, while ensuring the desired stability through a change of variables (an invertible Volterra or Fredholm-type transformation), and next, the inverse transformation is to transfer the desired stability property back to the original PDE system. This method leverages the system structure to facilitate the identification of suitable Lyapunov functions for assessing the stability of the closed-loop system. Additionally, it enhances control design and stability analysis. The backstepping method relies on constructing feedback controls based on the system current state by transforming it into a simpler system that is easier to control (see [54] for more details on bacsktepping method). This technique was utilized in [55] for the local stabilization of the Korteweg-de Vries (KdV) equation , which is a nonlinear PDE exhibiting the same type of nonlinearity as the Burgers equation. Furthermore, in [56], a more generalized integral transformation on the state was employed to demonstrate local stabilization of the KdV equation. The authors introduced a nonlinear Volterra transformation for the stabilization of nonlinear PDEs using a backstepping approach [39], [48].

1.2.2.2 Modal Decomposition approach

Linear parabolic PDE

As aforementioned, the modal decomposition approach relies on separating a finite-dimensional unstable part from a stable infinite-dimensional part of the PDE. This enables the design of simpler, more flexible, finite-dimensional controllers. The roots of modal decomposition can be traced back to seminal works like [57], [58], [59]. In [59], for instance, a Galerkin projection on a modal subspace is used to design a finite-dimensional control for the reduced order model of the reaction-diffusion equation so that the control law stabilizes the unstable part while maintaining the stability of the residual part. Using the modal decomposition approach, the authors of [60] proposed a finite-dimensional linear feedback control for the stabilization of a semilinear heat equation. They were the first to conduct the stability analysis of the closed-loop system by introducing a quadratic Lyapunov function. Later, in [61], a finite-dimensional linear predictor feedback control was designed for a 1-D reaction-diffusion equation with input delay, and the stability analysis of the closed-loop system was performed using a similar quadratic Lyapunov function. The methodology proposed in [61] was extended in [62] for a feedback stabilization of a class of diagonal infinite-dimensional systems with input delay. It followed [63], which addresses the output regulation of a one-dimensional reaction-diffusion equation with input delay. A sampled-data finite-dimensional linear feedback law is introduced in [64] via the modal decomposition approach. The proof of the stability of the closed-loop system is done using a small-gain approach.

In contrast to state-feedback stabilization or output-feedback stabilization using an infinite-dimensional observer, where the closed-loop system with a continuous linear state/output feedback law is a cascade system, the closed-loop system with a finite-dimensional observer-based feedback law is typically interconnected due to spillover. Several works [65], [66], [67], [68] have focused on the design of linear finite-dimensional observer-based controllers, yet they have not presented a constructive method for finding the dimension of the observer. Instead, they have proposed very restrictive and overly conservative conditions regarding the dimension of the observer. Recent research using Lyapunov and LMI-based techniques has revived interest in designing finite-dimensional observers and controllers (see, e.g., [23], [69], [70], [71]). Based on feasible LMI conditions and using a direct Lyapunov method, [23] constructed a finite-dimensional observer-based control for the stabilization of a parabolic PDE with unbounded control and observation operators, extending this approach to Dirichlet boundary control and measurement for the heat equation in [69]. They were inspired by [60] and [61] to propose a suitable Lyapunov function for studying the stability analysis of the closed-loop system. In [71] a finite-dimensional linear control is proposed for a semi-linear parabolic PDE. The work [23] was extended in [70] to the case of Dirichlet boundary control and either Dirichlet or Neumann boundary measurements. Finite-dimensional observer-based control has been considered for the 2D and 3D parabolic PDEs in [72]. In contrast to the Lyapunov method, in [73], a small-gain approach has been used to analyze the stability of the closed-loop system with a linear finite-dimensional observer-based control law. This strategy is used in [74] to tackle the problem of event-triggered finite-dimensional observer-based output feedback stabilization for a 1D reaction-diffusion equation.

Example: Backstepping vs modal decomposition

Let us consider the 1D reaction-diffusion equation

$$\begin{aligned} z_t(t,x) &= z_{xx}(t,x) + qz(t,x), && (1.2)\\ z(t,0) &= 0, \quad z(t,1) = U(t), && (1.3)\\ z(0,x) &= z_0(x), && (1.4) \end{aligned}$$

$(t,x) \in (0,+\infty) \times (0,1)$, $U(t)$ is the control input, $q > 0$ is the reaction coefficient, $z(t,\cdot)$ is the PDE state.

The following change of variable

$$w(t,x) = z(t,x) - \int_0^x k(x,y)z(t,y)dy, \tag{1.5}$$

accompanied by the feedback control

$$U_B(t) = \int_0^1 k(1,y)z(t,y)dy, \tag{1.6}$$

where $k(\cdot,\cdot)$ is given as follows:

$$k(x,y) = -qy\frac{I_1(\sqrt{q(x^2-y^2)})}{\sqrt{q(x^2-y^2)}}, \quad 0 \le y < x \le 1, \tag{1.7}$$

and $I_1(\cdot)$ the modified Bessel function of first kind, transforms the system (1.2)-(1.3) into the following one:

$$\begin{aligned} w_t(t,x) &= w_{xx}(t,x), && (1.8)\\ w(t,0) &= w(t,1) = 0, && (1.9) \end{aligned}$$

which is globally exponentially stable in $L^2(0,1)$. On the other hand, following the modal decomposition, we denote by $\phi_n(x) = \sqrt{2}\sin(n\pi x)$ the eigenvector of the Laplacian operator $z \mapsto z_{xx}$ associated to the eigenvalue $\lambda_n = -n^2\pi^2$. By multiplying the equation (1.2) by ϕ_n and integrating over the interval $(0,1)$ as follows:

$$\int_0^1 z_t(t,x)\phi_n(x)dx = \int_0^1 z_{xx}(t,x)\phi_n(x)dx + q\int_0^1 z(t,x)\phi_n(x)dx, \tag{1.10}$$

one derives

$$\frac{d}{dt}\int_0^1 z(t,x)\phi_n(x)dx = (\lambda_n + q)\int_0^1 z(t,x)\phi_n(x)dx - \phi_n'(1)U(t). \tag{1.11}$$

Let $N \geq 0$ such that $\lambda_n + q < 0$ for all $n \geq N+1$. By setting

$$z_n(t) = \int_0^L z(t,x)\phi_n(x)dx, \quad b_n = -\phi_n'(1), \quad Z(t) = (z_1(t), z_2(t), \cdots, z_N(t))^\top, \tag{1.12}$$

$$B = (b_1, \ldots, b_N)^\top, \quad A = \mathbf{diag}\{\lambda_n + q\}_{n=1}^N, \tag{1.13}$$

one obtains the following projected system:

$$\dot{Z}(t) = AZ(t) + BU(t), \tag{1.14}$$

$$\dot{z}_n(t) = (\lambda_n + q)z_n(t) + b_n U(t), \quad n > N. \tag{1.15}$$

Since $b_n \neq 0$ for all $1 \leq n \leq N$, then the pair $\{A, B\}$ is controllable; consequently, one can design a globally exponentially stabilizing feedback law for the subsystem (1.14) as follows:

$$U(t) = U(Z(t)) = KZ(t), \tag{1.16}$$

where $K \in \mathbb{R}^{1\times N}$ is the control gain such that the closed-loop system

$$z_t(t,x) = z_{xx}(t,x) + qz(t,x), \tag{1.17}$$

$$z(t,0) = 0, \quad z(t,1) = KZ(t) \tag{1.18}$$

$$z(0,x) = z_0(x), \tag{1.19}$$

with $Z(t) = \left(\int_0^1 z(t,x)\phi_1(x)dx, \cdots, \int_0^1 z(t,x)\phi_N(x)dx\right)^\top$ is globally exponentially stable in $L^2(0,1)$. Since the closed-loop system is in cascade, the exponential stability of the system can be established using the Lyapunov stability property and the attractivity of the origin.

In summary, the backstepping design impacts the entire spectrum of the Sturm–Liouville operator $SL : z \mapsto z'' + qz$ on the domain $H^2(0,L) \cap H_0^1(0,L)$, without specifically addressing individual eigenvalues. In contrast, the modal decomposition approach only affects the first N eigenvalues of the SL operator, where N represents the dimension of the finite-dimensional controlled subsystem. Consequently, modal decomposition results in the design of a finite-dimensional controller, while the backstepping method facilitates the design of infinite-dimensional controllers. Both approaches offer distinct advantages, and neither diminishes the value of the other.

1.2.2.3 Control Lyapunov function method

The control Lyapunov function method, as described in [75, Section 12.1], uses an appropriate Lyapunov function to design a feedback control law that ensures the desired

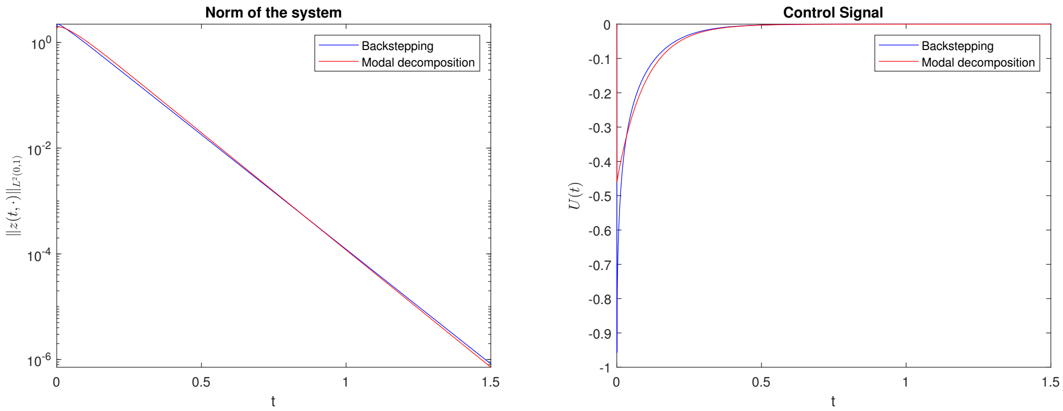

Figure 1.5: Left: Evolution of the norm $\|z(t,\cdot)\|_{L^2(0,1)}$ of the closed-loop system (1.2)-(1.3) with the backstepping controller (1.6) in blue line and finite-dimensional controller (1.16) in red line in a logarithmic scale. Right: Time evolution of the control signal of the backstepping controller (1.6) in blue line and the finite-dimensional controller(1.16) in red line

stability of a given system. The author in [76] applied this approach for the boundary stabilization of parabolic partial differential equations (PDEs). The issue of Lyapunov-based nonlinear boundary stabilization for a class of one-dimensional reaction-diffusion systems with a predefined rate of convergence is addressed in [77]. In [78], a nonlinear control strategy was introduced to achieve global stabilization of the Burgers equation, incorporating a quadratic Lyapunov function in its design. Additionally, the authors of [79], [80] further developed the approach from [78] to extend its application to the generalized Burgers equation.

Example 1.1
([76]) Let us consider the following reaction-diffusion equation

$$z_t(t,x) = z_{xx}(t,x) + 2\pi^2 z(t,x), \quad (1.20)$$
$$z(t,0) = 0, \quad z(t,1) = U(t), \quad (1.21)$$

where $z(t,x)$ is the PDE state and $U(t)$ is the control input. By introducing the following change of variable

$$w(t,x) = z(t,x) - \psi(x)U(t), \quad \psi(x) = \sin(\tfrac{5\pi}{2}x), \quad (1.22)$$

and the following virtual control input

$$\xi(t) = \dot{U}(t) + \tfrac{17\pi^2}{4}U(t), \quad (1.23)$$

one gets the following coupled ODE-PDE system

$$\dot{U}(t) = -\tfrac{17\pi^2}{4}U(t) + \xi(t), \tag{1.24}$$
$$w_t(t,x) = w_{xx}(t,x) + 2\pi^2 w(t,x) - \psi(x)\xi(t), \tag{1.25}$$
$$w(t,0) = w(t,1) = 0. \tag{1.26}$$

By multiplying the equation (1.25) *by the function* $\phi(x) = \sqrt{2}\sin(\pi x)$ *and then integrating over the interval* $(0,1)$*, one obtains:*

$$\tfrac{d}{dt}\int_0^1 w(t,x)\phi(x)dx = \pi^2 \int_0^1 w(t,x)\phi(x)dx + b\xi(t), \tag{1.27}$$

with $b = -\int_0^1 \psi(x)\phi(x)dx$*. A control Lyapunov function for the system* (1.24)*-*(1.26) *is*

$$V(t) = \tfrac{1-\alpha_1}{2}\Big(\int_0^1 w(t,x)\phi(x)dx\Big)^2 + \tfrac{\alpha_1}{2}\int_0^1 w^2(t,x)dx + \tfrac{\alpha_2}{2}U^2(t) \tag{1.28}$$

for some $\alpha_1, \alpha_2 > 0$ *and a feedback stabilizer for system* (1.24)*-*(1.26) *is*

$$\xi(t) = \int_0^1 k(x)w(t,x)dx, \tag{1.29}$$

with $k(x) = \frac{1}{b}(-\delta - \pi^2)\phi(x)$ *for any* $\delta > 0$*. The change of variable* (1.22) *can be used to estimate* $V(t)$ *and* $\xi(t)$ *in terms of the original state* $z(t,x)$ *and original control input* $U(t)$*.*

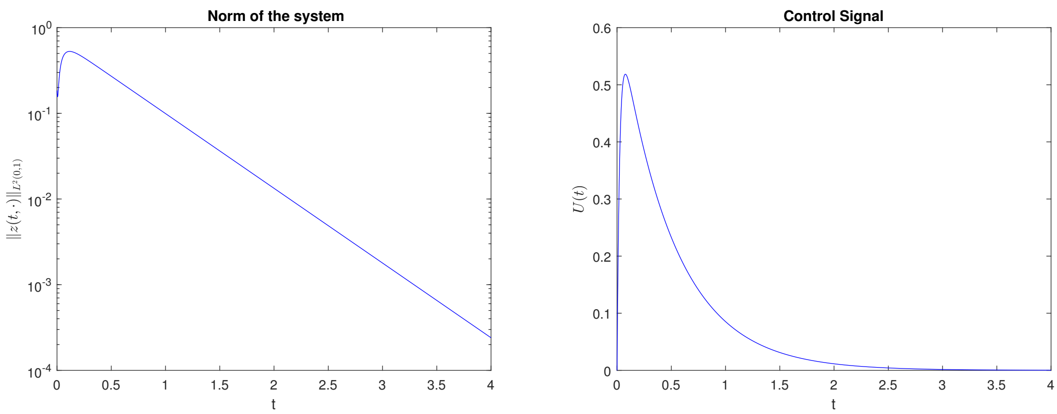

Figure 1.6: Left: Time Evolution of the Norm $\|z(t,\cdot)\|_{L^2(0,1)}$ for the Solution of the Closed-Loop System (1.20)-(1.21) and (1.24). Right: Control Signal $U(t)$ as Described in (1.24) and (1.29)

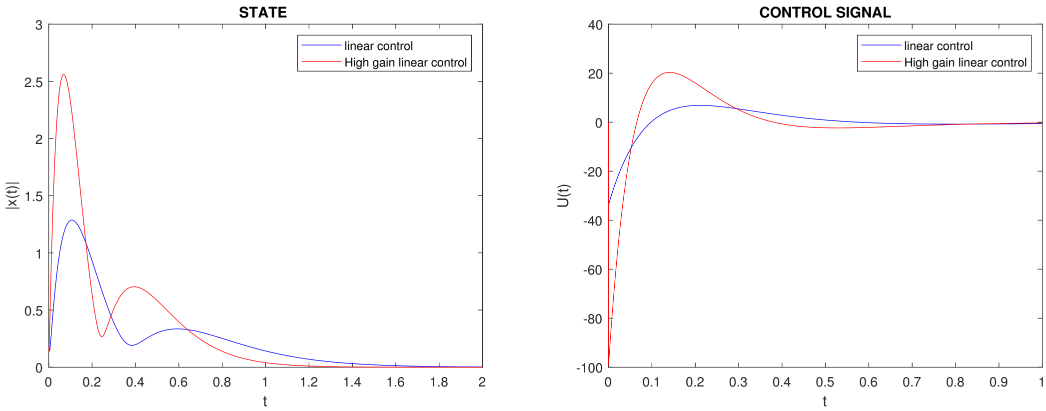

Figure 1.7: Peaking effect for a double integrator: Left: Norm of the state with linear control in blue line and high gain linear control in red line. Right: Linear control signal in blue line, high-gain linear control signal in red line .

On modal decomposition for nonlinear parabolic PDE

The main difficulty of modal decomposition for nonlinear PDEs lies in the coupling of the solution modes, which is introduced by the nonlinearity. The authors of [81] first solved the local stabilization problem for the 1D nonlinear Kuramoto–Sivashinsky equation using the modal decomposition approach and in parallel the problem of global stabilization of the semilinear heat equation with global Lipschitz nonlinearity [71] even with an input delay [82]. They showed the stability of the closed-loop system by constructing an appropriate Lyapunov functional combined with the Parseval identity. The Parseval identity helps for the compensation of the nonlinear terms in the nonlinear PDE system. A similar procedure is used in [83] for the output-feedback stabilization of the 1D nonlinear Kuramoto–Sivashinsky equation using a finite-dimensional observer. Recently, in [84], a finite-dimensional sampled data observer-based control has been designed for the Burgers equation.

In the aforementioned papers utilizing the modal decomposition approach, finite-dimensional linear controllers have been designed for the stabilization of both linear and nonlinear parabolic PDEs. The simplicity of linear controllers makes them well-suited for both application and implementation, while also facilitating a more straightforward well-posedness analysis of closed-loop systems. However, situations requiring robustness may benefit from alternative approaches like sliding-mode controllers [85], [86], [87], which can outperform linear controllers. In this context, the authors of the paper [88] proposed a finite-dimensional sliding mode output-feedback controller based on an infinite-dimensional linear observer for the stabilization of parabolic equations through modal decomposition. The stability analysis was conducted using the Lyapunov stability property and the attractivity of the origin due to the cascade structure of the closed-loop system. They also provided numerical simulations that illustrated the robustness of the constructed sliding mode controller in comparison to the linear controller. Linear controllers also have some other significant drawbacks, including the peaking effect, expressed in large overshoots during transient phases [89], [90], [91], [92]. The substantial overshoot of linear controllers (see Figure 1.7 for a numerical example) during the initial phase of stabilization—resulting in high energy consumption—can be particularly detrimental in applications such as active flow control, where the goal is to design control algorithms that stabilize flows (modeled by the viscous Burgers equation) with minimal energy use. The issues related to the peaking effect and large overshoot associated with linear controllers can be mitigated by transforming them (or as we often refer to as *"upgrading"*) into the so-called *homogeneous controllers*.

1.3 Homogeneity theory for nonlinear control design

Homogeneity is a type of symmetry of an object (function or set) with respect to a group transformation (dilation). This concept appears in many physical models and control systems. A dilation is a group action that scales variables, and various (isotropic and anisotropic) dilations have been studied in the literature. For example the following dilations stand out (see also Figure 1.8):

- Standard (isotropic) dilation introduced by Leonhard Euler in the 18th century.
- Weighted dilation introduced by by V.I. Zubov [93].
- Geometric dilation generated by unstable C^1 vector field [94].
- Linear dilation generated by an anti-Hurwitz matrix [95].

The *standard (Euler) homogeneity* means that a function $f(x)$ satisfies the scaling property:

$$f(e^s x) = e^{\nu s} f(x), \quad \forall s \in \mathbb{R},\ x \in \mathbb{R}^n,$$

where $\nu \in \mathbb{R}$ is the *degree of homogeneity* and $x \mapsto e^s x$ is the standard dilation.
Generalized homogeneity, introduced by Zubov in 1958 [93], uses a *weighted dilation* of the form:

$$(x_1, x_2, \dots, x_n) \mapsto (e^{r_1 s} x_1, e^{r_2 s} x_2, \dots, e^{r_n s} x_n), \quad s \in \mathbb{R},$$

where the weights $r_1, r_2, \dots, r_n > 0$ specify the dilation rate for each coordinate.

More general *nonlinear (geometric) dilations* have also been studied, including those generated by non-commutative vector fields or more complex transformation groups [96],

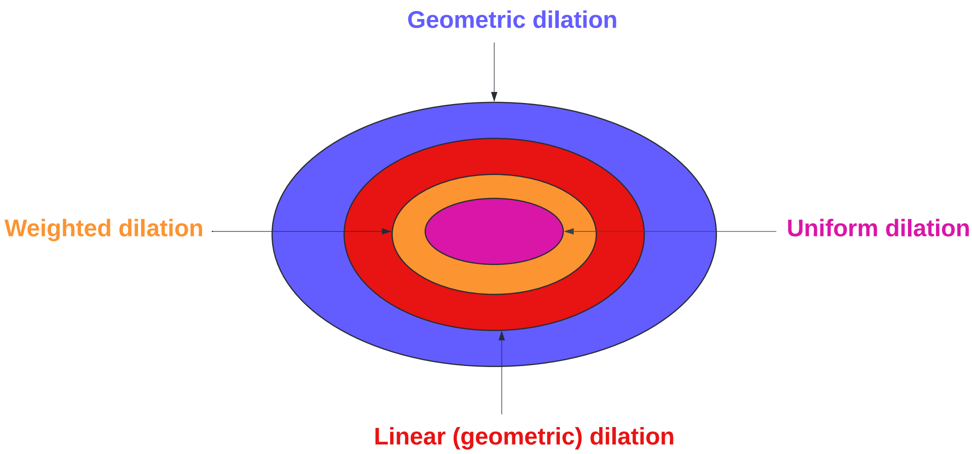

Figure 1.8: Uniform dilation ⊂ weighted dilation ⊂ linear dilation ⊂ geometric dilation

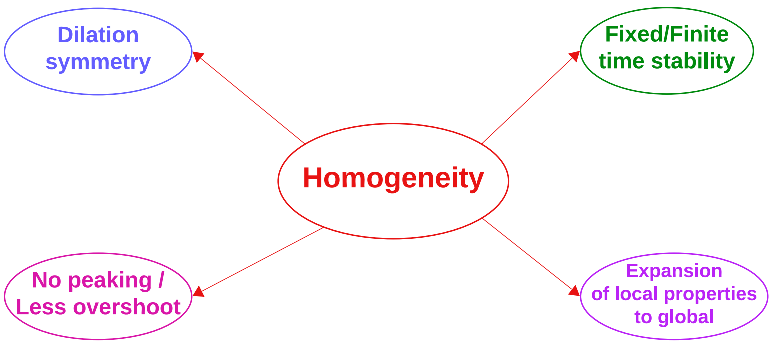

Figure 1.9: Homogeneity: Basic properties of homogeneous systems and advantages of homogeneous controllers.

[97], [98]. A specific case of geometric dilation is the *linear (geometric) dilation* [99], given by:

$$x \mapsto e^{G_{\mathbf{d}} s} x,$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ is an *anti-Hurwitz matrix* (i.e., $-G_{\mathbf{d}}$ is Hurwitz).

Homogeneous systems offer several valuable properties: local stability can often be extended globally, convergence rates can be inferred from the homogeneity degree [100], and such systems frequently yield more accurate approximations than linearizations [101], [102]. Even when a system lacks a controllable linear approximation, it may still admit a controllable homogeneous approximation.

In control theory, homogeneity is useful for stability and controllability analysis, and for the design of nonlinear controllers and observers [101], [103], [104], [105].

Homogeneity-based control bridges linear and sliding mode control and has been successfully applied to finite-dimensional systems [95], [105], [106], [107], [108].

For instance, homogeneous feedback laws solve classical problems like minimum-time control of double integrator [109].

While linear controllers are simple to implement and analyze, homogeneous nonlinear controllers provide significant performance benefits. In particular, they enable finite- or fixed-time stabilization [110], [111], faster convergence, and avoidance of the peaking effect. More details on this topic-core content of the thesis- will be given in Section 2.2 of Chapter 2.

1.4 Problems addressed in this thesis, structure and main contributions

This thesis focuses on designing homogeneous finite-dimensional controllers for infinite-dimensional systems modeled by some classes of PDEs using a modal decomposition approach. This approach allows to separate the PDE into two components: a controlled finite-dimensional part, which includes unstable modes, and an uncontrolled infinite-dimensional residual part that contains stable modes. The objective of this study was to design homogeneous controllers for the controlled finite-dimensional subsystem and to conduct a stability analysis using an appropriate Lyapunov function.

As we have discussed in Subsection 1.2.2.2, in the existing literature on the stabilization of parabolic PDEs through modal decomposition, several control design strategies are employed. For example, the classical linear control design, which enables exponential stabilization and whose stability analysis relies on the use of a quadratic Lyapunov function. Another approach utilizes nonlinear controllers, specifically sliding mode controllers, with stability analysis based on Lyapunov stability properties and the attractivity of the origin. Other approaches employ saturated controllers that lead to local stabilization, with stability analysis conducted using a quadratic Lyapunov function.

This thesis represents the first study on stabilizing linear and nonlinear parabolic PDEs by employing nonlinear homogeneous-based control strategies and conducting stability analysis based on suitable Lyapunov functions that feature switching and account for the degree of the homogeneity. Our main contributions are summarized as follows :

Part I

The first part of the thesis presents the design of finite-dimensional homogeneous control for the linear reaction-diffusion equation, as well as the design of a homogeneous

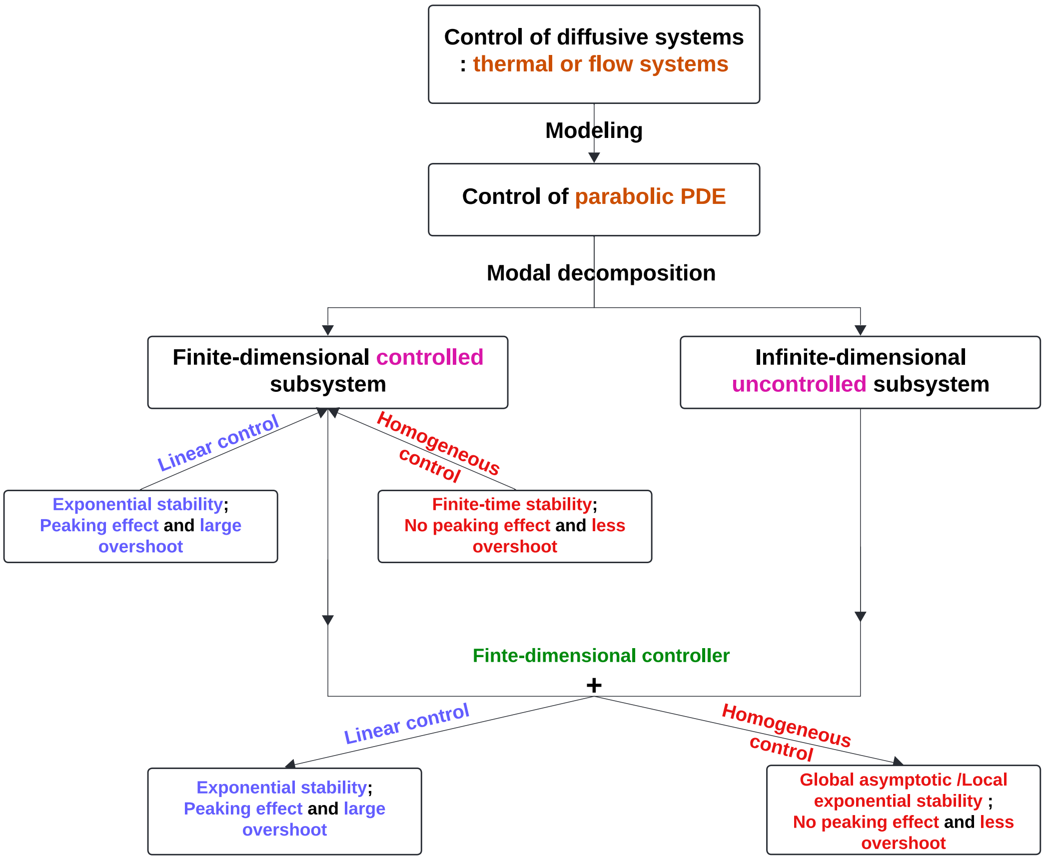

Figure 1.10: Methodologies used in this thesis.

predictor-based feedback for the reaction-diffusion equation with boundary input delay. In both scenarios, stability analysis is conducted through the construction of suitable Lyapunov functions. We put special emphasis on the nonlinear (locally non-Lipschitz) nature of the controllers and the well-posedness results.

C1). In Chapter 3, we consider the problem of boundary stabilization by means of homogeneous feedback for the following 1D reaction-diffusion PDE:

$$\begin{aligned} \partial_t z(t,x) &= \partial_{xx} z(t,x) + qz(t,x), \quad (t,x) \in (0,+\infty) \times (0,1) && (1.30)\\ z(t,0) &= 0, && (1.31)\\ z(t,1) &= U(t), && (1.32)\\ z(0,x) &= z_0(x) && (1.33) \end{aligned}$$

where $z(t,x)$ is the PDE state at time t and position x,q is the reaction coefficient, $U(t)$ is the control input and the initial condition is z_0.

We propose a boundary controller, designed via the modal decomposition approach, having the following form:

$$U(t) = \int_0^t \mathcal{H}[\mathcal{P}(U(s), z(s,\cdot))]ds, \qquad \forall t \geq 0, \tag{1.34}$$

where $\mathcal{H} : \mathbb{R}^{N+1} \to \mathbb{R}$ is the nonlinear homogeneous feedback law, $\mathcal{P}$ is the projection map given by

$$\mathcal{P} : \mathbb{R} \times L^2(0,1) \quad \to \mathbb{R}^{N+1}, \tag{1.35}$$

$$(U, z) \quad \mapsto BU + \Big(0, \langle z, \phi_1\rangle, \cdots, \langle z, \phi_N\rangle\Big)^\top \tag{1.36}$$

with

$$\phi_n(x) = \sqrt{2}\sin(n\pi x), \quad B = (1, b_1, \cdots, b_N)^\top \quad , b_n = \tfrac{\sqrt{2}(-1)^n}{n\pi}, \quad n \geq 1, \tag{1.37}$$

and N number of unstable modes. Next, by constructing a Lyapunov function defined as follows

$$V(t) = \Omega_1(\|\mathcal{P}(U(t), z(t,\cdot))\|) + \sum_{n>N} n^2\pi^2 |\langle z(t,\cdot), \phi_n(\cdot)\rangle + b_n U(t)|^2, \tag{1.38}$$

where $\Omega_1 \in \mathcal{K}_\infty$, we prove that the closed-loop system with the homogeneous state-feedback law is globally **AS** in $H^1(0,1)$, i.e, there exists $\beta_1 \in \mathcal{KL}$,

$$\forall t \geq 0, \quad \|z(t,\cdot)\|^2_{H^1(0,1)} \leq \beta_1(\|z_0\|^2_{H^1(0,1)}, t). \tag{1.39}$$

Next, we design a linear infinite-dimensional observer and study the output-feedback stabilization of the system using the homogeneous control output-feedback law

$$U(t) = \int_0^t \mathcal{H}[\mathcal{P}(U(s), \hat{z}(s,\cdot))]ds, \qquad \forall t \geq 0, \tag{1.40}$$

where $\hat{z}$ is the state of the observer. In this case, we also conduct stability analysis by considering the following Lyapunov function

$$V(t) = \Omega_1(\|\mathcal{P}(U(t), \hat{z}(t,\cdot))\|) + \textstyle\sum_{n>N} n^2\pi^2 |\langle \hat{z}(t,\cdot), \phi_n(\cdot)\rangle + b_n U(t)|^2 + \tag{1.41}$$

$$\Omega_2(\|z(t,\cdot) - \hat{z}(t,\cdot)\|_{H^1(0,1)}), \tag{1.42}$$

where $\Omega_2 \in \mathcal{K}_\infty$ and next we have shown that there exists $\beta_2 \in \mathcal{KL}$ such that

$$\forall t \geq 0, \quad \|\hat{z}(t,\cdot)\|^2_{H^1(0,1)} + \|z(t,\cdot) - \hat{z}(t,\cdot)\|^2_{H^1(0,1)} \leq \beta_2(\|z_0\|^2_{H^1(0,1)}, t). \tag{1.43}$$

In both scenarios—state feedback and output feedback—we analyze the well-posedness of the closed-loop system. Finally, we provide numerical simulations comparing the homogeneous controller and the linear controller in terms of closed-loop performance, particularly regarding the peaking phenomenon and the overshoot of the control signal.

C2). In Chapter 4, we consider the 1D reaction-diffusion equation with the input delay

$$\begin{aligned}
\partial_t z(t,x) &= \partial_{xx} z(t,x) + q z(t,x), \quad (t,x) \in (0,+\infty)\times(0,1) && (1.44)\\
z(t,0) &= 0, \quad z(t,1) = U(t-r), && (1.45)\\
z(0,x) &= z_0(x) && (1.46)
\end{aligned}$$

where $z(t,x)$ is the PDE state at time t and position x, q is the reaction coefficient, $U(t-r)$ is the delayed control input and the initial condition is z_0. We have shown in this part that under the following nonlinear homogeneous predictor feedback law

$$U(t-r) = \begin{cases} 0 & \text{if } \ t \le r, \\ \int_0^{t-r} \mathcal{H}[\bar{\mathcal{P}}(U(s), z(s,\cdot))]ds & \text{if } \ t > r, \end{cases} \tag{1.47}$$

where $\bar{\mathcal{P}}(U(s), z(s,\cdot))$ is the finite-dimensional predictor (using the Artstein transformation), the closed-loop system satisfies:

$$\forall t \ge 0, \quad \|z(t,\cdot)\|^2_{H^1(0,1)} \le \beta_3(\|z_0\|^2_{H^1(0,1)}, t), \tag{1.48}$$

and

$$U(t-r) = 0, \quad \langle z(t,\cdot), \phi_n \rangle = 0, \quad n = 1, \cdots, N, \quad \forall t \ge T + r, \tag{1.49}$$

with β_3 some $\mathcal{KL}$ function. The proof of the **AS** is based on the following Lyapunov function

$$\begin{aligned}
V(t) = \Omega_1(\|\bar{\mathcal{P}}(U(t-r), z(t,\cdot))\|) + \textstyle\int_{\max\{t-r,0\}}^{t} \psi(\|\bar{\mathcal{P}}(U(t-r), z(t,\cdot))\|) + && (1.50)\\
\textstyle\sum_{n>N} n^2\pi^2 |\langle z(t,\cdot), \phi_n(\cdot)\rangle + b_n U(t-r)|^2, \quad \forall t \ge 0, && (1.51)
\end{aligned}$$

where ψ is a $\mathcal{K}_\infty$. Furthermore, we provide numerical simulations that illustrate the closed-loop performance of the homogeneous predictor compared to the linear predictor. Notably, the homogeneous predictor feedback achieves faster convergence, successfully avoids peaking effects, and exhibits a smaller overshoot compared to the linear predictor feedback.

Part II

In this second part, we concentrate on the design of homogeneous controllers for nonlinear systems in both finite and infinite dimensions. Initially, we developed homogeneous controllers for the finite-time local stabilization of quasi-linear systems derived from viscous

Burgers' equation. Then, due to the fact that quasi-linear systems offer an approximation of the Burgers equation with less precision, they require very small discretization steps or a large number of modes when employing finite difference or Galerkin methods for satisfactory approximation. This can lead to a reduced model with very large matrices, complicating controller implementation. Thus, it is essential to directly propose finite-dimensional homogeneous controllers for the Burgers equation. Hence, drawing inspiration from our results of the first part of the thesis, we employ the modal decomposition approach to design homogeneous controllers for the local asymptotic stabilization of the Burgers equation. The stability analysis in this case is more complex due to the presence of nonlinearity. Unlike the previous scenario (homogeneous stabilization of the reaction diffusion equation), where global stability is achieved, the closed-loop nonlinear systems in this instance are only locally stable. The main challenge here is to identify techniques that can be employed for the compensation of the nonlinear terms.

Consider the 1D viscous Burgers equation

$$\begin{aligned}
\partial_t z(t,x) &= \nu\partial_{xx} z(t,x) - z(t,x)\partial_x z(t,x), \quad (t,x)\in(0,+\infty)\times(0,1) && (1.52)\\
z(t,0) &= 0, && (1.53)\\
z(t,1) &= U(t), && (1.54)\\
z(0,x) &= z_0(x) && (1.55)
\end{aligned}$$

$(t,x)\in\mathbb{R}_+\times[0,1]$, where $z(t,x)$ is the velocity of the flow at position x and time t, ν is the viscosity , $q>0$ is the reaction coefficient, z_0 the initial condition and $U(t)\in\mathbb{R}$ is the control input.

An ODE approximating Burgers equation using the finite-difference method can be defined as follows:

$$\begin{aligned}
\dot z(t,x_i) &= \nu\frac{z(t,x_{i+1})-2z(t,x_i)+z(t,x_{i-1})}{\Delta x^2} - z(t,x_i)\frac{z(t,x_{i+1})-z(t,x_{i-1})}{2\Delta x},\\
z(t,x_0) &= 0,\\
z(t,x_M) &= U(t), \quad i=1,\cdots,M-1,
\end{aligned}$$

where $M\geq 1$ is an integer and $\Delta_x = 1/M$ is the step of the discretization.Setting

$$Z(t) = (z(t,x_1),\cdots,z(t,x_{M-1}))^\top$$

one obtains the following quasi-linear system

$$\dot Z(t) = (A+W(Z(t)))Z(t) + (B+Q(Z(t)))U(t) \tag{1.56}$$

with

$$A\in\mathbb{R}^{(M-1)\times(M-1)}, \quad B\in\mathbb{R}^{(M-1)\times 1},$$

the maps $W : \mathbb{R}^{M-1} \to \mathbb{R}^{(M-1)\times(M-1)}$ and $Q : \mathbb{R}^{M-1} \to \mathbb{R}^{M-1}$ are linear with respect to their arguments.

C3). In Chapter 5, we study system (1.56). We exploit convex embedding techniques, to design an homogeneous feedback law having the following form:

$$U(t) = \mathcal{H}[Z(t)], \quad \forall t \geq 0, \tag{1.57}$$

$\mathcal{H} : \mathbb{R}^{M-1} \to \mathbb{R}$ is the homogeneous controller map such that the closed-loop system is locally stable and

$$Z(t) = 0, \quad \forall t \geq \tilde{T}, \tag{1.58}$$

where $\tilde{T}$ is the settling time. Existing methods for designing finite-time controllers for quasi-linear systems have employed approaches such as the controllability function, control Lyapunov function, and implicit Lyapunov function. However, these methods often yield controllers that are not easily implementable in experimental settings. In this section, we propose an LMI-based homogeneous control design for the finite-time stabilization of a quasi-linear system. Additionally, the presented LMIs enable a positive control signal, which is particularly advantageous for blowers used in turbulent flows. Utilizing blowers with a positive signal helps ensure efficient, stable, and controlled performance while minimizing undesirable effects such as pressure drops, noise, and vibrations.

C4). Chapter 6, deals directly with the control design of 1D viscous Burgers equation with a reaction term

$$\begin{aligned}
\partial_t z(t,x) &= \nu\partial_{xx} z(t,x) + qz(t,x) - z(t,x)\partial_x z(t,x), && (1.59)\\
z(t,0) &= 0, \quad z(t,1) = U(t), && (1.60)\\
z(0,x) &= z_0(x) && (1.61)
\end{aligned}$$

where $(t,x) \in \mathbb{R}_+ \times [0,1]$ and q is the reaction coefficient.

In contrast to previous work, where homogeneous controllers were designed for the reduced models of the Burgers equation, the objective of this chapter is to develop homogeneous controllers directly (i.e., late lumped approach) for the Burgers equation. We employed again the modal decomposition approach in synergy with homogeneity theory to propose a homogeneous controller having the following form:

$$U(t) = \int_0^t \mathcal{H}[\mathcal{P}(U(s), z(s,\cdot))]ds, \qquad \forall t \geq 0, \tag{1.62}$$

ensuring that the closed-loop system is locally asymptotically stable in $H^1(0,1)$. The proof of the asymptotic stability of the closed-loop system is established using the Lyapunov function

$$V(t) = \bar{\Omega}(\|\mathcal{P}(U(t), z(t,\cdot))\|) + \sum_{n>N} n^2\pi^2 |\langle z(t,\cdot), \phi_n(\cdot)\rangle + b_n U(t)|^2, \quad \forall t \geq 0. \tag{1.63}$$

and $\bar{\Omega} \in \mathcal{K}_\infty$ function. The Lyapunov function considered in this case is inspired by that used for the stabilization of the 1D reaction-diffusion equation.

Due to the nonlinear term zz_x present in the Burgers equation, the closed-loop system with a homogeneous feedback law is interconnected, in contrast to the reaction-diffusion case, where the closed-loop system with a homogeneous feedback law is a cascade system. We have shown in this part that given $r > 0$ sufficiently small, for any initial condition z_0 satisfying $\|z_0\|_{H^1(0,1)} \leq r$, any solution of the closed-loop system with the homogeneous feedback law satisfies

$$\forall t \geq 0, \quad \|z(t,\cdot)\|^2_{H^1(0,1)} \leq \beta_4(\|z_0\|^2_{H^1(0,1)}, t), \tag{1.64}$$

with $\beta_4 \in \mathcal{KL}$. A similar analysis was conducted for the output-feedback stabilization of the Burgers equation using homogeneous control, yielding the same results: namely, local stability of the closed-loop system and global existence of solutions.

1.5 Positioning of our contributions concerning existing results on control design via the modal decomposition approach for the considered class of systems.

In this section, we situate the contributions presented in this thesis within the framework of existing results on the stabilization of reaction-diffusion equations, both with and without input delay, as well as on the stabilization of the Burgers equation through a modal decomposition approach. As previously mentioned, most existing results on the stabilization of partial differential equations (PDEs) via modal decomposition involve the design of linear controllers, with stability analyses typically conducted using a quadratic Lyapunov function. There are very few results that utilize nonlinear control strategies, specifically sliding mode control or saturation control. In the case of sliding mode control, the stability analysis of the closed-loop system has employed the properties of Lyapunov stability and origin attractivity. Conversely, when using controls with saturation terms, only local stability can be achieved due to the inherently unstable nature of the parabolic systems considered. The stability analysis is done using a quadratic Lyapunov function. In this thesis, we propose a nonlinear control strategy (homogeneous control) that globally stabilizes the reaction-diffusion equations, accompanied by an analysis that includes the construction of a appropriate Lyapunov function.

Type of control / Type of system	Linear control strategy	Nonlinear control strategy
1D reaction-diffusion equation	[23], [69], [70], [73],[59], [65], [112]	Sliding mode control ([88]), Saturating control ([113], [114]), Homogeneous control (Chapter 3)
1D reaction-diffusion equation with input delay	[61], [63], [62], [115], [116], [117], [118],[119]	Homogeneous control (Chapter 4)
Nonlinear parabolic PDE with the same kind of nonlinearity as Burgers equation	[81], [83], [84]	Homogeneous control (Chapter 6)

Table 1.1: Positioning of our contributions concerning existing results on the stabilization of parabolic PDEs via the modal decomposition approach

Publications

- M. Ayamou, A. Polyakov, N. Espitia "Locally homogeneous finite time stabilization of quasilinear systems", Proc. of the 62nd IEEE Conference on Decision and Control (CDC), Singapore, 2023 [120]
- M. Ayamou, N. Espitia, A. Polyakov, E. Fridman, "Finite-dimensional homogeneous boundary control for a 1D reaction-diffusion equation", Proc. of the 63nd IEEE Conference on Decision and Control (CDC), Milan, 2024 [121]
- M. Ayamou, N. Espitia, A. Polyakov, E. Fridman, "Homogeneous predictor feedback for a 1D reaction-diffusion equation with input delay ", European Control Conference (ECC), 2025 [122] (accepted also for publication in a special issue of European Journal of Control https://doi.org/10.1016/j.ejcon.2025.101336)
- M. Ayamou, A. Polyakov, N. Espitia, E. Fridman, "Homogeneous boundary control for a 1D viscous Burgers equation : modal decomposition approach", IMA Journal of Mathematical Control and Information, https://doi.org/10.1093/imamci/dnaf033, [123]
- M. Ayamou, N. Espitia, A. Polyakov, E. Fridman, "Homogeneous boundary control for a 1D reaction-diffusion equation: modal decomposition approach ", regular paper, 16.5p, Automatica, (under review, 2nd round)

Chapter 2

Preliminaries

Contents

2.1 Stability Notions

The concept of stability, as introduced in Lyapunov's thesis [124], is one of the central concepts in modern control theory. The stability is a critical concept that refers to the behavior of dynamical systems over time in response to initial conditions and external inputs. The following definition is inspired by [124], [125],[126], [100], [127].

Definition 2.1
Let $\Omega \subset \mathbb{R}^n$ be an open neighborhood of the origin. The system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{2.1}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$ is said to be

- ***Lyapunov stable*** *if there exists $\alpha \in \mathcal{K}$ such that*

$$\|x(t)\| \leq \alpha(\|x_0\|), \quad \forall t > t_0, \forall x_0 \in \Omega; \tag{2.2}$$

- ***asymptotically stable*** *if it is Lyapunov stable and*

$$\lim_{t\to+\infty} \|x(t)\| = 0, \quad \forall x_0 \in \Omega; \tag{2.3}$$

- ***exponentially stable*** *if there exists some positive constants $M, \delta > 0$ such that*

$$\|x(t)\| \leq Me^{-\delta t}\|x_0\|, \quad \forall t > 0, \forall x_0 \in \Omega; \tag{2.4}$$

- ***finite-time stable*** *if it is Lyapunov stable and*

$$\forall x_0 \in \Omega, \exists T(x_0) \in (0, +\infty) \quad \textit{such that} \quad x(t) = 0, \quad \forall t \geq T(x_0); \tag{2.5}$$

- ***nearly fixed-time stable*** *if it is Lyapunov stable and*

$$\forall r > 0, \quad \exists T_r \in (0, +\infty) \quad \textit{such that} \quad \|x(t)\| \leq r, \quad \forall t \geq T_r, \quad \forall x_0 \in \Omega; \tag{2.6}$$

- ***fixed-time stable*** *if it is Lyapunov stable and*

$$\exists T \in (0, +\infty) \quad \textit{such that} \quad x(t) = 0, \quad \forall t \geq T, \quad \forall x_0 \in \Omega. \tag{2.7}$$

The set Ω is called the domain of attraction. If $\Omega = \mathbb{R}^n$, then the origin is globally stable.

Example 2.1

- *Lyapunov stability: A linear system*

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{2.8}$$

where $A \in \mathbb{R}^{n\times n}$ is such that $\mathcal{R}_e(A) \leq 0$, is Lyapunov stable;

- *Asymptotic stability: any solution of the system*

$$\dot{x}(t) = -x^3(t), \quad x(0) = x_0, \tag{2.9}$$

satisfies

$$|x(t)| \leq |x_0| \quad \textit{and} \quad \lim_{t\to+\infty} |x(t)| = 0. \tag{2.10}$$

- *Exponential stability: the unique solution of the following system*

$$\dot{x}(t) = -ax(t), \quad x(0) = x_0, \tag{2.11}$$

where $a > 0$, satisfies

$$|x(t)| \leq e^{-at}|x_0|, \quad \forall t \geq 0. \tag{2.12}$$

- *Finite-time stability: the unique solution of the following system*

$$\dot{x}(t) = -a|x(t)|^{1+\mu}\mathrm{sign}(x(t)), \quad x(0) = x_0, \tag{2.13}$$

where $a > 0, \mu \in (-1, 0)$ *satisfies*

$$|x(t)| \leq |x_0|, \quad \forall t \geq 0, \quad \textit{and} \quad x(t) = 0, \quad \forall t \geq T(x_0) = \tfrac{|x_0|^{-\mu}}{-\mu a}. \tag{2.14}$$

- *Nearly fixed-time stability: for* $\mu > 0$*, the unique solution of the system* (2.13) *satisfies*

$$|x(t)| \leq |x_0|, \quad \forall t \geq 0, \quad \textit{and} \quad \forall r > 0, \quad |x(t)| \leq r, \quad \forall t \geq \frac{1}{r^{\mu}a}, \quad \forall x_0 \in \mathbb{R}. \tag{2.15}$$

- *Fixed-time stability: the solutions of the following system*

$$\dot{x}(t) = -a(|x(t)|^{1+\mu_1} + |x(t)|^{1+\mu_2})\mathrm{sign}(x(t)) \tag{2.16}$$

where $a > 0, -1 < \mu_1 < 0 < \mu_2$ *satisfies*

$$|x(t)| \leq |x_0|, \quad \forall t \geq 0, \quad \textit{and} \quad x(t) = 0, \quad \forall t \geq \tfrac{-1}{a\mu_1} + \tfrac{1}{a\mu_2}. \tag{2.17}$$

2.2 Homogeneity in finite dimensional settings

2.2.1 Preliminaries on homogeneity

We provide an overview of the main concepts and revisit the idea of homogeneity with respect to *linear dilation*.

Dilation

The following definition of dilation is mainly inspired by [97], [94] . *A family of operators* $\mathbf{d}(s) : \mathbb{R}^n \to \mathbb{R}^n$ with $s \in \mathbb{R}$ is a continuous dilation in $\mathbb{R}^n$ if it satisfies:

- *group property* : $\mathbf{d}(0)x = x$, $\mathbf{d}(s) \circ \mathbf{d}(t)x = \mathbf{d}(s+t)x$, $\forall x \in \mathbb{R}^n, \forall s, t \in \mathbb{R}$;
- *continuity property* : the mapping $s \mapsto \mathbf{d}(s)x$ is continuous, $\forall x \in \mathbb{R}^n$;

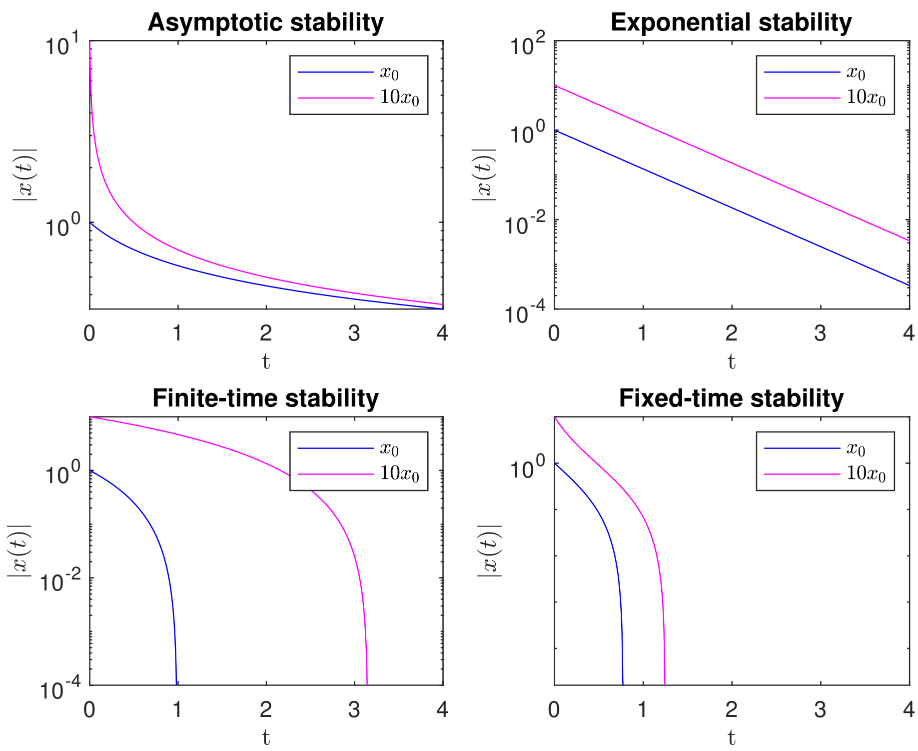

Figure 2.1: Plot of solutions for systems exhibiting asymptotic (2.9), exponential (2.11), finite-time (2.13), and fixed-time (2.16) stability .

- *limit property*: $\liminf\limits_{s\to+\infty}\|\mathbf{d}(s)x\| = +\infty$ and $\limsup\limits_{s\to-\infty}\|\mathbf{d}(s)x\| = 0$, $\forall x \neq \mathbf{0}$, where $\|\cdot\|$ is a norm of $\mathbb{R}^n$.

A dilation $\mathbf{d}$ is linear [99] if $\mathbf{d}(s) \in \mathbb{R}^{n\times n}$ is linear operator for any $s \in \mathbb{R}$. Being a Lie group [1] , any linear continuous dilation $\mathbf{d}$ in $\mathbb{R}^n$ admits the representation :

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} = \sum_{j=0}^{\infty} \frac{s^j G_{\mathbf{d}}^j}{j!}, \quad s \in \mathbb{R}, \tag{2.18}$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ is an anti-Hurwitz matrix called a generator of $\mathbf{d}$. In [98], [128], the dilation $\mathbf{d}$ is proposed to be generated as the flow of a C^1 vector field in the following manner:

$$\mathbf{d}(s)x = \phi_x(s), \quad \forall x \in \mathbb{R}^n, \tag{2.19}$$

[1] A Lie group is a group G which is also an analytic manifold such that the mapping $(x, y) \mapsto xy^{-1}$ of the product manifold $G \times G$ into G is analytic

where ϕ_x is the solution to the following system:

$$\dot{\phi}(s) = F(\phi(s)), \quad \phi(0) = x, \quad s \in \mathbb{R},$$

with the vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ such that the origin of the system $\dot{z} = -F(z)$ is globally uniformly asymptotically stable. In our case, we consider the so-called linear geometric dilation, where the vector field $F : x \mapsto G_{\mathbf{d}}x$ is linear, and the matrix $-G_{\mathbf{d}}$ is Hurwitz.

Example 2.2
In this example, we consider three classes of dilation, as introduced in Section 1.3, and give an numerical example (see Figure 2.2) on a sphere of radius 1.

- *Uniform dilation :* $\mathbf{d}(s) = e^s I_n$;
- *Weighted dilation :* $\mathbf{d}(s) = \mathrm{diag}\{e^{r_1 s}, e^{r_2 s}, \cdots, e^{r_n s}\}$, *with* $r_i > 0$ *for all* $1 \leq i \leq n$;
- *Linear (geometric) dilation:* $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$ *where* $G_{\mathbf{d}}$ *is an anti-Hurwitz matrix.*

Definition 2.2
A dilation $\mathbf{d}$ *is monotone if* $s \mapsto \|\mathbf{d}(s)x\|$ *is a monotone increasing function for any* $x \neq \mathbf{0}$.

A linear continuous dilation in $\mathbb{R}^n$ is monotone with respect to the weighted norm $\|\cdot\|_P$ with $0 \prec P = P^\top \in \mathbb{R}^{n\times n}$ if

$$PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \quad P \succ 0. \tag{2.20}$$

Any linear continuous and monotone dilation in a normed vector space also introduces an alternative norm topology.

A function $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \to [0, +\infty)$ having the properties [129]: $\|\mathbf{0}\|_{\mathbf{d}} = 0$ and

$$\|x\|_{\mathbf{d}} = e^{s_x}, \text{ where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\|_P = 1, \quad x \neq \mathbf{0}, \tag{2.21}$$

is a canonical $\mathbf{d}$-homogeneous norm in $\mathbb{R}^n$, where $\mathbf{d}$ is a linear continuous dilation being monotone with respect to the norm $\|\cdot\|_P$. Note that $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \to \mathbb{R}_+$ is single-valued, continuous on $\mathbb{R}^n$ and continuously differentiable on $\mathbb{R}^n \backslash \{\mathbf{0}\}$:

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \|x\|_{\mathbf{d}} \frac{x^\top \mathbf{d}^\top(-\ln \|x\|_{\mathbf{d}}) P \mathbf{d}(-\ln \|x\|_{\mathbf{d}})}{x^\top \mathbf{d}^\top(-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x}, \quad x \neq \mathbf{0}, \tag{2.22}$$

and for all $x \in \mathbb{R}^n$, one has

$$\|\mathbf{d}(-\ln(\|x\|_{\mathbf{d}}))x\|_P = 1. \tag{2.23}$$

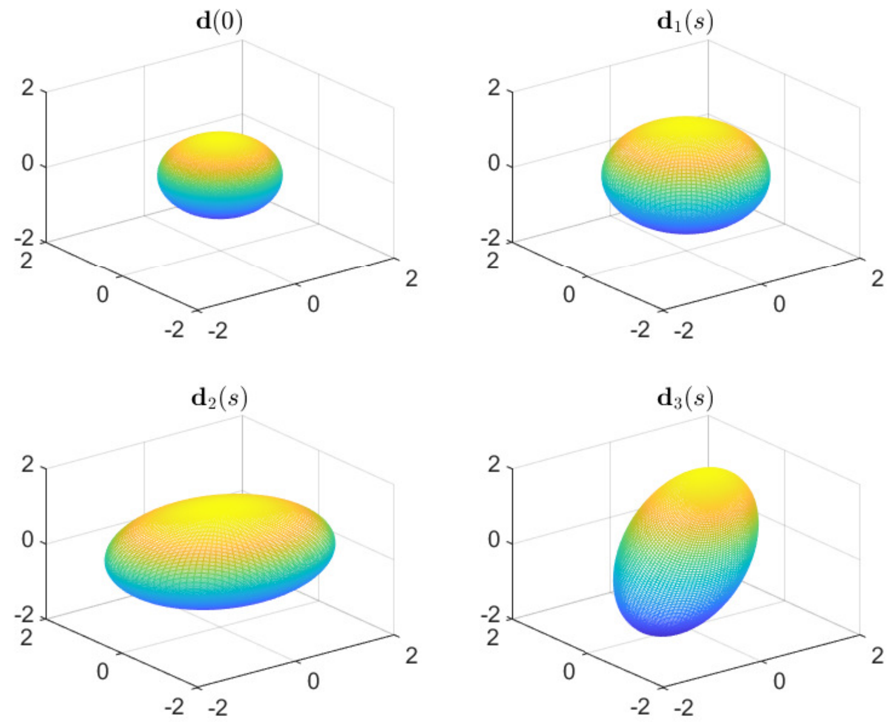

Figure 2.2: Illustration of different dilations on a sphere of radius 1. Coordinates of each marker are scaling by dilation $\mathbf{d}_i(s) = e^{sG_{\mathbf{d}_i}}, i = 1,2,3, G_{\mathbf{d}_1} = I_3, G_{\mathbf{d}_2} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}, G_{\mathbf{d}_3} = \begin{pmatrix} 1 & 0 & 1.5 \\ 1 & 0.5 & 0 \\ 0.5 & 1.5 & 1 \end{pmatrix}, s = 0.3$.

Proposition 2.1
[95] For the $\mathbf{d}$-homogeneous norm $\|\cdot\|_{\mathbf{d}}$ induced by the weighted Euclidean norm $\|\cdot\|_P$ one has :

$$\|x\|_{\mathbf{d}}^{\upsilon} \leq \|x\|_P \leq \|x\|_{\mathbf{d}}^{\tau}, \qquad \forall x \in \mathbb{R}^n : \|x\|_P \leq 1; \tag{2.24}$$

$$\|x\|_{\mathbf{d}}^{\tau} \leq \|x\|_P \leq \|x\|_{\mathbf{d}}^{\upsilon}, \qquad \forall x \in \mathbb{R}^n : \|x\|_P \geq 1, \tag{2.25}$$

with

$$\upsilon = \frac{\lambda_{\max}(P^{1/2}G_{\mathbf{d}}P^{-1/2}+P^{-1/2}G_{\mathbf{d}}^{\top}P^{1/2})}{2}, \quad \tau = \frac{\lambda_{\min}(P^{1/2}G_{\mathbf{d}}P^{-1/2}+P^{-1/2}G_{\mathbf{d}}^{\top}P^{1/2})}{2}. \tag{2.26}$$

In addition, for any $x \in \mathbb{R}^n$ one has

$$\tau\|x\|_P^2 \leq x^\top PG_{\mathbf{d}}x \leq \upsilon\|x\|_P^2. \tag{2.27}$$

If there exist $a, b > 0$ such that

$$2aP \prec PG_{\mathbf{d}}+G_{\mathbf{d}}^{\top}P \prec 2bP, \tag{2.28}$$

then $a < \tau < \upsilon < b$.

Proposition 2.2
[95] Let $\boldsymbol{d}$ be a dilation generated by an anti-Hurwitz matrix $G_{\boldsymbol{d}} \in \mathbb{R}^{n\times n}$ and let $0 \prec P = P^\top \in \mathbb{R}^{n\times n}$ satisfies (2.20). Then,

$$\begin{aligned} e^{\upsilon s} \leq \|\boldsymbol{d}(s)\|_P \leq e^{\tau s}, \quad s \leq 0,\\ e^{\tau s} \leq \|\boldsymbol{d}(s)\|_P \leq e^{\upsilon s}, \quad s \geq 0, \end{aligned} \tag{2.29}$$

with υ, τ defined in (2.26).

Homogeneous mappings

Vector fields that are homogeneous with respect to a dilation $\mathbf{d}$ possess many valuable properties for control design and state estimation in both linear and nonlinear systems. They also play a significant role in analyzing the convergence rate.

Definition 2.3
[94] A function $f : \mathbb{R}^n \to \mathbb{R}$ is $\boldsymbol{d}$-homogeneous of degree $\mu \in \mathbb{R}$ if

$$f(\boldsymbol{d}(s)x) = e^{\mu s} f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}. \tag{2.30}$$

Example 2.3
Let $p, q > 0$ and $\alpha, \beta > 0$ such that $\alpha p = \beta q$. The function $f(x, y) = x^p + y^q$ is $\boldsymbol{d}$-homogeneous of degree αp with the dilation $\boldsymbol{d}(s) = \begin{pmatrix} e^{\alpha s} & 0 \\ 0 & e^{\beta s}\end{pmatrix}$.

Definition 2.4
[94] Given a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, f is $\boldsymbol{d}$-homogeneous of degree $\mu \in \mathbb{R}$ if

$$f(\boldsymbol{d}(s)x) = e^{\mu s}\boldsymbol{d}(s) f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}. \tag{2.31}$$

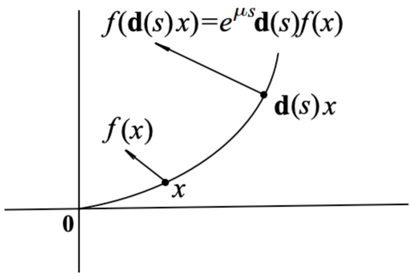

Figure 2.3: Homogeneous vector field f of degree μ with respect to the dilation $\mathbf{d}$

Homogeneity of a mapping implies homogeneity of any other mathematical object induced by this mapping. For example, a derivative of a smooth homogeneous function is homogeneous too, a flow generated by a homogeneous vector field is homogeneous as well. Being nonlinear, homogeneous dynamical systems have many properties useful for control systems design and analysis.

2.2.2 Homogeneous systems

Homogeneity is a property of mathematical objects that is akin to symmetry, defined by the preservation of a specific multiplicative scaling. In a homogeneous dynamical system, the key characteristic is the *scalability* of its solutions.

Symmetry of solutions

Let us consider the following system:

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{2.32}$$

with $x(t) \in \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$ the initial condition and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be homogeneous vector field of degree $\mu \in \mathbb{R}$ with respect to the standard dilation $\mathbf{d}(s) = e^s I$, i.e.,

$$f(e^s x) = e^{(\mu+1)s} f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}. \tag{2.33}$$

Any solution of the Cauchy problem (2.32) is symmetric with respect to the dilation of the initial state x_0 in the following sense

$$x(t, e^s x_0) = e^s x(e^{\mu s} t, x_0), \quad \forall t > 0, \quad \forall s \in \mathbb{R}. \tag{2.34}$$

Example 2.4
Consider the following system

$$\dot{x}(t) = -x^{2p}(t), \quad x(0) = x_0, \quad t > 0, \tag{2.35}$$

with the initial condition $x_0 > 0$ *and* $p > \frac{1}{2}$. *By setting* $f(x) = -x^{2p}$, *one has* $f(e^s x) = e^{(1+\mu)s} f(x)$ *with* $\mu = 2p - 1$ *for all* $s \in \mathbb{R}$. *The exact solution is given by*

$$x(t, x_0) = x_0 \Big(1 + (2p-1) x_0^{2p-1} t\Big)^{\frac{1}{1-2p}}, \quad t > 0. \tag{2.36}$$

For all $s \in \mathbb{R}$, *one has*

$$x(t, e^s x_0) = e^s x_0 \Big(1 + (2p-1) x_0^{2p-1} e^{(2p-1)s} t\Big)^{\frac{1}{1-2p}} = e^s x(e^{(2p-1)s} t, x_0), \quad \forall t > 0. \tag{2.37}$$

More generally, one has the following theorem borrowed by [93], [94], [110]:

Theorem 2.1
Given a linear continuous dilation $\boldsymbol{d}$ *and a* $\boldsymbol{d}$*-homogeneous continuous vector field* $f : \mathbb{R}^n \to \mathbb{R}^n$ *of degree* $\mu \in \mathbb{R}$ *,any solution* $x(t, x_0)$ *of the Cauchy problem*

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{2.38}$$

with $x_0 \in \mathbb{R}^n$ *the initial condition satisfies*

$$x(t, \boldsymbol{d}(s)x_0) = \boldsymbol{d}(s)x(e^{\mu s}t, x_0), \tag{2.39}$$

for all $s \in \mathbb{R}$ *and for all* $t > 0$*.*

This theorem has several corollaries that are crucial for the qualitative analysis of homogeneous systems. For example, local stability always implies global stability, and the existence of strictly invariant compact sets guarantees asymptotic stability.

Homogeneous Lyapunov function theorem

The Lyapunov method is a powerful tool that provides sufficient conditions for determining the stability of general nonlinear systems without the need to explicitly compute their solutions. It is the primary method used for stability analysis and stabilization of both finite and infinite-dimensional linear/nonlinear systems. This raises the natural question of whether the homogeneity of the Lyapunov function must be imposed for homogeneous systems. Vladimir Zubov [93] proved only the existence of a continuous homogeneous Lyapunov function, while Lionel Rosier [103] constructed a smooth one using Kurzweil's converse Lyapunov theorem.

Theorem 2.2
Let $\boldsymbol{d}$ *be a linear continuous dilation and* $f \in C(\mathbb{R}^n \backslash \{0\}, \mathbb{R}^n)$ *be a* $\boldsymbol{d}$*-homogeneous vector field of degree* $\mu \in \mathbb{R}$ *and* $f(0) = 0$*. The system*

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{2.40}$$

with $x(t) \in \mathbb{R}^n$ *is asymptotically stable if and only if there exists a positive definite homogeneous function* $V \in C(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \backslash \{0\})$ *of degree* m *such that*

$$\dot{V}(x) \leq -\rho V^{1+\frac{\mu}{m}}(x) \tag{2.41}$$

where $\rho > 0$ is some number.

Example 2.5
Let us consider the system

$$\dot{x}_i(t) = -\beta |x_i(t)|^p \mathrm{sign}(x_i(t)), \quad x_i(0) = x_{0,i}, \quad p > 0, \quad i = 1, \cdots, n \tag{2.42}$$

where $n \in \mathbb{N}^*, \beta > 0$ which is homogeneous of degree $p-1$ with respect to the standard dilation $\mathbf{d}(s) = e^s I_n$. Let us define the Lyapunov candidate

$$V(x_1, \cdots, x_n) = \tfrac{1}{2} \sum_{i=1}^{n} x_i^2(t), \tag{2.43}$$

which is $\mathbf{d}$-homogeneous of degree $m = 2$ provided that $V(e^s x_1, \cdots, e^s x_n) = e^{2s} V(x_1, \cdots, x_n)$ for all $s \in \mathbb{R}$. Along the solutions of the system (2.42) one has

$$\dot{V}(x_1, \cdots, x_n) = -\beta \sum_{i=1}^{N} |x_i(t)|^{p+1} \leq -\bar{\beta} V^{1+\frac{p-1}{2}}(x_1, \cdots, x_n), \tag{2.44}$$

with $\bar{\beta} = \beta n^{\frac{1-p}{2}}$ for $p > 1$ and $\bar{\beta} = \beta$ for $p \in (0, 1)$ and one obtains the claimed result in Theorem 2.2.

Characterization of convergence rates of homogeneous systems

The following theorem is borrowed by [100], [108],[95]:

Theorem 2.3
Let $\mathbf{d}$ be a linear continuous dilation and $f \in C(\mathbb{R}^n \backslash \{0\}, \mathbb{R}^n)$ be a $\mathbf{d}$-homogeneous vector field of degree $\mu \in \mathbb{R}$ and $f(0) = 0$. If the system (2.40) is locally asymptotically stable then

- *for $\mu < 0$ it is globally finite-time stable;*
- *for $\mu = 0$ it is globally exponentially stable;*
- *for $\mu > 0$ it is globally nearly fixed-time stable.*

The fixed-time stability of locally homogeneous systems was discovered in [104].

Example 2.6
We consider the following system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = -3 \begin{pmatrix} |x_1(t)|^p \mathrm{sign}(x_1(t)) \\ |x_2(t)|^p \mathrm{sign}(x_2(t)) \end{pmatrix} \tag{2.45}$$

with the initial condition $(x_1(0), x_2(0)) = x_0 = (1, -1)$ and $p \in \{0.2, 1.2\}$. Applying the previous theorem, this system is globally nearly fixed-time stable for $p = 1.2$ and globally finite-time stable for $p = 0.2$.

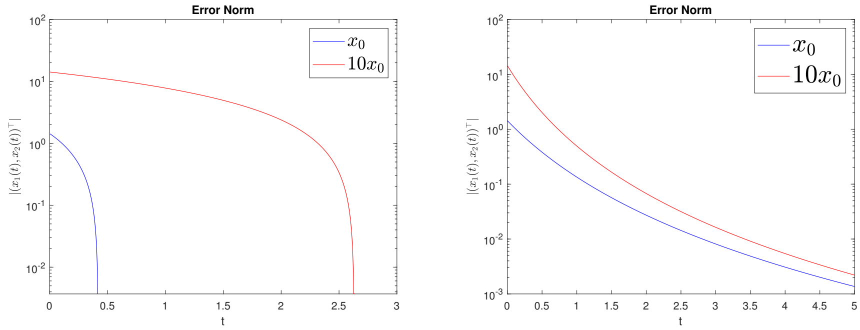

Figure 2.4: Left: Time evolution of the norm $|(x_1(t), x_2(t))^\top|$, in logarithmic scale of the system (2.45) with different initial conditions with $p = 0.2$. Right: Time evolution of the norm $|(x_1(t), x_2(t))^\top|$, in logarithmic scale of the system (2.45) with different initial conditions with $p = 1.2$.

2.2.3 Homogeneous control for LTI systems

Now, we give the result of the homogeneous stabilization of the following linear control system:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.46}$$

with the initial condition $x(0) = x_0$ where $x(t) \in \mathbb{R}^n$ is the state, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$.

Theorem 2.4
[95], [111] Let the pair $\{A, B\}$ be controllable. Then

- *any solution $Y_0 \in \mathbb{R}^{1\times n}$, $G_0 \in \mathbb{R}^{n\times n}$ of the linear algebraic equation*

$$AG_0 - G_0A + BY_0 = A, \quad G_0B = \mathbf{0}, \tag{2.47}$$

is such that the matrix $G_{\mathbf{d}} := I_n + \mu G_0$ is anti-Hurwitz for any $\mu \in (-1, \frac{1}{\tilde{n}})$ where $\tilde{n}$ is the minimal number such that $\mathrm{rank}[B, AB, \cdots, A^{\tilde{n}-1}B] = n$, and the matrix $A_0 = A + BK_0$ with $K_0 := Y_0(G_0 - I_n)^{-1}$ satisfies the identity

$$A_0G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I_n)A_0, \quad G_{\mathbf{d}}B = B; \tag{2.48}$$

- *the following Linear Matrix Inequalities (LMIs)*

$$A_0X + XA_0^\top + BY + Y^\top B^\top + \rho(G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top) \preceq 0, \tag{2.49}$$

$$G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top \succ 0, \quad X = X^\top \succ 0, \tag{2.50}$$

have solutions $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{1\times n}$ for any $\rho \in \mathbb{R}_+$;

- *the canonical homogeneous norm* $\|\cdot\|_{\mathbf{d}}$ *induced by the weighted norm* $\|\cdot\|_{X^{-1}}$ *is a Lyapunov function of the system* (2.46) *with the homogeneous feedback law*

$$u(x(t)) = K_0x(t) + \mathcal{N}(x(t)), \quad \mathcal{N}(x(t)) = \|x(t)\|_{\mathbf{d}}^{1+\mu} K\mathbf{d}(-\ln\|x(t)\|_{\mathbf{d}})x(t), \tag{2.51}$$

  where $K := YX^{-1}$, X, Y *solutions of the LMIs* (2.49)-(2.50) *and* $\mathbf{d}$ *is a dilation generated by the anti-Hurwitz matrix* $G_{\mathbf{d}}$*;*

- *the solution of the closed-loop system* (2.46) *and* (2.51) *satisfies*

$$\frac{d\|x(t)\|_{\mathbf{d}}}{dt} \leq -\rho\|x(t)\|_{\mathbf{d}}^{1+\mu}, \quad \forall t \geq 0. \tag{2.52}$$

Remark 2.1
The identity $A_0G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I_n)A_0$ *in* (2.48) *ensures that the vector field* $x \mapsto A_0x$ *is* $\mathbf{d}$*-homogeneous of degree* μ*, where* $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$*. Thus, the linear feedback law* $K_0x(t)$ *effectively "homogenizes" the linear system described by* (2.46)*. For* $\mu \in (-1, 0)$*, the inequality* (2.52) *indicates that the solution of the closed-loop system* (2.46) *and* (2.51) *satisfies*

$$x(t) = \mathbf{0}, \quad \forall t \geq T(x_0), \tag{2.53}$$

where $T(x_0) \leq \frac{\|x_0\|_{\mathbf{d}}^{-\mu}}{-\rho\mu}$ *represents the settling time.*

Example 2.7
We perform the numerical simulations on the system (2.46) *with*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{2.54}$$

Using Homogeneous Control Systems (HCS) Toolbox for MATLAB [130] , the homogeneous control parameters are :

$$\mu = -0.2, \quad K_0 = (0, 0), \quad K = (-12, -7), \quad G_{\mathbf{d}} = \begin{pmatrix} 1.2 & 0 \\ 0 & 1 \end{pmatrix}.$$

2.3 Motivation to design a homogeneous control for linear plant: peaking effect in linear control systems

In this section, we explain the concepts of overshoot and the peaking effect in closed-loop systems with linear control. We present existing results that address this phenomenon in

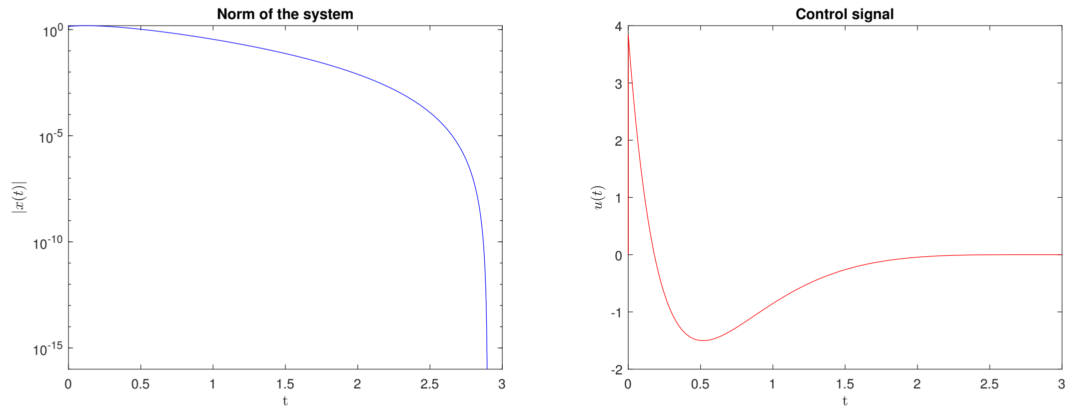

Figure 2.5: Left: Norm of state $|x(t)|$ of the closed-loop system (2.46) with the homogeneous feedback law (2.51) . Right: Homogeneous control signal (2.51) .

a broader context and provide examples to illustrate the notions of overshoot and peaking in linear systems. It is important to note that this section is essential for our thesis, as we will subsequently illustrate how the advantages of the homogeneous controller, in comparison to linear control, relate to these phenomena—specifically, the peaking effect and overshoot.

2.3.1 On the peaking problem in LTI systems

Consider the following SISO LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.55}$$

with the initial condition $x(0) = x_0$ where $x(t) \in \mathbb{R}^n$ is the state, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$. Assuming the pair $\{A, B\}$ satisfies the Kalman rank controllability condition, then $\{A, B\}$ can be assumed to be in the canonical controllability form without loss of generality. Hence, $A = J_n = ((0_{(n-1)\times 1}, I_{n-1})^\top, 0_{n\times 1})^\top$ and $B = (0, \ldots, 0, 1)^\top$.

Since the pair $\{A, B\}$ is controllable then there exists $K \in \mathbb{R}^{1\times n}$ such that $\mathcal{R}_e(A + BK) < 0$. The closed-loop system with linear state feedback is exponentially stable, with a decay rate that can be arbitrarily adjusted by suitably tuning K (the higher the gain, the faster the convergence). Assume now that the initial state is bounded as follows: $|x(0)| \leq 1$, and that the control aim is to stabilize the state of the system (2.55) into a ball of a small radius $\epsilon > 0$ in a prescribed time $T > 0$, i.e.,

$$|x(t)| \leq \epsilon, \quad \forall t \geq T. \tag{2.56}$$

As mentioned earlier, with a static linear controller (full state or output feedback), one can obtain a closed-loop system with arbitrarily fast decay. More precisely, we can derive the following properties:

$$\forall \epsilon > 0, \quad \exists K \in \mathbb{R}^{1\times n}: \quad \sup_{|x_0|=1} |x(t)| \le \epsilon, \quad t > T, \tag{2.57}$$

$$|x(t)| \le Me^{-\delta t}, \quad t > 0, \tag{2.58}$$

where $M \ge 1$ depends on the eigenvalues of the matrix $A+BK$, for which the real parts of its eigenvalues are less than $-\delta$ for some positive $\delta > 0$. Notice that the smaller $\epsilon > 0$ the larger $\delta > 0$ has to be assigned to solve the control problem (2.56). In particular, $\delta \to \infty$ as $\epsilon \to 0$ provided that T is fixed. According to [89], there exists $\gamma > 0$ independent of δ such that

$$\sup_{0\le t\le \frac{1}{\delta}} \sup_{|x(0)|=1} |x(t)| \ge \gamma\delta^{n-1}. \tag{2.59}$$

This means for $n \ge 2$,

$$\sup_{0\le t\le \frac{1}{\delta}} \sup_{|x(0)|=1} |x(t)| \to +\infty \quad \text{as} \quad \epsilon \to 0. \tag{2.60}$$

Therefore, using high gains to solve the problem (2.56) implies that the closed-loop linear system will have large deviations from the origin during the initial phase of stabilization. This phenomenon is called the “peaking effect" [2] and the large deviation is referred to as an “overshoot" (see, e.g., [92] and [89].).

Example 2.8
Let us consider the following double integrator system:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \quad (2.61)\\ \dot{x}_2(t) &= u(t), \quad (2.62) \end{aligned}$$

where $(x_1(t), x_2(t))^\top \in \mathbb{R}^2$ *is the state,* $u(t) \in \mathbb{R}$ *is the control input and the initial condition is given by* $(x_1(0), x_2(0)) = (1, 0)$*. By setting*

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.63}$$

one can rewrite (2.61)-(2.62) *as follows:*

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0. \tag{2.64}$$

We design a linear feedback law

$$u(t) = Kx(t) = k_1x_1(t) + k_2x_2(t), \tag{2.65}$$

[2] The peaking is a large deviation of the system state from the set-point during the initial phase of regulation/stabilization [90]

with $K = (k_1, k_2) \in \mathbb{R}^{1\times 2}$ such that the spectrum of the matrix $A + BK$ is equal to $\{-\delta\}$ for some $\delta > 0$. After some computations, one gets

$$k_1 = -\delta^2, \quad k_2 = -2\delta. \tag{2.66}$$

In this case, the exact solution of the closed-loop system (2.61)-(2.62) with the linear feedback law (2.65)-(2.66) is given by

$$x_1(t) = (\delta t + 1)e^{-\delta t}, \quad x_2(t) = -\delta^2 t e^{-\delta t}, \quad \forall t \geq 0. \tag{2.67}$$

The Euclidean norm of $x(t)$ is given by

$$|x(t)| = \sqrt{(\delta^4 + \delta^2)t^2 + 2\delta t + 1}e^{-\delta t}, \quad \forall t \geq 0, \tag{2.68}$$

and one has

$$\sup_{t\in[0,\frac{1}{\delta}]} |x(t)| \geq |x(\tfrac{1}{\delta})| = e^{-1}\sqrt{\delta^2 + 3}, \tag{2.69}$$

which leads to

$$\sup_{t\in[0,\frac{1}{\delta}]} |x(t)| \geq \gamma\delta^{2-1}, \tag{2.70}$$

with $\gamma = e^{-1}$ independent of δ. Moreover, the feedback control $u(t)$ is given explicitly by

$$u(t) = \delta^2(\delta t - 1)e^{-\delta t}, \quad t \geq 0, \tag{2.71}$$

and then $\sup_{t\in[0,\frac{1}{\delta}]} |u(t)| = \delta^2$. The norm of the state $|x(t)|$ and the amplitude of the control signal $u(t)$ during the initial stabilization phase grow like δ and δ^2, respectively. Consequently, a smaller value of ϵ implies a larger value of δ, which in turn results in a larger overshoot and peaking effect (see Figure 2.6).

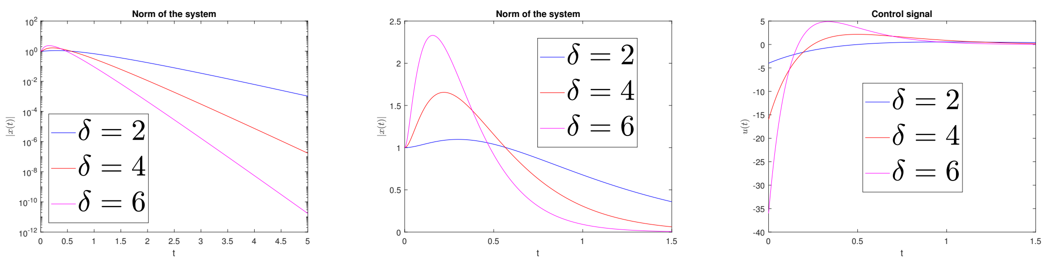

Figure 2.6: Left: Time evolution of $|x(t)|$ in logarithmic scale. Middle: Time evolution of $|x(t)|$ during the initial phase of stabilization. Right: Control signal $u(t)$ for different values of δ.

Example 2.9
Let us consider a triple chain of an integrator system

$$\dot{x}_1(t) = x_2(t), \tag{2.72}$$
$$\dot{x}_2(t) = x_3(t), \tag{2.73}$$
$$\dot{x}_3(t) = u(t), \tag{2.74}$$

with the initial condition $x_1(0) = 1, x_2(0) = 0, x_3(0) = 0$, which can be rewritten as follows

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{2.75}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{2.76}$$

The following linear controller:

$$u(t) = Kx(t), \tag{2.77}$$

with the control gain $K = (-\delta^3, -3\delta^2, -3\delta) \in \mathbb{R}^{1\times 3}$ and $\delta > 0$ is a globally stabilizer for the system (2.75) such that the spectrum of the matrix $A + BK$ is equal to $\{-\delta\}$. The exact solution of the closed-loop system (2.75)-(2.77) is given by

$$x_1(t) = (1 + \delta t + \tfrac{\delta^2 t^2}{2})e^{-\delta t}, \quad x_2(t) = -\tfrac{\delta^3 t^2}{2} e^{-\delta t}, \quad x_3(t) = \delta^3 t(-1 + \tfrac{\delta t}{2})e^{-\delta t}, \tag{2.78}$$

for all $t \geq 0$. One has the following estimate:

$$\sup_{t\in[0,\frac{1}{\delta}]} |x_3(t)| = \tfrac{1}{2e}\delta^2, \tag{2.79}$$

and then $\sup_{t\in[0,\frac{1}{\delta}]} |x(t)| \geq \gamma\delta^{3-1}$, with $\gamma = \frac{1}{2e}$ independent of δ. Using (2.77) and the expressions of the solutions $x_1(t), x_2(t)$ and $x_3(t)$ defined in (2.78), one gets $\sup_{t\in[0,\frac{1}{\delta}]} |u(t)| = \delta^3$. The norm of the state $|x(t)|$ and the amplitude of the control signal $u(t)$ during the initial stabilization phase grow like δ^2 and δ^3, respectively. As in the previous example, a smaller value of ϵ leads to a larger value of δ, which subsequently results in large overshoot and peaking effect (see Figure 2.7).

A natural question that may arise is whether the linear controller can be designed more effectively to minimize overshoot. This question is addressed in the next subsection which deals with minimization of peaking of linear control system by means of invariant ellipsoid method. However, another question arises: Can we do better to improve performance? Subsection 2.3.3 is devoted to give some insights to answer that question as it presents the design of the homogeneous controller, which aims to achieve a better reduction of overshoot.

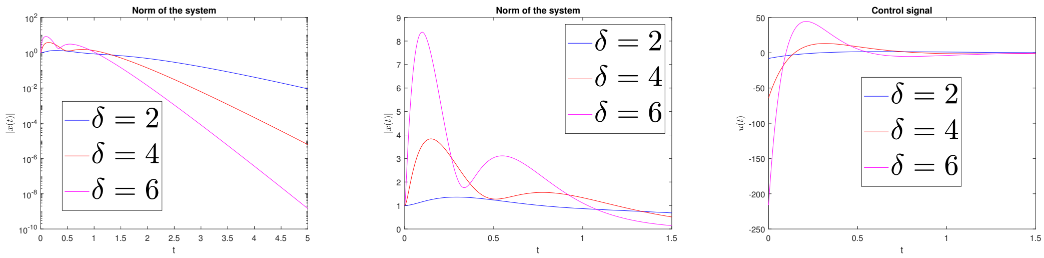

Figure 2.7: Left: Time evolution of $|x(t)|$ in logarithmic scale. Middle: Time evolution of $|x(t)|$ during the initial phase of stabilization. Right: Control signal $u(t)$ for different values of δ.

2.3.2 Minimization of peaking of linear control system by means of invariant ellipsoid method

Let us define a *magnitude of peaking* (also referred to as an overshoot) of the output control system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t > 0, \quad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times 1}, \tag{2.80}$$

$$z(t) = Hx(t), \quad H \in \mathbb{R}^{1\times n}, \tag{2.81}$$

by the following formula

$$\ell := \sup_{t\geq 0} |z(t)| - |z(0)| = \sup_{t\geq 0} |Hx(t)| - |Hx_0|, \tag{2.82}$$

where $x(t) \in \mathbb{R}^n$ is a solution of the system with a control $u(t) \in \mathbb{R}$ and with the initial condition

$$x(0) = x_0 \in \mathbb{R}^n. \tag{2.83}$$

The control aim is to stabilize the system and minimize ℓ for a given $x_0 \neq \mathbf{0}$. Inspired by [92], this problem can be solved in a suboptimal way using the invariant ellipsoid method. Let a feedback control

$$u = K_{\text{lin}}x, \tag{2.84}$$

stabilize the system (2.80). It is well known (see, e.g., [131], [132]) that the ellipsoid

$$\mathcal{E}(P,1) = \{x \in \mathbb{R}^n : x^\top Px \leq 1\}, \tag{2.85}$$

is an invariant set of the closed-loop system (2.80),(2.84) if and only if

$$P \succ 0, \quad P(A + BK_{\text{lin}}) + (A + BK_{\text{lin}})^\top P \preceq 0. \tag{2.86}$$

Moreover, $x_0 \in \mathcal{E}(P,1)$ if and only if

$$x_0^\top Px_0 \leq 1. \tag{2.87}$$

In this case, for any solution $x(t) \in \mathcal{E}(P,1)$ of the closed-loop system (2.80),(2.84) with $x(0) \in \mathcal{E}(P,1)$, it holds

$$z^2(t) = x^\top(t)H^\top Hx(t) = x^\top(t)P^{\frac{1}{2}}P^{-\frac{1}{2}}H^\top HP^{-\frac{1}{2}}P^{\frac{1}{2}}x(t) \leq$$

$$\lambda_{\max}\left(P^{-\frac{1}{2}}H^\top HP^{-\frac{1}{2}}\right)x^\top(t)Px(t) \leq \lambda_{\max}\left(P^{-\frac{1}{2}}H^\top HP^{-\frac{1}{2}}\right).$$

Therefore, the magnitude of peaking of the output can be reduced by minimization of $\lambda_{\max}\left(P^{-\frac{1}{2}}H^\top HP^{-\frac{1}{2}}\right)$ subject to constrains (2.86), (2.87). It can be proved that a solution of this optimization problem always exists if the system is stabilizable. However, the obtained closed-loop system may be just Lyapunov stable, but not asymptotically stable. In order to prescribe a convergence rate of the closed-loop system, let us modify the control aim as follows: *to reach a smaller ellipsoid $\mathcal{E}(P,\epsilon)$ with $\epsilon \in (0,1)$ in some fixed time $T > 0$:*

$$x(0) \in \mathcal{E}(P,1) \quad \Rightarrow \quad x(t) \in \mathcal{E}(P,\epsilon), \quad \forall t \geq T, \tag{2.88}$$

where P is a matrix of a minimal (in some sense) invariant ellipsoid . To fulfill the above constraint, the matrix inequality (2.86) has to be modified as follows

$$P \succ 0, \quad P(A+BK_{\text{lin}}) + (A+BK_{\text{lin}})^\top P - \frac{\ln\epsilon}{T}P \preceq 0. \tag{2.89}$$

Indeed, considering the quadratic Lyapunov function

$$V(x) = x^\top Px, \tag{2.90}$$

we derive the following estimate for the time derivative along the trajectories of the closed-loop system

$$\dot{V}(x) = x^\top\left(P(A+BK_{\text{lin}}) + (A+BK_{\text{lin}})^\top P\right)x \leq \frac{\ln\epsilon}{T}x^\top Px = \frac{\ln\epsilon}{T}V.$$

Hence, we conclude $V(t) \leq e^{\frac{t}{T}\ln\epsilon}V(0) \leq \epsilon V(0)$ for $t \geq T$, which implies (2.88). Therefore, the peaking of the system with a linear feedback having the prescribed convergence rate (2.88) can be optimized by solving the following problem

$$\lambda_{\max}\left(P^{-\frac{1}{2}}H^\top HP^{-\frac{1}{2}}\right) \to \min_{P,K_{\text{lin}}} \tag{2.91}$$

subject to (2.87), (2.89). This optimization problem is nonlinear, but it can be transformed to an equivalent *Semi-Definite Programming* (SDP) problem introducing the new scalar variable and the virtual constraint:

$$\lambda > 0, \quad \lambda I_n \succeq P^{-\frac{1}{2}}H^\top HP^{-\frac{1}{2}}. \tag{2.92}$$

The minimization of the parameter λ subject to (2.87), (2.89), (2.92) results in minimization of $\lambda_{\max}\left(P^{-\frac{1}{2}}H^\top HP^{-\frac{1}{2}}\right)$ subject to (2.87), (2.89). Making the change of variables

$$X = P^{-1}, \quad Y = K_{\text{lin}}P^{-1}, \tag{2.93}$$

and applying Schur complement, the constrains (2.87), (2.89), (2.92) can be transformed to the following system of LMIs:

$$X \succ 0, \quad XA + A^\top X + BY + Y^\top B^\top - \frac{\ln \epsilon}{T}X \preceq 0, \quad X \succeq x_0 x_0^\top, \quad \lambda \geq HXH^\top. \tag{2.94}$$

Therefore, the LMI-based solution of the considered stabilization problem can be formulated as follows

$$\lambda \to \min_{X,Y} \tag{2.95}$$

subject to LMI (2.94).

Lemma 2.1
If the pair $\{A, B\}$ is controllable then, for any $\epsilon \in (0,1)$, any $T > 0$ and $x_0 \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, the LMI (2.94) is feasible.

Proof. The feasibility of the LMI

$$X \succ 0, \quad XA + A^\top X + BY + Y^\top B^\top - \tfrac{\ln \epsilon}{T}X \preceq 0 \tag{2.96}$$

follows from controllability of the pair $\{A, B\}$ (see, e.g., [131], [132]). If the pair (X, Y) satisfies (2.96) then the pair $(\gamma X, \gamma Y)$ with $\gamma > 0$ also does. This means that a solution of the LMI (2.96) can always be selected such that the matrix inequality $X \succeq x_0 x_0^\top$ holds. Hence, the LMI (2.94) is feasible by taking a large enough λ. $\square$

Example 2.10
We perform numerical simulations with the following parameters:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 2.7324 & 4.5894 & 0 \\ -1.3662 & 0 & 0.1480 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -0.4502 \\ 0.2251 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

and the initial condition $x_0 = (0, 0.1, 0.1)^\top$. We compute the linear control [3] gain by solving the LMI (2.86), which resulted in a control gain K_{lin}, and solved the LMI (2.94) to obtain an optimal control gain K_{opt}:

$$K_{lin} = (-481.5839, -961.4387, -37.9778), \quad K_{opt} = (-88.5842, -167.8885, -5.2716).$$

[3]" Optimal " in the sense of control design based on minimization of the magnitude of the peaking.

Next, we compute the linear control $u_{lin}(x) = K_{lin}x$ and the optimal linear control $u_{opt}(x) = K_{opt}x$. Let

$$z_{lin}(t) = Hx_{lin}(t), \quad z_{opt}(t) = Hx_{opt}(t), \tag{2.97}$$

where x_{lin} is the solution of the closed-loop system with the linear feedback law $u_{lin}(x)$ and x_{opt} is the solution of the closed-loop system with the optimal linear feedback law $u_{opt}(x)$. As illustrated in Figure 2.8, we can see that

$$\sup_{t\geq 0} |z_{opt}(t)| \leq \sup_{t\geq 0} |z_{lin}(t)|,$$

while both the closed-loop system solution with the linear control $u_{lin}(x)$ and the optimal linear control $u_{opt}(x)$ converge at the same decay rate.

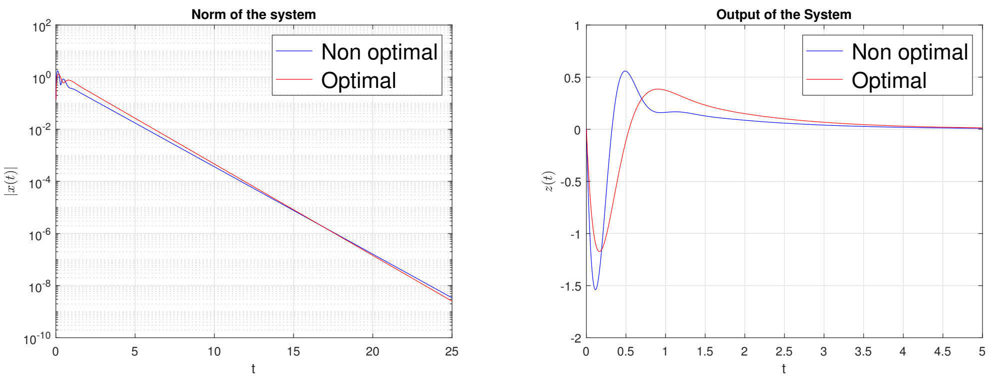

Figure 2.8: Left: Time evolution of $|x(t)|$ of the closed-loop system with $u_{lin}(x)$ in blue line, the closed-loop system with $u_{opt}(x)$ in red line . Right: The time evolution of the output $z_{lin}(t)$ in blue line, $z_{opt}(t)$ in red line.

Solving the optimization problem (2.95), (2.94) we obtain the linear feedback (2.84) with $K_{\text{lin}} = YX^{-1}$, which stabilizes the system with a prescribed convergence rate and minimized magnitude of peaking. In the proposed method, the minimal invariant ellipsoid $\mathcal{E}(X^{-1}, 1)$ characterizes the minimal magnitude of peaking where X is obtained by solving the above optimization problem. Below, we show that we can "upgrade" the obtained linear feedback to a homogeneous (nonlinear) control such that the minimal invariant ellipsoid is preserved, but the convergence rate is accelerated (e.g., the system is stabilized in a finite time to zero).

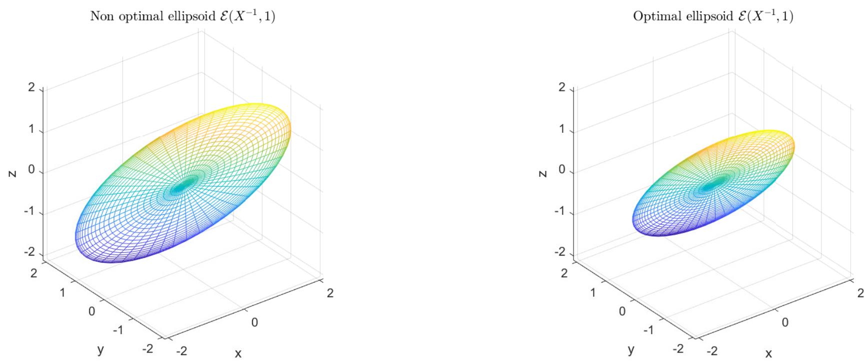

Figure 2.9: Left: Ellipsoid $\mathcal{E}(X^{-1},1)$ of the closed-loop system with a non-optimal linear control u_{lin}. Right: Ellipsoid $\mathcal{E}(X^{-1},1)$ of the closed-loop system with the optimal linear control u_{opt}.

2.3.3 Elimination of the peaking effect using homogeneous control

Upgrading a linear feedback to a homogeneous controller

An existing linear controller can be upgraded to a **d**-homogeneous controller, resulting in improved performance, such as faster convergence rates for the closed-loop system.

Let us assume that a linear control law is defined as follows:

$$u_{lin} = K_{lin}x(t), \quad K_{lin} \in \mathbb{R}^{1\times n}, \tag{2.98}$$

for the control system modeled by (2.46), where the control gain K_{lin} is selected such that the following Linear Matrix Inequality (LMI) is satisfied:

$$(A + BK_{lin})^\top P + P(A + BK_{lin}) + 2\rho P \le 0 \tag{2.99}$$

for some $P = P^\top > 0$ and $\rho > 0$. Let some (not yet selected) linear dilation $\mathbf{d}(s) = e^{sG_\mathbf{d}}$ in $\mathbb{R}^n$ be strictly monotone with respect to a weighted Euclidean norm $\|\cdot\|_P$ with some $0 \prec P = P^\top \in \mathbb{R}^{n\times n}$ satisfying $PG_\mathbf{d} + G_\mathbf{d}P \succ 0$. Let $\|\cdot\|_\mathbf{d}$ be the canonical homogeneous norm induced by $\|\cdot\|_P$. Let us define the homogeneous feedback upgraded from the linear controller (2.84) as follows

$$u_\mu(x) = K_0x + \|x\|_\mathbf{d}^{1+\mu}(K_{\text{lin}} - K_0)\mathbf{d}(-\ln\|x\|_\mathbf{d})x, \tag{2.100}$$

where $K_{\text{lin}} \in \mathbb{R}^{1\times n}$ is the linear control gain and $K_0 \in \mathbb{R}^{1\times n}$ is the homogeneous control gain. With such a definition, the homogeneous feedback coincides with the given linear controller on the unit sphere:

$$u_\mu(x) = K_{\text{lin}}x \quad \text{for} \quad x \in \mathbb{R}^n : \|x\|_P = 1.$$

According to [95], to complete the design of the homogeneous feedback (2.100), we just need to define a positive definite symmetric matrix P such that the unit ball $\mathcal{E}(P,1) = \{x \in \mathbb{R}^n : \|x\|_P \leq 1\}$ is the strictly positively invariant set of the linear dilation $\mathbf{d}$ monotone with respect to the $\|\cdot\|_P$. The following corollary proves that this is always possible if the pair $\{A, B\}$ is controllable and the requested homogeneity degree μ of the closed-loop system satisfies certain restrictions.

Corollary 2.1
Let the pair $\{A, B\}$ be controllable and the parameters $K_0 \in \mathbb{R}^{1\times n}$, $A_0 \in \mathbb{R}^{n\times n}$, $G_0 \in \mathbb{R}^{n\times n}$ and $\tilde{n} \in \mathbb{N}$ be defined as in Theorem 2.4. Let $W = (A + BK_{lin})^\top P + P(A + BK_{lin}) + 2\rho P$ and $Q = PG_0 + G_0^\top P$ and $\delta \in (0, 1)$. For any $\mu \in [\mu_{\min}, \mu_{\max}]$ satisfying

$$\frac{1}{\mu_{\min}} = \min\left\{-1, \frac{\max\left\{0, \lambda_{\max}\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right)\right\}}{-2\delta}, \frac{\rho\max\{0, -\lambda_{\min}(Q)\}}{\lambda_{\max}(W)}\right\} < 0, \tag{2.101}$$

$$\frac{1}{\mu_{\max}} = \max\left\{\tilde{n}, \frac{\max\left\{0, -\lambda_{\min}\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right)\right\}}{2\delta}, \frac{\rho\max\{0, \lambda_{\max}(Q)\}}{-\lambda_{\max}(W)}\right\} > 0, \tag{2.102}$$

1) the matrix $G_{\mathbf{d}} = I_n + \mu G_0$ is anti-Hurwitz and $PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0$;

2) the closed-loop system (2.80), (2.100) is globally uniformly asymptotically stable and

$$\frac{d}{dt}\|x\|_{\mathbf{d}} \leq -\rho\|x\|_{\mathbf{d}}^{1+\mu}, \quad \forall x \neq \mathbf{0}; \tag{2.103}$$

3) the ellipsoid $\mathcal{E}(P,1)$ is invariant for the closed-loop system (2.80), (2.100);

4) $\sup_{x\in\mathcal{E}(P,1)} |u_\mu(x) - K_0x|^2 = \sup_{x\in\mathcal{E}(P,1)} |K_{lin}x - K_0x|^2 = (K_{\text{lin}} - K_0)P^{-1}(K_{\text{lin}} - K_0)^\top$;

5) $\sup_{x\in\mathcal{E}(P,1)} |K_{lin}x|^2 = \sup_{x\in\mathcal{E}(P,1)} |u_\mu(x)|^2 = K_{lin}P^{-1}K_{lin}^\top$, at least, for μ being close enough to zero.

Proof. See the Appendix B.0.1. □

The above corollary also proves that any linear controller can be transformed to a homogeneous stabilizer such that

- the invariant ellipsoid $\mathcal{E}(P,1)$, which, in particular, characterizes the overshoot (see Section (2.3.2)), is preserved under this transformation;
- for $\mu < 0$, the obtained homogeneous controller stabilizes the system in a finite time avoiding the peaking discovered for linear feedback laws as $\epsilon \to 0$.

This corollary, as well as the examples given below, motivates the design of homogeneous controllers in the case when the precision or the response time of a linear control system is not sufficient, but further increase of feedback gain leads to an inadmissible overshoot.

Example 1 of upgrading Linear feedback to homogeneous feedback

We perform numerical simulations on the system described by (2.80) with the following parameters:

$$A = \begin{pmatrix} 0 & 0 \\ -8.8858 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -0.4502 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

and the initial condition $x_0 = (0, 1)^\top$. We compute the linear control gain by solving the LMI (2.86), which resulted in a control gain K_{lin}, and solved the LMI (2.94) to obtain an optimal control gain K_{opt}:

$$K_{lin} = (-1.1283, 6.8050), \quad K_{opt} = (-7.3607, 1.7252).$$

Next, we compute the linear feedback law $u_{lin}(x) = K_{lin}x$ and the optimal linear control $u_{opt}(x) = K_{opt}x$. Subsequently, we homogenize the two linear controllers to derive homogeneous controls:

$$u_h(x) = \|x\|_{\mathbf{d}}^{1+\mu} K_{lin}\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x, \quad u_{hopt}(x) = \|x\|_{\mathbf{d}}^{1+\mu} K_{opt}\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x,$$

where $K_0 = (0, 0)$. As shown in Figure 2.10, we can observe that

$$\sup_{t\geq 0} |u_{opt}(x)| \leq \sup_{t\geq 0} |u_{lin}(x)|,$$

while the solution of the closed-loop system with the optimal linear control $u_{opt}(x)$ converges to zero faster than the solution of the closed-loop system with the linear control $u_{lin}(x)$. A similar conclusion applies to the case of homogeneous control, where we can see that

$$\sup_{t\geq 0} |u_{hopt}(x)| \leq \sup_{t\geq 0} |u_h(x)|,$$

and the settling time of the closed-loop system with the homogeneous control $u_{hopt}(x)$ is shorter than that of the closed-loop system with the homogeneous control $u_h(x)$. In Figure 2.11, we observe that the overshoot (as defined by formula (2.82)) of the closed-loop system with linear optimal control u_{opt} is smaller than that of the closed-loop system with linear control u_{lin}. This is due to the optimal design of the linear control using the aforementioned optimization problem. Also, we can remark that upgrading the linear controls (both optimal and non-optimal) to homogeneous controls results in faster convergence (finite-time in this case) with a reduced overshoot.

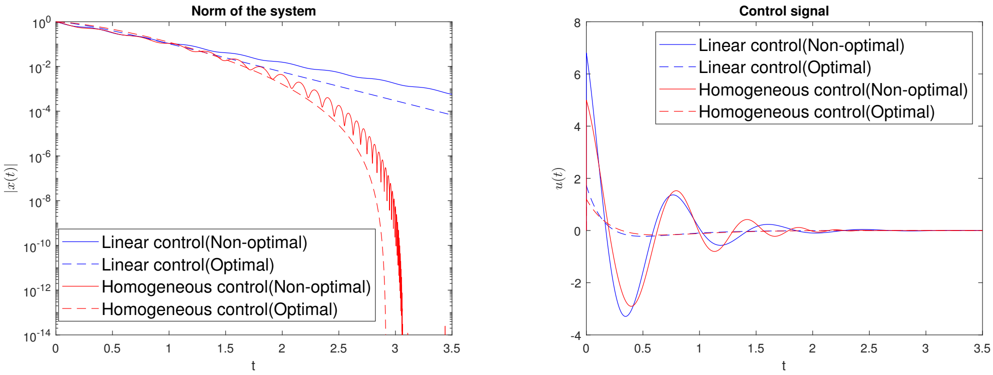

Figure 2.10: Left: Time evolution of $|x(t)|$ of the closed-loop system with $u_{lin}(x)$ in blue line, $u_{opt}(x)$ in blue dashed line and with homogeneous control $u_h(x)$ in red line, $u_{hopt}(x)$ in red dashed line. Right: Time evolution of the control signal $u_{lin}(x)$ in blue line, $u_{opt}(x)$ in blue dashed line, and homogeneous control $u_h(x)$ in red line and $u_{hopt}(x)$ red dashed line.

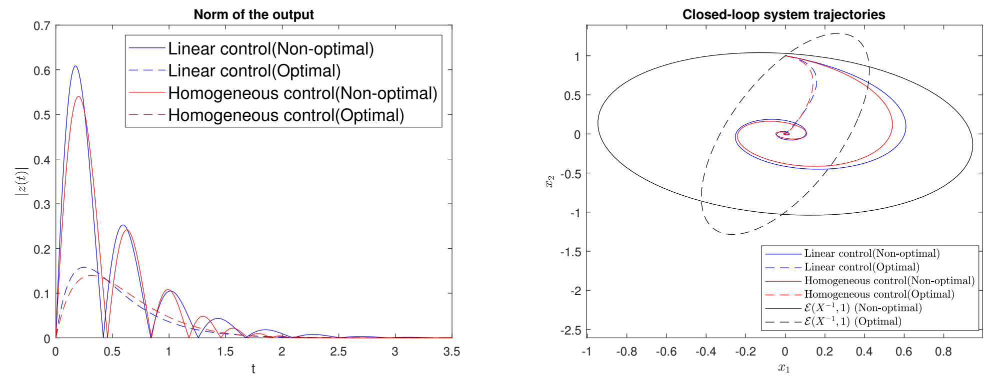

Figure 2.11: Left: Time evolution of the output $|z(t)|$ of the closed-loop system with $u_{lin}(x)$ in blue line, $u_{opt}(x)$ in blue dashed line and with homogeneous control $u_h(x)$ in red line, $u_{hopt}(x)$ in red dashed line. Right: the trajectories of the solutions of the closed-loop system with $u_{lin}(x)$ in blue line, $u_{opt}(x)$ in blue dashed line, and homogeneous control $u_h(x)$ in red line and $u_{hopt}(x)$ red dashed line.

Example 2 of transformation of linear to homogeneous controller

This example resembles the structure of the closed-loop systems addressed in the thesis. The system considered has two components: a subsystem that can be stabilized using linear control or homogeneous control, and another component that is already stable. The objective is to investigate how the control design for the stabilizable subsystem can influence the overall system in terms of peak and overshoot effects, as outlined in the previous examples.

Let us consider the following cascade system

$$\begin{aligned} \dot{x}_1(t) &= u(t), && (2.104)\\ \dot{x}_2(t) &= q_1 x_1(t) + b_1 u(t), && (2.105)\\ \dot{x}_3(t) &= -2\delta x_3(t) + q_2 x_1(t) + b_2 u(t), && (2.106)\\ x_1(0) &= 0, x_2(0) = 1, x_3(0) = 0, && (2.107)\end{aligned}$$

where $b_1, b_2, q_1, q_2 \in \mathbb{R}$, which can be rewritten as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), && (2.108)\\ \dot{x}_3(t) &= -2\delta x_3(t) + q_2 \mathbf{1} x(t) + b_2 u(t), && (2.109)\end{aligned}$$

with $x(t) = (x_1(t), x_2(t))^\top$ and the system parameters $A, B, \mathbf{1}$ are given by

$$A = \begin{pmatrix} 0 & 0 \\ q_1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ b_1 \end{pmatrix}, \quad \mathbf{1} = (1, 0), \tag{2.110}$$

and δ some positive parameter. The following linear feedback law

$$u(t) = kx(t), \quad k = (k_1, k_2), \quad k_1 = -2\delta + \tfrac{b_1 \delta^2}{q_1}, \quad k_2 = \tfrac{-\delta^2}{q_1}, \tag{2.111}$$

is a globally stabilizer for the system such that the spectrum of $A + Bk$ is $\{-\delta\}$. The exact solution of the closed-loop system is given by

$$x_1(t) = \tfrac{-\delta^2}{q_1} t e^{-\delta t}, \quad x_2(t) = (\bar{\delta} k_2 t + 1) e^{-\delta t}, \tag{2.112}$$

with $\bar{\delta} = -\frac{1}{k_2}(\delta + k_1) = b_1 - \frac{q_1}{\delta}$. Notice that

$$\sup_{t>0} |x_1(t)| = \sup_{t \in [0, \frac{1}{\delta}]} |x_1(t)| = \tfrac{\delta}{e q_1}. \tag{2.113}$$

From (2.109) and using the integration by parts formula, one derives

$$x_3(t) = (\bar{\alpha}_1 t + \bar{\alpha}_2) e^{-\delta t} + \bar{\alpha}_3 e^{-2\delta t}, \tag{2.114}$$

with

$$\alpha_1 = \tfrac{-\delta^2(b_2 k_1 + q_2)}{q_1} + b_2 \bar{\delta} k_2^2, \; \alpha_2 = b_2 k_2, \; \bar{\alpha}_1 = \tfrac{\alpha_1}{\delta}, \; \bar{\alpha}_2 = \tfrac{\alpha_2 \delta + \alpha_1}{\delta^2}, \; \bar{\alpha}_3 = -\tfrac{\alpha_1 + \delta \alpha_2}{\delta^2}. \tag{2.115}$$

From the expressions of $x_1(t), x_2(t)$ and $x_3(t)$ in (2.112) and (2.114) one gets

$$x_1^2(t) + x_2^2(t) + x_3^2(t) = (\beta_1 t^2 + \beta_2 t + \beta_3) e^{-2\delta t} + (\beta_4 t + \beta_5) e^{-3\delta t} + \beta_6 e^{-4\delta t}, \tag{2.116}$$

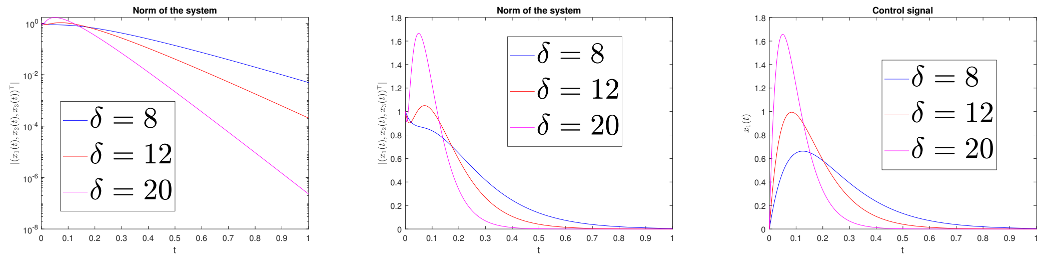

Figure 2.12: Time evolution of $|(x_1(t), x_2(t), x_3(t))^\top|$ in logarithmic scale and during the initial phase of stabilization and the state $x_1(t)$ for different values of δ.

with

$$\beta_1 = \tfrac{\delta^4}{q_1^2} + \bar{\delta}^2 k_2^2 + \bar{\alpha}_1^2, \quad \beta_2 = 2\bar{\delta}k_2 + 2\bar{\alpha}_1\bar{\alpha}_2, \quad \beta_3 = 1 + \bar{\alpha}_2^2, \tag{2.117}$$

$$\beta_4 = 2\bar{\alpha}_1\bar{\alpha}_3, \quad \beta_5 = 2\bar{\alpha}_2\bar{\alpha}_3, \quad \beta_6 = \bar{\alpha}_3^2; \tag{2.118}$$

By setting $r(t) = \sqrt{x_1^2(t) + x_2^2(t) + x_3^2(t)}$, for any $T > 0$, one has

$$r(T) \geq C(T)e^{-\delta T} \tag{2.119}$$

with

$$C(T) = 3\min\{\beta_1 T^2 + \beta_2 T + \beta_3, \beta_4 T + \beta_5, \beta_6\}. \tag{2.120}$$

Considering $\epsilon = \frac{C(T)e^{-\delta T}}{2}$ we conclude that the linear controller can never stabilize the system to the ϵ-neighborhood of the origin at time $T > 0$, i.e., $r(T) > \epsilon$ for any $T > 0$ provided that $\epsilon = \frac{C(T)e^{-\delta T}}{2}$. Let us show that upgrading the linear feedback to a homogeneous controller eliminates this drawback, at least, for large enough $T > 0$. Following the Corollary 2.1, we upgrade this linear feedback to the following homogeneous feedback law:

$$\xi(x(t)) = k\|x(t)\|_{\mathbf{d}}^{1+\mu}\mathbf{d}(-\ln\|x(t)\|_{\mathbf{d}})x(t) \tag{2.121}$$

where $\mu \in [-1, 0)$ is the homogeneity degree, $\mathbf{d}$ the dilation generated by an anti-Hurwitz matrix $G_{\mathbf{d}}$. Notice that such an upgrade results in a homogeneous control having the same estimate of an overshoot as the linear feedback. Along the solutions of the closed-loop system, one has

$$\tfrac{d}{dt}\|x(t)\|_{\mathbf{d}} \leq -\rho\|x(t)\|_{\mathbf{d}}^{1+\mu}, \tag{2.122}$$

for some $\rho > 0$ which implies that

$$x(t) = 0, \quad \forall t \geq T_0, \tag{2.123}$$

where $T_0 = \frac{\|x_0\|_{\mathbf{d}}^{-\mu}}{-\mu\rho}$ is the settling time. Then one has

$$|x_3(t)| = e^{-2\delta(t-T_0)}|x_3(T_0)|, \quad \forall t \geq T_0, \tag{2.124}$$

and

$$|x_3(T_0)| \leq Q(T_0) := \tfrac{1}{\delta}(b_2|k^\top|\lambda_{min}(X^{-1})^{\frac{-1}{2}}\|x_0\|_{\mathbf{d}}^{1+\mu} + q_2\sigma(\|x_0\|_{\mathbf{d}})), \tag{2.125}$$

with some $\sigma \in \mathcal{K}_\infty$.

Notice that for a large enough $T > 0$ we have $e^{\delta T}C(T) \geq 2Q(T_0)e^{2\delta T_0}$. Hence, we derive $Q(T_0)e^{-2\delta(T-T_0)} \leq 0.5C(T)e^{-\delta T} = \epsilon$ and

$$|(x_1(t), x_2(t), x_3(t))^\top| \leq \epsilon, \quad \forall t \geq T. \tag{2.126}$$

Therefore, we conclude that linear controller can never guarantee the stabilization of the state to the ϵ-neighborhood of the origin at time $T > 0$, while the homogeneous feedback ensures the prescribed-time practical stabilization, at least, in the case when the prescribed setting time T is large enough.

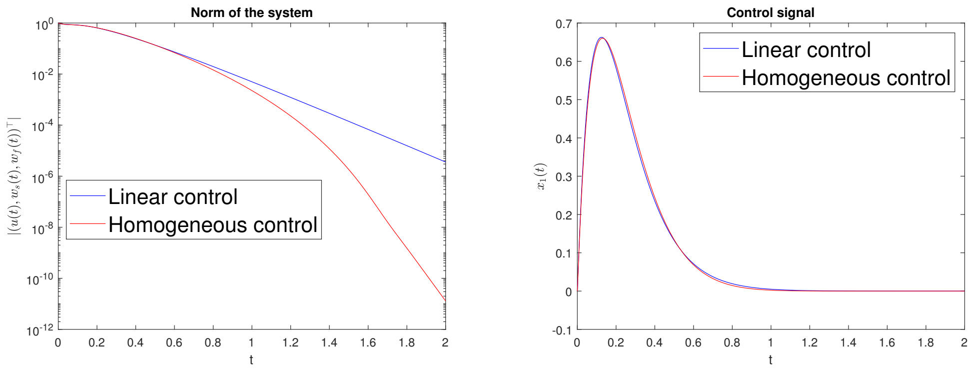

Figure 2.13: Left: Time evolution of the norm $|(x_1(t), x_2(t), x_3(t))^\top|$ of the closed-loop system with linear control in blue line and closed-loop system with homogeneous control signal in red line in logarithmic scale Right: The state $x_1(t)$ of the closed-loop system with linear control in the blue line and with the homogeneous control signal in the red line .

Part I

Homogeneous control design for linear parabolic partial differential equations (PDEs)

CHAPTER 3

Homogeneous boundary control for a 1D reaction-diffusion PDE: modal decomposition approach

Contents

This chapter addresses the problem of boundary stabilization for 1D linear reaction-diffusion PDE systems using the modal decomposition approach. We design nonlinear *homogeneous* full-state and output-feedback controllers to stabilize the unstable dynamics. We consider a linear infinite-dimensional observer for the design of the homogeneous output-feedback control. We investigate the well-posedness of the resulting closed-loop system and conduct a stability analysis by constructing a suitable Lyapunov function that accounts for the degree of homogeneity and a state-dependent switching condition based on the homogeneous norm. The properties of homogeneous-based controllers are exploited, and numerical simulations corroborate that the proposed controllers effectively mitigate peaking effects and control signal overshoot compared to classical linear boundary feedbacks designed via modal decomposition. Additional simulations are provided further to illustrate the performance of the proposed control laws.

3.1 Problem statement

Motivated by the discussion in Section 2.3.3, and the advantages that homogeneous controller may offer compared to static linear controller in terms of reducing peaking effect (and in some cases better performance such as faster convergence), we aim at studying if the same conclusion can be made for some infinite-dimensional system under a homogeneous-based control. In this chapter, we focus on a 1D reaction-diffusion PDE

$$\begin{aligned} \partial_t z(t,x) &= \partial_{xx} z(t,x) + qz(t,x), && (3.1)\\ a_{11}z(t,0) &+ a_{12}z_x(t,0) = 0, && (3.2)\\ z(t,1) &= U(t), && (3.3)\\ z(0,x) &= z_0(x), && (3.4) \end{aligned}$$

$(t,x) \in \mathbb{R}_+ \times [0,1]$, where $z(t,\cdot)$ is the reaction-diffusion PDE state at time t, $z_0 \in H^1(0,1)$ the initial condition, $q > 0$ is the reaction coefficient, $U(t) \in \mathbb{R}$ is the control input and the reals numbers a_{11}, a_{12} are such that

- $a_{11} = 1$ and $a_{12} = 0$: Dirichlet boundary conditions ,
- $a_{11} = 0$ and $a_{12} = 1$: Mixed Neumann-Dirichlet boundary conditions .

Assume that one has the following measurement: $y(t) = \int_0^1 c(x)z(t,x)dx$ where $c \in L^2(0,1)$ for non-local measurement or $y(t) = z(t,x^*)$ for the point measurement with $x^* \in (0,1)$.

As we have mentioned, we aim at developing homogeneous finite-time feedback laws for the system (3.1)-(3.4) using a modal decomposition approach. Moreover, we aim to investigate numerically if the use of homogeneous feedback results in a smaller peaking than linear feedback. Consider the following control problem: *given an initial state $z_0 \in H^1(0,1)$, a stabilization precision $\epsilon > 0$ and a prescribed time $T > 0$, the closed-loop system has the desired stabilization precision:*

$$\|z(t,\cdot)\|_{H^1(0,1)} \le \epsilon, \quad \forall t > T, \tag{3.5}$$

with the restricted control magnitude

$$\sup_{t>0} |U(t)| \le \bar{U}, \tag{3.6}$$

for some maximal control amplitude $\bar{U} > 0$. Alternatively, the control problem can read as follows: *given an initial state* $z_0 \in H^1(0,1)$*, a prescribed time* $T > 0$*, and a control amplitude* $\bar{U}$*, the solutions of the closed-loop system meet conditions* (3.5) *and* (3.6) *for some* $\epsilon > 0$. Two situations related to this problem stand out: i) Both a homogeneous controller and a *high-gain linear* controller may ensure that the solution of the closed-loop system satisfies the criterion specified in (3.5). However, the high-gain linear controller may not fulfill the control constraint outlined in (3.6), whereas the homogeneous controller satisfies both (3.5) and (3.6). ii) Both the linear and homogeneous controllers exhibit similar overshoot values, thereby satisfying (3.6). This situation can be achieved by what we refer to as an “upgrade" of the linear control (whose synthesis will be detailed in Subsection 3.2) into a homogeneous one, and that we develop in Subsection 3.3.1. In this case, however, the linear controller may violate condition (3.5), whereas the homogeneous controller does not. Moreover, the expected stabilization precision ϵ turns out to be smaller under a homogeneous controller. Hence, the advantage of the homogeneous controller lies in its ability to achieve faster convergence. These considerations can be assessed numerically. In the rest of the chapter, we focus on the second situation.

3.2 Modal decomposition and Linear finite-dimensional control design

We start by considering the following change of variable (see [52, Definition 3.3.2]):

$$w(t,x) = z(t,x) - r(x)U(t), \tag{3.7}$$

with $r(x) = x$ for Dirichlet boundary conditions and $r(x) = x^2$ for Mixed Neumann-Dirichlet boundary conditions (where $U(0) = 0$) to obtain the following equivalent PDE:

$$\begin{aligned}
\partial_t w(t,x) &= \partial_{xx} w(t,x) + qw(t,x) - r(x)\dot{U}(t) + (qr(x) + r''(x))U(t), && (3.8)\\
a_{11}w(t,0) &+ a_{12}w_x(t,0) = w(t,1) = 0, && (3.9)\\
w(0,x) &= z_0(x), && (3.10)
\end{aligned}$$

provided that $w(0,x) = z_0(x) - r(x)U(0) = z_0(x)$.

By introducing a new control input $\xi(t) = \dot{U}(t)$, one obtains a coupled ODE-PDE system:

$$\begin{aligned}
\dot{U}(t) &= \xi(t), && (3.11)\\
\partial_t w(t,x) &= \partial_{xx} w(t,x) + qw(t,x) - r(x)\xi(t) + (qr(x) + r''(x))U(t), && (3.12)\\
a_{11}w(t,0) &+ a_{12}w_x(t,0) = w(t,1) = 0, && (3.13)\\
w(0,x) &= z_0(x), && (3.14)
\end{aligned}$$

and

$$\tilde{y}(t) = \int_0^1 c(x)w(t,x)dx \quad \text{or} \quad \tilde{y}(t) = w(t,x^*), \tag{3.15}$$

with $\tilde{y}(t) = y(t) - \left(\int_0^1 r(x)c(x)dx\right)U(t)$ in the case of non-local measurement and $\tilde{y}(t) = y(t) - r(x^*)U(t)$ in the case of point measurement.

The solution to (3.12) can be represented as follows:

$$w(t,x) = \sum_{n=1}^{\infty} w_n(t)\phi_n(x), \quad w_n(t) = \langle w(t,\cdot), \phi_n \rangle, \tag{3.16}$$

with $\phi_n(x) = \sqrt{2}\sin(n\pi x)$ for Dirichlet boundary conditions and $\phi_n(x) = \sqrt{2}\cos((n-\frac{1}{2})\pi x)$ for Mixed Neumann-Dirichlet boundary conditions.
Projecting onto the basis $\{\phi_n\}_{n=1}^{\infty}$, one has

$$\begin{aligned}
\dot{U}(t) &= \xi(t), &(3.17)\\
\dot{w}_n(t) &= (\lambda_n + q)w_n(t) + b_n\xi(t) + q_nU(t), &(3.18)\\
\tilde{y}(t) &= \sum_{n\geq 1} c_n w_n(t), &(3.19)\\
w_n(0) &= \langle z_0, \phi_n \rangle, &(3.20)
\end{aligned}$$

where $n \geq 1$, $-\lambda_n$ is the eigenvalue of Sturm-Liouville eigenvalue problem [1] corresponding to the eigenvector ϕ_n and

$$b_n = -\langle r, \phi_n \rangle, \quad c_n = \langle c, \phi_n \rangle \quad \text{or} \quad c_n = \phi_n(x^*), \quad q_n = -qb_n + \langle r'', \phi_n \rangle. \tag{3.21}$$

Note that $\lambda_n = -n^2\pi^2$ in the case of Dirichlet boundary conditions and $\lambda_n = -(n-\frac{1}{2})^2\pi^2$ in the case of Mixed Neumann-Dirichlet boundary conditions. Since $\lambda_n \to -\infty$ when $n \to +\infty$, then there exists an integer $N \geq 1$ such that

$$\lambda_n + q < -\delta, \qquad \forall n \geq N+1, \tag{3.22}$$

with δ some positive constant. We obtain from (3.17)-(3.20) the following augmented system:

$$\begin{aligned}
\dot{W}_u(t) &= \tilde{A}W_u(t) + \tilde{B}\xi(t), &(3.23)\\
\dot{w}_n(t) &= (\lambda_n + q)w_n(t) + b_n\xi(t) + q_nU(t), \quad n > N, &(3.24)\\
\tilde{y}(t) &= CW(t) + \sum_{n\geq N+1} c_n w_n(t), &(3.25)\\
W_u(0) &= z_0^N, \quad w_n(0) = \langle z_0, \phi_n \rangle, \quad n \geq N+1, &(3.26)
\end{aligned}$$

[1] Recall that :

$$\phi''(x) + \lambda\phi(x) = 0, \quad a_{11}\phi(0) + a_{12}\phi'(0) = \phi(1) = 0,$$

with

$$\tilde{A} = \begin{pmatrix} 0 & 0_{1\times N} \\ Q & A \end{pmatrix}, \quad W_u(t) = (U(t), w_1(t), \cdots, w_N(t))^\top, \quad W(t) = (w_1(t), \cdots, w_N(t))^\top,$$
$$z_0^N = (0, \langle z_0, \phi_1\rangle, \cdots, \langle z_0, \phi_N\rangle)^\top, \quad \tilde{B} = (1, b_1, \cdots, b_N)^\top, \quad C = (c_1, \cdots, c_N),,$$
$$A = \mathrm{diag}\{\lambda_n + q\}_{n=1}^N, \quad Q = (q_1, \cdots, q_N)^\top,$$

where b_n, q_n and c_n are defined in (3.21).

Linear finite-dimensional control design

In this part, we present the design of a linear control that exponentially stabilizes the system (3.1)-(3.3).

Since $\{\tilde{A}, \tilde{B}\}$ is controllable (see [61] and [70]), the following LMIs hold[2]:

$$(1 - \tfrac{1}{2\alpha})\lambda_{N+1} + q + \delta \leq 0, \tag{3.27}$$
$$\tilde{A}X_c + X_c\tilde{A}^\top + \tilde{B}Y_c + Y_c^\top\tilde{B}^\top + 2\delta X_c + \alpha\|(b_n, q_n)\|_N \left(X_c\mathbf{1}_{N+1}^\top\mathbf{1}_{N+1}X_c + Y_c^\top Y_c\right) \prec 0 \tag{3.28}$$

where $\|(b_n, q_n)\|_N = \sum_{n\geq N+1}(|b_n| + |q_n|)^2, \mathbf{1}_{N+1} = (1, 0, \cdots, 0)$, and the solutions are $\alpha > \frac{1}{2}, X_c, Y_c$. It can be shown that the unique classical solution z of the closed-loop system (3.1)-(3.3) with the linear feedback law

$$U(t) = K_{lin}\int_0^t W_u(s)ds, \quad K_{lin} = Y_cX_c^{-1}, \tag{3.29}$$

satisfies

$$\|z(t,\cdot)\|^2_{H^1(0,1)} \leq Me^{-2\delta t}\|z_0\|^2_{H^1(0,1)}, \quad \forall t \geq 0, \tag{3.30}$$

with $M = 4\frac{\max\{1, \|r\|^2_{H^1(0,1)}\}|\lambda_N|\max\{\lambda_{max}(X_c^{-1}),1\}}{\min\{\lambda_{min}(X_c^{-1}),1\}}$. This can be established using the Lyapunov function ([61], [71])

$$V(t) = W_u(t)^\top X_c^{-1}W_u(t) + \sum_{n>N}|\lambda_n|w_n^2(t), \tag{3.31}$$

for all $t \geq 0$. Notice that if $\delta > \frac{1}{2T}\ln\left(\frac{M\|z_0\|^2_{H^1(0,1)}}{\epsilon^2}\right)$, then (3.5) is satisfied. This implies that for a fixed T, as $\epsilon \to 0$, it follows that $\delta \to +\infty$. Consequently, control gain synthesis through (3.28) yields a high-gain linear control K_{lin}, which ultimately results in significant overshoot when the linear control approach is applied.

As indicated in Section 2.3.3, instead of using a high-gain linear control to achieve faster convergence—at the expense of considerable overshoot of the control signal and

[2] similar to LMI (2.23) in [71]

peaking of the closed-loop solution—the designed linear control (3.29) can be upgraded to a homogeneous controller. This adjustment may lead to a faster convergence to zero with less overshoot of the control signal and without peaking of the closed-loop solution.

3.3 Homogeneous stabilization of 1-D heat system by full-state feedback

In this section, we consider the problem of full-state feedback stabilization of (3.23)-(3.24) using homogeneous control derived from upgrading the linear controller (3.29).

3.3.1 Upgrading a linear feedback to a homogeneous feedback

Since $\{\tilde{A}, \tilde{B}\}$ is controllable then according to Section 2.2.3, the linear algebraic equation

$$\tilde{A}G_0^c - G_0^c\tilde{A} + \tilde{B}Y_0^c = \tilde{A}, \quad G_0^c\tilde{B} = 0, \tag{3.32}$$

has solution Y_0^c, G_0^c such that the matrix $G_0^c - I_{N+1}$ is invertible.

Let (X_c, Y_c) be a solution of (3.28) and $P_c = X_c^{-1}$, $F_1 = (\tilde{A} + \tilde{B}K_{lin})^\top P_c + P_c(\tilde{A} + \tilde{B}K_{lin}) + 2\delta P_c$ and $F_2 = P_c G_0^c + {G_0^c}^\top P_c$. Assume that

$$F_2 - 2P_c \prec 0, \tag{3.33}$$

and let $\mu_{min} \in [-1, 0)$ be defined as follows:

$$\frac{1}{\mu_{\min}} = \min\left\{ -1, -\max\left\{0, \lambda_{\max}\left(2I_{N+1} + P_c^{-\frac{1}{2}} F_2 P_c^{-\frac{1}{2}}\right)\right\}, \right.$$
$$\left. -\lambda_{\max}\left(2I_{N+1} - P_c^{-\frac{1}{2}} F_2 P_c^{-\frac{1}{2}}\right), \frac{\delta \max\{0, -\lambda_{min}(F_2)\}}{\lambda_{max}(F_1)} \right\}. \tag{3.34}$$

According to Corollary 2.1, for any $\mu_c \in [\mu_{\min}, 0)$, the following statements hold:

1) the matrix $G_{\mathbf{d}_c} = I_{N+1} + \mu_c G_0^c$ is anti-Hurwitz ,the matrix $\tilde{A}_0^c = \tilde{A} + \tilde{B}K_0$ with $K_0 = Y_0^c(G_0^c - I_{N+1})^{-1}$ satisfies the identity

$$\tilde{A}_0^c G_{\mathbf{d}_c} = (G_{\mathbf{d}_c} + \mu_c I_{N+1})\tilde{A}_0^c, \quad G_{\mathbf{d}_c}\tilde{B} = \tilde{B}, \tag{3.35}$$

and the following LMIs hold:

$$
\begin{gathered}
X_c(\tilde{A}_0^c + \tilde{B}K)^\top + (\tilde{A}_0^c + \tilde{B}K)X_c + \delta(G_{\mathbf{d}_c}X_c + X_cG_{\mathbf{d}_c}^\top) \preceq 0, \\
2\max\{1+\mu_c, -\mu_c\}X_c \prec G_{\mathbf{d}_c}X_c + X_cG_{\mathbf{d}_c}^\top \prec 2(2+\mu_c)X_c, \\
X_c = X_c^\top \succ 0,
\end{gathered} \tag{3.36}
$$

where $K = K_{lin} - K_0$;

2) the system (3.23) with the continuous feedback law

$$
\xi(W_u(t)) = K_0 W_u(t) + \mathcal{N}(W_u(t)), \tag{3.37}
$$

$$
\mathcal{N}(W_u(t)) = \|W_u(t)\|_{\mathbf{d}_c}^{1+\mu_c} K\mathbf{d}_c(-\ln(\|W_u(t)\|_{\mathbf{d}_c}))W_u(t), \tag{3.38}
$$

where $\|\cdot\|_{\mathbf{d}_c}$ is the canonical homogeneous norm induced by the weighted norm $\|\cdot\|_{X_c^{-1}}$ satisfies

$$
W_u(t) = 0_{(N+1)\times 1}, \quad \forall t \geq T^c(z_0^N) = \frac{\|z_0^N\|_{\mathbf{d}_c}^{-\mu_c}}{-\delta\mu_c}. \tag{3.39}
$$

Remark 3.1
The finite-time convergence of the solution to the closed-loop system (3.23)-(3.37), as stated in (3.39), is due to:

$$
\frac{d\|W_u(t)\|_{\boldsymbol{d}_c}}{dt} \leq -\delta\|W_u(t)\|_{\boldsymbol{d}_c}^{1+\mu_c}, \tag{3.40}
$$

for all $t \geq 0$.

Notice that for μ_c close enough to zero, the following estimate holds:

$$
\sup_{\|W_u\|_{X_c^{-1}}\leq 1} |\xi(W_u)|^2 = \sup_{\|W_u\|_{X_c^{-1}}\leq 1} |K_{lin}W_u|^2 = Y_cX_c^{-1}Y_c^\top. \tag{3.41}
$$

where (X_c, Y_c) are solution of (3.28). This indirectly indicates that the linear controller and the homogeneous state-feedback controller exhibit the same overshoot. In contrast, the homogeneous controller stabilizes the finite-dimensional controlled subsystem (3.23) in a finite time (see (3.39)).

3.3.2 Well-posedness Analysis

The closed-loop system (3.11)-(3.14) with homogeneous feedback law (3.37) is then given by:

$$\dot{U}(t) = K_0 W_u(t) + \mathcal{N}(W_u(t)), \tag{3.42}$$
$$\partial_t w(t,x) = \partial_{xx} w(t,x) + q w(t,x) + (q r(x) + r''(x))U(t) - r(x)(K_0 W_u(t) + \mathcal{N}(W_u(t))),$$
$$a_{11} w(t,0) + a_{12} w_x(t,0) = w(t,1) = 0, \tag{3.43}$$
$$w(0,x) = z_0(x). \tag{3.44}$$

Lemma 3.1
For any initial condition $z_0 \in H^2(0,1)$ such that $a_{11} z_0(0) + a_{12} z_0'(0) = z_0(1) = 0$, the closed-loop system (3.42)-(3.44) admits

- *unique mild solution* [3] $(U,w) \in C([0,+\infty), \mathbb{R} \times \mathcal{H}_1) \cap H^1_{loc}([0,+\infty), \mathbb{R} \times L^2(0,1))$ *for* $\mu_c \in (-1, -\frac{1}{2})$,
- *unique classical solution* $(U,w) \in C([0,+\infty), \mathbb{R} \times \mathcal{H}_1) \cap C^1([0,+\infty), \mathbb{R} \times L^2(0,1))$ *for* $\mu_c \in [-\frac{1}{2}, 0)$,

with

$$\bar{\mathcal{H}} = \{f \in H^2(0,1) | a_{11} f(0) + a_{12} f'(0) = f(1) = 0\}. \tag{3.45}$$

Proof. See Appendix A.0.2 . □

Remark 3.2
Since the term $\mathcal{N}(\cdot)$ in (3.37) is non-Lipschitz at $0_{(N+1)\times 1}$, then [134, Theorem 1.6] commonly used to justify the well-posedness of infinite dimensional system with locally Lipschitz nonlinearities is inapplicable to the entire space $\mathcal{H} = \mathbb{R} \times L^2(0,1)$ in this case. Indeed, the control signal can be generated by an ODE (see (A.8)-(A.9)); therefore, we reformulate the closed-loop system as an abstract system that incorporates this control signal (see (A.14)-(A.15)) and subsequently utilize the results from [135], as demonstrated in [133]. For example, for $\mu_c = 0$, the homogeneous feedback control $\xi(W_u(t)) = K_{lin} W_u(t)$ is linear. In this case, the proposed methodology to study the well-posedness amounts to rewriting the closed-loop system as follows

$$\dot{Z}(t) = \mathcal{A} Z(t) + \mathcal{B} K_{lin} e^{(\tilde{A} + \tilde{B} K_{lin})t} z_0^N, \quad Z(0) = (0, z_0)^\top,$$

where $Z(t)$ and $\mathcal{A}, \mathcal{B}$ are defined in section A.0.2 and then used [135] to conclude the existence of a unique classical solution.

[3] in the sense of [133, Definition 1].

Another alternative ([136]) to study the well-posedness of the closed-loop system (A.5)-(A.6) involves considering the following system:

$$\dot{Z}(t) = (\mathcal{A} + \mathcal{B}K_0\mathcal{P})Z(t) + a_\gamma(\|\mathcal{P}Z(t)\|_{\mathbf{d}_c})\mathcal{B}\mathcal{N}(\mathcal{P}Z(t))$$

with the initial condition $Z(0) = (0, z_0)^\top$. *Here,* $a_\gamma \in C_c^\infty(\mathbb{R}_+, \mathbb{R})$ *and* $\gamma > 1$ *is an arbitrary real number such that* $a_\gamma(s) = 0$ *for* $s \notin \left(\frac{1}{2\gamma}, 2\gamma\right)$ *and* $a_\gamma(s) = 1$ *for* $s \in \left[\frac{1}{\gamma}, \gamma\right]$. *This system admits a unique mild solution* $Z_\gamma(t)$ *for any initial condition* $\mathcal{P}(0, z_0)^\top \in I(\gamma)$, *where* $I(\gamma) = \{x \in \mathbb{R}^{N+1} : \frac{1}{\gamma} < \|x\|_{\mathbf{d}_c} < \gamma\}$. *Furthermore,* $Z_\gamma(t)$ *coincides with the mild solution* $Z(t)$ *of (A.5)-(A.6) as long as* $\mathcal{P}Z(t) \in I(\gamma)$. *Similar methods based on control approximation to study the existence of a solution are also proposed in [88], [137], [138].*

3.3.3 Stability Analysis

For H^1 stability analysis of the closed-loop system (3.42)-(3.44), we consider the following Lyapunov functional:

$$V^h(t) = \gamma_1\Omega_1(V_1(t)) + V_2(t), \tag{3.46}$$

where

$$V_1(t) = \|W_u(t)\|_{\mathbf{d}_c}, \quad V_2(t) = \sum_{n>N} |\lambda_n| w_n^2(t),$$

$$\Omega_1(s) = \begin{cases} \frac{1}{2+\mu_c} s^{2+\mu_c} & \text{if} \quad s \leq 1, \\ \frac{s^{2\nu-\mu_c}}{2\nu-\mu_c} - \frac{1}{2\nu-\mu_c} + \frac{1}{2+\mu_c} & \text{if} \quad s \geq 1, \end{cases} \tag{3.47}$$

with $\gamma_1 > 0$ and ν being defined by:

$$\nu = \frac{\lambda_{\max}(X_c^{-1/2}G_{\mathbf{d}_c}X_c^{1/2} + X_c^{1/2}G_{\mathbf{d}_c}^\top X_c^{-1/2})}{2}. \tag{3.48}$$

Let

$$\tau = \frac{\lambda_{\min}(X_c^{-1/2}G_{\mathbf{d}_c}X_c^{1/2} + X_c^{1/2}G_{\mathbf{d}_c}^\top X_c^{-1/2})}{2}. \tag{3.49}$$

Notice that from (3.36) one has

$$\max\{1 + \mu_c, -\mu_c\} < \tau < \nu < 2 + \mu_c. \tag{3.50}$$

Since $1 + \mu_c < \tau$ then $2\nu - \mu_c > 2 + \mu_c$ and $\Omega_1 \in \mathcal{K}_\infty$.

Lemma 3.2
There exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ *such that the Lyapunov functional* $V^h(t)$ *defined in (3.46) satisfies*

$$\psi_1\left(U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)}\right) \leq V^h(t) \leq \psi_2\left(U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)}\right), \quad \forall t \geq 0. \tag{3.51}$$

Proof. See Appendix A.0.3 . □

Computing the time-derivative of $V^h(t)$ along the solutions of closed-loop system (3.42)-(3.44), one has from (3.18) and (3.40), for all $t \geq 0$,

$$\dot{V}_1(t) \leq -\delta V_1^{1+\mu_c}(t), \tag{3.52}$$

and

$$\dot{V}_2(t) = 2\sum_{n>N} |\lambda_n| w_n(t)\Big((\lambda_n + q)w_n(t) + f_n(W_u(t))\Big), \tag{3.53}$$

with

$$f_n(W_u(t)) = b_n(K_0 W_u(t) + \mathcal{N}(W_u(t))) + q_n U(t). \tag{3.54}$$

Using $|U(t)| \leq |W_u(t)|$ and $\|\mathbf{d}_c(-\ln \|W_u(t)\|_{\mathbf{d}_c})W_u(t)\|_{X_c^{-1}} = 1$, we obtain the following estimate:

$$|f_n(W_u(t))| \leq M_1(|b_n| + |q_n|)(\|W_u(t)\|_{X_c^{-1}} + \|W_u(t)\|_{\mathbf{d}_c}^{1+\mu_c}), \tag{3.55}$$

with $M_1 = \lambda_{min}(X_c^{-1})^{\frac{-1}{2}} \max\{|K_0^\top| + 1, |K^\top|\}$.

Using Young's inequality , for any $\gamma_2 > 0$ the equality (3.53) implies

$$\dot{V}_2(t) \leq 2\sum_{n>N} |\lambda_n|\Big((-1+\tfrac{1}{2\gamma_2})|\lambda_n|+q\Big)w_n^2(t)+\gamma_2 M_2(\|W_u(t)\|_{X_c^{-1}}+\|W_u(t)\|_{\mathbf{d}_c}^{1+\mu_c})^2, \tag{3.56}$$

with $M_2 = M_1^2 \sum_{n>N}(|b_n| + |q_n|)^2 < +\infty$.

Applying Proposition 2.1 and using the fact that $1 + \mu_c < \tau$ (from (3.50)), one has, for all $t \geq 0$,

$$\|W_u(t)\|_{X_c^{-1}} \leq \psi_3(V_1(t)), \tag{3.57}$$

with $\psi_3 \in \mathcal{K}_\infty$ defined as follows:

$$\psi_3(s) = \begin{cases} s^{1+\mu_c} & \text{if} \quad s \leq 1, \\ s^{\nu} & \text{if} \quad s \geq 1. \end{cases} \tag{3.58}$$

Then, from (3.56) we find

$$\dot{V}_2(t) \leq -2\theta_{N,1}V_2(t) + 4\gamma_2 M_2 \psi_3(V_1^2(t)), \quad \forall t \geq 0 \tag{3.59}$$

where $\theta_{N,1} = (1 - \frac{1}{2\gamma_2})|\lambda_{N+1}| - q > 0$ with $\gamma_2 > \frac{|\lambda_{N+1}|}{2(|\lambda_{N+1}|-q)}$.

From (3.47) and (3.52) , we obtain

$$\frac{d\Omega_1(V_1(t))}{dt} \leq -\delta\psi_3(V_1^2(t)), \quad \forall t \geq 0. \tag{3.60}$$

Therefore, from (3.46) one has

$$\dot{V}^h(t) = \gamma_1 \frac{d\Omega_1(V_1(t))}{dt} + \dot{V}_2(t), \tag{3.61}$$

and using (3.59)-(3.60), one derives, for all $t \geq 0$,

$$\dot{V}^h(t) \leq \max\{-2\theta_{N,1}, -\gamma_1\delta + 4\gamma_2 M_2\}(\psi_3(V_1^2(t)) + V_2(t)).$$

For $\gamma_1 > 4\gamma_2 M_2/\delta$, one has

$$\tilde{\delta} = \min\{2\theta_{N,1}, \delta\gamma_1 - 4\gamma_2 M_2\} > 0, \tag{3.62}$$

and thus

$$\dot{V}^h(t) \leq -\tilde{\delta}\psi_4(U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)}), \tag{3.63}$$

with ψ_4 some $\mathcal{K}_\infty$ function such that $\psi_3(V_1^2(t)) + V_2(t) \geq \psi_4(U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)})$. The proof of the existence of ψ_4 follows the same lines as the proof of Lemma 3.2.

Summarizing, one arrives at the first main result:

Theorem 3.1
There exists $\beta_1 \in \mathcal{KL}$ such that for any initial condition $z_0 \in \bar{\mathcal{H}}$, the solution of closed-loop system (3.1)-(3.3) with the homogeneous feedback law (3.11) and (3.37) satisfies

$$\|z(t,\cdot)\|^2_{H^1(0,1)} \leq \beta_1(\|z_0\|^2_{H^1(0,1)}, t), \quad \forall t \geq 0. \tag{3.64}$$

Moreover for all $t \geq T^c(z_0^N)$, the control input U and z satisfy

$$U(t) = 0, \quad \langle z(t,\cdot), \phi_n \rangle = 0, \quad n = 1, ..., N, \tag{3.65}$$

$$\|z(t,\cdot)\|^2_{H^1(0,1)} \leq e^{-2\delta(t-T^c(z_0^N))}\|z(T^c(z_0^N),\cdot)\|^2_{H^1(0,1)}, \tag{3.66}$$

with $T^c(z_0^N)$ is defined in (3.39), and δ such that (3.22) is verified. Additionally, there exists $\Delta_1 \in \mathcal{K}_\infty$ such that for any stabilization precision $\epsilon > 0$ and any prescribed time $T > T^c(z_0^N)$, if the parameter δ, characterized in (3.22), also satisfies[4]

$$\delta > \frac{1}{2T} \ln\left(\frac{\Delta_1\left(\|z_0\|^2_{H^1(0,1)}\right)}{\epsilon^2}\right), \tag{3.67}$$

then (3.5) is satisfied.

Proof. The inequality (3.63) with Lemma 3.2 imply that there exists $\bar{\beta}_1 \in \mathcal{KL}$ such that, for all $t \geq 0$,

$$U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)} \leq \bar{\beta}_1(\|z_0\|^2_{H^1(0,1)}, t). \tag{3.68}$$

[4] if $z_0 \equiv 0$ then (3.5) is satisfied for any $\delta > 0$.

Using (3.7), one derives for all $t \geq 0$,

$$\|z(t,\cdot)\|^2_{H^1(0,1)} \leq 2\max\{1, \|r\|^2_{H^1(0,1)}\}(\|w(t,\cdot)\|^2_{H^1(0,1)} + U^2(t)), \tag{3.69}$$

and with (3.68), one obtains (3.64) with $\beta_1 = 2\max\{1, \|r\|^2_{H^1(0,1)}\}\bar{\beta}_1$. Since for all $t \geq T^c(z_0^N)$, $W_u(t) = 0_{(N+1)\times 1}$ and using (3.7), one obtains

$$\begin{array}{ll} U(t)=0, \quad \langle z(t,\cdot), \phi_n\rangle = 0, \quad n=1,\cdots,N, \\ \frac{d}{dt}\langle z(t,\cdot),\phi_n\rangle = (\lambda_n + q)\langle z(t,\cdot),\phi_n\rangle, \quad n>N, \end{array} \quad \forall t \geq T^c(z_0^N). \tag{3.70}$$

Thus, using (3.22), one derives for all $t \geq T^c(z_0^N)$,

$$|\langle z(t,\cdot),\phi_n\rangle| \leq e^{-\delta(t-T^c(z_0^N))}|\langle z(T^c(z_0^N),\cdot),\phi_n\rangle|.$$

and one gets (3.66). From the equality (3.24), the estimate (3.55), and the fact that $\langle w(T^c(z_0^N),\cdot),\phi_n\rangle = \langle z(T^c(z_0^N),\cdot),\phi_n\rangle$, for all $n > N$ we obtain:

$$|\langle z(T^c(z_0^N),\cdot),\phi_n\rangle| \leq |\langle z_0,\phi_n\rangle| + \frac{2M_1(|b_n|+|q_n|)}{|\lambda_n + q|} \sup_{s\in[0,T^c(z_0^N)]} \psi_3(V_1(s)).$$

Since $V_1(s) \leq \|z_0^N\|_{\mathbf{d}_c}$ for any $s \in [0, T^c(z_0^N)]$, and by using Proposition 2.1, the estimate (3.66), along with the fact that $z(t,\cdot) \in \mathcal{H}_1$ for any $t \geq T^c(z_0^N)$, implies that there exists a function $\Delta_1 \in \mathcal{K}_\infty$ such that

$$\|z(t,\cdot)\|^2_{H^1(0,1)} \leq e^{-2\delta t}\Delta_1(\|z_0\|^2_{H^1(0,1)}), \quad \forall t \geq T^c(z_0^N).$$

The proof is complete. □

3.4 Homogeneous stabilization of 1-D heat system by output feedback

In this section, we propose constructing homogeneous output-feedback control to stabilize unstable modes. For the estimation of an infinite-dimensional state, we design a linear observer.

Assumption 3.1
([23], [69]) Assume that $c_n = \int_0^1 c(x)\phi_n(x)dx$ in the non-local measurement case, or $c_n = \phi_n(x^*)$ in the point measurement case satisfies

$$c_n \neq 0, \quad 1 \leq n \leq N, \tag{3.71}$$

where N is defined in (3.22).

In the case of Dirichlet boundary conditions and considering point measurement, Assumption 1 is satisfied for $N = 1$. For $N \geq 2$, it is equivalent to the condition that for all $2 \leq n \leq N$,

$$x^* \neq \tfrac{k}{n}, \quad \text{for all} \quad 1 \leq k \leq n-1. \tag{3.72}$$

Under this assumption, the pair $\{A, C\}$ is observable using the Hautus Lemma.

Remark 3.3
For very large N, selecting x^* in the case of point measurement, or choosing c for non-local measurement under this assumption, can lead to a considerably high observer gain. This may occur for a very large value of q.

3.4.1 Linear Observer Design

Since $\{A, C\}$ is observable then there exists $L_l = (l_1, \cdots, l_N)^\top \in \mathbb{R}^{N\times 1}$ and a symmetric positive definite matrix $P \in \mathbb{R}^{N\times N}$ such that

$$P(A + L_lC) + (A + L_lC)^\top P \preceq -2\delta_1 P, \tag{3.73}$$

with δ_1 some positive constant. Since in (3.11)-(3.14) U is known we construct an observer $\hat{w}$ for w only (as in [69], but in the form of linear PDE as in [117]) :

$$\begin{aligned} \partial_t \hat{w}(t,x) \quad &= \quad \partial_{xx}\hat{w}(t,x) + q\hat{w}(t,x) + (qr(x) + r''(x))U(t) - r(x)\xi(t) + \\ & \quad\quad l(x)\Big(\int_0^1 c(x)\hat{w}(t,x)dx - \tilde{y}(t)\Big), \end{aligned} \tag{3.74}$$

$$a_{11}\hat{w}(t,0) \quad + \quad a_{12}\hat{w}_x(t,0) = \hat{w}(t,1) = 0, \tag{3.75}$$

and $\hat{w}(0,x) = 0$ with $\tilde{y}(t)$ defined in (3.15) and $l(x)$ given by

$$l(x) = \sum_{n=1}^{N} l_n\phi_n(x), \tag{3.76}$$

where N satisfies (3.22). The error $e(t,x) = \hat{w}(t,x) - w(t,x)$ satisfies

$$\begin{aligned} \partial_t e(t,x) = \partial_{xx}e(t,x) + qe(t,x) + l(x)\textstyle\int_0^1 c(x)e(t,x)dx,& \\ a_{11}e(t,0) + a_{12}e_x(t,0) = e(t,1) = 0,& \\ e(0,x) = -z_0(x).& \end{aligned} \tag{3.77}$$

The observer $\hat{w}(t,\cdot)$ and the error $e(t,\cdot)$ can be represented as follows

$$\begin{aligned} \hat{w}(t,\cdot) = \textstyle\sum_{n=1}^\infty \hat{w}_n(t)\phi_n(\cdot), \quad & \hat{w}_n(t) = \langle \hat{w}(t,\cdot), \phi_n\rangle, \\ e(t,\cdot) = \textstyle\sum_{n=1}^\infty e_n(t)\phi_n(\cdot), \quad & e_n(t) = \langle e(t,\cdot), \phi_n\rangle. \end{aligned} \tag{3.78}$$

Next we design the homogeneous output-feedback law as follows

$$\xi(\hat{W}_u(t)) = K_0\hat{W}_u(t) + \mathcal{N}(\hat{W}_u(t)), \tag{3.79}$$

$\mathcal{N}$ defined in (3.38) and $\hat{W}_u(t) = (U(t), \hat{w}_1(t), \cdots, \hat{w}_N(t))^\top$.

The closed-loop system (3.11)-(3.14) and the observer (3.74) with the homogeneous output feedback (3.79) is given by:

$$\begin{aligned}
\dot{\hat{W}}_u(t) &= \tilde{A}\hat{W}_u(t) + \tilde{B}\xi(\hat{W}_u(t)) + \tilde{L}_l\,(CE(t)+\eta(t)), && (3.80)\\
\dot{E}(t) &= (A + L_lC)E(t) + L_l\eta(t), && (3.81)\\
\dot{e}_n(t) &= (\lambda_n + q)e_n(t), \quad n > N, && (3.82)\\
\dot{\hat{w}}_n(t) &= (\lambda_n+q)\hat{w}_n(t) + b_n\xi(\hat{W}_u(t)) + q_nU(t), \quad n>N, && (3.83)\\
e_n(0) &= -\langle z_0, \phi_n\rangle, \quad \hat{w}_n(0) = 0, \quad n \geq 1, && (3.84)
\end{aligned}$$

with $E(t) = (e_1(t), \cdots, e_N(t))^\top$, $\tilde{L}_l = (0, L_l^\top)^\top$ and $\eta(t)$ is given by

$$\eta(t) = \sum_{n\geq N+1} c_n e_n(t). \tag{3.85}$$

Lemma 3.3
For any initial condition $z_0 \in H^2(0,1)$ such that $a_{11}z_0(0) + a_{12}z_0'(0) = z_0(1) = 0$, the closed-loop system (3.80)-(3.83) admits a solution $(e, U, \hat{w})$ with $e(t,\cdot) \in \bar{\mathcal{H}}$, $(U, \hat{w}) \in C([0,+\infty), \mathbb{R}\times\mathcal{H}_1) \cap C^1([0,+\infty), \mathbb{R}\times L^2(0,1))$, where $\mathcal{H}_1$ is defined in (3.45).

Proof. See Appendix A.0.4. □

3.4.2 Stability analysis

Similarly to the proof of Theorem 3.1, for H^1 stability analysis of the closed-loop system (3.80)-(3.83) let us consider the following Lyapunov functional

$$V^{h,l}(t) = \gamma_3\Omega_1(V_1(t)) + V_2(t) + \gamma_4\Omega_2(V_3(t)), \tag{3.86}$$

where Ω_1 is defined in (3.47),

$$\begin{aligned}
V_1(t) &= \|\hat{W}_u(t)\|_{\mathbf{d}_c}, && (3.87)\\
V_2(t) &= \sum_{n>N} |\lambda_n|\hat{w}_n^2(t), && (3.88)\\
V_3(t) &= \|E(t)\|_P^2 + \gamma_5\sum_{n>N}|\lambda_n|e_n^2(t), && (3.89)\\
\Omega_2(s) &= \begin{cases} \frac{1}{\kappa_1}s^{\kappa_1}, & \text{if} \quad s\leq 1,\\ \frac{1}{\kappa_2}s^{\kappa_2} - \frac{1}{\kappa_2} + \frac{1}{\kappa_1}, & \text{if} \quad s \geq 1, \end{cases} && (3.90)\\
\kappa_1 &= \frac{1+\mu_c}{\nu+\mu_c}, \quad \kappa_2 = \frac{\nu}{\mu_c+\tau}, && (3.91)
\end{aligned}$$

$\gamma_3, \gamma_4, \gamma_5 > 0$ and P defined in (3.73). Since from (3.50) one has $\max\{1+\mu_c, -\mu_c\} < \tau$ then $0 < \kappa_1 < \kappa_2$ and $\Omega_2 \in \mathcal{K}_\infty$.

Lemma 3.4
There exist $\psi_5, \psi_6 \in \mathcal{K}_\infty$ such that the Lyapunov functional $V^{h,l}(t)$ defined in (3.86) satisfies:

$$\psi_5\Big(U^2(t) + \|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|e(t,\cdot)\|^2_{H^1(0,1)}\Big) \le V^{h,l}(t) \le \psi_6\Big(U^2(t) + \|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|e(t,\cdot)\|^2_{H^1(0,1)}\Big)$$

for all $t \ge 0$.

Proof. See Appendix A.0.5 . □

Computing the time derivatives along the solutions of closed-loop system (3.80)-(3.83), applying (2.22) and (3.40) one derives:

$$\dot{V}_1(t) \le -\delta V_1^{1+\mu_c}(t) + V_1(t)\frac{\hat{W}_u(t)^\top \mathbf{d}_c^\top(-\ln V_1(t))X_c^{-1}\mathbf{d}_c(-\ln V_1(t))\tilde{L}_l(CE(t)+\eta(t))}{\hat{W}_u(t)^\top \mathbf{d}_c^\top(-\ln V_1(t))X_c^{-1}G_{\mathbf{d}_c}\mathbf{d}_c(-\ln V_1(t))\hat{W}_u(t)},$$

which implies, using Proposition 2.1, that

$$\dot{V}_1(t) \le -\delta V_1^{1+\mu_c}(t) + M_3 V_1(t)|CE(t)+\eta(t)|\|\mathbf{d}_c(-\ln V_1(t))\|_{X_c^{-1}}$$

with $M_3 = \frac{\|\tilde{L}_l\|_{X_c^{-1}}}{\tau}$ and $\eta(t)$ defined in (3.85). Using the following estimate

$$|\eta(t)| \le M_4\Big(\sum_{n>N}|\lambda_n|e_n^2(t)\Big)^{\frac{1}{2}}, \quad M_4 = \Big(\sum_{n>N}^{\infty}\tfrac{c_n^2}{|\lambda_n|}\Big)^{\frac{1}{2}}, \tag{3.92}$$

combined with the Proposition 2.1 leads to

$$\dot{V}_1(t) \le -\delta V_1^{1+\mu_c}(t) + M_5 V_3^{\frac{1}{2}}(t)\begin{cases} V_1^{1-\nu}(t) & \text{if} \quad V_1(t) \le 1, \\ V_1^{1-\tau}(t) & \text{if} \quad V_1(t) > 1, \end{cases}$$

with $M_5 = M_3(\lambda_{min}(P)^{\frac{-1}{2}}|C^\top| + \frac{M_4}{\sqrt{\gamma_5}})$. Note that $M_4 \le \frac{\|c\|_{L^2}}{|\lambda_{N+1}|^{\frac{1}{2}}}$ in the case of nonlocal measurements and $M_4 \le \sqrt{2}\Big(\sum_{n>N}^{\infty}\frac{1}{|\lambda_n|}\Big)^{\frac{1}{2}}$ in the case of point measurements. Then

$$\frac{d\Omega_1(V_1(t))}{dt} \le -\delta\psi_3(V_1^2(t)) + M_5 V_3^{\frac{1}{2}}(t)\begin{cases} V_1^{2-\nu+\mu_c}(t), V_1(t) \le 1, \\ V_1^{2\nu-\mu_c-\tau}(t), V_1(t) > 1. \end{cases}$$

Repeating the same calculations as in the previous section, we get that the time-derivative of V_2 defined in (3.88) along the solutions of (3.80)-(3.83) is given by :

$$\dot{V}_2(t) \leq -2\theta_{N,2}V_2(t) + 4\gamma_6 M_2 \psi_3(V_1^2(t)), \tag{3.93}$$

with $\theta_{N,2} = (1 - \frac{1}{2\gamma_6})|\lambda_{N+1}| - q > 0$ and $\gamma_6 > \frac{|\lambda_{N+1}|}{2(|\lambda_{N+1}|-q)}$.

Along solutions of closed-loop system (3.80)-(3.83) , one has:

$$\dot{V}_3(t) \leq -2\delta_1\|E(t)\|_P^2 + 2\gamma_5(-|\lambda_{N+1}| + q)\sum_{n>N}|\lambda_n|e_n^2(t) + 2\|L_l\|_P|\eta(t)|\|E(t)\|_P.$$

Applying Young inequality to the term $\|L_l\|_P|\eta(t)|\|E(t)\|_P$, one has

$$\dot{V}_3(t) \leq -2\theta_{N,3}V_3(t), \quad \forall t \geq 0, \tag{3.94}$$

where

$$\theta_{N,3} = \min\{\delta_1 - \tfrac{\gamma_7}{2}\|L_l\|_P^2, \gamma_5(|\lambda_{N+1}| - q) - \tfrac{1}{2\gamma_7}M_4^2\} > 0$$

with $0 < \gamma_7 < \frac{2\delta_1}{\|L_l\|_P^2}$ and $\gamma_5 > \frac{M_4^2}{2\gamma_7(|\lambda_{N+1}|-q)}$. From (3.86), we obtain

$$\dot{V}^{h,l}(t) = \gamma_3\frac{d\Omega_1(V_1(t))}{dt} + \dot{V}_2(t) + \gamma_4\dot{V}_3(t)\Omega_2'(V_3(t)), \tag{3.95}$$

and therefore, one derives

$$\dot{V}^{h,l}(t) \leq -2\theta_{N,2}V_2(t) + \begin{cases} (-\gamma_3\delta + 4\gamma_6M_2)V_1^{2(1+\mu_c)}(t) + \gamma_3M_5V_3^{\frac{1}{2}}(t)V_1^{2-\nu+\mu_c}(t) \\ \quad -2\gamma_4\theta_{N,3}V_3^{\kappa_1}(t), \quad \text{if} \quad V_1(t) \leq 1, V_3(t) \leq 1 \\ (-\gamma_3\delta + 4\gamma_6M_2)V_1^{2(1+\mu_c)}(t) + \gamma_3M_5V_3^{\frac{1}{2}}(t)V_1^{2-\nu+\mu_c}(t) \\ \quad -2\gamma_4\theta_{N,3}V_3^{\kappa_2}(t), \quad \text{if} \quad V_1(t) \leq 1, V_3(t) \geq 1 \\ (-\gamma_3\delta + 4\gamma_6M_2)V_1^{2\nu}(t) + \gamma_3M_5V_3^{\frac{1}{2}}(t)V_1^{2\nu-\tau-\mu_c}(t) \\ \quad -2\gamma_4\theta_{N,3}V_3^{\kappa_1}(t), \quad \text{if} \quad V_1(t) \geq 1, V_3(t) \leq 1 \\ (-\gamma_3\delta + 4\gamma_6M_2)V_1^{2\nu}(t) + \gamma_3M_5V_3^{\frac{1}{2}}(t)V_1^{2\nu-\tau-\mu_c}(t) \\ \quad -2\gamma_4\theta_{N,3}V_3^{\kappa_2}(t), \quad \text{if} \quad V_1(t) \geq 1, V_3(t) \geq 1. \end{cases} \tag{3.96}$$

From (3.50) and using Young's inequality on the following cross terms $V_3^{\frac{1}{2}}(t)V_1^{2-\nu+\mu_c}(t)$ and $V_3^{\frac{1}{2}}(t)V_1^{2\nu-\tau-\mu_c}(t)$ appearing in (3.96) , one derives:

$$\begin{cases} V_3^{\frac{1}{2}}(t)V_1^{2-\nu+\mu_c}(t) \leq \frac{2-\nu+\mu_c}{2(1+\mu_c)}V_1^{2(1+\mu_c)}(t) + \frac{1}{2\kappa_1}V_3^{\kappa_1}(t), \\ V_3^{\frac{1}{2}}(t)V_1^{2\nu-\tau-\mu_c}(t) \leq \frac{2\nu-\tau-\mu_c}{2\nu}V_1^{2\nu}(t) + \frac{1}{2\kappa_2}V_3^{\kappa_2}(t), \end{cases} \tag{3.97}$$

with κ_1, κ_2 defined in (3.91). Then, from (3.96), one derives :

$$\dot{V}^{h,l}(t) \leq -2\theta_{N,2}V_2(t) - M_7\psi_3(V_1^2(t)) - M_8\psi_7(V_3(t)), \tag{3.98}$$

with ψ_3 defined in (3.58), $\psi_7 \in \mathcal{K}_\infty$ defined as follows:

$$\psi_7(s) = \begin{cases} s^{\kappa_1}, & \text{if} \quad s \leq 1, \\ s^{\kappa_2}, & \text{if} \quad s \geq 1, \end{cases} \tag{3.99}$$

$M_7 = \gamma_3(\delta - M_5M_6) - 4\gamma_6 M_2$, $M_8 = 2\gamma_4\theta_{N,3} - \gamma_3 M_5 M_6$ and

$$M_6 = \max\{\tfrac{2-\nu+\mu_c}{2(1+\mu_c)}, \tfrac{1}{2\kappa_1}, \tfrac{2\nu-\tau-\mu_c}{2\nu}\}. \tag{3.100}$$

For $\delta > M_5M_6$, $\gamma_3 > \frac{4\gamma_6 M_2}{(\delta - M_5M_6)}$ and $\gamma_4 > \frac{\gamma_3 M_5 M_6}{2\theta_{N,3}}$, one has

$$\bar{\delta} = \min\{2\theta_{N,2}, M_7, M_8\} > 0, \tag{3.101}$$

and thus

$$\dot{V}^{h,l}(t) \leq -\bar{\delta}\psi_8\Big(U^2(t) + \|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|e(t,\cdot)\|^2_{H^1(0,1)}\Big), \tag{3.102}$$

with ψ_8 some $\mathcal{K}_\infty$ function such that $\psi_3(V_1^2(t)) + \psi_7(V_3(t)) + V_2(t) \geq \psi_8(U^2(t) + \|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|e(t,\cdot)\|^2_{H^1(0,1)})$. The proof of the existence of ψ_8 follows the same lines as the proof of Lemma 3.4.

Summarizing, one arrives at the second main result:

Theorem 3.2
There exists $\beta_3 \in \mathcal{KL}$ such that for any initial condition $z_0 \in H^2(0,1)$ such that $a_{11}z_0(0) + a_{12}z_0'(0) = z_0(1) = 0$, any solution of closed-loop system (3.1)-(3.3) and the observer (3.74) with the homogeneous feedback law (3.11) and (3.79) satisfies:

$$\|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|z(t,\cdot)\|^2_{H^1(0,1)} \leq \beta_3(\|z_0\|^2_{H^1(0,1)}, t), \forall t \geq 0.$$

Proof. The inequality (3.102) with Lemma 3.4 imply that there exists $\beta_4 \in \mathcal{KL}$ such that for all $t \geq 0$

$$U^2(t) + \|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|e(t,\cdot)\|^2_{H^1(0,1)} \leq \beta_4(\|z_0\|^2_{H^1(0,1)}, t).$$

Using (3.7) one derives

$$\|z(t,\cdot)\|^2_{H^1(0,1)} \leq 3\max\{1, \|r\|^2_{H^1(0,1)}\}(\|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + U(t)^2 + \|e(t,\cdot)\|^2_{H^1(0,1)}), \tag{3.103}$$

which complete the proof. $\square$

3.5 Numerical Simulations

We perform numerical simulations on the system (3.11)-(3.15) with initial condition $z_0(x) = 1 - x^2$, Mixed Neumann-Dirichlet boundary conditions, point measurement $y(t) = z(t, 0)$ and $U(0) = 0$, the reaction coefficient $q = 22.5$. We selected $N = 2$, which corresponds to two unstable modes $(w_1(t), w_2(t))$. In this case, $W_u(t) = (U(t), w_1(t), w_2(t))^\top$.

We consider (3.16) with $M = 30$ truncated basis. We use the control toolbox on MATLAB to compute the linear control gains and the linear observer gain:

$$\begin{aligned} K_{l_1} &= (-539.3, -2852.3, -210.5), \\ K_{l_2} &= (-12147, -108190, -24788), \\ L_l &= (-77.4974, 25.6429)^\top, \end{aligned} \tag{3.104}$$

with which we compute the linear output-feedback control based on the linear observer (3.74) given by

$$\dot{U}(t) = \xi(\hat{W}_u(t)) = K_l \hat{W}_u(t). \tag{3.105}$$

On the other hand, we use the Homogeneous Control Systems Toolbox for MATLAB [130] to compute $\|\cdot\|_{\mathbf{d}}, \mu_c, \tau, \nu, K_0, K, G_{\mathbf{d}_c}$. This gives,

$$\mu_c = -0.3, \quad K_0 = (-51.6180, -183.4370, -0.0002), \quad K = (-487.7, -2668.9, 210.5)$$
$$\tau = 1, \quad \nu = 1.3, \quad G_{\mathbf{d}_c} = \begin{pmatrix} 0.5349 & -2.7058 & 0.0134 \\ 0.1301 & 1.7570 & -0.0038 \\ -0.1363 & 0.1746 & 1.6082 \end{pmatrix}.$$

We adopt the following notations: LC=Linear control $U(t)$ in (3.105) with K_{l_1}, High gain LC=High gain linear control $U(t)$ in (3.105) with K_{l_2}, HC=homogeneous control $U(t)$ in (3.11) and (3.79) , LO=Linear observer in (3.74).

Figure 3.1 shows the simulations the norm $\|z(t, \cdot)\|_{H^1(0,1)}$ of the closed-loop system with the linear output-feedback control (LC+LO in blue line and High gain LC+LO in black dashed line) and with the homogeneous output-feedback control (HC+LO in red line) in logarithmic scale and during the initial phase of stabilization. In Figure 3.2, the linear output-feedback control (3.105) and homogeneous output-feedback control (3.11) and (3.79) and the solution $z(t, x)$ of closed-loop system (3.1)-(3.3) with the homogeneous output-feedback law (3.11) and (3.79) are shown. We can observe from the numerical simulations (see Figure 3.1) that the closed-loop system with high gain linear control (High gain LC+LO) in black dashed line and the closed-loop system with homogeneous control (HC+LO) in red line achieve

$$\|z(t, \cdot)\|_{H^1(0,1)} \leq \epsilon = 10^{-7}, \quad \forall t \geq T = 1. \tag{3.106}$$

From Figure 3.2, closed-loop system with homogeneous control achieves (3.106) with the control restriction

$$\sup_{t>0} |U(t)| \leq \bar{U} = 18. \tag{3.107}$$

One can also observe in Figure 3.1 and Figure 3.2 that using high-gain linear control to achieve (3.106) implies a large deviation of the closed-loop system (a peaking) and an overshoot of the control signal during the initial phase of stabilization. Additionally, we can observe in Figure 3.2 that both the linear controller and the homogeneous one exhibit a similar overshoot value while verifying (3.107), but the linear controller violates condition (3.106) whereas the homogeneous one does not. Hence, we can observe the advantage of using a homogeneous control while making it possible to obtain faster convergence without a peaking effect and with a small overshoot of the control signal.

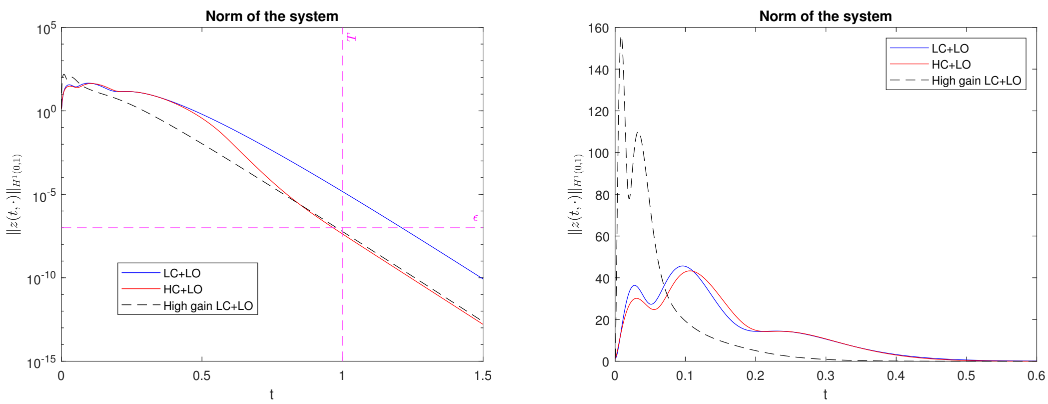

Figure 3.1: Evolution of the norm $\|z(t,\cdot)\|_{H^1(0,1)}$ of the closed-loop system (3.1)-(3.3) under linear output-feedback control (LC+LO in blue line and High gain LC+LO in black dashed line) and under homogeneous output-feedback control (HC+LO in red line) in logarithmic scale (at left) and during the initial phase of stabilization (at right).

3.5.1 Towards homogeneous observers

3.5.1.1 Homogeneous infinite-dimensional observer design

Since $\{A, C\}$ is observable then according to [139], [140] the linear algebraic equation

$$G_0^o A - A G_0^o + Y_0^o C = A, \quad C G_0^o = 0, \tag{3.108}$$

has a solution $Y_0^o \in \mathbb{R}^{N\times 1}$, $G_0^o \in \mathbb{R}^{N\times N}$ such that the matrix $G_0^o + I_N$ is invertible. Then the matrix

$$G_{\mathbf{d}_o} := I_N + \mu_o G_0^o \tag{3.109}$$

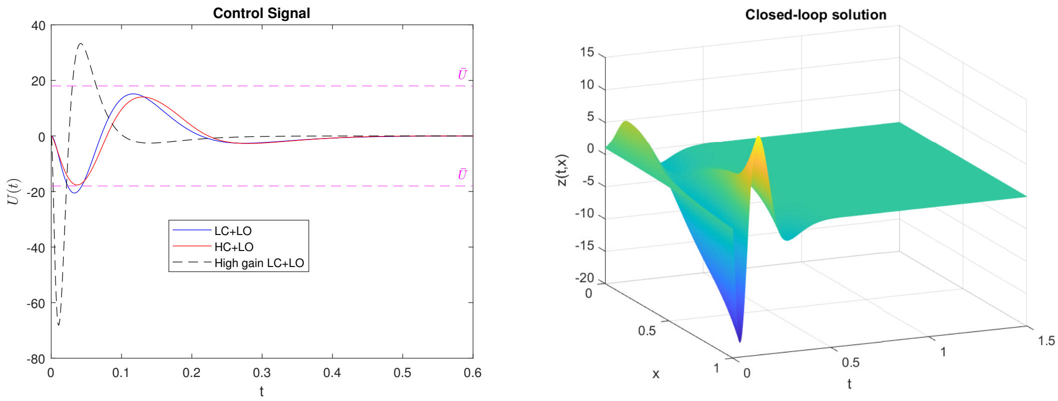

Figure 3.2: Time-evolution of the output-feedback control signal $U(t)$ during the initial phase of stabilization: the linear output-feedback control (LC+LO in blue line and High gain LC+LO in black dashed line) and homogeneous output-feedback control (HC+LO in red line) and the solution $z(t,x)$ of the closed-loop system (3.1)-(3.3) with the homogeneous output-feedback law (3.11) and (3.79)

is anti-Hurwitz for any $\mu_o \in (-\frac{1}{N+1}, 0)$ and the matrix $A_0^o = A - (I_N + G_0^o)^{-1} Y_0^o C$ satisfies the identity

$$A_0^o G_{\mathbf{d}_o} = (G_{\mathbf{d}_o} + \mu_o I_N) A_0^o, \quad C G_{\mathbf{d}_o} = C, \tag{3.110}$$

and the linear matrix inequalities

$$(A_0^o + \rho_o G_{\mathbf{d}_o}) X_o + X_o (A_0^o + \rho_o G_{\mathbf{d}_o})^\top + Y_o C + C^\top Y_o^\top \preceq 0, \tag{3.111}$$

$$G_{\mathbf{d}_o} X_o + X_o G_{\mathbf{d}_o}^\top \succ 0, \qquad X_o = X_o^\top \succ 0, \tag{3.112}$$

have solution $X_o \in \mathbb{R}^{N\times N}$, $Y_o \in \mathbb{R}^{N\times 1}$ for any $\rho_o > 0$. Let

$$L = X_o^{-1} Y_o, \quad L_0 = -(I_N + G_0^o)^{-1} Y_0^o, \tag{3.113}$$

with X_o, Y_o solution of (3.111)-(3.112) and let $\mathbf{d}_o$ the dilation generated by $G_{\mathbf{d}_o}$.

Let $h_i = (0, \ldots, 1, \ldots, 0)^\top$ the i-th unit Euclidean vector in $\mathbb{R}^N$. As in the previous section, we design a nonlinear observer $\hat{w}$ for w in the form of nonlinear PDE :

$$\partial_t \hat{w}(t,x) = \partial_{xx}\hat{w}(t,x) + q\hat{w}(t,x) + (qr(x) + r''(x))U(t) - r(x)\xi(t) + \tag{3.114}$$

$$\sum_{i=1}^{N} h_i^\top \mathcal{O}(\int_0^1 c(x)\hat{w}(t,x)dx - \tilde{y}(t))\phi_i(x), \tag{3.115}$$

$$a_{11}\hat{w}(t,0) + a_{12}\hat{w}_x(t,0) = \hat{w}(t,1) = 0, \tag{3.116}$$

and $\hat{w}(0,x) = 0$ with $\mathcal{O}(\cdot)$ defined by

$$\mathcal{O}(s) = L_0 s + |s|^{\mu_0 - 1}\mathbf{d}_o(\ln|s|)Ls, \quad \forall s \in \mathbb{R}. \tag{3.117}$$

The error $e(t,x) = \hat{w}(t,x) - w(t,x)$ satisfies

$$\begin{aligned} \partial_t e(t,x) &= \partial_{xx} e(t,x) + q e(t,x) + \sum_{i=1}^{N} h_i^\top \mathcal{O}(\int_0^1 c(x) e(t,x) dx) \phi_i(x), \quad (3.118) \\ a_{11} e(t,0) &+ a_{12} e_x(t,0) = e(t,1) = 0, \quad (3.119) \\ e(0,x) &= -z_0(x). \quad (3.120) \end{aligned}$$

The observer $\hat{w}(t,\cdot)$ and the error $e(t,\cdot)$ can be represented as follows

$$\begin{aligned} \hat{w}(t,\cdot) = \textstyle\sum_{n=1}^{\infty} \hat{w}_n(t)\phi_n(\cdot), \quad \hat{w}_n(t) = \langle \hat{w}(t,\cdot), \phi_n \rangle, \\ e(t,\cdot) = \textstyle\sum_{n=1}^{\infty} e_n(t)\phi_n(\cdot), \quad e_n(t) = \langle e(t,\cdot), \phi_n \rangle. \end{aligned} \quad (3.121)$$

Next, we design the homogeneous output feedback as follows:

$$\xi(\hat{W}_u(t)) = K_0 \hat{W}_u(t) + \mathcal{N}(\hat{W}_u(t)), \quad (3.122)$$

$\mathcal{N}$ defined in (3.38) , $\hat{W}_u(t) = (U(t), \hat{w}_1(t), \cdots, \hat{w}_N(t))^\top$ and N is defined in (3.22).

The closed-loop system (3.11)-(3.14) and the observer (3.114) with homogeneous output feedback (3.122) is given by:

$$\begin{aligned} \dot{\hat{W}}_u(t) &= \tilde{A}\hat{W}_u(t) + \tilde{B}\xi(\hat{W}_u(t)) + \tilde{\mathcal{O}}(CE(t) + \eta(t)), \quad (3.123) \\ \dot{E}(t) &= AE(t) + \mathcal{O}(CE(t) + \eta(t)), \quad (3.124) \\ \dot{e}_n(t) &= (\lambda_n + q)e_n(t), \quad n > N, \quad (3.125) \\ \dot{\hat{w}}_n(t) &= (\lambda_n + q)\hat{w}_n(t) + b_n \xi(\hat{W}_u(t)) + q_n U(t), \quad n > N, \quad (3.126) \\ e_n(0) &= -\langle z_0, \phi_n \rangle, \quad \hat{w}_n(0) = 0, \quad n \geq 1, \quad (3.127) \end{aligned}$$

with $\tilde{\mathcal{O}}(\cdot) = [0, \mathcal{O}(\cdot)]^\top$, $E(t) = (e_1(t), \cdots, e_N(t))^\top$ and $\eta(t) = \sum_{n>N} c_n e_n(t)$.

Since the closed-loop system is cascaded, the H^1 stability analysis of the closed-loop system (3.123)-(3.126) can be performed by demonstrating Lyapunov stability and asymptotic attractivity.

By using (3.125) combined with (3.22), we obtain

$$\sum_{n>N} |\lambda_n| e_n^2(t) \leq e^{-2\delta t} \sum_{n>N} |\lambda_n| |\langle z_0, \phi_n \rangle|^2, \quad \forall t \geq 0. \quad (3.128)$$

By following the same lines as in [95, Corrolary 8.8] , there exists $M_9 > 0$ and a dilation $\mathbf{d}_o$ such that

$$\|E(t)\|_{\mathbf{d}_o}^2 \leq \beta_o(\|W(0)\|_{\mathbf{d}_o}, t)^2 + M_9 \sup_{s\in[0,t]} \|\tilde{\eta}(s))\|_{\tilde{\mathbf{d}}_o}^2, \quad \forall t \geq 0, \quad (3.129)$$

with $\tilde{\eta}(t) = \tilde{L}_0 \eta(t)$, $\tilde{L}_0 = (1, L_0^\top)^\top$ and β_o a $\mathcal{GKL}-$function.

Using the fact that $\hat{W}_u(0) = 0_{(N+1)\times 1}$, one can be shown that there exists $M_{10} > 0$ and a dilation $\tilde{\mathbf{d}}_c$ fulfilling the inequality

$$\forall t \geq 0, \quad \|\hat{W}_u(t)\|^2_{\mathbf{d}_c} \leq M_{10} \sup_{s\in[0,t]} \|\tilde{\mathcal{O}}(CE(s)+\eta(s))\|^2_{\tilde{\mathbf{d}}_c}.$$

By using (3.59) combined with the fact that $\hat{w}_n(0) = 0$ one derives

$$\sum_{n>N} |\lambda_n| \hat{w}_n^2(t) \leq M_{11} \sup_{s\in[0,t]} \psi_3(\|\hat{W}_u(t)\|^2_{\mathbf{d}_c}), \forall t \geq 0, \tag{3.130}$$

for some $M_{11} > 0$ and ψ_3 is defined in (3.58).

Since, from (3.92), we have the following estimate: $|\eta(t)| \leq M_4 \left(\sum_{n>N} |\lambda_n| e_n^2(t)\right)^{\frac{1}{2}}$, it follows that $\tilde{\eta}(t)$ is bounded and converges to zero as $t \to +\infty$. Utilizing the converging input and converging state (CICS) property for ISS systems ([141]), we conclude that

$$\lim_{t\to+\infty} \|E(t)\|^2_{\mathbf{d}_o} = 0.$$

Hence, $\tilde{\mathcal{O}}(CE(t)+\eta(t)) \to 0$ as $t \to +\infty$ and, using the same arguments, we conclude

$$\lim_{t\to+\infty} \|\hat{W}_u(t)\|^2_{\mathbf{d}_c} = \lim_{t\to+\infty} \sum_{n>N} |\lambda_n| \hat{w}_n^2(t) = 0 \tag{3.131}$$

and then

$$\lim_{t\to+\infty} \left(\|\hat{W}_u(t)\|^2_{\mathbf{d}_c} + \sum_{n>N} |\lambda_n| \hat{w}_n^2(t) + \|E(t)\|^2_{\mathbf{d}_o} + \sum_{n>N} |\lambda_n| e_n^2(t) \right) = 0. \tag{3.132}$$

Using the fact that $|\eta(t)| \leq M_4 \Big(\sum_{n>N} |\lambda_n| |\langle z_0, \phi_n\rangle|^2 \Big)^{\frac{1}{2}}$ and using the estimates (3.128)-(3.130) combined with Proposition 2.1, there exists $\psi_9 \in \mathcal{K}_\infty$ such that

$$\begin{aligned} \|\hat{W}_u(t)\|^2_{\mathbf{d}_c} + \textstyle\sum_{n>N} |\lambda_n| \hat{w}_n^2(t) + \|E(t)\|^2_{\mathbf{d}_o} + \sum_{n>N} |\lambda_n| e_n^2(t) \leq \\ \psi_9 \Big(\|W(0)\|^2_{\mathbf{d}_o} + \textstyle\sum_{n>N} |\lambda_n| |\langle z_0, \phi_n\rangle|^2 \Big), \quad \forall t \geq 0. \end{aligned} \tag{3.133}$$

Summarizing, one arrives at the following third main result:

Theorem 3.3
There exists $\Delta_3 \in \mathcal{K}_\infty$ such that for any initial condition $z_0 \in \bar{\mathcal{H}}$, any solution of closed-loop system (3.1)-(3.3) and observer (3.114) with the homogeneous feedback law (3.11) and (3.122) satisfy:

$$\|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|z(t,\cdot)\|^2_{H^1(0,1)} \leq \Delta_3(\|z_0\|^2_{H^1(0,1)}), \quad \forall t \geq 0,$$

and

$$\lim_{t\to+\infty} \left(\|z(t,\cdot)\|^2_{H^1(0,1)} + \|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} \right) = 0. \tag{3.134}$$

Proof. By following the same lines as in the proof of Theorem 3.2, we achieve the claimed result. □

Remark 3.4
The estimate (3.129) is obtained by showing that there exists $\chi \in \mathcal{K}_\infty$ such that

$$\tfrac{d}{dt}\|E(t)\|_{\boldsymbol{d}_o} \leq \Big(-\tfrac{\rho_o}{2} + \tfrac{1}{\tau_o}\chi(\|\tilde{\boldsymbol{d}}_o(-\ln\|E(t)\|_{\boldsymbol{d}_o})\tilde{\eta}(t))\|_{\tilde{X}_o^{-1}}) \Big)\|E(t)\|_{\boldsymbol{d}_o}^{1+\mu_o},$$

for some τ_o and $\tilde{X}_o$. The construction of a Lyapunov function in this part requires the knowledge of χ and a Lyapunov function of the system with triple dilation $(\boldsymbol{d}_c(s), \boldsymbol{d}_o(s), e^s)$.

3.6 Conclusion

In this chapter, we have developed a nonlinear homogeneous state/output feedback controller to stabilize a 1D reaction-diffusion equation. The stability of the closed-loop system is established through the construction of a Lyapunov functional for both the closed system with state-feedback control and the closed-loop system with output-feedback control based on a linear infinite-dimensional observer. We also examine the well-posedness of the closed-loop system under both homogeneous state feedback and output feedback controls. Simulations indicate that the homogeneous controller outperforms the linear feedback controller in terms of closed-loop performance, demonstrating less overshoot and no peaking effect.

Chapter 4

Homogeneous predictor feedback for a 1D reaction-diffusion equation with input delay

Contents

This chapter deals with nonlinear boundary stabilization of a 1D reaction-diffusion equation with input delay. Using the modal decomposition approach, we propose a *homogeneous*-based predictor feedback for stabilizing the unstable modes. We prove the stability of the closed-loop system via the construction of a suitable Lyapunov functional. We present numerical simulations to support the analytical results and compare our proposed controller to linear predictor feedback regarding closed-loop performance and peaking effect.

4.1 Problem statement

Let us consider the 1D reaction-diffusion equation

$$\begin{aligned}
\partial_t z(t,x) &= \partial_{xx} z(t,x) + qz(t,x), && (4.1)\\
z(t,0) &= 0, && (4.2)\\
z(t,L) &= U(t-r), && (4.3)\\
z(0,x) &= z_0(x), && (4.4)
\end{aligned}$$

$(t,x) \in \mathbb{R}_+ \times [0,L]$, $z_0 \in H_0^1(0,L)$ the initial condition, where $z(t,\cdot)$ is the reaction-diffusion PDE state at time t, $q>0$ is the reaction coefficient, $U(t-r) \in \mathbb{R}$ is the control input with $r>0$ a constant delay where $U|_{[-r,0]} \equiv 0$.

This chapter aims to design a homogeneous-based predictor feedback for the system (4.1)-(4.4) using the modal decomposition approach ([61]) and to investigate numerically if the use of a homogeneous feedback results in a smaller peaking than a linear feedback. More precisely, as in Chapter 3, we consider the following control problem: *given initial state $z_0 \in H_0^1(0,L)$, a stabilization precision $\epsilon > 0$ and a prescribed time $T_p > 0$,the closed-loop system has the desired stabilization precision:*

$$\|z(t,\cdot)\|_{H^1(0,L)} \leq \epsilon, \quad \forall t > T_p, \tag{4.5}$$

with the restricted control magnitude

$$\sup_{t>0} |U(t-r)| \leq \bar{U}, \tag{4.6}$$

for some maximal control amplitude $\bar{U} > 0$. As discussed in Chapter 3, it may happen that a high-gain linear controller ensures that the solution of the closed-loop system meets the criteria specified in (4.5), but it does not fulfill the control constraint outlined in (4.6). In contrast, the homogeneous control meets both criteria (4.5) and (4.6). Another situation that can be considered is when both the linear controller and the homogeneous one exhibit a similar overshoot value e.g,. while verifying (4.6), but the linear controller may violate condition (4.5) whereas the homogeneous one does not. The advantage of the homogeneous controller lies then in its ability to achieve faster convergence. These considerations can be assessed numerically.

4.2 Modal decomposition

We start by considering the following trigonometric change of variable (see e.g., [71], [76]):

$$w(t,x) = z(t,x) - \kappa(x)U(t-r), \quad \kappa(x) = \sin(\sigma x), \tag{4.7}$$

with $\sigma = \frac{\pi}{2L}$ to obtain the following equivalent PDE:

$$\begin{aligned}
\partial_t w(t,x) &= \partial_{xx} w(t,x) + q w(t,x) - \kappa(x)\Big[\dot{U}(t-r) - (-\sigma^2+q)U(t-r)\Big], && (4.8)\\
w(t,0) &= w(t,L) = 0, && (4.9)\\
w(0,x) &= z_0(x). && (4.10)
\end{aligned}$$

By introducing a new control input $\xi(t)$ such that $\xi|_{[-r,0]} \equiv 0$ and

$$\xi(t) = \dot{U}(t) - (-\sigma^2+q)U(t), \quad \forall t \geq 0, \tag{4.11}$$

one obtains the following coupled ODE-PDE system:

$$\begin{aligned} \dot{U}(t-r) &= (-\sigma^2+q)U(t-r)+\xi(t-r), && (4.12)\\ \partial_t w(t,x) &= \partial_{xx}w(t,x)+qw(t,x)-\kappa(x)\xi(t-r), && (4.13)\\ w(t,0) &= w(t,L)=0, &&\\ w(0,x) &= z_0(x). && (4.14) \end{aligned}$$

The solution to (4.13) can be represented as follows:

$$w(t,x)=\sum_{n=1}^{\infty} w_n(t)\phi_n(x), \quad w_n(t)=\langle w(t,\cdot),\phi_n\rangle, \tag{4.15}$$

with $\phi_n(x)=\sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$, $n\geq 1$. Projecting onto the basis $\{\phi_n\}_{n=1}^{\infty}$, one has

$$\begin{aligned} \dot{U}(t-r) &= (-\sigma^2+q)U(t-r)+\xi(t-r), && (4.16)\\ \dot{w}_n(t) &= (-\lambda_n+q)w_n(t)+b_n\xi(t-r), && (4.17)\\ w_n(0) &= \langle z_0,\phi_n\rangle, \quad n\geq 1, && (4.18) \end{aligned}$$

where $n\geq 1$, $\lambda_n=\frac{n^2\pi^2}{L^2}$ is the eigenvalue of Sturm-Liouville eigenvalue problem

$$\phi''(x)+\lambda\phi(x)=0, \quad \phi(0)=\phi(L)=0, \tag{4.19}$$

corresponding to the eigenvector ϕ_n,

$$b_n=-\int_0^L \kappa(x)\phi_n(x)dx=\frac{(-1)^n\sqrt{2L}}{\pi}\left(\frac{1}{2n-1}+\frac{1}{2n+1}\right). \tag{4.20}$$

Since $\lambda_n\to+\infty$ when $n\to+\infty$, then there exists an integer $N\geq 1$ such that

$$-\lambda_n+q<-\delta, \quad \forall n\geq N+1, \tag{4.21}$$

with $\delta>0$ some positive constant. We obtain from (4.16)-(4.18), the following system:

$$\begin{aligned} \dot{W}(t) &= AW(t)+B\xi(t-r), && (4.22)\\ \dot{w}_n(t) &= (-\lambda_n+q)w_n(t)+b_n\xi(t-r), n>N, && (4.23)\\ w_n(0) &= \langle z_0,\phi_n\rangle, \quad n\geq 1, && (4.24) \end{aligned}$$

with

$$\begin{aligned} W(t)=(U(t-r),w_1(t),\cdots,w_N(t))^\top, \quad B=(1,b_1,\cdots,b_N)^\top, && (4.25)\\ A=\begin{pmatrix} -\sigma^2+q & 0_{1\times N}\\ 0_{N\times 1} & \mathrm{diag}\{-\lambda_n+q\}_{n=1}^N \end{pmatrix}. && (4.26) \end{aligned}$$

By introducing the Artstein transformation (see [142], [143]) as follows

$$Z(t)=e^{Ar}W(t)+\int_{t-r}^{t} e^{A(t-s)}B\xi(s)ds, \tag{4.27}$$

one derives

$$\dot{Z}(t)=AZ(t)+B\xi(t), \quad Z(0)=e^{Ar}W(0). \tag{4.28}$$

4.3 Homogeneous stabilization of 1-D heat system by full-state feedback

In this part, we stabilize (4.28) using homogeneous control and next study the stability of the entire system (4.22)-(4.24).

4.3.1 Stabilization of the finite-dimensional part

Since $\{A, B\}$ is controllable (see [71], [76, Lemma 2.1]) then according to Section 2.2.3, the linear algebraic equation

$$AG_0 - G_0A + BY_0 = A, \quad G_0B = 0, \tag{4.29}$$

has solutions $Y_0 \in \mathbb{R}^{1\times(N+1)}$, $G_0 \in \mathbb{R}^{(N+1)\times(N+1)}$ such that the matrix $G_0 - I_{N+1}$ is invertible , the matrix $G_{\mathbf{d}} := I_{N+1} + \mu G_0$ is anti-Hurwitz for any $\mu \in (-1, 0)$ and the matrix $A_0 = A + BK_0$ with $K_0 = Y_0(G_0 - I_{N+1})^{-1}$ satisfies the identity

$$A_0G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I_{N+1})A_0, \quad G_{\mathbf{d}}B = B. \tag{4.30}$$

Moreover, the linear matrix inequalities

$$\begin{aligned} &(A_0+\rho G_{\mathbf{d}})X + X(A_0+\rho G_{\mathbf{d}})^\top + BY + Y^\top B^\top \preceq 0, \\ &G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top \succ 2(1+\mu)X, \qquad X = X^\top \succ 0, \end{aligned} \tag{4.31}$$

have solutions X, Y for any $\rho > 0$. Recall that $\mathbf{d}$ is the dilation generated by $G_{\mathbf{d}}$. The system (4.28) with the continuous feedback law

$$\xi(Z(t)) = K_0Z(t) + \mathcal{N}(Z(t)), \tag{4.32}$$

$$\mathcal{N}(Z(t)) = \|Z(t)\|_{\mathbf{d}}^{1+\mu} K\mathbf{d}(-\ln(\|Z(t)\|_{\mathbf{d}}))Z(t), \tag{4.33}$$

where $K = YX^{-1}$, X, Y being solutions of (4.31) for some $\rho > 0$ is $\mathbf{d}$-homogeneous of degree μ and globally finite-time stable

$$Z(t) = 0, \quad \forall t \geq T := \frac{\|Z(0)\|_{\mathbf{d}}^{-\mu}}{-\rho\mu}. \tag{4.34}$$

Along the solutions of the closed-loop system (4.28) and (4.32)

$$\frac{d\|Z(t)\|_{\mathbf{d}}}{dt} \leq -\rho\|Z(t)\|_{\mathbf{d}}^{1+\mu}, \tag{4.35}$$

for all $t \geq 0$. Let

$$\nu = \frac{\lambda_{\max}(X^{-1/2}G_{\mathbf{d}}X^{1/2} + X^{1/2}G_{\mathbf{d}}^\top X^{-1/2})}{2}, \quad \tau = \frac{\lambda_{\min}(X^{-1/2}G_{\mathbf{d}}X^{1/2} + X^{1/2}G_{\mathbf{d}}^\top X^{-1/2})}{2}. \tag{4.36}$$

Since the control system (4.28) is uncontrolled for $t \le 0$, one considers the following feedback law

$$\xi(t) = \begin{cases} 0, & \text{if} \quad t \le 0, \\ K_0 Z(t) + \mathcal{N}(Z(t)) & \text{if} \quad t > 0. \end{cases} \tag{4.37}$$

Then, the closed-loop system (4.23),(4.24), (4.28) and (4.37) is as follows:

$$\dot{Z}(t) = AZ(t) + B\Big(K_0 Z(t) + \mathcal{N}(Z(t))\Big), \tag{4.38}$$

$$\begin{aligned} \dot{w}_n(t) &= (-\lambda_n + q)w_n(t) + \chi_{(r,+\infty)}(t) b_n \Big(K_0 Z(t-r) + \mathcal{N}(Z(t-r))\Big), \quad n > N, \\ Z(0) &= e^{Ar}W(0), \quad w_n(0) = \langle z_0, \phi_n \rangle, \quad n > N, \end{aligned} \tag{4.39}$$

with $\chi_{(r,+\infty)}(t) = 0$ for $t \le r$ and $\chi_{(r,+\infty)}(t) = 1$ otherwise.

4.3.2 Stability of entire system

For H^1 stability analysis of the closed-loop system (4.38), we define the following Lyapunov functional :

$$V(t) = \gamma_1 \Omega(\|Z(t)\|_{\mathbf{d}}) + \gamma_2 \int_{\max\{t-r,0\}}^{t} \Psi(\|Z(s)\|_{\mathbf{d}}) ds + \sum_{n>N} \lambda_n w_n^2(t), \quad \forall t \ge 0, \tag{4.40}$$

with

$$\Omega(s) = \begin{cases} \frac{1}{2+\mu} s^{2+\mu}, & \text{if} \quad s \le 1, \\ \frac{1}{2\nu-\mu} s^{2\nu-\mu} - \frac{1}{2\nu-\mu} + \frac{1}{2+\mu}, & \text{if} \quad s \ge 1, \end{cases}, \Psi(s) = \begin{cases} s^{2(1+\mu)}, & \text{if} \quad s \le 1, \\ s^{2\nu}, & \text{if} \quad s \ge 1. \end{cases} \tag{4.41}$$

and $\gamma_1, \gamma_2 > 0$, ν is defined in (4.36). Note that $\Psi \in \mathcal{K}_\infty$ and using $1 + \mu < \tau$ (from Proposition 2.1) then $2\nu - \mu > 2 + \mu$ and $\Omega \in \mathcal{K}_\infty$.

Lemma 4.1
There exists $\eta_1 \in \mathcal{K}_\infty$ such that for all $t \ge 0$,

$$\eta_1(U^2(t-r) + \|w(t,\cdot)\|^2_{H^1(0,L)}) \le V(t). \tag{4.42}$$

Proof. Since $w(t,\cdot) \in H_0^1(0,L)$ then there exists $C_1 > 0$ such that

$$\|w(t,\cdot)\|^2_{H^1(0,L)} \le C_1 \sum_{n=1}^{\infty} \lambda_n w_n^2(t), \tag{4.43}$$

which implies that

$$U^2(t-r) + \|w(t,\cdot)\|^2_{H^1(0,L)} \le C_2\Big(|W(t)|^2 + \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t)\Big), \tag{4.44}$$

with $C_2 = \max\{C_1, C_1\lambda_{N+1} + 1\}$. From (4.27), one has

$$W(t) = e^{-Ar}Z(t) - \textstyle\int_{\max\{t-r,0\}}^{t} e^{A(t-s-r)}B\Big(K_0 Z(s) + \mathcal{N}(Z(s))\Big)ds, \quad \forall t \ge 0. \tag{4.45}$$

From Proposition 2.1 (and using $1+\mu < \tau < \nu$), one has

$$\|Z(t)\|_{X^{-1}} \le \Psi^{\frac{1}{2}}(\|Z(t)\|_{\mathbf{d}}). \tag{4.46}$$

Using (4.46) and the fact that $\|\mathbf{d}(-\ln\|Z(t)\|_{\mathbf{d}})Z(t)\|_{X^{-1}} = 1$, one derives for all $t \ge 0$,

$$|W(t)|^2 \le M_1\Big(\Psi(\|Z(t)\|_{\mathbf{d}}) + \int_{\max\{t-r,0\}}^{t} \Psi(\|Z(s)\|_{\mathbf{d}})ds\Big), \tag{4.47}$$

with $M_1 = 2e^{2|A|r}\lambda_{min}(X^{-1})^{-1}\max\{1, r|B|^2M_2\}$ and $M_2 = 4\max\{|K_0^\top|^2, |K^\top|^2\}$. This means for all $t \ge 0$,

$$|W(t)|^2 \le M_1\Big(\Psi \circ \Omega^{-1}(\tfrac{1}{\gamma_1}V(t)) + \tfrac{1}{\gamma_2}V(t)\Big). \tag{4.48}$$

Using (4.44) and (4.48), the proof is complete. □

Let for all $t \ge 0$,

$$V_1(t) = \|Z(t)\|_{\mathbf{d}}, \quad V_2(t) = \sum_{n>N} \lambda_n w_n^2(t), \quad V_3(t) = \chi_{(r,+\infty)}(t)\|Z(t-r)\|_{\mathbf{d}}. \tag{4.49}$$

Computing the time-derivatives along the solutions of closed-loop system (4.38), for all $t > r$, one has from (4.35) and (4.41)

$$\frac{d\Omega(V_1(t))}{dt} \le -\rho\Psi(V_1(t)), \tag{4.50}$$

and

$$\tfrac{d}{dt}\int_{t-r}^{t} \Psi(V_1(s))ds = \Psi(V_1(t)) - \Psi(V_3(t)). \tag{4.51}$$

On other hand, along solution of closed-loop system (4.38) for all $t > r$,

$$\dot{V}_2(t) = 2\sum_{n>N} \lambda_n w_n(t)\Big((-\lambda_n + q)w_n(t) + b_n\xi(t-r)\Big). \tag{4.52}$$

Using the Young inequality, for all $\gamma_3 > \frac{\lambda_{N+1}}{2(\lambda_{N+1}-q)}$ the equality (4.52) implies for all $t > r$,

$$\dot{V}_2(t) \leq 2 \sum_{n>N} \lambda_n \Big((-1 + \tfrac{1}{2\gamma_3})\lambda_n + q \Big) w_n^2(t) + \gamma_3 M_3 \Psi(V_3(t)), \tag{4.53}$$

with $M_3 = \lambda_{min}(X^{-1})^{-1} M_2 \Big(\sum_{n>N} |b_n|^2 \Big)$. Using (4.47), one derives for all $t > r$,

$$\dot{V}_2(t) \leq -2\theta_{N,1} V_2(t) + \gamma_3 M_3 \Psi(V_3(t)), \tag{4.54}$$

with

$$\theta_{N,1} = (1 - \tfrac{1}{2\gamma_3})\lambda_{N+1} - q > 0. \tag{4.55}$$

From (4.40), along the solution of closed-loop system (4.38), one has for all $t > r$,

$$\dot{V}(t) = \gamma_1 \frac{d\Omega(V_1(t))}{dt} + \gamma_2 \tfrac{d}{dt} \int_{t-r}^{t} \Psi(V_1(s)) ds + \dot{V}_2(t). \tag{4.56}$$

Then using (4.50),(4.51) and (4.54) one derives for all $t > r$,

$$\dot{V}(t) \leq -2\theta_{N,1} V_2(t) + (\gamma_3 M_3 - \gamma_2) \Psi(V_3(t)) + (\gamma_2 - \gamma_1 \rho) \Psi(V_1(t)). \tag{4.57}$$

Since from (4.35) the map $t \mapsto V_1(t)$ is decreasing on $[0, +\infty)$ then for all $t > r$, and for all $s \in [t-r, t]$,

$$\Psi(V_1(t)) \leq \Psi(V_1(s)) \leq \Psi(V_3(t)). \tag{4.58}$$

This implies that for all $t > r$

$$\Psi(V_3(t)) \geq \tfrac{1}{r} \int_{t-r}^{t} \Psi(V_1(s)) ds. \tag{4.59}$$

In addition, for $\gamma_2 > \gamma_3 M_3$ and $\gamma_1 > \frac{\gamma_2}{\rho}$, one has

$$\tilde{\rho} = \min\{2\theta_{N,1}, \gamma_1 \rho - \gamma_2, \tfrac{\gamma_2 - \gamma_3 M_3}{r}\} > 0, \tag{4.60}$$

and then the inequality (4.57) implies that for all $t > r$

$$\dot{V}(t) \leq -\tilde{\rho} \Big(V_2(t) + \Psi(V_1(t)) + \int_{t-r}^{t} \Psi(V_1(s)) ds \Big). \tag{4.61}$$

By using the Lemma A.1 (in the Appendix), one has the following inequality

$$\int_{t-r}^{t} \Psi(V_1(s)) ds + V_2(t) + \Psi \circ \Omega^{-1}(\Omega(V_1(t))) \geq \eta_2(V(t)), \tag{4.62}$$

where η_2 is the $\mathcal{K}_\infty$ function given by

$$\eta_2(s) = \min\{\tfrac{s}{3}, \Psi \circ \Omega^{-1}\Big(\tfrac{s}{3\gamma_1}\Big), \tfrac{s}{3\gamma_2}\}, \quad \forall s \geq 0. \tag{4.63}$$

Finally, one concludes that, for all $t > r$,

$$\dot{V}(t) \leq -\tilde{\rho}\eta_2(V(t)), \tag{4.64}$$

and then there exists $\beta_1 \in \mathcal{KL}$ such that, for all $t \geq r$

$$V(t) \leq \beta_1(V(r), t-r). \tag{4.65}$$

Consider now the case $t < r$. Along the solution of closed-loop system (4.38), for all $t \in [0, r)$ one has:

$$\dot{V}(t) \leq (-\rho\gamma_1 + \gamma_2)\Psi(V_1(t)) + 2(-\lambda_{N+1} + q)\sum_{n>N}\lambda_n w_n^2(t). \tag{4.66}$$

Since from (4.60) $\rho\gamma_1 > \gamma_2$, one derives, for all $t \in [0, r), \dot{V}(t) \leq 0$, which implies that for all $t \in [0, r]$,

$$V(t) \leq \gamma_1\Omega(\|Z(0)\|_{\mathbf{d}}) + \sum_{n>N}\lambda_n w_n^2(0). \tag{4.67}$$

Since $\Omega \in \mathcal{K}_\infty$, using Proposition 2.1 and the fact that $Z(0) = e^{Ar}W(0)$ there exists $\eta_3 \in \mathcal{K}_\infty$ such that for all $t \in [0, r]$

$$V(t) \leq \eta_3(|W(0)|^2 + \sum_{n>N}\lambda_n w_n^2(0)). \tag{4.68}$$

Using the fact that $z_0 \in H_0^1(0, L)$ and $U(-r) = 0$, one has

$$\min\{\lambda_1, 1\}\Big(|W(0)|^2 + \sum_{n>N}\lambda_n w_n^2(0)\Big) \leq \|z_0\|^2_{H^1(0,L)}. \tag{4.69}$$

One concludes that there exists $\eta_4 \in \mathcal{K}_\infty$ such that

$$\forall t \in [0, r], \quad V(t) \leq \eta_4(\|z_0\|^2_{H^1(0,L)}). \tag{4.70}$$

From (4.65) and (4.70), there exists $\beta_2 \in \mathcal{KL}$ such that

$$\forall t \geq 0, \quad V(t) \leq \beta_2(\|z_0\|^2_{H^1(0,L)}, t). \tag{4.71}$$

Summarizing, one arrives at the following main result:

Theorem 4.1
There exists $\beta \in \mathcal{KL}$ such that for any initial condition $z_0 \in H_0^1(0, L)$, the solution of closed-loop system (4.1)-(4.3) with the homogeneous feedback law (4.12) and (4.37) where Z is given by (4.22) and (4.27) satisfies :

$$\|z(t, \cdot)\|^2_{H^1(0,L)} \leq \beta(\|z_0\|^2_{H^1(0,L)}, t), \quad \forall t \geq 0. \tag{4.72}$$

Moreover, for all $t \geq T_r$, the control input U and z satisfy

$$U(t-r) = 0, \quad \langle z(t,\cdot), \phi_n \rangle = 0, \quad n = 1, ..., N, \tag{4.73}$$

$$\|z(t,\cdot)\|^2_{H^1(0,L)} \leq e^{-2\delta(t-T_r)} \|z(T_r,\cdot)\|^2_{H^1(0,L)}, \tag{4.74}$$

with $T_r := \frac{\|e^{Ar} z_0^N\|_{\mathbf{d}}^{-\mu}}{-\rho\mu} + r$, $z_0^N = (0, \langle z_0, \phi_1 \rangle, \cdots, \langle z_0, \phi_N \rangle)^\top$.

Proof. The inequality (4.71) together with Lemma 4.1 imply that there exists $\beta_3 \in \mathcal{KL}$ such that

$$U^2(t-r) + \|w(t,\cdot)\|^2_{H^1(0,L)} \leq \beta_3(\|z_0\|^2_{H^1(0,L)}, t), \forall t \geq 0. \tag{4.75}$$

From (4.7), one has for all $t \geq 0$,

$$\|z(t,\cdot)\|^2_{H^1(0,L)} \leq 2\max\{1, \|\kappa\|^2_{H^1(0,L)}\}(\|w(t,\cdot)\|^2_{H^1(0,L)} + U^2(t-r)), \tag{4.76}$$

then, one concludes that there exists $\beta \in \mathcal{KL}$ such that

$$\|z(t,\cdot)\|^2_{H^1(0,L)} \leq \beta(\|z_0\|^2_{H^1(0,L)}, t), \quad \forall t \geq 0. \tag{4.77}$$

Since for all $t \geq T$, $Z(t) = 0$ then using (4.27) one obtains,

$$W(t) = 0, \quad \forall t \geq T_r := T + r. \tag{4.78}$$

Thus using (4.7), for all $t \geq T_r$,

$$\begin{aligned} &U(t-r) = 0, \quad \langle z(t,\cdot), \phi_n \rangle = 0, \quad n = 1, \cdots, N \\ &\tfrac{d}{dt}\langle z(t,\cdot), \phi_n \rangle = (-\lambda_n + q)\langle z(t,\cdot), \phi_n \rangle, \quad n > N. \end{aligned} \tag{4.79}$$

Using (4.21), one derives for all $t \geq T_r$,

$$\sum_{n>N} \lambda_n |\langle z(t,\cdot), \phi_n \rangle|^2 \leq e^{-2\delta(t-T_r)} \sum_{n>N} \lambda_n |\langle z(T_r,\cdot), \phi_n \rangle|^2, \tag{4.80}$$

which together the fact that for all $t \geq T_r, z(t,\cdot) \in H_0^1(0,L)$, the proof is complete. □

Remark 4.1

For $\mu = 0$, the homogeneous feedback control $\xi(Z(t)) = (K_0 + K)Z(t)$ is a linear one. In this case, $\mathbf{d}(s) = e^s I_{N+1}$,$G_{\mathbf{d}} = I_{N+1}$ and then $\nu = \tau = 1$. One recovers the classical Lyapunov functional considered in the linear case [61]:

$$V(t) = \tfrac{\gamma_1}{2} Z(t)^\top X^{-1} Z(t) + \sum_{n>N} \lambda_n w_n^2(t) + \gamma_2 \int_{\max\{t-r,0\}}^t Z(s)^\top X^{-1} Z(s) ds. \tag{4.81}$$

4.4 Numerical Simulations

We perform numerical simulations on the system (4.1)-(4.4), by using system (4.12)-(4.14) and transformation (4.7). The initial condition and parameters are: $L = \pi$, $z_0(x) = \frac{x}{L}(L - x)$, $r = 2$, $q = 1.25$ and $N = 1$. We consider (4.15) with $M = 10$ truncated basis. We use the control toolbox on Matlab to compute the linear control gains:

$$\begin{aligned} K_{l_1} &= (-26.6667 - 17.3114), \\ K_{l_2} &= (-34.3467 - 23.4025), \end{aligned} \tag{4.82}$$

with which we compute the linear control ([82]) given by

$$U(t-r) = \chi_{(r,+\infty)} K_l \int_0^{t-r} e^{(-\sigma^2+q)(t-s-r)} Z(s) ds. \tag{4.83}$$

We use the Homogeneous Control Systems (HCS) Toolbox for MATLAB [130] to compute $\|\cdot\|_{\mathbf{d}}, K_0, K, G_{\mathbf{d}}$:

$$\begin{aligned} &K_0 = (-1.3333, -0.0783), K = (-25.3333, -17.2331), \\ &\mu = -0.2, \tau = 1, \nu = 1.2, , G_{\mathbf{d}} = \begin{pmatrix} 0.9333 & -0.0627 \\ 0.2837 & 1.2667 \end{pmatrix}. \end{aligned} \tag{4.84}$$

Figure 4.1 shows the simulations in logarithmic scale of the norm $\|z(t,\cdot)\|_{H^1(0,L)}$ of closed-loop system (4.1)-(4.3) with linear control (4.83) (with K_{l_1}) in blue line, with high-gain linear control (4.83) (with K_{l_2}) in black dashed line and with homogeneous control (4.12) and (4.37) in red line (at the left) and during the initial phase of stabilization (at the right). Figure 4.2 shows the time-evolution of the linear predictor feedback law (4.83) with K_{l_1} in blue line, the linear predictor feedback law (4.83) with K_{l_2} in black dashed line and the with homogeneous control predictor feedback law (4.12) and (4.37) in red line (at the left) and the solution of the closed-loop system (4.1)-(4.3) with homogeneous control (4.12) and (4.37) (at the right).

We can observe the performance of the closed-loop system under both a high-gain linear control and homogeneous control while achieving a prescribed precision,e.g.,

$$\|z(t,\cdot)\|_{H^1(0,L)} \le \epsilon := 4.10^{-3}, \quad \forall t \ge T_p := 5.7. \tag{4.85}$$

with the control restriction

$$\sup_{t>0} |U(t-r)| \le \bar{U} := 3. \tag{4.86}$$

However, from Figure 4.2 we can observe that only the homogeneous predictor feedback allows to achieve (4.85) with the control restriction (4.86). Indeed, one can also observe in Figure 4.1 and Figure 4.2 the price to pay of using high-gain linear control: achieving

(4.85) implies a large deviation of the solution closed-loop system (a peaking) and an overshoot of the control signal during the initial phase of stabilization. Additionally, we can observe in Figure 4.2 that both the linear controller and the homogeneous one exhibit a similar overshoot value while verifying (4.86), but the linear controller violates condition (4.85), whereas the homogeneous one does not.

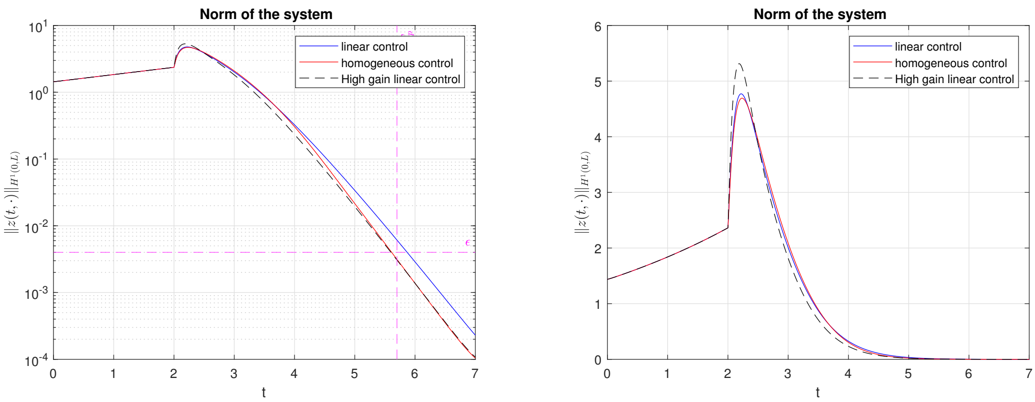

Figure 4.1: Evolution of $\|z(t,\cdot)\|_{H^1(0,L)}$ in a logarithmic scale (at the left) of the closed-loop system (4.1)-(4.3) with linear control (4.83) (blue and black dashed line) and homogeneous control (4.12) and (4.37) (red line) and during the initial phase of the stabilization (at the right)

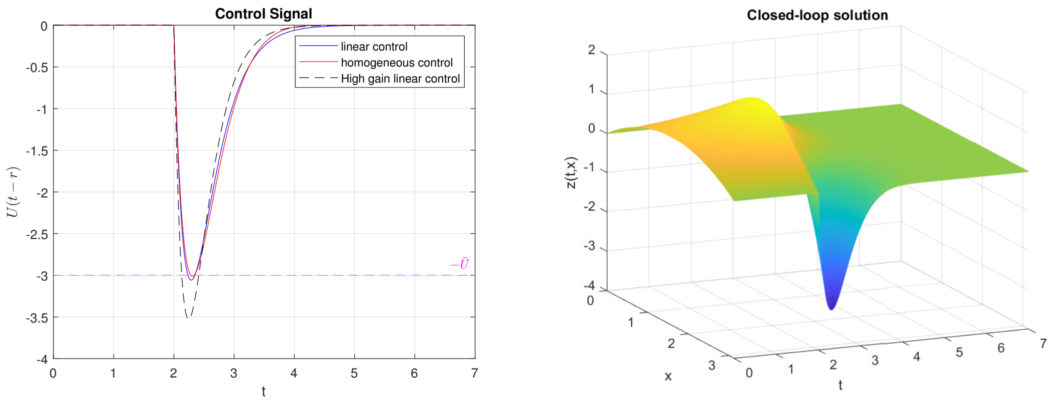

Figure 4.2: Time-evolution of the control signal $U(t-r)$ of the linear control (4.83) (black dashed and blue lines) and homogeneous control (4.12) and (4.37) (red line) and the solution $z(t,x)$ of closed-loop system (4.1)-(4.3) with homogeneous control (4.12) and (4.37).

Conclusion

In this chapter, we have designed a homogeneous boundary predictor feedback law for a 1D reaction-diffusion equation with input delay. We construct a suitable Lyapunov functional to prove the asymptotic stability of the closed-loop system. The simulations showed that the homogeneous controller makes it possible to obtain faster convergence without peaking and with less overshoot of the controller.

Part II

Homogeneous control design for nonlinear models in finite and infinite dimensional settings

CHAPTER 5

Locally homogeneous finite-time stabilization of quasi-Linear systems

Contents

In this chapter, an algorithm of a local finite-time control design is developed for a class of quasi-linear systems. The design procedure is essentially based on the concept of generalized homogeneity and the convex embedding technique. The adjustment of control parameters is realized by solving a system of linear matrix inequalities (LMIs). Theoretical results are supported by numerical simulations.

5.1 Introduction

In Chapter 1, we were motivated by models coming from fluid mechanics, namely the Navier-Stokes equation. Model reduction typically leads to the viscous Burgers equation, and in turn to bilinear systems. As shown in [144], the finite stabilization of affine-in-control systems can be realised using Sontag's universal formula [145]. However, it requires a design of an appropriate control Lyapunov function. The convex embedding approach for homogeneous affine-in-control systems has been introduced recently [146]. It essentially uses the homogeneity (symmetry) of the vector field to define a stability condition in terms of LMIs.

5.2 Problem statement

Let us consider the system

$$\dot{x} = (L + W(x))x + (F + Q(x))\xi + g, \quad x(0) = x_0, \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the system state, $\xi \in \bar{\mathbb{R}}^m_+$ is the control input, which is assumed to be positive (due to the physical constraints), $g \in \mathbb{R}^n$ is a known constant vector , $L \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times m}$ are known constant matrices and $W(x) \in \mathbb{R}^{n\times n}$ are state-dependent matrices, whose values can be computed for any $x \in \mathbb{R}^n$.

Assume that the matrix-valued functions $W : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ and $Q : \mathbb{R}^n \mapsto \mathbb{R}^{n\times m}$ are linear. The latter implies that $\exists \tilde{W} : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ and $\tilde{Q} : \mathbb{R}^m \mapsto \mathbb{R}^{n\times n}$ such that

$$W(x)y = \tilde{W}(y)x, \;\; Q(x)\xi = \tilde{Q}(\xi)x, \;\; \forall x, y \in \mathbb{R}^n, \;\; \forall \xi \in \mathbb{R}^m. \tag{5.2}$$

Assumption 5.1
Given $x^* \in \mathbb{R}^n$ there exists $\xi^* \in \mathbb{R}^m_+$ such that

$$(L + W(x^*))x^* + (F + Q(x^*))\xi^* + g = 0.$$

The state $x = x^*$ is an equilibrium (possibly unstable) of the system (5.1) with the control $\xi = \xi^*$. We consider x^* as a desired set point for tracking by the system.

Given $r > 0$, $x^* \in \mathbb{R}^n, \xi^* \in \mathbb{R}^m_+$, the *control aim* is to design a finite-time stabilizing feedback $\xi = \xi(x)$ such that for any $x_0 \in \mathbb{R}^n : |x_0 - x^*| \leq r$ the trajectory of the closed-loop system satisfies

$$x(t) \to x^* \quad \text{as} \quad t \to T(x_0), \tag{5.3}$$

$$\xi(x(t)) \in \bar{\mathbb{R}}^m_+, \quad \forall t \geq 0, \tag{5.4}$$

where $T : \mathbb{R}^n \mapsto \bar{\mathbb{R}}_+$ is the settling-time function.

5.3 Control design and stability analysis

Let us first derive the error equation and denote

$$\xi = \xi^* + u, \quad z = x - x^*, \tag{5.5}$$

where $u \in \mathbb{R}^m$ is a new (virtual) control such that $\xi^* + u \in \bar{\mathbb{R}}^m_+$. We derive

$$\begin{aligned}
\dot{z} &= (L + W(x))x + F\xi^* + Fu + Q(x)\xi^* + Q(x)u + g \\
&= Lx + W(x)x - Lx^* - W(x^*)x^* + Fu - Q(x^*)\xi^* + Q(x)\xi^* + Q(x)u \\
&= Lz + W(x - x^*)x + W(x^*)z + (F + Q(x))u + Q(x - x^*)\xi^*. \qquad (5.6)
\end{aligned}$$

Therefore, from (5.6) and using (5.2) we obtain the following bilinear system:

$$\dot{z} = Az + Bu + D(z, u)z, \qquad (5.7)$$

with

$$A = L + \tilde{W}(x^*) + W(x^*) + \tilde{Q}(\xi^*), \qquad (5.8)$$

$$B = F + Q(x^*), \qquad (5.9)$$

and

$$D(z, u) = W(z) + \tilde{Q}(u). \qquad (5.10)$$

According to Section 2.2.3, if the pair $\{A, B\}$ is controllable there exists a solution $Y_0 \in \mathbb{R}^{m\times n}$, $G_0 \in \mathbb{R}^{n\times n}$ of the linear algebraic equation

$$AG_0 - G_0A + BY_0 = A, \quad G_0B = 0 \qquad (5.11)$$

such that the matrix $G_{\mathbf{d}} := I_n + \mu G_0$ is anti-Hurwitz for any $\mu \in (-1, 0)$, and the matrix $A_0 = A + BK_0$ with $K_0 := Y_0(G_0 - I_n)^{-1}$ satisfies the identity

$$A_0G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I_n)A_0, \quad G_{\mathbf{d}}B = B. \qquad (5.12)$$

Theorem 5.1
Let the pair $\{A, B\}$ be controllable, the matrices K_0, $G_{\mathbf{d}}$ be defined by solving the equation (5.11) and $\mu \in [-1, 0)$. If

1) *$G_{\mathbf{d}}Q(z) = Q(z)$ for all $s \in \mathbb{R}$ and all $z \in \mathbb{R}^n$, where $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$ is the linear dilation;*

2) *for some $\rho, \beta, \omega_1, \omega_2 \in \mathbb{R}_+$ the matrices $X = X^\top \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{m\times n}$ satisfy the LMIs*

$$A_0X + XA_0^\top + BY + Y^\top B^\top + \rho(G_{\mathbf{d}}X + XG_{\mathbf{d}}) + (\omega_1 + \omega_2)X \preceq 0, \qquad (5.13)$$

$$\begin{pmatrix} \omega_1 X & rY^\top Q(h_k)^\top \\ rQ(h_k)Y & \omega_1 X \end{pmatrix} \succeq 0, \quad k = 1, \cdots, n \qquad (5.14)$$

$$\begin{pmatrix} \omega_2 X & rX(W(h_k) + Q(h_k)K_0)^\top \\ r(W(h_k) + Q(h_k)K_0)X & \omega_2 X \end{pmatrix} \succeq 0, \qquad (5.15)$$

$$\begin{pmatrix} 0.5\xi_j^* X & XK_0^\top e_j \\ e_j^\top K_0 X & 0.5\xi_j^* \end{pmatrix} \succeq 0, \quad \begin{pmatrix} 0.5\xi_j^* X & Y^\top e_j \\ e_j^\top Y & 0.5\xi_j^* \end{pmatrix} \succeq 0, \quad j=1,\cdots,m \tag{5.16}$$

$$2(2\beta-\mu)X \succ G_{\boldsymbol{d}}X + XG_{\boldsymbol{d}}^\top \succeq 2\beta X, \tag{5.17}$$

$$\tfrac{r^2}{n} I_n \succeq X \succ 0, \tag{5.18}$$

where $\lambda_j \geq 0$, $h_k = (0,\ldots,1,\ldots,0)^\top$ *is the unit Euclidean vector in* $\mathbb{R}^n$, $e_j = (0,\ldots,1,\ldots,0)^\top$ *is the unit Euclidean vector in* $\mathbb{R}^m$ *and* ξ_j^* *is the* j*-th element of the vector* $\xi^* \in \mathbb{R}_+^m$, *then the system* (5.7) *with the feedback control*

$$u = K_0 z + \|z\|_{\boldsymbol{d}}^{1+\mu} K\boldsymbol{d}(-\ln\|z\|_{\boldsymbol{d}})z, \tag{5.19}$$

$$K_0 := Y_0(G_0 - I_n)^{-1}, \quad K := YX^{-1},$$

is locally finite-time stable with the invariant ellipsoid

$$\mathcal{E}(X) := \{z \in \mathbb{R}^n : z^\top X^{-1} z \leq 1\}; \tag{5.20}$$

belonging to the attraction domain. Moreover, for $|z(0)| \leq r$, *we have*

$$0 \leq \xi^* + u(z(t)), \quad \forall t \geq 0,$$

$$z(t) \to 0 \quad as \quad t \to T(z(0)),$$

where $T(z(0)) \leq \frac{\|z(0)\|_{\boldsymbol{d}}^{-\mu}}{(-\mu\rho)}$ *with the canonical homogeneous norm* $\|\cdot\|_{\mathbf{d}}$ *induced by the weighted Euclidean norm* $\|z\| = \sqrt{z^\top X^{-1} z}$.

Remark 5.1
Given $r > 0$, *the system of LMIs* (5.13)-(5.17) *is feasible for* μ *sufficiently close to* 0 *provided that* $|rW(h_i)|$ *and* $|rQ(h_i)|$ *are small enough but* $\xi_j^* \in \mathbb{R}_+^n$ *is large enough.*

Indeed, taking $\beta = -3/4$ and any $X \succ 0$ one can see that the LMI (5.17) is fulfilled for μ sufficiently close to 0. The controllability of $\{A, B\}$ implies the feasibility of (5.13) (see, [111] for more details). For any fixed $\omega_1, \omega_2 \in \mathbb{R}_+$ and $X \succ 0$, the LMIs (5.14), (5.15) are feasible if $|rW(h_i)|$ and $|rQ(h_i)|$ are small enough. For any fixed $X \succ 0$, the LMI (5.16) is feasible if $\xi_j^* \in \mathbb{R}_+$ is large enough.

Therefore, the only critical restriction in the above theorem is $G_{\mathbf{d}}Q(z) = Q(z)$, but it may be fulfilled in some practical cases, for example, for a discretization of the one-dimensional viscous Burgers equation controlled on the boundary.

Proof. Using (2.22), for $V = \|z\|_{\mathbf{d}}$ we derive

$$\dot{V} = \|z\|_{\mathbf{d}} \frac{z^\top \mathbf{d}^\top(-\ln\|z\|_{\mathbf{d}})X^{-1}\mathbf{d}(-\ln\|z\|_{\mathbf{d}})(Az + Bu + D(z,u)z)}{z^\top \mathbf{d}^\top(-\ln\|z\|_{\mathbf{d}})X^{-1}G_{\mathbf{d}}\mathbf{d}(-\ln\|z\|_{\mathbf{d}})z}. \tag{5.21}$$

Using the homogeneity (namely, $A_0\mathbf{d}(s)=e^{\mu s}\mathbf{d}(s)A_0$ and $\mathbf{d}(s)B=e^sB$) we derive (see, [111] for more details)

$$\begin{aligned}&z^\top\mathbf{d}^\top(-\ln\|z\|_\mathbf{d})X^{-1}\mathbf{d}(-\ln\|z\|_\mathbf{d})(Az+Bu)\\&=\|z\|_\mathbf{d}^\mu z^\top\mathbf{d}^\top(-\ln\|z\|_\mathbf{d})X^{-1}(A_0+BK)\mathbf{d}(-\ln\|z\|_\mathbf{d})z,\end{aligned}\tag{5.22}$$

which in conjunction with condition (5.13), we obtain from (5.21) the following estimate:

$$\dot V\le-\rho V^{1+\mu}+\frac{-(\omega_1+\omega_2)\|z\|_\mathbf{d}^\mu+z^\top\mathbf{d}^\top(-\ln\|z\|_\mathbf{d})X^{-1}\mathbf{d}(-\ln\|z\|_\mathbf{d})D(z,u)z)}{z^\top\mathbf{d}^\top(-\ln\|z\|_\mathbf{d})X^{-1}G_\mathbf{d}\mathbf{d}(-\ln\|z\|_\mathbf{d})z}.\tag{5.23}$$

Notice that the term $D(z,u)z$ can be rewritten as follows:

$$\begin{aligned}D(z,u)z=&W(z)z+Q(z)u\\=&W(z)z+Q(z)K_0z+\|z\|_\mathbf{d}^{1+\mu}Q(z)K\mathbf{d}(-\ln\|z\|_\mathbf{d})z.\end{aligned}\tag{5.24}$$

Hence, using $\mathbf{d}(s)Q(z)=e^sQ(z)$ (since $G_\mathbf{d}Q(z)=Q(z)$) we obtain

$$\begin{aligned}z^\top\mathbf{d}^\top(-\ln\|z\|_\mathbf{d})X^{-1}\mathbf{d}(-\ln\|z\|_\mathbf{d})D(z,u)z=\|z\|_\mathbf{d}^\mu z^\top\mathbf{d}^\top(-\ln\|z\|_\mathbf{d})X^{-1}Q(z)K\mathbf{d}(-\ln\|z\|_\mathbf{d})z\\+z^\top\mathbf{d}^\top(-\ln\|z\|_\mathbf{d})X^{-1}\mathbf{d}(-\ln\|z\|_\mathbf{d})(W(z)+Q(z)K_0)z.\end{aligned}\tag{5.25}$$

Taking into account $z^\top\mathbf{d}^\top(-\ln\|z\|_\mathbf{d})X^{-1}\mathbf{d}(-\ln\|z\|_\mathbf{d})z=1$, and using Cauchy-Schwartz inequality we derive

$$z^\top\mathbf{d}^\top(-\ln\|z\|_\mathbf{d})X^{-1}Q(z)K\mathbf{d}(-\ln\|z\|_\mathbf{d})z\le\omega_1,\tag{5.26}$$

provided that $K^\top Q^\top(z)X^{-1}Q(z)K\le\omega_1^2X^{-1}$. Let us show that for $\|z\|_1\le r$, one has

$$K^\top Q^\top(z)X^{-1}Q(z)K\le\omega_1^2X^{-1}.\tag{5.27}$$

Indeed, using the Schur complement, inequality (5.27) can be rewritten as follows:

$$\begin{pmatrix}\omega_1X & rY^\top Q(z/r)^\top\\ rQ(z/r)Y & \omega_1X\end{pmatrix}\succeq0,\tag{5.28}$$

By Carathéodory lemma on convex hull, for any z such that $\|z\|_1\le1$, there exist scalars $\alpha_k^\pm\ge0$ such that $\sum_{k=1}^n\alpha_k^++\alpha_k^-=1$ and

$$z=\sum_{k=1}^n(\alpha_k^+-\alpha_k^-)h_k.\tag{5.29}$$

where $h_k=(0,\ldots,1,\ldots,0)^\top\in\mathbb{R}^n$ is the unit Euclidean vector. Hence, for $\|z\|_1\le r$ we derive

$$\begin{pmatrix}\omega_1X & rYQ(z/r)^\top\\ rQ(z/r)Y & \omega_1X\end{pmatrix}=\sum_{k=1}^n\alpha_k^+\begin{pmatrix}\omega_1X & rYQ(h_k)^\top\\ rQ(h_k)Y & \omega_1X\end{pmatrix}+\alpha_k^-\begin{pmatrix}\omega_1X & rYQ(-h_k)^\top\\ rQ(-h_k)Y & \omega_1X\end{pmatrix},\tag{5.30}$$

Taking into account $Q(-z) = -Q(z), \forall z \in \mathbb{R}^n$ we derive that (5.14) implies (5.26) for any $z : \|z\|_1 \leq r$.

Similarly, we derive

$$\frac{z^\top \mathbf{d}^\top(-\ln\|z\|_{\mathbf{d}})X^{-1}\mathbf{d}(-\ln\|z\|_{\mathbf{d}})(W(z)+Q(z)K_0)z}{\omega_2\|z\|_{\mathbf{d}}^{\mu}} \leq 1 \tag{5.31}$$

provided that

$$\left\|\frac{1}{\omega_2\|z\|_{\mathbf{d}}^{-\mu}}\mathbf{d}(-\ln\|z\|_{\mathbf{d}})(W(z)+Q(z)K_0)z\right\| \leq 1.$$

The inequality $2(2\beta-\mu)X \succ G_{\mathbf{d}}X+XG_{\mathbf{d}}^\top$ guarantees that the matrix $G_{\tilde{\mathbf{d}}} = (2\beta-\mu)I_n - G_{\mathbf{d}}$ is anti-Hurwitz and the dilation $\tilde{\mathbf{d}}(s) = e^{sG_{\tilde{\mathbf{d}}}}$ is monotone with respect to the norm $\|\cdot\|$. Hence, we have

$$\left\|\|z\|_{\mathbf{d}}^{2\beta-\mu}\mathbf{d}(-\ln\|z\|_{\mathbf{d}})\right\| = \|\tilde{\mathbf{d}}(\ln\|z\|_{\mathbf{d}})\| \leq 1 \tag{5.32}$$

for all $z : \|z\|_{\mathbf{d}} \leq 1$ (which is equivalent to $\|z\| \leq 1$). Thus, the inequality

$$\|\omega_2^{-1}(W(z/\|z\|_{\mathbf{d}}^{\beta}) + Q(z/\|z\|_{\mathbf{d}}^{\beta})K_0)z/\|z\|_{\mathbf{d}}^{\beta}\| \leq 1 \tag{5.33}$$

implies (5.31). The LMI $G_{\mathbf{d}}X+XG_{\mathbf{d}}^\top \succeq 2\beta X$ implies $\|z\| \leq \|z\|_{\mathbf{d}}^{\beta}$ provided that $\|z\|_{\mathbf{d}} \leq 1$. In this case, for $\|z\| \leq 1$ the inequality (5.33) follows from $\|\omega_2^{-1}(W(z)+Q(z)K_0)z\| \leq \|z\|^2 \Leftarrow$

$$(W(z)+Q(z)K_0)^\top(W(z)+Q(z)K_0) \leq \omega_2 X^{-1}. \tag{5.34}$$

Repeating the above consideration, we derive that LMI (5.14) implies (5.31) provided that $\|z\| \leq 1$ and $\|z\|_1 \leq r$. Therefore,

$$\dot{V} \leq -\rho V^{1+\mu}, \tag{5.35}$$

where $\|z\|_1 \leq r$ and $\|z\| \leq 1$. Since $r^2 n^{-1} I_n \succeq X$, then

$$\|z\|_1 r^{-1} \leq \tfrac{\sqrt{n}}{r}|z| = \sqrt{r^{-2}nz^\top z} \leq \sqrt{z^\top X^{-1}z} \leq 1,$$

for $\|z\| \leq 1$. Therefore, the closed-loop system is locally finite-time stable and the ellipsoid $\mathcal{E}(X) = \{z : \|z\| \leq 1\}$ belongs to the domain of attraction. Moreover, since $|z| \leq \|z\|_1 \leq r$ for $\|z\| \leq 1$ then $|z(0)| \leq r \Rightarrow z(t) \to 0$ with in the finite time $T(z(0)) \leq (-\mu\rho)^{-1}V(z(0))^{-\mu}$.

The LMI (5.16) and the inequality $z^\top X^{-1}z \leq 1$ guarantees

$$z^\top K_0^\top e_j e_j^\top K_0 z \leq (\xi_j^*)^2/4$$

or, equivalently, $|e_j^\top K_0 z(t)| \leq \xi_j^*/2$ for all $t \geq 0$. Similarly, taking into account

$$z^\top \mathbf{d}^\top(-\ln\|z\|_{\mathbf{d}})X^{-1}\mathbf{d}(-\ln\|z\|_{\mathbf{d}})z = 1,$$

we derive

$$z^\top \mathbf{d}^\top(-\ln\|z\|_{\mathbf{d}})X^{-1}Y^\top e_j e_j^\top Y X^{-1}\mathbf{d}(-\ln\|z\|_{\mathbf{d}})z \leq \tfrac{(\xi_j^*)^2}{4}$$

or, equivalently, $|e_j^\top K\mathbf{d}(-\ln\|z(t)\|_{\mathbf{d}})z(t)| \leq \xi_j^*/2$ for all $t \geq 0$. Since $\|z\|_{\mathbf{d}} \leq 1$ then $\xi_j^* + e_j^\top K_0 z + \|z\|_{\mathbf{d}} e_j^\top K\mathbf{d}(-\ln\|z(t)\|_{\mathbf{d}})z(t) \geq 0$ for all $t \geq 0$. The proof is complete.

The feasibility of the system (5.13)-(5.17) can be studied similarly to [111].

5.4 Simulations

Let us consider the error system (5.7) with the parameters

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}, \quad W(z) = \varepsilon \cdot \operatorname{diag}\left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} z \right\}, \tag{5.36}$$

and $B = (0.5, 0, 0, 0, 0)^\top$, $Q(z) = \varepsilon(-0.5z_1, 0, 0, 0, 0)^\top$ where $z = (z_1, z_2, z_3, z_4, z_5)^\top \in \mathbb{R}^5$. Solving the algebraic equation (5.11) we derive

$$K_0 = \begin{pmatrix} 20 & -88 & 220 & -330 & 264 \end{pmatrix}, \quad G_0 = \begin{pmatrix} 0 & -8 & 6 & -4 & 2 \\ 0 & -1 & -6 & 4 & -2 \\ 0 & 0 & -2 & -4 & 2 \\ 0 & 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}. \tag{5.37}$$

The dilation is given by $G_{\mathbf{d}} = I_5 + \mu G_0$ where $\mu \in [-1, 0)$. Obviously, $G_{\mathbf{d}} Q(z) = Q(z)$ so the first condition of the main theorem is fulfilled. The pair $\{A, B\}$ is controllable, so the LMIs (5.13)-(5.15),(5.17),(5.18) are feasible for small enough $\varepsilon > 0$ (in this example, the control restrictions are omitted). Solving (5.13),(5.17),(5.18) for $\mu = -0.1$ we derive

$$X = \begin{pmatrix} 148.1977 & 5.4546 & -28.9205 & -2.0518 & 9.0251 \\ 5.4546 & 25.7547 & 0.2926 & -4.9821 & 0.0481 \\ -28.9205 & 0.2926 & 6.3759 & 0.1808 & -2.3111 \\ -2.0518 & -4.9821 & 0.1808 & 1.0161 & -0.1000 \\ 9.0251 & 0.0481 & -2.3111 & -0.1000 & 1.0000 \end{pmatrix}, \tag{5.38}$$

$$K = \begin{pmatrix} -18.0000 & 65.0431 & -210.9065 & 272.6945 & -257.9888 \end{pmatrix}. \tag{5.39}$$

The simulations results for the open-loop ($u = 0$) and the closed-loop systems for $\varepsilon = 0.1$ are shown on the Figure 5.1. The simulation results are obtained using the explicit

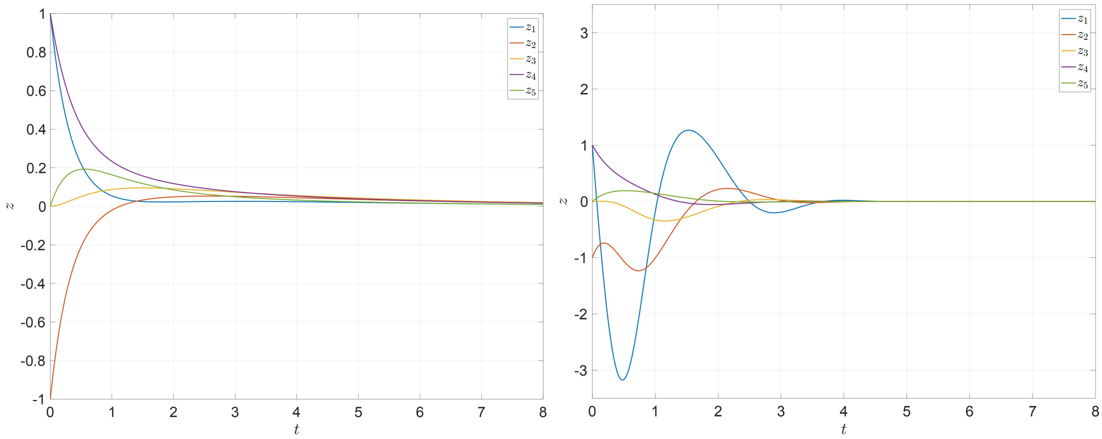

Figure 5.1: Left: Evolution of the states of the open-loop system. Right: The states of the closed-loop system.

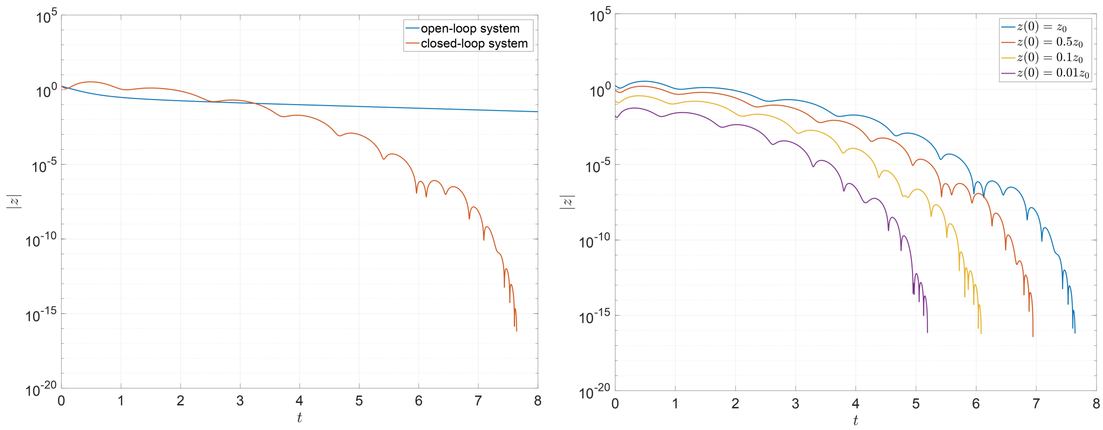

Figure 5.2: Left: Comparison of the decay rates of the open-loop and closed-loop systems in the logarithmic scale. Right: The dependence of the settling time on the variation of the norm of the initial state.

Euler method with the sampling period 10^{-3}. Application of the homogeneous control (5.19) requires a computation of the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ by means of the special numerical procedure [111]. Homogeneous Control Systems (HCS) Toolbox for MATLAB [130] is utilized for this purpose.

The comparison of the decay rates of the open-loop and the closed-loop systems is given on the Figure 5.2. The system with the homogeneous controller demonstrates the behavior typical for the finite-time stable system. According to simulation results the settling time for the chosen initial state $z(0) = z_0 := (1, -1, 0, 1, 0)^\top$ is about 8 units of time. The dependence of the settling time of the initial condition can be observed from the simulation results depicted on Figure 5.2.

Conclusion

In this chapter, an algorithm for the finite-time stabilizing control design is developed for the class of quasi-linear systems. The design procedure uses the linear approximation as the reference mode to design a homogeneous controller. The parameters of the controller are tuned in such a way that allow the stabilization of the original (quasi-linear) system as well. The convex embedding technique is utilized in order to derive the control parameters from linear matrix inequalities. The obtained LMIs are shown to be feasible under conditions of controllability of the linear approximation.

Chapter 6

Homogeneous boundary control design for a 1D viscous Burgers equation

Contents

In this chapter, we address the problem of nonlinear boundary control design for the 1D viscous Burgers equation. Using the modal decomposition approach, we design a finite-dimensional *homogeneous* controller for the controlled subsystem containing unstable modes. We prove that the closed-loop system with the homogeneous state-feedback control is locally asymptotically stable by constructing an appropriate Lyapunov functional depending on the canonical homogeneous norm and the degree of homogeneity. Next, we construct a linear infinite-dimensional observer and design the homogeneous output-feedback controller to achieve the local stability of the closed-loop system. We analyze the well-posedness of the closed-loop system with the finite-dimensional homogeneous state/output feedback control by using the Arzela–Ascoli theorem. Finally, we present numerical simulations to support the analytical results and compare our proposed controller to a linear one regarding closed-loop performance and the peaking effect.

6.1 Problem Statement

Let us consider the 1D viscous Burgers equation :

$$\begin{aligned}
\partial_t z(t,x) &= \nu\partial_{xx}z(t,x) + qz(t,x) - \tfrac{1}{2}\partial_x[z^2(t,x)], &(6.1)\\
z(t,0) &= 0, &(6.2)\\
z(t,1) &= U(t), &(6.3)\\
z(0,x) &= z_0(x), &(6.4)
\end{aligned}$$

$(t,x) \in \mathbb{R}_+ \times [0,1]$, where $z(t,x)$ is the velocity of the flow at position x and time t, ν is the viscosity , $q > 0$ is the reaction coefficient, z_0 the initial condition and $U(t) \in \mathbb{R}$ is the control input. Assume that one has the following measurement $y(t)$ given by

$$y(t) = \int_0^1 c(x)z(t,x)dx, \tag{6.5}$$

where $c \in L^2(0,1)$ for non-local measurement or $y(t) = z(t,x^*)$ for the point measurement with $x^* \in (0,1)$.

Notice that for $q < \nu\pi^2$, the open-loop system (i.e the equation (6.1)-(6.4) with $U(t) \equiv 0$) is globally exponentially stable in $L^2(0,1)$. This can be proven using the Lyapunov function given by $V(t) = \|z(t,\cdot)\|^2_{L^2(0,1)}$, which satisfies the inequality $\dot{V}(t) \leq 2(-\nu\pi^2 + q)V(t)$ along the solution of the open-loop system. The open-loop system may be unstable for $q > \nu\pi^2$. In this chapter, we

- design the homogeneous finite-time stabilizer $U(t)$ for the system (6.1)-(6.4) via the modal decomposition approach, achieving the H^1 local asymptotic stability of the closed-loop system and study the well-posedness of the closed-loop system with the homogeneous state-feedback law;
- construct an infinite-dimensional linear observer for (6.1)-(6.4) for an output feedback stabilization of (6.1)-(6.4) using the homogeneous control, and study the well-posedness of the closed-loop system with the homogeneous output-feedback law
- demonstrate through some numerical simulations that the designed homogeneous boundary feedback law for the 1D viscous Burgers equation has more advantages than the linear one: we compare numerically the homogeneous control with the linear control regarding closed-loop performance and show that the homogeneous feedback control has a smaller overshoot than the linear feedback control. More precisely, we consider the following control problem: *Given any initial state* $z_0 \in H^1(0,1)$, *a stabilization precision* $\epsilon > 0$ *and a prescribed time* $T_p > 0$ *the closed-loop system achieves the desired stabilization precision:*

$$\|z(t,\cdot)\|_{H^1(0,1)} \leq \epsilon, \quad \forall t > T_p, \tag{6.6}$$

with the restricted control magnitude

$$\sup_{t>0} |U(t)| \le \bar{U}, \tag{6.7}$$

for some maximal control amplitude $\bar{U} > 0$. A high-gain linear controller may ensure that the solution of the closed-loop system meets the criteria specified in (6.6), but it does not fulfill the control constraint outlined in (6.7). In contrast, the homogeneous control meets both criteria (6.6) and (6.7). Another situation that can be considered is when both the linear controller and the homogeneous one exhibit a similar overshoot value e.g., while verifying (6.7), but the linear controller may violate condition (6.6), whereas the homogeneous one does not. The advantage of the homogeneous controller lies then in its ability to achieve faster convergence. These considerations can be assessed numerically.

6.2 Modal decomposition

This section focuses on a modal decomposition of the linear model that facilitates the separation of the system into two distinct parts: the finite-dimensional controlled part, which encompasses the unstable modes, and the infinite-dimensional part, which includes the stable modes. To achieve this separation, it is convenient to reformulate the system (6.1)-(6.4) as an equivalent homogeneous Dirichlet problem.

We begin by introducing the following polynomial change of variables, as discussed in [52]:

$$w(t,x) = z(t,x) - xU(t), \tag{6.8}$$

where, as in [69], we set $U(0) = 0$. This leads us to the following equivalent PDE:

$$\begin{aligned}
\partial_t w(t,x) &= \nu\partial_{xx} w(t,x) + q w(t,x) - x\dot{U}(t) + qxU(t) - \frac{1}{2}\partial_x\left[(w(t,x)+xU(t))^2\right], && (6.9)\\
w(t,0) &= w(t,1) = 0, && (6.10)\\
w(0,x) &= z_0(x), && (6.11)\\
y(t) &= \int_0^1 c(x)w(t,x)\,dx + \left(\int_0^1 xc(x)\,dx\right)U(t), && (6.12)
\end{aligned}$$

(and $y(t) = w(t,x^*) + x^*U(t)$ for the point measurement case) provided that:

$$w(0,x) = z_0(x) - xU(0) = z_0(x). \tag{6.13}$$

We introduce a new control input $\xi(t)$ and a new output $\tilde{y}(t)$ given by

$$\xi(t) = \dot{U}(t), \quad \tilde{y}(t) = y(t) - \Big(\int_0^1 xc(x)dx\Big)U(t) \quad \text{or} \quad \tilde{y}(t) = y(t) - x^*U(t), \tag{6.14}$$

and obtain a following coupled ODE-PDE system

$$
\begin{aligned}
\dot{U}(t) &= \xi(t), && (6.15)\\
\partial_t w(t,x) &= \nu\partial_{xx} w(t,x) + q w(t,x) - x\xi(t) + qxU(t) - \tfrac{1}{2}\partial_x[(w(t,x)+xU(t))^2], && (6.16)\\
w(t,0) &= w(t,1) = 0, && (6.17)\\
\tilde{y}(t) &= \int_0^1 c(x)w(t,x)dx \quad \text{or} \quad \tilde{y}(t) = w(t,x^*), && (6.18)\\
w(0,x) &= z_0(x). && (6.19)
\end{aligned}
$$

The solution w to (6.16) can be represented by [69]:

$$
w(t,x) = \sum_{n=1}^{\infty} w_n(t)\phi_n(x), \quad w_n(t) = \langle w(t,\cdot), \phi_n\rangle, \tag{6.20}
$$

with $\phi_n(x) = \sqrt{2}\sin(n\pi x)$, $n = 1, 2, \dots$ and projecting onto the basis $\{\phi_n\}_{n=1}^{\infty}$, one has

$$
\begin{aligned}
\dot{U}(t) &= \xi(t) && (6.21)\\
\dot{w}_n(t) &= (-\nu\lambda_n + q)w_n(t) + b_n\xi(t) - qb_nU(t) + f_n(t), && (6.22)\\
\tilde{y}(t) &= \sum_{n=1}^{\infty} c_n w_n(t), && (6.23)\\
w_n(0) &= z_{0,n}, && (6.24)
\end{aligned}
$$

where $n \geq 1$, $\lambda_n = n^2\pi^2$ is the eigenvalue of Sturm-Liouville eigenvalue problem [1] corresponding to the eigenvector ϕ_n, and b_n, c_n, $f_n(t)$ are given by :

$$
f_n(t) = -\tfrac{1}{2}\int_0^1 \partial_x[(w(t,x) + xU(t))^2]\phi_n(x)dx, \tag{6.26}
$$

and

$$
b_n = -\int_0^1 x\phi_n(x)dx = \tfrac{(-1)^n\sqrt{2}}{n\pi}, c_n = \int_0^1 c(x)\phi_n(x)dx \quad \text{or} \quad c_n = \phi_n(x^*), z_{0,n} = \langle z_0, \phi_n\rangle. \tag{6.27}
$$

Since $\lambda_n \to +\infty$ when $n \to +\infty$, then there exists an integer $N \geq 1$ such that

$$
-\nu\lambda_n + q < -\delta, \quad \forall n \geq N+1, \tag{6.28}
$$

with δ some positive constant. Next by setting

$$
\begin{gathered}
W_u(t) = (U(t), w_1(t), \cdots, w_N(t))^\top, W(t) = (w_1(t), \cdots, w_N(t))^\top, \tilde{B} = (1, b_1, \cdots, b_N)^\top,\\
A = \mathrm{diag}\{-\nu\lambda_n + q\}_{n=1}^N, \quad \tilde{A} = \begin{pmatrix} 0 & 0_{1\times N} \\ -qB & A \end{pmatrix}, \quad C = (c_1, \cdots, c_N),\\
F_N(t) = (0, f_1(t), \cdots, f_N(t))^\top, \quad B = (b_n)_{n=1}^N,
\end{gathered} \tag{6.29}
$$

[1] Recall that :

$$
\phi''(x) + \lambda\phi(x) = 0, \quad \phi(0) = \phi(1) = 0, \tag{6.25}
$$

we obtain from (6.21)-(6.24), the following augmented system:

$$\begin{aligned}\dot{W}_u(t) &= \tilde{A}W_u(t) + \tilde{B}\xi(t) + F_N(t), && (6.30)\\ \dot{w}_n(t) &= (-\nu\lambda_n + q)w_n(t) + b_n\xi(t) - qb_nU(t) + f_n(t), \quad n > N, && (6.31)\\ \tilde{y}(t) &= CW(t) + \sum_{n=N+1}^{\infty} c_n w_n(t), && (6.32)\end{aligned}$$

with the initial data

$$(W_u(0), w_n(0)) = (z_0^N, z_{0,n}), \quad z_0^N = (0, z_{0,1}, \cdots, z_{0,N})^\top. \tag{6.33}$$

6.3 Homogeneous stabilization of 1D viscous Burgers equation by state feedback

In this part, we propose a nonlinear control following the concept of generalized homogeneity for the stabilization of the system (6.15)-(6.17) since the pair $\{\tilde{A}, \tilde{B}\}$ is controllable (see [61]).

Following the same lines as in Theorem 2.4 of Chapter 2, there exists a solution $Y_0 \in \mathbb{R}^{1\times(N+1)}$, $G_0 \in \mathbb{R}^{(N+1)\times(N+1)}$ of the linear algebraic equation

$$\tilde{A}G_0 - G_0\tilde{A} + \tilde{B}Y_0 = \tilde{A}, \quad G_0\tilde{B} = \mathbf{0}, \tag{6.34}$$

such that the matrix $G_{\mathbf{d}} := I_{N+1} + \mu G_0$ is anti-Hurwitz for any $\mu \in (-1, 0)$, and the matrix $\tilde{A}_0 = \tilde{A} + \tilde{B}K_0$ with $K_0 := Y_0(G_0 - I_{N+1})^{-1}$ satisfies the identity

$$\tilde{A}_0 G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I_{N+1})\tilde{A}_0, \quad G_{\mathbf{d}}\tilde{B} = \tilde{B}. \tag{6.35}$$

Let $X \in \mathbb{R}^{(N+1)\times(N+1)}$ and $Y \in \mathbb{R}^{1\times(N+1)}$ be the solutions of the LMIs

$$\begin{aligned}\tilde{A}_0X + X\tilde{A}_0^\top + \tilde{B}Y + Y^\top\tilde{B}^\top + \rho(G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top) &\preceq 0, && (6.36)\\ G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top \succ 0, \quad X = X^\top &\succ 0, && (6.37)\end{aligned}$$

for some $\rho \in \mathbb{R}_+$. We set

$$\zeta = \frac{\lambda_{\max}(X^{-1/2}G_{\mathbf{d}}X^{1/2} + X^{1/2}G_{\mathbf{d}}^\top X^{-1/2})}{2}, \quad \eta = \frac{\lambda_{\min}(X^{-1/2}G_{\mathbf{d}}X^{1/2} + X^{1/2}G_{\mathbf{d}}^\top X^{-1/2})}{2}, \tag{6.38}$$

and $\|\cdot\|_{\mathbf{d}}$ the canonical homogeneous norm induced by the weighted norm $\|\cdot\|_{X^{-1}}$. We design a homogeneous stabilizer for (6.30) as follows:

$$\xi(W_u(t)) = K_0W_u(t) + \mathcal{N}(W_u(t)), \quad \mathcal{N}(W_u(t)) = \|W_u(t)\|_{\mathbf{d}}^{1+\mu}K\mathbf{d}(-\ln\|W_u(t)\|_{\mathbf{d}})W_u(t), \tag{6.39}$$

where K_0 and $K = YX^{-1}$ are the homogeneous control gains. The closed-loop system (6.15)-(6.17) with the homogeneous state-feedback control (6.39) is then given by

$$\dot{U}(t) = K_0 W_u(t) + \mathcal{N}(W_u(t)), \tag{6.40}$$

$$\partial_t w(t,x) = \nu\partial_{xx}w(t,x) + qw(t,x) + qxU(t) - x\Big(K_0W_u(t) + \mathcal{N}(W_u(t))\Big) - \tfrac{1}{2}\partial_x[(w(t,x)+xU(t))^2], \tag{6.41}$$

$$w(t,0) = w(t,1) = 0, \tag{6.42}$$

$$w(0,x) = z_0(x), \tag{6.43}$$

or equivalently in the projected form

$$\begin{aligned}
\dot{W}_u(t) &= \tilde{A}W_u(t) + \tilde{B}\big(K_0W_u(t) + \mathcal{N}(W_u(t))\big) + F_N(t),\\
\dot{w}_n(t) &= (-\nu\lambda_n + q)w_n(t) + b_n\big(K_0W_u(t) + \mathcal{N}(W_u(t))\big) - qb_nU(t) + f_n(t), \quad n > N,\\
W_u(0) &= (0, \langle z_0, \phi_1\rangle, \cdots, \langle z_0, \phi_N\rangle)^\top, w_n(0) = \langle z_0, \phi_n\rangle, \quad n > N,
\end{aligned}$$

where $F_N(t)$ and $f_n(t)$ are defined in (6.29) and (6.26) respectively.

6.3.1 Well-posedness analysis

In this part, we analyze the well-posedness of the closed-loop system (6.40)-(6.43), which can be rewritten in the following abstract compact form:

$$\dot{Z}(t) = \Lambda Z(t) + F(Z(t)) + \mathcal{B}\big(K_0\mathcal{P}Z(t) + \mathcal{N}(\mathcal{P}Z(t))\big), \tag{6.44}$$

$$Z(0) = (0, z_0) \in \mathcal{H}, \tag{6.45}$$

where $\mathcal{H} = \mathbb{R} \times L^2(0,1)$ is a Hilbert space with inner product given by $\langle Z_1, Z_2\rangle_{\mathcal{H}} = U_1U_2 + \langle w_1, w_2\rangle$, with $Z_1 = (U_1, w_1) \in \mathcal{H}, Z_2 = (U_2, w_2) \in \mathcal{H}$, $Z(t) = (U(t), w(t,\cdot)) \in \mathcal{H}$ is the system state, the operators $\Lambda : \mathcal{D}(\Lambda) \subset \mathcal{H} \to \mathcal{H}$ and F are given by

$$\Lambda Z(t) = \begin{bmatrix} 0 & \mathbf{0} \\ q\kappa(x) & \nu\partial_{xx} + q\mathbf{1}\end{bmatrix} Z(t), \quad F(Z(t)) = -\frac{1}{2}\begin{pmatrix} 0 \\ \partial_x[(w(t,x)+xU(t))^2]\end{pmatrix}$$

$$\mathcal{D}(\Lambda) = \{(U,w) \in \mathbb{R}\times H^2(0,1) | w(0) = w(1) = 0\}, \tag{6.46}$$

$\kappa(x) = x$, using the identity operator $\mathbf{1}$ on $\mathcal{H}$ and null operator $\mathbf{0}$ on $\mathcal{H}$, the projection operator $\mathcal{P} : \mathcal{H} \to \mathbb{R}^{N+1}$ defined as $\mathcal{P}Z(t) = W_u(t)$ is linear and bounded and the operator $\mathcal{B} : \mathbb{R} \to \mathcal{H}$ defined as $\mathcal{B}\xi(t) = \begin{bmatrix} 1 \\ -\kappa(x)\end{bmatrix}\xi(t)$ is linear and bounded. The operator Λ generates a strongly continuous semigroup on $\mathcal{H}$ ([147, Therorem 3.2]) and the nonlinear map $F : \mathbb{R}\times H^1(0,1) \to \mathbb{R}\times L^2(0,1)$ is locally Lipschitz (see [81], [84]). However, since the nonlinear term $\mathcal{N}(\cdot)$ defined in (6.39) is non-Lipschitz at $\mathbf{0}$, the homogeneous state-feedback law considered is not Lipschitz on the entire state space $\mathcal{H}$, and then the classical methods [134], [148] commonly used to justify the well-posedness of infinite-dimensional systems with locally Lipschitz non-linearity are inapplicable to the entire state space.

To study the well-posedness in this case, we represent the closed-loop system (6.40)-(6.43) as follows:

$$\begin{aligned} \dot{w}_\infty(t) &= (2qI_\infty - \Lambda_\infty)w_\infty(t) + F_\infty(w_\infty(t)) + B_\infty\big(\xi(\mathcal{P}_\infty w_\infty(t)) - qU(t)\big), && (6.47)\\ w_\infty(0) &= (0, z_{0,1}, z_{0,2}, \cdots)^\top, && (6.48)\end{aligned}$$

where $w_\infty(t) = (U(t), w_1(t), \cdots)^\top$, I_∞ is the identity operator on ℓ^2, $F_\infty, \mathcal{P}_\infty, B_\infty$ given by

$$\begin{aligned} &\mathcal{P}_\infty w_\infty(t) = (U(t), w_1(t), \cdots, w_N(t))^\top, \quad F_\infty(w_\infty(t)) = (0, f_1(t), f_2(t), \cdots)^\top,\\ &B_\infty = (1, b_1, b_2, \cdots)^\top, \quad \Lambda_\infty = \mathrm{diag}\{\tilde{\lambda}_n\}_{n=0}^\infty, \quad \tilde{\lambda}_0 = q, \tilde{\lambda}_n = \nu\lambda_n + q, n \geq 1, \quad (6.49)\end{aligned}$$

and $b_n, f_n(\cdot)$ defined in (6.26).

The following theorem is a direct consequence of the results presented in Section C on the well-posedness of an infinite-dimensional system with a locally Lipschitz nonlinearity.

Theorem 6.1
For any initial condition $z_0 \in H_0^1(0,1)$, there exists $T > 0$ small enough such that the closed-loop system (6.1)-(6.4) with the homogeneous state-feedback law (6.40) has a mild solution $z \in C^1([0,T]; L^2(0,1)) \cap C([0,T]; H^1(0,1))$.

Proof. For any $n \geq 1$, we consider the following operator S_n defined as follows:

$$\begin{array}{llll} S_n: & \mathbb{R} \times H_0^1(0,1) & \to & \mathbb{R} \\ & (U_w, w) & \mapsto & -\int_0^1 (w'(x) + U_w)(w(x) + xU_w)\phi_n(x)dx. \end{array} \quad (6.50)$$

Then (6.26) reads

$$\forall n \geq 1, \quad \forall t \geq 0, \quad f_n(t) = S_n(U(t), w(t,\cdot)), \quad (6.51)$$

where $f_n(t)$ is defined in (6.26). Using the Parseval identity, the Young and Sobolev inequalities, for any $(U_w, w), (U_v, v) \in \mathbb{R} \times H_0^1(0,1)$, one has

$$\begin{aligned} \sum_{n=1}^\infty |S_n(U_w,w) - S_n(U_v,v)|^2 &= \sum_{n=1}^\infty \Big| \int_0^1 \Big[(w'(x) + U_w)\big(w(x) - v(x) + x(U_w - U_v)\big) + \\ &\quad \big(w'(x) - v'(x) + (U_w - U_v)\big)(v(x) + xU_v)\Big]\phi_n(x)dx\Big|^2 \\ &\leq \big(\|w'\|_{L^2(0,1)} + \|v'\|_{L^2(0,1)} + |U_w| + |U_v|\big)^2 \times \\ &\quad \big(\|w' - v'\|_{L^2(0,1)} + |U_w - U_v|\big)^2. \quad (6.52)\end{aligned}$$

On the other hand, using the expression of $\xi(\cdot)$ in (6.39) combined with the fact that $\|\mathbf{d}(-\ln\|\mathcal{P}_\infty w_\infty(t)\|_{\mathbf{d}})\mathcal{P}_\infty w_\infty(t)\|_{X^{-1}} = 1$ one has

$$|\xi(\mathcal{P}_\infty w_\infty(t))| \leq \lambda_{min}(X^{-1})^{\frac{-1}{2}} \max\{|K_0^\top|, |K^\top|\}(\|\mathcal{P}_\infty w_\infty(t)\|_{X^{-1}} + \|\mathcal{P}_\infty w_\infty(t)\|_{\mathbf{d}}^{1+\mu}). \tag{6.53}$$

Using (6.52), $F(\cdot)$ satisfies condition (C.6) and using the estimate (6.53) combined with Proposition 2.1 the homogeneous feedback control $\xi(\cdot)$ satisfies (C.10) . Applying the Theorem C.1, one concludes that there exists $T > 0$ small enough such that the closed-loop system (6.40)-(6.43) has a mild solution $(U, w) \in C^1([0, T]; \mathbb{R} \times L^2(0, 1)) \cap C([0, T]; \mathbb{R} \times H_0^1(0, 1))$. Using the relation (6.8), the proof is complete. □

6.3.2 Stability analysis

In the previous section, the dynamic extension of the PDE combined with the modal decomposition approach is employed to obtain a finite-dimensional subsystem containing the unstable modes and a residual containing the stable modes; next, we designed homogeneous state-feedback control for the controlled finite-dimensional subsystem by using the homogeneity notions. In this part, a stability analysis of the closed-loop system (6.40)-(6.43) in $H^1(0, 1)$ is studied; we prove the local asymptotic stability of the closed-loop system by constructing a suitable Lyapunov function which depends on the homogeneous norm and the degree of the homogeneity. For this, we consider the following Lyapunov functional:

$$V(t) = \Omega_1(V_1(t)) + V_2(t), \tag{6.54}$$

where

$$V_1(t) = \|W_u(t)\|_{\mathbf{d}}, \tag{6.55}$$

$$V_2(t) = \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t), \tag{6.56}$$

and Ω_1 defined by:

$$\Omega_1(s) = \tfrac{1}{2+\mu} s^{2+\mu}, \quad s \geq 0, \tag{6.57}$$

and $\mu \in (-1, 0)$ the degree of homogeneity. Notice that the function $\Omega_1 \in \mathcal{K}_\infty$ provided that $2 + \mu > 0$.

Lemma 6.1
There exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that the Lyapunov functional $V(t)$ defined in (6.54) satisfies

$$\psi_1(U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)}) \leq V(t) \leq \psi_2(U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)}), \quad \forall t \geq 0. \tag{6.58}$$

Proof. The proof follows the same lines as the proof of Lemma 3.2. □

The following lemma gives an estimate of the derivative $\dot{V}(t)$ along the solution of the closed-loop system (6.40)-(6.43).

Lemma 6.2
Given $r > 0$ sufficiently small, there exist $\tilde{\rho} > 0$ and $\psi_3 \in \mathcal{K}_\infty$ such that if the following estimate

$$U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)} \leq r, \quad \forall t \geq 0, \tag{6.59}$$

holds, then we have

$$\dot{V}(t) \leq -\tilde{\rho}\psi_3(U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)}), \quad \forall t \geq 0. \tag{6.60}$$

Proof. The proof is presented in Appendix B.0.2. □

In summary, combining the Lemmas 6.1 and 6.2 one arrives at the following main result

Theorem 6.2
Given $\bar{r} > 0$ sufficiently small, for any initial condition $z_0 \in H^1_0(0,1)$ such that $\|z_0\|_{H^1(0,1)} \leq \sqrt{\bar{r}}$, any solution z of the closed-loop system (6.1)-(6.4) with the homogeneous feedback law (6.40) can be extended to the interval $[0,+\infty)$ and $z(t,x)$ satisfies the following estimate

$$\|z(t,\cdot)\|^2_{H^1(0,1)} \leq \beta_1(\|z_0\|^2_{H^1(0,1)}, t), \quad \forall t \geq 0, \tag{6.61}$$

with β_1[2] *some $\mathcal{KL}$ function .*

Proof. In the view of the estimates (6.58) and (6.59)-(6.60), there exists $\beta_2 \in \mathcal{KL}$ such that

$$U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)} \leq \beta_2(\|z_0\|^2_{H^1(0,1)}, t), \quad \forall t \geq 0, \tag{6.62}$$

for any initial condition $z_0 \in H^1_0(0,1)$ such that $\|z_0\|_{H^1(0,1)} \leq \sqrt{\bar{r}}$ with $\bar{r} = \psi_2^{-1} \circ \psi_1(r)$. Using the relation (6.8), one has

$$\forall t \geq 0, \quad \|z(t,\cdot)\|^2_{H^1(0,1)} \leq \tfrac{8}{3}(\|w(t,\cdot)\|^2_{H^1(0,1)} + U^2(t)), \tag{6.63}$$

and then one concludes that there exists $\beta_1 \in \mathcal{KL}$ such that

$$\|z(t,\cdot)\|^2_{H^1(0,1)} \leq \beta_1(\|z_0\|^2_{H^1(0,1)}, t), \quad \forall t \geq 0, \tag{6.64}$$

for any initial condition $\|z_0\|_{H^1(0,1)} \leq \sqrt{\bar{r}}$. Since the Lyapunov function $V(t)$ given in (6.54) remains bounded on any time interval of finite length, then $z(t,\cdot)$ is well defined on the interval $[0,+\infty)$. The proof is complete. □

[2] independant of z_0

Remark 6.1
For $\mu = 0$ one obtains $\mathbf{d}(s) = e^s I_{N+1}$,$G_{\mathbf{d}} = I_{N+1}$, $\|\cdot\|_{\mathbf{d}} = \|\cdot\|_{X^{-1}}$. In this case, the homogeneous state-feedback control $\xi(W_u(t)) = (K + K_0)W_u(t)$ is a linear control, and one recovers the classical Lyapunov function considered in the linear case [81]

$$V(t) = \tfrac{1}{2}W_u(t)^\top X^{-1}W_u(t) + \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t). \tag{6.65}$$

Remark 6.2
An alternative approach to design the homogeneous state-feedback control for the 1D viscous Burgers equation consists of decomposing the nonlinear term $\langle w(t,\cdot)\partial_x w(t,\cdot), \phi_n\rangle$ as follows

$$\langle w(t,\cdot)\partial_x w(t,\cdot), \phi_n\rangle = \begin{cases} -\frac{\pi\sqrt{2}}{2}\sum_{k=1}^{\infty} w_k(t)w_{k+1}(t), & n = 1, \\ -\frac{n\pi\sqrt{2}}{2}\sum_{k=1}^{\infty} w_k(t)w_{k+n}(t) + \frac{\pi\sqrt{2}}{4}\sum_{k=1}^{n-1} w_k(t)w_{n-k}(t), & n \neq 1, \end{cases} \tag{6.66}$$

and next rewrite the projected system (6.30)-(6.32) as follows

$$\dot{W}_u(t) = \Big(\tilde{A} + F^1(W_u(t))\Big)W_u(t) + \tilde{B}\xi(t) + \sum_{k=1}^{\infty} w_k(t)F^2(W_u(t), w_k(t)), \tag{6.67}$$

$$\dot{w}_n(t) = (-\nu\lambda_n + q)w_n(t) + b_n\xi(t) - qb_nU(t) + f_n(t), \quad n > N, \tag{6.68}$$

where $F^1(\cdot)$ is linear with respect to its argument. Using similar arguments as in [120] due to the controllability of the pair $\{\tilde{A}, \tilde{B}\}$ and to the linearity of the term $F^1(\cdot)$, one can design a homogeneous feedback control for the quasi-linear controlled subsystem (6.67) based on the convex embedding technique ([146]). Next, by following the same lines as in the proof of Theorem 6.2 one can prove the local stability of the entire system by considering the same Lyapunov functional (6.54). Using this approach, the quasi-linear term $F^1(W_u(t))W_u(t)$ is rejected in finite time. This alternative involves more calculations (to treat the nonlinear term $\langle w(t,\cdot)\partial_x w(t,\cdot), \phi_n\rangle$), but more advantageous when there are many unstable modes by not requiring ρ to be large enough due to the rejection of the nonlinear term $F^1(W_u(t))W_u(t)$. It will therefore be sufficient in this case to choose ρ large such that the term $W_u(t)$ present in F^2 is compensated compared to the approach presented in this chapter for which ρ will be large enough to compensate both $F^1(W_u(t))W_u(t)$ and the term of $W_u(t)$ present in $F^2(W_u(t), w_k(t))$.

6.4 Homogeneous stabilization of the 1D viscous Burgers equation by output feedback

In the previous section, we constructed the full-state finite-dimensional homogeneous boundary controller to stabilize the 1D viscous Burgers equation. However, measuring a system's full internal state is sometimes difficult, expensive, or impossible. In these cases, the output feedback provides a way to achieve the control objectives while still using only the available output measurements. In this section, we study the output-feedback stabilization of the 1D viscous Burgers equation using the homogeneous control. To estimate an infinite-dimensional state, we design a linear infinite-dimensional observer where the observer gain is obtained following the same lines as in [117], [149]. Next, using the same procedures as done in the previous section (i.e., modal decomposition of the linear model and separation of unstable and stable modes), we design a finite-dimensional homogeneous output-feedback control based on LMI for the subsystem containing the unstable modes. Inspired by [81], we prove the local exponential stability of the error system by constructing a suitable Lyapunov function; the proof follows the same lines as in [150] by using the S-procedure to compensate the nonlinear term present in the error system; next using the homogeneity theory tools, we construct a suitable ISS Lyapunov function for the observer system and finally we derive the asymptotic stability of the closed-loop system with the homogeneous output-feedback control by combining the two Lyapunov functions constructed. The observability property of the finite-dimensional subsystem containing the unstable modes is obtained by making the following assumption:

Assumption 6.1
Assume that $c_n = \int_0^1 c(x)\phi_n(x)dx$ in the non-local measurement case or $c_n = \phi_n(x^*)$ in the point measurement case satisfies

$$c_n \neq 0, \quad 1 \leq n \leq N, \tag{6.69}$$

where N is defined in (6.28).

Under this assumption, the pair $\{A, C\}$ (A, C defined in (6.29)) is observable (see [69], [70]) and then there exists $L_l = (l_1, \cdots, l_N)^\top \in \mathbb{R}^{N\times 1}$ the observer gain and a symmetric positive definite matrix $P \in \mathbb{R}^{N\times N}$ such that

$$P(A + L_lC) + (A + L_lC)^\top P \preceq -2\delta_1 P, \quad P = P^\top \succ 0, \tag{6.70}$$

with δ_1 some positive constant. Since in (6.15)-(6.19) U is known we construct an observer $\hat{w}$ for w only as follows :

$$\begin{aligned}
\partial_t\hat{w}(t,x) &= \nu\partial_{xx}\hat{w}(t,x) + q\hat{w}(t,x) - \tfrac{1}{2}\partial_x[(\hat{w}(t,x) + xU(t))^2] && (6.71)\\
&\quad -x\xi(t) + qxU(t) + l(x)(\int_0^1 c(x)\hat{w}(t,x)dx - \tilde{y}(t)), && (6.72)\\
\hat{w}(t,0) &= \hat{w}(t,1) = 0, && (6.73)
\end{aligned}$$

and $\hat{w}(0,x) = 0$ with $\tilde{y}(t)$ defined in (6.18) and $l(x)$ given by

$$l(x) = \sum_{n=1}^{N} l_n\phi_n(x), \tag{6.74}$$

with $\{l_n\}_{n=1}^N$ the observer gain defined in (6.70) and N defined in (6.28) . This type of observer gain is suggested for reaction-diffusion PDEs in [117], [149]. Now using (6.16)-(6.17) and (6.71)-(6.73), the error $e(t,x)=\hat{w}(t,x)-w(t,x)$ satisfies

$$\begin{aligned}
\partial_t e(t,x) &= \nu\partial_{xx}e(t,x)+qe(t,x)-\partial_x[(\hat{w}(t,x)+xU(t))e(t,x)-\tfrac{1}{2}e^2(t,x)] && (6.75)\\
&+ l(x)\int_0^1 c(x)e(t,x)dx, && (6.76)\\
e(t,0) &= e(t,1)=0, && (6.77)\\
e(0,x) &= -z_0(x). && (6.78)
\end{aligned}$$

The observer $\hat{w}(t,\cdot)$ and the error $e(t,\cdot)$ can be represented as follows

$$\begin{aligned}
\hat{w}(t,\cdot) &= \sum_{n=1}^{\infty}\hat{w}_n(t)\phi_n(\cdot), \quad \hat{w}_n(t)=\langle\hat{w}(t,\cdot),\phi_n\rangle, && (6.79)\\
e(t,\cdot) &= \sum_{n=1}^{\infty}e_n(t)\phi_n(\cdot), \quad e_n(t)=\langle e(t,\cdot),\phi_n\rangle. && (6.80)
\end{aligned}$$

By setting $\hat{W}_u(t)=(U(t),\hat{w}_1(t),\cdots,\hat{w}_N(t))^\top$ and following the same lines as in the previous section, we design the homogeneous output-feedback control $\xi(\hat{W}_u(t))$ as follows:

$$\xi(\hat{W}_u(t))=K_0\hat{W}_u(t)+\mathcal{N}(\hat{W}_u(t)),\quad \mathcal{N}(\hat{W}_u(t))=\|\hat{W}_u(t)\|_{\mathbf{d}}^{1+\mu}K\mathbf{d}(-\ln\|\hat{W}_u(t)\|_{\mathbf{d}})\hat{W}_u(t), \tag{6.81}$$

where $\mu\in(-1,0)$ the degree of homogeneity and the homogeneous control gains $K_0,K\in\mathbb{R}^{1\times(N+1)}$ defined as in the previous section.

The closed-loop system (6.71)-(6.73) and (6.75)-(6.78) with the homogeneous output-feedback control (6.81) is then given by:

$$\begin{aligned}
\dot{\hat{W}}_u(t) &= \tilde{A}\hat{W}_u(t)+\tilde{B}\xi(\hat{W}_u(t))+\hat{F}_N(t)+\tilde{L}_l\,(CE(t)+\Lambda(t)), && (6.82)\\
\dot{E}(t) &= (A+L_lC)E(t)+L_l\Lambda(t)+O^N(t), && (6.83)\\
\dot{e}_n(t) &= (-\nu\lambda_n+q)e_n(t)+o_n(t), \quad n>N, && (6.84)\\
\dot{\hat{w}}_n(t) &= (-\nu\lambda_n+q)\hat{w}_n(t)+b_n\xi(\hat{W}_u(t))-qb_nU(t)+\hat{f}_n(t), \quad n>N, && (6.85)\\
e_n(0) &= -\langle z_0,\phi_n\rangle, \quad \hat{w}_n(0)=0, \quad n\geq 1, && (6.86)
\end{aligned}$$

with

$$\begin{aligned}
E(t) &= (e_1(t),\cdots,e_N(t))^\top, \tilde{L}_l=(0,l_1,\cdots,l_N)^\top, O^N(t)=(o_1(t),\cdots,o_N(t))^\top,\\
\hat{F}_N(t) &= (0,\hat{f}_1(t),\cdots,\hat{f}_N(t))^\top, \Lambda(t)=\sum_{n=N+1}^{\infty}c_ne_n(t),
\end{aligned}$$

and $\hat{f}_n(t), o_n(t)$ given by

$$\hat{f}_n(t) = -\tfrac{1}{2}\int_0^1 \partial_x[(\hat{w}(t,x)+xU(t))^2]\phi_n(x)dx, \tag{6.87}$$

$$o_n(t) = -\int_0^1 \partial_x[(\hat{w}(t,x)+xU(t))e(t,x)-\tfrac{1}{2}e^2(t,x)]\phi_n(x)dx. \tag{6.88}$$

6.4.1 Well-posedness analysis

The following theorem is a direct consequence of the results presented in Section C on the well-posedness of a coupled infinite-dimensional system with a locally Lipschitz nonlinearity.

Theorem 6.3
For any initial condition $z_0 \in H_0^1(0,1)$, there exists $T_o > 0$ small enough such that the closed-loop system (6.1)-(6.3) and the observer (6.71)-(6.73) with the homogeneous output-feedback law (6.15) and (6.81) admits a mild solution $(z,\hat{w})$ with

$$(z,\hat{w}) \in C([0,T_o]; H^1(0,1)\times H_0^1(0,1)) \cap C^1([0,T_o]; L^2(0,1)\times L^2(0,1)).$$

Proof. The proof is presented for the case $c \in L^2(0,1)$; a similar procedure can be adapted for the case of the point measurement by using [151, Lemma 4.1]. By setting

$$\hat{w}_\infty(t) = (U(t), \hat{w}_1(t), \cdots)^\top, \quad e_\infty(t) = (0, e_1(t), \cdots)^\top, \tag{6.89}$$

one can rewrite the closed-loop system (6.82)-(6.85) as follows

$$\frac{d}{dt}\begin{pmatrix}\hat{w}_\infty(t)\\ e_\infty(t)\end{pmatrix} = \begin{pmatrix}2qI_\infty - \Lambda_\infty & 0\\ 0 & 2qI_\infty - \Lambda_\infty\end{pmatrix}\begin{pmatrix}\hat{w}_\infty(t)\\ e_\infty(t)\end{pmatrix} + \begin{pmatrix}B_\infty\\ 0\end{pmatrix}\big(\xi(\mathcal{P}_\infty \hat{w}_\infty(t)) \\ -qU(t)\big) + G\begin{pmatrix}\hat{w}_\infty(t)\\ e_\infty(t)\end{pmatrix}, \tag{6.90}$$

with the initial condition

$$\hat{w}_\infty(0) = (0,0,\cdots)^\top, \quad e_\infty(0) = (0,-z_{0,1},-z_{0,2},\cdots)^\top \tag{6.91}$$

where $I_\infty, \Lambda_\infty, B_\infty, \mathcal{P}_\infty$ are defined previously and

$$L_\infty = (0, l_1, \cdots, l_N, 0, \cdots)^\top, \quad O(\hat{w}_\infty(t), e_\infty(t)) = (0, o_1(t), o_2(t), \cdots)^\top,$$
$$\hat{F}(\hat{w}_\infty(t)) = (0, \hat{f}_1(t), \hat{f}_2(t), \cdots)^\top, \quad C_\infty = (0, c_1, c_2, \cdots),$$
$$G\begin{pmatrix}\hat{w}_\infty(t)\\ e_\infty(t)\end{pmatrix} = \begin{pmatrix}\hat{F}(\hat{w}_\infty(t))\\ O(\hat{w}_\infty(t), e_\infty(t))\end{pmatrix} + \begin{pmatrix}L_\infty\\ L_\infty\end{pmatrix} C_\infty e_\infty(t),$$

$l_n, c_n, o_n(t), \hat{f}_n(t)$ defined previously. For any $n \geq 1$, we consider the operator G_n defined as follows: for any $(U_{\hat{w}}, \hat{w}, e_{\hat{w}}) \in \mathbb{R} \times H_0^1(0,1) \times H_0^1(0,1)$

$$G_n(U_{\hat{w}}, \hat{w}, e_{\hat{w}}) = \begin{pmatrix} -\int_0^1 \Big((\hat{w}'(x) + U_{\hat{w}})(\hat{w}(x) + xU_{\hat{w}})\phi_n(x) - l_n c(x) e_{\hat{w}}(x) \Big) dx \\ \int_0^1 \Big([-(\hat{w}(x) + xU_{\hat{w}}) e_{\hat{w}}(x) + \frac{1}{2} e_{\hat{w}}^2(x)]' \phi_n(x) + l_n c(x) e_{\hat{w}}(x) \Big) dx \end{pmatrix}$$

and satisfying : $\forall n \geq 1$, $\forall t \geq 0$,

$$G_n(U(t), \hat{w}(t,\cdot), e(t,\cdot)) = \begin{pmatrix} \hat{f}_n(t) + l_n \sum_{n=1}^{\infty} c_n e_n(t) \\ o_n(t) + l_n \sum_{n=1}^{\infty} c_n e_n(t) \end{pmatrix}. \tag{6.92}$$

Using the Parseval identity, the Young and Sobolev inequalities, for any

$$(U_{\hat{w}}, \hat{w}, e_{\hat{w}}), (U_{\hat{v}}, \hat{v}, e_{\hat{v}}) \in \mathbb{R} \times H_0^1(0,1) \times H_0^1(0,1)$$

one has

$$\begin{aligned} \sum_{n=1}^{\infty} |G_n(U_{\hat{w}}, \hat{w}, e_{\hat{w}}) - G_n(U_{\hat{v}}, \hat{v}, e_{\hat{v}})|^2 \leq 2\Big(\big(\|\hat{w}'\|_{L^2(0,1)} + \|\hat{v}'\|_{L^2(0,1)} + |U_{\hat{w}}| + |U_{\hat{v}}| \big)^2 \\ +2|L_l|^2 \|c\|^2_{L^2(0,1)} + \big(3\|e_{\hat{v}}'\|_{L^2(0,1)} + \|e_{\hat{w}}'\|_{L^2(0,1)} + 2\|\hat{w}'\|_{L^2(0,1)} + 2|U_{\hat{w}}| \big)^2 \Big) \Big(\|e_{\hat{w}}' - e_{\hat{v}}'\|_{L^2(0,1)} + \\ \|\hat{w}' - \hat{v}'\|_{L^2(0,1)} + |U_{\hat{w}} - U_{\hat{v}}| \Big)^2. \end{aligned} \tag{6.93}$$

Using (6.93), $G(\cdot)$ satisfies the condition (C.19) and using similar arguments as previously, the homogeneous feedback control $\xi(\cdot)$ satisfies (C.10). Applying the Theorem C.2, one concludes that there exists $T_o > 0$ small enough such that the closed-loop system (6.82)-(6.85) has a mild solution $(U, \hat{w}, e)$ with $(U, \hat{w}, e) \in C([0, T_o]; \mathbb{R} \times H_0^1(0,1) \times H_0^1(0,1)) \cap C^1([0, T_o]; \mathbb{R} \times L^2(0,1) \times L^2(0,1))$. Using the relation (6.8) we complete the proof . □

6.4.2 Stability Analysis

In this part, we will study the H^1 stability analysis of the closed-loop system (6.82)-(6.85). For this, we consider the following Lyapunov functional

$$V_o(t) = \Omega_1(V_1(t)) + V_2(t) + \Omega_2(V_3(t)), \tag{6.94}$$

where Ω_1 is defined in (6.57) and

$$V_1(t) = \|\hat{W}_u(t)\|_{\mathbf{d}}, \tag{6.95}$$

$$V_2(t) = \sum_{n=N+1}^{\infty} \lambda_n \hat{w}_n^2(t), \tag{6.96}$$

$$V_3(t) = \|E(t)\|_P^2 + \sum_{n=N+1}^{\infty} \lambda_n e_n^2(t), \tag{6.97}$$

$$\Omega_2(s) = \tfrac{1}{\kappa_1} s^{\kappa_1}, \quad s \geq 0, \tag{6.98}$$

$$\kappa_1 = \tfrac{1+\mu}{\zeta+\mu}, \tag{6.99}$$

with P defined in (6.70), ζ defined in (6.38) and $\mu \in (-1, 0)$ the degree of homogeneity.

Lemma 6.3
There exist $\psi_4, \psi_5 \in \mathcal{K}_\infty$ such that the Lyapunov functional $V_o(t)$ defined in (6.94) satisfies: $\forall t \geq 0$,

$$\psi_4(U^2(t)+\|\hat{w}(t,\cdot)\|^2_{H^1(0,1)}+\|e(t,\cdot)\|^2_{H^1(0,1)}) \leq V_o(t) \leq \psi_5(U^2(t)+\|\hat{w}(t,\cdot)\|^2_{H^1(0,1)}+\|e(t,\cdot)\|^2_{H^1(0,1)}). \tag{6.100}$$

Proof. The proof follows the same lines as the proof of Lemma 3.4. □

For the stability analysis, we will proceed as in the previous case by establishing a lemma to estimate the derivative $\dot{V}_o(t)$. The proof is based on the Poincaré inequality and the Parseval identity. The procedure used to estimate $\dot{V}_3(t)$ is similar to that proposed in [71], [81] for linear feedback stabilization of a nonlinear PDE, which consists of compensating the nonlinear term ($O^N(t)$ in this case) via the S-procedure.

Lemma 6.4
Given $r_1 > 0$ sufficiently small, there exist $\bar{\rho} > 0$ and $\psi_6 \in \mathcal{K}_\infty$ such that if the following estimate

$$U^2(t) + \|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|e(t,\cdot)\|^2_{H^1(0,1)} \leq r_1, \quad \forall t \geq 0, \tag{6.101}$$

holds, then

$$\dot{V}_o(t) \leq -\bar{\rho}\psi_6\Big(U^2(t) + \|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|e(t,\cdot)\|^2_{H^1(0,1)}\Big), \quad \forall t \geq 0. \tag{6.102}$$

Proof. The proof is presented in Appendix B.0.3. □

In summary, by combining the Lemmas 6.3 and 6.4 one arrives at the following main result:

Theorem 6.4
Given $\bar{r}_1 > 0$ sufficiently small, for any initial condition $z_0 \in H^1_0(0,1)$ such that $\|z_0\|_{H^1(0,1)} \leq \sqrt{\bar{r}_1}$, any solution of closed-loop system (6.1)-(6.3) and the observer (6.71)-(6.73) with the homogeneous output-feedback law (6.15) and (6.81) can be extended to the interval $[0, +\infty)$ and $(z(t,x), \hat{w}(t,x))$ satisfies the following inequality :

$$\|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|z(t,\cdot)\|^2_{H^1(0,1)} \leq \beta_3(\|z_0\|^2_{H^1(0,1)}, t), \quad \forall t \geq 0, \tag{6.103}$$

with β_3[3] some $\mathcal{KL}$ function .

[3] independant of z_0

Proof. Using the estimates (6.100)and (6.101)-(6.102), there exists $\beta_4 \in \mathcal{KL}$ such that

$$U^2(t) + \|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|e(t,\cdot)\|^2_{H^1(0,1)} \leq \beta_4(\|z_0\|^2_{H^1(0,1)}, t), \quad \forall t \geq 0, \tag{6.104}$$

for any initial condition z_0 such that $\|z_0\|_{H^1(0,1)} \leq \sqrt{\bar{r}_1}$ with $\bar{r}_1 = \psi_5^{-1} \circ \psi_4(r_1)$. From the relation (6.8), one has

$$z(t,x) = \hat{w}(t,x) + xU(t) + e(t,x) \tag{6.105}$$

which implies that

$$\|z(t,\cdot)\|^2_{H^1(0,1)} \leq 4(\|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + U^2(t) + \|e(t,\cdot)\|^2_{H^1(0,1)}), \tag{6.106}$$

and then one concludes that there exists $\beta_3 \in \mathcal{KL}$ such that

$$\|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|z(t,\cdot)\|^2_{H^1(0,1)} \leq \beta_3(\|z_0\|^2_{H^1(0,1)}, t), \quad \forall t \geq 0, \tag{6.107}$$

for any initial condition $\|z_0\|_{H^1(0,1)} \leq \sqrt{\bar{r}_1}$. Since the Lyapunov function $V_o(t)$ remains bounded on any time interval of finite length, then $z(t,\cdot)$ and $\hat{w}(t,\cdot)$ are well defined on the interval $[0, +\infty)$. The proof is complete. □

Remark 6.3
We discuss some limitations of the proposed methodologies presented concerning finite-dimensional observer design. Let $M \geq N$ be an integer (N defined in (6.28)), and let us consider the following finite-dimensional observer

$$\hat{w}(t,x) = \sum_{n=1}^{M} \hat{w}_n(t)\phi_n(x), \tag{6.108}$$

where $\hat{w}_n(t)$ with $1 \leq n \leq M$ satisfies

$$\dot{\hat{w}}_n(t) \;=\; (-\nu\lambda_n + q)\hat{w}_n(t) + b_n\xi(t) - qb_nU(t) + l_n(\sum_{n=1}^{M} c_n\hat{w}_n(t) - \tilde{y}(t)), \tag{6.109}$$

with the initial condition $\hat{w}_n(0) = 0$ and $l_n = 0$ for $N+1 \leq n \leq M$. The closed-loop system with the homogeneous output-feedback control is then given by

$$\begin{aligned}
\dot{\hat{W}}_u(t) &= \tilde{A}\hat{W}_u(t) + \tilde{B}\xi(\hat{W}_u(t)) + \tilde{L}(CE(t) + C_M E_M(t) + R(t)), && (6.110)\\
\dot{E}(t) &= (A + LC)E(t) + L(C_M E_M(t) + R(t)) - \tilde{F}_N(t), && (6.111)\\
\dot{E}_M(t) &= A_M E_M(t) - \tilde{F}_M(t) && (6.112)\\
\dot{\hat{W}}_M(t) &= A_M\hat{W}_M(t) + B_M\xi(\hat{W}_u(t)), && (6.113)\\
\dot{w}_n(t) &= (-\nu\lambda_n + q)w_n(t) + b_n\xi(\hat{W}_u(t)) - qb_nU(t) + f_n(t), n \geq M+1, && (6.114)
\end{aligned}$$

with $E_M(t), R(t), W_M(t), A_M, B_M$ defined as follows:

$$\begin{gathered}\tilde{F}_M(t) = (f_{N+1}(t), \cdots, f_M(t)), \tilde{F}_N(t) = (f_1(t), \cdots, f_N(t)), R(t) = \textstyle\sum_{n\geq M} c_n w_n(t),\\ B_M = (b_{N+1}, \cdots, b_M)^\top, C_M = (c_{N+1}, \cdots, c_M), A_M = \boldsymbol{diag}\{-\nu\lambda_n + q\}_{n=N+1}^{M},\\ E_M(t) = (e_{N+1}(t), \cdots, e_M(t))^\top, \quad \hat{W}_M(t) = (\hat{w}_{N+1}(t), \ldots, \hat{w}_M(t)).\end{gathered}$$

By setting $V_f(t) = \|\hat{W}_u(t)\|_{\boldsymbol{d}}, V_i(t) = \sum_{n\geq M+1} \lambda_n w_n^2(t)$ and following the same lines as in the proof of Theorem 6.4, along the solutions of the closed-loop system (6.110)-(6.114), if

$$V_f(t) \leq 1, \quad V_i(t) \leq r, \tag{6.115}$$

with $r > 0$ then one has

$$\begin{aligned}\tfrac{d}{dt}\Omega(V_f(t)) \quad &\leq \quad -\alpha_1 V_f^{2(1+\mu)}(t) + \alpha_2(|CE(t)|^{\kappa_2} + |C_M E_M(t)|^{\kappa_2}) + \alpha_3 V_i^{\kappa_1}(t)\\ \dot{V}_i(t) \quad &\leq \quad -\alpha_4 V_i(t) + \alpha_5 V_f^{2(1+\mu)}(t) + \alpha_6(|E(t)|^2 + |E_M(t)|^2),\end{aligned}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_6$ are some positive constants. Then one obtains

$$\begin{gathered}\tfrac{d}{dt}\Omega(V_f(t)) + \dot{V}_i(t) \leq (-\alpha_1 + \alpha_5)V_f^{2(1+\mu)}(t) + \Big(-\alpha_4 V_i(t) + \alpha_3 V_i^{\kappa_1}(t)\Big) +\\ \alpha_2(|CE(t)|^{\kappa_2} + |C_M E_M(t)|^{\kappa_2}) + \alpha_6(|E(t)|^2 + |E_M(t)|^2).\end{gathered}$$

In the case of linear control design, $\kappa_1 = 1$ and the term $V_i(t)$ can be rejected for M sufficiently large such that $\alpha_4 > \alpha_3$. In our case, $\kappa_1 < 1$ which implies that

$$V_i^{-1}(t)\Big(-\alpha_4 V_i(t) + \alpha_3 V_i^{\kappa_1}(t)\Big) = -\alpha_4 + \alpha_3 V_i^{\kappa_1 - 1}(t) \to +\infty \quad \text{as} \quad V_i(t) \to 0. \tag{6.116}$$

In other words, as we approach zero, the term $\alpha_3 V_i^{\kappa_1}(t)$ becomes the dominant term relative to $-\alpha_4 V_i(t)$. This indicates that we cannot consider the finite-dimensional observer in our context, or at least not with the Lyapunov function discussed in the chapter.

6.5 Numerical Simulations

We perform numerical simulations on the system (6.1)-(6.4) with the homogeneous output-feedback control (6.15) and (6.81), by using system (6.82)-(6.85). The initial condition z_0, the system parameters ν and q, as well as the point measurement $y(t)$, are defined as follows:

$$z_0(x) = x(1-x), \quad \nu = 0.15, \quad q = 4.1\nu\pi^2, \quad N = 2, \quad y(t) = z(t, 0.4), \tag{6.117}$$

and we set $U(0) = 0$. We consider (6.20) with $M = 10$ truncated basis. As highlighted in the chapter, the advantage of homogeneous control lies in its ability to achieve faster convergence with reduced peaking compared to linear control. Consequently, the linear control is optimally designed using Linear Matrix Inequalities (LMIs), to achieve faster convergence than that of classically designed linear control, while minimizing the overshoot of the control input (see Section 2.3.2 in Chapter 2). We utilize the control toolbox in MATLAB to compute the linear control gains:

$$K_{l_1} = (-140.0323, -272.0973, -14.2646), \quad K_{l_2} = (-202.8260, -416.7673, -38.0307), \tag{6.118}$$

and the observer gain is given by

$$L_l = (-9.2644, 2.4339),$$

with which we compute the linear output-feedback control expressed as:

$$U(t) = K_l \int_0^t \hat{W}_u(s)\, ds, \tag{6.119}$$

where $K_l = K_{l_1}$ corresponds to linear control, and $K_l = K_{l_2}$ corresponds to high-gain linear control. Next, we homogenize the linear output-feedback law to derive the homogeneous-output feedback law, defined as follows:

$$U(t) = \int_0^t \Big(K_0 \hat{W}_u(s) + \|\hat{W}_u(s)\|_{\mathbf{d}}^{1+\mu} (K_{l_1} - K_0) \mathbf{d}(-\ln \|\hat{W}_u(s)\|_{\mathbf{d}}) \hat{W}_u(s) \Big)\, ds, \tag{6.120}$$

using the following parameters: $\mu = -0.1$, $\eta = 1$, $\zeta = 1.1$, and

$$K_0 = (-19.4386, -32.6580, -0.0005), \quad G_{\mathbf{d}} = \begin{pmatrix} 0.6875 & -0.6886 & 0.0111 \\ 0.1846 & 1.4067 & -0.0067 \\ -0.0231 & 0.0517 & 1.2058 \end{pmatrix}. \tag{6.121}$$

The Homogeneous Control Systems (HCS) Toolbox for MATLAB [130] is used to obtain these parameters. Figure 6.1 shows the simulations in logarithmic scale of the norm $\|z(t,\cdot)\|_{H^1(0,1)}$ of closed-loop system (6.1)-(6.3) with linear output-feedback control (6.119) (with K_{l_1}) in blue line, with high-gain linear output-feedback control (6.119) (with K_{l_2}) in black dashed line and with homogeneous output-feedback control (6.15) and (6.81) in red line and during the initial phase of stabilization. In Figure 6.2, the linear output-feedback control (6.119), the homogeneous control (6.15) and (6.81) and the solution $z(t,x)$ of closed-loop system (6.1)-(6.3) with the homogeneous output-feedback law (6.15) and (6.81) are shown. One remark from Figure 6.1 is that the closed-loop systems with the homogeneous output-feedback control and the high-gain linear output-feedback control achieve

$$\|z(t,\cdot)\|_{H^1(0,1)} \leq \epsilon := 10^{-7}, \quad \forall t \geq T_P := 9. \tag{6.122}$$

From Figure 6.2, only the closed-loop system with the homogeneous output-feedback control achieves (6.122) with the control restriction

$$\sup_{t>0} |U(t)| \leq \bar{U} := 4. \tag{6.123}$$

In Figure 6.1 and Figure 6.2, one can observe that during the initial phase of stabilization, the high-gain linear closed-loop solution has a large deviation (peaking effect), and the high-gain linear control has a large overshoot. Additionally, we can observe in Figure 6.2 that both the linear controller and the homogeneous one exhibit a similar overshoot value while verifying (6.123), but the linear controller violates condition (6.122) whereas the homogeneous one does not. Hence, the advantage of using the homogeneous control is that it allows for better convergence without the peaking effect and with less overshoot.

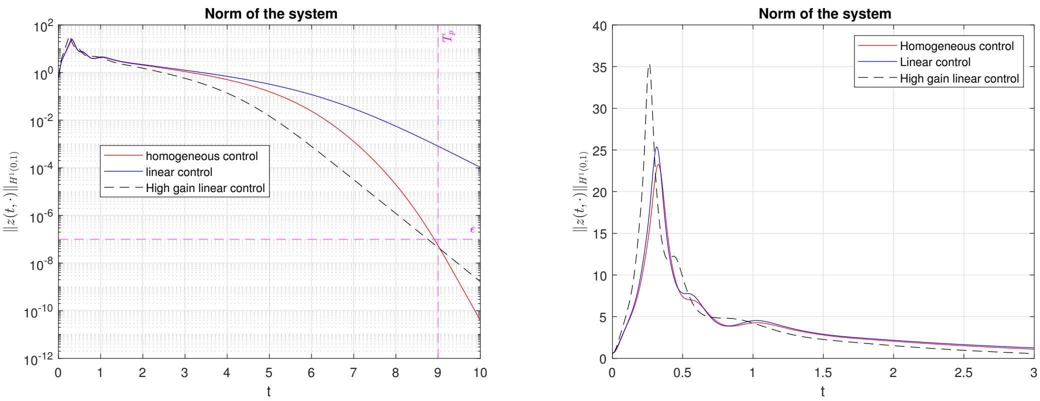

Figure 6.1: Evolution of the norm $\|z(t,\cdot)\|_{H^1(0,1)}$ in a logarithmic scale of the closed-loop system (6.1)-(6.3) under linear output-feedback control (6.119) (in blue line and black dashed line) and under homogeneous output-feedback control (6.15) and (6.81) in red line

Remark 6.4
For the numerical simulations, the norm of the initial condition is $\|z(0,\cdot)\|_{H^1(0,1)} \approx 0.7853$. *With this initial condition, the closed-loop system demonstrates asymptotic stability. However, upon conducting several tests, we observed instability in the closed-loop system when starting from* $\|z(0,\cdot)\|_{H^1(0,1)} \approx 1.5706$.

6.6 Conclusions

In this chapter, we have developed a homogeneous state/output feedback controller to stabilize the 1D viscous Burgers equation. The stability of the closed-loop system with

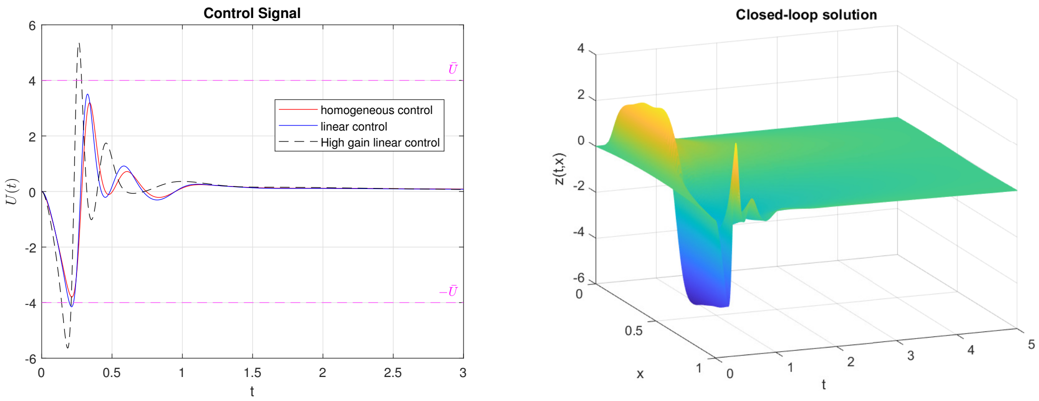

Figure 6.2: Time-evolution of the output-feedback control signal $U(t)$: the linear control (6.119) (in blue line and in black dashed line) and homogeneous control (6.15) and (6.81) in red line and the solution $z(t,x)$ of closed-loop system (6.1)-(6.3) with the homogeneous output-feedback law (6.15) and (6.81)

the homogeneous state/output feedback control is proven by constructing an appropriate Lyapunov functional, which utilizes the homogeneous canonical norm and the degree of homogeneity. The simulation results demonstrate that the homogeneous controller outperforms a linear feedback controller, exhibiting improved closed-loop performance characterized by reduced overshoot and smaller peaking.

Chapter 7

Conclusion and perspectives

7.1 Contributions

The research presented in this thesis primarily contributes to the field of finite-dimensional nonlinear control design for some classes of infinite-dimensional systems, including linear and nonlinear parabolic partial differential equations (PDEs). By bridging the gap between homogeneous control theory and distributed parameter systems, this work offers innovative strategies for finite-dimensional homogeneous control design for a 1D reaction-diffusion equation, both with and without input delay. Furthermore, it is demonstrated that the methodology developed for linear parabolic PDEs can be extended to certain nonlinear systems. For example, the thesis illustrates that finite-dimensional homogeneous control can be effectively designed to stabilize the 1D viscous Burgers equation.

The main novel contributions of this thesis to control theory are:

- A finite-dimensional nonlinear state/output feedback control design for the 1D reaction diffusion, with the stability analysis based on an appropriate Lyapunov function: In this part, we employed modal decomposition and subsequently designed a homogeneous control for the unstable component. Given that the closed-loop system is cascaded, demonstrating Lyapunov stability and the attractivity of the origin would have been sufficient to conclude asymptotic stability. The main contribution of this part lies in the use of the constructed Lyapunov function to establish the asymptotic stability of the closed-loop system.
- A finite-dimensional nonlinear predictor feedback control design for the 1D reaction-diffusion equation with input delay, with the stability analysis based on an appropriate Lyapunov function: we adopted the same approach as previously, employing modal decomposition in conjunction with homogeneous predictor feedback design. Subsequently, we analyzed the stability of the closed-loop system by proposing an appropriate Lyapunov function. To our knowledge, this contribution is the first nonlinear predictor feedback design for reaction-diffusion equations using the modal decomposition approach.

- A finite-dimensional nonlinear control design for nonlinear parabolic PDEs, precisely the 1D viscous Burgers equation, with the stability analysis based on an appropriate Lyapunov function: The linearized Burgers equation with reactive terms around the origin is a reaction-diffusion equation. For finite-dimensional systems, the global asymptotic stability of the linearized equation ensures the local asymptotic stability of the nonlinear system; however, this does not hold for infinite-dimensional systems. We applied modal decomposition to the linear part of the Burgers equation and then designed the homogeneous control for the finite-dimensional subsystem. It is essential to note that the presence of nonlinearity leads to an interconnected closed-loop system. To analyze the stability of the closed-loop system, we propose an appropriate Lyapunov function and employ the Parseval identity to compensate the nonlinear term.
- Well-posedness analysis of closed-loop system with non-Lipschitz nonlinearity: Since the homogeneous control is nonlinear and non-Lipschitz at the origin, classical techniques used to study well-posed systems cannot be applied to closed-loop systems considered in this thesis. We have developed specialized methods to analyze the well-posedness of closed-loop systems containing non-Lipschitz nonlinearities. Two methods are presented: The first involves studying the finite-dimensional system that generates the control signal, then considering an abstract system that incorporates this signal, and finally proving that a solution to this abstract system is also a solution to the original system. The second method involves using a continuous approximation of the non-Lipschitz controller and demonstrating that the closed-loop system with this approximation admits a solution that coincides with the solution of the original system on a specific interval.
- LMI-based finite-time control design for a quasi-linear system: We propose the design of homogeneous controllers based on Linear Matrix Inequalities (LMIs) for the local finite-time stabilization of quasi-linear systems. Quasi-linear systems are models derived from approximations of Burgers' equation using model reduction techniques, such as finite difference, Galerkin method, and finite element method. In most cases, experiments are conducted using these reduced models. The main contribution of this section is the design of finite-time controllers that are implementable for reduced models representing fluid flows.

7.2 Perspectives

This thesis contributes to finite-dimensional nonlinear boundary stabilization problems for various classes of infinite-dimensional systems. However, it is important to recognize that several issues remain open, and that deserve further investigation. Below are some potential avenues for future research that can be explored:

- Extension of the results presented in this thesis for 2D or 3D parabolic PDEs: The equations discussed in this thesis are in one-dimensional space. Although these 1D parabolic PDEs (including the reaction-diffusion and viscous Burgers equation) can effectively model thermal processes or flows under certain conditions, achieving higher precision in experiments necessitates the design of homogeneous control strategies for equations in two or three dimensions. This will be an extension of the methodology proposed in this thesis.

- Finite-dimensional homogeneous observer-based control design for the 1D reaction diffusion equation: In the case of state-feedback stabilization or output-feedback stabilization using an infinite-dimensional observer, the closed-loop system is structured as a cascade system. In contrast, for output-feedback stabilization employing a finite-dimensional observer, the closed-loop system is interconnected. In the work presented in this thesis, the specific form of the homogeneous controller, particularly the dilation $\mathbf{d}_c$, does not permit the design of a finite-dimensional homogeneous observer-based control for reaction-diffusion equations. Indeed, a sufficient condition for addressing this issue is that the observer gain $\tilde{L}$ must be an eigenvector of the dilation matrix $G_{\mathbf{d}_c}$ with eigenvalue 1 i.e $G_{\mathbf{d}_c}\tilde{L} = \tilde{L}$; however, this condition is exceedingly restrictive and practically infeasible. Therefore, the investigation of finite-dimensional homogeneous observer-based control design poses significant challenges.

- Implementation challenges, including sampled-data and event-triggered homogeneous control, and consistent discretization of a homogeneous control for infinite-dimensional systems: Designing homogeneous control laws is one aspect, but it is equally important to propose methods for implementing these controllers to facilitate their use during experiments. Therefore, it is crucial to develop homogeneous sampled-data or event-triggered controls. Most existing methods for event-triggered or sampled-data control rely on Lipschitz conditions, small-gain arguments, and descriptor methods; however, homogeneous control is not Lipschitz, and small-gain arguments are not applicable for homogeneous systems (at least for those considered in this thesis). Thus, it would be highly beneficial to develop specific tools to address this challenge.

Appendix A

Some technical proofs in Chapter 3

A.0.1 Technical Lemma

The following Lemma is similar to [152, Lemma 2] and is useful for lower bounding Lyapunov functions.

Lemma A.1
For any α_1, α_2 *and* $\alpha_3 \in \mathcal{K}_\infty$ *one has*

$$\alpha_1(s_1) + \alpha_2(s_2) + \alpha_3(s_3) \geq \bar{\alpha}(s_1 + s_2 + s_3), \tag{A.1}$$

for any $s_1, s_2, s_3 \geq 0$ *where* $\bar{\alpha} \in \mathcal{K}_\infty$ *is given by*

$$\bar{\alpha}(s) = \min\{\alpha_1(\tfrac{s}{3}), \alpha_2(\tfrac{s}{3}), \alpha_3(\tfrac{s}{3})\}, \tag{A.2}$$

for all $s \geq 0$.

Proof. Let $s_1, s_2, s_3 \geq 0$. By using the fact that $\bar{\alpha} \in \mathcal{K}_\infty$ one has the following inequality

$$\bar{\alpha}(s_1 + s_2 + s_3) \leq \bar{\alpha}(3s_1) + \bar{\alpha}(3s_2) + \bar{\alpha}(3s_3), \tag{A.3}$$

which combined with (A.2) implies that

$$\bar{\alpha}(s_1 + s_2 + s_3) \leq \sum_{i=1}^{3} \min\{\alpha_1(s_i), \alpha_2(s_i), \alpha_3(s_i)\}$$

and then

$$\bar{\alpha}(s_1 + s_2 + s_3) \leq \alpha_1(s_1) + \alpha_2(s_2) + \alpha_3(s_3). \tag{A.4}$$

The proof is complete. □

A.0.2 Proof of Lemma 3.1

Proof. The closed-loop system (3.42)-(3.44) is rewritten in the following compact form:

$$\dot{Z}(t) = (\mathcal{A}+\mathcal{B}K_0\mathcal{P})Z(t)+\mathcal{B}\mathcal{N}(\mathcal{P}Z(t)), \tag{A.5}$$
$$Z(0) = (0,z_0)^\top \in \mathcal{H}, \tag{A.6}$$

where $\mathcal{H}=\mathbb{R}\times L^2(0,1)$ is a Hilbert space with inner product given by $\langle Z_1,Z_2\rangle_{\mathcal{H}} = U_1U_2+\langle w_1,w_2\rangle$, with $Z_1=(U_1,w_1)^\top\in\mathcal{H}, Z_2=(U_2,w_2)^\top\in\mathcal{H}$, $Z(t)=(U(t),w(t,\cdot))^\top\in\mathcal{H}$ is the system state, the operator $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$:

$$\mathcal{A}Z(t)=[0,(qr+r'')U(t)+(\partial_{xx}+q)w(t,\cdot)]^\top, \tag{A.7}$$

$\mathcal{D}(\mathcal{A})=\mathbb{R}\times\mathcal{H}_1$ with $\mathcal{H}_1$ defined in (3.45), the operator $\mathcal{P}:\mathcal{H}\to\mathbb{R}^{N+1}$ defined as $\mathcal{P}Z(t)=W_u(t)$ is linear and bounded and the operator $\mathcal{B}:\mathbb{R}\to\mathcal{H}$ defined as $\mathcal{B}\xi(t)=\begin{bmatrix}1\\-r(x)\end{bmatrix}\xi(t)$ is linear and bounded. The operator $\mathcal{A}+\mathcal{B}K_0\mathcal{P}$ generates a strongly continuous semigroup $(\mathcal{S}(t))_{t\geq 0}$ on $\mathcal{H}$ and $\mathcal{B}$ is admissible (see [135, Section 4.2]) for $\mathcal{S}(t)$.

Let $\alpha(t)\in\mathbb{R}^{N+1}$ be a solution of

$$\dot{\alpha}(t)=\tilde{A}_0^c\alpha(t)+\tilde{B}\mathcal{N}(\alpha(t)), \tag{A.8}$$
$$\alpha(0)=\mathcal{P}(0,z_0)^\top\in\mathbb{R}^{N+1}, \tag{A.9}$$

where $\tilde{A}_0^c$ and $\tilde{B}$ are such that (3.35) is verified, $\mathcal{N}$ in (3.38). Notice that the solution $\alpha(t)$ of (A.8)-(A.9) is unique and continuously differentiable on $[0,+\infty)$ (see [95, Theorem 9.1]).

Using the fact that $\|\mathbf{d}_c(-\ln\|\alpha(t)\|_{\mathbf{d}_c})\|_{X_c^{-1}}=1$, for any $\mu_c\in(-1,0)$, the map $t\mapsto\mathcal{N}(\alpha(t))\in L^2([0,+\infty),\mathbb{R})$. Along the solution of (A.8)-(A.9) one derives

$$\tfrac{d}{dt}\mathcal{N}(\alpha(t))=K\left[\left(\tfrac{d\|\alpha(t)\|_{\mathbf{d}_c}}{dt}\right)\|\alpha(t)\|_{\mathbf{d}_c}^{\mu_c}\left[(1+\mu_c)I_{N+1}-G_{\mathbf{d}_c}\right]+\|\alpha(t)\|_{\mathbf{d}_c}^{1+2\mu_c}(\tilde{A}_0^c+\tilde{B}K)\right]$$
$$\times\mathbf{d}_c(-\ln\|\alpha(t)\|_{\mathbf{d}_c})\alpha(t), \tag{A.10}$$

provided that from (3.35), $\tilde{A}_0^c\mathbf{d}_c(s)=e^{\mu_c s}\mathbf{d}_c(s)\tilde{A}_0^c$ and $\mathbf{d}_c(s)\tilde{B}=e^s\tilde{B}$, $\forall s\in\mathbb{R}$ (see [111] for more details).

Let $\delta_2\geq\delta$ such that

$$(\tilde{A}_0^cX_c+X_c\tilde{A}_0^c)^\top+\tilde{B}Y_c+Y_c^\top\tilde{B}^\top\succeq-\delta_2(G_{\mathbf{d}_c}X_c+X_cG_{\mathbf{d}_c}^\top). \tag{A.11}$$

Then using (2.22), along the solution of system (A.8)-(A.9) one derives

$$-\delta_2\|\alpha(t)\|_{\mathbf{d}_c}^{1+\mu_c}\leq\tfrac{d\|\alpha(t)\|_{\mathbf{d}_c}}{dt}\leq-\delta\|\alpha(t)\|_{\mathbf{d}_c}^{1+\mu_c}. \tag{A.12}$$

From (A.10),(A.12) and using the fact that for all $t \geq T^c$, $\alpha(t) = 0_{(N+1)\times 1}$ one derives

$$\int_0^{\infty} \left|\tfrac{d}{ds}\mathcal{N}(\alpha(s))\right|^2 ds \leq M_{12}^2 \int_0^{T^c} \|\alpha(s)\|_{\mathbf{d}_c}^{2(1+2\mu_c)} ds, \tag{A.13}$$

with $M_{12} = |K^\top|\lambda_{min}(X_c^{-1})^{\frac{-1}{2}}(\|\tilde{A}_0^c + \tilde{B}K\|_{X_c^{-1}} + \delta_2\|\mu_c I_{N+1} - G_{\mathbf{d}_c}\|_{X_c^{-1}})$. Thus one concludes that

$$t \mapsto \mathcal{N}(\alpha(t)) \in \begin{cases} L^2([0,+\infty),\mathbb{R}), & \text{if} \quad \mu_c \in (-1, -\frac{1}{2}), \\ H^1([0,+\infty),\mathbb{R}), & \text{if} \quad \mu_c \in [\frac{-1}{2}, 0). \end{cases}$$

Using [135, Proposition 4.2.5 and 4.2.10], the system

$$\dot{\Gamma}(t) = (\mathcal{A} + \mathcal{B}K_0\mathcal{P})\Gamma(t) + \mathcal{B}\mathcal{N}(\alpha(t)), \tag{A.14}$$
$$\Gamma(0) = (0, z_0)^\top, \tag{A.15}$$

admits a unique solution

- $\Gamma \in C([0,+\infty), \mathbb{R}\times\mathcal{H}_1) \cap H^1_{loc}([0,+\infty), \mathbb{R}\times L^2(0,1))$ for $\mu_c \in (-1, \frac{-1}{2})$,
- $\Gamma \in C([0,+\infty), \mathbb{R}\times\mathcal{H}_1) \cap C^1([0,+\infty), \mathbb{R}\times L^2(0,1))$ for $\mu_c \in [\frac{-1}{2}, 0)$,

for any initial condition $z_0 \in \bar{\mathcal{H}}$.

Using modal decomposition on $\Gamma(t) \in \mathcal{D}(\mathcal{A})$, $\mathcal{P}\Gamma(t)$ satisfies $\frac{d}{dt}\mathcal{P}\Gamma(t) = \tilde{A}_0^c\mathcal{P}\Gamma(t) + \tilde{B}\mathcal{N}(\alpha(t))$ with $\mathcal{P}\Gamma(0) = \mathcal{P}(0, z_0)^\top$. Then $\alpha(t) = \mathcal{P}\Gamma(t)$ and therefore $\Gamma(t)$ satisfies

$$\dot{\Gamma}(t) = (\mathcal{A} + \mathcal{B}K_0\mathcal{P})\Gamma(t) + \mathcal{B}\mathcal{N}(\mathcal{P}\Gamma(t)), \tag{A.16}$$
$$\Gamma(0) = (0, z_0)^\top \in \mathcal{H}. \tag{A.17}$$

One concludes that (A.5)-(A.6) admits a solution for any initial condition $z_0 \in \bar{\mathcal{H}}$.

If there exists two solutions $Z_1 = (U_1, w_1)^\top$ and $Z_2 = (U_2, w_2)^\top$ of (A.5)-(A.6) such that $Z_1(0) = Z_2(0) = (0, z_0)^\top$, then $Z(t) = Z_1(t) - Z_2(t)$ is solution of

$$\begin{aligned} \dot{Z}(t) &= (\mathcal{A}+\mathcal{B}K_0\mathcal{P})Z(t)+\mathcal{B}(\mathcal{N}(\mathcal{P}Z_1(t))-\mathcal{N}(\mathcal{P}Z_2(t))), \\ Z(0) &= (0,0)^\top. \end{aligned}$$

Since $\alpha_1(t) = \mathcal{P}Z_1(t)$ and $\alpha_2(t) = \mathcal{P}Z_2(t)$ are solution of (A.8)-(A.9) and it admits unique solution, then $\mathcal{P}Z_1(t) = \mathcal{P}Z_2(t)$. Therefore $Z(t) = Z_1(t) - Z_2(t)$ satisfies $\dot{Z}(t) = (\mathcal{A}+\mathcal{B}K_0\mathcal{P})Z(t)$ with $Z(0) = (0,0)^\top$ and then $Z(t) = (0,0)^\top$ which implies that $Z_1(t) = Z_2(t)$. One concludes that the solution is unique. Therefore, the closed-loop system (3.42)-(3.44) admits the unique solution. □

A.0.3 Proof of Lemma 3.2

Proof. Using the integration by parts formula and using the Wirtinger inequality [153] one has

$$U^2(t) + \sum_{n\geq 1} |\lambda_n| w_n^2(t) \leq U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)} \leq (1 + \tfrac{4}{\pi^2})(U^2(t) + \sum_{n\geq 1} |\lambda_n| w_n^2(t)).$$

This implies

$$\begin{aligned} |W_u(t)|^2 + \textstyle\sum_{n>N} |\lambda_n| w_n^2(t) \leq U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)} \leq \\ (1 + \tfrac{4}{\pi^2})|\lambda_{N+1}|(|W_u(t)|^2 + \textstyle\sum_{n>N} |\lambda_n| w_n^2(t)). \end{aligned} \tag{A.18}$$

Let

$$\psi_{11}(s) = \begin{cases} \frac{1}{2+\mu_c} s^{\frac{2+\mu_c}{2}}, & \text{if} \quad s \leq 1, \\ \frac{1}{2\nu-\mu_c} s^{\frac{2\nu-\mu_c}{2}} - \frac{1}{2\nu-\mu_c} + \frac{1}{2+\mu_c}, & \text{if} \quad s \geq 1, \end{cases}$$

The function $\psi_{11} \in \mathcal{K}_\infty$ and satisfies $\psi_{11}(s^2) = \Omega_1(s)$. One can rewrite $V^h(t)$ as follows :

$$V^h(t) = \gamma_1 \psi_{11}(\|W_u(t)\|^2_{\mathbf{d}_c}) + \sum_{n\geq N+1} |\lambda_n| w_n^2(t).$$

From Proposition 2.1, there exists $\psi_{12}, \psi_{13} \in \mathcal{K}_\infty$ such that

$$\psi_{12}(|W_u(t)|^2) \leq \|W_u(t)\|^2_{\mathbf{d}_c} \leq \psi_{13}(|W_u(t)|^2).$$

Then there exists $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that

$$\psi_1(|W_u(t)|^2 + \sum_{n>N} |\lambda_n| w_n^2(t)) \leq V^h(t) \leq \psi_2(|W_u(t)|^2 + \sum_{n>N} |\lambda_n| w_n^2(t)), \tag{A.19}$$

and using (A.18) and (A.19), the proof is complete. □

A.0.4 Proof of Lemma 3.3

The proof for the case $c \in L^2(0,1)$ is presented and can be adapted for the case of the point measurement.

Proof. The closed-loop system (3.80)-(3.83) can be rewritten in the following compact form:

$$\begin{aligned} \partial_t e(t,x) &= \partial_{xx} e(t,x) + q e(t,x) + l(x)\tilde{e}(t), && \text{(A.20)} \\ \dot{\hat{Z}}(t) &= (\mathcal{A} + \mathcal{B}K_0\mathcal{P})\hat{Z}(t) + \mathcal{B}\mathcal{N}(\mathcal{P}\hat{Z}(t)) + \tilde{l}(x)\tilde{e}(t), && \text{(A.21)} \\ \hat{Z}(0) &= (0,0)^\top, e(0,x) = -z_0(x), && \text{(A.22)} \end{aligned}$$

where $\hat{Z}(t) = (U(t), \hat{w}(t,\cdot))^\top \in \mathcal{H}$, $\tilde{l}(x) = [0, l(x)]^\top$ and $\tilde{e}(t) = \int_0^1 c(x)e(t,x)dx$.

Let $\mathcal{H}_2 = \{f \in H^1(0,1) | f(1) = 0\}$. Since for all $e_1, e_2 \in \mathcal{H}_2$,

$$|\tilde{e}_1(t) - \tilde{e}_2(t)| \leq \tfrac{2\|c\|_{L^2}}{\pi} \|\partial_x(e_1(t,\cdot) - e_2(t,\cdot))\|_{L^2(0,1)}, \tag{A.23}$$

then using similar arguments as in [23], the system (3.77) admits unique solution in $C([0,+\infty), \bar{\mathcal{H}}) \cap C^1([0,+\infty), L^2(0,1))$ for any initial condition $z_0 \in \bar{\mathcal{H}}$.

Let $\alpha_e(t)$ be a solution of the following system:

$$\begin{aligned} \dot{\alpha}_e(t) &= \tilde{A}_0^c \alpha_e(t) + \tilde{B}\mathcal{N}(\alpha_e(t)) + \tilde{L}_l \tilde{e}(t), && \text{(A.24)} \\ \alpha_e(0) &= 0_{(N+1)\times 1}. && \text{(A.25)} \end{aligned}$$

Since $t \mapsto \mathcal{B}\mathcal{N}(\alpha_e(t)) + \tilde{l}(x)\tilde{e}(t) \in H^1_{loc}([0,+\infty), L^2(0,1))$ then according to [135, Theorem 4.1.6], the following system

$$\dot{\Gamma}_e(t) = (\mathcal{A} + \mathcal{B}K_0\mathcal{P})\Gamma_e(t) + \mathcal{B}\mathcal{N}(\alpha_e(t)) + \tilde{l}(x)\tilde{e}(t),$$

with $\Gamma_e(0) = (0,0)^\top$ admits a solution $\Gamma_e(t) \in C([0,+\infty), \mathcal{D}(\mathcal{A})) \cap C^1([0,+\infty), \mathcal{H})$.

Using modal decomposition on $\Gamma_e(t)$, $\mathcal{P}\Gamma_e(t)$ satisfies

$$\tfrac{d}{dt}\mathcal{P}\Gamma_e(t) = \tilde{A}_0^c \mathcal{P}\Gamma_e(t) + \tilde{B}\mathcal{N}(\alpha_e(t)) + \tilde{L}_l\tilde{e}(t),$$

with $\mathcal{P}\Gamma_e(0) = 0_{(N+1)\times 1}$, $\tilde{L}_l = (0, L_l^\top)^\top$, L_l in (3.73).

Then one obtains $\alpha_e(t) = \mathcal{P}\Gamma_e(t)$ and therefore $\Gamma_e(t)$ satifies

$$\dot{\Gamma}_e(t) = (\mathcal{A} + \mathcal{B}K_0\mathcal{P})\Gamma_e(t) + \mathcal{B}\mathcal{N}(\mathcal{P}\Gamma_e(t)) + \tilde{l}(x)\tilde{e}(t),$$

with $\Gamma_e(0) = (0,0)^\top$. Finally the closed-loop system (A.20) admits a solution $(e(t,\cdot), \Gamma_e(t))$ and thus (3.80)-(3.83) admits a solution. □

A.0.5 Proof of Lemma 3.4

Proof. The proof is similar to the previous one. Using the integration by parts formula and the Wirtinger inequality [153], we obtain

$$\sum_{n\geq 1} |\lambda_n| e_n^2(t) \leq \|e(t,\cdot)\|^2_{H^1(0,1)} \leq (1 + \tfrac{4}{\pi^2}) \sum_{n\geq 1} |\lambda_n| e_n^2(t).$$

Then,

$$\begin{gathered} \min\{\tfrac{1}{\lambda_{max}(P)}, \tfrac{1}{\gamma_5}\}\Big(\|E(t)\|_P^2 + \gamma_5 \textstyle\sum_{n\geq N+1} |\lambda_n| e_n^2(t)\Big) \leq \\ \|e(t,\cdot)\|^2_{H^1(0,1)} \leq \\ (1 + \tfrac{4}{\pi^2}) \max\{\tfrac{1}{\gamma_5}, \tfrac{|\lambda_{N+1}|}{\lambda_{min}(P)}\}\Big(\|E(t)\|_P^2 + \gamma_5 \textstyle\sum_{n\geq N+1} |\lambda_n| e_n^2(t)\Big). \end{gathered}$$

With Lemma 3.2 and using the fact that $\Omega_2 \in \mathcal{K}_\infty$, the proof is complete. □

Appendix B

Some technical proofs in Chapter 6

B.0.1 Proof of Corollary 2.1

Proof. Since (X, Y, λ) is a solution of the opimization problem (2.95) then

$$\lambda_{\max}(W) = \lambda_{\max}((A + BK_{\text{lin}})^\top P + P(A + BK_{\text{lin}}) + 2\rho P) < 0.$$

Hence, by construction, $-1 \leq \mu_{\min} < 0$ and $0 < \mu_{\max} \leq \frac{1}{\tilde{n}}$, where $\tilde{n} \in \mathbb{N}$ is defined in the formulation of Theorem 2.4.

1) Let us show that $PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0$. Indeed, if $\lambda_{\max}\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right) \leq 0$ then $\mu_{\min}P^{-\frac{1}{2}}QP^{-\frac{1}{2}} \succeq 0$ and

$$2I_n + \mu_{\min}P^{-\frac{1}{2}}QP^{-\frac{1}{2}} \succ 0, \tag{B.1}$$

If $\lambda_{\max}\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right) > 0$ then using the definition of $\mu_{\min}$ we derive

$$\mu_{\min}\lambda_{\max}\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right) \geq -2\delta \quad \Rightarrow \quad 2\delta I_n + \mu_{\min}P^{-\frac{1}{2}}QP^{-\frac{1}{2}} \succeq 0,$$

so the inequality (B.1) is fulfilled again.

Similarly, if $\lambda_{\min}\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right) \geq 0$ then $\mu_{\max}\lambda_{\min}\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right) \geq 0$ and

$$2I_n + \mu_{\max}P^{-\frac{1}{2}}QP^{-\frac{1}{2}} \succ 0. \tag{B.2}$$

If $\lambda_{\min}\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right) < 0$ then using the definition of $\mu_{\max}$ we derive

$$-\mu_{\max}\lambda_{\min}\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right) \leq 2\delta \quad \Rightarrow \quad 2\delta I_n + \mu_{\max}P^{-\frac{1}{2}}QP^{-\frac{1}{2}} \succeq 0,$$

so the inequality (B.2) is fulfilled again.

Using the convex embedding $\mu = \lambda\mu_{\min} + (1-\lambda)\mu_{\max}$ for some $\lambda \in [0, 1]$, from the LMIs (B.1), (B.2) we derive the LMI

$$2I_n + \mu P^{-\frac{1}{2}}(PG_0 + G_0^\top P)P^{-\frac{1}{2}} \succ 0. \tag{B.3}$$

Due to $G_{\mathbf{d}} = I_n + \mu G_0$, this LMI is equivalent to $PG_{\mathbf{d}} + G_{\mathbf{d}}P \succ 0$.

2) If $\lambda_{\min}(PG_0+G_0^\top P)\geq 0$ then $PG_0+G_0^\top P\succeq 0$, $\mu_{\min}(PG_0+G_0^\top P)\preceq 0$ and

$$(A + BK_{\text{lin}})^\top P + P(A + BK_{\text{lin}}) + 2\rho P + \mu_{\min}\rho(PG_0 + G_0^\top P) \preceq 0. \tag{B.4}$$

If $\lambda_{\min}(PG_0 + G_0^\top P) < 0$ then using the definition of $\mu_{\min}$ we derive

$$-\mu_{\min}\rho\lambda_{\min}(PG_0 + G_0^\top P) \geq \lambda_{\max}((A + BK_{\text{lin}})^\top P + P(A + BK_{\text{lin}}) + 2\rho P).$$

Hence, taking into account $-\mu_{\min}\rho(PG_0 + G_0^\top P) \succeq -\mu_{\min}\rho\lambda_{\min}(PG_0 + G_0^\top P)I_n$ we conclude that the inequality (B.4) is fulfilled again.

If $\lambda_{\max}(PG_0+G_0^\top P)\leq 0$ then $PG_0+G_0^\top P\preceq 0$, $\mu_{\max}(PG_0+G_0^\top P)\preceq 0$ and

$$(A + BK_{\text{lin}})^\top P + P(A + BK_{\text{lin}}) + 2\rho P + \mu_{\max}\rho(PG_0 + G_0^\top P) \preceq 0. \tag{B.5}$$

If $\lambda_{\max}(PG_0 + G_0^\top P) > 0$ then using the definition of $\mu_{\max}$ we derive

$$\mu_{\max}\rho\lambda_{\max}(PG_0 + G_0^\top P) \leq -\lambda_{\max}((A + BK_{\text{lin}})^\top P + P(A + BK_{\text{lin}}) + 2\rho P).$$

Hence, taking into account $\mu_{\max}\rho\lambda_{\max}(PG_0+G_0^\top P)I_n \succeq \mu_{\max}\rho(PG_0+G_0^\top P)$ we conclude that the inequality (B.5) is fulfilled again. Using the convex embedding $\mu = \lambda\mu_{\min} + (1-\lambda)\mu_{\max}$ for some $\lambda \in [0, 1]$, from the LMIs (B.4), (B.5) we derive that the LMI

$$(A + BK_{\text{lin}})^\top P + P(A + BK_{\text{lin}}) + 2\rho P + \mu\rho(PG_0 + G_0^\top P) \preceq 0, \tag{B.6}$$

holds for any $\mu \in [\mu_{\min}, \mu_{\max}]$.

Taking into account $G_{\mathbf{d}} = I_n + \mu G_0$ and $A + BK_{\text{lin}} = A_0 + BK$ for $K = K_{\text{lin}} - K_0$ we derive that the following LMI

$$(A_0 + BK)^\top P + P(A_0 + BK) + \rho(PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P) \preceq 0$$

is fulfilled. Hence, repeating the proof of Theorem 2.4, we conclude

$$\tfrac{d\|x\|_{\mathbf{d}}}{dt} = \|x\|_{\mathbf{d}}^{1+\mu} \tfrac{x^\top \mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})((A_0+BK)^\top P+P(A_0+BK))\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}{x^\top \mathbf{d}(-\ln\|x\|_{\mathbf{d}})(G_{\mathbf{d}}^\top P+PG_{\mathbf{d}})\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x} \leq -\rho\|x\|_{\mathbf{d}}^{1+\mu}.$$

3) The above inequality, in particular, implies that the set $\{x \in \mathbb{R}^n : \|x\|_{\mathbf{d}} \leq 1\}$ is invariant for the system. Taking into account that $\{x \in \mathbb{R}^n : \|x\|_P \leq 1\} = \{x \in \mathbb{R}^n : \|x\|_{\mathbf{d}} \leq 1\}$ we conclude that $\mathcal{E}(P, 1)$ is the invariant set of the closed-loop system with the homogeneous control.

4) The result follows from the following identity

$$\sup_{x\in\mathcal{E}(P,1)} |(K_{\text{lin}} - K_0)x|^2 = \sup_{x\in\partial\mathcal{E}(P,1)} |K_{\text{lin}}x - K_0x|^2 =$$

$$\sup_{x\in\partial\mathcal{E}(P,1)} \underbrace{\|x\|_{\mathbf{d}}^{1+\mu}}_{=1 \text{ for } \forall x\in\partial\mathcal{E}(P,1)} |(K_{\text{lin}}-K_0) \underbrace{\mathbf{d}(-\ln\|x\|_{\mathbf{d}})}_{=I_n \text{ for } \forall x\in\partial\mathcal{E}(P,1)} x| = \sup_{x\in\partial\mathcal{E}(P,1)} |u_\mu(x)-K_0x|^2 =$$

$$\sup_{x\in\mathcal{E}(P,1)} \underbrace{\|x\|_{\mathbf{d}}^{1+\mu}}_{\leq 1 \text{ for } \forall x\in\mathcal{E}(P,1)} |(K_{\text{lin}}-K_0) \underbrace{\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}_{\in\partial\mathcal{E}(P,1) \text{ for } \forall x\in\mathcal{E}(P,1)} |^2 = \sup_{x\in\mathcal{E}(P,1)} |u_\mu(x)-K_0x|$$

and the identity

$$\sup_{x\in\mathcal{E}(P,1)} |(K_{\text{lin}}-K_0)x|^2 = \sup_{x\in\mathcal{E}(P,1)} x^\top(K_{\text{lin}}-K_0)^\top(K_{\text{lin}}-K_0)x =$$

$$\sup_{x\in\mathcal{E}(P,1)} x^\top P^{\frac{1}{2}}P^{-\frac{1}{2}}(K_{\text{lin}}-K_0)^\top(K_{\text{lin}}-K_0)P^{-\frac{1}{2}}P^{\frac{1}{2}}x =$$

$$\sup_{y\in\mathbb{R}^n:y^\top y\leq 1} y^\top P^{-\frac{1}{2}}(K_{\text{lin}}-K_0)^\top(K_{\text{lin}}-K_0)P^{-\frac{1}{2}}y = \lambda_{\max}\left(P^{-\frac{1}{2}}(K_{\text{lin}}-K_0)^\top(K_{\text{lin}}-K_0)P^{-\frac{1}{2}}\right).$$

5) Notice that

$$\sup_{x\in\mathcal{E}(P,1)} |u_\mu(x)|^2 = \sup_{x\in\mathcal{E}(P,1)} \left|\left(K_0\mathbf{d}(\ln\|x\|_{\mathbf{d}})+\|x\|_{\mathbf{d}}^{1+\mu}(K_{\text{lin}}-K_0)\right)\underbrace{\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}_{\in\partial\mathcal{E}(P,1)}\right|^2 =$$

$$\sup_{\delta\in[0,1],y\in\partial\mathcal{E}(P,1)} \left|\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\text{lin}}-K_0)\right)y\right|^2 =$$

$$\sup_{\delta\in[0,1],y\in\mathbb{R}^n:|y|=1} |\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\text{lin}}-K_0)\right)P^{-\frac{1}{2}}y|^2 =$$

$$\sup_{\delta\in[0,1]} \lambda_{\max}\Big(P^{-\frac{1}{2}}\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\text{lin}}-K_0)\right)^\top\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\text{lin}}-K_0)\right)P^{-\frac{1}{2}}\Big) =$$

$$\sup_{\delta\in[0,1]} \left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\text{lin}}-K_0)\right)P^{-1}\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\text{lin}}-K_0)\right)^\top$$

Let us consider the function $q:[0,1]\to\mathbb{R}_+$ defined as follows

$$q(\delta) = \left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\text{lin}}-K_0)\right)P^{-1}\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\text{lin}}-K_0)\right)^\top$$

Let us show that $q'(\delta)>0$ for $\delta\in(0,1)$, at least, if μ is close enough to zero. Indeed, since

$$q'(\delta) = 2\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\text{lin}}-K_0)\right)P^{-1}\left(\frac{1}{\delta}K_0G_{\mathbf{d}}\mathbf{d}(\ln\delta)+(1+\mu)\delta^{\mu}(K_{\text{lin}}-K_0)\right)^\top$$

$$= \frac{2}{\delta}\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\text{lin}}-K_0)\right)P^{-1}\left(K_0G_{\mathbf{d}}\mathbf{d}(\ln\delta)+(1+\mu)\delta^{1+\mu}(K_{\text{lin}}-K_0)\right)^\top$$

$$= \frac{2}{\delta}\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\text{lin}}-K_0)\right)P^{-1}(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\text{lin}}-K_0)+\mu(K_0G_0\mathbf{d}(\ln\delta)$$

$$+\delta^{1+\mu}(K_{\text{lin}}-K_0)))^\top =$$

$$\frac{2}{\delta}\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\rm lin}-K_0)\right)P^{-1}\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\rm lin}-K_0)\right)^\top+$$
$$\frac{2\mu}{\delta}\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\rm lin}-K_0)\right)P^{-1}\left(K_0G_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\rm lin}-K_0)\right)^\top$$
$$\geq\frac{(2+\mu)}{\delta}\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\rm lin}-K_0)\right)P^{-1}\left(K_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\rm lin}-K_0)\right)^\top$$
$$+\frac{\mu}{\delta}\left(K_0G_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\rm lin}-K_0)\right)P^{-1}\left(K_0G_0\mathbf{d}(\ln\delta)+\delta^{1+\mu}(K_{\rm lin}-K_0)\right)^\top.$$

then taking into account $\mathbf{d}(\ln\delta) = \delta e^{\mu G_0 \ln\delta}$ we conclude that, at least, for μ being sufficiently close to zero we have $q'(\delta)>0$ for any $\delta\in(0,1)$. Hence, $q(\delta)\leq q(1)$ for all $\delta\in[0,1]$. □

B.0.2 Proof of Lemma 6.2

Proof. Let $r>0$ and assume that the following estimates

$$V_1(t)\leq 1, \tag{B.7}$$
$$\|\partial_x w(t,\cdot)\|^2_{L^2(0,1)}+U^2(t)\leq r, \tag{B.8}$$
$$\max\{1+\mu,-\mu\}<\eta<\zeta<2+\mu, \tag{B.9}$$

hold where η,ζ are defined in (6.38). Using the formula (2.22) and the Theorem 2.4 , along the solutions of the closed-loop system (6.40)-(6.43) one has:

$$\dot V_1(t)\leq -\rho V_1^{1+\mu}(t)+V_1(t)\tfrac{W_u(t)^\top\mathbf{d}^\top(-\ln V_1(t))X^{-1}\mathbf{d}(-\ln V_1(t))F_N(t)}{W_u(t)^\top\mathbf{d}^\top(-\ln V_1(t))X^{-1}G_{\mathbf{d}}\mathbf{d}(-\ln V_1(t))W_u(t)}, \tag{B.10}$$

which implies

$$\dot V_1(t)\leq -\rho V_1^{1+\mu}(t)+\tfrac{1}{\eta}V_1(t)\Big\|\mathbf{d}(-\ln V_1(t))F_N(t)\Big\|_{X^{-1}}, \tag{B.11}$$

provided that

$$\|\mathbf{d}(-\ln V_1(t))W_u(t)\|_{X^{-1}}=1,\quad W_u(t)^\top\mathbf{d}^\top(-\ln V_1(t))X^{-1}G_{\mathbf{d}}\mathbf{d}(-\ln V_1(t))W_u(t)\geq\eta, \tag{B.12}$$

from the Proposition 2.1. Using the estimate (B.11) one obtains

$$\dot V_1(t)\leq -\rho V_1^{1+\mu}(t)+\tfrac{1}{\eta}V_1(t)\|\mathbf{d}(-\ln V_1(t))\|_{X^{-1}}\|F_N(t)\|_{X^{-1}}, \tag{B.13}$$

which together the following estimate

$$\|\mathbf{d}(-\ln V_1(t))\|_{X^{-1}}\leq V_1^{-\zeta}(t), \tag{B.14}$$

(obtained using the Proposition 2.2 and (B.7)) give

$$\dot{V}_1(t) \le -\rho V_1^{1+\mu}(t) + M_2 V_1^{1-\zeta}(t)|F_N(t)|, \tag{B.15}$$

with $M_2 = \frac{\lambda_{max}(X^{-1})^{\frac{1}{2}}}{\eta}$. By using the expression of Ω_1 defined in (6.57) one has

$$\frac{d\Omega_1(V_1(t))}{dt} = \dot{V}_1(t) V_1^{1+\mu}(t), \tag{B.16}$$

and then from the previous inequality one gets

$$\frac{d\Omega_1(V_1(t))}{dt} \le -\rho V_1^{2(1+\mu)}(t) + M_2 V_1^{2+\mu-\zeta}(t)|F_N(t)|. \tag{B.17}$$

On the another hand, along the solution of the closed-loop system (6.40)-(6.43) one has

$$\dot{V}_2(t) = 2\sum_{n=N+1}^{\infty}(-\nu\lambda_n + q)\lambda_n w_n^2(t) + 2\sum_{n=N+1}^{\infty}\lambda_n w_n(t) f_n(t) + \tag{B.18}$$

$$2\sum_{n=N+1}^{\infty} b_n \lambda_n w_n(t)\big(\xi(W_u(t)) - qU(t)\big). \tag{B.19}$$

Next by using the Young inequality on the cross-terms

$$\sum_{n=N+1}^{\infty} \lambda_n w_n(t) f_n(t), \quad \sum_{n=N+1}^{\infty} b_n \lambda_n w_n(t)\big(\xi(W_u(t)) - qU(t)\big) \tag{B.20}$$

as follows

$$\sum_{n=N+1}^{\infty} \lambda_n w_n(t) f_n(t) \le \frac{\gamma_1}{2}\sum_{n=N+1}^{\infty} |f_n(t)|^2 + \frac{1}{2\gamma_1}\sum_{n=N+1}^{\infty} \lambda_n^2 w_n^2(t),$$

$$\sum_{n=N+1}^{\infty} b_n \lambda_n w_n(t)\big(\xi(W_u(t)) - qU(t)\big) \le \frac{\gamma_2}{2}\Big(\sum_{n=N+1}^{\infty} b_n^2\Big)|\xi(W_u(t)) - qU(t)|^2 \tag{B.21}$$

$$+\frac{1}{2\gamma_2}\sum_{n=N+1}^{\infty} \lambda_n^2 w_n^2(t), \tag{B.22}$$

one derives

$$\dot{V}_2(t) \le 2\sum_{n=N+1}^{\infty}\Big((-\nu + \tfrac{1}{2\gamma_1} + \tfrac{1}{2\gamma_2})\lambda_n + q\Big)\lambda_n w_n^2(t) + \gamma_1\sum_{n=N+1}^{\infty}|f_n(t)|^2 \tag{B.23}$$

$$+\gamma_2\big(\sum_{n=N+1}^{\infty} b_n^2\big)|\xi(W_u(t)) - qU(t)|^2, \tag{B.24}$$

$\gamma_1, \gamma_2 > 0$ will be defined later. Notice that since $1 + \mu < \eta$ from (B.9), by using Proposition 2.1 one obtains the following estimate :

$$|\xi(W_u(t)) - qU(t)| \le 2\lambda_{min}(X^{-1})^{\frac{-1}{2}} \max\{|K_0^\top| + q, |K^\top|\} V_1^{1+\mu}(t), \tag{B.25}$$

provided that

$$\|\mathbf{d}(-\ln V_1(t))W_u(t)\|_{X^{-1}} = 1 \quad \text{and} \quad |U(t)| \le |W_u(t)| \le \lambda_{min}(X^{-1})^{\frac{-1}{2}} V_1^{1+\mu}(t). \tag{B.26}$$

By using the expression of $V(t)$ is defined in (6.54) one has

$$\dot{V}(t) = \tfrac{d\Omega_1(V_1(t))}{dt} + \dot{V}_2(t), \tag{B.27}$$

and next by choosing $\gamma_1, \gamma_2 > \frac{\lambda_{N+1}}{\nu\lambda_{N+1}-q}$ and combining the estimates (B.17),(B.24)-(B.25) one gets

$$\dot{V}(t) \leq -(\rho - M_3)V_1^{2(1+\mu)}(t) - 2\Theta_{N,1}V_2(t) + \gamma_1 \sum_{n=N+1} |f_n(t)|^2 + M_2 V_1^{2+\mu-\zeta}(t)|F_N(t)|, \tag{B.28}$$

with

$$\Theta_{N,1} = (\nu - \tfrac{1}{2\gamma_1} - \tfrac{1}{2\gamma_2})\lambda_{N+1} - q > 0, \quad M_3 = \frac{4\gamma_2\big(\max\{|K_0^\top|+q, |K^\top|\}\big)^2}{\lambda_{min}(X^{-1})} \sum_{n=N+1}^{\infty} b_n^2. \tag{B.29}$$

Next, by using Parseval's identity, one has

$$\begin{aligned} \sum_{n=1}^{\infty} |f_n(t)|^2 &= \int_0^1 \Big|(w(t,x) + xU(t))(\partial_x w(t,x) + U(t))\Big|^2 dx, && \text{(B.30)}\\ &\leq 2\big(\|\partial_x w(t,\cdot)\|^2_{L^2(0,1)} + U^2(t)\big) \sup_{x\in[0,1]} |w(t,x) + xU(t)|^2, && \text{(B.31)}\\ \sum_{n=1}^{\infty} |f_n(t)|^2 &\leq 4\big(\|\partial_x w(t,\cdot)\|^2_{L^2(0,1)} + U^2(t)\big)^2, && \text{(B.32)} \end{aligned}$$

where we have used

$$\sup_{x\in[0,1]} |w(t,x) + xU(t)|^2 \leq 2 \sup_{x\in[0,1]} \Big|\int_0^x \partial_x w(t,x)dx\Big|^2 + 2U^2(t) \leq 2\|\partial_x w(t,\cdot)\|^2_{L^2(0,1)} + 2U^2(t),$$

and then using (B.8) one obtains

$$|F_N(t)| \leq 2\lambda_N |W_u(t)|^2 + 2V_2(t), \qquad \sum_{n=N+1}^{\infty} |f_n(t)|^2 \leq 4r\lambda_N |W_u(t)|^2 + 4rV_2(t). \tag{B.33}$$

From Proposition 2.1 and (B.7),(B.9), one has

$$|F_N(t)| \leq 2\lambda_N \lambda_{min}(X^{-1})^{-1} V_1^{2(1+\mu)}(t) + 2V_2(t), \tag{B.34}$$

and using the inequality (B.28) combined with the fact that

$$V_1^{2+\mu-\zeta+2(1+\mu)}(t) \leq V_1^{2(1+\mu)}(t)$$

(from (B.7) and (B.9)) one derives

$$\dot{V}(t) \leq -(\rho - M_4)V_1^{2(1+\mu)}(t) - 2(\Theta_{N,1} - 2\gamma_1 r)V_2(t) + 2M_2 V_1^{2+\mu-\zeta}(t)V_2(t), \tag{B.35}$$

with $M_4 = M_3 + 2\lambda_N \lambda_{min}(X^{-1})^{-1}(M_2 + 2\gamma_1 r)$. Next by using the Young's inequality on the cross term $V_1^{2+\mu-\zeta}(t)V_2(t)$ as follows

$$V_1^{2+\mu-\zeta}(t)V_2(t) \le \tfrac{\gamma_3^{\kappa_3}}{\kappa_3} V_1^{2(1+\mu)}(t) + \tfrac{1}{\kappa_2 \gamma_3^{\kappa_2}} V_2^{\kappa_2}(t), \tag{B.36}$$

where $\kappa_2, \kappa_3 > 1$ are given by:

$$\kappa_2 = \tfrac{2(1+\mu)}{\zeta+\mu}, \quad \kappa_3 = \tfrac{2(1+\mu)}{2+\mu-\zeta}, \tag{B.37}$$

one obtains

$$\dot{V}(t) \le -(\rho - M_4 - \tfrac{2M_2\gamma_3^{\kappa_3}}{\kappa_3})V_1^{2(1+\mu)}(t) - 2(\Theta_{N,1} - 2\gamma_1 r - \tfrac{M_2 r^{\kappa_2-1}}{\kappa_2\gamma_3^{\kappa_2}})V_2(t), \tag{B.38}$$

and $\gamma_3 > 0$ will be defined later. Now for r,γ_3 and ρ satisfying

$$r < \tfrac{\Theta_{N,1}}{2\gamma_1}, \quad \gamma_3 > \left(\tfrac{M_2 r^{\kappa_2-1}}{\kappa_2(\Theta_{N,1}-2\gamma_1 r)}\right)^{\frac{1}{\kappa_2}}, \quad \rho > M_4 + \tfrac{2M_2\gamma_3^{\kappa_3}}{\kappa_3}, \tag{B.39}$$

which is possible for r sufficiently small and ρ large enough, one gets the following estimate

$$\dot{V}(t) \le -\tilde{\rho}\psi_3(U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)}), \tag{B.40}$$

with $\tilde{\rho}$ given by

$$\tilde{\rho} = \min\{\rho - M_4 - \tfrac{2M_2\gamma_3^{\kappa_3}}{\kappa_3}, 2(\Theta_{N,1} - 2\gamma_1 r - \tfrac{M_2 r^{\kappa_2-1}}{\kappa_2\gamma_3^{\kappa_2}})\} > 0, \tag{B.41}$$

and with ψ_3 some $\mathcal{K}_\infty$ function such that

$$\psi_3(U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)}) \ge V_1^{2(1+\mu)} + V_2(t). \tag{B.42}$$

The proof of the existence of ψ_3 follows the same lines as the proof of Lemma 3.2. From the Proposition 2.1, one has the following estimates :

$$\begin{aligned} 2(2+\mu)X \succ G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top \succ 2\max\{1+\mu, -\mu\}X \quad &\Longrightarrow \quad \max\{-\mu, 1+\mu\} < \eta < \zeta < 2+\mu \\ X \succeq rI_{N+1}, \quad |W_u(t)|^2 \le r \quad &\Longrightarrow \quad V_1(t) \le 1, \end{aligned} \tag{B.43}$$

and then if the following LMIs

$$\tilde{A}_0 X + X\tilde{A}_0^\top + BY + Y^\top B^\top + \rho(G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top) \preceq 0, \tag{B.44}$$

$$2(2+\mu)X \succ G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top \succ 2\max\{1+\mu, -\mu\}X, \quad X = X^\top \succ rI_{N+1}, \tag{B.45}$$

hold for some ρ and r satisfying (B.39), the assumptions (B.7)-(B.9) can be reduced to the following one:

$$\forall t \ge 0, \quad \|\partial_x w(t,\cdot)\|^2_{L^2(0,1)} + U^2(t) \le r. \tag{B.46}$$

In conclusion, there exist $\tilde{\rho} > 0$ and $\psi_3 \in \mathcal{K}_\infty$ such that if $U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)} \le r$ then one gets

$$\dot{V}(t) \le -\tilde{\rho}\psi_3(U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)}). \tag{B.47}$$

The proof is complete. □

B.0.3 Proof of the Lemma 6.4

Proof. Let $r_1 > 0$ and assume that the following estimates

$$V_1(t) \leq 1, \tag{B.48}$$

$$\max\{1+\mu, -\mu\} < \eta < \zeta < 2+\mu, \tag{B.49}$$

$$U^2(t) + \|\partial_x \hat{w}(t,\cdot)\|^2_{L^2(0,1)} + \|\partial_x e(t,\cdot)\|^2_{L^2(0,1)} \leq r_1, \tag{B.50}$$

hold where η, ζ are defined in (6.38). We recall the expressions of $\kappa_1 > 0$ and $\kappa_2, \kappa_3 > 1$ given by

$$\kappa_1 = \tfrac{1+\mu}{\zeta+\mu}, \quad \kappa_2 = \tfrac{2(1+\mu)}{\zeta+\mu}, \quad \kappa_3 = \tfrac{2(1+\mu)}{2+\mu-\zeta}. \tag{B.51}$$

Along the solution of the closed-loop system (6.82)-(6.85) and following the same lines as in the proof of Lemma 6.2 one has

$$\begin{aligned}\tfrac{d\Omega_1(V_1(t))}{dt} + \dot{V}_2(t) \quad &\leq \quad -(\rho - M_6 - \tfrac{2M_2\gamma_6^{\kappa_3}}{\kappa_3})V_1^{2(1+\mu)}(t) - 2(\Theta_{N,2} - 2\gamma_4 r_1 - \tfrac{M_2 r_1^{\kappa_2-1}}{\kappa_2\gamma_6^{\kappa_2}})V_2(t)\\ &+ \quad V_1^{2+\mu}(t)\tfrac{\hat{W}_u(t)^\top \mathbf{d}^\top(-\ln V_1(t))X^{-1}\mathbf{d}(-\ln V_1(t))\tilde{L}_l(CE(t)+\Lambda(t))}{\hat{W}_u(t)^\top \mathbf{d}^\top(-\ln V_1(t))X^{-1}G_{\mathbf{d}}\mathbf{d}(-\ln V_1(t))\hat{W}_u(t)},\end{aligned}$$

with

$$\gamma_4, \gamma_5 > \tfrac{\lambda_{N+1}}{\nu\lambda_{N+1}-q}, \quad \Theta_{N,2} = (\nu - \tfrac{1}{2\gamma_4} - \tfrac{1}{2\gamma_5})\lambda_{N+1} - q > 0,$$

$$M_6 = M_7 + 2\lambda_N \lambda_{min}(X^{-1})^{-1}(M_2 + 2\gamma_4 r_1), \quad M_7 = \tfrac{4\gamma_5\left(\max\{|K_0^\top|+q, |K^\top|\}\right)^2}{\lambda_{min}(X^{-1})}\textstyle\sum_{n=N+1}^{\infty} b_n^2,$$

and $\gamma_6 > 0$ will be defined later. Using the fact that

$$\left|CE(t) + \Lambda(t)\right| \leq \left(|C^\top|\lambda_{min}(P)^{\frac{-1}{2}} + M_8\right)V_3^{\frac{1}{2}}(t), \quad M_8 = (\sum_{n=N+1}^{\infty} \tfrac{c_n^2}{\lambda_n})^{\frac{1}{2}}, \tag{B.52}$$

using the Propositions 2.1 and 2.2 one derives

$$\begin{aligned}\tfrac{d\Omega_1(V_1(t))}{dt} + \dot{V}_2(t) \quad &\leq \quad -(\rho - M_6 - \tfrac{2M_2\gamma_6^{\kappa_3}}{\kappa_3})V_1^{2(1+\mu)}(t) - 2(\Theta_{N,2} - 2\gamma_4 r_1 - \tfrac{M_2 r_1^{\kappa_2-1}}{\kappa_2\gamma_6^{\kappa_2}})V_2(t)\\ &+ \quad \tfrac{1}{\eta}\left(|C^\top|\lambda_{min}(P)^{\frac{-1}{2}} + M_8\right)\|\tilde{L}_l\|_{X^{-1}}V_1^{2+\mu-\zeta}(t)V_3^{\frac{1}{2}}(t).\end{aligned}$$

Notice that $M_8 \leq \frac{\|c\|_{L^2(0,1)}}{\sqrt{\lambda_{N+1}}}$ in the case of nonlocal measurements and $M_8 \leq \left(\sum_{n>N}\frac{2}{\lambda_n}\right)^{\frac{1}{2}}$ in the case of point measurements.

Next using the Young inequality one the cross term $V_1^{2+\mu-\zeta}(t)V_3^{\frac{1}{2}}(t)$ as done in (B.36) one derives

$$V_1^{2+\mu-\zeta}(t)V_3^{\frac{1}{2}}(t) \leq \tfrac{\gamma_7^{\kappa_3}}{\kappa_3}V_1^{2(1+\mu)}(t) + \tfrac{1}{2\kappa_1\gamma_7^{2\kappa_1}}V_3^{\kappa_1}(t), \tag{B.53}$$

where $\gamma_7 > 0$ will be defined later. Finally, one obtains

$$\frac{d\Omega_1(V_1(t))}{dt} + \dot{V}_2(t) \leq -(\rho - M_{10})V_1^{2(1+\mu)}(t) - 2(\Theta_{N,2} - M_{11})V_2(t) + 2M_{12}\Omega_2\big(V_3(t)\big), \quad \text{(B.54)}$$

with

$$M_{10} = M_6 + \frac{2M_2\gamma_6^{\kappa_3}}{\kappa_3} + \frac{M_9\gamma_7^{\kappa_3}}{\kappa_3}, \quad M_{11} = 2\gamma_4 r_1 + \frac{M_2 r_1^{\kappa_2-1}}{\kappa_2\gamma_6^{\kappa_2}}, \quad M_{12} = \frac{M_9}{4\gamma_7^{2\kappa_1}}, \quad \text{(B.55)}$$

$$M_9 = \frac{1}{\eta}\Big(|C^\top|\lambda_{min}(P)^{\frac{-1}{2}} + M_8\Big)\|\tilde{L}_l\|_{X^{-1}}. \quad \text{(B.56)}$$

Let's compute the time derivative $\dot{V}_3(t)$. Along the solutions of the closed-loop system (6.82)-(6.85) one has

$$\begin{aligned}\dot{V}_3(t) + 2\delta V_3(t) \quad = \quad & E(t)^\top\Big(P(A + L_lC) + (A + L_lC)^\top P + 2\delta P\Big)E(t) + 2\Big(O^N(t)^\top + \\ & \Lambda(t)L_l^\top\Big)PE(t) + \quad \text{(B.57)} \\ & 2\sum_{n=N+1}^{\infty}(-\nu\lambda_n + q + \delta)\lambda_n e_n^2(t) + 2\sum_{n=N+1}^{\infty}\lambda_n e_n(t)o_n(t), \quad \text{(B.58)}\end{aligned}$$

δ defined in (6.28). By using the Young inequality on the cross term $\Lambda(t)L_l^\top PE(t)$ combined with the fact that $|\Lambda(t)| \leq M_8\big(\sum_{n=N+1}^{\infty}\lambda_n e_n^2(t)\big)^{\frac{1}{2}}$ one obtains

$$\Lambda(t)L_l^\top PE(t) \leq \frac{1}{2\gamma_8}\sum_{n=N+1}^{\infty}\lambda_n e_n^2(t) + \frac{\gamma_8 M_8^2}{2}|L_l^\top PE(t)|^2 \quad \text{(B.59)}$$

with $\gamma_8 > 0$. Using the Parseval identity and the Sobolev inequality, one obtains the following estimate:

$$\begin{aligned}\sum_{n=1}^{\infty}o_n^2(t) \quad &= \quad \int_0^1\Big|\partial_x[(\hat{w}(t,x) + xU(t))e(t,x) - \tfrac{1}{2}e^2(t,x)]\Big|^2 dx, \quad \text{(B.60)} \\ &\leq \quad \Big[\big(\|\partial_x\hat{w}(t,\cdot)\|_{L^2(0,1)} + |U(t)|\big)\|\partial_x e(t,\cdot)\|_{L^2(0,1)} + \big(\|\partial_x\hat{w}(t,\cdot)\|_{L^2(0,1)} + |U(t)|\big) \\ &\qquad \times\|\partial_x e(t,\cdot)\|_{L^2(0,1)} + \|\partial_x e(t,\cdot)\|_{L^2(0,1)}^2\Big]^2, \quad \text{(B.61)} \\ \sum_{n=1}^{\infty}o_n^2(t) \quad &\leq \quad 12\big(\|\partial_x\hat{w}(t,\cdot)\|_{L^2(0,1)}^2 + \|\partial_x e(t,\cdot)\|_{L^2(0,1)}^2 + |U(t)|^2\big)\|\partial_x e(t,\cdot)\|_{L^2(0,1)}^2.\end{aligned}$$

Using (B.50) combined with the previous estimate, one obtains

$$\gamma_9\Big(\sum_{n=1}^{\infty}r_2\lambda_n e_n^2(t) - \sum_{n=1}^{\infty}o_n^2(t)\Big) \geq 0, \quad r_2 = 12r_1, \quad \text{(B.62)}$$

for any $\gamma_9 > 0$. Then using (B.58),(B.59) and (B.62) one derives

$$\begin{aligned}\dot{V}_3(t) + 2\delta V_3(t) &\leq E(t)^\top\Big(P(A+L_lC) + (A+L_lC)^\top P + 2r_2\gamma_9\Sigma_N + \gamma_8 M_8^2 P L_l L_l^\top P \\ &+ 2\delta P\Big)E(t) + 2O^N(t)^\top P E(t) - 2\gamma_9|O^N(t)|^2 + 2\sum_{n=N+1}^{\infty}\lambda_n e_n(t) o_n(t) \\ &- 2\gamma_9\sum_{n=N+1}^{\infty} o_n^2(t) + 2\sum_{n=N+1}^{\infty}(-\nu\lambda_n + q + \delta + \tfrac{1}{2\gamma_8} + \gamma_9 r_2)\lambda_n e_n^2(t).\end{aligned}$$

which can be rewritten as follows

$$\dot{V}_3(t) + 2\delta V_3(t) \leq \begin{bmatrix} E(t) \\ O^N(t)\end{bmatrix}^\top \begin{bmatrix}\Phi_P & P \\ P & -2\gamma_9 I_N\end{bmatrix}\begin{bmatrix} E(t) \\ O^N(t)\end{bmatrix} + \sum_{n=N+1}^{\infty}\begin{bmatrix} e_n(t) \\ o_n(t)\end{bmatrix}^\top \begin{bmatrix} 2\Phi_n & \lambda_n \\ \lambda_n & -2\gamma_9\end{bmatrix}\begin{bmatrix} e_n(t) \\ o_n(t)\end{bmatrix},$$

where

$$\Phi_n = \lambda_n\big(-\nu\lambda_n + q + \delta + \tfrac{1}{2\gamma_8} + \gamma_9 r_2\big), \quad \Sigma_N = \mathrm{diag}\{\lambda_n\}_{n=1}^N, \tag{B.63}$$
$$\Phi_P = P(A+L_lC) + (A+L_lC)^\top P + 2r_2\gamma_9\Sigma_N + \gamma_8 M_8^2 P L_l L_l^\top P + 2\delta P. \tag{B.64}$$

By applying the Schur complement Lemma, if γ_8, γ_9 and r_2 satisfy

$$\gamma_9 > \tfrac{\lambda_{N+1}}{4(\nu\lambda_{N+1} - q - \delta)}, \quad \gamma_8 > \tfrac{1}{(\nu - \frac{1}{4\gamma_9})\lambda_{N+1} - q - \delta}, \quad r_2 < \tfrac{1}{2\gamma_9}\Big((\nu - \tfrac{1}{4\gamma_9})\lambda_{N+1} - q - \delta\Big), \tag{B.65}$$

then $\begin{bmatrix} 2\Phi_n & \lambda_n \\ \lambda_n & -2\gamma_9\end{bmatrix} \leq 0$. Notice that if the following LMI

$$\begin{bmatrix} P(A+L_lC) + (A+L_lC)^\top P + 2\delta P & P & \gamma_8 M_8^2 P L_l & 2r_2\gamma_9\Sigma_N^{\frac{1}{2}} \\ * & -2\gamma_9 I_N & 0 & 0 \\ * & * & -\gamma_8 M_8^2 & 0 \\ * & * & * & -2r_2\gamma_9 I_N\end{bmatrix} \preceq 0, \tag{B.66}$$

holds, then applying the Schur complement Lemma once again, one derives

$$\begin{bmatrix}\Phi_P & P \\ P & -2\gamma_9 I_N\end{bmatrix} \preceq 0.$$

Due to (6.70) and following the same lines as in [81], [150], one can find $P = P^\top > 0$ such that the condition (B.66) is fulfilled. Finally one gets

$$\dot{V}_3(t) \leq -2\delta V_3(t), \quad \forall t \geq 0,$$

and using the expression of Ω_2 defined in (6.99) one derives

$$\tfrac{d\Omega_2(V_3(t))}{dt} \leq -2\delta\kappa_1\Omega_2(V_3(t)). \tag{B.67}$$

From the expression of $V_o(t)$ defined in (6.94) one has

$$\dot{V}_o(t) = \tfrac{d\Omega_1(V_1(t))}{dt} + \dot{V}_2(t) + \tfrac{d\Omega_2(V_3(t))}{dt}, \tag{B.68}$$

and using the estimates (B.54) and (B.67) one derives

$$\dot{V}_o(t) \leq -(\rho - M_{10})V_1^{2(1+\mu)}(t) - 2(\Theta_{N,2} - M_{11})V_2(t) - 2(\delta\kappa_1 - M_{12})\Omega_2(V_3(t)), \tag{B.69}$$

where

$$M_{10} = M_6 + \tfrac{2M_2\gamma_6^{\kappa_3}}{\kappa_3} + \tfrac{M_9\gamma_7^{\kappa_3}}{\kappa_3}, \quad M_{11} = 2\gamma_4 r_1 + \tfrac{M_2 r_1^{\kappa_2-1}}{\kappa_2\gamma_6^{\kappa_2}}, \quad M_{12} = \tfrac{M_9}{4\gamma_7^{2\kappa_1}}, \tag{B.70}$$

the parameters γ_6, γ_7 not yet defined and the other parameters defined above. Let r_1 sufficiently small, $\gamma_6, \gamma_7 > 0$ and $\rho > 0$ such that the following estimates

$$r_1 < \tfrac{\Theta_{N,2}}{2\gamma_4}, \quad \gamma_7 > \left(\tfrac{M_9}{4\kappa_1\delta}\right)^{\frac{1}{2\kappa_1}}, \quad \gamma_6 > \left(\tfrac{M_2 r_1^{\kappa_2-1}}{\kappa_2(\Theta_{N,2}-2\gamma_4 r_1)}\right)^{\frac{1}{\kappa_2}}, \quad \rho > M_{10}, \tag{B.71}$$

hold. In conclusion, one derives that

$$\dot{V}_o(t) \leq -\bar{\rho}\psi_6\Big(U^2(t) + \|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|e(t,\cdot)\|^2_{H^1(0,1)}\Big), \tag{B.72}$$

with

$$\bar{\rho} = \min\{\rho - M_{10}, 2(\kappa_1\delta - M_{12}), 2(\Theta_{N,2} - M_{11})\} > 0, \tag{B.73}$$

and ψ_6 some $\mathcal{K}_\infty$ function satisfying

$$V_1^{2(1+\mu)}(t) + V_2(t) + \Omega_2(V_3(t)) \geq \psi_6(U^2(t) + \|\hat{w}(t,\cdot)\|^2_{H^1(0,1)} + \|e(t,\cdot)\|^2_{H^1(0,1)}). \tag{B.74}$$

The proof of the existence of ψ_6 follows the same lines as the proof of Lemma 6.3. By following the same lines as in (B.43), if the following LMIs

$$\tilde{A}_0 X + X\tilde{A}_0^\top + BY + Y^\top B^\top + \rho(G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top) \preceq 0, \tag{B.75}$$

$$2(2+\mu)X \succ G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top \succ 2\max\{1+\mu, -\mu\}X, \quad X = X^\top \succ r_1 I_{N+1}, \tag{B.76}$$

hold for some ρ and r_1 satisfying (B.73), the assumptions (B.48)-(B.49) can be reduced to the following one:

$$U^2(t) + \|\partial_x\hat{w}(t,\cdot)\|^2_{L^2(0,1)} + \|\partial_x e(t,\cdot)\|^2_{L^2(0,1)} \leq r_1. \tag{B.77}$$

Finally, there exist $\bar{\rho} > 0$ and $\psi_6 \in \mathcal{K}_\infty$ such that if

$$U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)} + \|e(t,\cdot)\|^2_{H^1(0,1)} \leq r_1$$

then one gets

$$\dot{V}_o(t) \leq -\bar{\rho}\psi_6(U^2(t) + \|w(t,\cdot)\|^2_{H^1(0,1)} + \|e(t,\cdot)\|^2_{H^1(0,1)}). \tag{B.78}$$

The proof is complete. □

Appendix C

On Well-posedness of an infinite dimensional system with a locally Lipschitz nonlinearity

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Let us consider the following control system

$$\dot{w}_\infty(t) = (2qI_\infty - \Lambda_\infty)\, w_\infty(t) + F_\infty(w_\infty(t)) + B_\infty(\xi(t) - qw_0(t)), \quad t > 0, \quad w_\infty(0) \in \ell^2 \tag{C.1}$$

where

$$w_\infty(t) = (w_0(t), w_1(t), ...)^\top, \quad B_\infty = (b_0, b_1, b_2, ...)^\top, \quad B_\infty \in \ell^2, \tag{C.2}$$

are column vectors with an infinite number of elements, $q > 0$, $\Lambda_\infty = \mathrm{diag}\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ is the infinite-dimensional diagonal matrix with the elements

$$\forall n \geq 0, \quad \tilde{\lambda}_n \geq q, \tag{C.3}$$

on the main diagonal, I_∞ is the infinite-dimensional identity matrix, $F_\infty(w_\infty)$ is an infinite-dimensional vector, $\xi(\cdot) \in \mathbb{R}$ is the control input. The domain $\mathcal{D}(\Lambda_\infty)$ of the operator Λ_∞ is the space of real sequences $\{w_n\}_{n=0}^{\infty}$ quadratically summable with the weight $\tilde{\lambda}_n^2$ defined as follows:

$$\mathcal{D}(\Lambda_\infty) = \{(w_0, w_1, \cdots) \in \ell^2 / \sum_{n=0}^{\infty} \tilde{\lambda}_n^2 w_n^2 < \infty\}. \tag{C.4}$$

Let $\ell^2_\lambda \subset \ell^2$ be the space of real sequences $\{w_n\}_{n=0}^{\infty}$ quadratically summable with the weight $\tilde{\lambda}_n$. The space ℓ^2_λ is a Hilbert space with the inner product defined as follows

$$\langle w_\infty, v_\infty \rangle_{\ell^2_\lambda} = \sum_{n=0}^{\infty} \tilde{\lambda}_n w_n v_n, \quad w_\infty = (w_0, w_1, ...)^\top \in \ell^2_\lambda, \quad v_\infty = (v_0, v_1, ...)^\top \in \ell^2_\lambda. \tag{C.5}$$

The norm in ℓ^2_λ is $\|\cdot\|_{\ell^2_\lambda} = \sqrt{\langle \cdot, \cdot \rangle_{\ell^2_\lambda}}$. Assume that $F_\infty(\cdot)$ satisfies

$$\forall w^1_\infty, w^2_\infty \in \ell^2_\lambda, \quad \|F_\infty(w^1_\infty) - F_\infty(w^2_\infty)\|_{\ell^2} \leq C\left(\|w^1_\infty\|_{\ell^2_\lambda} + \|w^2_\infty\|_{\ell^2_\lambda}\right) \|w^1_\infty - w^2_\infty\|_{\ell^2_\lambda}. \tag{C.6}$$

for some $C > 0$. The existence of classical solutions of the evolution equation (C.1) in the space ℓ^2 can be studied using the conventional approach [134] based on strongly continuous semigroup of linear bounded operators. The operator $-\Lambda_\infty$ is an infinitesimal generator of the analytic semigroup of linear bounded operators given by

$$\Phi(t) = \mathrm{diag}\{e^{-\tilde{\lambda}_n t}\}_{n=0}^{\infty}, \quad t \geq 0,$$

on ℓ^2 and $\Phi(t)$ satisfies the following estimate

$$\forall t > 0, \quad \|\Lambda_\infty^{1/2}\Phi(t)\|_{\ell^2} \leq C_1 t^{-\frac{1}{2}}, \quad C_1 = \tfrac{1}{\sqrt{2e}}, \tag{C.7}$$

(see [134], [148] for more details).

C.0.1 Well-posedness analysis for (C.1) with a feedforward control $\xi \in C(\mathbb{R};\mathbb{R})$

Lemma C.1
For any $w_\infty(0) \in \ell^2_\lambda$ there exists $T > 0$ such that the system (C.1) has a unique mild solution $w_\infty(\cdot) = (w_0(\cdot), w_1(\cdot), ...)^\top \in C([0,T];\ell^2_\lambda)$ such that $w_n(\cdot) \in C^1((0,T);\mathbb{R})$ for any $n \geq 0$. Moreover, one has $\|w_\infty(t)\|_{\ell^2_\lambda} \leq r$ for all $t \in [0,T)$ provided that $T > 0$ and $r > \|w_\infty(0)\|_{\ell^2_\lambda}$ satisfy the inequality

$$e^{2qT}\|w_\infty(0)\|_{\ell^2_\lambda} + 2C_1 e^{2qT}\sqrt{T}\left(Cr^2 + \tfrac{q}{\sqrt{\tilde{\lambda}_0}}\|B_\infty\|_{\ell^2} r + \|B_\infty\|_{\ell^2}\|\xi\|_{C([0,T];\mathbb{R})}\right) \leq r.$$

Proof. The proof is, partially, inspired by [134, Chapter 6, Theorem 3.1]. Let us consider the operator $S : C([0,T];\ell^2) \to C([0,T];\ell^2)$ defined as follows

$$Sy_\infty(t) = e^{2qt}\Phi(t)\Lambda_\infty^{\frac{1}{2}} w_\infty(0) +$$

$$\int_0^t e^{2q(t-\tau)}\Lambda_\infty^{\frac{1}{2}}\Phi(t-\tau)\left(F_\infty(\Lambda_\infty^{-\frac{1}{2}} y_\infty(\tau)) + B_\infty\left(\xi(\tau) - \tfrac{q}{\sqrt{\tilde{\lambda}_0}} y_0(\tau)\right)\right) d\tau,$$

where $t \in [0,T]$, $y_\infty(\cdot) = (y_0(\cdot), y_1(\cdot), ...)^\top \in C([0,T];\ell^2)$ and $\Lambda_\infty^{\frac{1}{2}} = \mathrm{diag}\{\tilde{\lambda}_n^{1/2}\}_{n=0}^{\infty}$. The operator S is well defined on $C([0,T];\ell^2)$ since $F_\infty(\Lambda_\infty^{-1/2} y_\infty(\cdot)) \in C([0,T];\ell^2)$ (due to (C.6)), $B_\infty(\xi(\cdot) - \frac{q}{\sqrt{\tilde{\lambda}_0}} y_0(\cdot)) \in C([0,T];\ell^2)$ and $\|\Lambda_\infty^{1/2}\Phi(t-\tau)\|_{\ell^2} < \frac{C_1}{\sqrt{t-\tau}}, \forall \tau \in [0,t)$.

Indeed, one has

$$\begin{aligned}\|Sy_\infty(t)\|_{\ell^2} \le & e^{2qt}\underbrace{\|\Phi(t)\|_{\ell^2}}_{\le 1}\underbrace{\|\Lambda_\infty^{\frac{1}{2}}w_\infty(0)\|_{\ell^2}}_{=\|w_{\infty(0)}\|_{\ell^2_\lambda}} + C_1C\int_0^t \tfrac{e^{2q(t-\tau)}}{\sqrt{t-\tau}}d\tau \sup_{\tau\in[0,t]}\underbrace{\left\|\Lambda_\infty^{-\frac{1}{2}}y_\infty(\tau)\right\|^2_{\ell^2_\lambda}}_{=\|y_\infty(\tau)\|^2_{\ell^2}} \\ & + C_1\|B_\infty\|_{\ell^2}\int_0^t \tfrac{e^{2q(t-\tau)}}{\sqrt{t-\tau}}d\tau\left(\|\xi\|_{C([0,t];\mathbb{R})} + \tfrac{q}{\sqrt{\tilde{\lambda}_0}}\sup_{\tau\in[0,t]}\|y_\infty(\tau)\|_{\ell^2}\right) \\ \le & e^{2qt}\|w_\infty(0)\|_{\ell^2_\lambda} + \\ & 2C_1\sqrt{t}e^{2qt}(C\|y_\infty(\cdot)\|^2_{C([0,t];\ell^2)} + \tfrac{q}{\sqrt{\tilde{\lambda}_0}}\|B_\infty\|_{\ell^2}\|y_\infty(\cdot)\|_{C([0,t];\ell^2)} + \\ & \|B_\infty\|_{\ell^2}\|\xi\|_{C([0,t];\mathbb{R})}).\end{aligned}$$

for any $t > 0$. Given $r > \|w_\infty(0)\|_{\ell^2_\lambda}$ let us select a small enough $T > 0$ such that

$$e^{2qT}\|w_\infty(0)\|_{\ell^2_\lambda} + 2C_1e^{2qT}\sqrt{T}\left(Cr^2 + \tfrac{q}{\sqrt{\tilde{\lambda}_0}}\|B_\infty\|_{\ell^2}r + \|B_\infty\|_{\ell^2}\|\xi\|_{C([0,T];\mathbb{R})}\right) \le r$$

and $2C_1e^{2qT}\sqrt{T}(2Cr + \frac{q}{\sqrt{\tilde{\lambda}_0}}\|B_\infty\|_{\ell^2}) < 1$. In this case the operator S maps the complete metric space $K = \{y_\infty(\cdot) \in C([0,T];\ell^2) : \|y_\infty(\cdot)\|_{C([0,T];\ell^2)} \le r\}$ into K. Let us show that for small enough $T > 0$ this operator has a fixed point: $Sy_\infty(\cdot) = y_\infty(\cdot)$ on K. Since

$$\begin{aligned}\|Sy^1_\infty(t) - Sy^2_\infty(t)\|_{\ell^2} \le & C_1\int_0^t \tfrac{e^{2q(t-\tau)}}{\sqrt{t-\tau}}\left\|F_\infty(\Lambda_\infty^{-\frac{1}{2}}y^1_\infty(\tau)) - F_\infty(\Lambda_\infty^{-\frac{1}{2}}y^2_\infty(\tau))\right\|_{\ell^2}d\tau \\ & + C_1\int_0^t \tfrac{e^{2q(t-\tau)}}{\sqrt{t-\tau}}\tfrac{q}{\sqrt{\tilde{\lambda}_0}}\|B_\infty\|_{\ell^2}\|y^1_\infty(\tau) - y^2_\infty(\tau)\|_{\ell^2}d\tau\end{aligned}$$

then using (C.6) we derive

$$\begin{aligned}\|Sy^1_\infty(t) - Sy^2_\infty(t)\|_{\ell^2} \le & CC_1\int_0^t \tfrac{e^{2q(t-\tau)}}{\sqrt{t-\tau}}\left(\underbrace{\left\|\Lambda_\infty^{-\frac{1}{2}}y^1_\infty(\tau)\right\|_{\ell^2_\lambda}}_{=\|y^1_\infty(\tau)\|_{\ell^2}} + \underbrace{\left\|\Lambda_\infty^{-\frac{1}{2}}y^2_\infty(\tau)\right\|_{\ell^2_\lambda}}_{=\|y^2_\infty(\tau)\|_{\ell^2}}\right)\times \\ & \underbrace{\left\|\Lambda_\infty^{-\frac{1}{2}}(y^1_\infty(\tau) - y^2_\infty(\tau))\right\|_{\ell^2_\lambda}}_{=\|y^1_\infty(\tau)-y^2_\infty(\tau)\|_{\ell^2}}d\tau \\ & + C_1\int_0^t \tfrac{e^{2q(t-\tau)}}{\sqrt{t-\tau}}\tfrac{q}{\sqrt{\tilde{\lambda}_0}}\|B_\infty\|_{\ell^2}\left\|y^1_\infty(\tau) - y^2_\infty(\tau)\right\|_{\ell^2}d\tau.\end{aligned}$$

So, for $y^1_\infty(\cdot), y^2_\infty(\cdot) \in K$ we have

$$\|Sy^1_\infty(\cdot) - Sy^2_\infty(\cdot)\|_{C([0,T];\ell^2)} \le 2C_1e^{2qT}\sqrt{T}(2Cr + \tfrac{q}{\sqrt{\tilde{\lambda}_0}}\|B_\infty\|_{\ell^2})\|y^1_\infty(\cdot) - y^2_\infty(\cdot)\|_{C([0,T];\ell^2)}.$$

Since a small enough $T > 0$ is selected such that $2C_1 e^{2qT}\sqrt{T}(2Cr + \frac{q}{\sqrt{\tilde\lambda_0}}\|B_\infty\|_{\ell^2}) < 1$ then the operator S is a contraction, so, by Banach Fixed-Point Theorem, it has a unique fixed point $y_\infty(\cdot)$ on K, i.e., for all $t \in [0,T]$ one holds

$$y_\infty(t) = e^{2qt}\Phi(t)\Lambda_\infty^{\frac{1}{2}} w_\infty(0) +$$
$$\int_0^t e^{2q(t-\tau)}\Lambda_\infty^{\frac{1}{2}}\Phi(t-\tau)\left(F_\infty(\Lambda_\infty^{-\frac{1}{2}} y_\infty(\tau)) + B_\infty(\xi(\tau) - \frac{q}{\sqrt{\tilde\lambda_0}} y_0(\tau))\right) d\tau.$$

Hence, denoting $w_\infty(t) = \Lambda_\infty^{-\frac{1}{2}} y_\infty(t)$ we derive

$$w_\infty(t) = e^{2qt}\Phi(t)w_\infty(0) + \int_0^t e^{2q(t-\tau)}\Phi(t-\tau)\left(F_\infty(w_\infty(\tau)) + B_\infty(\xi(\tau) - qw_0(\tau))\right) d\tau,$$

i.e., $w_\infty(\cdot) \in C([0,T];\ell^2_\lambda)$ is the unique mild solution of (C.1). Moreover, for the unit vector $e_n = (0,...,0,1,0,...)^\top \in \ell^2$ we have

$$\begin{aligned} w_n(t) &= e_n^\top w_\infty(t) \\ &= e^{2qt}e^{-\tilde\lambda_n t}w_n(0) + \int_0^t e^{2q(t-\tau)}e^{-\tilde\lambda_n(t-\tau)}e_n^\top\left(F_\infty(w_\infty(\tau)) + B_\infty(\xi(\tau) - qw_0(\tau))\right) d\tau. \end{aligned}$$

Hence, we have

$$\begin{aligned} \dot w_n(t) =& (2q - \tilde\lambda_n)e^{2qt}e^{-\tilde\lambda_n t}w_n(0) + e_n^\top F_\infty(w_\infty(t)) + e_n^\top B_\infty(\xi(t) - qw_0(t)) + \\ &+ (2q - \tilde\lambda_n)\int_0^t e^{2q(t-\tau)}e^{-\tilde\lambda_n(t-\tau)}e_n^\top F_\infty(w_\infty(\tau)) + e_n^\top B_\infty(\xi(\tau) - qw_0(\tau))d\tau \end{aligned} \tag{C.8}$$

Since $\tau \mapsto w_0(\tau)$, $\tau \mapsto \xi(\tau)$ and $\tau \mapsto e_n^\top F_\infty(w_\infty(\tau))$ are a continuous functions then $w_n(\cdot) \in C^1([0,T];\mathbb{R})$. The proof is complete. □

C.0.2 Well-posedness analysis for (C.1) with a continuous finite-dimensional feedback control

Let the linear operator $\Pi : \ell^2 \to \mathbb{R}^{N+1}$ be a finite-dimensional projector ℓ^2 on $\mathbb{R}^{N+1}$ defined as follows

$$\Pi w_\infty = (w_0, w_1, ..., w_N)^\top \in \mathbb{R}^{N+1}.$$

Let

$$\beta(t) = \xi(\Pi w_\infty(t)), \quad t \geq 0, \tag{C.9}$$

where $\xi \in C(\mathbb{R}^{N+1};\mathbb{R})$ satisfies the estimate

$$\forall \alpha \in \mathbb{R}^{N+1}, \quad |\xi(\alpha)| \leq C_2|\alpha|^p, \tag{C.10}$$

for some positive constants C_2 and $p > 0$. The closed-loop system (C.1)-(C.9) is then given by

$$\dot{w}_\infty(t) = (2qI_\infty - \Lambda_\infty)\, w_\infty(t) + F_\infty(w_\infty(t)) + B_\infty(\beta(t) - qw_0(t)), \quad t > 0, \quad w_\infty(0) \in \ell^2. \tag{C.11}$$

Theorem C.1
For any $w_\infty(0) \in \ell^2_\lambda$ there exists $T > 0$ such that the closed-loop system (C.11) has a mild solution $w_\infty(\cdot) = (w_0(\cdot), w_1(\cdot), \ldots)^\top \in C([0,T]; \ell^2_\lambda)$ such that $w_n(\cdot) \in C^1([0,T]; \mathbb{R})$.

Proof. Inspired by [129], we consider the family of equations defined iteratively

$$\dot{w}_\infty^{[k]}(t) = (2qI_\infty - \Lambda_\infty) w_\infty^{[k]}(t) + F_\infty(w_\infty^{[k]}(t)) + B_\infty(\beta^{[k]}(t) - qw_0^{[k]}(t)), \quad k \geq 0, \tag{C.12}$$

with $w_\infty^{[k]}(0) = w_\infty(0)$ and

$$\beta^{[0]}(t) = \xi(\Pi w_\infty(0)), \quad \beta^{[k]}(t) = \xi(\Pi w_\infty^{[k-1]}(t)), \quad k \geq 1. \tag{C.13}$$

By Lemma C.1, the system considered has a unique solution $w_\infty^{[k]}(\cdot) \in C([0,T]; \ell^2_\lambda)$. Moreover, the small enough $T > 0$ can be selected independently of k. In fact, let $T > 0$ be such that the inequality

$$e^{2qT}\|w_\infty(0)\|_{\ell^2_\lambda} + 2C_1 e^{2qT}\sqrt{T}\left(Cr^2 + \frac{q}{\sqrt{\tilde{\lambda}_0}}\|B_\infty\|_{\ell^2} r + C_2 q^{-p/2} r^p \|B_\infty\|_{\ell^2}\right) \leq r,$$

and

$$2C_1 e^{2qT}\sqrt{T}(2Cr + \frac{q}{\sqrt{\tilde{\lambda}_0}}\|B_\infty\|_{\ell^2}) < 1,$$

is fulfilled for some $r > \|w_\infty(0)\|_{\ell^2_\lambda}$. In this case,

$$|\beta^{[0]}(t)| = |\xi(\Pi w_\infty(0))| \leq C_2 \|w_\infty(0)\|^p_{\ell^2} \leq C_2 q^{-p/2} r^p, \quad \forall t \in [0,T]$$

and, by Lemma C.1, we have $\|w_\infty^{[0]}(t)\|_{\ell^2_\lambda} \leq r$ for all $t \in [0,T]$. Similarly, by induction, if $\|w_\infty^{[k-1]}(t)\|_{\ell^2_\lambda} \leq r$ for all $t \in [0,T]$ then

$$|\beta^{[k]}(t)| = |\xi(\Pi w_\infty^{[k-1]}(t))| \leq C_2 \|w_\infty^{[k-1]}(t)\|^p_{\ell^2} \leq C_2 q^{-p/2} r^p, \quad \forall t \in [0,T]$$

and, by Lemma C.1, we have $\|w_\infty^{[k]}(t)\|_{\ell^2_\lambda} \leq r$ for all $t \in [0,T]$ and for any $k \geq 1$. Therefore, the sequence of functions $\Pi w_\infty^{[k]}(\cdot) \in C([0,T]; \mathbb{R}^{N+1})$ is uniformly bounded. Similarly (using the estimate (C.8)) it can be shown that the time derivative of the function $t \mapsto \Pi w_\infty^{[k]}(t)$ is uniformly bounded, so the sequence of functions $\Pi w_\infty^{[k]}(\cdot) \in C([0,T]; \mathbb{R}^{N+1})$ is equicontinuous. By Arzela–Ascoli theorem, there exists a uniformly convergent subsequence

$$\Pi w_\infty^{[k_m]}(\cdot) \to \pi(\cdot) \in C([0,T]; \mathbb{R}^{N+1}) \quad \text{as} \quad k_m \to \infty.$$

Since $|\xi(\Pi w_\infty^{[k_m-1]}(t))| \leq C_2 q^{-p/2} r^p$ and $\xi \in C(\mathbb{R}^{N+1};\mathbb{R})$ then $|\xi(\pi(t))| \leq C_2 q^{-p/2} r^p$.

Let us consider the system

$$\dot{w}_\infty(t) = (qI_\infty - \Lambda_\infty) w_\infty(t) + F_\infty(w_\infty(t)) + B_\infty \left(\xi(\pi(t)) - q w_0(t)\right), \tag{C.14}$$

which, by Lemma C.1, has the unique solution on $t \in [0,T]$ satisfying $\|w_\infty(t)\|_{\ell^2_\lambda} \leq r$ for all $t \in [0,T]$. Let us show that

$$\sup_{t\in[0,T]} \|w_\infty^{[k_m]}(t) - w_\infty(t)\|_{\ell^2_\lambda} \to 0 \quad \text{as} \quad k_m \to \infty.$$

Indeed, for $k_m \geq 1$ we have

$$\begin{aligned} w_\infty^{[k_m]}(t) - w_\infty(t) = &\int_0^t e^{2q(t-\tau)} \Phi(t-\tau) \left(F_\infty(w_\infty^{[k_m]}(\tau)) - F_\infty(w_\infty(\tau)) \right) d\tau \\ &+ \int_0^t e^{2q(t-\tau)} \Phi(t-\tau) B_\infty \left(\xi(\Pi w_\infty^{[k_m]}(\tau)) - \xi(\pi(\tau)) + q(w_0(\tau) - w_0^{[k_m]}(\tau)) \right) d\tau \end{aligned}$$

Since $\|w_\infty(t)\|_{\ell^2_\lambda} \leq r$ and $\|w_\infty^{[k_m]}(t)\|_{\ell^2_\lambda} \leq r$ for all $t \in [0,T]$ and for all $k_m \geq 0$ then we have

$$\begin{aligned} \|w_\infty^{[k_m]}(t) - w_\infty(t)\|_{\ell^2_\lambda} \leq & 2\sqrt{T} e^{2qT} C_1 (2Cr + \tfrac{q}{\sqrt{\tilde{\lambda}_0}} \|B_\infty\|_{\ell^2}) \sup_{\tau\in[0,t]} \|w_\infty(\tau) - w_\infty^{[k_m]}(\tau)\|_{\ell^2_\lambda} \\ & + 2\sqrt{T} e^{2qT} C_1 \sup_{\tau\in[0,t]} \left| \xi(\Pi w_\infty^{[k_m]}(\tau)) - \xi(\pi(\tau)) \right|, \end{aligned}$$

which implies that

$$\sup_{t\in[0,T]} \|w_\infty^{[k_m]}(t) - w_\infty(t)\|_{\ell^2_\lambda} \leq \frac{2\sqrt{T} e^{2qT} C_1}{1 - 2\sqrt{T} e^{2qT} C_1 \left(2Cr + \frac{q}{\sqrt{\tilde{\lambda}_0}} \|B_\infty\|_{\ell^2} \right)} \sup_{\tau\in[0,T]} \left| \xi(\Pi w_\infty^{[k_m]}(\tau)) - \xi(\pi(\tau)) \right|.$$

Since $\xi \in C(\mathbb{R}^{N+1};\mathbb{R})$ then $\sup_{\tau\in[0,T]} \left| \xi(\Pi w_\infty^{[k_m]}(\tau)) - \xi(\pi(\tau)) \right| \to 0$ as $k_m \to +\infty$ and $\sup_{t\in[0,T]} \|w_\infty^{[k_m]}(t) - w_\infty(t)\|_{\ell^2_\lambda} \to 0$ as $k_m \to +\infty$. This means that

$$\sup_{t\in[0,T]} \|\Pi w_\infty^{[k_m]}(t) - \Pi w_\infty(t)\|_{\ell^2_\lambda} \to 0 \quad \text{as} \quad k_m \to \infty \tag{C.15}$$

and $w_\infty(\cdot) \in C([0,T];\ell^2_\lambda)$ is a solution of

$$\dot{w}_\infty(t) = (2qI_\infty - \Lambda_\infty) w_\infty(t) + F_\infty(w_\infty(t)) + B_\infty \left(\xi(\Pi w_\infty(t)) - q w_0(t)\right). \tag{C.16}$$

The proof is complete. $\square$

C.0.3 Extension to a coupled infinite-dimensional system with a locally Lipschitz nonlinearity

Let us consider on $\ell^2 \times \ell^2$ the following system

$$\frac{d}{dt}\begin{pmatrix} w_\infty(t) \\ e_\infty(t) \end{pmatrix} = \begin{pmatrix} 2qI_\infty - \Lambda_\infty & 0 \\ 0 & 2qI_\infty - \Lambda_\infty \end{pmatrix}\begin{pmatrix} w_\infty(t) \\ e_\infty(t) \end{pmatrix} + \begin{pmatrix} B_\infty \\ 0 \end{pmatrix}\big(\xi(\Pi w_\infty(t)) - q w_0(t)\big) + G\begin{pmatrix} w_\infty(t) \\ e_\infty(t) \end{pmatrix}, \tag{C.17}$$

with

$$w_\infty(t) = (w_0(t), w_1(t), \cdots)^\top, \quad e_\infty(t) = (e_0(t), e_1(t), \cdots)^\top \tag{C.18}$$

and the infinite-dimensional matrices Λ_∞, I_∞ and the vector B_∞ defined as previously, G a nonlinear term that satisfies: $\forall (w^1_\infty, e^1_\infty)^\top, (w^2_\infty, e^2_\infty)^\top \in \ell^2_\lambda \times \ell^2_\lambda$,

$$\left\| G\begin{pmatrix} w^1_\infty \\ e^1_\infty \end{pmatrix} - G\begin{pmatrix} w^2_\infty \\ e^2_\infty \end{pmatrix} \right\|_{\ell^2\times\ell^2} \leq C_3\left(1 + \left\|\begin{pmatrix} w^1_\infty \\ e^1_\infty \end{pmatrix}\right\|_{\ell^2_\lambda\times\ell^2_\lambda} + \left\|\begin{pmatrix} w^2_\infty \\ e^2_\infty \end{pmatrix}\right\|_{\ell^2_\lambda\times\ell^2_\lambda}\right) \times \left\|\begin{pmatrix} w^1_\infty - w^2_\infty \\ e^1_\infty - e^2_\infty \end{pmatrix}\right\|_{\ell^2_\lambda\times\ell^2_\lambda}, \tag{C.19}$$

for some positive $C_3 > 0$. The norm $\|\cdot\|_{\ell^2_\lambda\times\ell^2_\lambda}$ is defined as follows

$$\forall (w_\infty, e_\infty)^\top \in \ell^2_\lambda \times \ell^2_\lambda, \quad \left\|\begin{pmatrix} w_\infty \\ e_\infty \end{pmatrix}\right\|_{\ell^2_\lambda\times\ell^2_\lambda} = (\|w_\infty\|^2_{\ell^2_\lambda} + \|e_\infty\|^2_{\ell^2_\lambda})^{\frac{1}{2}}. \tag{C.20}$$

The following result is a direct consequence of the Theorem C.1.

Theorem C.2
For any initial condition $(w_\infty(0), e_\infty(0))^\top \in \ell^2_\lambda \times \ell^2_\lambda$ there exists $T_o > 0$ such that the system (C.17) has a mild solution $w_\infty(\cdot) = (w_0(\cdot), w_1(\cdot), ...)^\top, e_\infty(\cdot) = (e_0(\cdot), e_1(\cdot), ...)^\top \in C([0, T_o]; \ell^2_\lambda)$ such that for any $n \geq 0$, $(w_n(\cdot), e_n(\cdot))^\top \in C^1([0, T_o]; \mathbb{R}^2)$.

Proof. The proof follows the same lines as the previous one. Since $-\Lambda_\infty$ is a generator of the analytical semigroup $(\Phi(t))_{t\geq 0}$ on ℓ^2 then the operator $-\Lambda_{1,\infty} = \begin{pmatrix} -\Lambda_\infty & 0 \\ 0 & -\Lambda_\infty \end{pmatrix}$ is a generator of the analytical semigroup $\Phi_1(t) = \begin{pmatrix} \Phi(t) & 0 \\ 0 & \Phi(t) \end{pmatrix}$ on $\ell^2 \times \ell^2$ and one has the estimate

$$\forall t > 0, \quad \|\Lambda^{1/2}_{1,\infty}\Phi_1(t)\|_{\ell^2\times\ell^2} \leq C_1 t^{-\frac{1}{2}}, \quad C_1 = \frac{1}{\sqrt{2e}}. \tag{C.21}$$

Let $r_1 > 0$ such that $\|(w_\infty(0), e_\infty(0))^\top\|_{\ell^2_\lambda \times \ell^2_\lambda} < r_1$ and let $T_o > 0$ small enough such that the estimates

$$e^{2qT_o}\|(w_\infty(0), e_\infty(0))^\top\|_{\ell^2_\lambda \times \ell^2_\lambda} + 2C_1 e^{2qT_o}\sqrt{T_o}(C_3(1+r_1)r_1 + \frac{q}{\sqrt{\bar{\lambda}_0}}\|B_\infty\|_{\ell^2} r_1 +$$

$$C_2 q^{-p/2} r_1^p \|B_\infty\|_{\ell^2}) \leq r_1$$

and

$$2C_1 e^{2qT_o}\sqrt{T_o}(C_3(1+2r_1) + \frac{q}{\sqrt{\bar{\lambda}_0}}\|B_\infty\|_{\ell^2}) < 1,$$

hold. Under these conditions and repeating the same procedure as in the proof of the Theorem C.1, the closed-loop system (C.17) has a mild solution $(w_\infty(\cdot), e_\infty(\cdot))^\top \in C([0, T_o); \ell^2_\lambda \times \ell^2_\lambda)$ such that $(w_n(\cdot), e_n(\cdot))^\top \in C^1([0, T_o]; \mathbb{R}^2)$. $\square$

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Titre : Contrôle homogène de dimension finie pour certaines classes de systèmes en dimension infinie

Résumé : Cette thèse se concentre sur la conception de contrôleurs non linéaires homogènes en dimension finie pour les phénomènes physiques en dimension infinie modélisés par example des équations différentielles partielles (EDP) paraboliques, telles que les processus thermiques et les écoulements. Pour stabiliser les systèmes de contrôle en dimension infinie étudiés, nous avons choisi la décomposition modale comme méthode de stabilisation, et nous avons considéré l'utilisation de contrôles homogènes comme outil de contrôle, lesquels constituent un intermédiaire entre le contrôle linéaire et le contrôle en modes glissants. La décomposition modale consiste à séparer l'équation aux dérivées partielles paraboliques en deux composants distincts : le premier est une partie contrôlée de dimension finie qui capture les modes instables, tandis que le second est une partie résiduelle de dimension infinie englobant les modes stables. En isolant les modes instables, toute technique de conception de contrôle pour les systèmes en dimension finie peut être utilisée pour élaborer des stratégies de contrôle pour ce sous-système. Dans notre étude, pour la stabilisation du sous-système en dimension finie, nous avons utilisé des contrôles homogènes, qui offrent des propriétés de convergence en temps fini ou en temps fixe pour ce sous-système, contrairement au contrôle linéaire qui ne garantit qu'une convergence exponentielle. De plus, nous avons opté pour le contrôle homogène, car il permet d'obtenir une meilleure convergence avec un faible débordement du signal de contrôle et moins de pics dans le système durant la phase initiale de stabilisation, par rapport au contrôle linéaire qui, tout en cherchant à garantir une meilleure convergence, entraîne un important débordement du signal de contrôle et large pic dans le système au cours de cette même phase. Cette thèse, fondée sur la conception de contrôle homogène combinée à la décomposition modale, propose une étude approfondie sur la conception de contrôleurs homogènes en dimension finie pour la stabilisation des EDP paraboliques modélisant des phénomènes d'écoulement ou des processus thermiques, en mettant particulièrement l'accent sur l'équation de réaction-diffusion et l'équation de Burgers 1D avec viscosité.

Mots clés : EDPs paraboliques, contrôle homogène, décomposition modale

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Title: Finite-dimensional homogeneous control for some classes of infinite-dimensional systems

Abstract: This thesis focuses on the design of finite-dimensional homogeneous controllers for physical phenomena modeled by infinite dimensional system, for example, by parabolic partial differential equations (PDEs), such as thermal processes and flows. To stabilize the studied infinite-dimensional control systems, we have chosen modal decomposition as the stabilization method, and we have considered the use of homogeneous controls as a control tool, which serves as an intermediary between linear control and sliding mode control. Modal decomposition involves separating the parabolic partial differential equation into two distinct components: the first is a controlled finite-dimensional part that captures the unstable modes, while the second is a residual infinite-dimensional part encompassing the stable modes. By isolating the unstable modes, any control design technique for finite-dimensional systems can be applied to develop control strategies for this sub-system. In our study, for the stabilization of the finite-dimensional subsystem, we utilized homogeneous controls, which provide properties of finite-time or fixed-time convergence for this subsystem, in contrast to linear control, which only guarantees exponential convergence. Furthermore, we opted for homogeneous control because it allows for better convergence with lower control signal overshoot and fewer peaks in the system during the initial stabilization phase, compared to linear control, which, while aiming to ensure better convergence, leads to significant control signal overshoot and large peaks in the system during the same phase. Thus, this

thesis, based on the design of homogeneous control combined with modal decomposition, offers an in-depth study on the design of finite-dimensional homogeneous controllers for the stabilization of parabolic PDEs modeling flow phenomena or thermal processes, with particular emphasis on the reaction-diffusion equation and the one-dimensional viscous Burgers equation.

Keywords: Parabolic PDEs, homogeneous control, modal decomposition