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3-group extensions of fusion 2-categories and quantum homotopy invariants of 4-manifolds

Extensions des 2-catégories de fusion graduées par les 3-groupes et invariants quantiques homotopiques des 4-variétés

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Introduction

Manifolds have played a central role in topology for over a century. The study of curves and surfaces—that is, manifolds of dimensions one and two—has been well understood since the nineteenth century. The theory of 3-dimensional manifolds is considerably more intricate, although it has seen steady progress throughout the twentieth century. In higher dimensions, a major breakthrough occurred in the 1960s with the development of surgery theory and the s-cobordism theorem, which provided powerful tools for addressing existence and uniqueness questions for manifolds of dimension at least 5.

Within this landscape, 4-dimensional manifolds occupy a particularly rich and singular position. A first major breakthrough came in 1982 with Freedman's work on topological 4-manifolds, culminating in the classification of all closed simply-connected topological 4-manifolds [Fre82, BKK$^+$21]. Shortly thereafter, Donaldson introduced revolutionary techniques from gauge theory into the study of smooth 4-manifolds, revealing striking differences from the higher-dimensional setting, see [Don83]. Notably, dimension 4 is the only dimension in which there exist infinite families of manifolds that are homeomorphic but not diffeomorphic. As a result, the study of 4-manifolds exhibits subtle and distinctive phenomena, with deep connections to areas such as symplectic topology, algebraic geometry, and quantum topology.

Quantum topology is a branch of mathematics that emerged at the end of the twentieth century following groundbreaking work by Jones, Drinfeld, Witten, and Kontsevich, which drastically renewed topology, particularly in low dimensions. Drawing on ideas from quantum physics, their framework has led to the construction of new manifold invariants based on quantum groups, called *quantum invariants*. One of the early successes in this direction is the Turaev–Viro–Barrett–Westbury invariant of closed 3-manifolds, defined as a state sum using quantum 6j-symbols associated with a fusion category, see [TV92, BW96].

The Turaev–Viro–Barrett–Westbury construction can be extended to 3-manifolds with boundary and the resulting invariants fit into a 3-dimensional topological quantum field theory (TQFT), which is a functorial assignment of modules and linear maps to surfaces and 3-dimensional cobordisms, respectively. While originally motivated by theoretical physics, TQFTs have come to play a central role in low-dimensional topology, in representation theory, and in higher category theory.

More recently, Douglas and Reutter [DR18] introduced a state-sum construction of invariants for 4-manifolds based on fusion 2-categories. Their framework unifies several previously known quantum invariants, including the Crane–Yetter–Kauffman invariant [CKY97], the (twisted) Yetter–Dijkgraaf–Witten invariant [Yet93, FMP07], the Mackaay invariant [Mac99], and the Cui invariant [Cui19]. Fusion 2-categories are fully dualizable objects in a suitable 4-category of tensor 2-categories, see [Déc24a]; by the cobordism hypothesis, they give rise to framed fully extended 4-dimensional TQFTs.

Homotopy quantum field theories (HQFTs) extend topological quantum field theories by incorporating additional structures on manifolds. Originally introduced by Segal [Seg88], they were later developed systematically by Turaev [Tur99, Tur00, Tur10], Porter [Por07], Porter–Turaev [PT08], Turaev–Virelizier [TV12, TV14, TV20], and Sözer–Virelizier [SV23, SV25]. HQFTs have been generalized for manifolds with corners (via the notion of extended HQFT) and these have been studied by Schweigert–Woike [SW20], Müller–Woike [MW20], and Sözer [Sö23]. These theories refine quantum invariants of manifolds to be sensitive to extra structures encoded by homotopy classes of maps to a fixed target space. The resulting invariants are known as *quantum homotopy invariants*.

It is classical in topology to associate to a connected space X the sequence $\{\pi_k(X)\}_{k\geqslant 1}$ of its homotopy groups. Roughly speaking, the group $\pi_k(X)$ records information about the "k-dimensional holes" in X. These groups provide invariants: for instance, Whitehead's theorem asserts that a map between CW complexes is a homotopy equivalence if and only if it induces isomorphisms on all homotopy groups. A major drawback, however, is that homotopy groups are notoriously difficult to compute. This motivates restricting attention to more tractable classes of spaces, such as *homotopy n-types*, namely spaces whose homotopy groups vanish in all

degrees $k > n$. The homotopy hypothesis, due to Grothendieck, predicts that ∞-groupoids model homotopy types, and in particular that n-groups correspond to connected homotopy n-types.

Homotopy quantum field theories with target a connected homotopy 0-type (that is, a contractible space) correspond to TQFTs. In general, the complexity of an HQFT increases both with the dimension of the source manifolds and with the homotopical complexity of the target space. For instance, Turaev and Virelizier generalized the Turaev–Viro–Barrett–Westbury invariant to 3-manifolds equipped with maps to the classifying space of a group G using the data of a G-fusion category, see [TV12]. This was further extended by Sözer and Virelizier to targets given by the classifying space of a 2-group, via 2-group extensions of fusion categories, see [SV23].

In higher dimensions and for more intricate target spaces, the construction of quantum homotopy invariants becomes more challenging. Recent advances in dimension 4 include Mochida's construction [Moc24] of quantum homotopy invariants with target the classifying space of a group G, based on Hopf G-algebras, as well as its generalization by Bridges and Cui [BC25] using Hopf G-triplets. The present work contributes to this line of research by providing a state-sum construction of 4-dimensional quantum homotopy invariants with target the classifying space of a 3-group. This uses 3-group extensions of fusion 2-categories as algebraic input. These quantum homotopy invariants of 4-manifolds generalize the invariant of Douglas–Reutter (corresponding to the trivial 3-group) and are expected to encompass, as special cases, the invariants of Mochida and Bridges–Cui.

In low degrees, the correspondence between n-groups and connected homotopy n-types is well understood: groups model connected homotopy 1-types, and crossed modules, introduced by Whitehead [Whi41], model connected homotopy 2-types. Several algebraic models for 3-groups exist, including 2-crossed modules defined by Conduché [Con84]. A 2-crossed module is a pair

$$\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega),$$

consisting of a complex of groups (in which H acts on L and E) and a Peiffer lifting $\omega\colon E \times E \to L$, satisfying certain compatibility conditions. The correspondence between 3-groups and homotopy 3-types is made explicit in the work of Faria Martins [FM11], who associates to a reduced CW complex its fundamental 2-crossed module.

In this work, building on the construction of Faria Martins, we introduce the fundamental 2-crossed module $\Pi_3(K)$ of a simplicial complex K, and we define the classifying space $B\sigma$ of a 2-crossed module σ. The classifying space $B\sigma$ is a connected CW complex whose homotopy groups are given by:

$$\pi_1(B\sigma) = H/\mathrm{Im}(\partial), \qquad \pi_2(B\sigma) = \mathrm{Ker}(\partial)/\mathrm{Im}(\delta), \qquad \pi_3(B\sigma) = \mathrm{Ker}(\delta), \qquad \pi_n(B\sigma) = 0, \text{ for all } n \geqslant 4.$$

Every connected homotopy 3-type is homotopy equivalent to the classifying space of a 2-crossed module. A first result in this thesis is a combinatorial description of the set of homotopy classes of maps $|K| \to B\sigma$, where $|K|$ is the underlying space of a simplicial complex K. To this end, we introduce the notion of *canonical simplicial complex*, in which every triangle has a distinguished vertex, and we define an equivalence relation on morphisms from $\Pi_3(K)$ to σ, called *gauge equivalence*. This leads to the following result.

Theorem (Homotopy classification theorem)**.** *Let K be a canonical simplicial complex and let σ be a 2-crossed module of groups. There is a canonical bijection*

$$[|K|, B\sigma] \cong [\Pi_3(K), \sigma]$$

where the left-hand side denotes homotopy classes of maps, and the right-hand side denotes gauge-equivalence classes of morphisms of 2-crossed modules.

This relies heavily on the work of Faria Martins [FM11] and provides a key tool for constructing quantum homotopy invariants with target a homotopy 3-type.

The study of homotopy classes of maps to a homotopy 3-type also has motivations from mathematical physics. In classical gauge theory, principal bundles describe how point-like particles transform along paths. Higher gauge theory extends this framework to strings and higher-dimensional objects via higher principal bundles and gerbes. For a discrete group G and a smooth manifold W, homotopy classes of maps $W \to BG$ correspond to isomorphism classes of flat G-bundles over W. Similarly, one can define the notion of flat 2-bundle with a structure 2-group, which are classified by homotopy classes of maps to the classifying space of the 2-group [BS09]. Adapting the work of Jurčo [Jur11], we extend this construction to 2-crossed modules and prove the following.

Theorem (Classification of flat 3-bundles)**.** *Let σ be a 2-crossed module of (discrete) groups and let W be a smooth manifold. There is a canonical bijection between the isomorphism classes of flat σ-bundles over W and the homotopy classes of maps $W \to B\sigma$.*

The Douglas–Reutter invariant of a smooth 4-manifold W is constructed via a state sum over a triangulation of W using the data of a fusion 2-category. Its well-definedness relies on Pachner's theorem [Pac91, Lic99], which states that two closed 4-manifolds are diffeomorphic if and only if their triangulations are related by a finite sequence of local transformations, called Pachner moves. Our objective is to extend the Douglas–Reutter state-sum construction to obtain quantum invariants of flat σ-bundles over 4-manifolds, or equivalently, by the above theorem, of *σ-manifolds*, which are pairs (W, g) where W is a closed, oriented smooth 4-manifold and g is a homotopy class of maps $W \to B\sigma$. We say that two σ-manifolds (W, g) and (W', g') are *equivalent* if there is an orientation preserving diffeomorphism $\varphi\colon W \xrightarrow{\sim} W'$ such that $g' \circ \varphi = g$. We model a σ-manifold (W, g) using a *σ-triangulation*, i.e., a pair (K, ϕ) where K is a triangulation of W and $\phi\colon \Pi_3(K) \to \sigma$ is a 2-crossed module morphism whose geometric realization is homotopic to g. In order to obtain invariants of σ-manifolds, we introduce σ-colored Pachner moves and prove:

Theorem (Colored Pachner theorem)**.** *Two σ-manifolds are equivalent if and only if their σ-triangulations are related by a finite sequence of σ-colored Pachner moves.*

We introduce the notion of *σ-graded monoidal 2-category*, which is the relevant algebraic data for the construction of quantum homotopy invariants of 4-manifolds. A σ-graded monoidal 2-category is a linear monoidal 2-category with objects, 1-morphisms, and 2-morphisms graded by the groups H, E, and L, respectively, in such a way that the tensor product is compatible with the action of H on L and E, and the interchange 2-isomorphism is compatible with the Peiffer lifting $\omega\colon E \times E \to L$. A *$\sigma$-fusion 2-category* is a σ-graded monoidal 2-category with duals satisfying suitable semisimplicity and finiteness conditions. Such a structure should be viewed as a 3-group extension of a fusion 2-category. In particular, its *neutral component* (the subcategory consisting of objects, 1-morphisms, and 2-morphisms that are all of degree 1) is itself a fusion 2-category.

Let W be a closed, oriented smooth 4-manifold equipped with a triangulation K. The Douglas–Reutter invariant of W is defined by summing the 10j-symbols over all states of K. In our setting, given a σ-manifold (W, g) together with a σ-triangulation (K, ϕ), we define an analogous state sum by restricting to the states of K that are compatible with ϕ and summing the associated colored 10j-symbols. This construction produces a scalar, denoted $\mathrm{HDR}_{\mathcal{C}}(W, g)$. Our main result is the following.

Theorem. *The state sum $\mathrm{HDR}_{\mathcal{C}}(W, g)$ is a scalar invariant of σ-manifolds.*

By what has been discussed above, $\mathrm{HDR}_{\mathcal{C}}$ can be seen as an invariant of flat σ-bundles over 4-manifolds. Note that if $g = *$ is the homotopy class of a constant map, then $\mathrm{HDR}_{\mathcal{C}}(W, *)$ is the Douglas–Reutter invariant of W constructed using the neutral component of $\mathcal{C}$. Moreover, we show through examples that $\mathrm{HDR}_{\mathcal{C}}$ provides nontrivial invariants of homotopy classes of maps. In particular, it can distinguish phantom maps, i.e., maps inducing trivial homomorphisms on homotopy groups. We predict that the invariants of Mochida and Bridges–Cui arise as special cases for 2-crossed modules of the form

$$\sigma = (1 \longrightarrow 1 \longrightarrow G,\ 1).$$

Also, we expect our construction to extend to σ-manifolds with boundary and to fit into the framework of a homotopy quantum field theory. Moreover, our state sum should recover the classical construction of cohomological HQFTs through suitable twistings of σ-fusion 2-categories by cocycles.

Thesis outline

This thesis is organized as follows. In Chapter I, we recall the notions of 2-crossed modules and 2-crossed complexes in the context of groupoids. We introduce the fundamental 2-crossed module of a simplicial complex and the classifying space of a 2-crossed module of groups. We then prove the homotopy classification theorem. In Chapter II, we introduce flat σ-bundles, and we show that isomorphism classes of flat σ-bundles are in bijective correspondence with homotopy classes of maps to the classifying space of σ. We then describe triangulations of σ-manifolds and show the colored Pachner theorem. Chapter III is devoted to the definition of σ-graded

monoidal 2-categories and σ-fusion 2-categories. Our state-sum quantum homotopy invariant is constructed in Chapter IV via the colored 10j-symbols of spherical σ-fusion 2-categories. We then compute examples and prove the invariance of the state sum. Chapter V concludes with potential directions for future research.

CHAPTER I

The classification of maps to a homotopy 3-type

Homotopy types are equivalence classes of spaces defined using the notion of homotopy. Two maps $f, g\colon X \to Y$ are *homotopic*, denoted $f \simeq g$, if f can be continuously deformed to g, that is, if there is a continuous family of maps $\{h_t\colon X \to Y\}_{t\in[0,1]}$, such that $h_0 = f$ and $h_1 = g$. Two spaces X and Y are *homotopy equivalent*, denoted $X \simeq Y$, if there are continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. A class of homotopy equivalent spaces is called a *homotopy type*.

Classifying all spaces up to homotopy equivalence is a very difficult problem, and it is natural to consider certain subclasses of spaces, such as CW complexes. CW complexes are spaces that are constructed by successively gluing cells of increasing dimension. Therefore, they come with a sequence of skeleta X_0, X_1, X_2... where X_n consists of all the cells of X that are of dimension at most n. Certain properties only depend on the homotopy type of one of the skeleta of X, and not on the homotopy type of X itself. Whitehead introduced the homotopy n-type of X, which is a homotopy invariant of X that only depends on X_{n+1}. There are two ways to present homotopy n-types: on the one hand they are certain equivalence classes of $(n+1)$-dimensional CW complexes, and on the other hand they are the homotopy types of certain spaces.

Let $\mathbf{CW_{n+1}}$ be the category of $(n+1)$-dimensional connected CW complexes with basepoint $* \in X_0$ and basepoint preserving cellular maps. We say that two morphisms $f, g\colon X_{n+1} \to Y_{n+1}$ in $\mathbf{CW_{n+1}}$ are *equivalent*, denoted $f \sim g$, if their restrictions $f_{|X_n}$ and $g_{|X_n}$ are pointed homotopic. A *connected homotopy n-type* (or simply a *homotopy n-type*) is an isomorphism type in the quotient category $\mathbf{CW_{n+1}}/\sim$.

A *CW-space* is a topological space that is homotopy equivalent to a CW complex. We denote by **Spaces** the category of pointed connected CW-spaces and basepoint preserving maps, and by $\mathbf{Spaces}/\simeq$ the corresponding homotopy category. The category **n-types** is the full subcategory

$$\mathbf{n\text{-}types} \subset \mathbf{Spaces}/\simeq$$

consisting of spaces X with trivial homotopy groups $\pi_i(X) = 0$ for all $i > n$. There is an equivalence of categories

$$\mathbf{n\text{-}types} \simeq \mathbf{CW_{n+1}}/\sim$$

and therefore the two notions of homotopy n-types coincide.

Connected homotopy 1-types are the same as classifying spaces of groups, and we can therefore identify a 1-type with a group. In fact, the fundamental group yields an equivalence of categories:

$$\pi_1\colon \mathbf{1\text{-}types} \xrightarrow{\sim} \mathbf{Gp},$$

where **Gp** denotes the category of groups and group morphisms. This functor turns a topological problem, the classification of homotopy 1-types, into an algebraic problem, the classification of groups. Homotopy n-types can thus be seen as higher algebraic structures extending groups.

The classification of connected homotopy 2-types is more involved than that of homotopy 1-types. There are several ways to encode the data of a connected homotopy 2-type X, such as

- via 3-cocycles $\pi_1(X)^3 \to \pi_2(X)$, see [EM49],
- via crossed modules, as introduced by Whitehead [Whi41].

We focus on crossed modules, as they are a convenient way of encoding a strict 2-group. A crossed module is a morphism of groups $\chi\colon E \to H$ together with an action of H on E satisfying certain compatibility conditions. In particular, any pointed CW complex X determines a crossed module

$$\pi_2(X, X_1) \to \pi_1(X_1)$$

called the *fundamental crossed module* of X, where X_1 is the 1-skeleton of X. Mac Lane and Whitehead show [MW50] that crossed modules are algebraic structures that fully classify connected homotopy 2-types.

There are many algebraic structures that classify connected homotopy 3-types, such as

- crossed squares, as defined by Guin-Waléry and Loday [GWL81] and studied by Ellis [Ell93],
- 2-crossed modules, as defined by Conduché [Con84] and studied by Faria Martins [FM11],
- simplicial groups with Moore complex of length 3,
- quadratic modules, as defined by Baues [Bau11],
- braided regular crossed modules, as defined by Brown and Gilbert [BG89].

In this chapter, we work in the setting of 2-crossed modules. A 2-crossed module is a pair

$$(L \longrightarrow E \longrightarrow H,\ \omega)$$

consisting of a complex of groups with actions of H on L and E, and a Peiffer lifting $\omega\colon E \times E \to L$, that satisfy certain compatibility conditions (see Section I.1.8).

Our aim is to give an algebraic description of the set $[|K|, X]$ of homotopy classes of maps from the underlying space of a simplicial complex K to a homotopy 3-type X. In the case where X is a homotopy 2-type, such a description is given by the homotopy classification theorem of Brown, Higgins, and Sivera [BHS11] in terms of homotopy classes of maps of crossed complexes. In this chapter, we build on the work of Faria Martins [FM11] concerning 2-crossed modules and pointed homotopy classes of maps. In order to avoid dealing with basepoints, we extend the definitions of 2-crossed module and 2-crossed complex to the context of groupoids. We introduce the fundamental 2-crossed module $\Pi_3(K)$ of a simplicial complex K and the classifying space $B\sigma$ of a 2-crossed module σ, and we show the following result:

Theorem 1 (Homotopy classification theorem)**.** *Let K be a canonical simplicial complex and let σ be a 2-crossed module of groups. There is a bijection*

$$[|K|, B\sigma] \cong [\Pi_3(K), \sigma]$$

where the left-hand side is the set of homotopy classes of maps from the underlying space of K to the classifying space of σ, and the right-hand side is the set of gauge-equivalence classes of morphisms of 2-crossed modules.

Its proof builds on the work of Brown–Loday [BL87], Berger [Ber99], Ellis [Ell93], and Faria Martins [FM11].

This chapter is organized as follows. In Section I.1, we extend the definitions of 2-crossed modules and 2-crossed complexes to the context of groupoids and we introduce notions of free 2-crossed modules. In Section I.2 we define the fundamental 2-crossed module $\Pi_3(K)$ of a simplicial complex K. Section I.3 is devoted to the construction of the classifying space $B\sigma$ of a 2-crossed module of groups σ. In Section I.4 we extend the notion of quadratic derivations to the context of groupoids and prove Theorem 1.

I.1. Crossed modules and 2-crossed modules

The notion of 2-crossed module was originally introduced by Conduché [Con84] as a model for pointed homotopy 3-types. In order to work in an unpointed setting, we extend the definition of a 2-crossed module to the context of groupoids. We then introduce free 2-crossed modules.

I.1.1. Notation. For any groupoid H, we denote by H_0 the set of objects of H, by $H(x, y)$ the set of morphisms from an object x to an object y, and by $H(x)$ the group of endomorphisms of an object x. In a groupoid, the composition of a morphism $h\colon x \to y$ and a morphism $k\colon y \to z$ is denoted $hk\colon x \to z$.

I.1.2. Crossed modules of groupoids. In this section, we recall some definitions due to Brown and Higgins [BH81].

A groupoid is *totally disconnected* if all of its morphisms are endomorphisms. If E is a totally disconnected groupoid, we write

$$E = (E(x))_{x \in E_0}.$$

A (left) *action* of a groupoid H on a groupoid E with the same set of objects as H is a family of maps

$$\{H(x, y) \times E(y) \to E(x),\ (h, e) \mapsto {}^h e\}_{x,y \in H_0}$$

such that

$${}^k(ef) = {}^k e\, {}^k f, \qquad {}^{hk}e = {}^h({}^k e), \qquad {}^1 e = e$$

for all $x, y, z \in H_0$, $h \in H(x, y)$, $k \in H(y, z)$, and $e, f \in E(z)$. We say that a groupoid H *acts on* a groupoid E if H and E have the same set of objects and there is an action of H on E.

A *pre-crossed module* is a morphism $(\partial\colon E \to H)$ from a totally disconnected groupoid E to a groupoid H together with an action of H on E such that ∂ is the identity on objects and preserves the action of H, where H acts on itself by conjugation.

Let E and E' be totally disconnected groupoids with actions of H and H' respectively, and let $\phi_1 : H \to H'$ be a morphism of groupoids. A morphism $\phi_2 : E \to E'$ is ϕ_1-*equivariant* if it coincides with ϕ_1 on objects and satisfies

$$\phi_2({}^h e) = {}^{\phi_1(h)}\phi_2(e)$$

for all $x, y \in H_0$, $h \in H(x, y)$, and $e \in E(y)$.

A *morphism* from a pre-crossed module $(\partial\colon E \to H)$ to a pre-crossed module $(\partial'\colon E' \to H')$ is pair of groupoid morphisms $\phi = (\phi_1\colon H \to H', \phi_2\colon E \to E')$ making the square

$$\begin{array}{ccc} E & \xrightarrow{\partial} & H \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} \\ E' & \xrightarrow{\partial'} & H' \end{array}$$

commutative and such that ϕ_2 is ϕ_1-equivariant.

A *crossed module* is a pre-crossed module $(\partial\colon E \to H)$ that also satisfies the *Peiffer identity*:

$${}^{\partial e} f = efe^{-1}$$

for all $x \in H_0$ and $e, f \in E(x)$. A *crossed module of groups* is a crossed module with set of objects a singleton.

I.1.3. Examples. 1) There is a natural inclusion of groupoids, which model homotopy 1-types, into the category of crossed modules, which model homotopy 2-types. If H is a groupoid and $\overline{H_0}$ denotes the discrete groupoid with set of objects H_0, then the morphism $\overline{H_0} \to H$ together with the trivial action of H define a crossed module. This construction is clearly functorial.

2) Crossed modules, as their name suggests, can be seen as a generalization of modules. A *module* over a groupoid H is a totally disconnected abelian groupoid with an action of H. If M is a module over a groupoid H, then the trivial morphism $M \to H$ together with the action of H define a crossed module.

3) Let H be a groupoid and let $E = (E(x))_{x \in H_0}$ be a totally disconnected groupoid such that $E(x)$ is a normal subgroup of $H(x)$ for all $x \in H_0$. Then, the faithful functor $E \to H$ defines a crossed module where H acts on E by conjugation.

4) A particularly important example of crossed module comes from topology. Let (X, A) be a pair of topological spaces, $A \subset X$, and let $X_0 \subset A$ be a set of basepoints. We denote by $\pi_1(A)$ the *fundamental groupoid* of A, that is, the groupoid with set of objects A and with morphisms from x to y given by the homotopy classes (relative to the endpoints) of paths in A from x to y. We denote by $\pi_1(A; X_0)$ the full subcategory of $\pi_1(A)$ with set of objects X_0. There is a boundary morphism ∂ from the totally disconnected groupoid

$$\pi_2(X, A; X_0) = \big(\pi_2(X, A; x)\big)_{x \in X_0}$$

to the groupoid $\pi_1(A; X_0)$. Together with the action of $\pi_1(A; X_0)$ on $\pi_2(X, A; X_0)$, this defines a crossed module. In the case where X is a CW complex and $A = X_1$ is its 1-skeleton, the crossed module

$$(\partial\colon \pi_2(X, X_1; X_0) \to \pi_1(X_1; X_0))$$

is called the *fundamental crossed module* of X.

I.1.4. The associated crossed module to a pre-crossed module. To any pre-crossed module

$$(\partial\colon E \to H),$$

we associate the groupoid E^{cr} with set of objects H_0 and morphisms given by:

$$E^{\mathrm{cr}}(x) = E(x)/\langle E(x), E(x)\rangle$$

for all $x \in H_0$, where $\langle E(x), E(x)\rangle$ denotes the normal subgroup of $E(x)$ generated by the elements of the form

$$efe^{-1}\,{}^{\partial e} f^{-1},$$

for all $e, f \in E(x)$. The action of H on E induces an action of H on E^{cr}, and the morphism $\partial\colon E \to H$ induces a morphism $\partial^{\mathrm{cr}}\colon E^{\mathrm{cr}} \to H$. The *associated crossed module* to a pre-crossed module $(\partial\colon E \to H)$ is the crossed module $(\partial^{\mathrm{cr}}\colon E^{\mathrm{cr}} \to H)$ where H acts on E^{cr} via the induced action. The associated crossed module

to a pre-crossed module $(\partial\colon E \to H)$ satisfies the following universal property: for any crossed module of the form $(\partial'\colon E' \to H)$ and for any morphism of pre-crossed modules $(\varphi, 1_H)\colon (\partial\colon E \to H) \to (\partial'\colon E' \to H)$, there exists a unique morphism of crossed modules:

$$(\phi, 1_H)\colon (\partial^{\mathrm{cr}}\colon E^{\mathrm{cr}} \to H) \to (\partial'\colon E' \to H),$$

such that $\phi \circ \pi = \varphi$, where $\pi\colon E \to E^{\mathrm{cr}}$ denotes the quotient morphism.

The *associated crossed module functor* $(-)^{\mathrm{cr}}\colon \mathbf{PXMod} \to \mathbf{XMod}$ sends a pre-crossed module to its associated crossed module, and sends a morphism of pre-crossed modules $(\phi, \psi)\colon (\partial\colon E \to H) \to (\delta\colon F \to G)$ to the associated morphism

$$(\phi^{\mathrm{cr}}, \psi)\colon (\partial^{\mathrm{cr}}\colon E^{\mathrm{cr}} \to H) \to (\delta^{\mathrm{cr}}\colon F^{\mathrm{cr}} \to G),$$

where $\phi^{\mathrm{cr}}\colon E^{\mathrm{cr}} \to F^{\mathrm{cr}}$ is the morphism induced by ϕ. The functor $(-)^{\mathrm{cr}}$ is left adjoint to the inclusion of categories $\mathbf{XMod} \to \mathbf{PXMod}$.

I.1.5. The coproduct of crossed modules. The *free product* of two totally disconnected groupoids E and F with $E_0 = F_0$ is the totally disconnected groupoid $E * F$ with objects $(E * F)_0 = E_0$ and morphisms

$$(E * F)(x) = E(x) * F(x),$$

for all $x \in E_0$, where $*$ denotes the free product of groups. The *free product* of two pre-crossed modules over H

$$(\partial\colon E \to H) \quad \text{and} \quad (\delta\colon F \to H)$$

is the pre-crossed module $(\partial * \delta\colon E * F \to H)$ with $\partial * \delta$ defined by:

$$(\partial * \delta)(ef) = \partial(e)\delta(f)$$

for all $e \in E(x)$ and $f \in F(x)$, together with the diagonal action of H on $E * F$.

The *coproduct* $(\partial \circ \delta\colon E \circ F \to H)$ of two crossed modules $(\partial\colon E \to H)$ and $(\delta\colon F \to H)$ over H is the associated crossed module to their free product:

$$(\partial \circ \delta\colon E \circ F \to H) = ((\partial * \delta)^{\mathrm{cr}}\colon (E * F)^{\mathrm{cr}} \to H).$$

I.1.6. The semidirect product of groupoids. The *semidirect product* of a groupoid E and a groupoid H that acts on E is the groupoid $E \rtimes H$ with objects $(E \rtimes H)_0 = H_0$, and morphisms

$$(E \rtimes H)(x, y) = \{(e, h) \mid e \in E(x),\ h \in H(x, y)\},$$

for all objects x and y with composition given by:

$$(e, h)(f, k) = (e\,{}^h f, hk) \in (E \rtimes H)(x, z),$$

for all $(e, h) \in (E \rtimes H)(x, y)$ and $(f, k) \in (E \rtimes H)(y, z)$.

I.1.7. Crossed complexes of groupoids. A *crossed complex* (of groupoids) is a complex of groupoids:

$$\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \cdots \xrightarrow{d_4} C_3 \xrightarrow{d_3} E \xrightarrow{\partial} H$$

over an object set X, together with actions of H on E and C_n for all $n \geqslant 3$ verifying the following conditions:

- $(\partial : E \to H)$ is a crossed module,
- C_n is a totally disconnected abelian groupoid for all $n \geqslant 3$,
- d_n is the identity on objects and is H-equivariant for all $n \geqslant 3$,
- $\partial \circ d_3 = 0$ and $d_n \circ d_{n+1} = 0$ for all $n \geqslant 3$,
- the groupoid $\partial(E) \subset H$ acts trivially on C_n for all $n \geqslant 3$.

We say that a crossed complex $\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \cdots \xrightarrow{d_4} C_3 \xrightarrow{d_3} E \xrightarrow{\partial} H$ is *of length* n, for some integer $n \geqslant 2$, if C_k is a discrete groupoid for all $k > n$.

A *morphism of crossed complexes*

$$f : (\cdots \xrightarrow{d_4} C_3 \xrightarrow{d_3} E \xrightarrow{\partial} H) \to (\cdots \xrightarrow{d'_4} C'_3 \xrightarrow{d'_3} E' \xrightarrow{\partial'} H')$$

is a family of groupoid morphisms

$$f_1 : H \to H', \qquad f_2 : E \to E', \qquad f_n : C_n \to C'_n \ \text{ for all } n \geqslant 3,$$

such that f_n is f_1-equivariant for all $n \geqslant 2$, and the (infinite) diagram

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{d_6} & C_5 & \xrightarrow{d_5} & C_4 & \xrightarrow{d_4} & C_3 & \xrightarrow{d_3} & E & \xrightarrow{\partial} & H \\
 & & \downarrow{\scriptstyle f_5} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\
\cdots & \xrightarrow{d'_6} & C'_5 & \xrightarrow{d'_5} & C'_4 & \xrightarrow{d'_4} & C'_3 & \xrightarrow{d'_3} & E' & \xrightarrow{\partial'} & H'
\end{array}$$

is commutative.

I.1.8. 2-crossed modules of groupoids. A *2-crossed module* is a tuple $\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega)$ where

- $(\partial\colon E \to H)$ is a pre-crossed module
- L is a totally disconnected groupoid and H acts on L
- δ is a morphism of groupoids that is the identity on objects and preserves the action of H,
- the *Peiffer lifting* ω is a collection of set maps

$$\omega = (\omega_x\colon E(x) \times E(x) \to L(x))_{x \in E_0}$$

that preserve the action of H,

satisfying the following conditions:

$$\partial \circ \delta(l) = 1_x, \qquad \delta\left(\omega_x(e,f)\right) = efe^{-1}\,{}^{\partial(e)}f^{-1}, \quad \omega_x(\delta(l),\delta(k)) = lkl^{-1}k^{-1},$$

$$\omega_x(\delta(l),e)\omega_x(e,\delta(l)) = l\,{}^{\partial(e)}l^{-1}, \quad \omega_x(ef,g) = \omega_x(e,fgf^{-1})\,{}^{\partial(e)}\omega_x(f,g), \quad \omega_x(e,fg) = \omega_x(e,f)({}^{\partial(e)}f) \triangleright \omega_x(e,g),$$

for all $x \in H_0$, $e, f, g \in E(x)$, and $l, k \in L(x)$, where we set

$$e \triangleright l = l\omega_x(\delta(l)^{-1}, e).$$

We call $(\partial\colon E \to H)$ the *underlying pre-crossed module* of σ. Notice that $\triangleright$ defines an action of E on L, and that together with this action, the morphism $\delta\colon L \to E$ defines a crossed module. A *2-crossed module of groups* is a 2-crossed module with set of objects a singleton.

I.1.9. Morphisms of 2-crossed modules. Let E and F be totally disconnected groupoids with objects $E_0 = F_0 = X$. The *fibered product* of E and F is the totally disconnected groupoid $E \times_X F$ with objects X and with morphisms

$$(E \times_X F)(x) = E(x) \times F(x),$$

for all $x \in X$.

A *morphism* of 2-crossed modules $\phi\colon \sigma \to \sigma'$, where

$$\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega) \quad \text{ and } \quad \sigma' = (L' \xrightarrow{\delta'} E' \xrightarrow{\partial'} H',\ \omega'),$$

is a triple $\phi = (\phi_1\colon H \to H', \phi_2\colon E \to E', \phi_3\colon L \to L')$ of groupoid morphisms making the diagram

$$\begin{array}{ccccccc}
E \times_{H_0} E & \xrightarrow{\omega} & L & \xrightarrow{\delta} & E & \xrightarrow{\partial} & H \\
\downarrow{\scriptstyle (\phi_2,\phi_2)} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} \\
E' \times_{H'_0} E' & \xrightarrow{\omega'} & L' & \xrightarrow{\delta'} & E' & \xrightarrow{\partial'} & H'
\end{array}$$

commutative and such that the morphisms ϕ_2 and ϕ_3 are ϕ_1-equivariant.

I.1.10. The underlying pre-crossed module of a 2-crossed module. In this section, we recall a construction from [CP16] and adapt it to the context of groupoids. Any 2-crossed module admits an underlying crossed module, see Section I.1.8. This induces a *truncation functor* **2XMod** $\to$ **PXMod**.

The *Peiffer square* on a pre-crossed module $(\partial\colon E \to H)$ is the totally disconnected groupoid

$$\{E, E\} = (\{E, E\}(x))_{x \in H_0},$$

where $\{E,E\}(x)$ denotes the free group on the set $\{(e,f) \mid e,f \in E\}$ quotiented by the normal subgroup generated by the elements

$$\{e,f\}\{g,h\}\{e,f\}^{-1}\{g,h\}^{-1}\{\langle e,f\rangle,\langle g,h\rangle\}^{-1} \qquad \{e,f\}\{{}^{\partial g}e,{}^{\partial g}f\}^{-1}\{g,\langle e,f\rangle\}^{-1}\{\langle e,f\rangle,g\}^{-1}$$
$$\{e,fgf^{-1}\}\{{}^{\partial e}f,{}^{\partial e}g\}\{ef,g\}^{-1} \qquad \{e,f\}\{e,g\}\{\langle e,g\rangle^{-1},{}^{\partial e}f\}\{e,fg\}^{-1},$$

for all $e,f,g,h \in E$, where $\langle e,f\rangle$ is shorthand for $efe^{-1}\,{}^{\partial e}f^{-1}$. From any pre-crossed module $(\partial\colon E \to H)$, we derive a 2-crossed module

$$(\{E,E\} \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega)$$

where $\omega(e,f) = \{e,f\}$ and $\delta(\{e,f\}) = \langle e,f\rangle$ for all $e,f \in E(x)$ and $x \in H_0$, and the action of H on $\{E,E\}$ is given by

$${}^{h}\{e,f\} = \{{}^{h}e,{}^{h}f\},$$

for all $e,f \in E(y)$, $h \in H(x,y)$, and $x,y \in H_0$. This defines a functor **PXMod** $\to$ **2XMod** that is left adjoint to the truncation functor.

I.1.11. Special cases. 1) If a 2-crossed module is of the form

$$\sigma = (\overline{H_0} \xrightarrow{1} E \xrightarrow{\partial} H,\ \omega),$$

where $\overline{H_0}$ denotes the discrete groupoid with set of objects H_0, then the truncation $(\partial\colon E \to H)$ is a crossed module. Conversely, any crossed module induces a 2-crossed module of the above form.

2) A *braided crossed module* is a crossed module of groups $(\delta\colon L \to E)$ together with a *braiding*

$$\omega\colon E \times E \to L,$$

such that

$$\delta\omega(e,f) = efe^{-1}f^{-1} \qquad \omega(\delta l,e) = l(e \triangleright l^{-1}) \qquad \omega(e,\delta l) = (e \triangleright l)l^{-1}$$
$$\omega(e,fg) = \omega(e,f)\,(f \triangleright \omega(e,g)) \qquad \omega(ef,g) = (e \triangleright \omega(f,g))\,\omega(e,g),$$

for all $e,f,g \in E$. If a 2-crossed module is of the form

$$\sigma = (L \xrightarrow{\delta} E \xrightarrow{1} 1,\ \omega),$$

where 1 denotes the trivial groupoid, then the morphism $\delta\colon L \to E$ together with the action $\triangleright$ of E on L and the braiding ω defines a braided crossed module. Conversely, any braided crossed module induces a 2-crossed module of the above form.

3) If a 2-crossed module is of the form

$$\sigma = (L \xrightarrow{1} \overline{H_0} \xrightarrow{1} H,\ 1),$$

where $\overline{H_0}$ denotes the discrete groupoid with set of objects H_0, then L is a module over H, i.e., a totally disconnected abelian groupoid with an action of H. Conversely, any module over H induces a 2-crossed module of the above form.

4) If a 2-crossed module is of the form

$$\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ 1),$$

where 1 denotes the trivial Peiffer lifting, then $L \xrightarrow{\delta} E \xrightarrow{\partial} H$ defines a crossed complex of length 3. Conversely, any crossed complex of length 3 induces a 2-crossed module with trivial Peiffer lifting.

I.1.12. The fourth cohomology of a group. In this section we relate certain 2-crossed modules of groups and the fourth cohomology of a group, see [EM49, Con84]. Let

$$\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega)$$

be a 2-crossed module of groups such that $\mathrm{Ker}(\partial)/\mathrm{Im}(\delta) = 1$. Then,

$$1 \longrightarrow \mathrm{Ker}(\delta) \xrightarrow{i} L \xrightarrow{\delta} E \xrightarrow{\partial} H \xrightarrow{p} \mathrm{Coker}(\partial) \longrightarrow 1$$

is an exact sequence of groups. This determines a cohomology class in $\mathrm{H}^4(\mathrm{Coker}(\partial); \mathrm{Ker}(\delta))$ as follows. Since p is surjective, there is a set map $u\colon \mathrm{Coker}(\partial) \to H$ such that $p \circ u = 1_{\mathrm{Coker}(\partial)}$ and $u(1) = 1$. Notice that

$$u(xy)u(y)^{-1}u(x)^{-1} \in \mathrm{Ker}(p) = \mathrm{Im}(\partial)$$

for all $x, y \in \mathrm{Coker}(\partial)$. Therefore, there exists a set map $v\colon \mathrm{Coker}(\partial)^2 \to E$ such that

$$\partial(v(x,y)) = u(xy)u(y)^{-1}u(x)^{-1}.$$

Notice that

$$v(x,yz)\,{}^{u(x)}v(y,z)v(x,y)^{-1}v(xy,z)^{-1} \in \mathrm{Ker}(\partial) = \mathrm{Im}(\delta)$$

for all $x, y, z \in \mathrm{Coker}(\partial)$. Therefore, there exists a set map $w\colon \mathrm{Coker}(\partial)^3 \to L$ such that

$$\delta(w(x,y,z)) = v(x,yz)\,{}^{u(x)}v(y,z)v(x,y)^{-1}v(xy,z)^{-1}.$$

We define a 4-cocyle $q\colon \mathrm{Coker}(\partial)^4 \to \mathrm{Ker}(\delta)$ by

$$\begin{aligned} q(x,y,z,t) = w(x,y,zt)\left(v(xy,zt) \triangleright \omega(v(x,y), {}^{u(x)u(y)}v(z,t))\right) w(xy,z,t) \\ \left(v(xyz,t) \triangleright w(x,y,z)^{-1}\right) w(x,yz,t)^{-1} \left(v(x,yzt) \triangleright {}^{u(x)}w(y,z,t)^{-1}\right), \end{aligned}$$

for all $x, y, z, t \in \mathrm{Coker}(\partial)$. This induces a class in $\mathrm{H}^4(\mathrm{Coker}(\partial); \mathrm{Ker}(\delta))$ that is independent of the choices made for the construction of u, v, and w. Therefore, the 2-crossed module σ defines a tuple

$$(\mathrm{Coker}(\partial), \mathrm{Ker}(\delta), \rho, [q]),$$

where $\rho\colon \mathrm{Coker}(\partial) \to \mathrm{Aut}(\mathrm{Ker}(\delta))$ is induced by the action of H on L, and $[q] \in \mathrm{H}^4(\mathrm{Coker}(\partial); \mathrm{Ker}(\delta))$ is defined above. In fact, this tuple uniquely determines the 2-crossed module σ up to homotopy, see [Con84].

I.1.13. Crossed squares. A *crossed square* (as defined by [GWL81]) is a commutative square of groupoids

$$\begin{array}{ccc} L & \xrightarrow{\lambda'} & N \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \nu} \\ M & \xrightarrow{\mu} & P \end{array}$$

over the same object set together with a *crossed product* $h\colon M \times_{P_0} N \to L$ and actions of P on the totally disconnected groupoids L, M, and N, satisfying the following conditions, where M and N act on L via P: h is the identity on objects, λ, λ', μ, ν, and $\mu \circ \lambda$ are crossed modules, λ and λ' are P-equivariant, and

$$\begin{aligned} h(mm',n) &= h({}^{m}m', {}^{m}n)h(m,n) & h(m,nn') &= h(m,n)h({}^{n}m, {}^{n}n') & \lambda\,(h(m,n)) &= m\,{}^{n}m^{-1} \\ \lambda'\,(h(m,n)) &= {}^{m}nn^{-1} & h(\lambda(l),n) &= l\,{}^{n}l^{-1} & h(m,\lambda'(l)) &= {}^{m}ll^{-1} \\ & & h({}^{p}m, {}^{p}n) &= {}^{p}h(m,n) & & \end{aligned}$$

for all $l \in L(x)$, $m, m' \in M(x)$, $n, n' \in N(x)$, $p \in P(y,x)$, and $x, y \in P_0$. A *morphism* of crossed squares

$$\phi\colon \begin{pmatrix} L & \longrightarrow & N \\ \downarrow & & \downarrow \\ M & \longrightarrow & P \end{pmatrix} \to \begin{pmatrix} L' & \longrightarrow & N' \\ \downarrow & & \downarrow \\ M' & \longrightarrow & P' \end{pmatrix}$$

consists of four morphisms $\phi_L\colon L \to L'$, $\phi_M\colon M \to M'$, $\phi_N\colon N \to N'$, and $\phi_P\colon P \to P'$ making the resulting cube commutative, such that ϕ_L ϕ_M and ϕ_N are ϕ_P-equivariant, and $\phi_L(h(m,n)) = h'(\phi_M(m), \phi_N(n))$ for all $m \in M(x)$, $n \in N(x)$, and $x \in P_0$. We denote by **XSq** the category of crossed squares.

There is a functor $\mathrm{F}\colon \mathbf{XSq} \to \mathbf{2XMod}$ that sends a crossed square

$$\begin{array}{ccc} L & \xrightarrow{\lambda'} & N \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \nu} \\ M & \xrightarrow{\mu} & P \end{array}$$

with crossed product $h\colon M \times_{P_0} N \to L$ to the 2-crossed module

$$L \xrightarrow{(\lambda,\lambda'^{-1})} M \rtimes N \xrightarrow{\mu\nu} P,$$

where $\mu\nu(m,n) = \mu(m)\nu(n)$, and with Peiffer lifting ω defined by

$$\omega((m,n),(m',n')) = h(m, nn'n^{-1})$$

for all $m, m' \in M(x)$, $n, n' \in N(x)$, and $x \in P_0$.

Similarly, there is a functor $\mathrm{G}\colon \mathbf{2XMod} \to \mathbf{XSq}$ defined as follows. Let

$$\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega)$$

be a 2-crossed module. We identify E with the subgroupoid $E \times \{1\} \subset E \rtimes H$ and we denote by $\overline{E}$ the subgroupoid of $E \rtimes H$ containing the elements of the form $(e, \partial(e)^{-1})$ for all $e \in E$. Then

$$\mathrm{G}(\sigma) = \begin{array}{ccc} L & \xrightarrow{(\delta,1)} & \overline{E} \\ \downarrow{\scriptstyle \delta} & & \downarrow \\ E & \hookrightarrow & E \rtimes H \end{array}$$

is a crossed square with crossed product $h\colon E \times_{H_0} \overline{E} \to L$ defined by

$$h(e,(f,\partial(f)^{-1})) = \omega(f, f^{-1}ef)$$

for all $e, f \in E(x)$, $x \in H_0$.

I.1.14. Topological example. The main example of a 2-crossed module arises from a triad (X, A, B) of topological spaces and $X_0 \subset A \cap B$ a non-empty set. The *fundamental 2-crossed module* of $(X, A, B; X_0)$ is the 2-crossed module

$$\Pi_3(X, A, B; X_0) = (\pi_3(X, A, B; X_0) \to \pi_2(A, A \cap B; X_0) \to \pi_1(B; X_0),\ \omega),$$

where $\pi_3(X, A, B; X_0) = (\pi_3(X, A, B; x))_{x \in X_0}$, and $\pi_3(X, A, B; x)$ is given by the homotopy classes of maps from a pointed 3-ball that send the southern hemisphere to A, the northern hemisphere to B, and the basepoint to x, the map

$$\pi_3(X, A, B; X_0) \to \pi_2(A, A \cap B; X_0)$$

is induced by the restriction to the southern hemisphere, and the map $\pi_2(A, A \cap B; X_0) \to \pi_1(B; X_0)$ is induced by the inclusion $A \cap B \subset B$. The Peiffer lifting ω is defined via the generalized Whitehead product, see [FM11, Ell93]. The details of this construction for certain triads of simplicial complexes can be found in Section I.2.4.

I.1.15. 2-crossed complexes of groupoids. A *2-crossed complex* is a complex of groupoids

$$\cdots \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} \cdots \xrightarrow{d_5} A_4 \xrightarrow{d_4} L \xrightarrow{\delta} E \xrightarrow{\partial} H,$$

together with actions of H on E, L, and A_n for all $n \geqslant 4$, and a *Peiffer lifting*:

$$\omega = (\omega_x : E(x) \times E(x) \to L(x))_{x \in H_0},$$

verifying the following conditions:

- $(L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega)$ is a 2-crossed module,
- A_n is a totally disconnected abelian groupoid with set of objects H_0 for all $n \geqslant 4$,
- d_n is H-equivariant for all $n \geqslant 4$,
- $d_n \circ d_{n+1} = 0$ for all $n \geqslant 4$, and $\delta \circ d_4 = 0$,
- if $e \in E(x)$, then $\partial(e)$ acts trivially on $A_n(x)$ for all $n \geqslant 4$ and $x \in H_0$.

A *morphism* of 2-crossed complexes

$$f : (\cdots \xrightarrow{d_5} A_4 \xrightarrow{d_4} L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega) \to (\cdots \xrightarrow{d'_5} A'_4 \xrightarrow{d_4} L' \xrightarrow{\delta} E' \xrightarrow{\partial} H',\ \omega')$$

is a family of morphisms

$$f_1 : H \to H', \qquad f_2 : E \to E', \qquad f_3 : L \to L', \qquad f_n : A_n \to A'_n \text{ for all } n \geqslant 4,$$

such that (f_3, f_2, f_1) is a morphism of 2-crossed modules, f_n is f_1-equivariant for all $n \geqslant 4$, and the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "d_6"] & A_5 \arrow[r, "d_5"] \arrow[d, "f_5"] & A_4 \arrow[r, "d_4"] \arrow[d, "f_4"] & L \arrow[r, "\delta"] \arrow[d, "f_3"] & E \arrow[r, "\partial"] \arrow[d, "f_2"] & H \arrow[d, "f_1"] \\
\cdots \arrow[r, "d'_6"] & A'_5 \arrow[r, "d'_5"] & A'_4 \arrow[r, "d_4"] & L' \arrow[r, "\delta"] & E' \arrow[r, "\partial"] & H'
\end{tikzcd}$$

is commutative.

I.1.16. Free groupoids. A *graph* (or *quiver*) is a 4-tuple $D = (V, E, s, t)$, with V a set of *vertices*, E a set of *edges*, $s\colon E \to V$ a *source* map, and $t\colon E \to V$ a *target* map. A *morphism* from a graph D to a graph D' is a pair of maps $\varphi = (\varphi_V, \varphi_E)$ where φ_V goes from the vertices of D to the vertices of D', φ_E goes from the edges of D to the edges of D', and such that φ_V and φ_E are compatible with the source and target maps.

Given a graph D, the *free groupoid* on D is the groupoid $\mathrm{F}(D)$ with objects the vertices of D, and with morphisms constructed recursively from the identity morphisms, the edges of D, and the formal inverses of the edges of D. This construction induces a functor F from the category of graphs to the category of groupoids. Conversely, there is a forgetful functor from the category of groupoids to the category of graphs that associates to any groupoid its underlying graph. The functor F is left adjoint to the forgetful functor.

I.1.17. Free pre-crossed modules. Throughout this section, we fix a groupoid H. An H-set is a pair (S, b), where S is a family of sets indexed by the objects of H

$$S = (S(x))_{x \in H_0},$$

and b is a family of functions

$$b = \{b_x\colon S(x) \to H(x)\}_{x \in H_0}.$$

If (S, b) is an H-set and $\sigma \in S$, then we denote by $v(\sigma)$ the object such that $\sigma \in S(v(\sigma))$.

Given an H-set (S, b) and an object $x \in H_0$, we denote by $F(S, b)(x)$ the free group generated by the symbols (h, σ), where $\sigma \in S$ and $h \in H(x, v(\sigma))$. These groups assemble to form a totally disconnected groupoid $F(S, b)$ with set of objects H_0. The groupoid H acts on $F(S, b)$ via the formula

$${}^h(k, \sigma) = (hk, \sigma)$$

for all $\sigma \in S$, $k \in H(y, v(\sigma))$ and $h \in H(x, y)$. The *free pre-crossed module* on an H-set (S, b) is the pre-crossed module $(\partial\colon F(S, b) \to H)$, where ∂ is defined by

$$\partial(h, \sigma) = hb(\sigma)h^{-1}$$

for any generator (h, σ).

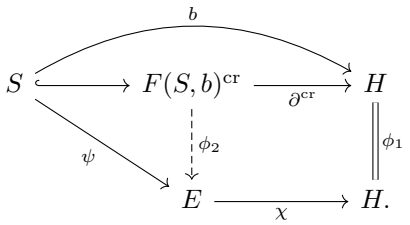
We fix (S, b) an H-set. The free pre-crossed module on (S, b) satisfies the following universal property. Let $(d\colon G \to H)$ be a pre-crossed module, and let $\psi = \{\psi_x\colon S(x) \to G(x)\}_{x \in H_0}$ be a family of maps such that $d \circ \psi = b$. Then there exists a unique morphism of pre-crossed modules $\phi = (\phi_1, \phi_2)$ from the free pre-crossed module on (S, b) to $(d\colon G \to H)$ such that $\phi_1\colon H \to H$ is the identity and $\phi_2(1, \sigma) = \psi(\sigma)$ for all $\sigma \in S$. We obtain the following commutative diagram

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[rr, bend left, "b"] \arrow[dr, "\psi"'] & F(S,b) \arrow[r, "\partial"'] \arrow[d, dashed, "\phi_2"] & H \arrow[d, equal, "\phi_1"] \\
& G \arrow[r, "d"'] & H.
\end{tikzcd}$$

We say that a free pre-crossed module $(\partial\colon F(S, b) \to H)$ is *totally free* if H is a free groupoid.

I.1.18. Free crossed modules. Throughout this section, we fix a groupoid H. Recall from Section I.1.4 the definition of the associated crossed module to a pre-crossed module. The *free crossed module* on an H-set (S, b) is the associated crossed module $(\partial^{\mathrm{cr}}\colon F(S,b)^{\mathrm{cr}} \to H)$ to the free pre-crossed module on (S, b).

We fix (S, b) an H-set. The free crossed module on (S, b) satisfies the following universal property. Let $(\chi\colon E \to H)$ be a crossed module, and let $\psi = \{\psi_x\colon S(x) \to E(x)\}_{x\in H_0}$ be a family of maps such that $\chi \circ \psi = b$. Then there exists a unique morphism of crossed modules $\phi = (\phi_1, \phi_2)$ from the free crossed module on (S, b) to $(\chi\colon E \to H)$ such that $\phi_1\colon H \to H$ is the identity and $\phi_2(1, \sigma) = \psi(\sigma)$ for all $\sigma \in S$. We obtain the following commutative diagram



We say that a free crossed module is *totally free* if H is a free groupoid.

I.1.19. Free 2-crossed modules. Throughout this section, we fix a pre-crossed module $(\partial\colon E \to H)$. We denote by $\mathrm{Ker}(\partial) = (\mathrm{Ker}(\partial)(x))_{x\in H_0}$ the totally disconnected groupoid where $\mathrm{Ker}(\partial)(x)$ is the kernel of $\partial_x\colon E(x) \to H(x)$. We now describe the construction of the free 2-crossed module on a $\mathrm{Ker}(\partial)$-set (S, b). Let $(\nu\colon C \to E \rtimes H)$ be the free crossed module on the $(E \rtimes H)$-set $(S, \hat{b})$, where $\hat{b}\colon S \to E \rtimes H$ is the map defined by

$$\hat{b}(s) = (b(s), 1)$$

for all $s \in S$. Notice that the image of ν is included in E. Recall from Section I.1.10 that the pre-crossed module $(\partial\colon E \to H)$ induces a 2-crossed module

$$(\{E, E\} \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega).$$

Let $(\hat{\delta}\colon \{E, E\} \to E \rtimes H)$ be the crossed module with

$$\hat{\delta}(l) = (\delta(l), 1)$$

for all $l \in \{E, E\}$, and where $E \rtimes H$ acts on $\{E, E\}$ via

$${}^{(e,h)}l = e \triangleright {}^{h}l.$$

for all $(e, h) \in (E \rtimes H)(x, y)$ and $l \in \{E, E\}(y)$. Let $\mathcal{F}(S, b)$ be the coproduct $\{E, E\} \circ C$ quotiented by the relations:

$$\iota\{\nu c, e\}\iota\{e, \nu c\} = \jmath(c)\jmath({}^{\partial e}c^{-1}), \qquad \iota\{\nu c, \nu c'\} = \jmath(cc'c^{-1}c'^{-1}), \qquad \jmath(c)\iota\{\nu c^{-1}, e\} = \jmath(e \triangleright c),$$

for all $e \in E(x)$, $c, c' \in C(x)$, and $x \in H_0$, where $\iota\colon \{E, E\} \to \{E, E\} \circ C$ and $\jmath\colon C \to \{E, E\} \circ C$ are the canonical inclusions. The *free 2-crossed module* on a $\mathrm{Ker}(\partial)$-set (S, b) is the 2-crossed module

$$(\mathcal{F}(S, b) \xrightarrow{d} E \xrightarrow{\partial} H,\ \hat{\omega})$$

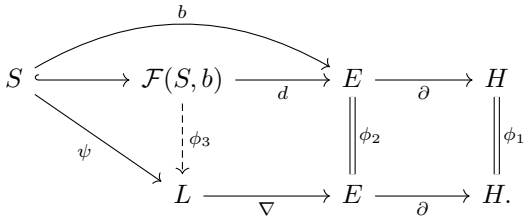
where d is induced by δ and ν, and $\hat{\omega}$ is induced by $\iota \circ \omega$.

Let (S, b) be a $\mathrm{Ker}(\partial)$-set. The free 2-crossed module on (S, b) satisfies the following universal property. Let

$$\sigma = (L \xrightarrow{\nabla} E \xrightarrow{\partial} H,\ w)$$

be a 2-crossed module with underlying pre-crossed module $(\partial\colon E \to H)$, and let $\psi = \{\psi_x\colon S(x) \to L(x)\}_{x\in H_0}$ be a family of maps such that $\nabla \circ \psi = b$. Then there exists a unique morphism of 2-crossed modules $\phi = (\phi_1, \phi_2, \phi_3)$ from the free 2-crossed module on (S, b) to σ such that $\phi_1\colon H \to H$ and $\phi_2\colon E \to E$ are the identity morphisms,

and $\phi_3([s]) = \psi(s)$ for all $s \in S$. We obtain the following commutative diagram



We say that a free 2-crossed module is *totally free* if its underlying pre-crossed module is totally free.

I.1.20. The model category of 2-crossed modules of groups. The *homotopy groups* of a 2-crossed module of groups $\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega)$ are the groups

$$\pi_1(\sigma) = \mathrm{Coker}(\partial), \qquad \pi_2(\sigma) = \mathrm{Ker}(\partial)/\mathrm{Im}(\delta), \qquad \pi_3(\sigma) = \mathrm{Ker}(\delta).$$

Cabello and Garzón [CG94] show that the category of 2-crossed modules of groups is a Quillen model category with *weak equivalences* given by the morphisms that induce isomorphisms at the level of homotopy groups. A *fibration* is a morphism of 2-crossed modules

$$\begin{array}{ccccc} L & \xrightarrow{\delta} & E & \xrightarrow{\partial} & H \\ \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} \\ L' & \xrightarrow{\delta'} & E' & \xrightarrow{\partial'} & H' \end{array}$$

such that ϕ_2 and ϕ_3 are surjective. A *cofibrant object* is a 2-crossed module of groups whose underlying pre-crossed module is totally free. The homotopy category of 2-crossed modules of groups is equivalent to the category **3-types**.

I.2. The fundamental 2-crossed module of a simplicial complex

While the fundamental groupoid (or group, in the connected case) completely characterizes homotopy 1-types, and the fundamental crossed module characterizes homotopy 2-types, the representation of homotopy 3-types requires a significant jump in categorical complexity. We therefore encounter the necessity of 3-groupoids, which we model via their semistrictification as 2-crossed modules.

Our primary objective in this section is to construct the fundamental 2-crossed module of a simplicial complex, elaborating on the work of Ellis [Ell93] and Faria Martins [FM11] in the case of reduced CW complexes. The intuition is geometrically straightforward, as the structure should be generated by the low-dimensional simplices of K:

- the base groupoid is freely generated by the edges of K,
- the middle groupoid is freely generated by the triangles of K,
- the top groupoid is generated by the tetrahedra of K, subject to the relations imposed by the 4-simplices of K.

Nevertheless, in order to define the boundary morphisms and the Peiffer lifting correctly, we need to endow the triangles of K with distinguished vertices to serve as basepoints. This leads us to the notion of *canonical simplicial complex* defined in Section I.2.2.

We then see that a σ-coloring of a canonical simplicial complex K, i.e., a morphism from the fundamental 2-crossed module of K to a 2-crossed module of groups σ, is uniquely determined by its values on the edges, triangles and tetrahedra of K. This yields a neat combinatorial description of the σ-colorings of K.

I.2.1. Simplicial complexes. In this section, we recall classical notions relating to simplicial complexes. An *n-simplex* (or simply a *simplex*), for some nonnegative integer n, is the convex hull $(v_0, \ldots, v_n)$ of $n+1$ affinely independent points $v_0, \ldots v_n$ in $\mathbb{R}^N$, for some $N \geqslant n$. The *standard n-simplex* is the convex hull Δ^n of the standard basis vectors of $\mathbb{R}^{n+1}$. An *m-face* (or simply a *face*) of an n-simplex s, for some nonnegative integer $m \leqslant n$, is the convex hull of a subset of size $m+1$ of the $n+1$ points that define s. In particular, an m-face of a simplex is an m-simplex. A *vertex* is a 0-simplex, an *edge* is a 1-simplex, a *triangle* is a 2-simplex, and a *tetrahedron* is a 3-simplex.

A *geometric simplicial complex* (or simply a *simplicial complex*) is a set K of simplices in $\mathbb{R}^N$, for some nonnegative integer N, such that every face of a simplex in K is also a simplex in K, the intersection of any two simplices in K is either empty or a face of both of these simplices, and K is locally finite, i.e., for every point x in a simplex in K, there is a neighborhood of x in $\mathbb{R}^N$ that intersects a finite number of simplices in K.

Let K be a simplicial complex in $\mathbb{R}^N$. We denote by $K^{(i)}$ the set of all i-simplices in K. A *simplicial subcomplex* of K is a simplicial complex L such that every simplex in L is a simplex in K. The *underlying space* of K is the union $|K|$ in $\mathbb{R}^N$ of all of the simplices in K

$$|K| = \bigcup_{s\in K} s,$$

with the topology induced by the topology of $\mathbb{R}^N$. Notice that a simplicial complex is finite if and only if its underlying space is compact. The *n-skeleton* of a simplicial complex K is the union K_n in $\mathbb{R}^N$ of all of the i-simplices in K for $0 \leqslant i \leqslant n$:

$$K_n = \bigcup_{i\leqslant n} \bigcup_{s\in K^{(i)}} s.$$

This induces a filtration of topological spaces $K_0 \subset K_1 \subset K_2 \subset \cdots \subset |K|$.

A *subdivision* of a simplicial complex K is a simplicial complex K' such that every simplex in K' is a subset of a simplex in K, and every simplex in K is a finite union of simplices in K'. Notice that a simplicial complex and any of its subdivisions have the same underlying space. A *piecewise linear map* from a simplicial complex K to a simplicial complex L is a continuous map $f\colon |K| \to |L|$ such that there exists a subdivision K' of K for which f is affine when restricted to each simplex in K'.

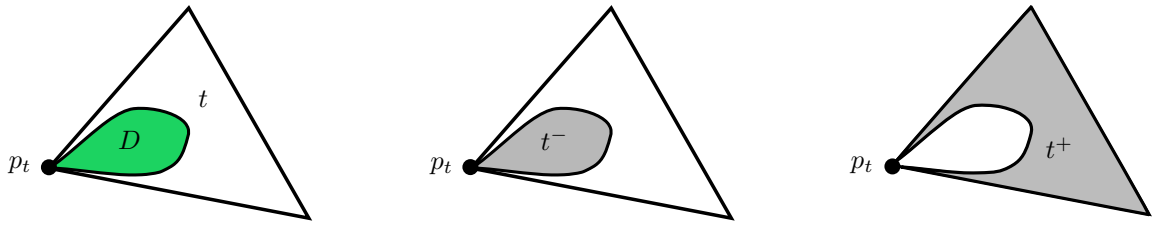
Let S be a subset of a simplicial complex K. The *star* of S is the set $\mathrm{St}(S)$ of simplices in K that have a face in S. The *closure* of S is the smallest simplicial subcomplex $\mathrm{Cl}(S)$ of K containing S. The *link* of S is

$$\mathrm{Lk}(S) = \mathrm{Cl}(\mathrm{St}(S)) \setminus \mathrm{St}(S).$$

I.2.2. Canonical simplicial complexes. A *canonical simplicial complex* is a simplicial complex K together with the choice, for every triangle t of K, of a distinguished vertex p_t and of a decomposition of t into subspaces t^+ and t^- as follows. Let D be a closed disk embedded in t such that $D \cap \partial t = \{p_t\}$. We set

$$t^- = \partial t \cup D \qquad t^+ = t \setminus \mathring{D},$$

where $\mathring{D}$ denotes the interior of D.



Let K be a canonical simplicial complex. Denote by K_2^- and K_2^+ the subspaces of K_2 defined by

$$K_2^- = K_1 \cup \left(\bigcup_{t\in K^{(2)}} t^- \right), \qquad K_2^+ = K_1 \cup \left(\bigcup_{t\in K^{(2)}} t^+ \right).$$

Notice that $K_2 = K_2^- \cup K_2^+$, and that the inclusions $K_1 \hookrightarrow K_2^-$ and $K_1 \hookrightarrow K_2^+$ induce homotopy equivalences.

I.2.3. Homotopy groupoids. The *fundamental groupoid* of a topological space X is the groupoid $\pi_1(X)$ with set of objects X and with morphisms from x to y given by the homotopy classes relative to the boundary of paths from x to y, where the composition is induced by the concatenation of paths. The *fundamental groupoid of X with chosen basepoints in A*, denoted by $\pi_1(X; A)$, for any subspace $A \subset X$, is the full subcategory of $\pi_1(X)$ with set of objects A.

The *n-th relative homotopy groupoid* of a triple of topological spaces $X_0 \subset A \subset X$ for $n \geqslant 2$ is the totally disconnected groupoid $\pi_n(X, A; X_0)$ with set of objects X_0 and with sets of morphisms:

$$\pi_n(X, A; X_0)(x) = \pi_n(X, A, x)$$

for all $x \in A$, where $\pi_n(X, A, x)$ denotes the n-th relative homotopy group of the pair $Y \subset A$ with basepoint $x \in X_0$. Notice that there is an action of the fundamental groupoid $\pi_1(A; X_0)$ on the n-th relative homotopy groupoid $\pi_n(X, A; X_0)$ for all $n \geqslant 2$.

The *n-th triad homotopy groupoid* of a quadruple of spaces $A \subset X$, $B \subset X$, $X_0 \subset A \cap B$ for $n \geqslant 3$ is the totally disconnected groupoid $\pi_n(X, A, B; X_0)$ with set of objects X_0 and sets of morphisms:

$$\pi_n(X, A, B; X_0)(x) = \pi_n(X, A, B, x)$$

for all $x \in X_0$, where $\pi_n(X, A, B, x)$ denotes the n-th triad homotopy group, see [BM49]. Notice that there is an action of the fundamental groupoid $\pi_1(A \cap B; X_0)$ on the n-th triad homotopy groupoid $\pi_n(X, A, B; X_0)$ for all $n \geqslant 3$.

I.2.4. The fundamental 2-crossed module of a canonical simplicial complex. In this section, we adapt the definition of Faria Martins [FM11] of the fundamental 2-crossed module of groups of a reduced CW complex to the context of simplicial complexes and groupoids. The *fundamental 2-crossed module* of a canonical simplicial complex K is the 2-crossed module:

$$\Pi_3(K) = (\pi_3(|K|, K_2^-, K_2^+; K_0) \xrightarrow{d_2} \pi_2(K_2^-, K_2^- \cap K_2^+; K_0) \xrightarrow{d_1} \pi_1(K_1; K_0),\ \omega)$$

where the maps d_1, d_2, and ω are defined as follows. The map $d_1 \colon \pi_2(K_2^-, K_2^- \cap K_2^+; K_0) \to \pi_1(K_1; K_0)$ is the composition of the connecting homomorphism

$$\pi_2(K_2^-, K_2^- \cap K_2^+; K_0) \to \pi_1(K_2^- \cap K_2^+; K_0)$$

from the long exact sequence of the pair $(K_2^-, K_2^- \cap K_2^+)$ with the morphism $\pi_1(K_2^- \cap K_2^+; K_0) \to \pi_1(K_1; K_0)$ induced by the inclusion $K_2^- \cap K_2^+ \subset K_2^+$ and the isomorphism $\pi_1(K_2^+; K_0) \cong \pi_1(K_1; K_0)$ induced by the inclusion $K_1 \subset K_2^+$. The map $d_2 \colon \pi_3(|K|, K_2^-, K_2^+; x) \to \pi_2(K_2^-, K_2^- \cap K_2^+; x)$ is the connecting homomorphism from the long exact sequence of the triad $(|K|, K_2^-, K_2^+)$ induced by the restriction to the southern hemisphere. The map

$$\omega = \big(\omega_x \colon \pi_2(K_2^-, K_2^- \cap K_2^+; x) \times \pi_2(K_2^-, K_2^- \cap K_2^+; x) \to \pi_3(|K|, K_2^-, K_2^+; x)\big)$$

is defined via the generalized Whitehead product for the triad $(|K|, K_2^-, K_2^+)$, see details in [FM11, Definition 3.1].

Theorem 2. *If X is a canonical simplicial complex, then the pre-crossed module*

$$(d_1 \colon \pi_2(K_2^-, K_2^- \cap K_2^+; K_0) \to \pi_1(K_1; K_0))$$

is totally free over the $\pi_1(K_1; K_0)$-set given by the triangles of K with their distinguished vertices. Moreover, if K is 3-dimensional, then its fundamental 2-crossed module is totally free over the $(\mathrm{Ker}(d_1))$-set given by the tetrahedra of K.

Proof. The groupoid $\pi_1(K_1; K_0)$ is clearly free on the 1-skeleton of K. We deduce from the discussion in [Ell93, Theorem 2] and from [BL87] that the fundamental 2-crossed module of a 3-dimensional canonical simplicial complex is totally free in the sense of Section I.1.19. □

We deduce from the freeness of the fundamental 2-crossed module that if K and K' are canonical simplicial complexes with the same underlying simplicial complex, then their fundamental 2-crossed modules are isomorphic.

I.2.5. Colorings. Let σ be a 2-crossed module of groups. A *σ-coloring* of a canonical simplicial complex K is a morphism of 2-crossed modules $\Pi_3(K) \to \sigma$. A *σ-colored simplicial complex* is a pair (K, ϕ), where K is a canonical simplicial complex and ϕ is a σ-coloring of K.

I.2.6. Ordered simplices. Let K be a canonical simplicial complex. An *order* of an i-simplex of K, for $i \neq 2$, is a choice of linear order on its vertices. An *order* of a triangle of K is a choice of cyclic order on its vertices. An i-simplex of K is *ordered* if it is endowed with an order.

A *pointing* of an ordered i-simplex s of K, for $i \geqslant 3$, is the choice, for every triangle t of s, of a homotopy class of paths in the boundary of t from the vertex of t that is minimal for the order of s to the distinguished vertex of t. A *pointed ordered* i-simplex, for $i \geqslant 3$, is an ordered i-simplex together with a choice of pointing. The *pointed vertex* of a pointed i-simplex, for $i \geqslant 3$, is the vertex that is minimal for the order.

I.2.7. Choices of ordered simplices of canonical simplicial complexes. A *choice of ordered simplices* of a canonical simplicial complex K is a sequence $B_* = (B_n)_{n\geqslant 1}$ such that

- for $i = 1, 2$, B_i is a subset of the ordered i-simplices of K containing exactly one choice of order for each i-simplex,
- for $i \geqslant 3$, B_i is a subset of the pointed ordered i-simplices of K containing exactly one choice of pointing and order for each i-simplex.

Let $B_* = (B_n)_{n\geqslant 1}$ be a choice of ordered simplices of a canonical simplicial complex K. There is a canonical injective map $B_1 \hookrightarrow \pi_1(K_1; K_0)$ that sends an ordered edge to its homotopy class. Therefore, we will often identify B_1 with its image in $\pi_1(K_1; K_0)$. There is an injective map $B_2 \hookrightarrow \pi_2(K_2^-, K_2^- \cap K_2^+; K_0)$ that sends an ordered triangle with distinguished vertex $p \in K_0$ to its homotopy class in $\pi_2(K_2^-, K_2^- \cap K_2^+; p)$, and we use this to identify B_2 with its image in $\pi_2(K_2^-, K_2^- \cap K_2^+; K_0)$. There is an injective map $B_3 \hookrightarrow \pi_3(K_3, K_2^-, K_2^+; K_0)$ that sends a pointed ordered tetrahedron with pointed vertex p to its homotopy class in $\pi_3(K_3, K_2^-, K_2^+; p)$, and we use this to identify B_3 with its image in $\pi_3(K_3, K_2^-, K_2^+; K_0)$. Similarly, we identify B_4 with its image in $\pi_4(K_4, K_3; K_0)$.

By Theorem 2, a σ-coloring of a canonical simplicial complex is uniquely determined by its values on a choice of ordered simplices. Conversely, a triple of maps $c = (c_1 \colon B_1 \to H, c_2 \colon B_2 \to E, c_3 \colon B_3 \to L)$ induces a σ-coloring of K if it verifies the *triangle identity*:

$$\partial(c_2(t)) = \widehat{c_1}(d_1(t))$$

for all $t \in B_2$, where $\widehat{c_1} \colon \pi_1(K_1; K_0) \to H$ is the unique morphism of groupoids that extends c_1, the *tetrahedron identity*:

$$\delta(c_3(\tau)) = \widehat{c_2}(d_2(\tau))$$

for all $\tau \in B_3$, where $\widehat{c_2} \colon \pi_2(K_2^-, K_2^- \cap K_2^+; K_0) \to E$ is the unique $\widehat{c_1}$-equivariant morphism of groupoids that extends c_2, and the *4-simplex identity*:

$$1 = \widehat{c_3}(d_3(s))$$

for all $s \in B_4$, where $\widehat{c_3} \colon \pi_3(K_3, K_2^-, K_2^+; K_0) \to L$ denotes the unique morphism of groupoids that coincides with c_3 such that $(\widehat{c_1}, \widehat{c_2}, \widehat{c_3})$ is a morphism of 2-crossed modules.

I.3. The classifying space of a 2-crossed module

This section deals with the construction of the classifying space $B\sigma$ of a 2-crossed module of groups σ. Conceptually, $B\sigma$ functions as the 3-dimensional analogue of the Eilenberg-MacLane space $K(G, 1)$ associated to a group G, or the classifying space $B\chi$ of a crossed module χ, see [BH91]. We adopt a fairly "hands-on" approach, where the classifying space of a 2-crossed module is defined directly using its generators. If

$$\sigma = (L \xrightarrow{\ \delta\ } E \xrightarrow{\ \partial\ } H,\ \omega)$$

is 2-crossed module of groups, then its classifying space $B\sigma$ is a connected CW complex verifying:

$$\pi_1(B\sigma) \cong \mathrm{Coker}(\partial), \quad \pi_2(B\sigma) \cong \mathrm{Ker}(\partial)/\mathrm{Im}(\delta), \quad \pi_3(B\sigma) \cong \mathrm{Ker}(\delta), \quad \pi_n(B\sigma) = \{0\} \text{ for all } n \geqslant 4.$$

Every connected homotopy 1-type is equivalent to the classifying space of a group, and every connected homotopy 2-type is equivalent to the classifying space of a crossed module. Similarly, we show that every connected homotopy 3-type is equivalent to the classifying space of a 2-crossed module of groups.

I.3.1. Canonical CW complexes. We refer the reader to Ellis [Ell93] and Faria Martins [FM11] for a more detailed presentation of canonical CW complexes. We denote

$$D = \{z \in \mathbb{C} \mid |z - 1| \leqslant 1\}, \qquad D^- = \{z \in \mathbb{C} \mid |z - \frac{1}{2}| \leqslant \frac{1}{2}\}, \qquad D^+ = D \setminus \mathrm{int}(D^-).$$

This defines a triad (D, D^-, D^+) with basepoint $z = 0$, as illustrated in Figure 1.

We also define a triad structure on the sphere S^2, taking $(S^2)^+$ to be the northern hemisphere and $(S^2)^-$ to be the southern hemisphere, with a basepoint on the equator, as illustrated in Figure 2.

If X is a CW complex, we denote by X_n its n-skeleton, for $n \geqslant 0$. A CW complex X is *canonical* if it has a unique 0-cell $*$, taken as its basepoint, the attaching map $S^1 \to X_1$ of every 2-cell is pointed, thus defining a triad (X_2, X_2^-, X_2^+) induced by the triad structure on D, and every 3-cell is attached via a pointed triad map $(S^2, (S^2)^-, (S^2)^+) \to (X_2, X_2^-, X_2^+)$.

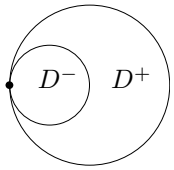


FIGURE 1. The triad structure on the disk

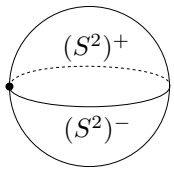


FIGURE 2. The triad structure on the sphere

I.3.2. The classifying space of a 2-crossed module of groups. The *fundamental 2-crossed module* of a canonical CW complex X, as defined by Faria Martins [FM11], is the 2-crossed module of groups

$$\Omega_3(X) = (\pi_3(X, X_2^-, X_2^+, *) \xrightarrow{\delta} \pi_2(X_2^-, X_2^- \cap X_2^+, *) \xrightarrow{\partial} \pi_1(X_1, *),\ \omega)$$

where the boundary maps δ and ∂ and the Peiffer lifting ω are defined analogously to the maps in Section I.2.4. For instance, if S^2 denotes the 2-sphere with the triad structure described above, then we get

$$\Omega_3(S^2) = (\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0,\ \omega),$$

where the Peiffer lifting is defined by $\omega(n, m) = nm$ for all $n, m \in \mathbb{Z}$. The following realization result is a direct consequence of the work of Faria Martins [FM11]:

Lemma I.3.1. *If σ is a 2-crossed module of groups whose underlying pre-crossed module is totally free, then there exists a canonical CW complex X whose fundamental 2-crossed module $\Omega_3(X)$ is isomorphic to σ.*

We recall the construction of the canonical CW complex X such that $\Omega_3(X) \cong \sigma$, as described by Ellis [Ell93]. Set $X_0 = *$. We suppose that $(\partial\colon E \to H)$ is totally free, therefore H is a free group on a set S_1. Set X_1 to be a wedge of circles, with one circle for every element of S_1, thus $\pi_1(X_1, *) = H$. Now suppose the pre-crossed module $(\partial\colon E \to H)$ is free on an H-set (S_2, b_2), see Section I.1.17 . For every element s of S_2, $b_2(s)$ defines a loop in X_1 with basepoint $*$. We construct X_2 from X_1 by gluing disks with attaching maps determined by b_2. Since the attaching maps are pointed, this defines a triad (X_2, X_2^-, X_2^+) induced by the triad structure on D, see Figure 1. Let V be a wedge of circles corresponding to the elements of L. Via the inclusion $\pi_2(X_2^-, X_2^- \cap X_2^+) \hookrightarrow \pi_1(X_2^- \cap X_2^+)$ and the map $\delta\colon L \to \pi_2(X_2^-, X_2^- \cap X_2^+)$, any element $l \in L$ determines a loop in $X_2^- \cap X_2^+$, and thus a we get a map

$$\beta\colon V \to X_2^- \cap X_2^+$$

that sends the circle associated to $l \in L$ to the loop associated to $\delta(l)$. Let C^-V and C^+V be two cones on V. The map β extends to a map

$$\beta^-\colon C^-V \to X_2^-,$$

and since $\partial \circ \delta(l) = 1$, the map β extends to a map

$$\beta^+\colon C^+V \to X_2^+.$$

Therefore, β extends to a map $\beta_S\colon SV \to X_2$, where SV is a wedge of 2-spheres corresponding to the elements of L. The space X_3 is obtained by attaching, for every $l \in L$, a 3-cell along the restriction of β_S to the 2-sphere corresponding to l. We obtain a 3-dimensional canonical CW complex X_3 such that $\Omega_3(X_3)$ is the free 2-crossed module on the $\mathrm{Ker}(\partial)$-set (L, δ), see Section I.1.19. Therefore, the identity map $L \to L$ induces a morphism

of 2-crossed modules

$$\begin{array}{ccccc} \mathcal{F}(L,\delta) & \xrightarrow{d} & E & \xrightarrow{\partial} & H \\ \downarrow{\scriptstyle \pi} & & \| & & \| \\ L & \xrightarrow{\delta} & E & \xrightarrow{\partial} & H. \end{array}$$

Pick S_4 a set of generators for the abelian group $\mathrm{Ker}(\pi)$. We build X by attaching a 4-cell to X_3 for every element $s \in S_4$ via the attaching map determined by $s \in \mathrm{Ker}(\delta \circ \pi) \cong \pi_3(X_3)$. This yields a canonical CW complex X such that $\Omega_3(X) \cong \sigma$. Notice that we may attach k-cells to X, for $k \geqslant 5$, in order to ensure that $\pi_n(X, *) = 0$ for all $n \geqslant 4$. This procedure does not affect the fundamental 2-crossed module of X. We've thus established that a 2-crossed module of groups is isomorphic to the fundamental 2-crossed module of a canonical CW complex strictly when its underlying pre-crossed module is totally free.

For an arbitrary 2-crossed module of groups σ that does not satisfy this freeness condition, we replace σ up to weak equivalence using the cofibrant replacement functor. Following the Quillen model structure on the category of 2-crossed modules of groups defined by Cabello and Garzón [CG94], we associate to a 2-crossed module of groups σ a cofibrant replacement $p\colon Q(\sigma) \xrightarrow{\sim} \sigma$, where p is a trivial fibration. A *classifying space* of a 2-crossed module of groups σ is a canonical CW complex $B\sigma$ such that $\pi_n(B\sigma) = 0$ for all $n > 3$ together with an isomorphism $\Omega_3(B\sigma) \cong Q(\sigma)$. The following is a consequence of the above discussion:

Lemma I.3.2. *Every 2-crossed module of groups admits a classifying space, and every connected CW complex such that $\pi_n(X) = 0$ for all $n \geqslant 4$ is homotopy equivalent to the classifying space of a 2-crossed module.*

I.3.3. Geometric realizations of a coloring. Let K be a canonical simplicial complex and let X be a canonical CW complex. A *bifiltered map* from K to X is a continuous map $f\colon |K| \to X$ such that

$$f(K_2^-) \subset X_2^- \qquad f(K_2^+) \subset X_2^+, \qquad f(K_n) \subset X_n \text{ for all } n \geqslant 0.$$

Any bifiltered map $K \to X$ clearly induces a morphism of 2-crossed modules $\Pi_3(K) \to \Omega_3(X)$. Now fix σ a 2-crossed module of groups. The *induced σ-coloring* of a bifiltered map f from K to a classifying space $B\sigma$ is the morphism $\Pi_3(f)\colon \Pi_3(K) \to \sigma$ obtained by composing the induced morphism $\Pi_3(K) \to \Omega_3(B\sigma)$ with the trivial fibration $\Omega_3(B\sigma) \xrightarrow{\sim} \sigma$. A *geometric realization* of a σ-coloring ϕ of K is a bifiltered map $B\phi\colon K \to B\sigma$ such that the induced σ-coloring $\Pi_3(f)$ is equal to ϕ in the homotopy category of 2-crossed modules of groups.

Lemma I.3.3. *Every σ-coloring of K admits a geometric realization, and any map from K to the classifying space of σ is homotopic to the geometric realization of a σ-coloring of K.*

Proof. If K is connected, it follows from [FM11] and the model structure on the category of 2-crossed modules of groups that if σ is a 2-crossed module of groups and $x \in K_0$, then any morphism $\phi\colon \Pi_3(K, x) \to \sigma$ admits a bifiltered map $f\colon K \to B\sigma$ that induces ϕ. If K is not connected, we simply apply the above construction to every connected component of K.

Now let f be a map from K to the classifying space $B\sigma$ of σ. By the bifiltered approximation theorem in [FM11], f is homotopic to a bifiltered map $\bar{f}$ that induces a morphism of 2-crossed modules

$$\Omega_3(\bar{f})\colon \Pi_3(K, x) \to \Omega_3(B\sigma).$$

By [FM11, Lemma 3.16], f is homotopic to the geometric realization of $\Omega_3(\bar{f})$. □

I.4. Homotopies of 2-crossed module maps

Section I.2 described the construction of the fundamental 2-crossed module $\Pi_3(K)$ of a canonical simplicial complex K, and Section I.3 described the construction of the classifying space $B\sigma$ of a 2-crossed module of groups σ. Moreover, we've shown that any morphism $\Pi_3(K) \to \sigma$ admits a geometric realization $K \to B\sigma$, see Lemma I.3.3. This section gives a combinatorial description of the set $[|K|, B\sigma]$ of homotopy classes of maps $K \to B\sigma$ in terms of morphisms of 2-crossed modules. To do so, we introduce the gauge groupoid of the pair (K, σ), defined via the notion of quadratic derivation, and we say that two σ-colorings of K are *gauge equivalent* if they are in the same component of the gauge groupoid.

The main result of this section is the homotopy classification theorem (see Theorem 1) A similar result is shown by Faria Martins [FM11] for pointed homotopy classes of maps. This will be an essential ingredient in

the proof of Theorem 4. Its proof builds on the work of Brown–Loday [BL87], Berger [Ber99], Ellis [Ell93], and Faria Martins [FM11].

I.4.1. Quadratic derivations. In this section, we extend the notion of quadratic derivation defined by Faria Martins [FM11] to the context of groupoids. Let

$$\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega) \quad \text{and} \quad \sigma' = (L' \xrightarrow{\delta'} E' \xrightarrow{\partial'} H',\ \omega')$$

be 2-crossed modules, and let $\phi = (\phi_1, \phi_2, \phi_3)$ be a morphism of 2-crossed modules from σ to σ'. A *quadratic ϕ-derivation* is a triple of set maps $(r\colon H_0 \to H', s\colon H \to E', t\colon E \to L')$ such that

$$r(x) \in H'(\phi(x)), \qquad s(h) \in E'(\phi(y)), \qquad t(e) \in L'(\phi(x)),$$

for all $x, y \in H_0$, $h \in H(x,y)$, and $e \in L(x)$, satisfying the following conditions:

$$s(hk) = \left({}^{\phi_1(k)^{-1}}s(h)\right) s(k),$$

for all $x, y, z \in H_0$, $h \in H(x,y)$, and $k \in H(y,z)$,

$$t(ef) = (\phi_2(f)s(d_1(f)))^{-1} \triangleright \left(\omega(\phi_2(f), {}^{\phi_1(d_1 f)^{-1}}s(d_1(e))^{-1})t(e)\right) t(f),$$

for all $x \in H_0$, $e, f \in E(x)$, and

$$t({}^h e) = {}^{\phi_1(h)}\Big(\Big(\big(s(h)s(d_1(e))^{-1}\big) \triangleright \omega\big(\phi_2(e)^{-1}, s(h)^{-1}\big)^{-1}\Big)\, \omega\big(s(h), s(d_1(e))^{-1}\phi_2(e)^{-1}\big)\, {}^{\partial(s(h))}t(e)\Big),$$

for all $x, y \in H_0$, $h \in H(x,y)$, and $e \in E(y)$.

The *action* of a quadratic ϕ-derivation (r,s,t) is the triple

$$\phi \cdot (r,s,t) = (\phi_1'\colon H \to H', \phi_2'\colon E \to E', \phi_3'\colon L \to L'),$$

such that

$$\phi_1'(h) = r(x)^{-1}\phi_1(h)\partial(s(h))r(y), \qquad \phi_2'(e) = {}^{r(x)^{-1}}\big(\phi_2(e)s(d_1(e))\delta(t(e))\big), \qquad \phi_3'(l) = {}^{r(x)^{-1}}\big(\phi_3(l)t(d_2(l))\big),$$

for all $x, y \in H_0$, $h \in H(x,y)$, $e \in E(x)$, and $l \in L(x)$. A simple computation shows that $\phi \cdot (r,s,t)$ is a morphism of 2-crossed modules (see [FM11, Theorem 2.13]). We say that two morphisms ϕ and ϕ' from σ to σ' are *homotopic* if there exists a quadratic ϕ-derivation (r,s,t) such that

$$\phi \cdot (r,s,t) = \phi'.$$

When H is a free groupoid, this defines an equivalence relation (see [GFM13] for the group case).

I.4.2. The gauge groupoid. Let K be a canonical simplicial complex with B_* a choice of ordered simplices, let $\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega)$ be a 2-crossed module of groups, and let ϕ be a σ-coloring of K. It follows from the freeness of $\Pi_3(K)$ that a quadratic ϕ-derivation (r,s,t) is uniquely determined by the restrictions $(r\colon K_0 \to H, s_{|B_1}\colon B_1 \to E, t_{|B_2}\colon B_2 \to L)$. Moreover, any triple of set maps

$$(r'\colon K_0 \to H, s'\colon B_1 \to E, t'\colon B_2 \to L)$$

defines a quadratic ϕ-derivation.

In order to define the composition of quadratic derivations, we introduce a map defined by Gohla and Faria Martins [GFM13]. Let s and s' denote maps $\pi_1(K_1; K_0) \to E$ such that

$$s(hk) = \left({}^{\phi_1(k)^{-1}}s(h)\right) s(k) \qquad \text{and} \qquad s'(hk) = \left({}^{\partial(s(k))^{-1}\phi_1(k)^{-1}}s'(h)\right) s'(k)$$

for all $x, y, z \in K_0$, $h \in \pi_1(K_1; K_0)(x,y)$, and $k \in \pi_1(K_1; K_0)(y,z)$. The *connecting map* of s and s' is the map

$$\omega^{(s,s')}\colon \pi_1(K_1; K_0) \to L$$

defined by $\omega^{(s,s')}(1_x) = 1$, $\omega^{(s,s')}(b) = 1$ for all $b \in B_1$, and

$$\omega^{(s,s')}(hk) = \omega^{(s,s')}(k)\left(s'(k)^{-1} \triangleright \left({}^{\phi_1(k)^{-1}}\omega\left({}^{\phi_1(k)}s(k)^{-1}, \delta\left(\omega^{(s,s')}(h)\right) s'(h)^{-1}\right)^{\partial(s(k))^{-1}\phi_1(k)^{-1}}\omega^{(s,s')}(h)\right)\right)$$

for all $x, y, z \in K_0$, $h \in \pi_1(K_1; K_0)(x,y)$, and $k \in \pi_1(K_1; K_0)(y,z)$.

Lemma I.4.1. *If (r,s,t) is a quadratic ϕ-derivation and (r',s',t') is a quadratic $\phi\cdot(r,s,t)$-derivation, then*

$$(\phi\cdot(r,s,t))\cdot(r',s',t') = \phi\cdot(r'',s'',t'')$$

where (r'',s'',t'') is the unique quadratic ϕ-derivation satisfying

$$r''(x) = r(x)r'(x) \qquad s''(b_1) = s(b_1)\,{}^{r(y)}s'(b_1) \qquad t''(b_2) = \omega^{(s,{}^{r}s')}(\partial(b_2))\left({}^{r(x)}s'(\partial(b_2))^{-1}\triangleright t(b_2)\right){}^{r(x)}t'(b_2)$$

for all $x,y,z\in K_0$, $b_1\in B_1(x,y)$, $b_2\in B_2(x)$, where ${}^{r}s'$ denotes the map defined by ${}^{r}s'(b) = {}^{r(y)}s'(b)$ for all $x,y\in K_0$, $b\in B_1(x,y)$, and

$$s'(hk) = \left({}^{\partial(s(k))^{-1}\phi_1(k)^{-1}r(y)}s'(h)\right){}^{r(z)}s'(k)$$

for all $x,y,z\in K_0$, $h\colon x\to y$, and $k\colon y\to z$.

Proof. Let $b_1\in B_1(x,y)$. We have

$$\begin{aligned}(\phi\cdot(r,s,t))\cdot(r',s',t')(b_1) &= r'(x)^{-1}r(x)^{-1}\phi_1(b_1)\partial(s(b_1))r(y)\partial\big({}^{r(y)}s'(b_1)\big)r'(y)\\ &= \phi\cdot(r'',s'',t'')(b_1)\end{aligned}$$

Let $b_2\in B_2(x)$. We have

$$\begin{aligned}(\phi\cdot(r,s,t))\cdot(r',s',t')(b_2) &= {}^{r'(x)^{-1}}\left({}^{r(x)^{-1}}(\phi_2(b_2)s(d_1(b_2))\delta(t(b_2)))s'(d_1(b_2))\delta(t'(b_2))\right)\\ &= \phi\cdot(r'',s'',t'')(b_2)\end{aligned}$$

Let $b_3\in\pi_3(K,K_2^-,K_2^+;x)$. We have

$$\begin{aligned}(\phi\cdot(r,s,t))\cdot(r',s',t'))(b_3) &= {}^{r'(x)^{-1}}\left({}^{r(x)^{-1}}\big(\phi_3(b_3)t(d_2(b_3))\big)t'(d_2(b_3))\right)\\ &= \phi\cdot(r'',s'',t'').\end{aligned}$$

□

Let $\mathcal{G}(K,\sigma)$ be the category with objects the σ-colorings of K and with morphisms from a σ-coloring ϕ to a σ-coloring ϕ' given by the quadratic ϕ-derivations (r,s,t) such that

$$\phi' = \phi\cdot(r,s,t).$$

The composition of morphisms $(r,s,t)\colon\phi\to\phi'$ and $(r',s',t')\colon\phi'\to\phi''$ is the quadratic ϕ-derivation

$$(r,s,t)\cdot(r',s',t')\colon\phi\to\phi''$$

defined in Lemma I.4.1.

Lemma I.4.2. *The category $\mathcal{G}(K,\sigma)$ is a groupoid.*

We call $\mathcal{G}(K,\sigma)$ the *gauge groupoid* of K with target σ. Two σ-colorings of K are *gauge equivalent* is they are in the same connected component of the gauge groupoid.

Proof. The composition of morphisms is clearly unital. Let's check that it is associative. Let

$$(r,s,t)\colon\phi\to\phi',\quad (r',s',t')\colon\phi'\to\phi'',\quad (r'',s'',t'')\colon\phi''\to\phi'''$$

be quadratic derivations. We set

$$((r,s,t)\cdot(r',s',t'))\cdot(r'',s'',t'') = (r_1,s_1,t_1)\quad\text{and}\quad (r,s,t)\cdot((r',s',t')\cdot(r'',s'',t'')) = (r_2,s_2,t_2).$$

Straightforward computations show that $r_1(x) = r_2(x)$ for all $x\in K_0$ and that $s_1(b_1) = s_2(b_1)$ for all $b_1\in B_1$.

For any morphism $\psi\colon\Pi_3(K)\to\sigma$, any quadratic ψ-derivation (ρ,σ,τ), and any quadratic $\psi\cdot(\rho,\sigma,\tau)$-derivation (ρ',σ',τ'), we denote by $\sigma\otimes\sigma'$ the map $\pi_1(K_1;K_0)\to E$ defined by $(\sigma\otimes\sigma')(b_1) = \sigma(b_1)\,{}^{\rho(y)}\sigma'(b_1)$ for all $x,y\in K_0$ and $b_1\in B_1(x,y)$ satisfying

$$(\sigma\otimes\sigma')(hk) = \left({}^{\phi(k)^{-1}}(\sigma\otimes\sigma')(h)\right)(\sigma\otimes\sigma')(k)$$

for all $x,y,z\in K_0$, $h\colon x\to y$, and $k\colon y\to z$. Let $x\in K_0$ and $b_2\in B_2(x)$.

$$t_1(b_2) = \omega^{(s\otimes s', {}^{rr'}s'')}(d_1(b_2)) \left({}^{r(x)r'(x)}s''(d_1(b_2))^{-1} \triangleright \omega^{(s, {}^{r}s')}(d_1(b_2))\right)$$
$$\left(\left({}^{r(x)r'(x)}s''(d_1(b_2))^{-1}\, {}^{r(x)}s'(d_1(b_2))^{-1}\right) \triangleright t(b_2)\right) \left({}^{r(x)r'(x)}s''(d_1(b_2))^{-1} \triangleright {}^{r(x)}t'(b_2)\right)^{r(x)r'(x)}t''(b_2),$$

and

$$t_2(b_2) = \omega^{(s, {}^{r}(s'\otimes s''))}(d_1(b_2))^{\,r(x)}\omega^{(s', {}^{r'}s'')}(d_1(b_2)) \left(\left({}^{r(x)r'(x)}s''(d_1(b_2))^{-1}\, {}^{r(x)}s'(d_1(b_2))\right) \triangleright t(b_2)\right)$$
$$\left({}^{r(x)r'(x)}s''(d_1(b_2))^{-1} \triangleright {}^{r(x)}t'(b_2)\right)^{r(x)r'(x)}t''(b_2).$$

Therefore, $t_1(b_2) = t_2(b_2)$ is equivalent to the equation

$$\omega^{(s\otimes s', {}^{rr'}s'')}(d_1(b_2)) \left({}^{r(x)r'(x)}s''(d_1(b_2))^{-1} \triangleright \omega^{(s, {}^{r}s')}(d_1(b_2))\right) = \omega^{(s, {}^{r}(s'\otimes s''))}(d_1(b_2))^{\,r(x)}\omega^{(s', {}^{r'}s'')}(d_1(b_2)).$$

By [GFM13, Equation (89)], it suffices to show that

$${}^{r(x)}\omega^{(s', {}^{r'}s)}(d_1(b_2)) = \omega^{({}^{r}s', {}^{rr'}s'')}(d_1(b_2)).$$

This equality can easily be shown by induction on the length of $d_1(b_2)$ written in the basis B_1.

Let's show that all morphisms in the gauge groupoid are isomorphisms. Let $(r,s,t)\colon \phi \to \phi'$. A right inverse of (r,s,t) is given by the ϕ'-derivation $(\bar{r},\bar{s},\bar{t})$ with

$$\bar{r}(x) = r(x)^{-1}, \qquad \bar{s}(b_1) = {}^{r(y)^{-1}}s(b_1)^{-1}, \qquad \bar{t}(b_2) = \left({}^{r^{-1}}\bar{s}(d_1(b_2))^{-1} \triangleright {}^{r(x)^{-1}}t(b_2)^{-1}\right)^{r(x)^{-1}}\omega^{(s,s^{-1})}(d_1(b_2))$$

for all $x, y \in K_0$, $b_1 \in B_1(x,y)$, and $b_2 \in B_2(x)$. Similarly, a left inverse of (r,s,t) is given by the quadratic ϕ'-derivation $(\bar{r},\bar{s},\tilde{t})$ with

$$\tilde{t}(b_2) = {}^{r(x)^{-1}}s(d_1(b_2)) \triangleright \left(\omega^{(\bar{s}, {}^{r^{-1}}s)}(d_1(b_2))^{\,r(x)^{-1}}t(b_2)^{-1}\right)$$

for all $x \in K_0$ and $b_2 \in B_2(x)$. The associativity tells us that in fact $\bar{t} = \tilde{t}$. □

I.4.3. Proof of the homotopy classification theorem. Let K be a canonical simplicial complex, let B_* be a choice of ordered simplices of K, and let σ be a 2-crossed module of groups with classifying space $B\sigma$. Without loss of generality, we can suppose that K is connected because

$$[|K|, B\sigma] = \prod_i [|K_{(i)}|, B\sigma] \quad \text{and} \quad [\Pi_3(K), \sigma] = \prod_i [\Pi_3(K_{(i)}), \sigma],$$

where $K_{(i)}$ denotes the i-th connected component of K.

Let ϕ and ϕ' be σ-colorings of K. By definition, ϕ and ϕ' are gauge-equivalent if and only if there exists a quadratic ϕ-derivation (r,s,t) such that $\phi' = \phi\cdot(r,s,t)$. We will say that ϕ and ϕ' are *pointed gauge-equivalent* if there is a point $x \in K_0$ such that the restrictions $\phi_x\colon \Pi_3(K,x) \to \sigma$ and $\phi'_x\colon \Pi_3(K,x) \to \sigma$ are homotopic in the sense of [FM11, Definition 2.14]. Notice that $(r,s,t) = (1,s,t)\cdot(r,1,1)$, and therefore ϕ and ϕ' are gauge equivalent if and only if there is a map $r\colon K_0 \to H$ such that ϕ and $\phi'\cdot(r,1,1)$ are pointed gauge-equivalent.

For any $h \in H$, and for any morphism $\psi\colon \Pi_3(K,x) \to \sigma$, we denote by $\psi\cdot h\colon \Pi_3(K,x) \to \sigma$ the morphism defined by

$$(\psi\cdot h)_1(a) = h^{-1}\psi_1(a)h \qquad (\psi\cdot h)_2(b) = {}^{h^{-1}}\psi_2(b) \qquad (\psi\cdot h)_3(c) = {}^{h^{-1}}\psi_3(c),$$

for all $a \in \pi_1(K_1;x)$, $b \in \pi_2(K_2^-, K_2^- \cap K_2^+; x)$, and $c \in \pi_3(K, K_2^-, K_2^+; x)$. This defines a right group action of H on the set of morphisms $\Pi_3(K,x) \to \sigma$. Notice that for any σ-coloring ϕ of K, for any map $r\colon K_0 \to H$, and for any $x \in K_0$, we have

$$(\phi\cdot(r,1,1))_x = \phi_x \cdot r(x).$$

Therefore, two σ-colorings ϕ and ϕ' of K are gauge-equivalent if and only if there exist $x \in K_0$ and $h \in H$ such that ϕ_x and $\phi'_x\cdot h$ are homotopic in the sense of [FM11, Definition 2.14]. By [FM11, Lemma 3.12], we know that ϕ_x and $\phi'_x\cdot h$ admit geometric realizations, denoted by $B(\phi_x)$ and $B(\phi'_x\cdot h)$ respectively. By [FM11, Lemma 3.16], we deduce that ϕ and ϕ' are gauge-equivalent if and only if there is an element $h \in H$ such that $B(\phi_x)$ and $B(\phi'_x\cdot h)$ are pointed homotopic maps.

Let $[|K|, B\sigma]_*$ denote the set of pointed homotopy classes of maps from $(|K|, x)$ to $(B\sigma, *)$. The fundamental group of $B\sigma$ acts on $[K, B\sigma]_*$ and

$$[|K|, B\sigma] \cong [|K|, B\sigma]_*/\pi_1(B\sigma; *).$$

Moreover, recall that $\pi_1(B\sigma, *) \cong H/\partial(E)$. A careful study of the construction of a geometric realization of a morphism $\phi'_x \colon \Pi_3(K, x) \to \sigma$ (see [Ell93, Proposition 9]) shows that for any $h \in H$, $B(\phi'_x \cdot h)$ and $B(\phi'_x) \cdot [h]$ are pointed homotopic. We deduce that two σ-colorings ϕ and ϕ' of K are gauge-equivalent if and only if there exists $h \in H$ such that $B\phi' \cdot h$ and $B\phi$ are pointed homotopic, where $B\phi$ and $B\phi'$ denote their geometric realizations. We conclude that ϕ and ϕ' and gauge-equivalent if and only if $B\phi'$ and $B\phi$ are homotopic.

CHAPTER II

Flat 3-bundles over 4-manifolds

Quantum homotopy invariants are invariants of manifolds equipped with extra structure encoded by a homotopy class of maps to a target topological space. This chapter is devoted to the study of this extra structure when the target space is a connected homotopy 3-type.

We have shown that any connected homotopy 3-type can be realized as the classifying space $B\sigma$ of a 2-crossed module of groups σ. In the simpler case where $B\sigma$ is a connected homotopy 1-type, i.e.,

$$\sigma = (1 \longrightarrow 1 \longrightarrow H,\ 1),$$

for some group H, the set of homotopy classes of maps from a manifold W to $B\sigma = BH$ corresponds to the set of isomorphism classes of flat H-bundles over W, where H has the discrete topology. Now suppose $B\sigma$ is a connected homotopy 2-type, i.e.,

$$\sigma = (1 \longrightarrow E \xrightarrow{\chi} H,\ 1),$$

for some crossed module $(\chi\colon E \to H)$. It follows from the work of Bartels [Bar04] and Baez–Stevenson [BS09] that the set of homotopy classes of maps from a manifold W to the classifying space $B\sigma = B\chi$ corresponds to the set of isomorphism classes of flat χ-bundles over W, where χ has the discrete topology.

In this chapter, we show that there is a correspondence between flat 3-bundles over smooth manifolds and homotopy classes of maps to the classifying space of a 3-group. To do so, we adapt the definition of nonabelian bundle 2-gerbe due to Jurčo [Jur11] to the context of discrete 2-crossed modules. We derive the notion of flat σ-bundle over a smooth manifold for any 2-crossed module σ of (discrete) groups. For any smooth manifold W, we show that there is a canonical bijection between the set of isomorphism classes of flat σ-bundles over W and the set of homotopy classes of maps from W to the classifying space of σ. This enables us to encode the data of a flat σ-bundle over a smooth 4-manifold as a *σ-manifold*, that is, a pair (W, g) where W is a closed, oriented smooth 4-manifold and g is a homotopy class of maps $W \to B\sigma$. Two σ-manifolds (W, g) and (W', g') are *equivalent* if there is an orientation preserving diffeomorphism $\varphi\colon W \xrightarrow{\sim} W'$ such that $g' \circ \varphi = g$.

The remainder of this chapter is devoted to the combinatorial description of σ-manifolds via their σ-triangulations. It is a classical result in topology, due to Pachner [Pac91, Lic99], that two closed 4-manifolds are diffeomorphic if and only if their triangulations are related by a finite sequence of local transformations called Pachner (or bistellar) moves. We generalize this result by introducing σ-colored Pachner moves on σ-triangulations of σ-manifolds. We show that two σ-manifolds are equivalent if and only if their σ-triangulations are related by a finite sequence of σ-colored Pachner moves.

This chapter is organized as follows. In Section II.1 we define flat 3-bundles and relate them to 3-cocycles. We show that flat 3-bundles are classified by the homotopy classes of maps to a homotopy 3-type. Section II.2 is devoted to the definition of 3-group-manifolds and their triangulations. In Section II.3 we introduce colored Pachner moves and prove a colored Pachner theorem for 3-group-manifolds.

II.1. Flat 3-bundles

Let H be a discrete group. A *flat H-bundle* over a smooth manifold W is a principal H-bundle $\pi\colon P \to W$. Let $\pi\colon P \to W$ be a flat H-bundle, and pick $x \in W$. Since H is discrete, the map π is in particular a covering of W, and so any closed loop $\gamma\colon [0,1] \to W$ based at x lifts to a unique path $\tilde{\gamma}\colon [0,1] \to P$ with chosen starting point $\tilde{\gamma}(0) \in \pi^{-1}(x)$. This determines an element $\mathrm{T}(\gamma) = \tilde{\gamma}(0)^{-1}\tilde{\gamma}(1) \in H$, and induces a morphism

$$\mathrm{T}\colon \pi_1(W, x) \to H,$$

called the *parallel transport* of π. In fact, there is a bijection between the set of isomorphism classes of flat H-bundles over W and the set of conjugacy classes of morphisms $\pi_1(W, x) \to H$. Moreover, if BH is the

classifying space of H, then there is a bijection between the set of isomorphism classes of flat H-bundles over W and the set of homotopy classes of maps $W \to BH$.

We wish to extend the above bijection to the case where σ is a (discrete) 2-crossed module of groups, and therefore its classifying space $B\sigma$ is a homotopy 3-type. We recall the notion of nonabelian bundle 2-gerbe, due to Jurčo [Jur11], which is a model for principal 3-bundles. In the case where σ has the discrete topology, we call these *flat σ-bundles*. For any smooth manifold W, we construct a canonical bijection between the set of isomorphism classes of flat σ-bundles over W and the set of homotopy classes of maps $W \to B\sigma$, see Theorem 3.

II.1.1. Open covers. In this section, we fix certain notations and definitions concerning open covers. Let X be a smooth manifold. The *n-th power* of an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X, for all $n \geqslant 1$, is the manifold

$$\mathcal{U}^{[n]} = \Big\{(x_1, \dots, x_n) \in \Big(\bigsqcup_{i\in I} U_i\Big)^n \mid p(x_1) = \dots = p(x_n)\Big\},$$

where $p\colon \sqcup_{i\in I} U_i \to X$ is the projection.

The *fibered product* of two open covers $\mathcal{U} = \{U_i\}_{i\in I}$, $\mathcal{V} = \{V_j\}_{j\in J}$ of X is the manifold

$$\mathcal{U} \times_X \mathcal{V} = \{(x, y) \in \big(\sqcup_{i\in I} U_i\big) \times \big(\sqcup_{j\in J} V_j\big) \mid p_{\mathcal{U}}(x) = p_{\mathcal{V}}(y)\}$$

where $p_{\mathcal{U}}\colon \sqcup_{i\in I} U_i \to X$ and $p_{\mathcal{V}}\colon \sqcup_{j\in J} V_j \to X$ are the projections. The *n-th power* of $\mathcal{U} \times_X \mathcal{V}$, for all $n \geqslant 1$, is the manifold

$$\big(\mathcal{U} \times_X \mathcal{V}\big)^{[n]} = \{(z_1, \dots, z_n) \in \big(\mathcal{U} \times_X \mathcal{V}\big)^n \mid p(z_1) = \dots = p(z_n)\},$$

where $p\colon \mathcal{U} \times_X \mathcal{V} \to X$ is the projection.

An open cover $\mathcal{U}$ of X is *good* if all finite non-empty intersections of elements of $\mathcal{U}$ are contractible. If $\mathcal{U} = \{U_i\}_{i\in I}$ is a good open cover, then denote by $I^{[n]}$ the set

$$I^{[n]} = \{(i_1, \dots, i_n) \in I^n \mid U_{i_1} \cap \dots \cap U_{i_n} \neq \emptyset\},$$

for all $n \geqslant 1$.

A *refinement* of an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover $\mathcal{V} = \{V_j\}_{j\in J}$ with a map $\phi\colon J \to I$, called the *refinement map*, such that $V_j \subset U_{\phi(j)}$ for all $j \in J$.

II.1.2. Flat bundles. Let H be a discrete group. A *flat H-bundle* over a smooth manifold W is a smooth map $\pi\colon P \to W$ together with a free and transitive action of H on P

$$\begin{array}{ccc} H \times P & \to & P \\ (h, p) & \mapsto & h \cdot p \end{array}$$

that preserves the fibers of π and admits a *trivializing cover*, that is, a pair $(\mathcal{U}, \psi)$, where $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of W and $\psi = \{\psi_i\colon \pi^{-1}(U_i) \xrightarrow{\sim} H \times U_i\}_{i\in I}$ is a family of H-equivariant diffeomorphisms such that the following diagram is commutative:

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\psi_i} & H \times U_i \\ & \searrow^{\pi} & \downarrow{\scriptstyle \mathrm{pr}_2} \\ & & U_i, \end{array}$$

for all $i \in I$, where pr_2 denotes the projection on the second coordinate. An *isomorphism* between flat H-bundles $\pi\colon P \to W$ and $\pi'\colon P' \to W$ over a manifold W is an H-equivariant diffeomorphism $\phi\colon P \xrightarrow{\sim} P'$ such that $\pi' \circ \phi = \pi$. Two flat H-bundles are *isomorphic* if there is an isomorphism between them. We say that a flat H-bundle over W is *trivial* if it is isomorphic to the projection $H \times W \to W$.

The *Whitney sum* of two flat H-bundles $\pi\colon P \to W$ and $\pi'\colon P' \to W$ over a manifold W is the flat H-bundle $P \oplus P' \to W$ with

$$P \oplus P' = \{(p, p') \in P \times P' \mid \pi(p) = \pi'(p')\}.$$

Let $\pi\colon P \to W$ be a flat H-bundle and let $\phi\colon H \to G$ be a group homomorphism. The *pushforward* of P along ϕ is the flat G-bundle $\phi_*(\pi)\colon G \times_\phi P \to W$, where $G \times_\phi P$ is the set of equivalence classes of elements $(g, p) \in G \times P$ for the equivalence relation $(g, p) \sim (\phi(h)g, h \cdot p)$ for all $h \in H$, with the action of G on $G \times_\phi P$ defined by $g \cdot (g', p) = (g'g^{-1}, p)$ and with projection $\phi_*(\pi)([g, p]) = \pi(p)$ for all $[g, p] \in G \times_\phi P$.

II.1.3. 1-cocycles. Let W be a smooth manifold and let H be a discrete group. A *1-cocycle* for a good open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of W is a map $h\colon I^{[2]} \to H$ such that $h_{ij}h_{jk} = h_{ik}$ for all $(i,j,k) \in I^{[3]}$. If h is a 1-cocycle for $\mathcal{U}$ and $\mathcal{T}$ is a good open cover that is a refinement of $\mathcal{U}$, then precomposing with the refinement map ϕ defines a 1-cocycle $\phi^*(h)$ for $\mathcal{T}$. A 1-cocycle for W is a pair $(\mathcal{U}, h)$ where $\mathcal{U}$ is a good open cover of W and h is a 1-cocycle for $\mathcal{U}$.

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a good open cover of W. Two 1-cocycles h, k for $\mathcal{U}$ are *gauge equivalent* if there exists a map $r\colon I \to H$ verifying

$$k_{ij} = r_i^{-1} h_{ij} r_j$$

for all $(i,j) \in I^{[2]}$. Two 1-cocycles $(\mathcal{U}, h)$ and $(\mathcal{V}, k)$ for W are *gauge equivalent* if there is a good open cover $\mathcal{T}$ of W that is a refinement of $\mathcal{U}$ and $\mathcal{V}$ and such that $\phi^*(h)$ and $\psi^*(k)$ are gauge equivalent as 1-cocycles for $\mathcal{T}$, where ϕ and ψ are the refinement maps.

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a good open cover of W and let $\pi\colon P \to W$ be a flat H-bundle with trivializing cover $(\mathcal{U}, \psi)$. Since H is discrete, the diffeomorphisms $\psi_i\colon \pi^{-1}(U_i) \to H \times U_i$ can be written $\psi_i = (s_i, \pi)$, where $s_i\colon \pi^{-1}(U_i) \to H$ is a locally constant H-equivariant map. The map $h\colon I^{[2]} \to H$ given by

$$h_{ij} = s_i(p)^{-1} s_j(p) \in H$$

defines a 1-cocycle for $\mathcal{U}$. Conversely, given a 1-cocycle for W, one can easily reconstruct a flat H-bundle. It is easy to check that there is a bijection between the set of isomorphism classes of flat H-bundles over W and the set of gauge-equivalence classes of 1-cocycles for W, or equivalently the first Čech cohomology $\check{\mathrm{H}}^1(W; H)$ with coefficients in H.

II.1.4. Flat 2-bundles. Throughout this section, $(\chi\colon E \to H)$ is a crossed module of groups with the discrete topology. Let X be a smooth manifold. A *χ-function* over X is a pair (P, m) where P is a flat E-bundle over X and $m\colon P \to H$ is an E-equivariant map, *i.e.*, $m(e\cdot p) = \chi(e)m(p)$ for all $e \in E$ and $p \in P$. In particular, if (P, m) is a χ-function over X, then the pushforward H-bundle $H \times_\chi P \to W$ is trivial with global section

$$\begin{cases} X & \to & H \times_\chi P \\ x & \mapsto & [m(p), p], \end{cases}$$

where p is any point in the fiber of x. Notice that a $(1 \to H)$-function over X is the same as a locally constant map $X \to H$.

An *isomorphism* between χ-functions (P, m) and (P', m') over X is an isomorphism $\phi\colon P \to P'$ of flat E-bundles over X such that $m'\phi = m$.

The *product* of two χ-functions (P, m) and (P', m') over X is the χ-function (PP', mm'), where PP' is the quotient of the Whitney sum $P \oplus P'$ by the equivalence relation $({}^{(m(p)}e)\cdot p, p') \sim (p, e\cdot p')$ for all $e \in E$, and $mm'(p, p') = m(p)m'(p')$. The action of E on PP' is defined by $e\cdot(p, p') = (e\cdot p, p')$ for all $e \in E$.

The *inverse* of a χ-function (P, m) over X is the χ-function (P^{-1}, m^{-1}), where P^{-1} has the same total space as P with E-action

$$\begin{cases} E \times P^{-1} & \to & P^{-1} \\ (e, p) & \mapsto & {}^{m(p)}e^{-1}\cdot p, \end{cases}$$

and $m^{-1}\colon P \to H$ is defined by $m^{-1}(p) = m(p)^{-1}$ for all $p \in P^{-1}$.

Let W be a smooth manifold. A *flat χ-bundle* over W is a triple $(\mathcal{U}, \mathcal{P}, e)$ where $\mathcal{U}$ is an open cover of W, $\mathcal{P}$ is a χ-function over $\mathcal{U}^{[2]}$ and $e\colon \mathcal{P}_{12}\mathcal{P}_{23} \xrightarrow{\sim} \mathcal{P}_{13}$ is an isomorphism of χ-functions over $\mathcal{U}^{[3]}$, where $\mathcal{P}_{ij}$ denotes the pullback of $\mathcal{P}$ along the projection $\mathcal{U}^{[3]} \to \mathcal{U}^{[2]}$ onto the i-th and j-th factors, such that the diagram of χ-functions over $\mathcal{U}^{[4]}$ is commutative

$$\begin{array}{ccc} \mathcal{P}_{12}\mathcal{P}_{23}\mathcal{P}_{34} & \xrightarrow{e_{234}} & \mathcal{P}_{12}\mathcal{P}_{24} \\ \downarrow{\scriptstyle e_{123}} & & \downarrow{\scriptstyle e_{124}} \\ \mathcal{P}_{13}\mathcal{P}_{34} & \xrightarrow{e_{134}} & \mathcal{P}_{14}, \end{array}$$

where $\mathcal{P}_{ij}$ denotes the pullback of $\mathcal{P}$ along the projection $\mathcal{U}^{[4]} \to \mathcal{U}^{[2]}$ to the i-th and j-th factors, and e_{ijk} denotes the pullback of e along the projection $\mathcal{U}^{[4]} \to \mathcal{U}^{[3]}$ to the i-th, j-th, and k-th factors. Notice that a flat $(1 \to H)$-bundle over W is the same as a flat H-bundle over W.

An *isomorphism* from a flat χ-bundle $(\mathcal{U},\mathcal{P},e)$ to a flat χ-bundle $(\mathcal{V},\mathcal{Q},f)$ over W is a pair $(\mathcal{R},\phi)$, where $\mathcal{R}$ is a χ-function over $\mathcal{U}\times_W\mathcal{V}$ and ϕ is an isomorphism of χ-functions

$$\phi\colon p_{\mathcal{U}}^*(\mathcal{P}) \xrightarrow{\sim} \mathcal{R}_1 p_{\mathcal{V}}^*(\mathcal{Q})\mathcal{R}_2^{-1}$$

over $\big(\mathcal{U}\times_W\mathcal{V}\big)^{[2]}$, where $\mathcal{R}_i$ denotes the pullback of $\mathcal{R}$ along the projection $(\mathcal{U}\times_W\mathcal{V})^{[2]}\to\mathcal{U}\times_W\mathcal{V}$ to the i-th factor, such that the diagram

$$\begin{array}{ccc} p_{\mathcal{U}}^*(\mathcal{P}_{12})p_{\mathcal{U}}^*(\mathcal{P}_{23}) & \xrightarrow{p_{\mathcal{U}}^*(e)} & p_{\mathcal{U}}^*(\mathcal{P}_{13}) \\ \downarrow{\scriptstyle \phi_{12}\phi_{23}} & & \downarrow{\scriptstyle \phi_{13}} \\ \mathcal{R}_1 p_{\mathcal{V}}^*(\mathcal{Q}_{12})p_{\mathcal{V}}^*(\mathcal{Q}_{23})\mathcal{R}_3^{-1} & \xrightarrow{p_{\mathcal{V}}^*(f)} & \mathcal{R}_1 p_{\mathcal{V}}^*(\mathcal{Q}_{13})\mathcal{R}_3^{-1} \end{array}$$

is commutative over $\big(\mathcal{U}\times_W\mathcal{V}\big)^{[3]}$, where $\mathcal{P}_{ij}$, $\mathcal{Q}_{ij}$, and ϕ_{ij} denote the pullbacks of $\mathcal{P}$, $\mathcal{Q}$, and ϕ, respectively, along the projection $(\mathcal{U}\times_W\mathcal{V})^{[3]}\to(\mathcal{U}\times_W\mathcal{V})^{[2]}$ to the i-th and j-th factors. Two flat χ-bundles over W are *isomorphic* if there is an isomorphism between them.

II.1.5. 2-cocycles. Throughout this section, $(\chi\colon E\to H)$ is a crossed module of groups with the discrete topology and W is a smooth manifold. A *2-cocycle* for a good open cover $\mathcal{U}=\{U_i\}_{i\in I}$ of W is a pair of maps $(h\colon I^{[2]}\to H, e\colon I^{[3]}\to E)$ such that

$$\chi(e_{ijk}) = h_{ik}h_{jk}^{-1}h_{ij}^{-1}, \qquad {e_{ijl}}^{h_{ij}}e_{jkl} = e_{ikl}e_{ijk},$$

for all $(i,j,k,l)\in I^{[4]}$. If (h,e) is a 2-cocycle for $\mathcal{U}$ and $\mathcal{T}$ is a good open cover that is a refinement of $\mathcal{U}$, then precomposing with the refinement map ϕ defines a 2-cocycle $(\phi^*(h),\phi^*(e))$ for $\mathcal{T}$. A *2-cocycle* for W is a triple $(\mathcal{U},h,e)$ where $\mathcal{U}$ is a good open cover of W and (h,e) is a 2-cocycle for $\mathcal{U}$.

Two 2-cocycles (h,e) and (k,f) for a good open cover $\mathcal{U}=\{U_i\}_{i\in I}$ of W are *gauge equivalent* if there is a pair of maps $(r\colon I^{[1]}\to H, s\colon I^{[2]}\to E)$ such that

$$k_{ij} = r_i^{-1}h_{ij}\chi(s_{ij})r_j \qquad f_{ijk} = {}^{r_i^{-1}}\big(e_{ijk}\,{}^{h_{ij}h_{jk}}s_{ik}\,{}^{h_{ij}h_{jk}}s_{jk}^{-1}\,{}^{h_{ij}}s_{ij}^{-1}\big),$$

for all $(i,j,k)\in I^{[3]}$. Two 2-cocycles $(\mathcal{U},h,e)$ and $(\mathcal{V},k,f)$ for W are *gauge equivalent* if there is a good open cover $\mathcal{T}$ of W that is a refinement of $\mathcal{U}$ and $\mathcal{V}$ and such that $(\phi^*(h),\phi^*(e))$ and $(\psi^*(k),\psi^*(f))$ are gauge equivalent as 2-cocycles for $\mathcal{T}$, where ϕ and ψ denote the refinement maps.

Any flat χ-bundle over W determines a *2-cocycle* for W, and conversely any 2-cocycle for W determines a flat χ-bundle over W. A direct computation shows that two flat χ-bundles over W are isomorphic if and only if their 2-cocycles are gauge equivalent. This yields a bijection between the set of isomorphism classes of flat χ-bundles over W and the set of gauge-equivalence classes of 2-cocycles for W, or equivalently the second Čech cohomology $\check{\mathrm{H}}^2(W;\chi)$ with coefficients in χ.

II.1.6. Flat 3-bundles. Throughout this section,

$$\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega)$$

is a 2-crossed module of groups with the discrete topology. Let X be a smooth manifold. Notice that σ induces a morphism

$$\mu\colon(\delta\colon L\to E)\to(1\to H)$$

of crossed modules. In particular, if $\mathcal{G}$ is a flat δ-bundle over X, then the pushforward $\mu(\mathcal{G})$ is a flat H-bundle over X.

A *σ-function* over X is a pair $(\mathcal{G},h)$, where $\mathcal{G}$ is a flat δ-bundle over X and $h\colon\mu(\mathcal{G})\to H$ is an H-equivariant map. In particular, if $(\mathcal{G},h)$ is a σ-function over X, then the flat H-bundle $\mu(\mathcal{G})$ over X is trivial. Notice that a $(1\to E\to H)$-function over X is the same as a flat $(E\to H)$-bundle over X.

An *isomorphism* from a σ-function $(\mathcal{G},h)$ to a σ-function $(\mathcal{G}',h')$ over X is an isomorphism $(\mathcal{R},\phi)\colon\mathcal{G}\xrightarrow{\sim}\mathcal{G}'$ of flat δ-bundles such that $h'\mu(\mathcal{R},\phi)=h$, where $\mu(\mathcal{R},\phi)$ denotes the pushforward of $(\mathcal{R},\phi)$ along μ. Let $(\mathcal{R},\phi)$ and $(\mathcal{S},\psi)$ be isomorphisms from $(\mathcal{G},h)$ to $(\mathcal{G}',h')$, where $\mathcal{G}=(\mathcal{U},\mathcal{P},l)$ and $\mathcal{G}'=(\mathcal{V},\mathcal{P}',l')$. A *2-isomorphism*

from $(\mathcal{R},\phi)$ to $(\mathcal{S},\psi)$ is an isomorphism $\xi\colon \mathcal{R} \tilde{\to} \mathcal{S}$ of δ-functions over $\mathcal{U}\times_X\mathcal{V}$ such that the following diagram of δ-functions over $(\mathcal{U}\times_X\mathcal{V})^{[2]}$ is commutative:

$$\begin{array}{ccc} p_{\mathcal{U}}^*(\mathcal{P}) & \xrightarrow{\phi} & \mathcal{R}_1 p_{\mathcal{V}}^*(\mathcal{P}')\mathcal{R}_2^{-1} \\ & \searrow^{\psi} & \downarrow{\scriptstyle \xi_1\xi_2^{-1}} \\ & & \mathcal{S}_1 p_{\mathcal{V}}^*(\mathcal{P}')\mathcal{S}_2^{-1}, \end{array}$$

where $\mathcal{R}_i$, $\mathcal{S}_i$, and ξ_i are the pullbacks of $\mathcal{R}$, $\mathcal{S}$, and ξ respectively along the projection $(\mathcal{U}\times_X\mathcal{V})^{[2]}\to\mathcal{U}\times_X\mathcal{V}$ to the i-th factor.

Let $(\mathcal{G},h)$ and $(\mathcal{G}',h')$ be two σ-functions over X with $\mathcal{G}=(\mathcal{U},(P,m),q)$ and $\mathcal{G}'=(\mathcal{U},(P',m'),q')$. Notice that h induces an H-valued function $\mathbf{n}$ on $\mathcal{U}^{[1]}$, and we denote by $\mathbf{n}_2$ the pullback of $\mathbf{n}$ along the projection to the second factor. We denote by $P\cdot_{\mathbf{n}}P'$ the set of equivalence classes of elements in the Whitney sum $P\oplus P'$ for the relation defined by

$$\big((m(p)\triangleright^{\mathbf{n}_2(x_1,x_2)}l)\cdot p,p'\big)\sim_{\mathbf{n}}(p,l\cdot p')$$

for all (p,p') in the fiber over $(x_1,x_2)\in\mathcal{U}^{[2]}$ and $l\in L$. The *product* of $(\mathcal{G},h)$ and $(\mathcal{G}',h')$ is the σ-function

$$(\mathcal{G},h)(\mathcal{G}',h')=((\mathcal{U},(P\cdot_{\mathbf{n}}P',m\,{}^{\mathbf{n}_2}m'),qq'),hh')$$

over X, where $m\,{}^{\mathbf{n}_2}m'\colon P\cdot_{\mathbf{n}}P'\to E$ is the L-equivariant map defined by

$$m\,{}^{\mathbf{n}_2}m'([p,p'])=m(p)\,{}^{\mathbf{n}_2(x_1,x_2)}m'(p')$$

for all $(x_1,x_2)\in\mathcal{U}^{[2]}$ and $[p,p']$ is the fiber above (x_1,x_2).

The *inverse* of a σ-function $((\mathcal{U},(P,m),q),h)$ is the σ-function $((\mathcal{U},(P,m),q),h)^{-1}=(((P^*,m^*),q^{-1}),h^{-1})$, where P^* is the flat L-bundle over $\mathcal{U}^{[2]}$ with the same total space as P and with left L-action

$$\begin{array}{ccc} L\times P & \to & P \\ (l,p) & \mapsto & (m(p)\triangleright^{\mathbf{n}_2(x_1,x_2)}l^{-1})\cdot p \end{array}$$

for all p in the fiber above (x_1,x_2) and $l\in L$, where $\mathbf{n}_2$ denotes the H-valued function on $\mathcal{U}^{[2]}$ defined above, and $m^*\colon P^*\to E$ is the L-equivariant map defined by

$$m^*(p)={}^{\mathbf{n}_2(x_1,x_2)}m(p)^{-1}$$

for all $(x_1,x_2)\in\mathcal{U}^{[2]}$ and p in the fiber above (x_1,x_2).

Let W be a smooth manifold. A *flat σ-bundle* over W is a tuple $(\mathcal{U},\mathcal{G},h,e,l)$ where $\mathcal{U}$ is an open cover of W, $(\mathcal{G},h)$ is a σ-function over $\mathcal{U}^{[2]}$,

$$e\colon (\mathcal{G},h)_{12}(\mathcal{G},h)_{23}\tilde{\to}(\mathcal{G},h)_{13}$$

is an isomorphism of σ-functions over $\mathcal{U}^{[3]}$ where $(\mathcal{G},h)_{ij}$ denotes the pullback of $(\mathcal{G},h)$ along the projection $\mathcal{U}^{[3]}\to\mathcal{U}^{[2]}$ to the i-th and j-th factors, and

$$l\colon e_{124}\circ e_{234}\tilde{\to}e_{134}\circ e_{123},$$

is a 2-isomorphism of isomorphisms of σ-functions over $\mathcal{U}^{[4]}$ where e_{ijk} denotes the pullback of e along the projection $\mathcal{U}^{[4]}\to\mathcal{U}^{[3]}$ to the i-th, j-th and k-th factors, such that

$$l_{1345}\circ l_{1235}=l_{1234}\circ l_{1245}\circ l_{2345}$$

over $\mathcal{U}^{[5]}$, where l_{ijkl} denotes the pullback of l along the projection $\mathcal{U}^{[5]}\to\mathcal{U}^{[4]}$ to the i-th, j-th, k-th, and l-th factors. Notice that a flat $(1\to E\to H)$-bundle is the same as a flat $(E\to H)$-bundle.

Two flat σ-bundles $(\mathcal{U},\mathcal{G},h,e,l)$ and $(\mathcal{V},\mathcal{G}',h',e',l')$ over W are *isomorphic* if there is a σ-function $\mathcal{Q}$ over $\mathcal{U}\times_W\mathcal{V}$, an isomorphism

$$\phi\colon p_{\mathcal{U}}^*(\mathcal{G},h)\tilde{\to}\mathcal{Q}_1 p_{\mathcal{V}}^*(\mathcal{G}',h')\mathcal{Q}_2^{-1}$$

of σ-functions over $(\mathcal{U}\times_W\mathcal{V})^{[2]}$, where $\mathcal{Q}_i$ is the pullback of $\mathcal{Q}$ along the projection $(\mathcal{U}\times_W\mathcal{V})^{[2]}\to\mathcal{U}\times_W\mathcal{V}$ to the i-th factor, and a 2-isomorphism

$$\psi\colon e'\phi_{12}\phi_{23}\tilde{\to}\phi_{13}e$$

over $(\mathcal{U}\times_W\mathcal{V})^{[3]}$, where ϕ_{ij} is the pullback of ϕ along the projection $(\mathcal{U}\times_W\mathcal{V})^{[3]}\to(\mathcal{U}\times_W\mathcal{V})^{[2]}$ to the i-th and j-th factors, such that

$$l_{1234}\psi_{124}\psi_{234}=\psi_{134}\psi_{123}l'_{1234}$$

over $(\mathcal{U} \times_W \mathcal{V})^{[4]}$, where ψ_{ijk} denotes the pullback of ψ along the projection $(\mathcal{U} \times_W \mathcal{V})^{[4]} \to (\mathcal{U} \times_W \mathcal{V})^{[3]}$ to the i-th, j-th, and k-th factors.

II.1.7. 3-cocycles. Throughout this section,

$$\sigma = (L \xrightarrow{\ \delta\ } E \xrightarrow{\ \partial\ } H,\ \omega)$$

is a 2-crossed module of groups with the discrete topology, and W is a smooth manifold. A *3-cocycle* for a good open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of W is a triple of maps $(h\colon I^{[2]} \to H, e\colon I^{[3]} \to E, l\colon I^{[4]} \to L)$ such that

$$\partial(e_{ijk})h_{ij}h_{jk} = h_{ik} \qquad \delta(l_{ijkl})e_{ikl}e_{ijk} = e_{ijl}\,{}^{h_{ij}}e_{jkl},$$

for all $(i,j,k,l) \in I^{[4]}$, and

$$l_{ijkm}e_{ikm} \triangleright \omega(e_{ijk}, {}^{h_{ij}h_{jk}}e_{klm})l_{iklm} = e_{ijm} \triangleright {}^{h_{ij}}l_{jklm}l_{ijlm}e_{ilm} \triangleright l_{ijkl},$$

for all $(i,j,k,l,m) \in I^{[5]}$. If (h,e,l) is a 3-cocycle for $\mathcal{U}$ and $\mathcal{T}$ is a good open cover that is a refinement of $\mathcal{U}$, then precomposing with the refinement map ϕ defines a 3-cocycle $(\phi^*(h), \phi^*(e), \phi^*(l))$ for $\mathcal{T}$. A 3-cocycle for W is a quadruple $(\mathcal{U}, h, e, l)$ where $\mathcal{U}$ is a good open cover of W and (h,e,l) is a 3-cocycle for $\mathcal{U}$.

Two 3-cocycles (h,e,l) and (k,f,m) for a good open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of W are *gauge equivalent* if there is a triple of maps $(r\colon I^{[1]} \to H, s\colon I^{[2]} \to E, t\colon I^{[3]} \to L)$ such that

$$h'_{ij} = r_i^{-1}h_{ij}\partial(s_{ij})r_j \qquad e'_{ijk} = {}^{r_i^{-1}}\big(e_{ijk}\,{}^{h_{ij}h_{jk}}s_{ik}\,{}^{h_{ij}h_{jk}}s_{jk}^{-1}\,{}^{h_{ij}}s_{ij}^{-1}\delta(t_{ijk})\big) \qquad l'_{ijkl} = {}^{r_i^{-1}}\big(l_{ijkl}t_{ijkl}\big),$$

for all $(i,j,k,l) \in I^{[4]}$, where t_{ijkl} denotes the element of L defined by the formula

$$\begin{aligned}
t_{ijkl} = {} & \left({}^{h_{il}}s_{il}\,{}^{h_{il}}s_{jl}^{-1}\,{}^{h_{il}h_{jl}^{-1}}s_{ij}^{-1}e_{ikl}e_{ijk}\,{}^{h_{ij}}e_{jkl}^{-1}\right) \triangleright \Big[\omega\big({}^{h_{ij}}e_{jkl}e_{ijk}^{-1}e_{ikl}^{-1}, {}^{h_{il}h_{jl}^{-1}}s_{ij}\,{}^{h_{il}}s_{jl}\,{}^{h_{il}}s_{il}^{-1}\big)t_{ijl}\Big] \\
& \left({}^{h_{il}}s_{il}\,{}^{h_{il}}s_{kl}^{-1}\,{}^{h_{il}h_{kl}^{-1}}s_{jk}^{-1}\,{}^{h_{il}h_{kl}^{-1}h_{jk}^{-1}}s_{ij}^{-1}e_{ikl}e_{ijk}\right) \triangleright \omega\big(e_{ijk}^{-1}e_{ikl}^{-1}, {}^{h_{il}h_{kl}^{-1}h_{jk}^{-1}}s_{ij}\,{}^{h_{il}h_{kl}^{-1}}s_{jk}\,{}^{h_{il}}s_{kl}\,{}^{h_{il}}s_{jl}^{-1}\,{}^{h_{il}h_{jl}^{-1}}s_{ij}^{-1}\big) \\
& \left({}^{h_{il}}s_{il}\,{}^{h_{il}}s_{kl}^{-1}\,{}^{h_{il}h_{kl}^{-1}}s_{jk}^{-1}\,{}^{h_{il}h_{kl}^{-1}h_{jk}^{-1}}s_{ij}^{-1}e_{ikl}e_{ijk}\,{}^{h_{ij}}s_{ij}\,{}^{h_{ij}h_{jk}}s_{jk}\,{}^{h_{ij}h_{jk}h_{kl}}s_{kl}\,{}^{h_{ij}h_{jk}h_{kl}}s_{jl}^{-1}\right) \triangleright \omega\big({}^{h_{ij}}e_{jkl}^{-1}, {}^{h_{ij}}s_{ij}^{-1}\big)^{-1} \\
& \left({}^{h_{il}}s_{il}\,{}^{h_{il}}s_{kl}^{-1}\,{}^{h_{il}h_{kl}^{-1}}s_{jk}^{-1}\,{}^{h_{il}h_{kl}^{-1}h_{jk}^{-1}}s_{ij}^{-1}e_{ikl}e_{ijk}\right) \triangleright \Big[\omega\big({}^{h_{ij}}s_{ij}, {}^{h_{ij}h_{jk}}s_{jk}\,{}^{h_{ij}h_{jk}h_{kl}}s_{kl}\,{}^{h_{ij}h_{jk}h_{kl}}s_{jl}^{-1}\,{}^{h_{ij}}e_{jkl}^{-1}\big)\,{}^{h_{ij}\partial s_{ij}}t_{jkl}\Big] \\
& \left({}^{h_{il}}s_{il}\,{}^{h_{il}}s_{kl}^{-1}\,{}^{h_{il}h_{kl}^{-1}}s_{ik}^{-1}e_{ikl}\right) \triangleright \omega\big(e_{ikl}^{-1}, {}^{h_{il}h_{kl}^{-1}h_{ik}^{-1}}\big({}^{h_{il}}s_{il}\,{}^{h_{il}}s_{kl}^{-1}\,{}^{h_{il}h_{kl}^{-1}}s_{ik}^{-1}\big)\big) \\
& \left({}^{h_{il}}s_{il}\,{}^{h_{il}}s_{kl}^{-1}\,{}^{h_{il}h_{kl}^{-1}}s_{ik}^{-1}e_{ikl}\,{}^{h_{ik}}s_{ik}\,{}^{h_{ik}}s_{jk}^{-1}\,{}^{h_{ik}h_{jk}^{-1}}s_{ij}^{-1}e_{ijk}\right) \triangleright \Big[t_{ijk}^{-1}\omega\big(e_{ijk}^{-1}, {}^{h_{ik}h_{jk}^{-1}}s_{ij}\,{}^{h_{ik}}s_{jk}\,{}^{h_{ik}}s_{ik}^{-1}\big)^{-1}\Big] \\
& \left({}^{h_{il}}s_{il}\,{}^{h_{il}}s_{kl}^{-1}\,{}^{h_{il}h_{kl}^{-1}}s_{ik}^{-1}e_{ikl}\right) \triangleright \Big[t_{ikl}^{-1}\omega\big(e_{ikl}^{-1}, {}^{h_{il}h_{kl}^{-1}}s_{ik}\,{}^{h_{il}}s_{kl}\,{}^{h_{il}}s_{il}^{-1}\big)^{-1}\Big].
\end{aligned}$$

Two 3-cocycles $(\mathcal{U}, h, e, l)$ and $(\mathcal{V}, k, f, m)$ for W are *gauge equivalent* if there is a good open cover $\mathcal{T}$ of W that is a refinement of $\mathcal{U}$ and $\mathcal{V}$ such that $(\phi^*(h), \phi^*(e), \phi^*(l))$ and $(\psi^*(k), \psi^*(f), \psi^*(m))$ are gauge equivalent as 3-cocycles for $\mathcal{T}$, where ϕ and ψ denote the refinement maps.

Any flat σ-bundle over W determines a 3-cocycle for W, and conversely, any 3-cocycle for W determines a flat σ-bundle. A direct computation shows that flat σ-bundles over W are isomorphic if and only if their 3-cocycles are gauge equivalent. This yields a bijection between the set of isomorphism classes of flat σ-bundles over W and the set of gauge-equivalence classes of 3-cocycles for W, or equivalently the third Čech cohomology $\check{\mathrm{H}}^3(W;\sigma)$ with coefficients in σ.

These constructions lead to the following result:

Theorem 3. *There is a canonical bijection between the set of isomorphism classes of flat σ-bundles over a smooth manifold W and the set $[W, B\sigma]$ of homotopy classes of maps from W to the classifying space of σ.*

We prove Theorem 3 in the following section.

II.1.8. Proof of Theorem 3. Let W be a smooth manifold and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a good open cover of W. The *open star* of a subset S of a simplicial complex K in $\mathbb{R}^N$ is the union $\mathrm{St}^\circ(S)$ in $\mathbb{R}^N$ of the vertices in S and of the interiors of all the simplices in K that have a face in S. Notice that if (K,φ) is a triangulation of W, then the open stars of the vertices of K define a good open cover of M.

Lemma II.1.1. *There exists a triangulation K of W such that the good open cover induced be the open stars of K is a refinement of $\mathcal{U}$.*

Proof. By the work of Whitehead, there exists a piecewise differentiable homeomorphism $\varphi\colon L \to W$ for some simplicial complex L. The pullback $\varphi^*(\mathcal{U})$ of $\mathcal{U}$ is an open cover of the metric space $|L|$. Let C be a compact subset of $|L|$. By the Lebesgue covering lemma, there exists a number d_C such that every subset of $C \cap |L|$ having diameter less than d_C is contained in some element of $\varphi^*(\mathcal{U})$. Therefore, after performing a finite number of barycentric subdivisions, we obtain a simplicial complex such that the open star of every vertex of that is in C is contained in an element of $\varphi^*(\mathcal{U}$. By paracompactness of $|L|$, we can perform these iterated barycentric subdivisions locally and obtain a subdivision K of L such that the open star of every vertex of K is included in an element of $\varphi^*(\mathcal{U})$. Therefore, the good open cover of W induced by the open stars of the vertices of K is a refinement of $\mathcal{U}$. □

Let $(\mathcal{U},\mathcal{G},h,e,l)$ be a flat σ-bundle over W. By the previous lemma, there exists a triangulation K of W such that the good open cover induced be the open stars of K is a refinement of $\mathcal{U}$. Moreover, the flat σ-bundle corresponds to a 3-cocycle (h,e,l) for $\mathcal{U}$. By considering the pullback of (h,e,l) along the refinement map, one obtains a 3-cocycle for the open cover given by the open stars of K and thus a σ-coloring of K, for any choice of canonical structure on K. Conversely, any σ-coloring of K determines a 3-cocycle for the open cover given by the open stars of the vertices of K. Moreover, two σ-colorings are gauge equivalent in the sense of Section I.4.2 if and only if the induced 3-cocycles are gauge equivalent in the sense of Section II.1.7. Therefore, by the homotopy classification theorem, there is a bijection between the set of homotopy classes of maps $W \to B\sigma$ and the set of isomorphism classes of flat σ-bundles over W.

II.2. 3-group-manifolds and their triangulations

In order to define invariants of manifolds, it is often useful to have a combinatorial presentation of them. For instance, the Douglas–Reutter invariant [DR18] of a closed, oriented smooth 4-manifold W is defined via a triangulation of W. Let σ be a 2-crossed module of groups. In this section, we introduce σ-manifolds and their triangulations.

II.2.1. Combinatorial manifolds. A *combinatorial n-ball* is a simplicial complex that is piecewise linear homeomorphic to the standard n-simplex. A *combinatorial $(n-1)$-sphere* is a simplicial complex that is piecewise linear homeomorphic to the boundary of the standard n-simplex. A *combinatorial n-manifold* (or simply a *combinatorial manifold*), for some positive integer n, is a simplicial complex, such that the link of every vertex is a combinatorial $(n-1)$-sphere. A *combinatorial n-manifold with boundary* (or simply a *combinatorial manifold with boundary*), for some positive integer n, is a simplicial complex, such that the link of every vertex is a combinatorial $(n-1)$-sphere or a combinatorial $(n-1)$-ball. The *boundary* of a combinatorial manifold with boundary is the subcomplex of simplices whose vertices have links that are combinatorial balls.

Manifolds can be considered in different categories based on their coordinate transformations. For instance, a manifold is *topological* if its coordinate transformations are continuous, *smooth* if its coordinate transformations are smooth, and *piecewise linear* (PL) if its coordinate transformations are piecewise linear. Clearly, combinatorial manifolds and PL manifolds correspond, thus we will use the notions of combinatorial manifold and PL manifold interchangeably.

II.2.2. Triangulations of smooth manifolds. A map from a simplicial complex K to a smooth manifold W is a *piecewise differentiable homeomorphism* (or simply a *PD homeomorphism*) if it is a homeomorphism and its restriction to every simplex of K is a smooth immersion. A *triangulation* of a smooth manifold W is a pair (K,φ), where K is a simplicial complex and $\varphi\colon K \to W$ is a PD homeomorphism. Clearly, triangulations are compatible with diffeomorphisms between smooth manifolds in the following sense: if $\psi\colon W \xrightarrow{\sim} W'$ is a diffeomorphism and $\varphi\colon K \to W$ is a triangulation of W, then $(K, \psi\circ\varphi)$ is a triangulation of W'.

It is a classical result, due to Cairns [Cai35] and Whitehead [Whi40], that any smooth manifold admits a triangulation. Moreover, if (K,φ) is a triangulation of a smooth manifold, then K is a combinatorial manifold.

In fact, the triangulation of a smooth manifold is unique up to PL homeomorphism. This induces a map from the set of diffeomorphism classes of smooth manifolds to the set of PL homeomorphism classes of combinatorial manifolds. In particular, this shows that one can define invariants of smooth manifolds using their triangulations.

However, two questions remain:

(1) Does every PL manifold admit a compatible smooth structure? In other words, is every combinatorial manifold the triangulation of a smooth manifold?
(2) Suppose two smooth manifolds W and W' have triangulations (K, φ) and (K, φ') with the same combinatorial manifold K. Are W and W' necessarily diffeomorphic?

The work of Moise [Moi52] answers both of these questions in the affirmative for manifolds of dimension less than 4. The work of Smale [Sma59], Munkres [Mun60], Hirsch [Hir63], Cerf [Cer68], and Hirsch–Mazur [HM74] shows that the answer to question (1) is yes in dimension less than 8 and the answer to question (2) is yes in dimension less than 7.

II.2.3. σ-manifolds. A *σ-manifold* is a pair (W, g) where W is a closed, oriented smooth 4-manifold and g is a homotopy class of maps from W to the classifying space $B\sigma$ of σ. An *equivalence* between two σ-manifolds (W, g) and (W', g') is an orientation preserving diffeomorphism $\varphi\colon W \xrightarrow{\sim} W'$ such that $g' \circ \varphi = g$. By an *invariant* of σ-manifolds, we mean a quantity that is invariant under this equivalence relation.

A *flat σ-bundle over a 4-manifold* is a pair $(W, \mathcal{P})$, where W is a closed, oriented smooth 4-manifold, and $\mathcal{P}$ is a flat σ-bundle over W. Two flat σ-bundles over 4-manifolds $(W, \mathcal{P})$ and $(W', \mathcal{P}')$ are *equivalent* if there is an orientation preserving diffeomorphism $\phi\colon W \xrightarrow{\sim} W'$ such that the flat σ-bundles $\mathcal{P}$ and $\phi^*(\mathcal{P}')$ over W are isomorphic (see Section II.1.6). By an *invariant* of flat σ-bundles over 4-manifolds, we mean a quantity that is invariant under this equivalence relation.

It follows from the canonical bijection in Theorem 3 that invariants of σ-manifolds and invariants of flat σ-bundles over 4-manifolds coincide.

II.2.4. Triangulations of σ-manifolds. A *canonical n-manifold* (or simply a *canonical manifold*), for some positive integer n, is a canonical simplicial complex (see Section I.2.2) whose underlying simplicial complex is a combinatorial n-manifold. A *canonical n-manifold with boundary* (or simply a *canonical manifold with boundary*), for some positive integer n, is a canonical simplicial complex whose underlying simplicial complex is a combinatorial n-manifold with boundary. An *orientation* of a canonical n-manifold K (with boundary) is a choice of orientation of every n-simplex of K, such that for every $(n-1)$-simplex of K (that isn't in the boundary), the orientations induced by the two adjacent n-simplices are opposite. We say that a canonical manifold (with boundary) is *oriented* if it is endowed with an orientation.

A *combinatorial σ-manifold* is a pair (K, ϕ), where K an oriented canonical 4-manifold, and ϕ is a σ-coloring of K. Two combinatorial σ-manifolds (K, ϕ) and (L, ψ) are *equivalent* if there is an orientation preserving PL homeomorphism $\varphi\colon K \xrightarrow{\sim} L$ such that $B(\psi) \circ \varphi$ and $B(\phi)$ are homotopic, where $B(\phi)$ and $B(\psi)$ are the geometric realizations of ϕ and ψ, respectively (see Section I.3.3). A *σ-triangulation* (or simply a *triangulation*) of a σ-manifold (W, g) is a combinatorial σ-manifold (K, ϕ), such that K is a triangulation of W and $B(\phi) \simeq g$. By Lemma I.3.3, every σ-manifold admits a triangulation.

II.3. Colored Pachner moves

In his seminal paper [Pac91], Pachner states a purely combinatorial theorem that plays a central role in low-dimensional topology. In dimension 4, it tells us that two closed 4-manifolds are diffeomorphic if and only if their triangulations are related by a finite sequence of Pachner moves.

Pachner moves (also called bistellar moves) are local changes in the triangulation of a simplicial complex. For triangulations of 4-manifolds, there are only 5 different types of Pachner moves. Therefore, in order to check that some quantity is a diffeomorphism invariant of closed 4-manifolds, it suffices to check that it doesn't change under the finite list of Pachner moves.

We are interested in invariants of σ-manifolds that we model via σ-triangulations (see Section II.2.3). In this section, we show a strengthening of Pachner's theorem:

Theorem 4 (Colored Pachner theorem)**.** *Two σ-manifolds are equivalent if and only if their σ-triangulations are related by a finite sequence of σ-colored Pachner moves.*

The aim of this section is to prove the colored Pachner theorem. We fix $\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega)$ a 2-crossed module of groups.

II.3.1. 4-dimensional Pachner moves. An *n-dimensional Pachner move* (or simply a *Pachner move*) replaces a combinatorial n-manifold K with a combinatorial manifold K' obtained by replacing a submanifold of K that is isomorphic to a subcomplex $I \subset \partial\Delta^{n+1}$ with the complementary subcomplex $J \subset \partial\Delta^{n+1}$. There are five types of 4-dimensional Pachner moves denoted by the cardinality of the 4-simplices in I and J: the Pachner $(1,5)$-move, the Pachner $(2,4)$-move, the Pachner $(3,3)$-move, and their inverses. Let $0,\dots,5$ denote the vertices of Δ^5. The following table lists the 4-simplices in I and J for the 4-dimensional Pachner moves:

	I	J
Pachner $(1,5)$-move	(01234)	(01235), (01245), (01345), (02345), (12345)
Pachner $(2,4)$-move	(01234), (12345)	(01235), (01245), (01345), (02345)
Pachner $(3,3)$-move	(01234), (01245), (02345)	(01235), (01345), (12345)

If the combinatorial manifold K is oriented, then we orient the 4-simplices in J such that the orientations of the 4-simplices outside of J stay unchanged. We refer the reader to [Lic99] for more details on Pachner moves.

II.3.2. Canonical Pachner moves. A *canonical Pachner move* on a canonical 4-manifold K is a Pachner move on the underlying combinatorial 4-manifold of K together with an arbitrary choice of distinguished vertex for every newly constructed triangle. Notice that any two canonical structures on a combinatorial 4-manifold are related by a finite sequence of canonical Pachner moves.

II.3.3. Colored Pachner moves. A *σ-colored Pachner move* (or simply a *colored Pachner move*) replaces a combinatorial σ-manifold (K,ϕ) with a combinatorial σ-manifold (L,ψ), where L is obtained by applying a canonical Pachner move to K, and ψ coincides with ϕ outside of the new simplices.

Let (K,ϕ) be a combinatorial σ-manifold, and let L be obtained by applying a canonical Pachner $(1,5)$-move to K. In other words, there is a 4-simplex (01234) in K such that L is obtained from K by replacing (01234) with the 4-simplices (01235), (01245), (01345), (02345), and (12345). Notice that all of the 0-, 1-, 2- and 3-simplices of K are also in L.

Lemma II.3.1. *If L is a canonical 4-manifold obtained by applying a canonical Pachner $(1,5)$-move to a canonical 4-manifold K with σ-coloring ϕ, then a σ-coloring ψ of L that coincides with ϕ on all pointed ordered 1-, 2-, and 3-simplices of K is uniquely specified by choosing:*

- *an element $h \in H$ such that $\psi(\langle 05\rangle) = h$,*
- *elements $e_1, e_2, e_3, e_4 \in E$ such that*

$$\psi(\langle 015\rangle) = e_1 \qquad \psi(\langle 025\rangle) = e_2$$
$$\psi(\langle 035\rangle) = e_3 \qquad \psi(\langle 045\rangle) = e_4$$

- *elements $l_1, l_2, l_3, l_4, l_5, l_6 \in L$ and choices of pointings and orders $\tau_1, \tau_2, \tau_3, \tau_4$, τ_5, and τ_6 on the simplices* (0125), (0135), (0145), (0235), (0245) *and* (0345), *respectively, such that $\psi(\tau_i) = l_i$ for all $i \in \{1,2,3,4,5\}$,*

with no compatibility conditions. Moreover, there is a one-to-one correspondence between the set of σ-colorings of L and the set

$$\{\sigma\text{-colorings of } K\} \times H \times E^4 \times L^6.$$

In particular, any σ-coloring of L induces a σ-coloring of K by restriction.

We give a proof of Lemma II.3.1 in Section II.3.5. The proofs of Lemmas II.3.2 and II.3.3 are very similar and are left to the reader.

Suppose K and L are canonical 4-manifolds such that L is obtained from K by applying a canonical Pachner $(2,4)$-move. In other words, there are two 4-simplices (01234) and (12345) in K such that L is obtained from K by replacing them with the 4-simplices (01235), (01245), (01345), and (02345). Notice that the only simplices of K that are not in L are (1234), (01234), and (12345).

Lemma II.3.2. *If L is a canonical 4-manifold obtained by applying a canonical Pachner $(2,4)$-move to a canonical 4-manifold K with σ-coloring ϕ, then a σ-coloring ψ of L that coincides with ϕ on all pointed ordered 1-, 2-, and 3-simplices of K is uniquely specified by choosing:*

- *an element $e \in E$ such that $\psi(\langle 015\rangle) = e$,*
- *elements $l_1, l_2, l_3 \in L$ and choices of pointings and orders τ_1, τ_2, and τ_3 on the simplices* (0125), (0135), *and* (0145), *respectively, such that $\psi(\tau_i) = l_i$ for $i \in \{1, 2, 3\}$,*

with no compatibility conditions. Moreover, there is a one-to-one correspondence between the set of σ-colorings of L and the set

$$\{\sigma\text{-colorings of } K\} \times E \times L^3.$$

In particular, any σ-coloring of L induces a σ-coloring of K by restriction.

Suppose K and L are canonical 4-manifolds such that L is obtained from K by applying a canonical Pachner $(3,3)$-move. In other words, there are three 4-simplices (01234), (01245), and (02345) in K such that L is obtained from K by replacing them with the 4-simplices (01235), (01345), and (12345).

Lemma II.3.3. *If L is a canonical 4-manifold obtained by applying a canonical Pachner $(3,3)$-move to a canonical 4-manifold K with σ-coloring ϕ, then a σ-coloring ψ of L that coincides with ϕ on K is uniquely specified by choosing an element $l \in L$ and a pointing and order τ on* (0135) *such that $\psi(\tau) = l$.*

II.3.4. Proof of the Colored Pachner Theorem. Let (K, ϕ) and (L, ψ) be combinatorial σ-manifolds. Our aim is to show that (K, ϕ) and (L, ψ) are equivalent if and only if (L, ψ) is obtained by applying a finite sequence of σ-colored Pachner moves to (K, ϕ). It is fairly easy to see that if (L, ψ) is obtained by applying a σ-colored Pachner move to (K, ϕ), then they are equivalent. Indeed, in that case there exists a PL homeomorphism $\varphi\colon K \to L$ such that the geometric realizations $B(\phi)$ and $B(\psi) \circ \varphi$ are homotopic because $\pi_4(B\sigma) = \{0\}$.

Conversely, let's show that if (K, ϕ) and (L, ψ) are equivalent, then (L, ψ) is obtained by applying a finite sequence of σ-colored Pachner moves to (K, ϕ). Notice that by the (classical) Pachner theorem, if (K, ϕ) and (L, ψ) are equivalent, then in particular L is obtained by applying a finite sequence of Pachner moves to K. Therefore, without loss of generality, we will suppose that $K = L$.

Moreover, the homotopy classification theorem (Theorem 1) tells us if (K, ϕ) and (K, ψ) are equivalent then ϕ and ψ are gauge-equivalent. Therefore, it suffices to show that if ϕ and ψ are gauge-equivalent then (K, ψ) is obtained by applying a finite sequence of σ-colored Pachner moves to (K, ϕ).

Lemma II.3.4. *Let (r, s, t) be a morphism in the gauge groupoid $\mathcal{G}(K, \sigma)$ (see Section I.4.2), and let B_* be a choice of ordered simplices of K (see Section I.2.7). There are non-negative integers k, l, m such that*

$$(r, s, t) = (r_1, 1, 1) \cdots (r_l, 1, 1) \cdot (1, s_1, 1) \cdots (1, s_m, 1) \cdot (1, 1, t_1) \cdots (1, 1, t_n)$$

where

- *for all $i \in \{1, \dots, l\}$, there exists a unique $x \in K_0$ such that $r_i(x) \neq 1$,*
- *for all $j \in \{1, \dots, m\}$, there exists a unique $b_1 \in B_1$ such that $s_j(b_1) \neq 1$,*
- *for all $k \in \{1, \dots, n\}$, there exists a unique $b_2 \in B_2$ such that $t_k(b_2) \neq 1$.*

Lemma II.3.4 may seem obvious at first sight, but is not at all immediate. In particular, contrary to what the notation might suggest, if $(1, s, 1)$ is a quadratic ϕ-derivation and $(1, s', 1)$ is a quadratic $(\phi \cdot (1, s, 1))$-derivation, we don't necessarily have

$$(1, s, 1) \cdot (1, s', 1) = (1, s \otimes s', 1).$$

Indeed, the composition of morphisms in $\mathcal{G}(K, \sigma)$ makes use of the connecting map $\omega^{(s,s')}$. Explicitly,

$$(1, s, 1) \cdot (1, s', 1) = (1, s \otimes s', 1 \otimes 1),$$

where $(1 \otimes 1)(b_2) = \omega^{(s,s')}(d_1(b_2))$ for all $b_2 \in B_2$.

Proof. Notice that for all $b_2 \in B_2$, we have $\omega^{(s,1)}(d_1(b_2)) = 1$. Therefore,

$$(r, s, t) = (r, 1, 1) \cdot (1, {}^{r^{-1}}s, 1) \cdot (1, 1, {}^{r^{-1}}t).$$

Moreover, for all $r, r'\colon K_0 \to H$ and $t, t'\colon B_2 \to L$, we have

$$(r, 1, 1) \cdot (r', 1, 1) = (rr', 1, 1) \quad \text{and} \quad (1, 1, t) \cdot (1, 1, t') = (1, 1, t''),$$

where t'' is defined in Lemma I.4.1. Finally, for all $s, s' \colon B_1 \to E$, we have

$$(1, s, 1) \cdot (1, s', 1) = (1, s \otimes s', 1) \cdot (1, 1, \omega^{(s,s')}(d_1)).$$

Therefore, we can iteratively apply the previous formulas to write (r, s, t) as a finite composition satisfying the required conditions. □

We call *generating quadratic derivation* a quadratic derivation that is nontrivial on a unique element b of B_*. Lemma II.3.4 tells us that any quadratic ϕ-derivation is the composition of generating quadratic derivations. We deduce that it suffices to show that if $\psi = \phi \cdot (r, s, t)$ with (r, s, t) a generating quadratic derivation, then (K, ψ) is obtained by applying a finite sequence of σ-colored Pachner moves to (K, ϕ).

Suppose $\psi = \phi \cdot (r, s, t)$ with (r, s, t) a generating quadratic derivation on $b \in B_*$. Without loss of generality, we can suppose that b has a vertex x whose link is $\partial\Delta^4$. Indeed, since K is a combinatorial 4-manifold, we know that the link of any vertex of K is PL homeomorphic to $\partial\Delta^4$. Therefore, by the classical Pachner Theorem, $\partial\Delta^4$ is obtained by applying a finite sequence of 3-dimensional Pachner moves to the link of x. In fact, these moves can be extended to a sequence of 4-dimensional Pachner moves denoted M such that the link of x in $M \cdot K$ is $\partial\Delta^4$, where $M \cdot K$ denotes the combinatorial manifold obtained by applying the moves M to K. If we arbitrarily extend the moves M to be σ-colored Pachner moves, our problem amounts to finding a finite sequence of σ-colored Pachner moves from $(M \cdot K, M \cdot \phi)$ to $(M \cdot K, M \cdot (\phi \cdot (r, s, t)))$, where $M \cdot \phi$ denotes the σ-coloring of $M \cdot K$ obtained by applying the moves M. We know that $M \cdot \phi$ and $M \cdot (\phi \cdot (r, s, t))$ are gauge equivalent, therefore there exists a quadratic $(M \cdot \phi)$-derivation (r', s', t') such that

$$(M \cdot \phi) \cdot (r', s', t') = M \cdot (\phi \cdot (r, s, t)),$$

and the maps r', s' and t' are trivial on all basis elements outside of the star of x. Moreover, (r', s', t') is a composition of generating quadratic derivations over simplices that all have a vertex whose link is $\partial\Delta^4$. Therefore, if we can find a finite sequence of σ-colored Pachner moves N such that $N \cdot (M \cdot \phi) = (M \cdot \phi) \cdot (r', s', t')$, we will have

$$M^{-1}NM \cdot \phi = M^{-1} \cdot ((M \cdot \phi) \cdot (r', s', t')) = M^{-1} \cdot (M \cdot (\phi \cdot (r, s, t))) = \phi \cdot (r, s, t).$$

Now suppose (r, s, t) is a generating quadratic ϕ-derivation on a simplex b that has a vertex x whose link is $\partial\Delta^4$. We denote the 3-simplices of $\partial\Delta^4$ as $\langle 1234\rangle$, $\langle 1235\rangle$, $\langle 1245\rangle$, $\langle 1345\rangle$, and $\langle 2345\rangle$. We give an explicit sequence of σ-colored Pachner moves going from (K, ϕ) to $(K, \phi \cdot (r, s, t))$. In the following, to simplify notation, we don't specify the distinguished vertices and pointings of simplices.

Case 1: $(r, s, t) = (r, 1, 1)$. In this case, $r(x') = 1$ for all $x' \in K_0$ such that $x' \neq x$.

(1) Apply the colored Pachner $(1, 5)$-move to $\langle x1234\rangle$. This yields the 4-simplices $\langle xy123\rangle$, $\langle xy124\rangle$, $\langle xy134\rangle$, $\langle xy234\rangle$, and $\langle y1234\rangle$ with σ-coloring defined by choosing $\phi\langle xy\rangle = r(x)$, $\phi\langle xyi\rangle = 1$ for all $i \in \{1, 2, 3, 4\}$, and $\phi\langle xyij\rangle = 1$ for all $i, j \in \{1, 2, 3, 4\}$ such that $i < j$.
(2) Apply the colored Pachner $(2, 4)$-move to $\langle x1235\rangle$ and $\langle xy123\rangle$. This yields the 4-simplices $\langle xy125\rangle$, $\langle xy135\rangle$, $\langle xy235\rangle$, and $y1235\rangle$ with σ-coloring defined by choosing $\phi\langle xy5\rangle = 1$ and $\phi\langle xyi5\rangle = 1$ for $i \in \{1, 2, 3\}$.
(3) Apply the colored Pachner $(3, 3)$-move to $\langle x1245\rangle$, $\langle xy124\rangle$, and $\langle xy125\rangle$. This yields the 4-simplices $\langle xy145\rangle$, $\langle xy245\rangle$, and $\langle y1245\rangle$ with σ-coloring defined by choosing $\phi\langle xy45\rangle = 1$.
(4) Apply the colored Pachner $(4, 2)$-move to $\langle xy134\rangle$, $\langle xy135\rangle$, $\langle xy145\rangle$, and $\langle x1345\rangle$. This yields the 4-simplices $\langle xy345\rangle$ and $\langle y1345\rangle$ with no new choices for the σ-coloring.
(5) Apply the colored Pachner $(5, 1)$-move to $\langle x2345\rangle$, $\langle xy234\rangle$, $\langle xy235\rangle$, $\langle xy245\rangle$, and $\langle xy345\rangle$. This yields the 4-simplex $\langle y2345\rangle$ with no new choices for the σ-coloring.

The resulting combinatorial σ-manifold is exactly $(K, \phi \cdot (r, 1, 1))$ (with the vertex x labeled y).

Case 2: $(r, s, t) = (1, s, 1)$. In this case, $s(b') = 1$ for all $b' \in B_1$ such that $b' \neq b$. We suppose for instance that $b = \langle x1\rangle$. We apply the same sequence of colored Pachner moves as is Case 1, with the σ-colorings determined by the following choices: $\phi\langle xy\rangle = 1$, $\phi\langle xy1\rangle = {}^{\phi\langle x1\rangle}s(\langle x1\rangle)^{-1}$, $\phi\langle xyi\rangle = 1$ for all $i \in \{2, 3, 4, 5\}$, and $\phi\langle xyij\rangle = 1$ for all $i, j \in \{1, 2, 3, 4, 5\}$ such that $i < j$. The resulting σ-colored combinatorial manifold is exactly $(K, \phi \cdot (1, s, 1))$ (with the vertex x labeled y).

Case 3: $(r, s, t) = (1, 1, t)$. In this case, $t(b') = 1$ for all $b' \in B_2$ such that $b' \neq b$. We suppose for instance that $b = \langle x12\rangle$. We apply the same sequence of colored Pachner moves as is Case 1, with the σ-colorings determined by the following choices: $\phi\langle xy\rangle = 1$, $\phi\langle xyi\rangle = 1$ for all $i \in \{1, 2, 3, 4, 5\}$,

$$\phi\langle xy12\rangle = \phi\langle x12\rangle \triangleright t(\langle x12\rangle) \quad \text{and} \quad \phi\langle xyij\rangle = 1$$

for all $i,j \in \{1,2,3,4,5\}$ such that $i<j$ and $(i,j)\neq(1,2)$. The resulting combinatorial σ-manifold is exactly $(K,\phi\cdot(1,1,t))$ (with the vertex x labeled y).

II.3.5. Proof of Lemma II.3.1. Let the canonical 4-manifold L be obtained by applying a Pachner $(1,5)$-move to a canonical 4-manifold K with σ-coloring c. Let's show that the map d specified above defines a unique σ-coloring of L.

First we show that the choices determine d. Recall that $d_{|K}=c$, therefore we only need to check that d is uniquely defined on simplices containing the vertex $\langle 5\rangle$. The triangle conditions applied to $\langle 015\rangle$, $\langle 025\rangle$, $\langle 035\rangle$, and $\langle 045\rangle$ determine the values of $d\langle 15\rangle$, $d\langle 25\rangle$, $d\langle 35\rangle$, and $d\langle 45\rangle$, respectively. Therefore d is uniquely defined on all edges of L. Similarly, the tetrahedron conditions applied to $\langle 0125\rangle$, $\langle 0135\rangle$, $\langle 0145\rangle$, $\langle 0235\rangle$, $\langle 0245\rangle$, and $\langle 0345\rangle$ determine the values of $d\langle 125\rangle$, $d\langle 135\rangle$, $d\langle 145\rangle$, $d\langle 235\rangle$, $d\langle 245\rangle$, and $d\langle 345\rangle$, respectively. Therefore, d is uniquely defined on all triangles of L. The 4-simplex conditions applied to $\langle 01235\rangle$, $\langle 01245\rangle$, $\langle 01345\rangle$, and $\langle 02345\rangle$ determine the values of $d\langle 1235\rangle$, $d\langle 1245\rangle$, $d\langle 1345\rangle$, and $d\langle 2345\rangle$, respectively. Therefore d is uniquely defined on all 1-, 2- and 3-simplices of L.

Now we check that the map d is a σ-coloring of L. Since $d_{|K}=c$ and c is a σ-coloring of K, d verifies the simplex conditions for all 2-, 3- and 4-simplices of K. By construction, d already verifies the simplex conditions for $\langle 015\rangle$, $\langle 025\rangle$, $\langle 035\rangle$, and $\langle 045\rangle$. For any choice of paths:

- γ_0 in $\langle 125\rangle$ from $\langle 1\rangle$ to the distinguished vertex of $\langle 125\rangle$,
- γ_1 in $\langle 015\rangle$ from $\langle 0\rangle$ to the distinguished vertex of $\langle 015\rangle$,
- γ_2 in $\langle 025\rangle$ from $\langle 0\rangle$ to the distinguished vertex of $\langle 025\rangle$,
- γ_3 in $\langle 012\rangle$ from $\langle 0\rangle$ to the distinguished vertex of $\langle 012\rangle$,

we get

$$\begin{aligned}\partial({}^{\gamma_0}d\langle 125\rangle) &= c\langle 01\rangle^{-1}\partial({}^{\gamma_1}e_1)^{-1}\partial({}^{\gamma_2}e_2)^{-1}\partial(c({}^{\gamma_3}\langle 012\rangle)c\langle 01\rangle\\ &= d\langle 15\rangle d\langle 25\rangle^{-1}d\langle 12\rangle^{-1}.\end{aligned}$$

Therefore d verifies the triangle condition for $\langle 125\rangle$. Similar computations show that d verifies the triangle conditions for $\langle 135\rangle$, $\langle 145\rangle$, $\langle 235\rangle$, $\langle 245\rangle$, and $\langle 345\rangle$. This proves that d verifies the triangle conditions for all triangles of L.

By construction, d already verifies the tetrahedron conditions for $\langle 0125\rangle$, $\langle 0135\rangle$, $\langle 0145\rangle$, $\langle 0235\rangle$, $\langle 0245\rangle$, $\langle 0245\rangle$, and $\langle 0345\rangle$. Let $(\langle 01235\rangle,\bar\gamma)$ be a pointed ordered 4-simplex, with

$$\bar\gamma=(\gamma_{01},\gamma_{02},\gamma_{03},\gamma_{04},\gamma_{12},\gamma_{13},\gamma_{14},\gamma_{23},\gamma_{24},\gamma_{34}).$$

For vertices $i,j,k,l\in\{0,1,2,3,5\}$, we denote by $(\langle ijkl\rangle,\bar\gamma)$ the pointed ordered tetrahedron $\langle ijkl\rangle$ induced by those of $(\langle 01235\rangle,\bar\gamma)$. By definition,

$$\begin{aligned}d^{\langle 01\rangle}(\langle 1235\rangle,\bar\gamma) = d({}^{\gamma_{23}}\langle 015\rangle)^{-1}\triangleright\Big[&d(\langle 0125\rangle,\bar\gamma)d({}^{\gamma_{13}}\langle 025\rangle)\triangleright\omega(d({}^{\gamma_{35}}\langle 012\rangle),d({}^{\langle 01\rangle\langle 12\rangle\gamma_{01}}\langle 235\rangle))d(\langle 0235\rangle,\bar\gamma)\\ &d({}^{\gamma_{12}}\langle 035\rangle)\triangleright d(\langle 0123\rangle,\bar\gamma)^{-1}d(\langle 0135\rangle,\bar\gamma)^{-1}\Big].\end{aligned}$$

Therefore,

$$\begin{aligned}\delta(d^{\langle 01\rangle}(\langle 1235\rangle,\bar\gamma) &= d({}^{\gamma_{23}}\langle 015\rangle)^{-1}\delta d(\langle 0125\rangle,\bar\gamma)d({}^{\gamma_{13}}\langle 025\rangle)\delta\omega(d({}^{\gamma_{35}}\langle 012\rangle),d({}^{\langle 01\rangle\langle 12\rangle\gamma_{01}}\langle 235\rangle))d({}^{\gamma_{13}}\langle 025\rangle)^{-1}\\ &\quad\delta d(\langle 0235\rangle,\bar\gamma)d({}^{\gamma_{12}}\langle 035\rangle)\delta d(\langle 0123\rangle,\bar\gamma)^{-1}d({}^{\gamma_{12}}\langle 035\rangle)^{-1}\delta d(\langle 0135\rangle,\bar\gamma)^{-1}d({}^{\gamma_{23}}\langle 015\rangle)\\ &= d({}^{\langle 01\rangle\gamma_{03}}\langle 125\rangle)d({}^{\langle 01\rangle\langle 12\rangle\gamma_{01}}\langle 235\rangle)d({}^{\langle 01\rangle\gamma_{05}}\langle 123\rangle)^{-1}d({}^{\langle 01\rangle\gamma_{02}}\langle 135\rangle)^{-1}.\end{aligned}$$

Similarly, d verifies the tetrahedron conditions for $\langle 1245\rangle$, $\langle 1345\rangle$, and $\langle 2345\rangle$.

By construction, d verifies the 4-simplex conditions for $\langle 01235\rangle$, $\langle 01245\rangle$, $\langle 01345\rangle$, and $\langle 02345\rangle$. It only remains to check that d verifies the 4-simplex condition for $\langle 12345\rangle$. Let

$$\bar\gamma=(\gamma_{01},\gamma_{02},\gamma_{03},\gamma_{04},\gamma_{12},\gamma_{13},\gamma_{14},\gamma_{23},\gamma_{24},\gamma_{34})$$

be a choice of pointing such that $(\langle 12345\rangle,\bar\gamma)$ is a pointed ordered 4-simplex. For ease of reading, we introduce the following notations:

- for any vertices $i,j\in\{0,1,2,3,4,5\}$ such that $i<j$, we write $[ij]=d\langle ij\rangle$,
- for any vertices $i,j,k,l,m\in\{0,1,2,3,4,5\}$ such that $i<j<k$ and $l<m$, we write

$$[ijk]={}^{d(\gamma_{lm})}d\langle ijk\rangle,$$

- for any vertices $i, j, k, l \in \{0, 1, 2, 3, 4, 5\}$ such that $i < j < k < l$, we write

$$[ijkl] = d(\langle ijkl\rangle, \bar{\gamma}),$$

where $(\langle ijkl\rangle, \bar{\gamma})$ denotes the ordered tetrahedron $\langle ijkl\rangle$ with pointing induced by $\bar{\gamma}$.

The 4-simplex condition amounts to $E = 1$, with

$$E = [1235][135] \triangleright \omega([123], {}^{[12][23]}[345])[1345][145] \triangleright [1234]^{-1}[1245]^{-1}[125] \triangleright {}^{[12]}[2345]^{-1}.$$

By construction,

$$\begin{aligned}
{}^{[01]}[1235] &= [015]^{-1} \triangleright \left([0125][025] \triangleright \omega([012], {}^{[01][12]}[235])[0235][035] \triangleright [0123]^{-1}[0135]^{-1}\right),\\
{}^{[01]}[1245] &= [015]^{-1} \triangleright \left([0125][025] \triangleright \omega([012], {}^{[01][12]}[245])[0245][045] \triangleright [0124]^{-1}[0145]^{-1}\right),\\
{}^{[01]}[1345] &= [015]^{-1} \triangleright \left([0135][035] \triangleright \omega([013], {}^{[01][13]}[345])[0345][045] \triangleright [0134]^{-1}[0145]^{-1}\right),\\
{}^{[01]}[2345] &= [025]^{-1} \triangleright \left([0235][035] \triangleright \omega([023], {}^{[02][23]}[345])[0345][045] \triangleright [0234]^{-1}[0245]^{-1}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
E = {}^{[01]^{-1}}[015]^{-1} \triangleright {}^{[01]^{-1}}\Big([0125][025] \triangleright \omega([012], {}^{[01][12]}[235])[0235][035] \triangleright [0123]^{-1}[0135]^{-1}\\
([015]\,{}^{[01]}[135]) \triangleright {}^{[01]}\omega\left([123], {}^{[12][23]}[345]\right)[0135][035] \triangleright \omega\left([013], {}^{[01][13]}[345]\right)[0345][045] \triangleright [0134]^{-1}\\
[0145]^{-1}([015]\,{}^{[01]}[145]) \triangleright {}^{[01]}[1234]^{-1}[0145][045] \triangleright [0124][0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}\\
[0125]^{-1}([015]\,{}^{[01]}[125]\,{}^{[01][12][02]^{-1}}[025]^{-1}) \triangleright {}^{[01][12][02]^{-1}}\Big([0245][045] \triangleright [0234][0345]^{-1}\\
[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1}[0235]^{-1}\Big)\Big).
\end{aligned}$$

Recall that $\delta[0135] = [015]\,{}^{[01]}[135][013]^{-1}[035]^{-1}$, therefore

$$[015]\,{}^{[01]}[135] = \delta[0135][035][013].$$

Similarly,

$$\begin{aligned}
[015]\,{}^{[01]}[145] &= \delta[0145][045][014],\\
[015]\,{}^{[01]}[125] &= \delta[0125][025][012].
\end{aligned}$$

Using the fact that for all $l, m \in L$, $\delta(l) \triangleright m = lml^{-1}$, we get

$$\begin{aligned}
E = {}^{[01]^{-1}}[015]^{-1} \triangleright {}^{[01]^{-1}}\Big([0125][025] \triangleright \omega\left([012], {}^{[01][12]}[235]\right)[0235][035] \triangleright [0123]^{-1}\\
([035][013]) \triangleright {}^{[01]}\omega\left([123], {}^{[12][23]}[345]\right)[035] \triangleright \omega\left([013], {}^{[01][13]}[345]\right)[0345][045] \triangleright [0134]^{-1}\\
([045][014]) \triangleright {}^{[01]}[1234]^{-1}[045] \triangleright [0124][0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}\\
([025][012]\,{}^{\partial[012]^{-1}}[025]^{-1}) \triangleright {}^{\partial[012]^{-1}}\Big([0245][045] \triangleright [0234][0345]^{-1}[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1}\\
[0235]^{-1}\Big)[0125]^{-1}\Big).
\end{aligned}$$

The equation $E = 1$ is equivalent to $E_1 = 1$ with

$$\begin{aligned} E_1 = {} & [025] \triangleright \omega\left([012], {}^{[01][12]}[235]\right)[0235][035] \triangleright [0123]^{-1}([035][013]) \triangleright {}^{[01]}\omega\left([123], {}^{[12][23]}[345]\right) \\ & [035] \triangleright \omega\left([013], {}^{[01][13]}[345]\right)[0345][045] \triangleright [0134]^{-1}([045][014]) \triangleright {}^{[01]}[1234]^{-1}[045] \triangleright [0124] \\ & [0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}([025][012]^{\,\partial[012]^{-1}}[025]^{-1}) \triangleright {}^{\partial[012]^{-1}}\Big([0245][045] \triangleright [0234] \\ & [0345]^{-1}[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1}[0235]^{-1}\Big) \end{aligned}$$

Recall that for all $e, f \in E$, we have $\delta(\omega(e,f)) = efe^{-1}\,{}^{\partial e}f^{-1}$. Therefore,

$$[025][012]^{\,\partial[012]^{-1}}[025]^{-1} = [012]\delta\omega\left([012]^{-1}, [025]\right).$$

Moreover, recall from [FM11, Lemma 2.2] that for all $e \in E$ and $l \in L$, we have

$${}^{\partial e}l = (e \triangleright l)\,\omega(e, \delta(l^{-1})).$$

Therefore,

$$\begin{aligned} & ([025][012]^{\,\partial[012]^{-1}}[025]^{-1}) \triangleright {}^{\partial[012]^{-1}}\left([0245][045] \triangleright [0234][0345]^{-1}[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1}[0235]^{-1}\right) \\ & \quad = [012] \triangleright \Big(\omega([012]^{-1}, [025])^{\,\partial[012]^{-1}}\left(0245][045] \triangleright [0234][0345]^{-1}[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1}[0235]^{-1}\right) \\ & \qquad \omega([012]^{-1}, [025])^{-1}\Big) \\ & \quad = [012] \triangleright \omega([012]^{-1}, [025])[0245][045] \triangleright [0234][0345]^{-1}[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1}[0235]^{-1} \\ & \qquad [012] \triangleright \omega\left([012]^{-1}, [025]\,{}^{[02]}[235]\,{}^{[02][23]}[345]\,{}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\right)[012] \triangleright \omega([012]^{-1}, [025])^{-1} \end{aligned}$$

We substitute in E_1:

$$\begin{aligned} E_1 = {} & [025] \triangleright \omega\left([012], {}^{[01][12]}[235]\right)[0235][035] \triangleright [0123]^{-1}([035][013]) \triangleright {}^{[01]}\omega\left([123], {}^{[12][23]}[345]\right) \\ & [035] \triangleright \omega\left([013], {}^{[01][13]}[345]\right)[0345][045] \triangleright [0134]^{-1}([045][014]) \triangleright {}^{[01]}[1234]^{-1}[045] \triangleright [0124] \\ & [0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}[012] \triangleright \omega([012]^{-1}, [025])[0245][045] \triangleright [0234][0345]^{-1} \\ & [035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1}[0235]^{-1}[012] \triangleright \omega\Big([012]^{-1}, [025]\,{}^{[02]}[235]\,{}^{[02][23]}[345]\,{}^{[02]}[234]^{-1} \\ & {}^{[02]}[245]^{-1}[025]^{-1}\Big)[012] \triangleright \omega([012]^{-1}, [025])^{-1} \end{aligned}$$

From the fact that c is a σ-coloring of K, we deduce that

$$1 = [0124][024] \triangleright \omega\left([012], {}^{[01][12]}[234]\right)[0234][034] \triangleright [0123]^{-1}[0134]^{1}[014] \triangleright {}^{[01]}[1234]^{-1}.$$

Therefore,

$$\begin{aligned} [045] \triangleright [0134]^{-1}([045][014]) \triangleright {}^{[01]}[1234]^{-1}[045] \triangleright [0124] = {} & ([045][034]) \triangleright [0123][045] \triangleright [0234]^{-1} \\ & ([045][024]) \triangleright \omega\left([012], {}^{[01][12]}[234]\right)^{-1}. \end{aligned}$$

We substitute in E_1:

$$\begin{aligned}E_1 = [025] \triangleright \omega\left([012], {}^{[01][12]}[235]\right)[0235][035] \triangleright [0123]^{-1}([035][013]) \triangleright {}^{[01]}\omega\left([123], {}^{[12][23]}[345]\right)\\ [035] \triangleright \omega\left([013], {}^{[01][13]}[345]\right)[0345]([045][034]) \triangleright [0123][045] \triangleright [0234]^{-1}\\ ([045][024]) \triangleright \omega\left([012], {}^{[01][12]}[234]\right)^{-1}[0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}\\ [012] \triangleright \omega([012]^{-1}, [025])[0245][045] \triangleright [0234][0345]^{-1}[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1}[0235]^{-1}\\ [012] \triangleright \omega\Big([012]^{-1}, [025]\,{}^{[02]}[235]\,{}^{[02][23]}[345]\,{}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big)[012] \triangleright \omega([012]^{-1}, [025])^{-1}.\end{aligned}$$

Recall that for all $l, m \in L$, we have $lm = (\delta(l) \triangleright m)l$, therefore

$$[0345]([045][034]) \triangleright [0123] = ([035]\,{}^{[03]}[345]) \triangleright [0123][0345].$$

This yields

$$\begin{aligned}E_1 = [025] \triangleright \omega\left([012], {}^{[01][12]}[235]\right)[0235][035] \triangleright [0123]^{-1}([035][013]) \triangleright {}^{[01]}\omega\left([123], {}^{[12][23]}[345]\right)\\ [035] \triangleright \omega\left([013], {}^{[01][13]}[345]\right)([035]\,{}^{[03]}[345]) \triangleright [0123][0345][045] \triangleright [0234]^{-1}\\ ([045][024]) \triangleright \omega\left([012], {}^{[01][12]}[234]\right)^{-1}[0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}\\ [012] \triangleright \omega([012]^{-1}, [025])[0245][045] \triangleright [0234][0345]^{-1}[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1}[0235]^{-1}\\ [012] \triangleright \omega\Big([012]^{-1}, [025]\,{}^{[02]}[235]\,{}^{[02][23]}[345]\,{}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big)[012] \triangleright \omega([012]^{-1}, [025])^{-1}.\end{aligned}$$

Recall from [FM11, Lemma 2.2] that for all $e, f, g \in E$ we have $\omega(e, fg) = \big((efe^{-1}) \triangleright \omega(e, g)\big)\,\omega(e, f)$, therefore

$$\begin{aligned}\omega\Big([012]^{-1}, [025]\,{}^{[02]}[235]\,{}^{[02][23]}[345]\,{}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big) = [012]^{-1}[025][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[235]\\ {}^{[02][23]}[345]\,{}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big)\omega([012]^{-1}, [025]).\end{aligned}$$

This yields

$$\begin{aligned}E_1 = [025] \triangleright \omega\left([012], {}^{[01][12]}[235]\right)[0235][035] \triangleright [0123]^{-1}([035][013]) \triangleright {}^{[01]}\omega\left([123], {}^{[12][23]}[345]\right)\\ [035] \triangleright \omega\left([013], {}^{[01][13]}[345]\right)([035]\,{}^{[03]}[345]) \triangleright [0123][0345][045] \triangleright [0234]^{-1}\\ ([045][024]) \triangleright \omega\left([012], {}^{[01][12]}[234]\right)^{-1}[0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}\\ [012] \triangleright \omega([012]^{-1}, [025])[0245][045] \triangleright [0234][0345]^{-1}[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1}[0235]^{-1}\\ ([025][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[235]\,{}^{[02][23]}[345]\,{}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big)\end{aligned}$$

Recall that by definition,

$${}^{[03]}[345] \triangleright [0123] = [0123]\omega\Big([023][012]\,{}^{[01]}[123]^{-1}[013]^{-1}, {}^{[03]}[345]\Big).$$

Therefore,

$$\begin{aligned}E_1 = {} & [025] \triangleright \omega\left([012], {}^{[01][12]}[235]\right)[0235][035] \triangleright [0123]^{-1}([035][013]) \triangleright {}^{[01]}\omega\left([123], {}^{[12][23]}[345]\right) \\ & [035] \triangleright \omega\left([013], {}^{[01][13]}[345]\right)[035] \triangleright [0123][035] \triangleright \omega\Big([023][012]\,{}^{[01]}[123]^{-1}[013]^{-1}, {}^{[03]}[345]\Big) \\ & [0345][045] \triangleright [0234]^{-1}([045][024]) \triangleright \omega\left([012], {}^{[01][12]}[234]\right)^{-1}[0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1} \\ & [012] \triangleright \omega([012]^{-1}, [025])[0245][045] \triangleright [0234][0345]^{-1}[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1}[0235]^{-1} \\ & ([025][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[235]\,{}^{[02][23]}[345]\,{}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big)\end{aligned}$$

Recall that for all $e, f, g \in E$, we have $\omega(e, fg) = \big((efe^{-1}) \triangleright \omega(e, g)\big)\,\omega(e, f)$, therefore

$$\begin{aligned}\omega\Big([012]^{-1}, {}^{[02]}[235]\,{}^{[02][23]}[345]\,{}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big) = {} & ([012]^{-1}\,{}^{[02]}[235][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02][23]}[345] \\ & {}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big)\omega\Big([012]^{-1}, {}^{[02]}[235]\Big).\end{aligned}$$

Moreover, recall from [FM11, Lemma 2.2] that for all $e, f \in E$, we have $\omega(e, f) = e \triangleright \omega(e^{-1}, {}^{\partial e} f)^{-1}$, therefore,

$$\omega\Big([012]^{-1}, {}^{[02]}[235]\Big) = [012]^{-1} \triangleright \omega\Big([012], {}^{[01][12]}[235]\Big)^{-1}.$$

By substituting, we get that $E_1 = 1$ is equivalent to $E_2 = 1$ with

$$\begin{aligned}E_2 = {} & [0235][035] \triangleright [0123]^{-1}([035][013]) \triangleright {}^{[01]}\omega\left([123], {}^{[12][23]}[345]\right)[035] \triangleright \omega\left([013], {}^{[01][13]}[345]\right) \\ & [035] \triangleright [0123][035] \triangleright \omega\Big([023][012]\,{}^{[01]}[123]^{-1}[013]^{-1}, {}^{[03]}[345]\Big)[0345][045] \triangleright [0234]^{-1} \\ & ([045][024]) \triangleright \omega\left([012], {}^{[01][12]}[234]\right)^{-1}[0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}[012] \triangleright \omega([012]^{-1}, [025]) \\ & [0245][045] \triangleright [0234][0345]^{-1}[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1}[0235]^{-1} \\ & ([025]\,{}^{[02]}[235][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02][23]}[345]\,{}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big).\end{aligned}$$

Recall that for all $l, m \in L$, we have $lm = (\delta(l) \triangleright m)l$, therefore

$$\begin{aligned}[0235]^{-1}([025]\,{}^{[02]}[235][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02][23]}[345]\,{}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big) & \\ = ([035][023][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02][23]}[345]\,{}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big)[0234]^{-1}.\end{aligned}$$

By substituting, we get that $E_2 = 1$ is equivalent to $E_3 = 1$ with

$$\begin{aligned}E_3 = {} & [035] \triangleright [0123]^{-1}([035][013]) \triangleright {}^{[01]}\omega\left([123], {}^{[12][23]}[345]\right)[035] \triangleright \omega\left([013], {}^{[01][13]}[345]\right) \\ & [035] \triangleright [0123][035] \triangleright \omega\Big([023][012]\,{}^{[01]}[123]^{-1}[013]^{-1}, {}^{[03]}[345]\Big)[0345][045] \triangleright [0234]^{-1} \\ & ([045][024]) \triangleright \omega\left([012], {}^{[01][12]}[234]\right)^{-1}[0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}[012] \triangleright \omega([012]^{-1}, [025]) \\ & [0245][045] \triangleright [0234][0345]^{-1}[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1} \\ & ([035][023][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02][23]}[345]\,{}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big).\end{aligned}$$

After applying the formula $lm = (\delta(l) \triangleright m)l$ with $l = [035] \triangleright [0123]^{-1}$ and

$$m = ([035][013]) \triangleright {}^{[01]}\omega\left([123], {}^{[12][23]}[345]\right)[035] \triangleright \omega\left([013], {}^{[01][13]}[345]\right),$$

we get

$$E_3 = ([035][023][012]^{[01]}[123]^{-1}) \triangleright^{[01]} \omega\left([123], {}^{[12][23]}[345]\right) ([035][023][012]^{[01]}[123]^{-1} [013]^{-1}) \triangleright \omega\left([013], {}^{[01][13]}[345]\right) [035] \triangleright \omega\Big([023][012]^{[01]}[123]^{-1}[013]^{-1}, {}^{[03]}[345]\Big)[0345][045] \triangleright [0234]^{-1} ([045][024]) \triangleright \omega\left([012], {}^{[01][12]}[234]\right)^{-1} [0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1} [012] \triangleright \omega([012]^{-1}, [025])[0245] [045] \triangleright [0234][0345]^{-1}[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1} ([035][023][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02][23]}[345]\, {}^{[02]}[234]^{-1}\, {}^{[02]}[245]^{-1}[025]^{-1}\Big).$$

Recall from [FM11, Lemma 2.2] that for all $e, f, g \in E$, we have $\omega(ef, g) = (e \triangleright \omega(f, g))\, \omega(e, {}^{\partial f} g)$ and that for all $e, f \in E$, we have $\omega(e, f) = e \triangleright \omega(e^{-1}, {}^{\partial e} f)^{-1}$, therefore

$$\omega\Big([023][012]^{[01]}[123]^{-1}[013]^{-1}, {}^{[03]}[345]\Big) = ([023][012]^{[01]}[123]^{-1}[013]^{-1}) \triangleright \omega\Big([013], {}^{[01][13]}[345]\Big)^{-1} \omega\Big([023][012]^{[01]}[123]^{-1}, {}^{[01][13]}[345]\Big).$$

This yields

$$E_3 = ([035][023][012]^{[01]}[123]^{-1}) \triangleright^{[01]} \omega\left([123], {}^{[12][23]}[345]\right) [035] \triangleright \omega\Big([023][012]^{[01]}[123]^{-1}, {}^{[03]}[345]\Big) [0345][045] \triangleright [0234]^{-1}([045][024]) \triangleright \omega\left([012], {}^{[01][12]}[234]\right)^{-1} [0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1} [012] \triangleright \omega([012]^{-1}, [025])[0245][045] \triangleright [0234][0345]^{-1}[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1} ([035][023][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02][23]}[345]\, {}^{[02]}[234]^{-1}\, {}^{[02]}[245]^{-1}[025]^{-1}\Big).$$

Similarly, we have

$$\omega\Big([023][012]^{[01]}[123]^{-1}, {}^{[03]}[345]\Big) = ([023][012]^{[01]}[123]^{-1}) \triangleright \omega\Big({}^{[01]}[123], {}^{[01][13]}[345]\Big)^{-1} \omega\Big([023][012], {}^{[01][12][23]}[345]\Big).$$

By substituting, we get

$$E_3 = [035] \triangleright \omega\Big([023][012], {}^{[01][12][23]}[345]\Big)[0345][045] \triangleright [0234]^{-1}([045][024]) \triangleright \omega\left([012], {}^{[01][12]}[234]\right)^{-1} [0245]^{-1} [025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1} [012] \triangleright \omega([012]^{-1}, [025])[0245][045] \triangleright [0234][0345]^{-1} [035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1} ([035][023][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02][23]}[345]\, {}^{[02]}[234]^{-1}\, {}^{[02]}[245]^{-1}[025]^{-1}\Big).$$

Recall that for all $e, f, g \in E$, we have $\omega(ef, g) = (e \triangleright \omega(f, g))\, \omega(e, {}^{\partial f} g)$, therefore

$$\omega\Big([023][012], {}^{[01][12][23]}[345]\Big) = [023] \triangleright \omega\Big([012], {}^{[01][12][23]}[345]\Big)\omega\Big([023], {}^{[02][23]}[345]\Big).$$

Recall that for all $e, f, g \in E$, we have $\omega(e, fg) = \big((efe^{-1}) \triangleright \omega(e, g)\big)\, \omega(e, f)$, therefore

$$\omega\Big([012]^{-1}, {}^{[02][23]}[345]\, {}^{[02]}[234]^{-1}\, {}^{[02]}[245]^{-1}[025]^{-1}\Big) = ([012]^{-1}\, {}^{[02][23]}[345][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[234]^{-1}\, {}^{[02]}[245]^{-1}[025]^{-1}\Big)\omega\Big([012]^{-1}, {}^{[02][23]}[345]\Big).$$

Recall that for all $e, f \in E$, we have $\omega(e, f) = e \triangleright \omega(e^{-1}, {}^{\partial e} f)^{-1}$, therefore

$$\omega\Big([012]^{-1}, {}^{[02][23]}[345]\Big) = [012]^{-1} \triangleright \omega\Big([012], {}^{[01][12][23]}[345]\Big)^{-1}.$$

By substituting the above formulas, we get that $E_3 = 1$ is equivalent to $E_4 = 1$ with

$$E_4 = [035] \triangleright \omega\Big([023], {}^{[02][23]}[345]\Big)[0345][045] \triangleright [0234]^{-1}([045][024]) \triangleright \omega\left([012], {}^{[01][12]}[234]\right)^{-1}[0245]^{-1}$$
$$[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}[012] \triangleright \omega([012]^{-1}, [025])[0245][045] \triangleright [0234][0345]^{-1}$$
$$[035] \triangleright \omega\left([023], {}^{[02][23]}[345]\right)^{-1}([035][023]\,{}^{[02][23]}[345][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big).$$

We apply the formula $lm = (\delta(l) \triangleright m)l$ with

$$l = [045] \triangleright [0234][0345]^{-1}[035] \triangleright \omega\Big([023], {}^{[02][23]}[345]\Big)^{-1}$$
$$m = ([035][023]\,{}^{[02][23]}[345][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big).$$

This yields that $E_4 = 1$ is equivalent to $E_5 = 1$ with

$$E_5 = ([045][024]) \triangleright \omega\left([012], {}^{[01][12]}[234]\right)^{-1}[0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}[012] \triangleright \omega([012]^{-1}, [025])$$
$$[0245]([045][024]\,{}^{[02]}[234][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big).$$

Recall that for all $e, f, g \in E$, we have $\omega(e, fg) = \big((efe^{-1}) \triangleright \omega(e, g)\big)\,\omega(e, f)$, therefore

$$\omega\Big([012]^{-1}, {}^{[02]}[234]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}\Big) = ([012]^{-1}\,{}^{[02]}[234]^{-1}[012]) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[245]^{-1}[025]^{-1}\Big)$$
$$\omega\Big([012]^{-1}, {}^{[02]}[234]^{-1}\Big).$$

Recall that for all $e, f \in E$, we have $\omega(e, f) = (efe^{-1}) \triangleright \omega(e, f^{-1})^{-1}$, therefore

$$\omega\Big([012]^{-1}, {}^{[02]}[234]^{-1}\Big) = ([012]^{-1}\,{}^{[02]}[234]^{-1}[012]) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[234]\Big)^{-1}.$$

Recall that for all $e, f \in E$, we have $\omega(e, f)^{-1} = e \triangleright \omega(e^{-1}, {}^{\partial e}f)$, therefore

$$\omega\Big([012]^{-1}, {}^{[02]}[234]\Big)^{-1} = [012]^{-1} \triangleright \omega\Big([012], {}^{[02]}[234]\Big).$$

By substituting, we get

$$E_5 = ([045][024]) \triangleright \omega\left([012], {}^{[01][12]}[234]\right)^{-1}[0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}[012] \triangleright \omega([012]^{-1}, [025])$$
$$[0245]([045][024][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[245]^{-1}[025]^{-1}\Big)([045][024]) \triangleright \omega\Big([012], {}^{[02]}[234]\Big).$$

Therefore $E_5 = 1$ is equivalent to $E_6 = 1$ with

$$E_6 = [0245]^{-1}[025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}[012] \triangleright \omega([012]^{-1}, [025])$$
$$[0245]([045][024][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[245]^{-1}[025]^{-1}\Big).$$

By applying the formula $lm = (\delta(l) \triangleright m)l$ with

$$l = [0245]$$
$$m = ([045][024][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[245]^{-1}[025]^{-1}\Big),$$

we get that $E_6 = 1$ is equivalent to $E_7 = 1$ with

$$E_7 = [025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1}[012] \triangleright \omega([012]^{-1}, [025])([025]\,{}^{[02]}[245][012]) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[245]^{-1}[025]^{-1}\Big).$$

Recall that for all $e, f, g \in E$, we have $\omega(e, fg) = \big((efe^{-1}) \triangleright \omega(e, g)\big)\,\omega(e, f)$, therefore

$$\omega\Big([012]^{-1}, {}^{[02]}[245]^{-1}[025]^{-1}\Big) = ([012]^{-1}\,{}^{[02]}[245]^{-1}[025]^{-1}[012]) \triangleright \omega\Big([012]^{-1}, [025]^{-1}\Big)\omega\Big([012]^{-1}, {}^{[02]}[245]^{-1}\Big).$$

Recall that for all $e, f \in E$, we have $\omega(e,f) = (efe^{-1}) \triangleright \omega(e, f^{-1})^{-1}$, therefore

$$\omega\Big([012]^{-1}, {}^{[02]}[245]^{-1}[025]^{-1}\Big) = ([012]^{-1\ [02]}[245]^{-1}[025]^{-1}[012]) \triangleright \omega\Big([012]^{-1}, [025]^{-1}\Big)$$
$$([012]^{-1\ [02]}[245]^{-1}[025]^{-1}) \triangleright \omega\Big([012]^{-1}, {}^{[02]}[245]\Big)^{-1}.$$

Recall that for all $e, f \in E$, we have $\omega(e,f)^{-1} = e \triangleright \omega(e^{-1}, {}^{\partial e} f)$, therefore

$$\omega\Big([012]^{-1}, {}^{[02]}[245]^{-1}[025]^{-1}\Big) = ([012]^{-1\ [02]}[245]^{-1}[025]^{-1}[012]) \triangleright \omega\Big([012]^{-1}, [025]\Big)^{-1}$$
$$([012]^{-1\ [02]}[245]^{-1}) \triangleright \omega\Big([012], {}^{[01][12]}[245]\Big).$$

By substituting, we get

$$\begin{aligned} E_7 = {} & [025] \triangleright \omega\left([012], {}^{[01][12]}[245]\right)^{-1} [012] \triangleright \omega\Big([012]^{-1}, [025]\Big)[012] \triangleright \omega\Big([012]^{-1}, [025]\Big)^{-1} \\ & [025] \triangleright \omega\Big([012], {}^{[01][12]}[245]\Big) \\ = {} & 1. \end{aligned}$$

Therefore the 4-simplex condition is verified for $\langle 12345 \rangle$, and d is a σ-coloring of L.

CHAPTER III

Monoidal 2-categories graded by a 3-group

Fusion categories are finite semisimple monoidal categories with duals and a simple unit. They can be seen as generalizations of finite groups. Indeed, over $\mathbb{C}$, the finite dimensional representations of a finite group form a fusion category. Quantum groups at roots of unity also provide many examples of fusion categories. Fusion categories serve as the algebraic input in the Turaev–Viro–Barrett–Westbury state-sum TQFT [TV92, BW96] and by the cobordism hypothesis they yield framed fully extended 3-dimensional TQFTs [DSS20]. We refer the reader to [EGNO15, TV17] for more details on fusion categories.

Fusion 2-categories were introduced by Douglas and Reutter [DR18] as a categorification of the notion of fusion category. They correspond to finite semisimple monoidal 2-categories with duals and a simple unit. These structures have quickly found numerous applications in the study of braided fusion categories [JR23] and of 4-dimensional TQFTs [DR18, Déc24a, Déc25]. Moreover, fusion 2-categories have been used to describe non-invertible symmetries in a variety of relevant physical theories, see [BBFP24, BP24, DT24, IO24].

In the 1-categorical setting, Sözer and Virelizier introduced [SV23] the notion of 2-group extensions of fusion categories, where 2-groups are modeled by crossed modules. For any crossed module $(\chi\colon E \to H)$, a χ-graded monoidal category is a monoidal category whose objects and morphisms have degrees in H and E, respectively. This grading is defined such that the monoidal product of objects is compatible with the product in H, the composition of morphisms is compatible with the product in E, and the monoidal product of morphisms is compatible with the action of H on E. A χ-fusion category is then a χ-graded monoidal category satisfying appropriate finiteness and semisimplicity conditions. While a χ-fusion category C need not be a fusion category itself, its neutral component (consisting of objects and morphisms all of degree 1) is a fusion category. Thus, χ-fusion categories can be viewed as χ-graded extensions of fusion categories. Using these structures, and generalizing the Turaev–Viro–Barrett–Westbury state-sum construction, Sözer and Virelizier derived state-sum invariants for pairs (M, g), where M is a closed, oriented 3-manifold and g is a homotopy class of maps from M to the classifying space of χ. Furthermore, they extended these invariants to 3-manifolds with boundary, and showed that they fit into a 3-dimensional HQFT (that is, roughly speaking, a 3-dimensional TQFT for cobordisms equipped with a homotopy class of maps to the classifying space of χ).

Let σ be a 2-crossed module of groups. The primary goal of this thesis is to construct invariants of flat σ-bundles over 4-manifolds, or equivalently (via Theorem 3) to construct invariants of σ-manifolds. Recall that a σ-manifold is a pair (W, g), where W is a closed oriented smooth 4-manifold and g is a homotopy class of maps from W to the classifying space $B\sigma$. This raises both the dimension and the homotopical complexity of the target space compared to the invariants defined by Sözer and Virelizier. To achieve this, we adapt the state-sum construction of Douglas and Reutter [DR18], making it sensitive to homotopy classes of maps. The aim of this chapter is to introduce the relevant algebraic data to carry out this construction. We define the notions of σ-graded monoidal 2-category and of σ-fusion 2-category. Throughout this chapter, $\Bbbk$ is a nonzero commutative ring.

In Section III.1, we recall the definition of monoidal 2-categories. In Section III.2, we recall the definition of linear 2-categories. Section III.3 is devoted to the definition of σ-graded monoidal 2-categories. In Section III.4, we define pivotal and spherical σ-graded 2-categories. Section III.5 is devoted to the definition of σ-fusion 2-categories. Section III.6 studies dimensions (of objects, 1-morphisms, Hom-categories...) in spherical σ-fusion 2-categories.

III.1. Monoidal 2-categories

The structure of what we now call a (strict) 2-category was first noted by Godement [God58]. He shows that natural transformations between functors can be composed in two distinct ways—either vertically (along a common functor) or horizontally (across composed functors)—and that these two compositions satisfy an

interchange law. A 2-category is a structure that formalizes this idea that 2-dimensional morphisms can be composed in two different directions and that these compositions should be compatible.

The notion of bicategory, due to Bénabou [Bén67], generalizes that of 2-category by allowing the composition of 1-morphisms to be unital and associative up to a coherent choice of isomorphisms. While this structure is less rigid than that of a 2-category and better describes many mathematical phenomena, it can sometimes become awkward to work with the coherence isomorphisms of a bicategory. Nevertheless, the work of Gordon, Power, and Street [GPS95] shows that every bicategory is equivalent to a 2-category. In practice, one can "act as if" the associator and unitor isomorphisms are identities, provided the diagram is only built from these coherence isomorphisms.

This situation breaks down when bicategories are equipped with a monoidal or tensor product: there are (weakly) monoidal bicategories that are not equivalent to strict monoidal 2-categories. We can still look for weaker versions of strictification where only certain coherence isomorphisms are strictified. This leads us to the notion of (semistrict) monoidal 2-category or Gray-monoid. Gordon, Power and Street [GPS95] show that every monoidal bicategory is equivalent to a monoidal 2-category, and that there is no stricter version of a monoidal bicategory with this property. Therefore, in order to simplify notation and without loss of generality, we work in the setting of monoidal 2-categories. Monoidal 2-categories have the added benefit that they have a particularly nice diagrammatic calculus, as shown by the work of Barrett, Meusburger, and Schaumann [BMS24]. In fact, the coherence data for monoidal 2-categories is precisely the part of the coherence data for monoidal bicategories that can be given a diagrammatic meaning.

In this section, we recall the definition of a monoidal 2-category. We refer the reader to the work of Johnson and Yau [JY20] for the full definition of a monoidal bicategory.

III.1.1. Notation. For any objects X, Y in a category C, we denote by $\mathrm{Hom}_C(X, Y)$ the set of morphisms from X to Y, and we denote by 1_X the identity of X.

III.1.2. 2-categories. A *2-category* is a category enriched over **Cat**, the category of small categories and functors with its cartesian product. More explicitly, a 2-category $\mathcal{C}$ consists of

- a class of objects $\mathrm{Ob}(\mathcal{C})$,
- for all objects X and Y, a *Hom-category* $\mathcal{C}(X, Y)$. Its objects $f, g : X \to Y$ are called *1-morphisms* and its morphisms $\alpha : f \Rightarrow g$ are called *2-morphisms*. The composition of 2-morphisms in $\mathcal{C}(X, Y)$ is called *vertical composition* and is denoted by $\cdot$,
- for any object X, a functor $\mathbf{1} \to \mathcal{C}(X, X)$, where $\mathbf{1}$ is the terminal category with one object and one morphism. This functor picks an *identity 1-morphism* denoted 1_X,
- for all objects X, Y, and Z, a *horizontal composition* functor:

$$\circ : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z),$$

  that is strictly associative and unital (with respect to the identity 1-morphisms).

Requiring that horizontal composition in a 2-category be a functor implies the *interchange law*:

$$(\beta' \circ \alpha') \cdot (\beta \circ \alpha) = (\beta' \cdot \beta) \circ (\alpha' \cdot \alpha),$$

for all 2-morphisms α, α', β, and β' such that this composition is well defined.

In a 2-category, we will usually denote objects with uppercase letters (X, Y, $Z \ldots$), 1-morphisms with lowercase letters (f, g, $h \ldots$), and 2-morphisms with lowercase Greek letters (α, β, $\gamma \ldots$).

III.1.3. Examples. 1) Strict monoidal categories are canonically identified with one-object 2-categories. We refer the reader to [TV17] for a detailed presentation of monoidal categories. Moreover, if $\mathcal{C}$ is a 2-category and X is an object of $\mathcal{C}$, then $\mathcal{C}(X, X)$ is a strict monoidal category.

2) Any crossed module $(\chi : E \to H -$ of groupoids defines a 2-category $\mathcal{G}_\chi$ as follows. The objects of $\mathcal{G}_\chi$ are given by the set H_0. For all objects $x, y \in H_0$, the category $\mathcal{G}_\chi(x, y)$ has objects given by the set $H(x, y)$ and morphisms

$$\mathrm{Hom}_{\mathcal{G}_\chi(x,y)}(h, k) = \{e \in E(x) \mid \chi(e)h = k\}$$

for all $h, k \colon x \to y$. The horizontal composition of 1-morphisms $h \colon x \to y$, $k \colon y \to z$ is given by their composition $k \circ h = hk$ in H. The vertical composition of 2-morphisms $e \colon h \Rightarrow k$, $f \colon k \Rightarrow l$ is given by their composition $f \cdot e = fe$ in E, and the horizontal composition of 2-morphisms is given by $f \circ e = e^{h} f \in E(x)$ for all $e \in \mathrm{Hom}_{\mathcal{G}_\chi(x,y)}(h, k)$ and $f \in \mathrm{Hom}_{\mathcal{G}_\chi(y,z)}(l, m)$.

III.1.4. Adjunctions and equivalences. Let $\mathcal{C}$ be a (strict) 2-category. An *adjunction* $f \dashv g$ in $\mathcal{C}$ is a quadruple $(f, g, \eta, \varepsilon)$ where $f\colon X \to Y$ is a *left adjoint* 1-morphism, $g\colon Y \to X$ is a *right adjoint* 1-morphism, $\eta\colon 1_X \Rightarrow g \circ f$ is a *unit* 2-morphism, and $\varepsilon\colon f \circ g \Rightarrow 1_Y$ is a *counit* 2-morphism, such that

$$(\varepsilon \circ 1_f) \cdot (1_f \circ \eta) = 1_f \quad \text{and} \quad (1_g \circ \varepsilon) \cdot (\eta \circ 1_g) = 1_g.$$

An *equivalence* in $\mathcal{C}$ is a 1-morphism $f : X \to Y$, such that there exists a 1-morphism $g : Y \to X$ and 2-isomorphisms $\eta : 1_X \xRightarrow{\sim} g \circ f$ and $\varepsilon : f \circ g \xRightarrow{\sim} 1_Y$. An adjunction $(f, g, \eta, \varepsilon)$ is called an *adjoint equivalence* if η and ε are 2-isomorphisms.

III.1.5. 2-functors. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. A *2-functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of

- an object $F(X)$ of $\mathcal{D}$ for every object X of $\mathcal{C}$,
- a functor of Hom-categories $F_{X,Y}\colon \mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$ for every pair of objects X, Y of $\mathcal{C}$,

such that the diagrams

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{1} & \mathcal{C}(X,X) \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle F_{X,X}} \\ & & \mathcal{D}(F(X),F(X)) \end{array} \qquad \begin{array}{ccc} \mathcal{C}(Y,Z)\times\mathcal{C}(X,Y) & \xrightarrow{\circ} & \mathcal{C}(X,Z) \\ \downarrow{\scriptstyle F_{Y,Z}\times F_{X,Y}} & & \downarrow{\scriptstyle F_{X,Z}} \\ \mathcal{D}(F(Y),F(Z))\times\mathcal{D}(F(X),F(Y)) & \xrightarrow{\circ} & \mathcal{D}(F(X),F(Z)) \end{array}$$

are commutative for all objects X,Y,Z of $\mathcal{C}$.

III.1.6. Monoidal 2-categories. The notion of monoidal 2-category (or Gray monoid) first appeared in the work of Gordon, Power, and Street [GPS95]. The definition we give here is taken from Douglas and Reutter [DR18]. A *monoidal 2-category* consists of:

- a 2-category $\mathcal{C}$,
- an *identity object* $\mathbb{1}$ in $\mathcal{C}$,
- *left* and *right tensor product* 2-functors
$$L_X \equiv X \otimes - : \mathcal{C} \to \mathcal{C}, \qquad R_X \equiv - \otimes X : \mathcal{C} \to \mathcal{C}$$
for every object X of $\mathcal{C}$
- *interchange* 2-isomorphisms
$$\phi_{f,g} : (f \otimes Y') \circ (X \otimes g) \xRightarrow{\sim} (X' \otimes g) \circ (f \otimes Y)$$
for all 1-morphisms $f\colon X \to X'$, $g\colon Y \to Y'$,

satisfying the following conditions:

- the left and right tensor products agree
$$L_X(Y) = X \otimes Y = R_Y(X),$$
are strictly unital
$$L_{\mathbb{1}} = \mathrm{Id}_{\mathcal{C}} = R_{\mathbb{1}},$$
and associative
$$L_X L_Y = L_{X\otimes Y}, \qquad R_Y R_X = R_{X\otimes Y}, \qquad L_X R_Y = R_Y L_X,$$
for all objects X and Y,
- the interchange 2-isomorphisms respect identities
$$\phi_{g,1_X} = 1_{g\otimes X}, \quad \phi_{1_X,g} = 1_{X\otimes g},$$
composition
$$\phi_{f'\circ f,g} = (\phi_{f',g} \circ (f \otimes Y)) \cdot ((f' \otimes Y') \circ \phi_{f,g}),$$
$$\phi_{f,g'\circ g} = ((X' \otimes g') \circ \phi_{f,g}) \cdot (\phi_{f,g'} \circ (X \otimes f)),$$
and tensor product
$$\phi_{X\otimes g,h} = X \otimes \phi_{g,h}, \quad \phi_{f\otimes Y,h} = \phi_{f,Y\otimes h}, \quad \phi_{f,g\otimes Z} = \phi_{f,g} \otimes Z,$$
for all 1-morphisms $f : X \to X'$, $f' : X' \to X''$, $g : Y \to Y'$, and $g' : Y' \to Y''$,

- the interchange 2-isomorphisms are natural

$$\phi_{f',g} \cdot ((\alpha \otimes Y') \circ (X \otimes g)) = ((X' \otimes g) \circ (\alpha \otimes Y) \cdot \phi_{f,g},$$
$$\phi_{f,g'} \cdot ((f \otimes Y') \circ (X \otimes \beta)) = ((X' \otimes \beta) \circ (f \otimes Y)) \cdot \phi_{f,g},$$

for all 1-morphisms $f, f' : X \to X'$ and $g, g' : Y \to Y'$, and 2-morphisms $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$.

Monoidal 2-categories are canonically identified with one-object Gray categories, see [GPS95].

III.1.7. Example. Any 2-crossed module of groups

$$\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega)$$

induces a monoidal 2-category $\widetilde{\mathcal{G}}_\sigma$ with set of objects the group H, with 1-morphisms

$$\widetilde{\mathcal{G}}_\sigma(h,k) = \{e \in E \mid \partial(e) = kh^{-1}\}$$

for any objects h and k, and with 2-morphisms

$$\mathrm{Hom}_{\widetilde{\mathcal{G}}_\sigma(h,k)}(e,f) = \{l \in L \mid \delta(l) = fe^{-1}\}$$

for all $e, f \colon h \to k$. The tensor product of two objects of $\widetilde{\mathcal{G}}_\sigma$ is given by their product in H, the composition of 1-morphisms $e \colon h \to k$ and $f \colon k \to l$ is given by their product $fe \colon h \to l$ in E, and the (vertical) composition of 2-morphisms $l \colon e \Rightarrow f$ and $m \colon f \Rightarrow g$ is given by their product $ml \colon e \Rightarrow g$ in L. The interchange 2-isomorphisms are given by $\phi_{e,f} = \omega(e, {}^h f)^{-1}$, for all $e \colon h \to k$ and $f \colon l \to m$.

III.2. Linear 2-categories

Recall that $\Bbbk$ is a nonzero commutative ring. In this section we recall the definitions of linear 2-categories, of direct sums, and of zero objects.

III.2.1. Linear categories. A category C is $\Bbbk$-*linear* (or simply *linear*) if its morphism sets carry the structure of $\Bbbk$-modules such that the composition of morphisms is bilinear. A *linear functor* $F \colon C \to D$ between $\Bbbk$-linear categories is a functor such that for all objects X, Y of C, the induced map

$$\mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_D(F(X), F(Y))$$

is a morphism of $\Bbbk$-modules. A $\Bbbk$-linear category C is $\Bbbk$-*additive* (or simply *additive*) if any finite (possibly empty) family of objects has a direct sum in C. A *zero object* in a linear category C is an object 0 that is both initial and terminal. Any additive category has a zero object that is unique up to isomorphism. An object X in an additive category C is a zero object if and only if $\mathrm{Hom}_C(X,X)$ is the zero $\Bbbk$-module.

III.2.2. Linear 2-categories. A 2-category is $\Bbbk$-*linear* (or simply *linear*) if its 2-morphism sets are $\Bbbk$-modules such that horizontal and vertical composition of 2-morphisms is bilinear. A *linear 2-functor* between linear 2-categories is a 2-functor that is linear at the level of Hom-categories and such that the horizontal composition is bilinear.

A *linear monoidal 2-category* is a monoidal 2-category whose underlying 2-category is linear such that the left and right tensor products are linear.

III.2.3. Direct sums in linear 2-categories. An object Z in a linear 2-category $\mathcal{C}$ is a *direct sum* of a finite family $\{X_i\}_{i\in I}$ of objects of $\mathcal{C}$ if there are *projection* 1-morphisms $\{p_i \colon Z \to X_i\}_{i\in I}$, *inclusion* 1-morphisms $\{q_i \colon X_i \to Z\}_{i\in I}$, and 2-isomorphisms $\{\varphi \colon p_i \circ q_i \xrightarrow{\sim} 1_{X_i}\}_{i\in I}$ such that $p_i \circ q_j$ is a zero 1-morphism for all $i \neq j \in I$ and the direct sum $\bigoplus_{i\in I}(q_i \circ p_i)$ exists and is isomorphic to 1_Z in $\mathcal{C}(Z,Z)$.

III.2.4. Zero objects and zero 1-morphisms. Let $\mathcal{C}$ be a linear 2-category. A *zero 1-morphism* in $\mathcal{C}$ is a 1-morphism $0 \colon X \to Y$ that is a zero object in the linear category $\mathcal{C}(X,Y)$. A *zero object* in $\mathcal{C}$ is an object 0 such that for every object Y of $\mathcal{C}$, the categories $\mathcal{C}(0,Y)$ and $\mathcal{C}(Y,0)$ are terminal. Notice that an object X of $\mathcal{C}$ is a zero object if and only if 1_X is a zero 1-morphism.

III.3. Graded monoidal 2-categories

The notion of monoidal category graded by a crossed module was introduced by Sözer and Virelizier [SV23, SV25] in order to construct 3-dimensional HQFTs with target a homotopy 2-type. We categorify their notion to obtain monoidal 2-categories graded by 2-crossed modules. Recall that $\Bbbk$ is a nonzero commutative ring.

III.3.1. Categories graded by a set. Let E be a set. An *E-graded category* (over $\Bbbk$) is a $\Bbbk$-additive category C endowed with a family $\{C_e\}_{e\in E}$ of full $\Bbbk$-additive subcategories such that every object of C is a finite direct sum of objects in $\bigcup_{e\in E} C_e$, and

$$\mathrm{Hom}_C(X,Y) = \{0\}$$

for all $X \in C_e$ and $Y \in C_f$ such that $e \neq f \in E$. In that case, we write

$$C = \bigoplus_{e\in E} C_e.$$

An object X of C is *homogeneous of degree e* if $X \in C_e$ for some $e \in E$. When X is nonzero, such an e is uniquely determined, denoted $|X|$, and is called the *degree* of X. If E is a group, the category C_1 is called the *neutral component* of C.

III.3.2. Hom-graded categories. In this section, we recall some definitions related to Hom-graded categories and we refer the reader to [SV23] for more details. Let L be a group. An *L-Hom-graded category* (over $\Bbbk$) is a category enriched over the monoidal category of L-graded $\Bbbk$-modules and $\Bbbk$-linear grading-preserving homomorphisms. In particular, in such a category C, the set of morphisms between two objects X,Y is an L-graded $\Bbbk$-module:

$$\mathrm{Hom}_C(X,Y) = \bigoplus_{l\in L} \mathrm{Hom}_C^l(X,Y).$$

The morphisms in $\mathrm{Hom}_C^l(X,Y)$ are *homogeneous of degree l*. The 1-*subcategory* of an L-Hom-graded category C is the category C^1 with the same objects as C and with morphisms

$$\mathrm{Hom}_{C^1}(X,Y) = \mathrm{Hom}_C^1(X,Y),$$

for all objects X and Y.

Let C be an L-Hom-graded category, and let $l \in L$. An object Z of C is an *l-direct sum* of a finite family $\{X_i\}_{i\in I}$ of objects of C if there is a family $\{p_i\colon Z \to X_i, q_i\colon X_i \to Z\}_{i\in I}$ of morphisms such that p_i is homogeneous of degree l^{-1}, q_i is homogeneous of degree l,

$$1_Z = \sum_{i\in I} q_i \circ p_i, \quad \text{and} \quad 1_{X_i} = p_i \circ q_i,$$

for all $i \in I$, and $p_i \circ q_j = 0$ for all $i \neq j \in I$. We denote such an l-direct sum by

$$Z = \bigoplus_{i\in I}^{l} X_i.$$

Such an l-direct sum, if it exists, is unique up to a homogeneous isomorphism of degree 1. An L-Hom-graded category is *L-additive* if any finite (possibly empty) family of objects has an l-direct sum for all $l \in L$.

III.3.3. Hom-graded 2-categories. Let $(\delta\colon L \to E)$ be a crossed module of groups with the action of E on L denoted by $\triangleright$. A *δ-Hom-graded 2-category* (over $\Bbbk$) is a $\Bbbk$-linear 2-category $\mathcal{C}$ such that the category $\mathcal{C}(X,Y)$ is L-Hom-graded for all objects X and Y, the 1-subcategory $\mathcal{C}(X,Y)^1$ of $\mathcal{C}(X,Y)$ is E-graded for all objects X and Y, and $\mathcal{C}$ satisfies the following conditions, where a 1-morphism $f\colon X \to Y$ is said to be *homogeneous* if it is homogeneous in $\mathcal{C}(X,Y)^1$:

- for all homogeneous 1-morphisms $f, g\colon X \to Y$ and for all $l \in L$ such that $\delta(l)|f| \neq |g|$,
$$\mathrm{Hom}^l_{\mathcal{C}(X,Y)}(f,g) = \{0\},$$
- if $f\colon X \to Y$ and $g\colon Y \to Z$ are homogeneous 1-morphisms, then the composite $g \circ f$ is homogeneous of degree $|g \circ f| = |g||f|$,
- all identity 1-morphisms are homogeneous of degree 1,
- the horizontal composition of 2-morphisms is grading preserving:
$$\circ\colon \mathrm{Hom}^l_{\mathcal{C}(Y,Z)}(h,i) \times \mathrm{Hom}^k_{\mathcal{C}(X,Y)}(f,g) \to \mathrm{Hom}^{l(|f|\triangleright k)}_{\mathcal{C}(X,Z)}(h \circ f, i \circ g)$$
for all objects X, Y, Z, for all $f, g\colon X \to Y$, $h, i\colon Y \to Z$, and $l, k \in L$, whenever f is homogeneous.

Notice that a one-object δ-Hom-graded 2-category is a δ-graded monoidal category in the sense of Sözer and Virelizier [SV23].

III.3.4. Equivalences in Hom-graded 2-categories. Let $(\delta\colon L \to E)$ be a crossed module of groups with the action of E on L denoted by $\triangleright$, and let $(l,e) \in L \times E$. An (l,e)-*equivalence* in a δ-Hom-graded 2-category $\mathcal{C}$ is an equivalence (f,g,η,ε) such that:

- $f\colon X \to Y$ is a homogeneous 1-morphism of degree e,
- $g\colon Y \to X$ is a homogeneous 1-morphism of degree $\delta(l)e^{-1}$,
- $\eta\colon 1_X \Rightarrow g \circ f$ is a homogeneous 2-isomorphism of degree l,
- $\varepsilon\colon f \circ g \Rightarrow 1_Y$ is a homogeneous 2-isomorphism of degree $e \triangleright l^{-1}$.

III.3.5. Direct sums in Hom-graded 2-categories. Let $(\delta\colon L \to E)$ be a crossed module of groups with the action of E on L denoted by $\triangleright$, and let $(l,e) \in L \times E$. An object Z of a δ-Hom-graded 2-category $\mathcal{C}$ is an (l,e)-*direct sum* of a finite family of objects $\{X_i\}_{i\in I}$ if there is a family $\{p_i, q_i, \varphi_i\}_{i\in I}$ such that for all $i,j \in I$:

- $p_i\colon Z \to X_i$ is a homogeneous 1-morphism of degree $e^{-1}\delta(l)^{-1}$,
- $q_i\colon X_i \to Z$ is a homogeneous 1-morphism of degree e,
- $\varphi_i\colon p_i \circ q_i \xRightarrow{\sim} 1_{X_i}$ is a homogeneous 2-isomorphism of degree $e^{-1} \triangleright l$,
- $p_i \circ q_j$ is a zero 1-morphism when $i \neq j$, and

$$1_Z = \bigoplus_{i\in I}^{l} (q_i \circ p_i),$$

in the L-Hom-graded category $\mathcal{C}(Z,Z)$. Such an (l,e)-direct sum, if it exists, is denoted by

$$Z = \boxplus_{i\in I}^{(l,e)} X_i.$$

If in addition $\mathcal{C}$ is locally L-additive (meaning that each Hom-category $\mathcal{C}(X,Y)$ is L-additive, see Section III.3.2) then such an (l,e)-direct sum is unique up to $(1,1)$-equivalence.

III.3.6. Subcategories of Hom-graded 2-categories. Let $(\delta\colon L \to E)$ be a crossed module of groups. The 1-*subcategory* of a δ-Hom-graded 2-category $\mathcal{C}$ is the 2-category $\mathcal{C}^1$ whose objects are the objects of $\mathcal{C}$ and such that $\mathcal{C}^1(X,Y) = \mathcal{C}(X,Y)^1$ is the 1-subcategory of $\mathcal{C}(X,Y)$ for all objects X and Y.

The $(1,1)$-*subcategory* of a δ-Hom-graded 2-category $\mathcal{C}$ is the 2-category $\mathcal{C}_1^1$ whose objects are the objects of $\mathcal{C}$ and such that $\mathcal{C}_1^1(X,Y) = \mathcal{C}^1(X,Y)_1$ is the neutral component of the E-graded category $\mathcal{C}^1(X,Y)$ for all objects X and Y.

III.3.7. Monoidal 2-categories graded by a group. Let $\mathcal{C}$ be a $\Bbbk$-linear monoidal 2-category. A *full subcategory* of $\mathcal{C}$ is a $\Bbbk$-linear 2-category $\mathcal{D}$ such that every object of $\mathcal{D}$ is an object of $\mathcal{C}$ and $\mathcal{D}(X,Y) = \mathcal{C}(X,Y)$ for all objects X and Y of $\mathcal{D}$, where the compositions in $\mathcal{D}$ are induced by the compositions in $\mathcal{C}$.

Let H be a group. An *H-graded 2-category* (over $\Bbbk$) is a $\Bbbk$-linear monoidal 2-category $\mathcal{C}$ endowed with a family $\{{}_h\mathcal{C}\}_{h\in H}$ of full subcategories such that

- every object of $\mathcal{C}$ is a direct sum of a finite family of objects in $\bigcup_{h\in H} {}_h\mathcal{C}$,
- if $X \in {}_h\mathcal{C}$ and $Y \in {}_k\mathcal{C}$ with $h \neq k \in H$, then $\mathcal{C}(X,Y)$ is the terminal category,
- if $X \in {}_h\mathcal{C}$ and $Y \in {}_k\mathcal{C}$, then $X \otimes Y \in {}_{hk}\mathcal{C}$,
- $\mathbb{1} \in {}_1\mathcal{C}$.

The monoidal 2-category ${}_1\mathcal{C}$ is called the *neutral component* of $\mathcal{C}$. We say that an object $X \in \mathcal{C}$ is *homogeneous* if X is nonzero and $X \in {}_h\mathcal{C}$ for some $h \in H$. Such an h is uniquely determined by X, denoted $|X|$, and is called the *degree* of X.

III.3.8. Monoidal 2-categories graded by a 2-crossed module. Let

$$\sigma = (L \xrightarrow{\ \delta\ } E \xrightarrow{\ \partial\ } H,\ \omega)$$

be a 2-crossed module of groups. A *σ-graded monoidal 2-category* (or simply a *σ-graded 2-category*) is a $\Bbbk$-linear monoidal 2-category $\mathcal{C}$ that is δ-Hom-graded, whose $(1,1)$-subcategory $\mathcal{C}_1^1$ is H-graded, and that satisfies the following conditions, where an object X of $\mathcal{C}$ is said to be *homogeneous* if it is homogeneous in $\mathcal{C}_1^1$:

- if X is a homogeneous object and f is a homogeneous 1-morphism, then $X \otimes f$ is homogeneous of degree ${}^{|X|}|f|$, and $f \otimes X$ is homogeneous of degree $|f|$;
- if X is a homogeneous object and α is a homogeneous 2-morphism, then $X \otimes \alpha$ is homogeneous of degree ${}^{|X|}|\alpha|$, and $\alpha \otimes X$ is homogeneous of degree $|\alpha|$;

- if $f\colon X \to X'$ and $g\colon Y \to Y'$ are homogeneous 1-morphisms and X is a homogeneous object, then the interchange 2-isomorphism

$$\phi_{f,g}\colon (f \otimes Y') \circ (X \otimes g) \Rightarrow (X' \otimes g) \circ (f \otimes Y)$$

  is homogeneous of degree $\omega(|f|, {}^{|X|}|g|)^{-1}$;
- if $f\colon X \to Y$ is a nonzero homogeneous 1-morphism, then $\partial(|f|)|X| = |Y|$ whenever X and Y are homogeneous objects.

III.3.9. Subcategories of graded 2-categories. Let σ be a 2-crossed module of groups. The *neutral component* of a σ-graded 2-category $\mathcal{C}$ is the $\Bbbk$-linear monoidal 2-category ${}_1\mathcal{C}_1^1$ whose objects are the objects of the neutral component of $\mathcal{C}_1^1$ and their $(1,1)$-direct sums in $\mathcal{C}$ (see Section III.3.6), and such that

$${}_1\mathcal{C}_1^1(X,Y) = \mathcal{C}_1^1(X,Y)$$

for all objects X and Y of ${}_1\mathcal{C}_1^1$.

III.3.10. Example. Any 2-crossed module of groups

$$\sigma = (L \xrightarrow{\ \delta\ } E \xrightarrow{\ \partial\ } H,\ \omega)$$

induces a $\Bbbk$-linear monoidal 2-category $\widehat{\mathcal{G}}_\sigma$ with set of objects the group H, with 1-morphisms

$$\widehat{\mathcal{G}}_\sigma(h,k) = \{e \in E \mid \partial(e) = kh^{-1}\}$$

for any objects h and k, and with 2-morphisms

$$\mathrm{Hom}_{\widehat{\mathcal{G}}_\sigma(h,k)}(e,f) = \Bbbk\{l \in L \mid \delta(l) = fe^{-1}\}$$

for any objects h,k and for all $e,f\colon h \to k$. The tensor product of two objects of $\widehat{\mathcal{G}}_\sigma$ is given by their product in H, the composition of 1-morphisms $e\colon h \to k$ and $f\colon k \to l$ is given by their product $fe\colon h \to l$ in E, and the (vertical) composition of 2-morphisms $l\colon e \Rightarrow f$ and $m\colon f \Rightarrow g$ is given by their product $ml\colon e \Rightarrow g$ in L. Let $\mathcal{G}_\sigma$ denote the locally additive completion of $\widehat{\mathcal{G}}_\sigma$. In other words, $\mathcal{G}_\sigma$ has the same objects as $\widehat{\mathcal{G}}_\sigma$, the 1-morphisms of $\mathcal{G}_\sigma$ from an object h to an object k are given by finite (possibly empty) sequences of 1-morphisms in $\widehat{\mathcal{G}}_\sigma(h,k)$, and the 2-morphisms of $\mathcal{G}_\sigma$ are matrices of 2-morphisms of $\widehat{\mathcal{G}}_\sigma$. We call $\mathcal{G}_\sigma$ the *linearization* of σ. The linearization of σ is a σ-graded 2-category.

III.3.11. Example. Let

$$\sigma = (L \xrightarrow{\ \delta\ } E \xrightarrow{\ \partial\ } H,\ \omega) \quad \text{and} \quad \sigma' = (L' \xrightarrow{\ \delta'\ } E' \xrightarrow{\ \partial'\ } H',\ \omega')$$

be 2-crossed modules of groups, let $\mathcal{C}$ be a σ-graded 2-category, and let

$$\phi = (\phi_1\colon H \to H', \phi_2\colon E \to E', \phi_3\colon L \to L')$$

be a morphism of 2-crossed modules from σ to σ'. The *pushforward* of $\mathcal{C}$ along ϕ is the σ'-graded 2-category $\phi_*(\mathcal{C})$ with the same underlying linear monoidal 2-category as $\mathcal{C}$ and with σ'-grading defined as follows. For all $l' \in L'$ and for all 1-morphisms $f,g\colon X \to Y$:

$$\mathrm{Hom}^{l'}_{\phi_*(\mathcal{C})(X,Y)}(f,g) = \bigoplus_{l \in \phi_3^{-1}(l')} \mathrm{Hom}^{l}_{\mathcal{C}(X,Y)}(f,g).$$

For all $e' \in E'$, the degree-e' homogeneous 1-morphisms in $\phi_*(\mathcal{C})$ are the homogeneous 1-morphisms in $\mathcal{C}$ of degree $e \in \phi_2^{-1}(e')$ and their 1-direct sums. For all $h' \in H'$, the degree-h' homogeneous objects in $\phi_*(\mathcal{C})$ are the homogeneous objects in $\mathcal{C}$ of degree $h \in \phi_1^{-1}(h')$.

III.3.12. Example. For any crossed module $(\chi\colon E \to H)$, Gainutdinov, Runkel, and Wang [GRW26] define a notion of χ-crossed braided monoidal category. A χ-crossed braided monoidal category is an E-graded monoidal category $C = \bigoplus_{e\in E} C_e$ together with an H-action on C such that $h \in H$ maps C_e to $C_{({}^h e)}$, and a χ-crossed braiding

$$\{c_{U,V}\colon U \otimes V \to (\chi(|U|)\cdot V)\otimes U\}_{U,V\in C},$$

satisfying compatibility conditions. To any χ-crossed braided category C we associate a σ-graded 2-category $\mathcal{C}$, with

$$\sigma = (1 \longrightarrow E \xrightarrow{\chi} H,\ 1).$$

The objects of $\mathcal{C}$ are the elements of H, and

$$\mathcal{C}(h,k) = \bigoplus_{e\in\chi^{-1}(kh^{-1})} C_e$$

for all $h, k \in H$. The composition of 1-morphisms in $\mathcal{C}$ is given by their tensor product in C, and the interchange 2-isomorphism in $\mathcal{C}$ is given by the crossed braiding in C.

III.4. Spherical graded 2-categories

Graphical calculus is a method, first suggested by Penrose [Pen71], where morphisms in a monoidal category are depicted by planar diagrams. Pivotal and spherical categories are defined in such a way that diagrams of morphisms are equivalent under isotopies of the plane and the sphere, respectively. This is particularly useful when applying categories to topological problems, see [BW99].

Similarly, Barrett, Meusburger and Schaumann introduced [BMS24] a diagrammatic calculus for monoidal 2-categories. Using this, many algebraic results on monoidal 2-categories can be understood via the geometry of the associated diagrams. In this section, we recall the definitions of planar pivotal, pivotal and spherical 2-categories from [DR18]. These structures are particularly well suited for diagrammatic calculus and thus for further topological applications. We then extend these notions to the context of graded monoidal 2-categories.

Throughout this section,

$$\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega)$$

is a 2-crossed module of groups, and $\delta = (\delta\colon L \to E, \triangleright)$ is the induced crossed module.

III.4.1. Planar pivotal 2-categories. Let $\mathcal{C}$ be a 2-category. A *planar pivotal structure* on $\mathcal{C}$ is a choice of right adjoint $f^*\colon Y \to X$ for every 1-morphism $f\colon X \to Y$, and a choice of unit $\eta_f\colon 1_X \Rightarrow f^*\circ f$ and counit $\varepsilon_f\colon f\circ f^* \Rightarrow 1_Y$ such that the choice of adjoint is functorial:

$$(1_X)^* = 1_X, \qquad (f\circ g)^* = g^*\circ f^*,$$

the choice of unit and counit is functorial:

$$\varepsilon_{1_X} = 1_{1_X}, \qquad \eta_{1_X} = 1_{1_X}, \qquad \varepsilon_{f\circ g} = \varepsilon_f\cdot(1_f\circ\varepsilon_g\circ 1_{f^*}), \qquad \eta_{f\circ g} = (g^*\circ\eta_f\circ g)\cdot\eta_g,$$

the adjoint is involutive:

$$f^{**} = f,$$

and the left and right mates agree:

$$\alpha^* := (1_{f^*}\circ\varepsilon_g)\cdot(1_{f^*}\circ\alpha\circ 1_{g^*})\cdot(\eta_f\circ 1_{g^*}) = (\varepsilon_{g^*}\circ 1_{f^*})\cdot(1_{g^*}\circ\alpha\circ 1_{f^*})\cdot(1_{g^*}\circ\eta_{f^*}) =: {}^*\alpha.$$

for any 2-morphism $\alpha\colon f \Rightarrow g$. A *planar pivotal 2-category* is a 2-category endowed with a planar pivotal structure.

The *left planar trace* and the *right planar trace* of a 2-endomorphism $\alpha\colon f \Rightarrow f$ of a 1-morphism $f\colon X \to Y$ in a planar pivotal category are the 2-morphisms $\mathrm{tr}_L(\alpha)\colon 1_X \Rightarrow 1_X$ and $\mathrm{tr}_R(\alpha)\colon 1_Y \Rightarrow 1_Y$ defined by:

$$\mathrm{tr}_L(\alpha) = \varepsilon_{f^*}\cdot(1_{f^*}\circ\alpha)\cdot\eta_f \quad\text{and}\quad \mathrm{tr}_R(\alpha) = \varepsilon_f\cdot(\alpha\circ 1_{f^*})\cdot\eta_{f^*},$$

respectively.

III.4.2. Monoidal planar pivotal 2-categories. A *monoidal planar pivotal 2-category* is a monoidal 2-category with a planar pivotal structure such that the adjoint of a tensor is the tensor of adjoints:

$$(X \otimes f \otimes Y)^* = X \otimes f^* \otimes Y$$

and the unit and counit for a tensor are the tensors of the units and counits:

$$\varepsilon_{X\otimes f\otimes Y} = X \otimes \varepsilon_f \otimes Y \quad \eta_{X\otimes f\otimes Y} = X \otimes \eta_f \otimes Y.$$

III.4.3. Pivotal 2-categories. A *right dual* of an object X in a monoidal 2-category $\mathcal{C}$ is an object $X^\#$ of $\mathcal{C}$ such that there exist *fold* 1-morphisms $i\colon \mathbb{1} \to X^\# \otimes X$ and $e\colon X \otimes X^\# \to \mathbb{1}$ verifying

$$(e \otimes X) \circ (X \otimes i) \cong 1_X \quad \text{and} \quad 1_{X^\#} \cong (X^\# \otimes e) \circ (i \otimes X^\#).$$

Let $\mathcal{C}$ be a monoidal 2-category with chosen right dual $X^\#$ and folds i_X, e_X for every object X. The *right* and *left mates* of a 1-morphism $f\colon X \to Y$ are the 1-morphisms

$$f^\# = (X^\# \otimes e_Y) \circ (X^\# \otimes f \otimes Y^\#) \circ (i_X \otimes Y^\#)\colon Y^\# \to X^\#,$$
$$^\# f = (e_{Y^\#} \otimes X^\#) \circ (Y^\# \otimes f \otimes X^\#) \circ (Y^\# \otimes i_{X^\#})\colon Y^\# \to X^\#,$$

respectively.

A *pivotal structure* on a monoidal planar pivotal 2-category $\mathcal{C}$ consists of the following data:

- a choice of right dual $X^\#$ for every object X of $\mathcal{C}$,
- a choice of folds $i_X\colon \mathbb{1} \to X^\# \otimes X$ and $e_X\colon X \otimes X^\# \to \mathbb{1}$,
- a choice of *cusp* 2-isomorphisms:

$$C_X\colon (e_X \otimes X) \circ (X \otimes i_X) \xRightarrow{\sim} 1_X \quad \text{and} \quad D_X\colon 1_{X^\#} \xRightarrow{\sim} (X^\# \otimes e_X) \circ (i_X \otimes X^\#),$$

subject to the following conditions:

- the cusps satisfy the *swallowtail equations*:

$$\big(e_X \circ (C_X \otimes X^\#)\big) \cdot \big(\phi_{e_X,e_X} \circ (X \otimes i_X \otimes X^\#)\big) \cdot (e_X \circ (X \otimes D_X)) = 1_{e_X},$$
$$\big((X^\# \circ C_X) \circ i_X\big) \cdot \big((X^\# \otimes e_X \otimes X) \circ \phi_{i_X,i_X}\big) \cdot ((D_X \otimes X) \circ i_X) = 1_{i_X},$$

- the choice of dual is compatible with tensor:

$$\mathbb{1}^\# = \mathbb{1}, \qquad (X \otimes Y)^\# = Y^\# \otimes X^\#,$$

- the choice of folds is compatible with tensor:

$$i_{\mathbb{1}} = e_{\mathbb{1}} = 1_{\mathbb{1}}, \quad i_{X\otimes Y} = (Y^\# \otimes i_X \otimes Y) \circ i_Y, \quad e_{X\otimes Y} = e_X \circ (X \otimes e_Y \otimes X^\#),$$

- the choice of cusps is compatible with tensor:

$$C_{\mathbb{1}} = D_{\mathbb{1}} = 1_{1_{\mathbb{1}}},$$
$$C_{X\otimes Y} = ((C_X \otimes Y) \circ (X \otimes C_Y)) \cdot ((e_X \otimes X \otimes Y) \circ (X \otimes \phi_{e_Y,i_X} \otimes Y) \circ (X \otimes Y \otimes i_Y)),$$
$$D_{X\otimes Y} = \big((Y^\# \otimes X^\# \otimes e_X) \circ (Y^\# \otimes \phi_{i_X,e_Y} \otimes X^\#) \circ (i_Y \otimes Y^\# \otimes X^\#)\big) \cdot \big((Y^\# \otimes D_X) \circ (D_Y \otimes X^\#)\big),$$

- the folds intertwine duality of objects and adjunction of 1-morphisms: $(e_X)^* = i_{X^\#}$,
- the cusps intertwine duality of objects and mates of 2-morphisms: $(D_X)^* = C_{X^\#}$,
- the dual is involutive: $X^{\#\#} = X$,
- the right-over and left-under twists between the left and right mates agree:

$$\theta_f = (\theta_{f^*})^*\colon {}^\# f \Rightarrow f^\#,$$

for any 1-morphism $f\colon X \to X$, where

$$\begin{aligned}\theta_f = &\Big((X^\# \otimes e_Y) \circ (X^\# \otimes f \otimes Y^\#) \circ (\varepsilon_{^\# f} \otimes X \otimes Y^\#) \circ (i_X \otimes Y^\#)\Big)\\ &\cdot \Big(\phi_{^\# f,e_Y\circ(f\otimes Y^\#)} \circ ((f^*)^\# \otimes X \otimes Y^\#) \circ (i_X \otimes Y^\#)\Big)\\ &\cdot \Big({}^\# f \circ (Y^\# \otimes e_Y) \circ (Y^\# \otimes f \otimes Y^\#) \circ (Y^\# \otimes e_X \otimes X \otimes Y^\#) \circ (\phi^{-1}_{(Y^\#\otimes f^*)\circ i_Y,i_X} \otimes Y^\#)\Big)\\ &\cdot \Big({}^\# f \circ (f \circ C_X^{-1} \circ f^*)^\#\Big) \cdot \Big({}^\# f \circ \eta^\#_{f^*}\Big) \cdot \Big({}^\# f \circ D_Y\Big).\end{aligned}$$

A *pivotal 2-category* is a monoidal planar pivotal 2-category equipped with a pivotal structure.

III.4.4. Spherical 2-categories. Let $\mathcal{C}$ be a pivotal 2-category. Recall from Section III.4.1 that the left and right planar traces of a 2-endomorphism $\alpha\colon f \Rightarrow f$ of a 1-morphism $f\colon X \to Y$ are 2-endomorphisms of 1_X and 1_Y, respectively. Thus, when $\mathcal{C}$ has multiple objects, it doesn't make sense to require these to be equal. The following result is shown in the preprint version of [BMS24].

Lemma III.4.1. *For any 2-endomorphism $\alpha\colon f \Rightarrow f$ of a 1-morphism $f\colon X \to Y$ in $\mathcal{C}$, the following left and right traces agree:*

$$\mathrm{tr}_R(1_{e_Y} \circ (\alpha \otimes Y^{\#})) = \mathrm{tr}_L((\alpha \otimes X^{\#}) \circ 1_{i_{X^{\#}}}).$$

The *back 2-spherical trace* and *front 2-spherical trace* of an endomorphism $\alpha\colon f \Rightarrow f$ of a 1-cell $f\colon X \to Y$ in a pivotal 2-category are the 2-morphisms $\mathrm{Tr}_B(\alpha)$ and $\mathrm{Tr}_F(\alpha)$ defined by

$$\mathrm{Tr}_B(\alpha) = \mathrm{tr}_R(1_{e_Y} \circ (\alpha \otimes Y^{\#})) = \mathrm{tr}_L((\alpha \otimes X^{\#}) \circ 1_{i_{X^{\#}}})\colon 1_{\mathbb{1}} \Rightarrow 1_{\mathbb{1}},$$
$$\mathrm{Tr}_F(\alpha) = \mathrm{tr}_R(1_{e_{Y^{\#}}} \circ (Y^{\#} \otimes \alpha)) = \mathrm{tr}_L((X^{\#} \otimes \alpha) \circ 1_{i_X})\colon 1_{\mathbb{1}} \Rightarrow 1_{\mathbb{1}}.$$

respectively.

A *spherical 2-category* is a pivotal 2-category such that the front 2-spherical trace and the back 2-spherical trace of any 2-endomorphism α are equal, then denoted $\mathrm{Tr}(\alpha)$ and called the *trace* of α.

III.4.5. Spherical graded 2-categories. A *pivotal σ-graded 2-category* is a σ-graded 2-category with a pivotal structure such that

- the dual $X^{\#}$ of a homogeneous object X is homogeneous of degree $|X^{\#}| = |X|^{-1}$,
- the adjoint f^* of a homogeneous 1-morphism f is homogeneous of degree $|f^*| = |f|^{-1}$,
- the units, counits, folds, and cusps are all homogeneous of degree 1.

Notice that the 1-subcategory, the $(1,1)$-subcategory, and the neutral component of a pivotal σ-graded 2-category are pivotal 2-categories (see Sections III.3.6 and III.3.9).

A *spherical* σ-graded 2-category is a pivotal σ-graded 2-category whose 1-subcategory is spherical. In other words, in a spherical σ-graded 2-category, the front 2-spherical trace and the back 2-spherical trace of any degree-1 homogeneous 2-endomorphism α are equal, then denoted $\mathrm{Tr}(\alpha)$ and called the *trace* of α.

III.4.6. Examples. 1) The linearization of σ is a spherical σ-graded 2-category, where the dual of an object $h \in H$ is the object h^{-1} and the adjoint of a 1-morphism $e \in E$ is the 1-morphism e^{-1}.

2) Notice that if $\mathcal{C}$ is a spherical σ-graded 2-category and $\phi\colon \sigma \to \sigma'$ is a morphism of 2-crossed modules of groups, then the pushforward $\phi_*(\mathcal{C})$ (see Section III.3.11) is a spherical σ'-graded 2-category.

III.5. Graded-fusion 2-categories

Let

$$\sigma = (L \xrightarrow{\ \delta\ } E \xrightarrow{\ \partial\ } H,\ \omega)$$

be a 2-crossed module of groups. In this section, we introduce σ-fusion 2-categories as a categorification of the notion of graded-fusion category defined by Sözer and Virelizier [SV23]. Recall that $\Bbbk$ is a nonzero commutative ring.

III.5.1. Semisimple categories. An object X in a $\Bbbk$-linear category D is *simple* if $\mathrm{Hom}_D(X,X)$ is a free $\Bbbk$-module of rank 1 (with basis $\{1_X\}$). A $\Bbbk$-linear category D is *semisimple* if every object of D is a direct sum of a finite family of simple objects, and for any non-isomorphic simple objects X and Y of D, $\mathrm{Hom}_D(X,Y) = \{0\}$.

III.5.2. Graded-semisimple categories. Let L be a group. Recall that the 1-subcategory of an L-Hom-graded category C is the category C^1 with the same objects as C and only the degree-1 homogeneous morphisms, see Section III.3.2. An L-Hom-graded category C is *L-semisimple* if C^1 is semisimple, and for all $l \in L$, every object of C is an l-direct sum of a finite family of simple objects of C^1. Notice that for $L = 1$, we recover the definition of a semisimple category. However, in general, an L-semisimple category need not be semisimple. In particular, a simple object in the 1-subcategory C^1 of an L-Hom-graded category C is not necessarily simple in C.

III.5.3. Simple objects and 1-morphisms. We say that a 1-morphism $f\colon X \to Y$ in a $\Bbbk$-linear 2-category $\mathcal{C}$ is *simple* if $\mathrm{Hom}_{\mathcal{C}(X,Y)}(f,f)$ is a free $\Bbbk$-module of rank 1 (with basis $\{1_f\}$). We say that an object X in a $\Bbbk$-linear 2-category is *simple* if 1_X is simple.

III.5.4. Semisimple 2-categories. A $\Bbbk$-linear 2-category $\mathcal{D}$ is *semisimple* if

- it is *locally semisimple*, that is, for all objects X and Y of $\mathcal{D}$, the category $\mathcal{D}(X,Y)$ is semisimple,
- it is *locally additive*, that is, for all objects X and Y of $\mathcal{D}$, the category $\mathcal{D}(X,Y)$ is additive,
- every 1-morphism of $\mathcal{D}$ admits a left adjoint and a right adjoint,
- every object of $\mathcal{D}$ is a direct sum of a finite family of simple objects.

Notice that a presemisimple 2-category in the sense of [DR18, Definition 1.2.7] is a semisimple 2-category in the above sense.

Let $\mathcal{D}$ be a semisimple 2-category and let X and Y be objects of $\mathcal{D}$. A set I of simple 1-morphisms of $\mathcal{D}(X,Y)$ is *representative* if every simple 1-morphism in $\mathcal{D}(X,Y)$ is isomorphic to a unique element of I. Each 1-morphism in $\mathcal{D}(X,Y)$ is then a direct sum of a finite family of 1-morphisms in I, and $\mathrm{Hom}_{\mathcal{D}(X,Y)}(i,j) = 0$ for any distinct elements $i, j \in I$.

A set J of simple objects of a semisimple 2-category $\mathcal{D}$ is *representative* if every simple object of $\mathcal{D}$ is equivalent to a unique element of J. Each object of $\mathcal{D}$ is then a direct sum of a finite family of objects in J.

Contrary to the case of semisimple categories, non-equivalent simple objects of a semisimple 2-category $\mathcal{D}$ may have nonzero 1-morphisms between them, see for instance [DR18, Example 1.4.14]. We say that two simple objects X and Y in $\mathcal{D}$ are *disconnected* if $\mathcal{D}(X,Y)$ is the terminal category, and we say that they are *connected* otherwise. A set K of disconnected simple objects of $\mathcal{D}$ is *representative* if every simple object of $\mathcal{D}$ is connected to a unique element of K.

III.5.5. Graded-semisimple 2-categories. Recall from Section III.3.6 that the 1-subcategory of a σ-graded 2-category $\mathcal{C}$ is the linear monoidal 2-category $\mathcal{C}^1$ with the same objects and 1-morphisms as $\mathcal{C}$, and with only degree-1 homogeneous 2-morphisms. A σ-graded 2-category $\mathcal{C}$ is *σ-semisimple* (over $\Bbbk$) if

- it is *locally L-semisimple*, that is, for all objects X and Y of $\mathcal{C}$, the category $\mathcal{C}(X,Y)$ is L-semisimple,
- it is *locally L-additive*, that is, for all objects X and Y of $\mathcal{C}$, the category $\mathcal{C}(X,Y)$ is L-additive (see Section III.3.2),
- every 1-morphism of $\mathcal{C}$ admits a left adjoint and a right adjoint in $\mathcal{C}^1$,
- for all $(l,e) \in L \times E$, every object of $\mathcal{C}$ is an (l,e)-direct sum of a finite family of simple objects of $\mathcal{C}^1$ (see Section III.3.5).

Note that for σ trivial, we recover the definition of a semisimple 2-category. The 1-subcategory $\mathcal{C}^1$, the $(1,1)$-subcategory $\mathcal{C}^1_1$, and the neutral component ${}_1\mathcal{C}^1_1$ of a σ-semisimple 2-category $\mathcal{C}$ are all semisimple (see Sections III.3.6 and III.3.9). However, in general, a σ-semisimple 2-category need not be semisimple. In particular, a simple object in $\mathcal{C}^1$ is not necessarily simple in $\mathcal{C}$.

Lemma III.5.1. *Let $\mathcal{C}$ be a σ-semisimple 2-category and let $\mathcal{C}^1$ denote its 1-subcategory. Any simple object in $\mathcal{C}^1$ is $(1,1)$-equivalent to a homogeneous object, and any simple 1-morphism in $\mathcal{C}^1$ is homogeneous.*

Proof. Let X be a simple object in $\mathcal{C}^1$. We know that X is a $(1,1)$-direct sum of a finite family of homogeneous objects $\{X_i\}_{i\in I}$. Let $\{p_i \colon X \to X_i,\ q_i \colon X_i \to X\}_{i\in I}$ denote the 1-morphisms of this direct sum. We have

$$1_X = \bigoplus_{i\in I}^{1} (q_i \circ p_i),$$

but 1_X is simple in $\mathcal{C}^1$, therefore there is a unique $i_0 \in I$ such that $q_{i_0} \circ p_{i_0}$ is nonzero. For $i \neq i_0$, $q_i \circ p_i = 0$, therefore $p_i \circ q_i \circ p_i = 0$. Moreover, for $i \neq i_0$, $1_{X_i} \cong p_i \circ q_i$, therefore $p_i = 0$, and therefore X_i is a zero object. We deduce that X is $(1,1)$-equivalent to a homogeneous object X_{i_0}.

Let f be a simple 1-morphism in $\mathcal{C}^1$. We know that f is a 1-direct sum of a finite family of homogeneous 1-morphisms $\{f_i\}_{i\in I}$. By simplicity of f, there is a unique $i_0 \in I$ such that $f \cong f_{i_0}$ in $\mathcal{C}^1$. Therefore f is a homogeneous 1-morphism. □

Let $\mathcal{C}$ be a σ-semisimple 2-category. Recall that for all objects X and Y, the category $\mathcal{C}^1(X,Y)$ is graded by (the set) E (see Section III.3.1), and so decomposes as

$$\mathcal{C}^1(X,Y) = \bigoplus_{e\in E} \mathcal{C}^1_e(X,Y),$$

where $\mathcal{C}_e^1(X,Y) = \mathcal{C}^1(X,Y)_e$ is the category whose objects are the degree-e homogeneous 1-morphisms from X to Y, and whose morphisms are the homogeneous 2-morphisms of degree 1. Notice that $\mathcal{C}_e^1(X,Y)$ is semisimple for all $e \in E$. Recall that the $(1,1)$-subcategory $\mathcal{C}_1^1$ of a σ-semisimple 2-category $\mathcal{C}$ is H-graded (see Section III.3.7), and for all $h \in H$, the 2-category ${}_h\mathcal{C}_1^1$ is semisimple.

We list some useful properties of σ-semisimple 2-categories:

Lemma III.5.2 [SV23]**.** *Let $\mathcal{C}$ be a σ-semisimple 2-category (or a locally L-semisimple 2-category). Then,*

(1) For any simple 1-morphism f of $\mathcal{C}^1$ and for all $l \in L$, there is a simple 1-morphism g of $\mathcal{C}^1$ and a degree-l 2-isomorphism from f to g.

(2) For any simple 1-morphisms $f, g\colon X \to Y$ of $\mathcal{C}^1$ and for all $l \in L$, $\mathrm{Hom}^l_{\mathcal{C}(X,Y)}(f,g) \cong \Bbbk$ if there is a degree-l homogeneous 2-isomorphism from f to g, and $\mathrm{Hom}^l_{\mathcal{C}(X,Y)}(f,g) = 0$ otherwise.

(3) For any 1-morphisms $f, g\colon X \to Y$ and for all $l, k \in L$, there is a $\Bbbk$-linear isomorphism

$$\mathrm{Hom}^{kl}_{\mathcal{C}(X,Y)}(f,g) \cong \bigoplus_{i \in I} \mathrm{Hom}^k_{\mathcal{C}(X,Y)}(i,g) \otimes \mathrm{Hom}^l_{\mathcal{C}(X,Y)}(f,i),$$

where I is any representative set of simple 1-morphisms of $\mathcal{C}^1(X,Y)$.

A σ-semisimple 2-category $\mathcal{C}$ is *finite* if

- for all objects X, Y of $\mathcal{C}$ and for all $e \in E$, any representative set of simple 1-morphisms of $\mathcal{C}_e^1(X,Y)$ is finite,
- for all $e \in \mathrm{Ker}(\partial)$, any representative set of simple 1-morphisms of $\mathcal{C}_e^1(\mathbb{1},\mathbb{1})$ is non-empty,
- for all $h \in H$, any representative set of simple objects of ${}_h\mathcal{C}_1^1$ is finite and non-empty.

We can rephrase the above definition as follows. A σ-semisimple 2-category $\mathcal{C}$ is *finite* if for all objects X and Y of $\mathcal{C}$, and for all $e \in E$, the set of degree-1 homogeneous 2-isomorphism classes of degree-e simple 1-morphisms of $\mathcal{C}^1$ from X to Y is finite; for all $e \in \mathrm{Ker}(\partial)$, the set of degree-$e$ homogeneous endomorphisms of $\mathbb{1}$ is non-empty; and for all $h \in H$, the set of $(1,1)$-equivalence classes of degree-h homogeneous simple objects of $\mathcal{C}^1$ is finite and non-empty.

III.5.6. Graded-fusion 2-categories. A *σ-fusion 2-category* (over $\Bbbk$) is a finite σ-semisimple 2-category $\mathcal{C}$ such that all objects have left and right duals in the $(1,1)$-subcategory of $\mathcal{C}$, and the monoidal unit of $\mathcal{C}$ is simple. A *spherical σ-fusion 2-category* is a σ-fusion 2-category that is spherical as a σ-graded 2-category.

Lemma III.5.3. *Let X, Y be objects in a σ-fusion 2-category $\mathcal{C}$, let $e \in E$, and $f \in \mathrm{Ker}(\partial)$. If $\mathcal{C}_e^1(X,Y)$ contains a nonzero 1-morphism, then so does $\mathcal{C}_{ef}^1(X,Y)$. In particular, for any simple object X of $\mathcal{C}^1$ and for any $e \in \mathrm{Ker}(\partial)$, there is a simple 1-morphism in $\mathcal{C}_e^1(X,X)$.*

Proof. We first suppose Y is simple in $\mathcal{C}^1$. Let $g\colon X \to Y$ be a nonzero degree-e homogeneous 1-morphism, and let $h\colon \mathbb{1} \to \mathbb{1}$ be a degree-f homogeneous 1-morphism. We can suppose that g and h are simple in $\mathcal{C}^1$. Then the 1-morphism $(h \otimes Y) \circ g\colon X \to Y$ is homogeneous of degree ef. By the proof of [DR18, Proposition 1.2.19], the equation $(h \otimes Y) \circ g = 0$ implies that $h \otimes Y = 0$. Let $Y^\#$ be a right dual of Y and let $e\colon Y \otimes Y^\# \to \mathbb{1}$ denote the associated fold. We have

$$\begin{aligned} h \otimes Y = 0 &\Rightarrow h \otimes Y \otimes Y^\# = 0 \\ &\Rightarrow (1 \otimes e) \circ (h \otimes Y \otimes Y^\#) = 0 \\ &\Rightarrow h \circ e = 0. \end{aligned}$$

By the proof of [DR18, Proposition 1.2.19] $h \circ e = 0$ implies that $h = 0$, and therefore $(h \otimes Y) \circ g$ is nonzero.

We no longer suppose that Y is simple, and we write $Y = \boxplus_{i \in I}^{(1,1)} Y_i$, where I is a finite set and Y_i is simple in $\mathcal{C}^1$ for all $i \in I$. We denote by $p_i\colon Y \to Y_i$ and $q_i\colon Y_i \to Y$ the projection and inclusion 1-morphisms for the $(1,1)$-direct sum. Let $g\colon X \to Y$ be a degree-e simple 1-morphism, and let $h\colon \mathbb{1} \to \mathbb{1}$ be a degree-f simple 1-morphism. Choose any $i \in I$ such that $p_i \circ g\colon X \to Y_i$ is nonzero. By the above discussion, since Y_i is simple, the 1-morphism

$$(h \otimes Y_i) \circ p_i \circ g\colon X \to Y_i$$

is nonzero and homogeneous of degree ef, and therefore $q_i \circ (h \otimes Y_i) \circ p_i \circ g\colon X \to Y$ is nonzero and homogeneous of degree ef. □

Notice that the 2-crossed module σ induces a crossed module $(\delta\colon L \to \mathrm{Ker}(\partial))$, where $\mathrm{Ker}(\partial)$ acts on L via $\triangleright$. Let X be a simple object in the 1-subcategory $\mathcal{C}^1$ of a σ-fusion 2-category $\mathcal{C}$. By Lemma III.5.3, the Hom-category $\mathcal{C}(X,X)$ is a $(\delta\colon L \to \mathrm{Ker}(\partial))$-fusion category in the sense of Sözer and Virelizier, see [SV23, Section 4.10]. In particular, $\mathcal{C}_1^1(X,X)$ is a fusion category. Moreover, up to additive and idempotent completion, the neutral component of a σ-fusion 2-category is a fusion 2-category in the sense of Douglas and Reutter when $\Bbbk$ is an algebraically closed field of characteristic 0, see [DR18, Definition 2.1.6]. However, in general, a σ-fusion 2-category is not necessarily a fusion 2-category.

III.5.7. The linearization of a 2-crossed module. The linearization $\mathcal{G}_\sigma$ of σ (see Section III.3.10) is a spherical σ-fusion 2-category. A representative set of simple objects of $(\mathcal{G}_\sigma)^1$ is given by H, and for all $h,k \in H$, a representative set of simple 1-morphisms of $(\mathcal{G}_\sigma)^1(h,k)$ is given by $\{e \in E \mid \partial(e) = kh^{-1}\}$.

III.5.8. The pushforward of a σ-fusion 2-category. Let

$$\sigma' = (L' \xrightarrow{\delta'} E' \xrightarrow{\partial'} H',\ \omega')$$

be a 2-crossed module of groups, and let $\phi = (\phi_1, \phi_2, \phi_3)$ be a morphism from σ to σ'. Recall from Section III.3.11 the definition of the pushforward $\phi_*(\mathcal{C})$ of a σ-graded 2-category $\mathcal{C}$ along ϕ. Notice that if $\phi\colon \sigma \to \sigma'$ is a morphism of 2-crossed modules such that

- $\phi_1\colon H \to H'$, $\phi_2\colon E \to E'$, and $\phi_3\colon L \to L'$ are surjective,
- $\mathrm{Ker}(\phi_1)$ and $\mathrm{Ker}(\phi_2)$ are finite,
- $\mathrm{Ker}(\phi_3) \cap \mathrm{Ker}(\delta) = 1$,

then the pushforward $\phi_*(\mathcal{C})$ of a σ-fusion 2-category $\mathcal{C}$ is a σ'-fusion 2-category.

III.6. Dimensions in graded-fusion 2-categories

In this section, we define the "quantum" dimensions of objects and 1-morphisms in spherical σ-fusion 2-categories. We use these to derive a notion of categorical dimension for spherical σ-fusion 2-categories and we prove some useful properties of dimensions. Throughout this section, let

$$\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega)$$

be a 2-crossed module of groups, and let $\mathcal{C}$ be a spherical σ-fusion 2-category over $\Bbbk$. Recall from Section III.3.6 that the 1-subcategory $\mathcal{C}^1$ of $\mathcal{C}$ has the same objects and 1-morphisms as $\mathcal{C}$, and only degree-1 homogeneous 2-morphisms. Recall that $\mathbb{1}$ is a simple object in $\mathcal{C}^1$, and therefore we identify $\mathrm{Hom}^1_{\mathcal{C}(\mathbb{1},\mathbb{1})}(1_\mathbb{1}, 1_\mathbb{1})$ and $\Bbbk$.

III.6.1. Dimensions of objects and 1-morphisms. The *dimension* of a 1-morphism f in $\mathcal{C}$ is

$$\dim(f) = \mathrm{Tr}(1_f) \in \Bbbk,$$

where $\mathrm{Tr}(1_f)$ denotes the trace of 1_f (see Sections III.4.4 and III.4.5). The *dimension* of an object X in $\mathcal{C}$ is

$$\dim(X) = \dim(1_X) = \mathrm{Tr}(1_{1_X}) \in \Bbbk.$$

Lemma III.6.1. *The dimensions of simple objects and simple 1-morphisms in $\mathcal{C}^1$ are invertible in $\Bbbk$.*

Proof. First, we show that if $f\colon X \to Y$ denotes a simple 1-morphism in $\mathcal{C}^1$ with Y a simple object in $\mathcal{C}^1$, then the right planar trace $\mathrm{tr}_R(1_f)$ is invertible in $\Bbbk$. We adapt the proof of [DR18, Lemma 2.3.10]. By definition, $\mathrm{tr}_R(1_f) = \varepsilon_f \cdot \eta_{f^*} = \lambda 1_{1_Y}$ for some $\lambda \in \Bbbk$. Note that by adjunction and by simplicity of f:

$$\mathrm{Hom}^1_{\mathcal{C}(Y,Y)}(f \circ f^*, 1_Y) \cong \mathrm{Hom}^1_{\mathcal{C}(X,Y)}(f,f) \cong \Bbbk,$$

and $\{\varepsilon_f\}$ is a basis of $\mathrm{Hom}^1_{\mathcal{C}(Y,Y)}(f \circ f^*, 1_Y)$. Similarly, $\mathrm{Hom}^1_{\mathcal{C}(Y,Y)}(1_Y, f \circ f^*) = \Bbbk\{\eta_{f^*}\}$. Moreover, by local L-semisimplicity of $\mathcal{C}$ and by simplicity of Y, 1_Y is a direct summand in the 1-direct sum decomposition of $f \circ f^*$ into simple 1-morphisms. Therefore, there are 2-morphisms $\alpha \in \mathrm{Hom}^1_{\mathcal{C}(Y,Y)}(f \circ f^*, 1_Y)$ and $\beta \in \mathrm{Hom}^1_{\mathcal{C}(Y,Y)}(1_Y, f \circ f^*)$ such that $\alpha \cdot \beta = 1_{1_Y}$. We write $\alpha = x\varepsilon_f$ and $\beta = y\eta_{f^*}$, for some scalars $x, y \in \Bbbk$. We have

$$1_{1_Y} = \alpha \cdot \beta = (xy)\varepsilon_f \cdot \eta_{f^*} = (xy\lambda)1_{1_Y},$$

therefore λ is invertible.

Now let $f\colon X \to Y$ be a simple 1-morphism in $\mathcal{C}^1$, where X and Y are not necessarily simple. By the duality between Y and $Y^\#$, we have

$$\mathrm{End}^1_{\mathcal{C}(X\otimes Y^\#,\mathbb{1})}(e_Y \circ (f \otimes Y^\#)) \cong \mathrm{End}^1_{\mathcal{C}(X,Y)}(f) \cong \Bbbk.$$

Therefore, $e_Y \circ (f \otimes Y^\#)\colon X \otimes Y^\# \to \mathbb{1}$ is a simple 1-morphism in $\mathcal{C}^1$. By the above discussion, the right trace $\mathrm{tr}_R(1_{e_Y\circ(f\otimes Y^\#)})$ is invertible, and the dimension of f is

$$\mathrm{Tr}(1_f) = \mathrm{tr}_R(1_{e_Y} \circ (1_f \otimes Y^\#)) = \mathrm{tr}_R(1_{e_Y\circ(f\otimes Y^\#)}).$$

Therefore $\dim(f)$ is invertible in $\Bbbk$. As a consequence, if X is a simple object in $\mathcal{C}^1$, then $\dim(X) = \dim(1_X)$ is invertible in $\Bbbk$. □

For any 1-morphisms $f, g\colon X \to Y$ in $\mathcal{C}$ and for all $l \in L$, the *trace pairing* is the map

$$\langle\cdot,\cdot\rangle\colon \mathrm{Hom}^l_{\mathcal{C}}(f,g) \otimes \mathrm{Hom}^{l^{-1}}_{\mathcal{C}}(g,f) \to \Bbbk, \qquad \alpha\otimes\beta \mapsto \langle\alpha,\beta\rangle = \mathrm{Tr}(\alpha\cdot\beta).$$

Lemma III.6.2. *The trace pairing $\langle\cdot,\cdot\rangle$ is nondegenerate.*

Proof. We adapt the proof of [DR18, Proposition 2.3.14]. Let $\{s_i\}_{i\in I}$ be a representative set of simple 1-morphisms in $\mathcal{C}^1(X,Y)$. By local L-semisimplicity of $\mathcal{C}$, we can write

$$f = \bigoplus_{i\in I}^{l} s_{k_i} \quad \text{and} \quad g = \bigoplus_{j\in J}^{1} s_{l_j}.$$

We denote by $\{\pi_i, \iota_i\}_{i\in I}$ the projection and inclusion 2-morphisms for f, and by $\{\pi'_j, \iota'_j\}_{j\in J}$ the projection and inclusion 2-morphisms for g.

Suppose $\alpha\colon f \Rightarrow g$ is a degree-l homogeneous 2-morphism such that for any degree-l^{-1} homogeneous 2-morphism $\beta\colon g \Rightarrow f$, we have $\langle\alpha,\beta\rangle = 0$. Every degree-$l^{-1}$ homogeneous 2-morphism $g \Rightarrow f$ is a linear combination of the 2-morphisms $\iota_i\cdot\pi'_j$ such that $k_i = l_j$. By assumption, $0 = \mathrm{Tr}(\alpha\cdot\iota_i\cdot\pi'_j) = \mathrm{Tr}(\pi'_j\cdot\alpha\cdot\iota_i)$. By simplicity $s_{k_i} = s_{l_j}$,

$$\pi'_j\cdot\alpha\cdot\iota_i = \lambda_{i,j} 1_{s_{k_i}}$$

for some scalar $\lambda_{i,j} \in \Bbbk$. Therefore $0 = \lambda_{i,j}\,\mathrm{Tr}(1_{s_{k_i}}) = \lambda_{i,j}\dim(s_{k_i})$. From Lemma III.6.1, we deduce that $\lambda_{i,j} = 0$ for all $i \in I$ and $j \in J$. Therefore,

$$\alpha = 1_g\cdot\alpha\cdot 1_f = \left(\sum_{j\in J}\iota'_j\cdot\pi'_j\right)\cdot\alpha\cdot\left(\sum_{i\in I}\iota_i\cdot\pi_i\right) = \sum_{i\in I,\ j\in J}\iota'_j\cdot\left(\pi'_j\cdot\alpha\cdot\iota_i\right)\cdot\pi_i = 0. \qquad \square$$

III.6.2. Weighted dimensions of objects. Recall that $\mathcal{C}^1$ denotes the 1-subcategory of $\mathcal{C}$, see Section III.3.6. Recall that for all objects X, Y of $\mathcal{C}$ and $e \in E$, $\mathcal{C}^1_e(X,Y)$ is the category whose objects are the degree-e homogeneous 1-morphisms from X to Y, and whose morphisms are homogeneous 2-morphisms of degree 1, see Section III.5.5. For simple objects X, Y of $\mathcal{C}^1$ and $e \in E$, the *dimension* of the category $\mathcal{C}^1_e(X,Y)$ is

$$\dim(\mathcal{C}^1_e(X,Y)) = \frac{1}{\dim(X)\dim(Y)}\sum_f \dim(f)^2,$$

where f runs over a representative set of simple 1-morphisms in $\mathcal{C}^1_e(X,Y)$. It is well defined, see Lemma III.6.1.

The *weighted dimension* of a simple object X of $\mathcal{C}^1$ is

$$\mathrm{d}(X) = \dim(X)\dim(\mathcal{C}^1_1(X,X))\,\mathrm{n}(X) \in \Bbbk,$$

where $\mathrm{n}(X)$ denotes the number of $(1,1)$-equivalence classes of simple objects that are connected to X in $\mathcal{C}^1_1$.

III.6.3. Categorical dimensions. Recall that the $(1,1)$-subcategory of $\mathcal{C}$ is H-graded (see Section III.3.7) and that ${}_h\mathcal{C}_1^1$ denotes the linear 2-category with degree-h homogeneous objects and degree-1 homogeneous 1-morphisms and 2-morphisms. Let $h \in H$ such that the dimensions $\dim(\mathcal{C}_1^1(X,X)) \in \Bbbk$ are invertible for all degree-h homogeneous simple objects X of $\mathcal{C}^1$. The *dimension* of the 2-category ${}_h\mathcal{C}_1^1$ is

$$\dim({}_h\mathcal{C}_1^1) = \sum_X \dim(\mathcal{C}_1^1(X,X))^{-1} \in \Bbbk,$$

where X runs over a representative set of disconnected simple objects in ${}_h\mathcal{C}_1^1$. From [DR18, Proposition 1.2.27], we deduce that if X and Y are simple objects of $\mathcal{C}^1$ that are connected in $\mathcal{C}_1^1$, then

$$\dim(\mathcal{C}_1^1(X,X)) = \dim(\mathcal{C}_1^1(Y,Y)).$$

Therefore, the dimension of ${}_h\mathcal{C}_1^1$ doesn't depend on the choice of representative set of disconnected simple objects of ${}_h\mathcal{C}_1^1$.

We say that $\mathcal{C}$ has *invertible dimensions*, if

- the dimensions $\dim(\mathcal{C}_1^1(X,X))$ are invertible in $\Bbbk$ for all simple objects X of $\mathcal{C}^1$,
- the categorical dimension $\dim({}_1\mathcal{C}_1^1)$ is invertible in $\Bbbk$,
- for every simple object X of $\mathcal{C}^1$, the number of $(1,1)$-equivalence classes of simple objects of $\mathcal{C}^1$ that are connected to X in $\mathcal{C}_1^1$ is invertible in $\Bbbk$.

By [ENO05, Theorem 2.3], any spherical σ-fusion 2-category has invertible dimensions when $\Bbbk$ is an algebraically closed field of characteristic zero.

III.6.4. Example. Let $\mathcal{G}_\sigma$ denote the linearization of σ (see Sections III.3.10 and III.5.7). For any $h \in H$,

$$\dim((\mathcal{G}_\sigma)_1^1(h,h)) = 1, \qquad \text{and} \qquad \dim({}_1(\mathcal{G}_\sigma)_1^1) = 1.$$

III.6.5. Dimension formulas. In this section, we suppose that $\mathcal{C}$ has invertible dimensions. We prove useful formulas relating the dimensions of objects and 1-morphisms in $\mathcal{C}$. These are adapted from the lemmas in [DR18, Appendix B].

Lemma III.6.3 [DR18]**.** *Let $g\colon X \to Y$ and $f\colon Y \to Z$ be 1-morphisms in $\mathcal{C}$, and assume Y is simple in $\mathcal{C}^1$. Then*

$$\dim(f \circ g) = \dim(f)\dim(g)\dim(Y)^{-1}.$$

Lemma III.6.4. *Let $f\colon X \to Y$ be a 1-morphism in $\mathcal{C}$. Then, for all $l \in L$,*

$$\sum_h \dim(\mathrm{Hom}^l_{\mathcal{C}(X,Y)}(h,f))\dim(h) = \dim(f),$$

where h runs over a representative set of simple 1-morphisms in $\mathcal{C}^1(X,Y)$.

Notice that we allow an infinite sum in Lemma III.6.4 because $\dim(\mathrm{Hom}^l_{\mathcal{C}(X,Y)}(h,f)) = 0$ for all but finitely many h.

Proof. We write $f = \bigoplus^l_{j\in J} f_j$ with f_j simple 1-morphisms in $\mathcal{C}^1(X,Y)$ for all $j \in J$. We denote by

$$\pi_j\colon f \Rightarrow f_j, \quad \text{and} \quad \iota_j\colon f_j \Rightarrow f$$

the projection and inclusion 2-morphisms for this l-direct sum. Let h be a simple 1-morphism in $\mathcal{C}^1(X,Y)$. Recall that

$$\mathrm{Hom}^l_{\mathcal{C}(X,Y)}(h,f) \cong \mathrm{Hom}^l_{\mathcal{C}(X,Y)}(h, \bigoplus_{j\in J}^l f_j) \cong \bigoplus_{j\in J} \mathrm{Hom}^1_{\mathcal{C}(X,Y)}(h,f_j).$$

We denote by $J_h \subset J$ the set of indices such that f_j is isomorphic to h in $\mathcal{C}^1(X,Y)$. Notice that

$$\dim(\mathrm{Hom}^l_{\mathcal{C}(X,Y)}(h,f)) = |J_h|.$$

Therefore,

$$\begin{aligned}\dim(f) = \operatorname{Tr}(1_f) &= \operatorname{Tr}(\sum_{j\in J} \iota_j \cdot \pi_j)\\ &= \sum_{j\in J} \operatorname{Tr}(\pi_j \cdot \iota_j)\\ &= \sum_{j\in J} \dim(f_j)\\ &= \sum_h |J_h| \dim(h),\end{aligned}$$

where in the last equality h runs over a representative set of simple 1-morphisms in $\mathcal{C}^1(X,Y)$. □

Lemma III.6.5. *Let $f\colon X \to Y$ be a 1-morphism in $\mathcal{C}$. Then, for all $e \in E$,*

$$\sum_W \sum_{g\colon W\to X} \dim(f\circ g)\dim(g)\,\mathrm{d}(W)^{-1} = \dim(f),$$

where W runs over a representative set of homogeneous simple objects in $\mathcal{C}^1$, and g runs over a representative set of degree-e simple 1-morphisms in $\mathcal{C}^1(W,X)$.

Notice that the sum over the W in Lemma III.6.5 is infinite, but this is not an issue because

$$\sum_{g\colon W\to X} \dim(f\circ g)\dim(g)\,\mathrm{d}(W)^{-1} = 0$$

for all but finitely many W.

Proof. Let W be a degree-h homogeneous simple object in $\mathcal{C}^1$. We decompose X into a $(1,e)$-direct sum

$$X = \boxplus_{i\in I}^{(1,e)} X_i,$$

with $p_i\colon X \to X_i$ homogeneous of degree e^{-1}, $q_i\colon X_i \to X$ homogeneous of degree e, and $\phi_i\colon p_i\circ q_i \xRightarrow{\sim} 1_{X_i}$ homogeneous of degree 1, for all $i \in I$. If $g\colon W \to X$ is a degree-e simple 1-morphism in $\mathcal{C}^1$, then there exists a unique index $\alpha(g) \in I$ such that $p_{\alpha(g)}\circ g \neq 0$. In fact, there is a bijection

$$\big\{\text{iso classes of simples in } \mathcal{C}^1_e(W,X)\big\} \leftrightarrow \bigsqcup_{i\in I} \big\{\text{iso classes of simples in } \mathcal{C}^1_1(W,X_i)\big\}$$

given by sending a degree-e simple 1-morphism g in $\mathcal{C}^1(W,X)$ to $p_{\alpha(g)}\circ g$, and a degree-1 simple 1-morphism h in $\mathcal{C}^1(W,X_i)$ to $q_i\circ h$. Therefore,

$$\begin{aligned}\sum_{g\colon W\to X} \dim(f\circ g)\dim(g) &= \sum_{i\in I}\sum_{h\colon W\to X_i} \dim(f\circ q_i\circ h)\dim(q_i\circ h)\\ &= \sum_{i\in I}\sum_{h\colon W\to X_i} \frac{\dim(q_i)\dim(h)}{\dim(X_i)}\dim(f\circ q_i\circ h),\end{aligned}$$

where the second equality follows from Lemma III.6.3. Therefore,

$$\sum_{g\colon W\to X} \dim(f\circ g)\dim(g) = \sum_{i\in I_W} \frac{\dim(q_i)}{\dim(X_i)}\dim(f\circ q_i)\dim(W)\dim(\mathcal{C}^1_1(W,W))$$

where $I_W \subset I$ is the set of indices such that X_i and W are connected. Notice that q_i is adjoint to p_i, therefore

$$\dim(q_i) = \dim(p_i),$$

and

$$\frac{\dim(q_i)}{\dim(X_i)}\dim(f\circ q_i) = \dim(f\circ q_i)\dim(p_i)\dim(X_i)^{-1} = \dim(f\circ q_i\circ p_i).$$

We conclude that

$$\begin{aligned}\sum_W \sum_{g\colon W\to X} \frac{\dim(f\circ g)\dim(g)}{\mathrm{d}(W)} &= \sum_W \frac{1}{\mathrm{n}(W)} \sum_{i\in I_W} \dim(f\circ q_i\circ p_i)\\ &= \sum_{i\in I} \dim(f\circ q_i\circ p_i)\\ &= \dim(f)\end{aligned}$$

where the last equality results from the fact that for any 1-morphisms h and k, $\dim(h\oplus k) = \dim(h)+\dim(k)$. □

Lemma III.6.6. *For all $e \in E$,*

$$\sum_X \sum_{f\colon X\to Y} \frac{\dim(f)^2}{\mathrm{d}(X)} = \dim(Y),$$

where X runs over a representative set of homogeneous simple objects in $\mathcal{C}^1$, and f runs over a representative set of degree-e simple 1-morphisms in $\mathcal{C}^1(X,Y)$.

Proof. Apply Lemma III.6.5 to the identity 1-morphism $1_Y\colon Y\to Y$. □

Lemma III.6.7. *Let Z be a homogeneous simple object in $\mathcal{C}^1$, and let $|Z| \in H$ denote its degree. Then, for all $h,k \in H$ and for all $e \in E$ such that*

$$\partial(e) = |Z|k^{-1}h^{-1},$$

we have

$$\sum_{X,Y} \sum_{f\colon X\otimes Y\to Z} \frac{\dim(f)^2}{\dim(Z)\,\mathrm{d}(X)\,\mathrm{d}(Y)} = \dim({}_1\mathcal{C}_1^1),$$

where X runs over a representative set of degree-h homogeneous simple objects in $\mathcal{C}^1$, Y runs over a representative set of degree-k homogeneous simple objects in $\mathcal{C}^1$, and f runs over a representative set of degree-e simple 1-morphisms in $\mathcal{C}^1(X\otimes Y, Z)$.

Proof. By sphericality and by Lemma III.6.6, the left-hand side of this equation is equal to

$$\sum_{X,Y} \sum_{f\colon X\to Z\otimes Y^\#} \frac{\dim(f)^2}{\dim(Z)\,\mathrm{d}(X)\,\mathrm{d}(Y)} = \sum_Y \frac{\dim(Z\otimes Y^\#)}{\dim(Z)\,\mathrm{d}(Y)} = \sum_Y \frac{1}{\mathrm{n}(Y)\dim(\mathcal{C}_1^1(Y,Y))} = \dim({}_k\mathcal{C}_1^1).$$

Similarly, the left-hand side is also equal to

$$\sum_{X,Y} \sum_{f\colon Y\to X^\#\otimes Z} \frac{\dim(f)^2}{\dim(Z)\,\mathrm{d}(X)\,\mathrm{d}(Y)} = \dim({}_h\mathcal{C}_1^1).$$

As a consequence, we have that $\dim({}_h\mathcal{C}_1^1) = \dim({}_1\mathcal{C}_1^1)$ for all $h \in H$. This proves the result. □

Lemma III.6.8. *Let $f\colon X\to Y$ be a 1-morphism in $\mathcal{C}$, let $e \in E$, and let $l \in L$. Then*

$$\sum_Z \sum_{h\colon X\to Z} \sum_{g\colon Z\to Y} \dim(\mathrm{Hom}^l_{\mathcal{C}(X,Y)}(g\circ h, f)) \frac{\dim(g)\dim(h)}{\mathrm{d}(Z)} = \dim(f),$$

where Z runs over a representative set of simple objects in $\mathcal{C}^1$, h runs over a representative set of simple 1-morphisms in $\mathcal{C}^1(X,Z)$, and g runs over a representative set of degree-e simple 1-morphisms in $\mathcal{C}^1(Z,Y)$.

Proof. By pivotality,

$$\mathrm{Hom}^l_{\mathcal{C}(X,Y)}(g\circ h, f) \cong \mathrm{Hom}^{e^{-1}\triangleright l}_{\mathcal{C}(X,Z)}(h, g^*\circ f).$$

Using Lemma III.6.4, we have

$$\sum_{h\colon X\to Z} \dim(\mathrm{Hom}^{e^{-1}\triangleright l}_{\mathcal{C}(X,Z)}(h, g^*\circ f))\dim(h) = \dim(g^*\circ f) = \dim(f^*\circ g).$$

Therefore, using Lemma III.6.5, the left-hand side of the equation becomes

$$\sum_{Z}\sum_{g\colon Z\to Y}\frac{\dim(g)\dim(f^*\circ g)}{\mathrm{d}(C)}=\dim(f^*),$$

and $\dim(f^*)=\dim(f)$.

□

Lemma III.6.9. *Let $f\colon X\to Y$ be a 1-morphism in $\mathcal{C}$, let $e\in E$ and $l\in L$. Then,*

$$\sum_{Z}\sum_{h\colon X\to Z}\sum_{g\colon Z\to Y}\frac{\dim(g)\dim(h)}{\mathrm{d}(Z)}\sum_{\gamma}\gamma\cdot\hat{\gamma}=1_f,$$

where Z runs over a representative set of simple objects in $\mathcal{C}^1$, h runs over a representative set of simple 1-morphisms in $\mathcal{C}^1(X,Z)$, and g runs over a representative set of degree-e simple 1-morphisms in $\mathcal{C}^1(Z,Y)$. Moreover, γ runs over a basis of the module $\mathrm{Hom}^{l}_{\mathcal{C}(X,Y)}(g\circ h,f)$, and $\hat{\gamma}$ denotes the dual of γ in the basis of $\mathrm{Hom}^{l^{-1}}_{\mathcal{C}(X,Y)}(f,g\circ h)$ with respect to the trace pairing.

Proof. We adapt the proof of [DR18, Corollary B.7]. Note that $\sum_\gamma\gamma\cdot\hat{\gamma}$ is independent of the choice of basis of $\mathrm{Hom}^{l}_{\mathcal{C}(X,Y)}(g\circ h,f)$. Let $\{s_i\}_{i\in I}$ be a representative set of simple 1-morphisms in $\mathcal{C}^1(X,Y)$. For each $i\in I$, let $\{\alpha_j^i\}_{j\in J_i}$ be a basis of $\mathrm{Hom}^1_{\mathcal{C}(X,Y)}(s_i,f)$, and let $\{\beta_k^i\}_{k\in K_i}$ be a basis of $\mathrm{Hom}^{l}_{\mathcal{C}(X,Y)}(g\circ h,s_i)$. By Lemma III.5.2, the set

$$\mathcal{B}=\bigsqcup_{i\in I}\left\{\alpha_j^i\cdot\beta_k^i\mid j\in J_i,\ k\in K_i\right\}$$

is a basis of $\mathrm{Hom}^{l}_{\mathcal{C}(X,Y)}(g\circ h,f)$.

Since s_i is simple, there are non-degenerate pairings

$$\mathrm{Hom}^1_{\mathcal{C}(X,Y)}(f,s_i)\otimes\mathrm{Hom}^1_{\mathcal{C}(X,Y)}(s_i,f)\to\Bbbk\quad\text{and}\quad\mathrm{Hom}^{l}_{\mathcal{C}(X,Y)}(g\circ h,s_i)\otimes\mathrm{Hom}^{l^{-1}}_{\mathcal{C}(X,Y)}(s_i,g\circ h)\to\Bbbk,$$

given by the vertical composition of 2-morphisms, where we identify $\mathrm{Hom}^1_{\mathcal{C}(X,Y)}(s_i,s_i)$ with $\Bbbk$. For these pairings, we denote by $\{\widehat{\alpha}_j^i\}_{j\in J_i}$ and $\{\widehat{\beta}_k^i\}_{k\in K_i}$ the dual bases of $\{\alpha_j^i\}_{j\in J_i}$ and $\{\beta_k^i\}_{k\in K_i}$, respectively. Direct calculation shows that

$$\bigsqcup_{i\in I}\left\{\frac{1}{\dim(s_i)}\widehat{\beta}_k^i\cdot\widehat{\alpha}_j^i\mid k\in K_i,\ j\in J_i\right\}$$

is a dual basis of $\mathcal{B}$ for the non-degenerate trace pairing (see Lemma III.6.2)

$$\mathrm{Hom}^{l}_{\mathcal{C}(X,Y)}(g\circ h,f)\otimes\mathrm{Hom}^{l^{-1}}_{\mathcal{C}(X,Y)}(f,g\circ h)\to\Bbbk.$$

With this choice of basis, the left-hand side of the equation becomes

$$\sum_{Z}\sum_{h\colon X\to Z}\sum_{g\colon Z\to Y}\frac{\dim(g)\dim(h)}{\mathrm{d}(Z)}\sum_{i\in I}\frac{1}{\dim(s_i)}\sum_{j\in J_i}\sum_{k\in K_i}\alpha_j^i\cdot\beta_k^i\cdot\widehat{\beta}_k^i\cdot\widehat{\alpha}_j^i.$$

Notice that $\sum_{k\in K_i}\beta_k^i\cdot\widehat{\beta}_k^i=\dim(\mathrm{Hom}^{l}_{\mathcal{C}(X,Y)}(g\circ h,s_i))1_{s_i}$. Therefore, by Lemma III.6.8, we get that the left-hand side of the equation is equal to

$$\sum_{i\in I}\frac{1}{\dim(s_i)}\sum_{j\in J_i}\alpha_j^i\cdot\widehat{\alpha}_j^i\dim(s_i),$$

and $\sum_{i\in I}\sum_{j\in J_i}\alpha_j^i\cdot\widehat{\alpha}_j^i=1_f$. □

CHAPTER IV

State-sum invariants of flat 3-bundles over 4-manifolds

In this chapter, given a 2-crossed module of groups σ, we use a spherical σ-fusion 2-category $\mathcal{C}$ with invertible dimensions to construct invariants of flat σ-bundles over 4-manifolds or, equivalently, of σ-manifolds.

Our construction is based on the topological and algebraic frameworks developed in the previous chapters and uses a state-sum method. We assign algebraic data from $\mathcal{C}$ to the simplices of a triangulation of a σ-manifold (W, g), thus adapting the construction of Douglas and Reutter and making it sensitive to homotopy classes of maps. More explicitly, we define a notion of $\mathcal{C}$-state that colors the edges of a triangulation with simple objects and the triangles with simple 1-morphisms, subject to compatibility conditions determined by the σ-coloring. From every such $\mathcal{C}$-state, we derive colored 10j-symbols, which are then summed over all states to obtain a scalar $\mathrm{HDR}_{\mathcal{C}}(W, g)$. Recall that two σ-manifolds are equivalent if and only if their triangulations are related by a finite sequence of σ-colored Pachner moves, see Theorem 4. Therefore, by verifying that the state sum is invariant with respect to the σ-colored Pachner moves, we obtain an invariant of σ-manifolds.

This chapter is organized as follows. In Section IV.1 we introduce the notion of $\mathcal{C}$-state on a σ-colored simplicial complex and define the 10j-symbols associated with these states. Section IV.2 is devoted to the definition of the state sum $\mathrm{HDR}_{\mathcal{C}}(W, g)$ of σ-manifolds and shows that this invariant is nontrivial. Finally, Section IV.3 provides the proof that the state sum is an invariant by showing that it is independent of the choices made for its construction.

Throughout this chapter,

$$\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega)$$

is a 2-crossed module of groups, and $\mathcal{C}$ is a spherical σ-fusion 2-category over a nonzero commutative ring $\Bbbk$.

IV.1. Colored 10j-symbols

Let (K, ϕ) be a σ-colored simplicial complex (see Section I.2.5). We introduce a notion of $\mathcal{C}$-state of (K, ϕ) that associates to each edge of K a simple object of $\mathcal{C}$ and to each triangle of K a simple 1-morphism of $\mathcal{C}$ in a way that is compatible with the coloring ϕ. From every such state, we derive colored 10j-symbols that generalize the 10j-symbols for spherical fusion 2-categories defined by Douglas and Reutter [DR18]. This enables us to obtain a scalar, called the 10j-action of the canonical associated state.

IV.1.1. Simplicial skeletons of spherical graded-fusion 2-categories. Recall that $\mathcal{C}^1$ denotes the 1-subcategory of $\mathcal{C}$, that is, the linear monoidal 2-category with the same objects and 1-morphisms as $\mathcal{C}$ and only the 2-morphisms of degree 1, see Section III.3.6. The spherical σ-fusion 2-category $\mathcal{C}$ determines a 2-truncated semisimplicial set $\Delta(\mathcal{C})$ with $\Delta(\mathcal{C})_0 = \{*\}$, $\Delta(\mathcal{C})_1 = \{\text{homogeneous simple objects of } \mathcal{C}^1\}$, and

$$\Delta(\mathcal{C})_2 = \{(X, Y, Z, f) \mid X, Y, Z \in \Delta(\mathcal{C})_1, f \text{ is a simple 1-morphism in } \mathcal{C}^1(X \otimes Y, Z)\}.$$

A *simplicial skeleton* of $\mathcal{C}$ is a subsemisimplicial set $\mathcal{S} \subset \Delta(\mathcal{C})$ such that:

- the tensor unit $\mathbb{1}$ is an element of $\mathcal{S}_1$,
- if $X \in \mathcal{S}_1$, then $X^{\#} \in \mathcal{S}_1$,
- every element of $\Delta(\mathcal{C})_1$ is $(1,1)$-equivalent to exactly one element of $\mathcal{S}_1$,
- for every element $X \in \mathcal{S}_1$, $(X, \mathbb{1}, X, 1_X)$ and $(\mathbb{1}, X, X, 1_X)$ are elements of $\mathcal{S}_2$,
- if $(X, Y, Z, f) \in \mathcal{S}_2$, then $(Z, Y^{\#}, X, (X \otimes e_Y) \circ (f^* \otimes Y^{\#})) \in \mathcal{S}_2$,
- for all $X, Y, Z \in \mathcal{S}_1$, and simple $f \in \mathcal{C}^1(X \otimes Y, Z)$, there exists a unique $(X, Y, Z, g) \in \mathcal{S}_2$ such that f and g are isomorphic in $\mathcal{C}^1$.

IV.1.2. States. Let $E(K)$ denote the set of ordered edges of K, and let $T(K)$ denote the set of ordered triangles of K, see Section I.2.6. A *$\mathcal{C}$-state* of (K,ϕ) is a pair of set maps

$$\Gamma = (\Gamma_1 \colon E(K) \to \Delta(\mathcal{C})_1,\ \Gamma_2 \colon T(K) \to \Delta(\mathcal{C})_2)$$

such that for every oriented edge $\langle v_0, v_1\rangle$ of K, the object $\Gamma_1\langle v_0, v_1\rangle$ is homogeneous of degree $\phi_1\langle v_0, v_1\rangle$ and

$$\Gamma_1\langle v_1, v_0\rangle = \Gamma_1\langle v_0, v_1\rangle^{\#},$$

and for every oriented triangle $\langle v_0, v_1, v_2\rangle$ of K with distinguished vertex v_0, the 1-morphism $\Gamma_2\langle v_0, v_1, v_2\rangle$ goes from $\Gamma_1\langle v_0, v_1\rangle \otimes \Gamma_1\langle v_1, v_2\rangle$ to $\Gamma_1\langle v_0, v_2\rangle$, is homogeneous of degree $\phi_2\langle v_0, v_1, v_2\rangle$, and verifies

$$\Gamma_2\langle v_0, v_2, v_1\rangle = \left(\Gamma_1\langle v_0, v_1\rangle \otimes e_{\Gamma_1\langle v_1, v_2\rangle}\right) \circ \left(\Gamma_2\langle v_0, v_1, v_2\rangle^* \otimes \Gamma_1\langle v_1, v_2\rangle^{\#}\right).$$

We say that a $\mathcal{C}$-state of (K,ϕ) *takes values* in a simplicial skeleton $\mathcal{S}$ of $\mathcal{C}$ if $\Gamma_1(E(K)) \subset \mathcal{S}_1$ and $\Gamma_2(T(K)) \subset \mathcal{S}_2$.

IV.1.3. Change of pointing for an oriented triangle. Let $\mathcal{S}$ be a simplicial skeleton of $\mathcal{C}$, and let Γ be a $\mathcal{C}$-state of (K,ϕ) taking values in $\mathcal{S}$. For any ordered triangle $\langle v_0, v_1, v_2\rangle$ of K and for any 1-morphism

$$f \colon \Gamma_1\langle v_0, v_1\rangle \otimes \Gamma_1\langle v_1, v_2\rangle \to \Gamma_1\langle v_0, v_2\rangle$$

in $\mathcal{S}_2$, we denote by ${}^{\Gamma_1\langle v_1, v_0\rangle}f \colon \Gamma_1\langle v_1, v_2\rangle \otimes \Gamma_1\langle v_2, v_0\rangle \to \Gamma_1\langle v_1, v_0\rangle$ the 1-morphism in $\mathcal{S}_2$ that is degree-1 isomorphic to

$$(\Gamma_1\langle v_1, v_0\rangle \otimes e_{\Gamma_1\langle v_0, v_2\rangle}) \circ (\Gamma_1\langle v_1, v_0\rangle \otimes f \otimes \Gamma_1\langle v_2, v_0\rangle) \circ (i_{\Gamma_1\langle v_0, v_1\rangle} \otimes \Gamma_1\langle v_1, v_2\rangle \otimes \Gamma_1\langle v_2, v_0\rangle).$$

Similarly, we denote by ${}^{\Gamma_1\langle v_2, v_0\rangle}f \colon \Gamma_1\langle v_0, v_2\rangle \otimes \Gamma_1\langle v_2, v_1\rangle \to \Gamma_1\langle v_0, v_1\rangle$ the 1-morphism in $\mathcal{S}_2$ that is degree-1 isomorphic to

$$(e_{\Gamma_1\langle v_2, v_0\rangle} \otimes \Gamma_1\langle v_2, v_1\rangle) \circ (\Gamma_1\langle v_2, v_0\rangle \otimes f \otimes \Gamma_1\langle v_2, v_1\rangle) \circ (\Gamma_1\langle v_2, v_0\rangle \otimes \Gamma_1\langle v_0, v_1\rangle \otimes i_{\Gamma_1\langle v_2, v_1\rangle}).$$

Notice that

$${}^{\Gamma_1\langle v_0, v_1\rangle\Gamma_1\langle v_1, v_0\rangle}f = f = {}^{\Gamma_1\langle v_0, v_2\rangle\Gamma_1\langle v_2, v_0\rangle}f.$$

This allows the definition of the 1-morphism ${}^{\Gamma_1(\gamma)}f$ for γ the homotopy class of a path in the boundary of $\langle v_0, v_1, v_2\rangle$ that ends at v_0 by decomposing γ into edges.

IV.1.4. Associator state spaces. Let Γ be a $\mathcal{C}$-state of (K,ϕ) taking values in a simplicial skeleton $\mathcal{S}$. For any pointed ordered tetrahedron $(\langle v_0, v_1, v_2, v_3\rangle, \gamma_0, \gamma_1, \gamma_2, \gamma_3)$ of K, we write:

$$[(v_0v_1v_2)v_3, \gamma_1, \gamma_3] = {}^{\Gamma_1(\gamma_1)}\Gamma_2\langle v_0, v_2, v_3\rangle \circ \left({}^{\Gamma_1(\gamma_3)}\Gamma_2\langle v_0, v_1, v_2\rangle \otimes \Gamma_1\langle v_2, v_3\rangle\right)$$
$$[v_0(v_1v_2v_3), \gamma_0, \gamma_2] = {}^{\Gamma_1(\gamma_2)}\Gamma_2\langle v_0, v_1, v_3\rangle \circ \left(\Gamma_1\langle v_0, v_1\rangle \otimes {}^{\Gamma_1(\gamma_0)}\Gamma_2\langle v_1, v_2, v_3\rangle\right).$$

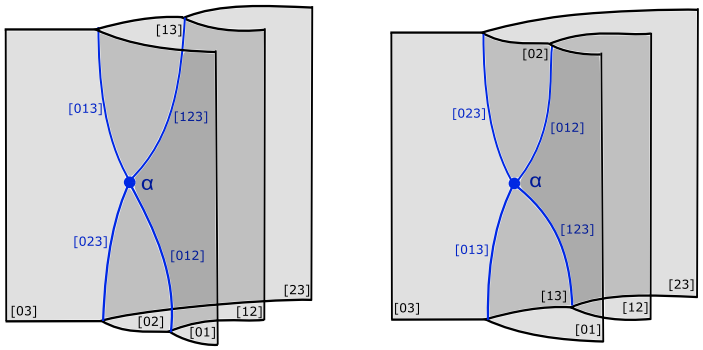
Let $\tau = (\langle v_0, v_1, v_2, v_3\rangle, \gamma_0, \gamma_1, \gamma_2, \gamma_3)$ be a pointed ordered tetrahedron of K. The *positive and negative associator state spaces of τ* are

$$V^+(\Gamma, \tau) = \mathrm{Hom}_{\mathcal{C}}^{\phi_3(\tau)}\Big([(v_0v_1v_2)v_3, \gamma_1, \gamma_3], [v_0(v_1v_2v_3), \gamma_0, \gamma_2]\Big)$$

and

$$V^-(\Gamma, \tau) = \mathrm{Hom}_{\mathcal{C}}^{\phi_3(\tau)^{-1}}\Big([v_0(v_1v_2v_3), \gamma_0, \gamma_2], [(v_0v_1v_2)v_3, \gamma_1, \gamma_3]\Big),$$

respectively. Elements in the associator state spaces are depicted graphically as in Figure 1, where we use the shorthand $[ij]$ to denote $\Gamma_1\langle v_i, v_j\rangle$, and $[ijk]$ to denote ${}^{\Gamma_1(\gamma_l)}\Gamma_2\langle v_i, v_j, v_k\rangle$. We refer the reader to the work of Barrett–Meusburger–Schaumann [BMS24] and Douglas–Reutter [DR18] for more details on graphical calculus in spherical 2-categories.



$\alpha \in V^+(\Gamma,\tau)$ $\alpha \in V^-(\Gamma,\tau)$

FIGURE 1. Elements in the graded associator state spaces

IV.1.5. The canonical associated state. Let Γ be a $\mathcal{C}$-state of (K,ϕ) taking values in a simplicial skeleton $\mathcal{S}$, and let B_* be a choice of ordered simplices of K (see Section I.2.7). The trace pairing

$$\langle\cdot,\cdot\rangle_{\Gamma,\tau}\colon V^+(\Gamma,\tau)\otimes V^-(\Gamma,\tau)\to\Bbbk$$

is nondegenerate for any pointed ordered tetrahedron τ (see Lemma III.6.2). Therefore, there is a canonical copairing

$$\cup_{\Gamma,\tau}\colon \Bbbk\to V^-(\Gamma,\tau)\otimes V^+(\Gamma,\tau).$$

The *canonical associated state* $\cup_\Gamma$ of Γ is the element

$$\cup_\Gamma = \bigotimes_{\tau\in B_3} \cup_{\Gamma,\tau}(1) \in \bigotimes_{\tau\in B_3} \left(V^+(\Gamma,\tau)\otimes V^-(\Gamma,\tau)\right),$$

where $\otimes$ denotes the unordered tensor product of $\Bbbk$-modules.

IV.1.6. The colored 10j-symbols. Let B_* be a choice of ordered simplices of K (see Section I.2.7) and let $\kappa\in B_4$. We denote by $B_3\cap\kappa$ the set of pointed ordered tetrahedra in B_3 that are included in κ. The *sign map*

$$\varepsilon\colon B_3\cap\kappa\to\{+,-\}$$

is defined as follows. Using the linear order on the vertices of κ, we write $\kappa=\langle 0,1,2,3,4\rangle$. Let $\tau\in B_3\cap\kappa$ and let $m\in\{0,1,2,3,4\}$ be the vertex of κ that is not in τ. Let $\partial_m(\kappa)$ be the m-th face of κ with pointing and order induced by restricting the pointing and order of κ. Let μ be the permutation that sends the order of $\partial_m(\kappa)$ to the order of τ. Then the sign of τ is defined by

$$\varepsilon(\tau)=(-1)^m\mathrm{s}(\mu),$$

where $\mathrm{s}(\mu)$ is the signature of μ. Moreover, there are isomorphisms of associator state spaces

$$\phi_+\colon V^{\varepsilon(\tau)}(\Gamma,\tau)\xrightarrow{\sim} V^{(-1)^m}(\Gamma,\partial_m(\kappa)) \qquad \phi_-\colon V^{-\varepsilon(\tau)}(\Gamma,\tau)\xrightarrow{\sim} V^{(-1)^{m+1}}(\Gamma,\partial_m(\kappa))$$

induced by the isomorphisms in [DR18, Lemma 4.2.4] and by the isomorphisms in Equation (IV.3.3) for the change of pointings.

Let Γ be a $\mathcal{C}$-state of (K,ϕ). The *colored 10j-symbol* associated to a pair (Γ,κ), where $\kappa\in B_4$, is the linear form $z(\Gamma,\kappa)$ defined as follows. If the orientation of κ (induced by the order on its vertices) coincides with the orientation of K, then

$$z(\Gamma,\kappa)\colon \bigotimes_{\tau\in B_3\cap\kappa} V^{\varepsilon(\tau)}(\Gamma,\tau)\to\Bbbk$$

is defined as the trace of the 2-morphism depicted on the left in Figure 2 composed with the isomorphisms ϕ_+ described above, where $[ijkl]$ denotes an element of $V^{(-1)^m}(\Gamma,\partial_m(\kappa))$. If the orientation of κ (induced by the order on its vertices) does not coincide with the orientation of K, then

$$z(\Gamma,\kappa)\colon \bigotimes_{\tau\in B_3\cap\kappa} V^{-\varepsilon(\tau)}(\Gamma,\tau)\to\Bbbk$$

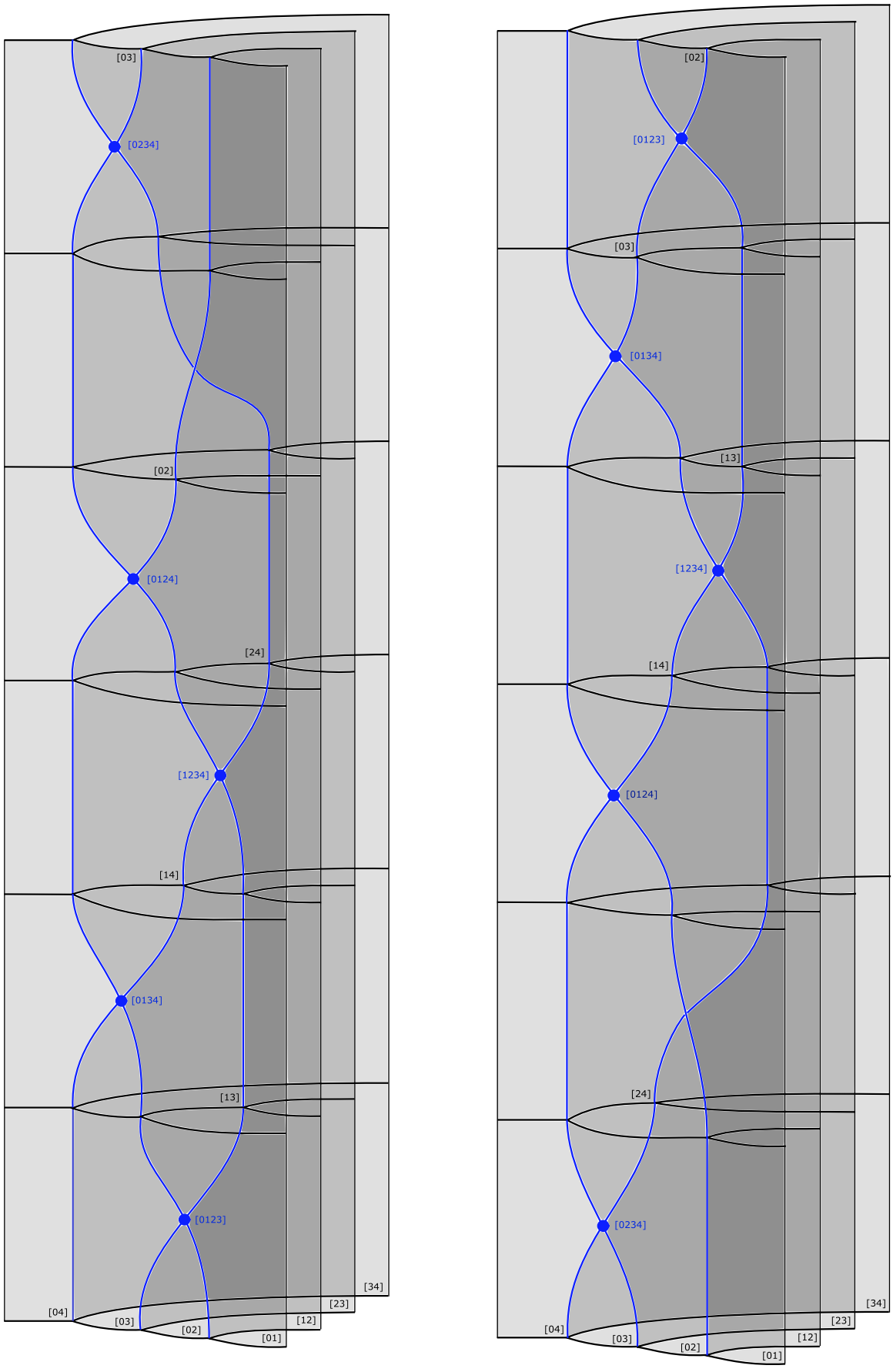


FIGURE 2. The colored 10j-symbols

is defined as the trace of the 2-morphism depicted on the right in Figure 2 composed with the isomorphisms ϕ_- described above, where $[ijkl]$ denotes an element of $V^{(-1)^{m+1}}(\Gamma, \partial_m(\kappa))$.

IV.1.7. The 10j-action of the canonical associated state. In this section, we assume that (K, ϕ) is a combinatorial σ-manifold (see Section II.2.4). Let Γ be a $\mathcal{C}$-state of (K, ϕ) taking values in a simplicial skeleton $\mathcal{S}$, and let B_* be a choice of ordered simplices of K. The *10j-action of the canonical associated state* of Γ is

$$Z(\Gamma) = \left(\bigotimes_{\kappa \in B_4} z(\Gamma, \kappa) \right) (\cup_\Gamma) \in \Bbbk.$$

Notice that for every $\tau \in B_3$, the associator state spaces $V^+(\Gamma, \tau)$ and $V^-(\Gamma, \tau)$ appear exactly once in the domain of $\otimes_{\kappa \in B_4} z(\Gamma, \kappa)$, and therefore the composition is well defined.

IV.2. The state sum for 3-group-manifolds

In this section, we assume that the spherical σ-fusion 2-category $\mathcal{C}$ has invertible dimensions (see Section III.6.3), and we derive from $\mathcal{C}$ an invariant of σ-manifolds (see Section II.2.3).

IV.2.1. The state sum. Let (W, g) be a σ-manifold. Pick a σ-triangulation (K, ϕ) of (W, g) (see Section II.2.4), a choice of ordered simplices B_* of K (see Section I.2.7), and a simplicial skeleton $\mathcal{S}$ of $\mathcal{C}$ (see Section IV.1.1). Consider the state sum

$$\mathrm{HDR}_{\mathcal{C}}(W, g) = \dim({}_1\mathcal{C}_1^1)^{-|K_0|} \sum_\Gamma \Big(\prod_{e \in B_1} \mathrm{d}(\Gamma_1(e))^{-1} \Big) \Big(\prod_{t \in B_2} \dim(\Gamma_2(t)) \Big) Z(\Gamma) \in \Bbbk,$$

where Γ runs over the $\mathcal{C}$-states of (K, ϕ) taking values in $\mathcal{S}$, e runs over the edges of K, and t runs over the triangles of K. Here, $\dim({}_1\mathcal{C}_1^1)$ denotes the dimension of the neutral component of $\mathcal{C}$ (see Section III.6.3), $\mathrm{d}(\Gamma_1(e))$ denotes the weighted dimension of the object $\Gamma_1(e)$ (see Section III.6.2), and $\dim(\Gamma_2(t))$ denotes the dimension of the 1-morphism $\Gamma_2(t)$ (see Section III.6.1).

Theorem 5. *The state sum* $\mathrm{HDR}_{\mathcal{C}}(W, g)$ *is an invariant of* σ*-manifolds.*

The invariant $\mathrm{HDR}_{\mathcal{C}}$ of σ-manifolds can be seen as an invariant of flat σ-bundles over 4-manifolds (see Section II.2.3). We show in Section IV.2.3 that this invariant is nontrivial.

We prove Theorem 5 in Section IV.3. The proof consists in showing that the state sum does not depend on the various choices, and it relies on the colored Pachner theorem (Theorem 4).

IV.2.2. Example: the Douglas–Reutter invariant. Let $(W, *)$ be a σ-manifold with $*$ the homotopy class of a constant map. Recall that the neutral component of $\mathcal{C}$ is the 2-category ${}_1\mathcal{C}_1^1$ with only degree-1 homogeneous objects, 1-morphisms, and 2-morphisms (see Section III.3.9). Notice that ${}_1\mathcal{C}_1^1$ is a fusion 2-category in the sense of Douglas and Reutter (up to additive and idempotent completion), and

$$\mathrm{HDR}_{\mathcal{C}}(W, *) = \mathrm{DR}_{{}_1\mathcal{C}_1^1}(W),$$

where $\mathrm{DR}_{{}_1\mathcal{C}_1^1}(W)$ is the Douglas–Reutter invariant of W defined using ${}_1\mathcal{C}_1^1$.

IV.2.3. Example: nontriviality of the homotopy invariant. In this section we fix $\Bbbk = \mathbb{Q}$, and we set

$$\sigma = (\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} 0,\ 0).$$

The classifying space $B\sigma$ is a $K(\mathbb{Z}/2\mathbb{Z}, 3)$ space. Let $W = (S^1)^4$ be the product of four circles. Then the set $[W, B\sigma]$ is in bijection with $\mathrm{H}^3(W; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^4$. Let $*$ be the trivial class in $[W, B\sigma]$, and pick g a nontrivial class. We construct a σ-fusion 2-category $\mathcal{C}$ such that

$$\mathrm{HDR}_{\mathcal{C}}(W, *) \neq \mathrm{HDR}_{\mathcal{C}}(W, g).$$

Let

$$\sigma' = (\mathbb{Z}/2\mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0),$$

and let $\phi = (\phi_1, \phi_2, \phi_3) \colon \sigma' \to \sigma$ be the 2-crossed module morphism with $\phi_3 = \mathrm{Id}$. Let $\mathcal{G}_{\sigma'}$ be the linearization of σ' (see Section III.5.7), and let $\mathcal{C} = \phi_*(\mathcal{G}_{\sigma'})$ be the pushforward of $\mathcal{G}_{\sigma'}$ (see Section III.5.8).

We first compute $\mathrm{HDR}_{\mathcal{C}}(W,*)$. We have

$$\dim(\mathcal{C}_1^1(\mathbb{1},\mathbb{1}))=2 \quad \text{and} \quad \dim({}_1\mathcal{C}_1^1)=\frac{1}{2}.$$

Let K be a triangulation of W, and let 1 be the trivial σ-coloring of K. Then $(K,1)$ is a σ-triangulation of $(W,*)$. Notice that for any $\mathcal{C}$-coloring Γ of $(K,1)$, we have $Z(\Gamma)=1$ if Γ induces a ρ-coloring of K, where

$$\rho=(0 \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} 0,\ 0),$$

and $Z(\Gamma)=0$ otherwise. Therefore,

$$\mathrm{HDR}_{\mathcal{C}}(W,*)=2^{|K^{(0)}|-|K^{(1)}|}\#\{\rho\text{-colorings of } K\}.$$

In fact, it is a consequence of Section IV.2.2 that $\mathrm{HDR}_{\mathcal{C}}(W,*)$ coincides with the (untwisted) Yetter–Dijkgraaf–Witten invariant of W associated to the crossed module $\chi=(\mathbb{Z}/2\mathbb{Z}\to 0)$. Therefore, by [FMP07, Theorem 1.21],

$$\mathrm{HDR}_{\mathcal{C}}(W,*)=\sum_{g\in[W,B\chi]}\frac{|\pi_2(\mathrm{TOP}(W,B\chi),g)|}{|\pi_1(\mathrm{TOP}(W,B\chi),g)|},$$

where $B\chi$ denotes the classifying space of χ (see [BHS11]) and $\mathrm{TOP}(W,B\chi)$ denotes the space of maps $W\to B\chi$ with the compact-open topology. For all $n\geqslant 1$ and $g\in[W,B\chi]$, we have

$$\begin{aligned}\pi_n(\mathrm{TOP}(W,B\chi),g)&\cong\pi_n(\mathrm{TOP}(W,B\chi),*)\\&\cong[S^n,\mathrm{TOP}(W,B\chi)]_*\\&\cong[(S^n\times W,\{*\}\times W),(B\chi,*)]\\&\cong \mathrm{H}^2(S^n\times W,\{*\}\times W;\mathbb{Z}/2\mathbb{Z}).\end{aligned}$$

Using the relative Künneth formula,

$$\mathrm{HDR}_{\mathcal{C}}(W,*)=|\mathbb{Z}/2\mathbb{Z}|^6\frac{|\mathbb{Z}/2\mathbb{Z}|}{|\mathbb{Z}/2\mathbb{Z}|^4}=8.$$

We now compute $\mathrm{HDR}_{\mathcal{C}}(W,g)$. Let (K,φ) be a σ-triangulation of (W,g). The state sum $\mathrm{HDR}_{\mathcal{C}}(W,g)$ is a sum over the $\mathcal{C}$-states of (K,φ). By considering the degrees in σ', any $\mathcal{C}$-state of K induces a σ'-coloring $\widetilde{\varphi}$ of K such that the diagram

$$\begin{array}{ccc}&&\sigma'\\&\overset{\widetilde{\varphi}}{\nearrow}&\big\downarrow{\scriptstyle\phi}\\\Pi_3(K)&\xrightarrow{\varphi}&\sigma\end{array}$$

commutes. Since $B\sigma'$ is contractible and g is not homotopically trivial, no such lifts can exist, and therefore

$$\mathrm{HDR}_{\phi_*(\mathcal{C})}(W,g)=0.$$

Therefore, $\mathrm{HDR}_{\phi_*(\mathcal{C})}(W,g)\neq\mathrm{HDR}_{\phi_*(\mathcal{C})}(W,*)$. Also, all the maps $W\to B\sigma$ are phantom maps (meaning that they induce trivial homomorphisms on homotopy groups), because $\pi_3(W)=0$ and $B\sigma$ is a $K(\mathbb{Z}/2\mathbb{Z},3)$ space. This shows that the invariant $\mathrm{HDR}_{\mathcal{C}}$ is nontrivial and can distinguish homotopy classes of phantom maps.

IV.2.4. Relation with other quantum homotopy invariants. Let G be a discrete group. In [Moc24], Mochida uses the data of a finite type involutory quasitriangular Hopf G-algebra to derive invariants of flat G-bundles over closed, oriented smooth 4-manifolds. These invariants are obtained via a 4-dimensional analogue of the Hennings-type construction using G-colored Kirby diagrams. We denote by σ the 2-crossed module

$$\sigma=(1\longrightarrow 1\longrightarrow G,\ 1).$$

Notice that the data of a flat G-bundle over a closed, oriented smooth 4-manifold is equivalent to that of a σ-manifold. We expect that in certain cases, the delooping of the category of representations of an involutory quasitriangular Hopf G-algebra should yield a spherical σ-fusion 2-category. Then, we conjecture that the invariants of Mochida could be recovered as the state-sum invariants from this σ-fusion 2-category.

The invariants of Mochida have been generalized by Bridges and Cui [BC25] using the data of a finite type involutory Hopf G-triplet. Their invariants are constructed via a 4-dimensional analogue of the Kuperberg-type

construction via G-colored trisection diagrams. We also expect that in certain cases, these should be recovered by our state-sum construction.

IV.2.5. Induced invariants of 4-manifolds. In this section, we assume that the 2-crossed module σ is *homotopy finite*, meaning that the groups $\pi_1(B\sigma)$, $\pi_2(B\sigma)$, and $\pi_3(B\sigma)$ are all finite. Let W be a closed, oriented smooth 4-manifold. In this case, the set of homotopy classes of maps $W \to B\sigma$ is also finite (see [Qui91] or [FMP23, Lemma 47]). Therefore, the state sum for σ-manifolds yields invariants of 4-manifolds by summing over all homotopy classes of maps to $B\sigma$:

$$|W| = \sum_{g \in [W, B\sigma]} \lambda(W, g)\, \mathrm{HDR}_{\mathcal{C}}(W, g),$$

where $\lambda(W, g) \in \Bbbk$ is an arbitrary scalar invariant of σ-manifolds.

In particular, let $\mathrm{TOP}(W, B\sigma)$ be the space of maps $W \to B\sigma$ with the compact-open topology, and set

$$\lambda(W, g) = \frac{|\pi_2(\mathrm{TOP}(W, B\sigma), g)|}{|\pi_1(\mathrm{TOP}(W, B\sigma), g)||\pi_3(\mathrm{TOP}(W, B\sigma), g)|} \in \mathbb{Q}$$

for all $g \in [W, B\sigma]$. These scalars are well defined invariants of σ-manifolds, see [Qui91, FMP23]. Let $\Bbbk = \mathbb{Q}$, and let $\mathcal{C} = \mathcal{G}_\sigma$ be the linearization of σ. Then,

$$\mathrm{HDR}_{\mathcal{C}}(W, g) = 1$$

for all $g \in [W, B\sigma]$, and

$$|W| = \sum_{g \in [W, B\sigma]} \frac{|\pi_2(\mathrm{TOP}(W, B\sigma), g)|}{|\pi_1(\mathrm{TOP}(W, B\sigma), g)||\pi_3(\mathrm{TOP}(W, B\sigma), g)|}.$$

Notice that since $B\sigma$ is a homotopy 3-type, $\pi_n(\mathrm{TOP}(W, B\sigma), g) = 0$ for all $n \geqslant 4$. Therefore, $|W|$ coincides with the invariant from Quinn's finite total homotopy TQFT, see Faria Martins–Porter [FMP23, Definition 51]. As a special case, when the Peiffer lifting of σ is trivial, we recover the Yetter invariant associated to the crossed complex defined by σ, see [FMP07, Theorem 2.33].

IV.3. Proof of Theorem 5

The aim of this section is to show that the state sum is an invariant of σ-manifolds. In Section IV.3.1 we show that the state sum does not depend on the choice of simplicial skeleton of $\mathcal{C}$. In Section IV.3.2 we show that it does not depend on the choice of ordered simplices. In Section IV.3.3 we extend the state sum to combinatorial σ-manifolds with boundary. In Section IV.3.4 we show that it does not depend on the choice of triangulation.

IV.3.1. The choice of simplicial skeleton. Let (K, ϕ) be a combinatorial σ-manifold with choice of ordered simplices B_*. Two $\mathcal{C}$-states Γ and Γ' of (K, ϕ) are *equivalent* if for every edge $e \in B_1$, there are inverse $(1,1)$-equivalences

$$h_e \colon \Gamma(e) \rightleftarrows \Gamma'(e) \colon k_e$$

and for every triangle $\langle ijk\rangle \in B_2$ with distinguished vertex i, there are degree-1 homogeneous 2-isomorphisms

$$\Gamma\langle ijk\rangle \cong k_{\langle ir\rangle} \circ \Gamma'(t) \circ (h_{\langle ij\rangle} \otimes \Gamma'\langle jk\rangle) \circ (\Gamma\langle ij\rangle \otimes h_{\langle jk\rangle}),$$

where $h_{\langle ab\rangle}$ denotes the right mate of $k_{\langle ba\rangle}$ if $\langle ba\rangle \in B_1$, and $k_{\langle ab\rangle}$ denotes the right mate of $h_{\langle ba\rangle}$ if $\langle ba\rangle \in B_1$.

Given two simplicial skeletons $\mathcal{S}$ and $\mathcal{S}'$ of $\mathcal{C}$, for each object X in $\mathcal{S}$, choose inverse $(1,1)$-equivalences

$$h_X \colon X \leftrightarrows X' \colon k_X$$

between X and the unique $(1,1)$-equivalent object X' in $\mathcal{S}'$. Similarly, for every 1-morphism $g \colon X \otimes Y \to Z$ in $\mathcal{S}$, choose a degree-1 homogeneous isomorphism between $h_Z \circ g \circ (k_X \otimes Y) \circ (X' \otimes k_y)$, and the unique degree-1 homogeneous isomorphic 1-morphism $g' \colon X' \otimes Y' \to Z'$ in $\mathcal{S}$. This construction provides a bijection between the $\mathcal{C}$-states with values in $\mathcal{S}$ and the $\mathcal{C}$-states with values in $\mathcal{S}'$, and this bijection takes each state to an equivalent state. Therefore, in order to show that the state sum is independent of the choice of simplicial skeleton, it is sufficient to show the following:

Lemma IV.3.1 [DR18]**.** *If two $\mathcal{C}$-states Γ and Γ' of (K, ϕ) are equivalent, then*

$$\Big(\prod_{e \in B_1} \mathrm{d}(\Gamma(e))^{-1}\Big)\Big(\prod_{t \in B_2} \dim(\Gamma(t))\Big) Z(\Gamma) = \Big(\prod_{e \in B_1} \mathrm{d}(\Gamma'(e))^{-1}\Big)\Big(\prod_{t \in B_2} \dim(\Gamma'(t))\Big) Z(\Gamma').$$

IV.3.2. The choice of ordered simplices. Let K be a canonical 4-manifold and let ϕ be a σ-coloring of K. We would like to show that the state sum doesn't depend on the choice of ordered simplices of K. It is sufficient to show the following:

Lemma IV.3.2. *If B_* and B'_* are choices of ordered simplices of K and Γ is a $\mathcal{C}$-state of (K,ϕ), then*

$$\Big(\prod_{e\in B_1}\mathrm{d}(\Gamma(e))^{-1}\Big)\Big(\prod_{t\in B_2}\dim(\Gamma(t))\Big)Z(\Gamma)=\Big(\prod_{e'\in B'_1}\mathrm{d}(\Gamma(e'))^{-1}\Big)\Big(\prod_{t'\in B'_2}\dim(\Gamma(t'))\Big)Z(\Gamma).$$

To prove Lemma IV.3.2, it suffices to check that the above quantity is invariant by the change of order and of pointing of 1-, 2-, 3- and 4-simplices of K.

Suppose that B_1 and B'_1 are related by the orientation reversal of an edge $\langle ij\rangle\in B_1$. It suffices to check that

$$\mathrm{d}(\Gamma\langle ij\rangle)=\mathrm{d}(\Gamma\langle ji\rangle).$$

Firstly, by sphericality, $\dim(\Gamma\langle ij\rangle)=\dim(\Gamma\langle ij\rangle^{\#})=\dim(\Gamma\langle ji\rangle)$. Secondly, the fusion category $\mathcal{C}_1^1(\Gamma\langle ij\rangle,\Gamma\langle ij\rangle)$ is equivalent to the category $\mathcal{C}_1^1(\Gamma\langle ij\rangle^{\#},\Gamma\langle ij\rangle^{\#})^{\mathrm{mop}}$ with opposite monoidal product, therefore

$$\dim\left(\mathcal{C}_1^1(\Gamma\langle ij\rangle,\Gamma\langle ij\rangle)\right)=\dim\left(\mathcal{C}_1^1(\Gamma\langle ji\rangle,\Gamma\langle ji\rangle)\right).$$

Finally, $\mathrm{n}(\Gamma\langle ij\rangle)=\mathrm{n}(\Gamma\langle ji\rangle)$ because simple objects are connected if and only if their duals are connected.

Suppose that B_2 and B'_2 are related by the orientation reversal of a triangle $\langle ijk\rangle\in B_2$ with distinguished vertex i. It suffices to check that $\dim(\Gamma\langle ijk\rangle)=\dim(\Gamma\langle ikj\rangle)$. Recall that

$$\Gamma\langle ikj\rangle=(\Gamma\langle ij\rangle\otimes e_{\Gamma\langle jk\rangle})\circ(\Gamma\langle ijk\rangle^*\otimes\Gamma\langle jk\rangle^{\#}),$$

and $\dim(\Gamma\langle ijk\rangle)=\dim(\Gamma\langle ijk\rangle^*)$, therefore $\dim(\Gamma\langle ijk\rangle)=\dim(\Gamma\langle ikj\rangle)$.

Suppose that B_3 and B'_3 are related by the change of pointing of a 3-simplex $\tau\in B_3$. For instance, suppose that $\tau=(\langle ijkl\rangle,\gamma_i,\gamma_j,\gamma_k,\gamma_l)$ and $\tau'=(\langle ijkl\rangle,\gamma'_i,\gamma_j,\gamma_k,\gamma_l)$. By induction, it suffices to show the case where

$$\gamma'_i=\langle jl\rangle\langle lk\rangle\langle kj\rangle\gamma_i.$$

The twist θ (see Section III.4.3) induces a 2-isomorphism

$$\phi\colon {}^{\Gamma_1(\gamma_0)}\Gamma\langle jkl\rangle\Rightarrow{}^{\Gamma_1(\gamma'_0)}\Gamma\langle jkl\rangle \tag{IV.3.3}$$

that induces isomorphisms between the associator state spaces. The 2-isomorphisms ϕ and ϕ^{-1} cancel in the expression of $Z(\Gamma)$, which proves the invariance. The same reasoning shows that $Z(\Gamma)$ is invariant by the change of pointing of a 4-simplex.

Suppose that B_3 and B'_3 are related by the change of an order on the vertices of a 3-simplex $\tau\in B_3$. It suffices to check that $Z(\Gamma)$ is invariant by the transposition of the order of two adjacent vertices of a pointed ordered 3-simplex. An explicit proof of this can be found in [DR18, Lemmas 4.2.4 and 4.2.5]. This also proves that $Z(\Gamma)$ is invariant by the change of order on the vertices of a 4-simplex.

IV.3.3. The state sum for combinatorial σ-manifolds with boundary. Following Douglas and Reutter, we extend the state sum to combinatorial σ-manifolds with boundary. First, we associate modules to σ-colored, closed, oriented combinatorial 3-manifolds. Let T be a closed, oriented canonical 3-manifold with choice of ordered simplices B_*, and let ϕ be a σ-coloring of T. Let $\varepsilon\colon B_3\to\{\pm1\}$ such that for any $\tau\in B_3$, $\varepsilon(\tau)=+1$ if and only if the orientation induced by the order on the vertices of τ coincides with the orientation induced by the orientation of T. To such a σ-colored combinatorial 3-manifold (T,ϕ) we assign the following module:

$$V_{\mathcal{C}}(T,\phi)=\bigoplus_{\Gamma}\bigotimes_{\tau\in B_3}V^{\varepsilon(\tau)}(\Gamma,\tau),$$

where Γ runs over the $\mathcal{C}$-states of (T,ϕ) with values in $\mathcal{S}$.

Let $(-T,\phi)$ be the σ-colored canonical 3-manifold with the opposite orientation. We define a nondegenerate pairing $\langle\cdot,\cdot\rangle_T\colon V_{\mathcal{C}}(-T,\phi)\otimes V_{\mathcal{C}}(T,\phi)\to\Bbbk$ as follows:

$$\langle\cdot,\cdot\rangle_T=\bigoplus_{\Gamma}\dim({}_1\mathcal{C}_1^1)^{-|T_0|}\left(\prod_{e\in B_1}\mathrm{d}(\Gamma_1(e))\right)^{-1}\left(\prod_{t\in B_2}\dim(\Gamma_2(t))\right)\bigotimes_{\tau\in B_3}\langle\cdot,\cdot\rangle_{\Gamma,\tau}$$

where Γ runs over the $\mathcal{C}$-states of (T,ϕ) with values in $\mathcal{S}$.

Let (K,ϕ) be a combinatorial σ-manifold with boundary. We denote by $i(K)$ the simplices in the interior of K, and let B_* be a choice of ordered simplices for $i(K)$. We define a linear map

$$Z_{\mathcal{C}}(K,\phi)\colon \Bbbk \to V_{\mathcal{C}}(-\partial K, \phi_{|\partial K})$$

as follows

$$Z_{\mathcal{C}}(K,\phi) = \bigoplus_{\Sigma} \dim({}_1\mathcal{C}_1^1)^{-|i(K)_0|} \sum_{\Gamma,\ \Gamma_{|\partial K}=\Sigma} \left(\prod_{e\in B_1} \mathrm{d}(\Gamma_1(e))\right)^{-1} \left(\prod_{t\in B_2} \dim(\Gamma_2(t))\right) \left(\bigotimes_{\kappa\in B_4} z(\Gamma,\kappa)\right) \circ \left(\bigotimes_{\tau\in B_3} \cup_{\Gamma,\tau}\right)$$

where Σ runs over the $\mathcal{C}$-states of $(\partial K, \phi_{|\partial K})$ with values in $\mathcal{S}$, Γ runs over the $\mathcal{C}$-states of (K,ϕ) that agree with Σ on ∂K, and the composition $\circ$ is over all the modules appearing both in the domain of $\bigotimes_{\kappa\in B_4} z(\Gamma,\kappa)$ and in the codomain of $\bigotimes_{\tau\in B_3} \cup_{\Gamma,\tau}$. Therefore, the codomain of $Z_{\mathcal{C}}(K,\phi)$ agrees with $V_{\mathcal{C}}(-\partial K, \phi_{|\partial K})$.

Lemma IV.3.4. *Let (K,ϕ) and (K',ϕ') be combinatorial σ-manifolds with boundary and let $f\colon \partial K \to \partial K'$ be an orientation reversing simplicial isomorphism such that $\phi' \circ f = \phi_{|\partial K}$. Then,*

$$Z_{\mathcal{C}}(K \cup_f K', \phi \cup_f \phi') = \langle Z_{\mathcal{C}}(K,\phi), Z_{\mathcal{C}}(K',\phi')\rangle_{\partial K}$$

where we used f to identify the modules $V_{\mathcal{C}}(-\partial K', \phi'_{|\partial K'})$ and $V_{\mathcal{C}}(\partial K, \phi_{|\partial K})$.

IV.3.4. The invariance under colored Pachner moves. Recall that a σ-colored Pachner move on a combinatorial σ-manifold (K,ϕ) replaces a codimension 0 submanifold of K simplicially isomorphic to a subcomplex $I \subset \partial\Delta^5$ with the complementary subcomplex $J \subset \partial\Delta^5$ and replaces the σ-coloring ϕ with a σ-coloring ψ that coincides with ϕ outside of I. To show the invariance of $\mathrm{HDR}_{\mathcal{C}}$ under such a move, it suffices by Lemma IV.3.4 to prove that

$$Z_{\mathcal{C}}(I,\phi) = Z_{\mathcal{C}}(J,\psi),$$

where ϕ and ψ are σ-colorings of I and J, respectively, such that $\phi_{|\partial I} = \psi_{|\partial J}$. Here we suppose that the subcomplex $I \subset \partial\Delta^5$ carries the orientation induced by the orientation on $\partial\Delta^5$ and that J carries the opposite orientation.

For any pointed ordered 4-simplex κ of a combinatorial σ-manifold (K,ϕ) and for any $\mathcal{C}$-state Γ of (K,ϕ), recall that the 10j-symbols are defined using the linear maps

$$z_+(\Gamma,\kappa)\colon \bigotimes_{m=0}^{4} V^{(-1)^n}(\Gamma,\partial_m(\kappa)) \to \Bbbk, \qquad z_-(\Gamma,\kappa)\colon \bigotimes_{m=0}^{4} V^{(-1)^{n+1}}(\Gamma,\partial_m(\kappa)) \to \Bbbk$$

depicted in Figure 2. Precomposing with the appropriate maps $\cup\colon \Bbbk \to V^+(\Gamma,\partial_m(\kappa)) \otimes V^-(\Gamma,\partial_m(\kappa))$ (determined by the nondegenerate pairings) leads to linear maps:

$$Z_+(\kappa)\colon V^+(\Gamma,\partial_4(\kappa)) \otimes V^+(\Gamma,\partial_2(\kappa)) \otimes V^+(\Gamma,\partial_0(\kappa)) \to V^+(\Gamma,\partial_3(\kappa)) \otimes V^+(\Gamma,\partial_1(\kappa)),$$
$$Z_-(\kappa)\colon V^+(\Gamma,\partial_3(\kappa)) \otimes V^+(\Gamma,\partial_1(\kappa)) \to V^+(\Gamma,\partial_4(\kappa)) \otimes V^+(\Gamma,\partial_2(\kappa)) \otimes V^+(\Gamma,\partial_0(\kappa)).$$

Using these maps, invariance under the σ-colored Pachner moves can be reexpressed as the following lemmas:

Lemma IV.3.5 (Invariance under the colored $(3,3)$-Pachner move)**.** *Let I and J be the subcomplexes of $\partial\Delta^5$ as defined in Figure 3 and let ϕ be a σ-coloring of $\partial\Delta^5$. Then the following holds for every $\mathcal{C}$-state of $(\partial I, \phi_{|\partial I})$:*

$$\sum_{[135]} \dim([135]) Z_+(01235) Z_+(01345) Z_+(12345) = \sum_{[024]} \dim([024]) Z_+(02345) Z_+(01245) Z_+(01234),$$

where $[135]$ *runs over degree-ϕ(135) 1-morphisms in $\mathcal{S}$ from* $[13]\otimes[35]$ *to* $[15]$*, and* $[024]$ *runs over degree-ϕ(024) 1-morphisms in $\mathcal{S}$ from* $[02]\otimes[24]$ *to* $[04]$*.*

Lemma IV.3.6 (Invariance under the colored $(2,4)$-Pachner move)**.** *Let I and J be the subcomplexes of $\partial\Delta^5$ as defined in Figure 4 and let ϕ be a σ-coloring of $\partial\Delta^5$. Then the following holds for every $\mathcal{C}$-state of $(\partial I, \phi_{|\partial I})$:*

$$Z_+(01235)Z_+(01345) = \sum_{[24]} \sum_{\substack{[024],[124],\\ [234],[245]}} \frac{\dim([024])\,\mathrm{d}([124])\,\dim([234])\,\dim([245])}{\mathrm{d}([24])} Z_+(01234)Z_+(01245)Z_+(02345)Z_+(12345)$$

where $[24]$ runs over degree-$\phi(24)$ objects in $\mathcal{S}$, and $[ijk]$ runs over degree-$\phi(ijk)$ 1-morphisms in $\mathcal{S}$ from $[ij]\otimes[jk]$ to $[ik]$.

Lemma IV.3.7 (Invariance under the colored $(1,5)$-Pachner move)**.** *Let I and J be the subcomplexes of $\partial\Delta^5$ as defined in Figure 5 and let ϕ be a σ-coloring of $\partial\Delta^5$. Then the following holds for every $\mathcal{C}$-state of $(\partial I, \phi_{|\partial I})$:*

$$Z_+(01235) = \dim({}_1\mathcal{C}_1^1)^{-1} \sum_{\substack{[04],[14],[24],\\ [34],[45]}} \sum_{\substack{[014],[024],[034],[045],[124],\\ [134],[145],[234],[245],[345]}} \Big(\prod \mathrm{d}([ij])^{-1}\Big)\Big(\prod \dim([ijk])\Big)$$
$$\mathrm{Tr}_{V^+(0345)}\big(Z_+(02345)Z_+(01245)Z_+(01234)Z_-(12345)Z_-(01345)\big),$$

where $[ij]$ runs over degree-$\phi(ij)$ objects in $\mathcal{S}$, and $[ijk]$ runs over degree-$\phi(ijk)$ 1-morphisms in $\mathcal{S}$ from $[ij]\otimes[jk]$ to $[ik]$. The trace $\mathrm{Tr}_{V^+(0345)}$ is over the module $V^+(0345)$.

Pachner $(3,3)$-move	I	J
4-simplices	(01234), (01245), (02345)	(01235), (01345), (12345)
Tetrahedra	(0124), (0234), (0245)	(0135), (1235), (1345)
Triangles	(024)	(135)

FIGURE 3. The simplices in the interior of I and J for the σ-colored Pachner $(3,3)$-move

Pachner $(2,4)$-move	I	J
4-simplices	(01235), (01345)	(01234), (01245), (02345), (12345)
Tetrahedra	(0135)	(0124), (0234), (0245), (1234), (1245), (2345)
Triangles		(024), (124), (234), (245)
Edges		(24)

FIGURE 4. The simplices in the interior of I and J for the σ-colored Pachner $(2,4)$-move

Pachner $(1,5)$-move	I	J
4-simplices	(01235)	(01234), (01245), (01345), (02345), (12345)
Tetrahedra		(0124), (0134), (0145), (0234), (0245), (0345), (1234), (1245), (1345), (2345)
Triangles		(014), (024), (034), (045), (124), (134), (145), (234), (245), (345)
Edges		(04), (14), (24), (34), (45)
Vertices		(4)

FIGURE 5. The simplices in the interior of I and J for the σ-colored Pachner $(1,5)$-move

The proofs of Lemmas IV.3.5, IV.3.6 and IV.3.7 follow directly from the proofs of [DR18, Lemmas 4.3.4, 4.3.5, and 4.3.6] by using the formulas shown in Section III.6.5. The details are left as an exercise for the reader.

CHAPTER V

Future research

The preceding chapters have established a framework for constructing quantum homotopy invariants of σ-manifolds using the algebraic data of spherical σ-fusion 2-categories. These constructions also open the door to a variety of categorical and topological questions. This final chapter outlines five primary directions for future research.

Section V.1 relates graded monoidal 2-categories and the framework of module categories. In Section V.2, we investigate the behavior of σ-fusion 2-categories under the pushforward along a morphism of 2-crossed modules and we discuss relations between the resulting invariants. In Section V.3, we explore the intersection between our quantum homotopy invariants and classical cohomological invariants via twistings by cocycles. Section V.4 proposes a generalization for the extension theory of fusion 2-categories due to Décoppet [Déc24b]. Finally, Section V.5 discusses the steps necessary to extend our invariants into a 4-dimensional HQFT.

V.1. Graded module categories

In this section, we suppose $\Bbbk$ is an algebraically closed field of characteristic zero. We refer the reader to [Ost03, DSS20] for details on module categories.

A *multifusion category* is a finite semisimple monoidal category whose objects have left and right duals. Recall that a fusion category is a multifusion category whose tensor unit is simple. Douglas and Reutter show [DR18] that the finite semisimple module categories over a multifusion category C form a finite semisimple 2-category $\mathrm{Mod}(C)$. Conversely, any finite semisimple (additive and idempotent complete) 2-category is equivalent to the 2-category of finite semisimple module categories over a multifusion category.

A fusion 2-category is *connected* if all of its simple objects are connected, see Section III.5.4. Décoppet introduces [Déc23] a notion of Morita equivalence on fusion 2-categories and shows [Déc25] that every fusion 2-category is Morita equivalent to a connected fusion 2-category. The connected (additive and idempotent complete) fusion 2-categories are exactly the 2-categories of finite semisimple module categories over braided fusion categories, see [DR18].

For any 2-crossed module of groups σ, we expect there to be a notion of *σ-crossed braided monoidal category* whose module categories form a σ-graded monoidal 2-category. This notion should generalize that of χ-crossed braided monoidal category due to Gainutdinov–Runkel–Wang [GRW26], where χ is a crossed module. The prototypical example of such a σ-crossed braided monoidal category would be the category $\mathcal{C}(\mathbb{1},\mathbb{1})$ of endomorphisms of the unit object of a σ-graded monoidal 2-category $\mathcal{C}$. Moreover, the finite semisimple modules over a σ-crossed braided fusion category should form a σ-fusion 2-category.

Using this framework, we hope to relate our state-sum invariant to the Mochida invariant of flat bundles, see [Moc24]. Recall that the Bridges–Cui invariant [BC25] generalizes the Mochida invariant and is a group-graded version of the invariant of closed 4-manifolds defined by Chaidez, Cotler, and Cui [CCC23]. Meusburger, Mulevicius, and Torzewska [MMT25] have recently generalized the Chaidez–Cotler–Cui invariant using the framework of module categories. We expect that a 3-group-graded version of their work could encompass, as a special case, the Bridges–Cui invariant.

V.2. Pushforwards of graded-fusion 2-categories

In this section, we first establish an algebraic relationship governing the categorical dimensions of graded-fusion 2-categories under the operation of pushforward along a morphism of 2-crossed modules. Then, we formulate a conjecture detailing how the resulting quantum homotopy invariants should be compatible with the geometric realization of the underlying morphism.

V.2.1. The categorical dimension of a pushforward. Let

$$\sigma = (L \xrightarrow{\delta} E \xrightarrow{\partial} H,\ \omega) \quad \text{and} \quad \sigma' = (L' \xrightarrow{\delta'} E' \xrightarrow{\partial'} H',\ \omega')$$

be 2-crossed modules of groups, and let

$$\phi = (\phi_1 \colon H \to H', \phi_2 \colon E \to E', \phi_3 \colon L \to L')$$

be a morphism from σ to σ' such that

- ϕ_1, ϕ_2, ϕ_3 are surjective,
- $\mathrm{Ker}(\phi_1)$, $\mathrm{Ker}(\phi_2)$ are finite,
- $\mathrm{Ker}(\phi_3) \cap \mathrm{Ker}(\delta) = 1$.

Let $\mathcal{C}$ be a spherical σ-fusion 2-category. Recall from Section III.5.8 that the pushforward $\phi_*(\mathcal{C})$ is a spherical σ'-fusion 2-category. Notice that the above conditions imply that the group $\mathrm{Ker}(\phi_3)$ is finite and that its cardinal $|\mathrm{Ker}(\phi_3)|$ divides $|\mathrm{Ker}(\phi_2) \cap \mathrm{Ker}(\partial)|$ in $\Bbbk$. Let X be a simple object of $\mathcal{C}^1$. Recall from Section III.5.6 that the Hom-category $\mathcal{C}(X,X)$ is a $(\delta \colon L \to \mathrm{Ker}(\partial))$-fusion category in the sense of Sözer and Virelizier. Therefore, by [SV23, Claim 7.5]:

$$\dim\left(\phi_*(\mathcal{C})_1^1(X,X)\right) = \frac{|\mathrm{Ker}(\phi_2) \cap \mathrm{Ker}(\partial)|}{|\mathrm{Ker}(\phi_3)|} \dim\left(\mathcal{C}_1^1(X,X)\right).$$

Suppose $\mathcal{C}$ has invertible dimensions, see Section III.6.3 (recall that this is the case if $\Bbbk$ is an algebraically closed field of characteristic zero).

Theorem 6. *Assume that $|\mathrm{Ker}(\phi_2)|$ is invertible in $\Bbbk$. Then, the dimension of the neutral component of $\phi_*(\mathcal{C})$ is computed by:*

$$\dim\left({}_1\phi_*(\mathcal{C})_1^1\right) = \frac{|\mathrm{Ker}(\phi_1)||\mathrm{Ker}(\phi_3)|}{|\mathrm{Ker}(\phi_2)|} \dim\left({}_1\mathcal{C}_1^1\right).$$

The proof of the above theorem is given in Section V.2.2.

Assume moreover that $\phi_*(\mathcal{C})$ has invertible dimensions. Recall that the data of $\mathcal{C}$ yields quantum homotopy invariants of σ-manifolds and the data of $\phi_*(\mathcal{C})$ yields quantum homotopy invariants of σ'-manifolds. We expect these invariants to be compatible with the geometric realization $B\phi \colon B\sigma \to B\sigma'$ of ϕ in the following sense:

Conjecture 1. *For any σ'-manifold (W, g'),*

$$\mathrm{HDR}_{\phi_*(\mathcal{C})}(W, g') = \sum_{B\phi \circ g = g'} \lambda_g \mathrm{HDR}_{\mathcal{C}}(W, g)$$

for some scalars λ_g, where the sum is taken over homotopy classes of maps $W \to B\sigma$ whose composition with $B\phi$ is g'.

We expect to express the scalars λ_g in terms of the homotopy groups of the mapping space $\mathrm{TOP}(W, B\sigma)$ based at g in a manner similar to that of Sözer and Virelizier in [SV23]. Recall that the neutral component $\mathcal{D} = {}_1\phi_*(\mathcal{C})_1^1$ of $\phi_*(\mathcal{C})$ is a fusion 2-category. Assuming Conjecture 1 holds, we get a decomposition of the Douglas–Reutter invariant of W:

$$\mathrm{DR}_{\mathcal{D}}(W) = \sum_{B\phi \circ g = *} \lambda_g \mathrm{HDR}_{\mathcal{C}}(W, g),$$

where $*$ denotes the homotopy class of a constant map $W \to B\sigma$.

V.2.2. Proof of Theorem 6. Let I be a representative set of homogeneous disconnected simple objects in $\mathcal{C}_1^1$. In particular,

$$I = \bigsqcup_{h \in H} I_h,$$

where I_h is a representative set of degree-h homogeneous disconnected simple objects in $\mathcal{C}_1^1$.

Lemma V.2.1. *There is a (left) action*

$$E \times I \to I,\ (e, X) \mapsto e \cdot X,$$

such that $e \cdot X = Y$ if and only if there is a degree-e homogeneous simple 1-morphism $f \colon X \to Y$ in $\mathcal{C}^1$.

PROOF. Let $e \in E$ and $X \in I$. First, we show that the object $e \cdot X$ exists. Pick a decomposition of X as a $(1, e^{-1})$-direct sum of homogeneous simple objects in $\mathcal{C}^1$:

$$X = \boxplus_{s\in S}^{(1,e^{-1})} X_s$$

with inclusion 1-morphisms $\{q_s \colon X_s \to X\}_{s\in S}$ and projection 1-morphisms $\{p_s \colon X \to X_s\}_{s\in S}$. Therefore,

$$1_X = \bigoplus_{s\in S}^{1} (q_s \circ p_s).$$

By simplicity of X, there is a unique $s_0 \in S$ such that $q_{s_0} \circ p_{s_0}$ is nonzero and $1_X \cong q_{s_0} \circ p_{s_0}$ in $\mathcal{C}^1$. Therefore $p_{s_0} \colon X \to X_{s_0}$ is nonzero and homogeneous of degree e. Moreover, X_{s_0} is connected in $\mathcal{C}_1^1$ to a unique $Y \in I$. Therefore $e \cdot X$ exists.

Now we show that $e \cdot X$ is uniquely defined. Let X,Y,$Z \in I$ and let $f \colon X \to Y$, $g \colon X \to Z$ be simple 1-morphisms in $\mathcal{C}^1$ that are homogeneous of degree e. Then $g^* \colon Z \to X$ is also simple in $\mathcal{C}^1$ and homogeneous of degree e^{-1}. By the categorical Schur lemma [DR18, Proposition 1.2.19] $f \circ g^*$ is nonzero, and therefore there are $h \colon Z \to Y$ that are simple in $\mathcal{C}^1$ and homogeneous of degree 1. Therefore, $Y = Z$. □

It follows from Lemma III.5.3 that $\mathrm{Ker}(\partial) \subset E$ acts trivially on I. Notice that $\mathrm{Ker}(\phi_2) \cap \mathrm{Ker}(\partial)$ is a normal subgroup of $\mathrm{Ker}(\phi_2)$. Let G be the quotient group

$$G = \mathrm{Ker}(\phi_2) / \left(\mathrm{Ker}(\phi_2) \cap \mathrm{Ker}(\partial)\right)$$

Lemma V.2.2. *The group G acts freely on I.*

PROOF. Let $X \in I$ and $e \in \mathrm{Ker}(\phi_2)$. The identity $e \cdot X = X$ implies that $\mathcal{C}_e^1(X, X)$ is nonzero. Moreover, since X is homogeneous, we get that $\partial(e) = 1$, and therefore $[e] = 1$. □

Let $J \subset I$ be a representative set of homogeneous disconnected simple objects in $\phi_*(\mathcal{C})_1^1$. In particular,

$$J = \bigsqcup_{h' \in H'} J_{h'}$$

where $J_{h'}$ is a representative set of degree-h' homogeneous disconnected simple objects of $\phi_*(\mathcal{C})_1^1$. Notice that J is a complete set of representatives for I/G.

Lemma V.2.3. *For all $h' \in H'$, the map*

$$\begin{cases} G \times J_{h'} & \to & \bigsqcup_{h \in \phi_1^{-1}(h')} I_h \\ ([e], X) & \mapsto & e \cdot X \end{cases}$$

is a bijection for all $h' \in H'$.

PROOF. We first show injectivity. Let $X, Y \in J_{h'}$ and $[e], [f] \in G$ such that $e \cdot X = f \cdot Y$. Therefore, there exists a nonzero homogeneous 1-morphism $p \colon X \to Y$ of degree $f^{-1}e$ in $\mathcal{C}$. Notice that p is homogeneous of degree 1 in $\phi_*(\mathcal{C})$, and therefore $X = Y$. Since G acts freely (see Lemma V.2.2), we have $[e] = [f]$. The surjectivity follows from the fact that J is a complete set of representatives for I/G. □

From Lemma V.2.3, we get

$$\begin{aligned} |\mathrm{Ker}(\phi_1)| \dim({}_1\mathcal{C}_1^1) &= \sum_{h \in \mathrm{Ker}(\phi_1)} \sum_{X \in I_h} \dim(\mathcal{C}_1^1(X, X))^{-1} \\ &= \sum_{[e] \in G} \sum_{X \in J_1} \dim(\mathcal{C}_1^1(X, X))^{-1} \\ &= \sum_{[e] \in G} \sum_{X \in J_1} \frac{|\mathrm{Ker}(\phi_2) \cap \mathrm{Ker}(\partial)|}{|\mathrm{Ker}(\phi_3)|} \dim(\phi_*(\mathcal{C})_1^1(X, X))^{-1} \\ &= \frac{|\mathrm{Ker}(\phi_2)|}{|\mathrm{Ker}(\phi_3)|} \dim({}_1\phi_*(\mathcal{C})_1^1). \end{aligned}$$

V.3. Relation to cohomological invariants

In this section, we explore the relationship between quantum homotopy invariants and classical cohomological invariants. We first recall the construction of cohomological homotopy invariants of maps and then we conjecture their relationship to the twistings of linearizations of 2-crossed modules by cocycles.

V.3.1. Cohomological homotopy invariants. Let θ be a class in $\mathrm{H}^4(X;\Bbbk^*)$ and let X be a connected topological space. An *X-manifold* is a pair (W,g), where W is a closed, oriented smooth 4-manifold and g is a homotopy class of maps $W \to X$. Two X-manifolds (W,g) and (W',g') are *equivalent* if there is an orientation-preserving diffeomorphism $\varphi\colon W \xrightarrow{\sim} W'$ such that $g' \circ \varphi = g$. The *cohomological homotopy invariant* of an X-manifold (W,g) is the scalar $\tau^\theta(W,g) \in \Bbbk$ defined by

$$\tau^\theta(W,g) = \langle g^*(\theta), [W]\rangle,$$

where $g^*(\theta) \in \mathrm{H}^4(W;\Bbbk^*)$ is the pullback of θ along g, $[W] \in \mathrm{H}_4(W;\mathbb{Z})$ is the fundamental class of W determined by its orientation, and $\langle\cdot,\cdot\rangle\colon \mathrm{H}^4(W;\Bbbk^*) \times \mathrm{H}_4(W;\mathbb{Z}) \to \Bbbk$ is the Kronecker pairing. These invariants extend to 4-manifolds with boundary and fit into a 4-dimensional homotopy quantum field theory with target X, called the *cohomological HQFT associated with θ*. In fact, Müller and Woike show [MW20] that these invariants extend to manifolds with corners and fit into an extended HQFT.

V.3.2. Twistings of σ-fusion 2-categories. Let

$$\sigma = (L \xrightarrow{\ \delta\ } E \xrightarrow{\ \partial\ } H,\ \omega)$$

be a 2-crossed module of groups, and let $B\sigma$ be its classifying space. Any class $\theta \in \mathrm{H}^4(B\sigma;\Bbbk^*)$ can be expressed as an algebraic cocycle

$$\theta\colon H^4 \times E^6 \times L^4 \to \Bbbk^*,$$

via a generalization of the formulas of Faria Martins and Porter [FMP07], by incorporating the Peiffer lifting ω. We expect this cocycle to induce a deformation $\mathcal{C}^\theta$ of any σ-fusion 2-category $\mathcal{C}$, similarly to the twisting defined in [SV23, Section 4.16].

Recall that the linearization $\mathcal{G}_\sigma$ of σ is a spherical σ-fusion 2-category with invertible dimensions (see Section III.5.7). Let (W,g) be a σ-manifold. A straightforward calculation shows that

$$\mathrm{HDR}_{\mathcal{G}_\sigma}(W,g) = 1.$$

Let $\mathcal{G}_\sigma^\theta$ be the twisting of $\mathcal{G}_\sigma$ by a class $\theta \in \mathrm{H}^4(B\sigma;\Bbbk^*)$. We expect the following:

Conjecture 2. *For any σ-manifold (W,g)*

$$\mathrm{HDR}_{\mathcal{G}_\sigma^\theta}(W,g) = \tau^\theta(W,g),$$

where $\tau^\theta(W,g)$ is the cohomological homotopy invariant of (W,g).

V.4. Classification of graded extensions of fusion 2-categories

In this section, $\mathcal{D}$ is a fusion 2-category. The aim is to classify graded extensions of $\mathcal{D}$. Let G be a group. We denote by $\underline{G}$ the 2-crossed module

$$\underline{G} = (1 \longrightarrow 1 \longrightarrow G,\ 1).$$

A *G-graded extension* of $\mathcal{D}$ is a $\underline{G}$-fusion 2-category $\mathcal{C}$ with neutral component ${}_1\mathcal{C}_1^1 = \mathcal{D}$. Note that when G is finite, any $\underline{G}$-fusion 2-category is a fusion 2-category. Décoppet shows [Déc24b], for finite G, that the G-graded extensions of $\mathcal{D}$ are classified by the homotopy classes of maps $BG \to B\mathcal{B}r\mathcal{P}ic(\mathcal{D})$, where $B\mathcal{B}r\mathcal{P}ic(\mathcal{D})$ is the delooping of the Brauer-Picard space of $\mathcal{D}$ (see [Déc24b, Definition 3.6]).

Recall that 2-groups can be modeled by crossed modules. Let $(\chi\colon E \to H)$ be a crossed module of groups. We denote by $\underline{\chi}$ the 2-crossed module

$$\underline{\chi} = (1 \longrightarrow E \xrightarrow{\ \chi\ } H,\ 1).$$

A *χ-graded extension* of $\mathcal{D}$ is a $\underline{\chi}$-fusion 2-category $\mathcal{C}$ with neutral component ${}_1\mathcal{C}_1^1 = \mathcal{D}$. Note that when E and H are finite, any $\underline{\chi}$-fusion 2-category is a fusion 2-category. We expect the following:

Conjecture 3. *The χ-graded extensions of a fusion 2-category $\mathcal{D}$ are classified by the homotopy classes of maps $B\chi \to B\mathcal{B}r\mathcal{P}ic(\mathcal{D})$.*

The case of 3-group extensions is more involved. Recall that a 3-group can be modeled by a 2-crossed module σ. In general a σ-fusion 2-category need not be a fusion 2-category. In particular, an object X in a σ-fusion 2-category $\mathcal{C}$ that is simple in the 1-subcategory $\mathcal{C}^1$ is not necessarily simple in $\mathcal{C}$. We expect that after adapting the definition of the Brauer-Picard space to the context of σ-fusion 2-categories, a similar classification result should hold.

V.5. Extension to HQFTs

For any 2-crossed module of groups σ, we expect the invariants defined in Theorem 5 to extend to σ-manifolds with boundary, and to fit into a 4-dimensional HQFT with target $B\sigma$. In this section, we sketch various steps involved in this construction. In this section, $\mathcal{C}$ is a spherical σ-fusion 2-category with invertible dimensions over a nonzero commutative ring $\Bbbk$.

V.5.1. The state sum for manifolds with boundary. Let M be a closed, oriented smooth 3-manifold, let T be a triangulation of M, and let ϕ be a σ-coloring of T. By the homotopy classification theorem, ϕ determines a homotopy class of maps $f \in [M, B\sigma]$. Let Γ be a $\mathcal{C}$-state of (T, ϕ). By Section IV.3.3, every (pointed ordered) tetrahedron τ of T determines a $\Bbbk$-module $V^{\varepsilon(\tau)}(\Gamma, \tau)$, where $\varepsilon(\tau) = \pm$ depends on whether the orientation of τ coincides with the orientation of M. Therefore, we associate to (M, T, ϕ) the $\Bbbk$-module

$$V(M, T, \phi) = \bigoplus_{\Gamma} \left(\bigotimes_{\tau \in T^{(3)}} V^{\varepsilon(\tau)}(\Gamma, \tau) \right),$$

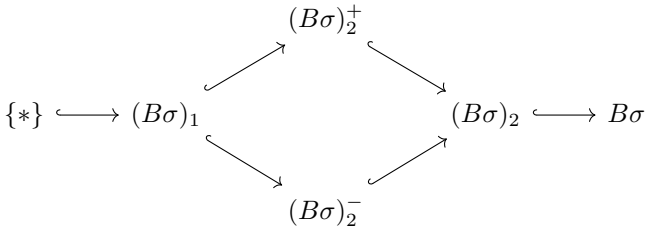
where Γ runs over the $\mathcal{C}$-states of (T, ϕ) and τ runs over the tetrahedra of T. Now let W be a compact, oriented smooth 4-manifold such that $\partial W = M$. Pick a triangulation K of W that coincides with T on the boundary. Notice that the tetrahedra of T bound a single 4-simplex (instead of two for the tetrahedra in the interior of W). As a consequence, the state-sum construction of Section IV.1.7 associates to any $g \in [W, B\sigma]$ an element

$$\mathrm{HDR}_{\mathcal{C}}(W, g) \in V(M, T, \phi).$$

This extends the state sum to σ-manifolds with boundary.

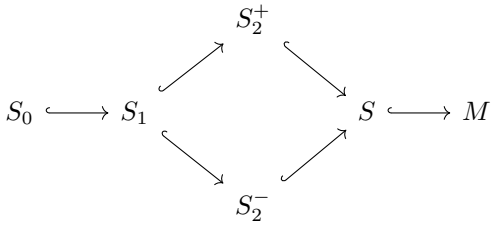
In principle, we would like the module $V(M, T, \phi)$ to be an invariant of the pair (M, f), but in practice, different σ-colorings of M may describe the same homotopy class. This makes the gluing of cobordisms quite difficult. Therefore, the definition of the module $V(M, T, \phi)$ will depend on some extra data (similarly to what is done by Sözer and Virelizier [SV23]): a σ-colored skeleton of M.

V.5.2. Colored skeletons of 3-manifolds. Recall from Section I.3.1 that a canonical CW complex is a CW complex with a single 0-cell, such that the attaching map of every 2-cell is pointed, and every 3-cell is attached via a pointed triad map. By construction, the classifying space $B\sigma$ is a canonical CW complex, see Section I.3.2. In particular, $B\sigma$ comes with a filtration:



Let M be a closed, oriented smooth 3-manifold. A *skeleton* of M is a 2-dimensional canonical simplicial complex S together with a piecewise differentiable embedding $S \to M$ such that $M \setminus S$ is piecewise differentiably

homeomorphic to a disjoint union of tetrahedra. A skeleton S of M determines a filtration



We say that a triangulation T of M is *compatible* with S if S is the 2-skeleton of T.

The idea is to use skeletons of 3-manifolds as basepoint replacements. A map $f\colon M \to B\sigma$ is *bifiltered* if it respects the above filtrations, and a homotopy $H\colon M \times [0,1] \to B\sigma$ is *bifiltered* if $H(-,t)$ is bifiltered for all $0 \leqslant t \leqslant 1$. By the work of Faria Martins [FM11], any map $M \to B\sigma$ is homotopic to a bifiltered map. Moreover, if S is a skeleton of M and T is a triangulation of M that is compatible with S, then any bifiltered map $M \to B\sigma$ determines a σ-coloring of T.

V.5.3. 4-dimensional HQFTs. We sketch the construction of a category Cob_4^σ of *4-dimensional σ-cobordisms*. The objects are *σ-skeletal 3-manifolds*, that is, triples (M,S,f), where M is a closed, oriented smooth 3-manifold, S is a skeleton of M, and f is a bifiltered homotopy class of bifiltered maps $M \to B\sigma$. The morphisms from (M_0,S_0,f_0) to (M_1,S_1,f_1) are equivalence classes of pairs (W,g), where W is a compact, oriented smooth 4-manifold with a diffeomorphism

$$\partial W \cong -M_0 \sqcup M_1,$$

and g is a homotopy class of maps $W \to B\sigma$ (relative to S_0 and S_1) that coincides with f_0 and f_1 on the boundary. Two such pairs (W,g) and (W',g') are *equivalent* if there is an orientation preserving diffeomorphism $\psi\colon W \xrightarrow{\sim} W'$ that is compatible with the boundary parametrizations and such that $g' \circ \psi = g$. A *4-dimensional HQFT with target $B\sigma$* is a symmetric monoidal functor $\mathrm{Cob}_4^\sigma \to \mathrm{Mod}_{\Bbbk}$.

Conjecture 4. *The state sum* $\mathrm{HDR}_{\mathcal{C}}$ *extends to a 4-dimensional HQFT with target $B\sigma$.*

As a consequence, the HQFT $\mathrm{HDR}_{\mathcal{C}}$ would induce a $\Bbbk$-linear representation of the *mapping class group of a σ-skeletal 3-manifold* (M,S,f) defined as the group of isotopy classes of orientation preserving diffeomorphisms $\psi\colon M \xrightarrow{\sim} M$ such that $\psi(S) = S$ and $\psi \circ f = f$.

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3-group extensions of fusion 2-categories and quantum homotopy invariants of 4-manifolds

Abstract. Quantum topology is a branch of mathematics that emerged in the late twentieth century following groundbreaking work by Jones, Drinfeld, Witten, and Kontsevich, which drastically renewed topology, particularly in low dimensions. Drawing on ideas from quantum physics, their framework has led to the construction of new manifold invariants, called quantum invariants, constructed from relevant algebraic data. One of the early successes in this direction is the Turaev–Viro invariant of 3-dimensional manifolds, defined as a state sum using the algebraic data of a fusion category (constructed for instance from the representations of a quantum group). More recently, Douglas and Reutter have constructed state-sum invariants of 4-dimensional manifolds using fusion 2-categories.

This thesis investigates quantum homotopy invariants, that is, quantum invariants of manifolds equipped with additional structure encoded by a map to a target topological space. More specifically, we generalize the Douglas–Reutter construction to derive invariants of flat 3-bundles over 4-manifolds or, equivalently, of 4-manifolds equipped with a homotopy class of maps to a homotopy 3-type. The relevant algebraic input for these quantum homotopy invariants is 3-group extensions of fusion 2-categories, where the 3-group models the target homotopy 3-type. Within this framework, maps to a homotopy 3-type are represented by colorings of triangulations. By proving a colored generalization of Pachner's theorem, we obtain that these state-sum invariants are independent of the chosen triangulation.

We show through examples that the resulting invariants are nontrivial and can even distinguish phantom maps, that is, maps that are trivial at the level of homotopy groups. Our invariants encompass several previously known 4-dimensional invariants, and are expected to fit into the framework of a homotopy quantum field theory.

Extensions des 2-catégories de fusion graduées par les 3-groupes et invariants quantiques homotopiques des 4-variétés

Résumé. La topologie quantique est un domaine des mathématiques apparu à la fin du XXe siècle à la suite des découvertes remarquables de Jones, Drinfeld, Witten et Kontsevich qui ont profondément renouvelé la topologie, en particulier en petite dimension. S'inspirant d'idées issues de la physique quantique, leur cadre théorique a permis la construction de nouveaux invariants de variétés, appelés invariants quantiques, à partir de certaines données algébriques. L'un des premiers succès dans cette direction est l'invariant de Turaev–Viro des 3-variétés, défini comme une somme d'états à partir de la donnée d'une catégorie de fusion (obtenue par exemple à partir des représentations d'un groupe quantique). Plus récemment, Douglas et Reutter ont construit des invariants de 4-variétés par somme d'états, en utilisant des 2-catégories de fusion.

Cette thèse porte sur les invariants quantiques homotopiques, c'est-à-dire les invariants quantiques de variétés munies d'une structure supplémentaire codée par une application vers un espace topologique. Plus précisément, nous généralisons la construction de Douglas–Reutter pour obtenir des invariants des 3-fibrés plats sur les 4-variétés ou, de manière équivalente, des 4-variétés munies d'une classe d'homotopie d'applications vers un 3-type d'homotopie. Les structures algébriques utiles pour cette construction sont les extensions des 2-catégories de fusion graduées par les 3-groupes, où le 3-groupe modélise le 3-type d'homotopie. Dans ce cadre, les applications vers un 3-type d'homotopie sont codées par des triangulations coloriées. En démontrant une généralisation coloriée du théorème de Pachner, nous obtenons que ces invariants sont indépendants de la triangulation choisie.

Nous montrons à l'aide d'exemples que les invariants qui en résultent sont non triviaux et même qu'ils permettent de distinguer des applications fantômes, c'est-à-dire des applications qui sont triviales au niveau des groupes d'homotopie. Notre invariant généralise plusieurs invariants déjà connus, et devrait s'intégrer dans une théorie quantique des champs homotopique.