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# A study of some remarkable properties of some classes of linear operators on Banach spaces: typicality of positive contractions and random products of operators

**Thesis supervised by** Sophie Grivaux

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# Étude de certaines propriétés remarquables de quelques classes d'opérateurs linéaires sur les espaces de Banach : typicité des contractions positives et produits aléatoires d'opérateurs

**Thèse dirigée par** Sophie Grivaux

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**Keywords:** Polish topologies, comeager sets, similar topologies, positive operators, typical properties, invariant subspaces, linear dynamics, ergodic theory, random operators, central limit theorems

**Mots clés :** Topologies polonaises, parties comaigres, topologies semblables, opérateurs positifs, propriétés typiques, sous-espaces invariants, dynamique linéaire, théorie ergodique, opérateurs aléatoires, théorèmes de la limite centrale

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**A study of some remarkable properties of some classes of linear operators on Banach spaces: typicality of positive contractions and random products of operators**

**Abstract**

The motivations of this thesis are, on the one hand, the Invariant Subspace Problem, and on the other hand, the asymptotic behavior of random products of matrices. We first study typical properties of positive contractions on a Banach space with a Schauder basis, the main goal being to shed some light on the question of whether a typical positive contraction on such a Banach space has a non-trivial invariant subspace. We will study this problem for different Polish operator topologies. In the second part, we investigate the linear dynamics of random products of operators of the form $T_n(\omega) := T(\tau^{n-1}\omega)\cdots T(\tau\omega)T(\omega)$. In other words, we study the linear dynamics of the sequence $(T_n(\omega))_{n\geq 1}$ for almost every $\omega \in \mathbb{T}$. These random products depend on an ergodic transformation $\tau : \mathbb{T} \to \mathbb{T}$ and a strongly measurable map $T : \mathbb{T} \to \mathcal{B}(X)$, taking values in the space of operators on a separable Fréchet space $X$. The aim is to study the influence of the ergodic transformation and of the operators $T(\omega)$ on the universality of the sequence $(T_n(\omega))_{n\geq 1}$ for almost every $\omega \in \mathbb{T}$. We will mainly consider the case where the operator $T(\omega)$ equals a fixed operator $T_1$ when $\omega \in A_1$, and another fixed operator $T_2$ when $\omega \in A_2$, where $(A_1, A_2)$ is a non-trivial measurable partition of $[0,1)$.

**Keywords:** Polish topologies, comeager sets, similar topologies, positive operators, typical properties, invariant subspaces, linear dynamics, ergodic theory, random operators, central limit theorems

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**Étude de certaines propriétés remarquables de quelques classes d'opérateurs linéaires sur les espaces de Banach : typicité des contractions positives et produits aléatoires d'opérateurs**

**Résumé**

Les motivations de cette thèse sont d'une part le Problème du Sous-espace Invariant, et d'autre part le comportement asymptotique des produits aléatoires de matrices. Nous nous intéressons, dans un premier temps, aux propriétés typiques des contractions positives sur un espace de Banach ayant une base de Schauder, notre but principal étant d'étudier si une contraction typique positive sur un tel espace de Banach a un sous-espace invariant non trivial. Nous étudierons ce problème pour différentes topologies polonaises d'opérateurs. Dans un second temps, on s'intéresse à la dynamique linéaire des produits aléatoires d'opérateurs de la forme $T_n(\omega) := T(\tau^{n-1}\omega)\cdots T(\tau\omega)T(\omega)$. En d'autres termes, on cherche à étudier la dynamique linéaire de la suite $(T_n(\omega))_{n\geq 1}$ pour presque tout $\omega \in \mathbb{T}$. Ces produits aléatoires dépendent d'une transformation ergodique $\tau : \mathbb{T} \to \mathbb{T}$ et d'une application fortement mesurable $T : \mathbb{T} \to \mathcal{B}(X)$ à valeurs opérateurs sur un espace de Fréchet séparable $X$. Il s'agira d'étudier l'influence de la transformation ergodique et des opérateurs $T(\omega)$ sur l'universalité de la suite $(T_n(\omega))_{n\geq 1}$ pour presque tout $\omega \in \mathbb{T}$. Nous étudierons principalement le cas où l'opérateur $T(\omega)$ est égal à un certain opérateur $T_1$ lorsque $\omega \in A_1$, et égal à un certain opérateur $T_2$ lorsque $\omega \in A_2$, où $(A_1, A_2)$ est une partition mesurable non triviale de $[0,1)$.

**Mots clés :** Topologies polonaises, parties comaigres, topologies semblables, opérateurs positifs, propriétés typiques, sous-espaces invariants, dynamique linéaire, théorie ergodique, opérateurs aléatoires, théorèmes de la limite centrale

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Chapter 1

# Introduction (in English)

The aims of this thesis are, on the one hand, the study of typical properties of positive contractions on a Banach space with a basis, and on the other hand, the study of the linear dynamics of random products of operators on separable Fréchet spaces.

The results that we obtain are the subject of three papers, listed below:

— Typical properties of positive contractions and the invariant subspace problem, published in *Positivity* [26].

— Similar operator topologies on the space of positive contractions, published in *Illinois Journal of Mathematics* [25].

— Linear dynamics of random products of operators, submitted [24].

The motivation of the first part of this thesis is the Invariant Subspace Problem. This problem can be stated as follows: given an infinite-dimensional separable real or complex Banach space $X$, does every bounded operator $T \in \mathcal{B}(X)$ have a non-trivial invariant subspace? The answer to this problem is negative in the non-reflexive setting: Enflo constructed a non-reflexive Banach space which supports an operator without non-trivial invariant subspaces (see [20]), and Read constructed operators on $\ell_1$ and on $c_0$ without non-trivial invariant subspaces (see [53] and [54]). This problem still remains open for reflexive Banach spaces and in particular for separable Hilbert spaces. It can also be restricted to positive operators as follows: given an infinite-dimensional real or complex Banach space $X$ with a Schauder basis, does every bounded positive operator on $X$ have a non-trivial invariant subspace? Despite important examples provided by Troitsky and Sirotkin, the answer to this problem is still unknown. Since this problem is very difficult, we will study it from the viewpoint of typicality. The main objective is to determine whether a typical positive contraction on a Banach space with a Schauder basis has a non-trivial invariant subspace. While this question remains open in full generality, we provide partial answers in the cases $X = \ell_1$ and $X = \ell_2$, and we investigate several more general settings.

The motivation of the second part of this thesis is the study of the asymptotic behavior of random products of matrices, first investigated by Bellman, and later by Furstenberg and Kesten, who proved laws of large numbers for these random products ([7], [22]). We study random products of operators on a separable Fréchet space. These random products will depend on an ergodic measure-preserving transformation on $\mathbb{T}$, equipped with the normalized

Lebesgue measure, and on different fixed operators $T_j$ on this Fréchet space. The aim will be to examine the influence of the underlying ergodic transformation and of the operators $T_j$ on the dynamics of these random products.

In this introduction, the original results are alphabetically enumerated (Proposition A, Theorem B, ...).

## 1.1 Typical properties of positive contractions

We detail here the content of the first part of this thesis on the typical properties of positive contractions, which corresponds to the articles [26] and [25].

Unless otherwise specified, all Banach spaces considered are assumed to be complex, separable, and infinite-dimensional.

### 1.1a Notion of a typical property of contractions

The general setting to study the notion of a typical property is provided by Baire spaces. These are topological spaces $M$ such that the intersection of countably many dense open sets in $M$ is dense. Every complete metric space is a Baire space. More generally, every Polish space is a Baire space.

**Definition 1.1.1.** A Polish space is a completely metrizable separable topological space.

Recall that a topological space $(M,\tau)$ is completely metrizable if there exists a metric $d$ on $M$ that induces the topology $\tau$ and such that $(M,d)$ is complete. Here are some examples of Polish spaces.

**Example 1.1.2.**
- Any set $M$ with the discrete topology is Polish. A compatible complete distance is provided by $d(x,y)=0$ if $x=y$ and $d(x,y)=1$ otherwise.
- Every closed subset of a Polish space is also Polish.
- Any open set $U$ in a complete metric space $(X,d)$ is Polish. A compatible complete distance is provided by $d'(x,y)=d(x,y)+\left|\frac{1}{d(x,X\setminus U)}-\frac{1}{d(y,X\setminus U)}\right|$.

There is also a description of the subsets of Polish spaces which are Polish. A subset of a Polish space is Polish if and only if it is a $G_\delta$, that is, a countable intersection of open sets ([39, Theorem 3.11]).

An important notion in Baire spaces is that of comeager sets. One advantage of comeager sets is that the intersection of two comeager sets in a Baire space is nonempty, thanks to the Baire category theorem — unlike the intersection of two dense sets, which can be empty.

**Definition 1.1.3.** Let $M$ be a topological space. A subset $A$ of $M$ is said to be comeager if it contains the intersection of a countable family of dense open sets. The complement of a comeager set is called meager. That is, a subset $A$ of $M$ is meager if it is contained in the union of a countable family of closed subsets, each with empty interior.

We give two examples of comeager sets. The second one shows that it is not always easy to prove that a set is comeager.

**Example 1.1.4.**
- The set $\mathbb{Q}$ of rational numbers is meager in $\mathbb{R}$, and hence $\mathbb{R}\setminus\mathbb{Q}$ is comeager in $\mathbb{R}$.

- The set of continuous nowhere differentiable real functions on $[0,1]$ is comeager in the space of continuous real functions endowed with the supremum norm.

We can now define the notion of a typical property.

**Definition 1.1.5.** Let $(M,\tau)$ be a topological space. A property (P) of elements of $M$ is typical for the topology $\tau$ if the set

$$\{x \in M : x \text{ has (P)}\}$$

is comeager in $(M,\tau)$. The property (P) is said to be atypical for $\tau$ if its negation is typical for $\tau$.

In the case where a property (P) of elements of a topological space $(M,\tau)$ is typical, we say that "a typical $x \in (M,\tau)$ has (P)". With Example 1.1.4, we obtain that a typical real number $x \in \mathbb{R}$ is irrational, and we also obtain that a typical continuous real function $f$ on $[0,1]$ is nowhere differentiable.

In the first part of this thesis, we will consider typical properties of contractions. Given a Banach space $X$, we denote by $\mathcal{B}_1(X)$ the set of contractions on $X$, that is,

$$\mathcal{B}_1(X) := \{T \in \mathcal{B}(X) : \|T\| \leq 1\}.$$

We will study typical properties of contractions for different Polish topologies on $\mathcal{B}_1(X)$. We won't do it for the operator norm topology $\|.\|$. Indeed, $(\mathcal{B}_1(X), \|.\|)$ is usually not separable and hence not Polish. And even if this topology is complete on $\mathcal{B}(X)$, it is difficult to study typical properties for this topology in general. We just mention that some results on typical properties for the operator norm topology were proved in [19] and [56]. We will instead consider the Strong Operator Topology, the Strong* Operator Topology, the Weak Operator Topology and the Dual Strong Operator Topology. The Strong Operator Topology is the pointwise convergence topology on $\mathcal{B}(X)$. The Strong* Operator Topology is the pointwise convergence topology for operators and their adjoints. The Weak Operator Topology is the weak pointwise convergence topology on $\mathcal{B}(X)$. Finally, the Dual Strong Operator Topology is the pointwise convergence topology for the adjoints on $\mathcal{B}(X)$. We will respectively write $\mathtt{SOT}$, $\mathtt{SOT}^*$, $\mathtt{WOT}$ and $\mathtt{SOT}_*$ for these topologies. We can summarize the convergence of a sequence for these topologies in the following way:

$$\begin{cases} T_i \xrightarrow[i]{\mathtt{SOT}} T \iff T_i x \xrightarrow[i]{\|.\|} Tx \quad \text{for every } x \in X, \\ T_i \xrightarrow[i]{\mathtt{SOT}^*} T \iff T_i \xrightarrow[i]{\mathtt{SOT}} T \quad \text{and} \quad T_i^* \xrightarrow[i]{\mathtt{SOT}} T^*, \\ T_i \xrightarrow[i]{\mathtt{WOT}} T \iff \langle y^*, T_i x\rangle \xrightarrow[i]{} \langle y^*, Tx\rangle \quad \text{for every } y^* \in X^* \text{ and every } x \in X, \\ T_i \xrightarrow[i]{\mathtt{SOT}_*} T \iff T_i^* \xrightarrow[i]{\mathtt{SOT}} T^*. \end{cases}$$

We have the following inclusions: $\mathtt{WOT} \subset \mathtt{SOT}$, $\mathtt{WOT} \subset \mathtt{SOT}_*$, $\mathtt{SOT} \subset \mathtt{SOT}^*$ and $\mathtt{SOT}_* \subset \mathtt{SOT}^*$. Unfortunately, these topologies are never metrizable on $\mathcal{B}(X)$ when $X$ is infinite-dimensional. However, under some assumptions on the Banach space $X$, their restrictions to $\mathcal{B}_1(X)$ are Polish. This is given by the following proposition.

**Proposition 1.1.6.** *Let $X$ be a Banach space.*

- *If $X$ is separable, then $(\mathcal{B}_1(X), \mathtt{SOT})$ is Polish.*
- *If $X^*$ is separable, then $(\mathcal{B}_1(X), \mathtt{SOT}_*)$ and $(\mathcal{B}_1(X), \mathtt{SOT}^*)$ are Polish.*

- *If $X$ is reflexive and separable, then $(\mathcal{B}_1(X), \mathtt{WOT})$ is Polish.*

For example, if $X$ is separable and $(x_n)_{n\geq 0}$ is a dense sequence in the unit sphere $S_X$ of $X$, then one can show that the following metric

$$d : (T,S) \in \mathcal{B}_1(X) \mapsto \sum_{n=0}^{\infty} 2^{-n} \|(T-S)x_n\|$$

is complete and compatible with $\mathtt{SOT}$ on $\mathcal{B}_1(X)$, and that the space $(\mathcal{B}_1(X), d)$ is separable. Under the assumptions of Proposition 1.1.6, one can also construct such metrics for the other topologies considered above.

The topology $\mathtt{SOT}^*$ is an interesting topology for the following reason. Let (P) be a property of operators on the Banach space $X = \ell_p$ with $p > 1$. If the set

$$\{T \in \mathcal{B}_1(\ell_p) : T \text{ has (P)}\}$$

is comeager in $(\mathcal{B}_1(\ell_p), \mathtt{SOT}^*)$, then the following set

$$\{T \in \mathcal{B}_1(\ell_{p'}) : T^* \text{ has (P)}\}$$

is also comeager in $(\mathcal{B}_1(\ell_{p'}), \mathtt{SOT}^*)$, where $p'$ is the conjugate exponent of $p$. This follows from the fact that the map given by $T \mapsto T^*$, from $(\mathcal{B}_1(\ell_p), \mathtt{SOT}^*)$ into $(\mathcal{B}_1(\ell_{p'}), \mathtt{SOT}^*)$, is a homeomorphism. This is obviously not true for the other topologies, but we can still say that if the set

$$\{T \in \mathcal{B}_1(\ell_p) : T \text{ has (P)}\}$$

is comeager in $(\mathcal{B}_1(\ell_p), \mathtt{SOT})$, then the following set

$$\{T \in \mathcal{B}_1(\ell_{p'}) : T^* \text{ has (P)}\}$$

is also comeager in $(\mathcal{B}_1(\ell_{p'}), \mathtt{SOT}_*)$.

The notion of a typical property of contractions was first initiated by Eisner in [18], where she showed that, with respect to the Weak Operator Topology, a typical contraction on a separable Hilbert space is unitary. This concept was further developed by Eisner and Mátrai in [19], who studied typical properties of contractions on separable Hilbert spaces for other topologies, such as the Strong Operator Topology and the Strong* Operator Topology. They proved that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ is unitarily equivalent to the infinite-dimensional unilateral backward shift on $\ell_2(\mathbb{Z}_+, \ell_2) := \{(u_n)_{n\geq 0} : u_n \in \ell_2 \quad \text{for every } n \geq 0 \text{ and } \sum_{n\geq 0} \|u_n\|^2 < \infty\}$, defined by $B_\infty(u_0, u_1, u_2, \ldots) = (u_1, u_2, \ldots)$. A few years later, Grivaux, Matheron and Menet investigated typical properties of contractions on $\ell_p$-spaces, for these Polish topologies as well ([28], [30], [29]). Their motivation was the Invariant Subspace Problem, and their main goal was to determine whether a typical contraction $T \in \mathcal{B}_1(\ell_p)$ has a non-trivial invariant subspace. Thanks to Eisner and Mátrai, it was known that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ (resp. $T \in (\mathcal{B}_1(\ell_2), \mathtt{WOT})$) has a non-trivial invariant subspace. Unfortunately, the result asserting unitary equivalence to the infinite-dimensional unilateral backward unilateral shift $B_\infty$ no longer holds for $\mathtt{SOT}^*$, since a typical contraction $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT}^*)$ is not a co-isometry. Nevertheless, the authors in [30] still managed to prove that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT}^*)$ has a non-trivial invariant subspace, by using the fact that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT}^*)$ has a rich spectrum. The authors in [30] also proved that a typical contraction $T \in (\mathcal{B}_1(\ell_1), \mathtt{SOT})$ has a non-trivial invariant subspace.

It is also worth noting that the authors of [30] proved the existence of a $0-1$ law for properties of contractions. More precisely, they showed the following result.

**Proposition 1.1.7.** *Let $X = \ell_p$ with $1 \leq p < \infty$. Let $\mathcal{A} \subset \mathcal{B}_1(\ell_p)$. If $\mathcal{A}$ has the Baire property for* SOT *and satisfies $J\mathcal{A}J^{-1} = \mathcal{A}$ for every surjective isometry $J$ of $\ell_p$, then $\mathcal{A}$ is either meager or comeager in* $(\mathcal{B}_1(\ell_p), \texttt{SOT})$.

Proposition 1.1.7 also holds for $\texttt{SOT}^*$ when $1 < p < \infty$. In Proposition 1.1.7, a subset $\mathcal{A} \subset \mathcal{B}_1(\ell_p)$ is said to have the Baire property for a topology $\tau$ if $\mathcal{A}$ equals a $\tau$-open set modulo a meager set, which means that there exists an open set $U$ of $(\mathcal{B}_1(\ell_p), \tau)$ such that $A \triangle U$ is meager. An important class of subsets with the Baire property for a topology $\tau$ is the class of analytic sets. A subset $\mathcal{A}$ is analytic in $(\mathcal{B}_1(\ell_p), \tau)$ if $\mathcal{A}$ is the projection of a Borel subset of a product space of the form $(\mathcal{B}_1(\ell_p), \tau) \times Y$ onto $\mathcal{B}_1(\ell_p)$, where $Y$ is a Polish space. Let us illustrate it on the following example.

**Example 1.1.8** ([30, Corollary 3.3])**.** Let $X = \ell_p$ with $1 \leq p < \infty$. Either a typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$ has a non-trivial invariant subspace, or a typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$ does not have a non-trivial invariant subspace. Indeed, let us denote by $\mathcal{A}$ the set of contractions on $X$ having a non-trivial invariant subspace. Then:

$$T \in \mathcal{A} \iff \exists (x, x^*) \in X \times B_{X^*} : (x \neq 0, x^* \neq 0 \text{ and } \forall n \geq 0, \langle x^*,\ T^n x\rangle = 0).$$

The condition under brackets defines a Borel subset of $(\mathcal{B}_1(X), \texttt{SOT}) \times X \times (B_{X^*}, w^*)$ and since the set $\mathcal{A}$ is the projection of this Borel subset onto $\mathcal{B}_1(X)$, it is analytic in $(\mathcal{B}_1(X), \texttt{SOT})$. The set $\mathcal{A}$ is easily seen to be invariant under the group of surjective isometries of $\ell_p$, and thus $\mathcal{A}$ is either meager or comeager in $(\mathcal{B}_1(X), \texttt{SOT})$.

Proposition 1.1.7 applies for most of the classical properties of operators, as in Example 1.1.8.

In this thesis, we study typical properties of positive contractions on a Banach space with a Schauder basis. Our motivation is also the Invariant Subspace Problem, but for positive operators. Since this problem remains open in the positive setting, it is natural to investigate it from the point of view of typicality. This is also worthwhile since there are techniques specific to positive operators.

### 1.1b Positive operators acting on a Banach space with a basis

We define positivity of operators on a Banach space with respect to a Schauder basis. We start with some reminders about Schauder basis. A Schauder basis is just a system of coordinates.

**Definition 1.1.9.** Let $X$ be a Banach space. A sequence $(e_n)_{n\geq 0}$ of non-zero vectors of $X$ is called a Schauder basis of $X$ if for every $x \in X$, there exists a unique sequence of complex numbers $(x_n)_{n\geq 0}$ such that

$$x = \sum_{n=0}^{\infty} x_n e_n,$$

where the series converges in the norm topology.

For example, orthonormal basis in Hilbert spaces are Schauder basis. The canonical basis of $\ell_p$ and $c_0$ (with $1 \leq p < \infty$), given by $e_n(k) = 1$ if $n = k$ and $e_n(k) = 0$ otherwise, is a Schauder basis of these spaces. The trigonometric system of $L^p(0,1)$ is an example of a Schauder basis of $L^p(0,1)$ when $1 < p < \infty$. There also exists a basis of $L^p(0,1)$ consisting of piecewise constant functions, called the Haar basis. Obviously, there are Banach spaces without Schauder basis, since it requires the space to be separable. Moreover, Enflo constructed a separable reflexive Banach space that does not admit any Schauder basis, showing that reflexivity and separability do not guarantee the existence of a basis, and Lindenstrauss proved that there exist Banach spaces $X$ with a Schauder basis and separable dual $X^*$, such that $X^*$ does not have a Schauder basis (see [44] for more details).

If $(e_n)_{n\geq 0}$ is a Schauder basis of a Banach space $X$, then the projections

$$P_N : x = \sum_{n=0}^{\infty} x_n e_n \mapsto \sum_{n=0}^{N} x_n e_n$$

are continuous, and $\sup_{N\geq 0}\|P_N\|$ is a finite constant, called the basis constant. The basis is said to be monotone if $\sup_{N\geq 0}\|P_N\| = 1$, which means that

$$\left\|\sum_{n=0}^{N} x_n e_n\right\| \leq \left\|\sum_{n=0}^{M} x_n e_n\right\|$$

for every $M \geq N$ and for all complex numbers $x_0, \dots, x_M$. In particular, when $(e_n)_{n\geq 0}$ is a Schauder basis of $X$, the biorthogonal functionals $(e_n^*)_{n\geq 0}$ associated to the sequence $(e_n)_{n\geq 0}$, given by $e_m^*\left(\sum_{n=0}^{\infty} x_n e_n\right) = x_m$, are continuous. However, the sequence $(e_n^*)_{n\geq 0}$ is not necessarily a Schauder basis of $X^*$. We say that a Schauder basis $(e_n)_{n\geq 0}$ of $X$ is shrinking if the sequence of biorthogonal functionals $(e_n^*)_{n\geq 0}$ is a Schauder basis of $X^*$. When the space $X$ is reflexive and separable, every Schauder basis is shrinking (see, for instance, [44, Theorem II.22]). For example, the canonical basis of $\ell_p$, $1 \leq p < \infty$, and of $c_0$, is monotone. The Haar basis is monotone in $L^p(0,1)$, $1 \leq p < \infty$.

We also define the notion of an unconditional basis, as it will be needed to define the modulus of a vector. A Schauder basis $(e_n)_{n\geq 0}$ of a Banach space $X$ is said to be unconditional if, for every $x = \sum_{n=0}^{\infty} x_n e_n$ in $X$, the series $\sum_{n\geq 0} x_{\sigma(n)} e_{\sigma(n)}$ converges in $X$ for every permutation $\sigma$ of $\mathbb{Z}_+$. For example, the canonical basis of $\ell_p$, $1 \leq p < \infty$, and of $c_0$, is unconditional. The Haar basis in $L^p(0,1)$ is unconditional only for $1 < p < \infty$, and the trigonometric system is unconditional only for $p = 2$. The space $L^1(0,1)$ is not contained in any Banach space that admits an unconditional basis. If the basis $(e_n)_{n\geq 0}$ is unconditional in $X$, then there exists a constant $K > 0$ such that for every $x = \sum_{n=0}^{\infty} x_n e_n$ in $X$ and for every bounded sequence of complex numbers $\lambda = (\lambda_n)_{n\geq 0}$, we have $\left\|\sum_{n=0}^{\infty} \lambda_n x_n e_n\right\| \leq 2K\|\lambda\|_\infty \|x\|$.

From now on, as we will only be dealing with Schauder basis, we shall omit the term "Schauder". We can now define the notion of a positive operator with respect to a basis.

**Definition 1.1.10.** Let $X$ be a Banach space with a basis $(e_n)_{n\geq 0}$.

- A vector $x = \sum_{n=0}^{\infty} x_n e_n$ is said to be positive (with respect to the basis $(e_n)_{n\geq 0}$) if $x_n \geq 0$ for every $n \geq 0$. In this case, we write $x \geq 0$.
- An operator $T \in \mathcal{B}(X)$ is said to be positive if $Tx \geq 0$ for every $x \geq 0$. In this case, we write $T \geq 0$.

We denote by $\mathcal{P}(X)$ the set of positive operators on $X$ and by $\mathcal{P}_1(X)$ the set of positive contractions on $X$.

The notion of positivity strongly depends on the choice of the basis. An operator may be positive with respect to one basis, but not positive with respect to another. It is very easy to recognize or to construct positive operators. Indeed, let $T$ be an operator on a Banach space $X$ with a basis $(e_n)_{n\geq 0}$. The operator $T$ is positive with respect to the basis $(e_n)_{n\geq 0}$ if and only if its matrix $(t_{i,j})_{i,j\geq 0}$ with respect to this basis has non-negative entries (with $t_{i,j} = \langle e_i^*, Te_j\rangle$ for every $i,j \geq 0$). In particular, every weighted forward or backward shift operator with non-negative weights is positive, and every diagonal operator with a non-negative diagonal is positive. When $X = \ell_p$ with $1 \leq p < \infty$, we always consider the canonical basis.

Let $X$ be a Banach space with a basis $(e_n)_{n\geq 0}$. The set $\mathcal{P}_1(X)$ is closed in $\mathcal{B}_1(X)$ for each one of the operator topologies $\mathtt{WOT}$, $\mathtt{SOT}$, $\mathtt{SOT}_*$ and $\mathtt{SOT}^*$, since the convergence for these topologies implies pointwise convergence of coordinates. In particular, we obtain the following result, which provides the appropriate setting to study typical properties of positive contractions.

**Proposition A.** Let $X$ be a Banach space with a basis. The set $(\mathcal{P}_1(X), \mathtt{SOT})$ is Polish. If $X^*$ is separable, then the sets $(\mathcal{P}_1(X), \mathtt{SOT}_*)$ and $(\mathcal{P}_1(X), \mathtt{SOT}^*)$ are Polish. Finally, if $X$ is reflexive, then the set $(\mathcal{P}_1(X), \mathtt{WOT})$ is Polish.

The study of typical properties of positive contractions is motivated by the invariant subspace problem for positive operators, which can be stated as follows: given a real or complex infinite-dimensional Banach space with a basis, does every positive operator on $X$ (with respect to this basis) have a non-trivial invariant subspace? The answer to this problem is still unknown today. But there are two important examples on this subject to keep in mind.

If $X$ has an unconditional basis $(e_n)_{n\geq 0}$ and if $x = \sum_{n=0}^{\infty} x_n e_n$ belongs to $X$, then the positive vector $|x| := \sum_{n=0}^{\infty} |x_n| e_n$ is well defined in $X$ and is called the *modulus* of $x$. Let $T$ be an operator on $X$ and let $(t_{i,j})_{i,j\geq 0}$ be the matrix of $T$ with respect to this basis. If the positive matrix $(|t_{i,j}|)_{i,j\geq 0}$ defines an operator on $X$, then we say that $T$ has a modulus on $X$ and the corresponding operator is called the *modulus* of $T$, denoted by $|T|$. On the space $\ell_1$, one can easily check that every operator has a modulus. The first important example is due to V. Troistky ([60]). He showed that the modulus of a Read operator on $\ell_1$ (considered as a real space), which is a positive operator on $\ell_1$, has a positive eigenvector and in particular has a non-trivial invariant subspace, although a Read operator has no non-trivial invariant subspace on $\ell_1$. The second important example is due to G. Sirotkin ([58]). He constructed an operator on $\ell_1$ without non-trivial invariant subspaces, whose matrix has all but one non-negative entries, which brings it closer to a negative solution of the invariant subspace problem for positive operators. It is still unknown today whether there exists a positive operator on $\ell_1$ without non-trivial invariant subspaces.

There is, however, a remarkable result regarding the existence of non-trivial invariant subspaces for positive operators. This result is due to Abramovich, Aliprantis and Burkinshaw.

**Theorem 1.1.11** ([2, Theorem 2.2])**.** *Let $X$ be a Banach space with a basis and let $T$ be a positive operator on $X$. If there exists a non-zero positive operator $A$ on $X$ in the commutant of $T$ which is quasinilpotent at a certain non-zero positive vector of $X$, then $T$ has a non-trivial invariant subspace.*

An operator $A$ on a Banach space $X$ is said to be quasinilpotent at a vector $x \in X$ if

$$\lim_{n\to\infty} \|A^n x\|^{1/n} = 0.$$

Obviously, any quasinilpotent operator is quasinilpotent at every vector of $X$. The converse is not true in general: the operator $T$ on $\ell_p$ (with $p > 1$), defined by $Te_0 = e_0 + e_1$ and $Te_j = \frac{1}{j} e_{j+1}$ for every $j \geq 1$, provides a counter-example, since it is quasinilpotent at $e_1$ but not quasinilpotent. Since Theorem 1.1.11 is a result of existence of non-trivial invariant subspaces for positive operators, it is natural to study the typicality of this property. We will thus say that a positive operator on a Banach space with a basis satisfies the *AAB criterion* if it satisfies the assumptions of Theorem 1.1.11. In the case of $\ell_p$-spaces, it will be easy to study the typicality of this criterion thanks to the notion of invariant ideals for positive operators.

Let $X$ be a Banach space with an unconditional basis. A vector subspace $V$ of $X$ is said to be an ideal of $X$ if $|x| \leq |y|$ and $y \in V$ imply $x \in V$, for every $x, y \in X$. This notion of ideal in a Banach space with an unconditional basis is interesting, since in $\ell_p$-spaces with $1 \leq p < \infty$, any positive operator satisfying the AAB criterion has a non-trivial closed invariant ideal ([1, Theorem 2.2]). To recognize whether a positive operator on a Banach space with an unconditional basis has a non-trivial invariant ideal, we use the following criterion, due to Troistky and Radjavi ([52, Proposition 1.2]).

**Proposition 1.1.12.** *Let $X$ be a Banach space with an unconditional basis $(e_n)_{n\geq 0}$. Let $T$ be a positive operator on $X$. Then $T$ has no non-trivial closed invariant ideals if and only if:*

$$\forall i \neq j \in \mathbb{Z}_+,\ \exists n \in \mathbb{Z}_+ : \langle e_j^*, T^n e_i \rangle > 0.$$

Thanks to this result, we will deduce the following one.

**Theorem B.** Let $1 \leq p < \infty$ and let $X = \ell_p$. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ and $T \in (\mathcal{P}_1(X), \texttt{WOT})$ when $1 < p < \infty$) does not satisfy the AAB criterion.

We will generalize Theorem B to Banach spaces with a monotone basis that is not necessarily unconditional, for the $\texttt{SOT}^*$ topology. This is given by the following result.

**Theorem C.** Let $X$ be a reflexive Banach space with a monotone basis. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ does not satisfy the AAB criterion.

Theorems B and C are proved in [26] and show in particular that the existence of a non-trivial invariant subspace for positive operators is a difficult problem. We will also consider other typical properties of positive contractions on $\ell_p$-spaces, with the aim of establishing the existence of a non-trivial invariant subspace for a typical positive contraction on such spaces. In particular, we will see that Proposition 1.1.7 works in $\mathcal{P}_1(\ell_p)$.

Let us recall that Eisner and Mátrai proved that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT})$ is unitarily equivalent to the infinite-dimensional unilateral backward shift on $\ell_2(\mathbb{Z}_+, \ell_2)$. To do so, they first proved that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT})$ is such that $T^*$ is an isometry, and then they applied the Wold decomposition to the adjoint of a typical contraction $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT})$. Moreover, they showed that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT})$ is such that $T^n x \underset{n\to\infty}{\longrightarrow} 0$ for every $x \in X$ and has an infinite dimensional kernel. With these properties, such a Wold decomposition for the adjoint of a typical $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ will have a trivial unitary part and an infinite-dimensional forward shift part. Unfortunately, this won't happen for a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$, due to the following result.

**Proposition D.** Let $1 < p < \infty$ and let $X = \ell_p$. A typical positive contraction $T \in (\mathcal{P}_1(X), \mathtt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \mathtt{SOT}^*)$ and $T \in (\mathcal{P}_1(X), \mathtt{WOT})$) is not a co-isometry.

Proposition D comes from the fact that a positive isometry of $\ell_p$ with $p > 1$ maps positive sequences with disjoint supports to positive sequences with disjoint supports. For this reason, the set of positive co-isometries won't be dense in $\mathcal{P}_1(\ell_p)$ for these operator topologies, and in particular it cannot be comeager. By the $0-1$ law, this set has to be meager in $\mathcal{P}_1(\ell_p)$. In particular, we cannot assert directly, as in the non-positive setting, that a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$ has a non-trivial invariant subspace. But we will see that a typical positive contraction $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT})$ (resp. $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT}^*)$ when $p > 1$) is such that $\sigma(T) = \overline{\mathbb{D}}$. So using the theorem of Brown, Chevreau and Pearcy, which states that every contraction on a Hilbert space whose spectrum contains the unit circle has a non-trivial invariant subspace ([11]), we will obtain the following result.

**Theorem E.** A typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$ (resp. $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT}^*)$) has a non-trivial invariant subspace.

Regarding the case $X = \ell_1$ and $\mathtt{SOT}$, the situation will be the same as for a typical contraction $T \in (\mathcal{B}_1(\ell_1), \mathtt{SOT})$.

**Theorem F.** A typical positive contraction $T \in (\mathcal{P}_1(\ell_1), \mathtt{SOT})$ is such that $T^*$ is a non-surjective isometry, and such that $\sigma_p(T) = \mathbb{D}$ with $\dim(\ker(T - \lambda)) = \infty$ for every $\lambda \in \mathbb{D}$. In particular, a typical positive contraction $T \in (\mathcal{P}_1(\ell_1), \mathtt{SOT})$ has a non-trivial invariant subspace.

We will also see that a typical positive contraction $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT}^*)$ is such that $T$ and $T^*$ have no eigenvalue for every $1 < p < \infty$. To do so, we will in fact prove that for every $1 \leq p < \infty$, a typical positive contraction $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT})$ (resp. $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT}^*)$ when $1 < p < \infty$) is such that $2T$ is hypercyclic, which means that there exists a vector $x \in \ell_p$ such that its orbit $\{(2T)^n x : n \geq 0\}$ under $2T$ is dense in $\ell_p$. Since the adjoint of a hypercyclic operator has no eigenvalue, one obtains that the adjoint of a typical positive contraction $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT}^*)$ and $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT})$ has no eigenvalue. For $\mathtt{SOT}^*$, one can obtain information on the point spectrum of a typical positive contraction $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT}^*)$, since the map $T \mapsto T^*$ is a homeomorphism for $\mathtt{SOT}^*$. The question of the point spectrum of a typical positive contraction $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT})$ will be more delicate. It is known that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ is such that $\sigma_p(T) = \mathbb{D}$, since it is unitarily equivalent to the infinite-dimensional unilateral backward shift operator on $\ell_2(\mathbb{Z}_+, \ell_2)$. We have seen that this is no longer the case in the positive setting. Moreover, the only remarkable known result on the existence of eigenvalues for positive operators is not sufficient for our study. Indeed, this result, which is a sort of Perron-Frobenius theorem, states that every positive compact operator on $\ell_p$, $1 \leq p < \infty$, with no non-trivial closed invariant ideals, has a positive eigenvector ([52, Theorem 5.7]). Even if a typical positive contraction on $\ell_2$ has no non-trivial closed invariant ideals, it is not compact and thus the result does not apply to a typical positive contraction. In order to study the point spectrum of a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$, we will use the points of continuity of the identity map on $\mathcal{P}_1(\ell_2)$.

### 1.1c Similar operator topologies

Given two topologies $\tau, \tau'$ on $\mathcal{P}_1(\ell_p)$ with $1 < p < \infty$, we write $\mathbf{i}_{\tau,\tau'}$ for the identity map from $(\mathcal{P}_1(\ell_p), \tau)$ into $(\mathcal{P}_1(\ell_p), \tau')$, and we denote by $\mathcal{C}(\tau, \tau')$ the set of points of continuity of this map.

If $M$ and $M'$ are two metrizable topological spaces, a map $f : M \to M'$ is said to be Borel-1 if $f^{-1}(U)$ is $F_\sigma$ in $M$ (i.e a countable union of closed sets) for any open set $U$ of $M'$. It is known that when $M, M'$ are metrizable and when $M'$ is separable, the set of points of continuity of any Borel-1 map $f : M \to M'$ is comeager in $(M, \tau)$ ([39, Theorem 21.14]). It can be shown, as in [29, Lemma 2.9], that the maps $\mathbf{i}_{\mathtt{WOT},\mathtt{SOT}}$, $\mathbf{i}_{\mathtt{WOT},\mathtt{SOT}^*}$, $\mathbf{i}_{\mathtt{WOT},\mathtt{SOT}_*}$ and $\mathbf{i}_{\mathtt{SOT},\mathtt{SOT}^*}$ are Borel-1. In particular, the set $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$ is comeager in $(\mathcal{P}_1(\ell_p), \mathtt{SOT})$, and the sets $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$, $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}^*)$ and $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ are comeager in $(\mathcal{P}_1(\ell_p), \mathtt{WOT})$. This is what motivates the use of points of continuity for the study of the point spectrum of a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$.

The authors in [29] proved that the points of continuity of both identity maps $(\mathcal{B}_1(\ell_2), \mathtt{SOT}) \to (\mathcal{B}_1(\ell_2), \mathtt{SOT}^*)$ and $(\mathcal{B}_1(\ell_2), \mathtt{WOT}) \to (\mathcal{B}_1(\ell_2), \mathtt{SOT}_*)$ is the set of co-isometries of $\ell_2$, whereas the set of points of continuity of the identity map $(\mathcal{B}_1(\ell_2), \mathtt{WOT}) \to (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ is the set of isometries of $\ell_2$. In view of our problem, it is natural to try to describe these sets in the positive setting. Obviously, these sets do not describe all the points of continuity in the positive setting, since we have seen that a typical positive contraction on $\ell_2$ is neither a co-isometry, nor an isometry. In order to describe these sets, we will use the notion of norming vectors for positive operators.

Let $T$ be an operator on a Banach space $X$. A non-zero vector $x \in X$ is said to be *norming* for $T$ if $\|Tx\| = \|T\|\|x\|$. If $T$ is a positive operator on $\ell_p$ with $1 \le p < \infty$ and if $x$ is a norming vector for $T$, then the positive vector $|x|$ will also be norming for $T$. The positive norming vectors for $T^*$ will be used to approximate the rows of the matrix of $T$ from approximations on the columns or on the individual entries. We only managed to give a partial description of the sets $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$, $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$ and $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$. The first result is the following.

**Proposition G.** Let $\mathcal{M}$ be the class of positive contractions $T$ on $\ell_2$ such that

- $\|T\| = 1$;
- there exists a family $(u_r)_{r\in J}$ of norming vectors for $T^*$ indexed by a set $J \subseteq \mathbb{Z}_+$ such that $u_r \ge 0$, $\|u_r\| = 1$ for every $r \in J$ and $\displaystyle\bigcup_{r\in J} \mathrm{Supp}(u_r) = \mathbb{Z}_+$.

Then every operator in $\mathcal{M}$ belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ and to $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$.

The second one is the following.

**Proposition H.** Let $\mathcal{M}'$ be the class of positive contractions $T$ on $\ell_2$ such that

- $\|T\| = 1$;
- there exists a family $(u_r)_{r\in J}$ of norming vectors for $T$ indexed by a set $J \subseteq \mathbb{Z}_+$ such that $u_r \ge 0$, $\|u_r\| = 1$ for every $r \in J$ and $\displaystyle\bigcup_{r\in J} \mathrm{Supp}(u_r) = \mathbb{Z}_+$.

Then every operator in $\mathcal{M}'$ belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$.

The classes $\mathcal{M}$ and $\mathcal{M}'$, introduced in Propositions G and H, respectively contain the co-isometries and the isometries of $\ell_2$. We will show that these sets are both $\mathtt{SOT}$-dense in $\mathcal{P}_1(\ell_2)$, which shows that they are large classes of points of continuity. But we will also show that a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$ and $T \in (\mathcal{P}_1(\ell_2), \mathtt{WOT})$ does not attain its norm, so in particular, the classes $\mathcal{M}$ and $\mathcal{M}'$ do not describe all the points of continuity.

Although the description of these points of continuity is somewhat disappointing, we will prove the following interesting result, again using the norming vectors.

**Theorem I.** Any Polish operator topology on $\mathcal{P}_1(\ell_2)$ lying between $\texttt{WOT}$ and $\texttt{SOT}^*$ has the same comeager sets as $\texttt{SOT}^*$ in $\mathcal{P}_1(\ell_2)$.

This is strikingly different from what happens on $\mathcal{B}_1(\ell_2)$, since a typical $T \in (\mathcal{B}_1(\ell_2), \texttt{WOT})$ is unitary, a typical $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT})$ is unitarily equivalent to the infinite-dimensional unilateral backward shift operator on $\ell_2(\mathbb{Z}_+, \ell_2)$, and a typical $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT}^*)$ is such that $T$ and $T^*$ have no eigenvalue. Thanks to Theorem I, we will be able to identify the point spectrum of a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \texttt{SOT})$, for example.

**Corollary J.** Let $\tau$ be any Polish operator topology on $\mathcal{P}_1(\ell_2)$ lying between $\texttt{WOT}$ and $\texttt{SOT}^*$. Then, a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \tau)$ is such that $T$ and $T^*$ have no eigenvalue.

Thanks to Theorem E, we will also be able to deduce the following results, which holds for $\texttt{WOT}$ in particular.

**Corollary K.** Let $\tau$ be any Polish operator topology on $\mathcal{P}_1(\ell_2)$ lying between $\texttt{WOT}$ and $\texttt{SOT}^*$. Then, a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \tau)$ has a non-trivial invariant subspace.

Propositions G, H, as well as Theorem I, are proved in [25]. The question of whether the topologies $\texttt{WOT}$,$\texttt{SOT}$, $\texttt{SOT}_*$ and $\texttt{SOT}^*$ have the same comeager sets in $\mathcal{P}_1(\ell_p)$ for $1 < p \neq 2 < \infty$ remains open.

## 1.2 Linear dynamics of random products of operators

We detail here the content of the second part of this thesis on the linear dynamics of random products of operators, which corresponds to the article [24].

### 1.2a Motivation

The aim of the second part of this thesis is to initiate the study of linear dynamics of random products of operators of the form

$$T_n(\omega) := T(\tau^{n-1}\omega)\cdots T(\tau\omega)T(\omega),\ n \geq 1,\ \omega \in \mathbb{T}, \tag{1.2.1}$$

where $\tau : \mathbb{T} \to \mathbb{T}$ is an ergodic measure-preserving transformation on $\mathbb{T}$, equipped with the normalized Lebesgue measure. Although the present work is set on the probability space $(\mathbb{T}, m)$, the setting readily generalizes to any probability space.

The motivation of this work comes from the study of the asymptotic behavior of random products of matrices, which come into play, for example, in considering solutions to differential or difference equations with random coefficients. Indeed, let us consider the example of a random sequence $(\xi_n)_{n\geq 0}$ satisfying $\xi_n = u_n\xi_{n-1} + v_n\xi_{n-2}$ for every $n \geq 2$, where $(u_n, v_n)_{n\geq 2}$ is a sequence of independent identically distributed random vectors. We can thus write

$$\begin{pmatrix} \xi_{n+1} \\ \xi_n \end{pmatrix} = A_n A_{n-1} \cdots A_1 \begin{pmatrix} \xi_1 \\ \xi_0 \end{pmatrix}, \quad \text{with } A_j = \begin{pmatrix} u_{j+1} & v_{j+1} \\ 1 & 0 \end{pmatrix}.$$

We then see that the asymptotic behavior of $(\xi_n)_{n\geq 0}$ is governed by that of the random products $A_n A_{n-1} \cdots A_1$. It is natural to investigate whether there exists a distribution describing the asymptotic behavior of the products $A_n A_{n-1} \cdots A_1$, where $(A_j)_{j\geq 1}$ is a sequence of independent identically distributed random matrices.

The first one to study such random products of matrices was Bellman, in the case of $2\times 2$ matrices with strictly positive entries ([7]). Precisely, he proved that if the random matrices $A_k$ take two values, have strictly positive entries, and share the same distribution, then the coefficients $x_{i,j}^{(n)}$ of the matrices $X_n := A_n \cdots A_1$ are such that $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[x_{i,j}^{(n)}]$ exists. In particular, he proved that the weak law of large numbers holds for the entries of such random matrices.

Furstenberg and Kesten improved Bellman's result by showing that if $A_1, A_2, \ldots$ is a sequence of independent random matrices with common distribution supported on a set of matrices whose coefficients are strictly positive and bounded away from 0 and $\infty$ in an appropriate sense, and if $x_{i,j}^{(n)}$ are the entries of $A_n A_{n-1} \cdots A_1$, then, for a certain constant $\alpha$,

$$\frac{1}{n}\log(x_{i,j}^{(n)}) \underset{n\to\infty}{\longrightarrow} \alpha$$

with probability 1 ([22]). They also showed that this result does not hold without the positivity assumption.

Furstenberg and Kesten also considered more general random products of matrices in [22]. They proved the following result.

**Theorem 1.2.1** ([22, Theorem 2])**.** *Let $X_n, n \geq 1$, be identically distributed independent random variables with distribution $\mu$ and with values in the space of $d \times d$ matrices. If $\mathbb{E}[\log^+(\|X_1\|)] < \infty$, then the limit*

$$\lim_{n\to\infty} \frac{1}{n}\log(\|X_n \cdots X_1\|) \tag{1.2.2}$$

*exists with probability 1 and is constant, depending only on $\mu$.*

Nowadays, Theorem 1.2.1 can be deduced from the following result, due to Kingman, and called the subadditive ergodic theorem.

**Theorem 1.2.2.** *Let $\tau : (\Omega, \mu) \to (\Omega, \mu)$ be an ergodic measure-preserving transformation on a probability space $(\Omega, \mu)$. Let $(f_n)_{n\geq 1}$ be a subadditive sequence of functions $f_n : \Omega \to \mathbb{R}$ (that is, $f_{n+m}(\omega) \leq f_n(\omega) + f_m(\tau^n\omega)$ for every $n \geq 1$, $m \geq 1$ and $\omega \in \Omega$) such that $f_1^+ := \max(0, f_1) \in L^1(\Omega)$. Then*

$$\frac{f_n(\omega)}{n} \underset{n\to\infty}{\longrightarrow} \inf_{k\geq 1} \frac{1}{k}\int_\Omega f_k \, d\mu \in [-\infty, \infty) \quad \textit{for almost every } \omega \in \Omega.$$

Let $(\Omega, \mathcal{F}, \mu, \tau)$ be an ergodic dynamical system and $A : \Omega \to GL_d(\mathbb{R})$ a measurable map such that $\log^+(\|A\|) \in L^1(\Omega)$. Applying Theorem 1.2.2 to $f_n(\omega) := \log(\|A(\tau^{n-1}\omega)\cdots A(\tau\omega)A(\omega)\|)$, we obtain that the sequence $(\frac{1}{n}\log(\|A_n(\omega)\|))_{n\geq 1}$ converges almost surely to a real number $\chi^+$, which is independent of $\omega$, called the dominant Lyapunov exponent of the sequence $(A_n(\omega))_{n\geq 1} = (A(\tau^{n-1}\omega)\cdots A(\tau\omega)A(\omega))_{n\geq 1}$ at $\omega \in \Omega$. It measures the asymptotic rate of growth of the random products $A(\tau^{n-1}\omega)\cdots A(\tau\omega)A(\omega)$.

While random products of matrices and operators have been extensively studied from a probabilistic and statistical viewpoint - most notably via Lyapunov exponents - the linear dynamics of such random products remains largely unexplored. This work aims to initiate such a study by examining the dynamical properties of random products of operators (1.2.1) on a separable Fréchet space.

### 1.2b Linear dynamics, ergodic theory

In view of our study, we introduce the necessary tools from linear dynamics. We refer to the books [6] and [31] for the classical results of this subsection, as well as the survey [32] for an overview on universality. We start with some basic definitions.

**Definition 1.2.3.** Let $X$ be a metric space and let $T_n$ be continuous maps from $X$ to itself. We say that the sequence $(T_n)_{n\geq 1}$ is universal if there exists an element $x \in X$ such that its orbit under $(T_n)_{n\geq 1}$,

$$\mathrm{orb}(x,(T_n)) := \{T_n x : n \geq 1\},$$

is dense in $X$. Such an element $x$ is said to be a universal vector for $(T_n)_{n\geq 1}$.

A classical example of universal family is provided by universal Taylor series. Fekete showed that there exists an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ such that, for any continuous function $g : \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$, there exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that

$$\sum_{l=0}^{n_k} \frac{f^{(l)}(0)}{l!} x^l \underset{k\to\infty}{\longrightarrow} g$$

uniformly on any compact subset of $\mathbb{R}$. In this case, $T_n$ is the map that associates to $f$ its Taylor polynomial of degree $n$ at 0.

If $T$ is an operator on a separable Fréchet space $X$, we say that $T$ is hypercyclic if the sequence $(T^n)_{n\geq 1}$ is universal.

An important notion in linear dynamics implying the universality is the notion of topological transitivity. This is defined as follows.

**Definition 1.2.4.** A sequence of continuous maps $(T_n)_{n\geq 1}$ from a space $X$ to itself is said to be topologically transitive if, for any pair $U, V \subset X$ of nonempty open sets, there exists an integer $n \geq 1$ such that $T_n(U) \cap V \neq \emptyset$. If this property holds for sufficiently large $n \geq 1$, then we say that the sequence $(T_n)_{n\geq 1}$ is topologically mixing. Finally, we say that the sequence $(T_n)_{n\geq 1}$ is topologically weakly mixing if, for any nonempty open sets $U_1, U_2, V_1, V_2$ of $X$, there is some $n \geq 1$ such that $T_n(U_1) \cap V_1 \neq \emptyset$ and $T_n(U_2) \cap V_2 \neq \emptyset$.

Every topologically mixing sequence is topologically weakly mixing, and every topologically weakly mixing sequence is topologically transitive. For a single operator $T$ on a separable Fréchet space $X$ without isolated point, being hypercyclic and being topologically transitive is the same, that is, $T$ is hypercyclic if and only if for any pair $U, V \subset X$ of nonempty open sets, there exists an integer $n \geq 1$ such that $T^n(U) \cap V \neq \emptyset$. For the case of a sequence of maps $(T_n)_{n\geq 1}$, this still holds provided that the operators $T_n$ commute and have dense range. A useful criterion to prove that a sequence of operators is universal in the linear setting is the following, called the Universality Criterion.

**Proposition 1.2.5.** *Let $X$ be a separable Fréchet space and $(T_n)_{n\geq 1}$ a sequence of operators on this space. Suppose that there are dense subsets $\mathcal{D}_1$ and $\mathcal{D}_2$ of $X$, a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers, and maps $S_{n_k} : \mathcal{D}_2 \to X$, $k \geq 1$, such that*

*(i)* $T_{n_k} x \underset{k\to\infty}{\longrightarrow} 0$ *for every* $x \in \mathcal{D}_1$.

*(ii)* $S_{n_k} y \underset{k\to\infty}{\longrightarrow} 0$ *for every* $y \in \mathcal{D}_2$.

*(iii)* $T_{n_k} S_{n_k} y \underset{k\to\infty}{\longrightarrow} y$ *for every* $y \in \mathcal{D}_2$.

*Then the sequence $(T_n)_{n\geq 1}$ is topologically weakly mixing, and in particular universal. If, in addition, there exists an integer $N \geq 1$ such that this criterion is satisfied for all $n \geq N$, then the sequence $(T_n)_{n\geq 1}$ is topologically mixing.*

For operators with many eigenvalues, we also have the following criterion, which is just a particular case of Proposition 1.2.5 and is the analogue of Godefroy-Shapiro criterion for sequences of operators.

**Proposition 1.2.6** ([9, Theorem 7]). *Suppose that $X$ is a separable Fréchet space and $(T_n)_{n\geq 1}$ is a sequence of operators on this space.*

*Suppose that there are two subsets $\mathcal{A}$ and $\mathcal{B}$ of $X$ satisfying*

*(a) Every element of $\mathcal{A} \cup \mathcal{B}$ is an eigenvector of $T_n$, $n \geq 1$, satisfying $|\lambda(T_n, a)| \underset{n\to\infty}{\longrightarrow} 0$ for every $a \in \mathcal{A}$ and $|\lambda(T_n, b)| \underset{n\to\infty}{\longrightarrow} \infty$ for every $b \in \mathcal{B}$.*

*(b) span$(\mathcal{A})$ and span$(\mathcal{B})$ are dense in $X$.*

*Then, the sequence $(T_n)_{n\geq 1}$ is topologically mixing.*

We now recall some notions from ergodic theory that will be used for our study ([50], [62]). We start with the notion of measure-preserving transformation.

**Definition 1.2.7.** Given a probability space $(\Omega, \mathcal{A}, \mu)$, we say that a measurable map $\tau : \Omega \to \Omega$ is $\mu$-preserving if

$$\mu(\tau^{-1}(B)) = \mu(B)$$

for every measurable set $B \in \mathcal{A}$. In this case, we say that $(\Omega, \mathcal{A}, \mu, \tau)$ is a measure-preserving system.

The random operators that we will consider in this thesis will depend on an ergodic transformation on $\mathbb{T}$. We define such transformations in the following way.

**Definition 1.2.8.** A measure-preserving transformation $\tau$ on $\Omega$ is said to be ergodic if it satisfies one of the following equivalent assertions.

1. Every set $A \in \mathcal{A}$ such that $\mu(A) > 0$ satisfies $\mu\left(\bigcup_{n\geq 1} \tau^{-n}(A)\right) = 1$.
2. Every set $A \in \mathcal{A}$ such that $\tau^{-1}(A) = A$ satisfies $\mu(A) = 0$ or $1$.
3. Every set $A \in \mathcal{A}$ such that $\tau(A) \subseteq A$ satisfies $\mu(A) = 0$ or $1$.

Finally, we introduce the notion of weak mixing and (strong) mixing.

**Definition 1.2.9.** A measure-preserving system $(\Omega, \mathcal{A}, \mu, \tau)$ is said to be mixing if

$$\mu(\tau^{-n}(A) \cap B) \underset{n\to\infty}{\longrightarrow} \mu(A)\,\mu(B)$$

for every measurable sets $A, B \subseteq \Omega$, and weakly mixing if

$$\frac{1}{n}\sum_{j=0}^{n-1} |\mu(\tau^{-j}(A) \cap B) - \mu(A)\mu(B)| \underset{n\to\infty}{\longrightarrow} 0$$

for every measurable sets $A, B \subseteq \Omega$.

Mixing is a property of asymptotic decorrelation. Every mixing system is weakly mixing and every weakly mixing system is ergodic. We will consider two important examples of ergodic transformations: irrational rotations and the doubling map on $\mathbb{T}$.

Given a real number $\alpha$ in $(0,1)$, the rotation of parameter $\alpha$ is defined by the transformation $R_\alpha : x \mapsto x+\alpha \pmod 1$ on $[0,1)$, where $x \pmod 1$ is the fractional part of the real $x$. This map can be seen as a map on $\mathbb{T}$ defined by $\tau(x) = x+\alpha$ in $\mathbb{T}$, and can also be seen as a map on the unit circle defined by $\tau(z) = e^{2i\pi\alpha}z$. This map is ergodic if and only if $\alpha$ is irrational. This map is never weakly mixing. The doubling map is the map on $[0,1)$ defined by $\tau(x) = 2x \pmod 1$. This map can also be seen as a map on $\mathbb{T}$ defined by $\tau(x) = 2x \in \mathbb{T}$, and also as a map on the unit circle defined by $\tau(z) = z^2$. Contrary to rotations, this map is mixing.

A main tool in our study of the linear dynamics of random sequences of products of operators will be Birkhoff's ergodic theorem, which is a generalization of the law of large numbers.

**Theorem 1.2.10.** *Let $(\Omega, \mathcal{A}, \mu, \tau)$ be a measure-preserving system. For any $\mu$-integrable function $f \in L^1(\Omega)$, there exists a function $g \in L^1(\Omega)$ satisfying $g\circ\tau = g$ and $\int_\Omega g\,d\mu = \int_\Omega f\,d\mu$, such that*

$$\frac{1}{N}\sum_{n=0}^{N-1} f(\tau^n\omega) \underset{N\to\infty}{\longrightarrow} g(\omega) \quad \textit{for almost every } \omega\in\Omega. \tag{1.2.3}$$

*Moreover, the system is ergodic if and only if $g(\omega) = \int_\Omega f\,d\mu$ for almost every $\omega$.*

Let us remark in particular that if the system is ergodic and if $A \subseteq \Omega$ is a measurable set, then, applying Theorem 1.2.10 to the function $f = \mathbb{1}_A$, we obtain

$$\frac{1}{N}\operatorname{Card}\{0 \le n \le N-1 : \tau^n\omega \in A\} \underset{N\to\infty}{\longrightarrow} \mu(A)$$

for almost every $\omega\in\Omega$.

If $f : \Omega \to \mathbb{R}$ is a real function, if $\tau$ is a measure-preserving transformation on $(\Omega, \mathcal{A}, \mu)$ and if $n \geq 1$, we will denote by

$$\mathbb{S}_n^\tau f(\omega) := \sum_{i=0}^{n-1} f(\tau^i\omega)$$

the $n$th Birkhoff sum associated to $f$ and $\tau$.

### 1.2c Random products of operators on separable Fréchet spaces

We describe here some of our results on the linear dynamics of the random operators of the form

$$T_n(\omega) := T(\tau^{n-1}\omega)\cdots T(\tau\omega)T(\omega),\ n\geq 1, \omega\in\mathbb{T}. \tag{1.2.4}$$

The transformation $\tau$ will be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$, where $m$ is the normalized Lebesgue measure on $\mathbb{T}$, and the map $T : \mathbb{T} \to \mathcal{B}(X)$ will be given by

$$T(\omega) = \begin{cases} T_1 & \text{if} \quad \omega\in A_1 \\ T_2 & \text{if} \quad \omega\in A_2 \end{cases}, \tag{1.2.5}$$

where $T_1, T_2$ are two operators on a separable Fréchet space $X$, and $A_1, A_2$ are two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. We will study the influence of the ergodic transformation $\tau$ and of the operators $T_1$ and $T_2$ on the universality of the sequence $(T_n(\omega))_{n\geq 1}$ for almost every $\omega \in \mathbb{T}$. More precisely, we would like to determine if the sequence $(T_n(.))_{n\geq 1}$ of the random operators is universal, or topologically weakly mixing, or topologically mixing, or none of it, in the sense that the sequence $(T_n(\omega))_{n\geq 1}$ has this property for almost every $\omega \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$, or none of these properties. In the case where $X$ is a separable Banach space, we can prove a $0-1$ law, which states that either the sequence $(T_n(\omega))_{n\geq 1}$ is universal for almost every $\omega \in \mathbb{T}$, or the sequence $(T_n(\omega))_{n\geq 1}$ is not universal for almost every $\omega \in \mathbb{T}$.

Let $\tau$ be an ergodic transformation on $(\mathbb{T}, m)$. The first natural example of random products of operators to consider is the case given by

$$T(\omega) = \begin{cases} \lambda_1 B & \text{if} \quad \omega \in A_1 \\ \lambda_2 B & \text{if} \quad \omega \in A_2 \end{cases},$$

where $\lambda_1, \lambda_2$ are two non-zero complex numbers and $B$ is the backward shift operator on $\ell_p(\mathbb{Z}_+)$ given by $B(x_0, x_1, \dots) = (x_1, x_2, \dots)$ for $1 \leq p < \infty$. The operator $\lambda B$ is hypercyclic precisely when $|\lambda| > 1$. By Theorem 1.2.10, there exists a Borel subset $E$ of $\mathbb{T}$ with $m(E) = 1$ such that for every $\omega \in E$,

$$\frac{a_1(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} m(A_1) \quad \text{and} \quad \frac{a_2(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} m(A_2), \tag{1.2.6}$$

where $a_k(n,\omega) := \mathrm{Card}\{0 \leq i \leq n-1 : \tau^i \omega \in A_k\}$. The operators $T_n(\omega)$ are given by

$$T_n(\omega) = \lambda_1^{a_1(n,\omega)} \lambda_2^{a_2(n,\omega)} B^n$$

for every $n \geq 1$.

Let us suppose first that $|\lambda_1|^{m(A_1)} |\lambda_2|^{m(A_2)} < 1$. Then, using (1.2.6), we remark that for every $\omega \in E$,

$$\|T_n(\omega)\| = |\lambda_1|^{a_1(n,\omega)} |\lambda_2|^{a_2(n,\omega)} \underset{n\to\infty}{=} \exp(n(\log(|\lambda_1|^{m(A_1)} |\lambda_2|^{m(A_2)}) + o(1))),$$

which converges to zero as $n \to \infty$. In particular each orbit under $(T_n(\omega))_{n\geq 1}$ is bounded, and the random sequence $(T_n(.))_{n\geq 1}$ is not universal.

Suppose now that $|\lambda_1|^{m(A_1)} |\lambda_2|^{m(A_2)} > 1$. Let us denote by $c_{00}$ the space of finitely supported sequences, which is dense in $\ell_p(\mathbb{Z}_+)$ for every $1 \leq p < \infty$. Let us notice that, for every $\omega \in E$, we have $T_n(\omega)x = 0$ for every $x \in c_{00}$, when $n$ is large enough. Let us denote by $S$ the forward shift on $\ell_p(\mathbb{Z}_+)$, defined by $S(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$. This operator satisfies $BS = I$. Let us set $S_n(\omega) = \lambda_1^{-a_1(n,\omega)} \lambda_2^{-a_2(n,\omega)} S^n$ for every $\omega \in E$ and $n \geq 1$. Then we have $T_n(\omega) S_n(\omega) = I$ for every $\omega \in E$ and every $n \geq 1$. And using (1.2.6) again, one can show that $S_n(\omega)x$ converges to zero for every $x \in \ell_p(\mathbb{Z}_+)$, for every $\omega \in E$. In particular, Proposition 1.2.5 applies and the random sequence $(T_n(.))_{n\geq 1}$ is universal (and even topologically mixing).

The question now is what happens if $|\lambda_1|^{m(A_1)} |\lambda_2|^{m(A_2)} = 1$. In this case, Theorem 1.2.10 is not enough to conclude on the universality of the random sequence $(T_n(.))_{n\geq 1}$. Since $|\lambda_2| = |\lambda_1|^{-\frac{m(A_1)}{m(A_2)}}$, we obtain

$$\|T_n(\omega)\| = |\lambda_1|^{a_1(n,\omega) - \frac{m(A_1)}{m(A_2)} a_2(n,\omega)} = |\lambda_1|^{\mathbb{S}_n^\tau f(\omega)}$$

and

$$\|S_n(\omega)\| = |\lambda_1|^{-(a_1(n,\omega) - \frac{m(A_1)}{m(A_2)} a_2(n,\omega))} = |\lambda_1|^{-\mathbb{S}_n^\tau f(\omega)},$$

where $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)} \mathbb{1}_{A_2}$. The behavior of the Birkhoff sums associated to this function $f$ plays an important role in determining the universality of the random sequence $(T_n(.))_{n\geq 1}$. In particular, we see that if the sequence $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ is bounded for almost every $\omega \in \mathbb{T}$, then the random sequence $(T_n(.))_{n\geq 1}$ is not universal. And in the case where $\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty$ and $\liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$ for almost every $\omega \in \mathbb{T}$, then the random sequence $(T_n(.))_{n\geq 1}$ is universal (and even topologically weakly mixing), thanks to Proposition 1.2.5.

The operator $\lambda B$ on $\ell_2(\mathbb{Z}_+)$ is unitarily equivalent to the adjoint of the multiplication operator by the function $\lambda z$ on the Hardy space $H^2(\mathbb{D})$. This space $H^2(\mathbb{D})$ is the space of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\sup_{0\leq r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt < \infty.$$

It is isometrically isomorphic to the space $\ell_2(\mathbb{Z}_+)$. We define multiplication operators on this space by $M_\phi : f \mapsto f\phi$, where $\phi$ is a function in $H^\infty(\mathbb{D})$, the space of bounded holomorphic functions on $\mathbb{D}$. A multiplication operator on $H^2(\mathbb{D})$ is never hypercyclic, and we will show that if both operators $T_1$ and $T_2$ are multiplication operators on this space, then the random product $(T_n(.))_{n\geq 1}$ defined by an ergodic transformation $\tau$ and by (1.2.5) is never universal too, as expected. However, the adjoint $(M_\phi)^*$ on $H^2(\mathbb{D})$ of the multiplication operator is hypercyclic if and only if $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, when $\phi$ is nonconstant in the space $H^\infty(\mathbb{D})$ of bounded holomorphic functions on $\mathbb{D}$ ([31, Theorem 4.42], [6, Example 1.11]). We will thus first focus on the case where both operators are given by $T_1 = (M_{\phi_1})^*$ and $T_2 = (M_{\phi_2})^*$, with $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ nonconstant. Since the operators $M_{\phi_1}$ and $M_{\phi_2}$ commute, we have

$$T_n(\omega) = (M_{\phi_1}^{a_1(n,\omega)})^* (M_{\phi_2}^{a_2(n,\omega)})^*$$

for every $n \geq 1$ and for every $\omega \in \mathbb{T}$.

The adjoint of a multiplication operator on $H^2(\mathbb{D})$ has many eigenvectors, namely the functions $k_\lambda$ defined by

$$k_\lambda(z) = \frac{1}{1 - \overline{\lambda} z}$$

for every $\lambda \in \mathbb{D}$. These are the reproducing kernels of $H^2(\mathbb{D})$. Using Proposition 1.2.6, we will obtain a first sufficient condition for the universality of the sequence $(T_n(.))_{n\geq 1}$.

**Theorem L.** Let $\tau$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1, 2$.

Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant on $\mathbb{D}$ and suppose that there exist $\lambda, \mu \in \mathbb{D}$ such that

$$|\phi_1(\lambda)|^{m(A_1)} |\phi_2(\lambda)|^{m(A_2)} < 1 \quad \text{and} \tag{1.2.7}$$

$$|\phi_1(\mu)|^{m(A_1)} |\phi_2(\mu)|^{m(A_2)} > 1. \tag{1.2.8}$$

Suppose that, for every $\omega \in \mathbb{T}$, the operator $T(\omega)$ is given by

$$T(\omega) = \begin{cases} (M_{\phi_1})^* & \text{if} \quad \omega \in A_1 \\ (M_{\phi_2})^* & \text{if} \quad \omega \in A_2 \end{cases}.$$

Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically mixing.

In the case where $m(A_1) = m(A_2)$, Conditions (1.2.7) and (1.2.8) are equivalent to $(\phi_1\phi_2)(\mathbb{D})\cap \mathbb{T} \neq \emptyset$ and $\phi_1\phi_2$ nonconstant, by the open mapping theorem. Thus, in the case where $m(A_1) = m(A_2) = 1/2$, we will have the cases $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{D}$ and $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{C}\setminus\overline{\mathbb{D}}$, that we will call limit cases. Our next goal will be to try to solve these cases. Obviously, if $m(A_1) = m(A_2) = 1/2$ and if $\phi_1(\mathbb{D}) \subset \mathbb{D}$ and $\phi_2(\mathbb{D}) \subset \mathbb{D}$, or $\phi_1(\mathbb{D}) \subset \mathbb{C}\setminus\overline{\mathbb{D}}$ and $\phi_2(\mathbb{D}) \subset \mathbb{C}\setminus\overline{\mathbb{D}}$, then the random sequence $(T_n(.))_{n\geq 1}$ won't be universal: in the first case, this will come from the fact that the operators $T_n(\omega)$ are contractions, and in the second case, the operators $T_n(\omega)$ will be invertible with a contractive inverse. The two cases should be called trivial situations of our limit cases. We will focus on other situations included in our limit cases. More precisely, we will treat the situation where $m(A_1) = m(A_2) = 1/2$ and $\phi_1\phi_2$ is an inner function on $\mathbb{D}$. An analytic function $f : \mathbb{D} \to \mathbb{C}$ is said to be *inner* if $f$ is bounded on $\mathbb{D}$ and if $|f^*| = 1$ almost everywhere on $\mathbb{T}$, where $f^*(e^{it}) := \lim_{r\to 1} f(re^{it})$. We will obtain the following result.

**Theorem M.** Let $\tau$ be an ergodic transformation on $(\mathbb{T}, m)$. Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and such that $m(A_1) = m(A_2) = 1/2$. Let $f$ be the function $f = \mathbb{1}_{A_1} - \mathbb{1}_{A_2}$. Suppose that the sequence $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ satisfies

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{or} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

for almost every $\omega \in \mathbb{T}$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that $\phi_1\phi_2$ is inner.
If $\phi_1(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ and $1/\phi_1 \in H^\infty(\mathbb{D})$, or $\phi_2(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ and $1/\phi_2 \in H^\infty(\mathbb{D})$, then the sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.

We will also remove the assumption on the images of $\phi_1$ and $\phi_2$ in the following way.

**Theorem N.** Let $\tau$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and such that $m(A_1) = m(A_2) = 1/2$. Let $f$ be the function $f = \mathbb{1}_{A_1} - \mathbb{1}_{A_2}$. Suppose that the sequence $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ satisfies

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

for almost every $\omega \in \mathbb{T}$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be two nonconstant functions, such that $\phi_1\phi_2$ is inner and nonconstant. Suppose also that either $\phi_1$ or $\phi_2$ is not inner.
Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.

In particular, Theorem N shows that it is possible for the random sequence $(T_n(.))_{n\geq 1}$ to be universal, even if both operators $T_1$ and $T_2$ are not hypercyclic.

We will see that when the sequence of Birkhoff sums $(\mathbb{S}_n^\tau f)_{n\geq 1}$ associated to an ergodic transformation $\tau$ and to our favorite function $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$ satisfies a Central Limit Theorem, which means that there exists a sequence $(a_n)_{n\geq 1}$ of positive numbers such that $a_n \underset{n\to\infty}{\longrightarrow} \infty$ and such that the sequence $(\frac{\mathbb{S}_n^\tau f}{a_n})_{n\geq 1}$ converges in distribution to a non-degenerate centered normal distribution, then the assumption

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

is satisfied for almost every $\omega \in \mathbb{T}$. The question of whether these Birkhoff sums satisfy a Central Limit Theorem will be examined for the case of the doubling map and for irrational rotations on $\mathbb{T}$. The case of the doubling map will be based on Kac's result [37], and the case of irrational rotations will be based on the articles [36], [15] and [16]. We will see that the case of irrational rotations strongly depends on the Diophantine properties of the irrational parameter of the rotation.

Thanks to these Central Limit Theorems, we will for example obtain the following results. The first one focuses on a case for the doubling map.

**Corollary O.** Let $\tau$ be the doubling map on $\mathbb{T}$. Let $A_1 = [0,1/2)$ and $A_2 = [1/2,1)$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that $\phi_1\phi_2$ is inner and nonconstant. Suppose also that either $\phi_1$ or $\phi_2$ is not inner. Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.

An analogous result holds for irrational rotations, as follows.

**Corollary P.** Let $\alpha$ be an irrational number in $(0,1)$ and let $\tau = R_\alpha$ on $\mathbb{T}$. Let $A_1 = [0,1/2)$ and $A_2 = [1/2,1)$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that $\phi_1\phi_2$ is inner and nonconstant. Suppose also that either $\phi_1$ or $\phi_2$ is not inner. Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.

We will also consider generalizations of Theorems M and N to Borel subsets $A_1, A_2$ such that $m(A_1) \neq m(A_2)$, but we will see that in this case this is a bit more subtle. For the doubling map, we will for example obtain the following result, obtained by a Central Limit Theorem.

**Proposition Q.** Let $\tau$ be the doubling map on $\mathbb{T}$ and let $A_1$ and $A_2$ be two disjoint intervals of $[0,1)$, such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant. Suppose that one of the two following conditions holds:

(i) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ almost everywhere on $\mathbb{T}$, and one of the images $\phi_1(\mathbb{D})$ or $\phi_2(\mathbb{D})$ meets the unit circle $\mathbb{T}$.

(ii) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ almost everywhere on $\mathbb{T}$, either $\phi_1$ or $\phi_2$ has a non-trivial inner part and either $\phi_1$ or $\phi_2$ is not inner.

Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.

For irrational rotations, an analogue of Proposition Q will be more delicate to obtain, as it strongly depends on the choice of the intervals $A_1$ and $A_2$ such that $m(A_1) \neq m(A_2)$, as well as on the Diophantine properties of the irrational number $\alpha \in (0,1)$.

We will also see that it is possible to have non-universality of the random sequence $(T_n(.))_{n\geq 1}$. This will be the case, for example, in certain situations where the Birkhoff sums are bounded almost everywhere. For example, for certain irrational rotations, we have the following result.

**Proposition R.** Let $\alpha$ be an irrational number in $(0,1)$ and let $\tau = R_\alpha$. Let $A_1 = [0,b)$ and $A_2 = [b,1)$, with $b \in (0,1)$. Suppose that $b \in \mathbb{Z}\alpha$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that

$$|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1 \quad \text{almost everywhere on } \mathbb{T}.$$

Then, for every $\omega \in \mathbb{T}$, the sequence $(T_n(\omega))_{n\geq 1}$ is not universal.

Finally, we will study the case where $T_1$ and $T_2$ are operators on the space of entire functions $H(\mathbb{C})$, equipped with the topology of uniform convergence on compact subsets of $\mathbb{C}$. We first study the case where $T_1$ and $T_2$ are entire functions of exponential type of the derivation operator $D$.

We say that an entire function $\varphi \in H(\mathbb{C})$ is of exponential type if there exist constants $M, A > 0$ such that

$$|\varphi(z)| \leq Me^{A|z|}, \quad \text{for every } z \in \mathbb{C}.$$

If $\varphi : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type, with $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$, the expression

$$\varphi(D) := \sum_{n\geq 0} a_n D^n$$

defines an operator on $H(\mathbb{C})$. We have a description of the operators on $H(\mathbb{C})$ that commute with $D$: an operator $T$ on $H(\mathbb{C})$ commute with $D$ if and only if there exists an entire function $\varphi$ of exponential type such that $T = \varphi(D)$. Moreover, these operators are always topologically mixing, provided they are not a scalar multiple of the identity ([31, Theorem 4.21], [6, Theorem 1.42]). We obtained the following result for this case, by using the eigenvectors $e_\lambda : z \mapsto e^{\lambda z}$ of the derivation operator $D$.

**Theorem S.** Let $\tau : \mathbb{T} \to \mathbb{T}$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Let $\varphi_1$ and $\varphi_2$ be two nonconstant entire functions of exponential type. Consider the map

$$T(\omega) := \begin{cases} \varphi_1(D) & \text{if } \omega \in A_1 \\ \varphi_2(D) & \text{if } \omega \in A_2 \end{cases}.$$

Suppose that there exist $z, w \in \mathbb{C}$ such that

$$|\varphi_1(z)|^{m(A_1)} |\varphi_2(z)|^{m(A_2)} < 1 \tag{1.2.9}$$

$$\text{and} \quad |\varphi_1(w)|^{m(A_1)} |\varphi_2(w)|^{m(A_2)} > 1. \tag{1.2.10}$$

Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically mixing.

In the case where $m(A_1) = m(A_2) = 1/2$ and where $\varphi_1$ and $\varphi_2$ are two nonconstant functions of exponential type, both Conditions 1.2.9 and 1.2.10 are simultaneously satisfied if and only if $\varphi_1\varphi_2$ is nonconstant. This comes from the fact that the image of a nonconstant entire function is dense in $\mathbb{C}$. We will study an example in the situation where $\varphi_1\varphi_2$ is constant.

Finally, we will study an example where the operators $T_1$ and $T_2$ on $H(\mathbb{C})$ do not commute. For $\lambda \in \mathbb{C}$ nonzero and $b \in \mathbb{C}$, we denote by $T_{\lambda,b}$ the operator on $H(\mathbb{C})$ defined by $T_{\lambda,b} f = f(\lambda . + b)$. The operator $T_{\lambda,b}$ is hypercyclic if and only if $\lambda = 1$ and $b \neq 0$. We obtained the following result.

**Theorem T.** Let $\tau : \mathbb{T} \to \mathbb{T}$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Consider the map

$$T(\omega) := \begin{cases} T_{\lambda,b} & \text{if } \omega \in A_1 \\ D & \text{if } \omega \in A_2 \end{cases},$$

with $\lambda \in \mathbb{C}$ nonzero and $b \in \mathbb{C}$. Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically mixing.

Theorem T gives an example of a map $\omega \mapsto T(\omega)$ which is hypercyclic for every $\omega \in A_2$ and not hypercyclic for every $\omega \in A_1$, for which the random sequence $(T_n(.))_{n\geq 1}$ is universal, and for which the maps $T(\tau^i \omega)$, $i \geq 0$, do not commute.

## Organization of the manuscript

The main part of this thesis is organized into three chapters (Chapters 3 to 5). Chapters 3 and 4 correspond to the first part of the introduction on the typical properties of positive contractions, while Chapter 5 corresponds to the second part on the linear dynamics of random products of operators.

Each chapter presents detailed statements and proofs of the original results, along with necessary technical material. At the end of these chapters, we discuss open questions and possible directions for future work.

# Chapter 2

# Introduction (en français)

Les objectifs de cette thèse sont, d'une part, l'étude des propriétés typiques de contractions positives sur un espace de Banach muni d'une base, et d'autre part, l'étude de la dynamique linéaire des produits aléatoires d'opérateurs sur les espaces de Fréchet séparables.

Les résultats obtenus font l'objet de trois articles, listés ci-dessous :

— Typical properties of positive contractions and the invariant subspace problem, publié dans *Positivity* [26].

— Similar operator topologies on the space of positive contractions, publié dans *Illinois Journal of Mathematics* [25].

— Linear dynamics of random products of operators, soumis [24].

La motivation de la première partie de cette thèse est le Problème du Sous-espace Invariant. Ce problème peut s'énoncer ainsi : étant donné un espace de Banach réel ou complexe séparable de dimension infinie $X$, est-ce que tout opérateur borné $T$ sur $X$ admet un sous-espace invariant non trivial ? La réponse à ce problème est négative dans le cadre non réflexif : Enflo a construit un espace de Banach non réflexif supportant un opérateur sans sous-espace invariant non trivial (voir [20]), et Read a construit des opérateurs sur $\ell_1$ et sur $c_0$ sans sous-espaces invariants non triviaux (voir [53] et [54]). Ce problème reste toujours ouvert pour les espaces de Banach réflexifs et en particulier pour les espaces de Hilbert séparables. Ce problème peut également être restreint aux opérateurs positifs de la manière suivante : étant donné un espace de Banach réel ou complexe de dimension infinie muni d'une base de Schauder, est-ce que tout opérateur borné positif sur $X$ possède un sous-espace invariant non trivial ? Malgré des exemples importants fournis par Troitsky et Sirotkin, la réponse à ce problème est toujours inconnue. Étant donné la difficulté de ce problème, nous l'étudierons du point de vue de la typicité. L'objectif principal est d'étudier si une contraction positive typique sur un espace de Banach muni d'une base de Schauder admet un sous-espace invariant non trivial. Bien que cette question reste ouverte en toute généralité, nous apportons des réponses partielles dans les cas $X = \ell_1$ et $X = \ell_2$, et nous étudions des cadres plus généraux.

La motivation de la seconde partie de cette thèse est l'étude du comportement asymptotique des produits aléatoires de matrices, d'abord étudiée par Bellman, puis par Furstenberg et Kesten, qui ont démontré des lois des grands nombres pour ces produits aléatoires ([7], [22]).

Nous étudions des produits aléatoires d'opérateurs sur un espace de Fréchet séparable. Ces produits aléatoires dépendent d'une transformation ergodique préservant la mesure sur $\mathbb{T}$, munie de la mesure de Lebesgue normalisée, ainsi que de différents opérateurs fixés $T_j$ sur cet espace de Fréchet. L'objectif sera d'étudier l'influence de la transformation ergodique sous-jacente ainsi que des opérateurs $T_j$ sur la dynamique de ces produits aléatoires.

Dans cette introduction, les résultats originaux sont énumérés alphabétiquement (Proposition A, Théorème B, ...).

## 2.1 Propriétés typiques de contractions positives

Nous détaillons ici le contenu de la première partie de cette thèse portant sur les propriétés typiques des contractions positives, qui correspond aux articles [26] et [25].

Sauf mention contraire, tous les espaces de Banach considérés seront supposés complexes, séparables et de dimension infinie.

### 2.1a Notion de propriété typique de contractions

Le cadre général pour étudier la notion de propriété typique est celui des espaces de Baire. Ce sont des espaces topologiques $M$ tels que l'intersection d'une famille dénombrable d'ouverts denses dans $M$ est dense. Tout espace métrique complet est un espace de Baire. Plus généralement, tout espace polonais est un espace de Baire.

**Définition 2.1.1.** Un espace polonais est un espace topologique séparable et complètement métrisable.

Rappelons qu'un espace topologique $(M,\tau)$ est complètement métrisable s'il existe une distance $d$ sur $M$ qui induit la topologie $\tau$ et telle que $(M,d)$ soit complet. Voici quelques exemples d'espaces polonais.

**Exemple 2.1.2.**
- Tout ensemble $M$ muni de la topologie discrète est polonais. Une distance complète compatible est donnée par $d(x,y)=0$ si $x=y$ et $d(x,y)=1$ sinon.
- Tout sous-ensemble fermé d'un espace polonais est aussi polonais.
- Tout ouvert $U$ d'un espace métrique complet $(X,d)$ est polonais. Une distance complète compatible est donnée par $d'(x,y)=d(x,y)+\left|\frac{1}{d(x,X\setminus U)}-\frac{1}{d(y,X\setminus U)}\right|$.

Il existe également une description des sous-ensembles polonais d'un espace polonais : un sous-ensemble d'un espace polonais est polonais si et seulement si c'est un $G_\delta$, c'est-à-dire une intersection dénombrable d'ouverts ([39, Théorème 3.11]).

Une notion importante dans les espaces de Baire est celle des parties comaigres. Un intérêt des parties comaigres est que l'intersection de deux parties comaigres dans un espace de Baire est non vide, grâce au théorème de Baire — contrairement à l'intersection de deux parties denses qui peut être vide.

**Définition 2.1.3.** Soit $M$ un espace topologique. Un sous-ensemble $A$ de $M$ est dit comaigre s'il contient l'intersection d'une famille dénombrable d'ouverts denses. Le complémentaire d'un ensemble comaigre est appelé un ensemble maigre. Autrement dit, un ensemble $A\subset M$ est maigre s'il est contenu dans une union dénombrable de fermés d'intérieur vide.

Nous donnons deux exemples d'ensembles comaigres. Le second montre qu'il n'est pas toujours évident de démontrer qu'un ensemble est comaigre.

**Exemple 2.1.4.**
- L'ensemble $\mathbb{Q}$ des nombres rationnels est maigre dans $\mathbb{R}$, et donc $\mathbb{R}\setminus\mathbb{Q}$ est comaigre dans $\mathbb{R}$.
- L'ensemble des fonctions continues réelles sur $[0,1]$ qui ne sont nulle part dérivables est comaigre dans l'espace des fonctions continues réelles muni de la norme infinie.

Nous pouvons maintenant définir la notion de propriété typique.

**Définition 2.1.5.** Soit $(M,\tau)$ un espace topologique. Une propriété (P) qui porte sur les éléments de $M$ est typique pour la topologie $\tau$ si l'ensemble

$$\{x \in M : x \text{ a la propriété (P)}\}$$

est comaigre dans $(M,\tau)$. La propriété (P) est dite atypique pour $\tau$ si sa négation est typique pour $\tau$.

Dans le cas où une propriété (P) qui porte sur les éléments d'un espace topologique $(M,\tau)$ est typique, on dit alors « un $x$ typique dans $(M,\tau)$ possède la propriété (P) ». Avec l'Exemple 2.1.4, on obtient qu'un nombre réel typique $x \in \mathbb{R}$ est irrationnel, et on obtient aussi qu'une fonction continue réelle typique $f$ sur $[0,1]$ est nulle part dérivable.

Dans la première partie de cette thèse, nous considérerons des propriétés typiques de contractions. Étant donné un espace de Banach $X$, on note $\mathcal{B}_1(X)$ l'ensemble des contractions sur $X$, c'est-à-dire

$$\mathcal{B}_1(X) := \{T \in \mathcal{B}(X) : \|T\| \le 1\}.$$

Nous étudierons des propriétés typiques de contractions pour différentes topologies polonaises sur $\mathcal{B}_1(X)$. Nous ne le ferons pas pour la topologie de la norme opérateur $\|\cdot\|$. En effet, $(\mathcal{B}_1(X),\|\cdot\|)$ n'est généralement pas séparable et donc pas polonais. Et même si cette topologie est complète sur $\mathcal{B}(X)$, il est difficile d'étudier des propriétés typiques pour cette topologie en général. Nous mentionnons tout de même que certains résultats sur les propriétés typiques pour la topologie de la norme opérateur ont été obtenus dans [19] et [56]. Nous considérerons plutôt la topologie opératorielle forte, la topologie opératorielle forte étoile, la topologie opératorielle faible et la topologie opératorielle forte duale. La topologie opératorielle forte est la topologie de la convergence simple sur $\mathcal{B}(X)$. La topologie opératorielle forte étoile est la topologie de la convergence simple pour les opérateurs et leurs adjoints. La topologie opératorielle faible est la topologie de la convergence faible sur $\mathcal{B}(X)$. Enfin, la topologie opératorielle forte duale est la topologie de la convergence simple des adjoints dans $\mathcal{B}(X)$. Nous noterons respectivement $\texttt{SOT}$, $\texttt{SOT}^*$, $\texttt{WOT}$ et $\texttt{SOT}_*$ ces topologies. Nous pouvons résumer la convergence d'une suite pour ces topologies de la manière suivante :

$$\begin{cases} T_i \xrightarrow[i]{\texttt{SOT}} T & \iff & T_i x \xrightarrow[i]{\|\cdot\|} Tx \quad \text{pour tout } x \in X, \\ T_i \xrightarrow[i]{\texttt{SOT}^*} T & \iff & T_i \xrightarrow[i]{\texttt{SOT}} T \quad \text{et} \quad T_i^* \xrightarrow[i]{\texttt{SOT}} T^*, \\ T_i \xrightarrow[i]{\texttt{WOT}} T & \iff & \langle y^*, T_i x\rangle \xrightarrow[i]{} \langle y^*, Tx\rangle \quad \text{pour tout } y^* \in X^* \text{ et tout } x \in X, \\ T_i \xrightarrow[i]{\texttt{SOT}_*} T & \iff & T_i^* \xrightarrow[i]{\texttt{SOT}} T^*. \end{cases}$$

On a les inclusions suivantes : $\texttt{WOT} \subset \texttt{SOT}$, $\texttt{WOT} \subset \texttt{SOT}_*$, $\texttt{SOT} \subset \texttt{SOT}^*$ et $\texttt{SOT}_* \subset \texttt{SOT}^*$. Malheureusement, ces topologies ne sont jamais métrisables sur $\mathcal{B}(X)$ lorsque $X$ est de dimension infinie.

Cependant, sous certaines hypothèses sur l'espace de Banach $X$, leurs restrictions à $\mathcal{B}_1(X)$ sont polonaises. Ceci est donné par la proposition suivante.

**Proposition 2.1.6.** *Soit $X$ un espace de Banach.*

- *Si $X$ est séparable, alors $(\mathcal{B}_1(X), \texttt{SOT})$ est polonais.*
- *Si $X^*$ est séparable, alors $(\mathcal{B}_1(X), \texttt{SOT}_*)$ et $(\mathcal{B}_1(X), \texttt{SOT}^*)$ sont polonais.*
- *Si $X$ est réflexif et séparable, alors $(\mathcal{B}_1(X), \texttt{WOT})$ est polonais.*

Par exemple, si $X$ est séparable et $(x_n)_{n\geq 0}$ est une suite dense dans la sphère unité $S_X$ de $X$, on peut montrer que la métrique suivante

$$d : (T,S) \in \mathcal{B}_1(X) \mapsto \sum_{n=0}^{\infty} 2^{-n} \|(T-S)x_n\|$$

est complète et compatible avec $\texttt{SOT}$ sur $\mathcal{B}_1(X)$, et que l'espace $(\mathcal{B}_1(X), d)$ est séparable. Sous les hypothèses de la Proposition 2.1.6, on peut aussi construire de telles métriques pour les autres topologies considérées ci-dessus.

La topologie $\texttt{SOT}^*$ est une topologie intéressante pour la raison suivante. Soit (P) une propriété d'opérateurs sur l'espace de Banach $X = \ell_p$ avec $p > 1$. Si l'ensemble

$$\{T \in \mathcal{B}_1(\ell_p) : T \text{ a la propriété (P)}\}$$

est comaigre dans $(\mathcal{B}_1(\ell_p), \texttt{SOT}^*)$, alors l'ensemble suivant

$$\{T \in \mathcal{B}_1(\ell_{p'}) : T^* \text{ a la propriété (P)}\}$$

est aussi comaigre dans $(\mathcal{B}_1(\ell_{p'}), \texttt{SOT}^*)$, où $p'$ est l'exposant conjugué de $p$. Cela découle du fait que l'application $T \mapsto T^*$, de $(\mathcal{B}_1(\ell_p), \texttt{SOT}^*)$ dans $(\mathcal{B}_1(\ell_{p'}), \texttt{SOT}^*)$, est un homéomorphisme. Cela n'est évidemment pas vrai pour les autres topologies, mais on peut tout de même affirmer que si l'ensemble

$$\{T \in \mathcal{B}_1(\ell_p) : T \text{ a la propriété (P)}\}$$

est comaigre dans $(\mathcal{B}_1(\ell_p), \texttt{SOT})$, alors l'ensemble

$$\{T \in \mathcal{B}_1(\ell_{p'}) : T^* \text{ a la propriété (P)}\}$$

est aussi comaigre dans $(\mathcal{B}_1(\ell_{p'}), \texttt{SOT}_*)$, et réciproquement.

La notion de propriété typique de contractions a été introduite par Eisner dans [18], où elle montre que, pour la topologie opératorielle faible, une contraction typique sur un espace de Hilbert séparable est unitaire. Ce concept a davantage été développé par Eisner et Mátrai dans [19], qui ont étudié des propriétés typiques de contractions sur des espaces de Hilbert séparables pour d'autres topologies, telles que la topologie opératorielle forte et la topologie opératorielle forte étoile. Ils ont démontré qu'une contraction typique $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT})$ est unitairement équivalente au « backward shift unilatéral » de multiplicité infinie sur $\ell_2(\mathbb{Z}_+, \ell_2) := \left\{(u_n)_{n\geq 0} : u_n \in \ell_2 \text{ pour tout } n \geq 0, \text{ et } \sum_{n\geq 0} \|u_n\|^2 < \infty\right\}$, défini par $B_\infty(u_0, u_1, u_2, \dots) = (u_1, u_2, \dots)$. Quelques années plus tard, Grivaux, Matheron et Menet ont étudié des propriétés typiques de contractions sur les espaces $\ell_p$ pour ces topologies polonaises ([28], [30], [29]). Leur motivation était le Problème du Sous-espace Invariant, et leur objectif principal était de déterminer si une contraction typique $T \in \mathcal{B}_1(\ell_p)$ possède un sous-espace invariant non trivial. Grâce aux travaux d'Eisner et Mátrai, il était déjà connu qu'une contraction typique $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ (resp. $T \in (\mathcal{B}_1(\ell_2), \mathtt{WOT})$) possède un sous-espace invariant non trivial. Malheureusement, le résultat d'équivalence unitaire avec le backward shift unilatéral $B_\infty$ n'est plus valable pour la topologie $\mathtt{SOT}^*$, car une contraction typique $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT}^*)$ n'est pas une co-isométrie. Néanmoins, les auteurs de [30] ont tout de même réussi à prouver qu'une contraction typique $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT}^*)$ possède un sous-espace invariant non trivial, en utilisant le fait qu'une telle contraction possède un spectre assez riche. Les auteurs de [30] ont également montré qu'une contraction typique $T \in (\mathcal{B}_1(\ell_1), \mathtt{SOT})$ admet un sous-espace invariant non trivial.

Il est également intéressant de noter que les auteurs de [30] ont établi l'existence d'une loi du 0-1 pour les propriétés de contractions. Plus précisément, ils ont démontré le résultat suivant :

**Proposition 2.1.7.** *Soit $X = \ell_p$ avec $1 \leq p < \infty$. Soit $\mathcal{A} \subset \mathcal{B}_1(\ell_p)$. Si $\mathcal{A}$ a la propriété de Baire pour* $\mathtt{SOT}$, *et si $\mathcal{A}$ est stable par conjugaison par isométrie surjective (i.e. $J\mathcal{A}J^{-1} = \mathcal{A}$ pour toute isométrie surjective $J$ de $\ell_p$), alors $\mathcal{A}$ est soit maigre, soit comaigre dans* $(\mathcal{B}_1(\ell_p), \mathtt{SOT})$.

La Proposition 2.1.7 reste valable pour la topologie $\mathtt{SOT}^*$ lorsque $1 < p < \infty$.

Dans cette proposition, un sous-ensemble $\mathcal{A} \subset \mathcal{B}_1(\ell_p)$ est dit avoir la propriété de Baire pour une topologie $\tau$ si $\mathcal{A}$ est égale à un ouvert modulo un ensemble maigre, c'est-à-dire qu'il existe un ouvert $U$ de $(\mathcal{B}_1(\ell_p), \tau)$ tel que $A \triangle U$ est maigre. Une classe importante d'ensembles ayant la propriété de Baire est celle des ensembles analytiques. Un ensemble $\mathcal{A}$ est analytique dans $(\mathcal{B}_1(\ell_p), \tau)$ s'il est la projection d'un borélien d'un produit $(\mathcal{B}_1(\ell_p), \tau) \times Y$ sur $\mathcal{B}_1(\ell_p)$, où $Y$ est un espace polonais. Illustrons cela avec l'exemple suivant.

**Exemple 2.1.8** ([30, Corollaire 3.3])**.** Soit $X = \ell_p$ avec $1 \leq p < \infty$. Alors, ou bien une contraction typique $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ possède un sous-espace invariant non trivial, ou bien une contraction typique $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ ne possède pas de sous-espace invariant non trivial. En effet, soit $\mathcal{A}$ l'ensemble des contractions sur $X$ ayant un sous-espace invariant non trivial. Alors

$$T \in \mathcal{A} \iff \exists (x, x^*) \in X \times B_{X^*} : (x \neq 0,\ x^* \neq 0, \text{ et } \forall n \geq 0,\ \langle x^*,\ T^n x \rangle = 0).$$

La condition entre parenthèses définit un borélien de $(\mathcal{B}_1(X), \mathtt{SOT}) \times X \times (B_{X^*}, w^*)$, et comme l'ensemble $\mathcal{A}$ est la projection de ce borélien sur $\mathcal{B}_1(X)$, il est analytique dans $(\mathcal{B}_1(X), \mathtt{SOT})$. De plus, $\mathcal{A}$ est clairement stable par conjugaison par isométrie surjective de $\ell_p$, donc d'après la Proposition 2.1.7, $\mathcal{A}$ est soit maigre, soit comaigre.

La Proposition 2.1.7 s'applique à la plupart des propriétés classiques d'opérateurs, comme dans l'Exemple 2.1.8.

Dans cette thèse, nous étudions des propriétés typiques de contractions positives sur un espace de Banach ayant une base de Schauder. Notre motivation est également le Problème du Sous-espace Invariant, mais dans le cadre des opérateurs positifs. Ce problème restant ouvert dans ce contexte, il est naturel de l'envisager sous l'angle de la typicité. L'intérêt de cette étude est renforcé par l'existence de méthodes propres aux opérateurs positifs.

### 2.1b Opérateurs positifs agissant sur un espace de Banach muni d'une base

Nous définissons la positivité des opérateurs sur un espace de Banach par rapport à une base de Schauder. Commençons par quelques rappels sur les bases de Schauder. Une base de Schauder est simplement un système de coordonnées.

**Définition 2.1.9.** Soit $X$ un espace de Banach. Une suite $(e_n)_{n\geq 0}$ de vecteurs non nuls de $X$ est appelée base de Schauder de $X$ si, pour tout $x \in X$, il existe une unique suite de nombres complexes $(x_n)_{n\geq 0}$ telle que

$$x = \sum_{n=0}^{\infty} x_n e_n,$$

où la série converge en norme.

Par exemple, une base orthonormée dans un espace de Hilbert est une base de Schauder. La base canonique de $\ell_p$ et de $c_0$ (pour $1 \leq p < \infty$), donnée par $e_n(k) = 1$ si $n = k$ et $e_n(k) = 0$ sinon, est une base de Schauder de ces espaces. Le système trigonométrique de $L^p(0,1)$ est un exemple de base de Schauder de $L^p(0,1)$ lorsque $1 < p < \infty$. Il existe aussi une base de $L^p(0,1)$ constituée de fonctions constantes par morceaux, appelée base de Haar. Évidemment, certains espaces de Banach ne possèdent pas de base de Schauder, car cela implique que l'espace soit séparable. De plus, Enflo a construit un espace de Banach réflexif séparable qui n'admet aucune base de Schauder, montrant ainsi que la réflexivité et la séparabilité ne garantissent pas l'existence d'une base, et Lindenstrauss a prouvé qu'il existe des espaces de Banach $X$ ayant une base de Schauder et un dual séparable, tels que $X^*$ ne possède pas de base de Schauder (voir [44] pour plus de détails).

Si $(e_n)_{n\geq 0}$ est une base de Schauder d'un espace de Banach $X$, alors les projections

$$P_N : x = \sum_{n=0}^{\infty} x_n e_n \mapsto \sum_{n=0}^{N} x_n e_n$$

sont continues, et $\sup_{N\geq 0} \|P_N\|$ est une constante finie, appelée constante de la base. La base est dite monotone si $\sup_{N\geq 0} \|P_N\| = 1$, ce qui signifie que

$$\left\| \sum_{n=0}^{N} x_n e_n \right\| \leq \left\| \sum_{n=0}^{M} x_n e_n \right\|$$

pour tout $M \geq N$ et pour tous nombres complexes $x_0, \ldots, x_M$. En particulier, lorsque $(e_n)_{n\geq 0}$ est une base de Schauder de $X$, les formes coordonnées $(e_n^*)_{n\geq 0}$ associés à la suite $(e_n)_{n\geq 0}$, définies par $e_m^*\left(\sum_{n=0}^{\infty} x_n e_n\right) = x_m$, sont continues. Cependant, la suite $(e_n^*)_{n\geq 0}$ n'est pas nécessairement une base de Schauder de $X^*$. On dit qu'une base de Schauder $(e_n)_{n\geq 0}$ de $X$ est contractante si la suite des formes coordonnées $(e_n^*)_{n\geq 0}$ est une base de Schauder de $X^*$. Lorsque l'espace $X$ est réflexif et séparable, toute base de Schauder est contractante (voir par exemple [44, Théorème II.22]). Par exemple, la base canonique de $\ell_p$, $1 \leq p < \infty$, et de $c_0$, est monotone. La base de Haar est monotone dans $L^p(0,1)$, $1 \leq p < \infty$.

Nous définissons aussi la notion de base inconditionnelle, car elle sera nécessaire pour définir le module d'un vecteur. Une base de Schauder $(e_n)_{n\geq 0}$ d'un espace de Banach $X$ est dite inconditionnelle si, pour tout $x = \sum_{n=0}^{\infty} x_n e_n$ dans $X$, la série $\sum_{n\geq 0} x_{\sigma(n)} e_{\sigma(n)}$ converge dans $X$ pour toute permutation $\sigma$ de $\mathbb{Z}_+$. Par exemple, la base canonique de $\ell_p$, $1 \leq p < \infty$, et de $c_0$, est inconditionnelle. La base de Haar dans $L^p(0,1)$ est inconditionnelle uniquement pour $1 < p < \infty$, et le système trigonométrique est inconditionnel seulement pour $p = 2$. L'espace $L^1(0,1)$ ne s'injecte dans aucun espace de Banach possédant une base inconditionnelle. Si la base $(e_n)_{n\geq 0}$ est inconditionnelle dans $X$, alors il existe une constante $K > 0$ telle que pour tout $x = \sum_{n=0}^{\infty} x_n e_n$ dans $X$ et toute suite bornée de nombres complexes $\lambda = (\lambda_n)_{n\geq 0}$, on ait

$$\left\| \sum_{n=0}^{\infty} \lambda_n x_n e_n \right\| \leq 2K \|\lambda\|_\infty \|x\|.$$

Dorénavant, comme nous ne travaillerons qu'avec des bases de Schauder, nous omettrons le terme « Schauder ». Nous pouvons maintenant définir la notion d'opérateur positif par rapport à une base.

**Définition 2.1.10.** Soit $X$ un espace de Banach muni d'une base $(e_n)_{n\geq 0}$.

- Un vecteur $x = \sum_{n=0}^{\infty} x_n e_n$ est dit positif (par rapport à la base $(e_n)_{n\geq 0}$) si $x_n \geq 0$ pour tout $n \geq 0$. Dans ce cas, on note $x \geq 0$.
- Un opérateur $T \in \mathcal{B}(X)$ est dit positif si $Tx \geq 0$ pour tout $x \geq 0$. Dans ce cas, on note $T \geq 0$.

On note $\mathcal{P}(X)$ l'ensemble des opérateurs positifs sur $X$, et $\mathcal{P}_1(X)$ l'ensemble des contractions positives sur $X$.

La notion de positivité dépend fortement du choix de la base. Un opérateur peut être positif par rapport à une base, mais ne pas l'être par rapport à une autre. Il est très facile de reconnaître ou de construire des opérateurs positifs. En effet, soit $T$ un opérateur sur un espace de Banach $X$ muni d'une base $(e_n)_{n\geq 0}$. L'opérateur $T$ est positif par rapport à la base $(e_n)_{n\geq 0}$ si et seulement si sa matrice $(t_{i,j})_{i,j\geq 0}$ relativement à cette base est à coefficients positifs (avec $t_{i,j} = \langle e_i^*, Te_j \rangle$ pour tous $i, j \geq 0$). En particulier, tout « forward shift » ou « backward shift » à poids, dont les poids sont positifs, est un opérateur positif, et tout opérateur diagonal à diagonale positive est positif. Lorsque $X = \ell_p$ avec $1 \leq p < \infty$, on considèrera toujours la base canonique.

Soit $X$ un espace de Banach muni d'une base $(e_n)_{n\geq 0}$. L'ensemble $\mathcal{P}_1(X)$ est fermé dans $\mathcal{B}_1(X)$ pour chacune des topologies d'opérateurs $\texttt{WOT}$, $\texttt{SOT}$, $\texttt{SOT}_*$ et $\texttt{SOT}^*$, car la convergence pour ces topologies implique la convergence coordonnée par coordonnée. En particulier, on obtient le résultat suivant, qui fournit le cadre approprié pour l'étude de propriétés typiques de contractions positives.

**Proposition A.** Soit $X$ un espace de Banach muni d'une base. L'ensemble $(\mathcal{P}_1(X), \texttt{SOT})$ est polonais. Si $X^*$ est séparable, alors les ensembles $(\mathcal{P}_1(X), \texttt{SOT}_*)$ et $(\mathcal{P}_1(X), \texttt{SOT}^*)$ sont polonais. Enfin, si $X$ est réflexif, alors l'ensemble $(\mathcal{P}_1(X), \texttt{WOT})$ est polonais.

L'étude des propriétés typiques de contractions positives est motivée par le Problème du Sous-espace Invariant pour les opérateurs positifs, que l'on peut formuler ainsi : étant donné un espace de Banach réel ou complexe de dimension infinie muni d'une base, est-ce que tout opérateur positif sur $X$ admet un sous-espace invariant non trivial ? La réponse à ce problème est encore inconnue à ce jour. Mais nous pouvons tout de même mentionner deux exemples importants à ce sujet.

Si $X$ est muni d'une base inconditionnelle $(e_n)_{n\geq 0}$ et si $x = \sum_{n=0}^{\infty} x_n e_n$ appartient à $X$, alors le vecteur positif $|x| := \sum_{n=0}^{\infty} |x_n| e_n$ est bien défini dans $X$ et est appelé le *module* de $x$. Soit $T$ un opérateur sur $X$, et soit $(t_{i,j})_{i,j\geq 0}$ la matrice de $T$ relativement à cette base. Si la matrice positive $(|t_{i,j}|)_{i,j\geq 0}$ définit un opérateur sur $X$, alors on dit que $T$ admet un module sur $X$, et dans ce cas l'opérateur correspondant est appelé le *module* de $T$, noté $|T|$. Sur l'espace $\ell_1$, on vérifie aisément que tout opérateur admet un module. Le premier exemple important est dû à V. Troitsky ([60]). Il a montré que le module d'un opérateur de Read sur $\ell_1$ (considéré comme espace réel), qui est un opérateur positif sur $\ell_1$, admet un vecteur propre positif et possède en particulier un sous-espace invariant non trivial, alors qu'un opérateur de Read n'a pas de sous-espace invariant non trivial sur $\ell_1$. Le second exemple important est dû à G. Sirotkin ([58]). Il a construit un opérateur sur $\ell_1$ sans sous-espace invariant non trivial, dont la matrice ne possède que des coefficients positifs sauf un seul, ce qui nous rapproche d'une solution négative au problème du sous-espace invariant pour les opérateurs positifs. On ne sait toujours pas aujourd'hui s'il existe un opérateur positif sur $\ell_1$ sans sous-espace invariant non trivial.

Il existe néanmoins un résultat remarquable concernant l'existence de sous-espaces invariants non triviaux pour les opérateurs positifs. Ce résultat est dû à Abramovich, Aliprantis et Burkinshaw.

**Théorème 2.1.11** ([2, Théorème 2.2]). *Soit $X$ un espace de Banach muni d'une base, et soit $T$ un opérateur positif sur $X$. S'il existe un opérateur positif non nul $A$ sur $X$, appartenant au commutant de $T$ et quasinilpotent en un certain vecteur positif non nul de $X$, alors $T$ admet un sous-espace invariant non trivial.*

Un opérateur $A$ sur un espace de Banach $X$ est dit quasinilpotent en un vecteur $x \in X$ si

$$\lim_{n\to\infty} \|A^n x\|^{1/n} = 0.$$

Évidemment, tout opérateur quasinilpotent est quasinilpotent en tout vecteur de $X$. La réciproque est fausse en général : l'opérateur $T$ sur $\ell_p$ (avec $p > 1$), défini par $Te_0 = e_0 + e_1$ et $Te_j = \frac{1}{j}e_{j+1}$ pour tout $j \geq 1$, fournit un contre-exemple, puisqu'il est quasinilpotent en $e_1$ mais n'est pas quasinilpotent. Le Théorème 2.1.11 étant un résultat d'existence de sous-espaces invariants non triviaux pour les opérateurs positifs, il est naturel d'étudier la typicité de cette propriété. On dira donc qu'un opérateur positif sur un espace de Banach muni d'une base satisfait le *critère AAB* s'il vérifie les hypothèses du Théorème 2.1.11. Dans le cas des espaces $\ell_p$, il sera facile d'étudier la typicité de ce critère grâce à la notion d'idéal invariant pour les opérateurs positifs.

Soit $X$ un espace de Banach muni d'une base inconditionnelle. Un sous-espace vectoriel $V$ de $X$ est un idéal de $X$ si $|x| \leq |y|$ et $y \in V$ impliquent $x \in V$, pour tous $x, y \in X$. La notion d'idéal dans un espace de Banach à base inconditionnelle est intéressante, car dans les espaces $\ell_p$ avec $1 \leq p < \infty$, tout opérateur positif satisfaisant le critère AAB possède un idéal fermé invariant non trivial ([1, Théorème 2.2]). Pour reconnaître si un opérateur positif sur un espace de Banach à base inconditionnelle possède un idéal invariant non trivial, on utilise le critère suivant dû à Troitsky et Radjavi ([52, Proposition 1.2]).

**Proposition 2.1.12.** *Soit $X$ un espace de Banach muni d'une base inconditionnelle $(e_n)_{n\geq 0}$. Soit $T$ un opérateur positif sur $X$. Alors $T$ ne possède aucun idéal invariant fermé non trivial si et seulement si :*

$$\forall i \neq j \in \mathbb{Z}_+, \exists n \in \mathbb{Z}_+ : \langle e_j^*, T^n e_i \rangle > 0.$$

Grâce à ce résultat, on déduit l'énoncé suivant.

**Théorème B.** Soit $1 \leq p < \infty$ et soit $X = \ell_p$. Une contraction typique $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ et $T \in (\mathcal{P}_1(X), \texttt{WOT})$ lorsque $1 < p < \infty$) ne satisfait pas le critère AAB.

On généralise ensuite le Théorème B aux espaces de Banach munis d'une base monotone (pas nécessairement inconditionnelle), pour la topologie $\mathtt{SOT}^*$. Cela est donné par le résultat suivant.

**Théorème C.** Soit $X$ un espace de Banach réflexif muni d'une base monotone. Une contraction typique $T \in (\mathcal{P}_1(X), \mathtt{SOT}^*)$ ne satisfait pas le critère AAB.

Les Théorèmes B et C sont démontrés dans [26] et montrent en particulier que l'existence d'un sous-espace invariant non trivial pour les opérateurs positifs est un problème difficile. On s'intéresse également à d'autres propriétés typiques de contractions positives sur les espaces $\ell_p$, dans le but d'établir l'existence d'un sous-espace invariant non trivial pour une contraction positive typique sur ces espaces. En particulier, la Proposition 2.1.7 reste valable dans $\mathcal{P}_1(\ell_p)$.

Rappelons qu'Eisner et Mátrai ont montré qu'une contraction typique $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ est unitairement équivalente au backward shift unilatéral de multiplicité infinie sur $\ell_2(\mathbb{Z}_+, \ell_2)$. Pour cela, ils ont d'abord prouvé qu'une contraction typique $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ est telle que $T^*$ est une isométrie, puis ils ont appliqué la décomposition de Wold à l'adjoint de $T$. De plus, ils ont montré qu'une contraction typique $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ vérifie $T^n x \underset{n\to\infty}{\longrightarrow} 0$ pour tout $x \in X$ et possède un noyau de dimension infinie. Avec ces propriétés, la décomposition de Wold de l'adjoint d'un tel $T$ typique n'aura qu'une partie « forward shift » de multiplicité infinie. Malheureusement, cela n'aura pas lieu pour une contraction positive typique $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$, à cause du résultat suivant.

**Proposition D.** Soit $1 < p < \infty$ et soit $X = \ell_p$. Une contraction typique $T \in (\mathcal{P}_1(X), \mathtt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \mathtt{SOT}^*)$ et $T \in (\mathcal{P}_1(X), \mathtt{WOT})$) n'est pas une co-isométrie.

La Proposition D provient du fait qu'une isométrie positive de $\ell_p$ avec $p > 1$ envoie des suites positives à supports disjoints sur des suites positives à supports disjoints. Pour cette raison, l'ensemble des co-isométries positives n'est pas dense dans $\mathcal{P}_1(\ell_p)$ pour ces topologies, et en particulier, il ne peut pas être comaigre. Par la loi du 0-1, cet ensemble est donc maigre dans $\mathcal{P}_1(\ell_p)$. En particulier, on ne peut pas affirmer directement, comme dans le cas non-positif, qu'une contraction positive typique $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$ possède un sous-espace invariant non trivial. Mais nous allons voir qu'une contraction positive typique $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT})$ (resp. $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT}^*)$ lorsque $p > 1$) vérifie $\sigma(T) = \overline{\mathbb{D}}$. En utilisant le théorème de Brown, Chevreau et Pearcy, qui affirme que toute contraction sur un espace de Hilbert dont le spectre contient le cercle unité possède un sous-espace invariant non trivial ([11]), on obtient le résultat suivant.

**Théorème E.** Une contraction typique $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$ (resp. $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT}^*)$) possède un sous-espace invariant non trivial.

Concernant le cas $X = \ell_1$ et la topologie $\mathtt{SOT}$, la situation est la même que pour une contraction typique $T \in (\mathcal{B}_1(\ell_1), \mathtt{SOT})$.

**Théorème F.** Une contraction typique $T \in (\mathcal{P}_1(\ell_1), \mathtt{SOT})$ est telle que $T^*$ est une isométrie non surjective, et telle que $\sigma_p(T) = \mathbb{D}$ avec $\dim(\ker(T - \lambda)) = \infty$ pour tout $\lambda \in \mathbb{D}$. En particulier, une contraction positive typique $T \in (\mathcal{P}_1(\ell_1), \mathtt{SOT})$ possède un sous-espace invariant non trivial.

Nous verrons également qu'une contraction positive typique $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT}^*)$ est telle que $T$ et $T^*$ n'ont pas de valeur propre pour tout $1 < p < \infty$. Pour ce faire, nous allons en fait démontrer que pour tout $1 \leq p < \infty$, une contraction positive typique $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT})$ (resp. $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT}^*)$ lorsque $1 < p < \infty$) est telle que $2T$ est hypercyclique, ce qui signifie qu'il existe un vecteur $x \in \ell_p$ dont l'orbite $\{(2T)^n x : n \geq 0\}$ sous l'action de $2T$ est dense dans $\ell_p$. Puisque l'adjoint d'un opérateur hypercyclique n'a pas de valeur propre, on obtient que l'adjoint d'une contraction positive typique $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT}^*)$ et $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT})$ n'a pas de valeur propre. Pour $\mathtt{SOT}^*$, on peut obtenir des informations sur le spectre ponctuel d'une contraction positive typique $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT}^*)$, puisque l'application $T \mapsto T^*$ est un homéomorphisme pour $\mathtt{SOT}^*$. La question du spectre ponctuel d'une contraction positive typique $T \in (\mathcal{P}_1(\ell_p), \mathtt{SOT})$ est plus délicate.

Il est connu qu'une contraction typique $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ est telle que $\sigma_p(T) = \mathbb{D}$, car elle est unitairement équivalente au backward shift unilatéral de multiplicité infinie sur $\ell_2(\mathbb{Z}_+, \ell_2)$. Nous avons vu que ce n'est plus le cas dans le cadre positif. De plus, le seul résultat remarquable connu sur l'existence de valeurs propres pour les opérateurs positifs n'est pas suffisant pour notre étude. En effet, ce résultat, qui est une sorte de théorème de Perron-Frobenius, affirme que tout opérateur compact positif sur $\ell_p$, $1 \le p < \infty$, sans idéal fermé invariant non trivial, possède un vecteur propre positif ([52, Théorème 5.7]). Bien qu'une contraction positive typique sur $\ell_2$ n'ait pas d'idéal fermé invariant non trivial, elle n'est pas compacte et donc ce résultat ne s'applique pas à une contraction positive typique. Pour étudier le spectre ponctuel d'une contraction positive typique $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$, nous utiliserons les points de continuité de l'application identité sur $\mathcal{P}_1(\ell_2)$.

## 2.1c Topologies semblables d'opérateurs

Étant donnée deux topologies $\tau, \tau'$ sur $\mathcal{P}_1(\ell_p)$ avec $1 < p < \infty$, on note $\mathbf{i}_{\tau,\tau'}$ l'application identité de $(\mathcal{P}_1(\ell_p), \tau)$ dans $(\mathcal{P}_1(\ell_p), \tau')$, et on désigne par $\mathcal{C}(\tau, \tau')$ l'ensemble des points de continuité de cette application.

Si $M$ et $M'$ sont deux espaces topologiques métrisables, une application $f : M \to M'$ est dite de première classe de Baire si pour tout ouvert $U$ de $M'$, l'ensemble $f^{-1}(U)$ est un $F_\sigma$ dans $M$ (c'est-à-dire une union dénombrable de fermés). Il est connu que lorsque $M, M'$ sont métrisables et que $M'$ est séparable, l'ensemble des points de continuité de toute application de première classe de Baire $f : M \to M'$ est comaigre dans $(M, \tau)$ ([39, Théorème 21.14]). On peut montrer, comme dans [29, Lemme 2.9], que les applications $\mathbf{i}_{\mathtt{WOT},\mathtt{SOT}}$, $\mathbf{i}_{\mathtt{WOT},\mathtt{SOT}^*}$, $\mathbf{i}_{\mathtt{WOT},\mathtt{SOT}_*}$ et $\mathbf{i}_{\mathtt{SOT},\mathtt{SOT}^*}$ sont de première classe de Baire. En particulier, l'ensemble $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$ est comaigre dans $(\mathcal{P}_1(\ell_p), \mathtt{SOT})$, et les ensembles $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$, $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}^*)$ et $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ sont comaigres dans $(\mathcal{P}_1(\ell_p), \mathtt{WOT})$. Cela motive l'utilisation des points de continuité pour l'étude du spectre ponctuel d'une contraction positive typique $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$.

Les auteurs de [29] ont montré que les points de continuité des applications identité $(\mathcal{B}_1(\ell_2), \mathtt{SOT}) \to (\mathcal{B}_1(\ell_2), \mathtt{SOT}^*)$ et $(\mathcal{B}_1(\ell_2), \mathtt{WOT}) \to (\mathcal{B}_1(\ell_2), \mathtt{SOT}_*)$ sont les co-isométries de $\ell_2$, tandis que l'ensemble des points de continuité de l'application identité $(\mathcal{B}_1(\ell_2), \mathtt{WOT}) \to (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ est l'ensemble des isométries de $\ell_2$. Au vu de notre problème, il est naturel d'essayer de décrire ces ensembles dans le cadre positif. Évidemment, ces ensembles ne décrivent pas tous les points de continuité dans le cadre positif, puisque nous avons vu qu'une contraction positive typique sur $\ell_2$ n'est ni une co-isométrie, ni une isométrie. Pour décrire ces ensembles, nous utiliserons la notion de vecteurs normants pour les opérateurs positifs.

Soit $T$ un opérateur sur un espace de Banach $X$. Un vecteur non nul $x \in X$ est dit *normant* pour $T$ si $\|Tx\| = \|T\|\|x\|$. Si $T$ est un opérateur positif sur $\ell_p$ avec $1 \le p < \infty$ et si $x$ est un vecteur normant pour $T$, alors le vecteur positif $|x|$ est également normant pour $T$. Les vecteurs positifs normants pour $T^*$ seront utilisés pour approximer les lignes de la matrice de $T$ à partir d'approximations sur les colonnes ou sur les coefficients. Nous n'avons réussi à donner qu'une description partielle des ensembles $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$, $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$ et $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$. Le premier résultat est le suivant.

**Proposition G.** Soit $\mathcal{M}$ la classe des contractions positives $T$ sur $\ell_2$ telles que

- $\|T\| = 1$ ;
- il existe une famille $(u_r)_{r\in J}$ de vecteurs normants pour $T^*$ indexée par un ensemble $J \subseteq \mathbb{Z}_+$ telle que $u_r \geq 0$, $\|u_r\| = 1$ pour tout $r \in J$ et $\bigcup_{r\in J} \mathrm{Supp}(u_r) = \mathbb{Z}_+$.

Alors tout opérateur de $\mathcal{M}$ appartient à $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ et à $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$.

Le second est le suivant.

**Proposition H.** Soit $\mathcal{M}'$ la classe des contractions positives $T$ sur $\ell_2$ telles que

- $\|T\| = 1$ ;
- il existe une famille $(u_r)_{r\in J}$ de vecteurs normants pour $T$ indexée par un ensemble $J \subseteq \mathbb{Z}_+$ telle que $u_r \geq 0$, $\|u_r\| = 1$ pour tout $r \in J$ et $\bigcup_{r\in J} \mathrm{Supp}(u_r) = \mathbb{Z}_+$.

Alors tout opérateur de $\mathcal{M}'$ appartient à $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$.

Les classes $\mathcal{M}$ et $\mathcal{M}'$, introduites dans les Propositions G et H, contiennent respectivement les co-isométries et les isométries de $\ell_2$. Nous montrerons que ces ensembles sont tous deux $\mathtt{SOT}$-denses dans $\mathcal{P}_1(\ell_2)$, ce qui montre qu'il s'agit de classes substantielles de points de continuité. Mais nous montrerons aussi qu'une contraction positive typique $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$ et $T \in (\mathcal{P}_1(\ell_2), \mathtt{WOT})$ n'atteint pas sa norme, donc en particulier, les classes $\mathcal{M}$ et $\mathcal{M}'$ ne décrivent pas tous les points de continuité.

Bien que la description de ces points de continuité soit quelque peu décevante, nous démontrons le résultat intéressant suivant, toujours en utilisant les vecteurs normants.

**Théorème I.** Toute topologie polonaise d'opérateurs sur $\mathcal{P}_1(\ell_2)$ plus fine que $\mathtt{WOT}$ et moins fine que $\mathtt{SOT}^*$ possède les mêmes parties comaigres que $\mathtt{SOT}^*$ dans $\mathcal{P}_1(\ell_2)$.

La situation est très différente de ce qui se passe sur $\mathcal{B}_1(\ell_2)$, puisqu'une contraction typique $T \in (\mathcal{B}_1(\ell_2), \mathtt{WOT})$ est unitaire, une contraction typique $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ est unitairement équivalente au backward shift unilatéral de multiplicité infinie sur $\ell_2(\mathbb{Z}_+, \ell_2)$, et une contraction typique $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT}^*)$ est telle que $T$ et $T^*$ n'ont pas de valeur propre. Grâce au Théorème I, nous pouvons par exemple identifier le spectre ponctuel d'une contraction positive typique $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$.

**Corollaire J.** Soit $\tau$ une topologie polonaise d'opérateurs sur $\mathcal{P}_1(\ell_2)$ plus fine que $\mathtt{WOT}$ et moins fine que $\mathtt{SOT}^*$. Alors, une contraction positive typique $T \in (\mathcal{P}_1(\ell_2), \tau)$ est telle que $T$ et $T^*$ n'ont pas de valeur propre.

Grâce au Théorème E, nous pouvons également déduire les résultats suivants, valables notamment pour $\mathtt{WOT}$.

**Corollaire K.** Soit $\tau$ une topologie polonaise d'opérateurs sur $\mathcal{P}_1(\ell_2)$ plus fine que $\mathtt{WOT}$ et moins fine que $\mathtt{SOT}^*$. Alors, une contraction positive typique $T \in (\mathcal{P}_1(\ell_2), \tau)$ possède un sous-espace invariant non trivial.

Les Propositions G, H, ainsi que le Théorème I, sont démontrés dans [25]. La question de savoir si les topologies $\mathtt{WOT}$, $\mathtt{SOT}$, $\mathtt{SOT}_*$ et $\mathtt{SOT}^*$ ont les mêmes parties comaigres dans $\mathcal{P}_1(\ell_p)$ pour $1 < p \neq 2 < \infty$ reste encore ouverte.

## 2.2 Dynamique linéaire des produits aléatoires d'opérateurs

Nous détaillons ici le contenu de la deuxième partie de cette thèse, consacrée à la dynamique linéaire des produits aléatoires d'opérateurs, qui correspond à l'article [24].

### 2.2a Motivation

Le but de cette deuxième partie est d'initier l'étude de la dynamique linéaire des produits aléatoires d'opérateurs de la forme

$$T_n(\omega) := T(\tau^{n-1}\omega)\cdots T(\tau\omega)T(\omega),\ n \geq 1,\ \omega \in \mathbb{T}, \tag{2.2.1}$$

où $\tau : \mathbb{T} \to \mathbb{T}$ est une transformation ergodique préservant la mesure sur $\mathbb{T}$, munie de la mesure de Lebesgue normalisée. Bien que le problème soit formulé ici pour l'espace probabilisé $(\mathbb{T}, m)$, le cadre se généralise aisément à tout espace probabilisé.

La motivation de ce travail provient de l'étude du comportement asymptotique des produits aléatoires de matrices, qui interviennent par exemple dans l'analyse des solutions d'équations différentielles ou aux différences à coefficients aléatoires. Considérons par exemple une suite aléatoire $(\xi_n)_{n\geq 0}$ satisfaisant $\xi_n = u_n\xi_{n-1} + v_n\xi_{n-2}$ pour tout $n \geq 2$, où $(u_n, v_n)_{n\geq 2}$ est une suite de vecteurs aléatoires indépendants et identiquement distribués. On peut alors écrire

$$\begin{pmatrix} \xi_{n+1} \\ \xi_n \end{pmatrix} = A_nA_{n-1}\cdots A_1 \begin{pmatrix} \xi_1 \\ \xi_0 \end{pmatrix}, \quad \text{avec } A_j = \begin{pmatrix} u_{j+1} & v_{j+1} \\ 1 & 0 \end{pmatrix}.$$

On voit donc que le comportement asymptotique de $(\xi_n)_{n\geq 0}$ est régi par celui des produits aléatoires $A_nA_{n-1}\cdots A_1$. Il est alors naturel de se demander s'il existe une loi décrivant le comportement asymptotique des produits $A_nA_{n-1}\cdots A_1$, où $(A_j)_{j\geq 1}$ est une suite de matrices aléatoires indépendantes et identiquement distribuées.

Le premier à avoir étudié de tels produits aléatoires de matrices est Bellman, dans le cas de matrices $2 \times 2$ à coefficients strictement positifs ([7]). Plus précisément, il montre que si les matrices aléatoires $A_k$ prennent deux valeurs, ont des coefficients strictement positifs et ont la même loi, alors les coefficients $x_{i,j}^{(n)}$ des matrices $X_n := A_n \cdots A_1$ sont tels que $\lim\limits_{n\to\infty} \frac{1}{n}\mathbb{E}[x_{i,j}^{(n)}]$ existe. En particulier, il établit que la loi faible des grands nombres s'applique aux coefficients de telles matrices aléatoires.

Furstenberg et Kesten ont amélioré le résultat de Bellman en montrant que, si $A_1, A_2, \ldots$ est une suite de matrices aléatoires indépendantes de même loi, dont le support est contenu dans un ensemble de matrices à coefficients strictement positifs bornés inférieurement et supérieurement, et si $x_{i,j}^{(n)}$ désigne les coefficients de $A_nA_{n-1}\cdots A_1$, alors, pour une certaine constante $\alpha$, on a

$$\frac{1}{n}\log(x_{i,j}^{(n)}) \underset{n\to\infty}{\longrightarrow} \alpha$$

presque sûrement ([22]). Ils ont également montré que ce résultat n'est pas valable sans l'hypothèse de positivité.

Furstenberg et Kesten ont aussi considéré des produits aléatoires plus généraux de matrices dans [22]. Ils y établissent le résultat suivant.

**Théorème 2.2.1** ([22, Théorème 2]). *Soit $X_n, n \geq 1$, une suite de variables aléatoires indépendantes et identiquement distribuées, à valeurs dans l'espace des matrices $d \times d$, de loi $\mu$. Si $\mathbb{E}[\log^+(\|X_1\|)] < \infty$, alors la limite*

$$\lim_{n\to\infty} \frac{1}{n} \log(\|X_n \cdots X_1\|) \tag{2.2.2}$$

*existe presque sûrement, et elle est constante, ne dépendant que de $\mu$.*

De nos jours, le Théorème 2.2.1 peut être déduit du résultat suivant, dû à Kingman, appelé le théorème ergodique sous-additif.

**Théorème 2.2.2.** *Soit $\tau : (\Omega, \mu) \to (\Omega, \mu)$ une transformation ergodique préservant la mesure sur un espace probabilisé $(\Omega, \mu)$. Soit $(f_n)_{n\geq 1}$ une suite sous-additive de fonctions $f_n : \Omega \to \mathbb{R}$ (c'est-à-dire telle que $f_{n+m}(\omega) \leq f_n(\omega) + f_m(\tau^n \omega)$ pour tous $n \geq 1$, $m \geq 1$ et $\omega \in \Omega$), et supposons que $f_1^+ := \max(0, f_1)$ appartienne à $L^1(\Omega)$. Alors*

$$\frac{f_n(\omega)}{n} \underset{n\to\infty}{\longrightarrow} \inf_{k\geq 1} \frac{1}{k} \int_\Omega f_k \, d\mu \in [-\infty, \infty) \quad \textit{pour presque tout } \omega \in \Omega.$$

Soit $(\Omega, \mathcal{F}, \mu, \tau)$ un système dynamique mesuré ergodique, et soit $A : \Omega \to GL_d(\mathbb{R})$ une application mesurable telle que $\log^+(\|A\|) \in L^1(\Omega)$. En appliquant le Théorème 2.2.2 à

$$f_n(\omega) := \log(\|A(\tau^{n-1}\omega) \cdots A(\tau\omega)A(\omega)\|),$$

on obtient que la suite $\left(\frac{1}{n}\log(\|A_n(\omega)\|)\right)_{n\geq 1}$ converge presque sûrement vers un réel $\chi^+$, indépendant de $\omega$, appelé exposant de Lyapunov dominant de la suite

$$(A_n(\omega))_{n\geq 1} = (A(\tau^{n-1}\omega) \cdots A(\tau\omega)A(\omega))_{n\geq 1}$$

en $\omega \in \Omega$. Celui-ci mesure la vitesse de croissance assymptotique des produits aléatoires

$$A(\tau^{n-1}\omega) \cdots A(\tau\omega)A(\omega).$$

Tandis que les produits aléatoires de matrices et d'opérateurs ont été largement étudiés d'un point de vue probabiliste — notamment via les exposants de Lyapunov — la dynamique linéaire de tels produits reste en grande partie inexplorée. Ce travail vise à initier une telle étude en examinant les propriétés dynamiques des produits aléatoires d'opérateurs (2.2.1) sur un espace de Fréchet séparable.

## 2.2b Dynamique linéaire, théorie ergodique

En vue de notre étude, nous introduisons les outils nécessaires issus de la dynamique linéaire. Nous renvoyons aux ouvrages [6] et [31] pour les résultats classiques de cette sous-section, ainsi qu'au survey [32] pour une vue d'ensemble sur l'universalité. Commençons par quelques définitions de base.

**Définition 2.2.3.** Soit $X$ un espace métrique, et soit $(T_n)_{n\geq 1}$ une suite d'applications continues de $X$ dans lui-même. On dit que la suite $(T_n)_{n\geq 1}$ est universelle s'il existe un élément $x \in X$ tel que son orbite sous $(T_n)_{n\geq 1}$,

$$\mathrm{orb}(x, (T_n)) := \{T_n x : n \geq 1\},$$

soit dense dans $X$. Un tel élément $x$ est appelé un vecteur universel pour $(T_n)_{n\geq 1}$.

Un exemple classique de famille universelle est donné par les séries de Taylor universelles. Fekete a montré qu'il existe une fonction $f : \mathbb{R} \to \mathbb{R}$ de classe $C^\infty$, vérifiant $f(0) = 0$, telle que pour toute fonction continue $g : \mathbb{R} \to \mathbb{R}$ vérifiant $g(0) = 0$, il existe une suite strictement croissante $(n_k)_{k\geq 1}$ d'entiers naturels telle que

$$\sum_{l=0}^{n_k} \frac{f^{(l)}(0)}{l!} x^l \underset{k\to\infty}{\longrightarrow} g$$

uniformément sur tout compact de $\mathbb{R}$. Dans ce cas, $T_n$ est l'application qui à $f$ associe son polynôme de Taylor d'ordre $n$ en 0.

Si $T$ est un opérateur sur un espace de Fréchet séparable $X$, on dit que $T$ est hypercyclique si la suite $(T^n)_{n\geq 1}$ est universelle. Une notion importante en dynamique linéaire, impliquant l'universalité, est la transitivité topologique, définie comme suit.

**Définition 2.2.4.** Une suite d'applications continues $(T_n)_{n\geq 1}$ d'un espace $X$ dans lui-même est dite topologiquement transitive si, pour toute paire d'ouverts non vides $U, V \subset X$, il existe un entier $n \geq 1$ tel que $T_n(U) \cap V \neq \emptyset$. Si cette propriété est satisfaite pour $n$ assez grand, on dit que la suite $(T_n)_{n\geq 1}$ est topologiquement mélangeante. Enfin, on dit que $(T_n)_{n\geq 1}$ est topologiquement faiblement mélangeante si, pour tout quadruplet d'ouverts non vides $U_1, U_2, V_1, V_2 \subset X$, il existe un entier $n \geq 1$ tel que $T_n(U_1) \cap V_1 \neq \emptyset$ et $T_n(U_2) \cap V_2 \neq \emptyset$.

Toute suite topologiquement mélangeante est faiblement mélangeante, et toute suite faiblement mélangeante est topologiquement transitive. Si $T$ est un opérateur sur un espace de Fréchet séparable sans point isolé, alors $T$ est hypercyclique si et seulement si il est topologiquement transitif, c'est-à-dire, pour tous ouverts non vides $U, V \subset X$, il existe $n \geq 1$ tel que $T^n(U) \cap V \neq \emptyset$. Dans le cas d'une suite $(T_n)_{n\geq 1}$ d'opérateurs, cette équivalence reste vraie à condition que les $T_n$ commutent deux à deux et aient une image dense. Un critère utile pour montrer l'universalité d'une suite d'opérateurs dans le cadre linéaire est le suivant, connu sous le nom de critère d'universalité.

**Proposition 2.2.5.** *Soit $X$ un espace de Fréchet séparable et soit $(T_n)_{n\geq 1}$ une suite d'opérateurs sur cet espace. Supposons qu'il existe des sous-ensembles denses $\mathcal{D}_1$ et $\mathcal{D}_2$ de $X$, une suite strictement croissante $(n_k)_{k\geq 1}$ d'entiers naturels, ainsi que des applications $S_{n_k} : \mathcal{D}_2 \to X$, $k \geq 1$, telles que :*

*(i)* $T_{n_k} x \underset{k\to\infty}{\longrightarrow} 0$ *pour tout* $x \in \mathcal{D}_1$.

*(ii)* $S_{n_k} y \underset{k\to\infty}{\longrightarrow} 0$ *pour tout* $y \in \mathcal{D}_2$.

*(iii)* $T_{n_k} S_{n_k} y \underset{k\to\infty}{\longrightarrow} y$ *pour tout* $y \in \mathcal{D}_2$.

*Alors la suite $(T_n)_{n\geq 1}$ est topologiquement faiblement mélangeante, et en particulier universelle. Si de plus ce critère est satisfait à partir d'un certain rang $N$, c'est-à-dire pour tous les entiers $n \geq N$, alors $(T_n)_{n\geq 1}$ est topologiquement mélangeante.*

Pour les opérateurs admettant beaucoup de valeurs propres, on dispose également du critère suivant, qui est un cas particulier de la Proposition 2.2.5 et constitue l'analogue du critère de Godefroy–Shapiro pour les suites d'opérateurs.

**Proposition 2.2.6** ([9, Théorème 7])**.** *Soit $X$ un espace de Fréchet séparable, et soit $(T_n)_{n\geq 1}$ une suite d'opérateurs sur cet espace.*

*Supposons qu'il existe deux sous-ensembles $\mathcal{A}$ et $\mathcal{B}$ de $X$ tels que :*

*(a) Tout élément de $\mathcal{A} \cup \mathcal{B}$ est un vecteur propre de $T_n$, $n \geq 1$, avec $|\lambda(T_n, a)| \underset{n\to\infty}{\longrightarrow} 0$ pour tout $a \in \mathcal{A}$, et $|\lambda(T_n, b)| \underset{n\to\infty}{\longrightarrow} \infty$ pour tout $b \in \mathcal{B}$.*

*(b) Les sous-espaces* $\operatorname{span}(\mathcal{A})$ *et* $\operatorname{span}(\mathcal{B})$ *sont denses dans $X$.*

*Alors la suite $(T_n)_{n\geq 1}$ est topologiquement mélangeante.*

Nous rappelons à présent quelques notions de théorie ergodique qui seront utilisées dans notre étude ([50], [62]). Nous commençons par la notion de transformation préservant la mesure.

**Définition 2.2.7.** Soit $(\Omega, \mathcal{A}, \mu)$ un espace probabilisé. Une application mesurable $\tau : \Omega \to \Omega$ est dite $\mu$-invariante (on dit aussi qu'elle préserve la mesure $\mu$) si

$$\mu(\tau^{-1}(B)) = \mu(B)$$

pour tout ensemble mesurable $B \in \mathcal{A}$. Dans ce cas, on dit que $(\Omega, \mathcal{A}, \mu, \tau)$ est un système dynamique mesuré.

Les opérateurs aléatoires que nous considérerons dans cette thèse dépendront d'une transformation ergodique sur $\mathbb{T}$. Nous définissons de telles transformations de la manière suivante.

**Définition 2.2.8.** Une transformation $\tau$ préservant la mesure sur $\Omega$ est dite ergodique si elle satisfait l'une des assertions équivalentes suivantes :

1. Pour tout ensemble $A \in \mathcal{A}$ tel que $\mu(A) > 0$, on a $\mu\left(\bigcup_{n\geq 1} \tau^{-n}(A)\right) = 1$.
2. Tout ensemble $A \in \mathcal{A}$ tel que $\tau^{-1}(A) = A$ vérifie $\mu(A) = 0$ ou $1$.
3. Tout ensemble $A \in \mathcal{A}$ tel que $\tau(A) \subseteq A$ vérifie $\mu(A) = 0$ ou $1$.

Nous introduisons enfin les notions de mélange faible et de mélange (fort).

**Définition 2.2.9.** Un système dynamique mesuré $(\Omega, \mathcal{A}, \mu, \tau)$ est dit mélangeant si

$$\mu(\tau^{-n}(A) \cap B) \underset{n\to\infty}{\longrightarrow} \mu(A)\,\mu(B)$$

pour tous ensembles mesurables $A, B \subseteq \Omega$, et faiblement mélangeant si

$$\frac{1}{n}\sum_{j=0}^{n-1} |\mu(\tau^{-j}(A) \cap B) - \mu(A)\mu(B)| \underset{n\to\infty}{\longrightarrow} 0$$

pour tous ensembles mesurables $A, B \subseteq \Omega$.

Le mélange est une propriété d'indépendance asymptotique. Tout système mélangeant est faiblement mélangeant, et tout système faiblement mélangeant est ergodique. Nous considérerons deux exemples importants de transformations ergodiques : les rotations irrationnelles et le doublement de l'angle sur $\mathbb{T}$.

Étant donné un réel $\alpha \in (0,1)$, la rotation de paramètre $\alpha$ est définie par la transformation $R_\alpha : x \mapsto x + \alpha \pmod 1$ sur $[0,1)$, où $x \pmod 1$ désigne la partie fractionnaire de $x$. Cette application peut être vue comme une application sur $\mathbb{T}$ définie par $\tau(x) = x + \alpha$ dans $\mathbb{T}$, ou encore comme une application sur le cercle unité définie par $\tau(z) = e^{2i\pi\alpha}z$. Cette transformation est ergodique si et seulement si $\alpha$ est irrationnel. Elle n'est jamais faiblement mélangeante. Le doublement de l'angle est l'application sur $[0,1)$ définie par $\tau(x) = 2x \pmod 1$. Elle peut également être vue comme une application sur $\mathbb{T}$ définie par $\tau(x) = 2x \in \mathbb{T}$, ou comme une application sur le cercle unité définie par $\tau(z) = z^2$. Contrairement aux rotations, cette transformation est mélangeante.

Un outil fondamental pour notre étude de la dynamique linéaire de suites aléatoires de produits d'opérateurs sera le théorème ergodique de Birkhoff, qui généralise la loi des grands nombres.

**Théorème 2.2.10.** *Soit $(\Omega, \mathcal{A}, \mu, \tau)$ un système dynamique mesuré. Pour toute fonction $f \in L^1(\Omega)$, il existe une fonction $g \in L^1(\Omega)$ vérifiant $g \circ \tau = g$ et*

$$\int_\Omega g\, d\mu = \int_\Omega f\, d\mu,$$

*telle que*

$$\frac{1}{N}\sum_{n=0}^{N-1} f(\tau^n\omega) \underset{N\to\infty}{\longrightarrow} g(\omega) \quad \textit{pour presque tout } \omega \in \Omega. \tag{2.2.3}$$

*De plus, le système est ergodique si et seulement si $g(\omega) = \int_\Omega f\, d\mu$ pour presque tout $\omega$.*

Remarquons en particulier que si le système est ergodique et si $A \subseteq \Omega$ est un ensemble mesurable, alors, en appliquant le Théorème 2.2.10 à la fonction $f = \mathbb{1}_A$, on obtient

$$\frac{1}{N}\mathrm{Card}\{0 \le n \le N-1 : \tau^n\omega \in A\} \underset{N\to\infty}{\longrightarrow} \mu(A)$$

pour presque tout $\omega \in \Omega$.

Si $f : \Omega \to \mathbb{R}$ est une fonction réelle, si $\tau$ est une transformation préservant la mesure sur $(\Omega, \mathcal{A}, \mu)$, et si $n \ge 1$, on note

$$\mathbb{S}_n^\tau f(\omega) := \sum_{i=0}^{n-1} f(\tau^i\omega)$$

la $n$-ième somme de Birkhoff associée à $f$ et $\tau$.

### 2.2c Produits aléatoires d'opérateurs sur des espaces de Fréchet séparables

Nous décrivons ici quelques-uns de nos résultats sur la dynamique linéaire des opérateurs aléatoires de la forme

$$T_n(\omega) := T(\tau^{n-1}\omega)\cdots T(\tau\omega)T(\omega),\ n \ge 1, \omega \in \mathbb{T}. \tag{2.2.4}$$

La transformation $\tau$ sera une transformation ergodique préservant la mesure sur $(\mathbb{T}, m)$, où $m$ est la mesure de Lebesgue normalisée sur $\mathbb{T}$, et l'application $T : \mathbb{T} \to \mathcal{B}(X)$ sera donnée par

$$T(\omega) = \begin{cases} T_1 & \text{si} \quad \omega \in A_1 \\ T_2 & \text{si} \quad \omega \in A_2 \end{cases}, \tag{2.2.5}$$

où $T_1, T_2$ sont deux opérateurs sur un espace de Fréchet séparable $X$, et $A_1, A_2$ sont deux parties boréliennes disjointes de $[0,1)$ telles que $A_1 \cup A_2 = [0,1)$ et $m(A_k) > 0$ pour $k = 1,2$. Nous étudierons l'influence de la transformation ergodique $\tau$ et des opérateurs $T_1$ et $T_2$ sur l'universalité de la suite $(T_n(\omega))_{n\geq 1}$ pour presque tout $\omega \in \mathbb{T}$. Plus précisément, nous souhaitons déterminer si la suite $(T_n(.))_{n\geq 1}$ des opérateurs aléatoires est universelle, ou topologiquement faiblement mélangeante, ou topologiquement mélangeante, ou n'a aucune de ces propriétés, au sens où la suite $(T_n(\omega))_{n\geq 1}$ a cette propriété pour presque tout $\omega \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$, ou rien de tout ça. Dans le cas où $X$ est un espace de Banach séparable, nous pouvons démontrer une loi du type $0-1$, qui affirme que soit la suite $(T_n(\omega))_{n\geq 1}$ est universelle pour presque tout $\omega \in \mathbb{T}$, soit elle ne l'est pas pour presque tout $\omega \in \mathbb{T}$.

Soit $\tau$ une transformation ergodique sur $(\mathbb{T}, m)$. Le premier exemple naturel de produits aléatoires d'opérateurs à considérer est le cas donné par

$$T(\omega) = \begin{cases} \lambda_1 B & \text{si} \quad \omega \in A_1 \\ \lambda_2 B & \text{si} \quad \omega \in A_2 \end{cases},$$

où $\lambda_1, \lambda_2$ sont deux nombres complexes non nuls et $B$ est le backward shift sur $\ell_p(\mathbb{Z}_+)$ défini par $B(x_0, x_1, \dots) = (x_1, x_2, \dots)$ pour $1 \leq p < \infty$. L'opérateur $\lambda B$ est hypercyclique si et seulement si $|\lambda| > 1$. D'après le Théorème 2.2.10, il existe un borélien $E \subset \mathbb{T}$ tel que $m(E) = 1$ et pour tout $\omega \in E$,

$$\frac{a_1(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} m(A_1) \quad \text{et} \quad \frac{a_2(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} m(A_2), \tag{2.2.6}$$

où $a_k(n,\omega) := \mathrm{Card}\{0 \leq i \leq n-1 : \tau^i \omega \in A_k\}$. Les opérateurs $T_n(\omega)$ sont donnés par

$$T_n(\omega) = \lambda_1^{a_1(n,\omega)} \lambda_2^{a_2(n,\omega)} B^n$$

pour tout $n \geq 1$.

Supposons d'abord que $|\lambda_1|^{m(A_1)} |\lambda_2|^{m(A_2)} < 1$. Alors, en utilisant (2.2.6), on remarque que pour tout $\omega \in E$,

$$\|T_n(\omega)\| = |\lambda_1|^{a_1(n,\omega)} |\lambda_2|^{a_2(n,\omega)} \underset{n\to\infty}{=} \exp(n(\log(|\lambda_1|^{m(A_1)} |\lambda_2|^{m(A_2)}) + o(1))),$$

qui tend vers zéro lorsque $n \to \infty$. En particulier, chaque orbite sous la suite $(T_n(\omega))_{n\geq 1}$ est bornée, et la suite aléatoire $(T_n(.))_{n\geq 1}$ n'est pas universelle.

Supposons maintenant que $|\lambda_1|^{m(A_1)} |\lambda_2|^{m(A_2)} > 1$. Notons $c_{00}$ l'espace des suites à support fini, qui est dense dans $\ell_p(\mathbb{Z}_+)$ pour tout $1 \leq p < \infty$. Remarquons que, pour tout $\omega \in E$, on a $T_n(\omega)x = 0$ pour tout $x \in c_{00}$ dès que $n$ est assez grand. Notons $S$ le forward shift sur $\ell_p(\mathbb{Z}_+)$, défini par $S(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$. Cet opérateur vérifie $BS = I$. Posons $S_n(\omega) = \lambda_1^{-a_1(n,\omega)} \lambda_2^{-a_2(n,\omega)} S^n$ pour tout $\omega \in E$ et tout $n \geq 1$. Alors on a $T_n(\omega) S_n(\omega) = I$ pour tout $\omega \in E$ et tout $n \geq 1$. En utilisant encore (2.2.6), on peut montrer que $S_n(\omega)x$ tend vers zéro pour tout $x \in \ell_p(\mathbb{Z}_+)$, pour tout $\omega \in E$. En particulier, la Proposition 2.2.5 s'applique et la suite aléatoire $(T_n(.))_{n\geq 1}$ est universelle (et même topologiquement mélangeante).

La question est maintenant de savoir ce qu'il advient lorsque $|\lambda_1|^{m(A_1)} |\lambda_2|^{m(A_2)} = 1$. Dans ce cas, le Théorème 2.2.10 ne suffit pas pour conclure sur l'universalité de la suite aléatoire

$(T_n(.))_{n\geq 1}$. Puisque $|\lambda_2| = |\lambda_1|^{-\frac{m(A_1)}{m(A_2)}}$, on obtient

$$\|T_n(\omega)\| = |\lambda_1|^{a_1(n,\omega)-\frac{m(A_1)}{m(A_2)}a_2(n,\omega)} = |\lambda_1|^{\mathbb{S}_n^\tau f(\omega)}$$

et

$$\|S_n(\omega)\| = |\lambda_1|^{-(a_1(n,\omega)-\frac{m(A_1)}{m(A_2)}a_2(n,\omega))} = |\lambda_1|^{-\mathbb{S}_n^\tau f(\omega)},$$

où $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$. Le comportement des sommes de Birkhoff associées à cette fonction $f$ joue un rôle important dans la détermination de l'universalité de la suite aléatoire $(T_n(.))_{n\geq 1}$. En particulier, on voit que si la suite $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ est bornée pour presque tout $\omega \in \mathbb{T}$, alors la suite aléatoire $(T_n(.))_{n\geq 1}$ n'est pas universelle. Et dans le cas où $\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty$ et $\liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$ pour presque tout $\omega \in \mathbb{T}$, alors la suite aléatoire $(T_n(.))_{n\geq 1}$ est universelle (et même topologiquement faiblement mélangeante), grâce à la Proposition 2.2.5.

L'opérateur $\lambda B$ sur $\ell_2(\mathbb{Z}_+)$ est unitairement équivalent à l'adjoint de l'opérateur de multiplication par la fonction $\lambda z$ sur l'espace de Hardy $H^2(\mathbb{D})$. Cet espace $H^2(\mathbb{D})$ est l'espace des fonctions holomorphes $f : \mathbb{D} \to \mathbb{C}$ telles que

$$\sup_{0\leq r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt < \infty.$$

Il est isométriquement isomorphe à l'espace $\ell_2(\mathbb{Z}_+)$. Nous définissons les opérateurs de multiplication sur cet espace par $M_\phi : f \mapsto f\phi$, où $\phi$ est une fonction de $H^\infty(\mathbb{D})$, l'espace des fonctions holomorphes bornées sur $\mathbb{D}$. Un opérateur de multiplication sur $H^2(\mathbb{D})$ n'est jamais hypercyclique, et nous montrerons que si les deux opérateurs $T_1$ et $T_2$ sont des opérateurs de multiplication sur cet espace, alors le produit aléatoire $(T_n(.))_{n\geq 1}$ défini par une transformation ergodique $\tau$ et par (2.2.5) n'est jamais universel non plus, comme prévu. Cependant, l'adjoint $(M_\phi)^*$ sur $H^2(\mathbb{D})$ de l'opérateur de multiplication est hypercyclique si et seulement si $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, lorsque $\phi$ est non constante dans l'espace $H^\infty(\mathbb{D})$ des fonctions holomorphes bornées sur $\mathbb{D}$ ([31, Théorème 4.42], [6, Exemple 1.11]). Nous nous concentrerons donc d'abord sur le cas où les deux opérateurs sont donnés par $T_1 = (M_{\phi_1})^*$ et $T_2 = (M_{\phi_2})^*$, avec $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ non constantes. Comme les opérateurs $M_{\phi_1}$ et $M_{\phi_2}$ commutent, on a

$$T_n(\omega) = (M_{\phi_1}^{a_1(n,\omega)})^* (M_{\phi_2}^{a_2(n,\omega)})^*$$

pour tout $n \geq 1$ et tout $\omega \in \mathbb{T}$.

L'adjoint d'un opérateur de multiplication sur $H^2(\mathbb{D})$ possède de nombreux vecteurs propres, à savoir les fonctions $k_\lambda$ définies par

$$k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$$

pour tout $\lambda \in \mathbb{D}$. Ce sont les noyaux reproduisants de $H^2(\mathbb{D})$. En utilisant la Proposition 2.2.6, nous obtiendrons une première condition suffisante d'universalité pour la suite $(T_n(.))_{n\geq 1}$.

**Théorème L.** Soit $\tau$ une transformation ergodique préservant la mesure sur $(\mathbb{T}, m)$. Soient $A_1, A_2$ deux parties boréliennes disjointes de $[0,1)$ telles que $A_1 \cup A_2 = [0,1)$ et $m(A_k) > 0$ pour $k = 1, 2$.

Soient $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ non constantes sur $\mathbb{D}$ et supposons qu'il existe $\lambda, \mu \in \mathbb{D}$ tels que

$$|\phi_1(\lambda)|^{m(A_1)}|\phi_2(\lambda)|^{m(A_2)} < 1 \quad \text{et} \tag{2.2.7}$$
$$|\phi_1(\mu)|^{m(A_1)}|\phi_2(\mu)|^{m(A_2)} > 1. \tag{2.2.8}$$

Supposons que, pour tout $\omega \in \mathbb{T}$, l'opérateur $T(\omega)$ soit donné par

$$T(\omega) = \begin{cases} (M_{\phi_1})^* & \text{si} \quad \omega \in A_1 \\ (M_{\phi_2})^* & \text{si} \quad \omega \in A_2 \end{cases}.$$

Alors la suite aléatoire $(T_n(.))_{n\geq 1}$ est topologiquement mélangeante.

Dans le cas où $m(A_1) = m(A_2)$, les conditions (2.2.7) et (2.2.8) sont équivalentes à $(\phi_1\phi_2)(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ et $\phi_1\phi_2$ non constante, par le théorème de l'application ouverte. Ainsi, dans le cas où $m(A_1) = m(A_2) = 1/2$, nous aurons à considérer les cas $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{D}$ et $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, que nous appelons cas limites. L'un de nos objectifs est d'étudier ces cas. Évidemment, si $m(A_1) = m(A_2) = 1/2$ et si $\phi_1(\mathbb{D}) \subset \mathbb{D}$ et $\phi_2(\mathbb{D}) \subset \mathbb{D}$, ou bien $\phi_1(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ et $\phi_2(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, alors la suite aléatoire $(T_n(.))_{n\geq 1}$ n'est pas universelle : dans le premier cas, cela provient du fait que les opérateurs $T_n(\omega)$ sont des contractions, et dans le second cas, les opérateurs $T_n(\omega)$ sont inversibles avec des inverses contractants. Ces deux cas sont appelés situations triviales de nos cas limites. Nous nous concentrerons sur d'autres situations incluses dans ces cas limites. Plus précisément, nous traitons la situation où $m(A_1) = m(A_2) = 1/2$ et où $\phi_1\phi_2$ est une fonction intérieure sur $\mathbb{D}$. Une fonction analytique $f : \mathbb{D} \to \mathbb{C}$ est dite *intérieure* si $f$ est bornée sur $\mathbb{D}$ et si $|f^*| = 1$ presque partout sur $\mathbb{T}$, où $f^*(e^{it}) := \lim_{r\to 1} f(re^{it})$. Nous obtenons le résultat suivant.

**Théorème M.** Soit $\tau$ une transformation ergodique sur $(\mathbb{T}, m)$. Soient $A_1, A_2$ deux parties boréliennes disjointes de $[0,1)$ telles que $A_1 \cup A_2 = [0,1)$ et telles que $m(A_1) = m(A_2) = 1/2$. Soit $f$ la fonction $f = \mathbb{1}_{A_1} - \mathbb{1}_{A_2}$. Supposons que la suite $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ vérifie

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{ou} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

pour presque tout $\omega \in \mathbb{T}$. Soient $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ non constantes, telles que $\phi_1\phi_2$ soit intérieure.

Si $\phi_1(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ et $1/\phi_1 \in H^\infty(\mathbb{D})$, ou $\phi_2(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ et $1/\phi_2 \in H^\infty(\mathbb{D})$, alors la suite $(T_n(.))_{n\geq 1}$ est topologiquement faiblement mélangeante.

Nous pouvons aussi supprimer l'hypothèse sur les images de $\phi_1$ et $\phi_2$ de la manière suivante.

**Théorème N.** Soit $\tau$ une transformation ergodique préservant la mesure sur $(\mathbb{T}, m)$. Soient $A_1, A_2$ deux parties boréliennes disjointes de $[0,1)$ telles que $A_1 \cup A_2 = [0,1)$ et telles que $m(A_1) = m(A_2) = 1/2$. Soit $f$ la fonction $f = \mathbb{1}_{A_1} - \mathbb{1}_{A_2}$. Supposons que la suite $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ vérifie

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{et} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

pour presque tout $\omega \in \mathbb{T}$. Soient $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ deux fonctions non constantes, telles que $\phi_1\phi_2$ soit une fonction intérieure non constante. Supposons également que $\phi_1$ ou $\phi_2$ n'est pas intérieure.

Alors, la suite aléatoire $(T_n(.))_{n\geq 1}$ est topologiquement faiblement mélangeante.

En particulier, le Théorème N montre qu'il est possible que la suite aléatoire $(T_n(.))_{n\geq 1}$ soit universelle, même si les deux opérateurs $T_1$ et $T_2$ ne sont pas hypercycliques.

Nous verrons que lorsque la suite des sommes de Birkhoff $(\mathbb{S}_n^\tau f)_{n\geq 1}$ associée à une transformation ergodique $\tau$ et à notre fonction de référence $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$ satisfait un théorème de la limite centrale, c'est-à-dire qu'il existe une suite $(a_n)_{n\geq 1}$ de nombres positifs telle que $a_n \underset{n\to\infty}{\longrightarrow} \infty$ et telle que la suite $(\frac{\mathbb{S}_n^\tau f}{a_n})_{n\geq 1}$ converge en loi vers une loi normale centrée non dégénérée, alors l'hypothèse

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{et} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

est satisfaite pour presque tout $\omega \in \mathbb{T}$. La question de savoir si ces sommes de Birkhoff satisfont un théorème de la limite centrale sera examinée pour le cas du doublement de l'angle et pour les rotations irrationnelles sur $\mathbb{T}$. Le cas du doublement de l'angle s'appuiera sur le résultat de Kac [37], et le cas des rotations irrationnelles s'appuiera sur les articles [36], [15] et [16]. Nous verrons que le cas des rotations irrationnelles dépend fortement des propriétés diophantiennes du paramètre irrationnel de la rotation.

Grâce à ces théorèmes de la limite centrale, nous obtenons par exemple les résultats suivants. Le premier porte sur le doublement de l'angle.

**Corollaire O.** Soit $\tau$ le doublement de l'angle sur $\mathbb{T}$. Soient $A_1 = [0,1/2)$ et $A_2 = [1/2,1)$. Soient $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ non constantes, telles que $\phi_1\phi_2$ soit intérieure et non constante. Supposons aussi que $\phi_1$ ou $\phi_2$ ne soit pas intérieure. Alors, la suite aléatoire $(T_n(.))_{n\geq 1}$ est topologiquement faiblement mélangeante.

Un résultat analogue pour les rotations irrationnelles est le suivant.

**Corollaire P.** Soit $\alpha$ un nombre irrationnel dans $(0,1)$ et soit $\tau = R_\alpha$ sur $\mathbb{T}$. Soient $A_1 = [0,1/2)$ et $A_2 = [1/2,1)$. Soient $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ non constantes, telles que $\phi_1\phi_2$ soit intérieure et non constante. Supposons également que $\phi_1$ ou $\phi_2$ ne soit pas intérieure. Alors, la suite aléatoire $(T_n(.))_{n\geq 1}$ est topologiquement faiblement mélangeante.

Nous considérerons également des généralisations des Théorèmes M et N pour des parties boréliennes $A_1, A_2$ telles que $m(A_1) \neq m(A_2)$, mais nous verrons que dans ce cas la situation est un peu plus subtile. Pour le doublement de l'angle, on obtient par exemple le résultat suivant, issu d'un théorème de la limite centrale.

**Proposition Q.** Soit $\tau$ le doublement de l'angle sur $\mathbb{T}$ et soient $A_1$ et $A_2$ deux intervalles disjoints de $[0,1)$, tels que $A_1 \cup A_2 = [0,1)$ et $m(A_k) > 0$ pour $k = 1,2$. Soient $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ non constantes. Supposons que l'une des deux conditions suivantes soit satisfaite :

(i) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ presque partout sur $\mathbb{T}$, et l'une des images $\phi_1(\mathbb{D})$ ou $\phi_2(\mathbb{D})$ rencontre le cercle unité $\mathbb{T}$.

(ii) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ presque partout sur $\mathbb{T}$, $\phi_1$ ou $\phi_2$ possède une partie intérieure non triviale et $\phi_1$ ou $\phi_2$ n'est pas intérieure.

Alors, la suite aléatoire $(T_n(.))_{n\geq 1}$ est topologiquement faiblement mélangeante.

Pour les rotations irrationnelles, un analogue de la Proposition Q est plus délicat à obtenir, car il dépend fortement du choix des intervalles $A_1$ et $A_2$ tels que $m(A_1) \neq m(A_2)$, ainsi que des propriétés diophantiennes du nombre irrationnel $\alpha \in (0,1)$.

Nous verrons aussi qu'il est possible que la suite aléatoire $(T_n(.))_{n\geq 1}$ ne soit pas universelle. Ce sera le cas, par exemple, dans certaines situations où les sommes de Birkhoff sont bornées presque partout. Par exemple, pour certaines rotations irrationnelles, nous avons le résultat suivant.

**Proposition R.** Soit $\alpha$ un nombre irrationnel dans $(0,1)$ et soit $\tau = R_\alpha$. Soient $A_1 = [0,b)$ et $A_2 = [b,1)$, avec $b \in (0,1)$. Supposons que $b \in \mathbb{Z}\alpha$. Soient $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ non constantes, telles que

$$|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1 \quad \text{presque partout sur } \mathbb{T}.$$

Alors, pour tout $\omega \in \mathbb{T}$, la suite $(T_n(\omega))_{n\geq 1}$ n'est pas universelle.

Enfin, nous étudierons le cas où $T_1$ et $T_2$ sont des opérateurs sur l'espace des fonctions entières $H(\mathbb{C})$, muni de la topologie de la convergence uniforme sur les compacts de $\mathbb{C}$. Nous étudions d'abord le cas où $T_1$ et $T_2$ sont des fonctions entières de type exponentiel de l'opérateur dérivation $D$.

On dit qu'une fonction entière $\varphi \in H(\mathbb{C})$ est de type exponentiel s'il existe des constantes $M, A > 0$ telles que

$$|\varphi(z)| \leq M e^{A|z|}, \quad \text{pour tout } z \in \mathbb{C}.$$

Si $\varphi : \mathbb{C} \to \mathbb{C}$ est une fonction entière de type exponentiel, avec $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$, l'expression

$$\varphi(D) := \sum_{n\geq 0} a_n D^n$$

définit un opérateur sur $H(\mathbb{C})$. On dispose d'une description des opérateurs sur $H(\mathbb{C})$ qui commutent avec $D$ : un opérateur $T$ sur $H(\mathbb{C})$ commute avec $D$ si et seulement si il existe une fonction entière de type exponentiel $\varphi$ telle que $T = \varphi(D)$. De plus, ces opérateurs sont toujours topologiquement mélangeants, dès qu'ils ne sont pas un multiple de l'identité ([31, Théorème 4.21], [6, Théorème 1.42]). Nous obtenons le résultat suivant dans ce cadre, en utilisant les vecteurs propres $e_\lambda : z \mapsto e^{\lambda z}$ de l'opérateur dérivation $D$.

**Théorème S.** Soit $\tau : \mathbb{T} \to \mathbb{T}$ une transformation ergodique préservant la mesure sur $(\mathbb{T}, m)$. Soient $A_1, A_2$ deux parties boréliennes disjointes de $[0,1)$ telles que $A_1 \cup A_2 = [0,1)$ et $m(A_k) > 0$ pour $k = 1,2$. Soient $\varphi_1$ et $\varphi_2$ deux fonctions entières non constantes de type exponentiel. Considérons l'application

$$T(\omega) := \begin{cases} \varphi_1(D) & \text{si } \omega \in A_1 \\ \varphi_2(D) & \text{si } \omega \in A_2 \end{cases}.$$

Supposons qu'il existe $z, w \in \mathbb{C}$ tels que

$$|\varphi_1(z)|^{m(A_1)}|\varphi_2(z)|^{m(A_2)} < 1 \tag{2.2.9}$$

$$\text{et} \quad |\varphi_1(w)|^{m(A_1)}|\varphi_2(w)|^{m(A_2)} > 1. \tag{2.2.10}$$

Alors, la suite aléatoire $(T_n(.))_{n\geq 1}$ est topologiquement mélangeante.

Dans le cas où $m(A_1) = m(A_2) = 1/2$ and où $\varphi_1$ and $\varphi_2$ sont deux fonctions entières de type exponentiel non constantes, les deux conditions 2.2.9 and 2.2.10 sont simultanément vérifiées si et seulement si $\varphi_1\varphi_2$ est non constante. Cela vient du fait que l'image d'une fonction entière non constante est dense dans $\mathbb{C}$. Nous étudierons un cas pour lequel $\varphi_1\varphi_2$ est constante.

Enfin, nous étudierons un exemple où les opérateurs $T_1$ et $T_2$ sur $H(\mathbb{C})$ ne commutent pas. Pour $\lambda \in \mathbb{C}$ non nul et $b \in \mathbb{C}$, on note $T_{\lambda,b}$ l'opérateur sur $H(\mathbb{C})$ défini par $T_{\lambda,b} f = f(\lambda . + b)$. L'opérateur $T_{\lambda,b}$ est hypercyclique si et seulement si $\lambda = 1$ et $b \neq 0$. Nous obtenons le résultat suivant.

**Théorème T.** Soit $\tau : \mathbb{T} \to \mathbb{T}$ une transformation ergodique préservant la mesure sur $(\mathbb{T}, m)$. Soient $A_1, A_2$ deux parties boréliennes disjointes de $[0,1)$ telles que $A_1 \cup A_2 = [0,1)$ et $m(A_k) > 0$ pour $k = 1,2$. Considérons l'application

$$T(\omega) := \begin{cases} T_{\lambda,b} & \text{si } \omega \in A_1 \\ D & \text{si } \omega \in A_2 \end{cases},$$

avec $\lambda \in \mathbb{C}$ non nul et $b \in \mathbb{C}$. Alors, la suite aléatoire $(T_n(.))_{n\geq 1}$ est topologiquement mélangeante.

Le Théorème T fournit un exemple d'application $\omega \mapsto T(\omega)$ hypercyclique pour tout $\omega \in A_2$ et non hypercyclique pour tout $\omega \in A_1$, pour laquelle la suite aléatoire $(T_n(.))_{n\geq 1}$ est universelle, et pour laquelle les applications $T(\tau^i \omega)$, $i \geq 0$, ne commutent pas.

## Organisation du manuscrit

La partie principale de cette thèse est organisée en trois chapitres (chapitres 3 à 5). Les chapitres 3 et 4 correspondent à la première partie de l'introduction sur les propriétés typiques de contractions positives, tandis que le chapitre 5 correspond à la seconde partie sur la dynamique linéaire des produits aléatoires d'opérateurs.

Chaque chapitre présente les énoncés détaillés et les démonstrations des résultats originaux, ainsi que les prérequis techniques nécessaires. En fin de chapitre, nous présentons certaines questions ouvertes ainsi que de possibles pistes de recherche.

Chapter 3

# Typical properties of positive contractions and the invariant subspace problem

*The results presented in this chapter are based on the manuscript [26], published in Positivity.*

**Abstract.** In this paper, we first study some elementary properties of a typical positive contraction on $\ell_q$ for the SOT and the SOT* topologies. Using these properties, we prove that a typical positive contraction on $\ell_1$ (resp. on $\ell_2$) has a non-trivial invariant subspace for the SOT topology (resp. the SOT and the SOT* topologies). We then focus on the case where $X$ is a Banach space with a basis. We prove that a typical positive contraction on a Banach space with an unconditional basis has no non-trivial closed invariant ideals for the SOT and the SOT* topologies. In particular, this shows that when $X = \ell_q$ with $1 \leq q < \infty$, a typical positive contraction $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ when $1 < q < \infty$) does not satisfy the Abramovich, Aliprantis and Burkinshaw criterion, that is, there is no non-zero positive operator in the commutant of $T$ which is quasinilpotent at a non-zero positive vector of $X$. Finally, we prove that, for the SOT* topology, a typical positive contraction on a reflexive Banach space with a monotone basis does not satisfy the Abramovich, Aliprantis and Burkinshaw criterion.

## 3.1 Introduction

Throughout this paper, $(X, \|.\|)$ will be a complex separable infinite-dimensional Banach space and $H$ will be a complex separable infinite-dimensional Hilbert space. The space of all bounded operators on $X$ equipped with the operator norm $\|.\|$ will be denoted by $\mathcal{B}(X)$.

If $M > 0$, we denote by $\mathcal{B}_M(X)$ the closed ball of radius M of $\mathcal{B}(X)$. With this notation the set $\mathcal{B}_1(X)$ is just the set of contractions on $X$. If $(e_n)_{n\geq 0}$ is a basis of $X$, we say that a vector

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$x = \sum_{n\geq 0} x_n e_n$ of $X$ is positive whenever $x_n \geq 0$ for all $n \geq 0$. We write $x \geq 0$ when $x$ is a positive vector of $X$. We say that an operator $T \in \mathcal{B}(X)$ is positive whenever $Tx \geq 0$ for every $x \geq 0$. The space of all bounded positive operators on $X$ will be denoted by $\mathcal{P}(X)$, and we denote by $\mathcal{P}_M(X) := \mathcal{B}_M(X) \cap \mathcal{P}(X)$ the positive operators in $\mathcal{B}_M(X)$.

Given a topology $\tau$ on $\mathcal{Y} = \mathcal{B}_M(X)$ or $\mathcal{Y} = \mathcal{P}_M(X)$ that turns $\mathcal{Y}$ into a Baire space, we say that a property (P) of operators on $X$ is typical for the topology $\tau$ if the set

$$\{T \in \mathcal{Y} : T \text{ satisfies the property (P)}\}$$

is comeager in $\mathcal{Y}$, that is, contains a dense $G_\delta$ subset of $\mathcal{Y}$. A property (P) of operators on $X$ is atypical for the topology $\tau$ if its negation is typical for $\tau$. Since the space $(\mathcal{B}_1(X), \|.\|)$ is usually not separable and so is not Polish, we will be focusing on the Strong Operator Topology and on the Strong* Operator Topology. Recall that the Strong Operator Topology is the topology on $\mathcal{B}(X)$ defined by the seminorms $\|.\|_x$, where $\|T\|_x = \|Tx\|$ for every $x \in X$. The Strong* Operator Topology is the topology on $\mathcal{B}(X)$ defined by the seminorms $\|.\|_x$ and $\|.\|_{x^*}$, where $\|T\|_x = \|Tx\|$ and $\|T\|_{x^*} = \|T^*x^*\|$ for every $x \in X$ and every $x^* \in X^*$. If $(T_i)$ is a net in $\mathcal{B}(X)$ and $T \in \mathcal{B}(X)$, we have

$$T_i \xrightarrow[i]{\texttt{SOT}} T \iff T_i x \xrightarrow[i]{\|.\|} Tx \quad \text{for every } x \in X,$$

and

$$T_i \xrightarrow[i]{\texttt{SOT}^*} T \iff \begin{cases} T_i x \xrightarrow[i]{\|.\|} Tx & \text{for every } x \in X, \\ T_i^* x^* \xrightarrow[i]{\|.\|} T^* x^* & \text{for every } x^* \in X^*. \end{cases}$$

We now denote by $\texttt{SOT}$ (resp. by $\texttt{SOT}^*$) the Strong Operator Topology (resp. the Strong* Operator Topology) on $\mathcal{B}(X)$. For every $M > 0$, the closed ball $(\mathcal{B}_M(X), \texttt{SOT})$ is Polish and when $X^*$ is separable, the space $(\mathcal{B}_M(X), \texttt{SOT}^*)$ is Polish.

The notion of a typical property of contractions was initiated by Eisner in [18]. She proved in [18] that a typical contraction on a separable Hilbert space is unitary for the Weak Operator Topology. This notion was studied in more depth by Eisner and Mátrai for operators on a separable Hilbert space for other topologies as, for example, the $\texttt{SOT}$ and the $\texttt{SOT}^*$ topologies. It is proved in [19] that if $H$ is a separable Hilbert space, a typical $T \in (\mathcal{B}_1(H), \texttt{SOT})$ is unitarily equivalent to the infinite-dimensional backward unilateral shift operator on $\ell_2(\mathbb{Z}_+ \times \mathbb{Z}_+)$ and in particular has a non-trivial invariant subspace, that is, a closed subspace $Y \subseteq X$ such that $Y \neq \{0\}, Y \neq X$ and $T(Y) \subseteq Y$. Grivaux, Matheron and Menet have also studied typical properties of contractions on $\ell_q$-spaces ([28], [30] and [29]). Their initial goal was to determine whether a typical $T \in (\mathcal{B}_1(\ell_q), \texttt{SOT})$ or $T \in (\mathcal{B}_1(\ell_q), \texttt{SOT}^*)$ has a non-trivial invariant subspace.

This motivation comes from the famous invariant subspace problem, which can be stated as follows: given an infinite-dimensional separable complex Banach space $Z$, does every bounded operator $T \in \mathcal{B}(Z)$ have a non-trivial invariant subspace? The answer to this problem is negative for some non-reflexive Banach spaces: Enflo constructed a non-reflexive Banach space which supports an operator without non-trivial invariant subspaces (see [20]), and Read constructed operators on $\ell_1$ and on $c_0$ (called Read's operators) without non-trivial invariant subspaces (see [53] and [54]). The invariant subspace problem still remains open for reflexive Banach spaces and in particular for separable Hilbert spaces. However, the problem has a positive answer for some classes of operators: for example, Lomonosov proved in [46] that if an operator $T$ contains a non-scalar operator in its commutant that commutes with a non-zero compact operator, then it has a non-trivial invariant subspace; Brown, Chevreau and Pearcy proved in [11] that every contraction on a Hilbert space whose spectrum contains the unit circle has a non-trivial invariant subspace. The invariant subspace problem can be restricted to positive operators as follows: given an infinite-dimensional complex Banach space $Z$ with a basis, does every bounded positive operator on $Z$ have a non-trivial invariant subspace? The answer to this problem is still unknown. Moreover, even if Read's operators give a counter-example to the invariant subspace problem on $Z = \ell_1$, it was proved in [60] that the modulus of a Read's operator on $\ell_1$ has a positive eigenvector and so the invariant subspace problem for positive operators still remains open for the case $Z = \ell_1$. A major result regarding the existence of a non-trivial invariant subspace for positive operators, due to Abramovich, Aliprantis and Burkinshaw ([2, Theorem 2.2]), is the following.

**Theorem 3.1.1.** *Let $X$ be a Banach space with a basis and $T$ be a positive operator on $X$. If there exists a non-zero positive operator $A$ on $X$ which is quasinilpotent at a certain non-zero positive vector of $X$ and such that $AT = TA$, then $T$ has a non-trivial invariant subspace.*

In particular, Theorem 3.1.1 implies the following corollary.

**Corollary 3.1.2.** *Let $X$ be a Banach space with a basis. Every positive operator on $X$ which is quasinilpotent at a non-zero positive vector of $X$ has a non-trivial invariant subspace.*

For more details on the invariant subspace problem and on known results on this subject, we refer to [12], [13] and [51].

The aim of this article is to study the invariant subspace problem from the point of view of typicality, that is, we are interested in the question of whether a typical $T \in (\mathcal{P}_1(X), \tau)$ has a non-trivial invariant subspace, when $\tau = \texttt{SOT}$ or $\tau = \texttt{SOT}^*$. We will mostly focus on the case $X = \ell_q$ with $1 \leq q < \infty$, but we will also generalize some results to Banach spaces with an unconditional basis or with a monotone basis. In the case where $X = \ell_q$, the basis $(e_n)_{n\geq 0}$ of $X$ will be the canonical basis, where we recall that $e_n(k) = 1$ if $n = k$ and $e_n(k) = 0$ if $n \neq k$, for every $n, k \geq 0$.

### 3.1a Notations

We introduce here some notation that will be used throughout the article.

- We denote by $\mathbb{Q}_{>0}$ the set of positive rational numbers.

- The open unit disk of $\mathbb{C}$ will be denoted by $\mathbb{D}$ and the unit circle of $\mathbb{C}$ will be denoted by $\mathbb{T}$.

- If $Z$ is a Banach space, we denote by $S_Z$ the unit sphere of $Z$.

- The closed linear span of $(x_i)_{i\in I} \subseteq X$ will be written as $[x_i : i \in I]$.

- If $X$ is a Banach space with a basis $(e_n)_{n\geq 0}$, we denote by $E_N$ the subspace $[e_0, \dots, e_N]$ for every $N \geq 0$ and by $F_N$ the subspace $[e_j : j > N]$ for every $N \geq 0$.

- If $X$ is a Banach space with a basis $(e_n)_{n\geq 0}$, we denote by $P_N$ the canonical projection onto $E_N$ and by $Q_N$ the canonical projection onto $F_N$. The biorthogonal functionals sequence associated to the basis $(e_n)_{n\geq 0}$ will be denoted by $(e_n^*)_{n\geq 0}$.

- If $T \in \mathcal{B}(X)$, we respectively write $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_p(T)$ and $\sigma_{ess}(T)$ for the spectrum, the approximate spectrum, the point spectrum and the essential spectrum of $T$.

### 3.1b Main results

We start by presenting in Section 3.2 some tools that will be useful throughout the article. We properly define the notion of a positive operator on a Banach space with a basis. In particular, Proposition 3.2.1 states that the set of positive contractions on a Banach space with a basis is Polish for both the $\texttt{SOT}$ and the $\texttt{SOT}^*$ topologies, providing the necessary setting for the study of typical properties of positive contractions.

Theorem 3.2.6 connects the comeager sets of $\mathcal{P}_1(X)$ for the topologies $\texttt{SOT}$ and $\texttt{SOT}^*$ in the case where $X = \ell_q$ with $q > 2$.

In Section 3.3, we put together some elementary properties of a typical positive contraction for the $\texttt{SOT}$ and the $\texttt{SOT}^*$ topologies. Proposition 3.3.1 will play an important role for Section 3.4 in order to study the invariant subspace problem for a typical positive contraction on $\ell_2$ and on $\ell_1$.

We also study the point spectrum of a typical positive contraction on $\ell_q$ (Corollaries 3.3.3 and 3.3.4).

Eisner and Mátrai proved in [19] that a typical $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT})$ is unitarily equivalent to the infinite-dimensional backward unilateral shift operator on $\ell_2(\mathbb{Z}_+ \times \mathbb{Z}_+)$ and in particular, this implies that a typical contraction on $\ell_2$ has a non-trivial invariant subspace. To do so, they proved first that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT})$ is such that $T^*$ is an isometry. It turns out that this is no longer the case for positive contractions, that is, the adjoint of a typical positive contraction on $\ell_2$ is not an isometry for the $\texttt{SOT}$ topology.

**Proposition 3.1.3.** *Let $X = \ell_q$ with $1 < q < \infty$. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$) is such that $T^*$ is not an isometry.*

The case $X = \ell_1$ is very different from the case $X = \ell_q$ with $q > 1$, as the following proposition shows.

**Proposition 3.1.4.** *If $X = \ell_1$, then a typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ is such that $T^*$ is a non-surjective isometry and such that $T - \lambda$ is surjective for every $\lambda \in \mathbb{D}$.*

Proposition 3.1.3 shows in particular that a property can be typical in $\mathcal{P}_1(X)$ but can be atypical in $\mathcal{B}_1(X)$ for a certain topology $\tau$ on $\mathcal{B}_1(X)$.

In Section 3.4, we first explain why a typical positive contraction on $\ell_1$ and on $\ell_2$ has a non-trivial invariant subspace, and then we focus on the case where $X$ is a Banach space with a basis. In particular, it includes the case $X = \ell_q$ with $1 < q \neq 2 < \infty$.

We say that a positive operator on a Banach space $X$ with a basis satisfies the Abramovich, Aliprantis and Burkinshaw criterion (abbreviated AAB criterion) if this operator satisfies the hypotheses of Theorem 3.1.1. Any positive operator on a Banach space $X$ with a basis satisfying the AAB criterion has a non-trivial invariant subspace and when $X = \ell_q$ with $1 \leq q < \infty$, any operator satisfying the AAB criterion has a non-trivial closed invariant ideal ([1, Theorem 2.2]), that is, a closed vector subspace $V$ of $X$ such that $|x| \leq |y|$ and $y \in V$ imply $x \in V$, for every $x, y \in X$. Here, $|x|$ is the positive vector of $X$ whose coordinates are the modulus of the coordinates of the vector $x \in X$. Troitsky and Radjavi gave the following characterization of positive operators on a Banach space with an unconditional basis admitting a non-trivial invariant closed ideal ([52, Proposition 1.2]).

**Proposition 3.1.5.** *Let $X$ be a Banach space with an unconditional basis $(e_n)_{n\geq 0}$ and let $T$ be a positive operator on $X$. The operator $T$ has no non-trivial closed invariant ideals if and only if the following property holds:*

$$\forall\, i \neq j \in \mathbb{Z}_+, \exists n \in \mathbb{Z}_+ : \langle e_j^*, T^n e_i \rangle > 0.$$

Thanks to this characterization, we obtain the following result.

**Proposition 3.1.6.** *Let $X$ be a Banach space with an unconditional basis. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ when $X^*$ is separable) has no non-trivial closed invariant ideals.*

In particular, Proposition 3.1.6 implies the following corollary.

**Corollary 3.1.7.** *Let $X = \ell_q$ with $1 \leq q < \infty$. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ when $1 < q < \infty$) does not satisfy the AAB criterion.*

The main result of this article is the following generalization of Corollary 3.1.7 for the $\texttt{SOT}^*$ topology to Banach spaces with a monotone basis.

**Theorem 3.1.8.** *Let $X$ be a reflexive Banach space with a monotone basis. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ does not satisfy the AAB criterion.*

Finally, we end Section 3.4 with the following result.

**Corollary 3.1.9.** *Let $X$ be a Banach space with a basis. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ when $X^*$ is separable) is not quasinilpotent at any non-zero positive vector of $X$.*

These results highlight the fact that the existence of invariant subspaces for typical operators is a delicate matter, even in the restricted setting of positive contractions.

## 3.2 Useful tools

### 3.2a Positive operators on $X$.

If $X$ is a Banach space with a basis $(e_n)_{n\geq 0}$, we denote by $\mathcal{C}^+$ the positive cone of $X$, that is:

$$\mathcal{C}^+ := \{x = \sum_{n\geq 0} x_n e_n : x_n \geq 0, \text{ for every } n \geq 0\}.$$

We have $\mathcal{C}^+ + \mathcal{C}^+ \subseteq \mathcal{C}^+$, $\alpha\mathcal{C}^+ \subseteq \mathcal{C}^+$ for every $\alpha \geq 0$ and $\mathcal{C}^+ \cap (-\mathcal{C}^+) = \{0\}$. If $x \in X$, we write $x \geq 0$ when $x \in \mathcal{C}^+$. We define a partial order on $\mathcal{C}^+$ by letting $x \leq y$ when $y - x \in \mathcal{C}^+$.

An operator $T : X \to X$ is said to be positive (with respect to the basis $(e_n)_{n\geq 0}$) whenever $T(\mathcal{C}^+) \subseteq \mathcal{C}^+$, that is, when $Tx \geq 0$ for every $x \geq 0$. We write $T \geq 0$ when $T$ is a positive operator, and we write $\mathcal{P}(X)$ the space of all bounded positive operators on $X$.

Let $(t_{i,j})_{i,j\geq 0}$ be the matrix of $T$ with respect to the basis $(e_n)_{n\geq 0}$ (with $t_{i,j} = \langle e_i^*, Te_j\rangle$ for every $i, j \geq 0$). Then $T$ is a positive operator if and only if $t_{i,j} \geq 0$ for every $i, j \geq 0$.

We notice that if $S, T$ are two positive operators on $X$, then for every $\lambda \geq 0$, the operators $ST$, $\lambda S$ and $S + T$ are also positive. Moreover, the projections $P_N$ are positive on $X$.

Recall that if $M > 0$, we write $\mathcal{P}_M(X)$ for the set of all positive operators on $X$ with norm at most equal to $M$. In particular, if $M = 1$, the set $\mathcal{P}_1(X)$ is just the positive contractions on $X$.

Our aim in this note is to investigate typical properties of positive contractions for the $\texttt{SOT}$ topology and the $\texttt{SOT}^*$ topology and, more precisely, we would like to know if the property to have a non-trivial invariant subspace is a typical property of positive contractions. The first step in this investigation is to check that $(\mathcal{P}_1(X), \texttt{SOT})$ and $(\mathcal{P}_1(X), \texttt{SOT}^*)$ are Polish spaces. We notice that for every $M > 0$, the space $\mathcal{P}_M(X)$ is closed in $\mathcal{B}_M(X)$ for the $\texttt{SOT}$ topology (and hence for the $\texttt{SOT}^*$ topology), and since $(\mathcal{B}_M(X), \texttt{SOT})$ is Polish when $X$ is separable and $(\mathcal{B}_M(X), \texttt{SOT}^*)$ is Polish when $X^*$ is separable (see [14, Page 256, Proposition 1.3]), we have the following result.

**Proposition 3.2.1.** *Let $X$ be a Banach space with a basis. For every $M > 0$, the space $(\mathcal{P}_M(X), \texttt{SOT})$ is Polish. If moreover $X^*$ is separable, then the space $(\mathcal{P}_M(X), \texttt{SOT}^*)$ is Polish for every $M > 0$.*

### 3.2b A tool for proving density results

When we study typical properties of positive contractions, we have to prove that certain sets of operators are dense in $\mathcal{P}_1(X)$. The following lemma is very useful for this.

**Lemma 3.2.2.** *Let $X$ be a Banach space with a monotone basis $(e_n)_{n\geq 0}$. Let $\mathcal{C}(X)$ be a class of operators on $X$ and define $\mathcal{C}_M(X) := \mathcal{C}(X) \cap \mathcal{P}_M(X)$ for every $M > 0$.*

*Let $M > 0$. Suppose that the following property holds: there exists an index $p \in \mathbb{Z}_+$ such that for every $\varepsilon > 0$, every $N \in \mathbb{Z}_+$ with $N \geq p$ and every positive operator $A \in \mathcal{P}(E_N)$ with $\|A\| < M$, there exists a positive operator $T \in \mathcal{C}_M(X)$ such that*

$$\|(T - A)e_k\| < \varepsilon \quad \textit{for every} \quad 0 \leq k \leq N. \tag{3.2.1}$$

*Then $\mathcal{C}_M(X)$ is dense in $(\mathcal{P}_M(X), \texttt{SOT})$. If $X$ has a shrinking monotone basis $(e_n)_{n\geq 0}$ and if the condition (3.2.1) is replaced by the following condition*

$$\|(T - A)e_k\| < \varepsilon \quad \textit{and} \quad \|(T - A)^* e_k^*\| < \varepsilon \quad \textit{for every} \quad 0 \leq k \leq N, \tag{3.2.2}$$

*then $\mathcal{C}_M(X)$ is dense in $(\mathcal{P}_M(X), \texttt{SOT}^*)$.*

*Proof.* We will prove the lemma for the $\texttt{SOT}^*$ topology.

Let $T_0 \in \mathcal{P}_M(X)$, let $\varepsilon > 0$, let $x_1, \dots, x_s \in X$ and let $y_1^*, \dots, y_s^* \in X^*$. Without loss of generality, we can suppose that $\|T_0\| < M$. We are looking for a positive operator $T \in \mathcal{C}_M(X)$ such that

$$\max_{1\leq j\leq s} \max\{\|(T - T_0)x_j\|, \|(T - T_0)^* y_j^*\|\} < \varepsilon. \tag{3.2.3}$$

For every $1 \leq j \leq s$, there exist two indices $N_j$ and $N_j'$ such that

$$\left\| x_j - \sum_{k=0}^{N_j} e_k^*(x_j)e_k \right\| < \frac{\varepsilon}{4M} \quad \text{and} \quad \left\| y_j^* - \sum_{k=0}^{N_j'} e_k^{**}(y_j^*)e_k^* \right\| < \frac{\varepsilon}{4M}.$$

Let $N_0 = \max_{1\leq j\leq s}\{N_j, N_j', p\} \in \mathbb{Z}_+$.

We claim that if the following inequality

$$\max_{0\leq k\leq N_0} \max\{\|(T - T_0)e_k\|, \|(T - T_0)^* e_k^*\|\} < \frac{\varepsilon}{2\alpha} \tag{3.2.4}$$

holds with

$$\alpha := \max_{1\leq j\leq s} \left\{ \sum_{k=0}^{N_0} \|x_j\|\|e_k^*\|, \sum_{k=0}^{N_0} \|y_j^*\|\|e_k^{**}\| \right\},$$

then the inequality (3.2.3) holds too. Indeed if (3.2.4) holds, we have for every $1 \leq j \leq s$:

$$\begin{aligned}
\|(T - T_0)x_j\| &\leq \|(T - T_0)(x_j - \sum_{k=0}^{N_j} e_k^*(x_j)e_k)\| + \|(T - T_0)\sum_{k=0}^{N_j} e_k^*(x_j)e_k\| \\
&< \frac{\varepsilon}{2} + \sum_{k=0}^{N_0} \|e_k^*(x_j)\|\|(T - T_0)e_k\| \\
&< \varepsilon
\end{aligned}$$

and likewise we have

$$\|(T-T_0)^* y_j^*\| < \varepsilon.$$

Now for every $N \geq N_0$, we consider the positive operator $A_N := P_N T_0 P_N$. Since the basis is monotone and $\|T_0\| < M$ we have that $\|A_N\| < M$, so there exists a positive operator $T \in \mathcal{C}_M(X)$ such that

$$\max_{0\leq k\leq N} \max\{\|(T-A_N)e_k\|, \|(T-A_N)^* e_k^*\|\} < \frac{\varepsilon}{4\alpha}.$$

Now, for every $0 \leq k \leq N$:

$$\begin{aligned}\|(T-T_0)e_k\| &\leq \|(T-A_N)e_k\| + \|(P_N T_0 P_N - T_0)e_k\| \\ &< \frac{\varepsilon}{4\alpha} + \|(P_N - I)T_0 e_k\|\end{aligned}$$

and

$$\|(T-T_0)^* e_k^*\| < \frac{\varepsilon}{4\alpha} + \|(P_N - I)^* T_0^* e_k^*\|.$$

Since $P_N \xrightarrow[N\to\infty]{\texttt{SOT}^*} I$, we can choose $N \in \mathbb{Z}_+$ large enough such that

$$\max_{0\leq k\leq N} \max\{\|(P_N - I)T_0\, e_k\|, \|(P_N - I)^* T_0^* e_k^*\|\} < \frac{\varepsilon}{4\alpha},$$

and the inequality (3.2.4) follows. This proves Lemma 3.2.2. $\square$

**Remark 3.2.3.** The hypothesis "for every positive operator $A \in \mathcal{P}_1(E_N)$ with $\|A\| < M$" in Lemma 3.2.2 can be replaced by "for every positive operator $A \in \mathcal{P}_1(E_N)$ with $\|A\| < M$ and with $\langle e_k^*, Ae_l\rangle > 0$ for every $0 \leq k, l \leq N$". Indeed, the operators $A_N$ in the proof of Lemma 3.2.2 can be approximated in the $\texttt{SOT}^*$-topology by operators on $E_N$ whose matrices have positive entries.

**Remark 3.2.4.** Lemma 3.2.2 requires the basis $(e_n)_{n\geq 0}$ to be monotone. Notice that the norm $\|\!|.|\!\|$ defined by $\|\!|x|\!\| = \sup_{N\geq 0} \|P_N x\|$ is equivalent to the norm $\|.\|$ and that when $X$ is equipped with this norm, the basis $(e_n)_{n\geq 0}$ becomes monotone.

### 3.2c Topological 0-1 law for positive operators

We assume in this subsection that $X = \ell_q$ with $1 \leq q < \infty$ and we consider the set

$$\mathrm{Iso}_+(X) := \{T \in \mathcal{P}_1(X) : T \text{ is a surjective isometry of } X\}.$$

It is a classical fact (see [45, Proposition 2.f.14]) that if $1 < q \neq 2 < \infty$, every surjective isometry $T$ of $X$ has the form

$$Tx = (\varepsilon_n x_{\sigma(n)})_{n\geq 0}, \quad \text{for every } x = (x_n)_{n\geq 0} \in X,$$

where $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+$ is a bijection of $\mathbb{Z}_+$ and where $(\varepsilon_n)_{n\geq 0}$ is a sequence of numbers such that $|\varepsilon_n| = 1$ for every $n \geq 0$ . It follows that every positive surjective isometry of $X$ has the form

$$Tx = (x_{\sigma(n)})_{n\geq 0}, \quad \text{for every } x = (x_n)_{n\geq 0} \in X,$$

where $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+$ is a bijection of $\mathbb{Z}_+$. In fact, every positive surjective isometry of $\ell_2$ also has this form. Indeed, if $T : \ell_2 \to \ell_2$ is a positive surjective isometry of $\ell_2$, then for every $i, j \geq 0$ with $i \neq j$, we have that

$$\langle Te_i, Te_j \rangle = \sum_{k \geq 0} \langle e_k^*, Te_i \rangle \langle e_k^*, Te_j \rangle = 0. \tag{3.2.5}$$

So using the fact that every coefficient in (3.2.5) is non-negative, we have that

$$\langle e_k^*, Te_i \rangle \langle e_k^*, Te_j \rangle = 0 \quad \text{for every } k \geq 0, \tag{3.2.6}$$

that is, $Te_i$ and $Te_j$ have disjoint supports. Now since we know that the vectors $Te_i$ for $i \geq 0$ have mutually disjoint supports, the proof given in [45, Proposition 2.f.14] also works for the positive surjective isometries of $\ell_2$.

From this description of the positive surjective isometries of $\ell_q$ with $1 \leq q < \infty$, it follows that $\mathrm{Iso}_+(X)$ is a group. We say that a subset $\mathcal{A}$ of $\mathcal{P}_1(X)$ is $\mathrm{Iso}_+(X)$-*invariant* if $J\mathcal{A}J^{-1} = \mathcal{A}$ for every $J \in \mathrm{Iso}_+(X)$. The following result shows that every property of positive contractions that we will consider in this article is either typical or atypical when $X = \ell_q$ with $1 \leq q < \infty$.

**Proposition 3.2.5.** *Let $X = \ell_q$ with $1 \leq q < \infty$. If $\mathcal{A} \subseteq (\mathcal{P}_1(X), \texttt{SOT})$ has the Baire property and is $\mathrm{Iso}_+(X)$-invariant, then $\mathcal{A}$ is either meager or comeager in $(\mathcal{P}_1(X), \texttt{SOT})$. If $1 < q < \infty$ and if $\mathcal{A} \subseteq (\mathcal{P}_1(X), \texttt{SOT}^*)$ has the Baire property and is $\mathrm{Iso}_+(X)$-invariant, then $\mathcal{A}$ is either meager or comeager in $(\mathcal{P}_1(X), \texttt{SOT}^*)$*

*Proof.* The proof given in [30, Proposition 3.2], which relies on [39, Theorem 8.46], works in exactly the same way for positive contractions. ☐

## 3.2d Similar topologies

The aim of this subsection is to link the topologies $\texttt{SOT}$ and $\texttt{SOT}^*$ on $\mathcal{P}_1(X)$ in terms of comeager sets when $X = \ell_q$ with $q > 2$. The main result of this subsection is the following.

**Theorem 3.2.6.** *Let $X = \ell_q$ with $q > 2$. The Baire spaces $(\mathcal{P}_1(X), \texttt{SOT})$ and $(\mathcal{P}_1(X), \texttt{SOT}^*)$ have the same comeager sets.*

The proof of Theorem 3.2.6 is very similar to the proof of [29, Theorem 3.4], but we have to adapt the proof given in [29] to positive contractions. Indeed, the proof given in [29] uses [29, Corollary 2.10], which works in $\mathcal{P}_1(X)$, and uses the two propositions [30, Proposition 5.15] and [30, Proposition 5.16], and these two propositions can easily be adapted to positive contractions as long as [30, Lemma 5.17] can be adapted to positive contractions.

We first introduce some terminology. We say that two topologies $\tau$ and $\tau'$ on $\mathcal{Y} = \mathcal{P}_1(\ell_q)$ are *similar* if they have the same dense sets. Similar topologies have the same comeager sets ([29, Lemma 2.1]), and simple examples show that the converse is not true in general ([29, Remark 2.2]). We write $\mathbf{i}_{\tau,\tau'}$ the identity map from $(\mathcal{Y}, \tau)$ to $(\mathcal{Y}, \tau')$ and $\mathcal{C}(\tau, \tau')$ the set of all points of continuity of this map.

A vector $x \in X$ is said to be *norming* for an operator $A \in \mathcal{B}(X)$ if $\|x\| = 1$ and $\|Ax\| = \|A\|$. Given $N \geq 0$, we will say that an operator $A \in \mathcal{P}(E_N)$ is *absolutely exposing* if the set of all norming vectors for $A$ consists only of unimodular multiples of a single vector $x_0 \in S_{E_N}$. We denote by $\mathcal{E}_1(E_N)$ the set of absolutely exposing positive operators $A \in \mathcal{P}_1(E_N)$.

The only part of the proof given in [29] that we have to adapt is [30, Lemma 5.17]. We have to slightly modify the expression of the operators $A_\delta$ involved in the proof of [30, Lemma 5.17] to obtain positive contractions. We recall that if $x$ is a vector of $X$, we denote by $|x|$ the positive vector of $X$ whose coordinates are the modulus of the coordinates of $x$. We thus have to prove the following lemma.

**Lemma 3.2.7.** *The set $\mathcal{E}_1(E_N)$ is dense in $\mathcal{P}_1(E_N)$.*

*Proof.* Let $A \in \mathcal{P}_1(E_N)$ with $A \neq 0$ and $\|A\| < 1$. Let $x_0 \in E_N$ be such that $\|x_0\| = 1$ and $\|Ax_0\| = \|A\|$. Since the vector $|x_0|$ is also a norming vector for $A$, we can suppose without loss of generality that $x_0 \geq 0$.

By the Hahn-Banach theorem, there exists a functional $x_0^* \in E_N^*$ such that $\|x_0^*\| = \langle x_0^*, x_0\rangle = 1$. We can also suppose that $x_0^*$ is a positive functional. Indeed if $x_0^* = \sum_{j=0}^{N} \beta_j e_j^*$, we consider the positive functional defined by $y_0^* = \sum_{j=0}^{N} |\beta_j| e_j^*$. Using Hölder's inequality, we can show that $\|y_0^*\| \leq 1$, and moreover, we have that $|\langle x_0^*, x_0\rangle| \leq \langle y_0^*, x_0\rangle \leq \|y_0^*\| \leq 1$, so $\|y_0^*\| = 1 = \langle y_0^*, x_0\rangle$.

Let $R_0$ be the positive rank 1 operator on $E_N$ defined by $R_0(x) := \langle x_0^*, x\rangle Ax_0$, for every $x \in E_N$; and for any $\delta > 0$, let $A_\delta$ be the operator defined by $A_\delta := A + \delta R_0$. The operators $A_\delta$ are now positive. As in the proof of [30, Lemma 5.17], we can prove that $A_\delta$ is absolutely exposing and given $\varepsilon > 0$, one can choose $\delta > 0$ so small that $\|A_\delta\| < 1$ and $\|A - A_\delta\| < \varepsilon$, because $\|A\| < 1$. This proves that $\mathcal{E}_1(E_N)$ is dense in $\mathcal{P}_1(E_N)$. □

Since the proof of [29, Theorem 3.4] can now be adapted to positive contractions thanks to Lemma 3.2.7, we can state the following results.

**Theorem 3.2.8.** *Let $X = \ell_q$ with $q > 2$. Then the topologies* SOT *and* SOT* *are similar on $\mathcal{P}_1(X)$.*

**Corollary 3.2.9.** *Let $X = \ell_q$ with $q > 2$. The Baire spaces $(\mathcal{P}_1(X), \texttt{SOT})$ and $(\mathcal{P}_1(X), \texttt{SOT}^*)$ have the same comeager sets.*

## 3.3 Some elementary properties of typical positive contractions

In this section, we study some elementary properties of a typical positive contraction on $\ell_q$ for the SOT and the SOT* topologies. These properties will be useful in the next section to prove that a typical positive contraction on $\ell_1$ and on $\ell_2$ has a non-trivial invariant subspace.

**Proposition 3.3.1.** *Let $X = \ell_q$ with $1 \leq q < \infty$. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ has the following properties:*

(a) $\|T\| = 1$;

(b) $\|T^n x\| \to 0$ *as* $n \to \infty$ *for all* $x \in X$;

(c) $T$ *is not invertible;*

(d) $\sigma_{ap}(T) = \sigma(T) = \overline{\mathbb{D}}$.

*If $1 < q < \infty$, a typical $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ also has these properties.*

*Proof.* The property (a) follows from the fact that the set

$$\mathcal{A} := \{T \in \mathcal{P}_1(X) : \|T\| = 1\}$$

can be written as
$$\mathcal{A} = \bigcap_{k\geq 1} \bigcup_{x\in S_X} \{T \in \mathcal{P}_1(X) : \|Tx\| > 1 - \frac{1}{k}\}.$$
So $\mathcal{A}$ is a $\texttt{SOT}$-$G_\delta$ subset of $\mathcal{P}_1(X)$ (and hence a $\texttt{SOT}^*$-$G_\delta$ subset of $\mathcal{P}_1(X)$).
Moreover, if $T$ is a positive contraction and if we set $T_N = P_N T P_N + Q_N$ for every $N \geq 0$, then $T_N$ belongs to $\mathcal{A}$ and we easily see that $T_N \xrightarrow[n\to\infty]{\texttt{SOT}^*} T$, so $\mathcal{A}$ is dense in $\mathcal{P}_1(X)$ for both the $\texttt{SOT}$ and the $\texttt{SOT}^*$ topologies.

The proof of properties (b), (c) and (d) can easily be adapted from [30, Propositions 3.7 to 3.9] to positive contractions. □

Recall that an operator $T \in \mathcal{B}(X)$ is said to be hypercyclic if there is a vector $x \in X$ such that the orbit $O(x,T) := \{T^n x : n \in \mathbb{Z}_+\}$ is dense in $X$. Such a vector is said to be hypercyclic for $T$. We refer to [6] and [31] for background on hypercyclicity.

**Proposition 3.3.2.** *Let $X = \ell_q$ with $1 \leq q < \infty$. For any $M > 1$, the set*
$$\{T \in \mathcal{P}_M(X) : T \text{ is hypercyclic}\}$$
*is comeager in $(\mathcal{P}_M(X), \texttt{SOT})$ and in $(\mathcal{P}_M(X), \texttt{SOT}^*)$. In particular, a typical positive contraction $T \in (\mathcal{P}_1(X), \texttt{SOT})$ is such that $2T$ is hypercyclic.*

*If $1 < q < \infty$, a typical $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ is such that $2T$ and $(2T)^*$ are hypercyclic.*

*Proof.* Using [6, Theorem 5.41] and replacing the property of being mixing in the Gaussian sense by the property of being hypercyclic in [28, Lemma 2.8], we notice that the proof given in [28, Proposition 2.3] works for any $1 \leq q < \infty$ in $\mathcal{P}_M(X)$. When $1 < q < \infty$, the map $T \mapsto T^*$ is a homeomorphism from $(\mathcal{P}_2(X), \texttt{SOT}^*)$ to $(\mathcal{P}_2(X), \texttt{SOT}^*)$, so an $\texttt{SOT}^*$-typical $T \in \mathcal{P}_1(X)$ is such that $(2T)^*$ is hypercyclic too. □

Using the fact that a hypercyclic operator $T$ is such that $\sigma_p(T^*) = \emptyset$, we have the following result.

**Corollary 3.3.3.** *Let $X = \ell_q$ with $1 \leq q < \infty$. An $\texttt{SOT}$-typical $T \in \mathcal{P}_1(X)$ is such that $T^*$ has no eigenvalue. If $1 < q < \infty$, an $\texttt{SOT}^*$-typical $T \in \mathcal{P}_1(X)$ is such that $T$ and $T^*$ have no eigenvalue.*

Using Corollary 3.2.9, we also have the following result.

**Corollary 3.3.4.** *Let $X = \ell_q$ with $q > 2$. An $\texttt{SOT}$-typical $T \in \mathcal{P}_1(X)$ is such that $T$ has no eigenvalue.*

The next corollary is an exact analogue of [30, Proposition 3.9].

**Corollary 3.3.5.** *If $X = \ell_q$ with $1 \leq q < \infty$, a typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ is such that $T - \lambda$ has dense range for every $\lambda \in \mathbb{C}$. If $q > 1$, a typical $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ is such that $T - \lambda$ has dense range for every $\lambda \in \mathbb{C}$.*

*Proof.* This is clear from Proposition 3.3.2 because a hypercyclic operator $T$ is such that $T - \lambda$ has dense range for every $\lambda \in \mathbb{C}$. □

Our next step is to investigate whether a typical $T \in \mathcal{P}_1(X)$ is such that $T^*$ is an isometry or not. The following fact that we already proved in Subsection 3.2c will be useful for this.

**Fact 3.3.6.** If $T$ is a positive isometry of $X = \ell_q$ with $1 \leq q < \infty$, then the vectors $Te_i$ for $i \geq 0$ have mutually disjoint supports.

By [19, Proposition 5.15] and [30, Corollary 3.5], we know that an SOT-typical $T \in \mathcal{B}_1(\ell_2)$ is such that $T^*$ is an isometry. We prove that this is no longer the case for a typical positive contraction on $\ell_2$.

**Proposition 3.3.7.** *Let $X = \ell_q$ with $1 < q < \infty$. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$) is such that $T^*$ is not an isometry.*

*Proof.* The arguments given in [30, Proposition 5.1] also apply in this case. We will detail a bit the denseness argument. Let

$$\mathcal{I}^* := \{T \in \mathcal{P}_1(X) : T^* \text{ is an isometry}\}.$$

By Fact 3.3.6, we have that

$$\mathcal{A} := \bigcup_{j \geq 0} \{T \in \mathcal{P}_1(X) : \langle e_0^*, Te_j \rangle \neq 0 \text{ and } \langle e_1^*, Te_j \rangle \neq 0\} \subseteq \mathcal{P}_1(X) \setminus \mathcal{I}^*$$

and the set $\mathcal{A}$ is SOT-open in $\mathcal{P}_1(X)$, so $\mathcal{A}$ is a SOT-$G_\delta$ of $\mathcal{P}_1(X)$ and hence SOT$^*$-$G_\delta$. It remains to prove that $\mathcal{A}$ is dense in $(\mathcal{P}_1(X), \texttt{SOT}^*)$. Let $\varepsilon > 0$, let $T \in \mathcal{P}_1(X)$ with $\|T\| < 1$ and let $x_1, \ldots, x_n \in X$ and $y_1^*, \ldots, y_n^* \in X^*$. We have to find a positive contraction $S$ in the set $\mathcal{A}$ such that

$$\max_{1 \leq l \leq n} \max\{\|(T - S)x_l\|, \|(T - S)^* y_l^*\|\} < \varepsilon. \tag{3.3.1}$$

Consider the positive operator $S_\delta$ defined by $S_\delta(x) = Tx + \delta\langle e_0^*, x\rangle(e_0 + e_1)$ for every $x \in X$, where $\delta$ is a positive number that we will define later on. We have that

$$\langle e_0^*, S_\delta e_0 \rangle \langle e_1^*, S_\delta e_0 \rangle \geq \delta^2 > 0.$$

For every $x \in X$, we have that

$$\|S_\delta x\| \leq (\|T\| + 2\delta)\|x\|$$

and for every $1 \leq l \leq n$, we have that

$$\|(T - S_\delta)x_l\| \leq 2\delta\|x_l\| \quad \text{and} \quad \|(T - S_\delta)^* y_l^*\| \leq 2\delta\|y_l^*\|.$$

If we choose $\delta > 0$ such that

$$\delta < \frac{1 - \|T\|}{2}, \quad 2\delta\|x_l\| < \varepsilon \quad \text{and} \quad 2\delta\|y_l^*\| < \varepsilon \quad \text{for every } 1 \leq l \leq n,$$

then the operator $S_\delta$ is a positive contraction of $\mathcal{A}$ satisfying (3.3.1). This concludes the proof of Proposition 3.3.7. □

The case $X = \ell_1$ is very different from the case $X = \ell_q$ with $q > 1$. We have the following result which will be useful in Section 3.4 to identify the point spectrum of an SOT-typical positive contraction on $\ell_1$.

**Proposition 3.3.8.** *Let $X = \ell_1$. An SOT-typical $T \in \mathcal{P}_1(X)$ is such that $T^*$ is a non-surjective isometry and such that $T - \lambda$ is surjective for every $\lambda \in \mathbb{D}$.*

*Proof.* The proof works exactly as in [30, Theorem 4.1]. Indeed, the set

$$\mathcal{I}^* := \{T \in \mathcal{P}_1(X) : T^* \text{ is an isometry}\}$$

is a SOT-$G_\delta$ of $\mathcal{P}_1(X)$ (see [30, Theorem 4.1]) and the set $\mathcal{I}^*$ is also SOT-dense in $\mathcal{P}_1(X)$ since all the operators involved in the proof of [30, Theorem 4.1] are positive. The second part of the proof immediately follows as in [30, Theorem 4.1]. □

Finally, we describe the essential spectrum of a typical positive contraction on $X = \ell_q$ with $1 \leq q < \infty$. This will be useful to prove that a typical positive contraction on $\ell_1$ has a non-trivial invariant subspace. Recall that an operator $T \in \mathcal{B}(X)$ is Fredholm if its kernel is finite-dimensional and its range has finite codimension, and it is upper semi-Fredholm if its range is closed and its kernel is finite-dimensional. Every Fredholm operator is upper semi-Fredholm. An operator is semi-Fredholm if it is upper semi-Fredholm or if its range has finite codimension. We denote by $\Phi_+(X)$ the set of all upper semi-Fredholm operators on $X$. The set $\Phi_+(X)$ is norm-open in $\mathcal{B}(X)$. We refer to [47] for background on Fredholm operators. The first lemma that we will use is the following analogue of [29, Fact 7.14].

**Lemma 3.3.9.** *Let $X = \ell_q$ with $1 \leq q < \infty$ and let $\lambda \in \overline{\mathbb{D}}$. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ when $1 < q < \infty$) has the following property:*

*For every $\varepsilon > 0$ and every $n \geq 1$, there exists a subspace $E$ of $X$ with $n < dim(E) < \infty$ such that $\|(T - \lambda)|_E\| < \varepsilon$.*

*Proof.* Let $\mathcal{G}$ be the set of all operators $T \in \mathcal{P}_1(X)$ satisfying this property. Then

$$\mathcal{G} = \bigcap_{\substack{p \geq 0 \\ n \geq 1}} \{T \in \mathcal{P}_1(X) : \exists\, E \text{ subspace of } X,\ n < \dim(E) < \infty,\ \|(T - \lambda)|_E\| < 2^{-p}\}.$$

For every subspace $E$ of $X$ satisfying $n < \dim(E) < \infty$, the set

$$\Lambda_{E,p} := \{T \in \mathcal{P}_1(X) : \|(T - \lambda)|_E\| < 2^{-p}\}$$

is SOT-open (see [29, Fact 7.14]), so $\mathcal{G}$ is SOT-$G_\delta$ in $\mathcal{P}_1(X)$ and hence SOT*-$G_\delta$ in $\mathcal{P}_1(X)$.

Moreover, if $T$ is a positive contraction and if we set $T_N = P_N T P_N + \lambda Q_N$ for every $N \geq 0$, then $T_N$ is a positive contraction which belongs to $\mathcal{G}$, and $T_N \xrightarrow[N \to \infty]{\texttt{SOT}^*} T$. So $\mathcal{G}$ is dense in $\mathcal{P}_1(X)$ for the SOT and the SOT* topologies. □

With Lemma 3.3.9, we obtain the following description of the essential spectrum of a typical positive contraction.

**Proposition 3.3.10.** *Let $X = \ell_q$ with $1 \leq q < \infty$. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ when $1 < q < \infty$) is such that $T - \lambda$ is not upper semi-Fredholm for every $\lambda \in \overline{\mathbb{D}}$ and such that $\sigma_{ess}(T) = \overline{\mathbb{D}}$.*

*Proof.* The proof given in [29, Proposition 7.13] works in our case since Lemma 3.3.9 is working in $\mathcal{P}_1(X)$ for the topologies SOT and SOT*. □

## 3.4 Invariant subspaces and typicality

We now come to our main goal in this paper, which is to investigate whether a typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$) has a non-trivial invariant subspace. We will in fact see that this is not an easy question to answer. A first observation is that the property of having a non-trivial invariant subspace is either typical or atypical ([30, Corollary 3.3]).

**Proposition 3.4.1.** *Let $X = \ell_q$ with $1 \leq q < \infty$. Either a typical $T \in (\mathcal{P}_1(X), \mathtt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \mathtt{SOT}^*)$ when $1 < q < \infty$) has a non-trivial invariant subspace, or a typical $T \in (\mathcal{P}_1(X), \mathtt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \mathtt{SOT}^*)$ when $1 < q < \infty$) does not have a non-trivial invariant subspace.*

An important result from Brown, Chevreau and Pearcy ([11]) states that every contraction on a Hilbert space whose spectrum contains the unit circle has a non-trivial invariant subspace. Since a typical $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$ (resp. $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT}^*)$) is such that $\sigma(T) = \overline{\mathbb{D}}$ by Proposition 3.3.1, we have the following result as in [30, Corollary 7.3].

**Theorem 3.4.2.** *A typical $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$ (resp. $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT}^*)$) has a non-trivial invariant subspace.*

## 3.4a Invariant subspace of typical positive contractions on $\ell_1$

In this subsection, we consider the case where $X = \ell_1$. We will prove that a typical positive contraction on $X = \ell_1$ has a non-trivial invariant subspace, and even has eigenvalues.

By Proposition 3.3.1 and Proposition 3.3.8, we know that a typical $T \in (\mathcal{P}_1(X), \mathtt{SOT})$ is such that $T - \lambda$ is surjective for every $\lambda \in \mathbb{D}$ and such that $\sigma(T) = \overline{\mathbb{D}}$. This implies that a typical $T \in (\mathcal{P}_1(X), \mathtt{SOT})$ is such that $T - \lambda$ is not injective for every $\lambda \in \mathbb{D}$. Thus, we have the following results.

**Theorem 3.4.3.** *Let $X = \ell_1$. A typical $T \in (\mathcal{P}_1(X), \mathtt{SOT})$ is such that $\sigma_p(T) = \mathbb{D}$ and such that $\dim(\mathrm{Ker}(T - \lambda)) = \infty$ for every $\lambda \in \mathbb{D}$.*

*Proof of Theorem 3.4.3.* The proof is motivated by [30, Remark 4.5]. By Propositions 3.3.8 and 3.3.10, a typical $T \in (\mathcal{P}_1(X), \mathtt{SOT})$ is not Fredholm and is surjective, so a typical $T \in (\mathcal{P}_1(X), \mathtt{SOT})$ is such that $\dim(\mathrm{Ker}(T)) = \infty$. But a typical $T \in (\mathcal{P}_1(X), \mathtt{SOT})$ is such that $T - \lambda$ is semi-Fredholm for every $\lambda \in \mathbb{D}$ (because it is surjective). By the continuity of the Fredholm index, $\mathrm{Ind}(T - \lambda)$ does not depend on $\lambda \in \mathbb{D}$. So a typical $T \in (\mathcal{P}_1(X), \mathtt{SOT})$ is such that $\mathrm{Ind}(T - \lambda) = \dim(\mathrm{Ker}(T - \lambda)) = \infty$ for every $\lambda \in \mathbb{D}$. □

**Corollary 3.4.4.** *Let $X = \ell_1$. A typical $T \in (\mathcal{P}_1(X), \mathtt{SOT})$ has a non-trivial invariant subspace.*

Hence, the cases $X = \ell_2$ and $X = \ell_1$ are fully understood. We will now see that the problem is more difficult in the other cases, which is not surprising because the problem is still open in $\mathcal{B}_1(X)$ when $X = \ell_q$ with $1 < q \neq 2 < \infty$ for the topologies $\mathtt{SOT}$ and $\mathtt{SOT}^*$.

## 3.4b Invariant subspace of typical positive contractions on a Banach space with a basis

We now focus on the case where $X$ is a Banach space with a basis. In particular, it includes the case $X = \ell_q$ with $1 < q \neq 2 < \infty$.

We start with a similar result to [30, Proposition 5.24] in the case where $X = \ell_q$ with $1 \leq q < \infty$.

Recall that an operator $T \in \mathcal{B}(X)$ is polynomially bounded if there exists $C > 0$ such that for every complex polynomial $P$:

$$\|P(T)\| \leq C \sup_{|z|=1} |P(z)|.$$

Every contraction on $\ell_2$ is polynomially bounded by Von Neumann's inequality. A remarkable result of Ambrozie and Müller [4] states that every polynomially bounded operator $T \in \mathcal{B}(X)$ such that $T^n \underset{n\to\infty}{\xrightarrow{\mathtt{SOT}}} 0$ and $\mathbb{T} \subseteq \sigma(T)$ has a non-trivial invariant subspace. We know that a typical

$T \in \mathcal{P}_1(X)$ is such that $T^n \xrightarrow[n\to\infty]{\texttt{SOT}} 0$ and $\sigma(T) = \overline{\mathbb{D}}$ by Proposition 3.3.1, so it is natural to ask whether a typical $T \in \mathcal{P}_1(X)$ is polynomially bounded or not.

**Proposition 3.4.5.** *Let $X = \ell_q$ with $1 \le q \ne 2 < \infty$. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ when $1 < q \ne 2 < \infty$ ) is not polynomially bounded.*

*Proof.* The proof given in [30, Proposition 5.24] can easily be adapted to positive contractions on $X$. ☐

Recall that a positive operator on a Banach space with a basis is said to be satisfying the AAB criterion if it satisfies the following theorem (already stated in the introduction) due to Abramovich, Aliprantis and Burkinshaw ([2, Theorem 2.2]).

**Theorem 3.4.6.** *Let $X$ be a Banach space with a basis and let $T$ be a positive operator on $X$. If there exists a non-zero positive operator $A$ on $X$ which is quasinilpotent at a certain non-zero positive vector of $X$ and such that $AT = TA$, then $T$ has a non-trivial invariant subspace.*

Recall also that by [1, Theorem 2.2], any positive operator on $\ell_q$ satisfying the AAB criterion has a non-trivial closed invariant ideal, that is, a closed vector subspace $V$ of $\ell_q$ such that $|x| \le |y|$ and $y \in V$ imply $x \in V$, for every $x, y \in \ell_q$. Moreover, by [52, Proposition 1.2], a positive operator on a Banach space with an unconditional basis $(e_n)_{n\ge 0}$ has no non-trivial closed invariant ideals if and only if it satisfies the following condition:

$$\forall\, i \ne j \in \mathbb{Z}_+, \exists n \in \mathbb{Z}_+ : \langle e_j^*, T^n e_i\rangle > 0.$$

The following result shows that a typical positive contraction on a Banach space with an unconditional basis has no non-trivial closed invariant ideals.

**Proposition 3.4.7.** *Let $X$ be a Banach space with an unconditional basis $(e_n)_{n\ge 0}$. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ when $X^*$ is separable) has no non-trivial closed invariant ideals.*

*Proof.* Consider the set

$$\mathcal{G} := \{T \in \mathcal{P}_1(X) : \forall\, i \ne j \in \mathbb{Z}_+, \exists\, n \in \mathbb{Z}_+ \text{ such that } \langle e_j^*, T^n e_i\rangle > 0\}.$$

Then

$$\mathcal{G} = \bigcap_{\substack{i,j\ge 0\\ i\ne j}} \bigcup_{n\ge 0} \{T \in \mathcal{P}_1(X) : \langle e_j^*, T^n e_i\rangle > 0\} = \bigcap_{\substack{i,j\ge 0\\ i\ne j}} \mathcal{G}_{i,j},$$

with

$$\mathcal{G}_{i,j} := \bigcup_{n\ge 0} \{T \in \mathcal{P}_1(X) : \langle e_j^*, T^n e_i\rangle > 0\} \quad \text{for every } i, j \ge 0 \text{ with } i \ne j.$$

Each $\mathcal{G}_{i,j}$ is easily seen to be $\texttt{SOT}$-open (and hence $\texttt{SOT}^*$-open) because the map $T \mapsto T^n$ is continuous on $\mathcal{P}_1(X)$ for the $\texttt{SOT}$ topology. Let us now show that every $\mathcal{G}_{i,j}$ is $\texttt{SOT}^*$-dense in $\mathcal{P}_1(X)$. Let $\varepsilon > 0$, let $T \in \mathcal{P}_1(X)$ with $\|T\| < 1$ and let $x_1, \dots, x_n \in X$ and $y_1^*, \dots, y_n^* \in X^*$. We have to find a positive contraction $S$ in the set $\mathcal{G}_{i,j}$ such that

$$\max_{1\le l\le n} \max\{\|(T-S)x_l\|, \|(T-S)^* y_l^*\|\} < \varepsilon. \tag{3.4.1}$$

Consider the positive operator $S_\delta$ defined by $S_\delta(x) = Tx + \delta\langle e_i^*, x\rangle e_j$ for every $x \in X$, where $\delta$ is a positive number that we will define later on. We have that

$$\langle e_j^*, S_\delta e_i\rangle \ge \delta > 0.$$

For every $x \in X$, we have that
$$\|S_\delta x\| \leq (\|T\| + \delta\|e_i^*\|\|e_j\|)\|x\|$$
and for every $1 \leq l \leq n$, we have that
$$\|(T - S_\delta)x_l\| \leq \delta\|x_l\|\|e_i^*\|\|e_j\| \quad \text{and} \quad \|(T - S_\delta)^* y_l^*\| \leq \delta\|y_l^*\|\|e_i^*\|\|e_j\|.$$
If we choose $\delta > 0$ such that
$$\delta < \frac{1 - \|T\|}{\|e_i^*\|\|e_j\|}, \quad \delta\|x_l\|\|e_i^*\|\|e_j\| < \varepsilon \quad \text{and} \quad \delta\|y_l^*\|\|e_i^*\|\|e_j\| < \varepsilon \quad \text{for every } 1 \leq l \leq n,$$
then the operator $S_\delta$ is a positive contraction of $\mathcal{G}_{i,j}$ satisfying (3.4.1). This proves that each $\mathcal{G}_{i,j}$ is $\texttt{SOT}^*$-dense in $\mathcal{P}_1(X)$ and this concludes the proof of Proposition 3.4.7. □

In particular, we obtain the following result in the case where $X = \ell_q$.

**Corollary 3.4.8.** *Let $X = \ell_q$ with $1 \leq q < \infty$. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ when $1 < q < \infty$) does not satisfy the AAB criterion.*

Let $X$ be a Banach space with a basis. Since any positive operator on $X$ that is quasinilpotent at a non-zero positive vector of $X$ satisfies the AAB criterion, we immediately get the following result.

**Corollary 3.4.9.** *Let $X = \ell_q$ with $1 \leq q < \infty$. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ when $1 < q < \infty$) is not quasinilpotent at any non-zero positive vector of $X$.*

We now would like to extend Corollary 3.4.8 to other Banach spaces with a basis. The following lemma will be very useful for what follows.

**Lemma 3.4.10.** *Let $X$ be a Banach space with a basis $(e_n)_{n\geq 0}$, let $A$ be a non-zero positive operator on $X$ and let $y \in X$ be such that $y \geq 0$ and $y \neq 0$. Let $j \in \mathbb{Z}_+$ be such that $\langle e_j^*, y\rangle := \alpha_j > 0$. If*
$$a_{j,j} := \langle e_j^*, Ae_j\rangle > 0,$$
*then the operator $A$ is not quasinilpotent at $y$.*

*Proof.* First, we remark that $Ae_j \geq a_{j,j}\, e_j$. Since
$$\langle e_j^*, Ay\rangle = \sum_{k\geq 0} \langle e_j^*, Ae_k\rangle \langle e_k^*, y\rangle,$$
we also have $\langle e_j^*, Ay\rangle \geq a_{j,j}\,\alpha_j$. It follows that $Ay \geq \alpha_j\, a_{j,j}\, e_j$ and by induction, we easily get that
$$A^k y \geq \alpha_j\, a_{j,j}^k\, e_j \quad \text{for every } k \geq 1.$$

In particular, we get that
$$\|A^k y\|\|e_j^*\| \geq \langle e_j^*, A^k y\rangle \geq \alpha_j\, a_{j,j}^k \quad \text{for every } k \geq 1$$
and thus
$$\liminf_{k\to\infty} \|A^k y\|^{1/k} \geq a_{j,j} > 0.$$
This shows that the operator $A$ cannot be quasinilpotent at $y$ and this concludes the proof of Lemma 3.4.10. □

The main result of this paper is the following generalization of Corollary 3.4.8.

**Theorem 3.4.11.** *Let $(X, \|.\|)$ be a reflexive Banach space with a monotone basis $(e_n)_{n\geq 0}$. A typical $T \in (\mathcal{P}_1(X), \mathtt{SOT}^*)$ does not satisfy the AAB criterion.*

*Proof.* We denote by $\mathcal{F}$ the set

$$\mathcal{F} = \{T \in \mathcal{P}_1(X) : T \text{ satisties the AAB criterion}\}.$$

By Lemma 3.4.10, we have

$$\begin{aligned}
\mathcal{F} &\subseteq \bigcup_{p\geq 0} \{T \in \mathcal{P}_1(X) : \exists A \in \mathcal{P}_1(X) \text{ such that } A \neq 0,\ AT = TA \text{ and } \langle e_p^*, Ae_p\rangle = 0\} \\
&\subseteq \bigcup_{p\geq 0} \bigcup_{i,j\geq 0} \bigcup_{\eta\in\mathbb{Q}_{>0}} \{T \in \mathcal{P}_1(X) : \exists A \in \mathcal{P}_1(X) \text{ such that } \langle e_j^*, Ae_i\rangle \geq \eta,\ AT = TA \text{ and } \langle e_p^*, Ae_p\rangle = 0\} \\
&\subseteq \bigcup_{p\geq 0} \bigcup_{i,j\geq 0} \bigcup_{\eta\in\mathbb{Q}_{>0}} \mathcal{F}_{i,j,\eta,p}
\end{aligned}$$

with

$$\mathcal{F}_{i,j,\eta,p} := \{T \in \mathcal{P}_1(X) : \exists A \in \mathcal{P}_1(X) \text{ such that } \langle e_j^*, Ae_i\rangle \geq \eta,\ AT = TA \text{ and } \langle e_p^*, Ae_p\rangle = 0\}.$$

We first prove the following fact.

**Fact 3.4.12.** For each $i, j, p \geq 0$ and $\eta \in \mathbb{Q}_{>0}$, the set $\mathcal{F}_{i,j,\eta,p}$ is $\mathtt{SOT}^*$-closed in $\mathcal{P}_1(X)$.

*Proof of Fact 3.4.12.* Let $(T_k)_{k\geq 0} \subseteq \mathcal{F}_{i,j,\eta,p}$ be such that $T_k \xrightarrow[k\to\infty]{\mathtt{SOT}^*} T$ with $T \in \mathcal{P}_1(X)$. For every $k \geq 0$, there exists an operator $A_k \in \mathcal{P}_1(X)$ such that $A_k T_k = T_k A_k$, $\langle e_j^*, A_k e_i\rangle \geq \eta$ and $\langle e_p^*, A_k e_p\rangle = 0$. Since $\mathcal{P}_1(X)$ is $\mathtt{WOT}$-compact (see [14, Page 275, Proposition 5.5]), we can suppose without loss of generality that there exists an operator $A \in \mathcal{P}_1(X)$ such that $A_k \xrightarrow[k\to\infty]{\mathtt{WOT}} A$. We immediately get that $\langle e_p^*, Ae_p\rangle = 0$ and that $\langle e_j^*, Ae_i\rangle \geq \eta$. Let us now show that $AT = TA$.

Let $y^* \in X^*$ and $x \in X$. We have

$$\langle y^*, A_k T_k x\rangle = \langle y^*, A_k(T_k - T)x\rangle + \langle y^*, A_k T x\rangle \quad \text{for every } k \geq 0.$$

Since the sequence $(A_k)_{k\geq 0}$ is bounded and since $T_k \xrightarrow[k\to\infty]{\mathtt{SOT}} T$, we have that

$$\langle y^*, A_k(T_k - T)x\rangle \xrightarrow[k\to\infty]{} 0,$$

and since $A_k \xrightarrow[k\to\infty]{\mathtt{WOT}} A$, we have that

$$\langle y^*, A_k T x\rangle \xrightarrow[k\to\infty]{} \langle y^*, ATx\rangle.$$

It follows that $\langle y^*, A_k T_k x\rangle \xrightarrow[k\to\infty]{} \langle y^*, ATx\rangle$.

Likewise, we have

$$\langle y^*, T_k A_k x\rangle = \langle (T_k^* - T^*)y^*, A_k x\rangle + \langle T^* y^*, A_k x\rangle \quad \text{for every } \ k \geq 0,$$

so using the fact that $T_k^* \xrightarrow[k\to\infty]{\texttt{SOT}} T^*$, that $A_k \xrightarrow[k\to\infty]{\texttt{WOT}} A$ and that the sequence $(A_k)_{k\geq 0}$ is bounded, we can prove by the same way that $\langle y^*, T_k A_k x\rangle \underset{k\to\infty}{\longrightarrow} \langle y^*, TAx\rangle$. It follows that

$$\langle y^*, ATx\rangle = \langle y^*, TAx\rangle$$

for every $y^* \in X^*$ and $x \in X$, and thus $AT = TA$. This proves that $T \in \mathcal{F}_{i,j,\eta,p}$ and concludes the proof of Fact 3.4.12. □

Let us now prove the following proposition.

**Proposition 3.4.13.** *For each $i, j, p \geq 0$ and $\eta \in \mathbb{Q}_{>0}$, the set $\mathcal{F}_{i,j,\eta,p}$ has empty interior in $\mathcal{P}_1(X)$ for the* $\texttt{SOT}^*$ *topology.*

*Proof of Proposition 3.4.13.* Using Lemma 3.2.2, we will prove that the set $\mathcal{P}_1(X)\setminus\mathcal{F}_{i,j,\eta,p}$ is dense in $\mathcal{P}_1(X)$ for the $\texttt{SOT}^*$ topology.

Let us fix $i, j, p \in \mathbb{Z}_+$ and $\eta \in \mathbb{Q}_{>0}$.

Let $N \in \mathbb{Z}_+$ be such that $N \geq p$ and let $M \in \mathcal{P}_1(E_N)$ with $\|M\| < 1$. Let $\varepsilon > 0$. Let also $u = e_0 + \ldots + e_{p+N+1}$ and let us write $P : X \to E_N$ the canonical projection instead of $P_N$. By Remark 3.2.3, we can suppose that $\langle e_k^*, Me_l\rangle > 0$ for all indices $0 \leq k, l \leq N$.

Let $T$ be the positive operator defined as follows:

$$Tx = MPx + \delta\langle e_{N+p+1}^*, x\rangle\, u + Sx \quad \text{for every } x \in X.$$

In this expression, $S$ is the operator on $X$ defined by

$$S\left(\sum_{k=0}^{\infty} e_k^*(x) e_k\right) = \sum_{k=0}^{N+p+1} \delta\, e_k^*(x)\, e_{k+N+1} + \sum_{k>N+p+1} \delta_{k-N-p-1}\, e_k^*(x)\, e_{k+N+1}\,,$$

where $(\delta_k)_{k\geq 1}$ is a sequence of positive real numbers satisfying

$$0 < \sum_{k>N+p+1} \delta_{k-N-p-1} \|e_k^*\| \|e_{k+N+1}\| < 1 - \|M\|,$$

where $\delta > 0$ is such that

$$0 < \delta < \frac{1 - \|M\| - \sum_{k>N+p+1} \delta_{k-N-p-1} \|e_k^*\| \|e_{k+N+1}\|}{\|u\| \|e_{N+p+1}^*\| + \sum_{k=0}^{N+p+1} \|e_k^*\| \|e_{k+N+1}\|},$$

and where

$$\delta\|e_{N+1+k}\| < \varepsilon, \quad \delta\|e_{N+1+p}^*\| < \varepsilon \quad \text{and} \quad \delta < \langle e_p^*, Me_p\rangle \quad \text{for every } 0 \leq k \leq N.$$

With these choices, we can easily show that $T$ is a positive contraction on $X$ and that

$$\|(T-M)e_k\| < \varepsilon, \quad \|(T-M)^* e_k^*\| < \varepsilon \quad \text{for every } 0 \leq k \leq N.$$

It remains to show that $T$ does not belong to $\mathcal{F}_{i,j,\eta,p}$. To do so, let $A$ be a positive operator such that $AT = TA$ and $\langle e_p^*, Ae_p\rangle = 0$. We will prove that $A = 0$. In particular, the condition $\langle e_j^*, Ae_i\rangle \geq \eta$ won't be fulfilled, and this will yield that $T$ does not belong to $\mathcal{F}_{i,j,\eta,p}$.

The equation $ATe_p = TAe_p$ implies that

$$AMe_p + \delta Ae_{p+N+1} = MPAe_p + \delta\langle e_{p+N+1}^*, Ae_p\rangle u + SAe_p. \tag{3.4.2}$$

We remark that

$$\langle e_{p+N+1}^*, SAe_p\rangle = \delta\langle e_p^*, Ae_p\rangle = 0$$

and that $MPAe_p \in E_N$, so we also have

$$\langle e_{p+N+1}^*, MPAe_p\rangle = 0.$$

Applying the functional $e_{p+N+1}^*$ to the equation (3.4.2) one gets that

$$\begin{aligned}\delta\langle e_{p+N+1}^*, Ae_p\rangle &= \langle e_{p+N+1}^*, AMe_p\rangle + \delta\langle e_{p+N+1}^*, Ae_{p+N+1}\rangle \\ &= \sum_{k=0}^{N}\langle e_k^*, Me_p\rangle\langle e_{p+N+1}^*, Ae_k\rangle + \delta\langle e_{p+N+1}^*, Ae_{p+N+1}\rangle\end{aligned}$$

and so

$$\begin{aligned}(\delta - \langle e_p^*, Me_p\rangle)\langle e_{p+N+1}^*, Ae_p\rangle = \sum_{\substack{0\leq k\leq N\\ k\neq p}}&\langle e_k^*, Me_p\rangle\langle e_{p+N+1}^*, Ae_k\rangle \\ &+ \delta\langle e_{p+N+1}^*, Ae_{p+N+1}\rangle.\end{aligned} \tag{3.4.3}$$

Using the non-negativity of the coefficients on the right hand side of (3.4.3), and using the fact that $\delta < \langle e_p^*, Me_p\rangle$ and that $\langle e_{p+N+1}^*, Ae_p\rangle \geq 0$, it follows that

$$\langle e_{p+N+1}^*, Ae_{p+N+1}\rangle = 0 \quad \text{and} \quad \langle e_{p+N+1}^*, Ae_k\rangle = 0 \quad \text{for every } 0 \leq k \leq N. \tag{3.4.4}$$

Thus one gets

$$ATe_k = AMe_k + \delta Ae_{k+N+1} \quad \text{for every } 0 \leq k \leq N, \tag{3.4.5}$$

$$\text{and} \quad TAe_k = MPAe_k + SAe_k \quad \text{for every } 0 \leq k \leq N. \tag{3.4.6}$$

The relations $TAe_k = ATe_k$ for $0 \leq k \leq N$ imply that

$$\left\{\begin{array}{lcl} MPAe_0 + SAe_0 & = & AMe_0 + \delta Ae_{N+1} \\ MPAe_1 + SAe_1 & = & AMe_1 + \delta Ae_{N+2} \\ & \vdots & \\ MPAe_N + SAe_N & = & AMe_N + \delta Ae_{2N+1} \end{array}\right. \tag{3.4.7}$$

and since $PSAe_l = 0$ for every $0 \le l \le N$, we obtain that

$$\begin{cases} MPAe_0 &= PAMe_0 + \delta PAe_{N+1} \\ MPAe_1 &= PAMe_1 + \delta PAe_{N+2} \\ &\vdots \\ MPAe_N &= PAMe_N + \delta PAe_{2N+1}. \end{cases} \tag{3.4.8}$$

Finally, rewriting the system (3.4.8), we obtain the following system:

$$\begin{cases} \sum_{k=0}^{N} \langle e_k^*, PAe_0\rangle Me_k &= \sum_{k=0}^{N} \langle e_k^*, Me_0\rangle PAe_k + \delta PAe_{N+1} \\ \sum_{k=0}^{N} \langle e_k^*, PAe_1\rangle Me_k &= \sum_{k=0}^{N} \langle e_k^*, Me_1\rangle PAe_k + \delta PAe_{N+2} \\ &\vdots \\ \sum_{k=0}^{N} \langle e_k^*, PAe_N\rangle Me_k &= \sum_{k=0}^{N} \langle e_k^*, Me_N\rangle PAe_k + \delta PAe_{2N+1}. \end{cases} \tag{3.4.9}$$

The system (3.4.9) is equivalent to the following matrix equation

$$BC = CB + \delta D, \tag{3.4.10}$$

where

$$B = \begin{pmatrix} \langle e_0^*, PAe_0\rangle & \langle e_1^*, PAe_0\rangle & \cdots & \langle e_N^*, PAe_0\rangle \\ \langle e_0^*, PAe_1\rangle & \langle e_1^*, PAe_1\rangle & \cdots & \langle e_N^*, PAe_1\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle e_0^*, PAe_N\rangle & \langle e_1^*, PAe_N\rangle & \cdots & \langle e_N^*, PAe_N\rangle \end{pmatrix},$$

$$C = \begin{pmatrix} \langle e_0^*, Me_0\rangle & \langle e_1^*, Me_0\rangle & \cdots & \langle e_N^*, Me_0\rangle \\ \langle e_0^*, Me_1\rangle & \langle e_1^*, Me_1\rangle & \cdots & \langle e_N^*, Me_1\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle e_0^*, Me_N\rangle & \langle e_1^*, Me_N\rangle & \cdots & \langle e_N^*, Me_N\rangle \end{pmatrix},$$

and

$$D = \begin{pmatrix} \langle e_0^*, PAe_{N+1}\rangle & \langle e_1^*, PAe_{N+1}\rangle & \cdots & \langle e_N^*, PAe_{N+1}\rangle \\ \langle e_0^*, PAe_{N+2}\rangle & \langle e_1^*, PAe_{N+2}\rangle & \cdots & \langle e_N^*, PAe_{N+2}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle e_0^*, PAe_{2N+1}\rangle & \langle e_1^*, PAe_{2N+1}\rangle & \cdots & \langle e_N^*, PAe_{2N+1}\rangle \end{pmatrix}.$$

We will now show that $D = 0$ using a similar idea to [10, Theorem 2.1].

Since $C$ is a matrix with positive entries, the spectral radius of $C$ and $C^\top$ is a positive eigenvalue of $C$ and $C^\top$ respectively associated to eigenvectors with positive entries. Let $\lambda$ be the spectral radius of $C$ and $C^\top$. Then there exist two vectors $x, y \in \mathbb{R}^{N+1}$ with positive entries such that

$$Cx = \lambda x \quad \text{and} \quad y^\top C = \lambda y^\top.$$

The equation (3.4.10) implies that

$$\lambda Bx = CBx + \delta Dx$$

and

$$\lambda y^\top Bx = \lambda y^\top Bx + \delta y^\top Dx,$$

so that

$$y^\top Dx = 0.$$

Using the fact that the vectors $x$ and $y$ have positive entries, it follows that $D = 0$.

Now because the vectors $PAe_l$ belong to $E_N$, we have just proved that

$$PAe_k = 0 \quad \text{for every } N+1 \le k \le 2N+1. \tag{3.4.11}$$

Using the last equation (3.4.11), the relation $ATe_{p+N+1} = TAe_{p+N+1}$ gives

$$\delta Au + \delta Ae_{p+2N+2} = SAe_{p+N+1},$$

so using the fact that

$$\langle e_k^*, SAe_{p+N+1}\rangle = 0 \quad \text{for every } \ 0 \le k \le N,$$

we get that

$$\delta PAu + \delta PAe_{p+2N+2} = PSAe_{p+N+1} = 0. \tag{3.4.12}$$

Now recall that $u = e_0 + \dots + e_{p+N+1}$. It follows from the non-negativity of the coefficients in the equation (3.4.12) that

$$PAe_k = 0 \quad \text{for every } 0 \le k \le 2N+1. \tag{3.4.13}$$

Let

$$a_{k,l} = \langle e_k^*, Ae_l\rangle \quad \text{and} \quad m_{k,l} = \langle e_k^*, Me_l\rangle \quad \text{for every } k,l \ge 0.$$

From the equations (3.4.4) and (3.4.13), we have:

$$a_{k,l} = 0 \quad \text{for every } 0 \le k \le N \text{ and every } 0 \le l \le 2N+1, \tag{3.4.14}$$

$$a_{p+N+1,l} = 0 \quad \text{for every } 0 \le l \le N, \tag{3.4.15}$$

$$a_{p+N+1,p+N+1} = 0. \tag{3.4.16}$$

An easy computation shows that

$$\langle e_k^*, ATe_0\rangle = \sum_{l=0}^{N} a_{k,l}\, m_{l,0} + \delta a_{k,N+1} \quad \text{for every } k \ge 0$$

and that

$$\langle e_k^*, TAe_0\rangle = \begin{cases} \delta\, a_{k-N-1,0} + \delta\, a_{N+p+1,0} & \text{if } N+1 \le k \le N+p+1, \\ \delta\, a_{k-N-1,0} & \text{if } N+p+1 < k \le 2N+p+2, \\ \delta_{k-2N-p-2}\, a_{k-N-1,0} & \text{if } k > 2N+p+2. \end{cases}$$

So using the equation

$$\langle e_k^*, ATe_0\rangle = \langle e_k^*, TAe_0\rangle,$$

one gets that

$$\sum_{l=0}^{N} a_{k,l}\, m_{l,0} + \delta\, a_{k,N+1} = 0 \quad \text{for every } N+1 \le k \le N+p+1,$$

and using the non-negativity of the coefficients and the fact that the coefficients $m_{l,0}$ are positive, we obtain that

$$a_{k,l} = 0 \quad \text{for every } 0 \le l \le N+1 \text{ and every } N+1 \le k \le N+p+1. \tag{3.4.17}$$

Proceeding by induction, we easily get that

$$a_{k,l} = 0 \quad \text{for every } 0 \le l \le N+1 \text{ and every } k \ge 0, \tag{3.4.18}$$

so in particular we have

$$Ae_l = 0 \quad \text{for every } 0 \le l \le N+1. \tag{3.4.19}$$

Now because we have

$$TAe_l = 0 \quad \text{for every } 0 \le l \le N+1 \tag{3.4.20}$$

and

$$ATe_l = AMPe_l + \delta\langle e^*_{N+p+1}, e_l\rangle Au + \delta Ae_{l+N+1} \quad \text{for every } 0 \le l \le N+1, \tag{3.4.21}$$

the equation (3.4.20) and the positivity of the vectors in the relation (3.4.21) give us

$$Ae_{l+N+1} = 0 \quad \text{for every } 0 \le l \le N+1. \tag{3.4.22}$$

If we put together the equations (3.4.19) and (3.4.22), we obtain that

$$Ae_l = 0 \quad \text{for every } 0 \le l \le 2N+2.$$

Proceeding by induction, we easily obtain that $Ae_k = 0$ for every $k \ge 0$ and thus $A = 0$. This concludes the proof of Proposition 3.4.13. □

The proof of Theorem 3.4.11 immediately follows from Fact 3.4.12 and Proposition 3.4.13. □

Finally, we extend Corollary 3.4.9 to Banach spaces with a basis. Observe that we do not require here that the basis be monotone, and hence Theorem 3.4.11 does not apply.

**Proposition 3.4.14.** *Let $X$ be a Banach space with a basis $(e_n)_{n\ge0}$. A typical $T \in (\mathcal{P}_1(X), \mathtt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \mathtt{SOT}^*)$ when $X^*$ is separable) is not quasinilpotent at any non-zero positive vector of $X$.*

*Proof.* Consider the following set

$$\mathcal{A} := \{T \in \mathcal{P}_1(X) : \exists\, y \in X, y \ge 0, y \ne 0 \text{ such that } T \text{ is quasinilpotent at } y\}.$$

By Lemma 3.4.10, we have

$$\mathcal{A} \subseteq \bigcup_{j\ge0} \mathcal{F}_j,$$

with

$$\mathcal{F}_j := \{T \in \mathcal{P}_1(X) : \langle e_j^*, T e_j \rangle = 0\} \quad \text{for every } j \geq 0.$$

Each $\mathcal{F}_j$ is SOT-closed in $\mathcal{P}_1(X)$ and hence SOT*-closed in $\mathcal{P}_1(X)$. Let us now prove that each $\mathcal{F}_j$ has empty interior in $(\mathcal{P}_1(X), \text{SOT}^*)$.

To do so, let $\varepsilon > 0$, let $T \in \mathcal{P}_1(X)$ with $\|T\| < 1$ and let $x_1, \dots, x_n \in X$ and $y_1^*, \dots, y_n^* \in X^*$. We have to find a positive contraction $S$ in the set $\mathcal{P}_1(X) \setminus \mathcal{F}_j$ such that

$$\max_{1 \leq l \leq n} \max\{\|(T - S)x_l\|, \|(T - S)^* y_l^*\|\} < \varepsilon. \tag{3.4.23}$$

Consider the positive operator $S_\delta$ defined by $S_\delta(x) = Tx + \delta \langle e_j^*, x \rangle e_j$ for every $x \in X$, where $\delta$ is a positive number that we will define later on. We have that

$$\langle e_j^*, S_\delta e_j \rangle \geq \delta > 0.$$

For every $x \in X$, we have that

$$\|S_\delta x\| \leq (\|T\| + \delta \|e_j^*\| \|e_j\|) \|x\|$$

and for every $1 \leq l \leq n$, we have that

$$\|(T - S_\delta)x_l\| \leq \delta \|x_l\| \|e_j^*\| \|e_j\| \quad \text{and} \quad \|(T - S_\delta)^* y_l^*\| \leq \delta \|y_l^*\| \|e_j^*\| \|e_j\|.$$

If we choose $\delta > 0$ such that

$$\delta < \frac{1 - \|T\|}{\|e_j^*\| \|e_j\|}, \quad \delta \|x_l\| \|e_j^*\| \|e_j\| < \varepsilon \quad \text{and} \quad \delta \|y_l^*\| \|e_j^*\| \|e_j\| < \varepsilon \quad \text{for every } 1 \leq l \leq n,$$

then the operator $S_\delta$ is a positive contraction of $\mathcal{P}_1(X) \setminus \mathcal{F}_j$ satisfying (3.4.23). This proves that each $\mathcal{P}_1(X) \setminus \mathcal{F}_j$ is SOT*-dense in $\mathcal{P}_1(X)$ and this concludes the proof of Proposition 3.4.14. □

## 3.5 Further remarks and questions

We end this article with some comments and open questions in relation to our previous results.

The first natural open question is, of course, the following.

**Question 3.5.1.** If $X = \ell_q$ with $1 < q \neq 2 < \infty$, does a typical $T \in (\mathcal{P}_1(X), \text{SOT})$ or $T \in (\mathcal{P}_1(X), \text{SOT}^*)$ have a non-trivial invariant subspace?

By [19, Corollary 5.3], the point spectrum of a typical contraction $T \in (\mathcal{B}_1(\ell_2), \text{SOT})$ is the open unit disk $\mathbb{D}$. This comes from the fact that an SOT-typical contraction on $\ell_2$ is unitarily equivalent to the infinite-dimensional backward unilateral shift operator on $\ell_2(\mathbb{Z}_+ \times \mathbb{Z}_+)$. The proof uses first the fact that a typical contraction on $\ell_2$ is a co-isometry for the SOT topology. Since this is no longer the case for an SOT-typical positive contraction on $\ell_2$ by Proposition 3.3.7, the proof given in [19] does not work for positive contractions. So the following question is still open.

**Question 3.5.2.** Is it still true that the point spectrum of an SOT-typical positive contraction on $\ell_2$ is $\mathbb{D}$?

The third question is motivated by Proposition 3.3.2.

**Question 3.5.3.** If $X = \ell_q$ with $1 < q < 2$, is it true that a typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ is such that $(2T)^*$ is hypercyclic?

Lemma 3.2.2 requires $X$ to have a monotone basis in order to be able to say that if $T_0$ is a positive operator on $X$ such that $\|T_0\| < 1$, then $\|P_N T_0 P_N\| < 1$ for every $N \geq 0$. This lemma was useful to prove Theorem 3.4.11. We thus have the following open question.

**Question 3.5.4.** Can Lemma 3.2.2 be generalized to Banach spaces admitting a basis which is not necessarily monotone?

Theorem 3.4.11 applies to the $\texttt{SOT}^*$ topology. Indeed, the proof of Fact 3.4.12 uses the $\texttt{SOT}^*$ topology to prove that each set $\mathcal{F}_{i,j,\eta,p}$ is closed is $\mathcal{P}_1(X)$. Since these sets are not necessarily $\texttt{SOT}$-closed, the following question is natural.

**Question 3.5.5.** Is the analogue of Theorem 3.4.11 still true for the $\texttt{SOT}$ topology?

It is proved in [30, Theorem 7.5] that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT}^*)$ does not commute with any non-zero compact operator on $\ell_2$. Since the proof uses unitary equivalence of operators, it does not extend to positive contractions. Thus, the following question is open.

**Question 3.5.6.** Does a typical $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ commute with a non-zero compact operator if $X = \ell_2$? And what about a typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$?

Finally, we have the following question. A positive answer to it would enlighten the situation very much.

**Question 3.5.7.** If $X = \ell_q$ with $1 < q \leq 2$, are the $\texttt{SOT}$ and the $\texttt{SOT}^*$ topologies similar on $\mathcal{P}_1(X)$?

Chapter 4

# Similar operator topologies on the space of positive contractions

*The results presented in this chapter are based on the manuscript [25], published in Illinois Journal of Mathematics.*

**Abstract.** In this article, we study the similarity of the Polish operator topologies $\texttt{WOT}$, $\texttt{SOT}$, $\texttt{SOT}_*$ and $\texttt{SOT}^*$ on the set of the positive contractions on $\ell_p$ with $p > 1$. Using the notion of norming vector for a positive operator, we prove that these topologies are similar on $\mathcal{P}_1(\ell_2)$, that is they have the same dense sets in $\mathcal{P}_1(\ell_2)$. In particular, these topologies will share the same comeager sets in $\mathcal{P}_1(\ell_2)$. We then apply these results to the study of typical properties of positive contractions on $\ell_p$-spaces in the Baire category sense. In particular, we prove that a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \texttt{SOT})$ has no eigenvalue. This stands in strong contrast to a result of Eisner and Mátrai, stating that the point spectrum of a typical contraction $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT})$ contains the whole unit disk. As a consequence of our results, we obtain that a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \texttt{WOT})$ (resp. $T \in (\mathcal{P}_1(\ell_2), \texttt{SOT}_*)$) has no eigenvalue.

## 4.1 Introduction

If $X$ is a complex Banach space, we denote by $\mathcal{B}(X)$ the space of all bounded operators on $X$ and by $\mathcal{B}_1(X)$ the set of all contractions on $X$, that is, bounded operators of norm at most 1.

If $X$ has a Schauder basis $(e_n)_{n\geq 0}$, a vector $x = \sum_{n\geq 0} x_n e_n$ of $X$ is said to be positive (written $x \geq 0$) whenever $x_n \geq 0$ for every $n \geq 0$. A bounded operator $T$ on $X$ is said to be positive (written $T \geq 0$) whenever $Tx \geq 0$ for every $x \geq 0$. If $X = \ell_p$ with $p \geq 1$ and if $x = \sum_{n\geq 0} x_n e_n$ is a vector of $X$, we denote by $|x|$ the positive vector $|x| := \sum_{n\geq 0} |x_n| e_n$, which also belongs to $X$. We call

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$|x|$ the modulus of the vector $x$. We denote by $\mathcal{P}(X)$ the set of all positive operators on $X$ and by $\mathcal{P}_1(X)$ the set of all positive contractions on $X$. We refer to [26] for more details on positive operators.

If $\tau$ is a topology on $\mathcal{Y} = \mathcal{B}_1(X)$ or $\mathcal{Y} = \mathcal{P}_1(X)$ such that $(\mathcal{Y}, \tau)$ is Polish, we say that a property $(P)$ of operators on $X$ is typical for $\tau$ if the set $\{T \in \mathcal{Y} : T \text{ satisfies } (P)\}$ is comeager in $\mathcal{Y}$, that is contains a dense $G_\delta$ subset of $\mathcal{Y}$. In this article, we will be working with the Strong Operator Topology, the Strong* Operator Topology, the Weak Operator Topology and the Dual Strong Operator Topology. The Strong Operator Topology is just the pointwise convergence topology on $\mathcal{B}(X)$, while the Strong* Operator Topology is the pointwise convergence topology for operators and their adjoints. The Weak Operator Topology is the weak pointwise convergence topology on $\mathcal{B}(X)$. Finally the Dual Strong Operator Topology is the pointwise convergence topology for the adjoints on $\mathcal{B}(X)$. We will respectively write $\texttt{SOT}$, $\texttt{SOT}^*$, $\texttt{WOT}$ and $\texttt{SOT}_*$ for these topologies. We can summarize the convergence of a sequence for these topologies in the following way:

$$\begin{cases} T_i \xrightarrow[i]{\texttt{SOT}} T \iff T_i x \xrightarrow[i]{\|.\|} Tx \quad \text{for every } x \in X, \\ T_i \xrightarrow[i]{\texttt{SOT}^*} T \iff T_i \xrightarrow[i]{\texttt{SOT}} T \quad \text{and} \quad T_i^* \xrightarrow[i]{\texttt{SOT}} T^*, \\ T_i \xrightarrow[i]{\texttt{WOT}} T \iff \langle y^*, T_i x\rangle \xrightarrow[i]{} \langle y^*, Tx\rangle \quad \text{for every } y^* \in X^* \text{ and every } x \in X, \\ T_i \xrightarrow[i]{\texttt{SOT}_*} T \iff T_i^* \xrightarrow[i]{\texttt{SOT}} T^*. \end{cases}$$

For more details on these topologies, see [30], [29], [14] and [26]. Also recall that when $X$ has a basis, the space $(\mathcal{P}_1(X), \texttt{SOT})$ is Polish, that the space $(\mathcal{P}_1(X), \texttt{SOT}^*)$ is Polish when $X^*$ is separable and that the spaces $(\mathcal{P}_1(X), \texttt{WOT})$ and $(\mathcal{P}_1(X), \texttt{SOT}_*)$ are Polish when $X$ is reflexive. This comes from the fact that $\mathcal{P}_1(X)$ is closed in $\mathcal{B}_1(X)$ for each one of these topologies.

Typical properties of Hilbert spaces contractions for the $\texttt{SOT}$, the $\texttt{SOT}^*$ and the $\texttt{WOT}$ topologies have been studied by Eisner in [18] and by Eisner and Mátrai in [19]. It was proved in [18] that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \texttt{WOT})$ is unitary, and it was proved in [19] that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT})$ is unitarily equivalent to the infinite-dimensional backward unilateral shift operator on $\ell_2(\mathbb{Z}_+ \times \mathbb{Z}_+)$. In particular, a typical contraction $T \in (\mathcal{B}_1(\ell_2), \texttt{WOT})$ (resp. $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT})$) has a non-trivial invariant subspace. Typical properties of $\ell_p$-spaces contractions for the $\texttt{SOT}$ and the $\texttt{SOT}^*$ topologies have been studied in depth in [28], [30] and [29]. In particular, it was proved that a typical contraction $T \in (\mathcal{B}_1(\ell_1), \texttt{SOT})$ (resp. $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT}^*)$) has a non-trivial invariant subspace. The study of typical properties of positive contractions on $\ell_p$-spaces for the $\texttt{SOT}$ and the $\texttt{SOT}^*$ topologies was initiated in [26], and different phenomena were exhibited there. Indeed, it was proved in [26, Proposition 3.7] that when $p > 1$, a typical positive contraction $T \in (\mathcal{P}_1(\ell_p), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(\ell_p), \texttt{SOT}^*)$) is not a co-isometry, whereas a typical contraction $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT})$ is a co-isometry. In particular, it cannot be asserted as in [19, Theorem 5.2] that a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \texttt{SOT})$ is unitarily equivalent to the infinite-dimensional backward unilateral shift operator on $\ell_2(\mathbb{Z}_+ \times \mathbb{Z}_+)$. The point spectrum of an $\texttt{SOT}$-typical (resp. $\texttt{WOT}$-typical) positive contraction on $\ell_2$ is still unknown, whereas a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \texttt{SOT}^*)$ is such that $2T$ and $2T^*$ are hypercyclic by [26, Proposition 3.2], so the point spectrum of a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \texttt{SOT}^*)$ is empty. Finally by [19, Corollary 5.3], a typical contraction $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT})$ has the open unit disk as point spectrum.

The notion of similar operator topologies was studied in [29] by Grivaux, Matheron and Menet. One advantage of similar topologies is that these topologies have the same comeager sets. It was proved in [29] that when $p > 2$, the topologies $\texttt{SOT}^*$ and $\texttt{SOT}$ are similar on $\mathcal{B}_1(\ell_p)$

and the topologies WOT and $\mathrm{SOT}_*$ are similar on $\mathcal{B}_1(\ell_p)$, and that when $1 < p < 2$, the topologies WOT and SOT are similar on $\mathcal{B}_1(\ell_p)$ and the topologies $\mathrm{SOT}_*$ and $\mathrm{SOT}^*$ are similar on $\mathcal{B}_1(\ell_p)$. It was pointed out in [26, Theorem 2.6] that this is still the case in $\mathcal{P}_1(\ell_p)$, since [30, Lemma 5.17] remains true for positive contractions and since the proof of [29, Theorem 3.4] is true for positive contractions as long as [30, Lemma 5.17] holds for positive contractions. The proof of [29, Theorem 3.4] uses appropriately the notion of norming vector for an operator.

The aim of this article is to link the comeager sets of the topologies WOT, SOT, $\mathrm{SOT}_*$ and $\mathrm{SOT}^*$ on $\mathcal{P}_1(\ell_2)$ by showing that these topologies are similar on $\mathcal{P}_1(\ell_2)$. This is strikingly different from what happens on $\mathcal{B}_1(\ell_2)$, where the topologies WOT, SOT, $\mathrm{SOT}_*$ and $\mathrm{SOT}^*$ are far from being similar. Similarity of these topologies on $\mathcal{P}_1(\ell_2)$ will allow us to deduce some properties of a typical positive contraction on $\ell_2$ for the SOT, the WOT and the $\mathrm{SOT}_*$ topologies. In particular, we will be able to identify the point spectrum of a typical positive contraction $T$ on $\ell_2$ for the WOT and the SOT topologies. This answers an open problem from [26, Question 5.2].

### 4.1a Notations

The following notation will be used throughout this article.

- The open unit disk of $\mathbb{C}$ will be denoted by $\mathbb{D}$.

- If $X$ is a Banach space, we denote by $S_X$ the unit sphere of $X$.

- The closed linear span of a family of vectors $(x_i)_{i\in I} \subseteq X$ will be written as $[x_i : i \in I]$.

- Let $(e_n)_{n\geq 0}$ be the canonical basis of $\ell_p$ for $p \geq 1$. We denote by $E_N$ the subspace $[e_0, \dots, e_N]$ for every $N \geq 0$ and by $F_N$ the subspace $[e_j : j > N]$ for every $N \geq 0$.

- For every $N \geq 0$, we denote by $P_N$ the canonical projection onto $E_N$ and by $Q_N$ the canonical projection onto $F_N$. The biorthogonal functionals sequence associated to the basis $(e_n)_{n\geq 0}$ will be denoted by $(e_n^*)_{n\geq 0}$.

- If $T \in \mathcal{B}(X)$, we respectively write $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_p(T)$ and $\sigma_{ess}(T)$ for the spectrum, the approximate spectrum, the point spectrum and the essential spectrum of $T$.

- If $T$ is an operator on $\ell_2$, we write $t_{i,j} := \langle e_i, Te_j\rangle$ for every $i, j \geq 0$ and if $u$ is a vector of $\ell_2$, we write $u_k := \langle e_k, u\rangle$ for every $k \geq 0$.

- If $u$ is a vector of $\ell_p$, we denote by $\mathrm{Supp}(u) := \{j \geq 0 : \langle e_j, u\rangle \neq 0\}$ the support of $u$.

### 4.1b Main results

Given two topologies $\tau$ and $\tau'$ on a topological space $\mathbf{Y}$, we say that these topologies are similar if they have the same dense sets in $\mathbf{Y}$. It is well-known (see for instance [29, Lemma 2.1]) that two similar topologies have the same comeager sets in $\mathbf{Y}$, but the converse is not always true as mentioned in [29, Remark 2.2]. The article [29] studies the similarity of the Polish operator topologies WOT, SOT, $\mathrm{SOT}_*$ and $\mathrm{SOT}^*$ in $\mathcal{B}_1(\ell_p)$ for $p > 1$. A main tool in this investigation is the use of continuity points of the identity map from $(\mathcal{B}_1(\ell_p), \tau)$ into $(\mathcal{B}_1(\ell_p), \tau')$, where $\tau \subseteq \tau'$ are two distinct topologies among the topologies WOT, SOT, $\mathrm{SOT}_*$ and $\mathrm{SOT}^*$.

One of the main results of the article [29] is the following one, which only concerns the cases $1 < p < 2$ and $p > 2$.

**Theorem 4.1.1** ([29, Theorem 3.4]). *Let $X = \ell_p$ with $p > 1$. If $1 < p < 2$, the topologies* WOT *and* SOT *are similar on $\mathcal{B}_1(X)$ and the topologies* $\mathrm{SOT}_*$ *and* $\mathrm{SOT}^*$ *are similar on $\mathcal{B}_1(X)$. If $p > 2$, the topologies* SOT *and* $\mathrm{SOT}^*$ *are similar on $\mathcal{B}_1(X)$ and the topologies* WOT *and* $\mathrm{SOT}_*$ *are similar on $\mathcal{B}_1(X)$.*

Regarding the case $p = 2$, the topologies $\mathtt{WOT}$, $\mathtt{SOT}$, $\mathtt{SOT}_*$ and $\mathtt{SOT}^*$ are far from being similar on $\mathcal{B}_1(\ell_2)$. Indeed, it was proved in [19, Corollary 5.3] that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ is such that $\sigma_p(T) = \mathbb{D}$, whereas a typical contraction $(\mathcal{B}_1(\ell_2), \mathtt{SOT}^*)$ is such that $\sigma_p(T) = \emptyset$ by [30, Proposition 7.1]. The latter property comes from the fact that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT}^*)$ is such that $2T^*$ is hypercyclic ([28, Proposition 2.3]), and it is very specific to the $\mathtt{SOT}^*$ topology. It was also proved in [19, Theorem 3.1] that a typical contraction $T \in (\mathcal{B}_1(\ell_2), \mathtt{WOT})$ is unitary, whereas a typical contraction $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ is not invertible by [30, Corollary 3.5].

The proof of Theorem 4.1.1 uses the following result, proved in [29, Corollary 7.4], and stated in a weaker form here.

**Proposition 4.1.2.** *Let $\mathbf{Y}$ be a set and let $\tau$ and $\tau'$ be two topologies on $\mathbf{Y}$. Suppose that the topologies $\tau$ and $\tau'$ are Polish on $\mathbf{Y}$ and such that $\tau \subseteq \tau'$. Then the following assertions are equivalent.*

*(a) The topologies $\tau$ and $\tau'$ are similar on $\mathbf{Y}$;*

*(b) The set of all points of continuity of the identity map from $(\mathbf{Y}, \tau)$ into $(\mathbf{Y}, \tau')$ is $\tau'$-dense in $\mathbf{Y}$.*

In this article, we study the similarity of the operator topologies $\mathtt{WOT}$, $\mathtt{SOT}$, $\mathtt{SOT}_*$ and $\mathtt{SOT}^*$ on $\mathcal{P}_1(\ell_p)$ for $p > 1$ and especially for $p = 2$. One of our tools will also be the points of continuity of the identity map on $\mathcal{P}_1(\ell_p)$ and, more precisely, Proposition 4.1.2.

The following result will also be used many times in this article and was proved in [30, Proposition 5.15] for contractions. It is mentioned in [26, Subsection 2d.] that this Proposition remains true for positive contractions, since [30, Lemma 5.17] remains true for positive contractions (see for instance [26, Lemma 2.7]). Before stating it, we will be using the following convention.

Let $X = \ell_p$ with $p > 1$. If $T$ is an operator on $X$, we identify the operator $P_N T P_N$ on $X$ with the operator $P_N T_{|E_N}$ on $E_N$ and if $A$ is an operator on $E_N$, we identify $A$ with the operator $P_N A P_N$ on $X$. We can now state the corresponding result.

**Proposition 4.1.3.** *Let $X = \ell_p$ with $p > 1$. Let $N \geq 0$ and let $A \in \mathcal{P}_1(E_N)$. For any $\varepsilon > 0$, there exists an operator $B \in \mathcal{P}_1(E_{2N+1})$ satisfying*

*(i) $\|B\| = 1$;*

*(ii) $B$ has a norming vector $u \in E_{2N+1}$ for which $Supp(u) = Supp(Bu) = \{0, ..., 2N+1\}$;*

*(iii) $\|BP_N - A\| < \varepsilon$.*

*Moreover, if $\|A\|$ is sufficiently close to 1, then one may require that in fact*

*(i') $\|B - A\| < \varepsilon$.*

Our main results in this article are the following. The first one is the exact analogue of Theorem 4.1.1 but in the positive setting.

**Theorem 4.1.4.** *Let $X = \ell_p$ with $p > 1$. If $1 < p < 2$, the topologies $\mathtt{WOT}$ and $\mathtt{SOT}$ are similar on $\mathcal{P}_1(X)$ and the topologies $\mathtt{SOT}_*$ and $\mathtt{SOT}^*$ are similar on $\mathcal{P}_1(X)$. If $p > 2$, the topologies $\mathtt{SOT}$ and $\mathtt{SOT}^*$ are similar on $\mathcal{P}_1(X)$ and the topologies $\mathtt{WOT}$ and $\mathtt{SOT}_*$ are similar on $\mathcal{P}_1(X)$.*

The second one concerns the case $p = 2$ and is not true for contractions on $\ell_2$. This result also highlights the fact that the notion of typical property in $\mathcal{P}_1(\ell_2)$ is very different from the notion of typical property in $\mathcal{B}_1(\ell_2)$ for each one of the topologies $\mathtt{WOT}$, $\mathtt{SOT}$, $\mathtt{SOT}_*$ and $\mathtt{SOT}^*$.

**Theorem 4.1.5.** *Any Polish operator topology lying between the topologies* $\texttt{WOT}$ *and* $\texttt{SOT}^*$ *is similar to* $\texttt{WOT}$ *and* $\texttt{SOT}^*$ *in* $\mathcal{P}_1(\ell_2)$*. In particular the topologies* $\texttt{WOT}$, $\texttt{SOT}_*$, $\texttt{SOT}$ *and* $\texttt{SOT}^*$ *are similar on* $\mathcal{P}_1(\ell_2)$.

Using Theorem 4.1.5, we give a negative answer to Question 5.2 in [26].

**Theorem 4.1.6.** *Let* $\tau$ *be any Polish topology on* $\mathcal{P}_1(\ell_2)$ *lying between* $\texttt{WOT}$ *and* $\texttt{SOT}^*$*. A typical positive contraction* $T \in (\mathcal{P}_1(\ell_2), \tau)$ *is such that* $T$ *and* $T^*$ *have no eigenvalue.*

The existence of a non-trivial invariant subspace for a typical positive contraction on $\ell_2$ for the $\texttt{SOT}$ and the $\texttt{SOT}^*$ topologies was investigated in [26]. Thanks to this investigation and thanks to Theorem 4.1.5, we also obtain the following result.

**Theorem 4.1.7.** *Let* $\tau$ *be any Polish topology on* $\mathcal{P}_1(\ell_2)$ *lying between* $\texttt{WOT}$ *and* $\texttt{SOT}^*$*. A typical positive contraction* $T \in (\mathcal{P}_1(\ell_2), \tau)$ *has a non-trivial invariant subspace.*

In Section 4.4, we propose a partial description of the points of continuity of the identity maps $\mathbf{i}_{\texttt{SOT},\texttt{SOT}^*} : (\mathcal{P}_1(\ell_2), \texttt{SOT}) \to (\mathcal{P}_1(\ell_2), \texttt{SOT}^*)$, $\mathbf{i}_{\texttt{WOT},\texttt{SOT}_*} : (\mathcal{P}_1(\ell_2), \texttt{WOT}) \to (\mathcal{P}_1(\ell_2), \texttt{SOT}_*)$ and $\mathbf{i}_{\texttt{WOT},\texttt{SOT}} : (\mathcal{P}_1(\ell_2), \texttt{WOT}) \to (\mathcal{P}_1(\ell_2), \texttt{SOT})$. The description of these sets of points of continuity is interesting to study typical properties of positive contractions since these sets are dense-$G_\delta$ in $\mathcal{P}_1(\ell_2)$ for the corresponding topologies (see [29, Corollary 2.10]). Although the full description of these sets in the non-positive setting is quite simple (see [29, Proposition 2.11]), we will see that such a description in the positive setting is more delicate.

Our main results in Section 4.4 are the following. The first one gives a partial description of the sets of points of continuity of the identity maps for the topologies $(\texttt{SOT}, \texttt{SOT}^*)$ and $(\texttt{WOT}, \texttt{SOT}_*)$ on $\mathcal{P}_1(\ell_2)$.

**Theorem 4.1.8.** *Let* $\mathcal{M}$ *be the class of every positive contractions* $T$ *on* $\ell_2$ *such that*

- $\|T\| = 1$;
- *there exists a family* $(u_r)_{r\in J}$ *of norming vectors for* $T^*$ *indexed by a set* $J \subseteq \mathbb{Z}_+$ *such that* $u_r \geq 0$, $\|u_r\| = 1$ *for every* $r \in J$ *and* $\displaystyle\bigcup_{r\in J} Supp(u_r) = \mathbb{Z}_+$.

*Then every operator in* $\mathcal{M}$ *is a point of continuity of both identity maps* $\mathbf{i}_{\texttt{WOT},\texttt{SOT}_*}$ *and* $\mathbf{i}_{\texttt{SOT},\texttt{SOT}^*}$ *on* $\mathcal{P}_1(\ell_2)$.

The second result concerns the topologies $(\texttt{WOT}, \texttt{SOT})$ on $\mathcal{P}_1(\ell_2)$ and will be obtained from Theorem 4.1.8.

**Theorem 4.1.9.** *Let* $\mathcal{M}'$ *be the class of every positive contractions* $T$ *on* $\ell_2$ *such that*

- $\|T\| = 1$;
- *there exists a family* $(u_r)_{r\in J}$ *of norming vectors for* $T$ *indexed by a set* $J \subseteq \mathbb{Z}_+$ *such that* $u_r \geq 0$, $\|u_r\| = 1$ *for every* $r \in J$ *and* $\displaystyle\bigcup_{r\in J} Supp(u_r) = \mathbb{Z}_+$.

*Then every operator in* $\mathcal{M}'$ *is a point of continuity of the identity map* $\mathbf{i}_{\texttt{WOT},\texttt{SOT}}$ *on* $\mathcal{P}_1(\ell_2)$.

We end Section 4.4 by proving that the classes $\mathcal{M}$ and $\mathcal{M}'$ introduced in Theorems 4.1.8 and 4.1.9 are large classes of points of continuity, but we also show that the elements of $\mathcal{M}$ (resp. $\mathcal{M}'$) are not all the points of continuity of the maps $\mathbf{i}_{\texttt{WOT},\texttt{SOT}_*}$ and $\mathbf{i}_{\texttt{SOT},\texttt{SOT}^*}$ (resp. $\mathbf{i}_{\texttt{WOT},\texttt{SOT}}$) on $\mathcal{P}_1(\ell_2)$.

## 4.2 Similar operator topologies and applications to typical properties of positive contractions

### 4.2a Points of continuity of the identity map, norming vector for a positive operator

The notion of points of continuity of the identity map on $\mathcal{B}_1(\ell_p)$ was used in [29] to study the similarity of the topologies $\mathtt{WOT}$, $\mathtt{SOT}$, $\mathtt{SOT}_*$ and $\mathtt{SOT}^*$ in $\mathcal{B}_1(\ell_p)$. In this article, Proposition 4.1.2 will also play an important role to prove Theorems 4.1.4 and 4.1.5.

Let $X = \ell_p$ with $p > 1$. If $\tau$ and $\tau'$ are two Polish topologies on $\mathcal{P}_1(X)$, we denote by $\mathcal{C}(\tau, \tau')$ the points of continuity of the identity map $\mathbf{i}_{\tau,\tau'} : (\mathcal{P}_1(X), \tau) \to (\mathcal{P}_1(X), \tau')$. If $\tau, \tau' \in \{\mathtt{WOT}, \mathtt{SOT}_*, \mathtt{SOT}, \mathtt{SOT}^*\}$ are such that $\tau \subseteq \tau'$, then $\mathcal{C}(\tau, \tau')$ is $\tau$-$G_\delta$ dense in $\mathcal{P}_1(X)$ (see, for instance, [29, Corollary 2.10]).

The notion of norming vector for an operator was also very useful in [29] and will also play an important role in this article. For example, this notion appears in Proposition 4.1.3.

Recall that a vector $x \neq 0$ of $X = \ell_p$ is a norming vector for an operator $T \in \mathcal{B}(X)$ if $\|Tx\| = \|T\|\|x\|$. If $x$ is a norming vector for $T$ and if $T$ is a positive operator, then $|x|$ is also a norming vector for $T$ with the same support as $x$. Indeed if $x \in S_X$ is a norming vector for $T$, then $\|T\| = \|Tx\| = \||Tx|\| \leq \|T|x|\| \leq \|T\|$, and so $\|T|x|\| = \|T\|$. Here we used the fact that the norm is increasing on $\ell_p$, as well as the fact that the norm of a vector is the norm of its modulus. The set of all norming vectors for $T \in \mathcal{B}(X)$ will be written $\mathcal{N}(T)$. If $\mathcal{N}(T) \neq \emptyset$, we say that the operator $T$ attains its norm. Also notice that there exist positive operators on $\ell_p$ which do not attain their norm. For example, if $a = (a_n)_{n\geq 0}$ is a sequence of real numbers in $(0, 1)$ such that $a_n \underset{n\to\infty}{\longrightarrow} 1$, then the positive diagonal operator $\Delta_a$ associated to the sequence $(a_n)_{n\geq 0}$ does not attain its norm.

The following proposition is a well-known fact and will be useful for our study. For the convenience of the reader, we recall here the proof of this fact.

**Proposition 4.2.1.** *Let $X = \ell_2$, let $T \in \mathcal{B}(X)$ and let $u \in X$ be a norming vector for $T$. Then $T^*Tu = \|T\|^2 u$. In particular if $\mathcal{N}(T) \neq \emptyset$, then $u + v \in \mathcal{N}(T)$ and $\lambda u \in \mathcal{N}(T)$ for every $u, v \in \mathcal{N}(T)$ and for every $\lambda \in \mathbb{C}$.*

*Proof.* We can suppose that $\|u\| = 1$.

We clearly have $\|T^*Tu\| \leq \|T\|^2$. Moreover, we have

$$\|T\|^2 = \|Tu\|^2 = \langle Tu, Tu\rangle = \langle T^*Tu, u\rangle \leq \|T^*Tu\|$$

and thus $\|T^*Tu\| = \|T\|^2$. By the equality case of the Cauchy–Schwarz inequality, there is a scalar $\alpha$ such that $T^*Tu = \alpha u$, and we must have $\alpha = \|T\|^2$. This proves Proposition 4.2.1. □

**Remark 4.2.2.** Conversely, if there exists a vector $u \in \ell_2$ such that $\|u\| = 1$ and $T^*Tu = \|T\|^2 u$, then $u \in \mathcal{N}(T)$ since $\|Tu\|^2 = \langle T^*Tu, u\rangle = \|T\|^2$. In particular if a vector $u \in \ell_2$ is norming for $T$, then $Tu$ is a norming vector for $T^*$.

### 4.2b Proof of Theorem 4.1.4

The proof of Theorem 4.1.1 uses the following proposition, which is a consequence of [30, Proposition 5.15]. It is the analog of Proposition 4.1.3 but for contractions instead of positive contractions.

**Proposition 4.2.3.** *Let $p > 1$ and let $X = \ell_p$. Let $\mathcal{U}$ be a non-empty* $\mathtt{SOT}^*$*-open set in $\mathcal{B}_1(X)$ and let $n_0 \geq 0$. Then one can find an index $M \geq n_0$ and an operator $B \in \mathcal{B}_1(E_M)$ such that $B \in \mathcal{U}$ and such that $B$ satisfies the following properties: $\|B\| = 1$ and $B$ has a norming vector $u \in E_M$ for which $Supp(Bu) = \{0,\dots,M\}$.*

The proof of Theorem 4.1.1 also uses the following result, which follows from [29, Lemma 3.3] and also works for positive contractions.

**Proposition 4.2.4.** *Assume that $p > 2$. Let $M \geq 0$ and let $B \in \mathcal{B}_1(E_M)$. Suppose that $\|B\| = 1$ and that $B$ has a norming vector $u \in E_M$ for which $Supp(Bu) = \{0,\dots,M\}$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that the following property holds: if $T \in \mathcal{B}_1(\ell_p)$ is such that $\|P_M(T-B)P_M\| < \delta$, then $\|P_M(T-B)\| < \varepsilon$.*

In order to adapt the proof of [29, Theorem 3.4] in the positive setting, that is in order to prove Theorem 4.1.4, we only have to show that Proposition 4.2.3 is working for positive contractions. This is the aim of the following proposition.

**Proposition 4.2.5.** *Let $p > 1$ and let $X = \ell_p$. Let $\mathcal{U}$ be a non-empty* $\mathtt{SOT}^*$*-open set in $\mathcal{P}_1(X)$ and let $n_0 \geq 0$. Then one can find an index $M \geq n_0$ and a positive operator $B \in \mathcal{P}_1(E_M)$ such that $B \in \mathcal{U}$ and such that $B$ satisfies the following properties: $\|B\| = 1$ and $B$ has a norming vector $u \in E_M$ for which $Supp(Bu) = \{0,\dots,M\}$.*

*Proof.* Since the set $S^+(X) := \{T \in \mathcal{P}_1(X) : \|T\| = 1\}$ is $\mathtt{SOT}^*$-dense in $\mathcal{P}_1(X)$ by [26, Proposition 3.1], there exists a positive operator $A$ in $\mathcal{U} \cap S^+(X)$. Let $\varepsilon > 0$. There exists a vector $x \in S_X$ such that $\|Ax\| > 1-\varepsilon$. Moreover, one can find an integer $n > n_0$ such that $A_n := P_nAP_n$ belongs to $\mathcal{U}$ and $\|A_nx\| > \|Ax\| - \varepsilon > 1-2\varepsilon$. Thus the norm of $A_n$ is sufficiently close to 1 and, applying Proposition 4.1.3, one can find an index $M > n_0$ and a positive operator $B \in \mathcal{P}_1(E_M)$ in $\mathcal{U}$ satisfying all the properties listed in Proposition 4.2.5. This concludes the proof of Proposition 4.2.5. □

By Propositions 4.2.4 and 4.2.5, the proof of Theorem 4.1.1 given in [29, Theorem 3.4] also applies to positive contractions, and thus one can state the following theorem.

**Theorem 4.2.6.** *Let $X = \ell_p$ with $p > 1$. If $1 < p < 2$, the topologies* $\mathtt{WOT}$ *and* $\mathtt{SOT}$ *are similar on $\mathcal{P}_1(X)$ and the topologies* $\mathtt{SOT}_*$ *and* $\mathtt{SOT}^*$ *are similar on $\mathcal{P}_1(X)$. If $p > 2$, the topologies* $\mathtt{SOT}$ *and* $\mathtt{SOT}^*$ *are similar on $\mathcal{P}_1(X)$ and the topologies* $\mathtt{WOT}$ *and* $\mathtt{SOT}_*$ *are similar on $\mathcal{P}_1(X)$.*

### 4.2c Proof of Theorem 4.1.5

We now focus on the case $p = 2$. As mentioned in the introduction, the topologies $\mathtt{WOT}$, $\mathtt{SOT}$, $\mathtt{SOT}_*$ and $\mathtt{SOT}^*$ are far from being similar on $\mathcal{B}_1(\ell_2)$. It appears that the situation is considerably different for positive contractions as we will see. This is not surprising since a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$ is not a co-isometry by [26, Proposition 3.7], and so we cannot apply the Wold decomposition to a typical positive contraction on $\ell_2$ as in [19, Theorem 5.2].

In order to prove Theorem 4.1.5, we need to adapt Proposition 4.2.4 to positive contractions on $\ell_2$. This is the aim of the next result.

**Proposition 4.2.7.** *Let $M \geq 0$ and let $B \in \mathcal{P}_1(E_M)$. Suppose that $\|B\| = 1$ and that $B^*$ has a positive norming vector $u \in E_M$ for which $Supp(u) = \{0,\dots,M\}$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that the following property holds: if $T \in \mathcal{P}_1(\ell_2)$ is such that $\max\limits_{0\leq k,l\leq M} |\langle e_k, (T-B)e_l\rangle| < \delta$, then $\max\limits_{0\leq k\leq M} \|(T-B)^*e_k\| < \varepsilon$.*

*Proof.* We can suppose that $\|u\| = 1$. We also set $u_l := \langle e_l, u\rangle$ for every $0 \le l \le M$ and $b_{i,j} := \langle e_i, Be_j\rangle$ for every $0 \le i, j \le M$.

Let $\varepsilon > 0$ and let $T \in \mathcal{P}_1(\ell_2)$ be such that $\max_{0\le k,l\le M} |\langle e_k, (T-B)e_l\rangle| < \delta$, where $\delta > 0$ is such that

$$b_{i,j} - \delta > 0 \quad \text{for every} \quad 0 \le i, j \le M \quad \text{such that } b_{i,j} > 0, \tag{4.2.1}$$

$$\text{and} \quad 2\delta \max_{0\le l\le M} \frac{1}{u_l^2}\left(\sum_{j=0}^{M} u_j^2 \sum_{k=0}^{M} b_{j,k} + \sum_{0\le i<j\le M} u_i u_j \sum_{k=0}^{M} (b_{i,k} + b_{j,k})\right) < \varepsilon. \tag{4.2.2}$$

Condition (4.2.2) makes sense since $u_l > 0$ for every $0 \le l \le M$, by the assumption on the support of the norming vector $u$.

In order to prove that $\max_{0\le k\le M} \|(T-B)^* e_k\|^2 < \varepsilon$, we first need to control each quantity $\|T^* e_l\|^2$, $0 \le l \le M$.

Using the parallelogram identity, one has

$$1 \ge \left\|T^*\left(\sum_{j=0}^{M} u_j e_j\right)\right\|^2 = \sum_{j=0}^{M} u_j^2 \|T^* e_j\|^2 + 2 \sum_{0\le i<j\le M} u_i u_j \langle T^* e_i, T^* e_j\rangle. \tag{4.2.3}$$

Since the operator $T$ is positive, each quantity $\langle T^* e_i, T^* e_j\rangle$, $0 \le i, j \le M$, is non-negative. By positivity and since $\max_{0\le k,l\le M} |\langle e_k, (T-B)e_l\rangle| < \delta$, one also has

$$\langle T^* e_i, T^* e_j\rangle \ge \sum_{\substack{0\le k\le M \\ b_{i,k}>0,\, b_{j,k}>0}} (b_{i,k} - \delta)(b_{j,k} - \delta) \tag{4.2.4}$$

for every $0 \le i, j \le M$.

Thus using (4.2.3), (4.2.4) and using the positivity of the vector $u$, we can control $\|T^* e_l\|^2$ for every $0 \le l \le M$ in the following way:

$$u_l^2 \|T^* e_l\|^2 \le 1 - \sum_{\substack{0\le j\le M \\ j\ne l}} u_j^2 \|T^* e_j\|^2 - 2 \sum_{0\le i<j\le M} u_i u_j \sum_{\substack{0\le k\le M \\ b_{i,k}>0,\, b_{j,k}>0}} (b_{i,k} - \delta)(b_{j,k} - \delta). \tag{4.2.5}$$

Exactly as (4.2.4), one has

$$\|T^* e_j\|^2 \ge \sum_{\substack{0\le k\le M \\ b_{j,k}>0}} (b_{j,k} - \delta)^2 \tag{4.2.6}$$

for every $0 \le j \le M$.

Thus using (4.2.5) and (4.2.6), one obtains for every $0 \le l \le M$

$$\begin{aligned} u_l^2 \|T^* e_l\|^2 \le 1 &- \sum_{\substack{0\le j\le M \\ j\ne l}} u_j^2 \sum_{\substack{0\le k\le M \\ b_{j,k}>0}} (b_{j,k} - \delta)^2 \\ &- 2 \sum_{0\le i<j\le M} u_i u_j \sum_{\substack{0\le k\le M \\ b_{i,k}>0,\, b_{j,k}>0}} (b_{i,k} - \delta)(b_{j,k} - \delta). \end{aligned} \tag{4.2.7}$$

In order to prove that $\max\limits_{0\le k\le M}\|(T-B)^*e_k\|^2<\varepsilon$, we also need to control $\langle T^*e_l,B^*e_l\rangle$ for every $0\le l\le M$. But as (4.2.4),

$$\langle T^*e_l,B^*e_l\rangle\ge\sum_{\substack{0\le k\le M\\ b_{l,k}>0}} b_{l,k}(b_{l,k}-\delta) \tag{4.2.8}$$

for every $0\le l\le M$.

Thus combining (4.2.7) and (4.2.8), the expression

$$u_l^2\|T^*e_l-B^*e_l\|^2=u_l^2\|T^*e_l\|^2-2u_l^2\langle T^*e_l,B^*e_l\rangle+u_l^2\|B^*e_l\|^2 \tag{4.2.9}$$

is bounded above for every $0\le l\le M$ by

$$\begin{aligned}1&-\sum_{\substack{0\le j\le M\\ j\neq l}}u_j^2\sum_{\substack{0\le k\le M\\ b_{j,k}>0}}(b_{j,k}-\delta)^2-2\sum_{0\le i<j\le M}u_iu_j\sum_{\substack{0\le k\le M\\ b_{i,k}>0,\,b_{j,k}>0}}(b_{i,k}-\delta)(b_{j,k}-\delta)\\ &-2u_l^2\sum_{\substack{0\le k\le M\\ b_{l,k}>0}}b_{l,k}(b_{l,k}-\delta)+u_l^2\sum_{0\le k\le M}b_{l,k}^2.\end{aligned} \tag{4.2.10}$$

Now putting together the terms in (4.2.10), the expression (4.2.9) is bounded above for every $0\le l\le M$ by

$$\begin{aligned}1&-\left(\sum_{j=0}^M u_j^2\sum_{k=0}^M b_{j,k}^2+2\sum_{0\le i<j\le M}u_iu_j\sum_{k=0}^M b_{i,k}b_{j,k}\right)\\ &+2\delta\left(\sum_{j=0}^M u_j^2\sum_{k=0}^M b_{j,k}+\sum_{0\le i<j\le M}u_iu_j\sum_{k=0}^M(b_{i,k}+b_{j,k})\right).\end{aligned} \tag{4.2.11}$$

Let us remark that

$$\begin{aligned}\sum_{k=0}^M\langle u,Be_k\rangle^2&=\sum_{k=0}^M\left(\sum_{j=0}^M u_jb_{j,k}\right)^2\\ &=\sum_{j=0}^M u_j^2\sum_{k=0}^M b_{j,k}^2+2\sum_{0\le i<j\le M}u_iu_j\sum_{k=0}^M b_{i,k}b_{j,k}.\end{aligned}$$

Thus using (4.2.11), the expression (4.2.9) can be bounded above for every $0\le l\le M$ by

$$\left(1-\sum_{j=0}^M\langle u,Be_j\rangle^2\right)+2\delta\left(\sum_{j=0}^M u_j^2\sum_{k=0}^M b_{j,k}+\sum_{0\le i<j\le M}u_iu_j\sum_{k=0}^M(b_{i,k}+b_{j,k})\right). \tag{4.2.12}$$

Now noticing that

$$\sum_{j=0}^M\langle u,Be_j\rangle^2=\|B^*u\|^2=1,$$

it follows that the expression (4.2.9) is bounded above by

$$2\delta\left(\sum_{j=0}^{M} u_j^2 \sum_{k=0}^{M} b_{j,k} + \sum_{0\le i<j\le M} u_i u_j \sum_{k=0}^{M}(b_{i,k}+b_{j,k})\right) \tag{4.2.13}$$

for every $0 \le l \le M$.

From (4.2.1) and (4.2.2), it easily follows that $\max_{0\le k\le M} \|(T-B)^* e_k\|^2 < \varepsilon$. This concludes the proof of Proposition 4.2.7. □

Now with Propositions 4.2.5 and 4.2.7, we are able to prove the following theorem regarding the case $p = 2$ in the positive setting.

**Theorem 4.2.8.** *Any Polish operator topology on $\mathcal{P}_1(\ell_2)$ lying between the topologies* $\mathtt{WOT}$ *and* $\mathtt{SOT}^*$ *is similar to* $\mathtt{WOT}$ *and* $\mathtt{SOT}^*$ *in $\mathcal{P}_1(\ell_2)$.*

*Proof.* In order to prove Theorem 4.2.8, it is enough to prove that the topologies $\mathtt{WOT}$ and $\mathtt{SOT}^*$ are similar on $\mathcal{P}_1(\ell_2)$.

If we prove that the set $\mathcal{C}(\mathtt{WOT},\mathtt{SOT}_*)$ is $\mathtt{SOT}^*$-dense in $\mathcal{P}_1(\ell_2)$, then the set $\mathcal{C}(\mathtt{SOT},\mathtt{SOT}^*)$ will also be $\mathtt{SOT}^*$-dense in $\mathcal{P}_1(\ell_2)$, since $\mathcal{C}(\mathtt{WOT},\mathtt{SOT}_*) \subseteq \mathcal{C}(\mathtt{SOT},\mathtt{SOT}^*)$. Since the map $T \mapsto T^*$ is a homeomorphism from $(\mathcal{C}(\mathtt{WOT},\mathtt{SOT}_*),\mathtt{SOT}^*)$ onto $(\mathcal{C}(\mathtt{WOT},\mathtt{SOT}),\mathtt{SOT}^*)$ (see Proposition 4.4.1 for more details), the set $\mathcal{C}(\mathtt{WOT},\mathtt{SOT})$ will also be $\mathtt{SOT}^*$-dense in $\mathcal{P}_1(\ell_2)$. Using Proposition 4.1.2, and using the fact that the relation "being similar to" is a transitive relation on the set of Polish topologies on $\mathcal{P}_1(\ell_2)$, the topologies $\mathtt{WOT}$ and $\mathtt{SOT}^*$ will be similar on $\mathcal{P}_1(\ell_2)$.

Hence to prove Theorem 4.2.8, it suffices to prove that $\mathcal{C}(\mathtt{WOT},\mathtt{SOT}_*)$ is $\mathtt{SOT}^*$-dense in $\mathcal{P}_1(\ell_2)$.

It is not difficult to show that the metric

$$d : (S,T) \mapsto \sum_{n=0}^{\infty} 2^{-n} \|(T-S)^* e_n\|$$

generates the $\mathtt{SOT}_*$ topology on $\mathcal{P}_1(\ell_2)$. Thus one has

$$\mathcal{C}(\mathtt{WOT},\mathtt{SOT}_*) = \bigcap_{\eta>0} \mathcal{V}_\eta = \bigcap_{k\ge 1} \mathcal{V}_{1/k},$$

where

$$\mathcal{V}_\eta := \{T \in \mathcal{P}_1(\ell_2) : \text{there exists a } \mathtt{WOT}\text{-neighborhood } \mathcal{W} \text{ of } T \text{ such that } diam(\mathcal{W}) < \eta\}$$

and where $diam(\mathcal{W})$ is the diameter of $\mathcal{W}$ for the distance $d$. Every set $\mathcal{V}_\eta$ is $\mathtt{WOT}$-open and hence $\mathtt{SOT}^*$-open in $\mathcal{P}_1(\ell_2)$. Thus in order to prove that $\mathcal{C}(\mathtt{WOT},\mathtt{SOT}_*)$ is $\mathtt{SOT}^*$-dense in $\mathcal{P}_1(\ell_2)$, it is enough to prove that each $\mathcal{V}_\eta$ is $\mathtt{SOT}^*$-dense in $\mathcal{P}_1(\ell_2)$ by the Baire category theorem.

Let $\mathcal{U}$ be a non-empty $\mathtt{SOT}^*$-open set of $\mathcal{P}_1(\ell_2)$ and let $n_0 \ge 0$ be such that

$$\sum_{n>n_0} 2^{-n} < \eta/8.$$

By Proposition 4.2.5, there exist $M > n_0$ and $B \in \mathcal{P}_1(E_M)$ such that $B \in \mathcal{U}$, $\|B\| = 1$, and $B$ has a positive norming vector $z \in E_M$ for which $\mathrm{Supp}(Bz) = \{0,\dots,M\}$. Thus by Remark 4.2.2, $B^*$ has a positive norming vector $u \in E_M$ whose support is $\{0,\dots,M\}$. By Proposition 4.2.7, there exists $\delta > 0$ such that the following property holds: if $T \in \mathcal{P}_1(\ell_2)$ is such that $\max_{0\le k,l\le M} |\langle e_k,(T-B)e_l\rangle| < \delta$, then $\max_{0\le k\le M}\|(T-B)^*e_k\| < \eta/8$. Now let us set $\mathcal{W} := \{T \in \mathcal{P}_1(\ell_2) : \max_{0\le k,l\le M} |\langle e_k, (T-B)e_l\rangle| < \delta\}$. The set $\mathcal{W}$ is a WOT-neighborhood of $B$ and $diam(\mathcal{W}) < \eta$ since for every $S, T \in \mathcal{W}$, we have

$$\begin{aligned} d(S,T) &\le d(S,B) + d(B,T) \\ &\le 2\sum_{n=0}^{M} 2^{-n}\eta/8 + 4\sum_{n>M} 2^{-n} \\ &< \eta. \end{aligned}$$

This proves that $\mathcal{U} \cap \mathcal{V}_\eta \neq \emptyset$ and this concludes the proof of Theorem 4.2.8. □

## 4.3 Applications and more typical properties of positive contractions

The purpose of this subsection is to apply the results of the previous section to the study of typical properties of positive contractions on $\ell_p$-spaces for $p > 1$, and for the topologies SOT, SOT$_*$ and WOT.

The fact that any Polish topology on $\mathcal{P}_1(\ell_2)$ lying between the topologies WOT and SOT$^*$ is similar to SOT$^*$ is a very interesting result for the following reason. If the set $\{T \in \mathcal{P}_1(\ell_2) :$ $T$ satisfies the property $(P)\}$ is comeager in $(\mathcal{P}_1(\ell_2), \texttt{SOT}^*)$, then the set

$$\{T \in \mathcal{P}_1(\ell_2) : T^* \text{ satisfies the property } (P)\}$$

is also comeager in $(\mathcal{P}_1(\ell_2), \texttt{SOT}^*)$. This follows from the fact that the map $T \mapsto T^*$ is a homeomorphism from $(\mathcal{P}_1(\ell_2), \texttt{SOT}^*)$ onto $(\mathcal{P}_1(\ell_2), \texttt{SOT}^*)$. This fact remains true on $\mathcal{P}_1(\ell_2)$ if we replace the topology SOT$^*$ by the topology SOT, thanks to Theorem 4.2.8. More generally, we have the following useful fact.

**Fact 4.3.1.** Let $p > 1$ and let (P) be a property of operators on $\ell_p$-spaces.

If a typical $T \in (\mathcal{P}_1(\ell_p), \texttt{SOT})$ satisfies (P), then a typical $T \in (\mathcal{P}_1(\ell_q), \texttt{SOT}_*)$ is such that $T^*$ satisfies (P), where $q$ is the conjugate exponent of $p$. This remains true if we replace the topologies SOT and SOT$_*$ both by the topology SOT$^*$.

*Proof.* This follows from the fact that the map $T \mapsto T^*$ is a homeomorphism from $(\mathcal{P}_1(\ell_p), \texttt{SOT})$ onto $(\mathcal{P}_1(\ell_q), \texttt{SOT}_*)$. □

We used this argument in [26, Proposition 3.2] to assert that a typical positive contraction on $(\mathcal{P}_1(\ell_p), \texttt{SOT}^*)$ is such that $2T^*$ is hypercyclic, since a typical positive contraction $(\mathcal{P}_1(\ell_p), \texttt{SOT}^*)$ is such that $2T$ is hypercyclic. In particular, this was useful to find the point spectrum of a typical positive contraction on $(\mathcal{P}_1(\ell_p), \texttt{SOT}^*)$, which is empty (we refer to [6] and [31] for background on hypercyclicity).

Also, we mention the fact that studying typical properties of positive contractions (resp. of contractions) on $\mathcal{P}_1(\ell_p)$ (resp. on $\mathcal{B}_1(\ell_p)$) for the WOT topology is something not always easy. It is also one of the reasons that the notion of similar topologies was studied in [29]. Thanks to Theorems 4.2.6 and 4.2.8, we will be able to deduce some typical properties of positive contractions on $\ell_p$-spaces for the WOT, the SOT and the SOT$_*$ topologies.

### 4.3a Applications of Theorem 4.2.8

The following consequence follows from Theorem 4.2.8. It concerns the case $p = 2$ and links the comeager sets of all the Polish operator topologies $\mathtt{WOT}$, $\mathtt{SOT}_*$, $\mathtt{SOT}$ and $\mathtt{SOT}^*$ on $\mathcal{P}_1(\ell_2)$.

**Corollary 4.3.2.** *Let $p = 2$ and let $\tau$ be a Polish operator topology on $\mathcal{P}_1(\ell_2)$ lying between the topologies $\mathtt{WOT}$ and $\mathtt{SOT}^*$. Then the topologies $\tau$ and $\mathtt{SOT}^*$ have the same comeager sets in $\mathcal{P}_1(\ell_2)$.*

Corollary 4.3.2 is far from being true in $\mathcal{B}_1(\ell_2)$ as we explained in subsection 4.1b.

The main goal of this article was to give an answer to the open question [26, Question 5.2], that is to determine the point spectrum of a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$. As noted earlier, the point spectrum of a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT}^*)$ is empty because a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT}^*)$ is such that $2T$ and $2T^*$ are hypercyclic. Moreover, there are not many results on the existence of an eigenvalue for positive operators on $\ell_p$-spaces. One of the few relevant results on this subject is [52, Theorem 5.7], which states that any positive compact operator on $\ell_p$ with no non-trivial closed invariant ideals has a unique (up to scaling) positive eigenvector. Although a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$ has no non-trivial closed invariant ideals (from [26, Proposition 4.7]), a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$ is not compact since a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT})$ is such that $2T$ is hypercyclic. Thanks to Corollary 4.3.2, we obtain the following result and we give a negative answer to [26, Question 5.2].

**Theorem 4.3.3.** *Let $\tau$ be any Polish operator topology on $\mathcal{P}_1(\ell_2)$ lying between $\mathtt{WOT}$ and $\mathtt{SOT}^*$. A typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \tau)$ is such that $T$ and $T^*$ have no eigenvalue.*

*Proof.* By [26, Corollary 3.3], a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT}^*)$ is such that $T$ and $T^*$ have no eigenvalue. By Corollary 4.3.2, the conclusion of Theorem 4.3.3 immediately follows. $\square$

Another consequence of Corollary 4.3.2 is the existence of a non-trivial invariant subspace for a typical positive contraction on $\ell_2$.

**Theorem 4.3.4.** *Let $\tau$ be any Polish operator topology on $\mathcal{P}_1(\ell_2)$ lying between $\mathtt{WOT}$ and $\mathtt{SOT}^*$. A typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \tau)$ has a non-trivial invariant subspace.*

*Proof.* By [26, Proposition 3.1], a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \mathtt{SOT}^*)$ is such that $\sigma(T) = \overline{\mathbb{D}}$. Thus by Corollary 4.3.2, a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \tau)$ is also such that $\sigma(T) = \overline{\mathbb{D}}$. Now an important result from Brown, Chevreau and Pearcy states that any contraction on a Hilbert space whose spectrum contains the unit circle has a non-trivial invariant subspace (see for instance [11]). Thus the conclusion of Theorem 4.3.4 immediately follows. $\square$

### 4.3b Applications of Theorem 4.2.6

The following consequence concerns the cases $1 < p < 2$ and $p > 2$ and follows from Theorem 4.2.6.

**Corollary 4.3.5.** *Let $X = \ell_p$ with $p > 1$. If $1 < p < 2$, the topologies $\mathtt{WOT}$ and $\mathtt{SOT}$ have the same comeager sets in $\mathcal{P}_1(X)$ and the topologies $\mathtt{SOT}_*$ and $\mathtt{SOT}^*$ have the same comeager sets in $\mathcal{P}_1(X)$. If $p > 2$, the topologies $\mathtt{SOT}$ and $\mathtt{SOT}^*$ have the same comeager sets in $\mathcal{P}_1(X)$ and the topologies $\mathtt{WOT}$ and $\mathtt{SOT}_*$ have the same comeager sets in $\mathcal{P}_1(X)$.*

Using Corollary 4.3.5, we can describe the point spectrum of a typical positive contraction on $\ell_p$ with $p > 2$ for the topologies $\mathtt{WOT}$ and $\mathtt{SOT}_*$. This description is analogous to the description given in [29, Corollary 3.6].

**Proposition 4.3.6.** *Let $p > 2$ and let $X = \ell_p$. A typical $T \in (\mathcal{P}_1(X), \texttt{WOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}_*)$) has no eigenvalue.*

*Proof.* By [26, Proposition 3.2], we know that a typical $T \in (\mathcal{P}_1(\ell_p), \texttt{SOT})$ is such that $2T$ is hypercyclic when $1 < p < 2$ and so a typical $T \in (\mathcal{P}_1(\ell_p), \texttt{SOT})$ is such that $T^*$ has no eigenvalue when $1 < p < 2$. By Fact 4.3.1, it follows that a typical $T \in (\mathcal{P}_1(\ell_p), \texttt{SOT}_*)$ is such that $T$ has no eigenvalue when $p > 2$. Since the topologies $\texttt{WOT}$ and $\texttt{SOT}_*$ have the same comeager sets in $\mathcal{P}_1(\ell_p)$ when $p > 2$, Proposition 4.3.6 follows. □

The following result concerns the essential spectrum of a typical positive contraction and is also analogous to [29, Corollary 3.7].

**Proposition 4.3.7.** *Let $p > 1$ and let $X = \ell_p$. A typical $T \in (\mathcal{P}_1(X), \texttt{WOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}_*)$) is such that $\sigma_{ess}(T) = \overline{\mathbb{D}}$, and so $\sigma(T) = \overline{\mathbb{D}}$.*

*Proof.* By [26, Proposition 3.10], we know that a typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$) is such that $\sigma_{ess}(T) = \overline{\mathbb{D}}$ when $p > 1$. But the property "$\sigma_{ess}(T) = \overline{\mathbb{D}}$" is self-adjoint, so using the same arguments as in the proof of Proposition 4.3.6 and using Corollaries 4.3.5 and 4.3.2, we obtain Proposition 4.3.7. □

## 4.3c Other typical properties of positive contractions for the topologies $\texttt{WOT}$ and $\texttt{SOT}_*$

We now focus on other $\texttt{WOT}$- and $\texttt{SOT}_*$-typical properties that have been studied in [26] for the $\texttt{SOT}$ and the $\texttt{SOT}^*$ topologies. The first property follows from [26, Proposition 3.7].

**Proposition 4.3.8.** *Let $X = \ell_p$ with $p > 1$. A typical $T \in (\mathcal{P}_1(X), \texttt{WOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}_*)$) is such that $T^*$ is not an isometry.*

*Proof.* As in the proof of [26, Proposition 3.7], the set

$$\mathcal{A} := \bigcup_{j\geq 0} \{T \in \mathcal{P}_1(X) : \langle e_0^*, Te_j\rangle \langle e_1^*, Te_j\rangle > 0\}$$

is contained in the set of positive contractions that are not co-isometries of $X$. The set $\mathcal{A}$ is easily seen to be a $\texttt{WOT}$-$G_\delta$ and hence a $\texttt{SOT}_*$-$G_\delta$ of $\mathcal{P}_1(X)$. Moreover, the set $\mathcal{A}$ is $\texttt{SOT}^*$-dense in $\mathcal{P}_1(X)$ and so it is $\texttt{WOT}$-dense and $\texttt{SOT}_*$-dense in $\mathcal{P}_1(X)$. This proves Proposition 4.3.8. □

Recall that an ideal of $X = \ell_p$ is a vector subspace $V$ of $X$ such that if $|x| \leq |y|$ and $y \in V$ then $x \in V$, for every $x, y \in X$. One can show that the closed ideals of $X = \ell_p$ are exactly the sets $[e_k : k \in A]$, where $A$ is a subset of $\mathbb{Z}_+$. As mentioned in [26], there exists a criterion to determine whether a positive operator on $X = \ell_p$ has a non-trivial closed invariant ideal or not. More precisely by [52, Proposition 1.2], a positive operator $T$ on $X = \ell_p$ has no non-trivial closed invariant ideals if and only if

$$\forall\, i \neq j \geq 0, \exists\, n \geq 0 \quad \text{such that} \quad \langle e_j^*, T^n e_i\rangle > 0.$$

From this criterion, we deduce the following result.

**Proposition 4.3.9.** *Let $X = \ell_p$ with $p > 1$. A typical $T \in (\mathcal{P}_1(X), \texttt{WOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}_*)$) has no non-trivial closed invariant ideals.*

*Proof.* Let us denote by $\mathcal{G}$ the set of positive contractions on $X$ having no non-trivial closed invariant ideals. The proof follows from the fact that the set

$$\bigcap_{\substack{i,j\geq 0\\ i\neq j}} \{T \in \mathcal{P}_1(X) : \langle e_j^*, Te_i\rangle > 0\}$$

is $\texttt{WOT}$-$G_\delta$ and hence $\texttt{SOT}_*$-$G_\delta$ in $\mathcal{P}_1(X)$, is contained in $\mathcal{G}$ and is $\texttt{SOT}^*$-dense in $\mathcal{P}_1(X)$ (see [26, Proposition 4.7] for more details). □

As mentioned in [26], the existence of non-trivial invariant subspaces for positive operators is a delicate matter. Moreover, it is still unknown whether there exists a positive operator on $\ell_p$ without non-trivial invariant subspaces for some $p \geq 1$. The most popular result regarding the existence of non-trivial invariant subspaces for positive operators is the following theorem, due to Abramovich, Aliprantis and Burkinshaw.

**Theorem 4.3.10** ([2, Theorem 2.2] and [1, Theorem 2.2])**.** *Let $X$ be a Banach space with a basis and let $T$ be a positive operator on $X$. Suppose that there exists a non-zero positive operator on $X$ in the commutant of $T$ which is quasinilpotent at a certain non-zero positive vector of $X$. Then $T$ has a non-trivial invariant subspace. Moreover, if $X = \ell_p$ and if $p \geq 1$, then $T$ has a non-trivial closed invariant ideal.*

As in [26], we say that a positive operator $T$ on $X = \ell_p$ satisfies the AAB criterion if it satisfies the hypotheses of Theorem 4.3.10. From Proposition 4.3.9, the following result immediately follows.

**Proposition 4.3.11.** *Let $X = \ell_p$ with $p > 1$. A typical $T \in (\mathcal{P}_1(X), \texttt{WOT})$ (resp. $T \in (\mathcal{P}_1(X), \texttt{SOT}_*)$) does not satisfy the AAB criterion.*

## 4.4 More on the points of continuity of the identity map on $\mathcal{P}_1(\ell_p)$

The aim of this section is to study in more depth the sets $\mathcal{C}(\texttt{WOT}, \texttt{SOT})$, $\mathcal{C}(\texttt{WOT}, \texttt{SOT}_*)$ and $\mathcal{C}(\texttt{SOT}, \texttt{SOT}^*)$. We will give examples of such points of continuity when $X = \ell_2$. We will also show that, when $X = \ell_2$, these sets are very different from the sets of the points of continuity of the identity maps on $\mathcal{B}_1(\ell_2)$ for the corresponding topologies.

As mentioned in Subsection 4.2a, the set $\mathcal{C}(\texttt{SOT}, \texttt{SOT}^*)$ is comeager in $(\mathcal{P}_1(\ell_p), \texttt{SOT})$ when $p > 1$, while the sets $\mathcal{C}(\texttt{WOT}, \texttt{SOT})$ and $\mathcal{C}(\texttt{WOT}, \texttt{SOT}_*)$ are comeager in $(\mathcal{P}_1(\ell_p), \texttt{WOT})$ when $p > 1$. So the description of these sets can be used to determine new typical properties of positive contractions on $\ell_p$-spaces.

In [29, Proposition 2.11], it was proved that the set of all points of continuity of the identity map from $(\mathcal{B}_1(\ell_2), \texttt{SOT})$ into $(\mathcal{B}_1(\ell_2), \texttt{SOT}^*)$ (resp. from $(\mathcal{B}_1(\ell_2), \texttt{WOT})$ into $(\mathcal{B}_1(\ell_2), \texttt{SOT}_*)$) is the co-isometries of $\ell_2$. It was also proved that the set of all points of continuity of the identity map from $(\mathcal{B}_1(\ell_2), \texttt{WOT})$ into $(\mathcal{B}_1(\ell_2), \texttt{SOT})$ is the isometries of $\ell_2$. These results are no longer true for positive contractions. Indeed, by [26, Proposition 3.7], a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \texttt{SOT})$ is not a co-isometry, and also a typical positive contraction $T \in (\mathcal{P}_1(\ell_2), \texttt{WOT})$ is not an isometry for the same reasons. We will see in this section that the situation is more complicated in the positive setting.

We first provide some observations concerning the sets $\mathcal{C}(\texttt{WOT}, \texttt{SOT}_*)$, $\mathcal{C}(\texttt{SOT}, \texttt{SOT}^*)$ and $\mathcal{C}(\texttt{WOT}, \texttt{SOT})$. If $p > 1$, we identify $(\ell_p)^*$ with $\ell_q$, where $q$ is the conjugate exponent of $p$.

**Proposition 4.4.1.** *Let $X = \ell_p$ with $p > 1$ and let $T$ be a positive contraction on $X$. Then $T \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ if and only if $T^* \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$.*

*Proof.* Let $T$ be a positive contraction on $X$. Suppose that $T$ belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ and let us show that $T^* \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$. Let $(T_n)_{n\geq 0}$ be a sequence of positive contractions such that $T_n \xrightarrow[n\to\infty]{\mathtt{WOT}} T^*$. Then $T_n^* \xrightarrow[n\to\infty]{\mathtt{WOT}} T$. Thus $T_n^* \xrightarrow[n\to\infty]{\mathtt{SOT}_*} T$ since $T \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$, and so $T_n \xrightarrow[n\to\infty]{\mathtt{SOT}} T^*$. This proves that $T^* \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$. The same arguments show that the converse is true and this concludes the proof of Proposition 4.4.1. $\square$

We now discuss some elementary properties that an element of $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$, $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ or $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$ must have. The first ones concern the norm of a point of continuity.

**Proposition 4.4.2.** *Let $T \in \mathcal{P}_1(\ell_2)$. If $T$ belongs to one of the sets $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*), \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$ or $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ , then $\|T\| = 1$.*

*Proof.* By Proposition 4.4.1, it is enough to prove that if $T$ belongs to $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$, then $\|T\| = 1$. Suppose that $\|T\| < 1$ and let $\delta > 0$ be such that $\|T\| + \delta < 1$. Let us consider the positive operator $T_n$ on $\ell_2$ defined by $T_n x = TP_n x + \delta \langle e_{n+1}, x\rangle e_0$ for every $x \in \ell_2$ and every $n \geq 0$. Our choice of $\delta$ implies that each $T_n$ is a positive contraction on $\ell_2$. We also have $T_n \xrightarrow[n\to\infty]{\mathtt{SOT}} T$. Moreover

$$\|T_n^* e_0 - T^* e_0\|^2 \geq (\delta - \langle e_{n+1}, T^* e_0\rangle)^2,$$

so $\liminf\limits_{n\to\infty} \|T_n^* e_0 - T^* e_0\|^2 \geq \delta^2 > 0$ and $T_n \notin \mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$. This proves Proposition 4.4.2. $\square$

Our second set of observations concern properties of the rows and the lines of the matrices representing points of continuity (in the canonical basis).

**Proposition 4.4.3.** *Let $T \in \mathcal{P}_1(\ell_2)$. If $T \in \mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$, then $T^* e_l \neq 0$ for every $l \geq 0$.*

*Proof.* Suppose that there exists an index $l \geq 0$ such that $T^* e_l = 0$. Then $T = P_{l-1}T + Q_l T$. Let us consider the positive operator $T_n$ on $\ell_2$ defined by $T_n x = TP_n x + \langle x, e_{n+1}\rangle e_l$ for every $x \in \ell_2$ and every $n \geq 0$. Each operator $T_n$ is a contraction since for every $x \in \ell_2$, we have

$$\begin{aligned}
\|T_n^* x\|^2 &= \|P_n T^* P_{l-1} x + \langle x, e_l\rangle e_{n+1} + P_n T^* Q_l x\|^2 \\
&= |\langle x, e_l\rangle|^2 + \|P_n T^* (P_{l-1} x + Q_l x)\|^2 \\
&\leq |\langle x, e_l\rangle|^2 + \|P_n T^*\|^2 \|P_{l-1} x + Q_l x\|^2 \\
&\leq |\langle x, e_l\rangle|^2 + \sum_{\substack{j\geq 0 \\ j\neq l}} |\langle x, e_j\rangle|^2 \\
&\leq \|x\|^2.
\end{aligned}$$

As in the proof of Proposition 4.4.2, the sequence $(T_n)$ converges to $T$ for the $\mathtt{SOT}$ topology but does not converge to $T$ for the $\mathtt{SOT}^*$ topology. This implies that $T \notin \mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$. $\square$

From Propositions 4.4.1 and 4.4.3, we deduce the following one.

**Proposition 4.4.4.** *Let $T \in \mathcal{P}_1(\ell_2)$. If $T \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ (resp. $T \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$), then $T^* e_l \neq 0$ for every $l \geq 0$ (resp. $Te_l \neq 0$ for every $l \geq 0$).*

Any co-isometry of $\ell_2$ belongs to $\mathcal{C}(\mathtt{WOT},\mathtt{SOT}_*)$ and any isometry of $\ell_2$ belongs to $\mathcal{C}(\mathtt{WOT},\mathtt{SOT})$ (the same arguments as in [29, Proposition 2.11] hold).

Our aim is now to exhibit new sets of points of continuity. To this aim, we need the following lemma.

**Lemma 4.4.5.** *Let $B \in \mathcal{P}_1(\ell_2)$. Then $B$ belongs to $\mathcal{C}(\mathtt{WOT},\mathtt{SOT}_*)$ if and only if the following property holds: for every $\varepsilon > 0$ and for every $r \geq 0$, there exist $\delta > 0$ and $m \geq 0$ such that*

$$\text{if } T \in \mathcal{P}_1(\ell_2) \text{ is such that } \max_{0\leq k,l\leq m} |\langle e_l,(T-B)e_k\rangle| < \delta, \qquad (4.4.1)$$
$$\text{then } \max_{0\leq k\leq r} \|(T-B)^* e_k\| < \varepsilon.$$

The proof of Lemma 4.4.5 uses similar arguments to the proof of [26, Lemma 2.2], that is uses local bases for the topologies $\mathtt{WOT}$ and $\mathtt{SOT}_*$ and the triangular inequality. For this reason, we will omit its proof. We now introduce two new classes of points of continuity, denoted by $\mathcal{M}$ and $\mathcal{M}'$. The set $\mathcal{M}$ strictly contains the co-isometries of $\ell_2$, whereas the set $\mathcal{M}'$ strictly contains the isometries of $\ell_2$.

**Theorem 4.4.6.** *Let $\mathcal{M}$ be the class of every positive contractions $T$ on $\ell_2$ such that*

- $\|T\| = 1$;
- *there exists a family $(u_r)_{r\in J}$ of norming vectors for $T^*$ indexed by a set $J \subseteq \mathbb{Z}_+$ such that $u_r \geq 0$, $\|u_r\| = 1$ for every $r \in J$ and $\bigcup_{r\in J} \mathit{Supp}(u_r) = \mathbb{Z}_+$.*

*Then every operator in $\mathcal{M}$ belongs to $\mathcal{C}(\mathtt{WOT},\mathtt{SOT}_*)$ and to $\mathcal{C}(\mathtt{SOT},\mathtt{SOT}^*)$.*

*Proof.* The proof of Theorem 4.4.6 is not very far from the proof of Proposition 4.2.7. The only difference is that here, the operators of the class $\mathcal{M}$ and the vectors $u_r$ are defined on $\ell_2$ and not just on $E_M$ for some $M \geq 1$. We must therefore pay attention to the fact that some vectors $u_r$ can have an infinite support.

Let $B \in \mathcal{M}$. For every $k \in J$ and for every $l \geq 0$, we set for this proof $u_k(l) := \langle e_l, u_k\rangle$. Also recall that $b_{i,j} = \langle e_i, Be_j\rangle$ for $i, j \geq 0$. Let $\varepsilon > 0$ and let $r \geq 0$. We have to find $\delta > 0$ and an integer $n \geq 0$ such that (4.4.1) holds.

By the condition on the supports, we can write $\{0,\dots,r\} = \bigcup_{k\in J_r}(\mathrm{Supp}(u_k)\cap\{0,\dots,r\})$, where $J_r$ is a certain finite subset of $J$. Let us now consider the following two sets

$$I := \{k \in J_r : \mathrm{Supp}(u_k) \text{ is infinite}\} \quad \text{and} \quad I' := \{k \in J_r : \mathrm{Supp}(u_k) \text{ is finite}\}.$$

For $k \in I'$, let us write $n_k = \max \mathrm{Supp}(u_k)$. For every $k \in I'$, one can find an integer $n'_k > r$ such that

$$\max_{\substack{0\leq l\leq r\\ u_k(l)>0}} \frac{1}{(u_k(l))^2}\left(1 - \|P_{n'_k} B^* u_k\|^2\right) < \varepsilon/4 \qquad (4.4.2)$$
$$\text{and} \quad \max_{0\leq l\leq r} \|Q_{n'_k} B^* e_l\|^2 < \varepsilon/4. \qquad (4.4.3)$$

Let us write $n = \max_{k\in I'} n_k$ and $n' = \max_{k\in I'} n'_k$.

Similarly for every $k \in I$, one can find an integer $N_k > r$ such that

$$\max_{\substack{0\le l\le r\\ u_k(l)>0}} \frac{1}{(u_k(l))^2}\,(1-\|P_{N_k}B^*P_{N_k}u_k\|^2) < \varepsilon/4 \tag{4.4.4}$$

$$\text{and}\quad \max_{0\le l\le r}\|Q_{N_k}B^*e_l\|^2 < \varepsilon/4. \tag{4.4.5}$$

We also write $N = \max_{k\in I} N_k$.

We choose $\delta > 0$ such that $b_{i,j} - \delta > 0$ for every $0 \le i, j \le \max(n, N, n') + 1$ such that $b_{i,j} > 0$,

$$2\delta \max_{\substack{0\le l\le r\\ k\in J_r\\ u_k(l)>0}} \frac{1}{(u_k(l))^2}\left(\sum_{j=0}^{n}(u_k(j))^2\sum_{m=0}^{n'} b_{j,m} + \sum_{0\le i<j\le n} u_k(i)u_k(j)\sum_{m=0}^{n'}(b_{i,m}+b_{j,m})\right) < \varepsilon/2 \tag{4.4.6}$$

and

$$2\delta \max_{\substack{0\le l\le r\\ k\in J_r\\ u_k(l)>0}} \frac{1}{(u_k(l))^2}\left(\sum_{j=0}^{N}(u_k(j))^2\sum_{m=0}^{N} b_{j,m} + \sum_{0\le i<j\le N} u_k(i)u_k(j)\sum_{m=0}^{N}(b_{i,m}+b_{j,m})\right) < \varepsilon/2. \tag{4.4.7}$$

Let $T \in \mathcal{P}_1(\ell_2)$ be such that

$$\max_{0\le k,l\le \max(n,N,n')+1} |\langle e_l, (T-B)e_k\rangle| < \delta.$$

Let us show that $\max_{0\le k\le r}\|(T-B)^*e_k\|^2 < \varepsilon$.

We will control $\|(T-B)^*e_l\|^2$ for every $l \in \mathrm{Supp}(u_k) \cap \{0,\dots,r\}$, when $k \in I$ and when $k \in I'$.

For every $k \in I'$, one has

$$\begin{aligned}1 \ge \left\|T^*\Big(\sum_{j=0}^{n_k} u_k(j)e_j\Big)\right\|^2 &= \sum_{j=0}^{n_k}(u_k(j))^2\|T^*e_j\|^2 \\ &\quad + 2\sum_{0\le i<j\le n_k} u_k(i)u_k(j)\langle T^*e_i, T^*e_j\rangle.\end{aligned} \tag{4.4.8}$$

Using positivity and the fact that

$$\max_{0\le k,l\le \max(n,N,n')+1} |\langle e_l, (T-B)e_k\rangle| < \delta,$$

we can say that

$$\langle T^*e_i, T^*e_j\rangle \ge \sum_{\substack{0\le m\le n'_k\\ b_{i,m}>0,\, b_{j,m}>0}} (b_{i,m}-\delta)(b_{j,m}-\delta) \tag{4.4.9}$$

for every $0 \leq i, j \leq n_k$.

Using (4.4.8), (4.4.9) and the positivity of the vectors $u_k$, we thus have for every $0 \leq l \leq n_k$ and for every $k \in I'$

$$(u_k(l))^2\|T^*e_l\|^2 \leq 1 - \sum_{\substack{0\leq j\leq n_k\\ j\neq l}} (u_k(j))^2 \sum_{\substack{0\leq m\leq n'_k\\ b_{j,m}>0}} (b_{j,m}-\delta)^2 - 2\sum_{0\leq i<j\leq n_k} u_k(i)u_k(j) \sum_{\substack{0\leq m\leq n'_k\\ b_{i,m}>0,\, b_{j,m}>0}} (b_{i,m}-\delta)(b_{j,m}-\delta). \tag{4.4.10}$$

As (4.4.9), one can say that

$$\langle T^*e_l, B^*e_l\rangle \geq \sum_{\substack{0\leq m\leq n'_k\\ b_{l,m}>0}} b_{l,m}(b_{l,m}-\delta) \tag{4.4.11}$$

for every $l \in \mathrm{Supp}(u_k) \cap \{0, \dots, r\}$ such that $k \in I'$.

Thus using (4.4.10) and (4.4.11), the quantity

$$\begin{aligned}&(u_k(l))^2\|T^*e_l - B^*e_l\|^2\\ &= (u_k(l))^2\|T^*e_l\|^2 - 2(u_k(l))^2\langle T^*e_l, B^*e_l\rangle + (u_k(l))^2\|B^*e_l\|^2\end{aligned} \tag{4.4.12}$$

can be bounded above for every $k \in I'$ and for every $l \in \mathrm{Supp}(u_k) \cap \{0, \dots, r\}$ by

$$\begin{aligned}&1 - \sum_{\substack{0\leq j\leq n_k\\ j\neq l}} (u_k(j))^2 \sum_{\substack{0\leq m\leq n'_k\\ b_{j,m}>0}} (b_{j,m}-\delta)^2\\ &- 2\sum_{0\leq i<j\leq n_k} u_k(i)u_k(j) \sum_{\substack{0\leq m\leq n'_k\\ b_{i,m}>0,\, b_{j,m}>0}} (b_{i,m}-\delta)(b_{j,m}-\delta)\\ &- 2(u_k(l))^2 \sum_{\substack{0\leq m\leq n'_k\\ b_{l,m}>0}} b_{l,m}(b_{l,m}-\delta) + (u_k(l))^2 \sum_{m\geq 0} b_{l,m}^2,\end{aligned} \tag{4.4.13}$$

that is by

$$\begin{aligned}&1 - \left(\sum_{j=0}^{n_k} (u_k(j))^2 \sum_{m=0}^{n'_k} b_{j,m}^2 + 2\sum_{0\leq i<j\leq n_k} u_k(i)u_k(j) \sum_{m=0}^{n'_k} b_{i,m}b_{j,m}\right)\\ &+2\delta\left(\sum_{j=0}^{n} (u_k(j))^2 \sum_{m=0}^{n'} b_{j,m} + \sum_{0\leq i<j\leq n} u_k(i)u_k(j) \sum_{m=0}^{n'} (b_{i,m}+b_{j,m})\right)\\ &+ (u_k(l))^2\|Q_{n'_k}B^*e_l\|^2\end{aligned} \tag{4.4.14}$$

since $n \geq n_k$ and $n' \geq n'_k$. Therefore, the quantity (4.4.12) can be bounded above by

$$(1 - \|P_{n'_k}B^*u_k\|^2) \tag{4.4.15}$$

$$+2\delta\left(\sum_{j=0}^{n}(u_k(j))^2\sum_{m=0}^{n'}b_{j,m}+\sum_{0\le i<j\le n}u_k(i)u_k(j)\sum_{m=0}^{n'}(b_{i,m}+b_{j,m})\right)$$
$$+(u_k(l))^2\|Q_{n'_k}B^*e_l\|^2,$$

since

$$\sum_{j=0}^{n_k}(u_k(j))^2\sum_{m=0}^{n'_k}b_{j,m}^2+2\sum_{0\le i<j\le n_k}u_k(i)u_k(j)\sum_{m=0}^{n'_k}b_{i,m}b_{j,m}=\sum_{m=0}^{n'_k}\langle u_k,P_{n_k}Be_m\rangle^2$$
$$=\|P_{n'_k}B^*u_k\|^2.$$

By the same way, replacing $n_k$ and $n'_k$ by $N_k$, we have for every $k\in I$ and for every $l\in \mathrm{Supp}(u_k)\cap\{0,\dots,r\}$,

$$\begin{aligned}&(u_k(l))^2\|T^*e_l-B^*e_l\|^2\\ &\le(1-\|P_{N_k}B^*P_{N_k}u_k\|^2)\\ &+2\delta\left(\sum_{j=0}^{N}(u_k(j))^2\sum_{m=0}^{N}b_{j,m}+\sum_{0\le i<j\le N}u_k(i)u_k(j)\sum_{m=0}^{N}(b_{i,m}+b_{j,m})\right)\\ &+(u_k(l))^2\|Q_{N_k}B^*e_l\|^2\end{aligned}\tag{4.4.16}$$

since $N\ge N_k$.

Using (4.4.2), (4.4.3), (4.4.4), (4.4.5), (4.4.15) and (4.4.16), we easily see that

$$\max_{0\le l\le r}\|(B-T)^*e_l\|^2<\varepsilon.$$

This concludes the proof of Theorem 4.4.6. □

From Proposition 4.4.1 and Theorem 4.4.6, we obtain the following result.

**Theorem 4.4.7.** *Let $\mathcal{M}'$ be the class of every positive contractions $T$ on $\ell_2$ such that*

- $\|T\|=1$;
- *there exists a family $(u_r)_{r\in J}$ of norming vectors for $T$ indexed by a set $J\subseteq\mathbb{Z}_+$ such that $u_r\ge0$, $\|u_r\|=1$ for every $r\in J$ and $\bigcup_{r\in J}\mathrm{Supp}(u_r)=\mathbb{Z}_+$.*

*Then every operator in $\mathcal{M}'$ belongs to $\mathcal{C}(\mathtt{WOT},\mathtt{SOT})$.*

We will now see that the classes $\mathcal{M}$ and $\mathcal{M}'$ are SOT-dense in $\mathcal{P}_1(\ell_2)$. They thus are large classes of continuity points. To do so, we will use the following approximation lemma, which is proved in [26, Lemma 2.2].

**Lemma 4.4.8.** *Let $X=\ell_p$ with $p>1$. Let $\mathcal{C}(X)$ be a class of operators on $X$ and define $\mathcal{C}_1(X):=\mathcal{C}(X)\cap\mathcal{P}_1(X)$.*

*Suppose that the following property holds: for every $\varepsilon>0$, every $N\in\mathbb{Z}_+$ and for every positive operator $A\in\mathcal{P}(E_N)$ with $\|A\|<1$, there exists a positive operator $T\in\mathcal{C}_1(X)$ such that*

$$\|(T-A)e_k\|<\varepsilon\quad\text{for every}\quad 0\le k\le N.\tag{4.4.17}$$

*Then $\mathcal{C}_1(X)$ is dense in $(\mathcal{P}_1(X), \texttt{SOT})$.*

With Lemma 4.4.8, we obtain the following result.

**Proposition 4.4.9.** *The classes $\mathcal{M}$ and $\mathcal{M}'$ are $\texttt{SOT}$-dense in $\mathcal{P}_1(\ell_2)$.*

*Proof.* In order to apply Lemma 4.4.8, let $N \geq 0$, let $\varepsilon > 0$ and let $A \in \mathcal{P}_1(E_N)$ be such that $\|A\| < 1$. By Proposition 4.1.3, one can find a positive operator $B$ on $E_{2N+1}$ such that $\|B\| = 1$, $B$ has a positive norming vector $u \in E_{2N+1}$ for which $\text{Supp}(u) = \{0, \ldots, 2N+1\}$ and $\|BP_N - A\| < \varepsilon$. Let us define a positive operator $T$ on $\ell_2$ by $T = BP_{2N+1} + Q_{2N+1}$. Then $\|T\| = 1$ and, since the operator $Q_{2N+1}$ has a positive norming vector on $F_{2N+1}$ with positive coordinates, we can easily obtain a positive norming vector for $T$ whose support is $\mathbb{Z}_+$. This shows that $T$ belongs to $\mathcal{M}'$. Moreover, we have $\max\limits_{0 \leq k \leq N} \|(T-A)e_k\| < \varepsilon$. This shows that $\mathcal{M}'$ is $\texttt{SOT}$-dense in $\mathcal{P}_1(\ell_2)$. Also, notice that by Proposition 4.1.3 and by Remark 4.2.2, the operator $B^*$ also has a positive norming vector whose support is $\{0, \ldots, 2N+1\}$ and so by the same way, the operator $T$ belongs to $\mathcal{M}$. This concludes the proof of Proposition 4.4.9. $\square$

Finally we end this article by showing that the class $\mathcal{M}$ is strictly contained in $\mathcal{C}(\texttt{WOT}, \texttt{SOT}_*)$ and in $\mathcal{C}(\texttt{SOT}, \texttt{SOT}^*)$, and that $\mathcal{M}'$ is strictly contained in $\mathcal{C}(\texttt{WOT}, \texttt{SOT})$. To do so, we need the following important proposition.

**Proposition 4.4.10.** *Let $X = \ell_p$ with $p > 1$. A typical $T \in (\mathcal{P}_1(X), \texttt{SOT}^*)$ does not attain its norm.*

*Proof.* Let $\mathcal{N}$ be the set of positive contractions on $X$ which attain their norm. Since the set $S^+(X) := \{T \in \mathcal{P}_1(X) : \|T\| = 1\}$ is comeager in $(\mathcal{P}_1(X), \texttt{SOT}^*)$ by [26, Proposition 3.1], it is enough to prove that $S^+(X) \setminus \mathcal{N}$ is comeager in $(S^+(X), \texttt{SOT}^*)$. As in the proof of [29, Theorem 6.1], the set $S^+(X) \setminus \mathcal{N}$ is a $G_\delta$ of $(S^+(X), \texttt{SOT}^*)$. This set is also $\texttt{SOT}^*$-dense in $S^+(X)$. Indeed, let $A \in S^+(X)$ and let $(\varepsilon_n)_{n \geq 0}$ be a sequence of positive real numbers tending to 0. For every $N \geq 0$, let $B_N$ be a positive operator of norm 1 on $F_N$ which does not attain its norm on $F_N$ (for example, we can take a diagonal operator on $F_N$ associated to a sequence of positive real numbers in $(0, 1)$ which converges to 1). If we set $T_N = (1 - \varepsilon_N) P_N A P_N + Q_N B_N Q_N$ for every $N \geq 0$, then each $T_N$ is a positive operator in $S^+(X)$, and we can show as in [29, Theorem 6.1] that these operators do not attain their norm. Since the sequence $(T_N)$ converges to $A$ for the $\texttt{SOT}^*$ topology, the set $S^+(X) \setminus \mathcal{N}$ is $\texttt{SOT}^*$-dense in $(S^+(X), \texttt{SOT}^*)$. This proves Proposition 4.4.10. $\square$

Corollary 4.3.2 has the following important consequence.

**Corollary 4.4.11.** *A typical $T \in (\mathcal{P}_1(\ell_2), \texttt{SOT})$ (resp. $T \in (\mathcal{P}_1(\ell_2), \texttt{WOT})$, $T \in (\mathcal{P}_1(\ell_2), \texttt{SOT}_*)$) does not attain its norm.*

With Corollary 4.3.5, we also obtain:

**Corollary 4.4.12.** *Let $X = \ell_p$ with $p > 1$. If $p > 2$, a typical $T \in (\mathcal{P}_1(X), \texttt{SOT})$ does not attain its norm. If $1 < p < 2$, a typical $T \in (\mathcal{P}_1(X), \texttt{SOT}_*)$ does not attain its norm.*

Theorem 4.4.6 (resp. Theorem 4.4.7) asserts that the class $\mathcal{M}$ (resp. $\mathcal{M}'$) is contained in the sets $\mathcal{C}(\texttt{WOT}, \texttt{SOT}_*)$ and $\mathcal{C}(\texttt{SOT}, \texttt{SOT}^*)$ (resp. in $\mathcal{C}(\texttt{WOT}, \texttt{SOT})$). By Corollary 4.4.11, we know that a typical positive contraction on $\ell_2$ for the $\texttt{SOT}$ and the $\texttt{WOT}$ topologies does not attain its norm, so in particular the class $\mathcal{M}$ can't be equal to $\mathcal{C}(\texttt{SOT}, \texttt{SOT}^*)$ or to $\mathcal{C}(\texttt{WOT}, \texttt{SOT}_*)$, and the class $\mathcal{M}'$ can't be equal to $\mathcal{C}(\texttt{WOT}, \texttt{SOT})$. This shows that the description of the sets $\mathcal{C}(\texttt{WOT}, \texttt{SOT}_*)$, $\mathcal{C}(\texttt{SOT}, \texttt{SOT}^*)$ and $\mathcal{C}(\texttt{WOT}, \texttt{SOT})$ in the positive setting may be an arduous problem, contrary to the description of these sets in the general setting of contractions on $\ell_2$.

## 4.5 Comments and open problems

We end this article with some comments and open questions related to our previous results.

As mentioned at the end of Section 4.4, the descriptions of the sets $\mathcal{C}(\mathtt{WOT},\mathtt{SOT}_*)$, $\mathcal{C}(\mathtt{SOT},\mathtt{SOT}^*)$ and $\mathcal{C}(\mathtt{WOT},\mathtt{SOT})$ are not yet complete. So the following problem is still open.

**Open Problem 4.5.1.** Describe completely the sets $\mathcal{C}(\mathtt{WOT},\mathtt{SOT}_*)$, $\mathcal{C}(\mathtt{SOT},\mathtt{SOT}^*)$ and $\mathcal{C}(\mathtt{WOT},\mathtt{SOT})$.

So far, we have proved that the topologies $\mathtt{WOT}$, $\mathtt{SOT}_*$, $\mathtt{SOT}$ and $\mathtt{SOT}^*$ are similar on $\mathcal{P}_1(\ell_2)$. Also, when $p > 2$, the topologies $\mathtt{SOT}$ and $\mathtt{SOT}^*$ are similar on $\mathcal{P}_1(\ell_p)$ and the topologies $\mathtt{WOT}$ and $\mathtt{SOT}_*$ are similar on $\mathcal{P}_1(\ell_p)$. And finally, when $1 < p < 2$, the topologies $\mathtt{SOT}_*$ and $\mathtt{SOT}^*$ are similar on $\mathcal{P}_1(\ell_p)$ and the topologies $\mathtt{WOT}$ and $\mathtt{SOT}$ are similar on $\mathcal{P}_1(\ell_p)$. The proof of Theorem 4.2.8 is very particular to the Hilbertian setting, since it uses the parallelogram identity in the proof of Proposition 4.2.7. This proof cannot be used for other values of $p$. Thus, the following problem is open.

**Open Problem 4.5.2.** Is it true that the topologies $\mathtt{WOT}$, $\mathtt{SOT}_*$, $\mathtt{SOT}$ and $\mathtt{SOT}^*$ are similar on $\mathcal{P}_1(\ell_p)$ for $1 < p < 2$ and for $p > 2$?

A positive answer to Open Problem 4.5.2 would not be surprising, and it would also not be surprising if the notion of norming vector helps to solve this problem. In particular, if the answer to Open Problem 4.5.2 is affirmative, then the point spectrum of a typical positive contraction on $\ell_p$ for $p > 1$ would be empty for the topologies $\mathtt{WOT}$, $\mathtt{SOT}$, $\mathtt{SOT}_*$ and $\mathtt{SOT}^*$.

Chapter 5

# Linear dynamics of random products of operators

*The results presented in this chapter are based on the manuscript [24], submitted.*

**Abstract.** We study the linear dynamics of the random sequence $(T_n(.))_{n\geq 1}$ of the operators $T_n(\omega) := T(\tau^{n-1}\omega)\cdots T(\tau\omega)T(\omega)$, $n \geq 1$. These products depend on an ergodic measure-preserving transformation $\tau : \mathbb{T} \to \mathbb{T}$ on the probability space $(\mathbb{T}, m)$ and on a strongly measurable map $T : \mathbb{T} \to \mathcal{B}(X)$, where $X$ is a separable Fréchet space. We will be focusing on the case where $T(\omega)$ is equal to an operator $T_1$ on $X$ for every $\omega \in A_1$ and equal to an operator $T_2$ on $X$ for every $\omega \in A_2$, where $A_1, A_2$ are two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. More precisely, we will be focusing on the case where the operators $T_1$ and $T_2$ are adjoints of multiplication operators on the Hardy space $H^2(\mathbb{D})$, as well as the case where $T_1$ and $T_2$ are entire functions of exponential type of the derivation operator on the space of entire functions. Finally, we will study the linear dynamics of a case of a random product $T_n(\omega)$ for which the operators $T(\tau^i\omega), i \geq 0$, do not commute. We will give particular importance to the case where the ergodic transformation is an irrational rotation or the doubling map on $\mathbb{T}$.

## 5.1 Introduction

### 5.1a Objectives of the article and motivation

The aim of this article is to study the linear dynamics of random sequences of products of operators on a separable Fréchet space. That is, we would like to determine if the sequence $(T_n(.))_{n\geq 1}$ of the random operators

$$T_n(\omega) = T(\tau^{n-1}\omega)\cdots T(\tau\omega)T(\omega) \tag{5.1.1}$$

**Key words and phrases:** Linear dynamics, ergodic theory, random operators, central limit theorems, multiplication operators, derivation operator.

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is universal, or topologically weakly mixing, or topologically mixing, or none of it, in the sense that the sequence $(T_n(\omega))_{n\geq 1}$ has this property for almost every $\omega \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$, or none of these properties. The probability space for our study will be the set $\mathbb{T}$, equipped with its Borel subsets and with the normalized Lebesgue measure $m$. The map $\tau : \mathbb{T} \to \mathbb{T}$ will be a measure-preserving transformation on $\mathbb{T}$, which means that

$$m(\tau^{-1}(B)) = m(B)$$

for every Borel subset $B$ of $\mathbb{T}$, and the map $T : \mathbb{T} \to \mathcal{B}(X)$ will be given by

$$T(\omega) = \begin{cases} T_1 & \text{if} \quad \omega \in A_1 \\ T_2 & \text{if} \quad \omega \in A_2 \end{cases},$$

with $T_1, T_2$ two operators on a separable Fréchet space $X$, and with $A_1, A_2$ two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Such types of products (5.1.1) appear when we consider linear cocycles $F : \mathbb{T} \times X \to \mathbb{T} \times X$, $(\omega, x) \mapsto (\tau\omega, T(\omega)x)$. In this case, we have $F^n(\omega, x) = (\tau^n x, T_n(\omega)x)$.

The motivation of our work comes from the study of the asymptotic behavior of random products of matrices, which come into play, for example, in considering solutions to differential or difference equations with random coefficients. Bellman was the first to study random products of $2 \times 2$ matrices with strictly positive entries ([7]). Roughly speaking, he proved that the weak law of large numbers holds for these kinds of random products. A few years later, Furstenberg and Kesten also considered random products of matrices ([22]). In particular, they extended Bellman's result by proving that the strong law of large numbers is applicable for random products of $2\times 2$ matrices with strictly positive entries, under some assumptions on the coefficients of these matrices.

While random products of matrices and operators have been extensively studied from a probabilistic and statistical viewpoint - most notably via Lyapunov exponents (see, for instance, [61]) - the linear dynamics of such random products remains largely unexplored. This work aims to initiate such a study by examining the dynamical properties (e.g., orbit structure, universality) of random products of operators of the form (5.1.1).

In particular, our study includes the case of random products of independent variables on $\mathbb{T}$ with a common distribution. Indeed, let $A_1 = [0, 1/2)$ and $A_2 = [1/2, 1)$. Let $\tau$ be the doubling map on $\mathbb{T}$ defined by $\tau(x) = 2x$. Let $T : \mathbb{T} \to \mathcal{B}(X)$ be defined by

$$T(\omega) = \begin{cases} T_1 & \text{if} \quad \omega \in A_1 \\ T_2 & \text{if} \quad \omega \in A_2 \end{cases},$$

with $T_1, T_2$ two bounded operators on a separable Fréchet space $X$. For each $n \geq 1$ and $\omega \in \mathbb{T}$, let $T_n(\omega) := T(\tau^{n-1}\omega)\cdots T(\tau\omega)T(\omega)$. Let us denote by $\varphi : \{0,1\}^{\mathbb{Z}_+} \to \mathbb{T}$ the map defined by $\varphi((x_n)_{n\geq 0}) = \sum_{n\geq 0} \frac{x_n}{2^{n+1}}$ in $\mathbb{T}$, and by $\sigma : \{0,1\}^{\mathbb{Z}_+} \to \{0,1\}^{\mathbb{Z}_+}$ the forward shift defined by $\sigma((x_n)_{n\geq 0}) = (x_{n+1})_{\geq 0}$. The map $\varphi$ is a bijection from the set $\Omega_0$ of sequences in $\{0,1\}^{\mathbb{Z}_+}$ not eventually equal to 1 onto $\mathbb{T}$, and $\tau\varphi = \varphi\sigma$. For every $j \geq 0$, let $\hat{X}_j : u = (u_k)_{k\geq 0} \in \{0,1\}^{\mathbb{Z}_+} \mapsto u_j$ be the $j$-th canonical projection. The random variables $\hat{X}_j$, $j \geq 0$, are independent on the probability space $(\{0,1\}^{\mathbb{Z}_+}, \mathcal{C}, \nu^{\otimes\mathbb{Z}_+})$ and have the same distribution $\nu := \frac{1}{2}(\delta_0 + \delta_1)$, where $\mathcal{C}$ denotes the $\sigma$-algebra generated by the cylinders and $\nu^{\otimes\mathbb{Z}_+}$ denotes the product measure of $\nu$. For every $\omega = \varphi(u) \in \mathbb{T}$ with $u \in \Omega_0$, and for every $j \geq 0$, we have $T(\tau^j\omega) = T(\varphi(\sigma^j u))$, which is equal to $T_1$ if and only if $\varphi(\sigma^j u) \in A_1$, that is, if and only if $\hat{X}_j(u) = 0$, and is equal to $T_2$ if and only if $\hat{X}_j(u) = 1$.

In particular, the random variables $T(\tau^j\omega), j \geq 0$, are independent on $\mathbb{T}$ and have the same distribution, namely $\frac{1}{2}(\delta_{T_1} + \delta_{T_2})$.

### 5.1b Linear dynamics

We now introduce the notions of linear dynamics that we will use for our study. For background on linear dynamics, we refer to Bayart's and Matheron's book [6], and to Grosse-Erdmann's and Peris Manguillot's book [31]. A good introduction to the notion of universality is given in [32].

A sequence of continuous maps $(T_n)_{n\geq 1}$ from a metric space $X$ to itself is said to be universal if there is an element $x \in X$ such that its orbit under $(T_n)_{n\geq 1}$,

$$\operatorname{orb}(x,(T_n)) := \{T_n x : n \geq 1\},$$

is dense in $X$. Fekete was the first to give an example of a universal family. He showed that there exists a real power series $\sum_{n\geq 1} a_n x^n$ on $[-1,1]$ that diverges at every point $x \neq 0$ in the worst possible way, in the sense that to every continuous function $g$ on $[-1,1]$ with $g(0) = 0$, there exists an increasing sequence of positive integers such that $\sum_{n=1}^{n_k} a_n x^n \underset{k\to\infty}{\longrightarrow} g$ uniformly on $[-1,1]$. In this case, $T_n$ is the map that associates to $f$ its partial Taylor series of order $n$ at 0. This example of universality shows two aspects of universality: an aspect of maximal divergence, and the aspect of the existence of a single object which allows us to approximate a maximal class of objects.

The notion of topological transitivity also plays an important role in topological and linear dynamics. We say that a sequence of continuous maps $(T_n)_{n\geq 1}$ from a space $X$ to itself is topologically transitive if, for any pair $U, V \subset X$ of nonempty open sets, there exists an integer $n \geq 1$ such that $T_n(U) \cap V \neq \emptyset$. If this property holds for sufficiently large $n \geq 1$, then we say that the sequence $(T_n)_{n\geq 1}$ is topologically mixing. Finally, we say that the sequence $(T_n)_{n\geq 1}$ is topologically weakly mixing if, for any nonempty open sets $U_1, U_2, V_1, V_2$ of $X$, there is some $n \geq 1$ such that $T_n(U_1) \cap V_1 \neq \emptyset$ and $T_n(U_2) \cap V_2 \neq \emptyset$. Every topologically mixing sequence is obviously topologically weakly mixing, and every topologically weakly mixing sequence is topologically transitive. If the sequence $(T_n)_{n\geq 1}$ is topologically transitive and if the maps $T_n$ are invertible, then the sequence $(T_n^{-1})_{n\geq 1}$ is also topologically transitive.

For a single continuous map $T$ on $X$, we say that $T$ is topologically transitive if the sequence $(T^n)_{n\geq 1}$ is topologically transitive. When the space $X$ is a separable complete metric space without isolated points, a continuous map $T$ is topologically transitive if and only if there is a point $x \in X$ with a dense orbit under $T$. In fact, in this situation, the map $T$ has a dense $G_\delta$-set of points with a dense orbit under $T$. This is Birkhoff's transitivity theorem ([6, Theorem 1.2] and [31, Theorem 1.16]). For a sequence of continuous and commuting maps $T_n$ on $X$, it is possible to generalize this result in the following way ([31, Exercise 1.6.2]).

**Proposition 5.1.1.** *Let $T_n : X \to X$, $n \geq 1$, be commuting continuous maps on a separable complete metric space $X$. Suppose that each $T_n$, $n \geq 1$, has a dense range. The following assertions are equivalent:*

*(i) $(T_n)_{n\geq 1}$ is topologically transitive.*

*(ii) there is a point $x \in X$ whose orbit under $(T_n)_{n\geq 1}$ is dense in $X$.*

*In this situation, the set of points in $X$ with a dense orbit under $(T_n)_{n\geq 1}$ is a dense $G_\delta$-set.*

A topologically transitive continuous linear map on a separable Fréchet space $T$ is also called a hypercyclic operator. In the linear setting, we have a more useful criterion than Birkhoff's transitivity theorem to prove that a sequence of operators on a separable Fréchet space $X$ is universal.

**Proposition 5.1.2** ([31, Theorem 3.24])**.** *Let $X$ be a separable Fréchet space and $(T_n)_{n\geq 1}$ a sequence of operators on this space. Suppose that there are dense subsets $\mathcal{D}_1$ and $\mathcal{D}_2$ of $X$, a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers, and maps $S_{n_k} : \mathcal{D}_2 \to X$, $k \geq 1$, such that*

*(i)* $T_{n_k} x \underset{k\to\infty}{\longrightarrow} 0$ *for every* $x \in \mathcal{D}_1$.

*(ii)* $S_{n_k} y \underset{k\to\infty}{\longrightarrow} 0$ *for every* $y \in \mathcal{D}_2$.

*(iii)* $T_{n_k} S_{n_k} y \underset{k\to\infty}{\longrightarrow} y$ *for every* $y \in \mathcal{D}_2$.

*Then the sequence $(T_n)_{n\geq 1}$ is topologically weakly mixing, and in particular universal. If moreover this criterion is satisfied with respect to the full sequence $(n)$, then the sequence $(T_n)_{n\geq 1}$ is topologically mixing.*

The criterion of Proposition 5.1.2 is called the Universality Criterion. This criterion will be a powerful tool for our study. Observe that the maps $S_n$ are not required to be linear nor continuous.

An important example of operator in this paper will be the adjoint of the multiplication operator $M_\phi$ on the Hardy space $H^2(\mathbb{D})$. Its adjoint $(M_\phi)^*$ is hypercyclic on $H^2(\mathbb{D})$ if and only if $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, when $\phi$ is a nonconstant bounded holomorphic function on $\mathbb{D}$ ([31, Theorem 4.42]). We will also consider entire functions of exponential type of the derivation operator $D$ on the space of entire functions. These operators are exactly the operators on the space of entire functions that commute with $D$, and they are always topologically mixing and hence hypercyclic, provided they are not a scalar multiple of the identity ([31, Theorem 4.21]).

## 5.1c Main results

We study the linear dynamics of random sequences $(T_n(.))_{n\geq 1}$ when the operators $T(\tau^i\omega), i \geq 0$, are adjoints of multiplication operators on the Hardy space $H^2(\mathbb{D})$, or entire functions of exponential type of the derivation operator $D$ on the space of entire functions. We also study a case of random products of operators on the space of entire functions, where the operators $T(\tau^i\omega)$ do not commute. The ergodic properties of the transformation $\tau : \mathbb{T} \to \mathbb{T}$ will play an important role in examining the universal behavior of the sequence $(T_n(\omega))_{n\geq 1}$, for almost every $\omega \in \mathbb{T}$. We put together here some of our main results. The notations will be defined in Subsection 5.1e.

Thanks to the fact that the adjoint of a multiplication operator has plenty of eigenvalues, we obtain a first sufficient criterion which gives the universality of the random sequence $(T_n(.))_{n\geq 1}$ in the case of adjoints of multiplication operators.

**Theorem 5.1.3.** *Let $\tau$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1, 2$.*

*Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant on $\mathbb{D}$ and suppose that there exist $\lambda, \mu \in \mathbb{D}$ such that*

$$|\phi_1(\lambda)|^{m(A_1)}|\phi_2(\lambda)|^{m(A_2)} < 1 \quad \textit{and} \tag{5.1.2}$$

$$|\phi_1(\mu)|^{m(A_1)}|\phi_2(\mu)|^{m(A_2)} > 1. \tag{5.1.3}$$

*Suppose that, for every $\omega \in \mathbb{T}$, the operator $T(\omega)$ is given by*

$$T(\omega) = \begin{cases} (M_{\phi_1})^* & \text{if} \quad \omega \in A_1 \\ (M_{\phi_2})^* & \text{if} \quad \omega \in A_2 \end{cases}.$$

*Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically mixing, that is, the sequence $(T_n(\omega))_{n\geq 1}$ is topologically mixing for almost every $\omega \in \mathbb{T}$.*

In the case $m(A_1) = m(A_2)$, Conditions (5.1.2) and (5.1.3) are equivalent to $(\phi_1\phi_2)(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ and $\phi_1\phi_2$ nonconstant, by the open mapping theorem. This leads to two limit cases when $m(A_1) = m(A_2)$: the case $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{D}$, and the case $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. We will see that the study of these limit cases is not that easy.

Under certain conditions, we establish that the random sequence of operators $(T_n(.))_{n\geq 1}$ is universal when the operators $T(\tau^i\omega)$ are entire functions of exponential type of the derivation operator $D$. This will follow from the fact that the derivation operator on $H(\mathbb{C})$ has suitable eigenvectors, namely the exponential functions $e_\lambda : z \mapsto e^{\lambda z}$, with $\lambda \in \mathbb{C}$.

**Theorem 5.1.4.** *Let $\tau : \mathbb{T} \to \mathbb{T}$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Let $\varphi_1$ and $\varphi_2$ be two nonconstant entire functions of exponential type. Consider the map*

$$T(\omega) := \begin{cases} \varphi_1(D) & \text{if } \omega \in A_1 \\ \varphi_2(D) & \text{if } \omega \in A_2 \end{cases}.$$

*Suppose that there exist $z, w \in \mathbb{C}$ such that*

$$|\varphi_1(z)|^{m(A_1)}|\varphi_2(z)|^{m(A_2)} < 1 \tag{5.1.4}$$

$$\text{and} \quad |\varphi_1(w)|^{m(A_1)}|\varphi_2(w)|^{m(A_2)} > 1. \tag{5.1.5}$$

*Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically mixing.*

In the case where $m(A_1) = m(A_2) = 1/2$, Conditions (5.1.4) and (5.1.5) are equivalent to the fact that $\varphi_1\varphi_2$ is nonconstant, when $\varphi_1$ and $\varphi_2$ are two nonconstant entire functions of exponential type. This comes from the fact that every nonconstant entire function has a dense range in $\mathbb{C}$. We will also study a situation where $\varphi_1\varphi_2$ is constant, showing in particular that the random sequence $(T_n(.))_{n\geq 1}$ can be non-universal even if $T(\omega)$ is hypercyclic for every $\omega \in \mathbb{T}$.

Until now, all the random products of operators $T_n(.)$ we introduced before have the property that the operators $T(\tau^i\omega), i \geq 0$, commute. We now consider a case of a random sequence of operators on $H(\mathbb{C})$ for which these operators do not commute:

**Theorem 5.1.5.** *Let $\tau : \mathbb{T} \to \mathbb{T}$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$, for $k = 1,2$. Consider the map*

$$T(\omega) := \begin{cases} T_{\lambda,b} & \text{if } \omega \in A_1 \\ D & \text{if } \omega \in A_2 \end{cases},$$

*where $T_{\lambda,b}$ is the operator on $H(\mathbb{C})$ defined by*

$$T_{\lambda,b} f = f(\lambda . + b),$$

*with $\lambda \in \mathbb{C}$ nonzero and $b \in \mathbb{C}$. Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically mixing, that is, the sequence $(T_n(\omega))_{n\geq 1}$ is topologically mixing for almost every $\omega \in \mathbb{T}$.*

The operator $T_{\lambda,b}$ is hypercyclic on $H(\mathbb{C})$ precisely when $\lambda = 1$ and $b \neq 0$ (see, for instance, [8]). An interest of Theorem 5.1.5 is that the product $T_n(\omega)$ can contain hypercyclic and non-hypercyclic operators $T(\tau^i\omega)$ that do not commute.

We will also consider the case of an irrational rotation and of the doubling map on $\mathbb{T}$. Given a real number $\alpha$ in $(0,1)$, the rotation of parameter $\alpha$ is defined by the transformation $R_\alpha : x \mapsto x+\alpha \pmod 1$ on $[0,1)$, where $x \pmod 1$ is the fractional part of the real $x$. This map can be seen as a map on $\mathbb{T}$ defined by $\tau(x) = x+\alpha$ in $\mathbb{T}$, and can also be seen as a map on the unit circle defined by $\tau(z) = e^{2i\pi\alpha}z$. This map is ergodic precisely when $\alpha$ is irrational. Moreover, this map is not weakly mixing. The doubling map is the map on $[0,1)$ defined by $\tau(x) = 2x \pmod 1$. This map can also be seen as a map on $\mathbb{T}$ defined by $\tau(x) = 2x \in \mathbb{T}$, and also as a map on the unit circle defined by $\tau(z) = z^2$. Contrary to rotations, this map is mixing. These two transformations will appear many times in this article.

When the ergodic transformation involved in Theorems 5.1.4 and 5.1.5 is an irrational rotation and when $A_1, A_2$ are intervals of $[0,1)$, the universality of the associated sequence $(T_n(\omega))_{n\geq 1}$ holds for every $\omega \in \mathbb{T}$. This comes from the uniform distribution mod 1 of the sequence $(n\alpha)_{n\geq 1}$.

Regarding the limit case $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{D}$ when $m(A_1) = m(A_2) = 1/2$, we studied the situation where $\phi_1\phi_2$ is an inner function. We also generalized it to the situation where $m(A_1) \neq m(A_2)$. In this context, for the doubling map on $\mathbb{T}$, we obtained the following result.

**Theorem 5.1.6.** *Let $\tau$ be the doubling map on $\mathbb{T}$ and let $A_1$ and $A_2$ be two disjoint intervals of $[0,1)$, such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant. Suppose that one of the two following conditions holds:*

*(i) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ almost everywhere on $\mathbb{T}$, and one of the images $\phi_1(\mathbb{D})$ or $\phi_2(\mathbb{D})$ meets the unit circle $\mathbb{T}$.*

*(ii) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ almost everywhere on $\mathbb{T}$, $\phi_1\phi_2$ is not outer and either $\phi_1$ or $\phi_2$ is not inner.*

*Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.*

The case of irrational rotations will be more harsh to study. This will strongly depend on the Diophantine properties of the irrational parameter $\alpha$. For example, we will establish the following result.

**Theorem 5.1.7.** *Let $\alpha$ be an irrational in $(0,1)$ and let $\tau = R_\alpha$. Let $b \in (0,1)$ and let $A_1 = [0,b)$ and $A_2 = [b,1)$. Suppose that the sequence $(q_k b)_{k\geq 1}$ is uniformly distributed mod 1, where $q_k$ are the denominators of the convergents of $\alpha$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant. Suppose that one of the two following conditions holds:*

*(i) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ almost everywhere on $\mathbb{T}$, and the image of either $\phi_1$ or $\phi_2$ intersects $\mathbb{T}$.*

*(ii) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ almost everywhere on $\mathbb{T}$, $\phi_1\phi_2$ is not outer and either $\phi_1$ or $\phi_2$ is not inner.*

*Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.*

However, in the case $A_1 = [0,1/2)$ and $A_2 = [1/2,1)$, we do not need any assumption on the irrational $\alpha$.

**Theorem 5.1.8.** *Let $\alpha$ be an irrational number in $(0,1)$ and let $\tau = R_\alpha$. Let $A_1 = [0,1/2)$ and $A_2 = [1/2,1)$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that $\phi_1\phi_2$ is inner and nonconstant. Suppose also that either $\phi_1$ or $\phi_2$ is not inner. Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.*

For some other intervals of $[0,1)$, it will also be possible for the random sequence $(T_n(.))_{n\geq 1}$ to be non-universal. This is, for example, the case of the following result that we obtained.

**Theorem 5.1.9.** *Let $\alpha$ be an irrational number in $(0,1)$ and let $\tau = R_\alpha$. Let $A_1 = [0,b)$ and $A_2 = [b,1)$, with $b \in (0,1)$. Suppose that $b \in \mathbb{Z}\alpha$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that*

$$|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1 \quad \textit{almost everywhere on } \mathbb{T}.$$

*Then, for every $\omega \in \mathbb{T}$, the sequence $(T_n(\omega))_{n\geq 1}$ is not universal.*

These results show in particular that the random sequence $(T_n(.))_{n\geq 1}$ can be universal, even if both operators $T_1$ and $T_2$ are not hypercyclic (see Remark 5.3.13). We will also generalize Theorems 5.1.6, 5.1.7 and 5.1.9 for some other ergodic transformations on $(\mathbb{T}, m)$ (see Section 5.3). A main tool in our study of the linear dynamics of random sequences of products of operators will be Birkhoff's ergodic theorem.

**Theorem 5.1.10.** *Let $(\Omega, \mathcal{A}, \mu, \tau)$ be a measure-preserving system. For any $\mu$-integrable function $f \in L^1(\Omega)$, there exists a function $g \in L^1(\Omega)$ satisfying $g \circ \tau = g$ and $\int_\Omega g\, d\mu = \int_\Omega f\, d\mu$, such that*

$$\frac{1}{N}\sum_{n=0}^{N-1} f(\tau^n\omega) \underset{N\to\infty}{\longrightarrow} g(\omega) \quad \textit{for almost every } \omega \in \Omega. \tag{5.1.6}$$

*Moreover, the system is ergodic if and only if $g(\omega) = \int_\Omega f\, d\mu$ for almost every $\omega$.*

We would like to point out the fact that the random variables $f(\tau^n\omega)$ involved in Theorem 5.1.10 are not necessarily independent. Thus, Birkhoff's ergodic theorem is a generalization of the law of large numbers. If the system is ergodic and if $A \subseteq \Omega$ is a measurable set, then, applying Theorem 5.1.10 to the function $f = \mathbb{1}_A$, we obtain

$$\frac{1}{N}\mathrm{Card}\{0 \leq n \leq N-1 : \tau^n\omega \in A\} \underset{N\to\infty}{\longrightarrow} \mu(A), \tag{5.1.7}$$

for almost every $\omega \in \Omega$. If $A_k$ is a subset of $[0,1)$, we denote by $a_k(n,\omega)$ the quantity

$$a_k(n,\omega) := \mathrm{Card}\{0 \leq i \leq n-1 : \tau^i\omega \in A_k\}.$$

Let $f : \mathbb{T} \to \mathbb{R}$ be a real function and let $\tau$ be a measure-preserving transformation on $(\Omega, \mathcal{A}, \mu)$. We denote by

$$\mathbb{S}_n^\tau f(\omega) := \sum_{i=0}^{n-1} f(\tau^i\omega)$$

the $n$th Birkhoff sum associated to $f$ and $\tau$. In particular, if $A_1, A_2$ are two disjoint Borel subsets of $[0,1)$ with $m(A_k) > 0$ for $k = 1,2$, we have

$$a_1(n,\omega) - \frac{m(A_1)}{m(A_2)}a_2(n,\omega) = \mathbb{S}_n^\tau f(\omega),$$

where $f$ is the real function on $\mathbb{T}$ defined by $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$. This function $f$ is centered, that is, $\displaystyle\int_{\mathbb{T}} f\,dm = 0$.

When $\Omega = \mathbb{T}$, when $\tau = R_\alpha$ with $\alpha$ an irrational and when $A = [a,b)$ with $0 \leq a < b \leq 1$, the convergence (5.1.7) holds for every $\omega \in \mathbb{T}$. This follows from the uniform distribution mod 1 of the sequence $(n\alpha)_{n\geq 1}$ (see, for instance, [50, Pages 49-50]).

### 5.1d Organization of the paper

The behavior of Birkhoff sums associated to the function $f := \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$ on $\mathbb{T}$, with $A_1$ and $A_2$ two disjoint Borel subsets of $[0,1)$, will play a crucial role in determining the universality of some random sequences of products of operators $(T_n(.))_{n\geq 1}$. The boundedness of these sums is linked to the so-called coboundary equation in $L^\infty(\mathbb{T})$ for the function $f$. We begin our study by presenting these results on the Birkhoff sums. We then focus on Birkhoff sums associated to this function $f$ and to the doubling map and to irrational rotations. The question of whether these Birkhoff sums satisfy a central limit theorem will be examined. This will be done in Section 5.2. After this study of the behavior of Birkhoff sums, we will study the linear dynamics of random sequences of products of operators $(T_n(.))_{n\geq 1}$, when the random operator $T(\omega)$ is equal to the adjoint of a multiplication operator $(M_{\phi_1})^*$ for every $\omega \in A_1$ and equal to the adjoint of a multiplication operator $(M_{\phi_2})^*$ for every $\omega \in A_2$, where $A_1, A_2$ are two disjoint Borel subsets of $\mathbb{T}$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$, and where $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ are nonconstant. This will be done in Section 5.3. Finally, we study random products of operators on the space of entire functions $H(\mathbb{C})$. First, we will study the situation where the operators $T(\tau^i\omega), i \geq 0$, are entire functions of exponential type of the derivation operator $D$, and afterwards, we will consider a case of random products where the operators $T(\tau^i\omega), i \geq 0$, do not commute. This will be done in Section 5.4.

### 5.1e Notation and definition

We put together here some important definitions and notations for the comprehension of this article.

**Basic ergodic facts** We denote by $\mathbb{T}$ the set $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, that we will often identify as $[0,1)$, and by $m$ the normalized Lebesgue measure on $\mathbb{T}$. A transformation $\tau : \mathbb{T} \to \mathbb{T}$ is measure-preserving if $m(\tau^{-1}(B)) = m(B)$ for every Borel subset $B$. A measure-preserving transformation $\tau$ is said to be ergodic on $(\mathbb{T}, m)$ if for every Borel subset $A$ of $\mathbb{T}$ such that $\tau^{-1}(A) = A$, we have $m(A) = 0$ or $m(A) = 1$. Moreover, this transformation $\tau$ is mixing whenever

$$m(\tau^{-n}(A) \cap B) \underset{n\to\infty}{\longrightarrow} m(A)\,m(B)$$

for every Borel sets $A, B$ of $\mathbb{T}$, and weakly mixing whenever

$$\frac{1}{n}\sum_{j=0}^{n-1} |m(\tau^{-j}(A) \cap B) - m(A)m(B)| \underset{n\to\infty}{\longrightarrow} 0$$

for every Borel subsets $A, B$ of $\mathbb{T}$. Mixing is a property of asymptotic decorrelation. Every mixing system is easily seen to be weakly mixing and ergodic. We refer to P. Walters's book [62] and to K. Petersen's book [50] for background on ergodic theory.

**Functions with bounded variation, continued fractions** We denote by $BV$ the space of real functions on $\mathbb{T}$ with bounded variation, and by $BV_0$ the space of real centered functions in $BV$, that is, the functions $f$ of $BV$ such that $\int_{\mathbb{T}} f\,dm = 0$. Any function defined on $\mathbb{T}$ can be seen as a 1-periodic function on $\mathbb{R}$. If $f \in BV_0$, we write $\|f\|_2 := \left(\int_0^1 f(t)^2\,dt\right)^{1/2}$ for the variance of $f$. For $f \in BV$, we denote by $c_r(f)$ the complex Fourier coefficients of $f$, with $r \in \mathbb{Z}$. When $f$ is in $BV_0$, we always have

$$c_r(f) = \frac{\gamma_r(f)}{r}, \quad \text{for } r \neq 0,$$

where $\gamma_r(f)$ are complex numbers such that $\sup_{r\neq 0}|\gamma_r(f)| \leq \frac{V(f)}{2\pi} < \infty$.

If $\alpha \in (0,1)$ is an irrational number, we write $\alpha = [0; a_1, a_2, \ldots]$ its continued fraction expansion, where the $a_j$'s are positive integers. We write $(\frac{p_n}{q_n})_{n\geq 0}$ for the sequence of the convergents associated to $\alpha$, where $p_n$ and $q_n$ are positive integers. The sequences $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ satisfy the following relations:

$$\begin{cases} p_0 = 0,\ p_1 = 1,\ q_0 = 1,\ q_1 = a_1 \\ q_{n+1} = a_{n+1}\,q_n + q_{n-1} \\ p_{n+1} = a_{n+1}\,p_n + p_{n-1} \\ p_{n-1}\,q_n - p_n\,q_{n-1} = (-1)^n \end{cases}, \ n \geq 1.$$

We say that $\alpha$ has bounded partial quotients if the sequence $(a_j)_{j\geq 1}$ is bounded. We refer to Khinchin's book [41] for more details on continued fractions.

**Hardy spaces** In what follows, we define Hardy spaces $H^p(\mathbb{D})$ for $p = 2$ or $p = \infty$, but we can also define them for $1 \leq p \leq \infty$. We will just need these two cases for our study. We refer to Nikolski's book [48] and to Koosis's book [42] for background on Hardy spaces.

The Hardy space $H^2(\mathbb{D})$ is the space of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\sup_{0\leq r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt = \lim_{r\nearrow 1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt < \infty.$$

For every $f, g \in H^2(\mathbb{D})$, we have

$$\langle f, g\rangle = \lim_{r\nearrow 1} \frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\overline{g(re^{it})}dt \quad \text{and} \quad \|f\|^2 = \sup_{0\leq r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt.$$

Moreover, $H^2(\mathbb{D})$ is isometrically isomorphic to the space $\ell_2(\mathbb{Z}_+)$. The space $H^\infty(\mathbb{D})$ is the space of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

We have $H^\infty(\mathbb{D}) \subset H^2(\mathbb{D})$. For any $\lambda \in \mathbb{D}$, the function $k_\lambda$ defined by

$$k_\lambda(z) = \frac{1}{1 - \overline{\lambda} z}$$

satisfies

$$f(\lambda) = \langle f, k_\lambda \rangle \tag{5.1.8}$$

for every $f \in H^2(\mathbb{D})$, and is called the reproducing kernel of $H^2(\mathbb{D})$. If $f \in H^p(\mathbb{D})$ with $p = 2$ or $p = \infty$, the limits

$$f^*(e^{it}) := \lim_{r \to 1} f(re^{it})$$

exist for almost every $e^{it} \in \mathbb{T}$. We call the function $f^*$ the boundary value of $f$. An analytic function $f : \mathbb{D} \to \mathbb{C}$ is said to be inner if $f$ is bounded on $\mathbb{D}$ and if $|f^*| = 1$ almost everywhere on $\mathbb{T}$. Any inner function $f$ satisfies $|f(z)| \leq 1$, for every $z \in \mathbb{D}$. An analytic function $F : \mathbb{D} \to \mathbb{C}$ is said to be outer if there is a constant $c \in \mathbb{T}$ and a positive function $h$ on $\mathbb{T}$ satisfying $\log h \in L^1(\mathbb{T})$, such that

$$F(z) = c \exp\left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log h(e^{it})\, dt \right), \quad \text{for every } z \in \mathbb{D}.$$

If $F, G$ are outer in $H^2(\mathbb{D})$, then $FG$ and $1/F$ are still outer. The product of two inner functions of $H^p(\mathbb{D})$, with $p = 2$ or $p = \infty$, remains an inner function. If $f \in H^p(\mathbb{D})$ with $p = 2$ or $p = \infty$, the function $Q_f$ defined on $\mathbb{D}$ by

$$Q_f(z) = \exp\left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log|f^*(e^{it})|\, dt \right)$$

is outer and is called the outer factor of $f$. An important result in Hardy spaces is the inner-outer factorization. This result states that if $f \in H^p(\mathbb{D})$ with $p = 2$ or $p = \infty$, then $f$ can be written as $f = IF$, with $I$ an inner function, and $F$ an outer function in $H^p(\mathbb{D})$. Moreover, we can take $F = Q_f$, and the decomposition becomes unique. As a consequence of the inner-outer decomposition, if $f \in H^\infty(\mathbb{D})$ and $1/f \in H^\infty(\mathbb{D})$, then $f$ is outer.

We define multiplication operators on $H^2(\mathbb{D})$ as follows. If $\phi$ is a function in $H^\infty(\mathbb{D})$, the multiplication operator $M_\phi$ on $H^2(\mathbb{D})$ is given by $M_\phi f = f\phi$. These operators satisfy $\|M_\phi\| = \|\phi\|_\infty = \sup_{z \in \mathbb{D}} |\phi(z)|$. Moreover, we have the relation

$$M_\phi^* k_\lambda = \overline{\phi(\lambda)}\, k_\lambda \quad \text{for every } \lambda \in \mathbb{D}. \tag{5.1.9}$$

It is well known that a multiplication operator $M_\phi$ is never hypercyclic on $H^2(\mathbb{D})$, but that its adjoint $M_\phi^*$ is hypercyclic if and only if $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, when $\phi \in H^\infty(\mathbb{D})$ is nonconstant. See, for instance, [31, Theorem 4.42] and [6, Example 1.11].

## 5.2 Behavior of Birkhoff sums associated to a centered real function

Birkhoff sums of centered real functions will appear a lot in Section 5.3. These Birkhoff sums will influence the dynamical behavior of the random sequence $(T_n(.))_{n \geq 1}$. The aim of Section 5.2 is to study the behavior of such random variables. First, we state a criterion for Birkhoff sums to be unbounded on a set of measure 1, and next we investigate central limit theorems for such sums.

### 5.2a Boundedness of Birkhoff sums

Let $(\Omega, \mathcal{A}, \mu, \tau)$ be an ergodic measure-preserving system and let $f \in L^1_0(\Omega)$ be a real centered function. It is known that the sequence of Birkhoff sums $(\mathbb{S}^\tau_n f(\omega))_{n\geq 1}$ changes sign infinitely often for almost every $\omega \in \Omega$, in the sense that it cannot be ultimately positive or negative ([34, Theorem 4]). Also, we have $\liminf\limits_{n\to\infty} |\mathbb{S}^\tau_n f(\omega)| = 0$ for almost every $\omega \in \Omega$ ([5], [57]), and $\limsup\limits_{n\to\infty} |\mathbb{S}^\tau_n f(\omega)| > 0$ for almost every $\omega \in \Omega$, if $f$ is nonzero ([38, Section 4.1, Remark 1]). More precisely, we have the following interesting result characterizing the boundedness of the sequence $(\mathbb{S}^\tau_n f(\omega))_{n\geq 1}$ for almost every $\omega \in \Omega$.

**Proposition 5.2.1** ([38, Section 4.1, Theorem 19])**.** *Let $(\Omega, \mathcal{A}, \mu, \tau)$ be an ergodic measure-preserving system and let $f \in L^1_0(\Omega)$ be a real centered function. The sequence $(\mathbb{S}^\tau_n f(\omega))_{n\geq 1}$ is bounded for almost every $\omega \in \Omega$ if and only if the coboundary equation $f = h - h \circ \tau$ has a solution $h$ in $L^\infty(\Omega)$.*

Thus, Proposition 5.2.1 implies that

$$\mu(\{\omega \in \Omega : \limsup_{n\to\infty} \mathbb{S}^\tau_n f(\omega) = \infty \quad \text{or} \quad \liminf_{n\to\infty} \mathbb{S}^\tau_n f(\omega) = -\infty\}) = 1$$

when the coboundary equation $f = h - h \circ \tau$ has no solution $h \in L^\infty(\Omega)$, thanks to the ergodicity of $\tau$. However, it is not clear if this implies that

$$\mu(\{\omega \in \Omega : \limsup_{n\to\infty} \mathbb{S}^\tau_n f(\omega) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \mathbb{S}^\tau_n f(\omega) = -\infty\}) = 1$$

in this case. The aim of the next subsection will be to give a sufficient condition to replace the "or" by an "and" in this set. But first, we would like to mention the two following important results. The first one is due to Adams and Rosenblatt.

**Proposition 5.2.2** ([3, Proposition 3.6])**.** *Let $f \in L^\infty(\mathbb{T})$ be a real centered function. Then, there exists an ergodic transformation $\tau$ on $(\mathbb{T}, m)$ such that the equation $f = h - h \circ \tau$ has a solution $h$ in $L^\infty(\mathbb{T})$.*

In the case of a real centered step function $f$ on $\mathbb{T}$, the authors of [3] even provide a description of this transformation $\tau$ ([3, Subsection 2.3]), of which they construct using Rokhlin towers. The second result studies when the sequence of Birkhoff sums associated to an irrational rotation and to a real step function is bounded at every point. If $f : \mathbb{T} \to \mathbb{R}$ is a step function, we define the number $\delta_f(x)$ for $x \in \mathbb{T}$ by

$$\delta_f(x) := f(x^+) - f(x^-) = \lim_{\varepsilon \searrow 0} f(x+\varepsilon) - f(x-\varepsilon),$$

which corresponds to the jump of $f$ at the point $x$. We also write $D_f := \{x \in \mathbb{T} : \delta_f(x) \neq 0\}$, which is a finite set, and we denote by $\Delta_f(x)$ the following finite number:

$$\Delta_f(x) := \sum_{k=-\infty}^{\infty} \delta_f(x + k\alpha).$$

The second result is due to I. Oren and is the following.

**Theorem 5.2.3** ([49, Theorem A])**.** *Let $\alpha$ be an irrational number, let $\tau = R_\alpha$ be the rotation of parameter $\alpha$ and let $f$ be a real centered step function. Then, the sequence $(\mathbb{S}^\tau_n f(\omega))_{n\geq 1}$ is bounded for some/every $\omega \in \mathbb{T}$ if and only if $\Delta_f(.) \equiv 0$.*

We now apply Theorem 5.2.3 to the function $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$, with $A_1 = [0,b)$ and $A_2 = [b,1)$.

**Proposition 5.2.4.** *Let $\alpha$ be an irrational number, let $\tau = R_\alpha$ and let $f$ be the real function*

$$f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2},$$

*where $A_1 = [0,b), A_2 = [b,1)$ and $b \in (0,1)$. Then, the sequence $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ is bounded for some/every $\omega \in \mathbb{T}$ if and only if $b \in \mathbb{Z}\alpha$.*

*Proof.* The function $f$ has two discontinuities: one at $x = 0$ and one at $x = b$. Moreover, we have

$$\delta_f(0) = f(0^+) - f(0^-) = 1 - (-\frac{b}{1-b}) = \frac{1}{1-b}$$
$$\delta_f(b) = f(b^+) - f(b^-) = -\frac{b}{1-b} - 1 = -\frac{1}{1-b}.$$

Suppose first that $b = l\alpha$ for some $l \in \mathbb{Z}$. If $x \notin \mathbb{Z}\alpha$, then $\Delta_f(x) = 0$. If $x \in \mathbb{Z}\alpha$, let's say $x = a\alpha$ for some $a \in \mathbb{Z}$, then $\Delta_f(x) = \delta_f(0) + \delta_f(b) = 0$. Thus $\Delta(.) \equiv 0$.

Suppose now that $b \notin \mathbb{Z}\alpha$. Then $\Delta_f(0) = \delta_f(0) \neq 0$. This proves Proposition 5.2.4. □

This shows in particular that when $b \in \mathbb{Z}\alpha$, for the function $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$ with $A_1 = [0,b)$ and $A_2 = [b,1)$, the sequence of Birkhoff sums is bounded at every point. In the case $b \notin \mathbb{Z}\alpha$, we know that at a given point $\omega \in \mathbb{T}$, we have

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{or} \quad \limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty,$$

but it does not seem clear whether both conditions can be satisfied simultaneously.

### 5.2b Central limit theorems for ergodic transformations

In this section, we discuss central limit theorems for the sequence $(\mathbb{S}_n^\tau f)_{n\geq 1}$ of Birkhoff sums associated to a real centered function $f$ on $\mathbb{T}$, where $\tau$ is an ergodic transformation.

Given an ergodic transformation $\tau$ on $(\mathbb{T}, m)$ and a real centered function $f$ on $\mathbb{T}$, we say that the sequence $(\mathbb{S}_n^\tau f)_{n\geq 1}$ satisfies a central limit theorem (CLT) if there exists a sequence $(a_n)_{n\geq 1}$ of positive real numbers with $a_n \underset{n\to\infty}{\longrightarrow} \infty$ such that

$$\frac{\mathbb{S}_n^\tau f}{a_n} \underset{n\to\infty}{\longrightarrow} \mathcal{N}(0,1) \quad \text{in distribution,}$$

where $\mathcal{N}(0,1)$ is the random variable with density $x \mapsto \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ on $\mathbb{R}$. Birkhoff sums satisfying a central limit theorem always satisfy $\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty$ and $\liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$, for almost every $\omega \in \mathbb{T}$.

**Proposition 5.2.5.** *Let $\tau$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$ and let $f \in L_0^1(\mathbb{T})$ be a real centered function on $\mathbb{T}$. Suppose that $(\mathbb{S}_n^\tau f)_{n\geq 1}$ has a subsequence satisfying a CLT. Then*

$$m(\{\omega \in \Omega : \limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \textit{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty\}) = 1.$$

*Proof.* Let $(n_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers such that

$$\frac{\mathbb{S}^\tau_{n_k} f}{a_{n_k}} \underset{k\to\infty}{\longrightarrow} \mathcal{N}(0,1) \quad \text{in distribution.}$$

Observe that the set

$$\mathcal{A}_1 := \{\omega \in \mathbb{T} : \limsup_{n\to\infty} \mathbb{S}^\tau_n f(\omega) = \infty\}$$

satisfies $m(\mathcal{A}_1) = 0$ or $m(\mathcal{A}_1) = 1$. Indeed, this set is easily seen to be $\tau$-invariant.

Now by the convergence in distribution to $\mathcal{N}(0,1)$, there exists an integer $k_1 \geq 1$ and a constant $C > 0$ such that for every $k \geq k_1$, we have

$$m\left(\frac{\mathbb{S}^\tau_{n_k} f}{a_{n_k}} \geq 1\right) \geq C.$$

For every $k_0 \geq k_1$, let $A_{k_0} := \{\exists\, k \geq k_0 : \frac{\mathbb{S}^\tau_{n_k} f}{a_{n_k}} \geq 1\}$. Since the sequence $(A_{k_0})_{k_0 \geq k_1}$ is decreasing, we have $m(\{\forall\, k_0 \geq k_1, \exists\, k \geq k_0 : \frac{\mathbb{S}^\tau_{n_k} f}{a_{n_k}} \geq 1\}) \geq C$.

In particular, this implies $m(\mathcal{A}_1) > 0$, and thus $m(\mathcal{A}_1) = 1$.

Since

$$-\frac{\mathbb{S}^\tau_{n_k} f}{a_{n_k}} \underset{k\to\infty}{\longrightarrow} \mathcal{N}(0,1) \quad \text{in distribution,}$$

we can also prove that $m(\{\liminf_{n\to\infty} \mathbb{S}^\tau_n f = -\infty\}) = 1$. This concludes the proof of Proposition 5.2.5. □

We will now give examples of such transformations $\tau$. First of all, Birkhoff sums associated to a real centered function $f$ on $\mathbb{T}$ and to an ergodic transformation $\tau$ do not necessarily satisfy a CLT. Indeed, if $\tau = R_\alpha$ is an irrational rotation with parameter $\alpha$ and $f$ is a real centered function on $\mathbb{T}$ with bounded variation, the Denjoy-Koksma inequality ([35, Theorem 3.1]) implies that $\|\mathbb{S}^\tau_{q_k} f\|_\infty \leq V(f)$ along the subsequence of denominators of the convergents of $\alpha$, where $V(f)$ is the total variation of $f$. Thus, the Birkhoff sums $(\mathbb{S}^\tau_n f)_{n\geq 1}$ cannot satisfy a CLT. In fact, for any aperiodic transformation $\tau$ on $(\mathbb{T}, m)$ and for any distribution $\nu$ on $\mathbb{R}$, there is a measurable function $f$ on $\mathbb{T}$ such that $\frac{\mathbb{S}^\tau_n f}{n}$ converges in distribution to $\nu$. This result is due to J-P. Thouvenot and B. Weiss ([59]).

However, under certain assumptions on the function $f$ and for an irrational rotation, there exist subsequences of $(\mathbb{S}^\tau_n f)_{n\geq 1}$ satisfying CLTs. An important result on this subject is the following, due to F. Huveneers ([36]).

**Theorem 5.2.6** ([36, Proposition 1]). *Let $\alpha$ be an irrational number in $(0,1)$, let $\tau = R_\alpha$ and let $f$ be the function*

$$f = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}.$$

*Then, there exists a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that $(\frac{\mathbb{S}^\tau_{n_k} f}{\sqrt{k}})_{k\geq 1}$ converges in distribution to $\mathcal{N}(0,1)$.*

In particular:

**Corollary 5.2.7.** *Let $\alpha$ be an irrational number in $(0,1)$, let $\tau = R_\alpha$ and let $f$ be the function*

$$f = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}.$$

*Then,*

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \textit{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

*for almost every $\omega \in \mathbb{T}$.*

It is important to note that Theorem 5.2.6 and Corollary 5.2.7 hold for any irrational $\alpha$ in $(0,1)$.

In [15], J-P. Conze, S. Isola, and S. Le Borgne also studied CLTs for rotations along subsequences of $(\mathbb{S}_n^\tau f)_{n\geq 1}$, when the parameter $\alpha$ of the rotation has unbounded partial quotients, and when $f$ is a certain real centered function with bounded variation. A few years later, J-P. Conze and S. Le Borgne continued the study of CLTs for irrational rotations and for a class of functions with bounded variation, which applies when the partial quotients of $\alpha$ are bounded ([16]). Their method is close to the method used in [36], and relies on an abstract CLT which is satisfied under some decorrelation conditions. It is also mentioned in [16] that when $\tau = R_\alpha$ with $\alpha$ an irrational, the behavior of the sequence $(\mathbb{S}_n^\tau f)_{n\geq 1}$ depends on the regularity of the function $f$ on $\mathbb{T}$. More precisely, under certain Diophantine conditions on $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, too much regularity on $f$ will force $f$ to be a coboundary in $L^\infty(\mathbb{T})$. This is one of the reasons why they considered less regular functions, but with bounded variation, as for example step functions.

We explore in more depth the case of irrational rotations and the case of the doubling map on $\mathbb{T}$. Recall that if $\alpha$ is an irrational number, we denote by $(a_j)_{j\geq 1}$ the partial quotients of $\alpha$, and by $(\frac{p_k}{q_k})_{k\geq 0}$ the sequence of convergents of $\alpha$.

### 5.2c CLTs for an irrational rotation $R_\alpha$ with $\alpha$ having bounded partial quotients.

In this subsection, we discuss CLTs for an irrational rotation $R_\alpha$, with $\alpha$ satisfying

$$\exists\, A > 1, \exists\, 0 \leq p < 1/8, \forall\, n \geq 1,\, a_n \leq A\, n^p. \tag{5.2.1}$$

In particular, the case of $\alpha$ having bounded partial quotients is obtained with $p = 0$.

The article [16] studies CLTs for irrational rotations, when the parameter $\alpha$ satisfies the assumption (5.2.1). An important point in the approach [16] is the control of the variance $\|\mathbb{S}_n^\tau f\|_2^2$. The authors of [16] proved that CLTs hold along certain subsets of integers with density one, on which the variance is large enough. In order to give a lower bound on the variance, they make the following assumption on the real centered functions $f$ defined on $\mathbb{T}$.

$$\exists\, M, \eta, \theta > 0 \quad \text{such that} \quad \mathrm{Card}\{0 \leq j \leq N : a_{j+1}|\gamma_{q_j}(f)| \geq \eta\} \geq \theta N,\ \forall N \geq M. \tag{5.2.2}$$

Under (5.2.2), and when $\alpha$ satisfies the assumption (5.2.1) (for example, when $\alpha$ has bounded partial quotients), we have the following CLT.

**Theorem 5.2.8.** *Let $f \in BV_0$ satisfy Condition (5.2.2). Suppose that the irrational number $\alpha$ satisfies (5.2.1), that is,*

$$\exists\, A > 1, \exists\, 0 \leq p < 1/8, \forall\, n \geq 1,\, a_n \leq A\, n^p.$$

*Then, there exists a constant $\eta_0 > 0$ such that the set*

$$W := \left\{ n \geq 1 : \|\mathbb{S}_n^\tau f\|_2 \geq \eta_0 (\log m(n))^{-\frac{1}{2}} m(n)^{\frac{1}{2}} \right\},$$

*where $m(n)$ is the unique positive integer such that $n \in [q_{m(n)}, q_{m(n)+1}[$, has density 1 in $\mathbb{N}$. Moreover, the sequence*

$$\left( \frac{\mathbb{S}_n^\tau f}{\|\mathbb{S}_n^\tau f\|_2} \right)_{n \in W}$$

*converges in distribution to $\mathcal{N}(0,1)$ as $n \to \infty$ along $W$.*

*In particular, if $\alpha$ has bounded partial quotients (that is, when $p = 0$), we can replace $m(n)$ by* $\log(n)$.

**Remark 5.2.9.** (a) Under the assumptions of Theorem 5.2.8, $m(n)$ is at least of order $\frac{\log(n)}{\log(\log(n))}$, up to a bounded factor (see [16, Remark 3.3]).

(b) The condition (5.2.1) holds for almost every irrational $\alpha$ (with respect to the Lebesgue measure): for every $p > 1$, for almost every irrational $\alpha$, there is a finite constant $A(\alpha, p)$ such that

$$a_n \leq A(\alpha, p)\, n^p, \text{ for every } n \geq 1.$$

See [16, Lemma 3.4] for more details.

Let us now apply Theorem 5.2.8 to the function $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)} \mathbb{1}_{A_2}$, when $A_1 = [0, b)$ and $A_2 = [b, 1)$, and when $b$ is a rational number in $(0,1)$. First, let us remark the following fact.

**Fact 5.2.10.** Let $(q_j)_{j \geq 0}$ be the denominators of the convergents of an irrational $\alpha$. Let $q' \geq 2$ be an integer. For every $k \geq 0$, if $q_k \equiv 0 \pmod{q'}$, then $q_{k+1} \not\equiv 0 \pmod{q'}$.

*Proof.* This follows from the fact that two consecutive $q_k$'s are coprime, since $p_{k-1} q_k - p_k q_{k-1} = (-1)^k$ for every $k \geq 1$. □

With this fact, we can prove the following lemma.

**Lemma 5.2.11.** *Let $\alpha$ be an irrational in $(0,1)$ satisfying (5.2.1) and let $\tau = R_\alpha$. Let $b = \frac{p'}{q'}$ be a rational in $(0,1)$ with $p'$ and $q'$ coprime, let $A_1 = [0, b)$ and let $A_2 = [b, 1)$. Consider the function $f \in BV_0$ defined by*

$$f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)} \mathbb{1}_{A_2}.$$

*Then, the function $f$ satisfies the assumption (5.2.2).*

*Proof.* The Fourier coefficients of $f$ are given by

$$c_r(f) = \frac{1}{1-b} \frac{1 - e^{-2i\pi r b}}{2i\pi r} = \frac{1}{1-b} \frac{\sin(\pi r \frac{p'}{q'})}{\pi r} e^{-i\pi r \frac{p'}{q'}}$$

for $r \neq 0$. Thus, for $j \geq 1$,

$$\gamma_{q_j}(f) = \frac{1}{1-b} \frac{\sin(\pi q_j \frac{p'}{q'})}{\pi} e^{-i\pi q_j \frac{p'}{q'}}. \tag{5.2.3}$$

Let us set $\eta := \min_{1\leq j\leq q'-1} \frac{1}{\pi(1-b)}|\sin(\pi j \frac{p'}{q'})| > 0$. Hence $a_{j+1}|\gamma_{q_j}(f)| \geq |\gamma_{q_j}(f)| \geq \eta$ for every $j \geq 1$ such that $q_j \not\equiv 0 \pmod{q'}$. Since $q_j \equiv 0 \pmod{q'}$ implies $q_{j+1} \not\equiv 0 \pmod{q'}$ for every $j \geq 1$ by Fact 5.2.10, it follows that

$$\frac{1}{N}\,\mathrm{Card}\{0 \leq j \leq N : a_{j+1}|\gamma_{q_j}(f)| \geq \eta\} \geq \frac{1}{4} \quad \text{for every } N \geq 2.$$

This shows that $f$ satisfies the assumption (5.2.2). □

We thus deduce the following result.

**Corollary 5.2.12.** *Let $\alpha$ be an irrational number in $(0,1)$ and let $\tau = R_\alpha$. Suppose that $\alpha$ satisfies the condition (5.2.1), that is,*

$$a_n \leq A\, n^p, \text{ for every } n \geq 1,$$

*for some $A > 0$ and $0 \leq p < 1/8$. Let $A_1, A_2$ be the two disjoint intervals of $[0,1)$ with rational endpoints, such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Let $f \in BV_0$ be the function defined by*

$$f = \mathbb{1}_{A_1} - \tfrac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}.$$

*Then, the sequence $\left(\frac{\mathbb{S}_n^\tau f}{\|\mathbb{S}_n^\tau f\|_2}\right)_{n\in W}$ converges in distribution to $\mathcal{N}(0,1)$.*

*In particular,*

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

*for almost every $\omega \in \mathbb{T}$.*

For some intervals $A_1$ and $A_2$ with irrational endpoints, it is also possible to apply Theorem 5.2.8 to the function $f$.

**Proposition 5.2.13.** *Let $\alpha$ be an irrational number in $(0,1)$ and let $\tau = R_\alpha$. Suppose that $\alpha$ satisfies Condition (5.2.1). Let $A_1 = [0,b)$ and $A_2 = [b,1)$. Suppose that the sequence $(b\,q_k)_{k\geq 1}$ is uniformly distributed mod 1, where the $q_k$'s are the denominators of the convergents of $\alpha$. Then, the function $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$ satisfies (5.2.2). In particular,*

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

*for almost every $\omega \in \mathbb{T}$.*

*Proof.* The proof follows from [16, Corollary 3.11]. □

It also mentioned in [16, Remark 3.13] that if $b = \sum_{n\geq 0} b_n q_n \alpha \bmod 1$ is the Ostrowski expansion of $b$ associated to the denominators of $\alpha$ and if $\lim_{n\to\infty} \frac{|b_n|}{a_{n+1}} = 0$, then $\lim_{k\to\infty} \|q_k b\| = 0$, where $\|x\|$ is the distance of $x$ to $\mathbb{Z}$ (see [33, Proposition 1]). In particular, under this assumption on $b$ and when the partial quotients of $\alpha$ are bounded, one can check that the sequence $(a_{j+1}|\gamma_{q_j}(f)|)_{j\geq 0}$ associated to our favorite function $f$ converges to zero, and Condition (5.2.2) is not satisfied. We thus cannot apply Theorem 5.2.8. This shows the limits of the techniques used in the article [16].

### 5.2d CLTs for an irrational rotation $R_\alpha$ with $\alpha$ having unbounded partial quotients.

In the case where the irrational number $\alpha$ does not satisfy (5.2.1), there are also CLTs for the rotation $R_\alpha$ along subsequences, but the techniques used in [15] are quite different. In this subsection, we focus on the case of an irrational number $\alpha$ with unbounded partial quotients, which means that the sequence $(a_j)_{j\geq 1}$ of the convergents of $\alpha$ is unbounded.

If $(t_k)_{k\geq 1}$ is an increasing sequence of positive integers, we set $L_0 := 0$ and $L_n = \sum_{k=1}^{n} q_{t_k}$ for $n \geq 1$. We have the following version of the CLT for an irrational rotation $R_\alpha$ when $\alpha$ is such that the sequence $(a_{j+1})_{j\geq 0}$ is growing fast enough along a subsequence.

**Theorem 5.2.14.** *Let $(t_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers. Assume the growth condition: there exists $\beta > 1$ such that $a_{t_k+1} \geq k^\beta$ for every $k \geq 1$. Then, for every function $f$ in $BV_0$ satisfying the condition*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{r\neq 0} \frac{|\gamma_{rq_k}(f)|^2}{r^2} > 0, \tag{5.2.4}$$

*we have $\|\mathbb{S}^\tau_{L_n} f\|_2^2 \sim \sum_{k=1}^{n} \sum_{r\neq 0} \frac{|\gamma_{rq_k}(f)|^2}{r^2}$ as $n \to \infty$, and the sequence $\left(\frac{\mathbb{S}^\tau_{L_n} f}{\|\mathbb{S}^\tau_{L_n} f\|_2}\right)_{n\geq 1}$ converges in distribution to $\mathcal{N}(0,1)$.*

**Remark 5.2.15.** (a) If the partial quotients of $\alpha$ are not bounded, we can always find a strictly increasing sequence of positive integers $(t_k)_{k\geq 1}$ such that $a_{t_k+1} \geq k^\beta$ with $\beta > 1$. But this doesn't mean that the condition (5.2.4) is satisfied.

(b) Condition (5.2.4) is satisfied in particular when $\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\gamma_{q_{t_k}}(f)|^2 > 0$.

(c) For a real function $f$ on $\mathbb{T}$ which is continuous and piecewise $C^1$, we know that $|\gamma_{rq_n}(f)|^2 = |r|^2 q_n^2 |c_{rq_n}(f)|^2 = |c_{rq_n}(f')|^2$, which converges to 0 as $n \to \infty$. In particular, $f$ cannot satisfy the condition (5.2.4).

We now apply Theorem 5.2.14 to our favorite function $f$.

**Proposition 5.2.16.** *Let $\alpha$ be an irrational in $(0,1)$ and let $\tau = R_\alpha$. Let $b = \frac{p'}{q'}$ be a rational in $(0,1)$ with $p'$ and $q'$ coprime, let $A_1 = [0,b)$ and let $A_2 = [b,1)$. Consider the function $f \in BV_0$ defined by*

$$f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)} \mathbb{1}_{A_2}.$$

*Suppose that $a_{t_k+1} \underset{k\to\infty}{\longrightarrow} \infty$ along a sequence $(t_k)_{k\geq 1}$ of positive integers such that $q_{t_k} \not\equiv 0 \pmod{q'}$ for every $k \geq 1$. Then, the sequence $\left(\frac{\mathbb{S}^\tau_n f}{\|\mathbb{S}^\tau_n f\|_2}\right)_{n\geq 1}$ has a subsequence converging in distribution to $\mathcal{N}(0,1)$ and for which $\|\mathbb{S}^\tau_n f\|_2 \underset{n\to\infty}{\longrightarrow} \infty$.*

*Proof.* Let $(t_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers such that $a_{t_k+1} \underset{k\to\infty}{\longrightarrow} \infty$ and $q_{t_k} \not\equiv 0 \pmod{q'}$ for every $k \geq 1$. The coefficients $\gamma_{q_{t_k}}(f)$ are given by (5.2.3). For every

$k \geq 1$, we have

$$|\gamma_{q_{t_k}}(f)|^2 \geq \frac{1}{(1-b)^2\pi^2} \min_{1 \leq j \leq q'-1} |\sin(\pi j \tfrac{p'}{q'})|^2 > 0,$$

and so Condition (5.2.4) is clearly satisfied in this case. The CLT follows from Theorem 5.2.14. □

We deduce the following consequence.

**Corollary 5.2.17.** *Let $\alpha$ be an irrational in $(0,1)$ and let $\tau = R_\alpha$. Let $b = \frac{p'}{q'}$ be a rational in $(0,1)$ with $p'$ and $q'$ coprime, let $A_1 = [0,b)$ and let $A_2 = [b,1)$. Consider the function $f \in BV_0$ defined by*

$$f = \mathbb{1}_{A_1} - \tfrac{m(A_1)}{m(A_2)} \mathbb{1}_{A_2}.$$

*Suppose that $a_{t_k+1} \underset{k\to\infty}{\longrightarrow} \infty$ along a sequence $(t_k)_{k\geq 1}$ of positive integers such that $q_{t_k} \not\equiv 0 \pmod{q'}$ for every $k \geq 1$. Then,*

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \textit{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

*for almost every $\omega \in \mathbb{T}$.*

Exactly as in Proposition 5.2.13, when the partial quotients of $\alpha$ are unbounded, we have the following result.

**Proposition 5.2.18.** *Let $\alpha$ be an irrational number in $(0,1)$ and $\tau = R_\alpha$. Suppose that there exists a sequence $(t_k)_{k\geq 1}$ of positive integers such that $a_{t_k+1} \geq k^\beta$ for every $k \geq 1$, with $\beta > 1$. Let $A_1 = [0,b)$ and $A_2 = [b,1)$. Suppose that the sequence $(bq_{t_k})_{k\geq 1}$ is uniformly distributed mod 1, where the $q_j$'s are the denominators of the convergents of $\alpha$. Then, the function $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$ satisfies (5.2.4). In particular,*

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \textit{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

*for almost every $\omega \in \mathbb{T}$.*

*Proof.* The proof follows from [15, Subsection 2.2]. □

The same remark as in the case of an irrational number $\alpha$ with bounded partial quotients holds: if $b = \sum_{n\geq 0} b_n q_n \alpha \bmod 1$ is the Ostrowski expansion of $b$ associated to the denominators of $\alpha$ and if $\lim_{n\to\infty} \frac{|b_n|}{a_{n+1}} = 0$, then Condition (5.2.4) is not satisfied for our favorite function $f$.

## 5.2e CLTs for the doubling map

For weakly mixing transformations, the situation is considerably different. The first one to consider CLTs for weakly mixing transformations is Kac ([37]), who studied CLTs for the doubling map and for integrable functions on $\mathbb{T}$ whose Fourier coefficients verify a certain growth condition. More precisely, he proved the following result.

**Theorem 5.2.19.** *Let $f$ be a centered function on $\mathbb{T}$ such that the Fourier coefficients of $f$ satisfy*

$$\exists\, \beta > 1/2, \forall\, n \neq 0,\ |c_n(f)| \leq \frac{C}{|n|^\beta}.$$

*Then, there is a constant $\sigma^2 \geq 0$ such that the sequence $(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(2^k.))_{n\geq 1}$ converges in distribution to $\mathcal{N}(0,\sigma^2)$. Moreover*

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} f(2^k.) \right\|_2^2,$$

*and if $\sigma^2 = 0$, then there exists a function $g$ in $L^2(\mathbb{T})$ such that*

$$f(t) = g(t) - g(2t), \quad \textit{for almost every } t \in \mathbb{T}. \tag{5.2.5}$$

Many improvements regarding CLTs for weakly mixing transformations have been obtained later. For a class of piecewise monotonic weakly mixing transformations on $\mathbb{T}$, S. Wong proved that CLTs hold for Hölder continuous functions defined on $[0,1)$ ([63]). G. Keller and J. Rousseau-Egele also proved the existence of CLTs for such transformations, when the function $f$ is in $BV$, by using spectral properties of the Perron-Frobenius operator associated to the weakly mixing transformation $\tau$ on $[0,1)$ ([40], [55]). J. Rousseau-Egele has even investigated the rate of convergence in the CLTs associated to such piecewise monotonic transformations and also proved a local limit theorem, when the function $f$ is in $BV$. For more information on CLTs for weakly mixing transformations, we refer to Denker's article ([17]).

We consider now the case of the doubling map on $\mathbb{T}$, that is, $\tau(x) = 2x \pmod 1$. We consider the case of the two intervals $A_1 = [0,b)$ and $A_2 = [b,1)$ of $[0,1)$. We denote by $f$ the following real function in $BV_0$ given by

$$f = \mathbb{1}_{A_1} - \tfrac{m(A_1)}{m(A_2)} \mathbb{1}_{A_2}.$$

We will show, using Kac's result (Theorem 5.2.19), that the sequence $(\mathbb{S}_n^\tau f)_{n\geq 1}$ satisfies a CLT with $\sigma^2 > 0$, when $b \in (0,1)$. Let us notice that this function $f$ satisfies the assumptions of Theorem 5.2.19. The only thing to check is that $\sigma^2 > 0$, that is, we have to prove that the coboundary equation (5.2.5) for $f$ and $\tau$ has no solution $g \in L^2(\mathbb{T})$.

**Lemma 5.2.20.** *Let $\tau$ be the doubling map. Let $A_1, A_2$ be the two intervals of $[0,1)$ defined by $A_1 = [0,b)$ and $A_2 = [b,1)$, where $b$ is a real number in $(0,1)$. Let $f$ be the function*

$$f = \mathbb{1}_{A_1} - \tfrac{b}{1-b} \mathbb{1}_{A_2}.$$

*Then, the coboundary equation $f = g - g\circ\tau$ has no solution $g$ in $L^2(\mathbb{T})$.*

*Proof.* Let $b \in (0,1)$ and let $q \geq 1$ be an odd integer such that $b \notin \frac{1}{q}\mathbb{Z}$. Suppose that there exists a function $g \in L^2(\mathbb{T})$ satisfying the coboundary equation.

First, let us notice that $c_{2k+1}(g\circ\tau) = 0$ for every $k \in \mathbb{Z}$. Indeed,

$$\begin{aligned} c_{2k+1}(g\circ\tau) &= \int_0^1 g(2t) e^{-2i\pi(2k+1)t} dt \\ &= \frac{1}{2}\left( \int_0^1 g(u) e^{-i\pi(2k+1)u} du + \int_1^2 g(u) e^{-i\pi(2k+1)u} du \right) \\ &= \frac{1}{2}\left( \int_0^1 g(u) e^{-i\pi(2k+1)u} du + \int_0^1 g(u) e^{-i\pi(2k+1)u} e^{-i\pi(2k+1)} du \right) \\ &= 0. \end{aligned}$$

We can also check that $c_{2k}(g\circ\tau) = c_k(g)$ for every $k \in \mathbb{Z}$. Thus, the function $g$ satisfies the equations

$$c_{2k+1}(f) = c_{2k+1}(g) \tag{5.2.6}$$
$$c_{2k}(f) = c_{2k}(g) - c_k(g) \tag{5.2.7}$$

for every $k \in \mathbb{Z}$.

The Fourier coefficients of $f$ are given by

$$\begin{aligned} c_n(f) &= \frac{1}{1-b}\,\frac{1-e^{-2i\pi nb}}{2i\pi n} \\ &= \frac{1}{\pi(1-b)}\,\frac{\sin(\pi nb)}{n}\,e^{-i\pi nb}. \end{aligned}$$

for $n \neq 0$. From equation (5.2.7), we have

$$c_{q.2^j}(f) = c_{q.2^j}(g) - c_{q.2^{j-1}}(g) \quad \text{for every } j \geq 1,$$

and thus

$$c_{q.2^k}(g) = \sum_{j=0}^{k} c_{q.2^j}(f) \quad \text{for every } k \geq 1,$$

since $c_q(g) = c_q(f)$. In particular

$$\operatorname{Im}(c_{q.2^k}(g)) = -\frac{1}{q\pi(1-b)}\sum_{j=0}^{k} 2^{-j}(\sin(\pi q 2^j b))^2 \leq -\frac{1}{q\pi(1-b)}(\sin(\pi q b))^2 < 0$$

for every $k \geq 1$, and thus

$$\operatorname{Im}(c_{q.2^k}(g)) \nrightarrow 0 \quad \text{as } k \to \infty.$$

This proves that such a function $g \in L^2(\mathbb{T})$ does not exist. □

We thus deduce the following CLT for the doubling map.

**Corollary 5.2.21.** *Let $\tau$ be the doubling map on $\mathbb{T}$ and let $A_1, A_2$ be two disjoint intervals of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Let $f \in BV_0$ be the function*

$$f = \mathbb{1}_{A_1} - \tfrac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}.$$

*Then, there exists a real number $\sigma^2 > 0$ such that the sequence $(\frac{1}{\sqrt{n}}\mathbb{S}_n^\tau f)_{n\geq 1}$ converges in distribution to $\mathcal{N}(0,\sigma^2)$.*

*In particular,*

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \textit{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

*for almost every $\omega \in \mathbb{T}$.*

## 5.3 Random products of adjoint multipliers on the Hardy space $H^2(\mathbb{D})$

Historically, the first examples of hypercyclic operators are Birkhoff's operators on the space of entire functions, MacLane's operator on the space of entire functions, and Rolewicz's operators on $X = \ell_p$ spaces, with $1 \leq p < \infty$, or on $X = c_0$. Precisely, Rolewicz's operators are the operators $T = \lambda B$, where $\lambda \in \mathbb{C}$ is nonzero and $B$ is the backward shift operator on $X$ defined by

$$B(x_0, x_1, \dots) = (x_1, x_2, \dots).$$

These operators are hypercyclic precisely when $|\lambda| > 1$. In the case $|\lambda| \leq 1$, the operator $T = \lambda B$ is just a contraction, so every orbit of $T$ is bounded. Consequently, the operator $T$ cannot be hypercyclic in this case. In the case $|\lambda| > 1$, an application of the Hypercyclicity Criterion (Theorem 5.1.2) with the set $\mathcal{D}_1 = c_{00}$ of finitely supported sequences as a dense subset of $X$, and with the operators $S_k = \lambda^{-k} S^k$, where $S$ the forward shift on $X$, shows that the operator $T = \lambda B$ is topologically mixing on $X$. The operator $T = \lambda B$ on $X = \ell_2$ is topologically conjugate to the adjoint multiplier operator $(M_{\lambda z})^*$ on $H^2(\mathbb{D})$. Moreover, we have seen in the introduction that the adjoint of a multiplication operator $(M_\phi)^*$ is hypercyclic if and only if $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, when $\phi$ is nonconstant. These operators are classic in linear dynamics and since their linear dynamics is well known, it is natural to investigate the universality of a sequence of random products of operators defined by adjoints of multiplication operators on $H^2(\mathbb{D})$. This section is devoted to this study. We will see that the study of the universality of random sequences $(T_n(.))_{n\geq 1}$ is not this simple compared to the study of the hypercyclicity of the adjoints of multiplication operators.

It is known that a multiplication operator $M_\phi$ is never hypercyclic on $H^2(\mathbb{D})$ (see, for instance, [31, Section 4.4]). For random sequences of products of multiplication operators, we have the following remark. Let us recall that if $A_k$ is a Borel subset of $[0,1)$, we denote by $a_k(n,\omega)$ the number

$$a_k(n,\omega) := \mathrm{Card}\{0 \leq i \leq n-1 : \tau^i \omega \in A_k\}.$$

**Proposition 5.3.1.** *Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Let $T(\omega)$ be defined by*

$$T(\omega) = \begin{cases} M_{\phi_1} & \text{if} \quad \omega \in A_1 \\ M_{\phi_2} & \text{if} \quad \omega \in A_2 \end{cases}, \tag{5.3.1}$$

*where $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ are nonconstant functions. Suppose that the transformation $\tau$ is ergodic on $(\mathbb{T}, m)$. Then, for almost every $\omega \in \mathbb{T}$, the sequence $(T_n(\omega))_{n\geq 1}$ is not universal.*

*Proof.* In this case, the operators $T_n(\omega)$ are given by

$$T_n(\omega) = (M_{\phi_1})^{a_1(n,\omega)} (M_{\phi_2})^{a_2(n,\omega)},$$

since the operators $M_{\phi_1}$ and $M_{\phi_2}$ commute. By Theorem 5.1.10 applied to the functions $\mathbb{1}_{A_1}$ and $\mathbb{1}_{A_2}$, there exists a subset $E$ of $\mathbb{T}$ with $m(E) = 1$ such that for every $\omega \in E$,

$$\frac{a_1(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} m(A_1) \quad \text{and} \quad \frac{a_2(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} m(A_2). \tag{5.3.2}$$

Suppose that there exists a function $h \in H^2(\mathbb{D})$ such that the set

$$\{T_n(\omega)h : n \geq 1\} = \{\phi_1^{a_1(n,\omega)}\phi_2^{a_2(n,\omega)}h : n \geq 1\}$$

is dense in $H^2(\mathbb{D})$. Then, for every $\lambda \in \mathbb{D}$, the set $\{\phi_1(\lambda)^{a_1(n,\omega)}\phi_2(\lambda)^{a_2(n,\omega)}h(\lambda) : n \geq 1\}$ is dense in $\mathbb{C}$, by continuity of point evaluations. In particular, the function $h$ does not vanish on $\mathbb{D}$. Let us notice that by (5.3.2), we have

$$|\phi_1(\lambda)|^{a_1(n,\omega)}|\phi_2(\lambda)|^{a_2(n,\omega)} \underset{n\to\infty}{=} e^{n[\log(|\phi_1(\lambda)|^{m(A_1)}|\phi_2(\lambda)|^{m(A_2)})+o(1)]}.$$

Thus, if there exists $\lambda \in \mathbb{D}$ such that $|\phi_1(\lambda)|^{m(A_1)}|\phi_2(\lambda)|^{m(A_2)} < 1$, or if there exists $\mu \in \mathbb{D}$ such that $|\phi_1(\mu)|^{m(A_1)}|\phi_2(\mu)|^{m(A_2)} > 1$, we obtain a contradiction.

Suppose now that we are in the case where $|\phi_1(\lambda)|^{m(A_1)}|\phi_2(\lambda)|^{m(A_2)} = 1$ for every $\lambda \in \mathbb{D}$. If $\phi_1(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, there exists $\alpha \in \mathbb{T}$ such that $|\phi_1(\alpha)| = 1$, and thus $|\phi_1(\alpha)^{a_1(n,\omega)}\phi_2(\alpha)^{a_2(n,\omega)}h(\alpha)| = |h(\alpha)|$ for every $n \geq 1$, which also contradicts the density of the set $\{\phi_1(\alpha)^{a_1(n,\omega)}\phi_2(\alpha)^{a_2(n,\omega)}h(\alpha) : n \geq 1\}$. We thus have $\phi_1(\mathbb{D}) \subset \mathbb{D}$ or $\phi_1(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, by the open mapping theorem.

Suppose that we are in the case $\phi_1(\mathbb{D}) \subset \mathbb{D}$. By the density of the set $\{T_n(\omega)h : n \geq 1\}$ in $H^2(\mathbb{D})$, we can find a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that $\phi_1^{a_1(n_k,\omega)}\phi_2^{a_2(n_k,\omega)}h \underset{k\to\infty}{\longrightarrow} z$ uniformly on each compact of $\mathbb{D}$. Let us remark that, for every $n \geq 1$,

$$|\phi_1(0)|^{a_1(n,\omega)}|\phi_2(0)|^{a_2(n,\omega)} = |\phi_1(0)|^{a_1(n,\omega)-\frac{m(A_1)}{m(A_2)}a_2(n,\omega)},$$

since $|\phi_2(0)| = |\phi_1(0)|^{-\frac{m(A_1)}{m(A_2)}}$. We recognize the Birkhoff sums

$$a_1(n,\omega) - \frac{m(A_1)}{m(A_2)}a_2(n,\omega) = \mathbb{S}_n^\tau f(\omega)$$

associated to the function $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$. Thus, looking at $z = 0$, the sequence $(|\phi_1(0)|^{\mathbb{S}_{n_k}^\tau f(\omega)})_{k\geq 1}$ converges to 0. This implies that $(\mathbb{S}_{n_k}^\tau f(\omega))_{k\geq 1}$ diverges to $\infty$, because $|\phi_1(0)| < 1$. But now if we look at $z = 1/2$, we obtain a contradiction. In the same way, the case $\phi_1(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ also leads to a contradiction. □

In this section, we investigate the linear dynamics of random sequences $(T_n(.))_{n\geq 1}$, with

$$T(\omega) = \begin{cases} (M_{\phi_1})^* & \text{if} \quad \omega \in A_1 \\ (M_{\phi_2})^* & \text{if} \quad \omega \in A_2 \end{cases}, \tag{5.3.3}$$

where $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ are nonconstant functions, $A_1, A_2$ are two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$, and where $\tau$ an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. In this case, the operators $T_n(\omega)$ are given by

$$T_n(\omega) = (M_{\phi_1}^{a_1(n,\omega)})^*(M_{\phi_2}^{a_2(n,\omega)})^*,$$

because the operators $M_{\phi_1}$ and $M_{\phi_2}$ commute. We will first start with the case where $m(A_1) = m(A_2) = 1/2$, simply because this case is simpler than the general case, and also because certain assumptions on the functions $\phi_1$ and $\phi_2$ will appear in this first case but will not necessarily appear in the general case. The behavior of the sequence $(T_n(\omega))_{n\geq 1}$ will depend very much on the ergodic transformation $\tau$, as we will see.

### 5.3a Case $m(A_1) = m(A_2) = 1/2$

In this subsection, we consider $A_1$ and $A_2$ two disjoint Borel subsets of $[0,1)$ satisfying $A_1 \cup A_2 = [0,1)$ and $m(A_1) = m(A_2) = 1/2$.

The computation of $\|T_n(\omega)\|$ gives a simple obstruction to the universality of the random sequence $(T_n(.))_{n\geq 1}$.

**Proposition 5.3.2.** *Let $\tau$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant. Suppose that $\|\phi_1\|_\infty \|\phi_2\|_\infty < 1$. Then, for almost every $\omega \in \mathbb{T}$, the sequence $(T_n(\omega))_{n\geq 1}$ is not universal. In particular, the random sequence $(T_n(.))_{n\geq 1}$ is not universal.*

*Proof.* By Theorem 5.1.10 applied to the functions $\mathbb{1}_{A_1}$ and $\mathbb{1}_{A_2}$, there exists a Borel subset $E$ of $\mathbb{T}$ with $m(E) = 1$ such that for every $\omega \in E$,

$$\frac{a_1(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} 1/2 \quad \text{and} \quad \frac{a_2(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} 1/2. \tag{5.3.4}$$

Let us observe that

$$\|T_n(\omega)\| = \|\phi_1^{a_1(n,\omega)} \phi_2^{a_2(n,\omega)}\|_\infty \leq \|\phi_1\|_\infty^{a_1(n,\omega)} \|\phi_2\|_\infty^{a_2(n,\omega)}. \tag{5.3.5}$$

But now since $\|\phi_1\|_\infty \|\phi_2\|_\infty < 1$, the upper bound of (5.3.5) converges to 0, since by (5.3.4), we have

$$\|\phi_1\|_\infty^{a_1(n,\omega)} \|\phi_2\|_\infty^{a_2(n,\omega)} \underset{n\to\infty}{=} e^{\frac{1}{2}n[\log(\|\phi_1\|_\infty \|\phi_2\|_\infty) + o(1)]}.$$

Thus, $\|T_n(\omega)\|$ converges to 0, and every orbit under $(T_n(\omega))_{n\geq 1}$ is bounded. In particular, the sequence $(T_n(\omega))_{n\geq 1}$ cannot be universal, and this concludes the proof. □

We will now see a sufficient condition for the random sequence $(T_n(.))_{n\geq 1}$ to be universal. The proof is based on the following useful criterion, which is mainly applied to operators with many eigenvalues. In what follows, if $e \in X$ is an eigenvector of $T$, we denote by $\lambda(T, e)$ the corresponding eigenvalue. The linear span of a set $\mathcal{A}$ of $X$ will be written as span$(\mathcal{A})$.

**Proposition 5.3.3** ([9, Theorem 7])**.** *Suppose that $X$ is a separable Fréchet space and $(T_n)_{n\geq 1}$ is a sequence of operators on this space.*

*Suppose that there are two subsets $\mathcal{A}$ and $\mathcal{B}$ of $X$ satisfying*

*(a) Every element of $\mathcal{A} \cup \mathcal{B}$ is an eigenvector of $T_n$, $n \geq 1$, satisfying $|\lambda(T_n, a)| \underset{n\to\infty}{\longrightarrow} 0$ for every $a \in \mathcal{A}$ and $|\lambda(T_n, b)| \underset{n\to\infty}{\longrightarrow} \infty$ for every $b \in \mathcal{B}$.*

*(b) span$(\mathcal{A})$ and span$(\mathcal{B})$ are dense in $X$.*

*Then, the sequence $(T_n)_{n\geq 1}$ is topologically mixing.*

The sufficient condition for the universality of the random sequence is the following.

**Theorem 5.3.4.** *Let $\tau$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be such that $\phi_1\phi_2$ is nonconstant on $\mathbb{D}$ and $(\phi_1\phi_2)(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$. Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically mixing.*

*Proof.* Using the identity (5.1.9) on the eigenvectors for the adjoint of a multiplication operator, we get

$$T_n(\omega)k_z = \overline{\phi_1(z)}^{a_1(n,\omega)}\,\overline{\phi_2(z)}^{a_2(n,\omega)}\,k_z, \tag{5.3.6}$$

for every $z \in \mathbb{D}$. Since the function $\phi_1\phi_2$ is nonconstant on $\mathbb{D}$, the open mapping theorem implies that there exist $\lambda, \mu \in \mathbb{D}$ such that $|\phi_1\phi_2(\lambda)| > 1$ and $|\phi_1\phi_2(\mu)| < 1$. Now, it is well-known (see [31, Lemma 4.39] for example) that if $\Lambda \subset \mathbb{D}$ is a set with an accumulation point in $\mathbb{D}$, then the set $\operatorname{span}\{k_\lambda : \lambda \in \Lambda\}$ is dense in $H^2(\mathbb{D})$. Thus, the sets

$$\mathcal{A}_1 := \{k_z : z \in \mathbb{D}, |\phi_1(z)\phi_2(z)| < 1\} \quad \text{and} \quad \mathcal{A}_2 := \{k_z : z \in \mathbb{D}, |\phi_1(z)\phi_2(z)| > 1\}$$

are such that $\operatorname{span}(\mathcal{A}_1)$ and $\operatorname{span}(\mathcal{A}_2)$ are dense in $H^2(\mathbb{D})$, since the sets $\{z \in \mathbb{D} : |\phi_1\phi_2(z)| < 1\}$ and $\{z \in \mathbb{D} : |\phi_1\phi_2(z)| > 1\}$ both have an accumulation point in $\mathbb{D}$. Finally, by Theorem 5.1.10, there exists a Borel subset $E$ of $\mathbb{T}$ with $m(E) = 1$ such that for every $\omega \in E$,

$$\frac{a_1(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} 1/2 \quad \text{and} \quad \frac{a_2(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} 1/2.$$

We thus have

$$|\lambda(T_n(\omega), k_z)| \underset{n\to\infty}{=} e^{\frac{1}{2}n[\log(|\phi_1(z)\phi_2(z)|)+o(1)]},$$

and it easily follows that $|\lambda(T_n(\omega), k_z)| \underset{n\to\infty}{\longrightarrow} 0$ for every $k_z \in \mathcal{A}_1$ and $|\lambda(T_n(\omega), k_w)| \underset{n\to\infty}{\longrightarrow} \infty$ for every $k_w \in \mathcal{A}_2$. Proposition 5.3.3 thus applies, and this concludes the proof of Theorem 5.3.4. □

**Remark 5.3.5.** If the ergodic transformation $\tau$ involved in Theorem 5.3.4 is an irrational rotation and if $A_1, A_2$ are intervals of $[0,1)$, the sequence $(T_n(\omega))_{n\geq 1}$ is in fact topologically mixing for every $\omega \in \mathbb{T}$. This comes from the uniform distribution of the sequence $(n\alpha)_{n\geq 1}$ mod 1.

Proposition 5.3.4 leads to two limit cases: the case $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{D}$ and the case $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, where $\phi_1\phi_2$ is nonconstant. We first treat some straightforward situations within these two cases.

**Proposition 5.3.6.** *Let $\tau$ be a measure-preserving transformation on $(\mathbb{T}, m)$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be such that $\phi_1(\mathbb{D}) \subseteq \overline{\mathbb{D}}$ and $\phi_2(\mathbb{D}) \subseteq \overline{\mathbb{D}}$. Then, for every $\omega \in \mathbb{T}$, the sequence $(T_n(\omega))_{n\geq 1}$ is not universal.*

*Proof.* This comes from the fact that $\|T_n(\omega)\| \leq 1$ for every $\omega \in \mathbb{T}$ and every $n \geq 1$. Every orbit under $(T_n(\omega))_{n\geq 1}$ is bounded in this case and the sequence cannot be universal. □

As a consequence of Proposition 5.3.6, we treat the second easy situation.

**Corollary 5.3.7.** *Let $\tau$ be a measure-preserving transformation on $(\mathbb{T}, m)$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be such that $\phi_1(\mathbb{D}) \subseteq \mathbb{C} \setminus \mathbb{D}$ and $\phi_2(\mathbb{D}) \subseteq \mathbb{C} \setminus \mathbb{D}$. Then, for every $\omega \in \mathbb{T}$, the sequence $(T_n(\omega))_{n\geq 1}$ is not universal.*

*Proof.* In this case, the operators $T_n(\omega)$ are invertible, with

$$T_n(\omega)^{-1} = (M^*_{\phi_1^{-1}})^{a_1(n,\omega)}\,(M^*_{\phi_2^{-1}})^{a_2(n,\omega)}.$$

Indeed, the functions $1/\phi_1$ and $1/\phi_2$ are well defined and belong to $H^\infty(\mathbb{D})$ under the assumptions of Corollary 5.3.7, and thus the operators $M_{\phi_1}$ and $M_{\phi_2}$ are invertible, with $M_{\phi_1}^{-1} = M_{1/\phi_1}$ and $M_{\phi_2}^{-1} = M_{1/\phi_2}$.

But since $(\phi_1^{-1})(\mathbb{D}) \subseteq \overline{\mathbb{D}}$ and $(\phi_2^{-1})(\mathbb{D}) \subseteq \overline{\mathbb{D}}$, the sequence $(T_n(\omega)^{-1})_{n\geq 1}$ is not universal, by Proposition 5.3.6. In particular, it cannot be topologically transitive (by Proposition 5.1.1), and the sequence $(T_n(\omega))_{n\geq 1}$ cannot be topologically transitive either. $\square$

In light of the forthcoming results, it is worth noting that if $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ are nonconstant such that $\|\phi_1\|_\infty \leq 1$, $\|\phi_2\|_\infty \leq 1$ and $\phi_1\phi_2$ is inner, then $\phi_1$ and $\phi_2$ are both inner. Indeed, in this case, we have that $1 \leq |\phi_1^*| = \frac{1}{|\phi_2^*|} \leq 1$. Thus $|\phi_1^*| = |\phi_2^*| = 1$. The same results holds if $\phi_1\phi_2$ is inner, $\phi_1(\mathbb{D}) \subseteq \mathbb{C} \setminus \mathbb{D}$ and $\phi_2(\mathbb{D}) \subseteq \mathbb{C} \setminus \mathbb{D}$.

We now focus on the case where $\phi_1\phi_2$ is an inner function. Depending on the properties of the ergodic transformation $\tau$ on $(\mathbb{T}, m)$, we obtain sufficient conditions for the random sequence $(T_n(.))_{n\geq 1}$ to be universal. We first make the following observation.

**Observation 5.3.8.** *Let $\varphi \in H^\infty(\mathbb{D})$ be an inner function. Then the multiplication operator $M_\varphi$ on $H^2(\mathbb{D})$ is an isometry. Indeed, for every $f \in H^2(\mathbb{D})$,*

$$\|M_\varphi f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\varphi^*(e^{it})|^2 |f^*(e^{it})|^2 dt = \int_0^{2\pi} |f^*(e^{it})|^2 dt = \|f\|^2.$$

*In particular, $M_\varphi^* M_\varphi = I$.*

We can now state the first sufficient condition.

**Theorem 5.3.9.** *Let $\tau$ be an ergodic transformation on $(\mathbb{T}, m)$. Let $f$ be the function $f = \mathbb{1}_{A_1} - \mathbb{1}_{A_2}$. Suppose that the coboundary equation $f = h - h \circ \tau$ has no solution $h \in L^\infty(\mathbb{T})$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that $\phi_1\phi_2$ is inner.*

*If $\phi_1(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ and $1/\phi_1 \in H^\infty(\mathbb{D})$, or $\phi_2(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ and $1/\phi_2 \in H^\infty(\mathbb{D})$, then the sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.*

*Proof.* Suppose for example that $\phi_1(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ and that $1/\phi_1 \in H^\infty(\mathbb{D})$. Then, since $\phi_1$ is nonconstant, there are two nonempty open sets $U, V$ in $\mathbb{D}$ such that

$$|\phi_1(z)| < 1 \quad \text{if} \quad z \in U \quad \text{and} \quad |\phi_1(w)| > 1 \quad \text{if} \quad w \in V.$$

Since $1/\phi_1 \in H^\infty(\mathbb{D})$, the function $\phi_1$ does not vanish on $U$.

Since the coboundary equation $f = h - h \circ \tau$ has no solution in $L^\infty(\mathbb{T})$, and since the function $f$ is a real centered function, there exists (by Proposition 5.2.1) a Borel subset $E \subset \mathbb{T}$ with $m(E) = 1$ such that for every $\omega \in E$,

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{or} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty.$$

Let $\omega \in E$. Suppose first that $\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty$, that is, there is a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that

$$\mathbb{S}_{n_k}^\tau f(\omega) \underset{k\to\infty}{\longrightarrow} \infty.$$

We will apply the Universality Criterion using the eigenvectors $k_z$ of $T_n(\omega)$. Recall that

$$T_n(\omega) k_z = \overline{\phi_1(z)}^{a_1(n,\omega)} \, \overline{\phi_2(z)}^{a_2(n,\omega)} k_z$$

for every $z \in \mathbb{D}$. Since $\phi_1\phi_2$ is inner, we have

$$|\phi_1(z)\phi_2(z)| \leq 1$$

for every $z \in U$. In particular, we have

$$|\phi_2(z)|^{a_2(n,\omega)} \leq |\phi_1(z)|^{-a_2(n,\omega)}$$

for every $z \in U$, and thus

$$\|T_{n_k}(\omega)k_z\| \leq |\phi_1(z)|^{a_1(n_k,\omega)-a_2(n_k,\omega)}\|k_z\|$$

for every $z \in U$. We recognize the Birkhoff sums $\mathbb{S}_n^\tau f(\omega)$ associated to the real centered function $f = \mathbb{1}_{A_1} - \mathbb{1}_{A_2}$, and we have

$$\|T_{n_k}(\omega)k_z\| \leq |\phi_1(z)|^{\mathbb{S}_{n_k}^\tau f(\omega)}\|k_z\| \tag{5.3.7}$$

for every $z \in U$. Let us denote by $Z_1$ the following dense subset of $H^2(\mathbb{D})$

$$Z_1 := \operatorname{span}\{k_z : z \in U\}.$$

Then by (5.3.7), we have $T_{n_k}(\omega)x \underset{k\to\infty}{\longrightarrow} 0$ for every $x \in Z_1$.

Let us set $\phi_1\phi_2 = \varphi$. Since $1/\phi_1 \in H^\infty(\mathbb{D})$, $M^*_{1/\phi_1}$ is a bounded operator on $H^2(\mathbb{D})$, and we have that

$$T_n(\omega) = (M^*_{\phi_1})^{a_1(n,\omega)}(M^*_{1/\phi_1})^{a_2(n,\omega)}(M^*_\varphi)^{a_2(n,\omega)}.$$

Let us denote by $Z_2$ the following dense subset of $H^2(\mathbb{D})$

$$Z_2 := \operatorname{span}\{k_z : z \in V\}.$$

Using Observation 5.3.8, we define an operator $S_{n_k}(\omega)$ on $Z_2$ by setting

$$S_{n_k}(\omega)k_z := \overline{\phi_1(z)}^{-\mathbb{S}_{n_k}^\tau f(\omega)}(M_\varphi)^{a_2(n_k,\omega)}k_z$$

for every $z \in V$ and extending it by linearity to $Z_2$. Using the fact that $\varphi$ is inner and using the expression of $T_n(\omega)$ above, we obtain

$$T_{n_k}(\omega)S_{n_k}(\omega)k_z = k_z$$

for every $z \in V$, and thus,

$$T_{n_k}(\omega)S_{n_k}(\omega)x = x$$

for every $x \in Z_2$. Finally, since $\|M_\varphi\| \leq 1$, we have

$$\|S_{n_k}(\omega)k_z\| \leq |\phi_1(z)|^{-\mathbb{S}_{n_k}^\tau f(\omega)}\|k_z\|.$$

It follows that $S_{n_k}(\omega)x \underset{k\to\infty}{\longrightarrow} 0$ for every $x \in Z_2$. Thus, in this case, Proposition 5.1.2 applies.

Now suppose that $\liminf\limits_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$, that is, there is a strictly increasing sequence $(m_k)_{k\geq1}$

of positive integers such that

$$\mathbb{S}^{\tau}_{m_k} f(\omega) \underset{k\to\infty}{\longrightarrow} -\infty.$$

We just have to exchange the roles of $Z_1$ and $Z_2$. We will have

$$T_{m_k}(\omega)x \underset{k\to\infty}{\longrightarrow} 0$$

for every $x \in Z_2$. We also define a right inverse $S_{m_k}(\omega)$ on $Z_1$ in the same way as before, and we will have

$$S_{m_k}(\omega)x \underset{k\to\infty}{\longrightarrow} 0$$

for every $x \in Z_1$. Observe that the definition of $S_{m_k}(\omega)$ makes sense in this case, since $\phi_1$ does not vanish on $U$.

This concludes the proof of Theorem 5.3.9. □

In the case where $f = \mathbb{1}_{A_1} - \mathbb{1}_{A_2}$ and the sequence $(\mathbb{S}^{\tau}_n f(\omega))_{n\geq 1}$ has a subsequence satisfying a CLT, we can omit the assumptions on the images of $\phi_1$ and $\phi_2$. The idea is to replace the dense subsets $Z_1$ and $Z_2$ of $H^2(\mathbb{D})$ for the operators $T_n(\omega)$, which are used to apply the Universality Criterion to the sequence $(T_n(\omega))_{n\geq 1}$, by another dense subset using model spaces. It can be seen as a particular case of Theorem 5.3.20, that we will present afterwards.

**Theorem 5.3.10.** *Let $\tau$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$ such that, for $f = \mathbb{1}_{A_1} - \mathbb{1}_{A_2}$, the sequence $(\mathbb{S}^{\tau}_n f(\omega))_{n\geq 1}$ satisfies*

$$\limsup_{n\to\infty} \mathbb{S}^{\tau}_n f(\omega) = \infty \quad \textit{and} \quad \liminf_{n\to\infty} \mathbb{S}^{\tau}_n f(\omega) = -\infty$$

*for almost every $\omega \in \mathbb{T}$. Let $\phi_1, \phi_2 \in H^{\infty}(\mathbb{D})$ be two nonconstant functions, such that $\phi_1\phi_2$ is inner. Suppose that either $1/\phi_1 \in H^{\infty}(\mathbb{D})$ and $\phi_2$ is not outer, or $1/\phi_2 \in H^{\infty}(\mathbb{D})$ and $\phi_1$ is not outer.*

*Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.*

*Proof.* We will again use the Universality Criterion (Proposition 5.1.2).

Let us set $\phi_1\phi_2 = \varphi$. Suppose for example that $1/\phi_1 \in H^{\infty}(\mathbb{D})$ and that $\phi_2$ is not outer. Let us denote by $I_2$ the inner part of $\phi_2$, which is nonconstant, and by $F_2 \in H^{\infty}(\mathbb{D})$ the outer part of $\phi_2$. Using a similar idea to the proof of [23, Theorem 3.1], we observe that the subspace

$$K := \bigcup_{n\geq 1} K_n,$$

with $K_n$ being the orthogonal of $I_2^n H^2(\mathbb{D})$ in $H^2(\mathbb{D})$, is dense in $H^2(\mathbb{D})$ (this follows from the inner-outer factorization). We will apply the Universality Criterion to the sequence $(T_n(\omega))_{n\geq 1}$ using this dense subspace.

Let $x \in K$ and let $n_x \geq 1$ be an integer such that $x \in K_{n_x}$. Let us remark that $(M^*_{\phi_2})^{n_x} x = 0$. Indeed, for every $y \in H^2(\mathbb{D})$,

$$\langle (M^*_{\phi_2})^{n_x} x, y\rangle = \langle x, I_2^{n_x} F_2^{n_x} y\rangle = 0. \tag{5.3.8}$$

We used the fact that $F_2^{n_x} y \in H^2(\mathbb{D})$ to assert that the dot product in (5.3.8) is equal to 0. This shows in particular that $T_n(\omega)x \underset{n\to\infty}{\longrightarrow} 0$ for every $x \in K$, for almost every $\omega \in \mathbb{T}$.

In order to define a right inverse of $T_n(\omega)$, we will again use the eigenvectors $k_z$ of $T_n(\omega)$.

First, let us denote by $E$ a subset of $\mathbb{T}$ with $m(E) = 1$ such that for every $\omega \in E$,

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty.$$

Let us fix $\omega \in E$.

Since $\phi_1$ is nonconstant, $\phi_1(\mathbb{D}) \cap \mathbb{D} \neq \emptyset$ or $\phi_1(\mathbb{D}) \cap \mathbb{C} \setminus \overline{\mathbb{D}} \neq \emptyset$. Suppose, to begin with, that $|\phi_1(z)| < 1$ on a nonempty open set $U$ of $\mathbb{D}$. Let us denote by $Z$ the dense subspace of $H^2(\mathbb{D})$ defined by

$$Z := \operatorname{span}\{k_z : z \in U\}.$$

Let $(n_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers such that $\mathbb{S}_{n_k}^\tau f(\omega) \underset{k\to\infty}{\longrightarrow} -\infty$. Exactly as in the proof of Theorem 5.3.9, we define a right inverse $S_{n_k}(\omega)$ on $Z$ by setting

$$S_{n_k}(\omega) k_z = \overline{\phi_1(z)}^{-\mathbb{S}_{n_k}^\tau f(\omega)} (M_\varphi)^{a_2(n_k,\omega)} k_z$$

for every $z \in U$ and extending it by linearity to $Z$. Using the fact that $M_{1/\phi_1}^*$ is bounded on $H^2(\mathbb{D})$, we have that

$$T_n(\omega) = (M_{\phi_1}^*)^{a_1(n,\omega)} (M_{1/\phi_1}^*)^{a_2(n,\omega)} (M_\varphi^*)^{a_2(n,\omega)}.$$

Since $\varphi$ is inner, we obtain that

$$T_{n_k}(\omega) S_{n_k}(\omega) x = x$$

for every $x \in Z$.

Finally, we have

$$\|S_{n_k}(\omega) k_z\| \leq |\phi_1(z)|^{-\mathbb{S}_{n_k}^\tau f(\omega)} \|k_z\|$$

for every $z \in U$, which proves that $S_{n_k}(\omega) x \underset{k\to\infty}{\longrightarrow} 0$ for every $x \in Z$.

If $\phi_1(\mathbb{D}) \cap \mathbb{D} = \emptyset$, we use another nonempty open set $V$ of $\mathbb{D}$ such that $|\phi_1(z)| > 1$ for $z \in V$ and we use the fact that $\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty$ to define a right inverse $S_n(\omega)$ along an appropriate subsequence on the dense subspace of $H^2(\mathbb{D})$

$$Z' := \operatorname{span}\{k_z : z \in V\}.$$

The conclusion then follows in the same way. This shows in particular the importance to have

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty,$$

and not just one of the two conditions. This concludes the proof of Theorem 5.3.10. □

We can deduce from Theorem 5.3.10 the following results regarding the cases of the doubling map and the irrational rotations. The first result holds for the doubling map and follows from Corollary 5.2.21.

**Corollary 5.3.11.** *Let $\tau$ be the doubling map on $\mathbb{T}$. Let $A_1 = [0, 1/2)$ and $A_2 = [1/2, 1)$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that $\phi_1\phi_2$ is inner. Suppose that either $1/\phi_1 \in H^\infty(\mathbb{D})$ and $\phi_2$ is not outer, or $1/\phi_2 \in H^\infty(\mathbb{D})$ and $\phi_1$ is not outer. Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.*

The second result holds for every irrational rotation and follows from Corollary 5.2.7.

**Corollary 5.3.12.** *Let $\alpha$ be an irrational number in $(0,1)$ and let $\tau = R_\alpha$ on $\mathbb{T}$. Let $A_1 = [0,1/2)$ and $A_2 = [1/2,1)$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that $\phi_1\phi_2$ is inner. Suppose that either $1/\phi_1 \in H^\infty(\mathbb{D})$ and $\phi_2$ is not outer, or $1/\phi_2 \in H^\infty(\mathbb{D})$ and $\phi_1$ is not outer. Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.*

**Remark 5.3.13.** Obviously, Theorem 5.3.10 shows that the converse of Theorem 5.3.9 is not true in general. Indeed, if the transformation $\tau$ is such that the sequence $(\mathbb{S}_n^\tau f)_{n\geq 1}$ has a subsequence satisfying a CLT, if $\phi_1(z) = \frac{3}{2} + \frac{1}{4}z$ and $\phi_2(z) = z(\phi_1(z))^{-1}$ for every $z \in \mathbb{D}$, then the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing, by Theorem 5.3.10. This follows from that fact that $\phi_2$ is not outer, and that $1/\phi_1 \in H^\infty(\mathbb{D})$. However, $\phi_1(\mathbb{D}) \cap \mathbb{T} = \emptyset$ and $\phi_2(\mathbb{D}) \cap \mathbb{T} = \emptyset$, so that neither $(M_{\phi_1})^*$ nor $(M_{\phi_2})^*$ is hypercyclic.

We have seen that, in the case where $\phi_1\phi_2$ is nonconstant and such that $(\phi_1\phi_2)(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, the sequence $(T_n(\omega))_{n\geq 1}$ is universal for almost every $\omega \in \mathbb{T}$. Moreover, by Theorem 5.3.9, if $\phi_1\phi_2$ is constant on $\mathbb{D}$ with $\phi_1\phi_2 \equiv \alpha$, $\alpha \in \mathbb{T}$, and if $\phi_1(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, then the random sequence $(T_n(.))_{n\geq 1}$ is universal. The following proposition shows that the converse is true in this case.

**Proposition 5.3.14.** *Let $\tau$ be a measure-preserving transformation on $(\mathbb{T}, m)$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that $\phi_1\phi_2 \equiv \alpha$ on $\mathbb{D}$, with $\alpha \in \mathbb{T}$. If the sequence $(T_n(\omega))_{n\geq 1}$ is universal for some $\omega \in \mathbb{T}$, then $\phi_1(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ (and $\phi_2(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$).*

The proof of Proposition 5.3.14 relies on an important result of F. León-Saavedra and V. Müller, which we state here for the convenience of the reader.

**Proposition 5.3.15** ([43, Theorem 1])**.** *Let $\mathcal{M} \subset \mathcal{B}(X)$ be a semigroup of operators on a Banach space $X$. Let $x \in X$ be such that the set $\{\mu Sx : \mu \in \mathbb{T}, S \in \mathcal{M}\}$ is dense in $X$. Suppose that there exists an operator $T \in \mathcal{B}(X)$ such that $T^*$ has no eigenvalue and that $T$ satisfies $ST = TS$ for every $S \in \mathcal{M}$. Then the set $\{Sx : S \in \mathcal{M}\}$ is dense in $X$.*

In Proposition 5.3.15, a semigroup of operators is a subset $\mathcal{M} \subset \mathcal{B}(X)$ containing the identity $I$, and such that $ST \in \mathcal{M}$ whenever $S, T \in \mathcal{M}$. We can now prove Proposition 5.3.14.

*Proof of Proposition 5.3.14.* Under the assumptions of Proposition 5.3.14, $1/\phi_1$ and $1/\phi_2$ belong to $H^\infty(\mathbb{D})$, and $M_{\phi_1}^*$ is invertible. We have

$$T_n(\omega) = \overline{\alpha}^{a_2(n,\omega)} (M_{\phi_1}^*)^{\mathbb{S}_n^\tau f(\omega)},$$

with $f = \mathbb{1}_{A_1} - \mathbb{1}_{A_2}$, and with $\mathbb{S}_n^\tau f(\omega) = a_1(n,\omega) - a_2(n,\omega)$ which is an integer. Let $\omega \in \mathbb{T}$ and let $h \in H^2(\mathbb{D})$ be such that the set

$$\{T_n(\omega)h : n \geq 1\}$$

is dense in $H^2(\mathbb{D})$. In particular, the set

$$\{\mu (M_{\phi_1}^*)^k h : \mu \in \mathbb{T}, k \in \mathbb{Z}\}$$

is dense in $H^2(\mathbb{D})$. We apply Proposition 5.3.15. Taking $\{(M_{\phi_1}^*)^k : k \in \mathbb{Z}\}$ for the semigroup and $T = M_{\phi_1}^*$ for the operator, whose adjoint has no eigenvalue in $H^2(\mathbb{D})$, we obtain that the set

$$\{(M_{\phi_1}^*)^k h : k \in \mathbb{Z}\}$$

is dense in $H^2(\mathbb{D})$. We will now show that there is a function $g \in H^2(\mathbb{D})$ such that the set

$$\{(M_{\phi_1}^*)^k g : k \geq 0\}$$

is dense in $H^2(\mathbb{D})$. Let us denote by $O_+(h)$ the positive orbit of $h$ and by $O_-(h)$ the negative orbit of $h$, that is,

$$O_+(h) := \{(M^*_{\phi_1})^k h : k \geq 0\} \quad \text{and} \quad O_-(h) := \{(M^*_{\phi_1})^k h : k \leq 0\}.$$

Since the set

$$\{(M^*_{\phi_1})^k h : k \in \mathbb{Z}\}$$

is dense, there is a sequence $(n_k)_{k\geq 1}$ of integers with $|n_k| \underset{k\to\infty}{\longrightarrow} \infty$, such that

$$(M^*_{\phi_1})^{n_k} h \underset{k\to\infty}{\longrightarrow} h.$$

For every integer $j \in \mathbb{Z}$, one has

$$(M^*_{\phi_1})^{n_k+j} h \underset{k\to\infty}{\longrightarrow} (M^*_{\phi_1})^j h.$$

But there are infinitely many positive integers $n_k$ or infinitely many negative integers $n_k$. So,

$$\{(M^*_{\phi_1})^k h : k \in \mathbb{Z}\} \subset \overline{O_+(h)} \quad \text{or} \quad \{(M^*_{\phi_1})^k h : k \in \mathbb{Z}\} \subset \overline{O_-(h)}.$$

Suppose that we are in the second case. Let $U, V$ be two non-empty open sets in $H^2(\mathbb{D})$. There exist then two integers $i < j < 0$ such that $(M^*_{\phi_1})^i h \in U$ and $(M^*_{\phi_1})^j h \in V$. Hence, $(M^*_{\phi_1})^{j-i}(U) \cap V \neq \emptyset$.

This proves that $M^*_{\phi_1}$ is topologically transitive, and there exists a function $g \in H^2(\mathbb{D})$ such that the set

$$\{(M^*_{\phi_1})^k g : k \geq 0\}$$

is dense in $H^2(\mathbb{D})$. But now, it is well known that this implies $\phi_1(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, because the adjoint of a multiplication operator $(M_\phi)^*$ associated to a nonconstant function $\phi \in H^\infty(\mathbb{D})$ is hypercyclic if and only if $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

This concludes the proof of Proposition 5.3.14. □

In the case where $m(A_1) \neq m(A_2)$, we will see that it is more difficult to identify limit cases, compared to the case where $m(A_1) = m(A_2) = 1/2$. However, we can still obtain sufficient conditions, similar to those of Theorems 5.3.9 and 5.3.10, for the sequence $(T_n(\omega))_{n\geq 1}$ to be universal for almost every $\omega \in \mathbb{T}$.

### 5.3b Case where possibly $m(A_1) \neq m(A_2)$

In this subsection, we consider $A_1$ and $A_2$ two disjoint Borel subsets of $[0,1)$ satisfying $A_1 \cup A_2 = [0,1)$ and $\min\{m(A_1), m(A_2)\} > 0$. We study the dynamics of $(T_n(\omega))_{n\geq 1}$ for almost every $\omega \in \mathbb{T}$, when $T(\omega)$ is given by (5.3.3).

Exactly as in Proposition 5.3.2, the computation of $\|T_n(\omega)\|$ provides a simple obstruction to the universality of random sequences $(T_n(.))_{n\geq 1}$.

**Proposition 5.3.16.** *Let $\tau$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant. Suppose that $\|\phi_1\|_\infty^{m(A_1)} \|\phi_2\|_\infty^{m(A_2)} < 1$. Then, the random sequence $(T_n(.))_{n\geq 1}$ is not universal.*

We also have an analog of Theorem 5.3.4 in the case $m(A_1)$ is not necessarily equal to $m(A_2)$. This is the following result.

**Theorem 5.3.17.** *Let $\tau$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant on $\mathbb{D}$ and suppose that there exist $\lambda, \mu \in \mathbb{D}$ such that*

$$|\phi_1(\lambda)|^{m(A_1)}|\phi_2(\lambda)|^{m(A_2)} < 1 \tag{5.3.9}$$

$$\text{and} \quad |\phi_1(\mu)|^{m(A_1)}|\phi_2(\mu)|^{m(A_2)} > 1. \tag{5.3.10}$$

*Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically mixing.*

*Proof.* The proof is very similar to the proof of Theorem 5.3.4. Let $E$ be a subset of $\mathbb{T}$ with $m(E) = 1$, such that

$$\frac{a_1(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} m(A_1) \quad \text{and} \quad \frac{a_2(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} m(A_2)$$

for every $\omega \in E$. We again use the relation on the eigenvectors

$$T_n(\omega)k_z = \overline{\phi_1(z)}^{a_1(n,\omega)}\overline{\phi_2(z)}^{a_2(n,\omega)}k_z$$

for every $z \in \mathbb{D}$. We thus have for $n \geq 1$, $z \in \mathbb{D}$ and $\omega \in E$,

$$|\lambda(T_n(\omega), k_z)| \underset{n\to\infty}{=} e^{n[\log(|\phi_1(z)|^{m(A_1)}|\phi_2(z)|^{m(A_2)})+o(1)]}.$$

By (5.3.9) and (5.3.10), the following sets

$$\mathcal{A}_1 := \{k_z : z \in \mathbb{D}, |\phi_1(z)|^{m(A_1)}|\phi_2(z)|^{m(A_2)} < 1\}$$

$$\text{and} \quad \mathcal{A}_2 := \{k_z : z \in \mathbb{D}, |\phi_1(z)|^{m(A_1)}|\phi_2(z)|^{m(A_2)} > 1\}$$

are such that $\operatorname{span}(\mathcal{A}_1)$ and $\operatorname{span}(\mathcal{A}_2)$ are dense in $H^2(\mathbb{D})$. Finally, $|\lambda(T_n(\omega), k_z)| \underset{n\to\infty}{\longrightarrow} 0$ for every $k_z \in \mathcal{A}_1$ and $|\lambda(T_n(\omega), k_w)| \underset{n\to\infty}{\longrightarrow} \infty$ for every $k_w \in \mathcal{A}_2$. Thus Proposition 5.3.3 applies, and this concludes the proof of Theorem 5.3.17. □

**Remark 5.3.18.** Again, if the ergodic transformation $\tau$ involved in Theorem 5.3.17 is an irrational rotation and if $A_1, A_2$ are intervals of $[0,1)$, then the sequence $(T_n(\omega))_{n\geq 1}$ is topologically mixing for every $\omega \in \mathbb{T}$.

In the case where $m(A_1) = m(A_2)$, if $\phi_1\phi_2$ is nonconstant, Conditions (5.3.9) and (5.3.10) are equivalent to $(\phi_1\phi_2)(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, by the open mapping theorem. In the situation where $m(A_1) \neq m(A_2)$, it is difficult to find a similar statement. The following two results are the analog of Theorems 5.3.9 and 5.3.10 respectively. The first statement requires an assumption on the images of $\phi_1$ and of $\phi_2$, as in Theorem 5.3.9. The only difference with Theorem 5.3.9 is that here, we require both $\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty$ and $\liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$ for almost every $\omega \in \mathbb{T}$, and not just one of the two conditions for almost every $\omega \in \mathbb{T}$.

**Theorem 5.3.19.** *Let $\tau$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$, such that the sequence $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ satisfies*

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

*for almost every $\omega \in \mathbb{T}$, with $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that*

$$|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1 \quad \textit{almost everywhere on } \mathbb{T}. \tag{5.3.11}$$

*If $\phi_1(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ or $\phi_2(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, then the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.*

*Proof.* Let us denote by $\phi_1 = I_1F_1$ and $\phi_2 = I_2F_2$ the inner-outer decomposition of $\phi_1$ and $\phi_2$, with $I_1, I_2$ inner and $F_1, F_2$ outer. We can take

$$F_1(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\log|\phi_1{}^*(e^{it})|\,dt\right),$$

and similarly for $F_2$ with $\phi_2$. Let us remark that

$$|F_1|^{m(A_1)}|F_2|^{m(A_2)} = 1 \tag{5.3.12}$$

on $\mathbb{D}$. Indeed, this follows from the expression of $F_1$ and $F_2$, and from (5.3.11). Moreover, let us remark that

$$|\phi_1(\lambda)|^{m(A_1)}|\phi_2(\lambda)|^{m(A_2)} \leq 1 \tag{5.3.13}$$

for every $\lambda \in \mathbb{D}$. This is a consequence of the inner-outer decomposition of $\phi_1$ and $\phi_2$, and of (5.3.12).

Suppose for example that $\phi_1(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$. Let us denote by $U$ and $V$ two nonempty open sets in $\mathbb{D}$ such that $|\phi_1(z)| < 1$ if $z \in U$ and $|\phi_1(w)| > 1$ if $w \in V$. Since the zeros of $\phi_1$ in $U$ are isolated, we can replace $U$ by $U \setminus \{z \in \mathbb{D} : \phi_1(z) \neq 0\}$ and suppose that $\phi_1$ does not vanish on $U$.

We will again use the Universality Criterion and the eigenvectors $k_z$ of $T_n(\omega)$. Let $E$ be a subset of $\mathbb{T}$ with $m(E) = 1$ such that for every $\omega \in E$,

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty.$$

Let us fix $\omega \in E$. Let $(n_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers such that $\mathbb{S}_{n_k}^\tau f(\omega) \underset{k\to\infty}{\longrightarrow} \infty$.

By (5.3.13) and since

$$T_n(\omega)k_z = \overline{\phi_1(z)}^{a_1(n,\omega)}\overline{\phi_2(z)}^{a_2(n,\omega)}k_z,$$

we have

$$\|T_{n_k}(\omega)k_z\| \leq |\phi_1(z)|^{a_1(n_k,\omega)-\frac{m(A_1)}{m(A_2)}a_2(n_k,\omega)}\|k_z\|$$

for every $z \in U$, that is

$$\|T_{n_k}(\omega)k_z\| \leq |\phi_1(z)|^{\mathbb{S}_{n_k}^\tau f(\omega)}\|k_z\|$$

for every $z \in U$, where

$$a_1(n_k,\omega) - \frac{m(A_1)}{m(A_2)}a_2(n_k,\omega) := \mathbb{S}_{n_k}^\tau f(\omega)$$

are the Birkhoff sums associated to the function $f$. In particular, we have $T_{n_k}(\omega)x \underset{k\to\infty}{\longrightarrow} 0$ for every $x$ in the dense subspace of $H^2(\mathbb{D})$

$$Z_1 := \operatorname{span}\{k_z : z \in U\}.$$

We define a right inverse $S_{n_k}(\omega)$ on the dense subspace of $H^2(\mathbb{D})$

$$Z_2 := \operatorname{span}\{k_z : z \in V\}$$

by setting

$$S_{n_k}(\omega)k_z = \overline{F_1(z)}^{-a_1(n_k,\omega)}\,\overline{F_2(z)}^{-a_2(n_k,\omega)}\,(M_{I_1})^{a_1(n_k,\omega)}\,(M_{I_2})^{a_2(n_k,\omega)}\,k_z$$

for every $z \in V$ and extending it by linearity to $Z_2$. This makes sense, since the operator $T_n(\omega)$ is given by

$$T_n(\omega) = (M_{F_1}^*)^{a_1(n,\omega)}(M_{F_2}^*)^{a_2(n,\omega)}(M_{I_1}^*)^{a_1(n,\omega)}(M_{I_2}^*)^{a_2(n,\omega)}.$$

Since the operators $M_{I_1}$ and $M_{I_2}$ are contractions, and by (5.3.12), we have

$$\|S_{n_k}(\omega)k_z\| \le |F_1(z)|^{-\mathbb{S}_{n_k}^\tau f(\omega)}\|k_z\|$$

for every $z \in V$. Now, we also have $|F_1(z)| > 1$ on $V$, since $|\phi_1(z)| > 1$. Thus $S_{n_k}(\omega)y \underset{k\to\infty}{\longrightarrow} 0$ for every $y \in Z_2$.

In the case where $\phi_1(\mathbb{D}) \cap \mathbb{T} = \emptyset$, that is where $\phi_2(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, the proof is the same. This is allowed because, if we denote by $g := \mathbb{1}_{A_2} - \frac{m(A_2)}{m(A_1)}\mathbb{1}_{A_1}$, then $-\mathbb{S}_n^\tau f(\omega) = \frac{m(A_1)}{m(A_2)}\mathbb{S}_n^\tau g(\omega)$, so the sequence $(\mathbb{S}_n^\tau g(\omega))_{n\ge1}$ has the same limsup and liminf as $(\mathbb{S}_n^\tau f(\omega))_{n\ge1}$, for every $\omega \in E$.

This concludes the proof of Theorem 5.3.19. □

The second theorem does not require any hypothesis on the images of $\phi_1$ and of $\phi_2$, as in Theorem 5.3.10, but requires $\phi_1\phi_2$ not to be outer. It is a more general situation than Theorem 5.3.10, since any function $\phi \in H^\infty(\mathbb{D})$ such that $1/\phi \in H^\infty(\mathbb{D})$ is outer.

**Theorem 5.3.20.** *Let $\tau$ be an ergodic measure-preserving transformation of $(\mathbb{T}, m)$, such that the sequence $(\mathbb{S}_n^\tau f(\omega))_{n\ge1}$ satisfies*

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

*for almost every $\omega \in \mathbb{T}$, with $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that $\phi_1\phi_2$ is not outer, either $\phi_1$ or $\phi_2$ is not inner, and*

$$|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1 \quad \text{almost everywhere on } \mathbb{T}.$$

*Then, the random sequence $(T_n(.))_{n\ge1}$ is topologically weakly mixing.*

*Proof.* The proof is very similar to the proof of Theorem 5.3.10.

Let us consider the inner-outer decomposition $\phi_1 = I_1F_1$ and $\phi_2 = I_2F_2$ of $\phi_1$ and $\phi_2$, with $I_1, I_2$ inner and $F_1, F_2$ outer. Since $\phi_1\phi_2$ is not outer, either $\phi_1$ or $\phi_2$ is not outer. Suppose for example that $I_1$ is nonconstant. As in the proof of Theorem 5.3.10, the subspace

$$K := \bigcup_{n\ge1} K_n,$$

with $K_n$ being the orthogonal of $I_1^n H^2(\mathbb{D})$ in $H^2(\mathbb{D})$, is dense in $H^2(\mathbb{D})$, and $T_n(\omega)x \underset{n\to\infty}{\longrightarrow} 0$ for every $x \in K$, for almost every $\omega \in \mathbb{T}$.

Let us suppose first that $\phi_1$ is not inner. In particular, we have that $F_1(\mathbb{D}) \cap \mathbb{D} \neq \emptyset$ or $F_1(\mathbb{D}) \cap \mathbb{C} \setminus \overline{\mathbb{D}} \neq \emptyset$. Suppose for example that we are in the first case, that is, $|F_1(z)| < 1$ on a nonempty open set $U$ of $\mathbb{D}$.

Let $E$ be a subset of $\mathbb{T}$ with $m(E) = 1$ such that for every $\omega \in E$,

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty.$$

Let us fix $\omega \in E$. Let $(n_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers such that $\mathbb{S}_{n_k}^\tau f(\omega) \underset{k\to\infty}{\longrightarrow} -\infty$.

We define a right inverse on the dense subspace of $H^2(\mathbb{D})$

$$Z := \operatorname{span}\{k_z : z \in U\}$$

by setting

$$S_{n_k}(\omega)k_z = \overline{F_1(z)}^{-a_1(n_k,\omega)}\, \overline{F_2(z)}^{-a_2(n_k,\omega)}\, (M_{I_1})^{a_1(n_k,\omega)}\, (M_{I_2})^{a_2(n_k,\omega)}\, k_z$$

for every $z \in U$, and by linear extension to $Z$. This makes sense, since the operator $T_n(\omega)$ is given by

$$T_n(\omega) = (M_{F_1}^*)^{a_1(n,\omega)}(M_{F_2}^*)^{a_2(n,\omega)}(M_{I_1}^*)^{a_1(n,\omega)}(M_{I_2}^*)^{a_2(n,\omega)}.$$

Since the operators $M_{I_1}$ and $M_{I_2}$ are contractions and since $|F_1|^{m(A_1)}|F_2|^{m(A_2)} = 1$ on $\mathbb{D}$, we obtain

$$\|S_{n_k}(\omega)k_z\| \leq |F_1(z)|^{-\mathbb{S}_{n_k}^\tau f(\omega)}\|k_z\|$$

for every $\omega \in E$ and $z \in U$, and thus $S_{n_k}(\omega)y \underset{k\to\infty}{\longrightarrow} 0$ for every $y \in Z$.

If we have $F_1(\mathbb{D}) \cap \mathbb{D} = \emptyset$, that is, if we have $F_1(\mathbb{D}) \cap \mathbb{C} \setminus \overline{\mathbb{D}} \neq \emptyset$, we use the fact that $\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty$. Finally, if $\phi_1$ is inner and thus $\phi_2$ is not inner, we reason in the same way, using the function $F_2$ and the Birkhoff sums $\mathbb{S}_n^\tau g(\omega)$, with $g := \mathbb{1}_{A_2} - \frac{m(A_2)}{m(A_1)}\mathbb{1}_{A_1}$.

This concludes the proof of Theorem 5.3.20. $\square$

We can apply Theorems 5.3.19 and 5.3.20 to the doubling map and to some irrational rotations. For the doubling map, this is the following result, which follows from Corollary 5.2.21.

**Corollary 5.3.21.** *Let $\tau$ be the doubling map on $\mathbb{T}$ and let $A_1$ and $A_2$ be two disjoint intervals of $[0,1)$, such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant. Suppose that one of the two following conditions holds:*

*(i)* $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ *almost everywhere on $\mathbb{T}$, and one of the images $\phi_1(\mathbb{D})$ or $\phi_2(\mathbb{D})$ meets the unit circle $\mathbb{T}$.*

*(ii)* $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ *almost everywhere on $\mathbb{T}$, $\phi_1\phi_2$ is not outer and either $\phi_1$ or $\phi_2$ is not inner.*

*Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.*

For irrational rotations, we first have the following result, which follows from Corollary 5.2.12 and holds for example if the irrational $\alpha$ has bounded partial quotients (in particular, for almost every irrational $\alpha$).

**Corollary 5.3.22.** *Let $\alpha$ be an irrational number satisfying (5.2.1), that is,*

$$a_n \leq A\, n^p, \quad \text{for every } n \geq 1\,,$$

*for some $A > 0$ and $0 \leq p < 1/8$. Let $\tau = R_\alpha$. Let $A_1$, $A_2$ be two disjoint intervals of $[0,1)$ with rational endpoints, such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant. Suppose that one of the two following conditions holds:*

*(i) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ almost everywhere on $\mathbb{T}$, and the image of either $\phi_1$ or $\phi_2$ intersects $\mathbb{T}$.*

*(ii) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ almost everywhere on $\mathbb{T}$, $\phi_1\phi_2$ is not outer and either $\phi_1$ or $\phi_2$ is not inner.*

*Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.*

The case where $\alpha$ as unbounded partial quotients is given by the following result, which follows from Corollary 5.2.17.

**Corollary 5.3.23.** *Let $\alpha$ be an irrational in $(0,1)$ and let $\tau = R_\alpha$. Let $b = \frac{p'}{q'}$ be a rational in $(0,1)$ with $p'$ and $q'$ coprime. Let $A_1 = [0,b)$ and let $A_2 = [b,1)$. Suppose that $a_{t_k+1} \underset{k\to\infty}{\longrightarrow} \infty$ along a sequence $(t_k)_{k\geq 1}$ of positive integers such that $q_{t_k} \not\equiv 0 \pmod{q'}$ for every $k \geq 1$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant. Suppose that one of the two following conditions holds:*

*(i) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ almost everywhere on $\mathbb{T}$, and the image of either $\phi_1$ or $\phi_2$ intersects $\mathbb{T}$.*

*(ii) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ almost everywhere on $\mathbb{T}$, $\phi_1\phi_2$ is not outer and either $\phi_1$ or $\phi_2$ is not inner.*

*Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.*

Finally, we have the following results for some irrational rotations and for some intervals of $[0,1)$, which follows from Propositions 5.2.13 and 5.2.18.

**Corollary 5.3.24.** *Let $\alpha$ be an irrational in $(0,1)$ and let $\tau = R_\alpha$. Let $b \in (0,1)$. Let $A_1 = [0,b)$ and let $A_2 = [b,1)$. Suppose that either $a_{t_k+1} \underset{k\to\infty}{\longrightarrow} \infty$ along a sequence $(t_k)_{k\geq 1}$ of positive integers such that $(bq_{t_k})_{k\geq 1}$ is uniformly distributed mod 1, or $\alpha$ has bounded partial quotients and the sequence $(bq_k)_{k\geq 1}$ is uniformly distributed mod 1, where the $q_k$'s are the denominators of the convergents of $\alpha$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant. Suppose that one of the two following conditions holds:*

*(i) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ almost everywhere on $\mathbb{T}$, and the image of either $\phi_1$ or $\phi_2$ intersects $\mathbb{T}$.*

*(ii) $|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1$ almost everywhere on $\mathbb{T}$, $\phi_1\phi_2$ is not outer and either $\phi_1$ or $\phi_2$ is not inner.*

*Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing.*

As we can see, the dynamics of the random sequence $(T_n(.))_{n\geq 1}$ is very dependent on the properties of the ergodic measure-preserving transformation $\tau$ on $(\mathbb{T}, m)$. We will now see that there exist ergodic transformations on $(\mathbb{T}, m)$ for which the random sequence $(T_n(.))_{n\geq 1}$ is never universal.

**Proposition 5.3.25.** *Let $\tau$ be a measure-preserving transformation on $(\mathbb{T}, m)$ such that the coboundary equation $f = h - h \circ \tau$ has a solution $h \in L^\infty(\mathbb{T})$ for $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that*

$$|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1 \quad \textit{almost everywhere on } \mathbb{T}.$$

*Then, for almost every $\omega \in \mathbb{T}$, the sequence $(T_n(\omega))_{n\geq 1}$ is not universal.*

*Proof.* Under the assumptions of Proposition 5.3.25, there exists a Borel subset $E$ of $\mathbb{T}$ with $m(E) = 1$ such that the sequence $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ is bounded when $\omega \in E$. Indeed, one has $\mathbb{S}_n^\tau f(\omega) = h(\omega) - h(\tau^n\omega)$ for every $n \geq 1$. Moreover, we have

$$T_n(\omega) = (M_{F_1}^*)^{a_1(n,\omega)}(M_{F_2}^*)^{a_2(n,\omega)}(M_{I_1}^*)^{a_1(n,\omega)}(M_{I_2}^*)^{a_2(n,\omega)},$$

where $\phi_1 = I_1F_1$ and $\phi_2 = I_2F_2$, with $I_1, I_2$ inner and $F_1, F_2$ outer. We have seen in the proof of Theorem 5.3.19 that

$$|F_1|^{m(A_1)}|F_2|^{m(A_2)} = 1$$

on $\mathbb{D}$. In particular, this implies that

$$\|F_2\|_\infty^{-\frac{m(A_2)}{m(A_1)}} \leq |F_1(z)| \leq \|F_1\|_\infty,$$

for every $z \in \mathbb{D}$, and similarly for $F_2$. For $\omega \in E$, we have

$$\begin{aligned}\|T_n(\omega)\| &\leq \sup_{z\in\mathbb{D}}\{|F_1(z)|^{a_1(n,\omega)}|F_2(z)|^{a_2(n,\omega)}\} \\ &\leq \sup_{z\in\mathbb{D}}\{|F_1(z)|^{\mathbb{S}_n^\tau f(\omega)}\}.\end{aligned}$$

Since the quantity $|F_1(z)|^{\mathbb{S}_n^\tau f(\omega)}$ is bounded from above for every $z \in \mathbb{D}$, $\omega \in E$ and $n \geq 1$ by

$$\exp(\max\{\mathbb{S}_n^\tau f(\omega)\log(\|F_1\|_\infty), -\tfrac{m(A_2)}{m(A_1)}\mathbb{S}_n^\tau f(\omega)\log(\|F_2\|_\infty)\}),$$

it follows that $\|T_n(\omega)\|$ is also bounded from above for every $n \geq 1$ and $\omega \in E$, since the sequence $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ is bounded.

In particular, every orbit under $(T_n(\omega))_{n\geq 1}$ is bounded and thus cannot be dense in $H^2(\mathbb{D})$. This concludes the proof of Proposition 5.3.25. ☐

**Remark 5.3.26.** It is clear that if the random sequence $(T_n(.))_{n\geq 1}$ associated to an ergodic transformation $\tau$ on $\mathbb{T}$ and to two operators $T_1$ and $T_2$ is universal, then the $\mathbb{Z}_+^2$-action generated by $T_1, T_2$ is hypercyclic, as soon as $T_1$ and $T_2$ commute. That is, there exists a vector $x \in X$ such that the set $\{T_1^nT_2^mx : n, m \geq 0\}$ is dense in $X$. The converse is not true in general. Indeed, let us consider $A_1 = [0, 1/2), A_2 = [1/2, 1)$, and let $\tau$ be an ergodic transformation such that the sequence of Birkhoff sums $(a_1(n,\omega) - a_2(n,\omega))_{n\geq 1}$ is bounded for almost every $\omega \in \mathbb{T}$. Let $T_1 = (M_{\phi_1})^*$ and $T_2 = (M_{\phi_2})^*$ on $H^2(\mathbb{D})$, where $\phi_1(z) = e^z$ and $\phi_2(z) = e^{-z}$ for every $z \in \mathbb{D}$. Then, for every $n, m \geq 0$, $T_1^nT_2^m = (M_{\phi_1}^*)^{n-m}$. In particular, since the operator $(M_{\phi_1})^*$ is hypercyclic on $H^2(\mathbb{D})$, the $\mathbb{Z}_+^2$-action generated by $T_1$ and $T_2$ is hypercyclic. However, by Proposition 5.3.25, the random sequence $(T_n(.))_{n\geq 1}$ is not universal.

In certain cases, for an irrational rotation, the sequence $(T_n(\omega))_{n\geq 1}$ can be non-universal, for every $\omega \in \mathbb{T}$.

**Corollary 5.3.27.** *Let $\alpha$ be an irrational number in $(0,1)$ and let $\tau = R_\alpha$. Let $A_1 = [0,b)$ and $A_2 = [b,1)$, with $b \in (0,1)$. Suppose that $b \in \mathbb{Z}\alpha$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that*

$$|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1 \quad \textit{almost everywhere on } \mathbb{T}.$$

*Then, for every $\omega \in \mathbb{T}$, the sequence $(T_n(\omega))_{n\geq 1}$ is not universal.*

*Proof.* This follows from Proposition 5.3.25 since in this case, the sequence $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$, where $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$, is bounded for every $\omega \in \mathbb{T}$, thanks to Proposition 5.2.4. □

Thanks to Theorem 5.2.2, we also deduce the following result.

**Corollary 5.3.28.** *Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that*

$$|\phi_1^*|^{m(A_1)}|\phi_2^*|^{m(A_2)} = 1 \quad \textit{almost everywhere on } \mathbb{T}.$$

*There exists an ergodic measure-preserving transformation on $(\mathbb{T}, m)$ such that for almost every $\omega \in \mathbb{T}$, the sequence $(T_n(\omega))_{n\geq 1}$ is not universal.*

In the same line, we have the following result.

**Proposition 5.3.29.** *Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Let $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that*

$$|\phi_1(z)|^{m(A_1)}|\phi_2(z)|^{m(A_2)} > 1 \quad \textit{for every } z \in \mathbb{D}.$$

*There exists an ergodic measure-preserving transformation on $(\mathbb{T}, m)$ such that for almost every $\omega \in \mathbb{T}$, the sequence $(T_n(\omega))_{n\geq 1}$ is not universal.*

*Proof.* Let us observe that the functions $\phi_1, \phi_2$ are bounded below and above on $\mathbb{D}$, and that $1/\phi_1, 1/\phi_2$ belong to $H^\infty(\mathbb{D})$. Indeed, we have

$$\|\phi_1\|_\infty^{-1} \leq |\phi_1(z)|^{-1} \leq \|\phi_2\|_\infty^{\frac{m(A_2)}{m(A_1)}}$$

for every $z \in \mathbb{D}$, and similarly for $\phi_2$.

By Proposition 5.2.2, there exists an ergodic measure-preserving transformation $\tau$ on $(\mathbb{T}, m)$ such that the sequence $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ is bounded for every $\omega$ in a certain Borel subset $E$ of $\mathbb{T}$ with $m(E) = 1$, where $f := \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$.

For this transformation $\tau$, the operators $T_n(\omega)$ are invertible, with

$$T_n(\omega)^{-1} = (M_{1/\phi_1}^*)^{a_1(n,\omega)}(M_{1/\phi_2}^*)^{a_2(n,\omega)}.$$

But

$$\|T_n(\omega)^{-1}\| \leq \sup_{z\in\mathbb{D}}\{|\phi_1(z)|^{-\mathbb{S}_n^\tau f(\omega)}\}.$$

The same arguments as in the proof of Proposition 5.3.25 show that each orbit under $(T_n(\omega)^{-1})_{n\geq 1}$ is bounded when $\omega \in E$, and this concludes the proof of Proposition 5.3.29. □

**Remark 5.3.30.** Let $\tau$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Let

$\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ be nonconstant, such that

$$|\phi_1^*|^{m(A_1)} |\phi_2^*|^{m(A_2)} = 1 \quad \text{almost everywhere on } \mathbb{T}. \tag{5.3.14}$$

We have seen in Theorems 5.3.19 and 5.3.20 that under some assumptions on the sequence of Birkhoff sums associated to the function $f := \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$ and under some assumptions on the functions $\phi_1$ and $\phi_2$, the random sequence $(T_n(.))_{n\geq 1}$ is topologically weakly mixing. Since $\liminf_{n\to\infty} |\mathbb{S}_n^\tau f(\omega)| = 0$ for almost every $\omega \in \mathbb{T}$, there is a subset $E$ of $\mathbb{T}$ with $m(E) = 1$ such that the sequence $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ has a bounded subsequence for every $\omega \in E$. In particular, for this subsequence $(\mathbb{S}_{n_k}^\tau f(\omega))_{k\geq 1}$, the sequence $(T_{n_k}(\omega))_{k\geq 1}$ is not universal by Proposition 5.3.25, and in particular the sequence $(T_n(\omega))_{n\geq 1}$ is not topologically mixing (see [31, Exercise 3.4.5]).

## 5.4 Other examples of random products on the space of entire functions

In this section, we study some natural examples of random products of operators defined on the space of entire functions. In particular, we will construct examples of universal random sequences $(T_n(.))_{n\geq 1}$ for which the operators $T(\tau^i\omega)$, $i \geq 0$, do not commute.

We denote by $H(\mathbb{C})$ the space of entire functions, equipped with the topology of uniform convergence on compact subsets of $\mathbb{C}$. This topology is generated by the seminorms $(p_n)_{n\geq 1}$, where $p_n(f) = \sup_{|z|\leq n} |f(z)|$. We denote by $D$ the derivation operator on $H(\mathbb{C})$ defined by $D : f \mapsto f'$. Using Cauchy estimates, one can show that $D$ is indeed an operator on $H(\mathbb{C})$, that is, a continuous linear map on $H(\mathbb{C})$.

We will first consider random products generated by entire functions of exponential type of $D$. Let us recall some facts regarding entire functions of exponential type.

An entire function $\varphi \in H(\mathbb{C})$ is said to be of exponential type if there exist constants $M, A > 0$ such that

$$|\varphi(z)| \leq M e^{A|z|}, \quad \text{for every } z \in \mathbb{C}.$$

If $\varphi : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type, with $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$, the expression

$$\varphi(D) := \sum_{n\geq 0} a_n D^n$$

defines an operator on $H(\mathbb{C})$ ([31, Proposition 4.19]). In particular, if $a \neq 0$ is a complex number and if we denote by $T_a : f \mapsto f(. + a)$ the translation by $a$, we have $T_a = \varphi(D)$ with $\varphi(z) = e^{az}$. It is also known that if $T$ is an operator on $H(\mathbb{C})$ commuting with $D$, then there exists an entire function $\varphi$ of exponential type such that $T = \varphi(D)$.

The linear dynamics of the operators $D$ and $T_a$, $a \neq 0$, is well known (see, for instance, [31, Examples 2.20 and 2.21]). It is thus natural to have a look at the linear dynamics of random sequences $(T_n(.))_{n\geq 1}$ composed of random products of the operators $D$ and $T_a$ with $a \neq 0$. This is the aim of the following result.

**Theorem 5.4.1.** *Let $\tau : \mathbb{T} \to \mathbb{T}$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $A_1, A_2$ be two disjoint Borel subsets of $[0, 1)$ such that $A_1 \cup A_2 = [0, 1)$ and $m(A_k) > 0$ for $k = 1, 2$. Let $\varphi_1$ and*

*$\varphi_2$ be two nonconstant entire functions of exponential type. Consider the map*

$$T(\omega) := \begin{cases} \varphi_1(D) & \text{if } \omega \in A_1 \\ \varphi_2(D) & \text{if } \omega \in A_2 \end{cases}.$$

*Suppose that there exist $z, w \in \mathbb{C}$ such that*

$$|\varphi_1(z)|^{m(A_1)}|\varphi_2(z)|^{m(A_2)} < 1 \tag{5.4.1}$$

$$\text{and} \quad |\varphi_1(w)|^{m(A_1)}|\varphi_2(w)|^{m(A_2)} > 1. \tag{5.4.2}$$

*Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically mixing.*

*Proof.* Let us set $\varphi_1(z) = \sum_{n\geq 0} a_n z^n$ and $\varphi_2(z) = \sum_{n\geq 0} b_n z^n$ for $z \in \mathbb{C}$. Since $\varphi_1(D)$ and $\varphi_2(D)$ commute, one has

$$T_n(\omega) = (\varphi_1(D))^{a_1(n,\omega)}(\varphi_2(D))^{a_2(n,\omega)}.$$

Our goal is to apply Proposition 5.3.3. It is well known that the functions $e_\lambda : z \mapsto e^{\lambda z}$ are eigenvectors for the operator $D$. It is thus natural to use these functions for the proof. More precisely, we have for every $\lambda \in \mathbb{C}$

$$T_n(\omega)e_\lambda = (\varphi_1(\lambda))^{a_1(n,\omega)}(\varphi_2(\lambda))^{a_2(n,\omega)}e_\lambda.$$

Let us denote by $E$ a Borel subset of $\mathbb{T}$ with $m(E) = 1$, such that

$$\frac{a_1(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} m(A_1) \quad \text{and} \quad \frac{a_2(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} m(A_2)$$

for every $\omega \in E$. We thus have,

$$\begin{aligned} |(\varphi_1(\lambda))^{a_1(n,\omega)}(\varphi_2(\lambda))^{a_2(n,\omega)}| &= e^{a_1(n,\omega)\log(|\varphi_1(\lambda)|)+a_2(n,\omega)\log(|\varphi_2(\lambda)|)} \\ &\underset{n\to\infty}{=} e^{n[\log(|\varphi_1(\lambda)|^{m(A_1)}|\varphi_2(\lambda)|^{m(A_2)})+o(1)]}. \end{aligned}$$

Under the assumptions of Theorem 5.4.1, the sets

$$\Lambda_1 := \{\lambda \in \mathbb{C} : |\varphi_1(\lambda)|^{m(A_1)}|\varphi_2(\lambda)|^{m(A_2)} > 1\}$$

$$\text{and} \quad \Lambda_2 := \{\lambda \in \mathbb{C} : |\varphi_1(\lambda)|^{m(A_1)}|\varphi_2(\lambda)|^{m(A_2)} < 1\}$$

are nonempty and both have an accumulation point in $\mathbb{C}$. Now let us denote by $\mathcal{A}_1$ and $\mathcal{A}_2$ the sets

$$\mathcal{A}_1 := \operatorname{span}\{e_\lambda : \lambda \in \Lambda_1\} \quad \text{and} \quad \mathcal{A}_2 := \operatorname{span}\{e_\lambda : \lambda \in \Lambda_2\}.$$

These sets are dense in $H(\mathbb{C})$. Moreover, denoting by $\lambda(T_n(\omega), f)$ the eigenvalue of $T_n(\omega)$ associated to the eigenfunction $f$, we have $|\lambda(T_n(\omega), f)| \underset{n\to\infty}{\longrightarrow} \infty$ for every $f \in \mathcal{A}_1$ and $|\lambda(T_n(\omega), g)| \underset{n\to\infty}{\longrightarrow} 0$ for every $g \in \mathcal{A}_2$, for every $\omega \in E$.

This proves that the sequence $(T_n(\omega))_{n\geq 1}$ is topologically mixing for almost every $\omega \in \mathbb{T}$ and this concludes the proof of Theorem 5.4.1. □

In the case where $m(A_1) = m(A_2) = 1/2$ and where $\varphi_1$ and $\varphi_2$ are two nonconstant functions of exponential type, both Conditions 5.4.1 and 5.4.2 are simultaneously satisfied if and only if $\varphi_1\varphi_2$ is nonconstant. Indeed, this comes from the fact that the image of a nonconstant entire function is dense in $\mathbb{C}$. In the situation where $\varphi_1\varphi_2$ is constant and nonzero, we must have $\varphi_1(z) = e^{\alpha z+\beta}$ for some $\alpha \neq 0$ and $\beta \in \mathbb{C}$. Indeed, suppose that $|\varphi_1(z)| \leq Me^{A|z|}$ for every $z \in \mathbb{C}$, where $A, M > 0$. Since $\varphi_1$ does not vanish on $\mathbb{C}$, we can write $\varphi_1 = e^{\varphi}$ with $\varphi$ an entire function. We thus obtain that $\mathrm{Re}(\varphi(z)) \leq \log(M)+A|z|$ for every $z \in \mathbb{C}$, and thus $\varphi$ is a polynomial of degree at most one.

In this latter case, we have that $\varphi_1(D) = e^{\beta} T_\alpha$ and $\varphi_2(D) = c\,e^{-\beta} T_{-\alpha}$ for some $c \neq 0$. We treat a case in the situation where $\varphi_1\varphi_2$ is constant on $\mathbb{C}$. This is again linked to the behavior of Birkhoff sums. This shows in particular that a random sequence $(T_n(.))_{n\geq 1}$ can be non-universal even if $T(\omega)$ is hypercyclic for every $\omega \in \mathbb{T}$.

**Proposition 5.4.2.** *Let $\tau : \mathbb{T} \to \mathbb{T}$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1, 2$. Let $\varphi_1(z) = e^z$ and $\varphi_2(z) = e^{-z}$ for every $z \in \mathbb{C}$. Consider the map*

$$T(\omega) := \begin{cases} \varphi_1(D) & \text{if } \omega \in A_1 \\ \varphi_2(D) & \text{if } \omega \in A_2 \end{cases}.$$

1. *If $m(A_1) \neq m(A_2)$, then $(T_n(\omega))_{n\geq 1}$ is topologically mixing for almost every $\omega \in \mathbb{T}$;*
2. *If $m(A_1) = m(A_2) = 1/2$, then $(T_n(\omega))_{n\geq 1}$ is universal for almost every $\omega \in \mathbb{T}$ if and only if the sequence $(\mathbb{S}^\tau_n f(\omega))_{n\geq 1}$ is unbounded for almost every $\omega \in \mathbb{T}$, where $f := \mathbb{1}_{A_1} - \mathbb{1}_{A_2}$.*

*Proof.* In the context of Proposition 5.4.2, the operators $T_n(\omega)$ are given by

$$T_n(\omega) = T_{a_1(n,\omega)-a_2(n,\omega)}.$$

The first point follows from Theorem 5.4.1. Let us prove the second point. We again use the eigenvectors $e_\lambda$ to obtain that

$$T_n(\omega)e_\lambda = e^{\lambda \mathbb{S}^\tau_n f(\omega)} e_\lambda$$

for every $\lambda \in \mathbb{C}$. Suppose first that $(\mathbb{S}^\tau_n f(\omega))_{n\geq 1}$ is unbounded for almost every $\omega \in \mathbb{T}$. Let us consider a subset $E$ of $\mathbb{T}$ with $m(E) = 1$ such that for every $\omega \in E$, we have that

$$\liminf_{n\to\infty} \mathbb{S}^\tau_n f(\omega) = -\infty \quad \text{or} \quad \limsup_{n\to\infty} \mathbb{S}^\tau_n f(\omega) = \infty.$$

Let $\omega \in E$, and suppose, for example, that $\limsup\limits_{n\to\infty} \mathbb{S}^\tau_n f(\omega) = \infty$. Let $(n_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers such that $\mathbb{S}^\tau_{n_k} f(\omega) \underset{k\to\infty}{\longrightarrow} \infty$. We define a right inverse of $T_{n_k}(\omega)$ on the eigenvectors $e_\lambda$ by setting $S_{n_k}(\omega)e_\lambda = e^{-\lambda \mathbb{S}^\tau_{n_k} f(\omega)} e_\lambda$ for $\mathrm{Re}\,\lambda > 0$ and extending it by linearity to the dense subspace $\mathrm{span}\{e_\lambda : \mathrm{Re}\,\lambda > 0\}$ of $H(\mathbb{C})$. We thus have $T_{n_k}(\omega)g \underset{k\to\infty}{\longrightarrow} 0$ for every $g \in \mathrm{span}\{e_\lambda : \mathrm{Re}\,\lambda < 0\}$ and $S_{n_k}(\omega)g \underset{k\to\infty}{\longrightarrow} 0$ for every $g \in \mathrm{span}\{e_\lambda : \mathrm{Re}\,\lambda > 0\}$, and the Universality Criterion applies. In the case where $\liminf\limits_{n\to\infty} \mathbb{S}^\tau_n f(\omega) = -\infty$, we proceed in the same way.

Suppose now that $(\mathbb{S}^\tau_n f(\omega))_{n\geq 1}$ is bounded for almost every $\omega \in \mathbb{T}$. Let $E$ be a subset of $\mathbb{T}$ with $m(E) = 1$ such that $(\mathbb{S}^\tau_n f(\omega))_{n\geq 1}$ is bounded for every $\omega \in E$. Let $\omega \in E$ and suppose that there exists a function $h \in H(\mathbb{C})$ such that the set $\{T_n(\omega)h : n \geq 1\}$ is dense in $H(\mathbb{C})$. Since $T_n(\omega)h = h(. + \mathbb{S}^\tau_n f(\omega))$ and since the map $g \mapsto g(0)$ is continuous on $H(\mathbb{C})$, the set $\{h(\mathbb{S}^\tau_n f(\omega)) : n \geq 1\}$ is dense in $\mathbb{C}$. This gives a contradiction, since the set $\{h(\mathbb{S}^\tau_n f(\omega)) : n \geq 1\}$ is bounded in $\mathbb{C}$. This concludes the proof of Proposition 5.4.2. □

We now study an example of random sequences $(T_n(.))_{n\geq 1}$ for which the operators $T(\tau^i\omega)$, $i \geq 0$, do not commute.

If $\lambda \in \mathbb{C}$ is nonzero and if $b \in \mathbb{C}$, we denote by $T_{\lambda,b}$ the operator on $H(\mathbb{C})$ defined by $T_{\lambda,b} f = f(\lambda . + b)$. Let us remark that

$$T_{\lambda,b}^n f = f(\lambda^n . + b \sum_{k=0}^{n-1} \lambda^k) \tag{5.4.3}$$

for every $n \geq 1$.

**Theorem 5.4.3.** *Let $\tau : \mathbb{T} \to \mathbb{T}$ be an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Let $A_1, A_2$ be two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. Consider the map*

$$T(\omega) := \begin{cases} T_{\lambda,b} & \text{if } \omega \in A_1 \\ D & \text{if } \omega \in A_2 \end{cases},$$

*with $\lambda \in \mathbb{C}$ nonzero and $b \in \mathbb{C}$. Then, the random sequence $(T_n(.))_{n\geq 1}$ is topologically mixing.*

**Remark 5.4.4.** It is known that the operator $T_{\lambda,b}$ is hypercyclic if and only if $\lambda = 1$ and $b \neq 0$, by [8, Proposition 2.4].

Thus, Theorem 5.4.3 gives an example of a map $\omega \mapsto T(\omega)$ which is hypercyclic for every $\omega \in A_2$ and not hypercyclic for every $\omega \in A_1$, for which the random sequence $(T_n(.))_{n\geq 1}$ is universal, and for which the maps $T(\tau^i\omega)$, $i \geq 0$, do not commute.

*Proof of Theorem 5.4.3 .* The operators $D$ and $T_{\lambda,b}$ do not commute, but we have the relation

$$DT_{\lambda,b} = \lambda T_{\lambda,b} D.$$

Thus, for $n \geq 1$, $\omega \in \mathbb{T}$ and $f \in H(\mathbb{C})$, we have

$$T_n(\omega) f = \lambda^{c(n,\omega)} f^{(a_2(n,\omega))}\left(\lambda^{a_1(n,\omega)} . + b \sum_{k=0}^{a_1(n,\omega)-1} \lambda^k\right), \tag{5.4.4}$$

where $c(n,\omega)$ is a random integer such that $0 \leq c(n,\omega) \leq n$ for every $n \geq 1$ and $\omega \in \mathbb{T}$. Let us denote by $r_n(\omega)$ the following number

$$r_n(\omega) := b \sum_{k=0}^{a_1(n,\omega)-1} \lambda^k,$$

and by $E \subseteq \mathbb{T}$ a Borel set of measure 1 such that

$$\frac{a_1(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} m(A_1) \quad \text{and} \quad \frac{a_2(n,\omega)}{n} \underset{n\to\infty}{\longrightarrow} m(A_2)$$

for every $\omega \in E$.

Our strategy will be to apply the Universality Criterion with respect to the full sequence $(n)$, with $Z := \mathbb{C}[z]$ as a dense subset of $H(\mathbb{C})$. As in the proof of [21, Theorem 1], we define a right inverse $S_n(\omega)$ on $Z$ by setting for every $k \geq 0$ and $n \geq 1$

$$S_n(\omega)z^k = \frac{k!}{(k+a_2(n,\omega))!}\,\lambda^{-c(n,\omega)}\lambda^{-k\,a_1(n,\omega)}\sum_{j=0}^{k}\binom{k+a_2(n,\omega)}{j}z^{k+a_2(n,\omega)-j}(-r_n(\omega))^j$$

if $b \neq 0$, and by

$$S_n(\omega)z^k = \frac{k!}{(k+a_2(n,\omega))!}\,\lambda^{-c(n,\omega)}\lambda^{-k\,a_1(n,\omega)}\,z^{k+a_2(n,\omega)}$$

if $b = 0$, and extending it by linearity to $Z$.

We clearly have, for every $f \in Z$, $T_n(\omega)f = 0$ when $n$ is sufficiently large, since $a_2(n,\omega) \underset{n\to\infty}{\longrightarrow} \infty$. For every $k \geq 0$, $n \geq 1$ and $\omega \in E$, we have

$$\begin{aligned} T_n(\omega)S_n(\omega)z^k &= \lambda^{-k\,a_1(n,\omega)}\sum_{j=0}^{k}\binom{k}{j}(\lambda^{a_1(n,\omega)}z + r_n(\omega))^{k-j}(-r_n(\omega))^j \\ &= \lambda^{-k\,a_1(n,\omega)}(\lambda^{a_1(n,\omega)}z)^k \\ &= z^k, \end{aligned}$$

and thus $T_n(\omega)S_n(\omega)f = f$ for every $f \in Z$. We now prove that $S_n(\omega)f \underset{n\to\infty}{\longrightarrow} 0$ for every $\omega \in E$. To this aim, we will distinguish the cases $|\lambda| < 1$ and $|\lambda| \geq 1$.

*Case* 1. Suppose first that $|\lambda| \geq 1$.

Let $k \geq 0$, $R > 0$, $n \geq 1$ and $0 \leq j \leq k$. Let $z \in \mathbb{C}$ such that $|z| < R$. We have

$$\begin{aligned} &\left|\frac{k!}{(k+a_2(n,\omega))!}\,\lambda^{-c(n,\omega)}\lambda^{-k\,a_1(n,\omega)}\binom{k+a_2(n,\omega)}{j}z^{k+a_2(n,\omega)-j}(-r_n(\omega))^j\right| \qquad (5.4.5)\\ &\leq \frac{k!\,|r_n(\omega)|^j\,R^{k+a_2(n,\omega)-j}}{j!\,(k+a_2(n,\omega)-j)!}|\lambda|^{-k\,a_1(n,\omega)} \\ &\leq \frac{k!\,R^{k+a_2(n,\omega)-j}}{j!\,(k+a_2(n,\omega)-j)!}|b|^j\,n^j\,|\lambda|^{nj}|\lambda|^{-k\,a_1(n,\omega)} \\ &\leq C_1\,\frac{R^{k+a_2(n,\omega)-j}}{(k+a_2(n,\omega)-j)!}(a_2(n,\omega))^j\left(\frac{n}{a_2(n,\omega)}\right)^j|\lambda|^{k(n-a_1(n,\omega))} \\ &\leq C_1\,\frac{R^{k+a_2(n,\omega)-j}}{(k+a_2(n,\omega)-j)!}(a_2(n,\omega))^j\left(\frac{n}{a_2(n,\omega)}\right)^j|\lambda|^{k\,a_2(n,\omega)}, \end{aligned}$$

where $C_1$ is a constant which depends on $k, b$ and $j$, but which is independent of $n$ and of $\omega \in E$. We used the fact that $|\lambda| \geq 1$ in the first inequality to bound $|\lambda|^{-c(n,\omega)}$, and we used that $|\lambda| \geq 1$ and that $a_1(n,\omega) \leq n$ in the second inequality to obtain that $|r_n(\omega)| \leq n|b||\lambda|^n$. Since

$$\frac{R^{k+n-j}}{(k+n-j)!}|\lambda|^{kn}n^j \underset{n\to\infty}{\longrightarrow} 0, \quad a_2(n,\omega) \underset{n\to\infty}{\longrightarrow} \infty \quad \text{and} \quad \left(\frac{n}{a_2(n,\omega)}\right)^j \underset{n\to\infty}{\longrightarrow} m(A_2)^{-j}$$

when $\omega \in E$, the upper bound of (5.4.5) converges to 0 as $n \to \infty$, for every $\omega \in E$. Now summing over $0 \leq j \leq k$ the left hand side of (5.4.5), we obtain that $S_n(\omega)z^k \underset{n\to\infty}{\longrightarrow} 0$ for every $k \geq 0$, when

$b \neq 0$. If $b = 0$, the convergence also holds because this case follows by taking $j = 0$ in the previous inequalities, using the convention that $a^0 = 1$ for every real number $a \geq 0$.

*Case 2.* We now consider the case where $|\lambda| < 1$.

Let $k \geq 0$, $R > 0$, $n \geq 1$ and $0 \leq j \leq k$. Let $z \in \mathbb{C}$ such that $|z| < R$. We have

$$\begin{aligned}
&\left| \frac{k!}{(k + a_2(n,\omega))!} \lambda^{-c(n,\omega)} \lambda^{-k a_1(n,\omega)} \binom{k + a_2(n,\omega)}{j} z^{k+a_2(n,\omega)-j} (-r_n(\omega))^j \right| \\
&\leq C_2\, |\lambda|^{-(k+1)n} \frac{R^{k+a_2(n,\omega)-j}}{(k + a_2(n,\omega) - j)!} \\
&\leq C_3\, |\lambda|^{-(k+1)n} \frac{R^{a_2(n,\omega)}}{a_2(n,\omega)!},
\end{aligned}$$

where $C_2$ and $C_3$ are two constants depending on $k, b$ and $j$, but that are independent of $\omega \in E$ and $n \geq 1$. For the first inequality, we used the fact that $|\lambda| < 1$, $c(n,\omega) \leq n$ and $a_1(n,\omega) \leq n$, and we also used the fact that $|\lambda| < 1$ to assert that

$$|r_n(\omega)|^j \leq |b|^j \left( \sum_{l \geq 0} |\lambda|^l \right)^j \leq (1 - |\lambda|)^{-j}\, |b|^j.$$

By Stirling's formula, there is a constant $C > 0$ such that

$$\frac{1}{n!} \leq C \left( \frac{e}{n} \right)^n n^{-1/2} \tag{5.4.6}$$

for every $n \geq 1$. Using (5.4.6), we deduce that there exists a constant $C_4$ which is independent of $\omega \in E$ and $n \geq 1$, such that for every $n \geq 1$, $k \geq 0$, $0 \leq j \leq k$, and $|z| < R$,

$$\begin{aligned}
&\left| \frac{k!}{(k + a_2(n,\omega))!} \lambda^{-c(n,\omega)} \lambda^{-k a_1(n,\omega)} \binom{k + a_2(n,\omega)}{j} z^{k+a_2(n,\omega)-j} (-r_n(\omega))^j \right| \\
&\leq C_4\, |\lambda|^{-(k+1)n} R^{a_2(n,\omega)} \left( \frac{e}{a_2(n,\omega)} \right)^{a_2(n,\omega)} (a_2(n,\omega))^{-1/2} \\
&\leq C_4 \exp(u_n(\omega)),
\end{aligned} \tag{5.4.7}$$

where

$$\begin{aligned}
u_n(\omega) := {}& a_2(n,\omega)(1 - \log(a_2(n,\omega))) - \tfrac{1}{2} \log(a_2(n,\omega)) \\
& - (k+1)\, n \log|\lambda| + a_2(n,\omega) \log(R)
\end{aligned}$$

for every $n \geq 1$ and $\omega \in E$. Finally, we have

$$\begin{aligned}
u_n(\omega) \underset{n \to +\infty}{=} {}& (n\, m(A_2) + o(n))(1 - \log(n\, m(A_2) + o(n))) - \tfrac{1}{2} \log(n\, m(A_2) + o(n)) \\
& - n(k+1) \log(|\lambda|) + n\, m(A_2) \log(R) + o(n) \\
\underset{n \to +\infty}{=} {}& n\, (m(A_2) + m(A_2) \log(R) - (k+1) \log(|\lambda|)) + o(n) \\
& - \log(n\, m(A_2) + o(n))(\tfrac{1}{2} + n\, m(A_2) + o(n)).
\end{aligned}$$

From this, one can check that

$$u_n(\omega) \underset{n\to\infty}{\sim} -n\,m(A_2)\log(n\,m(A_2)),$$

which in particular proves that $\exp(u_n(\omega))$ converges to 0 as $n \to \infty$ when $\omega \in E$. It follows that the left hand side of (5.4.7) converges to 0 as $n \to \infty$. Summing over $0 \le j \le k$ the left hand side of (5.4.7), we obtain that $S_n(\omega)z^k \underset{n\to\infty}{\longrightarrow} 0$ for every $k \ge 0$, when $b \ne 0$. If $b = 0$, the convergence also holds because this case follows by taking $j = 0$ in the previous inequalities, using again the convention that $a^0 = 1$ for every real number $a \ge 0$.

This proves that the sequence $(T_n(\omega))_{n\ge1}$ satisfies Proposition 5.1.2 for every $\omega \in E$. In particular, the sequence $(T_n(\omega))_{n\ge1}$ is topologically mixing for every $\omega \in E$. □

**Remark 5.4.5.** Again, the conclusions of Theorems 5.4.1 and 5.4.3 hold for every $\omega \in \mathbb{T}$ when the ergodic transformation $\tau$ is an irrational rotation and when $A_1, A_2$ are intervals of $[0,1)$, that is, the sequence $(T_n(\omega))_{n\ge1}$ is topologically mixing for every $\omega \in \mathbb{T}$.

## 5.5 Comments and open problems

We end this article with some comments and open questions related to our previous results. Since this is a first work on the linear dynamics of random products of operators, many problems are still open and we have a lot of remarks.

But before that, it is worth noting that many of the results stated in this paper remain valid for any measure-preserving ergodic dynamical system. This is the case, for example, of Propositions 5.2.5, 5.3.1, 5.3.2, 5.3.14, 5.3.16, 5.3.25, as well as Theorems 5.3.4, 5.3.9, 5.3.10, 5.3.17, 5.3.19, 5.3.20, 5.4.1 and 5.4.3.

### 5.5a A general remark on the universality of the random sequence $(T_n(.))_{n\ge1}$

Let us observe that in all of our results, the sequence of operators $(T_n(\omega))_{n\ge1}$ is either universal for almost every $\omega \in \mathbb{T}$, or not universal for almost every $\omega \in \mathbb{T}$. Most of the time, this comes from Birkhoff's Theorem 5.1.10, or this comes from the fact that the sequence of Birkhoff sums satisfies a CLT. We in fact have the following $0-1$ law.

**Proposition 5.5.1.** *Let $\tau : \Omega \to \Omega$ be an ergodic measure-preserving transformation on a Polish probability space $(\Omega, Bor(\Omega), m)$, where $Bor(\Omega)$ is the Borel $\sigma$-algebra of $\Omega$ and $m$ is a Borel probability measure on $\Omega$. Let $X$ be a separable Banach space. Let $T : \Omega \to \mathcal{B}(X)$ be strongly measurable, which means that for every $x \in X$, the map $\omega \mapsto T(\omega)x$ is $(Bor(\Omega), Bor(X))$-measurable for the Borel $\sigma$-algebras of $\Omega$ and $X$. Then, either the sequence $(T_n(\omega))_{n\ge1}$ is universal for almost every $\omega \in \Omega$, or the sequence $(T_n(\omega))_{n\ge1}$ is not universal for almost every $\omega \in \Omega$.*

*Proof.* Let us denote by $A$ the set

$$A := \{\omega \in \Omega : (T_n(\omega))_{n\ge1} \text{ is universal}\}.$$

Let $(V_j)_{j\ge1}$ be a countable basis of open sets of $X$. Let us remark that the set $A$ can be written as

$$A = \mathrm{proj}_\Omega\Big(\bigcap_{j\ge1}\bigcup_{n\ge1}\{(\omega,x) \in \Omega \times X : T_n(\omega)x \in V_j\}\Big),$$

where $\mathrm{proj}_\Omega$ is the projection onto $\Omega$. Now since the Banach space $X$ is separable, the strong measurability of the map $T:\Omega\to\mathcal{B}(X)$ is equivalent to the $(\mathrm{Bor}(\Omega)\otimes\mathrm{Bor}(X),\mathrm{Bor}(X))$-measurability of the map $(\omega,x)\mapsto T(\omega)x$. Moreover, the composition of strongly measurable maps is strongly measurable (see [27, Appendix A] for these remarks). In particular, for each $n\geq 1$, the map $(\omega,x)\mapsto T_n(\omega)x$ is measurable. Thus, the set $A$ is the projection of a Borel subset of $\Omega\times X$ and is an analytic subset of $\Omega$ (see [39] for background on analytic subsets). In particular, the set $A$ is universally measurable ([39, Theorem 21.10, page 155]) and belongs to the completion of the Borel $\sigma$-algebra of $\Omega$, that we denote by $\Sigma^*$. We also denote by $m^*$ the completion of the measure $m$ on $\Omega$. Every element of $\Sigma^*$ can be written as $B\cup N$, with $B$ a Borel subset of $\Omega$ and $N$ a subset of $\Omega$ such that $N\subset C$ with $C$ a Borel subset satisfying $m(C)=0$. Moreover, $m^*(B\cup N)=m(B)$. We can easily show that the transformation $\tau$ is measure-preserving for $m^*$. Let $D=B\cup N$ be in $\Sigma^*$ and satisfy $\tau^{-1}(D)=D$, with $N\subset C$ and $m(C)=0$. From $B\cup N=\tau^{-1}(B)\cup\tau^{-1}(N)$, it follows that $B\subset\tau^{-1}(C)\cup\tau^{-1}(B)$ and $\tau^{-1}(B)\subset C\cup B$, and thus $m(\tau^{-1}(B)\triangle B)=0$. Since $\tau$ is ergodic for $m$, it follows that $m(B)=m^*(D)=0$ or $m(B)=m^*(D)=1$, which proves that $\tau$ is ergodic for $m^*$.

Now let us notice that $\tau(A)\subset A$, since $T_{n+1}(\omega)=T_n(\tau\omega)T(\omega)$ for every $n\geq 1$ and every $\omega\in\Omega$. By ergodicity, we thus have $m^*(A)=0$ or $m^*(A)=1$, which in particular implies the conclusion of Proposition 5.5.1. □

## 5.5b Comments on random products of adjoints of multiplication operators

Let us consider the sequence $(T_n(\omega))_{n\geq 1}$ given by

$$T(\omega)=\begin{cases}(M_{\phi_1})^* & \text{if}\quad \omega\in A_1\\ (M_{\phi_2})^* & \text{if}\quad \omega\in A_2\end{cases}, \tag{5.5.1}$$

where $\phi_1,\phi_2\in H^\infty(\mathbb{D})$ are nonconstant functions, $A_1,A_2$ are two disjoint Borel subsets of $[0,1)$ such that $A_1\cup A_2=[0,1)$ and $m(A_1)=m(A_2)=1/2$, and with $\tau$ an ergodic measure-preserving transformation on $(\mathbb{T},m)$. For example, let us suppose that $A_1=[0,1/2)$ and $A_2=[1/2,1)$.

We know that, thanks to Theorem 5.3.4, if $\phi_1\phi_2$ is nonconstant and such that $(\phi_1\phi_2)(\mathbb{D})\cap\mathbb{T}\neq\emptyset$, then the random sequence $(T_n(.))_{n\geq 1}$ is universal. This situation thus leads to two cases that we should call limit cases: the case $(\phi_1\phi_2)(\mathbb{D})\subset\mathbb{D}$ and the case $(\phi_1\phi_2)(\mathbb{D})\subset\mathbb{C}\setminus\overline{\mathbb{D}}$. We have also studied some cases that lead to one of these two limit cases (Proposition 5.3.6 and Corollary 5.3.7). These two sub-cases can be seen as trivial situations of the two limit cases. We have also studied the situation where $\phi_1\phi_2$ is inner, which is included in the case $(\phi_1\phi_2)(\mathbb{D})\subset\mathbb{D}$ (Theorems 5.3.9 and 5.3.10). We can sum up our work on the two limit cases with the following table 5.1.

| | |
|---|---|
| $\phi_1\phi_2$ inner. $\phi_1(\mathbb{D})\cap\mathbb{T}\neq\emptyset$ and $1/\phi_1\in H^\infty(\mathbb{D})$, or $\phi_2(\mathbb{D})\cap\mathbb{T}\neq\emptyset$ and $1/\phi_2\in H^\infty(\mathbb{D})$. $\limsup_{n\to\infty}\mathbb{S}_n^\tau f=\infty$ or $\liminf_{n\to\infty}\mathbb{S}_n^\tau f=-\infty$. Universality of the random sequence | $\phi_1\phi_2$ inner and nonconstant. $\phi_1$ or $\phi_2$ not inner. $\limsup_{n\to\infty}\mathbb{S}_n^\tau f=\infty$ and $\liminf_{n\to\infty}\mathbb{S}_n^\tau f=-\infty$. Universality of the random sequence |
| $\phi_1(\mathbb{D})\subset\mathbb{D}$ and $\phi_2(\mathbb{D})\subset\mathbb{D}$. Non-universality of the random sequence | $\phi_1(\mathbb{D})\subset\mathbb{C}\setminus\overline{\mathbb{D}}$ and $\phi_2(\mathbb{D})\subset\mathbb{C}\setminus\overline{\mathbb{D}}$. Non-universality of the random sequence |

Table 5.1: Universality of the random sequence $(T_n(.))_{n\geq 1}$ in known limit cases

Certain situations are missing in our work. For example, we don't know if the sequence $(T_n(.))_{n\geq 1}$ is universal when $\phi_1,\phi_2\in H^\infty(\mathbb{D})$ are nonconstant such that $\phi_1(\mathbb{D})\subset\mathbb{D}$, $\phi_2(\mathbb{D})\subset\mathbb{C}\setminus\overline{\mathbb{D}}$, $(\phi_1\phi_2)(\mathbb{D})\subset\mathbb{D}$, $\phi_1\phi_2$ is nonconstant and not inner, and $\|\phi_1\|_\infty\|\phi_2\|_\infty\geq 1$. However, for certain types of functions $\phi_1$ and $\phi_2$ in this last case, we can observe that the sequence $(T_n(.))_{n\geq 1}$ is not universal. This is the case of the following example.

**Example 5.5.2.** Let $\phi_1(z)=e^{1-z}$ and $\phi_2(z)=e^{1/2(z-1)}$ for every $z\in\mathbb{D}$. In this case, we have $\phi_1(\mathbb{D})\subset\mathbb{C}\setminus\overline{\mathbb{D}}$, $\phi_2(\mathbb{D})\subset\mathbb{D}$, $\|\phi_1\|_\infty=e^2$, $\|\phi_2\|_\infty=1$, and $\|\phi_1\|_\infty\|\phi_2\|_\infty=e^2$. Also, $(\phi_1\phi_2)(\mathbb{D})\subseteq\mathbb{C}\setminus\overline{\mathbb{D}}$. In this case, the operator $T_n(\omega)$ is the adjoint of the multiplication operator by the function $e^{(1-z)(a_1(n,\omega)-\frac{1}{2}a_2(n,\omega))}$. Moreover, the map $T_n(\omega)$ is invertible and its inverse $S_n(\omega)$ is the adjoint of the multiplication operator by the function $e^{(z-1)(a_1(n,\omega)-\frac{1}{2}a_2(n,\omega))}$. Thus,

$$\|S_n(\omega)\|=\sup_{z\in\mathbb{D}}\{e^{(\mathrm{Re}(z)-1)(a_1(n,\omega)-\frac{1}{2}a_2(n,\omega))}\}=1$$

when $n\geq 1$ is sufficiently large, for almost every $\omega\in\mathbb{T}$, because $\frac{1}{n}(a_1(n,\omega)-\frac{1}{2}a_2(n,\omega))$ converges to $1/4$ for almost every $\omega\in\mathbb{T}$. This shows that both sequences $(T_n(\omega))_{n\geq 1}$ and $(S_n(\omega))_{n\geq 1}$ are non-universal, for almost every $\omega\in\mathbb{T}$.

Example 5.5.2 gives a nontrivial situation where the random sequence $(T_n(.))_{n\geq 1}$ is non-universal, and where the limit cases hold. But this example is quite simple because we could use the norm of the inverse of $\|T_n(\omega)\|$ to prove the non-universality. Unfortunately, it is not always possible to do it in general, since it depends heavily on the expression of the functions $\phi_1$ and $\phi_2$. Also, let us observe that the functions $\phi_1$ and $\phi_2$ in Example 5.5.2 are both outer. More generally, in the limit case $(\phi_1\phi_2)(\mathbb{D})\subset\mathbb{C}\setminus\overline{\mathbb{D}}$ and $\phi_1\phi_2$ nonconstant, both functions $\phi_1$ and $\phi_2$ are outer since $1/\phi_1$ and $1/\phi_2$ belong to $H^\infty(\mathbb{D})$. In this case, the inner parts $I_1$ and $I_2$ of $\phi_1$ and $\phi_2$ respectively are constant, and we thus cannot use the subspaces $K:=\bigcup_{n\geq 1}(I_1^nH^2(\mathbb{D}))^\perp$ and $K':=\bigcup_{n\geq 1}(I_2^nH^2(\mathbb{D}))^\perp$ as in the proof of Theorem 5.3.10 to apply the Universality Criterion. This also is what makes the limit case $(\phi_1\phi_2)(\mathbb{D})\subset\mathbb{C}\setminus\overline{\mathbb{D}}$ hard to solve.

Regarding the limit case $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{D}$ and $\phi_1\phi_2$ nonconstant, we believe that a good understanding of the following example will help us to solve the limit case $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{D}$.

**Example 5.5.3.** Let $\phi_1, \phi_2$ be the two functions defined by $\phi_1(z) = e^{-1}z$ and $\phi_2(z) = e^z$ for every $z \in \mathbb{D}$. In this case, $(\phi_1\phi_2)(z) = ze^{z-1}$, and thus $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{D}$. Moreover, we have $\phi_2(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, $\phi_1(\mathbb{D}) \subset \mathbb{D}$, $\|\phi_1\|_\infty = e^{-1}$, $\|\phi_2\|_\infty = e$, and $\|\phi_1\|_\infty\|\phi_2\|_\infty = 1$.The operator $T_n(\omega)$ is the adjoint of the multiplication operator by the function $e^{-a_1(n,\omega)}z^{a_1(n,\omega)}e^{a_2(n,\omega)z}$. The operator $T_n(\omega)$ is not invertible, but we can use the subspace $\mathbb{C}[z]$ of the complex polynomials as a dense subspace for the Universality Criterion, since $T_n(\omega)f$ converges to 0 for every complex polynomial $f$, for almost every $\omega \in \mathbb{T}$. However, defining a right inverse $S_n(\omega)$ on the eigenvectors $k_z$ as in the proof of Theorem 5.3.10 is not enough to apply the Universality Criterion here. Indeed, let us define a right inverse $S_n(\omega)$ on the eigenvectors $k_z$ by

$$S_n(\omega)k_z = e^{a_1(n,\omega)}\, e^{-a_2(n,\omega)\overline{z}} M_z^{a_1(n,\omega)} k_z$$

and extending it by linearity to the dense subspace $Z := \operatorname{span}\{k_z : z \in \mathbb{D}\}$ of $H^2(\mathbb{D})$. We can observe that

$$\|S_n(\omega)k_z\| = e^{a_1(n,\omega)-a_2(n,\omega)\operatorname{Re} z}\|k_z\|,$$

which diverges to $\infty$ for every $z \in \mathbb{D}$, for almost every $\omega \in \mathbb{T}$. This makes this case not that easy to solve.

Finally, let us observe that

$$\|T_n(\omega)\| = \sup_{z\in\mathbb{D}}\{e^{a_2(n,\omega)\operatorname{Re}(z)}e^{-a_1(n,\omega)}|z|^{a_1(n,\omega)}\} = e^{a_2(n,\omega)-a_1(n,\omega)}.$$

If the sequence $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ of Birkhoff sums $\mathbb{S}_n^\tau f(\omega) = a_2(n,\omega) - a_1(n,\omega)$ is bounded for almost every $\omega \in \mathbb{T}$, then the random sequence $(T_n(.))_{n\geq 1}$ associated to this transformation $\tau$ is not universal. But in the case of an irrational rotation or the doubling map, it is not clear if this remains true.

The following problem is thus still open.

**Question 5.5.4.** Let $\tau$ be an irrational rotation or the doubling map on $\mathbb{T}$. Let $A_1 = [0,1/2)$ and $A_2 = [1/2,1)$. Let $(T_n(.))_{n\geq 1}$ be the random sequence of operators defined by the map

$$T(\omega) = \begin{cases} (M_{\phi_1})^* & \text{if} \quad \omega \in A_1 \\ (M_{\phi_2})^* & \text{if} \quad \omega \in A_2 \end{cases},$$

where $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ are nonconstant functions. Suppose that we are in one of the two limit cases: $\phi_1\phi_2$ is nonconstant and either $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{D}$ or $(\phi_1\phi_2)(\mathbb{D}) \subset \mathbb{C}\setminus\overline{\mathbb{D}}$, and that no trivial situation on $\phi_1$ and $\phi_2$ holds. Suppose also that $\phi_1\phi_2$ is not inner. Is it true that the random sequence $(T_n(.))_{n\geq 1}$ is not universal?

When $A_1, A_2$ are two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$, Theorems 5.3.10, 5.3.19 and 5.3.20 require the Birkhoff sums $(\mathbb{S}_n^\tau f(\omega))_{n\geq 1}$ associated to the function $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$ to satisfy

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

for almost every $\omega \in \mathbb{T}$, in order to make the second part of the Universality Criterion on the right inverses work. This raises the following question.

**Question 5.5.5.** Do Theorems 5.3.10, 5.3.19 and 5.3.20 work if we replace the condition on the Birkhoff sums by the condition

$$\limsup_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = \infty \quad \text{or} \quad \liminf_{n\to\infty} \mathbb{S}_n^\tau f(\omega) = -\infty$$

for almost every $\omega \in \mathbb{T}$?

Corollaries 5.3.21, 5.3.22 and 5.3.23 hold only for certain intervals of $[0,1)$, since their proofs make use of the computation of the Fourier coefficients of the function $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$ in order to apply CLTs for the doubling map and for the irrational rotations. Thus, the following question is natural.

**Question 5.5.6.** Do Corollaries 5.3.21, 5.3.22 and 5.3.23 still hold if we replace the two intervals by disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$?

In the same vein, we have seen in Corollary 5.3.27 that for some irrational rotations, the random sequence $(T_n(.))_{n\geq 1}$ won't be universal. Thus, the following question is also natural.

**Question 5.5.7.** Let $\alpha$ be an irrational in $(0,1)$ and let $\tau = R_\alpha$. Let $A_1 = [0,b)$ and $A_2 = [b,1)$ with $b \in (0,1)$. Let $(T_n(.))_{n\geq 1}$ be the random sequence of operators defined by the map

$$T(\omega) = \begin{cases} (M_{\phi_1})^* & \text{if} \quad \omega \in A_1 \\ (M_{\phi_2})^* & \text{if} \quad \omega \in A_2 \end{cases},$$

where $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ are nonconstant functions. Do Corollaries 5.3.22 and 5.3.23 hold for these sets $A_1$ and $A_2$, if $b \notin \mathbb{Z}\alpha$?

In our work, we only use two disjoint Borel subsets $A_1$ and $A_2$ of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$. It is therefore natural to be interested in the case of more than two Borel subsets of $[0,1)$.

**Open Problem 5.5.8.** Generalize our work on the random products of adjoints of multiplication operators for a finite number of disjoint Borel subsets $A_k$ of $[0,1)$ covering $[0,1)$ such that $m(A_k) > 0$ for every $k \geq 1$, when the operator $T(\omega)$ is the adjoint of the multiplication operator $(M_{\phi_k})^*$ for every $\omega \in A_k$, with $\phi_k \in H^\infty(\mathbb{D})$ nonconstant.

Obviously, we can imagine similar results to Proposition 5.3.16 and Theorem 5.3.17 in the context of Problem 5.5.8. Finally, one can also look at the linear dynamics of random products $(T_n(.))_{n\geq 1}$ when the operator $T(\omega)$ is another type of operator on a space of holomorphic functions.

**Open Problem 5.5.9.** Study the linear dynamics of random sequences $(T_n(.))_{n\geq 1}$ when the operator $T(\omega)$ is given by

$$T(\omega) = \begin{cases} T_1 & \text{if} \quad \omega \in A_1 \\ T_2 & \text{if} \quad \omega \in A_2 \end{cases},$$

where $T_1, T_2$ are two operators belonging to some specific families of operators on separable Fréchet spaces, where $\tau$ is either the doubling map or an irrational rotation on $\mathbb{T}$, and where $A_1, A_2$ are two disjoint Borel subsets of $[0,1)$ such that $A_1 \cup A_2 = [0,1)$ and $m(A_k) > 0$ for $k = 1,2$.

For example, it would be natural to consider composition operators for $T_1$ and $T_2$ on a space of holomorphic functions and to study the linear dynamics of the corresponding sequence $(T_n(.))_{n\geq 1}$ in the situation of Problem 5.5.9, as well as weighted shifts on $\ell_p$-spaces, since the linear dynamics of these operators is rather well known and of interest for many problems in linear dynamics.

### 5.5c The case of a non-ergodic transformation

Almost all of our previous results in this work hold for an ergodic measure-preserving transformation on $(\mathbb{T}, m)$. Without the assumption on the ergodicity of the transformation, Birkhoff's theorem (Theorem 5.1.10) still holds, but the limit is not necessarily constant. Thus, our previous main results won't hold in this context, except Proposition 5.3.6, Corollary 5.3.7 and Propositions 5.3.14 and 5.3.25. In order to investigate the problem of universality of random sequences of products of operators $(T_n(.))_{n\geq 1}$ for non-ergodic transformations on $(\mathbb{T}, m)$, it is natural to start the investigation with the case of a rational rotation.

Let $\alpha = \frac{p}{q}$ be a rational number in $(0,1)$, with $p$ and $q$ coprime. Let $\tau$ be the rotation with parameter $\alpha$, which is non-ergodic. In this case, for $f$ a real centered bounded function on $\mathbb{T}$, the coboundary equation $f = h - h\circ\tau$ has a solution $h \in L^\infty(\mathbb{T})$ if and only if

$$\sum_{j=0}^{q-1} f(x + j\alpha) = 0 \tag{5.5.2}$$

for almost every $x \in [0,1)$. In this case, the function

$$h(x) := -\frac{1}{q}\sum_{j=0}^{q-1}(j+1)f(x+j\alpha)$$

provides a bounded solution of the coboundary equation. Thus, when Equation 5.5.2 has a solution for $f = \mathbb{1}_{A_1} - \frac{m(A_1)}{m(A_2)}\mathbb{1}_{A_2}$, the Birkhoff sums associated to the function $f$ are bounded. In particular, Proposition 5.3.25 applies in this case.

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# Table of Contents

**A study of some remarkable properties of some classes of linear operators on Banach spaces: typicality of positive contractions and random products of operators**

**Abstract**

The motivations of this thesis are, on the one hand, the Invariant Subspace Problem, and on the other hand, the asymptotic behavior of random products of matrices. We first study typical properties of positive contractions on a Banach space with a Schauder basis, the main goal being to shed some light on the question of whether a typical positive contraction on such a Banach space has a non-trivial invariant subspace. We will study this problem for different Polish operator topologies. In the second part, we investigate the linear dynamics of random products of operators of the form $T_n(\omega) := T(\tau^{n-1}\omega)\cdots T(\tau\omega)T(\omega)$. In other words, we study the linear dynamics of the sequence $(T_n(\omega))_{n\geq 1}$ for almost every $\omega \in \mathbb{T}$. These random products depend on an ergodic transformation $\tau : \mathbb{T} \to \mathbb{T}$ and a strongly measurable map $T : \mathbb{T} \to \mathcal{B}(X)$, taking values in the space of operators on a separable Fréchet space $X$. The aim is to study the influence of the ergodic transformation and of the operators $T(\omega)$ on the universality of the sequence $(T_n(\omega))_{n\geq 1}$ for almost every $\omega \in \mathbb{T}$. We will mainly consider the case where the operator $T(\omega)$ equals a fixed operator $T_1$ when $\omega \in A_1$, and another fixed operator $T_2$ when $\omega \in A_2$, where $(A_1, A_2)$ is a non-trivial measurable partition of $[0,1)$.

**Keywords:** Polish topologies, comeager sets, similar topologies, positive operators, typical properties, invariant subspaces, linear dynamics, ergodic theory, random operators, central limit theorems

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**Étude de certaines propriétés remarquables de quelques classes d'opérateurs linéaires sur les espaces de Banach : typicité des contractions positives et produits aléatoires d'opérateurs**

**Résumé**

Les motivations de cette thèse sont d'une part le Problème du Sous-espace Invariant, et d'autre part le comportement asymptotique des produits aléatoires de matrices. Nous nous intéressons, dans un premier temps, aux propriétés typiques des contractions positives sur un espace de Banach ayant une base de Schauder, notre but principal étant d'étudier si une contraction typique positive sur un tel espace de Banach a un sous-espace invariant non trivial. Nous étudierons ce problème pour différentes topologies polonaises d'opérateurs. Dans un second temps, on s'intéresse à la dynamique linéaire des produits aléatoires d'opérateurs de la forme $T_n(\omega) := T(\tau^{n-1}\omega)\cdots T(\tau\omega)T(\omega)$. En d'autres termes, on cherche à étudier la dynamique linéaire de la suite $(T_n(\omega))_{n\geq 1}$ pour presque tout $\omega \in \mathbb{T}$. Ces produits aléatoires dépendent d'une transformation ergodique $\tau : \mathbb{T} \to \mathbb{T}$ et d'une application fortement mesurable $T : \mathbb{T} \to \mathcal{B}(X)$ à valeurs opérateurs sur un espace de Fréchet séparable $X$. Il s'agira d'étudier l'influence de la transformation ergodique et des opérateurs $T(\omega)$ sur l'universalité de la suite $(T_n(\omega))_{n\geq 1}$ pour presque tout $\omega \in \mathbb{T}$. Nous étudierons principalement le cas où l'opérateur $T(\omega)$ est égal à un certain opérateur $T_1$ lorsque $\omega \in A_1$, et égal à un certain opérateur $T_2$ lorsque $\omega \in A_2$, où $(A_1, A_2)$ est une partition mesurable non triviale de $[0,1)$.

**Mots clés :** Topologies polonaises, parties comaigres, topologies semblables, opérateurs positifs, propriétés typiques, sous-espaces invariants, dynamique linéaire, théorie ergodique, opérateurs aléatoires, théorèmes de la limite centrale

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