Le groupe fondamental Tannakien et les modules différentiels

The Tannakian fundamental group and the differential modules

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1 Introduction

1.1 Résumé

Cette thèse est une compilation de plusieurs résultats indépendentes.

Dans Chapitre 3, nous montrons l'existence d'une connexion parabolique à poids rationnels, à partir d'une connexion logarithmique intégrable à monodromies quasi-unipotentes.

Le Chapitre 4 contient plusieurs résultats sur le groupe fondamental algébrique, sur un corps de caractéristique 0. Nous définissons le groupe fondamental demi-fini, et nous déduisons une suite exacte d'homotopie pour le groupe fondamental semi-fini. Nous prouvons une formule de Künneth pour le groupe fondamental de Rham, en utilisant l'extension universelle, d'après Nori. Nous provons un théorème de Lefschetz pour le groupe Tannakien des connexions intégrables sur une variété lisse et projective. Nous prouvons une formule de Künneth pour le groupe fondamental des connexions réguliers à l'infini et nous donnons une contre-example pour le groupe Tannakien de toutes les connexions.

Dans Chapitre 5, nous travaillons sur un corps de caractéristique p. Nous nous intéressons aux modules différentiels sur une variété lisse. En utilisant le point de vue des cristaux et des faiscaux F-divisés, nous prouvons un théorème de Lefschetz et nous montrons que la cohomologie infinitésimale de Grothendieck est de dimension finie sur une variété lisse et propre.

Dans Chapitre 6, nous incluons deux résultats partiels sur la génération de la catégorie dérivée d'une variété torique lisse, par la collection de Thomsen des fibrés en droites. C'est connu comme la conjecture de Bondal qui a été bien etabli. Nous la traitons du point de vue de la géométrie birationnelle. En particulier, nous la vérifions en cas de dimension 3, en unilisant le théorème de la factorisation faible.

1.2 Abstract

This thesis consists several parts which are independent among each other.

In Chapter 3, we prove the existence of a parabolic connection, starting from an integrable logarithmic connection with quasi-unipotent monodromies.

Chapter 4 consists of several results around algebraic fundamental groups in characteristic 0. We define the semi-finite fundamental group and prove a homotopy exact sequence for it. We prove the Künneth formula for the de Rham fundamental group, using the universal extension of Nori. Over a smooth projective variety, we prove a Lefschetz theorem for the Tannakian group of integrable connections. We prove a Künneth formula for the Tannakian group of regular singular connections and disprove it in the irregular case.

In Chapter 5, we work in characteristic p. We are interested in D-modules over a smooth variety. Using the point of view of crystals and of F-divided sheaves, we prove a Lefschetz theorem, and we prove that Grothendieck's infinitesimal cohomology is of finite dimension, over a smooth and proper variety.

In Chapter 6, we include 2 partial results on the generation of derived category of a smooth toric variety, using Thomsen collection of line bundles. This is known as Bondal's claim. In particular, we give a new and complete proof for smooth toric threefolds.

1.3 Overview of results

1.3.1 A homotopy exact sequence

We introduce the notion of semi-finite connections on a smooth geometrically connected scheme U over a field k of characteristic 0 and study them using stacks. The motivation is two-fold: on the one hand, Otabe introduces the notion of semi-finite bundles on a geometrically connected proper variety over k in [90], which is a mixture of the finite and unipotent theory developed in [85] by Nori. In particular, under certain assumptions, he proves in [90] a homotopy exact sequence for tannakian group schemes associated to semi-finite bundles. On the other hand, it has been proved in [27] that, there is a fundamental gerbe

$$\mathcal{G} \to \Pi^{\mathscr{C}}_{\mathcal{G}/k}$$

for any given well-founded class $\mathscr{C}$ of affine group schemes and any given fpqc stack $\mathcal{G}$ over k which is concentrated, geometrically reduced with $H^0(\mathcal{G}, \mathcal{O}) = k$. Here if we take $\mathscr{C}$ to be the class of virtually unipotent group schemes and $\mathcal{G}$ to be a geometrically connected proper variety over k, we recover Otabe's semi-finite bundles on $\mathcal{G}$ as the essential image of the pull-back functor

$$\mathrm{Vect}(\Pi^{\mathscr{C}}_{\mathcal{G}/k}) \to \mathrm{Vect}(\mathcal{G}).$$

In another extreme, if we take $\mathcal{G}$ to be the Tannakian gerbe of all regular singular integrable connections on U, and $\mathscr{C}$ still to be the class of virtually unipotent group schemes, the essential image of the pull-back functor, via tannakian duality, is the category of semi-finite connections. These connections can also be defined as iterated extensions of finite connections in the sense of [40]. With a k-point $x \in U$, the category of semi-finite connections is neutral Tannakian, and we denote the corresponding Tannakian group scheme by $\pi_1^{vu}(U, x)$. My result is a homotopy exact sequence showing some technical uniformity between étale fundamental groups and de Rham fundamental groups of nilpotent connections.

Theorem 1.3.1 (Corollary 4.1.22)**.** *Let $U \to S$ be a smooth morphism between geometrically connected smooth schemes over k with geometrically connected fibers. Let $X \to S$ be smooth proper together with an open immersion $U \to X$ of S-schemes, such that $D = X \setminus U$ is a strict normal crossing divisor of X relative to S. Fix a k-point $s \in S$ and a k-point $x \in U$ above s. We have a homotopy exact sequence of Tannakian group schemes:*

$$\pi_1^{vu}(U_s, x) \to \pi_1^{vu}(U, x) \to \pi_1^{vu}(S, s) \to 1.$$

Root stacks play a significant role in the proof. It is clear that Deligne's canonical extension is not compatible with tensor products. To remedy this we consider canonical extensions of semi-finite connections on root stacks $(\mathfrak{X}, \mathfrak{D})$ above (X, D) with suitable weights, allowing $\mathfrak{X} \to X$ to be ramified over D. We lift the category of semi-finite connections to a tower of tannakian categories of certain logarithmic connections on root stacks, and take advantage of the properness of $\mathfrak{X} \to S$.

1.3.2 Parabolic connections

Let X be a smooth scheme over an algebraically closed field k of characteristic 0 and D be a strict normal crossing divisor with irreducible components $D_1, ..., D_n$. Say $(\mathcal{E}, \nabla)$ is an integrable logarithmic connection with poles along D, with rational exponents along each D_i. It is known that we can construct a parabolic bundle $\mathcal{E}_\alpha$, with rational weights. For each i, the filtration over D_i and the weights are determined by the residue morphism $\mathrm{res}_{D_i}(\nabla)$. This construction, combined with the local monodromy theorem (for example, see [67]), leads to many works around Gauss-Manin connections, notably including [63] on the vanishing of certain parabolic Chern characters, and [71] on geometric local systems.

In some cases (for example, [8]) we ask more for a parabolic connection $(\mathcal{E}_\alpha, \nabla_\alpha)$. To be concrete we would like to ask if we have $\nabla(\mathcal{E}_\alpha) \subset \mathcal{E}_\alpha \otimes \Omega^1_X(log\, D)$ and we set ∇_α to be the connection map induced by ∇. For a smooth proper complex variety X this parabolic connection seems folklore ([63][8]), but to carry out a rigorous proof some analytic arguments are necessary. We make an algebraic proof for an arbitrary divisorial pair (X, D) and over an arbitrary ground field k:

Theorem 1.3.2 (Theorem 3.1.1)**.** *Let $(\mathcal{E}, \nabla)$ be an integrable logarithmic connection on X with poles along D with rational exponents. Then there exists a canonical parabolic logarithmic connection $(\mathcal{E}_\alpha, \nabla_\alpha)$ associated to it.*

This theorem can be seen as a parabolic incarnation of quasi-unipotent Riemann-Hilbert correspondence developed in [62] by Illusie, Kato, and Nakayama. Roughly speaking, by [24] we lift the obtained parabolic connection to a logarithmic connection on a root stack, and by [99] to a logarithmic connection over the Kummer étale site of the log scheme defined by the divisorial pair (X, D), as considered in [62].

1.3.3 Tannakian group schemes of $\mathcal{D}$-modules

Grothendieck introduces in [35] the sheaf of differential operators $\mathcal{D}_X$ on a smooth scheme X over a field k and defines a category of $\mathcal{D}_X$-modules. For k of characteristic 0, the sheaf $\mathcal{D}_X$ is the usual one and the category of $\mathcal{D}_X$-modules coincides with the category of flat connections. When X is geometrically connected and smooth over k, the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules is Tannakian, giving rise to a Tannakian group scheme $\pi_1^{diff}(X)$ where we leave the fiber functor implicit. In characteristic 0, we also consider the subcategory of regular singular flat connections on X, whose Tannakian group scheme is denoted by $\pi_1^{alg}(X)$. The natural morphism $\pi_1^{diff}(X) \to \pi_1^{alg}(X)$ is an isomorphism if X is proper over k.

My work serves as a complement to many existing works exploring properties of π_1^{diff} or π_1^{alg}, notably [106][37].

Theorem 1.3.3 (Proposition 4.3.6, Theorem 5.2.3)**.** *Let X be geometrically connected, smooth, and projective over a field k and H be a smooth hyperplane section of some ample line bundle. Assume k is perfect if k is of characteristic p. The morphism*

$$\pi_1^{diff}(H) \to \pi_1^{diff}(X),$$

is faithfully flat if the dimension of X is at least 2, and is an isomorphism if the dimension of X is at least 3.

Note Theorem 1.3.3 is non-trivial even in characteristic 0, as the base change

$$\pi_1^{diff}(X) \times_k k' \to \pi_1^{diff}(X \times_k k')$$

along a field extension $k \hookrightarrow k'$ is not an isomorphism in general. Therefore, there is no obvious way to reduce to the case $k = \mathbb{C}$ and then to the topological case. Strictly speaking, Theorem 1.3.3 is the combination of two theorems: in characteristic 0 the proof is Tannakian, with a spreading and specialization argument that relies on the case $k = \mathbb{C}$. In characteristic p, we proceed using infinitesimal crystals, following ideas from [44].

Using the spreading and specialization argument, I also prove the following result. Note that there is not a base change formula for π_1^{alg} neither.

Theorem 1.3.4 (Proposition 4.4.1, Example 4.4.3)**.** *Let X and Y be geometrically connected and smooth over a field k of characteristic 0. We have a Künneth formula*

$$\pi_1^{alg}(X \times_k Y) \xrightarrow{\sim} \pi_1^{alg}(X) \times_k \pi_1^{alg}(Y). \tag{1}$$

Moreover, the Künneth formula is wrong with π_1^{alg} replaced by π_1^{diff}.

Note that the second part of Theorem 4.1.23 complements results from [106][37] where it is proved that the Künneth formula for π_1^{diff} holds assuming at least one of X and Y to be proper. We see that the properness assumption cannot be removed. The counterexample is deduced by considering admissible formal decompositions for flat connections over higher dimensional bases ([81][69]).

1.3.4 F-divided bundles and moduli

We work over a perfect field k of characteristic p. Let X be smooth over k and then $\mathcal{D}_X$-modules can be interpreted as the so-called F-divided bundles. We use F to denote the absolute Frobenius morphism on X. By a F-divided bundle, we mean a sequence $(\mathcal{E}_n)_{n\geq 0}$ of vector bundles on X together with an isomorphism $\mathcal{E}_n \cong F^*\mathcal{E}_{n+1}$ for each n. All F-divided bundles form an abelian tensor category and Gieseker proves in [51] that there is an equivalence of categories between the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules and the category of F-divided bundles.

The category of F-divided bundles has been intensively studied from the tannakian point of view. The investigation from moduli point of view is initiated by Esnault and Mehta, in their groundbreaking paper [43], known as the solution to Gieseker's conjecture. For a F-divided bundle $(\mathcal{E}_n)_{n\geq 0}$ on a smooth projective variety over k, Esnault and Mehta identify $\mathcal{E}_n (n \gg 0)$ as points on a moduli space of sheaves. They lift F to a rational map on this moduli space and conclude by studying the dynamics of this rational map.

Now we have made sense that moduli techniques should play a role in the study of F-divided bundles. In this direction I prove

Theorem 1.3.5 (Proposition 5.4.3, Theorem 5.4.11)**.** *Let X be a proper scheme over a perfect field k of characteristic p. Let $(\mathcal{E}_n)_{n\geq 0}$ be a F-divided bundle on X. If X is projective over k, the sheaves $\mathcal{E}_n, n \geq 0$ are bounded in the sense of Kleiman. Without the assumption of X to be projective over k, we have cohomological boundedness*

$$\sup_n h^i(X, \mathcal{E}_n \otimes \mathcal{F}) < \infty$$

for any i.

The category of $\mathcal{D}_X$-modules has enough injectives and it makes sense to define the i-th $\mathcal{D}_X$-module cohomology $H^i_{\mathcal{D}_X}$ as the i-th right derived functor of the following left exact functor:

$$\mathrm{H}^0_{\mathcal{D}_X} : \mathcal{D}_X - \mathrm{mod} \to \mathrm{Vect}_k, \mathcal{E} \mapsto \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{E}),$$

where Vect_k is the category of all k-vector spaces, not necessarily of finite dimension. This cohomology is firstly defined using infinitesimal crystals and named infinitesimal cohomology by Grothendieck in [54].

Combining results from [86] by Ogus and Theorem 1.3.5, I prove

Theorem 1.3.6 (Theorem 5.5.3)**.** *Let X be a smooth proper scheme over a perfect field of characteristic p. For any $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module $\mathcal{E}$ and any cohomology degree i, the $\mathcal{D}_X$-module cohomology $\mathrm{H}^i_{\mathcal{D}_X}(X, \mathcal{E})$ is finite dimensional as a k-vector space.*

1.3.5 (Ongoing) Derived categories of sheaves

Let k be a field and X be a smooth toric variety over k. For each positive integer m, there exists an m-th Frobenius map, $[m]$, which restricts to the m-th power on the open torus. We fix a $\mathbb{Q}$-divisor $D = \sum c_i D_i$ where D_i are toric prime divisors on X. For sufficiently divisible m, we have that mD is integral and we look at $[m]_*(\mathcal{O}_X(mD))$. By [100] the pushforward splits into a direct sum of line bundles, and for m sufficiently divisible the collection of isomorphism classes of these line bundles becomes stable, which we call the generalized Thomsen collection associated to D, denoted by $T(X, D)$. For simplicity, we write $T(X) = T(X, 0)$.

For a smooth toric variety X, it is claimed without proof in [21] by Bondal that the Thomsen collection $T(X)$ generates the bounded derived cateogry of coherent sheaves $D^b_c(X)$. The claim is proved for 2-dimensional toric Deligne-Mumford stacks in [88]. The case of toric Fano threefolds is treated in [101], and for toric Fano fourfolds, the conjecture is settled in [93]. The claim is recently resolved completely in [57] as a consequence of their resolution of toric subvarieties in terms of line bundles from $T(X)$, and in [47][48] using explicit ideas from homological mirror symmetry.

We give a slight generalization of Bondal's claim. By Bondal's claim, we mean that for an arbitrary D, the generalized Thomsen collection $T(X, D)$ generates $D^b_c(X)$.

I aim to give an alternative proof of Bondal's claim, following [88], where the claim is proved for 2-dimensional toric Deligne-Mumford stacks, using the strong factorization theorem for toric surfaces. To extend their work, we use the toric weak factorization theorem ([82][103][1]), and Bondal's claim reduces readily to relating Bondal's claim of a toric variety and its blow-ups.

Theorem 1.3.7 (Proposition 6.5.3)**.** *Bondal's claim holds for smooth toric threefolds.*

Theorem 1.3.8 (Proposition 6.4.3)**.** *Let X be a smooth toric variety and Y be a smooth toric subvariety corresponding to a cone σ of dimension l obtained as the complete intersection $D_1 \cap ... \cap D_l$ of prime toric divisors. Use $\tilde{X} = Bl_Y X$ to denote the blow-up of X with center Y. Let Z be the union of all closed subvarieties of X which are both of codimension ≥ 2 in X and obtained as complete intersections of prime toric divisors not from the set $\{D_1, ..., D_l\}$. Use $\tilde{Z}$ to denote the strict inverse image of Z in $\tilde{X}$. Then Bondal's claim for $X \setminus Z$ implies that for $\tilde{X} \setminus \tilde{Z}$.*

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2 Preliminaries

In this chapter we review several standard facts, mainly around logarithmic connections, root stacks, parabolic sheaves, and tannakian categories.

Convention 2.0.1. *We fix a field k, and all schemes and stacks are separated and of finite type over k. More assumptions on k will be made in each subsection.*

2.1 Root stacks

Convention 2.1.1. *In this subsection, we fix a field k of characteristic 0.*

2.1.1 Introduction

In this section we review the construction of root stacks and basic properties. Root stacks are firstly useful due to the work [3] by Abramovich-Vistoli, where maps with source being stacky curves are used to compactify the space of stable maps to a proper Deligne-Mumford stack. Since then, the notion of root stacks has been clarified in many works: in [29] Cademan constructs a root stack over a scheme with respect to a generalized Cartier divisor and a weight. The construction is modified in [77] by Matsuki-Olsson to make sure that the resulting root stack is smooth, provided a smooth scheme and a normal crossing divisor. The Matsuki-Olsson construction is generalized in greater generality, in terms of logarithmic geometry, in [25] by Borne-Vistoli.

More recently root stacks turn out to be useful towards functorial resolution of singularties. See [12][13].

2.1.2 Main context

The stacky quotient $[\mathbb{A}^1/\mathbb{G}_m]$ is an Artin stack and classifies generalized Cartier divisors. Let X be a scheme and $(\mathcal{L}, s)$ be a generalized Cartier divisor on X, which is equivalent to giving a morphism

$$X \to [\mathbb{A}^1/\mathbb{G}_m].$$

Definition 2.1.2. *Let r be a positive integer. The r-th root stack associated to a scheme X and a generalized Cartier divisor, denote by $\sqrt[r]{(\mathcal{L}, s)/X}$, is defined via* *the following Cartesian diagram:*

$$\begin{tikzcd} \sqrt[r]{(\mathcal{L},s)/X} \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "\times r"] \\ X \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]} \end{tikzcd}$$

where via the moduli interpretation of $\mathbb{A}^1/\mathbb{G}_m$*, the lower horizontal morphism defines the generalized Cartier divisor* $(\mathcal{L}, s)$ *and the right vertical morphism is defined by raising an arbitrary generalized Cartier divisor to its* r*-th power.*

Fix a trivialization $\mathcal{L} \cong \mathcal{O}_X$ Zariski-locally, and s is identified with a local section of $\mathcal{O}_X$. The root stack is then defined as the stacky quotient $[\operatorname{Spec} \mathcal{O}_X[t]/(t^r - s)/\mu_r]$. In particular we see that it is a Deligne-Mumford stack (since we work in characteristic 0).

Now let X be smooth and D a divisor on X, with irreducible components $D_1, ..., D_n$. Fix an n-tuple of positive integers $\mathfrak{r} = (r_1, ...r_n)$. The smooth divisor D_i defines a line bundle $\mathcal{O}_X(D_i)$ and a section defining D_i, say s_{D_i}. We have now root stacks $\sqrt[r_i]{(\mathcal{O}_X(D_i), s_{D_i})/X}$ and we define

$$\sqrt[\mathfrak{r}]{D/X} = \prod_i \sqrt[r_i]{(\mathcal{O}_X(D_i), s_{D_i})/X}$$

where the fiber product on the right hand side is over X. By construction we have the following

Lemma 2.1.3. *Let* D_i $i = 1, 2$*, be strict normal crossing divisors on* X *with* $D_1 \cap D_2 = \emptyset$*. Let* $\mathfrak{r}_i$ *be a weight for* D_i*. There is an obvious weight for* $D_1 \cup D_2$*, denoted by* $\mathfrak{r}_1 \cup \mathfrak{r}_2$*. Then the natural morpshim*

$$\sqrt[\mathfrak{r}_1 \cup \mathfrak{r}_2]{D_1 \cup D_2/X} \to \sqrt[\mathfrak{r}_1]{D_1/X} \times_X \sqrt[\mathfrak{r}_2]{D_2/X}$$

is an isomorphism.

Notation 2.1.4. *To simplify notations. We use* $\mathfrak{X}_\mathfrak{r}$ *to denote the root stack* $\sqrt[\mathfrak{r}]{D/X}$ *when the divisorial pair* (X, D) *is clear from the context. We use* $\mathfrak{X}$ *to denote* $\mathfrak{X}_\mathfrak{r}$ *if furthermore the weight* $\mathfrak{r}$ *is clear from the context.*

Also observe that by construction, if we have two weights $\mathfrak{r}$ and $\mathfrak{r}'$ with $\mathfrak{r}|\mathfrak{r}'$, there is a natural morphism

$$\mathfrak{X}_{\mathfrak{r}'} \to \mathfrak{X}_\mathfrak{r}.$$

Proposition 2.1.5 (Proposition 2.4 [24]). *Let X be smooth over k, D be a simple normal crossing divisor with irreducible components $D_1, ...D_n$. Then for any n-tuple $\mathfrak{r} = (r_1, ..., r_n)$ the root stack $\mathfrak{X}_{\mathfrak{r}}$ is smooth over k.*

Keep the same assumption as in Proposition 2.1.5. Let $\pi_{\mathfrak{r}} : \mathfrak{X}_{\mathfrak{r}} \to X$ denote the structural map from the root stack to its coarse moduli space, which is X. There are tautological divisors $\mathfrak{D}_i$ on $\mathfrak{X}$, with $\pi_{\mathfrak{r}}^* D_i = r_i \mathfrak{D}_i$, and $\mathfrak{D}_{\mathfrak{r}} = \cup_i \mathfrak{D}_i$ is strict normal crossing.

Remark 2.1.6 (Clarification of notations). *Let X be smooth and D be a strict normal crossing divisor, with irreducible components $D_1, ..., D_n$. Let $\mathfrak{r} = (r_1, ..., r_n)$ be a weight with $r_1 = ... = r_n$. One the one hand, we have*

$$\sqrt[\mathfrak{r}]{D/X} = \prod_i \sqrt[r_1]{(\mathcal{O}_X(D_i), s_{D_i})/X}$$

where the fiber product is over X. One the other since D is effective we can take $\sqrt[r_1]{(\mathcal{O}_X(D), s_D)/X}$ according to Definition 2.1.2. In general $\sqrt[\mathfrak{r}]{D/X}$ is not isomorphic to $\sqrt[r_1]{(\mathcal{O}_X(D), s_D)/X}$. Indeed let x be a k-point of X through which exactly $D_1, ..., D_m$ pass, with $1 < m \leq n$. The residue gerbe of $\sqrt[\mathfrak{r}]{D/X}$ at x is $B(\mu_{r_1} \times ... \times \mu_{r_1})$ with m-copies of μ_{r_1} while the residue gerbe of $\sqrt[r_1]{(\mathcal{O}_X(D), s_D)/X}$ at x is isomorphic to $B\mu_{r_1}$.

2.2 Logarithmic connections

2.2.1 Introduction

Logarithmic structures serve to compactify. As an example, for a smooth variety U with a compactification X, with $D = X \setminus U$ being simple normal crossing, the logarithmic de Rham cohomology $\mathrm{H}^\bullet_{dR}(X, \Omega^\bullet_X(log\, D))$ computes the de Rham cohomology $\mathrm{H}^\bullet_{dR}(U, \Omega^\bullet_U)$. As an application, over $k = \mathbb{C}$, the de Rham cohomology is canonically isomorphic to the Betti cohomology $\mathrm{H}^\bullet_B(U^{an}, \mathbb{C})$ by [53], and Deligne defines in [31] the Hodge filtration on $\mathrm{H}^\bullet_B(U^{an}, \mathbb{C})$ as the Hodge-de Rham filtration on $\mathrm{H}^\bullet_{dR}(X, \Omega^\bullet_X(log\, D))$, which is a half of the mixed Hodge theory for smooth varieties.

There is a de Rham complex $\mathcal{E} \otimes \Omega^\bullet_U$ associated to each integrable connection $(\mathcal{E}, \nabla)$ on U, and now a natural question is that, is there a complex over X computing $\mathrm{H}^\bullet(U, \mathcal{E} \otimes \Omega^\bullet_U)$? This is partially answered by Deligne in [33] for those connetions with regular singularities along D. To be concrete, for a such a connection $(\mathcal{E}, \nabla)$ on U, it can be extended to a logarithmic connection in a canonical way (up to making a choice of exponents), and the associated logarithmic de Rham complex will do the job. We will explain Deligne's extension in details later in the main context.

2.2.2 Main context

The canonical reference is [33] by Deligne. Though he works over $k = \mathbb{C}$ and uses a lot of analytic arguments, results he obtains hold over an arbitrary k of characteristic 0, in principle. Our reference is [7].

Let $\mathfrak{X}$ be a smooth Deligne-Mumford stack with affine diagonal, and $\mathfrak{D}$ a strict normal crossing divisor with irreducible components $\mathfrak{D}_1, ...\mathfrak{D}_n$. We have the sheaf $\Omega^1_{\mathfrak{X}/k}$ of 1-forms and the sheaf $\Omega^1_{\mathfrak{X}/k}(log\,\mathfrak{D})$ of 1-forms with logarithmic poles along $\mathfrak{D}$. For each $\mathfrak{D}_i$ there is a residue map

$$res_{\mathfrak{D}_i} : \Omega^1_{\mathfrak{X}/k}(log\,\mathfrak{D}) \to \mathcal{O}_{\mathfrak{D}_i}$$

and the residue exact sequence

$$0 \longrightarrow \Omega^1_{\mathfrak{X}/k} \longrightarrow \Omega^1_{\mathfrak{X}/k}(log\,\mathfrak{D}) \xrightarrow{\oplus res_{D_i}} \bigoplus \mathcal{O}_{\mathfrak{D}_i} \longrightarrow 0$$

Definition 2.2.1. *A logarithmic connection on $\mathfrak{X}$ with poles along $\mathfrak{D}$ is a pair $(\mathcal{E}, \nabla)$ where $\mathcal{E}$ is a locally free sheaf of finite rank, and ∇ is a morphism*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\mathfrak{X}/k}(log\,\mathfrak{D}),$$

satisfying the usual Leibniz rule:

$$\nabla(ae) = a\nabla(e) + e \otimes da$$

for local sections a of $\mathcal{O}_{\mathfrak{X}}$ and e of $\mathcal{E}$. A connection is said to be integrable if

$$\nabla^2 = 0.$$

Notation 2.2.2. *We use $\mathrm{Conn}(\mathfrak{X}, \mathfrak{D})$ to denote the category consisting of integrable logarithmic connections on $\mathfrak{X}$ with poles along $\mathfrak{D}$.*

Remark 2.2.3. *Our definition of logarithmic connections is more restrictive than the definitions made in literature since for a logarithmic connection $(\mathcal{E}, \nabla)$ we assume $\mathcal{E}$ to be locally free. The same definition makes sense if we merely require*

$\mathcal{E}$ to be coherent. It is well-known that this makes no difference if the connection is regular even along $\mathfrak{D}$, i.e, it is of the form

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\mathfrak{X}/k},$$

since in this case one could prove $\mathcal{E}$ is automatically locally free. In the logarithmic case, this is not necessarily true. For example, let $\mathfrak{X}$ be $\operatorname{Spec} k[x]$, $\mathfrak{D}$ the divisor defined by $x = 0$ and $i : \mathfrak{D} \to \mathfrak{X}$ be the closed immersion. Let $\mathcal{E}$ be $i_*\mathcal{O}_{\mathfrak{D}}$ and ∇ be an arbitrary $\mathcal{O}_{\mathfrak{X}}$-linear morphism $i_*\mathcal{O}_{\mathfrak{D}} \to i_*\mathcal{O}_{\mathfrak{D}} \otimes \Omega^1_{\mathfrak{X}/k}(log\,\mathfrak{D})$. One readily checks the Leibniz rule but $\mathcal{E}$ is obviously not locally free.

Definition 2.2.4. *Let $(\mathcal{E}, \nabla)$ be a logarithmic connection on $\mathfrak{X}$ with poles along $\mathfrak{D}$. For each i, the composition*

$$\mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_{\mathfrak{X}/k}(log\,\mathfrak{D}) \xrightarrow{id \otimes res_{\mathfrak{D}_i}} \mathcal{E} \otimes \mathcal{O}_{\mathfrak{D}_i}$$

is $\mathcal{O}_{\mathfrak{X}}$-linear. We use $res_{\mathfrak{D}_i}(\nabla)$ to denote the resulting morphism

$$res_{\mathfrak{D}_i} : \mathcal{E}|_{\mathfrak{D}_i} \to \mathcal{E}|_{\mathcal{D}_i}$$

called the residue morphism along $\mathfrak{D}_i$.

Let X be a smooth scheme with a strict normal crossing divisor D of components $D_1, ..., D_n$ and $U = X \setminus D$. We write $\mathcal{T}_X(log\,D)$ to be the dual of $\Omega^1_X(log\,D)$. We use $\kappa(X)$ to denote the function field of X. For each D_i we use $\mathcal{O}_{X,D_i}$ to denote the local ring of X at the generic point of D_i, which is a discrete valuation ring. We write $\mathcal{T}_{X,D_i}(log\,D_i) = \mathcal{T}_X(log\,D) \otimes_{\mathcal{O}_X} \mathcal{O}_{X,D_i}$.

Definition 2.2.5 (Regularity for schemes)**.** *Over an algebraically closed ground field $k = \bar{k}$, a connection $(\mathcal{E}, \nabla)$ on U is regular along D, if for each D_i there is an $\mathcal{O}_{X,D_i}$-lattice in $\mathcal{E} \otimes \kappa(X)$ which is stable under $\mathcal{T}_{X,D_i}(log\,D_i)$. Over a general ground field k, a connection $(\mathcal{E}, \nabla)$ on U is regular along D if this is the case after base change to $\bar{k}$.*

From now on to the end of this subsection, we assume the ground field k to be algebraically closed.

Definition 2.2.6 (Exponents for schemes, 11.1.3 [7])**.** *Assume $k = \bar{k}$. Let $(\mathcal{E}, \nabla)$ be an integrable logrithmic connection on a smooth scheme X with poles along a strcit normal crossing divisor D with irreducible components $D_1, ..., D_n$. The exponents of $(\mathcal{E}, \nabla)$ along D_i are the eigenvalues of the $\kappa(\eta)$-linear endmoprhism $res_{D_i}(\nabla)_\eta$, for a schematic point $\eta \in D_i$.*

Note exponents are well-defined due to the integrability assumption (i.e. independent of the choice of η), and belong to k since we could just take η to be a closed point, i.e, a k-point. We will see later that it will be useful to take η to be the generic point of D_i.

Definition 2.2.7 (Exponents for stacks)**.** *Assume $k = \bar{k}$. Let $(\mathcal{E}, \nabla)$ be an integrable logarithmic connection on a smooth stack $\mathfrak{X}$ with poles along a strict normal crossing divisor $\mathfrak{D}$ with irreducible components $\mathfrak{D}_i$. The exponents along $\mathfrak{D}_i$ are defined as the exponents of $p^*(\mathcal{E}, \nabla)$ along E where $p : Y \to \mathfrak{X}$ is an arbitrary étale chart of $\mathfrak{X}$ and E is an arbitrary irreducible component of $p^{-1}\mathfrak{D}$ mapped to $\mathfrak{D}_i$. These exponents are well-defined by Proposition 10.3.3 [7].*

Deligne's canonical extension firstly appears in [33]. We also refer to [7] for a purely algebraic treatment. We state Deligne's canonical extension in stacky context (which is merely the canonical extension on schemes plus a descent argument).

Theorem 2.2.8 (Deligne's canonical extension)**.** *Assume $k = \bar{k}$. Let X be a smooth scheme, D be a strict normal crossing divisor with components $D_1, ..., D_n$. Let $(\mathcal{E}, \nabla)$ be an integrable connection on $U = X \setminus D$ which is regular along D. Let $s : k/\mathbb{Z} \to k$ be a section of the projection $k \to k/\mathbb{Z}$. There exists a unique (up to unique isomorphism) logarithmic connection $(\tilde{\mathcal{E}}, \tilde{\nabla})$ on X with poles along D such that*

1. $(\tilde{\mathcal{E}}|_U, \tilde{\nabla}|_U) \cong (\mathcal{E}, \nabla)$
2. *the exponents of $(\tilde{\mathcal{E}}, \tilde{\nabla})$ along irreducible components D_i are all in the image of s.*
3. *For two connections $(\mathcal{E}_1, \nabla_1)$ and $(\mathcal{E}_2, \nabla_2)$ on U which are regular along D, an arbitrary morphism $f : (\mathcal{E}_1, \nabla_1) \to (\mathcal{E}_2, \nabla_2)$ extends canonically to a moprhism $\tilde{f} : (\tilde{\mathcal{E}}_1, \tilde{\nabla}_1) \to (\tilde{\mathcal{E}}_2, \tilde{\nabla}_2)$. Therefore, the canonical extension defines functor from the category $\mathrm{Conn}(U)$ to $\mathrm{Conn}(X, D)$.*

Moreover the canonical extension as a functor is fully faithful.

Definition 2.2.9 (Regularity for stacks)**.** *Let $\mathfrak{X}$ be smooth and of affine diagonal, $\mathfrak{D}$ be a strict normal crossing divisor, and $U = \mathfrak{X} \setminus \mathfrak{D}$. Over an algebraically closed ground field $k = \bar{k}$, an integrable connection $(\mathcal{E}, \nabla)$ on U is said to be regular along $\mathfrak{D}$ if it extends to a logarithmic connection on $\mathfrak{X}$ with poles along $\mathfrak{D}$. Over a general ground field, a connection is said to be regular along $\mathfrak{D}$ if this is the case after base change to $\bar{k}$.*

We include an argument extending Deligne's canonical extension to stacks.

Corollary 2.2.10. *Let $\mathfrak{X}$ be smooth and of affine diagonal, $\mathfrak{D}$ be a strict normal crossing divisor, and $U = \mathfrak{X} \setminus \mathfrak{D}$. An integrable connection $(\mathcal{E}, \nabla)$ on U is regular over $\mathfrak{D}$ if and only if for one (and then any) étale chart $p : Y \to \mathfrak{X}$, the pull-back $p^*(\mathcal{E}, \nabla)$, as an integrable connection on $p^{-1}(U)$, is regular along $p^{-1}(\mathfrak{D})$.*

Proof. It suffices to assume $k = \bar{k}$ and it suffices to prove the if part. Fix a section $s : k/\mathbb{Z} \to k$. Choose an étale chart $p : Y \to \mathfrak{X}$. The pull back $p^*(\mathcal{E}, \nabla)$ is regular along $p^{-1}\mathfrak{D}$. By Deligne's canonical extension there is a logarithmic connection $(\tilde{\mathcal{E}}, \tilde{\nabla})$ on Y with poles along $p^{-1}\mathfrak{D}$ with exponents all in the image of s. Consider the fiber product $Y \times_{\mathfrak{X}} Y$ which is a scheme since by assumption $\mathfrak{X}$ is of affine diagonal, and the two natural projections $p_i, i \in \{1, 2\}$ to Y. Since Deligne's canonical extension is unique up to canonical isomorphism, there is a natural isomorphism $p_1^*(\tilde{\mathcal{E}}, \tilde{\nabla}) \xrightarrow{\sim} p_2^*(\tilde{\mathcal{E}}, \tilde{\nabla})$ which defines a descent data and $(\tilde{\mathcal{E}}, \tilde{\nabla})$ descends to $\mathfrak{X}$. □

Over a non-algebraically closed field k we fix $s : \bar{k}/\mathbb{Z} \to \bar{k}$. Suppose there is an integrable logarithmic connection $(\mathcal{E}, \nabla)$ on $U = \mathfrak{X} \setminus \mathfrak{D}$ which is regular along $\mathfrak{D}$. We consider the base change of $(\mathcal{E}, \nabla)$ to $\bar{k}$ and the canonical extension $(\tilde{\mathcal{E}}, \tilde{\nabla})$. The exponents are a priori in $\bar{k}$, but if they further lie in the $\mathrm{Gal}(\bar{k}/k)$-fixed set k, for any $\sigma \in \mathrm{Gal}(\bar{k}/k)$ we have a canonical isomorphism $\sigma^*(\tilde{\mathcal{E}}, \tilde{\nabla}) \xrightarrow{\sim} (\tilde{\mathcal{E}}, \tilde{\nabla})$. This defines a Galois descent data and $(\tilde{\mathcal{E}}, \tilde{\nabla})$ descends to a logarithmic connection on X with poles along D.

Corollary 2.2.11. *For a non-algebraically closed field k, fix a section $s : \bar{k}/\mathbb{Z} \to \bar{k}$. Let $(\mathcal{E}, \nabla)$ on $U = \mathfrak{X} \setminus \mathfrak{D}$ be an integrable connection which is regular along $\mathfrak{D}$. Suppose the base change of $(\mathcal{E}, \nabla)$ to $\bar{k}$ has exponents in $k \subset \bar{k}$. Then it admits a canonical extension to $\mathfrak{X}$ with poles along $\mathfrak{D}$. In particular, this corollary applies to connections with rational exponents.*

For a smooth scheme X and a strict normal crossing divisor D with components $D_1, ..., D_n$, by a stratum we mean a connected component of $D_I = \cap_{i \in I} D_i$ for a non-empty set $I \subset \{1, ..., n\}$. For example, each irreducible component D_i is a stratum by taking $I = \{i\}$. There is an absolute notion for connections to be regular singular, without referring to a particular divisor.

Definition 2.2.12. *Let X be a smooth scheme. Let $(\mathcal{E}, \nabla)$ be an integrable connection on X. We say that $(\mathcal{E}, \nabla)$ is regular singular if $(\mathcal{E}, \nabla)$ is regular along D, with $\bar{X}$ being a smooth compactification of X and $D = \bar{X} \setminus X$ being a strict normal crossing divisor.*

Proposition 2.2.13. *The notion of regular singular connections is independent from the compactification.*

Proof. It suffices to treat the case $k = \bar{k}$. Any two compactifications of X with strict normal crossing boundary is related by a zigzag of blow-ups and blow-downs with centers being stratums of codimension ≥ 2. It suffices to prove that, if we have a compactification $\bar{X}$ with $D = \bar{X} \setminus X$ being strict normal crossing, and $\tilde{X} = Bl_E \bar{X}$ with E being a stratum of codimension ≥ 2 in D with $\tilde{D}$ being the inverse image of D in $\tilde{X}$, a connection is regular along D if and only if it is regular along $\tilde{D}$.

Indeed, if $(\mathcal{E}, \nabla)$ is regular along D, by Theorem 2.2.8 it extends to a logarithmic connection on $\bar{X}$ with poles along D, and the pullback of this logarithmic connection to $\tilde{X}$ is a logarithmic connection on $\tilde{X}$ with poles along $\tilde{D}$. In particular $(\mathcal{E}, \nabla)$ is regular along $\tilde{D}$. Conversely, suppose $(\mathcal{E}, \nabla)$ is regular along $\tilde{D}$. We conclude by applying the definition 2.2.5 to each D_i and its strict inverse image on $\tilde{X}$. $\square$

2.2.3 Final remarks

Remark 2.2.14 (o-minimality). *Let X be a smooth complex algebraic variety. An o-minimal criterion for an integrable connection $(\mathcal{E}, \nabla)$ to be regular singular with unitary local monodromy eigenvalues at infinity is given in a very recent article [42].*

Remark 2.2.15. *Everything mentioned, including logarithmic strutures, differentials, de Rham complexes, etc, can be defined in much greater generalities. We refer to [87]. Comparisons among different cohomologies are available, by [66]. Notably for the Betti-de Rham comparison of an idealized smooth log scheme X over $\mathbb{C}$, we have $\mathrm{H}^\bullet(X^{KN}, \mathbb{C}) \cong \mathrm{H}^\bullet(X, \omega_X^\bullet)$, where X^{KN} is the associated Kato-Nakayama space which is a real manifold with boundaries, and $\omega_X^\bullet$ is the logarithmic de Rham complex.*

Remark 2.2.16. *In [5], the author introduces the notion of regular integrable logarithmic connections on an idealized smooth log scheme over $\mathbb{C}$. For a log scheme with trivial log structure we recover the notion of regular singular integrable connections in the classical sense. If the underlying scheme of the log scheme is proper, all logarithmic integrable connections are regular, as a nontrivial counterpart of the trivial fact that an integrable connection on a proper variety is regular singular.*

2.3 Parabolic bundles and the parabolic-stacky correspondence

2.3.1 Introduction

Parabolic bundles are introduced in [80], and the upshot is the so-called Mehta-Seshadri correspondence generalizing Narasimhan-Seshadri correspondence:

Theorem 2.3.1 ([80])**.** *Let X be a compact Riemann surface with points $x_1, ..., x_n$. There is an equivalence between irreducible unitary complex local systems on $X \setminus \{x_1, ..., x_n\}$ and stable parabolic bundles of degree 0, where parabolic structures are above $x_1, ..., x_n$.*

Since then, many facts around parabolic bundles and sheaves have been established in parallel with the non-parabolic story, especially their moduli by [80][76][104].

A nature of parabolic sheaves which does not have a counterpart in the non-parabolic world is that they are related to sheaves on certain orbifolds, which we refer as the parabolic-stacky correspondence. This firstly appears in the work [15] by Biswas. Later this is rephrased in terms of root stacks by Borne in [22][23]. The correspondence is further generalized using logarithmic geometry in [25] by Borne-Vistoli and in [99][98] by Talpo-Vistoli.

With the parabolic-stacky correspondence, we are suggested to consider moduli of sheaves on Deligne-Mumford stacks, notably on root stacks. Somewhat surprisingly, with moduli of parabolic sheaves and Higgs sheaves already developed in 1990s, the gap to study moduli of sheaves on stacks has not been filled until [84]. A study of vector bundles on stacky curves, without essentially relying on the language of parabolic bundles, is even latter, carried out recently in [30].

We will review parabolic bundles and sheaves with rational weights. Note this does not cover everything interesting: over $k = \mathbb{C}$ it is necessary to work with parabolic bundles or sheaves with real weights, for example, those from the Mehta-Seshadri correspondence. A parabolic-stacky correspondence is still available for parabolic sheaves with real weights, with stacks replaced by the Kato-Nakayama space. See [98].

Convention 2.3.2. *In this subsection, we fix a field k of characteristic 0.*

2.3.2 The main context

Let X be a smooth scheme over k and D be a strict normal crossing divisor with irreducible components $D_1, ..., D_n$. Fix a weight $\mathfrak{r}$, i.e, an n-tuple $\mathfrak{r} = (r_1, ..., r_n)$

of positive integers. We use $\frac{1}{\mathfrak{r}}\mathbb{Z}^n$ to denote the category, whose objects are n-tuple $(\alpha_1, ..., \alpha_n)$ where α_i is rational in $\frac{1}{r_i}\mathbb{Z}$ for any i. There is an arrow $\alpha = (\alpha_1, ..., \alpha_n) \to \beta = (\beta_1, ..., \beta_n)$ if and only if $\alpha_i \leq \beta_i$ for each i. We use δ_i to denote the tuple with 1 at the i-th place and 0 elsewhere.

Definition 2.3.3. *A parabolic sheaf of weight $\mathfrak{r}$ is the data of a functor*

$$\mathcal{E} : (\frac{1}{\mathfrak{r}}\mathbb{Z}^n)^{op} \to \mathrm{Coh}X,$$

together with a natural transformation $p_\gamma : \mathcal{E}_{*+\gamma} \xrightarrow{\sim} \mathcal{E}_* \otimes \mathcal{O}_X(-\gamma \cdot D)$ for any $\gamma \in \mathbb{Z}^n$, such that for $\gamma \geq 0$ the natural transformation fills in the following commutative diagram of functors:

$$\begin{tikzcd} \mathcal{E}_{*+\gamma} \arrow[d, "p_\gamma"] \arrow[dr] & \\ \mathcal{E}_* \otimes \mathcal{O}_X(-\gamma \cdot D) \arrow[r] & \mathcal{E}_*. \end{tikzcd}$$

A parabolic sheaf is said to be a parabolic bundle if $\mathcal{E}$ factors through the inclusion $\mathrm{Vect}X \to \mathrm{Coh}X$. The category of parabolic bundles of weight $\mathfrak{r}$ are denoted by $\mathrm{ParVect}_{\mathfrak{r}}X$.

Let $(\mathcal{E}, \nabla)$ be a logarithmic λ-connection on X with poles along D. For any n-tuple $\gamma = (\gamma_1, ..., \gamma_n)$, the locally free sheaf $\mathcal{E}(\gamma \cdot D) = \mathcal{E} \otimes \mathcal{O}_X(\gamma_1 D_1 + ... \gamma_n D_n)$ is canonically equipped with a λ-connection map, denoted by $\nabla(\gamma \cdot D)$. For the case of logarithmic connections (that is to say, $\lambda = 1$), see Lemma 2.7 [45].

Definition 2.3.4 (parabolic λ-connection)**.** *A parabolic λ-connection of weight $\mathfrak{r}$ is the data of a functor with target the category $\mathrm{Conn}_\lambda(X, D)$ of logarithmic λ-connections:*

$$\mathcal{E} : (\frac{1}{\mathfrak{r}}\mathbb{Z}^n)^{op} \to \mathrm{Conn}_\lambda(X, D),$$

together with a natural transformation $p_\gamma : (\mathcal{E}_{*+\gamma}, \nabla_{*+\gamma}) \xrightarrow{\sim} (\mathcal{E}_*(-\gamma \cdot D), \nabla_*(-\gamma \cdot D))$ for any $\gamma \in \mathbb{Z}^n$, such that for $\gamma \geq 0$ the natural transformation fills in the following commutative diagram of functors:

$$\begin{tikzcd} (\mathcal{E}_{*+\gamma}, \nabla_{*+\gamma}) \arrow[d, "p_\gamma"] \arrow[dr] & \\ (\mathcal{E}_*(-\gamma \cdot D), \nabla_*(-\gamma \cdot D)) \arrow[r] & (\mathcal{E}_*, \nabla_*). \end{tikzcd}$$

The category of parabolic λ-connections is denoted by $\mathrm{ParConn}_{\lambda,\mathfrak{r}}(X, D)$.

In particular, for $\lambda = 1$ we have the category of parabolic connections, denoted simply by $\mathrm{ParConn}_{\mathfrak{r}}(X, D)$, and for $\lambda = 0$ we have the category of parabolic Higgs bundles, denoted by $\mathrm{ParHigg}_{\mathrm{r}}(X, D)$.

Example 2.3.5. *Pick up a line bundle $\mathcal{L}$ on X and rational numbers $\beta_1, ..., \beta_n$ with $\beta_i \in \frac{1}{r_i}\mathbb{Z}$. We can define a parabolic line bundle by setting $\mathcal{L}_\alpha = \mathcal{L}(\sum\lfloor -\alpha_i + \beta_i \rfloor D_i)$, $\alpha = (\alpha_1, ..., \alpha_n)$. All parabolic line bundles in $\mathrm{ParVect}_{\mathfrak{r}} X$ arise in this way.*

Remark 2.3.6. *For a parabolic sheaf we look at $\mathcal{E}_\alpha$ for those α with only one nonzero entry. It is then clear that we obtain a filtration of $\mathcal{E}_0 \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i}$ for each i. We would like to ask if it is possible to recover the parabolic structure from these filtrations. This is clear for curves since divisors D_i have disjoint supports and these filtrations are independent from each others. See Remark 2.3.7. We will also give a quick proof of this for parabolic bundles. See Remark 2.3.10.*

Remark 2.3.7. *Suppose X is a curve together with a divisor $x_1 + ... + x_n$. For a coherent sheaf $\mathcal{E}$, to define a parabolic sheaf with $\mathcal{E}_0 = \mathcal{E}$ is equivalent to specifying a sequence $0 \leq \alpha_{i,1} < ...\alpha_{i,k_i} < 1$ of rational numbers and a filtration $0 \subsetneq \mathcal{E}_{x_i,k_i-1} \subsetneq ... \subsetneq \mathcal{E}_{x_i,1} \subsetneq \mathcal{E}_{x_i}$, for each x_i with $k_i \geq 1$.*

Indeed, for $\beta_i \in [0, 1)$, we define $\mathcal{E}_{\beta_i\delta_i}$ to be the inverse image under the projection $\mathcal{E} \to \mathcal{E}_{x_i}$ of $\mathcal{E}_{x_i}$ for $0 \leq \beta_i \leq \alpha_{i,1}$, of $\mathcal{E}_{x_i,j}$ if $\alpha_{i,j} < \beta_i \leq \alpha_{i,j+1}$ and $j \leq k_i - 1$, and of 0 if $\beta_i > \alpha_{i,k_i}$. For $\beta = (\beta_1, ..., \beta_n) \in [0, 1)^n$, we define $\mathcal{E}_\beta = \cap_i \mathcal{E}_{\beta_i\delta_i}$. This extends to the desired parabolic sheaf.

Example 2.3.8. *Let X be a smooth proper curve and $x_1, ..., x_n$ be closed points. Let $(\mathcal{E}, \nabla)$ be a logarithmic connection on X with poles along $x_1, .., x_n$, with rational exponents in $[0, 1)$. Such a logarithmic connection comes in different ways:*

1. *We work over $k = \mathbb{C}$. One can construct a local system on the analytification of $X \setminus \{x_1, ..., x_n\}$ by specifying local monodromies around each x_i, all of which are quasi-unipotent. By Riemann-Hilbert it corresponds to an analytic connection on $X \setminus \{x_1, ..., x_n\}$ which extends to an analytic logarithmic connection on X with poles along $x_1, ..., x_n$ by an analytic version of Deligne's canonical extension. The latter is actually algebraic by GAGA.*

2. *Notably, one can look at smooth proper families of varieties over $X \setminus \{x_1, ..., x_n\}$. The resulting Gauss-Manin connections are regular singular with rational exponents at infinity. See [67][61].*

At each x_i, we have rational numbers $0 \leq \alpha_{i,1} < ... < \alpha_{i,k_i} < 1$ as all eigenvalues of $res_{x_i}(\nabla)$. We then define $\mathcal{E}_{x_i,j}$ to be the summation of generalized eigenspaces of $res_{x_i}(\nabla)$ of eigenvalues $\alpha_{i,j+1}, ..., \alpha_{i,k_i}$.

The category $\mathrm{ParVect}_{\mathfrak{r}}X$ carries a tensor product structure. We refer to 2.1 [23] for details.

Theorem 2.3.9 (a version of the parabolic-stacky correspondence [23])**.** *There is a tensor equivalence between $\mathrm{Vect}\mathfrak{X}_\tau$ and $\mathrm{ParVect}_{\mathfrak{r}}X$. To be concrete, each $\mathcal{E}$ on $\mathrm{Vect}\mathfrak{X}_\tau$ defines a parabolic bundle by setting*

$$\mathcal{E}_\alpha = \pi_*(\mathcal{E} \otimes \mathcal{O}_{\mathfrak{X}_{\mathfrak{r}}}(-s_1\mathfrak{D}_1 - ... - s_n\mathfrak{D}_n)),$$

where $\alpha = (\frac{s_1}{r_1}, ..., \frac{s_n}{r_n})$ and we recall $\pi : \mathfrak{X}_{\mathfrak{r}} \to X$ is the canonical projection from a stack to its coarse moduli space and $\mathfrak{D}_1, ..., \mathfrak{D}_n$ are tautological divisors.

Remark 2.3.10. *Similar to the case of curves, we would like to know if we have $\mathcal{E}_\alpha = \bigcap_i \mathcal{E}_{\alpha_i\delta_i}$ with $\alpha = (\alpha_1, ..., \alpha_n)$ for a parabolic sheaf in higher dimensions. We give a quick proof using the stacky-parabolic correspondence though this should already be contained implicitly in homological algebra arguments from [23] . Note this property can be verified Zariski locally on X. Let us say the parabolic bundle corresponds to a vector bundle $\mathcal{E}$ on the root stack $\mathfrak{X}$. The argument in Lemma 2.3 [63] says that there exists a Zariski covering $\{X_i\}_i$ of X, such that $\mathcal{E}$ splits into a direct sum of line bundles on each $\mathfrak{X}_i = \mathfrak{X} \times_X X_i$. We reduce to verifying for parabolic line bundles, which is clear.*

We also have the parabolic-stacky correspondence for parabolic λ-connections. We refer to 4.6 [24] for a notion of strongly parabolic λ-connections.

Theorem 2.3.11 ([19][24])**.**

1. *There is a tensor equivalence between the category of strongly parabolic Higgs bundles $\mathrm{ParHigg}_{\mathfrak{r}}(X, D)$ and the category $\mathrm{Higg}(\mathfrak{X}_{\mathfrak{r}})$ of holomorphic Higgs bundles .*
2. *There is a tensor equivalence between the category of strongly parabolic connections $\mathrm{ParConn}_{\mathfrak{r}}(X, D)$ and the category $\mathrm{Conn}(\mathfrak{X}_{\mathfrak{r}})$ of holomorphic connections . Moreover, a correspondence for λ-connections for $\lambda \neq 0$ is obtained by rescaling.*

Remark 2.3.12. *The case of parabolic Higgs bundles is due to [19] and the case of parabolic connections is due to [24]. In the case $\lambda \neq 1$ the integrability assumption on λ-connections can be dropped, as in [24]. This is not natural for Higgs bundles since the Higgs fields are always assumed to be integrable.*

2.4 Aspects of Tannakian categories

Definition 2.4.1 (2.8 [34]). *A Tannakian category $\mathcal{C}$ is a k-linear abelian rigid tensor category with a unit object 1 and an isomorphism $\mathrm{End}(1) \cong k$, such that there exists a non-empty scheme S over k together with a fiber functor ω over S, which is a k-linear exact tensor functor from $\mathcal{C}$ to the k-linear tensor category of quasi-coherent sheaves on S.*

It is proved in 2.7 [34] that for any fiber functor ω on S, its essential image is contained in the category of locally free sheaves of finite rank on S.

For a Tannakian category $\mathcal{C}$, we define a category $\mathrm{Fib}(\mathcal{C})$ which is fibered in groupoids over the category of k-schemes. For any k-scheme S, we have $\mathrm{Fib}(\mathcal{C})(S)$ to be the groupoid whose objects are fiber functors on S and whose morphisms are natural isomorphisms between tensor functors. For a k-morphism $f : T \to S$, the pullback functor f^* defines a functor from $\mathrm{Fib}(\mathcal{C})(S)$ to $\mathrm{Fib}(\mathcal{C})(T)$.

Notation 2.4.2. *For a Tannakian category $\mathcal{C}$ and a set S of objects in $\mathcal{C}$, we use $\langle S \rangle$ to denote the smallest Tannakian subcategory of $\mathcal{C}$ containing S.*

Theorem 2.4.3. *The fiber category $\mathrm{Fib}(\mathcal{C})$ is an fpqc gerbe over the category of k-schemes, which is called the Tannakian gerbe of $\mathcal{C}$.*

Definition 2.4.4. *A Tannakian category is neutral if it admits a fiber functor ω over $\mathrm{Spec}\, k$. The Tannakian group scheme $\mathrm{Aut}(\mathcal{C}, \omega)$ of $\mathcal{C}$ with fiber functor ω is defined to be the inertia group scheme realized by the following 2-cartesian diagram:*

$$\begin{tikzcd}
\mathrm{Aut}(\mathcal{C},\omega) \arrow[r] \arrow[d] & k \arrow[d, "{(\omega,\omega)}"] \\
\mathrm{Fib}(\mathcal{C}) \arrow[r, "\Delta"] & \mathrm{Fib}(\mathcal{C}) \times_k \mathrm{Fib}(\mathcal{C}).
\end{tikzcd}$$

Remark 2.4.5. *For a Tannakian category $\mathcal{C}$ usually we do not know if there is any fiber functor ω over a given S nor there is a canonical choice of ω. The language of Tannakian gerbes has the advantage of not referring to any of them.*

Example 2.4.6. *Let G be an abstract group and consider the category of finite dimensional representations of G over k. This is clearly a Tannakian category, and it is neutral with the fiber functor given as the forgetful functor to the category of finite dimensional k-vector spaces. The resulting affine group scheme over k is called the algebraic envelope of G over k.*

Definition 2.4.7. *Let X be a geometrically connected smooth variety over a field k of characteristic 0. We have a Tannakian category $\mathrm{Conn}(X)$ of all integrable connections (resp. $\mathrm{Conn}^{reg}(X)$ of regular singular connections), which defines a Tannakian group scheme $\pi_1^{diff}(X,\omega)$ (resp. $\pi_1^{alg}(X,\omega)$) when a fiber functor over k is provided. For an example of such a fiber functor, we can pick up a k-point of X (if it exists) and look at fibers at x.*

Example 2.4.8 ([85])**.** *Let X be a geometrically connected, smooth and proper over k. Nori defines a category of essentially finite bundles on X which is Tannakian. Let k be algebraically closed, $x \in X$ be a k-point, and $\pi_1^N(X,x)$ denote the Tannakian group scheme. The pro-étale completion of $\pi_1^N(X,x)$ is the étale fundamental group $\pi_1^{et}(X,x)$.*

Remark 2.4.9. *The groupoid $\mathrm{Fib}(\mathcal{C})(k)$ can be subtle even if we pass to the set of isomorphism classes. For example, let $\mathcal{C}$ be the category of finite bundles over a geometrically connected smooth proper curve X over a number field k. The set of isomorphism classes of $\mathrm{Fib}(\mathcal{C})(k)$ consists of exactly sections. See [26].*

Now we move to nilpotent objects.

Definition 2.4.10 ([95])**.** *Let $\mathcal{C}$ be an abelian tensor category with 1 being a unit object. The category $\mathcal{NC}$ is defined to be the full subcategory consisting of nilpotent objects consisting of objects E admitting a filtration*

$$0 = E_0 \subset E_1 \subset ... \subset E_n = E$$

with $E_i/E_{i-1} \cong 1$ for all i.

Proposition 2.4.11 (Proposition 1.2.1 [95])**.** *Let $\mathcal{C}$ be an abelian tensor category with a unit 1. Assume $\mathrm{End}(1)$ is a field. Then $\mathcal{NC}$ is an abelian category.*

In particular, starting from a Tannakian category $\mathcal{C}$, the full subcategory $\mathcal{NC}$ is a Tannakian subcategory, and we have a morphism of gerbes $\mathrm{Fib}(\mathcal{C}) \to \mathrm{Fib}(\mathcal{NC})$. Suppose there is a fiber functor ω over k, we obtain a faithfully flat $\mathrm{Aut}(\mathcal{C},\omega) \to \mathrm{Aut}(\mathcal{NC},\omega)$. The latter is known as the pro-unipotent envelope of $\mathrm{Aut}(\mathcal{C},\omega)$.

Example 2.4.12. *As a continuation of Example 2.4.7. All nilpotent objects in $\mathrm{Conn}(X)$ are regular singular. We use $\pi_1^{dR}(X,\omega)$ to denote the unipotent envelope of $\pi_1^{diff}(X,\omega)$, or equivalently of $\pi_1^{alg}(X,\omega)$.*

Example 2.4.13. *Under the assumption of Example 2.4.8, the category of nilpotent bundles is Tannakian. For k of positive characteristic, nilpotent bundles are essentially finite. See [85].*

Example 2.4.14. *Let X be a geometrically connected smooth variety over a field k of characteristic 0, D be a strict normal crossing divisor and $U = X \setminus D$. We identify some Tannakian categories inside* $\mathrm{Conn}(X, D)$.

- *All integrable logarithmic connections with nilpotent residues form a Tannakian category, denoted by* $\mathrm{Conn}^{nil}(X, D)$.

  The only nontrivial thing to be checked is that, the kernel and cokernel of a morphism between these connections have locally free underlying sheaves. To check the locally freeness, it suffices to check after everything being pulled back to the completed local ring $\hat{\mathcal{O}}_{X,x}$ for each $x \in X$. Now all connections with nilpotent residues are nilpotent in the sense of Definition 2.4.10 and we apply Proposition 2.4.11 to conclude.

- *Clearly we can identify $\mathcal{N}\mathrm{Conn}(X, D)$ with $\mathcal{N}\mathrm{Conn}^{nil}(X, D)$.*

- *Fix a section s of the projection $k \to k/\mathbb{Z}$ such that $s(0+\mathbb{Z}) = 0$. Deligne's canonical extension is a priori a functor from $\mathcal{N}\mathrm{Conn}(U) \to \mathrm{Conn}^{nil}(X, D)$. We show the functor factors through $\mathcal{N}\mathrm{Conn}^{nil}(X, D) = \mathcal{N}\mathrm{Conn}(X, D)$. We apply induction on the rank of $(\mathcal{E}, \nabla)$ from $\mathcal{N}\mathrm{Conn}(U)$. By definition we have a morphism*

  $$0 \longrightarrow (\mathcal{O}_U, d) \xrightarrow{i} (\mathcal{E}, \nabla_{\mathcal{E}}) \longrightarrow (\mathcal{E}', \nabla_{\mathcal{E}'}) \longrightarrow 0.$$

  Since Deligne's canonical extension is fully faithful, the morphism i extends to $\tilde{i} : (\mathcal{O}_X, d) \to (\tilde{\mathcal{E}}, \tilde{\nabla}_{\tilde{\mathcal{E}}})$ between canonical extensions. The kernel of $\tilde{i}$ is 0 since it lies in $\mathrm{Conn}^{nil}(X, D)$ and restricts to 0 on U. The cokernel of $\tilde{i}$ is clearly the canonical extension of $(\mathcal{E}', \nabla_{\mathcal{E}'})$, which lies in $\mathcal{N}\mathrm{Conn}^{nil}(X, D)$ by induction hypothesis.

- *All above remain valid for a stacky divisorial pair $(\mathfrak{X}, \mathfrak{D})$.*

We include some Tannakian criteria concerning injectivity, surjectivity and homotopy exact sequence among affine group schemes in terms of the associated Tannakian categories of representations.

Theorem 2.4.15 (Theorem A.1 [41])**.** *Let*

$$L \xrightarrow{q} G \xrightarrow{p} A$$

be a sequence of homomorphisms of affine group schemes over k. It induces a sequence of functors

$$\mathrm{Rep}(A) \xrightarrow{p^*} \mathrm{Rep}(G) \xrightarrow{q^*} \mathrm{Rep}(L)$$

where Rep *denotes the category of finite dimensional representations. Then*

1. *The morphism $p : G \to A$ is faithfully flat if and only if the functor $p^* : \mathrm{Rep}(A) \to \mathrm{Rep}(G)$ is fully faithful and the essential image is closed under taking subquotients in $\mathrm{Rep}(G)$.*

2. *The morphism $q : L \to G$ is a closed immersion if and only if any object in $\mathrm{Rep}(L)$ is a subquotient of some object of the form q^*V for some V in $\mathrm{Rep}(G)$.*

3. *Assuming p is faithfully flat, the sequence $L \to G \to A$ is exact if and only if*

   (a) For any object $V \in \mathrm{Rep}(G)$, q^*V is a trivial object in $\mathrm{Rep}(L)$ if and only if V is of the form p^*W for some $W \in \mathrm{Rep}(A)$.

   (b) Let $W_0 \hookrightarrow q^*V$ be the maximal trivial subobject in $\mathrm{Rep}(L)$, then there exists $V_0 \hookrightarrow V$ in $\mathrm{Rep}(G)$ with $q^*V_0 \cong W_0$.

   (c) For any $W' \in \mathrm{Rep}(G)$ and any quotient $q^*W' \twoheadrightarrow W$ there exists $V \in \mathrm{Rep}(G)$ with an injection $W \hookrightarrow q^*V$.

2.4.1 Final remarks

Remark 2.4.16 (motives)**.** *One of the motivation to study nilpotent objects and the unipotent completion of a Tannakian gerbe or group scheme is motivic, as in [32]. To be precise assume k is a number field and $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Let ω be a fiber functor of the Tannakian category of regular singular connections on X defined by certain tangential base point. Since $\pi_1^{dR}(X, \omega)$ is pro-unipotent, we identify it with its Lie algebra, which is pro-nilpotent. Then $\pi^{dR}(X, \omega)$ is expected to be the de Rham realization of a pro-object in the category of mixed motives over k and this is why $\pi_1^{dR}(X, \omega)$ is said to be motivic. This picture is later made more clear by Voevodsky who constructs a triangulated category of mixed motives and by Levine who finds a t-structure on a triangulated subcategory and realizes the category of mixed Tate motives as the heart of this t-structure. Now $\pi_1^{dR}(X, \omega)$ is the de Rham realization of a (pro-)mixed Tate motive. The fundamental group*

$\pi_1^{dR}(X,\omega)$ may not be the largest possible motivic fundamental group but it is of suitable size for nontrivial applications. It seems we are still far from clarifying what motivic local systems are and defining a motivic fundamental group in full generality.

Of course one can avoid motives by constructing crystalline or Betti counterparts of π_1^{dR} by hand and study comparisons among them. This leads to arithmetic applications, for example, the multiple zeta values or the Chabauty-Kim program.

3 Parabolic connections

In this chapter we fix an algebraically closed field k of characteristic 0. Let X be a smooth scheme which is separated and of finite type over k, D be a strict normal crossing divisor with irreducible components $D_1, ..., D_n$, and $U = X \setminus D$.

3.1 Introduction

We have seen in the previous chapter the construction below, at least for curves.

Let $(\mathcal{E}, \nabla)$ be an integrable logarithmic connection on X with poles along D and we use $res_{D_i}(\nabla)$ to denote the residue morphism along D_i. Suppose now that $(\mathcal{E}, \nabla)$ has rational exponents in $[0, 1)$. For each n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of rationals in $[0, 1)$, we use $\mathcal{E}^i_{\alpha_i}$ to denote the inverse image of the sum of all generalized eigenspaces of $res_{D_i}(\nabla)$ with eigenvalues $\geq \alpha_i$, under the projection

$$\mathcal{E} \to \mathcal{E}|_{D_i}.$$

We then define $\mathcal{E}_\alpha = \cap_i \mathcal{E}^i_{\alpha_i}$. The definition of $\mathcal{E}_\alpha$ can be extended to all $\alpha \in \mathbb{R}^n$, giving rise to a parabolic sheaf. In this chapter we prove the following result:

Theorem 3.1.1. *The parabolic sheaf $\mathcal{E}_\alpha$ constructed above is a parabolic bundle. Moreover each $\mathcal{E}_\alpha$ is stable under ∇, i.e, $\nabla(\mathcal{E}_\alpha) \subset \mathcal{E}_\alpha \otimes \Omega^1_X(log\, D)$, so we obtain a parabolic connection $(\mathcal{E}_\alpha, \nabla_\alpha)$ with ∇_α induced by ∇.*

The theorem has been proved for curves by [17] (where the authors work over $k = \mathbb{C}$ but their argument is purely algebraic, known as Gabber's construction (for example, see [46]) and the ground field is irrelevant), and should be folklore at least for smooth proper complex varieties, as observed as Lemma 3.3 [63] or Example 1.7 [8].

We end the introduction by explaining the argument in [63][8]. A variant of Deligne's canonical extension says that, suppose we have $s_i : k/\mathbb{Z} \to k$, $i = 1, .., n$, each of which is a section of the natural projection $k \to k/\mathbb{Z}$. A connection on U which is regular along D admits a logarithmic extension with exponents in $\mathrm{Im}(s_i)$ along D_i. Over $k = \mathbb{C}$, for each real number α, the projection defines a bijection beween complex numbers with real part in $[\alpha, \alpha + 1)$ and $\mathbb{C}/\mathbb{Z}$, which defines a section $s_\alpha : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$. Now starting from a logarithmic connection $(\mathcal{E}, \nabla)$, for each $\alpha = (\alpha_1, ..., \alpha_n)$ with real entries , we define $(\mathcal{E}_\alpha, \nabla_\alpha)$ to be the canonical extension of $(\mathcal{E}, \nabla)|_U$ by specifying $s_i = s_{\alpha_i}$. To check $\mathcal{E}_\alpha$ is a parabolic bundle we have to provide a natural morphism $\mathcal{E}_\beta \to \mathcal{E}_\alpha$ for each $\alpha \leq \beta$, and this is where we use the assumption that X is proper and the ground field is $\mathbb{C}$: for simplicity let us find a morphism $\mathcal{E}_\alpha \to \mathcal{E}_0$, with entries of α in $[0, 1)$. GAGA essures that we can work in the complex analytic topology. Let $x \in X$. In an analytic neighborhood of x, there is a local coordinate system $(z_1, ..., z_m)$ such that D is defined by $z_1 z_2 ... z_l = 0$ and there is a basis $e_1, ..., e_r$ of $\mathcal{E}_0$ such that under this basis the connection can be written as

$$\nabla_0 = d + \sum_{1 \leq i \leq l} A_i \frac{dz_i}{z_i}, \tag{2}$$

with A_i all being constant and uppertriangular with entries $\lambda_i^1, ..., \lambda_i^r$ along the diagonal, with real part in $[0, 1)$. Now $\mathcal{E}_\alpha$ is spanned by $a_1 e_1, ..., a_r e_r$, with

$$a_j = \prod_{1 \leq i \leq l, \mathfrak{Re}(\lambda_i^j) < \alpha_i} z_i$$

and in this case we have $\mathcal{E}_\alpha \to \mathcal{E}_0$ as a subsheaf.

Without assuming X to be proper over $k = \mathbb{C}$. The first problem is that algebraically there may not be nice local form of ∇ as that in Equation (2). This can be remedied by passing to formal completion, but there is not a suitable algebraization technique, such as GAGA.

3.2 The proof

We start from the local case.

Lemma 3.2.1. *Let $R = \mathcal{O}_{X,\eta_i}$ be the local ring of X at η_i, which is a discrete valuation ring with residue field $\kappa = \kappa(\eta_i)$. Fix a uniformizer t of R. Let (M, ∇) be a logarithmic integrable connection on $\operatorname{Spec} R$ with poles along the unique closed point. Here we have that M is a R-lattice of finite rank, say of rank r, and a k-linear*

$$\nabla : M \to M \otimes \Omega^1_{R/k}(log\, t)$$

verifying the usual Leibniz rule. We assume all exponents are rational in $[0, 1)$ and are ordered as follows:

$$\lambda_1 > ... > \lambda_l.$$

For any i, we consider the κ-vector space which is the sum of generalized eigenspace corresponding to $\lambda_1, ..., \lambda_i$ for the residue morphism on $M \otimes \kappa$ and its inverse image N_i in M under the canonical projection $M \to M \otimes \kappa$. Then N_i is stabilized by ∇, i.e., $\nabla(N_i) \subset N_i \otimes \Omega^1_{R/k}$.

Proof. We can assume $l > 1$ for otherwise it is trivial: N_i will be M or $t \cdot M$. Choose a suitable basis $e_1, ..., e_r$ of M, we can assume $res(\nabla)$ is upper-triangular of the form

$$\begin{pmatrix} J_{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & J_{\lambda_l} \end{pmatrix}$$

where J_{λ_j} is an $r_j \times r_j$ upper-triangular matrice with λ_j along the diagonal and possibly 1 along the subdiagonal. All other entries are 0. We write $s = r_1 + ... + r_i$. We can write

$$\nabla = d + \omega$$

with $\omega = (\omega_{i,j})_{1 \leq i,j \leq r}$ a matrix of logarithmic 1-forms. Here is a simple observation: it follows from the residue exact sequence that we have $\omega_{p,q} \in \Omega^1_{R/k}$ whenever $q \neq p, p+1$.

It is clear N_i has a basis $e_1, ..., e_s, te_{s+1}, ..., te_r$. We have to prove that

$$\nabla(N_i) \subset N_i \otimes \Omega^1_{R/k}(log\, t).$$

It is clear that $\nabla(te_j) \in N_i \otimes \Omega^1_{R/k}(log\, t)$ for any $1 \leq j \leq r$. It remains to prove $\nabla(e_j) \in N_i \otimes \Omega^1_{R/k}(log\, t)$ for $1 \leq j \leq s$. We have

$$\nabla(e_j) = e_1 \otimes \omega_{1j} + ... + e_r \otimes \omega_{rj}.$$

Therefore we have to prove

$$\omega_{pq} \in t \cdot \Omega^1_{R/k}(log\, t),$$

for any $1 \leq p \leq s$ and $s < q \leq r$. Apply induction to $q - p$. When $q - p = 1$ we have exactly $p = s$ and $q = s + 1$. Note we have $\omega_{s,s+1} \in \Omega^1_{R/k}$ since the $(s, s+1)$-entry of $res(\nabla)$ is 0. From the integrability of ∇ we deduce that

$$d\,\omega_{s,s+1} + \sum_{1 \leq j \leq r} \omega_{s,j} \wedge \omega_{j,s+1} = 0,$$

from which we conclude

$$\omega_{s,s+1} \wedge (\omega_{s,s} - \omega_{s+1,s+1}) \in \Omega^2_{R/k}.$$

Note $\omega_{s,s} - \omega_{s+1,s+1} \notin \Omega^1_{R/k}$ since its image under the residue map is $\lambda_i - \lambda_{i+1} \neq 0$. Therefore we have

$$\frac{dt}{t} \wedge \omega_{s,s+1} \in \Omega^2_{R/k},$$

and we conclude that $\omega_{s,s+1} \in t \cdot \Omega^1_{R/k}(log\, t)$. For $q - p > 1$, still by the integrability of ∇ we can deduce the following:

$$(\omega_{pp} - \omega_{qq}) \wedge \omega_{pq} + \sum_{p<j<q} \omega_{p,j} \wedge \omega_{j,q} \in \Omega^2_{R/k}.$$

By induction hypothesis, we have $\omega_{p,j} \wedge \omega_{j,q} \in \Omega^2_{R/k}$ for $p < j < q$, since one of ω_{pj} and ω_{jq} belongs to $t \cdot \Omega^1_{R/k}(log\, t)$ by induction hypothesis. We conlude that

$$(\omega_{p,p} - \omega_{q,q}) \wedge \omega_{p,q} \in \Omega^2_{R/k}.$$

Note $\omega_{p,p} - \omega_{q,q} \notin \Omega^1_{R/k}$ and by an identical argument as that in the case $q - p = 1$, we conclude $\omega_{p,q} \in t \cdot \Omega^1_{R/k}(log\, t)$. □

Proof of Theorem 3.1.1. Fix an n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of rational numbers in $[0, 1)$. For each i, we restrict $(\mathcal{E}, \nabla)$ to $\eta_i = \operatorname{Spec} \mathcal{O}_{X,D_i}$ of D_i, and let $\bar{\mathcal{E}}_{\alpha_i}$ be the subsheaf of $\mathcal{E} \otimes \kappa(\eta_i)$ which is the sum of all generalized eigenspaces of $res_{D_i}(\nabla)$ with eigenvalues $\geq \alpha_i$. Consider the inverse image $\mathcal{E}_{\alpha_i}$ of $\bar{\mathcal{E}}_{\alpha_i}$ under the natural projection

$$\mathcal{E} \otimes \mathcal{O}_{X,\eta_i} \twoheadrightarrow \mathcal{E} \otimes \kappa(\eta_i).$$

We have that $\mathcal{E}_{\alpha_i}$ is a $\mathcal{O}_{X,\eta_i}$-lattice and by Lemma 3.2.1 it actually defines a subconnection $(\mathcal{E}_{\alpha_i}, \nabla_{\alpha_i})$ of $(\mathcal{E}, \nabla)$ restricted to η_i. We choose an open subscheme U_i of X so that U_i intersects with D_j if and only if $i = j$, and we can shrink U_i such that it still contains η_i and $(\mathcal{E}_{\alpha_i}, \nabla_{\alpha_i})$ extends to a logarithmic connection of U_i with poles along $U_i \cap D_i$, which is still denoted by $(\mathcal{E}_{\alpha_i}, \nabla_{\alpha_i})$. We write $V = \cup_i U_i \cup U$ and $j : V \hookrightarrow X$ the open immersion. Now we get a logarithmic connection on V with poles along $V \cap D$, denoted by $(\mathcal{E}_\alpha, \nabla_\alpha)$, whose restriction to U is the holomorphic connection $(\mathcal{E}, \nabla)|_U$ and whose restriction to U_i is the logarithmic connection $(\mathcal{E}_{\alpha_i}, \nabla_{\alpha_i})$. We consider the direct image

$$(j_*\mathcal{E}_\alpha, j_*\nabla_\alpha),$$

where $j_*\mathcal{E}_\alpha$ is a priori reflexive. To prove $j_*\mathcal{E}_\alpha$ is locally free it suffices to evoke Lemma 11.5.1 [7]. To verify the condition there, we have to check the spectrums of $res_{D_i}(j_*\nabla_\alpha)$ are rational in $[\alpha_i, \alpha+1)$. The spectra of $res_{D_i}(j_*\nabla_\alpha)$ are rational in $[\alpha_i, \alpha_i + 1)$ at the generic point η_i of D_i by construction, and since $j_*\mathcal{E}_\alpha|_{D_i}$ is torsion-free (Corollary 1.1.14 [60]), the spectra are in $[\alpha_i, \alpha_i + 1)$ everywhere along D_i.

The logarithmic connection $(j_*\mathcal{E}_\alpha, j_*\nabla_\alpha)$ is independent of the choice of V. For simplicity we write $(\mathcal{E}_\alpha, \nabla_\alpha)$ for $(j_*\mathcal{E}_\alpha, j_*\nabla_\alpha)$.

For a general α we write in a unique way $\alpha_i = \beta_i + \gamma_i$ with $\beta_i \in [0, 1)$ and γ_i integral. We then define $(\mathcal{E}_\alpha, \nabla_\alpha)$ to be

$$(\mathcal{E}_\beta(-\gamma \cdot D), \nabla_\beta(-\gamma \cdot D).$$

For $\alpha \leq \beta$, we have a canonical morphism $\mathcal{E}_\beta \to \mathcal{E}_\alpha$ since such a morphism exists when restricted to an open subset of X containing U and all η_i and extends to X by Hartogs' extension theorem. The morphism between bundles are compatible with connection maps and we obtain a parabolic connection $(\mathcal{E}_\alpha, \nabla_\alpha)$. $\square$

3.3 Final remarks

Remark 3.3.1. *Theorem 3.1.1 can also be obtained in the following ways:*

- *Let $(\mathcal{E}, \nabla)$ be an integrable connection on X with poles along D, with rational exponents. We pick up a weight $\mathfrak{r} = (r_1, ..., r_n)$ with r_i being sufficiently divisible, such that $(\mathcal{E}, \nabla)$ is pulled back to a logarithmic connection on the root stack $\mathfrak{X} = \mathfrak{X}_\mathfrak{r}$ with integral exponents. Then the stacky Deligne's canonical extension says that $(\mathcal{E}, \nabla)|_U$ extends to a logarithmic connection $(\tilde{E}, \tilde{\nabla})$ on $\mathfrak{X}$ with poles along $\mathfrak{D}$ and of exponent 0. Now apply the stacky-parabolic correspondence (Theorem 2.3.11).*

- *We work over $k = \mathbb{C}$ and complex analytically. The divisorial pair (X, D) defines a log analytic space $\mathcal{X}$ with underlying analytic space X. The restriction of $(\mathcal{E}, \nabla)$ to U defines a complex local system on U by Riemann-Hilbert and then a complex local system on the Kato-Nakayama space $\mathcal{X}^{KN}$ of $\mathcal{X}$, referred as a quasi-unipotent local system on $\mathcal{X}^{KN}$. (In our setting the Kato-Nakayama space $\mathcal{X}^{KN}$ can be understood as X with a tubular neighborhood of each D_i removed, and is homotopy equivalent to U. For example, for X being the affine line and D being the origin, the Kato-Nakayama space is the complex plane with an open disk centered at the oringin removed.) In [62] the authors construct a quasi-unipotent Riemann-Hilbert correspondence from the category of quasi-unipotent local systems to the category of certain integrable connections on the Kummer étale site of $\mathcal{X}$.*

- *The interpretations above using stacks and the Kummer étale site are bridged by [99]*

Remark 3.3.2 (Failure of Deligne's canonical extension to be a tensor functor)**.** *Deligne's canonical extension restricts to a functor from the category of integrable connections on* U *which are regular along* D *with rational exponents to the category of integrable logarithmic connections on* X *with poles along* D *and exponents in* $[0,1)$*. This functor is clearly not compatible with tensor products.*

To remedy this, one should further compose the canonical extension functor with the functor from the category of logarithmic connections with rational exponents in $[0,1)$*, to the category of parabolic connections (or connections on root stacks or the Kummer étale site) considered in this chapter, to obtain a functor compatible with tensor products.*

Remark 3.3.3 (Integrability)**.** *Though we only work with integrable (logarithmic) connections, we remark Theorem 3.1.1 may be wrong for some non-integrable logarithmic connections. The first thing we have to clarify is that, for a non-integrable logarithmic connection, its set of exponents may not be constant along irreducible divisors* D_i*, and it does not make sense to associate a parabolic connection, or merely a parabolic sheaf to it. In the case that the set of exponents is constant along irreducible* D_i*, there is indeed an associated parabolic sheaf, but not necessarily a parabolic connection. The following example is from Remark 4.17 [24].*

Let $X = \operatorname{Spec} k[x,y]$*,* $D = \{x = 0\}$*,* $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X$ *and*

$$\nabla = d + \begin{pmatrix} \frac{1}{2}\frac{dx}{x} & 0 \\ dy & 0 \end{pmatrix}.$$

It is clear that res_D *has two eigenvalues* 0 *and* $\frac{1}{2}$*. The eigenspace for* $\frac{1}{2}$ *is* $\mathcal{O}_D \oplus 0$ *and its inverse image under the projection* $\mathcal{E} \to \mathcal{E}|_D$ *is the subsheaf* $\mathcal{E}' = \mathcal{O}_X \oplus \mathcal{O}_X(-D)$ *of* $\mathcal{E}$*. We readily check that* $\nabla(\mathcal{E}') \not\subset \mathcal{E}' \otimes \Omega^1_X(log\, D)$*.*

4 Fundamental groups in characteristic 0

Convention 4.0.1. *In this subsection let k be a field of characteristic 0. All schemes are assumed to be of finite type and separated over k.*

4.1 Semi-finite connections and virtually unipotent fundamental groups

4.1.1 Introduction

We introduce the notion of semi-finite connections on a geometrically connected smooth scheme U over k, which is not necessarily proper, and study them via stacky technique. The motivation is two-fold: on the one hand Otabe introduces the notion of semi-finite bundles on a geometrically connected proper variety in [90], which is a kind of mixture of the finite and unipotent theory developed in [85] by Nori. In particular under certain assumptions, he proves a homotopy exact sequence for Tannakian group schemes associated to semi-finite bundles (Corollary 5.9 [90]). On the other hand it has been proved (Theorem 7.1 [27]) that, there is a fundamental gerbe

$$\mathcal{G} \to \Pi^{\mathscr{C}}_{\mathcal{G}/k}$$

for any given well-founded class $\mathscr{C}$ of affine group schemes and any given fpqc stack $\mathcal{G}$ over k which is concentrated, geometrically reduced and $H^0(\mathcal{G}, \mathcal{O}) = k$. Here if we take $\mathscr{C}$ to be the class of virtually unipotent group schemes and $\mathcal{G}$ to be a geometrically connected proper variety, we recover Otabe's semi-finite bundles on $\mathcal{G}$ as the essential image of the pull-back functor

$$\mathrm{Vect}(\Pi^{\mathscr{C}}_{\mathcal{G}/k}) \to \mathrm{Vect}(\mathcal{G}). \tag{3}$$

In another extreme, if we take $\mathcal{G}$ to be the Tannakian gerbe of all regular integrable connections on U, and $\mathscr{C}$ still to be the class of virtually unipotent group schemes, the essential image of the pull-back functor should give a subcategory of integrable connections, which is the category of semi-finite connections. Semi-finite connections form a rigid abelian tensor category, which is neutral, provided a fiber functor defined by a k-point $x \in U$. The corresponding Tannakian group scheme is denoted by $\pi_1^{vu}(U, x)$ and the main result will be a homotopy exact sequence for virtually unipotent fundamenal groups, which covers both the étale case and the unipotent case, showing some uniformity in techniques.

We also remark that, a more subtle reason to take $\mathscr{C}$ to be the class of virtually unipotent group schemes instead of other possible well-founded classes is that the functor (3) may not be fully faithful for general $\mathscr{C}$, and in this case we say that there is not a Tannakian recognition of $\mathrm{Vect}(\Pi^{\mathscr{C}}_{\mathcal{G}/k})$ as a subcategory of $\mathrm{Vect}(\mathcal{G})$.

Convention 4.1.1. *In this subsection, all logarithmic connections turn out to have rational exponents. By a canonical extension, we always mean the logarithmic model with rational exponents in* $[0, 1)$.

4.1.2 Foundations

Let U be a geometrically connected smooth scheme over k and X be a smooth compactification with strict normal crossing boundary divisor D. Let $D_1, ..., D_n$ be the irreducible components of D.

Recall the notion of finite connections

Definition 4.1.2 ([40])**.** *An integrable connection* $(\mathcal{E}, \nabla)$ *on* U *is finite if there exists distinct finite sequences* $(a_{n-1}, ..., a_0), (b_{m-1}, ..., b_0)$ *of non-negative integers such that*

$$(\mathcal{E}, \nabla)^{\otimes n} \oplus ((\mathcal{E}, \nabla)^{\otimes n-1})^{\oplus a_{n-1}} \oplus ... \oplus ((\mathcal{E}, \nabla)^{\otimes 0})^{\oplus a_0} \cong (\mathcal{E}, \nabla)^{\otimes m} \oplus ((\mathcal{E}, \nabla)^{\otimes m-1})^{\oplus b_{m-1}} \oplus ... \oplus ((\mathcal{E}, \nabla)^{\otimes 0})^{\oplus a_0}$$

where $(\mathcal{E}, \nabla)^{\otimes 0}$ *is understood to be the unit connection* $(\mathcal{O}_X, d)$.

Proposition 4.1.3 ([40])**.** *Assume* k *is algebraically closed. The category of finite connections, together with a* k*-point* $x \in X$ *defines a neutral Tannakian category, whose corresponding Tannakian group scheme coincides with the étale fundamental group* $\pi_1^{et}(X, x)$ *regarded as a pro-finite group scheme. In particular the category of finite connections is semi-simple, i.e, all short exact sequences split.*

It is also clear that, as a full subcategory of $\mathrm{Conn}(U)$, the category of finite connections are closed under taking subquotients.

We will consider finite connections on U and a variant called semi-finite connections motivated by semi-finite bundles defined in [90]. We will study certain logarithmic extensions of semi-finite connections on $(\mathfrak{X}, \mathfrak{D})$, which is the root stack associated to the divisorial pair (X, D) and some weights.

Definition 4.1.4. *A connection* $(\mathcal{E}, \nabla)$ *on* U *is said to be semi-finite if it is nilpotent after being pulled back to an étale covering of* U*. The category of all semi-finite connections on* U *is denoted by* $\mathrm{Conn}^{sf}(U)$.

The full subcategory of $\mathrm{Conn}(U)$ consisting of semi-finite connections is denoted by $\mathrm{Conn}^{sf}(U)$. It is a rigid abelian tensor category. The analogy with Otabe's semi-finite bundles is clarified in the following proposition:

Proposition 4.1.5. *A connection $(\mathcal{E}, \nabla)$ on U is semi-finite if and only if there exists a filtration*

$$(\mathcal{E}, \nabla) = (\mathcal{E}_0, \nabla_0) \supset ... \supset (\mathcal{E}_l, \nabla_l),$$

with $\mathcal{E}_i/\mathcal{E}_{i+1}$ equipped with the quotient connection, is finite for each i.

Proof. To simplify notations, the alphabet "∇" denoting connections is omitted. We remind that we are working with connections.

The "if" part is clear, since a connection is finite if and only if it becomes trivial after being pulled to an étale covering of U.

For the "only if" part, we proceed by induction on the rank of $\mathcal{E}$. Let $\pi : V \to U$ be an étale covering on which $\mathcal{E}$ becomes nilpotent. Say there is a filtration of $\pi^*\mathcal{E}$

$$\pi^*\mathcal{E} = \mathcal{E}_0 \supset ... \supset \mathcal{E}_l,$$

with $\mathcal{E}_i/\mathcal{E}_{i+1}$ being the trivial connection.

The natural morphism

$$\mathcal{E} \to \pi_*\pi^*\mathcal{E}$$

is non-zero, from which we deduce that for some i

$$\mathrm{Hom}(\mathcal{E}, \pi_*(\mathcal{E}_i/\mathcal{E}_{i+1})) \neq 0.$$

It follows that $\mathcal{E}$ surjects onto a subconnection of $\pi_*(\mathcal{E}_i/\mathcal{E}_{i+1})$ for some i. Note $\pi_*(\mathcal{E}_i/\mathcal{E}_{i+1})$ is finite and so are its subconnections. We deduce that $\mathcal{E}$ surjects onto a finite connection. We conclude by applying induction hypothesis to the kernel of this surjection. $\square$

Definition 4.1.6. *An étale covering of a Deligne-Mumford stack $\mathfrak{X}$ is a stack $\mathfrak{Y}$ together with a finite representable étale morphism $p : \mathfrak{Y} \to \mathfrak{X}$.*

Proposition 4.1.7. *Fix a semi-finite connection $(\mathcal{E}, \nabla)$ on U. For any n-tuple $\mathfrak{r} = (r_1, ..., r_n)$ with r_i being sufficiently divisible, the canonical extension of $(\mathcal{E}, \nabla)$ on $\mathfrak{X}_\mathfrak{r}$ has nilpotent residues and there exists an étale covering $p : \mathfrak{Y} \to \mathfrak{X}_r$ on which the pull-back of the canonical extension is a nilpotent logarithmic connection with poles along $p^{-1}(\mathfrak{D}_\mathfrak{r})$.*

Proof. Recall the following fact:

Lemma 4.1.8 (see 3.3 [23])**.** *Let $p : Y \to X$ be a ramified covering of X with ramifications along D. For any weight $\mathfrak{r} = (r_1, ..., r_n)$ with r_i being sufficiently divisible, there exists a covering $p' : \mathfrak{Y} \to \mathfrak{X}_{\mathfrak{r}}$ filling in the following (not necessarily cartesian) commutative diagram:*

$$\begin{tikzcd} \mathfrak{Y} \arrow[r, "\pi'"] \arrow[d, "p'"] & Y \arrow[d, "p"] \\ \mathfrak{X}_{\mathfrak{r}} \arrow[r, "\pi"] & X. \end{tikzcd}$$

We go back to the proof of the proposition. By definition, we can choose étale covering of U on which the pull-back of $(\mathcal{E}, \nabla)$ is nilpotent, and we extend it to a covering $p : Y \to X$ possibly ramified over D. We choose a weight $\mathfrak{r}$ such that r_i is sufficiently divisible to make sure a diagram as in Lemma 4.1.8 exists, and the canonical extension $(\tilde{\mathcal{E}}, \tilde{\nabla})$ of $(\mathcal{E}, \nabla)$ on $\mathfrak{X}_{\mathfrak{r}}$ has nilpotent residues. The pull-back $p'^*(\tilde{\mathcal{E}}, \tilde{\nabla})$ of $(\tilde{\mathcal{E}}, \tilde{\nabla})$ is the canonical extension of the nilpotent connection $\pi'^* p^*(\mathcal{E}, \nabla)$ on $\pi'^{-1} p^{-1}(U)$ along the divisor $\pi'^{-1} p^{-1} D = p'^{-1} \pi^{-1} D = p'^{-1} \mathfrak{D}_{\mathfrak{r}}$, which is nilpotent. $\square$

Example 4.1.9. *Let us work out an example. Say $X = \mathbb{P}^1$ and $D = (0) + (\infty)$. Consider the logarithmic connection $(\mathcal{O}_X, \nabla_m)$ defined by*

$$\nabla_m = d + \frac{1}{m}\frac{dx}{x}, m \geq 2.$$

with exponent $\frac{1}{m}$ at 0 and $-\frac{1}{m}$ at ∞. Consider the root stack $\mathfrak{X}_{\mathfrak{r}}$ with $\mathfrak{r} = (m, m)$. The pull-back $\pi_{\mathfrak{r}}^*(\mathcal{O}_X, \nabla_m)$ has exponent 1 at $\mathfrak{D}_0$ and exponent -1 at $\mathfrak{D}_\infty$. The logarithmic connection $\pi_{\mathfrak{r}}^*(\mathcal{O}_X, \nabla_m)$ gives rise to a logarithmic connection $(\mathcal{O}_{\mathfrak{X}_{\mathfrak{r}}}(-\mathfrak{D}_0 + \mathfrak{D}_\infty), \nabla)$ by twisting with $\mathcal{O}_{\mathfrak{X}_{\mathfrak{r}}}(-\mathfrak{D}_0 + \mathfrak{D}_\infty)$. This logarithmic connection has exponents all equal to 0. Now the morphism $[m] : \mathbb{G}_m \to \mathbb{G}_m$, $x \mapsto x^m$ extends to a ramified covering $[m] : \mathbb{P}^1 \to \mathbb{P}^1$. There is a morphism $p : \mathbb{P}^1 \to \mathfrak{X}_{\mathfrak{r}}$ with $[m] = \pi_{\mathfrak{r}} \circ p$. The morphism p is an étale covering of $\mathfrak{X}_{\mathfrak{r}}$, which is actually a μ_m-torsor. The pullback of $(\mathcal{O}_{\mathfrak{X}_{\mathfrak{r}}}(-\mathfrak{D}_0 + \mathfrak{D}_\infty), \nabla)$ along p becomes the unit logarithmic connection.

Definition 4.1.10. *An object in $\mathrm{Conn}(\mathfrak{X}_{\mathfrak{r}}, \mathfrak{D}_{\mathfrak{r}})$ is semi-finite if it is nilpotent after being pulled back to an étale covering. The full subcategory of $\mathrm{Conn}(\mathfrak{X}_{\mathfrak{r}}, \mathfrak{D}_{\mathfrak{r}})$ of semi-finite objects is denoted by $\mathrm{Conn}^{sf}(\mathfrak{X}_{\mathfrak{r}}, \mathfrak{D}_{\mathfrak{r}})$. One readily checks that this is a rigid abelian tensor category.*

Lemma 4.1.11. *The restriction functor*

$$res_{\mathfrak{r}} : \mathrm{Conn}^{sf}(\mathfrak{X}_{\mathfrak{r}}, \mathfrak{D}_{\mathfrak{r}}) \to \mathrm{Conn}^{sf}(U)$$

is fully faithuful and the essential image is stable under taking subobjects.

Proof. The functor is fully faithful since Deligne's canonical extension is fully faithul.

Now let $(\mathcal{E}, \nabla_{\mathcal{E}}) \hookrightarrow (\mathcal{F}, \nabla_{\mathcal{F}})|_U$ be a subconnection, with $(\mathcal{F}, \nabla_{\mathcal{F}}) \in \mathrm{Conn}^{sf}(\mathcal{X}_{\mathfrak{r}}, \mathfrak{D}_{\mathfrak{r}})$. We conclude by considering the canonical extension of $(\mathcal{E}, \nabla_{\mathcal{E}})$ on $\mathfrak{X}_{\mathfrak{r}}$. It has nilpotent residues and injects into $(\mathcal{F}, \nabla_{\mathcal{F}})$, so it is in $\mathrm{Conn}^{sf}(\mathfrak{X}_{\mathfrak{r}}, \mathfrak{D}_{\mathfrak{r}})$. □

Using the same idea, we can prove the following:

Lemma 4.1.12. *For two weights $\mathfrak{r}|\mathfrak{r}'$, the pull-back functor*

$$\mathrm{Conn}^{sf}(\mathfrak{X}_{\mathfrak{r}}, \mathfrak{D}_{\mathfrak{r}}) \to \mathrm{Conn}^{sf}(\mathfrak{X}_{\mathfrak{r}'}, \mathfrak{D}_{\mathfrak{r}'})$$

is fully faithful and the essential image is stable under taking subobjects.

The category $\mathrm{Conn}^{sf}(U)$ (resp. $\mathrm{Conn}^{sf}(\mathfrak{X}_{\mathfrak{r}}, \mathfrak{D}_{\mathfrak{s}})$) equipped with the fiber functor sending a connection (resp. logarithmic connection) $(\mathcal{E}, \nabla)$ to its fiber $\mathcal{E}_x$ over x, is a neutral Tannakian category, and we use $\pi_1^{vu}(U, x)$ (resp. $\pi_1^{vu}(\mathfrak{X}_{\mathfrak{r}}, \mathfrak{D}_{\mathfrak{r}}, x)$) to denote the corresponding Tannakian group scheme.

Proposition 4.1.13. *The morphism $\pi_1^{vu}(U, x) \to \pi_1^{vu}(\mathfrak{X}_{\mathfrak{r}}, \mathfrak{D}_{\mathfrak{r}}, x)$ defined by $res_{\mathfrak{r}}$ is faithfully flat. For $\mathfrak{r}|\mathfrak{r}'$ the morphism $\pi_1^{vu}(\mathfrak{X}_{\mathfrak{r}'}, \mathfrak{D}_{\mathfrak{r}'}, x)) \to \pi_1^{vu}(\mathfrak{X}_{\mathfrak{r}}, \mathfrak{D}_{\mathfrak{r}}, x)$ defined by pull-back is faithfully flat. Moreover the natural morphism*

$$\pi_1^{vu}(U, x) \to \varprojlim_{\mathfrak{r}} \pi_1^{vu}(\mathfrak{X}_{\mathfrak{r}}, \mathfrak{D}_{\mathfrak{r}}, x)$$

is an isomorphism.

Proof. It is a translation of Proposition 4.1.7, Lemma 4.1.11 and Lemma 13 into group theoretic language. □

4.1.3 A homotopy exact sequence

The following lemma will be used frequently.

Lemma 4.1.14. *Let S be a scheme and $\mathfrak{X} \to S$ be an algebraic stack over S with finite inertia, together with the morphism $\pi : \mathfrak{X} \to X$ to its coarse moduli space. Assume further that $\mathfrak{X}$ is tame in the sense of [2] .For any S-morphism $g : Y \to X$ between algebraic spaces we form the the Cartesian diagram*

$$\begin{tikzcd} \mathfrak{Y} \arrow[r, "\pi'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ \mathfrak{X} \arrow[r, "\pi"] & X. \end{tikzcd}$$

Then we have $g^*\pi_* \xrightarrow{\sim} \pi'_* g'^*$ as functors from the category of quasi-coherent sheaves on $\mathfrak{X}$ to the category of quasi-coherent sheaves on Y.

Definition 4.1.15 (relative strict normal crossing divisor)**.** *Let $X \to S$ be a smooth morphism. A closed subscheme D of X is said to be a strict normal crossing divisor relative to S if for any schematic point $x \in X$, there exists $f_1, ..., f_i$ in $\mathcal{O}_{X,x}$ such that*

1. *The closed subscheme D is defined by $\prod f_i = 0$ at x*
2. *Let s be the image of x in S. The subscheme $\operatorname{Spec} \mathcal{O}_{X,x}/(f_1, ..., f_i)$ of $\operatorname{Spec} \mathcal{O}_{X,x}$ is of codimension i and is smooth over $\operatorname{Spec} \mathcal{O}_{S,s}$.*

We examine the definition in the non-relative case. Let S be $\operatorname{Spec} k$. Let X be smooth over k of dimension n. Suppose D is a relative strict normal crossing divisor in the sense of Definition 4.1.15. At any $x \in X$ we have $f_1, ..., f_i$ such that $f_1 ... f_i = 0$ defines D. $\operatorname{Spec} \mathcal{O}_{X,x}/(f_1, ..., f_i)$ is smooth over k of dimention $n - i$ (since it has codimention i in X by definition). We can find $e_1, ..., e_{n-i}$ in $\mathcal{O}_{X,x}$ such that their images $\mathcal{O}_{X,x}/(f_1, ..., f_i)$ is a regular sequence for the maximal ideal of $\mathcal{O}_{X,x}/(f_1, ..., f_i)$ (which corresponds to x). Now we have $f_1, ..., f_i, e_1, ..., e_{n-i}$ a sequence of length n generating the maximal ideal of $\mathcal{O}_{X,x}$. Therefore it is a regular sequence and in particular $f_1, ..., f_i$ can be extended to a regular sequence at the maximal ideal of $\mathcal{O}_{X,x}$. We recover the definition of strict normal crossing divisors in the non-relative case.

Let $f : U \to S$ be a smooth morphism between connected smooth schemes over k with connected fibers. Let $\bar{f} : X \to S$ be smooth proper together with an open immersion $U \to X$ of S-schemes, such that $D = X \setminus U$ is a strict normal crossing divisor of X relative to S. Fix a k-point s of S. Let $D_1, ..., D_n$ be irreducible components of D and we fix a weight $\mathfrak{r} = (r_1, ..., r_n)$. The restriction of D to X_s is strict normal crossing, and there is a canonical weight $\mathfrak{r}_s$ which we describe as follows: by definition we have to associate a positive integer to each component of D_s. For each irreducible component E of D_s, there is a unique i such that E is an irreducible component of $D_{i,s}$, we associate r_i to E. Note the restriction $D_{i,s}$ of D_i to X_s may not be irreducible anymore and in this case it is a disjoint union of smooth divisors. By Lemma 2.1.3, the following lemma is clear.

Lemma 4.1.16. *There is a Cartesian diagram*

$$\begin{tikzcd} \sqrt[\mathfrak{r}_s]{(D_s/X_s)} \arrow[r] \arrow[d] & X_s \arrow[d] \\ \sqrt[\mathfrak{r}]{D/X} \arrow[r] & X \end{tikzcd}$$

Since the weight $\mathfrak{r}$ will be fixed (and then so is $\mathfrak{r}_s$) in the rest of this article, to simplify notations we omit it from the notaions by using $\mathfrak{X}$ to denote $\mathfrak{X}_\mathfrak{r}$, $\mathfrak{D}$ to denote the tautological divisor on $\mathfrak{X}_\mathfrak{r}$, π to denote the canonical morphism $\pi_\mathfrak{r} : \mathfrak{X}_\mathfrak{r} \to X$, and $\mathfrak{X}_s$ to denote $\mathfrak{X}_{s,\mathfrak{r}_s}$ which is indeed the restriction of $\mathfrak{X}$ to X_s as promised by Lemma 4.1.16. Also we work with connections so we always assume X and S to be smooth (and then so is $\mathfrak{X}$, by Proposition 2.1.5).

We shall need some facts concerning the 0-th Gauss-Manin connection from the category $\mathrm{Conn}(\mathfrak{X}, \mathfrak{D})$ to $\mathrm{Conn}(S)$. Everything is essentially contained in Chapter 8 of [67]. Since we are working in stacky context, for convenience and completeness we summarize results needed in our article and explain how proofs in [67] can be modified in our situation. Starting from a logarithmic connection $(\mathcal{E}, \nabla)$ on $\mathfrak{X}$ with poles along $\mathfrak{D}$ we consider the kernel of the composition of ∇ with the natural projection:

$$\mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_{\mathfrak{X}/k}(log\,\mathfrak{D}) \longrightarrow \mathcal{E} \otimes \Omega^1_{\mathfrak{X}/S}(log\,\mathfrak{D}) \,,$$

of which the pushforward along $\bar{f}\pi$ is a coherent $\mathcal{O}_S$-module and is equipped canonically with a structure of connection. The resulting connection on S will be denoted by $\mathrm{H}^0_{dR}(\mathcal{E}, \nabla)$, and note there is a natural morphism of logarithmic connections

$$\pi^*\bar{f}^*\mathrm{H}^0_{dR}(\mathcal{E}, \nabla) \to (\mathcal{E}, \nabla).$$

Lemma 4.1.17. *The Gauss-Manin functor commutes with arbitrary base change. Namely, whenever we have a Cartesian diagram*

$$\begin{tikzcd} \mathfrak{X}_T \arrow[r] \arrow[d, "g'"] & T \arrow[d, "g"] \\ \mathfrak{X} \arrow[r] & S, \end{tikzcd}$$

for any $(\mathcal{E},\nabla)\in Conn(\mathfrak{X},\mathfrak{D})$ *we have*

$$\mathrm{H}^0_{dR}g'^*(\mathcal{E},\nabla)\cong g^*\mathrm{H}^0_{dR}(\mathcal{E},\nabla),$$

where the left hand side is the Gauss-Manin connection of $g'^*(\mathcal{E},\nabla)\in Conn(\mathfrak{X}_T,\mathfrak{D}_T)$.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathfrak{X}_T \arrow[r] \arrow[d, "g'"] & X_T \arrow[r] \arrow[d] & T \arrow[d, "g"] \\
\mathfrak{X} \arrow[r] & X \arrow[r] & S,
\end{tikzcd}$$

where both of the two squares are Cartesian. Using Lemma 4.1.14, we reduce to the case in [67] (applying to the right square, with no stacks involved). $\square$

Lemma 4.1.18. *The Gauss-Manin connection associated to an object in* $\mathrm{Conn}^{sf}(\mathfrak{X},\mathfrak{D})$ *is in* $\mathrm{Conn}^{sf}(S)$.

Proof. Let $(\mathcal{E},\nabla)$ be in $\mathrm{Conn}^{sf}(\mathfrak{X},\mathfrak{D})$. Choose $p':\mathfrak{Y}\to\mathfrak{X}$ be a covering on which the pull-back of $(\mathcal{E},\nabla)$ is unipotent. Since we have an inclusion

$$(\mathcal{E},\nabla)\hookrightarrow p'_*p'^*(\mathcal{E},\nabla)$$

and semi-finite connections are closed under taking subobjects, quotients and extensions in $Conn(S)$, it suffices to prove that, the Gauss-Manin connection of $(\mathcal{O}_{\mathfrak{Y}},d)$ under the morphism $\bar{f}\pi p'$ is finite. We consider the Stein factorization of the composition

$$\mathfrak{Y}\to\mathfrak{X}\to S,$$

which is a commutative diagram

$$\begin{tikzcd}
\mathfrak{Y} \arrow[r] \arrow[d, "p'"] & T \arrow[d, "p"] \\
\mathfrak{X} \arrow[r, "\bar{f}\pi"] & S
\end{tikzcd},$$

where the resulting morphism p is finite étale (Proposition 2.1, Exposé X, [52]). The Gauss-Manin connection on S of $(\mathcal{O}_{\mathfrak{Y}},d)$ is trivialized after being pulled-backed to T, so it is a finite connection. $\square$

The following is our main theorem concerning the homotopy exact sequence.

Theorem 4.1.19. *Let $\bar{f} : X \to S$ be a smooth and proper morphism with connected geometric fibers between connected smooth schemes, and D be a strict normal crossing disivor relative to S. Let U denote the open subscheme $X \setminus D$. Fix a weight $\mathfrak{r}$ of D. We have a homotopy exact sequence:*

$$\pi_1^{vu}(\mathfrak{X}, \mathfrak{D}, x) \to \pi_1^{vu}(\mathfrak{X}, \mathfrak{D}, x) \to \pi_1^{vu}(S, s) \to 1.$$

Proof of faithful flatness in Theorem 4.1.19. By Proposition 2.4.15 this is equivalent to proving that the pull-back functor $\pi^* \bar{f}^*$ from $\mathrm{Conn}^{sf}(S)$ to $\mathrm{Conn}^{sf}(\mathfrak{X}, \mathfrak{D})$ is fully faithful and the essential image is closed under taking subobjects. The full faithfulness follows from the projection formula.

It remains to prove that the essential image is stable under taking subobjects. We follow closely the argument in the proof of Theorem 3.1 [106].

Suppose now we have $(\mathcal{E}, \nabla_{\mathcal{E}})$ in $\mathrm{Conn}^{sf}(S)$ and a subobject $(\mathcal{F}, \nabla_{\mathcal{F}}) \hookrightarrow \pi^* \bar{f}^*(\mathcal{E}, \nabla_{\mathcal{E}})$. For any $t \in S$ we consider the diagram where both squares are cartesian:

$$\begin{tikzcd} \mathfrak{X}_t \arrow[r, "\pi_t"] \arrow[d, "i'_t"] & X_t \arrow[r, "\bar{f}_t"] \arrow[d, "i_t"] & \operatorname{Spec} \kappa(t) \arrow[d] \\ \mathfrak{X} \arrow[r, "\pi"] & X \arrow[r, "\bar{f}"] & S. \end{tikzcd}$$

The connection $(\mathcal{F}, \nabla_{\mathcal{F}})$ restricts to a trivial connection on $\mathfrak{X}_t$ since it is a subobject of the restriction to $\mathfrak{X}_t$ of $\pi^* \bar{f}^*(\mathcal{E}, \nabla_{\mathcal{E}})$, which is trivial . In particular $\mathcal{F}$ restricts to a free sheaf on $\mathfrak{X}_t$. From Lemma 4.1.14 we deduce that the locally free sheaf $\pi_* \mathcal{F}$ restricts to a free sheaf on X_t. Since t is arbitrary, we deduce that the natural morphism $\pi^* \pi_* \mathcal{F} \to \mathcal{F}$ is an isomorphism since this is the case if we pull it back via i'_t for any t. We also conclude by II.5.Corollary 2 [83] that $\bar{f}_* \pi_* \mathcal{F}$ is locally free with the same rank as $\pi_* \mathcal{F}$, with

$$\bar{f}_* \pi_* \mathcal{F} \otimes \kappa(t) \xrightarrow{\sim} \mathrm{H}^0(X_t, i_t^* \pi_* \mathcal{F}).$$

The canonical morphism

$$\bar{f}^* \bar{f}_* \pi_* \mathcal{F} \xrightarrow{\sim} \pi_* \mathcal{F}$$

is an isomorphism since applying Lemma 4.1.14, after being pulled back via i_t for arbitrary t, it becomes the identity morphism beween two free sheaves of the same rank, and then we have

$$\pi^* \bar{f}^* \bar{f}_* \pi_* \mathcal{F} \xrightarrow{\sim} \pi^* \pi_* \mathcal{F} \xrightarrow{\sim} \mathcal{F}.$$

We also have a canonical morphism

$$\bar{f}_*\pi_*\mathcal{F} \to \mathcal{E}.$$

For any t we consider

$$\bar{f}_*\pi_*\mathcal{F} \otimes \kappa(t) \to \mathcal{E} \otimes \kappa(t)$$

which can be identified with $\mathrm{H}^0(X_t, i_t^*\pi_*\mathcal{F}) \to \mathrm{H}^0(X_t, i_t^*\bar{f}^*\mathcal{E})$ and then with $\mathrm{H}^0(\mathfrak{X}_t, i_t'^*\mathcal{F}) \to \mathrm{H}^0(\mathfrak{X}_t, i_t'^*\pi^*\bar{f}^*\mathcal{E})$ by Lemma 4.1.14. We conclude that

$$\bar{f}_*\pi_*\mathcal{F} \to \mathcal{E}$$

is injective and the quotient is a locally free sheaf as well.

We have the following commutative diagram

$$\begin{tikzcd}
\bar{f}_*\pi_*\mathcal{F} \arrow[drr] \arrow[dr] & & & \\
& \bar{f}_*\pi_*\mathcal{F} \otimes \Omega^1_{S/k} \arrow[r, hook] \arrow[d, hook] & \mathcal{E} \otimes \Omega^1_{S/k} \arrow[r, two heads] \arrow[d, hook] & \mathcal{E}/\bar{f}_*\pi_*\mathcal{F} \otimes \Omega^1_{S/k} \arrow[d, hook] \\
& \bar{f}_*\pi_*\mathcal{F} \otimes \bar{f}_*\pi_*\Omega^1_{\mathfrak{X}/k}(log\,\mathfrak{D}) \arrow[r, hook] & \mathcal{E} \otimes \bar{f}_*\pi_*\Omega^1_{\mathfrak{X}/k}(log\,\mathfrak{D}) \arrow[r, two heads] & \mathcal{E}/\bar{f}_*\pi_*\mathcal{F} \otimes \bar{f}_*\pi_*\Omega^1_{\mathfrak{X}/k}(log\,\mathfrak{D})
\end{tikzcd}$$

and some explanations are in order: the two horizontal sequences are exact, since we have proved that $\mathcal{E}/\bar{f}_*\pi_*\mathcal{F}$ is locally free. The vertical morphisms are induced by applying $\bar{f}_*\pi_*$ to the injection of sheaves of 1-forms

$$\pi^*\bar{f}^*\Omega^1_{S/k} \hookrightarrow \Omega^1_{\mathfrak{X}/k}(log\,\mathfrak{D})$$

and in particular they are all injective. The morphism $\bar{f}_*\pi_*\mathcal{F} \to \mathcal{E} \otimes \Omega^1_{S/k}$ is the composition of the injection $\bar{f}_*\pi_*\mathcal{F} \hookrightarrow \mathcal{E}$ with the connection map $\nabla_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_S$. The morphism $\bar{f}_*\pi_*\mathcal{F} \to \bar{f}_*\pi_*\mathcal{F} \otimes \bar{f}_*\pi_*\Omega^1_{\mathfrak{X}/k}(log\,\mathfrak{D})$ is obtained by applying $\bar{f}_*\pi_*$ to the connection map $\nabla_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_{\mathfrak{X}/k}(log\,\mathfrak{D})$ and using the projection formula as well as the established fact that $\pi^*\bar{f}^*\bar{f}_*\pi_*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$.

Now since the vertical morphisms are injective, it follows from the diagram that the morphism $\bar{f}_*\pi_*\mathcal{F} \to \mathcal{E} \otimes \Omega^1_{S/k}$ factors through $\bar{f}_*\pi_*\mathcal{F} \otimes \Omega^1_{S/k}$. Therefore there is a connection

$$\bar{f}_*\pi_*\mathcal{F} \to \bar{f}_*\pi_*\mathcal{F} \otimes \Omega^1_S$$

which is automatically semi-finite since it is a subobject of $(\mathcal{E}, \nabla_{\mathcal{E}})$ by construction, and we can verify that its pull-back by $\bar{f}\pi$ coincides with $(\mathcal{F}, \nabla_{\mathcal{F}})$. □

Proposition 4.1.20. *For any $(\mathcal{E}, \nabla_{\mathcal{E}}) \in \mathrm{Conn}^{sf}(\mathfrak{X}, \mathfrak{D})$, there exists a maximal subobject of it which is in the essential image of the pull-back functor $\pi^* \bar{f}^*$. Moreover it is given by $\pi^* \bar{f}^* \mathrm{H}^0_{dR}(\mathcal{E}, \nabla_{\mathcal{E}})$.*

Proof. The natural morphism $\pi^* \bar{f}^* \mathrm{H}^0_{dR}(\mathcal{E}, \nabla_{\mathcal{E}}) \to (\mathcal{E}, \nabla_{\mathcal{E}})$ is an injection. Indeed, since the essential image of $\pi^* \bar{f}^*$ is closed under taking subobjects, let its kernel be of the form $\pi^* \bar{f}^*(\mathcal{K}, \nabla_{\mathcal{K}})$ for some semi-finite $(\mathcal{K}, \nabla_{\mathcal{K}})$. Applying H^0_{dR} to the exact sequence

$$0 \to \pi^* \bar{f}^*(\mathcal{K}, \nabla_{\mathcal{K}}) \to \pi^* \bar{f}^* \mathrm{H}^0_{dR}(\mathcal{E}, \nabla_{\mathcal{E}}) \to (\mathcal{E}, \nabla_{\mathcal{E}})$$

we conclude $(\mathcal{K}, \nabla_{\mathcal{K}}) = 0$.

Now for any $\pi^* \bar{f}^*(\mathcal{F}, \nabla_{\mathcal{F}}) \hookrightarrow (\mathcal{E}, \nabla)$ we apply H^0_{dR} to obtain an injection $(\mathcal{F}, \nabla_{\mathcal{F}}) \hookrightarrow \mathrm{H}^0_{dR}(\mathcal{E}, \nabla)$ and by apply $\pi^* \bar{f}^*$ we see that $\pi^* \bar{f}^*(\mathcal{F}, \nabla_F)$ is a subobject of $\pi^* \bar{f}^* \mathrm{H}^0_{dR}(\mathcal{E}, \nabla_{\mathcal{E}})$.

□

Proof of the exactness in the middle of Theorem 4.1.19. By Proposition 2.4.15 we have to verify the following:

1. For any $(\mathcal{E}, \nabla) \in \mathrm{Conn}^{sf}(\mathfrak{X}, \mathfrak{D})$ such that its restriction to $\mathfrak{X}_s$ is trivial, it is in the essential image of $\pi^* \bar{f}^*$. To prove this, consider the injection $\pi^* \bar{f}^* \mathrm{H}^0_{dR}(\mathcal{E}, \nabla) \hookrightarrow (\mathcal{E}, \nabla)$ and it remains an injection restricted to $\mathfrak{X}_s$. Since by Lemma 4.1.17 H^0_{dR} commutes with arbitrary base change, we see by assumption that the restriction $\pi^* \bar{f}^* \mathrm{H}^0_{dR}(\mathcal{E}, \nabla)$ to $\mathfrak{X}_s$ is a trivial connection of rank $\mathrm{rk}\, \mathcal{E}$. We conclude that $\pi^* \bar{f}^* H^0_{dR}(\mathcal{E}, \nabla) \to (\mathcal{E}, \nabla)$ is an isomorphism.

2. For any $(\mathcal{E}, \nabla)$ we have to find a subobject which restricts to the maximal trivial subobject of $i_s^*(\mathcal{E}, \nabla)$. We can take $\pi^* \bar{f}^* \mathrm{H}^0_{dR}(\mathcal{E}, \nabla)$.

3. For any quotient $i_s^*(\mathcal{F}, \nabla_{\mathcal{F}}) \twoheadrightarrow (\mathcal{E}, \nabla_{\mathcal{E}})$ with $(\mathcal{F}, \nabla_{\mathcal{F}}) \in \mathrm{Conn}^{sf}(\mathfrak{X}, \mathfrak{D})$ and $(\mathcal{E}, \nabla_{\mathcal{E}}) \in \mathrm{Conn}^{sf}(\mathfrak{X}_s, \mathfrak{D}_s)$ we can find an injection $(\mathcal{E}, \nabla_{\mathcal{E}}) \hookrightarrow i_s^*(\mathcal{G}, \nabla_{\mathcal{G}})$ for some $(\mathcal{G}, \nabla_{\mathcal{G}}) \in \mathrm{Conn}^{sf}(\mathfrak{X}, \mathfrak{D})$. Note

$$(\mathcal{E}, \nabla_{\mathcal{E}}) \cong \det(\mathcal{E}, \nabla_{\mathcal{E}}) \otimes \wedge^{\mathrm{rk}\mathcal{E}-1}(\mathcal{E}, \nabla_{\mathcal{E}})^{\vee}.$$

   Since $\wedge^{\mathrm{rk}\mathcal{E}-1}(\mathcal{E}, \nabla_{\mathcal{E}})^{\vee}$ injects into $i_s^* \wedge^{\mathrm{rk}\mathcal{E}-1} (\mathcal{F}, \nabla_{\mathcal{F}})^{\vee}$ and $i_s^* \wedge^{\mathrm{rk}\mathcal{E}} (\mathcal{F}, \nabla_{\mathcal{F}})$ surjects onto $\det(\mathcal{E}, \nabla_{\mathcal{E}})$, we reduce to the case that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is of rank 1. From now on we assume this is the case. $(\mathcal{F}, \nabla_{\mathcal{F}})$ admits a filtration

$$(\mathcal{F}, \nabla_{\mathcal{F}}) = (\mathcal{F}_0, \nabla_{\mathcal{F}_0}) \supset ... \supset (\mathcal{F}_l, \nabla_{\mathcal{F}_l}) = 0$$

such that $(\mathcal{F}_i, \nabla_{\mathcal{F}_i})/(\mathcal{F}_{i+1}, \nabla_{\mathcal{F}_{i+1}})$ are all holomorphic and trivial after being pulled back to some étale covering of $\mathfrak{X}$. There exists i with a non-zero (and hence surjective) morphism $i_s^*(\mathcal{F}_i, \nabla_{\mathcal{F}_i})/i_s^*(\mathcal{F}_{i+1}, \nabla_{\mathcal{F}_{i+1}}) \twoheadrightarrow (\mathcal{E}, \nabla_{\mathcal{E}})$. This surjection has a section and we can take $(\mathcal{G}, \nabla_{\mathcal{G}})$ to be $(\mathcal{F}_i, \nabla_{\mathcal{F}_i})/(\mathcal{F}_{i+1}, \nabla_{\mathcal{F}_{i+1}})$.

$\square$

Remark 4.1.21. *The proof, especially the trick of reduction to the rank 1 case, is inspired by the proof of Theorem 5.10 [39].*

The following is clear from Theorem 4.1.19 and Corollary 4.1.11.

Corollary 4.1.22. *Let $f : U \to S$ be a smooth morphism between connected smooth schemes over k with connected fibers. Let $\bar{f} : X \to S$ be smooth proper together with an open immersion $U \to X$ of S-schemes, such that $D = X \setminus U$ is a strict normal crossing divisor of X relative to S. Fix a point $s \in S$ and a point $x \in U$ above s. We have a homotopy exact sequence of Tannakian group schemes:*

$$\pi_1^{vu}(U_s, x) \to \pi_1^{vu}(U, x) \to \pi_1^{vu}(S, s) \to 1.$$

Corollary 4.1.23. *Let X and Y be smooth connected schemes over k. We have a Künneth formula*

$$\pi_1^{vu}(X \times Y) \cong \pi_1^{vu}(X) \times \pi_1^{vu}(Y).$$

Proof. Let $\bar{X}$ be an arbitrary compactification of X with strict normal crossing. We conclude by applying the homotopy exact sequence to $X \times Y \to Y$, together with the relative compactification $\bar{X} \times Y \to Y$. $\square$

Corollary 4.1.24. *We have $\pi_1^{vu}(\mathbb{A}_k^n) = 1$.*

Proof. Indeed by Corollary 4.1.23 we reduce to the case $n = 1$. Note in this case all finite connections are trivial (since $\pi_1^{et}(\mathbb{A}_k^1) = 1$, as a simple application of Riemann-Hurwitz formula). There are no non-trivial unipotent connections as well, since $\mathrm{H}_{dR}^1(\mathbb{A}_k^1) = 0$. $\square$

Corollary 4.1.25. *Let X be a smooth connected scheme over k. Let $\mathcal{E}$ be a vector bundle over X and $E \to X$ be the total space of $\mathcal{E}$. Then we have*

$$\pi_1^{vu}(E) \cong \pi_1^{vu}(X).$$

Proof. We apply the homotopy exact sequence to $\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X) \to X$ from the total space of the projective bundle of $\mathcal{E} \oplus \mathcal{O}_X$, which is a relative compactification of $E \to X$. We conclude by Corollary 4.1.24. $\square$

One can work with nilpotent connections (in this case there is no need of using root stacks) or finite connections (in this case one replace the category $\mathrm{Conn}^{sf}(\mathfrak{X}_r, \mathfrak{D}_r)$ by the category of connections on $\mathfrak{X}_r$ which are trivial after being pulled back to a covering of $\mathfrak{X}_r$). We have corollaries for our main argument:

Corollary 4.1.26 (a special case of Proposition 4.1 VIII [52])**.** *Let $U \to S$ be a smooth morphism between smooth connected schemes over k and suppose it admits a good compactification. Fix a point $s \in S$ and a point $x \in U$ above s. We have a homotopy exact sequence of étale fundamental groups*

$$\pi_1^{et}(U_s, x) \to \pi_1^{et}(U, x) \to \pi_1^{et}(S, s) \to 1.$$

Corollary 4.1.27. *Let $U \to S$ be a smooth morphism between smooth connected schemes over k and suppose it admits a good compactification. Fix a point $s \in S$ and a point $x \in U$ above s. We have a homotopy exact sequence of de Rham fundamental group schemes of unipotent connections*

$$\pi_1^{dR}(U_s, x) \to \pi_1^{dR}(U, x) \to \pi_1^{dR}(S, s) \to 1.$$

4.1.4 Final Remarks

Remark 4.1.28. *See Theorem 8.3 [64] for a homotopy exact sequence for "relative" de Rham fundamental groups.*

Remark 4.1.29. *Very recently, on a compact connected Kähler manifold, a relation between the virtually unipotent tannkian group of semi-finite bundles, after Otabe-Borne-Vistoli, and the virtually unipotent Tannakian group of semi-finite connections, has been carried out in [6]. Too be concrete, the forgetful functor forgetting the connection map defines a closed immersion of Tannakian group schemes in the reverse direction.*

4.2 A proof of Künneth formula for π_1^{dR} using universal extensions

The notion of universal extensions is introduced by Nori in [85] and using it he proves a Künneth formula for the Tannakian group scheme of unipotent bundles. In this subsection we prove the Künneth formula for π_1^{dR} following the same strategy as Nori. For more applications of universal extensions we refer to [38].

Convention 4.2.1. *All schemes in this subsection are assumed to be geometrically connected, smooth, of finite type and separated over k*

Notation 4.2.2. *For an connection $(\mathcal{E}, \nabla)$ on a smooth scheme, we suppress the connection map ∇ from the notation by simply writing $\mathcal{E}$. In particular, with a slight abuse of notation by $\mathcal{O}_X$ we mean $(\mathcal{O}_X, d)$.*

Lemma 4.2.3. *Let X and Y be smooth. Let $\mathcal{F}$ (resp. $\mathcal{G}$) be an integrable connection on X (resp. Y). There is an integrable connection $\mathcal{F} \boxtimes \mathcal{G}$ on $X \times Y$ and we have a Künneth formula for de Rham cohomology*

$$\bigoplus_{i+j=n} \mathrm{H}^i_{dR}(X, \mathcal{F}) \otimes \mathrm{H}^j_{dR}(Y, \mathcal{G}) \xrightarrow{\sim} \mathrm{H}^n_{dR}(X \times Y, \mathcal{F} \boxtimes \mathcal{G}).$$

Proof. [97, Tag 0FLT] $\square$

Let X be smooth and $\mathcal{F}$ be an integrable connection. There is a universal integrable connection $U(\mathcal{F})$ which sits in the middle of

$$0 \to \mathrm{H}^1_{dR}(X, \mathcal{F}^\vee)^\vee \otimes \mathcal{O}_X \to U(\mathcal{F}) \to \mathcal{F} \to 0, \tag{4}$$

with the property that, dualizing the exact sequence (4) and take cohomology, the connecting homomorphism $\mathrm{H}^0(X, \mathrm{H}^1_{dR}(X, \mathcal{F}^\vee) \otimes \mathcal{O}_X) \to \mathrm{H}^1_{dR}(X, \mathcal{F}^\vee)$ is the identity. The connection is universal in the following sense: suppose $\mathcal{F}'$ is an another extension of $\mathcal{F}$ by $\mathcal{O}_X^n$, defined by a class $\alpha \in \mathrm{H}^1_{dR}(X, \mathcal{F}^\vee \otimes \mathcal{O}_X^n)$. The class α defines a morphism $\mathcal{O}_X^n \to \mathrm{H}^1_{dR}(X, \mathcal{F}^\vee) \otimes \mathcal{O}_X$ and then $\mathrm{H}^1_{dR}(X, \mathcal{F}^\vee)^\vee \otimes \mathcal{O}_X \to \mathcal{O}_X^n$ by dualizing, still denoted by α. The connection $\mathcal{F}'$ can be obtained as

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1_{dR}(X, \mathcal{F}^\vee)^\vee \otimes \mathcal{O}_X \arrow[r] \arrow[d, "\alpha"] & U(\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \mathcal{O}_X^n \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

where the left square is the pushout along α.

Lemma 4.2.4. *Let $\mathcal{F}$ (resp. $\mathcal{G}$) be an integrable connection on X (resp, Y). Then there is a quotient*

$$U(\mathcal{F}) \boxtimes U(\mathcal{G}) \twoheadrightarrow U(\mathcal{F} \boxtimes \mathcal{G}).$$

Proof. By construction we have $\mathrm{H}^1_{dR}(X, \mathcal{F}^\vee)^\vee \otimes \mathrm{H}^1_{dR}(Y, \mathcal{G}^\vee)^\vee \otimes \mathcal{O}_{X \times Y}$ to be a subconnection of $U(\mathcal{F}) \boxtimes U(\mathcal{G})$, whose quotient U' sits in the middle of the sequence

$$0 \to \mathrm{H}^1_{dR}(X, \mathcal{F}^\vee)^\vee \otimes \mathcal{O}_X \boxtimes \mathcal{G} \oplus \mathrm{H}^1_{dR}(Y, \mathcal{G}^\vee)^\vee \otimes \mathcal{F} \boxtimes \mathcal{O}_Y \to U' \to \mathcal{F} \boxtimes \mathcal{G} \to 0. \tag{5}$$

Note the canonical surjection

$$\mathcal{F} \twoheadrightarrow \mathrm{H}^0_{dR}(X, \mathcal{F}^\vee)^\vee \otimes \mathcal{O}_X$$

resp.

$$\mathcal{G} \twoheadrightarrow \mathrm{H}^0_{dR}(Y, \mathcal{G}^\vee)^\vee \otimes \mathcal{O}_Y.$$

Identifying $\mathrm{H}^1_{dR}(X, \mathcal{F}^\vee) \otimes \mathrm{H}^0_{dR}(Y, \mathcal{G}^\vee) \oplus \mathrm{H}^0_{dR}(X, \mathcal{F}^\vee) \otimes \mathrm{H}^1_{dR}(Y, \mathcal{G}^\vee)$ with $\mathrm{H}^1_{dR}(X \times Y, \mathcal{F}^\vee \boxtimes \mathcal{G}^\vee)$ by Lemma 4.2.3, we obtain a surjection

$$\mathrm{H}^1_{dR}(X, \mathcal{F}^\vee)^\vee \otimes \mathcal{O}_X \boxtimes \mathcal{G} \oplus \mathrm{H}^1_{dR}(Y, \mathcal{G}^\vee)^\vee \otimes \mathcal{F} \boxtimes \mathcal{O}_Y \twoheadrightarrow \mathrm{H}^1_{dR}(X \times Y, \mathcal{F}^\vee \boxtimes \mathcal{G}^\vee)^\vee \otimes \mathcal{O}_{X \times Y}. \tag{6}$$

Now $U(\mathcal{F} \boxtimes \mathcal{G})$ is obtained by pushout of the exact sequence (5) along the surjection (6). Now $U(\mathcal{F} \boxtimes \mathcal{G})$ is a quotient of U', so it is a quotient of $U(\mathcal{F}) \boxtimes U(\mathcal{G})$. $\square$

For a smooth X, we inductively define $U_1(X) = \mathcal{O}_X$ and $U_{n+1}(X) = U(U_n(X))$.

Lemma 4.2.5. *Let $\mathcal{F}$ be a nilpotent connection of rank n on $X \times Y$. Then it is a quotient of $(U_n(X) \boxtimes U_n(Y))^n$.*

Proof. Apply induction on the rank n of $\mathcal{F}$. The case $n = 1$ is trivial. Assume the lemma for rank $\leq n$ and let $\mathcal{F}$ be a nilpotent connection of rank $n + 1$. It sits in the middle of

$$0 \to \mathcal{O}_{X \times Y} \to \mathcal{F} \to \mathcal{F}' \to 0. \tag{7}$$

There is a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{X \times Y} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & (U_n(X) \boxtimes U_n(Y))^n \arrow[r] \arrow[d, "surj"] & 0 \\
0 \arrow[r] & \mathcal{O}_{X \times Y} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}' \arrow[r] & 0
\end{tikzcd}$$

as the pullback diagram along a surjection $(U_n(X) \boxtimes U_n(Y))^n \twoheadrightarrow \mathcal{F}'$ whose existence is by induction hypothesis.

Apply Lemma 4.2.4 to $U_n(X)$ and $U_n(Y)$ and the universal property of $U((U_n(X) \boxtimes$

$U_n(Y))^n)$ we have the following diagram

$$\begin{tikzcd}
 & & (U_{n+1}(X) \boxtimes U_{n+1}(Y))^n \arrow[r, "surj"] \arrow[d, "surj"] & (U_n(X) \boxtimes U_n(Y))^n \arrow[d, "id"] & \\
0 \arrow[r] & \mathcal{O}_{X\times Y}^m \arrow[r] \arrow[d] & U((U_n(X) \boxtimes U_n(Y))^n) \arrow[r] \arrow[d] & (U_n(X) \boxtimes U_n(Y))^n \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \mathcal{O}_{X\times Y} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & (U_n(X) \boxtimes U_n(Y))^n \arrow[r] \arrow[d, "surj"] & 0 \\
0 \arrow[r] & \mathcal{O}_{X\times Y} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}' \arrow[r] & 0.
\end{tikzcd}$$

Consider the image $\mathcal{F}''$ of $(U_{n+1}(X) \boxtimes U_{n+1}(Y))^n$ in $\mathcal{F}$. It has rank $\geq n$. If the rank is $n+1$ we are done. If the rank is n, since the composition $\mathcal{F}'' \to \mathcal{F} \to \mathcal{F}'$ is a surjection, it is an isomorphism and the exact sequence (7) splits. In this case we also conclude. □

Proposition 4.2.6. *Let X and Y be smooth. We have a Künneth formula*

$$\pi_1^{dR}(X \times_k Y) \xrightarrow{\sim} \pi_1^{dR}(X) \times_k \pi_1^{dR}(Y). \tag{8}$$

Proof. The morphism (8) is faithfully flat. To see this, let $x \in X$ (resp. $y \in Y$) be base points, and i_x (resp i_y) be the inclusion $i_x : Y = \{x\} \times Y \hookrightarrow X \times Y$ (resp. $i_y : X = X \times \{y\} \hookrightarrow X \times Y$). Let p_x (resp. p_y) be the projection $X \times Y \to X$ (resp. $X \times Y \to Y$). It suffices to observe the morphisms between Tannakian group schemes induced by $p_x i_x$ and $p_y i_y$ are trivial and the morphism between Tannakian group schemes induced by $p_y i_x$ and $p_x i_y$ are the identities.

To prove it is a closed immersion, we have to show that any nilpotent connection on $X \times Y$ is a subquotient of an object of the form $\mathcal{F} \boxtimes \mathcal{G}$, with $\mathcal{F}$ (resp. $\mathcal{G}$) being nilpotent on X (resp. Y). This follows from Lemma 4.2.5. □

4.3 Some results around the base change and the Lefschetz property

Notation 4.3.1. *For an abelian group A, we write $\mathbb{D}(A)$ to be the split multiplicative group scheme over k, with A being the group of characters.*

It is proved as 10.43 [32] that π_1^{dR} is compatible with base change, in the sense that for a smooth geometrically connected scheme X over k and a field extension $k \hookrightarrow k'$, we have an isomorphism

$$\pi_1^{dR}(X) \times_k k' \xrightarrow{\sim} \pi_1^{dR}(X \times_k k').$$

Therefore, for some results on π_1^{dR}, at least for the Künneth formula, we can make a quick proof by reduction to $\mathbb{C}$ using the base change formula, and then to the topological case.

However, we remark that the base change formula can be wrong for π_1^{alg} or π_1^{diff}.

Example 4.3.2 ([56]). *Assume $k = \bar{k}$. We have $\pi_1^{alg}(\mathbb{G}_m) \cong \mathbb{D}_k(k/\mathbb{Z}) \times_k \mathbb{G}_a$. This provides a counterexample to the base change formula. Namely suppose k' is an another algebraically closed field with $k \hookrightarrow k'$, we have*

$$\pi_1^{alg}(\mathbb{G}_m) \times_k k' \neq \pi_1^{alg}(\mathbb{G}_m \times_k k')$$

as group schemes over k'. Indeed the group of characters is $k/\mathbb{Z}$ on the left hand side and $k'/\mathbb{Z}$ on the right hand side.

A less well-known example is the following:

Example 4.3.3 (motivated by Remark 5.3 [10]). *Assume $k = \bar{k}$. Let X be a connected, smooth and proper curve over k with genus ≥ 1. In this case $\pi_1^{diff}(X) = \pi_1^{alg}(X)$ and we compute its group of characters, say $\mathbb{X}$. Characters are 1-dimensional representations of $\pi_1^{alg}(X)$, and by Tannakian duality they correspond bijectively to connections of rank 1. We have*

$$0 \to \mathrm{H}^0(X, \Omega_X^1) \to \mathbb{X} \to \mathrm{Pic}^0(X) \to 0.$$

To interprete this, a line bundle $\mathcal{L}$ carries a connection if and only if it is of degree 0, i.e, an element in $\mathrm{Pic}^0(X)$, and all connections on $\mathcal{L}$ form a torsor under the group $\mathrm{H}^0(X, \Omega_X^1)$, by sending $(\mathcal{L}, \nabla)$ to $(\mathcal{L}, \nabla) \otimes (\mathcal{O}_X, d + \omega)$, $\omega \in \mathrm{H}^0(X, \Omega_X^1)$.

We see that the base change formula does not hold for smooth proper curves X with genus ≥ 1.

Since there is not a base change formula, and there is not an immediate reduction to $k = \mathbb{C}$, the study of π_1^{diff} and π_1^{alg} is much more complicated. We have the following:

Theorem 4.3.4 ([106][37]). *Let $X \to S$ be a smooth proper morphism between smooth geometrically connected schemes over k, with geometrically connected fibers. Let $s \in S$ be a k-point and X_s be the inverse image of s. We have a homotopy exact sequence*

$$\pi_1^{diff}(X_s) \to \pi_1^{diff}(X) \to \pi_1^{diff}(S) \to 1.$$

In particular, we have a Künneth formula

$$\pi_1^{diff}(X \times_k Y) \xrightarrow{\sim} \pi_1^{diff}(X) \times_k \pi_1^{diff}(Y),$$

provided that X and Y are smooth and geometrically connected, and at least one of them is proper.

Some remarks on Theorem 4.3.4 are in order. We remark that the proof in [37] by dos Santos works in arbitrary characteristic, but we postpone until later chapter to make sense of a characteristic p-analogy of π_1^{diff}. We also remark that certain assertion from the proof in [106] by Zhang is still via reduction to $k = \mathbb{C}$. Later we will see that the case of $k = \mathbb{C}$ is still useful to attack the case of general k, combined with a spreading and specialization argument.

Now we move to Lefschetz theorem, an another remarkable theorem for fundamental groups.

Theorem 4.3.5. *Let X be a smooth projective complex variety and H be a smooth hyperplane section (of some ample line bundle). The morphism*

$$\pi_1^*(H) \to \pi_1^*(X), * \in \{top, diff\},$$

is faithfully flat if the dimension of X is at least 2, and is an isomorphism if the dimension of X is at least 3.

Proof. The topological case is classical. Pass to algebraic envelopes we conclude for π_1^{diff} (Example 2.4.6). $\square$

We shall prove Lefschetz theorem for π_1^{diff} over a general ground field of characteristic 0.

Proposition 4.3.6. *Let X be geometrically connected, smooth and projective over k and H be a smooth hyperplane section (of some ample line bundle). The morphism*

$$\pi_1^{diff}(H) \to \pi_1^{diff}(X),$$

is faithfully flat if the dimension of X is at least 2, and is an isomorphism if the dimension of X is at least 3.

Proof. For simplicity, we omit ∇ from the notation $(\mathcal{E}, \nabla)$ of a connection.

Since we have the Lefschetz theorem for topological fundamental groups, the proposition holds for $k = \mathbb{C}$ by passing to algebraic envelopes.

We first prove the faithfully flat part so let X be of dimension ≥ 2. We write $\mathrm{res}_k : \mathrm{Conn}(X/k) \to \mathrm{Conn}(H/k)$ the restriction functor. By Lemma 2.4.15, we have to prove that the functor res_k is fully faithful and the essential image is closed under taking subobjects. Now suppose we have a connection $\mathcal{E}$ and an injection $\mathcal{F} \hookrightarrow \mathcal{E}|_H$. We can find a subfield k' of k, which is finitely generated over $\mathbb{Q}$, such that x descends to a k'-point, H (resp. X) descends to H' (resp. X') over k', $\mathcal{E}$ (resp. $\mathcal{F}$) descends to a connection $\mathcal{E}'$ (resp. $\mathcal{F}'$) over X' (resp. H'), and the injection $\mathcal{F} \hookrightarrow \mathcal{E}|_H$ descends to $\mathcal{F}' \hookrightarrow \mathcal{E}'|_{H'}$. Now suppose we have a faithfully flat $\pi_1^{alg}(H') \to \pi_1^{alg}(X')$, by Lemma 2.4.15 we can find a connection $\tilde{\mathcal{F}'}$ on X' whose restriction to H' is $\mathcal{F}'$. Consider the base change of $\tilde{\mathcal{F}'}$ to X over k, so its restriction to H is $\mathcal{F}$. The same descent trick applies to prove res_k to be fully faithful. We reduce to the case that k is finitely generated over $\mathbb{Q}$.

Now fix a field extension $k \hookrightarrow \mathbb{C}$, and we have the following diagram of categories:

$$\begin{tikzcd}
\mathrm{Conn}(X/k) \arrow[r] \arrow[d, "\mathrm{res}_k"] & \langle \mathrm{Conn}(X/k) \rangle \arrow[r] \arrow[d, "\mathrm{res}_{\mathbb{C}}"] & \mathrm{Conn}(X_{\mathbb{C}}/\mathbb{C}) \arrow[d, "\mathrm{res}_{\mathbb{C}}"] \\
\mathrm{Conn}(H/k) \arrow[r] & \langle \mathrm{Conn}(H/k) \rangle \arrow[r] & \mathrm{Conn}(H_{\mathbb{C}}/\mathbb{C})
\end{tikzcd}$$

We use $\mathrm{Aut}(\langle \mathrm{Conn}(X/k) \rangle)$ (resp. $\mathrm{Aut}(\langle \mathrm{Conn}(H/k) \rangle))$) to denote the Tannakian group scheme of $\langle \mathrm{Conn}(X/k) \rangle$ (resp. $\langle \mathrm{Conn}(H/k) \rangle$), which is the full Tannakian subcategory of $\mathrm{Conn}(X_{\mathbb{C}}/\mathbb{C})$ (resp.$\mathrm{Conn}(H_{\mathbb{C}}/\mathbb{C})$) generated by objects obtained as extension of scalars from $\mathrm{Conn}(X/k)$ (resp. $\mathrm{Conn}(H/k)$). The morphism $\pi_1^{alg}(H_{\mathbb{C}}) \to \pi_1^{alg}(X_{\mathbb{C}})$ is faithfully flat, and it restricts to a faithfully flat morphism $\mathrm{Aut}(\langle \mathrm{Conn}(H/k) \rangle) \to \mathrm{Aut}(\langle \mathrm{Conn}(X/k) \rangle)$, which is the base change of the morpshim $\pi_1^{alg}(H) \to \pi_1^{alg}(X)$ over k to $\mathbb{C}$ (1.3.2 [68]). Therefore the morphism $\pi_1^{alg}(H) \to \pi_1^{alg}(X)$ is faithfully flat as well. The proof for the faithful flatness is finished.

Now we prove $\pi_1^{alg}(H) \to \pi_1^{alg}(X)$ to be a closed immersion, if X is of dimension ≥ 3. By Lemma 2.4.15, it suffices to prove that, for any connection $\mathcal{E}$ on H, there exists $\tilde{\mathcal{E}}$ on X with restriction to H being the given $\mathcal{E}$. Proceed as above, we reduce to the case that k is finitely generated over $\mathbb{Q}$ and fix an extension $k \hookrightarrow \mathbb{C}$.

The base change $\mathcal{E}_{\mathbb{C}}$ of $\mathcal{E}$ over $H_{\mathbb{C}}$ lifts to a connection $\tilde{\mathcal{E}}_{\mathbb{C}}$ over $X_{\mathbb{C}}$. We can find a spreading in the following sense: there exists $k \hookrightarrow R \hookrightarrow \mathbb{C}$ with R being a finitely generated k-algebra, together with a connection $\tilde{\mathcal{E}}_R \in \mathrm{Conn}(X_R/R)$ whose base change along $R \hookrightarrow \mathbb{C}$ is exactly $\tilde{\mathcal{E}}_{\mathbb{C}}$. Now consider the restriction $\tilde{\mathcal{E}}_R|_{H_R} \in \mathrm{Conn}(H_R/R)$. Since the base change of this restriction along $R \to \mathbb{C}$

is $\mathcal{E}_{\mathbb{C}}$, enlarging R if necessary we may assume the base change $\mathcal{E}_R$ of $\mathcal{E}$ along $k \hookrightarrow R$ is exactly the restriction $\tilde{\mathcal{E}}_R|_{H_R}$. Now we pick up $R \twoheadrightarrow l \hookrightarrow \bar{k}$ with l being a finite extension of k and consider the base change of $\tilde{\mathcal{E}}_R$ along $R \twoheadrightarrow l$. We see that $\mathcal{E}_l$ is the restriction to H_l of a connection on X_l. We pushforward this connection along $X_l \to X$, and then restrict it to H. We see that $\mathcal{E}$ is a subobject of the restriction to H of some connection on X, which shows that $\mathcal{E}$ is itself in the essential image of res_k, since the essential image of res_k is closed under taking subquotients. The proof for the closed immersion is finished. □

4.4 Some results around the Künneth formula for π_1^{diff} and π_1^{alg}

Using the same technique in the proof of the Lefschetz property of π_1^{diff}, we prove the following:

Proposition 4.4.1. *Let X and Y be smooth and geometrically connected. We have a Künneth formula*

$$\pi_1^{diff}(X \times_k Y) \xrightarrow{\sim} \pi_1^{diff}(X) \times_k \pi_1^{diff}(Y). \tag{9}$$

Proof. The proof proceeds along the same lines with Proposition 4.3.6. Still, the proposition holds in the case $k = \mathbb{C}$, as a consequence of the topological case and passage to algebraic envelopes.

The morphism (9) is faithfully flat. The morphism (9) is a closed immersion if and only if, any regular singular connection $\mathcal{E}$ is a subquotient of a connection of the form $\mathcal{F} \boxtimes \mathcal{G} = p_X^*\mathcal{F} \otimes p_Y^*\mathcal{G}$. We fix an arbitrary smooth compactification $(\bar{X}, D)$ (resp. $(\bar{Y}, E)$) of X (resp. Y) with strict nomal crossing boundary.

We readily reduce to the case that k is finitely generated over $\mathbb{Q}$ and fix an embedding $k \hookrightarrow \mathbb{C}$. The base change $\mathcal{E}_{\mathbb{C}}$ of $\mathcal{E}$ over $X_{\mathbb{C}} \times Y_{\mathbb{C}}$ is a subquotient $\mathcal{E}_{1,\mathbb{C}}/\mathcal{E}_{2,\mathbb{C}}$ with $\mathcal{E}_{2,\mathbb{C}} \hookrightarrow \mathcal{E}_{1,\mathbb{C}} \hookrightarrow \mathcal{F}_{\mathbb{C}} \boxtimes \mathcal{G}_{\mathbb{C}}$. The connection $\mathcal{F}_{\mathbb{C}}$ (resp. $\mathcal{G}_{\mathbb{C}}$) admits a logarithmic extension $\tilde{\mathcal{F}}_{\mathbb{C}}$ (resp. $\tilde{\mathcal{G}}_{\mathbb{C}}$) on $\bar{X}_{\mathbb{C}}$ (resp. $\bar{Y}_{\mathbb{C}}$) with poles along $D_{\mathbb{C}}$ (resp. $E_{\mathbb{C}}$). There is a spreading $k \hookrightarrow R \hookrightarrow \mathbb{C}$ with R being finitely generated over k, such that $\mathcal{F}_{\mathbb{C}}$ (resp. $\mathcal{G}_{\mathbb{C}}$) descends to a connection $\mathcal{F}_R$ (resp. $\mathcal{G}_R$) in $\mathrm{Conn}(X_R/R)$ (resp. $\mathrm{Conn}(Y_R/R)$), $\mathcal{E}_{1,\mathbb{C}}$ (resp. $\mathcal{E}_{2,\mathbb{C}}$) descends to $\mathcal{E}_{1,R}$ (resp. $\mathcal{E}_{2,R}$) in $\mathrm{Conn}(X_R \times_R Y_R/R)$, $\tilde{\mathcal{F}}_{\mathbb{C}}$ (resp. $\tilde{\mathcal{G}}_{\mathbb{C}}$) descends to $\tilde{\mathcal{F}}_R$ (resp. $\tilde{\mathcal{G}}_R$) in $\mathrm{Conn}(X_R/R, D_R)$ (resp. $\mathrm{Conn}(Y_R/R, E_R)$), the inclusion $\mathcal{E}_{2,\mathbb{C}} \hookrightarrow \mathcal{E}_{1,\mathbb{C}} \hookrightarrow \mathcal{F}_{\mathbb{C}} \boxtimes \mathcal{G}_{\mathbb{C}}$ descends to $\mathcal{E}_{2,R} \hookrightarrow \mathcal{E}_{1,R} \hookrightarrow \mathcal{F}_R \boxtimes \mathcal{G}_R$, and the base change $\mathcal{E}_R$ of $\mathcal{E}$ along $k \hookrightarrow R$ is isomorphic to $\mathcal{E}_{1,R}/\mathcal{E}_{2,R}$. By generic flatness we find a specialization $R \twoheadrightarrow l$ with l being finite over k, such that $\mathcal{E}_{2,R} \hookrightarrow \mathcal{E}_{1,R} \hookrightarrow \mathcal{F}_R \boxtimes \mathcal{G}_R$ specializes to an inclusion $\mathcal{E}_{2,l} \hookrightarrow \mathcal{E}_{1,l} \hookrightarrow \mathcal{F}_l \boxtimes \mathcal{G}_l$ in $\mathrm{Conn}(X_l \times_l Y_l/l)$, with

$\mathcal{F}_l \in \mathrm{Conn}(X_l/l)$ and $\mathcal{G}_l \in \mathrm{Conn}(Y_l/l)$. We have that the base change $\mathcal{E}_l$ of $\mathcal{E}$ along $k \hookrightarrow l$ is a subquotient of $\mathcal{F}_l \boxtimes \mathcal{G}_l$. Note $\mathcal{F}_l$ (resp $\mathcal{G}_l$) is regular singular, since we have a logarithmic extension $\tilde{\mathcal{F}}_l$ (resp. $\tilde{\mathcal{G}}_l$) as the specialization of $\tilde{\mathcal{F}}_R$ (resp. $\tilde{\mathcal{G}}_R$).

We write q_X (resp. q_Y) to be the canonical morphism $X_l \to X$ (resp. $Y_l \to Y$). Now by adjunction $\mathcal{E}$ is a subobject of $(q_X \times q_Y)_* \mathcal{E}_l$, from which we deduce that $\mathcal{E}$ is a subquotient of $(q_X \times q_Y)_*(\mathcal{F}_l \boxtimes \mathcal{G}_l)$. We conclude by noting the natural quotient $q_{X,*}\mathcal{F}_l \boxtimes q_{Y,*}\mathcal{G}_l \twoheadrightarrow (q_X \times q_Y)_*(\mathcal{F}_l \boxtimes \mathcal{G}_l)$, and $\mathcal{E}$ is then a subquotient of $q_{X,*}\mathcal{F}_l \boxtimes q_{Y,*}\mathcal{G}_l$. $\square$

The Künneth formula for π_1^{diff} is wrong in general, provided with two non-proper schemes over k. We need some preliminaries. Let k be algebraically closed, X be a smooth variety of dimension n over k, D be a strict normal crossing divisor and $U = X \setminus D$. For each point $x \in X$, we look at the formal local ring $R = \hat{\mathcal{O}}_{X,x} = k[[x_1, ..., x_n]]$ with D defined by $x_1 ... x_m = 0$. We write $S = R[\frac{1}{x_1}, ..., \frac{1}{x_m}]$. For an integrable connection $(\mathcal{E}, \nabla)$, an admissible formal decomposition around x, is an isomorphism

$$(\mathcal{E}, \nabla) \otimes_{\mathcal{O}_U} S \cong \bigoplus_{i \in I} \mathrm{Exp}(\phi_i) \otimes \mathcal{R}_i,$$

where $\phi_i \in S$, $\mathrm{Exp}(\phi_i)$ is the exponential connection $(\mathcal{O}_S, \nabla_{\phi_i})$ with

$$\nabla_{\phi_i} = \mathrm{d} + \mathrm{d}\phi_i,$$

and $\mathcal{R}_i$ is a connection on S which is regular along D. Note for $\phi_i \in R \subset S$ the exponential connection $\mathrm{Exp}(\phi_i)$ has no singularities along D and the admissible decomposition may not be unique. The existence of an admissible decomposition is known in the case $m = 1$, after possibly replacing R by a ramified extension, which is the so-called Levelt-Turritin decomposition. For $m \geq 2$ the existence of such a decomposition is not a priori clear, though this will be clear in our situation. Note that, whenever an admissible formal decomposition exists, we can look at the set $\{\phi_i + R | i \in I\}$ in S/R, and this set does not depend on the admissible decomposition we have chosen.

Remark 4.4.2. *For a given integrable connection $(\mathcal{E}, \nabla)$, after interating blow-ups with smooth centers supported on boundary divisors, we can even prove the existence of a good admissible formal decomposition at each point. This is proved firstly for X being a surface ([81]) and then in arbitrary dimensions ([69]).*

Example 4.4.3. *Suppose X and Y are connected smooth affine curves over an algebraically closed field k. We prove the natural morphism*

$$\pi_1^{diff}(X \times_k Y) \to \pi_1^{diff}(X) \times_k \pi_1^{diff}(Y) \tag{10}$$

is not a closed immersion. Therefore we disprove the Künneth formula for π_1^{diff} without assuming X or Y to be proper over k(compare Theorem 4.3.4).

Let $x \in \bar{X} \setminus X$ (resp. $y \in \bar{Y} \setminus Y$) be a boundary point, with local coordinate z (resp. w). Noting $\mathcal{O}_{\bar{X}}(x)$ (resp. $\mathcal{O}_{\bar{Y}}(y)$) is ample, there is a non-constant meromorphic function ϕ_x (resp. ϕ_y) on $\bar{X}$ (resp. $\bar{Y}$), which is not regular only at x (resp. y). Consider the connection $\mathcal{E}(\phi_x\phi_y)$ on $Z = X \times Y$ as $(\mathcal{O}_Z, \nabla_{\phi_x\phi_y})$, with

$$\nabla_{\phi_x\phi_y} = \mathrm{d} + \mathrm{d}(\phi_x\phi_y).$$

Suppose (10) *is an isomorphism. Consider morphisms to $\mathbb{G}_m$ of both sides of* (10)*, and we see that any integrable connection of rank* 1 *on Z is of the form $\mathcal{G} \boxtimes \mathcal{H}$. We will conclude by showing that this is not the case for $\mathcal{E}(\phi_x\phi_y)$. By contradiction, suppose $\mathcal{E}(\phi_x\phi_y)$ is of the form $\mathcal{F} \boxtimes \mathcal{G}$, with both $\mathcal{F}$ and $\mathcal{G}$ of rank* 1*. We consider the admissible formal decomposition of $\mathcal{F}$ (resp. $\mathcal{G}$) at the boundary point x (resp. y):*

$$\mathcal{F} \otimes_{\mathcal{O}_X} k((z)) \cong \mathcal{E}(\epsilon) \otimes \mathcal{R}_X, \epsilon \in k((z))$$

resp.

$$\mathcal{G} \otimes_{\mathcal{O}_Y} k((w)) \cong \mathcal{E}(\psi) \otimes \mathcal{R}_Y, \psi \in k((w)).$$

Consider the compactification of Z into $\bar{X} \times \bar{Y}$. At the point (x, y), the connection $\mathcal{F} \boxtimes \mathcal{G}$ has an admissible decomposition

$$\mathcal{F} \boxtimes \mathcal{G} \otimes_{\mathcal{O}_{X\times Y}} k[[z,w]][\frac{1}{z}, \frac{1}{w}] \cong \mathcal{E}(\epsilon + \psi) \otimes \mathcal{R}_X \otimes \mathcal{R}_Y,$$

Now an admissible formal decomposition of $\mathcal{E}(\phi_x\phi_y)$ is self-evident, and we see that

$$\phi_x\phi_y - \epsilon - \psi \in R = k[[z, w]].$$

This is a contradiction, as $\phi_x \in k((z)) \setminus k[[z]]$, $\phi_y \in k((w)) \setminus k[[w]]$, and $\phi_x\phi_y$ is never a summation of some $\epsilon \in k((z))$, some $\psi \in k((w))$ and some element from $k[[z, w]]$.

4.5 Final remarks

Though in this chapter we only work with fundamental groups in characteristic 0, we give a remark on Nori fundamental group in characteristic p, concerning base change, Künneth formula, homotopy exact sequence and Lefschetz theorem.

Remark 4.5.1. *Let X be a geometrically connected, smooth and proper variety over an algebraically closed field k of characteristic p. We are interested in properties of $\pi_1^N(X)$ where we leave the fiber functor implicit. The Künneth formula is proved in [79]. The base change is disproved in [91] (while there is already a conterexample in [79], but the example is a curve with cusps and is not smooth). The homotopy exact sequence is disproved in [105].*

Most interestingly, the Lefschetz theorem is wrong in general for a smooth hyperplane section H of an arbitrary ample line bundle $\mathcal{L}$ with a counterexample based on the failure of Kodaira vanishing in characteristic p. Nevertheless, the Lefschetz theorem holds for any smooth hyperplane section H of $\mathcal{L}^d$, with $d > d_0$ and d_0 being effective in terms of X and $\mathcal{L}$. For details see [18].

Concerning Lefschetz theorem we also have the following

Theorem 4.5.2 (Theorem 8.10 [58])**.** *Let X be a smooth projective variety of dimension ≥ 2 over an algebraically closed and uncountable field k, D be a strict normal crossing divisor and $U = X \setminus D$. Let $\mathcal{L}$ be an ample line bundle on X. Let $(\mathcal{E}, \nabla)$ be an integrable connection on U. There exists m_0 depending on $(\mathcal{E}, \nabla)$ such that for any $m \geq m_0$ and any generic hyperplane section H of $\mathcal{L}^m$, the natural morphism*

$$\pi_1^{diff}(\langle(\mathcal{E}, \nabla)|_{H\cap U}\rangle) \to \pi_1^{diff}(\langle(\mathcal{E}, \nabla)\rangle)$$

is an isomorphism.

5 $\mathcal{D}$-modules in characteristic $p > 0$

Convention 5.0.1. *We fix an algebraically closed field k of characteristic p. All schemes are assumed to be of finite type and separated over k.*

5.1 $\mathcal{D}$-modules and F-divided sheaves

We summarize some general facts concerning F-divided sheaves and $\mathcal{D}$-modules in positive characteristic, and the equivalence between them for smooth schemes. We refer to [51] [92] for more details.

Definition 5.1.1. *Let X be a scheme over k. The category $\mathrm{Fdiv}(X)$ of F-divided sheaves on X is the category whose objects are sequences $(\mathcal{E}_n, \sigma_n)_{n\geq 0}$ where $\mathcal{E}_n$ are coherent sheaves with isomorphisms $\sigma_n : F^*\mathcal{E}_{n+1} \cong \mathcal{E}_n$. Arrows between two objects $(\mathcal{E}_n, \sigma_n)_{n\geq 0}$ and $(\mathcal{F}_n, \tau_n)_{n\geq 0}$ are projective systems $(\alpha_n)_{n\geq 0} : \mathcal{E}_n \to \mathcal{F}_n$ of morphisms between $\mathcal{O}_X$-modules, verifying $\tau_n \circ F^*(\alpha_{n+1}) = \alpha_n \circ \sigma_n$.*

For a F-divided sheaf $(\mathcal{E}_n, \sigma_n)_{n\geq 0}$, the isomorphisms σ_n will not play a significant role in the sequel, and we will omit them from notions by simply writing $(\mathcal{E}_n)_{n\geq 0}$ for a F-divided sheaf.

Lemma 5.1.2 (Lemma 4.2 [16])**.** *Let $(\mathcal{E}_n)_{n\geq 0}$ be a F-divided sheaf. Then $\mathcal{E}_n$ are all vector bundles.*

Proof. We introduce one more notation: for a ring R with $pR = 0$, an ideal I of R, and a p-power p^n, we write $I^{[p^n]}$ the ideal generated by all r^{p^n} with $r \in I$.

We prove for $\mathcal{E}_0$. It suffices to prove that for any point $x \in X$, $\mathcal{E}_{0,x} = \mathcal{E}_0 \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}$ is free as an $\mathcal{O}_{X,x}$-module. By [97, Tag 07ZD] it suffices to show that, for any i if the i-th Fitting ideal $Fit_i(\mathcal{E}_x)$ of $\mathcal{E}_x$ is not $\mathcal{O}_{X,x}$, it is zero. Since $(F^n)^*\mathcal{E}_n \cong \mathcal{E}_0$, we have $Fit_i(\mathcal{E}_{0,x}) = Fit_i(\mathcal{E}_{n,x})^{[p^n]}$ and then $Fit_i(\mathcal{E}_{0,x}) \subset \mathfrak{m}_{X,x}^{p^n}$ for all $n \geq 0$. We conclude by noting that $\bigcap_{n\geq 0} \mathfrak{m}_{X,x}^n = 0$.

□

Definition 5.1.3 ([35])**.** *Let X be smooth over k. We inductively define an increasing sequence $\mathcal{D}_X^{\leq n}$ of k-linear differential operators of order at most n on X: let $\mathcal{D}_X^{\leq 0} = \mathcal{O}_X$, acting on $\mathcal{O}_X$ simply by multiplication. Once $\mathcal{D}_X^{\leq n}$ is defined, we define $\mathcal{D}_X^{\leq n+1}$ to be the subsheaf of $\mathcal{E}nd_k(\mathcal{O}_X)$ locally consisting of operators D, such that the operator D_a*

$$D_a(x) = D(ax) - aD(x)$$

is in $\mathcal{D}_X^{\leq n}$ for any local section $a \in \mathcal{O}_X$.

Example 5.1.4. *Suppose X is smooth of dimension 1, and let x be an étale local coordinate of X. Then $\mathcal{D}_X$ is generated over $k[x]$ by operators*

$$D_k, k \geq 0, D_k(x^l) = \binom{l}{k} x^{l-k},$$

verifying the relation

$$D_k D_l = \binom{k+l}{k} D_{k+l}.$$

In particular, $\mathcal{D}_X$ is not generated by its subsheaf $\mathcal{D}_X^{\leq 1}$ as an algebra over $\mathcal{O}_X$, contrary to the case in characteristic 0.

Definition 5.1.5. *A (left) $\mathcal{D}_X$-module on X is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ on X, together with a morphism between sheaves of algebras:*

$$\mathcal{D}_X \to \mathcal{E}nd_k(\mathcal{E}).$$

Remark 5.1.6. *There is an abvious notion of right $\mathcal{D}_X$-module. As in the case of characteritic 0, the functor sending $\mathcal{E}$ to $\mathcal{E} \otimes_{\mathcal{O}_X} \omega_X$ defines an equivalence from the category of left $\mathcal{D}_X$-modules to the category of right $\mathcal{D}_X$-modules. See [55].*

Remark 5.1.7. *It is proved [96] that for a smooth affine scheme X over k, the global dimension of the ring $\Gamma(X, \mathcal{D}_X)$ is equal to the Zariski dimension of X. From this one can prove that, any $\mathcal{D}_X$-module $\mathcal{E}$ admits a resolution*

$$\mathcal{P}^{\mathrm{dim}X} \to \mathcal{P}^{\mathrm{dim}X-1} \to ... \to \mathcal{P}^0 \to \mathcal{E} \to 0,$$

where one could choose $\mathcal{P}^i$ to be locally free as $\mathcal{D}_X$-modules for $i < \mathrm{dim}X$ and $\mathcal{P}^{\mathrm{dim}X}$ to be locally projective. Furthermore, if $\mathcal{E}$ is a coherent $\mathcal{D}_X$-module (in the sense of being locally finitely presented), all $\mathcal{P}^i$ can be chosen as coherent $\mathcal{D}_X$-modules.

The bad news is that, an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module is not likely to be coherent as a $\mathcal{D}_X$-module.

The category of $\mathcal{D}_X$-modules, denoted by $\mathcal{D}_X\text{-mod}$, is an abelian category and it has enough injective objects. For the existence of injective objects and injective resolutions, we refer to Proposition 1.6 [86].

Definition 5.1.8. *The i-th $\mathcal{D}_X$-module cohomology, denoted by $\mathrm{H}^i_{\mathcal{D}_X}(X, \cdot)$, is the i-th right derived functor of the following left exact functor :*

$$\mathrm{H}^0_{\mathcal{D}_X} : \mathcal{D}_X - \mathrm{mod} \to \mathrm{Vect}_k, \mathcal{E} \mapsto \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{E}).$$

Theorem 5.1.9 (Theorem 1.3 [51] or Theorem 1.2 [92]). *Let X be smooth over k. There is an equivalence between the full subcategory of $\mathcal{D}_X$-modules consisting of $\mathcal{O}_X$-coherent objects, and the category $\mathrm{Fdiv}(X)$. To be explicit, the functor going from the category of $\mathcal{D}_X$-modules to $Fdiv(X)$ is defined as follows: starting from a $\mathcal{D}_X$-module $\mathcal{E}$ we first set $\mathcal{E}_0 = \mathcal{E}$. For $n > 0$ consider the subsheaf $\mathcal{E}_n$ of $\mathcal{E}$ consisting of local sections annihilated by differential operators D of order $< p^n$ with $D(1) = 0$. The subsheaf $\mathcal{E}_n$ is equipped with a coherent $\mathcal{O}_X$-module structure by $a \cdot s = a^{p^n} s$, for $a \in \mathcal{O}_X$ and $s \in \mathcal{E}_n$, where the right hand side is the multiplication of a^{p^n} on s, by considering s as a local section of $\mathcal{E}$. We have natural isomorphisms $F^*\mathcal{E}_{n+1} \cong \mathcal{E}_n$.*

Remark 5.1.10. *We refer to [55] for a dual of Theorem 5.1.9 for right $\mathcal{D}_X$-modules.*

Lemma 5.1.11. *Let X be a reduced scheme which is proper over k. Let $(\mathcal{E}_n)_{n\geq 0}$ be a F-divided sheaf on X. The sequence $h^0(X, \mathcal{E}_n)$, $n \geq 0$, forms a decreasing (not necessarily strict) sequence.*

Proof. The natural morphism

$$\mathcal{E}_n \to F_*F^*\mathcal{E}_n \cong F_*\mathcal{E}_{n-1}$$

is injective, which can be checked Zariski locally. Take global sections we get an injection of cohomology groups

$$\mathrm{H}^0(X, \mathcal{E}_n) \hookrightarrow \mathrm{H}^0(X, \mathcal{E}_{n-1}).$$

This is not a morphism of k-vector spaces but we can k-linearize it by making a Frobenius twist

$$\mathrm{H}^0(X, \mathcal{E}_n) \otimes_{k,F} k \hookrightarrow \mathrm{H}^0(X, \mathcal{E}_{n-1}),$$

from which we conclude.

□

5.2 Fundamental groups

We include some results on $\mathcal{D}$-modules and F-divided sheaves from the point of view of fundamental groups.

Let X be a smooth scheme. Since the absolute Frobenius F is flat, $\mathrm{Fdiv}(X)$ is an abelian category. There is also a tensor structure for $\mathrm{Fdiv}(X)$ by taking tensor products componentwisely. There is a componentwise dual, and using Lemma 5.1.2, the category $\mathrm{Fdiv}(X)$ is Tannakian provided that X is connected. Fix a point $x \in X$ giving rise to a fiber functor, we use $\pi_1^{str}(X, x)$ to denote the corresponding Tannakian group scheme. For simplicity we will write $\pi_1^{str}(X)$ leaving the fiber functor implicit. This group scheme is clearly a characteristic p analogue of π_1^{diff} which we have discussed earlier. This group scheme has been studied intensively and we refer to [51][36][43][92][37][44] for this topic.

The following is a homotopy exact sequence, for which we have seen the characteristic 0 counterpart for π_1^{diff} (Theorem 4.3.4).

Theorem 5.2.1 ([37])**.** *Let $X \to S$ be a smooth projective morphism between smooth connected schemes over k with geometrically connected fibers. Pick up $s \in S$ and let X_s be the inverse image on X of s, we have a homotopy exact sequence*

$$\pi_1^{str}(X_s) \to \pi_1^{str}(X) \to \pi_1^{str}(S) \to 1.$$

To provide a uniform treatment to the homotopy exact sequence in arbitrary characteristic, as in [37], it is necessary to work from another point view, the stratified point of view, which is already introduced by Grothendieck in [54]. Related to it there is also another point of view using cystals.

Proposition 5.2.2 (Proposition 2.11 [14])**.** *Let X be smooth over k. There is an equivalence of categories between:*

1. *Stratified sheaves (Definition 2.10 [14]).*
2. *$\mathcal{D}_X$-modules.*
3. *Infinitesimal crystals on X. That is to say, a compatible family of quasi-coherent sheaves $\mathcal{E}_T$ on T for each nilpotent thickening $U \hookrightarrow T$ of an open set U in X, with transitive isomorphisms $u^*\mathcal{E}_T \xrightarrow{\sim} \mathcal{E}_{T'}$ for any $(U', T') \to (U, T)$.*

The point of view of crystals leads to a Lefschetz theorem for π_1^{str}. The following theorem should be arributed to Esnault-Srinivas-Bost. It is the analogue of Theorem 4.3.6 in characteristic p.

Theorem 5.2.3. *Let X be connected, smooth and projective over k and H be a smooth hyperplane section of some ample line bundle. The morphism*

$$\pi_1^{str}(H) \to \pi_1^{str}(X), \tag{11}$$

is faithfully flat if the dimension of X is at least 2, and is an isomorphism if the dimension of X is at least 3.

Proof. The faithfully flat part is Theorem 3.5 [44]. We make a complement on how to prove the morphism (11) is a closed immersion for X of dimension ≥ 3.

We construct a F-divided sheaf on X out of a F-divided sheaf $(\mathcal{E}_n)_{n\geq 0}$ on H, as the inverse to the restriction functor.

We use $H(m)$ to denote the m-th infinitesimal neighborhood of H in X. In particular H_0 is H itself. Now we fix a non-negative integer n. Interpreting F-divided sheaves as crystals, there is a locally free sheaf $\mathcal{E}_n(m)$ on $H(m)$ for each m, such that $\mathcal{E}_n(m)$ restricts to $\mathcal{E}_n(m')$ along $H(m) \hookrightarrow H(m')$ for any $m' \leq m$, and $\mathcal{E}_n(0) = \mathcal{E}_n$. By [97, Tag 0EL7], there is an open subset V_n of X containing H, and a locally free sheaf $\tilde{\mathcal{E}}_n$ on V_n restricting to $\mathcal{E}_n(m)$ along $H(m) \hookrightarrow V_n$ for any m (and this is where we use the assumption that X has dimension ≥ 3). The complement of V_n in X is of codimension at least 2. Indeed, since H is the hyperplane section of an ample line bundle, the complement $X \setminus H$ is an affine scheme. On the one hand, the complement $X \setminus V_n$ (with the reduced scheme structure) is clearly proper over k. On the other hand, $X \setminus V_n$ is a closed subscheme of the affine scheme $X \setminus H$, so it is affine. We conclude that $X \setminus V_n$ is in fact a scheme of 0-dimension. Now we consider the pushforward $j_{V_n *}\tilde{\mathcal{E}}_n$ on X where $j_{V_n} : V_n \to X$ is the open immersion. The pushforward $j_{V_n *}\tilde{\mathcal{E}}_n$ is coherent and reflexive.

We claim that the sheaf $j_{V_n *}\tilde{\mathcal{E}}_n$ on X does not depend on the choice of V_n. To prove this, we only have to notice that any two such pairs $(V_n, \mathcal{E}_n)$ $(V_n', \mathcal{E}_n')$ are isomorphic after being restricted to a small enough open subset $V \subset V_n \cap V_n'$ containing H, still by [97, Tag 0EL7]. Now it suffices to evoke [97, Tag 0E9I].

Now we prove that $(j_{V_n *}\tilde{\mathcal{E}}_n)_{n\geq 0}$ is F-divided, which is the lifting of $(\mathcal{E}_n)_{n\geq 0}$ as desired. Fix an n, we can shrink the open subschemes V_n and V_{n+1} with $V_n = V_{n+1}$ and $\tilde{\mathcal{E}}_n \cong F^*\tilde{\mathcal{E}}_{n+1}$. Then $j_{V_n *}\tilde{\mathcal{E}}_n \cong F^* j_{V_{n+1} *}\tilde{\mathcal{E}}_{n+1}$ follows from flat base change (hence the pullback along the absolute Frobenius commutes with pushforward along the open immersion). $\square$

5.2.1 Final remarks

Remark 5.2.4 (Remember we have omitted something?)**.** *As we have mentioned, the isomorphisms σ_n in a F-divided sheaf $(\mathcal{E}_n, \sigma_n)$ do not play a significant role for our purpose, so they are omitted .*

However we refer to [78], especially Theorem 7.3 and Lemma 7.4, to see how *these isomorphisms σ_n apply to an analogy of Abhyankar's conjecture for π_1^{str} over curves.*

5.3 Interlude: auxiliary results from intersection theory and moduli theory

5.3.1 Background in intersection theory

Following [50] faithfully we collect some results of intersection theory for later use. In this subsection let X be a scheme which is locally of finite type and seperated over k. We use $\mathrm{A}_m(X)_{\mathbb{Q}}$ to denote the Chow group, defined as the group generated by m-cycles with rational coefficients, modulo rational equivalence. We use $\mathrm{A}_\bullet(X)_{\mathbb{Q}}$ to denote the direct sum $\oplus_m A_\bullet(X)_{\mathbb{Q}}$. Let us say $\mathrm{A}_\bullet(X)_{\mathbb{Q}}$ is obviously graded where the degree-m piece is exactly $\mathrm{A}_m(X)_{\mathbb{Q}}$. If X is proper over k, we have a morphism

$$\int_X : \mathrm{A}_\bullet(X)_{\mathbb{Q}} \to \mathbb{Q},$$

by first projecting to $\mathrm{A}_0(X)$ and then taking the degree.

For us Chern classes and characters are defined as morphisms, which are well-defined in the greatest generality. Following Chapter 3 [50], for any locally free sheaf $\mathcal{E}$ over X of finite rank, we have the i-th chern class as a morphism

$$\mathrm{c}_i(\mathcal{E})\cap : \mathrm{A}_\bullet(X)_{\mathbb{Q}} \to \mathrm{A}_{\bullet-i}(X)_{\mathbb{Q}}.$$

The morphism $\mathrm{c}_i(\mathcal{E})$ sends $\mathrm{A}_m(X)_{\mathbb{Q}}$ to $\mathrm{A}_{m-i}(X)_{\mathbb{Q}}$ and it does not respect the grading on $A_\bullet(X)_{\mathbb{Q}}$. The (total) Chern character $\mathrm{ch}(\mathcal{E})$ can be defined in terms of c_i (Example 3.2.3 [50]) as well, still as a morphism from $\mathrm{A}_\bullet(X)_{\mathbb{Q}}$ to itself.

We shall need a Riemann-Roch theorem for singular schemes, from Chapter 18 [50] or the original paper [11].

Theorem 5.3.1 (Theorem 18.3, Corollary 18.3.1 [50])**.** *There exists a Todd class, denoted by $\mathrm{td}(X)$, defined as $\tau(\mathcal{O}_X)$ where τ is the morphism from the Grothendieck group of coherent sheaves $\mathrm{K}_\circ(X)$ to $\mathrm{A}_\bullet(X)_{\mathbb{Q}}$ from Theorem 18.3 [50]. If X is proper over k, for any vector bundle $\mathcal{E}$ we have*

$$\chi(X, \mathcal{E}) = \int_X \mathrm{ch}(\mathcal{E}) \cap \mathrm{td}(X).$$

We give the definition of numerically trivial vector bundles.

Definition 5.3.2. *Let X be proper over k. A vector bundle $\mathcal{E}$ on X is said to be numerically trivial, if for all $i > 0$, the homomorphism*

$$\int_X \mathrm{c}_i(\mathcal{E}) \cap : \mathrm{A}_\bullet(X)_{\mathbb{Q}} \to \mathbb{Q}$$

is zero.

5.3.2 Boundedness of sheaves

In this subsection let X be an integral normal scheme, which is projective over k with a fixed very ample line bundle $\mathcal{L}$. For a coherent sheaf $\mathcal{E}$ we use $\mathcal{E}(t)$ to denote the tensor product $\mathcal{E} \otimes \mathcal{L}^t$.

Definition 5.3.3. *A family of coherent sheaves $\mathcal{E}_\iota$, $\iota \in I$ is said to be bounded if there exists a scheme S of finite type over k, together with a coherent sheaf $\mathcal{E}$ on $X \times S$, such that each $\mathcal{E}_\iota$ is isomorphic to $\mathcal{E}|_{s \times X}$ for some closed point $s \in S$.*

Clearly, whenever there is a bounded family $\mathcal{E}_\iota$, $\iota \in I$, we obtain a new bounded family $\mathcal{E}_\iota \otimes \mathcal{F}$ by tensoring with an arbitrary coherent sheaf $\mathcal{F}$.

Lemma 5.3.4. *Let $\mathcal{E}_\iota, \iota \in I$ be a bounded family of coherent sheaves. For each degree i, the dimension $h^i(X, \mathcal{E}_\iota)$ is uniformly bounded for all $\iota \in I$ (that is to say, there is a constant c, such that $h^i(X, \mathcal{E}_\iota) < C$ for all $\iota \in I$).*

Proof. Let $\mathcal{E}$ be a coherent sheaf on $X \times S$ as promised by definition.

Passing to the flatenning stratification, we find an ascending sequence of closed subschemes of S:

$$\emptyset = S_{-1} \subset S_0 \subset \ldots \subset S_n = S$$

with $\mathcal{E}|_{S_i \setminus S_{i-1}}$ is flat over $S_i \setminus S_{i-1}$, where $S_i \setminus S_{i-1}$ is endowed with the reduced scheme structure. We immediately reduce to the case that $\mathcal{E}$ is flat over S, which we assume from now on. Now the uniform boundness of h^i follows from the upper-semicontinuity of h^i, and a similar noetherian induction argument. □

The following is known as Kleiman's criterion:

Theorem 5.3.5 (Théorème 1.13 [70], Theorem 1.7.8 [60])**.** *Let $\mathcal{E}_\iota, \iota \in I$ be a family of coherent sheaves on X. The following are equivalent:*

1. *The family of sheaves is bounded.*

2. *Only finitely many polynomials occur as the Hilbert polynomial $\chi(X, \mathcal{E}_\iota(t))$ for some $\mathcal{E}_\iota, \iota \in I$. For each of these polynomial, say P, of degree d, there are constants $c_0, ..., c_d$ such that for each $\mathcal{E}_\iota$ with Hilbert polynomial P, there is an $\mathcal{E}_\iota$-regular sequence of hyperplane sections $H_1, ..., H_d$, with $h^0(\mathcal{E}_\iota|_{\cap_{j\leq i} H_j}) \leq c_i$.*

5.4 Boundedness of sheaves and cohomology

The motivation to consider F-divided sheaves from moduli point of view is due to [43]. For a F-divided sheaf $(\mathcal{E}_n)_{n\geq 0}$ over a smooth projective variety over k, the sheaves $\mathcal{E}_n (n \gg 0)$ form a bounded family. The boundedness follows from Langer's work [73][72]. This consideration leads to a solution of Gieseker's conjecture, which is a characteristic p-analogue of Malčev's theorem saying that for a smooth complex variety, the triviality of π_1^{et} implies the triviality of π_1^{diff}.

Theorem 5.4.1 ([43]). *Let X be a smooth projective variety over k. Assume $\pi_1^{et}(X)$ is trivial. Then $\pi_1^{str}(X)$ is trivial.*

Remark 5.4.2. *Interestingly, some specializations of Theorem A [59] give rise to results in similar spirits. For example, as already pointed out in [59], if in Theorem A there we let $\mathcal{M}$ be the Deligne-Mumford stack of stable curves and $\mathcal{X}$ be a connected and simply connected variety, we conclude that there is no non-constant family of stable curves over $\mathcal{X}$ which can descend arbitrarily many times along the absolute Frobenius morphism.*

5.4.1 The projective case

The goal is to give a simple proof of the following:

Proposition 5.4.3. *Let X be a normal and integral scheme which is projective over k. For any F-divided sheaf $(\mathcal{E}_n)_{n\geq 0}$, the family of sheaves $\mathcal{E}_n$ parametrized by $n \geq 0$ is bounded.*

Note Proposition 5.4.3 is by no means new. A stronger version is as follows, with elaborated proofs:

Theorem 5.4.4 (Theorem 2.1 [44], Corollary 6.9 [74]). *Let X be integral, normal and projective over k, with $j : X^{sm} \hookrightarrow X$ be the smooth locus. For any F-divided sheaf $(\mathcal{E}_n)_{n\geq 0}$ on X^{sm}, the family of sheaves $j_*\mathcal{E}_n$, $n \geq 0$, is bounded.*

Our proof is based on the arguments in Lemma 2.2 [28], Lemma 2.1, Corollary 2.2 [43]. The only modification is that, since we work with possibly singular schemes, the usual Hirzebruch-Riemann-Roch theorem should be replaced by the singular Riemann-Roch theorem.

Lemma 5.4.5. *Let $(\mathcal{E}_n)_{n\geq 0}$ be a F-divided sheaf on X. All vector bundles $\mathcal{E}_n$ are numerically trivial in the sense of Definition 5.3.2*

Proof. We have to prove

$$\int_X \mathrm{c}_i(\mathcal{E}_n) \cap \alpha = 0$$

for an arbitrary i-cycle class $\alpha \in \mathrm{A}_i(X)_{\mathbb{Q}}$. Note $\int_X \mathrm{c}_i(\mathcal{E}_n) \cap \alpha$ have bounded denominators for $n \geq 0$. Indeed we choose a large d, such that $d \cdot \alpha$ is represented by a cycle with integral coefficients. Then $\int_X \mathrm{c}_i(\mathcal{E}_n) \cap \alpha \in \frac{1}{d}\mathbb{Z}$ for all $n \geq 0$.

Since F is a finite morphism, by the projection formula (Theorem 3.2 [50]) we have

$$F_*(\mathrm{c}_i(F^*\mathcal{E}_{n+1}) \cap \alpha) = \mathrm{c}_i(\mathcal{E}_{n+1}) \cap F_*\alpha.$$

Noting $F_*\alpha = p^i\alpha$([97, Tag 0CCF]), we have

$$\int_X \mathrm{c}_i(\mathcal{E}_n) \cap \alpha = p^i \int_X \mathrm{c}_i(\mathcal{E}_{n+1}) \cap \alpha.$$

Now $\int_X \mathrm{c}_i(\mathcal{E}_n) \cap \alpha = 0$ is necessarily 0 for otherwise it is arbitrarily p-divisible due to the formula

$$\int_X \mathrm{c}_i(\mathcal{E}_n) \cap \alpha = p^{im} \int_X \mathrm{c}_i(\mathcal{E}_{n+m}) \cap \alpha$$

for all $m \geq 0$. $\square$

Lemma 5.4.6. *Let $(\mathcal{E}_n)_{n\geq 0}$ be a F-divided sheaf on X. Then we have*

$$\chi(X, \mathcal{E}_n(t)) = \mathrm{rk}\,\mathcal{E}_n \cdot \chi(X, \mathcal{O}_X(t)).$$

Proof. This is a consequence of Theorem 5.3.1. We have

$$\chi(X, \mathcal{E}_n(t)) = \int_X \mathrm{ch}(\mathcal{E}_n) \cap (1 + \mathrm{c}_1(\mathcal{L}^t)) \cap \mathrm{td}(X).$$

Unfolding the definiton $\mathrm{ch}(\mathcal{E}) = \mathrm{rk}\,\mathcal{E}_n + \mathrm{c}_1 + \frac{1}{2}(\mathrm{c}_1^2 - \mathrm{c}_2) + \ldots$ by Lemma 5.4.5. only the leading term contributes. $\square$

Proof of Proposition 5.4.3. By lemma 5.4.6 all $\mathcal{E}_n$ have the same Hilbert polynomial. It suffices to look for constants $c_0, ..., c_d$ as in Theorem 5.3.5, and in the current setting d is the $\dim X$ as well. By Bertini ([65]) we choose a sequence of hyperplane sections $H_1, ..., H_d$, with $\bigcap_{j \leq i} H_j$ still being reduced for each i. The F-divided sheaf $(\mathcal{E}_n)_{n \geq 0}$ restricts to a F-divided sheaf on $\cap_{j \leq i} H_j$ for each i, and the uniform boundness of h^0 is promised by Lemma 5.1.11. $\square$

Corollary 5.4.7. *Let X be an integral normal scheme which is projective over an algebraically closed field k of characteristic p. Let $(\mathcal{E}_n)_{n \geq 0}$ be an F-divided sheaf on X and $\mathcal{F}$ be a coherent sheaf. Then we have*

$$\sup_n h^i(X, \mathcal{F} \otimes \mathcal{E}_n) < \infty$$

for any i.

Proof. Combine Lemma 5.3.4 and Proposition 5.4.3. $\square$

5.4.2 An alternative proof of the boundedness after Maruyama

We have used a very difficult Riemann-Roch as a black box to conclude Proposition 5.4.3. Here we give an alternative but self-contained proof for Proposition 5.4.3, following a similar argument in [75] by Maruyama.

Lemma 5.4.8. *Let X be smooth and projective over k, say of dimension $d \geq 2$. Let $\mathcal{L}$ be an ample line bundle. For any coherent sheaf $\mathcal{E}$ with S_2 property we have*

$$\mathrm{H}^i(X, \mathcal{E}(-l)) = 0$$

for $i = 0, 1$ and $l \gg 0$.

Proof. There is a locally free resolution of $\mathcal{E}$ of finite length. By Auslander-Buchsbaum formula and [97, Tag 00O5] we can further assume the resolution has length $\leq d - 2$:

$$0 \to \mathcal{P}_{d-2} \to \mathcal{P}_{d-3} \to ... \to \mathcal{P}_0 \to \mathcal{E} \to 0.$$

We write $\mathcal{K}_n = \ker(\mathcal{P}_n \to \mathcal{P}_{n-1})$ for $-1 \leq n \leq d - 3$, with $\mathcal{K}_{-1} = \mathcal{E}$. By Serre's duality, Serre's vanishing and the long exact sequence of cohomology associated to the short exact sequence

$$0 \to \mathcal{K}_n \to \mathcal{P}_n \to \mathcal{K}_{n-1} \to 0,$$

we conclude inductively that for any $-1 \leq n \leq d-3$ we have

$$\mathrm{H}^i(X, \mathcal{K}_n(-l)) = 0$$

for $0 \leq i \leq n+2$ and $l \gg 0$. □

Lemma 5.4.9. *Let X be integral, normal and projective over k, of dimension $d \geq 2$, with a very ample line bundle $\mathcal{L}$. Then for any $\mathcal{E}$ with S_2 property, we have*

$$\mathrm{H}^i(X, \mathcal{E}(-l)) = 0$$

for $i = 0, 1$ and $l \gg 0$.

Proof. The very ample line bundle $\mathcal{L}$ defines a closed embedding $X \hookrightarrow \mathbb{P}^n$ for some n with $d \leq n$. In the case $d < n$ we pick up a closed point x in $\mathbb{P}^n$ with $x \notin X$. Consider composition

$$p_x : X \hookrightarrow \mathbb{P}^n \setminus \{x\} \to \mathbb{P}^{n-1},$$

where the second morphism is defined by sending a point $y \in \mathbb{P}^n \setminus \{x\}$ to the unique line passing through x and y. The morphism p_x is clearly finite. Now replacing X by $p_x(X)$ if $d < n-1$ and iterating, we obtain a finite morphism

$$f : X \to \mathbb{P}^d,$$

with $f^*\mathcal{O}_{\mathbb{P}^d}(1) \cong \mathcal{L}$.

We claim $f_*\mathcal{E}$ has property S_2 on $\mathbb{P}^d$. Since $f_*\mathcal{E}$ is obviously torsion free there is an open subscheme V of $\mathbb{P}^d$ such that $\mathbb{P}^d \setminus V$ has codimension ≥ 2 in $\mathbb{P}^d$ and $f_*\mathcal{E}|_V$ is locally free. We write $U = f^{-1}(V)$ with the following Cartesian diagram

$$\begin{tikzcd} U \arrow[r, "f'"] \arrow[d, "j'"] & V \arrow[d, "j"] \\ X \arrow[r, "f"] & \mathbb{P}^d \end{tikzcd}$$

Note we have $j'_*j'^*\mathcal{E} = \mathcal{E}$ since $\mathcal{E}$ has property S_2. We have

$$j_*j^*f_*\mathcal{E} = j_*f'_*j'^*\mathcal{E} = f_*j'_*j'^*\mathcal{E} = f_*\mathcal{E}$$

from which we conclude.

Finally we have $\mathrm{H}^i(X, \mathcal{E}(-l)) = \mathrm{H}^i(\mathbb{P}^d, f_*\mathcal{E}(-l))$ and we finish the proof by Lemma 5.4.8. □

Alternative proof of Proposition 5.4.3. The case of dimension 1 is clear.

We apply induction on $d = \dim X \geq 2$ to prove the following: for any F-divided sheaf $(\mathcal{E}_n)_{n\geq 0}$ the family of sheaves $\mathcal{E}_n$ for $n \geq 0$ is bounded, and there is an $s \gg 0$ with $\mathrm{H}^i(X, \mathcal{E}_n(-l)) = 0$ for any $n \geq 0$, $i = 0, 1$ and $l \geq s$. For $d = 2$ and X being a normal surface, it suffices to prove that there are only finitely many Euler characteristic $\chi(X, \mathcal{E}_n)$. Actually all $\chi(X, \mathcal{E}_n)$ are the same since we can fix a resolution of singularties $f : \tilde{X} \to X$, and $\chi(\tilde{X}, f^*\mathcal{E}_n) = \chi(X, \mathcal{E}_n)$ (noting $Rf_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$). The vanishing for H^0 and H^1 follows from Serre's duality since normal surfaces are Cohen-Macaulay.

Now assume $d \geq 3$.

Using the same argument as Proposition 2.3 [43], we can drop finitely many leading terms of $(\mathcal{E}_n)_{n\geq 0}$ and assume $\mathcal{E}_n$ are all semistable of degree 0.

By Bertini, choose a hyperplane section $H \hookrightarrow X$ of a very ample line bundle $\mathcal{L}$ which is still integral normal with a very ample line bundle given by $\mathcal{L}|_H$. By induction hypothesis we know $(\mathcal{E}_n|_H)_{n\geq 0}$ is bounded, there exists a $t > 0$ such that $\mathrm{H}^i(H, \mathcal{E}_n|_H(l)) = 0$ for any $n \geq 0$, $i \geq 1$, and $l \geq t$. In particular this implies that $\mathrm{H}^i(X, \mathcal{E}_n(l)) = 0$ for any $n \geq 0$, $i \geq 2$ and $l \geq t$. To see this, consider the short exact sequence

$$0 \to \mathcal{E}_n(l-1) \to \mathcal{E}_n(l) \to \mathcal{E}_n|_H(l) \to 0, l > t$$

we conclude that for $i \geq 2$, $\mathrm{H}^i(X, \mathcal{E}_n(t)) = \mathrm{H}^i(X, \mathcal{E}_n(t+1)) = \ldots$, which should all be 0 by Serre's vanishing.

We prove the following: the dimention $h^0(X, \mathcal{E}_n(t))$ is bounded uniformly for $n \geq 0$ and there are only finitely many polynomials occuring as the Hilbert polynomial for some $\mathcal{E}_n(t)$. We also prove there is an $s \gg 0$ with $H^i(X, \mathcal{E}_n(-l)) = 0$ for any $n \geq 0$, $i = 0, 1$ and $l \geq s$. These are sufficient to conclude by invoking the induction hypothesis on H.

We write down the Hilbert polynomials of $\mathcal{E}_n(t)$ in a slightly different way. There are integral coefficients $a_0^n, \ldots, a_d^n$ with

$$\chi(X, \mathcal{E}_n(l+t)) = \sum_{i=0}^{d} a_i^n \binom{l+d-i}{d-i},$$

and now the difference polynomials look much simpler as

$$\chi(H, \mathcal{E}_n|_H(l+t)) = \sum_{i=0}^{d-1} a_i^n \binom{l+d-1-i}{d-1-i}.$$

By induction hypothesis only finite many sequences of integers occuring as $(a_0^n, ..., a_{d-1}^n)$ for some n. Since we have $h^0(X, \mathcal{E}_n(t)) - h^1(X, \mathcal{E}_n(t)) = \sum_{i=0}^d a_i^n$, to see only finitely many numbers appearing as some a_d^n, we only have to give uniform upperbounds for $h^0(X, \mathcal{E}_n(t))$ and $h^1(X, \mathcal{E}_n(t))$ among all $n \geq 0$.

We have $\mathrm{H}^0(X, \mathcal{E}_n(-l)) = 0$ for any $n \geq 0$ and $l > 0$ since $\mathcal{E}_n$ are all semistable of degree 0, and therefore we have $h^0(X, \mathcal{E}_n(t))$ uniformly bounded from above for $n \geq 0$ because of the inequality

$$h^0(X, \mathcal{E}_n(t)) \leq \sum_{i=0}^{t} h^0(H, \mathcal{E}_n|_H(i)).$$

It remains to prove that $h^1(X, \mathcal{E}_n(t))$ is uniformly bounded from above for $n \geq 0$. By induction hypothesis, there is a sufficiently large $s \gg 0$, with $\mathrm{H}^i(H, \mathcal{E}_n|_H(-l)) = 0$ for any $n \geq 0$, $i = 0, 1$ and $l \geq s$, which implies that

$$\mathrm{H}^1(X, \mathcal{E}_n(-s)) = \mathrm{H}^1(X, \mathcal{E}_n(-s-1)) = ...$$

By Lemma 5.4.9 we have $H^1(X, \mathcal{E}_n(-l)) = 0$ for sufficiently large l so we necesarily have $\mathrm{H}^1(X, \mathcal{E}_n(-l)) = 0$ for any $n \geq 0$, $l \geq s$. We have

$$\begin{aligned} h^1(X, \mathcal{E}_n(t)) &= h^1(X, \mathcal{E}_n(t)) - h^1(X, \mathcal{E}(-s) \\ &\leq \sum_{i=-s+1}^{t} (h^1(H, \mathcal{E}_n|_H(i)) + h^0(H, \mathcal{E}_n|_H(i))) \end{aligned}$$

which is bounded from above uniformly for $n \geq 0$. □

5.4.3 Cohomological boundedness in general

We will use the following naive bound for a spectral sequence. It will be later applied to Leray spectral sequences of maps.

Lemma 5.4.10. *Consider a spectral sequence at the second page in the first quadrant*

$$\mathrm{E}_2^{p,q} \Rightarrow \mathrm{H}^{p+q},$$

whose terms are all finite dimensional spaces over k, and that is bounded in the sense that $\mathrm{E}_2^{s,t} \neq 0$ only if $s \leq M$ and $t \leq N$. We have the inequalities

$$\mathrm{dim}\mathrm{H}^n \leq \mathrm{dim}\mathrm{E}_2^{n,0} + \sum_{0 \leq i \leq n-1} \mathrm{dim}\mathrm{E}_2^{i,(n-i)}$$

and

$$\mathrm{dim}\mathrm{E}_2^{n,0} \leq \mathrm{dim}\mathrm{H}^n + \sum_{2\leq i\leq N} \mathrm{dim}\mathrm{E}_2^{n-i,i-1}.$$

Proof. The first inequality is trivial. For the second note

$$\mathrm{dim}\mathrm{E}_{i+1}^{n,0} \geq \mathrm{dim}\mathrm{E}_i^{n,0} - \mathrm{dim}\mathrm{E}_i^{n-i,i-1} \geq \mathrm{dim}\mathrm{E}_i^{n,0} - \mathrm{dim}\mathrm{E}_2^{n-i,i-1}.$$

The spectral sequence necessarily degenerates at $(N+1)$-th page and we have

$$\mathrm{dim}\mathrm{H}^n \geq \mathrm{dim}\mathrm{E}_{N+1}^{n,0} \geq \mathrm{dim}\mathrm{E}_N^{n,0} - \mathrm{dim}\mathrm{E}_2^{n-N,N-1} \geq ... \geq \mathrm{dim}\mathrm{E}_2^{n,0} - \sum_{2\leq i\leq N} \mathrm{dim}\mathrm{E}_2^{n-i,i-1}.$$

□

Theorem 5.4.11. *Let X be a scheme which is proper over an algebraically closed field k of characteristic p. Let $(\mathcal{E}_n)_{n\geq 0}$ be an F-divided sheaf on X and $\mathcal{F}$ be a coherent sheaf on X. Then we have*

$$\sup_n h^i(X, \mathcal{F}\otimes\mathcal{E}_n) < \infty$$

for any i.

Proof. In the proof, we use

$$\lfloor X, (\mathcal{E}_n)_{n\geq 0}, \mathcal{F}\rfloor$$

to denote the property

$$\sup_{i\geq 0, n\geq 0} h^i(X, \mathcal{F}\otimes\mathcal{E}_n) < \infty$$

as in the statement of Theorem 5.4.11.

Step 1. *We have $\lfloor X, (\mathcal{E}_n)_{n\geq 0}, \mathcal{F}\rfloor$ whenver X is projective, normal and integral. This is Corollary 5.4.7.*

We apply induction on $d = \mathrm{dim}X$. The case of $d = 0$ is trivial.

Step 2. *It suffices to prove $\lfloor X, (\mathcal{E}_n)_{n\geq 0}, \mathcal{F}\rfloor$ for reduced schemes X. To prove this, let $\mathcal{I}$ be the sheaf of ideals defining the reduced scheme structure $r : X^{red} \hookrightarrow X$. We have a finite ascending sequence*

$$0 = \mathcal{I}^m\mathcal{F} \subset \mathcal{I}^{m-1}\mathcal{F} \subset ... \subset \mathcal{I}^0\mathcal{F} = \mathcal{F},$$

and it is clear that it suffices to check the property $\lfloor X, (\mathcal{E}_n)_{n\geq 0}, \mathcal{I}^l\mathcal{F}/\mathcal{I}^{l+1}\mathcal{F}\rfloor$ for $0 \leq l \leq m-1$, which is equivalent to the property $\lfloor X^{red}, (r^*\mathcal{E}_n)_{n\geq 0}, r^*(\mathcal{I}^l\mathcal{F}/\mathcal{I}^{l+1}\mathcal{F})\rfloor$ for $0 \leq l \leq m-1$.

Step 3. *It suffices to check $\lfloor X, (\mathcal{E}_n)_{n\geq 0}, \mathcal{F}\rfloor$ for normal integral schemes X. By Step 2 we assume X is reduced and consider the normalization map $\nu : X^\nu \to X$. The morphism ν is finite, since the normalization maps are finite for Nagata schemes ([97, Tag 035S]) and schemes of finite type over a field are Nagata ([97, Tag 035B]). By [97, Tag 0BXC] we have an open subset U of X containing all generic points of X over which ν is an isomorphism. We have $\lfloor X, (\mathcal{E}_n)_{n\geq 0}, \mathcal{F}\rfloor$ to be equivalent to $\lfloor X, (\mathcal{E}_n)_{n\geq 0}, \nu_*\nu^*\mathcal{F}\rfloor$ since the canonical morphism $\mathcal{F} \to \nu_*\nu^*\mathcal{F}$ restricts to an isomorphism on U and we apply induction to its kernel and cokernel, which have supports of dimension $< d$. The property $\lfloor X, (\mathcal{E}_n)_{n\geq 0}, \nu_*\nu^*\mathcal{F}\rfloor$ is exactly $\lfloor X^\nu, (\nu^*\mathcal{E}_n)_{n\geq 0}, \nu^*\mathcal{F}\rfloor$ since ν is finite.*

Step 4. *We assume X is integral and normal. By Chow's lemma there exists $f : X' \to X$ with X' being integral, normal and projective over k. There exists an open subset U of X over which f is an isomorphism. Consider the Leray spectral sequence for f and the sheaf $f^*\mathcal{F} \otimes f^*\mathcal{E}_n$:*

$$\mathrm{E}_2^{pq} = \mathrm{H}^p(X, R^q f_* f^*\mathcal{F} \otimes \mathcal{E}_n) \Rightarrow \mathrm{H}^{p+q}(X', f^*\mathcal{F} \otimes f^*\mathcal{E}_n).$$

We have that $\lfloor X, (\mathcal{E}_n)_{n\geq 0}, \mathcal{F}\rfloor$ is equivalent to $\lfloor X, (\mathcal{E}_n)_{n\geq 0}, f_*f^*\mathcal{F}\rfloor$ since the canonical morphism $\mathcal{F} \to f_*f^*\mathcal{F}$ restricts to an isomorphism on U and we apply induction to its kernel and cokernel, which have supports of dimension $< d$. Note we have $R^q f_* = 0$ over U for $q > 0$. Applying induction to $\lfloor X, (\mathcal{E}_n)_{n\geq 0}, R^q f_* f^*\mathcal{F}\rfloor$ for $q > 0$ and by Lemma 5.4.10 we see that $\lfloor X, (\mathcal{E}_n)_{n\geq 0}, f_*f^*\mathcal{F}\rfloor$ is equivalent to $\lfloor X', (f^*\mathcal{E}_n)_{n\geq 0}, f^*\mathcal{F}\rfloor$. Finally $\lfloor X', (f^*\mathcal{E}_n)_{n\geq 0}, f^*\mathcal{F}\rfloor$ holds by Step 1.

□

Corollary 5.4.12. *Let X be a smooth proper scheme over k. For any F-divided sheaf $(\mathcal{E}_n)_{n\geq 0}$ we have*

$$\sup_n h^i(X, \mathcal{E}_n) < \infty$$

for any i.

5.5 The finiteness of $\mathcal{D}$-module cohomology

Still let X be a scheme over k and let $(\mathcal{E}_n)_{n\geq 0}$ be a F-divided sheaf. We consider the n-fold Frobenius twist $H^i(X, \mathcal{E}_n) \otimes_{k,F^n} k$, with $\alpha x \otimes \beta = x \otimes \alpha^{p^n}\beta$ and the k-linear structure given by the right tensor factor. The natural morphism between cohomology

$$\mathrm{H}^i(X, \mathcal{E}_{n+1}) \to \mathrm{H}^i(X, F^*\mathcal{E}_{n+1}) \cong \mathrm{H}^i(X, \mathcal{E}_n),$$

can be k-linearized as a morphism

$$\mathrm{H}^i(X, \mathcal{E}_{n+1}) \otimes_{k,F^{n+1}} k \to \mathrm{H}^i(X, \mathcal{E}_n) \otimes_{k,F^n} k,$$

and we obtain an inverse system $\{\mathrm{H}^i(X, \mathcal{E}_n) \otimes_{k,F^n} k\}_{n \geq 0}$.

Theorem 5.5.1 (Theorem 2.4 [86])**.** *Let X be smooth over k. Let $\mathcal{E}$ be an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module and $(\mathcal{E}_n)_{n\geq 0}$ be the corresponding F-divided sheaf on X with $\mathcal{E}_0 = \mathcal{E}$ (Theorem 5.1.9). For any i, we have an exact sequence*

$$0 \to \mathrm{R}^1 \varprojlim_n \mathrm{H}^{i-1}(X, \mathcal{E}_n) \otimes_{k,F^n} k \to \mathrm{H}^i_{\mathcal{D}_X}(X, \mathcal{E}) \to \varprojlim_n \mathrm{H}^i(X, \mathcal{E}_n) \otimes_{k,F^n} k \to 0. \tag{12}$$

Remark 5.5.2. *The $\mathcal{D}$-module cohomology may not be finite dimensional for a non-proper smooth scheme. For example, by Theorem 5.5.1 it is not the case for* $\mathrm{H}^1_{\mathcal{D}}$ *of* $X = \operatorname{Spec} k[x]$.

We are ready to prove the main theorem.

Theorem 5.5.3. *Let X be a smooth proper scheme over an algebraically closed field of characteristic p. For any $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module $\mathcal{E}$ and any cohomology degree i, the $\mathcal{D}_X$-module cohomology $\mathrm{H}^i_{\mathcal{D}_X}(X, \mathcal{E})$ is finite dimensional as a k-vector space.*

Proof. We use $(\mathcal{E}_n)_{n\geq 0}$ to denote the F-divided sheaf corresponding to $\mathcal{E}$. Consider the exact sequence (12) in Theorem 5.5.1. Since $\mathrm{H}^{i-1}(X, \mathcal{E}_n)$ are all of finite dimension. The inverse system

$$\mathrm{H}^{i-1}(X, \mathcal{E}_n) \otimes_{k,F^n} k, n \geq 0$$

verifies the Mittag-Leffler condition (Definition 3.5.6 [102]) and the first derived limit $\mathrm{R}^1 \varprojlim_n \mathrm{H}^{i-1}(X, \mathcal{E}_n) \otimes_{k,F^n} k = 0$ (Proposition 3.5.7 [102]) .

The inverse limit $\varprojlim_n \mathrm{H}^i(X, \mathcal{E}_n) \otimes_{k,F^n} k$ is finite dimensional due to Corollary 5.4.12 and Lemma 5.5.4 below. □

Lemma 5.5.4. *Suppose we have an inverse system of finite dimensional vector spaces over k indexed by the category $\mathbb{N}$. Namely, we have a finite dimensional vector space V_i for each $i \in \mathbb{N}$ and an arrow $f_{ij} : V_i \to V_j$ whenver $i > j$. Suppose we have*

$$\sup_i \dim_k V_i < \infty,$$

then the dimension of $\varprojlim_i V_i$ is bounded by $\sup_i \dim_k V_i$.

Proof. For each i we use V_i^s to denote the intersection

$$V_i^s = \cap_{j>i} \mathrm{Im}\, f_{ji},$$

called the stable subspace of V_i. Then obviously the subspaces V_i^s form an inverse system indexed by $\mathbb{N}$ as well, with transition maps all surjective, and

$$\varprojlim_i V_i^s = \varprojlim_i V_i.$$

The transition map $V_{i+1}^s \to V_i^s$ is an isomorphism for any sufficiently large i, otherwise due to the surjectivity of these transition maps the dimension of V_i^s will be unbounded, contradicting to the assumption. The dimension of the inverse limit is also bounded by the dimension of V_i^s for sufficiently large i, hence bounded by $\sup_i \dim_k V_i$. $\square$

Remark 5.5.5. *The* 0*-th* $\mathcal{D}_X$*-module cohomology has a relative analogy named as the Gauss-Manin stratification in [92] where the author proves a finiteness result without even assuming the family is proper (see lemma 1.5 [92]).*

Remark 5.5.6. *We refer to [10] for some explicit computations for* $\mathcal{D}$*-module cohomology over curves.*

6 Derived categories of sheaves

Convention 6.0.1. *We fix an algebraically closed field k and all varieties are defined over k.*

Notation 6.0.2. *We fix some notations.*

Line bundle: *let X be a smooth variety and D be a Weil divisor. We use interchangeably $\mathcal{O}_X(D)$ or $\mathcal{L}(D)$ for the line bundle represented by D.*

Derived category: *for a variety X we use $D^b_c(X)$ to denote the bounded derived category of coherent sheaves.*

Triangulated subcategory: *let X be a variety. For a set S of $D^b_c(X)$, we use $\langle S \rangle$ to denote the smallest full triangulated subcategory of $D^b_c(X)$ containing S such that $\langle S \rangle$ is stable under taking direct summands and direct sums.*

Floor function: *for a real number x, we use $\lfloor x \rfloor$ to denote the largest integer which is not larger than x. We write $\{x\} = x - \lfloor x \rfloor$.*

Pairing: *we use $(v, u), v \in V, u \in V^*$ to denote the paring $V \times V^* \to \mathbb{R}$ for a finite dimensional real vector space V.*

6.1 Overview

Let X be a smooth toric variety over k. For each positive integer m, there exists an m-th Frobenius map, $[m]$, which restricts to the m-th power on the open torus. We fix a $\mathbb{Q}$-divisor $D = \sum c_i D_i$ where D_i are toric prime divisors on X. For sufficiently divisible m, we have that mD is integral and we look at $[m]_*(\mathcal{O}_X(mD))$. By [100] the pushforward splits into a direct sum of line bundles, and for m sufficiently divisible the collection of isomorphism classes of these line bundles becomes stable, which we call the generalized Thomsen collection associated to D, denoted by $T(X, D)$. For simplicity, we write $T(X) = T(X, 0)$.

For a smooth toric variety X, it is claimed without proof in [21] by Bondal that the Thomsen collection $T(X)$ generates the bounded derived category of coherent sheaves $D^b_c(X)$. The claim is proved for 2-dimensional toric Deligne-Mumford stacks in [88]. The case of toric Fano threefolds is treated in [101], and for toric Fano fourfolds, the conjecture is settled in [93]. The claim is recently resolved completely in [57] as a consequence of their resolution of toric subvarieties in terms of line bundles from $T(X)$, and in [47][48] using explicit ideas from homological mirror symmetry.

We aim to examine Bondal's claim from birational point of view. We include some partial results. We hope to settle Bondal's claim following presented ideas in the sequel.

6.2 Preliminaries in toric geometry

In this section we recall some general facts in toric geometry. We refer to [49] for more details. We also review some results around the Thomsen collection.

General construction

Let $N = \mathbb{Z}^n$ be a free abelian group of rank and we regard N as a lattice in the n-dimentional real Euclidean space $V = N \otimes_{\mathbb{Z}} \mathbb{R}$. We use M to denote the dual of N and still identify it as a lattice of the dual vector space V^*. A rational convex polyhedral cone σ in V is a cone

$$\sigma = \{a_1 v_1 + ... a_s v_s | a_i \geq 0, 1 \leq i \leq s\}$$

spanned by some $v_1, ..., v_s$ in N. The set $v_1, ..., v_s$ is said to be a set of generators for σ. A cone σ is said to be strictly convex if $\sigma \cap (-\sigma) = \{0\}$. For each cone σ we use $\sigma^\vee$ to denote

$$\sigma^\vee = \{u \in V^* | (\sigma, u) \geq 0\}.$$

For a cone σ, a face of it is a subset τ of the form

$$\tau = \{v \in \sigma | (v, u) = 0\}$$

for some $u \in \sigma^\vee$.

A fan Σ in V is a collection of strictly convex rational polyhedral cones $\Sigma = \{\sigma\}$, such that

1. For any cone $\sigma \in \Sigma$ all of its faces belong to Σ.

2. The intersection of two cones is a face of each other.

Starting from a fan Σ, we can construct a variety X_Σ, the so-called toric variety associated to Σ, with a torus T as an open dense subset and on which T acts. To be concrete, for each cone σ we have an affine variery $X_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$. In particular we can take $\sigma = \{0\}$ and the torus T is recovered as $X_{\{0\}}$. For a face τ

of σ, the natural morphism $X_\tau \to X_\sigma$ is an open immersion. We conclude that all X_σ can be patched into a single variety, which is X_Σ. The toric variety X_Σ arising from a fan Σ is necessarily normal. Note X_σ is itself a toric variety, which is the toric variety associated to the fan generated by all faces of σ.

Closed subvarieties

We refer to Chapter 3.1 [49] for more details. For each cone $\sigma \in \Sigma$ we consider the set $star(\sigma)$ consisting of cones in Σ admitting σ as a face. Considering the linear subspace V' of V spanned by σ and projecting $star(\sigma)$ onto V/V', we obtain a fan in V/V', and the associated toric variety Y_σ can be regarded as a torus invariant closed subvariety of X_Σ in the sense that there is a locally closed torus orbit O_σ associated to the cone σ in X_Σ and Y_σ is isomorphic to the Zariski closure of O_σ in X_Σ. Note Y_σ is of dimension $\mathrm{dim}X - \dim\sigma$.

Frobenius

Pick up an integer $m \geq 1$ and consider multiplication by m on the monoid $\sigma^\vee \cap M$, which defines a morphism

$$[m] : X_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M] \to \operatorname{Spec} k[\sigma^\vee \cap M]$$

which can still be patched into a morphism

$$[m] : X_\Sigma \to X_\Sigma.$$

The morphism is sometimes called a Frobenius morphism, as if the ground field is of characteristic p and we take $m = p$, the morphism $[m]$ can be identified with the relative Frobenius map in the usual sense.

Two well-known facts

Proposition 6.2.1 ([49])**.** *The toric variety X_Σ is proper over k if and only if $\cup_{\sigma\in\Sigma}\sigma = V$.*

Proposition 6.2.2 ([49])**.** *The toric variety X_Σ is smooth over k, if and only if, for any non-zero cone $\sigma \in \Sigma$, it has a set of generators which can be completed into a basis of N.*

Divisors

We consider a fan Σ giving rise to a smooth toric variety X_Σ. We also call 1-dimensional strictly convex cones in Σ rays. Let us say there are r rays $\rho_1, ..., \rho_r$. For each ρ_i there is a primitive generator v_i in the sense that the coordinates of v_i are coprime. By the discussion on closed subvarieties above, each ray ρ_i defines a Weil divisor D_i of X. The group $Div_T\, X$ of torus invariant Weil divisors are generated by D_i, $i = 1, ..., r$. The divisor classes of D_i, $i = 1, ..., r$, generate the Picard group $Pic\, X$ subject to the relation $\sum_{1\le i\le r}(v_i, u)D_i = 0$ for all $u \in M \subset V^*$.

Toric blow-up

Still fix a fan Σ giving rise to a smooth toric variety X_Σ. Consider a cone σ of dimension $s \geq 2$, the corresponding closed variety Y_σ and the blow-up $Bl_{Y_\sigma} X_\Sigma$. The blow-up is toric and is associated to a new fan $\widetilde{\Sigma}$. The exceptional divisor is torus invariant, hence corresponds to a ray in $\widetilde{\Sigma}$ we now describe. Say σ has generators $v_1, ..., v_s$ which can be completed into a basis of N, the ray in $\widetilde{\Sigma}$ corresponding to the exceptional divisor is $\mathbb{R}_{\geq 0}(v_1 + ... + v_s)$.

Thomsen collection

The theorem below firstly appears in [100] and we state it in a slightly different form following [4]. Suppose a fan Σ gives rise to a smooth toric variety X_Σ, with r rays. For each $1 \leq i \leq r$ there is the ray ρ_i with primitive generator v_i defining a Weil divisor D_i of X_Σ.

Theorem 6.2.3 ([100] [4])**.** *Let X be a smooth toric variety over k. For any integer $m \geq 1$ and a line bundle $\mathcal{L} \in Pic\, X$, we have*

$$[m]_*\mathcal{L} = \bigoplus_{\mathcal{L}' \in Pic\, X} \mathcal{L}'^{\oplus m(\mathcal{L}', \mathcal{L})},$$

where the multiplicity $m(\mathcal{L}', \mathcal{L})$ is the number of points $(l_1, ..., l_r)$ in the cube $\{0, 1, ..., m-1\}^r \subset \mathbb{Z}^r$ such that the divisor $l_1D_1 + ... + l_rD_r$ represents $\mathcal{L} - m\mathcal{L}'$.

Remark 6.2.4. *In this remark we assume the characteristic of k is $p > 0$. A partial converse to Theorem 6.2.3 is in [4]. Namely, for a smooth projective variety over k, the splitness of Frobenius pushforward of all line bundles implies that X is toric. Another interesting result is from [94], saying that a smooth projective* *variety X over k with non-negative Kodaira dimension is an ordinary abelian variety if and only if the pushforwards of $\mathcal{O}_X$ along all powers of the Frobenius map split into line bundles.*

Now we fix a $\mathbb{Q}$-divisor $\sum_{1\leq i\leq r} c_iD_i$. Choose sufficiently divisible m so that mD is integral. We take $\mathcal{L} = \mathcal{O}_X(mD)$ and unfold the description in Theorem 6.2.3, we see that a line bundle $\mathcal{L}(\sum_{1\leq i\leq r} a_iD_i)$ is a direct summand of $[m]_*\mathcal{O}_X$ if and only if there exists $l_1, ..., l_r$ in the cube $\{0, 1, ..., m-1\}^r \subset \mathbb{Z}^r$, $u \in M$ and an equality of divisors:

$$m\sum_{1\leq i\leq r} c_iD_i - m\cdot\sum_{1\leq i\leq r} a_iD_i = \sum_{1\leq i\leq r}((v_i, u) + l_i)D_i,$$

which is clearly equivalent to the equality

$$\sum_{1\leq i\leq r} a_iD_i = \sum_{1\leq i\leq r}((v_i, -\frac{u}{m}) + c_i - \frac{l_i}{m})D_i.$$

Summarizing the above discussion, we see that $\mathcal{L}(\sum_{1\leq i\leq r} a_iD_i)$ is a direct summand of $[m]_*\mathcal{O}_X$ if and only if there exists $u \in \frac{1}{m}M \subset V^*$, with $a_i = \lfloor (v_i, u)+c_i\rfloor$ for all $1 \leq i \leq r$. We see that, the set of line bundles appearing as direct summands of $[m]_*\mathcal{O}_X(mD)$ becomes stable provided m is sufficiently divisible. This set is called the generalized Thomsen collection of X and the $\mathbb{Q}$-divisor D, denoted by $T(X, D)$. Setting $D = 0$ we write simply $T(X)$ for $T(X, 0)$, which is the classical Thomsen collection. We also obtain the following description of $T(X, D)$ without referring to a particular m.

Corollary 6.2.5. *A line bundle is in $T(X, D)$ if and only if it is of the form $\mathcal{L}(D_u)$ with*

$$D_u = \sum_{1\leq i\leq r} \lfloor (v_i, u) + c_i\rfloor D_i$$

for some $u \in V^*$. Here it suffices to take u from $M \otimes \mathbb{Q} \subset M \otimes \mathbb{R} = V^*$ or even a lattice of the form $\frac{1}{m}M$ for sufficiently divisible m.

Remark 6.2.6. *We give a stacky interpretation of the appearance of an auxiliary D. All discussion made above is valid for toric stacks, for which we refer to 4.3 [88]. Our latter discussion will be logically independent from this remark.*

Still let $D = \sum_{1\leq i\leq r} c_iD_i$ be a $\mathbb{Q}$-divisor and choose m with mD being integral. Each c_i can be written as $\frac{m_i}{m}$ for some integer m_i. Consider the stacky fan

$$\beta : \mathbb{Z}^r \to N, \beta(e_i) = mv_i, 1 \leq i \leq r$$

where e_i is the standard basis of $\mathbb{Z}^r$ and we recall v_i is the primitive generator of the ray ρ_i corresponding to D_i. This gives rise to a toric stack $\mathcal{X}$, with $\pi : \mathcal{X} \to X$ being the canonical morphism to its coarse moduli space. Without referring to toric geometry, $\mathcal{X}$ can also be obtained by taking root stack construction along $D_1, ..., D_r$, with weights all equal to m. There are tautological divisors $\mathcal{D}_i$ on $\mathcal{X}$ with $\pi^* D_i = m\mathcal{D}_i$. Consider the divisor $\mathcal{D} = \sum_{1 \le i \le r} m_i \mathcal{D}_i$ and the computation made in 4.3 [88] shows that $T(\mathcal{X}, \mathcal{D})$ consists of $\mathcal{L}(\mathcal{D}_u)$ with

$$\mathcal{D}_u = \sum_{1 \le i \le r} \lfloor (mv_i, u) + m_i \rfloor \mathcal{D}_i.$$

We claim $\pi_* \mathcal{L}(\mathcal{D}_u) = \mathcal{L}(D_u)$ where we recall

$$D_u = \sum_{1 \le i \le r} \lfloor (v_i, u) + \frac{m_i}{m} \rfloor D_i.$$

Indeed we have

$$\pi_* \mathcal{L}(\mathcal{D}_u) = \mathcal{L}(\sum_{1 \le i \le r} \lfloor \frac{\lfloor (mv_i, u) + m_i \rfloor}{m} \rfloor D_i),$$

and we conclude by noting the simple fact that

$$\lfloor \frac{\lfloor mx \rfloor}{m} \rfloor = \lfloor x \rfloor$$

for any real number x and then

$$\lfloor \frac{\lfloor (mv_i, u) + m_i \rfloor}{m} \rfloor = \lfloor (v_i, u) + \frac{m_i}{m} \rfloor$$

for each i. We conclude that $T(X, D)$ is obtained by pushing forward $T(\mathcal{X}, \mathcal{D})$ along π.

Convention 6.2.7. *By Bondal's claim for a smooth toric variety, we mean $T(X, D)$ generates $D_c^b(X)$ for any $\mathbb{Q}$-divisor D.*

In particular we recover Bondal's oringinal claim by setting $D = 0$.

Example 6.2.8. *We check directly that Bondal's claim holds for $X = \mathbb{P}^n$. Note $\mathbb{P}^n$ arises from a fan decomposition with rays of primitive generators*

$$e_0 = (-1, ..., -1), e_1 = (1, .0, ..., 0), ..., e_n = (0, 0, ..., 1).$$

Fix a $\mathbb{Q}$-divisor $D = \sum_{0 \leq i \leq n} c_i D_i$. We can assume $c_1 = ... = c_n = 0$. Note line bundles represented by

$$D_u = \lfloor c_0 - u_1 - ... - u_n \rfloor, 0 \leq u_i < 1$$

are in $T(X, D)$. The degrees of these D_u are $n+1$ consecutive integers and these $\mathcal{L}(D_u)$ generate $D^b_c(X)$.

6.3 Interlude: Orlov's theorem on blow-ups

Theorem 6.3.1 ([89][20]). *Let X be a smooth variety and Y be a smooth subvariety of codimension l. We consider the blow-up $\tilde{X} = Bl_Y X$ and the cartesian diagram:*

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{\pi}"] & \tilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "i"] & X \end{tikzcd} .$$

We write $\mathcal{O}_{\tilde{Y}}(-1) = \tilde{i}^* \mathcal{O}_{\tilde{X}}(\tilde{Y})$ and then we have a semiorthogonal decomposition

$$\langle \tilde{\pi}^* D^b_c(Y) \otimes \mathcal{O}_{\tilde{Y}}(-l+1), \tilde{\pi}^* D^b_c(Y) \otimes \mathcal{O}_{\tilde{Y}}(-l+2), ..., \tilde{\pi}^* D^b_c(Y) \otimes \mathcal{O}_{\tilde{Y}}(-1), L\pi^* D^b_c(X) \rangle$$

for $D^b_c(\tilde{X})$.

6.4 General toric blow-ups

By a half space of a finite dimensional real space W, we mean the open subset defined by $f > 0$ for some non-constant affine function. Clearly the half space depends only on the class $[f]$ of f up to a positive constant. The affine hyperplane defined by $f = 0$ is said to be the boundary of this half space.

Lemma 6.4.1. *Consider the data $(W, W^+, W_i, D_i, 1 \leq i \leq s)$ consisting of*

1. *A real vector space W of dimension ≥ 2.*

2. *An open subset W^+ of W.*

3. *Half spaces $W_1, ..., W_s$ of W defined by $[f_1], ..., [f_s]$ respectively. We assume the following:*

   (a) *The half spaces $W_1, ..., W_s$ are distinct, i.e, $[f_i] \neq [f_j]$ for $1 \leq i \neq j \leq s$.*

(b) *For each i, the boundary ∂W_i of W_i intersects with W^+.*

(c) *We allow $i \neq j$ with $[f_i] = [-f_j]$ and in this case $W_i \cap W_j = \emptyset$.*

4. *A variety X.*

5. *An effective divisor E_i of X associated to W_i for each $1 \leq i \leq s$, such that E_i have disjoint supports.*

For each point $w \in W^+$, we use E_w to denote

$$E_w = \sum_{1 \leq i \leq s, w \in W_i} E_i.$$

Then the structural sheaf $\mathcal{O}_X$ lies in the triangulated subcategory $\langle \mathcal{O}_X(-E_w), w \in W^+ \rangle$ of $D^b_c(X)$.

Proof. Let us use T to denote $\langle \mathcal{O}_X(-E_w), w \in W^+ \rangle$.

Apply induction on s. Pick up a point $w \in W^+$ and a nearby point $w' \in W_s$ with $E_{w'} = E_w + E_s$. We have a naive Koszul exact sequence

$$0 \to \mathcal{O}_X(-E_{w'}) \to \mathcal{O}_X(-E_w) \oplus \mathcal{O}_X(-E_s) \to \mathcal{O}_X \to 0,$$

from which we conclude that the complex

$$... \to 0 \to \mathcal{O}_X(-E_s) \to \mathcal{O}_X \to 0 \to ...$$

is in T. Consider the data $(W, W^+, W_i, E_i, 1 \leq i \leq s-1)$, namely, with W_s and E_s removed, and write E'_w for the divisor corresponding to $w \in W^+$. By induction hypothesis we have $\mathcal{O}_X \in \langle \mathcal{O}_X(-E'_w), w \in W^+ \rangle$ and it suffices to prove that $\mathcal{O}_X(-E'_w) \in T$. For $w \notin W_s$ this is clear as $E'_w = E_w$. For $w \in W_s$ we have $E_w = E'_w + E_s$, and we conclude by the Koszul sequence

$$0 \to \mathcal{O}_X(-E_w) \to \mathcal{O}_X(-E'_w) \oplus \mathcal{O}_X(-E_s) \to \mathcal{O}_X \to 0.$$

□

Lemma 6.4.2. *Bondal's claim holds for smooth toric varieties with no torus stratums of codimension ≥ 2.*

Proof. Let Σ be a fan in V giving rise to such a toric variety X. Let ρ_i be all rays with corresponding primitive generator v_i and prime divisor D_i. Pick up a $\mathbb{Q}$-divisor $D = \sum_i c_i D_i$ and we have a generalized Thomsen collection $T(X, D)$. Note $X_\Sigma \setminus \cup_i D_i$ is affine and the derived category of a smooth affine variety is generated by an arbitrary line bundle. It suffices to prove that for each D_i, the derived category $D^b_c(D_i)$ is generated by $T(X, D)$. For a given i, we can pick-up $u \in V^*$ such that $(v_i, u) + c_i$ is integral and for any j we have $(v_j, u) + c_j$ not to be integral unless $v_j = v_i$ or $v_j = -v_i$. For a small pertubation u' of u with $(v_i, u') < (v_i, u)$, we have $D_{u'} = D_u - D_i$. Now we have that $\mathcal{L}(D_u)|_{D_i}$ is generated by $T(X, D)$. Since by assumption D_i is affine, the derived category $D^b_c(D_i)$ is generated by $\mathcal{L}(D_u)|_{D_i}$, and then by $T(X, D)$. $\square$

Proposition 6.4.3. *Let $X = X_\Sigma$ be a smooth toric variety and $Y = Y_\sigma$ be a smooth toric subvariety corresponding to a cone σ of dimension l obtained as the complete intersection $D_1 \cap ... \cap D_l$ of prime toric divisors. Use $\tilde{X} = Bl_Y X$ to denote the blow-up of X with center Y. Let Z be the union of all closed subvarieties of X which are both of codimension ≥ 2 in X and obtained as complete intersections of prime toric divisors not from the set $\{D_1, ..., D_l\}$. Use $\tilde{Z}$ to denote the strict inverse image of Z in $\tilde{X}$. Then Bondal's claim for $X \setminus Z$ implies that for $\tilde{X} \setminus \tilde{Z}$.*

.

Proof. Recall some notations. Suppose N is the lattice defining the integral structure of the underlying vector space $V = N \otimes_{\mathbb{Z}} \mathbb{R}$ of Σ. For each i there is a ray ρ_i with primitive generator v_i, corresponding to a torus invariant prime divisor D_i of X. All toric prime divisors arise in this way. With a slight abuse of notations, we still use D_i to denote its strict inverse image on $\tilde{X}$.

We need a suitable coordinate system, i.e, a basis of N.

Step 1. *We can choose a basis $e_1, ..., e_n$ of N, such that :*

1. *σ is spanned by $e_1, ..., e_l$ with $2 \leq l \leq n$. The case $l = 1$ is excluded since it is unnecessary to blow up a smooth divisor.*

2. *For any ray ρ_i, its projection to the first l-coordinates is not the ray $\mathbb{R}_{\geq 0}(1, ..., 1)$. Note we allow the projection to be 0 or $\mathbb{R}_{\leq 0}(1, ..., 1)$.*

We first choose an arbitrary basis such that σ is spanned by the first l standard vectors. For ρ_i and its primitive vector v_i, we write $v_i = (v_i^1, ..., v_i^n)$ its coordinate under this basis. We can choose a generic tuple $(k_{l+1}, ..., k_n) \in \mathbb{Z}^{n-l}$ such that for any i with $(v_i^{l+1}, ..., v_i^n) \neq 0$, we have $v_i^1 + \sum_{l<j\leq n} k_j v_i^j \neq v_i^2$ and there is a new basis under which the coordinate of v_i becomes $(v_i^1 + \sum_{l<j\leq n} k_j v_i^j, v_i^2, ..., v_i^n)$. We use this new coordinate system. Note σ is still spanned by the first l standard vectors. For each v_i with $(v_i^{l+1}, ..., v_i^n) \neq 0$, by construction its leading two coordinates are unequal and the projection of v_i to its first l coordinates can not be in $\mathbb{R}(1, ..., 1)$. For v_i with $(v_i^{l+1}, ..., v_i^n) = 0$, its projection to the first l coordinate can not be in in $\mathbb{R}_{>0}(1, .., 1)$ since $\mathbb{R}_{\geq 0}(1, ..., 1, 0, ..., 0)$ is not a ray of Σ. It is the new ray corresponding to the exceptional divisor $\tilde{Y}$. Step 1 is finished.

We fix a coordinate system as in Step 1. We identify V^* with V using the standard dot product and therefore we make sense of the pairing (v, u) for $u, v \in V$. For any $u \in V$ we write $u = (u_1, ..., u_n)$ its coordinate with respect to this basis, i.e, $u = \sum u_i e_i$. Looking at the first l and the last $n - l$ coordinates, we have a direct sum $V = \mathbb{R}^n = \mathbb{R}^l \oplus \mathbb{R}^{n-l}$. For any element in $v \in V$ we write $v = v^1 \oplus v^2$ respecting this decomposition. Since $e_i, 1 \leq i \leq l$ are primitive generators of rays we can assume $v_i = e_i$ for $1 \leq i \leq l$.

We have the following commutative diagram

$$\begin{tikzcd} \tilde{Y} \setminus \tilde{Z} \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{\pi}"] & \tilde{X} \setminus \tilde{Z} \arrow[d, "\pi"] \\ Y \setminus Z \arrow[r, "i"] & X \setminus Z \end{tikzcd}.$$

Replacing X by $X \setminus Z$ and Y by $Y \setminus Z$, we can assume

1. The toric variety Y has no torus stratum of codimension ≥ 2.

2. For any $j, j' \neq 1, ..., l$, we have $D_j \cap D_{j'} = \emptyset$.

Now we pick up a $\mathbb{Q}$-divisor $\tilde{D} = c_0\tilde{Y} + \sum_i c_i D_i$ on $\tilde{X}$. The ray corresponding to $\tilde{Y}$ has $e_1 + ... + e_l$ as the primitive generator and the following divisors $\tilde{D}_u = \lfloor u_1 + ... + u_l + c_0 \rfloor \tilde{Y} + \sum_i \lfloor (v_i, u) + c_i \rfloor D_i, u \in V$, represent lines bundles in $T(\tilde{X}, \tilde{D})$. For $\tilde{D}$ if we drop the term of $\tilde{Y}$ we have $D = \sum_i c_i D_i$, which we consider as a $\mathbb{Q}$-divisor on X, and we have divisors $D_u = \sum_i \lfloor (v_i, u) + c_i \rfloor D_i, u \in V$ representing line bundles in $T(X, D)$. Since $v_1, ..., v_l$ are linearly independent we can assume $c_1 = ... = c_l = 0$. We can assume $0 \leq c_i < 1$ for all c_i since twisting the Thomsen collection by a line bundle in common does not affect the generation of derived categories. We consider those u with $0 \leq u_i < 1$ for

$1 \le i \le l$. In this case we have the formula

$$\tilde{D}_u = \pi^* D_u + \lfloor u_1 + ... + u_l + c_0 \rfloor \tilde{Y}. \tag{13}$$

Note that l consecutive integers can be achieved as the value of $u_1 + ... + u_l + c_0$. For $c_0 = 0$ these could be $0, ..., l-1$ and for $c_0 > 0$ these could be $1, ..., l$.

The following claim is the key point:

Step 2. *For u with $u_1 + ... + u_l + c_0$ being integral, $0 < u_i < 1$, $1 \le i \le l$, we have*

$$\mathcal{L}(\tilde{D}_u - \tilde{Y}) \in \langle T(\tilde{X}, \tilde{D}) \rangle.$$

To do this we use Lemma 6.4.1. We take $W = \mathbb{R}^l$. The open subset W^+ is defined by $u_1 + ... + u_l > 0$. The smooth variety is taken to be $\tilde{X}$. For each class $[w] \in (\mathbb{R}^l \setminus \{0\})/\mathbb{R}_{>0}$ where $w \in \mathbb{R}^l \setminus \{0\}$ is a representative, there is a half space W_j defined by $(w, \cdot) > 0$ if and only if there exists i satisfying all of the following:

1. $v_i^1 \neq 0$.
2. $[v_i^1] = [w]$.
3. $(v_i, u) + c_i$ is integral.

with the corresponding divisor E_j being the summation of all prime divisors D_i for which i verifies the above. The assumption $0 < u_i < 1$ for $1 \le i \le l$ ensures that $D_1, ..., D_l$ never appear as a component of some E_j. Note by Claim 1 in this coordinate system we have $[v_i^1] \neq [(1, .., 1)]$ for any $v_i^1 \neq 0$, so W^+ is not equal to any of these W_j.

Now for each $w \in W^+$, we consider $u' = u - \epsilon w \oplus 0$. We claim that for $\epsilon > 0$ small enough, we have

$$\tilde{D}_{u'} = \tilde{D}_u - E_w - \tilde{Y}. \tag{14}$$

For the definition of E_w we refer to the statement of Lemma 6.4.1. Indeed the coefficient $\tilde{Y}$ decreases by 1 since $c_0 + u_1 + ... + u_l$ is an integer by assumption. We have

$$(v_i^1, u'^1) = (v_i^1, u^1) - \epsilon(v_i^1, w)$$

so for small $\epsilon > 0$ the value of $\lfloor (v_i, u') + c_i \rfloor$ is different from $\lfloor (v_i, u) + c_i \rfloor$ if and only if $(v_i, u) + c_i$ itself is an integer, and $(v_i^1, w) > 0$, and in this case we have $\lfloor (v_i, u') + c_i \rfloor = \lfloor (v_i, u) + c_i \rfloor - 1$. This proves Equation 14. We finish Step 2 by Lemma 6.4.1 and Equation 14.

Now there are two cases to consider.

Case 1: $c_0 = 0$.

We fix rational numbers $\bar{u}_1, ..., \bar{u}_l$ with $0 < \bar{u}_i < 1$, $1 \leq i \leq l$, and $d = \bar{u}_1 + ... + \bar{u}_l$ being an integer with $1 \leq d \leq l-1$ and consider those $u = u^1 \oplus u^2$ with $u^1 = \bar{u^1} = (\bar{u}_1, ..., \bar{u}_l)$. By Equation 13, Step 2 and the adjunction formula, we have

$$\tilde{i}_*(\tilde{\pi}^* i^* \mathcal{L}(D_u) \otimes \mathcal{O}_{\tilde{Y}}(-d)) = \tilde{i}_* \tilde{i}^* \mathcal{L}(\tilde{D}_u) \in \langle T(\tilde{X}, \tilde{D}) \rangle. \tag{15}$$

Note for each ρ_i which is not a ray of some cone in $star(\sigma)$, the intersection of D_i with Y is empty and $\mathcal{L}(D_i)$ is trivial restricted to Y. Now we have

$$i^* \mathcal{L}(D_u) = \mathcal{L}(\sum_{i, \rho_i \in star(\sigma) \setminus \sigma} \lfloor c_i' + (v_i^2, u^2) \rfloor D_i|_Y),$$

where the summation is taken over those i, such that ρ_i is a ray of some cone in $star(\sigma)$, but not of σ, with $c_i' = (v_i^1, \bar{u^1}) + c_i$. By construction of Y_σ, its fan decomposition is defined by projection of $star(\sigma)$ to $\mathbb{R}^{n-l}$, forgetting the first l coordinates, and its set of primitive generators of rays are exactly those v_i^2. Now let u^2 exhaust all vectors in $\mathbb{R}^{n-l}$ and by Lemma 6.4.2 we have

$$\langle i^* \mathcal{L}(D_u), u^1 = \bar{u^1}, u^2 \in \mathbb{R}^{n-l} \rangle = D_c^b(Y)$$

and then

$$\tilde{i}_*(\tilde{\pi}^* D_c^b(Y) \otimes \mathcal{O}_{\tilde{Y}}(-d)) \subset \langle T(\tilde{X}, \tilde{D}) \rangle, 1 \leq d \leq l-1 \tag{16}$$

by recalling Equation 15. By Theorem 6.3.1 it remains to prove $L\pi^* D_c^b(X) \subset \langle T(\tilde{X}, \tilde{D}) \rangle$. For u with $\lfloor u_1 + ... + u_l \rfloor = 0$ we have $\pi^* \mathcal{L}(D_u) = \mathcal{L}(\tilde{D}_u)$. For $1 \leq d = \lfloor u_1 + ... + u_l \rfloor \leq l-1$, we have $\tilde{i}_* \tilde{i}^* \mathcal{L}(\tilde{D}_u) \in \tilde{i}_*(\tilde{\pi}^* D_c^b(Y) \otimes \mathcal{O}_{\tilde{Y}}(-d)) \subset \langle T(\tilde{X}, \tilde{D}) \rangle$, from which we conclude that $\mathcal{L}(\tilde{D}_u - \tilde{Y}) \in \langle T(\tilde{X}, \tilde{D}) \rangle$. Repeat the same argument we finally conclude that $\pi^* \mathcal{L}(D_u) \in \langle T(\tilde{X}, \tilde{D}) \rangle$. Since Bondal's claim holds for X, we have $L\pi^* D_c^b(X) \subset D_c^b(\tilde{X})$.

Case 2: $0 < c_0 < 1$.

In this case the only thing to note is that for any $1 \leq d \leq l$ we can find $u_1, ..., u_l$ with $0 < u_i < 1, 1 \leq i \leq l$ such that $c_0 + u_1 + ... + u_l = d$. We conclude similarly that

$$\tilde{i}_*(\tilde{\pi}^* D_c^b(Y) \otimes \mathcal{O}_{\tilde{Y}}(-d)) \subset \langle T(\tilde{X}, \tilde{D}) \rangle, 1 \leq d \leq l, \tag{17}$$

which is slightly stronger than the similar Equation 16. All remaining arguments are identical to those in Case $c_0 = 0$.

6.5 The case of toric threefolds

Lemma 6.5.1. *Let $\pi : \tilde{X} = Bl_Y(X) \to X$ be a toric blow-up with smooth center between smooth toric varieties. Bondal's claim for $\tilde{X}$ implies that for X.*

Proof. Let D be an arbitrary $\mathbb{Q}$-divisor on X. Pick up m, with $\langle [m]_* \mathcal{O}_{\tilde{X}}(m\pi^* D)\rangle = \langle T(\tilde{X}, \pi^* D)\rangle = D^b_c(\tilde{X})$ and $\langle [m]_* \mathcal{O}_X(mD)\rangle = \langle T(X, D)\rangle$. Since $[m]$ commutes with π, we have

$$D^b_c(X) = R\pi_* D^b_c(\tilde{X}) = \langle R\pi_*[m]_* \mathcal{O}_{\tilde{X}}(m\pi^* D)\rangle = \langle [m]_* R\pi_* \mathcal{O}_{\tilde{X}}(m\pi^* D)\rangle = \langle [m]_* \mathcal{O}_X(mD)\rangle,$$

where the first equality is because $R\pi_* L\pi^* = id$ by projection formula. □

Proposition 6.5.2. *The converse of Lemma 6.5.1 holds for smooth toric threefolds. Namely, Bondal's claim for X implies that for $\tilde{X}$.*

Proof. Notations are consistent with those in Theorem 6.3.1.

The center of the blow-up is either a point or a $\mathbb{P}^1$. The blow-up a point can be treated as in [88], and we deal with the latter case. Let $v_1, v_2, ..., v_r$ be all primitive generators of rays ρ_i defining correpending toric prime divisors D_i of X. Without losing generality we choose an identification $V \cong \mathbb{R}^3$ such that the cone spanned by the standard basis e_1, e_2, e_3 of $\mathbb{R}^3$ is in Σ and the blow-up center corresponds to the cone σ spanned by e_1 and e_2. We assume $v_i = e_i$ for $1 \leq i \leq 3$. The cone σ is the face of exactly two cones of maximal dimension, with one of them being spanned by v_1, v_2, v_3. We assume v_1, v_2, v_4 span the another cone. For each v_i write $v_i = (v_i^1, v_i^2, v_i^3)$ for its coordinate, i.e, $v_i = v_i^1 e_1 + v_i^2 e_2 + v_i^3 e_3$. Let $\tilde{D} = c_0 \tilde{Y} + c_1 D_1 + ... + c_r D_r$ be a $\mathbb{Q}$-divisor on $\tilde{X}$ and we can assume $c_1 = c_2 = c_3 = 0$. We pick up u_3 and w_3 such that $\lfloor c_4 + u_3 v_4^3 \rfloor$ and $\lfloor c_4 + w_3 v_4^3 \rfloor$ are two consecutive integers, which can be achieved since in fact $v_4^3 = -1$. Now we pick up u_1, u_2 with $0 < u_i < 1$ and $u_1 + u_2 = 1$ such that

1. We have
$$\{u_1 v_4^1 + u_2 v_4^2\} + \{c_4 + u_3 v_4^3\} < 1$$
and
$$\{u_1 v_4^1 + u_2 v_4^2\} + \{c_4 + w_3 v_4^3\} < 1$$
where $\{x\} = x - \lfloor x \rfloor$ for a real number x.

2. For any i with $v_i^1 \neq v_i^2$, we have $c_i + (v_i, u)$ and $c_i + (v_i, w)$ not to be integral, where we use u (resp. w) to denote (u_1, u_2, u_3) (resp. (u_1, u_2, w_3)).

We explain how these u_1, u_2 can be chosen. Note for each $\epsilon > 0$, there is an open subset of the unit interval $0 < u_i < 1$ and $u_1 + u_2 = 1$, with $0 < \{u_1 v_4^1 + u_2 v_4^2\} < \epsilon$. It suffices to fix a priori an ϵ with $\epsilon + \{c_4 + u_3 v_4^3\} < 1$ and $\epsilon + \{c_4 + w_3 v_4^3\}$. These pairs (u_1, u_2) verify the first condition, and a generic pair verifies the second condition.

It remains to prove that $\mathcal{L}(\tilde{D}_u - \tilde{Y})$ and $\mathcal{L}(\tilde{D}_w - \tilde{Y})$ are both in $\langle T(\tilde{X}, \tilde{D})\rangle$. When this is proved, we have that $\tilde{i}_* L\tilde{i}^* \mathcal{L}(\tilde{D}_u)$ and $\tilde{i}_* L\tilde{i}^* \mathcal{L}(\tilde{D}_w)$ are in $\langle T(\tilde{X}, \tilde{D})\rangle$, from which we conclude that $\tilde{i}_*(D_c^b(\tilde{Y}) \otimes \mathcal{O}_{\tilde{Y}}(-1)) \in \langle T(\tilde{X}, \tilde{D})\rangle$. We then proceed in the same way as the end of Proposition 6.4.3 to draw the final conclusion. Now we prove $\mathcal{L}(\tilde{D}_u - \tilde{Y}) \in \langle T(\tilde{X}, \tilde{D})\rangle$ and by the same argument we have $\mathcal{L}(\tilde{D}_w - \tilde{Y}) \in \langle T(\tilde{X}, \tilde{D})\rangle$.

Consider those i, with $v_i^1 = v_i^2 > 0$. Note for these i we have $v_i^3 < 0$ for otherwise the ray lies in the interior of σ, a contradiction. For these i, we have $D_i \cap D_3 = \emptyset$ since there cannot be a cone with ρ_i and ρ_3 being rays at the same time. Let E be the summation of the corresponding prime divisors for those i with $v_i^1 = v_i^2 > 0$ and $c_i + (v_i, u)$ being integral. We have $E \cap D_3 = \emptyset$. There are two cases to consider:

1. u_3 is not integral. In this case, consider $u' = (u_1 - \epsilon, u_2 - \epsilon, u_3 - \mu)$ for small $\epsilon, \mu > 0$, such that

$$\epsilon(v_i^1 + v_i^2) + \mu v_i^3 < 0$$

   for all i with $v_i^1 = v_i^2 > 0$ and we find $\tilde{D}_{u'} = \tilde{D}_u - \tilde{Y}$ and $\mathcal{L}(\tilde{D}_u - \tilde{Y})$ clearly lies in $T(\tilde{X}, \tilde{D})$.

2. u_3 is integral. In this case, consider $u' = (u_1 - \epsilon, u_2 - \epsilon, u_3 - \mu)$ for suitable small $\epsilon, \mu \geq 0$ as follows:

   (a) Choose small $\epsilon, \mu > 0$, such that

$$\epsilon(v_i^1 + v_i^2) + \mu v_i^3 < 0$$

   for all i with $v_i^1 = v_i^2 > 0$, and we find $\tilde{D}_{u'} = \tilde{D}_u - D_3 - \tilde{Y}$.

   (b) Choose small $\epsilon, \mu > 0$, such that

$$\epsilon(v_i^1 + v_i^2) + \mu v_i^3 > 0$$

   for all i with $v_i^1 = v_i^2 > 0$, and we find $\tilde{D}_{u'} = \tilde{D}_u - D_3 - E - \tilde{Y}$.

   (c) Choose small $\epsilon > 0$ and $\mu = 0$, and we find $\tilde{D}_{u'} = \tilde{D}_u - E - \tilde{Y}$.

Since $E \cap D_3 = \emptyset$, by the Koszul exact sequence we have $\mathcal{L}(\tilde{D}_u - \tilde{Y}) \in \langle T(\tilde{X}, \tilde{D}) \rangle$.

□

Corollary 6.5.3. *Bondal's claim holds for all smooth toric threefolds.*

Proof. Any smooth toric variety admits a smooth toric compactification. We reduce to the proper case. By toric weak factorization theorem ([82][1][103]), the torus equivariant proper birational map $X \dashrightarrow \mathbb{P}^n$ can be factorized into a sequence of toric blow-ups and blow-downs with smooth centers. By Lemma 6.5.1 and Proposition 6.5.2 we reduce to the case that $X = \mathbb{P}^n$, which is Example 6.2.8. □

□

6.6 Final remarks

Remark 6.6.1. *A question posed in [57] is to ask, for which divisors D we have $[m]\mathcal{O}_X(D)$ generates $D^b_c(X)$ for suitable m on a smooth toric variety X?. A precise conjecture as formulated as 5.11 [57].*

Remark 6.6.2. *An interesting result appears in [9]. Namely, let X be a noetherian scheme with $p\mathcal{O}_X = 0$. For any compact generator G of $D(X)$, there exists a large e with F^e_*G generating $D^b_c(X)$.*

Remark 6.6.3. *To generalize Proposition 6.5.2 to higher dimensions, we need to analyze $star(\sigma)$ for the cone σ with Y_σ being the blow-up center. In general, the shape of $star(\sigma)$ can be very wild. We would like to examine arguments in [82][1][103], to extract extra properties of σ.*

7 References

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