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Jeux généralisés et applications en économie et finance : jeux avec contrainte partagée, stress tests et régulation bancaire

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Generalized games and applications in economics and finance: games with shared constraint, stress tests and bank regulation.

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Résumé

Cette thèse traite plusieurs problèmes de régulation en économie et finance, à travers le prisme des jeux généralisés comme instrument de modélisation. La thèse se compose de 4 parties.

Dans la première partie, on étudie les jeux généralisés avec une contrainte partagée générée par des contraintes individuelles. On prouve que l'ensemble des équilibres de Nash en contrainte partagée contient les équilibres de Nash du jeu avec contraintes individuelles générant la contrainte partagée. On étudie plusieurs propriétés de tels jeux en contrainte partagée et on démontre également quelques résultats pour les jeux dits non classiques. Deux exemples viennent illustrer l'intérêt économique de cette première partie : un modèle de régulation des émissions de gaz à effet de serre, ainsi qu'un modèle de financement de bien public.

Dans la deuxième partie, on traite l'étude de stress tests en régulation bancaire lors de ventes forcées. On montre que les jeux généralisés sont l'instrument naturel pour décrire les interactions stratégiques entre banques soumises à des contraintes de ratio de capitaux réglementaires (accords de Bâle III) et forcées de vendre des actifs pour réajuster leurs ratios suite à un choc financier exogène. La contrainte individuelle à laquelle est soumise chaque banque constitue une régulation microprudentielle, alors que la contrainte partagée générée par les contraintes individuelles peut être interprétée comme une régulation macroprudentielle. On prouve qu'il existe toujours au moins un équilibre de Nash qui vient minimiser les pertes en contrainte macroprudentielle, alors que l'existence est beaucoup plus difficile à obtenir en contrainte microprudentielle : on arrive à prouver l'existence d'un équilibre de Nash en contrainte microprudentielle seulement dans le cas où les marchés sont presque parfaitement liquides. Cette deuxième partie se termine avec une étude empirique et des stress tests pour les 4 banques françaises à importance systémique : on étudie les cascades de défaut suite à un choc exogène sur le banking book de ces banques.

Dans la troisième partie, on prolonge la deuxième partie en étudiant le cas des marchés parfaitement liquides (sans price impact) et en montrant que le théorème de Tarski permet de prouver l'existence d'au moins un équilibre de Nash en contrainte microprudentielle si les marchés sont assez liquides. On prouve également l'existence d'epsilon équilibres de Nash que l'on peut caractériser grâce au modèle sans price impact.

Dans la quatrième partie, on propose d'asseoir les fondations d'une théorie de la régulation optimale avec des jeux généralisés pour tout problème d'action collective où le rôle d'un régulateur est nécessaire, pertinent ou naturel. Le but du régulateur est de maximiser une (ou plusieurs) fonction(s) de bien-être collectif ou minimiser une (ou plusieurs) fonction(s) de coût social. Pour un jeu à N joueurs, une régulation correspond à la donnée des espaces de stratégies autorisées pour chaque joueur comme fonction des stratégies des autres joueurs, ce qui correspond parfaitement à un jeu généralisé. La régulation devient une variable du (ou des) critère(s) que le régulateur cherche à optimiser et on étudie l'existence d'une (ou plusieurs) régulation(s) optimale(s), ainsi que d'autres propriétés qui font sens du point de vue du régulateur. On fournit plusieurs résultats ainsi que des exemples et illustrations. En particulier, on étudie notre problème de stress tests et régulation bancaire lors de ventes forcées à travers le prisme de la régulation optimale pour répondre à plusieurs questions que pourrait se poser naturellement un régulateur bancaire.

Abstract

This Ph.D dissertation treats several problems of regulation in economics and finance, through the prism of generalized games as a modeling tool. The dissertation consists of four parts.

In the first part, we study generalized games with shared constraint generated from individual constraints. We prove that the set of Nash equilibria in shared constraint contains the Nash equilibria of the game with individual constraints that generates the shared constraint. We study several properties of such games with shared constraint and we also prove a few results for games that are called non-classical. Two examples illustrate the economic value of this first part: a model of regulation of greenhouse gas emissions, and a public good problem.

The second part of this dissertation treats the study of a stress test model in bank regulation during fire sales. We prove that generalized games are the natural tool to describe the strategic interactions between banks required to respect some regulatory capital ratios constraints (Basel III accords) while forced to sell assets during fire sales to rebalance their ratios above the regulatory threshold following an exogenous shock on the banking system. It appears that the individual constraint each bank is required to comply with is a microprudential regulation, while the shared constraint generated from the individual constraints can be interpreted as a macroprudential regulation. We prove that in macroprudential constraint there always exist at least one Nash equilibrium that minimizes the total losses in the financial system, while existence of a Nash equilibrium in microprudential constraint is much more difficult to get: we can prove existence of a Nash equilibrium in microprudential constraint only in the case where markets are almost perfectly liquid. This second part ends with an empirical study and stress testing the 4 French Global Systemically Important Banks (GSIBs): we study a default cascade following an exogenous shock on the banking book of these 4 French banks.

In the third part of the dissertation, we extend the second part: we study the case of perfectly liquid markets (with no price impact) and we prove that Tarski's theorem can be used to prove existence of at least one Nash equilibrium in microprudential constraint if the markets are liquid

enough. We also prove existence of epsilon Nash equilibria that we can characterize thanks to the model with no price impact.

In the fourth part, we propose to pave the way for the foundations of an optimal regulation theory with generalized games for any collective action problem where the role of a regulator is necessary, relevant or natural. The goal of the regulator is to maximize one (or several) social welfare function(s) or minimize one (or several) social cost function(s). For a game with N players, a regulation corresponds to the strategy set allowed for each player given what other players do, and this perfectly matches with a generalized game. The regulation becomes a variable of one or several criteria the regulator is seeking to optimize and we study existence of one (or several) optimal regulation(s), as well as some other properties that are meaningful from a regulatory perspective. We provide several results, examples and illustrations. In particular, we study our problem of stress tests in bank regulation during fire sales through the prism of optimal regulation to answer several questions a bank regulator could naturally ask.

Contents

Introduction

1 Chronology of events

My PhD started in September 2019. The initial project was to study indirect contagion of losses and/or defaults in the banking system following an exogenous shock on the banking system. I would focus on possible fire sales and/or liquidations with negative endogenous externality effects, while banks are required to satisfy some regulatory capital ratios requirements (Basel III). Contagion of losses and/or defaults happens through prices of assets held by the banks, which we call price-mediated contagion, and is an example of indirect contagion. Fire sales from one bank on a given asset would lower the price of the asset sold through price/market impact and therefore impact negatively all the banks holding this same asset. This gives birth to a strategic interaction between banks that we can model with game theory - in a game-theoretical framework. We would model the interaction between banks with a noncooperative static game and look for possible existence of Nash equilibria and other relevant results and configurations.

The initial idea was to generalize the results in the seminal model proposed by Yann Braouezec and Lakshithe Wagalath in their article *Strategic fire sales and price-mediated contagion in the banking system* [Braouezec and Wagalath, 2019]. They consider a model with p banks holding one single asset shared by all the banks and prove several theoretical results with some empirical applications. Among many results, they establish that such a strategic interaction is a game with strategic complementarities and it is therefore possible to prove existence of ordered Nash equilibria thanks to Tarki's fixed point theorem. A more general model with n assets would enable a more precise and sharper study: for instance the case $n = 2$ assets would enable a calibration of the model with on the one hand the banking book, and on the other hand the trading book.

It soon turned out that the study of a general model with p banks and n assets would not be a mere extension of the model with $n = 1$ as in its most general form, when $n \geq 2$, such a static game is not with strategic complementarities a priori. Moreover, while working on the way to model the strategic interactions between banks, I soon realized that this was not a classical game and could not be treated as a game in its usual form as we are used to treat in Economics and collective action problems. Indeed, the regulatory capital ratios constraints that each bank has to comply with create a constraint on each bank in the game-theoretical framework such that the set of possible/admissible strategies for a given bank i depends on the strategies picked by all other banks $j \neq i$, through the market impact of banks on the assets sold during the fire sales. It became blatant and indisputable that the accurate way to model this strategic interaction between banks was with a static game with constraints: each bank is seeking to minimize a cost function by picking a strategy in a strategy set which is dependent on the strategies picked by all other agents. And this is different from classical games we encounter in most of the literature, in the sense that usually in a game-theoretical framework the strategy set of a given agent/player is exogenous, constant and independent from what others do.

I started to set new definitions and find new results on such a framework, then after a few research realized this modeling framework already had a name - these games with constraints are called generalized games - and were first initiated in an article by Kenneth J. Arrow and Gerard Debreu, *Existence of an equilibrium for a competitive economy* [Arrow and Debreu, 1954]. See the survey paper by Francisco Facchinei and Christian Kanzow *Generalized Nash equilibrium problems* [Facchinei and Kanzow, 2007] to have a global overview on generalized games (also called Generalized Nash Equilibrium Problems) since the paper of K. J. Arrow and G. Debreu.

My PhD advisor and I realized that the initial model of strategic fire sales by Braouezec and Wagalath with $n = 1$ asset actually was a generalized game without naming it. But quite remarkably, no need to use the generalized games theory and framework when $n = 1$ asset since the use of games with strategic complementarities and Tarski's fixed point theorem are a shortcut to solve existence of Nash equilibria. However, this is no longer true for the game in its most general form when $n \geq 2$, and I could not escape the generalized game that naturally arised from the strategic interactions during fire sales between banks required to satisfy the Basel III accords.

At this stage of my PhD route started the first Covid lockdown. Yann proposed to me to pause this initial project to join his work on a second project that he had initiated and was in revise and resubmit at *Applied Economics*: *Target capital ratio and optimal channel(s) of adjustment: A simple model with empirical applications to European banks* [Braouezec and Kiani, 2021b]. I proved the second part of a proposition formulated by Yann with existence of a closed-form expression as one of the solutions of the problem which was modeled in the article. I also conducted a numerical study on two European banks. This project was accepted for publication in *Applied Economics* in August 2020, and can be consulted in Appendix of the present thesis.

The academic year 2020-2021 (with two lockdowns) was dedicated to the study of some results and applications of generalized games in Economics, as well as developments regarding the initial fire sales problem. In particular, we proved that in a generalized game, when agents share their initial individual constraints, this generates more Nash equilibria than the generalized game with individual constraints. To be specific, the set of Nash equilibria of the game with shared constraint contains the Nash equilibria of the game with individual constraints. In our article *Economic foundations of generalized games with shared constraint: Do binding agreements lead to less Nash equilibria?*, we show some applications of this result in an environmental problem with carbon emissions, as well as a public good problem. We moreover study existence of Nash equilibria for 2×2 generalized games, we establish a new classification of games including generalized games and a class of games that we call non-classical games. We prove a few existence and characterization results regarding Nash equilibria for non-classical games. This article constitutes the first chapter of my PhD thesis.

This is also during this academic year that I started to think of generalized games as a natural tool to model the action of any regulator/regulation in a game-theoretical framework. Consider a regulator who is seeking to maximize a social welfare function or minimize a social cost function on a set of n agents interacting with one another in a game theoretical framework. The regulator can decide to restrict the strategy picked by every agent i through a constraint that depends on what strategies all other agents $j \neq i$ pick. Let's call regulation the p-uple of such restriction functions (that are actually point-to-set maps), which defines a generalized game: the aim of the regulator is to seek if there is existence of one or several optimal regulation(s) that optimize the social function, the variable is the regulation itself, and we can look for the existence of one or several p-uple(s) of restriction functions that enables to optimize the social function. These considerations would be continued during the last year of my PhD...

The last year of my PhD, that is the academic year 2021-2022, was mainly dedicated to solving my initial problem of modeling with generalized games the interaction of p banks during fire sales following an exogenous shock. The strategy set of each bank i depends on the strategies picked by all other banks $j \neq i$ through the regulatory capital ratios that each bank has to satisfy and the effect of price/market impact (following fire sales) on these ratios. These considerations gave birth to two articles: *Strategic foundations of macroprudential regulation: preventing fire sales externalities*, which constitutes the second chapter of my PhD thesis, and *A Generalized Nash Equilibrium Problem arising in banking regulation: An existence result with Tarski's theorem*, which constitutes the third chapter of my PhD thesis.

In the article *Strategic foundations of macroprudential regulation: preventing fire sales externalities*, we show that the individual ratios constraints that banks are required to comply with are an example of microprudential regulation, while the regulation resulting from the shared ratios constraints can be interpreted as a form of macroprudential regulation. We study the best response functions in microprudential and macroprudential regulation. We prove that a Nash equilibrium in macroprudential constraint always exist, and moreover there exists a Nash equilbrium that minimizes the total cost function for the banks (among all admissible strategies), which is therefore Pareto optimal. However this result is no longer true in microprudential constraint as we prove that some Nash equilibria in macroprudential constraint might no longer be Nash equilibria in microprudential constraint. Existence of generic Nash equilibria in microprudential/individual constraints is usually hard to find. We however prove an existence result in microprudential constraint (under some quite strong assumptions) using Ichiishi's theorem. We complete our theoretical analysis with some empirical study on the set of the four French banks considered as Global Systemically Important Banks (GSIBs): BNP Paribas, Société Générale, Crédit Agricole and BPCE. We simulate some stress tests on these 4 banks to exhibit possible cascade of failures following an exogenous shock on the banking book of these four banks.

In the article *A Generalized Nash Equilibrium Problem arising in banking regulation: An existence result with Tarski's theorem*, we make a study of the case with no price impact where markets are perfectly liquid. We answer to one of the initial questions of my PhD route: we prove that if the markets are liquid enough (almost perfectly liquid) the generalized game is still with strategic complementarities and therefore we can use Tarski's theorem like in the seminal article by Braouezec and Wagalath to prove existence of a Nash equilibrium in microprudential constraints. This can be done thanks to the characterization of the case with no price impact. Moreover, we prove existence of a set of epsilon Nash equilibria when the markets are liquid enough, also thanks to the characterization of the case with no price impact.

After completion of these 2 articles, I resumed the work in progress that I had started on optimal regulation with generalized games: *Generalized games and Optimal Regulation in Economics and* *collective action problems. An application to bank regulation during fire sales.* This work in progress constitutes the fourth chapter of my PhD thesis.

2 Bank regulation since the Financial crisis of 2007-2008

Bank regulation has gone through many evolutions, improvements and changes since the Financial crisis of 2007-2008 with the rise of collective awareness about the flaws in the financial system and the necessity to build new regulatory tools to protect the global economy from these flaws. The many aspects of bank regulation have become a hot topic of research in academia since then in order to propose solutions and new regulatory tools to practitioners, central bankers, governments and regulators.

There has been plethora of articles and books published:

- to explain the orgins and causes of the financial crisis.
- to describe empirically the spread from the subrpime mortgage crisis in the United States to a crisis hitting the whole financial system and then the world economies.
- to propose new concepts, ideas, paradigms and models to apprehend the global financial system, and in particular the global banking system.
- to propose new solutions, policies and measures to protect the financial system from the flaws of the 2007-2008 crisis.
- And with a further step, to imagine solutions to any potential threats that might hit the financial system, in order to prevent the threats from happening.

If we look at the world economy, we could think about it with two medical metaphors.

It is a worldwide and interconnected population of economic (or financial) agents, each agent with its own characteristics, and if for any reason one of the agents gets sick, there might be some contagion effects and an epidemics in the financial and economic system, depending on the characteristics of the agents. So the role of people working on preventing financial crises could look similar to the work of an epidemiologist looking for measures and/or policies, and ideally a vaccine, to protect the financial system from an epidemics. And when it is not possible to prevent it from happening, at least to limit the damage on the system.

One could also think of the financial and economic system as a human body with different organs made of cells, and the whole interaction and coordination between the different organs, guaranteeing the well-being of the human body. But if one organ gets negatively hit such that its normal well functioning is impaired, this can affect other organs and be damageable for the human body and its health. So with this second metaphor, the role of people working on preventing financial crises could look similar to the job of a physician making regularly controls on the human body of its patients to check that everything goes well, a diagnostic pathologist when the human body is sick and one is looking for the origin, nature and characteristics of the sickness, and also must wear the hats of all physicians and disease specialists ranging from surgeons to cancer specialists to be able to fight against the diseases.

If you think of the financial system (and more generally the economic system), everything is on a constant move from the past to the future, it is rooted in a historical background that we inherit which gives the present system, with financial news happening at every moment, evolution of financial markets, new technologies, new institutions and instruments, and any kinds of evolution of legislation regarding economic and financial institutions/instruments. So the whole picture is quite hard to capture and it means a capacity to understand all the different components that interact between one another.

On the academic side, this means a capacity to understand all the research that has been done on the topic and all the literature available, plus a capacity to update one's knowledge to the present evolution of facts, to be able to draw ideas for the present and the future.

We remind and summarize the trigger and development of the financial crisis of 2007-2008 in a few sentences. See for instance [Mishkin, 2011] and [Brunnermeier, 2009]. The spark that started the fire was the US subprime mortgage market. The conjunction of a housing bubble in the real estate market in the US with the excessive subprime credit granting to households with low credit ratings created a tsunami of defaults ocurring at the same time in 2007. The households hit by a default of payment were unable to reimburse their loans by selling the house bought because of the burst of the housing bubble and a huge decrease in house prices at the same moment. The crisis began to spread through Collateralized Debt Obligations (CDOs) that were built by financial institutions granting subprime mortgage, and by Credit Default Swaps (CDS) that investors and financial institutions had bought to protect themselves against the risk of default of subprime loans and CDOs. These contractual links between financial agents can be called direct contagion effects.

On September 15, 2008, the American investment bank Lehman Brothers went bankrupt and the US decided to bail out AIG on September 16, 2008: the Federal Reserve provided a $85 billion two-year loan to AIG to prevent its bankruptcy and further stress on the global economy. The financial crisis spread to the whole world economies through contractual and non-contractual links between financial agents, that we will call direct contagion and indirect contagion effects. The intervention of Central Banks and the use of a high level of quantitative easing were necessary to stop the bleeding. As for the aftermath of the crisis on the real economy: the global financial meltdown had a cost over $20 trillion, foreclosures in the United States reached 6 million by early 2010 and tens of millions of jobs were destroyed worldwide.

2.1 Systemic Risk, Financial Stability, Microprudential and Macroprudential regulation

Systemic risk can be defined (IMF FSB 2009) as the risk of threats to financial stability that impair the functioning of a large part of the financial system with significant adverse effects on a broader economy. Financial stability, as defined by the ECB, is a state whereby the build-up of systemic risk is prevented.

- Microprudential regulation focuses on the resilience of each bank taken individually and isolated from the whole financial system, without any interconnections between financial institutions. Microprudential regulation was the main regulatory paradigm implemented before the financial crisis of 2007-2008.

- Macroprudential regulation, on the contrary, focuses on the resilience of the financial system as a whole taking into account all financial institutions and all possible contagion effects of losses or defaults between financial institutions on the global level. Macroprudential regulation aims at preventing systemic risk and ensuring financial stability.

For exhaustive developments on macroprudential regulation/policy, we refer to the excellent book by Xavier Freixas, Luc Laeven and José-Luis Peydró, *Systemic Risk, Crises, and Macroprudential Regulation* [Freixas et al., 2015], as well as the excellent book by Taryk Bennani, Laurent Clerc, Virginie Coudert, Marine Dujardin and Julien Idier *Politique macroprudentielle - Prévenir le risque systémique et assurer la stabilité financière* [Bennani et al., 2017].

We will give a few examples of macroprudential regulations in the next subsections.

2.2 Direct and Indirect contagion

Why do we need a macroprudential regulation?

Microprudential regulation is a sufficient paradigm if we only look to diseases that can affect financial agents and that are not contagious. A one-by-one study and control of financial agents satisfying the microprudential regulation implemented is sufficient to ensure the safeness and soundness of all financial institutions on an individual basis. But as soon as one or several financial institutions get hit by a disease that is contagious, microprudential regulation is no more sufficient as it does not take into account the possible contagion effects between the agents, and it is not relevant to prevent systemic risk.

We usually make the distinction between two types of contagion effects between financial institutions:

- Direct contagion of losses or defaults between banks and financial institutions occurs through contractual links between these institutions and can lead to a domino effect. Now this issue is in part tackled through the introduction of Central Clearing Counterparties (CCP) for standard and liquid OTC derivatives (see the European Market Infrastructure Regulation, EMIR [1]).
- Indirect contagion, on the contrary, is a type of contagion that does not imply contractual links and that can for instance be price mediated: a bank forced to sell assets during a crisis can create a market impact on this class of assets that affect the balance sheets of all banks holding this same class of assets

2.3 Different types of risks

The aim of the present thesis is not to draw an encyclopedia of all possible financial risks that can threaten the financial system, with the list of ideas to protect the financial system and financial institutions from these risks. This could be a further development and research project on the long term.

We can however classify the most common financial risks in 5 categories: Credit risk (also called default risk), Market risk, Counterparty risk, Operational risk and Liquidity risk.

There are different tools, instruments, models and methods that can help to protect the financial system against the different types of risks.

[1] https://www.amf-france.org/fr/actualites-publications/dossiers-thematiques/emir

Banks are supposed to hold enough capital to be protected against the four first categories of risk, i.e., Credit risk, Market risk, Counterparty risk and Operational risk.

And research on financial risks can improve our knowledge on potential threats and the way to protect the financial system from these potential threats.

2.4 Fire sales in the banking system

In this PhD thesis, we will focus on a specific type of financial risk, like a researcher in epidemiology would focus his/her study to a specific kind of epidemics, to describe its symptoms, transmission pattern and possible pills for healing or a vaccine that would prevent the epidemics from happening. We want to study the risk that follows fire sales on some class of assets on the financial system: this is an example of indirect contagion that happens through prices of assets - we call this phenomenon price-mediated contagion.

Fire sales happen when banks are forced to sell a large quantity of assets at dislocated prices : a given financial institution may be forced to sell some assets, pushing prices down and generating losses for all institutions holding the same assets, and therefore possibly generating new sales of assets. These fire sales typically occur when a stressed institution is willing to promptly liquidate part of its portfolio.

"In trying to make themselves safer, banks [...] can behave in a way that collectively undermines the system. Selling an asset when the price of risk increases, is a prudent response from the perspective of an individual bank. But if many banks act in this way, the asset price will collapse, forcing institutions to take yet further steps to rectify the situation." ([Brunnermeier et al., 2009])

There are several papers that document empirically such fire (forced) sales. See for instance, [Ellul et al., 2011], [Merrill et al., 2021] or [Chernenko and Sunderam , 2020]. We will propose a mathematical framework to model the interaction between banks hit by an exogenous shock and that enter into some fire sales to rebalance their regulatory ratios: we will study the possible effects of such fire sales on the banking system and conduct stress-tests to test its resilience, keeping in mind that we want to find some ideas and tools to guarantee the safeness and soudness of the financial system.

2.5 The Basel Accords and the rise of Macroprudential policy/regulation

The necessity of creating new regulatory instruments/tools became blatant and urgent following the Financial crisis of 2007 2008.

The microprudential policies implemented in the Basel I and II framework proved to be insufficient and incapable to prevent systemic risk in the financial system during the crisis.

Basel III (December 2010) proposed a new regulatory framework including a set of new regulatory instruments/measures, among which macroprudential instruments/measures, to reinforce the resistance of the financial system.

For instance, Basel III proposed the two following measures that we study in this thesis:

- A risk based capital ratio (RBC ratio) requirement for each bank that already existed in Basel I and Basel II) with additional capital requirements for banks identified as Global Systemically Important Banks (GSIBs). This can be seen as a macroprudential measure as its aim is to guarantee the stability of the financial system.

  $RBC = \frac{Tier1}{Risk-weightedassets} \geq 6\% +$ additional requirements

  We give below in Figure 1 the list of banks considered as GSIBs by the Finanicial Stability Board (FSB) as well as their systemic size and the additional RBC buffer requirements for each GSIB.

- A Leverage ratio such that:

  $L = \frac{Tier1}{Totalexposure} \geq 3\% + 0.5\times$ GSIB buffer requirement

2.6 Financial stress-tests

Stress-tests in Finance have become a hot research topic and method to assess the resilience of financial institutions since the Financial crisis of 2007-2008, and there is still a lot work to be done on this topic.

What is a stress-test?

“Stress testing is the analysis of how a generic entity (or object) such as a human body, a car, a bank... or a system of interacting entities such as a physical, biological or a financial system copes under pressure. For the specific case of a banking system, composed with interacting banks, there are actually various ways to design a stress test to assess its resilience."

Bucket[14]	G-SIBs in alphabetical order within each bucket
5 (3.5%)	(Empty)
4 (2.5%)	JP Morgan Chase
3 (2.0%)	BNP Paribas Citigroup HSBC
2 (1.5%)	Bank of America Bank of China Barclays China Construction Bank Deutsche Bank Goldman Sachs Industrial and Commercial Bank of China Mitsubishi UFJ FG
1 (1.0%)	Agricultural Bank of China Bank of New York Mellon Credit Suisse Groupe BPCE Groupe Crédit Agricole ING Bank Mizuho FG Morgan Stanley Royal Bank of Canada Santander Société Générale Standard Chartered State Street Sumitomo Mitsui FG Toronto Dominion UBS UniCredit Wells Fargo

Figure 1: List of GSIBs (published by the FSB) and additional RBC buffer requirements

In Finance, a stress-test is a simulation of shock scenario on a given financial institution or the financial system and study of the possible answers and reactions from the financial institution or the system to test its resilience. It can be compared to a crash test conducted on a car to ensure that the airbags are functioning.

Regulatory stress tests are conducted by the European Banking Authority (EBA) and the Eureopean Central Bank (ECB) in Europe and by the Federal Reserve in the United States. They assume no reaction from banks after the shock.

At the moment, to the best of our knowledge, there is no equilibrium macroprudential stress test, model or regulation preventing fire sales externalities in the banking system, and this is one major contribution of our thesis.

A macroprudential stress test on the banking system is a simulation of financial shock on the banking system as a whole, with a study of the reaction and resilience of the banking system as a whole.

In Chapter 2, we will study the case where fire sales occur after an exogenous shock on the financial system and can give birth to negative endogenous externalities.

See the book by [De Nicoló et al., 2012], *Externalities and macroprudential policy* for a global overview of such risks and the way to tackle them with macroprudential policy.

We will also study the case where some fire sales may be triggered by regulatory capital requirements themselves.

3 Generalized Games

3.1 New quantitative tools for new policies

It happens that research in financial regulation following the financial crisis of 2007-2008 have given birth to new concepts, methods, paradigms, and tools to apprehend the reality of the financial system and the interaction between the financial agents during periods of stress.

We give a diversified panel with a (non-exhaustive) list of articles that offer some landmark examples: [Glasserman and Young, 2016], [Amini et al., 2016], [Cont et al., 2010], [Castiglionesi and Navarro, 2008], [Laurent et al., 2011], [Laurent et al., 2016], [De Palma et al., 2008], [De Palma et al., 2009], [Cont and Wagalath, [Braouezec and Wagalath, 2018] and [Feinstein et al., 2017].

Regarding fire sales in the banking system with banks required to satisfy the RBC and Leverage ratio, we prove in this thesis that the natural tools to model the interaction between banks during such a phenomenon are generalized games, that is games where the set of possible strategies for a given agent depends upon the strategies chosen by the other agents. To the best of our knowledge, this is the first time that generalized games are used as a modeling tool in Finance.

3.2 Generalized games: historical background

Generalized games have first been introduced by Kenneth J. Arrow and Gerard Debreu in their famous article, *Existence of an Equilibrium for a Competitive Economy*, [Arrow and Debreu, 1954], with the premises by G. Debreu in his article *A Social Equilibrium Existence Theorem* [Debreu, 1952]. The very first name given to generalized games by K. J. Arrow and G. Debreu was "an abstract economy" (for a historical review see [Facchinei and Kanzow, 2007]).

They introduce the definition of an abstract economy as a game where the set of strategies available to each agent i depends on the vector of strategies chosen by the other agents. We keep the same notations as the ones in their article: "the concept of an abstract economy, a generalization of that of a game, will be introduced, and a definition of equilibrium given. A lemma giving conditions for the existence of equilibrium of an abstract economy will be stated."

Consider ν subsets of $\mathbb{R}^l$, $\mathfrak{A}_i$ $(i = 1, .., \nu)$. Let $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times ... \times \mathfrak{A}_\nu$. We consider the vector of strategies $a = (a_1, .., a_\nu)$ where $a_i \in \mathfrak{A}_i$ for $i = 1, .., \nu$. For each i, we suppose there is a real function f_i, defined over $\mathfrak{A}$. Let $\overline{\mathfrak{A}}_i = \mathfrak{A}_1 \times \mathfrak{A}_2 ... \times \mathfrak{A}_{i-1} \times \mathfrak{A}_{i+1} \times ... \times \mathfrak{A}_\nu$ and we denote $\overline{a}_i = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_\nu)$ where $a_{i'} \in \mathfrak{A}_{i'}$ for all $i' \neq i$. Let $A_i(\overline{a}_i)$ be a function defining for each point $\overline{a}_i \in \overline{\mathfrak{A}}_i$ a subset of $\mathfrak{A}_i$. Then the sequence $[\mathfrak{A}_1, ..., \mathfrak{A}_\nu, f_1, ...f_\nu, A_1(\overline{a}_1), ..., A_\nu(\overline{a}_\nu)]$ will be termed an *abstract economy.*

The definition of a Nash equilibrium proposed by John F. Nash in his famous article *Equilibrium Points in N-Person Games* [Nash Jr, 1950] can be extended to an abstract economy.

Definition: a^* is an equilibrium point of $[\mathfrak{A}_1, ..., \mathfrak{A}_\nu, f_1, ...f_\nu, A_1(\overline{a}_1), ..., A_\nu(\overline{a}_\nu)]$ if for all $i = 1, ..., \nu$, $a_i^* \in A_i(\overline{a}_i^*)$ and $f(\overline{a}_i^*, a_i^*) = max_{a_i \in A_i(\overline{a}_i^*)} f(\overline{a}_i^*, a_i)$.

The graph of $A_i(\overline{a}_i)$ is the set $\{a, a_i \in A_i(\overline{a}_i)\}$. The function $A_i(\overline{a}_i)$ is said to be continuous at $\overline{a}_i^0$ if for every $a_i^0 \in A_i(\overline{a}_i^0)$ and every sequence $\{\overline{a}_i^n\}$ converging to $\overline{a}_i^0$, there is a sequence $\{a_i^n\}$ converging to a_i^0 such that $a_i^n \in A_i(\overline{a}_i^n)$ for all n. K.J. Arrow and G. Debreu establish the following lemma, which is a generalization of Nash's theorem published in *Equilibrium Points in N-Person Games.*

Lemma: If, for each i, $\mathfrak{A}_i$ is compact and convex, $f_i(\overline{a}_i, a_i)$ is continuous on $\mathfrak{A}$ and quasi-concave in a_i for every $\overline{a}_i$, the set $A_i(\overline{a}_i)$ is convex and non-empty, then the abstract economy $[\mathfrak{A}_1, ..., \mathfrak{A}_\nu, f_1, ...f_\nu, A_1(\overline{a}_1), ..., A_\nu(\overline{a}_\nu)]$ has an equilibrium point.

We refer to the seminal survey paper *Generalized Nash equilibrium problems* by Francisco Facchinei and Christian Kanzow for a global description of the evolution of generalized games since the paper of K. J. Arrow and G. Debreu until 2007 (date of publication of the survey).

We propose to shed light on a few landmark articles that illustrate the evolution of generalized games since 1954.

In 1965, J. B. Rosen published his famous article *Existence and Uniqueness of Equilibrium Points for Concave N-Person Games* [Rosen, 1965] where he proves a new result for a specific class of generalized games (that will be later called games with shared constraint). J. B. Rosen focuses on games that are defined on a convex closed, and bounded set $R \subset E^{m_1} \times ... \times E^{m_n}$ (where E^{m_i} is an Euclidian space) and such that for all agents i, the payoff function $\varphi_i(x)$ is continuous in x and concave in x_i, and given the strategies picked by all other agents $j \neq i$, $(x_j)_{j \neq i}$, the strategies x_i available to agent i are such that $(x_1, .., x_i, .., x_n) \in R$. The constraint condition can be rewritten: for all agents i, x_i must belong to the set $X_i(x_{-i}) = \{x_i \in E^{m_i}$, such that $(x_1, .., x_i, .., x_n) \in R\}$. Such games are designated games with coupled constraint in the article by J. B. Rosen, in the sense that R is a coupled constraint set. They are moreover called concave n-person games in the sense that the payoff function of each agent i is concave in x_i. The main result by J. B. Rosen is stated in Theorem 1: an equilibrium point exists for every concave n-person game. That is to say, a game with coupled constraint satisfying the assumptions above always has a Nash equilibrium (that we will later call Nash equilibrium in shared constraint).

In 1983, Tatsuro Ichiishi published his famous book *Game Theory for Economic Analysis* [Ichiishi, 1983] and formulated a simpler and more general version of the result stated in the article by K. J. Arrow and G. Debreu in 1954. The terminology used in the book is quite similar: an abstract economy is formulated by introducing the concept of "feasability" to a game in normal form. Consider a set of players N, let X^j be the strategy set for player j and $X := \prod_{j \in N} X^j$. The feasable strategy correspondence of player j is a correspondence $F^j : X \longrightarrow X^j$. The preference relation of player j is represented by a utility function $u^j : grF^j \longrightarrow \mathbb{R}$. An *abstract economy* is a list of specified data $\{X^j, F^j, u^j\}_{j \in N}$. T. Ichiishi also employs the term *pseudo-game*. The definition of a Nash equilibrium of a classical game in normal formal can naturally be extended to an abstract economy and is called *social equilibrium* in the book. The theorem stated in the book is the following:

Theorem: Let $\{X^j, F^j, u^j\}_{j \in N}$ be an abstract economy. Assume for every $j \in N$, that X^j is a nonempty, convex, compact subset of a Euclidean space, that F^j is both upper semicontinuous and lower semicontinous in X, that $F^j(x)$ is nonempty, closed and convex for every $x \in X$, that u^j is continuous in grF^j, and that $u^j(x, \cdot)$ is quasi-concave in $F^j(x)$ given any $x \in X$. Then there exists a social equilibrium of the abstract economy.

Note that this theorem is a direct consequence of Kakutani's fixed point theorem (see the book of T. Ichiishi for full details of the proof).

In 1991, Patrick T. Harker published an article *Generalized Nash games and quasi-variational inequalities* [Harker, 1991] where he draws the history of links between generalized games and quasi-variational inequality theory. The formulation of Nash equilibria problems in classical games as variational inequalities (VIs) dates back to an article by J.L. Lions and G. Stampacchia [Lions and Stampacchia, 1967], while the formulation of the generalized Nash equilibrium problem as a quasi-variational inequality (QVI) is attributed to an article by A. Bensoussan [Bensoussan, 1974]. The results on the links between solving generalized Nash equilibria problems and solving quasi-variational inequalities usually make some convexity and differentiability assumptions on the payoff functions and some convexity and closedness assumptions on the strategy sets. These results are usually true under what we call the "jointly convex case" assumption (see the survey paper by F. Facchinei and C. Kanzow).

Since 2007, there has been an increasing number of papers developing generalized games (also called Generalized Nash Equilibrium Problems in the Operations Research community), proposing new existence and uniqueness results, new algorithms and new techniques to address generalized games, as well new problems in economics, social science or collective action problems for which generalized games appear to be the natural modeling framework. See for instance: [Le Cadre et al., 2020], [Fischer et al., 2014], [Aussel and Dutta, 2008], [Ardagna et al., 2012], [Facchinei and Sagra [Zhou et al., 2005], [Dreves et al., 2014], [Nagurney et al., 2016].

Our thesis sheds light on several problems in economics, collective action problems and financial regulation where generalized games are the natural modeling framework.

3.3 Generalized games with individual and/or shared constraints

In Chapter 1, we prove a simple result with many consequences regarding generalized games. Moreover this result and our framework in Chapter 1 is as general as possible and does not make any specific assumptions like "the jointly convex case" with differentiability assumptions that are commonly used when treating a generalized game with QVIs.

Consider a game such that all agents $i \in \{1, .., n\}$ are trying to maximize a payoff function $f_i(x_i, x_{-i})$ with $x_i \in X_i(x_{-i})$. $X_i()$ is a point-to-set map that associates a set of admissible strategies for player i given the vector of strategies of all other agents x_{-i}, and it is called an individual constraint. Now consider the constraint functions K_i such that for all i and x_{-i}, $K_i(x_{-i}) = \{x_i, x_j \in X_j(x_{-j}), j = 1, ..., n\}$. The constraints defined by K_i are called shared constraint generated from the individual constraints X_i. Our result is the following: the set of Nash equilibria in individual constraints is included in the set of Nash equilibria in shared constraint.

3.4 Modeling in Economics and collective action problems with generalized games

It happens that in many economic situations, generalized games are the natural framework to model the interaction between agents/players. We refer to the survey paper by F. Facchinei and C. Kanzow for several examples. We can moreover give the examples of [Le Cadre et al., 2020], [Nagurney et al., 2016], [Ardagna et al., 2012], to cite only a few.

In Chapter 1, we present and detail two examples of collective action problems that can be modeled with generalized games: a carbon emission problem and a public good problem. We show that for both examples there are some cases where no Nash equilibrium in individual constraints exist while there always exist at least one Nash equilibrium in shared constraint.

In Chapter 4, we model the action of a regulator in any game theory context through generalized games and study the existence of possible optimal regulations for a regulator who is seeking to maximize a social welfare function.

3.5 From static games with constraints to sequential and dynamic games with constraints

A frequent critics about static games, with or without constraints, is that they are not always representative of real-world problems in the sense that it is rather rare to meet with real-world problems where the agents act in a single shot and all at the same time. To this extent, static games offer an idealized and simplified modeling of interaction between agents that give a first grasp on the problem one is analyzing. We would like to defeat the critics about static games: an important point in our inquiry is that our frameworks and results are still true for any dynamic or sequential games with constraints, either discrete-time or continuous-time, either deterministic or with a stochastic component: the equilibria are the same in both cases. In this perspective, our study of problems with static games offer a first ground of study of real-world problems and the results we obtain on this simplified version of the problem we study can be transferred to the dynamic or sequential version.

The study of such dynamic generalized games offer enough substance for an article itself dedicated to this topic and we will not go further in this direction in the present thesis.

And there are many other considerations that we will not develop here but that will be the topic of discussion of a future article.

We can also quote the survey article from F. Facchinei and C. Kanzow:

"The point here is that one cannot imagine a game where the players make their choices simultaneously and then, for some reason, it happens that the constraints are satisfied. But indeed, this point of view appears to be rather limited, and severely undervalues

- the descriptive and explanatory power of the GNEP model;
- its normative value, i.e., the possibility to use GNEPs to design rules and protocols, set taxes and so forth, in order to achieve certain goals, a point of view that has been central to recent applications of GNEPs outside the economic field;
- the fact that in any case different paradigms for games can and have been adopted, where it is possible to imagine that, even in a noncooperative setting, there are mechanisms that make the satisfaction of the constraints possible."

Liste des travaux ayant contribué à la rédaction de la thèse

- *Economic foundations of generalized games with shared constraint: Do binding agreements lead to less Nash equilibria?*, accepté pour publication à *European Journal of Operational Research*;
- *Strategic foundations of macroprudential regulation: preventing fire sales externalities*, en revise and resubmit à *Journal of Mathematical Economics*;
- *A Generalized Nash Equilibrium Problem arising in banking regulation: An existence result with Tarski's theorem*, resoumis après une acceptation avec révision mineure à *Operations Research Letters*;
- *Generalized games and Optimal Regulation in Economics and collective action problems. An application to bank regulation during fire sales*, document de travail.
- *Target capital ratio and optimal channel(s) of adjustment: A simple model with empirical applications to European banks* publié dans *Applied Economics*.

Chapter 1

Economic foundations of generalized games with shared constraint: Do binding agreements lead to less Nash equilibria?

Abstract

A generalized game is a situation in which interaction between agents occurs not only through their objective function but also through their strategy sets; the strategy set of each agent depends upon the decision of the other agents and is called the individual constraint. As opposed to generalized games with exogenous shared constraint literature pioneered by [Rosen, 1965], we take the *individual constraints as the basic premises* and derive the shared constraint generated from the individual ones, a set K. For a profile of strategies to be a Nash equilibrium of the game with individual constraints, it must lie in K. But if, given what the others do, each agent agrees to restrict her choice in K, something that we call an endogenous shared constraint, this mutual restraint may generate new Nash equilibria. We show that the set of Nash equilibria in endogenous shared constraint contains the set of Nash equilibria in individual constraints. In particular, when there is no Nash equilibrium in individual constraints, there may still exist a Nash equilibrium in endogenous shared constraint. We also prove a few results for a specific class of generalized games that we call non-classical games. Finally, we give two economic applications of our results to collective action problems: carbon emission and public good problems.

Keywords: Game theory, generalized games, binding agreements, individual and shared constraints, collective action problems.

1 Introduction

It has long been recognized in Economics, Political sciences, International relations and more generally in Social sciences that Game theory is a useful tool to understand (and/or to predict) the outcome of a particular economic or social interaction between a set of agents (see [Fudenberg and Tirole, 1991], [Brams, 2011], [Schelling, 1980], [Moulin, 1986] for classical textbooks). A striking feature of most (not to say all) applied game theoretical models is that interaction between agents only takes place through the objective functions (utility or cost function). Given what the others choose, the aim of a given agent is to optimize her objective function by choosing the optimal strategy in a given set assumed to be invariant with respect to the choice of the other agents.

When each agent explicitly faces for instance a common binding constraint, the decision of a given agent may not only impact the objective function of the other agents but also their strategy set. Consider the well-known example of international environmental agreements (such as the Kyoto protocol) in which the total volume of emissions of greenhouse gas must be lower than a given threshold $\bar{e}$. From the point of view of a given country i, given the sum of emissions of greenhouse gas of the other countries e_{-i}, its total emission e_i explicitly depends upon the emissions of the other agents since the strategy set of country i is equal to $S_i(e_{-i}) = [0, \bar{e} - e_{-i}]$. Within such a simple and natural framework with a collective binding constraint, interaction between agents may not only take place through the objective function but also through their strategy set (see [Breton et al., 2006] and [Tidball and Zaccour, 2005] for early economic applications). Games in which the interaction takes place not only through the utility function (or cost function) but also through the strategy sets are called generalized games or Generalized Nash Equilibrium Problem (GNEP for short) (see [Facchinei and Kanzow, 2007], [Fischer et al., 2014]). Throughout this paper, we may call interchangeably GNEPs and generalized games.

Generalized games (GNEPs) are not recent and have received considerable attention in operational research. For instance, in their well-known survey, [Facchinei and Kanzow, 2007] offer a historical overview of GNEPs dating back to the seminal papers of [Arrow and Debreu, 1954], [Nash Jr, 1950], [Nash, 1951], and they provide interesting examples of applications of such games in telecommunication or in environmental pollution. In the applied maths literature more generally, there has been an abundant number of articles on GNEPs in recent years either proposing new methods, existence results or numerical algorithms to find a Nash equilibrium (e.g., [Facchinei et al., 2009], [Aussel and Dutta, 2008], [Fischer et al., 2014]). However, to the best of our knowledge, there has been only few papers trying to apply GNEPs in Economics (but see [Breton et al., 2006], [Elfoutayeni et al., 2012], [Le Cadre et al., 2020]).

A generalized game with *individual constraints* can be naturally defined as a game in which each agent has to satisfy her own individual constraint, that is, each agent i is required to pick a strategy x_i in a set which is (possibly) dependent on the strategies picked by the other agents. In this sense, classical games but also GNEPs can be seen as generalized games with individual constraints.

A generalized game with *shared constraint* is a specific category of games with individual constraints and has been introduced for the first time in [Rosen, 1965] although the term shared constraint does not appear[1]. In his seminal paper, [Rosen, 1965] pioneered such games and proves an existence result about concave games. Remarkably, Rosen's result is true not only on E defined as the classical Cartesian product of strategy sets but also on any closed and bounded convex subset $X \subset E$. Later on in the literature, X has been called the *shared constraint set.* Such a game is called generalized game with shared constraint in the sense that all the agents share the same constraint, that is, the profile of strategies $x := (x_1, ..., x_n)$ must always remain in the shared constraint set X: given x_{-i}, each agent i is required to pick a strategy $x_i \in E_i$ such that $x := (x_i, x_{-i}) \in X$.

The striking feature of the shared constraint approach developed in [Rosen, 1965] and the subsequent literature is that this shared constraint set X is *exogenously given* and bears no relationship with any possible individual constraints. Later on, [Bensoussan, 1974] introduced a variational formulation of the equilibrium of a generalized game with shared constraint and the literature on GNEPs now formulate the equilibrium as a (quasi) variational inequality (see also [Harker, 1984] and [Harker, 1991]). In its most basic version (see [Fischer et al., 2014] or [Facchinei and Kanzow, 2007] for excellent review papers), the variational formulation involves the partial derivative of the objective function of each agent with respect to her own strategy, which means that the underlying functions (objective function/constraints function) have to be differentiable.

It is the aim of this methodological paper to show the fruitfulness of such generalized games with shared constraints in the Economics of binding constraints[2]. We show how, by taking an alternative road-map to the traditional one encountered in the (mathematical) literature on GNEPs with shared

[1] Historically, generalized games were first introduced by [Debreu, 1952]. In [Harker, 1991] or in [Krawczyk, 2007], the authors note that a number of different names appeared in the literature to define these generalized games, abstract economy, social equilibria games, pseudo-Nash equilibria games or normalized equilibria ([Rosen, 1965]).

[2] It may be the result of an endogenous cooperation between the agents or the result of some constraining regulation.

constraints, our new methodological approach can be used to provide some foundations to strategic collective actions problems in Economics.

We re-consider in this paper GNEPs with shared constraints and our aim is to shed light on the relationships between the individual constraints on the one hand and the shared constraints on the other hand. Instead of considering a shared constraint set which is *exogenously given* and imposed to the set of agents, as in the literature on GNEPs, we consider these individual constraints as the basic premises of the game and derive an *endogenous shared constraint set* generated precisely by these individual constraints. The existing literature on generalized games with shared constraints has indeed two limitations. First, as said, the shared constraint is in general postulated and not generated from the individual constraints. Second, by formulating the equilibrium of the game as a variational inequality, the characterization of the equilibrium excludes the simplest games in which the strategy set of each agent is a finite set (e.g., the 2-2 games).

We prove in this paper the following simple result for which the economic applications are numerous: the set of Nash equilibria of a generalized game with individual constraints is *included* in the set of Nash equilibria of the generalized game with shared constraint generated from these individual constraints. Interestingly, [Feinstein and Rudloff, 2021] proved independently a similar result (see Theorem 3.2. in their paper) within a more abstract framework. This result, whose proof turns out to be simple, has two basic important consequences.

1. There are situations in which there is no Nash equilibrium in a game with individual constraints while such Nash equilibria exist in the game with shared constraint generated from the individual ones.

2. If there is no Nash equilibrium in shared constraint, then, no Nash equilibrium in the game with individual constraints can exist (the converse is however not true).

Note importantly that since we consider a game with shared constraint, our result requires some binding agreements or a constraining regulation; given what the others do, each agent i agrees or is constrained to pick a strategy not in her basic set of strategy (individual constraint) but in the shared constraint set that results from these individual constraints. Within our generalized game, sharing the constraint means in general that, given what the others do, compared with the primitive individual constraints X_i, each agent will now have to choose a strategy in a *smaller set*, that is, in $K_i \subset X_i$. Some strategies that were available in the game with individual (primitive) constraints are not anymore available in the game with shared constraint. Put it differently, introducing a shared constraint is equivalent to introducing restrictions and this kind of restriction exactly fits the notion of mutual restraint discussed in the well-known book of [Barrett, 2007] entitled *Why cooperate?*.

It is clear that a binding agreement requires some form of cooperation between the agents. In his well-known textbook, [Moulin, 1995] considers three modes of cooperation between a set of agents, direct agreements, decentralized behaviour and justice. In this paper, while not explicitly modeled, the mode of cooperation considered is the first one, direct agreements, and can be thought of as the result of preplay communication[3]. These direct binding agreements are particularly important when there is no Nash equilibrium in the game with individual constraints while (at least) one equilibrium exists in the game with shared constraint. We illustrate this idea in the second part of the paper to collective action problems, an externality-pollution and a public good problem. In each model, we show situation in which there is no Nash equilibrium in individual constraints while there may be (at least) one Nash equilibrium in shared constraint.

The remainder of the paper is structured as follows. In Section 2, we remind the definitions of generalized games as well as the corresponding notions of Nash equilibria. We then define the notion of generalized game with shared constraint generated from a game with individual constraints, something we call endogenous shared constraint. Moreover, we offer a classification of games depending upon the way the interaction between agents is introduced. A generalized game is a game where the interaction between agents occurs through the objective functions and the strategy sets. In section 3, to present our concept of endogenous shared constraint, we offer a gallery of 2×2 games, along the lines of [Fishburn and Kilgour, 1990], [Kilgour and Fraser, 1988]. In section 4, we prove that the set of Nash equilibria of a generalized game with individual constraints is included in the set of Nash equilibria of the generalized game with endogenous shared constraint. Moreover, we prove additional results regarding a specific class of generalized games called non-classical games in which interaction between agents occur only through the constraints. In Section 5, we offer two different collective action problem models which illuminate our results when no Nash equilibrium exists in individual constraints. The first model is a public good problem while the second one is an environment control problem and we show the economic usefulness of the introduction of an endogenous shared constraint in these problems. Section 6 concludes the paper.

[3] As observed in [Moulin, 1995], transaction cost is a drawback to this direct agreement mode. If this preplay communication is long and difficult, the transaction cost will be high. This problem of commitment is also discussed in the well-known book of [Ostrom, 1990]

2 Games with individual and shared constraints

2.1 Generalized games with individual constraints

We consider a game with $N \geq 2$ agents (or players) and we denote $J = \{1, ..., N\}$ the set of players. The decision (or control) variable of each player $i \in J$ is denoted by $x_i \in E_i$, where E_i is a subset of $\mathbb{R}^{n_i}$ called the strategy set. Let $E = \prod_{i=1}^{N} E_i = E_1 \times \ldots \times E_N$ and denote by $x \in E$ the vector formed by all these decision variables (strategies) which has dimension $n := \sum_{i=1}^{N} n_i$ so that $E \subset \mathbb{R}^n$. As usual in game theory, we denote by $x_{-i} \in E_{-i}$ the vector formed by all the players' decision variables except those of player i. To emphasize the i-the player's strategy, we sometimes write $(x_i, x_{-i}) \in E_i \times E_{-i}$ instead of $x \in E$. Each player i has an objective function $\theta_i : E \to \mathbb{R}$ that depends on both his own decision variables x_i as well as on the decision variables x_{-i} of all other players. We denote the objective function of player i by $\theta_i(x_i, x_{-i})$. For a given $x_{-i} \in E_{-i}$ and depending upon the game, the aim of agent i may be to maximize or to minimize its objective function $\theta_i(x_i, x_{-i})$. In general, when this objective function is a utility (or a payoff) function, it is the aim of the agent to maximize it while when it is a cost (or a loss) function, it is the aim of the agent to minimize it. Throughout the article, unless otherwise specified, we will assume that the objective function is a cost function so that each agent i, given the other players' strategies x_{-i}, is seeking a strategy x_i to minimize $\theta_i(x_i, x_{-i}) = \theta_i(x)$. Throughout the paper, we may use interchangeably the terms decision, decision variable and strategy and we only consider the case of pure strategy, that is, the situation in which each agent chooses a strategy with probability one.

In a generalized game, each player $i \in J$ must pick a strategy $x_i \in X_i(x_{-i}) \subseteq E_i$ where the set $X_i(x_{-i})$ explicitly depends upon the rival players' strategies. As in classical games in which the strategy set E_i of each agent i is given, in generalized games, the strategy set that we call the individual constraint function $X_i(x_{-i})$ (or simply the individual constraint) is also exogenously given. To define in full generality the individual constraint which depends upon the decision of others, let X_i be defined as follows:

$$X_i : E_{-i} \to \mathcal{P}(E_i) \quad i = 1, 2, ..., N \tag{1.1}$$

where $\mathcal{P}(E_i)$ is the power set of E_i. It is usual to call X_i a point-to-set map (ot a set-valued map) since it associates a subset of E_i to each point of E_{-i}. This means that for a given point $x_{-i} \in E_{-i}$, the strategy set also called the individual constraint of agent i is equal to $X_i(x_{-i}) \subseteq E_i$, a subset of E_i.

We want to focus on the economic foundation of generalized games. For instance, in a strategic interaction with two agents, given the choice x_2 of agent 2, agent 1 may have to choose x_1 in E_1 subject to a constraint of the form $g_1(x_1, x_2) = x_1 + bx_2 \leq r_1$. Put it differently, the strategy set of agent 1 explicitly depends upon the choice of agent 2. Let us consider two economic examples of such a situation that will be considered in detail in section 5.

- In an environmental problem with externalities, agent 1 (i.e., country 1) may be constrained to choose its emission of greenhouse gas x_1 subject to a constraint of the form $g_1(x_1, x_2) = x_1 + b_2 x_2 \leq r_1$ where r_1 is the maximum emission of country 1 and $b_2 x_2$ is the impact emission of country 2 on country 1.

- In a public good problem, following [Guttman, 1978], agent 1 has to choose a flat contribution x_1 plus a matching rate bx_2 (which is a function of the flat contribution of agent 2 where $b \in (0, 1]$) so that $g_1(x_1, x_2) = x_1 + bx_2 \leq r_1$ is the budget constraint of agent 1.

When thinking about collective action problems, there are various situations in which the decision of a given agent has an impact on the constraint of another agent.

Definition 1 *The* 4*-uplet* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *is called a generalized game with individual constraints.*

It should be clear that "classical games" encountered in Economics appear as a particular generalized games with individual constraints. When, for each $i \in J$, X_i is invariant with respect to the choice of the other agents, we are back to a classical game in which the interaction only takes place through the objective functions. In such a case, X_i reduces to E_i.

2.2 Generalized games with endogenous and exogenous shared constraint: definitions and economic foundations

Recall that for a classical game, the best response BR_i of an agent i is defined as

$$\begin{array}{rcccl} BR_i & : & E_{-i} & \to & E_i \\ & & x_{-i} & \mapsto & \{x_i^* \text{ such that } \theta_i(x_i^*, x_{-i}) = \min_{x_i \in E_i} \theta_i(x_i, x_{-i})\} \end{array}$$

that is, given a vector of strategies of other players x_{-i}, the best response gives the set of optimal strategie(s) for agent i. Unless mentioned otherwise, BR_i needs not be a singleton and is indeed in general a point-to-set map. As we shall now see, when studying a generalized game, different constraints will inevitably lead to different best response functions.

Some definitions. In a generalized game, as opposed to classical games, it may be the case that for some x_{-i}, the strategy set of agent i is simply empty, that is, $X_i(x_{-i}) = \emptyset$, which means that the objective function is undefined. As a result, the best response is also undefined so that the equilibrium can not exist. Such an empty set problem never occurs in classical games since the strategy set of each agent i is invariant with respect to x_{-i}. When $x \in E$ is such that $x_i \in X_i(x_{-i})$ for each agent $i \in J$, we say that the profile of strategies x is *admissible*.

Definition 2 *For a given generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$*, let* K *be the subset of* E *defined as follows.*

$$K = \{x \in E,\ \forall i \in J,\ x_i \in X_i(x_{-i})\} \tag{1.2}$$

K *is called the set of admissible strategies of the generalized game with individual constraints.*

The set K represents the set of profiles of strategies x for which the generalized game with individual constraints is defined for all agents. If x does not belong to K, it can not be a Nash equilibrium of the generalized game. Throughout the paper, for the sake of interest, we assume that K is not empty.

For a generalized game in individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$, the best response function of agent i is defined as

$$BR_i^{Ind} : x_{-i} \mapsto \{x_i^* \text{ such that } \theta_i(x_i^*, x_{-i}) = \min_{x_i \in X_i(x_{-i})} \theta_i(x_i, x_{-i})\} \tag{1.3}$$

A generalized Nash equilibrium problem (GNEP) is the given of N constrained optimization problems, that is, for each $i \in J$, given x_{-i}, agent i optimizes $\theta_i(x_i, x_{-i})$ subject to $x_i \in X_i(x_{-i})$. Note importantly that as opposed to a classical game, the best response for a generalized game depends upon the point-to-set map $X_i()$. A Nash equilibrium $x^* = (x_1^*, ..., x_N^*) \in K$ of the generalized game thus is such that no agent i wants to unilaterally deviate from her part of the equilibrium profile x^* but also such that the constraint of each agent $i \in J$ is satisfied, i.e., $x^* \in K$. The following definition makes clear this constraint.

Definition 3 *The profile of strategies $x^* \in E$ is a Nash equilibrium of the generalized game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ if, for each $i \in J$, $x_i^* \in BR_i^{Ind}(x_{-i}^*)$.*

From the above discussion, a necessary but not sufficient condition on the profile of strategies x to be a Nash equilibrium is that $x \in K$. Since K is assumed to be not empty, it makes economic sense to require from each agent that given what the other agents are choosing, i.e., x_{-i}, agent i should pick a strategy x_i such that the profile $x = (x_i, x_{-i})$ lies in K if such a x_i exists. Given x_{-i}, let $K_i(x_{-i})$ be the set of strategies of agent i defined as follows

$$K_i(x_{-i}) = \{x_i \in E_i : x \in K\} \tag{1.4}$$

We are now in a position to define a generalized game with endogenous shared constraint, that is, a generalized game in which the shared constraint is generated from the individual constraints.

Definition 4 *The 4-uplet $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ is called a generalized game with shared constraint generated from the game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. We call such a game a game with endogenous shared constraint.*

For a generalized game in (endogenous) shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$, the best response function of agent i is defined as

$$BR_i^{Sh} : x_{-i} \mapsto \{x_i^* \text{ such that } \theta_i(x_i^*, x_{-i}) = \min_{x_i \in K_i(x_{-i})} \theta_i(x_i, x_{-i})\} \tag{1.5}$$

The best response is similar to the one given for a generalized with individual constraints except that each agent i is required to choose a strategy in $K_i(x_{-i})$ rather than in $X_i(x_{-i})$. We are now in a position to give the definition of a Nash equilibrium in a game with endogenous shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$.

Definition 5 *The profile of strategies $x^* \in E$ is a Nash equilibrium for the generalized game with endogenous shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ if for each $i \in J$, $x_i^* \in BR_i^{Sh}(x_{-i}^*)$.*

To better understand the difference between a game with individual constraints and a game with endogenous shared constraint, assume that $X_i(x_{-i}) := \{x_i \in E_i : g_i(x) \leq 0\}$ is the strategy set (or constraint) of each agent i for some function (constraint) g_i. In the literature, it is standard to describe the constraint using such functions g_i. Making use of the functions g_i, $i = 1, 2, .., N$, the admissible set K (see equation (4.3)) can be rewritten as $K = \{x \in E,\ \forall i \in J, g_i(x) \leq 0\}$ so that $K_i(x_{-i}) := \{x_i \in E_i : \forall j \in J, g_j(x) \leq 0\}$.

- In the game with individual constraint, given x_{-i}, each agent optimizes its objective function with respect to x_i subject to its *own individual constraint* $g_i(x) \leq 0$, that is, $x_i \in X_i(x_{-i})$.

- In the game with shared constraint generated from the individual ones, given x_{-i}, each agent optimizes its objective function with respect to x_i subject to $g_1(x) \leq 0$, $g_2(x) \leq 0$,..., $g_N(x) \leq 0$, that is, $x_i \in K_i(x_{-i})$. Given x_{-i}, when each agent i chooses a strategy x_i, she takes not only into account its own constraint but also the *constraints of all the other agents* so that (x_i, x_{-i}) lies in K.

It should be clear that in a game with endogenous shared constraint, the set of strategies of each agent i may be *reduced* compared to the game with individual constraints, that is,

$$K_i(x_{-i}) \subseteq X_i(x_{-i}) \quad \forall i \in J \tag{1.6}$$

In general, the inclusion may be strict for some agents, that is, as long as $X_i(x_{-i})$ is not empty, $K_i(x_{-i}) \subset X_i(x_{-i})$. In what follows, to emphasize that the set K is the shared constraint, we may denote the game $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ as $(J, E, (\theta_i)_{i\in J}, K)$.

Let $\mathcal{N}_{Indiv}$ and $\mathcal{N}_{Shared}$ be respectively the set of Nash equilibria in individual constraints and with (endogenous) shared constraint and note that it may be the case that both sets are empty. We shall show later on that $\mathcal{N}_{Indiv} \subseteq \mathcal{N}_{Shared}$ and we shall provide various illustrations of this result (2-2 games, continuous games, linear or not...).

Remark 1 *Within our approach, we take the set $X_i(x_{-i})$ (the individual constraints) as the basic premises and we derive the set K (the endogenous shared constraint) from these individual constraints. This is in sharp contrast with the literature on generalized games in which the shared constraint is exogenous. Following [Rosen, 1965], authors typically start with a classical game for which $E = \prod_{i=1}^{N} E_i$ and what they call the shared constraint is an exogenous (mathematically convenient) prescribed set $X \subset E$, that is, the profile of strategies x must be located in X, see e.g., [Fischer et al., 2014] for a nice review paper.*

Let $(J, E, (\theta_i)_{i\in J}, X)$ be a generalized game with an exogenous shared constraint. The following well-known result about the existence of Nash equilibria for concave n-person games is due to

[Rosen, 1965]. Using our terminology, the following result is an existence result for a game with an exogenous shared constraint X.

Theorem 1 *([Rosen, 1965]) Let $(J, E, (\theta_i)_{i\in J}, X)$ be a game with an exogenous shared constraint where θ_i is a payoff function. If the set X is convex, closed and bounded and if each player's payoff function $\theta_i(x_i, x_{-i})$, $i \in J$ is continuous and concave in x_i, then, the generalized game has at least one Nash equilibrium.*

[Rosen, 1965] considers the simple example of a 2-person game in which X is a compact convex subset of the unit square such as an ellipse. This thus means that given the choice of agent 1, agent 2 has to choose a number between zero and one such that the couple of numbers chosen must be located in the ellipse. In many other papers, the authors consider the set X defined as $X = \{x \in E \mid G(x) \leq 0\}$, where $G : \mathbb{R}^n \to \mathbb{R}$ is a component-wise convex function called the shared constraint function, an assumption convenient for the mathematical analysis of the generalized game (see [Facchinei and Kanzow, 2010] or [Fischer et al., 2014] for review papers). An attractive feature of these games with shared constraint is that they are analytically convenient. Under particular assumptions on X and on the objective functions $\theta_i,\ i \in J$, it is often possible to use a fixed-point theorem (Kakutani, Tarski...) to prove the existence of a Nash equilibrium.

Shared constraint as binding agreements. As discussed in the introduction of this paper, sharing the constraints either requires a particular form of cooperation between agents that are called *direct agreements* in [Moulin, 1995] or an exogenous constraint that one may call a constraining regulation. In his seminal paper, [Harker, 1991] makes a similar observation. He notes that there are "two basic ways by which the constraints across players arise: the imposition of these constraints by an external 'player' in the game, or by the joint imposition by the set of N players on their own actions". Within our generalized game framework with shared constraint, given what the others do, each agent must restrict her choice in the shared constraint K, that is, given x_{-i}, each agent i must pick a strategy x_i in $K_i(x_{-i})$ rather than in $X_i(x_{-i})$, where $K_i(x_{-i})$ is included in $X_i(x_{-i})$. Given x_{-i}, it might be the case that the best response of a given agent i lies in $X_i(x_{-i})$ and not in $K_i(x_{-i})$. As a result, this restriction to the set of strategies $K_i(x_{-i})$ rather than $X_i(x_{-i})$ requires some binding agreements (or a regulation) that are however not explicitly modelled here. This point is identical to the discussion offered in the well-known book of [Ostrom, 1990] in which she notes that one (frequent theoretical) solution to this problem is *coercion*. Roughly speaking, one can make a (perhaps disputable) distinction between two types of coercion, external or internal.

- external (or exogenous) coercion corresponds to the situation in which each agent must comply with law or regulation and this means that there is an external enforcer, to use the terminology of ([Ostrom, 1990]). In such a situation, an agent who does not comply with the law or regulation can be sanctioned (i.e., fined) by the external enforcer.

- internal (or endogenous) coercion corresponds to the situation in which a group of agents (employees, firms, countries or even the overall society itself) must reach an agreement without any external enforcer. This clearly means that such an agreement is based on voluntarism since an agent who breach the (contractual) agreement can not be fined.

Internal coercion thus is an agreement based on self-restriction, which turns out to be very similar to the notion of *mutual restraint* discussed in [Barrett, 2007] (see chapter five). In [Barrett, 2007], the author explicitly considers the case in which agents are countries that seek to supply global public goods such as nuclear non-proliferation or climate change mitigation (e.g., limit carbon emission). For instance, when one considers a set of countries that try to reduce their individual pollution, mutual restraint can be reached through international treaties and these treaties can be thought of as an example of an internal coercion. Given x_{-i}, the restriction of each agent i to the set $K_i(x_{-i})$ can be seen as a possible formalization of the notion of mutual restraint discussed in [Barrett, 2007].

In what follows, we thus make the implicit assumption that, in shared constraint, agents succeeded to reach binding agreements so each agent (on a voluntary basis given x_{-i}) agrees to be restricted to $K_i(x_{-i})$. As we shall see, when no equilibrium exists in the game with individual constraints, this mutual restraint might be the unique solution to reach an equilibrium situation. In the last section of this paper, we shall offer two different models of collective actions in which the equilibrium only exists (depending upon parameters) in shared constraint.

2.3 A classification of games

In a generalized game, (see table 1), interaction occurs not only through the objective function but also through the strategy sets, that is, the objective function of a given agent $i \in J$ depends upon the decisions of the other players, i.e., $\theta_i(x_i, x_{-i})$ but the set of strategy of each agent i also depends upon the decisions of the other agents, that is, each agent i must choose $x_i \in X_i(x_{-i})$ where $X_i(x_{-i})$ is a subset of E_i. In table 1, we offer a fruitful classification of games, encountered in Economics and Operations Research literature, through the way interaction is introduced.

Dependence through the objective function and/or through the strategy sets

Objective functionStrategy set	Yes	No
Yes	Generalized game	Classical game
No	Non-Classical game	Non-strategic decision problem

Table 1.1: Types of interaction in games

For what we call *classical games*, those encountered in most economic textbooks and papers in economic theory (see e.g. classic textbooks [Fudenberg and Tirole, 1991], [Moulin, 1986], [Myerson, 2013], [Osborne and Rubinstein, 1994]), interaction occurs through the objective functions but not through the constraints. For what we call *non-classical games*, interaction occurs through the constraints but not through the objective functions. To the best of our knowledge, such an example of non-classical game appears for the first time in [Braouezec and Wagalath, 2019], and more recently in [Braouezec and Kiani, 2022], in a financial setting. While these non-classical games are fairly natural when one thinks to collective action problems, their application in the Economic literature is still limited at the moment. In the future, these kind of generalized games could be widely applied. Finally, when there is no interaction at all, that is, when the objective function only depends upon the decision variable of agent i, that is $\theta_i(x_i)$ and when E_i is exogenously given (e.g., it is a compact set), this gives rise to a non-strategic decision problem.

Remark 2 *[Banerjee and Feinstein, 2021] consider an extension of the game-theoretic model presented in [Braouezec and Wagalath, 2019] though none of the two articles mention the fact that they use generalized games (actually [Braouezec and Wagalath, 2019] consider a non-classical game).*

3 Equilibrium analysis of 2×2 generalized games: a gallery

Games with two agents and two strategies called 2×2 games are used in many introductory textbooks as a way to explain the basic concepts of game theory, but also to illustrate how in specific strategic situations, individual rationality may differ from the collective rationality (prisoners' dilemma). In [DeCanio and Fremstad, 2013], the authors apply 2×2 games to international climate negotiation problems where each agent has two strategies, Abate or Pollute. Various games such as the prisoners' dilemma and the chicken games are discussed. They also discuss cases in which there is no Nash equilibrium in pure strategy. In [Fishburn and Kilgour, 1990] (see also [Kilgour and Fraser, 1988] and [Walliser, 1988]), they offer a more theoretical analysis to 2×2

Cost for each agent	$\theta_A(x_A, x_B)$	$\theta_B(x_A, x_B)$
Constraint for each agent	$g_A(x_A, x_B)$	$g_B(x_A, x_B)$

Table 1.2: Cost and constraint for a given pair of strategies (x_A, x_B)

games.

Consider the following game with two agents such that each agent $i \in \{A, B\}$:

- seeks to minimize a cost function θ_i.
- can choose between two strategies 1 and 2: $E_i = \{1, 2\}$.
- is subject to the constraint $g_i(x_A, x_B) \leq c_i$

Generalized 2×2 games are more complex than classical 2×2 games encountered in economic theory (e.g., [DeCanio and Fremstad, 2013]) in which interaction occurs only through the objective function. In generalized games, whether or not the set strategies is finite, the choice of a given strategy by one agent directly impacts the objective function of the other agents but also their set of strategies (through their constraints). When one considers simple 2×2 generalized games, both the objective function and the constraint thus must be exhibited, as in table 2. In what follows, without loss of generality, we shall assume that $c_A = c_B = c$. For instance, in Fig.1.2, when agent A chooses strategy 1 and agent B chooses strategy 2, for agent A, the cost is equal to $\theta_A(1, 2) = 0$ and the constraint is equal to $g_A(1, 2) = 0.5$ while for agent B, the cost is equal to $\theta_B(1, 2) = 1$ and the constraint is equal to $g_B(1, 2) = 1.5$. We provide a simple taxonomy of generalized 2×2 games in which we consider the set of Nash equilibria in individual and in shared constraint. To make things easier, we consider generalized games with a simple structure: as long as agents are "coordinated", when they choose the same strategy, their cost is equal to one, that is, $\theta_i(1, 1) = \theta_i(0, 0) = 1$ for $i \in \{A, B\}$ and their constraint is satisfied. We consider three different games in which we only change the cost or the constraint when agents are not coordinated. Recall that $\mathcal{N}_{Indiv}$ and $\mathcal{N}_{Shared}$ are respectively the set of Nash equilibria in individual constraints and with (endogenous) shared constraint

1. $\mathcal{N}_{Indiv} = \mathcal{N}_{Shared}$. Consider Fig. 1.1 and assume that $c = 1$. There are two Nash equilibria in individual constraints, $\mathcal{N}_{Indiv} = \mathcal{N}_{Shared} = \{(1, 1); (2, 2)\}$. Knowing that agent A chooses strategy 1, the best response of agent B is to choose strategy 1. Conversely, knowing that agent B chooses strategy 1, the best response of agent A is to choose strategy 1 so that the pair

		B			
		1		2	
A	1	1	1	2	1
		0.5	0.5	0.5	1.5
	2	1	2	1	1
		1.5	0.5	0.5	0.5

Figure 1.1: $\mathcal{N}_{Indiv} = \mathcal{N}_{Shared} = \{(1,1);(2,2)\}$

		B			
		1		2	
A	1	1	1	0	1
		0.5	0.5	0.5	1.5
	2	1	0	1	1
		1.5	0.5	0.5	0.5

Figure 1.2: $\mathcal{N}_{Indiv} = \{(1,1)\}$ and $\mathcal{N}_{Shared} = \{(1,1);(2,2)\}$

		B			
		1		2	
A	1	1	1	2	0.75
		0.5	1.5	1.5	0.5
	2	1.5	0.5	1	1
		1.5	0.5	0.5	0.5

Figure 1.3: $\mathcal{N}_{Indiv} = \emptyset$ and $\mathcal{N}_{Shared} = \{(2,2)\}$

of strategy $(1,1)$ is a Nash equilibrium in individual constraints. The constraint of each agent is satisfied so that the pair of strategy $(1,1)$ is also a Nash equilibrium in shared constraint. A similar analysis can be done for the pair of strategies $(2,2)$. For such a game, everything is as if the constraint were not an issue since it is always satisfied. Such a game can be thought of as a coordination (generalized) game since there are two identical Pareto optimal Nash equilibria.

2. $\mathcal{N}_{Indiv} \subset \mathcal{N}_{Shared}$. Consider Fig. 1.2 and assume that the constraint is $c = 1$. In such a case, only $(1,1)$ is a Nash equilibrium in individual constraints so that $\mathcal{N}_{Indiv} = \{(1,1)\}$. To see this, assume that agent A chooses $x_A = 1$. Abstracting the constraint, agent B is indifferent between strategy 1 and 2. However, since $g_B(1,1) = 0.5$ and $g_B(1,2) = 1.5$, the constraint of agent B is only satisfied when she/he chooses strategy 1, i.e., $X_B(x_A = 1) = \{1\}$. It thus follows that the best response of agent B is $BR_B^{Ind}(x_A = 1) = 1$. Assume now that agent B chooses strategy 1. Since agent A only satisfies the constraint for decision 1, it thus follows that $BR_A^{Ind}(x_B = 1) = 1$ so that $(1,1)$ is a Nash equilibrium under individual constraints. Consider now the pair of strategies $(2,2)$. If agent B chooses $x_B = 2$, then, the best response in individual constraint is $BR_A^{Ind}(x_B = 2) = 1$ and this means that $(2,2)$ is not a Nash equilibrium under individual constraints. However, the pair of strategies $(2,2)$ is a Nash equilibrium in shared constraint since the best response in shared constraint are $BR_A^{Sh}(x_B = 2) = 2$ and $BR_B^{Sh}(x_A = 2) = 2$. To see this, it suffices to note that $X_A(x_B = 2) = \{1,2\}$ while $K_A(x_B = 2) = \{2\}$ and $X_B(x_A = 2) = \{1,2\}$ while $K_B(x_A = 2) = \{2\}$. It thus follows that $\mathcal{N}_{Shared} = \{(1,1);(2,2)\}$ while $\mathcal{N}_{Indiv} = \{(1,1)\}$

3. $\mathcal{N}_{Indiv} = \emptyset$ while $\mathcal{N}_{Shared} \neq \emptyset$. Along the same line of reasoning, it is easy to see in Fig. 3 that there is no Nash equilibrium in individual constraint while the pair of strategies $(2,2)$ is a Nash equilibrium in shared constraint. In the game in individual constraints, when agent A chooses strategy 2, agent B chooses strategy 1 so that $(2,2)$ is not a Nash equilibrium. However, in shared constraint, agent B can not choose strategy 1 because agent A would not satisfy her/his constraint anymore.

4 Theoretical results

4.1 An elementary general result

As we shall now see, a striking feature of a game with endogenous shared constraint, compared with the game with individual constraints, is that it may possess *additional Nash equilibria.* We already know that for a profile x to be a Nash equilibrium, x must be in K. But if each agent i, given x_{-i}, agrees to choose x_i such that $x = (x_i, x_{-i})$ is in K (i.e., $x_i \in K_i(x_{-i})$), the set of Nash equilibria may be *larger.*

Proposition 1 *The set of Nash equilibria of a game with individual constraints denoted $\mathcal{N}_{Indiv}$ is included in the set of the Nash equilibria of the game with shared constraint generated from the individual constraints, denoted $\mathcal{N}_{Shared}$; that is, $\mathcal{N}_{Indiv} \subseteq \mathcal{N}_{Shared}$.*

Proof. See the appendix.

Independently, [Feinstein and Rudloff, 2021] proved a similar result (see their Theorem 3.2). While easy to prove, it is important to point out that such a result only makes sense for generalized games in which the shared constraint is derived from the individual ones. As we shall see, this result is particularly relevant when $\mathcal{N}_{Indiv} = \emptyset$ while $\mathcal{N}_{Shared} \neq \emptyset$. To illustrate the above proposition, that is, to show that $\mathcal{N}_{Indiv} \subseteq \mathcal{N}_{Shared}$, consider the following game in which the set of strategies have the cardinality of the *continuum.*

Let $J = \{1, 2\}$ be the set of agents and let $E_1 = E_2 = [0, 1]$ be the strategy set of each agent so that $E = [0, 1] \times [0, 1]$ is the strategy space. The cost functions are $\theta_1(x_1, x_2) = x_1 + x_2$ and $\theta_2(x_1, x_2) = x_2 - x_1$. Assume now that the individual constraint for agent 1 is $g_1(x_1, x_2) = x_1 + x_2 \leq 1$ while it is equal to $g_2(x_1, x_2) = x_1 + x_2 \geq \frac{1}{2}$ for agent 2. In this example, the set K is defined as

$$K = \left\{ (x_1, x_2) \in [0, 1]^2 : x_1 + x_2 \leq 1 \ \text{ and } \ x_1 + x_2 \geq \frac{1}{2} \right\} \tag{1.7}$$

It is not difficult to see that K is a compact and convex set (see Fig. 1.4) and that the profile of strategies $(0, \frac{1}{2}) \in K$ is a Nash equilibrium of the game with individual constraints. Consider now the profile of strategies $(\frac{1}{2}, 0) \in K$.

- In the game with individual constraints, $(\frac{1}{2}, 0) \in K$ is *not* a Nash equilibrium. To see this, it suffices to note that when $x_2 = 0$, the best response of agent 1 is 0 and the constraint of

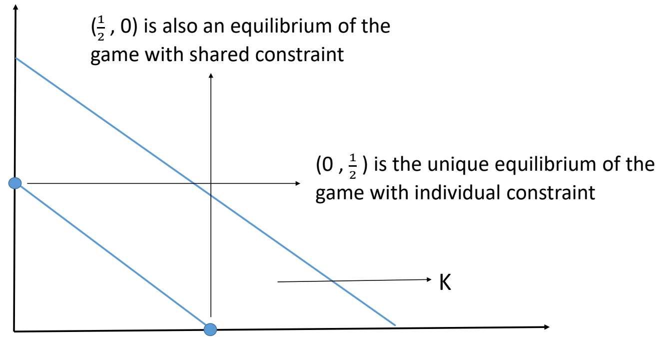

Figure 1.4: One equilibrium in individual constraints, an infinity of equilibria in shared constraint

agent 1 is fulfilled since $g_1(x_1, x_2) = x_1 + x_2 \leq 1$. Since $\frac{1}{2}$ is not the best response, $(\frac{1}{2}, 0) \in K$ thus is not an Nash equilibrium.

- In the game with shared constraint, $(\frac{1}{2}, 0) \in K$ is a Nash equilibrium. To see this, note that if $x_1 = \frac{1}{2}$, then, the best response of agent 2 is 0. This choice of 0 minimizes the cost of agent 2 and satisfies the constraint of agent 2 since $\frac{1}{2} + 0 \geq \frac{1}{2}$ but also the constraint of agent 1 since $\frac{1}{2} + 0 \leq 1$. If $x_2 = 0$, the best response of agent 1 is now to choose $\frac{1}{2}$ because agent 1 takes into account the constraint of agent 2, i.e., $g_2(x_1, x_2) = x_1 + x_2 \geq \frac{1}{2}$. As opposed to the game with individual constraints, agent 1 can not choose 0. As a result, $(\frac{1}{2}, 0)$ is a Nash equilibrium for the game with endogenous shared constraint. Actually all vectors (x_1, x_2) such that $x_1 + x_2 = \frac{1}{2}$ are Nash equilibria in shared constraint. Indeed, if $x_1 + x_2 = \frac{1}{2}$ both constraints are satisfied and no agent has an incentive and possibility to move to a better state. Let us prove that the converse is true. Consider (x_1, x_2) such that $1 \geq x_1 + x_2 > \frac{1}{2}$ then one of the two agents will have the incentive and possibility to decrease its cost. Therefore the set of Nash equilibria in endogenous shared constraint for this game is equal to: $\mathcal{N}_{Shared} = \{(x_1, x_2) \in E,\ x_1 + x_2 = \frac{1}{2}\}$.

- Let us now show that $\mathcal{N}_{Indiv} = \{(0, \frac{1}{2})\}$. From Proposition 1 we have that $\mathcal{N}_{Indiv} \subset \mathcal{N}_{Shared} = \{(x_1, x_2) \in E,\ x_1 + x_2 = \frac{1}{2}\}$. Consider $(x_1, x_2) \in \mathcal{N}_{Shared}$ such that $x_1 > 0$. Then agent 1 would have an incentive and possibility to take $x_1 = 0$ instead and therefore (x_1, x_2) would not be a Nash equilibrium in individual constraints. Therefore, $\mathcal{N}_{Indiv} = \{(0, \frac{1}{2})\}$.

This is an example of game in which there is a single equilibrium in individual constraints and an infinity of equilibria in shared constraint. As already said, by requiring from each agent i to choose a strategy in $K_i(x_{-i}) \subseteq X_i(x_{-i})$, this may *expand* the set of Nash equilibria. Proposition 1 thus yields the two following basic insights.

1. There may be situations in which there is no Nash equilibrium in the game with individual constraints, that is, $\mathcal{N}_{Indiv} = \emptyset$ while there exists a Nash equilibrium in the game with endogenous shared constraint (generated from the individual ones), that is $\mathcal{N}_{Shared} \neq \emptyset$.

2. If no Nash equilibrium exists in the game with endogenous shared constraint, that is, if $\mathcal{N}_{Shared} = \emptyset$, no Nash equilibrium exists in the game with individual constraints, that is, $\mathcal{N}_{Indiv} = \emptyset$, the converse is however not true.

It is interesting to note that proposition 1 does not require the underlying functions to be differentiable, as opposed to the variational formulation of the Nash equilibrium in generalized games. It can also be applied to generalized games in which the set of strategies of each agent is finite, for which the objective function needs not be a continuous function (see the gallery of 2×2 games).

Let us now provide an additional example in which $\mathcal{N}_{Indiv} = \emptyset$ while $\mathcal{N}_{Shared} \neq \emptyset$.

Let $J = \{1, 2, ..., N\}$ be the set of agents. For each $i \in J$, the strategy set of agent i is $E_i = [0, 1]$ so that $E = [0, 1]^N$. Assume moreover that the characteristics of the agents are as follows.

1. Each agent $i \in J$ has a cost function (to be minimized) equal to $\theta_i(x_i) = x_i$.

2. Each agent $i \in J$ has the following constraint function.

   - If $x_j \geq 0.9$ for every $j \in J \setminus \{i\}$, then, $X_i(x_{-i}) = [\frac{1}{2}, 1]$.
   - If $x_j < 0.9$ for at least one $j \in J \setminus \{i\}$, then, $X_i(x_{-i}) = \emptyset$.

This game thus differs from classical ones encountered in economic theory in that the interaction only occurs through the set of strategies but not through the objective functions. From the specification of the game, if a given agent i chooses a number $x_i \geq 0.9$, the cost of agent i is simply equal to the number x_i chosen. If there is one agent j who chooses a number $x_j < 0.9$, with $i \neq j$ then, the set of strategies of a given agent is empty and the objective function thus is undefined[4]. Before

[4]One can think of this abstract game theoretic framework to represent a tax competition problem in a fiscal union (see for instance [Zodrow, 2003]). State members may be committed to choose a tax rate greater than a given threshold. However, if one member state breaches the commitment, the problem becomes undefined for the other member states in that their strategy set is empty.

discussing the outcome of the game with individual constraints, let us consider the game with shared constraint generated from the individual constraints. In this game with shared constraint, the set of strategies of each agent $i \in J$ is equal to $K_i(x_{-i}) = [0.9, 1]$ if $x_{-i} \in [0.9, 1]^{N-1}$ (if $x_{-i} \notin [0.9, 1]^{N-1}$, then, $X_i(x_{-i}) = \emptyset$ so that $K_i(x_{-i}) = \emptyset$) so that K is not empty and equal to $K = [0.9, 1]^N$. Since agents minimize a cost function, the profile of strategies $x^* = (0.9, ..., 0.9)$ thus is the *unique* Nash equilibrium of the game with shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$. Therefore, from proposition 1, if the game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ has a Nash equilibrium, it must necessarily be $x^* = (0.9, ..., 0.9)$. But $x^* = (0.9, ..., 0.9)$ is *not* a Nash equilibrium of the game with individual constraints. If $x_{-i} = (0.9, ..., 0.9)$, the best response of agent i is 0.5 and not 0.9, which means that $x^* = (0.9, ..., 0.9)$ is not a Nash equilibrium of the game with individual constraints. When one agents picks a number lower than 0.9, the objective functions are undefined for the other agents and this means that there is no Nash equilibrium in such a game with individual constraints.

4.2 Existence and characterization results for non-classical games

We now provide an existence result which can be thought as a generalization of Rosen's theorem for non-classical game. As said earlier, such a non-classical game appears for the first time in [Braouezec and Wagalath, 2019] and more recently in [Braouezec and Kiani, 2022] in a financial stress test framework.

Proposition 2 *Consider a non-classical game $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ such that for each $i \in J$, $\theta_i(x) = \theta_i(x_i)$ and assume that for each $i \in J$, θ_i is continuous and that K is compact. The following results hold.*

1. *The game with shared constraint $(J, E, (\theta_i)_{i\in J}, K)$ admits at least one Nash equilibrium.*

2. *The set of Nash equilibria in shared constraint $\mathcal{N}_{Shared}$ exactly coincides with the set of minimizers of the total cost function $\sum_{i\in J} \theta_i(x_i)$ on K and each Nash equilibrium is Pareto optimal.*

Proof. See the appendix.

Note that from the above result, one cannot infer any result regarding the existence of Nash equilibria in individual constraints. As the next proposition shows, under these conditions, a Nash equilibrium in individual constraints does not always exist. Note interestingly that in proposition 2, we make no assumption regarding the evolution of θ_i with respect to $x_i \in \mathbb{R}^d$ with $d \geq 1$. We shall now show an example in which the non-classical game satisfies the assumptions of Proposition 2, and therefore the game with endogenous shared constraint $(J, E, (\theta_i)_{i \in J}, K)$ has at least one Nash equilibrium, but the initial game with individual constraints $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$ may or may not have a Nash equilibrium. The following result holds for non-classical games with positive linear constraints that will be illustrated in Section 5.1. Note that in the next result, the objective function is typically a profit function to be maximized.

Proposition 3 *Consider a non-classical game $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$ such that for each $i \in J$, $E_i = \mathbb{R}_+$ and let $S_i \in \mathbb{R}_+$ be the constraint of agent i. Assume that the payoff function $\theta_i(x) = \theta_i(x_i)$ is an increasing, continuous and concave function of x_i and that the constraint of agent i is given by a linear function $l_i(x) = \sum_{j \in J} a_{ij} x_j$ where $a_{ij} > 0$ for each i and j, such that $X_i(x_{-i}) = \{x_i \in E_i,\ l_i(x) \leq S_i\}$. Let $A = (a_{i,j})_{i,j \in J}$ be a matrix and $S = (S_i)_{i \in J}$ be a column vector.*

1) Game in individual constraints:

- *If the linear system $Ax = S$ admits no solution in $\mathbb{R}^N_+$, then there is no Nash equilibrium in individual constraints.*
- *If the matrix A is invertible, then, there exists a unique solution x^* to the linear system and if x^* is in $\mathbb{R}^N_+$, there exists a unique Nash equilibrium of the form $x^* = A^{-1}S$. If x^* is not in $\mathbb{R}^N_+$, there is no Nash equilibrium.*
- *If the matrix A is not invertible and the linear system $Ax = S$ admits at least one solution $x^{*,0} \in \mathbb{R}^N_+$, then the linear system admits infinitely many solutions, and there are infinitely many Nash equilibria in individual constraints given by the set $\{x^{*,0} + y, y \in \ker(A)\} \cap \mathbb{R}^N_+$.*

2) Game in shared constraint:

- *$\mathcal{N}_{Indiv} \subseteq \mathcal{N}_{Shared}$ (this is a direct consequence of Proposition 1)*
- *There always exists at least one Nash equilibrium.*

Proof. See the Appendix.

Remark 3 *This result is obviously true for the symmetric case of a game with agents seeking to minimize a cost function $Cost_i(x) = Cost_i(x_i)$ decreasing continuous and convex function of x_i and with individual constraints $X_i(x_{-i}) = \{x_i \in E_i,\ l_i(x) \geq S_i\}$ with $l_i(x) = \sum_{j \in J} a_{ij} x_j$.*

We provide in appendix two proofs of the second point of the above proposition regarding the existence of a Nash equilibrium in shared constraint (part 2). We rely on Rosen theorem for one proof but we use proposition 2 for the second proof. It suffices to note that the game is non-classical, the shared constraint K is a compact set and the cost functions are continuous. Therefore, this game satisfies the assumptions of Proposition 2, and a Nash equilibrium in shared constraint always exist.

5 Applications to collective action problems

We shall now discuss two applications of games with endogenous shared constraint, one applied to an environmental problem and another applied to a public good problem. A striking feature of these two economic applications is that a Nash equilibrium may not exist in individual constraints while it always exists in endogenous shared constraint. The first example is applied to an environmental problem (it is indeed formulated as a non-classical game) and the existence result can be proved either with the fundamental theorem of [Rosen, 1965] (see theorem 1 in this paper) or with Proposition 2. In the second example, applied to the financing problem of a public good, we consider a generalized game and show that when the dispersion of the optimal contribution of each agent is too large, no Nash equilibrium exists in individual constraints while a Nash equilibrium exists in endogenous shared constraint. To ease the mathematical analysis, we focus on the case in which the individual constraints are uni-dimensional.

5.1 Limiting global warming

Given the critical nature of the subject, the limitation of global warming of the earth, there is a large (game theoretical based) literature on the subject (see e.g., [Hoel and Schneider, 1997], [Carraro and Siniscalco, 1993], [Barrett, 2001]). We refer the reader to [Missfeldt, 1999], which is a survey of game theoretic models of trans-boundary pollution but note that this review paper does not mention generalized games. We found only few papers on the subject, [Tidball and Zaccour, 2005] and [Krawczyk, 2005], that analyze the pollution problem as a generalized game. In these models, the choice variable of a given country x_i is typically its pollution level (i.e., measured by the volume of emission of greenhouse gas) which, in the simplest case, is modeled as a linear function of its production. Such an environmental problem is interesting but challenging because the production of a given country typically generates *negative externalities* (i.e., pollution) to all the other countries.

In [Tidball and Zaccour, 2005], they consider a model where each country seeks to maximize a profit function of the type $w_i(x_1, ..., x_n) = f_i(x_i) - d_i(x_1 + ... + x_n)$, where x_i represent the emissions of country i and are assumed to be proportional to the production of country i, $f_i(x_i)$ is a non-negative, twice-differentiable, concave and increasing function, and the damage cost due to all the countries is denoted by a convex twice-differentiable increasing cost function $d_i(x_1 + ... + x_n)$. They consider three types of problems which correspond to three types of constraint:

- A *Generalized Nash equilibrium problem with individual constraints* where each agent is seeking to maximize $w_i(x_1, ..., x_n) = f_i(x_i) - d_i(x_1 + ... + x_n)$ subject to the constraint $x_i \leq T_i$ with T_i an exogenous given upper bound threshold on emissions.
- A *cooperative scenario*, where agents agree to jointly maximize the sum of their profit function: $\max_{x_1,...,x_n} \sum_{i=1}^{n} (f_i(x_i) - d_i(x_1 + x_2 + ... + x_n))$ subject to $\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} T_i$.
- A *Generalized Nash equilibrium problem with an exogenous shared constraint* where each agent is seeking to maximize $w_i(x_1, ..., x_n) = f_i(x_i) - d_i(x_1 + x_2 + ... + x_n)$ subject to the constraint $\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} T_i$.

The purpose of [Tidball and Zaccour, 2005] is to characterize and compare the solutions of these three different scenarios. They show that a Nash equilibrium in individual constraints may be better than a Nash equilibrium with (exogenous) shared constraint. In their framework, the shared constraint is actually not generated from the individual constraints[5]

In the same vein, [Krawczyk, 2005] proposes another model where three players $j = 1, 2, 3$ located along a river are engaged in an economic activity at a chosen level x_j and their joint production externalities must satisfy environmental constraints set by a local authority. It is assumed that player$j$ has a level of pollution $e_j x_j$, where e_j is the emission coefficient of player j. The pollution is expelled into the river and reaches a monitoring station in the amount of $\sum_{j=1}^{3} \delta_{jl} e_j x_j$ where δ_{jl} is the decay-and-transportation coefficient from player j to location l, and it is assumed that there are two monitoring stations, $l = 1, 2$, and the local authority has set maximum pollutant concentration levels C_l. It gives the following Generalized Nash equilibrium problem with shared constraint: each

[5] The individual constraint of country i does not depend upon the choices of the other countries, for each $i \in J$, $X_i(x_{-i}) = [0, T_i]$ no matter what x_{-i} is. It thus follows that $K = \prod_{i \in J} X_i$ while the shared constraint in [Tidball and Zaccour, 2005] is given by the set $S := \{x \in \mathbb{R}_+^n : \sum_{i \in J} x_i \leq \sum_{i \in J} T_i\}$. This shared constraint thus is *not* generated from the game with individual constraints since x can be in S but not in K. To see this within a numerical example, assume that $n = 2$ and that $T_1 = 3$ and $T_2 = 2$ so that $X_1 = [0, 3]$ while $X_2 = [0, 2]$. The point $(2, 3) \notin X_1 \times X_2$ while $(2, 3) \in S$.

player j is supposed to maximize its net profit $\phi_j(x) = d_1 - d_2(x_1 + x_2 + x_3) - (c_{1j} + c_{2j}x_j)x_j$ (where $d_1, d_2, c_{1j}c_{2j}$ are given constants) under the shared constraint $\sum_{j=1}^{3} \delta_{jl} e_j x_j \leq C_l$, $l = 1, 2$. [Krawczyk, 2005] proves existence of a unique Nash equilibrium under reasonable assumptions and exhibits an algorithm that converges towards the solution.

Inspired by [Tidball and Zaccour, 2005] and [Krawczyk, 2005], we now introduce a new environmental model formulated in terms of generalized game with individual constraints and with endegenous shared constraint.

Consider the following model with $N \geq 2$ countries/economies. Let $x_i \in \mathbb{R}_+$ be the quantity of non-renewable energy (i.e., fossil energy) chosen by country i where the carbon emissions C_i by country $i \in \{1, 2, ..., N\}$ are proportional to x_i (similar to [Tidball and Zaccour, 2005] and [Krawczyk, 2005]), that is $C_i = c_i \times x_i$, where $c_i > 0$ (and note that $E = \mathbb{R}_+^N$). Let $f_i(x_i)$ be the profit function of country i as a function of x_i where f_i is an increasing continuous and concave function. On this aspect, we differ from [Tidball and Zaccour, 2005] and [Krawczyk, 2005] since we assume that the payoff function of a country only depends on x_i, the quantity of non-renewable energy chosen by i. We assume that each country i has to satisfy an individual linear constraint of the form:

$$g_i(x) := a_{ii} \times c_i \times x_i + \sum_{j \neq i} a_{ji} \times c_j \times x_j \leq S_i \tag{1.8}$$

where a_{ji} is a coefficient measuring how the use of fossil fuels by country j is impacting the environment of country i and S_i is a threshold determined for each country i. The constraint S_i may come from a regulatory institution. Note that in the case in which $j = i$, the coefficient a_{ii} measures how the use of fossil fuels by country i is impacting its own environment. Given x_{-i}, the aim of country i is to maximize $f_i(x_i)$ subject to $g_i(x_i, x_{-i}) \leq S_i$ and note that, using our classification of games, such a strategic interaction is an example of a non-classical game.

The aim is now to study the possible existence of a Nash equilibrium for this game with individual constraints and its generated game with (endogenous) shared constraint. As seen earlier, it may indeed be the case that there is no Nash equilibrium for the game with individual constraints. To see this, assume that $N = 2$. Consider country 1 and assume that $a_{11} = c_1 = 1$ and that $a_{21} = 0$ so that its constraint is given by $x_1 \leq S_1$. Consider now country 2 and assume that $a_{22} = c_2 = 1$ and assume now that $a_{12} = 1$. From equation (1.8), it thus follows that the constraint of country 2 is given by $x_2 + x_1 \leq S_2$. Assume now that $S_1 > S_2$ and that this generalized game with individual constraints has at least a Nash equilibrium (x_1^*, x_2^*). Such an equilibrium must satisfy $x_1^* = S_1$. But in such a case, no matter what $x_2 \geq 0$ is, the constraint of country 2 is never satisfied which means that there is no Nash equilibrium in individual constraints.

We shall now derive a more general result about the (non) existence of a Nash equilibrium in this environmental game theoretical model with individual constraints. From the individual constraint given by equation (1.8), for a profile of strategy $x \in \mathbb{R}_+^N$ to be Nash equilibrium, the constraints must be satisfied, that is, $x \in \mathbb{R}_+^N$ must be such that $g_i(x) \leq S_i$ for $i = 1, 2, ..., N$. From the monotonicity of the profit function of each country i, the constraint of each country will be binding at equilibrium $x^* = (x_1^*, ..., x_N^*)$, which means that for $i = 1, 2, ..., N$, $g_i(x^*) = S_i$. The problem reduces to the analysis of a linear system of the form $A^\top \text{diag}(c)X = S$ where A is the matrix of the coefficient $(a_{ij})_{i,j=1,2,...,N}$ ($\top$ indicates the transpose), $\text{diag}(c) = \text{diag}(c_1, ..., c_N)$, $x = (x_1, ..., x_N) \in \mathbb{R}_+^N$ and S is the vector formed by S_i, $i = 1, 2, ..., N$. If $x^* = (x_1^*, ..., x_N^*)$ is a Nash equilibrium for this game with individual constraints, it must satisfy the linear system $A^\top \text{diag}(c)x^* = S$, that is:

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1N} \\ a_{22} & a_{22} & ... & a_{2N} \\ ... & ... & ... & ... \\ a_{N1} & a_{N2} & ... & a_{NN} \end{pmatrix}^\top \begin{pmatrix} c_1 & 0 & ... & 0 \\ 0 & c_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & c_N \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \\ ... \\ x_N^* \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ ... \\ S_N \end{pmatrix} \tag{1.9}$$

where $x^* \in \mathbb{R}_+^N$.

The next result exhibits conditions under which there is no Nash equilibrium in individual constraints. In particular, when the above linear system has no solution in $\mathbb{R}_+^N$, then, no Nash equilibrium of the game with individual constraints can exist. However, in such a case in which the linear system has no solution, a Nash equilibrium of the game with endogenous shared constraint still exists.

Proposition 4 *The game with individual constraints has (at least) one Nash equilibrium x^* if and only if the linear system as defined in equation (1.9) admits at least one solution $x^* \in \mathbb{R}_+^N$. If the linear system given by equation (1.9) admits no solution in $\mathbb{R}_+^N$, then, the game with individual constraints has no Nash equilibrium.*

Proof. See the appendix.

In appendix, we give a complete characterization of the existence or not of Nash equilibria for the game with individual constraints. It is interesting at this stage to compare our framework with the one offered in [Tidball and Zaccour, 2005] and [Krawczyk, 2005]. Within our approach, as opposed to the two mentioned papers, a Nash equilibrium does not always exist in individual constraints. However, as we shall see, a Nash equilibrium in shared constraint always exists. In [Tidball and Zaccour, 2005], they show that Nash equilibria in individual constraints may be better in some sense than the ones in shared constraint but, as already discussed, their shared constraint is exogenous since it is not generated from the individual ones.

Let us now consider the game with shared constraint generated from our game with individual constraints in which each country i is seeking to maximize $f_i(x_i)$ subject to the endogenous shared constraint defined as $g_k(x) := a_{kk}\times c_k\times x_k+\sum_{j\neq k} a_{jk} \times c_j \times x_j \leq S_k$, $k = 1, ..., N$. From proposition 1, we know that the Nash equilibria of the game with shared constraint contains the Nash equilibria of the game with individual constraints. But Nash equlibria of the game with individual constraints may not always exist, as seen in proposition 4. In the next result, we state that there always exists Nash equilibria for our game with endogenous shared constraint. This is a direct consequence of proposition 3.

Proposition 5 *The game with a shared constraint generated from the individual ones always has at least one Nash equilibrium. In particular, a Nash equilibrium in the game with endogenous shared constraint exists whether the linear system given by equation (1.9) admits a solution or not.*

Proposition 5 shows that a Nash equilibrium always exists in the game with shared constraint generated from the individual constraints while it may fail to exist in the game with individual constraints. By reinforcing the environmental constraints, i.e., by sharing the constraints, each country takes into account the individual constraints of the other countries, and this binding agreement always generates at least one Nash equilibrium.

5.2 Contributing to a public good

We now consider a model of collective action with $N \geq 2$ agents, similar to [Guttman, 1978] and to [Cornes and Hartley, 2007] (see also [Buchholz et al., 2011]) but in which each agent faces an individual constraint (see [Sandler, 2015] for a review paper on collective action models). Each agent i is asked to make a voluntary contribution to finance a public investment project such as a bridge, a street lighting or any public infrastructure equipment (water, gas, internet...) subject to an individual constraint. For concreteness, we assume that the higher the total contributions, the higher quality of the underlying investment. Following the standard terminology introduced in [Guttman, 1978] (see also ([Buchholz et al., 2011])), let $x_i \in E_i := \mathbb{R}_+$ be the flat (or direct) contribution of agent i and let $b \in [0,1]$ the known percentage of the sum of the flat contributions of all the other agents. The indirect contribution of a given agent i is by definition equal to $b\sum_{j\neq i} x_j$, so that the total contribution of agent i is equal to $x_i + b\sum_{j\neq i} x_j$. Following [Guttman, 1978], the utility of each agent i is assumed to be equal to

$$U_i(x_i, x_{-i}) = v_i(x_i + b\sum_{j\neq i} x_j) - (x_i + b\sum_{j\neq i} x_j) \tag{1.10}$$

where the function $v_i(.)$ measures the willingness-to-pay of an agent i for the public project and depends upon her own contribution but also upon the contributions of all the other agents. As in the literature on collective actions and aggregative games ([Guttman, 1978], [Cornes and Hartley, 2007] [Chen and Zeckhauser, 2018], [Cornes and Hartley, 2012], see also [Cornes, 2016] for a recent review paper), the willingness-to-pay $v_i(.)$ depends upon x_{-i} only through the sum of the contributions of the other agents. Given a profile of strategies $x = (x_i, x_{-i})$, let

$$z_i = x_i + b\sum_{j\neq i} x_j \tag{1.11}$$

be the total contribution of agent i and note that the utility function of agent i can be written as a function of the sole scalar z_i.

$$U_i(x_i, x_{-i}) = v_i(z_i) - z_i \tag{1.12}$$

We shall assume that $v_i(z_i)$ is a twice continuously differentiable increasing and strictly concave function of z_i (with $v_i(0) = 0$) so that U_i is also a strictly concave function[6]. An example of a functional form for $v_i(z_i)$ is $v_i(z_i) = \xi_i\sqrt{z_i}$ where ξ_i is a positive scalar.

Regarding now the individual constraints, it can be formulated as a *budget constraint* or as a *reservation utility*.

- The budget constraint of agent i can be given as an exogenous revenue of agent i, r_i, which

[6] Strict concavity is actually not required. What is actually required is only the single-peakedness of U with respect to z_i, that is, the strict quasi-concavity of U. Such a single-peakedness assumption is fairly natural and standard (see [Guttman, 1978], see also [Greenberg and Weber, 1993]).

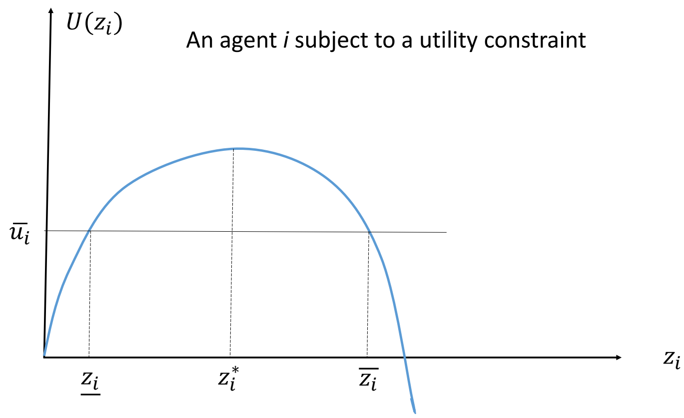

Figure 1.5: $U(z_i)$ and the two thresholds $\underline{z}_i$ and $\overline{z}_i$

means that it must be the case that $z_i \leq r_i$.

- The reservation utility constraint can be given as an exogenous reservation utility of agent i, $\overline{u}_i$, which means that it must be the case that $U_i(x_i, x_{-i}) \geq \overline{u}_i$. The threshold $\overline{u}_i$ can be interpreted as a minimal quality for the public good according to agent i.

The optimization problem of a given agent i can be formulated as a utility maximization problem subject to a budget constraint and/or subject to a reservation utility constraint. Let

- $\mathcal{B} \neq \emptyset$ be the subset of J that are subject to a budget constraint.
- $\mathcal{U} \neq \emptyset$ be the subset of J that are subject to a utility constraint.

where the $\mathcal{B}$ and $\mathcal{U}$ are assumed to form a partition of J, that is, $\mathcal{B} \cup \mathcal{U} = J$, with $\mathcal{B} \cap \mathcal{U} = \emptyset$. Since each agent i is endowed with a utility function assumed to be a concave function of z_i, it makes thus sense to consider her *ideal contribution* z_i^* to the public project.

Let $z_i^* = \arg\max_{z_i \geq 0} U_i(z_i) := v_i(z_i) - z_i$. Given the assumptions on the utility function, z_i^* is unique and for the sake if interest, we assume that $z_i^* > 0$ for each $i \in J$.

Let $\overline{z}_i$ be the critical threshold of each agent $i \in J$. If $i \in \mathcal{B}$, then $\overline{z}_i = r_i$. We shall assume that $r_i \geq z_i^*$. Consider now agent $i \in \mathcal{U}$. Given the assumptions on the utility function U_i (continuity, strict concavity), there exists two critical thresholds $\underline{z}_i < z_i^*$ and $\overline{z}_i > z_i^*$ such that $U_i(\underline{z}_i) = v_i(\underline{z}_i) - \underline{z}_i = \overline{u}_i = U_i(\overline{z}_i) = v_i(\overline{z}_i) - \overline{z}_i$. See Figure 1.5. As before, let us denote BR_i^{Ind} the best response function in individual constraints. The following fact holds and is an elementary consequence of the definition of z_i^*.

Fact 1 *The best response of agent $i \in J$ is given below.*

- *If $b\sum_{j\neq i} x_j \leq z_i^*$, then, $BR_i^{Ind}(x_{-i}) = z_i^* - b\sum_{j\neq i} x_j > 0$*
- *If $b\sum_{j\neq i} x_j \in (z_i^*, \overline{z}_i)$, then, $BR_i^{Ind}(x_{-i}) = 0$*
- *If $b\sum_{j\neq i} x_j > \overline{z}_i$ then, $BR_i^{Ind}(x_{-i}) = \emptyset$*

When $X_i(x_{-i}) = \emptyset$ for $i \in \mathcal{B}$, this simply means that this agent must pay *more* than her own revenue, which is impossible. As a result $BR_i^{Ind}(x_{-i}) = \emptyset$. One may interpret this as an exclusion, similar to the *violation of the no-bankruptcy condition* in aggregative games (see e.g., [Buchholz et al., 2011]). The situation is similar for agents of the group $\mathcal{U}$. When $b\sum_{j\neq i} x_j > \overline{z}_i$ for an agent $i \in \mathcal{U}$, her utility will be *lower* than her reservation utility. As a result $BR_i^{Ind}(x_{-i}) = \emptyset$.

Consider an agent $i \in \mathcal{B}$. Depending upon the willingness-to-pay and the revenue, an agent i may be such that its optimal contribution z_i^* may be equal to r_i or may be lower than r_i. When $z_i^* < r_i$, the budget constraint is not binding and this occurs when z_i^* solves the unconstrained maximization of $U_i(z_i)$. When $z_i^* = r_i$, the constraint is binding. Consider now an agent $j \in \mathcal{U}$ with a high z_j^*. Assume that given x_i with $i \in \mathcal{B}$, the best response of j equal to $BR_j^{Ind}(x_i) := x_j = z_j^* - bx_i$ is higher than $\frac{r_i}{b}$. In such a case, $bx_j > r_i = \overline{z}_i$. From fact 1, $BR_i^{Ind}(x_j) = \emptyset$, that is, no matter the choice of agent i, agent j will always choose a contribution so high that i is left with an empty set. Agent j has a dominant strategy which implies that $BR_i^{Ind}(x_j) = \emptyset$. The following result provides a simple illustration of this when $N = 2$ but nothing is changed in the general case of an arbitrary number N of agents.

Proposition 6 *Let $b \in (0, 1]$ and $J = \{1, 2\}$ where the group $\mathcal{B} = \{1\}$ and the group $\mathcal{U} = \{2\}$.*

A sufficient condition for the non-existence of a Nash equilibrium in individual constraints is $z_2^* > (b + \frac{1}{b})r_1$.

Assume moreover that $z_1^* = r_1 < z_2^*$ and that $z_1^* \geq b\underline{z}_2$ and let $\overline{x}_1 := \frac{r_1 - b\underline{z}_2}{(1-b^2)}$. Then, for any $\theta \in [0, 1]$, the pair of strategies $(x_1^* = \theta\overline{x}_1, x_2^* = \frac{r_1 - \theta\overline{x}_1}{b})$ is a Nash equilibrium in shared constraint.

Proof. See the appendix.

When agent 1 faces a budget constraint such that $z_1^* = r_1$, the existence of a Nash equilibrium in individual constraints critically depends upon the heterogeneity (or the dispersion) of the set of ideal contributions $(z_i^*)_{i\in\{1,2\}}$. If this dispersion is too high, that is, if $\frac{z_2^*}{z_1^*}$ is greater than a critical threshold, then, a Nash equilibrium in individual constraints does not exist.

Note that for any $\theta \in [0,1]$, the pair $(x_1^* = \theta\overline{x}_1, x_2^* = \frac{z_1^* - \theta\overline{x}_1}{b})$ is a Nash equilibrium and is such that $x_1^* + bx_2^* = z_1^*$, it thus follows that for any $\theta \in (0,1)$, the Nash equilibrium is Pareto optimal.

Corollary 1 *Each Nash equilibrium of the game with shared constraint is Pareto optimal.*

Within our particular model of collective action, when a Nash equilibrium in the game with individual constraints does not exist, there still exists a *continuum* of Nash equilibria in the game with shared constraint that are all Pareto-optimal.

Our analysis also reveals an important property of collective actions. If one only looks at the equilibrium in individual constraints when for instance $z_i^* = r_i$ for each $i \in \mathcal{B}$, the existence of such an equilibrium in individual constraints critically depends upon the dispersion of the ideal contribution $(z_i^*)_{i\in J}$. This thus suggests that for a collective action to be possible (in individual constraints), agents of the group J must have homogenous ideal contributions, something not required for the shared constraint problem. In the general case with $N \geq 2$ agents, let

$$K = \{x \in \mathbb{R}_+^N : x_i + b\sum_{j\neq i} x_j \leq r_i \ \forall i \in \mathcal{B} \text{ and } U_k(x_k + b\sum_{j\neq k} x_j) \geq \overline{u}_k \ \forall k \in \mathcal{U}\}$$

As in proposition 6, as long as the dispersion of $(z_i^*)_{i\in J}$ is "too high", K will be empty and no Nash equilibrium in individual constraints exists. In case in which K is a compact and convex subset of $\mathbb{R}_+^N$, from theorem 1 ([Rosen, 1965]), a Nash equilibrium in the game with shared constraint always exist (since the utility functions are concave) while a Nash equilibrium in individual constraint might not exist.

6 Conclusion

In this paper, we presented the notions of generalized games with individual constraints, generalized games with shared constraint, and generalized games with an endogenous shared constraint generated from individual constraints. We proved a result regarding the existence of Nash equilibria for a generalized game with an endogenous shared constraint generated from individual ones, that is the Nash equilibria of a generalized game with individual constraints are included in the set of Nash equilibria of the generalized game with endogenous shared constraint. We provided a taxonomy of

2×2 generalized games and established two results regarding non-classical games. We then studied different applications of this result, among which a public good problem and an environment control problem.

7 Appendix

Proof of Proposition 1. If $x^* = (x^{*,1}, ..., x^{*,N}) \in E$ is a Nash equilibrium for the game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$, then:

- $\forall i \in J,\ x_i^* \in X_i(x_{-i}^*)$ so that $x^* \in K$.
- $\forall i \in J,\ \forall x_i \in X_i(x_{-i}^*),\ \theta_i(x_i^*, x_{-i}^*) \leq \theta_i(x_i, x_{-i}^*)$ so $\forall i \in J,\ \forall x_i \in E_i$ such that $(x_i, x_{-i}^*) \in K,\ \theta_i(x_i^*, x_{-i}^*) \leq \theta_i(x_i, x_{-i}^*)$.

It thus follows that if the point x^* is a Nash equilibrium of the game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$, it is also a Nash equilibrium for the game with shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J}))$ but the converse is however not true. If the point x^* is a Nash equilibrium of the game with shared constraint generated from the individual constraints $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J}))$, it may be the case that the profile of strategies $x^* \in E$ is not a Nash equilibrium of the game with individual constraints because there may exist $i \in J$ such that the best response $x_i^* := BR_i(x_{-i}^*) \in X_i(x_{-i}^*)$ is such that $x^* \notin K$ where $x^* = (x_i^*, x_{-i}^*)$.

Examples in which a profile of strategies which is a Nash equilibrium in shared constraint but not a Nash equilibrium in individual constraints have been given. □

Proof of Proposition 2

Consider the application:

$$\theta : K \to \mathbb{R}$$
$$x \mapsto \sum_{i\in J} \theta_i(x_i)$$

Since K is compact and θ is continuous, there exists at least one $x^* \in K$ that minimizes θ, $\theta(x^*) = \min_{x\in K} \theta(x)$. Let us prove that x^* is a Nash equilibrium. If this is was not the case, there would exist an agent $j \in J$ and a strategy $x_j \in E_j$ such that $(x_j, x_{-j}^*) \in K$ and $\theta_j(x_j) < \theta_j(x_j^*)$ so that $\theta_j(x_j) + \sum_{i\neq j} \theta_i(x_i) < \sum_{i\in J} \theta_i(x_i^*)$ and this would contradict the definition of x^*. Therefore x^* is a Nash equilibrium and all Nash equilibria of $(J, E, (\theta_i)_{i\in J}, K)$ are minimizers of θ. As a result, a given Nash equilibrium x^* is Pareto optimal.

□

Proof of Proposition 3

1) Assume that this game has a Nash equilibrium in individual constraints $x^* = (x_1^*, ..., x_N^*)$. Such a Nash equilibrium must satisfy $\sum_j a_{ij} x_j^* = S_i$ for all i. Indeed, assume that for a given i we have $\sum_j a_{ij} x_j^* < S_i$, then agent i could still increase its payoff $\theta_i(x_i)$ with a x_i higher than x_i^*

still satisfying the constraint, and therefore x^* would not be a Nash equilibrium. Therefore a Nash equilibrium $x^* = (x_1^*, ..., x_N^*)$ for this problem with individual constraints must satisfy: $AX^* = S$ with $X^* \in \mathbb{R}_+^N$.

And this has a solution if and only if the linear system $Ax = S$ has a solution $x^* \in \mathbb{R}_+^N$.

- If A is invertible, we have an explicit formula for the Nash equilibrium candidate vector $x^* = A^{-1}S$
- If A is not invertible and the linear system $Ax = S$ admits one solution $x^{*,0} \in \mathbb{R}_+^N$, then $Ax = Ax^{*,0}$, which is equivalent to $A(x - x^{*,0}) = 0$, which is equivalent to $x \in \{x^{*,0} + y, y \in \ker(A)\} \cap \mathbb{R}_+^N$ and there are infinitely many Nash equilibria.
- If the linear system $Ax = S$ admits no solution in $\mathbb{R}_+^N$, then there is no Nash equilibrium.

2) We provide two different proofs of this result:

- First proof :

  It is easy to see that the game always satisfies the assumptions of the existence result of Rosen for n-person concave games (see theorem 1 in the text). Indeed, the payoff functions $\theta_i(x_i)$ are continuous and concave, and the shared constraint space K is clearly a convex compact space:

  $K = \{x \in \mathbb{R}_+^N : \sum_{j \in J} a_{ij} x_j \leq S_i, i = 1, 2, ..., N\}$

  Therefore, there always exist a Nash equilibrium for the game with shared constraint.

- Second proof :

  This is a non-classical game, K is a compact set and the cost functions are continuous, therefore this game satisfies the assumptions of Proposition 2, and a Nash equilibrium in shared constraint always exist. Moreover, $\mathcal{N}_{Shared}$ coincides exactly with the minimizers of the total cost function $\sum_{i \in J} \theta_i(x_i)$ on K and all its elements are Pareto optimal.

□

Proof of proposition 4.

We shall give here a result more detailed than the one stated in the text. In what follows, we actually offer a complete characterization of the existence or non-existence of the Nash equilibrium in individual strategies. The environmental problem formulated as a game with individual constraints has a Nash equilibrium if and only if the linear system $A^T \text{diag}(c)x = S$ (A^T denotes the transpose of A) admits at least one solution $x^* \in \mathbb{R}_+^N$.

PropositionA 4

- If the linear system $A^T \text{diag}(c)x = S$ admits no solution in $\mathbb{R}_+^N$, then there is no Nash equilibrium.

- If the matrix A is invertible, then, there exists a unique solution x^* to the linear system, and if x^* is in $\mathbb{R}_+^N$, there exists a unique Nash equilibrium of the form:

$$x^* = \left(x_1^* = \frac{1}{c_1} \sum_{j=1}^{N} b_{1j} S_j, ..., x_i^* = \frac{1}{c_i} \sum_{j=1}^{N} b_{ij} S_j, ..., x_N^* = \frac{1}{c_N} \sum_{j=1}^{N} b_{Nj} S_j \right)$$

  with $(B_{i,j})_{i,j \in J} = (A^T)^{-1}$. If x^* is not in $\mathbb{R}_+^N$, there is no Nash equilibrium.

- If the matrix A is not invertible and the linear system $A^T \text{diag}(c)x = S$ admits at least one solution $x^{*,0} \in \mathbb{R}_+^N$, then the linear system admits infinitely many solutions, and there are infinitely many Nash equilibria given by the set $\{x^{*,0} + y, y \in \ker(A^T \text{diag}(c))\} \cap \mathbb{R}_+^N$

Proof of propositionA 4

Let's assume that this game has a Nash equilibrium $x^* = (x_1^*, ..., x_N^*)$. Such a Nash equilibrium must satisfy $\sum_j a_{ji} \times c_j \times x_j^* = S_i$ for all i. Indeed, assume that for a given i we have $\sum_j a_{ji} \times c_j \times x_j^* < S_i$, then country i could still increase its profit $f_i(x_i)$ with a x_i higher than x_i^* still satisfying the constraint, and therefore x^* would not be a Nash equilibrium. Therefore a Nash equilibrium $x^* = (x_1^*, ..., x_N^*)$ for this problem with individual constraints must satisfy: $A^T \text{diag}(c)x^* = S$.

And this has a solution if and only if the linear system $A^T \text{diag}(c)x = S$ has a solution $x^* \in \mathbb{R}_+^N$.

- If A is invertible, we have an explicit formula for the Nash equilibrium candidate vector since:

$$\text{diag}(c)x^* = (\text{A}^T)^{-1} \begin{pmatrix} S_1 \\ S_2 \\ ... \\ S_N \end{pmatrix}$$

If we denote $B = (A^T)^{-1}$, we have:

$$\text{diag}(c)x^* = \begin{pmatrix} b_{11} & b_{12} & ... & b_{1N} \\ b_{21} & b_{22} & ... & b_{2N} \\ ... & ... & ... & ... \\ b_{N1} & b_{N2} & ... & b_{NN} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ ... \\ S_N \end{pmatrix}$$

And for all i $x_i^* = \frac{1}{c_i} \sum_{j=1}^N b_{ji} S_j$, and we have a unique Nash equilibrium candidate $x^* = (x_1^* = \frac{1}{c_1} \sum_{j=1}^N b_{1j} S_j, ..., x_i^* = \frac{1}{c_i} \sum_{j=1}^N b_{ij} S_j, ..., x_N^* = \frac{1}{c_N} \sum_{j=1}^N b_{Nj} S_j)$.

- If A is not invertible and the linear system $A^T \text{diag}(c)x = S$ admits one solution $X^{*,0} \in \mathbb{R}_+^N$, then $A^T \text{diag}(c)x = A^T \text{diag}(c)x^{*,0}$, which is equivalent to $A^T \text{diag}(c)(x - x^{*,0}) = 0$, which is equivalent to $x \in \{x^{*,0} + y, y \in \ker(A^T \text{diag}(c))\} \cap \mathbb{R}_+^N$ and there are infinitely many Nash equilibria.
- If the linear system $A^T \text{diag}(c)x = S$ admits no solution in $\mathbb{R}_+^N$, then there is no Nash equilibrium.

$\square$

Proof of proposition 6

Part i). Assume that $x_1 = r_1$. The best response of 2, x_2, is equal to $z_2^* - br_1$. As seen in the paragraph before stating Proposition 6, if $z_2^* - br_1 > \frac{r_1}{b}$, then, $x_2 > r_1$ so that the best response of agent 1 is empty. Solving $z_2^* - br_1 > \frac{r_1}{b}$ yields part (i) $\square$

Part ii). Let $K = \{(x_1, x_2) \in \mathbb{R}_2^+ : x_1 + bx_2 \leq r_1 \text{ and } U_2(x_1, x_2) \geq \overline{u}_2\}$. Let us denote BR_i^{Sh} the best response of agent i in shared constraint. For a Nash equilibrium with shared constraint to exist, we must first prove the non-vacuity of K, which in turn is equivalent to the existence of a solution that solves the following system.

$$x_1 + bx_2 \leq r_1 \tag{1.13}$$

$$U_2(x_2 + bx_1) \geq \overline{u}_2 \tag{1.14}$$

We know that for agent 2, given the assumptions on the utility function U_2 (continuity, strict concavity), there exists two critical thresholds $\underline{z}_2 < z_2^*$ and $\overline{z}_2 > z_2^*$ such that $U_2(\underline{z}_2) = v_2(\underline{z}_2) - \underline{z}_2 = \overline{u}_2 = U_2(\overline{z}_2) = v_2(\overline{z}_2) - \overline{z}_2$. It thus follows that equation (1.14) is equivalent to $z_2 \in [\underline{z}_2, \overline{z}_2]$, where $z_2 = x_2 + bx_1$.

For K to be non-empty, it thus suffices to find a solution to the following system

$$x_1 + bx_2 \leq r_1 \tag{1.15}$$

$$\underline{z}_2 \leq x_2 + bx_1 \leq \overline{z}_2 \tag{1.16}$$

First let us prove that (1.16) reduces to $\underline{z}_2 \leq x_2 + bx_1 \leq z_2^*$. Indeed, if we had $x_2 + bx_1 > z_2^*$, then this would imply $x_2 + bx_1 > (b + \frac{1}{b})r_1$, which would imply $bx_2 > r_1 = \overline{z}_1$, which would imply $BR_1^{Ind}(x_2) = \emptyset$. So inequation (1.16) becomes

$$\underline{z}_2 \leq x_2 + bx_1 \leq z_2^* \tag{1.17}$$

Solving inequation (1.15) in x_2 yields $x_2 \leq \frac{r_1 - x_1}{b}$ while solving inequation (1.17) yields $x_2 \geq \underline{z}_2 - bx_1$. For a solution in x_2 to exist, $\frac{r_1 - x_1}{b}$ must be higher than $\underline{z}_2 - bx_1$. Solving $\frac{r_1 - x_1}{b} \geq \underline{z}_2 - bx_1$ yields $r_1 - b\underline{z}_2 \geq x_1(1 - b^2)$. Then a necessary condition is $r_1 - b\underline{z}_2 \geq 0$.

Assuming that $r_1 - b\underline{z}_2 \geq 0$, x_1 must be in $\left[0, \frac{r_1 - b\underline{z}_2}{(1-b^2)}\right]$. Let $\overline{x}_1 := \frac{r_1 - b\underline{z}_2}{(1-b^2)}$ be the maximal value of x_1 and assume that $x_1 = \theta\overline{x}_1$ for some $\theta \in [0, 1]$. Consider x_2^* such that $\theta\overline{x}_1 + bx_2^* = r_1 = z_1^*$, which means that $x_2^* = \frac{r_1 - \theta\overline{x}_1}{b}$. Given $x_2^* = \frac{r_1 - \theta\overline{x}_1}{b}$, agent 1 will choose x_1 such that $x_1 + bx_2^* = z_1^*$ so that $BR_1^{Sh}(\frac{r_1 - \theta\overline{x}_1}{b}) = x_1^* = \theta\overline{x}_1$.

Let us prove that $x_2^* = BR_2^{Sh}(\theta\overline{x}_1)$. Given $\theta\overline{x}_1$, agent 2 is seeking to maximize x_2 subject to the two constraints (1.15) and (1.17). x_2^* is the maximum solution with (1.15). If the best response in shared constraint $BR_2^{Sh}(\theta\overline{x}_1)$ of agent 2 was higher than x_2^* then this would not satisfy the constraint (1.15).

Now, let us prove that $\underline{z}_2 < x_2^* + b\theta\overline{x}_1 < z_2^*$.

Note that $x_2^* + b\theta\overline{x}_1 = \frac{r_1 - \theta\overline{x}_1}{b} + b\theta\overline{x}_1 := h(\theta)$. The function $h(\theta)$ reaches its maximum when $\theta = 0$ and its maximum value is equal to $\frac{r_1}{b}$, which is lower than z_2^*. It reaches its minimum value when $\theta = 1$ and the minimum value is equal to $\frac{r_1 - \overline{x}_1}{b} + b\overline{x}_1 = \frac{r_1}{b} + \frac{b^2-1}{b}\overline{x}_1 = \frac{r_1}{b} + \frac{b^2-1}{b}\frac{r_1 - b\underline{z}_2}{1-b^2} = \frac{r_1}{b} - \frac{r_1}{b} + \underline{z}_2 = \underline{z}_2$. Therefore $\underline{z}_2 < x_2^* + b\theta\overline{x}_1 < z_2^*$ is always satisfied so $BR_2^{Sh}(\theta\overline{x}_1) = x_2^*$. It thus follows that for all $\theta \in [0,1]$, the pair of strategies $(x_1^* = \theta \overline{x}_1, x_2^* = \frac{r_1 - \theta \overline{x}_1}{b})$ is a Nash equilibrium in shared constraint. $\square$

Chapter 2

Strategic foundations of macroprudential regulation: preventing fire sales externalities

Abstract

We offer a stress test framework in which interaction between regulated banks occurs through pecuniary externalities when they delever. Since banks are constrained to maintain their capital ratio higher than a threshold, the deleveraging problem yields a generalized game in which the solvency constraint of each bank depends upon the decisions of the others. We analyze the game under microprudential but also under macroprudential regulation in which fire sales externalities are banned. We show that a Pareto optimal Nash equilibrium generically exists under macroprudential regulation while the existence under microprudential regulation requires strong conditions. An empirical analysis is also provided.

Keywords: Macroprudential regulation, fire sales, externalities, generalized games.

1 Introduction

In the new banking regulatory framework called Basel III published in the aftermath of the subprime financial crisis ([BCBS, 2010]), the Basel Committee points out that they have not only strengthened the classical microprudential regulatory framework (essentially the so-called risk-based capital ratio) but also introduced *a number of macroprudential elements into the capital framework to help contain systemic risks* such as the G-SIB buffer, which concerns institutions classified as systemic institutions (GSIBs) by the Financial Stability Board[1], defined as a capital surcharge (buffer) which depends upon the "systemicness" of the bank based on five public indicators such as size, interconnectedness or complexity[2].

The microprudential regulation focuses on the resilience of depositary institutions (i.e., banks) and its basic aim is to protect depositors by mitigating the incentive of banks to take excessive risk due to government-insured deposits ([Freixas et al., 2015], [Hanson et al., 2011]). By definition, the microprudential regulation adopts a partial equilibrium approach, which means that the impact of asset prices and markets on banks failures, something unrelated to government-insured deposits, is outside of its scope. The minimum capital requirement thus is designed as if each bank were isolated from the financial system and a "one size fits all" framework is adopted, that is, the minimum risk-based capital ratio is 8% for each bank.

The macroprudential regulation adopts a complementary point of view since it focuses on the resilience of the financial system as a whole and its aim is to safeguard it, that is, to ensure the resilience of the financial system to adverse shocks[3]. By definition, the macroprudential regulation adopts a general equilibrium approach, which means that the impact on asset prices due to fire sales on banks failures–negative externalities–falls explicitly within its scope ([Claessens, 2014], [De Nicoló et al., 2012], [Freixas et al., 2015] chapter 9, [Hanson et al., 2011]). While the Basel Committee notes that "greater resilience at the individual bank level reduces the risk of system-wide shocks" ([BCBS, 2010]), solvency of individual depositary institutions does however in general *not imply* the stability of the financial system as a whole (e.g., [Freixas et al., 2015], [Hanson et al., 2011]). As is well-known from the general theory of systems ([Bertalanffy, 1968]), the stability of a system, be it a social, physical, biological etc... critically depends upon the way

[1] See https://www.fsb.org/wp-content/uploads/P231121.pdf for the list of G-SIBs as of November, 2021.

[2] For European banks, see the European Banking Authority website, https://www.eba.europa.eu/risk-analysis-and-data/global-systemically-important-institutions.

[3] As noted in [Hanson et al., 2011], the macroprudential regulation is not related to the existence or not of deposit insurance and thus needs not be restricted to depositary institutions only.

its elements (agents, particles, cells) interact. For a financial system, its stability property depends thus upon the network of interconnection between financial institutions ([Acemoglu et al., 2015], [Allen and Gale, 2000], [Capponi and Larsson, 2015], [Elliott et al., 2014]), but also upon the various market imperfections (e.g., [Shleifer and Vishny, 2011], [Krishnamurthy, 2010]).

When financial markets are not perfectly competitive, i.e., perfectly liquid to use financial terminology, as in an oligopoly situation ([Vives, 2001]), a financial institution such as a large bank or a large hedge fund may have a positive price impact when selling large quantities of assets and this may result in a lower price. As opposed to other market participants such as hedge funds, banks are particular in that they are heavily regulated and must comply at all times with a minimum regulatory capital ratio. After an adverse shock, when a given bank does not anymore comply with its regulatory capital requirements, a simple way to restore its capital ratio consists in selling a portion of its assets (deleveraging), possibly at a dislocated price, something called fire sales in finance ([Shleifer and Vishny, 2011]). When more than one bank delevers because they are hit with the same adverse shock, at the aggregate level–the financial system–this may lead to an emergent effect called *generalized asset shrinkage* ([Hanson et al., 2011]). As recalled in the foreword of their early paper devoted to the fundamental principles of financial regulations, [Brunnermeier et al., 2009] observe that

> "In trying to make themselves safer, banks, and other highly leveraged financial intermediaries, can behave in a way that collectively undermines the system. Selling an asset when the price of risk increases, is a prudent response from the perspective of an individual bank. But if many banks act in this way, the asset price will collapse, forcing institutions to take yet further steps to rectify the situation." ([Brunnermeier et al., 2009])

Such a phenomenon occurs when many financial institutions such as banks sell a common asset at the same time so that the price of this asset will decrease (fire sales effect). As a result, they will be forced to sell more assets to restore their capital ratios and this will ultimately lead to a running for the exit– a kind of death spiral–observed in August 2007 in which the effect has been *disproportionally larger* than the initial shock ([Pedersen, 2009]). In the stationary state, a number of institutions may be insolvent not because of the initial shock, but because the (equilibrium) price is (much) lower than the price right after the initial exogenous shock ([Braouezec and Wagalath, 2019]). Such a contagion of failures is usually called price-mediated contagion and played an important role in the financial crisis of 2008 ([Brunnermeier, 2009], [Clerc et al., 2016]). A number of papers document empirically such fire (forced) sales. In [Ellul et al., 2011], the authors document forced sales of corporate bond by insurance companies while [Merrill et al., 2021] document a similar effect for the case of RMBS markets. In the same vein, [Chernenko and Sunderam, 2020] provide empirical evidence of fire sales externalities in the equity mutual fund industry (see [Choi et al., 2020]).

For banks or insurance companies, it is actually the regulatory constraint together with fair value accounting which leads to forced sales and thus to the (possible) destabilization of the financial through a generalized asset shrinkage, an emergent effect called negative externality and which is *not* taken into account by banks when they delever. The very foundation of macroprudential regulation precisely lies in the correction of these market imperfections–negative externalities– that give rise to systemic risk, in particular externalities related to fire sales ([Freixas et al., 2015], [Hanson et al., 2011]). According to [De Nicoló et al., 2012] and [Claessens, 2014], for the case of externalities related to fire sales, they suggest that they can be addressed either by capital surcharges, liquidity requirements, activities restriction or taxation (see table 1 in [De Nicoló et al., 2012]). Regulators also acknowledge this externality problem posed by large banks (GSIBs) and tackle it, as already discussed, through a GSIB-dependent capital surcharge (buffer) based on quantitative indicators.

> "The selected indicators are chosen to reflect the different aspects of what generates negative externalities and makes a bank critical for the stability of the financial system."
> (Basel Framework[4], p 17)

While such a methodology to design the GSIBs buffer is an interesting novelty[5] of Basel III, it remains unclear from a pure theoretical point of view in what sense the Basel Committee methodology takes into account these negative externalities with the GSIBs buffer. Indicators such as size, interconnectedness or complexity may give information about the possible magnitude of the externality but are not *per se* indicators to address the externality problem.

In this paper, to the best of our knowledge, we are the first to offer a theoretical strategic framework in which we explicitly address the externality problem posed by banks through macroprudential constraints (or regulation). We consider a fire sales oligopoly model of assets in the spirit of [Braouezec and Wagalath, 2019] but different from [Eisenbach and Phelan, 2022] in which banks compete à la Cournot, that is, through quantities (of assets) sold. In a Cournot oligopoly, and

[4]See https://www.bis.org/basel-framework.
[5]But see [Benoit et al., 2019] for concerns.

contrary to perfect competitive markets, each bank has a positive impact on the price of a given asset and thus is not a price taker. Our Cournot fire sales approach is novel in three aspects, and these novelties can be summarized as follows.

1. We take into account the solvency constraint of each bank (capital ratio) and this leads to a Cournot oligopoly which is more complex than the classical ones.

2. We consider both micro and macroprudential regulation. Under macroprudential regulation, each bank must take into account the capital ratio of all the other banks and not only its own capital ratio, as in microprudential regulation.

3. We show that under macroprudential regulation, there generically exists a Nash equilibrium that minimizes the total value of the asset sale. Such a result has no equivalent under microprudential regulation.

Due to banking regulation, each bank is constrained to sell a quantity of assets subject to a *solvency constraint*, that is, the deleveraging strategy must be chosen such that its Tier 1 capital ratio is greater than the minimum required, at least 8.5% in Basel III since the capital conservation buffer is 2.5%. Through the price impact, the solvency constraint of each bank turns out to explicitly depend upon the deleveraging decisions of all the other banks. Such a strategic situation, more complex than the classical one in Cournot oligopoly gives rise to a generalized game (see [Facchinei and Kanzow, 2010] or [Fischer et al., 2014] for review papers). As opposed to most game theoretic framework encountered in Economics, in a generalized game, the strategy set of a given bank, its solvency constraint, is *not invariant* with respect to the decisions of the other banks and this feature complicates the analysis. Our framework not only allows us to consider the classical approach to microprudential regulation in which each bank takes into account its own solvency constraint but, and more importantly, it also allows us to consider a macroprudential approach designed to prevent the fire sales externalities (i.e., price-mediated contagion) by explicitly constraining the way banks delever. Quite surprisingly, the analysis of the strategic interaction is easier under macroprudential regulation than under microprudential regulation.

Let us now explain in what sense macroprudential regulation prevents fire sales externalities and consider the point of view of given bank i. Given what banks $j \neq i$ are liquidating, bank i is not allowed to choose a deleveraging strategy such that one (or more than one) bank j would not comply with its regulatory constraint. Put it differently, when a given bank chooses its deleveraging strategy, given what the others are liquidating, it must not only consider its own regulatory constraint but also the regulatory constraint of all the other banks of the banking system. As a result, bank i may not be in a position to choose the cheapest delevaraging strategy because such a strategy might induce the failure of at least one bank. Bank i thus may have to choose a more expensive deleveraging strategy in order to avoid the failure of some banks, that is, bank i is explicitly constrained to internalize the externality it generates.

The organization of the paper is as follows. In the second section, we discuss the link of our paper with the various strands of literature. In the third section, we present the Cournot fire sale oligopoly while the fourth and fifth section are devoted to the microprudential and the macroprudential analysis. In the sixth section, we apply our model to the French systemic banks and calibrate our model to public data.

2 Related literature

The present paper, which deals with fire sales externalities and macroprudential regulation, is related to several strands of literature in Economics and Finance.

Stress testing banks. Since the financial crisis of 2007-2008, in Europe as in USA, public authority bodies such as the European Banking Authority and the European Central Bank, or the Federal Reserve, now implement on a regular basis regulatory stress tests under different scenarios. However, a striking feature of these regulatory stress tests is the *static balance sheet assumption*, that is, banks hit with an adverse shock are not allowed to react even if their capital ratio falls below the required minimum (see [Goldstein, 2017] a lucid criticism of regulatory stress tests). On the contrary, the academic literature on the subject explicitly considers such a reaction, fundamental if one wants to forecast systemic risk. An important paper on the subject is [Greenwood et al., 2015] (see also [Braouezec and Wagalath, 2019], [Cont and Schaanning, 2016], [Duarte and Eisenbach, 2021] for related models) in which they explicitly consider one round of deleveraging after an adverse shock. An interesting aspect of their model (as the framework offered in this paper) is that it is easy to calibrate to data and they offer an empirical analysis using data from the European Banking Authority.

Fire sales and pecuniary externalities in finance. In addition to non-market interdependence, the usual definition of an externality in Economics, [Scitovsky, 1954] notes that the concept of external economies also includes interdependence among firms (e.g., banks) through the *market* *mechanism*, called pecuniary externality. Such a concept of pecuniary externality has now been extensively applied in Economics (e.g., [Greenwald and Stiglitz, 1986]). In finance, when banks are hit with an adverse common shock and sell a portion of their assets at the same time to (try to) restore their capital ratios, this leads to fire sales, an example of *pecuniary externality*, and ultimately to a running for the exist ([Pedersen, 2009]), a kind of death spiral. In the recent literature on the subject, see for instance [Bichuch and Feinstein, 2019], [Bichuch and Feinstein, 2022], [Braouezec and Wagalath, 2019], [Caballero and Simsek, 2013], [Chernenko and Sunderam, 2020], [Duarte and Eisenbach, 2021], [Jeanne and Korinek, 2020], [Kara and Ozsoy, 2020], [Kuong, 2021] to cite few papers, all offer a framework in which the concept of pecuniary externality–fire sales– plays a central role. In a recent paper [Eisenbach and Phelan, 2022] consider, as we do here, a strategic model of fire sales called "Cournot fire sales" in which banks that face a liquidity (or productivity) shock need to delever. However, the authors do not consider capital ratios, which leads them to analyze a standard Cournot oligopoly model, that is, without solvency constraint.

Macroprudential regulation as a tool to mitigate pecuniary externalities. The literature on macroprudential regulation is now abundant, both in classical academic journals in Economics and Finance but also in policy-oriented journals such as IMF publication ([Brockmeijer et al., 2011]), BIS papers ([BIS, 2016]), ESRB papers ([(ESRB), 2014]). In [Brockmeijer et al., 2011], they recall that the objective of macroprudential policy is to limit the build-up of systemic risk and offer an interesting distinction between two types of macroprudential instruments, those that address the *time dimension* of systemic risk (which reflects to its buid-up over time) and those that address its *cross-sectional dimension* (which reflects the distribution of risk at one point in time). In [De Nicoló et al., 2012] and in [Claessens, 2014], they implicitly adopt the cross-sectional point of view and consider a number of tools designed to address the fire sales externalities problem; liquidity requirements, capital surcharges (currently implemented in Basel III, see [Benoit et al., 2019] for an appraisal), taxation and restrictions on activities. Both [De Nicoló et al., 2012] and in [Claessens, 2014] share the point of view that the foundation of macroprudential policy lies in the correction of the market failures–externalities–that give rise to systemic risk.

Generalized games. First developed by [Arrow and Debreu, 1954] under the terminology "abstract economy" (see [Harker, 1991], [Facchinei and Kanzow, 2010] or [Fischer et al., 2014]), in a generalized game, interaction between agents occurs not only through the payoff function but also through their strategy sets. In Economics, most game theoretic oligopoly à la Cournot are indeed not generalized games since each firm chooses a strategy in an *exogenous strategy set* which by assumption does not depend upon the choice of the other firms (e.g., [Vives, 2001], [Tirole, 1988], 5.4, see also [Ruffin, 1971] for an analysis of the competitive limit).

In a generalized game, the conditions under which a Nash equilibrium exists are particularly strong (e.g., [Ichiishi, 1983], see also [Dutang, 2013] for a nice review). To somehow circumvent this existence problem, in an influential paper, [Rosen, 1965] introduced the notion of a *shared constraint*. Under a shared constraint, each agent i, given what the others do, denoted x_{-i}, is constrained to pick a strategy x_i such that the profile of strategies (x_i, x_{-i}) lies in an exogenous set S called the shared constraint. [Rosen, 1965] shows that a Nash equilibrium with shared constraint exists under standard conditions (but see [Tóbiás, 2020]) and it is now very common in the generalized games literature to directly start with a shared constraint. In [Braouezec and Kiani, 2021a], we discuss the "micro foundation" of the shared constraint from an economic point of view, that is, the set S results from initial individual constraints. We shall here follow [Braouezec and Kiani, 2021a] and consider the *endogenous shared constraint*, which is the shared constraint that results from individual constraints. Within our financial model, the individual constraint called microprudential constraint is the solvency constraint of a given bank, while the endogenous shared constraint called macroprudential constraint is the solvency constraint of all banks. In the second case, each bank considers the constraint of all banks.

3 The Cournot fire sales oligopoly framework

In classic oligopoly models à la Cournot (e.g. [Tirole, 1988], [Vives, 2001]), the multi-product firm i offers a quantity $q_i = (q_{i,1}, ..., q_{i,n})$. In the simplest case of a mono-product firm ($q_i \in \mathbb{R}^+$) in which firms compete à la Cournot, the price of the good is a function of the sum of the quantities offered and is an example of an aggregative game (e.g. [Nocke and Schutz, 2018]). A striking feature of the classic Cournot oligopoly, as in most game theoretic frameworks used in Economics, is that the strategy set of each firm does *not depend* upon the decisions of the other firms. On the contrary, within our framework in which firms are regulated banks, the solvency constraint of a given bank i explicitly depends upon the quantities of assets sold by the other banks through the price mechanism, which gives rise to a negative pecuniary externality. We consider a banking system at a given time t and we assume that each bank complies with its regulatory ratio(s).

3.1 Banks' balance-sheets and regulatory constraints

We consider a banking system $B = \{1, 2, ..., p\}$ with $p \geq 2$ banks where each bank $i \in B$ invests in two types of risky assets; assets subject to credit risk (loans) and assets subject to market risk (traded securities). Banks may also invest in a riskless asset, cash, which represents the value of the bank account of that bank at the Central bank.

Consider the asset side of a given bank i and let V_{i0} be the value of the loans (asset 0) and let $V_{ij} := P_j \times q_{ij} \geq 0$ be the value (in currency) of each risky asset $j \in \{1, 2, ..., n\}$, where q_{ij} is the quantity (in shares) of risky asset j held by bank i and P_j is the price (or value) of the risky asset j at a given date t. Cash is denoted $v_i > 0$. Regarding now the liabilities, let D_i be the sum of the value of deposits and/or debt securities. The balance-sheet of the bank i at time t is as follows.

Balance-sheet of bank i at time t

Assets	Liabilities
Cash: v_i	Debt: D_i
Non-traded assets: V_{i0}	
Traded assets: $\sum_{j=1}^{n} q_{ij} P_j$	Equity: E_i
$A_i = v_i + V_{i0} + \sum_{j=1}^{n} q_{ij} P_j$	$E_i + D_i$

In general, loans (e.g., long-term consumers loans subject to credit risk) are illiquid contracts so that their resale value in the short-term is close to zero due to the so-called adverse selection problem. On the contrary, the risky assets subject to market or to counterparty risk (e.g., traded securities such as stocks, ETF, bonds, vanilla derivatives) can be resold in the short-term depending on their market liquidity. Some securities might be very liquid while others might be less liquid. By definition, the total value of the assets at time t is equal to the total value of the liabilities.

$$A_i = v_i + V_{i0} + \sum_{j=1}^{n} q_{ij} P_j = E_i + D_i$$

Assumption 1 *Banks are not directly interconnected through contractual obligation.*

In practice, banks are interconnected through a number of financial debt contracts such as derivatives, repurchase agreements or (long term) bonds. Unfortunately, this network of interconnections can not be retrieved from the observation of the annual reports of each bank. Assuming no contractual obligation between banks means that the debt of a given bank (liability) is either entirely composed with deposits or held by outside investors such as households or non-banks entities. As a result, a given bank A can not *directly* fail because bank B fails, that is, direct contagion (of default) can not arise. Considering such a network of interconnections would actually reinforce the contagion effects analyzed in this paper. From the above balance sheet, the value of equity (or capital) at time t of an operating (i.e., non-failed) bank i is equal to

$$E_i := A_i - D_i = v_i + V_i + \sum_{j=1}^{n} q_{ij} P_j - D_i > 0 \tag{2.1}$$

and is positive by assumption. Let $\alpha_{ij} > 0$ be the regulatory risk weight of bank i associated to risky asset j. By definition, abstracting operational risk, the total risk-weighted assets of bank i is equal to

$$\mathrm{RWA}_i = \alpha_{i0} V_{i0} + \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j \tag{2.2}$$

so that the global risk-based capital ratio (RBC) of that bank at time t is equal to

$$\theta_{i,t} := \frac{E_i}{\mathrm{RWA}_i} \tag{2.3}$$

From Basel III, the total value of equity of bank i at time t (ignoring regulatory adjustments) is equal to Tier 1 capital (going-concern capital) plus Tier 2 capital (gone-concern capital) and the minimum capital ratio is now bank-dependent in that it depends upon the activity of the bank. In this paper, we shall focus on Tier 1 capital ratio and $\theta_{i,t,min}$ denotes this minimum capital ratio at time t. For simplicity, we drop the time index and simply denote it $\theta_{i,min}$. By assumption

$$\theta_{i,t} \geq \theta_{i,min} \quad \text{for each } i = 1, 2, ..., p \tag{2.4}$$

For simplicity, we shall assume that each bank has only Tier 1 capital, that is, $E_i :=$ Tier 1_i so that the global risk-based capital ratio θ_i is a Tier 1 capital ratio.

3.2 Impact of an exogenous shock on banks' capital ratios

The timing of our model is as follows.

- At time t^+, assets are hit with an adverse shock and banks that do not anymore comply with their regulatory capital ratio sell a portion of their assets.

- At time $t+1$, equilibrium prices are disclosed and observed.

Banks that are required to react at time t^+ sell a portion of their assets by minimizing the value of the asset sale. More precisely, banks use the known price of each asset at time t^+ to choose their deleveraging strategy in order to be solvent at time $t+1$. Since no decisions are taken between time t^+ and $t+1$, the game is *static*. Throughout this section, we present and discuss the main assumptions of our Cournot fire sales oligopoly model.

Assume that a shock on all risky assets occurs at date t^+ and denote $\Delta = (\Delta_1, ..., \Delta_n) \in [0,1]^n$ the adverse shock vector, where Δ_j represents the size of the shock in percentage of P_j. The price (value) of risky asset j at time t^+ thus is equal to

$$P_{j,t^+} = P_j(1-\Delta_j) \qquad j = 0, 1, ..., n \tag{2.5}$$

Remark 4 *The vector of shocks $\Delta = (\Delta_1, ..., \Delta_n) \in [0,1]^n$ is naturally interpreted as a stress test scenario.*

Let $A^+_{i,t} = v_i + V_{i0}(1-\Delta_0) + \sum_{j=1}^n q_{ij} P_j (1-\Delta_j)$ be the value of the asset after the shock. The risk-based capital thus is equal to

$$\theta_{i,t^+}(\Delta) = \frac{\max\{A_{i,t^+} - D_i; 0\}}{\mathrm{RWA}_{i,t^+}} = \frac{\max\{E_{i,t} - V_{i0}\Delta_0 - \sum_{j=1}^n q_{ij} P_j \Delta_j; 0\}}{\alpha_{i0} V_{i0}(1-\Delta_0) + \sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1-\Delta_j)} \tag{2.6}$$

Let us now define the three following sets

$$\mathcal{Z}_i^{\emptyset} := \quad \{\Delta \in [0,1]^n : \ \theta_{i,t^+}(\Delta) \geq \theta_{i,min}\} \qquad \text{(no reaction)} \tag{2.7}$$

$$\mathcal{Z}_i^{sale} := \quad \{\Delta \in [0,1]^n : E_{i,t^+}(\Delta) > 0 \text{ and } \theta_{i,t^+}(\Delta) < \theta_{i,min}\} \qquad \text{(asset sale)} \tag{2.8}$$

$$\mathcal{Z}_i^{fail} := \quad \{\Delta \in [0,1]^n : E_{i,t^+}(\Delta) \leq 0\} \qquad \text{(insolvency and liquidation)} \tag{2.9}$$

and note that they form a partition of $[0,1]^n$. Of particular interest throughout this paper will be the sets $\mathcal{Z}_i^{sale}$ and $\mathcal{Z}_i^{fail}$. The following fact follows directly from equations (2.8) and (2.9).

Fact 2 *For each bank i, the critical sets $\mathcal{Z}_i^{sale}$ and $\mathcal{Z}_i^{fail}$ defined in equations (2.8) and (2.9) can* *be written as follows:*

$$\mathcal{Z}_i^{sale} = \{\Delta \in [0,1]^n : E_i - \theta_{min} \sum_{j=0}^{n} \alpha_{ij} q_{ij} P_j < \sum_{j=0}^{n} q_{ij} P_j \Delta_j (1 - \alpha_{ij} \theta_{i,min})\} \quad (2.10)$$

$$\mathcal{Z}_i^{fail} = \{\Delta \in [0,1]^n : E_i \leq \sum_{j=0}^{n} q_{ij} P_j \Delta_j\} \quad (2.11)$$

We shall adopt throughout the paper the following terminology.

Definition 6 *The shock Δ is said to be*

- *small to medium if, for each bank $i \in \mathcal{B}$, $E_i(\Delta) > 0$, but there exists at least one bank i' such that $\theta_{i'}(\Delta) < \theta_{i',min}$, i.e., $\Delta \in \mathcal{Z}_{i'}^{sale}$.*
- *severe if there exists at least one bank $i \in \mathcal{B}$ such that $\theta_i(\Delta) = 0$, i.e., $\Delta \in \mathcal{Z}_i^{fail}$.*

3.3 Deleveraging strategies, market liquidity and endogenous price impact

Since Δ is a common shock, it affects the balance-sheet of *all* banks that hold risky assets and may leave some of them undercapitalized, possibly insolvent. As observed in [Cohen and Scatigna, 2016], there are various channels of adjustment that can be used by an undercapitalized bank to restore its capital ratio. From a regulatory point of view, the best channel is clearly equity issuance although it is frequently considered as the most expensive one. Moreover, issuing new equity takes time, typically several months, which is certainly too long for undercapitalized banks. After an adverse shock, e.g., an event comparable to the failure of Lehman Brothers in 2008, banks liquidate a portion of their assets but do not issue equity ([Brunnermeier and Oehmke, 2014, Cifuentes et al., 2005, Greenlaw et al., 2012, Greenwood et al., 2015]). In the short-run, banks typically use one of the two following deleveraging strategies to reshuffle their assets side[6].

1. Asset shrinking strategy.
2. Risk-reduction strategy

In both cases, the bank will sell a portion of its risky assets. However, in the first case, the bank will use the proceeds to repay a portion of its debt while in the second case, it will use it to invest

[6]To the best of our knowledge, [Braouezec and Kiani, 2021b] is the unique formal (optimization based) model in which a bank can both issue new equity and/or liquidate assets in order to reach a target capital ratio from the current one.

in cash. As long as markets are perfectly liquid (i.e., perfectly competitive), whether the bank uses the asset shrinking strategy or the risk reduction strategy, its capital ratio will increase since the capital remains constant while the risk-weighted asset decrease. A number of recent papers (e.g., [Gropp et al., 2019], [Juelsrud and Wold, 2020]) document empirically that banks tend to decrease their risk-weighted assets in order to increase their risk-based capital ratio. In what follows, since we essentially focus on the risk-based capital ratio, we make the assumption that undercapitalized (but solvent) banks sell a portion of their risky assets and invest the proceeds in cash in order to (try to) restore their capital ratio. We make the implicit assumption that there are on the market financial institutions such as hedge funds that have no capital requirements (or other financial institutions such as pension funds, other banks...) that have the financial muscles to be the buy side. We shall discuss later on the situation in which the bank is required to manage more than one capital ratio.

Assumption 2 *Banks that do not comply with their regulatory risk-based capital ratio reshuffle their risk-weighted assets; they sell a portion of their risky assets and invest the proceeds in the riskless asset (cash).*

Let $x_{ij} \in [0, 1]$ be the proportion of risky assets j sold by bank i in reaction to the shock vector Δ at date t^+ and let

$$X_j := \sum_{i \in B} x_{ij} q_{ij} \tag{2.12}$$

be the total quantity of assets j sold by all banks. Note that $Q_j := \sum_{i \in B} q_{ij}$ is the maximum quantity of asset j that can be liquidated so that $X_j \leq Q_j$. As in [Banerjee and Feinstein, 2021] among others, we consider a fairly general price impact function denoted $I_j(.)$ that encompasses various price impact functions such as the linear or exponential ones but that only depends upon the quantity of asset j liquidated, that is, there is no cross price impact.

Assumption 3 *The price of the (risky) marketable asset $j = 1, ..., n$ at time $t + 1$ is equal to*

$$P_{j,t+1}(\Delta_j, X_j) = \underbrace{P_j \times (1 - \Delta_j)}_{P_{j,t^+}} \times I_j(X_j) \tag{2.13}$$

where the (price) impact function I_j is once again a twice continuously differentiable and decreasing function of X_j such that

$$I_j(0) = 1 \text{ and } I_j(Q_j) \leq 1 \tag{2.14}$$

Without loss of financial generality, we assume that the price impact function is a regular function (continuously differentiable).

Balance-sheet of bank i at date $t+1$ after deleveraging

Assets	Liabilities
Cash: $v_i + \sum_{j=1}^{n} x_{ij} P_{j,t+1}(.) q_{ij}$	Debt: D_i
Non-tradable assets: $V_{i0}(1-\Delta_0)$	
Tradable assets: $\sum_{j=1}^{n}(1-x_{ij})P_{j,t+1}(.)q_{ij}$	Equity: $E_{i,t+1}$
$A_{i,t+1} = v_i + V_{i0} + \sum_{j=1}^{n} P_{j,t+1}(.)q_{ij}$	$E_{i,t+1} + D_i$

It will be convenient to write the impact function as follows.

$$I_j(X_j) = 1 - \xi_j(X_j) \tag{2.15}$$

where $\xi_j(X_j)$ is a continuously differentiable and increasing function consistent with equation (2.14).

Remark 5 *When $\xi_j(X_j) = a_j X_j$, we will say as usual that the price impact is linear.*

After the adverse shock Δ_j but before any liquidation, the price of asset j is equal to $P_j^{\text{before}} = P_j(1-\Delta_j)$. When the price impact is linear, after liquidation, the price denoted P_j^{after} is equal to

$$P_j^{\text{after}} := P_j^{\text{before}} \times \left(1 - \frac{\sum_{i\in\mathcal{B}} x_{i,j} q_{i,j}}{\Phi_j}\right) \tag{2.16}$$

where Φ_j is called a *market depth* and measures the degree of liquidity of the market (of security) j. The greater the market depth, the greater the liquidity of security j. When $\Phi_j = \infty$, the market is *perfectly liquid* in that each market participant considers the price as exogenous. Throughout the paper, we assume that $\Phi_j > Q_j$ where $Q_j = \sum_{i\in\mathcal{B}} q_{i,j}$.

3.4 The deleveraging problem yields a Cournot fire sale oligopoly

Recall that within our framework, loans (i.e., asset 0) are non-tradable assets, that is, they are perfectly illiquid with no resale value. After the shock, banks can only resell assets of the trading book, that is, assets $j = 1, 2, ..., n$. Let

$$\mathcal{E}_i := [0,1]^n \quad \text{and} \quad \mathcal{E} = \prod_i \mathcal{E}_i \tag{2.17}$$

be respectively the set of liquidation strategies of bank $i \in \mathcal{B}$ and let the set of liquidation strategies of the overall banking system. As usual in game theory, let $x = (x_i, x_{-i}) \in \mathcal{E}_i \times \mathcal{E}_{-i}$ where $\mathcal{E}_{-i} := [0,1]^{n(p-1)}$ is a $n(p-1)$-dimensional vector and assume that x_{-i} is known to bank i. Let $V_{ij} = P_j \times q_{ij}$ be the value of asset j at time t for bank i, i.e., before the shock. Using equations (2.13) and (2.15), it is not difficult to show that, for a liquidation strategy x, the total capital of bank i is equal to

$$E_{i,t+1}(\Delta, x_i, x_{-i}) = \max\left\{E_{i,t} - V_{i0}\Delta_0 - \sum_{j=1}^{n} V_{ij} \times [\Delta_j + \xi_j(X_j)(1-\Delta_j)]\,;0\right\} \tag{2.18}$$

and note that the capital of each bank i depends upon the overall vector of liquidation $x \in \mathcal{E}$, which means that the deleveraging problem is a *strategic problem*. The total capital at time $t+1$ can be decomposed in three different terms, the initial capital of the bank $E_{i,t}$, the depletion of this capital due to an *exogenous* shock Δ_j and the further depletion of this capital due to the *endogenous* banks' reaction which involves the impact function $\xi_j(X_j)$. From the balance sheet, it is easy to see that for a given liquidation strategy x, the risk-weighted assets are equal to

$$\mathrm{RWA}_{i,t+1}(\Delta, x_i, x_{-i}) = \alpha_{i0}V_{i0}(1-\Delta_0) + \sum_{j=1}^{n} \alpha_{ij}V_{ij} \times (1-\Delta_j)\,(1-\xi_j(X_j))\,(1-x_{ij}) \tag{2.19}$$

so that the regulatory capital ratio of bank i at time $t+1$ is equal to

$$\theta_{i,t+1}(\Delta, x_i, x_{-i}) = \frac{E_{i,t+1}(\Delta, x_i, x_{-i})}{\mathrm{RWA}_{i,t+1}(\Delta, x_i, x_{-i})} \tag{2.20}$$

Note that we adopt the natural convention that $\theta_{i,t+1}(\Delta, x_i, x_{-i}) = 0$ when $(x_{i1}, x_{i2}, ..., x_{in}) = (1, 1, ..., 1)$ since $E_{i,t+1} = 0$ (i.e., when bank i is insolvent after the deleveraging process).

Fact 3 *For a given x_{-i}, if for each $i \in B$, $\alpha_{i0}V_{i0}(1-\Delta_0) > 0$, then, the risk-based capital ratio of each bank i as defined in equation (2.20) is a continuous function on the set $[0,1]^{np}$ and there exists $x_i^{max} \in [0,1]^n$ such that the function θ_i reaches its maximal value*

$$\theta_i^{max} := \sup_{x_i \in [0,1]^n} \theta_i(x_i, x_{-i}) \tag{2.21}$$

which may be higher or lower than $\theta_{i,min}$.

This continuity property follows from the fact that the price impact function $\xi_j(X_j)$ is assumed to be continuously differentiable for each j and the existence of θ_i^{max} follows from Weierstrass extreme value theorem. Due to the price impact, everything else equal, the numerator of the capital ratio of bank i given by equation (2.20) is a decreasing function of x_{ij} while its denominator may or may not be a decreasing function of x_{ij}. For the denominator of the capital ratio (i.e., the RWA) to be a decreasing function of x_{ij}, the function $h(x_{ij}) := (1 - \xi_j(X_j))\,(1 - x_{ij})$ should be decreasing, that is, $h'(x_{ij}) < 0$. Assuming even that the RWA is a decreasing function, since the capital is also a decreasing function of x_{ij}, the capital ratio needs not be an increasing function of x_{ij}. We shall come back to that point later on.

Following [Braouezec and Wagalath, 2019], we now introduce the *implied shock* for asset j, that is, the shock which is implied after the liquidation process. This implied shock denoted $\Delta_j(X_j)$ for the asset j is found by solving

$$P_{j,t+1}(\Delta_j, X_j) = P_j \times (1 - \Delta_j(X_j)) \tag{2.22}$$

and it is easy to show, using equations (2.13) and (2.15), that this implied shock for asset j is equal to

$$\Delta_j(X_j) := \Delta_j + \xi_j(X_j)(1 - \Delta_j) \tag{2.23}$$

As long as $X_j \neq 0$, $\Delta_j(X_j) > \Delta_j$ so that the rebalancing process of each asset j at date $t+1$ actually *reinforces* the underperformance of asset j caused by the initial shock Δ_j at date t^+. It is precisely because $\Delta_j(X_j)$, an endogenous quantity, is greater than Δ_j that there may be additional failures after the deleveraging process.

Remark 6 *Considering direct contagion within our framework (e.g., [Glasserman and Young, 2016], [Jackson and Pernoud, 2021] for review papers) would actually reinforce indirect contagion.*

Let $f_i(x_i)$ be the total value of the assets sold by bank i at time t^+ in order to restore its capital ratio. For a given bank i, as long as $x_{ij} > 0$, this cost is equal to

$$f_i(x_i) = \sum_{j=1}^{n} x_{ij} q_{ij} \underbrace{P_j(1 - \Delta_j)}_{=P_{jt^+}} \tag{2.24}$$

We make the fairly natural assumption that each bank i tries to minimize the total value of asset sold using the *known prices* of time t^+ subject to the constraint that the capital will be greater than the minimum required at time $t+1$, when the price of each asset j will be equal to $P_{j,t+1}(\Delta_j, X_j)$.

Assumption 4 *Given the shock* Δ *and what the other banks liquidate, i.e.,* $x_{-i} \in [0,1]^{n(p-1)}$*, each bank* i *such that* $\theta_{i,t^+}(\Delta) < \theta_{min}$ *must solve the following constrained optimization problem*

$$\min_{x_i \in [0,1]^n} f_i(x_i) \tag{2.25}$$

$$\text{subject to } \theta_{i,t+1}(x_i, x_{-i}, \Delta) \geq \theta_{i,min} \tag{2.26}$$

The optimization problem given by equations (2.25) and (2.26) yields a particular kind of game called *generalized game* since the set of strategies for a given bank i such that the constraint is satisfied explicitly depends upon x_{-i}. We do not exclude the vector $x_i = (1, 1, ..., 1)$ which is the situation in which bank i would have to liquidate 100% of its trading book assets, that is, assets $j = 1, 2, ..., n$. Note importantly that given the shock Δ and x_{-i}, the set of solutions of the optimization problem (given by equations (2.25) and (2.26)) may be empty. In such a case, the game is *undefined*. From a financial point of view, no solution simply means that bank i is insolvent. We shall allow later on such a possibility of insolvency.

Remark 7 *The formulation of the problem given by equations (2.25) and (2.26) yields a static optimization problem for each bank, given* x_{-i}*. In this paper, we are only interested in looking at the Nash equilibria of the generalized game. In practice, banks would liquidate sequentially their assets and it would be possible to analyze the underlying dynamical system generated by the best responses of banks over time. For instance, at time* $\tau + 1$*, a given bank* i *could form its best response based on the observed liquidation decisions at time* τ*, that is,* x^{τ}_{-i}*. The stationary states of this dynamical system would be the Nash equilibria of the static game.*

Assumption 5 *Complete information; all the quantities but also the structure of the game are known (indeed common knowledge) to every bank.*

Due to Basel III, banks must now disclose more information than before about their own activity[7] so that it makes sense to assume that each bank is aware of the positions of the other banks in the banking system. Since the structure of the game is itself common knowledge, each bank knows that each bank knows the structure of the game and so on and so forth. As a result, when a given bank i is insolvent after the shock, this failure is known to each bank (i.e., can be perfectly predicted) and this is common knowledge. But the consequences of this failure is also common

[7] Systemic banks are constrained to publicly disclose twelve indicators about their activity with the rest of the financial system on the website of the European Banking Authority (EBA). See https://www.eba.europa.eu/risk-analysis-and-data/global-systemically-important-institutions.

knowledge. Each bank is perfectly able to predict the consequences of the asset liquidation by bank i on asset prices and the (possible) resulting cascade of failures due to the existence of the price impact (negative fire sales externality). Within our model, we make the assumption that banks and regulators (or supervisors) share the same information because annual reports contain all the relevant information and are public, see section 7 devoted the empirical analysis. Admittedly, this assumption while disputable is reasonable. It is our belief that modeling the same game theoretic problem in asymmetric information (using Bayesian games) would raise additional issues that are beyond the scope of this paper.

3.5 Risk-based and non risk-based capital ratios in Basel III

In Basel III, banks must not only comply with the various risk-based capital ratios but also with a non-risk based capital ratio called the leverage ratio defined as Tier 1 capital divided by the total exposure. It turns out that for many European banks, the total value of the assets and the total exposures are very close and differ typically by 5% to 10%, which means that the total value of the assets is a fairly good approximation of the total exposure.

$$L_i \approx \frac{\text{Tier1 capital}}{\text{Total assets}} = \frac{E_i}{A_i} \tag{2.27}$$

Written as in equation (A.6), the leverage ratio can be thought of as the particular case of the risk-based ratio when each risk weight is equal to one, including cash. Since the denominator of the leverage ratio incorporates cash, the risk-reduction strategy is now inoperative. To increase its leverage ratio, bank i must adopt an asset shrinking strategy, that is, as already discussed, it must sell a portion of its risky assets and makes use of the proceed to pay back its debt[8]. Everything else equal, with the asset shrinking strategy, both the leverage ratio and the risk-based capital will increase. To see this, assume that the market of asset k is perfectly liquid (no price impact), that is, by selling this asset k at its fair value V_{ik}, the proceeds is equal to V_{ik}. The total value of the assets when asset k has been sold thus is equal to $A_i = \sum_{j \neq k} V_{ij} - V_{ik}$ and the total debt is equal to $D_i - V_{ik}$. Let θ_i and L_i be the risk-based and the leverage ratio before the asset sale.

- The leverage ratio after the sale is now equal to $L_i^{\text{after}} = \dfrac{E_i}{\sum_{j \neq k} V_{ij} - V_{ik}}$ and is greater than L_i since the denominator has decreased by V_{ik}.

[8] This might not always be possible. If a bank is essentially financed with deposits, it is difficult to see how deposits can be repaid...

- The risk-based capital ratio after the sale is now equal to $\theta_i^{\text{after}} = \dfrac{E_i}{\sum_{j\neq k} \alpha_{ij} V_{ij}}$ and is greater than θ_i since the denominator has decreased by $\alpha_{ik} V_{ik}$.

With low price impact, both capital ratios will yet increase. However, when the price impact is significant, things can more complex.

4 Nash Equilibrium analysis under microprudential regulation

4.1 No price impact as a non-strategic problem

The case in which there is no price impact is simple because the problem for each bank is non-strategic. For each $j = 1, 2, ..., n$, the function $\xi_j(.)$ is invariably equal to zero so that, from equation (2.18) and (2.19), the capital and the risk-based capital ratio of each bank i depends only upon x_i. For each bank i, the problem reduces to a standard optimization problem, that is, $\min_{x_i} f_i(x_i)$ subject to $\theta_i(x_i) \geq \theta_{i,min}$. The next proposition is simple to prove but gives an interesting insight regarding the optimal way to delever when there is no price impact. It also says that depending upon the exposure to loans, the bank might not be able to restore its capital ratio back above the required minimum.

Proposition 7 *Assume no price impact (i.e., for each j, $\mathcal{E}_j(X_j) = 0$ regardless of $X_j \geq 0$) and consider a given bank i such that $\Delta \in \mathcal{Z}_i^{sale}$.*

1. *For each $i \geq 1$ and each $j \geq 1$, the capital ratio $x_{ij} \to \theta_i(x_{ij}, ...)$ is an increasing function of x_{ij}.*

2. *When $V_{i0} = 0$, bank i is always able to restore its capital ratio back above the required minimum and it is optimal to first sell the security k with the highest regulatory weight k (i.e., the security k such that $\alpha_{ik} > \alpha_{ij}$ for $k \neq j$). If the proceeds of the sale is is not enough, it is optimal to sell the asset with next highest risk weight and so on and so forth.*

3. *When $V_{i0} > E_i$, bank i will always be able to restore its capital ratio back above the required minimum through the above optimal deleveraging strategy.*

Proof. See the appendix.

When markets are perfectly liquid (i.e., competitive), the numerator of the capital ratio of each bank is *invariant* with respect to the liquidation decision(s) since each asset can be resold at its current market price, without price impact case, (see equation 2.18). As a result, the capital ratio of bank i only depends upon its own decision x_i. Since the risk-weighted assets of each asset j decreases with x_{ij}, it thus follows that the capital ratio of bank i increases when x_{ij} increases. When $V_{i0} = 0$, the bank is not exposed to illiquid assets. As a result, by selling an arbitrarily high portion of each asset j, the capital ratio is arbitrarily high since the risk-weighted assets are arbitrarily close to zero. As a result, for a given constraint $\theta_{i,min}$, there exists x_{ij}, $j = 1, 2, ..., n$ possibly close to one) such that $\theta_i(x_{i1}, x_{i2}, ..., x_{in}) = \theta_{i,min}$. While there are possibly many deleveraging strategies to restore its capital ratio, bank i is assumed to choose the cheapest one, which leads to point 2 of the above proposition. To minimize the value of the asset sale, the bank sells first the asset (or security) with the highest risk weight. If this is not enough, it sells the asset with the next highest risk weight and so on and so forth. This liquidation strategy is in sharp contrast with the proportional liquidation rule used in [Greenwood et al., 2015] or in [Cont and Schaanning, 2016] in which the bank sells a proportion of each asset. Such a proportional liquidation strategy is suitable when the bank maintains a non-risk based capital ratio such as the leverage ratio but not a risk-based capital ratio.

4.2 Positive price impact as a generalized game problem

As long as the price impact is positive, from equation (2.26), the capital ratio of each bank i depends upon the decisions of all the other banks. Before discussing the best response properties, let us formulate the capital ratio constraint given in equation (2.26) as a microprudential (solvency) constraint.

Definition 7 *Given an adverse shock Δ and the liquidation decisions x_{-i} of all banks except bank i, the strategy set that results from the enforcement of the microprudential regulation of bank i defined as*

$$X_i(x_{-i}) = \{x_i \in [0,1]^n : \theta_{i,t+1}(x_i, x_{-i}, \Delta) \geq \theta_{i,min}\} \tag{2.28}$$

is called the microprudential constraint.

Note importantly that the strategy set of bank i denoted X_i for short explicitly depends upon x_{-i} but implicitly depends upon the price impact function $I_1(.), ..., I_n(.)$. It is natural to call the set $X_i(x_{-i})$ the *microprudential constraint* since it is the aim of each bank i to have a capital ratio greater (or equal) than the required minimum. If $X_i(x_{-i})$ is empty, then, bank i is insolvent and must be liquidated, that is, it must sell 100% of its asset, $x_i = (1, 1, ..., 1)$. As long as $X_i(x_{-i})$ is not empty, bank i can restore its capital ratio by choosing a strategy $x_i \in [0,1]^n$ to restore its capital ratio. As we shall see later on, allowing banks to be insolvent raises new issues. The optimization problem given by equations (2.25) and (2.26) can now simply be written as

$$\min_{x_i \in [0,1]^n} f_i(x_i) \quad s.t \quad x_i \in X_i(x_{-i}) \tag{2.29}$$

Within our framework, interaction between banks occurs through the strategy sets but *not* through their objective function.

Fact 4 *For each bank $i \in \mathcal{B}$, since the microprudential constraint $X_i(x_{-i})$ depends upon x_{-i}, it is usual to call $X_i()$ a point-to-set map. As a result, the Cournot fire sales oligopoly defines a generalized game ([Facchinei and Kanzow, 2010], [Fischer et al., 2014] for review papers).*

As already said, generalized games contrast with "classical" games frequently encountered in economic theory (e.g., [Fudenberg and Tirole, 1991], [Moulin, 1986], [Osborne and Rubinstein, 1994]) in which the strategy set $\mathcal{E}_i$ of each agent i does *not* depend upon the decisions of the other agents x_{-i}. In a generalized game, the strategy set of each agent depends upon the decisions of all the other agents and hence is denoted $X_i(x_{-i})$.

Let $BR_i(x_{-i})$ be the best response of bank i. We now provide a geometric characterization of the point-to-set map $X_i()$ and the strategy set $X_i(x_{-i})$ when not empty and we show that under linear price impact, the best response is unique.

Lemma 1 *Assume that the price impact is linear for each asset $j \in \{1, 2, ..., n\}$ and consider the situation of bank i for which $X_i(x_{-i}) \neq \emptyset$.*

1. *$X_i(x_{-i})$ is the intersection of a n-dimensional ellipsoid with the unit compact of $\mathbb{R}^n$, $[0,1]^n$.*

2. *The best response $BR_i(x_{-i})$ is either the unique tangency point between the n-dimensional ellipsoid $X_i(x_{-i})$ and a hyperplane or is a corner solution.*

Proof. See the appendix.

When $n = 2$, $X_i(x_{-i}) = \{x_i \in [0,1]^2 : \theta_i(x_i, x_{-i}) \geq \theta_{i,min}\}$ reduces to an *ellipse*. In Fig 2.1, for simplicity, we make the assumption that the ellipse is contained in the unit square but nothing is changed if we do not make this assumption. Let $\mathcal{F}_{\theta_{i,min}} = \{x_i \in [0,1]^2 : \theta_i(x_i, x_{-i}) = \theta_{i,min}\}$

be a level curve of the capital ratio, that is, all the deleveraging strategies x_i such that the capital ratio is equal to $\theta_{i,min}$. This level curve is depicted in yellow in Fig 2.1 and is the contour (or the boundary) of the ellipse. Let x_i^{max} be the solution of $\max_{x_i \in X_i(x_{-i})} \theta_i(x_i, x_{-i})$ and note that in general, $\theta_i(x_i^{max}, x_{-i}) := \theta_i^{max} > \theta_{i,min}$. A number of remarks are in order.

1. The point x_i^{max} is (generically) not the center c_i of the ellipse and is such that $x_i^{max} \geq c_i$ component-wise.

2. The two points $x_{i,A}$ and $x_{i,B}$ are located on the same level curve although $x_{i,B} > x_{i,A}$ component-wise.

3. The capital ratio of bank i is not an increasing function of $x_{i,1}$ and $x_{i,2}$. It can be directly seen from Fig 2.1 that on the north-east of x_i^{max}, the capital ratio of bank i *decreases* when $x_{i,1}$ or $x_{i,2}$ increases.

4. The triangle with a red contour represents the set of point x_i such that the capital of bank i is equal to zero. Depending upon x_{-i} and/or the capitalization of the bank, it may obviously be the case that such a set of points is empty. The equation of the red line follows from the linearity of the capital $E_i(x_i, x_{-i})$ with respect to $x_{i,1}$ and $x_{i,2}$. From equation (2.18) in the case of linear price impact, it reduces in the two dimensional case to an equation of the form $K - x_{i1}q_{i1} - x_{i2}q_{i2} = 0$ equivalent to $x_{i2} = \frac{-x_{i1}q_{i1}+K}{q_{i2}}$ whose slope is $-\frac{q_{i1}}{q_{i2}} < 0$

Consider point 3. When $x_{i,1}$ and $x_{i,2}$ increase, the risk-weighted assets decrease so that, everything else equal, the capital ratio increases. But everything is not equal. When $x_{i,1}$ and $x_{i,2}$ increase, this also decreases the price of the assets and thus the capital of the bank. By definition, the point $x_{i,A}$ is such that $\theta_i(x_{i,A}, x_{-i}) = \theta_{i,min}$. From Fig 2.1, since $x_{i,A} < x_i^{max}$ component-wise, by increasing each component of $x_{i,A}$, the capital ratio increases and reaches its maximum in x_i^{max}. Starting from $x_{i,A}$, this means that by selling more of each asset, the drop of the risk-weighted assets is more important than the drop of the capital of the bank so that the capital ratio increases. However, when one starts from the point x_i^{max}, when bank i sells more of each asset, the drop of the capital is more important than the drop of risk-weighted assets so that the capital ratio decreases and this explains point 3. If one continues to increase the sale of each asset after the point $x_{i,B}$, one may reach the region in which the capital of bank i is equal to zero.

Consider now the best response. On Fig 2.2, the strategy set of a given bank $X_i(x_{-i})$ is included in the unit compact of $\mathbb{R}^2$ while it is not on Fig 2.3. Since the level curves associated to the objective

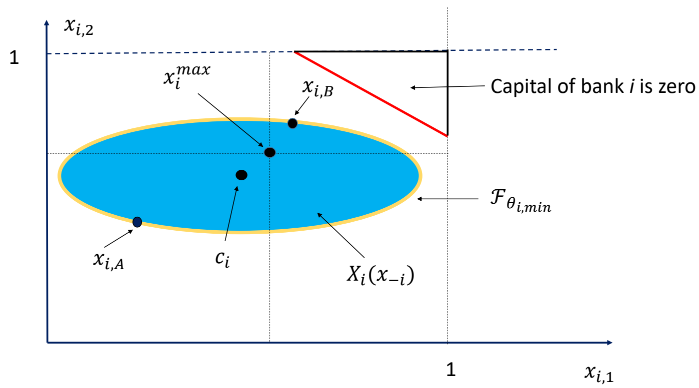

Figure 2.1: The strategy set $X_i(x_{-i})$ is an ellipse when the price impact is linear

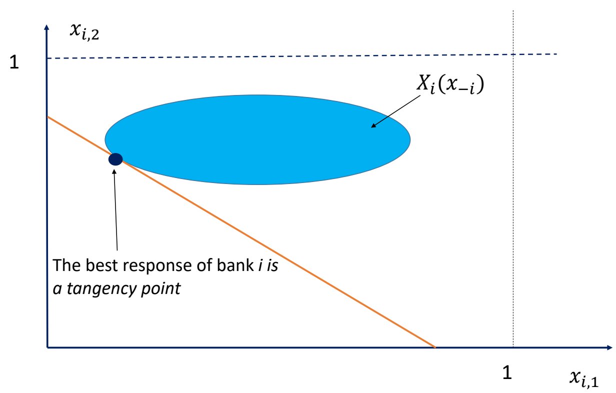

Figure 2.2: The strategy set $X_i(x_{-i})$ of a given bank

function (i.e., f_i) are lines, in Fig 2.2, the best response is a tangency point while in Fig 2.3, it is a corner solution, the best response is located on the boundary of the unit square $[0, 1]^2$.

To conclude this paragraph, let us now come back to the multiple ratios situation and let

$$Y_i(x_{-i}) = \{x_i \in [0, 1]^n : L_{i,t+1}(x_i, x_{-i}, \Delta) \geq L_{i,min}\} \tag{2.30}$$

be the microprudential constraint associated with the leverage ratio, where $L_{i,min}$ is the minimum leverage required. Since the leverage ratio is the particular case of the risk-based capital when all the risk weights are equal to one[9] (including cash), the strategy set $Y_i(x_{-i})$ also is (when included in the unit compact of $\mathbb{R}^n$) an n-dimensional ellipsoid. Let $H_i(x_{-i}) := X_i(x_{-i}) \bigcap Y_i(x_{-i})$ be the intersection of two n-dimensional ellipsoids and note that H_i is a convex set.

[9]This is approximately correct when the total exposures is close enough to the total value of the assets.

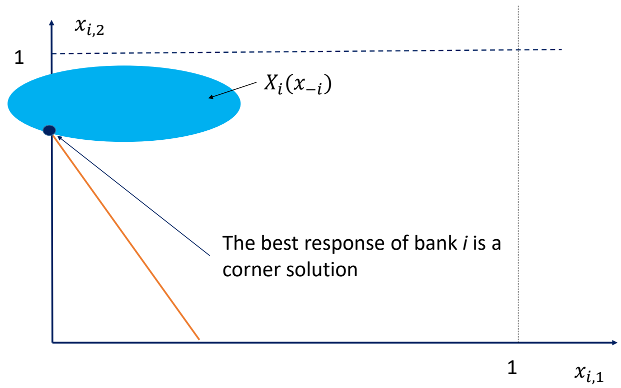

Figure 2.3: The strategy set $X_i(x_{-i})$ of a given bank

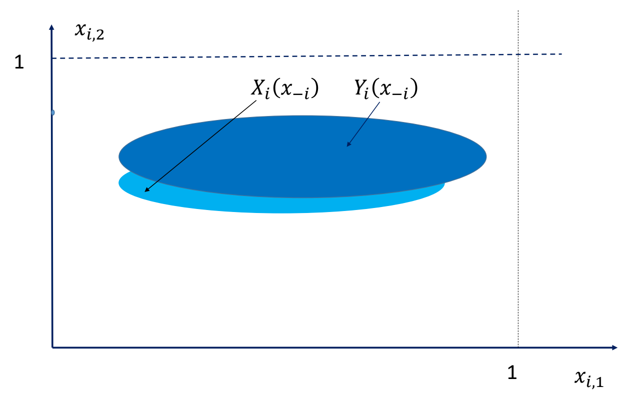

Figure 2.4: Risk-based capital ratio and leverage ratio

Fact 5 *Assume that price impact is linear and that each bank uses an asset shrinking strategy. When a given bank i manages both the risk-based capital ratio and leverage ratio, then, the overall Tier 1 microprudential constraint results from the intersection of two n-dimensional ellipsoids $H_i(x_{-i})$ with the unit compact of $\mathbb{R}^n$.*

To the best of our knowledge, this is the first paper that offers a description of the management of the two Tier 1 capital ratios in a fairly general framework.

4.3 Existence of a Nash equilibrium under microprudential constraints

Following the terminology introduced in [Braouezec and Kiani, 2021a], let K be a set *admissible strategies* defined as follows

$$K = \{x \in \mathcal{E} : \forall i \in \mathcal{B}, x_i \in X_i(x_{-i})\} \tag{2.31}$$

and note that if $K = \emptyset$, no equilibrium can exist. The non-vacuity of K thus is necessary for the Nash equilibrium to exist and we shall assume that K is not empty. Given the shock Δ and the price impact functions $I_j(.)$ $j = 1, 2, ..., n$, we make the assumption that the set of admissible strategies K is not empty. Note that when $x \notin K$, the game is *undefined* since at least one bank i does not satisfy its microprudential constraint.

Let $(\mathcal{B}, \mathcal{E}, (f_i)_{i\in\mathcal{B}}, (X_i)_{i\in\mathcal{B}})$ define the Cournot fire sale game *with microprudential constraints*. Recall that $\mathcal{E}_i = [0,1]^n$ and that $\mathcal{E} = \prod_{i\in\mathcal{B}} \mathcal{E}_i$ and note that $K \subset \mathcal{E}$.

Definition 8 *The profile of strategies $x^* \in K$ is a Nash equilibrium of the Cournot fire sale game with microprudential constraints $(\mathcal{B}, \mathcal{E}, (f_i)_{i\in\mathcal{B}}, (X_i)_{i\in\mathcal{B}})$ if, for each $i \in \mathcal{B}$ and each $x_i \in \mathcal{E}_i$ such that $x_i \in X_i(x^*_{-i})$, it holds true that $f_i(x^*_i, x^*_{-i}) \leq f_i(x_i, x^*_{-i})$.*

As already said, a necessary but not sufficient for a Nash equilibrium to exist in the game with microprudential constraints is $K \neq \emptyset$. But even when $K \neq \emptyset$, as long as the price impact is positive, the proof of the existence of a Nash equilibrium to the Cournot fire sale game with microprudential constraints $(\mathcal{B}, \mathcal{E}, (f_i)_{i\in\mathcal{B}}, (X_i)_{i\in\mathcal{B}})$ remains difficult because virtually nothing is known in general regarding the (topological) properties of the sets $X_i(x_{-i})$. For classical games (e.g., [Dasgupta and Maskin, 1986], theorem 1 and 2), it is well-known that for a Nash equilibrium in pure strategies to exist, the set $\mathcal{E}_i$ must be compact and convex. Within our generalized games framework, for a Nash equilibrium to exist, the point-to-set map $X_i()$ are such that for all banks i, $X_i(x_{-i})$ must be non-empty, compact and convex ([Ichiishi, 1983], see the review papers [Dutang, 2013] and [Facchinei and Kanzow, 2007]).

Theorem 2 *([Arrow and Debreu, 1954], [Ichiishi, 1983], [Facchinei and Kanzow, 2007]))*

Let $(J, E, (f_i)_{i\in J}, (X_i)_{i\in J})$ be a generalized game with individual constraints and suppose that:

- *There exists N nonempty convex and compacts sets $C_i \subset \mathbb{R}^{n_i}$ such that for all $x \in \mathbb{R}^n$ with $x_i \in C_i$ for every i, $X_i(x_{-i})$ is nonempty closed and convex, $X_i(x_{-i}) \subset C_i$, and $X_i()$ as a point-to-set map, is both upper and lower semi-continuous.*
- *For every player i, the function $f_i(, x_{-i})$ is quasiconvex on $X_i(x_{-i})$.*

Then a generalized Nash equilibrium exists.

Coming back to our model, we already know that, when non-empty, $X_i(x_{-i})$ is an ellipsoid when the price impact is linear, so that it is compact and convex. Applying theorem 2 for our model still requires the strategy sets to be non-empty for every x_{-i}. Note also that theorem 2 also requires the point-to-set map $X_i()$ to be both upper and lower semi-continuous, which is a notion of continuity for a set.

Proposition 8 *Assume that the price impact is linear for each asset* $j \in \{1, 2, ..., n\}$ *and assume that for all* i *and for all* $x_{-i} \in [0,1]^{(p-1)n}$, $X_i(x_{-i})$ *is nonempty. Under these assumptions, a generalized Nash equilibrium in microprudential constraints always exist.*

Proof. See the appendix.

The conditions under which a Nash equilibrium exists in microprudential constraints are extremely strong since they require in particular $X_i(x_{-i})$ to be non-empty for all $x_{-i} \in [0,1]^{(p-1)n}$. It requires not only the shock to be small but also the price impact to be low enough. To see this, assume one moment that each bank $k \neq i$ sells 99% of its trading book, that is, $x_k = (0.99, ..., 0.99)$ so that $x_{-i} = ((0.99, ..., 0.99), ..., (0.99, ..., 0.99))$. For the Nash equilibrium to exist in microprudential constraint, given this particular profile of strategies x_{-i}, $X_i(x_{-i})$ must be non-empty for each bank i, which is a very strong assumption. For this profile of strategies, we typically expect bank i to fail due to the price impact but such an insolvency is not yet allowed. Such a possibility will be discussed later on.

5 Nash equilibrium analysis under macroprudential regulation

5.1 How to address externalities related to fire sales ?

According to [Brockmeijer et al., 2011], the aim of the macroprudential regulation (or policy) is to address the two dimensions of systemic risk.

1. The time dimension;

2. The cross-sectional dimension.

The time dimension reflects the procyclical effect that operates over time within the financial system and the real economy. [Brockmeijer et al., 2011] observe that during the boom phase, procyclicality induces excessive leverage from financial institutions and this build up of aggregate risk increases the chance of financial distress. In that sense, the leverage ratio, which by definition limits the leverage of banks, is a time dimension macroprudential tool.

The cross-sectional dimension reflects the distribution of risk in the financial system at a given point in time and depends upon the links financial institutions may have (e.g., through contractual obligations, through identical exposures) but also, as noted in [Brockmeijer et al., 2011], upon the size of the institutions, concentration and substitutability of their activities. In that sense, the various buffers introduced in Basel III as well as the liquidity ratios are cross-sectional macroprudential tools. In this paper, we only consider the cross-sectional dimension of systemic risk seen as a restriction of banks deleveraging decisions.

Up to now, we focused on the microprudential approach to banking regulation, that is, each bank i only considers its own solvency constraint $X_i(x_{-i})$, that is, its set of deleveraging strategies to restore its capital ratio given what the other banks are liquidating x_{-i}. This solvency constraint is said to be microprudential because bank i takes into account its own solvency constraint only, that is, given x_{-i}, bank i chooses x_i so as to minimize its own cost function. In particular, bank i *does not take into account* the failure externality its decision x_i (i.e., its best response $BR_i(x_{-i})$) may generate. Such a best response of bank i might imply the failure of a subset of banks. In their table 1, [De Nicoló et al., 2012] consider four instruments that may be used to address externalities related to fire sales (see also [Claessens, 2014] for a discussion); capital surcharges, liquidity requirement, taxation and restrictions on activities. However, restrictions on activities is mentioned as a macroprudential tool but is not considered to address the fire sales problem and this is what we want to do here. It seems important to point out that within our approach, restrictions on activities arise ex post, that is, when banks must sell a portion of their assets after some systemic shock while liquidity ratios or buffers arise ex ante.

Within our model, each bank $i \in \mathcal{B}$ comes up with its solvency constraint $X_i(x_{-i})$ which depends upon its own characteristics and the decisions of other banks x_{-i}. Recall that K (see equation (2.31)) is the set of admissible deleveraging decisions. Following [Braouezec and Kiani, 2021a], assume that K is the endogenous shared constraint; given x_{-i}, the best response of bank i $BR_i(x_{-i})$ must be such that $(BR_i(x_{-i}), x_{-i})$ lies in K, which is a restriction of activity. Given x_{-i}, bank i is not anymore allowed to choose $x_i \in X_i(x_{-i})$ if $(x_i, x_{-i}) \notin K$. Such a restriction of activities may typically come from a social planner, indeed regulators or supervisors, and can be interpreted as a systemic constraint that we call a macroprudential regulation.

Definition 9 *Given x_{-i}, the strategy set of bank i that results from the enforcement of the macro-*

prudential regulation, the constraint K, is defined as

$$K_i(x_{-i}) = \{x_i \in \mathcal{E}_i : x \in K\} \tag{2.32}$$

It is clear that the macroprudential regulation yields an additional constraint for each bank i since

$$K_i(x_{-i}) \subseteq X_i(x_{-i}) \tag{2.33}$$

When each bank is subject to the macroprudential regulation, given x_{-i}, the choice x_i by bank i is restricted to $K_i(x_{-i})$ and not anymore to $X_i(x_{-i})$ and it is precisely in that sense that the macroprudential regulation introduces a *restriction of activities* compared to the sole microprudential constraint. By definition, as long as $x \in K$, all banks are able to comply with their regulatory constraint. It thus follows that when $x_i \in K_i(x_{-i})$, *all banks* comply with their regulatory capital ratio, i.e., they will all have their capital ratio greater than (or equal to) the minimum required. The optimization problem of each bank i can now simply be written as

$$\min_{x_i \in [0,1]^n} f_i(x_i) \quad s.t \quad x_i \in K_i(x_{-i}) \tag{2.34}$$

Given x_{-i}, under macroprudential regulation, each bank i must now choose x_i to minimize $f_i(x_i)$ subject to $\theta_i(x_i, x_{-i}) \geq \theta_{i,min}$ *and* subject to $\theta_k(x_i, x_{-i}) \geq \theta_{k,min}$ for $k \neq i$, that is, each bank i takes not only into account its own solvency constraint but also the solvency constraint of *all* the other banks. Under microprudential constraint, it may be the case that the best response $BR_i(x_{-i})$ is such that for some $k \in \mathcal{B}$, $\theta_k(BR_i(x_{-i}), x_{-i}) < \theta_{k,min}$, that is, bank i generates a negative (fire sales) externality on bank k which does not comply with its regulatory constraint. Our macroprudential regulation precisely forbids such negative externalities related to fire sales. We summarize this discussion in the following fact.

Fact 6 *When K is not empty, under the macroprudential regulation, each bank i minimizes $f_i(x_i)$ subject to the constraint that no bank will fail (i.e., solves problem 2.34), which means that the fire sales externalities problem is addressed.*

From a pure mathematical point of view, our model can be seen as a generalization of the strategic fire sale model introduced in [Braouezec and Wagalath, 2019] as we assume that the number of risky assets is finite and not restricted to one. This mathematical generalization allows us, from a financial point of view, to make a clear distinction between micro and macro prudential regulation. In [Braouezec and Wagalath, 2019], such a distinction is irrelevant since given x_{-i}, each bank has a unique deleveraging strategy to restore its capital ratio, because x_i is scalar. Within our framework in which $n \geq 2$, given x_{-i}, a given bank i may have *several deleveraging strategies* to restore its capital ratio. As a result, a given bank i may not be allowed to choose the cheapest one if it generates the failure of other bank(s). As we shall now show, when K is not empty, a Nash equilibrium always exists under macroprudential regulation, and among the Nash equilibria, one minimizes the total value of the asset sale, that is, this Nash equilibrium is Pareto optimal. Contrary to the basic intuition one may have, the analysis is much easier under macroprudential regulation than under microprudential regulation.

5.2 Existence of a Nash equilibrium that minimizes the value of asset sales

Assume that $x \in K$ and let

$$V(x) := \sum_{i \in \mathcal{B}} f_i(x_i) = \sum_{i \in \mathcal{B}} \sum_j f_i(x_{ij}) \tag{2.35}$$

be the total value of the asset resold by banks. Before we present an existence result of Nash equilibrium under macroprudential regulation, we offer once again a geometric description of $K_i(x_{-i})$, when not empty.

Lemma 2 *Assume that the price impact is linear for each asset $j \in \{1, 2, ..., n\}$ and consider the situation of bank i for which $K_i(x_{-i}) \neq \emptyset$.*

1. *$K_i(x_{-i})$ is the intersection of $X_i(x_{-i})$ with $p-1$ affine closed half-spaces. It is therefore the intersection of a n-dimensional ellipsoid with $p-1$ affine closed half-spaces and $[0,1]^n$.*

2. *The best response in macroprudential constraint $BR_i^M(x_{-i})$ is well defined and is either the unique point of tangency between the ellipsoid delimited by $X_i(x_{-i})$ intersected with $p-1$ closed affine half-spaces (that is, $K_i(x_{-i})$) and an affine hyperplane, or a corner solution.*

In Fig 2.3, we offer a geometric representation in the two-dimensional case. We give below the definition of a Nash equilibrium of the Cournot fire sale game under macroprudential constraint.

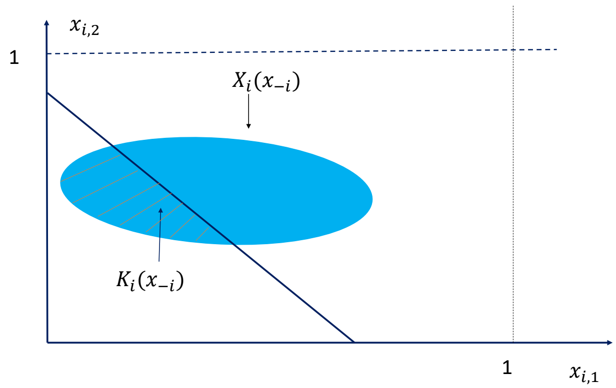

Figure 2.5: The strategy set $K_i(x_{-i}) \subset X_i(x_{-i})$

Definition 10 *The profile of strategies $x^* \in K$ is a Nash equilibrium of the Cournot fire sale game with macroprudential constraint $(\mathcal{B}, \mathcal{E}, (f_i)_{i\in\mathcal{B}}, (K_i)_{i\in\mathcal{B}})$ if, for each $i \in \mathcal{B}$ and each $x_i \in \mathcal{E}_i$ such that $x_i \in K_i(x^*_{-i})$, it holds true that $f_i(x_i^*, x^*_{-i}) \leq f_i(x_i, x^*_{-i})$.*

As the following result shows, as long as K is not empty, there always exists a Nash equilibrium under macroprudential constraint that minimizes the total value of asset sale, which is thus Pareto optimal.

Proposition 9 *Let Δ be a small to medium shock (i.e., for at least one bank i, $\Delta \in \mathcal{Z}_i^{sale}$). If $K \neq \emptyset$, then, there exists a Nash equilibrium under macroprudential regulation $x^{*,M} \in K$ such that $V(x^{*,M}) := \sum_{i\in\mathcal{B}} f_i(x_i^{*,M})$ is minimized (Pareto optimality) and such that for each $i \in \mathcal{B}$, $\theta_i(x_i^{*,M}, x_{-i}^{*,M}) \geq \theta_{i,min}$.*

Proof. See the appendix.

The Nash equilibrium is obviously Pareto optimal since the total value of the sales is minimized. It is worthwhile to note since the cost functions are all expressed in currency, the Nash equilibrium of interest is the one in which the total value of the asset sale is minimized. Our model is indeed similar to a transferable utility cooperative games for which interpersonal comparisons of utility between agents make sense.

Remark 8 *The Nash equilibrium under macroprudential regulation $x^{*,M} \in K$ is interesting for a number of reasons. From an economic point of view, it fully addresses the fire sales externalities problem in that it minimizes the total asset sale and hence constitutes the optimal solution from a* *regulatory point of view. From a technical point of view, as long as K is not empty, there always exists a Nash equilibrium under macroprudential regulation.*

Remark 9 *The existence of the Nash equilibrium under macroprudential regulation $x^{*,M} \in K$ that minimizes the total asset sale holds under general price impact function as long as it is a continuous function of x_j for all risky asset j.*

Recall that $H_i(x_{-i}) := Y_i(x_{-i}) \bigcap X_i(x_{-i})$ is the overall Tier 1 microprudential regulatory constraint. When $H_i(x_{-i}) \neq \emptyset$, given what the other banks liquidate, bank i is able to find a strategy such that it complies with the two Tier 1 capital ratios. Let

$$\mathcal{K} = \{x \in \mathcal{E} : \forall i \in \mathcal{B}, x_i \in H_i(x_{-i})\} \tag{2.36}$$

Fact 7 *If $\mathcal{K}$ is not empty, proposition 9 holds, that is, there exists a Nash equilibrium under macroprudential constraint that minimizes the total value of the resale and such that each bank both complies with the risk-based capital ratio and the leverage ratio.*

At equilibrium, it may obviously be the case that for some bank i, $\theta_i(x_i^{*,M}, x_{-i}^{*,M}) > \theta_{i,min}$. This depends upon the situation of that bank. For instance, a bank i which is extremely well-capitalized may be such that $\theta_i(\mathbf{0}, x_{-i}^{*,M}) > \theta_{i,min}$ where $\mathbf{0} := (0, 0, ..., 0)$ is a $n-$dimensional vector. In such a case, this bank i needs not to delever so that the resulting capital ratio at equilibrium is higher than the required minimum.

5.3 Macroprudential regulation as the natural benchmark

We have shown that a Nash equilibrium which minimizes the total value of asset sales exists in macroprudential constraint. From a theoretical point of view, it would clearly be interesting to compare the properties of a Nash equilibrium under macroprudential constraint $x^{*,M} \in K$ with the Nash equilibria under microprudential constraint $x^{*,m} \in K$ when it exists. However, as we have seen, the conditions under which a Nash equilibrium $x^{*,m}$ exists in microprudential constraints are extremely strong since the non-vacuity of the strategy set $X_i(x_{-i})$ for each i and each x_{-i} is required. On the contrary, as long as K is not empty, a Nash equilibrium under macroprudential regulation exists, which means that this equilibrium naturally defines the benchmark for regulation.

Definition 11 *The Nash equilibrium under macroprudential constraint $x^{*,M} \in K$ which minimizes the total asset resale is said be microprudentially incentive-compatible if, without any macroprudential constraint, for each bank $i \in \mathcal{B}$, $BR_i(x^{*,M}_{-i}) \in K_i(x^{*,M}_{-i})$.*

When the Nash equilibrium $x^{*,M}$ is microprudentially incentive-compatible, no bank has an incentive to choose a strategy such that another bank would fail to comply with its regulatory constraint. On the contrary, when the Nash equilibrium $x^{*,M}$ is not microprudentially incentive-compatible, at least one bank $i \in \mathcal{B}$ is such that $BR_i(x^{*,M}_{-i}) \notin K_i(x^{*,M}_{-i})$ (i.e., $BR_i(x^{*,M}_{-i}) \in X_i(x^{*,M}_{-i}) \backslash K_i(x^{*,M}_{-i})$). As a result, (at least) one bank does not comply with its constraint.

Numerical example. To understand our concepts in a simple framework, let us consider the case of two banks A and B in which each bank can either resell a "small" portion, a "medium" portion, or a "large" portion of each asset. Assume more specifically that 20% is the small portion, 40% is the medium portion and 70% is the large portion. As in the general model, each bank is assumed to be exposed to totally illiquid assets (loans) and to tradable assets. We assume that there are two tradable assets, asset 1 and 2, and we make the further assumption that asset 1 is perfectly liquid while asset 2 is imperfectly liquid. Assuming linear price impact, this means that the market Φ_2 is finite. Within our example, $\Phi_2 = 3000$. For simplicity, prices are normalized to one so that the value is equal to the quantity. Consider the balance sheets of the two banks A and B.

Bank A		**Bank B**	
Assets	Liabilities	Assets	Liabilities
$V_{0,a} = 80$ (credit) $V_{1,a} = 60$ (market) $V_{2,a} = 80$ (market)	$E_a = 10$	$V_{0,b} = 65$ (credit) $V_{2,b} = 30$ (market)	$E_b = 4.7$
$A_a = 220$	220	$A_b = 95$	95

From these two balance sheets, one can see that bank A is exposed to asset 1 and 2 while bank 2 is only exposed to asset 2. From a regulatory point of view, for each bank, the credit risk weight of the loans is equal to $\alpha_0 = 0.5$ while it is equal to $\alpha_2 = 0.6$ for the asset 2. The risk weight of the liquid asset is equal to $\alpha_1 = 0.2$. We also know that the required capital ratios are $\theta_{a,min} = 9\%$ and $\theta_{b,min} = 8\%$. Before the shock, the risk-based capital ratio of bank A is equal to $\theta_a = 10\%$ while the risk-based capital ratio of bank B is equal to $\theta_b = 9.3\%$. Each bank complies with the regulatory risk-based capital ratio.

Assume now that loans (and only loans) are hit with a shock, that is $\Delta = (\Delta_0, 0, 0)$. Assume that $\Delta_0 = 2\%$, which may lead to a large loss (in currency). After the shock, the value of the loans is equal to $V_{0,a} = 78.4$ for bank A and is equal to $V_{0,b} = 63.7$ for bank B. The capital ratios are respectively equal to $\theta_a(\Delta_0) \approx 8.47\%$ and $\theta_b = 6.82\%$, which means that each bank fails to comply with the regulatory capital ratio.

By assumption, under a macroprudential constraint, as long as K is not empty, bank A can not choose a deleveraging strategy so that bank B would not comply with its regulatory constraint. Table 2.6 provides the overall picture of what can happen. For instance, in the first cell, bank B sells 20% of asset 2 while bank A sells 20% of asset 1 and 2. For such deleveraging strategies, the cost for bank A is equal to 28 and the resulting capital ratio is equal to 8.98%. For bank B, the cost is equal to 6 and its resulting capital ratio is equal to 6.89%. Overall, the total cost is equal to 34.

From table 2.6, it is easy to see that there is a *unique* profile of strategies such that both banks comply with their regulatory capital ratio, which means that the Nash equilibrium under macroprudential constraint is unique. This profile of strategies corresponds to the cell (in blue in table 2.6) in which bank A sells 70% and 20% of asset 1 and 2 respectively and ends up with a capital ratio equal to 9.18% while bank B sells 70% of asset 2 and ends up with a capital ratio equal to 8.14%. The total cost for bank A is equal to 58.

Is this Nash equilibrium under macroprudential constraint microprudentially incentive-compatible when $\theta_{a,min} = 9\%$ and $\theta_{b,min} = 8\%$? The answer is negative. Knowing that bank B sells 70% of asset 2, the best response of bank A is to sell 20% of asset 1 and 40% of asset 2 and the resulting capital ratio and costs of bank A are equal to 9.06% and 44 respectively. Without any macroprudential regulation, when bank B sells 70% of asset 2, bank A is able to comply with the regulatory constraint for a cost of 44 instead of 58 with a macroprudential constraint. It thus follows that the profile of strategies $x = (x_a = (70\%, 20\%); x_b = 70\%)$ is *not* a Nash equilibrium when there is no macroprudential constraint. It should be pointed out that this non-existence of a Nash equilibrium in this example critically depends upon the required capital ratios.

Assume now that $\theta_{a,min} = 8.5\%$ and $\theta_{b,min} = 8\%$. From table 2.6, only three profile of strategies x are in K and in both cases, bank B sells once again 70%. The first one is when bank A sells 20% of each asset for a total cost equal to 28 and the resulting capital ratio is equal to 8.54%. Since the capital ratio of bank B is equal to 8.14%, both banks comply with their regulatory constraint. The second and third profile of strategies (in which both banks comply with their constraints) is when

		Bank B					
		20%		40%		70%	
Bank A	(20%,20%)	0.08989	0.06891	0.08813	0.07333	0.08548	0.08149
		28	6	28	12	28	21
		34		40		49	
	(40%,20%)	0.09245	0.06891	0.09063	0.07333	0.08791	0.08149
		40	6	40	12	40	21
		46		52		61	
	(70%,20%)	0.09656	0.06891	0.09467	0.07333	0.09183	0.08149
		58	6	58	12	58	21
		64		70		79	
	(20%,40%)	0.09564	0.06556	0.09364	0.06966	0.09063	0.07724
		44	6	44	12	44	21
		50		56		65	
	(40%,40%)	0.09871	0.06556	0.09664	0.06966	0.09354	0.07724
		56	6	56	12	56	21
		62		68		77	
	(70%,40%)	0.1037	0.06556	0.10153	0.06966	0.09828	0.07724
		74	6	74	12	74	21
		80		86		95	
	(20%,70%)	0.1073	0.0605	0.10476	0.06414	0.10101	0.07087
		68	6	68	12	68	21
		74		80		89	
	(40%,70%)	0.1115	0.0605	0.10892	0.06414	0.10502	0.07087
		80	6	80	12	80	21
		86		92		101	
	(70%,70%)	0.1186	0.0605	0.11581	0.06414	0.11168	0.07087
		98	6	98	12	98	21
		104		110		119	

Figure 2.6: Two banks example with three rebalancing strategies

bank A chooses either to sell 40% of asset 1 and 20% of asset 2 for a cost equal to 40 or to sell 70% of asset 1 and 20% of asset 2 for a cost equal to 58. For bank A, the cheapest cost is when it sells 20% of each asset, which means that the profile of strategies $x = (x_a = (20\%, 20\%); x_b = 70\%)$ is the unique Nash equilibrium under macroprudential constraint that is also microprudentially incentive-compatible.

Consider now the more realistic case in which each bank i is able to choose a quantity to resell a percentage of asset j $x_{ij} \in \{0\%, 1\%, 2\%, ..., 99\%, 100\%\}$. Using the numerical values as above with $\theta_{a,min} = \theta_{b,min} = 9\%$, we found numerically that the Nash equilibrium under macroprudential constraint is the profile of strategies $x^{*,M} = (x_a^{*,M}, x_b^{*,M}) = ((72\%; 25\%), 95\%)$. Since $BR_a(95\%) = (0\%; 57\%)$, such a Nash equilibrium is not microprudentially incentive-compatible since when bank a chooses to sell 57% of asset 2, bank b does not anymore comply with its constraint.

Policy implications. The foundation of macroprudential regulation is related to the prevention of negative externalities. When K is not empty, an equilibrium under macroprudential regulation that minimizes the total asset sale always exists. Depending upon the situation, this equilibrium prevents some banks to delever in a way that would be detrimental to other banks. By definition, under macroprudential regulation, a bank i is not allowed to delever in a way that a bank k would not anymore comply with its own regulatory constraint. Formally, since $x \in K$ by definition, given x_{-i}, bank i can not choose x_i such that $\theta_k(x_i, x_{-i}) < \theta_{k,min}$. Assume, as in the previous example, that bank a owns perfectly and imperfectly liquid assets while bank b only owns imperfectly liquid assets.

After an adverse shock, if bank a finds it cheaper to delever by selling imperfectly liquid assets, the asset price drop may be sufficiently large to generate the insolvency of bank b at equilibrium. Macroprudential regulation prevents such a failure that may arise due to such negative externality related to fire sales.

To analyze more deeply such a situation, let us build on the example given with two banks[10]. We consider the simplest case of two banks a and b in which the trading book contains at most two securities (assets) with different market liquidity.

- Security 1 is perfectly liquid (no price impact).
- Security 2 is imperfectly liquid (positive price impact).

As before, the banking book (loans) of each bank is assumed to be perfectly illiquid, without any resale value in the short-run. Let $\Delta := (\Delta_0, 0, 0)$ be a small to medium shock in the banking book only and Δ is such that for each $i \in \{a, b\}$, $\Delta \in \mathcal{Z}_i^{sale}$. Seen from 2021, such an adverse shock can be interpreted as a consequence of Covid 19 since the lock-down has sharply increased the cost of risk of banks. We make the assumption that after such a small to medium shock, neither bank a nor bank b comply with their regulatory capital ratio.

For the sake of financial interest, banks are assumed to be heterogeneous in terms of diversification, that is, bank b is *less diversified* than bank a. Concretely, bank b is only long security 2 while bank a is long security 1 and security 2. Since bank b has no position in security 1, the only way to delever is to sell asset 2. The situation is different for bank a since it can sell both security 1 and security 2, which means that bank a will always be in a position to increase its capital ratio by selling security 1 since there is no price impact. Throughout the discussion, we make as before the assumption that K is not empty. From proposition 9, we know a Nash equilibrium in macroprudential constraint $x^{*,M}$ that minimizes the total value of the resale exists and may or may not be microprudentially incentive-compatible. Such an incentive-compatible property is a fairly complex function of all the parameters of the models, the regulatory weights, the capital requirements, the positions on securities, the market depth etc...To now understand why, as a function of the market illiquidity of security 2, macroprudential regulation is required, let us make the following assumptions.

[10]This paragraph is based on computations that are not reproduced in the paper to focus on policy implication. Computations are available upon request.

1. If market of security 2 were perfectly liquid, bank b would be in a position to restore its capital ratio.

2. Bank a is in a position to restore its capital ratio by selling only security 1, that is, there exists $\overline{x}_{a1} < 1$ such that $\theta_a(\overline{x}_{a1}, 0) = \theta_{a,min}$.

To facilitate the discussion, assume that the price impact is linear for security 2 and let $\Phi_2 := \Phi$ be the market depth. From the above assumptions, since the capital ratio is a continuous function of the market depth, there exists a critical market depth (high enough) $\overline{\Phi}$ such that bank b will be able to restore its capital ratio independently of bank a. In such a situation, macroprudential regulation is not required. Assume now that the market depth for security 2 is low enough so that bank b, even alone, is not able to restore its capital ratio due to its own price impact. Let $\underline{\Phi}$ be this market depth[11]. Within such a framework, it thus follows from the discussion that as long as $\Phi \notin [\underline{\Phi}, \overline{\Phi}]$, macroprudential regulation is not justified. Consider now the situation in which $\Phi \in [\underline{\Phi}, \overline{\Phi}]$. Without macroprudential regulation, bank a may find cheaper (say because the regulatory weight of security 1 is much lower than the weight of security 2) to restore its capital ratio by essentially selling security 2. But such a decision may adversely impact bank b which might not be able to restore its capital ratio. In the worse scenario, given that bank b is insolvent (i.e., it must liquidate its position, $x_{b2} = 1$), bank a may still find cheaper to essentially sell security 2. Without macroprudential regulation, only bank a is solvent at equilibrium. On the contrary, with a macroprudential regulation, both bank will be solvent because bank a will not anymore have the possibility to choose the cheapest solution to delever. As a result, the externality related to fire sales disappears. We summarize the above discussion in the following policy implication fact.

Fact. *To prevent failures externalities, macroprudential regulation is justified for "intermediate" price impact but is unnecessary when the price impact is either low or high enough.*

In Basel III, banks are not only constrained to have idiosyncratic capital surcharge(s) that may significantly increase their Tier 1 capital but they now also have to maintain two types of liquidity ratios, called LCR and NSFR, that is, banks must have *sufficient high-quality liquid assets (HQLA) to survive a significant stress scenario lasting for 30 days*[12]. While these liquidity ratios have been introduced to avoid (funding) liquidity problems generated by maturity transformation, as observed during the subprime crisis, such liquidity ratios may also play an important role within

[11] It is easy to compute this critical market depth. For the failure of bank b, it suffices to compute $\underline{\Phi}$ such that $E_b(\Delta, x_{b2} = 1, \underline{\Phi}) = 0$. Note that since $x_{b2} = 1$, the capital ratio is undefined.

[12] See BCBS (2014), "Basel III: the net stable funding ratio", p. 1

our framework. By forcing banks to invest in liquid assets, banks are also more resilient after a shock since they can delever using these highly liquid assets. In that sense, liquidity ratios can be thought of as macroprudential instruments.

6 Severe shocks, cascade of failures and Nash equilibrium analysis

6.1 What happens if K is empty ? Forget about the conservation buffer!

Up to now, we made the assumption that K is not empty everything else equal, that is, given the positions of the banks but also the price impact functions and the capital requirement. A striking difference between Basel II and Basel III is precisely the capital requirements. In Basel III, the basic uniform Tier 1 capital requirement is 6% (of the (total) risk-weighted assets). However, all banks must comply with the capital conservation buffer (equal to 2.5%) and the countercyclical buffer among others and systemic banks must moreover comply with the GSIB buffer so the overall tier one capital requirement[13] in Basel III can be higher than 10%. Let β_{ih} be the buffer h for Tier 1 capital of bank i and et $\sum_{h \in H} \beta_{ih} := \beta_i$ be the total capital surcharge. The Tier 1 capital of a given bank i thus can be in general written as

$$\theta_{i,min} = 6\% + \beta_i \tag{2.37}$$

As noted on the website of the Bank for International settlements (BIS), the capital conservation buffer is supposed *to ensure that banks have an additional layer of usable capital that can be drawn down when losses are incurred*[14]. During a period of stress, say after a systemic shock, when K is empty, it makes no sense to require the capital conservation buffer of 2.5% so that the overall tier one capital requirement is equal to

$$\theta_{i,min} = 3.5\% + \beta_i \tag{2.38}$$

When banks are allowed to forget about the capital conservation buffer, K may be not empty and this means that we are back to what we did before. If K is still empty, then, under macroprudential constraints, failures can not be avoided, which means that the adverse shock is severe. In such a situation, that is, for some $i \in \mathcal{B}$, $\Delta \in \mathcal{Z}_i^{fail}$, some banks are insolvent right after the shock. Put it differently, given the shock size and/or the severity of the price impact, a subset of banks may be

[13] Note that some buffers such as the conservation buffer applies to CET 1 capital ratio.
[14] See the BIS document as of 2019 entitled "The capital buffers in Basel III – Executive Summary".

unable to satisfy their microprudential constraint given in equation (2.28). We thus have to extend the microprudential constraint X_i to the case in which bank i is insolvent.

6.2 Extended microprudential constraint, cascade of failures and Nash equilibrium

Given x_{-i}, let $\overline{X}_i(x_{-i})$ defines the extended microprudential constraint.

$$\overline{X}_i(x_{-i}) = \begin{cases} X_i(x_{-i}) & \text{when } X_i(x_{-i}) \neq \emptyset \\ (1, 1, ..., 1) := \mathbf{1} & \text{when } X_i(x_{-i}) = \emptyset \end{cases} \tag{2.39}$$

When for some $i \in \mathcal{B}$, $\Delta \in \mathcal{Z}_i^{fail}$ so that $X_i(x_{-i})$ is empty, these banks are insolvent and thus must be liquidated. They must thus sell 100% of their assets, that is, $x_i = (1, 1, ..., 1)$. From a resolution point of view, this means that there is no regulatory attempt to assist bank(s) in financial distress[15] (e.g., bail out) but also that there are, as already said, financial institutions such as solvent banks, hedge funds, institutional investors... that have once again the required financial muscles to purchase the assets sold at a discount ([Acharya and Yorulmazer, 2008]). Note that the extended microprudential constraint is the extension of [Braouezec and Wagalath, 2019] with $p \geq 2$ assets. Such an extension has also been considered recently in [Banerjee and Feinstein, 2021] and they show the existence of Nash equilibria in proposition 3.7.

When for some banks $i \in \mathcal{B}$, the shock $\Delta \in \mathcal{Z}_i^{fail}$, these banks are insolvent and must liquidate their assets. In a world without price impact, as long as $V_{i0} > E_i$ (see proposition 7 part 3), all the other banks are able to restore their regulatory capital ratio. However, this remains unclear when the price impact is positive. When these banks that are insolvent right after the initial shock sell their assets, this will depress the price of each asset and thus the capital of solvent banks. It may thus be the case that after these liquidations, a new subset of banks becomes insolvent and so on and so forth. Instead of directly considering the Nash equilibrium in which all banks are participating to the deleveraging process (as in [Braouezec and Wagalath, 2019]), we here disentangle the "pure" liquidation process (i.e., only banks that are insolvent sell their assets) from the deleveraging process of banks that are solvent after the liquidation process (i.e., banks that are still solvent after the

[15] In the European Union, since the 2014 banking unions, such a bank resolution procedure exists. We refer the reader to the excellent textbook of [Freixas et al., 2015] (paragraph 8.4.3) for an interesting and exhaustive discussion on the bank resolution procedure, see also [Acharya and Yorulmazer, 2007], [Acharya and Yorulmazer, 2008], [Acharya et al., 2011]

first stage and that sell a portion of their assets in order to remain solvent at equilibrium). The two-stage is as follows.

1. **Pure liquidation process** (see appendix B). Banks that are insolvent after the shock sell all their assets. If after this liquidation, some additional banks are insolvent, these banks also sell all their assets and so on and so forth.

2. **Nash equilibrium of the deleveraging game**. We look at the Nash equilibrium of this deleveraging game for banks that are still solvent after the liquidation process.

Recall that for the initial shock Δ, some banks are insolvent. By assumption, they liquidate all their asset and this will contribute to decrease the price of each asset j. Let $\Delta^{(1)} = (\Delta_1^{(1)}(.), ..., \Delta_n^{(1)}(.))$ be the implied shock after the first round of liquidation. If there are additional banks that are insolvent for the implied shock $\Delta^{(1)}$ (but were solvent for Δ), denoted by $F^{(1)}$, then there is a second round of liquidation and so on and so forth. Our liquidation process is in the spirit similar to the cascade of bankruptcies considered in many networks papers, e.g., [Amini et al., 2016], [Bernard et al.,], [Detering et al., 2021], [Caccioli et al., 2014] to quote few papers, see also [Jackson and Pernoud, 2021] or [Glasserman and Young, 2016] for insightful review papers. In appendix C, we describe precisely the algorithm and we show that this liquidation process ends after l liquidation rounds, with $l \leq p$.

Assume now that the liquidation process stops after $l < p$ rounds and that there is still a subset of banks that are solvent. The implied shock is equal to $\Delta^{(l)} = (\Delta_1^{(l)}(.), ..., \Delta_n^{(l)}(.))$. By definition, at round l, there are no more failure (i.e., $F^{(l)} = \emptyset$). Let $\Delta^{(l)} := \Delta_{Liq}$. Let $\mathcal{S}$ be the subset of solvent banks after the liquidation process (with a positive capital ratio).

$$\mathcal{S} = \{i \in \mathcal{B} : \Delta_{Liq} \notin \mathcal{Z}_i^{fail}\} \tag{2.40}$$

Note interestingly that the situation in which the set of solvent banks in $\mathcal{S}$ are hit by a shock Δ_{Liq} is equivalent to the initial situation when the set of solvent banks is $\mathcal{B}$ hit by a small to medium shock Δ. Since a number of banks have been liquidated, we only consider the set of banks $\mathcal{S} \subset \mathcal{B}$ as defined in equation (2.40). Let

$$K' = \{x \in \mathcal{E} : \forall i \in \mathcal{S}, x_i \in X_i(x_{-i})\} \tag{2.41}$$

When K' is not empty, we are back to the previous analysis and a Nash equilibrium under macroprudential constraint that minimizes the total value of the asset sale will exist. Assume now that due to the severity of the price impact, K' is empty. In such a situation, abstracting from the capital conservation buffer, one must now consider the set

$$\overline{K}' = \{x \in \mathcal{E} : \forall i \in \mathcal{S}, x_i \in \overline{X}_i(x_{-i})\} \tag{2.42}$$

By definition, since K' is empty, a number of solvent banks after the shock Δ_{Liq} will be insolvent after the deleveraging process. In the next result, we show that a Nash equilibrium exists.

Proposition 10 *Let Δ be a severe shock. There exists a Nash equilibrium under macroprudential regulation $x^{*,M} \in \overline{K}'$ such that $V(x^{*,M}) := \sum_{i \in \mathcal{B}} f_i(x_i^{*,M})$ is minimized on $\overline{K}'$ (Pareto optimality) and such that for each $i \in \mathcal{B}$, either $\theta_i(x_i^{*,M}, x_{-i}^{*,M}) \geq \theta_{i,min}$ or $x_i^{*,M} = 1$.*

Proof. See the appendix.

From a regulatory point of view, the "best" criterion to be used in such a situation of failures remains unclear. Should one minimize the number of failures or should one minimize the total value of the asset sales? This raises new issues that are beyond the scope of the paper.

7 Cascade of failures: application to French systemic banks

In this section, to illustrate how our model can be calibrated on data, we focus on the four French GSIBs as of 2020; BNP Paribas, BPCE, Crédit Agricole and Société Générale. Our approach boils down to manually collecting data and consists in extracting from the annual report the relevant disclosed quantities. The calibration of the model to real data raises however new issues as some inputs are not directly disclosed in annual reports, which means that one must find proxies. As in [Braouezec and Wagalath, 2018], we need to recover the banking book, the trading book and their associated risk weights.

7.1 Descriptive statistics and calibration methodology

Let θ and L respectively be the observed risk-capital capital ratio and leverage ratio as of December 2020. In the following table, except the total exposure, all the quantities come from annual reports. Except ratios, all the quantities are expressed in billion euros.

Risk-based capital ratio and leverage ratio (December, 2020)

Bank	Tier 1	RWA	θ	θ_{min}	E_{xp}	L	L_{min}	A	$\frac{E_{xp}}{A}$
BNP Paribas	98.8	695.52	0.142	0.1096	2266.86	0.0436	0.03	2488.49	0.915
Société Générale	56.18	351.85	0.160	0.1052	1188.5	0.047	0.03	1461.9	0.813
Crédit Agricole	50.02	336.04	0.149	0.0964	1861.5	0.027	0.03	1861	0.95
BPCE	68.98	431.22	0.160	0.12	1374.3	0.050	0.03	1446.26	0.95

In the above table, the total exposure reported comes from the banks individual templates available on the website of European Banking Authority[16] (EBA). Few remarks are in order.

- The total exposure incorporates on balance sheets items but also off balance sheets items. It is somehow surprising that this total exposure E_{xp} is always *lower* than the total value of the assets A, although fairly close to it. This means that A is a fairly good approximation of E_{xp}.
- As opposed to few years ago, the minimum capital required for each bank depends upon its own characteristics. Since the capital conservation is equal to 2.5% in Basel III, one can clearly see that the minimum required for Tier 1 risk-based capital ratio is higher than 8.5%, which means that these banks are subject to additional capital surcharges such as the GSIBs buffer.

Consider a given bank i and let $V_{i,BB}$ and $V_{i,TB}$ be the value of the banking and trading book and let and $\alpha_{i,BB}$ $\alpha_{i,TB}$ be their respective risk weight. We consider cash, denoted v in the model, as a separate item since it does not entail any capital. It thus follows that for each bank i, we have

$$A_i = v_i + V_{i,BB} + V_{i,TB} \tag{2.43}$$

In the same vein, the (total) risk-weighted assets RWA_i is equal to the risk-weighted asset of the banking book plus the risk-weighted assets of the trading book for each bank i, that is

$$\text{RWA}_i = \text{RWA}_{i,BB} + \text{RWA}_{i,TB} \tag{2.44}$$

Once $V_{i,BB}, V_{i,TB}$ and $\text{RWA}_{i,BB}, \text{RWA}_{i,TB}$ are recovered, the two risk weights $\alpha_{i,BB}$ and $\alpha_{i,TB}$

[16]See https://www.eba.europa.eu/risk-analysis-and-data/global-systemically-important-institutions. The reason lies in the fact that for some banks, Crédit Agricole, the total exposure reported in the annual report differs from the one found on the website of the EBA by 800 billion. This also means that the leverage ratio is not around 5%, as stated in the annual report, but around 2.8%.

are also known since

$$\text{RWA}_{i,BB} = \alpha_{i,BB} V_{i,BB} \qquad \text{RWA}_{i,TB} = \alpha_{TB} V_{i,TB} \tag{2.45}$$

Banking versus trading book. To construct the banking book and the trading book from the consolidated balance sheet, we make the assumption that the banking book is subject to credit risk while the trading book is subject to market risk and counterparty risk. For the banking book, we make the assumption that it is equal to the financial assets at amortized costs (loans to customers and to credit institutions). From equation (2.43), the trading book thus is equal to the total value of the assets minus the banking book (loans) and the cash. We thus implicitly incorporate in the trading book a number of items such as intangible assets and goodwill that are not traded assets and which might be difficult to resell in the short-run. Fortunately, as their value is small in percentage, around 1% or 2% of the total assets, this assumption is innocuous.

Partial risk-weighted assets assignment. The credit risk-weighted assets, by far the most important of all the risk-weighted assets, is assigned to the banking book risk-weighted assets. The securitization exposures is also assigned to the banking book risk-weighted assets. On the contrary, the market risk-weighted assets is assigned to the trading book and we decided to also assign the counterparty risk-weighted assets to the trading book. The difficulty concerns the operational risk-weighted assets and the settlement risk-weighted assets (although this last quantity is negligible). To reflect the fact that operational risk is both present in the banking book and in the trading book, we assign the operational risk-weighted assets as a proportion of the credit risk-weighted assets divided by the market risk-weighted assets plus the credit risk-weighted assets.

Proxied quantities as of December, 2020, (Phased In)

Bank	v	V_{BB}	V_{TB}	RWA_{BB}	RWA_{TB}	α_{BB}	α_{TB}	α_{Avg}
BNP Paribas	308.7	946.8	1232.96	625.32	70.2	0.660	0.057	0.319
Société Générale	168.18	502.14	791.6	306.63	45.22	0.611	0.057	0.272
Crédit Agricole	194.3	953.9	812.9	302.79	33.25	0.317	0.041	0.190
BPCE	153.4	836.82	456	402.74	28.48	0.481	0.062	0.334

One can clearly see from the table that the banking book risk weight is much higher than the trading book risk weight. While this in part depends upon our methodology, it essentially follows from the fact that the credit risk-weighted assets is by far the most important. Its contribution in the total risk-weighted assets varies from 70% (Société Générale) to 81% (BPCE), and this explains why the the banking book risk weight is much higher than the trading book risk weight. We also report the usual average risk weight α_{Avg}, used by regulators to calibrate the leverage ratio, defined here as the risk-weighted assets (RWA) divided by the total assets minus cash $(A - v)$.

7.2 Cascades of failures as a function of the severity of the stress test

We now consider the cascade of failures that may result after a given shock in the banking book, given a price impact, measured by Φ. We call a price impact of x% if, when all banks sell 100% of their trading book, the price decreases by x%. Overall, the severity of the stress test both depends upon the shock Δ and the price impact measured by x, which implicitly depends on the market depth Φ. Since it is difficult to assess the accuracy of the market depth Φ, in part because the trading book contains very different type of assets, this parameter can be actually seen as a measure of the severity of the stress test, just like the shock. Recall that when one bank fails, it sells 100% of its assets and this may generate new failures and so on and so forth.

Linear price impact: 1%.

- $\Delta = 6\%$: Crédit Agricole fails and there is no cascade of failures.
- $\Delta = 7\%$: idem
- $\Delta = 8\%$: idem
- $\Delta = 9\%$: Crédit Agricole and BPCE fail and there is no cascade of failures.

Linear price impact: 2%.

- $\Delta = 6\%$: Crédit Agricole fails and there is no cascade of failures.
- $\Delta = 7\%$: idem
- $\Delta = 8\%$: Crédit Agricole fails and BPCE fails after the liquidation of Crédit Agricole.
- $\Delta = 9\%$: Crédit Agricole and BPCE fail and there is no cascade of failures.
- $\Delta = 9.5\%$: Crédit Agricole and BPCE fail. BNP Paribas fails after their liquidation. Société Générale fails after the liquidation of BNP Paribas.

It is interesting to note that for a shock of 9.5%, Crédit Agricole and BPCE fail. After liquidation, BNP Paribas fail and Société Générale in turn fails after the liquidation of BNP.

Linear price impact: 4%**.**

- $\Delta = 6\%$: Crédit Agricole fails and there is no cascade of failures.
- $\Delta = 7\%$: idem
- $\Delta = 8\%$: Crédit Agricole fails and BPCE fails after the liquidation of Crédit Agricole.
- $\Delta = 9\%$: Crédit Agricole and BPCE fail and BNP Paribas and Société Générale fail after the liquidation.

Consider for instance the case in which the price impact is 2% and the shock is $\Delta = 8\%$. We know that Crédit Agricole fails and BPCE fails after the liquidation of Crédit Agricole. Since BNP Paribas and Société Générale are still solvent (with positive capital), we then consider the Nash equilibrium. It turns out that these two banks also fail at equlibrium.

It would certainly be interesting to consider the model with at least two risky assets in the trading book. However, due to the difficulty to obtain the various items of the trading book and the difficulty to estimate the price impact, it is our belief that such a comprehensive empirical analysis deserves a full paper.

8 Conclusion

We offer in this paper a new framework in which we make a clear distinction between microprudential and macroprudential regulation. We show that a Pareto optimal Nash equilibrium generically exists under macroprudential regulation while such an existence result under microprudential regulation requires a set of strong conditions. Our results clearly suggest to consider macroprudential regulation as the natural benchmark. An interesting aspect of our framework is that most of the parameters can be calibrated in a fairly easy way. While many theoretical extensions of our model could be done, it is our belief that an interesting and promising work would be to offer a more complete empirical analysis.

9 Technical proofs

Proof of proposition 7. Proof of part 1. When there is no price impact, the total capital of a bank i $E_{i,t+1}(.)$ given in equation (2.18) is invariant with respect to x_i while the risk-weighted assets $\text{RWA}_{i,t+1}(.)$ given in equation (2.19) is (for each j but also for each i) a decreasing function of x_{ij}. As a result, the risk-based capital ratio is an increasing function of each x_{ij} $\square$

Proof of part 2. Consider bank i and assume that for all $j \neq k$, $\alpha_{ij} \neq \alpha_{ik}$. One thus can define a permutation such that the asset index are ordered such that $j_1 > j_2 >,.., > j_n$ implies $\alpha_{ij_1} > \alpha_{ij_2} > ... > \alpha_{ij_n}$ (the case in which $\alpha_{ij_1} \geq \alpha_{ij_2} \geq ... \geq \alpha_{ij_n}$ will be discussed later on).

Claim 1 *Consider bank i and in addition to assuming $\Delta \in \mathcal{Z}_i^{sale}$ and no price impact, assume further that $V_{i0} = 0$. Then, there always exists an optimal strategy for bank i to restore its capital ratio back above the minimum required $\theta_{i,min}$. It is optimal for bank i to first sell a portion of the asset j_1 with the highest risk weight. If this is not enough to restore the regulatory capital ratio, it is optimal for the bank to sell 100% of the risky asset j_1 and a portion of risky asset j_2. If selling 100% of the asset j_1 and j_2 is not enough to restore the capital ratio, it is optimal to sell a portion of asset j_3 and so on and so forth.*

Proof of claim 11. For the optimal $(x_{i1}, ..., x_{in})$ it is clear that the constraint is binding, that is $\theta_{i,t+1}(x_i) = \theta_{min}$, that is,

$$\frac{E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j}{\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1 - \Delta_j)(1 - x_{ij})} = \theta_{min} \tag{2.46}$$

which implies that:

$$\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1 - \Delta_j) x_{ij} = \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1 - \Delta_j) - \frac{1}{\theta_{min}} (E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j) \tag{2.47}$$

If we rename $X_{ij} = q_{ij} P_j (1 - \Delta_j) x_{ij}$, and $K_i = \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1 - \Delta_j) - \frac{1}{\theta_{min}} (E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j)$, this is equivalent to:

$$\sum_{j=1}^{n} \alpha_{ij} X_{ij} = K_i \tag{2.48}$$

Each bank i seeks to minimize $\sum_{j=1}^{n} X_{ij}$. Therefore, it is optimal to start by selling the asset j_1 with the highest risk weight α_{i,j_1}, then asset j_2, ..., until asset j_k such that the capital ratio is restored $\square$

Remark. In the event of two risky assets j_k and j_{k+1} with the same risk weights, it is equivalent to sell one or the other first, or both at the same time. □

Proof of part 3. Assume now that there are illiquid asset (loans). Since $\Delta \in \mathcal{Z}_i^{sale}$, the total capital after the shock is positive, that is, $E_i - \Delta_0 V_{i0} - \sum_{j=1}^n q_{ij} P_j \Delta_j > 0$, which is equivalent to $\Delta_0 < \Delta_c := \frac{E_i - \sum_{j=1}^n q_{ij} P_j \Delta_j}{V_{i0}}$. Assume now that the bank resell 100% of the risky assets $j = 1, 2, ...n$, that is, $x_i = \mathbf{1}$. The risk-based capital ratio thus is equal to $\theta_i(\mathbf{1}) = \frac{E_i - \Delta_0 V_{i0} - \sum_{j=1}^n q_{ij} P_j \Delta_j}{\alpha_{i0} V_{i0}(1-\Delta_0)}$. If $\theta_i(\mathbf{1}) > \theta_{i,min}$, the bank will be able to restore its capital ratio. It is easy to show that $\theta_i(\mathbf{1}) > \theta_{i,min}$ is equivalent to $\Delta_0 < \Delta_i^c := \frac{E_i - \sum_{j=1}^n q_{ij} P_j \Delta_j - \alpha_{i0} V_{i0} \theta_{i,min}}{V_{i0}(1-\alpha_{i0}\theta_{i,min})}$ also easy to show that $\Delta_i^c > \Delta_{i,c}$ is equivalent to $V_{i0} > E_i - \sum_{j=1}^n q_{ij} P_j \Delta_j$ so that bank i will be able to restore its capital ratio. Since $V_{i0} > E_i$, $\Delta_0 < \Delta_{i,c} < \Delta_i^c$ so that bank i will always be in a position to restore its capital ratio □

Proof of lemma 1

Assume that $X_i(x_{-i}) \neq \emptyset$ and note that P_j is the price of asset j at time t (before the shock). We have:

Proof of part 1. Given x_{-i}, the capital ratio of bank i is equal to

$$\theta_i(x_i, x_{-i}) = \frac{E_{i,t} - q_{i0} P_0 \Delta_0 - \sum_{j=1}^n q_{ij} P_j (\Delta_j + \frac{\sum_{k \in \mathcal{B}} x_{kj} q_{kj}}{\Phi_j}(1 - \Delta_j))}{\alpha_{i0} q_{i0} P_0 (1 - \Delta_0) + \sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1 - \Delta_j) \left(1 - \frac{\sum_{k \in \mathcal{B}} x_{kj} q_{jk}}{\Phi_j}\right)(1 - x_{ij})}$$

By definition, $X_i(x_{-i})$ is given by the set of points $x_i \in [0,1]^n$ such that: $\theta_i(x_i, x_{-i}) \geq \theta_{i,min}$. It is easy to show that $\theta_i(x_i, x_{-i}) \geq \theta_{i,min}$ is equivalent to equation (2.49) lower (or equal) than equation (2.50):

$$\left(\alpha_{i0} q_{i0} P_0 (1 - \Delta_0) + \sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1 - \Delta_j) \left(1 - \frac{\sum_{k \in B} x_{kj} q_{kj}}{\Phi_j}\right)(1 - x_{ij})\right) \theta_{i,min} \tag{2.49}$$

$$E_{i,t} - q_{i0} P_0 \Delta_0 - \sum_{j=1}^n q_{ij} P_j (\Delta_j + \frac{\sum_{k \in B} x_{kj} q_{kj}}{\Phi_j}(1 - \Delta_j)) \tag{2.50}$$

which, given x_{-i}, is in turn equivalent to:

$$R_1(x_{i1}) + R_2(x_{i2}) + ... + R_n(x_{in}) \leq C \tag{2.51}$$

with

$$C = E_{i,t} - q_{i0}P_0\Delta_0 - \sum_{j=1}^{n} q_{ij}P_j(\Delta_j + \frac{\sum_{k\neq i} x_{kj}q_{kj}}{\Phi_j}(1-\Delta_j)) - \alpha_{i0}q_{i0}P_0(1-\Delta_0)$$
$$- \sum_{j=1}^{n} \alpha_{ij}q_{ij}P_j\,(1-\Delta_j)\left(1 - \frac{\sum_{k\neq i} x_{kj}q_{kj}}{\Phi_j}\right)\theta_{i,min} \quad (2.52)$$

and for all $j \in [1,..,n]$, $R_j(x_{ij})$ a polynomial of degree 2 with a positive leading coefficient of the form: $R_j(x_{ij}) = d_{ij} \times (x_{ij})^2 + e_{ij} \times (x_{ij})$

with for all $j \in [1,..,n]$,

$d_{ij} = \alpha_{ij}q_{ij}P_j\,(1-\Delta_j)\,\frac{q_{ij}}{\Phi_j}\theta_{i,min} > 0$

$e_{ij} = \alpha_{ij}q_{ij}P_j\,(1-\Delta_j)\,[-\left(1 - \frac{\sum_{k\neq i} x_{kj}q_{kj}}{\Phi_j}\right) - \frac{q_{ij}}{\Phi_j}]\theta_{i,min} + q_{ij}P_j\frac{q_{ij}}{\Phi_j}(1-\Delta_j)$

This can be rewritten: $R_j(x_{ij}) = d_{ij} \times (x_{ij} + \frac{e_{ij}}{2d_{ij}})^2 - \frac{e_{ij}^2}{4d_{ij}}$.

And (2.51) is equivalent to:

$$\sum_{j=1}^{n} d_{ij} \times (x_{ij} + \frac{e_{ij}}{2d_{ij}})^2 \leq C + \sum_{j=1}^{n} \frac{e_{ij}^2}{4d_{ij}} \quad (2.53)$$

If we denote $C' = C + \sum_{j=1}^{n} \frac{e_{ij}^2}{4d_{ij}}$, this is equivalent to:

$$\sum_{j=1}^{n} \frac{(x_{ij} + \frac{e_{ij}}{2d_{ij}})^2}{\frac{C'}{d_{ij}}} \leq 1 (2.54)$$

And this is equivalent to:

$$\sum_{j=1}^{n} \frac{(x_{ij} - c_{ij})^2}{(a_{ij})^2} \leq 1 \quad (2.55)$$

with $c_{ij} = \frac{e_{ij}}{2d_{ij}}$ and $a_{ij} = \sqrt{\frac{C'}{d_{ij}}} > 0$. Since the equation of a canonic ellipsoid in dimension n is given by $\sum_{j=1}^{n} \frac{x_j^2}{a_j^2} \leq 1$, it thus follows that equation (2.55) is the equation of an n-dimensional ellipsoid and this concludes the proof of part 1 □

Proof of part 2. Bank i aims to minimize $f_i(x_i) = \sum_{j=1}^{n} x_{ij}q_{ij}P_j$ subject to $x_i \in X_i(x_{-i})$. Let $\mathcal{F}_i^{(a)}(x_{-i}) = \{x_i \in X_i(x_{-i}) : f_i(x_i) = a\})$ be the level curve associated to the cost function $f_i = a$. Since $f_i(x_i) = \sum_{j=1}^{n} x_{ij}q_{ij}P_j$ is linear in each x_{ij}, each iso cost function defines an hyperplane. By definition of the best response $BR_i(x_{-i})$, it is minimum of the function f_i with respect to x_i subject to $x_i \in X_i(x_{-i})$. It thus follows that the best response $BR_i(x_{-i}) = x_i^*$ is such that the hyperplane is tangent to the ellipsoid delimited by $X_i(x_{-i})$ and thus is unique. When the best response is not a tangency point, it is a corner solution. Let $\mathcal{C}^n := [0,1]^n$ be the unit compact of $\mathbb{R}^n$ and let $\partial\mathcal{C}^n$ be its boundary and int$\mathcal{C}^n$ be its interior so that $\mathcal{C}^n := \partial\mathcal{C}^n \bigcup \text{int}\mathcal{C}^n$. A corner solution is defined as a best response which belongs to $\partial\mathcal{C}^n$ and which can not satisfy the tangency condition. □

Proof of proposition 8

Given i and $x_{-i} \in [0,1]^{(p-1)n}$, consider $X_i(x_{-i})$ and assume that it is nonempty. Since $X_i(x_{-i})$ is a n-dimensional ellipsoid, it is clearly compact and convex.

Let us prove that for all i, X_i is a lower and upper semi-continuous point-to-set map:

Indeed:

- X_i is lower semi-continuous: let us consider a sequence $(v_l) \in \left([0,1]^{(p-1)n}\right)^{\mathbb{N}}$ that converges to $v_\infty \in [0,1]^{(p-1)n}$. We consider $w \in X_i(v_\infty)$, that is $\theta_i(w, v_\infty) \geq \theta_{i,min}$. Let us prove that there exists a sequence (w_l) with $w_l \in X_i(v_l)$ for all l and such that (w_l) converges to w. We consider $\epsilon > 0$. Let us prove that there exists L_0 such that for all $l \geq L_0$ we have that $B'(w,\epsilon) \cap X_i(v_l) \neq \emptyset$. Indeed, if it was not the case, we would have for all $L_0 > 0$ existence of a $l > L_0$ such that $B'(w,\epsilon) \cap X_i(v_l) = \emptyset$. So we could build a subsequence $(v_{\phi(l)})$ that converges to v_∞ and such that $B'(w,\epsilon) \cap X_i(v_{\phi(l)}) = \emptyset$ for all l. This implies that $\theta_i(x_i, v_{\phi(l)}) < \theta_{i,min}$ for all l and for all $x_i \in S(w,\epsilon) = \partial(B'(w,\epsilon))$, and since θ_i is continuous we would have $\theta_i(x_i, v_\infty) \leq \theta_{i,min}$ for all $x_i \in S(w,\epsilon)$. Let us consider the vector $z_i \in S(w,\epsilon)$ that maximizes the distance to 0, that is $d(z_i, 0) = max_{x_i \in S(w,\epsilon)} d(x_i, 0)$. Since all coordinates of w are strictly lower than the coordinates of z_i, we would have $\theta_i(w, v_\infty) < \theta_i(z_i, v_\infty) \leq \theta_{i,min}$ since $\theta_i(x_i, v_\infty)$ is an increasing function of x_i. And $\theta_i(w, v_\infty) < \theta_{i,min}$ is a contradiction. Therefore, there exists L_0 such that for all $l \geq L_0$ we have that $B'(w,\epsilon) \cap X_i(v_l) \neq \emptyset$, and we can build a sequence $w_l \in B'(w,\epsilon) \cap X_i(v_l)$ that converges to w.

- X_i is upper semi-continuous: let us consider a sequence $(v_l) \in \left([0,1]^{(p-1)n}\right)^{\mathbb{N}}$ that converges to $v_\infty \in [0,1]^{(p-1)n}$, and a sequence $(w_l) \in X_i(v_l)$ for all l, that converges to $w_\infty \in [0,1]^{(p-1)n}$. $\theta_i(v_l, w_l) \geq \theta_{i,min}$ for all l, and since θ is continuous we have that $\theta_i(v_\infty, w_\infty) \geq \theta_{i,min}$, and therefore $w_\infty \in X_i(v_\infty)$.

Moreover, since for all x_{-i}, $f_i(, x_{-i})$ is linear in x_i, it is thus quasiconvex on $X_i(x_{-i})$.

Therefore, the assumptions of Theorem 2 are satisfied and under these assumptions there exists a Nash equilibrium in microprudential constraint. □

Proof of lemma 2

Proof of part 1. $K_i(x_{-i})$ is described by the set $\{x_i \in [0,1]^n$ such that $\theta_i(x_i, x_{-i}) \geq \theta_{i,min}$ and for all $l \neq i$, $\theta_l(x_i, x_{-i}) \geq \theta_{l,min}\}$.

$\theta_i(x_i, x_{-i}) \geq \theta_{i,min}$ gives us the equation of a n-dimensional ellipsoid similar to Proposition 1.

Moreover, for $l \neq i$ we have:

$$\theta_l(x_i, x_{-i}) = \frac{E_{l,t} - q_{l0}P_0\Delta_0 - \sum_{j=1}^n q_{lj}P_j(\Delta_j + \frac{\sum_{k\in B} x_{kj}q_{kj}}{\Phi_j}(1-\Delta_j))}{\alpha_{l0}q_{l0}P_0(1-\Delta_0) + \sum_{j=1}^n \alpha_{lj}q_{lj}P_j(1-\Delta_j)\left(1 - \frac{\sum_{k\in B} x_{kj}q_{kj}}{\Phi_j}\right)(1-x_{lj})}$$

For all $l \neq i$, $\theta_l(x_i, x_{-i}) \geq \theta_{l,min}$ is equivalent to:

$$\left(\alpha_{l0}q_{l0}P_0(1-\Delta_0) + \sum_{j=1}^n \alpha_{lj}q_{lj}P_j\,(1-\Delta_j)\left(1 - \frac{\sum_{k\in B} x_{kj}q_{kj}}{\Phi_j}\right)(1-x_{lj})\right)\theta_{l,min} \leq E_{l,t} - q_{l0}P_0\Delta_0 - \sum_{j=1}^n q_{lj}P_j(\Delta_j + \frac{\sum_{k\in B} x_{kj}q_{kj}}{\Phi_j}(1-\Delta_j))$$

which, given x_{-i}, is in turn equivalent to:

$\sum_{j=1}^n a_{ij}x_{ij} \leq C$ with for all $j \in [1,..,n]$, $a_{ij} \in \mathbb{R}$ and $C \in \mathbb{R}$.

And this is the equation of a closed affine half-space. □

Proof of part 2. A best response in macroprudential constraint $BR_i^M(x_{-i})$ satisfies $\theta_i(x_i, x_{-i}) = \theta_{i,min}$ and for all $l \neq i$, $\theta_l(x_i, x_{-i}) \geq \theta_{l,min}$. $\theta_i(x_i, x_{-i}) = \theta_{i,min}$ gives us the equation of the frontier of a n-dimensional ellipsoid. $\theta_l(x_i, x_{-i}) \geq \theta_{l,min}$ gives us the equations of $p-1$ closed affine half-spaces.

Bank i is seeking to minimize $f_i(x_i) = \sum_{j=1}^n x_{ij}q_{ij}P_j$ subject to $x_i \in K_i(x_{-i})$.

$\sum_{j=1}^n x_{ij}q_{ij}P_j = a$ is an isocost hyperplane, and the minimum for f_i, which is the best response $BR_i^M(x_{-i})$ is reached for a point of tangency of an affine hyperplane $\sum_{j=1}^n x_{ij}q_{ij}P_j = a_i$ with the frontier of the ellipsoid delimited by $X_i(x_{-i})$ intersected with the $p-1$ closed affine half-spaces defined by $\theta_l(x_i, x_{-i}) \geq \theta_{l,min}, l \neq i$. □

Proof of proposition 9

We shall prove the proposition with two lemma.

Lemma A 1 *K is a compact set.*

Proof. Since $K \subset [0,1]^{n\times k}$, it is clearly a bounded set. To show that K is compact, it remains to prove that K is closed. Let $\vec{\theta}(x) \geq \vec{\theta}_{min}$ be the vectorial notation for $\begin{pmatrix} \theta_1(x) \geq \theta_{1,min} \\ \vdots \\ \theta_p(x) \geq \theta_{p,min} \end{pmatrix}$. Recall that

$$K := \{x \in [0,1]^{n\times p} : \vec{\theta}(x) \geq \vec{\theta}_{min}\} \subset [0,1]^{n\times p} \tag{2.56}$$

and is not empty by assumption. Since $\Delta_0 < 1$, for all $i \in S$, $\alpha_{i0}V_{i0}(1-\Delta_0) > 0$, the denominator of the capital ratio θ_i is strictly positive so that θ_i is a continuous application on $[0,1]^{n\times p}$ (see fact 3). Therefore, K is the preimage of a closed set $[\theta_{1,min}, +\infty[\times ... \times [\theta_{p,min}, +\infty[$ by the continuous application $\theta = (\theta_1, ..., \theta_p)$, and thus K is closed. It thus follows that K is a closed and bounded set of $[0,1]^{n\times p}$ which means that K is a compact set. □

Consider now the application $V : [0,1]^{n\times p} \to \mathbb{R}$, i.e., for a given $x \in [0,1]^{n\times p}$, $V(x) = \sum_{i\in S} f_i(x_i)$. Since for all $i \in \mathcal{B}$, f_i is continuous, V is also continuous on the compact set K. From Weierstrass extreme value theorem, it admits at least one minimum $x^{*,M} \in K$. Let $\mathcal{M}_K \subset K$ be the set of minimizers of the function V on K, possibly a singleton.

Lemma A 2 *Each element of $\mathcal{M}_K$ is a Nash equilibrium of the game under macroprudential constraint.*

We shall prove that x^*_K is a Nash equilibrium of the game under macroprudential constraint, that is, for all i, $x_i^{*,M} = BR_i^M(x_{-i}^{*,M})$. Let us work by contradiction and assume that this is not the case, i.e., there exists $i_0 \in \mathcal{B}$ such that $x_{i_0}^{*,M} \neq BR_{i_0}(x_{-i_0}^{*,M})$. By definition, given $x_{-i_0}^{*,M}$, $x_{i_0}^{*,M}$ is not the cheapest deleveraging strategy. This means that there exists x'_{i_0} such that $f_i(x'_{i_0}) < f_i(x_{i_0}^{*,M})$ and $(x'_{i_0}, x_{-i_0}^{*,M})$ still in K so that $f_{i_0}(x'_{i_0}) + \sum_{i\neq i_0} f_i(x_i^{*,M}) < \sum_{i\in S} f_i(x_i^{*,M})$ and this yields the desired contradiction. Therefore, the minima of V are Nash equilibria □

This concludes the proof of proposition proposition 9 □

Proof of Proposition 10

The proof is quite similar to Proposition 9:

$$\overline{K}' = \{x \in \mathcal{E} : \forall i \in \mathcal{S}, x_i \in \overline{X}_i(x_{-i})\} \tag{2.57}$$

$$\overline{X}_i(x_{-i}) = \begin{cases} X_i(x_{-i}) & \text{when } X_i(x_{-i}) \neq \emptyset \\ (1,1,...,1) := \mathbf{1} & \text{when } X_i(x_{-i}) = \emptyset \end{cases} \tag{2.58}$$

Therefore:

$$\overline{K}' = \{x \in \mathcal{E} : \forall i \in \mathcal{S}, \theta_i(x_i, x_{-i}) \geq \theta_{i,min} \text{ or } x_i = (1,..,1)\} \tag{2.59}$$

Let us prove that $\overline{K}'$ is a compact set.

- $\overline{K}' \subset [0,1]^{np}$ is clearly a bounded set.
- We want to prove that $\overline{K}'$ is a closed set. Let $(x_m)_{m\in\mathbb{N}} = (x_{1,m}, ..., x_{p,m})_{m\in\mathbb{N}} \in (\overline{K}')^{\mathbb{N}}$ be a sequence which converges to a given $x_\infty \in [0,1]^{np}$. We will show that $x_\infty \in \overline{K}'$.

  Let $i \in \{1,..,p\}$.

  - either $x_{i,\infty} = (1,1,...,1)$
  - or $x_{i,\infty} \in [0,1]^n \setminus \{(1,1,...,1)\}$ and there exists $\epsilon > 0$ such that $B(x_{i,\infty}, \epsilon) \subset [0,1]^n \setminus \{(1,1,...,1)\}$ and there exists $m_0 \in \mathbb{N}$ such that $\forall m \geq m_0, x_{i,m} \in B(x_{i,\infty}, \epsilon)$. Therefore $\forall m \geq m_0, \theta_{i,t+1}(x_m) \geq \theta_{min}$. And since $\theta_{i,t+1}$ is continuous on $B(x_{i,\infty}, \epsilon)$, we have $\theta_{i,t+1}(x_\infty) \geq \theta_{min}$

  So $\overline{K}'$ is a closed set.

Therefore $\overline{K}'$ is a closed bounded set of $[0,1]^{np}$, so K is a compact set.

Same as in Proposition 9, we look at the minimizers of V on $\overline{K}'$, and these are Nash equilibria of the generalized game with shared constraint $\overline{K}'$.

□

10 Liquidation process

We now describe formally the algorithm associated to the liquidation process. Since the banking book has no value, when a given bank sells it, the proceeds is equal to zero.

Algorithm of the liquidation process with linear price impact

1. Let $F^{(1)} := \{i \in \mathcal{B} : \Delta \in \mathcal{Z}_i^{fail}\}$. If $F^{(1)} = \emptyset$, then, the liquidation process stops. If $F^{(1)} \neq \emptyset$, all banks $i \in F^{(1)}$ liquidate all their assets, that is, for each $i \in F^{(1)}$, $x_i = \mathbf{1}$. The resulting implied shock given by equation (4.16) for each asset $j = 1, 2, ..., n$ is equal to $\Delta_j^{(1)}(\sum_{i\in F^{(1)}} q_{ij}) := \Delta_j + \frac{\sum_{i\in F^{(1)}} q_{ij}}{\Phi_j}(1 - \Delta_j)$ so that the implied vector of shock after the first step is equal to $\Delta^{(1)} := (\Delta_1^{(1)}(.), ..., \Delta_n^{(1)}(.))$.

2. Let $F^{(2)} := \{i \in (\mathcal{B} \backslash F^{(1)}) : \Delta^{(1)} \in \mathcal{Z}_i^{fail}\}$. If $F^{(2)} = \emptyset$, then, the liquidation process stops. If $F^{(2)} \neq \emptyset$, all the bank $i \in F^{(2)}$ liquidate all their assets, that is, for each $i \in F^{(2)}$, $x_i = \mathbf{1}$. The resulting implied shock given by equation (4.16) for each asset $j = 1, 2, ..., n$ is equal to $\Delta_j^{(1)}(\sum_{i \in (F^{(1)} \bigcup (F^{(2)})} q_{ij}) := \Delta_j + \frac{\sum_{i \in (F^{(1)} \bigcup (F^{(2)})} q_{ij}}{\Phi_j}(1 - \Delta_j)$ so that the implied vector of shock after the first step is equal to $\Delta^{(2)} := (\Delta_1^{(2)}(.), ..., \Delta_n^{(2)}(.))$.

3. Repeat until $F^{(k)} := \{i \in (\mathcal{B} \backslash \cup_{a=1}^{k-1} F^{(a)}) : \Delta^{(k-1)} \in \mathcal{Z}_i^{fail}\}$ is not empty.

Fact 8 *The liquidation process stops after a finite number of liquidation rounds $l \leq p$.*

Proof. Assume that the process does not stop at step $k \geq 1$, which means that for an implied shock $\Delta^{(k-1)}$ at step $k-1$, $F^{(k)} \neq \emptyset$, that is, an additional subset of banks fail. Since $F^{(k)} \neq \emptyset$ is equivalent to $\mathrm{Card}(F^{(k)}) \geq 1$ and since $\sum_{k=1}^{p} \mathrm{Card}(F^{(k)}) \geq p$, the liquidation process must stop in at most $l \leq p$ steps such that $\sum_{k=1}^{l} \mathrm{Card}(F^{(k)}) \leq p$ $\square$

11 Appendix: Calibration result

Method of calculation of the RWAs Banking Book and RWAs Trading Book:

The RWAs in the annual reports of the four French Banks we study are partitioned in the following categories : RWAs Credit Risk, RWAs Counterparty risk, RWAs Settlement Risk, RWAs Securitization exposures in the Banking Book, RWAs Market Risk, RWAs Operational Risk and RWAs Amounts below the thresholds.

While RWAs Credit Risk and RWAs Securitization exposures in the Banking Book are clearly part of the RWAs Banking Book, and while RWAs Market Risk and RWAs Counterparty risk are clearly part of the RWAs Trading Book, it is not clear whether the other categories are in the Banking Book or the Trading Book. In our calibration we will make the following assumption: the RWAs Settlement Risk, RWAs Operational Risk and RWAs Amounts below the thresholds are partitioned between the Banking Book and Trading Book propotionally to the weight of the Credit Risk (for the Banking Book) and the Market Risk (for the Trading Book).

BNP Paribas:

RWAs Other risks = RWAs Settlement Risk + RWAs Operational Risk + RWAs Amounts below the thresholds = 70.63 +17.06 = 87.69

$\gamma_{BNP,BB} = \frac{RWAsCreditRisk}{RWAsCreditRisk+RWAsMarketRisk} = \frac{527.19}{527.19+25.21} = 95.4\%$

$\gamma_{BNP,TB} = \frac{RWAsMarketRisk}{RWAsCreditRisk+RWAsMarketRisk} = 1 - \gamma_{BNP,BB} = 4.6\%$

RWAs Banking Book = RWAs Credit Risk + RWAs Securitization exposures in the Banking Book + $\gamma_{BNP,BB}\times$ RWAs Other risks

RWAs Banking Book = 527.19 + 14.47 + 0.954×87.69 = 625.32

RWAs Trading Book = RWAs Market Risk + RWAs Counterparty risk + $\gamma_{BNP,TB}\times$ RWAs Other risks

RWAs Trading Book = Total RWAs - RWAs Banking Book = 695.52 - 625.32 = 70.2

To obtain the risk weights for the Banking Book and the Trading Book we compute the following:

$\alpha_{BNP,BB} = \frac{RWAsBankingBook}{BankingBook} = \frac{625.32}{946.8} = 0.660$

$\alpha_{BNP,TB} = \frac{RWAsTradingBook}{TradingBook} = \frac{70.2}{1232.96} = 0.057$

To obtain the average risk weight, we compute the following:

$\alpha_{BNP,Avg} = \frac{TotalRWA}{BankingBook+TradingBook} = \frac{695.52}{946.8+1232.96} = 0.319$

To obtain the RBC ratio and the Leverage ratio, we compute the following:

$RBC = \frac{Tier1}{RWA} = \frac{98.8}{695.52} = 0.142$

$L = \frac{Tier1}{TotalExposure} = \frac{98.8}{2266.86} = 0.044$

Société Générale:

RWAs Other risks = RWAs Settlement Risk + RWAs Operational Risk + RWAs Amounts below the thresholds = 0.07 + 49.19 + 8 = 57.26

$\gamma_{SG,BB} = \frac{RWAsCreditRisk}{RWAsCreditRisk+RWAsMarketRisk} = \frac{247.42}{247.42+15.34} = 94.2\%$

$\gamma_{SG,TB} = \frac{RWAsMarketRisk}{RWAsCreditRisk+RWAsMarketRisk} = 1 - \gamma_{BNP,BB} = 6.8\%$

RWAs Banking Book = RWAs Credit Risk + RWAs Securitization exposures in the Banking Book + $\gamma_{SG,BB}\times$ RWAs Other risks

RWAs Banking Book = 247.2 + 5.49 + 0.942×57.26 = 306.63

RWAs Trading Book = RWAs Market Risk + RWAs Counterparty risk + $\gamma_{SG,TB}\times$ RWAs Other risks

RWAs Trading Book = Total RWAs - RWAs Banking Book = 351.85 - 306.63 = 45.22

To obtain the risk weights for the Banking Book and the Trading Book we compute the following:

$\alpha_{SG,BB} = \frac{RWAsBankingBook}{BankingBook} = \frac{306.63}{502.14} = 0.611$

$\alpha_{SG,TB} = \frac{RWAsTradingBook}{TradingBook} = \frac{45.22}{791.6} = 0.057$

To obtain the average risk weight, we compute the following:

$\alpha_{SG,Avg} = \frac{TotalRWA}{BankingBook+TradingBook} = \frac{351.85}{502.14+791.6} = 0.272$

To obtain the RBC ratio and the Leverage ratio, we compute the following:

$RBC = \frac{Tier1}{RWA} = \frac{56.18}{351.85} = 0.160$

$L = \frac{Tier1}{TotalExposure} = \frac{56.18}{1188.5} = 0.047$

Crédit Agricole:

RWAs Other risks = RWAs Settlement Risk + RWAs Operational Risk + RWAs Amounts below the thresholds = 34.17 + 4.08 = 38.25

$\gamma_{CA,BB} = \frac{RWAsCreditRisk}{RWAsCreditRisk+RWAsMarketRisk} = \frac{257.2}{257.2+9.75} = 96.3\%$

$\gamma_{CA,TB} = \frac{RWAsMarketRisk}{RWAsCreditRisk+RWAsMarketRisk} = 1 - \gamma_{BNP,BB} = 3.7\%$

RWAs Banking Book = RWAs Credit Risk + RWAs Securitization exposures in the Banking Book + $\gamma_{CA,BB}\times$ RWAs Other risks

RWAs Banking Book = 257.2 + 8.76 + 0.963×38.25 = 302.79

RWAs Trading Book = RWAs Market Risk + RWAs Counterparty risk + $\gamma_{CA,TB}\times$ RWAs Other risks

RWAs Trading Book = Total RWAs - RWAs Banking Book = 336.04 - 302.79 = 33.25

To obtain the risk weights for the Banking Book and the Trading Book we compute the following:

$\alpha_{CA,BB} = \frac{RWAsBankingBook}{BankingBook} = \frac{302.79}{953.9} = 0.317$

$\alpha_{CA,TB} = \frac{RWAsTradingBook}{TradingBook} = \frac{33.25}{812.9} = 0.041$

To obtain the average risk weight, we compute the following:

$\alpha_{CA,Avg} = \frac{TotalRWA}{BankingBook+TradingBook} = \frac{336.04}{953.9+812.9} = 0.190$

To obtain the RBC ratio and the Leverage ratio, we compute the following:

$RBC = \frac{Tier1}{RWA} = \frac{50.02}{336.04} = 0.149$

$L = \frac{Tier1}{TotalExposure} = \frac{50.02}{1861.5} = 0.027$

BPCE:

RWAs Other risks = RWAs Settlement Risk + RWAs Operational Risk + RWAs Amounts below the thresholds = 38.32 + 11.33 = 49.65

$\gamma_{BPCE,BB} = \frac{RWAsCreditRisk}{RWAsCreditRisk+RWAsMarketRisk} = \frac{350.2}{350.2+14.44} = 96\%$

$\gamma_{BPCE,TB} = \frac{RWAsMarketRisk}{RWAsCreditRisk+RWAsMarketRisk} = 1 - \gamma_{BNP,BB} = 3\%$

RWAs Banking Book = RWAs Credit Risk + RWAs Securitization exposures in the Banking Book + $\gamma_{BPCE,BB}\times$ RWAs Other risks

RWAs Banking Book = 350.2 + 4.88 + 0.96×49.65 = 402.74

RWAs Trading Book = RWAs Market Risk + RWAs Counterparty risk + $\gamma_{BPCE,TB}\times$ RWAs Other risks

RWAs Trading Book = Total RWAs - RWAs Banking Book = 431.22 - 402.74 = 28.48

To obtain the risk weights for the Banking Book and the Trading Book we compute the following:

$\alpha_{BPCE,BB} = \frac{RWAsBankingBook}{BankingBook} = \frac{402.74}{836.82} = 0.481$

$\alpha_{BPCE,TB} = \frac{RWAsTradingBook}{TradingBook} = \frac{28.48}{456} = 0.062$

To obtain the average risk weight, we compute the following:

$\alpha_{BPCE,Avg} = \frac{TotalRWA}{BankingBook+TradingBook} = \frac{431.22}{836.82+456} = 0.334$

To obtain the RBC ratio and the Leverage ratio, we compute the following:

$RBC = \frac{Tier1}{RWA} = \frac{68.98}{431.22} = 0.160$

$L = \frac{Tier1}{TotalExposure} = \frac{68.98}{1374.3} = 0.050$

Chapter 3

A Generalized Nash Equilibrium Problem arising in banking regulation: An existence result with Tarski's theorem

Abstract

When hit with an adverse shock, banks that do not comply with capital regulation sell risky assets to satisfy their solvency constraint. When financial markets are imperfectly competitive, this naturally gives rise to a GNEP. We consider a new framework with an arbitrary number of banks and assets, and show that Tarski's theorem can be used to prove the existence of a Nash equilibrium when markets are sufficiently competitive. We also prove the existence of ϵ-Nash equilibria.

Keywords: Generalized games, banking regulation, Cournot oligopoly, asset sales, Tarki's fixed point theorem.

1 Introduction

Compared to other financial institutions such as hedge funds, banks are regulated; their main capital ratio called the risk-based capital ratio (RBC) defined as regulatory capital or equity E divided by the risk-weighted assets RWA, that is, $\frac{E}{\text{RWA}}$, must be greater than a critical threshold around 10% at all times (see [BCBS, 2017]). After an adverse systemic shock, as e.g., in mid-September 2008 when Lehman Brothers failed, some banks may fail to satisfy their capital requirement. To comply with regulation, the quickest solution consists in selling a portion of their risky assets to decrease their risk-weighted assets. However, when many banks sell the same asset at the same time, something called a generalized asset shrinkage ([Hanson et al., 2011]), the price of the asset will decrease through the market mechanism and will further deplete the capital of the bank, that is, this generates a kind of death spiral (e.g., [Brunnermeier and Pedersen, 2009], [Hanson et al., 2011]). It is usual to call such a phenomenon a price-mediated contagion problem (e.g., [Feinstein, 2020]), as opposed to direct contagion which is related to the network of exposures (e.g., [Feinstein, 2017], see [Glasserman and Young, 2016] for a review). In [Braouezec and Wagalath, 2019], they consider a game theoretic price-mediated contagion model in the particular case of one risky asset and show, using Tarski's theorem, that at least one Nash equilibrium exists. They however fail to recognize that this setting is indeed a generalized game. In this paper, we consider a new setting in the spirit of [Braouezec and Wagalath, 2019] in the general case of $n \geq 2$ assets and prove, under some conditions on the price impacts, the existence of a Nash equilibrium using Tarski's theorem. This theorem is widely applied in financial network models (e.g., [Glasserman and Young, 2016]) but not in generalized games (see [Dutang, 2013] for a review of existence theorems, see also [Arrow and Debreu, 1954] and [Facchinei and Kanzow, 2007]). Our contribution is intimately related to [Banerjee and Feinstein, 2021] and [Capponi and Weber, 2022] (see also [Capponi and Larsson, 2015]) since the authors, as we do here, consider a static strategic setting. [Banerjee and Feinstein, 2021] is the generalization of [Braouezec and Wagalath, 2019] to the case of an arbitrary number of assets and banks when each bank is subject to a risk-based capital requirement. [Capponi and Weber, 2022] consider a related model in which banks are subject to a non risk-based capital requirement (leverage ratio). Interestingly, as opposed to [Banerjee and Feinstein, 2021] and [Braouezec and Wagalath, 2019], in [Capponi and Weber, 2022] the asset allocation is not exogenous but is endogenous.

The aim of this paper is threefold. First, it is to show that due to both banking regulation and the market mechanism, a generalized game naturally occurs when one considers the asset sale problem between banks. Second, it is to show how Tarski's theorem can be used to prove the existence of a Nash equilibrium in a generalized game (see [Facchinei and Kanzow, 2007], [Dutang, 2013]). We also characterize the optimal liquidation strategy. Third, it is to generalize our existence result to ϵ-Nash equilibria.

2 The generalized game

We consider the extension of [Braouezec and Wagalath, 2019] with a finite number of risky assets. Let $\mathcal{B} = \{1, ..., p\}$ be the set of banks and $\mathcal{S} = \{1, ..., n\}$ be the set of risky securities. Each bank i holds a quantity q_{ij} of security j (stocks, bonds, index...) for which the price P_j has been hit with a shock Δ_j in percentage at time $t = 0$, i.e., it is equal to $P_j(1 - \Delta_j)$. Right after the shock $\Delta = (\Delta_1, ..., \Delta_n)$, the total value of the assets of each bank i is equal to $A_i(\Delta) := \sum_{j=1}^{n} P_j(1 - \Delta_j) q_{ij} = A_i - \sum_{j=1}^{n} q_{ij} P_j \Delta_j$ where $\sum_{j=1}^{n} q_{ij} P_j \Delta_j$ is the *loss* incurred by the bank. The balance sheet of bank i is given below where A_i is the total assets before the shock.

Balance-sheet of bank i

Assets	Liabilities
$\text{A}_{i1} - q_{i1} P_1 \Delta_1$ $\vdots$ $\text{A}_{in} - q_{in} P_n \Delta_n$	Debt: D_i Equity: $E_i(\Delta)$
$A_i - \sum_{j=1}^{n} q_{ij} P_j \Delta_j$	$E_i(\Delta) + D_i$

As long as the the bank is solvent, the total value of the assets is equal to the total value of the liabilities (i.e., $A_i(\Delta) = E_i(\Delta) + D_i$) so that the capital can absorb the loss, that is, $E_i(\Delta) = E_i - \sum_{j=1}^{n} q_{ij} P_j \Delta_j > 0$ where E_i is the capital before the shock. The positivity of the capital is however not enough. Banking regulation imposes a risk-based capital ratio for each bank i to be greater than $\theta_{i,min}$ (in practice $\theta_{i,min} \approx 10\%$). Let $\alpha_{ij} \in [0, 1]$ be the regulatory weight of asset j for bank i.

Assumption 6 *For each $i \in \mathcal{B}$*

$$\alpha_{i,1} > \alpha_{i,2} > ... > \alpha_{i,n} \tag{3.1}$$

The regulatory weights can be directly provided by regulators to banks (standardised approach) so that they are identical for each bank. But they can also be computed by the bank (internal model approach), which means that two different banks may have different estimates of the weight of a given asset. However, in the post-subprime crisis regulation called Basel III, banks have much less freedom than before to make use of internal models. The risk-weighted assets are equal to $\text{RWA}_i(\Delta) := \sum_j \alpha_{ij} P_j (1-\Delta_j) q_{ij}$. For the sake of financial interest, we assume that each bank must react after the shock.

Assumption 7 *For each $i \in \mathcal{B}$, $E_i(\Delta) > 0$ but*

$$\theta_i(\Delta) := \frac{E_i(\Delta)}{\text{RWA}_i(\Delta)} < \theta_{i,min}$$

To restore their capital ratio back above $\theta_{i,min}$, banks are assumed to sell a portion $x_{ij} \in [0,1]$ of security $j = 1, ..., n$. Let $\sum_{k\in\mathcal{B}} x_{kj} q_{kj}$ be the total quantity of security j sold by banks. When markets are imperfectly competitive, the price is impacted by such sales. For simplicity, we consider a linear price impact model (e.g., [Braouezec and Wagalath, 2019]) for which the price of security j after the asset sale is equal to

$$P_j^{\text{after}} = P_j^{\text{before}} \times \left(1 - \frac{\sum_{i\in\mathcal{B}} x_{kj} q_{kj}}{\Phi_j}\right) \tag{3.2}$$

where $\Phi_j >> \sum_{k\in\mathcal{B}} q_{kj}$ is called the market depth and measures the competitiveness of market j. The greater Φ_j, the more competitive the market of security j. At the limit, when Φ_j is infinite, it is perfectly competitive. Let $x_i \in [0,1]^n$ be the liquidation of bank i and $x \in [0,1]^{np}$ be a vector of liquidation of the set of banks. Let

$$L_i(x_i) := \sum_{j=1}^{n} x_{ij} q_{ij} P_j (1-\Delta_j)$$

be the total value of the assets sold by bank i. It is not difficult to show that the risk-based capital ratio of bank i is equal to

$$\theta_i(\Delta, x) := \frac{E_i(\Delta, x)}{\text{RWA}_i(\Delta, x)} = \frac{E_i - \sum_{j=1}^{n} q_{ij} P_j \times (\Delta_j + \frac{\sum_{k\in\mathcal{B}} x_{kj} q_{kj}}{\Phi_j}(1-\Delta_j))}{\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) \left(1 - \frac{\sum_{k\in\mathcal{B}} x_{kj} q_{jk}}{\Phi_j}\right)(1-x_{ij})} \tag{3.3}$$

As usual in game theory, let $x = (x_i, x_{-i})$. Since the capital ratio of bank i depends upon x_i but also upon $x_{-i} \in [0,1]^{n(p-1)}$ (what banks $k \neq i$ are liquidating), the asset sale problem is *strategic*.

Let

$$X_i(x_{-i}) = \{x_i \in [0,1]^n,\ \theta_i(x_i, x_{-i}) \geq \theta_{i,min}\}$$

be the solvency constraint of a given bank i. At time $t = 0$, each bank i chooses its liquidation strategy $x_i := (x_{ij})_{j \in \mathcal{S}} \in [0,1]^n$ and prices after liquidation are disclosed, i.e., $P^{\text{after}} = (P_1^{\text{after}}, ..., P_n^{\text{after}})$ at time $t = 1$.

Assumption 8 *Each bank $i \in \mathcal{B}$ solves the following constrained optimization problem.*

$$\min_{x_i} L_i(x_i) \quad s.t. \quad x_i \in X_i(x_{-i}) \tag{3.4}$$

The objective function is similar to [Braouezec and Wagalath, 2019] in that it only depends upon x_i, the decision of bank i. Interaction between banks thus takes place through the solvency constraint but not through the objective function. Let $\Phi := (\Phi_1, ..., \Phi_n)$ and let

$$K_{\Delta,\Phi} = K := \{x \in [0,1]^{np} : \forall i \in \mathcal{B}, x_i \in X_i(x_{-i})\} \subset [0,1]^{np}$$

be the set of admissible strategies of the generalized game. We now recall the definition of a Nash equilibrium for our game.

Definition 12 *The profile of strategies $x^* \in K$ is a Nash equilibrium of the asset sale game if, for each $i \in \mathcal{B}$ and each $x_i \in [0,1]^n$ such that $x_i \in X_i(x^*_{-i})$, it holds true that $L_i(x^*_i, x^*_{-i}) \leq L_i(x_i, x^*_{-i})$.*

3 Perfectly competitive markets

When markets are perfectly competitive, $\frac{1}{\Phi_j} = 0$ for all j. From equation (3.3), the risk-based capital ratio of bank i reduces to

$$\theta_i(\Delta, x_i) = \frac{E_{i,t} - \sum_{j=1}^n q_{ij} P_j \Delta_j}{\sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1 - \Delta_j)(1 - x_{ij})} \tag{3.5}$$

so that the decision problem is not anymore strategic; the capital ratio only depends upon x_i.

Lemma 3 *The risk-based capital ratio $\theta_i(\Delta, x_i)$ is an increasing function of x_{ij} for each $i \in \mathcal{B}$ and each $j \in \mathcal{S}$.*

Proof. Since the numerator of $\theta_i(\Delta, x_i)$ is invariant with respect to x_{ij} while the denominator decreases with x_{ij}, the result follows. □

The next proposition provides a characterization of the optimal liquidation strategy. Note that the optimization problem reduces to a linear programming problem.

Proposition 11 ***(Characterization of the optimal strategy)*** *Under assumptions 6, 7, 8, when markets are perfectly competitive, there is a unique optimal liquidation vector* $(x^*_{i,1}, ..., x^*_{i,2}, ..., x^*_{i,n}) \in [0,1]^n \backslash \{(1,1,...,1)\}$ *of the form* $(1,...,1,x^*_{i,h},0,...0)$ *where* $x^*_{i,h} \in (0,1)$ *for some integer* $h \in \{1,...,n\}$ *is such that*

$$x^*_{i,h} = \frac{1}{\alpha_{i,h} q_{i,h} P_h (1-\Delta_h)} \left[\sum_{j=h+1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) - \frac{1}{\theta_{i,min}} (E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j) \right]$$

Proof. The denominator of (3.5) tends to zero when x_{ij} tends to one for all j so that the capital ratio tends to infinity. Since $\theta_{i,min} < \infty$, a solution exists. For $x^*_i = (x^*_{i1}, ..., x^*_{in})$, the constraint is clearly binding; $\theta_{i,t+1}(\Delta, x_i) = \theta_{i,min}$, that is,

$$\frac{E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j}{\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j)(1-x^*_{ij})} = \theta_{i,min} \tag{3.6}$$

which implies that:

$$\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) x^*_{ij} = \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) - \frac{1}{\theta_{i,min}} (E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j)$$

Letting $X_{ij} = q_{ij} P_j (1-\Delta_j) x^*_{ij}$, and $C_i = \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) - \frac{1}{\theta_{i,min}} (E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j)$, equation (3.6) is equivalent to:

$$\sum_{j=1}^{n} \alpha_{ij} X_{ij} = C_i \tag{3.7}$$

Each bank i seeks to minimize $\sum_{j=1}^{n} X_{ij}$. Therefore, the unique solution consists in selling asset 1 with the highest risk weight α_{i1}, then asset 2 to asset $h \leq n$ until the capital ratio is restored. □

Proposition 11 says that it is optimal for each bank i to first sell the asset with the highest regulatory weight. If this is not enough to restore the capital ratio, it is now optimal to sell 100% of asset 1 and a portion of risky asset 2. If this is not enough, it is optimal to sell 100% of asset 1 and 2 and a portion of risky asset 3 and so on and so forth. The optimal liquidation strategy follows the order of the weights.

4 Imperfectly competitive markets: existence result with Tarski's theorem

4.1 Preliminary results and main assumptions

We shall prove a few preliminary results.

Assumption 9

$$For\ each\ i \in \mathcal{B}\ \ and\ each\ x \in [0,1]^{np}, E_i(\Delta, x) > 0 \tag{3.8}$$

Lemma 4 *Under assumption 9, regardless of* $x_{-i} \in [0,1]^{n(p-1)}$*,* $X_i(x_{-i}) \neq \emptyset$*.*

Proof. From assumption 9, $E_i(\Delta, 1) > 0$ where 1 is the np-dimensional vector. Let $E_i(\Delta, 1) := \xi_i > 0$ and note that $E_i(\Delta, x_i, x_{-i}) \geq \xi_i$ regardless of $x_{-i} \in [0,1]^{n(p-1)}$. Since for each i, $\text{RWA}_i(x_i)$ tends to zero when x_i tend to the n-dimensional vector 1, $\lim_{x_i \to 1} \frac{\xi_i}{\text{RWA}_i(x_i)} \to \infty$. Since $\theta_i(\Delta, x_i, x_{-i}) \geq \frac{\xi_i}{\text{RWA}_i(x_i)}$ regardless of $x_{-i} \in [0,1]^{n(p-1)}$ and since $\theta_{i,min} < \infty$, there exists $x_i \in [0,1]^n \backslash \{1, ..., 1\}$ such that $\theta_i(\Delta, x_i, x_{-i}) = \theta_{i,min}$, hence $X_i(x_{-i}) \neq \emptyset$. $\square$

For notations simplicity, we may denote $\theta_i(\Delta, x_i, x_{-i})$ by $\theta_i(.)$.

Lemma 5 *Given* x_i*,* $\theta_i(.)$ *is a decreasing function of* x_{kj} *for all* $k \neq i$ *and all* $j = 1, ..., n$*. In particular,* $\theta_i(\Delta, x_i, x_{-i})$ *is a decreasing function of* x_{-i}*.*

Proof. Let $N(x) := E_i(\Delta, x)$ be the numerator of the risk-based capital ratio and $D(x)$ be its denominator (see equation (3.3)). Consider $\frac{\partial \theta_i}{\partial x_{kj}}$ when $k \neq i$:

$$\frac{\partial \theta_i(.)}{\partial x_{kj}} = \frac{q_{ij} P_j q_{kj} (1 - \Delta_j)}{\Phi_j D(x)^2} [\alpha_{ij}(1 - x_{ij}) N(x) - D(x)]$$

which has same sign as $\alpha_{ij}(1-x_{ij})N(x) - D(x)$. We notice that $\alpha_{ij}(1-x_{ij}) \leq 1$, so if $N(x) < D(x)$, we have that $\frac{\partial \theta_i}{\partial x_{kj}} < 0$. Since $\theta_{i,min} << 1$, it is always true that $N(x) < D(x)$. Therefore, if $x_{-i} \leq y_{-i}$, we have that $\theta_i(x_i, x_{-i}) \geq \theta_i(x_i, y_{-i})$. $\square$

Lemma 6 *When the market depths* Φ *are large enough,* $\theta_i(.)$ *is an increasing function of* x_{ij} *for each* $j = 1, 2, ..., n$*.*

Proof. Consider $\frac{\partial \theta_i}{\partial x_{kj}}$ when $k = i$:

$$\frac{\partial \theta_i}{\partial x_{ij}} = \frac{q_{ij} P_j (1 - \Delta_j)}{D(x)^2} \left[\alpha_{ij} N(x) [1 + \frac{q_{ij}}{\Phi_j}(1 - x_{ij}) - \frac{\sum_{k \in B} x_{kj} q_{kj}}{\Phi_j}] - \frac{q_{ij}}{\Phi_j} D(x) \right] \tag{3.9}$$

which has same sign as $\alpha_{ij}N(x)[1+\frac{q_{ij}}{\Phi_j}(1-x_{ij})-\frac{\sum_{k\in B}x_{kj}q_{kj}}{\Phi_j}]-\frac{q_{ij}}{\Phi_j}D(x)$. From equation (3.8), $N(x)>0$ for all x. From equation (3.9), it thus follows that when market depths are high enough, $\frac{\partial\theta_i(.)}{\partial x_{ij}}>0$. □

Corollary 2 *There exists a critical smallest market depths vector* $\Phi' := (\Phi'_1,...,\Phi'_n)$ *such that, regardless of* $x_{-i}\in[0,1]^{n(p-1)}$, *for each* $i\in\mathcal{B}$ *and each* $j\in\mathcal{S}$, $\theta_i(.)$ *is an increasing function of* $x_{ij}\in[0,1]$.

Assumption 10 *The market depths satisfy* $\Phi\geq\Phi'$.

4.2 Main result

Before proving the main result, we show that when the market depths are high enough, the optimal liquidation strategy is identical to the one found in Proposition 11.

Lemma 7 *There exists a critical market depths vector* $\Phi^0 := (\Phi^0_1,...,\Phi^0_n)$ *such that, for all* $\Phi\geq\Phi^0$, *the optimal liquidation strategy of each bank* $i\in\mathcal{B}$ *is identical to the one in Proposition 11.*

Proof. See the appendix.

Assumption 11 *The market depths satisfy* $\Phi\geq\Phi^0$.

Lemma 8 *Under assumptions 6 to 11,* $BR_i(x_{-i})$ *is an non-decreasing function of* x_{-i}.

Proof. Let $x\in[0,1]^{np}$ and $y\in[0,1]^{np}$ such that $x\leq y$ so that $x_{-i}\leq y_{-i}$. When $\Phi\geq\Phi^0$, the best responses are given as in lemma 7. Let $BR_i(x_{-i})=(1,...,1,x_{ih},0,...0)$ and $BR_i(y_{-i})=(1,...,1,y_{il},0,...,0)$. Since $\theta_i(BR_i(x_{-i}),x_{-i})=\theta_i(BR_i(y_{-i}),y_{-i})=\theta_{i,min}$, using the properties of the best responses, $\theta_i((1,...,1,x_{ih},0,...,0),x_{-i})=\theta_i((1,...,1,y_{il},0,...,0),y_{-i})=\theta_{i,min}$. Since $x_{-i}\leq y_{-i}$, from Lemma 5 we have that $\theta_{i,min}=\theta_i((1,...,1,x_{ih},0,...0),x_{-i})\geq\theta_i((1,...,1,x_{ih},0,...0),y_{-i})$ and therefore $\theta_{i,min}=\theta_i((1,...,1,y_{il},0,...,0),y_{-i})\geq\theta_i((1,...,1,x_{ih},0,...,0),y_{-i})$. From Lemma 6, when $\Phi\geq\Phi'$, since given z_{-i}, $\theta_i(z_i,z_{-i})$ is an increasing function of z_{ij} for all $j\in[1,..,n]$, we necessarily have that $l\geq h$ and in the case that $l=h$ we have that $y_{il}\geq x_{il}$. Therefore $BR_i(x_{-i})=(1,...,1,x_{ih},0,...,0)\leq(1,...,1,y_{il},0,...,0)=BR_i(y_{-i})$, which concludes the proof □

If the market depths are not high enough, then, there may exist $x\leq y$ such that for instance given x_{-i}, $\alpha_{i1}(1-\delta_1(\Phi_1,x))>\alpha_{i2}(1-\delta_2(\Phi_2,x))$ for all x_i while given y_{-i}, $\alpha_{i2}(1-\delta_2(\Phi'_2,y))>\alpha_{i1}(1-\delta_1(\Phi'_1,y))$ for all y_i. In such a case, the best responses are no more increasing.

Tarski's theorem ([Tarski, 1955], see also [Vives, 1990]). *Let $(L, \geq)$ be a complete lattice and f a non decreasing function from L to L and $\mathcal{F}$ the set of fixed points of f. Then, $\mathcal{F}$ is non-empty and $(\mathcal{F}, \geq)$ is a complete lattice. In particular,* $\sup_x \mathcal{F}$ *and* $\inf_x \mathcal{F}$ *belong to* $\mathcal{F}$.

Consider the lattice $([0,1]^{np}, \leq)$ with $\leq$ defined by the natural order $x \leq y \Longleftrightarrow x_i \leq y_i$ for each $i = 1, ..., p$ where $x_i \leq y_i$ component-wise. Note that $[0,1]^{np}$ is the product of compact intervals and thus is a complete lattice. Consider a function f from $([0,1]^{np}, \leq)$ to $([0,1]^{np}, \leq)$. We shall now show that the function f

$$f(x) = (BR_1(x_{-1}), ..., BR_p(x_{-p}))$$

is non-decreasing from $([0,1]^{np}, \leq)$ to $([0,1]^{np}, \leq)$ and apply Tarski's theorem to f.

Proposition 12 *Under assumptions 6 to 11, there exists a smallest Nash equilibrium* $x^* = (x_1^*, ..., x_p^*) \in ([0,1]^n \backslash \{(1,1,...,1)\})^p$ *to the generalized game defined in (3.4).*

Proof of proposition 12

Let us consider $x \in [0,1]^{np}$ and $y \in [0,1]^{np}$ such that $x \leq y$. Therefore, $x_{-i} \leq y_{-i}$ so for all $z_i \in [0,1]^n$ we have that $\theta_i(z_i, x_{-i}) \geq \theta_i(z_i, y_{-i})$. This implies that $X_i(y_{-i}) = \{z_i \in [0,1]^n; \theta_i(z_i, y_{-i}) \geq \theta_{i,min}\} \subset X_i(x_{-i}) = \{z_i \in [0,1]^n; \theta_i(z_i, x_{-i}) \geq \theta_{i,min}\}$, and by assumption these two sets are not empty. Therefore, from Lemma 8, $BR_i(x_{-i}) \leq BR_i(y_{-i})$ for all i. Hence, f is a non-decreasing function from $([0,1]^{np}, \leq)$ to $([0,1]^{np}, \leq)$, and therefore it satisfies the assumptions of Tarski's theorem. As a consequence, the set of fixed points of f is not empty, so that there exists at least one Nash equilibrium. If there are multiple Nash equilibria, since they are ordered, by Tarski's theorem, $\inf_x \mathcal{F}$ belongs to $\mathcal{F}$, the set of Nash equilibrium and $x^* = \inf_x \mathcal{F}$ is the smallest Nash equilibrium. At equilibrium, each bank i is solvent so that it must be the case that $x_i^* \in [0,1]^n \backslash \{(1,1,...,1)\}$ $\square$

To the best of our knowledge, this is the first application of Tarski's theorem to a generalized game.

In [Banerjee and Feinstein, 2021], while they do not explicitly consider a generalized game, they offer a general extension of [Braouezec and Wagalath, 2019] in which each bank can go bankrupt. The strategy set of each bank i thus is extended and given by $\overline{X_i}(x_{-i}) = \{x_i \in [0,1]^n,\ \theta_i(x_i, x_{-i}) \geq$

$\theta_{i,min}\} \cup \{(1, .., 1)\}$. Using Berge maximum theorem, they prove in Proposition 3.7 the existence of (at least) one Nash equilibrium but they do not characterize the optimal liquidation rule.

5 Epsilon-Nash equilibria

We now consider epsilon-Nash equilibria, similar in the spirit to [Marinacci, 1997]. Let us rename and denote by $\bar{x}$ the optimal liquidation solution in case of no price impact as studied in Section 3, that is, a vector of the form $\bar{x} = (1, ..., 1, \bar{x}_{i,h}, 0, ..., 0)$. In this section, we prove the existence of epsilon-Nash equilibria and characterize a set of epsilon-Nash equilibria. The interest of these epsilon-Nash equilibria here is twofold: 1) Given $\epsilon > 0$ there always exists some market depths such that epsilon-Nash equilibria exist. 2) We can compute and describe a set of epsilon Nash equilibria, for any $\epsilon > 0$.

Definition 13 $x^* = (x_1^*, ..., x_p^*)$ *is an* ϵ*-Nash Equilibrium if:*

$\forall i \in \{1, ..., p\},\ \forall x_i \in [0,1]^n$ *s.t.* $\theta_{i,t+1}(x_i, x_{-i}^*) \geq \theta_{i,min} :\ L_i(x_i^*, x_{-i}^*) - \epsilon \leq L_i(x_i, x_{-i}^*)$

Definition 14 *Given the market depths* Φ_1, ..., Φ_n, *we denote the set of admissible strategies of the strategic problem* $K = K_{\Phi_1,...,\Phi_n}$.

Proposition 13 *Under assumptions 6 to 8:* $\forall \epsilon > 0$, *there exist a compact set containing* $\bar{x}$, $V(\bar{x}, \epsilon) = \prod [\bar{x}_{ij}, \bar{x}_{ij} + \tilde{\epsilon}_{ij}]$, *with for all* i, j, $\tilde{\epsilon}_{ij} = \frac{\epsilon}{n \times q_{ij} P_j (1-\Delta_j)}$, *and there exists* $\Phi_1^0, ..., \Phi_n^0 > 0$ *such that* $\forall \Phi_1 \geq \Phi_1^0$,..., $\forall \Phi_n \geq \Phi_n^0$, $V(\bar{x}, \epsilon) \cap K_{\Phi_1,...,\Phi_n}$ *is not empty and all its elements are* ϵ*-Nash equilibria.*

Proof of Proposition 13

We define $\tilde{\epsilon}_{ij} = \frac{\epsilon}{n \times q_{ij} P_j (1-\Delta_j)}$ and $\tilde{\epsilon} = (\tilde{\epsilon}_1, ..., \tilde{\epsilon}_p)$.

$\forall i,\ \theta_{i,t+1}(\bar{x}, \Phi = \infty) = \theta_{i,min}$, therefore $\theta_{i,t+1}(\bar{x} + \tilde{\epsilon}, \Phi = \infty) > \theta_{i,min}$, therefore since $\theta_{i,t+1}$ is continuous in Φ, there exist $\Phi^0 = (\Phi_1^0, ..., \Phi_n^0)$ such that $\forall \Phi \geq \Phi^0$, $\forall i,\ \theta_{i,t+1}(\bar{x} + \tilde{\epsilon}, \Phi) > \theta_{i,min}$. We consider such a Φ^0 and as from now we assume that $\Phi \geq \Phi^0$.

We notice that $L_i(\bar{x} + \tilde{\epsilon}) = \sum_{j=0}^{n} (\bar{x}_{ij} + \tilde{\epsilon}_{ij}) q_{ij} P_j (1-\Delta_j) = \sum_{j=0}^{n} \bar{x}_{ij} q_{ij} P_j (1-\Delta_j) + \epsilon = L_i(\bar{x}) + \epsilon$.

We also know that $\forall i \in \{1, ..., p\},\ \forall x_i \in [0,1]^n$ *s.t.* $\theta_{i,t+1}(x_i, \bar{x}_{-i} + \tilde{\epsilon}_{-i}) \geq \theta_{i,min} :\ L_i(x_i, \bar{x}_{-i} + \tilde{\epsilon}_{-i}) \geq L_i(\bar{x})$. Indeed, $L_i(\bar{x})$ is the infimum cost for bank i.

Therefore, $\forall i \in \{1, ..., p\},\ \forall x_i \in [0,1]^n$ *s.t.* $\theta_{i,t+1}(x_i, \bar{x}_{-i} + \tilde{\epsilon}_{-i}) \geq \theta_{i,min} :\ L_i(x_i, \bar{x}_{-i} + \tilde{\epsilon}_{-i}) \geq L_i(\bar{x} + \tilde{\epsilon}) - \epsilon$, and therefore $\bar{x} + \tilde{\epsilon}$ is an ϵ-Nash equilibrium.

Consider $V(\bar{x},\epsilon) = \prod [\bar{x}_{ij}, \bar{x}_{ij} + \tilde{\epsilon}_{ij}]$.

$\forall \Phi \geq \Phi^0$, $V(\bar{x},\epsilon) \cap K_{\Phi_1,\dots,\Phi_n}$ is not empty and for $a \in V(\bar{x},\epsilon) \cap K_{\Phi_1,\dots,\Phi_n}$, $f_i(a) \leq L_i(\bar{x}) + \epsilon$.

Also $\forall x_i \in [0,1]^n$ *s.t.* $\theta_{i,t+1}(x_i, a_{-i}) \geq \theta_{i,min}$: $L_i(x_i, a_{-i}) \geq L_i(\bar{x})$, which implies that $L_i(x_i, a_{-i}) \geq L_i(a) - \epsilon$ and therefore a is an ϵ-Nash equilibrium.

As a consequence, all elements of $V(\bar{x},\epsilon) \cap K_{\Phi_1,\dots,\Phi_n}$ are ϵ-Nash equilibria. □

6 Technical proofs

Proof of Lemma 7 Given x_{-i}, the best response $(x_{i,1},...,x_{i,n})$ is such that the constraint is binding, that is,

$$\frac{E_i - \sum_{j=1}^{n} q_{ij} P_j (\Delta_j + \frac{\sum_{k\in\mathcal{B}} x_{kj} q_{kj}}{\Phi_j}(1-\Delta_j))}{\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j \left(1-\Delta_j\right)\left(1 - \frac{\sum_{k\in\mathcal{B}} x_{kj} q_{jk}}{\Phi_j}\right)(1-x_{ij})} = \theta_{i,min} \tag{3.10}$$

We shall now show that, by suitably relabeling the quantities, equation (3.10) can be written, up to functions in $\frac{1}{\Phi_j}$, as (3.7). Define $\epsilon_j(\Phi_j, x) := \frac{\sum_{k\in\mathcal{B}} x_{kj} q_{kj}}{\Phi_j}$. From eq (3.10), we have

$$\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j)(1-\epsilon_j(\Phi_j,x)) x_{ij} =$$
$$\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j)(1-\epsilon_j(\Phi_j,x)) - \frac{E_i - \sum_{j=1}^{n} q_{ij} P_j (\Delta_j + \epsilon(\Phi_j,x))(1-\Delta_j))}{\theta_{i,min}}$$

Let $X_{ij} = q_{ij} P_j (1-\Delta_j) x_{ij}$ and $C_i = \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) - \frac{1}{\theta_{i,min}}(E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j)$. Equation (3.10) is therefore equivalent to:

$$\sum_{j=1}^{n} \alpha_{ij} (1-\epsilon_j(\Phi_j,x)) X_{ij} = C_i - \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) \epsilon_j(\Phi_j,x) + \frac{\sum_{j=1}^{n} q_{ij} P_j \epsilon_j(\Phi_j,x)(1-\Delta_j))}{\theta_{i,min}}$$

Define $\eta_j(\Phi_j, x_{-i}) = \epsilon_j(\Phi_j, x) - \frac{x_{ij} q_{ij}}{\Phi_j}$. Equation (3.10) is equivalent to:

$\sum_{j=1}^{n} \alpha_{ij}(1-\epsilon_j(\Phi_j,x)) X_{ij} - \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) \frac{x_{ij} q_{ij}}{\Phi_j} + \frac{\sum_{j=1}^{n} q_{ij} P_j \frac{x_{ij} q_{ij}}{\Phi_j}(1-\Delta_j))}{\theta_{i,min}} = C_i - \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) \eta_j(\Phi_j, x_{-i}) + \frac{\sum_{j=1}^{n} q_{ij} P_j \eta_j(\Phi_j, x_{-i})(1-\Delta_j))}{\theta_{i,min}}$

which is in turn equivalent to

$\sum_{j=1}^{n} \alpha_{ij}(1-\epsilon_j(\Phi_j,x)) X_{ij} - \sum_{j=1}^{n} \frac{\alpha_{ij} q_{ij}}{\Phi_j} X_{ij} + \sum_{j=1}^{n} X_{ij} \frac{q_{ij}}{\Phi_j \theta_{i,min}} = C_i - \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) \eta_j(\Phi_j, x_{-i}) + \frac{\sum_{j=1}^{n} q_{ij} P_j \eta_j(\Phi_j, x_{-i})(1-\Delta_j))}{\theta_{i,min}}$

To get a more compact expression, let

$$\eta(\Phi, x_{-i}) = -\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) \eta_j(\Phi_j, x_{-i}) + \frac{\sum_{j=1}^{n} q_{ij} P_j \eta_j(\Phi_j, x_{-i})(1-\Delta_j))}{\theta_{i,min}}$$

It thus follows that (3.10) is equivalent to:

$$\sum_{j=1}^{n}\alpha_{ij}(1-\epsilon_j(\Phi_j,x)-\frac{\alpha_{ij}q_{ij}}{\Phi_j}+\frac{q_{ij}}{\Phi_j\theta_{i,min}})X_{ij}=C_i+\eta(\Phi,x_{-i})$$

Letting now $\delta_j(\Phi_j,x)=\epsilon_j(\Phi_j,x)-\frac{\alpha_{ij}q_{ij}}{\Phi_j}+\frac{q_{ij}}{\Phi_j\theta_{i,min}}$, equation (3.10) is finally equivalent to:

$$\sum_{j=1}^{n}\alpha_{ij}(1-\delta_j(\Phi_j,x))X_{ij}=C_i+\eta(\Phi,x_{-i})$$

an expression, up to functions in $\frac{1}{\Phi_j}$, identical to equation (3.7) and note that $\delta_j(\Phi_j,x)\leq\delta_j(\Phi_j,1)$ for all $x\in[0,1]^{np}$.

By assumption, $\alpha_{i1}>\alpha_{i2}>...>\alpha_{in}$, i.e., for all $j\in[2,..,n]$, $\frac{\alpha_{ij}}{\alpha_{ij-1}}<1$ so that there exists $\delta^0_{ij}>0$ such that $\frac{\alpha_{ij}}{\alpha_{ij-1}}=1-\delta^0_{ij}$. We know that for all i and all j, $\lim_{\Phi_j=\infty}\delta_{ij}(\Phi_j,1)=0$. There thus exists $\Phi^{i,0}_j$ such that for all $\Phi_j\geq\Phi^{i,0}_j$, $\delta_{ij}(\Phi_j,1)<\delta^0_{ij}$. As a result, for all $\Phi^i\geq\Phi^{i,0}=(\Phi^{i,0}_1,..,\Phi^{i,0}_n)$ and all $x\in[0,1]^{np}$, we have

$$\alpha_{i1}(1-\delta_{i1}(\Phi^i_1,x))>\alpha_{i2}(1-\delta_{i2}(\Phi^i_2,x))>...>\alpha_{in}(1-\delta_{in}(\Phi^i_n,x))$$

Since bank i is seeking to minimize $\sum_{j=1}^{n}X_{ij}$ we are back in Proposition 11, that is, it is optimal to sell assets by decreasing risk weights. For this result to be true for all banks, it suffices to take $\Phi^0_j=\sup_i\Phi^{i,0}_j$, and $\Phi\geq\Phi^0=(\Phi^0_1,..,\Phi^0_n)$. $\square$

7 Appendix: calculation of Φ^0 and Φ'

In this section, we propose to establish the values of Φ^0 and Φ'.

- Φ^0 :

  We remind the two following definitions introduced in the proof of Lemma 7: $\epsilon_j(\Phi_j,x):=\frac{\sum_{k\in\mathcal{B}}x_{kj}q_{kj}}{\Phi_j}$ and $\delta_j(\Phi_j,x)=\epsilon_j(\Phi_j,x)-\frac{\alpha_{ij}q_{ij}}{\Phi_j}+\frac{q_{ij}}{\Phi_j\theta_{i,min}}$.
  δ^0_{ij} is defined such that $\frac{\alpha_{ij}}{\alpha_{ij-1}}=1-\delta^0_{ij}$.
  We solve the inequation $\delta_{ij}(\Phi_j,1)<\delta^0_{ij}$:
  $\delta_{ij}(\Phi_j,1)<\delta^0_{ij}\iff\epsilon_j(\Phi_j,1)-\frac{\alpha_{ij}q_{ij}}{\Phi_j}+\frac{q_{ij}}{\Phi_j\theta_{i,min}}<\delta^0_{ij}$
  $\iff\frac{\sum_{k\in\mathcal{B}}q_{kj}}{\Phi_j}-\frac{\alpha_{ij}q_{ij}}{\Phi_j}+\frac{q_{ij}}{\Phi_j\theta_{i,min}}<1-\frac{\alpha_{ij}}{\alpha_{ij-1}}$
  We can thus take: $\Phi^{i,0}_j=\frac{\sum_{k\in\mathcal{B}}q_{kj}-\alpha_{ij}q_{ij}+\frac{q_{ij}}{\theta_{i,min}}}{1-\frac{\alpha_{ij}}{\alpha_{ij-1}}}$

And $\Phi_j^0 = \sup_i \Phi_j^{i,0} = \sup_i \frac{\sum_{k\in\mathcal{B}} q_{kj} - \alpha_{ij} q_{ij} + \frac{q_{ij}}{\theta_{i,min}}}{1 - \frac{\alpha_{ij}}{\alpha_{ij-1}}}$ for all j, which gives us Φ^0.

- $\Phi^{'}$

  To find $\Phi^{'}$, we need to solve:

  $\alpha_{ij} N(x)[1 + \frac{q_{ij}}{\Phi_j}(1 - x_{ij}) - \frac{\sum_{k\in B} x_{kj} q_{kj}}{\Phi_j}] - \frac{q_{ij}}{\Phi_j} D(x) > 0$, for all i and j.

  We note that :

  $\alpha_{ij} N(x)[1 + \frac{q_{ij}}{\Phi_j}(1 - x_{ij}) - \frac{\sum_{k\in B} x_{kj} q_{kj}}{\Phi_j}] - \frac{q_{ij}}{\Phi_j} D(x) > \alpha_{ij} N(1)[1 - \frac{\sum_{k\in B} q_{kj}}{\Phi_j}] - \frac{q_{ij}}{\Phi_j} D(0)$

  We solve the inequation:

  $\alpha_{ij} N(1)[1 - \frac{\sum_{k\in B} q_{kj}}{\Phi_j}] - \frac{q_{ij}}{\Phi_j} D(0) > 0 \iff \Phi_j > \frac{q_{ij} D(0) + \sum_{k\in B} q_{kj}}{\alpha_{ij} N(1)}$

  $\iff \Phi_j > \frac{q_{ij} \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) + \sum_{k\in B} q_{kj}}{\alpha_{ij}[E_i - \sum_{j=1}^{n} q_{ij} P_j \times (\Delta_j + \frac{\sum_{k\in\mathcal{B}} q_{kj}}{\Phi_j}(1-\Delta_j))]}$

  We can thus take:

  $\Phi_j^{'} = \sup_i \frac{q_{ij} \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) + \sum_{k\in B} q_{kj}}{\alpha_{ij}[E_i - \sum_{j=1}^{n} q_{ij} P_j \times (\Delta_j + \frac{\sum_{k\in\mathcal{B}} q_{kj}}{\Phi_j}(1-\Delta_j))]}$ for all j, which gives us $\Phi^{'}$.

Note importantly that the two market depth Φ^0 and $\Phi^{'}$ only depend upon the input of the model. It would be easy to consider numerical examples within a two assets two banks framework.

Chapter 4

Generalized games and Optimal Regulation in Economics and collective action problems. An application to bank regulation during fire sales

Abstract

Spontaneously, a regulation on a set of economic agents boils down to the definition of the possible and restricted actions for each agent, which is therefore a variable strategy set. In classical games, interaction between economic agents only takes place through the payoff/cost function and the strategy set of each agent is given exogenously and constant. Generalized games are games where the strategy set of each agent depends on the strategy picked by all other agents, and this perfectly matches with the definition of a regulation/regulator. In this paper, we propose to model the role and action of a regulator in a game theory framework with generalized games: the regulator is seeking to maximize one or several social/collective welfare functions - which we call regulation criteria - for the collective well-being of the community of agents, and we look for existence of possible optimal regulations and other interesting regulatory characteristics. The interaction between the regulator and the economic agents happens through the strategy set X_i of each agent. We prove existence of an optimal regulation when the set E of all strategy vectors is finished. We illustrate our theory and results with an example of bank regulation during fire sales.

Keywords: Optimal regulation, game theory, generalized games, games with constraints, regulatory economics, bank regulation.

1 Introduction

Game theory has proven to be a usefool tool in many fields among which Economics, Political sciences, International relations and Social sciences (to name a few) since its introduction by [Morgenstern and Von Neumann, 1953], [Nash Jr, 1950] and [Nash, 1951]. It appears to be the natural mathematical framework one would use to quantify, understand and predict the outcome of a particular economic or social interaction between a set of agents who have a strategical connection with one another. Remarkably, it has also proven to be a fruitful theory to create new ideas and new ways of thinking in collective actions problems: it has permitted the creation of new concepts, take for instance the Nash equilibria or Pareto efficiency, that give a new grasp and point of view in Economics or Social sciences (and more broadly in every field it can be applied). It has frequently given birth to new paradigms, new theories that have profoundly marked these different disciplines. Game theory has not only a power of description of the reality in economics, social and political sciences, but it has also been the tool of creation of new economic, social and political realities. We refer to [Fudenberg and Tirole, 1991], [Brams, 2011], [Schelling, 1980], [Moulin, 1986] and [Demange and Ponssard, 1994] for classical textbooks.

In this article, we consider the point of view of a regulator, for instance a State or a Public policymaker, which has the ability to enforce a law and decisions on a group of N agents interacting with one another in a game theory framework: the goal of the regulator is to maximize a social welfare criterion (or minimize a social cost) for the community of agents. Indeed, it is often the case that when agents act individually in maximizing their own payoff function (or minimizing their own cost) without the help/action of a regulator, the consequences of isolated individual actions can lead to devastating consequences for the whole community of agents.

The Financial crisis of 2007-2008 is a striking example of a situation where the absence of regulation and coordination between banks, and particularly Global Systemically Important Banks (G-SIBs), on the global scale (on certain issues at least) has caused dramatic consequences on the whole financial system and world economies. Such absence of initial regulation has partly been resolved with the Basel Accords issued by the Basel Committee on Banking Supervision (BCBS).

The global environmental crisis we are currently facing and the difficulty in finding a regulation that satisfies all the countries and compatible with the economic challenges and ambitions of every country is another striking problem.

More generally, it is interesting to consider the point of view of a regulator in a framework where

N agents are interacting with one another in a game theory framework and to wonder what is the best decision from a regulator point of view. A prerequisite to answer this question is: what is the good criterion of social welfare from the regulatory point of view? And once this criterion has been defined, is there an optimal regulation that maximizes this criterion?

There has been an abundant literature on Optimal Regulation in the past decades, the books [Laffont and Tirole, 1993], [Train et al., 1991], [Laffont and Martimort, 2009] and the article [Laffont, 1994] give a good overview of the research that has been done on this topic. We take a new roadmap in our article in the sense that we consider directly a regulator on a game theory framework: for a given game, we define one or several regulation criteria and we study existence of an optimal regulation for each criterion.

What is a regulation? Spontaneously, we think of a regulation as the definition of a set of possible and restricted actions for an economic agent, taking into account the potential actions of all participants/agents interacting with one another, in order to fulfill one or several social/collective welfare criteria defined by the regulator.

It appears that generalized games seem to be the best tool to model the action of a regulator on a system of agents interacting with one another: a generalized game is a game where the set of strategies of each agent depends on the strategies taken by other agents, and this perfectly matches with our definition of a regulation on a set of economic agents. Generalized games, also called Generalized Nash Equilibrium Problems (GNEPs for short), have first been introduced by [Arrow and Debreu, 1954], and have gained an increasing attention in operational research over the past decades. In their well-known survey, [Facchinei and Kanzow, 2007] give a global overview of generalized games and remind interesting examples of applications of such games in environment policy or in telecommunication. However, to the best of our knowledge, there has been no paper trying to apply generalized games to optimal regulation. And only a few papers so far are trying to apply generalized games in Economics (but see [Breton et al., 2006], [Elfoutayeni et al., 2012], [Le Cadre et al., 2020], and see [Kulkarni, 2017] for a short review).

In our framework, the regulator is seeking to maximize a social welfare criterion (or minimize a social cost criterion), the variable is the regulation itself and we are looking for the possible optimal regulation that maximizes (respectively minimizes) this given criterion of social welfare (respectively social cost). Various examples of such criteria are given in the article.

The purpose of our article is to draw and propose an optimal regulation road map for any game theory model where the action of a regulator is relevant: first we consider the different regulation criteria of social welfare that are relevant from the regulator point of view, then we study the existence of an optimal regulation for each regulation criterion. Such a research roadmap gives answers to many questions on the regulatory point of view, among which: is there existence of at least an optimal regulation for each regulation criterion? Is there uniqueness of the optimal regulation? Are some regulation criteria equivalent: are there examples where a criterion A and a criterion B have the same optimal regulation? When can we say that it is equivalent to optimize a regulation criterion A and a regulation criterion B? It is precisely the aim of this paper to answer these different questions.

For instance, we will study the example of a financial regulator trying to implement a regulation (enforcing some laws and taking some decisions) to maintain the stability of the financial system during a financial crisis: what are the good criteria from a financial regulation perspective? Minimize the losses in the financial systems? Minimize the number of banks and companies going bankrupt? Minimize the number of jobs destroyed by the crisis? Can we prove that these minimization problems are equivalent and strongly correlated? Or are they not correlated at all?

Such a complete study gives a global overview of the questions a regulator can wonder and the levers of action at its disposal to complete its social goal and mission: it is the role of the regulator and policymaker to decide what is the good criterion of social welfare for the collective well-being of a community of economic agents, and therefore what is the optimal regulation. Our inquiry gives a radiography of the different mechanisms of action between the regulator and the agents, what is the best policy to enforce for the social welfare of a community of agents and what are the consequences of the decision of a regulator on this community.

The paper is structured as follows: in Section 2 we give two examples of regulation models as generalized games, we remind a few definitions and properties about generalized games, and we introduce the definitions of a regulation, regulation criterion and optimal regulation on a game. We study the particular case of existence of optimal vector(s). We also introduce the definitions of a surregulation, subregulation and correlated regulation criteria. We give a few examples and properties. In Section 3 we study in detail a regulation model of strategic fire-sales and price-mediated contagion in the banking system inspired from [Braouezec and Wagalath, 2019]: we consider different regulation criteria and we look for the existence of possible optimal regulations. Section 4 concludes the paper.

2 Regulation and Generalized games

2.1 Two examples of regulation models as generalized games

We can begin our inquiry by giving two relevant examples of regulation models using generalized games.

Our first example is inspired from environmental regulation: in papers like [Tidball and Zaccour, 2005] or [Krawczyk, 2005] they consider models of regulation of carbon emissions, which have been discussed in many articles (see for instance [Mathiesen, 2008]). [Tidball and Zaccour, 2005] consider a model where a regulator is seeking to cap the carbon emissions of each country, which is a function proportional to the production of each country: each country seeks to maximize a payoff function $w_i(x_1, ..., x_n) = f_i(x_i) - d_i(x_1 + ... + x_n)$, where x_i represent the emissions of country i and is proportional to the production of country i, $f_i(x_i)$ is a non-negative, twice-differentiable, concave and increasing function, and the damage cost due to all the countries is denoted by a convex twice-differentiable increasing cost function $d_i(x_1 + ... + x_n)$.

They consider 2 types of regulation:

- A first regulation where each country/agent has to satisfy the constraint $x_i \leq E_i$ with E_i an exogenous given upper bound on emissions.
- A second regulation in a cooperative scenario, where we assume that the countries/agents have to satisfy the common constraint $\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} E_i$, which is a generalized game.

Our second example is inspired from bank regulation: in [Braouezec and Wagalath, 2019] they consider a model of stress tests and fire sales in the banking system after an exogenous financial shock where each bank is seeking to minimize a cost function of the form $f_i(x) = x_i$ under a risk-based capital ratio constraint of the form $RBC_i(x) \geq RBC_{i,min}$ defined in the Basel agreements: this is a perfect example of a regulation model with a generalized game where the constraint $RBC_i(x) \geq RBC_{i,min}$ of each bank/agent i defines a set of possible strategies depending on the actions of all other banks/agents $j \neq i$.

Note that in both cases the regulator is in complete information: the regulator has all the relevant information about all the participants, and we will keep this assumption in the rest of the article. This is a reasonable assumption since if the players are in complete information, a regulator that has more information that the players themselves will also be in complete information.

Generalized games are the right concept to define the role of a regulator in a context where economic agents are interacting with one another in a game theory framework, through the constraints on the strategy set of each agent. We will therefore remind and discuss a few definitions about these games in the remaining part of this section before developing our inquiry on optimal regulations in the next section.

2.2 Generalized games and regulation: notations and definitions

We take the following classical notations throughout the paper: consider a Game with N agents/players, we denote $J = \{1, .., N\}$ the set of agents/players, each agent $i \in J$ controlling the variable $x_i \in E_i$, with E_i a subset of $\mathbb{R}^{n_i}$; x_i is called the strategy or decision or state of agent i and E_i is called the strategy set. We denote by $x \in E = E_1 \times \ldots \times E_N$ the vector formed by all these decision variables:

$$x := \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \in E \tag{4.1}$$

which has dimension $n := \sum_{i=1}^{N} n_i$ and such that $E \subseteq \mathbb{R}^n$. We denote by $x_{-i} \in E_{-i}$ the vector formed by all the agents' decision variables except those of agent i. To emphasize the i-th agent's variables within x, we sometimes write $(x_i, x_{-i}) \in E_i \times E_{-i}$ instead of $x \in E$.

Each agent has an objective function $\theta_i : \mathbb{R}^n \to \mathbb{R}$ that depends on both her own variable x_i as well as on the variables x_{-i} of all other agents. This mapping θ_i is often called the payoff function of agent i when the agents are seeking to maximize $\theta_i(x_i, x_{-i})$, or it can also be called the loss function or cost function of agent i when the agents are seeking to minimize $\theta_i(x_i, x_{-i})$, depending on the particular application in which the Game arises. Throughout the article, if not mentioned otherwise, we will say that θ_i is a payoff function and that each agent i, given the other agents' strategies x_{-i}, is seeking a strategy x_i to maximize $\theta_i(x_i, x_{-i}) = \theta_i(x)$.

A regulation (emanating from a regulator) is the given of n constraint functions $X_i : E_{-i} \to P(E_i)$ (where $P(E_i)$ denotes the set of all subsets of E_i), such that each agent's strategy x_i must belong to the set $X_i(x_{-i}) \subseteq E_i$ that depends on the rival agents' strategies and that we call the feasible set or strategy space of agent i. A regulation is therefore the set of all possible choices/decisions for every agent i, given what all others agents do.

In this article, we focus on static games for simplicity. We could obviously consider a dynamic or sequential game with a time dimension such that X_i is a function of x_{-i} and time t. However, for the sake of simplicity, we will only consider static games in this article, though most of our results could be extended to dynamic or sequential games. This will be the topic of a future article. The constraint functions X_i, $i \in J$ can be given for instance by an economic or social regulator, such as in the 2 examples in subsection 2.1..

If $x \in E$ is such that $x_i \in X_i(x_{-i})$ for a given agent i we will say that the strategy vector x is admissible for agent i or the constraint is satisfied on the individual level for i. And if $x_i \in X_i(x_{-i})$ for all $i \in J$ we will say that the strategy vector x is admissible or the constraint is satisfied on the global level. If there is $i \in J$ such that $x_i \notin X_i(x_{-i})$, then we will say that the strategy vector x is not admissible for agent i or the constraint is not satisfied on the individual level for i, and therefore we will also say that the strategy vector x is not admissible.

Each agent i, given the other agents' strategies x_{-i}, is seeking a strategy x_i to minimize $\theta_i(x_i, x_{-i}) = \theta_i(x)$ and respecting the constraint $x_i \in X_i(x_{-i})$: this perfectly matches with the definition of a generalized game, or Generalized Nash Equilibrium Problem (GNEP for short). Generalized games are therefore the natural tool to describe the action of a regulator in a game theoretical framework and we study the given of the N constrained maximization problems, for $i \in J = \{1, .., N\}$:

$$\text{maximize}_{x_i} \quad \theta_i(x_i, x_{-i}) \text{ subject to } x_i \in X_i(x_{-i}) \tag{4.2}$$

and it is defined in a unique way by the 4-uple $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

Actually, to emphasize the fact that each agent has a given exogenous individual constraint function X_i, we often find in the literature (see for instance [Fischer et al., 2014] or [Braouezec and Kiani, 2021a]) the denomination generalized game with individual constraints to name such generalized games $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

Definition 15 *The 4-uplet $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ is called a generalized game with individual constraints.*

Definition 16 *We consider the generalized game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.*

$(X_i)_{i\in J}$ is called a regulation on the Game $(J, E, (\theta_i)_{i\in J})$ and the agents $i \in J$.

We can also clearly define the set of admissible strategies of a generalized game.

Definition 17 *For a given generalized game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$, let K be the subset of E defined as follows.*

$$K = \{\boldsymbol{x} \in E,\ \forall i \in J,\ x_i \in X_i(x_{-i})\} \tag{4.3}$$

K is called the set of admissible strategies of the generalized game with individual constraints.

The set K represents the set of all the strategies $\boldsymbol{x}$ for which the generalized game with individual constraints is defined for all agents.

Now we can introduce the definition of a Nash equilibrium for a generalized game with individual constraints. A generalized Nash equilibrium is a profile of strategies $\boldsymbol{x}^* = (x_1^*, ..., x_N^*) \in K$ such that no agent i wants to unilaterally deviate from her part of the equilibrium profile $\boldsymbol{x}^*$. However, for $\boldsymbol{x}^* \in E$ to be an equilibrium profile, the constraint of each agent $i \in J$ should be satisfied. The following definition makes clear this constraint.

Definition 18 *The profile of strategies $\boldsymbol{x}^* \in E$ is a Nash equilibrium for the generalized game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ if for each $i \in J$ and each $x_i \in E_i$ such that $x_i \in X_i(x_{-i}^*)$, it holds true that $\theta_i(x_i^*, x_{-i}^*) \geq \theta_i(x_i, x_{-i}^*)$.*

We note that this definition is well consistent with the definition of a Nash equilibrium in a classical game.

In [Braouezec and Kiani, 2021a] they introduce the notion of endogenous shared constraint generated from individual constraints that will be useful in the present article. We remind a few definitions and properties.

Given x_{-i}, let $K_i(x_{-i})$ be the set of strategies of agent i defined as follows

$$K_i(x_{-i}) = \{x_i \in E_i : x \in K\} \tag{4.4}$$

Definition 19 *The 4-uplet $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ is called a generalized game with shared constraint generated from the game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. We call such a* *game a game with endogenous shared constraint. Such a game with shared constraint can also be denoted $(J, E, (\theta_i)_{i\in J}, K)$ where K is the shared constraint set.*

Note that for all $i \in J$ and for all x_{-i}, we have the following inclusion: $K_i(x_{-i}) \subseteq X_i(x_{-i})$.

We have the following resulting definition of a Nash equilibrium in endogenous shared constraint.

Definition 20 *The profile of strategies $\boldsymbol{x}^* \in E$ is a Nash equilibrium for the generalized game with endogenous shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ if for each $i \in J$ and each $x_i \in E_i$ such that $x_i \in K_i(x^*_{-i})$, it holds true that $\theta_i(x^*_i, x^*_{-i}) \geq \theta_i(x_i, x^*_{-i})$.*

The two following result are widely discussed in [Braouezec and Kiani, 2021a] and will be used in the following subsections and sections.

Proposition 14 *The set of Nash equilibria of a game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ denoted $\mathcal{N}_{Indiv}$ is included in the set of the Nash equilibria of the game with shared constraint generated from the individual constraints $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$, denoted $\mathcal{N}_{Shared}$; that is, $\mathcal{N}_{Indiv} \subseteq \mathcal{N}_{Shared}$.*

Proposition 15 *Consider a game $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ such that for each $i \in J$, $\theta_i(x) = \theta_i(x_i)$ and assume that for each $i \in J$, θ_i is continuous and that K is compact. The following results hold.*

1. *The game with shared constraint $(J, E, (\theta_i)_{i\in J}, K)$ admits at least one Nash equilibrium.*
2. *The set of Nash equilibria in shared constraint $\mathcal{N}_{Shared}$ exactly coincides with the set of minimizers of the total cost function $\sum_{i\in J} \theta_i(x_i)$ on K and each Nash equilibrium is Pareto optimal.*

2.3 Regulation and Regulation criterion

As seen previously, a regulation on a set of N agents interacting with one another in a game theory framework intuitively boils down to giving a $N-uple$ of individual constraints $(X_i)_{i\in J}$: each agent i has to satisfy the constraint X_i given by a regulator. For instance, as we have seen previously, in an environment policy problem X_i can be a threshold of emission of greenhouse gaz given by a regulator (see [Braouezec and Kiani, 2021a]) to a player i, or in a bank regulation problem X_i can be the minimum risk-based capital ratio each bank i has to satisfy (and which depends on the actions of all banks). In Section 4, we will detail this example in bank regulation where X_i is a threshold given to each bank i by the regulator via the Basel agreements.

Let us now move to the definition of a criterion of well-being - that we will call regulation criterion - for the community of agents that the regulator is seeking to maximize (or minimize) by choosing the right regulation(s). Intuitively, and staying as much general as we can in defining our concepts, a regulation criterion is a function of well-being that the regulator is seeking to maximize (or minimize) for the collective well-being of a group agents. The variable is the regulation itself. For instance, in environmental policy, the goal of the regulator is to minimize the emission of greenhouse gases. In bank regulation, the role played by the regulator can imply several criteria of social/collective welfare: minimizing the losses in the financial system, minimizing the number of banks going bankrupt during a financial crisis, maximizing the stability of the financial system... In Section 4, we will study in details the example in bank regulation where the regulator is either seeking to minimize the financial losses in the system, or minimize the number of banks going bankrupt. We will also see that in a period of crisis and instability the regulator can be seeking to get a Nash equilibrium as fast as possible which guarantees the end of instability, and therefore will prefer a regulation giving a large number of Nash equilibria that enable a diminution of volatilty and instability (a system of agents in a Nash equilibrium configuration does not move over time). The definition below makes clear the general concept of regulation criterion.

Definition 21 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. *We call regulation criterion on the Game* $(J, E, (\theta_i)_{i\in J})$ *any application* R *of the form:*

$$\begin{array}{rcccc} R & : & (F(E_{-i}, P(E_i)))_{i\in J} & \rightarrow & (V, \leq) \\ & & (X_i)_{i\in J} & \mapsto & R((X_i)_{i\in J}) \end{array}$$

with $(V, \leq)$ *a partially ordered set.*

Note that if R_1 and R_2 are two criteria on the Game $(J, E, (\theta_i)_{i\in J})$ with values in $(V, \leq)$ and g is an application from $(V, \leq)^2$ to a partially ordered set $(W, \leq)$, then $g(R_1, R_2)$ is also a criterion on the Game $(J, E, (\theta_i)_{i\in J})$. More generally, if R_1, R_2,..,R_p are p criteria on the Game $(J, E, (\theta_i)_{i\in J})$ with values in $(V, \leq)$ and g is an application from $(V, \leq)^p$ to a partially ordered set $(W, \leq)$, then $g(R_1, R_2, .., R_p)$ is also a criterion on the Game $(J, E, (\theta_i)_{i\in J})$.

We will study 4 examples of regulation criteria in this subsection, two criteria that we will call stability-type criteria (in the sense that the goal of the regulator is to maximize stability) and two payoff-type criteria (in the sense that the goal of the regulator is to maximize the global payoff).

We start with our 2 examples of stability-type criteria.

Example 1 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. $(X_i)_{i\in J}$ *is a regulation on the Game* $(J, E, (\theta_i)_{i\in J})$.

We can define the regulation criterion:

$$\begin{array}{rclcl} Nash & : & (F(E_{-i},P(E_i)))_{i\in J} & \to & (P(E), \subseteq) \\ & & (X_i)_{i\in J} & \mapsto & Nash((X_i)_{i\in J}) \end{array}$$

with $Nash((X_i)_{i\in J})$ *the set of Nash equilibria of the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

This regulation criterion is called the Nash-set stability criterion. If $(X_i)_{i\in J}$ *and* $(Y_i)_{i\in J}$ *are two regulations such that the set of Nash equilibria of the generalized game* $(J, E, (\theta_i)_{i\in J}, (Y_i)_{i\in J})$ *contains the set of Nash equilibria of the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$, *then* $Nash((X_i)_{i\in J}) \subseteq Nash((Y_i)_{i\in J})$. *We note that* $(P(E), \subseteq)$ *is not a totally ordered set.*

Example 2 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *and we assume that E is a finite set. Since E is a finite set, the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *has a finite number of Nash equilibria.*

We can define the regulation criterion:

$$\begin{array}{rclcl} Stab & : & (F(E_{-i},P(E_i)))_{i\in J} & \to & (\mathbb{N}, \leq) \\ & & (X_i)_{i\in J} & \mapsto & \text{number of Nash equilibria of } (J,E,(\theta_i)_{i\in J}, (X_i)_{i\in J}) \end{array}$$

This regulation criterion is called the stability criterion. The more Nash equilibria given by the regulation $(X_i)_{i\in J}$ *on the Game* $(J, E, (\theta_i)_{i\in J})$, *the higher* $R((X_i)_{i\in J})$, *and the higher the stability criterion: if* $(X_i)_{i\in J}$ *and* $(Y_i)_{i\in J}$ *are two regulations such that the generalized game* $(J, E, (\theta_i)_{i\in J}, (Y_i)_{i\in J})$ *has more Nash equilibria than the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$, *then* $Stab((Y_i)_{i\in J}) \geq Stab((X_i)_{i\in J})$.

Fact 9 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *and the generalized game with shared constraint* $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ *generated from* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

Then, $Nash((X_i)_{i\in J}) \subseteq Nash((K_i)_{i\in J})$.

Moreover, if E is finite set, $Stab((X_i)_{i\in J}) \leq Stab((K_i)_{i\in J})$.

Proof of Fact 9

This is a direct consequence of Proposition 14. □

We now study our two examples of payoff-type regulation criteria.

Example 3 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. $(X_i)_{i\in J}$ *is a regulation on the Game* $(J, E, (\theta_i)_{i\in J})$.

We can define the regulation criterion:

$$\begin{array}{rcl} T : (F(E_{-i},P(E_i)))_{i\in J} & \rightarrow & (\mathbb{R},\leq) \\ (X_i)_{i\in J} & \mapsto & sup_{x\in K((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x) \end{array}$$

with $K((X_i)_{i\in J})$ *the set of admissible strategies of the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. $T((X_i)_{i\in J})$ *is the supremum of the total payoffs among all the admissible strategies of the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. *If this supremum is higher for the regulation* $(Y_i)_{i\in J}$ *than for the regulation* $(X_i)_{i\in J}$, *then* $T((Y_i)_{i\in J}) \geq T((X_i)_{i\in J})$. *Of course, if there exists* x *such that* $T((X_i)_{i\in J}) = \sum_{i=1}^{N} \theta_i(x)$ *then such a vector* x *is Pareto Optimal.*

If θ_i is a cost function instead of a payoff function, we can define this regulation criterion in a symmetric way $T(X_i)_{i\in J}) = inf_{x\in K((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x)$, and here is the goal of a regulator is to minimize a global cost instead of maximizing a global payoff.

Fact 10 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *and the generalized game with shared constraint* $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ *generated from* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

Then, $T((X_i)_{i\in J}) = T((K_i)_{i\in J})$.

Proof of Fact 10

By definition, $K((X_i)_{i\in J}) = K((K_i)_{i\in J})$ and therefore $T((X_i)_{i\in J}) = T((K_i)_{i\in J})$. $\square$

Example 4 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

We can define the regulation criterion:

$$\begin{array}{rcl} TN : (F(E_{-i},P(E_i)))_{i\in J} & \rightarrow & (\mathbb{R},\leq) \\ (X_i)_{i\in J} & \mapsto & sup_{x^*\in Nash((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x^*) \end{array}$$

with $Nash((X_i)_{i\in J})$ *the set of Nash equilibria of the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

$TN((X_i)_{i\in J})$ *is the supremum of the total payoffs among all the Nash equilibria of the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. *If this supremum is higher for the regulation* $(Y_i)_{i\in J}$ *than for the regulation* $(X_i)_{i\in J}$, *then* $TN((Y_i)_{i\in J}) \geq TN((X_i)_{i\in J})$. *Of course, if there exists* x^* *such that* $TN((X_i)_{i\in J}) = \sum_{i=1}^{N} \theta_i(x^*)$ *then such a vector* x^* *is Pareto Optimal.*

Same remark as for Example 3, if θ_i is a cost function instead of a payoff function, we can define this regulation criterion in a symmetric way $TN(X_i)_{i\in J}) = inf_{x^* \in Nash((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x^*)$, and here is the goal of a regulator is to minimize a global cost instead of maximizing a global payoff.

Fact 11 *Given a generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$*, we have that* $T((X_i)_{i\in J}) \geq TN((X_i)_{i\in J})$

Proof of Fact 11

$sup_{x\in K((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x) \geq sup_{x^*\in Nash((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x^*)$ since $Nash((X_i)_{i\in J}) \subseteq K((X_i)_{i\in J})$.
□

Fact 12 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *and the generalized game with shared constraint* $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ *generated from* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

Then, $TN((X_i)_{i\in J}) \leq TN((K_i)_{i\in J})$.

Proof of Fact 12

From Proposition 14 we have that $Nash((X_i)_{i\in J}) \subseteq Nash((K_i)_{i\in J})$ and therefore $TN((X_i)_{i\in J}) \leq TN((K_i)_{i\in J})$. □

Fact 13 *Consider a generalized game with shared constraint* $(J, E, (\theta_i)_{i\in J}, X)$ *such that* X *is a compact set, and that* $\forall i \in J$, θ_i *is continuous and of the form* $\theta_i(x) = \theta_i(x_i)$ *. For such a game, we have that* $T(X) = TN(X)$.

Proof of Fact 13

This is a direct consequence of Proposition 15. □

2.4 Optimal Regulation

Now that we have defined what a regulation is and what a regulation criterion is, we can easily define a concept of optimal regulation, and study existence of one or several optimal regulation(s) for any game. In this subsection, we assume that the regulator is seeking to maximize a regulation criterion R, function of collective well-being (it is symmetric if we assume that the regulator is seeking to minimize a regulation criterion).

First, we explain what we mean when we say that a regulation $(Y_i)_{i\in J}$ is better than a regulation $(X_i)_{i\in J}$ for the criterion R, which is quite intuitive.

Definition 22 *We say that a regulation $(Y_i)_{i\in J}$ is better than a regulation $(X_i)_{i\in J}$ for the criterion R on the Game $(J, E, (\theta_i)_{i\in J})$ if $R((Y_i)_{i\in J}) \geq R((X_i)_{i\in J})$. We denote $(Y_i)_{i\in J} \geq (X_i)_{i\in J}$ for the criterion R.*

Of course, the definition of "better"' is symmetric if the goal of the regulator is to minimize a social cost rather than maximize a function of social welfare. If the goal of the regulator is to minimize a social cost, we say that a regulation $(Y_i)_{i\in J}$ is better than a regulation $(X_i)_{i\in J}$ for the criterion R on the Game $(J, E, (\theta_i)_{i\in J})$ if $R((Y_i)_{i\in J}) \leq R((X_i)_{i\in J})$.

The corollary below gives us a direct example of a regulation better than another regulation not only for one regulation criterion but actually for our four examples of regulation criteria, $Nash$, *Stab*, T and TN: we always have that the regulation with endogenous shared constraint is better than the one with individual constraints.

Corollary 3 *We consider the generalized game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ and the generalized game with shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ generated from $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. Then, from Fact 9, 10 and 12, $(K_i)_{i\in J}$ is a better regulation than $(X_i)_{i\in J}$ for the criteria $Nash$, Stab (if E is a finite set), T and TN.*

This corollary simply means that if a regulator is seeking to maximize one of the 4 criteria $Nash$, *Stab*, T and TN, this regulator should always choose the regulation with endogenous shared constraint rather than the regulation with individual constraints: the regulation with endogenous shared constraint is better than the regulation with individual constraints for these four criteria.

More generally, given a regulation criterion, a natural process of thinking is to compare regulations between one another (when such a comparison is possible) and study if there is one (or several) regulation which is better than the others. Now that we have defined what is a regulation, a regulation criterion, and what we mean when we say that a regulation is better than another for a given criterion, it consequently makes sense to define what is an optimal regulation in the following way.

Definition 23

- *We say that a regulation $(Y_i)_{i\in J}$ is optimal (in the strong sense) for the criterion R on the Game $(J, E, (\theta_i)_{i\in J})$ if for all regulation $(X_i)_{i\in J}$ we have $R((Y_i)_{i\in J}) \geq R((X_i)_{i\in J})$.*
- *We say that a regulation $(Y_i)_{i\in J}$ is optimal in the weak sense for the criterion R on the Game $(J, E, (\theta_i)_{i\in J})$ if there is no $(X_i)_{i\in J}$ such that $R((X_i)_{i\in J}) > R((Y_i)_{i\in J})$.*

Note that an optimal regulation in the strong sense is also an optimal regulation in the weak sense (but the converse is not true). And note that both notions coincide if $(V, \leq)$ is a totally ordered set. Also note that such optimal regulations may fail to exist.

We give a few cases of existence of optimal regulations below.

Fact 14 *If E is a finite set:*

- *$Stab$, T and TN and always have at least one optimal regulation (in the strong sense).*
- *$Nash$ always have at least one optimal regulation in the weak sense.*

Proof. See the appendix.

We can also state the more general result, which is an extension of Fact 14:

Fact 15 *If E is a finite set:*

- *For any criterion R, there exists at least one optimal solution in the weak sense.*
- *For any criterion R such that $(V, \leq)$ is a totally ordered set, there exists an optimal regulation (in the strong sense).*

Proof. See the appendix.

We add the following additional result, which concerns the criteria where the set V is finite:

Fact 16 *If V is a finite set:*

- *For any criterion R, there exists at least one optimal solution in the weak sense.*
- *For any criterion R such that $(V, \leq)$ is a totally ordered set, there exists an optimal regulation (in the strong sense).*

Proof. See the appendix.

Remark 10 *Note that if E or V can be approximated by a finite set (which is usually the case when modeling real-world problems) the above property is true and we have existence of optimal regulations. A good extension (that we will not develop in this paper) would be to study the limit of sequences of games with finite strategy set to see if we can draw an existence result asymptotically when considering the limit of such a sequence of games.*

Note that if the regulator can act as a social planner, the problem is trivial and reduces to a classical optimization problem: the social planner can choose directly the vector that maximizes the social welfare or minimizes the social cost. In the present article, we make the assumption that the regulator may be required (from a legal or natural requirement for instance) to respect a minimum of freedom for each agent and that for all i there exists a minimum freedom set L_i such that given x_{-i}, $L_i(x_{-i}) \subseteq X_i(x_{-i})$, that is to say $X_i(x_{-i})$ could not be reduced to a singleton and should at least contain $L_i(x_{-i}) \neq \emptyset$.

In the next subsection, we study the particular case of optimal vectors. And in the next subsections we will also see that it often makes sense to say that a regulation is optimal among a subset of regulations. It is convenient and fruitful to proceed considering if a regulation is optimal among either its surregulations or subregulations.

2.5 Optimal vector

In the particular case where the optimal regulation for a criterion meets with the choice of a vector $\overline{x}$, we say that the game has an optimal vector for this given criterion.

Definition 24 *If there exists an Optimal regulation $(\overline{X_i})_{i\in J}$ for the criterion R on the Game $(J, E, (\theta_i)_{i\in J})$ which is of the form*

$$\begin{array}{rccc} \overline{X_i}: & E_{-i} & \to & P(E_i) \\ & x_{-i} & \mapsto & \{\overline{x_i}\} \end{array}$$

with $\overline{x} = (\overline{x_1}, ..., \overline{x_1}) \in E$, then we say that $\overline{x}$ is an optimal vector (either in the strong sense or in the weak sense).

Or equivalently, if there exists a vector of strategies $\overline{x}$ such that the Optimal regulation $(\overline{X_i})_{i\in J}$ for the criterion R on the Game $(J, E, (\theta_i)_{i\in J})$ satisfies $R((\overline{X_i})_{i\in J}) = R(\overline{x})$, we say that $\overline{x}$ is an Optimal vector (either in the strong sense or in the weak sense).

In other words, it is optimal from a regulatory point of view to act as a social planner and choose the vector of strategies $\overline{x}$ to maximize the total payoff (or minimize the total cost) for the agents.

Fact 17 *If E or V are finite sets, T and TN always have at least one optimal vector (in the strong sense).*

Proof. See the appendix.

2.6 Product regulation, Surregulations and Subregulations

We now introduce 2 new intuitive concepts that are meaningful and essential in the development of our theory. First, the product of two regulations: given two different regulations $(X_i)_{i\in J}$ and $(W_i)_{i\in J}$ on a game, it makes sense to consider how the mixing of these two different regulations affect economic agents. Take for instance a citizen or company located in a country with different states: this citizen or company lives with both the laws of the country and those of the local state, this is a product regulation. The following definition makes this concept clear.

Definition 25 *We consider two regulations $(X_i)_{i\in J}$ and $(W_i)_{i\in J}$ on the Game $(J, E, (\theta_i)_{i\in J})$. We define the product regulation of $(X_i)_{i\in J}$ and $(W_i)_{i\in J}$, and we denote $(X_i, W_i)_{i\in J}$ the regulation on the Game $(J, E, (\theta_i)_{i\in J})$ such that each agent $i \in J$ is required to satisfy $x_i \in X_i(x_{-i}) \cap W_i(x_{-i})$.*

We now introduce the concepts of surregulation and subregulation. It is common to hear that a regulation can be stricter or weaker than another one. It makes sense to say that a regulation is stricter than another regulation if the conditions to satisfy the first regulation are stricter than those to satisfy the second one: an economic agent satisfying the first regulation would consequently also satisfy the second regulation. For instance, environmental regulation since the Kyoto protocol is stricter than environmental regulation before the Kyoto protocol; or another example, bank regulation since 2008 is stricter than bank regulation between 2000 and 2007. The definition below makes clear these concepts of surregulation and subregulation.

Definition 26 *We consider two regulations $(X_i)_{i\in J}$ and $(W_i)_{i\in J}$ on the Game $(J, E, (\theta_i)_{i\in J})$. We say that $(W_i)_{i\in J}$ is a surregulation of $(X_i)_{i\in J}$ on the Game $(J, E, (\theta_i)_{i\in J})$, or that regulation $(W_i)_{i\in J}$ is stronger (or stricter) than regulation $(X_i)_{i\in J}$, if for every agent i and for all x, $W_i(x_{-i}) \subseteq X_i(x_{-i})$. We will also say that $(X_i)_{i\in J}$ is a subregulation of $(W_i)_{i\in J}$ on the Game $(J, E, (\theta_i)_{i\in J})$, or that regulation $(X_i)_{i\in J}$ is weaker (or less strict) than regulation $(W_i)_{i\in J}$.*

For instance, the product of two regulations is obviously a surregulation of both initial regulations.

Example 5 *$(X_i, W_i)_{i\in J}$ is a surregulation of $(X_i)_{i\in J}$. It is obviously also a surregulation of $(W_i)_{i\in J}$.*

An endogenous shared constraint generated by exogenous individual constraints provides a natural example of surregulation.

Example 6 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *and the generalized game with shared constraint* $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ *generated from* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

Then, $(K_i)_{i\in J}$ *is a surregulation of* $(X_i)_{i\in J}$ *since by definition for every agent* i *and for all* x, $K_i(x_{-i}) \subseteq X_i(x_{-i})$.

We prove a few more results below.

Fact 18 *We consider two regulations* $(X_i)_{i\in J}$ *and* $(W_i)_{i\in J}$ *on the Game* $(J, E, (\theta_i)_{i\in J})$ *such that* $(W_i)_{i\in J}$ *is a surregulation of* $(X_i)_{i\in J}$. *Then, the set of admissible strategies of* $(J, E, (\theta_i)_{i\in J}, (W_i)_{i\in J})$ *is included in the set of admissible strategies of* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

Or in other words: $K((W_i)_{i\in J}) \subseteq K((X_i)_{i\in J})$.

Proof of Fact 18

$K((W_i)_{i\in J}) = \{x \in E,\ \forall i \in J,\ x_i \in W_i(x_{-i})\} \subseteq \{x \in E,\ \forall i \in J,\ x_i \in X_i(x_{-i})\} = K((X_i)_{i\in J})$.

□

Fact 19 *We consider two regulations* $(X_i)_{i\in J}$ *and* $(W_i)_{i\in J}$ *on the Game* $(J, E, (\theta_i)_{i\in J})$ *such that* $(W_i)_{i\in J}$ *is a surregulation of* $(X_i)_{i\in J}$. *Then,* $T((W_i)_{i\in J}) \leq T((X_i)_{i\in J})$.

Proof of Fact 19

$K((W_i)_{i\in J}) \subseteq K((X_i)_{i\in J})$ from Proposition 18.

Therefore $T((W_i)_{i\in J}) = sup_{x\in K((W_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x) \leq sup_{x\in K((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x) = T((X_i)_{i\in J})$.

□

Remark 11 *This result is not true for regulation criteria Stab,* $Nash$ *and* TN *a priori.*

Corollary 4 *We consider a regulation* $(X_i)_{i\in J}$ *on a Game* $(J, E, (\theta_i)_{i\in J})$. $(X_i)_{i\in J}$ *is optimal among all its surregulations for the criterion* T.

We give an example of existence of optimal vector among a set of surregulations:

Fact 20 *Consider a generalized game with shared constraint* $(J, E, (\theta_i)_{i\in J}, X)$ *such that* X *is a compact set, and that* $\forall i \in J,\ \theta_i$ *is continuous and of the form* $\theta_i(x) = \theta_i(x_i)$ *. Then such a game admits an optimal vector* x^* *among all the surregulations of* X *for the criteria* T *and* TN.

We now finish this subsection with an example of regulation which is optimal among a set of subregulations.

Fact 21 *Let* $(J, E, (\theta_i)_{i\in J}, X)$ *be a generalized game with shared constraint. Then the regulation* X *is optimal among all the subregulations* $(X_i)_{i\in J}$ *such that* $K((X_i)_{i\in J}) = X$ *for the criteria* T*,* $Nash$ *and* TN *on the Game* $(J, E, (\theta_i)_{i\in J})$*. Moreover, if* E *is finite set, this property is also true for the criterion* $Stab$*.*

Proof of Fact 21

If $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ is a generalized game with individual constraints such that $K((X_i)_{i\in J}) = X$ then, we have that $T((X_i)_{i\in J}) = T(X)$. From Proposition 1 in [Braouezec and Kiani, 2021a] we have that $Nash((X_i)_{i\in J})) \subseteq Nash(X)$ and therefore we also have $TN((X_i)_{i\in J})) \leq TN(X)$. If E is a finite set, we also have that $Stab((X_i)_{i\in J})) \leq Stab(X)$. □

2.7 Correlated regulation criteria and intersection of optimal solutions

We continue our development with comparing the regulation criteria with one another. The goal of this subsection is to answer to the theoretical question: to what extent is it similar for a regulator to optimize a criterion R_1 and a criterion R_2? Are the different criteria correlated with one another? Do they have common optimal solutions? This is what we inspect here.

Definition 27 *We say that two regulation criteria* R_1 *and* R_2 *are positively correlated if for any regulation* $(X_i)_{i\in J}$ *and* $(Y_i)_{i\in J}$ *such that* $R_1((Y_i)_{i\in J}) \geq R_1((X_i)_{i\in J})$*, we also have* $R_2((Y_i)_{i\in J}) \geq R_2((X_i)_{i\in J})$*. They are negatively correlated if for any regulation* $(X_i)_{i\in J}$ *and* $(Y_i)_{i\in J}$ *such that* $R_1((Y_i)_{i\in J}) \geq R_1((X_i)_{i\in J})$*, we have* $R_2((Y_i)_{i\in J}) \leq R_2((X_i)_{i\in J})$*.*

Example 7

- *If* g *is an increasing function then* R_1 *and* $R_2 = g(R_1)$ *are positively correlated.*
- *If* g *is a decreasing function then* R_1 *and* $R_2 = g(R_1)$ *are negatively correlated.*

Definition 28 *We say that two regulation criteria* R_1 *and* R_2 *are optimally identical if the set of optimal regulations for the criterion* R_1 *on the Game* $(J, E, (\theta_i)_{i\in J})$ *is equal to the set of optimal regulations for the criterion* R_2 *on the Game* $(J, E, (\theta_i)_{i\in J})$*, either in the strong or the weak sense. We say that the two criteria* R_1 *and* R_2 *have a common optimal regulation if the set of optimal regulations for the criterion* R_1 *and the set of optimal regulations for the criterion* R_2 *have at least one element in common, either in the strong or the weak sense.*

Note that if R_1 and R_2 are optimally identical then they have a common optical regulation (but the converse is not true).

Example 8 *If g is an increasing function then R_1 and $R_2 = g(R_1)$ are optimally identical.*

Fact 22 *We consider a Game $(J, E, (\theta_i)_{i\in J})$ with E a finite set. Then the regulations $Nash$ and $Stab$ have at least one common optimal regulation in the weak sense.*

Proof of Fact 22

Stab has at least one optimal regulation $(Y_i)_{i\in J}$ (in the strong sense). This means that there is no regulation $(X_i)_{i\in J}$such that $Stab((Y_i)_{i\in J}) < Stab((X_i)_{i\in J})$. And for such a $(Y_i)_{i\in J}$, there is no $(X_i)_{i\in J}$ such that $Nash((Y_i)_{i\in J}) \subsetneq Nash((X_i)_{i\in J})$. □

Fact 23 *We consider a generalized game with shared constraint $(J, E, (\theta_i)_{i\in J}, X)$. Then X is a common optimal regulation (in the strong sense) among the subregulations $(X_i)_{i\in J}$ of X such that $K((X_i)_{i\in J}) = X$ for the criteria T, TN and $Nash$ (and also $Stab$ if E is finite).*

Proof of Fact 23

See Fact 21. □

We finish this subsection with the example of two criteria that are optimally identical:

Fact 24 *Consider a generalized game with shared constraint $(J, E, (\theta_i)_{i\in J}, X)$ such that is X is a compact set, and that $\forall i \in J,\ \theta_i$ is continuous and of the form $\theta_i(x) = \theta_i(x_i)$. Then the criteria T and TN are optimally identical (in the strong sense) among all the surregulations of X and their set of optimal regulations is equal to the set of subregulations of the optimal vectors x^* of Nash equilibria that maximize the global payoff.*

In section 3, we will see an example where regulation criteria are not correlated and/or not optimally identical.

2.8 Other examples of regulation criteria

We give two additional examples of regulation criteria that can be useful in some contexts. In particular, these two criteria can be mixed to the four criteria already studied to define new and relevant criteria of collective social well-being from the regulatory point of view.

Example 9 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. $(X_i)_{i\in J}$ *is a regulation on the Game* $(J, E, (\theta_i)_{i\in J})$.

We can define the regulation criterion:

$$\begin{array}{llll} Diff & : & (F(E_{-i},P(E_i)))_{i\in J} & \to & (\mathbb{R},\leq) \\ & & (X_i)_{i\in J} & \mapsto & sup_{x\in K((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x) - inf_{x\in K((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x) \end{array}$$

with $K((X_i)_{i\in J})$ *the set of admissible strategies of the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

$Diff((X_i)_{i\in J})$ *is the difference between the maximum total payoff among all the admissible strategies of the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *and the minimum total payoff. If this difference is higher for the regulation* $(Y_i)_{i\in J}$ *than for the regulation* $(X_i)_{i\in J}$, *then* $Diff((Y_i)_{i\in J}) \geq Diff((X_i)_{i\in J})$.

Example 10 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. $(X_i)_{i\in J}$ *is a regulation on the Game* $(J, E, (\theta_i)_{i\in J})$.

We can define the regulation criterion:

$$\begin{array}{llll} DiffNash & : & (F(E_{-i},P(E_i)))_{i\in J} & \to & (\mathbb{R},\leq) \\ & & (X_i)_{i\in J} & \mapsto & sup_{x^*\in Nash((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x^*) - inf_{x^*\in Nash((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x^*) \end{array}$$

with $Nash((X_i)_{i\in J})$ *the set of Nash equilibria of the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

$DiffNash((X_i)_{i\in J})$ *is the is the difference between the maximum total payoff among all the Nash equilibria of the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *and the minimal total payoff among all the Nash equilibria. If this difference is higher for the regulation* $(Y_i)_{i\in J}$ *than for the regulation* $(X_i)_{i\in J}$, *then* $DiffNash((Y_i)_{i\in J}) \geq DiffNash((X_i)_{i\in J})$.

2.9 From static games with constraints to dynamic games with constraints

We would like to offer an extension of our theory and results to more general types of games with a time dimension. A frequent critics about static games, with or without constraints, is that they are not always representative of real-world problems in the sense that it is rather rare to meet with real-world problems where the agents act in a single shot and all at the same time. To this extent, static games offer an idealized and simplified modeling of interaction between agents that give a first grasp on the problem one is analyzing. We defeat the critics about static games in the context of our article: an important point in our theory on Optimal Regulation is that our modeling, framework and results are still true for any dynamic games, either discrete-time or continuous-time, either deterministic or with a stochastic component. In this perspective, our study of problems with static games offer a first ground of study of real-world problems and the results we obtain on this simplified version of the problem we study can be transferred to the dynamic version.

Indeed, consider the generalized game with constraint $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ and now consider that the strategy x_i chosen by agent i is also a function of time t and, therefore the strategy $x_i(t)$ of agent i can be reevaluated over time, either at some dates given by a deterministic law or at dates given by a law with a stochastic component, either at discrete times or continuous times. The goal of each agent is still to optimize its objective function $\theta_i(x(t))$. We observe that the definition of a Nash equilibrium for such a game is identical to the one in a static game and all the definitions and results of Section 2 can be extended to such dynamic games. The study of such dynamic generalized games offer enough substance for an article itself dedicated to this topic and we will not go further in this direction in the present article.

We also note that the introduction of the time dimension or a stochastic dimension in our study can also give birth to additional relevant definitions of regulation criteria: for instance, a regulator could be seeking to minimize the time before the agents get to a Nash equlibrium or another given configuration, or in a stochastic context the goal of a regulator could be to maximize the expected total payoff... And there are many other considerations that we will not develop here but that can also be the topic of discussion of a future article.

3 An application: strategic fire-sales in the banking system and optimal regulation

In this Section, we study applications of the theory and results developed in Section 2 to answer a problem in bank regulation during fire sales. Study of contagion of financial losses between financial institutions during a period of financial crisis has become a hot topic in financial regulation research since the global crisis of 2007-2008. In particular, Global Systemically Important Banks (G-SIBs), have raised an increasing attention over the years due to the risk of implosion of the financial system in case of bankruptcy of such a Bank. See [Krishnamurthy, 2010] and [Glasserman and Young, 2015] for review papers. A salient feature of the financial crisis of 2007-2008 is the role played by financial markets, which have been a vector of not only direct contagion through contractual links between institutions, but also indirect contagion of losses and/or bankruptcy between banks and financial institutions, through asset prices. See for instance [Clerc et al., 2016]. A financial institution hit by a shock during a financial crisis may be forced to sell an important quantity of assets to rebalance its portfolio and capital ratio, what we call fire sales, and this can cause a major decrease in the prices of the assets sold (through price impact) which therefore impacts the balance sheets of other banks.

In this article we study a model of contagion between banks through the prices of assets in the financial markets inspired from [Braouezec and Wagalath, 2019]. First we remind and summarize the model developed in their article: they present the situation of p banks which need to sell some of their assets after a financial shock to re-balance their risk-based capital ratio over a regulatory threshold (see the initial article for full details). Then, we take the regulator point of view and study different regulation criteria, and existence of possible optimal regulations: what is the relevant criterion to optimize for a regulator who seeks the collective welfare of the community of agents? For instance, a regulator can either choose to minimize the losses in the financial system or to minimize the number of financial institutions going bankrupt. Do we have existence of an optimal regulation for these two regulation criteria?

3.1 The model

Banks' balance-sheets and regulatory constraints

We take same notations as in [Braouezec and Wagalath, 2019]. We consider a set $B = \{1, 2, ..., p\}$ of $p \geq 2$ banks that can invest in a risky asset and in cash.

For each bank i, we denote by $v_i > 0$ the amount of cash (in dollars) and by $q_i P_t > 0$ the value (in dollars) of risky assets, where q_i is the quantity of risky assets held by the bank and P_t is the market price of the risky asset at a given date t. Let D_i be the sum of the value of deposits and/or debt securities, that have been issued by bank i.

The balance-sheet of the bank at time t is as follows.

Balance-sheet of bank i at time t

Assets	Liabilities
Cash: v_i	Debt: D_i
Risky assets: $q_i P_t$	Equity: $E_{i,t}$
$A_{i,t}$	$E_{i,t} + D_i$

We make the assumption that the risky asset is a financial security issued by a non-financial institution whose price is quoted on financial markets.

From the Basel accords on banking regulation, banks are required to hold enough capital as a *percentage* of their risk-weighted assets (RWA). Within our model, since there is a single risky asset, the risk-weighted asset of bank i is simply equal to

$$\mathrm{RWA}_{i,t} = \alpha_i q_i P_t \tag{4.5}$$

where α_i is the risk weight of bank i associated to the risky asset. Note that α_i may vary across banks.

We define the risk-based capital ratio (RBC) $\theta_{i,t}$ for a given bank i at time t:

$$\theta_{i,t} := \frac{E_{i,t}}{\mathrm{RWA}_{i,t}} \tag{4.6}$$

Note that $\theta_{i,t}$ here has nothing to do with a cost or payoff function as defined in Section 2. We choose to keep the notations of [Braouezec and Wagalath, 2019] and the cost functions in this model will be denoted f_i. For the sake of interest, we assume that all banks are solvent at date t, that is, $E_{i,t} > 0$ for all $1 \leq i \leq p$.

We denote by θ_{min} the minimum capital ratio imposed by the regulator, which is equal to 8%. For the sake of interest, we shall assume that, at date t, all banks comply with an individual regulatory constraint:

$$\theta_{i,t} \geq \theta_{i,min} \text{ for each } i = 1, 2, ..., p \tag{4.7}$$

Impact of an exogenous shock on banks' capital ratios

Assume that a shock on the risky asset occurs at date t^+ and denote $\Delta \in (0,1)$ the size of the adverse shock in percentage of P_t. The price of the risky asset at time t^+ thus is equal to

$$P_{t^+} = P_t(1 - \Delta) \tag{4.8}$$

At time t^+, right after the shock, the balance-sheet of bank i is given as follows.

Balance-sheet at time t^+

Assets	Liabilities
Cash: v_i	Debt: D_i
Risky assets: $q_i P_t(1-\Delta)$	Equity: E_{i,t^+}
A_{i,t^+}	$E_{i,t^+} + D_i$

It is the role of equity to absorb the shock, i.e., the loss which is equal to $q_i P_t \Delta$ in dollars. The RBC of bank i at date t^+ is equal to

$$\theta_{i,t^+}(\Delta) = \frac{\max\{A_{i,t^+} - D_i; 0\}}{\text{RWA}_{i,t^+}} = \frac{\max\{E_{i,t} - q_i P_t \Delta; 0\}}{\alpha_i q_i P_t(1-\Delta)} \tag{4.9}$$

In the remainder of this paper, we work under the following assumption.

Assumption 12 *At date t, each bank's equity is lower than the size of its risky assets, that is, for all $1 \leq i \leq p$:*

$$E_{i,t} < q_i P_t \tag{4.10}$$

This assumption is natural in the banking system as, in practice, banks' equities typically do not exceed 20% of their risky assets.

A given bank i may be in one of the three following situations, depending on the size of the shock Δ:

1. solvent and complying with regulatory capital requirement, that is $\theta_{i,t^+}(\Delta) \geq \theta_{i,min}$
2. solvent but not complying with regulatory capital requirement, that is $0 < \theta_{i,t^+}(\Delta) < \theta_{i,min}$
3. insolvent, that is $\theta_{i,t^+}(\Delta) = 0$, which is equivalent to $E_{i,t} - q_i P_t \Delta \leq 0$

Endogenous fire sales and feedback effects

Since Δ is a common shock, it affects the balance-sheet of *all* banks that hold the risky asset and may leave some of them undercapitalized. Banks that do not comply with the regulatory capital constraints consequently need to restore their capital ratio above the minimum required $\theta_{i,min}$ by selling assets and decrease the denominator of the risk-based capital ratio.

Such forced sales are usually called "fire sales". We will also assume that, as in most models (e.g., [Elliott et al., 2014, Caccioli et al., 2014]), a bank which is unable to restore its capital ratio above $\theta_{i,min}$ is fully liquidated at date $t+1$.

We denote by $x_i \in [0,1]$ the proportion of risky assets sold by bank i at date $t+1$, in reaction to the shock Δ at date t^+. When bank i does not need to liquidate assets, then $x_i = 0$. On the contrary, when the shock Δ is such that bank i is insolvent or unable to restore its capital ratio above $\theta_{i,min}$, then it is fully liquidated and $x_i = 1$. The volume (in shares) of liquidation by bank i is denoted by $x_i q_i$ and $\sum_{i\in B} x_i q_i$ denotes the total volume of fire sales in the banking system at date $t+1$.

Fire sales obviously impact the price of the asset at date $t+1$ and we assume here this price impact to be linear. We introduce the asset market depth Φ which is a linear measure of the asset liquidity [Kyle and Obizhaeva, 2016]. In [Cont and Wagalath, 2016], it is shown that the relevant quantity to capture the magnitude of feedback effects is $\frac{\sum_{i\in B} q_i}{\Phi}$.

The asset price at date $t+1$ thus depends on the vector of liquidations $\boldsymbol{x}(\Delta,\Phi) := \boldsymbol{x} = (x_1, x_2, ..., x_p) \in [0,1]^p$, and this vector of liquidation depends on both the shock Δ and the market depth Φ.

Assumption 13 *The price of the risky asset at time $t+1$ is equal to*

$$\begin{aligned} P_{t+1}(\boldsymbol{x},\Phi) &= P_t\,(1-\Delta)\left(1-\frac{\sum_{i\in B} x_i q_i}{\Phi}\right) && (4.11)\\ \frac{Q_{tot}}{\Phi} &< 1 && (4.12)\\ \text{where}\quad Q_{tot} &= \sum_{i\in B} q_i && (4.13) \end{aligned}$$

At time $t+1$, the balance-sheet of bank i that sold a portion x_i of the risky asset is given below:

Balance-sheet of bank i at date $t+1$ after deleveraging

Assets	Liabilities
Cash: $v_i + x_i q_i P_{t+1}(\boldsymbol{x},\Phi)$	Debt: D_i
Risky asset: $(1-x_i) q_i P_{t+1}(\boldsymbol{x},\Phi)$	Equity: $E_{i,t+1}$
$A_{i,t+1} = v_i + q_i P_{t+1}(\boldsymbol{x},\Phi)$	$E_{i,t+1} + D_i$

where $P_{t+1}(\boldsymbol{x},\Phi)$ is given in Assumption (13). Let $E_{i,t+1}(\boldsymbol{x})$ be the total capital at time $t+1$ after deleveraging. From the above balance-sheet, we have that

$$E_{i,t+1}(\boldsymbol{x},\Delta) = \max\left\{E_{i,t} - q_i P_t\left(\Delta + \frac{\sum_{j\in B} x_j q_j}{\Phi}(1-\Delta)\right);0\right\} \qquad (4.14)$$

and note that it is a *decreasing* function of x_i due to the existence of a price impact. The regulatory capital ratio of bank i at time $t+1$ (i.e., after deleveraging) thus is equal to

$$\theta_{i,t+1}(\boldsymbol{x}, \Delta) = \frac{E_{i,t+1}(\boldsymbol{x})}{\alpha_i q_i P_{t+1}(\boldsymbol{x}, \Phi)(1 - x_i)} \tag{4.15}$$

with the natural convention that $\theta_{i,t+1}(\boldsymbol{x}, \Delta) = 0$ when $x_i = 1$ and when $E_{i,t+1} = 0$.

We can also introduce the concept of the *implied shock*:

$$\Delta(\boldsymbol{x}) := \Delta + \frac{\sum_{j \in B} x_j q_j}{\Phi}(1 - \Delta) \tag{4.16}$$

associated to the vector of liquidation $\boldsymbol{x}$ such that the price of the risky asset at date $t+1$ can be written as follows

$$P_{t+1}(\boldsymbol{x}, \Phi) = P_t(1 - \Delta(\boldsymbol{x})) \tag{4.17}$$

Assumption 14 *Each bank $i = 1, 2, ..., p$ rebalances its portfolio of assets (i.e., deleverage) in order to minimize $x_i \in [0, 1]$ subject to the constraint*

$$\theta_{i,t+1}(\boldsymbol{x}, \Delta) \geq \theta_{i,min} \tag{4.18}$$

If the constraint can not be satisfied for some $x_i \in [0, 1)$, then bank i is insolvent and is costlessly liquidated at time $t+1$ so that $x_i = 1$.

3.2 Strategic fire sales and Regulation criteria

Now that we have reminded the model developed in the article [Braouezec and Wagalath, 2019], we can turn to the study from a regulator point of view of different regulation criteria, existence of possible optimal regulation, and possible correlation between different criteria.

We easily note that these strategic fire-sales with 1 asset are an example of generalized game with individual constraints:

- $J = \{1, ..., p\}$ a set of p banks
- $\forall i$, $E_i = [0, 1]$ and $E = [0, 1]^p$
- a cost function that we will denote f_i here $\begin{array}{rccc} f_i & : & E & \to & \mathbb{R} \\ & & x & \mapsto & x_i \end{array}$

- $$\begin{array}{rcl} \text{A regulation } X_i \; : \; [0,1]^{p-1} & \to & P([0,1]) \\ x_{-i} & \mapsto & \{x_i \in [0,1),\ \theta_{i,t+1}(x_i, x_{-i}) \geq \theta_{i,min}\} \cup \{1\} \end{array}$$

To avoid any confusion, we keep the notation θ_i of the article [Braouezec and Wagalath, 2019] for the RBC ratio and we will denote the cost function f_i in our presentation for this section. Note that this is an example of non-classical game (see classification of games in [Braouezec and Kiani, 2021a]) as the cost of each bank/agent i only depends on the choice of i.

In this section, we will consider the 3 criteria studied in Section 3, $Nash$ (a stability criterion), T, and TN (two cost criteria; obviously, as $f_i(x) = x_i$ here is a cost function, we adapt the corresponding definition of the criteria T and TN), plus two additional regulation criteria that we will call bankruptcy criteria:

$$\begin{array}{rcl} NB \; : \; (F(E_{-i}, \mathrm{P}(\mathrm{E}_i)))_{i\in J} & \to & ([0,\mathrm{N}], \leq) \\ (\mathrm{X}_i)_{i\in J} & \mapsto & \text{maximum}_{x\in K((X_i)_{i\in J})}\text{number of solvent banks(x) at } t+1 \end{array}$$

and

$$\begin{array}{rcl} NBN \; : \; (F(E_{-i}, \mathrm{P}(\mathrm{E}_i)))_{i\in J} & \to & ([0,\mathrm{N}], \leq) \\ (\mathrm{X}_i)_{i\in J} & \mapsto & \text{maximum}_{x\in Nash((X_i)_{i\in J})}\text{number of solvent banks(x) at } t+1 \end{array}$$

NB measures the minimum number of banks going bankrupt among all the admissible strategies of the generalized game $(J, E, (f_i)_{i\in J}, (X_i)_{i\in J})$ and it is equal to $min_x\ N-$ the number of banks going bankrupt for the strategy vector x.

NBN measures the minimum number of banks going bankrupt among all the Nash equilibria of the generalized game $(J, E, (f_i)_{i\in J}, (X_i)_{i\in J})$ and it is equal to $min_{x^*}\ N-$ the number of banks going bankrupt for the Nash equilibrium x^*.

Note that we immediately have that for any regulation $(X_i)_{i\in J}$, $NBN((X_i)_{i\in J}) \leq NB((X_i)_{i\in J})$.

It is the goal of our inquiry to study these 5 regulation criteria and to see if there is existence of optimal regulations and optimal vectors.

We also note that we immediately have the following results of existence since the set V is finished for NB and NBN.

Proposition 16 *The criteria NB and NBN have at least one optimal regulation and one optimal vector.*

3.3 Regulation with individual constraints or endogenous shared constraint

It is proven in [Braouezec and Wagalath, 2019] that for such a game with individual constraints, there is always existence of Nash equilibria thanks to Tarski's theorem.

We want to go further and consider the regulation with endogenous shared constraint generated from this regulation with individual constraints. We already know that the regulation with shared constraint is at least better than the initial regulation with individual constraints for criteria $Nash$, T and TN. We will moreover prove that with such a regulation in place, there is existence of a Nash equilbrium that minimizes the total cost and diffusion of the assets $\sum_{i=1}^{p} x_i$ in the banking system.

First, we introduce the generated shared constraint set:

$$K = \{x \in [0,1]^p;\ \forall i \in \{1,...,p\} : x_i \in [0,1(\textit{ and } \theta_i(x_i, x_{-i}) \geq \theta_{i,min}, \textit{ or } x_i = 1\} \tag{4.19}$$

Lemma 9 *K is a compact set.*

Proof See the Appendix.

So the regulation with shared constraint we introduced gives a generalized game with shared constraint that satisfies all the assumptions of Proposition 15 and therefore, as a corollary, such strategic fire-sales have a Nash equilibrium that minimizes the total cost and loss diffusion in the banking system if the regulator takes the regulation with shared constraint generated from the initial regulation with individual constraints, which is a new result.

Corollary 5 *The generalized game with shared constraint $(J, E, (f_i)_{i\in J}, K)$ admits at least one Nash equilibrium that minimizes the global cost $\sum_{i\in J} x_i$ on K: there exists a Nash equilibrium $x^* \in E$ such that $\sum_{i\in J} x_i^* = min_{x\in K} \sum_{i=1}^{N} x_i$. Therefore the regulation with shared constraint K is optimal among all the surregulations of K for the criteria T and TN. In addition, it is better than the regulation with individual constraints for the criteria T and TN, $K \leq (X_i)_{i\in J}$ for these two criteria. And obviously, from Fact 13, $T(K) = TN(K)$. Moreover, such a vector x^* is an optimal vector (in the strong sense) for the criteria T and TN.*

We will go further in our study and actually prove that this remarkable Nash equilibrium in shared constraint is also a Nash equilibrium in individual constraints.

First, we remind this intuitive assumption:

Assumption 15 *We make the following natural assumptions:*

- *we assume that the ratio θ_i of a given bank i is an increasing function of the quantity of assets its sells. Or in other words: for all $i \in J$, $\theta_i(x)$ is an increasing function of x_i.*

- *we assume that the ratio θ_i of a given bank i is a decreasing function of the quantity of assets sold by other banks $j \in J$. Or in other words: for all $i \in J$, for all $j \neq i$, $\theta_i(x)$ is a decreasing function of x_j.*

Lemma 10 *If x^* is a Nash equilibrium of the generalized game $(J, E, (f_i)_{i\in J}, (X_i)_{i\in J})$ then for all i such that $x_i^* < 1$, we have that $\theta_i(x^*) = \theta_{i,min}$.*

Proof See the appendix.

Proposition 17 *The generalized game with individual constraints $(J, E, (f_i)_{i\in J}, (X_i)_{i\in J})$ admits at least one Nash equilibrium that minimizes the global cost $\sum_{i\in J} x_i$ on $K((X_i)_{i\in J})$: there exists a Nash equilibrium $x^* \in E$ such that $\sum_{i\in J} x_i^* = min_{x\in K((X_i)_{i\in J})} \sum_{i=1}^{N} x_i$. Therefore, $T((X_i)_{i\in J}) = TN((X_i)_{i\in J})$, the regulation $(X_i)_{i\in J}$ is as good as the regulation K for the criteria T and TN and it is also optimal among all the surregulations of $(X_i)_{i\in J}$ for the criteria T and TN. Moreover, such a vector x^* is an optimal vector (in the strong sense) for the criteria T and TN.*

Proof See the Appendix.

Corollary 6 *T and TN have a common optimal regulation (in the strong sense) on the Game $(J, E, (f_i)_{i\in J})$ among the subregulations $(X_i)_{i\in J}$ such that $K((X_i)_{i\in J}) = K$. The criteria T and TN are optimally identical (in the strong sense) among all the surregulations of K and their set of optimal regulations is equal to the set of subregulations of the optimal vectors x^* of Nash equilibria that maximize the global payoff function.*

This is a new result that does not appear in [Braouezec and Wagalath, 2019]: they prove that Nash equilibria are ranked and that it is optimal to choose the smallest Nash equilibrium, we moreover prove that the smallest Nash equilibrium minimizes the total cost among all the admissible strategies of the game $(J, E, (f_i)_{i\in J})$. This Nash equilibrium is clearly the preferred solution of regulatory public institutions who seek to minimize the global losses in the financial system.

But what if the regulator is seeking to minimize the number of banks going bankrupt, or equivalently maximize the number of solvent banks? We answer to this question in the next subsection. We also note that we have the following result for the criterion NBN.

Proposition 18 *K is a better regulation than $(X_i)_{i\in J}$ for the criterion NBN.*

Proof See the Appendix.

3.4 Comparison of optimal regulations and optimal vectors for the different criteria

We know that we have existence of optimal regulations and optimal vectors for the criteria T, TN, NB and NBN. Now, one can wonder: are these optimal regulations/vectors identical? Is it the same from a regulatory point of view to say that a regulator is seeking to minimize the losses in the system or to minimize the number of banks going bankrupt? This is an important question from a regulatory point of view.

To answer this question, one should look for:

- The set $\{x_T^*\}$ of optimal vectors for the criterion T.
- The set $\{x_{TN}^*\}$ of optimal vectors for the criterion TN.
- The set $\{x_{NB}\}$ of optimal vectors for the criterion NB.
- The set $\{x_{NBN}\}$ of optimal vectors for the criterion NBN.

and look at the intersections of these four sets.

We consider two ways to describe these four sets:

- either it is analytically convenient to find these four sets, given the data in our stress-test model.
- or if it's not the case, one can do the following approximation and compute the following algorithm. Given the data in the model, we make the approximation that the set E is finite (which is actually the case in real life since x_i takes its value in a finite set in real life), and we compute an algorithm that tests all vectors $x \in K$ and find the optimal vectors for each criterion.

We can easily see that these vectors do not always coincide if for instance we have a world with one very large bank and (N-1) tiny banks with a quantity of assets which is for instance divided by 100: it is not equivalent from a regulatory point of view to minimize the losses in the system and to minimize the number of banks going bankrupt. When these sets do not coincide and therefore optimizing a given criterion leads to a different result than optimzing another criterion, the regulator should wonder: what criterion of social welfare should be chosen, given that optimizing the different criteria is not equivalent? It is interesting to think of these questions retrospectively with the financial crisis of 2007-2008 and figure out if the decisions taken were optimal for the different regulation criteria.

We end up this section with considering a new possibility from a regulatory point of view during stress-tests when enabling that the threshold $\theta_{i,min}$ given to each bank is variable.

3.5 Regulation with variable $\theta_{i,min}$

We go further in our inquiry by assuming that the thresholds $\theta_{i,min}$ asked by the regulator are also variable, so we get new variable regulations.

We consider the family of following regulations $((X_{i,s_i})_{s_i>0})_{i\in J}$ on the Game $(J, E, (f_i)_{i\in J}$:

$$\begin{array}{rcccl} X_{i,s_i} & : & [0,1]^{p-1} & \rightarrow & P([0,1]) \\ & & x_{-i} & \mapsto & \{x_i \in [0,1),\ \theta_{i,t+1}(x_i, x_{-i}) \geq s_i\} \cup \{1\} \end{array}$$

with $s_i > 0$. Note that we do not impose that all banks have the same ratio threshold s and the ratio threshold s_i can be customized by the regulator for each bank depending its size and own intrinsic characteristics.

Lemma 11 *If $0 < s_i \leq s'_i$, then for any $x_{-i} \in E_{-i}$, we have $X_{i,s'_i}(x_{-i}) \subseteq X_{i,s_i}(x_{-i})$.*

Proof See the Appendix.

Corollary 7 *If **s** and **s'** in $(\mathbb{R}_+^*)^N$ are such that for all $i \in J, 0 < s_i \leq s'_i$, we have that $K((X_{i,s'_i})_{i\in J}) \subseteq K((X_{i,s_i})_{i\in J})$.*

Corollary 8 *If **s** and **s'** in $(\mathbb{R}_+^*)^N$ are such that for all $i \in J, 0 < s_i \leq s'_i$, we have that:*

- $T((X_{i,s'_i})_{i\in J}) \geq T((X_{i,s_i})_{i\in J})$

- $TN((X_{i,s'_i})_{i\in J}) \geq TN((X_{i,s_i})_{i\in J})$
- $NB((X_{i,s'_i})_{i\in J}) \leq NB((X_{i,s_i})_{i\in J})$

Therefore $(X_{i,s_i})_{i\in J}$ is a better regulation than $(X_{i,s'_i})_{i\in J}$ for the criteria T, TN and NB. And therefore T and TN are positively correlated on the family of regulation $(X_{i,s_i})_{i\in J,s_i>0}$, and T and NB (and also TN and NB) are negatively correlated on the family of regulation $(X_{i,s_i})_{i\in J,s_i>0}$.

Proof See the Appendix.

Corollary 9 *The regulation $X_{i,0}(x_{-i}) = \{x_i \in [0,1(, \theta_{i,t+1}(x_i, x_{-i}) > 0\} \cup \{1\}$ is optimal (in the strong sense) for T, TN and NB among the family of regulations $(X_{i,s_i})_{i\in J,s_i\geq 0}$. Therefore these 3 criteria are optimally identical (in the strong sense) among the family of regulations $(X_{i,s_i})_{i\in J,s_i\geq 0}$.*

Therefore in a period of financial crisis following a financial shock our study provides the result that it is optimal for a regulator who is seeking to maximize either T, TN or NB to forget the RBC ratio for a while and choose the regulation $X_{i,0}(x_{-i}) = \{x_i \in [0,1(, \theta_{i,t+1}(x_i, x_{-i}) > 0\} \cup \{1\}$. In other words, a regulator who is seeking to minimize the losses in the system or the number of banks going bankrupt had better choose such a regulation $(X_{i,0})_{i\in J}$.

We also have the following result for the regulation criterion NBN:

Proposition 19 *There exists an optimal regulation for NBN among the family of regulations $(X_{i,s_i})_{i\in J,s_i>0}$.*

Proof See the Appendix.

One can wonder if the optimal regulation for NBN is same as NB and equal to $X_{i,0}(x_{-i}) = \{x_i \in [0,1(, \theta_{i,t+1}(x_i, x_{-i}) > 0\} \cup \{1\}$.

And same as in Subsection 3.4, one can wonder if there is existence of Optimal vectors x^*_T, x^*_{TN}, x_{NB} and x_{NBN} and if our four criteria are optimally identical or have at least some common optimal regulation/vector. The method to answer such questions is similar to the one in Subsection 3.4:

- either it is analytically convenient to find the four sets of optimal regulations and possible optimal vectors, given the data in our stress-test model.

- or if it's not the case, one can write the following approximation and compute the following algorithm. Given the data in the model, we make the approximation that the set E is finite (which is once again actually the case in real life since x_i takes its value in a finite set and also we note that the price of the asset is in a finite set of prices), and we compute an algorithm that tests all vectors $x \in K$ and find the optimal vectors for each criterion.

This gives us again a process of comparison of the different optimal regulations/vectors and the criteria.

4 Conclusion

In this paper, we developed a theory of optimal regulation in a game theory context using generalized games. We showed that generalized games are the natural tool and framework to express the role of a regulator in a context where some economic agents share strategic interactions: indeed, given that a regulation is the definition of a set of possible actions and restrictions for each agent depending on the strategies of all the agents, this perfectly matches with generalized games, which are games where the strategy set of each agent depends on the strategies picked by all other agents. In Section 2, we gave two natural examples of regulation models with generalized games and we reminded a few definitions and results about generalized games. We defined the concepts of regulation and regulation criterion: the regulation criterion is the function of social/collective welfare that the regulator is seeking to maximize for the well-being of the community of agents, and takes the regulation as a variable. We gave a few examples, we defined the concept of optimal regulation and studied existence of optimal regulations or optimal vector under reasonable hypotheses. We also defined a few concepts about correlated regulation criteria and common optimal regulations when the regulator can be seeking to maximize multiple social welfare functions. In Section 3, we studied applications of our theory and results in a bank regulation model with stress-tests during fire sales and proved existence of optimal regulations for several regulation criteria, which pave the road for new studies of bank regulation through the prism of generalized games and optimal regulation.

5 Technical proofs

Proof of Fact 14

- $Stab$, T and TN are applications from a finite set of regulations to $(\mathbb{R}, \leq)$, therefore they each admit a maximum $(Y_i)_{i \in J}$ and such a maximum is an optimal regulation in the strong sense.
- $Nash$ is an application from a finite set of regulations to $(P(E), \subseteq)$, therefore there exists at least one $(Y_i)_{i \in J}$ such that there is no $(X_i)_{i \in J}$ with $Nash((Y_i)_{i \in J}) \subsetneq Nash((X_i)_{i \in J})$. $\square$

Proof of Fact 15

- R is an application from a finite set of regulations to $(V, \leq)$, therefore there exists at least one $(Y_i)_{i \in J}$ such that there is no $(X_i)_{i \in J}$ with $R((Y_i)_{i \in J}) < R((X_i)_{i \in J})$.
- R is an application from a finite set of regulations to a totally ordered set $(V, \leq)$, therefore R has at least one maximum $(Y_i)_{i \in J}$ and such a maximum is an optimal regulation in the strong sense. $\square$

Proof of Fact 16

- R is an application to a finite set $(V, \leq)$, therefore there exists at least one $(Y_i)_{i \in J}$ such that there is no $(X_i)_{i \in J}$ with $R((Y_i)_{i \in J}) < R((X_i)_{i \in J})$.
- R is an application to a finite set $(V, \leq)$ which is totally ordered, therefore R has at least one maximum $(Y_i)_{i \in J}$ and such a maximum is an optimal regulation in the strong sense. $\square$

Proof of Fact 17

If E and V are finite set then then the maxima for T and TN are reached at some vectors that are optimal. $\square$

Proof of Lemma 9

- K is clearly a bounded set.
- We want to prove that K is a closed set. Let $(x_m)_{m \in \mathbb{N}} = (x_{1,m}, ..., x_{p,m})_{m \in \mathbb{N}} \in K^{\mathbb{N}}$ be a sequence which converges to a given $x_\infty \in [0,1]^p$. We will show that $x_\infty \in K$.

  Let $i \in \{1, .., p\}$.

- either $x_{i,\infty} = 1$
- or $x_{i,\infty} \in [0,1($ and there exists $\epsilon > 0$ such that $B(x_{i,\infty}, \epsilon) \subseteq [0,1] \setminus \{1\}$ and there exists $m_0 \in \mathbb{N}$ such that $\forall m \geq m_0, x_{i,m} \in B(x_{i,\infty}, \epsilon)$. Therefore $\forall m \geq m_0, \theta_i(x_m) \geq \theta_{i,min}$. And since θ_i is continuous on $B(x_{i,\infty}, \epsilon)$ (see [Braouezec and Wagalath, 2019]), we have $\theta_i(x_\infty) \geq \theta_{i,min}$

And this is true for all $i \in \{1, .., p\}$, so K is a closed set.

Therefore K is a closed bounded set of $[0,1]^p$, so K is a compact set. □

Proof of Lemma 10

From Assumption 15, if $\theta_i(x^*) > \theta_{i,min}$, then there would exist $y_i < x_i^*$ such that $(y_i, x_{-i}^*) \in K$, which is not possible.

Proof of Proposition 17

We consider the application:

$$\begin{array}{cccc} F & : & E & \to & \mathbb{R} \\ & & x & \mapsto & \sum_{i \in J} f_i(x) = \sum_{i \in J} f_i(x_i) = \sum_{i \in J} x_i \end{array}$$

F is continuous on our compact set $K \subseteq E$ therefore F has a minimum $x^* \in X$. Let's prove that x^* is a Nash equilibrium.

Indeed let's assume that there is $i \in J$ and $y_i \in E_i$ such that $y_i \in X(x_{-i}^*)$, and $f_i(y_i, x_{-i}^*) = y_i < x_i^* = f_i(x_i, x_{-i}^*)$.

- Either $x_i^* < 1$ and therefore since θ_i is an increasing function of x_i from Assumption 15 we have that $\theta_i(y_i, x_{-i}^*) < \theta_i(x_i^*, x_{-i}^*) = \theta_{i,min}$, which is not possible.
- Either $x_i^* = 1$ and therefore since for all $j \neq i$, $\theta_j(x)$ is a decreasing function of x_i, we have that $\theta_j(y_i, x_{-i}^*) \geq \theta_j(x_i^*, x_{-i}^*)$ for all j such that $x_j < 1$. And therefore, for all $j \neq i$ such that $x_j < 1$ we have that $x_{*,j} \in X_j(y_i, x_{*,-i,-j})$, and this is also the case if j is such that $x_j = 1$. Therefore, for all $j \neq i$, $x_{*,j} \in X_j(y_i, x_{*,-i,-j})$, and therefore (y_i, x_{-i}^*) is an admissible strategy and $(y_i, x_{-i}^*) \in K$. But we would have $F(y_i, x_{-i}^*) < F(x^{*,i}, x_{-i}^*) = F(x)$, which is not possible.

Therefore x_* is a Nash equilibrium of the generalized game $(J, E, (f_i)_{i \in J}, (X_i)_{i \in J})$. □

Proof of Proposition 18 We have that $Nash((X_i)_{i \in J}) \subseteq Nash(K)$, and therefore $NBN(K) \geq NBN((X_i)_{i \in J})$.

Proof of Lemma 11

If $s_i \leq s_i'$, from Assumption 15 we have that $\theta_i(x_i, x_{-i})$ is an increasing function of x_i and therefore $\{x_i \in [0,1),\ \theta_{i,t+1}(x_i, x_{-i}) \geq s_i' i\} \subseteq \{x_i \in [0,1),\ \theta_{i,t+1}(x_i, x_{-i}) \geq s_i\}$, so that $X_{i,s_i'}(x_{-i}) \subseteq X_{i,s_i}(x_{-i})$. $\square$

Proof of Corollary 8

$X_{i,s_i'}(x_{-i}) \subseteq X_{i,s_i}(x_{-i})$ for any $x_{-i} \in E_{-i}$ therefore $K((X_{i,s_i'})_{i\in J}) \subseteq K((X_{i,s_i})_{i\in J})$ and consequently, $T((X_{i,s_i'})_{i\in J}) \leq T((X_{i,s_i})_{i\in J})$ and $NB((X_{i,s_i'})_{i\in J}) \leq NB((X_{i,s_i})_{i\in J})$. From Proposition 17, $T((X_{i,s_i'})_{i\in J}) = TN((X_{i,s_i'})_{i\in J})$ and $T((X_{i,s_i})_{i\in J}) = TN((X_{i,s_i})_{i\in J})$ and therefore $TN((X_{i,s_i'})_{i\in J}) \leq TN((X_{i,s_i})_{i\in J})$ $\square$

Proof of Proposition 19

For the criterion NBN, the set V is equal to $\{0, ..., N\}$ and is therefore finite so we can apply Theorem 16. $\square$

Conclusion

We would like to conclude with a list of ideas and topics of investigation that naturally extend the contents of the present thesis and that may become topics of future work, possibly giving birth to future articles.

- One could extend the environment model with generalized games considered in Chapter 1, with a more general model, and with a calibration of the model with empirical data. One could for instance conduct an extension where each country has two sources of energy: one fossil energy with carbon emissions, and one green energy with no greenhouse gas emissions. We could imagine and study a setting where a regulator is asking for a maximum threshold on the carbon emissions and a minimum threshold for the green energy...

- A natural extension of the public good financing problem studied in Chapter 1 would be to consider different goods serving the same purpose but with different quality or with different quantities. If we consider the example of a public good with different qualities possible at different prices (from the cheapest to the most expensive) one could think for instance of the internet access in a village being the public good with different qualities of internet connection depending on price thresholds and the contributions of all citizens of the village. If the public good is a bridge for instance to cross a river in a village, one could think of having one, two, three or more bridges, depending on price thresholds and the monetary contributions of all citizens.

- A natural extension of Chapter 2 would be to consider a dynamic or sequential game where banks are allowed to update their strategy at given times depending the strategies picked by all other agents. One could study the dynamics of such a game and look at its asymptotic behavior: under which conditions does the system converge to a Nash equilibrium in finite time? Same question with the optimal Nash equilibrium that minimizes the total losses: under which conditions does the system converge to this point in finite time? How does the general system behave asymptotically? Does it have a limit? Or a cyclical limit? Can we imagine a system of incentives from a regulator that would enable moving the system of banks in finite time to the optimal Nash equilibrium? What regulation can we set from this analysis to avoid fire sales with negative endogenous externalities in real life in the future to prevent systemic risk?

- On the game-theoretical side, one could also wonder: do we have uniqueness of the Nash equilibrium? Or are there several Nash equilibria? Under which conditions do we have uniqueness? Can we characterize and clearly describe all Nash equilibria? Can we explicit an algorithm in polynomial time to find all Nash equilibria?

- A natural extension of the work in Chapter 2 would also be to carry out an empirical study on a larger scale with all GSIBs for instance (or say on the 50 main European Banks for instance). We could write an algorithm conducting the stress tests on the banking system given all the data that are public on the annual reports of the banks: cascade of defaults, endogenous shocks in the prices of assets, existence of optimal deleveraging solutions and Nash equilibria... And based on this empirical study one could give new regulatory tools, instruments and methods to the regulators to protect the system from the risks we assess in our stress-tests.

- Considering both direct contagion and indirect contagion through prices (which we call price-mediated contagion) in a same model seems quite challenging but it could be a topic for future investigation. The main difficulty is that the contractual links between banks are not public and it makes difficult to calibrate a model with real data: we do not have much information about the characteristics of the networks of links between banks. But considering that adding direct contagion effects can only worsen the situation that we have with our indirect contagion effects, we can say that the present model in our thesis gives an upper bound of what could happen in the case of an exogenous shock (and therefore the contractual links and direct contagion effects could only worsen the situation compared to the results here).

- A natural question that extends Chapter 3 is: can we prove the same result with Brouwer's fixed point theorem instead of Tarki's fixed point theorem? The answer seems to be positive and will be the topic of a future research note.

- Regarding the developments in Chapter 4, we have the ambition to replicate them to different settings where the role of a regulator in a game-theoretical framework is relevant or natural, for instance we could apply the theory and results of Chapter 4 to the two examples of generalized games in Chapter 1.

- We believe that optimal regulation can become a key instrument to conduct different stress-tests on a large variety of identified risks and be able to create new ideas, optimal regulatory tools and instruments to help the regulator protect the system from these risks. We come back to the metaphor in introduction with the work of an epidemiologist: our mission is to be able to find vaccines to vaccinate the financial system from any potential disease and threat. This means a high capacity of comprehension of the different mechanisms and interactions between the financial institutions as well as high capacity of imagination to propose new regulatory ideas. One can look back at the 2007-2008 Financial crisis and wonder: what should have been done to prevent the different stages of crisis from happening? And now, what can we do to prevent similar financial crises from happening in the future?

Appendix A

Target capital ratio and optimal channel(s) of adjustment: A simple model with empirical applications to European banks

Abstract

Why do banks decide to reach their target capital ratio by selling assets and/or issuing new shares? To answer this question, we offer a simple framework in which each channel of adjustment is costly; underwriting and dilution costs for equity issuance, profit reduction and price impact for asset sale. We make the assumption that the aim of the bank is to minimize the total adjustment cost subject to the target's constraint and we derive its optimal strategy. The solution is formulated in terms of two critical thresholds for which we give an explicit formula. We then compare our model's predictions to the decisions taken by two European systemic banks (Deutsche Bank and UniCredit) to issue new shares in 2017 and for which the target ratio was publicly disclosed. We show that the predictions of the model are consistent with the observed decisions.

Keywords: Equity issuance, asset sale, price impact, target capital ratio, systemic banks.

1 Introduction

In the aftermath of the global financial crisis of 2008, as a response to the deficiencies of the current banking regulation (Basel I or Basel II), regulators designed a new framework called Basel III (first published in 2010 and updated in 2011), intended to provide a *foundation for a resilient banking system that will help avoid the build-up of systemic vulnerabilities*. Under Basel III, regulators place a greater focus on going-concern loss-absorbing capital called Tier 1 capital, the best form of capital. Compared to Basel II, the minimum Tier 1 risk-based capital requirement defined as Tier 1 divided by the risk-weighted assets ($\frac{\text{Tier 1}}{\text{RWA}}$) is now higher and bank dependent due to the existence of buffers that depend upon the characteristics of each bank. A new Tier 1 risk-unweighted capital requirement, called the leverage ratio and defined as Tier 1 divided by the (total) exposure ($\frac{\text{Tier 1}}{\text{Exposure}}$) has also been introduced[1] and will be enforced in 2023. After this date, banks will thus have to comply with two Tier 1 capital ratios. In the empirical literature on the subject, it is shown that large banks maintain their risk-based capital ratios higher than the minimum required (e.g., [Berger et al., 2008, Memmel and Raupach, 2010]), which suggests the existence of a *target capital ratio*. Interestingly, some banks even disclose their target ratio. For instance, Deutsche Bank and Unicredit announced in March, 2017 a target Common Equity Tier 1 risk-based capital ratio equal to 14.1% and 12.9% respectively. To reach this target capital ratio, a given bank must thus adjust its balance sheet and there are two main (non-exclusive) channels of adjustments ([Gropp et al., 2019], [Cohen and Scatigna, 2016], [Juelsrud and Wold, 2020]). A bank can increase its regulatory capital ratio by increasing the numerator of the capital ratio (capital increase) or by decreasing the denominator of the capital ratio (asset shrinking/risk-reduction).

As all large corporations, a large bank can increase its capital ratio by issuing new equity, that is, by selling new shares through a capital increase. A bank can also sell a portion of its risky assets to repay its debt and this increases its capital ratio. As opposed to most corporations, a bank may also increase its capital ratio by using a risk-reduction strategy, that is, it can sell a portion of its risky assets in order to decrease its risk-weighted assets (RWA), the denominator of the capital ratio. Each channel of adjustment (capital increase, asset shrinking/risk-reduction) turns out to be costly. Consider the capital increase channel. Issuing new stocks via an underwritten rights offering is far from being free as there are direct costs (underwriter compensation, registration and listing fees...) and indirect costs (stock price reaction to the offering

[1] This ratio is designed to act as a backstop against risk-weighted assets that would be considered as too low by regulators.

announcement, see [Eckbo et al., 2007] for a comprehensive review and discussion). Moreover, this new issuance will also dilute existing shareholders since in percentage, their ownership will decrease. Consider now the asset shrinking/risk-reduction channel. Selling a large amount of risky assets may negatively impact the price of these assets due to the existence of a price impact (e.g., [French et al., 2010, Shleifer and Vishny, 2011, Greenwood et al., 2015]), and this may in turn decrease the capital of the bank. Moreover, when a bank sells a portion of its risky assets (and/or replace high risk weights assets with low risk weights assets), the expected profitability of that bank will also decrease.

It is the aim of the present paper to offer a theoretical model in which a large bank seeks to reach (in the short term) its target capital ratio by considering the two main channels of adjustment discussed above when each channel is costly. We make the natural assumption that the bank chooses the channel of adjustment (possibly a mix) that minimizes the overall cost of adjustment. The choice of the channel(s) of adjustment thus is formulated as an optimization problem which is in general non-linear. Depending upon the parameters, it may be more cost-efficient for the bank either to issue new shares or to sell the risky assets or even to do both, and we derive the optimal strategy of the bank. To facilitate the confrontation with observed decisions, we formulate the optimal strategy in terms of two critical thresholds (indeed critical spread) c_l and c_h (with $c_l < c_h$) where a critical spread is defined as the total issuance cost divided by the gross proceeds. We show that when the observed spread is lower than the lowest critical spread c_l, it is optimal for the bank to issue new shares only. On the other hand, when the observed spread is higher than the highest critical spread denoted c_h, it is optimal to sell risky assets only. In between, it is optimal to both issue new stocks and sell a portion of the risky assets.

The first contribution of this paper is theoretical in that we provide a complete and explicit solution to the optimization problem, that is, a closed form solution for the two critical thresholds c_l and c_h that are also of interest for regulators and supervisors. From a financial stability point of view, the greater these two thresholds are, the better it is. Everything else equal, when c_l is high, this increases the chance that the bank will issue new stocks only. To the best of our knowledge, it is the first paper that attempts to incorporate the possibility for the bank to issue new stocks when each channel of adjustment (asset sale, equity issuance) is costly. In most models, only deleveraging is considered (e.g., [Braouezec and Wagalath, 2018], [Duarte and Eisenbach, 2015], [Greenwood et al., 2015]).

The second contribution of this paper is empirical in that we exploit the recent capital increase of two European large banks to test the predictions of the model. In the first quarter of 2017, two European Global Systemically Important banks (G-SIBs), Deutsche Bank and UniCredit decided to issue new shares via an underwritten rights offering. Interestingly, they *publicly disclosed* their target capital ratio, expressed in terms of the CET1 capital ratio. Equipped with this information, the (model) parameters can be calibrated and we are now in a position to test whether or not our model is able to predict the optimal behavior of each bank. In both cases, the observed decisions are consistent with our theoretical predictions. The observed cost is lower than the lowest critical spread so that it is optimal for the bank to issue new shares only, which is exactly what they did. It may be worthwhile to mention that in June, 2016, the Eurosystem started to make purchases under its new corporate sector purchase programme (CSPP) in order to support the EU banking sector assets.

To complement the empirical analysis of these two banks, we also consider the smallest (non-systemic) banks listed on the website of the European Banking Authority (EBA) for which the exposure measure is between €200 billion and €250 billion. Assuming, as in [Altınkılıç and Hansen, 2000], that the spread paid is a U-shaped function of the gross proceeds, we show that our bank-by-bank result might explain the empirical finding of [De Jonghe and Öztekin, 2015], namely that it is optimal for small banks to sell their risky assets rather than to issue new shares.

This paper is related to the literature on bank's target capital ratio and/or bank's channels of adjustments, see for instance [Bakkar et al., 2019], [Berger et al., 2008],[Kok and Schepens, 2013], [Memmel and Raupach, 2010], [Öztekin and Flannery, 2012], [Shimizu, 2015] but departs from it in two respects. First, as opposed to this literature, we consider a theoretical model that we calibrate for *each* bank using public data and we compare the observed decision with its predictions, i.e., we do not test any statistical model. Second, we do not consider a partial adjustment model in which it may take several years for the bank, everything else equal, to reach the target capital ratio because the speed of adjustment is lower than one. Within our framework, we consider a two-period model, say date 0 and date 1, and we assume that the target should be reached at time 1, that is, we make the implicit assumption that the speed of adjustment is equal to one. Since we consider a short period (few months), we exclude the possibility to increase retained earnings to reach the target.

The rest of this paper is organized as follows. In the second section, we present a literature review on the subject. In the third section, we present our theoretical framework and then state our theoretical results. In the fourth section, we present in detail the way we calibrate the parameters of the model and discuss the capital increase done by two European systemic banks in the first quarter of 2017. In the fifth section, we briefly consider the case where the spread is a U-shaped function of the gross proceeds. The last section of the paper is devoted to a brief conclusion.

2 Literature review: banks capital requirements and channels of adjustment

Capital structure: debt versus equity. The literature on capital structure began with the so-called Modigliani-Miller theorem which states that the value of a firm is invariant with respect to its capital structure, that is, whether the firm is financed with equity only, debt only or a mix (i.e., debt and equity) has no impact on its value. In the Modigliani-Miller world, the capital structure only impacts the *distribution* of the surplus generated by the economic activity of the firm–the earning before interest and taxes (EBIT)–distributable to claim-holders (debt holders/equity holders). Consider the situation at a given point in time and let C be the financial expenses such as the coupon of a bond to be paid to bondholders at that time. When there is no default risk and no corporate income tax, bondholders receive C (for sure) while shareholders receive EBIT $-\ C$. The sum of the revenues (perceived by bondholders and shareholders) is equal to EBIT and this means that the capital structure, that is, whether the firm is financed with equity only or with a mix, has no impact on the value of the firm (here the EBIT) but only impacts its distribution[2]. This famous invariance result published in 1958 makes however a number of important assumptions–complete information, homemade leverage, risk class, no bankruptcy, no corporate income tax–that are not always realistic. We refer to [Stiglitz, 1988] for a lucid discussion of these assumptions. Since the publication of the Modigliani and Miller article, the aim of the literature on the subject has been precisely to relax these assumptions in order to inquire (theoretically and empirically) whether or not the capital structure of firms matters. This literature is now extremely large and contains different approaches and models, see for instance the review paper by [Harris and Raviv, 1991] or [Myers, 2001].

Tradeoff theory versus pecking order theory. In Modigliani-Miller analysis, it is assumed that there is no default risk and no corporate income tax. As long as there is a positive corporate income tax τ_{ax}, the sum of revenues perceived by shareholders and bondholders is now equal to

[2]See [Braouezec, 2010] for a classical and modern presentation of this result. The modern approach makes use of continuous-time models together with the no-arbitrage principle.

$C + (\text{EBIT} - C)(1 - \tau_{ax}) = \text{EBIT}(1 - \tau_{ax}) + \tau_{ax}C$ and is an increasing function of C due to the tax-shield effect. However, when C is too large, default risk increases so that the (expected) cost of financial distress also increases. Within this approach, called the *tradeoff theory*, the optimal capital structure (that results from the coupon that maximizes the value of the firm) is the best tradeoff between the tax advantage of the debt and the cost of possible financial distress. But this is only one possible explanation. Another approach, perhaps simpler, is called the *pecking order theory* and says basically that the firm will make use of debt when internal cash flow is not enough to finance the investment project. As the name suggests, the pecking order theory of capital structure leads to an order. Firms prefer internal to external finance but if external funds are needed to finance the investment project, the firm should issue debt before equity (see [Myers, 2001] for more on this subject).

Agency costs of debt. In the Modigliani-Miller world, it is implicitly assumed that there is no conflict of interest between the various stakeholders, in part because the investment policy of the firm is given. The only problem consists in choosing the financing policy, that is, debt, equity or both. However, in practice, there are potentially various conflicts of interest between managers, bondholders and shareholders precisely because the investment policy is not given. For instance, when an investment project is essentially financed with debt, due to limited liabilities, shareholders have an incentive to choose a project which is riskier than the one they would choose if the project was financed with equity only. This phenomenon is known in the literature as the asset substitution problem (see [Harris and Raviv, 1991], section I for more on this subject). An important body of literature on agency cost (in line with capital structure) began with the seminal paper by [Meckling and Jensen, 1976], in which the optimal capital structure of the firm is obtained by trading off the agency cost of debt against the benefit of debt. We refer the reader to the review theory papers by [Harris and Raviv, 1991] and [Myers, 2001] but also to the chapter two of [Tirole, 2010]. It should be noted to conclude that, as observed in [Myers, 2001], there is no universal theory of capital structure and even no reason to expect one.

Banks' capital requirements and channels of adjustment.

The vast majority of the literature on capital structure is devoted to non-financial firms and thus excludes financial institutions such as banks. However, banks are of particular interest not only because they are financial institutions, that is, with a particular balance sheet (securities both appear on the asset and liability side) but also because they are heavily *regulated*, which means that they are not allowed to freely choose their capital structure. Since the asset side of the banks' balance sheet is essentially composed with securities (subject to market risk and/or to counterparty risk) and loans (subject to credit risk), equity (or capital) is designed to absorb the asset losses as a going concern, that is, without impeding the usual activity of the bank. Regulators thus impose a minimal percentage of the banks' activities that must be financed with equity. After the 2008 financial crisis, to take into account the deficiencies of Basel I and II, the Basel Committee on Banking Supervision[3] (BCBS) published a document known as Basel 3, designed to strengthen the global capital framework. The BCBS makes a distinction between two types of capital, Tier 1 capital (going-concern capital), which is the sum of Common Equity Tier 1 (CET1) and additional Tier 1 (AT1) and Tier 2 (gone-concern capital). Beyond the classical risk-based capital ratios (e.g., Tier 1 divided by the risk-weighted assets (RWA)) now subject to various buffers (capital surcharge), the BCBS also introduced a new risk-unweighted capital ratio called the leverage ratio, defined as Tier 1 capital divided by the total exposure (a quantity which turns out to be close to the total assets). In 2023, banks will have to comply two Tier 1 capital ratios, the Tier 1 risk-based capital and the leverage ratio.

Assume now that a bank wants to increase its Tier 1 capital ratio because its current capital ratio is lower than its target capital ratio. What are the channels of adjustment that can be used? Following [Cohen and Scatigna, 2016], there are four main channels of adjustment that can be used by a bank to increase its risk-based capital ratio. Two channels are related to the liability side while the two others are related to the asset side.

1. The bank may increase over time its retained earnings by reducing its dividend policy (liability side). Progressively, everything else equal, the capital of the bank (Tier 1) as well as its Tier 1 capital ratio will increase.

2. The bank can issue new equity at one point in time (liability side). Right after this new issue, the capital (Tier 1) and thus the Tier 1 capital ratio of the bank will increase. Everything else equal, Tier 1 capital of the bank as well as its Tier 1 capital ratio will increase.

3. The bank can sell a portion of its loan portfolio and use the proceeds to pay back its debt (asset side). Everything else equal, Tier 1 capital of the bank as well as its Tier 1 capital ratio will increase.

[3] In December 2017, the BCBS published a document entitled Basel 3 Finalising post-crisis reforms. The complete document called the Basel framework is available since December, 2019.

4. The bank can reduce its risk-weighted assets by replacing riskier assets with safer ones (asset side). Everything else equal, the risk-weighted assets will decrease so that the capital ratio will also increase.

Note importantly that the first three channels of adjustment both increase the risk-based capital ratio and the leverage ratio. However, by replacing assets with high risk weights by assets with low risk weights, this adjustement will not contribute to increase the leverage ratio as long as the the total value of the risky assets remains identical.

In a crisis period, that is, when say the overall banking sector has been hit by a common shock, as in 2007-2008, issuing equity will be very expensive while cutting the dividend policy will take many years to increase the capital ratio. After such a shock, banks typically choose to adjust their asset side and quickly deleverage in order to increase their capital ratio, something which is acknowledged by the Basel Committee. In [BCBS, 2015] (see Graph 1 p. 9), they explicitly consider the case in which a bank may deleverage by selling tradable securities to increase its risk-based capital ratio and/or may cut-off balance sheet exposure to increase its leverage ratio. The reason for this choice of channel of adjustment is related to the well-known debt Myers (1977) overhang problem[4]. After a shock, a bank may be reluctant to raise new equity to fund its (profitable) investments because an important portion of the value created will then be siphoned off by senior creditors of the bank ([Hanson et al., 2011]). In such a debt overhang situation, by acting in the interest of shareholders, the bank will shrink its assets rather than issue new equity. When many banks deleverage by selling assets or by replacing risky assets with safe ones, this leads to a generalized asset shrinkage that is costly for the society. As observed in ((([Hanson et al., 2011]), the two primary costs are credit crunch (that is, banks may stop their lending activity and this can lead to an economic recession) and fire sales (that is, the price of the securities sold may sharply decrease). Interestingly and perhaps not so surprisingly, [Kapan and Minoiu, 2018] find that during the 2007/2008 financial crisis, banks with ex ante higher level of Tier 1 capital were able to maintain their credit supply when hit by a shock. [Brunnermeier, 2009] explains in detail the various origins of fire sales during the financial crisis and the vicious circle they generate. When asset prices fall, banks' capital ratio erodes and leads to fire sales to restore back their capital ratio which in turn push down the price and so on and so forth (see also [Braouezec and Wagalath, 2019] for a game theoretic equilibrium approach). These two costs for the society, credit crunch and fire sales, justify the implementation of a macro prudential regulation, which is by definition system-wide focused, and complements the

[4]See Myers, S. C. (1977). Determinants of corporate borrowing. *Journal of financial economics*, 5(2), 147-175.

micro prudential regulation, which is bank-by-bank focused. This macro prudential regulation leads to various capital buffers which in turn leads to higher capital requirements.

In [Gropp et al., 2019], the aim is precisely to inquire how banks adjust their balance sheets in response to higher capital requirement to understand the real consequence of these higher requirements. The authors exploit the 2011 capital exercise conducted by the European Banking Authority (see also [Greenwood et al., 2015]) and use a sample of 61 banks that were selected based on their total assets. Their main result is to show that banks decided to decrease their risk-weighted assets (by 16 percentage points) rather than to increase the capital by issuing new equity. More specifically, they show that banks reduced their exposure to corporate and retail borrowers. According to the authors, this lack of incentive to issue new equity can be partly explained by the debt overhang problem. In a related paper, [Bostandzic et al., 2018] also consider the EBA capital exercise as of 2011. Their results are consistent with the one [Gropp et al., 2019] in that they find that banks reduced their risk-weighted assets. Interestingly, they also use several (market) risk measures such as the Value-at-Risk, the expected shortfall or SRISK and all have increased, which suggests that the EBA capital exercise failed to increase bank solvency. In [Juelsrud and Wold, 2020], they consider a sample of 110-120 Norvegian banks and exclude foreign banks but also the second largest Norvegian bank Nordea (see also [Shimizu, 2015] for Japonese banks). An interesting feature of this article is that Norway is not a member of the EU. In Norway, the increase in capital requirements was proposed in March 2013 (and adopted in July 2013) and this new requirement was phased in over two years. Assuming that 2013 Q2 is the reform quarter, they document that banks increased their capital ratio by reducing the risk-weighted assets (through a reduction in credit supply), an empirical result which is consistent with [Gropp et al., 2019] and [Bostandzic et al., 2018]. In [Mayordomo and Rodríguez-Moreno, 2020], using a somewhat larger sample of 144 banks and with more recent data (i.e., from 2013 to 2017), they also show that the main channel through which banks increase their risk-based capital ratio is the risk reduction, that is, by rebalancing portfolios toward assets with lower risk-weights. Using a smaller sample of 35 banks but data from 2000 to 2016, [Klinac and Ercegovac, 2018] document a similar result, banks adapted to the new regulation by decreasing their credit (risk) exposure.

Overall, these recent papers provide evidence that the main channel of adjustment used by banks to increase their capital ratio is the asset side, either by reducing the exposure to risky assets or by rebalancing their portfolios toward safer assets.

3 A simple model of target capital ratio with costly channels of adjustment

We offer here a simple model for which the parameters are easy to calibrate using public data. We consider the case of a systemic universal bank which is subject to the (evolving) Basel regulation and that holds three types of assets that differ by their riskiness and their liquidity.

1. A safe asset, cash, which is the value of the bank account of the bank at the central bank. Cash is not risky and is thus not subject to capital requirement.
2. A risky traded asset such as a stock, an Exchange Traded Fund (ETF), a bond, (possibly a derivative) for which some capital is required because it is both subject to market risk and counterparty risk.
3. A set of non-traded loans for which capital is required because they are subject to credit risk.

Banks hold loans in their banking book for a non-negligible fraction (at least 30% of the total value of the assets) and also hold liquid traded assets in their trading book (stocks, bonds, ETF, derivatives). Due to the well-known adverse selection problem (e.g., [Diamond and Rajan, 2011]) loans are illiquid assets and thus are fairly difficult to resell in the short-term. Since we are interested to understand the conditions under which a bank will optimally issue new shares only, considering a model with more than one risky asset would only contribute to complicate the analysis without new financial insights. For the sake of simplicity, as in [Admati et al., 2018] (see their proposition 7), we thus focus on the simplest model in which there is a unique homogenous risky traded asset subject to capital requirements. Of course, the bank can also invest in a risk-free asset, its bank account at the central bank, which is thus not subject to capital requirement. In section 4 devoted to the empirical applications, we shall relax this assumption and we will explicitly consider the case of two risky assets, a liquid one and an illiquid one. We shall show that as long as the resale value of this illiquid asset is sufficiently small, nothing is fundamentally changed.

3.1 Bank's balance sheet and target capital ratio

Let $v > 0$ be the value of cash at time $t = 0$ and let P and q denote respectively the price (or the Mark-to-Market more generally) and the quantity of the risky asset held by the bank at date $t = 0$. The total value of the assets of the bank at time $t = 0$ is equal to $A = v + qP$. On the liability side, let D be the sum of deposits and total face value of bonds that have been issued by the bank. Let E denotes the capital of the bank at time $t = 0$. From limited liability of shareholders, the value of total equity at date $t = 0$ is equal to $E = \max\{A - D; 0\} = \max\{v + qP - D; 0\}$ and we shall assume that this total capital is positive at time $t = 0$, that is

$$E = v + qP - D > 0 \tag{A.1}$$

The total capital E is equal to Tier 1 capital K_1 plus Tier 2 capital K_2, that is $E = K_1 + K_2$. To facilitate the presentation, without loss of generality, we shall assume that $E = K_1$. This assumption is realistic in practice since the most important component of the total capital is by far Tier 1 capital. In any event, it is not difficult from a theoretical point of view to assume that Tier 2 capital is positive. As we shall see, the advantage of this assumption is that the target capital ratio can be formulated directly by using E rather than a fraction of E equal to $\frac{K_1}{E}$. The following balance-sheet represents the situation of the bank at time $t = 0$.

Balance sheet at time $t = 0$

Assets	Liabilities and Equity
Cash: v Risky asset: qP	Debt: D Equity : E
$A = v + qP$	$E + D$

Since there is only one risky asset, the risk-weighted asset (RWA) can thus be expressed as a percentage of the value of the risky asset qP. As cash is considered as risk-free by regulators, its weight is equal to zero and thus is not subject to any capital requirement. The risk-weighted assets thus is equal to

$$\text{RWA} = \alpha qP \tag{A.2}$$

where $\alpha < 1$ is the *risk-weight* associated to the risky asset. Since all the quantities involved, i.e., cash, the value of the risky assets and the risk-weighted assets are disclosed in the annual report of banks, α is easy to calibrate and is equal to

$$\alpha = \frac{\text{RWA}}{qP} \tag{A.3}$$

Let

$$\theta = \frac{v + qP - D}{\text{RWA}} = \frac{K_1}{\alpha qP} \tag{A.4}$$

be the regulatory capital ratio of the bank at time $t = 0$ and let respectively θ_{min} and θ^* be the minimum capital ratio and the target capital ratio. For the sake of interest, we shall assume that

$$\theta_{min} \leq \theta < \theta^* \tag{A.5}$$

that is, the current capital ratio is lower than the target capital ratio but is higher than the minimum required. The quantity defined as $\theta^* - \theta_{min} > 0$ can be interpreted as a safety spread (or margin). For the sake of interest, one may assume that at the current date, $\theta = \theta_{min}$ so that there is no safety margin.

In the Basel framework, another capital ratio, called the leverage ratio will complement the classical risk-based capital ratio. The striking feature of this leverage ratio is that it is a risk unweighted capital ratio defined as Tier 1 capital divided by the total exposure, a quantity close to the total assets. According to a document from the European Systemic Risk Board (ESRB)

> "*The leverage ratio increases the resilience of large, complex and interconnected institutions against higher model risk and uncertainty. Given that large and complex institutions are more likely to rely on internal rating-based approaches to set risk-weighted assets capital requirements and to have significant trading books with low measured risk, they are also more likely to be influenced by both model risk and uncertainty*". ([(ESRB), 2015] p. 6)

In practice, the leverage ratio is intended for (large) banks that make use of internal models and is designed to serve as a backstop against an average implied risk-weight that would be artificially low ([BoE, 2017]). Within our simple model, we shall proxy the total exposure by the total value of the risky asset qP. As a result, the leverage ratio is defined as

$$L = \frac{E}{qP} = \frac{K_1}{qP} \tag{A.6}$$

and must be greater than L_{min}, the minimum leverage ratio. The risk-based capital ratio and the leverage ratio turn out to be intimately related since they have the same numerator. The denominators are however different, it is the risk-weighted assets for the risk-based capital ratio and the total exposure for the leverage ratio. In practice, the requirements are also different since L_{min} is equal to 3% while θ_{min} is approximately equal to 8.5%. Within our framework, it is easy to see from equation (A.4) that

$$\theta = \frac{L}{\alpha} \tag{A.7}$$

that is, the leverage ratio and the risk-based capital ratio are equal up to the inverse of the implied risk-weight term equal to $\frac{1}{\alpha}$, something observed for instance in [(ESRB), 2015]. From an empirical point of view, the simple relation between the Tier 1 risk-based capital ratio and the leverage ratio (given in equation (A.7)) can be more complex since the value of the risky assets and the total exposure may not coincide. However, for most banks, total assets and total exposure do not differ by more than 5% and this means that equation (A.7) provides an accurate relation.

To understand why the leverage ratio acts as a backstop against an artificially low risk-weighted assets, we shall follow the presentation offered in [(ESRB), 2015] annex 1 and in [BoE, 2017]. Let $K_1^\theta = \theta_{min} \alpha q P$ be the capital of the bank required by the risk-based capital ratio and $K_1^L = L_{min} q P$ be the capital required by the leverage ratio. The leverage ratio and the risk-based capital ratio are said to be equally stringent when $K_1^L = K_1^\theta$, which is equivalent to an implied weight equal to $\alpha_c = \frac{L_{min}}{\theta_{min}}$ called the *critical average risk weight*. Note that since α_c depends upon L_{min} and θ_{min} only, the critical threshold α_c is a quantity which is fixed by regulators.

Assume that $K_1^L > K_1^\theta$, that is, the capital required by the leverage ratio is greater than the capital required by the risk-based capital ratio. It is easy to show that this is equivalent to $\alpha < \alpha_c$ and it is precisely in that sense that the leverage acts as a backstop. When the implied risk-weight α is considered as "too low" by regulators, that is, lower than the critical average risk weight α_c, the bank is constrained by the leverage ratio and not anymore by the risk-based capital ratio and it is in that sense that the leverage ratio acts as a backstop. It should be pointed out that the leverage ratio is currently (in 2020) disclosed in annual reports of banks but will be binding in principle in January, 2022, in practice in January 2023 due to Covid-19[5].

3.2 Equity issuance versus asset sale: what are the costs?

We have already seen in section 2 that there are four different (non-exclusive) ways through which a bank can increase its capital ratio. In this paper, we shall consider the two main channels of adjustment that can be used by a bank in the short term, equity issuance and/or asset sale (asset shrinking or risk-reduction). The bank can issue new equity with gross proceeds equal to $I > 0$ or

[5]See https://www.bis.org/press/p200327.htm. Note also that systemic banks identified as such by the financial stability board will have a leverage buffer requirement as a function of the G SIB buffer.

it can reduce its risk-weighted assets (RWA) by selling a portion $s \in [0, q]$ of the risky asset (with positive weights) and then investing the proceeds in cash (with no risk weight). Of course, the bank can also mix the channels, i.e., it can issue new shares (i.e, choose $I > 0$) *and* sell a portion of the risky asset (i.e, choose $s > 0$).

Consider first the possibility to issue new shares. While there are several flotation methods, i.e., various ways to issue such new shares (see e.g., [Eckbo et al., 2007] for a comprehensive review), we only mention the three following ones.

- *Firm commitment*, in which the set of underwriter(s) contractually commit to buy the new shares at a fixed price.
- *Standby rights*, in which existing shareholders are offered the right but not the obligation to buy the new shares at a discount and the set of underwriters are committed to buy the unexercised new shares.
- *Direct public offering*, in which the issuer sells directly the equity without any underwriters.

The issuance cost obviously depends on the choice of the flotation method. The cheapest one is clearly the direct public offering because the issuer has no guarantee to receive the desired gross proceeds while the most expensive one is the firm commitment.

In [Eckbo et al., 2007], they make a distinction between direct costs (e.g., underwriter(s) fees, registration and listing fees) and indirect costs (e.g., stock price reaction to the offering announcement, cost of offering delay/cancellations...). It is also common to split the total direct cost of issuance into two types of costs (see for instance [Altınkılıç and Hansen, 2000]); a fixed cost, related to various administrative costs (registration and listing fees) and a variable cost, related to the underwriter(s) compensation that critically depends upon the underwriting agreement and the gross proceeds. Since P denotes the asset price, we shall denote I the gross proceeds (I for investment) chosen by the bank. Following [Altınkılıç and Hansen, 2000], [Décamps et al., 2011] and [Gomes, 2001] among others, the total issuance cost function is assumed to be piece-wise linear:

$$\text{Cost of equity issuance} = K\,\mathbf{1}_{I>0} + cI \tag{A.8}$$

where $\mathbf{1}_{I>0} = 1$ if $I > 0$ (i.e., the bank issues new stocks) and $\mathbf{1}_{I>0} = 0$ if $I = 0$, K is the fixed (issuance) cost and $c \in (0, 1)$ is the (constant marginal) issuance cost. This means that when the bank decides to issue new equity for gross proceeds equal to I, it has to pay K and cI, which means that the net amount of cash received by the bank, called the net proceeds, is equal to $(1-c)I - K$. For the net proceeds to be positive, I must be high enough. The marginal cost c is the sum of the two following marginal costs m and d, that is, $c = m + d$ where

- m is the marginal cost of issuance and is related as already said for instance to underwriters compensation and
- d is the marginal cost of dilution assumed to linearly increase with I.

It should be noticed that the way dilution costs are encapsulated within our framework can be thought of as a reduced form approach. Dilution costs may be an issue when an institution decides to increase its capital. However, there is not a unique way to model these costs. Within our framework, when $K = c = 0$ but when $d > 0$, the cost of issuing equity is still positive due to the existence of dilution costs that are an increasing function of I. As we shall see in the empirical application, the right issue is precisely designed to avoid dilution for those shareholders who choose to exercise their rights.

It is usual to define the *spread* as the average cost of issuance per euro, i.e., it suffices to divide the rhs of equation (A.8) by the gross proceeds $I > 0$ (e.g., [Altınkılıç and Hansen, 2000]).

$$\text{Spread} = \frac{K}{I} + c \quad I > 0 \tag{A.9}$$

For systemic banks, as we shall see, the gross proceeds I is in billion while the fixed cost K is in (hundreds) thousands so that $\frac{K}{I}$ is negligible. We thus make the assumption that $\frac{K}{I} = 0$ so that c is the unique cost of issuance. It is actually well-known that the important source of issuance cost is the variable cost (e.g., [Altınkılıç and Hansen, 2000], [Calomiris and Tsoutsoura, 2010]).

Consider now the costs associated with replacing (i.e., selling) riskier assets with safe ones such as cash (or Government bonds that are considered as risk-free) in order to decrease the risk-weighted assets. Let $s \in [0, q]$ be the quantity of the risky asset sold by the bank and $V(s)$ be the proceeds of the asset sale placed on the bank account of the bank (cash). Since the risk-weighted assets $\text{RWA}(s) = \alpha P(q - s)$ is a decreasing function of s, everything else equal, the risk-based capital ratio of the bank will increase. For systemic banks, everything else is however not equal. Due to the existence of the price impact, the price of the risky asset will decrease with the quantity sold s and this will decrease the value of the assets and thus the total capital of the bank. Moreover, selling a portion of the risky asset will also reduce the (future) expected profit.

Price impact. For a systemic bank, called G-SIB (global systemically important bank), selling an important volume of assets in a short period of time may generate a positive price impact. Following the seminal paper of [Greenwood et al., 2015], see [Duarte and Eisenbach, 2015], we consider the simplest case of a linear price impact. When the bank sells a quantity $s \leq q$ of the risky asset, the sale proceeds $V(s)$ is not equal to sP but is lower due to the existence of a positive price impact. For a given price P at time $t = 0$, the price at time $t = 1$ thus is equal to

$$P\left(1 - \frac{s}{\Phi}\right) \tag{A.10}$$

where $\Phi < \infty$ is called the market depth. The lower (higher) the market depth, the more (less) important the price impact. With a linear price impact given by equation (A.10), the proceeds $V(s)$ is equal to $sP(1 - \frac{s}{\Phi})$ and we shall make the realistic assumption[6] that $V(s)$ increases with the quantity sold $s \in [0, q]$, i.e., $\frac{2q}{\Phi} < 1$. The cost due to the price impact of the bank is naturally measured as the difference between the proceeds without price impact and the proceeds with a positive price impact. This cost thus is equal to $sP(\frac{s}{\Phi})$ and increases with the quantity sold s. But this is not the only cost.

Reduction of the expected profit. Even without price impact, there is a cost associated to selling the risky asset, related to the fact that the expected profit will decrease with s. Assume that the expected profit is equal $\mathbb{E}\Pi(q) = \gamma qP$ (for some $\gamma < 1$ and some expectation operator $\mathbb{E}$) when the bank holds a quantity q of the risky asset, that is, the expected profit is a percentage of the total value of the position in the risky asset. When the bank resells a positive quantity s, it now holds a quantity $q - s$ and the resulting expected profit is equal to $\mathbb{E}\Pi(q - s) = \gamma P(q - s) = \mathbb{E}\Pi(q) - \gamma Ps$. Compared with the initial situation in which the bank held a quantity q, when it sells a quantity $s \leq q$, the reduction of its expected profit thus is equal to γsP, which constitutes the second opportunity cost of selling a portion of the risky asset. In the limiting case in which $s = q$, the bank only holds cash and the expected profit thus is equal to zero[7]. Overall, the cost of selling the risky

[6]We make the implicit assumption that when selling a portion of assets s, the price at which the bank can sell this quantity is equal to $P\left(1 - \frac{s}{\Phi}\right)$. As [Braouezec and Wagalath, 2018] among others, a more dynamic way to take into account the price impact would be to assume that the bank liquidate in a uniform way its assets from time zero to one and sells at the average price equal to $P\left(1 - \frac{s}{2\Phi}\right)$

[7]In Europe, the profit would actually be negative since the rate of the deposit facility administered by the ECB is negative, equal to -0.5%. The rate of the deposit facility is the interest which is applied by the ECB when a given

asset is the cost related to the price impact plus the reduction of the expected profit.

$$\text{Cost of selling the risky asset} = sP\left(\gamma + \frac{s}{\Phi}\right) \tag{A.11}$$

We make the natural assumption that the aim of the bank is to choose the channel(s) of adjustment in order minimize the sum of the adjustment costs given by equation (A.8) plus equation (A.11).

In the rest of this section, we shall state and solve the optimization problem of the bank as a function of the parameters. We make the assumption that the adjustment is *instantaneous*, that is, once the bank has chosen the channel(s) of adjustment, there is no delay for the implementation. In the empirical literature on partial adjustment (see e.g., [Flannery and Rangan, 2006] or [Öztekin and Flannery, 2012]) they make the assumption that there is a delay, that is, using our notations, the partial adjustment model for a given firm is written as $\theta_t - \theta_{t-1} = \lambda(\theta^* - \theta_{t-1}) + \epsilon$ where λ is the adjustment speed and ϵ is a noise term. Taking into account such an adjustment speed in an optimization model would be more difficult from a mathematical point of view since it would require to solve a dynamic optimization problem. It would also be more difficult to test assuming even that it yields clear predictions. Overall, we believe that the (purely) empirical partial adjustment literature and the static (optimization-based) model offered in this paper are complementary rather than substitutable. Since our static model yields clear prediction as a function of the observable) parameters, this prediction can be tested on a bank by bank basis. It is interesting to note that in the empirical partial adjustment models, the target is not assumed to be observable. On the contrary, within our approach, we explicitly use the fact that the target capital ratio θ^* is explicitly disclosed in annual reports of banks.

3.3 The bank's optimization problem

For a given choice of channel(s) of adjustment $(s, I) \in [0, q] \times \mathbb{R}^+$, the bank's balance-sheet at date $t = 1$ is given as follows:

Balance sheet at time $t = 1$

bank leaves its excess reserves on its bank account at the ECB. Since 30, October, 2019, there is a two tier system which "*exempts credit institutions from remunerating, at the negative rate currently applicable on the deposit facility, part of their excess reserve holdings (i.e. reserve holdings in excess of minimum reserve requirements").* See the website of the ECB, https://www.ecb.europa.eu/mopo/two-tier/html/index.en.html

Assets	Liabilities and Equity
Cash: $v + sP(1 - \frac{s}{\Phi}) + (1-c)I$	D
Risky asset: $(q-s)P(1-\frac{s}{\Phi})$	$E(s,I)$
$A(s,I) = v + qP(1-\frac{s}{\Phi}) + (1-c)I$	$E(s,I) + D$

and note that the total capital at time $t = 1$, $E(s, I) = \max\{A(s, I) - D; 0\}$, depends on (s, I) while the risk-weighted assets $\text{RWA}(s)$ depends only on s. Formally, the bank's optimization problem is as follows.

$$\min_{(s,I)\in[0,q]\times\mathbb{R}_+} C(s,I) = sP\left(\gamma + \frac{s}{\Phi}\right) + cI \tag{A.12}$$

$$s/c$$

$$\theta(s,I) := \frac{E(s,I)}{\text{RWA}(s)} = \frac{v - D + qP\left(1-\frac{s}{\Phi}\right) + (1-c)I}{\alpha(q-s)P(1-\frac{s}{\Phi})} = \theta^* \tag{A.13}$$

$$L(s,I) := \frac{E(s,I)}{A(s,I) - v} = \frac{v - D + qP\left(1-\frac{s}{\Phi}\right) + (1-c)I}{(q-s)P(1-\frac{s}{\Phi})} \geq L_{min} \tag{A.14}$$

$$s \geq 0, I \geq 0 \tag{A.15}$$

Note importantly that we make the assumption that the bank uses the *risk-reduction strategy*, that is, the bank sells a portion of its risky assets (with a positive risk weight) and invests the proceeds in reserves (that is, put the amount on its bank account at the central bank), which is an asset with no risk weight. Another solution for the bank is to use the *asset shrinking strategy*, that is, it also consists to sell a portion of the risky assets and to use the proceeds to pay back a portion of its debt.

Fact 25 *Whether the bank uses the risk-reduction strategy or the asset shrinking strategy, the two capital ratios given by equations (A.13) and (A.14) remain identical.*

The proof of this result is very simple. When the bank sells a quantity s of risky asset whose value is equal to sP, the proceeds are equal to $sP(1 - \frac{s}{\Phi})$ due to the price impact. Since the bank uses this amount to repay the debt, the value of the debt is reduced to $D - sP(1 - \frac{s}{\Phi})$ but all the other quantities except the total assets[8] remain unchanged, in particular the risk-weighted assets and the capital so that the two capital ratios given by equations (A.13) and (A.14) remain identical and this concludes the proof.

[8]When the bank replaces the risky asset by the non-risky one, the value of the total assets is equal to $v + qP(1 - \frac{s}{\Phi}) + (1-c)I$ while it is equal to $v + (q-s)P(1-\frac{s}{\Phi}) + (1-c)I$ when the bank repays a portion of its debt.

Let $(s^*, I^*) \in [0, q] \times \mathbb{R}^+$ be the optimal solution when it exists and note that it must be such that $s^* < q$ otherwise, the denominator of the capital ratio is equal to zero. Recall that it is assumed that $E = K_1$.

Proposition 20 *Assume that $L \geq L_{min}$, $\theta = \theta_{min}$ and $\theta^* > \theta_{min}$. If there exists an optimal solution $(s^*, I^*) \in [0, q) \times \mathbb{R}^+$ such that $\theta(s^*, I^*) = \theta^*$, then, $L(s^*, I^*) \geq L_{min}$*

The above result shows that as long as the target capital ratio can be reached under the optimal solution, then, if the leverage ratio was satisfied at time $t = 0$, it is *automatically* satisfied at time $t = 1$. To see this, recall that we make the assumption that $\theta = \theta_{min}$ so that $\alpha = \frac{L}{\theta_{min}}$. Since $\theta(s^*, I^*) = \frac{L(s^*,I^*)}{\alpha}$, by using the fact that $\alpha = \frac{L}{\theta_{min}}$ and that $\theta(s^*, I^*) = \theta^*$, we obtain that $L(s^*, I^*) = L \times \left(\frac{\theta^*}{\theta_{min}}\right) \geq L_{min}$ since $L \geq L_{min}$ and $\left(\frac{\theta^*}{\theta_{min}}\right) > 1$. If $\theta > \theta_{min}$, then, for some positive ζ, α can be written as $\alpha = \frac{L}{(1+\zeta)\theta_{min}}$. Everything else equal, the higher $\theta - \theta_{min}$, the higher ζ. By definition of a target capital ratio, the current capital ratio θ is lower than θ^* and rather close to the minimum required θ_{min}. If θ is much greater than θ_{min}, specifying a target makes no real sense. This means that ζ is close to one and as a consequence $\frac{\theta^*}{(1+\zeta)\theta_{min}} \geq 1$ so that $L(s^*, I^*) = \alpha\theta^* = L\frac{\theta^*}{(1+\zeta)\theta_{min}} \geq L_{min}$, that is, proposition 20 holds. Proposition 20 thus still holds when $\theta > \theta_{min}$ as long as, everything else equal, θ remains close to θ_{min}. So far, we make the assumption in proposition 20 that there is indeed an optimal solution (s^*, I^*). We shall now discuss the condition under which such an optimal solution exists.

Consider first the case in which there is no price impact, i.e., $\frac{1}{\Phi} = 0$. In such a case, since the total value of the assets as well as the total equity are *invariant* with respect to $s \in [0, q]$, a solution always exists in s (i.e., with $I = 0$) for each $\theta^* \in \mathbb{R}^+$. When the price impact is positive, the total value of the assets as well as the total capital of the bank are now *decreasing* functions of $s \in [0, q]$. As a result, a solution of the optimization problem in s only (i.e., with $I = 0$) may not always exist; reaching the target capital ratio may thus require from the bank to use the two channels of adjustment, i.e., asset sale and/or new issues. When the price impact is small enough, i.e., when $\frac{1}{\Phi}$ is close enough to zero, a solution always exists. Let $E(q, 0) = v - D + qP(1 - \frac{q}{\Phi})$ be the total capital at time $t = 1$ when the bank sells 100% of its risky asset and does not issue new shares. It is easy to show that

$$E(q, 0) > 0 \Longleftrightarrow \frac{q}{\Phi} < \frac{v + qP - D}{qP} \tag{A.16}$$

Let $\underline{\Phi} = \frac{q^2 P}{v+qP-D}$ and note that $\Phi > \underline{\Phi}$ is equivalent to $E(q,0) > 0$. When $\Phi > \underline{\Phi}$, a solution in $s < q$ always exists to reach the target capital ratio but it turns out that the risk-based capital ratio $\theta(s,0)$ may not be an increasing function of s for each $s \in [0,q]$. In appendix, we show the existence of a smallest market depth $\overline{\Phi}$ (greater than $\underline{\Phi}$) such that, when $\Phi > \overline{\Phi}$, the risk-based capital ratio $\theta(s,0)$ is an increasing function of s for each $s \in (0,q)$. While the existence of a solution only requires $\Phi > \underline{\Phi}$, we shall assume that $\Phi > \overline{\Phi}$ to avoid the somewhat pathological behavior of the risk-based capital ratio which may be locally decreasing with s.

Proposition 21 *(Existence of corner solutions)*

- *Assume that $s = 0$. If $c < 1$, then, for any $\theta^* \in \mathbb{R}^+$, there exists a unique $\overline{I}(\theta^*)$ such that $\theta(0, \overline{I}(\theta^*)) = \theta^*$*
- *Assume that $I = 0$. If $\Phi > \overline{\Phi}$, then, for any $\theta^* \in \mathbb{R}^+$, there exists a unique $\overline{s}(\theta^*)$ such that $\theta(\overline{s}(\theta^*), 0) = \theta^*$.*

Proof. See the appendix.

The above proposition shows that under rather mild assumptions, the bank is able to reach its target capital ratio by issuing new shares only or by selling a portion of the risky asset only.

3.4 Small banks, large banks and the optimal channel(s) of adjustment

In Europe, since few years, the European Banking Authority (EBA) provides a list of large institutions, defined as banks with an exposure measure higher than 200 billion (euros). In 2019, this list contains 36 banks and 11 are currently identified as Global Systemically Important Banks (G-SIBs). For instance, in France, four banks are identified as G-SIBs while in Germany and Italy, only one bank is identified as G-SIB, Deutsche Bank and Unicredit respectively. For such G-SIBs, the exposure measure is higher than 1000 billion (euros). As a result, when a G-SIB sells an important fraction of its total assets (say to reach its target capital ratio), the price impact is not negligible. On the contrary, for small banks, the price impact will be negligible and can be assumed equal to zero, that is, $\frac{1}{\Phi} = 0$.

In what follows, we first consider the case in which there is no price impact (i.e., small banks), that is, $\frac{1}{\Phi} = 0$. In such a situation, it is easy to see that the optimization program (A.12) subject to the constraints given by equations (A.13) and (A.15) reduces to a *linear programming problem.*

As a result, the optimal strategy is a corner solution, that is, either $(0, I^*)$ or $(s^*, 0)$. In particular, it is never optimal to both issue new stocks and sell a portion of the risky asset. We then consider the case of large banks such as G-SIBs for which the price impact is positive, that is, $\frac{1}{\Phi} > 0$. In such a case, this leads to a *non-linear programming problem* due to the presence of the price impact and it may be optimal for the bank to both issue new stocks and sell the risky asset. In both cases, i.e., with and without price impact, to facilitate the confrontation with observed data, we formulate the solution of the optimization problem in terms of *critical spreads*. In the no price impact case, there is a single critical spread that we denoted c_u, where u simply means unique or uniform. In the price impact case, there will be two critical spread that we shall denote c_l and c_h where l and h means as usual low and high.

Proposition 22 *(Small banks, linear programming problem)*

Assume that $\frac{1}{\Phi} = 0$ *(i.e., no price impact) and let*

$$c_u = \frac{\gamma}{\gamma + \alpha\theta^*} \tag{A.17}$$

be a critical spread. The optimal channel of adjustment is as follows.

- *When* $c < c_u$, $s^* = 0$ *and* $I^* = \frac{\theta^*\alpha q P - E}{1-c}$, *that is, it is optimal for the bank to issue equity only.*
- *When* $c > c_u$, $s^* = q - \frac{E}{\alpha P \theta^*}$ *and* $I^* = 0$, *that is, it is optimal for the bank to only sell a portion of the risky asset.*

Proof. See the appendix.

From the above proposition, when γ tends to zero, the cost of selling the risky asset also tends to zero since there is no price impact. As a result, the critical spread c_u tends to zero and it becomes highly likely that selling the risky asset will constitute the optimal channel of adjustment. From equation (A.17), when α tends to zero, the critical spread tends to one so that it becomes optimal to issue new equity only. This property comes from the fact that $\frac{\partial s^*}{\partial \alpha} > 0$, that is, the optimal quantity of risky asset to sell is an increasing function of α. Since s^* is negative when α is small enough, there exists $\overline{\alpha} > 0$ such that $s^* = 0$ for each α lower than $\overline{\alpha}$. The non-negativity of s^* comes from equation (A.15). From an empirical point of view, this property suggests that banks with a low α are more likely, everything else equal, to issue new equity when there is no price impact. In the same vein, when the target θ^* decreases (increases), everything else equal, banks are

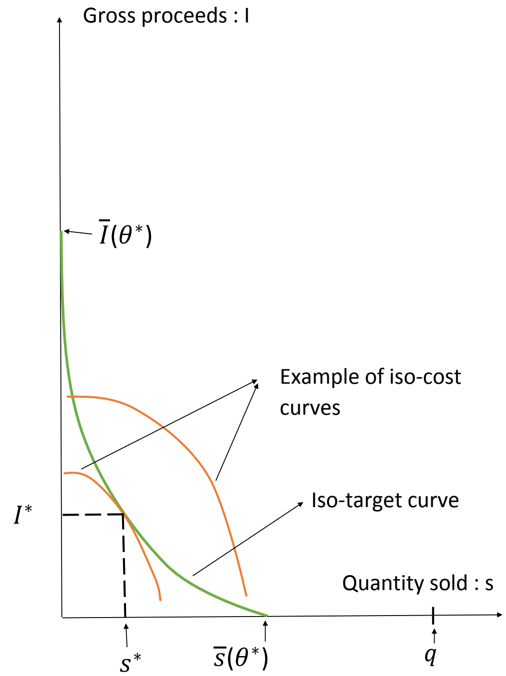

Figure A.1: Optimal solution: $s^* > 0$ and $I^* > 0$

more (less) likely to issue new shares since the critical spread increases (decreases) and tends to one.

We now come to the main result of our article and prove the existence and uniqueness of an optimal channel of adjustment in the general case, that is a solution to the optimization problem defined by equations (A.12) to (A.15). Consider Fig.A.1 and let us call iso-target curve the subset of points defined as follows

$$\Theta^* = \{(s, I) \in [0, \overline{s}(\theta^*)] \times [0, \overline{I}(\theta^*)] : \theta(s, I) = \theta^*\} \tag{A.18}$$

From proposition 21, when $s = 0$, the target can be reached by choosing $\overline{I}(\theta^*)$ while when $I = 0$, the target can be reached by choosing $\overline{s}(\theta^*)$. As both s and I can be positive, Θ^* provides the set of choices of s and I such that the target is reached. As one may expect, the marginal rate of substitution between the gross proceeds and the quantity of risky asset sold is negative. Starting from a given point $(s, I) \in \Theta^*$, if the bank decides to slightly increase the quantity sold from s to $s + \delta s$, it can thus decrease the gross proceeds from I to $I - \delta I$ for some positive δI. In appendix, we show that the iso-target curve (i.e., the constraint) is a decreasing and strictly convex function of s while an iso-cost curve is a decreasing and strictly concave function of s. The optimization problem thus is well-behaved in that when it is optimal to mix the channel of adjustment, i.e., $s^* > 0$ and $I^* > 0$, this optimal solution is found by using a classical tangency condition, see Fig. A.1. Depending upon the parameters of the model $\alpha, \gamma, c, \theta^*$, with $\frac{1}{\Phi} > 0$, the optimal solution belongs to one (and only one) type of the three types of solutions below.

$$\underbrace{s^* = 0 \text{ and } I^* > 0}_{\text{Issue only}} \quad \text{or} \quad \underbrace{s^* > 0 \text{ and } I^* > 0}_{\text{Mix}} \quad \text{or} \quad \underbrace{s^* > 0 \text{ and } I^* = 0}_{\text{Sell only}} \tag{A.19}$$

On Fig. A.1, we represent the unique solution $s^* > 0$ and $I^* > 0$ of the optimization problem, which implies for the bank to make use of the two channel of adjustment, asset sale and stock issuance so that $s^* > 0$ and $I^* > 0$ can be found using a classical tangency condition. The next proposition provides a characterization of the optimal solution in terms of two thresholds.

Proposition 23 *(Large banks, non-linear programming problem)*

Assume that $\frac{1}{\Phi} > 0$ *and that* $\Phi \geq \overline{\Phi}$. *There exists a couple of critical spreads* $(c_l, c_h) \in (0,1)^2$ *with* $c_l < c_h$ *such that the optimal channel of adjustment is as follows:*

- *When* $c < c_l$, *the bank will issue equity only so that* $s^* = 0$ *and* $I^* = \overline{I}(\theta^*) = \frac{\alpha q P \theta^* - E}{1-c}$
- *When* $c \in (c_l, c_h)$, *the bank will both sell a positive portion of the risky asset* $s^* > 0$ *and issue new equity* $I^* > 0$ *with* (s^*, I^*) *equal to*

$$s^* = \frac{(\alpha\theta^*(1 + \frac{q}{\Phi}) + \gamma - \frac{q}{\Phi})c - \gamma}{\frac{2}{\Phi}(\alpha\theta^* - 1)c + \frac{2}{\Phi}} \tag{A.20}$$

$$I^* = \frac{1}{1-c}(\theta^*\alpha(q - s^*)P(1 - \frac{s^*}{\Phi}) - E + \frac{qPs^*}{\Phi}) \tag{A.21}$$

- *When* $c > c_h$, *the bank will sell the risky asset only so that* $s^* = \overline{s}(\theta^*)$ *and* $I^* = 0$ *with*

$$\overline{s}(\theta^*) = \frac{\alpha P\theta^*(1 + \frac{q}{\Phi}) - \frac{qP}{\Phi} - \sqrt{\Delta}}{\frac{2\alpha P\theta^*}{\Phi}} \tag{A.22}$$

$$\Delta = \left(\alpha P\theta^*(1 + \frac{q}{\Phi}) - \frac{qP}{\Phi}\right)^2 - 4\frac{(\alpha P\theta^*)^2 q}{\Phi}\left(1 - \frac{\theta}{\theta^*}\right) \tag{A.23}$$

The critical spreads (c_l, c_h) *have the following expression:*

$$c_l = \frac{\gamma}{\gamma + \alpha\theta^* - \frac{q}{\Phi}(1-\alpha\theta^*)} \tag{A.24}$$

$$c_h = \frac{\gamma + \frac{2}{\Phi}\overline{s}(\theta^*)}{\gamma + \alpha\theta^* - \frac{q}{\Phi}(1-\alpha\theta^*) + \frac{2}{\Phi}\overline{s}(\theta^*)(1-\alpha\theta^*)} \tag{A.25}$$

Proof. See the appendix.

Recall that we make the assumption that the capital ratio is an increasing function of the quantity sold, that is, Φ is high enough. As a result, for any target ratio θ^*, there always exists $\overline{s}(\theta^*) < q$ such that the target is reached, which means that the discriminant Δ in equation (A.23) must be positive[9].

Consider first the lowest threshold c_l. Compared to the no-price impact case, it is now costlier to make use of the asset sale channel due to the existence of a price impact. As long as this price impact is positive, the term $-\frac{q}{\Phi}$ in the denominator of equation (A.24) is negative so that $c_l > c_u$. As expected, when $\frac{1}{\Phi} = 0$, the critical spreads coincide. As in the no price impact case, the following properties are true.

$$\frac{\partial c_l(\alpha,\theta^*,\gamma)}{\partial \alpha} < 0 \qquad \frac{\partial c_l(\alpha,\theta^*,\gamma)}{\partial \theta^*} < 0 \qquad \frac{\partial c_l(\alpha,\theta^*,\gamma)}{\partial \gamma} > 0 \tag{A.26}$$

As already discussed, when α (or θ^*) increases, the critical spread c_l decreases and this decreases the likelihood that the bank will issue new shares. On the contrary, when γ increases, this increases the critical spread c_l since the asset sale solution is now costlier.

Consider now the highest threshold c_h and note that this threshold depends upon $\overline{s}(\theta^*)$ given by equation (A.22). Since $\overline{s}(\theta^*)$ is positive, when there is a positive price impact, it can readily be seen from equation (A.25) that $c_h > c_l$ so that

$$c_u < c_l < c_h \tag{A.27}$$

When Φ tends to infinity, $\overline{s}(\theta^*)$ tends to a finite quantity, equal to the one as given in proposition 22, that is, the quantity that must be sold to reach the target capital ratio. Since the two thresholds differ by a term equal to $\frac{2}{\Phi}\overline{s}(\theta^*)$, when Φ tends to infinity, this term tends to zero so that $c_h - c_l$

[9] It is also easy to see that when Φ is high enough, the positive part of Δ will be close to $\alpha P\theta^*$ while the negative part will be close to zero so that Δ is positive.

tends to zero.

4 Empirical applications to two European systemic banks

In 2017, two European G-SIIs, the German Deutsche Bank and the Italian bank UniCredit, decided to issue new shares via an underwritten rights offering (standby rights). In such a right offer[10], existing shareholders are given the right but not the obligation during the subscription period (typically a couple of weeks) to buy the new shares on a prorata basis and at a pre-specified price (the subscription price) which is below the current market price. Moreover, the set of underwriter(s) have committed (in general under some conditions) to acquire all the new shares that would remain unsubscribed. Since these capital increase decisions have been taken in 2016, for the sake of interest, we shall apply the predictions of our model as if we were on December, 31, 2016 and we consider the following question: can we rationally explain the decision of Deutsche Bank and UniCredit to issue new stocks in the first quarter of 2017?

4.1 The case of Deusche Bank

We shall explain in detail the methodology followed to calibrate the various parameters for the case of Deusche Bank. All the information regarding the capital increase 2017 can be found on the website of Deutsche Bank (https://www.db.com/ir/en/capital-increase-2017.htm). Since the methodology is similar for UniCredit, we will be more brief.

Empirical analysis

Basic facts. In the beginning of April 2017, Deutsche Bank successfuly issued 687.5 million new shares stocks for a total value of €8 billion. In a Media Release as of March 5, 2017, Deutsche Bank announced a target Common Equity Tier 1 ratio, i.e., CET 1 divided by the risk-weighted assets (RWA), equal to 14.1%. The subscription period of the rights offer was from March 21, 2017 to April, 6, 2017, and the subscription price was €11.65 per new share (with no par value) while the market price was around €15 during this period, that is, the discount was approximately equal to 25%. Moreover, each new share carried the same dividend rights as all other outstanding shares of Deutsche Bank. The capital increase has been underwritten by thirty banks but Credit Suisse, Barclays, Goldman Sachs, BNP Paribas, Commerzbank, HSBC, Morgan Stanley and UniCredit

[10] See for instance the comprehensive review of [Eckbo et al., 2007], see also [Holderness and Pontiff, 2016] for a recent overview of rights offerings in USA.

are the main underwriters as each of them committed to subscribe from 6.09% to 8.28% of the new shares underwritten. The number of shares increased from 1.3793 billion to 2.066 billion and Deutsche Bank reports (in the media release as of April 7, 2017) that 98.9% of the subscription rights were exercised. According to Deutsche Bank (see the media release as of April 7, 2017).

> *Had the capital increase been completed on 31 December 2016, Deutsche Bank's Common Equity Tier 1 (CET1) ratio on that date would have been 14.1% on a pro forma CRD4 fully loaded basis rather than 11.8%.*

It is explicitly stated in the prospectus (see p. 113) that existing shareholders that exercise their subscription rights will continue to see their percentage share in the share capital of the Company nearly unchanged. However, for those who decided not to exercise their rights, their percentage ownership in the company's share and their voting rights will be diluted by 33%.

Estimation of the parameters. We need to estimate all the parameters of the model, that is, v, qP, $\frac{q}{\Phi}$, α, c, γ and θ^*.

From the interim report as of June 2017 (page 32), it is reported that the gross proceeds amount to €8 billion while the net proceeds amount to €7.9 billion. As a result, since the cost of issuance is equal to the difference between the gross proceeds and the net proceeds, i.e., it is equal to €0.1 billion[11], so that

$$c = \frac{0.1}{8} \approx 1.25\% \tag{A.28}$$

From the annual report as of December 2016 (see the balance sheet), the total value of the assets, A, is equal to €1591 billion while the value of the cash, v, is equal to €181 billion. As a result, the value of the risky asset $qP = A - v$ is equal to €1410 billion. From the liabilities side, since total equity is equal to €64.81 billion, it thus follows that $D = 1591 - 64.81 =$ €1526.2 billion. The balance sheet of Deutsche Bank at the end of December 2016 with only the two items of interest is provided below.

Deutsche Bank's Balance sheet as of December 2016 (in billion)

Assets	Liabilities and Equity
Cash: $v = 181$	Debt: $D = 1526.2$
Risky asset: $qP = 1410$	Equity : $E = 64.81$
$A = 1591$	$E + D = 1591$

[11] In the prospectus, it is explicitly stated that the maximal amount that Deutsche Bank will pay is €141 million. Due to the success of the right offer, this amount has been less than €141 million.

Since the risk-weighted assets (fully loaded) are equal to €357.5 billion, it thus follows from equation (A.3) that

$$\alpha = \frac{357.5}{1410} \approx 25.3\% \tag{A.29}$$

To apply proposition 23, we need an estimation $\frac{q}{\Phi}$. In [Greenwood et al., 2015], see also the related paper of [Duarte and Eisenbach, 2015] they make the assumption that $\frac{1}{\Phi}$ is of the order of 10^{-13}. Given equation (A.10), this means that selling for 10 billion (i.e., $s = 10^{10}$) of the risky asset leads to a price change of 10 bps (i.e., $\frac{s}{\Phi} = 10^{-3} = 0.1\%$). In this paper, since proposition 23 explicitly depends on $\frac{q}{\Phi}$, we shall proxy this quantity $\frac{q}{\Phi}$ by the total exposure of Deutsche Bank denoted V_{DB} divided by the sum of total exposure denoted V_{Sum}, that is $\frac{V_{DB}}{V_{Sum}}$. The ratio $\frac{V_{DB}}{V_{Sum}}$ is used by supervisors to compute the score of banks considered as G-SIBs (such as Deutsche Bank) for the size indicator. The greater the ratio $\frac{V_{DB}}{V_{Sum}}$, everything else equal, the higher the loss absorbency requirements (capital surcharge) for the bank. The total exposure of Deutsche Bank V_{DB} is available in the annual report but can also be found on the website of the European Banking Authority (EBA) while the sum of the total exposure V_{Sum} can be found on the website of the Bank for International Settlements (BIS). As of December 2016, V_{Sum}= €75900 billion while V_{DB}=€1363 billion. It thus follows that

$$\frac{q}{\Phi} = \frac{qP}{\Phi P} \approx \frac{V_{DB}}{V_{Sum}} = \frac{1363}{75900} = 1.8\% \tag{A.30}$$

This approach is obviously disputable but we have not found a natural way to estimate more precisely the price impact of a bank at an aggregate level. Interestingly, the proxy used for $\frac{1}{\Phi}$ is equal to 7.59 10^{-13}, and thus is consistent with the choice of [Greenwood et al., 2015].

The parameter γ is by definition equal to $\frac{\mathbb{E}\Pi(q)}{qP}$ and is more delicate to estimate as it depends upon the expected future profits. In 2015 and in 2016, (in part) due to litigation costs, Deutsche Bank made a *loss* equal respectively to €6.7 billion and to €1.35 billion. If we estimate statistically γ using years 2015 and 2016 only, γ will be negative and this might not correctly reflect the future expected profits. Moreover, since the capital increase has been successful, i.e., 98.9% of the rights were exercised, the new shares have been bought (by existing shareholders) in the expectation of positive future profit. We thus discard the years 2015 and 2016 and we estimate γ using the average of the years 2012, 2013 and 2014, that is, we consider the net income attributable to shareholders divided by the value of the risky assets (i.e., total assets minus cash). The three values found are equal to 0.013%, 0.04%, 0.098% respectively so that

$$\gamma \approx 0.05\% \tag{A.31}$$

Note that if one estimates this parameter γ by using years 2009 to 2016, i.e., including the two years where the bank makes losses, the prediction of our model remains unchanged.

From the annual report as of December 2016, p. 257, Common Equity Tier 1 (fully loaded) is equal to €42.28 billion, Additional Tier 1 is equal to €4.7 billion and Tier 2 is equal to €12.67 billion. The total capital *fully loaded* is equal to €59.6 billion and is thus lower than the €64.81 billion reported in the balance sheet because of few regulatory deductions. It thus follows that the Common Equity Tier 1 ratio is equal to $\frac{42.28}{357.5} \approx 11.8\%$ and the total capital ratio is equal to $\frac{59.6}{357.5} \approx 16.6\%$. If one adds the €8 billion of new shares to Common equity Tier 1, we obtain a Common Equity Tier 1 ratio equal to 14.1%, as predicted by Deutsche Bank in the media release as of April 7, 2017. Since the target θ^* is expressed in our model as Tier 1 plus Tier 2 divided by the risk-weighted assets, we now have to compute θ^* from the target CET 1 (fully loaded) announced by Deutsche Bank. From the target ratio CET 1 (fully loaded) equal to 14.1%, the target capital ratio θ^* thus is equal to

$$\theta^* = 14.1\% + \left(\frac{4.7 + 12.67}{357.5}\right) \approx 18.9\% \tag{A.32}$$

To sum-up, the value of the parameters are given below.

$$c = 1.25\%, \alpha = 25.3\%, \quad \gamma = 0.05\%, \quad \frac{q}{\Phi} = 1.8\%, \quad \theta^* = 18.9\% \tag{A.33}$$

Before computing the critical spread, let us check whether or not the condition given by equation (A.16) is satisfied. Since $v + qP - D = 59.6$, $qP = 1410$, it thus follows that $\frac{59.6}{1410} \approx 4.2\% > \frac{q}{\Phi} = 1.8\%$ so that the condition is satisfied.

Computation of the critical spreads. We are now in a position to compute the critical spread c_l provided by the rhs of equation (A.24) in proposition 23. By inserting the numerical values found in equation (A.33), we find a critical spread c_l equal to

$$c_l = \frac{0.05\%}{0.05\% + 25.3\% \times 18.9\% - 1.8\%(1 - 25.3\% \times 18.9\%)} \approx 1.6\% \tag{A.34}$$

Since $c = 1.25\%$, it thus follows that $c < c_l$ so that our model correctly predicts the decision of

Deutsche Bank to issue new shares only. Note interestingly that without price impact, the critical spread is equal to 1.03% and it is thus optimal to (only) sell assets. For Deutsche bank, the price impact thus seems an important factor to consider in the decision.

In proposition 23, we provide the two critical spreads c_l and c_h. While only c_l is useful for the empirical analysis, it is yet interesting to compute c_h. To do so, note that the value of the risky asset of Deutsche Bank is equal to $V_{DB} = qP$. Since only V_{DB} is observed in the annual report, we make the assumption that $P = 1$, which means that $\overline{s}(\theta^*)$ has to be lower than q. In appendix (see sub-section 7.3 in the appendix), we find that $\overline{s}(\theta^*) \approx 269$. The critical spread is equal to

$$c_h \approx 19,5\%$$

and is much higher than the critical spread for which the bank issues equity only.

Robustness check

A balance sheet with liquid and illiquid assets There are many types of robustness check that can be performed. As in [Admati et al., 2018], we shall here consider the case of asset heterogeneity, that is, some assets are liquid (i.e., marketable assets) and can be sold while others are not (i.e., typically loans) and can not be sold. With two risky assets, a liquid and an illiquid one, the balance sheet of Deutsche bank is now as follows.

Deutsche Bank's Balance sheet as of December 2016 (in billion)

Assets	Liabilities and Equity
Cash: $v = 181$ Liquid assets: $qP = 1001$ Illiquid assets : 409	Debt: $D = 1526.2$ Equity : $E = 64.81$
$A = 1591$	$E + D = 1591$

The illiquid assets, subject to credit risk, are loans accounted at fair value but it is important to note that this fair value is *not* equal to their resale value. As is well-known, due to the adverse selection problem, loans are difficult to resell in the short term even with a discount. The liquid assets, mainly subject to market risk and counterparty risk, include the financial assets through profit or loss (€743.8 billion), the financial assets available for sale (€56.22 billion) and various heterogeneous assets. In such a type of risky assets framework, to imply the weight of each risky asset, one must consider the RWA related to the risk of the asset and not the (total) RWA.

The RWA for each risk is disclosed in the annual report of each bank. The RWA for credit risk is equal to €220.3 billion, the RWA for counterparty risk is equal to €9.5 billion, the RWA for market risk for €33.8 billion and the RWA for operational risk for €92.6 billion. Of course, it remains unclear whether the operational risk should be related to the banking book (illiquid assets) or to the trading book (liquid assets). To imply the weight α of the liquid asset, we shall thus consider the two polar cases. One in which this operational risk is 100% related to the banking book and the other one in which it is 100% related to the trading book. As a result, we obtain a lower and an upper bound for α.

- Assume that operational risk is 100% related to the banking book. As a result, for the numerator, we only consider the RWA for market risk and counterparty risk. The minimum value of α thus is equal to $\underline{\alpha} = \frac{33.8+9.5}{1001} = 4.3\%$.
- Assume now that operational risk is 100% related to the trading book. As a result, we now add the RWA for operational risk. The maximum value of α thus is equal to $\overline{\alpha} = \frac{33.8+9.5+92.6}{1001} = 13.5\%$.

One can thus conclude that $\alpha \in [4.3\%, 13.5\%]$ so that it is always lower than 25.3%. From equation (A.26), we know that when α decreases, everything else equal, this increases the likelihood that the bank will issue new shares. As a result, abstracting the illiquid asset, it is still optimal to issue new shares and not to sell the liquid assets. Assuming now that the operational risk is 100% related to the trading book, the (implied) weight of the illiquid asset is equal to $\underline{\nu} = \frac{220.3}{409} \approx 54\%$, which means that it might be optimal to sell these illiquid assets rather than to issue new shares. However, if one assumes that the resale value of the loans is small enough, it won't be optimal to resell them. In such a case, the optimal liquidation strategy is equivalent to the one postulated in [Cifuentes et al., 2005]; the bank should first sell the liquid assets, and, if needed, the illiquid asset. For the case of Deutsche Bank, if it had to sell assets, liquidating a fraction of the liquid asset would be enough to reach the target. As a result, the illiquid asset plays no role, and we are back to the one risky asset model.

What happened since 2017?

We now provide a quick overview of the evolution of the situation of Deutsche Bank since 2017, the last stock issuance. We summarize the annual reports' data from 2015 to 2019 in Table A.1 and we provide the stock price evolution since the last four years in Fig. A.2. Overall, the CET1 ratio

Year	CET1 ratio	Leverage ratio	Total assets (in € bn)	Net income (loss) (in € m.)
2019	13,6%	4,2%	1298	(5265)
2018	13,6%	4,1%	1348	341
2017	14%	3,8%	1475	(735)
2016	11,8%	3,5%	1591	(1356)
2015	11,1%	3,5%	1629	(6772)

Table A.1: Deutsche Bank - CET1 ratio, Leverage ratio, Total assets and Net income

Figure A.2: Deutsche Bank stock price evolution

and the leverage ratio have always been satisfied and this observation is consistent with our results (see Proposition 20). From Table A.1, one can see an important increase of the CET1 ratio between 2016 and 2017 (explained by the issuance of new stocks) and an increase of the leverage ratio over the years (from 3,5% in 2015 to 4,2% in 2019). We can also notice an important asset shrinking over the years since the total assets of the bank decreased from 1629€bn in 2015 to 1298€bn in 2019. This seems to be a trend that can be explained by Deutsche Bank's strategy to increase over time its capital ratios. It is also interesting to point out that on May 11, 2020, Deutsche Bank launched an euro-denominated Tier 2 issuance (and announced a public tender offer for senior non-preferred-debt with a target acceptance volume of 2.0 billion euros) in order to increase its total capital ratio. From Fig. A.2, one can clearly see that the stock price is decreasing over time since 2018, which means that the performance of DB is perceived negatively by the market, something which can largely been explained by the fact that since 2015, the bank has been profitable only in 2018.

4.2 The case of UniCredit

The analysis of UniCredit is similar to the one of Deutsche Bank so that we do not repeat the way we calibrate the parameters.

Empirical analysis

On March 2, 2017, UniCredit[12] completed its capital increase for an overall amount of €13 billion and it is stated in a media release that their aim is bring their fully loaded CET 1 capital ratio above 12.9% at the end of December 2019. The subscription period of the rights issue was (in Italy) from February 6, 2017 to February 23, 2017 and the subscription price was equal to €8.09 while the market price was around €12.5 during this period. From the public (interim) report as of June 2017 p. 32, the cost of the capital increase is equal to €0.33 billion, the difference between the gross proceeds, equal to €13 billion and the net proceeds equal to €12.67 billion[13], so that $c = \frac{0.33}{13} = 2.53\%$. From page 42 of the annual report as of December 2016, the risk-weighted assets is equal to €387.15 billion, CET 1 is equal to €31.53 billion, Tier 1 is equal to €35 billion and the total capital, that includes Tier 2, is equal to €45.15 billion. The CET 1 capital ratio thus is equal to 8.15% while the total capital ratio is equal to 11.66%. Since the total value of the assets as of 2016 is equal to €859.5 billion while the cash is equal to €13.5 billion, the balance sheet is as follows.

UniCredit's Balance sheet (December 2016)

Assets	Liabilities and Equity
Cash: $v = 13.5$	Debt: $D = 814.35$
Risky asset: $qP = 846$	Equity : $E = 45.15$
$A = 859.5$	$E + D = 859.5$

The implied weight is equal to $\alpha = \frac{387.15}{846} = 45.7\%$ and note that it is almost two times the one of Deutsche Bank. Using years 2014 and 2015 yields $\gamma \approx 0.21\%$. From the website of the EBA, the total exposure is equal to €974.4 billion, so that $\frac{q}{\Phi} \approx \frac{974.4}{75900} = 1.3\%$. If one adds €13 billion to CET1, the CET 1 capital ratio is equal to 11.5% (and remains lower than the target of 12.9%, the

[12] All the information regarding the capital increase 2017 can be found on the web site of UniCredit, see https://www.unicreditgroup.eu/en/governance/capital-strengthening.html

[13] In a securities note as of February, 2017, the total amount of expenses have been estimated up to about €500 million, "including consulting expenses, out-of pocket expenses and underwriting fees calculated at the highest level". The real cost thus has been lower, equal to €330 million.

Year	CET1 ratio	Leverage ratio	Total assets (in € bn)	Net income (loss) (in € m.)
2019	13,2%	5,5%	856	3373
2018	12,1%	5,1%	832	3892
2017	13,7%	5,7%	837	5473
2016	8,2%	3,6%	860	(11790)
2015	10,6%	4,6%	860	1694

Table A.2: Unicredit - CET1 ratio, Leverage ratio, Total assets and Net income

target in 2019), then the risk-based capital ratio is equal to 15%, i.e., $\theta^* = 15\%$. By inserting the numerical values found in equation (A.33), we find a critical spread c_l equal to

$$c_l = \frac{0.21\%}{0.21\% + (45.7\% \times 15\%) - 1.28\%(1 - (45.7\% \times 15\%))} \approx 3.57\% \tag{A.35}$$

Since $c = 2.53\%$, it thus follows that $c < c_l$ so that our model correctly predicts the decision of Uni Credit to issue new shares only[14].

As we did for Deutsche Bank, we also provide the critical spread c_h and find (see sub-section 7.3 in the appendix) that $c_h \approx 14,07\%$. As for the case of Deutsche Bank c_h is much higher than c_l.

What happened since 2017?

We summarize the annual reports' data of Unicredit from 2015 to 2019 in Table A.2 and we report in Figure A.3 the stock price evolution. The CET1 ratio and the leverage ratio have always been satisfied for Unicredit and it is interesting to point out that the leverage ratio have increased over time, from 4,6% in 2015 to 5,5% in 2019 (far higher than the regulatory requirement as well). Regarding the stock price, after a significant drop between 2015 and the beginning of 2017, it has remained fairly stable.

[14]It should be noted that in December 2016, UniCredit signed binding agreement for the sale of Pioneer Investments to Amundi (for a price equal to 3.545 billion) in order to increase its fully loaded CET1 ratio of 78 basis points. If we make the assumption that the decision to increase the capital of UniCredit and the decision to sell Pioneer to Amundi have been taken simultaneously, then, our model wrongly predicts the decisions of UniCredit. Note finally that, as for the case of Deutsche Bank, there was no dilution for existing shareholders who decided to "fully subscribe the Offering to the extent to which they are entitled", see the document called the securities note as of February, 2017 (see p 89).

Figure A.3: Unicredit stock price evolution

5 The spread as a U-shaped function of the gross proceeds: an exploratory analysis

Up to now, we made the assumption that the spread is equal to $\frac{K}{I}+c$, where c is a constant, which means that the spread is a decreasing function of the gross proceeds I. However, in their paper, [Altınkılıç and Hansen, 2000] argue that, from the underwriting theories, the issuer's spread should be a U-shaped function. It should first decrease with I because of a fixed cost effect (i.e., because $\frac{K}{I}$ is decreasing with I) and should then increase due to a rising placement cost effect (i.e., finding more buyers becomes more difficult and thus riskier for the underlying set of underwriters). They find empirical results that are consistent with a family of U-shaped functions (spectrum theory). More recently, in [De Jonghe and Öztekin, 2015], the authors find empirically that, to reach a target capital ratio, smaller banks tend to sell assets while large banks tend to issue new stocks. They note that this is a good news for financial stability as smaller banks have by definition a total value of their assets (much) lower than systemic banks and are less interconnected. We shall now suggest that if one assumes that the spread function is a U-shaped function of the gross proceeds, then, our model might provide a theoretical explanation to the empirical findings of [De Jonghe and Öztekin, 2015].

From the 2017-list of 35 banks reported on the website of the EBA[15], 12 are considered as G-SIIs. By definition, a bank which is listed on the 2017-list has an overall exposure measure of more than €200 billion. When one inspects the 23 banks that are not considered as G-SIIs, some of them have a total value of their assets between €200 billion and €250 billion. Compared with many G-SIIs for which the total value of the assets typically exceeds €1500 billion, for these smaller banks, a capital

[15] http://www.eba.europa.eu/risk-analysis-and-data/global-systemically-important-institutions/2017

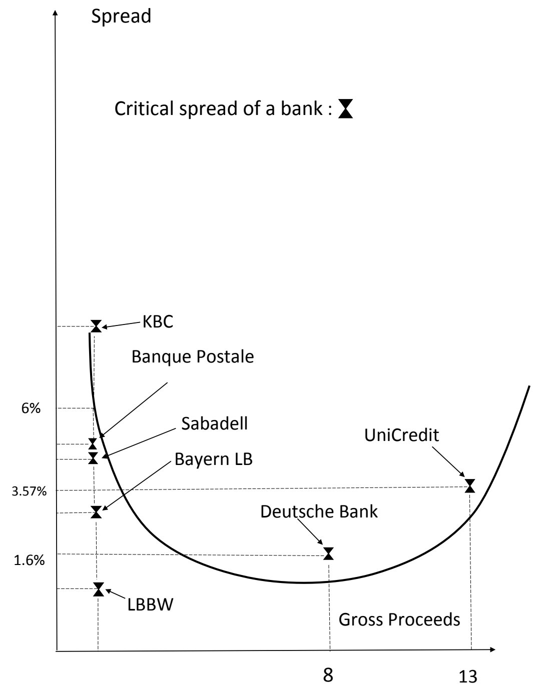

Figure A.4: Spread as a U-shaped function and critical spread

increase in million (i.e., less than one billion) might be enough to increase (significantly) their CET1 capital ratio. For instance, on June, 2016, Erste Group Bank decided to issue additional Tier 1 capital, more precisely non-cumulative bonds with an annual coupon rate equal to 8.875% and with a total nominal value of €500 billion. They also disclose in their annual report as of 2016 that their minimum target capital ratio is a CET1 capital ratio fully loaded at least equal to 12.75% for 2019 and they observe that the target has been reached since the CET1 capital ratio fully loaded is equal to 12.8% in 2016. To be able to derive the critical spread of these smaller European banks, i.e., to predict the optimal decision of those non-systemic banks, we shall make the assumptions that

$$I(\theta^*) = 500 \text{ million (euros)} \tag{A.36}$$

that is, we assume that given the target capital ratio which is not observed in general, the gross proceeds are equal €500 million. For these banks with a total value of the assets typically lower than €250 billion, the price impact is very low, i.e., close to $\frac{250}{75900} \approx 0.33\%$ as of 2016. To facilitate the exercise, we consider as negligible this price impact and we assume that $\frac{1}{\Phi} = 0$. In such a situation, as we have already seen, the optimization problem reduces to a linear programming problem for which it is never optimal to mix the channels of adjustment. We shall focus on banks that belong to countries of the Euro zone and that have total assets lower than €250 billion. Since Erste Group Bank reaches its target capital ratio in 2016, we have decided not to include this bank in our exercise. Overall, five banks are considered, Bayern LB (German), Banque Postale (French), Sabadell (Spanish), LBBW (German), and KBC (Belgium). The (unique) critical spread of each bank is given below (details related to the computation can be found in the appendix)

$$c_{uBayern} = 3.38\% \quad c_{uBPost} = 5.6\% \quad c_{uSabadell} = 5.4\% \quad c_{uLBBW} = 1.23\% \quad c_{uKBC} = 11.48\% \tag{A.37}$$

In our (small) sample of banks, the average spread is equal to 5.5%, which is a result consistent with the finding of [Boyson et al., 2016] since found a spread (for issuing common stocks) equal to 5.02%. Let

$$\text{Spread}(I) = \frac{K}{I} + c(I) \tag{A.38}$$

be the spread as a function of I, where $c(I)$ is the non constant marginal cost (of issuance) function. Assume now, as in Fig. A.4, that the spread is a U-shaped function of I. We already know that for Deutsche Bank and UniCredit, it was optimal to issue new shares only because the observed spread was higher than their critical spread, something that we reproduce on Fig. A.4. If one now assumes that the spread that must be paid by the bank is equal to 6% for gross proceeds of €0.5 billion, i.e., $c(0.5) = 6\%$, then, except for the Belgium bank BKC, it is optimal to sell assets only for the other banks. We do not claim that this approach in terms of U-shaped function constitutes the definitive answer to the empirical finding of [De Jonghe and Öztekin, 2015] but it provides a simple and credible explanation since the price impact should not be an issue for small banks.

6 Conclusion

We presented in this paper a simple model of optimal choice of channel(s) of adjustment when the aim of the bank is to reach a target capital ratio. We considered the case of two European systemic banks, Deutsche Bank and UniCredit, for which the optimal target is explicitly disclosed and we have shown that our model is able to predict the observed decision of these two banks to issue new shares only. We then considered a model in which the spread is U-shaped, and this simple approach might explain the empirical finding of [De Jonghe and Öztekin, 2015] in which large banks tend to issue new shares while small banks tend to sell assets.

Under Basel III, the minimum capital requirement will continue to increase so that large banks will have to increase their capital ratio in order to reach their target. The approach undertaken in this paper can be of interest for regulators and supervisors to forecast those banks that will most likely reach their target capital ratios by selling an important portion of their risky assets.

7 Mathematical appendix

7.1 Preliminary results

From equation (A.13) when $I = 0$, the risk-based capital ratio is equal to

$$\theta(s,0) := \frac{E(s,0)}{\text{RWA}(s)} = \frac{v - D + qP\left(1 - \frac{s}{\Phi}\right)}{\alpha(q-s)P(1 - \frac{s}{\Phi})} \tag{A.39}$$

which is a *non-linear function* of the quantity sold $s \in [0, q]$. We want to study the evolution of $\theta(s,0)$ as a function of s assuming that $\Phi \in (2q, \infty)$. Throughout this appendix, $E'(s,0)$, $\text{RWA}'(s)$ and $\theta'_s(s,0)$ will respectively denote the partial derivative of that function with respect to s. By definition of the derivative:

$$\theta'_s(s,0) = \frac{E'(s,0)\ \text{RWA}(s) - E(s,0)\ \text{RWA}'(s)}{(\text{RWA}(s))^2} \tag{A.40}$$

The sign of $\theta'_s(s,0)$, denoted $\text{Sgn}(\theta'_s(s,0))$, thus is the sign of the numerator of equation (A.40)

$$\text{Sgn}(\theta'_s(s,0)) = \text{Sgn}[E'(s,0)\ \text{RWA}(s) - E(s,0)\ \text{RWA}'(s)] \tag{A.41}$$

From equation (A.39), it is not difficult to show that

$$\text{Sgn}(\theta'_s(s,0)) = \text{Sgn}\left(-\frac{qP}{\Phi}\ \alpha(q-s)P\left(1 - \frac{s}{\Phi}\right) + \alpha P\left[1 - \left(\frac{2s-q}{\Phi}\right)\right]\left[v - D + qP\left(1 - \frac{s}{\Phi}\right)\right]\right) \tag{A.42}$$

Note importantly that $\theta'_s(s,0)$ is a *quadratic equation* in s so that $\theta'_s(s,0)$ needs not be a monotonic function of $s \in [0, q]$.

Lemma A 3 *If $\Phi > \underline{\Phi}$ (or equivalently if $E(q,0) > 0$), then, $\theta'_s(0,0) > 0$*

Proof. Letting $s = 0$ in equation (A.42) leads to $\text{Sgn}(\theta'_s(0,0)) = v - D + qP + (v-D)\frac{q}{\Phi}$. Since $v - D + qP > 0$, if $(v - D) > 0$, then, it follows immediately that $\theta'_s(0,0) > 0$. However, in general, $(v - D) < 0$. Since $E(q,0) = v - D + qP(1 - \frac{q}{\Phi}) > 0$ by assumption, it is easy to show that $v - D + qP + (v-D)\frac{q}{\Phi} > v - D + qP(1 - \frac{q}{\Phi})$ is equivalent to $\frac{q}{\Phi}(v - D + qP) > 0$ so that $\theta'_s(0,0) > 0\square$

Lemma A 4 *There exists a smallest $\overline{\Phi} > 0$ such that for all $\Phi > \overline{\Phi}$ and all $s \in (0,q)$, $\theta'_s(s,0) > 0$.*

Proof. From equation (A.42), when Φ tends to infinity, for all $s \in (0, q)$, the rhs of equation (A.42) tends to $\alpha P(v + qP - D) > 0$. Since the rhs of equation (A.42) is a continuous function of Φ, there thus exists a smallest market depth denoted $\overline{\Phi}$ such that for all $\Phi > \overline{\Phi}$ and all $s \in (0, q)$, $\theta'_s(s, 0) > 0$ $\square$

Remark. *It is actually not difficult to show that a necessary but not sufficient condition for $\theta'_s(s, 0) > 0$ is $E(s, 0) > 0$.*

We now provide an upper bound for $\overline{\Phi}$.

Lemma A 5 $\overline{\Phi} \leq q\left(1 + \frac{qP}{E(0,q)}\right)$

Proof. From equation (A.42) to the sign of $\theta'_s(s, 0)$ is equal to

$$\underbrace{\left(-\frac{qP}{\Phi}\ \alpha(q-s)P(1-\frac{s}{\Phi})\right)}_{A(s)} + \underbrace{\alpha P\left[1-\left(\frac{2s-q}{\Phi}\right)\right]\left[v - D + qP\left(1-\frac{s}{\Phi}\right)\right]}_{B(s)} \tag{A.43}$$

We want to find a sufficient condition such that equation (A.43) is positive for all s and note that $A(s) < 0$ while $B(s) > 0$ for each s since $E(0, q) = v - D + qP\left(1 - \frac{q}{\Phi}\right) > 0$. Note also that $B(s) > \alpha P\left[1 - \left(\frac{2s-q}{\Phi}\right)\right] E(0, q)$ for each s. In the same vein, $A(s) < -\frac{qP}{\Phi}\alpha qP\left(1 - \frac{s}{\Phi}\right)$ for each s. Simplifying by αP, this leads to $\left[1 - \left(\frac{2s-q}{\Phi}\right)\right] E(0, q) > \frac{qP}{\Phi} q\left(1 - \frac{s}{\Phi}\right)$ which in turn is always true if $\left[1 - \frac{q}{\Phi}\right] E(0, q) > \frac{q^2P}{\Phi}$ and is in turn equivalent to $\Phi > q\left(1 + \frac{qP}{E(0,q)}\right)$. We thus have shown that for each s, $\alpha P\left[1 - \left(\frac{2s-q}{\Phi}\right)\right] E(0, q) > \frac{qP}{\Phi}\alpha qP\left(1 - \frac{s}{\Phi}\right)$, so that for each s $B(s) > A(s)$. Since we found a sufficient condition for which $\theta'_s(s, 0)$ for each s, it may be the case that $\overline{\Phi} < q\left(1 + \frac{qP}{E(0,q)}\right)$ $\square$

Lemma A 6 *If $\Phi \in (\underline{\Phi}, \overline{\Phi})$, then, there exists two roots $\widehat{s}_1$ and $\widehat{s}_2$ (with $\widehat{s}_1 < \widehat{s}_2$) such that $\theta'_s(\widehat{s}_i, 0) = 0$ for $i = 1, 2$.*

Proof. When $\Phi \in (\underline{\Phi}, \overline{\Phi})$, by definition of $\overline{\Phi}$, the risk-based capital ratio $\theta(s, 0)$ cannot be an increasing function of s for each $s \in (0, q)$. From lemma A 3, we know that when $\Phi > \underline{\Phi}$ (or equivalently when $E(q, 0) > 0$), $\theta'_s(0, 0) > 0$. Since $E(q, 0) > 0$ and $\lim_{s\to q} \mathrm{RWA}(s) \to 0$ so that $\lim_{s\to q} \theta(s, 0) \to \infty$, there thus exists two roots $\widehat{s}_1$ and $\widehat{s}_2$, with $0 < \widehat{s}_1 < \widehat{s}_2 < q$, such that $\theta'_s(\widehat{s}_i, 0) = 0$ for $i = 1, 2$ and such that $\theta'_s(s, 0) > 0$ for $s \in (0, \widehat{s}_1)$, $\theta'_s(s, 0) < 0$ for $s \in (\widehat{s}_1, \widehat{s}_2)$ and $\theta'_s(s, 0) > 0$ for $s \in (\widehat{s}_2, q)$ $\square$ Figure A.5 illustrates the lemma.

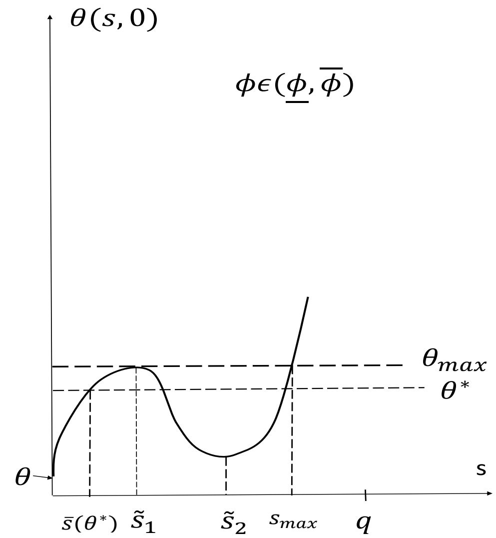

Figure A.5: The risk-based capital is not an increasing function of s

7.2 Proofs

Proof of proposition 20. The leverage ratio is defined as $L = \frac{E}{qP}$. Since $\theta = \frac{E}{\alpha qP}$, it thus follows that $\theta = \frac{L}{\alpha}$. Since $\theta = \theta_{min}$ by assumption, it thus follows that $\alpha = \frac{L}{\theta_{min}}$. Under the optimal solution, we know that $\theta(s^*, I^*) = \theta^*$. Noting that for each $(s, I) \in [0, q) \times \mathbb{R}^+$, $\theta(s, I) = \frac{L(s,I)}{\alpha}$, it thus follows that $\theta(s^*, I^*) = \frac{L(s^*,I^*)}{\alpha}$. Using now the fact that $\alpha = \frac{L}{\theta_{min}}$, $\theta(s^*, I^*) = \frac{L(s^*,I^*)}{\alpha}$ is equivalent to $L(s^*, I^*) = L\ \frac{\theta^*}{\theta_{min}}$ since $\theta(s^*, I^*) = \theta^*$. Recalling that $\frac{\theta^*}{\theta_{min}} > 1$, it thus follows that $L(s^*, I^*) > L \geq L_{min}$ $\square$

Proof of proposition 21. When $c < 1$ and $s = 0$, the risk-based capital ratio $\theta(0, I)$ is an increasing function of I. As a result, for each $\theta^* \in \mathbb{R}^+$, there exists a unique $\overline{I}(\theta^*)$ such that $\theta(0, \overline{I}(\theta^*)) = \theta^*$ and this concludes the first part of the proof. For the second part of the proof, note first that $\lim_{s \to q} \text{RWA}(s) \to 0$. Since $\Phi > \underline{\Phi} \iff E(q, 0) > 0$, it thus follows that $\lim_{s \to q} \theta(s, 0) := \frac{E(s,0)}{\text{RWA}(s)} \to \infty$. Since $\Phi > \overline{\Phi}$, $\theta(s, 0) = \frac{E(s,0)}{\text{RWA}(s)}$ is a continuous function of $s \in [0, q)$ and for each $\theta^* \in \mathbb{R}^+$, there exists a unique $\overline{s}(\theta^*) < q$ such that $\theta(\overline{s}(\theta^*), 0) = \theta^*$ $\square$

We have defined $\overline{s}(\theta^*)$ such that $\theta(\overline{s}(\theta^*), 0) = \theta^*$. Let us now define the function $I(s, \theta^*) \in [0, \overline{I}(\theta^*)]$ such that $\theta(s, I(s, \theta^*)) = \theta^*$ for each $s \in [0, \overline{s}(\theta^*)]$. Let $C(s, I) = sP(\gamma + \frac{s}{\Phi}) + cI = k$, where $k > 0$ be a level curve of the cost function and let $I(s, k)$ be such that $C(s, I(s, k)) = k$ for each s.

Proposition A 1 *The function $I(s,\theta^*)$ is a decreasing and strictly convex function of s while the function $I(s,k)$ is a decreasing and strictly concave function of s.*

The proof will consist in few simple results.

Claim A 1 *Assume that $I_2 > I_1$. Then, for all $s \in [0,q)$, $\theta(s,I_2) > \theta(s,I_1)$.*

Proof. Since by definition $\theta(s,I) = \frac{v-D+qP\left(1-\frac{s}{\Phi}\right)+(1-c)I}{\alpha(q-s)P(1-\frac{s}{\Phi})}$, it is elementary to show that if $I_2 > I_1$, then, for all $\Phi > \underline{\Phi}$ (but this is true for all $\Phi > 0$) and all $s \in [0,q)$, $\theta(s,I_2) > \theta(s,I_1)$. $\square$

From equation (A.13), it is easy to show that the iso-target curve $I(s,\theta^*)$ is equal to

$$I(s,\theta^*) := I(s) = \frac{\theta^*\alpha(q-s)P(1-\frac{s}{\Phi}) - \left[v - D + qP\left(1-\frac{s}{\Phi}\right)\right]}{1-c} \tag{A.44}$$

and note that from claim A 1, if $I(s,\theta^*) > 0$, then $s < \overline{s}(\theta^*)$.

Claim A 2 *For each $s \in (0,\overline{s}(\theta^*))$, $\theta_s'(s,0) > 0$ is equivalent to $I'(s,\theta^*) < 0$*

Proof. Along the iso target curve, i.e., as long as s is such that $\theta(s,I(s,\theta^*)) = \theta^*$, $d\theta(s,I) = 0$, which is equivalent to $d\theta(s,I) = \frac{\partial\theta(s,I)}{\partial s}ds + \frac{\partial\theta(s,I)}{\partial I}dI = 0$ and finally yields $\frac{dI}{ds} := I'(s) = -\frac{\frac{\partial\theta(s,I)}{\partial s}}{\frac{\partial\theta(s,I)}{\partial I}}$ where $\frac{\partial\theta(s,I)}{\partial I} > 0$. Since $\frac{\partial\theta(s,I)}{\partial I} > 0$, for each $s \in (0,q)$, $I'(s) < 0$ is equivalent to $\frac{\partial\theta(s,I)}{\partial s} > 0$ for any $I \geq 0$ $\square$

It is easy to show that $C(s,I(s,k)) = k$ is equivalent to

$$I(s,k) = \frac{k}{c} - \left(\frac{sP(\gamma + \frac{s}{\Phi})}{c}\right) \tag{A.45}$$

We are now in a position to complete the proof. From claim A 2, we already know that $I'(s) < 0$ and it is easy to see that $I''(s) = (\frac{1}{c})\frac{\alpha\theta^* 2P}{\Phi} > 0$ for each $s \in (0,\overline{s}(\theta^*))$, which shows that $I(s)$ is a decreasing and strictly convex function of s. From equation (A.45), it is easy to show that $I'(s,k) = -\frac{1}{c}\left(\gamma P + \frac{2sP}{\Phi}\right) < 0$ while $I''(s,k) = -(\frac{1}{c})\frac{2P}{\Phi} < 0$. As a result, the function $I(s,k)$ is a decreasing and strictly concave function of s for any $k > 0$ $\square$

Proof of proposition 22. Since the optimization program is a linear programming problem, the solution is either $(0,\overline{I})$ or $(\overline{s},0)$ and note that $\theta < \theta^*$ is equivalent to $-v + D + qP(\theta^*\alpha - 1) > 0$. Consider the pure equity solution. It is easy to show that $\overline{I} = \frac{qP(\alpha\theta^*-1)-(v-D)}{1-c} > 0$ so that

$C(0,\overline{I}) = \frac{c}{1-c}(qP(\alpha\theta^* - 1) - (v - D)) > 0$. Consider now the pure asset sale solution. It is easy to show that $\overline{s} = \frac{qP(\alpha\theta^*-1)-(v-D))}{\alpha P\theta^*} > 0$ so that $C(\overline{s},0) = \frac{\gamma[qP(\alpha\theta^*-1)-(v-D)]}{\alpha\theta^*} > 0$. Let c_u be the critical spread for which $C(\overline{s},0) = C(0,\overline{I})$. Since $C(\overline{s},0) = C(0,\overline{I}) \iff \frac{c_u}{1-c_u} = \frac{\gamma}{\alpha\theta^*}$, this yields the desired critical spread $c_u = \frac{\gamma}{\gamma+\alpha\theta^*}$. It is easy to show that $c < c_u$ is equivalent to $C(\overline{s},0) > C(0,\overline{I})$ so that it is optimal to issue new equity only, i.e., $(s^*, I^*) = (0, \overline{I})$. Solving $\theta(0, I^*) = \theta^*$ yields $I^* = \frac{\alpha qP\theta^*-E}{1-c}$. When $c > c_u$, it is optimal to only sell assets. Solving $\theta(s^*, 0) = \theta^*$ yields $s^* = q - \frac{E}{\alpha P\theta^*}$ □

Proof of proposition 23. Instead of solving the optimization problem using Kuhn and Tucker, we make use of the specific problem under consideration and we use the fact that we have only two variables s and I. By inserting equation (A.44) into the cost function $C(s, I)$, we thus obtain a cost function $C(s, I(s, \theta^*)) \equiv C(s)$ that only depends on s. The optimization problem thus reduces to a uni-dimensional minimization problem. Since $C''(s) > 0$ for each $s \in (0, q)$, i.e., the cost function is convex in s, it thus follows that s^* such that $C'(s^*) = 0$ is a global minimum. Computations of $C'(s^*) = 0$ is equivalent to

$$s^* = \frac{\frac{\alpha\theta^* c}{1-c}(1 + \frac{q}{\Phi}) - \left(\gamma + \frac{cq}{(1-c)\Phi}\right)}{\frac{2}{\Phi}(1 + \frac{\alpha\theta^* c}{1-c})} = \frac{(\alpha\theta^*(1 + \frac{q}{\Phi}) + \gamma - \frac{q}{\Phi})c - \gamma}{\frac{2}{\Phi}(\alpha\theta^* - 1)c + \frac{2}{\Phi}} \tag{A.46}$$

We notice that and it suffices now to solve $s^*(c_l) = 0$, i.e., to solve $\frac{\alpha\theta^* c_l}{1-c_l}(1 + \frac{q}{\Phi}) - \left(\gamma + \frac{c_l q}{(1-c_l)\Phi}\right) = 0$ to obtain the desired critical spread c_l. When $c < c_l$, since s^* is non-negative, $s^* = 0$ and it is thus optimal to only issue new equity, $I^* = \overline{I}(\theta^*) = \frac{\alpha qP\theta^*-E}{1-c}$ this concludes the first part of the proof.

Consider now the highest critical spread c_h. We first note that $s^*(c)$ is a strictly increasing function and

$$I(s) = \frac{1}{1-c}(\theta^*\alpha(q-s)P(1 - \frac{s}{\Phi}) - E + \frac{qPs}{\Phi}) \tag{A.47}$$

is a strictly decreasing function of s between 0 and q. So $I(s^*(c))$ is a strictly decreasing function of c.

To prove the existence and uniqueness of c_h, assume that $c = 1$. In such a case, $\frac{\partial\theta(s,I)}{\partial I} = 0$ while $\frac{\partial C(s,I)}{\partial I} > 0$ for each $\in [0, q)$ and each $I \geq 0$ so that $I^* = 0$ when $c = 1$. Therefore there exists a critical spread $c_h \in (0, 1)$ such that $I(s^*(c_h)) = 0$ and for all $c > c_h$ we have $s^* = \overline{s}(\theta^*)$ and $I^* = 0$. We find the expression of $\overline{s}(\theta^*)$ by solving $I(\overline{s}(\theta^*)) = 0$ (it is the smaller root of the polynomial $I(s)$):

$$I(s) = 0 \iff \alpha P \frac{\theta^*}{\Phi} s^2 + \left(\frac{qP}{\Phi} - \alpha P \theta^* (1 + \frac{q}{\Phi}) \right) s + (D - v - qP + \alpha P \theta^* q) = 0 \tag{A.48}$$

The discriminant Δ is equal to:

$$\Delta = \left(\alpha P \theta^* (1 + \frac{q}{\Phi}) - \frac{qP}{\Phi} \right)^2 - 4 \frac{(\alpha P \theta^*)^2 q}{\Phi} \left(1 - \frac{\theta}{\theta^*} \right) \tag{A.49}$$

And the left root $\overline{s}(\theta^*)$ is equal to:

$$\overline{s}(\theta^*) = \frac{\alpha P \theta^* (1 + \frac{q}{\Phi}) - \frac{qP}{\Phi} - \sqrt{\Delta}}{\frac{2\alpha P \theta^*}{\Phi}} \tag{A.50}$$

To find the expression of c_h, we solve: $s^*(c_h) = \overline{s}(\theta^*) \iff (\alpha\theta^*(1 + \frac{q}{\Phi}) + \gamma - \frac{q}{\Phi})c_h - \gamma = \overline{s}(\theta^*) \frac{2}{\Phi} (\alpha\theta^* - 1) c_h + \frac{2}{\Phi}$.

Which gives:

$$c_h = \frac{\gamma + \frac{2}{\Phi} \overline{s}(\theta^*)}{\gamma + \alpha\theta^* - \frac{q}{\Phi}(1 - \alpha\theta^*) + \frac{2}{\Phi} \overline{s}(\theta^*)(1 - \alpha\theta^*)} \tag{A.51}$$

7.3 Computation of the critical threshold c_h

The case of Deutsche Bank

First, we compute Δ:

$$\Delta = (25,3\% \times 18,9\% \times (1 + 1,8\%) - 1,8\%)^2 - 4 \times (25,3\% \times 18,9\%)^2 \times 1,8\% \times (1 - \frac{16,6\%}{18,9\%}) \tag{A.52}$$

$\approx 9,21 \times 10^{-4}$

Then, we can compute $\overline{s}(\theta^*)$:

$$\overline{s}(\theta^*) = \frac{25,3\% \times 18,9\% \times (1+1,8\%) - 1,8\% - \sqrt{9,21 \times 10^{-4}}}{2 \times 25,3\% \times 18,9\% \times \frac{1,8\%}{1410}} \approx 269 \tag{A.53}$$

And finally we can compute c_h:

$$c_h = \frac{0.05\% + 2 \times \frac{1,8\%}{1410} \times 269}{0.05\% + 25.3\% \times 18.9\% - 1.8\%(1 - 25.3\% \times 18.9\%) + 2 \times \frac{1,8\%}{1410} \times 269 \times (1 - 25,3\% \times 18,9\%)} \approx 19,5\% \tag{A.54}$$

The case of Unicredit

First, we compute Δ:

$$\Delta = (45,7\% \times 15\% \times (1+1,3\%) - 1,3\%)^2 - 4 \times (45,7\% \times 15\%)^2 \times 1,3\% \times (1 - \frac{11,66\%}{15\%}) \tag{A.55}$$

$\approx 3,131 \times 10^{-3}$

Then, we can compute $\overline{s}(\theta^*)$:

$$\overline{s}(\theta^*) = \frac{45,7\% \times 15\% \times (1+1,3\%) - 1,3\% - \sqrt{3,131 \times 10^{-3}}}{2 \times 45,7\% \times 15\% \times \frac{1,3\%}{846}} \approx 230 \tag{A.56}$$

And finally we can compute c_h:

$$c_h = \frac{0.21\% + 2 \times \frac{1,3\%}{846} \times 230}{0.21\% + 45,7\% \times 15\% - 1.8\%(1 - 45,7\% \times 15\%) + 2 \times \frac{1,3\%}{846} \times 230 \times (1 - 45,7\% \times 15\%)} \approx 14,07\% \tag{A.57}$$

7.4 Critical spread for European banks with total assets between €200 billion and €300 billion

Bayern LB (Germany). As of December 2016, the RWA is equal to €65.20 billion, CET1 is equal to €9.54 billion while the total capital is equal to €11.05 billion. The total capital ratio thus is equal to 17%. The total value of the assets is equal to €212.15 billion and the value of cash is equal to €2.1 billion so that $qP = 210.05$. From these data, $\alpha = 31\%$. Since CET1 is equal to 14.7%, when the bank raises €0.5 billion, the CET1 capital ratio moves to 15.45%, i.e., it increases by 75 bps. As a result, the total capital ratio, which becomes the target, is now is equal to $\theta^* = 17.7\%$. By estimating γ using the years 2015, 2016, we find respectively a value equal to 0.20% and 0.184% so that the average is equal to $\gamma = 0.192\%$. It thus follows from equation (A.17) that

$$c_{uBayern} = 3.38\%. \tag{A.58}$$

Banque Postale (France). As of December 2016, the RWA is equal to €59.53 billion, CET1 is equal to €8.17 billion while the total capital is equal to €11.55 billion. The total capital ratio thus is equal to 19.4%. The total value of the assets is equal to €229.6 billion and the value of cash is equal to €2.7 billion so that $qP = 226.9$. From these data, $\alpha = 26.2\%$. Since CET1 is equal to 13.7%, when the bank raises €0.5 billion, the CET1 capital ratio moves to 14.5%, i.e., it increases by almost 80 bps. As a result, the total capital ratio, which becomes the target, is now is equal to $\theta^* = 20.24\%$. By estimating γ using the years 2015, 2016, we find respectively a value equal to 0.325% and 0.3% so that the average is equal to $\gamma = 0.3125\%$. It thus follows from equation (A.17) that

$$c_{uBP} = 5.6\% \tag{A.59}$$

SABADELL (Spain). As of December[16] 2016, the RWA is equal to €86.07 billion, CET1 is equal to €10.33 billion while the total capital is equal to €11.851 billion. The total capital ratio thus is equal to %. The total value of the assets is equal to €212.5 billion and the value of cash is equal to €11.68 billion so that $qP = 200.82$. From these data, $\alpha = 42.8\%$. Since CET1 is equal to 12 %, when the bank raises €0.5 billion, the CET1 capital ratio moves to 12.6%, i.e., it increases by 60 bps. As a result, the total capital ratio, which becomes the target, is now is equal to $\theta^* = 14.35\%$. By estimating γ using the years 2015, 2016, we find respectively a value equal to 0.351% and 0.353%

[16] They placed 500 billion of Tier 2 capital in 2016.

so that the average is equal to $\gamma = 0.352\%$. It thus follows from equation (A.17) that

$$c_{uSabadell} = 5.4\%. \tag{A.60}$$

LBBW (Germany). As of December 2016, the RWA is equal to €77.4 billion, CET1 (fully loaded) is equal to €11.76 billion while the total capital is equal to €16.64 billion. The total capital ratio thus is equal to 21.5%. The total value of the assets is equal to €243.6 billion and the value of cash is equal to €13.53 billion so that $qP = 230.07$. From these data, $\alpha = 33.6\%$. Since CET1 is equal to 15.2 %, when the bank raises €0.5 billion, the CET1 capital ratio moves to 15.83 %, i.e., it increases by 63 bps. As a result, the total capital ratio, which becomes the target, is now is equal to $\theta^* = 22.15\%$. By estimating γ using the years 2015, 2016, we find respectively a value equal to 0.18% and 0.0043% so that the average[17] is equal to $\gamma = 0.093\%$. It thus follows from equation (A.17) that

$$c_{uLBBW} = 1.23\% \tag{A.61}$$

KBC (Belgium). As of December 2016, the RWA is equal to €78.48 billion, CET1 is equal to €12.65 billion while the total capital is equal to 16.24 €billion. The total capital ratio thus is equal to 20.7%. The total value of the assets is equal to €239.33 billion and the value of cash is equal to €20.14 billion so that $qP = 219.2$. From these data, $\alpha = 35.8\%$. Since CET1 is equal to 14.3%, when the bank raises €0.5 billion, the CET1 capital ratio moves to 14.95%, i.e., it increases by 43 bps. As a result, the total capital ratio, which becomes the target, is now is equal to $\theta^* = 21.33\%$. By estimating γ using the years 2015, 2016, we find respectively a value equal to 1.06% and 0.924% so that the average is equal to $\gamma = 0.99\%$. It thus follows from equation (A.17) that

$$c_{uKBC} = 11.48\% \tag{A.62}$$

[17] The net consolidated profit is equal to 0.142 billion in 2016 while it was equal to 0.531 billion in 2015. This explains why the value of γ is very low for the year 2016.

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