

THESE DE DOCTORAT

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Dualité de Poincaré partielle pour les hypersurfaces tropicales singulières et bornes des nombres de Betti des patchworks combinatoires non-primitifs

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Abstract

This thesis deals with the topology of real algebraic varieties. In the spirit of Hilbert's sixteenth problem, a fundamental and still largely open question is to classify, for a fixed degree, non-singular real projective hypersurfaces up to homeomorphism. This classification problem naturally has two aspects: construction and obstruction. On the construction side, Viro's combinatorial patchworking method, introduced in the 1980s, is one of the most powerful known techniques. Recently, Renaudineau and Shaw established upper bounds on Betti numbers for patchworkings in the primitive case. Their proof relies on tropical geometry, in particular on theorems analogous to Poincaré duality and the Lefschetz hyperplane section theorem for tropical homology, again in the primitive case. This thesis extends this approach to the non-primitive case. To this end, we prove a partial Poincaré duality theorem for tropical homology in the singular case and we derive upper bounds on the Betti numbers of non-primitive patchworks.

Key words

Real algebraic geometry, Algebraic topology, Poincaré duality, Toric geometry, Tropical geometry, Combinatorial patchwork.

Résumé

Cette thèse porte sur la topologie des variétés algébriques réelles. Dans la lignée du seizième problème de Hilbert, une question fondamentale et encore largement ouverte est de classifier, à degré fixé, les hypersurfaces projectives réelles non-singulières à homéomorphisme près. Ce problème de classification a naturellement deux volets : la construction et l'obstruction. Du côté de la construction, le patchwork combinatoire de Viro, apparu dans les années 1980, est l'une des méthodes les plus puissantes connues. Récemment, Renaudineau et Shaw ont obtenu des bornes supérieures sur les nombres de Betti pour les patchworks dans le cas primitif. Leur démonstration s'appuie sur la géométrie tropicale et notamment sur des théorèmes de type dualité de Poincaré et théorème de la section hyperplane de Lefschetz pour l'homologie tropicale, toujours dans le cas primitif. Cette thèse généralise cette approche dans le cas non primitif. Pour ce faire, nous démontrons un théorème de dualité de Poincaré partielle pour l'homologie tropicale dans le cas singulier et nous en déduisons des bornes supérieures sur les nombres de Betti de patchworks non primitifs.

Mots clefs

Géométrie algébrique réelle, Topologie algébrique, Dualité de Poincaré, Géométrie torique, Géométrie tropicale, Patchwork combinatoire.

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Chapter 1

Introduction

1.1 Historique

La première partie du seizième problème de Hilbert [Hi 1935] concerne le nombre de branches réelles d'une courbe algébrique et leur disposition. Dans le plan projectif réel $\mathbb{R}P^2$, une courbe algébrique réelle de degré d est le lieu d'annulation d'un polynôme homogène de degré d. Dans le cas des courbes algébriques réelles de $\mathbb{R}P^2$, on peut reformuler la première partie du seizième problème de Hilbert de la façon suivante :

Problème 1.1.1. *Soit $d \geq 1$ un entier. Déterminer l'ensemble des topologies réalisables par la paire $(\mathbb{R}P^2, \mathbb{R}X)$ où $\mathbb{R}X$ est une courbe algébrique réelle de degré d de $\mathbb{R}P^2$.*

Dans le cas d'une courbe algébrique réelle non-singulière, les composantes connexes sont homéomorphes à des cercles. Un cercle peut être plongé de deux manières dans $\mathbb{R}P^2$: soit son image sépare $\mathbb{R}P^2$ en deux composantes connexes (comme un plongement d'un cercle dans $\mathbb{R}^2$), soit le complémentaire de l'image n'a qu'une seule composante connexe (comme $\mathbb{R}P^1 \subset \mathbb{R}P^2$). Ainsi les composantes connexes d'une courbe algébrique réelle non-singulière du plan projectif sont de deux types : on dit que ce sont des ovales si elles séparent $\mathbb{R}P^2$, et que ce sont des pseudo-droites sinon. On peut montrer qu'une courbe algébrique réelle de $\mathbb{R}P^2$ a au plus une seule pseudo-droite parmi ses composantes connexes, les autres étant des ovales, car deux pseudo-droites se coupent nécessairement. De plus, on peut également montrer que la présence d'une pseudo-droite parmi les composantes connexes est équivalente à ce que le degré de la courbe algébrique soit impair. Comme la pseudo-droite ne sépare pas $\mathbb{R}P^2$, il suffit alors de connaître l'arrangement des ovales pour connaître la topologie de la paire $(\mathbb{R}P^2, \mathbb{R}X)$. Dans le cas d'une courbe algébrique non-singulière de $\mathbb{R}P^2$, la première partie du seizième problème de Hilbert peut donc être reformulée de la manière suivante :

Problème 1.1.2. *Déterminer, pour $d \geq 1$, les nombres d'ovales possibles pour une courbe algébrique réelle non-singulière de degré d dans $\mathbb{R}P^2$, et déterminer les arragements possibles pour ces ovales.*

Ce problème comporte donc deux aspects : le nombre d'ovales possibles d'une part, et leurs arrangements possibles d'autre part. Concernant le nombre d'ovales possibles, le problème était déjà résolu depuis 1876 par Harnack ([Ha 1876]):

Theorème 1.1.3. *Le nombre de composantes connexes d'une courbe algébrique réelle non-singulière de degré d de $\mathbb{R}P^2$ est au plus $m = (d-1)(d-2)/2+1$. De plus pour tout entier $0 \leq l \leq m$ si d est pair, et tout entier $1 \leq l \leq m$ si d est impair, il existe une courbe algébrique réelle non-singulière de degré d dans $\mathbb{R}P^2$ ayant exactement l composantes connexes.*

Ce théorème peut être compris en s'intéressant à la complexifiée de la courbe algébrique réelle : une courbe algébrique complexe non-singulière de degré d de $\mathbb{C}P^2$ est une surface de Riemann de genre $g = (d-1)(d-2)/2$. Le lieu réel de cette courbe est donc l'intersection de cette surface de Riemann de genre g avec le plan projectif réel $\mathbb{R}P^2$, son nombre de composantes connexes est alors au plus $g+1$.

Si le nombre d'ovales possibles d'une courbe algébrique réelle non-singulière de degré fixé de $\mathbb{R}P^2$ est connu depuis la fin du XIXème siècle, la détermination des arragements possibles est un problème plus difficile. Durant la première moitié du XXème siècle, cette question a été étudiée par L. Brusotti ([Br 1914] et [Br 1952]) et I. Petrowski ([Pe 1938]) et V. Ragsdale ([Ra 1906]).

Pour classifier les arrangements d'ovales des courbes algébriques réelles non-singulières de $\mathbb{R}P^2$, remarquons que chaque ovale sépare $\mathbb{R}P^2$ en deux composantes connexes, l'une orientable (homéomorphe à un disque, appelée intérieur de l'ovale) et l'autre non-orientable (homéomorphe à une bande Möbius, appelée extérieur de l'ovale). Considérons une courbe algébrique réelle non-singulière de degré pair $d = 2k$ de $\mathbb{R}P^2$, définie par un polynôme homogène f. Comme f est de degré pair, le signe de f est bien défini, et quitte à prendre $-f$ à la place de f, on peut supposer que f est de signe négatif sur l'intersection des extérieurs de tous les ovales. Cela permet de définir les ovales pairs et les ovales impairs :

Definition 1.1.4. *Un ovale pair d'une courbe algébrique réelle non-singulière de degré pair $d = 2k$ de $\mathbb{R}P^2$ est un ovale dont les points intérieurs immédiats sont des points de $\{x \in \mathbb{R}P^2, f(x) > 0\}$, et un ovale impair est un ovale dont les points intérieurs immédiats sont des points de $\{x \in \mathbb{R}P^2, f(x) < 0\}$, où f est un polynôme homogène définissant la courbe algébrique réelle tel que $f(x) < 0$ pour tout point x de l'intersection des extérieurs de tous les ovales.*

I. Petrowski démontra alors l'inégalité suivante ([Pe 1938]):

Theorème 1.1.5. *Considérons une courbe algébrique réelle non-singulière de degré pair $d = 2k$ dans $\mathbb{R}P^2$. Alors le nombre d'ovales pairs P et le nombre d'ovales impairs N vérifient l'inégalité suivante :*

$$-\frac{3}{2}k(k-1) \leq P - N \leq \frac{3}{2}k(k-1) + 1$$

Ragsdale formula la conjecture suivante ([Ra 1906]), un peu plus forte :

Conjecture 1.1.6. *(Ragsdale) Considérons une courbe algébrique réelle non-singulière de degré pair $d = 2k$ dans $\mathbb{R}P^2$. Alors le nombre d'ovales pairs P et le nombre d'ovales impairs N vérifient les inégalités :*

$$P \leq \frac{3}{2}k(k-1) + 1 \quad ; \quad N \leq \frac{3}{2}k(k-1).$$

A partir des années 1970, d'importantes avancées eurent lieu avec notamment D.A. Gudkov ([Gu 1969]), V. Arnold ([Ar 1971]) ou encore V.A. Rokhlin ([Ro 1972], [Ro2 1972], [Ro 1973] et [Ro 1974]). En particulier, D.A. Gudkov compléta la classification des arrangements d'ovales de courbes algébriques réelles non-singulières de degré $d \leq 6$ dans $\mathbb{R}P^2$. C'est à cette période aussi que la généralisation en dimension supérieure de la première partie du seizième problème de Hilbert commença à suciter de l'intérêt. Cette généralisation peut s'énoncer de la façon suivante :

Problème 1.1.7. *Soit $d \geq 1$ et $n \geq 1$ des entiers. Déterminer l'ensemble des topologies réalisables par la paire $(\mathbb{R}P^n, \mathbb{R}X)$ où $\mathbb{R}X$ est une hypersurface algébrique réelle de degré d de $\mathbb{R}P^n$.*

De même que dans le cas des courbes, ce problème comporte un aspect plus simple qui est de déterminer les topologies possibles pour l'hypersurface $\mathbb{R}X$ seule, et non plus les topologies de la paire $(\mathbb{R}P^n, \mathbb{R}X)$:

Problème 1.1.8. *Soit $d \geq 1$ et $n \geq 1$ des entiers. Déterminer l'ensemble des topologies réalisables par $\mathbb{R}X$, où $\mathbb{R}X$ est une hypersurface algébrique réelle de degré d de $\mathbb{R}P^n$.*

V. M. Kharlamov répondit à ce dernier problème dans le cas des surfaces de degré 4. Il réussit également à déterminer les topologies possibles pour la paire $(\mathbb{R}P^3, \mathbb{R}X)$ où $\mathbb{R}X$ est une surface algébrique réelle non-singulière de degré 4 de $\mathbb{R}P^3$ ([Kh 1976] et [Kh 1978]).

Plusieurs résultats contraignent la topologie des hypersurfaces algébriques réelles en fonction de la topologie de leur complexification. Il y a notamment l'inégalité de Smith-Thom :

Theorème 1.1.9. *(Inégalité et congruence de Smith-Thom) Considérons une variété réelle $\mathbb{R}X$ et sa complexifiée $\mathbb{C}X$. Si $\mathbb{C}X$ est compacte, alors les nombres de Betti sur le corps $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ de $\mathbb{R}X$ et $\mathbb{C}X$ vérifient :*

$$\sum_i b_i(\mathbb{R}X; \mathbb{F}_2) \leq \sum_j b_j(\mathbb{C}X; \mathbb{F}_2) \quad ; \quad \sum_i b_i(\mathbb{R}X; \mathbb{F}_2) \equiv \sum_j b_j(\mathbb{C}X; \mathbb{F}_2) \mod 2$$

Il ya aussi les inégalités de Petrovsky-Oleinik-Kharlamov :

Proposition 1.1.10. *(Inégalités de Petrovsky-Oleinik-Kharlamov) Soit $\mathbb{R}X$ une variété algébrique réelle dont la complexification $\mathbb{C}X$ est une variété compacte de Kähler de dimension $4n$. Alors la caractéristique d'Euler $\chi(\mathbb{R}X)$ de cette variété réelle vérifie :*

$$2 - h^{n,n}(\mathbb{C}X) \leq \chi(\mathbb{R}X) \leq h^{n,n}(\mathbb{C}X)$$

où $h^{p,q}(\mathbb{C}X)$ désigne les nombres de Hodge de $\mathbb{C}X$.

Ou encore la congruence de Rokhlin :

Proposition 1.1.11. *(Congruence de Rokhlin) Soit $\mathbb{R}X$ une variété algébrique réelle dont la complexification $\mathbb{C}X$ est une variété compacte de dimension réelle $4n$. Si cette variété est maximale pour l'inégalité de Smith-Thom alors :*

$$\chi(\mathbb{R}X) \equiv \sigma(\mathbb{C}X) \mod 16$$

où $\chi(\mathbb{R}X)$ est la caractéristique d'Euler de $\mathbb{R}X$ et $\sigma(\mathbb{C}X)$ est la signature de $\mathbb{C}X$.

En 1980, O. Viro émet la conjecture suivante :

Conjecture 1.1.12. *(Viro) Soit $\mathbb{R}X$ une surface algébrique réelle non-singulière et simplement connexe de $\mathbb{R}P^3$. Alors :*

$$b_1(\mathbb{R}X;\mathbb{F}_2) \leq h^{1,1}(\mathbb{C}X)$$

où $b_1(\mathbb{R}X;\mathbb{F}_2)$ est le premier nombre de Betti sur $\mathbb{F}_2$ de $\mathbb{R}X$, et où $h^{1,1}(\mathbb{C}X)$ est le nombre de Hodge $h^{1,1}$ de la complexification $\mathbb{C}X$ de $\mathbb{R}X$.

Par ailleurs, O. Viro construisit dans [Vi 1980] des exemples de courbes algébriques réelles non-singulières de degré $2k$ dans $\mathbb{R}P^2$ telles que le nombres d'ovales impairs soit $N = \frac{3}{2}k(k-1)+1$, réfutant ainsi la conjecture de Ragsdale. Il proposa alors la conjecture de Ragsdale modifiée :

Conjecture 1.1.13. *(Ragsdale modifiée par Viro) Considérons une courbe algébrique réelle non-singulière de degré pair $d = 2k$ dans $\mathbb{R}P^2$. Alors le nombre d'ovales pairs P et le nombre d'ovales impairs N vérifient les inégalités :*

$$P \leq \frac{3}{2}k(k-1)+1 \quad ; \quad N \leq \frac{3}{2}k(k-1)+1.$$

A la fin des années 1970, O. Viro découvrit une puissante méthode de construction des hypersurfaces algébriques réelles avec un contrôle sur leur topologie. Cette méthode fonctionne en toute dimension et en tous degrés, et peut même se généraliser aux hypersurfaces réelles dans des variétés toriques. Elle permit à Viro de terminer la classification des arrangements d'ovales de courbes algébriques réelles non-singulières de degré $d = 7$ dans $\mathbb{R}P^2$. A ce jour, le degré 7 reste toujours le dernier degré pour lequel cette classification est entièrement connue. La méthode de Viro permit également à Itenberg de réfuter cette Conjecture 1.1.13 de Ragsdale modifiée par Viro, en montrant la proposition suivante :

Proposition 1.1.14. *(Itenberg) Soit $k \geq 5$. Alors il existe une courbe algébrique réelle non-singulière de degré $2k$ dans $\mathbb{R}P^2$ dont le nombre d'ovales pairs est :*

$$P = \frac{3}{2}k(k-1)+1+\left\lfloor \frac{(k-3)^2+4}{8} \right\rfloor$$

ainsi qu'une courbe algébrique réelle non-singulière de degré $2k$ dans $\mathbb{R}P^2$ dont le nombre d'ovales impairs est :

$$N = \frac{3}{2}k(k-1)+\left\lfloor \frac{(k-3)^2+4}{8} \right\rfloor$$

La méthode de Viro, ou patchwork de Viro, ([Vi 2006] et [Ri 1993]) est une méthode permettant de construire une hypersurface algébrique réelle de polytope de Newton P dans une variété torique projective construite également à partir de ce polytope P. Cette hypersurface est construite à partir d'une triangulation à sommets entiers de ce polytope P et d'une distribution de signes $+$ ou $-$ fixés sur les sommets de la triangulation. Dans le cas de la construction d'hypersurfaces algébriques réelles de degré d dans $\mathbb{R}P^n$, le polytope à trianguler est le simplexe de sommets $(0,...,0)$, $(d,0,...,0)$, ..., $(0,...,0,d)$. Un choix

astucieux de la triangulation et de la distribution de signes permet d'obtenir des hypersurfaces avec des topologies intéressantes, et c'est comme ceci que Viro a pu terminer la classification des arrangements d'ovales des courbes non-singulières de degré 7 dans $\mathbb{R}P^2$, ou que Itenberg a pu démontrer sa proposition réfutant la conjecture de Ragsdale. Un nouveau problème est alors apparu :

Problème 1.1.15. *Soit P un polytope à sommets entiers, de dimension n dans $\mathbb{R}^n$. Déterminer les topologies possibles pour la paire $(\mathbb{R}Y, \mathbb{R}X)$, où $\mathbb{R}Y$ est le lieu réel de la variété torique de polytope de Newton P, et $\mathbb{R}X$ est une hypersurface algébrique réelle de polytope de Newton P dans $\mathbb{R}Y$, construite par Patchwork à partir d'une triangulation à sommets entiers de P.*

Les propriétés de la triangulation de départ influencent les topologies atteignables par l'hypersurface algébrique réelle ainsi construite. Par exemple, lorsque la triangulation est maximale, c'est-à-dire lorsque tous le points entiers du polytope sont des sommets de la triangulation, Itenberg a démontré ([It 1997]):

Proposition 1.1.16. *Soit $\mathbb{R}X$ une surface algébrique réelle de $\mathbb{R}P^3$ construite par Patchwork de Viro à partir d'une triangulation maximale. Alors $\mathbb{R}X$ vérifie la conjecture de Viro :*

$$b_1(\mathbb{R}X; \mathbb{F}_2) \leq h^{1,1}(\mathbb{C}X)$$

où $\mathbb{C}X$ est la complexification de $\mathbb{R}X$.

Une propriété intéressante pour une triangulation à sommets entiers d'un polytope à sommets entiers de dimension n de $\mathbb{R}^n$ est d'imposer que tous les n-simplexes soient de volume $1/n!$. Dans ce cas on parle de triangulation primitive. De nombreuses propriétés topologiques ont été découvertes pour les hypersurfaces construites à partir de Patchworks primitifs. Il y a notamment un théorème de Bertrand ([Be 2010]) :

Theorème 1.1.17. *Une hypersurface algébrique réelle $\mathbb{R}X$ d'une variété torique projective $\mathbb{R}Y$ de polytope de Newton P, construite par Patchwork à partir d'une triangulation primitive de P vérifie $\chi(\mathbb{R}X) = \sigma(\mathbb{C}X)$, où $\mathbb{C}X$ est sa complexification.*

Et un résultat d'Itenberg pour les surfaces ([It 1997]) :

Theorème 1.1.18. *Soit $\mathbb{R}X$ une surface algébrique réelle de $\mathbb{R}P^3$ construite par Patchwork de Viro à partir d'une triangulation primitive. Alors $\mathbb{R}X$ vérifie :*

$$b_0(\mathbb{R}X; \mathbb{F}_2) \leq 1 + h^{2,0}(\mathbb{C}X) \quad ; \quad b_1(\mathbb{R}X; \mathbb{F}_2) \leq h^{1,1}(\mathbb{C}X)$$

où $\mathbb{C}X$ est la complexification de $\mathbb{R}X$.

La découverte du Patchwork par Viro a aussi été le début de la géométrie tropicale, développée notamment par G. Mikhalkin ([Mi 2004], [Mi 2005] et [Mi 2006]). En effet, la triangulation d'un polytope à sommets entiers utilisée pour constuire une hypersurface réelle peut être vue comme la subdivision duale d'une hypersurface tropicale, et l'étude des propriétés de cette hypersurface tropicale permet d'obtenir des informations sur la topologie de l'hypersurface réelle. C'est ainsi que le Patchwork de Viro devient un pont entre géométrie tropicale et géométrie réelle. Dans ce contexte, Renaudineau et Shaw généralisèrent le Théorème 1.1.18 d'Itenberg en toute dimension ([RS 2023]) :

Theorème 1.1.19. *Soit $n \geq 1$. Soit P un polytope à sommets entiers de dimension n dans $\mathbb{R}^n$, et soit $\mathbb{R}Y$ le lieu réel de la variété torique projective construite à partir du polytope P. Soit $\mathbb{R}X$ une hypersurface algébrique réelle dans $\mathbb{R}Y$, construite par Patchwork de Viro à partir d'une triangulation primitive de P. Alors les nombres de Betti de $\mathbb{R}X$ vérifient :*

$$b_q(\mathbb{R}X;\mathbb{F}_2) \leq \begin{cases} h^{q,q}(\mathbb{C}X) & si\ q = (n-1)/2 \\ h^{q,n-q}(\mathbb{C}X) + h^{q,q}(\mathbb{C}X) & sinon \end{cases}$$

où $h^{p,q}(\mathbb{C}X)$ sont les nombres de Hodge de la complexification $\mathbb{C}X$ de $\mathbb{R}X$.

Pour prouver ce théorème, Renaudineau et Shaw trouvèrent une suite spectrale qui dégénère vers l'homologie de l'hypersurface réelle, mais dont la première page est égale à l'homologie tropicale de l'hypersurface tropicale duale à la triangulation utilisée pour ce Patchwork. Ensuite ils utilisèrent un théorème de type Lefschetz tropical, un théorème de dualité de Poincaré tropical (prouvé dans [JRS 2018]), et des identités de caractéristiques d'Euler, pour prouver que les dimensions des groupes d'homologie tropicale sont égales aux nombres de Hodge de $\mathbb{C}X$. Leurs inégalités viennent alors du fait que les dimensions des termes de la page infinie d'une suite spectrale sont plus petites que les dimensions des termes correspondants de la première page.

Toutes les triangulations à sommets entiers d'un polytope à sommets entiers de dimension n ne permettent pas de construire par Patchwork une hypersurface algébrique réelle. Le théorème de Patchwork de Viro demande que la triangulation de départ soit convexe, c'est-à-dire qu'il existe une fonction à valeurs réelles, convexe, définie sur le polytope triangulé, dont la restriction à chaque n-simplexe soit affine, et dont la restriction à chaque réunion de deux n-simplexes distincts ne soit pas affine. C'est également cette condition de convexité qui est requise pour qu'il existe une hypersurface tropicale duale à la triangulation. Si l'on s'intéresse à une triangulation non-convexe d'un polytope à sommets entiers, l'hypersurface combinatoire obtenue par le processus de Patwork de Viro peut tout de même être définie, mais il n'existe pas forcément d'hypersurface algébrique réelle homéomorphe à elle. Brugallé, Lopez de Medrano et Rau ont remarqué dans [BMR 2024] que les outils de la géométrie tropicale, tels que l'homologie tropicale, peuvent également être définis pour une triangulation non-convexe, même s'il n'existe pas d'hypersurface tropicale duale à la triangulation. C'est ainsi que l'étude de la topologie des hypersurfaces combinatoires construites par Patchwork, qu'elles aient une interprétation en géométrie réelle ou non, est devenue un problème en soi :

Problème 1.1.20. *Soit P un polytope à sommets entiers, de dimension n dans $\mathbb{R}^n$. Déterminer les topologies possibles pour les hypersurfaces combinatoires obtenues par Patchwork de Viro de ce polytope, à partir d'une triangulation non-nécéssairement convexe*

Dans [BMR 2024] est prouvé :

Theorème 1.1.21. *Soit $n \geq 1$. Soit P un polytope à sommets entiers de dimension n dans $\mathbb{R}^n$, et soit $\mathbb{C}Y$ la variété torique projective construite à partir du polytope P. Soit X une hypersurface combinatoire construite par Patchwork de Viro à partir d'une triangulation primitive non-nécéssairement convexe de P. Soit $\mathbb{C}\tilde{X}$ une hypersurface algébrique non-dégénérée quelconque de la variété torique $\mathbb{C}Y$. Alors les nombres de Betti de X vérifient*

:

$$b_q(X;\mathbb{F}_2) \leq \begin{cases} h^{q,q}(\mathbb{C}\tilde{X}) & si\ q = (n-1)/2 \\ h^{q,n-q}(\mathbb{C}\tilde{X}) + h^{q,q}(\mathbb{C}\tilde{X}) & sinon \end{cases}$$

Pour prouver ce théorème Brugallé, Lopez de Medrano et Rau ont généralisé dans [BMR 2024] l'approche de Renaudineau et Shaw dans un cadre purement combinatoire, sans interprétation en géométrie tropicale. C'est dans cette lignée que s'inscrit cette thèse : nous obtenons une généralisation non-primitive de la dualité de Poincaré tropicale qui s'applique aux Patchworks de triangulations non-nécéssairement convexes, et une généralisation non-primitive des bornes des nombres de Betti des Patchworks qui s'applique également aux triangulations non-convexes.

1.2 Notions préliminaires

Dans le Chapitre 2 nous expliquons toutes les notions classiques intervenant dans les énoncés de nos résultats et dans leurs preuves, et nous présentons les travaux de nos prédesseurs.

Nous commençons par décrire la construction des variétés toriques projectives à partir d'un polytope à sommet entier, car c'est dans ces variétés toriques que vivent les hypersurfaces que l'on étudie dans cette thèse.

Ensuite nous décrivons la méthode du patchwork de Viro permettant de construire des hypersurfaces réelles. Nous rentrons à nouveau dans le détail de cette méthode dans la Section 4.1 du Chapitre 4, pour la nécessité de notre preuve. Ainsi, pour donner au lecteur deux présentations différentes du patchwork, nous avons choisi de commencer par le présenter pour la construction d'hypersurfaces de l'espace projectif réel $\mathbb{R}P^n$ dans la Section 2.2 des préliminaires. Dans la Section 4.1 du Chapitre 4 nous le présentons dans le cadre plus général de la construction d'hypersurfaces d'une variété torique projective réelle, cadre choisi pour cette thèse.

Puis nous rappelons la définition d'une triangulation maximale apparaissant dans l'énoncé du Théorème 1.1.16 d'Itenberg, et la définition d'une triangulation primitive apparaissant dans l'énoncé du Théorème 1.1.17 de Bertrand, du Théorème 1.1.18 d'Itenberg, du Théorème 1.1.19 de Renaudineau et Shaw, et du Théorème 1.1.21 de Brugallé, Lopez de Medrano et Rau. Nous présentons alors comment l'inégalité de Smith-Thom (Théorème 1.1.9) combinée à une identité de caractéristique d'Euler permet d'obtenir des bornes sur les nombres de Betti des surfaces de $\mathbb{R}P^3$ de degré d fixé obtenues par patchwork d'une triangulation primitive (Théorème 1.1.18 d'Itenberg).

Renaudineau et Shaw ont montré dans [RS 2023] que l'homologie d'une hypersurface réelle construite par Patchwork de Viro d'une triangulation primitive convexe pouvait être calculée à partir de l'homologie d'un certain cofaisceau cellulaire qu'ils ont introduit (le cofaisceau des signes), un cofaisceau défini sur l'hypersurface tropicale duale à la triangulation utilisée pour le patchwork. Ainsi nous présentons dans la Section 2.5 les faisceaux et cofaisceaux cellulaires, définis sur un ensemble partiellement ordonné fini, et nous expliquons la construction de l'homologie de ces cofaisceaux et de la cohomologie de ces faisceaux, sous certaines hypothèses. Cela nous permet alors de présenter dans la Section 2.6 les grandes lignes de la preuve du Théorème 1.1.19 de Renaudineau et Shaw

donnant des bornes sur les nombres de Betti des patchworks primitifs en toutes dimensions. Pour obtenir leurs bornes, Renaudineau et Shaw ont introduit une suite spectrale dégénérant vers l'homologie de l'hypersurface réelle construite par patchwork mais dont la première page calcule l'homologie tropicale de l'hypersurface tropicale duale à la triangulation utilisée pour ce patchwork. Ils ont alors utilisé un théorème de dualité de Poincaré tropicale (de [JRS 2018]), un théorème de Lefschetz tropical (de [ARS 2021]) et une identité de caractéristique d'Euler.

Ce théorème de Lefschetz tropical a par la suite été généralisé par Chenal dans [Ch 2025] au cadre des triangulations non-primitives d'un polytope simple. Cette généralisation nous est utile dans le Chapitre 4 de cette thèse pour généraliser au cas non-primitif les bornes sur les nombres de Betti des patchworks. Ainsi dans la Section 2.7 nous présentons le théorème de Lefschetz tropical de Chenal.

Ce théorème de dualité de Poincaré tropicale de [JRS 2018] a été expliqué par Brugallé, Lopez de Medrano et Rau dans [BMR 2024] dans le cadre des triangulations primitives non-nécessairement convexes d'un polytope à sommets entiers non-singulier. Nous généralisons ce théorème au cas des triangulations non-primitives dans le Chapitre 3 de cette thèse. Ainsi nous avons choisi de présenter dans la Section 2.8 la preuve de ce théorème de dualité de Poincaré tropicale telle qu'expliquée dans [BMR 2024]. L'idée de cette preuve vient de [JRS 2018], c'est une preuve par récurrence appliquant l'identité de Mayer-Vietoris. Pour notre généralisation de ce théorème dans le Chapitre 3, notre stratégie est différente. Nous utilisons une subdivision cellulaire plus fine pour l'hypersurface tropicale, faisant apparaître une structure de complexe double pour les cofaisceaux et faisceaux tropicaux. De ces complexes doubles apparaissent des suites spectrales dégénérant vers l'homologie et la cohomologie tropicale, et la preuve de notre généralisation de la dualité de Poincaré tropicale vient de l'établissement de relations entre ces suites spectrales.

1.3 Nos Résultats

1.3.1 Chapitre 3 : Dualité de Poincaré pour les hypersurfaces tropicales singulières

Dans le cas d'une triangulation primitive d'un polytope à sommets entiers, [RS 2023], utilisèrent le théorème de dualité de Poincaré tropicale de [JRS 2018] pour trouver des bornes sur les nombres de Betti des Patchworks de cette triangulation. Pour généraliser ces bornes sur les nombres de Betti au cas des Patchworks non-primitifs, il était nécéssaire de généraliser aussi la dualité de Poincaré tropicale au cas des triangulations non-primitives.

Afin de rester général, nous avons choisi de démontrer nos théorèmes de dualité de Poincaré tropicale partielle et complète pour les cofaisceaux tropicaux à coefficients dans un anneau intègre R quelconque, même si c'est finalement $R = \mathbb{F}_2$ qui est utile pour généraliser les bornes sur les nombres de Betti des Patchworks. Mais cette généralité est intéressante en particulier dans le cas $R = \mathbb{Q}$ où nous obtenons une dualité de Poincaré tropicale complète pour toute triangulation d'un polytope à sommets entiers, fournissant une réciproque à un théorème de [Ak 2023] qui affirme que pour avoir une dualité de Poincaré tropicale à coefficients dans $\mathbb{Q}$ il est nécéssaire que la subdivision du polytope soit une triangulation. Par ailleurs, notre théorème de dualité de Poincaré tropicale complète à coefficients dans $\mathbb{Q}$ peut aussi être utile dans le monde de la géométrie tropicale car

certains résultats de géométrie tropicale demandent comme hypothèse que la dualtié de Poincaré tropicale à coefficients dans $\mathbb{Q}$ soit satisfaite ([AP 2023]). Ce Chapitre de la thèse est l'objet de notre article [De 2025].

Dans ce chapitre, nous commençons par définir les cofaisceaux tropicaux $\mathcal{F}_p^R$ et les faisceaux tropicaux $\mathcal{F}_R^p$ que nous utilisons, et la subdivision sur laquelle ils sont définis. Nous faisons ensuite apparaître une famille de suites spectrales dégénérant vers les homologies des cofaisceaux $\mathcal{F}_p^R$, et une famille de suites spectrales dégénérant vers les cohomologies des faisceaux $\mathcal{F}_R^p$.

Ensuite nous définissons une classification des triangulations non-primitives par niveau de primitivité. Notre notion de primitivité n'est plus absolue, mais dépend de l'anneau R sur lequel nous travaillons, nous parlons de R-primitivité. Ainsi, un n-simplexe à sommets entiers dans $\mathbb{R}^n$ est dit R-primitif s'il peut se ramener, par homothétie, à un simplexe dont le volume est de la forme $r/n!$, avec r un entier inversible dans l'anneau R. Une triangulation (k, R)-primitive d'un polytope à sommets entiers dans $\mathbb{R}^n$ est alors une triangulation dont tous les simplexes de dimension $\leq k$ sont R-primitifs. Nous définissons également la R-non-singularité d'un polytope simple à sommets entiers dans $\mathbb{R}^n$ comme la propriété selon laquelle pour chaque sommet de ce polytope le déterminant des vecteurs directeurs primitifs des arêtes adjacentes à ce sommet soit inversible dans l'anneau R.

Ensuite vient la construction d'un cap produit pour ces complexes de cofaisceaux et de faisceaux, puis la construction d'une classe fondamentale. Enfin vient la partie où, utilisant plusieurs lemmes, nous démontrons nos théorèmes de dualité de Poincaré partielle et complète :

Theorème 1.3.1. *(Dualité de Poincaré patielle). Soit R est un anneau intègre, et P un polytope de dimension n à sommets entiers R-non-singulier dans $\mathbb{R}^n$, muni d'une triangulation (k, R)-primitive, avec $2 \leq k \leq n-1$. Alors le cap-produit avec la classe fondamentale induit les isomorphismes suivants :*

$$\forall p + q \geq 2n - k, \qquad H^q(\mathcal{F}_R^p) \xrightarrow{\sim} H_{n-1-q}(\mathcal{F}_{n-1-p}^R).$$

De plus, pour $k = n$ (c'est-à-dire pour une triangulation R-primitive), nous avons le théorème suivant :

Theorème 1.3.2. *(Dualité de Poincaré complète) Soit R est un anneau intègre, et P un polytope à sommets entiers R-non-singulier de dimension n dans $\mathbb{R}^n$, muni d'une triangulation R-primitive. Alors le cap-produit avec la classe fondamentale induit les isomorphismes suivants :*

$$\forall p, q, \qquad H^q(\mathcal{F}_R^p) \xrightarrow{\sim} H_{n-1-q}(\mathcal{F}_{n-1-p}^R).$$

Dans le cas $R = \mathbb{Q}$ on peut remarquer que les polytopes $\mathbb{Q}$-non-singuliers sont exactement les polytopes simples, et on peut remarquer aussi que toutes les triangulations sont $\mathbb{Q}$-primitives, ce qui nous donne alors comme corollaire :

Corollaire 1.3.3. *(Dualité de Poincaré tropicale à coefficients dans $\mathbb{Q}$) Soit P un polytope simple à sommets entiers de dimension n dans $\mathbb{R}^n$, muni d'une triangulation à sommets*

entiers. Alors le cap-produit avec la classe fondamentale induit les isomorphismes suivants :

$$\forall p, q, \qquad H^q(\mathcal{F}^p_{\mathbb{Q}}) \xrightarrow{\sim} H_{n-1-q}(\mathcal{F}^{\mathbb{Q}}_{n-1-p}).$$

Pour démontrer ces théorèmes nous montrons que le cap produit avec la classe fondamentale induit des isomorphismes entre certains termes de la première page de la suite spectrale filtrant la cohomologie de $\mathcal{F}^p_R$ et certains termes de la première page de la suite spectrale filtrant l'homologie de $\mathcal{F}^R_{n-1-p}$. Le niveau de primitivité va déterminer le nombre de ces isomorphismes. En utilisant certains lemmes, nous montrons que ces isomorphismes sont conservés en tournant les pages de ces suites spectrales, pour aboutir, à la page infinie, à des isomorphismes entre la cohomologie de $H^q(\mathcal{F}^p_R)$ et l'homologie de $H_{n-1-q}(\mathcal{F}^{\mathbb{Q}}_{n-1-p})$. La section 3.8 est alors consacrée à la preuve des lemmes que nous avons utilisés pour prouver nos théorèmes. Enfin, nous énonçons dans la dernière section une généralisation non-compacte de nos théorèmes de dualité de Poincaré tropicale non-primitive.

1.3.2 Chapitre 4 : Bornes sur les nombres de Betti des patworks non-primitifs

Dans ce Chapitre, nous généralisons l'approche de [RS 2023] et [BMR 2024] pour obtenir des bornes sur les nombres de Betti des Patchworks non-primitifs. Comme [BMR 2024], nous nous plaçons dans le cadre des triangulations non-nécéssairement convexes d'un polytope P à sommets entiers de $\mathbb{R}^n$. La triangulation que l'on considère est donc une triangulation non-nécéssairement convexe et non-nécéssairement primitive. Ce Chapitre de la thèse est l'objet de notre article [De 2026].

Dans ce Chapitre, nous commençons par généraliser la définition de la suite spectrale introduite par [RS 2023] dégénérant vers l'homologie de l'hypersurface réelle. Comme dans le cas primitif, nos bornes sur les nombres de Betti proviennent donc de l'inégalité de dimension entre les termes de la page infinie et les termes de la page 1 de cette suite spectrale. Mais dans le cas non-primitif général, les termes de la première page de cette suite spectrale ne s'interprêtent pas tous commes des groupes d'homologie tropicale $H_q(\mathcal{F}_p)$. Cependant, on peut tout de même définir un cofaisceau $\tilde{\mathcal{F}}_p$ tel que les termes de la première page de cette suite spectrale coïncident avec les groupes d'homologie $H_q(\tilde{\mathcal{F}}_p)$. Ce cofaisceau $\tilde{\mathcal{F}}_p$ ne dépend pas que de la triangulation du polytope P, comme le cofaisceau tropical $\mathcal{F}_p$, mais il dépend aussi de la distribution de signes fixés sur les sommets de la triangulation.

L'objectif pour obtenir nos bornes sur les nombres de Betti de l'hypersurface combinatoire construite par ce Patchwork non-nécessairement primitif est de comprendre nos groupes d'homologie $H_q(\tilde{\mathcal{F}}_p)$. Pour cela, nous montrons d'abord que, pour une triangulation $(k, \mathbb{F}_2)$-primitive d'un polytope P à sommets entiers de dimension n dans $\mathbb{R}^n$, une partie des groupes $H_q(\tilde{\mathcal{F}}_p)$ s'identifie aux groupes d'homologie tropicale $H_q(\mathcal{F}_p)$ (leur nombre dépend de k, et si $k = n$ ils s'identifient tous). Nous utilisons ensuite un théorème de Lefschetz de Chenal ([Ch 2025]) et notre théorème de dualité de Poincaré partielle pour identifier les dimensions de certains de ces groupes d'homologie tropicale $H_q(\mathcal{F}_p)$ avec les nombres de Hodge $h^{p,q}(\mathbb{C}\tilde{X})$ d'une hypersurface algébrique non-dégénérée quelconque $\mathbb{C}\tilde{X}$ de la variété torique projective de polytope P. Enfin, nous utilisons une identité

de caractéristiques d'Euler pour exprimer les groupes d'homologie tropicale $H_q(\mathcal{F}_p)$ où $p+q=n-1$.

Le fait d'utiliser un théorème de type Lefschetz partiel, une dualité de Poincaré tropicale partielle, et une comparaison de caractéristiques d'Euler nous permet autant que possible de remplacer les dimensions des groupes d'homologie $H_q(\tilde{\mathcal{F}}_p)$ ou $H_q(\mathcal{F}_p)$ par des nombres de Hodge $h^{p,q}(\mathbb{C}\tilde{X})$ dans nos expressions des bornes des nombres de Betti.

Dans le cas d'un Patchwork d'une triangulation $\mathbb{F}_2$-primitive, ce procédé nous permet de faire complètement disparaître les termes $\dim H_q(\tilde{\mathcal{F}}_p)$ ou $\dim H_q(\mathcal{F}_p)$ de nos bornes :

Theorème 1.3.4. *Soit P un polytope $\mathbb{F}_2$-non-singulier à sommets entiers de dimension n dans $\mathbb{R}^n$, muni d'une triangulation $\mathbb{F}_2$-primitive Γ à sommets entiers telle que toutes les arêtes de cette triangulation soient de longueur entière impaire. Alors les nombres de Betti d'une hypersurface combinatoire X construite par Patchwork de Viro à partir de cette triangulation de ce polytope P vérifient :*

$$\forall\ q,\quad b_q(X;\mathbb{F}_2) \leq (-1)^q \hat{\chi}_{n-1-q} + (1+(-1)^n) h^{q,q}(\mathbb{C}\tilde{X})$$

où $\hat{\chi}_p = \sum_{(F,\sigma)\in\Xi^1} (-1)^{\dim F - \dim\sigma} \left[\binom{\dim F}{p} - \binom{\dim F - \dim\sigma}{p-\dim\sigma} \right]$, avec $\Xi^1 := \{(F,\sigma),\quad F$ face de P, $\sigma\in\Gamma$, $\sigma\subseteq F$, $\dim\sigma\geq 1\}$, et où $\mathbb{C}\tilde{X}$ est une hypersurface algébrique non-dégénérée quelconque de la variété torique de polytope P.

Dans le cas d'un Patchwork d'une triangulation $(k,\mathbb{F}_2)$-primitive, certains termes $H_q(\tilde{\mathcal{F}}_p)$ ou $H_q(\mathcal{F}_p)$ subsistent dans l'expression de nos bornes :

Theorème 1.3.5. *Soit P un polytope $\mathbb{F}_2$-non-singulier à sommets entiers de dimension n dans $\mathbb{R}^n$, muni d'une triangulation (k,$\mathbb{F}_2$)-primitive Γ à sommets entiers telle que toutes les arêtes de cette triangulation soient de longueur entière impaire. Considérons une distribution de signes sur les sommets de la triangulation Γ. Si $k\leq n-1$, alors les nombres de Betti de l'hypersurface combinatoire X construite par Patchwork à partir de cette triangulation Γ de P et de cette distribution de signes vérifient :*

$$\forall q\geq n-k/2 \text{ ou } k/2-1\geq q\geq n-k,\qquad b_q(X;\mathbb{F}_2)\leq (1+(-1)^n)h^{q,q}(\mathbb{C}\tilde{X})$$
$$+\sum_{p=n-q}^{2n-k-1-q}\dim H_q(\mathcal{F}_p) - \sum_{r=q+1}^{n-k+q}(-1)^{q+r}\dim H_r(\mathcal{F}_{n-1-q}) + (-1)^q\hat{\chi}_{n-1-q}\ ;$$

$$\forall (n-1)/2 > q\geq \max((k-1)/2, n-k),\qquad b_q(X;\mathbb{F}_2)\leq h^{q,q}(\mathbb{C}\tilde{X})$$
$$+\sum_{p=n-q}^{2n-k-1-q}\dim H_q(\mathcal{F}_p) - \sum_{r=q+1}^{\mathrm{n-k+q}}(-1)^{q+r}\dim H_r(\mathcal{F}_{n-1-q}) + (-1)^q\hat{\chi}_{n-1-q}\ ;$$

$$\forall q\in[\max(n-k, n/2), n-(k+1)/2],\qquad b_q(X;\mathbb{F}_2)\leq (-1)^n h^{q,q}(\mathbb{C}\tilde{X})$$
$$+\sum_{p=n-q}^{2n-k-1-q}\dim H_q(\mathcal{F}_p) - \sum_{r=q+1}^{\mathrm{n-k+q}}(-1)^{q+r}\dim H_r(\mathcal{F}_{n-1-q}) + (-1)^q\hat{\chi}_{n-1-q}\ ;$$

$$\forall q=(n-1)/2\geq n-k,\quad b_{(n-1)/2}(X;\mathbb{F}_2)\leq \sum_{p=(n+1)/2}^{2n-k-1-(n-1)/2}\dim H_{(n-1)/2}(\mathcal{F}_p)$$
$$-\sum_{r=q+1}^{\mathrm{n-k+(n-1)/2}}(-1)^{(n-1)/2+r}\dim H_r(\mathcal{F}_{(n-1)/2})+(-1)^{(n-1)/2}\hat{\chi}_{(n-1)/2}\ ;$$

$$\forall q<\min(k,n-k),\quad b_q(X;\mathbb{F}_2)\leq h^{q,q}(\mathbb{C}\tilde{X})+\sum_{p=k}^{n}\dim H_q(\tilde{\mathcal{F}}_p)\ ;$$

$$\forall q\in[k,n-k-1],\quad b_q(X;\mathbb{F}_2)\leq \sum_{p=k}^{n}\dim H_q(\tilde{\mathcal{F}}_p)$$

où $\hat{\chi}_p=\sum_{(F,\sigma)\in\Xi^1}(-1)^{\dim F-\dim\sigma}\left[\binom{\dim F}{p}-\binom{\dim F-\dim\sigma}{p-\dim\sigma}\right]$, *avec* $\Xi^1:=\{(F,\sigma),\quad F$ *face de* P, $\sigma\in\Gamma$, $\sigma\subseteq F$, $\dim\sigma\geq 1\}$, *et où* $\mathbb{C}\tilde{X}$ *est une hypersurface algébrique non-dégénérée quelconque de la variété torique de polytope* P.

Le cas particulier $k=n-1$ est intéressant, car dans ce cas on arrive à exprimer tous les termes $\dim H_q(\tilde{\mathcal{F}}_p)$ à partir de termes $\dim H_q(\mathcal{F}_p)$ et du nombre N_{np} de n-simplexes non-$\mathbb{F}_2$-primitifs et du nombre N_c de n-simplexes non-$\mathbb{F}_2$-primitifs de la triangulation dont le produit des signes fixés sur les sommets vaut -1 :

Theorème 1.3.6. *Soit* $n\geq 3$. *Soit* P *un polytope* $\mathbb{F}_2$*-non-singulier à sommets entiers de dimension* n *dans* $\mathbb{R}^n$, *muni d'une triangulation (n-1,*$\mathbb{F}_2$*)-primitive* Γ *à sommets entiers telle que toutes les arêtes de cette triangulation soient de longueur entière impaire. Considérons une distribution de signes sur les sommets de la triangulation* Γ. *Alors les nombres de Betti de l'hypersurface combinatoire* X *construite par Patchwork à partir de cette triangulation* Γ *de* P *et de cette distribution de signes vérifient :*

$$\begin{aligned}
&\forall q\notin\{0,n/2-1,n/2\},\\
&b_q(X;\mathbb{F}_2)\leq(1+(-1)^n)h^{q,q}(\mathbb{C}\tilde{X})+\dim H_q(\mathcal{F}_{n-q})+\dim H_{q+1}(\mathcal{F}_{n-1-q})+(-1)^q\hat{\chi}_{n-1-q}\ ;\\
&\forall q\in\{n/2-1,n/2\},\\
&b_q(X;\mathbb{F}_2)\leq h^{q,q}(\mathbb{C}\tilde{X})+\dim H_q(\mathcal{F}_{n-q})+\dim H_{q+1}(\mathcal{F}_{n-1-q})+(-1)^q\hat{\chi}_{n-1-q}\ ;\\
&b_0(X;\mathbb{F}_2)\leq 1+(-1)^n+\hat{\chi}_{n-1}+\dim H_1(\mathcal{F}_{n-1})-N_{np}+2N_c
\end{aligned}$$

où N_c *est le nombre de* n*-simplexes non-*$\mathbb{F}_2$*-primitifs* σ *de la triangulation* Γ *tels que le produit des signes fixés aux sommets de* σ *soit égal à* -1, *où* N_{np} *est le nombre de* n*-simplexes non-*$\mathbb{F}_2$*-primitifs de la triangulation* Γ, *et où* $\hat{\chi}_p=\sum_{(F,\sigma)\in\Xi^1}(-1)^{\dim F-\dim\sigma}\left[\binom{\dim F}{p}-\binom{\dim F-\dim\sigma}{p-\dim\sigma}\right]$, *avec* $\Xi^1:=\{(F,\sigma),\quad F$ *face de* P, $\sigma\in\Gamma$, $\sigma\subseteq F$, $\dim\sigma\geq 1\}$, *et où* $\mathbb{C}\tilde{X}$ *est une hypersurface algébrique non-dégénérée quelconque de la variété torique de polytope* P.

Dans la Section 4.6 du Chapitre 4, nous nous intéressons aux applications de ces théorèmes. Notamment, dans le cas des surfaces, le Théorème 1.3.6 nous permet d'obtenir le corollaire suivant :

Corollaire 1.3.7. *Soit P un polytope $\mathbb{F}_2$-non-singulier à sommets entiers de dimension 3 dans $\mathbb{R}^3$, muni d'une triangulation Γ à sommets entiers telle que toutes les arêtes de cette triangulation soient de longueur entière impaire. Considérons une distribution de signes sur les sommets de cette triangulation Γ. Alors la surface combinatoire X construite par Patchwork à partir de cette triangulation et de cette distribution de signe vérifie :*

$$b_0(X;\mathbb{F}_2) \leq 1 + h^{0,2}(\mathbb{C}\tilde{X}) - \mathrm{Card}\Big\{v \in \mathbb{Z}^3 \cap \overset{\circ}{P} \text{ s.t. } v \text{ is not a vertex of } \Gamma\Big\} + \min(\mathrm{dim}H_2(\mathcal{F}_1); \mathrm{dim}H_1(\mathcal{F}_2) - N_{np} + 2N_c).$$

De plus, si on suppose que la triangulation est maximale, alors :

$$b_1(X;\mathbb{F}_2) \leq \mathrm{dim}H_1(\mathcal{F}_2) + \mathrm{dim}H_2(\mathcal{F}_1) + h^{1,1}(\mathbb{C}\tilde{X}) + (f_3(\Gamma) - f_3(\Gamma_p))$$

où N_c est le nombre de 3-simplexes non-$\mathbb{F}_2$-primitifs σ de la triangulation Γ tels que le produit des signes fixés aux sommets de σ soit égal à -1, où N_{np} est le nombre de 3-simplexes non-$\mathbb{F}_2$-primitifs de la triangulation Γ, et où $\hat{\chi}_p = \sum_{(F,\sigma)\in\Xi^1}(-1)^{\mathrm{dim}F-\mathrm{dim}\sigma}\Big[\binom{\mathrm{dim}F}{p} - \binom{\mathrm{dim}F-\mathrm{dim}\sigma}{p-\mathrm{dim}\sigma}\Big]$, avec $\Xi^1 := \{(F,\sigma),\quad F \text{ face de } P,\ \sigma \in \Gamma,\ \sigma \subseteq F,\ \ \mathrm{dim}\sigma \geq 1\}$, où $f_3(\Gamma)$ désigne le nombre de 3-simplexes de la triangulation Γ, où $f_3(\Gamma_p)$ désigne le nombre de 3-simplexes d'une triangulation primitive Γ_p quelconque du polytope P, et où $\mathbb{C}\tilde{X}$ est une hypersurface algébrique non-dégénérée quelconque de la variété torique de polytope P.

Pour que ces bornes soient meilleures que $b_1(\mathbb{R}X;\mathbb{F}_2) \leq h^{1,1}(\mathbb{C}X)$ (Proposition 1.1.16, obtenue par [It 1997] pour les surfaces de $\mathbb{R}P^3$ construites à partir de Patchwork de triangulations maximales), et que $b_0(\mathbb{R}X;\mathbb{F}_2) \leq 1 + h^{0,2}(\mathbb{C}X) + N_c$, obtenue par [It 1997] et écrite dans [Ar 2020] (également pour les surfaces de $\mathbb{R}P^3$ construites à partir de Patchwork de triangulations maximales), il suffirait de montrer la conjecture suivante :

Conjecture 1.3.8. *Soit P un polytope $\mathbb{F}_2$-non-singulier à sommets entiers de dimension 3 dans $\mathbb{R}^3$, muni d'une triangulation Γ à sommets entiers telle que toutes les arêtes de cette triangulation soient de longueur entière impaire. Alors :*

$$\mathrm{dim}H_2(\mathcal{F}_1) + \mathrm{dim}H_1(\mathcal{F}_2) \leq N_{np}$$

où N_{np} désigne le nombre de 3-simplexe non-$\mathbb{F}_2$-primitifs de la triangulation.

Une autre application du Theorème 1.3.6 est le corollaire suivant :

Corollaire 1.3.9. *Soit $n \geq 3$. Soit P un polytope $\mathbb{F}_2$-non-singulier à sommets entiers de dimension n dans $\mathbb{R}^n$, muni d'une triangulation (n-1,$\mathbb{F}_2$)-primitive Γ à sommets entiers telle que toutes les arêtes de cette triangulation soient de longueur entière impaire. Considérons une distribution de signes sur les sommets de la triangulation Γ. Alors le nombre de Betti $b_0(X;\mathbb{F}_2)$ de l'hypersurface combinatoire X construite par Patchwork à partir de cette triangulation Γ de P et de cette distribution de signes vérifie :*

$$b_0(X;\mathbb{F}_2) \leq 1 + h^{0,n-1}(\mathbb{C}\tilde{X}) - \mathrm{Card}\Big\{v \in \mathbb{Z}^n \cap \overset{\circ}{P} \text{ s.t. } v \text{ is not a vertex of } \Gamma\Big\} + \min(\mathrm{dim}H_{n-1}(\mathcal{F}_1); \mathrm{dim}H_1(\mathcal{F}_{n-1}) - N_{np} + 2N_c)$$

où N_c est le nombre de n-simplexes non-$\mathbb{F}_2$-primitifs σ de la triangulation Γ tels que le produit des signes fixés aux sommets de σ soit égal à -1, où N_{np} est le nombre de n-simplexes non-$\mathbb{F}_2$-primitifs de la triangulation Γ, et où $\mathbb{C}\tilde{X}$ est une hypersurface algébrique non-dégénérée quelconque de la variété torique de polytope P.

Pour que le dernier terme $\min(\dim H_{n-1}(\mathcal{F}_1); \dim H_1(\mathcal{F}_{n-1}) - N_{np} + 2N_c)$ de cette borne soit remplacé par N_c, il suffirait de montrer la conjecture suivante :

Conjecture 1.3.10. *Soit P un polytope $\mathbb{F}_2$-non-singulier à sommets entiers de dimension $n \geq 3$ dans $\mathbb{R}^3$, muni d'une triangulation Γ à sommets entiers telle que toutes les arêtes de cette triangulation soient de longueur entière impaire. Alors :*

$$\dim H_{n-1}(\mathcal{F}_1) + \dim H_1(\mathcal{F}_{n-1}) \leq N_{np}$$

où N_{np} désigne le nombre de 3-simplexe non-$\mathbb{F}_2$-primitifs de la triangulation.

Voici une application du Théorème 1.3.4 :

Corollaire 1.3.11. *Soit $n \geq 3$. Soit P un polytope $\mathbb{F}_2$-non-singulier à sommets entiers de dimension n dans $\mathbb{R}^n$, muni d'une triangulation $\mathbb{F}_2$-primitive Γ à sommets entiers telle que toutes les arêtes de cette triangulation soient de longueur entière impaire. Considérons une distribution de signes sur les sommets de la triangulation Γ. Alors le nombre de Betti $b_0(X; \mathbb{F}_2)$ de l'hypersurface combinatoire X construite par Patchwork à partir de cette triangulation Γ de P et de cette distribution de signes vérifie :*

$$b_0(X; \mathbb{F}_2) \leq 1 + h^{0,n-1}(\mathbb{C}\tilde{X}) - \mathrm{Card}\Big\{ v \in \mathbb{Z}^n \cap \overset{\circ}{P} \text{ s.t. } v \text{ is not a vertex of } \Gamma \Big\}$$

où $\mathbb{C}\tilde{X}$ est une hypersurface algébrique non-dégénérée quelconque de la variété torique de polytope P.

On peut remarquer que cette borne est plus petite que la borne obtenue dans le cas d'une triangulation primitive. Intuitivement on peut le comprendre par le fait que dans une triangulation $\mathbb{F}_2$-primitive le nombre d'éléments de la triangulation est inférieur au nombre d'éléments d'une triangulation primitive, et donc la topologie de l'hypersurface combinatoire associée serait moins riche.

Chapter 2

Preliminaries

In the preliminaries of this thesis we explain the mathematical context of our research.

2.1 Projective toric varieties

Topology of real algebraic hypersurfaces were initially studied mainly in the case of hypersurfaces of projective spaces. In this context, [Wi 1978] is a survey of researches about the topology of real algebraic curves in $\mathbb{R}P^2$, and [It 1997] studied bounds on the Betti numbers of real algebraic surfaces of $\mathbb{R}P^3$ constructed by Viro's method. However, Viro's method to construct real algebraic hypersurfaces can be easily adapted to construct real hypersurfaces of projective toric varieties, replacing the triangulation of a simplex by a triangulation of a lattice polytope. It is one of the reason why projective toric varieties are now an often used framework to study topology of real algebraic varieties. [RS 2023] and [BMR 2024] have obtained bounds on the Betti numbers of Viro's combinatorial patchworkings starting from a primitive triangulation of a lattice polytope.

In this section, we recall how to construct a toric variety from a lattice polytope. More details can be found in [Fu 1993], [GKZ 1994], [Od 1988],[Da 1978] and [CLS 2011]. Then we define the real locus of a projective toric variety and we describe it with an homeomorphism with copies of the lattice polytope, glued with specific rules.

Consider $M \simeq \mathbb{Z}^n$ unimodular lattice of dimension $n \geq 1$ and consider $M_{\mathbb{R}} := M \otimes \mathbb{R} \simeq \mathbb{R}^n$. Consider $N := \mathrm{Hom}(M; \mathbb{Z})$ the dual lattice of M, and $N_{\mathbb{R}} := N \otimes \mathbb{R} = \mathrm{Hom}(M_{\mathbb{R}}; \mathbb{R})$. We can understand N as a subset of $N_{\mathbb{R}}$.

A lattice polytope of $M_{\mathbb{R}}$ is the convex hull in $M_{\mathbb{R}}$ of a finite subset of $M \subseteq M_{\mathbb{R}}$. If the finite subset is not included into an hyperplan, the lattice polytope is of dimension n. In the following, we consider a lattice polytope P of $M_{\mathbb{R}}$ of dimension n.

We can see $\mathbb{C}^*$ as a multiplicative group, and we have $\mathrm{Hom}(M; \mathbb{C}^*) \simeq (\mathbb{C}^*)^n$. For any $z \in \mathrm{Hom}(M; \mathbb{C}^*)$ and $w \in M$ we have an element $z(w) \in \mathbb{C}^*$. Then we can define the projective troic variety associated to the lattice polytope P:

Definition 2.1.1. *Consider $w_0, ..., w_r$ the integral points of the lattice polytope P. The toric variety associated to the polytope P is the closure of the following set in $\mathbb{C}P^r$:*

$$\{[z(w_0) : \cdots : z(w_r)],\ z \in \mathrm{Hom}(M; \mathbb{C}^*)\} \subseteq \mathbb{C}P^r\ .$$

The toric variety associated to the lattice polytope P is a compact algebraic variety of dimension n. In the following we will denote it by $\mathbb{C}Y$. We have a natural continuous action of the torus $\mathrm{Hom}(M;\mathbb{C}^*) \simeq (\mathbb{C}^*)^n$ on this toric variety $\mathbb{C}Y$, defined by:

$$(z,[x_1:\cdots:x_r]) \mapsto [z(w_0)x_1:\cdots:z(w_r)x_r]$$

A face F of the lattice polytope P is a subset of P such that there exists a linear form $f \in N$ and $a \in \mathbb{Z}$ satisfying:

$$\forall x \in F, \quad f(x) = a \text{ and } \forall x \in P\backslash F, \quad f(x) > a.$$

The dimension of a face F is the rank of its tangent lattice $T_{\mathbb{Z}}F$. The vertices of the polytope are the faces of dimension 0, they contain only one point. The facets are the faces of dimension $n-1$.

The orbits of the action of the torus $\mathrm{Hom}(M;\mathbb{C}^*) \simeq (\mathbb{C}^*)^n$ on the toric variety are in one-to-one correspondence with the faces of the lattice polytope P, the corresponding orbit of a face F of P for this action is:

$$\{[z(w_0)\delta_{w_0\in F}:\cdots:z(w_n)\delta_{w_n\in F}],\ z \in \mathrm{Hom}(M;\mathbb{C}^*)\}$$

where $\delta_{w_i\in F}$ is the Kronecker-type symbol, equal to 1 if $w_i \in F$ and 0 otherwise.

Definition 2.1.2. *We say that the polytope P is simple if for every vertex of P there exist exactly n facets of P containing it.*

If a polytope P is simple, then every singularity of the toric variety $\mathbb{C}Y$ is orbifold. To characterize the polytopes P for which $\mathbb{C}Y$ is smooth, consider for any facet F of P a generator u_F of $(T_{\mathbb{Z}}F)^{\perp}$.

Definition 2.1.3. *We say that the polytope P is smooth if for any vertex s, the family $\{u_F, \quad F \text{ facet of } P \text{ containing } s\}$ forms a basis of the lattice N.*

Proposition 2.1.4. *The toric variety $\mathbb{C}Y$ is smooth if and only if P is smooth. In particular, the smoothness of P implies that P is simple.*

The real locus of the toric variety $\mathbb{C}Y$ associated to the polytope P is its intersection with the real locus $\mathbb{R}P^r$ of $\mathbb{C}P^r$. We denote it by $\mathbb{R}Y$, it is the compactification in $\mathbb{R}P^r$ of $\{[z(w_0):\cdots:z(w_r)],\ z \in \mathrm{Hom}(M;\mathbb{R}^*)\}$.

The topological group isomorphism $\mathbb{R}^* = \mathbb{R}_+^* \times \mathbb{F}_2$ (with the discret topology on $\mathbb{F}_2$) induces a topological group isomorphism $\mathrm{Hom}(M;\mathbb{R}^*) = \mathrm{Hom}(M;\mathbb{R}_+^*) \times \mathrm{Hom}(M;\mathbb{F}_2) \simeq (\mathbb{R}_+^*)^n \times \mathbb{F}_2^n$. Then we obtain that the real locus of the orbit of P of the toric variety $\mathbb{C}Y$ has 2^n connected components, labeled by the elements of $\mathrm{Hom}(M;\mathbb{F}_2)$. For an element $x \in \mathrm{Hom}(M;\mathbb{F}_2)$ the corresponding connected component is:

$$\{[(-1)^{x(w_0)}z(w_0):\cdots:(-1)^{x(w_r)}z(w_r)],\ z \in \mathrm{Hom}(M;\mathbb{R}_+^*)\}$$

The compactification in $\mathbb{R}P^r$ of this connected component associated to an element $x \in \mathrm{Hom}(M;\mathbb{F}_2)$ is homeomorphic to a ball, and it can be intersected with each orbit of the toric variety to obtain a subdivision of this compactification. We can prove that

this subdivision is a finite cellular complex of dimension n homeomorphic to the polyhedral subdivision of the polytope P: the interior of a face F of P is sent through this homeomorphism to the intersection of the compactification of $\{[(-1)^{x(w_0)}z(w_0) : \cdots : (-1)^{x(w_r)}z(w_r)],\ z \in \mathrm{Hom}(M;\mathbb{R}_+^*)\}$ with the orbit associated to F, that is the set $\{[(-1)^{x(w_0)}\delta_{w_0\in F}z(w_0) : \cdots : (-1)^{x(w_r)}\delta_{w_r\in F}z(w_r)],\ z \in \mathrm{Hom}(M;\mathbb{R}_+^*)\}$. Moreover we remark that for a given face F of the polytope P and two $x, y \in \mathrm{Hom}(M;\mathbb{F}_2)$, the set $\{[(-1)^{x(w_0)}\delta_{w_0\in F}z(w_0) : \cdots : (-1)^{x(w_r)}\delta_{w_r\in F}z(w_r)],\ z \in \mathrm{Hom}(M;\mathbb{R}_+^*)\}$ and the set $\{[(-1)^{y(w_0)}\delta_{w_0\in F}z(w_0) : \cdots : (-1)^{y(w_r)}\delta_{w_r\in F}z(w_r)],\ z \in \mathrm{Hom}(M;\mathbb{R}_+^*)\}$ are equal if $\forall w \in F\cap M,\ x(w) = y(w)$, and disjoint otherwise. However, denoting by $N_{\mathbb{F}_2} := N\otimes\mathbb{F}_2 = \mathrm{Hom}(M;\mathbb{F}_2)$, $T_{\mathbb{F}_2}F := T_{\mathbb{Z}}F\otimes\mathbb{F}_2$ and $T_{\mathbb{F}_2}F^\perp := \{x \in N_{\mathbb{F}_2} \mid \forall w \in T_{\mathbb{F}_2}F,\ x(w) = 0\}$ we have the equivalence:

$$\forall w \in F\cap M,\ \ x(w) = y(w) \iff x - y \in T_{\mathbb{F}_2}F^\perp$$

Then we obtain the following proposition:

Proposition 2.1.5. *The real locus $\mathbb{R}Y$ of the toric variety $\mathbb{C}Y$ admits the following description:*

$$\mathbb{R}Y \simeq \frac{P\times N_{\mathbb{F}_2}}{(z,x)\sim(z,y) \text{ if } \exists F \text{ face of } P \text{ s.t. } z\in F \text{ and } x-y\in T_{\mathbb{F}_2}F^\perp}$$

Example 2.1.6. *If the polytope P is a simplex Δ of vertices $(0,...,0)$, $(d,0,...,0)$, ..., $(0,...,0,d)$ in $\mathbb{R}^n$, with $d \geq 1$ an integer, then the associated projective toric variety $\mathbb{C}Y$ is $\mathbb{C}P^n$. The real locus is $\mathbb{R}P^n$ and we have the following homeomorphism:*

$$\mathbb{R}P^n \simeq \frac{\Delta\times N_{\mathbb{F}_2}}{(z,x)\sim(z,y) \text{ if } \exists F \text{ face of } \Delta \text{ s.t. } z\in F \text{ and } x-y\in T_{\mathbb{F}_2}F^\perp}\ .$$

However, in this particular case we can identify $N_{\mathbb{F}_2}$ with the set of orthants of $\mathbb{R}^n$, and if $x \in N_{\mathbb{F}_2}$ we can identify $\Delta\times\{x\}$ with the simplex of vertices $(-1)^{x(0,...,0)}(0,...,0)$, $(-1)^{x(1,0,...,0)}(d,0,...,0)$, ..., $(-1)^{x(0,...,0,1)}(0,...,0,d)$, a symmetry of Δ in the orthant x. The previous glueings of $\Delta\times N_{\mathbb{F}_2}$ are compatible with the intersections two by two of the symmetric copies of Δ in the orthants. We obtain that $\mathbb{R}P^n$ is homeomorphic to the cross-polytope of vertices $(-d,0,...,0)$, ..., $(0,...,0,-d)$, $(d,0,...,0)$, $(0,...,0,d)$ with the identification $x \sim -x$ for any point x of the boundary of this cross-polytope. This agrees with the homeomorphism described in [Ri 1993].

Before ending this section, let us recall a proposition from [DK 1987] (Corollary 2.5) which will be useful in the following:

Proposition 2.1.7. *Consider a simple lattice polytope P of dimension n in $\mathbb{R}^n$. Then the Hodge numbers $h^{q,p}(\mathbb{C}Y)$ of the associated toric variety $\mathbb{C}Y$ can be defined and we have:*

$$h^{q,p}(\mathbb{C}Y) = \begin{cases} \sum_{i=0}^{p}(-1)^{p-i}\binom{n-i}{p-i}f_{n-i}(P) & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases} \tag{2.1}$$

where $f_{n-i}(P)$ denotes the number of faces of dimension $n-i$ of the polytope P.

Definition 2.1.8. *Consider a lattice polytope P of dimension n in $\mathbb{R}^n$. Consider the associated projective toric variety $\mathbb{C}Y$. A hypersurface $\mathbb{C}X$ of Newton polytope P of the toric variety $\mathbb{C}Y$ is the closure in $\mathbb{C}P^r$ of a set of the form:*

$$\{[z(w_0) : \cdots : z(w_r)],\ z \in \mathrm{Hom}(M; \mathbb{C}^*), \sum_{x \in M} a_x z(x) = 0\} \subseteq \mathbb{C}P^r$$

where $(a_x)_{x \in M}$ is a family of complex numbers with finite support satisfying:

$$\mathrm{Conv}(\mathrm{Supp}((a_x)_{x \in M})) = P$$

The following definition comes from [DK 1987]:

Definition 2.1.9. *Consider a lattice polytope P of dimension n in $\mathbb{R}^n$. Consider the associated projective toric variety $\mathbb{C}Y$. A hypersurface $\mathbb{C}X$ of Newton polytope P of the toric variety $\mathbb{C}Y$ is said to be non-degenerate (or generic) if it intersects transversely each orbit of the toric variety.*

We have the following result from [DK 1987]:

Proposition 2.1.10. *Consider a simple lattice polytope P of dimension n in $\mathbb{R}^n$. Consider the associated projective toric variety $\mathbb{C}Y$. If $\mathbb{C}X$ is a non-degenerate hypersurface of Newton polytope P of the toric variety $\mathbb{C}Y$, then the Hodge numbers of $\mathbb{C}X$ can be defined in this case and they satisfy the following Lefschetz hyperplane section theorem:*

$$\forall p + q < n - 1, \quad h^{p,q}(\mathbb{C}X) = h^{p,q}(\mathbb{C}Y)$$

In the case of a non-singular lattice polytope P, this theorem was known as Lefschetz hyperplane section theorem.

Let us now present Viros'method to construct real algebraic hypersurfaces of projective toric varieties with a control on their topology.

2.2 Viro's method

Viro's combinatorial patchworking is a powerful method to construct real algebraic hypersurfaces in real projective toric varieties with a control on their topology. In Chapter 4, we will recall the general Viro's Patchworking method to obtain a real algebraic hypersurface of Newton polytope P in a projective toric variety of polytope P, from a triangulation with integral vertices of P and a sign distribution on the vertices of the triangulation. In this section we present the particular case of $\mathbb{R}P^n$: we will explain how to construct a real algebraic hypersurface of degree $d \geq 1$ in $\mathbb{R}P^n$ from a triangulation of the simplex of vertices $(0, ..., 0)$, $(d, 0, ..., 0)$, ..., $(0, ..., 0, d)$ in $\mathbb{R}^n$ (of Example 2.1.6) and from a sign distribution on the vertices of the triangulation, following Viro's Patchworking method. See [Vi 1984], [Vi 2006] and [Ri 1993] for more details.

Consider the simplex Δ of vertices $(0, ..., 0)$, $(d, 0, ..., 0)$, ..., $(0, ..., 0, d)$ in $\mathbb{R}^n$. Consider the symmetries of Δ by each coordinate hyperplane to obtain at the end 2^n copies of Δ (one in each orthant). The union of these copies forms a cross-polytope, and we identify each point x of the boundary of this cross-polytope with $-x$. We denote Δ_* the resulting topological object, and according to Example 2.1.6 it is homeomorphic to $\mathbb{R}P^n$.

Consider a triangulation Γ with integral vertices of the simplex Δ of vertices $(0,...,0)$, $(d,0,...,0)$, ..., $(0,...,0,d)$ in $\mathbb{R}^n$. Considering the same symmetries and the same glueings, it induces a triangulation Γ_* of Δ_*.

Fix a sign $\epsilon(v) \in \{-1,1\}$ for each vertex v of Γ and symmetrize these signs on each vertex of Γ_*, following the rule: $\epsilon(x_1,...,x_n) = \epsilon(|x_1|,...,|x_n|).(\text{sign } x_1)^{x_1}...(\text{sign } x_n)^{x_n}$. Consider the union X of barycentric cells $\text{Conv}(\text{bar}(\sigma^a), \text{bar}(\sigma^{a+1}), ..., \text{bar}(\sigma^b))$ where $\sigma^a \subseteq \sigma^{a+1} \subseteq ... \subseteq \sigma^b$ are incident simplices of Γ_* with $\dim \sigma^i = i$ and such that every σ^i has different signs on its vertices.

Recall that the triangulation of Δ is said to be convex if there exists a convex continuous piecewise affine function $\nu : \Delta \to \mathbb{R}$, affine on each n-simplex of the triangulation and non-affine on each union of two distinct n-simplices. Then the combinatorial Viro's patchworking theorem is:

Theorem 2.2.1. *(Combinatorial Viro's patchworking theorem) If the triangulation Γ is convex there exists a real algebraic hypersurface $\mathbb{R}X$ of degree d of $\mathbb{R}P^n$ satisfying the following homeomorphism of pairs:*

$$(\mathbb{R}P^n, \mathbb{R}X) \simeq (\Delta_*, X).$$

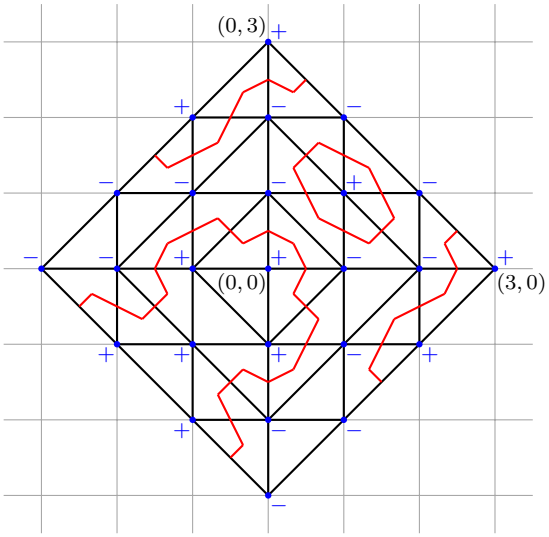

Figure 2.1: A real algebraic curve of $\mathbb{R}P^2$ of degree 3 obtained by Viro's patchworking.

Example 2.2.2. *Figure 2.1 shows an example of algebraic curve of $\mathbb{R}P^2$ of degree 3 obtained by Viro's patchworking. In this example Δ is the triangle of vertices $(0,0)$, $(0,3)$ and $(3,0)$. Δ_* is the square of vertices $(-3,0)$, $(0,3)$, $(0,-3)$ and $(3,0)$ where the line segments $[(3,0);(0,3)]$ and $[(-3,0);(0,-3)]$ are glued together, as well as the line segments $[(3,0);(0,-3)]$ and $[(-3,0);(0,3)]$. A triangulation and a sign distribution are fixed on Γ and symmetrized by the rules stated above to obtain a triangulation and a sign distribution on Δ_*. X is the red curve with the glueing inherited from Δ_*. X is an union of some barycentric cells of the triangulation with the rule stated above, it separates the signs "+" and "−". According to Viro's theorem there exists a real algebraic curve $\mathbb{R}X$ of degree 3 in $\mathbb{R}P^2$ such that the pair $(\mathbb{R}P^2, \mathbb{R}X)$ is homeomorphic to this pair (Δ_*, X).*

Consider the union $|X|$ of barycentric cells $\text{Conv}(\text{bar}(\sigma^a), \text{bar}(\sigma^{a+1}), ..., \text{bar}(\sigma^b))$ where $\sigma^a \subseteq \sigma^{a+1} \subseteq ... \subseteq \sigma^b$ are incident simplices of Γ with $\dim \sigma^i = i$. The cells of the cellular complex X are copies of the cells of $|X|$ in the appropriate orthants. Then the homology of X can be understood with introducing a cellular cosheaf on $|X|$ which takes into account the orthants in which each cell of $|X|$ is copied to obtain X. This idea was introduced in [RS 2023] and we will describe it in Chapter 4. If the triangulation Γ is convex, then there exists a tropical hypersurface $\mathbb{T}X$ of degree d of the tropical projective space $\mathbb{T}P^n$ such that we have the following homeomorphism of pairs:

$$(\Delta, |X|) \simeq (\mathbb{T}P^n, \mathbb{T}X) \ .$$

That is why in the case of a convex triangulation, the study of the homology of the real hypersurface $\mathbb{R}X$ obtained by Viro's Patchwork can be done studying properties of this corresponding tropical hypersurface $\mathbb{T}X$ (done by [RS 2023]). That is why Viro's Patchwork has strong links with tropical geometry.

In 2024, [BMR 2024] have generalized the approach of [RS 2023] to non-convex triangulations: they study the homology of the object X even if it is not obtained from a convex triangulation, that is even if X cannot be interpreted as a real algebraic hypersurface. Then $|X|$ is not necessarily homeomorphic to a tropical hypersurface, and tropical geometry completely disappears of their language. In this thesis we focus on this general case of X obtained from non-necessarily convex patchworkings. That is why we do not use the language of tropical geometry.

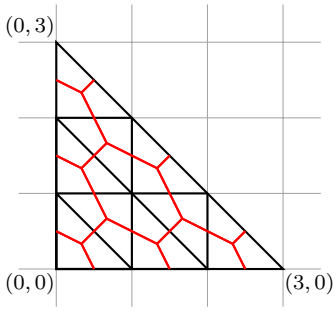

Figure 2.2: Cellular complex $|X|$ (in red) homeomorphic to a tropical hypersurface of $\mathbb{T}P^2$ of degree 3 from Example 2.2.2

[It 1997], [RS 2023] and [BMR 2024] have studied topological properties of patchworks. Their results depend on properties of the triangulation Γ such as its primitivity or maximality. In the next section we define the primitivity and the maximality of the triangulation.

2.3 Maximality and primitivity

Consider a lattice polytope P of dimension n in $\mathbb{R}^n$. Consider a triangulation Γ of P with integral vertices. In this section we define the notion of maximality and primitivity of the triangulation Γ. These definitions were introduced by Ilia Itenberg in [It 1997].

Definition 2.3.1. *The triangulation Γ of P is said to be maximal if every integral point of P is a vertex of Γ.*

Remark 2.3.2. *The terminology maximal comes from the fact that we can see the set of triangulation with integral vertices of the polytope P as a poset, ordered by inclusion. Maximal triangulations are precisely the maximal elements of this poset.*

Definition 2.3.3. *The triangulation Γ of P is said to be primitive if any n-simplex of it is of volume $1/n!$*

Remark 2.3.4. *Equivalently, a primitive triangulation is a one for which any n-simplex σ of vertices $a_0, ...a_n$, the family $\{a_1 - a_0, ..., a_n - a_0\}$ forms a basis of $\mathbb{Z}^n$*

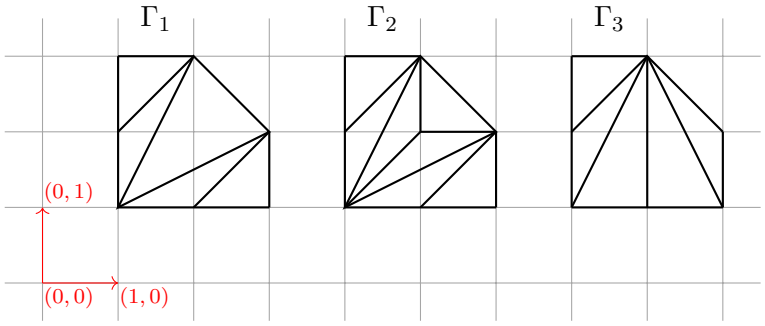

Figure 2.3: Different triangulation of a lattice polygon

In Figure 2.3 the triangulation Γ_1 and Γ_3 are not maximal and not primitive. The triangulation Γ_2 is maximal and primitive.

For any dimension n of the lattice polytope, the primitivity implies the maximality of the triangulation. Indeed, $1/n!$ is the minimal volume of a n-simplex with integral vertices in $\mathbb{R}^n$. In the case of a polytope of dimension $n = 2$, the maximality of the triangulation is equivalent to its primitivity. However, for some polytopes of dimension $n \geq 3$, there exist maximal triangulations which are not primitive.

There are many properties on primitive triangulations. For instance, if Γ and Γ' are two primitive triangulations of a lattice polytope P, for any integer $0 \leq k \leq n$ the number of k-simplices in Γ is the same as the number of k-simplices of Γ'. This is the reason why patchworkings of primitive triangulations were much studied. In [RS 2023] and [BMR 2024], primitivity is needed. In [It 1997], Ilia Itenberg studied the case of real algebraic hypersurfaces obtained by Viro's Patchwork from primitive triangulations, but he generalizes some of his results to maximal triangulations.

2.4 Bounds of Betti numbers of patchworkings : the case of curves and surfaces

Viro's combinatorial patchworking is one of the most efficient known method to construct real algebraic hypersurfaces with a control on their topology. That is why the study of hypersurfaces obtained by patchworks became a field of research on itself. The problem of bounding the Betti numbers of hypersurfaces obtained by Viro's combinatorial patchworking has been studied in [It 1997], [RS 2023] and [BMR 2024]. [It 1997] focused the case of surfaces. [RS 2023] studied the case of more general hypersurfaces. [BMR 2024]

generalized [RS 2023] in the case of non-convex triangulations. In this section we present the work of [It 1997], before presenting that of [RS 2023] and [BMR 2024] in the Section 2.6.

The first idea to bound the Betti numbers of a real hypersurface is to use the Smith-Thom inequality:

Theorem 2.4.1. *(Smith-Thom) Consider a compact complex algebraic variety $\mathbb{C}X$. Let $\mathbb{R}X$ its real locus. Then the Betti numbers $b_i(\mathbb{R}X;\mathbb{F}_2)$ over the field $\mathbb{F}_2$ of $\mathbb{R}X$, and the Betti numbers $b_j(\mathbb{C}X;\mathbb{F}_2)$ over the field $\mathbb{F}_2$ of $\mathbb{C}X$ satisfy the following relation:*

$$\sum_i b_i(\mathbb{R}X;\mathbb{F}_2) \leq \sum_j b_j(\mathbb{C}X;\mathbb{F}_2)$$

In the case of a real algebraic curve $\mathbb{R}X$ obtained by Viro's combinatorial patchworking, there is only one Betti number $b_0(\mathbb{R}X;\mathbb{F}_2) = b_1(\mathbb{R}X;\mathbb{F}_2)$ to bound, and the Smith-Thom inequality is enough to obtain a bound of it:

$$2b_0(\mathbb{R}X;\mathbb{F}_2) \leq 2b_0(\mathbb{C}X;\mathbb{F}_2) + b_1(\mathbb{C}X;\mathbb{F}_2) = 2 + 2g$$

where g is the genus of the complex algebraic curve $\mathbb{C}X$. This is exactly Harnack's Inequality ($b_0(\mathbb{R}X;\mathbb{F}_2) \leq 1+g$, from [Ha 1876]) described in Introduction. In the case of an algebraic curve of the projective plane, this bound depends only on the degree (we have $g = (d-1)(d-2)/2$). Moreover, Harnack has given a construction of a real algebraic curve of $\mathbb{R}P^2$ maximizing this inequality for any given degree. In the case of real algebraic curves of $\mathbb{R}P^2$ obtained by Patchworks, Ilia Itenberg has proven in [It 1996] that Harnack's inequality remains optimal.

In the case of real algebraic surfaces obtained by Viro's combinatorial patchworking, there are two different Betti numbers to bound ($b_0(\mathbb{R}X;\mathbb{F}_2) = b_2(\mathbb{R}X;\mathbb{F}_2)$ and $b_1(\mathbb{R}X;\mathbb{F}_2)$). Thus Smith-Thom inequality is not enough to obtain a bound for $b_0(\mathbb{R}X;\mathbb{F}_2) = b_2(\mathbb{R}X;\mathbb{F}_2)$ and a bound for $b_1(\mathbb{R}X;\mathbb{F}_2)$. However, when the triangulation is maximal or when the triangulation is primitive, [It 1997] established the following theorem which links the Euler characteristic $\chi(\mathbb{R}X)$ of a real algebraic surface obtained by Viro's Patchwork and the signature $\sigma(\mathbb{C}X)$ of its complexification:

Theorem 2.4.2. *A real algebraic surface $\mathbb{R}X$ of $\mathbb{R}P^3$ obtained by combinatorial patchwork starting with a primitive triangulation satisfies $\chi(\mathbb{R}X) = \sigma(\mathbb{C}X)$. In the case of a maximal triangulation we have $\chi(\mathbb{R}X) \geq \sigma(\mathbb{C}X)$.*

In [Be 2010], this theorem was later generalized in higher dimension, for primitive triangulations of a lattice polytope:

Theorem 2.4.3. *A real algebraic hypersurface $\mathbb{R}X$ obtained by combinatorial patchworking starting with a primitive triangulation of a lattice polytope satisfies $\chi(\mathbb{R}X) = \sigma(\mathbb{C}X)$.*

For a primitive triangulation, Theorem 2.4.2 and Smith Thom inequality combined give the following system:

$$\begin{cases} 2b_0(\mathbb{R}X;\mathbb{F}_2) - b_1(\mathbb{R}X;\mathbb{F}_2) = \chi(\mathbb{R}X) = \sigma(\mathbb{C}X) = 2 - h^{1,1}(\mathbb{C}X) + 2h^{2,0}(\mathbb{C}X) \\ 2b_0(\mathbb{R}X;\mathbb{F}_2) + b_1(\mathbb{R}X;\mathbb{F}_2) \leq \sum_{j=0}^{4} b_j(\mathbb{C}X;\mathbb{F}_2) = 2 + h^{1,1}(\mathbb{C}X) + 2h^{2,0}(\mathbb{C}X) \end{cases}$$

where $h^{p,q}(\mathbb{C}X)$ are the Hodge numbers of $\mathbb{C}X$. It enables Itenberg to prove the following theorem (see [It 1997]):

Theorem 2.4.4. *A real algebraic surface $\mathbb{R}X$ of $\mathbb{R}P^3$ obtained by combinatorial patchworking starting with a primitive triangulation of the simplex of vertices $(0,...,0)$, $(d,0,...,0)$, ..., $(0,...,0,d)$ of $\mathbb{R}^n$, with $d \geq 1$ an integer, satisfies:*

$$b_0(\mathbb{R}X;\mathbb{F}_2) \leq 1 + h^{2,0}(\mathbb{C}X);$$

$$b_1(\mathbb{R}X;\mathbb{F}_2) \leq h^{1,1}(\mathbb{C}X)$$

where $\mathbb{C}X$ is the complexification of $\mathbb{R}X$.

In the case of a maximal triangulation, the system formed by the identity $\chi(\mathbb{R}X) \geq \sigma(\mathbb{C}X)$ of Theorem 2.4.2 and Smith-Thom inequality enables Itenberg to obtain the following theorem in [It 1997]:

Theorem 2.4.5. *A real algebraic surface $\mathbb{R}X$ of $\mathbb{R}P^3$ obtained by combinatorial patchworking starting with a maximal triangulation of the simplex of vertices $(0,0,0)$, $(d,0,0)$, $(0,d,0)$ and $(0,0,d)$ of $\mathbb{R}^n$, with $d \geq 1$ an integer, satisfies:*

$$b_1(\mathbb{R}X;\mathbb{F}_2) \leq h^{1,1}(\mathbb{C}X) \ ,$$

where $\mathbb{C}X$ is the complexification of $\mathbb{R}X$.

In the case of surfaces obtained from a maximal triangulations, Smith-Thom inequality and the inequality $\chi(\mathbb{R}X) \geq \sigma(\mathbb{C}X)$ are not enough to obtain a bound on $b_0(\mathbb{R}X;\mathbb{F}_2)$. However in [It 1997], Itenberg achieved to obtain such a bound on in the case of surfaces obtained from a maximal triangulation, studying the behavior of the non-primitive simplices in Viro's method. This bound is not directly written in [It 1997] but it can be deduced from it, Arnal has written it in his PhD thesis ([Ar 2020], Chapter 9):

Theorem 2.4.6. *A real algebraic surface $\mathbb{R}X$ of $\mathbb{R}P^3$ obtained by combinatorial patchworking starting with a maximal triangulation of the simplex of vertices $(0,0,0)$, $(d,0,0)$, $(0,d,0)$ and $(0,0,d)$ of $\mathbb{R}^n$, with $d \geq 1$ an integer, satisfies*

$$b_0(\mathbb{R}X;\mathbb{F}_2) \leq 1 + h^{2,0}(\mathbb{C}X) + N_c \ ,$$

where $\mathbb{C}X$ is the complexification of $\mathbb{R}X$, and N_c is the number of non-primitive 3-simplices τ of the triangulation such that the product of the fixed signs on the vertices of τ is -1.

Remark 2.4.7. *Unlike the behavior of the bound of $b_1(\mathbb{R}X;\mathbb{F}_2)$, and unlike the behavior of the bound of $b_0(\mathbb{R}X;\mathbb{F}_2)$ when the triangulation is primitive, this bound in the case of a maximal triangulation depends on the fixed signs on the vertices, because the number N_c depends on these fixed signs. The number N_c can also be understood as the number of 3-simplices τ of the triangulation such that the cellular complex X constructed by Viro's patchwork intersects all of the 2^3 copies of τ (the copy of τ in each orthant).*

Remark 2.4.8. *In his PhD thesis [Ar 2020], Arnal has given an Example (Example 9.1.1) of a real algebraic surface $\mathbb{R}X$ obtained by combinatorial patchworking starting with a maximal triangulation of the simplex of vertices $(0,...,0)$, $(d,0,...,0)$, ..., $(0,...,0,d)$ of $\mathbb{R}^n$, with $d \geq 1$ and integer, such that $b_0(\mathbb{R}X;\mathbb{F}_2) > 1 + h^{2,0}(\mathbb{C}X)$. It proves that the additional term N_c in the maximal case cannot be removed.*

In the case of a real algebraic hypersurface of dimension 3, the theorem of [Be 2010] is useless because the Euler characteristic $\chi(\mathbb{R}X)$ is $0 \times b_0(\mathbb{R}X;\mathbb{F}_2) + 0 \times b_1(\mathbb{R}X;\mathbb{F}_2)$ and we do not obtain any information on the Betti numbers of $\mathbb{R}X$. In the case of a real algebraic hypersurface of dimension at least 4, the number of Betti numbers which can be different is at least 3. Since Smith-Thom inequality and the theorem of [Be 2010] provide only two inequations, they are not enough to produce bounds of each Betti numbers $b_i(\mathbb{R}X;\mathbb{F}_2)$.

2.5 Cellular sheaves and cosheaves

In Section 2.2 we explained that two cellular complexes X and $|X|$ naturally appear in the patchwork of a triangulation. We also explained that in the case of a convex triangulation, the cellular complex $|X|$ is homeomorphic to a tropical hypersurface (the tropical hypersurface dual to the triangulation), and the cellular complex X is homeomorphic to a real algebraic hypersurface. The cells of the cellular complex X are copies of cells of the cellular complex $|X|$. Then [RS 2023] provided a way to compute the homology of X using a cellular cosheaf on $|X|$. Their cosheaf encodes the orthants in which each cell of $|X|$ are copied to obtain X, and encodes the adjacency relations.

The notion of cellular sheaf or cosheaf appears many times in the work of [RS 2023] and [BMR 2024]. Thus, before presenting their work in Section 2.6, we introduce the notion of cellular sheaf or cosheaf on a polyhedral complex.

Consider a finite poset Ω, with a partial order $\leq$. If $\sigma, \tau \in \Omega$ satisfy $\sigma \neq \tau$ and $\sigma \leq \tau$, we denote it by $\sigma < \tau$.

Definition 2.5.1. *We say that two elements σ, τ of the finite poset Ω with $\sigma < \tau$ are immediately incident if there does not exist any element $\gamma \in \Omega$ satisfying $\sigma < \gamma < \tau$. The set of immediately incident pairs of Ω is denoted by $\mathcal{I}(\Omega)$.*

Definition 2.5.2. *We say that the finite poset Ω is graded if there exists a map $\dim : \Omega \to \mathbb{Z}_{\geq 0}$ such that for any immediately incident elements $\sigma < \tau$ of Ω we have $\dim\tau = \dim\sigma + 1$.*

Example 2.5.3. *A finite polyhedral complex is a finite poset, graded by the dimension of its cells.*

Definition 2.5.4. *We say that a graded finite poset Ω satisfies the diamond property if for any $\sigma < \tau$ satisfying $\dim\tau = \dim\sigma + 2$ we have:*

$$\mathrm{Card}\{\gamma \in \Omega \mid \sigma < \gamma < \tau\} = 2$$

Proposition 2.5.5. *Any finite polyhedral complex satisfies the diamond property.*

Definition 2.5.6. *A balancing signature on a finite graded poset Ω satisfying the diamond property is a map $\rho : \mathcal{I}(\Omega) \to \{-1, 1\}$ such that for any diamond $\sigma < \gamma_1, \gamma_2 < \tau$ we have:*

$$\rho(\sigma, \gamma_1)\rho(\gamma_1, \tau) + \rho(\sigma, \gamma_2)\rho(\gamma_2, \tau) = 0$$

Proposition 2.5.7. *Any finite polyhedral complex admits a balancing signature. The balancing signature ρ is obtained by fixing an orientation $o(\sigma)$ on each polyhedron σ and then defining for every immediately incident polyhedrons $\sigma < \tau$:*

$$\rho(\sigma, \tau) := \begin{cases} 1 & \text{if } \mu_{\sigma,\tau}(o(\tau)) = o(\sigma) \\ -1 & \text{if } \mu_{\sigma,\tau}(o(\tau)) = -o(\sigma) \end{cases}$$

where $\mu_{\sigma,\tau}$ is an orientation morphism who gives an orientation on the polyhedron σ from an orientation of the polyhedron τ. For each polyhedron τ, the choices of the orientation morphisms $\mu_{\sigma,\tau}$ have to be compatible for all of the facets σ of τ.

Example 2.5.8. *If we consider a lattice polytope P of dimension n of $\mathbb{R}^n$, and a triangulation Γ of P with integral vertices, then the poset $\Xi^1 := \{(F,\sigma),\ F \text{ face of } P,\ \sigma \in \Gamma \text{ such that } \sigma \subseteq F,\ \dim\sigma \geq 1\}$ introduced by [BMR 2024] to parametrize the cells of $|X|$ is a finite poset ordered by $(F,\sigma) \leq (G,\tau) \iff F \subseteq G$ and $\tau \subseteq \sigma$ and graded by the map $\dim(F,\sigma) := \dim F - \dim\sigma$. Moreover, it satisfies the diamond property and it admits a balancing signature. Indeed, this poset parametrizes the cellular complex $|X|$ which is a polyhedral complex, the ordering and the grading come from the ordering and the grading on $|X|$ as a cellular complex.*

Example 2.5.9. *If we consider a lattice polytope P of dimension n of $\mathbb{R}^n$ and a triangulation Γ of P with integral vertices, then the poset $\Omega := \{(\sigma,\tau) \in \Gamma^2 \mid \sigma \subseteq \tau,\ \dim\sigma \geq 1\}$ introduced by Chenal in [Ch 2025] who calls it the cubical subdivision of $|X|$, which ordering is $(\sigma,\tau) \leq (\sigma',\tau') \iff \sigma' \subseteq \sigma$ and $\tau \subseteq \tau'$, is graded by $\dim(\sigma,\tau) = \dim\tau - \dim\sigma$. Moreover, it satisfies the diamond property and admits a balancing signature. Indeed, this poset labels the cubical cells of the triangulation Γ which appears in $|X|$, and then it labels a subdivision of $|X|$. Its ordering and grading come from the ordering and grading of this subdivision of $|X|$. As this subdivision of $|X|$ is polyhedral, we conclude that this poset also admits a balancing signature.*

A finite poset can be considered as a category, where the objects are its elements. The set of morphisms from an object σ to an object τ of this category is a singleton $\{*\}$ if $\sigma \leq \tau$, and is empty otherwise. Now we can define cellular sheaves and cosheaves:

Definition 2.5.10. *Consider an integral domain R and a finite poset Ω. A cellular cosheaf of R-modules is a contravariant functor from Ω (seen as a category) to the category of R-modules. A cellular sheaf in R-modules is a covariant functor from Ω to the category of R-modules.*

Example 2.5.11. *For any integer $p \geq 0$, the tropical cosheaf ${}^{\mathbb{Q}}\mathcal{F}_p$ introduced by [IKMZ 2019] is a cellular cosheaf in $\mathbb{Q}$-vector spaces on tropical hypersurfaces (which can be considered as polyhedral complexes and then as posets)*

Example 2.5.12. *In the presence of a non-necessarily convex primitive triangulation of a lattice polytope P, the object $|X|$ is non-necessarily homeomorphic to a tropical hypersurface. However, [BMR 2024] has extended the definition of tropical cosheaf to this context. Denote by $M := \mathbb{Z}^n \subseteq \mathbb{R}^n$ the lattice defining P, $N := \mathrm{Hom}(M;\mathbb{Z})$ its dual, and $N_{\mathbb{F}_2} := N \otimes \mathbb{F}_2 = \mathrm{Hom}(M;\mathbb{F}_2)$. For any polyhedron F of $\mathbb{R}^n$ with integral vertices in M denote also $T_{\mathbb{Z}}F := T_{\mathbb{R}}F \cap M$ the integral tangent space and $T_{\mathbb{F}_2}F := T_{\mathbb{Z}}F \otimes \mathbb{F}_2$ the $\mathbb{F}_2$-tangent space. The orthogonal space $T_{\mathbb{F}_2}F^{\perp}$ is seen as a vector subspace of $N_{\mathbb{F}_2}$. With these notations, [BMR 2024] define for any integer p:*

$$\forall (F,\sigma) \in \Xi^1,\quad \mathcal{F}_p(F,\sigma) := \sum_{\tau \subseteq \sigma,\ \dim\tau = 1} \bigwedge^p T_{\mathbb{F}_2}\tau^{\perp}/T_{\mathbb{F}_2}F^{\perp}\ .$$

Any incident cells $(F,\sigma) \geq (F',\sigma')$ of the poset Ξ^1 define a morphism $\mathcal{F}_p(F,\sigma) \to \mathcal{F}_p(F',\sigma')$ induced by the inclusion $\mathcal{F}_p(F,\sigma) \subseteq \mathcal{F}_p(F,\sigma')$ and a morphism $\mathcal{F}_p(F,\sigma) \twoheadrightarrow \mathcal{F}_p(F',\sigma)$ coming from the projection $N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp} \twoheadrightarrow N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F'^{\perp}$. The poset Ξ^1 is the one of Example 2.5.8. With this definition, for any integer p, $\mathcal{F}_p$ is a contravariant functor from Ξ^1 to the category of $\mathbb{F}_2$-vector-spaces. Thus it is a cellular cosheaf in $\mathbb{F}_2$-vector-spaces on the poset Ξ^1.

Its dual $\mathcal{F}^p$ is defined by:

$$\forall (F,\sigma) \in \Xi^1, \quad \mathcal{F}^p(F,\sigma) := \mathrm{Hom}(\mathcal{F}_p(F,\sigma); \mathbb{F}_2)$$

and whose morphisms are the duals of the morphisms of $\mathcal{F}_p$. Thus it is a covariant functor from Ξ^1 to the category of $\mathbb{F}_2$-vector-spaces. In particular, it is a cellular sheaf in $\mathbb{F}_2$-vector-spaces on the poset Ξ^1.

In the presence of a cellular cosheaf on a finite graded poset Ω satisfying the diamond property and admitting a balancing signature, we can define a cosheaf chain complex and then a cosheaf homology in the following way:

Definition 2.5.13. *If Ω is a finite graded poset Ω satisfying the diamond property and admitting a balancing signature ρ, if $\mathcal{F}$ is a cellular cosheaf of Ω, we define:*

$$C_q(\mathcal{F}) := \bigoplus_{\sigma\in\Omega,\ \dim\sigma=q} \mathcal{F}(\sigma) \ ;$$

$$d_q^{\mathcal{F}} : C_q(\mathcal{F}) \to C_{q-1}(\mathcal{F}) \ \ ; \ \bigoplus_{\sigma\in\Omega,\ \dim\sigma=q} \alpha_\sigma \mapsto \bigoplus_{\tau\in\Omega,\ \dim\tau=q-1} \sum_{\sigma\supseteq\tau,\ \dim\sigma=q} \rho(\tau,\sigma)\mathcal{F}_{\tau,\sigma}(\alpha_\sigma) \ ,$$

where $\mathcal{F}_{\tau,\sigma}$ denotes the cosheaf morphism $\mathcal{F}(\sigma) \to \mathcal{F}(\tau)$.

Proposition 2.5.14. *Let Ω be a finite graded poset satisfying the diamond property and admitting a balancing signature. If $\mathcal{F}$ is a cellular cosheaf of Ω, then the object $(C_q(\mathcal{F}) \ ; \ d_q^{\mathcal{F}})$, for $q \in \mathbb{Z}$ defined above is a chain complex. It defines the homology $H_*(\mathcal{F})$. This homology does not depend on the choice of the balancing signature.*

Remark 2.5.15. *In the case of a finite graded poset Ω satisfying the diamond property and admitting a balancing signature, if $\mathcal{F}$ is a cellular sheaf of Ω, we can similarly define a cochain complex $C^q(\mathcal{F})$, and then the cohomology $H^*(\mathcal{F})$ of the sheaf $\mathcal{F}$.*

We have yet explained the notions of cellular sheaves and cosheaves on a finite poset. These notions were needed to understand the work of [RS 2023] and [BMR 2024] to obtain bounds on Betti numbers of primitive patchworkings in any dimension. This is the objective of the next section.

2.6 Bounds of Betti numbers of primitive Patchworkings in any dimension

In this section we present the works of [RS 2023] and [BMR 2024] which provide bounds on Betti numbers of primitive Patchworkings in any dimension. In [RS 2023], Renaudineau

and Shaw obtain bounds of the Betti numbers of real algebraic hypersurfaces $\mathbb{R}X$ constructed by combinatorial patchworking starting with a convex primitive triangulation of a non-singular lattice polytope. In [BMR 2024], Brugallé, Lopez de Medrano and Rau generalized these bounds to the cellular complex X constructed by combinatorial patchworking starting with a primitive and non-necessarily convex triangulation of a non-singular lattice polytope.

In our Section 2.2 we explained that a cellular complex $|X|$ naturally appears in the patchwork of a triangulation, and in the case of a convex triangulation, this cellular complex is homeomorphic to a tropical hypersurface (the tropical hypersurface dual to the triangulation). To obtain their bounds, in [RS 2023], Renaudineau and Shaw used the language of tropical geometry and worked on this tropical hypersurface homeomorphic to $|X|$. In [BMR 2024], Brugallé, Rau and Shaw studied directly the object $|X|$ and did not use the language of tropical geometry. In this section we will use the language of [BMR 2024].

Consider a non-singular lattice polytope P of dimension $n \geq 1$ in $\mathbb{R}^n$. Consider a primitive triangulation Γ of P with integral vertices. Consider a sign distribution $\epsilon(v) \in \{-1, 1\}$ on each vertex v of the triangulation. Consider the cellular complexes X and $|X|$ (defined in Section 2.2 of this chapter) associated to Viro's combinatorial patchworking of this polytope P induced by this triangulation and this sign distribution. In [BMR 2024] the cells of the cellular complex $|X|$ are labeled by the following poset:

$$\Xi^1 := \{(F, \sigma),\ F \text{ face of } P,\ \sigma \in \Gamma \text{ such that } \sigma \subseteq F, \dim \sigma \geq 1\}$$

The order relation of this poset Ξ^1 is inherited from the ordering on the faces of $|X|$, we have $(F, \sigma) \leq (G, \tau)$ if and only if $F \subseteq G$ and $\tau \subseteq \sigma$.

An important idea of [RRS 2022] and [RS 2023] was to introduce the notion of a real phase structure $\mathcal{E}$ defined for each facet of $|X|$ (a facet of $|X|$ corresponds to an element of the form $(P; \sigma)$ with $\dim \sigma = 1$):

$$\forall \sigma \in \Gamma \text{ s.t. } \dim \sigma = 1,\ \ \mathcal{E}(P, \sigma) := \{\text{orthants in which } X \text{ intersects } \sigma\}$$

This data of the family $\mathcal{E}(P, \sigma)$, for $\sigma \in \Gamma$ s.t. $\dim \sigma = 1$ allows to reconstruct the cellular complex X from the cellular complex $|X|$. The cellular complex X is completely encoded in this family.

Denoting by $M := \mathbb{Z}^n \subseteq \mathbb{R}^n$ the lattice in which P is a lattice polytope, and $N := \mathrm{Hom}(M; \mathbb{Z})$, we have seen in Example 2.1.6 that $N_{\mathbb{F}_2} := N \otimes \mathbb{F}_2 = \mathrm{Hom}(M; \mathbb{F}_2) \simeq \mathbb{F}_2^n$ can be identified to the set of orthants. With this consideration, [RS 2023] proved that for any edge σ of Γ the real phase structure $\mathcal{E}(P, \sigma)$ is an affine hyperplane of $N_{\mathbb{F}_2}$. To compute the homology of X from the family $\mathcal{E}(P, \sigma)$ with σ edge of the triangulation Γ, [RS 2023] introduced the sign cosheaf, a cellular cosheaf on $|X|$:

$$\forall (F, \tau) \in \Xi^1,\ \ \mathcal{S}(F, \tau) = \bigoplus_{w \in \frac{\bigcup_{\sigma \text{ edge of } \Gamma,\ \sigma \subseteq \tau} \mathcal{E}(P,\sigma)}{T_{\mathbb{F}_2} F^\perp}} \mathbb{F}_2\ .$$

Here, the cosheaf morphisms for incident cells $(F, \tau) \leq (F, \gamma)$ of Ξ^1 are the inclusions $\mathcal{S}(F, \gamma) \subseteq \mathcal{S}(F, \tau)$, and the cosheaf morphisms for incident cells $(F, \tau) \leq (G, \tau)$ of Ξ^1

are induced by the projections $\frac{\bigcup_{\sigma \text{ edge of } \Gamma,\ \sigma\subseteq\tau} \mathcal{E}(P,\sigma)}{T_{\mathbb{F}_2} G^\perp} \twoheadrightarrow \frac{\bigcup_{\sigma \text{ edge of } \Gamma,\ \sigma\subseteq\tau} \mathcal{E}(P,\sigma)}{T_{\mathbb{F}_2} F^\perp}$. In [RS 2023], Renaudineau and Shaw proved that this sign cosheaf has the same homology as that of the cellular complex X (which is homeomorphic to the real hypersurface $\mathbb{R}X$ in the case of a convex triangulation):

Proposition 2.6.1. *Consider a real hypersurface $\mathbb{R}X$ obtained by Viro's combinatorial patchworking starting with a convex primitive triangulation of a non-singular lattice polytope P. Then the associated sign cosheaf $\mathcal{S}$ of this patchwork satisfies:*

$$H_*(\mathbb{R}X;\mathbb{F}_2) = H_*(\mathcal{S}) \ .$$

For any edge σ of the triangulation Γ, the real phase structure $\mathcal{E}(P,\sigma)$ is an affine hyperplane of $N_{\mathbb{F}_2} \simeq \mathbb{F}_2^n$. Then in [RS 2023] Renaudineau and Shaw noticed that by considering a linear isomorphism $\mathcal{S}(P;\sigma) \simeq \mathbb{F}_2[V]$ where V is the linear vectorial space of $N_{\mathbb{F}_2}$ (of dimension $n-1$)), parallel to $\mathcal{E}(P,\sigma)$ we can find a filtration on the sign cosheaf $\mathcal{S}$:

$$0 = \mathcal{K}_n \subseteq ... \subseteq \mathcal{K}_0 \subseteq \mathcal{S} \tag{2.2}$$

This filtration comes from the filtration of the algebra $\mathbb{F}_2[V]$ by the powers of the augmentation ideal $\mathcal{I}_V^p$ of this algebra (the augmentation ideal $\mathcal{I}_V$ is the kernel of the morphism $\mathbb{F}_2[V] \to \mathbb{F}_2$ which send $\sum_{\epsilon\in V} a_\epsilon w_\epsilon$ to $\sum_{\epsilon\in V} a_\epsilon \in \mathbb{F}_2$). This filtration was used by [BFMH 2006] and [Ho 2008] to give a criteria for a toric variety to be maximal in the sense of the Smith-Thom inequality. It is often called Kalinin's filtration.

The article [RS 2023] extended the definition of $\mathcal{K}_p(P,\sigma)$ for each edge σ edge of the triangulation Γ to any cell $(F,\tau) \in \Xi^1$ to obtain a cosheaf on $|X|$. It proved that the cosheaves morphisms of two cosheaves $\mathcal{K}_p$ and $\mathcal{K}_{p+1}$ commute with the inclusion $\mathcal{K}_{p+1} \subseteq \mathcal{K}_p$. Thus, it enables Renaudineau and Shaw to define their spectral sequence:

Definition 2.6.2. *The Renaudineau-Shaw spectral sequence of a Patchwork of a non-singular lattice polytope starting from a primitive convex triangulation is the spectral sequence associated to the filtration $0 = \mathcal{K}_n \subseteq ... \subseteq \mathcal{K}_0 = \mathcal{S}$ whose first page is $E^1_{q,p} = H_q(\mathcal{K}_p/\mathcal{K}_{p+1})$, degenerating on the homology $H_q(\mathcal{S})$*

Using Proposition 2.6.1, this spectral sequence degenerates to the homology $H_q(\mathbb{R}X;\mathbb{F}_2)$ of the real algebraic hypersurface $\mathbb{R}X$ constructed by this patchwork.

Remark 2.6.3. *This spectral sequence can be generalized to non-convex triangulation, and in this case it degenerates on the homology of the cellular complex X. It was done in [BMR 2024]. Moreover, in the case of a primitive triangulation Chenal has found in [Ch2 2025] a criterion for the degeneracy of the Renaudineau-Shaw spectral sequence at the second page.*

Remark 2.6.4. *We can extend the definition of Renaudineau-Shaw spectral sequence to every patchwork of a lattice polytope with a non-necessarily primitive triangulation with integral vertices. It is what we do in Chapter 4. We define the filtration $\mathcal{K}_p$ to be adapted in this context.*

The bounds of the Betti numbers of the real hypersurface $\mathbb{R}X$ constructed from this patchwork are then obtained using the inequality $\dim E^{\infty}_{q,p} \leq \dim E^{1}_{q,p}$:

$$b_q(\mathbb{R}X;\mathbb{F}_2) = b_q(\mathcal{S}) = \sum_p \dim E^{\infty}_{q,p} \leq \sum_p \dim E^{1}_{q,p} \ .$$

To this end, we need now to compute $E^1_{q,p} = H_q(\mathcal{K}_p/\mathcal{K}_{p+1})$. It is in that context that the tropical cosheaves $\mathcal{F}_p$ presented in Example 2.5.12 become useful. [RS 2023] proved the following proposition:

Proposition 2.6.5. *For a patchwork of a non-singular lattice polytope starting from a primitive convex triangulation, the cosheaves $\mathcal{K}_p,\ p \geq 0$ (from Equation 2.2), and the tropical cosheaves $\mathcal{F}_p$ (from Example 2.5.12) satisfy the following relation:*

$$\forall p, q \quad H_q(\mathcal{K}_p/\mathcal{K}_{p+1}) = H_q(\mathcal{F}_p)$$

All that remains is to understand the homology $H_q(\mathcal{F}_p)$ of the tropical cosheaves $\mathcal{F}_p$ to compute bounds on the Betti numbers of the real hypersurface $\mathbb{R}X$ obtained by patchworking starting from a primitive convex triangulation. To this end, [ARS 2021] have proven Lefschetz-type theorem:

Theorem 2.6.6. *Consider a projective toric variety $\mathbb{C}Y$ constructed from a non-singular lattice polytope P of dimension n in $\mathbb{R}^n$. For a primitive convex triangulation of P, the associated cosheaf $\mathcal{F}_p$ satisfies:*

$$\forall p + q < n - 1, \quad \dim H_q(\mathcal{F}_p) = h^{p,q}(\mathbb{C}Y)$$

where $h^{p,q}(\mathbb{C}Y)$ denote the Hodge numbers of the toric variety $\mathbb{C}Y$.

In [JRS 2018], Jell, Rau, Shaw also have proven a tropical Poincaré duality theorem, adapted by [ARS 2021] in the context of hypersurfaces obtained by primitive Patchworks :

Theorem 2.6.7. *Consider a projective toric variety $\mathbb{C}Y$ constructed from a non-singular lattice polytope P of dimension n in $\mathbb{R}^n$. For a primitive convex triangulation of P, the associated cosheaves $\mathcal{F}_p,\ p \geq 0$ and their duals $\mathcal{F}^p,\ p \geq 0$ there exist the following natural isomorphisms:*

$$\forall p, q, \quad H^q(\mathcal{F}^p) \xrightarrow{\sim} H_{n-1-q}(\mathcal{F}_{n-1-p}) \ .$$

And an identity of Euler characteristic also appears in [ARS 2021]:

Theorem 2.6.8. *Consider a projective toric variety $\mathbb{C}Y$ constructed from a non-singular lattice polytope P of dimension n in $\mathbb{R}^n$. Consider a real algebraic $\mathbb{R}X$ in the real locus $\mathbb{R}Y$ of this toric variety, obtained by combinatorial patchworking starting from a primitive convex triangulation of the polytope P. Denote by $\mathbb{C}X$ the complexification of $\mathbb{R}X$. Then, the associated cosheaves $\mathcal{F}_p,\ p \geq 0$ of this triangulation satisfy:*

$$\forall p, \quad \sum_q (-1)^q H_q(\mathcal{F}_p) = \sum_q (-1)^q h^{p,q}(\mathbb{C}X)$$

As $\mathbb{F}_2$ is a field and $\mathcal{F}_p$ are cellular cosheaves of $\mathbb{F}_2$-modules, we have the identity $\dim H^q(\mathcal{F}^p) = \dim H_q(\mathcal{F}_p)$. Moreover the complex algebraic hypersurface $\mathbb{C}X$ is smooth and satisfies Poincaré duality: $h^{p,q}(\mathbb{C}X) = h^{n-1-p,n-1-q}(\mathbb{C}X)$. It also satisfies Lefshetz hyperplane section theorem:

$$\forall p+q < n-1, \quad h^{p,q}(\mathbb{C}X) = h^{p,q}(\mathbb{C}Y).$$

However, using Proposition 2.1, the Hodge numbers $h^{p,q}(\mathbb{C}Y)$ of the toric variety $\mathbb{C}Y$ vanish except when $p = q$, and then the Hodge numbers $h^{p,q}(\mathbb{C}X)$ of the hypersurface vanish except when $p = q$ of $p + q = n - 1$. Using the identity $\dim H^q(\mathcal{F}^p) = \dim H_q(\mathcal{F}_p)$, it enables [RS 2023] to deduce the following Corollary from Theorems 2.6.6, 2.6.7 and 2.6.8:

Corollary 2.6.9. *Consider a projective toric variety $\mathbb{C}Y$ constructed from a non-singular lattice polytope P of dimension n in $\mathbb{R}^n$. Consider a real algebraic $\mathbb{R}X$ in the real locus $\mathbb{R}Y$ of this toric variety, obtained by combinatorial patchworking starting from a primitive convex triangulation of the polytope P. Denote $\mathbb{C}X$ the complexification of $\mathbb{R}X$. Then the associated cosheaves $\mathcal{F}_p$, $p \geq 0$ of this triangulation satisfy:*

$$\forall q, p, \quad \dim H_q(\mathcal{F}_p) = h^{p,q}(\mathbb{C}X) \ ;$$

$$\forall p, q \text{ s.t. } p \neq q \text{ and } p+q \neq n-1, \quad \dim H_q(\mathcal{F}_p) = h^{p,q}(\mathbb{C}X) = 0 \ ;$$

Then [RS 2023] deduce their main theorem:

Theorem 2.6.10. *Consider a real algebraic hypersurface $\mathbb{R}X$ obtained by combinatorial patchworking of a non-singular lattice polytope P of dimension n in $\mathbb{R}^n$, starting with a convex primitive triangulation. Then the Betti numbers of $\mathbb{R}X$ admit the following bounds:*

$$b_q(\mathbb{R}X; \mathbb{F}_2) \leq \begin{cases} h^{q,q}(\mathbb{C}X) + h^{n-1-q,q}(\mathbb{C}X) & \text{if } q \neq (n-1)/2 \\ h^{q,q}(\mathbb{C}X) & \text{if } q = (n-1)/2 \end{cases}$$

In [BMR 2024], Brugallé, Lopez de Medrano and Rau generalized Theorems 2.6.6, 2.6.7 and 2.6.8 to primitive non-necessarily convex triangulations of a non-singular lattice polytope P. They also generalized the construction of Renaudineau-Shaw spectral sequence and Proposition 2.6.5 in the context of primitive non-necessarily convex patchworks. In this context, there is no real algebraic hypersurface $\mathbb{R}X$ homeomorphic to the cellular complex X, because hypotheses of Viro's theorem are not satisfied. However, the Hodge numbers of any generic hypersurface $\mathbb{C}\tilde{X}$ of Newton polytope P of the toric variety $\mathbb{C}Y$ do not depend on the choice of $\mathbb{C}\tilde{X}$, and [BMR 2024] used these Hodge numbers instead of the Hodge numbers of the real hypersurface $\mathbb{R}X$ (which does not necessarily exist) in their generalization of Theorems 2.6.6 and 2.6.8. They finally obtain the following theorem:

Theorem 2.6.11. *Consider a cellular complex X obtained by combinatorial patchworking of a non-singular lattice polytope P of dimension n in $\mathbb{R}^n$, starting from a primitive non-necessarily convex triangulation. Then the Betti numbers of X admit the following bounds:*

$$b_q(\mathbb{R}X; \mathbb{F}_2) \leq \begin{cases} h^{q,q}(\mathbb{C}\tilde{X}) + h^{n-1-q,q}(\mathbb{C}\tilde{X}) & \text{if } q \neq (n-1)/2 \\ h^{q,q}(\mathbb{C}\tilde{X}) & \text{if } q = (n-1)/2 \end{cases} .$$

Here $\mathbb{C}\tilde{X}$ is a generic complex algebraic hypersurface of Newton polytope P in the projective toric variety constructed with the lattice polytope P.

2.7 Non-primitive generalization of Lefschetz theorem

In this section we recall what was known about non-primitive triangulations of a lattice polytope before this PhD thesis. [RS 2023] and [BMR 2024] worked on primitive triangulations and used a Poincaré duality theorem (Theorem 2.6.7), a Lefschetz hyperplane section theorem (Theorem 2.6.6) and an identity of Euler characteristics (Theorem 2.6.8) in this framework of primitive triangulations. Before our partial and complete Poincaré duality theorems (Theorems 1.3.1 and 1.3.2), there were no non-primitive generalization of Poincaré duality. Similarly, generalization of the identity of Euler characteristics in the non-primitive case were not studied before. However, a generalization of the Lefschetz theorem (Theorem 2.6.6) was already found by Chenal before the publication of this thesis (in [Ch 2025]).

In this section we present Theorem 2 of [Ch 2025] which is a non-primitive generalization of Theorem 2.6.6 of [RS 2023] (generalized to non-necessarily convex triangulation by [BMR 2024]). We consider a lattice polytope P of dimension n in $\mathbb{R}^n$ and we consider a triangulation of P with integral vertices, non-necessarily convex and non-necessarily primitive. We consider the poset Ξ^1 introduces by [BMR 2024] for this triangulation. We denote $M := \mathbb{Z}^n \subseteq \mathbb{R}^n$ the lattice in which is defined the lattice polytope P, and $N := \mathrm{Hom}(M;\mathbb{Z})$ the dual of M. For any polyhedron F of $\mathbb{R}^n$ we denote its integral tangent space $T_{\mathbb{Z}}F := T_{\mathbb{R}}F \cap M$. Its orthogonal $T_{\mathbb{Z}}F^{\perp}$ is considered as a sub-lattice of N.

Definition 2.7.1. *For any $p \geq 0$ we define the cosheaf $\mathcal{F}_p^{\mathbb{Z}}$ in $\mathbb{Z}$-modules on the poset Ξ^1 in the following way:*

$$\forall (F,\sigma) \in \Xi^1, \quad \mathcal{F}_p^{\mathbb{Z}}(F,\sigma) := \sum_{\tau \subseteq \sigma,\ \dim\tau = 1} \bigwedge^p T_{\mathbb{Z}}\tau^{\perp}/T_{\mathbb{Z}}F^{\perp} \ .$$

For any incident cells $(F,\sigma) \geq (F',\sigma')$ of the poset Ξ^1 the cosheaf morphism $\mathcal{F}_p^{\mathbb{Z}}(F,\sigma) \to \mathcal{F}_p^{\mathbb{Z}}(F',\sigma')$ is induced by the inclusion $\mathcal{F}_p^{\mathbb{Z}}(F,\sigma) \subseteq \mathcal{F}_p^{\mathbb{Z}}(F,\sigma')$ and the projection $\mathcal{F}_p^{\mathbb{Z}}(F,\sigma) \twoheadrightarrow \mathcal{F}_p^{\mathbb{Z}}(F',\sigma)$ coming from $N/T_{\mathbb{Z}}F^{\perp} \twoheadrightarrow N/T_{\mathbb{Z}}F'^{\perp}$.

Then in [Ch 2025], Jules Chenal introduced the saturated cosheaf $\hat{\mathcal{F}}_p^{\mathbb{Z}}$ of the cosheaf $\mathcal{F}_p^{\mathbb{Z}}$:

Definition 2.7.2. *For any integer $p \geq 0$, the saturated cosheaf $\hat{\mathcal{F}}_p^{\mathbb{Z}}$ of the cosheaf $\mathcal{F}_p^{\mathbb{Z}}$ is defined on the poset Ξ^1 in the following way:*

$$\forall (F,\sigma) \in \Xi^1, \quad \hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma) := \left\{ \alpha \in \bigwedge^p N/T_{\mathbb{Z}}F^{\perp} \mid \exists \lambda \in \mathbb{Z},\ \lambda\alpha \in \mathcal{F}_p^{\mathbb{Z}}(F,\sigma) \right\}$$

The morphisms of this cosheaf are induced by the inclusions $\hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma) \hookrightarrow \hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\tau)$ for $\sigma,\tau \in \Gamma,\ \sigma \subseteq \tau \subseteq F$, and the projections $\bigwedge^p N/T_{\mathbb{Z}}F^{\perp} \twoheadrightarrow \bigwedge^p N/T_{\mathbb{Z}}G^{\perp}$ for $G \subseteq F$ faces of the polytope P.

Remark 2.7.3. *The cosheaves morphisms of $\hat{\mathcal{F}}_p^{\mathbb{Z}}$ are extensions of the morphism of the cosheaf $\mathcal{F}_p^{\mathbb{Z}}$. We can prove that for any ring R, $\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes R$ is a cellular cosheaf in R-modules. Moreover, in the case of a primitive triangulation we can check that $\hat{\mathcal{F}}_p^{\mathbb{Z}} = \mathcal{F}_p^{\mathbb{Z}}$.*

In [Ch 2025], Chenal also defined an integer $\delta(P)$ for any simple lattice polytope P:

Definition 2.7.4. *The integer $\delta(P)$ of a simple lattice polytope P is defined as being the least common multiple of the exponents of the finite groups $\Delta(F)$ for any face F of the polytope P, where:*

$$\Delta(F) := T_{\mathbb{Z}}F^{\perp}/\bigoplus_{Q\supseteq F,\ Q \text{ face of } P,\ \dim Q=n-1} T_{\mathbb{Z}}Q^{\perp}\ .$$

Example 2.7.5. *In the case of a non-singular lattice polytope P, we can prove that $\delta(P) = 1$.*

Now we can state the generalization of Lefschetz theorem by Chenal (Theorem 2 of [Ch 2025]):

Theorem 2.7.6. *Consider a triangulation with integral vertices of a simple lattice polytope P of dimension n in $\mathbb{R}^n$. Consider a ring R in which $\delta(P)$ is invertible. Then the saturated cosheaves $\hat{\mathcal{F}}_p^{\mathbb{Z}}$ of this triangulation satisfy that:*

$$\forall p+q<n-1,\quad H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}\otimes R) \text{ is a free } R\text{-module}$$

$$\forall p+q<n-1,\quad \mathrm{rk}H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}\otimes R) = \begin{cases} \sum_{i=0}^{p}(-1)^{p-i}\binom{n-i}{p-i}f_{n-i}(P) & \text{if } p=q \\ 0 & \text{if } p\neq q \end{cases}$$

where $f_{n-i}(P)$ denotes the number of faces of dimension $n-i$ of the polytope P.

Using proposition 2.1, established in [DK 1987], we obtain the following Corollary:

Corollary 2.7.7. *Consider a triangulation with integral vertices of a simple lattice polytope P of dimension n in $\mathbb{R}^n$. Consider a ring R in which $\delta(P)$ is invertible. Denote $\mathbb{C}Y$ the toric variety associated with this lattice polytope P. Then the saturated cosheaves $\hat{\mathcal{F}}_p^{\mathbb{Z}}$ of this triangulation satisfy that:*

$$\forall p+q<n-1,\quad H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}\otimes R) \text{ is a free } R\text{-module}$$

$$\forall p+q<n-1,\quad \mathrm{rk}H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}\otimes R) = h^{p,q}(\mathbb{C}Y)$$

where $h^{p,q}(\mathbb{C}Y)$ are the generalized Hodge number defined by [DK 1987] in the case of a toric variety constructed from a simple polytope (non-necessarily smooth).

This generalization of Lefschetz theorem to non-primitive triangulation is useful in Chapter 4 to generalize the bounds of Betti numbers of [RS 2023] and [BMR 2024] to non-primitive patchworks. To generalize these bounds we also need to generalize tropical Poincaré duality to non-primitive triangulation. It is the goal of Chapter 4. However, before proving this non-primitive generalization, we present in the next section the proof of [BMR 2024] of tropical Poincaré duality in the primitive case.

2.8 Poincaré duality for primitive Patchworks

Before proving our non-primitive generalization of tropical Poincaré duality in Chapter 4, needed to generalize bounds of the Betti numbers of non-primitive patchworks, in this section we present the proof of [BMR 2024] of tropical Poincaré duality for primitive Patchworks. The first proof of tropical Poincaré duality theorem was given by [JRS 2018], formulated in a local framework. [BMR 2024] adapted this proof in the framework of a patchwork of a primitive non-necessarily convex triangulation of a lattice polytope. As our non-primitive generalization of tropical Poincaré duality is also formulated in the framework of a patchwork of a non-necessarily convex triangulation of a lattice polytope, we have chosen to present in this section the proof of given in [BMR 2024] of tropical Poincaré dualtiy. Their theorem is the following:

Theorem 2.8.1. *Consider a projective toric variety $\mathbb{C}Y$ constructed from a non-singular lattice polytope P of dimension n in $\mathbb{R}^n$. For a primitive non-necessarily convex triangulation of P, the associated cosheaves $\mathcal{F}_p,\ p \geq 0$ satisfy the following isomorphisms:*

$$\forall p, q, \quad H_q(\mathcal{F}_p) \simeq H_{n-1-q}(\mathcal{F}_{n-1-p}) \ .$$

[BMR 2024] use the definition of an open of a finite poset:

Definition 2.8.2. *We say that a subset U of a finite poset Ω is open if for every $x \in U$, if $y \geq x$ then $y \in U$.*

Definition 2.8.3. *Given a finite poset Ω, its poset of orders $O(\Omega)$ is its partially ordered set of flags:*

$$O(\Omega) = \{\sigma_1 < ... < \sigma_k, \quad k \geq 1,\ \forall 1 \leq i < k,\ (\sigma_i, \sigma_{i+1}) \in \mathcal{I}(\Omega)\}$$

where $\mathcal{I}(\Omega)$ is the set of immediately incident couples of Ω. The last element of a flag $\sigma_1 < ... < \sigma_k$ of $O(\Omega)$ is σ_k, we will use the notation $\text{last}(\sigma_1 < ... < \sigma_k) = \sigma_k$ in the following.

Definition 2.8.4. *Given a finite poset Ω and an open subset U, we write:*

$$K(U) := O(\Omega) \backslash O(\Omega \backslash U)$$

Proposition 2.8.5. *Given a finite poset Ω and an open subset U of Ω, we have that $O(U)$ and $K(U)$ are open subsets of $O(\Omega)$, and we have the inclusion $O(U) \subseteq K(U)$.*

Given a cellular cosheaf $\mathcal{F}$ on a finite poset Ω, its restriction $\mathcal{F}_U$ to an open subset U of Ω remains a cellular cosheaf. Moreover it induces a cellular cosheaf $\mathcal{F}_{O(\Omega)}$ on $O(\Omega)$ with for any flag $\sigma_1 < ... < \sigma_k$ the definition $\mathcal{F}_{O(\Omega)}(\sigma_1 < ... < \sigma_k) := \mathcal{F}(\sigma_k)$. Then, as $O(U)$ and $K(U)$ are open subsets of $O(\Omega)$, the restrictions $\mathcal{F}_{O(U)}$ and $\mathcal{F}_{K(U)}$ are cellular cosheaves on $O(U)$ and $K(U)$. We can make similiar considerations for cellular sheaves.

[BMR 2024] have proven the following proposition:

Proposition 2.8.6. *If $\mathcal{F}$ is a cellular cosheaf on a finite graded poset Ω satisfying the diamond property and admitting a balancing signature, then for any open U of Ω we have an isomorphism:*

$$H_q(\mathcal{F}_U) \simeq H_q(\mathcal{F}_{K(U)}) \ .$$

Then, considering the poset Ξ^1 and the cosheaves $\mathcal{F}_p$ and their duals $\mathcal{F}^p$, [BMR 2024] have defined a cap product for any integer p and any open subset U of Ξ^1:

$$\cap : C^q(\mathcal{F}^p_{O(U)}) \times C_{q'}(\mathcal{F}_{p',K(U)}) \to C_{q'-q}(\mathcal{F}_{p'-p,K(U)}) \ ,$$

where $\mathcal{F}^p := \mathrm{Hom}(\mathcal{F}_p; \mathbb{F}_2)$ denotes the cellular sheaf dual to the cellular cosheaf $\mathcal{F}_p$. They prove that it induces a cap product in homology:

$$\cap : H^q(\mathcal{F}^p_{O(U)}) \times H_{q'}(\mathcal{F}_{p',K(U)}) \to H_{q'-q}(\mathcal{F}_{p'-p,K(U)}) \ .$$

In [BMR 2024] the authors defined also an element $\Omega^1 \in C_{n-1}(\mathcal{F}_{n-1})$ which they called fundamental class, and they proved that it satisfies $\partial\Omega^1 = 0$, where ∂ denotes the differential of the coheaf complex $C_*(\mathcal{F}_{n-1})$. Thus, it belongs to the homology group $H_{n-1}(\mathcal{F}_{n-1})$. Using proposition 2.8.6, we can consider its image in $H_{n-1}(\mathcal{F}_{n-1,O(\Xi^1)})$, and then its restriction to $C_*(\mathcal{F}_{n-1,K(U)})$ for any open subset U of Ξ^1. In [BMR 2024] this restriction is denoted $\Omega^1_{|U}$.

Then they proved the following proposition by induction:

Proposition 2.8.7. *Consider a projective toric variety constructed from a non-singular lattice polytope P of dimension n in $\mathbb{R}^n$. For a primitive convex triangulation of P, consider the associated cellular cosheaves $\mathcal{F}_p$, $p \geq 0$ and their duals $\mathcal{F}^p$, defined on the poset Ξ^1 of this triangulation. Then for any open subset U of Ξ^1, the cap product with $\Omega^1_{|U}$ induces an isomorphism:*

$$\cap\Omega^1_{|U} : H^q(\mathcal{F}^p_{O(U)}) \xrightarrow{\sim} H_{n-1-q}(\mathcal{F}_{n-1-p,K(U)})$$

This proposition is proven by induction on the cardinality of the minimal subset $\tau \subseteq U$ which satisfies $U = \{y \in \Xi^1 \mid \exists x \in \tau,\ x \leq y\}$. As U is open, such a subset τ exists (for instance $\tau = U$ works), and as Ξ^1 is finite, the cardinality of τ is defined and we can find τ minimalizing it.

Then [BMR 2024] initialize their induction by proving the result for open subsets U of the form $U_x := \{y \in \Xi^1,\ y \geq x\}$ with $x \in \Xi^1$. Then the inductive step comes from a commutative diagram of two Mayer-Vietoris exact sequences (idea of [JRS 2018]):

$$\begin{array}{ccccc}
0 & \to & C^q(\mathcal{F}^p_{O(U\cup V)}) & \to & C^q(\mathcal{F}^p_{O(U)}) \oplus C^q(\mathcal{F}^p_{O(V)}) \\
 & & \cap\Omega^1_{|U\cup V}\downarrow & & \cap\Omega^1_{|U} \oplus \cap\Omega^1_{|V}\downarrow \\
0 & \to & C_{n-1-q}(\mathcal{F}_{n-1-p,K(U\cup V)}) & \to & C_{n-1-q}(\mathcal{F}_{n-1-p,K(U)}) \oplus C_{n-1-q}(\mathcal{F}_{n-1-p,K(V)})
\end{array}$$

$$\begin{array}{ccccc}
\to & C^q(\mathcal{F}^p_{O(U\cap V)}) & \to & 0 \\
 & \cap\Omega^1_{|U\cap V}\downarrow & & \\
\to & C_{n-1-q}(\mathcal{F}_{n-1-p,K(U\cap V)}) & \longrightarrow & 0
\end{array}$$

where U and V are two open subsets of Ξ^1. This diagram proves the inductive step because for any open subset W, considering the minimal subset τ of W which satisfies $W = \{y \in \Xi^1 \mid \exists x \in \tau,\ x \leq y\}$, if $\mathrm{Card}\ \tau \geq 2$ we can fix an element $z \in \tau$ and we can define $U := \{y \in \Xi^1 \mid \exists x \in \tau\backslash\{z\},\ x \leq y\}$ and $V := \{y \in \Xi^1 \mid z \leq y\}$ open subset of Ξ^1. If we assume by induction that the proposition is true for U and V, as we have $W = U \cup V$

we obtain that the previous commutative diagram implies that the proposition is true for W. The result then follows by induction.

Then, [BMR 2024] applied the proposition to the case $U = \Xi^1$ and obtain an isomorphism $H^q(\mathcal{F}^p_{O(\Xi^1)}) \overset{\sim}{\to} H_{n-1-q}(\mathcal{F}_{n-1-p,O(\Xi^1)})$. However, $\mathbb{F}_2$ is a field, and then $H^q(\mathcal{F}^p_{O(\Xi^1)}) \simeq H_q(\mathcal{F}_{p,O(\Xi^1)})$. Finally, they use Proposition 2.8.6 to obtain $H_q(\mathcal{F}_{p,O(\Xi^1)}) \simeq H_q(\mathcal{F}_p)$ and $H_{n-1-q}(\mathcal{F}_{n-1-p,O(\Xi^1)}) = H_{n-1-q}(\mathcal{F}_{n-1-p})$, which concludes the proof of their tropical Poincaré duality theorem.

We have generalized this theorem to the case of non-primitive triangulations in the next Chapter. In our proof we also use a cap product with a fundamental class. However, we do not prove by induction, and we do not use the poset of orders $O(\Omega)$ of a finite poset Ω. In our proof we do not work on the poset Ξ^1 introduced by [BMR 2024], but we use the poset Ω of Example 2.5.9 introduced by [Ch 2025]. We define the cosheaves $\mathcal{F}_p$ and their duals $\mathcal{F}^p$ in this context. In our proof we introduce a filtration of the cosheaves chains $C_q(\mathcal{F}_p)$ and a filtration of the dual sheaves cochains $C^q(\mathcal{F}^p)$, which induce spectral sequences on the homology side and spectral sequence on the cohomology side. We prove that the cap product with the fundamental class induces isomorphisms from some of the terms of page 1 of the spectral sequence of the cohomological side to some terms of page 1 of the spectral sequence of the homological side. These isomorphisms will induce our partial Poincaré duality (Theorem 1.3.1) and our complete Poincaré duality (Theorem 1.3.2).

Chapter 3

Poincaré Duality for singular tropical hypersurfaces

In this chapter we prove our partial Poincaré duality theorem (Theorem 1.3.1) and our complete Poincaré duality theorem (Theorem 1.3.2). These theorems are non-primitive generalizations of the tropical Poincaré duality explained in the Section 2.8 of the Preliminaries. These theorems are proven in the general framework of tropical cosheaves and sheaves with coefficient in a general integral domain R. The application of these theorems in the case $R = \mathbb{F}_2$ is useful in Chapiter 4 to obtain bounds of the Betti numbers of non-primitive Patchworks. The case $R = \mathbb{Q}$ (our Corollary 1.3.3) is interesting because it gives a converse of a Theorem of [Ak 2023], and has applications in tropical geometry (see the Introduction of the thesis for more explanations).

3.1 Tropical cosheaves

Consider a convex lattice polytope P of dimension $n \geq 2$ living in $\mathbb{R}^n$ endowed with the canonical unimodular lattice $M := \mathbb{Z}^n$. It defines a projective toric variety $\mathbb{C}Y$. Consider a triangulation Γ of P with integral vertices (non-necessarily convex). For convenience, in this Chapter, when we use exponents to denote the dimension of a simplex of the triangulation Γ. For instance, a simplex $\sigma^a \in \Gamma$ denotes a simplex of dimension a of Γ.

There exist different subdivisions associated to the triangulation Γ. In their article [BMR 2024], Brugallé, Lopez de Medrano and Rau use a subdivision labeled by the poset $\Xi := \{(F, \sigma),\ F \text{ face of } P,\ \sigma \in \Gamma\}$. However for the convenience of our proof in this Chapter we use a finer subdivision, named the cubical subdivision (or dihomologic subdivision), which has more symetries than the one labeled by Ξ, leading to a better definition of the cap product. The use of the cubical subdivision of the triangulation to compute tropical homology and cohomology was first introduced by Jules Chenal in his PhD thesis [Ch 2024]. He has proven that the resulting tropical homology and cohomology are the same as for the subdivision labeled by the poset Ξ, which justifies the possibility of this choice.

The cubical subdivision associated to the triangulation Γ is the polyhedral complex

whose cells are of the form:

$$c(\sigma^a, \sigma^b) := \mathrm{Conv}\left(\left\{\mathrm{bar}(\tau), \quad \tau \in \Gamma, \quad \sigma^a \subseteq \tau \subseteq \sigma^b\right\}\right), \qquad \sigma^a, \sigma^b \in \Gamma,\ \sigma^a \subseteq \sigma^b.$$

Among these cubical cells, we can consider the union of the ones which starts from a simplex of dimension ≥ 1:

$$|X| := \bigcup_{\sigma^a, \sigma^b \in \Gamma,\ \sigma^a \subseteq \sigma^b,\ a \geq 1} c(\sigma^a, \sigma^b) \tag{3.1}$$

This object $|X|$ is the same as the one discribed in Section 2.2 of the Preliminaries, we can prove that it is compact. In the case of a convex triangulation, it is homeomorphic to the tropical hypersurface dual to the triangulation. The cubical cells of $|X|$ are labeled by the following poset :

$$\Omega := \{(\sigma^a, \sigma^b) \in \Gamma^2, \quad \sigma^a \subseteq \sigma^b, \quad a \geq 1\}.$$

The cubical cells of the $|X|$ are partially ordered by inclusion, and graded by their dimension. It induces on the set Ω a structure of graded poset with the following partial order and dimension:

$$(\sigma_1, \sigma_2) \leq (\tau_1, \tau_2) \iff \sigma_1 \subseteq \tau_1 \text{ and } \sigma_2 \subseteq \tau_2 \ ;$$

$$\dim\,(\sigma^a, \sigma^b) := b - a.$$

This poset Ω is the one described in Example 2.5.9 of Section 2.5 of Preliminaries. As it labels a polyhedral complex, it satisfies the diamond property and it admits a balancing signature. Applying Proposition 2.5.14 and Remark 2.5.15 of Section 2.5, it implies that the cellular cosheaves on Ω admit a homology, and that the cellular sheaves on Ω admit a cohomology. These homology and cohomology do not depend on the choice of the balancing signature.

Remark 3.1.1. *As the elements of Ω are couples of elements of the triangulation Γ, it is possible to define a balancing signature η on Ω from a balancing signature ρ of the triangulation Γ:*

$$\begin{aligned}
&\forall (\sigma, \gamma), (\epsilon, \gamma) \in \Omega, \quad \text{if } (\sigma, \epsilon) \in \mathcal{I}(\Gamma), \quad \eta((\epsilon, \gamma), (\sigma, \gamma)) := (-1)^{\dim\,(\epsilon, \gamma)} \rho(\sigma, \epsilon) \ ; \\
&\forall (\sigma, \epsilon), (\sigma, \gamma) \in \Omega, \quad \text{if } (\epsilon, \gamma) \in \mathcal{I}(\Gamma), \quad \eta((\sigma, \epsilon), (\sigma, \gamma)) := \rho(\epsilon, \gamma) \ .
\end{aligned}$$

where $\mathcal{I}(\Gamma)$ denotes the immediately incident couples of the triangulation Γ (Definition 2.5.1). In the remainder of this Chapter it will be sometimes useful to consider this balancing signature η of Ω, obtained from a balancing signature ρ of Γ.

In the remainder of this Chapter, we fix an integral domain R. Let $N := \mathrm{Hom}(M, \mathbb{Z})$ the dual of the lattice $M = \mathbb{Z}^n$, and let $M_R = M \otimes R$ and $N_R = N \otimes R = \mathrm{Hom}_{\mathbb{Z}}(M, R) = \mathrm{Hom}_R(M_R, R)$. For any $\tau \in \Gamma$, we denote by F_τ the minimal face of the polytope P containing τ. We can consider the tangent unimodular lattice $T_{\mathbb{Z}} F_\tau := T_{\mathbb{R}} F_\tau \cap M$. Its tensorisation by the integral domain R is called the tangent over R of F_τ: $T_R F_\tau :=$

$T_{\mathbb{Z}}F_\tau \otimes R \subseteq M_R$. Similarly, for any element σ of the triangulation we consider the tangent over R of σ: $T_R\sigma := T_{\mathbb{Z}}\sigma \otimes R \subseteq M_R$. As M_R has for dual N_R, we can also consider the orthogonal tangent spaces over R of F_τ and σ: $T_RF_\tau^\perp$ and $T_R\sigma^{1\perp}$, living in N_R. With these notations we can write for any integer $p \geq 0$ and $(\sigma, \tau) \in \Omega$:

$$\mathcal{F}_p^R(\sigma,\tau) := \sum_{\sigma^1 \subseteq \sigma} \bigwedge^p \frac{T_R\sigma^{1\perp}}{T_RF_\tau^\perp}. \tag{3.2}$$

In tropical geometry, these objects are often called tropical cosheaves, or multi-tangent cosheaves, their were first introduced by [IKMZ 2019] for $R = \mathbb{R}$ and in the case of convex triangulation. Renaudineau and Shaw, and Brugallé, Lopez de Medrano and Rau used them for $R = \mathbb{F}_2$, for a coaser subdivision than the cubical subdivision. Chenal is the first one to use them for cubical subdivisions.

There are canonical morphisms of inclusion and projection:

$$\forall (\sigma,\tau),(\gamma,\tau) \in \Omega,\ \sigma \subseteq \gamma, \quad \begin{array}{l} \iota_{\sigma,\tau}^{\gamma,\tau} : \mathcal{F}_p^R(\sigma,\tau) \hookrightarrow \mathcal{F}_p^R(\gamma,\tau) \\ \pi_{\sigma,\tau}^{\sigma,\gamma} : \mathcal{F}_p^R(\sigma,\tau) \twoheadrightarrow \mathcal{F}_p^R(\sigma,\gamma) \end{array}.$$

These morphisms induce by composition morphisms $\mathcal{F}_p^R(\sigma,\tau) \to \mathcal{F}_p^R(\gamma,\epsilon)$ for every ordered element $(\gamma,\epsilon) \leq (\sigma,\tau)$. Then $\mathcal{F}_p^R$ is a contravariant functor from the poset (seen as a category whose objects are its elements, and whose morphisms are defined by the partial order), to the category of R-modules. Thus it is a cellular cosheaf of R-modules, in the sense of Section 2.5 of Preliminaries. Then we can take the direct sum over the elements of Ω with the graduation, to define the following complex:

$$C_q(\mathcal{F}_p^R) := \bigoplus_{(\sigma,\tau)\in\Omega,\ \dim\,(\sigma,\tau)=q} \mathcal{F}_p^R(\sigma,\tau) = \bigoplus_{b-a=q,\ \ 1\leq a\leq b\leq n} \bigoplus_{\sigma^a\subseteq\sigma^b} \mathcal{F}_p^R(\sigma^a,\sigma^b).$$

The homology of the complex $C_*(\mathcal{F}_p^R)$ is constructed using a differential d_q induced by the cosheaves morphisms, with respect to a balancing signature fixed on the poset Ω. We denote $H_*(\mathcal{F}_p^R)$ this homology. We observe that a double complex $C_{a,b}(\mathcal{F}_p^R) := \bigoplus_{\sigma^a\subseteq\sigma^b} \mathcal{F}_p^R(\sigma^a,\sigma^b)$ filters the complex $C_*(\mathcal{F}_p^R)$. We notice that the differential d_q induces by restrictions and projections some morphisms on this double complex:

$$\begin{aligned} d_{a,b}^1 &: C_{a,b}(\mathcal{F}_p^R) \to C_{a+1,b}(\mathcal{F}_p^R), \\ d_{a,b}^2 &: C_{a,b}(\mathcal{F}_p^R) \to C_{a,b-1}(\mathcal{F}_p^R), \end{aligned}$$

and these morphisms satisfy:

$$d_q = \left(\bigoplus_{b-a=q} d_{a,b}^1\right) \oplus \left(\bigoplus_{b-a=q} d_{a,b}^2\right).$$

Then we see that two other maps appear: $d_q^1 := \bigoplus_{b-a=q} d_{a,b}^1$ and $d_q^2 := \bigoplus_{b-a=q} d_{a,b}^2$. They satisfy $d_q = d_q^1 + d_q^2$, and we can check that d_q^1 and d_q^2 are both differentials for the complex $C_*(\mathcal{F}_p^R)$. The fact that these differentials are direct sums of the morphisms $d_{a,b}^1$, and the morphisms $d_{a,b}^2$ respectively make them compatible with the structure of double complex $C_{a,b}(\mathcal{F}_p^R)$, then they define two spectral sequences $E_{*,*}^*(d^1)$ and $E_{*,*}^*(d^2)$ whose 0th

pages are $E^0_{*,*}(d^1) = E^0_{*,*}(d^2) = C_{*,*}(\mathcal{F}^R_p)$, whose 0th differentials are $\partial^0_{*,*}(d^1) := d^1_{*,*}$ and $\partial^0_{*,*}(d^2) := d^2_{*,*}$ respectively, and whose differentials of the successive pages are induced by the differential $d_* = d^1_* + d^2_*$ of the complex $C_*(\mathcal{F}^R_p)$. These two spectral sequences both degenerate to the homology $H_*(\mathcal{F}^R_p)$ of the complex $C_*(\mathcal{F}^R_p)$ for the differential d_*.

Moreover we can notice that the double complex $C_{*,*}(\mathcal{F}^R_p)$ can be refined in the two following ways:

$$C_{a,b}(\mathcal{F}^R_p) := \bigoplus_{\sigma^b \in \Gamma} \bigoplus_{\sigma^a \subseteq \sigma^b} \mathcal{F}^R_p(\sigma^a, \sigma^b) = \bigoplus_{\sigma^a \in \Gamma} \bigoplus_{\sigma^a \subseteq \sigma^b} \mathcal{F}^R_p(\sigma^a, \sigma^b).$$

Then we introduce the following complexes:

$$\forall \sigma^b \in \Gamma, \quad C_{b-a}(\mathcal{F}^R_p(*, \sigma^b)) := \bigoplus_{\sigma^a \subseteq \sigma^b} \mathcal{F}^R_p(\sigma^a, \sigma^b),$$
$$\forall \sigma^a \in \Gamma, \quad C_{b-a}(\mathcal{F}^R_p(\sigma^a, *)) := \bigoplus_{\sigma^a \subseteq \sigma^b} \mathcal{F}^R_p(\sigma^a, \sigma^b).$$

For every $\sigma^b \in \Gamma$, the differential d^1_q of $C_q(\mathcal{F}^R_p)$ induces a differential d^{1,σ^b}_{b-a} on the complex $C_{b-a}(\mathcal{F}^R_p(*, \sigma^b))$, and we obtain an homology $H_{b-a}(\mathcal{F}^R_p(*, \sigma^b))$. Likewise, for every $\sigma^a \in \Gamma$, the differential d^2_q of $C_q(\mathcal{F}^R_p)$ induces a differential d^{2,σ^a}_{b-a} of the complex $C_{b-a}(\mathcal{F}^R_p(\sigma^a, *))$, and we obtain a homology $H_{b-a}(\mathcal{F}^R_p(\sigma^a, *))$. We obtain that the first pages of the spectral sequences $E^*_{*,*}(d^1)$ and $E^*_{*,*}(d^2)$ can be expressed by direct sums of these homology groups:

$$E^1_{a,b}(d^1) = \bigoplus_{\sigma^b \in \Gamma} H_{b-a}(\mathcal{F}^R_p(*, \sigma^b)),$$
$$E^1_{a,b}(d^2) = \bigoplus_{\sigma^a \in \Gamma} H_{b-a}(\mathcal{F}^R_p(\sigma^a, *)).$$

To introduce the homology $H_q(\mathcal{F}^R_p)$ of the cosheaves $\mathcal{F}^R_p$ it was necessary to define a lot of objects. However, the partial Poincaré duality concerns both a cohomology and an homology. Let us now construct the dual objects, the sheaves $\mathcal{F}^p_R$ and their cohomology. For any integer $p \geq 0$ and $(\sigma, \tau) \in \Omega$ we define:

$$\mathcal{F}^p_R(\sigma, \tau) := \mathrm{Hom}_R(\mathcal{F}^R_p(\sigma, \tau), R) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{F}^{\mathbb{Z}}_p(\sigma, \tau), R).$$

The sheaves morphisms are dual to the cosheaves morphisms, then we obtain dual complexes of sheaves $C^q(\mathcal{F}^R_p)$ with dual differential $\delta^q := (d_{q+1})^*$, which induces cohomologies $H^q(\mathcal{F}^p_R)$. There are dual double complexes $C^{a,b}(\mathcal{F}^p_R)$ with dual morphisms $\delta^{a,b}_1 := (d^1_{a-1,b})^*$ and $\delta^{a,b}_2 := (d^2_{a,b+1})^*$ which induce by direct sums the differentials $\delta^q_1 := (d^q_1)^*$ and $\delta^q_2 := (d^q_2)^*$ satisfying $\delta^q_1 + \delta^2_q = \delta^q$. It defines also two spectral sequences $E^{*,*}_*(\delta_1)$ and $E^{*,*}_*(\delta_2)$ whose 0th page are $E^{*,*}_0(\delta_1) = E^{*,*}_0(\delta_2) = C^{*,*}(\mathcal{F}^p_R)$, whose 0th differentials are $\partial^{*,*}_0(\delta_1) := \delta^{*,*}_1$ and $\partial^{*,*}_0(\delta_2) := \delta^{*,*}_2$ respectively, and whose differentials of the successive pages are induced by the differential $\delta^* = \delta^*_1 + \delta^*_2$ of the complex $C^*(\mathcal{F}^p_R)$. Those two spectral sequences both degenerate on the cohomology $H^*(\mathcal{F}^p_R)$ of the complex $C^*(\mathcal{F}^R_p)$ for the differential δ^*.

Moreover, we can also construct for every $\sigma^b \in \Gamma$ the dual complex $C^{b-a}(\mathcal{F}_R^p(*,\sigma^b))$, with the dual differential $\delta_{1,\sigma^b}^{b-a} := (d_{b-a+1}^{1,\sigma^b})^*$ inducing the cohomology $H^{b-a}(\mathcal{F}_R^p(*,\sigma^b))$, and for every $\sigma^a \in \Gamma$ the dual complex $C^{b-a}(\mathcal{F}_R^p(\sigma^a,*))$, with the dual differential $\delta_{2,\sigma^a}^{b-a} := (d_{b-a+1}^{2,\sigma^a})^*$ inducing the cohomology $H^{b-a}(\mathcal{F}_R^p(\sigma^a,*))$. We obtain that the first pages of the spectral sequences $E_*^{*,*}(\delta_1)$ and $E_*^{*,*}(\delta_2)$ can be expressed by direct sums of these homology groups:

$$E_1^{a,b}(\delta_1) = \bigoplus_{\sigma^b \in \Gamma} H^{b-a}(\mathcal{F}_R^p(*,\sigma^b)),$$
$$E_1^{a,b}(\delta_2) = \bigoplus_{\sigma^a \in \Gamma} H^{b-a}(\mathcal{F}_R^p(\sigma^a,*)).$$

We have constructed tropical homology and cohomology and their associated spectral sequences. Let us describe the notion of R-primitivity, R-non-singularity and related considerations on the triangulation before defining an associated cap product and a fundamental class.

3.2 Around primitivity and non-singularity

Consider an integral domain R. In this section we explain the notion of R-primitivity for the triangulation Γ and the notion of R-non-singularity for the lattice polytope P. These notions will be useful for our partial and complete Poincaré theorems. They will be also useful in Chapter 4 in the case $R = \mathbb{F}_2$ to obtain bounds on the Betti numbers of X.

Definition 3.2.1. *(Global R-non-singularity) For any vertex s of the lattice polytope P, let us denote by L_s^1 the family of edges of P which have s as a vertex. For any edge e of P, let us denote by u_e a generator of $T_R e$. Then we say that the lattice polytope P is R-non-singular if for every vertex s of P the family $\{u_e, \quad e \in L_s^1\}$ forms a basis of M_R. In particular it implies that the polytope P is simple.*

Definition 3.2.2. *(Local R-non-singularity) We say that the lattice polytope P is R-non-singular on a face F of P if there exists a free family $\{e_1, ..., e_{n-\dim F}\}$ of vectors of M_R such that for every face G of the polytope for which F is a face of G, there exists a subfamily of $\{e_{i_1}, ..., e_{i_{\dim G - \dim F}}\}$ of $\{e_1, ..., e_{n-\dim F}\}$ such that:*

$$T_R G = T_R F \oplus Re_{i_1} \oplus ... \oplus Re_{i_{\dim G - \dim F}}.$$

Remark 3.2.3. *This definition of local R-non-singularity is compatible with the definition of global R-non-singularity. Indeed, the polytope P is R-non-singular if and only if it is R-non-singular on every one of its faces.*

In Figure 3.1 the polygon P_1 is R-non-singular for any integral domain R. The polygon P_2 is $\mathbb{Q}$-non-singular but it is $\mathbb{Z}$-singular and $\mathbb{F}_2$-singular, because it is $\mathbb{Z}$-singular on the vertices A_2 and B_2, and $\mathbb{F}_2$-singular on the vertex A_2. However, it is $\mathbb{F}_2$-non-singular on the vertex B_2 even if it is $\mathbb{Z}$-singular on this vertex. The polygon P_3 is $\mathbb{Z}$-singular (because $\mathbb{Z}$-singular on the vertices A_3 and B_3), but it is $\mathbb{Q}$-non-singular and $\mathbb{F}_2$-non-singular.

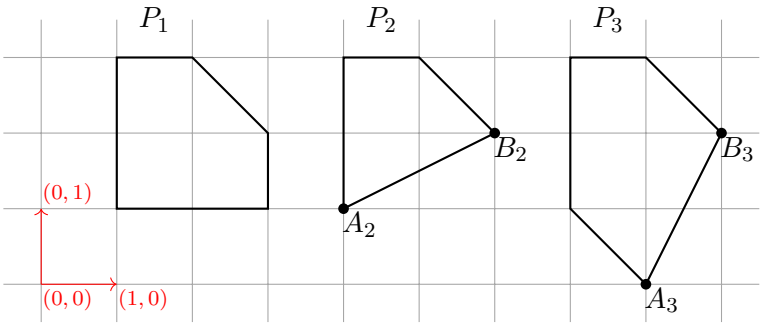

Figure 3.1: R-non-singularity of polytopes

Remark 3.2.4. *The definition of local R-non-singularity implies that if the polytope P is R-non-singular along a face F, then the number of faces G of P which have F as a face is at most $2^{n-\dim F}$. However, as a general property of convex polytopes, this number is at least $2^{n-\dim F}$, thus it is equal to $2^{n-\dim F}$, and we obtain conversely that for any subfamily $\{e_{i_1}, ..., e_{i_{k-\dim F}}\}$ of $\{e_1, ..., e_{n-\dim F}\}$, there exists a face G of dimension k of P which has F as a face such that:*

$$T_R G = T_R F \oplus Re_{i_1} \oplus ... \oplus Re_{i_{\dim G - \dim F}}.$$

Definition 3.2.5. *(Integral length of an edge, index of a simplex) For any edge σ^1 of vertices $s_1, s_2 \in \mathbb{Z}^n$ we define the integral length $|\sigma^1|$ as being the greatest common divisor of the coordinates of $s_2 - s_1$ in a basis of $\mathbb{Z}^n$. It does not depend on the choice of a basis of $\mathbb{Z}^n$ and on the choice of the ordering of s_1, s_2. For any simplex τ with vertices in $\mathbb{Z}^n$ we define the index $|\tau|$ of the simplex τ as being the greatest common divisor of the integral lengths $|\sigma^1|$ for σ^1 edge of τ.*

Definition 3.2.6. *(R-primitivity of a simplex) A k-simplex σ of $\mathbb{Z}^n$ with integral vertices $a_0, a_1, ..., a_k$ is said to be R-primitive if for any $0 \leq i \leq k$ the family of integral vectors $\{\frac{a_j - a_i}{|\sigma|}, \quad j \in \{0, ..., r\} \backslash \{i\}\}$ forms a basis of $T_R\sigma$ ($|\sigma|$ is the index of σ defined just above). Equivalently, it means that the volume of the k-simplex σ is of the form $m|\sigma|^k/k!$ with m invertible in the integral domain R.*

Remark 3.2.7. *In the previous Definition 3.2.6, the existence of one integer $0 \leq i \leq k$ such that $\{\frac{a_j - a_i}{|\sigma|}, \quad j \in \{0, ..., r\} \backslash \{i\}\}$ forms a basis of $T_R\sigma$ is equivalent to the fact that for every integer $1 \leq i \leq k$ the family $\{\frac{a_j - a_i}{|\sigma|}, \quad j \in \{0, ..., r\} \backslash \{i\}\}$ forms a basis of $T_R\sigma$. Then R-primitivity could be defined in this way, and practically it will be often checked like that.*

Remark 3.2.8. *In the case $R = \mathbb{Z}$, a simplex is $\mathbb{Z}$-primitive if and only if it is a dilatation of a primitive simplex. In the case $R = \mathbb{Q}$, every simplex is $\mathbb{Q}$-primitive.*

In Figure 3.2 the triangles T_3, T_4 and T_5 are R-primitive for any integral domain R. The triangles T_1, T_2 and T_6 are not $\mathbb{Z}$-primitive but they are $\mathbb{Q}$-primitive. The triangle T_2 is not $\mathbb{F}_2$-primitive but the triangles T_1 and T_6 are $\mathbb{F}_2$-primitive.

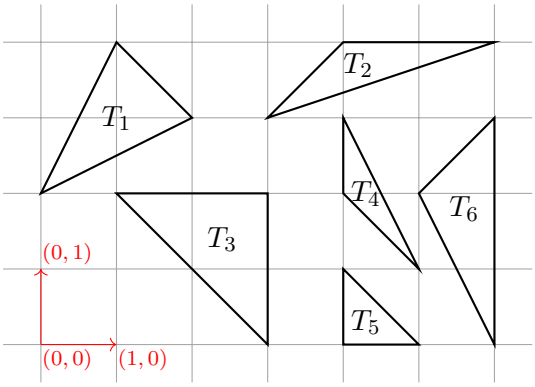

Figure 3.2: R-primitivity of triangles

Definition 3.2.9. *The triangulation Γ is said to be (k, R)-primitive if any k-simplex is R-primitive. Similarly, a simplex with vertices in $\mathbb{Z}^n$ is said to be (k, R)-primitive if any of its k-faces is R-primitive. The triangulation Γ is said to be R-primitive if it is (n, R)-primitive. It implies that any of its simplex is R-primitive.*

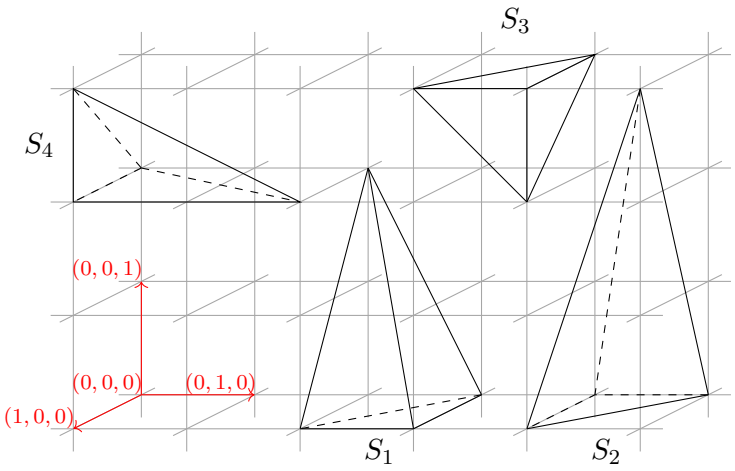

Figure 3.3: R-primitivity for 3-simplices

In Figure 3.3 the simplex S_3 is R-primitive for any integral domain R. The simplices S_2, S_1 and S_4 are not $\mathbb{Z}$-primitive but they are $\mathbb{Q}$-primitive. Moreover, any proper faces of the simplices S_1 and S_2 are R-primitive, then S_1 and S_2 are $(2, R)$-primitive for any integral domain R. However, the simplex S_4 is not $(2, \mathbb{F}_2)$-primitive because the facet containing the edge of integral length 2 are not $\mathbb{F}_2$-primitive, then we obtain also that S_4 is not $\mathbb{F}_2$-primitive. The simplex S_1 is also not $\mathbb{F}_2$-primitive but the simplex S_2 is $\mathbb{F}_2$-primitive.

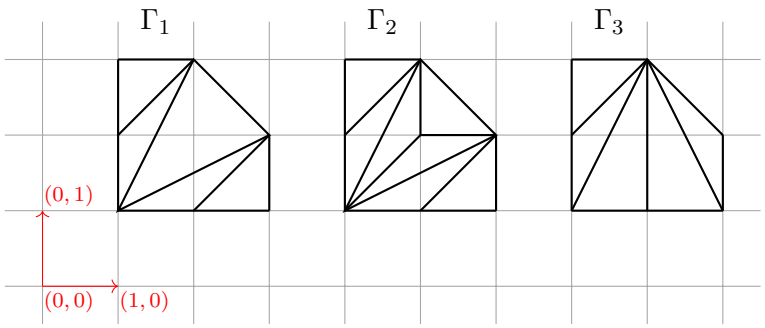

Figure 3.4: Different triangulations of a lattice polygon

Figure 3.4 shows three triangulations Γ_1, Γ_2 and Γ_3 of a lattice polygon. The triangulation Γ_2 is R-primitive for any integral domain R. The triangualtion Γ_1 is not $\mathbb{Z}$-primitive but it is $\mathbb{Q}$-primitive and $\mathbb{F}_2$-primitive. The triangulation Γ_3 is not $\mathbb{Z}$-primitive and not $\mathbb{F}_2$-primitive but it is $\mathbb{Q}$-primitive.

3.3 Hypotheses weaker than primitivity

The complete Poincaré duality theorem (Theorem 1.3.2) is stated for R-primitive triangulations of a R-non-singular lattice polytope of dimension $n \geq 2$. The partial Poincaré duality theorem (Theorem 1.3.1) is stated for (k, R)-primitive triangulations of a R-non-singular lattice polytope, with $2 \leq k \leq n-1$. However, many lemmas are true under weaker hypothesis. For instance, the existence of a fundamental class requires weaker hypothesis.

For this reason we will define two hypotheses 1 and 2 for the triangulation Γ, weaker than the (k, R)-primitivity for $2 \leq k \leq n$ but enough to prove several lemmas needed for complete or partial Poincaré duality. To do that let us define the index $|\Gamma|$ of the triangulation Γ and the reduced length $|\sigma^1|_r$ of any edge of it:

Definition 3.3.1. *We define the index $|\Gamma|$ of the triangulation Γ as the greatest common divisor of the integral lengths $|\sigma^1|$ for $\sigma^1 \in \Gamma$. Then we define the reduced length $|\sigma^1|_r$ of each edge $\sigma^1 \in \Gamma$ as the quotient of the length by this greatest common divisor:*

$$|\sigma^1|_r := \frac{|\sigma^1|}{|\Gamma|}.$$

Then we can define Hypotheses 1 and 2:

Definition 3.3.2. *(hypotheses 1 and 2)*

Hypothesis 1: We say that the triangulation Γ satisfy Hypothesis 1 for the integral domain R if for any edge $\sigma^1 \in \Gamma$ the reduced length $|\sigma^1|_r$ is non-zero in the integral domain R.

Hypothesis 2: We say that the triangulation Γ satisfies Hypothesis 2 for the integral domain R if the reduced length $|\sigma^1|_r$ is invertible in R.

In Figure 3.4 the triangulations Γ_1 and Γ_2 satisfies hypotheses 1 and 2 for any integral domain R. The triangulation Γ_3 satisfies hypotheses 1 and 2 for $\mathbb{Q}$ but it satisfies only Hypothesis 1 for $\mathbb{Z}$ and it does not satisfy any of Hypotheses 1 and 2 for $\mathbb{F}_2$. In the next section we will prove that if Hypothesis 1 is satisfied, the homology group $H_{n-1}(\mathcal{F}^R_{n-1})$ is a free R-module of rank 1 and we define the fundamental class as a generator. This homology group can be of rank greater than 1 if Hypothesis 1 is not satisfied and then the fundamental class does not exist. It is the case for the triangulation Γ_3 of Figure 3.4, the homology $H_1(\mathcal{F}_1^{\mathbb{F}_2})$ is of rank 2 and the fundamental class cannot be defined.

Hypothesis 1 is weaker than Hypothesis 2. If the characteristic of the integral domain R is non-zero, Hypotheses 1 and 2 are equivalent. If the characteristic of the integral domain R is zero, Hypothesis 1 is always satisfied. To prove that these hypotheses are weaker that Hypothesis of (k, R)-primitivity of the triangulation for any $2 \leq k \leq n$, it is enough to prove that the $(2, R)$-primitivity implies Hypothesis 2, which is the following proposition:

Proposition 3.3.3. *If the triangulation Γ is $(2, R)$-primitive, then it satisfies Hypothesis 2 for the integral domain R (and thus also Hypothesis 1).*

Before proving this proposition, we need an intuitive lemma:

Lemma 3.3.4. *If σ^1_a, σ^1_b are two edges of the triangulation Γ of the polytope P, then there exists a sequence $\tau^1_1, \tau^1_2, ..., \tau^1_k$ of edges of the triangulation Γ such that $\tau^1_1 = \sigma^1_a$, $\tau^1_k = \sigma^1_b$ and for all $1 \leq i \leq k-1$ the edges τ^1_i and τ^1_{i+1} belongs to the same triangle of the triangulation.*

This lemma is quite intuitive, it comes from the convexity of the lattice polytope P, but it is a bit technical to prove it. To obtain it, we have to use the following lemma (which states that the triangulation Γ of P is strongly connected):

Lemma 3.3.5. *(Strong connexity of Γ) Let $\sigma^n_1, \sigma^n_2 \in \Gamma$. Then there exists an integer $k \geq 1$ and τ^n_1, τ^n_2,...,τ^n_k a sequence of n-dimensional simplices of Γ such that $\sigma^n_1 = \tau^n_1$, $\sigma^n_2 = \tau^n_k$ and for every $1 \leq i \leq k-1$ there exists $\tau^{n-1}_i \in \Gamma$ a codimension 1 face of both τ^n_i and τ^n_{i+1}.*

Proof. Proof of Lemma 3.3.5: Let $x \in \mathbb{R}^n$ a point of the interior of the simplex σ^n_1. For any $\mu^a \in \Gamma$, $a \leq n-2$, there exists an affine hyperplane $\mathcal{H}_{x,\mu^a}$ of $\mathbb{R}^n$ which contains x and μ^a. Let us consider the following union:

$$A := \bigcup_{\mu^a \in \Gamma,\ a \leq n-2} \mathcal{H}_{x,\mu^a}.$$

It is a finite union of affine hyperplane of $\mathbb{R}^n$, thus its interior is empty. The interior of the simplex σ^n_2 is a non-empty open subset of $\mathbb{R}^n$. Thus, it is not included in A. Therefore, there exists a point y of the interior of the simplex σ^n_2 such that $y \notin A$. Then for any hyperplane $\mathcal{H}_{x,\mu^a}$, the line (xy) is transverse to it and the intersection is $\{x\}$. Then the intersection of the line (xy) with A is $\{x\}$. Thus the intersection of A with the open line segment $]xy[$ is empty. As A contains the union of the $\mu^a \in \Gamma$, $a \leq n-2$, we obtain that the simplices of Γ crossed by $]xy[$ are only simplices of dimension $n-1$ or n. Moreover, as the line (xy) does not cross any $\mu^a \in \Gamma$, $a \leq n-2$, it means that it does not cross

any proper face of the crossed simplices of dimension $n-1$ of Γ, thus any simplex of Γ of dimension $n-1$ crossed by the line (xy) is crossed transversely. Let $a_1, ..., a_{k-1}$ the intersection points of $]xy[$ with the simplices of dimension $n-1$, ordered by their distance from x, and let τ_1^{n-1},...,τ_{k-1}^{n-1} the associated simplices of dimension $n-1$.

By convexity of the polytope P, as x and y belong to P, the line segment $[xy]$ is contained in P. Then the open line segments $]xa_1[$, $]a_1a_2[$, ..., $]a_{k-1}y[$ are contained in P. However they don't intersect any simplex of dimension $n-1$ of Γ. Then they are contained in the union of the interiors of the n-simplices of Γ. As these interiors of n-simplices are pairwise disjoint open subsets of $\mathbb{R}^n$, each open line segment $]xa_1[$, $]a_1a_2[$, ..., $]a_{k-1}y[$ must be included into the interior of one n-simplex of Γ. Let us denote by τ_1^n,...,τ_k^n these n-simplices of Γ whose interior contains $]xa_1[$, $]a_1a_2[$, ..., $]a_{k-1}y[$ respectively. Then we have for every $1 \leq i \leq k-1$ that τ_i^{n-1} is a face of both τ_i^n and τ_{i+1}^n. Moreover, as x and y belong respectively to the interior of σ_1^n and σ_2^n, we have that $\tau_1^n = \sigma_1^n$ and $\tau_k^n = \sigma_2^n$. It concludes the proof of the lemma. □

Now we can prove Lemma 3.3.4:

Proof. Proof of Lemma 3.3.4: Let σ_a^1, σ_b^1 edges of the triangulation Γ. Then there exist some n-simplices σ_1^n, σ_2^n of Γ containing σ_a^1 and σ_b^1 respectively. Applying Lemma 3.3.5, there exists an integer k and a sequence $\tau_1^n, ..., \tau_k^n$ of n-simplices of Γ such that $\sigma_1^n = \tau_1^n$, $\sigma_2^n = \tau_k^n$ and for every $1 \leq i \leq k-1$ there exists $\tau_i^{n-1} \in \Gamma$ a codimension 1 face of both τ_i^n and τ_{i+1}^n.

Let us fix an edge $\tau_i^1 \subseteq \tau_i^{n-1}$ for any $1 \leq i \leq k-1$. Let us also fix the notations $\tau_0^1 = \sigma_a^1$ and $\tau_k^1 = \sigma_b^1$. Then for every $0 \leq i \leq k-1$, the edges τ_i^1 and τ_{i+1}^1 belong to the same n-simplex σ_{i+1}^n. If $\tau_i^1 \neq \tau_{i+1}^1$ there exists a vertex s of τ_i^1 and a vertex t of τ_{i+1}^1 such that $s \neq t$. But s and t are vertices of the same n-simplex σ_{i+1}^n, thus there exists an edge $\epsilon_i^1 \subseteq \sigma_{i+1}^n$ of vertices s and t. Then τ_i^1 and ϵ_i^1 are edges of the same n-simplex σ_{i+1}^n and they share the vertex s, thus they belong to a same triangle. Similarly, ϵ_i^1 and τ_{i+1}^1 belong also to a same triangle. In the case $\tau_i^1 = \tau_{i+1}^1$ we let $\epsilon_i^1 := \tau_i^1 = \tau_{i+1}^1$ and we also have that τ_i^1 and ϵ_i^1 belong to a same triangle and τ_{i+1}^1 and ϵ_i^1 belong to a same triangle.

Finally the sequence $\tau_0^1, \epsilon_0^1, \tau_1^1, \epsilon_1^1, ..., \tau_{k-1}^1 \epsilon_{k-1}^1, \tau_k^1$ satisfies that each consecutive edge belong to a same triangle of Γ, and $\sigma_a^1 = \tau_0^1$, $\sigma_b^1 = \tau_k^1$. It concludes the proof of Lemma 3.3.4 □

We are now ready to prove Proposition 3.3.3:

Proof. Proof of Proposition 3.3.3: Let us consider a triangle $\sigma^2 \in \Gamma$, of edges σ_a^1, σ_b^1, σ_c^1. As the triangulation Γ is $(2, R)$-primitive, this triangle σ^2 is R-primitive. Then the volume of this triangle in a basis of $T_{\mathbb{Z}}\sigma^2$ is of the form $|\sigma^2|^2 x/2!$ with x an invertible element of R. Then the product of the lengths of two edges (for instance $|\sigma_a^1||\sigma_b^1|$) divides $|\sigma^2|^2 x$. Thus there exists an integer y such that:

$$x = y \frac{|\sigma_a^1|}{|\sigma^2|} \frac{|\sigma_b^1|}{|\sigma^2|}.$$

As x is invertible in R, it implies that the three integers y, $|\sigma_a^1|/|\sigma^2|$ and $|\sigma_b^1|/|\sigma^2|$ are all invertible in R. Then, denoting by 1_R the unit element of R, there exists $u, v \in R^*$

such that:

$$\begin{cases} |\sigma_a^1|1_R = |\sigma^2|u \\ |\sigma_b^1|1_R = |\sigma^2|v \end{cases}.$$

Finally we have:

$$|\sigma_a^1|1_R = |\sigma_b^1|v^{-1}u.$$

We have proven that if two edges σ_a^1 and σ_b^1 of the triangulation Γ are edges of a same triangle of Γ, then there exists an element $w \in R^*$ such that $|\sigma_a^1|1_R = |\sigma_b^1|w$. However, according to Lemma 3.3.4, for any edges σ_a^1 and σ_a^2 of the triangulation Γ, there exists a sequence $\tau_1^1, \tau_2^1, ..., \tau_k^1$ of edges of the triangulation such that $\tau_1^1 = \sigma_a^1$, $\tau_k^1 = \sigma_b^1$ and for all $1 \leq i \leq k-1$ the edges τ_i^1 and τ_{i+1}^1 share a vertex. Applying by induction the property that we just have proven, we obtain that there exists $z \in R^*$ such that $|\sigma_a^1|1_R = |\sigma_b^1|z$.

Let us fix an edge $\sigma_0^1 \in \Gamma$. For every edge $\tau^1 \in \Gamma$, let us denote by z_{τ^1} the element of R^* such that $|\tau^1|1_R = |\sigma_0^1|z_{\tau^1}$. By definition, the index $|\Gamma|$ of the triangulation is the greatest common divisor of the lengths $|\tau^1|$ for $\tau^1 \in \Gamma$. The reduced lengths are obtained by dividing the lengths by this index $|\Gamma|$, thus we obtain that the reduced lengths also satisfy:

$$\forall \tau^1 \in \Gamma, \quad |\tau^1|_r 1_R = |\sigma_0^1|_r z_{\tau^1}.$$

There exists a family of integers $\lambda_{\tau^1} \in \mathbb{Z}, \quad \tau^1 \in \Gamma$ such that the reduced lengths satisfy the Bezout relation:

$$1 = \sum_{\tau^1 \in \Gamma} \lambda_{\tau^1} |\tau^1|_r.$$

Thus we obtain:

$$1_R = \sum_{\tau^1 \in \Gamma} \lambda_{\tau^1} |\tau^1|_r 1_R = |\sigma_0^1|_r \sum_{\tau^1 \in \Gamma} \lambda_{\tau^1} z_{\tau^1}.$$

Therefore $|\sigma_0^1|_r$ is invertible in R, and using again the equation $|\tau^1|_r 1_R = |\sigma_0^1|_r z_{\tau^1}$, we obtain that for any $\tau^1 \in \Gamma$ the reduced length $|\tau^1|_r$ is invertible in R. It is exactly the purpose of hypothesis 2. It ends the proof. □

In the following of this Chapter, we precise for each lemma or proposition the assumption we make. It can be nothing, it can be the R-non-singularity of the polytope, or the local R-non-singularity on one of its faces. From the weakest to the strongest, it can be Hypothesis 1, Hypothesis 2, the (k, R)-primitivity for some integer $2 \leq k \leq n$, or the R-primitivity. The purpose of the following subsection is to construct the cap product.

3.4 The cap product

Definition 3.4.1. *If V is a R-module and $0 \leq p \leq p'$ integers, the contraction (denoted by a dot) is the unique bilinear operator $\bigwedge^p V^* \times \bigwedge^{p'} V \to \bigwedge^{p'-p} V$ such that:*

$$\forall \alpha \in \bigwedge^{p'-p} V^*, \quad \forall w \in \bigwedge^{p} V^*, \quad \forall v \in \bigwedge^{p'} V, \quad \alpha(w.v) = (\alpha \wedge w)(v).$$

If V is a R-module, I a finite subset of $\mathbb{Z}$ and $(v_i)_{i\in I}$ a family of elements of V, the notation $\bigwedge_{i\in I} v_i$ may be ambiguous because the sign depends on the order of computation of the wedges of the v_i, $I \in I$. But as $\mathbb{Z}$ is totally ordered, I inherits a total order. In the following $\bigwedge_{i\in I} v_i$ denotes the wedge of the family $(v_i)_{i\in I}$ with respect to the total order on I induced by the total order on $\mathbb{Z}$. That is, if $I = \{i_1, ..., i_k\}$ with $i_1 < i_2 < ... < i_k$ we let:

$$\bigwedge_{i\in I} v_i := v_{i_1} \wedge v_{i_2} \wedge ... \wedge v_{i_k}.$$

Before introducing the cap product of sheaves and cosheaves, let us state the following property satisfied by the contraction: if W is a sub-R-module of V and if $\pi : \bigwedge^p V^* \twoheadrightarrow \bigwedge^p W^*$ is the canonical projection, we have:

$$\forall p' \geq p, \quad \forall \beta \in \bigwedge^p V^*, \quad \forall \alpha \in \bigwedge^{p'} W, \quad \beta.\alpha = \pi(\beta).\alpha. \tag{3.3}$$

Now we can define the cap product of tropical sheaves and cosheaves by the following proposition:

Proposition 3.4.2. *(Cap product of sheaves and cosheaves) For every $\sigma \subseteq \gamma \subseteq \tau$ incident simplices of Γ, with $\dim \sigma \geq 1$, and for every integers $0 \leq p \leq p'$, we name cap product the unique bilinear map:*

$$\cap : \begin{array}{ccc} \mathcal{F}^p_R(\gamma,\tau) \times \mathcal{F}^R_{p'}(\sigma,\tau) & \to & \mathcal{F}^R_{p'-p}(\sigma,\gamma) \\ (\beta,\alpha) & \mapsto & \beta \cap \alpha \end{array}$$

which satisfies the following identity:

$$\forall \overline{\beta} \in \bigwedge^p T_R F_\tau, \quad \forall \alpha \in \mathcal{F}^R_{p'}(\sigma,\tau), \quad [(\iota^\tau_{\sigma,\tau})^*(\overline{\beta})] \cap \alpha = \pi^\gamma_\tau(\overline{\beta}.\alpha)$$

where the dot $\overline{\beta}.\alpha$ denotes the contraction (Definition 3.4.1), where π^γ_τ is the canonical projection $\bigwedge^{p'-p} \frac{N_R}{T_R F^\perp_\tau} \twoheadrightarrow \bigwedge^{p'-p} \frac{N_R}{T_R F^\perp_\gamma}$, and $(\iota^\tau_{\sigma,\tau})^* : \bigwedge^p T_R F_\tau \twoheadrightarrow \mathcal{F}^p_R(\sigma,\tau)$ is the dual of the canonical inclusion $\iota^\tau_{\sigma,\tau} : \mathcal{F}^R_p(\sigma,\tau) \hookrightarrow \bigwedge^p \frac{N_R}{T_R F^\perp_\tau}$

Proof. As the projection $(\iota^\tau_{\sigma,\tau})^*$ is surjective, we just need to prove two statements. Firstly that $\pi^\gamma_\tau(\overline{\beta}.\alpha)$ belongs to $\mathcal{F}^R_{p'-p}(\sigma,\gamma)$, and secondly that if we have two $\overline{\beta}_1, \overline{\beta}_2 \in \bigwedge^p \frac{N_R}{T_R F^\perp_\tau}$ such that $(\iota^\tau_{\sigma,\tau})^*(\overline{\beta}_1) = (\iota^\tau_{\sigma,\tau})^*(\overline{\beta}_2)$ then $\pi^\gamma_\tau(\overline{\beta}_1.\alpha) = \pi^\gamma_\tau(\overline{\beta}_2.\alpha)$.

Let us begin by proving the first statement. Let us write $\alpha = \sum_{\sigma^1 \subseteq \sigma} \alpha_{\sigma^1}$ with $\alpha_{\sigma^1} \in \bigwedge^{p'} \mathcal{F}^R_1(\sigma^1,\tau)$. Then, as $\mathcal{F}^R_1(\sigma^1,\tau)$ is a sub-R-module of $\frac{N_R}{T_R F^\perp_\tau}$, we can apply Equation (3.3) and we get:

$$\forall \sigma^1 \subseteq \sigma, \qquad \overline{\beta}.\alpha_{\sigma^1} = (\iota^\tau_{\sigma^1,\tau})^*(\overline{\beta}).\alpha_{\sigma^1} \in \mathcal{F}^R_{p'-p}(\sigma^1,\tau).$$

It proves that $\pi^\gamma_\tau(\overline{\beta}.\alpha)$ belongs to $\sum_{\sigma^1\subseteq\sigma} \mathcal{F}^R_{p'-p}(\sigma^1,\tau) = \mathcal{F}^R_{p'-p}(\sigma,\gamma)$.

Let us take now $\overline{\beta}_1, \overline{\beta}_2 \in \bigwedge^p T_R F_\tau$ such that $(\iota^\tau_{\sigma,\tau})^*(\overline{\beta}_1) = (\iota^\tau_{\sigma,\tau})^*(\overline{\beta}_2)$. We have for every $\sigma^1 \subseteq \sigma$ and $\alpha_{\sigma^1} \in \mathcal{F}^R_{p'}(\sigma^1,\tau)$:

$$\begin{aligned} \overline{\beta}_1.\alpha_{\sigma^1} &= (\iota^\tau_{\sigma^1,\tau})^*(\overline{\beta}_1).\alpha_{\sigma^1} = (\iota^{\sigma,\tau}_{\sigma^1,\tau})^* \circ (\iota^\tau_{\sigma,\tau})^*(\overline{\beta}_1).\alpha_{\sigma^1} \\ &= (\iota^{\sigma,\tau}_{\sigma^1,\tau})^* \circ (\iota^\tau_{\sigma,\tau})^*(\overline{\beta}_2).\alpha_{\sigma^1} = \overline{\beta}_2.\alpha_{\sigma^1}. \end{aligned}$$

Then using that every $\alpha \in \mathcal{F}^R_{p'}(\sigma,\tau)$ admits a decomposition $\alpha = \sum_{\sigma^1 \subseteq \sigma} \alpha_{\sigma^1}$ with $\alpha_{\sigma^1} \in \mathcal{F}^R_{p'}(\sigma^1,\tau)$, we conclude the proof of the second statement. □

The cap product of cosheaves and sheaves can be extended to a cap product at the level of complexes. For every integers $1 \leq a \leq c \leq b$ and $0 \leq p \leq p'$, for every $\sigma^c \in \Gamma$ we have the following canonical cap product:

$$\cap : C^{b-c}(\mathcal{F}^p_R(\sigma^c,*)) \times C_{a,b}(\mathcal{F}^R_{p'}) \to C_{c-a}(\mathcal{F}^R_{p'-p}(*,\sigma^c))$$

where for $\sigma^a \subseteq \sigma^c$, the $\mathcal{F}^R_{p'-p}(\sigma^a,\sigma^c)$-component of the cap product $\beta \cap \alpha$ of a cochain β of $C^{b-c}(\mathcal{F}^p_R(\sigma^c,*))$ and a chain α of $C_{a,b}(\mathcal{F}^R_{p'})$ is:

$$[\beta \cap \alpha]_{\sigma^a,\sigma^c} := \sum_{\sigma^c \subseteq \sigma^b} \beta_{\sigma^c,\sigma^b} \cap \alpha_{\sigma^a,\sigma^b}.$$

We can even extend it to bigger complexes, fixing equal to zero the other coordinates of the direct sums. We obtain for every integers $0 \leq q \leq q'$ and $0 \leq p \leq p'$ a cap product:

$$\cap : C^q(\mathcal{F}^p_R) \times C_{q'}(\mathcal{F}^R_{p'}) \to C_{q'-q}(\mathcal{F}^R_{p'-p})$$

where for every $c-a = q'-q$ and for every $(\sigma^a,\sigma^c) \in \Omega$, the $\mathcal{F}^R_{p'-p}(\sigma^a,\sigma^c)$-component of the cap product $\beta \cap \alpha$ of a cochain β of $C^q(\mathcal{F}^p_R)$ and a chain α of $C_{q'}(\mathcal{F}^R_{p'})$ is also:

$$[\beta \cap \alpha]_{\sigma^a,\sigma^c} := \sum_{\sigma^c \subseteq \sigma^{c+q}} \beta_{\sigma^c,\sigma^{c+q}} \cap \alpha_{\sigma^a,\sigma^{c+q}}.$$

3.5 The fundamental class

We want to prove Poincaré Duality theorems. We already have a cap product. As usually for Poincaré duality, we have to define a fundamental class, and the Poincaré duality will come from the cap product with this fundamental class. As usual in topology, we want this fundamental class to belong to the chains of the top degree, and more precisely to be a generator of the top degree homology group. In particular the fondamental class can exist only if the top degree homology group is of rank 1.

Here the maximal cubic cells of the hypersurface are of dimension $n-1$, labeled by some $(\sigma^1,\sigma^n) \in \Omega$. We can consider the $C_{n-1}(\mathcal{F}^R_p)$ for p an integer. However, for $p < n-1$, the homology group $H_{n-1}(\mathcal{F}^R_p)$ is a R-module of rank different from 1 in general. In the following we prove that the cycles of $C_{n-1}(\mathcal{F}^R_{n-1})$ are a R-module of rank 1 if we assume that Hypothesis 1 is satisfied by the triangulation Γ for the integral domain R (see Definition 3.3.2), and then the fundamental class will be defined as a generator.

Lemma 3.5.1. *For every $(\sigma^1,\sigma^n) \in \Omega$ we have $\mathcal{F}^R_{n-1}(\sigma^1,\sigma^n) \simeq R$, and in particular we have $\mathcal{F}^{\mathbb{Z}}_{n-1}(\sigma^1,\sigma^n) \simeq \mathbb{Z}$. Moreover we also have $\mathcal{F}^R_{n-1}(\sigma^1,\sigma^n) = \mathcal{F}^{\mathbb{Z}}_{n-1}(\sigma^1,\sigma^n) \otimes R$.*

Proof. For every $(\sigma^1,\sigma^n) \in \Omega$ we have $\mathcal{F}^{\mathbb{Z}}_{n-1}(\sigma^1,\sigma^n) = \bigwedge^{n-1} T_{\mathbb{Z}}\sigma^{1,\perp}$, and $T_{\mathbb{Z}}\sigma^{1,\perp}$ is of rank $n-1$, thus $\mathcal{F}^{\mathbb{Z}}_{n-1}(\sigma^1,\sigma^n) \simeq \mathbb{Z}$.

Moreover, $T_{\mathbb{Z}}\sigma^{1\perp}$ is saturated in N, and then $\bigwedge^{n-1} T_{\mathbb{Z}}\sigma^{1\perp}$ is saturated in $\bigwedge^{n-1} N$. It precisely means that $\bigwedge^{n-1} T_R\sigma^{1\perp} = \bigwedge^{n-1} T_{\mathbb{Z}}\sigma^{1\perp} \otimes R$ and then we obtain $\mathcal{F}^R_{n-1}(\sigma^1,\sigma^n) = \mathcal{F}^{\mathbb{Z}}_{n-1}(\sigma^1,\sigma^n) \otimes R$. □

To construct a fundamental class, we need the following lemma:

Lemma 3.5.2. *Let us denote by 1_R the unit element of the integral domain R. Then there exists a family $(1_{\sigma^1,\sigma^n})_{(\sigma^1,\sigma^n)\in\Omega}$ of generators of the $\mathcal{F}^{\mathbb{Z}}_{n-1}(\sigma^1,\sigma^n)$, $(\sigma^1,\sigma^n)\in\Omega$ such that:*

$$d_{n-1}\left(\bigoplus_{(\sigma^1,\sigma^n)\in\Omega} |\sigma^1|_r 1_{\sigma^1,\sigma^n}\otimes 1_R\right)=0.$$

Proof. The cubical subdivision associated to the poset Ω is a polyhedral complex, satisfying the diamond property. It has for balancing signature a map η used to construct the differential d_{n-1} of the cosheaf $\mathcal{F}^{\mathbb{Z}}_{n-1}$. However we can also define the classical chain complex $C_*(\Omega;\mathbb{Z})$ of Ω over $\mathbb{Z}$, with the classical differential constructed with the balancing signature η and defining the classical polyhedral homology $H_*(\Omega;\mathbb{Z})$ of Ω over $\mathbb{Z}$.

Let us fix $o(M)$ an orientation of the lattice M. For any $\sigma^1\in\Gamma$ of vertices s_1,s_2, an orientation of σ^1 is equivalent to the data of a generator u_{σ^1} of $T_{\mathbb{Z}}\sigma^1$. Then this generator can be completed into a basis $\mathcal{B}$ of M of orientation $o(M)$. Considering the dual basis and removing the dual vector of u_{σ^1} we obtain a basis $\mathcal{B}\backslash\{u^*_{\sigma^1}\}$ of $T_{\mathbb{Z}}\sigma^{1\perp}$. Then we have that the orientation of this basis $\mathcal{B}\backslash\{u^*_{\sigma^1}\}$ of $T_{\mathbb{Z}}\sigma^{1\perp}$ does not depend on the choice of the base $\mathcal{B}$ of orientation $o(M)$ which completes u_{σ^1} in M. It is a canonical way to obtain an orientation on $T_{\mathbb{Z}}\sigma^{1\perp}$ from an orientation of $T_{\mathbb{Z}}\sigma^1$. As an orientation of $T_{\mathbb{Z}}\sigma^{1\perp}$ is equivalent to the data of a generator of $\bigwedge^{n-1}T_{\mathbb{Z}}\sigma^{1\perp}$, we obtain a canonical way to associate a generator of $\bigwedge^{n-1}T_{\mathbb{Z}}\sigma^{1\perp}$ from an orientation of σ^1. We will denote $g(o(\sigma^1))$ this generator.

For any simplex τ, and any facet σ of τ, let us fix an orientation morphism $f_{(\sigma,\tau)}$ associating an orientation of τ from an orientation of σ in the following way: we can consider an ordering of the vertices of τ in which the vertices of σ occupy the first positions, we obtain an ordering of the vertices of σ by restriction. Now we can prove that if we quotient the orderings by even permutations, this map remains well defined, thus it associates an orientation of σ from an orientation of τ. As a property of a balancing signature on simplicial complexes, the balancing signature ρ of Γ can be obtained by fixing a family $o(\sigma),\ \sigma\in\Gamma$ of orientations of simplices of Γ such that for every immediately incident couple $(\sigma,\tau)\in\mathcal{I}(\Gamma)$ we have $f_{(\sigma,\tau)}(o(\tau))=\rho(\sigma,\tau)o(\tau)$. However we also have fixed an orientation $o(M)$ on the lattice M, it defines orientations on each n-simplex of Γ, which may be not equal to the fixed orientations $o(\sigma^n),\ \sigma^n\in\Gamma$. Then we can introduce the following map μ:

$$\forall\sigma^n\in\Gamma,\quad \mu(\sigma^n):=\begin{cases}1 & \text{if } o(\sigma^n)=o(M)\\ -1 & \text{if } o(\sigma^n)=-o(M)\end{cases}.$$

Finally, we can define the following generator of $\mathcal{F}^{\mathbb{Z}}_{n-1}(\sigma^1,\sigma^n)=\bigwedge^{n-1}T_{\mathbb{Z}}\sigma^{1\perp}$:

$$1_{\sigma^1,\sigma^n}:=\mu(\sigma^n)g(o(\sigma^1)).$$

It leads us to define the following direct sum:

$$\alpha:=\bigoplus_{(\sigma^1,\sigma^n)\in\Omega}|\sigma^1|_r 1_{\sigma^1,\sigma^n}\otimes 1_R.$$

Let us begin by proving that α is a cycle for the differential d^2. Let $(\sigma^1, \sigma^{n-1}) \in \Omega$. We have two possibilities: either the minimal face $F_{\sigma^{n-1}}$ of P containing σ^{n-1} is a facet, or it is P itself.

In the first case $\frac{T_R\sigma^{1\perp}}{T_R F^{\perp}_{\sigma^{n-1}}}$ is of rank $n-2$ and then $\mathcal{F}^R_{n-1}(\sigma^1, \sigma^{n-2}) = 0$, which implies that the coordinate over (σ^1, σ^{n-1}) of $d^2_{n-1}(\alpha)$ vanishes:

$$[d^2_{n-1}(\alpha)]_{\sigma^1,\sigma^{n-1}} = 0.$$

In the second case, there exist exactly two simplices $\sigma^n_1, \sigma^n_2 \in \Gamma$ such that σ^{n-1} is a face of both σ^n_1, σ^n_2. Moreover if we apply the orientation morphisms $f_{(\sigma^{n-1},\sigma^n_1)}$ and $f_{(\sigma^{n-1},\sigma^n_2)}$ on the orientation $o(M)$ of the lattice M, we get opposite orientations on σ^{n-1}:

$$f_{(\sigma^{n-1},\sigma^n_1)}(o(M)) = -f_{(\sigma^{n-1},\sigma^n_2)}(o(M)).$$

Thus we have:

$$\rho(\sigma^{n-1}, \sigma^n_1)\mu(\sigma^n_1) = -\rho(\sigma^{n-1}, \sigma^n_2)\mu(\sigma^n_2).$$

However we can use for balancing signature η of Ω a balancing signature inherited from this balancing signature ρ of Γ (see Remark 3.1.1). Therefore we obtain in the second case:

$$\begin{aligned}
&[d^2_{n-1}(\alpha)]_{\sigma^1,\sigma^{n-1}} \\
&= \rho(\sigma^{n-1}, \sigma^n_1)\mu(\sigma^n_1)g(o(\sigma^1))|\sigma^1|_r \otimes 1_R + \rho(\sigma^{n-1}, \sigma^n_2)\mu(\sigma^n_2)g(o(\sigma^1))|\sigma^1|_r \otimes 1_R \\
&= 0.
\end{aligned}$$

Then we have that α is a cycle for the differential d^2:

$$d^2_{n-1}(\alpha) = 0.$$

To conclude the proof, we have now to prove that it is also a cycle for the differential d^1. Let $\sigma^2 \in \Gamma$. It is a triangle, let $\sigma^1_1, \sigma^1_2, \sigma^1_3 \in \Gamma$ its edges. For $i = 1, 2, 3$ let us denote by $u_{\sigma^1_i}$ the generator of $T_{\mathbb{Z}}\sigma^1$ associated to the orientation $f_{(\sigma^1_i,\sigma^2)}(o(\sigma^2))$ of σ^1. Then, the orientation morphisms $f_{(\sigma^1_1,\sigma^2)}$, $f_{(\sigma^1_2,\sigma^2)}$ and $f_{(\sigma^1_3,\sigma^2)}$ satisfy the following property:

$$|\sigma^1_1|_r u_{\sigma^1_1} + |\sigma^1_2|_r u_{\sigma^1_2} + |\sigma^1_3|_r u_{\sigma^1_3} = 0.$$

It implies that the associated generators $g \circ f_{(\sigma^1_i,\sigma^2)}(o(\sigma^2))$ of $\bigwedge^{n-1} T_{\mathbb{Z}}\sigma^{1\perp}_i$ for $i = 1, 2, 3$ satisfy:

$$|\sigma^1_1|_r \; g \circ f_{(\sigma^1_1,\sigma^2)}(o(\sigma^2)) + |\sigma^1_2|_r \; g \circ f_{(\sigma^1_2,\sigma^2)}(o(\sigma^2)) + |\sigma^1_3|_r \; g \circ f_{(\sigma^1_3,\sigma^2)}(o(\sigma^2)) = 0.$$

However, for all $i = 1, 2, 3$ we have:

$$g \circ f_{(\sigma^1_i,\sigma^2)}(o(\sigma^2)) = \rho(\sigma^1_i, \sigma^2)g(o(\sigma^1_i)).$$

Therefore we obtain for every $\sigma^n \in \Gamma, \; \sigma^2 \subseteq \sigma^n$:

$$[d^1_{n-1}(\alpha)]_{\sigma^2,\sigma^n} = (-1)^{n-1}\mu(\sigma^n)\left[\sum_{i=1}^{3} |\sigma^1_i|_r \rho(\sigma^1_i, \sigma^2)g(o(\sigma^1_i))\right] \otimes 1_R = 0.$$

The cochain α is then also a cycle for the differential d^1_{n-1}:

$$d^1_{n-1}(\alpha) = 0.$$

As the differential d_{n-1} is the sum of the differentials d^1_{n-1} and d^2_{n-1}, we obtain that it is a cycle for the differential d_{n-1}, which concludes the proof:

$$d_{n-1}\left(\bigoplus_{(\sigma^1,\sigma^n)\in\Omega} |\sigma^1|_r 1_{\sigma^1,\sigma^n} \otimes 1_R\right) = 0.$$

□

Corollary 3.5.3. *The rank of $H_{n-1}(\mathcal{F}^R_{n-1})$ is at least 1.*

Proof. If the characteristic of the integral domain R is zero, then every coordinates of the element $\bigoplus_{(\sigma^1,\sigma^n)\in\Omega} |\sigma^1|_r 1_{\sigma^1,\sigma^n} \otimes 1_R$ constructed in the previous lemma is non-zero, thus this element itself is non-zero, and then $H_{n-1}(\mathcal{F}^R_{n-1})$ is non-zero.

If the characteristic of the integral domain R is a prime number $p > 0$, then as the reduced lengths $|\sigma^1|_r,\ \sigma^1 \in \Gamma$ are coprime, there exists at least one $\sigma^1 \in \Gamma$ such that p does not divide $|\sigma^1|_r$, and thus for this particular edge σ^1 we have that $|\sigma^1|_r 1_{\sigma^1,\sigma^n} \otimes 1_R$ is non-zero. Then the element $\bigoplus_{(\sigma^1,\sigma^n)\in\Omega} |\sigma^1|_r 1_{\sigma^1,\sigma^n} \otimes 1_R$ constructed in the previous lemma is non-zero, and thus $H_{n-1}(\mathcal{F}^R_{n-1})$ is non-zero. □

We have proven that without hypothesis $H_{n-1}(\mathcal{F}^R_{n-1})$ is of rank at least 1. However we will see in the following that we need more to prove that the rank is exactly 1, we need to assume that the triangulation Γ satisfies Hypothesis 1 for the integral domain R (Definition 3.3.2). Let us prove some lemma needed for that.

Lemma 3.5.4. *Assume that the triangulation Γ satisfies Hypothesis 1 for the integral domain R (Definition 3.3.2). Let $\alpha \in C_{n-1}(\mathcal{F}^R_{n-1})$ such that $d_{n-1}(\alpha) = 0$. Let $\sigma^2 \in \Gamma$ a triangle of edges σ^1_1, σ^1_2 and σ^1_3. For every $\sigma^n \in \Gamma,\ \sigma^2 \subseteq \sigma^n$, if $\alpha_{\sigma^1_1,\sigma^n} = 0$, then $\alpha_{\sigma^1_2,\sigma^n} = 0$ and $\alpha_{\sigma^1_3,\sigma^n} = 0$.*

Proof. Let $\sigma^n \in \Gamma,\ \sigma^2 \subseteq \sigma^n$ such that $\alpha_{\sigma^1_1,\sigma^n} = 0$. Then we have:

$$[d_{n-1}(\alpha)]_{\sigma^2,\sigma^n} = [d^1_{n-1}(\alpha)]_{\sigma^2,\sigma^n} = (-1)^{n-1}\left[\rho(\sigma^1_2,\sigma^2)\alpha_{\sigma^1_2,\sigma^n} + \rho(\sigma^1_3,\sigma^2)\alpha_{\sigma^1_3,\sigma^n}\right].$$

However we know that $[d_{n-1}(\alpha)]_{\sigma^2,\sigma^n} = 0$, thus we obtain:

$$\rho(\sigma^1_2,\sigma^2)\alpha_{\sigma^1_2,\sigma^n} = -\rho(\sigma^1_3,\sigma^2)\alpha_{\sigma^1_3,\sigma^n}.$$

This element both belongs to $\mathcal{F}^R_{n-1}(\sigma^1_2,\sigma^n)$ and $\mathcal{F}^R_{n-1}(\sigma^1_3,\sigma^n)$. However, as a consequence of Hypothesis 1, the intersection of the R-modules $T_R\sigma^{1\perp}_2$ and $T_R\sigma^{1\perp}_3$ is $\{0\}$, and then the intersection of the R-modules $\mathcal{F}^R_{n-1}(\sigma^1_2,\sigma^n)$ and $\mathcal{F}^R_{n-1}(\sigma^1_3,\sigma^n)$ is also $\{0\}$. Then the previous element is zero. It concludes the proof of the lemma. □

Corollary 3.5.5. *Assume that the triangulation Γ satisfies Hypothesis 1 for the integral domain R (Definition 3.3.2). Let $\alpha \in C_{n-1}(\mathcal{F}^R_{n-1})$ such that $d_{n-1}(\alpha) = 0$. For every $\sigma^n \in \Gamma$, if there exists $\sigma^1 \subseteq \sigma^n$ such that $\alpha_{\sigma_1,\sigma^n} = 0$, then for every $\sigma^1 \subseteq \sigma^n$ we have $\alpha_{\sigma_1,\sigma^n} = 0$.*

Proof. Let $\sigma^n \in \Gamma$ and assume that there exists $\sigma^1 \subseteq \sigma^n$ such that $\alpha_{\sigma_1,\sigma^n} = 0$. Let s a vertex of σ^1. For any $\tau^1 \in \Gamma$, $s \subseteq \tau^1 \subseteq \sigma^n$, as τ^1 and σ^1 share the vertex s, they are equals or they define a unique triangle $\sigma^2 \subseteq \sigma^n$, which satisfies $\sigma^1 \subseteq \sigma^2$ and $\tau^1 \subseteq \sigma^2$. Thus, according to the previous lemma, for any $\tau^1 \in \Gamma$, $s \subseteq \tau^1 \subseteq \sigma^n$ we have $\alpha_{\tau^1,\sigma^n} = 0$.

Let $\tau^1 \subseteq \sigma^n$ such that s is not a vertex of τ^1. Let r, t the vertices of τ^1. Then rts forms a triangle of σ^n. The edge st has s for vertex, then according to what we just have said, we have $\alpha_{st,\sigma^n} = 0$. Thus, using the previous lemma for the triangle rst we finally obtain that $\alpha_{\tau^1,\sigma^n} = 0$

We have proven that no matter if s is a vertex of an edge τ^1 of σ^n, we have in any case $\alpha_{\tau^1,\sigma^n} = 0$. It concludes the proof of this corollary. □

Lemma 3.5.6. *Let $\alpha \in C_{n-1}(\mathcal{F}^R_{n-1})$ such that $d_{n-1}(\alpha) = 0$. For every $\sigma^n_1, \sigma^n_2 \in \Gamma$ such that there exists $\tau^{n-1} \in \Gamma$ a common face of codimension 1 of σ^n_1 and σ^n_2, for every $\tau^1 \subseteq \tau^{n-1}$, if $\alpha_{\tau^1,\sigma^n_1} = 0$, then $\alpha_{\tau^1,\sigma^n_2} = 0$.*

Proof. Let $\sigma^n_1, \sigma^n_2 \in \Gamma$ such that there exists $\tau^{n-1} \in \Gamma$ a common face of codimension 1 of σ^n_1 and σ^n_2. Let $\tau^1 \subseteq \tau^{n-1}$. Assume that $\alpha_{\tau^1,\sigma^n_1} = 0$. If $\sigma^n_1 = \sigma^n_2$, it is clear that $\alpha_{\tau^1,\sigma^n_2} = 0$. Assume that $\sigma^n_1 \neq \sigma^n_2$. Then the simplices of dimension n of Γ which have τ^{n-1} as a face are exactly σ^n_1 and σ^n_2. Then we have:

$$[d_{n-1}(\alpha)]_{\tau^1,\tau^{n-1}} = \rho(\tau^{n-1}, \sigma^n_2)\pi^{\tau^1,\tau^{n-1}}_{\tau^1,\sigma^n_2}(\alpha_{\tau^1,\sigma^n_2}).$$

With $\pi^{\tau^1,\tau^{n-1}}_{\tau^1,\sigma^n_2} : \mathcal{F}^R_{n-1}(\tau_1, \sigma^n_2) \twoheadrightarrow \mathcal{F}^R_{n-1}(\tau_1, \tau^n)$ canonical cosheaf morphism of projection. However, as $d_{n-1}(\alpha)$ vanishes, we obtain that $\pi^{\tau^1,\tau^{n-1}}_{\tau^1,\sigma^n_2}(\alpha_{\tau^1,\sigma^n_2}) = 0$.

Moreover, the minimal face $F_{\tau^{n-1}}$ of the polytope P containing the simplex τ^{n-1} is P itself, because it is a face of two distinct n-dimensional simplices of the triangulation (if it was contained in the border of P, it would have been a face of only one n-dimensional simplex). We have $F_{\tau^{n-1}} = P = F_{\sigma^n_2}$. Therefore we obtain $\mathcal{F}^R_{n-1}(\tau^1, \tau^{n-1}) = \mathcal{F}^R_{n-1}(\tau^1, \sigma^n_2)$. The morphism of R-module $\pi^{\tau^1,\tau^{n-1}}_{\tau^1,\sigma^n_2}$ is a projection from $\mathcal{F}^R_{n-1}(\tau^1, \sigma^n_2)$ into $\mathcal{F}^R_{n-1}(\tau^1, \tau^{n-1})$ which are identical, thus it is the identity morphism. Then we obtain $\alpha_{\tau^1,\sigma^n_2} = 0$, which concludes the proof of the lemma. □

Now the previous lemma and the previous corollary enable us to define the fundamental class for a triangulation which satisfies Hypothesis 1 for the integral domain R:

Proposition 3.5.7. *(Existence of a fundamental class) Assume that the triangulation Γ satisfies Hypothesis 1 for the integral domain R (Definition 3.3.2). Then the homology group $H_{n-1}(\mathcal{F}^R_{n-1})$ is a free R-module of rank 1. Moreover, there exists $(1_{\sigma^1,\sigma^n})_{(\sigma^1,\sigma^n)\in\Omega}$ a family of generators of the $\mathcal{F}^{\mathbb{Z}}_{n-1}(\sigma^1, \sigma^n)$, $(\sigma^1, \sigma^n) \in \Omega$ such that the direct sum $[\Omega] := \bigoplus_{\sigma^1,\sigma^n} |\sigma^1|_r 1_{\sigma^1,\sigma^n} \otimes 1_r$ is a generator of $H_{n-1}(\mathcal{F}^R_{n-1})$. We call it the fundamental class over R of the poset Ω. It is unique if R is of characteristic 2, and it is unique up to a multiplication by -1 otherwise.*

Proof. Let $\alpha \in C_{n-1}(\mathcal{F}^R_{n-1})$ such that $d_{n-1}(\alpha) = 0$. Assume that there exists $(\sigma^1_1, \sigma^n_1) \in \Omega$ such that $\alpha_{\sigma^1_1,\sigma^n_1} = 0$. Let us prove that it implies $\alpha = 0$.

Let $(\sigma_2^1, \sigma_2^n) \in \Omega$. If $\sigma_1^n = \sigma_2^n$, it is clear from the last corollary that $\alpha_{\sigma_2^1,\sigma_2^n} = 0$. Assume now that $\sigma_1^n \neq \sigma_2^n$. According to the Lemma 3.3.5, there exists τ_1^n, τ_2^n,...,τ_k^n a sequence of n-dimensional simplices of Γ such that $\sigma_1^n = \tau_1^n$, $\sigma_2^n = \tau_k^n$ and for every $1 \leq i \leq k-1$ there exists $\tau_i^{n-1} \in \Gamma$ a codimension 1 face of both τ_i^n and τ_{i+1}^n. As $\sigma_1^n \neq \sigma_2^n$, we must have $k \geq 2$.

For every $1 \leq i \leq k-1$, let $\tau_i^1 \in \Gamma$ such that $\tau_i^1 \subseteq \tau_i^{n-1}$ (it exists because $n \geq 2$). Let us prove by induction that for every $1 \leq i \leq k-1$ we have $\alpha_{\tau_i^1,\tau_i^n} = 0$.

For $i = 1$, as $\tau_1^{n-1} \subseteq \tau_1^n$ and $\tau_1^n = \sigma_1^n$, we have that τ_1^1 and σ_1^1 are edges of the same simplex τ_1^n. However $\alpha_{\sigma_1^1,\sigma_1^n} = 0$. Thus, according to the last corollary we also have $\alpha_{\tau_1^1,\tau_1^n} = 0$.

Let $1 \leq i \leq k-2$ such that $\alpha_{\tau_i^1,\tau_i^n} = 0$. As $\tau_i^1 \subseteq \tau_i^{n-1}$, as τ_i^{n-1} is a face of both τ_i^n and τ_{i+1}^n, we have $\alpha_{\tau_i^1,\tau_{i+1}^n} = 0$ from Lemma 3.5.6. Moreover, as τ_i^{n-1} and τ_{i+1}^{n-1} are both faces of τ_{i+1}^n, we have that τ_i^1 and τ_{i+1}^1 are edges of the same simplex τ_{i+1}^n. Therefore, according to the last corollary, as $\alpha_{\tau_i^1,\tau_{i+1}^n} = 0$, we also have $\alpha_{\tau_{i+1}^1,\tau_{i+1}^n} = 0$.

Then we have by induction that $\alpha_{\tau_{k-1}^1,\tau_{k-1}^n} = 0$. The simplex τ_{k-1}^1 is an edge of the simplex τ_{k-1}^{n-1} which is a face of both τ_{k-1}^n and σ_2^n. Then we obtain $\alpha_{\tau_{k-1}^1,\sigma_2^n} = 0$ from Lemma 3.5.6. However, σ_2^1 is also an edge of σ_2^n, thus we obtain $\alpha_{\sigma_2^1,\sigma_2^n} = 0$ from the last corollary.

We have proven that for every $\alpha \in C_{n-1}(\mathcal{F}_{n-1}^R)$ such that $d_{n-1}(\alpha) = 0$, if there exists $(\sigma^1, \sigma^n) \in \Omega$ such that $\alpha_{\sigma^1,\sigma^n} = 0$, then $\alpha = 0$. We also know from Lemma 3.5.2 that there exists $(1_{\sigma^1,\sigma^n})_{(\sigma^1,\sigma^n)\in\Omega}$ a family of non-zero elements of the $\mathcal{F}_{n-1}^R(\sigma^1, \sigma^n)$, $(\sigma^1, \sigma^n) \in \Omega$ such that the direct sum $[\Omega] := \bigoplus_{\sigma^1,\sigma^n} |\sigma^1|_r 1_{\sigma^1,\sigma^n} \otimes 1_R$ satisfies $d_{n-1}([\Omega]) = 0$. Let $\alpha \in C_{n-1}(\mathcal{F}_{n-1}^R)$ such that $d_{n-1}(\alpha) = 0$ and let us prove that α is proportional to $[\Omega]$.

If $\alpha = 0$ then we have clearly $\alpha = 0.[\Omega]$. Assume that $\alpha \neq 0$. There exists $(\sigma_1^1, \sigma_1^n) \in \Omega$ such that $\alpha_{\sigma_1^1,\sigma_1^n} \neq 0$. Then, as $1_{\sigma_1^1,\sigma_1^n} \otimes 1_R$ is a generator of $\mathcal{F}_{n-1}^R(\sigma_1^1, \sigma_1^n)$, there exists a non-zero element $\lambda \in R$ such that $\alpha_{\sigma_1^1,\sigma_1^n} = \lambda.(1_{\sigma_1^1,\sigma_1^n} \otimes 1_R)$. Thus we also have:

$$|\sigma_1^1|_r \alpha_{\sigma_1^1,\sigma_1^n} = \lambda.(|\sigma_1^1|_r 1_{\sigma_1^1,\sigma_1^n} \otimes 1_R).$$

Then the chain $|\sigma_1^1|_r \alpha - \lambda[\Omega]$ satisfies:

$$\big(|\sigma_1^1|_r \alpha - \lambda[\Omega]\big)_{\sigma_1^1,\sigma_1^n} = 0.$$

According to our conclusion in the first part of the current proof, as $d_{n-1}(|\sigma_1^1|_r \alpha - \lambda[\Omega]) = 0$ we obtain that the whole chain $|\sigma_1^1|_r \alpha - \lambda[\Omega]$ is zero:

$$|\sigma_1^1|_r \alpha = \lambda[\Omega].$$

Let us denote for every $(\sigma^1, \sigma^n) \in \Omega$, the coefficient $\omega_{\sigma^1,\sigma^n} \in R$ such that $\alpha_{\sigma^1,\sigma^n} = \omega_{\sigma^1,\sigma^n}.(1_{\sigma^1,\sigma^n} \otimes 1_R)$. We have the identities:

$$\forall (\sigma^1, \sigma^n) \in \Omega, \quad |\sigma_1^1|_r \omega_{\sigma^1,\sigma^n} = |\sigma^1|_r \lambda.$$

As the integers $|\sigma^1|_r$, $\sigma^1 \in \Gamma$ are coprime, there exists a family $\epsilon_{\sigma^1} \in \mathbb{Z}$, $\sigma^1 \in \Gamma$ satisfying the Bezout relation:

$$\sum_{\sigma^1 \in \Gamma} \epsilon_{\sigma^1} |\sigma^1|_r = 1.$$

Then, choosing for every $\sigma^1 \in \Gamma$ a particular $\sigma^n_{\sigma^1} \in \Gamma$ satisfying that $(\sigma^1, \sigma^n_{\sigma^1}) \in \Omega$, we obtain:

$$\lambda = \sum_{\sigma^1 \in \Gamma} \epsilon_{\sigma^1} |\sigma^1_1|_r . \omega_{\sigma^1, \sigma^n_{\sigma^1}} = |\sigma^1_1|_r \sum_{\sigma^1 \in \Gamma} \epsilon_{\sigma^1} . \omega_{\sigma^1, \sigma^n_{\sigma^1}} .$$

We have proven that $|\sigma^1_1|_r$ divides λ in the integral domain R. Moreover, $|\sigma^1_1|_r$ is non-zero according to hypothesis 1 satisfied by the triangulation for the integral domain R (Definition 3.3.2). Thus denoting by $\mu := \sum_{\sigma^1 \in \Gamma} \epsilon_{\sigma^1} . \omega_{\sigma^1, \sigma^n_{\sigma^1}}$ the quotient, we obtain:

$$\alpha = \mu[\Omega].$$

Finally we have proven that $[\Omega]$ is a generator of $H_{n-1}(\mathcal{F}^R_p)$, which concludes the proof. □

3.6 The cap product with the fundamental class

We have proven in Lemma 3.5.2 that without hypothesis, we can define the element $[\Omega] := \bigoplus_{\sigma^1, \sigma^n} |\sigma^1|_r 1_{\sigma^1, \sigma^n} \otimes 1_R$, with 1_R the unit element of R, and the $1_{\sigma^1, \sigma^n}$ generators of the $\mathcal{F}^{\mathbb{Z}}_{n-1}(\sigma^1, \sigma^n)$ such that $d_{n-1}([\Omega]) = 0$ (see Definition 3.3.1 for the definition of the reduced length $|\sigma^1|_r$ of an edge $\sigma^1 \in \Gamma$). However, without Hypothesis 1 (see Definition 3.3.2) this element $[\Omega]$ will not be called the fundamental class because it is not a generator of $H_{n-1}(\mathcal{F}^R_{n-1})$. This homology group can be of rank higher than 1, and the definition of $[\Omega]$ is not unique up to a multiplication by -1. However, the cap product with the element $[\Omega]$ is defined without supposing Hypothesis 1. In the following we will fix $[\Omega]$ without supposing Hypothesis 1, except if we precise it, and in this case $[\Omega]$ would be the fundamental class.

Definition 3.6.1. *(Cap product with $[\Omega]$) For any n-dimensional polytope P and triangulation Γ with integral vertices, we define the following linear maps:*

$$\forall \sigma^{n-q} \in \Gamma, \quad \phi_{\sigma^{n-q}} : \begin{array}{ccc} C^q(\mathcal{F}^p_R(\sigma^{n-q}, *)) & \to & C_{n-1-q}(\mathcal{F}^R_{n-1-p}(*, \sigma^{n-q})) \\ \beta & \mapsto & \beta \cap [\Omega] \ ; \end{array}$$

$$\phi^q : \begin{array}{ccc} C^q \mathcal{F}^p_R) & \to & C_{n-1-q}(\mathcal{F}^R_{n-1-p}) \\ \beta & \mapsto & \beta \cap [\Omega] \end{array} .$$

By definition of the cap product, these two maps satisfy the following relation:

$$\phi^q \left(\bigoplus_{b-a=q} \bigoplus_{\sigma^a \subseteq \sigma^b} \beta_{\sigma^a, \sigma^b} \right) = \bigoplus_{\sigma^{n-q} \in \Gamma} \phi_{\sigma^{n-q}} \left(\bigoplus_{\sigma^{n-q} \subseteq \sigma^n} \beta_{\sigma^{n-q}, \sigma^n} \right).$$

The map ϕ^q is defined at the level of complexes. With good hypotheses, it induces isomorphisms at the level of cohomology and homology. In the following section, we state several lemmas and we prove the partial Poincaré duality and the Poincaré duality using them.

3.7 Proof of the partial and complete Poincaré duality

3.7.1 The lemmas and their consequences

From the beginning of this Chapter we omit the index p of the cosheaves $\mathcal{F}_p^R$ and the sheaves $\mathcal{F}_R^p$ in the notations of the differentials d^1, d^2, d, δ_1, δ_2, δ, in the notations of the map ϕ^q and in the notation of the spectral sequences $E^*_{*,*}(d^1)$, $E^*_{*,*}(d^2)$, $E_*^{*,*}(\delta_1)$, $E_*^{*,*}(\delta_2)$. The reason is that the index p was mostly constant and there was no possible ambiguity. However in the next statements there will be possible ambiguity. Thus, if necessary, we will precise the index p by using the notations $d^{1,(p)}$, $d^{2,(p)}$, $d^{(p)}$, $\delta_{1,(p)}$, $\delta_{2,(p)}$, $\delta_{(p)}$ for the differentials and $E^*_{*,*}(d^{1,(p)})$, $E^*_{*,*}(d^{2,(p)})$, $E_*^{*,*}(\delta_{1,(p)})$, $E_*^{*,*}(\delta_{2,(p)})$ for the spectral sequences. For the map ϕ^q , we will also use the notation $\phi^q_{(p)}$ and it will denote the map starting from $C^q(\mathcal{F}_R^p)$.

To prove partial and complete Poincaré duality theorem, let us announce three lemmas about the first pages of the spectral sequences $E^*_{*,*}(d^{1,(p)})$ and $E_*^{*,*}(\delta_{2,(p)})$. We will detail the proof of those lemmas in the next section.

Lemma 3.7.1. *(For partial Poincaré duality) If the triangulation is (k,R)-primitive for an integer $2 \leq k \leq n-1$, then for any integer $p < k$ the first page of the spectral sequence $E^*_{*,*}(d^{1,(p)})$ satisfies:*

$$E^1_{a,b}(d^{1,(p)}) = \begin{cases} \ker(d^{1,(p)}_{1,b}) & \text{if } a = 1 \\ \bigoplus_{\sigma^b \in \Gamma} H_{b-a}(\mathcal{F}_p^R(*, \sigma^b)) & \text{if } 1 < a \leq p+1 \text{ and } k < b \leq n \\ 0 & \text{otherwise} \end{cases} .$$

Lemma 3.7.2. *(For complete Poincaré duality) If the triangulation is R-primitive, then for any integer p the first page of the spectral sequences $E^*_{*,*}(d^{1,(p)})$ satisfies:*

$$E^1_{a,b}(d^{1,(p)}) = \begin{cases} \ker(d^{1,(p)}_{1,b}) & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases} .$$

Lemma 3.7.3. *(For partial and complete Poncaré duality) If the polytope P is R-non-singular, then for any integer p the first page of the spectral sequence $E_*^{*,*}(\delta_{2,(p)})$ satisfies the following property:*

$$E_1^{a,b}(\delta_{2,(p)}) = \begin{cases} \operatorname{coker}(\delta^{a,n-1}_{2,(p)}) & \text{if } b = n \\ 0 & \text{otherwise} \end{cases} .$$

For a R-non-singular polytope, this last lemma means that for any p the spectral sequence $E_*^{*,*}(\delta_{2,(p)})$ vanishes at the first page except on a line. As the degree of the r-th page is $(-r, 1-r)$, it implies that it degenerates at most at the second page, where it is reduced to a line. As it converges toward the cohomology $H^*(\mathcal{F}_R^p)$, we get the following proposition:

Proposition 3.7.4. *If the polytope P is R-non-singular, then for any integer p we have the following identity:*

$$H^q(\mathcal{F}_R^p) = E_2^{n-q,n}(\delta_{2,(p)}).$$

Similarly, for a R-primitive triangulation, Lemma 3.7.2 means that for any p the spectral sequence $E^*_{*,*}(d^{1,(p)})$ vanishes at the first page except on a column. As the degree of the r-th page is $(1-r,-r)$, it implies that it degenerates at most at the second page, where it is reduced to a column. As it converges toward the cohomology $H_*(\mathcal{F}_p^R)$, we get the following proposition:

Proposition 3.7.5. *If the triangulation is R-primitive, then for any integer p we have the following identity:*

$$H_q(\mathcal{F}_p^R) = E^2_{1,q+1}(d^{1,(p)}).$$

If the triangulation is (k,R)-primitive for an integer $2 \leq k \leq n-1$, then for an integer $p < k$, the first page of the spectral sequence $E^*_{*,*}(d^{1,(p)})$ vanishes except on the half column $a = 1, b \leq k$ and on the rectangle $1 \leq a \leq p+1$; $k < b \leq n$. As the differential of the r-th page is of degree $(1-r,-r)$, then for any $r \geq 2$ the differential starting from a term $(1,b)$ with $b < k-p$ arrives to a zero (the term $(2-r, b-r)$), and the differential arriving to a term $(1,b)$ with $b < k-p$ starts from a zero (the term $(r, b+r)$). Indeed, as $r \geq 2$, the term $(r, b+r)$ can be non-zero only if $b + r > k$ and $r \leq p+1$, which implies $b \geq k-p$, in contradiction with $b < k-p$. Thus we obtain the identity $E^2_{1,b}(d^{1,(p)}) = E^\infty_{1,b}(d^{1,(p)})$ for any $b < k-p$. Moreover, for any $b_0 < k-p$, at the second page this term $(1,b_0)$ is the only one which is non-zero on the diagonal of terms (a,b) satisfying $a+b = b_0 - 1$. However it is this diagonal which contributes to the homology group $H_{b_0-1}(\mathcal{F}_p^R)$, and then this homology group is equal to this single non-zero term $E^\infty_{1,b_0}(d^{1,(p)}) = E^2_{1,b_0}(d^{1,(p)})$. We sum up this conclusion in the following proposition:

Proposition 3.7.6. *If the triangulation is (k,R)-primitive for an integer $2 \leq k \leq n-1$, then we have:*

$$\forall p < k, \quad \forall q < k-p-1, \quad H_q(\mathcal{F}_p^R) = E^2_{1,q+1}(d^{1,(p)}).$$

Let us announce now tree lemmas concerning the map ϕ^q. As for the previous lemma, we will also prove them in a following part.

Lemma 3.7.7. *(For partial and complete Poincaré duality) If the triangulation is (k,R)-primitive for an integer $2 \leq k \leq n$, then the image of the map $\phi^q_{(p)}$ is the following:*

$$\forall q \geq n-k, \qquad \mathrm{im}(\phi^q_{(p)}) = \ker(d^{1,(n-1-p)}_{1,n-q}).$$

Moreover, for $q < n-k$ we still have $d^{1,(n-1-p)}_{1,n-q} \circ \phi^q_{(p)} = 0$.

Lemma 3.7.8. *If the polytope P is R-non-singular and if the triangulation satisfies Hypothesis 2 for the integral domain R (see Definition 3.3.2), then we have the following identity:*

$$\ker(\phi^q_{(p)}) \cap C^{n-q,n}(\mathcal{F}^p_R) = \mathrm{im}(\delta^{n-q,n-1}_{2,(p)}).$$

Lemma 3.7.9. *For any lattice polytope and triangulation with integral vertices, we have the following relation of commutation:*

$$\phi^{q+1}_{(p)} \circ \delta^q_{1,(p)} = (-1)^{q+1} d^{2,(n-1-p)}_{n-q-1} \circ \phi^q_{(p)}.$$

3.7.2 Proof of the complete Poincaré duality

Corollary 3.7.10. *(Corollary of Lemmas 3.7.2, 3.7.3 3.7.7 and 3.7.8). If the polytope P is R-non-singular and if the triangulation is R-primitive, then the maps ϕ^* induce the following isomorphisms between the terms of the first pages of the spectral sequences $E^{*,*}_*(\delta_{2,(p)})$ and $E^*_{*,*}(d^{1,(n-1-p)})$:*

$$\tilde{\phi}^q_{1,(p)} : E_1^{n-q,n}(\delta_{2,(p)}) \xrightarrow{\sim} E^1_{1,n-q}(d^{1,(n-1-p)}).$$

Proof. As a R-primitive triangulation is a n-primitive triangulation, we have that the statement of the Lemma 3.7.8 is valid for any q in the case of a R-primitive triangulation:

$$\forall p, \forall q, \quad \mathrm{im}(\phi^q_{(p)}) = \ker(d^{1,(n-1-p)}_{1,n-q}).$$

Moreover the Lemma 3.7.7 is also valid because the R-primitivity of the triangulation implies Hypothesis 2 for the integral domain R (see Proposition 3.3.3). Then it implies that the restriction of $\phi^q_{(p)}$ to $C^{n-q,n}(\mathcal{F}^p_R)$ can be factorised into an isomorphism defined from $\mathrm{coker}(\delta^{n-q-1,n}_{2,(p)})$ to its image. However from the last equation we know that its image is $\ker(d^{1,(n-1-p)}_{1,n-q})$, thus we have an isomorphism from $\mathrm{coker}(\delta^{n-q-1,n}_{2,(p)})$ to $\ker(d^{1,(n-1-p)}_{1,n-q})$. Then we can conclude the proof using Lemmas 3.7.2 and 3.7.3. □

Proposition 3.7.11. *If the polytope P is R-non-singular and if the triangulation is R-primitive, then the maps ϕ^* induce the following isomorphisms between the terms of the second pages of the spectral sequences $E^{*,*}_*(\delta_{2,(p)})$ and $E^*_{*,*}(d^{1,(n-1-p)})$:*

$$\tilde{\phi}^q_{2,(p)} : E_2^{n-q,n}(\delta_{2,(p)}) \xrightarrow{\sim} E^2_{1,n-q}(d^{1,(n-1-p)}).$$

Proof. A consequence of Lemmas 3.7.7, 3.7.8 and 3.7.9 is the following global relation of commutation valid for all p and q:

$$\phi^{q+1}_{(p)} \circ \delta^q_{(p)} = (-1)^{q+1} d^{(n-1-p)}_{n-q-1} \circ \phi^q_{(p)}.$$

As the spectral sequences $E_*^{*,*}(\delta_{2,(p)})$ and $E_{*,*}^*(d^{1,(n-1-p)})$ come from a filtration of the cohomology $H^*(\mathcal{F}_R^p))$ and a filtration of the homology $H_*(\mathcal{F}_{n-1-p}^R))$ respectively, the differentials of their successive pages are induced by the associated differentials of complexes $\delta_{(p)}^*$ and $d_*^{(n-1-p)}$ respectively. Let us then denote $\partial_{r,(p)}^{*,*}$ and $\partial_{*,*}^{r,(n-1-p)}$ the differentials of the r-th pages $E_r^{*,*}(\delta_{2,(p)})$ and $E_{*,*}^r(d^{1,(n-1-p)})$ respectively. Thus, as we know from the last proposition that the maps $\phi_{(p)}^*$ induce isomorphisms $\tilde{\phi}_{1,(p)}^*$ between the first pages of the corresponding spectral sequences, we get the following relation between the differential of the first pages $\partial_{1,(p)}^{*,*}$ and $\partial_{*,*}^{1,(n-1-p)}$:

$$\partial_{1,n-q}^{1,(n-1-p)} = (-1)^{q+1}\tilde{\phi}_{1,(p)}^{q+1} \circ \partial_{1,(p)}^{n-q,n} \circ (\tilde{\phi}_{1,(p)}^q)^{-1}.$$

Then the isomorphisms $\tilde{\phi}_{1,(p)}^*$ are not only isomorphisms between the terms of the first pages of the spectral sequences, but they are also compatible with the differentials $\partial_{1,(p)}^{*,*}$ and $\partial_{*,*}^{1,(n-1-p)}$. Therefore they induce isomorphisms at the second pages:

$$\bar{\phi}_{2,(p)}^q : E_2^{n-q,n}(\delta_{2,(p)}) \xrightarrow{\sim} E_{1,n-q}^2(d^{1,(n-1-p)}).$$

□

To conclude, the Theorem 1.3.2 of Complete Poincaré duality comes directly from the Propositions 3.7.4, 3.7.5 and 3.7.11.

3.7.3 Proof of the partial Poincaré duality

Corollary 3.7.12. *(Corollary of Lemmas 3.7.1, 3.7.3 3.7.7 and 3.7.8). If the polytope P is R-non-singular and if the triangulation is (k,R)-primitive for an integer $2 \leq k \leq n-1$, then for any $p \geq n-k$ the maps $\phi_{(p)}^*$ induce the following isomorphisms between some of the terms of the first pages of the spectral sequences $E_*^{*,*}(\delta_{2,(p)})$ and $E_{*,*}^*(d^{1,(n-1-p)})$:*

$$\forall p \geq n-k, \quad \forall q \geq n-k, \quad \tilde{\phi}_{1,(p)}^q : E_1^{n-q,n}(\delta_{2,(p)}) \xrightarrow{\sim} E_{1,n-q}^1(d^{1,(n-1-p)}).$$

Proof. The Lemma 3.7.8 implies that the restriction of $\phi_{(p)}^q$ to $C^{n-q,n}(\mathcal{F}_R^p)$ can be factorised into an isomorphism defined from $\mathrm{coker}(\delta_{2,(p)}^{n-q-1,n})$ to its image. However the Lemma 3.7.7 is also valid because the (k,R)-primitivity for an integer $k \geq 2$ implies Hypothesis 2 (see Proposition 3.3.3). Then for $q \geq n-k$ the image is $\ker(d_{1,n-q}^{1,(n-1-p)})$, thus if $q \geq n-k$ we have an isomorphism from $\mathrm{coker}(\delta_{2,(p)}^{n-q-1,n})$ to $\ker(d_{1,n-q}^{1,(n-1-p)})$. Then we can conclude the proof using Lemmas 3.7.1 and 3.7.3 (Lemma 3.7.1 has to be applied for $n-1-p$ instead of p, that is why we get the condition $n-1-p < k$ and thus $p \geq n-k$). □

Proposition 3.7.13. *If the polytope P is R-non-singular and if the triangulation is R-primitive, then for $p \geq n-k$ the maps $\phi_{(p)}^*$ induce the following isomorphisms between some of the terms of the second pages of the spectral sequences $E_*^{*,*}(\delta_{2,(p)})$ and $E_{*,*}^*(d^{1,(n-1-p)})$:*

$$\forall p \geq n-k, \quad \forall q \geq n-k+1, \quad \tilde{\phi}_{2,(p)}^q : E_2^{n-q,n}(\delta_{2,(p)}) \xrightarrow{\sim} E_{1,n-q}^2(d^{1,(n-1-p)}).$$

Proof. A consequence of Lemma 3.7.7, 3.7.8 and 3.7.9 is the following global relation of commutation:

$$\phi_{(p)}^{q+1} \circ \delta_{(p)}^{q} = (-1)^{q+1} d_{n-q-1}^{(n-1-p)} \circ \phi_{(p)}^{q}.$$

As the spectral sequences $E_*^{*,*}(\delta_{2,(p)})$ and $E_{*,*}^{*}(d^{1,(n-1-p)})$ come from a filtration of the cohomology $H^*(\mathcal{F}_R^p)$) and a filtration of the homology $H_*(\mathcal{F}_{n-1-p}^R)$) respectively, the differentials of their successive pages are induced by the associated differentials of complexes $\delta_{(p)}^*$ and $d_*^{(n-1-p)}$ respectively. Let us then denote $\partial_{r,(p)}^{*,*}$ and $\partial_{*,*}^{r,(n-1-p)}$ the differentials of the r-th pages $E_r^{*,*}(\delta_{2,(p)})$ and $E_{*,*}^{r}(d^{1,(n-1-p)})$ respectively. Thus, as we know from the last proposition that for $p \geq n-k$ the maps $\phi_{(p)}^*$ induce isomorphisms $\tilde{\phi}_{1,(p)}^*$ between the terms $(n-q,n)$ and $(1,n-q)$ for $q \geq n-k$ of the first pages of these spectral sequences, we get the following relation between the differential of the first pages $\partial_{1,(p)}^{*,*}$ and $\partial_{*,*}^{1,(n-1-p)}$:

$$\forall p \geq n-k, \quad \forall q \geq n-k, \quad \partial_{1,n-q}^{1,(n-1-p)} = (-1)^{q+1} \tilde{\phi}_{1,(p)}^{q+1} \circ \partial_{1,(p)}^{n-q,n} \circ (\tilde{\phi}_{1,(p)}^{q})^{-1}.$$

Let $p \geq n-k$. For any $q \geq n-k$, the first differential starting from $E_1^{n-q,n}(\delta_{2,(p)})$ arrives on $E_1^{n-q-1,n}(\delta_{2,(p)})$ which is also a term in the "isomorphism zone" because $q+1 \geq n-k$. Moreover, for $q \geq n-k$, the first differential arriving on $E_1^{n-q,n}(\delta_{2,(p)})$ starts from $E_1^{n-q+1,n}(\delta_{2,(p)})$ which is a term in the "isomorphism zone" if and only if $q-1 \geq n-k$. Therefore we finally have that the maps $\tilde{\phi}_{1,(p)}^*$ induce these following isomorphisms at the second page:

$$\forall p \geq n-k, \quad \forall q \geq n-k+1, \quad \forall \bar{\phi}_{2,(p)}^{q} : E_2^{n-q,n}(\delta_{2,(p)}) \xrightarrow{\sim} E_{1,n-q}^2(d^{1,(n-1-p)}).$$

□

Proof. Proof of Theorem 1.3.1 of partial Poincaré duality:

From Proposition 3.7.6 we have:

$$\forall p \geq n-k, \quad \forall n-1-q < k-n+p, \quad H_{n-1-q}(\mathcal{F}_p^R) = E_{1,n-q}^2(d^{1,(n-1-p)}).$$

And from Proposition 3.7.4 we also have:

$$\forall p, \quad \forall q, \quad H^q(\mathcal{F}_R^p) = E_2^{n-q,n}(\delta_{2,(p)}).$$

However, $n-1-q < k-n+p$ means $2n-k \leq p+q$. If we restrict us to the non-zero zone $0 \leq p < n-1$ and $0 \leq q < n-1$, it implies $q \geq n-k+1$, and $p \geq n-k+1 \geq n-k$. Then we can apply the last proposition and conclude the proof of partial Poincaré duality theorem. □

3.8 Proof of the lemmas

To prove the lemmas, we use many times the notion of a saturated sub-module. Let us recall the definition:

Definition 3.8.1. *(Saturation) If R is an integral domain, if L is a R-module, and S a sub-module of L, we say that S is saturated in L if:*

$$\forall y \in L, \forall \lambda \in R\backslash\{0\}, \quad \lambda y \in S \implies y \in S.$$

3.8.1 The homological spectral sequence: proof of Lemma 3.7.1 and 3.7.2

Let us begin by recalling the following identity satisfied by the first page of the spectral sequence $E^{*}_{*,*}(d^1)$:

$$E^1_{a,b}(d^1) = \bigoplus_{\sigma^b \in \Gamma} H_{b-a}(\mathcal{F}^R_p(*, \sigma^b)).$$

It shows that to prove Lemmas 3.7.1 and 3.7.2 we just need to study the properties of the homology groups $H_{b-a}(\mathcal{F}^R_p(*, \sigma^b))$. For Lemma 3.7.2, we just need to prove that $H_{b-a}(\mathcal{F}^R_p(*, \sigma^b)) = 0$ for any $a \neq 1$ and R-primitive simplex σ^b, because then we get that the first page of the spectral sequence vanishes at the first page for $a \neq 1$. The fact that the terms of the column $a = 1$ correspond to the kernel of the differentials $d^1_{1,b}$ is just a consequence of the definition of the spectral sequence $E^{*}_{*,*}(d^1)$ (the differentials of the 0-th page corresponds to d^1) and the fact that at the 0-th page we have $E^0_{0,b} = C_{0,b}(\mathcal{F}^R_p) = 0$.

For Lemma 3.7.1, for a (k, R)-primitive triangulation with $2 \leq k \leq n-1$, we also need to prove that $H_{b-a}(\mathcal{F}^R_p(*, \sigma^b)) = 0$ for any $a \neq 1$ and any R-primitive simplex σ^b, which will imply that $E^1_{a,b}(d^1) = 0$ for $a \neq 1$ and $b \leq k$ (because in a (k, R)-primitive triangulation, every b-simplex for $b \leq k$, is R-primitive). However we also need to prove that if $p < k$ and $b > k$, any (k, R)-primitive simplex σ^b satisfies $H_{b-a}(\mathcal{F}^R_p(*, \sigma^b)) = 0$ for any $a > p + 1$.

Finally, the proof of Lemma 3.7.1 and 3.7.2 is a consequence of the two following propositions that we will prove in this section:

Proposition 3.8.2. *Let $\sigma^b \in \Gamma$. Let $r := \text{rank } T_R F_{\sigma^b}$. Assume that the simplex σ^b is R-primitive (see Definition 3.2.6). Then, for any integer $p \geq 0$, the homology of the chain complex $C_q(\mathcal{F}_p(*, \sigma^b))$ equipped with the differential d_q^{*,σ^b} is torsion-free and satisfies:*

$$\forall q \neq b-1, \quad H_q(\mathcal{F}^R_p(*, \sigma^b)) = 0,$$

$$\text{rank}_R \; H_{b-1}(\mathcal{F}^R_p(*, \sigma^b)) = \binom{r}{p+1} - \binom{r-b}{p+1}.$$

Proposition 3.8.3. *Let σ^b a (k, R)-primitive simplex for an integer $2 \leq k \leq n-1$, with $b > k$. Then we have:*

$$\forall p < k, \quad \forall a > p+1, \quad H_{b-a}(\mathcal{F}^R_p(*, \sigma^b)) = 0.$$

Proposition 3.8.2 can be deduced from the following proposition due to Philipp Jell, Johanes Rau and Kris Shaw:

Proposition 3.8.4. *(Philipp Jell, Johanes Rau, Kris Shaw) If a b-simplex Δ^b with integral vertices in $\mathbb{Z}^n$ is primitive (that is that the directing vectors of the adjacent edges of a vertex form a basis of the integral tangent lattice $T_{\mathbb{Z}}\Delta^b$), if this b-simplex Δ^b belongs to a triangulation of a lattice polytope P, then the resulting cosheaves $\mathcal{F}_p^{\mathbb{Z}}$ of this triangulation satisfy:*

$$\forall q \neq b-1, \quad H_q(\mathcal{F}_p^{\mathbb{Z}}(*, \Delta^b)) = 0.$$

This proposition was formulated in their article [JRS 2018] in terms of matroidal fans. In their proof of their Proposition 5.5, they write that the tropical Borel-Moore homology $H_{p,q}^{BM}(V;\mathbb{Z})$ of a matroidal fan V of dimension r of $\mathbb{R}^s$ vanishes except if $q = r$. Here a primitive b-simplex Δ^b living in $T_{\mathbb{R}}F_{\Delta^b}$ (F_{Δ^b} denoting by the minimal face of the polytope P containing Δ^b) is dual to a matroidal fan of the form $V := \frac{N_{\mathbb{R}}}{T_{\mathbb{R}}\Delta^{b\perp}} \oplus H_{trop}(\Delta^b)$ where $H_{trop}(\Delta^b) \subseteq \frac{T_{\mathbb{R}}\Delta^{b\perp}}{T_{\mathbb{R}}F_{\Delta^b}^{\perp}}$ is the tropical hyperplane dual to $\Delta^b \subseteq T_{\mathbb{R}}\Delta^b$. This matroidal fan lives in $\frac{N_{\mathbb{R}}}{T_{\mathbb{R}}\Delta^{b\perp}} \oplus \frac{T_{\mathbb{R}}\Delta^{b\perp}}{T_{\mathbb{R}}F_{\Delta^b}^{\perp}} = \frac{N_{\mathbb{R}}}{T_{\mathbb{R}}F_{\Delta^b}^{\perp}} \simeq \mathbb{R}^{\dim F_{\Delta^b}}$. This matroidal fan is of dimension $\dim F_{\Delta^b} - 1$, then according to [JRS 2018], its tropical Borel Moore Homology $H_{p,q}^{BM}(V;\mathbb{Z})$ vanishes for all $q \neq \dim F_{\Delta^b} - 1$.

However, looking at equation 5.1 in [JRS 2018], we can identify the cellular complex $C_{p,*}^{BM,Cell}(V;\mathbb{Z})$ with the complex $C_*(\mathcal{F}_p^{\mathbb{Z}}(*, \Delta^b))$ in the following way:

$$C_{p,q}^{BM,Cell}(V;\mathbb{Z}) = C_{q-\dim F_{\Delta^b}+b}(\mathcal{F}_p^{\mathbb{Z}}(*, \Delta^b)).$$

With the identity $H_{p,q}^{BM}(V;\mathbb{Z}) = H_{q,p}^{BM,Cell}(V;\mathbb{Z})$ from equation 5.1 of [JRS 2018], it leads to:

$$H_{q-\dim F_{\Delta^b}+b}(\mathcal{F}_p^{\mathbb{Z}}(*, \Delta^b)) = H_{p,q}^{BM}(V;\mathbb{Z}).$$

Finally we obtain that in our case the claim $H_{p,q}^{BM}(V;\mathbb{Z}) = 0$ for all $q \neq \dim F_{\Delta^b} - 1$ of Philipp Jell, Johanes Rau, Kris Shaw can be reformulated into the Proposition 3.8.4. As a consequence of this Proposition 3.8.4, we can prove the Proposition 3.8.2:

Proof. Proof of Proposition 3.8.2: Consider a R-primitive simplex σ^b of the triangulation Γ of our lattice polytope P and denote $r = \text{rank } T_R F_{\sigma^b}$ (where F_{σ^b} is the minimal face of P that contains σ^b). Now denoting by $(e_1, ..., e_n)$ the canonical basis of $\mathbb{Z}^n$, let us define the following objects:

$$\begin{aligned}
&\Delta^b := \text{Conv}(0, e_1, ..., e_b),\\
&P' := \text{Conv}(-e_1, ..., -e_r, e_1, ..., e_n),\\
&\Gamma' := \{\text{faces of Conv}(0, \epsilon_1 e_1, ..., \epsilon_r e_r, e_{r+1}, ..., e_n); \quad \epsilon_1, ..., \epsilon_r \in \{-1, +1\}\}.
\end{aligned}$$

Δ^b is a primitive b-simplex, P' is a lattice polytope of dimension n of $\mathbb{Z}^n$, Γ' is a triangulation of P', Δ^b is an element of this triangulation Γ' and $F'_{\Delta^b} = \text{Conv}(-e_1, ..., -e_r, e_1, ..., e_r)$ is the minimal face of P' containing Δ^b, it is of dimension r. We can apply Proposition 3.8.4 to this simplex Δ^b with the cosheaves $\mathcal{F}_p'^{\mathbb{Z}}$ of this triangulation Γ':

$$\forall q \neq b-1, \quad H_q(\mathcal{F}_p'^{\mathbb{Z}}(*, \Delta^b)) = 0.$$

According to the thesis of Jules Chenal [Ch 2024], Proposition 3.1.14, the primitivity of the simplex Δ^b implies that for any face Δ^a of Δ^b the $\mathbb{Z}$-module $\mathcal{F}_p'^{\mathbb{Z}}(\Delta^a, \Delta^b)$ is saturated in $\bigwedge^p \frac{N}{T_{\mathbb{Z}}F'^{\perp}_{\Delta^b}}$ (see Definition 3.8.1 for the saturation of a sub-module).

However, if $A_1, ..., A_k$ are saturated $\mathbb{Z}$-sub-module of a $\mathbb{Z}$-module B, then for any integral domain R the tensor products $A_1 \otimes R$, ..., $A_k \otimes R$ can be identified as R-sub-modules of $B \otimes R$, and if $A_1 + ... + A_k$ is saturated in B, then the tensor product $(A_1 + ... + A_k) \otimes R$ can also be identified as a sub-module of $B \otimes R$ and it satisfies the identity $A_1 \otimes R + ... + A_k \otimes R = (A_1 + ... + A_k) \otimes R$. For any $\Delta^a \subseteq \Delta^b$, we can apply this result to $\mathcal{F}_p'^{\mathbb{Z}}(\Delta^a, \Delta^b) = \sum_{\Delta^1 \subseteq \Delta^a} \mathcal{F}_p'^{\mathbb{Z}}(\Delta^1, \Delta^b)$, knowing that all of the $\mathcal{F}_p'^{\mathbb{Z}}(\Delta^1, \Delta^b)$ and also $\mathcal{F}_p'^{\mathbb{Z}}(\Delta^a, \Delta^b)$ are saturated in $\bigwedge^p \frac{N}{T_{\mathbb{Z}}F'^{\perp}_{\Delta^b}}$. We obtain the following identity of sub-modules of $\bigwedge^p \frac{N_R}{T_R F'^{\perp}_{\Delta^b}}$:

$$\forall \Delta^a \subseteq \Delta^b, \quad \mathcal{F}_p'^{R}(\Delta^a, \Delta^b) = \mathcal{F}_p'^{\mathbb{Z}}(\Delta^a, \Delta^b) \otimes R. \tag{3.4}$$

Let us denote by $d_{b-a}'^{\mathbb{Z},*,\Delta^b}$ and $d_{b-a}'^{R,*,\Delta^b}$ the differentials of the complexes $C_{b-a}(\mathcal{F}_p'^{\mathbb{Z}}(*, \Delta^b))$ and $C_{b-a}(\mathcal{F}_p'^{R}(*, \Delta^b))$ respectively. As they are induced by the inclusions $\mathcal{F}_p'^{\mathbb{Z}}(\Delta^a, \Delta^b) \hookrightarrow \mathcal{F}_p'^{\mathbb{Z}}(\Delta^{a+1}, \Delta^b)$ and $\mathcal{F}_p'^{R}(\Delta^a, \Delta^b) \hookrightarrow \mathcal{F}_p'^{R}(\Delta^{a+1}, \Delta^b)$ respectively, it implies that they commute with the morphisms $x \mapsto x \otimes \lambda$, for any $\lambda \in R$. Then the image $\text{im } d_{b-a}'^{R,*,\Delta^b}$ satisfies:

$$\text{im } d_{b-a}'^{R,*,\Delta^b} = \{y_1 \otimes \lambda_1 + ... + y_k \otimes \lambda_k, \ k \geq 1, \ y_1, ..., y_k \in \text{im } d_{b-a}'^{\mathbb{Z},*,\Delta^b}, \ \lambda_1, ..., \lambda_k \in R\}.$$

In the previous identity the sums of the tensor products $y_i \otimes \lambda_i$ are computed as elements of $\bigoplus_{\Delta^{a+1} \subseteq \Delta^b} \bigwedge^p \frac{N}{T_{\mathbb{Z}}F'^{\perp}_{\Delta^b}} \otimes R = \bigoplus_{\Delta^{a+1} \subseteq \Delta^b} \bigwedge^p \frac{N_R}{T_R F'^{\perp}_{\Delta^b}}$. To conclude that it is equal to $\text{im } d_{b-a}'^{\mathbb{Z},*,\Delta^b} \otimes R$, we need to have the saturation of $\text{im } d_{b-a}'^{\mathbb{Z},*,\Delta^b}$ in $\bigoplus_{\Delta^{a+1} \subseteq \Delta^b} \bigwedge^p \frac{N}{T_{\mathbb{Z}}F'^{\perp}_{\Delta^b}}$. The saturation of $\mathcal{F}_p'^{\mathbb{Z}}(\Delta^a, \Delta^b)$ in $\bigwedge^p \frac{N}{T_{\mathbb{Z}}F'^{\perp}_{\Delta^b}}$ implies the saturation of $C_{b-a}(\mathcal{F}_p'^{\mathbb{Z}}(*, \Delta^b))$ in $\bigoplus_{\Delta^a \subseteq \Delta^b} \bigwedge^p \frac{N}{T_{\mathbb{Z}}F'^{\perp}_{\Delta^b}}$ which implies also the saturation of $\ker d_{b-a}'^{\mathbb{Z},*,\Delta^b}$ in $\bigoplus_{\Delta^a \subseteq \Delta^b} \bigwedge^p \frac{N}{T_{\mathbb{Z}}F'^{\perp}_{\Delta^b}}$. However the homology vanishes for $b - a \neq b - 1$, then the image $\text{im } d_{b-a}'^{\mathbb{Z},*,\Delta^b}$ is also saturated in $\bigoplus_{\Delta^{a+1} \subseteq \Delta^b} \bigwedge^p \frac{N}{T_{\mathbb{Z}}F'^{\perp}_{\Delta^b}}$ if $a \neq 0$. However, for $a = 0$ the complex is zero and the image of the differential is also zero. Then for any a the image $\text{im } d_{b-a}'^{\mathbb{Z},*,\Delta^b}$ is saturated in $\bigoplus_{\Delta^{a+1} \subseteq \Delta^b} \bigwedge^p \frac{N_R}{T_R F'^{\perp}_{\Delta^b}}$ and we obtain:

$$\forall a, \quad \text{im } d_{b-a}'^{R,*,\Delta^b} = \text{im } d_{b-a}'^{\mathbb{Z},*,\Delta^b} \otimes R. \tag{3.5}$$

In particular this identity implies an equality of rank:

$$\forall a, \quad \text{rank}_{\mathbb{Z}} \text{ im } d_{b-a}'^{\mathbb{Z},*,\Delta^b} = \text{rank}_R \text{ im } d_{b-a}'^{R,*,\Delta^b}. \tag{3.6}$$

However from equation 3.4 we also have $C_{b-a}(\mathcal{F}_p'^{R}(*, \Delta^b)) = C_{b-a}(\mathcal{F}_p'^{\mathbb{Z}}(*, \Delta^b)) \otimes R$ which also leads to an equality of rank:

$$\forall a, \quad \text{rank}_{\mathbb{Z}} \ C_{b-a}(\mathcal{F}_p'^{\mathbb{Z}}(*, \Delta^b)) = \text{rank}_R \ C_{b-a}(\mathcal{F}_p'^{R}(*, \Delta^b)). \tag{3.7}$$

Then using the rank theorem, we obtain:

$$\forall a, \quad \text{rank}_R \text{ ker } d'^{R,*,\Delta^b}_{b-a} = \text{ rank}_{\mathbb{Z}} \text{ ker } d'^{\mathbb{Z},*,\Delta^b}_{b-a}.$$

As the homology $H_{b-a}(\mathcal{F}'^{\mathbb{Z}}_p(*,\Delta^b))$ vanishes for $a \neq 1$, and using equation 3.6, we obtain:

$$\forall a \neq 1, \quad \text{rank}_R \text{ ker } d'^{R,*,\Delta^b}_{b-a} = \text{rank}_{\mathbb{Z}} \text{ im } d'^{\mathbb{Z},*,\Delta^b}_{b-a-1} = \text{rank}_R \text{ im } d'^{R,*,\Delta^b}_{b-a-1}.$$

To conclude that the homology $H_{b-a}(\mathcal{F}'^{R}_p(*,\Delta^b))$ vanishes for all $a \neq 1$, it is now enough to prove that the R-module im $d'^{R,*,\Delta^b}_{b-a-1}$ is saturated in $\bigoplus_{\Delta^a \subseteq \Delta^b} \bigwedge^p \frac{N_R}{T_R F'^{\perp}_{\Delta^b}}$. However, the fact that im $d'^{\mathbb{Z},*,\Delta^b}_{b-a-1}$ is saturated in $\bigoplus_{\Delta^a \subseteq \Delta^b} \bigwedge^p \frac{N}{T_{\mathbb{Z}} F'^{\perp}_{\Delta^b}}$, combined with the identity 3.5 proves it (and it even proves it in the case $a = 1$). Then we obtain that the homology vanishes for all $a \neq 1$, and it is torsion-free for $a = 1$:

$$\forall a \neq 1, \qquad H_{b-a}(\mathcal{F}'^{R}_p(*,\Delta^b)) = 0,$$
$$H_{b-1}(\mathcal{F}'^{R}_p(*,\Delta^b)) \text{ is torsion-free.}$$

We can compute the rank of $H_{b-1}(\mathcal{F}'^{R}_p(*,\Delta^b))$ using the Euler characteristic. However, from the identity 3.7 this Euler characteristic does not depend on the integral domain R. As the simplex Δ^b is primitive, any face $\Delta^a \subseteq \Delta^b$ is also primitive. Then we can apply the Lemma 2.4 of [BMR 2024] (applied for $k := 1$, $F := F'_{\Delta^b}$ and $\sigma := \Delta^a$), and we obtain:

$$\text{rank}_{\mathbb{F}_2} \mathcal{F}'^{\mathbb{F}_2}_p(\Delta^a, \Delta^b) = \binom{r}{p} - \binom{r-a}{p-a}.$$

As the number of a-dimensional faces of a b-simplex is $\binom{b+1}{a+1}$, we obtain that the Euler characteristic is the following:

$$\begin{aligned}
\chi(\mathcal{F}'^{R}_p(*,\Delta^b)) &= \sum_{a=1}^{b} (-1)^{b-a} \binom{b+1}{a+1} \left[\binom{r}{p} - \binom{r-a}{p-a} \right] \\
&= \sum_{a=1}^{b} (-1)^{b-a} \left[\binom{b}{a} + \binom{b}{a+1} \right] \left[\binom{r}{p} - \binom{r-a}{p-a} \right] \\
&= \sum_{a=1}^{b} (-1)^{b-a} \binom{b}{a} \left[\binom{r}{p} - \binom{r-a}{p-a} - \binom{r}{p} + \binom{r-a+1}{p-a+1} \right] \\
&= \sum_{a=1}^{b} (-1)^{b-a} \binom{b}{a} \binom{r-a}{p+1-a} \\
&= \sum_{a=1}^{b} (-1)^{b-a} \sum_{D \in \mathcal{P}_a(\{1,\dots,b\})} \text{Card} \bigcap_{i \in D} \{X \in \mathcal{P}_{p+1}(\{1,\dots,r\}), i \in X\} \\
&= (-1)^{b-1} \text{Card} \bigcup_{i=1}^{b} \{X \in \mathcal{P}_{p+1}(\{1,\dots,r\}), i \in X\} \\
&= (-1)^{b-1} \left[\binom{r}{p+1} - \binom{r-b}{p+1} \right]
\end{aligned}$$

where $\mathcal{P}_{p+1}(\{1,...,r\})$ denotes the set of subsets of cardinal $p+1$ of $\{1,...,r\}$. Then we obtain the rank of $H_{b-1}(\mathcal{F}_p'^R(*,\Delta^b))$:

$$\text{rank } H_{b-1}(\mathcal{F}_p'^R(*,\Delta^b)) = (-1)^{b-1}\chi(\mathcal{F}_p'^R(*,\Delta^b)) = \binom{r}{p+1} - \binom{r-b}{p+1}.$$

Let us denote by $s, a_1, ..., a_b$ the vertices of the simplex σ^b. By definition of the R-primitivity (see Definition 3.2.6), the family $\mathcal{B}_1 = \{\frac{sa_1}{|\sigma^b|}, ..., \frac{sa_b}{|\sigma^b|}\}$ forms a R-basis of the tangent space $T_R\sigma^b$ (Recall that $|\sigma^b|$ denotes the greatest common divisor of the integral lengths of the edges of σ^b, see Definition 3.2.5). Moreover, as the $\mathbb{Z}$-module $T_{\mathbb{Z}}\sigma^b$ is saturated in $T_{\mathbb{Z}}F_{\sigma^b}$ and $T_{\mathbb{Z}}F_{\sigma^b}$ is saturated in $\mathbb{Z}^n$, there exist integral vectors $u_{b+1}, ..., u_n$ of $\mathbb{Z}^n$ such that:

$$T_{\mathbb{Z}}\sigma^b \oplus \bigoplus_{i=b+1}^{r} \mathbb{Z}u_i = T_{\mathbb{Z}}F_{\sigma^b} \quad ; \quad T_{\mathbb{Z}}F_{\sigma^b} \oplus \bigoplus_{i=r+1}^{n} \mathbb{Z}u_i = \mathbb{Z}^n.$$

Then the family $\mathcal{B}_2 := \mathcal{B}_1 \cup \{u_{b+1}, ..., u_r\}$ forms a R-basis of $T_RF_{\sigma^b}$ and the family $\mathcal{B}_3 := \mathcal{B}_2 \cup \{u_{r+1}, ..., u_n\}$ forms a R-basis of $\mathbb{Z}^n \otimes R$. We can consider the $\mathbb{Z}$-linear map $\psi : \mathbb{Z}^n \to \mathbb{Z}^n$ sending each e_i, $1 \leq i \leq b$ to $\frac{sa_1}{|\sigma^b|}$, and each e_i, $b+1 \leq i \leq n$ to u_i. Then as $\mathcal{B}_3$ is a R-basis of $\mathbb{Z}^n \otimes R$, it induces by tensorisation an isomorphism of R-module $\psi_R : \mathbb{Z}^n \otimes R \xrightarrow{\sim} \mathbb{Z}^n \otimes R$. This isomorphism satisfies $\psi_R(T_RF'_{\Delta^b}) = T_RF_{\sigma^b}$, then its dual satisfies $\psi_R^*(T_RF_{\sigma^b}^{\perp}) = T_RF_{\Delta^b}'^{\perp}$, and then it induces the following factorization:

$$\tilde{\psi}_R^* : \frac{N_R}{T_RF_{\sigma^b}^{\perp}} \xrightarrow{\sim} \frac{N_R}{T_RF_{\Delta^b}'^{\perp}}.$$

Moreover, the morphism ψ maps the b-simplex Δ^b to a dilation of the b-simplex σ^b (by the factor $1/|\sigma^b|$), then for each edge of $\Delta^1 \subseteq \Delta^b$, denoting by $\sigma^1 \subseteq \sigma^b$ the corresponding edge of σ^b, the map ψ induces by restriction and co-restriction an isomorphism $T_{\mathbb{Z}}\Delta^1 \xrightarrow{\sim} T_{\mathbb{Z}}\sigma^1$. Then ψ_R induces an isomorphism $T_R\Delta^1 \xrightarrow{\sim} T_R\sigma^1$ and thus its dual ψ_R^* induces an isomorphism $T_R\sigma^{1\perp} \xrightarrow{\sim} T_R\Delta^{1\perp}$. Finally $\tilde{\psi}_R^*$ induces by restriction and co-restriction isomorphisms $\frac{T_R\sigma^{1\perp}}{T_RF_{\sigma^b}^{\perp}} \xrightarrow{\sim} \frac{T_R\Delta^{1\perp}}{T_RF_{\Delta^b}'^{\perp}}$ which induce also isomorphims $\bigwedge^p \frac{T_R\sigma^{1\perp}}{T_RF_{\sigma^b}^{\perp}} \xrightarrow{\sim} \bigwedge^p \frac{T_R\Delta^{1\perp}}{T_RF_{\Delta^b}'^{\perp}}$. By sums, it induces between any face $\Delta^a \subseteq \Delta^b$ and its corresponding $\sigma^a \subseteq \sigma^b$ an isomorphism $\mathcal{F}_p'^R(\Delta^a, \Delta^b) \xrightarrow{\sim} \mathcal{F}_p^R(\sigma^a, \sigma^b)$. As these family of isomorphims came from a same map ψ sending Δ^b to σ^b compatible with the combinatorial of the simplices Δ^b and σ^b, these isomorphisms induces isomorphisms between the chain complexes which commute with the differential (as soon as we have chosen compatible balancing signature between the poset of the faces of Δ^b and the poset of the faces of σ^b). However, the homology does not depend on the choice of a balancing signature, then we can choose compatible ones, and we obtain at the end that their homology are isomorphic. It proves the proposition: the homology $H_{b-a}(\mathcal{F}_p^R(*,\sigma^b))$ is torsion-free for all a, and we have:

$$\begin{aligned} &\forall a \neq 1, \quad H_{b-a}(\mathcal{F}_p^R(*,\sigma^b)) = 0, \\ &\text{rank}_R\ H_{b-1}(\mathcal{F}_p^R(*,\sigma^b)) = \binom{r}{p+1} - \binom{r-b}{p+1}. \end{aligned}$$

□

Proof. Proof of Proposition 3.8.3: Let us fix σ^b a (k,R)-primitive simplex for $2 \leq k \leq n-1$ and $b > k$. Then any a-face σ^a of σ^b with $a \leq k$ is a R-primitive simplex. Let us fix a a-face σ^a of σ^b, with $a \leq k$. Similarly as in the proof of Proposition 3.8.2, we can construct a primitive a-dimensional simplex Δ^a which belongs to a primitive triangulation Γ' of a non-singular lattice polytope P', such that the minimal face F'_{Δ^a} of P' containing Δ^a is of dimension dim F_{σ^b} (F_{σ^b} denotes the minimal face of the lattice polytope P containing the simplex σ^b). We also have an injective morphism of $\mathbb{Z}$-module $\psi : \mathbb{Z}^n \to \mathbb{Z}^n$ which sends the simplex Δ^a to a dilatation of the simplex σ^a such that it induces an isomorphism of R-module $\psi_R : \mathbb{Z}^n \otimes R \to \mathbb{Z}^n \otimes R$ satisfying $\psi_R(T_R F'_{\Delta^a}) = T_R F_{\sigma^b}$. Then we obtain similarly that its dual ψ_R^* induces isomophisms $\mathcal{F}_p^{'R}(\sigma^a, \sigma^b) \xrightarrow{\sim} \mathcal{F}_p^R(\Delta^a, \Delta^a)$. However, we obtain also a similar equation as 3.4: $\mathcal{F}_p^{'R}(\Delta^a, \Delta^a) = \mathcal{F}_p^{'\mathbb{Z}}(\Delta^a, \Delta^a) \otimes R$ which implies that the rank of the R-module $\mathcal{F}_p^R(\Delta^a, \Delta^a)$ does not depend on the integral domain R. Then we obtain:

$$\text{rank}_R\ \mathcal{F}_p^R(\sigma^a, \sigma^b) = \text{rank}_R\ \mathcal{F}_p^R(\Delta^a, \Delta^a) = \text{rank}_{\mathbb{F}_2}\ \mathcal{F}_p^{'\mathbb{F}_2}(\Delta^a, \Delta^a).$$

However, as the polytope P' is non-singular and the triangulation Γ' is primitive we can use Lemma 2.4 of [BMR 2024] (applied for $k := 1$, $F := F'_{\Delta^a}$ and $\sigma := \Delta^a$):

$$\text{rank}_{\mathbb{F}_2}\ \mathcal{F}_p^{'\mathbb{F}_2}(\Delta^a, \Delta^a) = \binom{\text{dim} F'_{\Delta^a}}{p} - \binom{\text{dim} F'_{\Delta^a} - a}{p - a}.$$

Therefore using $\text{dim} F'_{\Delta^a} = \text{dim} F_{\sigma^b}$ we finally obtain for every face σ^a of σ^b with $a \leq k$:

$$\text{rank}_R\ \mathcal{F}_p^R(\sigma^a, \sigma^b) = \binom{\text{dim} F_{\sigma^a}}{p} - \binom{\text{dim} F_{\sigma^a} - a}{p - a}.$$

Then we obtain in the case $p < a \leq k$, for every $\sigma^a \subseteq \sigma^b$:

$$\text{rank}_R\ \mathcal{F}_p^R(\sigma^a, \sigma^b) = \binom{\text{dim} F_{\sigma^b}}{p} = \text{rank}\ \bigwedge^p \frac{N_R}{T_R F_{\sigma^b}^{\perp}}. \tag{3.8}$$

However, using again the simplex Δ^a of the triangulation Γ' of the polytope P', and the related morphism $\psi : \mathbb{Z}^n \to \mathbb{Z}^n$, we have similarly as in the proof of Proposition 3.8.2 that the $\mathbb{Z}$-module $\mathcal{F}_p^{'\mathbb{Z}}(\Delta^a, \Delta^a)$ is saturated in $\bigwedge^p \frac{N}{T_{\mathbb{Z}} F_{\sigma^b}^{\perp}}$ (see Definition 3.8.1 for the saturation of a sub-module). Using again the similar identity as equation 3.4 ($\mathcal{F}_p^{'R}(\Delta^a, \Delta^a) = \mathcal{F}_p^{'\mathbb{Z}}(\Delta^a, \Delta^a) \otimes R$), it implies that $\mathcal{F}_p^{'R}(\Delta^a, \Delta^a)$ is a R-module saturated in $\bigwedge^p \frac{N_R}{T_R F_{\sigma^b}^{\perp}}$. Then using the isomorphism $\psi_R^* : \frac{N_R}{T_R F_{\sigma^b}^{\perp}} \xrightarrow{\sim} \frac{N_R}{T_R F_{\Delta^a}^{'\perp}}$ we finally obtain that $\mathcal{F}_p^R(\sigma^a, \sigma^b)$ is a R-module saturated in $\bigwedge^p \frac{N_R}{T_R F_{\sigma^b}^{\perp}}$. However equation 3.8 shows that they are of same rank, thus they are equal. We obtain for any $p < a \leq k$ and every $\sigma^a \subseteq \sigma^b$:

$$\mathcal{F}_p^R(\sigma^a, \sigma^b) = \bigwedge^p \frac{N_R}{T_R F_{\sigma^b}^{\perp}}.$$

However, for any face $\sigma^a \subseteq \sigma^b$ with $a \geq k$ there exists a face $\sigma^k \subseteq \sigma^a$ and then if $p < k$, we have $\mathcal{F}_p^R(\sigma^a, \sigma^b) \supseteq \mathcal{F}_p^R(\sigma^k, \sigma^b) = \bigwedge^p \frac{N_R}{T_R F_{\sigma^b}^{\perp}}$. As we also know that $\mathcal{F}_p^R(\sigma^a, \sigma^b) \subseteq \bigwedge^p \frac{N_R}{T_R F_{\sigma^b}^{\perp}}$,

we finally get:

$$\forall p < k, \quad \forall p < a \leq b, \quad \forall \sigma^a \subseteq \sigma^b, \quad \mathcal{F}_p(\sigma^a, \sigma^b) = \bigwedge^p \frac{N_R}{T_R F^{\perp}_{\sigma^b}} \simeq R^{\binom{\dim F_{\sigma^b}}{p}}.$$

We have obtained that for any $p < k$, $\mathcal{F}_p(*, \sigma^b)$ is a constant cosheaf for the $\sigma^a \subseteq \sigma^b$ with $p < a \leq b$. We have the following isomorphisms:

$$\forall p < k, \quad \forall p < a \leq b, \quad C_{b-a}(\mathcal{F}^R_p(*, \sigma^b)) = C_{b-a}(\sigma^b; R^{\binom{\dim F_{\sigma^b}}{p}}).$$

where $C_*(\sigma^b; R^{\binom{\dim F_{\sigma^b}}{p}})$ is the classical simplicial chains of the simplex σ^b with coefficients in $R^{\binom{\dim F_{\sigma^b}}{p}}$. These isomorphisms commute with the differential and we obtain isomorphisms with the simplicial homology of σ^b:

$$\forall p < k, \quad \forall p + 1 < a \leq b, \quad H_{b-a}(\mathcal{F}^R_p(*, \sigma^b)) = H_{b-a}(\sigma^b; R^{\binom{r}{p}}) = 0.$$

□

3.8.2 The cohomological spectral sequence: proof of Lemma 3.7.3

Let us begin by recalling the following identity:

$$E_1^{a,b}(\delta_2) := \bigoplus_{\sigma^a \in \Gamma} H^{b-a}(\mathcal{F}^p_R(\sigma^a, *)).$$

It shows that to prove Lemma 3.7.1, we just need to study the properties of the cohomology groups $H^{b-a}(\mathcal{F}^p_R(\sigma^a, *))$. In particular, as the Lemma 3.7.3 states that $E_1^{a,b}(\delta_2) = 0$ for every $b \neq n$, we have to prove that $H^{b-a}(\mathcal{F}^p_R(\sigma^a, *)) = 0$ for every $b \neq n$. To prove that we need an assumption of local R-non-singularity on the lattice polytope P. It is our following proposition:

Proposition 3.8.5. *Let $\sigma^a \in \Gamma$. Recall that F_{σ^a} denotes the minimal face of the polytope P containing σ^a. If the polytope P is R-non-singular on its face F_{σ^a} (see Definition 3.2.2), then the cohomology $H^*(\mathcal{F}^p_R(\sigma^a, *))$ is torsion-free and satisfies:*

$$H^{b-a}(\mathcal{F}^p_R(\sigma^a, *)) = \begin{cases} 0 & \text{if } b \neq n \\ \mathcal{F}^{p-n+\dim F_{\sigma^a}}_R(\sigma^a, \sigma^a) & \text{if } b = n \end{cases}.$$

The isomorphism $H^{n-a}(\mathcal{F}^p_R(\sigma^a, *)) = \mathcal{F}^{p-n+\dim F_{\sigma^a}}_R(\sigma^a, \sigma^a)$ is canonical up to a choice of a generator of $\bigwedge^{n-\dim F_{\sigma^a}} \frac{M_R}{T_R F_{\sigma^a}}$.

Brugallé, Lopez de Medrano and Rau have proven this proposition in [BMR 2024] (in their proof of Lemma 3.1), in the case of a non-singular lattice polytope, for the integral domain $R = \mathbb{F}_2$ and for homology:

Proposition 3.8.6. *[BMR 2024] For a non-singular lattice polytope P with a triangulation Γ, we have:*

$$H_{b-a}(\mathcal{F}^{\mathbb{F}_2}_p(\sigma^a, *)) = \begin{cases} 0 & \text{if } b \neq n \\ \mathcal{F}^{\mathbb{F}_2}_{p-n+\dim F_{\sigma^a}}(\sigma^a, \sigma^a) & \text{if } b = n \end{cases}.$$

As their article [BMR 2024] concerns only primitive triangulations, their proof is written for a primitive triangulation Γ. However the primitivity of Γ is used nowhere in their proof, so it is also valid for a non-necessarily primitive triangulation Γ, like it is stated in Proposition 3.8.6. Their cosheaves $\mathcal{F}_p^{\mathbb{F}_2}$ are also defined for a coarser cellular complex, labeled by the poset $\Xi = \{(F, \sigma),\ F \text{ face of } P,\ \sigma \in \Gamma,\ \dim\sigma \geq 1\}$. However, Jules Chenal has proven in his PhD thesis [Ch 2024] that the resulting homology is the same as with the cubical subdivision, labeled by the poset $\Omega = \{(\sigma, \tau),\ \sigma, \tau \in \Gamma,\ \sigma \subseteq \tau,\ \dim\sigma \geq 1\}$ that we use in this Chapter. It is the reason why we have written this Proposition 3.8.6 of Brugallé, Lopez de Medrano and Rau in terms of the cubical subdivision.

Some arguments of the proof of Brugallé, Lopez de Medrano and Rau of Proposition 3.8.6 can be adapted to our context to prove our Proposition 3.8.5. Hypothesis of R-non-singularity of the polytope on the face F_{σ^a} can be used instead of their stronger hypothesis of non-singularity of P and we obtain the following identity:

$$H^{b-a}(\mathcal{F}_R^p(\sigma^a, *)) = \bigoplus_{r+t=p} \mathcal{F}_R^r(\sigma^a, \sigma^a) \otimes H^b(\mathcal{G}_{R,\sigma^a}^t) \tag{3.9}$$

where $\mathcal{G}_{R,\sigma^a}^t$ is a cellular sheaf on the poset $L_{\sigma^a} := \{F \text{ face of } P,\ \ F \supseteq \sigma^a\}$ graded by the dimension of the faces of P, defined by $\mathcal{G}_{R,\sigma^a}^t(F) := \bigwedge^t \frac{T_R F}{T_R F_{\sigma^a}}$, whose sheaves morphisms are induced by the inclusions $T_R F \subseteq T_R G$ for any incident faces $F \subseteq G$ of P containing σ^a.

Now to conclude the proof of our Proposition 3.8.5, we have to prove that the cohomology $H^b(\mathcal{G}_{R,\sigma^a}^t)$ vanishes except if $b = n = t + \dim F_{\sigma^a}$, which is the purpose of the following lemma:

Lemma 3.8.7. *If the polytope P is non-singular of the face F_{σ^a}, then we have for any integers t, b:*

$$H^b(\mathcal{G}_{R,\sigma^a}^t) = \begin{cases} 0 & \text{if } b \neq n \text{ or } n \neq t + \dim F_{\sigma^a} \\ R & \text{if } b = n = t + \dim F_{\sigma^a} \end{cases} .$$

We cannot use the same strategy as Brugallé, Lopez de Medrano, Rau in [BMR 2024], who use some properties of non-singular projective toric varieties, because here the polytope P is not supposed to be non-singular but only R-non-singular on the face F_{σ^a}. However, fixing a basis of $\mathbb{Z}^n \otimes R$ adapted to the definition of R-non-singularity of the polytope P on the face F_{σ^a} (see Definition 3.2.2), we are able to compute this cohomology by comparison with the simplicial cohomology of several abstract simplices. It is the purpose of the following proof:

Proof. Proof of Lemma 3.8.7: Let us fix the set of vectors $\{e_1, ..., e_{n-\dim F_{\sigma^a}}\}$ of M_R adapted for the definition of R-non-singularity of the polytope P on the face F_{σ^a} (see Definition 3.2.2). Using Definition 3.2.2 of local R-non-singularity and Remark 3.2.4, we have a bijection of sets:

$$\begin{aligned} \gamma : \{F \text{ face of } P,\ F \supseteq \sigma^a\} &\simeq \mathcal{P}(\{1, ..., n - \dim F_{\sigma^a}\}) \\ F \qquad &\mapsto \quad I \text{ such that } T_R F = T_R F_{\sigma^a} \oplus \bigoplus_{i \in I} R e_i \end{aligned}$$

where $\mathcal{P}(\{1,...,n-\dim F_{\sigma^a}\})$ denotes the power set of $\{1,...,n-\dim F_{\sigma^a}\}$.

Let us denote by $\{e'_1,...,e'_{n-\dim F_{\sigma^a}}\}$ the image of $\{e_1,...,e_{n-\dim F_{\sigma^a}}\}$ by the projection $M_R \twoheadrightarrow \frac{M_R}{T_R F_{\sigma^a}}$. Then for any face F of P containing σ^a the family $(\bigwedge_{i\in I} e'_i)_{I\subseteq\gamma(F),\ \mathrm{Card}I=t}$ forms a basis of $\mathcal{G}^t_{R,\sigma^a}(F)$. It induces the following decomposition of the cochain complex $C^b(\mathcal{G}^t_{R,\sigma^a})$:

$$C^b(\mathcal{G}^t_{R,\sigma^a}) = \bigoplus_{I\subseteq\{1,...,n-\dim F_{\sigma^a}\},\ \mathrm{Card}I=t} C^b(D_{\gamma^{-1}(I)};R)\otimes\bigwedge_{i\in I} e'_i. \tag{3.10}$$

where $D_{\gamma^{-1}(I)} := \{F \text{ face of } P \text{ containing } \gamma^{-1}(I)\}$ and $C^b(D_{\gamma^{-1}(I)};R)$ are the cochains with coefficients in R over $D_{\gamma^{-1}(I)}$ ordered by inclusion and graded by the dimension, with a balancing signature inherited from the balancing signature of the polytope P as a polyhedral complex. Using the bijection γ, $D_{\gamma^{-1}(I)}$ is isomorphic as a poset to the poset of the subsets of $\{1,...,n-\dim F_{\sigma^a}\}$ which contain I. However, if $n > \dim F_{\sigma^a}+t$, the reduced simplicial cochains of an abstract simplex $\Delta^{n-\dim F_{\sigma^a}-t-1}$ of dimension $n-\dim F_{\sigma^a}-t-1$ can be labeled by the poset of subsets of $\{1,...,n-\dim F_{\sigma^a}-t\}$, which is isomorphic to the poset of the subsets of $\{1,...,n-\dim F_{\sigma^a}\}$ which contains I (because $\mathrm{Card}I=t$). Then if $n > \dim F_{\sigma^a}+t$, we can compute the cohomology of the polyhedral complex $D_{\gamma^{-1}(I)}$ using the reduced simplicial cohomology of an abstract simplex $\Delta^{n-\dim F_{\sigma^a}-\mathrm{Card}I-1}$ of dimension $n-\dim F_{\sigma^a}-t-1$:

$$\forall n > \dim F_{\sigma^a}+t,\quad H^b(D_{\gamma^{-1}(I)};R) = \tilde{H}^{b-\dim F_{\sigma^a}-t-1}(\Delta^{n-\dim F_{\sigma^a}-t-1};R) = 0.$$

The difference between the degrees b and $b-\dim F_{\sigma^a}-t-1$ of the two cohomologies comes from the fact that there is a shift between the gradings of the corresponding cochain complexes.

If $n < \dim F_{\sigma^a}+t$, the cohomology $H^b(D_{\gamma^{-1}(I)};R)$ also vanishes because $D_{\gamma^{-1}(I)}$ is empty. In the case $n = \dim F_{\sigma^a}+t$, we have $D_{\gamma^{-1}(I)} = \{P\}$, and P is of dimension n, then we obtain that the cohomology groups $H^b(D_{\gamma^{-1}(I)};R)$ vanish in every degree b except in degree $b=n$ where the cohomology is R. Finally we obtain the general identity:

$$H^b(D_{\gamma^{-1}(I)};R) = \begin{cases} R & \text{if } b = n = \dim F_{\sigma^a}+t \\ 0 & \text{if } b\neq n \text{ or } n \neq \dim F_{\sigma^a}+t \end{cases}.$$

It concludes the proof using equation 3.10. □

Now we can prove Proposition 3.8.5 using Lemma 3.8.7:

Proof. Proof of Proposition 3.8.5: From equation 3.9 and Lemma 3.8.7, we obtain:

$$H^{b-a}(\mathcal{F}^p_R(\sigma^a,*)) = \begin{cases} 0 & \text{if } b\neq n \\ \mathcal{F}^{p-n+\dim F_{\sigma^a}}_R(\sigma^a,\sigma^a)\otimes H^n(\mathcal{G}^{n-\dim F_{\sigma^a}}_{R,\sigma^a}) & \text{if } b=n \end{cases}.$$

In the case $b=n$, we can choose a generator of $H^n(\mathcal{G}^{n-\dim F_{\sigma^a}}_{R,\sigma^a}) = R$ (which corresponds to a choice of a generator of $\bigwedge^{n-\dim F_{\sigma^a}}\frac{M_R}{T_R F_{\sigma^a}}$), which concludes the proof. □

We can now conclude the proof of Lemma 3.7.3, which was the purpose of this subsection. As we assume that the polytope P is R-non-singular, then it is R-non-singular on every one of its faces. Then we obtain:

$$E_1^{a,b}(\delta_2) := \bigoplus_{\sigma^a \in \Gamma} H^{b-a}(\mathcal{F}_R^p(\sigma^a, *)) = \begin{cases} \bigoplus_{\sigma^a \in \Gamma} \mathcal{F}_R^{p-n+\dim F_{\sigma^a}}(\sigma^a, \sigma^a) & \text{if } b = n \\ 0 & \text{if } b \neq n \end{cases}$$
$$= \begin{cases} \text{coker } \delta_2^{a,n-1} & \text{if } b = n \\ 0 & \text{if } b \neq n \end{cases}.$$

3.8.3 The cap product with the fundamental class: proof of Lemmas 3.7.7, 3.7.8 and 3.7.9

Lemmas 3.7.7, 3.7.8 and 3.7.9 concern the relations between the map $\phi^q_{(p)}$ (defined in Definition 3.6.1) and the differentials of the chain complexes of cosheaves $\mathcal{F}_p^R$ and the cochain complex of sheaves $\mathcal{F}_R^p$. We already know from the last section the rank of ker $d_{1,n-q}^{1,(n-1-p)}$ if the triangulation is R-primitive (Proposition 3.8.2), and the rank of coker $\delta_2^{n-q,n-1}$ if the polytope P is R-non-singular (Proposition 3.8.5). Thus to prove the Lemma 3.7.7, we want to compute the rank of im $\phi^q_{(p)}$, to prove that it is included into ker $d_{1,n-q}^{1,(n-1-p)}$, and to prove the Lemma 3.7.8 we want to compute the rank of ker $\phi^q_{(p)} \cap C^{n-q,n}(\mathcal{F}_R^p)$ and to prove that it contains im $\delta_{2,(p)}^{n-q,n-1}$. However, those two computations are related by the rank theorem, and we just need to compute one of the two. Let us prove the following proposition:

Proposition 3.8.8. *Let us fix a simplex $\sigma^a \in \Gamma$. Assume that the triangulation satisfies Hypothesis 1 for the integral domain R (see Definition 3.3.2). Let us denote by $V = \sum_{\sigma^1 \subseteq \sigma^a} T_R\sigma^1$. Then the map $\phi_{\sigma^a,(p)}$ (see Definition 3.6.1) satisfies:*

$$\text{rank } \phi_{\sigma^a,(p)} = \binom{\dim F_{\sigma^a}}{p-n+\dim F_{\sigma^a}} - \binom{\dim F_{\sigma^a} - \dim V}{p-n+\dim F_{\sigma^a}-\dim V}.$$

Moreover, if we assume that the triangulation satisfies Hypothesis 2 for the integral domain R (see Definition 3.3.2), the image of $\phi_{\sigma^a,(p)}$ is a saturated sub-R-module of $C_{a-1}(\mathcal{F}^R_{n-1-p}(*, \sigma^a))$ (see Definition 3.8.1 for the saturation of a sub-module).

In the following, we fix a simplex $\sigma^a \in \Gamma$. For any $\sigma^n \supseteq \sigma^a$, we have the surjective map $(\iota^{\sigma^n}_{\sigma^a,\sigma^n})^* : \bigwedge^p M_R \twoheadrightarrow \mathcal{F}_R^p(\sigma^a, \sigma^n)$. Then the image of $\phi_{\sigma^a,(p)}$ is equal to the image of the following map:

$$\psi : \bigoplus_{\sigma^n \supseteq \sigma^a} \bigwedge^p M_R \to C_{a-1}(\mathcal{F}_{n-1-p}(*, \sigma^a))$$
$$\bigoplus_{\sigma^n \supseteq \sigma^a} \beta_{\sigma^n} \mapsto \phi_{\sigma^a,(p)}\left(\bigoplus_{\sigma^n \supseteq \sigma^a} (\iota^{\sigma^n}_{\sigma^a,\sigma^n})^*(\beta_{\sigma^n}) \right).$$

By definition of $\phi_{\sigma^a,(p)}$ (Definition 3.6.1) and by definition of the cap product (Proposition 3.4.2), we have:

$$\psi\left(\bigoplus_{\sigma^n\supseteq\sigma^a}\beta_{\sigma^n}\right)=\bigoplus_{\sigma^1\subseteq\sigma^a}\sum_{\sigma^n\supseteq\sigma^a}\pi^{\sigma^a}(\beta_{\sigma^n}.[\Omega]_{\sigma^1,\sigma^n})=\bigoplus_{\sigma^1\subseteq\sigma^a}\pi^{\sigma^a}\left(\sum_{\sigma^n\supseteq\sigma^a}\beta_{\sigma^n}.[\Omega]_{\sigma^1,\sigma^n}\right)$$

where the dot denotes the contraction morphism (Definition 3.4.1), $[\Omega]_{\sigma^1,\sigma^n}$ denotes the (σ^1,σ^n)-coordinate of $[\Omega]$, and π^{σ^a} denotes the projection $\bigwedge^{n-1-p}N_R\twoheadrightarrow\bigwedge^{n-1-p}\frac{N_R}{T_RF_{\sigma^a}^{\perp}}$.

Recall that the element $[\Omega]$ is defined by $[\Omega]:=\bigoplus_{\sigma^1,\sigma^n}|\sigma^1|_r1_{\sigma^1,\sigma^n}\otimes 1_R$, with the $1_{\sigma^1,\sigma^n}$ generators of the $\mathcal{F}^R_{n-1}(\sigma^1,\sigma^n)$, and satisfying $d_{n-1}([\Omega])=0$. Moreover, in the proof of Lemma 3.5.2, the element $1_{\sigma^1,\sigma^n}$ was defined in the following way:

$$1_{\sigma^1,\sigma^n}=\mu(\sigma^n)g(o(\sigma^1))$$

where $o(\sigma),\ \sigma\in\Gamma$ is a family of orientation fixed on the simplices of Γ; compatible with the balancing signature of the simplicial complex Γ ; where an orientation $o(M)$ is fixed on the ambient lattice M and $\mu(\sigma^n)$ is defined by $\mu(\sigma^n)=1$ if $o(\sigma^n)=o(M)$, and $\mu(\sigma^n)=-1$ if $o(\sigma^n)=-o(M)$; $g(o(\sigma^1))$ is a generator of $\bigwedge^{n-1}T_{\mathbb{Z}}\sigma^{1\perp}$ obtained canonically from the orientation $o(\sigma^1)$ of σ^1 and the fixed orientation $o(M)$ of M.

We obtain:

$$\psi\left(\bigoplus_{\sigma^n\supseteq\sigma^a}\beta_{\sigma^n}\right)=\bigoplus_{\sigma^1\subseteq\sigma^a}\pi^{\sigma^a}\left(\left[\sum_{\sigma^n\supseteq\sigma^a}\mu(\sigma^n)\beta_{\sigma^n}\right].(|\sigma^1|_rg(o(\sigma^1))\otimes 1_R)\right).$$

It shows that the map ψ admits a factorisation $\psi=\overline{\psi}\circ\epsilon$ where:

$$\epsilon:\bigoplus_{\sigma^n\supseteq\sigma^a}\bigwedge^p M_R\twoheadrightarrow\bigwedge^p M_R\qquad;\quad\overline{\psi}:\bigwedge^p M_R\twoheadrightarrow C_{a-1}(\mathcal{F}_{n-1-p}(*,\sigma^a))$$

$$\bigoplus_{\sigma^n\supseteq\sigma^a}\beta_{\sigma^n}\mapsto\sum_{\sigma^n\supseteq\sigma^a}\mu(\sigma^n)\beta_{\sigma^n}\qquad\alpha\mapsto\bigoplus_{\sigma^1\subseteq\sigma^a}\pi^{\sigma^a}\left(\alpha.(|\sigma^1|_rg(o(\sigma^1))\otimes 1_R)\right).$$

Since for every $\sigma^n\in\Gamma$ the term $\mu(\sigma^n)$ is invertible in R (because it is $+1$ or -1), the map ϵ is surjective and we have $\text{im }\overline{\psi}=\text{im }\psi=\text{im }\phi_{\sigma^a,(p)}$. Thus it is enough to study the map $\overline{\psi}$ to prove proposition 3.8.8:

Proof. Proof of Proposition 3.8.8.

Let us fix a basis $(f_1,...,f_{\text{rank}V})$ of V, and let $\{f_{\text{rank}V+1},...,f_{\dim F_{\sigma^a}}\}$ a set of vectors which completes it, to obtain a basis $(f_1,...,f_{\dim F_{\sigma^a}})$ of $T_RF_{\sigma^a}$. Let $\{e_1,...,e_{n-\dim F_{\sigma^a}}\}$ a set of vectors which complete $(f_1,...,f_{\dim F_{\sigma^a}})$, to obtain a basis of M_R. Let us denote by $(f_1^*,...,f_{\dim F_{\sigma^a}}^*,e_1^*,...,e_{n-\dim F_{\sigma^a}}^*)$ the dual basis of N_R. For every $\sigma^1\subseteq\sigma^a$, let us denote by u_{σ^1} a generator of $T_R\sigma^1$. We can complete it to obtain a basis $(u_{\sigma^1},v_1^{\sigma^1},...,v_{\text{rank}V-1}^{\sigma^1})$ of V, and then the following basis of M_R:

$$(u_{\sigma^1},v_1^{\sigma^1},...,v_{\text{rank}V-1}^{\sigma^1},f_{\text{rank}V+1},...,f_{\dim F_{\sigma^a}},e_1,...e_{n-\dim F_{\sigma^a}}).$$

The dual basis is of the form:

$$(u^*_{\sigma^1}, (v_1^{\sigma^1})^*, ..., (v^{\sigma^1}_{\text{rank}V-1})^*, f^*_{\text{rank}V+1}, ..., f^*_{\dim F_{\sigma^a}}, e^*_1, ...e^*_{n-\text{rank}V})$$

where the vectors $f^*_{\text{rank}V+1}, ..., f^*_{\dim F_{\sigma^a}}, e^*_1, ...e^*_{n-\dim F_{\sigma^a}}$ are the same as the ones that appear in the dual basis $(f^*_1, ..., f^*_{\dim F_{\sigma^a}}, e^*_1, ..., e^*_{n-\dim F_{\sigma^a}})$.

Then $((v_1^{\sigma^1})^*, ..., (v^{\sigma^1}_{\dim F_{\sigma^a}-1})^*, f^*_{\text{rank}V+1}, ..., f^*_{\dim F_{\sigma^a}}, e^*_1, ..., e^*_{n-\dim F_{\sigma^a}})$ is a basis of $T_R\sigma^{1\perp}$. Then $(v_1^{\sigma^1})^* \wedge ... \wedge (v^{\sigma^1}_{\dim F_{\sigma^a}-1})^* \wedge f^*_{\text{rank}V+1} \wedge ... \wedge f^*_{\dim F_{\sigma^a}} \wedge e^*_1 \wedge ... \wedge e^*_{n-\dim F_{\sigma^a}}$ and $g((o(\sigma^1)) \otimes 1_R$ are both generators of $\bigwedge^{n-1} T_R\sigma^{1\perp}$, thus there exists an invertible element λ_{σ^1} of R such that:

$$(v_1^{\sigma^1})^* \wedge ... \wedge (v^{\sigma^1}_{\dim F_{\sigma^a}-1})^* \wedge f^*_{\text{rank}V+1} \wedge ... \wedge f^*_{\dim F_{\sigma^a}} \wedge e^*_1 \wedge ... \wedge e^*_{n-\dim F_{\sigma^a}} = \lambda_{\sigma^1} g(o(\sigma^1)) \otimes 1_R.$$

Let $\alpha \in \bigwedge^p M_R$. We can write:

$$\alpha = \sum_{k=0}^{p} \sum_{A \in \mathcal{P}_{p-k}(\{1,...,\dim F_{\sigma^a}\})} \sum_{B \in \mathcal{P}_k(\{1,...,n-\dim F_{\sigma^a}\})} \alpha_{A,B} \bigwedge_{i \in A} f_i \wedge \bigwedge_{j \in B} e_i$$

where $\mathcal{P}_l(\{1, ..., s\})$ denotes the set of subsets of cardinal l of $\{1, ..., s\}$. However, for every $k \neq n - \dim F_{\sigma^a}$, $B \in \mathcal{P}_k(\{1, ..., n - \dim F_{\sigma^a}\})$, and $A \in \mathcal{P}_{p-k}(\{1, ..., \dim F_{\sigma^a}\})$, for every $\sigma^1 \subseteq \sigma^a$ we have:

$$\pi^{\sigma^a}\left(\left[\bigwedge_{i \in A} f_i \wedge \bigwedge_{j \in B} e_i\right] . \left[\bigwedge_{i=1}^{\dim F_{\sigma^a}-1} (v_i^{\sigma^1})^* \wedge \bigwedge_{i=\text{rank}V+1}^{\dim F_{\sigma^a}} f_i^* \wedge \bigwedge_{j=1}^{n-\dim F_{\sigma^a}} e_j^*\right]\right) = 0.$$

Thus every coordinate of α contributes to zero to $\overline{\psi}(\alpha)$ except the ones for which $k = n - \dim F_{\sigma^a}$, which corresponds to $B = \{1, ..., n - \dim F_{\sigma^a}\}$ Thus we obtain:

$$\overline{\psi}(\alpha) = \overline{\psi}\left(\sum_{A \in \mathcal{P}_{p-n+\dim F_{\sigma^a}}(\{1,...,\dim F_{\sigma^a}\})} \alpha_{A,\{1,...,n-\dim F_{\sigma^a}\}} \bigwedge_{i \in A} f_i \wedge \bigwedge_{j=1}^{n-\dim F_{\sigma^a}} e_i\right).$$

Let us denote by $\overline{\alpha} := \sum_{A \in \mathcal{P}_{p-n+\dim F_{\sigma^a}}(\{1,...,\text{rank}V\})} \alpha_{A,\{1,...,n-\dim F_{\sigma^a}\}} \bigwedge_{i \in A} f_i$. As we assume the triangulation to satisfy Hypothesis 1 for the integral domain R, we have for every $\sigma^1 \subseteq \sigma^a$ that $|\sigma^1|_r$ does not vanish in the integral domain R. Thus we have:

$$\overline{\psi}(\alpha) = 0 \iff \forall \sigma^1 \subseteq \sigma^a,\ \overline{\alpha}.\left[\bigwedge_{i=1}^{\text{rank}V-1} (v_i^{\sigma^1})^* \wedge \bigwedge_{i=\text{rank}V+1}^{\dim F_{\sigma^a}} f_i^*\right] = 0.$$

Let us identify the vectors $(v_i^{\sigma^1})^*$ and the vector $u^*_{\sigma^1}$, for every $\sigma^1 \subseteq \sigma^a$, with their projection in $\frac{N_R}{V^\perp} = V^*$. Then for every $\sigma^1 \subseteq \sigma^a$, we have that $(u^*_{\sigma^1}, (v_1^{\sigma^1})^*, ..., (v^{\sigma^1}_{\text{rank}V-1})^*)$ forms a basis of V^*, dual to the basis $(u_{\sigma^1}, v_1^{\sigma^1}, ..., v^{\sigma^1}_{\dim F_{\sigma^a}})$ of V. Then the vector $\bigwedge_{i=1}^{\text{rank}V-1}(v_i^{\sigma^1})^*$ is uniquely determined from the vector u_{σ^1}, up to an invertible factor of R. We have the following equality of rank:

$$\text{rank} \sum_{\sigma^1 \subseteq \sigma^a} \bigwedge_{i=1}^{\text{rank}V-1} (v_i^{\sigma^1})^* = \text{rank} \sum_{\sigma^1 \subseteq \sigma^a} Ru_{\sigma^1} = \text{rank}\ V = \text{rank} \bigwedge^{\text{rank}V-1} V^*.$$

Thus we obtain that the family $(\bigwedge_{i=1}^{\mathrm{rank}V-1}(v_i^{\sigma^1})^*)_{\sigma^1\subseteq\sigma^a}$ generates $\bigwedge^{\mathrm{rank}V-1}V^*$. Then we have:

$$\overline{\psi}(\alpha)=0 \iff \forall x\in \bigwedge^{\mathrm{rank}V-1}\left(\sum_{i=1}^{\mathrm{rank}V} Rf_i^*\right),\ \ \overline{\alpha}.\left[x\wedge \bigwedge_{i=\mathrm{rank}V+1}^{\dim F_{\sigma^a}} f_i^*\right]=0$$

$$\iff \forall j\in\{1,...,\mathrm{rank}\ V\},\ \ \overline{\alpha}.\left[\bigwedge_{i\in\{1,...,\mathrm{rank}V\}\backslash\{j\}} f_i^*\wedge \bigwedge_{i=\mathrm{rank}V+1}^{\dim F_{\sigma^a}} f_i^*\right]=0$$

$$\iff \forall A\in\mathcal{P}_{p-n+\dim F_{\sigma^a}}(\{1,...,\dim F_{\sigma^a}\}), A\not\supseteq\{1,...,\mathrm{rank}V\} \implies \alpha_{A,\{1,...n-\dim F_{\sigma^a}\}}=0.$$

Then, we obtain:

$$\mathrm{rank\ ker}\ \overline{\psi}=\mathrm{rank}\bigwedge^p M_R-\left[\binom{\dim F_{\sigma^a}}{p-n+\dim F_{\sigma^a}}-\binom{\dim F_{\sigma^a}-\mathrm{rank}V}{p-n+\dim F_{\sigma^a}-\mathrm{rank}V}\right].$$

Finally, we obtain:

$$\mathrm{rank}\ \phi_{\sigma^a,(p)}=\mathrm{rank}\ \overline{\psi}=\binom{\dim F_{\sigma^a}}{p-n+\dim F_{\sigma^a}}-\binom{\dim F_{\sigma^a}-\mathrm{rank}V}{p-n+\dim F_{\sigma^a}-\mathrm{rank}V}.$$

Now let us assume that the triangulation satisfies Hypothesis 2 for the integral domain R, and let us prove that the image of $\phi_{\sigma^a,(p)}$ is saturated in $C_{a-1}(\mathcal{F}^R_{n-1-p}(*,\sigma^a))$. We just need to prove that the image of $\overline{\psi}$ is saturated. Let $y\in C_{a-1}(\mathcal{F}^R_{n-1-p}(*,\sigma^a))$ and $\mu\in R\backslash\{0\}$ such that $\overline{\psi}(\alpha)=\mu y$. For $\sigma^1\subseteq\sigma^a$, using the properties of contraction, we have:

$$\mu y_{\sigma^1}=\pi^{\sigma^a}\left(\alpha.(|\sigma^1|_r g(o(\sigma^1))\otimes 1_R)\right)=\pm\overline{\alpha}.\left[\frac{|\sigma^1|_r}{\lambda_{\sigma^1}}\bigwedge_{i=1}^{\mathrm{rank}V-1}(v_i^{\sigma^1})^*\wedge\bigwedge_{i=\mathrm{rank}V+1}^{\dim F_{\sigma^a}} f_i^*\right].$$

However we have already proven that the family $(\bigwedge_{i=1}^{\mathrm{rank}V-1}(v_i^{\sigma^1})^*)_{\sigma^1\subseteq\sigma^a}$ generates the R-module $\bigwedge^{n-1}\sum_{i=1}^{\mathrm{rank}V}Rf_i^*$. Then for any $1\le i\le\dim V$ there exists a family of coefficients $(a^i_{\sigma^1})_{\sigma^1\subseteq\sigma^a}$ such that:

$$\bigwedge_{j\in\{1,...,\mathrm{rank}V\}\backslash\{i\}} f_j^*=\sum_{\sigma^1\subseteq\sigma^a}a^i_{\sigma^1}\bigwedge_{i=1}^{\mathrm{rank}V-1}(v_i^{\sigma^1})^*.$$

Then for any $A\in\mathcal{P}_{p-n+\dim F_{\sigma^a}}(\{1,...,\dim F_{\sigma^a}\})$, if $A\not\supseteq\{1,...,\mathrm{rank}V\}$, there exists $i\in\{1,...,\mathrm{rank}V\}$ such that $i\notin A$ and we have:

$$\alpha_{A,\{1,...,n-\dim F_{\sigma^a}\}}\bigwedge_{j\in\{1,...\mathrm{rank}V\}\backslash(A\cup\{i\})} f_j^*=\sum_{\sigma^1\subseteq\sigma^a}\pm a^i_{\sigma^1}\frac{\lambda_{\sigma^1}}{|\sigma^1|_r}\mu y_{\sigma^1}.$$

We have used here the assumption that the triangulation satisfies Hypothesis 2 for the integral domain R, which enables us to consider an inverse of $|\sigma^1|_r$ in R.

From the previous equation, we have that μ divides $\alpha_{A,\{1,...,n-\dim F_{\sigma^a}\}}$ for any subset $A\in\mathcal{P}_{p-n+\dim F_{\sigma^a}}(\{1,...,\dim F_{\sigma^a}\})$ such that $A\not\supseteq\{1,...,\mathrm{rank}V\}$. However, we already

know that if $A \supseteq \{1, ..., \text{rank}V\}$, then the term $\alpha_{A,\{1,...,n-\dim F_{\sigma^a}\}}$ contributes to zero in $\phi_{\sigma^a,(p)}(\alpha)$. Then, for any $A \in \mathcal{P}_{p-n+\dim F_{\sigma^a}}(\{1, ..., \dim F_{\sigma^a}\})$ such that $A \not\supseteq \{1, ..., \text{rank}V\}$ we can define β_A the element of R such that $\mu\beta_A = \alpha_{A,\{1,...,\dim F_{\sigma^a}\}}$ and we obtain:

$$\phi_{\sigma^a,(p)}\left(\sum_{A\in\mathcal{P}_{p-n+\dim F_{\sigma^a}}(\{1,...,\dim F_{\sigma^a}\}),\ A\not\supseteq\{1,...\text{rank}V\}} \beta_A \bigwedge_{i\in A} f_i^* \wedge \bigwedge_{j=1}^{n-\dim F_{\sigma^a}} e_j^*\right) = y.$$

It proves that the image of $\phi_{\sigma^a,(p)}$ is saturated in $C_{a-1}(\mathcal{F}^R_{n-1-p}(*, \sigma^a))$. □

Lemma 3.8.9. *Without hypothesis on the triangulation, we have:*

$$d_{a-1}^{*,\sigma^a,(n-1-p)} \circ \phi_{\sigma^a,(p)} = 0.$$

Proof. It is enough to prove that $d_{a-1}^{*,\sigma^a,(n-1-p)} \circ \psi = 0$, with the map ψ defined previously. We have:

$$\begin{aligned} d_{a-1}^{*,\sigma^a,(n-1-p)} \circ \psi\left(\bigoplus_{\sigma^n\supseteq\sigma^a} \beta_{\sigma^n}\right) &= \bigoplus_{\sigma^2\subseteq\sigma^a} (-1)^{a-1} \sum_{\sigma^1\subset\sigma^2} \sum_{\sigma^n\supseteq\sigma^a} \rho(\sigma^1, \sigma^2)\pi^{\sigma^a}_{\sigma^n}\left(\beta_{\sigma^n}.[\Omega]_{\sigma^1,\sigma^n}\right) \\ &= \bigoplus_{\sigma^2\subseteq\sigma^a} \pi^{\sigma^a}\left(\sum_{\sigma^n\supseteq\sigma^a}\left[\beta_{\sigma^n}.\left(d_{n-1}^{1,(n-1)}([\Omega])\right)_{\sigma^2,\sigma^n}\right]\right). \end{aligned}$$

However, we know that without any hypothesis (Proposition 3.5.2) we have:

$$d_{n-1}^{1,(n-1)}([\Omega]) = 0.$$

It concludes the proof. □

Lemma 3.8.10. *Without hypothetis on the triangulation we have:*

$$\phi_{\sigma^a,(p)} \circ \delta^{n-a-1}_{\sigma^a,*,(p)} = 0.$$

Proof. Let $\alpha \in C^{n-a-1}(\mathcal{F}_p(\sigma^a, *))$ We have:

$$\delta^{n-a-1}_{\sigma^a,*,(p)}(\alpha) = \bigoplus_{\sigma^a\subseteq\sigma^n} \sum_{\sigma^a\subseteq\sigma^{n-1}\subset\sigma^n} \rho(\sigma^{n-1}, \sigma^n)(\pi^{\sigma^a,\sigma^{n-1}}_{\sigma^a,\sigma^n})^*(\alpha_{\sigma^a,\sigma^{n-1}}).$$

For every $\sigma^a \subseteq \sigma^{n-1} \subseteq \sigma^n$, we know that $(\iota^{\sigma^{n-1}}_{\sigma^a,\sigma^{n-1}})^* : \bigwedge^p T_R F_{\sigma^{n-1}} \twoheadrightarrow \mathcal{F}^p_R(\sigma^a, \sigma^{n-1})$ is a surjective map. Thus there exists $\beta_{\sigma^{n-1}} \in \bigwedge^p T_R F_{\sigma^{n-1}}$ such that $(\iota^{\sigma^{n-1}}_{\sigma^a,\sigma^{n-1}})^*(\beta_{\sigma^{n-1}}) = \alpha_{\sigma^a,\sigma^{n-1}}$. Moreover, as $\pi^{\sigma^{n-1}}_{\sigma^n}$ is just the inclusion $\bigwedge^p T_R F_{\sigma^{n-1}} \hookrightarrow \bigwedge^p M_R$, we have:

$$\begin{aligned} (\pi^{\sigma^a,\sigma^{n-1}}_{\sigma^a,\sigma^n})^*(\alpha_{\sigma^a,\sigma^n}) &= (\pi^{\sigma^a,\sigma^{n-1}}_{\sigma^a,\sigma^n})^* \circ (\iota^{\sigma^{n-1}}_{\sigma^a,\sigma^{n-1}})^*(\beta_{\sigma^n}) \\ &= (\iota^{\sigma^n}_{\sigma^a,\sigma^n})^* \circ (\pi^{\sigma^{n-1}}_{\sigma^n})^*(\beta_{\sigma^n}) = (\iota^{\sigma^n}_{\sigma^a,\sigma^n})^*(\beta_{\sigma^n}). \end{aligned}$$

Then, we obtain:

$$\begin{aligned}
\phi_{\sigma^a,(p)} \circ \delta^{n-a-1}_{\sigma^a,*,(p)}(\alpha) &= \bigoplus_{\sigma^1 \subseteq \sigma^a} \sum_{\sigma^n \supseteq \sigma^a} \pi^{\sigma^a}\left(\left[\sum_{\sigma^a \subseteq \sigma^{n-1} \subset \sigma^n} \rho(\sigma^{n-1}, \sigma^n) \beta_{\sigma^{n-1}} \right] . [\Omega]_{\sigma^1, \sigma^n} \right) \\
&= \bigoplus_{\sigma^1 \subseteq \sigma^a} \sum_{\sigma^a \subseteq \sigma^{n-1}} \pi^{\sigma^a}_{\sigma^{n-1}} \circ \pi^{\sigma^{n-1}} \left(\beta_{\sigma^{n-1}} . \left[\sum_{\sigma^n \supset \sigma^{n-1}} \rho(\sigma^{n-1}, \sigma^n) [\Omega]_{\sigma^1, \sigma^n} \right] \right) \\
&= \bigoplus_{\sigma^1 \subseteq \sigma^a} \sum_{\sigma^a \subseteq \sigma^{n-1}} \pi^{\sigma^a}_{\sigma^{n-1}} \left(\beta_{\sigma^{n-1}} . \left[\sum_{\sigma^n \supset \sigma^{n-1}} \rho(\sigma^{n-1}, \sigma^n) \pi^{\sigma^{n-1}} ([\Omega]_{\sigma^1, \sigma^n}) \right] \right) \\
&= \bigoplus_{\sigma^1 \subseteq \sigma^a} \sum_{\sigma^a \subseteq \sigma^{n-1}} \pi^{\sigma^a}_{\sigma^{n-1}} \left(\beta_{\sigma^{n-1}} . \left[d^{2,(n-1)}_{n-1}([\Omega]) \right]_{\sigma^1, \sigma^{n-1}} \right).
\end{aligned}$$

However, we know that without any hypothesis (Lemma 3.5.2) we have:

$$d^{2,(n-1)}_{n-1}([\Omega]) = 0.$$

It concludes the proof. □

Lemma 3.8.11. *If the polytope P is R-non-singular on the face F_{σ^a}, and if the triangulation satisfies Hypothesis 2 for the integral domain R, then we have:*

$$\ker\, \phi_{\sigma^a,(p)} = \text{im}\, \delta^{n-1-a}_{\sigma^a,*,(p)}.$$

Proof. Firstly, let us notice that the cohomology group $H^{n-a}(\mathcal{F}^p_R(\sigma^a, *)) \simeq \mathcal{F}^p_R(\sigma^a, \sigma^a)$ is torsion-free. Thus the image of $\delta^{n-a-1}_{\sigma^a,*,(p)}$ is saturated in $C^{n-a}(\mathcal{F}^p_R(\sigma^a, *))$ (see Definition 3.8.1 for the saturation of a sub-module). Let us denote by $V := \sum_{\sigma^1 \subseteq \sigma^a} T_R \sigma^1$. Then, using the saturation of the image of $\delta^{n-a-1}_{\sigma^a,*,(p)}$, we have:

$$\begin{aligned}
\text{rank}\, \delta^{n-a-1}_{\sigma^a,*,(p)} &= \text{rank}\, C^{n-a}(\mathcal{F}^p_R(\sigma^a, *)) - \text{rank}\, \mathcal{F}^p_R(\sigma^a, \sigma^a) \\
&= \text{rank}\, C^{n-a}(\mathcal{F}^p_R(\sigma^a, *)) \\
&\quad - \left[\binom{\dim F_{\sigma^a}}{p-n+\dim F_{\sigma^a}} - \binom{\dim F_{\sigma^a} - \text{rank} V}{p-n+\dim F_{\sigma^a} - \text{rank} V} \right] \\
&= \text{rank}\, \ker\, \phi_{\sigma^a,(p)}.
\end{aligned}$$

Now we conclude the proof using again that the image of $\delta^{n-a-1}_{\sigma^a,*,(p)}$ is a saturated sub-module of $C^{n-a}(\mathcal{F}^p_R(\sigma^a, *))$. □

We are now ready to prove the Lemma 3.7.8.

Proof. Proof of the Lemma 3.7.8.

If the polytope P is R-non-singular, then it is R-non-singular on every one of its faces. Thus we have:

$$\ker\, \phi^q_{(p)} \cap C^{n-q,n}(\mathcal{F}^p_R) = \bigoplus_{\sigma^{n-q} \in \Gamma} \ker\, \phi_{\sigma^{n-q},(p)} = \bigoplus_{\sigma^{n-q} \in \Gamma} \text{im}\, \delta^{q-1}_{\sigma^{n-q},*,(p)} = \text{im}\, \delta^{n-q,n-1}_{2,(p)}.$$

□

Lemma 3.8.12. *If the simplex σ^a is R-primitive and if the triangulation satisfies Hypothesis 2 for the integral domain R, then we have:*

$$\ker\, d_{a-1}^{*,\sigma^a,(n-1-p)} = \text{im}\ \phi_{\sigma^a,(p)}.$$

Proof. From Proposition 3.8.2, we have:

$$\text{rank ker}\, d_{a-1}^{*,\sigma^a,(n-1-p)} = \binom{\dim F_{\sigma^a}}{n-p} - \binom{\dim F_{\sigma^a} - a}{n-p}.$$

However, as we assume the simplex σ^a to be R-primitive, we have:

$$\text{rank} \sum_{\sigma^1 \subseteq \sigma^a} T_R\sigma^1 = \text{rank}\ T_R\sigma^a = a.$$

Thus we obtain by Proposition 3.8.8:

$$\text{rank ker}\, d_{a-1}^{*,\sigma^a,(n-1-p)} = \text{rank}\ \phi_{\sigma^a,(p)}.$$

Now, as the Proposition 3.8.8 also announces that the image of $\phi_{\sigma^a,(p)}$ is saturated in $C_{a-1}(\mathcal{F}_p^R(*,\sigma^a))$, it concludes the proof (see Definition 3.8.1 for the saturation of a submodule). □

We are now ready to prove the Lemma 3.7.7.

Proof. Proof of the Lemma 3.7.7.

If the triangulation is (k,R)-primitive, then each of its $n-q$-simplices for $q \geq n-k$ is R-primitive. Moreover, Hypothesis 2 is also a consequence of the (k,R)-primitivity (see Proposition 3.3.3). Then we can apply the lemma above for any k-simplex, and we get for $q \geq n-k$:

$$\text{im}\ \phi_{(p)}^q = \bigoplus_{\sigma^{n-q}\in\Gamma} \text{im}\ \phi_{\sigma^{n-q},(p)} = \bigoplus_{\sigma^{n-q}\in\Gamma} \ker\, d_{n-q-1}^{*,\sigma^{n-q},(n-1-p)} = \ker\, d_{1,n-q}^{1,(n-1-p)}.$$

□

Now let us prove the Lemma 3.7.9:

Proof. Proof of the Lemma 3.7.9.

We have previously defined a map $\psi : \bigoplus_{\sigma^n \supseteq \sigma^a} \bigwedge^p M_R \to C_{a-1}(\mathcal{F}_{n-1-p}(*,\sigma^a))$ which depends on a fixed simplex σ^a. Let us then denote it ψ_{σ^a}. Then we can define:

$$\psi^q := \bigoplus_{\sigma^{n-q}\in\Gamma} \psi_{\sigma^{n-q}}.$$

This map satisfies $\psi^q = \phi_{(p)}^q \circ \gamma^q$, where $\gamma^q : \bigoplus_{(\sigma^{n-q},\sigma^n)\in\Omega} \bigwedge^p M_R \twoheadrightarrow C^{n-q,n}(\mathcal{F}_R^p)$ is the canonical projection.

Let $\alpha \in C^{n-q,n}(\mathcal{F}_R^p)$. We have:

$$\psi(\alpha) = \bigoplus_{\sigma^1\subseteq\sigma^{n-q}} \sum_{\sigma^n\supseteq\sigma^{n-q}} \pi^{\sigma^{n-q}}(\alpha_{\sigma^a,\sigma^n}.[\Omega]_{\sigma^1,\sigma^n}).$$

Then we have:

$$
\begin{aligned}
&d_{1,n-q}^{2,(n-1-p)} \circ \psi^q(\alpha) \\
&= \bigoplus_{(\sigma^1,\sigma^{n-q-1})\in\Omega} \sum_{\sigma^{n-q-1}\subset\sigma^{n-q}\subseteq\sigma^n} \rho(\sigma^{n-q-1},\sigma^{n-q})\pi_{\sigma^{n-q}}^{\sigma^{n-q-1}} \circ \pi^{\sigma^{n-q}}(\alpha_{\sigma^{n-q},\sigma^n}.[\Omega]_{\sigma^1,\sigma^n}) \\
&= \bigoplus_{(\sigma^1,\sigma^{n-q-1})\in\Omega} \sum_{\sigma^{n-q-1}\subset\sigma^n} \pi^{\sigma^{n-q-1}}\left(\left[\sum_{\sigma^{n-q-1}\subset\sigma^{n-q}\subseteq\sigma^n} \rho(\sigma^{n-q-1},\sigma^{n-q})\alpha_{\sigma^{n-q},\sigma^n}\right].[\Omega]_{\sigma^1,\sigma^n}\right).
\end{aligned}
$$

However, we also have:

$$
\begin{aligned}
&\delta_{1,(p)}^{n-q,n} \circ \gamma^q(\alpha) \\
&= \bigoplus_{(\sigma^{n-q-1},\sigma^n)\in\Omega} (-1)^{q+1} \sum_{\sigma^{n-q-1}\subset\sigma^{n-q}} \rho(\sigma^{n-q-1},\sigma^{n-q})(\iota_{\sigma^{n-q-1},\sigma^n}^{\sigma^{n-q},\sigma^n})^* \circ (\iota_{\sigma^{n-q},\sigma^n}^{\sigma^n})^*(\alpha_{\sigma^{n-q},\sigma^n}) \\
&= \bigoplus_{(\sigma^{n-q-1},\sigma^n)\in\Omega} (\iota_{\sigma^{n-q-1},\sigma^n}^{\sigma^n})^*\left((-1)^{q+1} \sum_{\sigma^{n-q-1}\subset\sigma^{n-q}} \rho(\sigma^{n-q-1},\sigma^{n-q})\alpha_{\sigma^{n-q},\sigma^n}\right) \\
&= \gamma^{q+1}\left(\bigoplus_{(\sigma^{n-q-1},\sigma^n)\in\Omega} (-1)^{q+1} \sum_{\sigma^{n-q-1}\subset\sigma^{n-q}} \rho(\sigma^{n-q-1},\sigma^{n-q})\alpha_{\sigma^{n-q},\sigma^n}\right).
\end{aligned}
$$

Thus we have:

$$
\begin{aligned}
&\phi_{(p)}^{q+1} \circ \delta_{1,(p)}^{n-q,n} \circ \gamma^q(\alpha) \\
&= \phi_{(p)}^{q+1} \circ \gamma^{q+1}\left(\bigoplus_{(\sigma^{n-q-1},\sigma^n)\in\Omega} (-1)^{q+1} \sum_{\sigma^{n-q-1}\subset\sigma^{n-q}} \rho(\sigma^{n-q-1},\sigma^{n-q})\alpha_{\sigma^{n-q},\sigma^n}\right) \\
&= \psi^{q+1}\left(\bigoplus_{(\sigma^{n-q-1},\sigma^n)\in\Omega} (-1)^{q+1} \sum_{\sigma^{n-q-1}\subset\sigma^{n-q}} \rho(\sigma^{n-q-1},\sigma^{n-q})\alpha_{\sigma^{n-q},\sigma^n}\right) \\
&= \bigoplus_{(\sigma^1,\sigma^{n-q-1})\in\Omega} \sum_{\sigma^{n-q-1}\subset\sigma^n} \pi^{\sigma^{n-q-1}}\left(\left[(-1)^{q+1} \sum_{\sigma^{n-q-1}\subset\sigma^{n-q}\subseteq\sigma^n} \right.\right. \\
&\qquad \left.\left. \rho(\sigma^{n-q-1},\sigma^{n-q})\alpha_{\sigma^{n-q},\sigma^n}\right].[\Omega]_{\sigma^1,\sigma^n}\right) \\
&= (-1)^{q+1} d_{1,n-q}^{2,(n-1-p)} \circ \phi_{(p)}^q \circ \gamma^q(\alpha).
\end{aligned}
$$

By surjectivity of the map γ^q, we obtain:

$$\phi_{(p)}^{q+1} \circ \delta_{1,(p)}^{n-q,n} = (-1)^{q+1} d_{1,n-q}^{2,(n-1-p)} \circ \phi_{(p)}^q.$$

However, we also know that im $\phi_{(p)}^q \subseteq C_{1,n-q}(\mathcal{F}_{n-1-p}^R(*,\sigma^{n-q}))$, thus $d_{a+1+q-n,a}^{2,(n-1-p)} \circ \phi_{(p)}^q = 0$ for every $a \neq n-q$. Moreover $\phi_{(p)}^{q+1}$ vanishes on every $C^{a,a+q+1}(\mathcal{F}_R^p)$ for $a \neq n-q-1$, and im $\delta_{1,(p)}^{a+1,a+1+q} \subseteq C^{a,a+q+1}(\mathcal{F}_R^p)$ thus we have $\phi_{(p)}^{q+1} \circ \delta_{1,(p)}^{a+1,a+1+q} = 0$ for every $a \neq n-q-1$. Therefore, the previous equations were enough to conclude the proof:

$$\phi_{(p)}^{q+1} \circ \delta_{1,(p)}^q = (-1)^{q+1} d_{n-q-1}^{2,(n-1-p)} \circ \phi_{(p)}^q.$$

□

3.9 Non-compact generalization

In this section we present a generalization of our partial Poincaré duality theorem (Theorem 1.3.1) and our complete Poincaré duality theorem (Theorem 1.3.2).

For any face F of the lattice polytope P, consider its interior $\text{int}(F) := F \backslash \partial F$, where ∂F denotes the union of proper faces of F. Notice that with these notations, the interior of a vertex v of P is itself, and the family $\{\text{int}(F), \quad F \text{ face of } P\}$ is a partition of P.

Among the unions of elements of this partition $\{\text{int}(F), \quad F \text{ face of } P\}$ of P, several of thems are non-empty open subsets of P. If Q is a subset of faces of P, we can prove that $\cup_{F \in Q} \text{int}(F)$ is an open subset of P if and only if Q satisfies the following condition:

$$\forall F \in Q, \ \forall G \text{ face of P}, \quad F \text{ face of } G \implies G \in Q \tag{3.11}$$

Consider such a subset Q of faces of P and denote $P_Q := \cup_{F \in Q} \text{int}(F)$ the corresponding open union of interiors of faces of P. Recall that the poset Ω associated to the triangulation Γ of P labels the cubical cells of an object $|X| \subseteq P$ (defined in Equation 3.1). The object $|X| \subseteq P$ is compact but its intersection $|X| \cap P_Q$ for some Q satisfying Equation 3.11 can be not compact. For any Q satisfying Equation 3.11, the compactification of $|X| \cap P_Q$ in P is $|X|$. Thus, our generalization of our partial and complete tropical Poincaré duality theorems in the non-compact framework will be tropical Poincaré duality theorems for these intersections $|X| \cap P_Q$ for some Q satisfying Equation 3.11.

Recall that for any element σ of the triangulation Γ of P, F_σ denotes the minimal face of P containing σ. Consider the following subset $\bar{\Gamma}$ of elements of the triangulation Γ:

$$\bar{\Gamma} := \{\sigma \in \Gamma \mid F_\sigma \in Q\}$$

It induces a sub-poset $\bar{\Omega}$ of the poset Ω:

$$\bar{\Omega} := \{(\sigma, \tau) \in \Omega \mid \sigma \in \bar{\Gamma}\}$$

This poset $\bar{\Omega}$ labels the cubical cells of $|X|$ whose interiors are included in $|X| \cap P_Q$. A consequence of Equation 3.11 is that the posets $\bar{\Gamma}$ and $\bar{\Omega}$ inherits the diamond property and the existence of a balancing signature from the posets Γ and Ω. Then the restriction to $\bar{\Omega}$ of the sheaves $\mathcal{F}_R^p$ admit sheaf cochain complexe and a cohomology. They are the standard cochains of $\mathcal{F}_R^p$, denoted $C^q_{\text{Stand}}(\mathcal{F}_R^p), \ q \geq 0$, and the standard cohomology $H^q_{\text{Stand}}(\mathcal{F}_R^p), \ q \geq 0$.

In this non-compact framework, the chain complexe of the cosheaf $\mathcal{F}_p^R$ over the poset Ω (that is the one already considered in this chapter) are the Borel-Moore chains of $\mathcal{F}_p^R$, denoted $C_q^{BM}(\mathcal{F}_p^R), \ q \geq 0$, and its homology are the Borel-Moore homology, denoted $H_q^{BM}(\mathcal{F}_p^R), \ q \geq 0$.

For Poincaré duality we need a cap product with a fundamental class. However, the Borel-Moore chains $C_{n-1}^{BM}(\mathcal{F}_{n-1}^R)$ are equals to the chains $C_{n-1}(\mathcal{F}_{n-1}^R)$ studied in the rest of this chapter (there are both the chains of the cosheaf $\mathcal{F}_{n-1}^R$ on the poset Ω), and we know from Proposition 3.5.7 that under Hypothesis 1 the fundamental class exists (see Definition 3.3.2 for Hypothesis 1). Then this fundamental class also belongs to $C_{n-1}^{BM}(\mathcal{F}_{n-1}^R)$. Thus, under Hypothesis 1 we can consider the following restriction of the cap product with the

fundamental class $[\Omega]$:

$$\begin{array}{rccc} \bar{\phi}^q : & C^q_{\mathrm{Stand}}(\mathcal{F}^p_R) & \to & C^{BM}_{n-1-q}(\mathcal{F}^R_{n-1-p}) \\ & \beta & \mapsto & \beta \cap [\Omega] \end{array}$$

Similarly as in the rest of this chapter, we can consider double complexes filtrating $C^q_{\mathrm{Stand}}(\mathcal{F}^p_R)$ and double complexes filtrating $C^{BM}_{n-1-q}(\mathcal{F}^R_{n-1-p})$, they induced spectral sequences, and all of the lemmas and propositions demonstrated in the compact framework for the map ϕ^q can be adapted for this map $\bar{\phi}^q$. At the end, we obtain partial an complete Poincaré duality theorems in this non-compact framework:

Theorem 3.9.1. *(Partial Poincaré Duality). Consider R an integral domain, and P a lattice polytope of dimension n in $\mathbb{R}^n$. Consider a triangulation with integral vertices Γ of P. Consider a subset Q of faces of P such that the union P_Q of their interiors is an non-empty open subset of P. Assume that the polytope P is R-non-singular and that the triangulation Γ is (k,R)-primitive, with $2 \leq k \leq n-1$. Then the standard tropical cohomology and the Borel-Moore tropical homology over R of this triangulation Γ of P and this subset Q of faces of P admit the following canonical isomorphisms:*

$$\forall p+q \geq 2n-k, \qquad H^q_{\mathrm{Stand}}(\mathcal{F}^p_R) \xrightarrow{\sim} H^{BM}_{n-1-q}(\mathcal{F}^R_{n-1-p}).$$

Moreover, for $k = n$ (that is for R-primitive triangulations), we have the following theorem:

Theorem 3.9.2. *(Complete Poincaré duality theorem) Consider R an integral domain, and P a lattice polytope of dimension n in $\mathbb{R}^n$. Consider a triangulation with integral vertices Γ of P. Consider a subset Q of faces of P such that the union P_Q of their interiors is an non-empty open subset of P. Assume that the polytope P is R-non-singular and that the triangulation Γ is R-primitive. Then the standard tropical cohomology and the Borel-Moore tropical homology over R of this triangulation Γ of P and this subset Q of faces of P admit the following canonical isomorphisms: c*

Similarly as in the compact framework, we also obtain the following Corollary for the case $R = \mathbb{Q}$:

Corollary 3.9.3. *(Tropical Poincaré Duality over $\mathbb{Q}$) Consider P a lattice polytope of dimension n in $\mathbb{R}^n$. Consider a triangulation with integral vertices Γ of P. Consider a subset Q of faces of P such that the union P_Q of their interiors is an non-empty open subset of P. Then the standard tropical cohomology and the Borel-Moore tropical homology over $\mathbb{Q}$ of this triangulation Γ of P and this subset Q of faces of P admit the following canonical isomorphisms:*

$$\forall p, q, \qquad H^q_{\mathrm{Stand}}(\mathcal{F}^p_{\mathbb{Q}}) \xrightarrow{\sim} H^{BM}_{n-1-q}(\mathcal{F}^{\mathbb{Q}}_{n-1-p}).$$

Chapter 4

Bounds on the Betti numbers of non-primitive patchworks

In this chapter we present our non-primitive generalization of the bounds of the Betti numbers obtained by [RS 2023] and [BMR 2024] for hypersurfaces constructed by combinatorial patchwork. We present the proof of our Theorems 1.3.4, 1.3.5 and 1.3.6 provided in the Introduction. We also state corollaries and applications in particular cases.

4.1 Patchwork and real phase structure

In this section we describe Viro's Patchwork of a lattice polytope P, starting with a non-necessarily primitive and non-necessarily convex triangulation, and we describe the generalization of the real phase structure to this framework. Real phase structures were firstly introduced by [RRS 2022] and [RS 2023] in the case of primitive triangulations. Their definitions were presented in Section 2.6 of this thesis. Viro's Patchwork was already presented in Section 2.2, in the particular case of hypersurfaces in projective spaces. The reader will find figures illustrating Viro's Patchwork in Section 2.2.

Consider a convex simple lattice polytope P of dimension $n \geq 1$ living in $\mathbb{R}^n$. We endow $\mathbb{R}^n$ with the canonical unimodular lattice $M := \mathbb{Z}^n$. It defines a projective toric variety $\mathbb{C}Y$. Consider a triangulation Γ of P with integral vertices (non-necessarily convex). For convenience, in the following, when use an exponent on a given simplex of the triangulation Γ, it denots its dimension. For instance, $\sigma^a \in \Gamma$ will design a simplex of dimension a.

Consider the dual lattice $N := \mathrm{Hom}(M, \mathbb{Z})$, and $N_{\mathbb{F}_2} := N \otimes \mathbb{F}_2$ its tensor product by $\mathbb{F}_2$. Similarly, consider $M_{\mathbb{F}_2} = M \otimes \mathbb{F}_2$. For any face F of P we denote the tangent lattice $T_{\mathbb{Z}}F := T_{\mathbb{R}}F \cap M$ and $T_{\mathbb{F}_2}F := T_{\mathbb{Z}}F \otimes \mathbb{F}_2 \subseteq M_{\mathbb{F}_2}$.

For every $x \in N_{\mathbb{F}_2}$, consider a copy P_x of the lattice polytope P, and for any face F of P we denote by F_x the corresponding face of P_x. The glueing of the copies $\{P_x,\ x \in N_{\mathbb{F}_2}\}$ with the identifications $F_x = F_y$ whenever $x - y \in T_{\mathbb{F}_2}F^{\perp}$ for any F face of P forms a topological space homeomorphic to the real locus $\mathbb{R}Y$ of the projective toric variety $\mathbb{C}Y$. We denote it by P_* (see Proposition 2.1.5 in the preliminaries for more details). The triangulation Γ of P induces triangulations on its copies and then a triangulation Γ_* on P_*. The set of cells of the simplicial complex Γ_* is labeled by the set of pairs $\{(\sigma, x),\ \sigma \in \Gamma,\ x \in N_{\mathbb{F}_2}\}$ with the identifications $(\sigma, x) = (\sigma, y)\ \ \forall x - y \in T_{\mathbb{F}_2}F_{\sigma}^{\perp}$, where F_σ denotes the

minimal face of P containing σ. Therefore, in a more direct manner, the cells of Γ_* may be regarded as the set of pairs $\{(\sigma, x),\ \sigma \in \Gamma,\ x \in N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F_\sigma^\perp\}$. In this parametrization, the faces of a cell (σ, x) are the pairs (τ, x_τ) for τ face of σ and x_τ the projection of x in $N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F_\tau^\perp$.

For every vertex v of the triangulation Γ we fix a sign $\epsilon_v \in \mathbb{F}_2$. Each vertex v of Γ has an image $v_{\mathbb{F}_2} := v \otimes 1_{\mathbb{F}_2} \in M_{\mathbb{F}_2}$. This allows us to define the sign $\epsilon(v, x) := \epsilon_v + x(v_{\mathbb{F}_2}) \in \mathbb{F}_2$ for any vertex v of Γ and any $x \in N_{\mathbb{F}_2}$.

In their article [BMR 2024], Brugallé, Lopez de Medrano and Rau introduced a subdivision of Γ, coarser than the barycentric subdivision of Γ, labeled by the poset $\Xi := \{(F, \sigma),\ F \text{ face of } P,\ \sigma \in \Gamma \text{ such that } \sigma \subseteq F\}$. For F face of P, and $\sigma \in \Gamma$, the cell labeled by (F, σ) can be defined as the union of the barycentric cells of Γ containing the barycenter of σ and contained in F. This can be extend to obtain a subdivision of Γ_*, coarser than the barycentric subdivision of Γ_*, labeled by the poset $\Xi_* := \{(F, \sigma, x),\ F \text{ face of } P,\ \sigma \in \Gamma \text{ such that } \sigma \subseteq F,\ x \in N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp\}$.

For a face F of P and two vertices v_1, v_2 of Γ such that $v_1, v_2 \in F$, we have $\epsilon(v_1, x) - \epsilon(v_2, x) = \epsilon_{v_1} - \epsilon_{v_2}$ for every $x \in T_{\mathbb{F}_2}F$. Then for any $x, y \in N_{\mathbb{F}_2}$ such that $x - y \in T_{\mathbb{F}_2}F$, we have $\epsilon(v_1, x) = \epsilon(v_2, x)$ if and only if $\epsilon(v_1, y) = \epsilon(v_2, y)$. Thus for any $x \in N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp$ we can give a meaning to the identity $\epsilon(v_1, x) = \epsilon(v_2, x)$. [JRS 2018] and [RS 2023] introduced the real phase structure $\mathcal{E}(F, \sigma)$ for any $(F, \sigma) \in \Xi$ with $\dim \sigma \geq 1$:

$$\mathcal{E}(F, \sigma) := \{x \in N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp \mid \exists v_1, v_2 \text{ vertices of } \sigma \text{ s.t. } \epsilon(v_1, x) \neq \epsilon(v_2, x)\}.$$

They introduced it in the case of a primitive triangulation Γ, but the primitivity is not needed for the definition (see Section 2.6 of the preliminaries for more details about the ideas behind the real phase structure introduced by [RS 2023] and [JRS 2018]).

Then we obtain that the subcomplex labeled by $\Xi_*^1 := \{(F, \sigma, x),\ F \text{ face of } P,\ \sigma \in \Gamma \text{ such that } \sigma \subseteq F,\ \dim \sigma \geq 1,\ x \in \mathcal{E}(F, \sigma)\}$ of the subdivision of Γ_* labeled by Ξ_* is a topological hypersurface of $P_* \simeq \mathbb{R}Y$, we denote it X. Recall that the triangulation Γ of P is said to be convex if there exists a convex piecewise linear function $\nu : P \to \mathbb{R}$, affine on each n-simplex of the triangulation Γ and non-affine on each union of two distinct n-simplices. Then the combinatorial Viro's patchworking theorem is:

Theorem 4.1.1. *If the triangulation Γ is convex, there exists a real algebraic hypersurface $\mathbb{R}X$ of Newton polytope P of the projective toric variety $\mathbb{R}Y$ constructed from the polytope P, satisfying the following homeomorphism of pairs:*

$$(\mathbb{R}Y, \mathbb{R}X) \simeq (P_*, X).$$

In the general case of a triangulation which is non-necessarily convex, we can still be interested on the properties of the topological hypersurface X. [BMR 2024] have given bounds on its Betti numbers, assuming that the triangulation is primitive. In this chapter we generalize these bounds to non-primitive triangulations.

The dimension of a cell $(F, \sigma, x) \in \Xi_*^1$ is the dimension of the underlying union of barycentric cells of Γ_*, which are the barycentric cells containing the barycenter of (σ, x_σ) and contained in F_x (it means that $\dim(F, \sigma, x) = \dim F - \dim \sigma$). Then the homology over $\mathbb{F}_2$ of the topological hypersurface X can be computed with the following chain complex :

$$C_q(\Xi_*^1, \mathbb{F}_2) := \bigoplus_{(F,\sigma,x)\in\Xi_*^1,\ \dim F - \dim \sigma = q} \mathbb{F}_2$$

The differentials of this chain complex are induced by the inclusions of cells of the related cellular complex of X, which corresponds in Ξ^1_* to the inclusion of (F,σ,x) in (F',σ',x') if and only if $F \subseteq F'$, $\sigma \supseteq \sigma'$ and the projection of x in $N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F'^{\perp}$ is x'. There is no need of a balancing signature because the integral domain on which we are working is $\mathbb{F}_2$.

Consider the subcomplex labeled by $\Xi^1 := \{(F,\sigma),\ F \text{ face of } P,\ \sigma \in \Gamma \text{ such that } \sigma \subseteq F,\ \dim \sigma \geq 1\}$ of the subdivision of Γ labeled by Ξ. The union of the cells labeled by Ξ^1 forms a topological space (denoted $|X|$). In the case of a convex triangulation Γ, there exists a tropical hypersurface $\mathbb{T}X$ of Newton polytope P of the tropical projective toric variety $\mathbb{T}Y$ of Polytope P, satisfying the following homeomorphism of pairs:

$$(\mathbb{T}Y, \mathbb{T}X) \simeq (P, |X|)$$

The polynomial which defines the tropical hypersurface $\mathbb{T}X$ is the same as the polynomial which defines the real algebraic hypersurface $\mathbb{R}X$ in Viro's combinatorial patchworking theorem. It was the point of view used in [RS 2023] to obtain bounds on Betti numbers of a real hypersurfaces obtained by patchworking of a convex primitive triangulation of a non-singular polytope.

Here we are in the non-necessarily convex framework and there is no tropical interpretation of the topological space $|X|$. However, we can still define tropical cosheaves and sheaves on the poset Ξ^1 (done in [BMR 2024]) and we will use them to understand the homology of the hypersurface X. Let us begin by notice that $\Xi^1_* = \{(F,\sigma,x),\ \ (F,\sigma) \in \Xi^1,\ x \in \mathcal{E}(F,\sigma)\}$. It leads to the following rewriting of $C_q(\Xi^1_*, \mathbb{F}_2)$:

$$C_q(\Xi^1_*, \mathbb{F}_2) = \bigoplus_{(F,\sigma)\in\Xi^1,\ \dim F - \dim \sigma = q} \left(\bigoplus_{x\in\mathcal{E}(F,\sigma)} \mathbb{F}_2 \right)$$

It leads to the introduction of the sign cosheaf $\mathcal{S}$ on Ξ^1 (introduced by [RS 2023]) defined by:

$$\mathcal{S}(F,\sigma) := \bigoplus_{x\in\mathcal{E}(F,\sigma)} \mathbb{F}_2 \tag{4.1}$$

where the cosheaves morphisms are induced by the inclusions $\mathcal{E}(F,\sigma) \subseteq \mathcal{E}(F,\sigma')$ if $\sigma \subseteq \sigma'$ and the projections $\mathcal{E}(F,\sigma) \twoheadrightarrow \mathcal{E}(F',\sigma)$ if $F \supseteq F'$ (the reader can check that these projections are well-defined). Then we obtain an identity of complexes :

$$C_q(\Xi^1_*, \mathbb{F}_2) = C_q(\mathcal{S}) := \bigoplus_{(F,\sigma)\in\Xi^1,\ \dim F - \dim \sigma = q} \mathcal{S}(F,\sigma)$$

Moreover, the inclusion of cells of Ξ^1_* is compatible with the cosheaves morphisms of $\mathcal{S}$, thus we obtain an identity of homology:

$$H_q(\mathcal{S}) = H_q(\Xi^1_*, \mathbb{F}_2) = H_q(X, \mathbb{F}_2)$$

4.2 Kalinin's filtration

In this section we generalize to the non-primitive case the filtration of the sign cosheaf $\mathcal{S}$ introduced by [RS 2023] and described in Section 2.6 of the preliminaries of this thesis. Then we show that it induces a generalization of Renaudineau-Shaw spectral sequence.

Let us notice that for any $\sigma \in \Gamma$ and any face F of P containing σ, as $\mathcal{E}(F,\sigma)$ is contained in $N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}$, we have that $\mathcal{S}(F,\sigma)$ is a vector sub-space of $\bigoplus_{x\in N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}}\mathbb{F}_2$. However, $\bigoplus_{x\in N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}}\mathbb{F}_2$ can also be considered as the group algebra $\mathbb{F}_2[N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}]$, with the relations $w_\alpha w_\beta = w_{\alpha+\beta}$, where $\{w_\alpha,\ \alpha \in N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}\}$ denotes the canonical generators of $\mathbb{F}_2[N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}]$ as a vector space. There is an associated augmentation morphism:

$$\varphi_F : \begin{array}{ccc} \mathbb{F}_2[N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}] & \to & \mathbb{F}_2 \\ \sum a_\alpha w_\alpha & \mapsto & \sum a_\alpha \end{array}$$

The kernel of φ_F is an ideal $\mathcal{I}_F$ of $\mathbb{F}_2[N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}]$, called the augmentation ideal, then the sequence of powers $(\mathcal{I}_F^p)_{p\geq 0}$ is a decreasing sequence of ideals, it filtrates $\mathbb{F}_2[N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}]$. Consider the following map :

$$\begin{array}{ccc} (N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp})^p & \to & \mathcal{I}_F^p \\ (\alpha_1,...,\alpha_p) & \mapsto & (w_0+w_{\alpha_1})...(w_0+w_{\alpha_p}) \end{array} \tag{4.2}$$

According to [RS 2023] (section 4), this map descends to a linear isomorphism $\mu_{F,p} : \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp} \simeq \mathcal{I}_F^p/\mathcal{I}_F^{p+1}$.

Using the inclusion $\mathcal{S}(F,\sigma) \subseteq \mathbb{F}_2[N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}]$, we can define :

$$\forall p \geq 0, \qquad \mathcal{K}_p(F,\sigma) := \mathcal{S}(F,\sigma) \cap \mathcal{I}_F^p$$

For any $p \geq 0$, we can identify $\mathcal{K}_p(F,\sigma)/\mathcal{K}_{p+1}(F,\sigma)$ as a subspace of $\mathcal{I}_F^p/\mathcal{I}_F^{p+1}$. Then we can consider the pre-image through the isomorphism μ:

$$\forall p \geq 0, \qquad \tilde{\mathcal{F}}_p(F,\sigma) := \mu_{F,p}^{-1}\Big(\mathcal{K}_p(F,\sigma)/\mathcal{K}_{p+1}(F,\sigma)\Big) \subseteq \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}$$

Notice that for any face τ of the simplex σ, we can identify $\mathcal{K}_p(F,\tau)/\mathcal{K}_{p+1}(F,\tau)$ as a sub-space of $\mathcal{K}_p(F,\sigma)/\mathcal{K}_{p+1}(F,\sigma)$. Then we have a canonical inclusion morphism $\tilde{\mathcal{F}}_p(F,\tau) \hookrightarrow \tilde{\mathcal{F}}_p(F,\sigma)$ and the following diagram commmutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_{p+1}(F,\tau) & \hookrightarrow & \mathcal{K}_p(F,\tau) & \twoheadrightarrow & \tilde{\mathcal{F}}_p(F,\tau) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K}_{p+1}(F,\sigma) & \hookrightarrow & \mathcal{K}_p(F,\sigma) & \twoheadrightarrow & \tilde{\mathcal{F}}_p(F,\sigma) & \longrightarrow & 0 \end{array} \tag{4.3}$$

Now if we fix a simplex σ of the triangulation Γ and if we consider two faces F, G of the polytope P satisfying $\sigma \subseteq F \subseteq G$, we have a natural projection $\pi : \mathbb{F}_2[N_{\mathbb{F}_2}/T_{\mathbb{F}_2}G^{\perp}] \twoheadrightarrow \mathbb{F}_2[N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}]$, which commutes with the maps φ_F and φ_G. Thus the kernels $\mathcal{I}_F$ and $\mathcal{I}_G$ satisfy $\pi(\mathcal{I}_G) = \mathcal{I}_F$ and then as π is a morphism of group algebra they satisfy for any integer p the relation $\pi(\mathcal{I}_G^p) = \mathcal{I}_F^p$. Then π induces a projection $\tilde{\pi} : \mathcal{I}_G^p/\mathcal{I}_G^{p+1} \twoheadrightarrow \mathcal{I}_F^p/\mathcal{I}_F^{p+1}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_G^{p+1} & \hookrightarrow & \mathcal{I}_G^p & \twoheadrightarrow & \mathcal{I}_G^p/\mathcal{I}_G^{p+1} & \longrightarrow & 0 \\ & & \pi\downarrow & & \pi\downarrow & & \tilde{\pi}\downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_F^{p+1} & \hookrightarrow & \mathcal{I}_F^p & \twoheadrightarrow & \mathcal{I}_F^p/\mathcal{I}_F^{p+1} & \longrightarrow & 0 \end{array}$$

Moreover, the isomorphisms $\mu_{G,p}$ and $\mu_{F,p}$ descend from the maps $(N_{\mathbb{F}_2}/T_{\mathbb{F}_2}G^\perp)^p \to \mathcal{I}_G^p$ and $(N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp)^p \to \mathcal{I}_F^p$ which commute with the canonical projection $(N_{\mathbb{F}_2}/T_{\mathbb{F}_2}G^\perp)^p \twoheadrightarrow (N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp)^p$ and $\pi : \mathcal{I}_G^p \to \mathcal{I}_F^p$, then we obtain the following commutative diagram:

$$\begin{array}{ccc} \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}G^\perp & \xrightarrow{\mu_{G,p}} & \mathcal{I}_G^p/\mathcal{I}_G^{p+1} \\ \downarrow & & \tilde{\pi}\downarrow \\ \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp & \xrightarrow{\mu_{F,p}} & \mathcal{I}_F^p/\mathcal{I}_F^{p+1} \end{array}$$

Finally, we can observe that $\pi(\mathcal{K}_p(G,\sigma)) = \mathcal{K}_p(F,\sigma)$ for any integer p. Then we obtain the following commutative diagram, by restriction of the two previous ones:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_{p+1}(G,\sigma) & \hookrightarrow & \mathcal{K}_p(G,\sigma) & \twoheadrightarrow & \tilde{\mathcal{F}}_p(G,\sigma) & \longrightarrow & 0 \\ & & \pi\downarrow & & \pi\downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K}_{p+1}(F,\sigma) & \hookrightarrow & \mathcal{K}_p(F,\sigma) & \twoheadrightarrow & \tilde{\mathcal{F}}_p(F,\sigma) & \longrightarrow & 0 \end{array} \tag{4.4}$$

where the projection $\tilde{\mathcal{F}}_p(G,\sigma) \twoheadrightarrow \tilde{\mathcal{F}}_p(F,\sigma)$ is inherited by restriction from the natural projection $\bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}G^\perp \twoheadrightarrow \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp$.

The vertical morphisms of the diagrams 4.3 and 4.4 induce a structure of cellular cosheaf on $\mathcal{K}_{p+1}$, $\mathcal{K}_p$ and $\tilde{\mathcal{F}}_p$ on the poset Ξ^1, since they commute if we consider chain of incident elements of the poset Ξ^1. The cosheaves chains of degree q are:

$$\forall p \geq 0, \quad C_q(\mathcal{K}_p) := \bigoplus_{(F,\sigma)\in\Xi^1,\ \dim\ F-\dim\ \sigma\ =\ q} \mathcal{K}_p(F,\sigma);$$

$$\forall p \geq 0, \quad C_q(\tilde{\mathcal{F}}_p) := \bigoplus_{(F,\sigma)\in\Xi^1,\ \dim\ F-\dim\ \sigma\ =\ q} \tilde{\mathcal{F}}_p(F,\sigma)$$

As the lines of the commutative diagrams 4.3 and 4.4 are short exact sequences, we obtain a long exact sequence in homology:

$$... \longrightarrow H_{q+1}(\tilde{\mathcal{F}}_p) \longrightarrow H_q(\mathcal{K}_{p+1}) \longrightarrow H_q(\mathcal{K}_p) \longrightarrow H_q(\tilde{\mathcal{F}}_p) \longrightarrow H_{q-1}(\mathcal{K}_{p+1}) \longrightarrow ...$$

For any face F of the lattice polytope P, we can observe the following identity :

$$\sum_{p=0}^{\dim F} \dim\ \mathcal{I}_F^p/\mathcal{I}_F^{p+1} = \sum_{p=0}^{\dim F} \dim \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp = 2^{\dim F} = \dim\ \mathbb{F}_2[N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp]$$

As a consequence, we obtain $\mathcal{I}_F^{\dim F+1} = 0$. The dimension of F is at most n, in the case $F = P$. Then $(\mathcal{K}_p)_{p\geq 0}$ is a finite filtration of $\mathcal{S}$:

$$0 = \mathcal{K}_{n+1} \subseteq \mathcal{K}_n \subseteq ... \subseteq \mathcal{K}_1 \subseteq \mathcal{K}_0 = \mathcal{S}$$

This filtration defines a spectral sequence degenerating on the homology $H_*(\mathcal{S})$ whose first page is $E^1_{q,p} = H_q(\mathcal{K}_p/\mathcal{K}_{p+1}) = H_q(\tilde{\mathcal{F}}_p)$. It is the spectral sequence of Renaudineau-Shaw, introduced in [RS 2023]. Bounds on the Betti numbers of the hypersurface X can be obtained using the inequalities $\dim\ E^\infty_{q,p} \leq \dim\ E^1_{q,p}$:

$$b_q(X;\mathbb{F}_2) := \dim\ H_q(X;\mathbb{F}_2) = \dim\ H_q(\mathcal{S}) = \sum_p \dim\ E^\infty_{q,p} \leq \sum_p \dim\ E^1_{q,p} = \sum_p \dim\ H_q(\tilde{\mathcal{F}}_p) \tag{4.5}$$

4.3 Tropical cosheaves

In this Section we establish the relations between our cosheaves $\tilde{\mathcal{F}}_p$ introduced in the previous section, and the tropical cosheaves $\mathcal{F}_p$ (see Example 2.5.8 in the preliminaries for a definition of tropical cosheaves). We begin by a describtion of our cosheaves $\tilde{\mathcal{F}}_p$ in the following proposition:

Proposition 4.3.1. *Assume that any edge of the triangulation Γ is of odd integral length. For any face F of the polytope P, for any σ element of the triangulation Γ contained in F, we have the identity:*

$$\tilde{\mathcal{F}}_p(F,\sigma) = \begin{cases} \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp & \text{if } \mathcal{E}(F,\sigma) = N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp \\ \sum_{\sigma^1\subseteq\sigma} \bigwedge^p T_{\mathbb{F}_2}\sigma^{1\perp}/T_{\mathbb{F}_2}F^\perp & \text{otherwise} \end{cases}$$

Proof. If $\mathcal{E}(F,\sigma) = N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp$, then $\mathcal{S}(F,\sigma) = N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp$ and then for any $p \geq 0$ we have $\mathcal{K}_p(F,\sigma) = \mathcal{I}_F^p$ and then $\tilde{\mathcal{F}}_p(F,\sigma) = \mu_{F,p}^{-1}(\mathcal{I}_F^p/\mathcal{I}_F^{p+1}) = \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp$.

Now assume that $\mathcal{E}(F,\sigma) \neq N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp$. Denote $a_0,...,a_k$ the vertives of σ. We have the equivalences:

$$\begin{aligned} x \in \mathcal{E}(F,\sigma) &\iff \exists 0 \leq i,j \leq k \text{ s.t. } \epsilon(a_i,x) \neq \epsilon(a_j,x) \\ &\iff \exists 1 \leq i \leq k \text{ s.t. } \epsilon(a_0,x) \neq \epsilon(a_i,x) \\ &\iff \exists \{a_0\} \subseteq \sigma^1 \subseteq \sigma \mid x \in \mathcal{E}(F,\sigma^1) \end{aligned}$$

Then we have the identity $\mathcal{E}(F,\sigma) = \bigcup_{\{a_0\}\subseteq\sigma^1\subseteq\sigma} \mathcal{E}(F,\sigma^1)$. Consider an edge $\{a_0\} \subseteq \sigma^1 \subseteq \sigma$ an edge of Γ. Its vertices are a_0, a_i for some $1 \leq i \leq k$. For any $x \in N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp$, we have the equivalences:

$$\begin{aligned} x \in \mathcal{E}(F,\sigma^1) &\iff \epsilon(a_0,x) \neq \epsilon(a_i,x) \iff x((a_0-a_i)_{\mathbb{F}_2}) \neq \epsilon_{a_0} - \epsilon_{a_i} \\ &\iff x((a_0-a_i)_{\mathbb{F}_2}) = \epsilon_{a_0} - \epsilon_{a_i} + 1 \end{aligned}$$

Since every edge of Γ is of odd integral length, we have $(a_0-a_i)_{\mathbb{F}_2} \neq 0$, then we obtain that $\mathcal{E}(F,\sigma^1)$ is an affine hyperplane of direction $T_{\mathbb{F}_2}\sigma^{1\perp}/T_{\mathbb{F}_2}F^\perp$. Consider a subfamily $a_{i_1},...,a_{i_s}$ of $a_1,...,a_k$ such that $((a_0-a_{i_1})_{\mathbb{F}_2},...,(a_0-a_{i_s})_{\mathbb{F}_2})$ forms a basis of $T_{\mathbb{F}_2}\sigma$. Consider $\{a_0\} \subseteq \sigma^1 \subseteq \sigma$ such that the vertices of σ^1 are a_0 and a certain a_j with $j \notin \{i_1,...,i_s\}$. Let us write $(a_0-a_j)_{\mathbb{F}_2} = \sum_{r=1}^s \alpha_r(a_0-a_{i_r})_{\mathbb{F}_2}$.

Our assumption is that $\mathcal{E}(F,\sigma) \neq N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp$. In particular, there exists an element $x \notin \mathcal{E}(F,\sigma^1) \cup \bigcup_{r=1}^s \mathcal{E}(F,a_0a_{i_r})$. It implies that $x((a_0-a_j)_{\mathbb{F}_2}) = \epsilon_{a_0} - \epsilon_{a_j} + 1$, and that for any $1 \leq r \leq s$ we have $x((a_0-a_{i_r})_{\mathbb{F}_2}) = \epsilon_{a_0} - \epsilon_{a_{i_r}}$. Therefore, we obtain the identity:

$$\sum_{r=1}^s \alpha_r(\epsilon_{a_0} - \epsilon_{a_{i_r}}) = \epsilon_{a_0} - \epsilon_{a_j}$$

In particular, this identity implies the following inclusion:

$$\mathcal{E}(F,a_0a_j) \subseteq \bigcup_{r=1}^s \mathcal{E}(F,a_0a_{i_r})$$

Applying this idea for any $j \notin \{i_1, ..., i_s\}$, we finally obtain:

$$\mathcal{E}(F, \sigma) = \bigcup_{r=1}^{s} \mathcal{E}(F, a_0 a_{i_r})$$

As $((a_0 - a_{i_1})_{\mathbb{F}_2}, ..., (a_0 - a_{i_s})_{\mathbb{F}_2})$ forms a basis of $T_{\mathbb{F}_2}\sigma$, the affine hyperplanes $\mathcal{E}(F, a_0 a_{i_r})$ of direction $(a_0 - a_{i_r})^{\perp}_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}$ are in generic position. Then there exists an element $\theta \in \bigcap_{1 \leq i \leq r} \mathcal{E}(F, a_0 a_{i_r})$. We have:

$$\mathcal{S}(F, \sigma) = \sum_{r=1}^{s} \mathcal{S}(F, a_0 a_{i_r}) = w_\theta . \sum_{r=1}^{s} \mathbb{F}_2[(a_0 - a_{i_r})^{\perp}_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}]$$

Since $\mathcal{I}_F^p$ is an ideal, we obtain:

$$\mathcal{K}_p(F, \sigma) = \mathcal{I}_F^p \cap \mathcal{S}(F, \sigma) = w_\theta . \Big(\mathcal{I}_F^p \cap \sum_{r=0}^{s} \mathbb{F}_2[(a_0 - a_{i_r})^{\perp}_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}] \Big)$$

Recall that the isomorphism $\mu_{F,p} : \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp} \simeq \mathcal{I}_F^p/\mathcal{I}_F^{p+1}$ descends from the map $(N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp})^p \to \mathcal{I}_F^p$ defined in equation 4.2. Let $(\alpha_1, ..., \alpha_p)$ an element of $(N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp})^p$ such that its image $(w_0 + w_{\alpha_1})...(w_0 + w_{\alpha_p})$ by this map $(N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp})^p \to \mathcal{I}_F^p$ belongs to $\sum_{r=0}^{s} \mathbb{F}_2[(a_0 - a_{i_r})^{\perp}_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}]$. We can assume that the family $\alpha_1, ..., \alpha_p$ is free because otherwise the image vanishes. We have:

$$(w_0 + w_{\alpha_1})...(w_0 + w_{\alpha_p}) = \sum_{z \in \mathrm{Vec}_{\mathbb{F}_2}(\alpha_1, ..., \alpha_p)} w_z \in \sum_{r=0}^{s} \mathbb{F}_2[(a_0 - a_{i_r})^{\perp}_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}]$$

It means that the vector sub-space spanned by $\alpha_1, ..., \alpha_p$ is contained in the union of vector hyperplanes $\bigcup_{r=1}^{s}(a_0 - a_{i_r})^{\perp}_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}$. Equivalently, it means that $\alpha_1 \wedge ... \wedge \alpha_p \in \sum_{r=1}^{s} \bigwedge^p (a_0 - a_{i_r})^{\perp}_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}$. Finally, descending to the isomorphism $\mu_{F,p}$, we obtain:

$$\begin{aligned}
\tilde{\mathcal{F}}_p(F, \sigma) &= \mu_{F,p}^{-1} \Big(\mathcal{I}_F^p \cap \sum_{r=0}^{s} \mathbb{F}_2[(a_0 - a_{i_r})^{\perp}_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}] / \mathcal{I}_F^{p+1} \cap \sum_{r=0}^{s} \mathbb{F}_2[(a_0 - a_{i_r})^{\perp}_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}] \Big) \\
&= \sum_{r=1}^{s} \bigwedge^p (a_0 - a_{i_r})^{\perp}_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}
\end{aligned}$$

However, as $((a_0 - a_{i_1})_{\mathbb{F}_2}, ..., (a_0 - a_{i_s})_{\mathbb{F}_2})$ forms a basis of $T_{\mathbb{F}_2}\sigma$, we have the identity:

$$\sum_{r=1}^{s} \bigwedge^p (a_0 - a_{i_r})^{\perp}_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp} = \sum_{\sigma^1 \subseteq \sigma} \bigwedge^p T_{\mathbb{F}_2}\sigma^{1\perp}/T_{\mathbb{F}_2}F^{\perp}$$

It concludes that in the case $\mathcal{E}(F, \sigma) \neq N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}$ we have:

$$\tilde{\mathcal{F}}_p(F, \sigma) = \sum_{\sigma^1 \subseteq \sigma} \bigwedge^p T_{\mathbb{F}_2}\sigma^{1\perp}/T_{\mathbb{F}_2}F^{\perp}$$

□

The proposition above is the motivation to use the definition of tropical cosheaves $\mathcal{F}_p$ in this context. Example 2.5.8 in the preliminaries gives the definition of tropical cosheaves in the case of a primitive triangulation. For non-primitive triangulations, we have already given a definition of tropical cosheaves $\mathcal{F}_p^R$ in equation 3.2 in Chapter 3, it was a general definition over an integral domain R. It was a definition over the cubical subdivision Ω of the triangulation Γ, and not over the poset Ξ^1 used in this chapter 4. However the posets Ω and Ξ^1 subdivide the same topological object X, the subdivision Ξ^1 is coarser than Ω, and according to a result of [Ch 2024], the homology of a cellular cosheaf on Ξ^1 will be te same as the homology of the induced cosheaf on the finer poset Ω. Then the tropical coheaves that we are about to define here on Ξ^1 for non-necessarily primitive triangulations will have the same homology as the homology of the tropical cosheaves defined in Chapter 3, which will be useful to apply our Theorems 1.3.1 and 1.3.2.

The integral domain R that interests us in this chapter is $\mathbb{F}_2$, and for simplicity we will denote this cosheaf by $\mathcal{F}_p$ instead of $\mathcal{F}_p^{\mathbb{F}_2}$. It is defined on every cell (F,σ) of Ξ^1 by:

$$\mathcal{F}_p(F,\sigma) := \sum_{\sigma^1 \subseteq \sigma} \bigwedge^p T_{\mathbb{F}_2}\sigma^{1\perp}/T_{\mathbb{F}_2}F^{\perp}$$

The cosheaves morphisms are induced by the inclusions $\mathcal{F}_p(F,\sigma) \subseteq \mathcal{F}_p(F,\sigma')$ if $\sigma \subseteq \sigma'$ and the projections $\mathcal{F}_p(F,\sigma) \twoheadrightarrow \mathcal{F}_p(F',\sigma)$ if $F \subseteq F'$. It commutes with the identities $\tilde{\mathcal{F}}_p(F,\sigma) = \mathcal{F}_p(F,\sigma)$, if they are satisfied. It leads to the following proposition:

Proposition 4.3.2. *Assume that any edge of the triangulation Γ is of odd integral length. Consider an integer $1 \leq k \leq n$ such that any k-simplex σ of the triangulation Γ satisfies $\mathcal{E}(P,\sigma) \neq N_{\mathbb{F}_2}$. Then we have the identity:*

$$\forall q \geq n-k, \quad H_q(\mathcal{F}_p) = H_q(\tilde{\mathcal{F}}_p)$$

Proof. Let (F,σ) a cell of Ξ^1 of dimension $\geq n-k$. We have $\dim F - \dim\sigma \geq n-k$ and thus $\dim\sigma \leq k$.

We can consider a k-simplex $\tau \in \Gamma$ which has σ as a face. According to Hypothesis of the proposition, we have $\mathcal{E}(P,\tau) \neq N_{\mathbb{F}_2}$. Moreover, we have the inclusion $\mathcal{E}(P,\sigma) \subseteq \mathcal{E}(P,\tau)$. It proves that $\mathcal{E}(P,\sigma) \neq N_{\mathbb{F}_2}$. According to the proof of the last proposition, it implies that $\mathcal{E}(P,\sigma)$ is an union of affine hyperplans in generic position. Denoting by $a_0,...,a_l$ the vertices of σ, there exists $i_1,...,i_s$ such that these affine hyperplans in generic position are of direction $(a_0 - a_{i_r})^{\perp}_{\mathbb{F}_2}$ for $1 \leq r \leq s$. Each one of these directions contains $T_{\mathbb{F}_2}F^{\perp}$. Then the quotients of these hyperplans by $T_{\mathbb{F}_2}F^{\perp}$ remain hyperplans of $N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}$ in generic position. $\mathcal{E}(F,\sigma)$ is the union of these quotients, thus $\mathcal{E}(F,\sigma) \neq N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}$. According to the last proposition, we obtain:

$$\tilde{\mathcal{F}}_p(F,\sigma) = \mathcal{F}_p(F,\sigma)$$

Thus the cosheaves $\tilde{\mathcal{F}}_p$ and $\mathcal{F}_p$ are equals on every cell of dimension $\geq n-k$. Moreover, for two incident cells $(F,\sigma) \hookrightarrow (F',\sigma')$ of dimension $\geq n-k$, the cosheaves morphisms are the same. To conclude the proposition, we just have to study the morphisms for two

incident cells $(F,\sigma) \hookrightarrow (F',\sigma')$ in the case $\dim(F,\sigma) = n-k$ and $\dim(F',\sigma') = n-k-1$. In this case we have:

$$\tilde{\mathcal{F}}_p(F',\sigma') = \begin{cases} \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F'^{\perp} & \text{if } \mathcal{E}(F',\sigma) = N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F'^{\perp} \\ \mathcal{F}_p(F',\sigma') & \text{otherwise} \end{cases}$$

However, in any case we have:

$$\tilde{\mathcal{F}}_p(F',\sigma') \subseteq \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F'^{\perp}$$

and the cosheaves morphisms of $\mathcal{F}_p$ and $\tilde{\mathcal{F}}_p$ between the cells $(F,\sigma) \hookrightarrow (F',\sigma')$ are the same. It proves that the homology groups are the same in every degree $\geq n-k$. □

Corollary 4.3.3. *Assume that any edge of the triangulation Γ is of odd integral length. In the case where any n-simplex σ of Γ satisfies $\mathcal{E}(P,\sigma) \neq N_{\mathbb{F}_2}$, we obtain:*

$$H_*(\mathcal{F}_p) = H_*(\tilde{\mathcal{F}}_p)$$

Proposition 4.3.4. *Assume that any edge of the triangulation Γ is of odd integral length. Assume that the triangulation Γ is $(k,\mathbb{F}_2)$-primitive with $0 \leq k \leq n$ (see definition 3.2.9). Then we have the identity :*

$$\forall p,q, \quad \text{if } q \geq n-k \text{ or } p < k, \quad H_q(\tilde{\mathcal{F}}_p) = H_q(\mathcal{F}_p)$$

Proof. If $k=0$, the statement is trivial because the homology of these cosheaves vanishes for $q \geq n$ or $p < 0$.

Consider $k \geq 1$. Consider σ a k-simplex of the triangulation Γ. Let $a_0, a_1, ..., a_k$ its vertices. The $(k,\mathbb{F}_2)$-primitivity and the assumption that any edge of the triangulation is of odd integral length implies that $((a_0-a_1)_{\mathbb{F}_2}, ..., (a_0-a_k)_{\mathbb{F}_2})$ forms a basis of $T_{\mathbb{F}_2}\sigma$. Then the affine hyperplans $\mathcal{E}(P,a_0a_1), ..., \mathcal{E}(P,a_0a_k)$ of $N_{\mathbb{F}_2}$ are in generic position. Thus their union $\mathcal{E}(P,\sigma)$ is different from $N_{\mathbb{F}_2}$

For every k-simplex σ of the triangulation Γ, we have $\mathcal{E}(P,\sigma) \neq N_{\mathbb{F}_2}$. Then we can apply Proposition 4.3.2 and we obtain:

$$\forall q \geq n-k, \quad H_q(\mathcal{F}_p) = H_q(\tilde{\mathcal{F}}_p)$$

Consider an integer $p < k$. Consider a simplex σ of the triangulation Γ. If $\dim\sigma \leq k$, we know from the proof of Proposition 4.3.2 that for any face F of the lattice polytope P containing σ, we have $\tilde{\mathcal{F}}_p(F,\sigma) = \mathcal{F}_p(F,\sigma)$. Now assume that $\dim\sigma > k$. Consider a face τ of σ of dimension k. This face is $\mathbb{F}_2$-primitive. Denoting by $a_0, a_1, ..., a_k$ its vertices, we have that $((a_0-a_1)_{\mathbb{F}_2}, ..., (a_0-a_k)_{\mathbb{F}_2})$ forms a basis of $T_{\mathbb{F}_2}\tau$. It leads to the fact that $\mathcal{F}_p(F,\tau)$ has the same dimension as if τ would be a primitive k-simplex. In the Lemma 2.4 of [BMR 2024], this dimension is computed. We obtain:

$$\dim\mathcal{F}_p(F,\tau) = \binom{\dim F}{p} - \binom{\dim F - \dim\tau}{p - \dim\tau}$$

However τ is of dimension k and $p < k$, thus we obtain:

$$\dim\mathcal{F}_p(F,\tau) = \binom{\dim F}{p} = \dim \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}$$

Therefore we have $\mathcal{F}_p(F,\tau) = \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}$. However, as τ is a face of σ we have $\mathcal{F}_p(F,\tau) \subseteq \mathcal{F}_p(F,\sigma)$ and thus $\mathcal{F}_p(F,\sigma) = \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^{\perp}$. Using Proposition 4.3.1 we finally obtain $\tilde{\mathcal{F}}_p(F,\sigma) = \mathcal{F}_p(F,\sigma)$. Then no matter the dimension of σ, we obtain the identity $\tilde{\mathcal{F}}_p(F,\sigma) = \mathcal{F}_p(F,\sigma)$. The morphisms are also compatible, then we finally obtain that the cosheaves $\tilde{\mathcal{F}}_p$ and $\mathcal{F}_p$ are equal if $p < k$. It concludes the proof:

$$\forall p < k, \quad H_q(\mathcal{F}_p) = H_q(\tilde{\mathcal{F}}_p)$$

□

Corollary 4.3.5. *Assume that any edge of the triangulation Γ is of odd integral length. Assume that the triangulation is $(n-1,\mathbb{F}_2)$-primitive. Then we have :*

$$\forall (p,q) \notin \{(n-1,0),(n,0)\}, \quad H_q(\tilde{\mathcal{F}}_p) = H_q(\mathcal{F}_p) \ ;$$

$$\dim H_0(\tilde{\mathcal{F}}_{n-1}) = \dim H_0(\mathcal{F}_p) + N_c \ ;$$

$$\dim H_0(\tilde{\mathcal{F}}_n) = N_c \ .$$

where N_c is the number of n-simplices σ of the triangulation Γ such that $\mathcal{E}(P,\sigma) = N_{\mathbb{F}_2}$.

Proof. The first equation is an immediate consequence of the last proposition. The cosheaves $\mathcal{F}_{n-1}$ and $\mathcal{F}_{n-1}$ are equals on every cell except on the cells of dimension 0. These cells are of the form (P,σ) with $\dim\sigma = n$. If $\mathcal{E}(P,\sigma) = N_{\mathbb{F}_2}$, then we have:

$$\dim\tilde{\mathcal{F}}_{n-1}(P,\sigma) = \dim \bigwedge^{n-1} N_{\mathbb{F}_2} = n$$

Let $a_0,...,a_n$ the vertices of σ. The fact that $\mathcal{E}(P,\sigma) = N_{\mathbb{F}_2}$ implies that $(a_0-a_1)^{\perp}_{\mathbb{F}_2},...,(a_0-a_n)^{\perp}_{\mathbb{F}_2}$ are not in generic position and then $((a_0-a_1)_{\mathbb{F}_2},...(a_0-a_n)_{\mathbb{F}_2})$ does not form a basis of $N_{\mathbb{F}_2}$. However any facet of σ is $\mathbb{F}_2$-primitive (because the triangulation Γ is $(n-1,\mathbb{F}_2)$-primitive). Consider τ the facet of vertices $a_0,...,a_{n-1}$. Then $((a_0-a_1)_{\mathbb{F}_2},...(a_0-a_{n-1})_{\mathbb{F}_2})$ is a basis of $T_{\mathbb{F}_2}\tau$. As $(a_0-a_n)_{\mathbb{F}_2}$ is a linear combination of the elements of this basis we obtain:

$$\mathcal{F}_p(P,\sigma) = \mathcal{F}_p(F,\tau) \ .$$

Moreover, as τ is a $\mathbb{F}_2$-primitive simplex, the dimension of $\mathcal{F}_p(F,\tau)$ is the same as if τ were primitive. Applying the Lemma 2.4 of [BMR 2024], we obtain:

$$\dim\mathcal{F}_{n-1}(P,\tau) = \binom{n}{n-1} - \binom{n-(n-1)}{n-1-(n-1)} = n-1 \ .$$

Then in the case of a cell of the form (P,σ) with $\dim\sigma = n$ and $\mathcal{E}(P,\sigma) = N_{\mathbb{F}_2}$, we have:

$$\dim\tilde{\mathcal{F}}_{n-1}(P,\sigma) - \mathcal{F}_p(P,\sigma) = n-(n-1) = 1 \ .$$

For any other cell of Ξ^1, this difference in dimension is zero; because the cosheaves $\tilde{\mathcal{F}}_{n-1}$ and $\mathcal{F}_{n-1}$ are equal on these cells. Comparing the Euler-characteristic of $\tilde{\mathcal{F}}_{n-1}$ and $\mathcal{F}_{n-1}$ we obtain the result:

$$\dim H_0(\tilde{\mathcal{F}}_{n-1}) = \dim H_0(\mathcal{F}_p) + N_c \ .$$

To prove the last equation, it suffices to notice that the cosheaf $\tilde{\mathcal{F}}_n$ is zero on every cell of Ξ^1 except those of the form (P,σ) with $\dim\sigma = n$ and $\mathcal{E}(P,\sigma) = N_{\mathbb{F}_2}$ (it is a consequence of the $(n-1,\mathbb{F}_2)$-primitivity). Then, the chain complex of this cosheaf $\tilde{\mathcal{F}}_n$ vanishes in every degree except in degree 0, and thus its homology is equal to it. However, for each cell of the form (P,σ) with $\dim\sigma = n$ and $\mathcal{E}(P,\sigma) = N_{\mathbb{F}_2}$, we have:

$$\dim \mathcal{F}_n(P,\sigma) = \dim \bigwedge^n N_{\mathbb{F}_2} = 1$$

Thus we obtain the last equation of the proposition:

$$\dim H_0(\tilde{\mathcal{F}}_n) = N_c \ .$$

□

Corollary 4.3.6. *Assume that any edge of the triangulation Γ is of integral length 1. Assume that the triangulation Γ is $\mathbb{F}_2$-primitive. Then we have:*

$$\forall q,p, \quad H_q(\tilde{\mathcal{F}}_p) = H_q(\mathcal{F}_p)$$

Proof. A $\mathbb{F}_2$-primitive triangulation is a $(n,\mathbb{F}_2)$-primitive triangulation, then it is a immediate consequence of Proposition 4.3.4. □

4.4 The partial Poincaré duality and the partial Lefschetz theorem for $\tilde{\mathcal{F}}_p$

In this section we use our partial Poincaré duality theorem (Theorem 1.3.1) and our complete Poincaré duality theorem (Theorem 1.3.2). We use also the Lefschetz theorem of Chenal (Theorem 2 of [Ch 2025], recalled in Section 2.7 of the preliminaries of this thesis). It enables us to obtain a partial Poincaré duality theorem and a partial Lefschetz theorem for the cosheaves $\tilde{F}_p$. As the homology of these cosheaves are the terms of the first page of the Renaudineau-Shaw spectral sequence, these results will be useful to obtain our bounds of the Betti numbers in the non-primitive case.

Theorem 4.4.1. *(Partial Poincaré Duality for $\tilde{\mathcal{F}}_p$) Assume that any edge of the triangulation Γ is of odd integral length. Assume that the triangulation Γ is $(k,\mathbb{F}_2)$-primitive with $0 \leq k \leq n$. Assume that the lattice polytope P is $\mathbb{F}_2$-non-singular (see Definitions 3.2.1 and 3.2.2). Then, we have:*

$$\forall p+q \geq 2n-k, \quad \dim H_q(\tilde{\mathcal{F}}_p) = \dim H_{n-1-q}(\tilde{\mathcal{F}}_{n-1-p})$$

Proof. Consider $p+q \geq 2n-k$. If $q < n-k$, then $p > n$ and thus $H_q(\tilde{\mathcal{F}}_p) = 0 = H_q(\mathcal{F}_p)$. If $q \geq n-k$, then applying Proposition 4.3.4 we also have $H_q(\tilde{\mathcal{F}}_p) = H_q(\mathcal{F}_p)$. Then in any case we have $H_q(\tilde{\mathcal{F}}_p) = H_q(\mathcal{F}_p)$. Moreover, if $n-1-p \geq k$, then we have $n-1-q \leq p+k-n-1 \leq -2$ and then $H_{n-1-q}(\tilde{\mathcal{F}}_{n-1-p}) = 0 = H_{n-1-q}(\mathcal{F}_{n-1-p})$. If $n-1-p < k$, then applying Proposition 4.3.4 we also have $H_{n-1-q}(\tilde{\mathcal{F}}_{n-1-p}) = H_{n-1-q}(\mathcal{F}_{n-1-p})$. Then in any case we have $H_{n-1-q}(\tilde{\mathcal{F}}_{n-1-p}) = H_{n-1-q}(\mathcal{F}_{n-1-p})$.

However, if $2 \leq k \leq n-1$, the cosheaf $\mathcal{F}_p$ satisfies the partial Poincaré duality Theorem 1.3.1. In reality, we have proven this Theorem 1.3.1 for the cubical subdivision of the triangulation Γ, which is finer that the subdivision Ξ^1, but the homology does not depend on the choice of the subdivision (see for instance the PhD thesis of Jules Chenal [Ch 2024] for a proof of the fact that the homologies are the same). Then our partial Poincaré duality theorem can be applied here, for the integral domain $R = \mathbb{F}_2$, and denoting by $\mathcal{F}^p$ the dual sheaf of the cosheaf $\mathcal{F}_p$, we obtain that for $p+q \geq 2n-k$ there is an isomorphism:

$$H^q(\mathcal{F}^p) \simeq H_{n-1-q}(\mathcal{F}_{n-1-p})$$

Moreover, if $2 \leq k = n$ the cosheaf $\mathcal{F}_p$ satisfies the complete Poincaré duality theorem (Theorem 1.3.2): for every integer p, q, there is an isomorphism:

$$H^q(\mathcal{F}^p) \simeq H_{n-1-q}(\mathcal{F}_{n-1-p})$$

Then this isomorphism exists in the particular case $p+q \geq 2n-k$ and we obtain that for every $2 \leq k \leq n$:

$$\forall p+q \geq 2n-k, \quad H^q(\mathcal{F}^p) \simeq H_{n-1-q}(\mathcal{F}_{n-1-p})$$

Moreover, if $0 \leq k \leq 1$ and $n \geq 1$, we have :

$$p+q \geq 2n-k \implies \begin{cases} p > n-1 \text{ or } q > n-1 \\ n-1-p < 0 \text{ or } n-1-q < 0 \end{cases} \implies \begin{cases} H^q(\mathcal{F}^p) = 0 \\ H_{n-1-q}(\mathcal{F}_{n-1-p}) = 0 \end{cases}$$

Thus, we obtain for any $n \geq 1$ and $0 \leq k \leq n$, the existence of the following isomorphisms:

$$\forall p+q \geq 2n-k, \quad H^q(\mathcal{F}^p) \simeq H_{n-1-q}(\mathcal{F}_{n-1-p})$$

However, as $\mathbb{F}_2$ is a field, we have the following identity of dimension:

$$\forall p, q, \quad \dim H^q(\mathcal{F}^p) = \dim H_q(\mathcal{F}_p)$$

It concludes the proof of this theorem. □

For a smooth complex algebraic variety and a generic hypersurface, the Lefschetz hyperplane section theorem gives the identification between the Hodge numbers $h^{q,p}$ of the hypersurface and the ambient variety, for $p+q$ strictly less than the dimension of the hypersurface. Here we assume that the polytope P is $\mathbb{F}_2$-non-singular (see Definitions 3.2.1 and 3.2.2), then the associated toric variety $\mathbb{C}Y$ is not necessarily smooth. However, the assumption of $\mathbb{F}_2$-non-singularity of the polytope P implies that the polytope is simple, and then the associated toric variety $\mathbb{C}Y$ is quasi-smooth and according to [Da 1978] Corollary 14.4 we can generalize the Hodge numbers $h^{p,q}(\mathbb{C}Y)$ in this case. Moreover, according to [DK 1987] Corollary 2.5, these Hodge numbers are:

$$h^{q,p}(\mathbb{C}Y) = \begin{cases} \sum_{i=0}^{p} (-1)^{p-i} \binom{n-i}{p-i} f_{n-i}(P) & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases} \tag{4.6}$$

where $f_{n-i}(P)$ denotes the number of faces of dimension $n-i$ of the polytope P.

Moreover, if we consider a non-degenerate algebraic hypersurface $\mathbb{C}\tilde{X}$ of the toric variety $\mathbb{C}Y$ (that is an hypersurface which intersects each orbit of $\mathbb{C}Y$ transversely), according to [DK 1987] Theorem 3.7, we have a Lefschetz hyperplane section theorem and the Hodge numbers $h^{p,q}(\mathbb{C}\tilde{X})$ of the hypersurface satisfy:

$$\forall p+q<n-1, \quad h^{p,q}(\mathbb{C}\tilde{X})=h^{p,q}(\mathbb{C}Y) \tag{4.7}$$

Using a theorem of [Ch 2025], this result will lead to a partial Lefschetz theorem:

Theorem 4.4.2. *(Partial Lefschetz theorem) Assume that any edge of the triangulation Γ is of odd integral length. Assume that the triangulation Γ is $(k,\mathbb{F}_2)$-primitive with $0\leq k\leq n$. Assume that the polytope P is $\mathbb{F}_2$-non-singular (see Definitions 3.2.1 and 3.2.2). Then we have:*

$$\forall p<\min(k,n-1-q), \quad \dim H_q(\tilde{\mathcal{F}}_p)=h^{p,q}(\mathbb{C}Y)$$

where $h^{p,q}(\mathbb{C}Y)=\begin{cases}\sum_{i=0}^{p}(-1)^{p-i}\binom{n-i}{p-i}f_{n-i}(P) & \text{if } p=q\\ 0 & \text{if } p\neq q\end{cases}$ *(with $f_{n-i}(P)$ the number of faces of dimension $n-i$ of the polytope P) are the Hodge numbers of the complex projective toric variety associated to the lattice polytope P.*

Proof. If $k=0$ the result is trivial because $H_p(\tilde{\mathcal{F}}_p)$ and $h^{p,q}(\mathbb{C}\tilde{X})$ both vanish for $p<0$. In the following, we assume that $k\geq 1$.

From Proposition 4.3.4 we know that for every $q\geq n-k$ or $p<k$ we have $H_q(\tilde{\mathcal{F}}_p)=H_q(\mathcal{F}_p)$.

In [Ch 2025], Jules Chenal introduced the saturated cosheaf $\hat{\mathcal{F}}_p^{\mathbb{Z}}$ of the cosheaf $\mathcal{F}_p^{\mathbb{Z}}$:

$$\forall (F,\sigma)\in\Xi^1, \quad \hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma):=\left\{\alpha\in\bigwedge^p N/T_{\mathbb{Z}}F^{\perp}\mid \exists\lambda\in\mathbb{Z},\ \lambda\alpha\in\mathcal{F}_p^{\mathbb{Z}}(F,\sigma)\right\}$$

The morphisms of this cosheaf are induced by the inclusions $\hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma)\hookrightarrow\hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\tau)$ for $\sigma,\tau\in\Gamma,\ \sigma\subseteq\tau\subseteq F$, and the projections $\bigwedge^p N/T_{\mathbb{Z}}F^{\perp}\twoheadrightarrow\bigwedge^p N/T_{\mathbb{Z}}G^{\perp}$ for $G\subseteq F$ faces of the polytope P. They are extensions of the morphism of the cosheaf $\mathcal{F}_p^{\mathbb{Z}}$.

In his Theorem 2 of [Ch 2025], Jules Chenal considers an integer $\delta(P)$ depending on the polytope P. He defines $\delta(P)$ as the least common multiple of the exponents of the finite groups $\Delta(F)$ for F face of the polytope P, where:

$$\Delta(F):=T_{\mathbb{Z}}F^{\perp}/\bigoplus_{Q\supseteq F,\ Q\text{ face of }P,\ \dim Q=n-1}T_{\mathbb{Z}}Q^{\perp}$$

However, we assume that the polytope P is $\mathbb{F}_2$-non-singular (see Definitions 3.2.1 and 3.2.2). Then it is $\mathbb{F}_2$-non-singular on any one of its faces, that is that for any face F, there exists a linearly independent family $\{e_1,...,e_{n-\dim F}\}$ of vectors of $M_{\mathbb{F}_2}$ such that for every face G of P for which F is a face of G, there exists a subfamily $\{e_{i_1},...,e_{i_{n-\dim G}}\}$ of $\{e_1,...,e_{n-\dim F}\}$ such that:

$$T_{\mathbb{F}_2}G=T_{\mathbb{F}}F\oplus\mathrm{Vec}_{\mathbb{F}_2}\{e_{i_1},...,e_{i_{n-\dim G}}\}$$

Considering $\{e_{n-\dim F+1},...,e_n\}$ a basis of $T_{\mathbb{F}_2}F$, we can dualize the base $(e_1,...,e_n)$ to obtain a basis $(e_1^*,...,e_n^*)$ of $N_{\mathbb{F}_2}$ and we obtain:

$$T_{\mathbb{F}_2}F^\perp = \mathrm{Vec}_{\mathbb{F}_2}\{e^*_{n-\dim F+1},...,e^*_n\}\ ;$$

$$\bigoplus_{Q\supseteq F,\ Q \text{ face of } P,\ \dim Q=n-1} T_{\mathbb{F}_2}Q^\perp = \bigoplus_{i=n-\dim F+1}^{n} \mathrm{Vec}_{\mathbb{F}_2}\{e_i^*\} = T_{\mathbb{F}_2}F^\perp.$$

Thus we obtain:

$$\Delta(F)\otimes\mathbb{F}_2 = T_{\mathbb{F}_p}F^\perp / \bigoplus_{Q\supseteq F,\ Q \text{ face of } P,\ \dim Q=n-1} T_{\mathbb{F}_2}Q^\perp = 0$$

Then the $\mathbb{F}_2$-non-singularity of the polytope P implies that for any face F of P the exponent of the finite group $\Delta(F)$ is odd, and then their least common multiple $\delta(P)$ is also odd.

It proves that the Lefschetz theorem (Theorem 2 of [Ch 2025]) can be applied for $\hat{\mathcal{F}}_p^{\mathbb{Z}}\otimes\mathbb{F}_2$, and applying also his Proposition 3.9, we obtain:

$$\forall p+q<n-1,\ \ \dim\ H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}\otimes\mathbb{F}_2) = \begin{cases} \sum_{i=0}^{p}(-1)^{p-i}\binom{n-i}{p-i}f_{n-i}(P) & \text{if } p=q \\ 0 & \text{if } p\neq q \end{cases}$$
$$= h^{p,q}(\mathbb{C}Y)$$

Now, to conclude the proof, we have to prove that $H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}\otimes\mathbb{F}_2) = H_q(\mathcal{F}_p)$ for all $p<k$.

Consider the case $p<k$. Consider a cell $(F,\sigma)\in\Xi^*$ such that $\dim\sigma\geq k$. Then the simplex σ has a $\mathbb{F}_2$-primitive k-face τ and applying lemma 2.4 of [BMR 2024], we obtain :

$$\dim\mathcal{F}_p(F,\sigma)\geq\dim\mathcal{F}_p(F,\tau) = \binom{\dim F}{p} - \binom{\dim F-k}{p-k} = \binom{\dim F}{p} = \dim\bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp.$$

Then we have $\mathcal{F}_p(F,\sigma) = \bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp$. Moreover, by definition of $\hat{\mathcal{F}}_p^{\mathbb{Z}}$ we have the inclusions:

$$\mathcal{F}_p(F,\sigma)\subseteq\hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma)\otimes\mathbb{F}_2\subseteq\bigwedge^p N_{\mathbb{F}_2}/T_{\mathbb{F}_2}F^\perp$$

Thus we have $\mathcal{F}_p(F,\sigma) = \hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma)$.

Consider a cell $(F,\sigma)\in\Xi^*$ such that $\dim\sigma\leq k$ (and no assumption on p). Then the simplex σ is $\mathbb{F}_2$-primitive and we have:

$$\dim\mathcal{F}_p(F,\sigma) = \mathrm{rk}\mathcal{F}_p^{\mathbb{Z}}(F,\sigma) = \mathrm{rk}\hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma) = \dim\hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma)\otimes\mathbb{F}_2$$

Thus, we obtain:

$$\mathcal{F}_p(F,\sigma) = \hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma)\otimes\mathbb{F}_2$$

These considerations prove the following identity of complexes:

$$\forall p<k \text{ or } q\geq n-k,\quad C_q(\mathcal{F}_p) = C_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}\otimes\mathbb{F}_2) \tag{4.8}$$

However, the cosheaves morphisms of $\mathcal{F}_p$ are the restriction and co-restriction of the cosheaves morphism of $\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2$, thus these identities of complexes imply the following identities in homology:

$$\forall p < k, \text{ or } q \geq n-k, \quad H_q(\mathcal{F}_p) = H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2) \tag{4.9}$$

However, we have the equivalences:

$$\left\{ \begin{array}{c} p < k \text{ or } q \geq n-k \\ p+q < n-1 \end{array} \right. \iff \left\{ \begin{array}{c} p < k \\ p+q < n-1 \end{array} \right. \iff p < \min(k, n-1-q)$$

Moreover, if $q < 0$, we have obviously for any p the identity $H_q(\mathcal{F}_p) = 0 = H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2)$, and then for any $p+q < k$ we have $H_q(\mathcal{F}_p) = H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2)$. Using identity 4.6 it implies the partial Lefschetz theorem:

$$\forall p < \min(k, n-1-q), \quad \dim H_q(\tilde{\mathcal{F}}_p) = h^{p,q}(\mathbb{C}Y)$$

□

Corollary 4.4.3. *Assume that any edge of the triangulation Γ is of odd integral length. Assume that the triangulation Γ is (k,R)-primitive. Assume that the lattice polytope P is $\mathbb{F}_2$-non-singular (see Defintions 3.2.1 and 3.2.2). Then we have:*

$$\forall p < \min(k, n-1-q), \quad \dim H_q(\tilde{\mathcal{F}}_p) = \left\{ \begin{array}{ll} \sum_{i=0}^{p} (-1)^{p-i} \binom{n-i}{p-i} f_{n-i}(P) & \text{if } p = q \\ 0 & \text{if } p \neq q \end{array} \right. ;$$

$$\forall p+q > 2n-k-1, \quad \dim H_q(\tilde{\mathcal{F}}_p) = \left\{ \begin{array}{ll} \sum_{i=0}^{n-1-p} (-1)^{n-1-p-i} \binom{n-i}{n-1-p-i} f_{n-i}(P) & \text{if } p = q \\ 0 & \text{if } p \neq q \end{array} \right.$$

where $f_{n-i}(P)$ denotes the number of faces of P of dimension $n-i$. Moreover if we consider $\mathbb{C}\tilde{X}$ a non-degenerate complex algebraic hypersurface in the toric variety $\mathbb{C}Y$, we have:

$$\forall p \notin [\min(k, n-1-k), 2n-k-1-q], \quad \dim H_q(\tilde{\mathcal{F}}_p) = h^{p,q}(\mathbb{C}\tilde{X})$$

Proof. Applying Partial Lefschetz Theorem, we obtain:

$$\forall p < \min(k, n-1-q), \quad \dim H_q(\tilde{\mathcal{F}}_p) = h^{p,q}(\mathbb{C}Y)$$

However, as the lattice polytope P is simple, we can apply Equation 4.7 and we obtain:

$$\forall p, q, \quad h^{p,q}(\mathbb{C}Y) = \left\{ \begin{array}{ll} \sum_{i=0}^{p} (-1)^{p-i} \binom{n-i}{p-i} f_{n-i}(P) & \text{if } p = q \\ 0 & \text{if } p \neq q \end{array} \right.$$

However, applying Equation 4.6, for a non-degenerate complex algebraic hypersurface $\mathbb{C}\tilde{X}$ of the toric variety $\mathbb{C}Y$, we have:

$$\forall p+q < n-1, \quad h^{p,q}(\mathbb{C}\tilde{X}) = h^{p,q}(\mathbb{C}Y)$$

Moreover, the non-degenerate hypersurface $\mathbb{C}\tilde{X}$ satisfies the classical Poincaré duality:

$$\forall p,q, \quad h^{p,q}(\mathbb{C}\tilde{X}) = h^{n-1-p,n-1-q}(\mathbb{C}\tilde{X})$$

and the partial Poincaré duality theorem gives:

$$\forall p+q > 2n-k-1, \quad \dim H_q(\tilde{\mathcal{F}}_p) = \dim H_{n-1-q}(\tilde{\mathcal{F}}_{n-1-p})$$

If $p+q > 2n-k-1$ then $(n-1-p) < \min(k, n-1-(n-1-q))$, then applying the five last equations, we can conclude the proof. □

Corollary 4.4.4. *Assume that any edge of the triangulation Γ is of odd integral length. Assume that the triangulation Γ is (k,R)-primitive. If the polytope P is a the n-simplex of vertices $(0,...,0)$, $(d,0,...,0)$, ..., $(0,...,0,d)$ for $d \geq 1$ a fixed integer, then we have:*

$$\forall p \notin [\min(k, n-1-q), 2n-k-1-q], \quad \dim H_q(\tilde{\mathcal{F}}_p) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$$

Proof. As the polytope P is a n-simplex, we have:

$$f_{n-i}(P) = \binom{n+1}{n-i+1}$$

Then we have for any $p \geq 0$:

$$\sum_{i=0}^{p} (-1)^{p-i} \binom{n-i}{p-i} f_{n-i}(P) = \sum_{i=0}^{p} (-1)^{p-i} \binom{n-i}{p-i} \binom{n+1}{n-i+1}$$

Applying the Pascal triangle, an Abel transformation, and the Pascal triangle again, we obtain:

$$\begin{aligned} \sum_{i=0}^{p} (-1)^{p-i} \binom{n-i}{p-i} \binom{n+1}{n-i+1} &= \sum_{i=0}^{p} (-1)^{p-i} \binom{n-i}{p-i} \left[\binom{n}{n-i} + \binom{n}{n-i+1} \right] \\ &= \sum_{i=0}^{p} (-1)^{p-i} \left[\binom{n-i}{p-i} - \binom{n-i-1}{p-i-1} \right] \binom{n}{n-i} \\ &= \sum_{i=0}^{p} (-1)^{p-i} \binom{n-i-1}{p-i} \binom{n}{n-i} \end{aligned}$$

We can notice that we obtain the same expression replacing n by $n-1$. We can iterate this process to replace at the end n by p:

$$\begin{aligned} \sum_{i=0}^{p} (-1)^{p-i} \binom{n-i}{p-i} f_{n-i}(P) &= \sum_{i=0}^{p} (-1)^{p-i} \binom{p-i}{p-i} \binom{p+1}{p-i+1} \\ &= 1 + \sum_{i=0}^{p+1} (-1)^{p-i} \binom{p+1}{p-i+1} \\ &= 1 - (1-1)^{p+1} = 1 \end{aligned}$$

It concludes the proof applying the previous corollary, using the fact that the simplex P is non-singular and then also $\mathbb{F}_2$-non-singular. □

4.5 Bounds on the Betti numbers of X

In this section we use our partial Poincaré duality theorem and partial Lefschetz theorem established in the previous section to deduce expressions of the bounds of the Betti numbers of non-necessarily primitive patchworks. We state and prove our Theorems 1.3.4, 1.3.5 and 1.3.6 stated in the Introduction of this thesis. At the end of this section, we also interpret our bounds in terms of the 2-torsion of the homology groups $H_q(\hat{\mathcal{F}}_p)$ (these homology groups are defined in Section 2.7 of preliminaries).

Lemma 4.5.1. *Assume that any edge of the triangulation Γ is of odd integral length. Assume that the triangulation is $(k, \mathbb{F}_2)$-primitive. Assume that the lattice polytope P is $\mathbb{F}_2$-non-singular. If $k \leq n-1$, the Betti numbers of X are bounded in the following way:*

$$\forall q \geq n-k/2 \text{ or } (n-1)/2 > q \geq n-k, \quad b_q(X;\mathbb{F}_2) \leq h^{q,q}(\mathbb{C}\tilde{X}) + \sum_{p=n-1-q}^{2n-k-1-q} \dim H_q(\mathcal{F}_p) \ ; \tag{4.10}$$

$$\forall q < \min(k, n-k), \quad b_q(X;\mathbb{F}_2) \leq h^{q,q}(\mathbb{C}\tilde{X}) + \sum_{p=k}^{n} \dim H_q(\tilde{\mathcal{F}}_p) \ ; \tag{4.11}$$

$$\forall q \in [\max(n-k, (n-1)/2), n-(k+1)/2], \quad b_q(X;\mathbb{F}_2) \leq \sum_{p=n-1-q}^{2n-k-1-q} \dim H_q(\mathcal{F}_p) \ ; \tag{4.12}$$

$$\forall q \in [k, n-k-1], \quad b_q(X;\mathbb{F}_2) \leq \sum_{p=k}^{n} \dim H_q(\tilde{\mathcal{F}}_p) \ . \tag{4.13}$$

If $k = n$, the triangulation Γ is $\mathbb{F}_2$-primitive and the bounds of the Betti numbers of X are the following:

$$\forall\ q, p, \quad b_q(X;\mathbb{F}_2) \leq \begin{cases} h^{q,q}(\mathbb{C}\tilde{X}) + \dim H_q(\mathcal{F}_{n-1-q}) & \text{if } q \neq (n-1)/2 \\ \dim H_{(n-1)/2}(\mathcal{F}_{(n-1)/2}) & \text{if } q = (n-1)/2 \end{cases} \tag{4.14}$$

Remark 4.5.2. *The cosheaves $\mathcal{F}_p$ do not depend on the choice of the signs ϵ_v for v vertex of the triangulation Γ. They depend only on the polytope P and the triangulation Γ. Then if every edge of the triangulation is of odd integral length and if the tirangulation is $(k, \mathbb{F}_2)$-primitive with $k \leq n-1$, the bounds of the Betti numbers $b_q(X;\mathbb{F}_2)$ for $q \geq n-k$ depend only on the polytope P and the triangulation Γ (Equations 4.10 and 4.12). The bounds of the Betti numbers $b_q(X;\mathbb{F}_2)$ for $q < n-k$ can depend also on the signs $\epsilon_v \in \mathbb{F}_2$ for v vertex of the triangulation, because the dimension of the homology groups of some $\tilde{\mathcal{F}}_p$ appear, and these cosheaves can depend on the signs (Equations 4.11 and 4.13). In the $\mathbb{F}_2$-primitive case, every bound depends only on the polytope P and the triangulation Γ (Equation 4.14).*

Proof. Proof of the Lemma: If $k \leq n-1$, Corollary 4.4.3 gives:

$$\forall p \notin [\min(k, n-1-q), 2n-k-1-q], \dim H_q(\tilde{\mathcal{F}}_p) = \begin{cases} h^{q,q}(\mathbb{C}\tilde{X}) & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$$

Using Proposition 4.3.4 to identify $\dim H_q(\mathcal{F}_p)$ and $\dim H_q(\tilde{\mathcal{F}}_p)$ for any $q \geq n-k$ or $p < k$, we obtain the following bounds from Equation 4.5:

$$\begin{aligned} b_q(X;\mathbb{F}_2) &\leq \sum_{p=0}^{n} \dim H_q(\tilde{\mathcal{F}}_p) \\ &= \sum_{p \notin [\min(k,n-1-q),2n-k-1-q]} h^{p,q}(\mathbb{C}\tilde{X}) + \sum_{p \in [\min(k,n-1-q),2n-k-1-q]} \dim H_q(\tilde{\mathcal{F}}_p) \end{aligned}$$

However we know from Corollary 4.4.3 that $h^{q,p}(\mathbb{C}\tilde{Y})$ vanishes except if $p = q$. Moreover, we have the equivalences:

$$\begin{cases} p \notin [\min(k, n-1-q), 2n-k-1-q] \\ q = p \end{cases} \iff \begin{cases} q \notin [\min(k, (n-1)/2), n-(k+1)/2] \\ q = p \end{cases}$$

Then, we obtain:

$$\begin{aligned} &\forall q \notin [\min(k,(n-1)/2), n-(k+1)/2], \\ &b_q(X;\mathbb{F}_2) \leq h^{q,q}(\mathbb{C}\tilde{X}) + \sum_{p \in [\min(k,n-1-q),2n-k-1-q]} \dim H_q(\tilde{\mathcal{F}}_p) \ ; \\ &\forall q \in [\min(k,(n-1)/2), n-(k+1)/2], \\ &b_q(X;\mathbb{F}_2) \leq \sum_{p \in [\min(k,n-1-q),2n-k-1-q]} \dim H_q(\tilde{\mathcal{F}}_p) \ . \end{aligned}$$

Now, we can apply Proposition 4.3.4 to identify $\dim H_q(\mathcal{F}_p)$ and $\dim H_q(\tilde{\mathcal{F}}_p)$ for any $q \geq n-k$ or $p < k$. We have the equivalences:

$$\begin{cases} q \notin [\min(k,(n-1)/2), n-(k+1)/2] \\ q \geq n-k \end{cases} \iff q \geq n-k/2 \text{ or } (n-1)/2 > q \geq n-k \ ;$$

$$\begin{cases} q \notin [\min(k,(n-1)/2), n-(k+1)/2] \\ q < n-k \end{cases} \iff q < \min(k, n-k) \ ;$$

$$\begin{cases} q \in [\min(k,(n-1)/2), n-(k+1)/2] \\ q \geq n-k \end{cases} \iff q \in [\max(n-k, (n-1)/2), n-(k+1)/2] \ ;$$

$$\begin{cases} q \in [\min(k,(n-1)/2), n-(k+1)/2] \\ q < n-k \end{cases} \iff q \in [k, n-k-1] \ .$$

Moreover, if $q < n-k$, we have $\min(k, n-1-q) = k$ and $2n-k-1-q \geq n$. And if $q \geq n-k$ we have $\min(k, n-1-q) = n-1-q$ and $2n-k-1-q < n$. Therefore, we

obtain:

$$\begin{aligned}
&\forall q \geq n-k/2 \text{ or } (n-1)/2 > q \geq n-k,\\
&b_q(X;\mathbb{F}_2) \leq h^{q,q}(\mathbb{C}\tilde{X}) + \sum_{p\in[n-1-q,2n-k-1-q]} \dim H_q(\mathcal{F}_p)\ ;\\
&\forall q < \min(k, n-k),\\
&b_q(X;\mathbb{F}_2) \leq h^{q,q}(\mathbb{C}\tilde{X}) + \sum_{p\in[k,n]} \dim H_q(\tilde{\mathcal{F}}_p)\ ;\\
&\forall q \in [\max(n-k,(n-1)/2), n-(k+1)/2],\\
&b_q(X;\mathbb{F}_2) \leq \sum_{p\in[n-1-q,2n-k-1-q]} \dim H_q(\mathcal{F}_p)\ ;\\
&\forall q \in [k, n-k-1],\\
&b_q(X;\mathbb{F}_2) \leq \sum_{p\in[k,n]} \dim H_q(\tilde{\mathcal{F}}_p)\ .
\end{aligned}$$

If $k = n$ (the $\mathbb{F}_2$-primitive case), Corollary 4.4.3 gives:

$$\forall p+q \neq n-1,\quad H_q(\tilde{\mathcal{F}}_p) = \begin{cases} h^{q,q}(\mathbb{C}\tilde{X}) & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$$

Then, applying Corollary 4.3.6, the bounds in the $\mathbb{F}_2$-primitive case can be obtained from Equation 4.5 □

Corollary 4.5.3. *Assume that $n \geq 3$. Assume that any edge of the triangulation Γ is of odd integral length. Assume that the triangulation Γ is $(n-1,\mathbb{F}_2)$-primitive. Assume that the polytope P is $\mathbb{F}_2$-non-singular. Then we obtain:*

$$\begin{aligned}
&\forall q \geq (n+1)/2 \text{ or } (n-1)/2 > q \geq 1,\\
&b_q(X;\mathbb{F}_2) \leq h^{q,q}(\mathbb{C}\tilde{X}) + \dim H_q(\mathcal{F}_{n-1-q}) + \dim H_q(\mathcal{F}_{n-q})\ ;\\
&\forall q \in [(n-1)/2, n/2],\\
&b_q(X;\mathbb{F}_2) \leq \dim H_q(\mathcal{F}_{n-1-q}) + \dim H_q(\mathcal{F}_{n-q})\ .
\end{aligned}$$

And also:

$$b_0(X;\mathbb{F}_2) \leq 1 + \dim H_0(\mathcal{F}_{n-1}) + 2N_c\ .$$

where N_c is the number of n-simplices σ of the triangulation Γ such that $\mathcal{E}(P,\sigma) = N_{\mathbb{F}_2}$.

Remark 4.5.4. *For $n \geq 3$, if every edge of the triangulation is of odd integral length and if the triangulation is $(n-1,\mathbb{F}_2)$-primitive, for any $q \neq 0$, the bound of $b_q(X;\mathbb{F}_2)$ depend only on the polytope P and the triangulation Γ. The bound of $b_0(X;\mathbb{F}_2)$ can depend also on the choice of the signs $\epsilon_v \in \mathbb{F}_2$ on the vertices of the triangulation Γ, because the number N_c appears in its expression.*

Proof. proof of the corollary: We have to apply the previous Lemma in the case $k = n-1$ and $n \geq 3$. In this case we have the equivalences:

$$q \geq n-k/2 \text{ or } (n-1)/2 > q \geq n-k \iff q \geq (n+1)/2 \text{ or } (n-1)/2 > q \geq 1\ ;$$

$$q < \min(k, n-k) \iff q \leq 0 \ ;$$
$$q \in [\max(n-k, (n-1)/2), n-(k+1)/2] \iff q \in [(n-1)/2, n/2] \ ;$$
$$q \in [k, n-k-1] \iff q \in \emptyset \ ;$$

Then applying the previous lemma, we obtain:

$$\begin{aligned}
&\forall q \geq (n+1)/2 \text{ or } (n-1)/2 > q \geq 1,\\
&b_q(X;\mathbb{F}_2) \leq h^{q,q}(\mathbb{C}\tilde{X}) + \dim H_q(\mathcal{F}_{n-1-q}) + \dim H_q(\mathcal{F}_{n-q}) \ ;\\
&\forall q \leq 0,\\
&b_q(X;\mathbb{F}_2) \leq h^{q,q}(\mathbb{C}\tilde{X}) + \dim H_q(\tilde{\mathcal{F}}_{n-1}) + \dim H_q(\tilde{\mathcal{F}}_n) \ ;\\
&\forall q \in [(n-1)/2, n/2],\\
&b_q(X;\mathbb{F}_2) \leq \dim H_q(\mathcal{F}_{n-1-q}) + \dim H_q(\mathcal{F}_{n-q}) \ .
\end{aligned}$$

Moreover, applying Equation 4.6, we have $h^{0,0}(\mathbb{C}\tilde{X}) = (-1)^0\binom{n}{0} f_n(P) = 1$, and applying Corollary 4.3.5, we have $\dim H_0(\tilde{\mathcal{F}}_{n-1}) = \dim H_0(\mathcal{F}_{n-1}) + N_c$ and $\dim H_0(\tilde{\mathcal{F}}_n) = N_c$ and then we obtain the bound of $b_0(X;\mathbb{F}_2)$, which concludes the proof. □

In the proof of the partial Lefschetz theorem Theorem 4.4.2 we use the cosheaf $\hat{\mathcal{F}}_p^{\mathbb{Z}}$ introduced by Jules Chenal in [Ch 2025], defined as being the saturation of the cosheaf $\mathcal{F}_p^{\mathbb{Z}}$:

$$\forall (F,\sigma) \in \Xi^1, \quad \hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma) := \left\{ \alpha \in \bigwedge^p N/T_{\mathbb{Z}}F^{\perp} \mid \exists \lambda \in \mathbb{Z},\ \lambda\alpha \in \mathcal{F}_p^{\mathbb{Z}}(F,\sigma) \right\}$$

We precised that the morphisms of this cosheaf are induced by the inclusions $\hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma) \hookrightarrow \hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\tau)$ for $\sigma, \tau \in \Gamma$, $\sigma \subseteq \tau \subseteq F$, and the projections $\bigwedge^p N/T_{\mathbb{Z}}F^{\perp} \twoheadrightarrow \bigwedge^p N/T_{\mathbb{Z}}G^{\perp}$ for $G \subseteq F$ faces of the polytope P. We precised also that these morphisms are extensions of the morphisms of the cosheaf $\mathcal{F}_p^{\mathbb{Z}}$. In this proof of Theorem 4.4.2 we have established that if every edge of the triangulation Γ is of odd integral length and if the triangulation Γ is $(k, \mathbb{F}_2)$ with $0 \leq k \leq n$, then we have Equation 4.9:

$$\forall p < k, \text{ or } q \geq n-k, \qquad H_q(\mathcal{F}_p) = H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2)$$

However, the homology of $\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2$ can be understood using universal coefficient theorem which gives the following short exact sequence:

$$0 \to H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}) \otimes \mathbb{F}_2 \to H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2) \to \mathrm{Tor}(H_{q-1}(\hat{\mathcal{F}}_p^{\mathbb{Z}}); \mathbb{F}_2) \to 0$$

We obtain:

$$\begin{aligned}
\dim H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2) &= \dim H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}) \otimes \mathbb{F}_2 + \dim \mathrm{Tor}(H_{q-1}(\hat{\mathcal{F}}_p^{\mathbb{Z}}); \mathbb{F}_2)\\
&= \dim H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}) \otimes \mathbb{Q} + \dim \mathrm{Tor}(H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}); \mathbb{F}_2) + \dim \mathrm{Tor}(H_{q-1}(\hat{\mathcal{F}}_p^{\mathbb{Z}}); \mathbb{F}_2)\\
&= \dim H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{Q}) + \dim \mathrm{Tor}(H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}); \mathbb{F}_2) + \dim \mathrm{Tor}(H_{q-1}(\hat{\mathcal{F}}_p^{\mathbb{Z}}); \mathbb{F}_2)
\end{aligned}$$

However, applying Theorem 2 of [Ch 2025] for $\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{Q}$, we obtain:

$$\forall p+q < n-1, \quad \dim H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{Q}) = h^{p,q}(\mathbb{C}\tilde{Y}) = h^{p,q}(\mathbb{C}\tilde{X})$$

where $\mathbb{C}\tilde{X}$ is a non-degenerate hypersurface of the toric variety $\mathbb{C}Y$.

However we have the identity $\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{Q} = \mathcal{F}_p^{\mathbb{Q}}$, and using our Poincaré duality theorem over $\mathbb{Q}$ (Corollary 1.3.3) and Poincaré duality for the hypersurface $\mathbb{C}\tilde{X}$ we obtain:

$$\forall p+q > n-1, \quad \dim H_q(\mathcal{F}_p^{\mathbb{Q}}) = \dim H_{n-1-q}(\mathcal{F}_{n-1-p}^{\mathbb{Q}}) = h^{n-1-p,n-1-q}(\mathbb{C}\tilde{X}) = h^{p,q}(\mathbb{C}\tilde{X})$$

Then we have:

$$\forall p+q \neq n-1, \quad \dim H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{Q}) = h^{p,q}(\mathbb{C}\tilde{X})$$

And thus, we obtain:

$$\forall p+q \neq n-1,$$
$$\dim H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2) = h^{p,q}(\mathbb{C}\tilde{X}) + \dim \mathrm{Tor}(H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}); \mathbb{F}_2) + \dim \mathrm{Tor}(H_{q-1}(\hat{\mathcal{F}}_p^{\mathbb{Z}}); \mathbb{F}_2)$$

Finally, we have:

$$\forall p < k \text{ or } q \geq n-k, \quad \text{if } p+q \neq n-1, \tag{4.15}$$
$$\dim H_q(\mathcal{F}_p) = h^{p,q}(\mathbb{C}\tilde{X}) + \dim \mathrm{Tor}(H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}}); \mathbb{F}_2) + \dim \mathrm{Tor}(H_{q-1}(\hat{\mathcal{F}}_p^{\mathbb{Z}}); \mathbb{F}_2) \tag{4.16}$$

Then, if we want to express the bounds of the Betti numbers of X in terms of the 2-torsion of the homology of $\hat{\mathcal{F}}_p^{\mathbb{Z}}$ and the Hodge numbers of the non-degenerate hypersurface $\mathbb{C}\tilde{X}$, we have to remove the terms $\dim H_q(\mathcal{F}_p)$ for $p+q = n-1$. To do that, we can use Euler characteristics:

$$\hat{\chi}_p := \sum_{q=0}^{n-1} (-1)^q \dim H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2) = \sum_{(F,\sigma)\in\Xi^1} (-1)^{\dim F - \dim\sigma} \dim \hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma) \otimes \mathbb{F}_2$$

However as $\hat{\mathcal{F}}_p^{\mathbb{Z}}$ is the saturation of $\mathcal{F}_p^{\mathbb{Z}}$, the rank of $\hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma)$ is equal to the rank in the primitive case. Using Lemma 2.4 of [BMR 2024], we obtain:

$$\mathrm{rk}\hat{\mathcal{F}}_p^{\mathbb{Z}}(F,\sigma) = \binom{\dim F}{p} - \binom{\dim F - \dim\sigma}{p - \dim\sigma} \tag{4.17}$$

We obtain:

$$\hat{\chi}_p = \sum_{(F,\sigma)\in\Xi^1} (-1)^{\dim F - \dim\sigma} \left[\binom{\dim F}{p} - \binom{\dim F - \dim\sigma}{p - \dim\sigma} \right] \tag{4.18}$$

It enables us to obtain an expression of $\dim H_q(\hat{\mathcal{F}}_{n-1-q} \otimes \mathbb{F}_2)$:

$$\dim H_q(\hat{\mathcal{F}}_{n-1-q} \otimes \mathbb{F}_2) = (-1)^q \hat{\chi}_{n-1-q} - \sum_{r\neq q} (-1)^{q+r} \dim H_r(\hat{\mathcal{F}}_{n-1-q}^{\mathbb{Z}} \otimes \mathbb{F}_2) \tag{4.19}$$

Using this last equation and Lemma 4.5.1 we can know prove our Theorem 1.3.5 stated in the Introduction of this thesis:

Theorem 4.5.5. *(Theorem 1.3.5) Assume that every edge of the triangulation Γ is of odd integral length. Assume that the triangulation is $(k, \mathbb{F}_2)$-primitive. Assume that the polytope P is $\mathbb{F}_2$-non-singular. If $k \leq n-1$, the Betti numbers of X are bounded in the following way:*

$$\forall q \geq n-k/2 \text{ ou } k/2-1 \geq q \geq n-k, \quad b_q(X;\mathbb{F}_2) \leq (1+(-1)^n)h^{q,q}(\mathbb{C}\tilde{X}) + \sum_{p=n-q}^{2n-k-1-q} \dim H_q(\mathcal{F}_p) - \sum_{r=q+1}^{n-k+q} (-1)^{q+r} \dim H_r(\mathcal{F}_{n-1-q}) + (-1)^q \hat{\chi}_{n-1-q} \ ;$$

$$\forall (n-1)/2 > q \geq \max((k-1)/2, n-k), \quad b_q(X;\mathbb{F}_2) \leq h^{q,q}(\mathbb{C}\tilde{X}) + \sum_{p=n-q}^{2n-k-1-q} \dim H_q(\mathcal{F}_p) - \sum_{r=q+1}^{n-k+q} (-1)^{q+r} \dim H_r(\mathcal{F}_{n-1-q}) + (-1)^q \hat{\chi}_{n-1-q} \ ;$$

$$\forall q \in [\max(n-k, n/2), n-(k+1)/2], \quad b_q(X;\mathbb{F}_2) \leq (-1)^n h^{q,q}(\mathbb{C}\tilde{X}) + \sum_{p=n-q}^{2n-k-1-q} \dim H_q(\mathcal{F}_p) - \sum_{r=q+1}^{n-k+q} (-1)^{q+r} \dim H_r(\mathcal{F}_{n-1-q}) + (-1)^q \hat{\chi}_{n-1-q} \ ;$$

$$\forall q = (n-1)/2 \geq n-k, \quad b_{(n-1)/2}(X;\mathbb{F}_2) \leq \sum_{p=(n+1)/2}^{2n-k-1-(n-1)/2} \dim H_{(n-1)/2}(\mathcal{F}_p) - \sum_{r=q+1}^{n-k+(n-1)/2} (-1)^{(n-1)/2+r} \dim H_r(\mathcal{F}_{(n-1)/2}) + (-1)^{(n-1)/2} \hat{\chi}_{(n-1)/2} \ ;$$

$$\forall q < \min(k, n-k), \quad b_q(X;\mathbb{F}_2) \leq h^{q,q}(\mathbb{C}\tilde{X}) + \sum_{p=k}^{n} \dim H_q(\tilde{\mathcal{F}}_p) \ ;$$

$$\forall q \in [k, n-k-1], \quad b_q(X;\mathbb{F}_2) \leq \sum_{p=k}^{n} \dim H_q(\tilde{\mathcal{F}}_p)$$

where $\hat{\chi}_p = \sum_{(F,\sigma)\in\Xi^1} (-1)^{\dim F - \dim\sigma} \left[\binom{\dim F}{p} - \binom{\dim F - \dim\sigma}{p-\dim\sigma} \right]$ *is the Euler characteristic of the cosheaf* $\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2$.

Proof. In this theorem, we have not modified the Equations 4.11 and 4.13 of Lemma 4.5.1. Then for the proof, we just have to study Equations 4.10 and 4.12. These Equations 4.10 and 4.12 are the case $q \geq n-k$. In this case $q \geq n-k$, we also have $n-1-q < k$, and then applying Equation 4.9, Equation 4.19 becomes:

$$\forall q \geq n-k, \quad \dim H_q(\mathcal{F}_{n-1-q}) = (-1)^q \hat{\chi}_{n-1-q} - \sum_{r\neq q} (-1)^{q+r} \dim H_r(\mathcal{F}_{n-1-q}) \qquad (4.20)$$

We have the equivalences:

$$\left\{ \begin{array}{c} n-1-q \in [\min(k, n-1-r), 2n-k-1-r] \\ q \geq n-k \\ r \neq q \end{array} \right. \iff \left\{ \begin{array}{c} r \in [q+1, n-k+q] \\ q \geq n-k \end{array} \right.$$

Then applying Corollary 4.4.3, we obtain:

$$\begin{aligned} \forall q \geq n-k, \quad \dim H_q(\mathcal{F}_{n-1-q}) =& (-1)^q \hat{\chi}_{n-1-q} - \sum_{r=q+1}^{n-k+q} (-1)^{q+r} \dim H_r(\mathcal{F}_{n-1-q}) \\ & - \sum_{r \notin [q, n-k+q]} (-1)^{q+r} h^{r,n-1-q}(\mathbb{C}\tilde{X}) \end{aligned}$$

However, we also know that if $r \neq q$, the Hodge number $h^{r,n-1-q}(\mathbb{C}\tilde{X})$ is zero except if $r = n-1-q$. However, we have the equivalences:

$$n-1-q \notin [q, n-k+q] \iff q \notin [(k-1)/2, (n-1)/2]$$

Then we have:

$$\begin{aligned} & \forall q \geq n-k, \text{ if } q \notin [(k-1)/2, (n-1)/2], \\ & \dim H_q(\mathcal{F}_{n-1-q}) = (-1)^q \hat{\chi}_{n-1-q} - \sum_{r=q+1}^{n-k+q} (-1)^{q+r} \dim H_r(\mathcal{F}_{n-1-q}) \\ & \qquad + (-1)^n h^{n-1-q,n-1-q}(\mathbb{C}\tilde{X}) \end{aligned}$$

and also:

$$\begin{aligned} & \forall q \geq n-k, \text{ if } q \in [(k-1)/2, (n-1)/2], \\ & \dim H_q(\mathcal{F}_{n-1-q}) = (-1)^q \hat{\chi}_{n-1-q} - \sum_{r=q+1}^{n-k+q} (-1)^{q+r} \dim H_r(\mathcal{F}_{n-1-q}) \end{aligned}$$

We can even replace $h^{n-1-q,n-1-q}(\mathbb{C}\tilde{X})$ by $h^{q,q}(\mathbb{C}\tilde{X})$ using Poincaré duality on $\mathbb{C}\tilde{X}$. Now to conclude the proof, we just have to compute the intersections of the sets $q \notin [(k-1)/2, n/2-1]$ or $q \in [(k-1)/2, n/2-1]$ with the different sets $q \geq n-k/2$, $(n-1)/2 > q \geq n-k$, or $q \in [\max(n-k, (n-1)/2), n-(k+1)/2]$. We have the equivalences:

$$\left\{ \begin{array}{c} q \notin [(k-1)/2, (n-1)/2] \\ q \geq n-k/2 \text{ or } (n-1)/2 > q \geq n-k \end{array} \right. \iff q \geq n-k/2 \text{ or } (k-1)/2 > q \geq n-k \ ;$$

$$\left\{ \begin{array}{c} q \notin [(k-1)/2, (n-1)/2] \\ q \in [\max(n-k, n/2), n-(k+1)/2] \end{array} \right. \iff q \in [\max(n-k, (n-1)/2), n-(k+1)/2] \ ;$$

$$\left\{ \begin{array}{c} q \in [(k-1)/2, (n-1)/2] \\ q \geq n-k/2 \text{ or } (n-1)/2 > q \geq n-k \end{array} \right. \iff (n-1)/2 > q \geq \max(k-1)/2, n-k) \ ;$$

$$\left\{ \begin{array}{c} q \in [(k-1)/2, (n-1)/2] \\ q \in [\max(n-k, (n-1)/2), n-(k+1)/2] \end{array} \right. \iff q = (n-1)/2 \geq n-k \ .$$

It concludes the proof for the case $k \leq n-1$. □

Applying the same method, using Lemma 4.5.1, and Equation 4.19, we prove our Theorem 1.3.4 stated in the Introduction of this thesis:

Theorem 4.5.6. *(Theorem 1.3.4) Assume that every edge of the triangulation Γ is of odd integral length. Assume that the triangulation is $\mathbb{F}_2$-primitive. Assume that the polytope P is $\mathbb{F}_2$-non-singular. Then the Betti numbers of X are bounded in the following way:*

$$\forall\ q,\quad b_q(X;\mathbb{F}_2) \leq (-1)^q\hat{\chi}_{n-1-q} + (1+(-1)^n)h^{q,q}(\mathbb{C}\tilde{X})$$

where $\hat{\chi}_p = \sum_{(F,\sigma)\in\Xi^1}(-1)^{\dim F-\dim\sigma}\left[\binom{\dim F}{p} - \binom{\dim F-\dim\sigma}{p-\dim\sigma}\right]$ is the Euler characteristic of the cosheaf $\hat{\mathcal{F}}_p^{\mathbb{Z}}\otimes\mathbb{F}_2$.

Remark 4.5.7. *In the case of a primitive triangulation and a non-singular lattice polytope P, we have $(-1)^{(n-1)/2}\hat{\chi}_{(n-1)/2} = h^{(n-1)/2,(n-1)/2}(\mathbb{C}\tilde{X})$, and for $q \neq (n-1)/2$ we have $(-1)^q\hat{\chi}_{n-1-q} = (-1)^{n-1}h^{q,q}(\mathbb{C}\tilde{X}) + h^{n-1-q,q}(\mathbb{C}\tilde{X})$. Then, in the case of a primitive triangulation and a non-singular lattice polytope P, we obtain the same bounds as the bounds established by Renaudineau an Shaw in [RS 2023] and generalized by [BMR 2024] in the non-convex case. Indeed, in the case of a primitive triangulation and a non-singular lattice polytope P, the Euler characteristic $\hat{\chi}_p$ is equal to $\sum_q(-1)^q h^{p,q}(\mathbb{C}\tilde{X})$ (proposition 3.12 of [BMR 2024]).*

Proof. The $\mathbb{F}_2$-primitivity is the $(k,\mathbb{F}_2)$-primitivity for $k=n$. Then, the condition $q \geq n-k$ required to apply Equation 4.9 is satisfied by any non-zero homology group $H_q(\mathcal{F}_p)$. Applying Equation 4.19 we obtain:

$$\forall q,\quad \dim H_q(\mathcal{F}_{n-1-q}) = (-1)^q\hat{\chi}_{n-1-q} - \sum_{r\neq q}(-1)^{q+r}\dim H_r(\mathcal{F}_{n-1-q})$$

Moreover, we also have from Corollary 4.3.6 the identity $H_r(\mathcal{F}_{n-1-q}) = H_r(\tilde{\mathcal{F}}_{n-1-q})$. Then, applying Corollary 4.4.3 for $k=n$, we have:

$$\forall r\neq q,\quad \dim H_r(\mathcal{F}_{n-1-q}) = \begin{cases} h^{n-1-q,n-1-q}(\mathbb{C}\tilde{X}) & \text{if } r = n-1-q \\ 0 & \text{if } r \neq n-1-q \end{cases}$$

Then we obtain:

$$\forall q\neq(n-1)/2,\quad \dim H_q(\mathcal{F}_{n-1-q}) = (-1)^q\hat{\chi}_{n-1-q} + (-1)^n h^{n-1-q,n-1-q}(\mathbb{C}\tilde{X})$$

And for $q=(n-1)/2$:

$$\dim H_{(n-1)/2}(\mathcal{F}_{(n-1)/2}) = (-1)^{(n-1)/2}\hat{\chi}_{(n-1)/2}$$

Using Poincaré Duality for the hypersurface $\mathbb{C}\tilde{X}$, we can replace $h^{n-1-q,n-1-q}(\mathbb{C}\tilde{X})$ by $h^{q,q}(\mathbb{C}\tilde{X})$. Then using Equation 4.14, we obtain:

$$\forall\ q,\quad b_q(X;\mathbb{F}_2) \leq \begin{cases} (-1)^q\hat{\chi}_{n-1-q} + (1+(-1)^n)h^{q,q}(\mathbb{C}\tilde{X}) & \text{if } q\neq(n-1)/2 \\ (-1)^{(n-1)/2}\hat{\chi}_{(n-1)/2} & \text{if } q=(n-1)/2 \end{cases}$$

However, in the case $q=(n-1)/2$, the integer n is odd, and then $0 = (1+(-1)^n)h^{q,q}(\mathbb{C}\tilde{X})$. Thus we obtain the same expression of the bound of $b_q(X;\mathbb{F}_2)$ in the case $q\neq(n-1)/2$ and in the case $q=(n-1)/2$. It concludes the proof. □

Now we prove our Theorem 1.3.6 stated in the Introduction of this thesis:

Theorem 4.5.8. *(Theorem 1.3.6) Assume that $n \geq 3$. Assume that every edge of the triangulation Γ is of odd integral length. Assume that the triangulation is $(n-1, \mathbb{F}_2)$-primitive. Assume that the polytope P is $\mathbb{F}_2$-non-singular. Then the Betti numbers of X are bounded in the following way:*

$$\forall q \notin \{0, n/2-1, n/2\},$$
$$b_q(X;\mathbb{F}_2) \leq (1+(-1)^n)h^{q,q}(\mathbb{C}\tilde{X}) + \mathrm{dim}H_q(\mathcal{F}_{n-q}) + \mathrm{dim}H_{q+1}(\mathcal{F}_{n-1-q}) + (-1)^q\hat{\chi}_{n-1-q} \ ;$$
$$\forall q \in \{n/2-1, n/2\},$$
$$b_q(X;\mathbb{F}_2) \leq h^{q,q}(\mathbb{C}\tilde{X}) + \mathrm{dim}H_q(\mathcal{F}_{n-q}) + \mathrm{dim}H_{q+1}(\mathcal{F}_{n-1-q}) + (-1)^q\hat{\chi}_{n-1-q} \ ;$$
$$b_0(X;\mathbb{F}_2) \leq 1 + (-1)^n + \hat{\chi}_{n-1} + \mathrm{dim}H_1(\mathcal{F}_{n-1}) - N_{np} + 2N_c$$

where N_c is the number of n-simplices σ of the triangulation Γ such that $\mathcal{E}(P,\sigma) = N_{\mathbb{F}_2}$, where N_{np} is the number of non-$\mathbb{F}_2$-primitive simplices of the triangulation Γ, and where $\hat{\chi}_p = \sum_{(F,\sigma)\in\Xi^1}(-1)^{\mathrm{dim}F-\mathrm{dim}\sigma}\left[\binom{\mathrm{dim}F}{p} - \binom{\mathrm{dim}F-\mathrm{dim}\sigma}{p-\mathrm{dim}\sigma}\right]$ is the Euler characteristic of the cosheaf $\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2$.

Proof. Notice that in the case $q = n/2$, the integer n is even and then $(-1)^n h^{q,q}(\mathbb{C}\tilde{X}) = h^{q,q}(\mathbb{C}\tilde{X})$. Remark that in the case $q = (n-1)/2$, the integer n is odd and then $0 = (1+(-1)^n)h^{q,q}(\mathbb{C}\tilde{X})$. Then the bounds of $b_q(X;\mathbb{F}_2)$ for $q \neq 0$ stated in this theorem are the direct consequence of Theorem 4.5.6 in the case $k = n-1$ and $n \geq 3$. We just need to compute the bound of $b_0(X;\mathbb{F}_2)$. Using Corollary 4.5.3 we have:

$$b_0(X;\mathbb{F}_2) \leq 1 + \mathrm{dim}H_0(\mathcal{F}_{n-1}) + 2N_c$$

where N_c is the number of n-simplices σ of the triangulation Γ such that $\mathcal{E}(P,\sigma) = N_{\mathbb{F}_2}$. However, applying Equation 4.9 and Equation 4.8 in our case $k = n-1$, we have:

$$\forall q \neq 0, \quad H_q(\mathcal{F}_p) = H_q(\hat{\mathcal{F}}_{n-1}^{\mathbb{Z}} \otimes \mathbb{F}_2) \quad ; \quad C_q(\mathcal{F}_{n-1}) = C_q(\hat{\mathcal{F}}_{n-1}^{\mathbb{Z}} \otimes \mathbb{F}_2)$$

Then we have:

$$\begin{aligned}
\mathrm{dim}H_0(\mathcal{F}_{n-1}) &= \mathrm{dim}H_0(\hat{\mathcal{F}}_{n-1}^{\mathbb{Z}} \otimes \mathbb{F}_2) + \sum_q (-1)^q \mathrm{dim}H_q(\mathcal{F}_{n-1}) \\
&\quad - \sum_q (-1)^q \mathrm{dim}H_q(\hat{\mathcal{F}}_{n-1}^{\mathbb{Z}} \otimes \mathbb{F}_2) \\
&= \mathrm{dim}H_0(\hat{\mathcal{F}}_{n-1}^{\mathbb{Z}} \otimes \mathbb{F}_2) + \sum_q (-1)^q \mathrm{dim}C_q(\mathcal{F}_{n-1}) \\
&\quad - \sum_q (-1)^q \mathrm{dim}C_q(\hat{\mathcal{F}}_{n-1}^{\mathbb{Z}} \otimes \mathbb{F}_2) \\
&= \mathrm{dim}H_0(\hat{\mathcal{F}}_{n-1}^{\mathbb{Z}} \otimes \mathbb{F}_2) + \mathrm{dim}C_0(\mathcal{F}_{n-1}) - \mathrm{dim}C_0(\hat{\mathcal{F}}_{n-1}^{\mathbb{Z}} \otimes \mathbb{F}_2)
\end{aligned}$$

Consider a cell $(F,\sigma) \in \Xi^1$ such that σ is a non-$\mathbb{F}_2$-primitive simplex of the triangulation Γ. Then σ must be a n-simplex, because the triangulation is $(n-1,\mathbb{F}_2)$-primitive which means that ever non-$\mathbb{F}_2$-primitive simplex of Γ is of dimension n. And we must have

$F = P$ because $\dim F \geq \dim\sigma = n$. Consider a $(n-1)$-face τ of σ. According to the $(n-1, \mathbb{F}_2)$-primitivity of the triangulation Γ, this face τ is $\mathbb{F}_2$-primitive. Then applying lemma 2.4 of [BMR 2024], we have:

$$\dim\mathcal{F}_{n-1}(P,\tau) = \binom{n}{n-1} - \binom{n-(n-1)}{n-1-(n-1)} = n-1$$

Consider an edge σ^1 of σ. If σ^1 is an edge of τ then $\mathcal{F}_{n-1}(P,\sigma^1) \subseteq \mathcal{F}_{n-1}(P,\tau)$. If σ^1 is not an edge of τ, consider the vertex s of intersection of σ^1 with τ. Consider the other vertices $a_1, ..., a_{n-1}$ of τ and the other vertex a_n of σ^1. The $\mathbb{F}_2$-primitivity of τ and the fact that every edge of Γ is of odd integral length implies that the family $(a_1 - s) \otimes 1_{\mathbb{F}_2}, ..., (a_{n-1} - s) \otimes 1_{\mathbb{F}_2}$ forms a basis of $T_{\mathbb{F}_2}\tau$, and the non-$\mathbb{F}_2$-primitivity of σ leads to the fact that the family $(a_1 - s) \otimes 1_{\mathbb{F}_2}, ..., (a_n - s) \otimes 1_{\mathbb{F}_2}$ is not free in $N_{\mathbb{F}_2}$. Therefore $(a_n - s) \otimes 1_{\mathbb{F}_2} \in T_{\mathbb{F}_2}\tau$, that is $T_{\mathbb{F}_2}\sigma^1 \subseteq T_{\mathbb{F}_2}\tau$. Then we have:

$$\mathcal{F}_{n-1}(P,\sigma^1) = \bigwedge^{n-1} T_{\mathbb{F}_2}\sigma^{1\perp} \subseteq \sum_{i=1}^{n-1} \bigwedge^{n-1} [(a_i - s) \otimes 1_{\mathbb{F}_2}]^\perp \subseteq \mathcal{F}_{n-1}(P,\tau)$$

Therefore, in any case $\mathcal{F}_{n-1}(P,\sigma^1) \subseteq \mathcal{F}_{n-1}(P,\tau)$. Thus, we have:

$$\mathcal{F}_{n-1}(P,\sigma) \subseteq \sum_{\sigma^1 \subseteq \sigma} \mathcal{F}_{n-1}(P,\sigma^1) \subseteq \mathcal{F}_{n-1}(P,\tau)$$

As we also know that $\mathcal{F}_{n-1}(P,\tau) \subseteq \mathcal{F}_{n-1}(P,\sigma)$ we obtain $\mathcal{F}_{n-1}(P,\tau) = \mathcal{F}_{n-1}(P,\sigma)$, and then:

$$\dim\mathcal{F}_{n-1}(P,\sigma) = \dim\mathcal{F}_{n-1}(P,\tau) = n-1$$

Moreover, we know from Equation 4.17 that for any n-simplex σ of the triangulation Γ:

$$\dim\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}(P,\sigma) \otimes \mathbb{F}_2 = \mathrm{rk}\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}(P,\sigma) = \binom{n}{n-1} - \binom{n-n}{n-1-n} = n$$

We have proven that for any cell $(F,\sigma) \in \Xi^1$ such that σ is a non-$\mathbb{F}_2$-primitive simplex we have the identity:

$$\dim\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}(P,\sigma) \otimes \mathbb{F}_2 - \dim\mathcal{F}_{n-1}(P,\sigma) = n - (n-1) = 1$$

Now if we consider a cell $(F,\sigma) \in \Xi^1$ such that σ is a $\mathbb{F}_2$-primitive simplex of the triangulation Γ, we have from Lemma 2.4 of [BMR 2024]:

$$\dim\mathcal{F}_{n-1}(F,\sigma) = \binom{\dim F}{n-1} - \binom{\dim F - \dim\sigma}{n-1-\dim\sigma}$$

and from Equation 4.17, we also have:

$$\dim\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}(F,\sigma) \otimes \mathbb{F}_2 = \mathrm{rk}\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}(F,\sigma) = \binom{\dim F}{n-1} - \binom{\dim F - \dim\sigma}{n-1-\dim\sigma}$$

Then, if σ is a $\mathbb{F}_2$-primitive simplex of the triangulation Γ, the difference $\dim\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}(F,\sigma) \otimes \mathbb{F}_2 - \dim\mathcal{F}_{n-1}(F,\sigma)$ is zero.

Therefore, denoting by N_{np} the number non-$\mathbb{F}_2$-primitive n-simplices of the triangulation Γ we obtain:

$$\begin{aligned}&\dim C_0(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}\otimes\mathbb{F}_2)-\dim C_0(\mathcal{F}_{n-1})\\&=\sum_{(F,\sigma)\in\Xi^1,\ \dim F-\dim\sigma=0}\dim\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}(F,\sigma)\otimes\mathbb{F}_2-\dim\mathcal{F}^{\mathbb{Z}}_{n-1}(F,\sigma)\otimes\mathbb{F}_2=N_{np}\end{aligned}$$

Thus, we obtain:

$$\dim H_0(\mathcal{F}_{n-1})=\dim H_0(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}\otimes\mathbb{F}_2)-N_{np}$$

However, using Equation 4.19, we have:

$$\dim H_0(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}\otimes\mathbb{F}_2)=\hat{\chi}_{n-1}-\sum_{r=1}^{n-1}(-1)^r\dim H_r(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}\otimes\mathbb{F}_2)$$

Moreover, applying Proposition 4.3.4 and Equation 4.9 for $k=n-1$, we obtain the identity $H_r(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}\otimes\mathbb{F}_2)=H_r(\mathcal{F}_{n-1})=H_r(\tilde{\mathcal{F}}_{n-1})$ for any $r\geq 1$. Then, applying Corollary 4.4.3 for $k=n-1$, we have:

$$\forall r\geq 2,\quad \dim H_r(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}\otimes\mathbb{F}_2)=\begin{cases}h^{n-1,n-1}(\mathbb{C}\tilde{X})\text{ if }r=n-1\\0\text{ if }r\neq n-1\end{cases}.$$

However, using Poincré Duality for the hypersurface $\mathbb{C}\tilde{X}$, we also have from Equation 4.6:

$$h^{n-1,n-1}(\mathbb{C}\tilde{X})=h^{0,0}(\mathbb{C}\tilde{X})=\sum_{i=0}^{0}(-1)^{0-i}\binom{n-i}{0-i}f_{n-0}(P)=1$$

where $f_n(P)$ is the number of n-faces of P (it is equal to 1 because there is only P). Then we obtain:

$$\dim H_0(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}\otimes\mathbb{F}_2)=\hat{\chi}_{n-1}-(-1)^{n-1}+\dim H_1(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}\otimes\mathbb{F}_2)$$

Moreover, from Equation 4.9 for $k=n-1$, we have $\dim H_1(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1}\otimes\mathbb{F}_2)=\dim H_1(\mathcal{F}_{n-1})$. It leads to our expression of the bound of $b_0(X;\mathbb{F}_2)$:

$$b_0(X;\mathbb{F}_2)\leq 1+(-1)^n+\hat{\chi}_{n-1}+\dim H_1(\mathcal{F}_{n-1})-N_{np}+2N_c$$

□

In the three last theorems (our Theorems 1.3.4, 1.3.5 and 1.3.6), the bounds of the Betti numbers $b_q(X;\mathbb{F}_2)$ for $q\geq n-k$ are stated in terms of the homology of the cosheaves $\mathcal{F}_p$. However, we have proven that in some cases this homology can be understood in terms the 2-torsion of the homology of the cosheaf $\hat{\mathcal{F}}^{\mathbb{Z}}_p$ (Equation 4.16). Then, we can also state the bounds of the Betti numbers $b_q(X;\mathbb{F}_2)$ for $q\geq n-k$ in terms of this 2-torsion of the homology of the cosheaf $\hat{\mathcal{F}}^{\mathbb{Z}}_p$, which is the subject of the following corollaries:

Corollary 4.5.9. *Assume that every edge of the triangulation Γ is of odd integral length. Assume that the triangulation is $(k, \mathbb{F}_2)$-primitive. Assume that the polytope P is $\mathbb{F}_2$-non-singular. If $k \leq n-1$, the Betti numbers of X are bounded in the following way:*

$$\begin{aligned}
&\forall q \geq n-k, \\
&b_q(X;\mathbb{F}_2) \leq (1+(-1)^n)h^{q,q}(\mathbb{C}\tilde{X}) + (-1)^q \hat{\chi}_{n-1-q} + \mathrm{dimTor}(H_q(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1-q});\mathbb{F}_2) \\
&\qquad\qquad + \sum_{p=n-q}^{2n-k-1-q} [\mathrm{dimTor}(H_q(\hat{\mathcal{F}}^{\mathbb{Z}}_p);\mathbb{F}_2) + \mathrm{dimTor}(H_{q-1}(\hat{\mathcal{F}}^{\mathbb{Z}}_p);\mathbb{F}_2)] \ ;
\end{aligned}$$

$$\forall q < \min(k, n-k), \quad b_q(X;\mathbb{F}_2) \leq h^{q,q}(\mathbb{C}\tilde{X}) + \sum_{p=k}^{n} \mathrm{dim}H_q(\tilde{\mathcal{F}}_p) \ ;$$

$$\forall q \in [k, n-k-1], \quad b_q(X;\mathbb{F}_2) \leq \sum_{p=k}^{n} \mathrm{dim}H_q(\tilde{\mathcal{F}}_p)$$

where $\hat{\chi}_p = \sum_{(F,\sigma)\in\Xi^1} (-1)^{\dim F - \dim\sigma} \left[\binom{\dim F}{p} - \binom{\dim F - \dim\sigma}{p-\dim\sigma} \right]$ *is the Euler characteristic of the cosheaf* $\hat{\mathcal{F}}^{\mathbb{Z}}_p \otimes \mathbb{F}_2$.

Proof. If $q \geq n-k$, we obtain from equation 4.16:

$$\begin{aligned}
\forall p \in [n-k, 2n-k-1-q], \quad \mathrm{dim}H_q(\mathcal{F}_p) &= h^{p,q}(\mathbb{C}\tilde{X}) + \mathrm{dimTor}(H_q(\hat{\mathcal{F}}^{\mathbb{Z}}_p);\mathbb{F}_2) \\
&\quad + \mathrm{dimTor}(H_{q-1}(\hat{\mathcal{F}}^{\mathbb{Z}}_p);\mathbb{F}_2) \ ;
\end{aligned}$$

$$\begin{aligned}
\forall r \geq q+1, \quad \mathrm{dim}H_r(\mathcal{F}_{n-1-q}) &= h^{n-1-q,r}(\mathbb{C}\tilde{X}) + \mathrm{dimTor}(H_r(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1-q});\mathbb{F}_2) \\
&\quad + \mathrm{dimTor}(H_{r-1}(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1-q});\mathbb{F}_2) \ ;
\end{aligned}$$

Then if $q \geq n-k$, we obtain:

$$\begin{aligned}
\sum_{p=n-q}^{2n-k-1-q} \mathrm{dim}H_q(\mathcal{F}_p) = &\sum_{p=n-q}^{2n-k-1-q} [\mathrm{dimTor}(H_q(\hat{\mathcal{F}}^{\mathbb{Z}}_p);\mathbb{F}_2) + \mathrm{dimTor}(H_{q-1}(\hat{\mathcal{F}}^{\mathbb{Z}}_p);\mathbb{F}_2)] \\
&+ \sum_{p=n-q}^{2n-k-1-q} h^{p,q}(\mathbb{C}\tilde{X}) \ ;
\end{aligned}$$

However, we also know from Equation 4.6 that for all $q \geq n-k$:

$$\sum_{p=n-q}^{2n-k-1-q} h^{p,q}(\mathbb{C}\tilde{X}) = \begin{cases} h^{q,q}(\mathbb{C}\tilde{X}) & \text{if } n/2 \leq q \leq n-(k+1)/2 \\ 0 & \text{otherwise} \end{cases}$$

Applying Equations 4.10 and 4.12 of Lemma 4.5.1, we obtain for $q \geq n-k$:

$$\begin{aligned}
\text{if } q \neq (n-1)/2, \quad b_q(X;\mathbb{F}_2) &\leq h^{q,q}(\mathbb{C}\tilde{X}) + \dim H_q(\mathcal{F}_{n-1-q}) \\
&\quad + \sum_{p=n-q}^{2n-k-1-q} [\dim\mathrm{Tor}(H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}});\mathbb{F}_2) + \dim\mathrm{Tor}(H_{q-1}(\hat{\mathcal{F}}_p^{\mathbb{Z}});\mathbb{F}_2)] \\
\text{if } q = (n-1)/2, \quad b_q(X;\mathbb{F}_2) &\leq \dim H_q(\mathcal{F}_{n-1-q}) \\
&\quad + \sum_{p=n-q}^{2n-k-1-q} [\dim\mathrm{Tor}(H_q(\hat{\mathcal{F}}_p^{\mathbb{Z}});\mathbb{F}_2) + \dim\mathrm{Tor}(H_{q-1}(\hat{\mathcal{F}}_p^{\mathbb{Z}});\mathbb{F}_2)]
\end{aligned}$$

However, applying Equation 4.20, we have:

$$\forall q \geq n-k, \quad \dim H_q(\mathcal{F}_{n-1-q}) = (-1)^q \hat{\chi}_{n-1-q} - \sum_{r\neq q} (-1)^{q+r} \dim H_r(\mathcal{F}_{n-1-q})$$

And applying Theorem 4.4.2, we obtain:

$$\begin{aligned}
-\sum_{r\neq q} (-1)^{q+r} \dim H_r(\mathcal{F}_{n-1-q}) &= -\sum_{r\leq q-1} (-1)^{q+r} h^{r,n-1-q}(\mathbb{C}\tilde{X}) \\
&\quad - \sum_{r\geq q+1} (-1)^{q+r} \dim H_r(\mathcal{F}_{n-1-q})
\end{aligned}$$

Then we have:

$$\begin{aligned}
\forall q \geq n-k, \quad \dim H_q(\mathcal{F}_{n-1-q}) &= (-1)^q \hat{\chi}_{n-1-q} - \sum_{r\leq q-1} (-1)^{q+r} h^{r,n-1-q}(\mathbb{C}\tilde{X}) \\
&\quad - \sum_{r\geq q+1} (-1)^{q+r} h^{r,n-1-q}(\mathbb{C}\tilde{X}) \\
&\quad - \sum_{r\geq q+1} (-1)^{q+r} \dim\mathrm{Tor}(H_r(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1-q});\mathbb{F}_2) \\
&\quad - \sum_{r\geq q+1} (-1)^{q+r} \dim\mathrm{Tor}(H_{r-1}(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1-q});\mathbb{F}_2) \\
&= (-1)^q \hat{\chi}_{n-1-q} - \sum_{r\neq q} (-1)^{q+r} h^{r,n-1-q}(\mathbb{C}\tilde{X}) \\
&\quad + \dim\mathrm{Tor}(H_q(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1-q});\mathbb{F}_2)
\end{aligned}$$

Moreover, we also have from Equation 4.6, for all $q \geq n-k$:

$$-\sum_{r\neq q} (-1)^{q+r} h^{r,n-1-q}(\mathbb{C}\tilde{X}) = \begin{cases} (-1)^n h^{n-1-q,n-1-q}(\mathbb{C}\tilde{X}) & \text{if } q \neq (n-1)/2 \\ 0 & \text{if } q = (n-1)/2 \end{cases}$$

However, Poincaré duality on the hypersurface $\mathbb{C}\tilde{X}$ gives $(-1)^n h^{n-1-q,n-1-q}(\mathbb{C}\tilde{X}) =$

$(-1)^n h^{q,q}(\mathbb{C}\tilde{X})$. Therefore we obtain for $q \geq n-k$:

$$\begin{aligned}
&\text{if } q \neq (n-1)/2,\\
&b_q(X;\mathbb{F}_2) \leq (1+(-1)^n)h^{q,q}(\mathbb{C}\tilde{X}) + (-1)^q\hat{\chi}_{n-1-q} + \mathrm{dimTor}(H_q(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1-q});\mathbb{F}_2)\\
&\qquad\qquad + \sum_{p=n-q}^{2n-k-1-q} [\mathrm{dimTor}(H_q(\hat{\mathcal{F}}^{\mathbb{Z}}_p);\mathbb{F}_2) + \mathrm{dimTor}(H_{q-1}(\hat{\mathcal{F}}^{\mathbb{Z}}_p);\mathbb{F}_2)];\\
&\text{if } q = (n-1)/2,\\
&b_q(X;\mathbb{F}_2) \leq (-1)^q\hat{\chi}_{n-1-q} + \mathrm{dimTor}(H_q(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1-q});\mathbb{F}_2)\\
&\qquad\qquad + \sum_{p=n-q}^{2n-k-1-q} [\mathrm{dimTor}(H_q(\hat{\mathcal{F}}^{\mathbb{Z}}_p);\mathbb{F}_2) + \mathrm{dimTor}(H_{q-1}(\hat{\mathcal{F}}^{\mathbb{Z}}_p);\mathbb{F}_2)]
\end{aligned}$$

However, in the case $q = (n-1)/2$, the integer n is odd, thus we have $0 = (1+(-1)^n)h^{n-1-q,n-1-q}(\mathbb{C}\tilde{X})$. Then we obtain the same expression for every bound of the Betti numbers $b_q(X;\mathbb{F}_2)$ for $q \geq n-k$. It concludes the proof. □

In the particular case $n \geq 3$ and $k = n-1$, we have the following corollary:

Corollary 4.5.10. *Assume that $n \geq 3$. Assume that every edge of the triangulation Γ is of odd integral length. Assume that the triangulation is $(n-1,\mathbb{F}_2)$-primitive. Assume that the polytope P is $\mathbb{F}_2$-non-singular. Then the Betti numbers of X are bounded in the following way:*

$$\begin{aligned}
\forall q \geq 1,\quad b_q(X;\mathbb{F}_2) &\leq (1+(-1)^n)h^{q,q}(\mathbb{C}\tilde{X}) + (-1)^q\hat{\chi}_{n-1-q} + \mathrm{dimTor}(H_q(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1-q});\mathbb{F}_2)\\
&\quad + \mathrm{dimTor}(H_q(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-q});\mathbb{F}_2) + \mathrm{dimTor}(H_{q-1}(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-q});\mathbb{F}_2)\ ;\\
\textit{if } q=0,\quad b_0(X;\mathbb{F}_2) &\leq 1+(-1)^n + \hat{\chi}_{n-1} - N_{np} + 2N_c\\
&\quad + \mathrm{Tor}(H_1(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1});\mathbb{F}_2) + \mathrm{Tor}(H_0(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1});\mathbb{F}_2)\ .
\end{aligned}$$

where N_c is the number of n-simplices σ of the triangulation Γ such that $\mathcal{E}(P,\sigma) = N_{\mathbb{F}_2}$, where N_{np} is the number of non-$\mathbb{F}_2$-primitive simplices of the triangulation Γ, and where $\hat{\chi}_p = \sum_{(F,\sigma)\in\Xi^1}(-1)^{\dim F-\dim\sigma}\left[\binom{\dim F}{p} - \binom{\dim F-\dim\sigma}{p-\dim\sigma}\right]$ is the Euler characteristic of the cosheaf $\hat{\mathcal{F}}^{\mathbb{Z}}_p \otimes \mathbb{F}_2$.

Proof. We can apply the previous corollary in the case $k = n-1$. We obtain:

$$\begin{aligned}
\forall q \geq 1,\quad b_q(X;\mathbb{F}_2) &\leq (1+(-1)^n)h^{q,q}(\mathbb{C}\tilde{X}) + (-1)^q\hat{\chi}_{n-1-q} + \mathrm{dimTor}(H_q(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1-q});\mathbb{F}_2)\\
&\quad + \mathrm{dimTor}(H_q(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-q});\mathbb{F}_2) + \mathrm{dimTor}(H_{q-1}(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-q});\mathbb{F}_2)\ .
\end{aligned}$$

From Theorem 4.5.8, we have the following bound of $b_0(X;\mathbb{F}_2)$:

$$b_0(X;\mathbb{F}_2) \leq 1+(-1)^n + \hat{\chi}_{n-1} + \mathrm{dim}H_1(\mathcal{F}_{n-1}) - N_{np} + 2N_c$$

However, we have from Equation 4.16 applied for $k = n-1$:

$$\mathrm{dim}H_1(\mathcal{F}_{n-1}) = h^{1,n-1}(\mathbb{C}\tilde{X}) + \mathrm{dimTor}(H_1(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1});\mathbb{F}_2) + \mathrm{Tor}(H_0(\hat{\mathcal{F}}^{\mathbb{Z}}_{n-1});\mathbb{F}_2)$$

And from Equation 4.6, as $n \geq 3$ we have $1 \neq n-1$ and then $h^{1,n-1}(\mathbb{C}\tilde{X}) = 0$. Thus, we obtain the bound of $b_0(X;\mathbb{F}_2)$, it concludes the proof. □

4.6 Examples and particular cases

Before the particular cases let us prove a preliminar lemma:

Lemma 4.6.1. *The Euler characteristic* $\hat{\chi}_0 = \sum_{(F,\sigma)\in\Xi^1}(-1)^{\dim F-\dim\sigma}$ *of the cosheaf* $\hat{\mathcal{F}}_0^{\mathbb{Z}}\otimes\mathbb{F}_2$ *satisfies the following property:*

$$\hat{\chi}_0 = 1-(-1)^n\mathrm{Card}\Big\{v \text{ vertex of } \Gamma \text{ s.t. } v\in \overset{\circ}{P}\Big\}$$

where $\overset{\circ}{P}$ *is the interior of the polytope* P*.*

Proof. Consider $\Xi = \{(F,\sigma),\ F \text{ face of } P,\ \sigma\in\Gamma,\ \sigma\subseteq F\}$. Each cell (F,σ) corresponds to the union of the barycentric cells of the triangulation Γ, contained in F and containing σ, this union is homeomorphic to a ball of dimension $\dim F-\dim\sigma$. These unions of barycentric cells forms a cellular subdivision of the polytope P, thus we have:

$$\sum_{(F,\sigma)\in\Xi}(-1)^{\dim F-\dim\sigma} = \chi(P) = 1$$

Moreover, for any vertex v of Γ, considering F_v the minimal face of P containing v, the quantity $\sum_{F \text{ face of } P,\ v\in F}(-1)^{\dim F}$ is equal to $(-1)^{\dim F_v}$ times the Euler characteristic of the affine convex polyhedral cone $\{x\in N \mid \forall w\in P\cap M,\ x(w-v)\geq 0\}$ dual to the face F_v. This cone is of dimension $n-\dim F_v$, and the Euler characteristic of an affine convex polyhedral cone is 0 if its dimension is ≥ 1 and 1 if its dimension is zero. Then, we obtain:

$$\sum_{F \text{ face of } P,\ v\in F}(-1)^{\dim F} = \begin{cases} 0 & \text{if } F_v\neq P \\ (-1)^n & \text{if } F_v = P\end{cases}$$

Therefore, we obtain:

$$\begin{aligned}\hat{\chi}_0 &= \sum_{(F,\sigma)\in\Xi^1}(-1)^{\dim F-\dim\sigma} = \sum_{(F,\sigma)\in\Xi}(-1)^{\dim F-\dim\sigma} - \sum_{v \text{ vertex of } P}\ \sum_{F \text{ face of } P,\ v\in F}(-1)^{\dim F} \\ &= 1-(-1)^n\mathrm{Card}\Big\{v \text{ vertex of } \Gamma \text{ s.t. } v\in\overset{\circ}{P}\Big\}\end{aligned}$$

□

Corollary 4.6.2. *The Euler characteristic* $\hat{\chi}_0 = \sum_{(F,\sigma)\in\Xi^1}(-1)^{\dim F-\dim\sigma}$ *of the cosheaf* $\hat{\mathcal{F}}_0^{\mathbb{Z}}\otimes\mathbb{F}_2$ *satisfies the following property:*

$$\hat{\chi}_0 = 1+(-1)^{n-1}h^{0,n-1}(\mathbb{C}\tilde{X}) - (-1)^{n-1}\mathrm{Card}\Big\{v\in M\cap\overset{\circ}{P} \text{ s.t. } v \text{ is not a vertex of } \Gamma\Big\}$$

where $\overset{\circ}{P}$ *is the interior of the polytope* P*, and* $\mathbb{C}\tilde{X}$ *a non-degenerate hypersurface of polytope* P *in the toric variety* $\mathbb{C}Y$*.*

In the case of a maximal triangulation Γ *(that is that any point of* $M\cap P$ *is a vertex of* Γ*, then we have:*

$$\hat{\chi}_0 = 1+(-1)^{n-1}h^{0,n-1}(\mathbb{C}\tilde{X})$$

Proof. If the triangulation Γ is primitive, then according to Proposition 3.12 of [BMR 2024] we have $\hat{\chi}_0 = 1+(-1)^{n-1}h^{0,n}(\mathbb{C}\tilde{X})$. However, it implies also $\left\{v \text{ vertex of } \Gamma \text{ s.t. } v \in \overset{\circ}{P}\right\} = M \cap \overset{\circ}{P}$. Then using the previous lemma, we obtain the following identity:

$$1 + (-1)^{n-1}h^{0,n-1}(\mathbb{C}\tilde{X}) = 1 - (-1)^n \text{Card} M \cap \overset{\circ}{P}$$

Then for any triangulation Γ of P we can conclude the proof using this last equation and the last lemma. □

4.6.1 Case of a polytope P of dimension $n = 1$

In the case of a polytope P of dimension $n = 1$, if we assume that any edge of the triangulation Γ is of odd integral length, then the triangulation is $\mathbb{F}_2$-primitive. Moreover, in this case the polytope P is a line segment, then it is smooth and the $\mathbb{F}_2$-non-singular. Then, applying Theorem 4.5.6, our bounds of the Betti numbers of X are:

$$\begin{aligned} b_0(X;\mathbb{F}_2) \leq (-1)^0\tilde{\chi}_0 &= \sum_{\sigma\in\Gamma,\ \dim\sigma=1} (-1)^{1-1}\left[\binom{1}{1} - \binom{1-1}{0-1}\right] \\ &= \text{Card}\{\sigma \in \Gamma,\ \dim\sigma = 1\} \end{aligned}$$

Moreover, in this particular case $\Xi^1 = \{(P,\sigma),\ \sigma \in \Gamma,\ \dim\sigma = 1\}$ and the real phase structure $\mathcal{E}(F,\sigma)$ for any $\sigma \in \Gamma$ edge of vertices v_1, v_2 is the following:

$$\mathcal{E}(F,\sigma) = \{x \in N_{\mathbb{F}_2} \mid \epsilon_{v_1} + x(v_{1\mathbb{F}_2}) \neq \epsilon_{v_2} + x(v_{2\mathbb{F}_2})\}$$

However, σ must be of odd integral length and then $v_{1\mathbb{F}_2} - v_{2\mathbb{F}_2} \neq 0$. Thus, as $N_{\mathbb{F}_2} \simeq \mathbb{F}_2$, we obtain that the equation $\epsilon_{v_1} + x(v_{1\mathbb{F}_2}) \neq \epsilon_{v_2} + x(v_{2\mathbb{F}_2})$ has a unique solution $x \in N_{\mathbb{F}_2}$. We obtain that $\mathcal{E}(F,\sigma)$ is of cardinal 1. Then $\mathcal{S}(P,\sigma)$ is of dimension 1, and therefore the chains $C_0(\mathcal{S})$ are of dimension $\text{Card}\{\sigma \in \Gamma,\ \dim\sigma = 1\}$. Finally as the chains of the cosheaf $\mathcal{S}$ are zero except in degree 0, we obtain:

$$b_0(X;\mathbb{F}_2) = \dim H_0(\mathcal{S}) = \dim C_0(\mathcal{S}) = \text{Card}\{\sigma \in \Gamma,\ \dim\sigma = 1\}$$

It proves that our bound $\text{Card}\{\sigma \in \Gamma,\ \dim\sigma = 1\}$ of the Betti number $b_0(X;\mathbb{F}_2)$ is equal to $b_0(X;\mathbb{F}_2)$ itself.

4.6.2 Case of a polytope P of dimension $n = 2$

In the case of a polytope P of dimension $n = 2$, if we assume that any edge of the triangulation Γ is of odd integral length, then the triangulation is $\mathbb{F}$-primitive. Indeed, if we consider a triangle of integral vertices $v_1, v_2, v_3 \in M$ such that any edge is of odd integral length, considering $v_{1\mathbb{F}_2}, v_{2\mathbb{F}_2}, v_{3\mathbb{F}_2}$ the projection of these vertices in $M_{\mathbb{F}_2}$, we have:

$$0 \neq v_{3\mathbb{F}_2} - v_{2\mathbb{F}_2} = v_{3\mathbb{F}_2} - v_{1\mathbb{F}_2} - (v_{2\mathbb{F}_2} - v_{1\mathbb{F}_2})$$

However, in a $\mathbb{F}_2$-vectorial space, the identity $v_{3\mathbb{F}_2} - v_{1\mathbb{F}_2} \neq (v_{2\mathbb{F}_2} - v_{1\mathbb{F}_2})$ is equivalent to the fact that the family $\{v_{3\mathbb{F}_2} - v_{1\mathbb{F}_2}, v_{2\mathbb{F}_2} - v_{1\mathbb{F}_2}\}$ is free, which means in this case that it is a basis of $M_{\mathbb{F}_2}$ and then that the triangle of integral vertices v_1, v_2, v_3 is $\mathbb{F}_2$-primitive.

Assume that $n = 2$, that any edge of the triangulation Γ is of odd integral length, and that the polytope P is $\mathbb{F}_2$-non-singular. Then, applying Theorem 9, our bounds of the Betti numbers of X are:

$$b_0(X;\mathbb{F}_2) \leq (-1)^0\hat{\chi}_1 + 2h^{0,0}(\mathbb{C}\tilde{X})$$
$$b_1(X;\mathbb{F}_2) \leq (-1)^1\hat{\chi}_0 + 2h^{1,1}(\mathbb{C}\tilde{X})$$

However, using Poincaré duality (Corollary 1.3.3) for the cosheaf $\mathcal{F}_1^{\mathbb{Q}}$ which has the same Euler characteristic as $\hat{\mathcal{F}}_1^{\mathbb{Z}} \otimes \mathbb{F}_2$, we obtain $\hat{\chi}_1 = -\hat{\chi}_0$, and using Poincaré duality for the hypersurface $\mathbb{C}\tilde{X}$ we obtain $h^{1,1}(\mathbb{C}\tilde{X}) = h^{0,0}(\mathbb{C}\tilde{X})$. Thus the bounds we have for $b_0(X;\mathbb{F}_2)$ and $b_1(X;\mathbb{F}_2)$ are the same. Moreover, according to 4.6 we have $h^{0,0}(\mathbb{C}\tilde{X}) = 1$. We can also compute $\hat{\chi}_0$ using Lemma 4.6.1:

$$\hat{\chi}_0 = \sum_{(F,\sigma)\in\Xi^1} (-1)^{\dim F - \dim\sigma} = 1 - \mathrm{Card}\Big\{v \text{ vertex of } \Gamma \text{ s.t. } v \in \overset{\circ}{P}\Big\}$$

Then, we obtain that our bound of $b_0(X;\mathbb{F}_2) = b_1(X;\mathbb{F}_2)$ is:

$$\begin{aligned} b_0(X;\mathbb{F}_2) &\leq \mathrm{Card}\Big\{v \text{ vertex of } \Gamma \text{ s.t. } v \in \overset{\circ}{P}\Big\} - 1 + 2 \\ &= 1 + \mathrm{Card}\Big\{v \text{ vertex of } \Gamma \text{ s.t. } v \in \overset{\circ}{P}\Big\} \end{aligned}$$

4.6.3 Case of a polytope P of dimension $n = 3$

Consider the case $n = 3$. Assume that any edge of the triangulation Γ is of odd integral length. According to the last subsection, it implies that any triangle is $\mathbb{F}_2$-primitive. Then the triangulation Γ is $(2,\mathbb{F}_2)$-primitive. Assume that the polytope P is $\mathbb{F}_2$-non-singular. Then, applying Theorem 4.5.8, our bounds of the Betti numbers of X are the following:

$$\begin{aligned} &b_2(X;\mathbb{F}_2) \leq \dim H_2(\mathcal{F}_1) + \hat{\chi}_0 \ ; \\ &b_1(X;\mathbb{F}_2) \leq \dim H_1(\mathcal{F}_2) + \dim H_2(\mathcal{F}_1) - \hat{\chi}_1 \ ; \\ &b_0(X;\mathbb{F}_2) \leq \hat{\chi}_2 + \dim H_1(\mathcal{F}_2) - N_{np} + 2N_c. \end{aligned}$$

where N_c is the number of n-simplices σ of the triangulation Γ such that $\mathcal{E}(P,\sigma) = N_{\mathbb{F}_2}$, where N_{np} is the number of non-$\mathbb{F}_2$-primitive simplices of the triangulation Γ, and where $\hat{\chi}_p = \sum_{(F,\sigma)\in\Xi^1}(-1)^{\dim F-\dim\sigma}\left[\binom{\dim F}{p} - \binom{\dim F-\dim\sigma}{p-\dim\sigma}\right]$ is the Euler characteristic of the cosheaf $\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2$.

In this case, the bounds of $b_2(X;\mathbb{F}_2)$ and $b_0(X;\mathbb{F}_2)$ may be different, then the better bound we have for $b_2(X;\mathbb{F}_2) = b_0(X;\mathbb{F}_2)$ is the minimum of these two bounds.

Using our Poincaré duality theorem for the cosheaves $\mathcal{F}_p^{\mathbb{Q}}$ (Corollary 1.3.3) and using the fact that the cosheaves $\mathcal{F}_p^{\mathbb{Q}}$ have the same Euler characteristic as the cosheaves $\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2$, we obtain $\hat{\chi}_2 = \hat{\chi}_0$. Therefore using Lemma 4.6.1 and Corollary 4.6.2, we get:

$$\begin{aligned} \hat{\chi}_2 = \hat{\chi}_0 &= \sum_{(F,\sigma)\in\Xi^1} (-1)^{\dim F-\dim\sigma} = 1 + \mathrm{Card}\Big\{v \text{ vertex of } \Gamma \text{ s.t. } v \in \overset{\circ}{P}\Big\} \\ &= 1 + h^{0,2}(\mathbb{C}\tilde{X}) - \mathrm{Card}\Big\{v \in M \cap \overset{\circ}{P} \text{ s.t. } v \text{ is not a vertex of } \Gamma\Big\} \end{aligned}$$

We obtain the following bound for $b_0(X;\mathbb{F}_2) = b_2(X;\mathbb{F}_2)$:

$$b_0(X;\mathbb{F}_2) \leq 1 + h^{0,2}(\mathbb{C}\tilde{X}) - \text{Card}\Big\{v \in M \cap \overset{\circ}{P} \text{ s.t. } v \text{ is not a vertex of } \Gamma\Big\} + \min(\dim H_2(\mathcal{F}_1); \dim H_1(\mathcal{F}_2) - N_{np} + 2N_c).$$

Then in the case of a maximal triangulation (that is that any point of $M \cap P$ is a vertex of the triangulation Γ), we obtain the following bound:

$$b_0(X;\mathbb{F}_2) \leq 1 + h^{0,2}(\mathbb{C}\tilde{X}) + \min(\dim H_2(\mathcal{F}_1); \dim H_1(\mathcal{F}_2) - N_{np} + 2N_c).$$

It is unclear if this bound is better or worse than the bound of [It 1997] stated in [Ar 2020] (Chapter 9): they stated that if the triangulation is convex and maximal and if P is a simplex of vertices $(0,...,0),(d,0,...,0),...,(0,...,0,d)$, the bound of $b_0(X;\mathbb{F}_2)$ is $1+h^{0,2}(\mathbb{C}\tilde{X})+N_c$. Then, to prove that our bound is better than their one, we would have to prove the following conjecture:

Conjecture 4.6.3. *If $n = 3$, if any edge of the triangulation Γ is of odd integral length, if the lattice polytope P is $\mathbb{F}_2$-non-singular, then we have:*

$$\min(\dim H_2(\mathcal{F}_1); \dim H_1(\mathcal{F}_2) - N_{np} + 2N_c) \leq N_c$$

We cannot have $\dim H_2(\mathcal{F}_1) = 0$ for any maximal triangulation Γ because otherwise a bound of $b_0(X;\mathbb{F}_2)$ would be $1+h^{0,2}(\mathbb{C}\tilde{X})$, but this bound is false for some examples (For instance Example 9.1.1 of [Ar 2020]). However, for a fixed triangulation Γ, $1+h^{0,2}(\mathbb{C}\tilde{X})+\dim H_2(\mathcal{F}_1)$ does not depend on the sign distribution $\epsilon_v \in \mathbb{F}_2$ on each vertex v of Γ, but N_c depends on this sign distribution, and we can even choose a sign distribution such that $N_c = 0$ (for instance the sign distribution $\epsilon_v = 0\ \forall v$). It proves that the inequality $1 + h^{0,2}(\mathbb{C}\tilde{X}) + \dim H_2(\mathcal{F}_1) \leq 1 + h^{0,2}(\mathbb{C}\tilde{X}) + N_c$ is false in general.

Similarly, we may not have in general $1 + h^{0,2}(\mathbb{C}\tilde{X}) + \dim H_1(\mathcal{F}_2) - N_{np} + 2N_c \leq 1 + h^{0,2}(\mathbb{C}\tilde{X}) + N_c$, chosing a sign disctribution such that $N_c = N_{np}$.

However, to prove conjecture 4.6.3, it is enough to prove the following conjecture 4.6.4:

Conjecture 4.6.4. *If $n = 3$, if any edge of the triangulation Γ is of odd integral length, if the lattice polytope P is $\mathbb{F}_2$-non-singular, then we have:*

$$\dim H_2(\mathcal{F}_1) + \dim H_1(\mathcal{F}_2) \leq N_{np}$$

Indeed, if $\dim H_2(\mathcal{F}_1) + \dim H_1(\mathcal{F}_2) \leq N_{np}$, then we have:

$$\min(\dim H_2(\mathcal{F}_1); \dim H_1(\mathcal{F}_2) - N_{np} + 2N_c) \leq \frac{1}{2}(\dim H_2(\mathcal{F}_1) + \dim H_1(\mathcal{F}_2) - N_{np} + 2N_c) \leq N_c$$

Thus, if Conjecture 4.6.4 is true, our bound of $b_0(X;\mathbb{F}_2)$ would be better than the one of [It 1997] stated in [Ar 2020] (Chapter 9).

Now let us study our bound of $b_1(X;\mathbb{F}_2)$. Assume that the triangulation Γ is maximal. Then the restriction of the triangulation to any facet of P is primitive. Consider Γ_p a

fixed primitive triangulation of the lattice polytope P. According to Proposition 3.12 of [BMR 2024], the quantity $\hat{\chi}_1$ for Γ_p (denoted $\hat{\chi}_1(\Gamma_p)$ satisfies $\hat{\chi}_1(\Gamma_p) = -h^{1,1}(\mathbb{C}\tilde{X})$. Then in a general case of a maximal triangulation Γ, we have:

$$\begin{aligned}\hat{\chi}_1 + h^{1,1}(\mathbb{C}\tilde{X}) &= \hat{\chi}_1 - \hat{\chi}_1(\Gamma_p) \\ &= \sum_{\sigma\in\Gamma}(-1)^{3-\dim\sigma}\Big[\binom{3}{1} - \binom{3-\dim\sigma}{1-\dim\sigma}\Big] \\ &\quad - \sum_{\sigma\in\Gamma_p}(-1)^{3-\dim\sigma}\Big[\binom{3}{1} - \binom{3-\dim\sigma}{1-\dim\sigma}\Big] \\ &= 3f_3(\Gamma) - 3f_2(\Gamma) + 2f_1(\Gamma) - 3f_3(\Gamma_p) + 3f_2(\Gamma_p) - 2f_1(\Gamma_p)\end{aligned}$$

where $f_i(\Gamma)$ and $f_i(\Gamma_p)$ denote respectively the number of i-faces of the triangulation Γ and Γ_p. However, as the restrictions of the triangulation Γ and Γ_p to the facets are both primitives, the number of simplex of fixed dimension which are included in the boundary of P is the same for Γ and Γ_p. Therefore, as each tetrahedron of the triangulation has 4 facets and as each facet not included in the boundary of P is shared with another tetrahedron we obtain:

$$f_2(\Gamma) - f_2(\Gamma_p) = \frac{4}{2}(f_3(\Gamma) - f_3(\Gamma_p)) = 2(f_3(\Gamma) - f_3(\Gamma_p)).$$

Moreover, the assumption that the triangulation Γ is maximal means that $f_0(\Gamma) - f_0(\Gamma_p) = 0$. We also know that the Euler characteristic of the triangulation Γ and Γ_p are equals (equal to 1, the Euler characteristic of the polytope P) Then, we obtain:

$$\begin{aligned}0 &= (f_0(\Gamma) - f_0(\Gamma_p)) - (f_1(\Gamma) - f_1(\Gamma_p)) + (f_2(\Gamma) - f_2(\Gamma_p)) - (f_3(\Gamma) - f_3(\Gamma_p)) \\ &= (f_3(\Gamma) - f_3(\Gamma_p)) - (f_1(\Gamma) - f_1(\Gamma_p))\end{aligned}$$

Therefore, we obtain:

$$\begin{aligned}\hat{\chi}_1 &= -h^{1,1}(\mathbb{C}\tilde{X}) + (3 - 3\times 2 + 2)(f_3(\Gamma) - f_3(\Gamma_p)) \\ &= -h^{1,1}(\mathbb{C}\tilde{X}) - (f_3(\Gamma) - f_3(\Gamma_p))\end{aligned}$$

Finally, for a maximal triangulation Γ of a $\mathbb{F}_2$-non-singular lattice polytope P of dimension $n = 3$, denoting by $f_3(\Gamma)$ the number of 3-simplices of Γ, if $f_3(\Gamma_p)$ denotes the number of 3-simplices in a fixed primitive triangulation of the same lattice polytope P, our bound of the Betti number $b_1(X;\mathbb{F}_2)$ of the the hypersurface X constructed from the triangulation Γ and from a distribution of signs on the vertices of Γ is the following:

$$b_1(X;\mathbb{F}_2) \leq \dim H_1(\mathcal{F}_2) + \dim H_2(\mathcal{F}_1) + h^{1,1}(\mathbb{C}\tilde{X}) + (f_3(\Gamma) - f_3(\Gamma_p))$$

Moreover, if we assume that the lattice polytope P is the simplex of vertices $(0,...,0)$, $(d,0,...,0)$, ..., $(0,...,0,d)$, we have $f_3(\Gamma_p) = d^3$ and the bound of $b_1(X;\mathbb{F}_2)$ is the following:

$$b_1(X;\mathbb{F}_2) \leq \dim H_1(\mathcal{F}_2) + \dim H_2(\mathcal{F}_1) + h^{1,1}(\mathbb{C}\tilde{X}) + f_3(\Gamma) - d^3$$

We obtain the same bound as the one conjectured by [Ar 2020] (Chapiter 9). As noticed by [Ar 2020], it is unclear if this bound is better or not than the bound of [It 1997] which is $h^{1,1}(\mathbb{C}\tilde{X})$. Then, to prove that our bound is better than the one of [It 1997], we would have to prove the following conjecture:

Conjecture 4.6.5. *If $n = 3$, if the lattice polytope P is $\mathbb{F}_2$-non-singular, then we have:*

$$\dim H_2(\mathcal{F}_1) + \dim H_1(\mathcal{F}_2) \leq f_3(\Gamma_p) - f_3(\Gamma)$$

where $f_3(\Gamma_p)$ denotes the number of 3-simplices of an arbitrary primitive triangulation Γ_p of P.

Moreover, we can notice that the volume of a non-$\mathbb{F}_2$-primitive 3-simplex is an even number, thus it is at least twice the volume of a primitive 3-simplex. N_{np} is the number of non-$\mathbb{F}_2$-primitive 3-simplex of Γ. Therefore, we have $f_3(\Gamma) + N_{np} \leq f_3(\Gamma_p)$. Thus Conjecture 4.6.4 implies our Conjecture 4.6.5 and then if Conjecture 4.6.4 is true, our bounds for $b_0(X;\mathbb{F}_2)$ and $b_1(X;\mathbb{F}_2)$ are better than the ones of [It 1997].

Now consider the case of a $\mathbb{F}_2$-primitive triangulation Γ of a $\mathbb{F}_2$-non-singular lattice polytope P of dimension $n = 3$ and assume that every edge of the triangulation Γ is of odd integral length. Applying Theorem 4.5.6, the bounds of the Betti numbers of X are:

$$\begin{aligned} b_2(X;\mathbb{F}_2) &\leq \hat{\chi}_0 \\ b_1(X;\mathbb{F}_2) &\leq -\hat{\chi}_1 \\ b_0(X;\mathbb{F}_2) &\leq \hat{\chi}_2 \end{aligned}$$

where $\hat{\chi}_p = \sum_{(F,\sigma)\in\Xi^1} (-1)^{\dim F - \dim\sigma} \left[\binom{\dim F}{p} - \binom{\dim F - \dim\sigma}{p - \dim\sigma} \right]$ is the Euler characteristic of the cosheaf $\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2$.

However, using our Poincaré duality theorem (Corollary 1.3.3) for the cosheaves $\mathcal{F}_p^{\mathbb{Q}}$, as the Euler characteristic of $\mathcal{F}_p^{\mathbb{Q}}$ is equal to the Euler characteristic $\hat{\chi}_p$ of $\hat{\mathcal{F}}_p^{\mathbb{Z}} \otimes \mathbb{F}_2$, we obtain the identity $\hat{\chi}_2 = \hat{\chi}_0$ and then the bound of $b_0(X;\mathbb{F}_2)$ is the same as the bound of $b_2(X;\mathbb{F}_2)$. Using Lemma 4.6.1 and Corollary 4.6.2, we obtain the following bound for $b_0(X;\mathbb{F}_2) = b_2(X;\mathbb{F}_2)$:

$$\begin{aligned} b_0(X;\mathbb{F}_2) &\leq 1 + \text{Card}\Big\{ v \text{ vertex of } \Gamma \text{ s.t. } v \in \overset{\circ}{P} \Big\} \\ &= 1 + h^{0,2}(\mathbb{C}\tilde{X}) - \text{Card}\Big\{ v \in M \cap \overset{\circ}{P} \text{ s.t. } v \text{ is not a vertex of } \Gamma \Big\} \end{aligned}$$

Then, if we also assume that the triangulation Γ is maximal, the set $\Big\{ v \in M \cap \overset{\circ}{P} \text{ s.t. } v \text{ is not a vertex of } \Gamma \Big\}$ is empty, and then the bound of $b_0(\mathbb{C}\tilde{X})$ is:

$$b_0(X;\mathbb{F}_2) \leq 1 + h^{0,2}(\mathbb{C}\tilde{X})$$

Moreover, applying our earlier calculations, we obtain that in the case of a $\mathbb{F}_2$-primitive and maximal triangulation Γ of a $\mathbb{F}_2$-non-singular lattice polytope P of dimension $n = 3$, the bound of $b_1(X;\mathbb{F}_2)$ is:

$$b_1(X;\mathbb{F}_2) \leq h^{1,1}(\mathbb{C}\tilde{X}) + (f_3(\Gamma) - f_3(\Gamma_p))$$

where $f_3(\Gamma)$ is the number of 3-simplices of Γ, and $f_3(\Gamma_p)$ is the number of 3-simplices of an arbitrary fixed primitive triangulation of the lattice polytope P.

4.6.4 Case of a polytope P of dimension $n \geq 4$

Assume that any edge of the triangulation Γ is of odd integral length, and assume that the lattice polytope P is $\mathbb{F}_2$-non-singular. We can generalize the previous computation for the bound of $b_0(X;\mathbb{F}_2)$ done in dimension 2, using Lemma 4.6.1 of 4.6.2 to compute $\hat{\chi}_{n-1} = (-1)^{n-1}\hat{\chi}_0$, and using Theorem 4.5.6 in the case of a $\mathbb{F}_2$-primitive triangulation, using Theorem 4.5.8 in the case of a $(n-1,\mathbb{F}_2)$-primitive triangulation, or using Theorem 4.5.5 in the case of a $(k,\mathbb{F}_2)$-primitive triangulation with $k \notin \{n-1, n-2\}$. For instance, for a $\mathbb{F}_2$-primitive triangulation, we obtain:

$$b_0(X;\mathbb{F}_2) \leq 1 + h^{0,n-1}(\mathbb{C}\tilde{X}) - \mathrm{Card}\Big\{v \in M \cap \overset{\circ}{P} \text{ s.t. } v \text{ is not a vertex of } \Gamma\Big\}$$

For a $(n-1,\mathbb{F}_2)$-primitive triangulation we obtain:

$$\begin{aligned} b_0(X;\mathbb{F}_2) \leq 1 + h^{0,n-1}(\mathbb{C}\tilde{X}) - \mathrm{Card}\Big\{v \in M \cap \overset{\circ}{P} \text{ s.t. } v \text{ is not a vertex of } \Gamma\Big\} \\ + \min(\dim H_{n-1}(\mathcal{F}_1); \dim H_1(\mathcal{F}_{n-1}) - N_{np} + 2N_c). \end{aligned}$$

If we also assume that the triangulation is maximal, the term $-\mathrm{Card}\Big\{v \in M \cap \overset{\circ}{P} \text{ s.t. } v \text{ is not a vertex of } \Gamma\Big\}$ disappears from the last two bounds. Moreover, similarly as for a polytope of dimension $n = 3$, a conjecture can be formulated:

Conjecture 4.6.6. *For $n \geq 4$, if any edge of the triangulation Γ is of odd integral length, if Γ is $(n-1,\mathbb{F}_2)$-primitive, and if the lattice polytope P is $\mathbb{F}_2$-non-singular, then we have:*

$$\dim H_{n-1}(\mathcal{F}_1) + \dim H_1(\mathcal{F}_{n-1}) \leq N_{np}$$

If this conjecture is true, then we would have for any $(n-1,\mathbb{F}_2)$-primitive triangulation:

$$b_0(X;\mathbb{F}_2) \leq 1 + h^{0,n-1}(\mathbb{C}\tilde{X}) - \mathrm{Card}\Big\{v \in M \cap \overset{\circ}{P} \text{ s.t. } v \text{ is not a vertex of } \Gamma\Big\} + N_c.$$

For the bounds of the other Betti numbers, it is more complicate to find a simple expression of the Euler characteristics $\hat{\chi}_p$. In the case $n = 3$, we achieved to do it with adding another assumption: the maximality of the triangulation Γ. The maximality has given the identity $f_0(\Gamma) = f_0(\Gamma_p)$ between the number of vertices of the triangulation Γ and the number of vertices of a primitive arbitrary triangulation Γ_p of the same lattice polytope P. Then a comparison of Euler characteristic of Γ and Γ_p was enough to compute all of the differences $f_i(\Gamma) - f_i(\Gamma_p)$, in terms of $f_3(\Gamma) - f_3(\Gamma_p)$. However, in the case $n \geq 4$ the assumption of maximality ($f_0(\Gamma) = f_0(\Gamma_p)$) and the comparison of Euler characteristic $\chi(\Gamma) = \chi(\Gamma_p)$ is not enough to compute all of the differences $f_i(\Gamma) - f_i(\Gamma_p)$, in terms of $f_n(\Gamma) - f_n(\Gamma_p)$. We can compute them modulo adding some parameters. For instance in the case $n = 4$, with the assumption of maximality of the triangulation, one added parameter is necessary, for instance we can decide that the quantity $f_1(\Gamma) - f_1(\Gamma_p)$ is this parameter. In general, the theory of Erhardt polynomials enables us to determine the number of parameters that need to be fixed (see [St 1990]).

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