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New observers for nonlinear dynamical systems via Lie theory with application to robotics
Nouveaux observateurs pour des systèmes dynamiques non-linéaires via théorie de Lie avec des applications à la robotique

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Abstract

This thesis aims to introduce and study a class of continuous-time dynamical systems characterized by the existence of symmetries on the state, input and time. The concept of time symmetry can be encompassed by more general definitions found in the litterature, however their properties were never explored in generality through the lenses of dynamical anaysis and observer design, a shortcoming that this thesis adresses. Their importance is well justified due to their generalization of homogeneous systems and their display of exclusive features such as finite-time stability. Examples of systems possessing symmetries are considered and studied for the purposes of observer and control design, namely an extended class of Persidskii systems and a Unycicle Mobile Robot.

Résumé

Cette thèse vise à présenter et à étudier une classe de systèmes dynamiques à temps continu caractérisés par l'existence de symétries dans l'état, l'entrée et le temps. Le concept de symétrie temporelle peut être englobé dans des définitions plus générales présentes dans la littérature ; toutefois, leurs propriétés n'ont jamais été étudiées de manière générale du point de vue de l'analyse dynamique et de la conception d'observateurs, une lacune que cette thèse se propose de remédier. Leur importance est bien justifiée par leur généralisation des systèmes homogènes ainsi que par leurs caractéristiques uniques, telles que la stabilité en temps fini. Des exemples de systèmes possédant des symétries sont étudiés à des fins de conception de lois de contrôle et d'observateurs, notamment une classe étendue de systèmes de Persidskii et un robot mobile de type unicycle.

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Contents

Abstract **3**

Résumé **5**

Acknowledgement **7**

Notation **11**

List of Figures **13**

1 Introduction **15**

1.1 Motivation and State-of-the-Art 15

1.2 Preliminaries 20

2 The dynamics of θ-invariant systems **35**

2.1 θ-invariant systems 36

2.2 Normed actions 44

2.3 Convergence rate of scalar systems 50

2.4 Conclusion 52

3 Canonical form of θ-invariant observers **53**

3.1 θ-invariant input-output systems 53

3.2 θ-invariant pre-observers 55

3.3 Error dynamics 62

3.4 Conclusion 67

4 Observer design for bilinear Persidskii systems **69**

4.1 Bilinear Persidskii Systems 69

4.2 Observer design 73

4.3 SIIOS Analysis 82

4.4 Conclusion 89

5 Finite-time control using symmetries: unicycle mobile robot **91**

5.1 Models of interest 91

5.2 Control of the canonical forms 95

5.3 Control of the UMR 98

5.4 Simulations 99

5.5 Conclusion 108

Bibliography **109**

Notation

- The set of reals is denoted by $\mathbb{R}$ and $\mathbb{R}_+ := \{s \in \mathbb{R} \mid s \geq 0\}$.
- The usual Euclidean norm in $\mathbb{R}^n$ is denoted by $\|\cdot\|$.
- For a Lebesgue measurable function $u : \mathbb{R}_+ \to \mathbb{R}^\ell$, define the norm $\|u\|_{[t_1,t_2)} = \text{ess sup}_{t\in[t_1,t_2)}\|u(t)\|$ for $[t_1,t_2) \subset \mathbb{R}_+$. We denote by $\mathscr{L}_\infty^\ell$ the set of functions u such that $\|u\|_\infty := \|u\|_{[0,+\infty)} < +\infty$.
- For a nonempty set $\mathscr{X}$, we denote its identity function by $\text{Id}_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}$, that is, the function such that $\text{Id}_{\mathscr{X}}(x) = x$, for every $x \in \mathscr{X}$;
- A closed (respectively open) ball of radius $r > 0$ centered in $x_0 \in \mathbb{R}^n$ will be denoted by $B_r[x_0] := \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$ (respectively $B_\delta(x_0) := \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$).
- A matrix $A \in \mathbb{R}^{n\times n}$ is said to be positive semidefinite (denoted $A \geq 0$) if $x^T Ax \geq 0$, for every $x \in \mathbb{R}^n$. If, in addition, $x^T Ax > 0$ for every $x \neq 0$, A is said to be positive definite and such property is denoted by $A > 0$.
- The set of $n \times n$ diagonal matrices with nonnegative elements on the diagonal is denoted by $\mathbb{D}_+^n$.

List of Figures

4.1 State trajectory $M(t)$ . . . 81

4.2 Norm of the estimation error . . . 82

4.3 Norm of the estimation error for noisy output. . . . 82

4.4 Trajectory of the chemical reactor dynamic model for $w(t) = 0.2 + 0.1\cos(t)$ and three different initial conditions. . . . 88

4.5 Norm of estimation error for the trajectory with gains $L_0 = [1\,0\,0]^\top$, $R_1^2 = [-0.4439\ -0.4315\ 0.3351]^\top$ and three different initial conditions. . . . 89

4.6 Norm of error for noisy output and three distinct pairs of initial conditions. 89

5.1 Coordinate plot, phase portrait and control law of the UMR trajectories for the control law described in Theorem 20 for parameters $\gamma = 1$ and $\beta = 1$. 100

5.2 Coordinate plot, phase portrait and control law of the UMR trajectories for the control law described in Theorem 21. . . . 101

5.3 Coordinate plot, phase portrait and control law of the UMR trajectories for the control law described in Theorem 22. . . . 102

5.4 Coordinate plot, phase portrait and control law of the UMR trajectories for the control law described in Theorem 23. The parameters chosen are $a = -0.1; p_1 = p_2 = \zeta = 1; \alpha_2 = 0.001; \zeta_0 = 0.1; q_1 = 1.1; q_2 = 0.5\gamma = 0.5; T_{c_1} = 0.1; T_{c_2} = 0.9; T_{c_3} = 5$. . . . 103

5.5 Coordinate plot, phase portrait and control law of the UMR following the control of Theorem 20 with disturbance. . . . 104

5.6 Coordinate plot, phase portrait and control law of the UMR following the control of Theorem 21 with disturbance. . . . 105

5.7 Coordinate plot, phase portrait and control law of the UMR following the control of Theorem 22 with disturbance. . . . 106

5.8 Coordinate plot, phase portrait and control law of the UMR following the control of Theorem 23 with disturbance. 107

Chapter 1

Introduction

1.1 Motivation and State-of-the-Art

Estimating the state of a dynamical system is a task of extreme importance for practical applications. This is the case because often we do not have direct access to the full state of a system at a given moment in time, since it can be technically difficult or expensive to install the sensors measuring the whole state. This measuring issue permeates through the use of chemical, robotic, biological and epidemiological models [1–6]. In such situations, a measured partial state must be considered from the sensors available, together with the system model, parameters and (some or all) inputs. Having this as motivation, several techniques have been constructed to infer the full state from the partial measurements. The most fruitful technique developed in this direction is that of state observers [7–9]. A state observer is an auxiliary system that uses the outputs and inputs of the original system to track its state. The combined systems generate an error dynamics, given by the difference between the true and the observed states. The error dynamics itself can be written as a differential equation with inputs and parametrized in order to achieve convergence. This strategy has its limitations. First, not all systems are observable (for all or some inputs) or detectable. Then, second, for the ones that are observable, there are no univer-

sal algorithms to design an observer [10–13]. Nonetheless, observer design techniques have been developed for several classes of systems of practical interest, such as linear (parameter-varying) systems, Lypschitz systems, homogeneous systems and Persidskii systems [14–17].

Thus, the design of observers can be decomposed on several steps: 1) select the observer structure, which can be related with the type of the observability presented in the system [18, 19], 2) tuning of the observer gains, 3) analysis of the estimation error dynamics with assessment of the performances that are presented (as the convergence rate, robustness to delays and measurement noises, external perturbations and unmodeled dynamics). For the indicated above classes of systems these steps have been performed and some canonical forms of the observers and the respective Lyapunov functions, that can be used for the analysis, have been indicated. Frequently, the structure of the observer is chosen as the copy of the system with an additive injection term, as in the foundational works presenting the Luenberger observer [14] or the Kalman filter [20], but it is a straightforward approach and its restrictiveness is always in questions, especially for highly nonlinear systems.

In parallel to this, the theory of symmetries in differential equations was developed. It was first proposed by Sophus Lie [21], as a technique to reduce the complexity of systems and render analysis simpler. Thus was born the concept of Lie groups, which are the most useful tool to classify the symmetries of continuous systems. In its full generality, the symmetry method allow for a differential equation of order two or higher, with any number of dependent and independent variables, to have its local symmetries implicitly given as solutions of a simpler system of partial differential equations [22]. The first order case was dealt with later through different techniques [23]. The study of symmetries in dynamical systems allows to deduce stronger links between different types of stability, such as infering global behaviour from local, or insuring fast convergence of trajectories. The efficiency of real-world applications rely on those basic dynamical properties, justifying the growing interest in the symmetric approach.

The symmetry approach was also proposed as a tool for the design of observers

[24, 25]. It consists of searching for observers that share the same symmetries of the original system, the so called invariant observers. After being used for specific models with symmetries, a general method was proposed, making use of the Cartan moving frame method [26] (see also [27]). More than one symmetry method was proposed for observer design, according to whether the symmetry group dimension is smaller or larger than the dimension of the state space [28, 29]. The general method allows a full characterization of the possible invariant observers of a system, provided their symmetries are given. Nonetheless, the methods provided do not account for symmetries in the time coordinate, which leaves behind important examples like the class of homogeneous systems [16]. Another disadvantage of current methods is the fact that the error dynamics itself, although simplified in order, does not necessarily share the same symmetries as the system. This is a feature that naturally arises in linear and homogeneous systems and make analysis simpler [14]. In Chapters 2 and 3 we build upon previous works to improve this two aspects, by allowing for an extra symmetry on the time coordinate and slighlty changing the error measurement to render its dynamics invariant [LimaCDC24, LimaCDC25].

Some of the models used in the context of invariant observer design are also studied under more traditional methods [2, 3, 27]. In the later chapters, we have pursued a deeper study of those.

The first is a class of models, called the Persidskii sytems. They were first introduced in [30], [31] for stability analysis, and they are also related to Lur'e systems studied in the absolute stability theory [32], [33] (the difference is for Persidskii systems the presence of the linear part is not obligatory). The benefit of Persidskii systems lies in the existence of well-investigated candidate Lyapunov functions containing the integrals of sector nonlinearities, used in many cases [34], [17], [35], usually resulting in stability conditions that can be formulated as linear matrix inequalities. This class of models has been practical in representing numerous physical and biological phenomena, and has been studied from many different viewpoints, including the observer design in the IOS framework [17], see also [36–38].

We will then consider here a significant extension of the class of systems investi-

gated in [17], motivated by biological and physical examples [39], [1], [27], including the addition of bilinear cross-products. For bilinear systems with cross-terms in the state and input only, many solutions exist in the literature, see for instance, pioneering works [40], [41], [42] (this case can be easily reduced to analysis of linear time-varying systems). Frequently, the state cross-terms can also be hidden in the state matrix, transforming the system into a similar linear time-varying form, as in [43], [44] or to a quasi-LPV model [45] and nonlinear cases are rarely studied. In Chapter 4, we aim to propose a state observer using the IOS theory, assuming the presence of products of the states and nonlinear functions and also that nonlinear state measurements are possible. Two kinds of design are given, providing different IOS characteristics. First, we will establish local stability conditions not based on linearization. Then, we will develop a global condition assuming that the trajectories of the system have positive coordinates (this restriction is motivated by chemical and biological applications) [LimaECC24, LimaIJC25].

The last part of the thesis is devoted to the analysis of importance of symmetries for a robotic application, where however the control problem is studied [LimaTAC25]. To this end, the Unicycle Mobile Robot (UMR) system is considered, that corresponds to a car-like robot. It is one of the simplest robotic models and yet it possesses a rich dynamics, being invariant under the special euclidean group $SE(2)$. The UMR has been a subject of control theory research for the past decades due to the simplicity of its model and its practicality in real-world locomotion problems [46]. Despite the straightforwardness of its equations, regulating its dynamics can be challenging, notably by the presence of a non-holonomic constraint, which imposes difficulty in designing continuous and robust control laws [47].

The UMR system is an example of a non-holonomic dynamic model that represents a car-like robot [3]. Such models dispose of natural canonical forms, called *chained forms* [48, 49], which can be obtained through a smooth one-to-one change of coordinates in the state and input spaces. In some cases, these forms make the control strategy easier [4, 50] since they possess different kinds of symmetries that can be utilized in the analysis and control design. It is important to note that both the unicycle model and the

canonical forms of non-holonomic integrators have topological obstacles when developing continuous control feedbacks to stabilize these systems at the origin, as shown with the failure of Brocket's conditions on stabilizability [47, 51]. It was also observed that the UMR dynamics is equivalent to another classical system that has been exhaustively studied in the literature, namely the Heisenberg system [52], which possesses a similar kind of symmetry.

This type of robots can be required to perform numerous tasks, from achieving a target point in the plane to tracking a given trajectory [3, 53]. Here, however, we are interested in the first task: going from any point in the configuration space to the origin in finite time. This goal requires additional effort in terms of feedback design. It is quite simple to find a control strategy that stabilizes this kind of system at the origin in a finite time (or asymptotically) if the problem is divided into a list of subtasks, which can be solved subsequently in time (see Theorem 18). Nevertheless, providing a united control law is much more challenging, mainly if we impose certain restrictions on its regularity [54].

There are many methods for designing asymptotically or exponentially stabilizing controls for the UMR model or the non-holonomic integrators [54]. In Chapter 5, we focus on recently obtained results [4, 50, 55] that ensure finite-time stability for the closed-loop system. Some of these methods have extensions establishing the robustness of these control laws with respect to disturbances and measurement noises. This robustness will be illustrated only by simulations. Our main focus is on how the convergence to the origin is guaranteed and the methods' suitability for robotic applications. In addition, since several transformations of coordinates and inputs bring the UMR model to its canonical chained representation, the role of these transformations is highlighted.

Therefore, this thesis is organized as follows: Next section gives preliminary definitions and results that will be used throughout the chapters. Chapter 2 introduces the notion of θ-invariant systems and normed actions in the context of autonomous differential equations and provides some dynamical properties of such systems. Chapter 3 extends this notion for systems with inputs and outputs and deals with the problem of observer design. Chapter 4 is dedicated to the design of observers for the class of Persidskii sys-

tems using an LMI approach and applying it to three real world models. Finally, Chapter 5 deals with a control design problem for the Unicycle Mobile Robot (UMR), by comparing different controls obtained through change of coordinates. Each chapter can be read independently, as it is based on research papers published during the thesis period. Nonetheless the existing connections between the chapters are highlighted whenever they appear.

1.2 Preliminaries

As mentioned in the introduction, our goal is to analyse dynamical systems possessing symmetries in order to understand their asymptotical behaviour and propose a framework for observer design. In this section, we present the fundamental concepts and key results necessary to understand the material in the subsequent chapters.

1.2.1 Dynamical systems and stability

To the extent of our research, the notions studied in this work are associated to continuous-time dynamical systems represented by a differential equations. Rigorously, we consider $\mathscr{X} \subset \mathbb{R}^n$ and $\mathscr{U} \subset \mathbb{R}^m$ to be open sets representing respectively the state and input space, and we consider as our object of study the dynamical system given by the differential equation

$$\dot{x}(t) = f(x(t), u(t)), \tag{1.1}$$

where $x(t) \in \mathscr{X}$, $u \in \mathscr{L}_\infty^m$ and $f : \mathscr{X} \times \mathscr{U} \to \mathbb{R}^n$ is a locally Lipschitz function. This property implies the existence of local solutions passing though any initial condition.

We will also assume that the system (1.1) is forward-complete. That means that, for every $x_0 \in \mathscr{X}$, $u \in \mathscr{L}_\infty^m(\mathbb{R})$, there exists a solution curve $x(t)$ such that $x(0) = x_0$ and equation (1.1) is satisified for every $t \geq 0$. We will denote this solution curve at time t by

$x(t, x_0, u)$.

An important particular case of Equation (1.1) is that on which the function f depends only on the state x:

$$\dot{x}(t) = f(x(t)). \tag{1.2}$$

These systems are refered to as *autonomous* and will undergo a more specific analysis. In this case the solution at time $t \geq 0$ for initial condition $x_0 \in \mathscr{X}$ will be denoted as $x(t, x_0)$.

Remark 1. *Whenever the time reference is not necessary we will omit it in favor of a clearer notation, in which case equation (1.1) will read simply* $\dot{x} = f(x, u)$.

We study the above general systems for purposes of stability and symmetry analysis, as well as observer and control design. We also study the intertwining between the above concepts. For instance, a whole chapter is dedicated to building a framework for observer design upon knowledge of the symmetries of a system. Another is dedicated to studying the role those symmetries play on stability analysis.

We are mostly interested in equilibrium points of Equation (1.2) and their stability, as defined below.

Definition 1. *A point* $x_0 \in \mathscr{X}$ *is an equilibrium of the autonomous system (1.2) if* $f(0) = 0$. *The equilibrium* x_0 *is said to be*

- *Stable, if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $\|x(t, x_0)\| < \varepsilon$ *for every* $t \geq 0$ *and every* $x_0 \in B_\delta(0)$;
- *Attractive, if there exists* $\delta > 0$ *such that* $\lim_{t \to \infty} x(t, x_0) = 0$, *for every* $x_0 \in B_\delta(0)$;
- *Asymptotically stable if it is attractive and stable;*
- *Finite-time stable (FTS) if there exists a so called time-settling function* $T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *such that* $x(t, x_0) = 0$, *for every* $x_0 \in \mathbb{R}^n$ *and every* $t \geq T(x_0)$.
- *Fixed-time stable (FxTS) if it is finite-time stable in a neighborhood* U *of zero and* $\sup_{x_0 \in U} T(x_0) < +\infty$.

Definition 2. *The basin of attraction of an attractive equilibrium point $x_0 \in \mathscr{X}$ is defined as the set* $\{z_0 \in \mathscr{X} : \lim_{t\to\infty} x(t,z_0) = x_0\}$.

We will often assume that the equilibrium point is $x_0 = 0$, as any other case can be obtained through translation of the variable, without significant change to the dynamics or the symmetries of the system.

Next we define important concepts of stability associated with subsets of $\mathbb{R}^n$ that will be useful to infer stability of equilibria.

Definition 3. *Let $\mathscr{F}_t : \mathbb{R}^n \to \mathbb{R}^n$ denote the time-t flow of the system (1.2) (that is, $\mathscr{F}_t(x_0) = x(t,x_0)$) and let $K \subset \mathbb{R}^n$. We say that K is*

- *positively invariant under f, if $\mathscr{F}_t(K) \subseteq K$ for every $t \geq 0$;*
- *strictly positively invariant under f, if $\mathscr{F}_t(K) \subset \text{int}K$ for every $t \geq 0$;*
- *invariant under f, if $\mathscr{F}_t(K) = K$ for every $t \geq 0$;*

Next, we introduce the classical uniform stability notions of the systems with inputs (1.1). Before doing so, we need the concept of comparison functions.

Definition 4. *A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathscr{K}$ if it is strictly increasing and $\sigma(0) = 0$. If in addition $\lim_{t\to+\infty} \sigma(t) = +\infty$, we say that σ belongs to the class $\mathscr{K}_\infty$. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathscr{K}\mathscr{L}$ if $\beta(\cdot,r) \in \mathscr{K}$ and $\beta(r,\cdot)$ is a decreasing function going to zero for any fixed $r > 0$.*

Definition 5. *A forward complete system (1.1) is said to be*

1. uniformly stable (uS) *if there exists $\beta \in \mathscr{K}\mathscr{L}$ such that*

$$\|x(t,x_0,d)\| \leq \beta(\|x_0\|,t), \forall t \in \mathbb{R}_+, \forall x_0 \in \mathbb{R}^n, \forall d \in \mathscr{D}.$$

2. uniformly finite-time stable (uFTS) *if it is* uS *and for every $x_0 \in \mathbb{R}^n$, there exists $T_{x_0} \in \mathbb{R}_+$ such that $x(t,x_0,d) = 0$, $\forall t \geq T_{x_0}$ and $\forall d \in \mathscr{D}$.*

3. uniformly fixed-time stable (uFxTS) *if it is* uFTS *and* $\sup_{x_0\in\mathbb{R}^n}\{T_{x_0}\}<+\infty$.

The most used tool to analyse stability of continuous-time systems is that of Lyapunov functions. Next we definte it and restate classical results that show how each type of stability can be checked through the design of a Lyapunov function of its associated type.

Definition 6. *Suppose that system (1.2) has $x_0=0$ as an equilibrium point. Let $U\subset\mathscr{X}$ be an open neighborhood of the origin. A function $V:U\to\mathbb{R}$ is called a Lyapunov function for system (1.2) if it satisfies*

1. $V(0)=0$ *and* $V(x)>0$, *for every* $x\in U\setminus\{0\}$;
2. $\nabla V(x)\cdot f(x)\leq 0$, *for every* $x\in U$.

The next result highlights the importance of the Lyapunov functions in the stability analysis of autonomous systems.

Theorem 1. *([56], [57]) If system (1.2) possesses a Lyapunov function, then the origin is a stable equilibrium point. If in addition we have*

$$\nabla V(x)\cdot f(x)<0$$

in $\mathscr{X}\setminus\{0\}$, then the origin is asymptotically stable.

If there exists $\tau,\gamma>0$ and $\beta\in[0,1)$ such that, for every $x\in\mathscr{X}\setminus\{0\}$,

$$\dot{V}(x)\leq-\tau V^{\beta}(x)+\gamma V(x),$$

then the system is Finite-Time Stable, with basin of attraction $\{x\in\mathscr{X}:V^{1-\beta}(x)<\tau/\gamma\}$ and settling time

$$T(x_0)\leq\frac{\log(1-(\gamma/\tau)V^{1-\beta}(x))}{\gamma(\beta-1)}$$

While analysing a system like (1.1), we often do not have access to its full-state $x\in\mathscr{X}$, but only to a measured partial state. This limitation is implemented in our model by

the introduction of an output or measurement function. That makes the system take the following form:

$$\begin{aligned} \dot{x}(t) &= f(x(t),u(t)), \\ y(t) &= h(x(t)), \end{aligned} \tag{1.3}$$

where $y(t) \in \mathscr{Y} \subset \mathbb{R}^p$, and $h : \mathscr{X} \to \mathscr{Y}$ is a continuous function. The output trajectory generated by the initial condition $x(0) = x_0$ and input $u \in \mathscr{L}_\infty^m$ will be denoted by $y(t,x_0,u)$.

Input-output systems have rather similar stability notions as the input systems.

Definition 7. *A forward complete system (1.3) is said to be*

1. uniformly output stable (uOS) *if there exists* $\beta \in \mathscr{K}\mathscr{L}$ *such that*

$$\|y(t,x_0,d)\| \leq \beta(\|x_0\|,t), \forall t \in \mathbb{R}_+, \forall x_0 \in \mathbb{R}^n, \forall d \in \mathscr{D}.$$

2. uniformly finite-time output stable (uFTOS) *if it is* uOS *and for every* $x_0 \in \mathbb{R}^n$, *there exists* $T_{x_0} \in \mathbb{R}_+$ *such that* $y(t,x_0,d) = 0$, $\forall t \geq T_{x_0}$ *and* $\forall d \in \mathscr{D}$.

3. uniformly fixed-time output stable (uFxTOS) *if it is* uFTOS *and* $\sup_{x_0 \in \mathbb{R}^n} \{T_{x_0}\} < +\infty$.

If $h(x) = x$, then uFTOS and uFxTOS properties become conventional uniform finite/fixed-time stability notions at the origin (uFTS/uFxTS). If $\mathscr{D}$ is a singleton, then the term *uniform* can be omitted.

It is of interest often to estimate the norm of the output trajectory not only as a function of the initial condition but also the amplitude of the input, ,i.e., to consider not uniform stability concepts. The next two definitions encompass this idea.

Definition 8. *[58] We say that a forward complete system (1.3) is* input-to-output stable

(IOS) *if there exist functions* $\beta \in \mathscr{K}\mathscr{L}$ *and* $\gamma \in \mathscr{K}$ *such that*

$$\|y(t,x_0,u)\| \leq \beta(\|x_0\|,t) + \gamma(\|u\|_\infty),$$

for every $t \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ *and* $u \in \mathscr{L}_\infty^m$.

Definition 9. *[58] We say that a forward complete system (1.3) is* state-independent input-to-output stable (SIIOS) *if there exist functions* $\beta \in \mathscr{K}\mathscr{L}$ *and* $\gamma \in \mathscr{K}$ *such that*

$$\|y(t,x_0,u)\| \leq \beta(\|h(x_0)\|,t) + \gamma(\|u\|_\infty),$$

for every $t \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ *and* $u \in \mathscr{L}_\infty^m$.

The difference between these stability characteristics is that the former accepts the output transient deviations and the convergence time governed by the initial norm of the whole state $\|x_0\|$, while for the latter, everything depends on the output initial deviation $\|h(x_0)\|$ only. The Lyapunov function method can be used to check these stability properties:

Definition 10. *[58] A continuously differentiable function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *is called an* IOS-Lyapunov function *for (1.3) if there exist functions* α_1, $\alpha_2 \in \mathscr{K}_\infty$, $\chi \in \mathscr{K}$ *and* $\alpha_3 \in \mathscr{K}\mathscr{L}$ *such that*

$$\alpha_1(\|h(x)\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{1.4}$$

$$\text{and } V(x) \geq \chi(\|u\|) \implies \nabla V(x) f(x,u) \leq -\alpha_3(V(x), \|x\|) \tag{1.5}$$

for all $x \in \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$. *We say that* V *is a* SIIOS-Lyapunov function *for (1.3) if there exist functions* α_1, $\alpha_2 \in \mathscr{K}_\infty$ *and* χ, $\alpha_3 \in \mathscr{K}$ *such that*

$$\alpha_1(\|h(x)\|) \leq V(x) \leq \alpha_2(\|h(x)\|) \tag{1.6}$$

$$\text{and } V(x) \geq \chi(\|u\|) \implies \nabla V(x) f(x,u) \leq -\alpha_3(V(x)), \tag{1.7}$$

for all $x \in \mathbb{R}^m$ *and* $u \in \mathbb{R}^m$.

Note that the right-hand side in (1.5) is not a definite function of V if the state norm is not bounded. The next property will be used to avoid this issue:

Definition 11. *The forward complete system (1.3) is said to be* uniformly bounded-input-bounded-state stable (UBIBS) *if there exists a function $\sigma \in \mathscr{K}$ such that*

$$\|x(t,x_0,u)\| \leq \max\{\sigma(\|x_0\|), \sigma(\|u\|_\infty)\},$$

for every $t \in \mathbb{R}_+$, every $x_0 \in \mathbb{R}^n$ and every $u \in \mathscr{L}_\infty^m$.

The key results connecting these stability properties and the corresponding Lyapunov function are given below.

Theorem 2. *[58] A* UBIBS *system (1.3) is* IOS *if and only if it admits an* IOS*-Lyapunov function.*

Theorem 3. *[58] A forward-complete system (1.3) is* SIIOS *if and only if it admits a* SIIOS*-Lyapunov function.*

The IOS property and the IOS-Lyapunov function are notions defined globally. The local counterparts can also be formulated by restricting the domain of validity. For many canonical forms of dynamical systems like linear, homogeneous and Persidskii systems the design and analysis of IOS and SIIOS-Lyapunov functions are often reduced to Linear Matrix Inequalities (LMIs) [59], that is, inequalities of the form

$$A(x) = A_0 + \sum_{i=1}^{n} x_i A_i > 0,$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$ is the variable, $A_0, A_1, ..., A_n \in \mathbb{R}^{n\times n}$ are symmetric matrices and the inequality $A(x) > 0$ means that $y^T A(x) y > 0$, for every nonzero $y \in \mathbb{R}^n$. Reducing a problem to an LMI is convenient due to the abundant number of mathematical techniques and softwares dedicated to solve them, as opposed to the problem determining for which parameters a general nonlinear function is positive. This reduction will be useful in Chapter 4.

1.2.2 Measurements and observers

An important question concerning system (1.3) is whether we can estimate its full state from its measured output and input. In this section we formalize what this means and, in the cases where estimation is possible, we present the main techniques to achieve it.

Definition 12. *Let $U \subset \mathscr{X} \subset \mathbb{R}^n$ be an open set. Let $x_1, x_2 \in \mathscr{X}$ be two states of System (1.3). They are said to be U-undistinguishable if for every $u \in \mathscr{U}$ and $T \geq 0$ the following implication holds:*

$$x(t;x_1;u), x(t;x_2;u) \in U, \forall t \in [0,T)$$

$$\implies y(t;x_1;u) = y(t;x_2;u), \forall t \in [0,T)$$

We denote by $\mathscr{I}_U(x_0)$ the set of points that are U-undistinguishable from x_0.

It is easy to see that if two points are $\mathscr{X}$-undistinguishable then there is no hope of reconstructing the full initial state for that initial condition, since any input yields the same output for all times. Therefore, for observation purposes we need to insure that the class $\mathscr{I}_U(x)$ to be as small as possible.

Definition 13. *System (1.3) is said to be*

- *locally observable at $x_0 \in \mathscr{X}$ if $\mathscr{I}_U(x_0) = \{x_0\}$ for every open neighborhood U of x_0;*
- *weakly observable at $x_0 \in \mathscr{X}$ if there exists an open neighborhood U of x_0 such that $\mathscr{I}(x_0) \cap U = \{x_0\}$;*
- *locally weakly observable if there exists an open neighborhood U of x_0 such that, for every open set V with $x_0 \in V \subset U$, we have $\mathscr{I}_V(x_0) = \{x_0\}$*

Definition 14. *A pair of states $x_1, x_2 \in \mathscr{X}$ are said to be uniformly distinguishable for system (1.3) if for every input $u \in \mathscr{L}_\infty^m$, there exists $t \geq 0$ such that*

$$y(t, x_1, u) \neq y(t, x_2, u).$$

A system is said to be locally uniformly observable if each $x_1 \in \mathbb{R}^n$ has a neighborhood U such that every $x_2 \in U$ is uniformly distinguishable from x_1. The system is said to be uniformly observable if any pair of distinct states is uniformly distinguishable.

A common way of estimating an observable system's state is to design a second system that uses the output of the previous one as an input. The design is made so that the observer's trajectory and the original trajectory converge asymptotically.

Definition 15. *An observer of system (1.3) is a system*

$$\dot{\hat{x}}(t) = F(\hat{x}(t), y(t), u(t))$$

such that for every $x_0, \hat{x}_0 \in \mathbb{R}^n$ and every $u \in \mathscr{L}^m_\infty$, the state estimation error trajectory

$$e(t, x_0, \hat{x}_0, u) = x(t, x_0, u) - \hat{x}(t, \hat{x}_0, h(x(t, x_0)), u)$$

tends to zero, as $t \to +\infty$.

Example 1. *The linear system*

$$\dot{x} = Ax + Bu,$$

$$y = Cx,$$

for $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ is know to be observable if and only the pair (A, C) is observable, meaning the matrix

$$\mathscr{O}(A, C) = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

is full-rank. In this case, it is shown that the system

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

is an observer for some $L \in \mathbb{R}^{n\times p}$, with error dynamics given by

$$\dot{e} = (A - LC)e.$$

This simple design technique features elements that are present in more complex designs, namely a copy of the system with the addition of the gain function L applied on the output error $E(\hat{x}, y) = y - C\hat{x}$.

In the cases where the input u represents the system's perturbation, and therefore is unknown, the error cannot be expected to converge to zero for every pair of trajetories. Thus, another definition of observers using the IOS property becomes useful.

Definition 16. *We say that the system*

$$\dot{\hat{x}}(t) = \widetilde{f}(\hat{x}(t), y(t)) \tag{1.8}$$

is an observer *for the system (1.3) if the combined system $\begin{pmatrix} \dot{x} & \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} f(x,u) & \widetilde{f}(\hat{x},y) \end{pmatrix}$ is IOS with respect to the output $e(x,\hat{x}) = x - \hat{x}$ (the state estimation error).*

1.2.3 Symmetries and smooth group actions

The most useful tool to analyse symmetries of differential equations is that of smooth Lie group actions. In this section we establish the main definitions related to this tool and important concepts that are going to play an important part in the symmetry method for design of observers.

Definition 17. *A* group *is a set G together with a closed binary operation $\cdot : G \times G \to G$ satisfying the following axioms:*

1. **(existence of identity)** *There exists $e \in G$ such that $e \cdot g = g \cdot e = g$, for every $g \in G$;*
2. **(existence of inverse)** *For every $g \in G$, there exists $h \in G$ such that $gh = hg = e$ (notation: $h = g^{-1}$);*

3. **(associativity)** *For every $g,h,i \in G$, we have $g \cdot (h \cdot i) = (g \cdot h) \cdot i$.*

We say that G is a Lie group *if it is endowed with the structure of a smooth manifold and the group operations of product and inverse, $(a,b) \mapsto ab$ and $a \mapsto a^{-1}$, are smooth (i.e., their differential maps exist in every point and vary continuously).*

In what follows we will always denote the group identity by e.

Definition 18. *Let G be a group and $\mathscr{X}$ a nonempty set. A group action of G over $\mathscr{X}$ is a function $\phi : G \times \mathscr{X} \to \mathscr{X}$ satisfying the following properties:*

1. *$\phi(e,x) = x$, for every $x \in \mathscr{X}$;*
2. *$\phi(gh,x) = \phi(g,\phi(h,x))$.*

It is common to denote $\phi(g,x)$ as $\phi_g(x)$ when more convenient. With this notation we can consider an action as a family of invertible functions $(\phi_g)_{g\in G}$ of $\mathscr{X}$ into $\mathscr{X}$, satisfying $\phi_e = Id_{\mathscr{X}}$ and $\phi_{gh} = \phi_g \circ \phi_h$.

In what follows we are going to consider G as a Lie group, $\mathscr{X} = \mathbb{R}^n$ and ϕ a differentiable action.

Since G is a smooth manifold of dimension $r \geq 1$ (see Definition 17), up to composition with a chart, we can regard any function $\phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ as a smooth function between open sets of $\mathbb{R}^{r+n}$ and $\mathbb{R}^n$, written $\phi(g,x) = \phi(g_1,...,g_r,x_1,...,x_n)$. We can then consider the differential $D_g\phi(g,x)$ as the transformation associated to the $n \times r$-matrix

$$\left(\frac{\partial \phi_i}{\partial g_j}\right)_{1\leq i\leq n, 1\leq j\leq r} \in \mathbb{R}^{n\times r} \tag{1.9}$$

which is a submatrix of $D\phi(g,x) \in \mathbb{R}^{n\times(r+n)}$. We say that the application $g \mapsto \phi_g(x)$ is *full-rank* in x_0 if the matrix $D_g\phi(e,x_0)$ has full rank. In this case we say that ϕ satisfies the *full-rank condition* on x_0.

The importance of this property is that it allows us to parametrize the state space using group coordinates. More formally, if ϕ is full-rank in x_0, then there exists a neighborhood $\mathscr{O}$ of x_0 and a function $\gamma : \mathscr{O} \to G$ satisfying

$$\gamma(\phi_g(x))\, g = \gamma(x) \tag{1.10}$$

for every $g \in G$ and $x \in \mathcal{O}$.

We can give a constructive proof of the moving frame γ.

Lemma 1. *[27] Let $\phi : G \times \mathscr{X} \to \mathscr{X}$ be a group action and suppose $r \leq n$. Let $x_0 \in \mathscr{X}$ be such that $g \mapsto \phi_g(x)$ has full-rank in x_0. Then there exists a neighborhood $\mathcal{O}$ of x_0 and a function $\gamma : \mathcal{O} \to G$ such that*

$$\gamma(\phi_g(x))\, g = \gamma(x) \tag{1.11}$$

for every $g \in G$ and $x \in \mathcal{O}$.

Proof. Since $\operatorname{rank}(D_g\phi(e,x)) = r$, we can write

$$\phi_g = \left(\phi_g^a, \phi_g^b\right), \tag{1.12}$$

where the map $g \mapsto \phi_g^a(x) \in \mathscr{X}$ is invertible in a neighborhood of the identity $e \in G$, for every $x \in \mathcal{O}$. Let $c \in \operatorname{Im}\left(\phi_g^a\right)$ and define $\gamma : \mathcal{O} \to G$ implicitly by

$$\phi^a_{\gamma(x)}(x) = c. \tag{1.13}$$

Due to the compatibility property, we have

$$\phi^a_{\gamma\left(\phi_g(x)\right)g}(x) = \phi^a_{\gamma(\phi_g(x))}(\phi_g(x)) = c,$$

by definition of γ. By uniqueness of $\gamma(x)$ and its defining property, we have

$$\gamma(\phi_g(x))g = \gamma(x).$$

□

Equation (1.13) is sometimes called the *normalization equation*, and the function γ is called the *moving frame*.

The importance of the moving frame is to provide a simple and constructive way to characterize al the functions that are invariant under the action ϕ. Once we have them, we can do the same for output errors and vector fields, which play an important part in producing the canonical form of invariant observers.

Definition 19. *[27] We say that a function $J : \mathscr{X} \to \mathbb{R}^k$ is invariant for an action $\phi : G \times \mathscr{X} \to \mathscr{X}$ if*

$$J(\phi_g(x)) = J(x)$$

for every $x \in \mathscr{X}$ and $g \in G$.

If J has the additional property that any other invariant function I can be written as

$$I(x) = \mathscr{F}(J(x)),$$

for some function $\mathscr{F} : \mathbb{R}^k \to \mathbb{R}^k$, then J is called a complete set of invariant functions.

Definition 20. *[27] Let $\mathscr{X} \subset \mathbb{R}^n$, $\mathscr{U} \subset \mathbb{R}^m$ and $\mathscr{Y} \subset \mathbb{R}^p$ be open sets. An* invariant output error *is a function $E : \mathscr{X} \times \mathscr{U} \times \mathscr{Y} \to \mathbb{R}^p$ such that*

1. *The map $y \mapsto E(\widehat{x}, u, y)$ is invertible for every $\widehat{x} \in \mathscr{X}$ and every $u \in \mathscr{U}$,*
2. *$E(\widehat{x}, u, h(\widehat{x})) = 0$, for every $\widehat{x} \in \mathscr{X}$ and every $u \in \mathscr{U}$,*
3. *$E(\phi_g(\widehat{x}), \psi_g(u), \rho_g(y)) = E(\widehat{x}, u, y)$, for every $\widehat{x} \in \mathscr{X}, u \in \mathscr{U}$ and $y \in \mathscr{Y}$.*

Definition 21. *An invariant frame for a space $\mathscr{X}$ and an action $\phi : G \times \mathscr{X} \to \mathscr{X}$ is a set $\{w_1, \ldots, w_n\}$ of vector fields such that the vectors $w_1(x), \ldots, w_n(x)$ form a basis of the tangent space $T_x\mathscr{X}$, for every $x \in \mathscr{X}$ and satisfy*

$$w_i(\phi_g(x)) = D\phi_g(x)[w_i(x)].$$

In what follows, we show how the moving frame can be used to explicitly compute a complete set of invariant functions, the invariant state error and the invariant frame. For a more detailed exposition, we refer to [27].

Proposition 1. *[27] Let $x_0 \in \mathscr{X}$ be such that* $\operatorname{rank}(D_g\phi(e,x_0)) = r = \dim(G)$. *Define $J(x) = \phi^b_{\gamma(x)}(x)$, where ϕ^b_g is as in (1.12). Then, $J(x)$ is a complete set of invariant functions in a neighborhood of x_0.*

Proposition 2. *[27] The function*

$$E(\widehat{x},u,y) = \rho_{\gamma(\widehat{x})}(h(\widehat{x})) - \rho_{\gamma(\widehat{x})}(y)$$

is an invariant output error. Furthermore, every invariant output error $\widetilde{E}$ can be written as

$$\widetilde{E}(\widehat{x},u,y) = \mathscr{L}(I(\widehat{x},u),E(\widehat{x},u,y)),$$

where I is an invariant function of the actions ϕ and ψ and $\mathscr{L}$ is a smooth function such that $\mathscr{L}(I,0) = 0$.

Proposition 3. *[22] Let $\mathscr{X}$ be an open subset of $\mathbb{R}^n$ and G a Lie group of dimension $r \leq n$. Given an action $\phi : G \times \mathscr{X} \to \mathscr{X}$ that is full-rank at every point, there always exists an invariant frame $\{w_1,\ldots,w_n\}$, which is given by the columns of the matrix $(D\phi_{\gamma(x)}(x))^{-1}$.*

Chapter 2

The dynamics of θ-invariant systems

This chapter aims to rigorously define θ-invariant systems and prove their main properties. The main motivation for introducing this concept is to find a natural extension of the observer design framework proposed by Bonnabel [27] to a larger class of systems, including, among others, homogeneous systems of arbitrary degree. In doing so through the introduction of a time-rescaling, we realize that other fundamental homogeneous-like properties may be proven as well, and that G-invariant and non-trivial θ-invariant autonomous systems can behave very differently, as shown by the results in the present chapter. The observer design framework itself will be the theme of the subsequent chapter.

The main results of this chapter were presented at Conference of Decision and Control (CDC) 2025 [LimaCDC25].

2.1 θ-invariant systems

Although the notion of θ-invariance can be applied to systems with inputs, this chapter deals exclusively with autonomous systems, given by equation

$$\dot{x}(t) = f(x(t)), \tag{2.1}$$

where $x \in \mathscr{X} \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz. The system is assumed to be forward complete and a solution of the initial value problem $\dot{x}(t) = f(x(t)); x(0) = x_0$ will be denoted by $x(t;x_0)$.

Definition 22. *We say that (2.1) is θ-invariant if there is a connected non-trivial Lie group G, an action $\phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ (also denoted $\phi_g : \mathbb{R}^n \to \mathbb{R}^n$, for a given $g \in G$) and a function $\theta : G \to \mathbb{R}_{>0}$, such that*

$$f(\phi_g(x)) = \theta(g) D\phi_g(x)[f(x)], \tag{2.2}$$

for every $x \in \mathbb{R}^n$ and $g \in G$, where $D\phi_g(x)$ is the Jacobian of ϕ_g at the point $x \in \mathbb{R}^n$. We say that (2.1) is G-invariant if the property (2.2) is satisfied for $\theta \equiv 1$.

The case $\theta \equiv 1$ was already dealt with in the literature, and the nomenclature G-invariant comes from S. Bonnabel [27]. Although the class of θ-invariant systems can be encompassed by general symmetry studies like Lie symmetry analysis (see [60]), a concise definition specific to ordinary differential equations, as well as the derivation of the specific dynamical properties that follows, were not found in the literature except for the particular case of homogeneous systems [61].

The class of θ-invariant systems demonstrates a very important phenomenon of scaling of trajectories.

Proposition 4. *System (2.1) is θ-invariant if and only if*

$$x(t;\phi_g(x_0)) = \phi_g(x(\theta(g)t;x_0)), \tag{2.3}$$

for every $g \in G$ and every solution curve $x(t;x_0) \in \mathbb{R}^n$ with initial condition $x_0 \in \mathbb{R}^n$ and $t \geq 0$.

Proof. Suppose that property (2.2) holds and $x(t)$ is a trajectory of the system. We need to show that

$$\frac{d}{dt}\phi_g(x(\theta(g)t)) = f(\phi_g(x(\theta(g)t))) \tag{2.4}$$

also holds for every $t \in \mathbb{R}_{\geq 0}$. We compute the left-hand side of (2.4), obtaining:

$$\frac{d}{dt}\phi_g(x(\theta(g)t)) = D\phi_g\left(x(\theta(g)t)\right)\left[\theta(g)\dot{x}(\theta(g)t)\right]$$

$$= \theta(g)D\phi_g\left(x(\theta(g)t)\right)\left[f(x(\theta(g)t))\right],$$

for every $t \in \mathbb{R}_{\geq 0}$. By invariance, the last term equals $f(\phi_g(x(\theta(g)t)))$, as desired.

Conversely, suppose that equation (2.4) holds. By the above calculation, we have

$$\theta(g)D\phi_g(x(\theta(g)t))[f(x(\theta(g)t))]$$

$$= f(\phi_g(x(\theta(g)t))),$$

for every curve $x(t)$ that is a trajectory of the system. Making $t = 0$, we have that $x_0 = x(\theta(g)t;x_0)$, for any $g \in G$. Therefore, we can omit the argument $\theta(g)t$ and the above equation is equivalent to the time-independent equation (2.2), which proves the other side of the equivalence. □

We remark that there are some restrictions on the function $\theta : G \to \mathbb{R}_{>0}$ so that (3.2) is satisfied:

Proposition 5. *Suppose that (3.2) holds for every $x \in \mathscr{X}, u \in \mathscr{U}$, $g \in G$, and that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is not identically zero. Then $\theta : G \to \mathbb{R}_{>0}$ is a group homomorphism. In particular, the mapping $(t,g) \mapsto \theta(g)t$ is an action of G over the set of strictly positive real numbers.*

Proof. Using the property that ϕ_e and ψ_e are the identity transformations in their respective domains, we have

$$D\phi_e(x)[f(x,u)] = \theta(e)f(\phi_e(x),\psi_e(u)) = \theta(e)f(x,u),$$

where $e \in G$ is also the identity. Since $D\phi_e$ is the identity, that implies $\theta(e) = 1$.

The proof that $\theta(gh) = \theta(g)\theta(h)$ follows from a direct computation using the fact that each ϕ_g is a diffeomorphism, meaning that $D\phi_g(x)$ is invertible for every $g \in G$ and every $x \in \mathscr{X}$. We have

$$\begin{aligned} D\phi_{gh}(x)[f(x,u)] &= \theta(gh)f(\phi_{gh}(x),\psi_{gh}(u)) \\ &= \theta(gh)f(\phi_g(\phi_h(x)),\psi_g(\psi_h(u))) \\ &= \frac{\theta(gh)}{\theta(g)}\theta(g)f(\phi_g(\phi_h(x)),\psi_g(\psi_h(u))) \\ &= \frac{\theta(gh)}{\theta(g)}D\phi_g(\phi_h(x))[f(\phi_h(x),\psi_h(u))] \end{aligned}$$

By the chain rule, we have:

$$D\phi_{gh}(x)[f(x,u)] = D\phi_g(\phi_h(x))[D\phi_h(x)[f(x,u)]]$$

Applying the inverse of $D\phi_g(\phi_h(x))$ to both sides yields

$$D\phi_h(x)[f(x,u)] = \frac{\theta(gh)}{\theta(g)}f(\phi_h(x),\psi_h(u)),$$

for every $x \in \mathscr{X}$ and every $u \in \mathscr{U}$. By the definition of θ-invariance, we must have $\frac{\theta(gh)}{\theta(g)} = \theta(h)$, or $\theta(gh) = \theta(g)\theta(h)$, as desired. □

Since the homomorphism θ represents a change solely on the time coordinate, it is natural to ask under which conditions a G-invariant system can be made into θ-invariant one without changing its trajectories. To answer that question, we make use of the full-rank property and the invariant elements discussed in Section 1.2.3 (see also [22, 27]).

Proposition 6. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be such that $\dot{x} = f(x)$ is a G-invariant system for an action $\phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying the full-rank property in every $x \in \mathbb{R}^n$. Then the function $\gamma : \mathbb{R}^n \to G$ is defined everywhere, the system $\dot{x} = \theta(\gamma(x))^{-1} f(x)$ is G-invariant and has the same trajectories as $\dot{x} = f(x)$, up to time-scaling.*

Proof. It is clear that if $t \mapsto x(t)$ is a solution of $\dot{x} = f(x)$, then $t \mapsto x(\theta(\gamma(x))^{-1}t)$ is a solution of $\dot{x} = \theta(\gamma(x))^{-1} f(x)$, and both curves have the same image.

Let us prove θ-invariance. Define $\tilde{f}(x) = \theta(\gamma(x)) f(x)$. Then

$$\tilde{f}(\phi_g(x)) = \theta(\gamma(\phi_g(x)))^{-1} f(\phi_g(x))$$

Using G-invariance of f, the property of the moving frame and the fact that θ is a homomorphism, we have

$$\theta(\gamma(\phi_g(x))) f(\phi_g(x)) = \theta(g^{-1})^{-1} \theta(\gamma(x))^{-1} D\phi_g(x)[f(x)]$$

$$= \theta(g) D\phi_g(x)[\theta(\gamma(x))^{-1} f(x)] = \theta(g) D\phi_g(x)[\tilde{f}(x)],$$

so property 2.2 holds for $\tilde{f}$. □

Indeed there are challenges and limitations in the construction of an invariant frame. One of them is posed by the existence of equilibria.

Proposition 7. *If $x_0 \in \mathbb{R}^n$ is an isolated equilibrium point of a θ-invariant system (2.1) with $\dim G \geq 1$, then the action $\phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ cannot satisfy the full-rank condition at x_0.*

Proof. Let x_0 be an equilibrium point of (2.1). Consider the G-orbit of x_0, $\{\phi_g(x_0) : g \in G\}$. It is a connected set containing x_0 and made up entirely of equilibrium points (thanks to Lemma 2). Then, since x_0 is isolated as an equilibrium point, such a G-orbit must be equal to the unitary set $\{x_0\}$, which means that the image of a neighborhood

of the identity e through the application $g \mapsto \phi_g(x_0)$ is a dimension zero set. Therefore, $\text{rank}\,(D_g\phi_g(x_0)) = 0 < \dim(G)$. □

Such limitations prevent a complete duality between G-invariant and non-trivial θ-invariant systems. Thus, there are systems that fundamentally belong to one class or the other, and so do some dynamical features that we will explore in the next results.

We begin by making some basic observations on how periodic orbits are transformed under a symmetry action, and then making a concluding remark about isolated periodic orbits as examples of features that cannot happen in θ-invariant systems.

Lemma 2. *Suppose $x_0 \in \mathbb{R}^n$ generates a periodic solution for system (2.1) with period $\tau > 0$ (that is $x(t;x_0) = x(t+\tau;x_0)$ for all $t \geq 0$) and let $g \in G$. Then the point $y_0 = \phi_g(x_0)$ generates a periodic solution with period $\theta(g)^{-1}\tau$.*

Proof. A direct computation using the formula (2.3) for the scaling of trajectories gives us

$$x(\theta(g)^{-1}\tau, y_0) = \phi_g(x(\theta(g)\theta(g)^{-1}\tau, x_0))$$

$$= \phi_g(x(\tau, x_0)) = \phi_g(x_0) = y_0.$$

□

Proposition 8. *System (2.1) cannot have a non-trivial isolated periodic orbit (that is, one that is different from an equilibrium point) if $\theta \neq 1$.*

Proof. Suppose that there exists such orbit $\mathscr{O} \subset \mathbb{R}^n$, and let $x_0 \in \mathscr{O}$. Consider the G-orbit of x_0, that is $Gx_0 = \{\phi_g(x_0) : g \in G\}$. It is a connected set that contains x_0. Furthermore, by Lemma 2, every point in it is periodic. Therefore, since $\mathscr{O}$ is isolated, we must have $Gx_0 \subseteq \mathscr{O}$. But that is a contradiction, since for any $g \in G$ such that $\theta(g) \neq 1$, the point $y_0 = \phi_g(x_0)$ has period $\theta(g)^{-1}\tau \neq \tau$. □

The Propositions 6 and 8 show that some G-invariant systems can be easily turned

into θ-invariant while its trajectories are preserved, while others cannot. It is still unclear what is the exact condition for a G-invariant system to be transformable into a θ-invariant one, for non-trivial θ.

As we will see in the next results, the latter type of invariance may be necessary to obtain accelerated rates of convergence, so pursuing that question might be a fruitful direction in the research of invariant systems. The next criteria, for instance, states that certain G-invariant systems cannot be Finite-Time Stable (FTS), under a topological hypothesis over the symmetry action ϕ.

Theorem 4. *Suppose that* 0 *is FTS for a* θ*-invariant system (2.1), and assume that there exists a compact set* $K \subset \mathbb{R}^n$ *such that* $0 \notin K$ *and* $G \cdot K = \mathbb{R}^n \setminus \{0\}$. *Then* θ *cannot be the trivial homomorphism* $\theta \equiv 1$.

Proof. Suppose by contradiction that $\theta \equiv 1$. Let T be the settling-time function of the FTS system (2.1). Then we have $T(\phi_g(x)) = T(x)$, for every $g \in G$ and $x \in \mathbb{R}^n$. We notice that the function T is lower semi-continuous: indeed, if $x_n \to x_0 \in \mathbb{R}^n$ and $T = \liminf_{n\to\infty} T(x_n)$, then $x(T_n;x_0) \to 0$ as $n \to \infty$ and, by continuity of the flow $x(\cdot;\cdot)$, we have $x(T;x_0) = 0$, implying that $T(x_0) \leq T$.

Therefore, since K is compact, there must exist $x_0 \in K$ such that

$$T(x_0) = \inf_{x\in K} T(x).$$

Since $G \cdot K = \mathbb{R}^n \setminus \{0\}$ and $T(\phi_g(x)) = T(x)$, we also have

$$\inf_{x\in K} T(x) = \inf_{x\in\mathbb{R}^n\setminus\{0\}} T(x) = 0$$

Therefore, we must have $T(x_0) = 0$. This can only be possible if $x_0 = 0$, a contradiction with the fact that $0 \notin K$. □

Despite depending on a topological condition rather than a direct computation, there

is a considerable generality in Theorem 4, and as we will see in next section, it still encompasses a wide range of symmetries.

Now we turn our attention to regular stability and attractiveness. In particular, we are interested in whether stability is inherited by attractiveness. In order to formulate those stability conditions we need the following result:

Lemma 3. *[[61], Lemma 2.2] Let system (2.1) be such that the origin is an attractive equilibrium point. Then for every compact set $K \subset \mathbb{R}^n$, there is $M_K > 0$ such that $\|x(t;x_0)\| \leq M_K$, for every $t \in \mathbb{R}_{\geq 0}$ and every $x_0 \in K$.*

We are now ready to prove our result connecting attractiveness to stability, through some extra topological hypotheses on the action ϕ.

Theorem 5. *Let system (2.1) be such that the origin is an attractive equilibrium point. Suppose also that f is θ-invariant for an action $\phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ and a homomorphism $\theta : G \to \mathbb{R}_{>0}$ (that is, $x(\theta(g)t, \phi_g(x_0)) = \phi_g(x(t,x_0))$) satisfying the following properties:*

1. *There exists $\delta_0, \delta_1 > 0$ such that for every $x \neq 0$ with $\|x\| < \delta_1$, there is $g \in G$ such that $\|\phi_g(x)\| \geq \delta_0$;*
2. *For every $M > 0$ and every $\delta > 0$, there exists a class $\mathscr{K}$ function $\alpha_{M,\delta} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for every $x \in S_\delta$*

$$\phi_g(B_M[0]) \subset B_{\alpha_{M,\delta}(\|\phi_g(x)\|)}[0]. \tag{2.5}$$

Then, the origin is a stable equilibrium.

Proof. For every $x_0 \in \mathbb{R}^n$, define

$$S(x_0) = \sup_{t \geq 0} \|x(t;x_0)\|.$$

Let δ_2 be such that $\|x(t;x_0)\| \to 0$ for every $x \in B_{\delta_2}(0)$, let $\delta = \min\{\delta_0, \delta_1, \delta_2\}$ and finally let $M = \sup_{\|x\|=\delta} S(x)$. Using Lemma 3, we have that M is finite. We are going to prove that $S(x_0) \leq \alpha_{M,\delta}(\|x_0\|)$, for all $x_0 \in B_\delta(0)$, and consequently that the system is stable at the origin.

Since G is connected, by Hypothesis (1) we can chose, for each $x_0 \in B_\delta(0)$, a $g \in G$ such that $\|\phi_{g^{-1}}(x_0)\| = \delta$. By invariance, we have

$$S(x_0) = \sup_{t\geq 0} \|x(t;x_0)\| = \sup_{t\geq 0} \|\phi_g(x(\theta(g)t;\phi_{g^{-1}}(x_0)))\|$$

$$= \sup_{t\geq 0} \|\phi_g(x(t;\phi_{g^{-1}}(x_0)))\|.$$

Since $\phi_{g^{-1}}(x_0) \in S_\delta$, we have by definition that $x(t;\phi_{g^{-1}}(x_0)) \in B_M[0]$, for every $t \geq 0$. Therefore

$$\sup_{t\geq 0} \|\phi_g(x(t;\phi_{g^{-1}}(x_0)))\| \leq \sup_{\|z\|\in B_M[0]} \|\phi_g(z)\|.$$

But the latter is bounded by $\alpha_{M,\delta}(\|\phi_g(\phi_{g^{-1}}(x_0))\|) = \alpha_{M,\delta}(\|x_0\|)$, by Hypothesis (2).

Therefore $S(x_0) \leq \alpha_{M,\delta}(\|x_0\|)$, for every $x_0 \in B_\delta(0)$, as we wanted. □

One class of systems of particular interest that falls into the definition of θ-invariance is that of **r**-homogeneous systems.

Example 2. *Let* $\mathbf{r} = \begin{pmatrix} r_1 & \dots & r_n \end{pmatrix}^\top \in \mathbb{R}^n_{>0}$. *For* $g > 0$, *define* $\Lambda^r_g = \operatorname{diag}(g^{r_1}, \dots, g^{r_n})$. *We say that system (2.1) is an* **r**-*homogeneous system of degree* μ *if*

$$f(\Lambda^r_g x) = g^\mu \Lambda^r_g f(x)$$

for every $x \in \mathscr{X}$ *and every* $g \in G$.

A direct computation shows that the multiplicative group of strictly positive real numbers $(\mathbb{R}_{>0}, \cdot)$ *together with the action* $\phi_g(x) = \Lambda^r_g x$ *and the homomorphism* $\theta(g) = g^\mu$

makes the system θ-invariant:

$$f(\phi_g(x)) = f(\Lambda_g^r x) = g^\mu \Lambda_g^r f(x)$$

$$= g^\mu D\phi_g(x)[f(x)] = \theta(g) D\phi_g(x)[f(x)].$$

Therefore, by Proposition 2.4, for every trajectory $x(t)$, the curve $\Lambda_g^r x(g^\mu t)$ is also a trajectory.

The orbits of the action ϕ are the homogeneous rays $R_x = \{\Lambda_\varepsilon^r x : \varepsilon > 0\}$. Therefore, to understand the system dynamics, it suffices to know the trajectories starting on the homogeneous unit sphere

$$\mathbb{S}_r = \{x \in \mathbb{R}^n : \|x\|_r = 1\},$$

where $\|x\|_r = (|x_1|^{\kappa/r_1} + \cdots + |x_n|^{\kappa/r_n})^{1/\kappa}$ with $\kappa \geq \max_{1\leq i\leq n}\{r_i\}$ is the **r***-homogeneous norm. In particular, since the homogeneous norm is topologically equivalent to the euclidean norm, local properties can be made global for this type of system.*

Notice that a homogeneous system does not fall into the regular definition of invariance unless the defining vector field has degree zero.

2.2 Normed actions

The conditions imposed on the action ϕ in Theorems 4 and 5 are rather sophisticated, and they can be simplified by introducing an associated norm.

Definition 23. *Let $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ denote the usual Euclidean norm. We say that a continuous function $\|\cdot\|_{inv} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an invariant norm for the action $\phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ if there exists a non-constant function $\xi : G \to \mathbb{R}_{>0}$ and $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ such that*

- $\alpha_1(\|x\|) \leq \|x\|_{inv} \leq \alpha_2(\|x\|)$, *for every $x \in \mathbb{R}^n$;*

- $\|\phi_g(x)\|_{inv} = \xi(g)\|x\|_{inv}$, *for every* $g \in G$ *and* $x \in \mathbb{R}^n$

An action posessing an invariant norm will be called a normed action.

As we will demonstrate below, replacement of an action by a normed one allows us to get all the properties used in the proof of Theorems 4 and 5. Moreover, existence of such a norm is standard for the homogeneous systems [62], which form an important class of examples for θ-invariant dynamics.

Although the invariant norm might not satisfy the condition $\|\lambda x\| = |\lambda|\|x\|$ for every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$ associated to the usual definition of norm, we show in the next Lemma that the function ξ from Definition 23 ranges over all $\mathbb{R}_{>0}$. That means that large initial conditions can always be obtained from small ones through the group symmetry, which is crucial to understand global behavior.

Lemma 4. *Let G be a connected Lie group, $\phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ a normed action and $\xi : G \to \mathbb{R}_{>0}$ be such that $\|\phi_g(x)\|_{inv} = \xi(g)\|x\|_{inv}$. Then ξ is a surjective group homomorphism.*

Proof. First, we note that continuity of ϕ, together with the continuity of the invariant norm, implies the continuity of the function ξ, since $\xi(g)$ can be written as $\|\phi_g(x)\|_{inv}/\|x\|_{inv}$ for any fixed $x \neq 0$.

Let us show that ξ is a homomorphism. We clearly have $\xi(e) = 1$, since ϕ_e is the identity transformation. If $g, h \in G$ and $x \in \mathbb{R}^n$, then

$$\|\phi_g(\phi_h(x))\|_{inv} = \xi(g)\|\phi_h(x)\|_{inv} = \xi(g)\xi(h)\|x\|_{inv}$$

But since $\phi_g \circ \phi_h = \phi_{gh}$, we have

$$\|\phi_g(\phi_h(x))\|_{inv} = \|\phi_{gh}(x)\|_{inv} = \xi(gh)\|x\|_{inv}$$

Therefore, the righ-hand side of last two equation are the same and, chosing any $x \neq 0$, we have the equality $\xi(gh) = \xi(g)\xi(h)$, showing that ξ is a homomorphism.

Since ξ is non-constant by hypothesis, there exists $g \in G$ such that $\xi(g) \neq 1$. Then, $\xi(g^n) = \xi(g)^n$ for every $n \in \mathbb{Z}$ and the set $\{\xi(g^n) : n \in \mathbb{Z}\}$ is contained in $\xi(G)$ and unbounded. By continuity of ξ and connectedness of G, the set $\xi(G)$ is connected. Therefore, the set $\xi(G)$ is a connected unbounded subgroup of $(\mathbb{R}_{>0}, \cdot)$, which is only possible if $\xi(G) = \mathbb{R}_{>0}$. □

Now we reproduce Theorem 4 under a much simpler hypothesis of the existence of an invariant norm.

Theorem 6. *Suppose that 0 is FTS for the θ-invariant system (2.1), and suppose that the action $\phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ of equation (2.3) is normed. Then θ cannot be the trivial homomorphism $\theta \equiv 1$.*

Proof. Let $K = S^1_{inv} = \{x \in \mathbb{R}^n : \|x\|_{inv} = 1\}$. Then K is a compact set, $0 \notin K$ and $G \cdot K = \mathbb{R}^n \setminus \{0\}$. Since ξ is a non-constant homomorphism (see Lemma 4) and G is connected, we have $\xi(G) = \mathbb{R}_{>0}$. Then for every $x \in \mathbb{R}^n \setminus \{0\}$, we can chose $g \in \xi^{-1}(\|x\|^{-1}_{inv})$ and compute $\|\phi_g(x)\|_{inv} = \|x\|^{-1}_{inv}\|x\|_{inv} = 1$. Therefore, $y = \phi_g(x) \in K$ and $\phi_{g^{-1}}(y) = x$, which means that $G \cdot K = \mathbb{R}^n \setminus \{0\}$. The result then follows from Theorem 4. □

As an application, we show that we cannot represent any dilation symmetry of a homogeneous system of negative degree using only an action on the state space. Thus, we prove rigorously that θ-invariance is indeed necessary.

Recall that, given $r = (r_1, \ldots, r_n) \in \mathbb{R}^n, r_i > 0$ and $\nu \in \mathbb{R}$, we denote $\Lambda^r_\varepsilon = \mathrm{diag}(\varepsilon^{r_1}, \ldots, \varepsilon^{r_n})$ and define a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ to be r-homogeneous of degree ν if it satifies the equation

$$f(\Lambda^r_\varepsilon x) = \varepsilon^\nu \Lambda^r_\varepsilon f(x),$$

for every $\varepsilon > 0$ and $x \in \mathbb{R}^n$. We also say that the associated system $\dot{x} = f(x)$ is r-homogeneous of degree ν in this case.

To make matters more precise, we define what we mean by a dilation symmetry.

Definition 24. *Let G be a connected Lie group. We call an action $\phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ a dilation if for every $x \in \mathbb{R}^n$ there exists a curve $\gamma : \mathbb{R}_{>0} \to G$ such that $\lim_{t\to+\infty} \|\phi_{\gamma(t)}(x)\| = +\infty$ and $\lim_{t\to 0^+} \phi_{\gamma(t)}(x) = 0$*

Corollary 1. *Let system (2.1) be r-homogeneous of degree $\nu < 0$. Then there is no dilation $\phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ that makes the system G-invariant.*

Proof. It has been proven that homogeneous systems of negative degree are finite-time stable [63].

It is also clear that for a dilation ϕ, the set $K = \{x \in \mathbb{R}^n : \|x\|_{inv} = 1\}$ has the property $G \cdot K = \mathbb{R}^n \setminus \{0\}$. The result then follows by Theorem 4. □

In the case of homogeneous systems we also have an associated homogeneous norm, given by

$$\|x\|_{r,\nu} = (|x_1|^{\frac{\nu}{r_1}} + \ldots + |x_n|^{\frac{\nu}{r_n}})^{\frac{1}{\nu}}$$

and first introduced in [62]. This norm fits the definition of invariant norm, since it satisfies

$$\|\Lambda_\varepsilon^r x\|_{r,\nu} = \varepsilon \|x\|_{r,\nu},$$

for every $\varepsilon > 0$ and $x \in \mathbb{R}^n$.

Now we reproduce Theorem 5 using again the hypothesis of existence of an invariant norm.

Theorem 7. *Let system (2.1) be such that the origin is an attractive equilibrium point. Suppose also that f is G-invariant for a normed action $\phi : G \times \mathbb{R}^n \to \mathbb{R}^n$. Then the origin is a stable equilibrium point.*

Proof. We only need to ensure that hypotheses (1) and (2) from Theorem 5 are satisfied.

For Hypothesis (1) we remind that, since ξ is surjective (see Lemma 4), for every $\delta \in \mathbb{R}$ and every $x_0 \neq 0$, we can choose $g \in G$ such that $\xi(g) \geq \delta / \|x\|_{inv}$ and consequently

$\|\phi_g(x)\|_{inv} \geq \delta$. Since $\|\phi_g(x)\| \geq \alpha_2^{-1}(\|\phi_g(x)\|_{inv})$ for some $\alpha_2 \in \mathscr{K}_\infty$ (after the Definition 23), we can choose δ such that $\alpha_2^{-1}(\delta) \geq \delta_0$, and thus the corresponding g will satisfy $\|\phi_g(x)\| \geq \delta_0$, as wanted.

Now we prove Hypothesis (2). Let $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ be the functions of Definition 23. Let $M, \delta > 0$, and $x \in S_\delta$. Then we have

$$\phi_g(B_M[0]) \subset \phi_g(B^{inv}_{\alpha_2(M)}[0]) = B_{\frac{\|\phi_g(x)\|_{inv}}{\|x\|_{inv}}\alpha_1(M)}[0]$$

We can estimate

$$\frac{\|\phi_g(x)\|_{inv}}{\|x\|_{inv}} \leq \frac{\alpha_2(\|\phi_g(x)\|)}{\alpha_2(\|x\|)} = \frac{\alpha_2(\|\phi_g(x)\|)}{\alpha_2(\delta)}$$

Thus, by choosing $\alpha_{M,\delta}(\varepsilon) = \alpha_1(\varepsilon)\alpha_2(M)/\alpha_2(\delta)$, we can see that $\alpha_{M,\delta} \in \mathscr{K}$ and we have that

$$\phi_g(B_M[0]) \subset B_{\alpha_{M,\delta}(\|\phi_g(x)\|)}[0],$$

for every $x \in S_\delta$, proving Hypothesis (2). □

Next result shows that we can infer global behavior from local, following what should intuitively extend from the theory of linear and homogeneous systems.

Corollary 2. *Let system (2.1) be θ-invariant for a normed action $\phi : G \times \mathbb{R}^n \to \mathbb{R}^n$. If the origin is a locally attractive equilibrium point, then it is also globally asymptotically stable.*

Proof. By Theorem 7, the origin is stable. So all we need to prove is that $\lim_{t\to\infty} x(t;x_0) = 0$, for every $x_0 \in \mathbb{R}^n$.

Let $x_0 \in \mathbb{R}^n \setminus \{0\}$ and $\delta > 0$ be such that $x(t;y_0) \to 0$, for every $y_0 \in B^{inv}_\delta(0)$. If we chose $g \in G$ such that $\xi(g) < \delta/\|x_0\|_{inv}$ and define $y_0 = \phi_g(x_0)$, then $\|y_0\|_{inv} = \|\phi_g(x_0)\|_{inv} < \delta$, therefore $x(t;y_0) \to 0$. But, by scaling of trajectories, we have

$$x(t;x_0) = \phi_{g^{-1}}(x(\theta(g)t;y_0)$$

Since $\theta(g) \neq 0$ and ϕ is continuous, we have that the righ-hand side tends to 0, showing that $x(t;x_0) \to 0$ for an arbitrary $x_0 \in \mathbb{R}^n$. □

Another useful condition for the asymptotically stability of an equilibrium point is given in terms of the existence of invariant sets. The next result is a generalization of Theorem 6.1 of [63] for θ-invariant systems with normed actions.

Theorem 8. *Let system (2.1) be θ-invariant for a normed action $\phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ and such that the origin is an equilibrium point. Suppose that the system possesses a strictly positively invariant compact set A. Then the origin is globally asymptotically stable.*

Proof. Let $K \subset A$ be the maximal compact invariant set contained in A. By ([61], Lemma 2.1), this set is contained in $int(A)$ and it is attractive for all initial conditions belonging to A. Therefore, if we prove that $K = \{0\}$, then by Theorem 2 the result will follow.

By θ-invariance, we have that $\phi_g(K)$ is also an invariant set under f for all $g \in G$. By continuity of ϕ and the fact that $K \subset int(A)$, there exists a neighborhood U_e of the identity $e \in G$ such that $\phi_g(K) \subset A$, for all $g \in U_e$. But since K is the maximal invariant subset of A, it follows that $\phi_g(K) \subset K$, for all $g \in U_e$.

Suppose, by contradiction, that $K \neq \{0\}$, let $M = \max_{x \in K} \|x\|_{inv}$ and $z \in K$ such that $\|z\|_{inv} = M$. Let $\xi : G \to \mathbb{R}_{>0}$ be the homomorphism such that $\|\phi_g(x)\|_{inv} = \xi(g)\|x\|_{inv}$ for every $x \in \mathbb{R}^n$. Then we have that $\xi(g) = \|\phi_g(z)\|_{inv}/\|z\|_{inv} \leq 1$, for every $g \in U_e$, since $\phi_g(z) \in K$ and z has maximal norm in K, which can be written as $\xi(U_e) \subset]0,1]$. Using the Lie group property that every element of G is a finite product of elements of U_e, and the fact that ξ is a homomorphism, we have that $\xi(G) \subset]0,1]$ and therefore $\xi(G) = \{1\}$, contradicting the definition of normed action. Therefore $K = \{0\}$ as we wanted. □

2.3 Convergence rate of scalar systems

When dealing with systems having $\mathbb{R}$ as state space, we can search further for easily verifiable algebraic conditions over ϕ permitting invariant systems to demonstrate finite-time convergence. In this section we give one such condition.

Lemma 5. *Consider the one-dimensional G-invariant system (2.1) with $f \in C^0(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{0\})$ and action $\phi \in C^2(\mathbb{R}_{>0}\times\mathbb{R})$. For each $x \in \mathbb{R}\setminus\{0\}$, define*

$$\zeta_\phi(x) = x\left[\frac{\partial}{\partial g\partial x}\phi(g,x)\cdot\frac{\partial}{\partial g}\phi(g,x)^{-1}\right]_{g=1}. \tag{2.6}$$

Let $L_\phi^- = \liminf_{x\to 0}|\zeta_\phi(x)|$ and $L_\phi^+ = \limsup_{x\to 0}|\zeta_\phi(x)|$. If $L_\phi^+ < 1$, then $f'(x)$ is unbounded around 0. If $L_\phi^- > 1$, then f is differentiable at 0 and $f'(0) = 0$.

Proof. Since system (2.1) is invariant, we have

$$f(\phi_g(x)) = \phi_g'(x)[f(x)], \text{ for every } g \in \mathbb{R}_{>0} \text{ and } x \in \mathbb{R}.$$

Differentiation on the variable g yields

$$f'(\phi_g(x))\frac{\partial}{\partial g}\phi(g,x) = \frac{\partial}{\partial g\partial x}\phi(g,x)\cdot f(x),$$

for every $g \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}$. Making $g = 1$ and rearranging the terms, we have

$$|f'(x)| = |\zeta_\phi(x)|\left|\frac{f(x)}{x}\right|, \text{for every } x \in \mathbb{R}.$$

Let $x_0 > 0$. By the Mean Value Theorem, we can find $x_1 \in [0, x_0]$ such that

$$|f'(x_1)| = \frac{|f(x_0)|}{|x_0|} = \frac{1}{|\zeta_\phi(x_0)|}|f'(x_0)|.$$

Proceeding recursively, we can produce a decreasing sequence x_n such that $\lim_{n\to\infty} x_n = 0$

(using transfinite induction if needed), such that

$$|f'(x_n)| = \prod_{k=0}^{n-1} \frac{1}{|\zeta_\phi(x_k)|} |f'(x_0)|.$$

If $L_\phi^+ < 1$ and we choose x_0 small enough, then

$$|f'(x_n)| \geq \left(\frac{1}{L_\phi^+}\right)^n |f'(x_0)| \to \infty, \text{ as } n \to \infty,$$

proving that f' is unbounded near 0.

Likewise, if $L_\phi^- > 1$ and we choose x_0 small enough, we have

$$|f'(x_n)| \leq \left(\frac{1}{L_\phi^-}\right)^n |f'(x_0)| \to 0, \text{ as } n \to \infty.$$

Since $f \in C^1(\mathbb{R}\setminus\{0\}) \cap C^0(\mathbb{R})$, we have that $\lim_{x\to 0^+} f'(x) = 0$. The proof that $\lim_{x\to 0^-} f'(x) = 0$ is entirely analogous. □

It is well-known that finite-time stable smooth dynamical systems have right-hand sides with unbounded derivatives at the equilibrium. This result establishes when we can have this analytical feature purely from looking at the action ϕ. This allows us to formulate the following result:

Theorem 9. *If a scalar system (2.1) is G-invariant for an action ϕ such that $L_\phi^- > 1$, then the origin cannot be FTS.*

We hope to extend this result to systems in higher dimensions through differentiation in the direction of the group infinitesimal generators, in order to provide easy-to-compute necessary conditions for a system to be FTS purely through its symmetries.

2.4 Conclusion

In this chapter we made a dynamical analysis of θ-invariant autonomous systems with equilibrium solutions. We compared its dynamical features with those of G-invariant systems. We then established stability properties that extend naturally from similar ones encountered in homogeneous systems. Finally, we introduced the concept of invariant norm, which brings together the conditions necessary to establish those stability results.

Chapter 3

Canonical form of θ-invariant observers

In this chapter we extend our definition and properties of θ-invariance to allow for systems with inputs, and we propose an extention of the framework for observer design proposed initially for G-invariant systems [27]. This extention includes homogeneous systems of arbitrary degree. It also allows for a richer range of choice on the measurement of state errors, while guaranteeing that its dynamics itself will be θ-invariant.

The main results of this chapter were presented in the Conference on Decision and Control (CDC) 2024 (see [LimaCDC24]]).

3.1 θ-invariant input-output systems

The object of study in this chapter is a general system with inputs and outputs. We consider $\mathscr{X} \subset \mathbb{R}^n, \mathscr{U} \subset \mathbb{R}^m$ and $\mathscr{Y} \subset \mathbb{R}^p$ to be open sets and the system given by the differential equation

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \\ y(t) &= h(x(t)), \end{aligned} \tag{3.1}$$

where $x(t) \in \mathscr{X}$ is the state, $u : \mathbb{R}_+ \to \mathscr{U}$ is a measurable function regarded as input and $y(t) \in \mathscr{Y}$ is the measured output. We assume the function $h : \mathscr{X} \to \mathscr{Y}$ to be continuous, the function $f : \mathscr{X} \times \mathscr{U} \to \mathbb{R}^n$ to be locally Lipschitz and the system (3.1) to be forward-complete, denoting the solutions for $x(0) = x_0$ and $u \in \mathscr{L}_\infty^m$ by $x(t; x_0; u)$.

The notion of θ-invariance can easily be applied to systems with inputs, by the addition of an action over the input space.

Definition 25. *Let $\mathscr{X} \subset \mathbb{R}^n$ and $\mathscr{U} \subset \mathbb{R}^m$ be open sets and G be a Lie group, with differentiable actions $\phi : G \times \mathscr{X} \to \mathscr{X}$ and $\psi : G \times \mathscr{U} \to \mathscr{U}$. Let $\theta : G \to \mathbb{R}_{>0}$ be a differentiable function. The system (3.1) is said to be* θ-invariant *with respect to G and the actions ϕ and ψ if the following holds:*

$$D\phi_g(x)\,[f(x,u)] = \theta(g) f(\phi_g(x), \psi_g(u)), \tag{3.2}$$

for every $x \in \mathscr{X}, u \in \mathscr{U}$ and $g \in G$.

As expected, the θ-invariance of systems with inputs also implies a scaling of trajectories, entirely analogous to the autonomous version with the addition of the u coordinate scaling. To avoid repetition, we make the convention that ϕ always denotes an action of G over the state space $\mathscr{X} \subset \mathbb{R}^n$ and ψ denotes an action over the input space $\mathscr{U} \subset \mathbb{R}^m$.

Definition 26. *A* trajectory *of the system (3.1) is a curve $\delta : \mathbb{R}_{\geq 0} \to \mathscr{X} \times \mathscr{U}$ given by $\delta(t) = (x(t), u(t)) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $\dot{x}(t) = f(x(t), u(t))$.*

Proposition 9. *The system (3.1) is θ-invariant with respect to the actions ϕ and ψ and the homomorphism θ if, and only if, for every trajectory $t \mapsto (x(t), u(t))$ of (3.1), the curve $t \mapsto (\phi_g(x(\theta(g)t)), \psi_g(u(\theta(g)t)))$ is also a trajectory.*

The proof is analogous to the one given in Proposition 2.4.

3.2 θ-invariant pre-observers

We start this section by briefly recalling the concept of pre-observers and their canonical forms for invariant systems from [27].

Definition 27. *A system* $\dot{\hat{x}} = F(\hat{x},u,y)$ *is called a* pre-observer *of* (3.1) *if for all* $x \in \mathscr{X}$ *and* $u \in \mathscr{U}$,

$$F(x,u,h(x)) = f(x,u). \tag{3.3}$$

The pre-observer dynamics possesses one extra variable in comparison to the original system. Therefore, before rigorously defining the concept of G-invariant pre-observers, we need to introduce a symmetry in the output space.

Definition 28. *Let* (3.1) *be* θ*-invariant for actions* ϕ, ψ *and a homomorphism* θ. *The output* $y = h(x)$ *is said to be compatible if there exists an action* $\rho : G \times \mathscr{Y} \to \mathscr{Y}$ *such that*

$$h(\phi_g(x)) = \rho_g(h(x)),$$

for every $g \in G$ *and* $x \in \mathscr{X}$.

Definition 29. *Let* (3.1) *be a* θ*-invariant system with a* G*-compatible output, with respect to group actions* $\phi : G \times \mathscr{X} \to \mathscr{X}$, $\psi : G \times \mathscr{U} \to \mathscr{U}$ *and* $\rho : G \times \mathscr{Y} \to \mathscr{Y}$. *A pre-observer* $\dot{\hat{x}} = F(\hat{x},u,y)$ *is called a* θ-invariant pre-observer *for* (3.1) *if for all* $x \in \mathscr{X}$, $u \in \mathscr{U}$, $y \in \mathscr{Y}$, *and* $g \in G$,

$$D\phi_g(x)\,[F(x,u,y)] = \theta(g) F(\phi_g(x), \psi_g(u), \rho_g(y)). \tag{3.4}$$

If θ *is the constant homomorphism* $\theta \equiv 1$, *we simply say that* $\dot{\hat{x}} = F(\hat{x},u,y)$ *is a* G*-invariant pre-observer.*

It is known that for a G-invariant system, any G-invariant pre-observer has the form

$$\dot{\hat{x}} = F(\hat{x},u,y) = f(\hat{x},u) + \sum_{i=1}^{n} \mathscr{L}_i\left(I(\hat{x},u), E(\hat{x},u,y)\right) w_i(\hat{x}), \tag{3.5}$$

where I is an invariant function, E is the invariant output error and $\{w_1,...,w_n\}$ is an invariant frame [27].

As we will see in the main result of this chapter, the formula is repeated for θ-invariant systems and many of the elements featuring in the formula for θ-invariant pre-observers coincide with those appearing in the G-invariant version [27] - such as invariant functions and invariant output errors - while others have to be corrected by a scaling factor - namely the invariant frame and, later, the state error. The former elements were introduced in preliminary Section 1.2.3. It is left to define and prove the existence of an adapted version of the invariant frame.

Definition 30. *A θ-invariant frame for a space $\mathscr{X}$ and an action $\phi : G \times \mathscr{X} \to \mathscr{X}$ is a set $\{w_1,\dots,w_n\}$ of θ-invariant vector fields such that $\{w_1(x),\dots,w_n(x)\}$ forms a basis of the tangent space $T_x\mathscr{X}$, for every $x \in \mathscr{X}$.*

Lemma 6. *Let $\mathscr{X}$ be an open subset of $\mathbb{R}^n$ and G a Lie group of dimension $r \leq n$. Given an action $\phi : G \times \mathscr{X} \to \mathscr{X}$ that is full-rank at every point, there always exists a θ-invariant frame $\{\widetilde{w}_1,\dots,\widetilde{w}_n\}$.*

Proof. Let $\{w_1,\dots,w_n\}$ be an invariant frame for the action ϕ (the existence of which is guaranteed by Proposition 3) and define $\widetilde{w}_i(x) = \theta(\gamma(x))^{-1}w_i(x)$, where $\gamma(x)$ is a moving frame for ϕ. Since $\{w_1(x),\dots,w_n(x)\}$ is a linearly independent set and $\theta(\gamma(x))$ is a non-zero real number, it follows that $\{\widetilde{w}_1,\dots,\widetilde{w}_n\}$ is also linearly independent. To prove the θ-invariance of each w_i, a direct computation using the property $\gamma(\phi_g(x))g = \gamma(x)$ gives us

$$\begin{aligned}
\widetilde{w}_i(\phi_g(x)) &= \theta(\gamma(\phi_g(x))^{-1}w_i(\phi_g(x)) \\
&= \theta(g)\theta(\gamma(x))^{-1}D\phi_g(x)[w_i(x)] \\
&= \theta(g)D\phi_g(x)[\theta(\gamma(x))^{-1}w_i(x)] \\
&= \theta(g)D\phi_g(x)[\widetilde{w}_i(x)]
\end{aligned}$$

which is the definition of a θ-invariant frame. □

With the proper frame we can finally classify all possible θ-invariant pre-observers, given a θ-invariant system with compatible output.

Theorem 10. *Suppose that (3.1) is θ-invariant with a compatible output. Assume that for each $x \in \mathscr{X}$, the mapping $g \to \phi_g(x)$ is full-rank. Then, $\dot{\widehat{x}} = F(\widehat{x},u,y)$ is a θ-invariant pre-observer if and only if*

$$F(\widehat{x},u,y) = f(\widehat{x},u) + \sum_{i=1}^{n} \mathscr{L}_i\left(I(\widehat{x},u), E(\widehat{x},u,y)\right) w_i(\widehat{x}), \tag{3.6}$$

where I is an invariant function (that is, $I(\phi_g(x),\psi_g(u)) = I(x,u)$, for all $x \in \mathscr{X}, u \in \mathscr{U}$ and $g \in G$), E is an output error, the $\mathscr{L}_i$ are smooth functions such that for all $\widehat{x} \in \mathscr{X}$ and $u \in \mathscr{U}$, $\mathscr{L}_i(I(\widehat{x},u),0) = 0$, and $\{w_1,\ldots,w_n\}$ is a θ-invariant frame.

Proof. Let $\dot{\widehat{x}} = F(\widehat{x},u,y)$ be a θ-invariant pre-observer and $\{\widetilde{w}_i,\ldots,\widetilde{w}_n\}$ a θ-invariant frame. We start by decomposing $F(\widehat{x},u,y) = \sum_{i=1}^{n} F_i(\widehat{x},u,y)\widetilde{w}_i(\widehat{x})$ and we want to prove that each F_i is a G-invariant scalar function. For that, we compute

$$\begin{aligned}
&\sum_{i=1}^{n} F_i(\phi_g(\widehat{x}),\psi_g(u),\rho_g(y))\widetilde{w}_i(\phi_g(x)) \\
&\quad = F(\phi_g(\widehat{x}),\psi_g(u),\rho_g(y)) \\
&\quad = \frac{1}{\theta(g)} D\phi_g(\widehat{x})[F(\widehat{x},u,y)] \\
&\quad = \frac{1}{\theta(g)} \sum_{i=1}^{n} F_i(\widehat{x},u,y) D\phi_g(\widehat{x})[\widetilde{w}_i(\widehat{x})] \\
&\quad = \frac{1}{\theta(g)} \sum_{i=1}^{n} F_i(\widehat{x},u,y)\theta(g)\widetilde{w}_i(\widehat{x}) \\
&\quad = \sum_{i=1}^{n} F_i(\widehat{x},u,y)\widetilde{w}_i(\widehat{x}).
\end{aligned}$$

Since $\{\widetilde{w}_1(\phi_g(\widehat{x})),\ldots,\widetilde{w}_n(\phi_g(\widehat{x}))\}$ is a basis of $T_{\phi_g(x)}\mathscr{X}$, we have the equality

$$F_i(\phi_g(\widehat{x}),\psi_g(u),\rho_g(y)) = F_i(\widehat{x},u,y),$$

for $i = 1,\ldots,n$.

Define

$$\widetilde{E}_i(\widehat{x},u,y) = F_i(\widehat{x},u,y) - F_i(\widehat{x},u,h(\widehat{x})).$$

The invariance of the functions F_i implies that the function $\widetilde{E} = (\widetilde{E}_1,\ldots,\widetilde{E}_n)$ is an invariant output error, and therefore by Proposition 2, we can write

$$\widetilde{E}_i(\widehat{x},u,y) = \mathscr{L}_i(I(\widehat{x},u),E(\widehat{x},u,y)),$$

where the functions $\mathscr{L}_i, I$ and E are as in the Theorem's statement. To conclude, we write

$$\begin{aligned} & F(\widehat{x},u,y) = f(\widehat{x},u) + (F(\widehat{x},u,y) - f(\widehat{x},u)) \\ = \;& f(\widehat{x},u) + \sum_{i=1}^{n} (F_i(\widehat{x},u,y) - F_i(\widehat{x},u,h(\widehat{x})))\,\widetilde{w}_i(\widehat{x}) \\ = \;& f(\widehat{x},u) + \sum_{i=1}^{n} \mathscr{L}_i(I(\widehat{x},u),E(\widehat{x},u,y))\widetilde{w}_i(\widehat{x}). \end{aligned}$$

□

We recall that the previous definition of G-invariance [27] fails to include the important class of homogeneous systems and weighted homogeneous systems, to which a large body of the theory on observer design and stabilization techniques have been proposed [64, 65].

We now explicitly construct all possible θ-invariant pre-observers for a homogeneous system.

Example 3. *Suppose that system (3.1) is* **r***-homogeneous of degree* μ *having* $y = h(x) = x_1$ *as the system output. Example 2 together with the addition of the identity action on the input show that the group of symmetries* $G = \mathbb{R}_{>0}$ *with actions* $\phi_g(x) = \Lambda_g x$, $\psi_g(u) = u$ *and*

$\theta(g) = g^{\mu}$ make the system θ-invariant, and a direct computation shows that $\rho_g(y) = g^{r_1} y$ makes the output G-compatible.

The condition that $g \mapsto \phi_g(x)$ is full-rank is met for every $x \neq 0$, so Theorem 10 applies in $\mathbb{R}^n \setminus \{0\}$.

We can write $x = (\omega, \widetilde{x})$, where $\omega = \|x\|_r$ and $\widetilde{x} = \Lambda^r_{\omega^{-1}} x$, so that the action ϕ becomes

$$\phi_g(x) = (g\omega, \widetilde{x})$$

The moving frame in this case is the solution to the normalization equation

$$1 = \gamma(x)\omega$$

therefore $\gamma(x) = \frac{1}{\omega} = \frac{1}{\|x\|_r}$.

The invariant functions defined on $\mathbb{R}^n \setminus \{0\}$ are given by

$$I(x,u) = \phi^b_{\gamma(x)}(x) = \Lambda^r_{\|x\|_r^{-1}} x = (\|x\|_r^{-r_1} x_1, \ldots, \|x\|_r^{-r_n} x_n)$$

Following Lemma 6, the θ-invariant frame is the set of vector fields formed by the columns of the matrix

$$w(x) = \theta(\gamma(x))^{-1} (D\phi_{\gamma(x)}(x))^{-1}$$

$$= (\|x\|_r^{-\mu} \operatorname{diag}(\|x\|_r^{-r_1}, \ldots, \|x\|_r^{-r_n}))^{-1}$$

$$= \operatorname{diag}(\|x\|_r^{\mu+r_1}, \ldots, \|x\|_r^{\mu+r_n})$$

The invariant output error is

$$E(\widehat{x}, u, y) = \rho_{\gamma(\widehat{x})}(y) - \rho_{\gamma(\widehat{x})}(h(\hat{x}))$$

$$= \|\hat{x}\|_r^{-r_1} (x_1 - \hat{x}_1)$$

Therefore, by Theorem 10 any invariant pre-observer reads

$$\begin{aligned} F(\widehat{x},u,y) &= f(\widehat{x},u)+\sum_{i=1}^{n}\mathscr{L}_i(I(\widehat{x},u),E(\widehat{x},u,y))\widetilde{w}_i(x) \\ &= f(\widehat{x},u)+\mathrm{diag}(\|\widehat{x}\|_r^{\mu+r_1},...,\|\widehat{x}\|_r^{\mu+r_n})\cdot \\ &\quad L\left(I(\hat{x},u),\rho_{\gamma(\widehat{x})}(y)-\rho_{\gamma(\widehat{x})}(h(\hat{x}))\right) \\ &= f(\widehat{x},u)+\mathrm{diag}(\|\widehat{x}\|_r^{\mu+r_1},...,\|\widehat{x}\|_r^{\mu+r_n})\cdot \\ &\quad L\left(\|\hat{x}\|_r^{-r_1}\hat{x}_1,...,\|\hat{x}\|_r^{-r_n}\hat{x}_n,u,\|\hat{x}\|_r^{-r_1}(x_1-\hat{x}_1)\right) \end{aligned}$$

for some $L\in\mathbb{R}^{n\times p}$ such that $L(0)=0$ and some $\mu\in\mathbb{R}$. The last property allows us to simplify even further, defining a new smooth function $\overline{L}:\mathbb{R}^n\to\mathbb{R}^n$ such that $L(E(\hat{x},y))=\overline{L}(E(\hat{x},y))\cdot E(\hat{x},y)$ (see [27]). So any invariant pre-observer writes as

$$\begin{aligned} \dot{\hat{x}} &= f(\widehat{x},u)+(x_1-\hat{x}_1)\mathrm{diag}(\|\widehat{x}\|_r^{\mu},...,\|\widehat{x}\|_r^{\mu+r_n-r_1})\cdot \\ &\quad \overline{L}\left(\|\hat{x}\|_r^{-r_1}\hat{x}_1,...,\|\hat{x}\|_r^{-r_n}\hat{x}_n,u,\|\hat{x}\|_r^{-r_1}(x_1-\hat{x}_1)\right). \end{aligned}$$

The simplest example of homogeneous systems are the linear ones. For them, we can explicitly compute the largest possible group of symmetries given by linear actions.

Theorem 11. *Consider a linear time-invariant system with output*

$$\begin{aligned} \dot{x} &= Ax+Bu \\ y &= Cx \end{aligned} \tag{3.7}$$

for $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$ and $C\in\mathbb{R}^{p\times n}$. Suppose that the system is G-invariant under linear actions $\phi:G\times\mathscr{X}\to\mathscr{X}$, $\psi:G\times\mathscr{U}\to\mathscr{U}$, and the output is G-compatible under an action $\rho:G\times\mathscr{Y}\to\mathscr{Y}$ where G is a subgroup of $GL(n)$. Denote

$$\begin{aligned} G_{(A,B,C)}=\{g\in GL(n) : gA=Ag, g(\mathrm{Im}\,B)\subset \mathrm{Im}\,B \\ \text{and } \ker C\subset g^{-1}(\ker C)\}. \end{aligned} \tag{3.8}$$

Then, G is a subgroup of $G_{(A,B,C)}$.

Proof. The invariance equation gives us

$$\begin{aligned} Agx + B\psi_g(u) &= f(\phi_g(x), \psi_g(u)) \\ &= D\phi_g(x)[f(x,u)] = gAx + gBu \end{aligned}$$

for every $x \in \mathscr{X}$ and $u \in \mathscr{U}$. Making $u = 0$, that implies

$$Agx = gAx \text{ for every } x \in \mathscr{X}$$

and therefore $Ag = gA$, for every $g \in G$.

Making $x = 0$, we have $gBu = B\psi_g(u)$, which has a solution $\psi_g(u)$ for each u if and only if $g(\mathrm{Im}B) \subset \mathrm{Im}B$ and if it is the case, we can choose $\psi_g(u)$ to be linear in u.

By the equation that defines compatibility, we have a unique choice of output action:

$$\rho_g(Cx) = C\phi_g(x) = Cgx$$

This equation is well defined if and only if for any pair $x_1, x_2 \in \mathscr{X}$ such that $Cx_1 = Cx_2$ we also have $Cgx_1 = Cgx_2$. By linearity, this is equivalent to the implication $Cx = 0 \implies Cgx = 0$, for every $x \in \mathscr{X}$. In other words, we must have $\ker C \subset \ker Cg = g^{-1}(\ker C)$, for every $g \in G$.

Hence, G is a subgroup of $G_{(A,B,C)}$. □

Similar characterizations of symmetries were considered for systems with no inputs to determine when they are homogeneous [65].

3.3 Error dynamics

This canonical form is particularly useful because, by combining it with a more suitable definition of the state estimation, an error dynamic equation with fewer variables can be obtained, making it simpler for analysis. For instance, it was proven that a particular choice of state error yields a dynamical equation that depends only on itself and an invariant function of x and u.

Theorem 12. *[27] The invariant state error dynamics*

$$\eta(x,\widehat{x}) = \phi_{\gamma(x)}(\widehat{x}) - \phi_{\gamma(x)}(x)$$

can be written as

$$\dot{\eta} = \Upsilon(\eta, I(x,u))$$

where Υ is a smooth function and I is an invariant function.

Therefore, the shape of pre-observers proposed in Proposition 2 keeps its meaning for θ-invariant systems.

With a similar rescaling, we can adjust Theorem 12 to obtain an invariant state error whose dynamics are simplified.

Theorem 13. *Let $\dot{x} = f(x,u)$ be a θ-invariant system with a compatible output $y = h(x)$ and $\dot{\hat{x}} = F(\hat{x},u,h(x))$ be a θ-invariant pre-observer. Let $\eta : \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ be such that $\eta_x = \eta(x,\cdot)$ is invertible for every $x \in \mathscr{X}$ and $\varphi : G \times \mathscr{X} \to \mathscr{X}$ be a group action. Define*

$$\eta(x,\hat{x}) = \varphi_{\gamma(x)^{-1}}(\phi_{\gamma(x)}(x) - \phi_{\gamma(x)}(\hat{x})) \tag{3.9}$$

Then, for every trajectory $t \mapsto (x(t),u(t))$ of the system, we have

$$\frac{d}{dt}\eta(x(t),\hat{x}(t)) = \sigma(\eta,x,u),$$

where $\sigma : \mathscr{X} \times \mathscr{X} \times \mathscr{U} \to \mathbb{R}^n$ *satisfies*

$$\sigma(\varphi_g(\eta), \phi_g(x), \psi_g(u)) = \theta(g) D\varphi_g(\eta)[\sigma(\eta, x, u)], \tag{3.10}$$

for every $g \in G$, $x, \hat{x} \in \mathscr{X}$ *and* $u \in \mathscr{U}$.

Proof. First, a simple computation using the property $\gamma(\phi_g(x)) = \gamma(x)g^{-1}$ give us that

$$\eta(\phi_g(x), \phi_g(\hat{x})) = \varphi_g(\eta(x, \hat{x})), \tag{3.11}$$

for every $g \in G$ and $x, \hat{x} \in \mathscr{X}$.

Let $\overline{\sigma}(x, \hat{x}, u) = \frac{d}{dt}\eta(x, \hat{x})$. A simple application of the chain rule leads to

$$\overline{\sigma}(x, \hat{x}, u) = \frac{\partial \eta}{\partial x}(x, \hat{x}) f(x, u) + \frac{\partial \eta}{\partial \hat{x}}(x, \hat{x}) F(\hat{x}, h(x), u).$$

Equation (3.11) gives us

$$\frac{d}{dt}\eta(\phi_g(x), \phi_g(\hat{x})) = \frac{d}{dt}\varphi_g(\eta(x, \hat{x}))$$

The left-hand side is computed as

$$\begin{aligned}
&\frac{\partial \eta}{\partial x}(\phi_g(x), \phi_g(\hat{x}))(x, \hat{x}) D\phi_g(x)[f(x, u)] \\
&+ \frac{\partial \eta}{\partial \hat{x}}(\phi_g(x), \phi_g(\hat{x})) D\phi_g(\hat{x})[F(\hat{x}, h(x), u)] \\
&= \frac{\partial \eta}{\partial x}(\phi_g(x), \phi_g(\hat{x}))\theta(g)^{-1} f(\phi_g(x), \psi_g(u)) \\
&+ \frac{\partial \eta}{\partial \hat{x}}(\phi_g(x), \phi_g(\hat{x}))\theta(g)^{-1} F(\phi_g(\hat{x}), h(\phi_g(x))) \\
&= \theta(g)^{-1} \overline{\sigma}(\phi_g(x), \phi_g(\hat{x}), \psi_g(u)),
\end{aligned}$$

and the right-hand side as

$$D\varphi_g(\eta)[\frac{d}{dt}\eta(x,\hat{x})] = D\varphi_g(\eta)[\overline{\sigma}(x,\hat{x},u)],$$

so we have the equality

$$\overline{\sigma}(\phi_g(x),\phi_g(\hat{x}),\psi_g(u)) = \theta(g)D\varphi_g(\eta)[\overline{\sigma}(x,\hat{x},u)].$$

Since η is invertible in $\hat{x}$ and invariant, we can write

$$\hat{x} = v(\eta,x),$$

for a function v such that $\phi_g(v(\eta,x)) = v(\varphi_g(\eta),\phi_g(x))$, for every $g \in G$ and $\eta,x,\hat{x} \in \mathscr{X}$.

Thus, we can define

$$\sigma(\eta,x,u) = \overline{\sigma}(x,v(\eta,x),u).$$

By definition we have

$$\frac{d}{dt}\eta(x,\hat{x}) = \sigma(\eta,x,u),$$

so we are left to prove property (3.10). For that we compute directly

$$\sigma(\varphi_g(\eta),\phi_g(x),\psi_g(u))$$

$$= \overline{\sigma}(\phi_g(x),\phi_g(v(x,\eta)),\psi_g(u))$$

$$= \theta(g)D\varphi_g(\eta)[\overline{\sigma}(x,v(x,\eta),u)]$$

$$= \theta(g)D\varphi_g(\eta)[\sigma(\eta,x,u)].$$

□

We remark that the case $\varphi_g \equiv Id$ corresponds to Theorem 12.

Example 4. *Consider the SIR model with output, given by*

$$\begin{aligned}\dot{S} &= \frac{-\beta SI}{S+I+R}\\ \dot{I} &= \frac{\beta SI}{S+I+R} - \nu I\\ \dot{R} &= \nu I - \alpha R\\ h(S,I,R) &= R,\end{aligned}$$

where S, I, $R \in \mathbb{R}_{\geq 0}$ are the states defining the sizes of susceptible, infected and recovered populations, respectively, and β, ν, $\alpha > 0$ are parameters. The variable R is a natural choice for the output since it makes the system observable. The system possesses as a symmetry the simple action $\phi : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ given by $\phi_g(S,I,R) = (gS, gI, gR)$ and the action $\rho_g(R) = gR$ makes the output G-compatible. It has also been shown [66] that the variable S dynamics can be written as a second order ordinary differential equation independent from I and R, given by

$$\ddot{S} = S^{-1}\left((\alpha - \nu)S\dot{S} + 2\dot{S}^2\right)$$

which possesses 8 one-dimensional symmetries.

Next, we will write the invariant pre-observer together with the respective error dynamics. For simplicity we are going to denote the state as $x = (S,I,R)$, the observed state as $\hat{x} = (\hat{S},\hat{I},\hat{R})$ and the dynamical equation (3.12) as $\dot{x} = f(x); h(x) = R = y$.

The first step is to obtain the moving frame $\gamma(x)$, of the action $\phi_g(S,I,R) = (gS, gI, gR)$. For that we nottice that, through a change of variables we can write the state as (N,I,R), where $N = S+I+R$. That allows us to split the action into its invertible part $\phi^a_g(x) = gN$ and the remainder part $\phi^b_g(x) = (gI, gR)$. Setting $c = 1 \in Im\phi^a_g$, the normalization equation gives us $\gamma(x)N = 1 \implies \gamma(x) = N^{-1}$.

The invariant functions $I(x)$ are given as functions of the variables $\phi^b_{\gamma(x)}(x) = (I/N, R/N)$.

The invariant output error is given by the formula

$$E(\hat{x},y) = \rho_{\gamma(\hat{x})}(y) - \rho_{\gamma(\hat{x})}(h(\hat{x})) = \hat{N}^{-1}(\hat{R}-R)$$

The invariant frame is given by the columns of the matrix

$$w(x) = (D\phi_{\gamma(x)}(x))^{-1} = N \cdot Id,$$

so we have $w_i(x) = N \cdot e_i$, *for* $i = 1,2,3$.

Now, using Theorem 10, we have that any invariant pre-observer of the SIR-model writes

$$\dot{\hat{x}} = f(\hat{x}) + \hat{N}L\left(\frac{\hat{I}}{\hat{N}}, \frac{\hat{R}}{\hat{N}}, \frac{\hat{R}-R}{\hat{N}}\right),$$

for a function $L : \mathbb{R}^3 \to \mathbb{R}^3$ *such that*

$$L\left(\frac{\hat{I}}{\hat{N}}, \frac{\hat{R}}{\hat{N}}, 0\right) = 0$$

Using the last property we can rewrite it as

$$\dot{\hat{x}} = f(\hat{x}) + \hat{N}\left(\frac{\hat{R}-R}{\hat{N}}\right)\overline{L}\left(\frac{\hat{I}}{\hat{N}}, \frac{\hat{R}}{\hat{N}}, \frac{\hat{R}-R}{\hat{N}}\right)$$

$$= f(\hat{x}) + (\hat{R}-R)\overline{L}\left(\frac{\hat{I}}{\hat{N}}, \frac{\hat{R}}{\hat{N}}, \frac{\hat{R}-R}{\hat{N}}\right)$$

By choosing $\varphi_g(\eta) = g\eta$ *in Theorem 13, we have that the dynamically invariant state error is the simple subtraction*

$$\eta(x,\hat{x}) = (\eta_S, \eta_I, \eta_R) = (S-\hat{S}, I-\hat{I}, R-\hat{R}).$$

Defining $\eta_N = N - \hat{N}$, *we write the error dynamics as*

$$\dot{\eta} = f(x) - f(x+\eta) - \eta_R \cdot \overline{L}\left(\frac{I+\eta_I}{N+\eta_N}, \frac{R+\eta_R}{N+\eta_N}, \frac{\eta_R}{N+\eta_N}\right),$$

which is guaranteed to be G-invariant in the state variable η and input variable x for the actions φ and ϕ.

3.4 Conclusion

In chapter we characterized all the possible θ-invariant pre-observers of a given θ-invariant system, thus providing a framework for observer design. Additionally, we have explicitly computed the canonical form of a θ-invariant pre-observer for the class of weighted homogeneous systems and proved that for such a class the error dynamics can be made θ-invariant as well. The stability analysis of such systems is a theme of our ongoing study.

Chapter 4

Observer design for bilinear Persidskii systems

4.1 Bilinear Persidskii Systems

The class of Persidskii systems was introduced in 1961 [30]. Its main interest is in that the stability analysis can be made rather simply through the design of Lyapunov functions given by integrals of the nonlinear terms [31]. In this chapter we extend it by adding a bilinear term, and design an IOS-based observer for the extended class. The new class include three models of interest in the litterature [1, 2, 39]. One of these models is also considered for its simplicity in the design of observers using the invariance framework [2, 27].

Some results of this chapter were presented at European Control Conference (ECC) 2024 (see [LimaECC24]]) and some were accepted for publication at International Journal of Control (IJC) [LimaIJC25].

4.1.1 Bilinear Persidskii systems

We start by defining the class of systems of interest, and then providing the models that motivate the study of such systems.

Let us consider the following class of systems:

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^{M} A_j f^j\left(H_j x(t)\right) + Gw(t) + \sum_{i=1}^{n} x_i(t) \sum_{j=1}^{M} B_j^i f^j(H_j x(t)) + \sum_{i=1}^{n} x_i(t) D_i w(t), \tag{4.1}$$

$$y(t) = \begin{pmatrix} C_0 x(t) \\ C_1 f^1(H_1 x(t)) \\ \vdots \\ C_M f^M\left(H_M x(t)\right) \end{pmatrix}, \ t \in \mathbb{R}_+,$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^\ell$ is the unknown input with $w \in \mathscr{L}_\infty^\ell$, $y(t) \in \mathbb{R}^z$ is the measured output, $A_0 \in \mathbb{R}^{n\times n}$, $A_j \in \mathbb{R}^{n\times k_j}$, $H_j \in \mathbb{R}^{k_j\times n}$, $B_j^i \in \mathbb{R}^{n\times k_j}$, $C_0 \in \mathbb{R}^{z_0\times n}$, $C_j \in \mathbb{R}^{z_j\times k_j}$ and $G, D_i \in \mathbb{R}^{n\times \ell}$ for $i = 1,...,n$ and $j = 1,...,M$, $z = \sum_{s=0}^{M} z_s$, $f^j : \mathbb{R}^{k_j} \to \mathbb{R}^{k_j}$ for $j = 1,...,M$ with the hypothesis that $f^j(s) = \begin{pmatrix} f_1^j(s_1) & \dots & f_{k_j}^j(s_{k_j}) \end{pmatrix}^\top$ for $s \in \mathbb{R}^{k_j}$ and

$$\tau f_i^j(\tau) > 0, \text{ for every } \tau \neq 0 \text{ and } i = 1,\dots,k_j. \tag{4.2}$$

For brevity of notation, we will use the convention $k_0 = n$, $H_0 = I_n$, $f^0(x) = x$, and $K = \sum_{j=1}^{M} k_j$.

After a suitable rearrangement of the nonlinearities f_i^j, we can consider $0 \leq \mu \leq m \leq M$ be such that for all $s = 1,\dots,m$ and $i = 1,\dots,k_s$, we have $\lim_{r\to\pm\infty} f_i^s(r) = \pm\infty$ and for every $s = 1,\dots,\mu$ and $i = 1,\dots,k_s$, we have $\lim_{r\to\pm\infty} \int_0^\nu f_i^s(r)\,dr = +\infty$. Those indices ($\mu$ and m) will be useful in determining the radial unboundedness of the considered Lyapunov function and its time derivative.

The objective of this paper is to design a robust observer for the system (4.1) in the sense of Definition 16.

The system (4.1) includes the generalized Persidskii systems studied in [34], [17], [35] (the first line in (4.1)), with additional bilinear terms multiplied by the state components (the second line) [42], [41], [44], [40]. The motivation for considering this class is that it encapsulates some models of interest, which have been studied in the context of observer design and stabilization [1], [27].

Example 5. *The following model describes the bacterial growth of two distinct species inside a chemostat with a single limiting substrate [1]:*

$$\begin{aligned} \dot{x_1} &= (\mu_1(S) - D)x_1, \\ \dot{x_2} &= (\mu_2(S) - D)x_2, \\ \dot{S} &= (S_{\text{in}} - S)D - \rho_1\mu_1(S)x_1 - \rho_2\mu_2(S)x_2, \end{aligned} \tag{4.3}$$

where $x_i \geq 0$ is the concentration of the ith species ($i = 1, 2$), $S \geq 0$ is the concentration of the substrate, $D \geq 0$ is the diffusion rate, $S_{\text{in}} \geq 0$ is the nutrient inflow concentration, ρ_1 and ρ_2 are positive constants and the functions μ_1, μ_2 are given by $\mu_1(S) = \frac{a_1 S}{b_1+S}$ and $\mu_2(S) = \frac{a_2 S}{b_2+S}$ with a_1, b_1, a_2, b_2 positive parameters.

Setting x_3 as S, define the state vector $x(t) = \begin{pmatrix} x_1(t) & x_2(t) & x_3(t) \end{pmatrix}^\top \in \mathbb{R}^3_+$. Now, we reparametrize the input as $w = \begin{pmatrix} D & u \end{pmatrix}^\top$, where $u = S_{\text{in}}D$, so we can see that (4.3) fits the model (4.1) for $n = 3$ and $M = 2$, by replacing $A_0 = A_1 = A_2 = 0_{3\times 3}$, $H_1 = H_2 = I_3$,

$$G = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, D_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}, D_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{pmatrix},$$

$$B_1^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -\rho_1 \end{pmatrix}, B_2^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\rho_2 \end{pmatrix},$$

$B_1^2 = B_2^1 = B_1^3 = B_2^3 = 0_{3\times 3}$, *and* $f^j(x) = \begin{pmatrix} \mu_j(x_1) & \mu_j(x_2) & \mu_j(x_3) \end{pmatrix}^\top$, $j = 1, 2$.

Example 6. *The following is a classical model of a chemical exothermic reactor [27]:*

$$\begin{aligned}\dot{X}_{in} &= 0,\\ \dot{X} &= D(X_{\text{in}} - X) - k\exp\left(\frac{-E}{RT}\right)X,\\ \dot{T} &= D(T_{in} - T) + c\exp\left(\frac{-E}{RT}\right)X + w,\end{aligned}$$

where $\begin{pmatrix} X_{\text{in}}(t) & X(t) & T(t)\end{pmatrix}^\top \in \mathbb{R}^3_+$ *is the state, representing the inlet composition, the reactor composition and the temperature inside the reactor, respectively;* $\begin{pmatrix} E & R & k & c & D\end{pmatrix}^\top \in \mathbb{R}^5_+$ *is the vector of constant parameters, and* $w(t) \in \mathbb{R}$ *is a nonnegative input.*

Setting $x = \begin{pmatrix} x_1 & x_2 & x_3\end{pmatrix}^\top = \begin{pmatrix} X_{\text{in}} & X & T\end{pmatrix}^\top$, *we have*

$$\dot{x} = \begin{pmatrix} 0 & Dx_1 - Dx_2 & -Dx_3\end{pmatrix}^\top + x_2\begin{pmatrix} 0 & -k\exp\left(-\frac{E}{Rx_3}\right) & c\exp\left(-\frac{E}{Rx_3}\right)\end{pmatrix}^\top + \begin{pmatrix} 0 & 0 & w + DT_{in}\end{pmatrix}^\top.$$

As before, this system is obtained from (4.1) for $n = 3$ *and* $M = 1$, *by replacing* $H_1 = I_3$, $D_1 = D_2 = D_3 = 0$, $A_0 = \begin{pmatrix} 0 & 0 & 0\\ D & -D & 0\\ 0 & 0 & -D\end{pmatrix}$, $B_1^2 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -k\\ 0 & 0 & c\end{pmatrix}$, $G = \begin{pmatrix} 0\\ 0\\ 1\end{pmatrix}$, $B_1^1 = B_1^3 = 0_{3\times 3}$ *and* $f^1(x) = \begin{pmatrix} \varphi(x_1) & \varphi(x_2) & \varphi(x_3)\end{pmatrix}^\top$, *where* $\varphi(x_i) = \exp\left(-\frac{E}{Rx_i}\right)$.

Example 7. *The following system represents a kind of consumer-resource interaction dynamics [39]:*

$$\begin{aligned}\dot{M}_1 &= M_1\left(r_1 + c_1\frac{M_2}{h_1 + M_2} + c_2\frac{M_2}{h_2 + M_1} - c_3M_1\right) + w_1,\\ \dot{M}_2 &= M_2\left(r_2 + c_4\frac{M_1}{h_2 + M_1} + c_5\frac{M_1}{h_1 + M_2} - c_6M_2\right) + w_2,\end{aligned} \tag{4.4}$$

where $M_1(t)$, $M_2(t) \in \mathbb{R}_+$ *are the population compartments,* r_1, r_2, h_1, h_2, $c_i \in \mathbb{R}$ $(i = 1,\ldots,6)$ *and* $w_1(t)$, $w_2(t) \in \mathbb{R}_+$ *are bounded inputs.*

This system is also in the form (4.1) for $n = 2$ *and* $M = 2$, *with* $x = \begin{pmatrix} M_1 & M_2\end{pmatrix}^\top$,

$A_0 = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$, $A_1 = A_2 = 0_{2\times 2}$, $B_1^1 = \begin{pmatrix} -c_3 & 0 \\ 0 & 0 \end{pmatrix}$, $B_1^2 = \begin{pmatrix} 0 & 0 \\ 0 & -c_6 \end{pmatrix}$, $B_2^1 = \begin{pmatrix} 0 & c_1 \\ 0 & c_5 \end{pmatrix}$, $B_2^2 = \begin{pmatrix} c_2 & 0 \\ c_4 & 0 \end{pmatrix}$, $f^1(M) = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$, $f^2(M) = \begin{pmatrix} \frac{M_1}{h_2+M_1} \\ \frac{M_2}{h_1+M_2} \end{pmatrix}$, *and* $H_1 = H_2 = G = I_2$, *other matrices are zero.*

Remark 2. *It is important to note that in all the examples of bilinear Persidskii systems (4.1), the state vector takes nonnegative values.*

Beyond their practical interest, these systems exhibit intrinsic symmetry, which can be further exploited as concrete examples of the theory of equivariant systems [27], [1]. Through this analysis, we can understand the role of this type of symmetry in the design of simpler and more effective observers, as discussed in the previous chapter.

4.2 Observer design

The proposed observer for the system (4.1) contains a copy of the system with linear and nonlinear output injections:

$$\begin{aligned} \dot{\hat{x}}(t) = A_0\hat{x}(t) + \sum_{j=1}^{M} A_j f^j(H_j\hat{x}(t)) + \sum_{i=1}^{n} \hat{x}_i(t) \sum_{j=1}^{M} B_j^i f^j(H_j\hat{x}(t)) + L_0(y_0(t) - C_0\hat{x}(t)) \\ + \sum_{j=1}^{M} L_j(y_j(t) - C_j f^j(H_j\hat{x}(t))) + \sum_{i=1}^{n} \hat{x}_i(t) \sum_{j=1}^{M} R_j^i(y_j(t) - C_j f^j(H_j\hat{x}(t))), \end{aligned} \tag{4.5}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the observer state and the estimate of $x(t)$, and L_0, L_j, $R_j^i \in \mathbb{R}^{n\times z_j}$ for $j = 1,\ldots,M$ and $i = 1,\ldots,n$ are matrices to be designed.

Let us denote $e = x - \hat{x}$ and $\delta f^j(x,\hat{x}) = f^j(H_j x) - f^j(H_j\hat{x})$ (for brevity, we will use

the notation δf^j), so the error dynamics is given by

$$\dot{e} = (A_0 - L_0C_0)e + \sum_{j=1}^{M}(A_j - L_jC_j)\delta f^j + \sum_{i=1}^{n}\sum_{j=1}^{M} B_j^i\left(x_i f^j(H_jx) - \hat{x}_i f^j(H_j\hat{x})\right) + Gw + \sum_{i=1}^{n} D_i x_i w - \sum_{i=1}^{n}\hat{x}_i \sum_{j=1}^{M} R_j C_j \delta f^j.$$

The dynamics of the joint state variable $X = \begin{pmatrix} x & \hat{x} \end{pmatrix}^\top$ for systems (4.1) and (4.5) is given by the equation

$$\dot{X} = \tilde{A}_0 X + \sum_{j=1}^{M} \tilde{A}_j F^j(\tilde{H}_j X) + \sum_{i=1}^{2n} X_i \sum_{j=1}^{M} \tilde{B}_j^i F_j(\tilde{H}_j X) + \tilde{G}w + \sum_{i=1}^{2n} X_i \tilde{D}_i w \tag{4.6}$$

where $X_j = x_j$ and $X_{n+j} = \hat{x}_j$, for $j = 1, \ldots, n$ and

$$\tilde{A}_j = \begin{pmatrix} A_j & 0_{n\times k_j} \\ L_jC_j & A_j - L_jC_j \end{pmatrix}, \tilde{H}_j = \begin{pmatrix} H_j & 0_{k_j\times n} \\ 0_{k_j\times n} & H_j \end{pmatrix}, \tilde{B}_j^i = \begin{pmatrix} B_j^i & 0_{n\times k_j} \\ 0_{n\times k_j} & 0_{n\times k_j} \end{pmatrix},$$

$$\tilde{B}_j^{n+i} = \begin{pmatrix} 0_{n\times k_j} & 0_{n\times k_j} \\ R_j^i C_j & B_j^i - R_j^i C_j \end{pmatrix},$$

$$F^j(\tilde{H}_j X) = \begin{pmatrix} f^j(H_jx) \\ f^j(H_j\hat{x}) \end{pmatrix}, \tilde{G} = \begin{pmatrix} G \\ 0 \end{pmatrix} \tilde{D}_i = \begin{pmatrix} D_i \\ 0 \end{pmatrix}, \tilde{D}_{n+i} = 0$$

for $j = 1, \ldots, M$ and $i = 1, \ldots, n$. Remark that the system (4.5) still belongs to the considered class of bilinear Persidskii systems, and that the error can be written as $e(X) = \Gamma X$, where $\Gamma = \begin{pmatrix} I_n & -I_n \end{pmatrix}$.

Now, consider the following Lyapunov function candidate:

$$V(X) = X^\top P_\Gamma X + 2\sum_{j=1}^{M}\sum_{i=1}^{2k_j} \Lambda_i^j \int_0^{\tilde{H}_j^i X} F_i^j(\tau)\, d\tau, \tag{4.7}$$

where $\tilde{H}_j^i$ is the ith row of the matrix $\tilde{H}_j$, $P_\Gamma = \Gamma^\top P_1 \Gamma + P_2$, for positive semidefinite matrices $P_1 \in \mathbb{R}^{n\times n}$, $P_2 \in \mathbb{R}^{2n\times 2n}$ and $\Lambda^j = \text{diag}(\Lambda_1^j, \ldots, \Lambda_{2k_j}^j)$ to be defined. The time

derivative of V along the trajectories of the system (4.6) can be computed as follows:

$$\dot{V}(X) = \dot{X}^\top P_\Gamma X + X^\top P_\Gamma \dot{X} + 2\sum_{j=1}^{M} \dot{X}^\top \tilde{H}_j^\top \Lambda^j F^j(\tilde{H}_j X) = X^\top (\tilde{A}_0^\top P_\Gamma + P_\Gamma \tilde{A}_0) X + \left(\sum_{j=1}^{M} F^j(\tilde{H}_j X)^\top \tilde{A}_j^\top \right.$$

$$\left. + \sum_{i=1}^{2n} X_i \sum_{j=1}^{M} F^j(\tilde{H}_j X)^\top (\tilde{B}_i^j)^\top \right) P_\Gamma X + X^\top P_\Gamma \left(\sum_{j=1}^{M} \tilde{A}_j F^j(\tilde{H}_j X) + \sum_{i=1}^{2n} X_i \sum_{j=1}^{M} \tilde{B}_i^j F^j(\tilde{H}_j X) \right)$$

$$+ 2\sum_{j=1}^{M} [X^\top \tilde{A}_0{}^\top \tilde{H}_j^\top \Lambda^j F^j(\tilde{H}_j X) + \left(\sum_{s=1}^{M} F^s(\tilde{H}_s X)^\top \tilde{A}_s{}^\top \right) \tilde{H}_j^\top \Lambda^j F^j(\tilde{H}_j X)$$

$$+ \left(\sum_{i=1}^{2n} X_i \sum_{s=1}^{M} F^s(\tilde{H}_s X)^\top (\tilde{B}_i^s)^\top \right) \tilde{H}_j^\top \Lambda^j F^j(\tilde{H}_j X)$$

$$+ \left(\tilde{G} w + \sum_{i=1}^{2n} X_i \tilde{D}_i w \right)^\top \tilde{H}_j^\top \Lambda^j F^j(\tilde{H}_j X)] + 2X^\top P_\Gamma \left(\tilde{G} w + \sum_{i=1}^{2n} X_i \tilde{D}_i w \right).$$

Regrouping terms, we can write $\dot{V}$ as a quadratic form acting on the vector

$$Y = \begin{pmatrix} X^\top & F^1(\tilde{H}_1 X)^\top & \dots & F^M(\tilde{H}_M X)^\top & w^\top \end{pmatrix}^\top,$$

given by

$$\dot{V} = Y^\top \left(Q + \sum_{i=1}^{2n} X_i S^{(i)} \right) Y,$$

where the symmetric matrix $Q = Q^\top \in \mathbb{R}^{(2n+2K+\ell)\times(2n+2K+\ell)}$ is defined by the blocks

$$Q_{1,1} = \tilde{A}_0^\top P_\Gamma + P_\Gamma \tilde{A}_0,$$

$$Q_{1,j+1} = P_\Gamma \tilde{A}_j + \tilde{A}_0^\top \tilde{H}_j^\top \Lambda^j,$$

$$Q_{1,M+2} = P_\Gamma G,\ Q_{k+1,j+1} = \tilde{A}_k^\top \tilde{H}_j^\top \Lambda^j + \Lambda^k \tilde{H}_k \tilde{A}_j,$$

$$Q_{M+2,j+1} = G^\top \tilde{H}_j^\top \Lambda^j \text{ and } Q_{M+2,M+2} = 0,$$

for $j,\ k = 1,\dots,M$. The symmetric matrices $S^{(1)},\dots,S^{(2n)} \in \mathbb{R}^{(2n+2K+\ell)\times(2n+2K+\ell)}$ are defined by $S_{1,1}^{(i)} = 0,\ S_{1,j+1}^{(i)} = P_\Gamma \tilde{B}_i^j,\ S_{j+1,j+1}^{(i)} = 2\tilde{B}_i^j \tilde{H}_j \Lambda^j,\ S_{1,M+2}^{(i)} = P_\Gamma \tilde{D}_i,\ S_{M+2,j+1}^{(i)} = D_i^\top \tilde{H}_j^\top \Lambda^j,\ S_{M+2,M+2}^{(i)} = 0$, for $i = 1,\dots,2n$ and $j = 1,\dots,M$, and $S_{k+1,j+1}^{(i)} = (\tilde{B}_i^k)^\top \tilde{H}_j^\top \Lambda^j +$

$\Lambda^k \tilde{H}_k \tilde{B}_i^j$, $j = 1,\dots,M$, $k = 1,\dots,j-1$. Using this decomposition, we are able to show that (4.5) is an observer for trajectories staying near the origin:

Theorem 14. *Suppose that the system (4.1) is UBIBS and that there exist matrices* $\Xi^j, \Upsilon_{0,j} \in \mathbb{D}_+^{2k_j}$ $(j = 1,...,M)$, $\Psi \in \mathbb{D}_+^n$, $\Upsilon_{k,j} \in \mathbb{D}_+^{2n}$ $(k = 1,...,M-1, j = k+1,...,M)$, $0 < \Phi = \Phi^\top \in \mathbb{R}^{2n\times 2n}$, $\Theta \in \mathbb{D}_+^{2n}$ *and* $\xi > 0$, $\rho > 0$ *satisfying the inequalities*

$$P_1 > 0 \text{ or } P_2^{11} - 2P_2^{12} + P_2^{22} - \rho P_1 > 0 \tag{4.8}$$
$$\text{or } \sum_{j=1}^{\mu} \tilde{\Lambda}^j + \rho P_1 > 0$$

where P_2^{11},P_2^{12}, *and* $P_2^{22} \in \mathbb{R}^{n\times n}$ *are such that*

$$P_2 = \begin{pmatrix} P_2^{11} & P_2^{12} \\ P_2^{12} & P_2^{22} \end{pmatrix}$$

and

$$\tilde{\Lambda}^j = \tilde{H}_j^\top \operatorname{diag}(\tilde{\Lambda}_1^j,...,\tilde{\Lambda}_{k_j}^j)\tilde{H}_j + \tilde{H}_j^\top \operatorname{diag}(\tilde{\Lambda}_{k_j+1}^j,...,\tilde{\Lambda}_{2k_j}^j)\tilde{H}_j$$

satisfy

$$P_1 \le \xi\Psi, \tag{4.9}$$

$$P_2 + \sum_{j=1}^{M} \tilde{H}_j^\top \Lambda^j \tilde{H}_j$$
$$\le \xi\Big(\sum_{k=0}^{m} \tilde{H}_k^\top \Xi^k \tilde{H}_k + 2\sum_{r=1}^{m} \tilde{H}_r^\top \Upsilon_{0,r} \tilde{H}_r + 2\sum_{s=1}^{m-1}\sum_{r=s+1}^{m} \tilde{H}_s^\top \tilde{H}_s \Upsilon_{s,r} \tilde{H}_r^\top \tilde{H}_r\Big) \tag{4.10}$$

and such that

$$Q + \tilde{Q} < 0, \tag{4.11}$$

where $\tilde{Q}$ *is the symmetric matrix defined by the blocks:*

$$\tilde{Q}_{1,1} = \Gamma^\top \Psi \Gamma + \Xi^0,$$

$$\tilde{Q}_{j+1,j+1} = \Xi^{j+1}, j = 1,...,M,$$

$$\tilde{Q}_{k+1,j+1} = \tilde{H}_k \Upsilon_{k,j} \tilde{H}_j^\top, k = 1,\dots,M-1, j = k+1,\dots,M-1,$$

$$\tilde{Q}_{1,j+1} = \tilde{H}_j^\top \Upsilon_{0,j}, j = 1,\dots,M,$$

$$\tilde{Q}_{M+2,j} = 0, j = 1,\dots,M+1, \tilde{Q}_{M+2,M+2} = -\Phi.$$

Then there exists $\delta > 0$ *such that the system (4.6) is IOS with respect to the output* $e = x - \hat{x}$*, for* $\|X(0)\| \leq \delta$ *and* $\|w\|_\infty < \delta$.

Proof. Consider the candidate Lyapunov function (4.7). The inequality (4.8) assures that if $V(X) = 0$ then $e(X) = \Gamma X = 0$ [17], which fulfills condition (1.4) with $h(X) = \Gamma X$. The upper bound α_2 in (1.4) exists by the continuity of V, the passivity of the nonlinearities and due to all matrices are nonnegative definite.

Once there exists $\tilde{Q} \in \mathbb{R}^{M+2\times M+2}$ verifying the inequality $Q + \tilde{Q} < 0$, we can find $\varepsilon > 0$ such that

$$Q + \tilde{Q} + \sum_{i=1}^{2n} X_i S^{(i)} \leq 0 \tag{4.12}$$

provided that $|X_i| < \varepsilon$ for all $i = 1,\dots,2n$. Therefore, we can write

$$\dot{V} = Y^T \left(Q + \tilde{Q} + \sum_{i=1}^{2n} X_i S^{(i)} \right) Y - Y^T \tilde{Q} Y$$

$$\leq -Y^T \tilde{Q} Y.$$

The inequalities (4.9),(4.10) make sure that

$$Y^T \tilde{Q} Y \geq \alpha(V(X)) - w^T \Phi w,$$

for some function $\alpha \in \mathscr{K}_\infty$. The computations to prove this fact are rather direct and are

analogous to the ones performed in [17]. For completeness, we reproduce them here:

$$\begin{aligned}
\dot{V} &= Y^\top QY - X^T(\Gamma^T\Psi\Gamma + \Xi^0)X \\
&\quad - \sum_{j=1}^{M} F^j(\tilde{H}_jX)^T\Xi^jF^j(\tilde{H}_jX) - 2\sum_{j=1}^{M} X^T\tilde{H}_j^T\Upsilon_{0,j}F^j(\tilde{H}_jX) \\
&\quad - 2\sum_{s=1}^{M-1}\sum_{r=s+1}^{M} F^s(\tilde{H}_sX)^T\tilde{H}_s^TH_{s,r}\tilde{H}_rH_rF^r(\tilde{H}_rX) + w^\top\Phi w \\
&\leq -X^T(\Gamma^T\Psi\Gamma + \Xi^0)X - \sum_{j=1}^{M} F^j(\tilde{H}_jX)^T\Xi^jF^j(\tilde{H}_jX) \\
&\quad - 2\sum_{j=1}^{M} X^T\tilde{H}_j^T\Upsilon_{0,j}F^j(\tilde{H}_jX) \\
&\quad - 2\sum_{s=1}^{M-1}\sum_{r=s+1}^{M} F^s(H_sX)^TH_s^TH_{s,r}H_rF^r(H_rX) + w^\top\Phi w.
\end{aligned}$$

Hence, it follows that

$$\dot{V} \leq -\alpha(V(X)) + w^T\Phi w,$$

which leads to

$$\dot{V} \leq -\frac{1}{2}\alpha(V(X)), \text{ for } V(X) \geq 2\alpha^{-1}(w^T\Phi w),$$

showing that V is an IOS-Lyapunov function for the system. Hence, by Theorem 2 this system is IOS, and there exist related functions $\beta \in \mathscr{KL}$ and $\gamma \in \mathscr{K}$ given in Definition 8.

Let us demonstrate the validity of (4.12), i.e., that the state X can be made arbitrarily bounded in the conditions of the theorem. Since (4.1) is UBIBS, there is $\sigma \in \mathscr{K}$ such that for any $\varepsilon' > 0$ we can chose $\delta > 0$ providing

$$\|x(t)\| \leq \max\{\sigma(\delta), \sigma(\delta)\} \leq \varepsilon', \ \forall t \geq 0$$

for any $\|x(0)\| \leq \delta$ and $\|w\|_\infty \leq \delta$. Further, this means that the established IOS property of the system (4.6) with respect to $e(X) = x - \hat{x}$ is satisfied for small enough times, so this

constant δ can be chosen sufficiently small guaranteeing

$$\|e(t)\| \leq \beta(\alpha_2(2\delta),0) + \gamma(\delta) < \varepsilon', \ \forall t \in [0,T]$$

for all $\|\hat{x}(0)\| \leq \delta$, $\|x(0)\| \leq \delta$ and $\|w\|_\infty \leq \delta$ and some $T > 0$. Repeating the argument for $t = T$ and combining these inequalities we obtain the boundedness of the observer state:

$$\|\hat{x}(t)\| \leq \|x(t)\| + \|e(t)\| < 2\varepsilon',$$

and subsequently

$$\|X(t)\| < 3\varepsilon',$$

for all $t \in \mathbb{R}_+$.

Therefore if we chose $\varepsilon' > 0$ small enough, for the corresponding δ, we have $|X_i(t)| < \varepsilon$ for all $i = 1,...,2n$, and all $t \in \mathbb{R}_+$, provided that $\|X(0)\| < \delta$ and $\|w\|_\infty < \delta$, so the inequality (4.12) is satisfied. □

Note that for local analysis, the indexes m and μ in the LMIs above can be substituted with M. Moreover, the size of the domain of validity of the design (the value of δ) is regulated not by the linearization arguments, but by relation between the matrices Q and $S^{(i)}$ for $i = 1,\ldots,2n$, hence, if the latter are sufficiently small comparing with the former, the value of δ can be large.

Now, we show that it can be made as a global observer by adding some new positivity assumptions, all of which can be put into linear matrix inequalities.

Theorem 15. *Suppose that the system (4.1) is UBIBS, that $X_i(t) \geq 0$ for any initial condition $X_i(0) \geq 0$ and that $S^{(i)} \leq 0$, for every $i = 1,...,2n$. Suppose also that there exist matrices $\Xi^j, \Upsilon_{0,j} \in \mathbb{D}_+^{2k_j}$ $(j = 1,...,M)$, $\Psi \in \mathbb{D}_+^n$, $\Upsilon_{k,j} \in \mathbb{D}_+^{2n}$ $(k = 1,...,M-1, j = k+1,...,M)$, $0 < \Phi = \Phi^\top \in \mathbb{R}^{2n\times 2n}$, $\Theta \in \mathbb{D}_+^{2n}$ and $\xi > 0$, $\rho > 0$ such that the LMIs (4.8)–(4.11) are verified. Then the system (4.6) is IOS with respect to the output $e = x - \hat{x}$.*

Proof. Consider the candidate Lyapunov function (4.7). The inequality (4.8) assures that if $V(X) = 0$ then $e(X) = \Gamma X = 0$ [17], which fulfills condition (1.4) with $h(X) = \Gamma X$. The upper bound α_2 in (1.4) exists by the continuity of V, the sector property of the nonlinearities and due to all matrices are nonnegative. Also, (4.8) assures that the function V is radially unbounded.

The combined hypotheses $Q + \tilde{Q} < 0, S^{(i)} \leq 0$ and $X_i \geq 0$ $(i = 1, ..., 2n)$ give us

$$Q + \tilde{Q} + \sum_{i=1}^{2n} X_i S^{(i)} \leq 0. \tag{4.13}$$

Therefore, using (4.11) we can write

$$\dot{V} = Y^\top \left(Q + \tilde{Q} + \sum_{i=1}^{2n} X_i S^{(i)} \right) Y - Y^\top \tilde{Q} Y \leq -Y^\top \tilde{Q} Y.$$

The inequalities (4.9), (4.10) make sure that

$$Y^\top \tilde{Q} Y \geq \alpha(V(X)) - w^\top \Phi w,$$

for some function $\alpha \in \mathscr{K}_\infty$. Thus $\dot{V} \leq -\alpha(V(X)) + w^\top \Phi w$, which leads to $\dot{V} \leq -\frac{1}{2}\alpha(V(X))$ for $V(X) \geq 2\alpha^{-1}(w^\top \Phi w)$, showing that V is an IOS-Lyapunov function for the system. Hence, by Theorem 2 this system is IOS. □

The conditions given in the formulation of Theorem 15 are sufficient for the existence of a Lyapunov function that establishes convergence of the estimation error to zero in the disturbance-free case.

Note that the conditions of Theorems 14 and 15 are almost the same: smallness of initial conditions and inputs is replaced with negative definiteness of the matrices $S_i, i = 1, \ldots, 2n$ and nonnegativity of the state vector $X(t)$. The latter property should also be verified for the observer state; hence, it has to be guaranteed by the design. Unlike Theorem 14, Theorem 15 enables the global analysis of systems with states that have positive coordinates, such as the following example.

Example 8. *If we consider Example 7 with parameters* $r_1 = r_2 = -1$ *and* $C_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}$, *then the inequalities (4.10) and (4.11) become*

$$P_2 + \Lambda^1 + \Lambda^2 \leq \xi(\Xi^0 + 2(\Upsilon_{0,1} + \Upsilon_{0,2}) + \Upsilon_{1,2}),$$

and

$$\begin{pmatrix} \tilde{A_0}^\top P_\Gamma + P_\Gamma \tilde{A_0} + \Xi^0 & \Upsilon_{0,1} \\ 2P_\Gamma^\top + \Upsilon_{0,1} & -\Phi \end{pmatrix} < 0,$$

which, together with (4.8) and (4.9), constitute a feasible system of inequalities. Furthermore, through the introduction of the Lyapunov function $V(M_1, M_2) = M_1 + M_2$, *a direct computation shows that*

$$\dot{V} = \dot{M}_1 + \dot{M}_2 = -c_3 M_1^2 - c_6 M_2^2 + B(M_1, M_2) + w_1 + w_2,$$

where $B(M_1, M_2)$ *has a linear growth rate. The system is then clearly UBIBS. Hence, Theorem 15 can be applied, and the observer can be chosen in the form (4.5).*

For simulations, all system parameters are set to 1, *and we chose the gain matrix as* $L_0 = \begin{pmatrix} 20.43 & -5.01 \end{pmatrix}^\top$, *with* $L_j = R_j^i = 0_{2\times 2}$, *for* $i = 1,2$ *and* $j = 1,2$. *Figs. 4.1 and 4.3 show the behavior of a trajectory of the system and the evolution of the error for* $w_i(t) = 0.5\sin(t)$ *for* $i = 1,2$ *and three different initial conditions, illustrating that the error remains small for relatively small inputs and small initial conditions.*

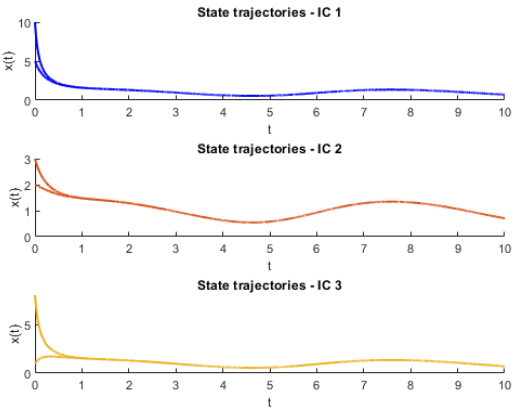


Figure 4.1: State trajectory $M(t)$

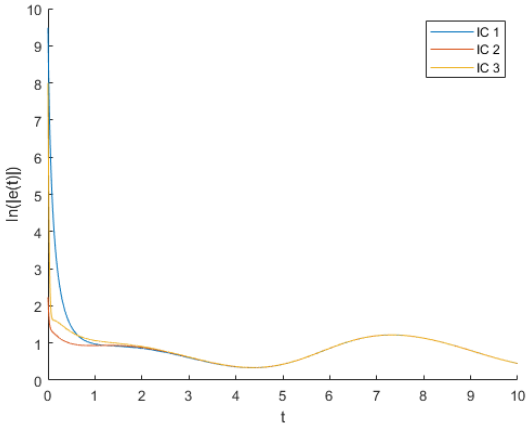


Figure 4.2: Norm of the estimation error

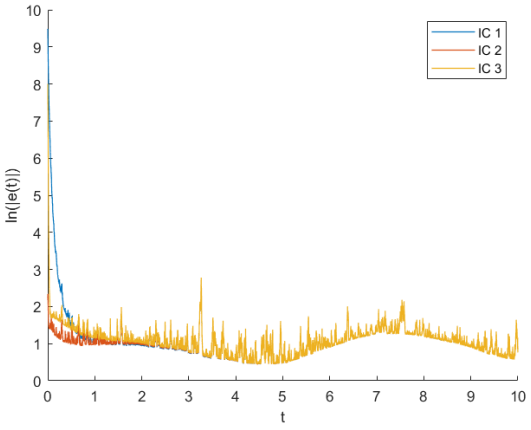


Figure 4.3: Norm of the estimation error for noisy output.

In Fig. 4.3 the results of simulations are shown for the same scenarios and with random output noise centered at zero and of amplitude 2. They give numerical evidence that the IOS property still holds in this case.

4.3 SIIOS Analysis

In this section, we again suppose that the system (4.1) is nonnegative, meaning that given an initial condition $x(0)$ such that $x_i(0) \geq 0$ for every $i = 1,...,n$ and positive inputs, the resulting trajectory also has nonnegative coordinates (i.e., $x_i(t) \geq 0$ for every $i = 1,...,n$ and all $t \geq 0$). However, we would like to avoid the restriction of nonnegativity of the

state of the observer $\hat{x}(t)$.

To this end, we also assume from now on that the functions $f^j, j = 1,...,M$ are bounded (it is the case of the considered motivating examples 5–7).

The error dynamics with respect to the observer (4.5) can be rewritten as

$$\begin{aligned}\dot{e} = (A_0 - L_0C_0)e + \sum_{j=1}^{M}(A_j - L_jC_j)\delta f^j(x,\hat{x}) + v(x,\hat{x}) \\ + \sum_{i=1}^{n} x_i \sum_{j=1}^{M}(B_j^i - R_j^iC_j)\delta f^j(x,\hat{x}) + Gw + \sum_{i=1}^{n} x_iD_iw\end{aligned} \tag{4.14}$$

where

$$v(x,\hat{x}) = \sum_{i=1}^{n} e_i \sum_{j=1}^{M} B_j^i f^j(H_jx) - \sum_{i=1}^{n} e_i \sum_{j=1}^{M}(B_j^i + R_j^iC_j)\delta f^j(x,\hat{x}),$$

so that we have the bound $\|v(x,\hat{x})\| \leq c\|e\|$ for some $c > 0$ (boundedness of f^j implies the same property of δf^j). Following [17], we consider the function

$$V(e) = e^\top Pe + 2\sum_{j=1}^{M}\sum_{i=1}^{k_j}\int_0^{H_j^ie_i} f_j^i(\tau)\,d\tau \tag{4.15}$$

where H_j^i is the ith row of the matrix H_j, as the SIIOS-Lyapunov function candidate for the system (4.14). Its derivative is computed as

$$\dot{V}(e) = \dot{e}^\top Pe + e^\top P\dot{e} + 2\dot{e}^\top \sum_{j=1}^{M} H_j^\top \Lambda^j f^j(H_je).$$

Substituting $\dot{e}$ by its expression and rearranging terms, we have

$$\dot{V}(e) = e^\top[(A_0 - L_0C_0)^\top P + P(A_0 - L_0C_0)]e + 2e^\top Pv +$$

$$\sum_{j=1}^{M}(\delta f^j)^\top(A_j - L_jC_j)^\top Pe + \sum_{j=1}^{M} e^\top P(A_j - L_jC_j)\delta f^j$$
$$+2\sum_{k=1}^{M}[e^\top(A_0 - L_0C_0)H_k\Lambda^k f^k(H_ke) + v^\top H_k\Lambda^k f^j(H_je)]$$
$$+2\sum_{j=1}^{M}\sum_{k=1}^{M}(\delta f^j)^\top(A_j - L_jC_j)^\top H_k^\top\Lambda^k f^k(H_ke)$$
$$+w^\top G^\top Pe + e^\top PGw + 2\sum_{k=1}^{M} w^\top G^\top H_k^\top\Lambda^k f^k(H_ke)$$
$$+\sum_{i=1}^{n} x_i[(\delta f^j)^\top(B_j^i - R_j^iC_j)^\top Pe + e^\top P(B_j^i - R_j^iC_j)\delta f^j$$
$$+2\sum_{k=1}^{M}\sum_{j=1}^{M}(\delta f^j)^\top(B_j^i - R_j^iC_j)^\top H_k^\top\Lambda^k f^k(H_ke)]$$

Despite its long expression, $\dot{V}$ can be written as a quadratic form:

$$\dot{V} = W^\top\left(Q + \sum_{i=1}^{n} x_iT_i\right)W,$$

where $W = [e^\top(\delta f^1)^\top \ldots (\delta f^M)^\top f^1(H_1e)^\top \ldots f^M(H_Me)^\top v^\top w^\top]^\top$ and to write Q and T_i simply, we introduce the following useful notations $\tilde{A}_j = (A_j - L_jC_j)$, $j = 0,\ldots,M$, $\tilde{A} = \text{diag}(\tilde{A}_1,\ldots,\tilde{A}_M)$, $\Lambda = [H_1^\top\Lambda^1 \ldots H_M^\top\Lambda^M]^\top$, $\tilde{B}_k^i = B_k^i - R_k^iC_k$; $\tilde{B}_i = \text{diag}(\tilde{B}_1^i,\ldots,\tilde{B}_M^i)$. Then we can now write Q and T_i as block matrices $Q = Q^\top = (Q_{a,b})_{a,b=1}^5$, where

$$Q_{1,1} = \tilde{A}_0^\top P + P\tilde{A}_0;\ Q_{1,2} = P\tilde{A};\ Q_{1,3} = \tilde{A}_0\Lambda;\ Q_{1,4} = P;\ Q_{1,5} = PG;\ Q_{2,3} = \tilde{A}^\top\Lambda;\ Q_{3,4} = \Lambda\ Q_{3,5} = G^\top\Lambda;$$

and $T_i = T_i^\top = (T_i^{a,b})_{a,b=1}^5$, with $T_i^{1,2} = P\tilde{B}_i$, $T_i^{1,5} = PD_i$, $T_i^{2,3} = (\tilde{B}_i\Lambda + \Lambda\tilde{B}_i^\top)/2$, $T_i^{3,5} = (D_i^\top\Lambda + \Lambda D_i)/2$. The indices a,b not mentioned above correspond to zero matrices of their respective block size.

Before stating and proving the main result, we need the following assumptions for the

nonlinearities in the system:

Assumption 1. *Assume* $(x-\hat{x})^\top H_j^\top \delta f^j(x,\hat{x}) > 0$ *for all* $x \neq \hat{x} \in \mathbb{R}^n$.

Denote by $\mathscr{M}(H_z,H_s) \subset \mathbb{R}^{k_z \times k_s}$ a set of matrices, whose ij-th element is *i*) nonnegative if $H_z^{\mathfrak{i}}$ and $H_s^{\mathfrak{j}}$ are co-directional, *ii*) nonpositive if $H_z^{\mathfrak{i}}$ and $H_s^{\mathfrak{j}}$ are anti-directional, and *iii*) zero otherwise, for $\mathfrak{i} = \overline{1,k_z}$, $\mathfrak{j} = \overline{1,k_s}$, $z,s = 1,...,M$.

Assumption 2. *There exist matrices* $S_0^j, S_1^j, S_2^j, \Sigma_0^j, \Sigma_1^j, \Sigma_2^j \in \mathbb{D}_+^{k_j}$ *and* $S_3^{j,q}, \Sigma_3^{j,q} \in \mathscr{M}(H_j,H_q)$, *for* $j,q = 1,...,M$ *such that*

$$
\begin{aligned}
&(\delta f^j)^\top \delta f^j \\
&\leq e^\top H_j^\top S_0^j H_j e + 2e^\top H_j^\top S_1^j(\delta f^j) + 2e^\top H_j^\top S_2^j f^j(H_j e) + 2\sum_{q=1}^{M} (\delta f^j)^\top S_3^{j,q} f^q(H_q e); \\
&f^j(H_j e)^\top f^j(H_j e) \\
&\leq e^\top H_j^\top \Sigma_0^j H_j e + 2e^\top H_j^\top \Sigma_1^j(\delta f^j) + 2e^\top H_j^\top \Sigma_2^j f^j(H_j e) + 2\sum_{q=1}^{M} (\delta f^j)^\top \Sigma_3^{j,q} f^q(H_q e)
\end{aligned}
$$

for all $x,\hat{x} \in \mathbb{R}^n$ *with* $e = x - \hat{x}$.

The former requires incremental sector (incremental passivity) property to be verified for the nonlinearities in the system (i.e., they should be monotonously growing). In particular, this property implies that integrals of all nonlinearities become unbounded, as well as the products $sf_i^j(s)$ or $(s-\hat{s})^\top \delta f_i^j(s-\hat{s})$ for $s,\hat{s} \to \pm\infty$ with $j = 1,\ldots,M$ and $i = 1,\ldots,k_j$. The latter assumption is a quadratic constraint upper bound on the nonlinearities.

In the examples dealt in this paper, Assumption 1 can be verified by direct computation, while Assumption 2 comes from the fact that the nonlinearities f^j are uniformly Lipschitz continuous and incrementally passive. This means, in particular, that the functions $f^j(H_j e)^\top f^j(H_j e)$ and $(\delta f^j)^T \delta f^j$ are bounded by some quadratic forms of the error. Thus, we can choose S_0^j, Σ_0^j such that $(\delta f^j)^\top \delta f^j \leq e^\top H_j^\top S_0^j H_j e$ and $f^j(H_j e)^\top f^j(H_j e) \leq e^\top H_j^\top \Sigma_0^j H_j e$, which is suficient for Assumption 2, since all other right-hand side terms can be made nonnegative.

Theorem 16. *Suppose that the system (4.1) is forward complete, that Assumptions 1 and 2 hold, that the state trajectory is nonnegative with $x_i(t) \geq 0$ for all $t \geq 0$ and that $T_i \leq 0$, for $i = 1,...,n$. Suppose that there exist $\rho \in \mathbb{R}$, $\eta, \gamma, \xi, d > 0$, matrices $0 \leq P = P^\top \in \mathbb{R}^{n\times n}, \Xi^0 \in \mathbb{D}^n_+, \Lambda^j = \operatorname{diag}(\Lambda^j_1,...,\Lambda^j_n), \Gamma_j, \Omega_j \in \mathbb{D}^{k_j}_+$, $\Upsilon_{j,k} \in \mathscr{M}(H_j, H_k)$, $j,k = 1,...,M$ such that*

$$P + \rho \sum_{j=1}^{M} H_j^\top \Lambda^j H_j > 0; \tag{4.16}$$

$$\Xi^0 - c^2 \xi I_n - \gamma \sum_{j=1}^{M} H_j^\top S_0^j H_j - \eta \sum_{j=1}^{M} H_j \Sigma_0^j H_j \geq 0; \tag{4.17}$$

$$\Gamma_j - \gamma S_1^j - \eta \Sigma_1^j \geq 0; \tag{4.18}$$

$$\Omega_j - \gamma S_2^j - \eta \Sigma_2^j \geq 0; \tag{4.19}$$

$$\Upsilon_{j,k} - \gamma S_3^{j,k} - \eta \Sigma_3^{j,k} \geq 0; \tag{4.20}$$

$$\begin{aligned} &\Xi^0 - c^2 \xi I_n - \gamma \sum_{j=1}^{M} H_j^\top S_0^j H_j - \eta \sum_{j=1}^{M} H_j \Sigma_0^j H_j + \sum_{j=1}^{M} H_j^\top (\Gamma_j - \gamma S_1^j - \eta \Sigma_1^j) H_j \\ &+ \sum_{j=1}^{M} H_j^\top (\Omega_j - \gamma S_2^j - \eta \Sigma_2^j) H_j + \sum_{j=1}^{M} \sum_{k=1}^{M} H_j^\top (\Upsilon_{j,k} - \gamma S_3^{j,k} - \eta \Sigma_3^{j,k}) H_k > 0 \end{aligned} \tag{4.21}$$

and a matrix $\tilde{Q} = \tilde{Q}^\top = (Q_{a,b})_{a,b=1}^5$ such that

$$\tilde{Q} < 0 \tag{4.22}$$

defined by $\tilde{Q}_{1,1} = \Xi^0$, $\tilde{Q}_{2,2} = -\gamma I_{nM}$, $\tilde{Q}_{3,3} = -\eta I_{nM}$, $\tilde{Q}_{4,4} = -\xi I_n$, $\tilde{Q}_{5,5} = -dI_n$, $\tilde{Q}_{1,2} = \overline{\Gamma}$, $\tilde{Q}_{1,3} = \Omega$, $\tilde{Q}_{2,3} = \Upsilon$, where the $\tilde{Q}_{i,j}$ with omitted indices equal to zero, $\overline{\Gamma} = [H_1^\top \Gamma_1 ... H_M^\top \Gamma_M]$, $\Omega = [H_1^\top \Omega_1 ... H_M^\top \Omega_M]$ and $\Upsilon = (H_j \Upsilon_{j,k} H_k^T)_{j,k=1}^M$. Then the system (4.1) with the observer (4.5) is SIIOS with respect to the output $e(x,\hat{x}) = x - \hat{x}$.

Proof. Since $Q + \tilde{Q} < 0$, $T_i \leq 0$ and $x_i(t) \geq 0$, we have

$$Q + \tilde{Q} + \sum_{i=1}^{n} x_i T_i < 0.$$

Considering V to be as defined in (4.15), which is a positive definite and radially un-

bounded function of e due to (4.16) with Finsler's lemma, and following the computations thereafter, we conclude that

$$\dot{V} = W^\top (Q + \sum_{i=1}^{n} x_i T_i) W = W^\top (Q + \tilde{Q} + \sum_{i=1}^{n} x_i T_i) W - W^\top \tilde{Q} W \leq -W^\top \tilde{Q} W,$$

where the last inequality holds due to hypothesis (4.22). Writing down the full expression for the last term we have

$$\dot{V} \leq -e^\top \Xi^0 e + \gamma \sum_{j=1}^{M} (\delta f^j)^\top \delta f_j + \xi v^\top v + dw^\top w - \sum_{j=1}^{M} e^\top H_j \Gamma_j \delta f^j$$

$$- \sum_{j=1}^{M} e^\top H_j \Omega_j f^j(H_j e) - \sum_{j=1}^{M} \sum_{k=1}^{M} (\delta f^j)^\top \Upsilon_{j,k} f^k(H_k e)$$

due to assumptions 1, 2 and the existence of the upper bound on v in terms of the estimation error, we have the estimate:

$$\dot{V} \leq -e^\top (\Xi^0 - c^2 \xi I_n - \gamma \sum_{j=1}^{M} H_j^\top S_0^j H_j - \eta \sum_{j=1}^{M} H_j \Sigma_0^j H_j) e$$

$$-2 \sum_{j=1}^{M} e^\top H_j^\top (\Gamma_j - \gamma S_1^j - \eta \Sigma_1^j) \delta f^j - 2 \sum_{j=1}^{M} e^\top H_j^\top (\Omega_j - \gamma S_2^j - \eta \Sigma_2^j) f^j(H_j e)$$

$$-2 \sum_{j=1}^{M} \sum_{k=1}^{M} (\delta f^j)(\Upsilon_{j,k} - \gamma S_3^{j,k} - \eta \Sigma_3^{j,k}) f^k(H_k e) + dw^\top w,$$

where each term is nonnegative due to (4.17)–(4.20), and $-\dot{V} + dw^\top w$ can be lower bounded by a class $\mathscr{K}_\infty$ function of the error e following the inequality (4.21). Therefore, V is an SIIOS Lyapunov function for the system (4.1), (4.5), which possesses the unboundedness observability property that can replace the forward completeness requirement. By Theorem 3 the required conclusion follows. □

Compared with Theorem 15, here it is not required that the observer state be nonnegative, simplifying the verification of the conditions and design, and the UBIBS property of (4.1) is reduced to forward completeness.

Example 9. *Let us consider Example 6 with* $C_0 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$. *To see that the system satis-*

fies the UBIBS property we introduce a simple Lyapunov function $V(X,T) = (k/c)X + T$, *then direct computation yields*

$$\dot{V} = \frac{k}{c}\dot{X} + \dot{T} = -D\frac{k}{c}X - DT + (D\frac{k}{c}X_{in} + DT_{in}) + w,$$

which is negative for large state values and bounded input. We have $n = 3$, $M = 1$, *inequality (4.16) becomes* $P + \rho\Sigma_0^j > 0$ *and the inequality (4.22) becomes*

$$\begin{pmatrix} \Xi^0 + \mathscr{A}_0^\top P + P\mathscr{A}_0 & \Gamma_1 & \mathscr{A}_0\Lambda^1 + \Omega^1 & P & PG \\ \Gamma_1 & -\gamma I_3 & \Upsilon_{1,1} & 0 & 0 \\ -\Lambda^1\mathscr{A}_0^\top + \Omega^1 & \Upsilon_{1,1} & -\eta I_3 & \Lambda^1 & G^\top\Lambda^1 \\ P & 0 & \Lambda^1 & -cI_3 & 0 \\ G^\top P & 0 & \Lambda^1 G & 0 & -dI_3 \end{pmatrix} < 0$$

Assuming all parameters in the model are 1 *and adding to that the inequalities* $T_i \leq 0$ *for* $i = 1,2,3$, *we obtain the gains* $L_0 = [1,\ 0,\ 0], R_1^2 = [-0.4439,\ -0.4315,\ 0.3351]^\top$ *through solving the LMIs from Theorem 16. Figs. 4.4 and 4.5 show a state trajectory example and the estimation norm behavior for three distinct initial conditions and input* $w(t) = 0.2 + 0.1\cos(t)$.

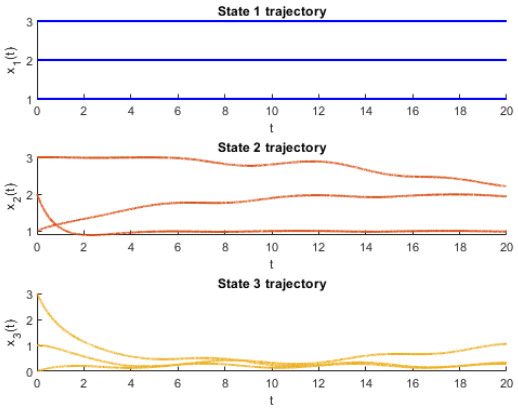


Figure 4.4: Trajectory of the chemical reactor dynamic model for $w(t) = 0.2 + 0.1\cos(t)$ and three different initial conditions.

We also generated a noisy output version of the simulation for the example of the chemical reactor. The noise was chosen to be centered at zero and of amplitude 6, the

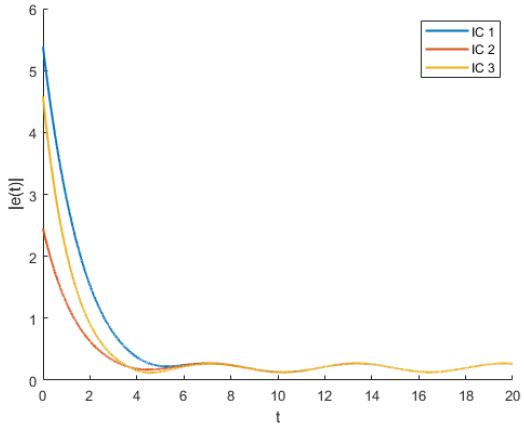


Figure 4.5: Norm of estimation error for the trajectory with gains $L_0 = [1\,0\,0]^\top$, $R_1^2 = [-0.4439\ -0.4315\ 0.3351]^\top$ and three different initial conditions.

results are shown in Fig. 4.6.

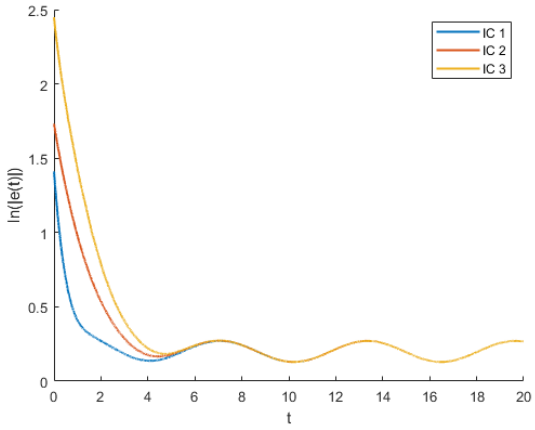


Figure 4.6: Norm of error for noisy output and three distinct pairs of initial conditions.

4.4 Conclusion

In this chapter, we introduced a new class of systems that generalizes the Persidskii dynamics while incorporating bilinear cross-product terms. This extension of the model is essential in many applications based on the mass-balance principle. For this class of systems, we proposed a state estimator using the concepts of IOS and SIIOS. The efficiency of this estimator is illustrated through examples, including a consumer-resource

interaction model and a chemical exothermic reactor. We formulated conditions for the observer's applicability in terms of linear matrix inequalities, which is an advancement considering the complexity of the system. Moreover, one of the observer designs, based on the nonnegativity of the state of the plant, offers a global solution, a rare achievement for bilinear systems.

Chapter 5

Finite-time control using symmetries: unicycle mobile robot

The main results of this chapter were submitted for publication at Transactions on Automatic Control (TACON) [LimaTAC25].

5.1 Models of interest

A UMR is a device that moves on a plane so that its orientation is coupled with its direction of motion, preventing the robot from drifting. Its state can be entirely characterized by the variables x, y, and θ, the first two corresponding to the position of the robot on the plane and the third corresponding to its orientation, more precisely, the angle between a given fixed axis attached to the robot and the x-axis, measured in the counter-clockwise sense. The coupling of the orientation with the velocity plus the freedom to intervene with control inputs in the linear and angular velocity makes it so that the dynamic model

takes the following form:

$$\begin{aligned} \dot{x} &= v_1 \cos(\theta) \\ \dot{y} &= v_1 \sin(\theta) \\ \dot{\theta} &= v_2, \end{aligned} \tag{5.1}$$

where $\begin{pmatrix} x(t) & y(t) \end{pmatrix}^\top \in \mathbb{R}^2$, $\theta(t) \in \mathbb{S}^1$ (the unit ball in $\mathbb{R}^2$) are the state variables representing the position of the robot on the plane and its orientation, respectively, and $v = \begin{pmatrix} v_1(t) & v_2(t) \end{pmatrix}^\top \in \mathbb{R}^2$ is the control input to be designed.

Our goal in this work is to examine different finite-time stabilizing control strategies recently proposed in the literature for the system (5.1) and compare their effectiveness.

In the following subsections, we will present, following [3, 4, 52], alternative coordinates in which the UMR dynamic equation can be written. This approach can facilitate the control design task to achieve a target in a finite time.

5.1.1 Heisenberg System

The Heisenberg system is named after the Heisenberg algebra [67]. This model is a well-studied non-holonomic system. Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and consider the system

$$\begin{aligned} \dot{X}(t) &= u(t), \ \forall t \in \mathbb{R}_+, \\ \dot{z}(t) &= u^\top(t) J X(t), \end{aligned} \tag{5.2}$$

where $X(t) \in \mathbb{R}^2, z(t) \in \mathbb{R}$ are the states and $u(t) \in \mathbb{R}^2$ is the input.

Proposition 10. [52] *Assume the variables* $X = \begin{pmatrix} X_1 & X_2 \end{pmatrix}^\top \in \mathbb{R}^2$, $z \in \mathbb{R}$ *and* $\begin{pmatrix} u_1 & u_2 \end{pmatrix}^\top \in \mathbb{R}^2$ *satisfy* (5.2) *and consider the transformations* $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ *and* $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ *given by* $\phi(X,z) = (x,y,\theta)$ *and* $\psi(u_1,u_2) = (v_1,v_2)$ *with*

$$\begin{aligned} x &= -2X_2\cos(X_1) + (z - X_1X_2)\sin(X_1) \\ y &= 2X_2\sin(X_1) + (z - X_1X_2)\cos(X_1) \\ \theta &= -X_1 \\ v_1 &= (z - X_1X_2)u_1 - 2u_2 \\ v_2 &= -u_1. \end{aligned} \tag{5.3}$$

Then the variables (x,y,θ) *and* (v_1,v_2) *satisfy* (5.1).

It is easy to verify that the transformation (5.3) has a well-defined inverse given by:

$$\begin{aligned} \xi_1 &= -\frac{1}{2}\left(x\cos(\theta) - y\sin(\theta)\right) \\ \xi_2 &= \frac{1}{2}\theta\left(x\cos(\theta) - y\sin(\theta)\right) + x\sin(\theta) + y\cos(\theta) \\ \xi_3 &= -\theta \\ u_1 &= -v_2 \\ u_2 &= -\frac{1}{2}\left(v_1 + (z - X_1X_2)v_2\right) \end{aligned}$$

with $\phi^{-1}(x,y,\theta) = \xi \in \mathbb{R}^2 \times \mathbb{R}$.

5.1.2 The chained form

As a non-holonomic robot model, the UMR dynamics has a canonical chained form, given by the equations

$$\begin{aligned} \dot{x_1} &= u_1 \\ \dot{x_2} &= u_2 \\ \dot{x_3} &= u_1 x_2, \end{aligned} \tag{5.4}$$

where $x = \begin{pmatrix} x_1(t) & x_2(t) & x_3(t) \end{pmatrix}^\top \in \mathbb{R}^3$ is the state and $u = \begin{pmatrix} u_1(t) & u_2(t) \end{pmatrix}^\top \in \mathbb{R}^2$ is the control input. Two one-to-one transformations send the UMR model into this chained form, as stated in the two following results:

Proposition 11. [3] *Assume* $x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^\top \in \mathbb{R}^3$ *and* $\begin{pmatrix} u_1 & u_2 \end{pmatrix}^\top \in \mathbb{R}^2$ *satisfy* (5.4) *and consider the transformations* $\phi : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ *and* $\psi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ *given by* $\phi(x_1,x_2,x_3) = (x,y,\theta)$ *and* $\psi(u_1,u_2) = (v_1,v_2)$ *with*

$$\begin{aligned} x &= x_2\cos(x_1) + x_3\sin(x_1) \\ y &= -x_2\sin(x_1) + x_3\cos(x_1) \\ \theta &= -x_1 \\ v_1 &= x_3 u_1 + u_2 \\ v_2 &= -u_1. \end{aligned} \tag{5.5}$$

Then the variables (x,y,θ) *and* (v_1,v_2) *satisfy* (5.1).

Proposition 12. *[49] Let* $x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^\top \in \mathbb{R}^3$ *and* $\begin{pmatrix} u_0 & u_1 \end{pmatrix}^\top \in \mathbb{R}^2$ *satisfy equation* (5.4) *and consider the transformations* $\phi : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ *and* $\psi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ *given by* $\phi(x_1,x_2,x_3) = (x,y,\theta)$ *and* $\psi(u_0,u_1) = (v,w)$ *with*

$$\begin{aligned} x &= x_1 \\ y &= x_3 \\ \theta &= \arctan(x_2) \\ v &= u_0\sec(\theta) \\ w &= u_1\cos^2(\theta). \end{aligned} \tag{5.6}$$

Then the variables (x,y,θ) *and* (v,w) *satisfy equation* (5.1).

That means that for each control law designed for the system (5.4), there are two natural ways of transforming it into a control law for the UMR while preserving specific properties, such as stability in finite time. It is important to highlight here that the variable θ takes values in $\mathbb{S}^1$, while the variables X_1 in (5.2) and x_1 in (5.4), which are transformed into θ using (5.3) and (5.5), respectively, take values in $\mathbb{R}$. The controls designed for (5.2) and (5.4), whether globally or locally, stabilize these models in their respective spaces, which is $\mathbb{R}$ for the mentioned variables.

Therefore, depending on initial conditions, even if the trajectory for θ (X_1 or x_1) is initiated in $\mathbb{S}^1$, the closed-loop controlled system may not guarantee the invariance of $\mathbb{S}^1$, and in such a case a careful projection of the states of (5.2) and (5.4) into the framework for (5.1) is required. Unfortunately, such a projection cannot be continuous since the points π and $-\pi$ coincide in $\mathbb{S}^1$ but significantly differ in $\mathbb{R}$. Note that (5.6) does not possess this drawback thanks to using the arctan function. Therefore, (5.6) is generally preferable.

In the following session, we will explore the controls designed for the chained form and the Heisenberg system to take advantage of these observations.

5.2 Control of the canonical forms

The first result provides a local control algorithm for (5.2). Additionally, the estimation of the domain of attraction and the extension to the global stabilization, along with a robustness analysis with respect to the external disturbances, can also be found in [50].

Theorem 17. *[55] There exist γ, $\beta > 0$ such that the control law*

$$u(t) = \begin{cases} -\frac{\gamma}{\|X\|} R\left(\arctan\left(\frac{2\beta \lceil z \rfloor^{1/2}}{\|X\|}\right)\right) X & X \neq 0 \\ 0 & X = 0, \end{cases} \tag{5.7}$$

where $R(\theta) \in \mathbb{R}^{2\times 2}$ is the rotation matrix of the angle θ, makes system (5.2) locally uFTS

at the origin.

Note that equation (5.7) utilizes the SO(2) symmetry of the model (5.2) along with the Heisenberg algebra in its design. The following control algorithm is global, but it uses no symmetry-based idea. It proposes the simplest strategy when the overall problem of stabilization at the origin is decomposed into two sub-tasks. These tasks are solved sequentially over two different intervals of time: first, the variables x_2 and x_3 are stabilized at the price of a growing coordinate x_1; second, x_1 is pushed to zero while keeping the rest of the system at the origin.

Theorem 18. *[4] The control law defined by*

$$u_1 = \begin{cases} a & \text{if } t \le T_{c_1} + T_{c_2} \\ \dfrac{\sqrt{\pi}}{T_{c_3}} \exp |x_1| \lceil x_1 \rfloor^{\frac{1}{2}} - \zeta_0 \operatorname{sgn}(x_1) & \text{otherwise} \end{cases}$$
$$u_2 = \begin{cases} \dfrac{\varphi(x_2, a x_3)}{a} & \text{if } t \le T_{c_1} + T_{c_2} \\ -\gamma \operatorname{sgn}(x_3) & \text{otherwise}\,, \end{cases} \tag{5.8}$$

where

$$\varphi(x_2, x_3) = -\alpha_2^{\frac{\beta_2 q_2 - 1}{p_2}} \Gamma\left(\frac{1-\beta_2 q_2}{p_2}\right) \exp \alpha_2 |\sigma|^{p_2} \lceil \sigma \rfloor^{\beta_2 q_2}$$
$$-\zeta \operatorname{sgn}(\sigma) - \left(\frac{2^{\frac{1-q_1/2}{p_1}} \Gamma\left(\frac{1-q_1/2}{p_1}\right)}{p_1 T_{c_1}} \right)^2 (q_1 + p_1 \lceil x_2 \rfloor^{p_1}) |x_2|^{q_1 - 1}$$

and $a \in \mathbb{R}$, T_{c_1}, T_{c_2}, $T_{c_3} > 0$, $\zeta_0 \ge \delta_0$, $\zeta \ge a\delta_1$, $\gamma \ge \gamma_1$, p_1, p_2, α_2, $\beta_2 > 0$, $1 \le q_1 < 2$, $q_2 < \frac{1}{\beta_2}$ *and*

$$\sigma = x_3 + \left\lceil \lceil x_3 \rfloor^2 + 2 \left(\frac{2^{\frac{1-q_1/2}{p_1}} \Gamma\left(\frac{1-q_1/2}{p_1}\right)}{p_1 T_{c_1}} \right) \exp\left(|x_2|^{p_1}\right) \lceil x_2 \rfloor^{q_1} \right\rfloor^{\frac{1}{2}}$$

makes system (5.4) UFxTS *with the settling time estimate* $T_c = T_{c_1} + T_{c_2} + T_{c_3}$.

Note that this control algorithm contains many parameters, and it is rather difficult to

tune it.

The last control strategy is also based on symmetry (or homogeneity) concepts. It uses an implicitly defined Lyapunov function ψ, depending on the generator dynamics. This approach comes from the idea of applying periodic time-varying control to stabilize non-holonomic systems [54]. Therefore, for this closed-loop system, the state includes the state x of the system (5.4) and the state of the generator dynamics. Thus, x can be considered as an output for the complete system:

Theorem 19. [50] *System* (5.4) *is* uFTOS *with respect to the output x under the control laws*

$$\begin{aligned} u_1(x,z) &= \psi^{0.5}(x,z)(-x_1 - \psi(x,z)x_2(\psi(x,z)x_3 \\ &\qquad -(x_2 - \psi(x,z)x_3x_3z_1)z_1)); \\ u_2(x,z) &= \psi^{0.5}(x,z)(-x_2 + \psi(x,z)x_3(z_1+z_2)), \end{aligned} \tag{5.9}$$

where

$$\dot{z} = \psi^{0.5}(x,z)\begin{pmatrix} z_2 \\ -z_1 \end{pmatrix}$$

and $\psi(x,z)$ *is the solution in* ψ *of the equation*

$$\psi^2[x_1^2 + (x_2 - \psi x_3 z_1)^2 + \psi^2 x_3^2] = 2.$$

It is worth mentioning that the control (5.9) is a particular example of a set of controls that were designed to chained forms of arbitrary orders [50] and therefore modified through appropriate transformations to be used as a control law for any non-holonomic robot model having these chains as canonical forms.

5.3 Control of the UMR

In this section, we will use the fact that invertible changes of coordinates can transform the UMR model into either the $(2,3)$-chain model [3] or into the Heisenberg system [4], i.e., into to the models (5.2) and (5.4). By applying the controls presented in the previous section, we can explicitly derive controls for the UMR that will preserve the respective stability properties of their matching model (these results are given without a proof since they follow the theory given in the previous section):

Theorem 20. *The control law* $(v_1(x,y,\theta), v_2(x,y,\theta)) = \psi(u_1(\phi^{-1}(x,y,\theta)), u_2(\phi^{-1}(x,y,\theta)))$, *where* $\phi : \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{S}^1$ *and* $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ *are defined in* (5.3) *and the control law* (u_1, u_2) *is defined in* (5.7) *makes the system* (5.1) *finite-time stable.*

Theorem 21. *The control law* $(v_1(x,y,\theta), v_2(x,y,\theta)) = \psi(u_1(\phi^{-1}(x,y,\theta)), u_2(\phi^{-1}(x,y,\theta)))$, *where* $\phi : \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{S}^1$ *and* $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ *are defined in* (5.5) *and the control law* (u_1, u_2) *is defined in* (5.9) *makes the system* (5.1) *finite-time stable.*

Theorem 22. *The control law* $(v_1(x,y,\theta), v_2(x,y,\theta)) = \psi(u_1(\phi^{-1}(x,y,\theta)), u_2(\phi^{-1}(x,y,\theta)))$, *where* $\phi : \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{S}^1$ *and* $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ *are defined in* (5.6) *and the control law* (u_1, u_2) *is defined in* (5.9) *makes the system* (5.1) *finite-time stable.*

Theorem 23. *The control law* $(v_1(x,y,\theta), v_2(x,y,\theta)) = \psi(u_1(\phi^{-1}(x,y,\theta)), u_2(\phi^{-1}(x,y,\theta)))$, *where* $\phi : \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{S}^1$ *and* $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ *are defined in* (5.6) *and the control law* (u_1, u_2) *is defined in* (5.8) *makes the system* (5.1) *finite-time stable.*

The results of theorems 21 and 22 will be used to demonstrate the role of the transformations of coordinates (5.5) and (5.6) (preserving or not the topology of (5.1), respectively) on the quality of transients in the closed-loop system.

5.4 Simulations

In this section, we illustrate the finite-time stabilization achieved by different control laws, as stated in the previous Theorems, through the simulation of a particular trajectory for each of the proposed controls. For each one of the pictures, we chose the initial conditions to be $x(0) = 1, y(0) = 0.5$ and $\theta(0) = 7$. The closed-loop systems (5.1) is integrated using the explicit Euler method with a step $h = 0.0001$. The results of simulation are presented in Figures 5.1-5.4 (the used values of parameters of control algorithms (if any) are given in the captions of the figures). As we can conclude from these plots, the control amplitudes were not normalized, and the time of convergence is very different being in general dependent on the control norm. The control presented in Fig. 5.1 demonstrates chattering. The control from Fig. 5.2 has much bigger amplitude than others, however, due to utilization of a transformation of coordinates that does not preserve the structure of UMR state space, the transients are rather long having a complex shape, which may be inadmissible from the practical point of view. The control shown in Fig. 5.3 stabilizes the value of variable θ around the value 2π (which is equivalent to zero in $\mathbb{S}^1$). Finally, the control and the transients plotted in Fig. 5.4 demonstrate clear two-stage motion (for $t \leq 1$ and $t > 1$) with phase path being rather far from a straight line connecting the initial and final positions.

It is also worth mentioning that each of the controls presented in Section 5.2 were designed to be robust to perturbations in their respective model dynamics. Although we do not explicitly mention how each type of robustness is translated into the UMR, we do observe it in simulations, as Figures 5.5-5.8 illustrate. For each one, the initial conditions and parameters were kept as the ones in Figures 5.1-5.4, and the control u was replaced by ud, with $d(t) = 1 + 0.7\sin(20t)$ being a periodic multiplicative disturbance that influences the efficiency of the control law application. As we can conclude from these results, all controls successfully passed such a robustness test, and the presence of perturbation does influence much the control form and the transients.

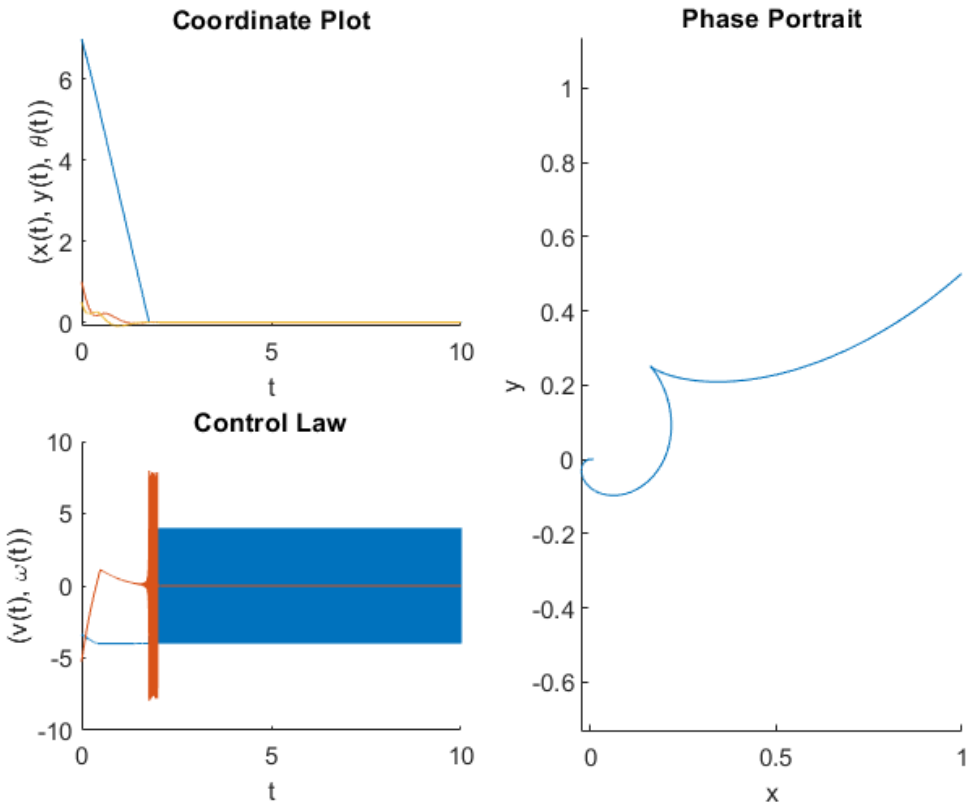


Figure 5.1: Coordinate plot, phase portrait and control law of the UMR trajectories for the control law described in Theorem 20 for parameters $\gamma = 1$ and $\beta = 1$.

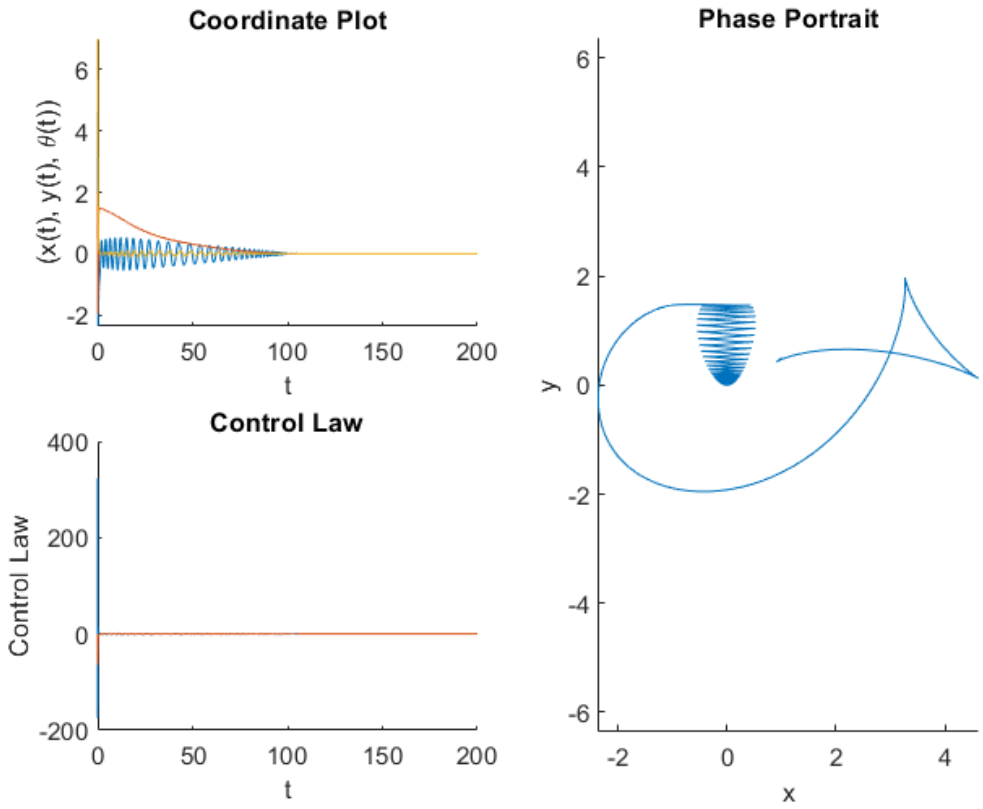


Figure 5.2: Coordinate plot, phase portrait and control law of the UMR trajectories for the control law described in Theorem 21.

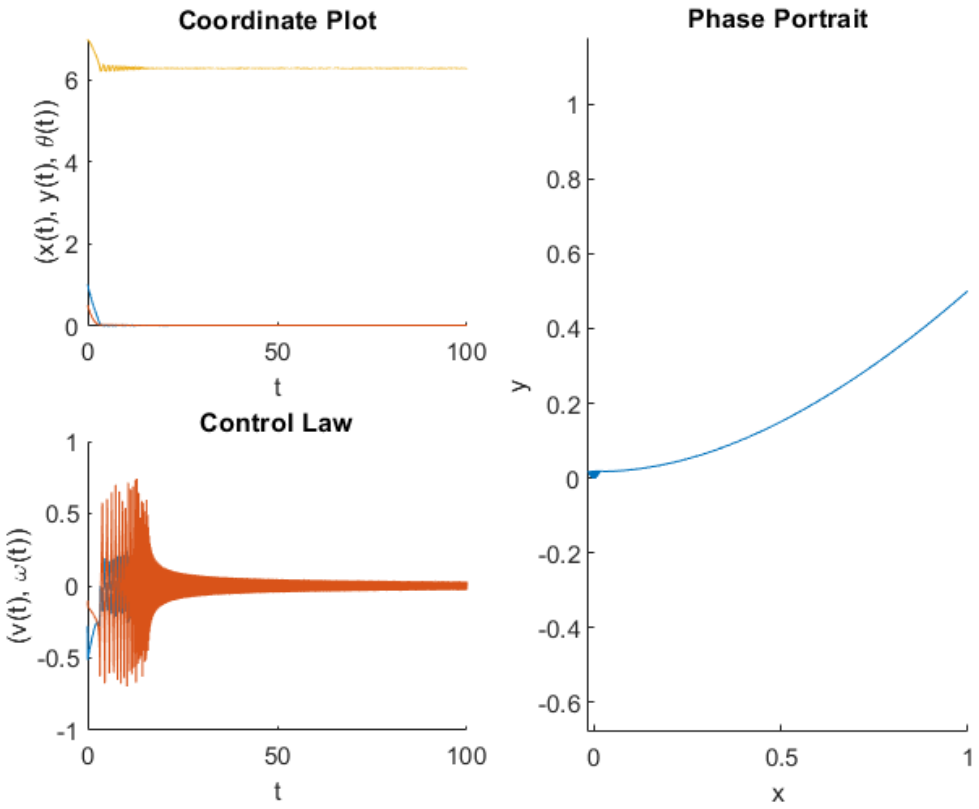


Figure 5.3: Coordinate plot, phase portrait and control law of the UMR trajectories for the control law described in Theorem 22.

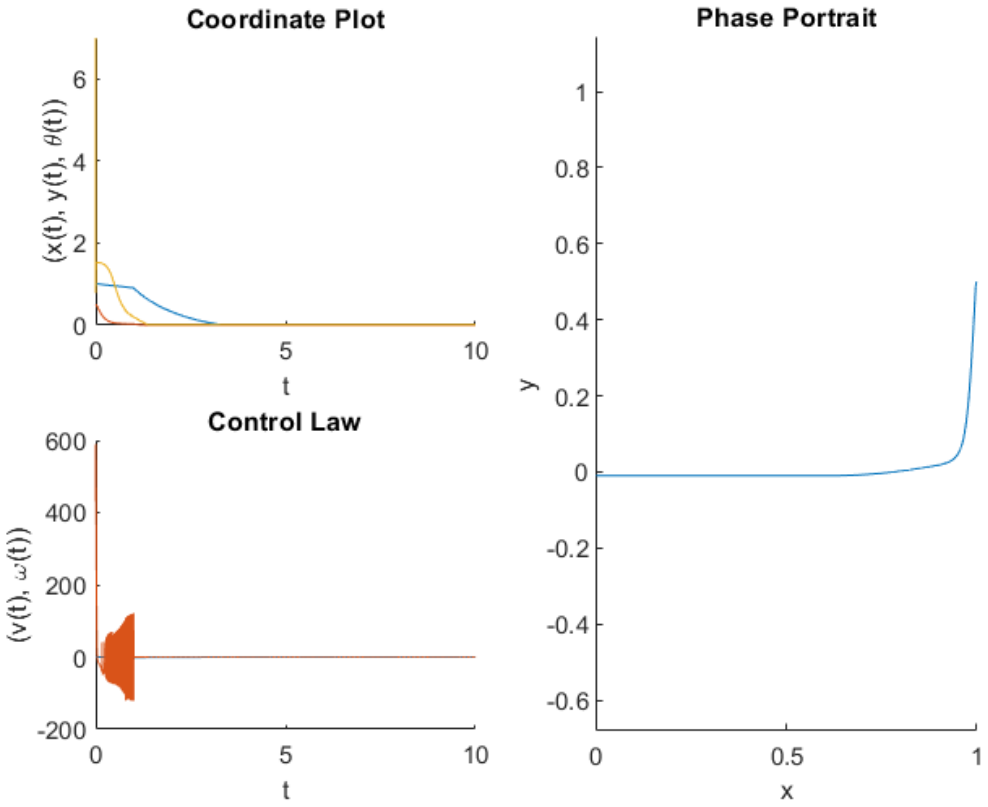


Figure 5.4: Coordinate plot, phase portrait and control law of the UMR trajectories for the control law described in Theorem 23. The parameters chosen are $a = -0.1; p_1 = p_2 = \zeta = 1; \alpha_2 = 0.001; \zeta_0 = 0.1; q_1 = 1.1; q_2 = 0.5 \gamma = 0.5; T_{c_1} = 0.1; T_{c_2} = 0.9; T_{c_3} = 5$.

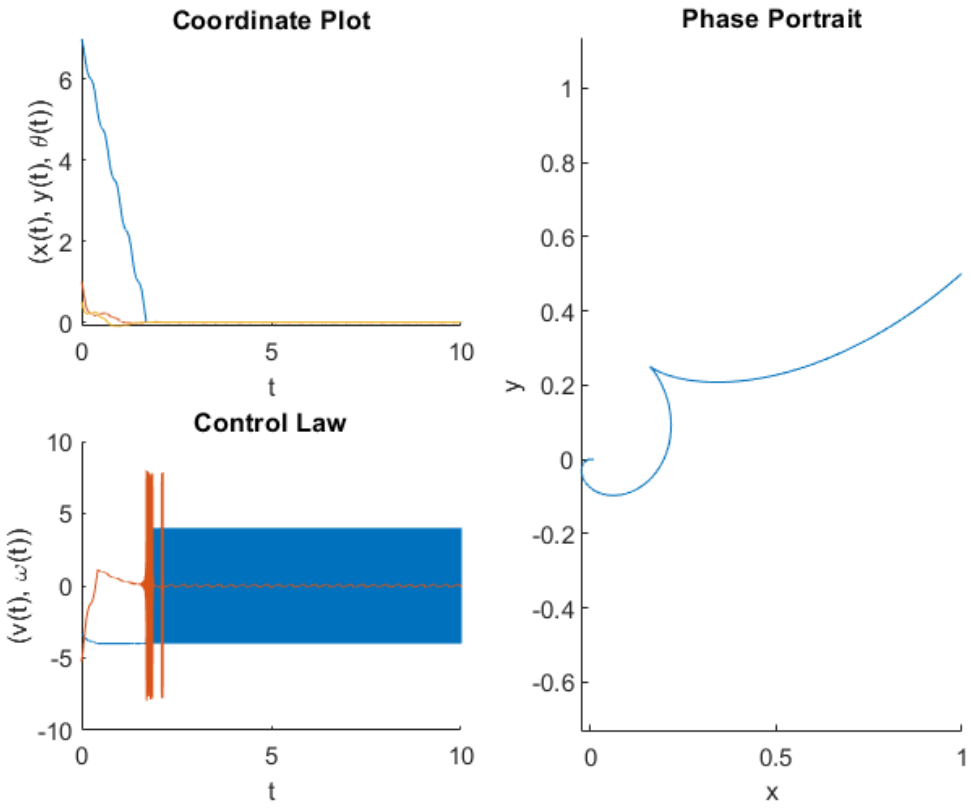


Figure 5.5: Coordinate plot, phase portrait and control law of the UMR following the control of Theorem 20 with disturbance.

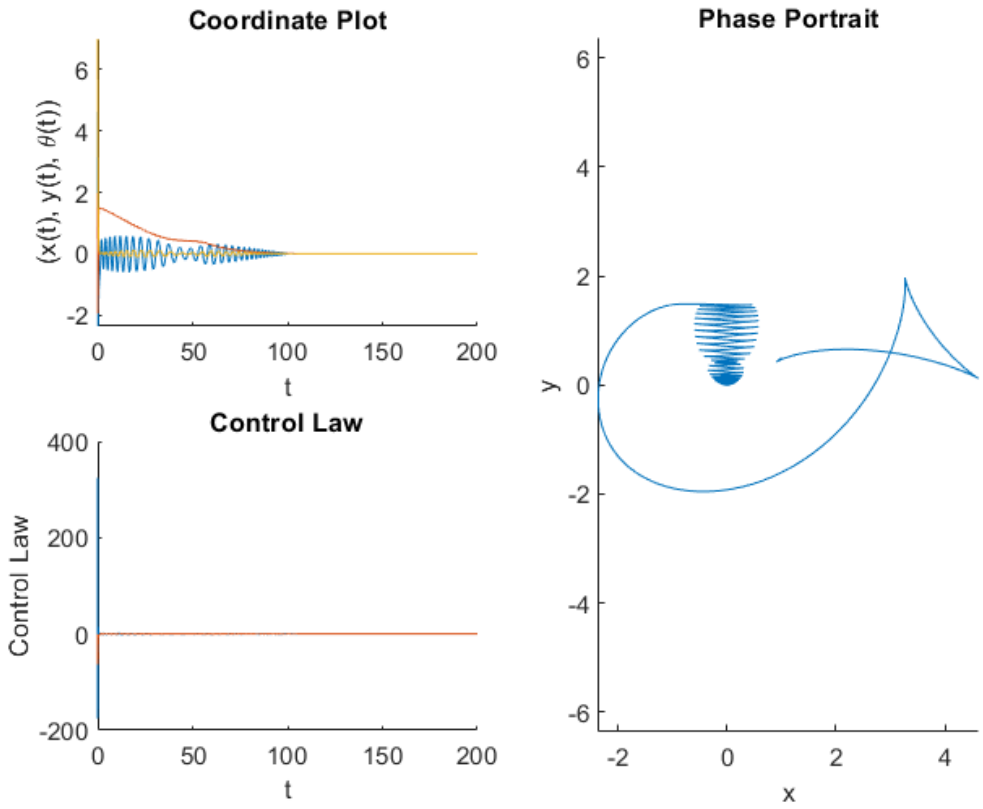


Figure 5.6: Coordinate plot, phase portrait and control law of the UMR following the control of Theorem 21 with disturbance.

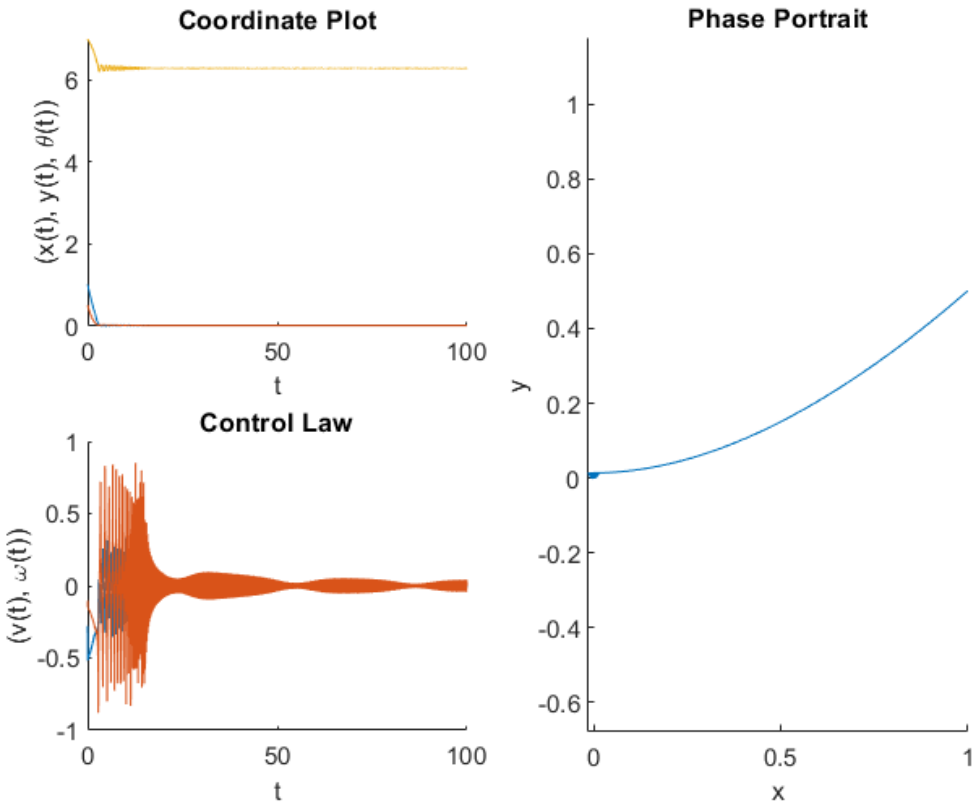


Figure 5.7: Coordinate plot, phase portrait and control law of the UMR following the control of Theorem 22 with disturbance.

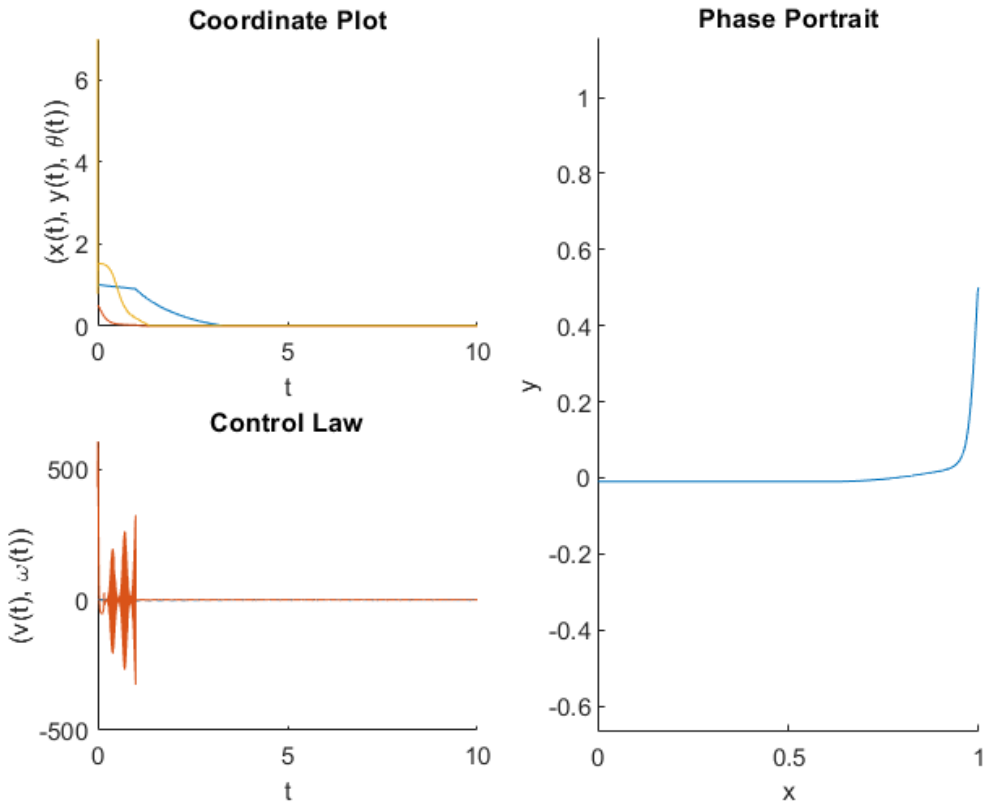


Figure 5.8: Coordinate plot, phase portrait and control law of the UMR following the control of Theorem 23 with disturbance.

5.5 Conclusion

In this chapter, we compared four different control strategies to regulate the dynamics of a UMR in finite time. For that, we used smooth one-to-one transformations of the UMR into simpler canonical forms and developed control laws specifically designed for those systems. The results highlight the importance of utilizing a suitable transformation, demonstrating both the complexity of the problem and the diversity of transients under the benchmarked control laws. The complexity of tuning was not discussed, nor were the challenges of implementation, which we intend to study in further work.

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