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MODULI OF CONNECTIONS

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Contents

Introduction

The main objective of the thesis is the study of the relations between various moduli spaces arising from connections on vector bundles. Let X be a complete scheme of finite type over k or a compact complex space (then $k = \mathbb{C}$), D an effective Cartier divisor on X. A connection on X with divisor of poles D is a pair $(\mathcal{E}, \nabla)$ consisting of a vector bundle $\mathcal{E}$ of rank r on X and a meromorphic connection ∇ on $\mathcal{E}$, such that for any trivialization $\boldsymbol{e} = (e_1, \ldots, e_r)$ of $\mathcal{E}$ over some open set $U \subset X$, the connection matrix $A_{\boldsymbol{e}}$ of ∇ with respect to $\boldsymbol{e}$ has its entries in $\Gamma(U, \Omega^1_X(D))$.

In my PhD thesis, the moduli spaces for the following classes of connections with fixed divisor of poles D are studied: all such connections, as well as for its subclasses of integrable, integrable logarithmic and integrable logarithmic connections with a parabolic structure over D. I am mainly interested in a description of their local structure and their relation to the moduli space of the underlying vector bundles $\mathcal{E}$.

Another kind of moduli space related to connections are moduli spaces of representations of the fundamental group $\pi_1(X \setminus D, x_0)$. There is a natural map from the moduli space of integrable connections to the moduli space of representations, induced by taking the monodromy of a connection. This map is a form of the Riemann-Hilbert correspondence and will be denoted by RH.

The classical Riemann-Hilbert problem is a question on the recovery of a Fuchsian system of ordinary differential equations on a Riemann sphere from its monodromy. Andrey Bolibruch proved that any irreducible representation

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \{a_1, \ldots, a_n\}) \longrightarrow GL(r, \mathbb{C})$$

can be realized as the monodromy representation of a Fuchsian system with logarithmic poles in $\{a_1, \ldots, a_n\}$.

There are two natural ways to generalize the classical problem: towards Riemann surfaces of higher genera, and towards varieties of higher dimensions. It is also natural to replace "Fuchsian system" by "logarithmic connection". One can interprete the Fuchsian system as a logarithmic connection on the trivial bundle, and a logarithmic connnection on a nontrivial vector bundle can be seen as a Fuchsian system with apparent singularities.

The generalized Riemann-Hilbert correspondence relates the integrable logarithmic (or Fuchsian) connections over an algebraic variety X to the representations of the fundamental group $\pi_1(X \setminus D)$, where D denotes the divisor of poles of a connection. Deligne [De] proved its bijectivity, provided D is a fixed normal crossing divisor and the eigenvalues λ of the residues of the logarithmic connections are only allowed to vary in the band $0 \leq \Re\lambda < 1$. Nevertheless, Deligne's solution is not effective in the sense that it does not imply any formulas to compute the Riemann-Hilbert correspondence. Therefore, it is important to have on hand a stock of examples that can be solved explicitly. Chapter 1 of the thesis provides such examples: these are rank-2 logarithmic connections on an elliptic curve obtained as direct images of regular connections on a genus-2 double cover of the elliptic curve. These examples are used in the end of Chapter 3 as a testing ground for the method of determining the local structure of the moduli space of connections developed in Chapter 2 and the first part of Chapter 3. This method consists in computing the versal deformation of the connection ∇ via the Kuranishi map and then using the Luna slice theorem, which implies that the germ of the moduli space of connections at ∇ is the quotient of the base of the versal deformation by $\mathrm{Aut}(\nabla)$.

The idea of constructing connections on curves by pushing them down from finite covers has been already used in literature. The authors of [Kor-2], [EG] constructed logarithmic connections of rank n over $\mathbb{P}^1$ with quasi-permutation monodromy in terms of theta functions on a ramified cover of $\mathbb{P}^1$ of degree n. Korotkin in [Kor-1] considers a class of generalized connections, called connections with constant twists, and constructs such twisted connections of rank 2 with logarithmic singularities on an elliptic curve E via theta functions on a double cover C of E.

In Chapter 1, we obtain genuine (non-twisted) rank-2 connections on E from its double cover C by a different method, similar to the method applied in [LvdPU] to the double covers of $\mathbb{P}^1$. We consider a genus-2 cover $f : C \to E$ of degree 2 with two branch points p_+, p_- and a regular connection $\nabla_{\mathcal{L}}$ on a line bundle $\mathcal{L}$ over C. Then the sheaf-theoretic direct image $\mathcal{E} = f_*(\mathcal{L})$ is a rank-2 vector bundle carrying the connection $\nabla_{\mathcal{E}} := f_*(\nabla_{\mathcal{L}})$ with logarithmic poles at p_+ and p_-. We explicitly parameterize all such connections and their monodromy representations $\rho : \pi_1(E \setminus \{p_-, p_+\}) \to GL(2, \mathbb{C})$. We also investigate the abstract group-theoretic structure of the obtained monodromy groups as well as their Zariski closures in $GL(2, \mathbb{C})$, which are the differential Galois groups of the connections $\nabla_{\mathcal{E}}$.

Establishing a bridge between the analytic and algebro-geometric counterparts of the problem is one of the main objectives of Chapter 1. We show that the underlying vector bundle $\mathcal{E}$ of $\nabla_{\mathcal{E}}$ is stable of degree -1 for generic values of parameters and identify the special cases where it is unstable and is the direct sum of two line bundles.

We also illustrate the following Bolibruch–Esnault–Viehweg Theorem [EV-2]: any irreducible logarithmic connection over a curve can be converted by a sequence of Gabber's transforms into a logarithmic connection with the same singular points and same monodromy on a semistable vector bundle of degree 0. Bolibruch has established this result in the genus-0 case, in which "semistable of degree 0" means just "trivial" [AB], so that this result of Bolibruch gives a positive answer to the classical Riemann-Hilbert problem for *irreducible* representations on $\mathbb{CP}^1$. It is worth mentioning here that Bolibruch provided an example showing that the Riemann-Hilbert problem has no solution for certain *reducible* representations.

We explicitly indicate a Gabber's transform of the above direct image connection $(\mathcal{E}, \nabla_{\mathcal{E}})$ which satisfies the conclusion of the Bolibruch–Esnault–Viehweg Theorem. The importance of results of this type is that they allow us to consider maps from the moduli space of connections to the moduli spaces of vector bundles, for only semistable bundles have a consistent moduli theory. Another useful feature of the elementary transforms is that they permit to change arbitrarily the degree, and this enriches our knowledge of the moduli space of connections providing maps to moduli spaces of vector bundles of different degrees, which may be quite different and even have different dimensions.

A part of Chapter 1 is devoted to algebro-geometric tools used in the sequel. One of them is the relation between ruled surfaces and rank-2 vector bundles on curves. This relation is particularly useful in finding line subbundles of rank-2 vector bundles and hence in the study of the question of their stability. This is classical, see [LN]. Another tool is the reconstruction of a vector bundle from the singularities of a given connection on it. Though it is known as a theoretical method ([EV-1], [EV-2]), it has not been used for a practical calculation of vector bundles underlying a given meromorphic connection over a Riemann surface different from the sphere. For the Riemann sphere, any vector bundle is the direct sum of the line bundles $\mathcal{O}(k_i)$, and Bolibruch developed the method of valuations (see [AB]) serving to calculate the integers k_i for the underlying vector bundles of connections. He exploited extensively this method, in particular in his construction of counter-examples to the Riemann–Hilbert problem for reducible representations.

Genus-2 double covers of elliptic curves is a classical subject, originating in the work of Legendre and Jacobi [J]. We provide several descriptions of them, based on a more recent work,[F-K], [Di]. We determine the locus of their periods, a result which we could not find elsewhere in the literature and which we need for finding the image of the Riemann-Hilbert correspondence on our direct image connections.

In Chapter 2, we construct the Kuranishi space, or in other words, the versal deformation, of connections belonging to each one of the following classes:
meromorphic connections with fixed divisor of poles D;
integrable meromorphic connections with fixed divisor of poles D;
integrable logarithmic connections with fixed divisor of poles D;
integrable logarithmic connections on curves with parabolic structure at singular points.

The interest in versal deformations is twofold. First, a versal deformation is a kind of a local moduli space which exists in a much wider range of situations than the moduli spaces in the proper sense do. Second, versal deformations are usually easier to write down than the moduli spaces, and one can use the versal deformation to determine the germ of the moduli space up to analytic, formal or étale equivalence.

Historically, versal deformations were introduced for the first time in late 50's in the work of Kodaira and Spencer ([KS-1],[KS-2]), and ([Ku-1],[Ku-2]). In the beginning, this theory was only concerned with deformations of compact complex manifolds and was viewed as a replacement for Riemman's insight of moduli of compact complex curves in higher dimensions. But since then the theory has been significantly formalized and extended to a much wider range of domains: singularities [Ar], [Schl], [AGZV], vector bundles and sheaves [Rim-1], [Rim-2], [Artam-1], [Artam-2], singular complex spaces [Gro-2], [Illu-1],[Illu-2], [Pa-1],[Pa-2], and morphisms of varieties or complex spaces [Fl], [Bi], [Ran-1], [Ran-2].

All the constructions are enclosed in the paradigm of the Kuranishi space associated to a "good" deformation theory. A "good" deformation theory for some type of object X consists in determining a triple (T^1_X, T^2_X, f), where T^1_X is the tangent space to deformations of X, T^2_X is the obstruction space, $f : \hat{T^1}_X \to \hat{T^2}_X$ a formal map without linear terms, called the Kuranishi map ($\hat{}$ denotes the formal completion at zero). Then the formal scheme $f^{-1}(0)$ is the Kuranishi space, or a formal germ of the versal deformation of X.

We provide the triples (T^1_X, T^2_X, f) for the above four classes of connections. In all the 4 cases, $T^i_X = \mathbb{H}^i(\mathcal{C}^\bullet)$, the hypercohomology of an appropriate complex of sheaves, and the initial component f_2 of f is the Yoneda square map. For instance, in the case $X = (\mathcal{E}, \nabla)$ is a meromorphic connection with fixed divisor of poles D, the complex $\mathcal{C}^\bullet$ is a two-term one and is

$$\mathcal{C}^\bullet = [\mathcal{E}nd(\mathcal{E}) \xrightarrow{\nabla} \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)].$$

A similar situation occurs in the deformation theory of Higgs bundles or Hitchin pairs [B-R], where $T^1_X = \mathbb{H}^1(\mathcal{C}^\bullet)$ with complex

$$\mathcal{C}^\bullet = [\mathcal{E}nd(\mathcal{E}) \xrightarrow{\operatorname{ad}\varphi} \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)]$$

defined by the Higgs field $\varphi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(D)$; contrary to our case, $\operatorname{ad}\varphi$ is $\mathcal{O}_X$-linear.

In Chapter 3, we recall the GIT construction of moduli spaces of sheaves and construct moduli spaces of connections belonging to the first three of the four classes, mentioned above. For parabolic connections on curves, we limit ourselves by quoting the results of Inaba-Saito-Iwasaki [I-Iw-S], [I].

In order to state the results on moduli spaces, we first introduce the relevant terminology: stable, semistable objects, representable and corepresentable functors, coarse and fine moduli spaces, Mumford's m-regularity and Grothendieck's Quot-scheme. We base essentially upon the monograph [H-L], but also use [Sim], [Ma-1] and [Ma-2].

To construct the moduli spaces of connections, we follow the approach of [Sim]. He introduces the notion of a sheaf of rings of differential operators Λ over a projective scheme X and constructs quasi-projective moduli schemes of (semi)-stable coherent Λ-modules. Plugging in different Λ's, one obtains moduli spaces of sheaves ($\Lambda = \mathcal{O}_X$), integrable regular connections ($\Lambda = \mathcal{D}_X$, the standard sheaf of differential operators), integrable logarithmic connections ($\Lambda \subset \mathcal{D}_X$ is the sheaf of subrings generated by logarithmic vector field $\mathcal{T}_X\langle D\rangle$, where X is assumed to be smooth and D is a normal crossing divisor), Higgs bundles ($\Lambda = \operatorname{gr}_\bullet \mathcal{D}_X = \oplus_{m=0}^\infty S^m \mathcal{T}_X$) and others. Note that the case of integrable logarithmic connections was treated earlier in [Ni].

We extended the approach of Simpson to the case of non-integrable connections, regular or meromorphic with fixed divisor of poles D. To this end, we had to slightly generalize Simpson's notion of a sheaf of rings of differential operators. As in Simpson's definition, our Λ is a filtered $\mathcal{O}_X$-bialgebra, satisfying a bunch of axioms (see Sect. 3.2 of Ch. 3), but contrary to Simpson, we do not assume that the graded ring $\operatorname{gr}_\bullet \Lambda$ is commutative. Thus to obtain moduli of regular connections on X, we set Λ to be the sheaf $\mathbf{D}_X$ of noncommutative differential operators (the basic vector fields $\frac{\partial}{\partial x_i}$ associated to some coordinates $(x_1, \dots, x_n)$ do not satisfy the commutativity relation $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}$). To deal with the meromorphic connections with fixed divisor of poles D, we choose for Λ the sheaf of subalgebras of $\mathfrak{D}_X$ generated by the subsheaf $\mathcal{T}_X(-D)$ of $\mathcal{T}_X$, consisting of vector field vanishing on D.

This change does not require new techniques. We make a routine verification that all the ingredients of Simpson's proofs are easily adapted to such rings Λ. The crucial points are the bound of the slope of an $\mathcal{O}_X$-destabilizing subsheaf of a coherent semistable Λ-module (Lemma 3.2.2, Sect. 3.2 of Chap 3) and the resulting boundness of coherent semistable Λ-modules with fixed Hilbert polynomial.

Next we state the Luna slice theorem and show that, in our definition of the moduli space of Λ-modules, it provides a versal deformation of a Λ-module. This fact was known and used for moduli spaces of sheaves [O'G], [Dr], [M-T], [L-S]. In the end of Ch. 3, we apply the Luna slice theorem to compute, in some examples, the germ of a moduli space of connections. It is determined as the GIT quotient $f^{-1}(0)//H$, where $f : \hat{T^1}_X \to \hat{T^2}_X$ is the Kuranishi map discussed above, and H is the automorphism group of the connection ∇ represented by the origin $O \in \hat{T^1}_X$. We put in evidence the situations when the map from the moduli space of connections to that of underlying vector bundles is either undefined, or is not a locally trivial fibration. We illustrate by an example the point of view that the moduli space of semistable connections is a partial compactification of the locally trivial affine bundle of connections over the stable locus $M^s_X(r,d)$ of the moduli space of vector bundles "in the limit $\nabla \to \infty$."

CHAPTER 1

A FAMILY OF CONNECTIONS ON AN ELLIPTIC CURVE

INTRODUCTION

In this chapter, we give an explicit parameterization of all the logarithmic rank-2 connections on an elliptic curve that are obtained as the direct image of regular rank-1 connections on genus-2 double covers of the elliptic curve. We determine the periods of such double covers, the monodromy and the differential Galois groups of the connections, study the question on the (semi)stability of the underlying rank-2 vector bundles of the connections and some of their elementary transforms.

In Section 1.1, we describe the genus-2 covers of elliptic curves of degree 2 and determine their periods. In Section 1.2, we investigate rank-1 connections on C and discuss the dependence of the Riemann-Hilbert correspondence for these connections on the parameters of the problem: the period of C and the underlying line bundle $\mathcal{L}$. In Section 1.3, we compute, separately for the cases $\mathcal{L} = \mathcal{O}_C$ and $\mathcal{L} \neq \mathcal{O}_C$, the matrix of the direct image connection $\nabla_{\mathcal{E}}$ on $\mathcal{E} = f_*\mathcal{L}$. For $\mathcal{L} = \mathcal{O}_C$, we also provide two different forms for a scalar ODE of order 2 equivalent to the 2×2 matrix equation $\nabla_{\mathcal{E}}\varphi = 0$. In Section 1.4, we determine the fundamental matrices and the monodromy of connections $\nabla_{\mathcal{E}}$ and discuss their isomonodromy deformations. Section 1.5 introduces the elementary transforms of rank-2 vector bundles, relates them to birational maps between ruled surfaces and states a criterion for (semi)-stability of a rank-2 vector bundle. In Section 1.6, we apply the material of Section 1.5 to describe $\mathcal{E}$ as a result of a series of elementary transforms starting from $\mathcal{E}_0 = f_*\mathcal{O}_C$ and prove its stability or instability depending on the value of parameters. We also describe Gabber's elementary transform which illustrates the Bolibruch–Esnault–Viehweg Theorem and comment briefly on the twisted connections of [Kor-1]. In Section 1.7, we give a description of the structure of the monodromy and differential Galois groups for $\nabla_{\mathcal{E}}$. The results of Chapter 1 have been published in [Machu].

TERMINOLOGY. If not specified otherwise, a curve will mean a nonsingular complex projective algebraic curve, which we will not distinguish from the associated analytic object, a compact Riemann surface.

1.1 GENUS-2 COVERS OF AN ELLIPTIC CURVE

In this section, we will describe the degree-2 covers of elliptic curves which are curves of genus 2.

Definition 1.1.1. Let $\pi : C \to E$ be a degree-2 map of curves. If E is elliptic, then we say that C is bielliptic and that E is a degree-2 elliptic subcover of C.

Legendre and Jacobi [J] observed that any genus-2 bielliptic curve has an equation of the form

$$y^2 = c_0x^6 + c_1x^4 + c_2x^2 + c_3 \qquad (c_i \in \mathbb{C}) \tag{1.1}$$

in appropriate affine coordinates (x, y). It immediately follows that any bielliptic curve C has two elliptic subcovers $\pi_i : C \to E_i$,

$$\begin{array}{ll} E_1: & y_1^2 = c_0x_1^3 + c_1x_1^2 + c_2x_1 + c_3, \quad \pi_1 : (x, y) \mapsto (x_1 = x^2, y_1 = y), \text{ and} \\ E_2: & y_2^2 = c_3x_2^3 + c_2x_2^2 + c_1x_2 + c_0, \quad \pi_2 : (x, y) \mapsto (x_2 = 1/x^2, y_2 = y/x^3). \end{array} \tag{1.2}$$

This description of bielliptic curves, though very simple, depends on an excessive number of parameters. To eliminate unnecessary parameters, we will represent E_i in the form

$$E_i : y_i^2 = x_i(x_i - 1)(x_i - t_i), \;\; (t_i \in \mathbb{C} \setminus \{0,\ 1\},\ t_1 \neq t_2). \tag{1.3}$$

Note that any pair of elliptic curves (E_1, E_2) admits such a representation even if $E_1 \simeq E_2$.

We will describe the reconstruction of C starting from (E_1, E_2) following [Di]. This procedure will allow us to determine the periods of bielliptic curves C in terms of the periods of their elliptic subcovers E_1, E_2.

In somewhat more abstract terms, let $f : C \longrightarrow E$ be a double cover, where E is an elliptic curve and C is a curve of genus 2. Then we have the inclusion morphism $f^* : JE = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z} \hookrightarrow JC = \mathbb{C}^2/\mathbb{Z}^2 \oplus \mathbb{Z}^2$. We will identify E with JE and JE with $f^*(JE)$. Further, C can be embedded into JC by the Abel-Jacobi map $\alpha : C \longrightarrow JC$, and we can construct another elliptic curve E' together with a double cover $f' : C \longrightarrow E'$, defined by

$$C \xrightarrow{\alpha} JC \xrightarrow{\beta} JC/E = E'.$$

We obtain the following commutative diagram

$$\begin{tikzcd} C \arrow[r, hook, "\alpha"] \arrow[d, "f \times f'"'] & JC \\ E \times E' \arrow[ur, "f^* + f'^*"'] & \end{tikzcd}$$

in which $f^* + f'^*$ is an isogeny of degree 2 and $f \times f'(C)$ is the graph of a $(2, 2)$ correspondence between E, E'. Conversely, given two elliptic curves E, E', we can construct a genus 2 curve C as the graph of a $(2, 2)$ correspondence between E, E'.

We come to the analytic construction. Let $\varphi_i : E_i \to \mathbb{P}^1$ be the double cover map $(x_i, y_i) \mapsto x_i$ ($i = 1, 2$). Recall that the fibered product $E_1 \times_{\mathbb{P}^1} E_2$ is the set of pairs $(P_1, P_2) \in E_1 \times E_2$ such that $\varphi_1(P_1) = \varphi_2(P_2)$. It can be given by two equations with respect to three affine coordinates (x, y_1, y_2):

$$\bar{C} := E_1 \times_{\mathbb{P}^1} E_2 : \begin{cases} y_1^2 = x(x-1)(x-t_1) \\ y_2^2 = x(x-1)(x-t_2) \end{cases} \tag{1.4}$$

It is easily verified that $\bar{C}$ has nodes over the common branch points $0, 1, \infty$ of φ_i and is nonsingular elsewhere. For example, locally at $x = 0$, we can choose y_i as a local parameter on E_i, so that x has a zero of order two on E_i; equivalently, we can write $x = f_i(y_i)y_i^2$ where f_i is holomorphic and $f_i(0) \neq 0$. Then eliminating x, we obtain that $\bar{C}$ is given locally by a single equation $f_1(y_1)y_1^2 = f_2(y_2)y_2^2$. This is the union of two smooth transversal branches $\sqrt{f_1(y_1)}y_1 = \pm\sqrt{f_2(y_2)}y_2$.

Associated to $\bar{C}$ is its normalization (or desingularization) C obtained by divorcing the two branches at each singular point. Thus C has two points over $x = 0$, whilst the only point of $\bar{C}$ over $x = 0$ is the node, which we will denote by the same symbol 0. We will also denote by 0_+, 0_- the two points of C over 0. Each function y_1, y_2 defines a local parameter at $0_\pm$. In a similar way, we introduce the points $1, \infty \in \bar{C}$ and $1_\pm$, $\infty_\pm \in C$.

Proposition 1.1.2. *Given a genus-*2 *bielliptic curve* C *with its two elliptic subcovers* $\pi_i : C \to E_i$*, one can choose affine coordinates for* E_i *in such a way that* E_i *are given by the equations (*1.3*),* C *is the normalization of the nodal curve* $\bar{C} := E_1 \times_{\mathbb{P}^1} E_2$*, and* $\pi_i = \mathrm{pr}_i \circ \nu$*, where* $\nu : C \to \bar{C}$ *denotes the normalization map and* pr_i *the projection onto the* i*-th factor.*

Proof. See [Di]. □

It is curious to know, how the descriptions given by (1.1) and Proposition 1.1.2 are related to each other. The answer is given by the following proposition.

Proposition 1.1.3. *Under the assumptions and in the notation of Proposition* 1.1.2*, apply the following changes of coordinates in the equations of the curves* E_i*:*

$$(x_i, y_i) \to (\tilde{x}_i, \tilde{y}_i), \quad \tilde{x}_i = \frac{x_i - t_j}{x_i - t_i}, \quad \tilde{y}_i = \frac{y_i}{(x_i - t_i)^2}\sqrt{\frac{(t_j - t_i)^3}{t_i(1-t_i)}},$$

where $j = 3 - i$*,* $i = 1, 2$*, so that* $\{i, j\} = \{1, 2\}$*. Then the equations of* E_i *acquire the form*

$$\begin{aligned} E_1 : \ \tilde{y}_1^2 &= \left(\tilde{x}_1 - \frac{t_2}{t_1}\right)\left(\tilde{x}_1 - \frac{1-t_2}{1-t_1}\right)(\tilde{x}_1 - 1), \\ E_2 : \ \tilde{y}_2^2 &= \left(1 - \frac{t_2}{t_1}\tilde{x}_2\right)\left(1 - \frac{1-t_2}{1-t_1}\tilde{x}_2\right)(1 - \tilde{x}_2). \end{aligned} \tag{1.5}$$

Further, C *can be given by the equation*

$$\eta^2 = \left(\xi^2 - \frac{t_2}{t_1}\right)\left(\xi^2 - \frac{1-t_2}{1-t_1}\right)(\xi^2 - 1), \tag{1.6}$$

and the maps $\pi_i : C \to E_i$ *by* $(\xi, \eta) \mapsto (\tilde{x}_i, \tilde{y}_i)$*, where*

$$(\tilde{x}_1, \tilde{y}_1) = (\xi^2, \eta), \qquad (\tilde{x}_2, \tilde{y}_2) = (1/\xi^2, \eta/\xi^3).$$

Proof. We have the following commutative diagram of double cover maps

$$\begin{tikzcd}
& C \arrow[dl, "\pi_1"'] \arrow[d, "f"'] \arrow[dr, "\pi_2"] & \\
E_1 \arrow[dr, "\varphi_1"'] & \mathbb{P}^1 \arrow[d, "\tilde{\varphi}"'] & E_2 \arrow[dl, "\varphi_2"] \\
& \mathbb{P}^1 &
\end{tikzcd} \tag{1.7}$$

in which the branch loci of $\tilde{\varphi}$, φ_i, f, π_i are respectively $\{t_1, t_2\}$, $\{0, 1, t_i, \infty\}$, $\tilde{\varphi}^{-1}(\{0, 1, \infty\})$, $\varphi_i^{-1}(t_j)$ $(j = 3 - i)$. Thus the $\mathbb{P}^1$ in the middle of the diagram can be viewed as the Riemann surface of the function $\sqrt{\frac{x-t_2}{x-t_1}}$, where x is the coordinate on the bottom $\mathbb{P}^1$. We introduce a coordinate ξ on the middle $\mathbb{P}^1$ in such a way that $\tilde{\varphi}$ is given by $\xi \mapsto x$, $\xi^2 = \frac{x-t_2}{x-t_1}$. Then C is the double cover of $\mathbb{P}^1$ branched in the 6 points $\tilde{\varphi}^{-1}(\{0, 1, \infty\}) = \{\pm 1, \pm\sqrt{\frac{1-t_2}{1-t_1}}, \pm\sqrt{\frac{t_2}{t_1}}\}$, which implies the equation (1.6) for C. Then we deduce the equations of E_i in the form (1.5) following the recipe of (1.2), and it is an easy exercise to transform them into (1.3). $\square$

The locus of bielliptic curves in the moduli space of all the genus-2 curves is 2-dimensional, hence is a hypersurface. In [SV], an explicit equation of this hypersurface is given in terms of the Igusa invariants of the genus-2 curves. We will give a description of the same locus in terms of periods. We start by recalling necessary definitions.

Let a_1, a_2, b_1, b_2 be a symplectic basis of $H_1(C, \mathbb{Z})$ for a genus-2 curve C, and ω_1, ω_2 a basis of the space $\Gamma(C, \Omega^1_C)$ of holomorphic 1-forms on C.

Definition 1.1.4. Let us introduce the 2×2-matrices $A = (\int_{a_i} \omega_j)$ and $B = (\int_{b_i} \omega_j)$. Their concatenation $\Pi = (A|B)$ is a 2×4 matrix, called the period matrix of the 1-forms ω_1, ω_2 with respect to the basis a_1, a_2, b_1, b_2 of $H_1(C, \mathbb{Z})$. The period of C is the 2×2-matrix $Z = A^{-1}B$. If $A = I$ is the identity matrix, the basis ω_1, ω_2 of $\Gamma(C, \Omega^1_C)$ and the corresponding period matrix $\Pi_0 = (I|Z)$ are called normalized.

The period lattice $\Lambda = \Lambda(C)$ is the $\mathbb{Z}$-submodule of rank 4 in $\Gamma(C, \Omega^1_C)^*$ generated by the 4 linear forms $\omega \mapsto \int_{a_i} \omega$, $\omega \mapsto \int_{b_i} \omega$. A choice of the basis ω_i identifies $\Gamma(C, \Omega^1_C)^*$ with $\mathbb{C}^2$, and Λ is then generated by the 4 columns of Π.

The period Z_C of C is determined modulo the discrete group Sp$(4, \mathbb{Z})$ acting by symplectic base changes in $H_1(C, \mathbb{Z})$.

Riemann's bilinear relations. The period matrix of any genus-2 curve C satisfies the conditions

$$Z^t = Z \quad \text{and} \quad \Im Z > 0\,.$$

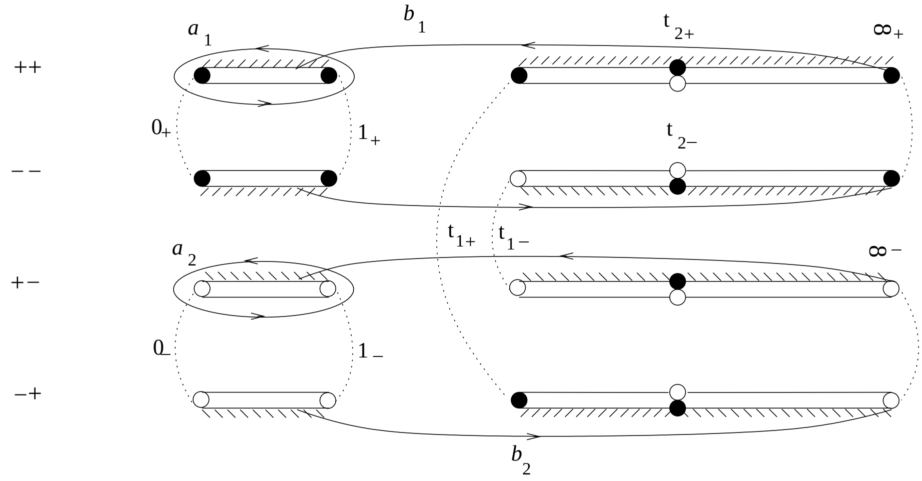

Figure 1.1: The 4 sheets of C. The segments of two edges of the cuts are glued together if they are: (1) situated one under the other, and (2) hatched by dashes of the same orientation. Thus, the upper edge of the cut on Σ_{++} between t_1, t_2 is glued to the lower edge of the cut on Σ_{-+} between t_1, t_2. Four black points over t_2 glue together to give one point $t_{2+} \in C$, and similarly four white ones give $t_{2-} \in C$. The 4 preimages of each one of the points $0, 1, t_1, \infty$ are glued in pairs, as shown by the colors black/white and by dotted lines, and give 8 points of C denoted by $0_\pm, 1_\pm, t_{1\pm}, \infty_\pm$.

To determine the periods of bielliptic curves C, it is easier to use the representation from Proposition 1.1.2 rather than the standard equation of a genus-2 curve (1.6). This is due to the fact that we can choose $\omega_1 = dx/y_1$, $\omega_2 = dx/y_2$ as a basis of the space $\Gamma(C, \Omega^1_C)$ of holomorphic 1-forms on C, and the periods of these 1-forms are easily related to the periods on E_i.

To fix the ideas, we assume for a while that t_1 , t_2 are real and $1 < t_1 < t_2$ (the general case is obtained by a deformation moving the points t_i). E_i can be represented as the result of gluing two sheets Σ_{i+}, Σ_{i-}, which are Riemann spheres with cuts along the segments $[0, 1]$ and $[t_i, \infty]$. Then C, parameterizing the pairs of points (P_1, P_2) with $P_i \in E_i$ and with the same x-coordinate, is the result of gluing 4 sheets, which are copies of the Riemann sphere with cuts along the segments $[0, 1]$ and $[t_1, \infty]$ labelled by $++, --, +-, -+$. For example, the sheet Σ_{+-} is formed by the pairs (P_1, P_2) where P_1 lies on Σ_{1+} and P_2 on Σ_{2-}. Figure 1.1 shows the gluings of the edges of the cuts with the help of hatching and fixes the choice of the cycles a_i, b_i. Black points on one vertical are identified, the same for the white ones.

Proposition 1.1.5. *Let C, E_1, E_2 be as in Proposition 1.1.2, and a_i, b_i as on Fig. 1.1. Then the period matrix of C is*

$$Z_C = \begin{pmatrix} \frac{1}{2}(\tau_1 + \tau_2) & \frac{1}{2}(\tau_1 - \tau_2) \\ \frac{1}{2}(\tau_1 - \tau_2) & \frac{1}{2}(\tau_1 + \tau_2) \end{pmatrix},$$

where τ_i is the period of E_i with respect to the basis $\gamma_i = \pi_{i*}(a_1), \delta_i = \pi_{i*}(b_1)$ of $H_1(E_i, \mathbb{Z})$.

Proof. Let k_i, l_i be the periods of the differential dx/y_i on E_i along the cycles γ_i, δ_i respectively. Take $\omega_i = \pi_i^*(dx/y_i)$ as a basis of $\Gamma(C, \Omega_C)$. We have

$$\int_{a_1} \pi_j^*(dx/y_j) = \int_{\pi_{j*}(a_1)} dx/y_j = k_j.$$

But when calculating the integral over a_2, we have to take into account the fact that a positively oriented loop around a cut on Σ_{+-} projects to a positively oriented loop on Σ_{2-}, and the latter defines the cycle $-\gamma_2$ on E_2. Thus $\pi_{2*}(a_2) = -\gamma_2$, and the correspoding period acquires an extra sign:

$$\int_{a_2} \pi_j^*(dx/y_j) = \int_{\pi_{j*}(a_2)} dx/y_j = (-1)^{j+1} k_j.$$

The integrals over b_j are transformed in a similar way. We obtain the period matrix of C in the form

$$\Pi = \left(\begin{array}{cc|cc} k_1 & k_1 & l_1 & l_1 \\ k_2 & -k_2 & l_2 & -l_2 \end{array} \right).$$

Multiplying by the inverse of the left 2×2-block and using the relations $\tau_i = l_i/k_i$, we obtain the result. $\square$

Corollary 1.1.6. *The locus $\mathcal{H}$ of periods of genus-2 curves C with a degree-2 elliptic subcover is the set of matrices*

$$Z_C = \begin{pmatrix} \frac{1}{2}(\tau+\tau') & \frac{1}{2}(\tau-\tau') \\ \frac{1}{2}(\tau-\tau') & \frac{1}{2}(\tau+\tau') \end{pmatrix} \qquad (\Im\tau > 0, \Im\tau' > 0).$$

Equivalently, $\mathcal{H}$ is the set of all the matrices of the form $Z = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ ($a, b \in \mathbb{C}$) such that $\Im Z > 0$.

1.2 Rank-1 connections on C and their monodromy

We start by recalling the definition of a connection. Let V be a curve or a complement of a finite set in a curve C. Let $\mathcal{E}$ be a vector bundle of rank $r \geq 1$ on V. We denote by $\mathcal{O}_V$, Ω_V^1 the sheaves of holomorphic functions and 1-forms on V respectively. By abuse of notation, we will denote in the same way vector bundles and the sheaves of their sections. A *connection* on $\mathcal{E}$ is a $\mathbb{C}$-linear map of sheaves $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_V^1$ which satisfies the Leibnitz rule: for any open $U \subset V$, $f \in \Gamma(U, \mathcal{O})$ and $s \in \Gamma(U, \mathcal{E})$, $\nabla(fs) = f\nabla(s) + s\,df$. If $\mathcal{E}$ is trivialized by a basis of sections $\boldsymbol{e} = (e_1, \ldots, e_r)$ over U, then we can write $\nabla(e_j) = \sum_i a_{ij} e_i$, and the matrix $A(\boldsymbol{e}) = (a_{ij})$ of holomorphic 1-forms is called the connection matrix of ∇ with respect to the trivialization $\boldsymbol{e}$. If there is no ambiguity with the choice of a trivialization, one can write, by abuse of notation, $\nabla = d + A$.

Given r meromorphic sections $\boldsymbol{s} = (s_1, \ldots, s_r)$ which span $\mathcal{E}$ over an open subset, the matrix $A(\boldsymbol{s})$ defined as above is a matrix of meromorphic 1-forms on V. Its poles in V are called *apparent* singularities of the connection with respect to the meromorphic trivialization $\boldsymbol{s}$. The apparent singularities arise at the points $P \in V$ in which either some of the s_i are non-regular, or all the s_i are regular but $s_i(P)$ fail to be linearly independent. They are not singularities of the connection, but those of the chosen connection matrix.

In the case when the underlying vector bundle is defined not only over V, but over the whole compact Riemann surface C, we can speak about singularities at the points of $C \setminus V$ of the connection itself. To this end, choose local trivializations $\boldsymbol{e}_P$ of $\mathcal{E}$ at the points $P \in C \setminus V$, and define the local connection matrices $A(\boldsymbol{e}_P)$ as above, $\nabla(\boldsymbol{e}_P) = \boldsymbol{e}_P A(\boldsymbol{e}_P)$. The connection ∇, regular on V, is said to be meromorphic on C if $A(\boldsymbol{e}_P)$ has at worst a pole at P for all $P \in C \setminus V$. If, moreover, $A(\boldsymbol{e}_P)$ can be represented in the form $A(\boldsymbol{e}_P) = B(\tau_P)\frac{d\tau_P}{\tau_P}$, where τ_P is a local parameter at P and $B(\tau_P)$ is a matrix of holomorphic functions in τ_P, then P is said to be a logarithmic singularity of ∇. A connection is called logarithmic, or Fuchsian, if it has only logarithmic singularities.

To define the *monodromy* of a connection ∇, we have to fix a reference point $P_0 \in V$ and a basis $\boldsymbol{s} = (s_1, \ldots, s_r)$ of solutions of $\nabla s = 0$, $s \in \Gamma(U, \mathcal{E})$ over a small disc U centered at P_0. The analytic continuation of the s_i along any loop γ based at P_0 provides a new basis $\boldsymbol{s}^\gamma = (s_1^\gamma, \ldots, s_r^\gamma)$, and the monodromy matrix M_γ is defined by $\boldsymbol{s}^\gamma = \boldsymbol{s}M_\gamma$. The monodromy matrix depends only on the homotopy class of a loop, and the monodromy ρ_∇ of ∇ is the representation of the fundamental group of V defined by

$$\rho = \rho_\nabla : \pi_1(V, P_0) \longrightarrow GL_r(\mathbb{C}), \qquad \gamma \mapsto M_\gamma.$$

Let now $C = V$ be a genus-2 bielliptic curve with an elliptic subcover $\varphi : C \to E$. Our objective is the study of rank-2 connections over E which are direct images of rank-1 connections over C. We first study the rank-1 connections over C and their monodromy representations.

Let $\mathcal{L}$ be a line bundle on C and e a meromorphic section of $\mathcal{L}$ which is not identically zero. Then a connection $\nabla_{\mathcal{L}}$ on $\mathcal{L}$ can be written as $\mathrm{d} + \omega$, where ω is a meromorphic 1-form on C defined by $\nabla_{\mathcal{L}}(e) = \omega e$. The apparent singularities are simple poles with integer residues at the points where e fails to be a basis of $\mathcal{L}$. We will start by considering the case when $\mathcal{L}$ is the trivial line bundle $\mathcal{O} = \mathcal{O}_C$. Then the natural trivialization of $\mathcal{L}$ is $e = 1$, and ω is a regular 1-form. The vector space $\Gamma(C, \Omega_C^1)$ of regular 1-forms on C is 2-dimensional; let ω_1, ω_2 be its basis. We can write $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$ with λ_1, λ_2 in $\mathbb{C}$.

The horizontal sections of $\mathcal{O}$ are the solutions of the equation $\nabla_{\mathcal{O}}\varphi = 0$. To write down these solutions, we can represent C as in Proposition 1.1.2 and introduce the multi-valued functions $z_1 = \int \omega_1$ and $z_2 = \int \omega_2$, normalized by $z_1(\infty_+) = z_2(\infty_+) = 0$. We denote by the same symbols z_1, z_2 the flat coordinates on the Jacobian $JC = \mathbb{C}^2/\Lambda$ associated to the basis (ω_1, ω_2) of $\Gamma(C, \Omega_C^1)$, and C can be considered as embedded in its Jacobian via the Abel-Jacobi map $AJ : C \to JC$, $P \longmapsto ((z_1(P), z_2(P))$ modulo Λ.

To determine the monodromy, we will choose $P_0 = \infty_+$ and fix some generators α_i, β_i of $\pi_1(C, \infty_+)$ in such a way that the natural epimorphism $\pi_1(C, \infty_+) \longrightarrow H_1(C, \mathbb{Z})$ (where $H_1(C, \mathbb{Z}) = \pi_1(C, \infty_+)/[\pi_1(C, \infty_+), \pi_1(C, \infty_+)]$) is given by $\alpha_i \mapsto a_i$, $\beta_i \mapsto b_i$.

The following lemma is obvious:

Lemma 1.2.1. *The general solution of $\nabla_{\mathcal{O}}\varphi = 0$ is given by $\varphi = ce^{-\lambda_1 z_1 - \lambda_2 z_2}$, where c is a complex constant. The monodromy matrices of $\nabla_{\mathcal{O}}$ are $M_{\alpha_i} = \exp(-\oint_{a_i} \omega)$, $M_{\beta_i} = \exp(-\oint_{b_i} \omega)$ ($i = 1, 2$).*

Now we turn to the problem of Riemann-Hilbert type: determine the locus of the representations of G which are monodromies of connections $\nabla_{\mathcal{L}}$. Since any rank-1 representation ρ of G is determined by 4 complex numbers $\rho(\alpha_i), \rho(\beta_i)$, we can take $(\mathbb{C}^*)^4$ for the moduli space of representations of G in which lives the image of the Riemann-Hilbert correspondence.

Before solving this problem on C, we will do a similar thing on an elliptic curve E. The answer will be used as an auxiliary result for the problem on C.

Any rank-1 representation $\rho : \pi_1(E) \to \mathbb{C}^*$ is determined by the images $\rho(a), \rho(b)$ of the generators a, b of the fundamental group of E, so that the space of representations of $\pi_1(E)$ can be identified with $\mathbb{C}^* \times \mathbb{C}^*$. We will consider several spaces of rank-1 connections. Let $\mathcal{C}(E, \mathcal{L})$ be the space of all the connections $\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega^1_E$ on a line bundle $\mathcal{L}$ on E. It is non empty if and only if $\deg \mathcal{L} = 0$, and then $\mathcal{C}(E, \mathcal{L}) \simeq \Gamma(E, \Omega^1_E) \simeq \mathbb{C}$. Further, $\mathcal{C}(E)$ will denote the moduli space of pairs $(\mathcal{L}, \nabla)$, that is, $\mathcal{C}(E) = \cup_{[\mathcal{L}] \in J(E)} \mathcal{C}(E, \mathcal{L})$. We will also define the moduli space $\mathcal{C}$ of triples $(E_\tau, \mathcal{L}, \nabla)$, $\mathcal{C} = \cup_{\Im\tau > 0} \mathcal{C}(E_\tau)$, and $\mathcal{C}_{triv} = \cup_{\Im\tau > 0} \mathcal{C}(E_\tau, \mathcal{O}_{E_\tau})$, where $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. For any of these moduli spaces, we can consider the Riemann-Hilbert correspondence map

$$RH : (E_\tau, \mathcal{L}, \nabla) \longmapsto (\rho_\nabla(a), \rho_\nabla(b)),$$

where ρ_∇ is the monodromy representation of ∇, and (a, b) is a basis of $\pi_1(E)$ corresponding to the basis $(1, \tau)$ of the period lattice $\mathbb{Z} + \mathbb{Z}\tau$. Remark that $RH \mid_{\mathcal{C}(E,\mathcal{L})}$ cannot be surjective by dimensional reasons. The next proposition shows that $RH \mid_{\mathcal{C}_{triv}}$ is dominant, though non-surjective, and that $RH \mid_{\mathcal{C}}$ is surjective.

Proposition 1.2.2. *In the above notation,*

$$RH(\mathcal{C}_{triv}) = (\mathbb{C}^* \times \mathbb{C}^* \setminus \{S^1 \times S^1\}) \cup \{(1,1)\},\ RH(\mathcal{C}(E)) = \mathbb{C}^* \times \mathbb{C}^*.$$

Proof. Let $\nabla = \mathrm{d} + \omega$ be a connection on an elliptic curve E, where $\omega \in \Gamma(E_\tau, \Omega^1_{E_\tau})$, $A = \oint_a \omega$, $B = \oint_b \omega = \tau A$. By analytic continuation of solutions of the equation $\nabla\varphi = 0$ along the cycles in E, we obtain $\rho(a) = e^{-A}$ and $\rho(b) = e^{-\tau A}$. The pair $(-A, -B) = (-A, -\tau A)$ is an element of $(0,0) \cup \mathbb{C}^* \times \mathbb{C}^*$. By setting $z = -A$, we deduce $RH(\mathcal{C}_{triv}) = \{(e^z, e^{z\tau}) \mid (z, \tau) \in \mathbb{C} \times \mathbb{H}\}$. The map $\exp : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ is surjective, so for all $w_1 \in \mathbb{C}^*$, we can solve the equation $e^z = w_1$, and once we have fixed z, it is possible to solve $e^{\tau z} = w_2$ with respect to τ if and only if $(w_1, w_2) \notin S^1 \times S^1 \setminus \{(1,1)\}$. This ends the proof for $RH(\mathcal{C}_{triv})$. The proof for $RH(\mathcal{C})$ is similar to the genus-2 case, see Proposition 1.2.3 below. $\square$

From now on, we turn to the genus-2 case. We define the moduli spaces $\mathcal{C}_2(C, \mathcal{L}), \mathcal{C}_2(C), \mathcal{C}_2, \mathcal{C}_{2,triv}$ similarly to the above, so that $\mathcal{C}_2(C) = \cup_{[\mathcal{L}] \in J(C)} \mathcal{C}_2(C, \mathcal{L})$, $\mathcal{C}_2 = \cup_{Z \in \mathcal{H}} \mathcal{C}_2(C_Z)$, and $\mathcal{C}_{2,triv} = \cup_{Z \in \mathcal{H}} \mathcal{C}_2(C_Z, \mathcal{O}_{C_Z})$. Here $\mathcal{H}$ is the locus of periods introduced in Corollary 1.1.6, C_Z is the genus-2 curve with period Z, $J_2(C) = \mathbb{C}^2/\Lambda$, where $\Lambda \simeq \mathbb{Z}^4$ is the lattice generated by the column vectors of the full period matrix $(1 \mid Z)$ of C. The Riemann-Hilbert correspondence is the map

$$RH : (C_Z, \mathcal{L}, \nabla) \longmapsto (\rho_\nabla(\alpha_1), \rho_\nabla(\alpha_2), \rho_\nabla(\beta_1), \rho_\nabla(\beta_2)) \in (\mathbb{C}^*)^4,$$

where the generators α_i, β_i of $\pi_1(C)$ correspond to the basis of the lattice Λ.

Proposition 1.2.3. *In the above notation,*

$$RH(\mathcal{C}_{2,triv}) = \{w \in (\mathbb{C}^*)^4 \mid (w_1w_2, w_3w_4) \in W, (\frac{w_1}{w_2}, \frac{w_3}{w_4}) \in W\}, \tag{1.8}$$

$$RH(\mathcal{C}_2(C_Z)) = (\mathbb{C}^*)^4, \tag{1.9}$$

where W denotes the locus $RH(\mathcal{C}_{triv})$ determined in Proposition 1.2.2.

Proof. Let $\nabla = \mathrm{d} - \omega$, $\omega \in \Gamma(C_Z, \Omega^1_{C_Z})$. We can consider C_Z in its Abel-Jacobi embedding in JC, then $\omega = \lambda_1 \mathrm{d}z_1 + \lambda_2 \mathrm{d}z_2$, where (z_1, z_2) are the standard flat coordinates on $\mathbb{C}^2/\Lambda$. Therefore,

$$RH(C_Z, \mathcal{O}_Z, \nabla) = (e^{\lambda_1 z_1}, e^{\lambda_2 z_2}, e^{\frac{1}{2}(\tau+\tau')\lambda_1 z_1 + \frac{1}{2}(\tau-\tau')\lambda_2 z_2}, e^{\frac{1}{2}(\tau-\tau')\lambda_1 z_1 + \frac{1}{2}(\tau+\tau')\lambda_2 z_2}).$$

Denoting the latter 4-vector by w, we see that $(w_1w_2, w_3w_4) = (e^z, e^{\tau z})$ with $z = \lambda_1 z_1 + \lambda_2 z_2$, and $(\frac{w_1}{w_2}, \frac{w_3}{w_4}) = (e^{z'}, e^{\tau' z'})$ with $z' = \lambda_1 z_1 - \lambda_2 z_2$.

Then Proposition 1.2.2 implies the answer for $RH(\mathcal{C}_{2,triv})$. Now, we will prove the surjectivity of $RH\ |_{\mathcal{C}_2}$. On a genus-2 curve, any line bundle of degree 0 can be represented in the form $\mathcal{L} = \mathcal{O}_C(P_1 + P_2 - Q_1 - Q_2)$ for some 4 points $P_i, Q_i \in C$. It is defined by its stalks: for any $P \in C$, $\mathcal{L}_P = \mathcal{O}_P$ if $P \notin \{P_1, P_2, Q_1, Q_2\}$, $\mathcal{L}_{P_i} = \frac{1}{\tau_{P_i}}\mathcal{O}_{P_i}$, $\mathcal{L}_{Q_i} = \tau_{Q_i}\mathcal{O}_{Q_i}$, where τ_P denotes a local parameter at P for any $P \in C$. This implies that the constant function $e = 1$ considered as a section of $\mathcal{L}$ has simple zeros at P_i and simple poles at Q_i, that is, for its divisor we can write: $(e) = P_1 + P_2 - Q_1 - Q_2$. According to [At-2], any line bundle of degree 0 admits a connection, and two connections differ by a holomorphic 1-form. Hence any connection on $\mathcal{L}$ can be written in the form $\nabla = d + \omega$, $\omega = \nu + \lambda_1 dz_1 + \lambda_2 dz_2$, where ν is a meromorphic 1-form with simple poles at P_i, Q_i such that $\mathrm{Res}_{P_i}\,\nu = 1, \mathrm{Res}_{Q_i}\,\nu = -1$ (these are apparent singularities of ∇ with respect to the meromorphic trivialization $e = 1$).

We can choose the coefficients λ_1, λ_2 in such a way that ω will have zero a-periods. Let us denote the periods of ω by N_i:

$$N_1 = \int_{a_1} \omega, \quad N_2 = \int_{a_2} \omega, \quad N_3 = \int_{b_1} \omega, \quad N_4 = \int_{b_2} \omega. \tag{1.10}$$

Then $N_1 = N_2 = 0$ by the choice of ω, and

$$N_{2+j} = 2\pi i \sum_k \mathrm{Res}_{s_k}(\omega) \int_{s_0}^{s_k} dz_j, \;\; j = 1,\ 2,$$

by the Reciprocity Law for differentials of 1st and 3rd kinds ([GH], Sect. 2.2), where $\sum_k s_k$ is the divisor of poles $(\omega)_\infty$ of ω, and s_0 is any point of C. Taking into account that $(\omega)_\infty = (\nu)_\infty = P_1 + P_2 + Q_1 + Q_2$, $\mathrm{Res}_{P_i}\,\nu = 1, \mathrm{Res}_{Q_i}\,\nu = -1$, and $z_j(P) = \int_{P_0}^P dz_j$, we can rewrite:

$$N_{2+j} = 2\pi i[z_j(P_1) - z_j(Q_1) + z_j(P_2) - z_j(Q_2)].$$

Hence the components of the vector $\dfrac{1}{2\pi i}\begin{pmatrix} N_3 \\ N_4 \end{pmatrix}$ are the 2 coordinates on JC of the class $[\mathcal{L}]$ of the line bundle $\mathcal{L}$, which is the same as the divisor class $[P_1 + P_2 - Q_1 - Q_2]$. Now, we can finish the proof.

Let $(w_i) \in (\mathbb{C}^*)^4$. Then, we can find a 1-form η_1 of a connection on a degree-0 line bundle $\mathcal{L}_1$ with monodromy $(1, 1, w_3, w_4)$ in choosing $\mathcal{L}_1$ with coordinates $-\frac{1}{2\pi i}(\log w_3, \log w_4)$ on JC. In interchanging the roles of a- and b-periods, we will find another 1-form of connection η_2 on another degree-0 line bundle $\mathcal{L}_2$, with monodomy $(w_1, w_2, 1, 1)$. Then $\omega = \eta_1 + \eta_2$ is the form of a connection on $\mathcal{L}_1 \otimes \mathcal{L}_2$ with monodromy $(w_i) \in (\mathbb{C}^*)^4$. $\square$

1.3 Direct images of rank-1 connections

We will determine the direct image connections $f_*(\nabla_{\mathcal{L}}) = \nabla_{\mathcal{E}}$ on the rank-2 vector bundle $\mathcal{E} = f_*\mathcal{L}$, where $f : C \to E$ is an elliptic subcover of degree 2 of C. From now on, we will stick to a representation of C in the classical form $y^2 = F_6(\xi)$, where F_6 is a degree-6 polynomial. We want that E is given the Legendre equation $y^2 = x(x-1)(x-t)$, but F_6 is not so bulky as in (1.6). Of course, this can be done in many different ways. We will fix for C and f the following choices:

$$\begin{array}{rcl} f : C = \{y^2 = (t' - \xi^2)(t' - 1 - \xi^2)(t' - t - \xi^2)\} & \to & E = \{y^2 = x(x-1)(x-t))\} \\ (\xi, y) & \mapsto & (x, y) = (t' - \xi^2, y) \end{array} \tag{1.11}$$

Lemma 1.3.1. *For any bielliptic curve C with an elliptic subcover $f : C \to E$ of degree 2, there exist affine coordinates ξ, x, y on C, E such that f, C, E are given by* (1.11) *for some $t, t' \in \mathbb{C} \setminus \{0, 1\}$, $t \neq t'$.*

Proof. By Proposition 1.1.2, it suffices to verify that the two elliptic subcovers E, E' of the curves C given by (1.11), as we vary t, t', run over the whole moduli space of elliptic curves independently from each other. E' can be determined from (1.2). It is a double cover of $\mathbb{P}^1$ ramified at $\frac{1}{t'}, \frac{1}{t'-1}, \frac{1}{t'-t}, \infty$. This quadruple can be sent by a homographic transformation to $0, 1, t, t'$, hence E' is given by $y^2 = x(x-1)(x-t)(x-t')$. If we fix t and let vary t', we will obviously obtain all the elliptic curves, which ends the proof. $\square$

The only branch points of f in E are $p_\pm = (t', \pm y_0)$, where $y_0 = \sqrt{t'(t'-1)(t'-t)}$, and thus the ramification points of f in C are $\tilde{p}_\pm = (0, \pm y_0)$. In particular, f is non-ramified at infinity and the preimage of $\infty \in E$ is a pair of points $\infty_\pm \in C$. E is the quotient of C by the involution $\iota : C \to C$, called the Galois involution of the double covering f. It is given in coordinates by $\iota : (\xi, y) \mapsto (-\xi, y)$.

We first deal with the case when $\mathcal{L}$ is the trivial bundle $\mathcal{O}_C$, in which we write $\nabla_{\mathcal{O}}$ instead of $\nabla_{\mathcal{L}}$. The direct image $\mathcal{E}_0 = f_*\mathcal{O}_C$ is a vector bundle of rank 2 which splits into the direct sum of the ι-invariant and anti-invariant subbundles: $\mathcal{E}_0 = (f_*\mathcal{O}_C)^+ \oplus (f_*\mathcal{O}_C)^-$. The latter subbundles are defined as sheaves by specifying their sections over any open subset U of E:

$$\Gamma(U, (f_*\mathcal{O}_C)^\pm) = \{s \in \Gamma(f^{-1}(U), \mathcal{O}_C) \mid \iota^*(s) = \pm s\}.$$

Obviously, the ι-invariant sections are just functions on E, so the first direct summand $(f_*\mathcal{O}_C)^+$ is the trivial bundle $\mathcal{O}_E$. The second one is generated over the affine set $E \setminus \{\infty\}$ by a single generator ξ, one of the two coordinates on C. Thus, we can use $(1, \xi)$ as a basis trivializing $\mathcal{E}_0$ over $E \setminus \{\infty\}$ and compute $\nabla = f_*(\nabla_{\mathcal{O}})$ in this basis. We use, of course, the constant function 1 to trivialize $\mathcal{O}_C$ and write $\nabla_{\mathcal{O}}$ in the form

$$\nabla_{\mathcal{O}} = d + \omega, \quad \omega = \nabla_{\mathcal{O}}(1) = \lambda_1 \frac{\mathrm{d}\xi}{y} + \lambda_2 \frac{\xi \mathrm{d}\xi}{y}. \tag{1.12}$$

Re-writing $\nabla_{\mathcal{O}}(1) = \omega$ in terms of the coordinate $x = t' - \xi^2$, we get:

$$\nabla_{\mathcal{O}}(1) = -\frac{\lambda_1}{2(t'-x)} \frac{\mathrm{d}x}{y} \xi - \frac{\lambda_2}{2} \frac{\mathrm{d}x}{y} 1.$$

Likewise,

$$\nabla_{\mathcal{O}}(\xi) = -\frac{\lambda_1}{2} \frac{\mathrm{d}x}{y} 1 - \frac{\lambda_2}{2y} \frac{\mathrm{d}x}{y} \xi - \frac{\mathrm{d}x}{2(t'-x)} \xi.$$

We obtain the matrix of $\nabla = f_*(\nabla_{\mathcal{O}})$ in the basis $(1, \xi)$:

$$A = \begin{pmatrix} -\frac{\lambda_2}{2y}\mathrm{d}x & -\frac{\lambda_1}{2y}\mathrm{d}x \\ -\frac{\lambda_1}{2(t'-x)y}\mathrm{d}x & -(\frac{\lambda_2}{2y} + \frac{1}{2(t'-x)})\mathrm{d}x \end{pmatrix}. \tag{1.13}$$

This matrix has poles at the branch points $p_\pm$ with residues

$$\operatorname{Res}_{p_+} A = \begin{pmatrix} 0 & 0 \\ \frac{\lambda_1}{2y_0} & \frac{1}{2} \end{pmatrix}, \quad \operatorname{Res}_{p_-} A = \begin{pmatrix} 0 & 0 \\ -\frac{\lambda_1}{2y_0} & \frac{1}{2} \end{pmatrix}. \tag{1.14}$$

As the sum of residues of a meromorphic 1-form on a compact Riemann surface is zero, we can evaluate the residue at infinity:

$$\operatorname{Res}_{p_-} A + \operatorname{Res}_{p_+} A = -\operatorname{Res}_\infty(A) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is nonzero, hence A is not regular at ∞ and has exactly 3 poles on E. In fact, the pole at ∞ is an apparent singularity due to the fact that $(1, \xi)$ fails to be a basis of $f_*\mathcal{O}_C$ at ∞, which follows from the following proposition:

Proposition 1.3.2. *Let $f : C \to E$ be the bielliptic cover* (1.11)*, and $\nabla_{\mathcal{O}} = d + \omega$ a regular connection on the trivial bundle $\mathcal{O}_C$ with connection form $\omega = \lambda_1 \frac{\mathrm{d}\xi}{y} + \lambda_2 \frac{\xi\mathrm{d}\xi}{y}$. Then the direct image $\nabla = f_*(\nabla_{\mathcal{O}})$ is a logarithmic connection on a rank-2 vector bundle $\mathcal{E}_0$ over E, whose only poles are the two branch points $p_\pm$ of f. In an appropriate trivialization of $\mathcal{E}_0$ over $E \setminus \{\infty\}$, ∇ is given by the connection matrix* (1.13)*, and the residues at $p_\pm$ are given by* (1.14)*.*

Proof. If $P \in E$ is not a branch point, than we can choose a small disk U centered at P such that $f^{-1}(U)$ is the disjoint union of two disks $U_\pm$. Let $e_\pm$ be a nonzero $\nabla_{\mathcal{O}}$-flat section of $\mathcal{O}_C$ over $U_\pm$.

Then (e_+, e_-) is a basis of $\mathcal{E}_0$ over U consisting of ∇-flat sections. This implies the regularity of ∇ over U (the connection matrix of ∇ in this basis is zero).

We have shown that the only points where the direct image of a regular connection might have singularities are the branch points of the covering. In particular, ∞ is not a singularity of ∇. The fact that the branch points are logarithmic poles follows from the calculation preceding the statement of the proposition. $\square$

At this point, it is appropriate to comment on the horizontal sections of ∇, which are solutions of the matrix ODE $d\Phi + A\Phi = 0$ for the vector $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$. We remark that the matrix ODE is equivalent to one scalar equation of second order which we have not encountered in the literature. It is obtained as follows: the first line of the matrix equation gives

$$\Phi_2 = \frac{2\lambda_2}{\lambda_1}\Phi_1' - \frac{\lambda_2}{\lambda_1}\Phi_1,$$

where Φ_1, Φ_2 denote the components of a single 2-vector Φ, and the prime denotes the derivative with respect to x. The second equation gives:

$$\Phi_2' = \frac{\lambda_1}{2y(t'-x)}\Phi_1 + \Big(\frac{\lambda_2}{2y} + \frac{1}{2(t'-x)}\Big)\Phi_2.$$

By substituing here Φ_2 in terms of Φ_1, we get one second order equation for Φ_1. By setting $y^2 = P_3(x) = x(x-1)(x-t)$, we have $y' = \frac{P_3'}{2y}$ and the differential equation for Φ_1 takes the form

$$\Phi_1'' + \Big[\frac{P_3'(x)}{2P_3(x)} - \frac{\lambda_2}{y} + \frac{1}{2(x-t')}\Big]\Phi_1' + \Big[\frac{\lambda_1^2}{4P_3(x)(x-t')} + \frac{\lambda_2^2}{4P_3(x)} - \frac{\lambda_2}{4(x-t')y}\Big]\Phi_1 = 0.$$

We can also write out the second order differential equation for Φ_1 with respect to the flat coordinate $z = \int \frac{\mathrm{d}x}{y}$ on E. Now, set up the convention that the prime denotes $\frac{\mathrm{d}}{\mathrm{d}z}$. Then, after an appropriate scaling, $x = \wp(z) + \frac{t+1}{3}, y = \frac{\wp'(z)}{2}$. Let $z_0, -z_0$ be the solutions of $\wp(z) = t' - \frac{t+1}{3}$ modulo the lattice of periods. Then we have the following equation for Φ_1:

$$\Phi_1'' + \Big[-\lambda_2 + \frac{\wp'(z)}{2(\wp(z) - \wp(z_0))}\Big]\Phi_1' + \Big[\frac{\lambda_2^2}{4} + \frac{2\lambda_1^2 - \lambda_2\wp'(z)}{8(\wp(z) - \wp(z_0))}\Big]\Phi_1 = 0.$$

We now go over to the general case, in which $\mathcal{L}$ is any line bundle of degree 0 on C endowed with a regular connection $\nabla_{\mathcal{L}}$. Then $f_*\mathcal{L} = \mathcal{E}$ is a vector bundle of rank 2 on E endowed with a logarithmic connection $\nabla_{\mathcal{E}} = f_*\nabla_{\mathcal{L}}$. We can represent $\mathcal{L}$ in the form $\mathcal{L} = \mathcal{O}(\tilde{q}_1 + \tilde{q}_2 - \infty_+ - \infty_-)$ with $\tilde{q}_1 = (\xi_1, y_1)$ and $\tilde{q}_2 = (\xi_2, y_2)$ some points of C. Their images on E will be denoted by q_i, or (x_i, y_i) in coordinates. We will use $1 \in \Gamma(\mathcal{O}_C)$ as a meromorphic trivialization of $\mathcal{L}$ as in the proof of Proposition 1.2.3. In this trivialization, the connection form of $\nabla_{\mathcal{L}}$ has simple poles at the 4 points $\tilde{q}_i, \infty_\pm$ with residues $+1$ at $\tilde{q}_i$ and -1 at $\infty_\pm$. It is easy to invent one example of such a form: $\nu = \frac{1}{2}\left(\frac{y+y_1}{\xi-\xi_1} + \frac{y+y_2}{\xi-\xi_2}\right)$. Hence the general form of $\nabla_{\mathcal{L}}$ is as follows:

$$\nabla_{\mathcal{L}} = \mathrm{d} + \omega = \mathrm{d} + \frac{1}{2}\left(\frac{y+y_1}{\xi-\xi_1} + \frac{y+y_2}{\xi-\xi_2}\right)\frac{\mathrm{d}\xi}{y} + \lambda_1\frac{\mathrm{d}\xi}{y} + \lambda_2\frac{\xi\mathrm{d}\xi}{y}. \tag{1.15}$$

We compute $\nabla_{\mathcal{L}}(1)$ and $\nabla_{\mathcal{L}}(\xi)$ and express the result in the coordinates (x, y) of E. This brings us to formulas for the connection $\nabla_{\mathcal{E}}$ on E. We obtain:

$$\nabla_{\mathcal{L}}(1) = \omega.1 = \left[\frac{1}{2}\left(\frac{y+y_1}{\xi(\xi-\xi_1)} + \frac{y+y_2}{\xi(\xi-\xi_2)}\right) + \frac{\lambda_1}{\xi} + \lambda_2\right]\frac{\xi\mathrm{d}\xi}{y}.$$

Splitting $\frac{1}{\xi-\xi_i}$ into the invariant and anti-invariant parts, we get:

$$\begin{aligned}\nabla_{\mathcal{L}}(\xi) = \xi\nabla_{\mathcal{L}}(1) + \mathrm{d}\xi\cdot 1 = \Bigg[\frac{1}{2}\left(\frac{(y+y_1)\xi_1}{\xi^2-\xi_1^2} + \frac{(y+y_2)\xi_2}{\xi^2-\xi_2^2}\right) + \lambda_1 \\ + \frac{1}{2}\left(\frac{y+y_1}{\xi^2-\xi_1^2} + \frac{y+y_2}{\xi^2-\xi_2^2}\right)\xi + \lambda_2\xi + \frac{y}{\xi^2}\xi\Bigg]\frac{\xi\mathrm{d}\xi}{y}.\end{aligned}$$

By using the relations $x = t' - \xi^2$, $\mathrm{d}x = -2\xi\mathrm{d}\xi$, we determine the connection $\nabla_{\mathcal{E}} = d + A$, where A is the matrix of $\nabla_{\mathcal{E}}$ in the basis $(1, \xi)$:

$$\begin{pmatrix} -\frac{1}{2}\left(\frac{1}{2}\left(\frac{y+y_1}{x_1-x} + \frac{y+y_2}{x_2-x}\right) + \lambda_2\right)\frac{\mathrm{d}x}{y} & -\frac{1}{2}\left(\frac{1}{2}\left(\frac{(y+y_1)\xi_1}{x_1-x} + \frac{(y+y_2)\xi_2}{x_2-x}\right) + \lambda_1\right)\frac{\mathrm{d}x}{y} \\ -\frac{1}{2}\left(\frac{1}{2}\left(\frac{(y+y_1)\xi_1}{(x_1-x)(t'-x)} + \frac{(y+y_2)\xi_2}{(x_2-x)(t'-x)}\right) + \frac{\lambda_1}{t'-x}\right)\frac{\mathrm{d}x}{y} & -\frac{1}{2}\left(\frac{1}{2}\left(\frac{y+y_1}{x_1-x} + \frac{y+y_2}{x_2-x}\right) + \lambda_2 + \frac{y}{t'-x}\right)\frac{\mathrm{d}x}{y} \end{pmatrix}. \tag{1.16}$$

We compute $\operatorname{Res}_{p_\pm} A$, where $p_\pm$ are the only singularities of $\nabla_{\mathcal{E}}$:

$$\operatorname{Res}_{p_\pm} A = \begin{pmatrix} 0 & 0 \\ \frac{1}{4}\left(\frac{(y_1\pm y_0)\xi_1}{x_1-t'} + \frac{(y_2\pm y_0)\xi_2}{x_2-t'}\right) \pm \frac{\lambda_1}{2y_0} & \frac{1}{2} \end{pmatrix}. \tag{1.17}$$

Proposition 1.3.3. *Let $f : C \to E$ be the bielliptic cover* (1.11), $\mathcal{L} = \mathcal{O}(\tilde{q}_1 + \tilde{q}_2 - \infty_+ - \infty_-)$ *with $\tilde{q}_i = (\xi_i, y_i) \in C$ $(i = 1, 2)$, and $\nabla_{\mathcal{L}} = d + \omega$ a regular connection on $\mathcal{L}$ with connection form ω defined by* (1.15)*. Assume that $\xi_i \neq 0$, that is $\tilde{q}_i \neq \tilde{p}_\pm$. Then the direct image $\nabla_{\mathcal{E}} = f_*(\nabla_{\mathcal{L}})$ is a logarithmic connection on a rank-2 vector bundle $\mathcal{E}$ over E whose only poles are the two branch points $p_\pm$ of f. In the meromorphic trivialization of $\mathcal{E}$ defined by $(1, \xi)$, $\nabla_{\mathcal{E}}$ is given by the connection matrix* (1.16)*, and the residues at $p_\pm$ are given by* (1.17)*.*

Remark that the points $q_i = f(\tilde{q}_i) = (x_i, y_i)$ are apparent singularities of $\nabla_{\mathcal{E}}$. We write down the residues of A at these points for future use:

$$\operatorname{Res}_{q_1} A = \begin{pmatrix} \frac{1}{2} & \frac{\xi_1}{2} \\ \frac{1}{2\xi_1} & \frac{1}{2} \end{pmatrix}, \quad \operatorname{Res}_{q_2} A = \begin{pmatrix} \frac{1}{2} & \frac{\xi_2}{2} \\ \frac{1}{2\xi_2} & \frac{1}{2} \end{pmatrix}. \tag{1.18}$$

We can also compute $\operatorname{Res}_\infty A$. First homogenize the equation of E via the change $x = \frac{x_1}{x_0}$, and $y = \frac{x_2}{x_0}$. The homogeneous equation is $x_0x_2^2 = x_1^3 - (1+t)x_1^2x_0 + tx_1x_0^2$. Then, setting $v = \frac{x_0}{x_2}$, $u = \frac{x_1}{x_2}$, we obtain the equation $v = u^3 - (1+t)u^2v + tuv^2$ in the neighborhood of ∞. Near $\infty = (0,0)$, we have $v \sim u^3$, $\frac{\mathrm{d}x}{y} \sim -2\mathrm{d}u$. Therefore, $\operatorname{Res}_\infty A$ is:

$$\operatorname{Res}_\infty A = \operatorname{Res}_{u=0} A = \begin{pmatrix} -1 & -\frac{\xi_1+\xi_2}{2} \\ 0 & -2 \end{pmatrix}. \tag{1.19}$$

1.4 Monodromy of direct image connections

We are using the notation of the previous section. We will calculate the monodromy of the direct image connections $\nabla_{\mathcal{E}}$. We will start by choosing generators of the fundamental group $\pi_1(E)$, $\pi_1(E \setminus \{p_+, p_-\})$. To express the monodromy of ∇ in terms of periods of C, we will first introduce generators a_i, b_i, c_i for $\pi_1(C \setminus \{\tilde{p}_+, \tilde{p}_-\})$, and then descend some of them to E by applying f_*. We start by representing a generating system a_i, b_i, c_i (i=1, 2) of $\pi_1(C \setminus \{\tilde{p}_\pm\}, \infty_+)$ by simple loops meeting at the reference point ∞_+, one of the two points over ∞ on E. We choose them in such a way that the following relation is satisfied: $[a_1, b_1]c_1[a_2, b_2]c_2 = 1$; see Fig. 1.2. To fix the ideas, we assume

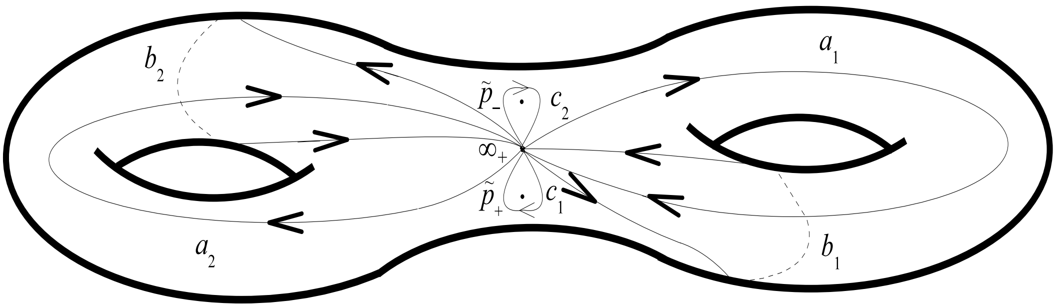

Figure 1.2: Generators of $\pi_1(C \setminus \{\tilde{p}_\pm\}, \infty_+)$.

for a while that t , t' are real and $1 < t < t'$ (for general t, t', the loops a_i, b_i, c_i are defined up to an isotopy bringing t, t' onto the real axis so that $1 < t < t'$). We deform the basic loops so that they follow intervals of the real or imaginary axes or arcs of small circles around the marked point. C can be represented as the result of glueing of two copies of the Riemann sphere along three cuts. We call these copies of the Riemann sphere upper and lower sheets, and the cuts are realized along the rectilinear segments $[-\sqrt{t'}, -\sqrt{t'-1}]$, $[-\sqrt{t'-t}, \sqrt{t'-t}]$ and $[\sqrt{t'-1}, \sqrt{t'}]$. The sheets are glued together in such a way that the upper edge of each cut on the upper sheet is identified with the lower edge of the respective cut on the lower sheet, and vice versa. The result is the pretzel indicated on Fig. 1.3, where the neighborhoods of the cuts are streched out into the necks of the pretzel for the sake of the illustration.

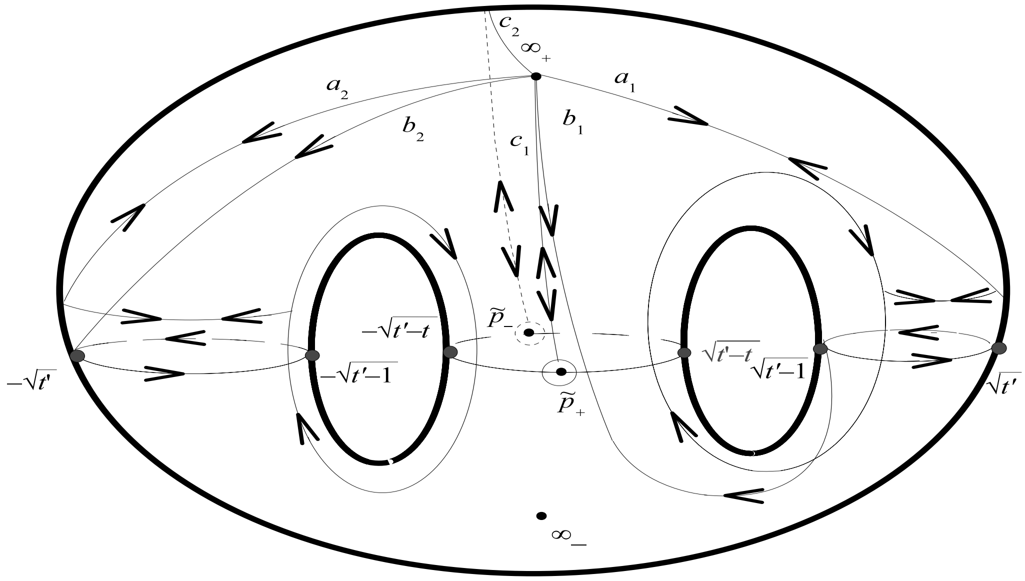

Figure 1.3: C as the result of glueing two copies of the Riemann sphere and deformed loops a_i, b_i, c_i on it.

Let ∞_+ be on the upper sheet, singled out by the condition $\Im y > 0$ when $\xi \in \mathbb{R}$, $\xi \to +\infty$. Fig. 1.3 also shows an intermediate stage of the deformation of the basic loops bringing them from the positions shown on Fig. 1.2 to the desired one. This implies that the values of $\Re y, \Im y$ on $\mathbb{R}$ are as on Fig. 1.4, where the loops a_i, b_i, c_i generating $\pi_1(C \setminus \{\tilde{p}_+, \tilde{p}_-\})$ are shown. Remark that the loops c_i are chosen in the form $c_i = d_i \tilde{c}_i d_i^{-1}$, where d_i is a path joining ∞_+ with some point close to $\tilde{p}_\pm$ and $\tilde{c}_i$ is a small circle around $\tilde{p}_\pm$ (the values $i = 1, 2$ correspond to $\tilde{p}_+$, $\tilde{p}_-$ respectively). The paths d_i follow the imaginary axis of the upper sheet.

Now, we go over to E. We can construct E from the genus-2 Riemann surface C as the quotient by the involution $\iota : C \longrightarrow C$ defined by $(\xi, y) \longmapsto (-\xi, y)$. In the representation of C given by Fig. 1.3, we can realize ι as the symmetry with respect to the axis $\tilde{p}_+\tilde{p}_-$. Then, to construct $E = C/\iota$, we can cut C into halves by the longitudinal plane passing through four points $\tilde{p}_\pm, \infty_\pm$, discard the left hand half, and the right hand one will give E after we glue the upper part of the cut with its lower part, in identifying pairs of points, symmetric with respect to the axis $\tilde{p}_+\tilde{p}_-$. This quotient procedure is shown on Fig. 1.5 and 1.6.

Set $a = f_*(a_1)$, $b = f_*(b_1)$, and define the closed paths running round the branch points $p_\pm$ as follows: $\gamma_i = f(d_i)\tilde{\gamma}_i f(d_i)^{-1}$, where $\tilde{\gamma}_i$ are small circles around $p_\pm$ running in the same direction as $f(\tilde{c}_i)$ (but $f(\tilde{c}_i)$ makes two revolutions around $p_\pm$, whilst $\tilde{\gamma}_i$ only one).

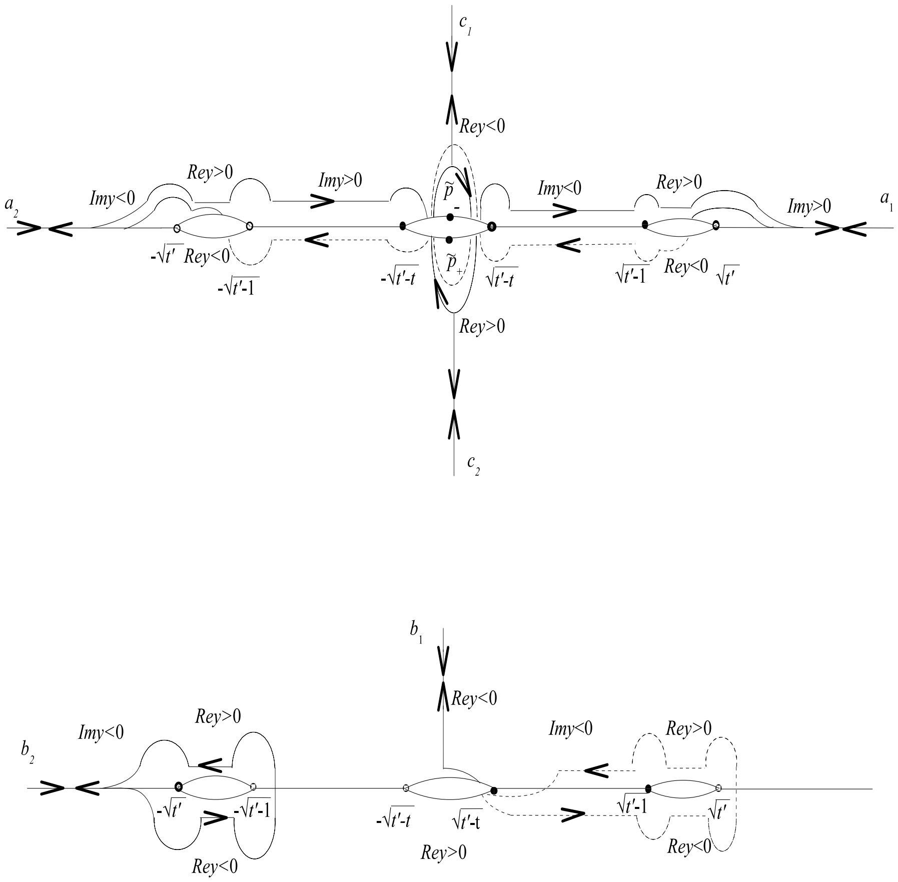

Figure 1.4: The final position of the generating loops of $\pi_1(C \setminus \{\tilde{p}_\pm\}, \infty_+)$. The parts of the arcs represented in solid (resp. dash) lines are on the upper (resp. lower) sheet.

One can verify that the thus defined generators of both fundamental groups satisfy the relations $[a_1, b_1]c_1[a_2, b_2]c_2 = 1$ and $[a, b]\gamma_1\gamma_2 = 1$ and that the group morphism $f_* : \pi_1(C \setminus \{\tilde{p}_\pm\}, \infty_+) \longrightarrow \pi_1(E \setminus \{p_\pm\}, \infty)$ is given by the formulas

$$f_*(a_1) = a,\ f_*(b_1) = b,\ f_*(a_2) = \gamma_1^{-1} a \gamma_1,\ f_*(b_2) = \gamma_1^{-1} b \gamma_1,\ f_*(c_i) = \gamma_i^2\ (i = 1, 2).$$

As $\nabla_{\mathcal{L}}$ is regular at $\tilde{p}_\pm$, it has no monodromy along c_i, and this together with the above formulas for f_* immediately implies that the monodromy matrices M_{γ_i} of $\nabla_{\mathcal{E}}$ are of order 2.

We first assume that $\mathcal{L} = \mathcal{O}_C$ is trivial, in which case $\nabla_{\mathcal{L}}$ is denoted $\nabla_{\mathcal{O}}$, and $\nabla_{\mathcal{E}}$ just ∇. As in the previous section, we trivialize $\mathcal{E}_0 = f_*(\mathcal{O}_C)$ by the basis $(1, \xi)$ over $E \setminus \{\infty\}$. Splitting the solution $\varphi = e^{-\lambda_1 z_1 - \lambda_2 z_2}$ of $\nabla_{\mathcal{O}}\varphi = 0$ into the ι-invariant and anti-invariant parts, we represent φ by

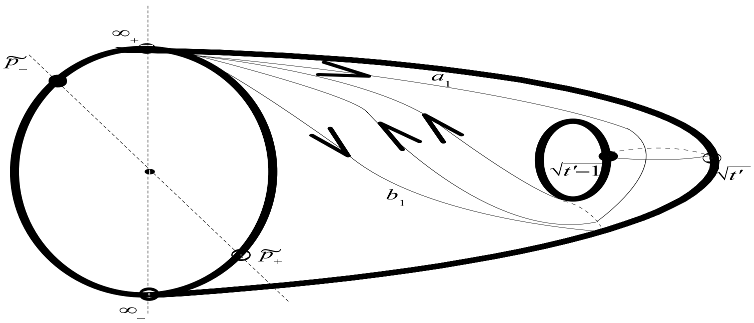

Figure 1.5: The right hand half of C.

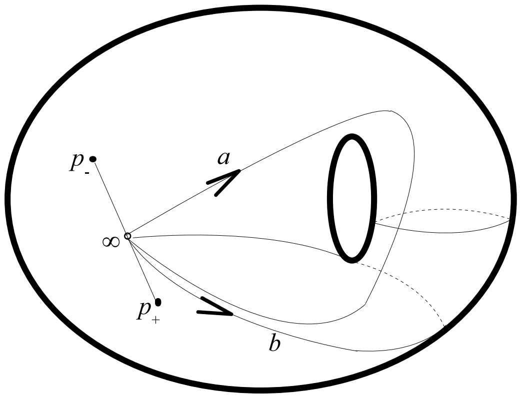

Figure 1.6: The quotient C/ι.

a 2-component vector in the basis $(1, \xi)$:

$$\Phi = \begin{pmatrix} e^{-\lambda_2 z_2} \cosh(\lambda_1 z_1) \\ -\frac{e^{-\lambda_2 z_2}}{\xi} \sinh(\lambda_1 z_1) \end{pmatrix}.$$

We have to complete Φ to a fundamental matrix $\mathbf{\Phi}$, and then we can define the monodromy M_γ along a loop γ by $T_\gamma(\mathbf{\Phi}) = \mathbf{\Phi} M_\gamma$, where T_γ denotes the analytic continuation along γ. We already know the first column of $\mathbf{\Phi}$: this is just Φ. Denote it also by $\mathbf{\Phi_1}$, the column vector $\begin{pmatrix} \Phi_{1,1} \\ \Phi_{2,1} \end{pmatrix}$. It remains to find $\mathbf{\Phi_2} = \begin{pmatrix} \Phi_{1,2} \\ \Phi_{2,2} \end{pmatrix}$ so that

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{pmatrix}$$

is a fundamental matrix. By Liouville's theorem, the matrix equation $\mathbf{\Phi}' + A\mathbf{\Phi} = 0$ implies the following scalar equation for $\Psi = \det \Phi$: $\Psi' + Tr(A)\Psi = 0$. In our case, $Tr(A) = -\frac{\lambda_2}{y} - \frac{1}{2(t'-x)}$, and we get a solution in the form: $\Psi = \frac{e^{-2\lambda_2 z_2}}{\sqrt{t'-x}} = \frac{e^{-2\lambda_2 z_2}}{\xi}$. Thus we can determine $\mathbf{\Phi}_2$ from the system:

$$\begin{aligned} \cosh(\lambda_1 z_1)\Phi_{2,2} + \frac{1}{\xi}\sinh(\lambda_1 z_1)\Phi_{1,2} &= \frac{e^{-2\lambda_2 z_2}}{\xi} \\ \Phi'_{1,2} &= \frac{1}{2y}(\lambda_2\Phi_{1,2} + \lambda_1\Phi_{2,2}). \end{aligned}$$

Eliminating $\Phi_{2,2}$, we obtain an inhomogeneous first order linear differential equation for $\Phi_{1,2}$. Finally, we find:

$$\mathbf{\Phi_2} = \begin{pmatrix} -e^{-\lambda_2 z_2}\sinh(\lambda_1 z_1) \\ \frac{e^{-\lambda_2 z_2}}{\xi}\cosh(\lambda_1 z_1) \end{pmatrix}.$$

Now we can compute the monodromies of $\nabla_{\mathcal{E}}$ along the loops a, b, γ_i. It is convenient to represent the result in a form, in which the real and imaginary parts of all the entries are visible as soon as $t, t' \in \mathbb{R}$ and $1 < t < t'$. Under this assumption, the entries of the period matrix $\Pi = ((a_{ij})|(b_{ij}))$ of C are real or imaginary and can be expressed in terms of hyperelliptic integrals along the real segments joining branch points.

Thus reading the cycles of integration from Fig. 1.4, we obtain:

$$a_{1,1} = -a_{1,2} = 2iK, K = \int_{\sqrt{t'-t}}^{\sqrt{t'-1}} \frac{\mathrm{d}\xi}{\sqrt{(t'-\xi^2)(t'-1-\xi^2)(t-t'+\xi^2)}} > 0,$$

$$a_{2,1} = a_{2,2} = 2iK', K' = \int_{\sqrt{t'-t}}^{\sqrt{t'-1}} \frac{\xi\mathrm{d}\xi}{\sqrt{(t'-\xi^2)(t'-1-\xi^2)(t-t'+\xi^2)}} > 0,$$

$$b_{1,1} = -b_{1,2} = -2L, L = \int_{\sqrt{t'}}^{\sqrt{t'-1}} \frac{\mathrm{d}\xi}{|y|} > 0,$$

$$b_{2,1} = b_{2,2} = -2L', L' = \int_{\sqrt{t'-1}}^{\sqrt{t'}} \frac{\xi\mathrm{d}\xi}{|y|} > 0.$$

Proposition 1.4.1. *The monodromy matrices of the connection $\nabla = f_*(\nabla_{\mathcal{O}})$, where $\nabla_{\mathcal{O}}$ is the rank-*1 *connection* (1.12)*, are given by*

$$M_a = \begin{pmatrix} e^{-2i\lambda_2 K'}\cos(2\lambda_1 K) & -e^{-2i\lambda_2 K'} i\sin(2\lambda_1 K) \\ -e^{-2i\lambda_2 K'} i\sin(2\lambda_1 K) & e^{-2i\lambda_2 K'}\cos(2\lambda_1 K) \end{pmatrix}, \tag{1.20}$$

$$M_b = \begin{pmatrix} e^{2\lambda_2 L'}\cosh(2\lambda_1 L) & e^{2\lambda_2 L'}\sinh(2\lambda_1 L) \\ e^{2\lambda_2 L'}\sinh(2\lambda_1 L) & e^{2\lambda_2 L'}\cosh(2\lambda_1 L) \end{pmatrix}, \tag{1.21}$$

$$M_{\gamma_i} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (i = 1, 2). \tag{1.22}$$

Now we turn to the general case of nontrivial $\mathcal{L}$. Our computations done in the special case allows us to guess the form of the fundamental matrix of solutions to $\nabla_{\mathcal{E}}\Phi = 0$, where $\nabla_{\mathcal{E}} = f_*\nabla_{\mathcal{L}}$ and $\nabla_{\mathcal{L}}$ is given by (1.15) (remark, it would be not so easy to find it directly from (1.16)):

$$\mathbf{\Phi} = \frac{1}{2}\begin{pmatrix} e^{-\int\omega} + e^{-\int\omega^*} & e^{-\int\omega} - e^{-\int\omega^*} \\ \frac{1}{\xi}(e^{-\int\omega} - e^{-\int\omega^*}) & \frac{1}{\xi}(e^{-\int\omega} + e^{-\int\omega^*}) \end{pmatrix}, \tag{1.23}$$

where $\omega^* := \iota^*(\omega)$ is obtained from ω by the change $\xi \mapsto -\xi$. We deduce the monodromy:

Proposition 1.4.2. *The monodromy matrices of the connection $\nabla_{\mathcal{E}}$ given by* (1.16) *are the following:*

$$M_a = \frac{1}{2}\begin{pmatrix} e^{-N_1} + e^{-N_2} & e^{-N_1} - e^{-N_2} \\ e^{-N_1} - e^{-N_2} & e^{-N_1} + e^{-N_2} \end{pmatrix},$$

$$M_b = \frac{1}{2}\begin{pmatrix} e^{-N_3} + e^{-N_4} & e^{-N_3} - e^{-N_4} \\ e^{-N_3} - e^{-N_4} & e^{-N_3} + e^{-N_4} \end{pmatrix},$$

$$M_{\gamma_i} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (i = 1, 2).$$

*Here (N_1, N_2, N_3, N_4) are the periods of ω as defined in (*1.10*).*

Proof. By a direct calculation using the observation that the periods of ω^* are (N_2, N_1, N_4, N_3). □

Here the connection (1.16) depends on 6 independent parameters $\tilde{q}_1, \tilde{q}_2, \lambda_1, \lambda_2, t, t'$, and the monodromy is determined by the 4 periods N_i. Hence, it is justified to speak about the isomonodromic deformations for this connection. The problem of isomonodromic deformations is easily solved upon an appropriate change of parameters. Firstly, change the representation of ω: write $\omega = \omega_0 + \lambda_1\omega_1 + \lambda_2\omega_2$, where $\omega_0 = \nu + \lambda_{10}\omega_1 + \lambda_{02}\omega_2$ is chosen with zero a-periods, as in the proof of Proposition 1.2.3, and assume that (ω_1, ω_2) is a normalized basis of differentials of first kind on C. Secondly, replace the 2 parameters $\tilde{q}_1, \tilde{q}_2$ by the coordinates $z_1[\mathcal{L}], z_2[\mathcal{L}]$ of the class of $L = \mathcal{O}(\tilde{q}_1 + \tilde{q}_2 - \infty_+ - \infty_-)$ in JC. Thirdly, replace (t, t') by the period Z of C. Then an isomonodromic variety $N_i = \text{const}$ $(i = 1, \dots, 4)$ is defined, in the above parameters, by the equations

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \text{const}, \begin{pmatrix} z_1[\mathcal{L}] \\ z_2[\mathcal{L}] \end{pmatrix} + Z\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \text{const}. \tag{1.24}$$

Thus the isomonodromy varieties can be considered as surfaces in the 4-dimensional relative Jacobian $J(\mathcal{C}/\mathcal{H})$ of the universal family of bielliptic curves $\mathcal{C}{\to}\mathcal{H}$ over the bielliptic period locus $\mathcal{H}$ introduced in Corollary 1.1.6. The fiber C_Z of $\mathcal{C}$ over a point $Z \in \mathcal{H}$ is a genus-2 curve with period Z, and $J(\mathcal{C}/\mathcal{H}){\to}\mathcal{H}$ is the family of the Jacobians of all the curves C_Z as Z runs over $\mathcal{H}$. The isomonodromy surfaces $S_{\lambda_1,\lambda_2,\mu_1,\mu_2}$ in $J(\mathcal{C}/\mathcal{H})$ depend on 4 parameters λ_i, μ_i. Every isomonodromy surface is a cross-section of the projection $J(\mathcal{C}/\mathcal{H}){\to}\mathcal{H}$ defined by

$$S_{\lambda_1,\lambda_2,\mu_1,\mu_2} = \left\{(Z, [\mathcal{L}]) \mid Z \in \mathcal{H},\ [\mathcal{L}] \in JC_Z,\ \begin{pmatrix} z_1[\mathcal{L}] \\ z_2[\mathcal{L}] \end{pmatrix} = -Z\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}\right\}.$$

1.5 ELEMENTARY TRANFORMS OF RANK-2 VECTOR BUNDLES

In this section, we will recall basic facts on elementary transforms of vector bundles in the particular case of rank 2, the only one needed for application to the underlying vector bundles of the direct image connection in the next section. The ubiquity of the elementary transforms is twofold. First, they provide a tool of identification of vector bundles. If we are given a vector bundle $\mathcal{E}$ and if we manage to find a sequence of elementary transforms which connect $\mathcal{E}$ to some “easy" vector bundle $\mathcal{E}_0$ (like $\mathcal{O} \oplus \mathcal{O}(-p)$ for a point p), we provide an explicit construction of $\mathcal{E}$ and at the same time we determine, or identify $\mathcal{E}$ via this construction. Second, the elementary transforms permit to change the vector bundle endowed with a connection without changing the monodromy of the connection. The importance of such applications is illustrated in the article [EV-2], in which the authors prove that any irreducible representation of the fundamental group of a Riemann surface with punctures can be realized by a logarithmic connection on a *semistable* vector bundle of degree 0 (see Theorem 1.6.12). On one hand, this is a far-reaching generalization of Bolibruch's result [AB] which affirms the solvability of the Riemann-Hilbert problem over the Riemann sphere with punctures, and on the other hand, this theorem gives rise to a map from the moduli space of connections to the moduli space of vector bundles, for only the class of *semistable* vector bundles has a consistent moduli theory. We will illustrate this feature of elementary transforms allowing us to roll between stable, semistable and unstable bundles in the next section.

Let E be a curve. As before, we identify locally free sheaves on E with associated vector bundles. Let $\mathcal{E}$ be a rank-2 vector bundle on E, p a point of E, $\mathcal{E}_{|p} = \mathcal{E} \otimes \mathbb{C}_p$ the fiber of $\mathcal{E}$ at p. Here $\mathbb{C}_p$ is the sky-scraper sheaf whose only nonzero stalk is the stalk at p, equal to the 1-dimensional vector space $\mathbb{C}$. We emphasize that $\mathcal{E}_{|p}$ is a $\mathbb{C}$-vector space of dimension 2, not to be confused with the stalk $\mathcal{E}_p$ of $\mathcal{E}$ at p, the latter being a free $\mathcal{O}_p$-module of rank 2. Let e_1, e_2 be a basis of $\mathcal{E}_{|p}$. We extend e_1, e_2 to sections of $\mathcal{E}$ in a neighborhood of p, keeping for them the same notation. We define the elementary transforms $\mathcal{E}^+$ and $\mathcal{E}^-$ of $\mathcal{E}$ as subsheaves of $\mathcal{E} \otimes \mathbb{C}(E) \simeq \mathbb{C}(E)^2$, in giving their stalks at all the points of E:

$$\begin{gathered}\mathcal{E}^- = elm^-_{p,e_2}(\mathcal{E}),\ \mathcal{E}^-_p = \mathcal{O}_p \tau_P e_1 + \mathcal{O}_p e_2,\\ \mathcal{E}^+ = elm^+_{p,e_1}(\mathcal{E}),\ \mathcal{E}^+_p = \mathcal{O}_p \frac{1}{\tau_P} e_1 + \mathcal{O}_p e_2,\\ \mathcal{E}^\pm_z = \mathcal{E}_z,\ \ \forall\ z \in E \setminus \{p\},\end{gathered} \tag{1.25}$$

where τ_P denotes a local parameter at p. The thus obtained sheaves are locally free of rank 2. They fit into the exact triples:

$$0 \to \mathcal{E}^- \to \mathcal{E} \xrightarrow{\gamma} \mathbb{C}_p \to 0\ , \quad 0 \to \mathcal{E} \to \mathcal{E}^+ \to \mathbb{C}_p \to 0. \tag{1.26}$$

Remark, that the surjection γ restricted to $\mathcal{E}_{|p}$ is a projection parallel to the e_2 axis; this is the reason for which we included in the notation of elm^- its dependence on e_2. Thus, if we vary e_1, in keeping e_2 (or in keeping the proportionality class of e_2), the isomorphism class of $elm^-_{e_2}$ will not change, but it can change if we vary the proportionality class $[e_2]$ in the projective line $\mathbb{P}(\mathcal{E}_{|p})$.

For degrees, we have $\deg \mathcal{E}^{\pm} = \deg \mathcal{E} \pm 1$. We can give a more precise version of this equality in terms of the determinant line bundles: $\det \mathcal{E}^{\pm} = \det \mathcal{E}(\pm p)$. Here and further on, given a line bundle $\mathcal{L}$ and a divisor $D = \sum n_i p_i$ on E, we denote by $\mathcal{L}(D)$ ("$\mathcal{L}$ twisted by D") the following line bundle, defined as a sheaf by its stalks at all the points of E: $\mathcal{L}(D)_z = \mathcal{L}_z$ if z is not among the p_i, and $\mathcal{L}(D)_{p_i} = \tau_{p_i}^{-n_i}\mathcal{L}_z$. For example, the regular sections of $\mathcal{L}(p)$ can be viewed as meromorphic sections of $\mathcal{L}$ with at most simple pole at p, whilst the regular sections of $\mathcal{L}(-p)$ are regular sections of $\mathcal{L}$ vanishing at p. For the degree of a twist, we have $\deg \mathcal{L}(D) = \deg \mathcal{L} + \deg D = \deg \mathcal{L} + \sum n_i$, so that $\deg \mathcal{L}(\pm p) = \deg \mathcal{L} \pm 1$.

A similar notion of twists applies to higher-rank bundles $\mathcal{E}$: the twist $\mathcal{E}(D)$ can be defined either as $\mathcal{E} \otimes \mathcal{O}(D)$, or via the stalks in replacing $\mathcal{L}$ by $\mathcal{E}$ in the above definition. For degrees, we have $\deg \mathcal{E}(D) = \deg \mathcal{E} + \mathrm{rk}\, \mathcal{E} \cdot \deg D$. Coming back to $\mathrm{rk}\, \mathcal{E} = 2$ and twisting $\mathcal{E}$ by $\pm p$, we obtain some more exact triples:

$$0 \to \mathcal{E}^{+} \to \mathcal{E}(p) \to \mathbb{C}_p \to 0, \qquad 0 \to \mathcal{E}(-p) \to \mathcal{E}^{-} \to \mathbb{C}_p \to 0.$$

They are easily defined via stalks, as $\mathcal{E}(p) = \mathcal{E} \otimes \mathcal{O}_E(p)$ is spanned by $\frac{1}{\tau_P}e_1, \frac{1}{\tau_P}e_2$ at p, and $\mathcal{E}(-p)$ by $\tau_P e_1, \tau_P e_2$.

A basis-free description of elms can be given as follows: Let $W \subset \mathcal{E}_{|p}$ be a 1-dimensional vector subspace. Then $elm^-_{(p,W)}(\mathcal{E})$ is defined as the kernel of the composition of natural maps $\mathcal{E} \to \mathcal{E}_{|p} \to \mathcal{E}_{|p}/W$ (here $\mathcal{E}_{|p}$, $\mathcal{E}_{|p}/W$ are considered as sky-scraper sheaves, i. e. vector spaces placed at p). The positive elm is defined via the duality:

$$elm^+_{p,W}(\mathcal{E}) := (elm^-_{p,W^\perp}(\mathcal{E}^\vee))^\vee.$$

To set a correspondence with the previous notation, we write:

$$elm^+_{p,e_1}(\mathcal{E}) = elm^+_{p,\mathbb{C}e_1}(\mathcal{E}),\ elm^-_{p,e_2}(\mathcal{E}) = elm^-_{p,\mathbb{C}e_2}(\mathcal{E}).$$

One can also define elm^+ as an appropriate elm^-, applied not to $\mathcal{E}$, but to $\mathcal{E}(p)$:

$$elm^+_{p,e_1} = elm^-_{p,e_1}(\mathcal{E}(p)). \tag{1.27}$$

We will now interprete the elementary transforms in terms of ruled surfaces. For a vector bundle $\mathcal{E}$ over E, we denote by $\mathbb{P}(\mathcal{E})$ the projectivization of $\mathcal{E}$, whose fiber over $z \in E$ is the projective line $\mathbb{P}(\mathcal{E}|_z)$ parameterizing vector lines in $\mathcal{E}_{|z}$. It has a natural projection $\mathbb{P}(\mathcal{E}) \to E$ with fibers isomorphic to $\mathbb{P}^1$ and is therefore called a ruled surface. We will see that the elementary transforms of vector bundles correspond to birational maps between associated ruled surfaces which split into the composition of one blowup and one blowdown. The transfer to ruled surfaces allows us to better understand the structure of $\mathcal{E}$, for it replaces all the line subbundles $\mathcal{L} \subset \mathcal{E}$ by cross-sections of the fiber bundle $\mathbb{P}(\mathcal{E}) \to E$; the latter cross-sections being curves in a surface, we can use the intersection theory on the surface to study them. As an example, we will give a criterion of (semi)stability of $\mathcal{E}$ in terms of the intersection theory on $\mathbb{P}(\mathcal{E})$.

Let us return to the setting of the description of elms via bases. We can assume that e_1, e_2 are rational sections of $\mathcal{E}$, regular and linearly independent at p. Let $S = \mathbb{P}(\mathcal{E})$ and let $\pi : S \longrightarrow E$ be the natural projection. Then e_1, e_2 define two global cross-sections of π, which will be denoted $\overline{e_1}, \overline{e_2}$. If $\mathcal{E}^- = elm^-_{p,e_2}(\mathcal{E})$, then the natural map $\mathcal{E}^- \longrightarrow \mathcal{E}$ gives rise to the birational isomorphism of ruled surfaces $S \longrightarrow S^- = \mathbb{P}(\mathcal{E}^-)$ which splits into the composition of one blowup and one blowdown, as shown on Fig. 1.7.

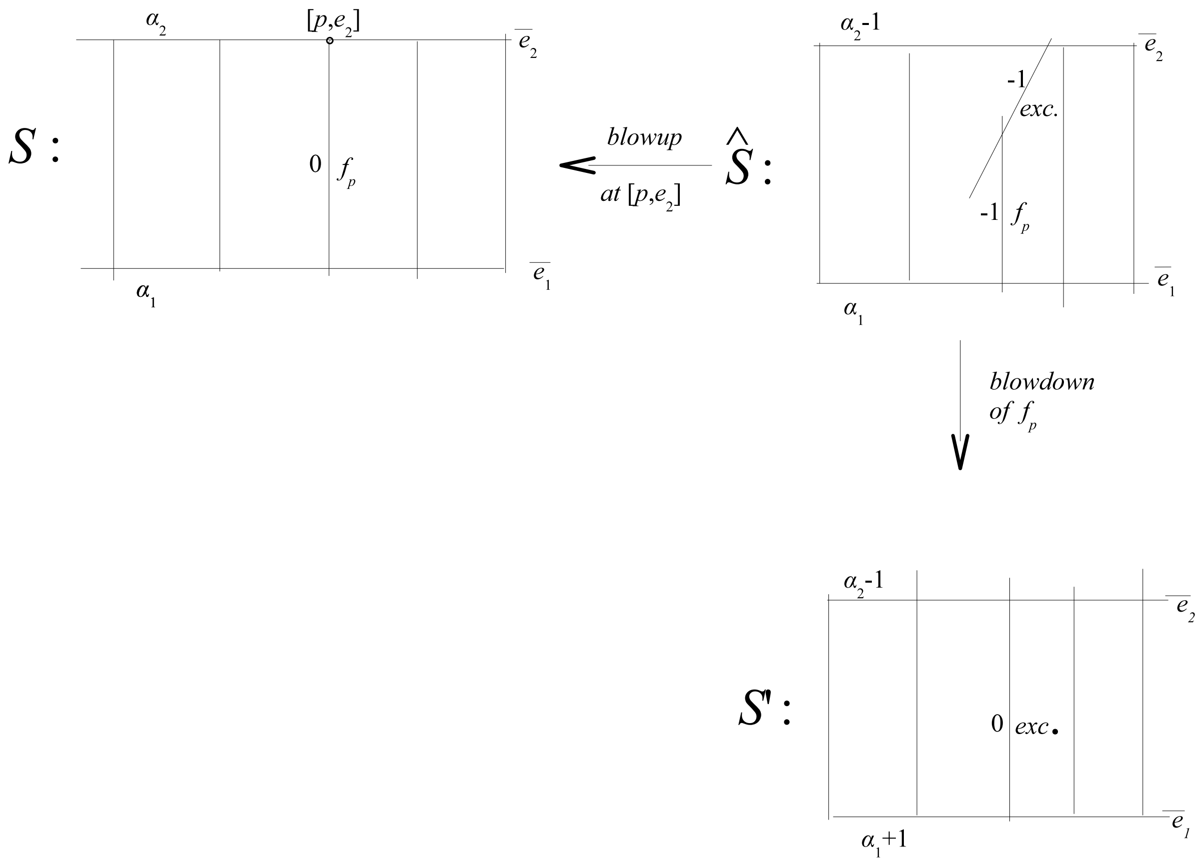

Figure 1.7: Decomposition of elm^- in a blowup followed by a blowdown.

Let f_p denote the fiber $\pi^{-1}(p) \simeq \mathbb{P}^1$ of π; we keep the same notation for curves and their proper transforms in birational surfaces. We label some of the curves by their self-intersection; for example, $(\overline{e_1}^2)_S = \alpha_1, (f_p^2)_S = 0, (f_p^2)_{\hat{S}} = -1$. For any vector $v \in \mathcal{E}_{|_p} \setminus \{0\}$, we denote by $[p, v]$ the point of $f_p = \mathbb{P}(\mathcal{E}_{|_p})$ which is the vector line spanned by v. Remark that the cross-sections $\overline{e_1}, \overline{e_2}$ are disjoint in the neighborhood of p where e_1, e_2 is a basis of $\mathcal{E}$, but $\overline{e_1}$ can intersect $\overline{e_2}$ at a finite number of points where e_1, e_2 fail to generate $\mathcal{E}$.

The positive elm has a similar description. Basically, as $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ for any invertible sheaf $\mathcal{L}$ on E, we have $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{E}(p))$. Hence, in view of (1.27), elm^+ and elm^- have the same representation on the level of ruled surfaces. There exists also an elegant way to define elm^- in using $\pi : S \longrightarrow E$:

$$elm^-_{p,v}(\mathcal{E}) = \pi_*(I_{S,[p,v]}(1)).$$

Here, $I_{S,[p,v]}$ is the ideal sheaf of the point $[p,v]$, and $F(1)$ denotes the twist of a sheaf F by $\mathcal{O}_{\mathbb{P}(\mathcal{E})/E}(1)$. We have the natural exact triple of an ideal sheaf on $S = \mathbb{P}(\mathcal{E})$:

$$0 \rightarrow I_{S,[p,v]}(1) \rightarrow \mathcal{O}_{S/E}(1) \rightarrow \mathbb{C}_{[p,v]} \rightarrow 0.$$

By a basic property of the tautological sheaf $\mathcal{O}_{S/E}(1)$, we have $\pi_*\mathcal{O}_{S/E}(1) \simeq \mathcal{E}$. By applying π_*, we get the exact triple

$$0 \rightarrow \pi_* I_{S,[p,v]}(1) \rightarrow \mathcal{E} \rightarrow \mathbb{C}_p \rightarrow 0.$$

One can prove that in this way we recover the first exact triple (1.26).

Now, we will say a few words about the (semi)-stability in terms of ruled surfaces.

Definition 1.5.1. A rank-2 vector bundle on a curve E is stable (resp. semistable) if for any line subbundle $\mathcal{L} \subset \mathcal{E}$, $\deg \mathcal{L} < \frac{1}{2} \deg \mathcal{E}$ (resp., $\deg \mathcal{L} \leq \frac{1}{2} \deg \mathcal{E}$), or equivalently, if for any surjection $\mathcal{E} \rightarrow \mathcal{M}$ onto a line bundle $\mathcal{M}$, $\deg \mathcal{M} > \frac{1}{2} \deg \mathcal{E}$ (resp., $\deg \mathcal{M} \geq \frac{1}{2} \deg \mathcal{E}$). A vector bundle is called unstable if it is not semistable. It is called strictly semistable if it is semistable, but not stable.

Definition 1.5.2. Let $\mathcal{E}$ be a rank-2 vector bundle on a curve E. The index of the ruled surface $\pi : S = \mathbb{P}(\mathcal{E}) \longrightarrow E$ is the minimal self-intersection number of a cross-section of π:

$$i(S) = \min\{(e)^2_S \mid e \subset S \text{ is a cross-section of } \pi\}.$$

The assertion of the following proposition is well-known, see e. g. [LN], p. 55. For the reader's convenience, we provide a short proof of it.

Proposition 1.5.3. *$\mathcal{E}$ is stable (resp. semi-stable) iff $i(S) > 0$ (resp $i(S) \geq 0$).*

Proof. The cross-sections of $\mathbb{P}(\mathcal{E}) \longrightarrow E$ are in 1-to-1 correspondence with the exact triples

$$0 \rightarrow \mathcal{L}_1 \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{L}_2 \rightarrow 0, \tag{1.28}$$

where $\mathcal{L}_1, \mathcal{L}_2$ are line bundles over E. The cross-section e associated to such a triple is $\mathbb{P}(\mathcal{L}_1) \subset \mathbb{P}(\mathcal{E})$. It is the zero locus of $\pi^*\beta \circ \pi^*\alpha \in Hom(\pi^*\mathcal{L}_1, \pi^*\mathcal{L}_2) \simeq H^0(S, \pi^*(\mathcal{L}_2 \otimes \mathcal{L}_1^{-1}))$. Hence, the normal bundle $N_{e/S}$ is isomorphic to $\mathcal{L}_2 \otimes \mathcal{L}_1^{-1}$. The stability (resp. semi-stability) of $\mathcal{E}$ is equivalent to the fact that $\deg \mathcal{L}_1 < \deg \mathcal{L}_2$ (resp $\deg \mathcal{L}_1 \leq \deg \mathcal{L}_2$) for any triple (1.28). As $(e^2)_S = \deg N_{e/S} = \deg \mathcal{L}_2 - \deg \mathcal{L}_1$, this ends the proof. □

We will end this section by two lemmas which help to identify vector bundles via the geometry of the associated ruled surfaces.

Lemma 1.5.4. *Let $\mathcal{E}$ be a rank-2 vector bundles over a curve X such that the associated ruled surface $S = \mathbb{P}(\mathcal{E})$ has two disjoint cross-sections s_1, s_2. Then $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_i$ are line subbundles of $\mathcal{E}$ corresponding to s_i: $s_i = \mathbb{P}(\mathcal{L}_i)$, $i = 1, 2$. Further, for the self-intersection numbers of s_i, we have $(s_1)^2 = -(s_2)^2 = \deg \mathcal{L}_2 - \deg \mathcal{L}_1$.*

Proof. The first assertion is obvious, and the second one follows from the formula for $(e^2)_S$ in the proof of Proposition 1.5.3, in taking into account that $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ fits into an exact triple of the form (1.28). $\square$

Lemma 1.5.5. *Let $\mathcal{E}$ be a rank-2 vector bundle over a curve X, $\mathcal{L}$ a line subbundle of $\mathcal{E}$ and $s = \mathbb{P}(\mathcal{L})$ the associated cross-section of the ruled surface $S = \mathbb{P}(\mathcal{E})$. Let $p \in X$, $[p, v] \in f_p$, where f_p denotes the fiber of S over p. Let $S^\pm = \mathbb{P}(\mathcal{E}^\pm)$, where $\mathcal{E}^\pm = elm^\pm_{p,v}$, $\pi^\pm : S \dashrightarrow S^\pm$ the natural birational map, $s^\pm$ the proper transform of s in $S^\pm$ under $\pi^\pm$ (that is, the closure of $\pi^\pm(s \setminus \{[p, v]\})$), and $\mathcal{L}^\pm$ the line subbundle of $\mathcal{E}^\pm$ such that $s^\pm = \mathbb{P}(\mathcal{L}^\pm)$. Then we have:*

(i) If $[p, v] \in s$, then $(s^\pm)^2_{S^\pm} = (s^2)_S - 1$, $\deg \mathcal{L}^+ = \deg \mathcal{L} + 1$, and $\deg \mathcal{L}^- = \deg \mathcal{L}$. Moreover, $\mathcal{L}^+ \simeq \mathcal{L}(p)$ and $\mathcal{L}^- \simeq \mathcal{L}$.

(ii) If $[p, v] \notin s$, then $(s^\pm)^2_{S^\pm} = (s^2)_S + 1$, $\deg \mathcal{L}^+ = \deg \mathcal{L}$, and $\deg \mathcal{L}^- = \deg \mathcal{L} - 1$. Moreover, $\mathcal{L}^+ \simeq \mathcal{L}$ and $\mathcal{L}^- \simeq \mathcal{L}(-p)$.

Proof. The formulas for $(s^\pm)^2_{S^\pm}$ follow from the behavior of the intersection indices as shown on Fig. 1.7, and those for $\deg \mathcal{L}^\pm$ are easily deduced directly from the definition of elementary transforms (1.25) by choosing for e_1 or e_2 a rational trivialization of $\mathcal{L}$. $\square$

1.6 UNDERLYING VECTOR BUNDLES OF DIRECT IMAGE CONNECTION

Let us go over again to the setting of Sect. 1.3. Consider first the case when $\mathcal{L}$ is the trivial bundle, $\mathcal{L} = \mathcal{O}_C$. The following fact is well known:

Lemma 1.6.1. *Let $f : X \to Y$ be a finite morphism of smooth varieties of degree 2 and Δ the class of its branch divisor in $Pic(Y)$. Then Δ is divisible by two in $Pic(Y)$, and there exists $\delta \subset Pic(Y)$ such that 2δ is linearly equivalent to Δ and $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-\delta)$.*

Proof. See [Mu], Sect. 1. $\square$

Applying this lemma to $f : C \to E$, we find that $f_*\mathcal{O}_C = \mathcal{O}_E \oplus \mathcal{O}_E(-\delta)$, where $2\delta \simeq p_+ + p_-$. This property determines δ only modulo $E[2]$, but as we saw in Sect. 1.3, $\mathcal{O}_E(-\delta)$ is trivialized by a section ξ over $E \setminus \{\infty\}$, thus $\delta = \infty$ and $f_*\mathcal{O}_C = \mathcal{O}_E \oplus \mathcal{O}_E(-\infty)$. We deduce:

Proposition 1.6.2. *If $\mathcal{L} = \mathcal{O}_C$, then the direct image connection $\nabla_\mathcal{E} = f_*(\nabla_\mathcal{L})$, determined by formula (1.16), is a logarithmic connection on the vector bundle $\mathcal{E}_0 = \mathcal{O}_E \oplus \mathcal{O}_E(-\infty)$ with two poles at p_+, p_-.*

Let now $\mathcal{L}$ be an arbitrary line bundle over C of degree 0. By continuity, $\deg f_*\mathcal{L} = \deg f_*\mathcal{O}_C = -1$. To determine $f_*\mathcal{L}$, we use the following lemma:

Lemma 1.6.3. *Let X be a nonsingular curve, p a point in X, and z a local parameter at p. Let $\mathcal{E}$ be a rank-2 vector bundle on X with a meromorphic connection ∇, regular at p. Let s_1, s_2 be a pair of meromorphic sections of $\mathcal{E}$, linearly independent over $\mathbb{C}(X)$ and $\mathcal{F} =< s_1, s_2 >$ the subsheaf of $\mathcal{E} \otimes \mathbb{C}(X)$ generated by s_1, s_2 as a $\mathcal{O}_X$-module. Let $\mathcal{A}$ be the matrix of ∇ with respect to the $\mathbb{C}(X)$-basis s_1, s_2 and $A = res_{z=0}\mathcal{A}$. Assume that $\mathcal{E}_{|p}$ has a basis v_1, v_2 consisting of eigenvectors of A. Then v_1, v_2 extend to a basis of the stalk $\mathcal{E}_p$, the corresponding eigenvalues n_1, n_2 of A are integers and we have the following relations between the stalks of subsheaves of $\mathcal{E} \otimes \mathbb{C}(X)$ at p:*

$$\begin{array}{l} \mathcal{E}_p =< v_1, v_2 >,\ \mathcal{F}_p =< z^{n_1}v_1, z^{n_2}v_2 >, \\ \textit{if } n_1 = 1, n_2 = 0,\ \mathcal{E}_p = elm^+_{p,v_1}(F_p), \\ \textit{if } n_1 = -1, n_2 = 0,\ \mathcal{E}_p = elm^-_{p,v_2}(F_p), \\ \textit{if } n_1 = n_2,\ \mathcal{E}_p = ((F(n_1))_p. \end{array}$$

Proof. Straightforward. $\square$

Let us apply this lemma to the connection $\nabla_{\mathcal{E}}$, given by formula (1.16) in the basis $(1, \xi)$. We have: $\mathcal{E}_p \neq \mathcal{F}_p \iff p \in \{q_1, q_2, \infty\}$,

$$A_i = \operatorname{Res}_{q_i}\mathcal{A} = \begin{pmatrix} \frac{1}{2} & \frac{\xi_1}{2} \\ \frac{1}{2\xi_1} & \frac{1}{2} \end{pmatrix} (i = 1, 2),\ A_\infty = \operatorname{Res}_\infty \mathcal{A} = \begin{pmatrix} -1 & -\frac{\xi_1+\xi_2}{2} \\ 0 & -2 \end{pmatrix}.$$

We list the eigenvectors $v_j^{(i)}, v_j^{(\infty)}$ together with the respective eigenvalues for the matrices $A_i,\ A_\infty$:

$$v_1^{(i)} = \begin{pmatrix} -\xi_i \\ 1 \end{pmatrix}, \eta_1^i = 0,\ v_2^{(i)} = \begin{pmatrix} \xi_i \\ -1 \end{pmatrix}, \eta_2^i = -1,\ v_1^{(\infty)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \eta_1^\infty = 1,$$

$$v_2^{(\infty)} = \begin{pmatrix} -\frac{\xi_1+\xi_2}{2} \\ 1 \end{pmatrix}, \eta_2^\infty = -2.$$

Applying Lemma 1.6.3 (twice at ∞), we obtain the following corollary:

Corollary 1.6.4. *Let $\mathcal{L} = \mathcal{O}_C(\tilde{q}_1 + \tilde{q}_2 - \infty_+ - \infty_-)$, $\tilde{q}_i = (\xi_i, y_i)$, $q_i = f(\tilde{q}_i)$, $i = 1, 2$, as in Proposition 1.3.3, and let $v_j^{(i)}$ be the eigenvectors of A_i as above. Then*

$$\mathcal{E} = elm^+_{q_1, v_2^{(1)}} elm^+_{q_2, v_2^{(2)}}(\mathcal{E}_0(-\infty)).$$

Remark 1.6.5. Note that though the sheaf-theoretic direct image $f_*\mathcal{L}$ does not depend on the choice of a connection $\nabla_{\mathcal{L}}$ on $\mathcal{L}$, our method of computation of $f_*\mathcal{L}$, given by Corollary 1.6.4, uses the direct image connection $\nabla_{\mathcal{E}} = f_*\nabla_{\mathcal{L}}$ for some $\nabla_{\mathcal{L}}$.

Proposition 1.6.6. *For generic $\mathcal{L} \in Pic(C)$, the rank-2 vector bundle $\mathcal{E}$ is stable.*

Proof. Starting from the ruled surface $S_0 = \mathbb{P}(\mathcal{E}_0) = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-\infty))$, we apply two elementary transforms $S_0 \to S_1 \to S_2 = \mathbb{P}(\mathcal{E})$, and we have to prove that any cross-section of S_2 has strictly positive self-intersection, provided that $\tilde{q}_i = (\xi_i, y_i)$ are sufficiently generic. For a rational section s of $\mathcal{E}_0$, let us denote by $\overline{s}$ the associated cross-section of S_0. S_0 is characterized by the existence of two distinguished sections $\overline{s}_1, \overline{s}_2$ associated to $s_1 = 1, s_2 = \xi$ with self-intersections $\overline{s}_1^2 = -1, \overline{s}_2^2 = 1$, and we have the relations $\overline{s}_1 \overline{s}_2 = 0$, $\overline{s}_2 \sim \overline{s}_1 + f_\infty$, where $f_p = \pi^{-1}(p)$ is the fiber of the structure projection $\pi : S_0 \longrightarrow E$. When there is no risk of confusion, we will keep the same notation for curves and their proper transforms in birational surfaces. Any cross-section $\overline{s}$ is linearly equivalent to $\overline{s}_1 + f_{p_1} + \ldots + f_{p_r}$ for some points $p_1, \ldots, p_r$ in E, and $\overline{s}^2 = 2r - 1$. In particular, $i(S_0) = -1$, attained on $\overline{s}_1$. Remark that $\overline{s}_1$ is rigid, whilst $\overline{s}_2$ moves in a pencil $|\overline{s}_1 + f_\infty|$. Let us apply $elm^+_{q_1, v_2^{(1)}}$. First, we blow up $P_1 = [q_1, v_2^{(1)}]$. Let $\overline{e}_1$ be the corresponding (-1)-curve and $\hat{S}_0$ the blown up surface. For the self-intersection numbers of the cross-sections, we have the following relations: $(\overline{s}^2)_{\hat{S}_0} = (\overline{s}^2)_{S_0}$ if $P_1 \notin \overline{s}$ and $(\overline{s}^2)_{\hat{S}_0} = (\overline{s}^2)_{S_0} - 1$ if $P_1 \in \overline{s}$. Hence, $\hat{S}_0$ has only one cross-section for each one of the self-intersection numbers -1, 0, and $(\overline{s}^2)_{S_0} \geq 1$ for all the other cross-sections. The cross-section with self-intersection -1 is $\overline{s}_1$ and the one with self-intersection 0 is the proper transform of the unique member $\overline{s}_{P_1}$ of the pencil $|\ \overline{s}_1 + f_\infty\ |$ on S_0 going through P_1, see Fig. 1.8. The next step is the blowdown of $f_{q_1} \subset \hat{S}_0$. The self-intersection number of all the cross-sections of $\hat{S}_0 \to E$ that meet f_{q_1} goes up by 1. We conclude that $S_1 = \mathbb{P}(elm^+_{q_1, v_2^1}(\mathcal{E}_0))$ has two cross-sections $\overline{s}_{P_1}, \overline{s}_1$ with square 0, and $(\overline{s}^2)_{S_1} \geq 2$ for any other cross-section of S_1. In the language of vector bundles, this means that $\mathcal{E}_1$ is the direct sum of two line bundles of degree 0. More precisely, $\mathcal{E}_1 = \mathcal{O}_E \oplus \mathcal{O}_E(q_1 - \infty)$ by Lemma 1.5.4, the first summand corresponding to $\overline{s}_1$ and the second one to $\overline{s}_{P_1}$.

The second elementary transform is performed at $P_2 \in S_1$. As $P_2 \notin \overline{s}_{P_1} \cup \overline{s}_1$, the minimal self-intersection number of a cross-section in S_1 passing through P_2 is 2. The elementary transform decreases by 1 the self-intersection of such cross-sections and increases by 1 the self-intersection of all other cross-sections (Lemma 1.5.5). Hence, $i(S_2) = 1$, the value attained on many cross-sections, for example, $\overline{s}_{P_2}, \overline{s}_{P_1}, \overline{s}_1$. This ends the proof. □

Theorem 1.6.7 (Atiyah, [At-2]). *For any line bundle $\mathcal{N}$ of odd degree over an elliptic curve E, there exists one and only one stable rank-2 vector bundle on E with determinant $\mathcal{N}$.*

Using Atiyah's theorem in our case, we have $\deg \mathcal{N} = -1$, so that $\mathcal{N}$ can be represented in the form $\mathcal{N} = \mathcal{O}_E(-q)$ for some $q \in E$. $\mathcal{E}$ is obtained as the unique non-trivial extension of vector bundles:

$$0 \to \mathcal{O}_E(-q) \to \mathcal{E} \to \mathcal{O}_E \to 0.$$

Moreover, the correspondence $\mathcal{E} \leftrightarrow q$ identifies the moduli space $\mathcal{M}^s_E(2, -1)$ of rank-2 stable vector bundles of degree -1 over E with E itself. We deduce:

Corollary 1.6.8. *Under the above identification $\mathcal{M}^s_E(2, -1) \simeq E$, the rational map:*

$$f : JC \dashrightarrow \mathcal{M}^s_E(2, -1)$$

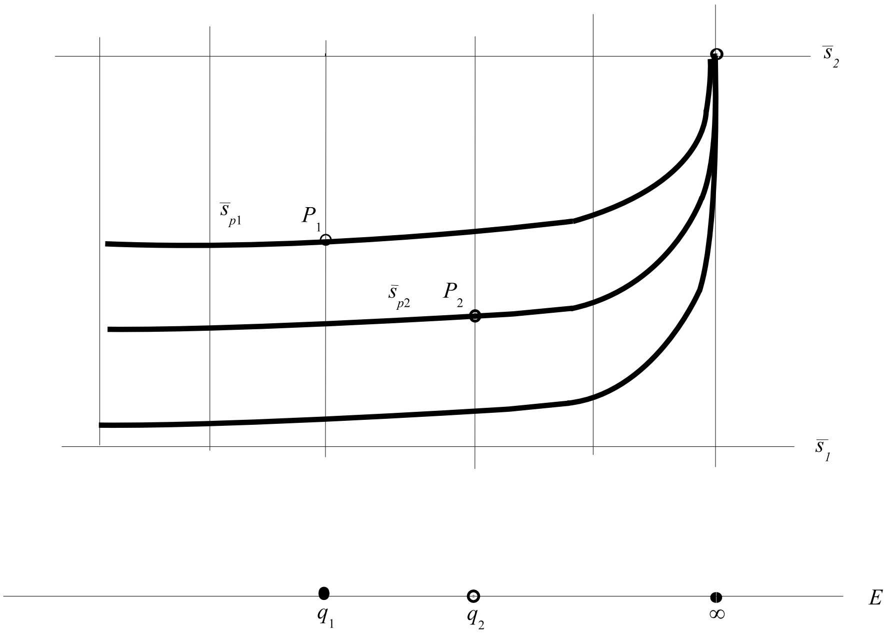

Figure 1.8: The ruled surface S_0. The pencil $|\ \bar{s}_1 + f_\infty\ |$ has a unique member passing through P_i for each $i = 1, 2$.

$$\mathcal{L} = \mathcal{O}_C(\tilde{q}_1 + \tilde{q}_2 - \infty_+ - \infty_-) \quad \mapsto \quad f_*(\mathcal{L})$$

can be given by

$$[\tilde{q}_1 + \tilde{q}_2 - \infty_+ - \infty_-] \mapsto [q_1 + q_2 - 2\infty].$$

Now we go over to the nongeneric line bundles $\mathcal{L}$. The direct image $f_*\mathcal{L}$ can be unstable for special $\mathcal{L}$. This may happen when either the argument of Prop. 1.6.6 does not work anymore, or when formulas (1.16)-(1.19) are not valid. We list the cases which need a separate analysis in the next proposition.

Proposition 1.6.9. *Let $\mathcal{L} = \mathcal{O}_C(\tilde{q}_1 + \tilde{q}_2 - \infty_+ - \infty_-)$, $\mathcal{E} = f_*(\mathcal{L})$, $\mathcal{E}_0 = f_*\mathcal{O}_C$, as above. Whenever $\tilde{q}_i$ is finite, it will be represented by its coordinates: $\tilde{q}_i = (\xi_i, y_i)$. The following assertions hold:*

(a) If $\tilde{q}_1 + \tilde{q}_2$ is a divisor in the hyperelliptic linear series $g^1_2(C)$ (that is $\xi_1 = \xi_2$, $y_1 = -y_2$, or $\{\tilde{q}_1, \tilde{q}_2\} = \{\infty_+, \infty_-\}$), then $\mathcal{E} \simeq \mathcal{E}_0$, and hence $\mathcal{E}$ is unstable.

(b) If $\tilde{q}_1 = \tilde{q}_2 \neq \infty_\pm$, then $\mathcal{E} \simeq \mathcal{O}_E(-\infty) \oplus \mathcal{O}_E(2q_1 - 2\infty)$ is unstable.

(c) If $\tilde{q}_i = \infty_\pm$ for at least one value $i \in \{1, 2\}$, then $\mathcal{E} \simeq \mathcal{O}_E(-2\infty + q_{3-i}) \oplus \mathcal{O}_E$ is unstable.

(d) If $\tilde{q}_i = \tilde{p}_\pm$ for exactly one value $i \in \{1, 2\}$, then $\mathcal{E}$ is a stable bundle of degree -1 with $\det \mathcal{E} \simeq \mathcal{O}_E(q_{3-i} + p_\pm - 3\infty)$.

Proof. (a) In this case, $s_{p_1} = s_{p_2}$, $\tilde{q}_1 + \tilde{q}_2 \sim \infty_+ + \infty_-$, then $\mathcal{L} \simeq \mathcal{O}_C$, and $\mathcal{E} \simeq \mathcal{E}_0$.

(b) Let, for example, $i=1$. Then $\mathcal{L}=\mathcal{O}_C(\tilde{q}_1+\tilde{q}_2-\infty_+-\infty_-)$ degenerates to $\mathcal{L}=\mathcal{O}_C(2\tilde{q}_1-\infty_+-\infty_-)$. In this case, the matrix of a regular connection on $\mathcal{L}$ in the rational basis 1 of $\mathcal{L}=\mathcal{O}_C(2\tilde{q}_1-\infty_+-\infty_-)\hookrightarrow\mathcal{O}_C(2\tilde{q}_1)\hookleftarrow\mathcal{O}_C\ni 1$ is a rational 1-form with residues 2 at $\tilde{q}_1$ and -1 at points $\infty_\pm$. Such a 1-form can be written by the same formula $\omega=\frac{1}{2}(\frac{y+y_1}{\xi-\xi_1}+\frac{y+y_2}{\xi-\xi_2})\frac{\mathrm{d}\xi}{y}+\lambda_1\frac{\mathrm{d}\xi}{y}+\lambda_2\frac{\xi\mathrm{d}\xi}{y}$, as in the general case, but now we substitute $\xi_2=\xi_1, y_2=y_1$ in it:

$$\omega=\frac{y+y_1}{\xi-\xi_1}\frac{\mathrm{d}\xi}{y}+\lambda_1\frac{\mathrm{d}\xi}{y}+\lambda_2\frac{\xi\mathrm{d}\xi}{y}.$$

Assume that $\xi_1=\xi_2\neq 0$; the case when $\tilde{q}_1=\tilde{q}_2=\tilde{p}_\pm$ should be treated separately as in (d). Then we get the matrix A of $\nabla_{\mathcal{E}}$ in substituting $\xi_2=\xi_1, y_2=y_1, x_2=x_1$ into formulas (1.16)-(1.19). We obtain the following residues:

$$\operatorname{Res}_{p\pm}A=\begin{pmatrix}0 & 0\\ \frac{1}{2}\frac{(y_1\pm y_0)\xi_1}{(x_1-t')}\pm\frac{\lambda_1}{2y_0} & \frac{1}{2}\end{pmatrix},\ \operatorname{Res}_{q_1}A=\begin{pmatrix}1 & \xi_1\\ \xi_1 & 1\end{pmatrix},$$

$$Res_\infty A=\operatorname{Res}_{u=0}A=\begin{pmatrix}-1 & -\xi_1\\ 0 & -2\end{pmatrix}.$$

As in the proof of Prop. 1.6.6, we can describe $\mathcal{E}$ as the result of two successive positive elm's applied to $\mathcal{E}_0(-\infty)$. In contrast to the general case, considered in Lemma 1.6.3, the second elm has for its center the point $\tilde{P}_1=\overline{s}_{P_1}\cap\tilde{f}_{q_1}\subset S_1$, where $\tilde{f}_{q_1}$ is the fiber of $S_1\to E$ over q_1. As $(\overline{s}_{P_1})^2_{S_1}=0$, the resulting surface S_2 has a cross-section with self-intersection -1, thus $i(S_2)=-1$, and consequently $\mathcal{E}$ is unstable. Applying Lemmas 1.5.4 and 1.5.5, we can identify it with $\mathcal{O}_E(-\infty)\oplus\mathcal{O}_E(2q_1-2\infty)$.

(c) Let, for example, $\tilde{q}_2=\infty_-$. Then $\mathcal{L}$ degenerates to $\mathcal{O}_C(\tilde{q}_1-\infty_+)$, and we can again write the connection in the same way as in the previous case. $\mathcal{E}$ is obtained from $\mathcal{E}_0(-\infty)$ by 2 positive elms. From Lemmas 1.5.4 and 1.5.5, we deduce that $\mathcal{E}\simeq\mathcal{O}_E(-2\infty+q_1)\oplus\mathcal{O}_E$.

(d) One of the points $\tilde{p}_\pm$ collides with $\tilde{q}_i$. This corresponds to $\xi_i=0$. So, we assume that $\xi_2=0$, $\xi_1\neq 0$ $(\tilde{q}_2=\tilde{p}_+)$. Hence, the 1-form of the connection can be written as follows:

$$\omega=\frac{1}{2}\left(\frac{y+y_0}{\xi}+\frac{y+y_1}{\xi-\xi_1}\right)\frac{\mathrm{d}\xi}{y}+\lambda_1\frac{\mathrm{d}\xi}{y}+\lambda_2\frac{\xi\mathrm{d}\xi}{y}.$$

The matrix A is given by

$$A=\begin{pmatrix}-\frac{1}{2}\big(\frac{1}{2}(\frac{y+y_1}{x_1-x}+\frac{y+y_0}{t'-x})+\lambda_2\big)\frac{\mathrm{d}x}{y} & -\frac{1}{2}\big(\frac{1}{2}(\frac{(y+y_1)\xi_1}{x_1-x})+\lambda_1\big)\frac{\mathrm{d}x}{y}\\ -\frac{1}{2}\big(\frac{1}{2}(\frac{(y+y_1)\xi_1}{(x_1-x)(t'-x)})+\frac{\lambda_1}{t'-x}\big)\frac{\mathrm{d}x}{y} & -\frac{1}{2}\big(\frac{1}{2}(\frac{y+y_1}{x_1-x}+\frac{y+y_0}{t'-x})+\lambda_2+\frac{y}{t'-x}\big)\frac{\mathrm{d}x}{y}\end{pmatrix}.$$

Its residues at finite points are:

$$\operatorname{Res}_{p_+}A=\begin{pmatrix}\frac{1}{2} & 0\\ \frac{1}{4}\frac{(y_0+y_1)\xi_1}{(x_1-t')}+\frac{\lambda_1}{2y_0} & 1\end{pmatrix}$$

$$\operatorname{Res}_{p_-} A = \begin{pmatrix} 0 & 0 \\ \frac{1}{4}\frac{(y_1-y_0)\xi_1}{(x_1-t')} - \frac{\lambda_1}{2y_0} & \frac{1}{2} \end{pmatrix}$$

$$\operatorname{Res}_{q_1} A = \begin{pmatrix} \frac{1}{2} & \frac{\xi_1}{2} \\ \frac{1}{2\xi_1} & \frac{1}{2} \end{pmatrix}$$

Here $\mathcal{E}_0 = \mathcal{O}_E \oplus \mathcal{O}_E(-\infty)$, the first elm applied to $\mathcal{E}_0$ gives $\mathcal{E}_1 = \mathcal{O}_E \oplus \mathcal{O}_E(p_+ - \infty)$, and the second one transforms $\mathcal{E}_1$ into a stable vector bundle $\mathcal{E}_2$ which fits into the exact triple

$$0 \to \mathcal{O}_E \to \mathcal{E}_2 \to \mathcal{O}_E(q_1 + p_+ - \infty) \to 0.$$

Thus, the resulting vector bundle $\mathcal{E} = f_*(\mathcal{L}) = \mathcal{E}_2(-\infty)$ behaves exactly as in the general case $(\tilde{p_+} \neq \tilde{q_2})$. $\square$

Next we will discuss Gabber's elementary transforms as defined by Esnault and Viehweg [EV-2]. Gabber's transform of a pair $(\mathcal{E}, \nabla)$, consisting of a vector bundle $\mathcal{E}$ over a curve and a logarithmic connection on $\mathcal{E}$ is another pair $(\mathcal{E}', \nabla')$, where $\mathcal{E}'$ is an elementary transform of $\mathcal{E}$ at some pole p of ∇, and one of the eigenvalues of $\operatorname{Res}_p \nabla'$ differs by 1 from the respective eigenvalue of $\operatorname{Res}_p \nabla$, whilst the other eigenvalues as well as the other residues remain unchanged. We adapt the definition of Esnault–Viehweg to the rank-2 case and to our notation:

Definition 1.6.10. Let $\mathcal{E}$ be a rank-2 vector bundle on a curve X, ∇ a logarithmic connection on $\mathcal{E}$, $p \in X$ a pole of ∇, and $v \in \mathcal{E}|_p$ an eigenvector of the residue $\operatorname{Res}_p(\nabla) \in \operatorname{End}(\mathcal{E}|_p)$. The Gabber transform $elm_{p,v}(\mathcal{E}, \nabla)$ is a pair $(\mathcal{E}', \nabla')$ constructed as follows:

(i) $\mathcal{E}' = elm^+_{p,v}(\mathcal{E})$.

(ii) ∇' is identified with ∇ under the isomorphism $\mathcal{E}|_{X-p} \simeq \mathcal{E}'|_{X-p}$ as a meromorphic connection over $X - p$, and this determines ∇' as a meromorphic connection over X.

By a local computation of ∇' at p one proves:

Lemma 1.6.11. *In the setting of Definition* 1.6.10, *let us complete v to a basis $(e_1 = v, e_2)$ of $\mathcal{E}$ near p, so that $\mathcal{E}'_p = \mathcal{O}_p \cdot \frac{1}{\tau_p} v + \mathcal{O}_p \cdot e_2$ and the matrix R of $\operatorname{Res}_p(\nabla)$ has the form $R = \begin{pmatrix} \lambda_1 & * \\ 0 & \lambda_2 \end{pmatrix}$. Then ∇' is a logarithmic connection on $\mathcal{E}'$ and the matrix R' of its residue at p computed with respect to the basis $(e'_1, e'_2) = (\frac{v}{\tau_p}, e_2)$ of $\mathcal{E}'$ has the form $R' = \begin{pmatrix} \lambda_1 - 1 & 0 \\ * & \lambda_2 \end{pmatrix}$.*

Theorem 1.6.12 (Bolibruch–Esnault–Viehweg [AB], [EV-2])**.** *Let $\mathcal{E}$ be a rank-r vector bundle on a curve X, ∇ a logarithmic connection on $\mathcal{E}$, and assume that the pair $(\mathcal{E}, \nabla)$ is irreducible in the following sense: $\mathcal{E}$ has no ∇-invariant subundles $\mathcal{F} \subset \mathcal{E}$. Then there exists a sequence of Gabber's transforms that replaces $(\mathcal{E}, \nabla)$ by another pair $(\mathcal{E}', \nabla')$, in which $\mathcal{E}'$ is a semistable vector bundle of degree 0 and ∇' is a logarithmic connection on $\mathcal{E}'$ with the same singular points and the same monodromy as ∇.*

We are illustrating this theorem by presenting explicitly one elementary Gabber's transform which transforms our bundle $\mathcal{E} = f_*\mathcal{L}$ of degree -1 into a semistable bundle $\mathcal{E}'$ of degree 0:

Proposition 1.6.13. *Let $\mathcal{E}$, ∇ be as in Proposition 1.6.6. Let v be an eigenvector of $\mathrm{Res}_{p_+}(\nabla)$ with eigenvalue $\frac{1}{2}$ (see formula (1.17)). Then the Gabber transform $(\mathcal{E}', \nabla') = elm^+_{p_+,v}(\mathcal{E}, \nabla)$ satisfies the conclusion of the Bolibruch–Esnault–Viehweg Theorem: $\mathcal{E}'$ is semistable of degree 0 and ∇' is a logarithmic connection with the same singularities and the same monodromy as ∇. Furthermore, $\mathcal{E}' \simeq \mathcal{O}_E(p_+ - \infty) \oplus \mathcal{O}_E(q_1 + q_2 - 2\infty)$.*

Proof. By Corollary 1.6.4, $\mathcal{E}'$ is the result of application of three positive elms to $\mathcal{E}_0(-\infty) = \mathcal{O}_E(-\infty) \oplus \mathcal{O}_E(-2\infty)$:

$$\mathcal{E}' = elm^+_{p_+,v} elm^+_{q_1,v_2^{(1)}} elm^+_{q_2,v_2^{(2)}} (\mathcal{E}_0(-\infty)).$$

The surface $S_0 = \mathbb{P}(\mathcal{E}_0(-\infty))$ can be decomposed as the open subset $S_0 \setminus \bar{s}_1$ (see Fig. 1.8), which is a line bundle over E with zero section $\bar{s}_2$, plus the "infinity section" $\bar{s}_1$. The line bundle is easily identified as the normal bundle to $\bar{s}_2$ in S_0: $S_0 \setminus \bar{s}_1 \simeq \mathcal{N}_{\bar{s}_2/S_0} \simeq \mathcal{O}_E(\infty)$. Then the pencil $|\bar{s}_2| = |\bar{s}_1 + f_\infty|$ is the projective line which naturally decomposes into the affine line $H^0(E, \mathcal{O}(\infty))$ and the infinity point representing the reducible member of the pencil $\bar{s}_1 + f_\infty$ (the curves $\bar{s}_{P_1}, \bar{s}_{P_2}$ shown on Fig. 1.8 are members of this pencil). The fact that all the global sections $s \in H^0(\mathcal{O}_E(\infty))$ come from $H^0(\mathcal{O}_E)$={constants} under the embedding $\mathcal{O}_E \hookrightarrow \mathcal{O}_E(\infty)$ implies that they all vanish at ∞. Thus all the $\bar{s} \in |\bar{s}_2|$ pass through the point $f_\infty \cdot \bar{s}_2$ which is the zero of the fiber of the line bundle $\mathcal{O}_E(\infty)$ over ∞.

Using this representation of S_0, we can prove the existence of a cross-section $\bar{r} \subset S_0, \bar{r} \in |\bar{s}_2 + f_{p_+}|$ passing through the three points $P_0 = [p_+, v]$ and $P_i = [q_i, v_2^{(i)}]$ $(i = 1, 2)$. Namely, the curves from the linear system $|\bar{s}_2 + f_{p_+}|$ are the sections of $\mathcal{O}_E(\infty + p_+)$ considered as sections of $\mathcal{O}_E(\infty)$ having a simple pole at p_+. The fact that they have a simple pole at p_+ means that they meet $\bar{s}_1$ at $f_{p_+} \cdot \bar{s}_1$. The vector space $H^0(\mathcal{O}_E(\infty + p_+))$ is 2-dimensional, so we can find r in it taking the values $v_2^{(1)}, v_2^{(2)}$ at q_1, resp q_2.

We have $\bar{s}_1^2 = -1, \bar{s}_2^2 = 1, \bar{r}^2 = 3, \bar{s}_1.\bar{r} = 1$, and $\bar{s}_1 \cap \bar{r} = P_0$. After we perform the 3 elementary transforms at P_i $(i = 0, 1, 2)$, the self-intersection $\bar{r}^2$ goes down by 3. At the same time $\bar{s}_1^2$ goes up by 2 when making elms P_1, P_2 and descends by 1 after the elm at P_0. Hence in $S' = \mathbb{P}(\mathcal{E}')$, we have two disjoint sections $\bar{r}, \bar{s}_1$ with self-intersection 0. Thus, by Lemma 1.5.4, $\mathcal{E}' = \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1, \mathcal{L}_2$ are line bundles of the same degree. By Lemma 1.5.5, $\deg \mathcal{E}' = \deg \mathcal{E}_0(-\infty) + 3 = 0$, hence $\deg \mathcal{L}_1 = \deg \mathcal{L}_2 = 0$. The direct sum of line bundles of the same degree is strictly semistable.

Next, $\bar{s}_1$ (in S_0) corresponds to the line subbundle $\mathcal{O}_E(-\infty)$. It remains $\mathcal{O}_E(-\infty)$ after elms in P_1, P_2, and becomes $\mathcal{O}_E(p_+ - \infty)$ after the elm in P_0. Hence $\mathcal{L}_1 = \mathcal{O}_E(p_+ - \infty)$ and $\mathcal{L}_2 = \det \mathcal{E}' \otimes \mathcal{L}_1^{-1}$. But $\det \mathcal{E}' = \det \mathcal{E}_0(-\infty)(q_1 + q_2 + p_+) = \mathcal{O}_E(q_1 + q_2 + p_+ - 3\infty)$. Thus $\mathcal{L}_2 = \mathcal{O}_E(q_1 + q_2 - 2\infty)$. □

Remark 1.6.14. If we fix E and let vary p_+, q_1, q_2, then we see that the generic direct sum $\mathcal{L}_1 \oplus \mathcal{L}_2$ of two line bundles of degree 0 occurs as the underlying vector bundle of ∇'. According to [Tu], the moduli space of semistable rank-2 vector bundles on E is isomorphic to the symmetric square $E^{(2)}$ of E, and its open set parameterizes, up to an isomorphism, the direct sums $\mathcal{L}_1 \oplus \mathcal{L}_2$. Thus we obtain a natural map from the parameter space of our direct image connections to the the symmetric square $E^{(2)}$, whilst using the *stable* bundles $\mathcal{E}$ of degree -1 provides a natural map onto E (Corollary 1.6.8).

Remark 1.6.15. Korotkin [Kor-1] considers twisted rank-2 connections on E with connection matrices A satisfying the transformation rule

$$T_a(A) = QAQ^{-1},\ T_b(A) = RAR^{-1} \tag{1.29}$$

for some 2×2 matrices Q, R. In the case when Q, R commute, such a twisted connection can be undertood as an ordinary connection on a nontrivial vector bundle $\mathcal{E}$ over E that can be described as follows: let $E = \mathbb{C}/\Lambda$ where Λ is the period lattice of E with basis $(1, \tau)$, and let z be the flat coordinate on E (or on the universal cover $\mathbb{C}$ of E) such that $T_a(z) = z + 1, T_b(z) = z + \tau$. Let us make Λ act on $\mathbb{C}^2 \times \mathbb{C}$ by the rule

$$(v, z) \overset{a}{\mapsto} (Qv, z + 1),\ (v, z) \overset{b}{\mapsto} (Rv, z + \tau).$$

Then $\mathcal{E} \to E$ is obtained as the quotient $\mathbb{C}^2 \times \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ of the trivial vector bundle $\mathbb{C}^2 \times \mathbb{C} \xrightarrow{pr_2} \mathbb{C}$.

However, the twisted connections obtained in [Kor-1] satisfy (1.29) with non-commuting Q, R, given by Pauli matrices:

$$Q = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad R = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This follows from the relation $A = \mathrm{d}\Psi\Psi^{-1}$, where Ψ is a fundamental matrix of the connection, and from the transformation law for Ψ: $T_a(\Psi) = i\sigma_1\Psi,\ T_b(\Psi) = i\sigma_3\Psi e^{-2i\pi\lambda\sigma_3}$, where $\lambda \in \mathbb{C}$ is a parameter (see (3.74) in loc. cit). Hence Korotkin's connections are really twisted and have no underlying vector bundles. This is a major difference between the result of [Kor-1] and that of the present paper. Another difference, concerning the method, is that the starting point in [Kor-1] is an ad hoc expression for Ψ in terms of Prym theta functions of the double cover $C \to E$, and the connection matrix is implicit.

1.7 MONODROMY AND DIFFERENTIAL GALOIS GROUPS

Let G be the monodromy group of the connection $\nabla_{\mathcal{E}}$ on $f_*\mathcal{L} = \mathcal{E}$ defined by formula (1.16). It is the subgroup of $GL(2, \mathbb{C})$ generated by M_a, M_b, M_{γ_1}. We will first consider the case of generic values of the parameters $(\lambda_1 K, \lambda_1 L, \lambda_2 K', \lambda_2 L')$. Here, *generic* means that the point belongs to the complement of a countable union of affine $\mathbb{Q}$-subspaces of $\mathbb{C}^4$. More exactly, we require that the triples $(i\lambda_2 K', \lambda_2 L', i\pi)$ and $(\lambda_1 K, i\lambda_1 L, \pi)$ are free over $\mathbb{Q}$. Let

$$R^\theta = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}, \quad H^\theta = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \quad (H^{i\theta} = R^\theta).$$

Let N be the normal subgroup of G defined by

$$N = \{X \in G \mid \det X = \pm 1\}. \tag{1.30}$$

We have:

$$N = \{\prod_{i=1}^{r} M_a^{j_i} M_b^{k_i} M_{\gamma_1}^{\epsilon_i} \mid r \geq 0, j_i \in \mathbb{Z}, k_i \in \mathbb{Z}, \epsilon_i \in \{0,1\}, \sum_i k_i = \sum_i j_i = 0\}. \tag{1.31}$$

We can write G as the semi-direct product of N with the subgroup of G generated by M_a, M_b. The latter is identified with $\mathbb{Z} \times \mathbb{Z}$, so $G = N \rtimes (\mathbb{Z} \times \mathbb{Z})$. Let N_1 be the subgroup of N generated by $R^{4\lambda_1 K}, H^{4\lambda_1 L}$. As $M_a = e^{-2i\lambda_2 K'} R^{-2\lambda_1 K}$, and $M_b = e^{2\lambda_2 L'} H^{2\lambda_1 L}$, we have $[M_a, M_{\gamma_1}] = R^{-4\lambda_1 K}, [M_b, M_{\gamma_1}] = H^{4\lambda_1 L}, [M_a, M_b] = 1$. Hence N is the semi-direct product $N = N_1 \rtimes \mu_2$, where $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$ denotes a cyclic group of order n, and the factor μ_2 of the semi-direct product is generated by M_{γ_1}. Finally, we obtain a normal sequence $1 \lhd N_1 \lhd N \lhd G$ with successive quotients $\mathbb{Z} \times \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$, all of whose levels are semi-direct products. We can write:

$$G \simeq ((\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}) \rtimes (\mathbb{Z} \times \mathbb{Z}).$$

We have also $N_1 = D(G)$, the commutator subgroup of G. As $D(G) \simeq \mathbb{Z} \times \mathbb{Z}$ is abelian, G is solvable of height 2.

From now on, we go over to the general case. The formulas (1.30), (1.31) are no more equivalent. Let us define N by (1.31), and N_1 by the same formula with the additional condition $\sum_i \epsilon_i \equiv 0(2)$. We have again the normal sequence $1 \lhd N_1 \lhd N \lhd G$. Its first level is a semidirect product, $N = N_1 \rtimes \mu_2$, but the upper one may be a nonsplit extension. Define two group epimorphisms

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{Z} & \xrightarrow{\varphi_1} & N_1 \\ (n_1, n_2) & \longmapsto & \frac{\sigma(n_1,n_2)^2}{\det \sigma(n_1,n_2)} \end{array} \quad , \quad \begin{array}{ccc} \mathbb{Z} \times \mathbb{Z} & \xrightarrow{\varphi_2} & G/N \\ (n_1, n_2) & \longmapsto & \sigma(n_1,n_2)N \end{array} \quad , \tag{1.32}$$

where $\sigma(n_1, n_2) = M_a^{n_1} M_b^{n_2}$.

Thus both N_1 and G/N are quotients of $\mathbb{Z} \times \mathbb{Z}$. We want to find out, which pairs Q_1, Q_2 of quotients of $\mathbb{Z} \times \mathbb{Z}$ can be realized as the pair N_1, G/N for some monodromy group G. We will denote by π_N the canonical epimorphism $G \rightarrow G/N$, and the maps $\bar{\varphi}_1$, $\bar{\varphi}_2$ are defined by the following commutative diagram:

$$\begin{tikzcd}
& \mathbb{Z} \times \mathbb{Z} \arrow[dl, "\varphi_2"'] \arrow[d, "\sigma"] \arrow[dr, "\varphi_1"] & \\
G/N & <M_a, M_b> \arrow[l, two heads, "\bar{\varphi}_2"'] \arrow[r, two heads, "\bar{\varphi}_1"] \arrow[d, hook] & N_1 \\
& G \arrow[ul, "\pi_N"] &
\end{tikzcd}$$

One can also give $\bar{\varphi}_1$ by the formulas

$$\bar{\varphi}_1(X) = \frac{1}{\det X} X^2 = [X, M_{\gamma_1}] \text{ for all } X \in <M_a, M_b> .$$

Proposition 1.7.1. *For any connection* (1.16)*, its monodromy group G fits into a normal sequence $N_1 \lhd N \lhd G$ in such a way, that the following properties are verified:*

1. *Both N_1 and G/N are quotients of $\mathbb{Z} \times \mathbb{Z}$, and $N/N_1 \simeq \mu_2$.*
2. *The extension $N_1 \lhd N$ is always split: $N \simeq N_1 \rtimes \mu_2$, the generator $h \in \mu_2$ acting on N_1 via the map $g \mapsto g^{-1}$.*
3. *The subgroup $<M_a, M_b>$ of G provides a splitting of the extension $N \lhd G$ if and only if $\bar{\varphi}_2$ is an isomorphism. In this case, the action of G/N on N defining the split extension is given by $x : g \mapsto g$ and $x : h \mapsto \bar{\varphi}_1\bar{\varphi}_2^{-1}(x)h$ for any $x \in G/N$, $g \in N_1$.*

Conversely, let (Q_1, Q_2) be a pair of group quotients of $\mathbb{Z} \times \mathbb{Z}$. Then (Q_1, Q_2) can be realized as the pair $(N_1, G/N)$ for the monodromy group G of a connection (1.16) *if and only if (Q_1, Q_2) occurs in the following table:*

N°	rk Q_1	rk Q_2	Q_1	Q_2	Restrictions
1*	2	2	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}$	—
2	2	1	$\mathbb{Z}\times\mathbb{Z}$	$\mu_d\times\mathbb{Z}$	$2\|d$
3	2	0	$\mathbb{Z}\times\mathbb{Z}$	$\mu_2\times\mu_d$	$2\|d$
4*	1	2	$\mu_d\times\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}$	$d\geq 1$
5	1	1	$\mu_d\times\mathbb{Z}$	$\mu_{d'}\times\mathbb{Z}$	*if* $2\|d$, *then* $2\|d'$
6	1	0	$\mu_d\times\mathbb{Z}$	$\mu_{d'}$	$2\nmid d$, $2\|d'$
7	1	0	$\mu_d\times\mathbb{Z}$	$\mu_2\times\mu_{d'}$	$d\geq 1$, $2\|d'$
8*	0	2	μ_d	$\mathbb{Z}\times\mathbb{Z}$	$d\geq 1$
9	0	1	μ_d	$\mu_{d'}\times\mathbb{Z}$	$d\equiv d' \bmod 2$
10	0	0	μ_d	$\mu_{d'}$	$d\geq 1$, $d'\geq 1$
11	0	0	μ_d	$\mu_2\times\mu_{d'}$	$2\|d$, $2\|d'$

The items whose numbers are marked with an asterisk correspond to the pairs that always give a split extension $N \lhd G$.

Proof. The first part, resuming the properties of the tower of group extensions $N_1 \lhd N \lhd G$, is an easy exercise, and we go over to the second one. Given a pair (Q_1, Q_2), we find out whether it is possible to choose epimorphisms $\mathbb{Z}\times\mathbb{Z} \xrightarrow{\varphi_1} Q_1$ and $\mathbb{Z}\times\mathbb{Z} \xrightarrow{\varphi_2} Q_2$ and identify them as the morphisms defined in (1.32) for a suitable choice of matrices M_a, M_b. The proof follows a case by case enumeration of different types of kernels of φ_1 and φ_2. To shorten the notation, let us write $M_a = e^{\alpha_1}H^{\beta_1}$, $M_b = e^{\alpha_2}H^{\beta_2}$. The case $\operatorname{rk}\ker\varphi_1 = \operatorname{rk}\ker\varphi_2 = 0$, corresponding to $\operatorname{rk}_{\mathbb{Q}}(\alpha_1, \beta_1, \pi i) = \operatorname{rk}_{\mathbb{Q}}(\alpha_2, \beta_2, \pi i) = 3$, has been treated before the statement of the proposition. It gives item 1 of the table.

The proofs of all the other cases resemble each other, and we will give only one example of this type of argument, say, when both kernels are of rank 1. Under this assumption, there exist $(d, k_1, k_2) \in \mathbb{Z}^3$ and $(d', k_1', k_2') \in \mathbb{Z}^3$ such that

$$d \geq 1,\ \ d' \geq 1,\ \ \gcd(k_1, k_2) = 1,\ \ \gcd(k_1', k_2') = 1, \tag{1.33}$$

and

$$\ker\varphi_1 = <d(k_1,k_2)>, \quad \ker\varphi_2 = <d'(k'_1,k'_2)>.$$

For $(n_1,n_2)\in\mathbb{Z}^2$, we have:

$$(n_1,n_2)\in\ker\varphi_1 \iff \exists\, m\in\mathbb{Z} \mid n_1\beta_1+n_2\beta_2=\pi i m\ ; \tag{1.34}$$

$$(n_1,n_2)\in\ker\varphi_2 \iff \exists\,(m_1,m_2)\in\mathbb{Z}^2 \ \mid\ M_a^{n_1}M_b^{n_2}=\left(H^{2\beta_1}\right)^{m_1}\left(H^{2\beta_2}\right)^{m_2}. \tag{1.35}$$

The latter equality can be written in the form $e^{\alpha}H^{\beta}=1$, where

$$\alpha=n_1\alpha_1+n_2\alpha_2,\ \ \beta=(n_1-2m_1)\beta_1+(n_2-2m_2)\beta_2.$$

As $e^{\alpha}H^{\beta}=1$ if and only if $e^{\alpha}=H^{\beta}=\pm1$, we see that the condition of (1.35) is equivalent to the existence of an integer vector $(m_0,m_1,m_2,m_3)\in\mathbb{Z}^4$ such that

$$m_0\equiv m_3 \mod 2 \tag{1.36}$$
$$n_1\alpha_1+n_2\alpha_2=\pi i m_0, \tag{1.37}$$
$$(n_1-2m_1)\beta_1+(n_2-2m_2)\beta_2=\pi i m_3. \tag{1.38}$$

Substituting the generators of $\ker\varphi_i$ for (n_1,n_2), we obtain the following system of equations:

$$dk_1\beta_1+dk_2\beta_2=\pi i m \tag{1.39}$$
$$d'k'_1\alpha_1+d'k'_2\alpha_2=\pi i m_0, \tag{1.40}$$
$$(d'k'_1-2m_1)\beta_1+(d'k'_2-2m_2)\beta_2=\pi i m_3. \tag{1.41}$$

The condition that $d(k_1,k_2)$, $d'(k'_1,k'_2)$ are not just elements of the corresponding kernels, but their generators, is transcribed as follows:

$$\gcd(m,d)=\gcd(d',m_0,2m_1,2m_2,m_3)=1 \tag{1.42}$$

for any (m_1,m_2,m_3) satisfying (1.36), (1.41).

As $\operatorname{rk}\ker\varphi_1=1$, the equations (1.39) and (1.41) have to be proportional. If d' is odd, but d is even, then at least one of the coefficients of β_i in (1.41) is odd. But both coefficients in (1.39) are even, and this contradicts (1.42). We get the restriction from item 5 of the table: if d is even, then d' is even, too. This leaves possible three combinations of parities of d,d', and it is easy to see that a solution to (1.33), (1.39)-(1.42) exists for any of them. For example, if $d\equiv d' \bmod 2$, then we can choose k_i,k'_i in such a way that $k_i\equiv k'_i \bmod 2$ $(i=1,2)$, $k_1k'_1\neq 0$. We get a solution to the problem as follows:

$$m_i=\frac{1}{2}d(k'_i-k_i)\ (i=1,2),\ m=m_0=m_3=1,\ \alpha_2=\beta_2=1,$$
$$\alpha_1=\frac{\pi i-d'k'_2}{d'k'_1},\ \beta_1=\frac{\pi i-dk_2}{dk_1}.$$

Our choice for α_2,β_2 is explained by the observation that we should have $\operatorname{rk}_{\mathbb{Q}}(\alpha_1,\beta_1,\pi i)=\operatorname{rk}_{\mathbb{Q}}(\alpha_2,\beta_2,\pi i)=2$, and 1 is the simplest complex number which is not a rational multiple of πi. $\square$

Remark 1.7.2. In the above proof, if $\ker\varphi_2 \not\subset \ker\varphi_1$, then any solution of (1.39)-(1.41) satisfies the condition $(m_1, m_2) \neq 0$, which means that $d'(k_1', k_2') \notin \ker\sigma$. Hence $\sigma(d'(k_1', k_2'))$ is a nonzero element of $\ker\bar\varphi_2$, and $\bar\varphi_2$ is not an isomorphism. This implies that the extension $N \lhd G$ is nonsplit. Hence it is never split, unless $d|d'$. In this case, it can be occasionally split, if $\ker\varphi_2 \subset \ker\varphi_1$.

We can deduce from Proposition 1.7.1 a description of all the finite monodromy groups; they correspond to lines 10 and 11 of the table. This description is only partial, because we do not determine completely the extension data.

Corollary 1.7.3. *All the finite monodromy groups G of connections (1.16) are obtained as extensions*

$$D_d \hookrightarrow G \twoheadrightarrow \mu_{d'} \ \ (d \geq 1,\ d' \geq 1)$$

or

$$D_d \hookrightarrow G \twoheadrightarrow \mu_2 \times \mu_{d'} \ \ (2 \mid d,\ 2 \mid d'),$$

where $D_d = \mu_d \rtimes \mu_2$ is the dihedral group.

Corollary 1.7.4. *The only finite abelian groups occuring as the monodromy groups of connections (1.16) are μ_2 and $\mu_2 \times \mu_d$ ($d \geq 2$).*

We add a few examples of infinite monodromy groups with nongeneric parameters $(\lambda_1 K, \lambda_1 L, \lambda_2 K', \lambda_2 L')$.

Example 1.7.5. It is easy to select the parameters to get for G one of the groups $D_n \times \mathbb{Z}^i$ or $D_n \rtimes \mathbb{Z}^i$, where $n \in \mathbb{N} \cup \{\infty\}$, $i = 0, 1, 2$. For example, to get $D_n \rtimes \mathbb{Z}$, we can set $M_a = R^{\frac{2\pi}{n}}$, $M_b = R^1$, and to get $D_n \times \mathbb{Z}$, we can set $M_a = R^{\frac{2\pi}{n}}$, $M_b = 1$.

Now that we have described the structure of the monodromy group of $\nabla_{\mathcal{E}}$, we can ask the question on its Zariski closure. According to ([Ka], Proposition 5.2), the Zariski closure of G is the differential Galois group $\mathrm{DGal}(\nabla_{\mathcal{E}})$. For the reader's convenience, we recall its definition.

Let $(K,\ ')$ be a differential field with field of constants $\mathbb{C}$. This means that K is endowed with a $\mathbb{C}$-linear derivation $\ ' : K \to K$.

Definition 1.7.6. Let $(K,\ ') \subset (L,\ ')$ be an extension of differential fields with field of constants $\mathbb{C}$. The differential Galois group $\mathrm{DGal}(L/K)$ is the group consisting of all the K-automorphisms σ of L such that $\sigma(f') = (\sigma(f))'$ for all $f \in L$.

If L is finitely generated as a K-algebra, say, by p elements, then $\mathrm{DGal}(L/K)$ can be embedded onto $GL(p, \mathbb{C})$, and it is an algebraic group if considered as a subgroup of $GL(p, \mathbb{C})$ in this embedding.

We apply this definition to $K = \mathbb{C}(E)$, the derivation $'$ being the differentiation with respect to some nonconstant function $z \in K$. Given a connection $\nabla_{\mathcal{E}}$ on E, we can consider a fundamental matrix Φ of its solutions, and set L to be the field generated by all the matrix elements of Φ. The group $\mathrm{DGal}(\nabla_{\mathcal{E}})$ is defined to be $\mathrm{DGal}(L/K)$. See [vdP, vdPS] for more details.

Remark that the monodromy group G lies in the subgroup $\mathbb{G}$ of $GL(2,\mathbb{C})$ defined by

$$\mathbb{G}=\left\{\begin{pmatrix} C\alpha & C\epsilon\beta \\ C\beta & C\alpha\epsilon \end{pmatrix} \mid C\in\mathbb{C}^*, (\alpha,\beta)\in\mathbb{C}^2, \epsilon\in\{-1,1\}, \alpha^2-\beta^2=1\right\}.$$

Denote by $\mathbb{G}_0$ the connected component of unity in $\mathbb{G}$, singled out by the condition $\epsilon=1$. The Zariski closure $\bar{G}$ of G is contained in $\mathbb{G}$ and is not contained in $\mathbb{G}_0$. The following statement is obvious:

Lemma 1.7.7. *Let* $\psi:\mathbb{C}^*\times\mathbb{C}^*\rtimes\{-1,1\}\longrightarrow\mathbb{G}$ *be defined by*

$$(\lambda,\mu,\epsilon)\longmapsto\begin{pmatrix} \lambda\alpha & \lambda\beta\epsilon \\ \lambda\beta & \lambda\alpha\epsilon \end{pmatrix}$$

with $\alpha=\frac{1}{2}(\mu+\frac{1}{\mu})$, $\beta=\frac{1}{2}(\mu-\frac{1}{\mu})$. *Then* ψ *is a surjective morphism with kernel* $\{(1,1,1),(-1,-1,-1)\}$.

We see that $\mathbb{G}_0=\psi(\mathbb{C}^*\times\mathbb{C}^*)$ is identified with the quotient $\mathbb{C}^*\times\mathbb{C}^*/\{-1,1\}$, and the latter is isomorphic to $\mathbb{C}^*\times\mathbb{C}^*$ via the map $(z_1,z_2)\bmod\{-1,1\}\longmapsto(z_1z_2,\frac{z_1}{z_2})$. Thus we get an explicit isomorphism $\mathbb{G}_0\simeq\mathbb{C}^*\times\mathbb{C}^*$. Using this identification, one can easily determine the Zariski closure $\overline{G_0}$ of the subgroup $G_0=G\cap\mathbb{G}_0=<M_a,M_b>$ of $\mathbb{C}^*\times\mathbb{C}^*$, and $\mathrm{DGal}(\nabla_{\mathcal{E}})=\overline{G_0}\rtimes<M_{\gamma_1}>$.

We can use the following observations:

a) If a pair $(s,t)\in\mathbb{C}^*\times\mathbb{C}^*$ is such that $\mathrm{rk}_{\mathbb{Q}}(\ln(s),\ln(t),i\pi)=1$ (that is, s and t are roots of unity), then the group generated by the pair (s,t) is finite and coincides with its closure.

b) If a pair $(s,t)\in\mathbb{C}^*\times\mathbb{C}^*$ is such that $\mathrm{rk}_{\mathbb{Q}}(\ln(s),\ln(t),i\pi)=2$, and $k_1\ln(s)+k_2\ln(t)+2k_3i\pi=0$ is a $\mathbb{Z}$-linear relation with relatively prime k_i, then $\overline{<(s,t)>}$ is the subgroup V of $\mathbb{C}^*\times\mathbb{C}^*$ defined by $z_1^{k_1}z_2^{k_2}=1$, isomorphic to $\mathbb{C}^*\times\mu_d$, where $d=\gcd(k_1,k_2)$, and μ_d is the cyclic group of order d.

c) If the triple $(\ln(s),\ln(t),\pi i)$ is free over $\mathbb{Q}$, then the closure of $<(s,t)>$ is $\mathbb{C}^*\times\mathbb{C}^*$.

Apply this to pairs (s,t) belonging to the subgroup generated by two pairs (s_1,t_1), (s_2,t_2) which are the images of M_a, resp. M_b. Then if $(\ln(s_j),\ln(t_j),\pi i)$ is free over $\mathbb{Q}$ for at least one value of $j=1$ or 2, then $\overline{G_0}=\mathbb{C}^*\times\mathbb{C}^*$ and $\mathrm{DGal}(\nabla_{\mathcal{E}})=\mathbb{G}$. In the case when both triples $(\ln(s_1),\ln(t_1),\pi i)$, $(\ln(s_2),\ln(t_2),\pi i)$ are not free over $\mathbb{Q}$, the necessary and sufficient condition for $\overline{<(s_1,t_1),(s_1,s_2)>}$ to be $\mathbb{C}^*\times\mathbb{C}^*$ is the following: $\mathrm{rk}_{\mathbb{Q}}(\ln(s_j),\ln(t_j),i\pi)=2$ for both values $j=1,2$, and if $a_{j1}\ln(s_j)+a_{j2}\ln(t_j)+a_{j3}\pi i=0$ $(j=1,2)$ are nontrivial $\mathbb{Q}$-linear relations in these triples, then $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\neq 0$. This condition can be easily formulated in terms of the epimorphisms φ_i defined in (1.32): $\ker\varphi_1$, $\ker\varphi_2$ are both of rank 1 and $\ker\varphi_1\cap\ker\varphi_2=0$. In this case we have the same conclusion: $\mathrm{DGal}(\nabla_{\mathcal{E}})=\mathbb{G}$.

We obtain the following description of possible differential Galois groups of connections (1.16):

Proposition 1.7.8. *Let* $r_i=\mathrm{rk}_{\mathbb{Q}}\ker\varphi_i$ $(i=1,2)$.

(i) $\mathrm{DGal}(\nabla_{\mathcal{E}})=\mathbb{G}$ *if and only if one of the following condition is verified: either* $\min\{r_1,r_2\}=0$, *or* $r_1=r_2=1$ *and* $\ker\varphi_1\cap\ker\varphi_2=0$.

(ii) $\mathrm{DGal}(\nabla_{\mathcal{E}})$ *is a* 1*-dimensional subgroup of* $\mathbb{G}$ *if and only if* $\min\{r_1, r_2\} = 1$ *and the condition of (i) is not satisfied. Then there exists a one-parameter subgroup* V_0 *and a finite cyclic subgroup* μ_d *in* $\mathbb{G}$ *such that* $\mathrm{DGal}(\nabla_{\mathcal{E}}) = (V_0\mu_d) \rtimes <M_{\gamma_1}>$.

(iii) $\mathrm{DGal}(\nabla_{\mathcal{E}})$ *is finite if and only if* $r_1 = r_2 = 2$*, and then* $\mathrm{DGal}(\nabla_{\mathcal{E}}) = G$.

CHAPTER 2

KURANISHI SPACES

INTRODUCTION

Let X be a complete scheme of finite type over k or a compact complex space (then $k = \mathbb{C}$). The existence of a versal deformation and the theoretical approach to its construction are known for coherent sheaves on X. The construction of the Kuranishi space ($=$ versal deformation) for coherent sheaves is done in using the injective resolutions. We are studying vector bundles $\mathcal{E}$ with an additional structure (a connection ∇), and in this case the deformation theory of both $\mathcal{E}$ and $(\mathcal{E}, \nabla)$ can be stated in terms of the Čech cohomology of a sufficiently fine open covering of X. This approach is easier than the one via injective resolutions. We start by the construction of the Kuranishi space of vector bundles serving as a model for that of the pairs $(\mathcal{E}, \nabla)$. This is done in Sect. 2.1, where it is also explained how the versal deformations can be used to construct analytic moduli spaces of simple vector bundles. In Sect. 2.2, we introduce connections with fixed divisor of poles and show that their isomorphism classes of first order deformations are classified by the hypercohomology $\mathbb{H}^1(\mathcal{C}^\bullet)$ of some two-term complex of sheaves. In Sect. 2.3, we show that the first obstruction to lifting the first order deformation is given by the Yoneda square and construct the Kuranishi space. We also define several versions of the Atiyah class. In Sect. 2.4, we describe the construction of the Kuranishi space for integrable and integrable logarithmic connections. The last Sect. 2.5 treats the Kuranishi space of parabolic connections.

2.1 CONSTRUCTION OF THE KURANISHI SPACE IN THE CASE OF VECTOR BUNDLES OVER ANY BASE.

Let X be a complete scheme of finite type over k or a complex space (then $k = \mathbb{C}$), $\mathfrak{U} = (U_\alpha)$ be an open covering of X, e_α a trivialization of $\mathcal{E}_{|U_\alpha}$. The transition functions $g_{\alpha\beta}$ relate the trivializations by the formula $e_\beta = e_\alpha g_{\alpha\beta}$ over $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and satisfy the following relations

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1},\ g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1. \tag{2.1}$$

In other words, $(g_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathrm{GL}(r, \mathcal{O}_X))$ is a skew-symmetric multiplicative 1-cocycle.

2.1.1 First order deformations

Deform the transition functions: $\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon g_{\alpha\beta,1}$, where $g_{\alpha\beta,1} \in \Gamma(U_{\alpha\beta}, M_r(\mathcal{O}_X))$ and $\epsilon^2 = 0$. We have $g_{\alpha\beta,1} = \frac{\mathrm{d}\tilde{g}_{\alpha\beta}}{\mathrm{d}\epsilon}$. Differentiating (2.1), we obtain:

$$g_{\beta\alpha,1} = \frac{\mathrm{d}\tilde{g}_{\alpha\beta}^{-1}}{\mathrm{d}\epsilon} = -g_{\alpha\beta}^{-1} g_{\alpha\beta,1} g_{\alpha\beta}^{-1}, \tag{2.2}$$

$$g_{\alpha\beta,1} g_{\beta\gamma} g_{\gamma\alpha} + g_{\alpha\beta} g_{\beta\gamma,1} g_{\gamma\alpha} + g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha,1} = 0,$$

and by (2.2), $g_{\gamma\alpha,1} = -g_{\alpha\gamma}^{-1} g_{\alpha\gamma,1} g_{\alpha\gamma}^{-1}$. Plugging this into the previous formula, we get

$$g_{\alpha\beta,1} g_{\beta\gamma} g_{\gamma\alpha} + g_{\alpha\beta} g_{\beta\gamma,1} g_{\gamma\alpha} = g_{\alpha\beta} g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\gamma,1} g_{\alpha\gamma}^{-1}.$$

Multiply by $g_{\alpha\gamma}$ on the right:

$$g_{\alpha\beta,1} g_{\beta\gamma} + g_{\alpha\beta} g_{\beta\gamma,1} = g_{\alpha\gamma,1}. \tag{2.3}$$

We want to represent this in the form $a_{\alpha\beta} + a_{\beta\gamma} = a_{\alpha\gamma}$ for an appropriate additive 1-cocycle $a = (a_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))$, associated with $(g_{\alpha\beta,1})$ and skew-symmetric: $a_{\alpha\beta} = -a_{\beta\alpha}$. Define $a_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathcal{E}nd(\mathcal{E}))$ by its matrix: $g_{\alpha\beta}^{-1} g_{\alpha\beta,1}$ in the basis e_β and $g_{\alpha\beta,1} g_{\alpha\beta}^{-1}$ in the basis e_α. Then (2.2) gives $g_{\alpha\beta} g_{\beta\alpha,1} + g_{\alpha\beta,1} g_{\alpha\beta}^{-1} = 0$, written in terms of matrices with respect to the basis e_α, and (2.3) amounts to $a_{\alpha\beta} + a_{\beta\gamma} = a_{\alpha\gamma}$. Thus the first order deformations of $\mathcal{E}$ are classified by the 1-cocycles $a = (a_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))$. Such a deformation is trivial if the vector bundle $\tilde{\mathcal{E}}$ defined over $X \times \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ by the 1-cocycle $\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon g_{\alpha\beta,1}$ is isomorphic to $\mathrm{pr}_1^*(\mathcal{E})$, where $\mathrm{pr}_1 : X \times \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2) \to X$ is the natural projection. This means that there exists a change of basis $e_\alpha \mapsto \tilde{e}_\alpha = e_\alpha(1 + \epsilon h_\alpha)$ which transforms $\tilde{g}_{\alpha\beta}$ into $g_{\alpha\beta}$. We compute $\tilde{e}_\beta = e_\beta(1+\epsilon h_\beta) = e_\alpha g_{\alpha\beta}(1+\epsilon h_\beta) = \tilde{e}_\alpha(1-\epsilon h_\alpha) g_{\alpha\beta}(1+\epsilon h_\beta)$ and we want that this coincides with $\tilde{e}_\beta = \tilde{e}_\alpha \tilde{g}_{\alpha\beta}$. That is: $g_{\alpha\beta} + \epsilon g_{\alpha\beta,1} = (1 - \epsilon h_\alpha) g_{\alpha\beta}(1 + \epsilon h_\beta)$, or $g_{\alpha\beta,1} = -h_\alpha g_{\alpha\beta} + g_{\alpha\beta} h_\beta$. Interpreting h_α as the matrix of $b_\alpha \in \Gamma(U_\alpha, \mathcal{E}nd(\mathcal{E}))$ with respect to the basis e_α, we obtain $a_{\alpha,\beta} = -b_\alpha + b_\beta$ which is written in the basis e_α in the form $g_{\alpha\beta}^{-1} g_{\alpha\beta,1} = -h_\alpha + g_{\alpha\beta} h_\beta g_{\alpha\beta}^{-1}$. Thus the equivalence classes of first order deformations of $\mathcal{E}$ over $V = \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ are classified by

$$\check{H}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E})) = \frac{\{\text{1-cocycles } (a_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))\}}{\{\text{coboundaries } a_{\alpha\beta} = b_\beta - b_\alpha, \text{ where } (b_\alpha) \in \check{C}^0(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))\}}.$$

2.1.2 First obstruction

We denote $V_k = \operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon)^{k+1}$. We will investigate the following question: which of the deformations of $\mathcal{E}$ over V_1 lift to V_2? Let $G_{\alpha\beta} = g_{\alpha\beta,0} + \epsilon g_{\alpha\beta,1} + \epsilon^2 g_{\alpha\beta,2}$ be a deformation of the cocycle $g_{\alpha\beta} = g_{\alpha\beta,0}$ over V_2. Assume that $G_{\alpha\beta} \mod \epsilon^2$ is a 1-cocycle, then (2.2) and (2.3) are verified, and compute the coefficient $K_{\alpha\beta\gamma,2}$ of ϵ^2 in $G_{\alpha\beta}G_{\beta\gamma}G_{\gamma\alpha}$, which will be denoted $K_{\alpha\beta\gamma,2}$:

$$\begin{aligned} K_{\alpha\beta\gamma,2} = {} & g_{\alpha\beta,0}g_{\beta\gamma,1}g_{\gamma\alpha,1} + g_{\alpha\beta,1}g_{\beta\gamma,0}g_{\gamma\alpha,1} + g_{\alpha\beta,1}g_{\beta\gamma,1}g_{\gamma\alpha,0} \\ & + g_{\alpha\beta,2}g_{\beta\gamma,0}g_{\gamma\alpha,0} + g_{\alpha\beta,0}g_{\beta\gamma,2}g_{\gamma\alpha,0} + g_{\alpha\beta,0}g_{\beta\gamma,0}g_{\gamma\alpha,2} \end{aligned} \tag{2.4}$$

Similar to the above, introduce the sections $a_{\alpha\beta,i}$, $(i = 1, 2)$ of the endomorphism sheaf $\mathcal{E}nd(\mathcal{E}_{|(U_{\alpha\beta})})$ having $g_{\alpha\beta,i}g_{\alpha\beta}^{-1}$ for their matrices in the bases e_α. Then, as above, $g_{\alpha\beta,2}g_{\beta\gamma,0}g_{\gamma\alpha,0} + g_{\alpha\beta,0}g_{\beta\gamma,2}g_{\gamma\alpha,0} + g_{\alpha\beta,0}g_{\beta\gamma,0}g_{\gamma\alpha,2}$ is the matrix of $a_{\alpha\beta,2} + a_{\beta\gamma,2} + a_{\gamma\alpha,2}$ in the basis e_α, and $g_{\alpha\beta,0}g_{\beta\gamma,1}g_{\gamma\alpha,1} + g_{\alpha\beta,1}g_{\beta\gamma,0}g_{\gamma\alpha,1} + g_{\alpha\beta,1}g_{\beta\gamma,1}g_{\gamma\alpha,0}$ is the matrix of

$$a_{\beta\gamma,1}a_{\gamma\alpha,1} + a_{\alpha\beta,1}a_{\gamma\alpha,1} + a_{\alpha\beta,1}a_{\beta\gamma,1} \tag{2.5}$$

in the basis e_α. Let a_1 denote the cocycle $(a_{\alpha\beta,1})$ and $[a_1]$ its class in $\check{H}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))$. Then $a_{\beta\gamma,1}a_{\gamma\alpha,1} = c_{\beta\gamma\alpha}$ represents the Yoneda product $[a_1] \circ [a_1] = [c] \in \check{H}^2(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))$; see for instance $10.1.1.$ of [H-L] for the definition of the Yoneda product

$$\check{H}^i(\mathfrak{U}, \mathcal{E}nd(\mathcal{E})) \times \check{H}^j(\mathfrak{U}, \mathcal{E}nd(\mathcal{E})) \to \check{H}^{i+j}(\mathfrak{U}, \mathcal{E}nd(\mathcal{E})).$$

The whole expression (2.5) is the skew-symmetrization $\hat{c}_{\alpha\beta\gamma}$ of $c_{\beta\gamma\alpha}$, hence it represents the same cohomology class $[c]$. Let also a_2 denote the Čech cochain $(a_{\alpha\beta,2})$. We can rewrite $K_2 = (K_{\alpha\beta\gamma,2})$ in the form

$$K_2 = \hat{c} + \check{d}a_2. \tag{2.6}$$

We now see that we can find a_2 in such a way that $(G_{\alpha\beta})$ is a cocycle over V_2 if and only if $\hat{c}$ is $\check{d}$-exact. We have proved:

Proposition 2.1.1. *Let X be a complete scheme of finite type over k or a complex space (and then $k = \mathbb{C}$), $\mathcal{E}$ a vector bundle on X, $[a] \in H^1(X, \mathcal{E}nd(\mathcal{E}))$. Then the first order deformation of $\mathcal{E}$ over V_1 defined by $[a]$ lifts to a deformation over V_2 if and only if the Yoneda square $[a] \circ [a]$ is zero in $H^2(X, \mathcal{E}nd(\mathcal{E}))$.*

Definition 2.1.2. The map

$$\begin{array}{ccc} H^1(X, \mathcal{E}nd(\mathcal{E})) & \rightarrow & H^2(X, \mathcal{E}nd(\mathcal{E})) \\ ([a]) & \mapsto & [a] \circ [a] \end{array} \tag{2.7}$$

will be called first obstruction, and denoted $ob^{(2)}$.

Thus $ob^{(2)}$ is the map of taking the Yoneda square. We will now construct a universal first order deformation of $\mathcal{E}$ on X. Let $W = H^1(X, \mathcal{E}nd(\mathcal{E}))$, $t_1, \dots, t_N$ a coordinate system on W, $W_k = \operatorname{Spec} k[t_1, \dots, t_N]/(t_1, \dots, t_N)^{k+1}$ the k-th infinitesimal neighborhood of the origin in W. The universal first order deformation $\mathcal{E}_1$ of $\mathcal{E}$ over W_1 can be described as follows.

Choose an open covering of X as above, so that $\mathcal{E}$ is defined by a 1-cocyle $(g_{\alpha\beta})$. We deform $\mathcal{E}$ by specifying a family $G_{\alpha\beta}(t_1, \dots, t_N)$ of 1-cocyles over $X \times W_1$. Pick up N cocycles $a_i = (a^{(i)}_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))$ whose cohomology classes $[a_1], \dots, [a_N]$ form a basis of W dual to the coordinates $t_1, \dots, t_N$. Then we set $g^{(i)}_{\alpha\beta} = a^{(i)}_{\alpha\beta} g_{\alpha\beta}$, where $a^{(i)}_{\alpha\beta}$ is represented by its matrix in the basis e_α and write $G_{\alpha\beta}(t_1, \dots, t_N) = g_{\alpha\beta} + \sum_{i=1}^N g^{(i)}_{\alpha\beta} t_i$. Then $G_{\alpha\beta}$ is a 1-cocycle and defines a vector bundle $\mathcal{E}_1$ over $X \times W_1$ called a universal first order deformation of $\mathcal{E}$. The whole universal deformation over W_1 cannot be lifted to a deformation on W_2. Proposition 2.1.1 implies:

Proposition 2.1.3. *There is a maximal subscheme $K_2 \subset W_2$ with the property that $\mathcal{E}_1$ extends as a vector bundle from $X \times W_1$ to $X \times K_2$. This maximal subscheme K_2 is the (second infinitesimal neighborhood of the origin in the cone) defined by the equation $ob^{(2)}(z) = 0$ in W_2.*

We will now prove the following theorem, providing a construction of the formal Kuranishi space:

Theorem 2.1.4. *Let $X, \mathcal{E}$ be as above, $W = H^1(X, \mathcal{E}nd(\mathcal{E}))$, $(\delta_1, \dots, \delta_N)$ a basis of W and $(t_1 \dots, t_N)$ the dual coordinates on W. Let $W_k = \operatorname{Spec} k[t_1, \dots, t_N]/(t_1, \dots, t_N)^{k+1}$ be the k-th infinitesimal neighborhood of the origin in W, $\mathcal{E}_1$ a universal first order deformation of $\mathcal{E}$ over $X \times W_1$ as above. Then there exists a formal power series*

$$f(t_1, \dots, t_N) = \sum_{k=2}^{\infty} f_k(t_1 \dots, t_N) \in H^2(X, \mathcal{E}nd(\mathcal{E}))[[t_1, \dots, t_N]],$$

where f_k is homogeneous of degree k, with the following property. Let I be the ideal of $k[[t_1, \dots, t_N]]$ generated by the image of the map $f^* : H^2(X, \mathcal{E}nd(\mathcal{E}))^* \to k[[t_1, \dots, t_N]]$, adjoint to f. Then for any $k \geq 2$, the universal first deformation $\mathcal{E}_1$ of $\mathcal{E}$ over $X \times W_1$ extends to a vector bundle $\mathcal{E}_k$ on $X \times K_k$, where K_k is a closed subscheme of W_k defined by the ideal $I \otimes k[[t_1, \dots, t_N]]/(t_1, \dots, t_N)^{k+1}$.

Definition 2.1.5. The inverse limit $\mathbb{K} = \varprojlim K_k$ is called the formal Kuranishi space of $\mathcal{E}$, and $\boldsymbol{\mathcal{E}} = \varprojlim \mathcal{E}_k$ the formal universal bundle over $\mathbb{K}$.

Proof. Let $\mathfrak{U} = (U_k)$ be an open covering, sufficiently fine so that $\mathcal{E}_{|U_\alpha}$ is trivialized by a basis e_α, and the groups $H^i(X, \operatorname{End}(\mathcal{E}))$ are computed by the $\check{C}$ech complex $(\check{C}^\bullet(\mathfrak{U}, \mathcal{E}nd(\mathcal{E})), \check{d})$. Let $\check{Z}^i(\mathfrak{U}, \mathcal{E}nd(\mathcal{E})), \check{B}^i(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))$ denote the subspaces of cocycles and coboundaries in $\check{C}^i(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))$ respectively. Let us fix some cross-sections $\sigma_i : H^i(X, \mathcal{E}nd(\mathcal{E})) \to \check{Z}^i(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))$ and $\tau : \check{B}^2(\mathfrak{U}, \mathcal{E}nd(\mathcal{E})) \to \check{C}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))$ of the natural maps in the opposite direction. Let $a_i = (a^{(i)}_{\alpha\beta}) = \sigma_1(\delta_i)$, and denote, as above, by $(g_{\alpha\beta})$ the 1-cocycle defining $\mathcal{E}$, so that $e_\beta = e_\alpha g_{\alpha\beta}$. We will construct by induction on $k \geq 0$ the homogeneous forms of degree k in $t_1, \dots, t_N$

$$G_{\alpha\beta,k}(t_1, \dots, t_N) \in \Gamma(U_{\alpha\beta}, M_r(\mathcal{O}_X)) \otimes k[t_1, \dots, t_N], \tag{2.8}$$

$$F_{\alpha\beta\gamma,k}(t_1,\ldots,t_N)\in\Gamma(U_{\alpha\beta\gamma},\mathcal{E}nd(\mathcal{E}))\otimes k[t_1,\ldots,t_N],$$
$$f_k(t_1,\ldots,t_N)\in H^2(X,\mathcal{E}nd(\mathcal{E}))\otimes k[t_1,\ldots,t_N]$$

with the following properties:
(i) $G_{\alpha\beta,0}=g_{\alpha\beta}, G_{\alpha\beta,1}=\sum_{i=1}^N a^{(i)}_{\alpha\beta}g_{\alpha\beta}t_i$, where $a^{(i)}_{\alpha\beta}$ are represented by their matrices in the basis e_α.
(ii) $f_k=0, F_{\alpha\beta\gamma,k}=0$ for $k=0,1$.
(iii) For each k ≥ 1, let $f^{(k)}=\sum_{i\leq k}f_i$, and let $I^{(k+1)}$ be the ideal generated by $(t_1,\ldots,t_N)^{k+2}$ and the image of the adjoint map $f^{(k)*}: H^2(X,\mathcal{E}nd(\mathcal{E}))^*\to k[t_1,\ldots,t_N]$. Then $(F_{\alpha\beta\gamma,k+1})$ is a cocycle modulo $I^{(k+1)}$ and f_{k+1} is a lift to $H^2(X,\mathcal{E}nd(\mathcal{E}))\otimes k[t_1,\ldots,t_N]$ of the cohomology class $[(F_{\alpha\beta\gamma,k+1} \mod I^{(k+1)})]\in H^2(X,\mathcal{E}nd(\mathcal{E}))\otimes k[t_1,\ldots,t_N]/I^{(k+1)}$.
(iv) For any $k\geq 1$, set $G^{(k)}_{\alpha\beta}=\sum_{i\leq k}G_{\alpha\beta,i}$. Then $G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha}\equiv(1+F_{\alpha\beta\gamma,k+1}) \mod I^{(k+1)}$.

Properties $(i),(ii)$ determine $G_{\alpha\beta,k},F_{\alpha\beta\gamma,k}$ for $k\leq 1$. The proof of Proposition 2.1.1 allows us to see that $(iii),(iv)$ are verified for $k=1$ with

$$F_{\alpha\beta\gamma,2}=\sum_{i,j=1}^N(a^{(i)}_{\beta\gamma}a^{(j)}_{\gamma\alpha}+a^{(i)}_{\alpha\beta}a^{(j)}_{\gamma\alpha}+a^{(i)}_{\alpha\beta}a^{(j)}_{\beta\gamma})t_it_j,$$

and to determine $G_{\alpha\beta,2}$ we proceed as follows. Let $f_2=[(F_{\alpha\beta\gamma,2})]$, and $I^{(2)}$ be the ideal of K_2, that is the ideal generated by $(t_1,\ldots,t_N)^3$ and the image of the adjoint map $f^{(2)*}: H^2(X,\mathrm{End}(\mathcal{E}))^*\to k_2[t_1,\ldots,t_N]=\mathrm{Sym}^2(W^*)$ (the degree-2 homogeneous part of $k[t_1,\ldots,t_N]$). Then the reduction $\mod I_2$ of $F_2=(F_{\alpha\beta\gamma,2})$ is an element $\bar{F}_2=(F_{\alpha\beta\gamma,2}) \mod I^{(2)}\in\check{B}^2(\mathfrak{U},\mathcal{E}nd(\mathcal{E}))\otimes(\mathrm{Sym}^2(W^*)/I^{(2)}\cap\mathrm{Sym}^2(W^*))$. We define a skew-symmetric 1-cochain $a_2=a_{\alpha\beta,2}\in\check{C}^1(\mathfrak{U},\mathcal{E}nd(\mathcal{E}))\otimes\mathrm{Sym}^2(W^*)$ as an arbitrary lift of $(\tau\otimes\mathrm{id})(\bar{F}_2)\in\check{C}^1(\mathfrak{U},\mathcal{E}nd(\mathcal{E}))\otimes(\mathrm{Sym}^2(W^*)/I^{(2)}\cap\mathrm{Sym}^2(W^*))$ under the quotient map. Next we define $G_{\alpha\beta,2}$ by $G_{\alpha\beta,2}=a_{\alpha\beta,2}g_{\alpha\beta}$, where the matrix of $a_{\alpha\beta,2}$ is taken in the basis e_α.

Likewise, assuming that $G^{(k-1)}_{\alpha\beta}$, $F^{(k)}_{\alpha\beta}$ are already fixed, we can choose $F_{\alpha\beta\gamma,k+1}$ and $G_{\alpha\beta,k}$ as follows. By the induction hypothesis, we have $G^{(k-1)}_{\alpha\beta}G^{(k-1)}_{\beta\gamma}G^{(k-1)}_{\gamma\alpha}\equiv(1+F_{\alpha\beta\gamma,k}) \mod I^{(k)}$. Then $(F_{\alpha\beta\gamma,k})$ is a cocycle modulo $I^{(k)}$, and is a coboundary modulo $I^{(k+1)}$: $\bar{F}_k=(F_{\alpha\beta\gamma,k} \mod I^{(k+1)})\in\check{B}^2(\mathfrak{U},\mathcal{E}nd(\mathcal{E}))\otimes(\mathrm{Sym}^k(W^*)/I^{(k+1)}\cap\mathrm{Sym}^k(W^*))$. We define $G_{\alpha\beta,k}=a_{\alpha\beta,k}g_{\alpha\beta}$ with $(a_{\alpha\beta,k})\in\check{C}^1(\mathfrak{U},\mathrm{End}(\mathcal{E}))\otimes\mathrm{Sym}^k(W^*)$ an arbitrary skew-symmetric lift to $\mathrm{Sym}^k(W^*)$ of $(\tau\otimes\mathrm{id})(\bar{F}_k)$. Then $G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha}\equiv 1 \mod (I^{(k+1)}+(t_1,\ldots,t_N)^{(k+1)})$, and we can define $F_{\alpha\beta\gamma,k+1}$ as the degree-$(k+1)$ homogeneous component of $G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha}$. To end this inductive construction of the sequences $G_{\alpha\beta,k}$, $F_{\alpha\beta\gamma,k+1}$, we need only to prove that $F_{k+1}=(F_{\alpha\beta\gamma,k+1})$ is a 2-cocycle modulo $I^{(k+1)}$ with values in $\mathcal{E}nd(\mathcal{E})$. $\square$

The latter is proved in Lemma 2.1.6 below.

Lemma 2.1.6. *The* 2*-cochain* $(F_{\alpha\beta\gamma,k+1})$*, constructed in the proof of Theorem* 2.1.4 *as the degree-*$(k+1)$ *homogeneous component of* $G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha}$*, is a* 2*-cocycle modulo* $I^{(k+1)}$ *with values in* $\mathcal{E}nd(\mathcal{E})$*.*

Proof. The hypotheses, under which we have to prove the assertion of lemma 2.1.6, are the following: $G^{(k)}_{\alpha\beta} = \sum_{i=0}^{k} G_{\alpha\beta,i} \in \Gamma(U_{\alpha\beta}, M_r(\mathcal{O}_X)) \otimes k[t_1, \dots, t_N]$ are the matrix polynomials of degree $\leq k$ in $t_1, \dots, t_N$ and there is an ideal $J \subset (t_1, \dots, t_N)^2$ such that $G^{(k)}_{\alpha\beta} G^{(k)}_{\beta\alpha} \equiv 1 \mod J$ and $G^{(k)}_{\alpha\beta} G^{(k)}_{\beta\gamma} G^{(k)}_{\gamma\alpha} \equiv 1 \mod (J + (t_1, \dots, t_N)^{k+1})$. The ideal J in Theorem 2.1.4 is $I^{(k+1)}$. The collection $(F_{\alpha\beta\gamma,k})$ is considered not as a 2-cochain in $M_r(\mathcal{O}_X)$, but as a 2-cochain in $\mathcal{E}nd(\mathcal{E})$, $\mathcal{E}$ being defined by the multiplicative cocycle $(g_{\alpha\beta}) = G_{\alpha\beta,0} \in \check{Z}^1(\mathfrak{U}, \mathrm{GL}_r(\mathcal{O}_X))$. Thus $F_{\alpha\beta\gamma} = F_{\alpha\beta\gamma,k+1}$ is a certain section of $\mathcal{E}nd(\mathcal{E})$ over $U_{\alpha\beta\gamma}$ given by its matrix in the basis e_α of $\mathcal{E}_{|U_{\alpha\beta\gamma}}$. We want to show that

$$F_{\alpha\beta\gamma} - F_{\alpha\beta\delta} + F_{\alpha\gamma\delta} - F_{\beta\gamma\delta} \equiv 0 \mod J \tag{2.9}$$

We will replace it by a slightly different identity

$$F_{\alpha\beta\gamma} + F_{\alpha\gamma\delta} + F_{\alpha\delta\beta} + F_{\beta\delta\gamma} \equiv 0 \mod J, \tag{2.10}$$

which is the same as (2.9) as soon as we know that $(F_{\alpha\beta\gamma})$ is skew symmetric. We have:

$$\begin{aligned} F_{\alpha\beta\gamma} = [G_{\alpha\beta} G_{\beta\gamma} G_{\gamma\alpha}]_{k+1}, F_{\alpha\gamma\delta} = [G_{\alpha\gamma} G_{\gamma\delta} G_{\delta\alpha}]_{k+1}, F_{\alpha\delta\beta} = [G_{\alpha\delta} G_{\delta\beta} G_{\beta\alpha}]_{k+1}, \\ F_{\beta\delta\gamma} = G_{\alpha\beta,0}([G_{\beta\delta} G_{\delta\gamma} G_{\gamma\beta}]_{k+1}) G^{-1}_{\alpha\beta,0} = [G_{\alpha\beta} G_{\beta\delta} G_{\delta\gamma} G_{\gamma\beta} G_{\beta\alpha}]_{k+1}, \end{aligned}$$

where we omitted the superscript k in $G^{(k)}_{\alpha\beta}$, $[\dots]_{k+1}$ stands for the homogeneous component of degree $k+1$ in $t_1, \dots, t_N$, and all the four terms are given by their matrices in the basis e_α. Now

$$\begin{aligned} F_{\alpha\beta\gamma} + F_{\alpha\gamma\delta} + F_{\alpha\delta\beta} + F_{\beta\delta\gamma} = [G_{\alpha\beta} G_{\beta\gamma} G_{\gamma\alpha} + G_{\alpha\gamma} G_{\gamma\delta} G_{\delta\alpha} + G_{\alpha\delta} G_{\delta\beta} G_{\beta\alpha} + \\ G_{\alpha\beta} G_{\beta\delta} G_{\delta\gamma} G_{\gamma\beta} G_{\beta\alpha}]_{k+1} \equiv [G_{\alpha\beta} G_{\beta\gamma} G_{\gamma\alpha} \times G_{\alpha\gamma} G_{\gamma\delta} G_{\delta\alpha} \times G_{\alpha\delta} G_{\delta\beta} G_{\beta\alpha} \\ \times G_{\alpha\beta} G_{\beta\delta} G_{\delta\gamma} G_{\gamma\beta} G_{\beta\alpha}]_{k+1} \equiv 0 \mod J. \end{aligned}$$

The skew symmetry of $(F_{\alpha\beta\gamma})$ is a particular case of (2.10) when $\delta = \gamma$. □

The vector bundle $\mathcal{E}_k$ defined by the cocycle $G^{(k)}_{\alpha\beta}$ over $X \times K_k$ satisfies the conditions of theorem.

Definition 2.1.7. A vector bundle $\mathcal{E}$ on X is simple if and only if $H^0(X, \mathcal{E}nd(\mathcal{E})) = k\,\mathrm{id}$.

In the case of a simple vector bundle, the versal deformation is in fact universal and this is a local version of the moduli space:

Proposition 2.1.8. *Let $\mathcal{E}$ be a simple vector bundle on a scheme X of finite type on k or a complex space (in which case $k = \mathbb{C}$). Then there exists an analytic space $M(\mathcal{E})$ with a reference point $*$ and a vector bundle E on $X \times M(\mathcal{E})$ which satisfy the following properties:*
(1) $E|_{X \times *} \simeq \mathcal{E}$.
(2) *If T is an analytic space with a reference point $*$ and E' a vector bundle on $X \times T$ such that $E'|_{X \times *} \simeq \mathcal{E}$, then there is a holomorphic mapping $\Phi : T \to M(\mathcal{E})$ such that $\Phi(*) = *$ and $E' \simeq (1 \times \Phi)^*(E)$.*
(3) *The above mapping Φ is unique as a germ of a holomorphic mapping from $(T, *)$ to $(M(\mathcal{E}), *)$. $(M(\mathcal{E}), *)$ and E are called the Kuranishi space and the Kuranishi family of $\mathcal{E}$, respectively.*

Proof. See [Mu-1]. □

We define SV_X as the set of isomorphism classes of simple vector bundles on X. Using Proposition 2.1.8, we can endow it with an analytic structure so that SV_X has a universal family only locally in the étale or classical topology. Then there exists a sufficiently small open set U of SV_X in the classical or étale topology and a vector bundle E on $X \times U$ satisfiying the following property: For any analytic space S, there exists a functorial bijection between the sets $\{$morphisms $S \to U\} \to \{$vector bundles E on $X \times S$ such that $\forall s \in S$, $E_s = E\mid_{X \times s}$ is simple and its class belongs to $U\}/\sim$ given by $\varphi \mapsto (1 \times \varphi)^*(E)$.

Proposition 2.1.9. *Let $X, \mathcal{E}$ be as in Proposition 2.1.8. Every obstruction to the smoothness of SV_X at $[\mathcal{E}]$ lies in $\ker(H^2(\mathrm{Tr}) : H^2(X, \mathcal{E}nd(\mathcal{E})) \to H^2(X, \mathcal{O}_X))$. In particular, SV_X is smooth at $[\mathcal{E}]$ if $H^2(\mathrm{Tr})$ is injective.*

Proof. See [Mu-1]. □

Note, however, that SV_X, even if it is smooth, is not a nice concept of moduli space: it is non-separated in many examples.

2.2 Connections

Let $X, \mathcal{E}$ be as above. A rational (or meromorphic in the case when X is a complex space) connection on $\mathcal{E}$ is a k-linear morphism of sheaves $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X(D)$ satisfying the Leibniz rule:

$$\forall p \in X, \forall f \in \mathcal{O}_p, \forall s \in \mathcal{E}_p, \nabla(fs) = f\nabla s + s \otimes \mathrm{d}f.$$

We assume that D is an effective Cartier divisor and call D the divisor of poles of ∇. We can extend ∇ in a natural way to

$$\mathcal{E} \otimes \Omega^\bullet(*D) = \varinjlim_n \oplus_{i \geq 0} \mathcal{E} \otimes \Omega^i(nD)$$

as a k-linear map $\nabla : \mathcal{E} \otimes \Omega^i(*D) \to \mathcal{E} \otimes \Omega^{i+1}(*D)$ satisfying the Leibniz rule $\nabla(s \otimes \omega) = \nabla s \wedge \omega + s \otimes \mathrm{d}\omega$. The connection is integrable if $\nabla^2 = 0$. In this case, ∇ defines the generalized de Rham complex

$$0 \to \mathcal{E}(*D) \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1(*D) \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^2(*D) \xrightarrow{\nabla} \ldots, \tag{2.11}$$

If X is smooth at all the points of $X \setminus D$, then this complex is exact over $X \setminus D$ in all degrees different from 0 by the Poincaré lemma. Under the same assumption, the subsheaf $\mathcal{E}^h$ of sections s of $\mathcal{E}|_{X \setminus D}$ satisfying $\nabla(s) = 0$ is a local system of rank r, that is a vector bundle with constant transition functions, and $\mathcal{E}|_{X \setminus D} = \mathcal{E}^h \otimes \mathcal{O}_{X \setminus D}$; the sections of $\mathcal{E}^h$ are called horizontal sections of $(\mathcal{E}, \nabla)$. The complex defined above, when restricted to $X \setminus D$, is a resolution of $\mathcal{E}^h$.

A connection ∇ on $\mathcal{E}$ induces natural connections on $\mathcal{E}^*$, $\mathcal{E}nd(\mathcal{E})$, $(\mathcal{E}^*)^{\otimes m} \otimes \mathcal{E}^{\otimes n}$, and more generally, on any Schur functor of $\mathcal{E}$ or $\mathcal{E}^*$. We will use in the sequel the induced connection $\nabla_{\mathcal{E}nd(\mathcal{E})}$ on

$\mathcal{E}nd(\mathcal{E})$. Taking a local section φ of $\mathcal{E}nd(\mathcal{E})$, we can think of φ as a sheaf homomorphism $\mathcal{E}\to\mathcal{E}$ over an open set $U \subset X$, and $\nabla_{\mathcal{E}nd(\mathcal{E})}$ is defined by

$$\nabla_{\mathcal{E}nd(\mathcal{E})}(\varphi) = \nabla \circ \varphi - \varphi \circ \nabla$$
$$\nabla_{\mathcal{E}nd(\mathcal{E})} : \mathcal{E}nd(\mathcal{E})\to\mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)$$

If ∇ is integrable, then $\nabla_{\mathcal{E}nd(\mathcal{E})}$ is also integrable, and $\mathcal{E}nd(\mathcal{E})^h = \mathcal{E}nd(\mathcal{E}^h)$.

Let now $\mathfrak{U} = (U_\alpha)$ be a sufficiently fine open covering of X, e_α a trivialization of $\mathcal{E}$ over U_α, $(g_{\alpha\beta})$ the transition functions of $\mathcal{E}$ with respect to the trivilizations (e_α). The connection matrices $A_\alpha \in \Gamma(U_\alpha, M_r(\mathcal{O}_X) \otimes \Omega^1(D))$ of ∇ are defined by $\nabla(e_\alpha) = e_\alpha A_\alpha$. The transition rule for the matrices A_α is

$$A_\beta = g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} + g_{\alpha\beta}^{-1}A_\alpha g_{\alpha\beta} \tag{2.12}$$

over $U_{\alpha\beta}$. This equation can be given a cohomological interpretation. To this end, introduce the cochains $\mathcal{A} = (\mathcal{A}_\alpha) \in \check{C}^0(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D))$, $\mathcal{G} = (\mathcal{G}_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1)$ by saying that the matrix of $\mathcal{A}_\alpha$ (resp. $\mathcal{G}_{\alpha\beta}$) in the basis e_α is A_α (resp. $\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1}$). Then $\mathcal{G}$ is a cocycle.

Definition 2.2.1. The cohomology class $[\mathcal{G}]$ of $\mathcal{G}$ in $H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1)$ does not depend on the choice of trivializations (e_α) and is called the Atiyah class of $\mathcal{E}$. We will denote this class by $\mathrm{At}(\mathcal{E})$ and its image in $H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D))$, in $H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(*D))$ by $\mathrm{At}^D(\mathcal{E})$,(resp. $\mathrm{At}^{*D}(\mathcal{E})$).

Now we can write (2.12) in the form

$$\mathcal{G} = \check{d}\mathcal{A},$$

and we get the following assertion:

Proposition 2.2.2. *Let $X, \mathcal{E}$ be as above, D an effective Cartier divisor in X. Then $\mathcal{E}$ admits a connection with divisor of poles D if and only if $\mathrm{At}^D(\mathcal{E})$ vanishes in $H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D))$.*

Informally speaking, this property is expressed by saying that the Atiyah class is the obstruction to the existence of a connection on a vector bundle. For future use, we also provide the integrability condition of ∇ in terms of the local data A_α:

$$\mathrm{d}A_\alpha + A_\alpha \wedge A_\alpha = 0 \tag{2.13}$$

2.2.1 First order deformations of connections with fixed divisor of poles D

Let $(\mathcal{E}, \nabla)$ be defined as above and $V_1 = \mathrm{Spec}\, k[\epsilon]/(\epsilon^2)$. We represent the deformed pair $(\tilde{\mathcal{E}}, \tilde{\nabla})$ over V_1 by the local data

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon g_{\alpha\beta,1},\ \tilde{A}_\alpha = A_\alpha + \epsilon A_{\alpha,1}$$

We have already studied the compatibility conditions which guarantee that $\tilde{g}_{\alpha\beta}$ ia a cocycle; they can be stated by saying that the cochain $a = (a_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))$, defined over $U_{\alpha\beta}$ by the matrix

$g_{\alpha\beta,1}g_{\alpha\beta}^{-1}$ in the basis e_α, is a cocycle. Now, we fix this cocycle and search for a cochain $(\mathcal{A}_{\alpha,1})$ compatible with a. Expanding (2.12) to order 1, we obtain:

$$A_{\beta,1} = g_{\beta\alpha,1}\mathrm{d}g_{\alpha\beta} + g_{\beta\alpha}\mathrm{d}g_{\alpha\beta,1} + g_{\beta\alpha,1}A_\alpha g_{\alpha\beta} + g_{\beta\alpha}A_{\alpha,1}g_{\alpha\beta} + g_{\beta\alpha}A_\alpha g_{\alpha\beta,1} \tag{2.14}$$

Lemma 2.2.3. *Define the* 0*-cochain* $\mathcal{A}_1 = (\mathcal{A}_{\alpha,1})$ *in* $\mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D)$ *whose matrix over* U_α *is* $A_{\alpha,1}$ *in the basis* e_α*. Then* (2.14) *implies:*

$$(\check{d}\mathcal{A}_1)_{\alpha\beta} = \mathcal{A}_{\beta,1} - \mathcal{A}_{\alpha,1} = \mathrm{d}a_{\alpha\beta} + [A_\alpha, a_{\alpha\beta}] \tag{2.15}$$

Proof. Conjugate (2.14) by $g_{\alpha\beta}$:

$$g_{\alpha\beta}A_{\beta,1}g_{\alpha\beta}^{-1} = g_{\beta\alpha}^{-1}g_{\beta\alpha,1}\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1} + \mathrm{d}g_{\alpha\beta,1}g_{\alpha\beta}^{-1} + g_{\alpha\beta}g_{\beta\alpha,1}A_\alpha + A_{\alpha,1} + A_\alpha g_{\alpha\beta,1}g_{\alpha\beta}^{-1} \tag{2.16}$$

□

Then $g_{\alpha\beta}A_{\beta,1}g_{\alpha\beta}^{-1}$, $A_{\alpha,1}$ are the matrices of $\mathcal{A}_{\beta,1}, \mathcal{A}_{\alpha,1}$ respectively in the basis e_α; we will also interprete all the remaining terms of (2.16) as matrices of some sections of $\mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)$. We have

$$g_{\beta\alpha}^{-1}g_{\beta\alpha,1} = a_{\beta\alpha} = -a_{\alpha\beta}; \, g_{\alpha\beta,1}g_{\beta\alpha}^{-1} = a_{\alpha\beta}, \tag{2.17}$$

so that

$$g_{\alpha\beta}g_{\beta\alpha,1}A_\alpha + A_\alpha g_{\alpha\beta,1}g_{\alpha\beta}^{-1} = [A_\alpha, a_{\alpha\beta}]. \tag{2.18}$$

Next, $g_{\alpha\beta,1} = a_{\alpha\beta}g_{\alpha\beta}$, so that

$$\mathrm{d}g_{\alpha\beta,1} = \mathrm{d}a_{\alpha\beta}g_{\alpha\beta} + a_{\alpha\beta}\mathrm{d}g_{\alpha\beta}. \tag{2.19}$$

Further, by (2.17),

$$g_{\beta\alpha}^{-1}g_{\beta\alpha,1}\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1} = -a_{\alpha\beta}\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1} \tag{2.20}$$

Combining (2.19), (2.20), we obtain

$$g_{\beta\alpha}^{-1}g_{\beta\alpha,1}\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1} + \mathrm{d}g_{\alpha\beta,1}g_{\alpha\beta}^{-1} = -a_{\alpha\beta}\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1} + \mathrm{d}a_{\alpha\beta} + a_{\alpha\beta}\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1} = \mathrm{d}a_{\alpha\beta} \tag{2.21}$$

Substituing (2.18), (2.21) into (2.16), we obtain (2.15).

Corollary 2.2.4. *The pair* $(\tilde{g}_{\alpha\beta}), (\tilde{\mathcal{A}_\alpha})$ *defines a first order deformation of* $(\mathcal{E}, \nabla)$ *if and only if the cochains* $a = (a_{\alpha\beta}) = (g_{\alpha\beta,1}g_{\alpha\beta}^{-1}), \mathcal{A}_{\alpha,1} = A_{\alpha,1}$ *(both given in the basis* e_α*) satisfy the relations* $\check{d}(a_{\alpha\beta}) = 0, \check{d}(\mathcal{A}_{\alpha,1}) = (\mathrm{d}a_{\alpha\beta} + [A_\alpha, a_{\alpha\beta}])$.

We will interprete the latter result in terms of the induced connection on $\mathcal{E}nd(\mathcal{E})$. As we saw, given a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(D)$ on $\mathcal{E}$, we can define a connection $\nabla_{\mathcal{E}nd(\mathcal{E})} : \mathcal{E}nd(\mathcal{E}) \to \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)$ by $\nabla_{\mathcal{E}nd(\mathcal{E})}(\varphi) = \nabla \circ \varphi - \varphi \circ \nabla$. If we represent φ by its matrix M_α in the basis e_α, then $\nabla_{\mathcal{E}nd(\mathcal{E})}(\varphi) = \mathrm{d}M_\alpha + [A_\alpha, M_\alpha]$. Now, we can reformulate Corollary 2.2.4 as follows.

Proposition 2.2.5. *The first order deformations of* $(\mathcal{E}, \nabla)$ *with fixed divisor of poles* D *are classified by the pairs* $(a, \mathcal{A}_1) \in \check{C}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E})) \times \check{C}^0(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D))$ *such that*

$$\check{d}(a) = 0, \check{d}(\mathcal{A}_1) = \nabla_{\mathcal{E}nd(\mathcal{E})}(a). \tag{2.22}$$

Now, let us assume in addition that the initial connection is integrable. Then the condition that the deformed connection $(\tilde{\mathcal{E}}, \tilde{\nabla})$, given by the data $(a, \mathcal{A}_1)$ as in Proposition 2.2.5 , remains integrable, can be written in the form:

$$\mathrm{d}A_{\alpha,1} = -A_{\alpha,1} \wedge A_\alpha - A_\alpha \wedge A_{\alpha,1}, \tag{2.23}$$

or in an invariant form, $\nabla_{\mathcal{E}nd(\mathcal{E})}(A_1) = 0$. We remark that here we consider $\nabla_{\mathcal{E}nd(\mathcal{E})}$ extended to $\mathcal{E}nd(\mathcal{E}) \otimes \Omega^\bullet(*D)$ in the same way as was explained for $\nabla = \nabla_{\mathcal{E}}$.

Proposition 2.2.6. *The first order deformations of integrable connections* $(\mathcal{E}, \nabla)$ *with fixed divisor of poles* D *are classified by the pairs* $(a, \mathcal{A}_1)$ *as above satisfying three relations*

$$\check{d}(a) = 0, \check{d}(\mathcal{A}_1) = \nabla_{\mathcal{E}nd(\mathcal{E})}(a), \nabla_{\mathcal{E}nd(\mathcal{E})}(\mathcal{A}_1) = 0. \tag{2.24}$$

2.2.2 Hypercohomology

Let $K^\bullet = (K^p, d_K)$ be a complex of sheaves over X, and $\mathfrak{U} = (U_\alpha)$ a sufficiently fine open covering of X. The $\check{C}$ech complex of $K^\bullet$ is the double complex

$$(\check{C}^p(\mathfrak{U}, K^q), \check{d}, (-1)^p d_K). \tag{2.25}$$

The hypercohomology group $\mathbb{H}^i(X, K^\bullet)$ is by definition the i-th cohomology of the simple complex $(L^\bullet, D)$ associated to (2.25):

$$L^n = \oplus_{p+q=n} \check{C}^p(\mathfrak{U}, K^q), D_{|_{\check{C}^p(\mathfrak{U},K^q)}} = \check{d} + (-1)^p d_K,$$

$$\mathbb{H}^i(X, K^\bullet) := H^i(L^\bullet, D).$$

A hypercohomology class $c \in \mathbb{H}^i(X, K^\bullet)$ is represented by a cocycle $c \in L^i$, $c = (\dots, c^{p-1,q+1}, c^{p,q}, c^{p+1,q-1}, \dots)$, where $p + q = i$, and the cocycle condition is $(\dots, \check{d}c^{p-1,q+1} + (-1)^p d_K c^{p,q} = 0, \check{d}c^{p,q} + (-1)^{p+1} d_K c^{p+1,q-1} = 0, \dots)$. A cocycle $(c^{p,q})_{p+q=n}$ is a coboundary if there exists a cochain $(b^{p,q})_{p+q=n-1}$ such that

$$c^{p,q} = \check{d}b^{p-1,q} + (-1)^p d_K b^{p,q-1}.$$

We denote the i-cocycles $\check{Z}^i(\mathfrak{U}, K^\bullet)$ and the i-coboundaries $\check{B}^i(\mathfrak{U}, K^\bullet)$, so that

$$\mathbb{H}^i(X, K^\bullet) = \check{Z}^i(\mathfrak{U}, K^\bullet)/\check{B}^i(\mathfrak{U}, K^\bullet).$$

Let now come back to the setting of Proposition 2.2.5. Define the two-term complex of sheaves

$$\mathcal{C}^\bullet = [\mathcal{C}^0 \to \mathcal{C}^1], \tag{2.26}$$

where $\mathcal{C}^0 = \mathcal{E}nd(\mathcal{E})$, $\mathcal{C}^1 = \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)$, and differential $d_{\mathcal{C}} = \nabla_{\mathcal{E}nd(\mathcal{E})}$. Then the equations (2.22) express the fact that $(a, \mathcal{A}_1) \in \check{Z}^1(\mathfrak{U}, \mathcal{C}^\bullet)$. Changing the bases e_α over $V_1 = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$ by the rule $\tilde{e}_\alpha = e_\alpha(1 + \epsilon h_\alpha)$, where $h = (h_\alpha) \in \check{C}^0(\mathfrak{U}, \mathcal{E}nd(\mathcal{E})) = \check{C}^0(\mathfrak{U}, \mathcal{C}^0)$, we obtain the transformation rule of the cocycle $(a, \mathcal{A}_1)$ in the following form: $(a, \mathcal{A}_1) \rightarrow (a + \check{d}h, \mathcal{A}_1 + d_{\mathcal{C}}h)$, so that isomorphic first order deformations differ by a 1-coboundary. We deduce:

Theorem 2.2.7. *Let X be a complete scheme of finite type over k or a complex space (then $k = \mathbb{C}$). Let $\mathcal{E}$ be a vector bundle on X and ∇ a rational (or meromorphic) connection on $\mathcal{E}$ with divisor of poles D. Then the isomorphism classes of first order deformations of $(\mathcal{E}, \nabla)$ with fixed divisor of poles are classified by $\mathbb{H}^1(X, \mathcal{C}^\bullet)$.*

In order to characterize the first order deformations of integrable connections, we introduce two other complexes:

$$\mathcal{R}^\bullet = [\mathcal{E}nd(\mathcal{E}) \rightarrow \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D) \rightarrow \mathcal{E}nd(\mathcal{E}) \otimes \Omega^2(*D) \rightarrow \dots]$$

with differential $d_{\mathcal{R}} = \nabla_{\mathrm{End}(\mathcal{E})}$, and

$$\mathcal{F}^\bullet = [\mathcal{F}^0 \xrightarrow{d_{\mathcal{F}}} \mathcal{F}^1], \tag{2.27}$$

where $\mathcal{F}^0 = \mathcal{E}nd(\mathcal{E})$, $d_{\mathcal{F}} = \nabla_{\mathcal{E}nd(\mathcal{E})}$, and $\mathcal{F}^1 = \ker(\mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)) \rightarrow \mathcal{E}nd(\mathcal{E}) \otimes \Omega^2(*D))$. It is easy to see that these complexes have the same 1-cocycles and 1-coboundaries, so that

$$\mathbb{H}^1(X, \mathcal{F}^\bullet) = \mathbb{H}^1(X, \mathcal{R}^\bullet).$$

The formulas (2.20) express the fact that the pair $(a, \mathcal{A}_1)$ is a 1-cocycle in either one of the complexes $\mathcal{F}^\bullet, \mathcal{R}^\bullet$.

Theorem 2.2.8. *Let X be a scheme of finite type over k or a complex space (then $k = \mathbb{C}$). Let $\mathcal{E}$ a vector bundle on X and ∇ a rational (or meromorphic) integrable connection on $\mathcal{E}$ with fixed divisor of poles D. Then the isomorphism classes of first order deformations of $(\mathcal{E}, \nabla)$ in the class of integrable connections with fixed divisor of poles D are classified by*

$$\mathbb{H}^1(X, \mathcal{F}^\bullet) = \mathbb{H}^1(X, \mathcal{R}^\bullet).$$

2.3 Obstructions

2.3.1 First obstruction

Let $X, \mathcal{E}, \nabla, (a, \mathcal{A}_1)$ be as in Theorem 2.2.7, and let $(\mathcal{E}_1, \nabla_1)$ be the first order deformation of $(\mathcal{E}, \nabla)$ over V_1 associated to $(a, \mathcal{A}_1)$. We want to determine the obstruction to extend $(\mathcal{E}_1, \nabla_1)$ to $(\mathcal{E}_2, \nabla_2)$ over $V_2 = \operatorname{Spec} k[\epsilon]/(\epsilon^3)$. As before, we only consider deformations with fixed divisor of poles D. We search for the extended data

$$G_{\alpha\beta} = (1 + \epsilon a_{\alpha\beta} + \epsilon^2 a_{\alpha\beta,2}) g_{\alpha\beta} = g_{\alpha\beta} + \epsilon g_{\alpha\beta,1} + \epsilon^2 g_{\alpha\beta,2}$$

$$\tilde{A_\alpha} = A_\alpha + \epsilon A_{\alpha,1} + \epsilon^2 A_{\alpha,2},\ \mathcal{A}_{\alpha,1} = A_{\alpha,1},$$

with respect to the basis e_α. We assume that they satisfy the cocycle condition modulo ϵ^2. Then the cocycle condition modulo ϵ^3 has two counterparts: the one expressing the extendability of $\mathcal{E}_1$, which we have already treated in Section 2, and the other expressing the extendability of the connection. The latter has the following form:

$$\begin{aligned} A_{\beta,2} = g_{\beta\alpha,2}\mathrm{d}g_{\alpha\beta} + g_{\beta\alpha,1}\mathrm{d}g_{\alpha\beta,1} + g_{\beta\alpha}\mathrm{d}g_{\alpha\beta,2} \\ +g_{\beta\alpha,2}A_\alpha g_{\alpha\beta} + g_{\beta\alpha}A_{\alpha,2}g_{\alpha\beta} + g_{\beta\alpha}A_\alpha g_{\alpha\beta,2} \\ +g_{\beta\alpha,1}A_{\alpha,1}g_{\alpha\beta} + g_{\beta\alpha,1}A_\alpha g_{\alpha\beta,1} + g_{\beta\alpha}A_{\alpha,1}g_{\alpha\beta,1} \end{aligned} \tag{2.28}$$

Introduce the cochain $\mathcal{A}_2 \in \check{C}^0(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D))$ given over U_α by the matrix $A_{\alpha,2}$ in the basis e_α. By transformations similar to those used in the proof of (10), and in using formulas (22) and $a_{\beta\alpha,2} - (a_{\alpha\beta,1})^2 + a_{\alpha\beta,2} = 0$, we reduce (2.28) to the following equation:

$$\begin{aligned} \nabla_{\mathcal{E}nd(\mathcal{E})}(a_{\alpha\beta,2}) - \nabla_{\mathcal{E}nd(\mathcal{E})}(a_{\alpha\beta,1})a_{\alpha\beta,1} - [a_{\alpha\beta,1}, \mathcal{A}_{\beta,1}] \\ = \nabla_{\mathcal{E}nd(\mathcal{E})}(a_{\alpha\beta,2}) + \mathcal{A}_{\alpha,1}a_{\alpha\beta,1} - a_{\alpha\beta,1}\mathcal{A}_{\beta,1} = \mathcal{A}_{\beta,2} - \mathcal{A}_{\alpha,2} \end{aligned} \tag{2.29}$$

Let us denote

$$k_{\alpha\beta} = \nabla_{\mathcal{E}nd(\mathcal{E})}(a_{\alpha\beta,2}) + \mathcal{A}_{\alpha,1}a_{\alpha\beta,1} - a_{\alpha\beta,1}\mathcal{A}_{\beta,1}. \tag{2.30}$$

We consider $k = (k_{\alpha\beta})$ as a cochain in $\check{C}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D))$.

Lemma 2.3.1. *k is a skew-symmetric cocycle.*

Proof. A straightforward calculation using the relations

$$a_{\alpha\beta,2} + a_{\beta\gamma,2} + a_{\gamma\alpha,2} = -a_{\alpha\beta,1}a_{\beta\gamma,1} - a_{\beta\gamma,1}a_{\gamma\alpha,1} - a_{\alpha\beta,1}a_{\gamma\alpha,1} \tag{2.31}$$

and $\nabla_{\mathcal{E}nd(\mathcal{E})}(XY) = \nabla_{\mathcal{E}nd(\mathcal{E})}(X)Y + Y\nabla_{\mathcal{E}nd(\mathcal{E})}(X)$, for any local sections X, Y of $\mathcal{E}nd(\mathcal{E})$ □

Proposition 2.3.2. *Let $(a, \mathcal{A}_1) \in \check{Z}^1(\mathfrak{U}, \mathcal{C}^\bullet)$, and let $(\mathcal{E}_1, \nabla_1)$ be the deformation of $(\mathcal{E}, \nabla)$ over V_1 defined by $(a, \mathcal{A}_1)$. Then $(\mathcal{E}_1, \nabla_1)$ extends to a deformation $(\mathcal{E}_2, \nabla_2)$ over V_2 if and only if the following two conditions are verified:*
(i) The Yoneda square $[a_1] \circ [a_1] \in H^2(X, \mathcal{E}nd(\mathcal{E}))$ vanishes.
(ii) Provided (i) holds, let $a_2 = (a_{\alpha\beta,2}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))$ be a solution of (2.31), and let $k = (k_{\alpha\beta})$ be the cocycle (2.30) determined by this choice of a_2. Then $[k] \in H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D))$ vanishes.

The expression $\mathcal{A}_{\alpha,1}a_{\alpha\beta,1} - a_{\alpha\beta,1}\mathcal{A}_{\beta,1}$ entering (2.30) is a component $c^{1,1}$ of the $\check{C}$ech cocycle $(c^{1,1}, c^{2,0}) \in \check{Z}^2(\mathfrak{U}, \mathcal{C}^\bullet)$ representing the Yoneda square $[a_1, \mathcal{A}_1] \circ [a_1, \mathcal{A}_1]$. The other component is $c^{2,0}_{\alpha\beta\gamma} = a_{\alpha\beta,1}a_{\beta\gamma,1} + a_{\beta\gamma,1}a_{\gamma\alpha,1} + a_{\alpha\beta,1}a_{\gamma\alpha,1}$. Hence we have:

Proposition 2.3.3. *Under the assumptions of Prop. (2.3.2), $(\mathcal{E}_1, \nabla_1)$ extends to $(\mathcal{E}_2, \nabla_2)$ over V_2 with fixed divisor of poles D if and only if the Yoneda square $[a_1, \mathcal{A}_1] \circ [a_1, \mathcal{A}_1]$ vanishes in $\mathbb{H}^2(X, \mathcal{C}^\bullet)$.*

2.3.2 Infinitesimal deformations of the Atiyah class

We fix a vector bundle $\mathcal{E}$ on X given by a cocycle $g_{\alpha\beta}$. Recall that we defined the Atiyah class of $\mathcal{E}$ as the cohomology class of the cocycle $\mathcal{G}_{\alpha\beta} = \mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1}$ (here $\mathcal{G}_{\alpha\beta}$ is considered as a section of $\mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)$ given by the matrix $\mathrm{d}g_{\alpha\beta}g_{\alpha\beta}^{-1}$ in the basis e_α).

If $\mathcal{E}_i$ is an extension of $\mathcal{E}$ (as a vector bundle) to $X \times V_i$, where $V_i = \mathrm{Spec}\, k[\epsilon]/(\epsilon^{i+1})$, then we can define the Atiyah class $\mathrm{At}(\mathcal{E}_i) \in H^1(X, \mathcal{E}nd(\mathcal{E}_i) \otimes \Omega^1)$ by the cocycle $\mathcal{G}_{i,\alpha\beta} = \mathrm{d}g_{i,\alpha\beta}g_{i,\alpha\beta}^{-1}$, where $(g_{i,\alpha\beta})$ is a cocycle defining $\mathcal{E}_i$, $g_{i,\alpha\beta} \in \Gamma(U_{\alpha\beta}, M_r(\mathcal{O}_X) \otimes k[\epsilon]/(\epsilon^{i+1}))$. The following assertion is obvious.

Lemma 2.3.4. *Assume that $\mathcal{E}$ admits a connection ∇ with fixed divisor of poles D. Then ∇ extends to a connection ∇_i on $\mathcal{E}_i$ with fixed divisor of poles D if and only if the image $\mathrm{At}^D(\mathcal{E}_i)$ of $\mathrm{At}(\mathcal{E}_i)$ in $H^1(X, \mathcal{E}nd(\mathcal{E}_i) \otimes \Omega^1(D))$ is zero.*

Corollary 2.3.5. *Let $j > 0$, and assume $\mathcal{E}$ extends to a vector bundle $\mathcal{E}_j$ over $X \times V_j$. For any $i \geq 0, i \leq j$, denote by $\mathcal{E}_i$ the restriction of $\mathcal{E}_j$ to $X \times V_i$. The following assertions hold:*
(i) if ∇_j is a connection with fixed divisor D of poles on $\mathcal{E}_j$, then $\nabla_i = \nabla_j|_{\mathcal{E}_i}$ is such a connection on $\mathcal{E}_i$. Thus $\mathrm{At}^D(\mathcal{E}_j) = 0 \Rightarrow \mathrm{At}^D(\mathcal{E}_i) = 0 (i \leq j)$.
(ii) Let $\mathrm{At}^D(\mathcal{E}_j) = 0$. Introduce the natural restriction map

$$\mathrm{res}_{ji} : H^0(\mathcal{E}nd(\mathcal{E}_j) \otimes \Omega^1(D)) \to H^0(\mathcal{E}nd(\mathcal{E}_i) \otimes \Omega^1(D))$$

$$\varphi \mapsto \varphi \otimes k[\epsilon]/(\epsilon^{i+1})$$

Then any connection with fixed divisor of poles D on $\mathcal{E}_i$ extends to such a connection on $\mathcal{E}_j$ if and only if res_{ji} is surjective.

Proof. (i) is obvious. To prove (ii), we use the following observation: for two connections ∇_j, ∇'_j on $\mathcal{E}_j$ with fixed divisor D of poles, the difference $\nabla_j - \nabla'_j$ is an element of $H^0(\mathcal{E}nd(\mathcal{E}_j) \otimes \Omega^1(D))$ and $(\nabla_j - \nabla'_j)|_{\mathcal{E}_i} = \mathrm{res}_{ji}(\nabla_j - \nabla'_j) \in H^0(\mathcal{E}nd(\mathcal{E}_i) \otimes \Omega^1(D))$. □

In this Corollary, it is possible that both $\mathcal{E}_i, \mathcal{E}_j$ admit connections with fixed divisor of poles D, but not every connection with the same D on $\mathcal{E}_i$ extends to such a connection on $\mathcal{E}_j$. To produce an example, set $D = 0, i = 0, j = 1, X$ an elliptic curve, $\mathcal{E} = \mathcal{O}_X^{\oplus 2}$. Define $\mathcal{E}_1$ as a nontrivial extension of vector bundles

$$0 \to \mathcal{O}_{X \times V_1} \xrightarrow{\mu} \mathcal{E}_1 \xrightarrow{\nu} \mathcal{O}_{X \times V_1} \to 0 \tag{2.32}$$

Such extensions are classified by $\mathrm{Ext}^1(\mathcal{O}_{X \times V_1}, \mathcal{O}_{X \times V_1}) = H^1(\mathcal{O}_{X \times V_1}) \simeq k[\epsilon]/(\epsilon^2)$, and we choose an extension class in the form $\epsilon[f]$, so that the extension is trivial modulo ϵ. We can describe $[f]$ and the associated extension explicitly as follows. Let $\mathfrak{U} = \{U_{+-}\}$ be an open covering of X, and $f \in \Gamma(U_\pm, \mathcal{O}_X)$ a function whose cohomology class $[f]$ generates $H^1(X, \mathcal{O}_X)$. Let $e_\pm = (e_{\pm 1}, e_{\pm 2})$ be a basis of $\mathcal{E}|_{U_{+-}}$, and define the transition matrix over U_{+-} by

$$\begin{pmatrix} 1 & \epsilon f \\ 0 & 1 \end{pmatrix}. \tag{2.33}$$

Define the maps μ, ν in (2.32) by $\mu : 1 \mapsto e_{\pm 1}$, $\nu : (e_{\pm 1}, e_{\pm 2}) \mapsto (0,1)$. To be more explicit, we will give X by the Legendre equation

$$y^2 = x(x-1)(x-t) \ (t \in k \setminus \{0,1\}),$$

and define an open covering $\mathfrak{U}$ of X by $U_+ = X \setminus \{\infty\}, U_- = X \setminus \{0\}$. Then we can choose $f = \frac{y}{x}$ as a function having two simple poles at 0 and ∞ and no other singularities. The Residue Theorem implies that it is impossible to represent f as the difference of two functions, one regular on U_+ and the other on U_-, so the cohomology class of f considered as a $\check{C}$ech cocycle of the covering $\mathfrak{U}$ with coefficients in $\mathcal{O}_X$ is nonzero. We now verify that $\mathrm{At}(\mathcal{E}_1) = 0$. It is represented by the cocycle

$$\mathrm{d}g_{+-}g_{+-}^{-1} = \begin{pmatrix} 0 & \epsilon df \\ 0 & 0 \end{pmatrix}, \tag{2.34}$$

and

$$df = \mathrm{d}(\frac{y}{x}) = \frac{\mathrm{d}y}{x} - y\frac{\mathrm{d}x}{x^2} = \omega_+ - \omega_-,$$

where

$$\omega_+ = 2\frac{\mathrm{d}y}{x} - y\frac{\mathrm{d}x}{x^2}, \ \omega_- = \frac{\mathrm{d}y}{x},$$

ω_+ (resp. w_-) being regular on U_+ (resp. U_-). Hence,

$$\mathrm{d}g_{+-}g_{+-}^{-1} = \begin{pmatrix} 0 & \epsilon\omega_+ \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \epsilon\omega_- \\ 0 & 0 \end{pmatrix} \tag{2.35}$$

is a Čech coboundary, and $\mathrm{At}(\mathcal{E}_1) = 0$. Thus $\mathcal{E}_1$ has a regular connection.
Now, we will show that the map res_{10} defined in the last corollary is not surjective, so not every regular connection on $\mathcal{E}$ extends to a regular connection on $\mathcal{E}_1$. Remark that in our case Ω^1_X is trivial, $D = 0$, so res_{10} is just the restriction map $\mathrm{res}_{10} : H^0(\mathcal{E}nd(\mathcal{E}_1)) \to H^0(\mathcal{E}nd(\mathcal{E}_0))$. Consider $\mathcal{E}_1$ as an extension of another kind:

$$0 \to \epsilon\mathcal{E} \to \mathcal{E}_1 \to \mathcal{E} \to 0,$$

where $\epsilon\mathcal{E} \simeq \mathcal{O}_X^{\oplus 2}$ and $\mathcal{E} \simeq \mathcal{E}_1/\epsilon\mathcal{E} \simeq \mathcal{O}_X^{\oplus 2}$. Apply to it $\mathcal{H}om(\mathcal{E}_1, .)$(the Hom-sheaf as $\mathcal{O}_{X\times V_1}$-modules):

$$0 \to \mathcal{H}om(\mathcal{E}_1, \mathcal{E}) \to \mathcal{E}nd(\mathcal{E}_1) \to \mathcal{H}om(\mathcal{E}_1, \mathcal{E}) \to 0.$$

As $\mathcal{E} \simeq \mathcal{O}_X^{\oplus 2}$, the first and the third terms of the last triple are described as follows:

$$\mathcal{H}om(\mathcal{E}_1, \mathcal{E}) \simeq \mathcal{E}nd(\mathcal{E}) = M_2(\mathcal{O}_X).$$

Take an element in $H^0(\mathcal{H}om(\mathcal{E}_1, \mathcal{E})) \simeq M_2(k)$ given by the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.36}$$

(as above, $\mathcal{E}_1, \mathcal{E}$ are trivialized by the bases $e_\pm = (e_{\pm 1}, e_{\pm 2})$). We will see that it is not in the image of the restriction map $\mathrm{res}_{1,0}$.
Indeed, assume there is a lift of

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.37}$$

to $H^0(\mathcal{E}nd(\mathcal{E}_1))$. Then it is given in the basis e_+ by a matrix of the form

$$A_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon B, \tag{2.38}$$

$B \in M_2(k[U_+])$. Transforming it to the basis e_-, we obtain the matrix

$$A_- = \begin{pmatrix} 0 & -\epsilon f \\ 0 & 1 \end{pmatrix} + \epsilon B, \tag{2.39}$$

which has to be regular in U_-. Thus $\epsilon f = \epsilon b_{12} - a_{-12}$, where b_{12} is regular in U_+ and a_{-12} is regular in U_-. This contradicts the fact that f is not a $\check{C}$ech coboundary in $\check{C}(\mathfrak{U}, \mathcal{O}_X)$, and this ends the proof.

2.3.3 Kuranishi space for deformations of connections

Theorem 2.3.6. *Let X be a complete scheme of finite type over k or a complex space (in which case $k = \mathbb{C}$), $\mathcal{C}^\bullet$ the 2-term complex of sheaves on X defined by (2.26), $W = \mathbb{H}^1(X, \mathcal{C}^\bullet)$,$(\delta_1 \dots, \delta_N)$ a basis of W and $(t_1, \dots, t_N)$ the dual coordinates on W. Let W_k denote the k-th infinitesimal neighborhood of 0 in W, and $(\mathcal{E}_1, \nabla_1)$ the universal first order deformation over $X \times W_1$ of a connection $(\mathcal{E}, \nabla)$ on X with fixed divisor of poles D. Then there exists a formal power series*

$$f(t_1, \dots, t_N) = \sum_{k=2}^{\infty} f_k(t_1 \dots, t_N) \in \mathbb{H}^2(X, \mathcal{C}^\bullet)[[t_1, \dots, t_N]],$$

where f_k is homogeneous of degree k ($k \geq 2$), with the following property. Let I be the ideal of $k[[t_1, \dots, t_N]]$ generated by the image of the map $f^* : \mathbb{H}^2(X, \mathcal{C}^\bullet)^* \to k[[t_1, \dots, t_N]]$, adjoint to f. Then for any $k \geq 2$, the pair $(\mathcal{E}_1, \nabla_1)$ extends to a connection $(\mathcal{E}_k, \nabla_k)$ on $X \times V_k$, where V_k is the closed subscheme of W_k defined by the ideal $I \otimes k[[t_1, \dots, t_N]]/(t_1, \dots, t_N)^{k+1}$.

Proof. We will start by fixing a particular choice of coordinates $(t_1, \dots, t_N)$, coming from the spectral sequence $E_1^{p,q} = H^q(\mathcal{C}^p) \Rightarrow \mathbb{H}^{p+q}(\mathcal{C}^\bullet)$. The latter is supported on two vertical strings $p = 0$ and $p = 1$ (see Fig. 2.1).

Thus the spectral sequence degenerates in the second term E_2, and we have the long exact sequence

$$0 \longrightarrow \mathbb{H}^0(X, \mathcal{C}^\bullet) \longrightarrow H^0(X, \mathcal{E}nd(\mathcal{E})) \xrightarrow{d_1} H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D))$$

$$\longrightarrow \mathbb{H}^1(X, \mathcal{C}^\bullet) \longrightarrow H^1(X, \mathcal{E}nd(\mathcal{E})) \xrightarrow{d_1} H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D))$$
$$\longrightarrow \mathbb{H}^2(X, \mathcal{C}^\bullet) \longrightarrow H^2(X, \mathcal{E}nd(\mathcal{E})) \xrightarrow{d_1} H^2(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D)) \to \dots,$$

We deduce the exact triple

$$0 \to W' \to W \to W'' \to 0,$$

with

$$W' = \frac{H^0(X, \mathrm{End}(\mathcal{E}) \otimes \Omega^1_X(D))}{\operatorname{im} d_1}, W = \mathbb{H}^1(X, \mathcal{C}^\bullet),$$
$$W'' = \ker(H^1(X, \mathrm{End}(\mathcal{E})) \to H^1(X, \mathrm{End}(\mathcal{E}) \otimes \Omega^1_X(D)).$$

Let $N' = \dim W'$, $N'' = \dim W''$; choose $t_1, \dots, t_N$ in such a way that $s_1 = t_{N'+1}, \dots, s_{N''} = t_{N'+N''}(N = N' + N'')$, are coordinates on W'' and $t_1, \dots, t_{N'}$ restrict to W' as coordinates on W'. We will construct by induction on $k \geq 0$ the homogeneous forms

$$\begin{aligned} G_{\alpha\beta,k}(s_1, \dots, s_{N''}) &\in \Gamma(U_{\alpha\beta}, \mathcal{E}nd(\mathcal{E})) \otimes k[s_1, \dots, s_{N''}], \\ F_{\alpha\beta\gamma,k}(s_1, \dots, s_{N''}) &\in \Gamma(U_{\alpha\beta\gamma}, \mathcal{E}nd(\mathcal{E})) \otimes k[s_1, \dots, s_{N''}], \\ \bar{f}_k(s_1, \dots, s_{N''}) &\in H^2(X, \mathcal{E}nd(\mathcal{E})) \otimes k[s_1, \dots, s_{N''}], \\ A_{\alpha,k}(t_1 \dots, t_N) &\in \Gamma(U_\alpha, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D)) \otimes k[t_1, \dots, t_N], \\ \kappa_k(t_1 \dots, t_N) &\in H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D)) \otimes k[t_1, \dots, t_N], \\ K_{\alpha\beta,k}(t_1 \dots, t_N) &\in \Gamma(U_{\alpha\beta}, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D)) \otimes k[t_1, \dots, t_N] \end{aligned} \tag{2.40}$$

with the following properties:
(i) $G_{\alpha\beta,0} = g_{\alpha\beta}$, and $A_{\alpha,0}$ define $\mathcal{E}$ and resp. ∇ with respect to the local trivializations e_α of $\mathcal{E}$ on U_α.
(ii) $\bar{f}_k = 0, F_{\alpha\beta\gamma,k} = 0$ for $k = 0, 1$, and $K_{\alpha\beta,0} = 0$.
(iii) For each $k \geq 1$, let $\bar{f}^{(k)} = \sum_{i \leq k} \bar{f}_i$, and let $\bar{I}^{(k+1)}$ be the ideal generated by $(s_1, \dots, s_{N''})^{k+2}$ and the image of the adjoint map $\bar{f}^{(k)*} : H^2(X, \mathcal{E}nd(\mathcal{E}))^* \to k[s_1, \dots, s_{N''}]$. Then $(F_{\alpha\beta\gamma,k+1})$ is a cocycle modulo $\bar{I}^{(k+1)}$ and $\bar{f}_{k+1}$ is a lift to $W'' \otimes k[s_1, \dots, s_{N''}]$ of the cohomology class

$$[(F_{\alpha\beta\gamma,k+1} \mod \bar{I}^{(k+1)})] \in W'' \otimes k[s_1, \dots, s''_N]/\bar{I}^{(k+1)}.$$

(iv) For any $k \geq 1$, set $G^{(k)}_{\alpha\beta} = \sum_{i \leq k} G_{\alpha\beta,i}$. Then

$$G^{(k)}_{\alpha\beta} G^{(k)}_{\beta\gamma} G^{(k)}_{\gamma\alpha} \equiv (1 + F_{\alpha\beta\gamma,k+1}) \mod \bar{I}^{(k+1)}. \tag{2.41}$$

(v) For each $k \geq 1$, set $\kappa^{(k)} = \sum_{i \leq k} \kappa_i$, and let $J^{(k+1)}$ be the ideal generated by $(t_1, \dots, t_N)^{k+2}$ and by the image of the adjoint map $\kappa^{(k)*} : H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D))^* \to k[t_1, \dots, t_N]$. Then $(K_{\alpha\beta,k+1})$ is a cocycle modulo $J^{(k+1)} + \bar{I}^{(k+2)}$ and κ_{k+1} is a lift of the cohomology class

$$[(K_{\alpha\beta,k+1} \mod (J^{k+1} + \bar{I}^{(k+2)})] \in H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)) \otimes k[[t_1, \dots, t_N]]/(J^{k+1} + \bar{I}^{(k+1)})$$

in $H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)) \otimes k[t_1, \dots, t_N]$.
(vi) For any $k \geq 0$, set $A_\alpha^{(k)} = \sum_{i \leq k} A_{\alpha,i}$. Then

$$K_{\alpha\beta,k+1} \equiv \mathrm{d}G_{\alpha\beta}^{(k+1)} - G_{\alpha\beta}^{(k+1)} A_\beta^{(k)} + A_\alpha^{(k)} G_{\alpha\beta}^{(k+1)} \mod (J^{k+1} + \bar{I}^{(k+2)}). \tag{2.42}$$

In these properties, $G_{\alpha\beta}^{(k)}$ is considered as an endomorphism of $\mathcal{E}_k$ over $U_{\alpha\beta} \times V_k$ given by its matrix with respect to two bases: e_α for the source, e_β for the target, where $\mathcal{E}_k$ is the vector bundle over $X \times V_k$ defined by the 1-cocycle $(G_{\alpha\beta}^{(k)})$. Similarly $(A_\alpha^{(k)})$ is understood as a 1-cochain with values in $\mathcal{E}nd(\mathcal{E}_k) \otimes \Omega^1(D)$, and in formula (2.42), $A_\alpha^{(k)}$ (resp $A_\beta^{(k)}$) is represented by its matrix in the basis e_α (resp e_β). The base changes $G_{\alpha\beta,k+1}$ acting on both sides of (2.42), reduce to $G_{\alpha\beta,0}$, since the only nonzero terms in (2.42) are of degree $k+1$, and everything is reduced modulo $(t_1, \dots, t_N)^{k+2}$. Thus (2.42) defines $(K_{\alpha\beta,k+1})$ as a 1-cochain with values in $\mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)$. Going over to the proof, we first remark that $G_{\alpha\beta,0}, A_{\alpha,0}$ are already known, and we have to indicate the choice of $G_{\alpha\beta,k}, A_{\alpha,k}$ inductively on $k \geq 0$, the other data $F_{\alpha\beta\gamma,k}, \bar{f}_k, K_{\alpha\beta,k}, \kappa_k$ being recovered via formulas (2.41),(2.42). To initialize the induction, first look at (2.41) with $k = 0$. Then $F_{\alpha\beta\gamma,1} = 0$ by (ii), which implies

$$G_{\alpha\beta,1} G_{\beta\gamma,0} G_{\gamma\alpha,0} + G_{\alpha\beta,0} G_{\beta\gamma,1} G_{\gamma\alpha,0} + G_{\alpha\beta,0} G_{\beta\gamma,0} G_{\gamma\alpha,1} = 0 \tag{2.43}$$

The latter equation expresses the fact that $(G_{\alpha\beta,1})$ is a 1-cocycle with values in $\mathcal{E}nd(\mathcal{E}) \otimes (W'')^*$. As in Section 2, we can write $G_{\alpha\beta,1} = \sum a_{\alpha\beta}^{(i)} g_{\alpha\beta} s_i$, where $[(a_{\alpha\beta}^{(i)})]$ for $i = 1, \dots, N''$ form the basis of W'' dual to $s_1, \dots, s_{N''}$. Here and further on, we adopt the following convention: all the $G_{\alpha\beta,k}$ (resp. $G_{\alpha\beta}^{(k)}$) are regarded as 1-cochains with values in $\mathcal{E}nd(\mathcal{E})$ (resp. $\mathcal{E}nd(\mathcal{E}_k)$) given by matrices with respect to two bases: e_α for the source, e_β for the target. We denote by $\mathcal{E}_k$ the vector bundle over $X \times V_k$ defined by the cocycle $G_{\alpha\beta}^{(k)}$.

Hence, looking at the first term $G_{\alpha\beta,1} G_{\beta\gamma,0} G_{\gamma\alpha,0}$ of the sum in (2.43), we see that it represents the matrix of $G_{\alpha\beta,1}$ with respect to one and the same basis e_α for the source and the target. The same applies to the other two summands in (2.43), thus (2.43) is the cocycle condition

$$a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha} = 0$$

put down via matrices of the three summands in the basis e_α.

We will adopt the same convention for cochains with values in $\mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)$ or in $\mathcal{E}nd(\mathcal{E}_k) \otimes \Omega^1(D)$. The $A_{\alpha,k}$ (resp $A_\alpha^{(k)}$) will be considered as matrices representing cochains in $\mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)$ (resp. $\mathcal{E}nd(\mathcal{E}_k) \otimes \Omega^1(D)$) in the basis e_α over U_α. Now write (2.42) for $k = 0$:

$$K_{\alpha\beta,1} = \mathrm{d}G_{\alpha\beta,1} - G_{\alpha\beta,1} A_{\beta,0} + A_{\alpha,0} G_{\alpha\beta,1}; \tag{2.44}$$

we take into account that $I^{(1)} = J^{(1)} = 0$ and that $\mathrm{d}G_{\alpha\beta,0} - G_{\alpha\beta,0} A_{\beta,0} + A_{\alpha,0} G_{\alpha\beta,0} = 0$, the latter equation being a form of (2.12) in which $G_{\alpha\beta,0}$ are considered as matrices of endomorphisms of $\mathcal{E}$ written with respect to two bases: e_α for the source, e_β for the target, and $(\mathrm{d}G_{\alpha\beta,0})$ is a cocycle representing $\mathrm{At}^D(\mathcal{E})$.

The r.h.s of (2.44), with the same convention that $G_{\alpha\beta,1}$ are matrices of endormorphisms of $\mathcal{E}$ with respect to the two bases, is just the cochain $(\mathrm{d}a_{\alpha\beta}+[A_\alpha,a_{\alpha\beta}])\in\check{C}^1(\mathfrak{U},\mathcal{E}nd(\mathcal{E})\otimes\Omega^1(D))$. As in (2.22), we can rewrite it as $\nabla_{\mathcal{E}nd(\mathcal{E})}(a)$, where $a=(G_{\alpha\beta,1})$, and this representation makes obvious that $(K_{\alpha\beta,1})$ is a 1-cocycle. The differential d_1 of the spectral sequence being induced by $\nabla_{\mathcal{E}nd(\mathcal{E})}$, we see that the cocycle $(K_{\alpha\beta,1})$ is a coboundary if and only if

$$[a]=[G_{\alpha\beta,1}]\in\ker(H^1(X,\mathcal{E}nd(\mathcal{E}))\otimes(W'')^*\to H^1(X,\mathcal{E}nd(\mathcal{E})\otimes\Omega^1(D))\otimes(W'')^*).$$

Assuming that $(K_{\alpha\beta,1})$ is a coboundary, we choose $(A_{\alpha,1})$ as a solution to

$$\tilde{K}_{\alpha\beta,1}=G_{\alpha\beta,0}A_{\beta,1}-A_{\alpha,1}G_{\alpha\beta,0}\tag{2.45}$$

Such a solution can be chosen as a linear form in $s_1,\dots,s_{N''}$. Single out one such solution and denote it $(A''_{\alpha,1})=(A''_{\alpha,1}(s_1,\dots,s_{N''}))$. Let $(A'^{(i)}_{\alpha,1}), i=1,\dots,N'$ be a basis of $H^0(\mathfrak{U},\mathcal{E}nd(\mathcal{E})\otimes\Omega^1(D))$ dual to the coordinates $t_1,\dots,t_{N'}$ on W'. Then set

$$A_{\alpha,1}=A''_{\alpha,1}(s_1,\dots,s_{N''})+\sum_{i=1}^{N'}A'^{(i)}_{\alpha,1}t_i.$$

Now assume that the forms (2.40) have been constructed up to degree $k\geq 0$ and define them for degree $k+1$. Start by $F_{\alpha\beta\gamma,k+1}$, which we define, as in the proof of Theorem 2.1.4, to be a lift to $\check{Z}^2(\mathfrak{U},\mathcal{E}nd(\mathcal{E}))\otimes k[s_1,\dots,s_{N''}]$, of the homogeneous component of degree $k+1$ in $G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha}$, which is a cocycle modulo $\bar{I}^{(k+1)}+(s_1,\dots,s_{N''})^{k+1}$ by the proof of Lemma 2.1.6.

Then we set $\bar{f}_{k+1}$ equal to any lift of the cohomology class $(F_{\alpha\beta\gamma,k+1})\in H^2(X,\mathcal{E}nd(\mathcal{E}))\otimes k[[s_1,\dots,s_{N''}]]/\bar{I}^{(k+1)}$ to $H^2(X,\mathcal{E}nd(\mathcal{E}))\otimes k[s_1,\dots,s_{N''}]$. By construction, $(F_{\alpha\beta\gamma,k+1})$ is a coboundary modulo $\bar{I}^{(k+2)}+(s_1,\dots,s_{N''})^{k+2}$, so there exists a cochain in

$$\check{C}^1(\mathfrak{U},\mathcal{E}nd(\mathcal{E}))\otimes k[s_1,\dots,s_{N''}]/(\bar{I}^{(k+2)}+(s_1,\dots,s_{N''})^{k+2})$$

whose coboundary is $(F_{\alpha\beta\gamma,k+1}) \mod (\bar{I}^{(k+2)}+(s_1,\dots,s_{N''})^{k+2})$, and $(G_{\alpha\beta,k+1})$ is defined as any lift of this cochain to $\check{C}^1(\mathfrak{U},\mathcal{E}nd(\mathcal{E}))\otimes k[s_1,\dots,s_{N''}]$ which is homogeneous of degree $k+1$ in $s_1,\dots,s_{N''}$. Consider now the expression

$$\begin{aligned}\tilde{K}_{\alpha\beta,k+1}&=\mathrm{d}G^{(k+1)}_{\alpha\beta}-G^{(k+1)}_{\alpha\beta}A^{(k)}_{\beta}+A^{(k)}_{\alpha}G^{(k+1)}_{\alpha\beta}=\mathrm{d}G^{(k)}_{\alpha\beta}-G^{(k)}_{\alpha\beta}A^{(k-1)}_{\beta}+A^{(k-1)}_{\alpha}G^{(k)}_{\alpha\beta}\\&\quad+\mathrm{d}G_{\alpha\beta,k+1}-G_{\alpha\beta,k+1}A^{(k-1)}_{\beta}+A^{(k-1)}_{\alpha}G_{\alpha\beta,k+1}-G^{(k+1)}_{\alpha\beta}A_{\beta,k}+A_{\alpha,k}G^{(k+1)}_{\alpha\beta},\end{aligned}\tag{2.46}$$

By the induction hypothesis, $\tilde{K}_{\alpha\beta,k}=\mathrm{d}G^{(k)}_{\alpha\beta}-G^{(k)}_{\alpha\beta}A^{(k-1)}_{\beta}+A^{(k-1)}_{\alpha}G^{(k+1)}_{\alpha\beta}$ is a cocycle modulo $J^k+\bar{I}^{(k+1)}$ and is a coboundary modulo $J^{k+1}+\bar{I}^{(k+1)}+(t_1,\dots,t_N)^{k+1}$. From (2.42), in order that $\tilde{K}_{\alpha\beta,k+1}$ has no homogeneous components of order $<k+1$ modulo $J^{k+1}+\bar{I}^{(k+1)}+(t_1,\dots,t_N)^{k+1}$, we have to set $(A_{\alpha,k})$ to be a solution of

$$G^{(k+1)}_{\alpha\beta}A_{\beta,k}-A_{\alpha,k}G^{(k+1)}_{\alpha\beta}\equiv\tilde{K}_{\alpha\beta,k}\mod (J^{k+1}+\bar{I}^{k+1}+(t_1,\dots,t_N)^{k+1}),\tag{2.47}$$

where $G^{(k+1)}_{\alpha\beta}$ can be replaced by $G_{\alpha\beta,0}$, so that (2.47) is an equation for the cochain $(G_{\alpha\beta,0}A_{\beta,k})$ with values in $\mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)$.

Thus we come to the following inductive procedure: define $K_{\alpha\beta,k+1}$ as the homogeneous form of degree $k+1$ in $\tilde{K}_{\alpha\beta,k+1}$. Assuming it is a cocycle modulo $(J^{k+1} + \bar{I}^{(k+2)})$, we define κ_{k+1} as a lift to $H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)) \otimes k[t_1, \dots, t_N]$ of the cohomology class $[(K_{\alpha\beta,k+1}) \mod J^{k+1} + \bar{I}^{(k+2)}]$. Then $J^{(k+2)}$ is well-defined and $(K_{\alpha\beta,k+1})$ becomes a coboundary modulo $J^{(k+2)} + \bar{I}^{(k+2)} + (t_1, \dots, t_N)^{k+2}$. Hence we can construct $(A_{\alpha,k+1})$ as a lift to $\check{C}^0(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)) \otimes k[t_1, \dots, t_N]$ of a solution $(A_{\alpha,k+1})$ of the equation

$$G_{\alpha\beta,0}A_{\beta,k+1} - A_{\alpha,k+1}G_{\alpha\beta,0} \equiv \tilde{K}_{\alpha\beta,k+1} \mod (J^{k+2} + \bar{I}^{(k+2)} + (t_1, \dots, t_N)^{k+2}).$$

Thus we have to verify that $(K_{\alpha\beta,k+1})$ is a cocycle. □

Lemma 2.3.7. $(K_{\alpha\beta,k+1})$ *defined as the homogeneous component of degree* $k+1$ *of* $\tilde{K}_{\alpha\beta,k+1}$*, is a* 1*-cocycle modulo* $J^{k+1} + \bar{I}^{(k+2)}$.

Proof. By the induction hypothesis, we have

$$\mathrm{d}G^{(k)}_{\alpha\beta} \equiv G^{(k)}_{\alpha\beta}A^{(k-1)}_{\beta} - A^{(k-1)}_{\alpha}G^{(k)}_{\alpha\beta} \mod (J^k + \bar{I}^{(k+1)}),$$

$$G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha} \equiv 1 + F_{\alpha\beta\gamma,k+1} \mod \bar{I}^{(k+1)},$$

and by construction,

$$\begin{aligned} G_{\alpha\beta,k+1}G^{(k)}_{\beta\gamma}G^{(k)}_{\gamma\alpha} + G^{(k)}_{\alpha\beta}G_{\beta\gamma,k+1}G^{(k)}_{\gamma\alpha} + G^{(k)}_{\alpha\beta}G^{(k)}_{\beta\gamma}G_{\gamma\alpha,k+1} \\ \equiv -F_{\alpha\beta\gamma,k+1} \mod (\bar{I}^{(k+2)} + (s_1, \dots, s_{N''})^{k+2}), \end{aligned}$$

$$K_{\alpha\beta,k+1} \equiv \mathrm{d}G^{(k+1)}_{\alpha\beta} - G^{(k+1)}_{\alpha\beta}A^{(k)}_{\beta} + A^{(k)}_{\alpha}G^{(k+1)}_{\alpha\beta} \mod (J^{k+1} + \bar{I}^{(k+1)}).$$

Denote $G^{(k+1)}_{\alpha\beta}, G^{(k)}_{\alpha\beta}, G_{\alpha\beta,k+1}, A^{(k)}_{\alpha}, K_{\alpha\beta,k+1}$ by $G_{\alpha\beta}, G'_{\alpha\beta}, G''_{\alpha\beta}, A_\alpha, K_{\alpha\beta}$ respectively. We have

$$\begin{aligned} K_{\alpha\beta}G_{\beta\gamma}G_{\gamma\alpha} + G_{\alpha\beta}K_{\beta\gamma}G_{\gamma\alpha} + G_{\alpha\beta}G_{\beta\gamma}K_{\gamma\alpha} \equiv \mathrm{d}G_{\alpha\beta}G_{\beta\gamma}G_{\gamma\alpha} + G_{\alpha\beta}\mathrm{d}G_{\beta\gamma}G_{\gamma\alpha} \\ +G_{\alpha\beta}G_{\beta\gamma}\mathrm{d}G_{\gamma\alpha} - G_{\alpha\beta}A_\beta G_{\beta\gamma}G_{\gamma\alpha} + A_\alpha G_{\alpha\beta}G_{\beta\gamma}G_{\gamma\alpha} - G_{\alpha\beta}G_{\beta\gamma}A_\gamma G_{\gamma\alpha} + G_{\alpha\beta}A_\beta G_{\beta\gamma}G_{\gamma\alpha} \\ -G_{\alpha\beta}\mathrm{d}G_{\beta\gamma}G_{\gamma\alpha}A_\alpha + G_{\alpha\beta}\mathrm{d}G_{\beta\gamma}A_\gamma G_{\gamma\alpha} - \mathrm{d}G'_{\alpha\beta}G'_{\beta\gamma}G'_{\gamma\alpha} \equiv G'_{\alpha\beta}\mathrm{d}G'_{\beta\gamma}G'_{\gamma\alpha} + G'_{\alpha\beta}G'_{\beta\gamma}\mathrm{d}G'_{\gamma\alpha} \\ +\mathrm{d}G''_{\alpha\beta}G'_{\beta\gamma}G'_{\gamma\alpha} + G'_{\alpha\beta}\mathrm{d}G''_{\beta\gamma}G'_{\gamma\alpha} + G'_{\alpha\beta}G'_{\beta\gamma}\mathrm{d}G''_{\gamma\alpha} + \mathrm{d}G'_{\alpha\beta}G''_{\beta\gamma}G'_{\gamma\alpha} + \mathrm{d}G'_{\alpha\beta}G'_{\beta\gamma}G''_{\gamma\alpha} + G''_{\alpha\beta} \\ \mathrm{d}G'_{\beta\gamma}G'_{\gamma\alpha} + G'_{\alpha\beta}\mathrm{d}G'_{\beta\gamma}G''_{\gamma\alpha} + G''_{\alpha\beta}G'_{\beta\gamma}\mathrm{d}G'_{\gamma\alpha} \equiv \mathrm{d}(G'_{\alpha\beta}G'_{\beta\gamma}G'_{\gamma\alpha}) - G'_{\alpha\beta}G''_{\beta\gamma}\mathrm{d}G'_{\gamma\alpha} \\ +\mathrm{d}(G''_{\alpha\beta}G'_{\beta\gamma}G'_{\gamma\alpha} + G'_{\alpha\beta}G''_{\beta\gamma}G'_{\gamma\alpha} + G'_{\alpha\beta}G'_{\beta\gamma}G''_{\gamma\alpha}) \equiv \mathrm{d}(F_{\alpha\beta\gamma,k+1}) - \mathrm{d}(F_{\alpha\beta\gamma,k+1}) \\ \equiv 0 \mod (J^{k+1} + \bar{I}^{(k+2)}) \end{aligned} \tag{2.48}$$

This ends the proof. □

Coming back to the proof of the Theorem, we define f_k as any lift to $\mathbb{H}^2(\mathcal{C}^\bullet) \otimes k[t_1, \ldots, t_N]$, homogeneous of degree k in $t_1, \ldots, t_N$, of the cohomology class of the cochain

$$((K_{\alpha\beta,k}), (F_{\alpha\beta\gamma,k})) \mod (J^k + \bar{I}^{k+1}) \in \check{C}^2(\mathfrak{U}, \mathcal{C}^\bullet) \otimes k[[t_1, \ldots, t_N]]/(J^k + \bar{I}^{(k+1)}), \qquad (2.49)$$

which we are assuming to be a cocycle. Then quotienting by I makes (2.49) a coboundary of $((A_{\alpha,k}), (G_{\alpha\beta,k}))$, and the pair $(G^{(k)}_{\alpha\beta}, (A^{(k)}_\alpha))$ defines $(\mathcal{E}_k, \nabla_k)$ over $X \times V_k$.

It remains to prove that (2.49) is a cocycle with values in $\mathcal{C}^\bullet \otimes k[t_1, \ldots, t_N]/(J^k + \bar{I}^{k+1})$. One part of this, namely, the equation

$$\check{d}(K_{\alpha\beta,k}) = \nabla_{\mathcal{E}nd(\mathcal{E})}(F_{\alpha\beta\gamma,k})$$

is verified by the computation (2.48). The second part $\check{d}(F_{\alpha\beta\gamma,k}) = 0$ is guaranteed by Lemma 2.1.6.

2.4 Integrable connections

2.4.1 Higher order deformations of integrable connections

From now on, we take into account the fact that $(\mathcal{E}, \nabla)$ is an integrable connection with fixed divisor of poles D and consider deformations of $(\mathcal{E}, \nabla)$ preserving the integrability and the divisor of poles. In Theorem 2.2.8, we characterized the first order deformations of $(\mathcal{E}, \nabla)$ in terms of the hypercohomology group $\mathbb{H}^1(X, \mathcal{F}^\bullet) = \mathbb{H}^1(X, \mathcal{R}^\bullet)$. Now we will consider the second order deformation and respectively the first obstruction. So, we search for the extension

$$\begin{aligned} \tilde{g}_{\alpha\beta} &= (1 + \epsilon a_{\alpha\beta,1} + \epsilon^2 a_{\alpha\beta,2}) g_{\alpha\beta} \\ \tilde{A}_\alpha &= A_\alpha + \epsilon A_{\alpha,1} + \epsilon^2 A_{\alpha,2} \end{aligned} \qquad (2.50)$$

of $(g_{\alpha\beta}, A_\alpha)$ to $V = \operatorname{Spec} k[\epsilon]/(\epsilon^3)$. To order 1, we have the conditions (2.24):

$$\check{d}(a_{\alpha\beta,1}) = 0, \check{d}(A_{\alpha,1}) = \nabla(a_{\alpha\beta,1}), \nabla(A_{\alpha,1}) = 0. \qquad (2.51)$$

Expanding (2.13) to order 2, we obtain in addition to (2.6) and (2.23), the equation

$$\nabla A_{\alpha,2} = -A_{\alpha,1} \wedge A_{\alpha,1}, \qquad (2.52)$$

Note that $\nabla(A_{\alpha,1}) = 0$ implies that $\nabla(A_{\alpha,1} \wedge A_{\alpha,1}) = 0$. One easily verifies the following relations

$$\begin{aligned} \nabla(A_{\alpha,1} \wedge A_{\alpha,1}) &= 0 \\ \check{d}(A_{\alpha,1} \wedge A_{\alpha,1}) &= -\nabla(A_{\alpha,1} a_{\alpha\beta,1} - a_{\alpha\beta,1} A_{\beta,1}) \\ \check{d}(A_{\alpha,1} a_{\alpha\beta,1} - a_{\alpha\beta,1} A_{\beta,1}) &= \nabla(a_{\alpha\beta,1} a_{\beta\gamma,1} \circlearrowleft), \end{aligned}$$

where $\circlearrowleft$ denotes the skew-symmetrization on the subscripts α, β, γ. These three equations express the fact that the triple

$$((a_{\alpha\beta,1}a_{\beta\gamma,1} \circlearrowleft), (A_{\alpha,1}a_{\alpha\beta,1} - a_{\alpha\beta,1}A_{\beta,1}), (A_{\alpha,1} \wedge A_{\alpha,1})) \in \check{C}^2(\mathfrak{U}, \mathcal{R}^\bullet)$$

is a cocycle with respect to the differential $D = \nabla \pm \check{d}$. Then the conditions saying that (2.50) is an integrable connection with fixed divisor of poles D modulo ϵ^3, that is, formulas (2.29), (2.31) and (2.52), mean that the cocycle defined above is the coboundary of the cochain $((a_{\alpha\beta,2}), (\mathcal{A}_{\alpha,2}))$:

$$D(a_2, \mathcal{A}_2) = ((a_{\alpha\beta,1}a_{\beta\gamma,1} \circlearrowleft), (A_{\alpha,1}a_{\alpha\beta,1} - a_{\alpha\beta,1}A_{\beta,1}), (A_{\alpha,1} \wedge A_{\alpha,1})).$$

As the cocycle (2.53) represents the Yoneda square of $[a_1, \mathcal{A}_1]$, we deduce:

Proposition 2.4.1. *The first order deformation $(\mathcal{E}_1, \nabla_1)$ of $(\mathcal{E}, \nabla)$ defined by the cocycle $((a_{\alpha\beta,1}), (\mathcal{A}_{\alpha,1}))$ extend to an integrable connection $(\mathcal{E}_2, \nabla_2)$ over $X \times V_2$ with fixed divisor of poles D if and only if the Yoneda square $[a_1, \mathcal{A}_1] \circ [a_1, \mathcal{A}_1]$ is zero in $\mathbb{H}^2(\mathcal{R}^\bullet)$.*

Thus the integrable case looks similar to the non-integrable one (compare to Prop 2.1.3), provided we replace the 2-term complex $\mathcal{C}^\bullet$ by $\mathcal{R}^\bullet$. As far as only the hypercohomology $\mathbb{H}^1$ and $\mathbb{H}^2$ are concerned, we can also truncate $\mathcal{R}^\bullet$ at the level 2: $\mathbb{H}^i(\mathcal{R}^\bullet) = \mathbb{H}^i(\tilde{\mathcal{R}}^\bullet)$, for $i = 0, 1, 2$, where $\tilde{\mathcal{R}}^\bullet = [\mathcal{R}^0 \to \mathcal{R}^1 \to \ker(\mathcal{R}^2 \to \mathcal{R}^3)]$.

2.4.2 Kuranishi space of integrable connections

Now, we turn to the construction of the Kuranishi space of integrable connections with fixed divisor of poles D. Its construction is completely similar to the one in the non-integrable case, so instead of giving a proof of the next theorem, we will only supply some remarks indicating modifications that should be brought to the proof of Theorem 2.3.6 in order to get the proof in the integrable case.

The spectral sequence $E_1^{p,q} = H^q(X, \mathcal{R}^p)$ converging to $\mathbb{H}^\bullet(\mathcal{R}^\bullet)$ is not concentrated on two vertical strings, so here $\mathbb{H}^2(\mathcal{R}^\bullet)$ has a filtration consisting of three nonzero summands which are subquotients of $H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega_X^2(*D)), H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega_X^1(D), H^2(X, \mathcal{E}nd(\mathcal{E}))$. Hence, we have to add to the forms (2.40) two more homogeneous forms of degree k, say

$$\begin{aligned} L_{\alpha,k}(t_1, \dots, t_N) &\in \Gamma(U_\alpha, \mathcal{E}nd(\mathcal{E}) \otimes \Omega_X^2(*D)) \otimes k[t_1, \dots, t_N], \\ l_k(t_1, \dots, t_N) &\in H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega_X^2(*D)) \otimes k[t_1, \dots, t_N], \end{aligned} \tag{2.53}$$

and modify according the conditions $(i), \dots, (vi)$ to which the forms (2.40),(2.53) should satisfy. Remark also that the long exact cohomology sequence for $\mathcal{C}^\bullet$ introduced in the proof of Theorem 2.3.6 remains exact only in its 4 terms when $\mathcal{C}^\bullet$ is replaced by $\mathcal{R}^\bullet$.

Theorem 2.4.2. *Let X be a complete scheme of finite type over k or a complex space (in which case $k = \mathbb{C}$), ∇ an integrable connection on $\mathcal{E}$ with fixed divisor of poles D, $\mathcal{R}^\bullet$ the complex of sheaves on*

X defined above, $W = \mathbb{H}^1(X, \mathcal{R}^\bullet)$, $(\delta_1 \ldots, \delta_N)$ a basis of W and $(t_1, \ldots, t_N)$ the dual coordinates on W. Let W_k denote the k-th infinitesimal neighborhood of 0 in W, and $(\mathcal{E}_1, \nabla_1)$ the universal first deformation of $(\mathcal{E}, \nabla)$ over $X \times W_1$ in the class of integrable connections with fixed divisor of poles D. Then there exists a formal power series

$$f(t_1, \ldots, t_N) = \sum_{k=2}^{\infty} f_k(t_1 \ldots, t_N) \in \mathbb{H}^2(X, \mathcal{R}^\bullet)[[t_1, \ldots, t_N]],$$

where f_k is homogeneous of degree k ($k \geq 2$), with the following property. Let I be the ideal of $k[[t_1, \ldots, t_N]]$ generated by the image of the map $f^* : \mathbb{H}^2(X, \mathcal{R}^\bullet)^* \to k[[t_1, \ldots, t_N]]$, adjoint to f. Then for any $k \geq 2$, the pair $(\mathcal{E}_1, \nabla_1)$ extends to an integrable connection $(\mathcal{E}_k, \nabla_k)$ on $X \times V_k$, where V_k is the closed subscheme of W_k defined by the ideal $I \otimes k[[t_1, \ldots, t_N]]/(t_1, \ldots, t_N)^{k+1}$.

Remark 2.4.3. The complex $\mathcal{R}^\bullet$ may be replaced by its subcomplex $0 \to \mathcal{E}nd(\mathcal{E}) \to \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D) \to \mathcal{E}nd(\mathcal{E}) \otimes \Omega^2_X(2D) \to \ldots$. Theorem 2.3.6 will remain valid if we replace $\mathcal{R}^\bullet$ in its statement by this smaller complex.

In the case where ∇ is an integrable logarithmic connection, we can reduce $\mathcal{R}^\bullet$ further to $\mathcal{L}^\bullet = [0 \to \mathcal{E}nd(\mathcal{E}) \to \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(\log D) \to \mathcal{E}nd(\mathcal{E}) \otimes \Omega^2_X(\log D) \to \ldots]$. We now go over to integrable logarithmic connections.

2.4.3 Integrable logarithmic connections

Definition 2.4.4. Let X be a nonsingular complex projective variety, S a normal crossing divisor with smooth components. An integrable logarithmic connection E on X is a pair $(\mathcal{E}, \nabla)$ where $\mathcal{E}$ is a torsion free coherent sheaf of $\mathcal{O}_X$-modules on X and $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X(\log S)$ is $\mathbb{C}$-linear and satisfies the Leibniz rule and the integrability condition $\nabla^2 = 0$ (see in the beginning of Sect. 2.2).

Let $\mathcal{D}_X$ be the sheaf of algebraic differential operators on X and let $\mathcal{D}_X[\log S]$ be the $\mathcal{O}_X$-subalgebra generated by the germs of tangent vector fields which preserve the ideal sheaf of the reduced scheme S. According to [Ni], a logarithmic connection on X with singularities over S can be interpreted as a $\mathcal{D}_X[\log S]$-module which is coherent and torsion free as an $\mathcal{O}_X$-module.

Remark 2.4.5. A nonsingular integrable connection on X is simply a $\mathcal{D}_X$-module which is coherent as an $\mathcal{O}_X$-module.

Definition 2.4.6. An infinitesimal deformation of an integrable logarithmic connection $\mathcal{E}$ is a pair $(\mathcal{E}_V, \alpha)$, where $\mathcal{E}_V$ is a family of logarithmic connections parametrized by $V = \mathrm{Spec}(\mathbb{C}[\epsilon])/\epsilon^2$, with an isomorphism $\alpha : \mathcal{E}_V/\epsilon\mathcal{E}_V \to \mathcal{E}$.

We define $T_{\mathcal{E}}$ as the set of all equivalence classes of infinitesimal deformations of $\mathcal{E}$. Let the sheaf $\mathcal{K}_{\mathcal{E}}$ be the kernel of $\nabla : \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(\log S) \to \mathcal{E}nd(\mathcal{E}) \otimes \Omega^2(\log S)$. As the curvature of ∇ is 0, the image of $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(\log S)$, is contained in $\mathcal{K}_{\mathcal{E}}$. If $A \in H^0(X, \mathcal{K}_{\mathcal{E}})$, then $\nabla + \epsilon A$ is a family of logarithmic connections on the underlying sheaf $\mathcal{E}$ parametrized by V. This gives a linear map $p : H^0(X, \mathcal{K}_{\mathcal{E}}) \to T_{\mathcal{E}}$.

Theorem 2.4.7. *If an integrable logarithmic connection $\mathcal{E}$ is locally free, the vector space $T_{\mathcal{E}}$ of infinitesimal deformations of $\mathcal{E}$ (which equals the tangent space at $[\mathcal{E}]$ to the moduli scheme $\mathcal{M}$ of stable integrable logarithmic connections when $\mathcal{E}$ is stable) is canonically isomorphic to the first hypercohomology $\mathbb{H}^1(\mathcal{C}_{\mathcal{E}})$ of the complex $\mathcal{C}_{\mathcal{E}} = (\nabla : \mathcal{E}nd(\mathcal{E}) \to \mathcal{K}_{\mathcal{E}})$, which is in turn equal to the first hypercohomology of the logarithmic de Rham complex $\mathcal{L}^\bullet = (\mathcal{E}nd(\mathcal{E}) \otimes \Omega_X^\bullet(\log S), \nabla)$ associated to $\mathcal{E}nd(\mathcal{E})$.*

Proof. See [Ni]. $\square$

We deduce the construction of the Kuranishi space of integrable logarithmic connections over X.

2.4.4 Kuranishi space of integrable logarithmic connections

Theorem 2.4.8. *Let X be a smooth projective variety over an algebraically closed field k (or on $\mathbb{C}$), $\mathcal{E}$ a vector bundle on X, ∇ an integrable logarithmic connection on $\mathcal{E}$, $\mathcal{L}^\bullet$ the complex of sheaves on X defined in Theorem 2.4.7, $W = \mathbb{H}^1(X, \mathcal{L}^\bullet)$, $(\delta_1 \dots, \delta_N)$ a basis of W and $(t_1, \dots, t_N)$ the dual coordinates on W. Let W_k denote the k-th infinitesimal neighborhood of 0 in W, and $(\mathcal{E}_1, \nabla_1)$ the universal first order deformation of $(\mathcal{E}, \nabla)$ over $X \times W_1$ in the class of integrable logarithmic connections with fixed divisor of poles D. Then there exists a formal power series*

$$f(t_1, \dots, t_N) = \sum_{k=2}^{\infty} f_k(t_1 \dots, t_N) \in \mathbb{H}^2(X, \mathcal{L}^\bullet)[[t_1, \dots, t_N]],$$

where f_k is homogeneous of degree k ($k \geq 2$), with the following property. Let I be the ideal of $k[[t_1, \dots, t_N]]$ generated by the image of the map $f^* : \mathbb{H}^2(X, \mathcal{L}^\bullet)^* \to k[[t_1, \dots, t_N]]$, adjoint to f. Then for any $k \geq 2$, the pair $(\mathcal{E}_1, \nabla_1)$ extends to an integrable logarithmic connection $(\mathcal{E}_k, \nabla_k)$ on $X \times V_k$, where V_k is the closed subscheme of W_k defined by the ideal $I \otimes k[[t_1, \dots, t_N]]/(t_1, \dots, t_N)^{k+1}$.

2.5 Parabolic connections

Let X be a smooth projective curve of genus g. We set

$$T_n := \left\{ (t_1, \dots, t_n) \in \overbrace{X \times \cdots \times X}^{n} \,\middle|\, t_i \neq t_j \text{ for } i \neq j \right\}$$

for a positive integer n. For integers d, r with $r > 0$, we set

$$\Lambda_r^{(n)}(d) := \left\{ (\lambda_j^{(i)})_{0 \leq j \leq r-1}^{1 \leq i \leq n} \in \mathbb{C}^{nr} \,\middle|\, d + \sum_{i,j} \lambda_j^{(i)} = 0 \right\}.$$

Take an element $t = (t_1, \dots, t_n) \in T_n$ and $\lambda = (\lambda_j^{(i)})_{1 \leq i \leq n, 0 \leq j \leq r-1} \in \Lambda_r^{(n)}(d)$.

Definition 2.5.1. $(E, \nabla, \{l_*^{(i)}\}_{1 \le i \le n})$ is said to be a (t, λ)-parabolic connection of rank r if
(1) E is a rank r algebraic vector bundle on X, and
(2) $\nabla : E \to E \otimes \Omega_C^1(\log(t_1 + \cdots + t_n)$ is a connection, and
(3) for each t_i, $l_*^{(i)}$ is a filtration of $E|_{t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ such that $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$ and $(\mathrm{Res}_{t_i}(\nabla) - \lambda_j^{(i)} \mathrm{id}_{E|_{t_i}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $j = 0, \ldots, r-1$.

Remark 2.5.2. By condition (3) above and [EV-1], we have

$$\deg E = \deg(\det(E)) = -\sum_{i=1}^{n} \mathrm{Tr}\,\mathrm{Res}_{t_i}(\nabla) = -\sum_{i=1}^{n} \sum_{j=0}^{r-1} \lambda_j^{(i)} = d.$$

Let T be a smooth algebraic scheme which is a covering of the moduli stack of n-pointed smooth projective curves of genus g over $\mathbb{C}$ and take a universal family $(\mathcal{C}, \tilde{t}_1, \ldots, \tilde{t}_n)$ over T.

Definition 2.5.3. We denote the pull-back of $\mathcal{C}$ and $\tilde{t}$ with respect to the morphism $T \times \Lambda_r^{(n)}(d) \to T$ by the same characters $\mathcal{C}$ and $\tilde{t} = (\tilde{t}_1, \ldots, \tilde{t}_n)$. Then $D(\tilde{t}) := \tilde{t}_1 + \cdots + \tilde{t}_n$ becomes a family of Cartier divisors on $\mathcal{C}$ flat over $T \times \Lambda_r^{(n)}(d)$. We also denote by $\tilde{\lambda}$ the pull-back of the universal family on $\Lambda_r^{(n)}(d)$ by the morphism $T \times \Lambda_r^{(n)}(d) \to \Lambda_r^{(n)}(d)$. We define a functor $\mathcal{M}_{\mathcal{C}/T}^{\boldsymbol{\alpha}}(\tilde{t}, r, d)$ from the category of locally noetherian schemes over $T \times \Lambda_r^{(n)}(d)$ to the category of sets by

$$\mathcal{M}_{\mathcal{C}/T}^{\boldsymbol{\alpha}}(\tilde{t}, r, d)(S) := \left\{ (E, \nabla, \{l_j^{(i)}\}) \right\} / \sim,$$

where

1. E is a vector bundle on $\mathcal{C}_S = \mathcal{C} \times_{T \times \Lambda_r^{(n)}(d)} S$ of rank r,
2. $\nabla : E \to E \otimes \Omega^1_{\mathcal{C}_S/S}(D(\tilde{t})_S)$ is a relative connection,
3. $E|_{(\tilde{t}_i)_S} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ is a filtration by subbundles such that $(\mathrm{Res}_{(\tilde{t}_i)_S}(\nabla) - (\tilde{\lambda}_j^{(i)})_S)(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $0 \le j \le r-1$, $i = 1, \ldots, n$,
4. for any geometric point $s \in S$, $\dim(l_j^{(i)}/l_{j+1}^{(i)}) \otimes k(s) = 1$ for any i, j and $(E, \nabla, \{l_j^{(i)}\}) \otimes k(s)$ is α-stable.

Here $(E, \nabla, \{l_j^{(i)}\}) \sim (E', \nabla', \{l_j'^{(i)}\})$ if there exist a line bundle $\mathcal{L}$ on S and an isomorphism $\sigma : E \xrightarrow{\sim} E' \otimes \mathcal{L}$ such that $\sigma|_{t_i}(l_j^{(i)}) = l_j'^{(i)}$ for any i, j and the diagram

$$\begin{array}{ccc}
E & \xrightarrow{\nabla} & E \otimes \Omega^1_{\mathcal{C}/T}(D(\tilde{t})) \\
{\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma\otimes\mathrm{id}} \\
E' \otimes \mathcal{L} & \xrightarrow{\nabla'} & E' \otimes \Omega^1_{\mathcal{C}/T}(D(\tilde{t})) \otimes \mathcal{L}
\end{array}$$

commutes.

Let $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ be a universal family on $\mathcal{C} \times_T M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{t}, r, d)$. We define a complex $\mathcal{G}^\bullet$ by

$$\begin{aligned}
\mathcal{G}^0 &:= \left\{ s \in \mathcal{E}nd(\tilde{E}) \,\middle|\, s|_{\tilde{t}_i \times M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{t},r,d)}(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \text{ for any } i, j \right\} \\
\mathcal{G}^1 &:= \left\{ s \in \mathcal{E}nd(\tilde{E}) \otimes \Omega^1_{\mathcal{C}/T}(D(\tilde{t})) \,\middle|\, \mathrm{Res}_{\tilde{t}_i \times M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{t},r,d)}(s)(\tilde{l}_j^{(i)}) \subset \tilde{l}_{j+1}^{(i)} \text{ for any } i, j \right\} \\
\nabla_{\mathcal{G}^\bullet} &: \mathcal{G}^0 \longrightarrow \mathcal{G}^1; \quad \nabla_{\mathcal{G}^\bullet}(s) = \tilde{\nabla} \circ s - s \circ \tilde{\nabla}.
\end{aligned}$$

As in the previous section, we can construct the Kuranishi space of (t, λ)-parabolic connections on a smooth projective curve in using the hypercohomology of $\mathcal{G}^\bullet$.

Theorem 2.5.4. *Let X be a smooth projective curve over k, $(\mathcal{E}, \nabla, \{l_*^{(i)}\})$ a (t, λ)-parabolic connection on X, $\mathcal{G}^\bullet$ the complex of sheaves on X defined above, $W = \mathbb{H}^1(X, \mathcal{G}^\bullet)$, $(\delta_1 \ldots, \delta_N)$ a basis of W and $(t_1, \ldots, t_N)$ the dual coordinates on W. Let W_k denote the k-th infinitesimal neighborhood of 0 in W, and $(\mathcal{E}_1, \nabla_1, \{l_*^{(i)}\}_1)$ the universal first order deformation of $(\mathcal{E}, \nabla, \{l_*^{(i)}\})$ over $X \times W_1$ in the class of (t, λ)-parabolic connections. Then there exists a formal power series*

$$f(t_1, \ldots, t_N) = \sum_{k=2}^{\infty} f_k(t_1 \ldots, t_N) \in \mathbb{H}^2(X, \mathcal{G}^\bullet)[[t_1, \ldots, t_N]],$$

where f_k is homogeneous of degree k ($k \geq 2$), with the following property. Let I be the ideal of $k[[t_1, \ldots, t_N]]$ generated by the image of the map $f^* : \mathbb{H}^2(X, \mathcal{G}^\bullet) \to k[[t_1, \ldots, t_N]]$, adjoint to f. Then for any $k \geq 2$, the triple $(\mathcal{E}_1, \nabla_1, \{l_*^{(i)}\}_1)$ extends to a (t, λ)-parabolic connection $(\mathcal{E}_k, \nabla_k, \{l_*^{(i)}\}_k)$ on $X \times V_k$, where V_k is the closed subscheme of W_k defined by the ideal $I \otimes k[[t_1, \ldots, t_N]]/(t_1, \ldots, t_N)^{k+1}$.

We now want to construct the Kuranishi space of T-parabolic bundles. Let T be a finite set of smooth points $\{P_1, \ldots, P_n\}$ of X and W a vector bundle on X.

Definition 2.5.5. By a quasi-parabolic structure on a vector bundle W at a smooth point P of X, we mean a choice of a flag

$$W_P = F_1(W)_P \supset F_2(W)_P \supset \ldots \supset F_l(W)_P = 0,$$

in the fibre W_P of W at P. A parabolic structure at P is a pair consisting of a flag as above and a sequence $0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_l < 1$ of weights of W at P.

The integers $k_1 = \dim F_1(W)_P - \dim F_2(W)_P, \ldots, k_l = \dim(F_l(W)_P)$ are called the multiplicities of $\alpha_1, \ldots, \alpha_l$. A T-parabolic structure on W is the triple consisting of a flag at P, some weights α_i, and their multiplicities k_i. A vector bundle W endowed with a T-parabolic structure is called a T-parabolic bundle.

Definition 2.5.6. A T-parabolic bundle W_1 on X is a T-parabolic subbundle of a T-parabolic bundle W_2 on X, if W_1 is a subbundle of W_2 and at each smooth point P of T, the weights of W_1 are a subset of those of W_2. Further, if we take the weight α_{j_0} such that $1 \leq j_0 \leq m$, and the weight β_{k_0} for the greatest integer k_0 such that $F_{j_0}(W_1)_P \subset F_{k_0}(W_2)_P$, then $\alpha_{j_0} = \beta_{k_0}$.

Definition 2.5.7. The parabolic degree of a T-parabolic vector bundle W on X is

$$\operatorname{par}\deg(W) := \deg(W) + \sum_{P\in I}\sum_{i=1}^{r} k_i(P)\alpha_i(P).$$

Definition 2.5.8. A T-parabolic bundle W is stable (resp. semistable) if for any proper nonzero T-parabolic subbundle $W' \subset W$ the inequality

$$\operatorname{par}\deg W' < (\text{resp. } \leq) \frac{\operatorname{par}\deg \mathrm{W}\operatorname{rk}(W')}{\operatorname{rk} W}$$

holds.

We have a forgetful map g from (t,λ) parabolic connections to T-parabolic bundles. We thus can construct the Kuranishi space of T-parabolic bundles by following an analogous argument to the one given above. We first introduce the Higgs field $\Phi : \mathcal{E}\to\mathcal{E}\otimes\Omega^1_X(D)$ defined as follows:

$$\forall p \in X, \forall f \in \mathcal{O}_{X,p}, \forall s \in \mathcal{E}_P, \Phi(fs) = f\Phi(s).$$

We afterwards consider a parabolic bundle $\mathcal{E}$ with fixed weights and parabolic points $P_1,\ldots,P_N$. We set $L = K\otimes\mathcal{O}(P_1,\ldots,P_N)$, the line bundle associated to the canonical divisor together with the divisor of poles $D = P_1+\cdots+P_N$. The sheaf of rational 1-forms on X is identified with the sheaf of rational sections of the canonical bundle having single poles at points $P_1,\ldots,P_N$. We replace t_i by P_i, for $i=1,\ldots,N$ and $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{t},r,d)$ by M^s_T. We define a complex $\mathcal{B}^\bullet$ by

$$\begin{aligned}
\mathcal{B}^0 &:= \left\{ s\in \mathcal{E}nd(\tilde{E}) \,\middle|\, s|_{\tilde{P}_i\times M^s_{Z,\mathcal{C}/T}(\tilde{P},r,d)}(\tilde{l}^{(i)}_j) \subset \tilde{l}^{(i)}_j \text{ for any } i,j \right\}\\
\mathcal{B}^1 &:= \left\{ s\in \mathcal{E}nd(\tilde{E})\otimes\Omega^1_{\mathcal{C}/T}(D(\tilde{P}i)) \,\middle|\, \operatorname{Res}_{\tilde{P}_i\times M^s_{Z,\mathcal{C}/T}(\tilde{P},r,d)}(s)(\tilde{l}^{(i)}_j) \subset \tilde{l}^{(i)}_{j+1} \text{ for any } i,j \right\}\\
\operatorname{ad}\Phi_{\mathcal{B}^\bullet} &: \mathcal{B}^0 \longrightarrow \mathcal{B}^1; \quad \operatorname{ad}\Phi_{\mathcal{B}^\bullet}(s) = \tilde{\Phi}\circ s - s\circ\tilde{\Phi}.
\end{aligned}$$

From this, we deduce the construction of the Kuranishi space of T-parabolic bundles on a smooth projective curve.

Theorem 2.5.9. *Let X be a smooth projective curve over k or a complex space (in which case $k=\mathbb{C}$), $\mathcal{E}$ a T-parabolic bundle on X, $\mathcal{B}^\bullet$ the complex of sheaves on X defined as above, $W = \mathbb{H}^1(X,\mathcal{B}^\bullet)$, $(\delta_1\ldots,\delta_N)$ a basis of W and $(t_1,\ldots,t_N)$ the dual coordinates on W. Let W_k denote the k-th infinitesimal neighborhood of 0 in W, and $\mathcal{E}_1$ the universal first order deformation of $\mathcal{E}$ over $X\times W_1$. Then there exists a formal power series*

$$f(t_1,\ldots,t_N) = \sum_{k=2}^{\infty} f_k(t_1\ldots,t_N) \in \mathbb{H}^2(X,\mathcal{B}^\bullet)[[t_1,\ldots,t_N]],$$

where f_k is homogeneous of degree k ($k\geq 2$), with the following property. Let I be the ideal of $k[[t_1,\ldots,t_N]]$ generated by the image of the map $f^* : \mathbb{H}^2(X,\mathcal{B}^\bullet)^*\to k[[t_1,\ldots,t_N]]$, adjoint to f. Then for any $k\geq 2$, $\mathcal{E}_1$ extends to a T-parabolic bundle $\mathcal{E}_k$ on $X\times V_k$, where V_k is the closed subscheme of W_k defined by the ideal $I\otimes k[[t_1,\ldots,t_N]]/(t_1,\ldots,t_N)^{k+1}$.

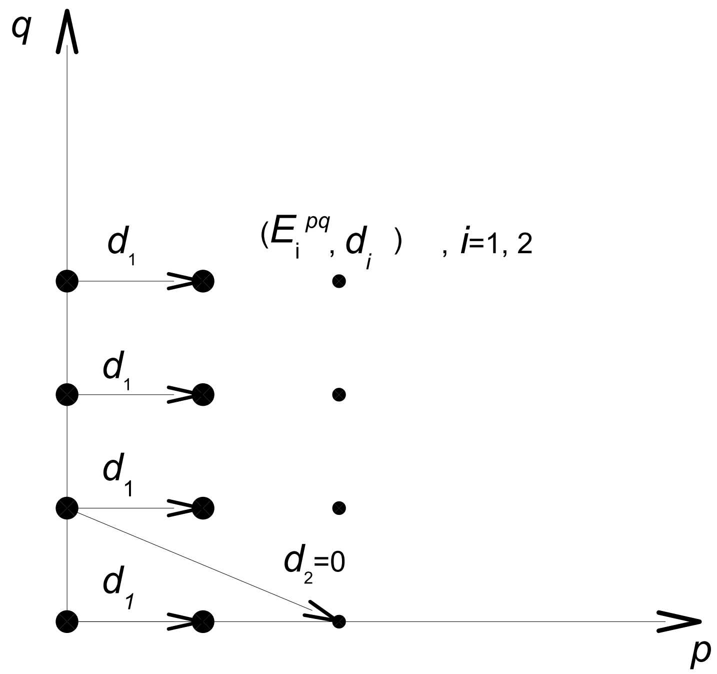

Figure 2.1: The spectral sequence is supported on 2 vertical strings $p = 0, p = 1$.

CHAPTER 3

LOCAL STRUCTURE OF MODULI SPACES

INTRODUCTION

We consider, as before, the four classes of connections: all the connections with fixed divisor of poles D, integrable ones, integrable logarithmic connections and integrable logarithmic ones with a parabolic structure over D. In addition to the above assumptions, we assume that X is projective, and in the logarithmic case, X is smooth and D is a simple normal crossing divisor. In each of these classes, there exists an appropriate notion of stability of $(\mathcal{E}, \nabla)$, and the moduli space of stable objects can be constructed as a GIT quotient under an action of $GL(k)$ for some (big) k. This can be seen for the integrable case by an easy modification of the proof of Simpson [Sim], originally written for regular integrable connections (that is, $D = 0$). For the logarithmic case, the moduli space was constructed by Nitsure [Ni]. The moduli space for logarithmic connections with parabolic structure at poles was constructed in [I-Iw-S] for the case $\dim X = 1$. The nonintegrable case is completely similar to the integrable one. We explain, which modifications one should introduce in the proof of Simpson in order to get the moduli space of meromorphic connections with fixed divisor of poles, either integrable or not.

In all these cases a standard argument using the Luna slice theorem provides a versal deformation of $(\mathcal{E}, \nabla)$ whose base $\mathcal{W}$ is an affine scheme endowed with an action of the group $H = \mathrm{Aut}(\mathcal{E}, \nabla) = \mathrm{Stab}_{GL(k)}(\mathcal{E}, \nabla)$, and the germ of the moduli space at $[(\mathcal{E}, \nabla)]$ is isomorphic to the germ of the GIT quotient $\mathcal{W}//H$. On the other hand, the Kuranishi space $\mathcal{K}$ of $(\mathcal{E}, \nabla)$ is the formal completion of $\mathcal{W}$; it carries a natural action of H, and the quotient $\mathcal{K}//H$ is the formal neighborhood of $[(\mathcal{E}, \nabla)]$ in the moduli space.

We use this method to determine the singularities of the moduli space of connections in some examples.

3.1 MODULI OF SHEAVES

In this section, we introduce basic notions on the moduli space of sheaves. We will follow [H-L], [Ma-1], [Ma-2], [Sim]. Throughout the section, X will be a polarized projective scheme over a field k, $\mathcal{O}_X(1)$ some fixed ample invertible sheaf on X. We will identify vector bundles with their associated locally free sheaves.

3.1.1 Stability

For any coherent sheaf E on X, one defines:

Definition 3.1.1. The support of E is the closed set $\mathrm{Supp}(E) = \{x \in X \mid E_x \neq 0\}$. Its dimension is called the dimension of the sheaf E and is denoted by $\dim(E)$.

Definition 3.1.2. E is pure of dimension d if $\dim(F) = d$ for all non-trivial coherent subsheaves $F \subset E$.

We only consider the equidimensional coherent sheaves on a polarized projective scheme (or variety) over k.

Recall that the Euler characteristic of a pure coherent sheaf E of $\mathcal{O}_X$-modules on X is

$$\chi(E) := \sum_{i=0}^{\dim(E)} (-1)^i h^i(X, E),$$

where $h^i(X, E) = \dim H^i(X, E)$. If we fix an ample line bundle $\mathcal{O}_X(1)$ on X, then the Hilbert polynomial P_E is given by

$$m \mapsto \chi(E \otimes \mathcal{O}_X(m));$$

In particular, $P(E)$ can be uniquely written in the form

$$P(E, m) = \sum_{i=0}^{\dim(E)} \alpha_i(E) \frac{m^i}{i!},$$

with integer coefficients $\alpha_i(E)$. Futhermore, if $E \neq 0$, the leading coefficient $\alpha_{\dim(E)}$, called the multiplicity, is always positive.

Definition 3.1.3. If E is a pure coherent sheaf of $\mathcal{O}_X$-modules on X of dimension $d = \dim(X)$, then

$$\mathrm{rk}(E) := \frac{\alpha_d(E)}{d!}$$

is called the rank of E and

$$\deg(E) := \alpha_{d-1}(E) - \mathrm{rk}(E)(d-1)!$$

is called the degree of E. The slope of E is

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}.$$

Remark 3.1.4. On a smooth projective variety, the Hirzebruch-Riemann-Roch formula shows that $\deg(E) = c_1(E).H^{d-1}$, where H is an ample divisor defined by a section of $\mathcal{O}_X(1)$.

Definition 3.1.5. The reduced Hilbert polynomial $p(E)$ ($E \neq 0$) is defined by

$$p(E, m) := \frac{P(E, m)}{\alpha_d(E)}.$$

Definition 3.1.6. A pure coherent sheaf E of $\mathcal{O}_X$-modules on a scheme X of dimension $d = \dim(X)$ is semistable (resp. stable) if and only if for any proper subsheaf F of E, one has $p(F) \leq p(E)$ (resp. $p(F) < p(E)$).

Remark 3.1.7. We here have used the relation of lexicographic order of their coefficients on the polynomial ring $\mathbb{Q}[m]$. Explicitly, $f \leq g$ if and only if $f(m) \leq g(m)$ for $m >> 0$. Analogously, $f < g$ if and only if $f(m) < g(m)$ for $m >> 0$.

Definition 3.1.8. A pure coherent sheaf E of $\mathcal{O}_X$-modules on X of dimension $d = \dim(X)$ is μ-stable (resp. μ-semistable) if $\mu(F) < \mu(E)$ (resp. $\mu(F) \leq \mu(E)$) for all subsheaves $F \subset E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$.

One easily proves

Corollary 3.1.9. *If E is a pure coherent sheaf E of $\mathcal{O}_X$-modules on X of dimension $d = \dim(X)$, then one has the following chain of implications*

$$E \textit{ is } \mu\textit{-stable} \Rightarrow E \textit{ is stable} \Rightarrow E \textit{ is semistable} \Rightarrow E \textit{ is } \mu\textit{-semistable}.$$

3.1.2 Representable and corepresentable functors

Let $\mathcal{C}$ be a category, $\mathcal{C}^0$ the opposite category, i.e. the category with the same objects and reversed arrows, and let $\mathcal{C}'$ be the functor category whose objects are the functors $\mathcal{C}^0 \to \mathrm{Sets}$ and whose morphisms are the natural transformations between functors. The Yoneda Lemma states that the functor $\mathcal{C} \to \mathcal{C}'$ which associates to $x \in \mathrm{Ob}(\mathcal{C})$ the functor $\underline{x} : y \mapsto \mathrm{Mor}_{\mathcal{C}}(y, x)$ embeds $\mathcal{C}$ as a full subcategory into $\mathcal{C}'$. A functor in $\mathcal{C}'$ of the form $\underline{x}$ is said to be represented by the object x.

Let X be a projective scheme, $\mathcal{C} = \mathrm{Sch}\,/k$. For a fixed polynomial $P \in \mathbb{Q}[z]$ define a functor

$$\mathcal{M}'_X(P) : \mathcal{C}^0 \to \mathrm{Sets}$$

as follows. If $S \in \mathrm{Ob}(\mathrm{Sch}\,/k)$ (that is S is a scheme with a morphism $S \to \mathrm{Spec}(k)$) , let $\mathcal{M}'(S)$ be the set of S-flat families $\mathcal{F} \to X \times S$ of vector bundles on X all of whose fibres have Hilbert polynomial P.

And if $f : S' \to S$ is a morphism in (Sch/k), let $\mathcal{M}'(f)$ be the map obtained by pulling back sheaves via $f_X = id_X \times f$:

$$\mathcal{M}'_X(P)(f) : \mathcal{M}'_X(P)(S) \to \mathcal{M}'_X(P)(S'), [F] \to [f_X^* F].$$

If $\mathcal{F} \in \mathcal{M}'_X(P)(S)$ is an S-flat family of vector bundles on X with Hilbert polynomial P, and if L is an arbitrary line bundle on S, then $\mathcal{F} \otimes p^*(L)$ is also an S-flat family of vector bundles on X with a Hilbert polynomial, where p is the natural projection from $X \times S$ to S. It is therefore reasonable to consider the quotient $\mathcal{M}_X(P) = \mathcal{M}'_X(P)/\sim$, where $\sim$ is the equivalence relation:

$$\mathcal{F} \sim \mathcal{F}' \iff \mathcal{F} \simeq \mathcal{F}' \otimes p^* L \text{ for some } L \text{ in } \mathrm{Pic}(S).$$

Definition 3.1.10. A functor $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}')$ is corepresented by $F \in \mathrm{Ob}(\mathcal{C})$ if there is a $\mathcal{C}'$-morphism $\alpha : \mathcal{F} \to \underline{F}$ such that any morphism $\alpha' : \mathcal{F} \to \underline{F}'$ factors through a unique morphism $\beta : \underline{F} \to \underline{F}'$.

Assume now that $\mathcal{C}$ admits fiber products, then so does $\mathcal{C}'$. A functor $\mathcal{F}$ is universally corepresented by $\alpha : \mathcal{F} \to \underline{F}$ if for any $T \in \mathrm{Ob}(\mathcal{C})$ and any morphism $\varphi : \underline{T} \to \underline{F}$, the fibre product $\Gamma = \underline{T} \times_{\underline{F}} \mathcal{F}$ is corepresented by T.

Definition 3.1.11. A coarse moduli scheme of vector bundles on a polarized projective scheme X over k with Hilbert polynomial P is a scheme $M_X(P)$ such that the functor $\mathcal{M}_X(P) = \mathcal{M}'_X(P)/\sim$ is universally corepresented by $M_X(P)$.

Definition 3.1.12. $\mathcal{F}$ is represented by F if $\alpha : \mathcal{F} \to \underline{F}$ is a $\mathcal{C}'$-isomorphism. We can rephrase that definition by saying that F represents $\mathcal{F}$ if $\mathrm{Mor}_{\mathcal{C}}(\mathcal{Y}, F) = \mathrm{Mor}_{\mathcal{C}'}(\underline{Y}, \mathcal{F})$ for all $\mathcal{Y} \in \mathrm{Ob}(\mathcal{C})$.

Definition 3.1.13. A fine moduli space of vector bundles on a polarized projective scheme X over k with Hilbert polynomial P is a scheme $M_X(P)$ which represents the functor $\mathcal{M}_X(P)$. In this case, $\mathcal{M}_X(P)(M_X(P))$ contains a universal vector bundle $\mathcal{U}$ over $X \times M_X(P)$ with the following property: for any $S \in \mathrm{Sch}/k$ and any $\mathcal{E} \in \mathcal{M}_X(P)(S)$, there exists a unique morphism $\varphi : S \to M_X(P)$ such that $(id_X \times \varphi)^*(\mathcal{U}) \simeq \mathcal{E}$.

Remark 3.1.14. If a fine moduli space $M_X(P)$ exists, it is unique up to an isomorphism. Nevertheless, in general, the functor $\mathcal{M}_X(P)$ is not representable. In fact, there are very few classification problems for which a fine moduli scheme exists. To get, at least, a coarse moduli scheme, we must somehow restrict the class of vector bundles that we consider. In [Ma-1] and [Ma-2], M. Murayama found an answer: (semi)stable vector bundles.

If we take families of (semi)stable locally free sheaves with respect to $H = \mathcal{O}_X(1)$ only, we get open subfunctors $(\mathcal{M}')^{ss}_X(P) \subset \mathcal{M}'_X(P)$, resp $\mathcal{M}'_X(P)^s \subset \mathcal{M}'_X(P)$ and $\mathcal{M}^{ss}_X(P) \subset \mathcal{M}_X(P)$, resp $\mathcal{M}^s_X(P) \subset \mathcal{M}_X(P)$, and $\mathcal{M}^s_X(P)$ is open in $\mathcal{M}^{ss}_X(P)$.

3.1.3 Construction of moduli space

A necessary condition for the existence of a coarse moduli space for the functor $\mathcal{M}^{ss}_X(P)$ as a scheme of finite type on k is the boundedness of the family of all semistable vector bundles on X with Hilbert polynomial P.

Definition 3.1.15. A family of isomorphism classes of coherent sheaves of $\mathcal{O}_X$-modules on X is bounded if there exists a k-scheme S of finite type and a coherent sheaf F of $\mathcal{O}_{X\times S}$-modules on $X \times S$ such that the given family is contained in the set $\{F_s | s$ a closed point in $S.\}$

To present S and an $\mathcal{O}_{X\times S}$-modules F providing the boundedness for $\mathcal{M}_X^{ss}(P)$, we need the following definition.

Definition 3.1.16. Let m be an integer. A coherent sheaf F is said to be m-regular, if

$$H^i(X, F(m-i)) = 0 \text{ for all } i > 0.$$

Lemma 3.1.17. *For any semistable sheaf F with Hilbert polynomial P, there is an integer m such that F is m-regular.*

Proof. Follows Serre's vanishing Theorem. □

Definition 3.1.18. Let $(X, \mathcal{O}_X(1))$ be a polarized projective scheme over k, S a k-scheme of finite type, $\mathcal{C} = (\mathrm{Sch}\,/S)$, $\mathcal{H}$ a S-flat coherent sheaf of $\mathcal{O}_X$-modules with Hilbert polynomial P, then we define the functor

$$\mathbf{Quot}_{\boldsymbol{X/S}}(\boldsymbol{\mathcal{H}}, \boldsymbol{P}) : \mathcal{C}^0 \to \text{Sets}$$

as follows: If T is a S-scheme, then $\mathbf{Quot}_{\boldsymbol{X/S}}(\boldsymbol{\mathcal{H}}, \boldsymbol{P})(T)$ is the set of T-flat coherent quotient sheaves F of the sheaf $\mathcal{H}_T = \mathcal{H} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ such that the fibers of F over all the geometric points of the Grassmann projective S-scheme $\mathcal{G}^r$ have Hilbert polynomial P.

Theorem 3.1.19. *The functor $\mathbf{Quot}_{\boldsymbol{X/S}}(\boldsymbol{\mathcal{H}}, \boldsymbol{P})$ defined above is represented by a projective S-scheme $\mathrm{Quot}_{X/S}(\mathcal{H}, P)$ with the universal quotient sheaf $\mathcal{U}$.*

Proof. See Theorem 2.2.4 of [H-L]. □

Lemma 3.1.20. *As a family representing all the semistable sheaves from $\mathcal{M}_X^{ss}(P)(k)$, one can take the universal quotient sheaf $\mathcal{U}$ over $\mathrm{Quot}_{X/S}(\mathcal{H}, P)$, where $\mathcal{H} = k^{\oplus P(m)} \otimes \mathcal{O}_X(-m)$ and m is such that F is m-regular for all the semistable sheaves F on X with Hilbert polynomial P.*

Corollary 3.1.21. *The family of semistable sheaves with fixed Hilbert polynomial P on a smooth projective variety X is bounded or, in other words, the functor $\mathcal{M}$ is bounded.*

Proof. Follows from Lemma 3.1.20. □

Definition 3.1.22. The S-equivalence classes are the same as Jordan classes. They are defined for sheaves (or vector bundles) on X/k as the classes of semistable sheaves with graded objects which are stable and having the same reduced Hilbert polynomial with respect to the Harder-Narasimham filtration. Let F and F' be semistable sheaves with filtration (F_i) and (F'_i), then F is S-equivalent to F' if and only if
(1) $\mathrm{rk}(F) = \mathrm{rk}(F')$
(2) The quotients $F_i/F_{i-1} \simeq F'_i/F'_{i-1}$ up to an appropriate permutation.

Theorem 3.1.23. *The functor $\mathcal{M}_X^{ss}(P)$ has a coarse moduli scheme $M_X^{ss}(P)$ which is quasi-projective over k, and the points of $M_X^{s.s}(P)$ represent the S-equivalence classes of semistable sheaves with Hilbert polynomial P. There exists an open subscheme $M_X^{s}(P)$ of $M_X^{ss}(P)$ which is quasi-projective and whose points represent the isomorphism classes of stable sheaves with Hilbert polynomial P.*

Proof. This is Theorem 1.21 of [Sim] □

Remark 3.1.24. Stable vector bundles are a class of vector bundles with the property that families of stable bundles over $\mathrm{Spec}(K) \subset \mathrm{Spec}(R)$, where R is a discrete valuation ring with quotient field K, have at most one extension to families of stable bundles over $\mathrm{Spec}(R)$. Hence, by the valuative criterion, the moduli space is separated if it exists. In higher dimension, if we want to represent the functor by a projective moduli space, we have to consider not just (semi)stable vector bundles, but (semi)stable torsion-free sheaves.

Remark 3.1.25. (1) If a coarse moduli space exists for a given classification problem, then it is unique (up to an isomorphism). (2) A fine moduli space for a given classification problem is always a coarse moduli space for this problem, but, in general not vice versa. In fact, there is no a priori reason why the map

$$\Phi(S) : \mathcal{M}^s(S) \to \mathrm{Hom}(S, M^s)$$

should be bijective for varieties S other than a point.

We refer to [H-L] for general facts on the infinitesimal structure of the moduli space M^s. Just let me recall that if E is a stable vector bundle on X with Hilbert polynomial P, represented by a point $[E] \in M^s$, then the Zariski tangent space of M^s at $[E]$ is given by $T_{[E]}M^s \simeq \mathrm{Ext}^1(E, E)$. If $\mathrm{Ext}^2(E, E) = 0$, then M^s is smooth at $[E]$. In general, we have the following bounds:

$$\dim_k \mathrm{Ext}^1(E, E) \geq \dim_{[E]} M^s \geq \dim_k \mathrm{Ext}^1(E, E) - \dim_k \mathrm{Ext}^2(E, E).$$

We rely on Lemma 3.1.20 to describe the family of stable resp. semistable sheaves on X. Thus, $F(m)$ is generated by global sections. If we set $V = k^{\oplus P(m)}$, $\mathcal{H} := V \otimes \mathcal{O}_X(-m)$, then there exists a surjection $\rho : \mathcal{H} \to F$ obtained by composing the canonical evaluation map $H^0(F(m)) \otimes \mathcal{O}_X(-m) \to F$ with an isomorphism $V \to H^0(F(m))$. This defines a closed point $[\mathcal{H} \to F] \in \mathrm{Quot}(\mathcal{H}, P)$, more precisely this point is contained in the open subscheme R of all those coherent quotient sheaves $[\mathcal{H} \to E]$, where E is semistable and the induced map $H^0(\mathcal{H}(m)) \simeq H^0(E(m))$ is an isomorphism. The family of stable sheaves is parametrized by an open subscheme R^s of R. The family of S-equivalence classes of semistable sheaves is parametrized by the quotient of R by $\mathrm{GL}(V)$.
The next lemma relates the moduli problem to that of finding a quotient for the group action.

Theorem 3.1.26. *If $R \to M_X^{ss}(P)$ is a categorical quotient for the $GL(V)$-action, then $M_X^{ss}(P)$ corepresents the functor $\mathcal{M}_X^{ss}(P)$. Conversely, if $M_X^{ss}(P)$ corepresents $\mathcal{M}_X^{ss}(P)$ then the morphism $R \to M_X^{ss}(P)$, induced by the universal quotient module on $R \times X$, is a categorical quotient . Similarly, $R^s \to M_X^s(P)$ is a categorical quotient if and only if $M_X^s(P)$ corepresents $\mathcal{M}_X^s(P)$. Therefore, we have $M_X^{ss}(P) = R//GL(V)$ and $M_X^s(P) = R^s//GL(V)$.*

Proof. See [H-L]. □

3.2 Moduli of connections

Basics for constructing moduli spaces of connections were developed by Simpson [Sim] and Nitsure [Ni]. Nitsure constructed the coarse moduli space of integrable logarithmic connections with poles on a normal crossing divisor D in a smooth projective manifold. Simpson provided a more general approach covering not only regular integrable connections and Nitsure's case of integrable logarithmic connections, but also Higgs bundles, Hitchin pairs, integrable connections along a foliation, Deligne's τ- connnections and so on. Simpson handled all these objects on equal basis as Λ-modules for an appropriate sheaf of rings of differential operators Λ, which is just the standard sheaf of differential operators $\mathcal{D}_X$ for regular integrable connections, and its associated graded $\mathrm{gr}_\bullet\mathcal{D}_X = \oplus_{m=0}^{\infty} S_m\mathcal{T}_X$ in the case of Higgs bundles, where $S_m\mathcal{T}_X$ denotes the m-th symmetric power of the tangent bundle on X.

3.2.1 Sheaf Λ of rings of differential operators

We will recall Simpson's definition of Λ. Let X be a scheme of finite type over k, an algebraically closed field of characteristic 0. Then a sheaf of rings of differential operators on X is a sheaf Λ of associative rings with unity together with a filtration by subsheaves of abelian groups $\Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \dots$ satisfying the following properties:
(1) $\Lambda_0 = \mathcal{O}_X$, $\Lambda_i \cdot \Lambda_j \subset \Lambda_{i+j}$, and $\Lambda = \cup_{i=0}^{\infty}\Lambda_i$. In particular, Λ and each Λ_i are $\mathcal{O}_X$-bimodules.
(2) The image of the constant sheaf $\mathbb{C}_X$ in $\mathcal{O}_X = \Lambda_0$ is contained in the center of Λ.
(3) The left and the right structures of $\mathcal{O}_X$-modules on the i-th graded piece $\mathrm{gr}_i(\Lambda) = \Lambda_i/\Lambda_{i-1}$ are equal.
(4) The sheaves $\mathrm{gr}_i(\Lambda)$ are $\mathcal{O}_X$-coherent.
(5) The $\mathcal{O}_X$-algebra $\mathrm{gr}_\bullet(\Lambda)$ is generated by its component $\mathrm{gr}_1(\Lambda)$ of degree 1.
(6) There is a left $\mathcal{O}_X$-linear map $\iota : \mathrm{gr}_1(\Lambda)\to\Lambda_1$ providing a splitting of the triple $0\to\Lambda_0\to\Lambda_1\to \mathrm{gr}_1(\Lambda)\to 0$.
(7) $\mathrm{gr}_\bullet(\Lambda)$ is the symmetric algebra $S_\bullet(\mathrm{gr}_1(\Lambda))$ over $\mathrm{gr}_1(\Lambda)$.

As we have mentioned above, $\Lambda = \mathcal{D}_X$ is an example. If X is smooth, $\mathcal{E}$ a vector bundle and ∇ an integrable connection over X, then we can consider $\mathcal{E}$ as a left Λ-module, setting $v \cdot s = \nabla_v(s)$ for any local vector field v and any local section s of $\mathcal{E}$. To include the case of non-integrable connections, we will replace the axiom (7) by the following one
$(7')$ $\mathrm{gr}_.(\Lambda)$ is the tensor algebra $T_\bullet(\mathrm{gr}_1(\Lambda)) = \oplus_{i=0}^{\infty} T_i(\mathrm{gr}_1(\Lambda))$ over $\mathrm{gr}_1(\Lambda)$, where $T_i(V) = V^{\otimes i}$ for any module V.

Simpson's construction of moduli spaces of semistable Λ-modules works in all details whether Λ satisfies the axioms $(1) - (7)$ or $(1) - (6), (7')$. We will specialize it to the case of meromorphic connections as in Chapter 2, and will briefly describe the steps of the construction of the moduli space. In the sequel, Λ satisfies the axioms (7) or $(7')$ depending on whether we are working with integrable or arbitrary connections.

Let X be a smooth variety over k and $\mathfrak{D}_X$ the sheaf of non commutative rings of differential operators on X. It can be defined as follows. As a left $\mathcal{O}_X$-module, it is just the tensor algebra

$T_\bullet(\mathcal{T}_X)$ over the tangent bundle. To determine the multiplicative structure on it, it suffices to define the products $v \cdot f$, where $v \in \mathcal{T}_{X,p}$, $f \in \mathcal{O}_{X,p}$, $p \in X$. We set

$$v \cdot f = fv + v(f),$$

where $v(f)$ denotes the derivative of f in the direction of v. This rule allows us to transform the product of two elements of $T_\bullet(\mathcal{T}_X)$

$$fv_1 \otimes \cdots \otimes v_r \cdot gw_1 \otimes \cdots \otimes w_s$$

into an element of $T.(\mathcal{T}_X)$ in a finite number of steps.

Now, let us fix an effective divisor D on X. We define Λ as the subsheaf of rings in $\mathfrak{D}_X$ generated by $\mathcal{O}_X$ and $\mathcal{T}_X(-D)$, the latter sheaf being considered as the subsheaf of $\mathcal{T}_X \subset \mathfrak{D}_{X1}$ consisting of vector fields vanishing on D. Then for a rank r connection $(\mathcal{E}, \nabla)$ with divisor of poles D, we endow $\mathcal{E}$ with a structure of a left Λ-module by setting

$$v \cdot s = \nabla_v s = v \lrcorner \nabla s$$

for any $v \in \mathcal{T}_{X,p}, s \in \mathcal{E}_p, p \in X$. Conversely, we can completely recover ∇ from a structure of a Λ-module in applying the above formula to the vector field v ranging over some basis of $\mathcal{T}_{X,p}$ as an $\mathcal{O}_{X,p}$-module. In the sequel of this section, we will think of connections $(\mathcal{E}, \nabla)$ with divisor of poles D as left Λ-modules that are locally free of rank r when considered as $\mathcal{O}_X$-modules. For intermediate steps of this construction, we also need to consider coherent $\mathcal{O}_X$-modules with a structure of a left Λ-module.

From now on, Λ is any sheaf of rings of differential operators satisfying either axioms $(1)-(7)$ or $(1)-(6),(7')$. The moduli space of semistable Λ-modules that we are going to construct is interpreted as the moduli space of a class of connections in the following cases.

(a) $\Lambda \subset \mathfrak{D}_X$, generated by $\mathcal{O}_X$ and $\mathcal{T}_X(-D)$ and satisfying axiom $(7')$, correspond to meromorphic connections with fixed divisor of poles D.

(b) $\Lambda \subset \mathcal{D}_X$, generated by $\mathcal{O}_X$ and $\mathcal{T}_X(-D)$ and satisfying axiom (7), correspond to integrable meromorphic connections with fixed divisor of poles D.

(c) $\Lambda \subset \mathfrak{D}_X$, generated by $\mathcal{O}_X$ and $\mathcal{T}_X < D >= \mathcal{T}_X(\log D)$ and satisfying axiom $(7')$, correspond to logarithmic connections with a simple normal crossing divisor of poles D.

(d) $\Lambda \subset \mathcal{D}_X$, generated by $\mathcal{O}_X$ and $\mathcal{T}_X < D >= \mathcal{T}_X(\log D)$ and satisfying axiom (7), correspond to integrable logarithmic connections with a simple normal crossing divisor of poles D.

The sheaf $\mathcal{T}_X < D >$ is dual to $\Omega^1_X(\log D)$ and can be defined as the subsheaf of $\mathcal{T}_X$ preserving the ideal subsheaf $\mathcal{I}_D$ of D in X.

3.2.2 Moduli space of semistable Λ-modules

To speak about quasi-projective moduli spaces, we have to start with defining the notions of stability and semistability.

Definition 3.2.1. Let X be a smooth projective variety with a very ample sheaf $\mathcal{O}_X(1)$. A coherent $\mathcal{O}_X$-module $\mathcal{E}$ of rank $r > 0$ endowed with a structure of a left Λ-module is called a semistable Λ-module of rank r, if it is torsion free and for any Λ-submodule $\mathcal{F} \neq 0$ of $\mathcal{E}$,

$$\frac{P(\mathcal{F}, n)}{\mathrm{rk}(\mathcal{F})} \leq \frac{P(\mathcal{E}, n)}{\mathrm{rk}(\mathcal{E})} \; \forall n >> 0,$$

where $P(\mathcal{F}, n) = \chi(\mathcal{F}, n)$ denotes the Hilbert polynomial of $\mathcal{F}$. If the inequality is strict for all $\mathcal{F}$, then $\mathcal{E}$ is called a stable Λ-module.

The following lemma is crucial for the boundedness of the family of all semistable Λ-modules of rank r with fixed Hilbert polynomial P on X.

Lemma 3.2.2. *Let m be an integer such that $gr_1(\Lambda) \otimes \mathcal{O}_X(m)$ is generated by global sections. Then for any semistable Λ-module $\mathcal{E}$ of rank r, and any $\mathcal{O}_X$ submodule $\mathcal{F} \neq 0$, we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E}) + mr$, where $\mu(\mathcal{F})$ denotes the slope of $\mathcal{F}$, $\mu(\mathcal{F}) = \frac{\deg_{\mathcal{O}_X(1)} c_1(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}$.*

Proof. See the proof of Lemma 3.3 of [Sim], which works perfectly for our definition of Λ. □

Corollary 3.2.3. *The set of semistable Λ-modules on X with given Hilbert polynomial P is bounded.*

Proof. We remark that semistable $\Rightarrow$ μ-semistable and refer to the proof of Corollary 3.4 of [Sim]. □

The following assertion realizes the boundedness property for semistable Λ-modules with given Hilbert polynomial P: it provides a scheme parametrizing all of them.

Theorem 3.2.4. *For fixed P, there exists $N_0 \in \mathbb{N}$ depending on Λ and P such that for any $N \geq N_0$ and any S-flat semistable Λ-module $\mathcal{E}$ with Hilbert polynomial P on X such that $\forall s \in S$, we have $H^i(X, \mathcal{E}_s(N)) = 0$ if $i > 0$, $\dim H^0(X, \mathcal{E}_s(N)) = P(N)$ and $\mathcal{E}_s(N)$ is generated by global sections.*

Pick up any $N \geq N_0$. Then the functor which associates to each k-scheme S the set of isomorphism classes of pairs $(\mathcal{E}, \alpha)$, where $\mathcal{E}$ is a semistable Λ-module with Hilbert polynomial P on $X_S = X \times S$ and α is an isomorphism $\mathcal{O}_S^{P(N)} \rightarrow H^0(X_S/S, \mathcal{E}(N))$, is represented by a quasi-projective scheme Q over k.

Proof. See Corollary 3.6 and Theorem 3.8 of [Sim]. □

The scheme Q is constucted in Theorem 3.2.4 in several steps. First, take the Grothendieck Quot scheme $\tilde{Q} = \mathrm{Quot}^P(\mathcal{O}_X^{P(N)}(-N))$ parametrizing the quotients $\mathcal{O}_X^{P(N)}(-N) \rightarrow \mathcal{E} \rightarrow 0$ with Hilbert polynomial P. Over $\tilde{Q}$, one considers the family $\tilde{\tilde{Q}}$ of morphisms $\Lambda_1 \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \mathcal{E}$ defining on the quotients $\mathcal{E}$ structures of Λ-modules. This is a family with affine fibers (e.g. affines spaces in the case (a) of non-integrable connections). And finally, Q is the open subscheme of $\tilde{\tilde{Q}}$ parametrizing the Λ-modules which are semistable.

Let now $\mathcal{M}(\Lambda, P)$ denote the functor on schemes over k which associates to a k-scheme S the set of isomorphism classes of semistable Λ-modules with Hilbert polynomial P. We are now ready to construct the moduli space for this functor as a GIT quotient.

Theorem 3.2.5. *Under the hypotheses and in the notation of Theorem* 3.2.4, Q *is invariant under* $G = SL(P(N))$ *and carries a G-linearized very ample line bundle* $\mathcal{L}$ *such that all the points of* Q *are* $\mathcal{L}$*-semistable.*

Let $M(\Lambda, P) = Q//G$ *be the GIT quotient. Then* $M(\Lambda, P)$ *universally corepresents* $\mathcal{M}(\Lambda, P)$*. The following properties hold:*
(1) $M(\Lambda, P)$ *is a quasi-projective variety.*
(2) *The geometric points of* $M(\Lambda, P)$ *represent* S*-equivalence classes of semistables* Λ*-modules with Hilbert polynomial* P*.*
(3) *The closed orbits are in* 1*-to-*1 *correspondence with the semisimple objects.*
(4) *The geometric points of the open set* $M^s(\Lambda, P) = Q^s//G$*, where* Q^s *parametrizes stable* Λ*-modules, are in* 1*-to-*1 *correspondence with the isomorphism classes of stable* Λ*-modules with Hilbert polynomial* P*.*

Proof. This is Theorem 4.7 of [Sim]. □

Remark that the notion of S-equivalence and semisimple objects are defined exactly as in the case of moduli of sheaves. Namely, any semistable Λ-module $\mathcal{E}$ has a Harder-Narisimham filtration

$$\mathcal{E}_0 = 0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_t = \mathcal{E}$$

with the property that all the factors $gr_i\mathcal{E} = \mathcal{E}_i/\mathcal{E}_{i-1}$ are stable Λ-modules with the same reduced Hilbert polynomial, equal to the reduced Hilbert polynomial $\frac{P(\mathcal{E},n)}{\mathrm{rk}\,\mathcal{E}}$ of $\mathcal{E}$.
Two semistable Λ-modules are called S-equivalent if the associated graded objects of their Harder-Narasimhan filtrations are isomorphic. Further, $\mathcal{E}$ is semisimple if $\mathcal{E} \simeq gr(\mathcal{E})$.

Finally, if we assert the additional restriction that our Λ-modules are locally free as $\mathcal{O}_X$-modules, we will obtain the open subvarieties $M_X^0(\Lambda, P) \subset M(\Lambda, P)$ and $M^{0,s}(\Lambda, P) \subset M^s(\Lambda, P)$, moduli spaces of vector bundles with a structure of a Λ-module

3.3 MODULI OF STABLE PARABOLIC CONNECTIONS, ACCORDING TO INABA-IWASAKI-SAITO

For the sake of completeness, we set out here the main results of [I] and [I-Iw-S]. We consider vector bundles and connections over the base $X = C$ which is a smooth projective curve over k (or a compact Riemann surface if $k = \mathbb{C}$).

Let X be a smooth projective curve of genus g. We set

$$T_n := \left\{ (t_1, \dots, t_n) \in \overbrace{X \times \cdots \times X}^{n} \;\middle|\; t_i \neq t_j \text{ for } i \neq j \right\}$$

for a positive integer n. For integers d, r with $r > 0$, we set

$$\Lambda_r^{(n)}(d) := \left\{ (\lambda_j^{(i)})_{0\le j\le r-1}^{1\le i\le n} \in \mathbb{C}^{nr} \;\middle|\; d + \sum_{i,j} \lambda_j^{(i)} = 0 \right\}.$$

Take an element $t = (t_1, \dots, t_n) \in T_n$ and $\lambda = (\lambda_j^{(i)})_{1\le i\le n, 0\le j\le r-1} \in \Lambda_r^{(n)}(d)$. Let T be a smooth algebraic scheme which is a covering of the moduli stack of n-pointed smooth projective curves of genus g over $\mathbb{C}$ and take a universal family $(\mathcal{C}, \tilde{t}_1, \dots, \tilde{t}_n)$ over T. Take rational numbers

$$0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \cdots < \alpha_r^{(i)} < 1$$

for $i = 1, \dots, n$ satisfying $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$ for $(i,j) \neq (i',j')$. We choose $\boldsymbol{\alpha} = (\alpha_j^{(i)})$ sufficiently generic.

Definition 3.3.1. A parabolic connection $(E, \nabla, \{l_*^{(i)}\}_{1\le i\le n})$ is $\boldsymbol{\alpha}$-stable (resp. $\boldsymbol{\alpha}$-semistable) if for any proper nonzero subbundle $F \subset E$ satisfying $\nabla(F) \subset F \otimes \Omega_C^1(t_1 + \cdots + t_n)$, the inequality

$$\frac{\deg F + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim((F|_{t_i} \cap l_{j-1}^{(i)})/(F|_{t_i} \cap l_j^{(i)}))}{\operatorname{rk} F} \underset{(\text{resp. } \le)}{<} \frac{\deg E + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim(l_{j-1}^{(i)}/l_j^{(i)})}{\operatorname{rk} E}$$

holds.

We describe briefly the construction of the moduli space of stable parabolic connections. On a curve C, we recall the definition of a parabolic Λ_D^1-triple following [I-Iw-S]. Let D be an effective divisor on a curve C. We define Λ_D^1 as $\mathcal{O}_C \oplus \Omega_C^1(D)^\vee$ with the bimodule structure given by

$$\begin{aligned} f(a,v) &= (fa, fv) \quad (f, a \in \mathcal{O}_C, v \in \Omega_C^1(D)^\vee) \\ (a,v)f &= (fa + v(f), fv) \quad (f, a \in \mathcal{O}_C, v \in \Omega_C^1(D)^\vee). \end{aligned}$$

Definition 3.3.2. $(E_1, E_2, \Phi, F_*(E_1))$ is said to be a parabolic Λ_D^1-triple on C of rank r and degree d if

1. E_1 and E_2 are vector bundles on C of rank r and degree d,
2. $\Phi : \Lambda_D^1 \otimes E_1 \to E_2$ is a nonzero left $\mathcal{O}_C$-homomorphism,
3. $E_1 = F_1(E_1) \supset F_2(E_1) \supset \cdots \supset F_l(E_1) \supset F_{l+1}(E_1) = E_1(-D)$ is a filtration by coherent subsheaves.

We take positive integers β_1, β_2, γ and rational numbers $0 < \alpha_1' < \cdots < \alpha_l' < 1$. We assume $\gamma \gg 0$. We denote $D(t) = t_1 + \cdots + t_n$.

Definition 3.3.3. A parabolic Λ_D^1-triple $(E_1, E_2, \Phi, F_*(E_1))$ is $(\boldsymbol{\alpha}, \beta, \gamma)$-stable (resp. $(\boldsymbol{\alpha}, \beta, \gamma)$-semistable) if for any subbundles $(F_1, F_2) \subset (E_1, E_2)$ satisfying $(0,0) \neq (F_1, F_2) \neq (E_1, E_2)$ and $\Phi(\Lambda_{D(t)}^1 \otimes F_1) \subset F_2$, the inequality

$$\frac{\beta_1 \deg F_1(-D) + \beta_2(\deg F_2 - \gamma \operatorname{rk} F_2) + \beta_1 \sum_{j=1}^l \alpha_j' \operatorname{length}((F_j(E_1) \cap F_1)/(F_{j+1}(E_1) \cap F_1))}{\beta_1 \operatorname{rk} F_1 + \beta_2 \operatorname{rk} F_2}$$

$$\begin{array}{c} < \\ \text{(resp. } \leq) \end{array} \frac{\beta_1 \deg E_1(-D) + \beta_2(\deg E_2 - \gamma \operatorname{rk} E_2) + \beta_1 \sum_{j=1}^{l} \alpha'_j \operatorname{length}(F_j(E_1)/F_{j+1}(E_1))}{\beta_1 \operatorname{rk} E_1 + \beta_2 \operatorname{rk} E_2}$$

holds.

Theorem 3.3.4. ([I-Iw-S], Theorem 7.1) *Let S be a noetherian scheme, $\mathcal{C}$ a flat family of smooth projective curves of genus g and D an effective Cartier divisor on $\mathcal{C}$ flat over S. Then there exists a coarse moduli scheme $\overline{M_{\mathcal{C}/S}^{D,\boldsymbol{\alpha}',\beta,\gamma}}(r, d, \{1\}_{1 \leq i \leq nr})$ of $(\boldsymbol{\alpha}', \beta, \gamma)$-stable parabolic Λ_D^1-triples on $\mathcal{C}$ over S. If α' is generic, it is projective over S.*

Definition 3.3.5. We denote the pull-back of $\mathcal{C}$ and $\tilde{t}$ with respect to the morphism $T \times \Lambda_r^{(n)}(d) \to T$ by the same characters $\mathcal{C}$ and $\tilde{t} = (\tilde{t}_1, \ldots, \tilde{t}_n)$. Then $D(\tilde{t}) := \tilde{t}_1 + \cdots + \tilde{t}_n$ becomes a family of Cartier divisors on $\mathcal{C}$ flat over $T \times \Lambda_r^{(n)}(d)$. We also denote by $\tilde{\lambda}$ the pull-back of the universal family on $\Lambda_r^{(n)}(d)$ by the morphism $T \times \Lambda_r^{(n)}(d) \to \Lambda_r^{(n)}(d)$. We define a functor $\mathcal{M}_{\mathcal{C}/T}^{\boldsymbol{\alpha}}(\tilde{t}, r, d)$ from the category of locally noetherian schemes over $T \times \Lambda_r^{(n)}(d)$ to the category of sets by

$$\mathcal{M}_{\mathcal{C}/T}^{\boldsymbol{\alpha}}(\tilde{t}, r, d)(S) := \left\{ (E, \nabla, \{l_j^{(i)}\}) \right\} / \sim,$$

where

1. E is a vector bundle on $\mathcal{C}_S = \mathcal{C} \times_{T \times \Lambda_r^{(n)}(d)} S$ of rank r,

2. $\nabla : E \to E \otimes \Omega^1_{\mathcal{C}_S/S}(D(\tilde{t})_S)$ is a relative connection,

3. $E|_{(\tilde{t}_i)_S} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ is a filtration by subbundles such that $(\operatorname{Res}_{(\tilde{t}_i)_S}(\nabla) - (\tilde{\lambda}_j^{(i)})_S)(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $0 \leq j \leq r-1$, $i = 1, \ldots, n$,

4. for any geometric point $s \in S$, $\dim(l_j^{(i)}/l_{j+1}^{(i)}) \otimes k(s) = 1$ for any i, j and $(E, \nabla, \{l_j^{(i)}\}) \otimes k(s)$ is α-stable.

Here $(E, \nabla, \{l_j^{(i)}\}) \sim (E', \nabla', \{l_j'^{(i)}\})$ if there exist a line bundle $\mathcal{L}$ on S and an isomorphism $\sigma : E \xrightarrow{\sim} E' \otimes \mathcal{L}$ such that $\sigma|_{t_i}(l_j^{(i)}) = l_j'^{(i)}$ for any i, j and the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \otimes \Omega^1_{\mathcal{C}/T}(D(\tilde{t})) \\ \sigma \downarrow & & \downarrow \sigma \otimes \mathrm{id} \\ E' \otimes \mathcal{L} & \xrightarrow{\nabla'} & E' \otimes \Omega^1_{\mathcal{C}/T}(D(\tilde{t})) \otimes \mathcal{L} \end{array}$$

commutes.

We now can construct the moduli space of this functor.

Theorem 3.3.6. *There exists a relative fine moduli scheme*

$$M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{t}, r, d) \to T \times \Lambda^{(n)}_r(d)$$

of $\boldsymbol{\alpha}$-stable parabolic connections of rank r and degree d, which is smooth, irreducible and quasi-projective and has an algebraic symplectic structure. The fiber $M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(\tilde{t}_x, \lambda)$ over $(x, \lambda) \in T \times \Lambda^{(n)}_r(d)$ is the irreducible moduli space of $\boldsymbol{\alpha}$-stable $(\tilde{t}_x, \lambda)$ parabolic connections whose dimension is $2r^2(g-1) + nr(r-1) + 2$ if it is non-empty.

Proof. See [I]. □

3.4 LUNA SLICE THEOREM

In Chapter 2, we have constructed the formal versal deformations, called also formal Kuranishi spaces, for 4 types of connections: all the connections with fixed divisor of poles D, integrable ones, integrable logarithmic connections and integrable logarithmic ones with a parabolic structure over D. It is quite easy to see that our formal Kuranishi spaces lift to germs of complex analytic spaces, that is, our formal series have nonzero radius of convergence. The Luna slice theorem allows us to go further and produce an affine scheme with a marked point whose germ at the marked point is the base of a versal deformation. The Luna slice theorem is stated in the general framework of a reductive algebraic group acting on a k-scheme X of finite type.

Definition 3.4.1. Let G be an affine algebraic group over k acting on a k-scheme X. A morphism $\varphi : X \to Y$ is a good quotient, if
(1) φ is affine, surjective and open.
(2) The natural homorphism $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism.
(3) If W is an invariant closed subset of X, then its image $\varphi(W)$ is also a closed subset of Y. If W_1 and W_2 are disjoint invariant closed subsets of X, then $\varphi(W_1) \cap \varphi(W_2) = \varnothing$.

Definition 3.4.2. In the situation of Definition 3.4.1, φ is a geometric quotient, if it is a good quotient, and the geometric fibers of φ are the orbits of geometric points of X. We will denote a good quotient of X, if it exists, by $X//G$.

In the constructions of moduli spaces, described in the previous sections, the quotient of the semistable locus of the Quot scheme by $SL(P(N))$ is a good quotient, and the quotient of the stable one is a geometric quotient.

Definition 3.4.3. Let G be an algebraic group , $H \subset G$ an algebraic subgroup, V a k-scheme of finite type with an action of H. Make H act on the product $G \times V$ according to the rule

$$h : (g, v) \mapsto (gh^{-1}, hv).$$

Then there exists a geometric quotient $G \times V//H$ such that the natural map $g : G \times V \to G \times V//H$ is a H-principal bundle. We denote $G \times V//H$ by $G \times^H V$. It has a natural (left) action of G, and we say that $G \times^H V$ is obtained from V by extending the action from H to G.

Definition 3.4.4. Let G be an affine algebraic group acting on a k-scheme X of finite type, $x_0 \in X$, $\mathcal{O}(x_0) = G \cdot x_0$ the orbit of x_0. A normal slice to $\mathcal{O}(x_0)$ at x_0 is an affine scheme $S \subset X$ with the following properties:
(1) $x_0 \in S$ and S is invariant under the action of $G_{x_0} = \mathrm{Stab}_G(x_0)$, the stabilizer of x_0 in G.
(2) The natural morphism $\varphi : G \times^{G_{x_0}} S \to X$ has an open image and is étale over its image.

Luna Slice Theorem 3.4.5 ([Ln])**.** *Let X be a k-scheme of finite type, G a reductive algebraic group acting on X, and let $\pi : X \to X//G$ be a good quotient. Let $x \in X$ be a point such that $\mathcal{O}(x)$ is closed. Then there exists a normal slice S to $\mathcal{O}(x)$ at x and the stabilizer G_x is a reductive algebraic group so that there exists a good quotient $S//G_x$. Moreover, the induced morphism of good quotients $S//G_x \to X//G$ has an affine open image and is étale over the image. Furthermore the following diagram is commutative.*

$$\begin{tikzcd} S \times^{G_x} G \arrow[r] \arrow[d] & X \arrow[d] \\ S//G_x \arrow[r] & X//G \end{tikzcd}$$

If X is normal (resp. smooth) at x, then S can be also taken normal (resp. smooth).

Proof. See [Ln]. □

The fact that the Luna normal slices are versal deformations is known for moduli of sheaves and was used by several authors for computing the local structure of the moduli space of sheaves at a strictly semistable point [O'G], [Dr], [L-S], [M-T].

Now, we will prove that a similar property holds for moduli of connections.

Theorem 3.4.6. *Let Λ be as in one of the cases $(a)-(d)$ of Sect.* 3.2*, and set the hypotheses and the notation as in Theorems* 3.2.4,3.2.5 *of Sect.* 3.2*. Let $\mathcal{E}$ be a polystable Λ-module with Hilbert polynomial P, $z = [\mathcal{E}]$ the corresponding point of Q. Assume that $\mathcal{E}$ is locally free as an $\mathcal{O}_X$-module, then the orbit $\mathcal{O}(z) = G \cdot z$ is closed, so that there is a normal slice V at z. Let $\boldsymbol{\mathcal{E}}$ be the restriction to $X \times V$ of the tautological quotient Λ-module over Q. Then the couple $(V, \boldsymbol{\mathcal{E}})$ is a versal deformation of $\mathcal{E}$.*

Proof. The versality of $(V, \boldsymbol{\mathcal{E}})$ means that the following two properties are verified:
(1) Any (flat) deformation $\mathcal{F}$ of $\mathcal{E}$ over any k-scheme S is induced from $(V, \boldsymbol{\mathcal{E}})$ via some morphism $S \to V$.
(2) The Kodaira-Spencer map

$$\kappa : T_z V \to \mathrm{Ext}^1_\Lambda(\mathcal{E}, \mathcal{E})$$

is an isomorphism.

The property (1) follows easily from the universal property of the Quot scheme. We will prove (2) for the case (a); the other cases are treated similarly. First remark that a structure of a Λ-module is the same as a connection ∇ with divisor of poles D, so $\mathrm{Ext}^1_\Lambda(\mathcal{E}, \mathcal{E}) = \mathbb{H}^1(\mathcal{C}^\bullet)$, where

$$\mathcal{F}^\bullet = [\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \xrightarrow{\nabla} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes_{\mathcal{O}_X} \Omega^1_X(D)],$$

is the complex introduced in Theorem 2.9. Thus, we have the following exact triple:

$$0 \to H^0(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes_{\mathcal{O}_X} \Omega^1_X(D))/\nabla(H^0(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})) \to \mathrm{Ext}^1_\Lambda(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \to 0. \tag{3.1}$$

Now, a tangent vector from T_zQ is an infinitesimal deformation of the quotient map $\mathcal{O}_X(-N)^{P(N)} \to \mathcal{E} \to 0$, or equivalently, of the exact triple

$$0 \to \mathcal{K} \to \mathcal{O}_X(-N)^{P(N)} \to \mathcal{E} \to 0, \tag{3.2}$$

followed by an infinitesimal deformation of the connection ∇ on $\mathcal{E}$. The Quot-scheme parametrizing the triple (3.2) was denoted $\tilde{Q}$ (Sect. 2, before Theorem 3.2.5), and

$$T_z\tilde{Q} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K},\mathcal{E}).$$

Next, the deformations of ∇ over the algebra of dual numbers $k[\epsilon]/\epsilon^2$ that fix $\mathcal{E}$ are obviously parametrized by $H^0(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes \Omega^1_X(D))$; two such deformations are isomorphic if they differ by an element of $\nabla(H^0(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})))$, so we obtain the exact triple

$$0 \to H^0(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes_{\mathcal{O}_X} \Omega^1_X(D))/\nabla(H^0(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})) \to T_zQ \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K},\mathcal{E}) \to 0. \tag{3.3}$$

The Kodaira-Spencer map κ induces a morphism of triples (3.2), (3.3), which is the identity on the left hand side of the triples. The right hand side counterpart $\bar{\kappa}$ of κ can be identified, as in the proof of Prop. $1.2.3$ of [O'G], with the homomorphism in the long exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E},\mathcal{E}) \to \mathrm{Hom}(\mathcal{O}_X(-N)^{P(N)},\mathcal{E}) \xrightarrow{\alpha} \mathrm{Hom}(\mathcal{K},\mathcal{E}) \xrightarrow{\bar{\kappa}} \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \dots,$$

obtained by applying $\mathrm{Hom}(\cdot,\mathcal{E})$ to (3.2). In loc. cit., it is proved that $\bar{\kappa}$ is surjective. Hence, κ is also surjective. $\square$

Corollary 3.4.7. *Let $(\mathcal{E},\nabla)$ be a connection arising in one of the 4 cases $(a)-(d)$ of Section 2 with locally free sheaf $\mathcal{E}$, and let $M = M_X(\Lambda,P)$ be the corresponding moduli space. We denote by $(K,0)$ the formal Kuranishi space of $(\mathcal{E},\nabla)$ constructed in Chapter 2. Let $z \in M$ be the point representing $(\mathcal{E},\nabla)$ in M, and $H = \mathrm{Aut}(\mathcal{E},\nabla)$. Then*

$$(K,0)//H = (M,z).$$

We will use this corollary in the next section to describe the local structure of moduli spaces of connections in some examples.

3.5 Examples

Let X be a curve, D an effective divisor on X, $r \in \mathbb{N}$ and $d \in \mathbb{Z}$. We will use the following notation: $\mathcal{C}_X(r,d;D)$, the moduli space of semistable pairs $(\mathcal{E},\nabla)$, where $\mathcal{E}$ is a vector bundle on X of rank r

and degree d, and ∇ is a meromorphic connection on $\mathcal{E}$ with fixed divisor of poles D.
$M_X(r,d)$, the moduli space of semistable vector bundles of rank r and degree d on X.
$M_X^s(r,d)$, the locus of stable vector bundles in $M_X(r,d)$.
$\mathcal{C}_X^s(r,d;D)$, the locus of stable pairs $(\mathcal{E},\nabla)$ in $\mathcal{C}_X(r,d;D)$.
$\mathcal{C}_X^0(r,d;D)$, the locus of pairs $(\mathcal{E},\nabla)\in\mathcal{C}_X(r,d;D)$ with stable $\mathcal{E}$.
$\tilde{\mathcal{C}}_X(r,d;D)$, the locus of pairs $(\mathcal{E},\nabla)\in\mathcal{C}_X(r,d;D)$ with semistable $\mathcal{E}$.
$\tilde{\mathcal{C}}_X^s(r,d;D)$, the locus of pairs $(\mathcal{E},\nabla)\in\mathcal{C}_X^s(r,d;D)$ with semistable $\mathcal{E}$.

In the sequel, we will determine some of the relations between these moduli loci and produce examples of computing their local structure, based on the Luna slice theorem and Corollary 3.4.7.

3.5.1 The case when the underlying vector bundle is stable

Let $(\mathcal{E},\nabla)\in\mathcal{C}^0(r,d;D)$. Then $\mathcal{E}$ is automatically stable. The map of forgetting the second component of a pair is a well-defined morphism $\pi:\mathcal{C}^0(r,d;D)\rightarrow M^s(r,d)$. Moreover, given two connections ∇,∇' with divisor of poles D on the same vector bundle $\mathcal{E}$, we have $\nabla-\nabla'\in H^0(X,\mathcal{E}nd(\mathcal{E})\otimes\Omega_X^1(D))$. Thus the fiber of π over a point $[E]\in M^s(r,d)$, if nonempty, is the affine space $H^0(X,\mathcal{E}nd(\mathcal{E})\otimes\Omega_X^1(D))$. To determine the image $\operatorname{im}\pi\subset M^s(r,d)$, recall that $\mathcal{E}$ admits a connection with fixed divisor of poles D if and only if the Atiyah class $\operatorname{At}^D(\mathcal{E})$ vanishes in $H^1(X,\mathcal{E}nd(\mathcal{E})\otimes\Omega_X^1(D))$. By Serre duality,

$$H^1(X,\mathcal{E}nd(\mathcal{E})\otimes\Omega_X^1(D))^*=H^0(X,\mathcal{E}nd(\mathcal{E})\otimes\mathcal{O}_X(-D)),$$

and by stability, $h^0(X,\mathcal{E}nd(\mathcal{E}))=1$, the only global endomorphisms of $\mathcal{E}$ being the homotheties. Thus $H^1(X,\mathcal{E}nd(\mathcal{E})\otimes\Omega_X^1(D))=0$ whenever $D>0$, and $h^1(X,\mathcal{E}nd(\mathcal{E})\otimes\Omega_X^1)=1$. $\mathcal{E}$ may fail to be in the image of π only when $D=0$. In this case, we refer to a theorem of Atiyah:

Theorem 3.5.1 ([At-1])**.** *A holomorphic vector bundle $\mathcal{E}$ on a curve X admits a holomorphic connection if and only if $\mathcal{E}$ is semistable of degree 0.*

Remark also that the dimension of fibres of π is constant, as follows from the Riemann-Roch theorem. This implies that π is a locally trivial fiber bundle with fiber an affine space and with structure group the affine group. The local triviality follows only in the classical and the étale topologies, for $\mathcal{C}_X^0(r,d;D)$ is not a fine moduli space in general and possesses universal connections $(\boldsymbol{\mathcal{E}},\boldsymbol{\nabla})$ only in the classical or étale topologies. Summarizing the above, we state the following theorem.

Theorem 3.5.2. *Let X be curve of genus $g\geq 1$, D an effective divisor on X, $r\in\mathbb{N}$ and $d\in\mathbb{Z}$. Then the following assertions hold:*

(i) *Assume $D=0$. Then $\mathcal{C}_X(r,d;0)=\varnothing$ if $d\neq 0$, and $\mathcal{C}_X(r,0;0)=M_X^s(r,0)=\varnothing$ if $r>1$ and $g=1$. In the case when either $g\geq 2$, or $r=g=1$, the map $\pi:\mathcal{C}_X^0(r,0;0)\rightarrow M_X^s(r,0)$ is an affine bundle, locally trivial in the étale topology, with fiber $\mathbb{C}^{r^2(g-1)+1}$.*

(ii) *Assume $D>0$ and $M_X^s(r,d)\neq\varnothing$, that is either $g\geq 2$, or $g=1$ and $g.c.d.(r,d)=1$. Then the map $\pi:\mathcal{C}_X^0(r,d;D)\rightarrow M_X^s(r,d)$ is an affine bundle, locally trivial in the étale topology, with fiber $\mathbb{C}^{r^2(g-1+\deg(D))}$.*

3.5.2 The case $g = 1, r = 2, d = 0$: local structure of $\tilde{\mathcal{C}}_X(2,0;0)$ at the most degenerate point

Let X be an elliptic curve, $\nabla = d$ the trivial connection, equal to the de Rham differential. We refer to $(\mathcal{E}, \nabla)$ as the most degenerate point of $\tilde{\mathcal{C}}_X(2,0;0)$, the moduli space of regular connections on semistable rank-2 vector bundles of degree 0 over an elliptic curve. Let $\mathcal{C}^\bullet$ be the complex of sheaves

$$\mathcal{E}nd(\mathcal{E}) \xrightarrow{\nabla} \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X.$$

Consider the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathbb{H}^0(X, \mathcal{C}^\bullet) \longrightarrow H^0(X, \mathcal{E}nd(\mathcal{E})) \xrightarrow{d_1} H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X) \\ &\longrightarrow \mathbb{H}^1(X, \mathcal{C}^\bullet) \longrightarrow H^1(X, \mathcal{E}nd(\mathcal{E})) \xrightarrow{d_1} H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X) \\ &\longrightarrow \mathbb{H}^2(X, \mathcal{C}^\bullet) \longrightarrow H^2(X, \mathcal{E}nd(\mathcal{E})) = 0 \end{aligned} \tag{3.4}$$

coming from the spectral sequence $E_1^{p,q} = H^q(\mathcal{C}^p) \Rightarrow \mathbb{H}^{p+q}(\mathcal{C}^\bullet)$, supported on two vertical strings $p = 0$ and $p = 1$. The maps d_1 are commutators $B \mapsto [A, B] = A \circ B - B \circ A$, where $\circ$ denotes the Yoneda composition. The first map d_1 is zero, for it is induced by $\nabla_{\mathcal{E}nd(\mathcal{E})} = d$ (de Rham differential), and $H^0(X, \mathcal{E}nd(\mathcal{E})) = \mathbb{C}^4 \,\mathrm{id}_{\mathcal{E}}$, and the second map d_1 is the adjoint of the first one with respect to the Serre duality, so it is zero, too. We conclude that $\mathbb{H}^2(X, \mathcal{C}^\bullet) \simeq H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X) \simeq H^1(X, \mathcal{O}_X^{\oplus 4} \otimes \Omega^1_X) \simeq \mathbb{C}^4$, and

$$0 \to H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X) \to \mathbb{H}^1(X, \mathcal{C}^\bullet) \to H^1(X, \mathcal{E}nd(\mathcal{E}) \to 0 \tag{3.5}$$

is an exact triple, so that $\mathbb{H}^1(X, \mathcal{C}^\bullet) \simeq \mathbb{C}^8$. We can represent the elements of $\mathbb{H}^1(X, \mathcal{C}^\bullet)$ as the pairs (A, a), where $a \in H^1(X, \mathcal{E}nd(\mathcal{E}))$ and $A \in H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X)$. As $\Omega^1_X \simeq \mathcal{O}_X$, both a, A can be represented by 2×2-matrices, which we will write down in the following form:

$$A = \begin{pmatrix} x & x_{12} \\ x_{21} & -x \end{pmatrix}, \quad a = \begin{pmatrix} y & y_{12} \\ y_{21} & -y \end{pmatrix} \delta,$$

where δ is a generator of $H^1(X, \Omega^1_X)$.

The identification of $\mathbb{H}^1(X, \mathcal{C}^\bullet)$ with $H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X) \oplus H^1(X, \mathcal{E}nd(\mathcal{E}))$ assumes that some cross-section of the epimorphism $\mathbb{H}^1(X, \mathcal{C}^\bullet) \to H^1(X, \mathcal{E}nd(\mathcal{E}))$ is fixed. In the case under consideration, (3.5) is just the standard Dolbeault cohomology triple

$$0 \to H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C}) = \mathbb{H}^1(X, \Omega^\bullet_X) \to H^1(X, \mathcal{O}_X) \to 0 \tag{3.6}$$

tensored by $\mathrm{End}(\mathbb{C}^2) = H^0(X, \mathcal{E}nd(\mathcal{E}))$. Thus $H^1(X, \mathcal{O}_X)$ can be identified with $H^{0,1}(X, \mathbb{C}) = \overline{H^{1,0}(X, \mathbb{C})}$ in $H^1(X, \mathbb{C})$. This fixes the choice of a splitting of (3.6) and hence of (3.5). The first obstruction map is then given by the commutator:

$$\mathrm{ob}_2 : (A, a) \mapsto [A, a].$$

As the identity matrix commutes with any other matrix, we will split the summands $x_0\,\mathrm{id}$ and $y_0\,\mathrm{id}$ out of A, a. In fact, it is easy to prove that all the components of the Kuranishi map are independent of x_0, y_0, so that the Kuranishi space is in this case of the form $K \simeq \mathbb{C}^2 \times \tilde{K}$. Here $\mathbb{C}^2$ has x_0, y_0 as coordinates and $\tilde{K}$ is the zero locus of the reduced Kuranishi map $\tilde{ob} = \tilde{\mathrm{ob}}_2 + \tilde{\mathrm{ob}}_3 + \cdots : \mathbb{C}^6 \to \mathbb{C}^3$, where $\mathbb{C}^6$ has $(x, x_{12}, x_{21}, y, y_{12}, y_{21})$ as coordinates, and $\mathbb{C}^3$ in the target of $\tilde{\mathrm{ob}}$ is the traceless part of $H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X)$. Computing in coordinates, we obtain the initial term:

$$\tilde{\mathrm{ob}}_2 : \left(A = \begin{pmatrix} x & x_{12} \\ x_{21} & -x \end{pmatrix}, a = \begin{pmatrix} y & y_{12} \\ y_{21} & -y \end{pmatrix} \delta\right) \mapsto [A, a] = \begin{pmatrix} q_1 & q_2 \\ q_3 & -q_1 \end{pmatrix} \delta,$$

where $q_1 = x_{12}y_{21} - x_{21}y_{12}, q_2 = 2xy_{12} - 2x_{12}y, q_3 = 2x_{21}y - 2y_{21}x$. Thus $\tilde{\mathrm{ob}}_2^{-1}(0)$ is a complete intersection of three quadrics of rank 4 in $\mathbb{C}^6$, $q_1 = q_2 = q_3 = 0$. One might show by an argument of "equivariant deformation to the normal cone", similar to the one used in [L-S], that $\tilde{\mathrm{ob}}_2^{-1}(0)$ is isomorphic to $\tilde{\mathrm{ob}}^{-1}(0)$, by an isomorphism, equivariant under the action of $\mathbb{P}\,\mathrm{Aut}(\mathcal{E}, \nabla) \simeq \mathrm{PGL}(2, \mathbb{C})$, so that one can use $\tilde{\mathrm{ob}}_2^{-1}(0)$ instead of $\tilde{\mathrm{ob}}^{-1}(0)$ in order to determine the local structure of the moduli space. But, we will use another approach, proving directly that all the higher obstructions vanish in our case: $\tilde{\mathrm{ob}}_3 = \tilde{\mathrm{ob}}_4 = \cdots = 0$ (see Sect. 3.5.3).

Now we will determine the GIT quotient $Q := \tilde{\mathrm{ob}}_2^{-1}(0)//\,\mathrm{PGL}(2, \mathbb{C})$. We will proceed to a change of notation for the coordinates in $\mathbb{C}^6$: $(x, x_{12}, x_{21}) = (s_0, s_1, s_2)$ and $(y, y_{12}, y_{21}) = (t_0, t_1, t_2)$. Then the three quadrics $q_1 = q_2 = q_3 = 0$ become $s_i t_j = s_j t_i$. These equations express the proportionality $s = (s_0, s_1, s_2) \sim t = (t_0, t_1, t_2)$. The space of solutions is of dimension 4; it can be expressed as the affine cone over the image of Segre $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$:

$$\begin{array}{rcl} \mathrm{Cone}\left(\mathbb{P}^2 \times \mathbb{P}^1\right) & \to & \tilde{\mathrm{ob}}_2^{-1}\{0\} \\ ((\xi_0, \xi_1, \xi_2), (\lambda_0, \lambda_1) & \mapsto & (s, t) = (\lambda_0 \xi, \lambda_1 \xi). \end{array}$$

To determine the quotient of this cone by $\mathrm{PGL}(2, \mathbb{C})$, remark that the latter acts by simultaneous conjugation on the two components:

$$\mathrm{PGL}(2, \mathbb{C}) \ni g : (A, a) \mapsto (gAg^{-1}, gag^{-1}).$$

Consider the map

$$\begin{array}{rcl} \varphi : \tilde{\mathrm{ob}}_2^{-1}(0) & \to & \mathbb{C}^2 \\ (A, a) & \mapsto & (z_1, z_2) = (\mathrm{Tr}(A^2), \mathrm{Tr}(a^2)). \end{array}$$

It is given by $PGL(2, \mathbb{C})$-invariant functions, hence descends to a morphism $\psi : Q = \tilde{\mathrm{ob}}_2^{-1}(0)//\,\mathrm{PGL}(2) \to \mathbb{C}^2$. For a fixed pair (z_1, z_2) with $z_1 z_2 \neq 0$, there are two orbits of pairs of commuting traceless matrices (A, a) such that $(\mathrm{Tr}(A^2), \mathrm{Tr}(a^2)) = (z_1, z_2)$: these are orbits of two pairs

$$(A, a) = \left\{ \begin{pmatrix} \frac{\sqrt{z_1}}{\sqrt{2}} & 0 \\ 0 & -\frac{\sqrt{z_1}}{\sqrt{2}} \end{pmatrix}, \pm \begin{pmatrix} \frac{\sqrt{z_2}}{\sqrt{2}} & 0 \\ 0 & -\frac{\sqrt{z_2}}{\sqrt{2}} \end{pmatrix} \right\}.$$

They are distinguished by the $\mathrm{PGL}(2)$-invariant function $z = \mathrm{Tr}(Aa)$, taking opposite signs on them. There is only one orbit in the fiber of ψ if $z_1 z_2 = 0$. Thus the map

$$\begin{aligned} \tilde{\psi} : Q = \tilde{\mathrm{ob}}_2^{-1}(0)//PGL(2,\mathbb{C}) &\to Q_0 = \{(z, z_1, z_2) | z^2 = z_1 z_2\} \\ (A, a) &\mapsto (\mathrm{Tr}(Aa), \mathrm{Tr}(A^2), \mathrm{Tr}(a^2)), \end{aligned}$$

is a regular morphism which is bijective. As Q_0 is a normal variety, $\tilde{\psi}$ is an isomorphism. We have proved the following result:

Theorem 3.5.3. *Let X be an elliptic curve, $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X$, $\nabla = d$ the de Rham connection on $\mathcal{E}$. Then the germ of $\tilde{\mathcal{C}}_X(2,0;0)$ at $(\mathcal{E},\nabla)$ is analytically isomorphic to the germ $(\mathbb{C}^2 \times Q_0, 0)$, where Q_0 is the quadratic cone in $\mathbb{C}^3$ with equation $z^2 = z_1 z_2$.*

We will end this subsection by determining the fiber of the natural map $\pi : \tilde{\mathcal{C}}_X(2,0;0) \to M_X(2,0)$ over $[\mathcal{E}]$, the point representing the isomorphism class of $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X$. We represent the germ of $M_X(2,0)$ at $[\mathcal{E}]$ in the same manner as we did for $\tilde{\mathcal{C}}_X(2,0;0)$: this is the quotient of $sl_2(\mathbb{C}) = \mathrm{End}(\mathbb{C}^2)_0$ (the a-component of the pairs (A,a)) by the action of $\mathrm{PGL}(2,\mathbb{C})$.

$$\mathrm{PGL}(2,\mathbb{C}) \ni g : a \mapsto gag^{-1}.$$

The quotient is just the affine space $\mathbb{C}^2$ with coordinates

$$\frac{1}{2}\mathrm{Tr}(a) = y_0, \mathrm{Tr}(a^2) = z_2.$$

Hence the germ of π at $(\mathcal{E},\nabla)$ is given by

$$\begin{aligned} \pi : \mathbb{C}^2 \times Q_0 &\to \mathbb{C}^2 \\ \Big((x_0, y_0), (z, z_1, z_2)\Big) &\mapsto (y_0, z_2), \end{aligned}$$

and the fiber over $[\mathcal{E}] = (0,0) \in \mathbb{C}^2$ is

$$\mathbb{C}^2 = \{z = z_2 = y_0 = 0\} \tag{3.7}$$

taken with multiplicity 2, for the equation $z^2 = z_1 z_2$ reduces to z^2 modulo z_2. The fact that it occurs with multiplicity 2 implies that π is not a locally trivial fibration near $[\mathcal{E}]$, though it is still equidimensional.

Remark also that the formulas (3.7) describe the fiber in $\pi^{-1}([\mathcal{E}])$ only locally near $(\mathcal{E},\nabla)$, but it is obvious that this fiber is globally identified with $H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X)_0/\mathbb{P}(\mathrm{Aut}(\mathcal{E})) \simeq \mathrm{End}(\mathbb{C}^2)_0/\,\mathrm{PGL}(2,\mathbb{C}) \simeq \mathbb{C}^2$. We conclude:

Proposition 3.5.4. *In the situation of Theorem 3.5.2, let $\pi : \tilde{\mathcal{C}}_X(2,0;0) \to M_X(2,0)$ be the natural map of forgetting the connection component of each pair $(\mathcal{E},\nabla)$. Then the fiber of π over the most degenerate point $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X$ of $M_X(2,0)$ is isomorphic to the affine plane $\mathbb{C}^2$ taken with multiplicity 2.*

3.5.3 Vanishing of higher obstructions

In Sect. 3.5.2, we stated that when determining the local structure of the moduli space $\tilde{\mathcal{C}}_X(2,0;0)$ at $(\mathcal{E},\nabla) = (\mathcal{O}_X \oplus \mathcal{O}_X, d)$, one can replace the obstruction map $\mathrm{ob} : \mathbb{H}^1(X, \mathcal{C}^\bullet) \to \mathbb{H}^2(X, \mathcal{C}^\bullet)$ by its initial term ob_2, which is quadratic on $\mathbb{H}^1(X, \mathcal{C}^\bullet)$. We noted that this might be proved by an argument of [L-S], but here we provide a simpler proof of a stronger assertion: the higher obstructions ob_k vanish for all $k \geq 3$.

As in Sect. 3.5.2, we will denote by x_0, x, x_{12}, x_{21} coordinates on $H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X)$ and by y_0, y, y_{12}, y_{21} coordinates on $H^1(X, \mathcal{E}nd(\mathcal{E}))$. Let us fix a flat complex coordinate z on X and denote by P the point $z = 0$. To compute the cohomology of (complexes of) coherent sheaves on X, we will use the two-element Stein open covering $X = U_0 \cup U_1$, where $U_0 = \{|z| < \epsilon\}$ is a small disc centered at P, and $U_1 = X \setminus \{P\}$. We denote a basis of $H^1(X, \mathcal{O}_X)$ by δ, as in Sect. 3.5.2, and will represent it by the Čech cocycle $\delta_{\alpha\beta} = \frac{1}{z} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$. Here and in the sequel, $\alpha = 0, \beta = 1$. We will also specify a section $\sigma : H^1(X, \mathcal{O}_X) \to \mathbb{H}^1(X, \Omega^\bullet_X)$ on the level of Čech cocycles. A cocycle defining $\sigma(\delta)$ is given by $\varphi_\gamma \in \Gamma(U_\gamma, \Omega^1_X)$ $(\gamma = 0, 1)$ such that

$$\varphi_\beta - \varphi_\alpha = \mathrm{d}(\delta_{\alpha\beta}) = -\frac{\mathrm{d}z}{z^2}.$$

Lemma 3.5.5. *For any $k \in \mathbb{Z} \setminus \{-1\}$, there exists a 0-cochain $\varphi_k = (\varphi_{\gamma,k}) \in \check{C}^0(\mathfrak{U}, \Omega^1_X)$ such that*

$$\varphi_{\beta,k} - \varphi_{\alpha,k} = \frac{\mathrm{d}z}{z^k}$$

on $U_\alpha \cap U_\beta$, where $\mathfrak{U} = (U_0, U_1)$ denotes the two-element covering.

Proof. The Weierstrass $\wp$-function has the following Laurent expansion at 0:

$$\wp(z) = \frac{1}{z^2} + c_2 z^2 + c_4 z^4 + \dots .$$

It is regular on U_1. We can set

$$\varphi_{\beta,k} = \frac{(-1)^k}{(k-1)!}\wp^{(k-2)}(z)\mathrm{d}z, \quad \varphi_{\alpha,k} = -\frac{\mathrm{d}z}{z^k} + \varphi_{\beta,k} \quad (k \geq 2).$$

□

The choice of $(\varphi_{\gamma,k})$ is not unique. Let us fix such choices for all $k \geq 2$ once and for all; set $\varphi_\gamma = -\varphi_{\gamma,2}$. Now we are ready to fix a choice of coordinates on $\mathbb{H}^1(X, \mathcal{C}^\bullet)$: these are 8 linear forms $x_0, x, x_{12}, x_{21}, y_0, y, y_{12}, y_{21}$ assembled into two matrices

$$T = \begin{pmatrix} x_0 + x & x_{12} \\ x_{21} & x_0 - x \end{pmatrix}, \qquad Y = \begin{pmatrix} y_0 + y & y_{12} \\ y_{21} & y_0 - y \end{pmatrix},$$

The pair $(A, a) = (T\mathrm{d}z, Y\delta)$ corresponds to the cohomology class, represented by the cocycle

$$\big((T-\varphi_{\alpha,2}Y), (T-\varphi_{\beta,2}Y), \frac{1}{z}Y\big) \in \Gamma(U_\alpha, \mathcal{E}nd(\mathcal{E})\otimes\Omega^1_X)\oplus\Gamma(U_\beta, \mathcal{E}nd(\mathcal{E})\otimes\Omega^1_X)\oplus\Gamma(U_\alpha\cap U_\beta, \mathcal{E}nd(\mathcal{E})).$$

An order-k deformation of the trivial bundle $\mathcal{E} = \mathcal{O}\oplus\mathcal{O}$ will be represented by a cocycle $G^{(k)}_{\alpha\beta} = G_{\alpha\beta,0}+G_{\alpha\beta,1}+\cdots+G_{\alpha\beta,k}$, where $G_{\alpha\beta,0} = \mathrm{id}_2$ and $G_{\alpha\beta,k} \in \Gamma(U_\alpha\cap U_\beta, \mathcal{E}nd(\mathcal{E}))\otimes k[y_0, y, y_{12}, y_{21}]$ is homogeneous of degree k in y_0, y, y_{12}, y_{21}. The forms $G_{\alpha\beta,k}$ are not subject to any constraints (cocycle condition), because our covering has no triple intersections. Similarly, an order-k deformation of ∇ will be given by connection matrices $A^{(k)}_\gamma$,

$$A^{(k)}_\gamma = A_{\gamma,0} + A_{\gamma,1} + \cdots + A_{\gamma,k}\ (\gamma = 0, 1),$$

where $A_{\gamma,0} = 0$, and $A_{\gamma,k} \in \Gamma(U_\gamma, \mathcal{E}nd(\mathcal{E})\otimes\Omega^1_X)\otimes k[x_0, x, x_{12}, x_{21}, y_0, y, y_{12}, y_{21}]$ is homogeneous of degree k.

Theorem 3.5.6. *There exist sequences* $(G_{\alpha\beta,k})_{k\geq 0}, (A_{0,k})_{k\geq 0}, (A_{1,k})_{k\geq 0}$, *where* $G_{\alpha\beta,k} \in \Gamma(U_\alpha\cap U_\beta, \mathcal{E}nd(\mathcal{E}))\otimes k[x_0, x, x_{12}, x_{21}]$, $A_{\gamma,k} \in \Gamma(U_\gamma, \mathcal{E}nd(\mathcal{E})\otimes\Omega^1_X)\otimes k[x_0, x, x_{12}, x_{21}, y_0, y, y_{12}, y_{21}]$, *are homogeneous of degree* k *(*$\alpha = 0, \beta = 1, \gamma\in\{0,1\}$*), with the following properties*
(i) $G_{\alpha\beta,0} = \mathrm{id}_2,\ G_{\alpha\beta,1} = \frac{1}{z}Y.$
(ii) $A_{\gamma,0} = 0,\ A_{\gamma,1} = T - \varphi_{\gamma,2}Y\ \ (\gamma = 0, 1).$
(iii) *For any* $k\geq 1$, *the polynomials* $G^{(k)}_{\alpha\beta} = \sum_{i=0}^k G_{\alpha\beta,i}$ *and* $A^{(k)}_\gamma = \sum_{i=0}^k A_{\gamma,i}$ *satisfy the following congruence:*

$$\mathrm{d}G^{(k)}_{\alpha\beta} \equiv G^{(k)}_{\alpha\beta}A^{(k)}_\beta - A^{(k)}_\alpha G^{(k)}_{\alpha\beta} \mod I_k,$$

where $I_k = (q_1, q_2, q_3) + \mathfrak{m}^{k+1}$, $q_1 = x_{12}y_{21} - x_{21}y_{12}, q_2 = 2xy_{12} - 2x_{12}y, q_3 = 2x_{21}y - 2y_{21}x$, $\mathfrak{m} = (x_0, x, x_{12}, x_{21}, y_0, y, y_{12}, y_{21})$.

In other words, there are only quadratic obstructions to extending $(\mathcal{E}, \nabla)$ to any order $k\geq 2$.

Proof. As

$$[T, Y] = \begin{pmatrix} q_1 & q_2 \\ q_3 & -q_1 \end{pmatrix}, \tag{3.8}$$

we can handle T, Y as commuting matrices. A solution to $(i) - (iii)$ can be represented by the following power series:

$$G_{\alpha\beta} = \sum_{k=0}^\infty G_{\alpha\beta,k} = e^{\frac{1}{z}Y}, A_\gamma = \sum_{k=0}^\infty A_{\gamma,k} = T - \varphi_{\gamma,2}Y$$

(so that $A_{\gamma,k} = 0$ for all $k\neq 1$.) Then we have

$$\mathrm{d}G_{\alpha\beta} = -\frac{1}{z^2}Ye^{\frac{1}{z}Y}\mathrm{d}z = (\varphi_{\alpha,2} - \varphi_{\beta,2})Ye^{\frac{1}{z}Y}$$

$$\begin{aligned} &= e^{\frac{1}{z}Y}(T - \varphi_{\beta,2}Y) - (T - \varphi_{\alpha,2}Y)e^{\frac{1}{z}Y} \\ &= G_{\alpha\beta}A_\beta - A_\alpha G_{\alpha\beta}, \end{aligned}$$

(the computation is done modulo (q_1, q_2, q_3)), so T and Y commute.) $\square$

Remark 3.5.7. One might also set $G_{\alpha\beta,1} = \frac{1}{z}Y$, $G_{\alpha\beta,k} = 0\ \forall k \geq 2$, and construct $A_{\alpha,k}$ by induction on k, using the $\varphi_{\gamma,k}$ from Lemma 3.5.5.

3.5.4 The local structure of $\tilde{\mathcal{C}}_X(2,0;0)$ for $g = 1$: mildly degenerate case

In the setting of Sect. 3.5.2. and 3.5.3, let now $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X$ and $\nabla = d + A$, with $A \in H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X)$ generic. Changing a basis for $\mathcal{E}$, we can assume A diagonal:

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathrm{d}z,$$

$\lambda_1\lambda_2 \neq 0, \lambda_1 \neq \lambda_2$. Then the map d_1 on both $H^i(X, \mathcal{E}nd(\mathcal{E}))$ in the exact sequence (3.4) is given by $B \mapsto [A, B] = A \circ B - B \circ A$, where $\circ$ denotes the Yoneda composition, and it has a two-dimensional kernel and 2-dimensional image. Thus we have the exact triple

$$0 \to M_2(\mathbb{C})\mathrm{d}z / \begin{pmatrix} 0 & \mathbb{C}\mathrm{d}z \\ \mathbb{C}\mathrm{d}z & 0 \end{pmatrix} \to \mathbb{H}^1(X, \mathcal{C}^\bullet) \to \begin{pmatrix} \mathbb{C}\delta & 0 \\ 0 & \mathbb{C}\delta \end{pmatrix} \to 0$$

and

$$\mathbb{H}^0(X, \mathcal{C}^\bullet) = \begin{pmatrix} \mathbb{C}\mathrm{d}z & 0 \\ 0 & \mathbb{C}\mathrm{d}z \end{pmatrix}$$

$$\mathbb{H}^2(X, \mathcal{C}^\bullet) = M_2(\mathbb{C})\delta\mathrm{d}z / \begin{pmatrix} 0 & \mathbb{C}\delta\mathrm{d}z \\ \mathbb{C}\delta\mathrm{d}z & 0 \end{pmatrix}.$$

The obstruction map always takes values in the traceless part $\mathbb{H}^2(X, \mathcal{C}^\bullet_0)$, where

$$\mathcal{C}^\bullet_0 = [\mathcal{E}nd_0(\mathcal{E}) \xrightarrow{\nabla} \mathcal{E}nd_0(\mathcal{E}) \otimes_{\mathcal{O}_X} \Omega^1_X],$$

which is 1-dimensional in the case under consideration. Thus we might expect that the base of the versal deformation is a hypersurface in the 4-dimensional space $\mathbb{H}^1(X, \mathcal{C}^\bullet)$. But this is impossible, for we can construct by hand a 4-dimensional family of connections, which is a deformation of $(\mathcal{E}, \nabla)$ and all of whose members are pairwise non-isomorphic. Indeed, $\mathcal{O}_X \oplus \mathcal{O}_X$ deforms to $\mathcal{L}_1 \oplus \mathcal{L}_2$ with $\mathcal{L}_i \in \mathrm{Pic}^0(X)$; each of the L_i has a connection d_i, and we get two more parameters in adding arbitrary multiples of $\mathrm{d}z$ to d_i. The family obtained $(\mathcal{L}_1 \oplus \mathcal{L}_2, (d_1 + \alpha_1\mathrm{d}z) \oplus (d_2 + \alpha_2)\mathrm{d}z)$ is the wanted 4-parameterized deformation of $(\mathcal{O}_X \oplus \mathcal{O}_X, \nabla)$. We conclude that the obstruction map is zero, and moreover, as the members of our family are polystable connections, the base of the family injects into the moduli space $\tilde{\mathcal{C}}_X(2,0;0)$. This implies that though the automorphism group $\mathbb{P}(\mathrm{Aut}(\mathcal{E}, \nabla)) \subset \mathrm{PGL}(2, \mathbb{C})$ is non-trivial here, it acts trivially on $\mathbb{H}^1(X, \mathcal{C}^\bullet)$, so that the neighborhood of 0 in $\mathbb{H}^1(X, \mathcal{C}^\bullet)$ is a local chart for $\tilde{\mathcal{C}}_X(2,0;0)$ at $(\mathcal{E}, \nabla)$ (in the classical or étale topology). We summarize this in the following statement.

Theorem 3.5.8. *Let X be an elliptic curve, $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X$, $\nabla = d + A$ with $A \in H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X)$ generic. Then $\dim \mathbb{H}^1(X, \mathcal{C}^\bullet) = 4$, and the neighborhood of 0 in $\mathbb{H}^1(X, \mathcal{C}^\bullet)$ is a local chart for $\tilde{\mathcal{C}}_X(2,0;0)$ at $(\mathcal{E}, \nabla)$, so that $\tilde{\mathcal{C}}_X(2,0;0)$ is smooth of dimension 4 at $(\mathcal{E}, \nabla)$. In particular, this means that the obstruction map $\mathrm{ob} : \mathbb{H}^1(X, \mathcal{C}^\bullet) \to \mathbb{H}^2(X, \mathcal{C}^\bullet)$ vanishes and that $\mathbb{P}\,\mathrm{Aut}(\mathcal{E}, \nabla)$ acts trivially on $\mathbb{H}^1(X, \mathcal{C}^\bullet)$.*

We remark that the map $\pi : \tilde{\mathcal{C}}_X(2,0;0) \to M_X(2,0)$ is not locally trivial near $(\mathcal{E}, \nabla)$, for $(\mathcal{E}, \nabla)$ belongs to the same fiber $\pi^{-1}([\mathcal{O}_X \oplus \mathcal{O}_X])$ which we showed to be a multiple one. The failure of local triviality is due to the fact that $\mathrm{Aut}(\mathcal{E})$ is bigger than $\mathrm{Aut}(\mathcal{E}, \nabla)$.

3.5.5 Direct images from bielliptic curves

Let E be an elliptic curve, $f : C \to E$ a bielliptic cover as in Sect. 1.1, $\mathcal{L}$ a line bundle on C and $(\mathcal{E}, \nabla) = (f_*(\mathcal{L}), f_*(\nabla_{\mathcal{L}}))$ a direct image connection on E. We will use the notation from Sect. 1.3 of Chapter 1. Let $\mathcal{C}_X$ denote the moduli space $\mathcal{C}_X(2, -1; p_+ + p_-)$, where $p_\pm$ are branch points of f. We will investigate $\mathcal{C}_X$ in the neighborhood of $(\mathcal{E}, \nabla)$. We will first assume that $\mathcal{E}$ is stable.

Lemma 3.5.9. *If $\mathcal{E}$ is stable, then $\dim H^i(E, \mathcal{E}nd(\mathcal{E})) = 1$ $(i = 0, 1)$, and $\dim H^0(E, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D)) = 8$, $\dim H^1(E, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D)) = 0$, where $D = p_+ + p_-$.*

Proof. $\mathcal{E}$ can be obtained as an extension

$$0 \to \mathcal{O}_E(-p) \to \mathcal{E} \to \mathcal{O}_E \to 0, \tag{3.9}$$

where $p \in E$ is such that $\det(\mathcal{E}) \simeq \mathcal{O}_E(-p)$. The Bockstein homomorphism $\partial : H^0(E, \mathcal{O}_E) \to H^1(E, \mathcal{O}_E(-p))$ is the product with the extension class of (3.9), hence an isomorphism, otherwise the extension would be trivial and $\mathcal{E}$ would be unstable. Twisting (3.9) by $\mathcal{O}_E(D)$ and $\mathcal{O}_E(D+p)$, we find $\dim H^0(E, \mathcal{E}(D)) = 3$, $\dim H^0(E, \mathcal{E}(D+p)) = 5$, and $\dim H^1(E, \mathcal{E}(D)) = \dim H^0(E, \mathcal{E}(D+p)) = 0$. Finally, tensoring (3.9) by $\mathcal{E}^*(D) \simeq \mathcal{E}(D+p)$, we obtain the exact triple

$$0 \to \mathcal{E}(D) \to \mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_E(D) \to \mathcal{E}(D+p) \to 0,$$

which implies the result since $\Omega^1_E \simeq \mathcal{O}_E$. $\square$

Thus the exact sequence (3.4) implies that $\mathbb{H}^2(E, \mathcal{C}^\bullet) = 0$, so that the infinitesimal deformations are unobstructed. Further,

$$\mathbb{H}^0(E, \mathcal{C}^\bullet) \simeq H^0(E, \mathcal{E}nd(\mathcal{E})) \simeq \mathbb{C},$$

(by stability), and the triple

$$0 \to H^0(E, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D)) \to \mathbb{H}^1(E, \mathcal{C}^\bullet) \to H^1(E, \mathcal{E}nd(\mathcal{E})) \to 0 \tag{3.10}$$

is exact. Hence $\dim \mathbb{H}^1(E, \mathcal{C}^\bullet) = 9$, the base of the versal deformation is smooth, and $\mathrm{Aut}(\mathcal{E}, \nabla)$ is trivial. Similarly, deformations of $\mathcal{E}$ are unobstructed and $\mathrm{Aut}\,\mathcal{E}$ is trivial. We deduce:

Proposition 3.5.10. *In the above notation, the neighborhood of zero in $\mathbb{H}^1(E, \mathcal{C}^\bullet)$ (resp. $H^1(X, \mathcal{E}nd(\mathcal{E}))$) is a local chart for $\mathcal{C}_E^0 = \mathcal{C}_E^0(2, -1; D)$ (resp. $M_E(2, -1)$), and the natural projection $\pi : \mathcal{C}_E^0 \to M_E(2, -1)$, $(\mathcal{E}, \nabla) \mapsto [\mathcal{E}]$, is represented in these charts by the natural epimorphism $\mathbb{H}^1(E, \mathcal{C}^\bullet) \to H^1(E, \mathcal{E}nd(\mathcal{E}))$ coming from the exact triple (3.10). We have $M_E(2, -1) \simeq E$, and π is an affine bundle over E with fiber $\mathbb{C}^8$, locally trivial in the Zariski topology.*

Proof. The existence of universal families of pairs $(\mathcal{E}, \nabla)$ over Luna slices locally in the étale topology implies that π is an affine bundle, locally trivial in the étale topology. But according to Grothendieck [Gro-1], over a curve, any affine bundle, locally trivial in the étale topology is also locally trivial in the Zariski topology; this follows from the vanishing of the Brauer group of any smooth projective curve. □

Now, we turn to the case where $\mathcal{E}$ is unstable. Then $\pi : \mathcal{C}_E \dashrightarrow M_E(2, -1)$ exists as a rational map in a neighborhood of $(\mathcal{E}, \nabla)$, and it is undefined at $(\mathcal{E}, \nabla)$. Then there exists $\mathcal{L} \in \mathrm{Pic}^0(E)$ (see Prop. 1.6.9) such that $\mathcal{E} = \mathcal{O}_E(-\infty) \oplus \mathcal{L}$. We have $\mathcal{E}nd(\mathcal{E}) \simeq \mathcal{O}^{\oplus 2} \oplus \mathcal{L}^*(-\infty) \oplus \mathcal{L}(\infty)$, hence $\dim H^i(\mathcal{E}nd(\mathcal{E})) = 3$ $(i = 0, 1)$, $\dim H^0(E, \mathcal{E}nd(\mathcal{E}) \otimes \Omega_X^1(D)) = 8$ and $\dim H^1(E, \mathcal{E}nd(\mathcal{E}) \otimes \Omega_X^1(D)) = 0$. Hence $\mathbb{H}^2(E, \mathcal{C}_E^\bullet) = 0$ and the deformations of $(\mathcal{E}, \nabla)$ are unobstructed. The description of $\mathbb{H}^1(E, \mathcal{C}_E^\bullet)$ depends on the rank of the map $d_1 : H^0(\mathcal{E}nd(\mathcal{E})) \to H^0(E, \mathcal{E}nd(\mathcal{E}) \otimes \Omega_X^1(D))$ in the long exact sequence (3.4). For instance, if ∇ is the direct image of the regular connection $\mathrm{d} + \omega$ on the trivial bundle $\mathcal{O}_C$, we see from formulas (1.12)–(1.14) that

$$d_1 : B = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} \mapsto [A, B]$$

is of rank 1 if $\lambda_1 = 0$ and of rank 2 otherwise. But if $\lambda_1 = 0$, then $(\mathcal{E}, \nabla)$ is the direct sum of rank-1 connections and hence is unstable. Thus it does not represent a point of $\mathcal{C}_E$. Similarly, using formulas from Prop. 1.6.9, we can show that $\mathrm{rk}\, d_1 = 2$ in all the remaining cases in which $(\mathcal{E}, \nabla)$ is stable, but $\mathcal{E}$ is not. We conclude that there is the exact triple

$$\begin{aligned} 0 \to\ & H^0(E, \mathcal{E}nd(\mathcal{E}) \otimes \Omega_X^1(D))/d_1(H^0(E, \mathcal{E}nd(\mathcal{E})) \simeq \mathbb{C}^6 \to \mathbb{H}^1(E, \mathcal{C}_E^\bullet) \simeq \mathbb{C}^9 \\ \to\ & H^0(E, \mathcal{E}nd(\mathcal{E})) \simeq \mathbb{C}^3 \to 0. \end{aligned}$$

Thus $\mathcal{C}_E$ is smooth of dimension 9, as in the case when $\mathcal{E}$ is stable. Note that in the base of the versal deformation of $(\mathcal{E}, \nabla)$, there is a hypersurface parameterizing the deformations of $(\mathcal{E}, \nabla)$ with $\mathcal{E}$ unstable. It is defined by the vanishing of the component corresponding to the factor $H^1(E, \mathcal{L}^*(-\infty)) \subset H^1(E, \mathcal{E}nd(\mathcal{E}))$. The complement to this 8-dimensional hypersurface corresponds to stable $\mathcal{E}$'s, so that $\mathcal{C}_E$ can be viewed as a partial compactification of $\mathcal{C}_E^0$ at infinity (at the limit "$\nabla \to \infty$"). Moreover as $\det : M_E(2, -1) \to E$ is an isomorphism, and $\det$ is well-defined on $\mathcal{C}_E$, so that $\mathcal{C}_E \setminus \mathcal{C}_E^0$ is a removable singularity of π, locally near $(\mathcal{E}, \nabla)$. We conclude:

Theorem 3.5.11. *Let $(\mathcal{E}, \nabla)$ be a stable direct image connection, as in Sect. 1.3, with $\mathcal{E}$ unstable. Then the rational map $\pi : \mathcal{C}_E \dashrightarrow M_E(2, -1)$ is regular at $(\mathcal{E}, \nabla)$ and is a smooth partial compactification of the affine bundle $\mathcal{C}_E^0 \to M_E(2, -1)$ with fiber $\mathbb{C}^8$.*

The infinitesimal deformations of $(\mathcal{E}, \nabla)$ are unobstructed, and $\mathbb{H}^1(E, \mathcal{C}^\bullet) \simeq \mathbb{C}^9$ is a local chart of $\mathcal{C}_E$ near $(\mathcal{E}, \nabla)$.

We remark that the local triviality of π is not global. For example, $\mathcal{C}_E$ contains a pair $(\mathcal{E}, \nabla)$ with $\mathcal{E} = (\mathcal{O}_E(-\infty), \mathcal{L})$ as above, and

$$A = \begin{pmatrix} 0 & \mathrm{d}z \\ 0 & 0 \end{pmatrix}.$$

This pair is stable, since the unique destabilizing subbundle $\mathcal{L}$ in $\mathcal{E}$ is not ∇-invariant, but $\operatorname{rk} d_1 = 1$ and $\dim \mathbb{H}^1(\mathcal{C}^\bullet) = 10 > \dim \mathcal{C}_E$.

List of Figures

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RÉSUMÉ

La correspondance de Riemann-Hilbert, comprise comme la correspondance de monodromie entre les connections logarithmiques intégrables sur une variété algébrique X avec diviseur de pôles D et les représentations de $\pi_1(X \setminus D)$, est bijective selon Deligne [De], à condition que D est un diviseur à croisements normaux et des restrictions sont imposées sur les résidus de la connection le long des composantes de D. Néanmoins, la solution de Deligne n'est pas effective dans le sens que cela n'implique pas de formules pour calculer la correspondance de Riemann-Hilbert. Par conséquent, il est important d'avoir sous sa main un nombre d'exemples qui peuvent être résolus explicitement pour avoir une meilleure compréhension des propriétés générales de cette correspondance. Le travail de trouver explicitement des exemples calculables avait été commencé par d'autres personnes parmi lesquelles je cite Korotkin, Enolskiy, Grava, Loray, van der Put, Ulmer. Comme un exemple nouveau, je donne dans le premier Chapitre une description complète de toutes les connections logarithmiques de rang 2 sur les courbes elliptiques E avec 2 pôles qui sont les images directes des connections régulières de rang 1 sur des revêtements doubles C de genre 2 de E (de telles courbes C sont appelées bielliptiques).

Theorem 3.5.12. *Pour un choix approprié de coordonnées affines ξ, x, y, tout revêtement double de genre 2 $f : C \to E$ peut être écrit sous la forme suivante:*

$$f : C = \{y^2 = (t' - \xi^2)(t' - 1 - \xi^2)(t' - t - \xi^2)\} \to E = \{y^2 = x(x-1)(x-t)\},$$
$$(\xi, y) \mapsto (x, y) = (t' - \xi^2, y). \tag{3.11}$$

Les connections $(f_*\mathcal{L}, f_*\nabla_{\mathcal{L}})$, où $(\mathcal{L}, \nabla_{\mathcal{L}})$ parcourt toutes les connections régulières de rang 1 sur les courbes de genre 2 bielliptiques, dépendent de 6 paramètres $(t, t', \lambda_1, \lambda_2, \tilde{q}_1, \tilde{q}_2)$, parmi lesquels les quatre premiers sont des nombres complexes ($t, t' \in \mathbb{C} \setminus \{0, 1\}$, $t \neq t'$) et $\tilde{q}_i = (\xi_i, y_i)$ sont deux points de C tels que $\mathcal{L} = \mathcal{O}_C(\tilde{q}_1 + \tilde{q}_2 - \infty_+ - \infty_-)$, où $\infty_\pm$ sont les deux points à l'infini de C correspondant à $\xi = \infty$. Ces connections ont des pôles logarithmiques aux deux points de branchement $p_\pm$ de f et peuvent être representées sous la forme $d + A$ dans une trivialisation méromorphe appropriée de $\mathcal{E}$, où

$$A = \begin{pmatrix} -\frac{1}{2}\left(\frac{1}{2}\left(\frac{y+y_1}{x_1-x} + \frac{y+y_2}{x_2-x}\right) + \lambda_2\right)\frac{\mathrm{d}x}{y} & -\frac{1}{2}\left(\frac{1}{2}\left(\frac{(y+y_1)\xi_1}{x_1-x} + \frac{(y+y_2)\xi_2}{x_2-x}\right) + \lambda_1\right)\frac{\mathrm{d}x}{y} \\ -\frac{1}{2}\left(\frac{1}{2}\left(\frac{(y+y_1)\xi_1}{(x_1-x)(t'-x)} + \frac{(y+y_2)\xi_2}{(x_2-x)(t'-x)}\right) + \frac{\lambda_1}{t'-x}\right)\frac{\mathrm{d}x}{y} & -\frac{1}{2}\left(\frac{1}{2}\left(\frac{y+y_1}{x_1-x} + \frac{y+y_2}{x_2-x}\right) + \lambda_2 + \frac{y}{t'-x}\right)\frac{\mathrm{d}x}{y} \end{pmatrix}. \tag{3.12}$$

Les fibrés vectoriels sous-jacents $\mathcal{E} = f_*\mathcal{L}$ de toutes ces connections sont determinés en termes de transformées élémentaires, appliquées au fibré vectoriel initial $\mathcal{E}_0 = f_*\mathcal{O}_C \simeq \mathcal{O}_E \oplus \mathcal{O}_E(-\infty)$, où ∞ désigne le point à l'infini $x = \infty$ de E. Pour terminer ceci, l'ancienne technique de Lange–Narasimhan reliant certaines propriétés des fibrés vectoriels de rang 2 sur une courbe à la théorie de l'intersection sur les surfaces réglées associées $\mathbb{P}(\mathcal{E})$ est utilisée. Ces calculs impliquent:

Theorem 3.5.13. *Dans la situation du théorème précédent, $\mathcal{E} = f_*\mathcal{L}$ est un fibré vectoriel de rang 2, de degrée -1 sur E avec déterminant $\mathcal{O}_E(q_1 + q_2 - 3\infty)$, où $q_i = f(\tilde{q}_i)$. Il est instable et une*

somme directe de deux fibrés en droites si et seulement si l'un des trois cas suivants se présente: (i) $\tilde{q}_1 + \tilde{q}_2$ est un diviseur dans la série linéaire hyperelliptique $g_2^1(C)$ (c'est à dire $\xi_1 = \xi_2$, $y_1 = -y_2$, ou $\{\tilde{q}_1, \tilde{q}_2\} = \{\infty_+, \infty_-\}$), (ii) $\tilde{q}_1 = \tilde{q}_2$, (iii) $\tilde{q}_i = \infty_\pm$ pour au moins une valeur $i \in \{1, 2\}$.

Dans tous les autres cas , $\mathcal{E}$ est l'unique fibré vectoriel de rang 2 avec déterminant donné; il peut être obtenu comme l'unique extension non-triviale

$$0 \longrightarrow \det \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

Dans le cadre de travail du problème de Riemann–Hilbert , il est intéressant de savoir: Quels sont les fibrés vectoriels de degré 0 qui réalisent les mêmes représentations de monodromies? Selon un théorème de Bolibruch–Esnault–Viehweg [AB, EV-2], toute connection logarithmique irréductible $(\mathcal{E}, \nabla)$ sur une courbe peut être transformée par une suite de tranformées élémentaires de Gabber à une autre paire $(\mathcal{E}', \nabla')$, dans laquelle $\mathcal{E}'$ est un fibré vectoriel semistable de degré 0 et ∇' est une connection logarithmique sur $\mathcal{E}'$ avec les mêmes points singuliers et la même monodromie que ∇. J'illustre ce résultat pour la famille de connections $(\mathcal{E}, \nabla) = (f_*\mathcal{L}, f_*\nabla_{\mathcal{L}})$ définies au-dessus en fournissant une transformée explicite de Gabber:

Proposition 3.5.14. *Soient ($\mathcal{E}$, ∇) définis comme au Théorème 1. Soit p_+ l'un des deux points de branchement $p_\pm$ de f et v un vector propre de $\mathrm{Res}_{p_+}(\nabla)$ avec valeur propre $\frac{1}{2}$. Alors la transformée de Gabber $(\mathcal{E}', \nabla') = elm^+_{p_+,v}(\mathcal{E}, \nabla)$ satisfait à la conclusion du théorème de Bolibruch–Esnault–Viehweg: $\mathcal{E}'$ est un fibré vectoriel semistable de degré 0 et ∇' est une connection logarithmique avec les mêmes points singuliers et la même monodromie que ∇. De plus, $\mathcal{E}' \simeq \mathcal{O}_E(p_+ - \infty) \oplus \mathcal{O}_E(q_1 + q_2 - 2\infty)$.*

L'importance de ce type de résultats est qu'il nous autorise à considérer des applications de l'espace de modules des connections à l'espace de modules des fibrés vectoriels, puisque uniquement les fibrés semistables ont une théorie de l'espace des modules conséquente. Une autre caractéristique utile des transformées élémentaires est de changer arbitrairement le degré, et ceci enrichit notre connaissance de l'espace des modules des connections produisant des applications vers des espaces de modules des fibrés vectoriels de différents degrés, lesquels peuvent être assez différents et avoir des dimensions différentes.

Les autres résultats concernant cette famille de connections contenus dans ma thèse sont les descriptions des déformations isomonodromiques et des possibles groupes de monodromie et de Galois différentiels.

Dans le deuxième chapitre , nous construisons les espaces de Kuranishi. Soit X un schéma complet de type fini sur k ou un espace complexe (alors $k = \mathbb{C}$). L'existence d'une déformation verselle et l'approche théorique à sa construction sont connues pour les faisceaux cohérents sur X. La construction de l'espace de Kuranishi (= déformation verselle) est faite pour les faisceaux en utilisant les résolutions injectives. Nous faisons l'étude des fibrés vectoriels $\mathcal{E}$ avec une structure supplémentaire (une connection ∇), et dans ce cas, la théorie de la déformation à la fois de $\mathcal{E}$ et $(\mathcal{E}, \nabla)$ peut être énoncée en termes de cohomologie de Čech d'un recouvrement par des ouverts assez fin de X. Cette approche est plus facile que celle qui utilise les résolutions injectives. Nous commençons avec la construction de l'espace de Kuranishi de fibrés vectoriels servant comme un modèle pour celui des paires $(\mathcal{E}, \nabla)$. Nous introduirons des notations supplémentaires et énoncerons le théorème qui résulte de cette approche.

Soit $\mathfrak{U} = (U_\alpha)$ un recouvrement de X, afin que la cohomologie de Čech $\check{H}^i(\,\mathcal{E}nd\,(\mathcal{E}))$ $(i = 0, 1, 2)$ puisse être calculée sur $\mathfrak{U}$ et que $\mathcal{E}_{|U_\alpha}$ admet une trivialisation e_α pour tout α. Les fonctions de transition $g_{\alpha\beta}$ relient les trivialisations par la formule $e_\beta = e_\alpha g_{\alpha\beta}$ sur $U_{\alpha\beta} = U_\alpha \cap U_\beta$.

Les classes d'équivalence des déformations du premier ordre de $\mathcal{E}$ sur $V = \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ sont classifiées par $W = \check{H}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))$. Nous construisons une déformation universelle du premier ordre de $\mathcal{E}$ sur X. Soit $t_1, \ldots, t_N$ les coordonnées linéaires sur W, $W_k = \operatorname{Spec} k[t_1, \ldots, t_N]/(t_1, \ldots, t_N)^{k+1}$ le k-ième voisinage infinitésimal de l'origine dans W. Nous déformons $\mathcal{E}$ sur la base W_1 en spécifiant une famille $G_{\alpha\beta}(t_1, \ldots, t_N)$ de fonctions de transitions sur $X \times W_1$. Collectons N cocycles $a_i = (a^{(i)}_{\alpha\beta}) \in \check{C}^1(\mathfrak{U}, \mathcal{E}nd(\mathcal{E}))$ dont les classes de cohomologie $[a_1], \ldots, [a_N]$ forment une base de W duale aux coordonnées $t_1, \ldots, t_N$. Alors nous posons $g^{(i)}_{\alpha\beta} = a^{(i)}_{\alpha\beta} g_{\alpha\beta}$, où $a^{(i)}_{\alpha\beta}$ est representée par sa matrice dans la base e_α et écrivons $G_{\alpha\beta}(t_1, \ldots, t_N) = g_{\alpha\beta} + \sum_{i=1}^N g^{(i)}_{\alpha\beta} t_i$. Alors $G_{\alpha\beta}$ est un 1-cocycle et definit un fibré vectoriel $\mathcal{E}_1$ sur $X \times W_1$ appelé une déformation universelle du premier ordre de $\mathcal{E}$.

Theorem 3.5.15. *Soient $X, \mathcal{E}$ définies comme ci-dessus, $W = H^1(X, \mathcal{E}nd(\mathcal{E}))$, $(\delta_1, \ldots, \delta_N)$ une base de W et $(t_1 \ldots, t_N)$ les coordonnées duales sur W. Soit $W_k = \operatorname{Spec} k[t_1, \ldots, t_N]/(t_1, \ldots, t_N)^{k+1}$ le k-ième voisinage infinitésimal de l'origine dans W, $\mathcal{E}_1$ la déformation universelle du premier ordre définie comme ci-dessus de $\mathcal{E}$ sur $X \times W_1$ avec $[a_i] = \delta_i$. Alors il existe une série formelle*

$$f(t_1, \ldots, t_N) = \sum_{k=2}^{\infty} f_k(t_1 \ldots, t_N) \in H^2(X, \mathcal{E}nd(\mathcal{E}))[[t_1, \ldots, t_N]],$$

où f_k est homogène de degré k, avec la propriété suivante. Soit I l'idéal de $k[[t_1, \ldots, t_N]]$ engendré par l'image de l'application $f^* : H^2(X, \mathcal{E}nd(\mathcal{E}))^* \to k[[t_1, \ldots, t_N]]$, adjointe de f. Alors pour tout $k \geq 2$, la déformation universelle du premier ordre de $\mathcal{E}_1$ of $\mathcal{E}$ on $X \times W_1$ s'étend à un fibré vectoriel $\mathcal{E}_k$ sur $X \times K_k$, où K_k est le sous-schéma fermé de W_k défini par l'idéal $I \otimes k[[t_1, \ldots, t_N]]/(t_1, \ldots, t_N)^{k+1}$.

La limite inverse $\mathbb{K} = \varprojlim K_k$ est appelée l'espace de Kuranishi formel de $\mathcal{E}$, et $\boldsymbol{\mathcal{E}} = \varprojlim \mathcal{E}_k$ le fibré universel formel sur $\mathbb{K}$.

Il résulte que l'espace de Kuranishi pour les connections peut être construit exactement de la même manière, mais au lieu de la cohomologie du faisceau $\mathcal{E}nd(\mathcal{E})$, on doit considérer l'hypercohomologie d'un complexe approprié $\mathcal{C}^\bullet$ de faisceaux. Pour la théorie de la déformation de la classe entière des connections $(\mathcal{E}, \nabla)$ avec diviseur de pôles fixé D, ceci est le complexe à deux termes $\mathcal{C}^\bullet = [\mathcal{C}^0 \to \mathcal{C}^1]$, où $\mathcal{C}^0 = \mathcal{E}nd(\mathcal{E})$, $\mathcal{C}^1 = \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1(D)$, et la différentielle $d_{\mathcal{C}} = \nabla_{\mathcal{E}nd(\mathcal{E})}$ est la connection sur $\mathcal{E}nd\,(\mathcal{E})$ induites par ∇. Les déformations du premier ordre de $(\mathcal{E}, \nabla)$ avec diviseur de pôles fixé sont classifiées par $W = \mathbb{H}^1(X, \mathcal{C}^\bullet)$, et il y a une déformation universelle du premier ordre $(\mathcal{E}_1, \nabla_1)$ de $(\mathcal{E}, \nabla)$ sur $X \times W_1$.

Theorem 3.5.16. *Soient $W = \mathbb{H}^1(X, \mathcal{C}^\bullet)$, $(\delta_1 \ldots, \delta_N)$ une base de W et $(t_1, \ldots, t_N)$ les coordonnées duales sur W. Soit $W_k = \operatorname{Spec} k[t_1, \ldots, t_N]/(t_1, \ldots, t_N)^{k+1}$ le k-ième voisinage infinitésimal de* *l'origine dans W, $(\mathcal{E}_1, \nabla_1)$ la déformation universelle du premier ordre de $(\mathcal{E}, \nabla)$ sur $X \times W_1$ avec $[a_i] = \delta_i$. Alors il existe une série formelle*

$$f(t_1, \ldots, t_N) = \sum_{k=2}^{\infty} f_k(t_1 \ldots, t_N) \in \mathbb{H}^2(X, \mathcal{C}^\bullet)[[t_1, \ldots, t_N]],$$

où f_k est homogène de degré k ($k \geq 2$), avec la propriété suivante. Soit I l'idéal de $k[[t_1, \ldots, t_N]]$ engendré par l'image de l'application $f^* : \mathbb{H}^2(X, \mathcal{C}^\bullet)^* \to k[[t_1, \ldots, t_N]]$, adjointe de f. Alors pour tout $k \geq 2$, la paire $(\mathcal{E}_1, \nabla_1)$ s'étend à une connection $(\mathcal{E}_k, \nabla_k)$ sur $X \times V_k$, où V_k est le sous-schéma fermé de W_k défini par l'idéal $I \otimes k[[t_1, \ldots, t_N]]/(t_1, \ldots, t_N)^{k+1}$.

Changer à d'autres classes de connections est équivalent à juste changer le complexe $\mathcal{C}^\bullet$. Disons dans la classe de toutes les connections intégrables avec diviseur de pôles fixé D, nous pouvons choisir

$$\mathcal{C}^\bullet = [\mathcal{E}nd(\mathcal{E}) \to \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D) \to \mathcal{E}nd(\mathcal{E}) \otimes \Omega^2_X(D')],$$

où D' est n'importe quel diviseur telle que $\nabla_{\mathcal{E}nd(\mathcal{E})}(\mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D)) \subset \mathcal{E}nd(\mathcal{E}) \otimes \Omega^2_X(D')$ (on peut choisir $D' = 2D$).

Remark 3.5.17. Dans tous les cas, l'application première obstruction (la quadratique) est le carré de Yoneda $H^1(X, \mathcal{E}nd(\mathcal{E})) \to H^2(X, \mathcal{E}nd(\mathcal{E}))$ (resp. $\mathbb{H}^1(X, \mathcal{C}^\bullet) \to \mathbb{H}^2(X, \mathcal{C}^\bullet)$) induit par la composition $a \mapsto a \circ a$ sur $\mathcal{E}nd(\mathcal{E})$. Ceci était connu pour les déformations de fibrés vectoriels (voir [Mu-1]) et a été observé dans ma thèse pour toutes les autres théories de déformations prises sous considération.

Dans le chapitre 3, nous abordons l'étude de la structure locale de l'espace des modules. Nous considérons comme auparavant les quatre classes de connections: toutes les connections avec diviseur de pôles D fixé, connections intégrables, connections intégrables logarithmiques et celles qui sont intégrables logarithmiques avec une structure parabolique sur D. En plus des hypothèses ci-dessus, nous supposons que X soit projective, et dans le cas logarithmique, X soit lisse et D soit un diviseur à croisements normaux simples. Dans chacune de ces classes, il existe une notion appropriée de la stabilité de $(\mathcal{E}, \nabla)$, et l'espace des modules des objects stables peut être construit comme un GIT quotient sous une action de $GL(k)$ pour un k assez grand. Ceci peut être vu pour le cas intégrable par une modification facile de la preuve de Simpson [Sim], originalement écrite pour les connections intégrables régulières (ceci est, $D = 0$). Pour le cas logarithmique, l'espace des modules avait été construit par Nitsure [Ni]. L'espace des modules pour les connections logarithmiques avec une structure parabolique aux pôles avait été construit dans [I-Iw-S] pour le cas $\dim X = 1$. Nous étendons l'approche de Simpson aux cas des connections non-intégrables, régulières ou méromorphes avec diviseur des pôles D fixé.

Dans tous les cas un argument standard utilisant le Théorème des slices de Luna fournit une déformation verselle de $(\mathcal{E}, \nabla)$ dont la base $\mathcal{W}$ est un schéma affine muni d'une action du groupe $H = \mathrm{Aut}(\mathcal{E}, \nabla) = \mathrm{Stab}_{GL(k)}(\mathcal{E}, \nabla)$, et le germe de l'espace des modules au point $[(\mathcal{E}, \nabla)]$ est isomorphe au germe du GIT quotient $\mathcal{W}//H$. D'autre part, l'espace de Kuranishi $\mathcal{K}$ de $(\mathcal{E}, \nabla)$ est le complété formel of $\mathcal{W}$; il est muni d'une action naturelle de H, et le quotient $\mathcal{K}//H$ est le voisinage formel de $[(\mathcal{E}, \nabla)]$ dans l'espace des modules.

Nous utilisons cette méthode pour déterminer les singularités de l'espace des modules des connections dans des exemples. Nous décrivons l'un d'entre eux. Soit X une courbe de genre $g > 0$. Remarquons que si nous supposons que $\mathcal{E}$ est stable, alors $\mathbb{H}^2(X, \mathcal{C}^\bullet) = H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D))$ est de dimension 1 ou 0. En réalité, l'image de l'application d'obstruction est dans le noyau de l'application trace $H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X(D)) \to H^1(X, \Omega^1_X(D))$. Ceci implique que l'application d'obstruction s'annule et l'espace des modules des connections est lisse au point $[(\mathcal{E}, \nabla)]$. Ainsi, nous pouvons nous attendre à obtenir des singularités uniquement sur les $\mathcal{E}$ non-stables.

Example 3.5.18. Soient $D = 0$, $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}$, et supposons que la connection régulière $\nabla = d + A$ sur $\mathcal{E}$ soit suffisamment générique, pour que la paire $(\mathcal{E}, \nabla)$ soit stable, mais $\mathcal{E}$ soit strictement semistable. Considérons la suite exacte longue

$$\begin{aligned} 0 \longrightarrow \mathbb{H}^0(X, \mathcal{C}^\bullet) \longrightarrow H^0(X, \mathcal{E}nd(\mathcal{E})) &\xrightarrow{d_1} H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X) \\ \longrightarrow \mathbb{H}^1(X, \mathcal{C}^\bullet) \longrightarrow H^1(X, \mathcal{E}nd(\mathcal{E})) &\xrightarrow{d_1} H^1(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X) \\ \longrightarrow \mathbb{H}^2(X, \mathcal{C}^\bullet) \longrightarrow H^2(X, \mathcal{E}nd(\mathcal{E})) &= 0 \end{aligned}$$

provenant de la suite spectrale $E_1^{p,q} = H^q(\mathcal{C}^p) \Rightarrow \mathbb{H}^{p+q}(\mathcal{C}^\bullet)$, supporté sur deux droites verticales $p = 0$ et $p = 1$. Les applications d_1 sont les commmutateurs $B \mapsto [A, B] = A \circ B - B \circ A$, où $\circ$ dénote la composition de Yoneda. Sous l'hypothèse que A est générique, d_1 est de rang 2, ainsi la partie sans trace $\mathbb{H}^2(X, \mathcal{C}^\bullet)_0$ de $\mathbb{H}^2(X, \mathcal{C}^\bullet)$ est de dimension 1 or l'application d'obstruction est zéro. Par conséquent, la déformation verselle de $(\mathcal{E}, \nabla)$ est lisse.

Considérons maintenant le cas le plus dégénéré: $A = 0$, pour que ∇ soit juste la différentielle de de Rham. Alors les deux applications d_1 sont zéro, et $\mathbb{H}^2(X, \mathcal{C}^\bullet)_0 \simeq \mathbb{C}^3$. Le germe de l'espace des modules au point $(\mathcal{O} \oplus \mathcal{O}, d)$ est ainsi le quotient $f^{-1}(0)// \operatorname{PGL}(2)$, où $f^{-1}(0)$ est donné par trois équations dans $\mathbb{H}^1(X, \mathcal{C}^\bullet) \simeq \mathbb{C}^{8g}$. Les termes initiaux de ces équations sont des quadriques indépendantes de rang $4g$, et la méthode de [L-S] de déformation equivariante du cône normal implique qu'en réalité, nous avons $f^{-1}(0)// \operatorname{PGL}(2) \simeq f_2^{-1}(0)// \operatorname{PGL}(2)$, ceci étant, on peut se limiter à la première obstruction f_2 pour déterminer la structure locale de l'espace des modules.

Un certain nombre d'exemples supplémentaires de calculs des déformations verselles sont traités pour les connections sur une courbe elliptique avec deux points singuliers décrites au Chapitre 1.

SUMMARY

The logarithmic connections studied in Chapter 1 are direct images of regular connections on line bundles over genus-2 double covers of the elliptic curve. We give an explicit parameterization of all such connections, determine their monodromy, differential Galois group and the underlying rank-2 vector bundle. The latter is described in terms of elementary transforms. The question of its (semi)-stability is addressed. In Chapter 2, we construct the Kuranishi spaces (or versal deformations) for the four connection classes: the class of meromorphic connections with fixed divisor of poles D and its subclasses of integrable, integrable logarithmic and integrable logarithmic connections with a parabolic structure over D. In Chapter 3, we use the Kuranishi spaces to describe the local structure of the moduli spaces of connections and their relation to the moduli spaces of underlying vector bundles.

FRENCH SUMMARY

Les connections logarithmiques étudiées dans le Chapitre 1 sont les images directes des connections régulières sur des fibrés en droites au-dessus de revêtements doubles de genre 2 de la courbe elliptique. Nous donnons une paramétrisation explicite de telles connections, déterminons leurs monodromies, leurs groupes de Galois différentiels et les fibrés vectoriels sous-jacents de rang 2. Ce dernier est décrit en termes de transformées élémentaires. La question de sa stabilité est examinée. Dans le Chapitre 2, nous construisons les espaces de Kuranishi (ou les déformations verselles) pour les quatre classes de connections: la classe des connections méromorphes avec un diviseur fixé de pôles D et ses sous-classes de connections intégrables, connections logarithmiques intégrables et connections logarithmiques intégrables avec une structure parabolique sur D. Dans le Chapitre 3, nous utilisons les espaces de Kuranishi pour décrire la structure locale des espaces de modules des connections et leur relation aux espaces de modules des sous-fibrés vectoriels sous-jacents.

Key words: elliptic curve, ramified covering, logarithmic connection, bielliptic curve, genus-2 curve, monodromy, Riemann-Hilbert problem, differential Galois group, elementary transformation, stable bundle, vector bundle, versal deformations, Kuranishi spaces, moduli of sheaves, moduli of connections, Luna slice.

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