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Analyse d'Erreurs d'Estimateurs des Dérivées de Signaux Bruités et Applications

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Contents

General Introduction

Context

This Ph.D. work was carried out in Project-team ALIEN (ALgèbre pour Identification et Estimation Numériques) supported by the INRIA Lille-Nord Europe. The ALIEN project-team was created in June 2004 and is continued in January 2011 by the present Project-team Non-A (Non-Asymptotic estimation for online systems).

For engineers, a wide variety of information is not directly accessible to measurement. Some parameters (constants of a magnetic machine, delay time in communication, etc.) or internal variables (mechanical torques in a robot, etc.) are unknown. Similarly, more often than not, signals from sensors are distorted and tainted by measurement noises. To control such processes, and to extract information conveyed by the signals, one often has to identify a system and estimate parameters. Among the unknown variables to be reconstructed are derivatives of a signal. This problem to reconstruct numerical derivatives from noisy observational data arises in several practical applications such as image processing, identification, state observation and much more. This numerical differentiation problem is well known to be ill-posed in the sense that a small noise in measurement data can induce a huge error in the approximated derivatives.

The ALIEN project-team has developed an estimation theory, built around differential algebra and operational calculation. It has resulted in relatively simple, rapid algorithms: solutions are provided by explicit formulae, with straightforward implementation, using standard tools from computational mathematics. Unlike traditional methods, the majority of which pertain to asymptotic statistics, the ALIEN estimators are "non-asymptotic". In many application sectors, the response time parameter is crucial. Using this approach, computations are performed as the application is running: the "real-time" computing is targeted, as opposed to processing that occurs after the event.

The identification of linear systems, in the sense of automatic control, is benefiting from the algebraic module theory and from operational calculus. It permits to work in real time, *i.e.*, to simultaneously identify and control, a fact which is often indispensable in practice. The nonlinear generalization is based on a long-standing problem, *i.e.*, the estimation of the derivatives of various order of a noisy signal, in a way which is easy to implement. Works in progress demonstrate that we are not only able to identify the poorly known parameters, but also to estimate the state: this renewed perspectives yield for the first time a systematic procedure for obtaining non-linear observers.

In what concerns the signal processing, similar methods yield answers to denoising, to the detection of abrupt changes, to demodulation, blind equalization and compression, even for transient signals in a quite noisy environment. Two patents related to those techniques, which are of utmost industrial significance, are pending. The extension to image and video signals yields remarkable results for denoising, compression, edge and motion detection.

Objective of the thesis

The first objective of this thesis is to extend the derivative estimators introduced by M. Mboup, M. Fliess and C. Join in [Mboup 2009b]. We apply the extended estimators to non linear observation. The second one is to provide some parameter estimators for noisy sinusoidal signals. We also compare the estimation results obtained by a classical lock-in amplifier and our estimators. This classical technique is used by Atomic Force Microscope in tapping mode.

Both of these estimators are obtained by using the algebraic parametric techniques. Hence, they depend on some parameters. This thesis also aims at the analysis of the influence of these parameters on our estimators so as to minimize the estimation errors by choosing the "optimal" parameters.

Outline of the thesis

Part I is devoted to the theme of numerical differentiation in finite time of noisy signals and the application to non linear observation.

Chapter 1 gives several classes of differentiation estimators, *namely Jacobi estimators*, without considering noises. These estimators are based on the ones originally introduced by Mboup, Fliess and Join [Mboup 2007, Mboup 2009b] by using the algebraic parametric techniques. We generalize them by taking the truncated Jacobi orthogonal series expansion and by taking the scalar product of Jacobi polynomials so as to extend the parameters defining these estimators from $\mathbb{N}$ to $\mathbb{R}$. They can be used for on-line or off-line estimations. Since Jacobi estimators depend on a set of parameters, by providing some error bounds for the associated truncation errors we study the corresponding convergence rate and the influence of parameters on the estimation errors. This gives us a guide of how to choose parameters for Jacobi estimators. Then, by using the algebraic parametric techniques we show how to obtain a general form for Jacobi estimators. Finally, we show that by using the algebraic parametric techniques we can also obtain some estimators for the fractional order derivatives.

In **Chapter 2**, we study Jacobi estimators obtained in the noisy case. We consider mainly three different types of noises: integrable noises, non independent stochastic process noises and independent stochastic process noises. By providing some error bounds, we study the influence of parameters on the noise contribution errors. Finally, by choosing a set of appropriate parameters we give some numerical examples to show the efficiency and the stability of Jacobi estimators.

In **Chapter 3**, we focus on the applications of Jacobi estimators to non linear observation. Firstly, we recall some results of observability for a nonlinear system within the differential geometric framework and also in the differential algebraic framework. Secondly, by taking the ball and beam system we compare our Jacobi estimators to high gain observers and high order sliding mode differentiators.

Part II is devoted to the theme of finite time numerical parameter estimations for noisy sinusoidal signals. We compare the on-line results of our estimators with the ones obtained by a lock-in amplifier system which is classically used by an AFM in tapping mode.

Chapter 4 concentrates on the parameters estimation for noisy sinusoidal signals with time-varying amplitudes. We use the algebraic parametric techniques and the modulating functions method to obtain some useful equations and linear systems. Then, by solving these equations and linear systems we estimate the wanted parameters: frequency, amplitude and phase.

Chapter 5 studies the estimation errors of the previous parameter estimators: the numerical error due to a numerical integration method and the noise error contribution due to an integrable noise or a stochastic process noise. Since these estimators depend on a set of parameters, we give some error bounds which permit us to choose the optimal ones.

Chapter 6 begins by showing how to use these error bounds to choose some appropriate parameters for our estimators. Then, some comparisons between the algebraic parametric techniques and the modulating functions method are given by taking different signal models.

In **Chapter 7**, we give some on-line experimental results obtained at the Laboratoire National de métrologie et d'Essais (LNE) by applying our amplitude estimators. These results are based on the comparison of our results with respect to a DSP lock-in amplifier which is usually used as an amplitude detector for the atomic force microscopy in tapping mode.

Finally, this thesis is completed by some conclusions and perspectives.

Part I

Numerical differentiation

Chapter 1

Numerical point-wise derivative estimations

Contents

1.1 Introduction

Numerical differentiation is concerned with the numerical estimation of derivatives of an unknown function from its discrete, potentially noisy measurement data. It has attracted a lot of attention from different points of view:

- observer design in the control literature [Chitour 2002, Ibrir 2004, Levant 2003, Diop 1994, Diop 2000],
- digital filter in signal processing [Chen 1995, Rader 2006, Tseng 2005],
- Volterra integral equation of the first kind [Cheng 2004, Gorenflo 1991],
- identification [Hanke 1999, Wang 2008].

The problem of numerical differentiation is ill-posed in the sense that a small error in measurement data can induce a large error in the approximate derivatives. Therefore, various numerical methods have been developed to obtain stable algorithms more or less sensitive to additive noise. They mainly fall into eight categories:

- the finite difference methods [Khan 2000, Rahul 2006, Qu 1996, Ramm 2001],
- the Savitzky Golay methods [Savitzky 1964, Gorry 1990, Barak 1995, Diop 1994]
- the wavelet differentiation methods [Shao 2003, Nie 2002, Shao 2000, Leung 1998, Diop 2000]
- the Fourier transform methods [Fu 2010, Dou 2010, Y. 2008, Qian 2006a, Qian 2006b, Kauppinen 1981]
- the mollification methods [Hào 1995, Murio 1993, Murio 1998],
- the Tikhonov regularization methods [Cullum 1971, Hanke 2001, Nakamura 2008, Wei 2005, Wang 2002],
- the algebraic methods [Mboup 2009b, Mboup 2007, Liu 2011c, Liu 2009, Liu 2011b, Liu 2011a],

- the differentiation by integration methods [Lanczos 1956, Rangarajana 2005, Wang 2010], i.e. using the Lanczos generalized derivatives.

Recent algebraic parametric estimation techniques for linear systems [Fliess 2003b, Fliess 2007] have been extended to various problems in signal processing (see, e.g., [Fliess 2003a, Fliess 2004a, Mboup 2009a, Neves 2006, Neves 2007, Trapero 2007a, Trapero 2007b, Trapero 2008, Liu 2008]). Let us emphasize that those methods, which are algebraic and non-asymptotic, exhibit good robustness properties with respect to corrupting noises, without the need of knowing their statistical properties (see [Fliess 2006, Fliess 2008] for more theoretical details). The robustness properties have already been confirmed by numerous computer simulations and several laboratory experiments. It appears that these techniques can also be used to derive numerical differentiation algorithms exhibiting similar properties (see [Mboup 2009b, Mboup 2007, Liu 2011c, Liu 2009, Liu 2011b, Liu 2011a]). Such techniques are used in [Fliess 2004c, Fliess 2004b, Barbot 2007] for state estimation.

In this chapter, we aim at constructing high order precise numerical derivative estimators of smooth functions from an algebraic frame work. For this, we consider three cases. In the two first cases, we use the sampling data given before (resp. after) the point at which the derivative value we want to estimate. The such obtained estimator is called *causal estimator* (resp. *anti-causal estimator*). In the last case, the point at which the derivative value we want to estimate is the middle point of the time window used for data. Hence, we get *central estimator*. We will show in the next chapter the robustness of these estimators when the used discrete data are corrupted by noises. In the two following sections, we are going to present the algebraic parametric technique for obtaining anti-causal derivative estimators.

1.1.1 An introductory example

In the classical numerical differentiation methods, an interpolating polynomial (see [Anderssen 1998], [Brown 1992]) or a least-squares polynomial (see [Gorry 1990], [Savitzky 1964]) is generally used to approximate a function, the derivatives of which we want to estimate. Then the derivatives of this polynomial is closely linked to the coefficients of this polynomial. In the recent papers [Mboup 2007, Mboup 2009b], a new algebraic parametric differentiation method is presented where an elimination technique such as the one introduced in [Herceg 1986] was used to calculate the useful coefficients.

Let us start to illustrate this algebraic parametric technique method with a simple example. Let $p_1(t) = a_0 + a_1 t$ be a first order polynomial known on $\mathbb{R}^+$, where a_0 and a_1 are unknown. We are going to calculate the first order derivative of p_1 which is the coefficient a_1. For this, we apply an elimination technique in the operational domain. By applying the Laplace transform to $t^\alpha, \alpha \in \mathbb{R}$ (recalled in Appendix (7.11)), we obtain $\hat{p}_1 = \frac{a_0}{s} + \frac{a_1}{s^2}$, where $\hat{p}_1$ is the Laplace transform of p_1. Then, by multiplying both sides by s, we get $s\hat{p}_1 = a_0 + \frac{a_1}{s}$. Thus, we can annihilate the coefficient a_0 by deriving with respect to s the last equation

$$s\hat{p}_1^{(1)} + \hat{p}_1 = -\frac{1}{s^2} a_1. \tag{1.1}$$

Such that it only remains a_1 and $\hat{p}_1$ in (1.1). We need to return into the time domain in order to calculate a_1 by using the knowledge of p_1. Since the inverse Laplace transform of $s\hat{p}_1^{(1)}$ contains the derivative of p_1 which is unknown, we multiply both sides of (1.1) by s^{-2}. Then, by applying (7.11) and (7.13) (given in Appendix) we obtain an integral which only depends on p_1:

$$a_1 = \frac{3!}{t^3} \int_0^t (2\tau - t)\, p_1(\tau) d\tau, \quad t > 0. \tag{1.2}$$

In the previous computations, we used the following differential operator:

$$\Pi_{0,0} = \frac{1}{s^2} \cdot \frac{d}{ds} \cdot s, \tag{1.3}$$

which permits us to annihilate a_0 and to calculate a_1 by an integral. Consequently, we call such annihilator *integral annihilator*. This method is aptly called a method of *differentiation by integration*. An advantage of this presented method is that a quite short time window $[0, t]$ is sufficient for obtaining accurate value of a_1.

1.1.2 Annihilator - Estimator

The extension to polynomial functions of higher order is straightforward. For derivatives estimates up to some finite order of a given smooth function, we take a suitable truncated Taylor series expansion around a given time instant, to which we apply some similar computations to the ones in the example of Subsection 1.1.1. Moreover, using sliding time windows permits to estimate derivatives at any sampled time instant. Precisely, let x be a real valued analytical function defined on a finite time open interval $I \subset \mathbb{R}^+$. Let $n \in \mathbb{N}$, we are going to estimate the n^{th} order derivative of x. For any $t_0 \in I$, we take the N^{th} $(N \geq n)$ order truncated Taylor series expansion of x at t_0

$$x_N(t + t_0) = \sum_{i=0}^{N} \frac{t^i}{i!} x^{(i)}(t_0), \tag{1.4}$$

where we want to calculate $x^{(n)}(t_0)$. Then by applying the Laplace transform to (1.4) and using (7.11) given in Appendix we get

$$\hat{x}_N(s) = \sum_{i=0}^{N} s^{-(i+1)} x^{(i)}(t_0),$$

where $\hat{x}_N(s)$ is the Laplace transform of $x_N(t)$ with a variable $s \in \mathbb{C}$. The next step is to give an integral annihilator so as to annihilate the terms containing $x^{(i)}(t_0)$ with $i \neq n$ and calculate $x^{(n)}(t_0)$ with an integral. A differential operator rooted in [Mboup 2007] of the following form

$$\Pi_{k,\mu}^{N,n} = \frac{1}{s^{N+1+\mu}} \cdot \frac{d^{n+k}}{ds^{n+k}} \cdot \frac{1}{s} \cdot \frac{d^{N-n}}{ds^{N-n}} \cdot s^{N+1}, \quad \text{with } k, \mu \in \mathbb{N} \tag{1.5}$$

is used. Then, the coefficient $x^{(n)}(t_0)$ in the right side of (1.4) is kept in such a way: being multiplied by s^{N+1}, $\hat{x}_N(s)$ becomes a polynomial of degree N. Then the terms of degree lower than $N - n$, which include $x^{(i)}(t_0)$, $n < i < N$, are annihilated by applying $N - n$ times derivations. In order to preserve the term including $x^{(n)}(t_0)$, we multiply the remaining polynomial by $\frac{1}{s}$. Then we apply more than n times derivations with respect to s so as to annihilate the other terms including $x^{(i)}(t_0)$ with $0 < i < n$.

Finally, we multiply by $\frac{1}{s^{N+1+\mu}}$ to return into the time domain at instant t_0 with a sliding window of length T ($T > 0$ such that $[t_0, t_0+T] \subset I$) which only depends on function x. Hence, by applying (7.11) and (7.13), we obtain an integral expression of $x^{(n)}(t_0)$. By replacing x_N by x, the approximation of the remainder coefficient $x^{(n)}(t_0)$ is taken as an estimator for $x_{t_0}^{(n)}$. Since this estimator is determined by the parameters T, N, k and μ, we denote it by $\tilde{x}_{t_{0+}}^{(n)}(k, \mu, T, N)$. Consequently, by using a sliding integration window $[t_0, t_0 + T]$ we can estimate the derivative values of x for all point $t_0 \in I$ verifying the condition $[t_0, t_0 + T] \subset I$. This derivative estimator contains a truncated term error which comes from the truncation of the Taylor series expansion of x. It is clear that when we estimate the n^{th} order derivative of x, we can reduce the truncated term error by increasing the truncated order N. If we take $N = n$ in (1.4) and (1.5), we call the such obtained estimator *minimal estimator* and we denote it by $\tilde{x}_{t_{0+}}^{(n)}(k, \mu, T)$. It was shown in [Mboup 2007, Mboup 2009b] that the estimator $\tilde{x}_{t_{0+}}^{(n)}(k, \mu, T, N)$ obtained by using N with $N > n$ can be written as an affine combination of some minimal estimators. Hence, it corresponds to a point in the $\mathbb{Q}$-affine hull of the set

$$S_{k,\mu,T,q} = \left\{ \tilde{x}_{t_{0+}}^{(n)}(k + q, \mu, T), \dots, \tilde{x}_{t_{0+}}^{(n)}(k, \mu + q, T) \right\} \text{ with } q = N - n. \tag{1.6}$$

In this case, we call it *affine estimator*. Moreover, it was shown in [Mboup 2009b] that this estimator $\tilde{x}_{t_{0+}}^{(n)}(k, \mu, T, N)$ ($N \geq n$) can be also obtained by taking the q^{th} ($q = N - n \leq n + k$) order truncated Jacobi orthogonal series expansion of $x^{(n)}(t_0 + T\xi)$ at $\xi = 0$:

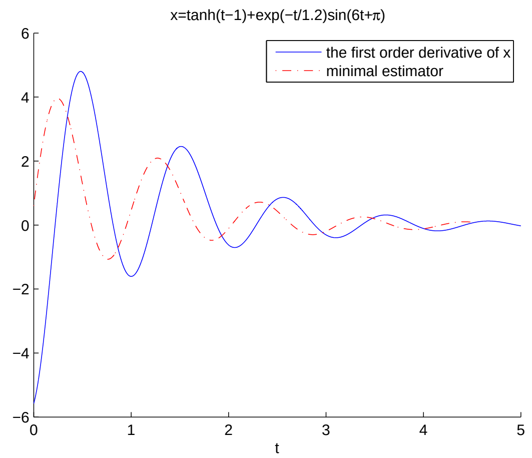
$$\tilde{x}_{t_{0+}}^{(n)}(k, \mu, T, N) = \sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+n,k+n)}(\cdot), x^{(n)}(t_0 + T\cdot) \right\rangle_{\mu+n,k+n}}{\|P_i^{(\mu+n,k+n)}\|_{\mu+n,k+n}^2} \; P_i^{(\mu+n,k+n)}(\xi), \quad \text{with } \xi = 0. \tag{1.7}$$

By taking $\xi \in [0, 1]$ rather than 0, the $\mathbb{Q}$-affine hull $S_{k,\mu,T,q}$ is extended to a $\mathbb{R}$-affine hull. Hence, it is clear that any point in this set represents an estimator for $x_{t_0}^{(n)}$, in some meaningful sense. Characterizing these points which minimize a given distance to $x^{(n)}(t_0)$ is an important question. A judicious choice was introduced in [Mboup 2009b] by taking ξ as the smallest root of $P_{q+1}^{(\mu+n,k+n)}$. However, it corresponds to take the $(q + 1)^{th}$ order truncated Jacobi series expansion of $x^{(n)}(t_0 + T\xi)$, which produces a time-drift. In this case, we denote this time-drift estimator by $\tilde{x}_{t_{0+}}^{(n)}(k, \mu, T, N, \xi)$ and the estimation error comes from the truncation of the Jacobi orthogonal series expansion.

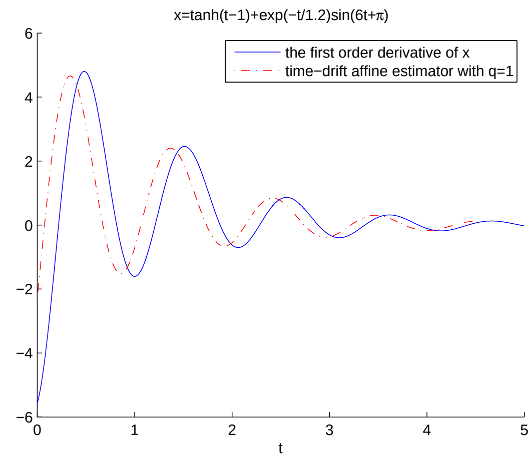
In order to show the efficiency of the previous estimators, we give a simple example. Let us consider the following function

$$x(t_i) = \tanh(t_i - 1) + \exp(-\frac{t_i}{1.2}) \sin(6t_i + \pi), \tag{1.8}$$

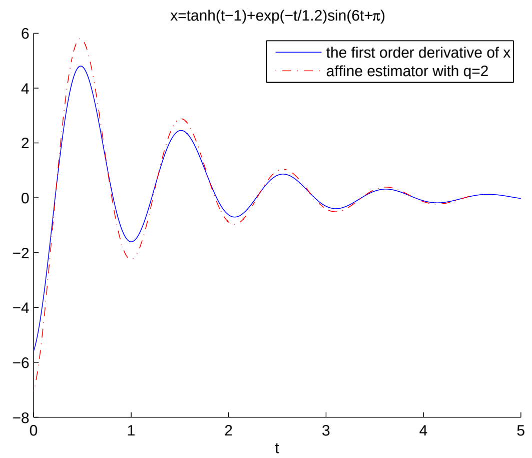
where $t_i \in [0, 5]$. We estimate the first order derivative of x by using the presented minimal and affine estimators. In this example and more generally in this chapter, we assume that x is given in discrete case with an equidistant sampling period $T_s = \frac{1}{2000}$. Then, we apply the trapezoidal numerical integration method to approximate the integrals in the estimators $\tilde{x}_{t_{0+}}^{(n)}(k, \mu, T, N, \xi)$. In this chapter, we always set the length of the moving integration window $[t_i, t_i + T]$ to be equal to $\frac{1}{2}$, *i.e.* there are 1001 sampling data in each integration window. Hence, by using the well-known error bound given in [Ralston 1965] for numerical integration error, we can verify that the numerical error produced by the



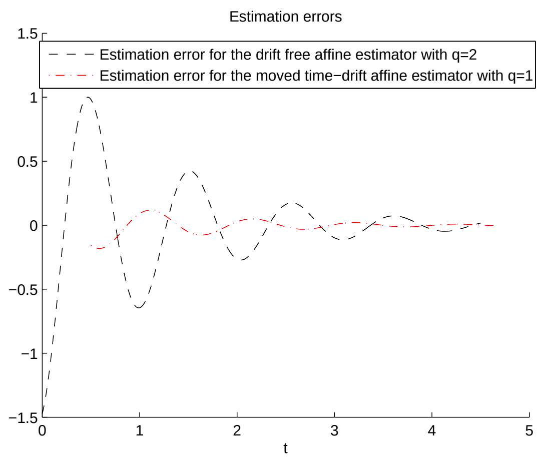
(a) Minimal estimator $\tilde{x}^{(n)}_{t_0+}(k, \mu, T)$ with $n = 1$, $k = \mu = 0$ and $T = \frac{1}{2}$.



(b) Time-drift affine estimator $\tilde{x}^{(n)}_{t_0+}(k, \mu, T, N, \xi)$ with $n = 1$, $k = \mu = 0$, $T = \frac{1}{2}$, $N = 2\,(q = 1)$ and $\xi = 0.2764$.



(c) Drift free affine estimator $\tilde{x}^{(n)}_{t_0+}(k, \mu, T, N)$ with $n = 1$, $k = \mu = 0$, $T = \frac{1}{2}$, $N = 3\,(q = 2)$.



(d) Truncated term error for $\tilde{x}^{(1)}_{t_0+}(0, 0, \frac{1}{2}, 3)$ and amplitude error for $\tilde{x}^{(1)}_{t_0+}(0, 0, \frac{1}{2}, 2, 0.2764)$.

Figure 1.1: Estimations by using minimal and affine estimators.

trapezoidal rule is negligible. This numerical integration problem and the analysis of the choice of the parameters will be addressed in Chapter 2. Consequently, in this chapter, we only consider for each estimator the truncated error part.

The estimation obtained by using the minimal estimator $\tilde{x}^{(n)}_{t_0+}(k, \mu, T)$ is given in Figure 1.1(a). We can see that there is not only a drift error but also an amplitude error for this estimator. The estimation obtained by the time-drift affine estimator $\tilde{x}^{(n)}_{t_0+}(k, \mu, T, N, \xi)$ with $N = 1$ is given in Figure 1.1(b), where $\xi = \min\left(\frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}\right)$ is the smallest root of $P_2^{(\mu+1,k+1)}$ with $a = \frac{1}{2}(k^2 + \mu^2 + 2\mu k + 7\mu + 7k + 12)$, $b = -(k^2 + \mu k + 2\mu + 5k + 6)$ and $c = \frac{1}{2}(k^2 + 3k + 2)$. We can see that the drift error $T\xi$ and the amplitude error are improved. In Figure 1.1(c), we give the estimation obtained by using the affine estimator $\tilde{x}^{(n)}_{t_0+}(k, \mu, T, N)$ defined by (1.7) with $N = 2$. In this case $\xi = 0$

and there is no drift. Let us compare the amplitude error between these two kind of estimators. In Figure 1.1(b), we know that the time-drift of the affine estimator $\tilde{x}^{(1)}_{t_{0+}}(k, \mu, T, 1, \xi)$ is equal to $T\xi$. Hence, by translating the so obtained estimation we can calculate the amplitude error by subtracting the exact derivative values. We can see in Figure 1.1(d) the difference between this amplitude error and the truncated term error issued from the drift free affine estimator $\tilde{x}^{(n)}_{t_{0+}}(k, \mu, T, N)$ with $N = 3$. Consequently, it is shown that by admitting a time-drift the affine estimator is significantly improved.

The aim of this chapter is to give some extended estimators by using the algebraic parametric technique so as to improve the truncated term error and the drift error. In the next subsection, we introduce the Lanczos generalized derivative estimator which was originally introduced by Cioranescu [Cioranescu 1938] in 1938 and also developed by Lanczos in 1956. We show that these estimators are a particular case of (1.7) in the causal or anti-causal case.

1.1.3 Lanczos derivative generalized estimators

The Lanczos generalized derivative estimator $D_T x$ for the central case estimation, defined in [Lanczos 1956] (p. 324) is given by

$$\forall t_0 \in I,\ D_T x(t_0) = \frac{3}{2T^3} \int_{-T}^{T} \tau\, x(t_0 + \tau)\, d\tau = \frac{3}{2T} \int_{-1}^{1} \tau\, x(t_0 + T\tau)\, d\tau, \tag{1.9}$$

where I is an open interval of $\mathbb{R}$ and $2T > 0$ is the length of the integral window $[t_0 - T, t_0 + T] \subset I$. It generalizes the ordinary derivative in the following two senses: Firstly, if x is assumed to belong to $\mathcal{C}^3(I)$, then by using the Taylor series expansion of x at t_0 in (1.9) we obtain $|D_T x(t_0) - \dot{x}(t_0)| = \mathcal{O}(T^2)$. Secondly, if we assume that x has both the right and left derivatives $\dot{x}(t_{0-})$ and $\dot{x}(t_{0+})$ at t_0, then we have

$$\lim_{T \to 0} D_T x(t_0) = \frac{\dot{x}(t_{0-}) + \dot{x}(t_{0+})}{2}. \tag{1.10}$$

It is also called a method of *differentiation by integration*. Rangarajana and al. [Rangarajana 2005] generalized it for higher order derivatives by taking the n^{th} order truncated Taylor expansion of x at t_0

$$x_n(T\tau + t_0) = \sum_{i=0}^{n} \frac{(T\tau)^i}{i!} x^{(i)}(t_0), \tag{1.11}$$

where $\tau \in [-1, 1]$ and $T > 0$ such that $[t_0 - T, t_0 + T] \subset I$. Then by taking the scalar product of x_n by a Legendre polynomial P_n of degree n, the terms containing $x^{(j)}(t_0)$ with $j < n$ are annihilated by the following property

$$\int_{-1}^{1} P_n(\tau)\, \tau^j\, d\tau = 0, \quad \text{for } 0 \leq j < n.$$

Thus, they introduced the following estimator

$$\forall t_0 \in I,\ D_T^{(n)} x(t_0) = \frac{1}{T^n} \int_{-1}^{1} \gamma_n P_n(\tau)\, x(t_0 + T\tau)\, d\tau,\ n \in \mathbb{N}, \tag{1.12}$$

where the coefficient γ_n is equal to $\frac{1\times 3\times 5\times\cdots\times(2n+1)}{2}$. If x is assumed to belong to $\mathcal{C}^{n+2}(I)$, then by using $\int_{-1}^{1} P_n(t)\, t^{n+1}\, dt = 0$, they showed that $\left| D_T^{(n)} x(t_0) - x^{(n)}(t_0) \right| = \mathcal{O}(T^2)$. Recently, by using Richardson extrapolation [Joyce 1971] Wang and al. [Wang 2010] improved the convergence rate for high order Lanczos derivative estimators with the following affine schemes

$$\forall t_0 \in I,\ D_{T,\lambda_n}^{(n)} x(t_0) = \frac{1}{T^n} \int_{-1}^{1} P_n(\tau)\,(a_n\, x(t_0 + T\tau) + b_n\, x(t_0 + \lambda_n T\tau))\, d\tau. \tag{1.13}$$

If x is assumed to belong to $\mathcal{C}^{n+4}(I)$, then a_n, b_n and λ_n are chosen such that

$$\int_{-1}^{1} (a_n + b_n \lambda^{n+2}) P_n(\tau)\, \tau^{n+2}\, d\tau = 0$$

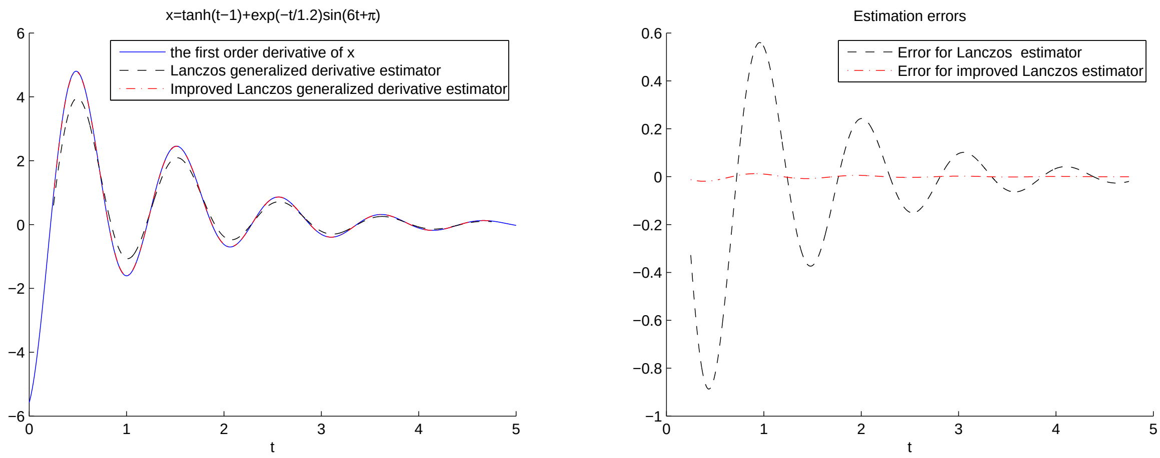
in order to obtain $\left| D_{T,\lambda_n}^{(n)} x(t_0) - x^{(n)}(t_0) \right| = \mathcal{O}(T^4)$. Moreover, if we assume that $x \in \mathcal{C}^{n-1}(I)$, $x_R^{(n)}$ and $x_L^{(n)}$ exist at t_0, then

$$\lim_{T\to 0} D_{T,\lambda_n}^{(n)} x(t_0) = \frac{x_R^{(n)}(t_0) + x_L^{(n)}(t_0)}{2}. \tag{1.14}$$

In order to show the efficiency of the Lanczos estimator $D_T x(t_0)$ and the improved Lanczos estimator $D_{T,\lambda_n}^{(n)} x(t_0)$, let us estimate the first derivative of the function defined by (1.8). Similarly to the example given in the previous section, we apply the trapezoidal numerical integration method, where the length of the sliding integration window $[t_i - T, t_i + T]$ is equal to $\frac{1}{2}$, *i.e.* $T = \frac{1}{4}$. Then, we can see in Figure 1.2(a) the estimations obtained by using $D_T x(t_0)$ and $D_{T,\lambda_n}^{(n)} x(t_0)$. Since, these estimators are central estimators, there are no drift errors but there are amplitude errors. They are given in Figure 1.2(b).

Now, let us consider the function $x \equiv |\cdot|$ defined on $[-\frac{5}{2}, \frac{5}{2}]$ the first derivative of which is discontinuous at 0. We assume that x is given in discrete case with an equidistant sampling period $T_s = \frac{1}{2000}$. Then, we can see in Figure 1.3 the estimations obtained by using $D_T x(t_0)$ and $D_{T,\lambda_1}^{(1)} x(t_0)$ where $\lambda_1 = \frac{1}{2}$ and $T = \frac{1}{4}$.

Unlike the previously presented algebraic parametric technique which uses an elimination technique in the operational domain, the Lanczos differentiation by integration method uses directly the orthogonality of Legendre polynomials defined on $[-1, 1]$ to the truncated Taylor series expansion of x with an integration window $[t_0 - T, t_0 + T]$. Let us recall that the minimal estimator presented in the previous subsection can be also obtained by using the Jacobi polynomial $P_n^{(\mu+n,\kappa+n)}$ defined on $[0, 1]$ with $\kappa, \mu \in]-1, \infty[$. Contrary to [Mboup 2009b] where $\kappa, \mu \in \mathbb{N}$ we can extend their domain to $]-1, \infty[$. Since the affine estimator is an affine combination of minimal estimators, the values of κ and μ can be also extended to $]-1, \infty[$. For this, we use the truncated Jacobi orthogonal series. However, it is difficult to choose judicious parameters κ and μ, as well as T and N. Similarly to the Lanczos generalized derivative estimator, the second aim of this chapiter is to analyze the truncated term error for the affine estimators by giving the convergence rate and the corresponding error bounds. Since these error bounds also depend on parameters κ, μ, T and N, we study the influence of these parameters on the truncated term error. This allows us to reduce this error by choosing judicious such parameters. The third aim of this chapiter is then to introduce some other estimators by using truncated Taylor expansion. The effect of the smoothness condition for x will be also discussed.



(a) Lanczos estimator $D_T x(t_0)$ and Improved Lanczos estimator $D^{(1)}_{T,\lambda_1} x(t_0)$ with $\lambda_1 = \frac{1}{2}$ where $T = \frac{1}{4}$.

(b) Associated estimation errors.

Figure 1.2: Estimations and associated estimation errors of $D_T x(t_0)$ and $D^{(1)}_{T,\lambda_1} x(t_0)$.

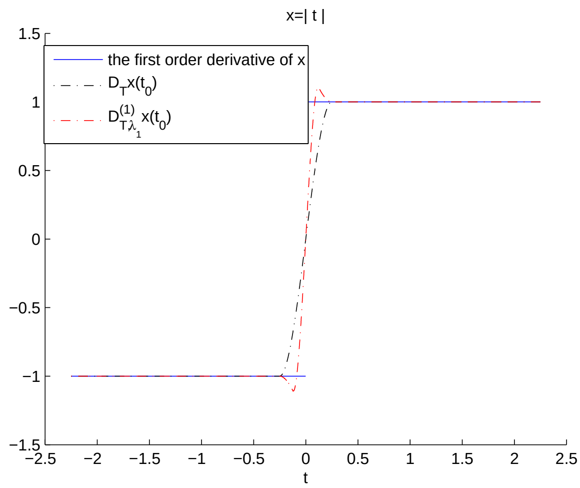


Figure 1.3: Estimations by using $D_T x(t_0)$ and $D^{(1)}_{T,\lambda_1} x(t_0)$.

1.1.4 Organization of the chapter

In this chapter, we consider the numerical differentiation of a smooth function x only known at discrete time, the n^{th} ($n \in \mathbb{N}$) order derivative of which we want to estimate. We assume that x belongs to $\mathcal{C}^n(I)$ where I is an open interval of $\mathbb{R}$ and the discrete data are not corrupted by some noises. This chapter is organized as follows.

We start by studying the minimal estimator which is previously presented in the anti-causal case, *i.e.* it is obtained by using the integration window $[t_0, t_0 + T]$. We show in Section 1.2 that it can be obtained by three ways in the causal case, *i.e.* it is obtained by using the integration window

$[t_0 - T, t_0]$, as well as in the anti-causal case. Firstly, in Subsection 1.2.1 we apply the algebraic parametric technique to the n^{th} truncated Taylor expansion of $x(t_0 + \beta t)$ for any $t_0 \in I$ with $\beta \in \mathbb{R}^*$ and $t > 0$ such that $t_0 + \beta t \in I$. For this, we apply the integral annihilator $\Pi_{k,\mu}^{N,n}$ defined by (1.5) with $N = n$. From now on, by using the Riemann-Liouville fractional integral (see (7.10) in Appendix), we can take the value of μ in $]-1, +\infty[$ for $\Pi_{k,\mu}^{n,n}$ rather than in $\mathbb{N}$. Then, we obtain a family of causal minimal estimators $\tilde{x}_{t_{0-}}^{(n)}(k, \mu, \beta T)$ with $\beta < 0$ and a family of anti-causal minimal estimators $\tilde{x}_{t_{0+}}^{(n)}(k, \mu, \beta T)$ with $\beta > 0$. Secondly, similarly to the way for obtaining (1.12), we show in Subsection 1.2.2 that these minimal estimators can be also obtained by applying the classical orthogonal properties of the Jacobi polynomials defined on $[0, 1]$. In this way, we can extend the value of k to $]-1, +\infty[$. Then, we denote the extended minimal estimators by $D_{\kappa,\mu,\beta T}^{(n)} x(t_0)$ with $\kappa, \mu \in]-1, +\infty[$. By applying the recurrence relations of the Jacobi polynomials, a simple recurrence relation between the minimal estimators for $x^{(n)}(t_0)$ and $x^{(n-1)}(t_0)$ is given. Thirdly, we show in Subsection 1.2.3 that the extended minimal estimators $D_{\kappa,\mu,\beta T}^{(n)} x(t_0)$ are equal to the first term in the Jacobi orthogonal series expansion of $x^{(n)}$. Hence, taking the n^{th} order truncated Taylor expansion of x corresponds to take the 0^{th} order truncated Jacobi expansion of $x^{(n)}$. Then, by using the Rodrigues formula and the definition of the Jacobi polynomials a recurrence relation between the minimal estimators for $x^{(n)}(t_0)$ and $x^{(0)}(t_0)$ is given. In subsection 1.2.4, we analyze the truncated term error for minimal estimators. A precise local error bound shows that the convergence rate for the minimal estimators is $\mathcal{O}(T)$ as $T \to 0$. By this way, we show the influence of parameters κ, μ and T on the truncated term error.

We investigate in Section 1.3 the extension of the affine estimator $\tilde{x}_{t_{0+}}^{(n)}(k, \mu, T, N)$ presented in Subsection 1.1.2 which is originally introduced in [Mboup 2007, Mboup 2009b] with $k, \mu \in \mathbb{N}$. In Subsection 1.3.1, by applying the algebraic parametric technique with the integral annihilator $\Pi_{k,\mu}^{N,n}$ given by (1.5) with $N \geq n$ and $\mu \in]-1, +\infty[$, we obtain a family of anti-causal estimators $\tilde{x}_{t_{0+}}^{(n)}(k, \mu, T, N)$ and a family of causal estimators $\tilde{x}_{t_{0-}}^{(n)}(k, \mu, -T, N)$. Then, by giving the relation between $\Pi_{k,\mu}^{N,n}$ and $\Pi_{k,\mu}^{n,n}$ we show that $\tilde{x}_{t_{0\pm}}^{(n)}(k, \mu, \beta T, N)$ can be written as affine combination of some minimal estimators with $k \in \mathbb{N}$ and $\mu \in]-1, +\infty[$. Then, by assuming that $x \in \mathcal{C}^{N+1}(I)$ we give a global error bound for the truncated term errors for these affine estimators which shows that the convergence rate is $\mathcal{O}(T^{N-n+1})$ as $T \to 0$. In Subsection 1.3.2, we denote $D_{\kappa,\mu,\beta T,q}^{(n)} x(\beta T \xi + t_0)$ as the q^{th} ($q = N - n \in \mathbb{N}$) order truncated Jacobi orthogonal series expansion of $x^{(n)}$, where $\xi \in [0, 1]$. Then, we extend the affine estimators $\tilde{x}_{t_{0\pm}}^{(n)}(k, \mu, \beta T, N)$ by $D_{\kappa,\mu,\beta T,q}^{(n)} x(\beta T \xi + t_0)$. Firstly, we recall the main result of [Mboup 2009b]. It was shown that if $q \leq k + n$ then we have $\tilde{x}_{t_{0\pm}}^{(n)}(k, \mu, \beta T, N) = D_{k,\mu,\beta T,q}^{(n)} x(\beta T \xi + t_0)$ with $\xi = 0$, $k, \mu \in \mathbb{N}$. Moreover, it was shown that for any $\xi \in [0, 1]$, $D_{k,\mu,\beta T,q}^{(n)} x(\beta T \xi + t_0)$ could be written as an affine combination of some minimal estimators $\tilde{x}_{t_{0\pm}}^{(n)}(k, \mu, \beta T)$, where the associated coordinates are given by solving a linear system. Secondly, we shown that for any integer q, $D_{\kappa,\mu,\beta T,q}^{(n)} x(\beta T \xi + t_0)$ can be written as an affine combination of some extended minimal estimators $D_{\kappa,\mu,\beta T}^{(n)} x(t_0)$ with $\kappa, \mu \in]-1, +\infty[$, where the associated coordinates are explicitly given. Hence, we obtain extended affine estimators. If we take $q = 0$, then $D_{\kappa,\mu,\beta T,q}^{(n)} x(\beta T \xi + t_0)$ becomes minimal estimators $D_{\kappa,\mu,\beta T}^{(n)} x(t_0)$. Hence, $D_{\kappa,\mu,\beta T,q}^{(n)} x(\beta T \xi + t_0)$ give a general presentation for minimal estimators and affine estimators. We call them *Jacobi estimators*. In particular, we show the relation between $D_{\kappa,\mu,\beta T,q}^{(n)} x(\beta T \xi + t_0)$ with $q = 1$ and $D_{\kappa,\mu,\beta T}^{(n)} x(t_0)$.

Thirdly, we show that the Jacobi estimators $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi+t_0)$ for $x^n(t_0)$ are in fact connected to the n^{th} order derivative of the Jacobi estimators $D^{(0)}_{\kappa,\mu,\beta T,q+n}x(\beta T\xi+t_0)$ for $x(t_0)$. Then, a formula for the Jacobi estimators is given. Hence, the $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi+t_0)$ are calculated by an integral of x. Subsection 1.3.3 is devoted to study the truncated term error for the Jacobi estimators. The truncated term error for the Jacobi estimators can be divided into two parts. The first part is considered as bias term error which produces an error between $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi+t_0)$ and $x^{(n)}(t_0+\beta T\xi)$ in the horizontal direction and the second part is considered as a drift term error which produces an error in the vertical direction with a value of $T\xi$. On the one hand, by assuming that $x\in\mathcal{C}^{N+1}(I)$, we show that the convergence rate for the bias term error is $\mathcal{O}(T^{q+1})$ as $T\to 0$. Moreover, if we take ξ_q^{min} as the smallest root of the $P^{(\mu+n,k+n)}_{q+1}$, then we get $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi_q^{min}+t_0)=D^{(n)}_{\kappa,\mu,\beta T,q+1}x(\beta T\xi_q^{min}+t_0)$. Hence, the convergence rate for the bias term error is improved to $\mathcal{O}(T^{q+2})$ as $T\to 0$. On the other hand, we show that the minimum value of the time-drift $T\xi_q^{min}$ occurs when κ and μ are negative. This is one reason why we extend the parameters' domain. Finally, two local error bounds for the truncated term error of the affine estimators $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi+t_0)$ with $q=1$ are precisely given.

In Section 1.4, we recall some well-known approximation theories so as to explain our approximation method. Then we study the truncated term error by assuming that the smooth function x belongs to the Beppo-Levi space. In Subsection 1.4.3, we consider the case where $x\in\mathcal{C}^{n-1}(I)$ and the right and left hand derivatives for the n^{th} order exist. Then, by using the local Taylor formula with the Peano remainder term we show that the Jacobi estimators can be considered as generalized derivative estimators for $x^{(n)}$ which converge to these one-sided derivatives.

In Section 1.5, by applying the orthogonality of the Jacobi polynomials we introduce two new types of estimators which are based on the minimal Jacobi estimators so as to improve the convergence rate for the minimal Jacobi estimators. The first type estimators are obtained by applying the Richardson extrapolation technique to an affine scheme of minimal estimators, which are the extension of (1.13) in the causal and anti-causal cases. We call them *Richardson estimators*. The associated convergence rate is $\mathcal{O}(T^2)$ as $T\to 0$. Moreover, the Richardson estimators can be considered as generalized derivative estimators for $x^{(n)}$. The second type estimators are based on a modified Taylor expansion introduced in [Poffald 1990], which improve the convergence rate to $\mathcal{O}(T^3)$.

In the three previous sections, all the estimators are studied in the causal case with the time window $[t_0-T,t_0]$ for all $t_0\in I$ or in the anti-causal case with the time window $[t_0,t_0+T]$. These estimators produce a time-drift in order to get a small bias term error. In Section 1.6, we introduce some drift-free estimators by using the integration window $[t_0-T,t_0+T]$. We call them *central estimators*. In Subsection 1.6.1, we consider the functions $X^{\pm}(t+t_0)=\frac{1}{2}(x(t+t_0)\pm x(t_0-t))$. Firstly, by applying the algebraic parametric technique to the N^{th} order truncated Taylor series expansions of $X^{\pm}$ we give a family of estimators which are based on a combination of causal and anti-causal Jacobi estimators. Secondly, we extend these central estimators by taking the q^{th} $(N-n)$ order truncated Jacobi series expansion of $(X^{\pm})^{(n)}$. Thirdly, since X_N^- (resp. X^+) only contains the values of the odd (reps. even) order derivatives, we show that the convergence rate for these central estimators is $\mathcal{O}(T^{q+2})$ as $T\to 0$. In Subsection 1.6.2, we give a family of central estimators by taking the truncated Jacobi orthogonal series expansion of $x^{(n)}(t_0-T+2T\xi)$ with $\xi=\frac{1}{2}$. Since they are modified Jacobi estimators of $x^{(n)}(t_0)$, the convergence rate is $\mathcal{O}(T^{q+1})$ as $T\to 0$. Moreover, if we take $\kappa=\mu$, then $\frac{1}{2}$ is a common root of all the odd order Jacobi polynomials. Hence, the convergence rate can be improved to $\mathcal{O}(T^{q+2})$ as $T \to 0$. Finally, we show that these central estimators can also be given by taking the truncated Jacobi orthogonal series expansion defined on $[-1, 1]$. Thus, we call them *central Jacobi estimators*. In Subsection 1.6.3, we give a family of central Richardson estimators by using central minimal Jacobi estimators. Then, a family of improved central Richardson estimators is given in the case where $\kappa = \mu$, such that the convergence rate can be improved to $\mathcal{O}(T^{q+4})$ as $T \to 0$. These improved central Richardson estimators are exactly the extension of the ones given in (1.13). In Subsection 1.6.4, we show that the introduced central estimators can be considered as generalized derivative estimators for $x^{(n)}$ which converge to the average value of the one-sided derivatives of $x \in \mathcal{C}^{n-1}$.

In Section 1.7, by applying the algebraic parametric technique we give a general form for the derivative estimators which are affine combination of estimators with different integration window lengths. For this, we give a general differential operator parameterized by a set of parameters. Sufficient and necessary conditions on this set are given to obtain such an integral annihilator and it is shown that such set of parameters is always exists.

In Section 1.8, we talk about some new non-asymptotic estimators for the derivative with fractional order. Firstly, we apply the algebraic parametric technique to a truncated fractional order Taylor series. Secondly, we use the previous Jacobi estimators in the two considered definitions of fractional order derivative where we need to calculate the integer order derivative. At the end, we give a table which gives the trends of the convergence rate for each cases.

1.2 Minimal estimators

Let $x \in \mathcal{C}^n(I)$ with $n \in \mathbb{N}$ be a smooth function defined in an open interval $I \subset \mathbb{R}$. We investigate in this section some detailed properties and performances of a class of point-wise derivative estimators for $x^{(n)}$. In the following subsection, we show that these estimators are derived from recent algebraic parametric techniques applied to the truncated Taylor series expansion of x.

1.2.1 Algebraic parametric derivative estimations

For any $t_0 \in I$, we introduce the set $D_{t_0} := \{t \in \mathbb{R}_+; t_0 + \beta t \in I\}$ where $\beta > 0$. Let us take the Taylor series expansion of x at t_0. Then by using Taylor's formula formulated by Hardy ([Hardy 1952] p. 293), we obtain that

$$\forall t \in D_{t_0},\ x(t_0 + \beta t) = \sum_{j=0}^{n} \frac{(\beta t)^j}{j!} x^{(j)}(t_0) + \mathcal{O}(t^n), \quad \text{as } t \to 0. \tag{1.15}$$

Then, we consider the following truncated Taylor series expansion of x on $\mathbb{R}_+$

$$\forall t \in \mathbb{R}_+,\ x_n(t_0 + \beta t) := \sum_{j=0}^{n} \frac{(\beta t)^j}{j!} x^{(j)}(t_0). \tag{1.16}$$

Since x_n is a polynomial defined on $\mathbb{R}_+$ of degree n, we take the Laplace transform of x_n. Then, by applying (7.11) (see Appendix) we get

$$\hat{x}_n(s) = \sum_{j=0}^{n} \beta^j s^{-(j+1)} x^{(j)}(t_0), \tag{1.17}$$

where $\hat{x}_n(s)$ is the Laplace transform of $x_n(t_0 + \beta t)$ and s is the Laplace variable. We consider $\hat{x}_n$ as the n^{th} order truncated Taylor series expansion of x in the operational domain.

In all the sequel, the Laplace transform of a signal $u(\cdot)$ will be denoted by $\hat{u}(s)$. To simplify the notation, the argument s will be dropped and we write it as $\hat{u}$ for short.

From now on, we give the estimates of the n^{th} order derivative of x at point t_0. The basic step is to calculate the coefficient $x^{(n)}(t_0)$ from $\hat{x}_n$. Hence, all the terms $\beta^j s^{-(j+1)} x^{(j)}(t_0)$ in (1.17) with $j \neq n$, are consequently considered as undesired terms which we proceed to annihilate. For this, it suffices to find a linear differential operator of the form

$$\Pi = \sum_{\text{finite}} \left(\prod_{\text{finite}} \varrho_l(s) \frac{d^l}{ds^l} \right), \quad \varrho_l(s) \in \mathbb{C}(s), \tag{1.18}$$

such that

$$\Pi \left(\hat{x}_n \right) = \varrho(s)\, x^{(n)}(t_0), \tag{1.19}$$

for some rational function $\varrho(s) \in \mathbb{C}(s)$. Such a linear differential operator is subsequently called an *annihilator* for $x^{(n)}(t_0)$, originally defined in [Mboup 2009b]. When the summation in (1.18) is reduced to a single term, we give the following annihilator which was introduced in [Mboup 2009b] with $\mu \in \mathbb{N}$

$$\Pi^n_{k,\mu} = \frac{1}{s^{n+1+\mu}} \cdot \frac{d^{n+k}}{ds^{n+k}} \cdot s^n, \quad \text{where } -1 < \mu,\ k \in \mathbb{N}. \tag{1.20}$$

Then, we obtain the following proposition for $k \in \mathbb{N}$ and $-1 < \mu$. This result was given in [Mboup 2009b] with $\mu \in \mathbb{N}$ and $k \in \mathbb{N}$.

Proposition 1.2.1 *Let $x \in \mathcal{C}^n(I)$, then a family of estimators for $x^{(n)}(t_0)$ at any point $t_0 \in I$ is given by*

$$\tilde{x}^{(n)}_{t_{0\pm}}(k,\mu,\beta T) = \frac{(-1)^n}{(\beta T)^n} \Gamma(\mu + k + 2n + 2) \int_0^1 \sum_{i=0}^{n} d_{k,\mu,n,i}\, w_{\mu+n-i,i+k}(\tau)\, x(\beta T \tau + t_0) d\tau, \tag{1.21}$$

where $w_{\mu+n-i,i+k}$ are defined by (7.19) and

$$d_{k,\mu,n,i} = \frac{(-1)^i}{\Gamma(\mu + n + 1 - i)(i+k)!} \binom{n}{i}. \tag{1.22}$$

The anti-causal estimator $\tilde{x}^{(n)}_{t_0}(k,\mu,\beta T)$ ($\beta > 0$) (resp. causal estimator $\tilde{x}^{(n)}_{t_0}(k,\mu,\beta T)$ ($\beta < 0$)) is obtained by using the integral window $[t_0, t_0 + \beta T] \subset I$ (resp. $[t_0 + \beta T, t_0] \subset I$) with $k \in \mathbb{N}$, $-1 < \mu \in \mathbb{R}$, $T \in D_{t_0}$.

Proof. We proceed to annihilate the terms including $x^{(i)}(t_0)$, $i \neq n$ in the right hand side of equation (1.17) by applying the annihilator $\Pi^n_{k,\mu}$, then we obtain

$$\Pi^n_{k,\mu}\left(\hat{x}_n\right) = \frac{\beta^n(-1)^{n+k}(n+k)!}{s^{2+2n+k+\mu}}\; x^{(n)}(t_0). \tag{1.23}$$

By applying the inverse of the Laplace transform to (1.23) and by using (7.11) given in Appendix, we obtain

$$(-1)^{n+k}\beta^n(n+k)!\frac{T^{1+2n+k+\mu}}{\Gamma(2+2n+k+\mu)}x^{(n)}(t_0) = \mathcal{L}^{-1}\left\{\Pi^n_{k,\mu}\left(\hat{x}_n\right)\right\}(T),$$

where $T \in D_{t_0}$. Since $n+1+\mu > 0$, by applying (7.13) and (7.8) (given in Appendix) we obtain

$$\begin{aligned}\mathcal{L}^{-1}\left\{\Pi^n_{k,\mu}\left(\hat{x}_n\right)\right\}(T) &= \frac{(-1)^{n+k}}{\Gamma(n+1+\mu)}\int_0^T (T-\tau)^{n+\mu}\tau^{n+k}\mathcal{L}^{-1}\left\{s^n\hat{x}_n\right\}(\tau)d\tau \\ &= \frac{(-1)^{n+k}}{\Gamma(n+1+\mu)}\int_0^T (T-\tau)^{n+\mu}\tau^{n+k}\beta^n x_n^{(n)}(\beta\tau+t_0)\,d\tau.\end{aligned} \tag{1.24}$$

Thus, we have

$$x^{(n)}(t_0) = \frac{1}{T^{1+2n+k+\mu}}\frac{\Gamma(2+2n+k+\mu)}{(n+k)!\Gamma(n+1+\mu)}\int_0^T (T-\tau)^{n+\mu}\tau^{n+k}x_n^{(n)}(\beta\tau+t_0)\,d\tau. \tag{1.25}$$

By applying a change of variable $\tau \to T\tau$ and n times integrations by parts, we get

$$x^{(n)}(t_0) = \frac{(-1)^n}{(\beta T)^n}\frac{\Gamma(2+2n+k+\mu)}{(n+k)!\Gamma(n+1+\mu)}\int_0^1 \frac{d^n}{d\tau^n}\left\{(1-\tau)^{n+\mu}\tau^{n+k}\right\}x_n(\beta T\tau+t_0)\,d\tau. \tag{1.26}$$

By substituting x_n in (1.26) by x, we obtain two families of estimators for $x^{(n)}(t_0)$

$$\tilde{x}^{(n)}_{t_{0\pm}}(k,\mu,\beta T) = \frac{(-1)^n}{(\beta T)^n}\frac{\Gamma(\mu+k+2n+2)}{(n+k)!\Gamma(\mu+n+1)}\int_0^1 \frac{d^n}{d\tau^n}\left\{(1-\tau)^{\mu+n}\tau^{k+n}\right\}\, x(\beta T\tau+t_0)d\tau. \tag{1.27}$$

Finally, this proof can be completed applying the Leibniz formula. □

In the above proof, we apply the annihilator $\Pi^n_{k,\mu}$ to (1.17). On one hand, being multiplied by s^n the terms in the right side of (1.17), which include $x^{(i)}(t_0)$, with $i \neq n$, are annihilated by applying $n+k$ times derivations. On the other hand, the multiplication by $\frac{1}{s^{n+1+\mu}}$ allows us to return into the time domain and we obtain a Riemann-Liouville fractional integral (see (7.10) in Appendix) in the left side. Consequently, we obtain two families (causal and anti-causal) of estimators for $x^{(n)}(t_0)$, which are based on integrals of x. In this case, $\Pi^n_{k,\mu}$ is called an *integral annihilator*. Moreover, $\Pi^n_{k,\mu}$ seems to be the simplest integral annihilator which can be obtained from the n^{th} order truncated Taylor series expansion of x. Hence, we give the following definition.

Definition 1 *For any $t_0 \in I$, the estimators defined by (1.21) obtained from the n^{th} order truncated Taylor series expansion of function $x \in \mathcal{C}^n(I)$ are called minimal estimators.*

From now on, we assume that $\beta = \pm 1$. If $\beta = 1$, then we denote the anti-causal minimal estimator by $\tilde{x}^{(n)}_{t_{0+}}(k,\mu,T)$, and $\tilde{x}^{(n)}_{t_{0-}}(k,\mu,-T)$ as the causal minimal estimator else.

1.2.2 Derivative estimations by using the Jacobi polynomials

In this subsection, we show that the minimal estimators obtained by using the algebraic parametric techniques in the previous subsection can be extended by using the Jacobi orthogonal polynomials.

Lemma 1.2.2 *Let $x \in \mathcal{C}^n(I)$, then for any $t_0 \in I$ the minimal estimators for $x^{(n)}(t_0)$ can be expressed as follows*

$$\tilde{x}_{t_0\pm}^{(n)}(k,\mu,\beta T) = \frac{n!}{(\beta T)^n} \frac{\Gamma(\mu+k+2n+2)}{(n+k)!\Gamma(\mu+n+1)} \int_0^1 w_{\mu,k}(\tau) P_n^{(\mu,k)}(\tau)\, x(\beta T\tau + t_0) d\tau, \tag{1.28}$$

where $T \in D_{t_0}$, $k \in \mathbb{N}$, $-1 < \mu$, $P_n^{(\mu,k)}$ being the n^{th} order Jacobi polynomial (defined by (7.15) in Appendix) and $w_{\mu,k}$ being the associated weighted function defined by (7.19).

Proof. It is sufficient to apply the Rodrigues formula (given by (7.22) in Appendix) to (1.27). □

We can observe in Lemma 1.2.2 that the integral in (1.28) is in fact the scalar product of the Jacobi polynomial $P_n^{(\mu,k)}$ and x. Hence, by applying the following classical orthogonal properties of the Jacobi polynomials (given by (7.23) in Appendix) to the n^{th} order truncated Taylor series expansion $x_n(t_0+\beta T\tau) = \sum_{j=0}^{n} \frac{(\beta T\tau)^j}{j!} x^{(j)}(t_0)$ with $\tau \in [0,1]$, we obtain an exact expression of $x^{(n)}(t_0)$. Then, by substituting x_n by x we get the minimal estimators given in (1.28). Thus, the minimal estimators can also be obtained by using the Jacobi orthogonal polynomials $P_n^{(\mu,k)}$. Since $P_n^{(\mu,k)}$ is defined with $\mu, k \in]-1,+\infty[$, we can extend in Lemma 1.2.2 the value of k for the minimal estimators $\tilde{x}_{t_0\pm}^{(n)}(k,\mu,\beta T)$ to $]-1,+\infty[$. In this case, we denote these extended minimal estimators by $D_{\kappa,\mu,\beta T}^{(n)} x(t_0)$. We obtain

$$D_{\kappa,\mu,\beta T}^{(n)} x(t_0) = \gamma_{\mu,\kappa,\beta T,n} \int_0^1 w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau)\, x(\beta T\tau + t_0) d\tau, \tag{1.29}$$

where

$$\gamma_{\mu,\kappa,\beta T,n} = \frac{n!}{(\beta T)^n} \frac{1}{\mathrm{B}(n+\kappa+1,\mu+n+1)} \tag{1.30}$$

with $T \in D_{t_0}$, $\mu, k \in]-1,+\infty[$ and $\mathrm{B}(\cdot,\cdot)$ is the classical Beta function (see [Abramowitz 1965] p. 258).

By a direct adaptation from [Mboup 2009b], we obtain the following proposition by using some recurrence relations of the Jacobi orthogonal polynomials.

Proposition 1.2.3 *[Liu 2011c] Let $D_{\kappa,\mu,\beta T}^{(n)} x(t_0)$ be the extended minimal estimators defined by (1.29) with $n \geq 1$, then we have*

$$D_{\kappa,\mu,\beta T}^{(n)} x(t_0) = \frac{2n+\mu+\kappa+1}{\beta T} \left(D_{\kappa+1,\mu,\beta T}^{(n-1)} x(t_0) - D_{\kappa,\mu+1,\beta T}^{(n-1)} x(t_0) \right). \tag{1.31}$$

Proof. Let us recall the recurrence relations given by (7.27) and (7.28) in Appendix. Then, by subtracting (7.27) from (7.28), we get

$$P_n^{(\mu,\kappa)}(\tau) = \frac{\mu-\kappa}{2n} P_{n-1}^{(\mu,\kappa)}(\tau) + \frac{2n+\kappa+\mu}{2n} \left[\tau P_{n-1}^{(\mu,\kappa+1)}(\tau) - (1-\tau) P_{n-1}^{(\mu+1,\kappa)}(\tau) \right]. \tag{1.32}$$

By using (1.32), (1.29) becomes

$$\begin{aligned} D^{(n)}_{\kappa,\mu,\beta T} x(t_0) = & \frac{\mu-\kappa}{2n}\,\gamma_{\mu,\kappa,\beta T,n} \int_0^1 w_{\mu,\kappa}(\tau) P^{(\mu,\kappa)}_{n-1}(\tau)\, x(\beta T\tau + t_0) d\tau \\ & + \frac{2n+\kappa+\mu}{2n}\,\gamma_{\mu,\kappa,\beta T,n} \int_0^1 w_{\mu,\kappa+1}(\tau) P^{(\mu,\kappa+1)}_{n-1}(\tau)\, x(\beta T\tau + t_0) d\tau \\ & - \frac{2n+\kappa+\mu}{2n}\,\gamma_{\mu,\kappa,\beta T,n} \int_0^1 w_{\mu+1,\kappa}(\tau) P^{(\mu+1,\kappa)}_{n-1}(\tau)\, x(\beta T\tau + t_0) d\tau. \end{aligned}$$

Observe that

$$\begin{aligned} \gamma_{\mu,\kappa,\beta T,n} &= \frac{n}{\beta T} \frac{(\mu+\kappa+2n+1)(\mu+\kappa+2n)}{(n+\kappa)\,(\mu+n)}\,\gamma_{\mu,\kappa,\beta T,n-1}, \\ \gamma_{\mu,\kappa,\beta T,n} &= \frac{n}{\beta T}\frac{\mu+\kappa+2n+1}{n+\mu}\,\gamma_{\mu,\kappa+1,\beta T,n-1}, \quad \gamma_{\mu,\kappa,\beta T,n} = \frac{n}{\beta T}\frac{\mu+\kappa+2n+1}{n+\kappa}\,\gamma_{\mu+1,\kappa,\beta T,n-1}. \end{aligned}$$

By using (1.29) with $n-1$ in place of n, we get

$$D^{(n)}_{\kappa,\mu,\beta T} x(t_0) = -(A+B)\, D^{(n-1)}_{\kappa,\mu,\beta T} x(t_0) + A\, D^{(n-1)}_{\kappa+1,\mu,\beta T} x(t_0) + B\, D^{(n-1)}_{\kappa,\mu+1,\beta T} x(t_0), \tag{1.33}$$

where $A = \frac{\alpha_{\mu,\kappa,\beta T,n}}{n+\mu}$ and $B = -\frac{\alpha_{\mu,\kappa,\beta T,n}}{n+\kappa}$ with $\alpha_{\mu,\kappa,\beta T,n} = \frac{(\mu+\kappa+2n+1)(\mu+\kappa+2n)}{2\beta T}$.
By applying the Rodrigues formula to (1.29) and applying n times integrations by parts, we obtain

$$\begin{aligned} D^{(n)}_{\kappa,\mu,\beta T} x(t_0) &= \frac{(-1)^n}{(\beta T)^n} \frac{1}{\mathrm{B}(n+\kappa+1,\mu+n+1)} \int_0^1 \frac{d^n}{d\tau^n}\left\{w_{\mu+n,\kappa+n}(\tau)\right\}\, x(\beta T\tau + t_0)\, d\tau \\ &= \frac{1}{\mathrm{B}(n+\kappa+1,\mu+n+1)} \int_0^1 w_{\mu+n,\kappa+n}(\tau)\, x^{(n)}(\beta T\tau + t_0)\, d\tau. \end{aligned} \tag{1.34}$$

Hence, $D^{(n-1)}_{\kappa,\mu,\beta T} x(t_0)$ can be written as

$$\begin{aligned} D^{(n-1)}_{\kappa,\mu,\beta T} x(t_0) &= \frac{1}{\mathrm{B}(n+\kappa,\mu+n)} \int_0^1 w_{\mu+n-1,\kappa+n-1}(\tau)(1-\tau+\tau)\, x^{(n-1)}(\beta T\tau + t_0)\, d\tau \\ &= \frac{1}{\mathrm{B}(n+\kappa,\mu+n)} \int_0^1 w_{\mu+n,\kappa+n-1}(\tau)\, x^{(n-1)}(\beta T\tau + t_0)\, d\tau \\ &+ \frac{1}{\mathrm{B}(n+\kappa,\mu+n)} \int_0^1 w_{\mu+n-1,\kappa+n}(\tau)\, x^{(n-1)}(\beta T\tau + t_0)\, d\tau. \end{aligned}$$

Thus, we have

$$D^{(n-1)}_{\kappa,\mu,\beta T} x(t_0) = \frac{n+\mu}{2n+\kappa+\mu} D^{(n-1)}_{\kappa,\mu+1,\beta T} x(t_0) + \frac{n+\kappa}{2n+\kappa+\mu} D^{(n-1)}_{\kappa+1,\mu,\beta T} x(t_0). \tag{1.35}$$

Then, this proof can be completed by using the two following equalities.

$$\begin{aligned} A - (A+B)\frac{n+\kappa}{2n+\kappa+\mu} &= \frac{2n+\mu+\kappa+1}{\beta T}, \\ B - (A+B)\frac{n+\mu}{2n+\kappa+\mu} &= -\frac{2n+\mu+\kappa+1}{\beta T}. \end{aligned}$$

□

1.2.3 Derivative estimations by using Jacobi orthogonal series

It is shown, in the previous subsection, that the minimal estimators obtained by the algebraic parametric techniques can be extended by using the Jacobi polynomials. Now, let us take the Jacobi orthogonal series expansion of $x^{(n)}(\beta T \cdot +t_0)$ with $\beta = \pm 1$ and $T \in D_{t_0}$

$$\forall \xi \in [0,1],\ x^{(n)}(t_0 + \beta T \xi) = \sum_{i \geq 0} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), x^{(n)}(t_0 + \beta T \cdot) \right\rangle_{\mu+n,\kappa+n}}{\| P_i^{(\mu+n,\kappa+n)} \|^2_{\mu+n,\kappa+n}} P_i^{(\mu+n,\kappa+n)}(\xi), \tag{1.36}$$

where the scalar product $\langle \cdot, \cdot \rangle_{\mu+n,\kappa+n}$ is defined in (7.18) in Appendix with $\kappa, \mu \in]-1, +\infty[$.

In the following lemma, we can see that the extended minimal estimators can be also obtained by taking the first term in (1.36).

Lemma 1.2.4 *[Mboup 2009b] Let $x \in \mathcal{C}^n(I)$, then for any $t_0 \in I$ the extended minimal estimators $D^{(n)}_{\kappa,\mu,\beta T} x(t_{0\pm})$ given by (1.29) can be also written as follows*

$$D^{(n)}_{\kappa,\mu,\beta T} x(t_0) = \frac{\left\langle P_0^{(\mu+n,\kappa+n)}(\cdot), x^{(n)}(t_0 + \beta T \cdot) \right\rangle_{\mu+n,\kappa+n}}{\| P_0^{(\mu+n,\kappa+n)} \|^2_{\mu+n,\kappa+n}} P_0^{(\mu+n,\kappa+n)}(\xi), \tag{1.37}$$

with $\xi \in [0,1]$. Moreover, we have

$$\forall t_0 \in I,\ D^{(n)}_{\kappa,\mu,\beta T} x(t_0) = D^{(0)}_{\kappa+n,\mu+n,\beta T} x^{(n)}(t_0). \tag{1.38}$$

Proof. By using (1.29), (1.34) and the following formulae given in Appendix

$$P_0^{(\mu+n,\kappa+n)}(\tau) \equiv 1 \text{ and } \| P_0^{(\mu+n,\kappa+n)} \|^2_{\mu+n,\kappa+n} = \mathrm{B}(n+\mu+1, n+\kappa+1),$$

we can achieve this proof. □

This lemma shows that the extended minimal estimators $D^{(n)}_{\kappa,\mu,\beta T} x(t_0)$ can be obtained by taking the truncated Jacobi orthogonal series expansion of $x^{(n)}$. If we take the n^{th} order truncated Taylor series expansion x_n (defined by (1.16)), then by taking the Jacobi orthogonal series expansion of $x_n^{(n)}(\beta T \tau + t_0)$ with $\tau \in [0,1]$ at point $\tau = 0$, we obtain

$$x_n^{(n)}(t_0) = \frac{\left\langle P_0^{(\mu+n,\kappa+n)}(\cdot), x_n^{(n)}(t_0 + \beta T \cdot) \right\rangle_{\mu+n,\kappa+n}}{\| P_0^{(\mu+n,\kappa+n)} \|^2_{\mu+n,\kappa+n}} P_0^{(\mu+n,\kappa+n)}(0). \tag{1.39}$$

Since $x^{(n)}(t_0) = x_n^{(n)}(t_0)$ in (1.16) at $t = 0$, $D^{(n)}_{\kappa,\mu,\beta T} x(t_0)$ can also be obtained by substituting $x_n^{(n)}$ in (1.39) by $x^{(n)}$.

By using the recurrence relations of the Jacobi polynomials, it is shown in Proposition 1.2.3 the relation existing between extended minimal estimators of $x^{(n)}(t_0)$ and the ones of $x^{(n-1)}(t_0)$ with $n \geq 1$. In the following proposition, we give the relation existing between n^{th} order ($n \in \mathbb{N}$) extended minimal estimators and the ones of 0^{th} order.

Proposition 1.2.5 *Let $x \in \mathcal{C}^n(I)$, then for any $t_0 \in I$ we have*

$$D^{(n)}_{\kappa,\mu,\beta T} x(t_0) = \frac{1}{(\beta T)^n} \frac{\Gamma(\mu+\kappa+2n+2)}{\Gamma(\mu+\kappa+n+2)} \sum_{j=0}^{n} (-1)^{n+j} \binom{n}{j} \, D^{(0)}_{\kappa_j,\mu_{nj},\beta T} x(t_0), \tag{1.40}$$

where $\mu_{nj} = \mu + n - j$ and $\kappa_j = \kappa + j$.

In order to prove this proposition, let us give the following lemma.

Lemma 1.2.6 *For any $i \in \mathbb{N}$, we have*

$$\forall t_0 \in I, \ \frac{\left\langle P_i^{(\mu,\kappa)}(\tau), x(\beta T \tau + t_0) \right\rangle_{\mu,\kappa}}{\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}} = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \frac{2i+\mu+\kappa+1}{i+\mu+\kappa+1} \, D^{(0)}_{\kappa_j,\mu_{ij},\beta T} x(t_0), \tag{1.41}$$

where $\mu_{ij} = \mu + i - j$ and $\kappa_j = \kappa + j$.

Proof. Observing from the expression of the Jacobi polynomials (defined by (7.15) in Appendix) that

$$P_i^{(\mu,\kappa)}(\tau) \, w_{\mu,\kappa}(\tau) = \sum_{j=0}^{i} \binom{i+\mu}{j} \binom{i+\kappa}{i-j} (-1)^{i-j} w_{\mu_{ij},\kappa_j}(\tau), \tag{1.42}$$

we get

$$\left\langle P_i^{(\mu,\kappa)}(\tau), x(\beta T \tau + t_0) \right\rangle_{\mu,\kappa} = \sum_{j=0}^{i} \binom{i+\mu}{j} \binom{i+\kappa}{i-j} (-1)^{i-j} \int_0^1 w_{\mu_{ij},\kappa_j}(\tau) \, x(\beta T \tau + t_0) \, d\tau. \tag{1.43}$$

Then, by using (1.37) with $n = 0$, we obtain

$$\frac{\left\langle P_i^{(\mu,\kappa)}(\tau), x(\beta T \tau + t_0) \right\rangle_{\mu,\kappa}}{\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}} = \sum_{j=0}^{i} \binom{i+\mu}{j} \binom{i+\kappa}{i-j} (-1)^{i-j} \frac{\mathrm{B}(\kappa_j+1, \mu_{ij}+1)}{\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}} D^{(0)}_{\kappa_j,\mu_{ij},\beta T} x(t_0). \tag{1.44}$$

Consequently, this proof can be given by using the expression of $\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}$ (given in (7.20) in Appendix) in (1.44). □

Proof of Proposition 1.2.5. From (1.29), it is easy to show after some calculations that

$$D^{(n)}_{\kappa,\mu,\beta T} x(t_0) = \frac{1}{(\beta T)^n} \frac{\Gamma(\mu+\kappa+2n+1)}{\Gamma(\mu+\kappa+n+1)} \frac{\left\langle P_n^{(\mu,\kappa)}(\tau), x(\beta T \tau + t_0) \right\rangle_{\mu,\kappa}}{\|P_n^{(\mu,\kappa)}\|^2_{\mu,\kappa}}. \tag{1.45}$$

Hence, (1.40) can be given by using Lemma 1.2.6 with $i = n$ in (1.45). □

1.2.4 Analysis of the truncated term error

In the previous subsections, we study the minimal estimators, which are obtained by three methods. However, in each method, the main idea is to use the n^{th} order truncated Taylor series expansion of x. Hence, the errors for these estimators come from the remainder term in the Taylor series expansion of x. By writing $x(t) = x_n(t) + R_n(t)$ for $t \in I$, we obtain from (1.29)

$$\begin{aligned} D^{(n)}_{\kappa,\mu,\beta T} x(t_0) &= \gamma_{\mu,\kappa,\beta T,n} \int_0^1 w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau) \left\{ x_n(\beta T\tau + t_0) + R_n(\beta T\tau + t_0) \right\} d\tau \\ &= x^{(n)}(t_0) + e^{\beta}_{R_n}(t_0;\kappa,\mu,T), \end{aligned} \tag{1.46}$$

where $e^{\beta}_{R_n}(t_0;\kappa,\mu,T)$ are the corresponding truncated term errors. It is clear that for any function $x \in \mathcal{P}_n(I)$ where $\mathcal{P}_n(I)$ is the space of all the polynomials defined on I of degree below or equal to n, we have $\forall t_0 \in I$, $D^{(n)}_{\kappa,\mu,\beta T} x(t_0) = x^{(n)}(t_0)$ with $e^{\beta}_{R_n}(t_0;\kappa,\mu,T) = 0$. For any other function x in $\mathcal{C}^{n+1}(I)$, we study the errors $e^{\beta}_{R_n}(t_0;\kappa,\mu,T)$ for $D^{(n)}_{\kappa,\mu,\beta T} x(t_0)$ in the following proposition.

Proposition 1.2.7 *Let $x \in \mathcal{C}^{n+1}(I)$ and $D^{(n)}_{\kappa,\mu,\beta T} x(t_0)$ be the minimal estimators defined by* (1.29) *for $x^{(n)}(t_0)$, then by assuming that there exists $M_{n+1} \in \mathbb{R}_+^*$ such that $\left\| x^{(n+1)} \right\|_\infty \le M_{n+1}$, we have*

$$\left\| D^{(n)}_{\kappa,\mu,\beta T} x(t_{0\pm}) - x^{(n)}(t_0) \right\|_\infty \le M_{n+1} C_{\kappa,\mu,n} T, \tag{1.47}$$

where $C_{\kappa,\mu,n} = \frac{1}{(n+1)\mathrm{B}(n+\kappa+1,\mu+n+1)} \int_0^1 \left| w_{\mu,\kappa+n+1}(\tau) P_n^{(\mu,\kappa)}(\tau) \right| d\tau.$

Proof. From (1.46), we have

$$\forall t_0 \in I,\ e^{\beta}_{R_n}(t_0;\kappa,\mu,T) = \gamma_{\mu,\kappa,\beta T,n} \int_0^1 w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau) R_n(\beta T\tau + t_0) d\tau.$$

Since $x \in \mathcal{C}^{n+1}(I)$, by applying Taylor's formula (see [Abramowitz 1965] p.14) we get

$$x(t_0 + \beta T\tau) = \sum_{j=0}^{n} \frac{(\beta T\tau)^j}{j!} x^{(j)}(t_0) + \frac{(\beta T\tau)^{n+1}}{(n+1)!} x^{(n+1)}(\theta^{\beta}_{n,t_0}), \tag{1.48}$$

where $\theta^{\beta}_{n,t_0} \in]t_0, t_0 + \beta T\tau[$ if $\beta > 0$ (resp. $\theta^{\beta}_{n,t_0} \in]t_0 + \beta T\tau, t_0[$ if $\beta < 0$).

Then by using (1.48), we get

$$\forall t_0 \in I,\ e^{\beta}_{R_n}(t_0;\kappa,\mu,T) = \gamma_{\mu,\kappa,\beta T,n} \int_0^1 w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau) \frac{(\beta T\tau)^{n+1}}{(n+1)!} x^{(n+1)}(\theta^{\beta}_{n,t_0}) d\tau.$$

Finally, this proof can be easily completed by taking the norm $\|\cdot\|_\infty$ with respect to t_0. □

In the above proposition, we have given a global bound for the truncated term errors $e^{\beta}_{R_n}(t_0;\kappa,\mu,T)$ for the minimal estimators. In the next proposition, we give local lower and upper error bounds for $e^{\beta}_{R_n}(t_0;\kappa,\mu,T)$.

Proposition 1.2.8 *[Liu 2011c] Let $x \in \mathcal{C}^{n+1}(I)$ and $D^{(n)}_{\kappa,\mu,\beta T}x(t_{0\pm})$ be the minimal estimators defined by (1.29) for $x^{(n)}(t_0)$, then the corresponding truncated term errors $e^{\beta}_{R_n}(t_0;\kappa,\mu,T)$ can be bounded by*

$$\begin{aligned} C^{\mu,\kappa}_{T} m^{+}_{l} &\leq e^{+}_{R_n}(t_0;\kappa,\mu,T) \leq C^{\mu,\kappa}_{T} m^{+}_{u}, \\ C^{\mu,\kappa}_{-T} m^{-}_{l} &\leq e^{-}_{R_n}(t_0;\kappa,\mu,T) \leq C^{\mu,\kappa}_{-T} m^{-}_{u}, \end{aligned} \tag{1.49}$$

where $C^{\mu,\kappa}_{\beta T} = \beta T \frac{\kappa+n+1}{\mu+\kappa+2n+2}$ and

$$\begin{aligned} m^{+}_{l} &= \inf_{t_0<\hat{\theta}^{+}_{n,t_0}<t_0+T} x^{(n+1)}(\hat{\theta}^{+}_{n,t_0}),\ m^{+}_{u} = \sup_{t_0<\hat{\theta}^{+}_{n,t_0}<t_0+T} x^{(n+1)}(\hat{\theta}^{+}_{n,t_0}), \\ m^{-}_{l} &= \sup_{t_0-T<\hat{\theta}^{-}_{n,t_0}<t_0} x^{(n+1)}(\hat{\theta}^{-}_{n,t_0}),\ m^{-}_{u} = \inf_{t_0-T<\hat{\theta}^{-}_{n,t_0}<t_0} x^{(n+1)}(\hat{\theta}^{-}_{n,t_0}). \end{aligned} \tag{1.50}$$

Proof. By using (1.37) and (1.39), we obtain that

$$e^{\beta}_{R_n}(t_0;\kappa,\mu,T) = \frac{1}{\mathrm{B}(n+\kappa+1,\mu+n+1)} \int_0^1 w_{\mu+n,\kappa+n}(\tau) \left(x^{(n)}(\beta T\tau+t_0) - x^{(n)}_n(\beta T\tau+t_0)\right) d\tau.$$

As $x \in \mathcal{C}^{n+1}(I)$, $x^{(n)}(\beta T\tau+t_0) - x^{(n)}_n(\beta T\tau+t_0)$ represents the remainder terms of the Taylor series expansion of $x^{(n)}$, we obtain by applying Taylor's formula (see [Abramowitz 1965] p.14) that

$$x^{(n)}(T\tau+t_0) - x^{(n)}_n(\beta T\tau+t_0) = x^{(n+1)}(\hat{\theta}^{+}_{n,t_0})T\tau, \text{ with } t_0 < \hat{\theta}^{+}_{n,t_0} < t_0+T\tau, \tag{1.51}$$

$$x^{(n)}(-T\tau+t_0) - x^{(n)}_n(\beta T\tau+t_0) = -x^{(n+1)}(\hat{\theta}^{-}_{n,t_0})T\tau, \text{ with } t_0-T\tau < \hat{\theta}^{-}_{n,t_0} < t_0. \tag{1.52}$$

Thus, the truncated term errors are given by

$$e^{\beta}_{R_n}(t_0;\kappa,\mu,T) = \frac{\beta T}{\mathrm{B}(n+\kappa+1,\mu+n+1)} \int_0^1 w_{\mu+n,\kappa+n+1}(\tau) x^{(n+1)}(\hat{\theta}^{\pm}_{n,t_0}) d\tau. \tag{1.53}$$

Then, this proof can be easily completed by taking the Beta function and the extreme values of $x^{(n+1)}(\hat{\theta}^{\pm}_{n,t_0})$. □

1.3 Affine estimators

In the previous section, we study $\mathcal{O}(T)$ minimal estimators which are obtained from the n^{th} order truncated Taylor series expansion of smooth functions. In this section, we improve these estimators by taking higher order truncated Taylor series expansion.

1.3.1 Algebraic parametric derivative estimations

In (1.20), we introduce a simple integral annihilator denoted by $\Pi^{n}_{k,\mu}$. Let us give the following differential operator

$$\Pi^{N,n}_{k,\mu} = \frac{1}{s^{N+1+\mu}} \cdot \frac{d^{n+k}}{ds^{n+k}} \cdot \frac{1}{s} \cdot \frac{d^{N-n}}{ds^{N-n}} \cdot s^{N+1}, \quad \text{where } -1<\mu,\ k \in \mathbb{N}. \tag{1.54}$$

This operator was originally introduced in [Mboup 2009b] with $k, \mu \in \mathbb{N}$. Moreover, if we set $N = n$ in (1.54), then we have $\Pi^{N,n}_{k,\mu} = \Pi^{n}_{k,\mu}$. By using the algebraic parametric techniques with $\Pi^{N,n}_{k,\mu}$, we give a new family of derivative estimators in the following proposition.

Proposition 1.3.9 *[Liu 2009] Let $x \in \mathcal{C}^n(I)$, then a family of estimators for $x^{(n)}(t_0)$ at any point $t_0 \in I$ is given by*

$$\tilde{x}_{t_{0\pm}}^{(n)}(k,\mu,\beta T,N) = \frac{1}{(\beta T)^n}\int_0^1 a_{k,\mu,n,N}\sum_{i=0}^{N-n} b_{n,N,i}K_{k,\mu,n,N,i}(\tau)\,x(\beta T\tau + t_0)d\tau, \tag{1.55}$$

where

$$a_{k,\mu,n,N} = (-1)^{n+k}\frac{\Gamma(N+n+k+\mu+2)}{(n+k)!(N-n)!},\ b_{n,N,i} = \binom{N-n}{i}\frac{(N+1)!}{(n+i+1)!},$$
$$K_{k,\mu,n,N,i}(\tau) = \sum_{j=\max(0,k-i)}^{n+k}\frac{(-1)^{i+j}}{\Gamma(N+1+\mu+k-i-j)}\binom{n+k}{j}\frac{(n+i)!}{(i+j-k)!}\ w_{N+\mu+k-i-j,i+j}(\tau).$$

The anti-causal estimator $\tilde{x}_{t_0}^{(n)}(k,\mu,\beta T,N)$ $(\beta = 1)$ (resp. causal estimator $\tilde{x}_{t_0}^{(n)}(k,\mu,\beta T,N)$ $(\beta=-1)$) is obtained by using the integral window $[t_0, t_0+T] \subset I$ (resp. $[t_0-T, t_0] \subset I$), with $k \in \mathbb{N}$, $-1<\mu$, $T \in D_{t_0}$.

Proof. We apply $\Pi_{k,\mu}^{N,n}$ to $\hat{x}_n$ which is defined by (1.17). Firstly, the terms including $x^{(i)}(t_0)$, $i \neq n$ in the right side of (1.17) are annihilated by $\Pi_{k,\mu}^{N,n}$, we obtain

$$\begin{aligned}\Pi_{k,\mu}^{N,n}(\hat{x}_n) &= \frac{1}{s^\nu}\frac{d^{n+k}}{ds^{n+k}}\sum_{i=0}^{n}\beta^i\frac{(N-i)!}{(n-i)!}s^{n-i-1}x^{(i)}(t_0)\\ &= \frac{\beta^n(N-n)!(-1)^{n+k}(n+k)!}{s^{1+n+k+\nu}}\ x^{(n)}(t_0),\end{aligned} \tag{1.56}$$

where $\nu = N+1+\mu$. Secondly, in the left side of (1.17), we have

$$\begin{aligned}\Pi_{k,\mu}^{N,n}(\hat{x}_n) &= \frac{1}{s^\nu}\frac{d^{n+k}}{ds^{n+k}}\sum_{i=0}^{N-n}\frac{\binom{N-n}{i}(N+1)!}{(n+i+1)!}s^{n+i}(\hat{x}_n)^{(i)}\\ &= \sum_{i=0}^{N-n}\frac{\binom{N-n}{i}(N+1)!}{(n+i+1)!}\sum_{j=\max(0,k-i)}^{n+k}\frac{\binom{n+k}{j}(n+i)!}{(i+j-k)!}\frac{1}{s^{\nu-i-j+k}}(\hat{x}_n)^{(i+j)}.\end{aligned} \tag{1.57}$$

Hence, we get

$$\frac{x^{(n)}(t_0)}{s^{\nu+n+k+1}} = \frac{(-1)^{n+k}}{\beta^n(n+k)!(N-n)!}\sum_{i=0}^{N-n}\frac{\binom{N-n}{i}(N+1)!}{(n+i+1)!}\sum_{j=\max(0,k-i)}^{n+k}\frac{\binom{n+k}{j}(n+i)!}{(i+j-k)!}\frac{(\hat{x}_n)^{(i+j)}}{s^{\nu+k-i-j}}. \tag{1.58}$$

As $\nu+k-i-j>0$, we can express (1.58) back into the time domain by using (7.11) and (7.13) given in Appendix and by denoting by $T \in D_{t_0}$ the length of the estimation time window we obtain

$$\begin{aligned}x^{(n)}(t_0) &= \frac{(-1)^{n+k}}{\beta^n T^{\nu+n+k}}\frac{\Gamma(\nu+n+k+1)}{(n+k)!(N-n)!}\mathcal{L}^{-1}\left\{\Pi_{k,\mu}^{N,n}(\hat{x}_n)\right\}\\ &= \frac{(-1)^{n+k}}{\beta^n T^{\nu+n+k}}\frac{\Gamma(\nu+n+k+1)}{(n+k)!(N-n)!}\sum_{i=0}^{N-n}\sum_{j=\max(0,k-i)}^{n+k}\frac{\binom{N-n}{i}(N+1)!}{(n+i+1)!}A_{i,j},\end{aligned} \tag{1.59}$$

where

$$A_{i,j} = \frac{(-1)^{i+j}}{\Gamma(\nu+k-i-j)} \frac{\binom{n+k}{j}(n+i)!}{(i+j-k)!} \int_0^T (T-\tau)^{\nu+k-i-j-1} \tau^{i+j} x_n(\beta\tau + t_0) d\tau.$$

By substituting x_n in (1.59) by x, a family of estimators is obtained, which is parameterized by k, μ, T and N. Finally, we can achieve the proof by applying the following change of the variable: $\tau \to T\tau$. □

Consequently, $\Pi_{k,\mu}^{N,n}$ is also an integral annihilator. Moreover, if we take $N = n$ in Proposition 1.3.9, then the minimal estimators $\tilde{x}_{t_{0\pm}}^{(n)}(k,\mu,\beta T)$ given by Proposition 1.2.1 can also be obtained. In the following lemma similarly to [Mboup 2009b], we show the relation between $\Pi_{k,\mu}^{N,n}$ and the integral annihilator $\Pi_{k,\mu}^{n}$ defined by (1.20).

Lemma 1.3.10 *[Mboup 2009b] Let $\Pi_{k,\mu}^{N,n}$ be the integral annihilator defined by (1.54) and $\Pi_{k,\mu}^{n}$ be the integral annihilator defined by (1.20), then for any function f defined on I with Laplace transform $\hat{f}$ (existence is assumed), we have*

$$\Pi_{k,\mu}^{N,n}(\hat{f}) = \sum_{j=0}^{q} \sum_{i=m(j)}^{j} a_{i,j} \Pi_{k_j,\mu_j}^{n}(\hat{f}) \tag{1.60}$$

where $a_{i,j} = \binom{q}{i}\binom{p}{j-i}\frac{(q+1)!}{(q+1-i)(q-j)!}$, $k_j = k+q-j$, and $\mu_j = \mu + j$ with $q = N-n$ and $p = n+k$. The index function $m(j)$ is defined as follows

$$m(j) = \begin{cases} 0, & \textit{if } \ p \geq q, \\ \max(0, j-p), & \textit{else.} \end{cases} \tag{1.61}$$

Proof. Let us denote by $\nu = N+1+\mu$, $q = N-n$ and $p = n+k$, and apply $\Pi_{k,\mu}^{N,n}$ to $\hat{f}$. Then by using the Leibniz formula, we obtain

$$\begin{aligned} s^{\nu}\Pi_{k,\mu}^{N,n}(\hat{f}) &= \frac{d^p}{ds^p}\frac{1}{s}\frac{d^q}{ds^q}\left(s^{N+1-n}(s^n\hat{f})\right) \\ &= \frac{d^p}{ds^p}\left(\sum_{i=0}^{q}\binom{q}{i}\frac{(q+1)!}{(q+1-i)!}s^{q-i}(s^n\hat{f})^{(q-i)}\right) \\ &= \sum_{i=0}^{q}\binom{q}{i}\frac{(q+1)!}{(q+1-i)!}\frac{d^p}{ds^p}(s^{q-i}(s^n\hat{f})^{(q-i)}) \\ &= \sum_{i=0}^{q}\binom{q}{i}\frac{(q+1)!}{(q+1-i)!}\left(\sum_{j=0}^{\min(p,q-i)}\binom{p}{j}\frac{(q-i)!}{(q-i-j)!}s^{q-i-j}(s^n\hat{f})^{(q-i+p-j)}\right). \end{aligned}$$

By applying a change of index: $j \to j-i$, we obtain

$$\begin{aligned} \Pi_{k,\mu}^{N,n}(\hat{f}) &= \sum_{i=0}^{q}\binom{q}{i}\frac{(q+1)!}{(q+1-i)!}\left(\sum_{j=i}^{\min(p,q-i)+i}\binom{p}{j-i}\frac{(q-i)!}{(q-j)!}\frac{1}{s^{\nu+j-q}}(s^n\hat{f})^{(q+p-j)}\right) \\ &= \sum_{i=0}^{q}\sum_{j=i}^{\min(p,q-i)+i} a_{i,j}\Pi_{k_j,\mu_j}^{n}(\hat{f}), \end{aligned}$$

where $a_{i,j} = \binom{q}{i}\binom{p}{j-i}\frac{(q+1)!}{(q+1-i)(q-j)!}$, $k_j = k + q - j$, and $\mu_j = \mu + j$. Finally, by rearranging the terms in the above summation, this proof can be completed. □

By using the above lemma, we can see in the following proposition the relation between the minimal estimators $\tilde{x}^{(n)}_{t_{0\pm}}(k, \mu, \beta T)$ given in Proposition 1.2.1 and the estimators $\tilde{x}^{(n)}_{t_{0\pm}}(k, \mu, \beta T, N)$ given in Proposition 1.3.9.

Proposition 1.3.11 *[Mboup 2009b] Let $x \in \mathcal{C}^n(I)$, $\tilde{x}^{(n)}_{t_{0\pm}}(k, \mu, \beta T)$ be the minimal estimators given in Proposition 1.2.1 and $\tilde{x}^{(n)}_{t_{0\pm}}(k, \mu, \beta T, N)$ be the estimators given in Proposition 1.3.9, then we have*

$$\tilde{x}^{(n)}_{t_{0\pm}}(k, \mu, \beta T, N) = \sum_{j=0}^{q} \lambda_{j,k}\ \tilde{x}^{(n)}_{t_{0\pm}}(k_j, \mu_j, \beta T), \tag{1.62}$$

where $k_j = k + q - j$, $\mu_j = \mu + j$, and $\lambda_{j,k} = \sum_{i=m(j)}^{j} (-1)^{q-j}\binom{q+1}{i}\binom{p}{j-i}\binom{n+k+q-j}{q-j}$ with $m(\cdot)$ being the index function defined by (1.61). *Moreover, we have*

$$\sum_{j=0}^{q} \lambda_{j,k} = 1. \tag{1.63}$$

These estimators, obtained as an affine combination of minimal estimators, are called affine estimators.

Proof. By applying Lemma 1.3.10 to x, we get

$$\Pi^{N,n}_{k,\mu}(\hat{x}) = \sum_{j=0}^{q}\sum_{i=m(j)}^{j} a_{i,j}\Pi^{n}_{k_j,\mu_j}(\hat{x}). \tag{1.64}$$

Then, by applying the inverse Laplace transform to (1.64), we get

$$\mathcal{L}^{-1}\left\{\Pi^{N,n}_{k,\mu}(\hat{x})\right\}(T) = \sum_{j=0}^{q}\sum_{i=m(j)}^{j} a_{i,j}\mathcal{L}^{-1}\left\{\Pi^{n}_{k_j,\mu_j}(\hat{x})\right\}(T). \tag{1.65}$$

By substituting $\hat{x}_n$ by $\hat{x}$ in (1.59), we obtain

$$\tilde{x}^{(n)}_{t_{0\pm}}(k, \mu, \beta T, N) = \frac{(-1)^{n+k}}{\beta^n T^{\nu+n+k}}\frac{\Gamma(\nu+n+k+1)}{(n+k)!(N-n)!}\mathcal{L}^{-1}\left\{\Pi^{N,n}_{k,\mu}(\hat{x})\right\}(T). \tag{1.66}$$

Since $\tilde{x}^{(n)}_{t_{0\pm}}(k, \mu, \beta T, N) = \tilde{x}^{(n)}_{t_{0\pm}}(k, \mu, \beta T)$ when $N = n$, then by taking $N = n$ and substituting k by k_j, μ by μ_j in (1.66), we obtain

$$\tilde{x}^{(n)}_{t_{0\pm}}(k_j, \mu_j, \beta T) = \frac{(-1)^{n+k_j}}{\beta^n T^{1+2n+k_j+\mu_j}}\frac{\Gamma(2+2n+k_j+\mu_j)}{(n+k_j)!}\mathcal{L}^{-1}\left\{\Pi^{n}_{k_j,\mu_j}(\hat{x})\right\}(T).$$

As $1 + 2n + k_j + \mu_j = n + k + \nu$, then by using (1.65) we get

$$\tilde{x}^{(n)}_{t_{0\pm}}(k, \mu, \beta T, N) = \sum_{j=0}^{q}\sum_{i=m(j)}^{j} b_{i,j}\,\tilde{x}^{(n)}_{t_{0\pm}}(k_j, \mu_j, \beta T),$$

where $b_{i,j} = (-1)^{q-j}\binom{q+1}{i}\binom{p}{j-i}\binom{n+k+q-j}{q-j}$. Hence, (1.62) is obtained. By applying Lemma 1.3.10 to x_n, we obtain

$$\Pi_{k,\mu}^{N,n}(\hat{x}_n) = \sum_{j=0}^{q} \sum_{i=m(j)}^{j} a_{i,j} \Pi_{k_j,\mu_j}^{n}(\hat{x}_n). \tag{1.67}$$

By using (1.56) and (1.23) in (1.67), we obtain

$$\frac{\beta^n (N-n)!(-1)^{n+k}(n+k)!}{s^{1+n+k+\nu}}\, x^{(n)}(t_0) = \sum_{j=0}^{q} \sum_{i=m(j)}^{j} a_{i,j} \frac{\beta^n (-1)^{n+k_j}(n+k_j)!}{s^{2+2n+k_j+\mu_j}}\, x^{(n)}(t_0).$$

Then, we have

$$x^{(n)}(t_0) = \sum_{j=0}^{q} \sum_{i=m(j)}^{j} b_{i,j}\, x^{(n)}(t_0).$$

Hence, (1.63) is obtained. □

It is shown in the previous section that the convergence rate for minimal estimators is $\mathcal{O}(T)$ as $T \to 0$. In the following proposition, we show that the affine estimators have an improved convergence rate $\mathcal{O}\left(T^{N-n+1}\right)$.

Proposition 1.3.12 *If $x \in \mathcal{C}^{N+1}(I)$ with $N \geq n$ and $\forall t_0 \in I$, $\tilde{x}_{t_{0\pm}}^{(n)}(k,\mu,\beta T,N)$ are affine estimators for $x^{(n)}(t_0)$, then we have*

$$\forall t_0 \in I,\ \tilde{x}_{t_{0\pm}}^{(n)}(k,\mu,\beta T,N) = x^{(n)}(t_0) + \mathcal{O}\left(T^{N-n+1}\right). \tag{1.68}$$

Moreover, by assuming that there exists $M_{N+1} \in \mathbb{R}_+^*$ such that $\left\|x^{(N+1)}\right\|_\infty \leq M_{N+1}$, then we have

$$\left\|\tilde{x}_{t_{0\pm}}^{(n)}(k,\mu,\beta T,N) - x^{(n)}(t_0)\right\|_\infty \leq C_{\kappa,\mu,n,N} M_{N+1} T^{N-n+1}, \tag{1.69}$$

where

$$C_{\kappa,\mu,n,N} = \frac{1}{(N+1)!}\int_0^1 \left| \tau^{N+1} a_{k,\mu,n,N} \sum_{i=0}^{N-n} b_{n,N,i} K_{k,\mu,n,N,i}(\tau) \right| d\tau, \tag{1.70}$$

and $a_{k,\mu,n,N} \sum_{i=0}^{N-n} b_{n,N,i} K_{k,\mu,n,N,i}(\tau)$ is defined by (1.55).

Proof. Let us take the Taylor series expansion of x at t_0, then by using Taylor's formula (see [Abramowitz 1965] p. 14) it yields for any $t \in D_{t_0}$ that there exists $\theta_{N,t_0}^{+} \in]t_0, t_0+t[$ with $\beta = 1$ (resp. $\theta_{N,t_0}^{-} \in]t_0 - t, t_0[$ with $\beta = -1$) such that

$$x(t_0 + \beta t) = \sum_{j=0}^{N} \frac{(\beta t)^j}{j!} x^{(j)}(t_0) + \frac{(\beta t)^{N+1}}{(N+1)!} x^{(N+1)}(\theta_{N,t_0}^{\beta}). \tag{1.71}$$

Let $x_N(t_0+\beta\cdot)$ be the N^{th} order truncated Taylor series expansion of x, then by taking the Laplace transform of x_N we have

$$\hat{x}_N(s) = \sum_{j=0}^{N} \beta^j s^{-(j+1)} x^{(j)}(t_0). \tag{1.72}$$

Then, the operator $\Pi_{k,\mu}^{N,n}$ applied to (1.72) corresponds to an elimination technic. Being multiplied by s^{N+1}, $\hat{x}_N$ becomes a polynomial of degree N. Then the terms of degree lower than $N-n$, which include $x^{(i)}(t_0)$, $n < i < N$, are annihilated by applying $N-n$ times derivations with respect to s. In order to preserve the term including $x^{(n)}(t_0)$, we multiply the remaining polynomial by $\frac{1}{s}$. Then we apply more than n times derivations with respect to s such that the other terms including $x^{(i)}(t_0)$, $0 < i < n$, are annihilated. Finally, we multiply by $\frac{1}{s^{N+1+\mu}}$ to return into time domain. Consequently, similarly to (1.59), we obtain

$$x^{(n)}(t_0) = \frac{1}{(\beta T)^n} \int_0^1 a_{k,\mu,n,N} \sum_{i=0}^{N-n} b_{n,N,i} K_{k,\mu,n,N,i}(\tau)\, x_N(\beta T\tau + t_0) d\tau. \tag{1.73}$$

Thus, by using (1.55) the truncated term error due to the truncated Taylor series expansion can be given by

$$\tilde{x}_{t_0\pm}^{(n)}(k,\mu,\beta T,N) - x^{(n)}(t_0) = \frac{1}{(\beta T)^n} \int_0^1 a_{k,\mu,n,N} \sum_{i=0}^{N-n} b_{n,N,i} K_{k,\mu,n,N,i}(\tau)\, \frac{(\beta T)^{N+1}}{(N+1)!} x^{(N+1)}(\theta_{N,t_0}^{\beta}).$$

Then, this proof is completed by taking the norm $\|\cdot\|_\infty$ with respect to t_0. □

We can observe in the previous proof that if $x \in \mathcal{C}^{N+1}(I)$, then the affine estimators are obtained in fact by applying the annihilator $\Pi_{k,\mu}^{N,n}$ to the N^{th} order truncated Taylor series expansion of x. It explains why the convergence rate can be improved from $\mathcal{O}(T)$ to $\mathcal{O}(T^{N-n+1})$ as $T \to 0$, as soon as $\left\|x^{(N+1)}\right\|_\infty$ is bounded.

1.3.2 Derivative estimations by using the Jacobi orthogonal series

In the previous section, by using the Jacobi orthogonal series we extend the minimal estimators obtained by applying the algebraic parametric techniques. In this subsection, we extend the affine estimators given by Proposition 1.3.9 in a similar way.

By taking the $q+1$ ($q \in \mathbb{N}$) first terms in the Jacobi orthogonal series expansion of $x^{(n)}$ defined in (1.36) and denoting it by $D_{\kappa,\mu,\beta T,q}^{(n)} x(\beta T\xi + t_0)$, we have

$$D_{\kappa,\mu,\beta T,q}^{(n)} x(\beta T\xi + t_0) := \sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), x^{(n)}(t_0+\beta T\cdot)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|_{\mu+n,\kappa+n}^2}\, P_i^{(\mu+n,\kappa+n)}(\xi). \tag{1.74}$$

If we consider $D_{\kappa,\mu,\beta T,q}^{(n)} x(\beta T\xi + t_0)$ as the estimates of $x^{(n)}(t_0)$, then Mboup and al. [Mboup 2009b] have obtained the following Theorem.

Theorem 1.3.13 *[Mboup 2009b] Let $D^{(n)}_{k,\mu,\beta T,q}x(\beta T\xi + t_0)$ be the estimates defined by (1.74) and $\tilde{x}^{(n)}_{t_0}(k,\mu,\beta T,N)$ be the affine estimators given in Proposition 1.3.9. Assume that $k,\mu \in \mathbb{N}$ and $q \leq k+n$ with $q = N-n$, then for $\xi = 0$, we have*

$$\forall t_0 \in I,\ D^{(n)}_{k,\mu,\beta T,q}x(t_0) = \tilde{x}^{(n)}_{t_{0\pm}}(k,\mu,\beta T,N). \tag{1.75}$$

Moreover, for any $\xi \in [0,1]$, there exists a unique set of real coordinates $\lambda_l(\xi) \in \mathbb{R}$, for $l = 0,\ldots,q$, such that

$$D^{(n)}_{k,\mu,\beta T,q}x(\beta T\xi + t_0) = \sum_{l=0}^{q} \lambda_l(\xi)\ \tilde{x}^{(n)}_{t_{0\pm}}(k_l,\mu_l,\beta T), \tag{1.76}$$

where $\tilde{x}^{(n)}_{t_{0\pm}}(k_l,\mu_l,\beta T)$ are the minimal estimators defined by Proposition 1.2.1 with $k_l = k+q-l$, $\mu_l = \mu + l$, and these coordinates satisfy $\sum_{l=0}^{q} \lambda_l(\xi) = 1$.

The calculation of $\lambda_l(\xi)$ for $l = 0,\cdots,q$ is given in [Mboup 2009b] by the following formula

$$\lambda(\xi) = \Phi^{-1}B^{-1}b_q(\xi), \tag{1.77}$$

where

$$\lambda(\xi) = \begin{bmatrix} \lambda_0(\xi) \\ \vdots \\ \lambda_q(\xi) \end{bmatrix},\ b_q(\xi) = \begin{bmatrix} b_{0,q}(\xi) \\ \vdots \\ b_{q,q}(\xi) \end{bmatrix},\ \Phi = \begin{bmatrix} \Phi_0 & & 0 \\ & \ddots & \\ 0 & & \Phi_q \end{bmatrix}, \tag{1.78}$$

with $B_{i,j} = \binom{q}{i} \left\| P_0^{k+2q-(i+j),\mu+(i+j)} \right\|^2 \binom{q}{j}$, for $0 \leq i,j \leq q$, $b_{l,q}(\xi) = \binom{q}{l}\xi^{q-l}(1-\xi)^l$, $\Phi_l = \frac{\gamma_{k_l,\mu_l,n}}{\binom{q}{l}}$, and $\gamma_{k_l,\mu_l,n} = \frac{(\mu_l+k_l+2n+1)!}{(\mu_l+n)!(k_l+n)!}$.

Hence, it is shown in the previous theorem that for any $\xi \in [0,1]$ $D^{(n)}_{k,\mu,\beta T,q}x(\beta T\xi + t_0)$ are affine estimators when $k,\mu \in \mathbb{N}$ and $q \leq k+n$. We show in the following proposition that $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi + t_0)$ can be also written as an affine combination of the extended minimal estimators defined by (1.29) with $\kappa,\mu \in]-1,+\infty[$. Thus, $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi + t_0)$ can be considered as the extended affine estimators for $x^{(n)}(t_0)$.

Proposition 1.3.14 *Let $x \in \mathcal{C}^n(I)$ and for any $t_0 \in I$, $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi + t_0)$ be the estimates of $x^{(n)}(t_0)$ defined in (1.74), then we have for any $\xi \in [0,1]$*

$$D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi + t_0) = \sum_{i=0}^{q} P_i^{(\mu+n,\kappa+n)}(\xi)\frac{2i+\mu+\kappa+2n+1}{i+\mu+\kappa+2n+1}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\ D^{(n)}_{\kappa_j,\mu_{ij},\beta T}x(t_0), \tag{1.79}$$

where $D^{(n)}_{\kappa_j,\mu_{ij},\beta T}x(t_0)$ is defined by (1.29) with $T \in D_{t_0}$, $\mu_{ij} = \mu+i-j$, $\kappa_j = \kappa+j$ with $\kappa,\mu \in]-1,+\infty[$. Moreover, we have

$$\sum_{i=0}^{q} P_i^{(\mu+n,\kappa+n)}(\xi)\frac{2i+\mu+\kappa+2n+1}{i+\mu+\kappa+2n+1}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j} = 1. \tag{1.80}$$

If $\beta = -1$ (resp. if $\beta = 1$), estimators given in (1.79) are called causal (resp. anti-causal) Jacobi estimators.

Proof. Replacing μ by $\mu+n$, κ by $\kappa+n$ and $x(\beta T\tau+t_0)$ by $x^{(n)}(\beta T\tau+t_0)$ in Lemma 1.2.6, we get

$$\frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\tau), x^{(n)}(\beta T\tau+t_0)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\mu+n}} = \sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\frac{2i+\mu+\kappa+2n+1}{i+\mu+\kappa+2n+1}\, D^{(0)}_{\kappa_j+n,\mu_{ij}+n,\beta T}x^{(n)}(t_0).$$

Then, (1.79) can be obtained by using (1.38) and (1.74). If $q=0$, then as $P_0^{(\mu+n,\kappa+n)} \equiv 1$ relation (1.80) can be easily given. If $q>0$, then by using the Binomial Theorem we get

$$\begin{aligned}\sum_{i=1}^{q} P_i^{(\mu+n,\kappa+n)}(\xi)\frac{2i+\mu+\kappa+2n+1}{i+\mu+\kappa+2n+1}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j} =& \sum_{i=1}^{q} P_i^{(\mu+n,\kappa+n)}(\xi)\frac{2i+\mu+\kappa+2n+1}{i+\mu+\kappa+2n+1}(1-1)^i \\ =&0.\end{aligned}$$

Hence, this proof can be completed. □

It is clear that if $q=0$, then $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi+t_0) = D^{(n)}_{\kappa,\mu,\beta T}x(t_0)$. Hence, $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi+t_0)$ gives a general presentation for minimal estimators and affine estimators. For $q=1$, we can give the following corollary.

Corollary 1.3.15 *Let $D^{(n)}_{\kappa,\mu,\beta T,1}x(\beta T\xi+t_0)$ be the Jacobi estimators given by (1.74) with $q=1$, then for any $t_0 \in I$ we have*

$$D^{(n)}_{\kappa,\mu,\beta T,1}x(\beta T\xi+t_0) = \lambda_{\kappa,\mu,n,\xi} D^{(n)}_{\kappa,\mu+1,\beta T}x(t_0) + (1-\lambda_{\kappa,\mu,n,\xi}) D^{(n)}_{\kappa+1,\mu,\beta T}x(t_0), \tag{1.81}$$

where $\lambda_{\kappa,\mu,n,\xi} = (\kappa+n+2)-(2n+\kappa+\mu+3)\xi$.

Proof. Let us take $q=1$ in (1.79), then we get

$$\begin{aligned}D^{(n)}_{\kappa,\mu,\beta T,1}x(\beta T\xi+t_0) =& P_0^{(\mu+n,\kappa+n)}(\xi) D^{(n)}_{\kappa,\mu,\beta T}x(t_0) \\ &+ P_1^{(\mu+n,\kappa+n)}(\xi)\frac{\mu+\kappa+2n+3}{\mu+\kappa+2n+2}\left(D^{(n)}_{\kappa+1,\mu,\beta T}x(t_0) - D^{(n)}_{\kappa,\mu+1,\beta T}x(t_0)\right).\end{aligned}$$

By substituting $n-1$ in (1.35) by n, we get

$$D^{(n)}_{\kappa,\mu,\beta T}x(t_{0\pm}) = \frac{n+\mu+1}{2n+\kappa+\mu+2}D^{(n)}_{\kappa,\mu+1,\beta T}x(t_0) + \frac{n+\kappa+1}{2n+\kappa+\mu+2}D^{(n)}_{\kappa+1,\mu,\beta T}x(t_0).$$

Since $P_0^{(\mu+n,\kappa+n)}(\xi) = 1$ and $P_1^{(\mu+n,\kappa+n)}(\xi) = (\kappa+\mu+4)\xi-(\kappa+2)$, we have

$$\begin{aligned}\frac{n+\mu+1}{2n+\kappa+\mu+2} - P_1^{(\mu+n,\kappa+n)}(\xi)\frac{\mu+\kappa+2n+3}{\mu+\kappa+2n+2} &= \lambda_{\kappa,\mu,n,\xi}, \\ \frac{n+\kappa+1}{2n+\kappa+\mu+2} + P_1^{(\mu+n,\kappa+n)}(\xi)\frac{\mu+\kappa+2n+3}{\mu+\kappa+2n+2} &= 1-\lambda_{\kappa,\mu,n,\xi}.\end{aligned}$$

Then, this proof can be completed. □

Remark 1 *If we take* $\xi = \frac{\kappa+n+2}{2n+\kappa+\mu+3}$, *then* $\lambda_{\kappa,\mu,n,\xi} = 0$ *and we obtain* $D^{(n)}_{\kappa,\mu,\beta T,1} x(\beta T\xi + t_0) = D^{(n)}_{\kappa,\mu+1,\beta T,1} x(t_0)$.

It is shown in Proposition 1.2.5 the relation between minimal estimators for $x^{(n)}(t_0)$ and the ones for $x^{(0)}(t_0)$ with $n \geq 0$. We can see in the following proposition that the Jacobi estimators $D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0)$ are in fact connected to the n^{th} order derivative of $D^{(0)}_{\kappa,\mu,\beta T,q+n} x(\beta T\xi + t_0)$ which are the $(q+n)^{th}$ order truncated Jacobi orthogonal series expansion of x

$$\forall \xi \in [0,1],\ D^{(0)}_{\kappa,\mu,\beta T,q+n} x(\beta T\xi + t_0) = \sum_{i=0}^{q+n} \frac{\left\langle P_i^{(\mu,\kappa)}(\cdot), x(\beta T \cdot + t_0)\right\rangle_{\mu,\kappa}}{\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}}\ P_i^{(\mu,\kappa)}(\xi). \tag{1.82}$$

Proposition 1.3.16 *[Liu 2011a] Let* $x \in \mathcal{C}^n(I)$, $D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0)$ *and* $D^{(0)}_{\kappa,\mu,\beta T,q+n} x(\beta T\xi + t_0)$ *be the Jacobi estimators defined in (1.74) and (1.82) respectively, then we have*

$$\forall \xi \in [0,1],\ D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) = \frac{1}{(\beta T)^n} \frac{d^n}{d\xi^n} \left\{ D^{(0)}_{\kappa,\mu,\beta T,q+n} x(\beta T\xi + t_0) \right\}. \tag{1.83}$$

Moreover, we have

$$D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) = \frac{1}{(\beta T)^n} \int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau) x(\beta T\tau + t_0) d\tau, \tag{1.84}$$

where

$$Q_{\kappa,\mu,n,q,\xi}(\tau) = w_{\mu,\kappa}(\tau) \sum_{i=0}^{q} C_{\kappa,\mu,n,i} P_i^{(\mu+n,\kappa+n)}(\xi) P_{n+i}^{(\mu,\kappa)}(\tau), \tag{1.85}$$

with $C_{\kappa,\mu,n,i} = \frac{(\mu+\kappa+2n+2i+1)\Gamma(\kappa+\mu+2n+i+1)\Gamma(n+i+1)}{\Gamma(\kappa+n+i+1)\Gamma(\mu+n+i+1)}$.

Proof. By applying n times the derivation operator to (1.82), we obtain from the formula (7.25) (given in Appendix) that

$$\begin{aligned} \frac{d^n}{d\xi^n} \left\{ D^{(0)}_{\kappa,\mu,\beta T,q+n} x(\beta T\xi + t_0) \right\} &= \sum_{i=0}^{q} \frac{\left\langle P_{i+n}^{(\mu,\kappa)}(\cdot), x(\beta T \cdot + t_0)\right\rangle_{\mu,\kappa}}{\|P_{i+n}^{(\mu,\kappa)}\|^2_{\mu,\kappa}} \frac{d^n}{d\xi^n} \left\{ P_{i+n}^{(\mu,\kappa)}(\xi) \right\} \\ &= \sum_{i=0}^{q} \frac{\left\langle P_{i+n}^{(\mu,\kappa)}(\cdot), x(\beta T \cdot + t_0)\right\rangle_{\mu,\kappa}}{\|P_{i+n}^{(\mu,\kappa)}\|^2_{\mu,\kappa}} \frac{\Gamma(\mu+\kappa+2n+i+1)}{\Gamma(\mu+\kappa+n+i+1)} P_i^{(\mu+n,\kappa+n)}(\xi). \end{aligned} \tag{1.86}$$

By applying two times the Rodrigues formula and by taking n integrations by parts, we get

$$\begin{aligned} \left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), x^{(n)}(t_0 + \beta T \cdot)\right\rangle_{\mu+n,\kappa+n} &= \int_0^1 w_{\mu+n,\kappa+n}(\tau) P_i^{(\mu+n,\kappa+n)}(\tau)\, x^{(n)}(t_0 + \beta T\tau) d\tau \\ &= \int_0^1 \frac{(-1)^i}{i!} w^{(i)}_{\mu+n+i,\kappa+n+i}(\tau)\, x^{(n)}(t_0 + \beta T\tau) d\tau \\ &= \frac{1}{(\beta T)^n} \int_0^1 \frac{(-1)^{i+n}}{i!} w^{(n+i)}_{\mu+n+i,\kappa+n+i}(\tau)\, x(t_0 + \beta T\tau) d\tau \\ &= \frac{1}{(\beta T)^n} \int_0^1 \frac{(n+i)!}{i!} w_{\mu,\kappa}(\tau) P_{n+i}^{(\mu,\kappa)}(\tau)\, x(t_0 + \beta T\tau) d\tau. \end{aligned}$$

Then, by using (7.20) given in Appendix we obtain after some calculations

$$\frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), x^{(n)}(t_0+\beta T\cdot)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}} = \frac{\left\langle P_{i+n}^{(\mu,\kappa)}(\cdot), x(t_0+\beta T\cdot)\right\rangle_{\mu,\kappa}}{(\beta T)^n \|P_{n+i}^{(\mu,\kappa)}\|^2_{\mu,\kappa}} \frac{\Gamma(\mu+\kappa+2n+i+1)}{\Gamma(\mu+\kappa+n+i+1)}. \quad (1.87)$$

Finally, by taking (1.74) and (1.86) we obtain

$$\forall \xi \in [0,1],\ D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) = \frac{1}{(\beta T)^n} \frac{d^n}{d\xi^n} \left\{ D^{(0)}_{\kappa,\mu,\beta T,q+n} x(\beta T\xi + t_0) \right\}. \quad (1.88)$$

Then, this proof can be completed. □

When we apply the Jacobi estimators, we need to calculate $Q_{\kappa,\mu,n,q,\xi}$ which is a sum of $q+1$ terms. Since the computational complexity of $P_{n+i}^{(\mu,\kappa)}$ is $\mathcal{O}(n^2)$, then the one of $Q_{\kappa,\mu,n,q,\xi}$ is also $\mathcal{O}(n^2)$.

In the next subsection, we are going to analyze the truncated errors for these Jacobi estimators.

1.3.3 Analysis on the truncated term error

Let $D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0)$ be a Jacobi estimator. Then, its associated truncated errors can be decomposed into two sources of errors as follows

$$e^{\beta}_{R_n}(t_0;\kappa,\mu,T,\xi,q) = \left(D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) - x^{(n)}(t_0+\beta T\xi) \right) + \left(x^{(n)}(t_0+\beta T\xi) - x^{(n)}(t_0) \right). \quad (1.89)$$

The first error part can be considered as a bias term error which produces an amplitude error estimation and the second error can be considered as a drift error. In the next proposition, we study the bias term error.

Proposition 1.3.17 *[Liu 2011a] If $x \in \mathcal{C}^{N+1}(I)$ with $N \geq n$ and $\forall t_0 \in I$, $D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0)$ is a Jacobi estimator of $x^{(n)}(t_0)$ defined by (1.84), then we have*

$$\forall t_0 \in I,\ D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) = x^{(n)}(t_0+\beta T\xi) + \mathcal{O}\left(T^{q+1}\right), \quad (1.90)$$

with $q = N-n$. Moreover, by assuming that there exists $M_{N+1} \in \mathbb{R}^*_+$ such that $\left\|x^{(N+1)}\right\|_\infty \leq M_{N+1}$, then we have for any $\xi \in [0,1]$

$$\left\| D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) - x^{(n)}(t_0+\beta T\xi) \right\|_\infty \leq M_{N+1} C_{\kappa,\mu,n,q,\xi} T^{q+1}, \quad (1.91)$$

where $C_{\kappa,\mu,n,q,\xi} = \frac{1}{(n+1+q)!} \int_0^1 \left| Q_{\kappa,\mu,n,q,\xi}(\tau) \tau^{n+1+q} \right| d\tau + \frac{\xi^{q+1}}{(q+1)!}$.

Proof. Let us take the Taylor series expansion of x at t_0, which is given by (1.71)

$$x(t_0+\beta T\xi) = x_N(t_0+\beta T\xi) + \frac{(\beta T\xi)^{N+1}}{(N+1)!} x^{(N+1)}(\theta^{\beta}_{N,t_0}), \quad (1.92)$$

where $x_N(t_0+\beta T\xi) = \sum_{j=0}^{N} \frac{(\beta T\xi)^j}{j!} x^{(j)}(t_0)$. Substituting $x(\beta T\xi + t_0)$ in (1.74) by $x_N(\beta T\xi + t_0)$, then we get

$$D^{(n)}_{\kappa,\mu,\beta T,q} x_N(\beta T\xi + t_0) = \sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), x_N^{(n)}(t_0+\beta T\cdot)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}} P_i^{(\mu+n,\kappa+n)}(\xi). \quad (1.93)$$

Since $x_N^{(n)}(\beta T \cdot +t_0)$ is a polynomial of degree q^{th} $(q = N - n)$, it can be written using the Jacobi series expansion given in (1.93). Hence, we get

$$D^{(n)}_{\kappa,\mu,\beta T,q} x_N(\beta T\xi + t_0) = x_N^{(n)}(\beta T\xi + t_0). \tag{1.94}$$

Then by using Proposition 1.3.16 we obtain

$$x_N^{(n)}(\beta T\xi + t_0) = \frac{1}{(\beta T)^n} \int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau)\, x_N(t_0 + \beta T\tau)\, d\tau. \tag{1.95}$$

Hence, we get

$$\left| D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) - x_N^{(n)}(\beta T\xi + t_0) \right| \leq \frac{M_{N+1} T^{N-n+1}}{(N+1)!} \int_0^1 \left| Q_{\kappa,\mu,n,q,\xi}(\tau) \tau^{N+1} \right| d\tau.$$

Let us take the Taylor series expansion of $x^{(n)}$ at t_0, then by using the well known Taylor's formula it yields for any $T\xi \in D_{t_0}$ with ξ set in $[0,1]$ that if $\beta > 0$ (resp. if $\beta < 0$) then there exists $\hat{\theta}^+_{N,t_0} \in]t_0, t_0 + \beta T\xi[$ (resp. $\hat{\theta}^-_{N,t_0} \in]t_0 + \beta T\xi, t_0[$) such that

$$x^{(n)}(t_0 + \beta T\xi) = x_N^{(n)}(t_0 + \beta T\xi) + \frac{(\beta T\xi)^{N-n+1}}{(N-n+1)!} x^{(N+1)}(\hat{\theta}^\beta_{N,t_0}). \tag{1.96}$$

Then we get

$$\left| x^{(n)}(t_0 + \beta T\xi) - x_N^{(n)}(t_0 + \beta T\xi) \right| \leq M_{N+1} \frac{(T\xi)^{N-n+1}}{(N-n+1)!}.$$

Finally, this proof can be completed by the following inequality

$$\begin{aligned} &\left| D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) - x^{(n)}(\beta T\xi + t_0) \right| \leq \\ &\quad \left| D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) - x_N^{(n)}(\beta T\xi + t_0) \right| + \left| x^{(n)}(t_0 + \beta T\xi) - x_N^{(n)}(t_0 + \beta T\xi) \right|. \end{aligned}$$

□

Remark 2 *According to [Poffald 1990], we can deduce the asymptotic behavior of the number* $\hat{\theta}^\beta_{N,t_0}$ *when* $T \to 0^+$

$$\lim_{T \to 0^+} \frac{|\hat{\theta}^\beta_{N,t_0} - t_0|}{T} = \frac{1}{N+2}. \tag{1.97}$$

We can see that the term $C_{\kappa,\mu,n,q,\xi}$ in the obtained error bound depends on parameters κ, μ, T and ξ. Since we extend the values of κ, μ from $\mathbb{N}$ to $]-1, +\infty[$, we obtain a higher degree of freedom so as to minimize the bias term error on our estimators.

It is shown in Subsection 1.3.1 that if $x \in \mathcal{C}^{N+1}(I)$, by applying the algebraic parametric techniques with the annihilator $\Pi^{N,n}_{k,\mu}$ to the N^{th} order truncated Taylor series expansion of x, the convergence rate for the affine estimators $\tilde{x}^{(n)}_{t_{0\pm}}(k, \mu, \beta T, N)$ is improved to $\mathcal{O}(T^{N-n+1})$ as $T \to 0$. Let us remark that if we take $\xi = 0$ in Proposition 1.3.17, then the convergence rate for the Jacobi estimators $D^{(n)}_{\kappa,\mu,\beta T,q} x(t_0)$ is also equal to $\mathcal{O}(T^{q+1})$ with $q = N - n$ as $T \to 0$.

If we take $\xi = 0$ in the previous proof, then we can deduce that the Jacobi estimators $D^{(n)}_{\kappa,\mu,\beta T,q}x(t_0)$ obtained by taking the q^{th} order truncated Jacobi orthogonal series expansion of $x^{(n)}$ can be also obtained by taking the $(n+q)^{th}$ order truncated Taylor series expansion of x by applying the scalar product with Jacobi polynomials. Moreover, for any $\xi \in [0,1]$ let $x_N(t_0+\beta T\xi) = x_n(t_0+\beta T\xi) + r_q(t_0+\beta T\xi)$ with $r_q(t_0+\beta T\xi) = \sum_{j=n+1}^{n+q} \frac{(T\tau)^j}{j!} x^{(j)}(t_0)$ for $q \geq 1$ and $r_q(t_0+\beta T\xi) = 0$ for $q = 0$, then by using (1.94) and (1.93) with $\xi = 0$ we get

$$x_N^{(n)}(t_0) = \sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), x_n^{(n)}(t_0+\beta T\cdot)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}} \, P_i^{(\mu+n,\kappa+n)}(0) + R,$$

where $R = \sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), r_q^{(n)}(t_0+\beta T\cdot)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}} \, P_i^{(\mu+n,\kappa+n)}(0)$.

Since $x_n^{(n)}(t_0+\beta T\cdot)$ is a 0^{th} order polynomial, then by using the orthogonal properties of $P_i^{(\mu+n,\kappa+n)}$ we get

$$\sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), x_n^{(n)}(t_0+\beta T\cdot)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}} \, P_i^{(\mu+n,\kappa+n)}(0) =$$
$$\frac{\left\langle P_0^{(\mu+n,\kappa+n)}(\cdot), x_n^{(n)}(t_0+\beta T\cdot)\right\rangle_{\mu+n,\kappa+n}}{\|P_0^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}} \, P_0^{(\mu+n,\kappa+n)}(0) = x_n^{(n)}(t_0).$$

By calculating the values of $x_N^{(n)}$ and $x_n^{(n)}$ at t_0, we obtain $x_N^{(n)}(t_0) = x_n^{(n)}(t_0) = x^{(n)}(t_0)$, and consequently $R = 0$. Hence, we can deduce that

$$R = \frac{1}{(\beta T)^n} \int_0^1 Q_{\kappa,\mu,n,q,0}(\tau) r_q(t_0+\beta T\tau) d\tau = 0, \tag{1.98}$$

where $Q_{\kappa,\mu,n,q,0}$ is given in (1.85) with $\xi = 0$.

Consequently, relation (1.98) explains why the convergence rate can be improved from $\mathcal{O}(T)$ to $\mathcal{O}(T^{q+1})$ when the function x is smoother.

Now, let us remark that when we estimate $x^{(n)}(t_0)$ by the Jacobi estimators $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi + t_0)$, then according to (1.89) the drift error produces a time-drift of value $T\xi$. If ξ_q is one root of the Jacobi polynomial $P_{q+1}^{(\mu+n,\kappa+n)}$, then by using (1.74) we get $D^{(n)}_{\kappa,\mu,-T,q}x(-T\xi_q + t_0) = D^{(n)}_{\kappa,\mu,-T,q+1}x(-T\xi_q + t_0)$. According to Proposition 1.3.17, if we estimate the derivative value of $x^{(n)}(t_0 - T\xi)$ by $D^{(n)}_{\kappa,\mu,-T,q}x(-T\xi_q + t_0)$, then the corresponding convergence rate is improved from $\mathcal{O}(T^{q+1})$ to $\mathcal{O}(T^{q+2})$ as $T \to 0$.

Corollary 1.3.18 *[Liu 2011a] If $x \in \mathcal{C}^{N+2}(I)$ with $N \geq n$ and ξ_q is a root of the Jacobi polynomial $P_{q+1}^{(\mu+n,\kappa+n)}$, then we have*

$$\forall t_0 \in I, \; D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi_q + t_0) = x^{(n)}(t_0 + \beta T\xi_q) + \mathcal{O}\left(T^{q+2}\right), \tag{1.99}$$

with $q = N - n$. Moreover, if we assume that there exists $M_{N+2} \in \mathbb{R}_+^*$ such that $\left\| x^{(N+2)} \right\|_\infty \leq M_{N+2}$, then we have

$$\left\| D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T \xi_q + t_0) - x^{(n)}(t_0 + \beta T \xi_q) \right\|_\infty \leq M_{N+2} C_{\kappa,\mu,n,q+1,\xi_q} T^{q+2}, \tag{1.100}$$

where $C_{\kappa,\mu,n,q+1,\xi_q}$ is given by (1.91).

It is clear that we can choose ξ_q as the smallest root of the Jacobi polynomial $P_{q+1}^{(\mu+n,\kappa+n)}$ so as to reduce the time-drift. We denote it by ξ_q^{min}. Moreover, it was shown in [Mboup 2009b] that if the value of T is small enough then the bias term error for $D^{(n)}_{\kappa,\mu,-T,q} x(t_0 - T\xi_q^{min})$ is smaller than the truncated term error for $D^{(n)}_{\kappa,\mu,-T,q+1} x(t_0)$. These Jacobi estimators are then significantly improved by admitting a minimal time-drift given by $T\xi_q^{min}$. Hence, they are called time-drift estimators.

Minimal estimators can be given by $D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T \xi + t_0)$ with $q = 0$. Since, the root of the Jacobi polynomial $P_1^{\mu+n,\kappa+n}$ is equal to $\frac{\kappa+n+1}{\mu+\kappa+2n+2}$, then the time-drift for the associated minimal estimator is equal to $C_T^{\mu,\kappa}$ which is defined in (1.49). Consequently, if the variation of the $(n+1)^{th}$ order derivative of the signal x is small inside the time observation window, then by using Proposition 1.2.8 we deduce that we can reduce the bias term error by making the time-drift $C_T^{\mu,\kappa}$ as small as possible.

Corollary 1.3.19 *[Liu 2011c] If $m_l^+ \simeq m_u^+$ and $m_l^- \simeq m_u^-$ with $m_l^\pm$ and $m_u^\pm$ being given in* 1.50*, then by minimizing the time-drift $C_T^{\mu,\kappa}$ we can also minimize the bias term error for the minimal estimators.*

When $\kappa, \mu \in]-1, +\infty[$, the value of $\frac{\kappa+n+1}{\mu+\kappa+2n+2}$ increases with respect to κ and it decreases with respect to μ. The negative values of κ produce smaller bias term errors than the ones produced by integer values of κ. This is why we extend the values of κ from $\mathbb{N}$ to $]-1, +\infty[$ in our minimal estimators. It is clear that one can achieve a given bias term error by increasing μ and reducing T (even choosing κ, μ as integer). However, we will see in Chapter 2 that this increases the noise error contribution. When $n = 1$, we can see the variation of $\frac{\kappa+2}{\kappa+\mu+4}$ with respect to $(\kappa, \mu) \in]-1, 1]^2$ in Figure 1.4.

When we use the affine estimators $D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T \xi_q + t_0)$ with $q = 1$, we take the value of ξ_1^{min} as the smaller root of $P_2^{(\mu+n,\kappa+n)}$. Since ξ_1^{min} is a function of κ, μ and n, we denote it by $\xi(\kappa, \mu, n)$. Hence, we can see the variation of $\xi(\kappa, \mu, n)$ with respect to $(\kappa, \mu) \in]-1, 1]^2$ for $n = 1, 2, 3, 4$ in Figure 1.5. We can observe that the extended parameters values of κ give smaller values for $\xi(\kappa, \mu, n = 1)$ in the extended affine estimators. Moreover, we give in Figure 1.6 the variation of $C_{\kappa,\mu,n,2,\xi_1}$ given by (1.91) with $q = 1$ so as to study the parameters' influence on the associated amplitude error. Consequently, it is shown that we can increase the value of μ and decrease the value of κ so as to reduce the truncated error for the affine estimator $D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T \xi_q + t_0)$ with $q = 1$.

Let us take the affine estimator $D^{(n)}_{\kappa,\mu,\beta T,q} x(t_0)$ with $q = 2$, which is a drift-free estimator. The associated truncated error bounded contains the term $C_{\kappa,\mu,n,2,0}$ in (1.91). We can see in Figure 1.7 the variation of $C_{\kappa,\mu,n,2,0}$ with $q = 1$. Hence, we can increase the value of μ and decrease the value of κ so as to reduce the truncated error. By comparing with Figure 1.6, we can observe that the value of $C_{\kappa,\mu,n,2,0}$ is much larger than the one of $C_{\kappa,\mu,n,2,\xi_1}$. Hence, the truncated error bound for $D^{(n)}_{\kappa,\mu,\beta T,q} x(t_0)$ with $q = 2$ is much larger than the one for $D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T \xi_q + t_0)$ with $q = 1$. Consequently, the

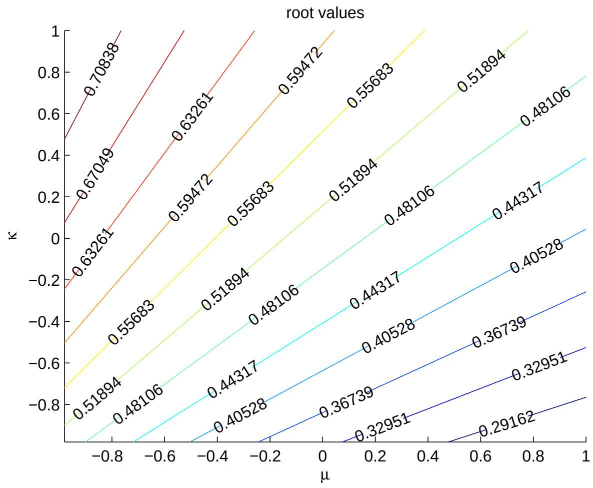
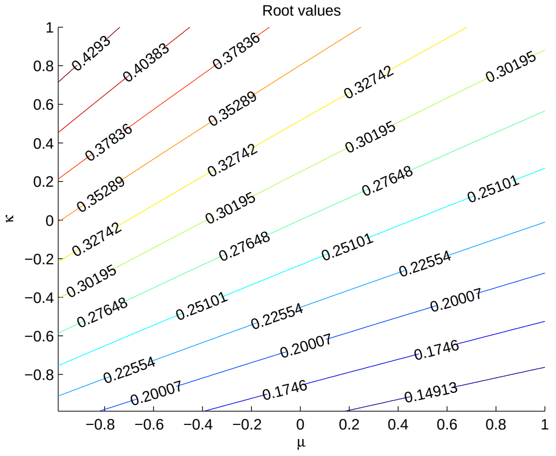
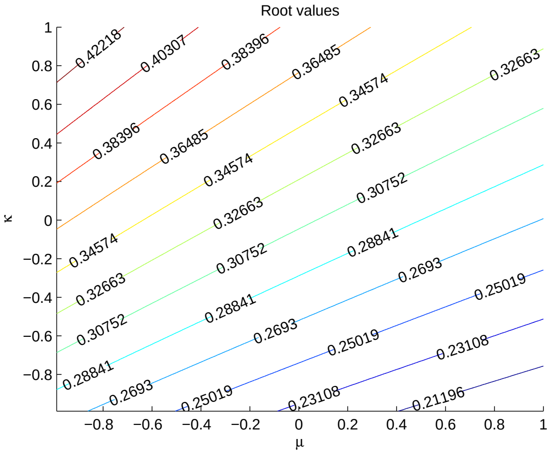


Figure 1.4: Variation of $\frac{\kappa+2}{\kappa+\mu+4}$ with respect to κ and μ.

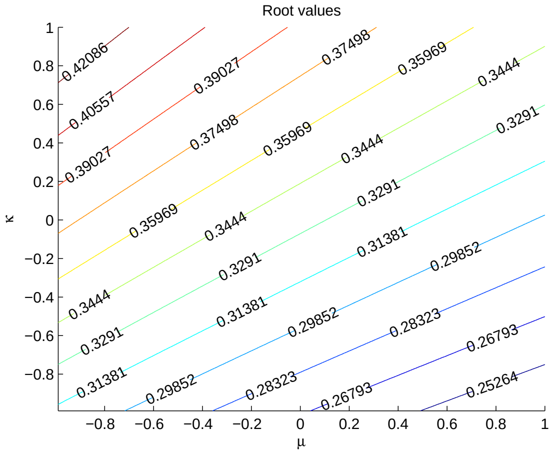
truncated error for $D^{(n)}_{\kappa,\mu,\beta T,q}x(t_0)$ with $q=2$ can be larger than the one for $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi_q+t_0)$ with $q=1$, independently of the value of T.



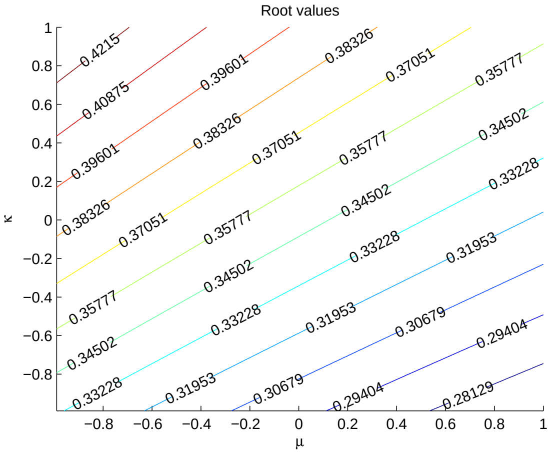
(a) $\xi(\kappa, \mu, n = 1)$.



(b) $\xi(\kappa, \mu, n = 2)$.

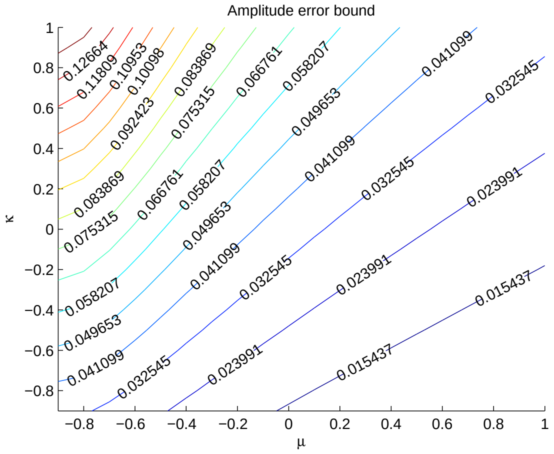


(c) $\xi(\kappa, \mu, n = 3)$.

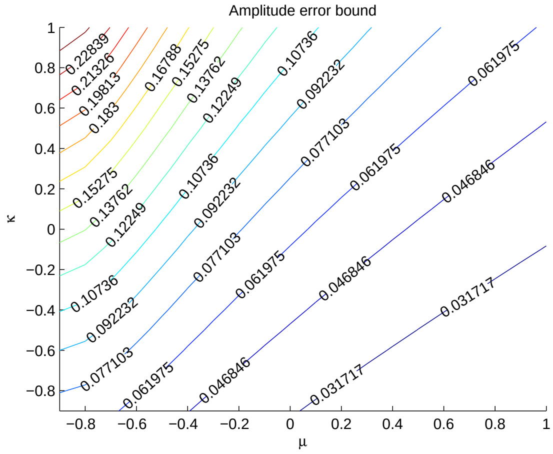


(d) $\xi(\kappa, \mu, n = 4)$.

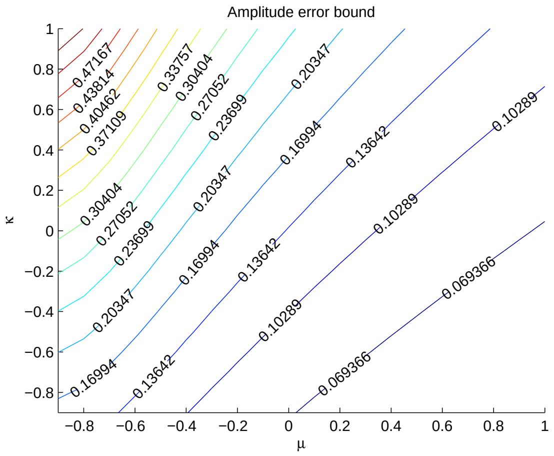
Figure 1.5: Variation of $\xi(\kappa, \mu, n)$ with respect to κ and μ for $n = 1, 2, 3, 4$.



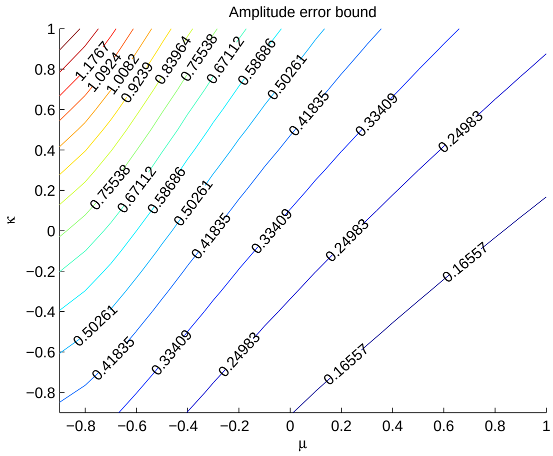
(a) $C_{\kappa,\mu,n,2,\xi(\kappa,\mu,n)}$ with $n = 1$.



(b) $C_{\kappa,\mu,n,2,\xi(\kappa,\mu,n)}$ with $n = 2$.

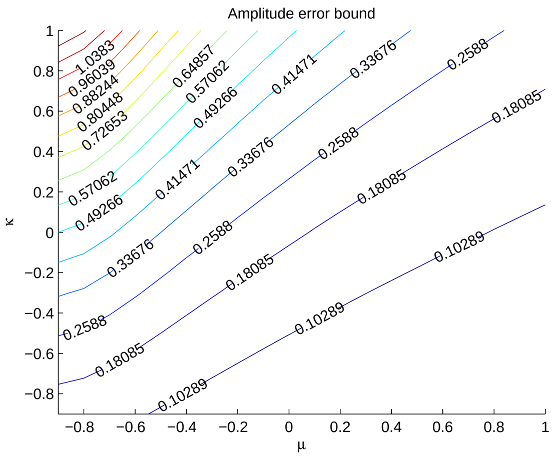


(c) $C_{\kappa,\mu,n,2,\xi(\kappa,\mu,n)}$ with $n = 3$.

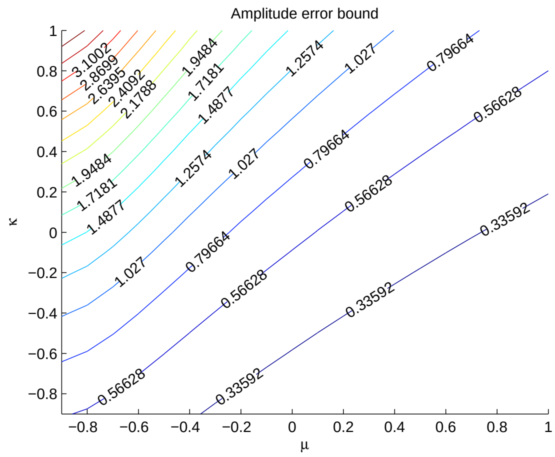


(d) $C_{\kappa,\mu,n,2,\xi(\kappa,\mu,n)}$ with $n = 4$.

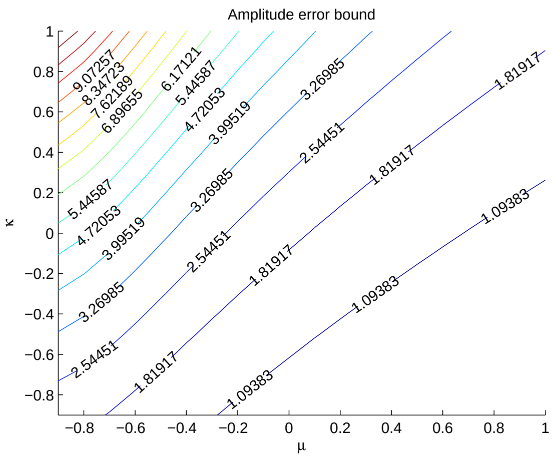
Figure 1.6: Variation of $C_{\kappa,\mu,n,2,\xi(\kappa,\mu,n)}$ with respect to κ and μ for $n = 1, 2, 3, 4$.



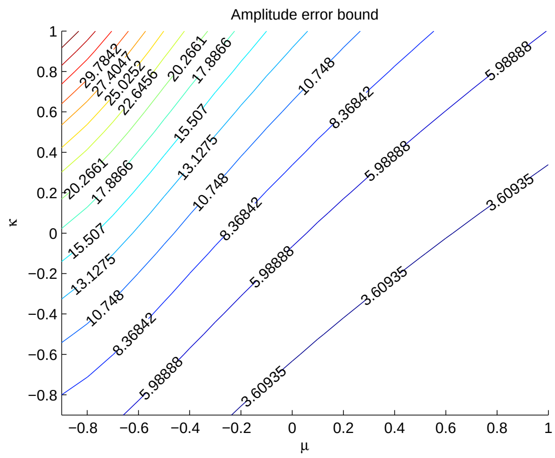
(a) $C_{\kappa,\mu,n,2,0}$ with $n = 1$.



(b) $C_{\kappa,\mu,n,2,0}$ with $n = 2$.



(c) $C_{\kappa,\mu,n,2,0}$ with $n = 3$.



(d) $C_{\kappa,\mu,n,2,0}$ with $n = 4$.

Figure 1.7: Variation of $C_{\kappa,\mu,n,2,0}$ with respect to κ and μ for $n = 1, 2, 3, 4$.

For numerical estimations, we give in Proposition 1.2.8 local lower and upper error bounds for the truncated term errors. Similarly, we give error bounds in the following proposition for affine estimators $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi+t_0)$ given in Corollary 1.3.15 with $q=1$. To do so, let us denote

$$\forall \tau\in[0,1],\ p_{\kappa,\mu,n,\xi}(\tau)=(n+\kappa+1)\lambda_{\kappa,\mu,n,\xi}+[(n+\mu+1)-(2n+\kappa+\mu+2)\lambda_{\kappa,\mu,n,\xi}]\tau, \tag{1.101}$$

the root of which is $\tau_0=\frac{-(n+\kappa+1)\lambda_{\kappa,\mu,n,\xi}}{(n+\mu+1)-(2n+\kappa+\mu+2)\lambda_{\kappa,\mu,n,\xi}}$ with $\lambda_{\kappa,\mu,n,\xi}=(\kappa+n+2)-(2n+\kappa+\mu+3)\xi$, and $I_\alpha=\int_0^\alpha p_{\kappa,\mu,n,\xi}(\tau)w_{\mu+n,\kappa+n+1}d\tau$ with $0<\alpha\leq 1$. Moreover, we denote by

$$m^+_{\alpha,l}=\inf_{t_0<\hat\theta^+_{n,t_0}<t_0+T\alpha}x^{(n+1)}(\hat\theta^+_{n,t_0}),\quad m^+_{\alpha,u}=\sup_{t_0<\hat\theta^+_{n,t_0}<t_0+T\alpha}x^{(n+1)}(\hat\theta^+_{n,t_0}), \tag{1.102}$$

$$m^-_{\alpha,l}=\sup_{t_0-T\alpha<\hat\theta^-_{n,t_0}<t_0}x^{(n+1)}(\hat\theta^-_{n,t_0}),\quad m^-_{\alpha,u}=\inf_{t_0-T\alpha<\hat\theta^-_{n,t_0}<t_0}x^{(n+1)}(\hat\theta^-_{n,t_0}). \tag{1.103}$$

Proposition 1.3.20 *Let $x\in\mathcal{C}^{n+1}(I)$ and $e^\beta_{R_n}(t_0;\kappa,\mu,T,\xi,1)$ be the truncated term errors for the Jacobi estimators $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi+t_0)$ with $q=1$, then we have*

$$\pm\frac{\Gamma(2n+\kappa+\mu+3)}{\Gamma(n+\kappa+2)\Gamma(\mu+n+2)}TM^\pm_l\leq e^\beta_{R_n}(t_0;\kappa,\mu,T,\xi,1)\leq\pm\frac{\Gamma(2n+\kappa+\mu+3)}{\Gamma(n+\kappa+2)\Gamma(\mu+n+2)}TM^\pm_u, \tag{1.104}$$

where

$$M^\pm_l=\begin{cases} I_{\tau_0}m^\pm_{\tau_0,l}+(I_1-I_{\tau_0})\,m^\pm_{1,u}, & \textit{if}\quad 0\leq\xi<\frac{\kappa+n+1}{\kappa+\mu+2n+3},\\ I_1m^\pm_{1,l}, & \textit{if}\quad \frac{\kappa+n+1}{\kappa+\mu+2n+3}\leq\xi<\frac{\kappa+n+2}{\kappa+\mu+2n+3},\\ I_{\tau_0}m^\pm_{\tau_0,u}+(I_1-I_{\tau_0})\,m^\pm_{1,l}, & \textit{if}\quad \frac{\kappa+n+2}{\kappa+\mu+2n+3}\leq\xi\leq 1,\end{cases} \tag{1.105}$$

and

$$M^\pm_u=\begin{cases} I_{\tau_0}m^\pm_{\tau_0,u}+(I_1-I_{\tau_0})\,m^\pm_{1,l}, & \textit{if}\quad 0\leq\xi<\frac{\kappa+n+1}{\kappa+\mu+2n+3},\\ I_1m^\pm_{1,u}, & \textit{if}\quad \frac{\kappa+n+1}{\kappa+\mu+2n+3}\leq\xi<\frac{\kappa+n+2}{\kappa+\mu+2n+3},\\ I_{\tau_0}m^\pm_{\tau_0,l}+(I_1-I_{\tau_0})\,m^\pm_{1,u}, & \textit{if}\quad \frac{\kappa+n+2}{\kappa+\mu+2n+3}\leq\xi\leq 1.\end{cases} \tag{1.106}$$

Proof. According to (1.81), the truncated term errors $e^\beta_{R_n}(t_0;\kappa,\mu,T,\xi,1)$ for $D^{(n)}_{\kappa,\mu,\beta T,1}x(\beta T\xi+t_0)$ can be written as

$$e^\beta_{R_n}(t_0;\kappa,\mu,T,\xi,1)=\lambda_{\kappa,\mu,n,\xi}e^\beta_{R_n}(t_0;\kappa,\mu+1,T)+(1-\lambda_{\kappa,\mu,n,\xi})e^\beta_{R_n}(t_0;\kappa+1,\mu,T), \tag{1.107}$$

where $e^\beta_{R_n}(t_0;\kappa,\mu+1,T)$ (resp. $e^\beta_{R_n}(t_0;\kappa+1,\mu,T)$) are the truncated term errors for $D^{(n)}_{\kappa,\mu+1,\beta T}x(t_0)$ (resp. $D^{(n)}_{\kappa+1,\mu,\beta T}x(t_0)$). Then, by using (1.53) we get

$$\begin{aligned} e^\beta_{R_n}(t_0;\kappa,\mu,T,\xi,1)=&\frac{\lambda_{\kappa,\mu,n,\xi}\beta T}{\mathrm{B}(n+\kappa+1,\mu+n+2)}\int_0^1 w_{\mu+n+1,\kappa+n+1}(\tau)x^{(n+1)}(\hat\theta^\pm_{n,t_0})d\tau\\ &+\frac{(1-\lambda_{\kappa,\mu,n,\xi})\beta T}{\mathrm{B}(n+\kappa+2,\mu+n+1)}\int_0^1 w_{\mu+n,\kappa+n+2}(\tau)x^{(n+1)}(\hat\theta^\pm_{n,t_0})d\tau\\ =&\frac{\Gamma(2n+\kappa+\mu+3)}{\Gamma(n+\kappa+2)\Gamma(\mu+n+2)}\beta T\int_0^1 p_{\kappa,\mu,n,\xi}(\tau)w_{\mu+n,\kappa+n+1}(\tau)x^{(n+1)}(\hat\theta^\pm_{n,t_0})d\tau,\end{aligned}$$

where $\hat{\theta}^+_{n,t_0} \in]t_0, t_0 + T\tau[$, $\hat{\theta}^-_{n,t_0} \in]t_0 - T\tau, t_0[$, $p_{\kappa,\mu,n,\xi}$ is defined by (1.101).

The first derivative of $p_{\kappa,\mu,n,\xi}$ is $(n+\mu+1) - (2n+\kappa+\mu+2)\lambda_{\kappa,\mu,n,\xi}$. Then, we study the variation of $p_{\kappa,\mu,n,\xi}$ in the following three cases:

1. If $\lambda_{\kappa,\mu,n,\xi} = \frac{n+\mu+1}{2n+\kappa+\mu+2}$, then $(n+\mu+1) - (2n+\kappa+\mu+2)\lambda_{\kappa,\mu,n,\xi} = 0$. Hence, $p_{\kappa,\mu,n,\xi} \equiv (n+\kappa+1)\lambda_{\kappa,\mu,n,\xi} > 0$ for any $\tau \in [0,1]$.

2. If $\lambda_{\kappa,\mu,n,\xi} < \frac{n+\mu+1}{2n+\kappa+\mu+2}$, then $(n+\mu+1) - (2n+\kappa+\mu+2)\lambda_{\kappa,\mu,n,\xi} > 0$. Hence, $p_{\kappa,\mu,n,\xi}$ increases on $[0,1]$. Let us recall that

$$\tau_0 = \frac{-(n+\kappa+1)\lambda_{\kappa,\mu,n,\xi}}{(n+\mu+1) - (2n+\kappa+\mu+2)\lambda_{\kappa,\mu,n,\xi}}.$$

   Hence, if $0 < \lambda_{\kappa,\mu,n,\xi} < \frac{n+\mu+1}{2n+\kappa+\mu+2}$, then $\tau_0 < 0$. In this case, $p_{\kappa,\mu,n,\xi}$ is strictly positive on $[0,1]$. If $\lambda_{\kappa,\mu,n,\xi} \leq 0$, then $0 \leq \tau_0 < 1$. In this case, $p_{\kappa,\mu,n,\xi}(\tau) \geq 0$ for any $\tau \in [\tau_0, 1]$ and $p_{\kappa,\mu,n,\xi}(\tau) < 0$ for any $\tau \in [0, \tau_0[$.

3. If $\lambda_{\kappa,\mu,n,\xi} > \frac{n+\mu+1}{2n+\kappa+\mu+2}$, then $(n+\mu+1) - (2n+\kappa+\mu+2)\lambda_{\kappa,\mu,n,\xi} < 0$. Hence, $p_{\kappa,\mu,n,\xi}$ decreases on $[0,1]$. If $\frac{n+\mu+1}{2n+\kappa+\mu+2} < \lambda_{\kappa,\mu,n,\xi} \leq 1$, then $\tau_0 \geq 1$. In this case, we get $p_{\kappa,\mu,n,\xi}(\tau) > 0$ for any $\tau \in [0,1]$. If $\lambda_{\kappa,\mu,n,\xi} > 1$, then $0 < \tau_0 < 1$. In this case, $p_{\kappa,\mu,n,\xi}(\tau) \leq 0$ for any $\tau \in [\tau_0, 1]$ and $p_{\kappa,\mu,n,\xi}(\tau) > 0$ for any $\tau \in [0, \tau_0[$.

For $\xi \in [0,1]$, $\lambda_{\kappa,\mu,n,\xi} = (\kappa+n+2) - (2n+\kappa+\mu+3)\xi$, then we conclude that

			$\tau : 0 \longrightarrow \tau_0 \longrightarrow 1$
a)	$\lambda_{\kappa,\mu,n,\xi} > 1$	$0 \leq \xi < \frac{\kappa+n+1}{\kappa+\mu+2n+3}$	$+ \quad 0 \quad -$
b)	$0 < \lambda_{\kappa,\mu,n,\xi} \leq 1$	$\frac{\kappa+n+1}{\kappa+\mu+2n+3} \leq \xi < \frac{\kappa+n+2}{\kappa+\mu+2n+3}$	$+$
c)	$\lambda_{\kappa,\mu,n,\xi} \leq 0$	$\frac{\kappa+n+2}{\kappa+\mu+2n+3} \leq \xi \leq 1$	$- \quad 0 \quad +$

Table 1.1: Variation of $p_{\kappa,\mu,n,\xi}$ in three cases.

Let us consider the case when $0 \leq \xi < \frac{\kappa+n+1}{\kappa+\mu+2n+3}$. Then we obtain that for $0 \leq \tau \leq \tau_0$,

$$\begin{aligned} &\pm p_{\kappa,\mu,n,\xi}(\tau) w_{\mu+n,\kappa+n+1}(\tau) m^{\pm}_{\tau_0,l} \leq \\ &\quad \pm p_{\kappa,\mu,n,\xi}(\tau) w_{\mu+n,\kappa+n+1}(\tau) x^{(n+1)}(\hat{\theta}^{\pm}_{n,t_0}) \leq \pm p_{\kappa,\mu,n,\xi}(\tau) w_{\mu+n,\kappa+n+1}(\tau) m^{\pm}_{\tau_0,u}, \end{aligned} \tag{1.108}$$

and for $\tau_0 < \tau \leq 1$,

$$\begin{aligned} &\pm p_{\kappa,\mu,n,\xi}(\tau) w_{\mu+n,\kappa+n+1}(\tau) m^{\pm}_{1,u} \leq \\ &\quad \pm p_{\kappa,\mu,n,\xi}(\tau) w_{\mu+n,\kappa+n+1}(\tau) x^{(n+1)}(\hat{\theta}^{\pm}_{n,t_0}) \leq \pm p_{\kappa,\mu,n,\xi}(\tau) w_{\mu+n,\kappa+n+1}(\tau) m^{\pm}_{1,l}. \end{aligned} \tag{1.109}$$

By integrating (1.108) on $[0, \tau_0]$, we obtain

$$\pm I_{\tau_0} m^{\pm}_{\tau_0,l} \leq \pm \int_0^{\tau_0} p_{\kappa,\mu,n,\xi}(\tau) w_{\mu+n,\kappa+n+1}(\tau) x^{(n+1)}(\hat{\theta}^{\pm}_{n,t_0}) d\tau \leq \pm I_{\tau_0} m^{\pm}_{\tau_0,u},$$

and by integrating (1.109) on $[\tau_0, 1]$, we obtain

$$\pm(I_1 - I_{\tau_0})m_{1,u}^{\pm} \leq \pm \int_{\tau_0}^{1} p_{\kappa,\mu,n,\xi}(\tau) w_{\mu+n,\kappa+n+1}(\tau) x^{(n+1)}(\hat{\theta}_{n,t_0}^{\pm}) d\tau \leq \pm(I_1 - I_{\tau_0})m_{1,l}^{\pm}.$$

Consequently, when $0 \leq \xi \leq \frac{k+2}{k+\mu+5}$, we obtain two bounds for the truncated term errors $e_{R_n}^{\beta}(t_0; \kappa, \mu, T, \xi, 1)$:

$$\pm \frac{\Gamma(2n+\kappa+\mu+3)}{\Gamma(n+\kappa+2)\Gamma(\mu+n+2)} T M_l^{\pm} \leq e_{R_n}^{\beta}(t_0; \kappa, \mu, T, \xi, 1) \leq \pm \frac{\Gamma(2n+\kappa+\mu+3)}{\Gamma(n+\kappa+2)\Gamma(\mu+n+2)} T M_u^{\pm}, \tag{1.110}$$

where $M_l^{\pm} = I_{\tau_0} m_{\tau_0,l}^{\pm} + (I_1 - I_{\tau_0})\, m_{1,u}^{\pm}$ and $M_u^{\pm} = I_{\tau_0} m_{\tau_0,u}^{\pm} + (I_1 - I_{\tau_0})\, m_{1,l}^{\pm}$.

Then, this proof can be completed by applying a similar analysis for the cases b) and c) (see Table 1.1). □

1.3.4 Some numerical examples

We finish this section with some numerical examples. We can observe that when κ is negative then the integral given in the Jacobi estimator in (1.84) is an improper integral. Hence, there will be a singular value at $\tau = 0$ when we apply a numerical integration method. In order to avoid this problem, we apply the following change of variable $\tau \to t^{\frac{1}{1+\kappa}}$ in (1.84). Thus, we get

$$D_{\kappa,\mu,\beta T,q}^{(n)} x(\beta T\xi + t_0) = \frac{1}{(\beta T)^n} \int_0^1 Q_{\kappa,\mu,n,q,\xi}(t^{\frac{1}{1+\kappa}}) x(\beta T t^{\frac{1}{1+\kappa}} + t_0) dt, \tag{1.111}$$

where

$$Q_{\kappa,\mu,n,q,\xi}(t) = \frac{1}{1+\kappa}(1 - t^{\frac{1}{1+\kappa}})^{\mu} \sum_{i=0}^{q} C_{\kappa,\mu,n,i} P_i^{(\mu+n,\kappa+n)}(\xi) P_{n+i}^{(\mu,\kappa)}(t^{\frac{1}{1+\kappa}}), \tag{1.112}$$

with $C_{\kappa,\mu,n,i} = \frac{(\mu+\kappa+2n+2i+1)\Gamma(\kappa+\mu+2n+i+1)\Gamma(n+i+1)}{\Gamma(\kappa+n+i+1)\Gamma(\mu+n+i+1)}$.

If μ is negative in (1.84), then we apply an another change of variable $\tau \to 1 - \tau$. Hence, we get

$$D_{\kappa,\mu,\beta T,q}^{(n)} x(\beta T\xi + t_0) = \frac{1}{(\beta T)^n} \int_0^1 Q_{\kappa,\mu,n,q,\xi}(1-\tau)\, x(\beta T(1-\tau) + t_0) d\tau, \tag{1.113}$$

where

$$Q_{\kappa,\mu,n,q,\xi}(1-\tau) = (1-\tau)^{\kappa} \tau^{\mu} \sum_{i=0}^{q} C_{\kappa,\mu,n,i} P_i^{(\mu+n,\kappa+n)}(\xi) P_{n+i}^{(\mu,\kappa)}(1-\tau). \tag{1.114}$$

Then, by substituting τ by $t^{\frac{1}{1+\kappa}}$ in (1.113) we avoid the singular value at $\tau = 0$ when we apply a numerical integration method. In case where both κ and μ are negative, we decompose the integral given in (1.84) into two parts

$$D_{\kappa,\mu,\beta T,q}^{(n)} x(\beta T\xi + t_0) = \frac{1}{(\beta T)^n} \int_0^{\frac{1}{2}} Q_{\kappa,\mu,n,q,\xi}(\tau)\, x(\beta T\tau + t_0) d\tau + \frac{1}{(\beta T)^n} \int_{\frac{1}{2}}^{1} Q_{\kappa,\mu,n,q,\xi}(\tau)\, x(\beta T\tau + t_0) d\tau. \tag{1.115}$$

Then, we respectively apply the previous changes of variable in order to avoid the singular values in the two integrals given in (1.115).

Let us remark that in practice the function x is usually known at an equidistant sampling period. Hence, if we apply a numerical integration method to (1.111), then $t_i^{\frac{1}{1+\kappa}} = \frac{i}{m}$ for $i = 0, \ldots, m$ where $m+1$ is the number of sampling data in the sliding integration window. Thus, the sampling period for the numerical integration method becomes $h_i = \left(\frac{i+1}{m}\right)^{1+\kappa} - \left(\frac{i}{m}\right)^{1+\kappa}$ for $i = 0, \ldots, m-1$.

By now on, we will show that taking negative values for κ improves the quality of Jacobi estimators. We take the sampling data of function x defined by (1.8) with an sampling period $T_s = \frac{1}{2000}$. Then, we use minimal estimators and affine estimators to estimate the first order derivative of x. We show in Figure 1.8 the estimations obtained by using causal minimal estimators given in (1.29) with $\kappa = -0.8$ and $\kappa = 0$ respectively. We can see that the time-delay (time-drift) for $D^{(1)}_{\kappa,\mu,-T}x(t_0)$ with $\kappa = -0.8$ is smaller than the one for $D^{(1)}_{\kappa,\mu,-T}x(t_0)$ with $\kappa = 0$. The associated absolute truncated term errors are shown in Figure 1.9(a). Indeed, the time-delay $T\frac{\kappa+2}{\kappa+\mu+4}$ for these estimations is equal to 0.094 and 0.125. Hence, by removing these estimations we can calculate the associated amplitude errors in Figure 1.9(b). Consequently, the estimation obtained with $\kappa = -0.8$ produces a smaller time-delay and a smaller truncated term error but a larger amplitude error than the one obtained with $\kappa = 0$. The estimations obtained by using the causal affine estimators defined in (1.84) with $\kappa = -0.8$ and $\kappa = 0$ are given in Figure 1.10. By calculating $T\xi_1^{min}$ where ξ_1^{min} is the smaller root of $P_2^{(\mu+1,\kappa+1)}$, the time-delay for these estimations are equal to 0.046 and 0.069 respectively. By observing Figure 1.11(a) and Figure 1.11(b), we can see that the estimation obtained with $\kappa = -0.8$ produces a smaller truncated term error than the one obtained with $\kappa = 0$.

Secondly, let us show the improvement of the Jacobi estimators by taking larger value for q. In Section 1.1 we compare the amplitude error for the time-delay estimator $D^{(1)}_{\kappa,\mu,-T,q}x(-T\xi_1^{min} + t_0)$ with $\kappa = \mu = 0$ and $q = 1$ to the one for the delay-free estimator $D^{(1)}_{\kappa,\mu,-T,q}x(t_0)$ with $\kappa = \mu = 0$ and $q = 2$. Here, we compare the amplitude error for the time-delay estimator $D^{(1)}_{\kappa,\mu,-T,q}x(-T\xi_1^{min} + t_0)$ with $\kappa = -0.8$, $\mu = 0$ and $q = 1$ to the ones for the delay-free estimator $D^{(1)}_{\kappa,\mu,-T,q}x(t_0)$ with $\kappa = \mu = 0$, and $q = 3, 4$ respectively. The obtained estimations are given in Figure 1.12(a) and Figure 1.13(a). The associated amplitude errors are given in Figure 1.12(b) and Figure 1.13(b). Hence, it is shown that the amplitude error for $D^{(1)}_{-0.8,0,-T,1}x(-T\xi_1^{min} + t_0)$ is smaller than the one for the time-delay estimator with $q = 3$. Nevertheless, it is larger than the one obtained for the time-delay estimator with $q = 4$. We will show in Chapter 2 that when q is equal to 3 or 4, the noise error contribution increases. Consequently, the Jacobi estimator is significantly improved by admitting a time-delay. Finally, let us recall that the analysis for the choice of parameters will be addressed in Chapter 2.

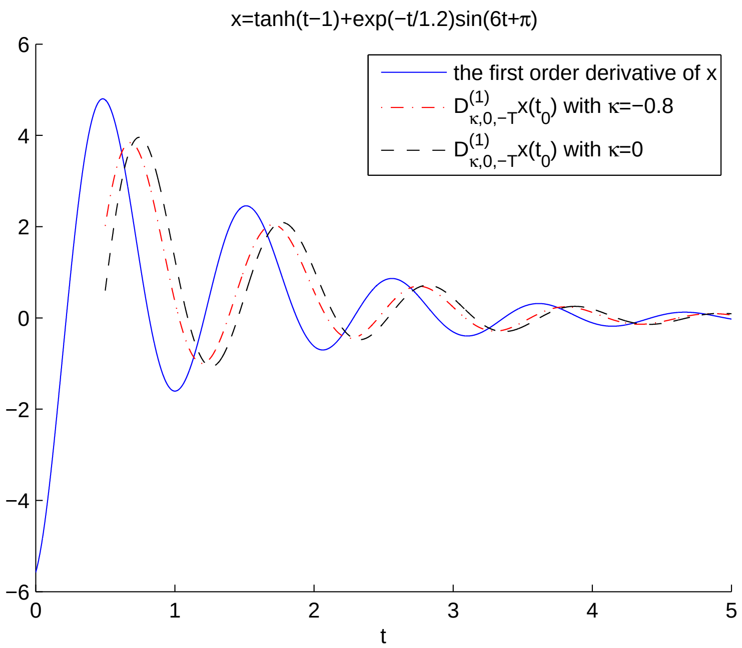
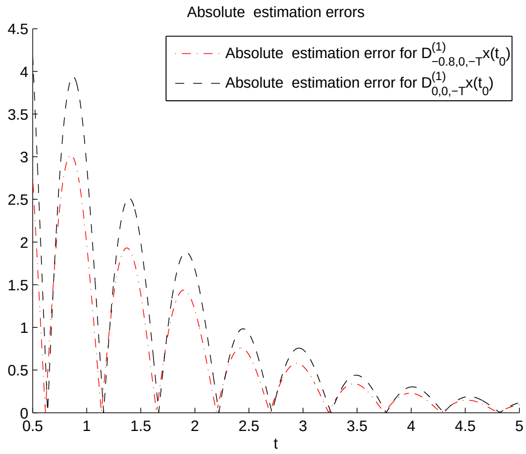
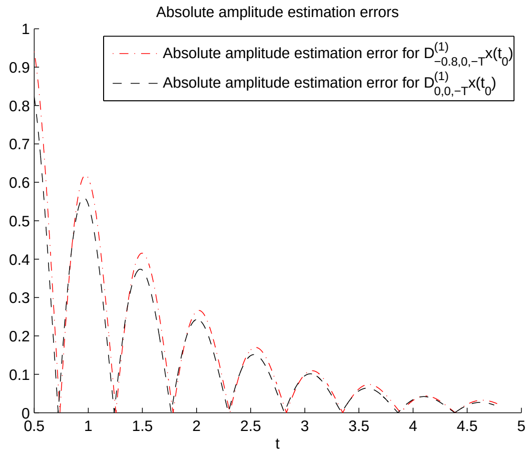


Figure 1.8: Estimations using $D^{(1)}_{\kappa,\mu,-T}x(t_0)$ with $\kappa = -0.8$ and $D^{(1)}_{\kappa,\mu,-T}x(t_0)$ with $\kappa = 0$ where $\mu = 0$ and $T = \frac{1}{2}$.



(a) Associated absolute truncated term errors.



(b) Associated absolute amplitude errors.

Figure 1.9: Estimation error for $D^{(1)}_{\kappa,0,-\frac{1}{2}}x(t_0)$ with $\kappa = -0.8$ and $D^{(1)}_{\kappa,0,-\frac{1}{2}}x(t_0)$ with $\kappa = 0$.

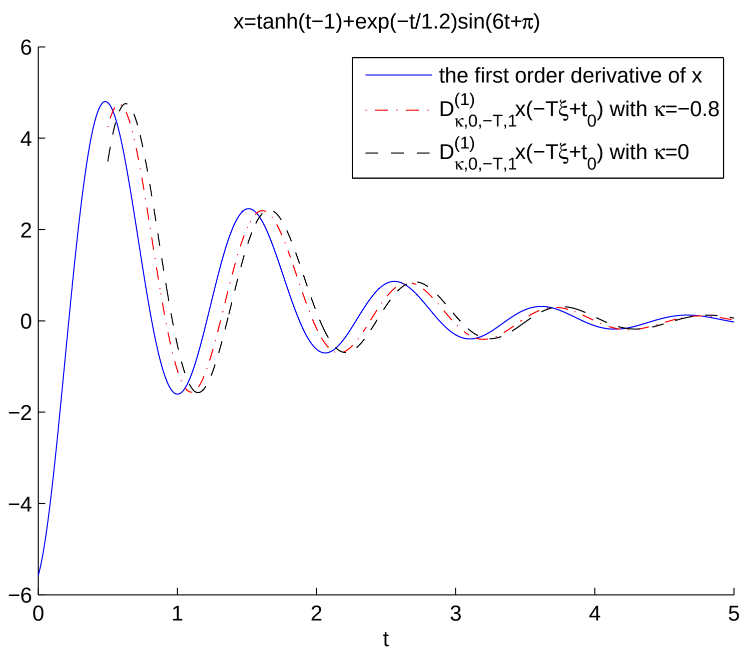
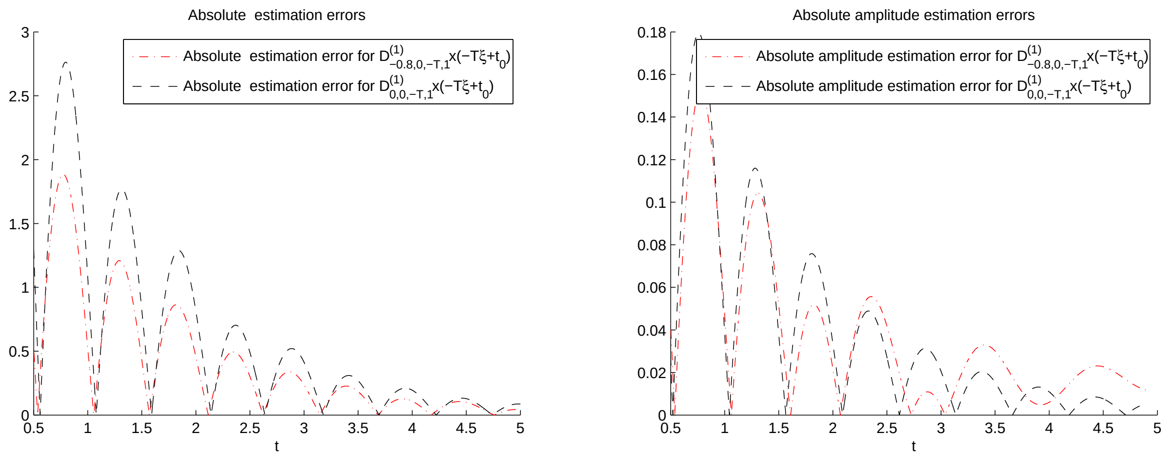


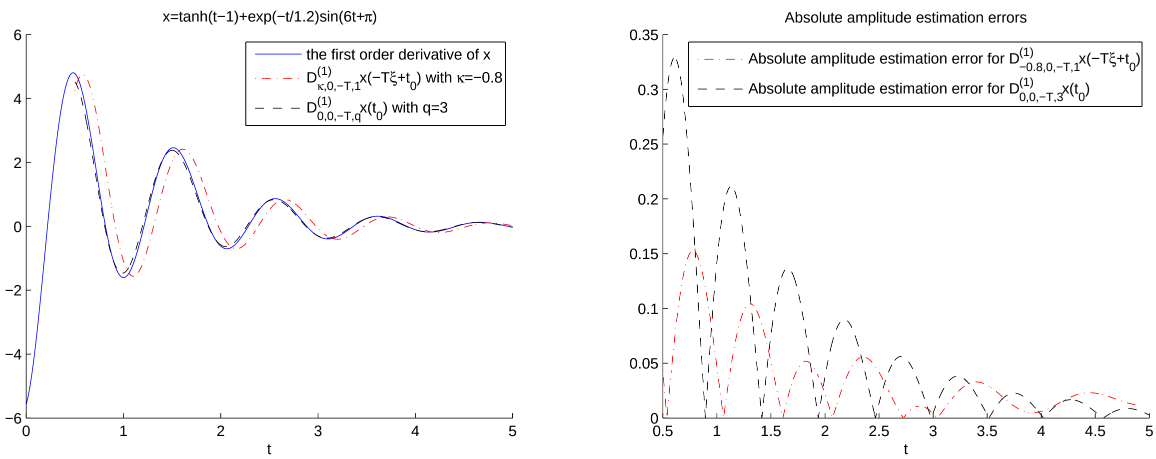
Figure 1.10: Estimations using $D^{(1)}_{\kappa,\mu,-T,q}x(-T\xi_1^{min}+t_0)$ with $\kappa=-0.8$, $\xi_1^{min}=0.182$ and $D^{(1)}_{\kappa,\mu,-T,q}x(-T\xi_1^{min}+t_0)$ with $\kappa=0$, $\xi_1^{min}=0.2764$ where $\mu=0$, $T=\frac{1}{2}$ and $q=1$.



(a) Associated absolute truncated term errors.

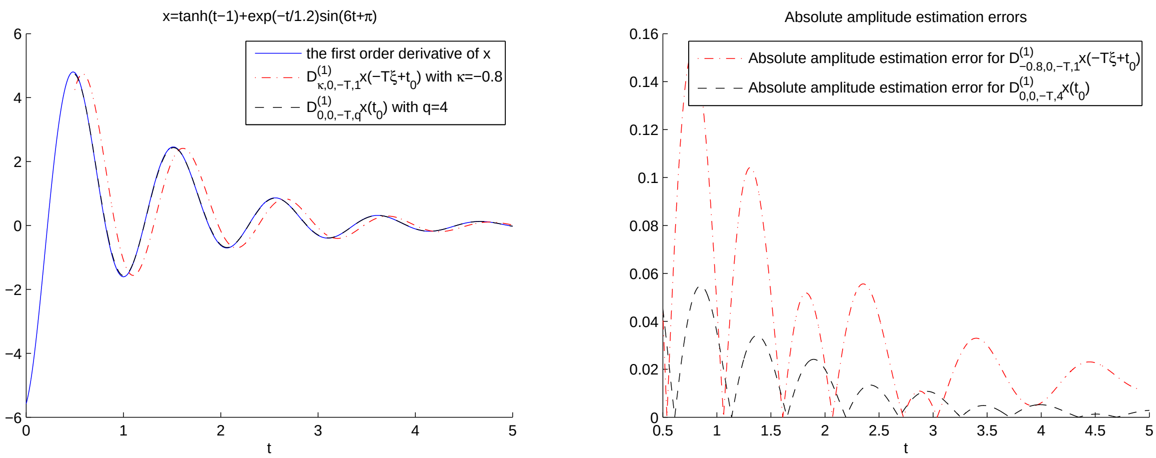
(b) Associated absolute amplitude errors.

Figure 1.11: Estimation error for $D^{(1)}_{\kappa,\mu,-T,q}x(-T\xi_1^{min}+t_0)$ with $\kappa=-0.8$, $\xi_1^{min}=0.182$ and $D^{(1)}_{\kappa,\mu,-T,q}x(-T\xi_1^{min}+t_0)$ with $\kappa=0$, $\xi_1^{min}=0.2764$ where $\mu=0$, $T=\frac{1}{2}$ and $q=1$.



(a) Estimations. (b) Associated absolute amplitude errors.

Figure 1.12: Comparison between $D^{(1)}_{\kappa,\mu,-T,q}x(-T\xi_1^{min}+t_0)$ with $\kappa=-0.8$, $\xi_1^{min}=0.182$, $q=1$ and $D^{(1)}_{\kappa,\mu,-T,q}x(t_0)$ with $\kappa=0$, $q=3$ where $\mu=0$, $T=\frac{1}{2}$.



(a) Estimations. (b) Associated absolute amplitude errors.

Figure 1.13: Comparison between $D^{(1)}_{\kappa,\mu,-T,q}x(-T\xi_1^{min}+t_0)$ with $\kappa=-0.8$, $\xi_1^{min}=0.182$, $q=1$ and $D^{(1)}_{\kappa,\mu,-T,q}x(t_0)$ with $\kappa=0$, $q=4$ where $\mu=0$, $T=\frac{1}{2}$.

1.4 Approximation theory

In this section, we recall some well-known approximation theories, and then we consider some special spaces to which the function x may belongs to.

1.4.1 Some contexts

Let us recall some well-known facts. Let us consider the subspace of $\mathcal{C}([0,1])$, defined by

$$\mathcal{P}_q(I) = \text{span}\left\{P_0^{(\mu+n,\kappa+n)}, P_1^{(\mu+n,\kappa+n)}, \cdots, P_q^{(\mu+n,\kappa+n)}\right\}, \tag{1.116}$$

where $\mathcal{P}_q(I)$ is the space of all the polynomials defined on I of degree below or equal to q. Equipped with the inner product $\langle\cdot,\cdot\rangle_{\mu+n,\kappa+n}$, $\mathcal{P}_q(I)$ is clearly an Hilbert reproducing kernel space with the reproducing kernel

$$\mathcal{K}_q(\tau,\xi) = \sum_{i=0}^{q} \frac{P_i^{(\mu+n,\kappa+n)}(\tau)P_i^{(\mu+n,\kappa+n)}(\xi)}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}}. \tag{1.117}$$

The reproducing property implies that for any function $x^{(n)}(t_0+\beta T\cdot)$ belonging to $\mathcal{C}([0,1])$, we have

$$\left\langle \mathcal{K}_q(\cdot,\xi), x^{(n)}(t_0+\beta T\cdot)\right\rangle_{\mu+n,\kappa+n} = D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi+t_0), \tag{1.118}$$

where $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\cdot+t_0)$ stands for the orthogonal projection of $x^{(n)}(t_0+\beta T\cdot)$ on $\mathcal{P}_q(I)$. Consequently, it is the best approximation of $x^{(n)}(t_0+\beta T\cdot)$ in $\mathcal{P}_q(I)$.

Hence, Similar to the classical approximation theory, the Jacobi estimator $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\cdot+t_0)$ denotes in fact a polynomial approximation of $x^{(n)}(t_0+\beta T\cdot)$. Parameters κ and μ give the coefficients of these polynomials, q is the order for these polynomials, βT determines the interval on which $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\cdot+t_0)$ approximate $x^{(n)}(t_0+\beta T\cdot)$. Parameter ξ determines at which point on this interval we take this estimate. Thus, our method is a point-wise derivative approximation.

1.4.2 Beppo-Levi space

In the previous sections, we study the convergence rate for the Jacobi estimators by considering the space $\mathcal{C}^{n+1+q}(I)$ with $n,q\in\mathbb{N}$. In this subsection, we consider the Beppo-Levi space which is defined as follows

$$\mathcal{H}^{n+1+q}(I) := \left\{x\in\mathcal{C}^{n+q}(I) \ \text{ such that } \ \int_I |x^{(n+1+q)}(\tau)|^2 d\tau < \infty\right\}, \tag{1.119}$$

with $n,q\in\mathbb{N}$. Then, we get the following relation

$$D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\cdot+t_0) \in \mathcal{P}_q(I) \subset \mathcal{C}^{1+q}(I) \subset \mathcal{H}^{1+q}(I) \subset \mathcal{C}(I). \tag{1.120}$$

Hence, the convergence rate for $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\cdot+t_0)$ depends on which space the function $x^{(n)}$ belongs to.

Now, we give the following proposition.

Proposition 1.4.21 *If $x \in \mathcal{H}^{n+1+q}(I)$ with $q \in \mathbb{N}$, then the Jacobi estimator of $x^{(n)}(t_0)$ by (1.84) satisfies:*

$$\forall t_0 \in I,\ D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) = x^{(n)}(t_0 + \beta T\xi) + \mathcal{O}\left(T^{q+\frac{1}{2}}\right). \tag{1.121}$$

Moreover, we have for any $\xi \in [0,1]$

$$\left\|D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) - x^{(n)}(t_0 + \beta T\xi)\right\|_\infty \leq \check{M}_{n+1+q}\check{C}_{\kappa,\mu,n,q,\xi} T^{q+\frac{1}{2}}, \tag{1.122}$$

where $\check{C}_{\kappa,\mu,n,q,\xi} = \frac{\left(\int_0^1 |Q_{\kappa,\mu,n,q,\xi}(\tau)\,\tau^{n+q+\frac{1}{2}}|d\tau\right)^{\frac{1}{2}}}{(n+q)!(2n+2q+1)^{\frac{1}{2}}} + \frac{\xi^{q+\frac{1}{2}}}{q!(2q+1)^{\frac{1}{2}}}$ *and* $\check{M}_{n+1+q} = \int_I \left|x^{(n+1+q)}(t)\right|^2 dt$.

Proof. Since $x \in \mathcal{H}^{n+1+q}(I)$, we have for any $t_0 \in I$

$$\forall \tau \in [0,1],\ x(t_0 + \beta T\tau) = x_{n+q}(t_0 + \beta T\tau) + \int_{I_{\beta T}} \frac{(t_0 + \beta T\tau - t)^{n+q}}{(n+q)!} x^{(n+1+q)}(t)dt, \tag{1.123}$$

where for $\beta = 1$ $I_{\beta T} = [t_0, t_0 + T\tau]$ (resp. $\beta = -1$, $I_{\beta T} = [t_0 - T\tau, t_0]$), $T \in D_{t_0}$, $x_{n+q}(t_0 + \beta T\tau) = \sum_{j=0}^{n+q} \frac{(\beta T\tau)^j}{j!} x^{(j)}(t_0)$. Then, by using (1.95) with $N = n + q$ we get

$$D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) - x^{(n)}(t_0 + \beta T\xi) = \frac{1}{(\beta T)^n} \int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau) \int_{I_{\beta T}} \frac{(t_0 + \beta T\tau - t)^{n+q}}{(n+q)!} x^{(n+1+q)}(t)dt\, d\tau.$$

By using the Cauchy-Schwarz inequality we get

$$\left|\int_{I_{\beta T}} \frac{(t_0 + \beta T\tau - t)^{n+q}}{(n+q)!} x^{(n+1+q)}(t)dt\right| \leq \frac{(T\tau)^{n+q+\frac{1}{2}}}{(n+q)!(2n+2q+1)^{\frac{1}{2}}} \left(\phi(\tau, t_0)\right)^{\frac{1}{2}},$$

where $\phi(\tau, t_0) = \int_{I_{\beta T}} \left|x^{(n+1+q)}(t)\right|^2 dt$. Hence, we get

$$\begin{aligned} &\left\|D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) - x^{(n)}_{n+q}(t_0 + \beta T\xi)\right\|_\infty \\ &\leq \frac{T^{q+\frac{1}{2}}}{(n+q)!(2n+2q+1)^{\frac{1}{2}}} \int_0^1 |Q_{\kappa,\mu,n,q,\xi}(\tau)|\ \tau^{n+q+\frac{1}{2}} \left(\phi(\tau, t_0)\right)^{\frac{1}{2}}\ d\tau. \end{aligned}$$

Since

$$\left\|x^{(n)}(t_0 + \beta T\xi) - x^{(n)}_{n+q}(t_0 + \beta T\xi)\right\|_\infty = \left\|\int_{I_{\beta T}} \frac{(t_0 + \beta T\xi - t)^q}{q!} x^{(n+1+q)}(t)dt\right\|_\infty \leq \frac{(T\xi)^{q+\frac{1}{2}}}{q!(2q+1)^{\frac{1}{2}}} \left(\phi(\xi, t_0)\right)^{\frac{1}{2}},$$

this proof can be completed by using the fact that $\phi(\tau, t_0) \leq \int_I \left|x^{(n+1+q)}(t)\right|^2 dt$. □

1.4.3 Generalized derivative estimators

In this subsection, we consider the case where $x \in \mathcal{C}^{n-1}(I)$ and the right and left hand derivatives for the n^{th} order exist. Then, we introduce some generalized derivative estimators for $x^{(n)}$ which converge to these one-sided derivatives.

Proposition 1.4.22 *Let $x \in \mathcal{H}^n(I)$ such that for any $t_0 \in I$ the right derivative $x^{(n)}(t_{0+})$ (resp. the left derivative $x^{(n)}(t_{0-})$) exists. Consequently, the Jacobi estimator $D^{(n)}_{\kappa,\mu,\beta T,q} x(t_0 + \beta T\xi)$ defined by (1.74) satisfies:*

$$\lim_{T \to 0^+} D^{(n)}_{\kappa,\mu,T,q} x(t_0 + T\xi) = x^{(n)}(t_{0+}), \quad \lim_{T \to 0^+} D^{(n)}_{\kappa,\mu,-T,q} x(t_0 - T\xi) = x^{(n)}(t_{0-}), \tag{1.124}$$

where q is an even integer.

Proof. Let us recall the local Taylor formula with the Peano remainder term (see [Zorich 2004] p. 219-232). For any given $\varepsilon' > 0$, there exists $\delta > 0$ such that for $0 < T\tau < \delta$ we have

$$\left| x(t_0 - T\tau) - x_{n-1}(t_0 - T\tau) - \frac{x^{(n)}(t_{0-})}{n!}(-T\tau)^n \right| < \varepsilon'(T\tau)^n, \tag{1.125}$$

and

$$\left| x(t_0 + T\tau) - x_{n-1}(t_0 + T\tau) - \frac{x^{(n)}(t_{0+})}{n!}(T\tau)^n \right| < \varepsilon'(T\tau)^n, \tag{1.126}$$

where $x_{n-1}(t_0 + T\tau)$ is the $(n-1)^{th}$ order truncated Taylor series expansion of $x(t_0 + T\tau)$.

Let us consider the function $g(t) = t^n$ the n^{th} order derivative of which is equal to $(n!)$. As g is an n^{th} order polynomial, then by substituting x_N by g in (1.95) and taking $t_0 = 0$ we get

$$\frac{1}{(\beta T)^n} \int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau)\, g(\beta T\tau)\, d\tau = (n!). \tag{1.127}$$

Hence, we have

$$\frac{1}{(-T)^n} \int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau) \frac{x^{(n)}(t_{0-})}{n!}(-T\tau)^n\, d\tau = x^{(n)}(t_{0-}), \tag{1.128}$$

and

$$\frac{1}{T^n} \int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau) \frac{x^{(n)}(t_{0+})}{n!}(T\tau)^n\, d\tau = x^{(n)}(t_{0+}). \tag{1.129}$$

Since x_{n-1} is an $(n-1)^{th}$ order polynomial, it is easy to obtain that

$$\forall t_0 \in I,\ \frac{1}{(\beta T)^n} \int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau)\, x_{n-1}(t_0 + \beta T\tau)\, d\tau = 0. \tag{1.130}$$

Hence, by using (1.128), (1.129) and (1.130) we obtain

$$\begin{aligned}
&\left|D^{(n)}_{\kappa,\mu,-T,q}x(t_0-T\xi)-x^{(n)}(t_{0-})\right| \\
\leq&\frac{1}{T^n}\int_0^1\left|Q_{\kappa,\mu,n,q,\xi}(\tau)\left(x(t_0-T\tau)-x_{n-1}(t_0-T\tau)-\frac{x^{(n)}(t_{0-})}{n!}(-T\tau)^n\right)\right|d\tau, \\
&\left|D^{(n)}_{\kappa,\mu,T,q}x(t_0+T\xi)-x^{(n)}(t_{0+})\right| \\
\leq&\frac{1}{T^n}\int_0^1\left|Q_{\kappa,\mu,n,q,\xi}(\tau)\left(x(t_0+T\tau)-x_{n-1}(t_0+T\tau)-\frac{x^{(n)}(t_{0+})}{n!}(T\tau)^n\right)\right|d\tau,
\end{aligned} \tag{1.131}$$

By using the expression of $Q_{\kappa,\mu,n,q,\xi}$ given in (1.85), we get

$$\int_0^1 |Q_{\kappa,\mu,n,q,\xi}(\tau)\tau^n|\,d\tau \leq \int_0^1\sum_{i=0}^{q} C_{\kappa,\mu,n,i}P_i^{(\mu+n,\kappa+n)}(\xi)\left|w_{\mu,\kappa}(\tau)P_{n+i}^{(\mu,\kappa)}(\tau)\tau^n\right|d\tau<\infty. \tag{1.132}$$

Consequently, for any $\varepsilon>0$, by using (1.131), (1.125) and (1.126) with $\varepsilon=\varepsilon'\int_0^1|Q_{\kappa,\mu,n,q,\xi}(\tau)\tau^n|\,d\tau$, there exists δ such that $0<T<\delta$ and

$$\left|D^{(n)}_{\kappa,\mu,-T,q}x(t_0-T\xi)-x^{(n)}(t_{0-})\right|<\varepsilon,\quad \left|D^{(n)}_{\kappa,\mu,T,q}x(t_0+T\xi)-x^{(n)}(t_{0+})\right|<\varepsilon.$$

Then, this proof can be completed. □

1.5 Some modified estimators

In Section 1.2, we introduce the minimal estimators $D^{(n)}_{\kappa,\mu,\beta T}x(t_0)$, the convergence rate of which is equal to $\mathcal{O}(T)$. In Section 1.3, by taking higher order truncated Taylor series expansion we improve this convergence rate by giving affine estimators $D^{(n)}_{\kappa,\mu,\beta T,q}x^{\delta}(\beta T\xi+t_0)$ where $q>0$. In this section, by studying on Taylor series expansion we give two new families of estimators which improve also the convergence rate of the minimal estimators.

1.5.1 Richardson extrapolation technique

We show in Subsection 1.2.2 that the minimal estimators $D^{(n)}_{\kappa,\mu,\beta T}x(t_0)$ given by (1.29) are obtained by applying the orthogonal properties of the Jacobi polynomials to the Taylor series expansion of x. In [Wang 2010], a family of derivative estimators were introduced by using Legendre orthogonal polynomials and by applying Richardson extrapolation technique. The Richardson extrapolation was proposed in 1927 and its historical background can be found in [Joyce 1971]. Similarly, we propose in the following proposition a new affine scheme by applying Richardson extrapolation technique.

Proposition 1.5.23 *Let $x\in\mathcal{C}^n(I)$, then a family of estimators for the derivative value $x^{(n)}(t_0)$ at any point $t_0\in I$ is given by*

$$D^{(n)}_{\kappa,\mu,\beta T,\lambda}x(t_0)=a_\lambda D^{(n)}_{\kappa,\mu,\beta T}x(t_0)+b_\lambda D^{(n)}_{\kappa,\mu,\beta\lambda T}x(t_0), \tag{1.133}$$

where $a_\lambda = \frac{-\lambda}{1-\lambda}$, $b_\lambda = \frac{1}{1-\lambda}$ *with* $\lambda \in \mathbb{R}_+/\{1\}$, $D^{(n)}_{\kappa,\mu,\beta T} x(t_0)$ *are the minimal estimators defined by* (1.29) *with* $T \in D_{t_0}$ *and* $\mu, \kappa \in]-1, +\infty[$. *If we assume that* $x \in \mathcal{C}^{n+2}(I)$, *then we have*

$$\forall t_0 \in I,\ D^{(n)}_{\kappa,\mu,\beta T,\lambda} x(t_0) = x^{(n)}(t_0) + \mathcal{O}\left(T^2\right). \tag{1.134}$$

Moreover, if there exists $M_{n+2} \in \mathbb{R}^*_+$ *such that* $\left\|x^{(n+2)}\right\|_\infty \leq M_{n+2}$, *then we have*

$$\left\| D^{(n)}_{\kappa,\mu,\beta T,\lambda} x(t_0) - x^{(n)}(t_0) \right\|_\infty \leq M_{n+2} C_{\kappa,\mu,n,\lambda} T^2, \tag{1.135}$$

where $C_{\kappa,\mu,n,\lambda} = \frac{|a_\lambda|+|b_\lambda|\lambda^2}{(n+1)(n+2)\mathrm{B}(n+\kappa+1,n+\mu+1)} \int_0^1 \left| w_{\mu,\kappa}(\tau) P^{(\mu,\kappa)}_n(\tau)\tau^{n+2}\right| d\tau$. *If* $\beta = -1$, *we call* $D^{(n)}_{\kappa,\mu,\beta T,\lambda} x(t_0)$ *causal Richardson-Jacobi estimators (resp. anti-causal Richardson-Jacobi estimators if* $\beta = 1$*).*

Proof. Assume that $x \in \mathcal{C}^{n+2}(I)$, then we take the Taylor series expansion of x at $t_0 \in I$. By using the well known Taylor's formula we have for any $T \in D_{t_0}$ there exists $\theta^\beta_{n+1,t_0} \in]t_0, t_0 + T[$ if $\beta = 1$ (resp. $\theta^\beta_{n+1,t_0} \in]t_0 - T, t_0[$ if $\beta = -1$) such that

$$\forall \tau \in [0,1],\ x(t_0 + \beta T\tau) = \sum_{j=0}^{n+1} \frac{(\beta T\tau)^j}{j!} x^{(j)}(t_0) + \frac{(\beta T\tau)^{n+2}}{(n+2)!} x^{(n+2)}(\theta^\beta_{n+1,t_0}). \tag{1.136}$$

Then, by using (1.29), we get

$$D^{(n)}_{\kappa,\mu,\beta T,\lambda} x(t_0) = \gamma_{\mu,\kappa,\beta T,n} \int_0^1 w_{\mu,\kappa}(\tau) P^{(\mu,\kappa)}_n(\tau) \left(a_\lambda x(\beta T\tau + t_0) + \frac{b_\lambda}{\lambda^n} x(\beta\lambda T\tau + t_0)\right) d\tau, \tag{1.137}$$

where $\gamma_{\mu,\kappa,\beta T,n} = \frac{n!}{(\beta T)^n} \frac{1}{\mathrm{B}(n+\kappa+1,\mu+n+1)}$. Substituting (1.136) into (1.137) and using the orthogonal properties of the Jacobi polynomial, we get

$$\gamma_{\mu,\kappa,\beta T,n} \int_0^1 w_{\mu,\kappa}(\tau) P^{(\mu,\kappa)}_n(\tau) \left(a_\lambda \frac{(\beta T\tau)^j}{j!} + \frac{b_\lambda}{\lambda^n} \frac{(\beta\lambda T\tau)^j}{j!}\right) d\tau = 0,\ \forall j < n.$$

Since $a_\lambda + b_\lambda = 1$ and $a_\lambda + b_\lambda\lambda = 0$, then we obtain

$$\gamma_{\mu,\kappa,\beta T,n} \int_0^1 w_{\mu,\kappa}(\tau) P^{(\mu,\kappa)}_n(\tau) \left(a_\lambda \frac{(\beta T\tau)^n}{n!} + b_\lambda \frac{(\beta T\tau)^n}{n!}\right) d\tau = 1,$$
$$\gamma_{\mu,\kappa,\beta T,n} \int_0^1 w_{\mu,\kappa}(\tau) P^{(\mu,\kappa)}_n(\tau) \left(a_\lambda \frac{(\beta T\tau)^{n+1}}{(n+1)!} + b_\lambda\lambda \frac{(\beta T\tau)^{n+1}}{(n+1)!}\right) d\tau = 0.$$

Thus, we get

$$\begin{aligned} & D^{(n)}_{\kappa,\mu,\beta T,\lambda} x(t_0) - x^{(n)}(t_0) \\ =& \gamma_{\mu,\kappa,\beta T,n} \int_0^1 w_{\mu,\kappa}(\tau) P^{(\mu,\kappa)}_n(\tau) \left(a_\lambda \frac{(\beta T\tau)^{n+2}}{(n+2)!} x^{(n+2)}(\theta^\beta_{n+1,t_0}) + b_\lambda\lambda^2 \frac{(\beta T\tau)^{n+2}}{(n+2)!} x^{(n+2)}(\bar{\theta}^\beta_{n+1,t_0})\right) d\tau, \end{aligned}$$

where $\bar{\theta}^\beta_{n+1,t_0} \in]t_0, t_0+\lambda T[$ if $\beta = 1$ (resp. $\bar{\theta}^\beta_{n+1,t_0} \in]t_0-\lambda T, t_0[$ if $\beta = -1$). Consequently, $D^{(n)}_{\kappa,\mu,\beta T,\lambda} x(t_0)$ can be considered as a family of estimators for $x^{(n)}(t_0)$. Moreover, we have

$$\left\| D^{(n)}_{\kappa,\mu,\beta T,\lambda} x(t_{0\pm}) - x^{(n)}(t_0) \right\|_\infty \leq M_{n+2} \left(|a_\lambda| + |b_\lambda|\lambda^2\right) \frac{T^{n+2}}{(n+2)!} \gamma_{\mu,\kappa,\beta T,n} \int_0^1 \left| w_{\mu,\kappa}(\tau) P^{(\mu,\kappa)}_n(\tau)\tau^{n+2}\right| d\tau.$$

Thus, this proof is completed. □

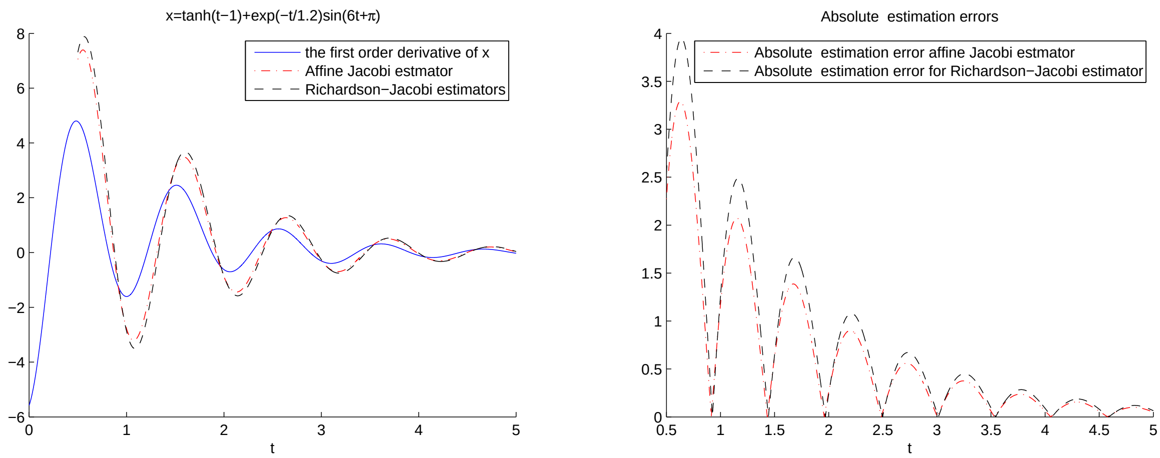
Since the Richardson-Jacobi estimators $D^{(n)}_{\kappa,\mu,\beta T,\lambda}x(t_0)$ are an affine combination of minimal Jacobi estimators with $a_\lambda + b_\lambda = 1$. Then, by applying Proposition 1.4.22, we get the following corollary.

Corollary 1.5.24 *Let $x \in \mathcal{C}^{n-1}(I)$, then the Richardson-Jacobi estimator $D^{(n)}_{\kappa,\mu,\beta T,\lambda}x(t_0)$ defined by (1.133) in Proposition 1.5.23 with $T \in D_{t_0}$, $\kappa, \mu \in]-1, +\infty[$ is a generalized derivative estimator for $x^{(n)}$. Moreover, if $x^{(n)}(t_{0+})$ and $x^{(n)}(t_{0-})$ exist at any point $t_0 \in I$, then we have*

$$\lim_{T\to 0^+} D^{(n)}_{\kappa,\mu,T,\lambda}x(t_0) = x^{(n)}(t_{0+}), \quad \textit{and} \quad \lim_{T\to 0^+} D^{(n)}_{\kappa,\mu,-T,\lambda}x(t_0) = x^{(n)}(t_{0-}), \tag{1.138}$$

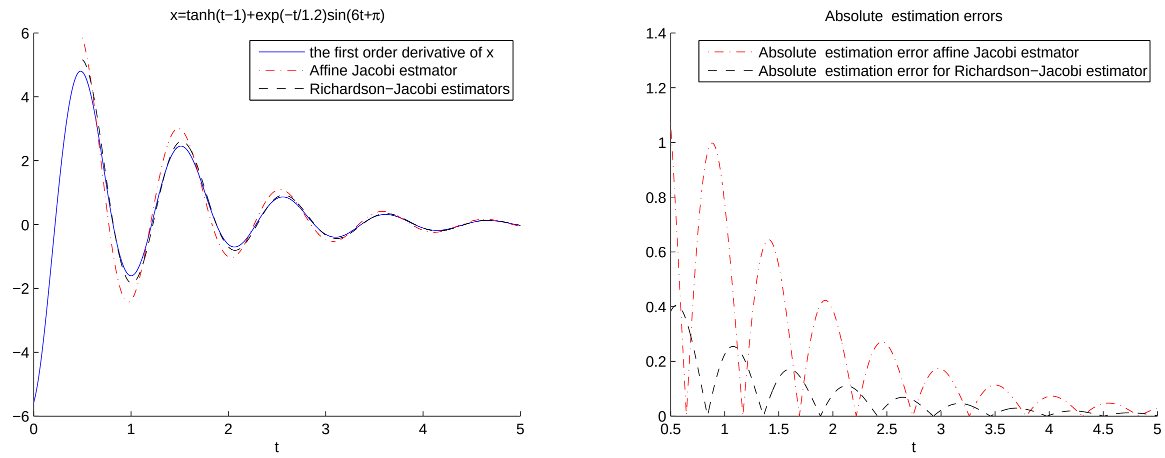
where $x^{(n)}(t_{0+})$ (resp. $x^{(n)}(t_{0-})$) denotes the right (resp. left) hand-side derivative for the n^{th} order.

Now, let us take the sampling data of function x defined by (1.8) with a sampling period $T_s = \frac{1}{2000}$. Then, we use causal affine Jacobi estimators with $q = 1, 2$ respectively and causal Richardson-Jacobi estimator to estimate the first order derivative of x. We can see the obtained estimations in Figure 1.14(a) and Figure 1.15(a). The associated estimation errors are given in Figure 1.14(b) and Figure 1.15(b). We can observe that when $\lambda = 0.8$ the estimation error for the Richardson-Jacobi estimator is larger than the one for the affine Jacobi estimator with $q = 1$. When $\lambda = 0.08$, the estimation error for the Richardson-Jacobi estimator is smaller than the one for the affine Jacobi estimator with $q = 2$. Consequently, the Richardson-Jacobi estimator can be improved by reducing the value of λ. However, we will see in Chapter 2, this can increase the noise error contribution.



(a) Estimations. (b) Associated absolute estimation errors.

Figure 1.14: Affine Jacobi estimator $D^{(1)}_{\kappa,\mu,-T,q}x(t_0)$ with $q = 1$ and Richardson-Jacobi estimator $D^{(n)}_{\kappa,\mu,-T,\lambda}x(t_0)$ with $\lambda = 0.8$, where $\kappa = \mu = 0$ and $T = \frac{1}{2}$.



(a) Estimations. (b) Associated absolute estimation errors.

Figure 1.15: Affine Jacobi estimator $D^{(1)}_{\kappa,\mu,-T,q}x(t_0)$ with $q = 2$ and Richardson-Jacobi estimator $D^{(n)}_{\kappa,\mu,-T,\lambda}x(t_0)$ with $\lambda = 0.08$, where $\kappa = \mu = 0$ and $T = \frac{1}{2}$.

1.5.2 A new Remainder in Taylor's Formula

We introduce in the following proposition a family of modified minimal estimators which are given by using the following new Remainder in Taylor's Formula [Poffald 1990]:

$$x(\beta T\tau + t_0) = x_n(\beta T\tau + t_0) + \frac{(\beta T\tau)^{n+1}}{(n+1)!}x^{(n+1)}\left(t_0 + \frac{\beta T\tau}{n+2}\right) + \frac{n+1}{2(n+2)}\frac{(\beta T\tau)^{n+3}}{(n+3)!}x^{(n+3)}(\theta^{\beta}_{n+3,t_0}), \tag{1.139}$$

where $x_n(\beta T\tau + t_0) = \sum_{i=0}^{n} \frac{(\beta T\tau)^i}{i!}x^{(i)}(t_0)$ with $\tau \in [0,1]$, $T \in D_{t_0}$, and $\theta^{\beta}_{n+3,t_0} \in]t_0 - T\tau, t_0[$ if $\beta = -1$ (resp. $\theta^{\beta}_{n+3,t_0} \in]t_0, T\tau + t_0[$ if $\beta = -1$).

Proposition 1.5.25 *Let $x \in \mathcal{C}^n(I)$, then a family of estimators for the derivative value $x^{(n)}(t_0)$ at any point $t_0 \in I$ is given by*

$$E^{(n)}_{\kappa,\mu,\beta T}x(t_0) = D^{(n)}_{\kappa,\mu,\beta T}x(t_0) + E_{\kappa,\mu,\beta T,n}x(t_0), \tag{1.140}$$

where $D^{(n)}_{\kappa,\mu,\beta T}x(t_0)$ are the minimal estimators defined by (1.29) with $T \in D_{t_0}$, $n < \mu \in \mathbb{R}$, $-1 < \kappa \in \mathbb{R}$. Consequently,

$$E_{\kappa,\mu,\beta T,n}x(t_0) = \gamma_{\mu,\kappa,\beta T,n}\frac{(n+2)^{n+1}}{(n+1)!}\int_0^1 \sum_{j=0}^{n}\sum_{l=0}^{n+1} C_{\kappa,\mu,n+1,j,l}\, w_{\mu+n-j-l,\kappa+j+l}(\tau)\, x\left(t_0 + \frac{\beta T\tau}{n+2}\right) d\tau,$$

where $\gamma_{\mu,\kappa,\beta T,n}$ is defined by (1.30), and $C_{\kappa,\mu,n+1,j,l} = (-1)^{n-j+l}\binom{n+\mu}{j}\binom{n+\kappa}{n-j}\frac{\Gamma(\mu+n-j+1)}{(l!)^2}\frac{\Gamma(\kappa+n+j+2)}{((n+1-l)!)^2}$. If we assume that $x \in \mathcal{C}^{n+3}(I)$, then we have

$$\forall t_0 \in I,\ E^{(n)}_{\kappa,\mu,\beta T}x(t_0) = x^{(n)}(t_0) + \mathcal{O}\left(T^3\right). \tag{1.141}$$

Moreover, if there exists $M_{n+3} \in \mathbb{R}_+^*$ such that $\left\| x^{(n+3)} \right\|_\infty \leq M_{n+3}$, then we have

$$\left\| E_{\kappa,\mu,\beta T}^{(n)} x(t_0) - x^{(n)}(t_0) \right\|_\infty \leq M_{n+3} C_{\kappa,\mu,n} T^3, \tag{1.142}$$

where $C_{\mu,\kappa,n} = \frac{1}{2(n+2)^2(n+3)\mathrm{B}(n+\kappa+1,\mu+n+1)} \int_0^1 \left| w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau) \tau^{n+3} \right| d\tau.$

Proof. Assume that $x \in \mathcal{C}^{n+3}(I)$, then by substituting (1.139) into (1.29) and using the orthogonal properties of the Jacobi polynomial, we get

$$\begin{aligned} & D_{\kappa,\mu,\beta T}^{(n)} x(t_0) - x^{(n)}(t_0) + E_{\kappa,\mu,\beta T,n} x(t_0) \\ = & \gamma_{\mu,\kappa,\beta T,n} \frac{n+1}{2(n+2)} \frac{(\beta T)^{n+3}}{(n+3)!} \int_0^1 w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau) \tau^{n+3} x^{(n+3)}(\theta_{n+3,t_0}^{\beta}) d\tau, \end{aligned} \tag{1.143}$$

where

$$E_{\kappa,\mu,\beta T,n} x(t_0) = \gamma_{\mu,\kappa,\beta T,n} \frac{(\beta T)^{n+1}}{(n+1)!} \int_0^1 w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau) \tau^{n+1} x^{(n+1)} \left(t_0 + \frac{\beta T \tau}{n+2} \right) d\tau. \tag{1.144}$$

Then, we apply $n+1$ times integrations by parts in (1.144) so as to get $x(\cdot)$ in the integral of $E_{\kappa,\mu,\beta T,n} x(t_0)$. Let us calculate the i^{th} order derivative of $w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau) \tau^{n+1}$ for $i = 0, \cdots, n+1$, then by using (1.42), we obtain

$$\begin{aligned} & \frac{d^i}{d\tau^i} \left\{ w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau) \tau^{n+1} \right\} \\ = & \frac{d^i}{d\tau^i} \left\{ \sum_{j=0}^{n} \binom{n+\mu}{j} \binom{n+\kappa}{n-j} (-1)^{n-j} w_{\mu+n-j,\kappa+j+n+1}(\tau) \right\} \\ = & \sum_{j=0}^{n} \binom{n+\mu}{j} \binom{n+\kappa}{n-j} (-1)^{n-j} \frac{d^i}{d\tau^i} \left\{ w_{\mu+n-j,\kappa+j+n+1}(\tau) \right\}. \end{aligned}$$

By assuming that $n < \mu \in \mathbb{R}$, we get

$$\frac{d^i}{d\tau^i} \left\{ w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau) \tau^{n+1} \right\} = \sum_{j=0}^{n} \sum_{l=0}^{i} C_{\kappa,\mu,i,j,l} (1-\tau)^{\mu+n-j-l} \tau^{\kappa+n+1+j+l-i}, \tag{1.145}$$

where $C_{\kappa,\mu,i,j,l} = (-1)^{n-j+l} \binom{n+\mu}{j} \binom{n+\kappa}{n-j} \binom{i}{l} \frac{\Gamma(\mu+n-j+1)}{l!} \frac{\Gamma(\kappa+n+j+2)}{(i-l)!}$.

Let us apply $n+1$ times integrations by parts in (1.144). Since $n < \mu \in \mathbb{R}$, $\mu + n - j - l > 0$ and $\kappa + n + 1 + j + l - i > 0$ for $i = 0, \cdots, n$, such that all the boundary values are equal to zeros. Hence, we obtain

$$E_{\kappa,\mu,\beta T,n} x(t_0) = \gamma_{\mu,\kappa,\beta T,n} \frac{(n+2)^{n+1}}{(n+1)!} \int_0^1 \sum_{j=0}^{n} \sum_{l=0}^{n+1} C_{\kappa,\mu,n+1,j,l} w_{\mu+n-j-l,\kappa+j+l}(\tau) x \left(t_0 + \frac{\beta T \tau}{n+2} \right) d\tau.$$

Since $E_{\kappa,\mu,\beta T,n} x(t_0)$ is an integral of $x(\cdot)$, we define $E_{\kappa,\mu,\beta T}^{(n)} x(t_0) = D_{\kappa,\mu,\beta T}^{(n)} x(t_0) + E_{\kappa,\mu,\beta T,n} x(t_0)$ as the estimators for $x^{(n)}(t_0)$, the truncated term error for which is given by (1.143). Then, this proof can be easily completed.

□

1.6 Central estimators

In the previous sections, by applying algebraic parametric techniques to truncated Taylor series, we introduce causal Jacobi estimators (resp. anti-causal Jacobi estimators) which are based on the integration window $[t_0-T, t_0]$ (resp. $[t_0, t_0+T]$) for any $t_0 \in I$ and $T \in D_{t_0}$. They are extended by taking truncated Jacobi orthogonal series expansion. Let us recall that the Jacobi estimators produce a time-drift so as to get a small bias term error. The aim of this section is to introduce some drift-free estimators by using the integration window $[t_0-T, t_0+T]$ for any $t_0 \in I$ and $T \in \hat{D}_{t_0} = \{t \in \mathbb{R}_+; [t_0-t, t_0+t] \in I\}$.

1.6.1 Combination of causal and anti-causal estimators

It is shown in Subsection 1.3.3 that we estimate $x^{(n)}(t_0)$ by the causal Jacobi estimators $D^{(n)}_{\kappa,\mu,-T,q}x(-T\xi+t_0)$ (resp. anti-causal Jacobi estimators $D^{(n)}_{\kappa,\mu,T,q}x(T\xi+t_0)$), the drift term errors produce a time-delay (resp. time-advance) of value $T\xi$. In this subsection, we give a family of estimators which are based on a combination of causal and anti-causal Jacobi estimators so as to reduce the bias term errors by avoiding a time-drift.

Let $x \in \mathcal{C}^{N+1}(I)$, where $n \leq N \in \mathbb{N}$. For any $t_0 \in I$, we consider the two following functions

$$\forall t \in \hat{D}_{t_0},\ X^-(t+t_0) = \frac{1}{2}(x(t+t_0) - x(t_0-t)) \quad \text{and} \quad X^+(t+t_0) = \frac{1}{2}(x(t+t_0) + x(t_0-t)). \tag{1.146}$$

By taking the N^{th} order truncated Taylor series expansion of x, we obtain

$$X_N^-(t+t_0) = \frac{1}{2}\left(x_N(t+t_0) - x_N(-t+t_0)\right) \quad \text{and} \quad X_N^+(t+t_0) = \frac{1}{2}\left(x_N(t+t_0) + x_N(-t+t_0)\right),$$

where $x_N(t_0 \pm t) = \sum_{j=0}^{N} \frac{(\pm t)^j}{j!} x^{(j)}(t_0)$. Hence, if N is odd, then we have

$$X_N^-(t+t_0) = \sum_{i=0}^{\frac{N-1}{2}} \frac{t^{2i+1}}{(2i+1)!} x^{(2i+1)}(t_0) \quad \text{and} \quad X_N^+(t+t_0) = \sum_{i=0}^{\frac{N-1}{2}} \frac{t^{2i}}{(2i)!} x^{(2i)}(t_0), \tag{1.147}$$

and if N is even, then we have

$$X_N^-(t+t_0) = \sum_{i=0}^{\frac{N}{2}-1} \frac{t^{2i+1}}{(2i+1)!} x^{(2i+1)}(t_0) \quad \text{and} \quad X_N^+(t+t_0) = \sum_{i=0}^{\frac{N}{2}} \frac{t^{2i}}{(2i)!} x^{(2i)}(t_0). \tag{1.148}$$

Thus, X_N^- only contains the values of the odd order derivatives at t_0 and X_N^+ only contains the values of the even order derivatives at t_0. Hence, similarly to Proposition 1.3.9, by applying the algebraic parametric techniques we can give a family of estimators as follows.

Proposition 1.6.26 *Let $x \in \mathcal{C}^{N+1}(I)$, where $n \leq N \in \mathbb{N}$, then a family of estimators $\tilde{x}^{(n)}_{t_0}(k, \mu, T, N)$ for the derivative value of $x^{(n)}$ at any point $t_0 \in I$ is given by*

$n \setminus N$	even
even	$\frac{1}{2}\left(\tilde{x}^{(n)}_{t_0+}(k,\mu,T,N)+\tilde{x}^{(n)}_{t_0-}(k,\mu,-T,N)\right)$
odd	$\frac{1}{2}\left(\tilde{x}^{(n)}_{t_0+}(k-1,\mu,T,N-1)+\tilde{x}^{(n)}_{t_0-}(k-1,\mu,-T,N-1)\right)$

$n \setminus N$	odd
even	$\frac{1}{2}\left(\tilde{x}^{(n)}_{t_0+}(k,\mu,T,N-1)+\tilde{x}^{(n)}_{t_0-}(k,\mu,-T,N-1)\right)$
odd	$\frac{1}{2}\left(\tilde{x}^{(n)}_{t_0+}(k-1,\mu,T,N)+\tilde{x}^{(n)}_{t_0-}(k-1,\mu,-T,N)\right)$

where $\tilde{x}^{(n)}_{t_{0\pm}}(k,\mu,\beta T,N)$ *are the affine estimators defined by Proposition 1.3.9 with* $T \in \hat{D}_{t_0}$*,* $k \in \mathbb{N}$ *and* $-1 < \mu \in \mathbb{R}$*.*

Proof. Let us consider the case where n is odd and N is even. Thus, the Laplace transform of X_N^- is given by

$$\hat{X}_N^-(s) = s^{-2}x^{(1)}(t_0) + s^{-4}x^{(3)}(t_0) + \cdots + s^{-N}x^{(N-1)}(t_0). \tag{1.149}$$

We proceed to annihilate in (1.149) the terms containing $x^{(i)}(t_0)$ with $j \neq n$ and preserve the term containing $x^{(n)}(t_0)$. Since n and N have not the same parity, X_N^- which does not contain the term $x^{(N)}(t_0)$ is equal to an $(N-1)^{th}$ order truncated Taylor series expansion of x. Thus, if we multiply $\hat{X}_N^-$ by s^N, $\hat{X}_N^-$ becomes a polynomial of degree $N-2$. Then the terms of degree lower than $N-n-1$, which include $x^{(i)}(t_0)$ with $n < i < N$, are annihilated by applying $N-1-n$ times derivations. In order to preserve the term including $x^{(n)}(t_0)$, we multiply the remaining polynomial by $\frac{1}{s}$. Then we apply more than $n-1$ times derivations with respect to s such that the other terms including $x^{(i)}(t_0)$ with $0 < i < n-1$, are annihilated. Finally, we multiply by $\frac{1}{s^{N+\mu}}$ to return into the time domain where $-1 < \mu \in \mathbb{R}$. For this, we apply the following annihilator

$$\Pi^{N-1,n}_{k-1,\mu} = \frac{1}{s^{N+\mu}} \cdot \frac{d^{(n-1)+k}}{ds^{(n-1)+k}} \cdot \frac{1}{s} \cdot \frac{d^{(N-1)-n}}{ds^{(N-1)-n}} \cdot s^N. \tag{1.150}$$

Similar to the proof of Proposition 1.3.9, we can obtain

$$\tilde{x}^{(n)}_{t_0}(k,\mu,T,N) = \frac{1}{T^n}\int_0^1 a_{k-1,\mu,n,N-1}\sum_{i=0}^{N-1-n} b_{n,N-1,i}K_{k-1,\mu,n,N-1,i}(\tau)\frac{1}{2}\left(x(T\tau+t_0)-x(-T\tau+t_0)\right)d\tau,$$

where $a_{k-1,\mu,n,N-1}$, $b_{n,N-1,i}$ and $K_{k-1,\mu,n,N-1,i}$ are defined in Proposition 1.3.9. By using (1.55), we get

$$\begin{aligned}\tilde{x}^{(n)}_{t_0}(k,\mu,T,N) &= \frac{1}{2}\left(\tilde{x}^{(n)}_{t_0+}(k-1,\mu,T,N-1)-(-1)^n\tilde{x}^{(n)}_{t_0-}(k-1,\mu,-T,N-1)\right)\\ &= \frac{1}{2}\left(\tilde{x}^{(n)}_{t_0+}(k-1,\mu,T,N-1)+\tilde{x}^{(n)}_{t_0-}(k-1,\mu,-T,N-1)\right), \quad \text{since } n \text{ is odd.}\end{aligned}$$

The calculations in the other cases are similar with the following annihilators: $\Pi^{N,n}_{k-1,\mu}$ (if n and N are odd), $\Pi^{N-1,n}_{k,\mu}$ (if n is even and N is odd), $\Pi^{N,n}_{k,\mu}$ (if n and N are even). □

Remark 3 *When* $k = 0$*, since* $n \geq 1$*, we can observe that* $\tilde{x}^{(n)}_{t_{0\pm}}(-1,\mu,\pm T,N)$ *are well defined in (1.55).*

The estimators $\tilde{x}_{t_0}^{(n)}(k,\mu,T,N)$ are calculated on the interval $[t_0-T,t_0+T]$ with $T\in\hat{D}_{t_0}$. Hence they are central estimators.

Let us take the same parity for n and N in $\tilde{x}_{t_0}^{(n)}(k,\mu,T,N)$ defined in Proposition 1.6.26, then we have

$$\tilde{x}_{t_0}^{(n)}(k,\mu,T,N)=\begin{cases}\frac{1}{2}\left(\tilde{x}_{t_{0+}}^{(n)}(k,\mu,T,N)+\tilde{x}_{t_{0-}}^{(n)}(k,\mu,-T,N)\right), & \text{if } n \text{ and } N \text{ are even},\\ \frac{1}{2}\left(\tilde{x}_{t_{0+}}^{(n)}(k-1,\mu,T,N)+\tilde{x}_{t_{0-}}^{(n)}(k-1,\mu,-T,N)\right), & \text{if } n \text{ and } N \text{ are odd}.\end{cases} \tag{1.151}$$

If we take $n=N$ in (1.151), then by denoting $\tilde{x}_{t_0}^{(n)}(k,\mu,T,N)$ by $\tilde{x}_{t_0}^{(n)}(k,\mu,T)$ we get

$$\tilde{x}_{t_0}^{(n)}(k,\mu,T)=\frac{1}{2}\left(\tilde{x}_{t_{0+}}^{(n)}(k_n,\mu,T)+\tilde{x}_{t_{0-}}^{(n)}(k_n,\mu,-T)\right), \tag{1.152}$$

where $k_n=k-\frac{1}{2}(1-(-1)^n)$ and $\tilde{x}_{t_{0\pm}}^{(n)}(k_n,\mu,\pm T)$ are the minimal estimators defined in Proposition 1.3.9 by taking $N=n$ as follows

$$\tilde{x}_{t_{0\pm}}^{(n)}(k_n,\mu,\beta T)=\frac{1}{(\beta T)^n}\int_0^1 a_{k_n,\mu,n}K_{k_n,\mu,n}(\tau)\,x(\beta T\tau+t_0)d\tau, \tag{1.153}$$

where $a_{k_n,\mu,n}=(-1)^{n+k_n}\frac{\Gamma(2n+\mu+k_n+2)}{(n+k_n)!}$,

$$K_{k_n,\mu,n}(\tau)=\sum_{j=\max(0,k_n)}^{n+k_n}\frac{(-1)^j}{\Gamma(\mu+k_n+n+1-j)}\binom{n+k_n}{j}\frac{n!}{(j-k_n)!}\;w_{\mu+k_n+n-j,j}(\tau), \tag{1.154}$$

with $k\in\mathbb{N}$, $-1<\mu\in\mathbb{R}$ and $T\in\hat{D}_{t_0}$. Let us remark that if n is even, then by applying a change of index in (1.154): $j\to j+k$, we can obtain the same formula for the minimal estimators given by Proposition 1.2.1. If n is odd, then the estimators $\tilde{x}_{t_{0\pm}}^{(n)}(-1,\mu,\beta T)$ are well defined by (1.153), which are not defined by Proposition 1.2.1. Then, by using Proposition 1.3.11 we get the following corollary.

Corollary 1.6.27 *Let $x\in\mathcal{C}^{N+1}(I)$ with $n\le N\in\mathbb{N}$ and $(-1)^{n+N}=1$, $\tilde{x}_{t_0}^{(n)}(k,\mu,T,N)$ be the central estimators given in Proposition 1.6.26 and $\tilde{x}_{t_0}^{(n)}(k,\mu,T)$ be the estimators given in (1.152), then we have*

$$\tilde{x}_{t_0}^{(n)}(k,\mu,T,N)=\sum_{j=0}^{q}\lambda_{j,k_n}\;\tilde{x}_{t_0}^{(n)}(k_j,\mu_j,T), \tag{1.155}$$

where $k_j=k+q-j$, $\mu_j=\mu+j$, and λ_{j,k_n} defined in Proposition 1.3.11.

Consequently, $\tilde{x}_{t_0}^{(n)}(k,\mu,T,N)$ can be considered as affine central estimators which can be written as an affine combination of some minimal central estimators $\tilde{x}_{t_0}^{(n)}(k,\mu,T)$. Let us recall that the affine estimators $\tilde{x}_{t_{0\pm}}^{(n)}(k,\mu,\beta T,N)$ obtained by using the algebraic parametric techniques are extended in Subsection 1.3.2 by $D_{\kappa,\mu,\beta T,q}^{(n)}x(\beta T\xi+t_0)$ obtained by taking truncated Jacobi series expansion. Similarly, we extend $\tilde{x}_{t_0}^{(n)}(k_n,\mu,T,N)$ by taking the following truncated Jacobi series expansions

$$D^{(n)}_{\kappa,\mu,T,q,\xi}x(t_0) := \begin{cases} \displaystyle\sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), (X^-)^{(n)}(T\cdot+t_0)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}} P_i^{(\mu+n,\kappa+n)}(\xi), & \text{if } n \text{ is odd,} \\ \displaystyle\sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), (X^+)^{(n)}(T\cdot+t_0)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}} P_i^{(\mu+n,\kappa+n)}(\xi), & \text{if } n \text{ is even,} \end{cases} \tag{1.156}$$

Then, we give the following proposition.

Proposition 1.6.28 *Let $x \in \mathcal{C}^n(I)$ and $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi+t_0)$ be the Jacobi estimators defined by (1.74), then a family of central estimators are given by*

$$D^{(n)}_{\kappa,\mu,T,q,\xi}x(t_0) = \frac{1}{2}\left(D^{(n)}_{\kappa,\mu,T,q}x(T\xi+t_0) + D^{(n)}_{\kappa,\mu,-T,q}x(-T\xi+t_0)\right), \tag{1.157}$$

where $T \in \hat{D}_{t_0}$, $\mu,\kappa \in]-1,+\infty[$ and $\xi \in [0,1]$.

Proof. If n is an odd integer, then by using (1.156) and (1.146) we get

$$\begin{aligned} D^{(n)}_{\kappa,\mu,T,q,\xi}x(t_0) =& \sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), \frac{1}{2}\left(x^{(n)}(t_0+T\cdot) - (-1)^n x^{(n)}(t_0-T\cdot)\right)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}} P_i^{(\mu+n,\kappa+n)}(\xi) \\ =& \frac{1}{2}\sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), x^{(n)}(t_0+T\cdot)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}} P_i^{(\mu+n,\kappa+n)}(\xi) \\ &+ \frac{(-1)^{n+1}}{2}\sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), x^{(n)}(t_0-T\cdot)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}} P_i^{(\mu+n,\kappa+n)}(\xi) \\ =& \frac{1}{2}\left(D^{(n)}_{\kappa,\mu,T,q}x(T\xi+t_0) + D^{(n)}_{\kappa,\mu,-T,q}x(-T\xi+t_0)\right). \end{aligned}$$

Hence, this proof can be completed by similar calculations for the case where n is an even integer. □

Consequently, $D^{(n)}_{\kappa,\mu,T,q,\xi}x(t_0)$ is the extension of $\tilde{x}^{(n)}_{t_0}(k,\mu,T,N)$ in the case where n and N are even integers, it is the extension of $\tilde{x}^{(n)}_{t_0}(k+1,\mu,T,N)$ in the case where n and N are odd integers. Then, by applying Proposition 1.3.16 we get the expression of $D^{(n)}_{\kappa,\mu,T,q,\xi}x(t_0)$ in the following corollary.

Corollary 1.6.29 *Let $D^{(n)}_{\kappa,\mu,T,q,\xi}x(t_0)$ be the estimators for $x^{(n)}(t_0)$ given by (1.157), then for any $t_0 \in I$ we have*

$$D^{(n)}_{\kappa,\mu,T,q,\xi}x(t_0) = \frac{1}{2T^n}\int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau)\left(x(t_0+T\tau) + (-1)^n x(t_0-T\tau)\right)d\tau, \tag{1.158}$$

where $T \in \hat{D}_{t_0}$, $\kappa,\mu \in]-1,+\infty[$, $q \in \mathbb{N}$, $\xi \in [0,1]$ and $Q_{\kappa,\mu,n,q,\xi}(\tau)$ is defined by (1.85).

According to Proposition 1.3.17, the convergence rate of the Jacobi estimators $D^{(n)}_{\kappa,\mu,\beta T,q}x(t_0)$ for $x^{(n)}$ at $t = t_0$ is $\mathcal{O}\left(T^{q+1}\right)$. Since the Taylor series expansion of X^- (resp. X^+) contains only the odd (resp. even) order derivative values, similarly to Proposition 1.3.17 we can show in the following proposition that the convergence rate for the central estimator $D^{(n)}_{\kappa,\mu,T,q,\xi}x(t_0)$ is $\mathcal{O}\left(T^{q+2}\right)$.

Corollary 1.6.30 *Let $D^{(n)}_{\kappa,\mu,T,q,0}x(t_0)$ be the Jacobi estimators for $x^{(n)}(t_0)$ given by (1.156) with $\xi = 0$. If $x \in \mathcal{C}^{N+2}(I)$ with $N \geq n$ and $(-1)^{n+N} = 1$, then we have*

$$\forall t_0 \in I,\ D^{(n)}_{\kappa,\mu,T,q,0}x(t_0) = x^{(n)}(t_0) + \mathcal{O}\left(T^{q+2}\right), \tag{1.159}$$

with $q = N - n$. Moreover, by assuming that there exists $M_{N+2} \in \mathbb{R}^*_+$ such that $\left\|x^{(N+2)}\right\|_\infty \leq M_{N+2}$, then we have

$$\left\|D^{(n)}_{\kappa,\mu,T,q,0}x(t_0) - x^{(n)}(t_0)\right\|_\infty \leq M_{N+2}C_{\kappa,\mu,n,q}T^{q+2}, \tag{1.160}$$

where $C_{\kappa,\mu,n,q} = \frac{1}{(n+2+q)!}\int_0^1 \left|Q_{\kappa,\mu,n,q,0}(\tau)\tau^{n+2+q}\right| d\tau$ and $Q_{\kappa,\mu,n,q,0}$ is defined by (1.85) with $\xi = 0$.

Proof. This proof is similar to the one of Proposition 1.3.17. If n is an odd integer, then let us take the Taylor series expansion of X^- at t_0. By using the well known Taylor's formula, we get

$$\forall \xi \in [0,1],\ X^-(t_0 + T\xi) = X^-_N(t_0 + T\xi) + \frac{(T\xi)^{N+2}}{(N+2)!}x^{(N+2)}(\theta^+_{N+1,t_0}), \tag{1.161}$$

where $T \in \hat{D}_{t_0}$, $X^-_N(t_0+T\xi) = \sum_{i=0}^{\frac{N-1}{2}} \frac{(T\xi)^{2i+1}}{(2i+1)!}x^{(2i+1)}(t_0)$, $\theta^+_{N+1,t_0} \in]t_0, t_0+T\xi[$ and N is an odd integer. Similarly to (1.95) we can get

$$\left(X^-_N\right)^{(n)}(t_0) = \frac{1}{T^n}\int_0^1 Q_{\kappa,\mu,n,q,0}(\tau)\, X^-_N(t_0 + T\tau)\, d\tau. \tag{1.162}$$

By using (1.158) with $\xi = 0$ and (1.146), we get

$$D^{(n)}_{\kappa,\mu,T,q,0}x(t_0) = \frac{1}{T^n}\int_0^1 Q_{\kappa,\mu,n,q,0}(\tau)X^-(t_0 + T\tau)d\tau. \tag{1.163}$$

Since $\left(X^-_N\right)^{(n)}(t_0) = x^{(n)}(t_0)$, then this proof can be completed by using (1.161), (1.162) and (1.163). The case where n is an even integer can be proven similarly. □

Similarly, it is easy to verify that the convergence rate of the central estimator $\tilde{x}^{(n)}_{t_0}(0,\mu,T,N) = \frac{1}{2}\left(\tilde{x}^{(n)}_{t_{0+}}(-1,\mu,T,N) + \tilde{x}^{(n)}_{t_{0-}}(-1,\mu,-T,N)\right)$ with n and N being odd integers is also $\mathcal{O}\left(T^{N-n+2}\right)$ as $T \to 0$.

1.6.2 Central Jacobi estimators

In this subsection, we give a family of estimators which are easily obtained from the modified Jacobi estimators so as to eliminate the time-drift in the Jacobi estimators. These estimators can be also obtained by using the Jacobi polynomials defined on $[-1,1]$, which extend the derivative estimations introduced in [Lanczos 1956] and [Rangarajana 2005], where the Legendre polynomials were used.

Let us assume that $x \in \mathcal{C}^n(I)$, then for any $t_0 \in I$ the Jacobi orthogonal series expansion of $x^{(n)}(t_0 - T + 2T\cdot)$ with $T \in \hat{D}_{t_0}$ is given as follows

$$\forall \xi \in [0,1],\ x^{(n)}(t_0 - T + 2T\xi) = \sum_{i\geq 0} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), x^{(n)}(t_0 - T + 2T\cdot)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}} \ P_i^{(\mu+n,\kappa+n)}(\xi). \tag{1.164}$$

By taking the $q+1$ ($q \in \mathbb{N}$) first terms in (1.164) with $\xi = \frac{1}{2}$ and denoting it by $\hat{D}^{(n)}_{\kappa,\mu,\beta T,q}x(t_0)$, we have

$$\hat{D}^{(n)}_{\kappa,\mu,T,q}x(t_0) := \sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), x^{(n)}(t_0 - T + 2T\cdot)\right\rangle_{\mu+n,\kappa+n}}{\|P_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}} \ P_i^{(\mu+n,\kappa+n)}(\frac{1}{2}). \tag{1.165}$$

Hence, $\hat{D}^{(n)}_{\kappa,\mu,T,q}x(t_0)$ is an estimator of $x^{(n)}$ with a found time-shift T. Thus, $\hat{D}^{(n)}_{\kappa,\mu,T,q}x(t_0)$ is a drift-free estimator for $x(t_0)$, which is based on the integration window $[t_0 - T, t_0 + T]$. Then, we give the following definition.

Definition 2 (Central Jacobi estimator) *Let us assume that $x \in \mathcal{C}^n(I)$, then for any $t_0 \in I$ the central estimator $\hat{D}^{(n)}_{\kappa,\mu,T,q}x(t_0)$ defined in (1.165) is called central Jacobi estimator for $x^{(n)}(t_0)$.*

Similarly to Proposition 1.3.16 we get

$$\forall t_0 \in I,\ \hat{D}^{(n)}_{\kappa,\mu,T,q}x(t_0) = \frac{1}{(2T)^n}\int_0^1 Q_{\kappa,\mu,n,q,\frac{1}{2}}(\tau)\, x(t_0 - T + 2T\tau)\, d\tau, \tag{1.166}$$

where $T \in \hat{D}_{t_0}$ and $Q_{\kappa,\mu,n,q,\frac{1}{2}}$ is defined in (1.85) with $q \in \mathbb{N}$, $\kappa, \mu \in]-1, +\infty[$.

Let us apply a change of variable in (1.166): $\tau \to \frac{\tau+1}{2}$, then we get

$$\forall t_0 \in I,\ \hat{D}^{(n)}_{\kappa,\mu,T,q}x(t_0) = \frac{1}{2^{n+1}T^n}\int_{-1}^1 Q_{\kappa,\mu,n,q,\frac{1}{2}}(\frac{\tau+1}{2})\, x(t_0 + T\tau)\, d\tau, \tag{1.167}$$

where

$$Q_{\kappa,\mu,n,q,\frac{1}{2}}(\frac{\tau+1}{2}) = w_{\mu,\kappa}(\frac{\tau+1}{2})\sum_{i=0}^{q} C_{\kappa,\mu,n,i}P_i^{(\mu+n,\kappa+n)}(\frac{1}{2})P_{n+i}^{(\mu,\kappa)}(\frac{\tau+1}{2}), \tag{1.168}$$

with $C_{\kappa,\mu,n,i} = \frac{(\mu+\kappa+2n+2i+1)\Gamma(\kappa+\mu+2n+i+1)\Gamma(n+i+1)}{\Gamma(\kappa+n+i+1)\Gamma(\mu+n+i+1)}$.

Since $P_n^{(\mu,\kappa)}(\frac{\tau+1}{2}) = \hat{P}_n^{(\mu,\kappa)}(\tau)$ and $w_{\mu,\kappa}(\frac{\tau+1}{2}) = \frac{1}{2^{\mu+\kappa}}\hat{w}_{\mu,\kappa}(\tau)$ where $\hat{P}_n^{(\mu,\kappa)}$ is the n^{th} order Jacobi polynomial defined on $[-1,1]$ (defined by (7.14) in Appendix) with the weight function $\hat{w}_{\mu,\kappa}$, (1.167) can be written as follows

$$\forall t_0 \in I,\ \hat{D}^{(n)}_{\kappa,\mu,T,q}x(t_0) = \frac{1}{T^n}\int_{-1}^1 \hat{Q}_{\kappa,\mu,n,q}(\tau)\, x(t_0 + T\tau)\, d\tau, \tag{1.169}$$

where

$$\hat{Q}_{\kappa,\mu,n,q}(\tau) = \frac{1}{2^{\mu+\kappa+n+1}}\hat{w}_{\mu,\kappa}(\tau)\sum_{i=0}^{q} C_{\kappa,\mu,n,i}\hat{P}_i^{(\mu+n,\kappa+n)}(0)\hat{P}_{n+i}^{(\mu,\kappa)}(\tau). \tag{1.170}$$

Let us recall that if we take $\kappa = \mu$ in the Jacobi polynomials, then the Jacobi polynomials become ultraspherical polynomials (see [Szegö 1967] p. 80). In particular, if we take $\kappa = \mu$ in (1.169) and q to be an even integer, then according to (7.26) $\hat{P}_i^{(\kappa+n,\kappa+n)}(0) = 0$ for any odd integer i. Hence, we get

$$\forall t_0 \in I,\ \hat{D}^{(n)}_{\kappa,T,q} x(t_0) := \frac{1}{T^n} \int_{-1}^{1} \hat{Q}_{\kappa,n,q}(\tau)\, x(t_0 + T\tau)\, d\tau, \tag{1.171}$$

where

$$\hat{Q}_{\kappa,n,q}(\tau) = \frac{1}{2^{2\kappa+n+1}} \hat{w}_{\kappa,\kappa}(\tau) \sum_{i=0}^{\frac{q}{2}} C_{\kappa,\kappa,n,2i} \hat{P}_{2i}^{(\kappa+n,\kappa+n)}(0) \hat{P}_{n+2i}^{(\kappa,\kappa)}(\tau). \tag{1.172}$$

Moreover, it is easy to verify that if we take the following Jacobi orthogonal series expansion defined on $[-1, 1]$ of x^n

$$\forall \hat{\xi} \in [-1, 1],\ x^{(n)}(T\hat{\xi} + t_0) = \sum_{i \geq 0} \frac{\left\langle \hat{P}_i^{(\mu+n,\kappa+n)}(\cdot), x^{(n)}(t_0 + T\cdot) \right\rangle_{\mu+n,\kappa+n}}{\|\hat{P}_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}}\ \hat{P}_i^{(\mu+n,\kappa+n)}(\hat{\xi}), \tag{1.173}$$

where the scalar product $\langle \cdot, \cdot \rangle_{\mu+n,\kappa+n}$ is defined by (7.16) in Appendix, then similarly to the calculations done in Subsection 1.3.2, the estimator $\hat{D}^{(n)}_{\kappa,\mu,T,q} x(t_0)$ can be obtained by taking the $(q+1)^{th}$ order truncation of (1.173) with $\hat{\xi} = 0$:

$$\forall \hat{\xi} \in [-1, 1],\ \hat{D}^{(n)}_{\kappa,\mu,T,q} x(t_0) = \sum_{i=0}^{q} \frac{\left\langle \hat{P}_i^{(\mu+n,\kappa+n)}(\cdot), x^{(n)}(t_0 + T\cdot) \right\rangle_{\mu+n,\kappa+n}}{\|\hat{P}_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}}\ \hat{P}_i^{(\mu+n,\kappa+n)}(0), \tag{1.174}$$

where the expression of $\|\hat{P}_i^{(\mu+n,\kappa+n)}\|^2_{\mu+n,\kappa+n}$ is given in (7.17) in Appendix.

If $q = 0$ in (1.169), then we get a family of minimal central Jacobi estimators

$$\forall t_0 \in I,\ \hat{D}^{(n)}_{\kappa,\mu,T} x(t_0) := \frac{1}{T^n} \int_{-1}^{1} \hat{\rho}_{n,\mu,\kappa}(\tau)\, x(t_0 + T\tau)\, d\tau, \tag{1.175}$$

where $T \in \hat{D}_{t_0}$ and $\hat{\rho}_{n,\mu,\kappa}(\tau) = \frac{2^{-n-1-\mu-\kappa} n!}{\mathrm{B}(n+\kappa+1, n+\mu+1,)} \hat{P}_n^{(\mu,\kappa)}(\tau)\, \hat{w}_{\mu,\kappa}(\tau)$ with $\mu, \kappa \in]-1, +\infty[$. Hence, by observing the expression of $\hat{D}^{(n)}_{\kappa,\mu,T} x(t_0)$ it is easy to verify that similarly to Subsection 1.2.2 these estimators can be also obtained by applying the orthogonal properties of Jacobi polynomial defined on $[-1, 1]$ to the Taylor series expansion of x. Hence, the minimal central Jacobi estimator with $\kappa = \mu = 0$ is in fact the Lanczos generalized derivative estimator defined in (1.12).

These central Jacobi estimators are drift-free estimators. Similarly to Proposition 1.3.17 the bias term error bounds are given in the following proposition.

Proposition 1.6.31 *[Liu 2011b] Let $\hat{D}^{(n)}_{\kappa,\mu,T,q} x(t_0)$ be the central Jacobi estimator of $x^{(n)}(t_0)$ defined by (1.166). If we assuming that $x \in \mathcal{C}^{q+n+1}(I)$ with $q \in \mathbb{N}$, then we have*

$$\forall t_0 \in I,\ \hat{D}^{(n)}_{\kappa,\mu,T,q} x(t_0) = x^{(n)}(t_0) + \mathcal{O}\left(T^{q+1}\right). \tag{1.176}$$

Moreover, if there exists $M_{q+n+1} \in \mathbb{R}_+^*$ such that $\left\| x^{(q+n+1)} \right\|_\infty \leq M_{q+n+1}$, then we have

$$\left\| \hat{D}^{(n)}_{\kappa,\mu,\beta T,q} x(t_0) - x^{(n)}(t_0) \right\|_\infty \leq M_{q+n+1} \hat{C}_{\kappa,\mu,n,q} T^{q+1}, \tag{1.177}$$

where $\hat{C}_{\kappa,\mu,n,q} = \frac{1}{(n+1+q)!} \int_{-1}^{1} \left| \hat{Q}_{\kappa,\mu,n,q}(\tau) \tau^{n+1+q} \right| d\tau$.

It is shown in Corollary 1.3.18 that if we take the value of ξ as the smallest root of the Jacobi polynomial $P_{q+1}^{(\mu+n,\kappa+n)}$ in the Jacobi estimators, then the corresponding convergence rate can be improved from $\mathcal{O}(T^{q+1})$ to $\mathcal{O}(T^{q+2})$ as $T \to 0$. Similarly, since $\hat{P}_i^{(\kappa+n,\kappa+n)}(0) = 0$ for any odd integer i, we have $\hat{D}_{\kappa,T,q}^{(n)} x(t_0) = \hat{D}_{\kappa,T,q+1}^{(n)} x(t_0)$ with q being an even integer. Hence, we improve the convergence rate for the central Jacobi estimators in the following corollary.

Corollary 1.6.32 *[Liu 2011b] Let $x \in \mathcal{C}^{q+n+2}(I)$ where q is an even integer and $\hat{D}_{\kappa,T,q}^{(n)} x(t_0)$ be the central jacobi estimator defined by (1.171), then we have*

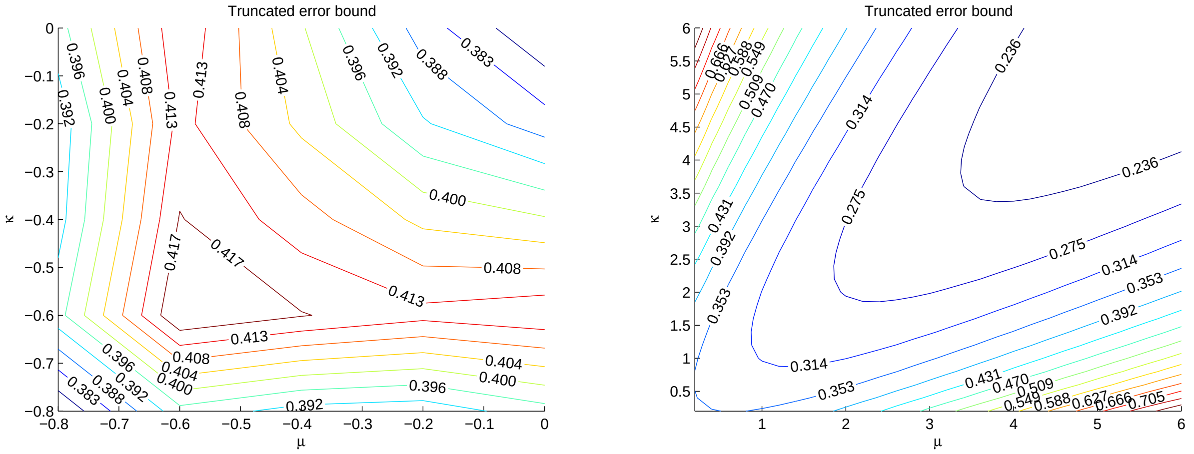
$$\forall t_0 \in I,\ \hat{D}_{\kappa,T,q}^{(n)} x(t_0) = x^{(n)}(t_0) + \mathcal{O}\left(T^{q+2}\right). \tag{1.178}$$

Moreover, by assuming that there exists $M_{q+n+2} \in \mathbb{R}_+^*$ such that $\left\|x^{(q+n+2)}\right\|_\infty \leq M_{q+n+2}$, then we have

$$\left\|\hat{D}_{\kappa,T,q}^{(n)} x(t_0) - x^{(n)}(t_0)\right\|_\infty \leq M_{q+n+2} \hat{C}_{\kappa,n,q} T^{q+2}, \tag{1.179}$$

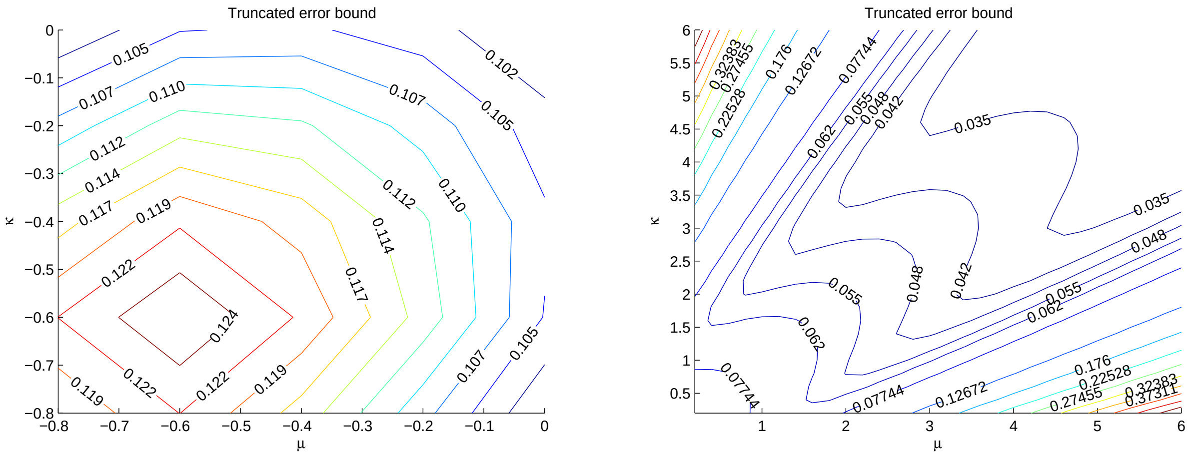
where $\hat{C}_{\kappa,n,q} = \frac{1}{(n+2+q)!} \int_{-1}^{1} \left|\hat{Q}_{\kappa,n,q}(\tau)\tau^{n+2+q}\right| d\tau$.

In Figure 1.16-1.19 we give the variations of $\hat{C}_{\kappa,\mu,n,q=0}$ defined in Proposition 1.6.31 with respect to κ and μ for $n = 1$ and $q = 0, 1, 2, 3$. We can see that $\hat{C}_{\kappa,\mu,n,q}$ is decreasing with respect to κ and μ. Hence, we can increase the value of κ and μ so as to reduce the truncated error for the central Jacobi estimator.



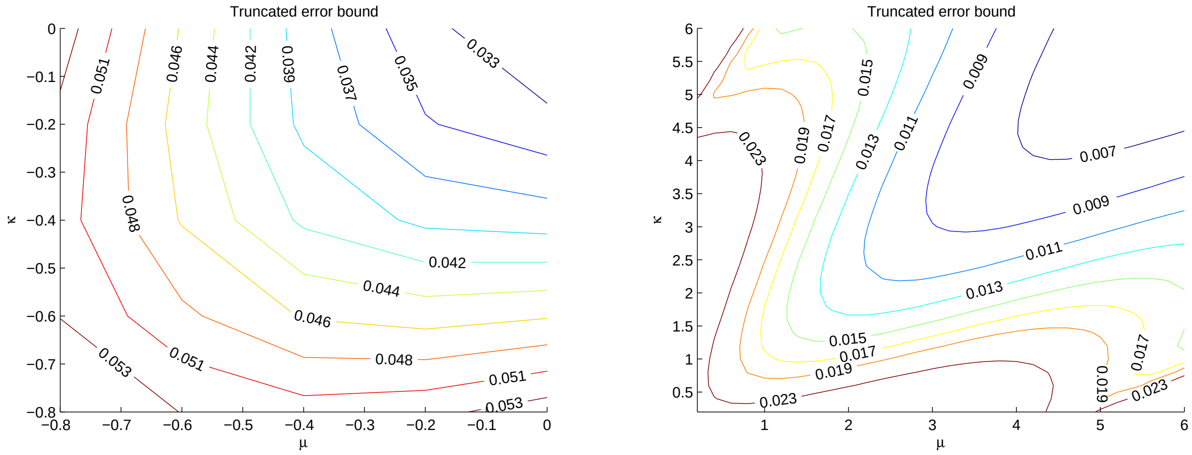
(a) $\hat{C}_{\kappa,\mu,1,q}$ with $q = 0$, $-1 < \kappa \leq 0$ and $-1 < \mu \leq 0$. (b) $\hat{C}_{\kappa,\mu,1,q}$ with $q = 0$, $0 < \kappa \leq 6$ and $0 < \mu \leq 6$.

Figure 1.16: Variation of $\hat{C}_{\kappa,\mu,n,q=0}$ with respect to κ and μ for $n = 1$.



(a) $\hat{C}_{\kappa,\mu,1,q}$ with $q = 1$, $-1 < \kappa \leq 0$ and $-1 < \mu \leq 0$. (b) $\hat{C}_{\kappa,\mu,1,q}$ with $q = 1$, $0 < \kappa \leq 6$ and $0 < \mu \leq 6$.

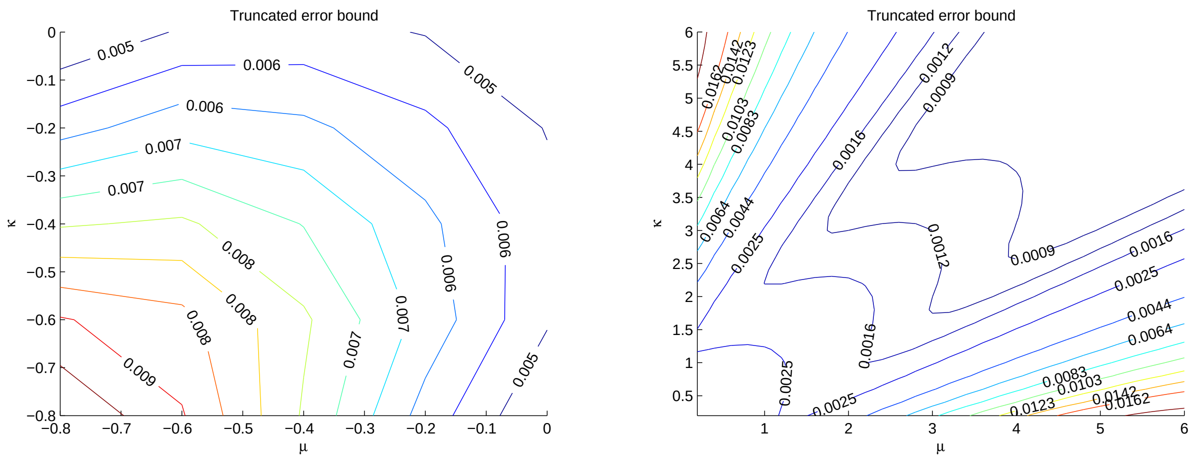
Figure 1.17: Variation of $\hat{C}_{\kappa,\mu,n,q=1}$ with respect to κ and μ for $n = 1$.



(a) $\hat{C}_{\kappa,\mu,1,q}$ with $q = 2$, $-1 < \kappa \leq 0$ and $-1 < \mu \leq 0$.

(b) $\hat{C}_{\kappa,\mu,1,q}$ with $q = 2$, $0 < \kappa \leq 6$ and $0 < \mu \leq 6$.

Figure 1.18: Variation of $\hat{C}_{\kappa,\mu,n,q=2}$ with respect to κ and μ for $n = 1$.



(a) $\hat{C}_{\kappa,\mu,1,q}$ with $q = 3$, $-1 < \kappa \leq 0$ and $-1 < \mu \leq 0$.

(b) $\hat{C}_{\kappa,\mu,1,q}$ with $q = 3$, $0 < \kappa \leq 6$ and $0 < \mu \leq 6$.

Figure 1.19: Variation of $\hat{C}_{\kappa,\mu,n,q=3}$ with respect to κ and μ for $n = 1$.

1.6.3 Richardson extrapolation technique

It is shown in Subsection 1.5.1 that the convergence rate for the minimal estimators $D^{(n)}_{\kappa,\mu,\beta T}$ can be improved by applying the Richardson extrapolation technique. Two families of Richardson-Jacobi estimators are obtained in the causal and anti-causal cases respectively. In this subsection, we propose a family of central Richardson-Jacobi estimators which are exactly the extension for the one introduced by using Legendre polynomial in [Wang 2010].

Let us consider the minimal central Jacobi estimator $\hat{D}^{(n)}_{\kappa,\mu,T}x(t_0)$ defined in (1.175), then similarly to Proposition 1.5.23 we give the following proposition.

Proposition 1.6.33 *Let $x \in \mathcal{C}^n(I)$, then a family of central Richardson-Jacobi estimators for $x^{(n)}(t_0)$ at any point $t_0 \in I$ is given by*

$$\hat{D}^{(n)}_{\kappa,\mu,T,\lambda}x(t_0) = a_\lambda \hat{D}^{(n)}_{\kappa,\mu,T}x(t_0) + b_\lambda \hat{D}^{(n)}_{\kappa,\mu,\lambda T}x(t_0), \tag{1.180}$$

where $a_\lambda = \frac{-\lambda}{1-\lambda}$, $b_\lambda = \frac{1}{1-\lambda}$ with $\lambda \in \mathbb{R}_+/\{1\}$, $\hat{D}^{(n)}_{\kappa,\mu,T}x(t_0)$ is minimal central Jacobi estimator defined in (1.175) with $T \in \hat{D}_{t_0}$ and $\mu, \kappa \in]-1,+\infty[$. If we assume that $x \in \mathcal{C}^{n+2}(I)$, then we have

$$\forall t_0 \in I,\ \hat{D}^{(n)}_{\kappa,\mu,T,\lambda}x(t_0) = x^{(n)}(t_0) + \mathcal{O}\left(T^2\right). \tag{1.181}$$

Moreover, if there exists $M_{n+2} \in \mathbb{R}^*_+$ such that $\left\|x^{(n+2)}\right\|_\infty \leq M_{n+2}$, then we have

$$\left\|\hat{D}^{(n)}_{\kappa,\mu,T,\lambda}x(t_0) - x^{(n)}(t_0)\right\|_\infty \leq M_{n+2}\hat{C}_{\kappa,\mu,n,\lambda}T^2, \tag{1.182}$$

where $\hat{C}_{\kappa,\mu,n,\lambda} = \frac{|a_\lambda|+|b_\lambda|\lambda^2}{(n+2)!}\int_{-1}^{1}\left|\rho_{n,\kappa,\mu}(\tau)\tau^{n+2}\right| d\tau$.

It is shown in the previous subsection that if we take $\kappa = \beta$ and q to be an even integer in the central Jacobi estimators, then we can improve the convergence rate. Similarly, let us take $\kappa = \beta$ in the central minimal Jacobi estimators. Consequently, we can give a family of improved central Richardson-Jacobi estimators.

Corollary 1.6.34 *Let $x \in \mathcal{C}^n(I)$, then a family of improved central Richardson-Jacobi estimators for $x^{(n)}(t_0)$ at any point $t_0 \in I$ is given by*

$$\hat{D}^{(n)}_{\kappa,T,\lambda}x(t_0) = c_\lambda \hat{D}^{(n)}_{\kappa,T}x(t_0) + d_\lambda \hat{D}^{(n)}_{\kappa,\lambda T}x(t_0), \tag{1.183}$$

where $c_\lambda = \frac{-\lambda^2}{1-\lambda^2}$, $d_\lambda = \frac{1}{1-\lambda^2}$ with $\lambda \in \mathbb{R}_+/\{1\}$, $\hat{D}^{(n)}_{\kappa,T}x(t_0)$ is the minimal central Jacobi estimator defined in (1.175) with $T \in \hat{D}_{t_0}$ and $-1 < \mu = \kappa \in \mathbb{R}$. Moreover, if we assume that $x \in \mathcal{C}^{n+4}(I)$ and there exists $M_{n+4} \in \mathbb{R}^*_+$ such that $\left\|x^{(n+4)}\right\|_\infty \leq M_{n+4}$, then we have

$$\left\|\hat{D}^{(n)}_{\kappa,T,\lambda}x(t_0) - x^{(n)}(t_0)\right\|_\infty \leq M_{n+4}\hat{C}_{\kappa,n,\lambda}T^4, \tag{1.184}$$

where $\hat{C}_{\kappa,n,\lambda} = \dfrac{|c_\lambda| + |d_\lambda|\lambda^4}{(n+4)!}\displaystyle\int_{-1}^{1}\left|\rho_{n,\kappa,\mu}(\tau)\tau^{n+4}\right| d\tau.$

Let us give the following lemma so as to proof the previous corollary.

Lemma 1.6.35 *[Liu 2011b] Let $\hat{P}_n^{(\kappa,\kappa)}$ be the n^{th} order ultraspherical polynomial with the weight function $\hat{w}_{\kappa,\kappa}$, then we have*

$$\int_{-1}^{1} \hat{P}_n^{(\kappa,\kappa)}(\tau)\hat{w}_{\kappa,\kappa}(\tau)\,\tau^{n+l}\,d\tau = 0, \tag{1.185}$$

where l is an odd integer.

Proof. By using the Rodrigues formula (given in (7.21) in Appendix) in (1.185) and applying n times integrations by parts, we get

$$\int_{-1}^{1} \hat{P}_n^{(\kappa,\kappa)}(\tau)\hat{w}_{\kappa,\kappa}(\tau)\,\tau^{n+l}\,d\tau = \frac{(n+l)!}{2^n(n!)^2}\int_{-1}^{1} \hat{w}_{\kappa+n,\kappa+n}(\tau)\,\tau^{l}\,dt. \tag{1.186}$$

If l is an odd number then $w_{\kappa+n,\kappa+n}(\tau)\,\tau^l$ is an odd function. Hence, this proof is completed. □

Proof. This proof is similar to the one of Proposition 1.5.23. Here, we take the Taylor series expansion of x at $t_0 \in I$,

$$\forall \tau \in [-1,1],\ x(t_0+T\tau) = \sum_{j=0}^{n+3} \frac{(T\tau)^j}{j!}x^{(j)}(t_0) + \frac{(T\tau)^{n+4}}{(n+4)!}x^{(n+4)}(\theta_{n+3,t_0}), \tag{1.187}$$

where $T \in \hat{D}_{t_0}$ and $\theta_{n+3,t_0} \in]t_0-T, t_0+T[$. By applying the orthogonal properties of the Jacobi polynomial obtained in $\hat{D}_{\kappa,T}^{(n)}x(t_0)$, the terms containing $x^{(j)}(t_0)$ with $0 \leq j \leq n-1$ are annihilated. By using Lemma 1.6.35, the terms containing $x^{(n+1)}(t_0)$ and $x^{(n+3)}(t_0)$ are annihilated. The relation $c_\lambda + d_\lambda \lambda^2 = 0$ is used to annihilate the terms containing $x^{(n+2)}(t_0)$, and the relation $c_\lambda + d_\lambda = 1$ is used to calculate $x^{(n)}(t_0)$. Then, this proof can be easily completed. □

1.6.4 Generalized derivative estimators

In this subsection, we assume that $x \in \mathcal{C}^{n-1}(I)$ and the right and left hand derivatives for the n^{th} order exist. Then, we introduce some generalized derivative estimators for $x^{(n)}$ which converge to the average value of these one-sided derivatives.

In Subsection 1.4.3, we define two families of generalized derivative estimators by the Jacobi estimators. Since the central estimator $D_{\kappa,\mu,T,q,\xi}^{(n)}x(t_0)$ is the average of the causal and anti-causal Jacobi estimators, then by using Proposition 1.4.22 we get easily the following corollary.

Corollary 1.6.36 *[Liu 2011b] Let $x \in \mathcal{H}^n(I)$ such that for any $t_0 \in I$ the right derivative $x^{(n)}(t_{0+})$ (resp. the left derivative $x^{(n)}(t_{0-})$) exists. Then we define the central estimator $D_{\kappa,\mu,T,q,\xi}^{(n)}x(t_0)$ given by (1.156) as a generalized derivative estimator for $x^{(n)}$, where $T \in \hat{D}_{t_0}$, $\kappa, \mu \in]-1,+\infty[$. Moreover, we have*

$$\lim_{T\to 0^+} D_{\kappa,\mu,T,q,\xi}^{(n)}x(t_0) = \frac{1}{2}\left(x^{(n)}(t_{0+}) + x^{(n)}(t_{0-})\right), \tag{1.188}$$

where $x^{(n)}(t_{0+})$ (resp. $x^{(n)}(t_{0-})$) denotes the right (resp. left) hand derivative for the n^{th} order.

For the family of central Jacobi estimators $\hat{D}_{\kappa,T,q}^{(n)}x(t_0)$, we give the following proposition.

Proposition 1.6.37 *[Liu 2011b] Let $x \in \mathcal{H}^n(I)$ such that for any $t_0 \in I$ the right derivative $x^{(n)}(t_{0+})$ (resp. the left derivative $x^{(n)}(t_{0-})$) exists. Then we define the central Jacobi estimator $\hat{D}^{(n)}_{\kappa,T,q}x(t_0)$ given by (1.171) as a generalized derivative estimator for $x^{(n)}$, where $T \in \hat{D}_{t_0}$, $-1 < \kappa \in \mathbb{R}$ and q is an even integer. Moreover, we have*

$$\lim_{T\to 0^+} \hat{D}^{(n)}_{\kappa,T,q}x(t_0) = \frac{1}{2}\left(x^{(n)}(t_{0+}) + x^{(n)}(t_{0-})\right). \tag{1.189}$$

Proof. Let us recall the local Taylor formula with the Peano remainder term (see [Zorich 2004] p. 219-232). For any given $\varepsilon' > 0$, there exists $\delta > 0$ such that

$$\left| x(t_0+T\tau) - x_{n-1}(t_0+T\tau) - \frac{x^{(n)}(t_{0-})}{n!}(T\tau)^n \right| < \varepsilon' |T\tau|^n, \text{ for } \delta < T\tau < 0, \tag{1.190}$$

and

$$\left| x(t_0+T\tau) - x_{n-1}(t_0+T\tau) - \frac{x^{(n)}(t_{0+})}{n!}(T\tau)^n \right| < \varepsilon' (T\tau)^n, \text{ for } 0 < T\tau < \delta, \tag{1.191}$$

where $x_{n-1}(t_0+T\tau)$ is the $(n-1)^{th}$ order truncated Taylor series expansion of $x(t_0+T\tau)$.

Let us consider the function $g(t) = t^n$ the n^{th} order derivative of which is equal to $(n!)$. Thus, by using (1.174) we have

$$\forall t_0 \in I,\ \hat{D}^{(n)}_{\kappa,T,q}g(t_0) = (n!).$$

Then, by applying (1.171) we get

$$\forall t_0 \in I,\ \hat{D}^{(n)}_{\kappa,T,q}g(t_0) = \frac{1}{T^n}\int_{-1}^{1} \hat{Q}_{\kappa,n,q}(\tau)\, g(t_0+T\tau)\, d\tau = (n!).$$

In particular, by taking $t_0 = 0$ we get $\frac{1}{T^n}\int_{-1}^{1} \hat{Q}_{\kappa,n,q}(\tau)\,(T\tau)^n\, d\tau = (n!)$. By using (7.26) in (1.172), it is easy to obtain that $\hat{Q}_{\kappa,n,q}(-\tau) = (-1)^n \hat{Q}_{\kappa,n,q}(\tau)$ for any $\tau \in [-1,1]$, which leads that $\tau^n \hat{Q}_{\kappa,n,q}(\tau)$ is an even function. Hence, we get

$$\frac{1}{T^n}\int_{-1}^{0} \hat{Q}_{\kappa,n,q}(\tau)\,(T\tau)^n\, d\tau = \frac{1}{T^n}\int_{0}^{1} \hat{Q}_{\kappa,n,q}(\tau)\,(T\tau)^n\, d\tau = \frac{n!}{2}. \tag{1.192}$$

Then,

$$\frac{1}{T^n}\int_{-1}^{0} \hat{Q}_{\kappa,n,q}(\tau)\frac{x^{(n)}(t_{0-})}{n!}(T\tau)^n\, d\tau = \frac{1}{2}x^{(n)}(t_{0-}), \tag{1.193}$$

and

$$\frac{1}{T^n}\int_{0}^{1} \hat{Q}_{\kappa,n,q}(\tau)\frac{x^{(n)}(t_{0+})}{n!}(T\tau)^n\, d\tau = \frac{1}{2}x^{(n)}(t_{0+}). \tag{1.194}$$

By using (1.174) and (1.171), we get

$$\forall t_0 \in I,\ \hat{D}^{(n)}_{\kappa,T,q}x_{n-1}(t_0) = \frac{1}{T^n}\int_{-1}^{1} \hat{Q}_{\kappa,n,q}(\tau)\, x_{n-1}(t_0+T\tau)\, d\tau = 0. \tag{1.195}$$

Hence, by using (1.193), (1.194) and (1.195) we obtain

$$\begin{aligned}&\left|\hat{D}^{(n)}_{\kappa,T,q}x(t_0)-\frac{1}{2}\left(x^{(n)}(t_{0-})+x^{(n)}(t_{0+})\right)\right|\\ \leq&\frac{1}{T^n}\int_{-1}^{0}\left|\hat{Q}_{\kappa,n,q}(\tau)\left(x(t_0+T\tau)-x_{n-1}(t_0+T\tau)-\frac{x^{(n)}(t_{0-})}{n!}(T\tau)^n\right)\right|d\tau\\ +&\frac{1}{T^n}\int_{0}^{1}\left|\hat{Q}_{\kappa,n,q}(\tau)\left(x(t_0+T\tau)-x_{n-1}(t_0+T\tau)-\frac{x^{(n)}(t_{0+})}{n!}(T\tau)^n\right)\right|d\tau.\end{aligned} \tag{1.196}$$

By using (1.172), we get

$$\int_0^1\left|\hat{Q}_{\kappa,n,q}(\tau)\tau^n\right|d\tau\leq\frac{1}{2^{2\kappa+n+1}}\sum_{i=0}^{\frac{q}{2}}C_{\kappa,\kappa,n,2i}\hat{P}^{(\kappa+n,\kappa+n)}_{2i}(0)\int_0^1\left|\hat{w}_{\kappa,\kappa}(\tau)\hat{P}^{(\kappa,\kappa)}_{n+2i}(\tau)\tau^n\right|d\tau<\infty. \tag{1.197}$$

Since $\hat{Q}_{\kappa,n,q}(\tau)\,\tau^n$ is an even function, we get

$$\int_{-1}^{0}\left|\hat{Q}_{\kappa,n,q}(\tau)\tau^n\right|d\tau=\int_0^1\left|\hat{Q}_{\kappa,n,q}(\tau)\tau^n\right|d\tau<\infty$$

Consequently, for any $\varepsilon>0$, by using (1.196), (1.190) and (1.191) with $\varepsilon=2\varepsilon'\int_0^1\left|\hat{Q}_{\kappa,n,q}(\tau)\,\tau^n\right|d\tau$, there exists δ such that $0<T<\delta$ and

$$\left|\hat{D}^{(n)}_{\kappa,T,q}x(t_0)-\frac{1}{2}\left(x^{(n)}(t_{0+})+x^{(n)}(t_{0-})\right)\right|<\varepsilon.$$

Then, this proof can be completed. □

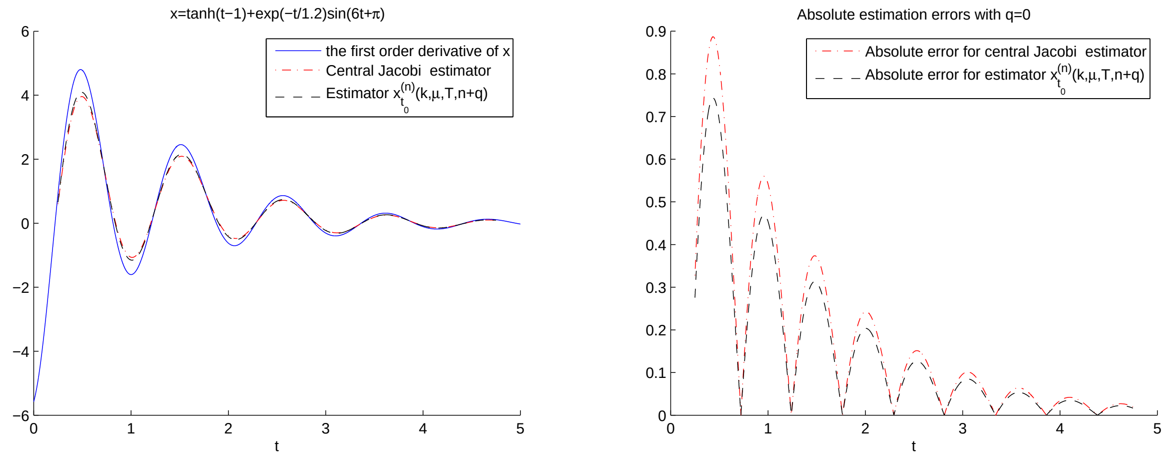
Since the central Richardson-Jacobi estimator $\hat{D}^{(n)}_{\kappa,T,\lambda}x(t_0)$ defined by Corollary 1.6.34 is an affine combination of minimal central Jacobi estimators with $c_\lambda+d_\lambda=1$. Then, by applying the previous proposition, we get the following corollary.

Corollary 1.6.38 *Let $x\in\mathcal{H}^n(I)$ such that for any $t_0\in I$ the right derivative $x^{(n)}(t_{0+})$ (resp. the left derivative $x^{(n)}(t_{0-})$) exists. Then we define the central Richardson-Jacobi estimator $\hat{D}^{(n)}_{\kappa,T,\lambda}x(t_0)$ defined by Corollary 1.6.34 as a generalized derivative estimator for $x^{(n)}$, where $T\in\hat{D}_{t_0}$, $-1<\kappa\in\mathbb{R}$. Moreover, we have*

$$\lim_{T\to0^+}\hat{D}^{(n)}_{\kappa,T,\lambda}x(t_0)=\frac{1}{2}\left(x^{(n)}(t_{0+})+x^{(n)}(t_{0-})\right). \tag{1.198}$$

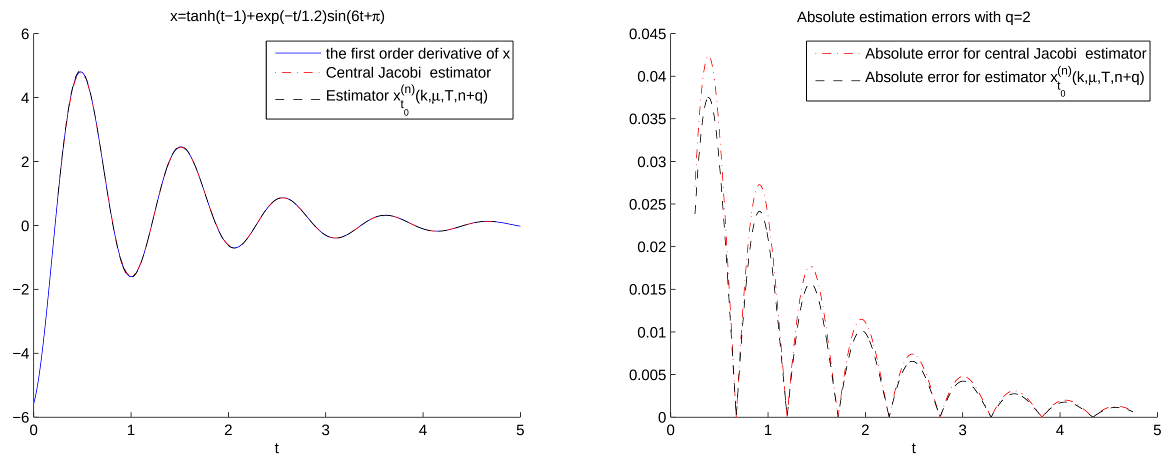
1.6.5 Some numerical examples

We give in this section some numerical examples. Let us take the sampling data of function x defined by (1.8) with a sampling period $T_s=\frac{1}{2000}$. Then, we use the central estimator to estimate the first order derivative of x. For each estimator, we set $\kappa=\mu=0$ and $T=\frac{1}{4}$. According to Corollary 1.6.30 and Corollary 1.6.32, we take q as an even integer. The analysis for the choice of parameters κ, μ and T will be addressed in Chapter 2. The estimations obtained by using the central estimator $\tilde{x}^{(1)}_{t_0}(\kappa,\mu,T,n+q)$ given by (1.151) and central Jacobi estimator $\hat{D}^{(1)}_{\kappa,T,q}x(t_0)$ given by (1.171) are given in Figure 1.20(a) and Figure 1.21(a) with $q = 0$ and $q = 2$ respectively. The associated absolute errors are given in Figure 1.20(b) and Figure 1.21(b). We can see that there is no time-drift for the central estimators and the amplitude error can be improved by increasing the truncation order q. Then, we take the central Richardson-Jacobi estimator defined in (1.183) with different value of λ. The obtained absolute errors are given in Figure 1.22(a) and Figure 1.22(b). In order to compare the estimations, we also give the absolute estimation errors for the central Jacobi estimator with $q = 2$. Hence, the central Richardson-Jacobi estimator can be improved by reducing the value of λ.



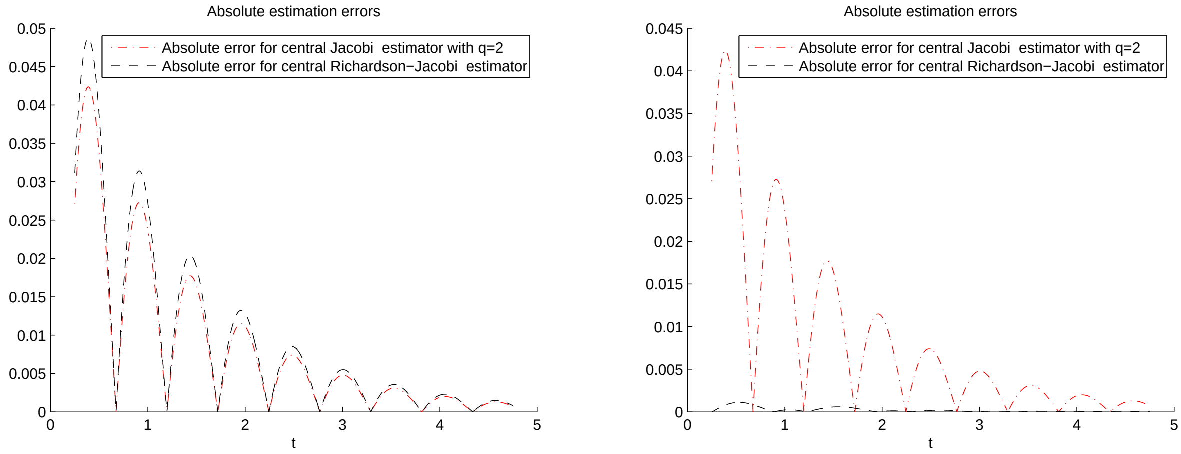
(a) Estimations. (b) Associated absolute errors.

Figure 1.20: Central estimator $\tilde{x}_{t_0}^{(1)}(\kappa, \mu, T, n + q)$ and central Jacobi estimator $\hat{D}_{\kappa,T,q}^{(1)} x(t_0)$, where $\kappa = \mu = 0$, $T = \frac{1}{4}$ and $q = 0$.



(a) Estimations. (b) Associated absolute errors.

Figure 1.21: Central estimator $\tilde{x}_{t_0}^{(1)}(\kappa, \mu, T, n + q)$ and central Jacobi estimator $\hat{D}_{\kappa,T,q}^{(1)} x(t_0)$, where $\kappa = \mu = 0$, $T = \frac{1}{4}$ and $q = 2$.



(a) Associated absolute estimation errors. (b) Associated absolute estimation errors.

Figure 1.22: Absolute estimation errors for central Jacobi estimator $\hat{D}^{(1)}_{\kappa,T,q}x(t_0)$ with $q = 2$ and central Richardson-Jacobi estimator $\hat{D}^{(1)}_{\kappa,T,\lambda}x(t_0)$ with $\lambda = 0.8$ and $\lambda = 0.08$ respectively, where $\kappa = 0$ and $T = \frac{1}{4}$.

1.7 General estimator

In Subsection 1.5.1, we give two families of Richardson-Jacobi estimators which are the combination of minimal Jacobi estimators with different lengths for integration windows. In Subsection 1.6.1, by taking the combination of causal and anti-causal Jacobi estimators we introduce a family of central estimators. In this section, by applying the algebraic parametric technique we give a general form of the derivative estimators which are an affine combination of estimators with different lengths for integration windows. For this, we give a general differential operator parameterized by a set of parameters. Sufficient and necessary conditions on this set are given to obtain such an integral annihilator. It is proven that such set of parameters is non empty.

1.7.1 Operational domain

Let us assume that $x \in \mathcal{C}^N(I)$ with $n \leq N \in \mathbb{N}$. Then, for any $t_0 \in I$, we introduce the following function

$$X(t) = \sum_{i=0}^{L} a_i\, x(t_0 + \beta_i t), \tag{1.199}$$

where $L \in \mathbb{N}, a_i \in \mathbb{R}^*, \beta_i \in \mathbb{R}^*$, $\beta_0 < \beta_1 < \cdots < \beta_L$, $t \in D := \{t \in \mathbb{R}^+; \forall i \in \{1, ..., L\}, t_0 + \beta_i t \in I\}$ and $\sum_{i=0}^{L} a_i \beta_i^n \neq 0$. This function X will be used to perform the estimation of $x^{(n)}$ in a general framework. Actually, if all the $\beta_i < 0$ (resp. $\beta_i > 0$), then we can obtain causal estimators (resp. anti-causal estimators). In the other cases, we can obtain "finite difference" type estimators. Let us consider the

N^{th} order truncated Taylor series expansion of X on $\mathbb{R}^+$

$$\forall t \in \mathbb{R}^+,\ X_N(t) = \sum_{i=0}^{L} a_i \sum_{j=0}^{N} \frac{(\beta_i t)^j}{j!} x^{(j)}(t_0) = \sum_{j=0}^{N} \left(\sum_{i=0}^{L} a_i \beta_i^j \right) \frac{t^j}{j!} x^{(j)}(t_0). \tag{1.200}$$

Since X_N is an N^{th} order polynomial defined on $\mathbb{R}^+$, we can apply the Laplace transform to (1.200) (s being the Laplace variable)

$$\hat{X}_N = \sum_{j=0}^{N} c_j s^{-(j+1)} x^{(j)}(t_0), \tag{1.201}$$

where $\hat{X}_N$ is the Laplace transform of $X_N(t)$, $c_j = \sum_{i=0}^{L} a_i \beta_i^j$ with $c_n \neq 0$.

1.7.2 Annihilators

Let us recall that the basic step for the algebraic parametric technique towards the estimation of $x^{(n)}$ is to estimate the coefficient $x^{(n)}(t_0)$ from $\hat{X}_N$ by using a linear differential operator. A general form for this operator is given in (1.18). When the sum in (1.18) is reduced to a single term, we obtain a particular case where the linear differential operator is a finite product of length $\Theta \in \mathbb{N}$. If for all indexes l, the rational function ϱ_l is of the following form $\varrho_l(s) = \frac{1}{s^{m_l}}$, then the linear differential operator defined by (1.18) can be parameterized by a set $E = \{(n_l, m_l)\}_{l=1}^{\Theta}$:

$$\Pi_E = \prod_{l=1}^{\Theta} \frac{1}{s^{m_l}} \frac{d^{n_l}}{ds^{n_l}} = \frac{1}{s^{m_1}} \frac{d^{n_1}}{ds^{n_1}} \cdots \frac{1}{s^{m_\Theta}} \frac{d^{n_\Theta}}{ds^{n_\Theta}}. \tag{1.202}$$

Note that $m_l \in \mathbb{Z}^*$ for $l = 2, \ldots, \Theta$, except for $m_1 \in \mathbb{R}^*$, and $n_l \in \mathbb{N}^*$ for $l = 1, \ldots, \Theta - 1$, except for $n_\Theta \in \mathbb{N}$. In the following proposition, we give conditions on the integers m_l and n_l so as to calculate the value of $\Pi_E\left(\hat{X}_N\right)$ where Π_E preserves only the term containing $x^{(n)}(t_0)$.

Proposition 1.7.39 *Let $\hat{X}_N$ be defined by* (1.201) *and Π_E be the linear differential operator defined by (1.202). If E satisfies the following conditions*

(C_1): $\forall l \in \{1, \ldots, \Theta - 1\}$: either $n + 1 + r_l > 0$ or $n + 1 + r_l \leq -n_l$ is true,

(C_2): for each $j \in \mathbf{J} = \{k; k \in \{0, \ldots, n-1, n+1, \ldots, N\}, c_k x^{(k)}(t_0) \neq 0\}$, there exists a $l_j \in \{1, ..., \Theta - 1\}$, such that $0 \leq -(j+1) - r_{l_j} < n_{l_j}$,

with $r_l = \sum_{i=l+1}^{\Theta} n_i + m_i$ for $l = 0, \ldots, \Theta - 1$ and $r_\Theta = 0$. Then, Π_E is an annihilator and

$$\Pi_E\left(\hat{X}_N\right) = c_n x^{(n)}(t_0) \frac{\hat{c}}{s^{n+1+r_0}}, \tag{1.203}$$

where $c_n = \sum_{i=0}^{L} a_i \beta_i^n \neq 0$, $\hat{c} = \prod_{l=1}^{\Theta} \hat{c}_l$, *with*

$$\hat{c}_l = \begin{cases} \frac{(-1)^{n_l}(n_l+n+r_l)!}{(n+r_l)!}, & \text{if } n+1+r_l > 0, \\ \frac{(|n+1+r_l|)!}{(|n+1+r_l|-n_l)!}, & \text{if } n+1+r_l \leq -n_l. \end{cases} \tag{1.204}$$

Moreover, if $c_n x^{(n)}(t_0) \neq 0$*, then the conditions* (C_1) *and* (C_2) *are also necessary.*

Proof. By applying the linear differential operator defined by (1.202) to the right hand side of (1.201), one obtains

$$\Pi_E \left(\sum_{j=0}^{N} c_j s^{-(j+1)} x^{(j)}(t_0) \right) = \sum_{j=0}^{N} c_j x^{(j)}(t_0)\ \Pi_E(s^{-(j+1)}).$$

• Sufficiency: the computation is divided into two parts: one concerning the term $\Pi_E(s^{-(j+1)})$ with $j = n$, and one concerning the others with $j \neq n$. Recall firstly the following formulae: for $k \in \mathbb{N}$ and $m \in \mathbb{Z}^*$, $\frac{d^k(s^m)}{ds^k}$ is given by

$$\frac{m!}{(m-k)!} s^{m-k} \ \text{ if } 0 \leq k < m, \tag{1.205a}$$

$$0 \ \text{ if } 0 \leq m < k, \tag{1.205b}$$

$$\frac{(-1)^k (k-m-1)!}{(-m-1)!} s^{m-k} \ \text{ if } m < 0 \leq k. \tag{1.205c}$$

Computation of $\Pi_E(s^{-(n+1)})$: by induction, we want to prove that

$$\prod_{l=J}^{\Theta} \frac{1}{s^{m_l}} \frac{d^{n_l}}{ds^{n_l}} s^{-(n+1)} = \frac{1}{s^{n+1+r_{J-1}}} \prod_{l=J}^{\Theta} \hat{c}_l \tag{1.206}$$

holds for any $J \in \{1, \ldots, \Theta\}$, where $\hat{c}_l = \frac{(-1)^{n_l}(n_l+n+r_l)!}{(n+r_l)!}$, if $n+1+r_l > 0$, and $\hat{c}_l = \frac{(|n+1+r_l|)!}{(|n+1+r_l|-n_l)!}$, if $n+1+r_l \leq -n_l$.

Initial step: when $J = \Theta$, using (1.205c) one obtains

$$\frac{1}{s^{m_\Theta}} \frac{d^{n_\Theta}}{ds^{n_\Theta}} s^{-(n+1)} = \frac{\hat{c}_\Theta}{s^{n+1+r_{\Theta-1}}}, \quad \text{with } \hat{c}_\Theta = \frac{(-1)^{n_\Theta}(n_\Theta+n)!}{n!}. \tag{1.207}$$

Assume now that (1.206) holds for $1 < J \leq \Theta$, this leads to

$$\prod_{l=J-1}^{\Theta} \frac{1}{s^{m_l}} \frac{d^{n_l}}{ds^{n_l}} s^{-(n+1)} = \frac{1}{s^{m_{J-1}}} \frac{d^{n_{J-1}}}{ds^{n_{J-1}}} \cdot \frac{1}{s^{n+1+r_{J-1}}} \prod_{l=J}^{\Theta} \hat{c}_l.$$

We distinguish the two following cases in the condition (C_1):

1. If $n+1+r_{J-1} > 0$, then by using (1.205c) we get

$$\prod_{l=J-1}^{\Theta} \frac{1}{s^{m_l}} \frac{d^{n_l}}{ds^{n_l}} s^{-(n+1)} = \frac{1}{s^{n+1+r_{J-2}}} \prod_{l=J-1}^{\Theta} \hat{c}_l, \text{ with } \hat{c}_{J-1} = \frac{(-1)^{n_{J-1}}(n_{J-1}+n+r_{J-1})!}{(n+r_{J-1})!}.$$

2. If $n+1+r_{J-1} \leq -n_{J-1}$, then by using (1.205a) we get

$$\prod_{l=J-1}^{\Theta} \frac{1}{s^{m_l}} \frac{d^{n_l}}{ds^{n_l}} s^{-(n+1)} = \frac{1}{s^{n+1+r_{J-2}}} \prod_{l=J-1}^{\Theta} \hat{c}_l, \text{ with } \hat{c}_{J-1} = \frac{(|n+1+r_{J-1}|)!}{(|n+1+r_{J-1}| - n_{J-1})!}.$$

Hence, (1.206) is true for $J-1$. Consequently, we conclude by induction that (1.206) is true for any $J \in \{1, \ldots, \Theta\}$.

Computation of $\Pi_E(s^{-(j+1)})$ with $j \in \mathbf{J}$:

For any $l_j \in \{1, ..., \Theta - 1\}$, we have

$$\prod_{l=l_j+1}^{\Theta} \frac{1}{s^{m_l}} \frac{d^{n_l}}{ds^{n_l}} s^{-(j+1)} = \frac{\tilde{c}_{l_j}}{s^{j+1+r_{l_j}}}, \text{ with } \tilde{c}_{l_j} \in \mathbb{Q}.$$

From the condition (C_2), we know that $0 \leq -(j+1) - r_{l_j} < n_{l_j}$, then we obtain

$$\Pi_E\left(s^{-(j+1)}\right) = \frac{d^{n_{l_j}}}{ds^{n_{l_j}}} \left(\prod_{l=l_j+1}^{\Theta} \frac{1}{s^{m_l}} \frac{d^{n_l}}{ds^{n_l}} s^{-(j+1)} \right) = 0. \tag{1.208}$$

Hence, we conclude that

$$\Pi_E\left(\sum_{j=0}^{N} c_j s^{-(j+1)} x^{(j)}(t_0) \right) = c_n x^{(n)}(t_0) \Pi_E(s^{-(n+1)}) = c_n x^{(n)}(t_0) \frac{\hat{c}}{s^{n+1+r_0}}.$$

Consequently, we shown that the conditions (C_1) and (C_2) are sufficient conditions for (1.203) to hold.

• Necessity: we are going to prove that the conditions (C_1) and (C_2) are also necessary as soon as $c_n x^{(n)}(t_0) \neq 0$. In order to do this, we assume that (1.203) is true, then we have

$$\sum_{j=0}^{N} c_j x^{(j)}(t_0) \Pi_E\left(s^{-(j+1)}\right) = c_n x^{(n)}(t_0) \frac{\hat{c}}{s^{n+1+r_0}}. \tag{1.209}$$

By doing similar calculations leading to (1.206), we obtain (without using the conditions (C_1) and (C_2)): for $0 \leq j \leq N$, $\Pi_E\left(s^{-(j+1)}\right) = \frac{\bar{c}_j}{s^{j+1+r_0}}$, with $\bar{c}_j \in \mathbb{Q}$. Thus, (1.209) becomes

$$\forall s \in \mathbb{C} \text{ with } \Re(s) > 0, \;\; \sum_{j \in \mathbf{J}} c_j x^{(j)}(t_0) \frac{\bar{c}_j}{s^{j+1+r_0}} + c_n x^{(n)}(t_0) \frac{\hat{c} - \bar{c}_n}{s^{n+1+r_0}} = 0. \tag{1.210}$$

Therefore, we obtain that $c_n x^{(n)}(t_0)(\hat{c} - \bar{c}_n) = 0$ and $c_j x^{(j)}(t_0)\bar{c}_j = 0$ for all $j \in \mathbf{J}$. As $c_j x^{(j)}(t_0) \neq 0$ for all $j \in \mathbf{J} \cup \{n\}$, we have $\hat{c} = \bar{c}_n$ and $\bar{c}_j = 0$ for all $j \in \mathbf{J}$. Hence, $\Pi_E\left(s^{-(n+1)}\right) = \frac{\hat{c}}{s^{n+1+r_0}}$ and $\Pi_E\left(s^{-(j+1)}\right) = 0$ for all $j \in \mathbf{J}$. Since $\Pi_E\left(s^{-(n+1)}\right) \neq 0$, $s^{-(n+1)}$ is not annihilated after each derivation, we conclude that $\forall l \in \{1, \ldots, \Theta - 1\}$, either $n+1+r_l > 0$ or $n+1+r_l \leq -n_l$ is true. On the other hand, $s^{-(j+1)}$ is annihilated for all $j \in \mathbf{J}$. Hence, there exists a $l_j \in \{1, ..., \Theta - 1\}$, such that $0 < -(j+1) - r_{l_j} < n_{l_j}$. □

We can see in the previous proof that the condition (C_2) is used to annihilate all the undesired terms: $c_j s^{-(j+1)} x^{(j)}(t_0)$ in (1.201) for all $j \in \mathbf{J}$, and the condition (C_1) is used to keep the term $c_n s^{-(n+1)} x^{(n)}(t_0)$. In the following proposition, we give a new conditions on the set E such that the annihilator Π_E is an integral annihilator: the estimator of $x^{(n)}(t_0)$ only depends on a unique integral of the measured signal in the time domain. Before doing so, we propose the following lemma.

Lemma 1.7.40 *Let $\hat{f}$ be the Laplace transform of a function f defined by $\mathbb{R}^+$, the Laplace transform of a function of which exists, and Π_E be an operator defined by (1.202). We split each m_l into two terms: $m_l = \hat{m}_l + \bar{m}_l$ with $\hat{m}_l \in \mathbb{Z}^*$ and $\bar{m}_l \in \mathbb{Z}$ for $l = 1, \cdots, \Theta$, except for $\hat{m}_1 \in \mathbb{R}^*$. Let $\mathbf{j} = (j_1, ..., j_\Theta)$ be a multi-index of length Θ and $\bar{E}_{\mathbf{j}} = \{(n_l - j_l, \bar{m}_l)\}_{l=1}^{\Theta}$ be a subset of $\mathbb{N} \times \mathbb{Z}$, then Π_E can be written as follow*

$$\Pi_E\left(\hat{f}\right) = \sum_{j_\Theta=0}^{I_\Theta} \cdots \sum_{j_1=0}^{I_1} C_1 \frac{1}{s^{\gamma_1}} \Pi_{\bar{E}_{\mathbf{j}}}\left(\hat{f}\right), \tag{1.211}$$

where $\gamma_l = \sum_{i=l}^{\Theta} \hat{m}_i + j_i$ and $C_l = \prod_{i=l}^{\Theta} e_{j_i}$ for $l = 1, \cdots, \Theta$, with the values

$$e_{j_l} = \begin{cases} \binom{n_l}{j_l} \frac{(-1)^{j_l}(j_l+\gamma_{l+1}-1)!}{(\gamma_{l+1}-1)!}, & \text{if } \gamma_{l+1} > 0, \\ \binom{n_l}{j_l} \frac{|\gamma_{l+1}|!}{(|\gamma_{l+1}|-j_l)!}, & \text{else,} \end{cases} \quad \text{and} \quad I_l = \begin{cases} n_l, & \text{if } \gamma_{l+1} > 0, \\ \min(n_l, |\gamma_{l+1}|), & \text{else,} \end{cases}$$

for $l = 1, \cdots, \Theta - 1$. For $l = \Theta$ we have $I_\Theta = 0$, $e_{j_\Theta} = 1$.

Proof. We prove the following relation by induction: for $J = 1, \cdots, \Theta$,

$$\prod_{l=J}^{\Theta} \frac{1}{s^{m_l}} \frac{d^{n_l}}{ds^{n_l}}\left(\hat{f}(s)\right) = \sum_{j_\Theta=0}^{I_\Theta} \cdots \sum_{j_J=0}^{I_J} C_J \frac{1}{s^{\gamma_J}} \prod_{l=J}^{\Theta} \frac{1}{s^{\bar{m}_l}} \frac{d^{n_l-j_l}}{ds^{n_l-j_l}}\left(\hat{f}(s)\right). \tag{1.212}$$

<u>Initial step:</u> For $J = \Theta$, we have

$$\frac{1}{s^{m_\Theta}} \frac{d^{n_\Theta}}{ds^{n_\Theta}}\left(\hat{f}(s)\right) = \frac{1}{s^{\hat{m}_\Theta}} \frac{1}{s^{\bar{m}_\Theta}} \frac{d^{n_\Theta}}{ds^{n_\Theta}}\left(\hat{f}(s)\right).$$

Hence, the relation (1.212) is true for $J = \Theta$ with $I_\Theta = 0$, $C_\Theta = e_{j_\Theta} = 1$ and $\gamma_\Theta = \hat{m}_\Theta$.
Now assume that the relation (1.212) is true for $1 < J \leq \Theta$, this leads to

$$\begin{aligned}
\prod_{l=J-1}^{\Theta} \frac{1}{s^{m_l}} \frac{d^{n_l}}{ds^{n_l}}\left(\hat{f}(s)\right) &= \frac{1}{s^{m_{J-1}}} \frac{d^{n_{J-1}}}{ds^{n_{J-1}}} \cdot \left(\sum_{j_\Theta=0}^{I_\Theta} \cdots \sum_{j_J=0}^{I_J} C_J \frac{1}{s^{\gamma_J}} \prod_{l=J}^{\Theta} \frac{1}{s^{\bar{m}_l}} \frac{d^{n_l-j_l}}{ds^{n_l-j_l}}\left(\hat{f}(s)\right) \right) \\
&= \sum_{j_\Theta=0}^{I_\Theta} \cdots \sum_{j_J=0}^{I_J} C_J \frac{1}{s^{m_{J-1}}} \frac{d^{n_{J-1}}}{ds^{n_{J-1}}} \cdot \left(\frac{1}{s^{\gamma_J}} \prod_{l=J}^{\Theta} \frac{1}{s^{\bar{m}_l}} \frac{d^{n_l-j_l}}{ds^{n_l-j_l}}\left(\hat{f}(s)\right) \right) \\
&= \sum_{j_\Theta=0}^{I_\Theta} \cdots \sum_{j_J=0}^{I_J} C_J \frac{1}{s^{m_{J-1}}} \sum_{j_{J-1}=0}^{n_{J-1}} \binom{n_{J-1}}{j_{J-1}} \frac{d^{j_{J-1}}}{ds^{j_{J-1}}} \left(\frac{1}{s^{\gamma_J}}\right) \frac{d^{n_{J-1}-j_{J-1}}}{ds^{n_{J-1}-j_{J-1}}} \left(\prod_{l=J}^{\Theta} \frac{1}{s^{\bar{m}_l}} \left(\hat{f}(s)\right) \right).
\end{aligned}$$

If $\gamma_J > 0$, we obtain

$$\prod_{l=J-1}^{\Theta} \frac{1}{s^{m_l}} \frac{d^{n_l}}{ds^{n_l}} \left(\hat{f}(s)\right)$$
$$= \sum_{j_\Theta=0}^{I_\Theta} \cdots \sum_{j_J=0}^{I_J} C_J \sum_{j_{J-1}=0}^{n_{J-1}} \binom{n_{J-1}}{j_{J-1}} \frac{(-1)^{j_{J-1}}(j_{J-1}+\gamma_J-1)!}{(\gamma_J-1)!s^{m_{J-1}+\gamma_J+j_{J-1}}} \frac{d^{n_{J-1}-j_{J-1}}}{ds^{n_{J-1}-j_{J-1}}} \left(\prod_{l=J}^{\Theta} \frac{1}{s^{\bar{m}_l}} \frac{d^{n_l-j_l}}{ds^{n_l-j_l}} \left(\hat{f}(s)\right)\right)$$
$$= \sum_{j_\Theta=0}^{I_\Theta} \cdots \sum_{j_{J-1}=0}^{I_{J-1}} C_{J-1} \frac{1}{s^{\gamma_{J-1}}} \prod_{l=J-1}^{\Theta} \frac{1}{s^{\bar{m}_l}} \frac{d^{n_l-j_l}}{ds^{n_l-j_l}} \left(\hat{f}(s)\right),$$

where $I_{J-1} = n_{J-1}$, $C_{J-1} = C_J \cdot e_{j_{J-1}}$, $e_{j_{J-1}} = \binom{n_{J-1}}{j_{J-1}} \frac{(-1)^{j_{J-1}}(j_{J-1}+\gamma_J-1)!}{(\gamma_J-1)!}$ and $\gamma_{J-1} = \gamma_J + \hat{m}_{J-1} + j_{J-1} = \sum_{i=J-1}^{\Theta} \hat{m}_i + j_i$.

If $\gamma_J \leq 0$, we obtain

$$\prod_{l=J-1}^{\Theta} \frac{1}{s^{m_l}} \frac{d^{n_l}}{ds^{n_l}} \left(\hat{f}(s)\right)$$
$$= \sum_{j_\Theta=0}^{I_\Theta} \cdots \sum_{j_J=0}^{I_J} C_J \sum_{j_{J-1}=0}^{\min(n_{J-1},|\gamma_J|)} \binom{n_{J-1}}{j_{J-1}} \frac{|\gamma_J|!s^{-(m_{J-1}+\gamma_J+j_{J-1})}}{(|\gamma_J|-j_{J-1})!} \frac{d^{n_{J-1}-j_{J-1}}}{ds^{n_{J-1}-j_{J-1}}} \left(\prod_{l=J}^{\Theta} \frac{1}{s^{\bar{m}_l}} \frac{d^{n_l-j_l}}{ds^{n_l-j_l}} \left(\hat{f}(s)\right)\right)$$
$$= \sum_{j_\Theta=0}^{I_\Theta} \cdots \sum_{j_{J-1}=0}^{I_{J-1}} C_{J-1} \frac{1}{s^{\gamma_{J-1}}} \prod_{l=J-1}^{\Theta} \frac{1}{s^{\bar{m}_l}} \frac{d^{n_l-j_l}}{ds^{n_l-j_l}} \left(\hat{f}(s)\right),$$

where $I_{J-1} = \min(n_{J-1}, |\gamma_J|)$, $C_{J-1} = C_J \cdot e_{j_{J-1}}$, $e_{j_{J-1}} = \binom{n_{J-1}}{j_{J-1}} \frac{\gamma_J!}{(\gamma_J-j_{J-1})!}$ and $\gamma_{J-1} = \gamma_J + \hat{m}_{J-1} + j_{J-1} = \sum_{i=J-1}^{\Theta} \hat{m}_i + j_i$.

Hence, the relation (1.212) is true for $J-1$. Consequently, we conclude by induction that (1.212) is true for any $J \in \{1, \ldots, \Theta\}$. Finally, (1.211) holds with $J = 1$. □

Remark 4 *Here $\Pi_{\bar{E}_j}$ is an operator that can or not be also an annihilator. The criterion is that $\Pi_{\bar{E}_j}\left(\hat{X}_N\right) = \varrho(s)\, x^{(n)}(t_0)$, holds for some $\rho(s)$.*

Proposition 1.7.41 *Let $\hat{X}_N$ be defined in (1.201) and Π_E be the annihilator defined in (1.202), where the m_l satisfies the following condition (C_3): $\sum_{l=1}^{\Theta} m_l > 0$. Then, we have*

$$\mathcal{L}^{-1}\left\{\Pi_E\left(\hat{X}_N\right)\right\}(t) = \int_0^t p_{t,\Theta}(\tau) X_N(\tau) d\tau, \tag{1.213}$$

where $t \in D$, $p_{t,\Theta}(\tau) = \sum_{j_\Theta=0}^{I_\Theta} \cdots \sum_{j_1=0}^{I_1} C_1 \frac{(-1)^{N_\Theta}}{\Gamma(\gamma_1)} (t-\tau)^{\gamma_1-1} \tau^{N_\Theta}$ with $N_\Theta = \sum_{l=1}^{\Theta} n_l - j_l$, I_l for $l = 1, \cdots, \Theta$, C_1 and γ_1 is defined in Lemma 1.7.40.

Proof. We apply Lemma 1.7.40 by taking $\bar{m}_l = 0$ for $l = 1, \cdots, \Theta$ and $f = X_N$. As $m_l = \hat{m}_l$ for $l = 1, \cdots, \Theta$, we get

$$\Pi_E\left(\hat{X}_N\right) = \sum_{j_\Theta=0}^{I_\Theta} \cdots \sum_{j_1=0}^{I_1} C_1 \frac{1}{s^{\gamma_1}} \frac{d^{N_\Theta}}{ds^{N_\Theta}}\left(\hat{X}_N(s)\right). \tag{1.214}$$

Since $\gamma_1 = \sum_{l=1}^{\Theta} m_l + j_l$, the condition (C_3) implies that $\gamma_1 \geq \sum_{l=1}^{\Theta} m_l > 0$. Then, by applying (7.13), we get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\Pi_E\left(\hat{X}_N\right)\right\}(t) &= \sum_{j_\Theta=0}^{I_\Theta} \cdots \sum_{j_1=0}^{I_1} C_1 \mathcal{L}^{-1}\left\{\frac{1}{s^{\gamma_1}} \frac{d^{N_\Theta}}{ds^{N_\Theta}}\left(\hat{X}_N(s)\right)\right\}(t) \\ &= \sum_{j_\Theta=0}^{I_\Theta} \cdots \sum_{j_1=0}^{I_1} C_1 \frac{(-1)^{N_\Theta}}{(\gamma_1-1)!} \int_0^t (t-\tau)^{\gamma_1-1}\, \tau^{N_\Theta} X_N(\tau) d\tau.\end{aligned}$$

□

Now, we can give the following corollary.

Corollary 1.7.42 *Let $x \in \mathcal{C}^N(I)$ and Π_E be the integral annihilator defined by (1.202) where the set E satisfies conditions (C_1) and (C_2) of Proposition 1.7.39 and condition (C_3) of Proposition 1.7.41. Then, a family of estimators for $x^{(n)}$ is given by*

$$\forall t_0 \in I,\ \tilde{x}^{(n)}(t_0) = \frac{\Gamma(r_0+n+1)}{c_n\, \hat{c}\, T^{r_0+n}} \sum_{i=0}^{L} a_i \int_0^T p_{T,\Theta}(\tau) x(t_0 + \beta_i \tau) d\tau, \tag{1.215}$$

where $T \in D$, $\hat{c}$ is defined by (1.204), $r_0 = \sum_{l=1}^{\Theta} m_l + n_l$ and the polynomial $p_{T,\Theta}$ is defined in Proposition 1.7.41.

Proof. We start by applying the annihilator Π_E to the relation (1.201) and then to go back into the time domain. Firstly, by applying Proposition 1.7.41 we get

$$\mathcal{L}^{-1}\left\{\Pi_E\left(\hat{X}_N(s)\right)\right\}(t) = \int_0^t p_{t,\Theta}(\tau) X_N(\tau) d\tau. \tag{1.216}$$

Secondly, by Proposition 1.7.39, we obtain

$$\Pi_E\left(\sum_{j=0}^{N} c_j s^{-(j+1)} x^{(j)}(t_0)\right) = c_n x^{(n)}(t_0) \frac{\hat{c}}{s^{n+1+r_0}}.$$

As $r_0 = \sum_{l=1}^{\Theta} n_l + m_l \geq \sum_{l=1}^{\Theta} m_l > 0$ (condition (C_3)), by applying (7.11) we get

$$\mathcal{L}^{-1}\left\{\Pi_E\left(\sum_{j=0}^{N} c_j s^{-(j+1)} x^{(j)}(t_0)\right)\right\}(t) = c_n\, \hat{c}\, x^{(n)}(t_0) \frac{t^{r_0+n}}{\Gamma(r_0+n+1)}. \tag{1.217}$$

Finally, since $c_n \neq 0$, we have

$$x^{(n)}(t_0) = \frac{\Gamma(r_0+n+1)}{c_n\, \hat{c}\, t^{r_0+n}} \int_0^t p_{t,\Theta}(\tau) X_N(\tau) d\tau, \text{ with } t \in D. \tag{1.218}$$

Recall that $X_N(\tau) = \sum_{i=0}^{L} a_i x_N(t_0 + \beta_i \tau)$, then by substituting x_N by x in (1.218) we get

$$\tilde{x}^{(n)}(t_0) = \frac{\Gamma(r_0+n+1)}{c_n\, \hat{c}\, t^{r_0+n}} \sum_{i=0}^{L} a_i \int_0^t p_{t,\Theta}(\tau) x(t_0 + \beta_i \tau) d\tau. \tag{1.219}$$

Here, the variable t is the length of the estimation time interval. The equation (1.219) has therefore to be considered for fixed t, say $t = T \in D$. □

Remark 5 *We can consider a general family of linear differential operators defined by* (1.18)*, where* $\Pi = \sum_{j=1}^{W} \rho_j \Pi_{E_j}$*, with* $W \in \mathbb{N}^*, \rho_j \in \mathbb{R}^*$ *and* Π_{E_j} *being defined by (1.202). Moreover, we assume that the set* E_j *satisfies the conditions* (C_1), (C_2) *and* (C_3). *Such affine annihilators help us to estimate* $x^{(n)}(t_0)$*, which have the following integral form*

$$\tilde{x}^{(n)}(t_0) = \sum_{j=1}^{W} \frac{\Gamma(r_0+n+1)}{c_n\, \hat{C}_W\, T^{r_0+n}} \sum_{i=0}^{L} a_i \int_0^T p_{T,\Theta,j}(\tau) y(t_0 + \beta_i \tau) d\tau, \tag{1.220}$$

where $\hat{C}_W = \sum_{j=1}^{W} \rho_j \hat{c}(j)$, $\hat{c}(j)$ *and each polynomial* $p_{T,\Theta,j}$ *defined by* (1.215) *is associated to* Π_{E_j}.

Three conditions on E are given such that the linear differential operator defined by (1.202) can be an integral annihilator. We show in the following proposition that we can build some sets E such that the conditions (C_1), (C_2) and (C_3) are satisfied.

Proposition 1.7.43 *There exists the sets* $E = \{(n_l, m_l)\}_{l=1}^{\Theta}$ *for* $\Theta \geq 3, \Theta \in \mathbb{N}$ *that meet the conditions* (C_1) *and* (C_2) *given in Proposition* 1.7.39 *and the condition* (C_3) *given in Proposition* 1.7.41.

Proof. We prove firstly that there exists the sets E which meet the conditions (C_1) and (C_2) given in Proposition 1.7.39. Each of these sets give us an annihilator by annihilating all the undesired terms: $c_j s^{-(j+1)} x^{(j)}(t_0)$ in (1.201) with $j \neq n$ and keeping the term $c_n s^{-(n+1)} x^{(n)}(t_0)$ at the same time. The construction of these sets depends on the way of annihilating the undesired terms, but in any case they can be found. We are going to give such a set.

In order to annihilate the undesired terms, it is necessary to let the degree of s be positive. Hence, by taking in particular $n_\Theta = 0$ and $m_\Theta = -n$, we get

$$\begin{aligned} &\frac{1}{s^{m_\Theta}} \frac{d^{n_\Theta}}{ds^{n_\Theta}} \left(\sum_{j=0}^{N} c_j s^{-(j+1)} x^{(j)}(t_0) \right) \\ =&c_0 s^{n-1} x(t_0) + \cdots + c_{n-1} x^{(n-1)}(t_0) + c_n s^{-1} x^{(n)}(t_0) + \cdots + c_N s^{-N-1+n} x^{(N)}(t_0). \end{aligned} \tag{1.221}$$

By taking $n_{\Theta-1} = n+k$ with $k \in \mathbb{N}$, we can annihilate the terms $c_j s^{-(j+1)} x^{(j)}(t_0)$ for $j = 0, \cdots, n-1$. We get $0 \leq -(j+1) - (n_\Theta + m_\Theta) \leq n_{\Theta-1}$ for $j = 0, \cdots, n-1$ and

$$\frac{d^{n_{\Theta-1}}}{ds^{n_{\Theta-1}}} \frac{1}{s^{m_\Theta}} \frac{d^{n_\Theta}}{ds^{n_\Theta}} \left(\sum_{j=0}^{N} c_j s^{-(j+1)} x^{(j)}(t_0) \right) = \sum_{j=n}^{N} c_j \frac{(-1)^{n+k}(j+k)!}{(j-n)!} s^{-1-j-k} x^{(j)}(t_0). \tag{1.222}$$

Then, with the same reason we can annihilate the terms $c_j s^{-(j+1)} x^{(j)}(t_0)$ for $j = n+1, \cdots, N$ by taking $m_{\Theta-1} = -(N+1+k)$ and $n_{\Theta-2} = N-n$. We get $0 \leq -(j+1) - (n_\Theta + m_\Theta + n_{\Theta-1} + m_{\Theta-1}) \leq n_{\Theta-2}$ for $j = n+1, \cdots, N$ and

$$\frac{d^{n_{\Theta-2}}}{ds^{n_{\Theta-2}}} \prod_{i=\Theta-1}^{\Theta} \frac{1}{s^{m_\Theta}} \frac{d^{n_\Theta}}{ds^{n_\Theta}} \left(\sum_{j=0}^{N} c_j s^{-(j+1)} x^{(j)}(t_0) \right) = c_n (-1)^{n+k} (N-n)!(k+n)! x^{(n)}(t_0). \tag{1.223}$$

Utile here we give an annihilator meeting the conditions (C_1) and (C_2). In order to meet the condition (C_3), we can choose $m_{\Theta-2} = |m_\Theta| + |m_{\Theta-1}| + \mu + 1 = N + n + k + \mu + 2$ with $-1 < \mu \in \mathbb{R}$. Finally, we construct the set $\{(n_l, m_l)\}_{l=\Theta-2}^{\Theta}$ which meets the conditions (C_1), (C_2) and (C_3). The associated annihilator is

$$\frac{1}{s^{N+n+k+\mu+2}} \frac{d^{N-n}}{ds^{N-n}} s^{N+1+k} \frac{d^{n+k}}{ds^{n+k}} s^n. \tag{1.224}$$

For $1 \leq l \leq \Theta - 3$, let us take $n_l = m_l = 1$, then the conditions (C_1), (C_2) and (C_3) hold and we have

$$\Pi_E \left(\sum_{j=0}^{N} c_j s^{-(j+1)} x^{(j)}(t_0) \right) = \frac{c_n (-1)^{n+k+\Theta-3} (N-n)!(k+n)! x^{(n)}(t_0)}{s^{N+n+k+\mu+2\Theta-4}} \prod_{i=1}^{\Theta-3} (N+n+k+\mu+2i). \tag{1.225}$$

□

If we take $\Theta = 3$ in the previous proof, then the conditions (C_1), (C_2) and (C_3) are satisfied. For example, the integral annihilator $\Pi_{k,\mu}^{N,n}$ defined in (1.54) is parameterized by the set $E = \{(n_l, m_l)\}_{l=1}^{3}$ where $m_1 = \nu$, $n_1 = n+k$, $m_2 = 1$, $n_2 = N-n$, $m_3 = -(N+1)$, $n_3 = 0$. It is easy to verify that This set meets also the conditions (C_1), (C_2) and (C_3).

It is shown in Lemma 1.3.10 that the annihilator $\Pi_{k,\mu}^{N,n}$ can be written as an affine combination of different annihilators Π_{k_j,μ_j}^{n} defined in (1.20). Inspired by this, let us consider the relation given in (1.211) where Π_E is an annihilator if E meets conditions (C_1), (C_2) and (C_3). We wonder if the $\Pi_{\bar{E}_\mathbf{j}}$ are also annihilators similar to the annihilators Π_{k_j,μ_j}^{n}. By assumption, the operators $\bar{E}_\mathbf{j}$ meet the conditions (C_1) and (C_2). Moreover, from (C_1) and (C_2), as $\gamma_1 + \sum_{l=1}^{\Theta} \bar{m}_l \geq \sum_{l=1}^{\Theta} m_l$, then the condition (C_3) holds automatically. Consequently, the annihilator Π_E applied to (1.201) with $N > n$ will be an affine annihilator of annihilators $\Pi_{\bar{E}_\mathbf{j}}$ applied to (1.201) with $N = n$.

1.8 Fractional derivative estimators

In this section, we discuss some estimators for the derivative with fractional order. Firstly, we apply the algebraic parametric technique to a truncated fractional order Taylor series. Secondly, we apply the Jacobi estimators in the definitions of fractional order derivative where we need to calculate the integer order derivative.

1.8.1 Fractional order Taylor's Formula

It was previously shown that by applying the algebraic parametric technique to the truncated Taylor series expansion we can give some families of derivative estimators, where the order of the estimated derivative is an integer. In the subsection, we show that by applying the algebraic parametric technique to a truncated fractional order Taylor series we can estimate fractional order derivatives.

A generalized Taylor series expansion of fractional order is given in [Jumarie 2006] as follows: let $x \in \mathcal{C}^n(I)$ and $n < \alpha \leq n+1$ with $n \in \mathbb{N}^*$, then we have

$$\forall t_0 \in I, t \in D_{t_0},\ x(t_0+t) = \sum_{j=0}^{n} \frac{t^j}{j!} x^{(j)}(t_0) + \sum_{j=1}^{+\infty} \frac{t^{(j\gamma+n)}}{\Gamma(j\gamma+n+1)} x^{(j\gamma+n)}(t_0), \tag{1.226}$$

where $\gamma = \alpha - n$.

Then, by using this fractional order Taylor series expansion we can give the following proposition.

Proposition 1.8.44 *Let $x \in \mathcal{C}^n(I)$ and $n < \alpha \leq n+1$ with $n \in \mathbb{N}^*$, then a family of anti-causal estimators for the α^{th} order derivative of x is given by*

$$\forall t_0 \in I,\ \tilde{x}_{t_0+}^{(\alpha)}(k,\mu,T) = \frac{(n+1)!}{T^\alpha} \frac{\Gamma(\alpha-n)}{\mathrm{B}(\alpha+1+k, n+\mu+2)} \int_0^1 w_{\mu,k}(\tau) P_{n+1}^{(\mu,k)}(\tau)\, x(t_0+T\tau)\, d\tau, \tag{1.227}$$

where $T \in D_{t_0}$, $k \in \mathbb{N}$ and $-1 < \mu \in \mathbb{R}$.

Proof. By taking the truncation of the fractional Taylor series expansion given in (1.226) on $\mathbb{R}^+$, we get

$$\forall t_0 \in I, t \in \mathbb{R}^+,\ x_\alpha(t_0+t) = \sum_{j=0}^{n} \frac{t^j}{j!} x^{(j)}(t_0) + \frac{t^\alpha}{\Gamma(\alpha+1)} x^{(\alpha)}(t_0). \tag{1.228}$$

Applying the Laplace transform to (1.228), we get

$$\hat{x}_\alpha = \sum_{j=0}^{n} s^{-(j+1)} x^{(j)}(t_0) + s^{-(\alpha+1)} x^{(\alpha)}(t_0). \tag{1.229}$$

Then, by apply the annihilator $\Pi_{k,\mu}^{n+1}$ defined in (1.20) to (1.229) we get

$$\begin{aligned} \Pi_{k,\mu}^{n+1}(\hat{x}_\alpha) &= \frac{1}{s^{n+\mu+2}} \frac{d^{n+1+k}}{ds^{n+1+k}} \frac{1}{s^{\alpha-n}} x^{(\alpha)}(t_0) \\ &= (-1)^{n+1+k} \frac{\Gamma(\alpha+1+k)}{\Gamma(\alpha-n)} \frac{1}{s^{n+\alpha+3+k+\mu}} x^{(\alpha)}(t_0), \end{aligned} \tag{1.230}$$

where $k \in \mathbb{N}$ and $-1 < \mu \in \mathbb{R}$. Then by returning to the time domain, we get

$$\begin{aligned} &(-1)^{n+1+k} \frac{\Gamma(\alpha+1+k)}{\Gamma(\alpha-n)} \frac{T^{n+\alpha+2+k+\mu}}{\Gamma(n+\alpha+k+\mu+3)} x^{(\alpha)}(t_0) \\ =&\frac{(-1)^{n+1+k}}{\Gamma(n+\mu+2)} \int_0^T (T-\tau)^{n+\mu+1} \tau^{n+k+1} x_\alpha^{(n+1)}(t_0+\tau) d\tau. \end{aligned} \tag{1.231}$$

By applying a change of variable $\tau \to T\tau$ and $n+1$ times integrations by parts, we get

$$x^{(\alpha)}(t_0) = \frac{(-1)^{(n+1)}}{T^\alpha} \frac{\Gamma(\alpha-n)}{\mathrm{B}(\alpha+1+k, n+\mu+2)} \int_0^1 \frac{d^{n+1}}{d\tau^{n+1}} \left\{(1-\tau)^{n+\mu+1}\tau^{n+k+1}\right\} x_\alpha(t_0+T\tau)\, d\tau. \tag{1.232}$$

Finally, this proof can be completed by substituting x_α in (1.232) by x and applying the Rodrigues formula to (1.232). □

Remark 6 *If we take $\alpha = n+1$ in (1.227), then it is easy to obtain that*

$$\tilde{x}_{t_{0+}}^{(\alpha)}(k,\mu,T) = \tilde{x}_{t_{0+}}^{(n+1)}(k,\mu,T), \tag{1.233}$$

where $\tilde{x}_{t_{0+}}^{(n+1)}(k,\mu,T)$ are the minimal Jacobi estimators for $x^{(n+1)}(t_0)$ given in Proposition 1.2.1. Moreover, if we take the annihilator $\Pi_{k,\mu}^{N+1,n+1}$ defined in (1.54) with $n \leq N \in \mathbb{N}$, then two families of affine estimators for $\tilde{x}^{(\alpha)}(t_0)$ can be also given similarly.

1.8.2 Application of Jacobi estimators

In this subsection, we give a family of causal estimators for the fractional order derivative.

Let $x \in \mathcal{C}^n(I)$ with $n \in \mathbb{N}^*$, then the α^{th} $(n-1<\alpha<n)$ order derivative of x can be defined as follows

- First method:

$$\forall t_0 \in I,\ D_L^{(\alpha)} x(t_0) := \frac{d^n}{dt_0^n}\left\{\frac{1}{\Gamma(n-\alpha)}\int_{t_0-T}^{t_0} \frac{x(s)}{(t_0-s)^{\alpha+1-n}} ds\right\}, \tag{1.234}$$

- Second method:

$$\forall t_0 \in I,\ D_R^{(\alpha)} x(t_0) := \frac{1}{\Gamma(n-\alpha)}\int_{t_0-T}^{t_0} \frac{x^{(n)}(s)}{(t_0-s)^{\alpha+1-n}} ds, \tag{1.235}$$

where $T \in D_{t_0}$.

Denote $F(t_0) := \frac{1}{\Gamma(n-\alpha)}\int_{t_0-T}^{t_0} \frac{x(s)}{(t_0-s)^{\alpha+1-n}} ds$. Hence, we need the n^{th} order derivative values of F (resp. x) to calculate the α^{th} order derivative of x by using (1.234) (resp. (1.235)). For this, we use the causal Jacobi estimators. Thus, by using (1.84) we give the following estimators

$$\begin{aligned} \forall t_0 \in I,\ \tilde{D}_L^{(\alpha)} x(t_0) :&= \frac{1}{(-T)^n}\int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau) F(t_0 - T\tau) d\tau, \\ &= \frac{1}{(-T)^n\Gamma(n-\alpha)}\int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau)\int_{t_0-T\tau-T}^{t_0-T\tau} \frac{x(s)}{(t_0-T\tau-s)^{\alpha+1-n}} ds\, d\tau, \end{aligned} \tag{1.236}$$

and

$$\begin{aligned} \forall t_0 \in I,\ \tilde{D}_R^{(\alpha)} x(t_0) :=& \frac{1}{\Gamma(n-\alpha)}\int_{t_0-T}^{t_0} \frac{1}{(t_0-s)^{\alpha+1-n}} D_{\kappa,\mu,\beta T,q}^{(n)} x(s-T\xi)\, ds, \\ =& \frac{1}{(-T)^n\Gamma(n-\alpha)}\int_{t_0-T}^{t_0} \frac{1}{(t_0-s)^{\alpha+1-n}}\int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau) x(-T\tau+s) d\tau\, ds, \end{aligned} \tag{1.237}$$

where $Q_{\kappa,\mu,n,q,\xi}$ is given by (1.85) with $\kappa, \mu \in]-1, +\infty[$, $q \in \mathbb{N}$ and $\xi \in [0,1]$.

By applying a change of variable in (1.236): $s \to s - T\tau$, we get

$$\forall t_0 \in I,\ \tilde{D}_L^{(\alpha)} x(t_0) := \frac{1}{(-T)^n \Gamma(n-\alpha)} \int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau) \int_{t_0-T}^{t_0} \frac{x(s-T\tau)}{(t_0-s)^{\alpha+1-n}} ds\, d\tau. \tag{1.238}$$

Hence, if $\kappa, \mu \in \mathbb{N}$, then $Q_{\kappa,\mu,n,q,\xi}$ is a polynomial. By applying the Fubini's theorem it is easy to get that $\tilde{D}_L^{(\alpha)} x(t_0) = \tilde{D}_R^{(\alpha)} x(t_0)$. Let us denote $D_{\kappa,\mu,T,q}^{(\alpha)} x(t_0) := \tilde{D}_L^{(\alpha)} x(t_0) = \tilde{D}_R^{(\alpha)} x(t_0)$, then we give the following proposition.

Proposition 1.8.45 *Let $x \in \mathcal{C}^n(I)$ and $n-1 < \alpha \leq n$ with $n \in \mathbb{N}^*$, then a family of estimators for the α^{th} order derivative of x is given by*

$$\forall t_0 \in I,\ D_{\kappa,\mu,T,q}^{(\alpha)} x(t_0) = \frac{(-1)^n}{T^\alpha \Gamma(n-\alpha)} \int_{-1}^1 p_{\kappa,\mu,n,q,\xi}(u) x(Tu + t_0 - T) du, \tag{1.239}$$

where

$$p_{\kappa,\mu,n,q,\xi}(u) = \sum_{i=0}^{q} C_{\kappa,\mu,n,i} P_i^{(\mu+n,\kappa+n)}(\xi) \sum_{j=0}^{n+i} \binom{n+i+\mu}{j} \binom{n+i+\kappa}{n+i-j} (-1)^{n+i-j} I_{\kappa,\mu,i,j}(u), \tag{1.240}$$

with $I_{\kappa,\mu,i,j}(u) = \displaystyle\sum_{l=0}^{\mu+n+i-j} \sum_{k=0}^{\kappa+j} \binom{\mu+n+i-j}{l} \binom{\kappa+j}{k} \mathrm{B}(\mu+2n+i-j-l-\alpha-1, \kappa+j-k)(-1)^k u^{l+k}$, $T \in D_{t_0}$, $\kappa, \mu \in \mathbb{N}$, $q \in \mathbb{N}$ *and* $\xi \in [0,1]$.

Lemma 1.8.46 *Let $\hat{\kappa}, \hat{\mu} \in \mathbb{N}$, then we have the following integral value*

$$\int_0^1 \frac{w_{\hat{\mu},\hat{\kappa}}(v-u)}{(1-v)^{\alpha+1-n}} dv = \sum_{i=0}^{\hat{\mu}} \sum_{j=0}^{\hat{\kappa}} \binom{\hat{\mu}}{i} \binom{\hat{\kappa}}{j} \mathrm{B}(\hat{\mu}-i+n-\alpha, \hat{\kappa}-j+1)(-1)^j\, u^{i+j}. \tag{1.241}$$

Proof. By using the Binomial theorem, we get

$$\begin{aligned} \frac{w_{\hat{\mu},\hat{\kappa}}(v-u)}{(1-v)^{\alpha+1-n}} &= (1-v+u)^{\hat{\mu}} (v-u)^{\hat{\kappa}} (1-v)^{n-\alpha-1} \\ &= \left(\sum_{i=0}^{\hat{\mu}} \binom{\hat{\mu}}{i} (1-v)^{\hat{\mu}-i+n-\alpha-1} u^i \right) \left(\sum_{j=0}^{\hat{\kappa}} \binom{\hat{\kappa}}{j} v^{\hat{\kappa}-j} (-u)^j \right) \\ &= \sum_{i=0}^{\hat{\mu}} \sum_{j=0}^{\hat{\kappa}} \binom{\hat{\mu}}{i} \binom{\hat{\kappa}}{j} (1-v)^{\hat{\mu}-i+n-\alpha-1} v^{\hat{\kappa}-j} (-1)^j u^{i+j}. \end{aligned} \tag{1.242}$$

Then, this proof can be completed by using the classical beta function. □

Proof of Proposition 1.8.45. By applying a change of variable in (1.238): $s \to sT + (t_0 - T)$, we get

$$\forall t_0 \in I,\ \tilde{D}_L^{\alpha} x(t_0) = \frac{(-1)^n}{T^\alpha \Gamma(n-\alpha)} \int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau) \left(\int_0^1 \frac{x(sT + (t_0 - T\tau - T))}{(1-s)^{\alpha+1-n}} ds \right) d\tau. \tag{1.243}$$

By apply the following change of variables: $(\tau, s) = (v-u, v) = \Phi(u,v)$ in (1.243), we get

$$\begin{aligned}\forall t_0 \in I,\ \tilde{D}_L^{\alpha} x(t_0) &= \frac{(-1)^n}{T^{\alpha}\Gamma(n-\alpha)} \int_{-1}^{1} Q_{\kappa,\mu,n,q,\xi}(v-u) \left(\int_0^1 \frac{x(Tu+t_0-T)}{(1-v)^{\alpha+1-n}} \left| det(J_\Phi) \right| dv \right) du \\ &= \frac{(-1)^n}{T^{\alpha}\Gamma(n-\alpha)} \int_{-1}^{1} x(Tu+t_0-T) \left(\int_0^1 \frac{Q_{\kappa,\mu,n,q,\xi}(v-u)}{(1-v)^{\alpha+1-n}} dv \right) du.\end{aligned} \tag{1.244}$$

Using (1.42) in (1.85) we get

$$Q_{\kappa,\mu,n,q,\xi}(\tau) = \sum_{i=0}^{q} C_{\kappa,\mu,n,i} P_i^{(\mu+n,\kappa+n)}(\xi) \sum_{j=0}^{n+i} \binom{n+i+\mu}{j} \binom{n+i+\kappa}{n+i-j} (-1)^{n+i-j} w_{\mu_{ij},\kappa_j}(\tau), \tag{1.245}$$

where $\mu_{ij} = \mu + n + i - j$ and $\kappa_j = \kappa + j$. Then, this proof can be completed by using Lemma 1.8.46. □

1.9 Conclusion

In this chapter, by taking truncated Taylor series expansion and truncated Jacobi orthogonal series expansion we have given some different derivative estimators. The associated truncated term errors have been studied by giving some corresponding error bounds which showed the parameters' influence on truncated term errors. Let us recall these derivative estimators in Table 1.9, where $\kappa, \mu \in]-1, +\infty[$, $T \in D_{t_0}$, $q \in \mathbb{N}$, $\xi \in [0,1]$, $\lambda \in \mathbb{R}_+/\{1\}$ and ξ_q is one root of the Jacobi polynomial $P_{q+1}^{(\mu+n,\kappa+n)}$. In the following chapter, we will study our derivative estimators in the case where the smooth function is corrupted by a noise.

Name	Estimator	Convergence rate	Needed smoothness on x	Equation
Causal minimal Jacobi estimator	$D^{(n)}_{\kappa,\mu,-T}x(t_0)$	$\mathcal{O}(T)$	$\mathcal{C}^{n+1}$	(1.29)
Anti-causal minimal Jacobi estimator	$D^{(n)}_{\kappa,\mu,T}x(t_0)$	$\mathcal{O}(T)$	$\mathcal{C}^{n+1}$	(1.29)
Causal Jacobi estimator	$D^{(n)}_{\kappa,\mu,-T,q}x(t_0)$	$\mathcal{O}(T^{q+1})$	$\mathcal{C}^{n+1+q}$	(1.84)
Time-delay estimator	$D^{(n)}_{\kappa,\mu,-T,q}x(-T\xi_q+t_0)$	$\mathcal{O}(T^{q+2})$	$\mathcal{C}^{n+q+2}$	(1.84)
Anti-causal Jacobi estimator	$D^{(n)}_{\kappa,\mu,T,q}x(t_0)$	$\mathcal{O}(T^{q+1})$	$\mathcal{C}^{n+1+q}$	(1.84)
Time-advance estimator	$D^{(n)}_{\kappa,\mu,T,q}x(T\xi_q+t_0)$	$\mathcal{O}(T^{q+2})$	$\mathcal{C}^{n+2+q}$	(1.84)
Causal Richardson-Jacobi estimator	$D^{(n)}_{\kappa,\mu,-T,\lambda}x(t_0)$	$\mathcal{O}(T^2)$	$\mathcal{C}^{n+2}$	(1.133)
Anti-causal Richardson-Jacobi estimator	$D^{(n)}_{\kappa,\mu,T,\lambda}x(t_0)$	$\mathcal{O}(T^2)$	$\mathcal{C}^{n+2}$	(1.133)
Causal modified estimator	$E^{(n)}_{\kappa,\mu+n+1,-T}x(t_0)$	$\mathcal{O}(T^3)$	$\mathcal{C}^{n+3}$	(1.140)
Anti-causal modified estimator	$E^{(n)}_{\kappa,\mu+n+1,T}x(t_0)$	$\mathcal{O}(T^3)$	$\mathcal{C}^{n+3}$	(1.140)
Central estimator	$D^{(n)}_{\kappa,\mu,T,q,\xi}x(t_0)$	$\mathcal{O}(T^{q+1})$	$\mathcal{C}^{n+1+q}$	(1.156)
Improved with even integer q	$D^{(n)}_{\kappa,\mu,T,q,\xi=0}x(t_0)$	$\mathcal{O}(T^{q+2})$	$\mathcal{C}^{n+2+q}$	(1.156)
Improved with odd integer n	$\tilde{x}^{(n)}_{t_0}(k=0,\mu,T,n+q)$	$\mathcal{O}(T^{q+2})$	$\mathcal{C}^{n+2+q}$	(1.151)
Central Jacobi estimator	$\hat{D}^{(n)}_{\kappa,\mu,T,q}x(t_0)$	$\mathcal{O}(T^{q+1})$	$\mathcal{C}^{n+1+q}$	(1.169)
Improved with even integer q	$\hat{D}^{(n)}_{\kappa,\mu=\kappa,T,q}x(t_0)$	$\mathcal{O}(T^{q+2})$	$\mathcal{C}^{n+2+q}$	(1.171)
Central Richardson-Jacobi estimator	$\hat{D}^{(n)}_{\kappa,\mu,T,\lambda}x(t_0)$	$\mathcal{O}(T^2)$	$\mathcal{C}^{n+2}$	(1.180)
Improved with $\kappa=\mu$	$\hat{D}^{(n)}_{\kappa,T,\lambda}x(t_0)$	$\mathcal{O}(T^4)$	$\mathcal{C}^{n+4}$	(1.183)

Chapter 2

Error analysis for the Jacobi estimators

Contents

2.1 Introduction

2.1.1 Context

In practical identification, an observed signal is usually obtained from a sensor, which is quantitized and discretized. If the sensor is not ideal, a random noise can be observed, the amplitude of which can be assumed to be finite. Hence, we consider in this chapter the numerical differentiation for noisy signals.

2.1.2 Noise error contribution

We assume that $x^\delta = x + \varpi$ is a noisy function on I, where $x \in \mathcal{C}^n(I)$ and ϖ is an additive corrupting noise which is usually a rapid oscillation. From now on, we consider the noisy function x^δ in the Jacobi estimators. Then, it is sufficient to replace $x(t_0 + \beta T \cdot)$ in (1.84) by $x^\delta(t_0 + \beta T \cdot)$ so as to estimate $x^{(n)}(t_0)$. Then, we have the following definition.

Definition 3 *Let x^δ is a noisy function on I, then, for any $t_0 \in I$ a family of Jacobi estimators for $x^{(n)}(t_0)$ is defined as follows*

$$D^{(n)}_{\kappa,\mu,\beta T,q} x^\delta(\beta T\xi + t_0) = \frac{1}{(\beta T)^n} \int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau)\, x^\delta(t_0 + \beta T \tau)\, d\tau, \tag{2.1}$$

where $T \in D_{t_0}$, $\beta = \pm 1$, $Q_{\kappa,\mu,n,q,\xi}$ is defined by (1.85) with $\kappa, \mu \in]-1, +\infty[$, $q \in \mathbb{N}$ and ξ is a fixed value on $[0,1]$.

Hence, the associated estimation errors are given as follows

$$\begin{aligned}
& D^{(n)}_{\kappa,\mu,\beta T,q} x^\delta(\beta T\xi + t_0) - x^{(n)}(t_0) \\
=& \Big(D^{(n)}_{\kappa,\mu,\beta T,q} x^\delta(\beta T\xi + t_0) - D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) \Big) + \Big(D^{(n)}_{\kappa,\mu,\beta T,q} x(\beta T\xi + t_0) - x^{(n)}(t_0)) \Big) \\
=& e^\beta_\varpi(t_0; n, \kappa, \mu, T, \xi, q) + e^\beta_{R_n}(t_0; \kappa, \mu, T, \xi, q).
\end{aligned} \tag{2.2}$$

Hence, the Jacobi estimators $D^{(n)}_{\kappa,\mu,\beta T,q} x^\delta(\beta T\xi + t_0)$ are corrupted by two sources of errors:

- the truncated term error $e^\beta_{R_n}(t_0; \kappa, \mu, T, \xi, q)$,
- the noise error contribution $e^\beta_\varpi(t_0; n, \kappa, \mu, T, \xi, q)$.

Now, we assume that the noisy function x is given in discrete case. Let $x^\delta(t_i) = x(t_i) + \varpi(t_i)$ be a noisy measurement of x with an equidistant sampling period T_s.

Since x^δ is a discrete measurement, we need to use a numerical integration method to approximate the integral value in (1.84) for Jacobi estimators. Let $t_i = \frac{i}{m}$ and $w_i > 0$ for $i = 0, \ldots, m$ with $m = \frac{T}{T_s} \in \mathbb{N}$ (except for $w_0 \geq 0$ and $w_m \geq 0$) be respectively the abscissas and the weights for a given numerical integration method. Weight w_0 (resp. w_m) is set to zero in order to avoid the infinite values when κ (resp. μ) is negative. Then, we have

$$D^{(n)}_{\kappa,\mu,\beta T,q} x^\delta(\beta T\xi + t_0) \approx \frac{1}{(\beta T)^n} \sum_{i=0}^{m} \frac{w_i}{m} Q_{\kappa,\mu,n,q,\xi}(t_i)\, x^\delta(t_0 + \beta T t_i). \tag{2.3}$$

The noise error contribution $e^\beta_\varpi(t_0; \kappa, \mu, T, \xi, q)$ can be written in discrete cases as follows

$$e^\beta_{\varpi,m}(t_0; n, \kappa, \mu, T, \xi, q) := \frac{1}{(\beta T)^n} \sum_{i=0}^{m} \frac{w_i}{m} Q_{\kappa,\mu,n,q,\xi}(t_i)\, \varpi(t_0 + \beta T t_i). \tag{2.4}$$

This numerical integration method also implies a numerical error. Hence, the Jacobi estimators lead to

$$D^{(n)}_{\kappa,\mu,\beta T,q} x^\delta(\beta T\xi + t_0) = x^{(n)}(t_0) + e_m(t_0) + e^\beta_{R_n,m}(t_0; \kappa, \mu, T, \xi, q) + e^\beta_{\varpi,m}(t_0; n, \kappa, \mu, T, \xi, q), \tag{2.5}$$

where $e^{\beta}_{R_n,m}(t_0;\kappa,\mu,T,\xi,q)$ is the bias term error in discrete case and $e_m(t_0)$ is the numerical integration error.

The truncated term error was studied in the previous chapter, we study in this chapter the noise error contribution only for the causal and anti-causal Jacobi estimators. The ones for the central Jacobi estimators and the Richardson-Jacobi estimators (central or not) can be similarly studied. In Section 2.2, we recall the result of nonstandard analysis of noise given in [Fliess 2006]. In Section 2.3, the noise ϖ is a bounded and integrable function such as biased sinusoidal functions with high frequency. In Section 2.4, the noise ϖ is a non-independent stochastic process such as the Brownian process and the Poisson process, which is bounded with certain probability and integrable in the sense of convergence in mean square. In Section 2.4, the noise ϖ is an independent stochastic process such as the White Gaussian noise and the Poisson noise. In Section 2.6, we give some numerical simulations to demonstrate the efficiency and the stability of Jacobi estimators.

2.2 Nonstandard analysis of noise

Thanks to the nonstandard formalization of fast oscillating functions, due to P. Cartier and Y. Perrin [Cartier 1995], M. Fliess proposed in [Fliess 2006, Fliess 2008] an appropriate mathematical framework for the algebraic parametric techniques methods, which exhibit good robustness properties with respect to corrupting noises, without the need of knowing their statistical properties. In other words, to assume that the noise is Gaussian, or that its statistics are known, is not required at all. This assumption is common in other well-known methods like maximum likelihood, minimum least squares or Kalman filtering approach to parameter estimation. More precisely, according to the nonstandard theory of noise in [Fliess 2006], the noise ϖ is a S-integrable fast oscillating. In this case, x^{δ} is S-integrable, *i.e.* the sum of the Lebesgue integrable function x and ϖ. According to Proposition 3.2 in [Fliess 2006], by choosing an appreciable length T for integration window, the noise error contribution $e^{\beta}_{\varpi}(t_0;n,\kappa,\mu,T,\xi,q)$ can be very small even for an unbounded noise. Nevertheless, when compared to classical approaches in communication engineering (see, e.g., [Proakis 2001]), a weakness of these methods was a lack of any precise error analysis, when they are implemented in practice. To carry out such analysis and comparison, we only consider here often bounded noises which are in practice the most frequent situation we encounter. However, as mentioned after we can deal with noises "polynomial in time".

2.3 Integrable noises

In this section, we assume that the noise ϖ is a bounded and integrable function, which is written as follows

$$\forall\, t\in I,\quad \varpi(t)=\varrho(t)+\varpi_0(t), \tag{2.6}$$

where $\varrho(t)=\sum_{j=0}^{n-1}\nu_j\,t^j$ and ϖ_0 is a bounded noise with a noise level δ *i.e.* $\delta=\sup_{t\in I}|\varpi_0(t)|$. Then, according to [Fliess 2003a, Fliess 2004a] the polynomial ϱ of degree $n-1$ is an $(n-1)^{th}$ order *structured* *perturbation* which, like a polynomial perturbation of unknown amplitude, is solutions of a given homogeneous linear differential equation, and ϖ_0 is an *unstructured noise*, which is understood as high frequency perturbations.

We show in the following lemma that the $(n-1)^{th}$ order structured perturbation can be annihilated in the estimation for n^{th} order derivatives by Jacobi estimators.

Lemma 2.3.47 *Let $e_{\varrho}^{\beta}(t_0;n,\kappa,\mu,T,\xi,q)$ be the noise error contribution due to an $(n-1)^{th}$ order structured perturbation for the Jacobi estimator $D_{\kappa,\mu,\beta T,q}^{(n)}x^{\delta}(\beta T\xi+t_0)$ defined in (2.1), then we have*

$$e_{\varrho}^{\beta}(t_0;n,\kappa,\mu,T,\xi,q) := \frac{1}{(\beta T)^n}\int_0^1 Q_{\kappa,\mu,n,q,\xi}(\tau)\,\varrho(t_0+\beta T\tau)\,d\tau = 0. \tag{2.7}$$

Proof. By using the orthogonality of the Jacobi polynomials with the expression of $Q_{\kappa,\mu,n,q,\xi}$ given in (1.85), we obtain that

$$\int_0^1 \tau^j\, Q_{\kappa,\mu,n,q,\xi}(\tau)\,d\tau = 0, \quad \text{for any } j\in\{0,\dots,n-1\}. \tag{2.8}$$

Then, this proof can be easily completed. □

We study in the previous chapter the convergence rate for the Jacobi estimators, which is studied in the following proposition in the noise case.

Proposition 2.3.48 *[Liu 2011a] Let x^{δ} be a noisy function where $x\in\mathcal{C}^{n+1+q}(I)$ ($q\in\mathbb{N}$) and noise ϖ is given in (2.6). Assume that there exists $M_{n+1+q}\in\mathbb{R}_+^*$ such that $\left\|x^{(N+1)}\right\|_\infty \le M_{n+1+q}$, then we have*

$$\left\|D_{\kappa,\mu,\beta T,q}^{(n)}x^{\delta}(\beta T\xi+t_0)-x^{(n)}(t_0+\beta T\xi)\right\|_\infty \le M_{N+1}C_{\kappa,\mu,n,q,\xi}T^{q+1}+E_{\kappa,\mu,n,q,\xi}\frac{\delta}{T^n}, \tag{2.9}$$

where $C_{\kappa,\mu,n,q,\xi}$ is given in (1.91) and $E_{\kappa,\mu,n,q,\xi}=\int_0^1 |Q_{\kappa,\mu,n,q,\xi}(\tau)|\,d\tau$. Moreover, if we choose $T=\left[\frac{nE_{\kappa,\mu,n,q,\xi}}{(q+1)M_{N+1}C_{\kappa,\mu,n,q,\xi}}\delta\right]^{\frac{1}{n+q+1}}$, then we have

$$\left\|D_{\kappa,\mu,\beta T,q}^{(n)}x^{\delta}(\beta T\xi+t_0)-x^{(n)}(t_0+\beta T\xi)\right\|_\infty = \mathcal{O}(\delta^{\frac{q+1}{n+1+q}}). \tag{2.10}$$

Proof. By applying Lemma 2.3.47, the noise error contributions for $D_{\kappa,\mu,\beta T,q}^{(n)}x^{\delta}(\beta T\xi+t_0)$ are bounded by

$$\begin{aligned}
&\left\|D_{\kappa,\mu,\beta T,q}^{(n)}x^{\delta}(\beta T\xi+t_0)-D_{\kappa,\mu,\beta T,q}^{(n)}x(\beta T\xi+t_0)\right\|_\infty\\
=&\left\|D_{\kappa,\mu,\beta T,q}^{(n)}\left[x^{\delta}(\beta T\xi+t_0)-x(\beta T\xi+t_0)\right]\right\|_\infty\\
\le&\frac{\delta}{T^n}\int_0^1 |Q_{\kappa,\mu,n,q,\xi}(\tau)|\,d\tau.
\end{aligned}$$

Then, by using (1.91) we get

$$\begin{aligned}
&\left\|D_{\kappa,\mu,\beta T,q}^{(n)}x^{\delta}(\beta T\xi+t_0)-x^{(n)}(t_0+\beta T\xi)\right\|_\infty\\
&\quad\le \left\|D_{\kappa,\mu,\beta T,q}^{(n)}x^{\delta}(\beta T\xi+t_0)-D_{\kappa,\mu,\beta T,q}^{(n)}x(\beta T\xi+t_0)\right\|_\infty\\
&\quad\quad+\left\|D_{\kappa,\mu,\beta T,q}^{(n)}x(\beta T\xi+t_0)-x^{(n)}(t_0+\beta T\xi)\right\|_\infty\\
&\quad\le M_{N+1}C_{\kappa,\mu,n,q,\xi}T^{q+1}+E_{\kappa,\mu,n,q,\xi}\frac{\delta}{T^n},
\end{aligned}$$

where $E_{\kappa,\mu,n,q,\xi} = \int_0^1 |Q_{\kappa,\mu,n,q,\xi}(\tau)|\, d\tau$.

Let us denote the error bound by $\psi(T) = M_{N+1} C_{\kappa,\mu,n,q,\xi} T^{q+1} + E_{\kappa,\mu,n,q,\xi} \frac{\delta}{T^n}$. Consequently, we can calculate its minimum value. It is obtained for $T^* = \left[\frac{n E_{\kappa,\mu,n,q,\xi}}{(q+1) C_{\kappa,\mu,n,q,\xi}} \delta \right]^{\frac{1}{n+q+1}}$ and

$$\psi(T^*) = \frac{n+1+q}{q+1} \left(\frac{q+1}{n} \right)^{\frac{n}{n+1+q}} C_{\kappa,\mu,n,q,\xi}^{\frac{n}{n+1+q}} E_{\kappa,\mu,n,q,\xi}^{\frac{q+1}{n+1+q}} \delta^{\frac{q+1}{n+1+q}}. \tag{2.11}$$

Then, the proof is completed. □

Let us remark that the error bound $E_{\kappa,\mu,n,q,\xi} \frac{\delta}{T^n}$ obtained in the previous proof depends on the parameters κ, μ, T and ξ which can help us in minimizing the noise error contributions. From the previous chapter, we extend the values of κ, μ from $\mathbb{N}$ to $]-1, +\infty[$. Hence, we obtain a higher degree of freedom so as to minimize the noise effects on our estimators, as well as the minimum value $\psi(T^*)$ obtained in (2.11).

It is clear that we can increase the value of T to decrease the the value of $E_{\kappa,\mu,n,q,\xi} \frac{\delta}{T^n}$ so as to decrease the noise error contributions for Jacobi estimators. Consequently, according to the expression of Richardson-Jacobi estimators we can increase the value of λ so as to decrease the associated noise error contributions.

If we take $q = 0$ in the previous proposition, then the convergence rate for the minimal estimators $D^{(n)}_{\kappa,\mu,\beta T} x^\delta(t_0)$ is equal to $\mathcal{O}(\delta^{\frac{1}{n+1}})$ as $T \to 0$. If we take $\xi = 0$ in the previous proposition, then the convergence rate for the estimators $D^{(n)}_{\kappa,\mu,\beta T,q} x^\delta(t_0)$ is equal to $\mathcal{O}(\delta^{\frac{q+1}{n+1+q}})$ as $T \to 0$. In Proposition 1.3.17, we improve the convergence rate from $\mathcal{O}(T)$ to $\mathcal{O}(T^{q+1})$ ($q \in \mathbb{N}$) for the exact function x by taking an affine combination of minimal estimators of $x^{(n)}$. Here, the convergence rate is also improved for noisy functions. It passes from $\mathcal{O}(\delta^{\frac{1}{n+1}})$ to $\mathcal{O}(\delta^{\frac{q+1}{n+1+q}})$ if we choose $T = c\,\delta^{\frac{1}{n+1+q}}$, where c is a constant.

Similarly, we can calculate the convergence rate for noisy functions in the other cases, where the value of ξ is equal to the smallest root of the Jacobi polynomial $P^{(\mu+n,\kappa+n)}_{q+1}$ by using Corollary 1.3.18, the function x belongs to the Beppo-Levi space by using Proposition 1.4.21.

Since there is a numerical error in the discrete case, we always set the value of T larger than the optimal one calculated in the previous proof.

2.4 Non-independent stochastic process noises

2.4.1 Integrability of stochastic process

In this section, we assume that the noise ϖ is a continuous parameter stochastic process (see [Parzen 1962]). Before analyzing the noise error contribution of such noises, let us study the existence of these integrals in the expressions of Jacobi estimators. As the function x^δ is the sum of x and the noise ϖ, the Jacobi estimators are well defined if and only if the noise ϖ is integrable. Indeed, according to (1.85), if ϖ is integrable then integrability of $w_{\mu,\kappa}(\cdot)\varpi(t_0 + \beta T \cdot)$ holds for $\mu, \kappa \in]-1, +\infty[$ and $T \in \mathcal{D}_{t_0}$ with $\beta = \pm 1$. Thus, in that case, the integrals in the Jacobi estimators exist. The next result (Lemma 2.4.49) proves the existence of these integrals and thus justifies (1.84), as soon as the integral are understood in the sense of convergence in mean square (see Proposition 2.4.50). For this, the stochastic process $\{\varpi(\tau), \tau \geq 0\}$ should satisfy the following condition

(C_1) : $\{\varpi(\tau), \tau \geq 0\}$ is a continuous parameter stochastic process with finite second moments, whose mean value function and covariance kernel are continuous functions.

Lemma 2.4.49 *[Liu 2011c] Let $\{\varpi(\tau), \tau \geq 0\}$ be a stochastic process satisfying condition (C_1). Then for any $t_0 \in I$ and $T \in \mathcal{D}_{t_0}$, the integral $\int_0^1 w_{\mu,\kappa}(\tau)\varpi(t_0+T\tau)d\tau$ (with $\mu, \kappa \in]-1, +\infty[$) is well defined as a limit in mean square of the usual approximating sum of the following form*

$$\int_0^1 Y(\tau)d\tau = \lim_{m\to\infty} \sum_{l=1}^{m} (\tau_l - \tau_{l-1})\, Y_l, \tag{2.12}$$

where $Y(\tau) = w_{\mu,\kappa}(\tau)\varpi(t_0 + T\tau)$, $Y_l = Y(\xi_l)$ for any $\xi_l \in]\tau_{l-1}, \tau_l[$ and $0 = \tau_0 < \tau_1 < \cdots < \tau_m = 1$ is a subdivision of the interval $]0, 1[$, such that $\max_{l=1,\cdots,m} (\tau_l - \tau_{l-1})$ tends to 0 when m tends to infinite.

Proof. For any fixed $t_0 \in D$, it was shown in [Loève 1963] (p. 472) that if $\{Y(\tau), 0 < \tau < 1\}$, where $Y(\tau) = w_{\mu,\kappa}(\tau)\varpi(t_0 + T\tau)$, is a continuous parameter stochastic process with finite second moments, then a necessary and sufficient condition such that the family of approximating sums on the right-hand side of (2.12) has a limit in the sense of convergence in mean square is that the double integral $\int_0^1 \int_0^1 E[Y(s)Y(\tau)]\, ds\, d\tau$ exists.

Since for any $\tau \in]0, 1[$, $(1-\tau)^\alpha\, \tau^\beta < \infty$, and $\{\varpi(\tau), \tau \geq 0\}$ is a continuous parameter stochastic process with finite second moments, so does $\{Y(\tau), 0 < \tau < 1\}$ for any $t_0 \in I$. Moreover, since the mean value function and covariance kernel of $\varpi(\tau)$ are continuous functions, so does $E[\varpi(t_0+T\tau)\,\varpi(t_0+Ts)]$ for all $\tau, s \in [0, 1]$. Hence, $E[\varpi(t_0 + T\tau)\,\varpi(t_0 + Ts)]$ is bounded for all $\tau, s \in [0, 1]$.

Consequently, $\int_0^1 \int_0^1 w_{\mu,\kappa}(\tau)w_{\mu,\kappa}(s)\, E[\varpi(t_0 + T\tau)\,\varpi(t_0 + Ts)]\, ds\, d\tau$ exists when $\kappa, \mu \in]-1, +\infty[$, which implies that (2.12) holds. □

If we take x^δ instead of ϖ in the previous lemma, then we can obtain the following proposition.

Proposition 2.4.50 *[Liu 2011c] If $x \in C^n(I)$, and the noise ϖ satisfies condition (C_1), then for any $t_0 \in I$, the integrals in the Jacobi estimators exist in the sense of convergence in mean square.*

2.4.2 Error bounds for noise error contribution

From now on, we can investigate the noise error contribution for the Jacobi estimator. To simplify our notations, we denote $e^{\beta}_{\varpi}(t_0; n, \kappa, \mu, T, \xi, q)$ by $e^{\beta T}_{\varpi}(t_0)$. However, ϖ satisfying condition (C_1) is usually not bounded. In order to study the convergence rate as it is done in Proposition 2.3.48, we use the Bienaymé-Chebyshev inequality to give an error bounds for this noise error. Then, we have for any real number $\gamma > 0$

$$Pr\left(\left|e^{\beta T}_{\varpi}(t_0) - E[e^{\beta T}_{\varpi}(t_0)]\right| < \gamma\sqrt{Var[e^{\beta T}_{\varpi}(t_0)]}\right) > 1 - \frac{1}{\gamma^2}, \tag{2.13}$$

i.e. the probability for $e^{\beta T}_{\varpi}(t_0)$ to be within the interval $]M_l, M_h[$ is higher than $1 - \frac{1}{\gamma^2}$, where $M_l = E[e^{\beta T}_{\varpi}(t_0)] - \gamma\sqrt{Var[e^{\beta T}_{\varpi}(t_0)]}$ and $M_h = E[e^{\beta T}_{\varpi}(t_0)] + \gamma\sqrt{Var[e^{\beta T}_{\varpi}(t_0)]}$. Then, we give two error bounds as follows

$$M_l \overset{p_r}{<} e^{\beta T}_{\varpi}(t_0) \overset{p_r}{<} M_h, \tag{2.14}$$

where $a \overset{p_r}{<} b$ means that the probability for a real number b to be larger than an other real number a is equal to p_r with $p_r > 1 - \frac{1}{r^2}$. Thus, we have

$$\left| e_{\varpi}^{\beta T}(t_0) \right| \overset{p_r}{<} M_{max}, \tag{2.15}$$

where $M_{max} = \max(|M_l|, |M_h|)$. Consequently, we can use M_{max} in Proposition 2.3.48 as the error bound for the noise error part so as to study the convergence rate for the Jacobi estimators.

In order to obtain these bounds we need to compute the mean and variance of $e_{\varpi}^{\beta T}(t_0)$. To simplify our notations, we denote the function $Q_{\kappa,\mu,n,q,\xi}$ associated to the Jacobi estimator by Q. Then by applying Theorem 3A in [Parzen 1962] (p. 79) the means, variances and covariances of the noise error contributions for the Jacobi estimator are given as follows

$$E\left[e_{\varpi}^{\beta T}(t_0)\right] = \frac{1}{(\beta T)^n} \int_0^1 Q(\tau)\; E\left[\varpi(t_0 + \beta T \tau)\right] d\tau, \tag{2.16}$$

$$Cov\left[e_{\varpi}^{\beta T_1}(t_0), e_{\varpi}^{\beta T_2}(t_0)\right] = \frac{1}{T_1^n T_2^n} \int_0^1 \int_0^1 Q(s)\, Q(\tau)\, Cov\left[\varpi(t_0 + \beta T_1 s),\, \varpi(t_0 + \beta T_2 \tau)\right] ds\, d\tau, \tag{2.17}$$

$$Var\left[e_{\varpi}^{\beta T}(t_0)\right] = Cov\left[e_{\varpi}^{\beta T}(t_0), e_{\varpi}^{\beta T}(t_0)\right], \tag{2.18}$$

where $T, T_1, T_2 \in D_{t_0}$.

By using Lemma 2.3.47, we show in the following theorem that the Jacobi estimator can deal with a large class of noises for which the mean and covariance are polynomials in time satisfying the following conditions

(C_2) : $\forall (t_0 + \tau) \in I$, the following holds

$$E[\varpi(t_0 + \tau)] = \sum_{i=0}^{n-1} \nu_i\, t_0^{k_1(i)} \tau^i + E[\varpi(\tau)], \tag{2.19}$$

$$Cov[\varpi(t_0 + s), \varpi(t_0 + \tau)] = \sum_{i=0}^{n_1} \eta_i\, t_0^{k_2(i)} \tau^i \sum_{i=0}^{n_2} \eta_i'\, t_0^{k_3(i)} s^i + Cov[\varpi(s), \varpi(\tau)], \tag{2.20}$$

where $k_1(i) \in \mathbb{N}$, $k_2(i) \in \mathbb{N}$, $k_3(i) \in \mathbb{N}$, $\nu_i \in \mathbb{R}$, $\eta_i \in \mathbb{R}$, $\eta_i' \in \mathbb{R}$ and $n_1 \in \mathbb{N}$, $n_2 \in \mathbb{N}$ such that $\min(n_1, n_2) \leq n - 1$.

(C_3) : $\forall \tau \in I$, the following holds

$$E[\varpi(\tau)] = \sum_{i=0}^{n-1} \bar{\nu}_i\, \tau^i, \tag{2.21}$$

$$Cov[\varpi(s), \varpi(\tau)] = \sum_{i=0}^{n_1} \bar{\eta}_i\, \tau^i \sum_{i=0}^{n_2} \bar{\eta}_i'\, s^i, \tag{2.22}$$

where $\bar{\nu}_i \in \mathbb{R}, \bar{\eta}_i, \bar{\eta}_i' \in \mathbb{R}$ and $\min(n_1, n_2) \leq n - 1$

Theorem 2.4.51 *[Liu 2011c] Let $e_{\varpi}^{\beta T}(t_0)$ be the noise error contribution for the Jacobi estimator $D_{\kappa,\mu,\beta T,q}^{(n)} x^{\delta}(\beta T \xi + t_0)$ where the noise $\{\varpi(\tau), \tau \geq 0\}$ satisfies conditions (C_1) and (C_2). If $n \in \mathbb{N}^*$, then the mean, variance and covariance of $e_{\varpi}^{\beta T}(t_0)$ do not depend on t_0. If in addition the noise $\{\varpi(\tau), \tau \geq 0\}$ satisfies conditions (C_3) then $E[e_{\varpi}^{\beta T}(t_0)] = 0$, $Cov[e_{\varpi}^{\beta T_1}(t_0), e_{\varpi}^{\beta T_2}(t_0)] = 0$ and $Var[e_{\varpi}^{\beta T}(t_0)] = 0$.*

Proof. Then by applying (2.8), (2.16) and (2.17) with the conditions given in (2.19) and (2.20) we obtain

$$E\left[e_{\varpi}^{\beta T}(t_0)\right] = \frac{1}{(\beta T)^n}\int_0^1 Q(\tau)\,E\left[\varpi(\beta T\tau)\right]\,d\tau, \tag{2.23}$$

$$Cov\left[e_{\varpi}^{\beta T_1}(t_0), e_{\varpi}^{\beta T_2}(t_0)\right] = \frac{1}{T_1^n T_2^n}\int_0^1\int_0^1 Q(\tau)\,Q(s)\,Cov\left[\varpi(\beta T_1\tau),\,\varpi(\beta T_2 s)\right]\,ds\,d\tau. \tag{2.24}$$

Consequently the mean and covariance of $e_{\varpi}^{\beta T}(t_0)$ do not depend on t_0. If we take $T_1 = T_2$ in (2.24), then the variance of $e_{\varpi}^{\beta T}(t_0)$ do not depend on t_0. Moreover, if $E[\varpi(\tau)] = \sum_{i=0}^{n-1} \bar{\nu}_i\,\tau^i$, then by applying (2.8) to (2.23), we obtain $E[e_{\varpi}^{\beta T}(t_0)] = 0$. If $Cov[\varpi(s), \varpi(\tau)] = \sum_{i=0}^{n_1} \bar{\eta}_i\,\tau^i \sum_{i=0}^{n_2} \bar{\eta}_i'\,s^i$ with $\min(n_1, n_2) \leq n-1$ then by applying (2.8) to (2.24), we obtain $Cov[e_{\varpi}^{\beta T_1}(t_0), e_{\varpi}^{\beta T_2}(t_0)] = 0$. Then if we take $T_1 = T_2$ in (2.24), we get $Var[e_{\varpi}^{\beta T}(t_0)] = 0$. □

From which the following important theorem is obtained.

Theorem 2.4.52 *[Liu 2011c] Let $e_{\varpi}^{\beta T}(t_0)$ be the noise error contribution for the Jacobi estimator $D_{\kappa,\mu,\beta T,q}^{(n)} x^{\delta}(\beta T\xi + t_0)$ where the noise $\{\varpi(\tau), \tau \geq 0\}$ satisfies conditions (C_1) to (C_3), then*

$$e_{\varpi}^{\beta T}(t_0) = 0 \text{ almost surely.} \tag{2.25}$$

Proof. If the noise $\{\varpi(\tau), \tau \geq 0\}$ satisfies conditions (C_1) to (C_3), then we have $E[e_{\varpi}^{\beta T}(t_0)] = 0$ and $Var[e_{\varpi}^{\beta T}(t_0)] = 0$. Since

$$E\left[\left(e_{\varpi}^{\beta T}(t_0)\right)^2\right] = Var\left[e_{\varpi}^{\beta T}(t_0)\right] + \left(E\left[e_{\varpi}^{\beta T}(t_0)\right]\right)^2,$$

we get $E\left[\left(e_{\varpi}^{\beta T}(t_0)\right)^2\right] = 0$. Consequently, we have $e_{\varpi}^{\beta T}(t_0) = 0$ almost surely.

□

Two stochastic processes, the Wiener process (also known as the Brownian motion) and the Poisson process (cf [Parzen 1962]), play a central role in the theory of stochastic processes. These processes are valuable, not only as models of many important phenomena, but also as building blocks to model other complex stochastic processes. They are characterized by:

- let $\{W(t), t \geq 0\}$ be the Wiener process with parameter σ^2, then

$$E\left[W(t)\right] = 0,\ Cov\left[W(t), W(s)\right] = \sigma^2 \min(t, s); \tag{2.26}$$

- let $\{N(t), t \geq 0\}$ be the Poisson process with intensity $\nu \in \mathbb{R}^+$, then

$$E\left[N(t)\right] = \nu t,\ Cov\left[N(t), N(s)\right] = \nu \min(t, s). \tag{2.27}$$

Thus, these processes satisfy conditions (C_1) and (C_2). Hence, we can characterize the noise error contributions due to these two stochastic processes for the Jacobi estimators, and calculate the corresponding means and variances. If the noise is a Wiener process, then it is clear that $E[e_{\varpi}^{\beta T}(t_0)] = 0$. If the noise is a Poisson process, then we have

Proposition 2.4.53 *[Liu 2011c] The mean of the noise error contribution due to a Poisson process for the Jacobi estimator is given by*

$$\begin{cases} E\left[e_{\varpi}^{\beta T}(t_0)\right] = 0, & \text{if } n \geq 2, \\ E\left[e_{\varpi}^{\beta T}(t_0)\right] = \nu, & \text{if } n = 1. \end{cases} \tag{2.28}$$

Proof. For $n \geq 2$, this can be simply proven by using Theorem 2.4.51. Thus we only need to compute the mean of the noise error contribution for the estimates of $\dot{x}$. Let $n = 1$ in (1.29), then the minimal estimators are given by

$$D^{(1)}_{\kappa,\mu,\beta T} x^{\delta}(t_0) = \frac{1}{\beta T} \int_0^1 Q(\tau)\; x^{\delta}(\beta T\tau + t_0)\; d\tau, \tag{2.29}$$

where $Q(\tau) = \frac{1}{\mathrm{B}(\kappa+2,\mu+2)} \left((\mu+\kappa+2)\tau - (\kappa+1)\right)(1-\tau)^{\mu}\tau^{\kappa}$. Then, according to (2.23) we obtain

$$E\left[e_{\varpi}^{\beta T}(t_0)\right] = \nu \int_0^1 \tau\, Q(\tau)\; d\tau.$$

By using integration by parts and the classical Beta function, we obtain $E\left[e_{\varpi}^{\beta T}(t_0)\right] = \nu$.

According to Proposition 1.3.14, we can deduce that the noise error contribution for the affine Jacobi estimator is an affine combination of the ones for the minimal Jacobi estimators. Thus, by using (1.80) this proof can be completed. □

Now, we calculate the variance. Since the covariance kernels of the Wiener process and the Poisson process are determined by the same function $\min(\cdot,\cdot)$, the variance of the noise error contributions due to a Wiener process or a Poisson process for the Jacobi estimators is given by (Using (2.24) with $T = T_1 = T_2$)

$$Var\left[e_{\varpi}^{\beta T}(t_0)\right] = \frac{\eta}{T^{2n}} \int_0^1 \int_0^1 Q(\tau)\, Q(s)\, \min(\beta T s,\, \beta T\tau)\, ds\, d\tau.$$

Using the symmetry property of function $\min(\cdot,\cdot)$ and the fact that $\int_{\tau}^1 Q(s)\, ds = -\int_0^{\tau} Q(s)\, ds$, we obtain

$$Var\left[e_{\varpi}^{\beta T}(t_0)\right] = \frac{2\eta}{T^{2n-1}} \int_0^1 Q(\tau)\, \tau \int_{\tau}^1 Q(s)\, ds\, d\tau. \tag{2.30}$$

Let us denote by $e_{\varpi}^{\beta}(t_0; n, \kappa, \mu)$ as the noise error contribution in the minimal Jacobi estimators given by (1.29). Then, we have

$$Var\left[e_{\varpi}^{\beta}(t_0; n, \kappa, \mu)\right] = \frac{2\eta}{T^{2n-1}} \int_0^1 Q_n(\tau)\, \tau \int_{\tau}^1 Q_n(s)\, ds\, d\tau, \tag{2.31}$$

where $Q_n(\tau) = \frac{n!}{(\beta T)^n} \frac{w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau)}{\mathrm{B}(n+\kappa+1,\mu+n+1)}$ with $T \in D_{t_0}$, $\mu, k \in]-1, +\infty[$.

By applying the Rodrigues formula, we get

$$\begin{aligned} &\int_0^1 w_{\mu,\kappa+1}(\tau) P_n^{(\mu,\kappa)}(\tau) \int_{\tau}^1 w_{\mu,\kappa}(s) P_n^{(\mu,\kappa)}(s) ds\, d\tau \\ =&\frac{(-1)^n}{n!} \int_0^1 w_{\mu,\kappa+1}(\tau) P_n^{(\mu,\kappa)}(\tau) \int_{\tau}^1 w^{(n)}_{\mu+n,\kappa+n}(s) ds\, d\tau \\ =&\frac{1}{n} \int_0^1 w_{2\mu+1,2\kappa+2}(\tau)\, P_n^{\mu,\kappa}(\tau)\, P_{n-1}^{\mu+1,\kappa+1}(\tau)\, d\tau. \end{aligned}$$

Then, we obtain

$$Var\left[e_{\varpi}^{\beta}(t_0;n,\kappa,\mu)\right] = \frac{2\eta\; n!(n-1)!}{T^{2n-1}\mathrm{B}^2(\kappa+n+1,\mu+n+1)} I(\mu,\kappa,n). \tag{2.32}$$

with

$$I(\mu,\kappa,n) = \int_0^1 w_{2\mu+1,2\kappa+2}(\tau)\, P_n^{\mu,\kappa}(\tau)\, P_{n-1}^{\mu+1,\kappa+1}(\tau)\, d\tau. \tag{2.33}$$

Let us stress that $Var\left[e_{\varpi}^{\beta}(t_0;n,\kappa,\mu)\right] \sim \frac{1}{T^{2n-1}}$.

Let us denote by $e_{\varpi}^{\beta}(t_0;n,\kappa,\mu,q)$ as the noise error contribution in the affine Jacobi estimators $D^{(n)}_{\kappa,\mu,\beta T,q} x^{\delta}(\beta T\xi + t_0)$ given by Corollary 1.3.15 with $q=1$. For $n=1$, we have the following results:

Proposition 2.4.54 *[Liu 2011c] The variances of the noise error contributions for the Jacobi estimators of the first order derivative of x are given by*

$$Var\left[e_{\varpi}^{\beta}(t_0;1,\kappa,\mu)\right] = \frac{2\eta}{T}\frac{\mu+1}{2\mu+2\kappa+5}\frac{\mathrm{B}(2\mu+2,2\kappa+3)}{\mathrm{B}^2(\kappa+2,\mu+2)}, \tag{2.34}$$

for minimal estimators and by

$$\begin{aligned} &Var\left[e_{\varpi}^{\beta}(t_0;1,\kappa,\mu,1)\right] = \\ &\quad \lambda^2_{\kappa,\mu,n,\xi}\frac{2\eta}{T}\frac{\mu+2}{2\mu+2\kappa+7}\frac{\mathrm{B}(2\mu+4,2\kappa+3)}{\mathrm{B}^2(\kappa+2,\mu+3)} \\ &\quad + \hat{\lambda}^2_{\kappa,\mu,n,\xi}\frac{2\eta}{T}\frac{\mu+1}{2\mu+2\kappa+7}\frac{\mathrm{B}(2\mu+2,2\kappa+5)}{\mathrm{B}^2(\kappa+3,\mu+2)} \\ &\quad + \lambda_{\kappa,\mu,n,\xi}\hat{\lambda}_{\kappa,\mu,n,\xi}\frac{2\eta}{T}\frac{\mathrm{B}(2\mu+4,2\kappa+4)}{\mathrm{B}(\kappa+2,\mu+3)\mathrm{B}(\kappa+3,\mu+2)} \end{aligned} \tag{2.35}$$

for affine estimators $D^{(n)}_{\kappa,\mu,\beta T,q} x^{\delta}(\beta T\xi + t_0)$ with $q=1$, where $\lambda_{\kappa,\mu,n,\xi} = (\kappa+n+2)-(2n+\kappa+\mu+3)\xi$ and $\hat{\lambda}_{\kappa,\mu,n,\xi} = 1-\lambda_{\kappa,\mu,n,\xi}$. The value η is equal to σ^2, if the noise is a Wiener process, and η is equal to ν, if the noise is a Poisson process.

Proof. By using (2.32) with

$$I(\mu,\kappa,n=1) = \frac{(\mu+1)B(2\mu+2,2\kappa+3)}{2\mu+2\kappa+5},$$

we get the desired result (2.34). Similarly, we can obtain $Var\left[e_{\varpi}^{\beta}(t_0;1,\kappa,\mu,1)\right]$ by using

$$e_{\varpi}^{\beta}(t_0;1,\kappa,\mu,1) = \lambda_{\kappa,\mu,n,\xi} e_{\varpi}^{\beta}(t_0;1,\kappa,\mu+1) + \hat{\lambda}_{\kappa,\mu,n,\xi} e_{\varpi}^{\beta}(t_0;1,\kappa+1,\mu).$$

□

As a consequence, since $E[e_{\varpi}^{\beta}(t_0;1,\kappa,\mu)] = E[e_{\varpi}^{\beta}(t_0;1,\kappa,\mu,1)] = \nu$ for a Wiener process ($\nu=0$) or Poisson process ($\nu \neq 0$, where ν is the intensity parameter of the Poison Process), by using the Bienaymé-Chebyshev (2.13) we obtain the error bounds for the noise error contributions for the Jacobi estimators of the first order derivative of x.

Theorem 2.4.55 *[Liu 2011c] [First order derivative estimation] Let $n = 1$. Let the noise be a Wiener process or Poisson process, then for any real number $\gamma > 0$,*

$$Pr\left(\left|e_{\varpi}^{\beta}(t_0;1,\kappa,\mu)-\nu\right| < \gamma\sqrt{Var[e_{\varpi}^{\beta}(t_0;1,\kappa,\mu)]}\right) > 1-\frac{1}{\gamma^2}, \tag{2.36}$$

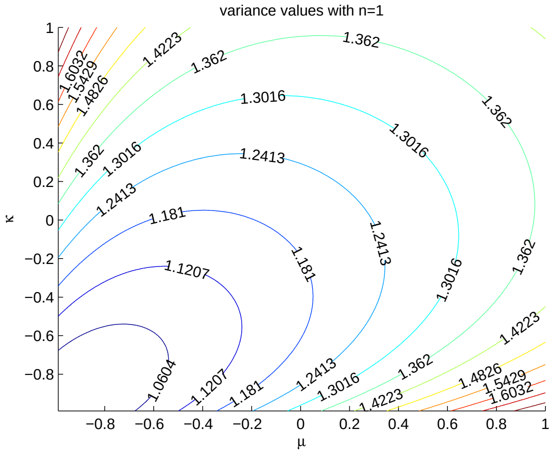
$$Pr\left(\left|e_{\varpi}^{\beta}(t_0;1,\kappa,\mu,1)-\nu\right| < \gamma\sqrt{Var[e_{\varpi}^{\beta}(t_0;1,\kappa,\mu,1)]}\right) > 1-\frac{1}{\gamma^2}, \tag{2.37}$$

where $\nu = 0$ for a Wiener process; $\nu \neq 0$ for a Poisson process and $Var[e_{\varpi}^{\beta}(t_0;1,\kappa,\mu)]$, $Var[e_{\varpi}^{\beta}(t_0;1,\kappa,\mu,1)]$ are given respectively by (2.34) *and* (2.35).

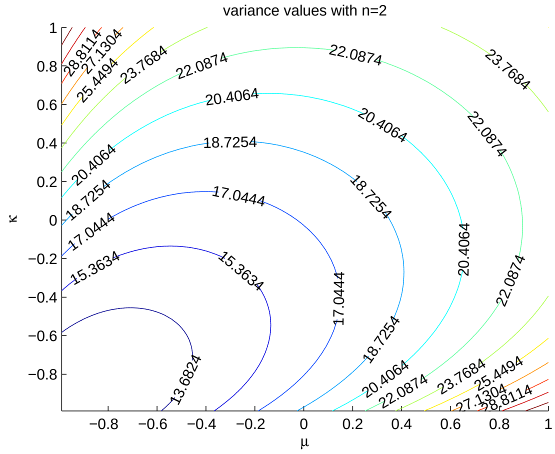
For the case $n = 1$, the bounds given by Theorem 2.4.55 characterize the noise error contribution $e_{\varpi}^{\beta}(t_0;1,\kappa,\mu)$ (respectively $e_{\varpi}^{\beta}(t_0;1,\kappa,\mu,1)$) for the minimal Jacobi estimator $D_{\kappa,\mu,\beta T}^{(n)}x^{\delta}(t_0)$ (respectively the affine Jacobi estimator $D_{\kappa,\mu,\beta T,q}^{(n)}x^{\delta}(\beta T\xi+t_0)$ with $q = 1$). They depend on $Var\left[e_{\varpi}^{\beta}(t_0;1,\kappa,\mu)\right]$ given by (2.34) (respectively $Var\left[e_{\varpi}^{\beta}(t_0;1,\kappa,\mu)\right]$ given by (2.35)). Similar results can be obtained for $n = 2$ since

$$\begin{aligned}
2I(\mu,\kappa,n=2) = &-(\kappa+2)^2(\kappa+1)\mathrm{B}(2\mu+5,2\kappa+3)\\
&+(\kappa+2)(\mu+2)(3\kappa+5)\mathrm{B}(2\mu+4,2\kappa+4)\\
&-(\kappa+2)(\mu+2)(3\mu+5)\mathrm{B}(2\mu+3,2\kappa+5)\\
&+(\mu+2)^2(\mu+1)\mathrm{B}(2\mu+2,2\kappa+6).
\end{aligned}$$

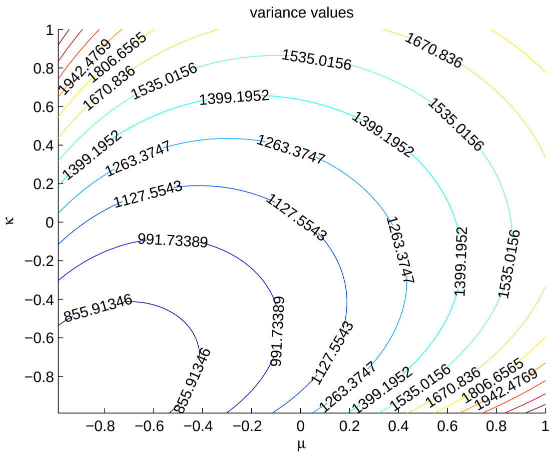
and of course for higher values of n. Remember that, for fixed T, we have $Var\left[e_{\varpi}^{\beta}(t_0;n,\kappa,\mu)\right] \sim \frac{1}{T^{2n-1}}$. Since all these variance functions decrease with respect to T independently of κ and μ, it is sufficient to observe the influence of κ and μ. In the minimal Jacobi estimator case one can get a direct computation (result is reported in Figure 2.1 by taking $\eta = T = 1$) whereas in the affine case it is not difficult to obtain a 3-D plot as in Figure 2.2 and where $\eta = T = 1$, $\xi = \xi(\kappa,\mu)$ is the smaller root of $P_2^{\mu+1,\kappa+1}$. From this analysis, we should take negative values for κ and μ so as to minimize the noise error contribution. Moreover, we can observe that the variance of $e_{\varpi}^{\beta}(t_0;1,\kappa,\mu,1)$ is larger than the one of $e_{\varpi}^{\beta}(t_0;1,\kappa,\mu)$ if we take same value for κ and μ, hence we should take the value of T for affine estimator $D_{\kappa,\mu,\beta T,1}^{(n)}x^{\delta}(\beta T\xi+t_0)$ larger than the one for $D_{\kappa,\mu,\beta T}^{(n)}x^{\delta}(t_0)$ so as to obtain the same noise effect.



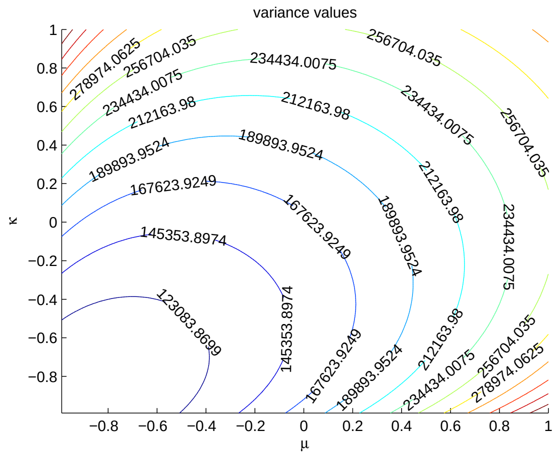
(a) $n = 1$



(b) $n = 2$

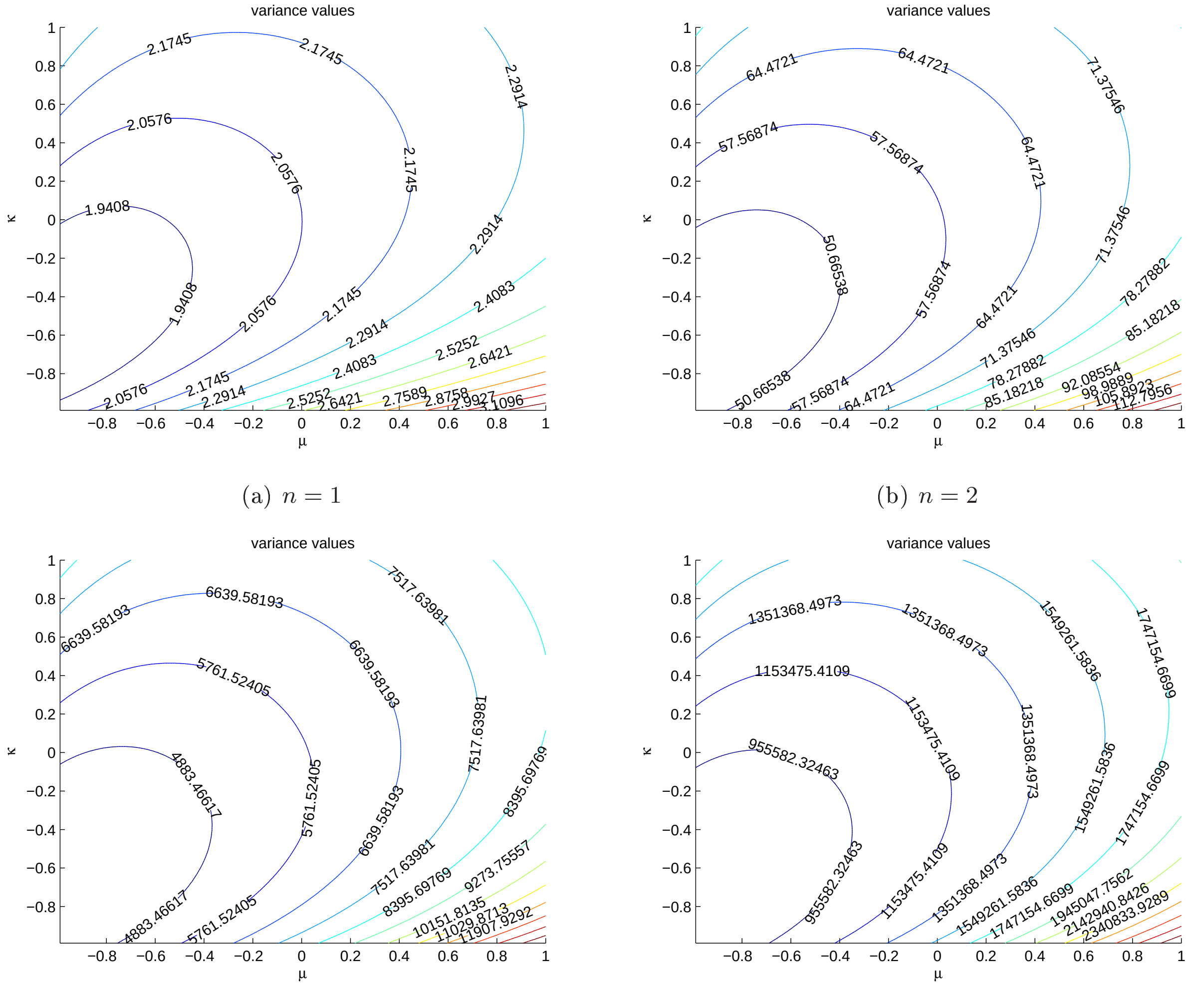


(c) $n = 3$



(d) $n = 4$

Figure 2.1: Variances of the noise errors for the minimal estimators.



(a) $n = 1$

(b) $n = 2$

(c) $n = 3$

(d) $n = 4$

Figure 2.2: Variances of the noise errors for the affine estimators.

Usually, the observation function x^δ is only known on discrete values. To simplify our notations, we denote the noise error contribution $e^{\beta}_{\varpi,m}(t_0;\kappa,\mu,T,\xi,q)$ defined in (2.4) by $e^{\beta T}_{\varpi,m}(t_0)$. By applying the properties of the mean, variance and covariance, we have

$$E\left[e^{\beta T}_{\varpi,m}(t_0)\right] = \frac{1}{(\beta T)^n}\frac{1}{m}\sum_{i=0}^{m} w_i\, Q(t_i)\, E\left[\varpi(t_0+\beta T t_i)\right], \tag{2.38}$$

$$\begin{aligned} Var\left[e^{\beta T}_{\varpi}(t_0)\right] &= \frac{1}{T^{2n}}\frac{1}{m^2}\sum_{i=0}^{m} w_i^2\,(Q(t_i))^2\, Var\left[\varpi(t_0+\beta T t_i)\right] \\ &+ \frac{1}{T^{2n}}\frac{2}{m^2}\sum_{i=0}^{m-1}\sum_{j=i+1}^{m} w_i\, w_j\, Q(t_i)\, Q(t_j)\, Cov\left[\varpi(t_0+\beta T t_i),\varpi(t_0+\beta T t_j)\right]. \end{aligned} \tag{2.39}$$

Moreover, for any $T_1>0$ and $T_2>0$

$$Cov\left[e^{\beta T_1}_{\varpi}(t_0), e^{\beta T_2}_{\varpi}(t_0)\right] = \frac{1}{T_1^n T_2^n}\frac{1}{m^2}\sum_{i=0}^{m}\sum_{j=0}^{m} w_i\, w_j\, Q(t_i)\, Q(t_j)\, Cov\left[\varpi(t_0+\beta T_1 t_i),\varpi(t_0+\beta T_2 t_j)\right]. \tag{2.40}$$

Then, by using Bienaymé-Chebyshev (2.13) and the previous formulae, we can derive similar results than the ones obtained in the continuous subsection and which coincide if $m\to\infty$. However this is true with some few additional assumptions as detailed below.

In order to show the bridge with the continuous case, we can use the following properties, where T, T_1 and T_2 are given (finite), and T_s tends to 0, i.e. m tends to infinite.

$$\lim_{m\to\infty} E\left[e^{\beta T}_{\varpi,m}(t_0)\right] = E\left[e^{\beta T}_{\varpi}(t_0)\right], \tag{2.41}$$

$$\lim_{m\to\infty} Var\left[e^{\beta T}_{\varpi,m}(t_0)\right] = Var\left[e^{\beta T}_{\varpi}(t_0)\right], \tag{2.42}$$

$$\lim_{m\to\infty} Cov\left[e^{\beta T_1}_{\varpi,m}(t_0), e^{\beta T_2}_{\varpi,m}(t_0)\right] = Cov\left[e^{\beta T_1}_{\varpi}(t_0), e^{\beta T_2}_{\varpi}(t_0)\right]. \tag{2.43}$$

Hence, by using Theorem 2.4.51 and the fact that $E\left[(Y_m-c)^2\right] = Var\left[Y_m\right] + \left(E\left[Y_m\right]-c\right)^2$ for any sequence of random variables Y_m, we can get the following theorem.

Theorem 2.4.56 *Let $\{\varpi(\tau),\tau\geq 0\}$ be a continuous parameter stochastic process satisfying conditions (C_1) to (C_3), $\varpi(t_i)$ be a sequence of $\{\varpi(\tau),\tau\geq 0\}$ with an equidistant sampling period T_s. Then, $e^{\beta T}_{\varpi,m}(t_0)$ converges in mean square to 0 when $T_s\to 0$.*

2.5 Independent stochastic process noises

In the section, let us consider a family of noises which are continuous parameter stochastic processes satisfying the following conditions

(C_4) : for any $s,t\geq 0$, $s\neq t$, $\varpi(s)$ and $\varpi(t)$ are independent;

(C_5) : the mean value function of $\{\varpi(\tau), \tau \in \Omega\}$ belongs to $\mathcal{L}(I)$;

(C_6) : the variance function of $\{\varpi(\tau), \tau \in \Omega\}$ is bounded on I.

Note that white Gaussian noise and Poisson noise satisfy these conditions. Then, we can give the following theorem.

Lemma 2.5.57 *Let $\{\varpi(\tau), \tau \geq 0\}$ be a continuous parameter stochastic process satisfying conditions (C_4) to (C_6). Let $\varpi(t_i)$ be a sequence of $\{\varpi(\tau), \tau \geq 0\}$ with an equidistant sampling period T_s. If $Q \in \mathcal{L}^2(I)$, then we have*

$$\lim_{m\to\infty} E\left[e^{\beta T}_{\varpi,m}(t_0)\right] = \frac{1}{(\beta T)^n}\int_0^1 Q(\tau)\, E[\varpi(t_0+\beta T\tau)]\, d\tau, \tag{2.44}$$

$$\lim_{m\to\infty} Var\left[e^{\beta T}_{\varpi,m}(t_0)\right] = 0, \tag{2.45}$$

where $e^{\beta T}_{\varpi,m}(t_0)$ is the associated noise error contribution for the Jacobi estimators.

Proof. Since $\varpi(t_i)$ is a sequence of independent random variables, by using (2.39) and (2.38) we have

$$E\left[e^{\beta T}_{\varpi,m}(t_0)\right] = \frac{1}{(\beta T)^n}\frac{1}{m}\sum_{i=0}^{m} w_i\, Q(t_i)\, E\left[\varpi(t_0+\beta T t_i)\right], \tag{2.46}$$

$$Var\left[e^{\beta T}_{\varpi,m}(t_0)\right] = \frac{1}{T^{2n}}\frac{1}{m^2}\sum_{i=0}^{m} w_i^2\, (Q(t_i))^2\, Var\left[\varpi(t_0+\beta T t_i)\right]. \tag{2.47}$$

According to condition (C_6) the variance function of ϖ is bounded. Hence, we have

$$0 \leq \frac{1}{m^2}\sum_{i=0}^{m} w_i^2\, (Q(t_i))^2\, \left|Var\left[\varpi(t_0+\beta T t_i)\right]\right| \leq U\frac{w(m)}{m}\sum_{i=0}^{m}\frac{w_i}{m}\,(Q(t_i))^2, \tag{2.48}$$

where $w(m) = \max\limits_{0\leq i\leq m} w_i$ and $U = \sup\limits_{0\leq t\leq 1}\left|Var\left[\varpi(t_0+\beta T t)\right]\right| < \infty$. Moreover,

$$\lim_{m\to\infty} E\left[e^{\beta T}_{\varpi,m}(t_0)\right] = \frac{1}{(\beta T)^n}\int_0^1 Q(\tau)\, E[\varpi(t_0+\beta T\tau)]\, d\tau, \tag{2.49}$$

$$\lim_{m\to\infty}\sum_{i=0}^{m}\frac{w_i}{m}\,(Q(t_i))^2 = \int_0^1 (Q(t))^2\, dt. \tag{2.50}$$

Since all w_i are bounded and $Q \in \mathcal{L}^2(I)$, we have

$$\lim_{m\to\infty} U\frac{w(m)}{m}\sum_{i=0}^{m}\frac{w_i}{m}\,(p^{\beta T}(t_i))^2 = 0.$$

Thus this proof is completed. □

By using the previous lemma, the Bienaymé-Chebyshev inequality implies that if the value of T is set then $e^{\beta T}_{\varpi,m}(t_0)$ converges in probability to $\frac{1}{(\beta T)^n}\int_0^1 Q(\tau)\, E[\varpi(t_0+\beta T\tau)]\, d\tau$ when $T_s \to 0$. Moreover, similarly to Theorem 2.4.56, we can get the convergence in mean square.

Theorem 2.5.58 *[Liu 2011c] Let* $\{\varpi(\tau), \tau \geq 0\}$ *be a continuous parameter stochastic process satisfying conditions* (C_4) *to* (C_6)*,* $\varpi(t_i)$ *be a sequence of* $\{\varpi(\tau), \tau \geq 0\}$ *with an equidistant sampling period* T_s*. If* $\kappa, \mu > -\frac{1}{2}$ *and the value of* T *is set, then* $e_{\varpi,m}^{\beta T}(t_0)$ *converges in mean square to* $\frac{1}{(\beta T)^n}\int_0^1 Q(\tau)\, E[\varpi(t_0+\beta T\tau)]\, d\tau$ *when* $T_s \to 0$*, where* Q *is defined in* (1.85)*. Moreover, if* $E[\varpi(\tau)] = \sum_{i=0}^{n-1} \bar{\nu}_i\, \tau^i$ *with* $\bar{\nu}_i \in \mathbb{R}$*, then* $e_{\varpi,m}^{\beta T}(t_0)$ *converges in mean square to* 0 *when* $T_s \to 0$*.*

Proof. If $E[\varpi(\tau)] = \sum_{i=0}^{n-1} \bar{\nu}_i\, \tau^i$ with $\bar{\nu}_i \in \mathbb{R}$, then similarly to Theorem 2.4.51 we can obtain

$$\frac{1}{(\beta T)^n}\int_0^1 Q(\tau)\, E[\varpi(t_0+\beta T\tau)]\, d\tau = 0.$$

Hence, this proof is completed. □

When the sampling period and the value of T are set, the noise error contribution dose not converge to zero. In this case, similarly to (2.14) we can use the Bienaymé-Chebyshev inequality to give two error bounds for this noise error. Then, we can study the associated convergence rate by using Proposition 2.3.48. In particular, if ϖ is a white Gaussian noise, then according to the tree-sigma rule, we have

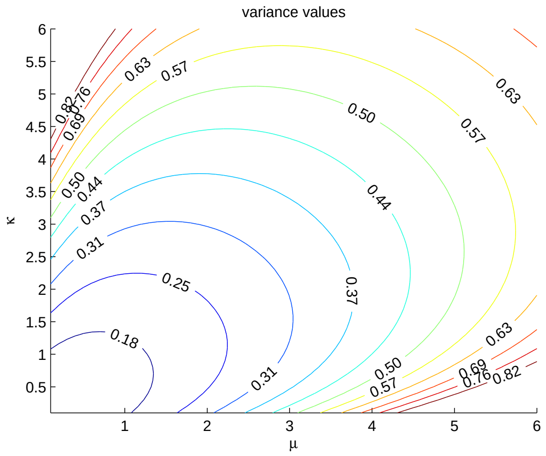
$$M_l \overset{p_r}{\leq} e_{\varpi,m}^{\beta T} \overset{p_r}{\leq} M_h, \tag{2.51}$$

where $p_1 = 68.26\%$, $p_2 = 95.44\%$ and $p_3 = 99.73\%$.

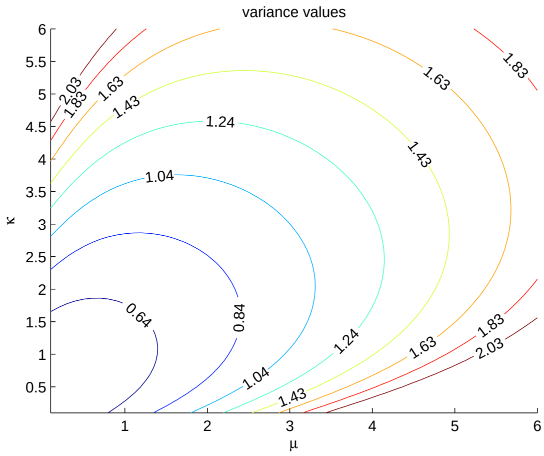
Now, we are going to study the influence of parameters κ, μ, q and n on the variance $Var\left[e_{\varpi,m}^{\beta T}(t_0)\right]$ given by (2.47). This study is done in the case where κ and μ are positif. The case where κ and μ are negative will be considered later. Let us denote $Var\left[e_{\varpi,m}^{\beta T}(t_0)\right]$ by $V(\kappa,\mu,q,n,\xi)$. Then, we assume that $T=1$, $m=100$, and the variance of ϖ is a constant which is equal to 1. We take trapezoidal rue as the used numerical integration method.

We can see in Figure 2.3 the variations of $V(\kappa,\mu,q,1,\xi)$ corresponding to the noise error contributions for the Jacobi estimators $D_{\kappa,\mu,\beta T,q}^{(1)} x^\delta(t_0)$ with $q = 0,2,3$ and the time-drift Jacobi estimator $D_{\kappa,\mu,\beta T,q=1}^{(1)} x^\delta(\beta T\xi + t_0)$ with $\xi = \xi_1^{min}$. It is clear that $V(\kappa,\mu,q,n,\xi)$ is increasing with respect to q and increasing with respect to κ and μ when $q = 0,1$. Hence, we can decrease the value of κ and μ so as to reduce the noise error contribution. Moreover, the noise error contribution for $D_{\kappa,\mu,\beta T,2}^{(1)} x^\delta(t_0)$ can be much larger than the one for $D_{\kappa,\mu,\beta T,1}^{(1)} x^\delta(\beta T\xi_1^{min} + t_0)$. We can obtain similar results in case where $n \neq 1$.

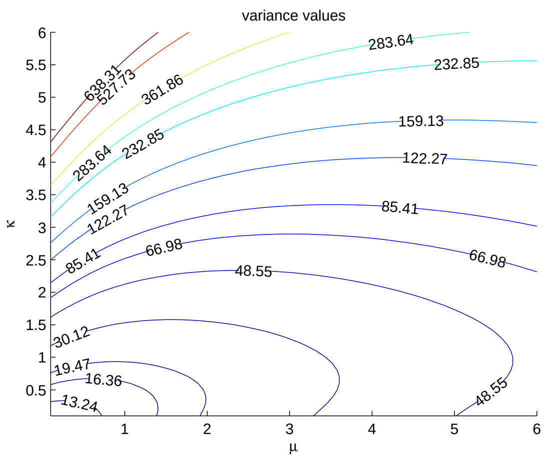
According to (1.166), we can use $V(\kappa,\mu,q,n,\xi)$ with $\xi = \frac{1}{2}$ to study the parameters' influence on the noise error contribution for the central Jacobi estimators. The variations of $V(\kappa,\mu,q,n,\frac{1}{2})$ are given in Figure 2.4 with $q = 0,\ldots,5$. We can see that $V(\kappa,\mu,q,n,\frac{1}{2})$ is increasing with respect to κ, μ and q. Hence, we can decrease the value of κ, μ and q so as to reduce the noise error contribution.



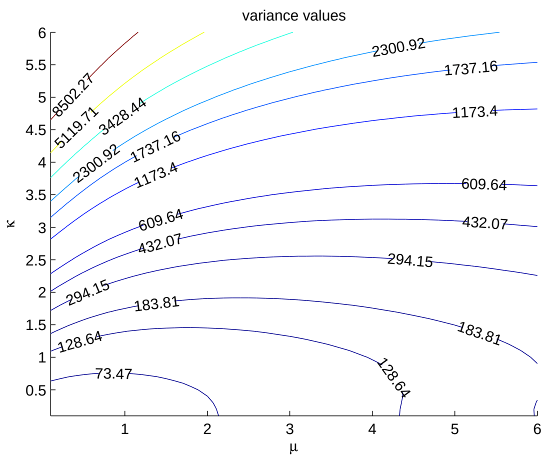
(a) $V(\kappa, \mu, q, 1, \xi)$ with $q = \xi = 0$, $0 \leq \kappa \leq 6$ and $0 \leq \mu \leq 6$.



(b) $V(\kappa, \mu, q, 1, \xi)$ with $q = 1$, $\xi = \xi_q^{min}$, $0 \leq \kappa \leq 6$ and $0 \leq \mu \leq 6$.

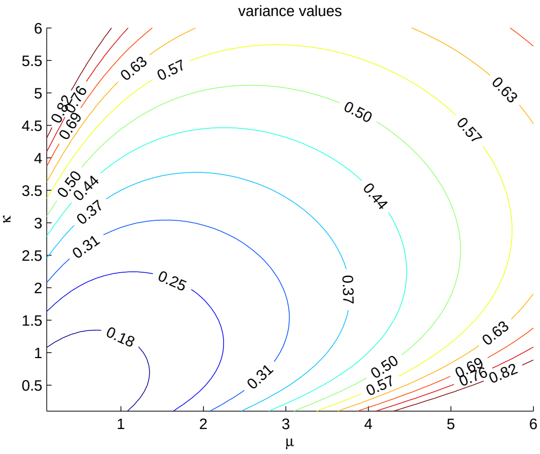


(c) $V(\kappa, \mu, q, 1, \xi)$ with $q = 2$, $\xi = 0$, $0 \leq \kappa \leq 6$ and $0 \leq \mu \leq 6$.

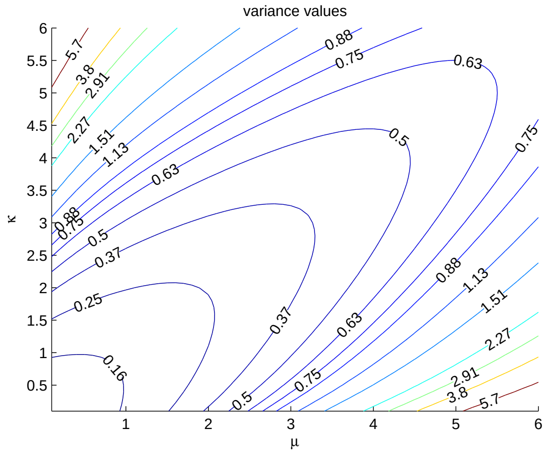


(d) $V(\kappa, \mu, q, 1, \xi)$ with $q = 3$, $\xi = 0$, $0 \leq \kappa \leq 6$ and $0 \leq \mu \leq 6$.

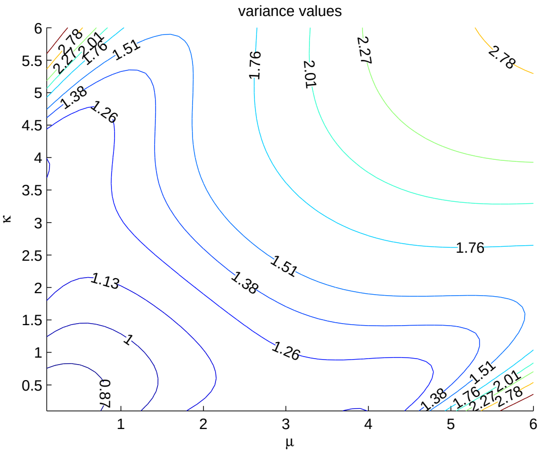
Figure 2.3: Variation of $V(\kappa, \mu, q, n, \xi)$ with respect to κ and μ for $n = 1$ and $q = 0, 1, 2, 3$.



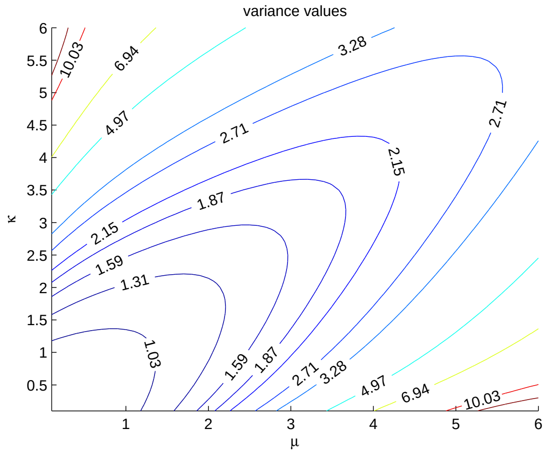
(a) $V(\kappa, \mu, q, 1, \frac{1}{2})$ with $q = 0$, $0 \leq \kappa \leq 6$ and $0 \leq \mu \leq 6$.



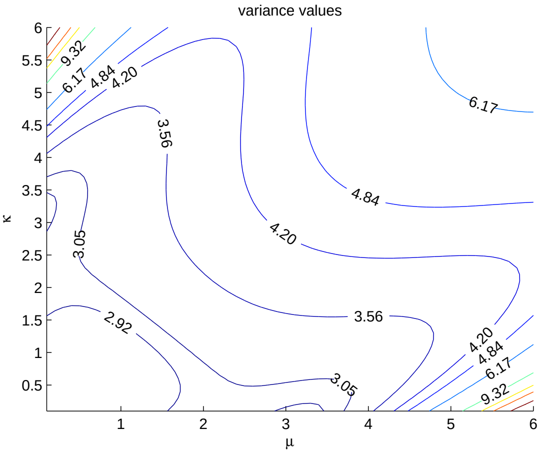
(b) $V(\kappa, \mu, q, 1, \frac{1}{2})$ with $q = 1$, $0 \leq \kappa \leq 6$ and $0 \leq \mu \leq 6$.



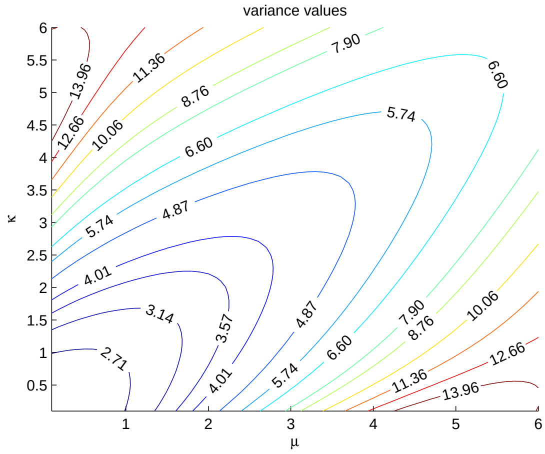
(c) $V(\kappa, \mu, q, 1, \frac{1}{2})$ with $q = 2$, $0 \leq \kappa \leq 6$ and $0 \leq \mu \leq 6$.



(d) $V(\kappa, \mu, q, 1, \frac{1}{2})$ with $q = 3$, $0 \leq \kappa \leq 6$ and $0 \leq \mu \leq 6$.



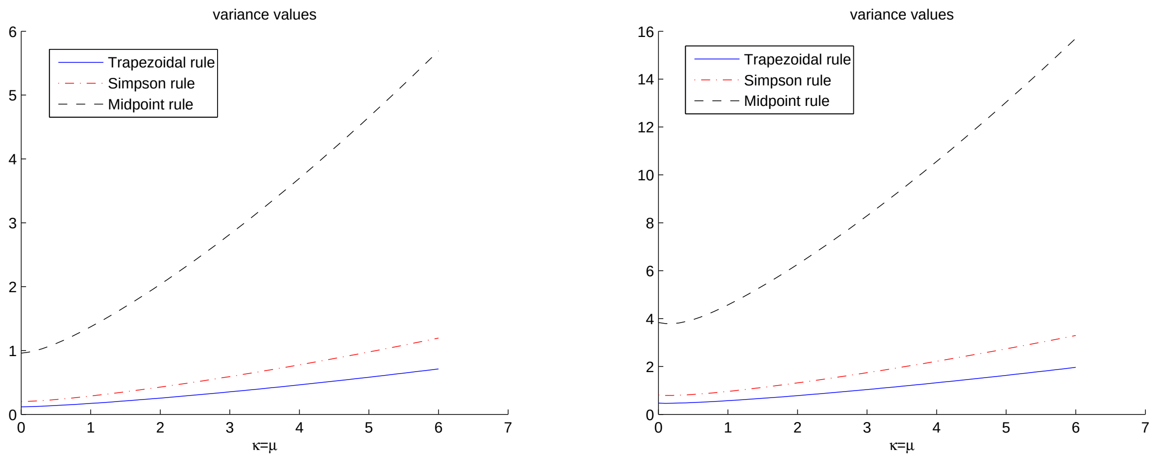
(e) $V(\kappa, \mu, q, 1, \frac{1}{2})$ with $q = 4$, $0 \leq \kappa \leq 6$ and $0 \leq \mu \leq 6$.



(f) $V(\kappa, \mu, q, 1, \frac{1}{2})$ with $q = 5$, $0 \leq \kappa \leq 6$ and $0 \leq \mu \leq 6$.

Figure 2.4: Variation of $V(\kappa, \mu, q, n, \frac{1}{2})$ with respect to κ and μ for $n = 1$ and $q = 0, 1, 2, 3, 4, 5$.

We can also consider the other numerical integration methods so as to calculate the variance. It is clear that if $\kappa > 0$ and $\mu > 0$ then we have $Q(0) = Q(1) = 0$. Hence, by applying trapezoidal rue, right rectangle rule and left rectangle rule we obtain the same value for $V(\kappa, \mu, q, n, \xi)$. Then, we take Simpson's rule and midpoint rule. We can see in Figure 2.5 the variation of $V(\kappa, \kappa, 1, 1, \xi)$ with $\xi = 0.5$ and $\xi = \xi_1^{min}$ respectively. Consequently, trapezoidal rule is the optimal numerical integration method to reduce the noise error contribution for Jacobi estimators (central or not). We can obtain similar results for other values of q and n.



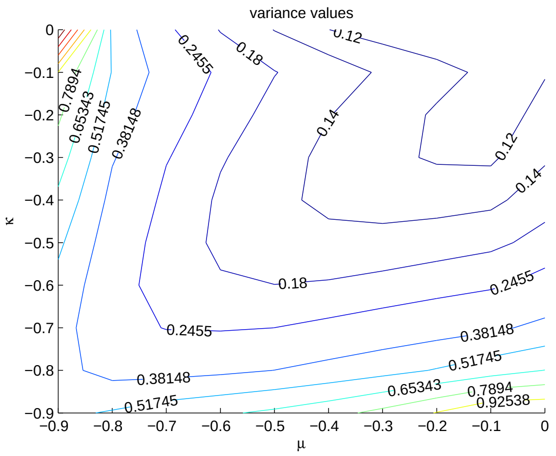
(a) $V(\kappa, \mu, 1, 1, \xi)$ with $\xi = 0.5$ and $0 \leq \kappa = \mu \leq 6$.

(b) $V(\kappa, \mu, q, 1, \xi)$ with $\xi = \xi_q^{min}$ and $0 \leq \kappa = \mu \leq 6$.

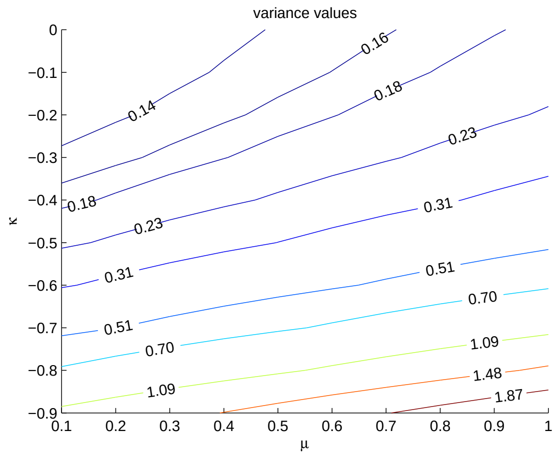
Figure 2.5: Variation of $V(\kappa, \mu, q, n, \xi)$ obtained by different numerical integration methods with $q = n = 1$ and $0 \leq \kappa = \mu \leq 6$.

Finally, let us consider the case where $\kappa < 0$ and $\mu < 0$. We take the formulae given in (1.111), (1.113) and (1.115) to calculate the variance $V(\kappa, \mu, q, n, \xi)$. Then, we can see in Figure 2.6 the variation of $V(\kappa, \mu, 0, 1, 0)$ for $-1 < \kappa \leq 0$ and $-1 < \mu \leq 1$. Hence, $V(\kappa, \mu, 0, 1, 0)$ is decreasing with respect to κ and μ when $\kappa < 0$ and $\mu < 0$. Consequently, the negative values of κ produce larger noise error contributions for the Jacobi estimators given by (1.111), (1.113) and (1.115).

Let us recall that the formulae (1.111), (1.113) and (1.115) are given to avoid singular values in Jacobi estimators in discrete case. However, these formulae produce larger noise error contributions when κ or μ is negative. We give a new way to avoid the singular values. If $\kappa < 0$ (resp. $\mu < 0$), then we set the weight w_0 (w_m) equal to 0. Hence, there is not singular values at $\tau = 0$ and $\tau = 1$ when $\kappa < 0$ and $\mu < 0$. We can see the variation of so obtained $V(\kappa, \mu, 0, 1, 0)$ and $V(\kappa, \mu, 1, 1, \xi_q^{min})$ in Figure 2.7. Then, we can observe that $V(\kappa, \mu, 0, 1, 0)$ and $V(\kappa, \mu, 1, 1, \xi_q^{min})$ are increasing with respect to κ and μ. Consequently, the negative values of κ and μ can reduce the noise error contributions for Jacobi estimators. However, this choice of the weight w_0 can produce a numerical error. We can see in the next section that if the function the first order derivative of which we want to estimate satisfies a differential equation then this numerical error can reduce the truncated term error for minimal Jacobi estimators.

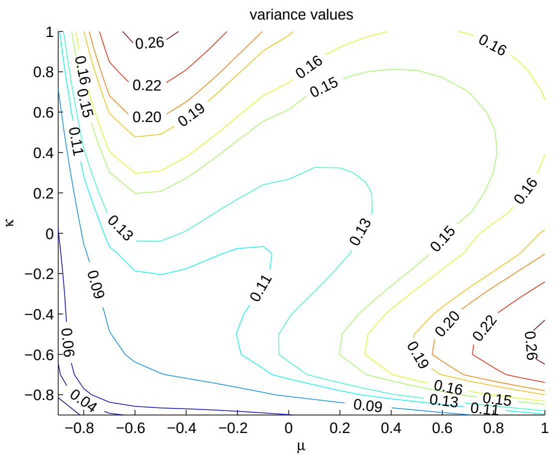


(a) $V(\kappa, \mu, 0, 1, 0)$ with $-1 < \kappa \leq 0$ and $-1 < \mu \leq 0$.

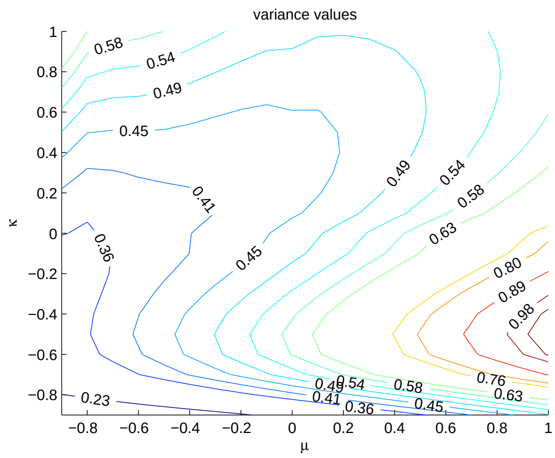


(b) $V(\kappa, \mu, 0, 1, 0)$ with $-1 < \kappa \leq 0$ and $0 < \mu \leq 1$.

Figure 2.6: Variation of $V(\kappa, \mu, q, n, \xi)$ with respect to κ and μ for $n = 1$ and $q = \xi = 0$.



(a) $V(\kappa, \mu, 0, 1, 0)$ with $-1 < \kappa \leq 1$ and $-1 < \mu \leq 0$.



(b) $V(\kappa, \mu, 1, 1, \xi_q^{min})$ with $-1 < \kappa \leq 1$ and $-1 < \mu \leq 1$.

Figure 2.7: Variation of $V(\kappa, \mu, q, n, \xi)$ with respect to κ and μ for $n = 1$, $q = 0, 1$ and $\xi = 0, \xi_q^{min}$.

2.6 Numerical simulations

In order to demonstrate the efficiency and the stability of Jacobi estimators, we present some numerical results in this section. The influence of parameters for these estimators is studied in Subsection 1.3.3, Subsection 1.6.2, Subsection 2.4 and Subsection 2.5. We recall the results in the two following tables, where the notations $a \uparrow$, $b \nearrow$ and $c \searrow$ mean that if we increase the value for the parameter a then the error b increases and the error c decreases.

$D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi_q + t_0)$	Amplitude error	Time-drift	Noise error contribution
$\kappa \uparrow$	$\nearrow$	$\nearrow$	$\nearrow$
$\mu \uparrow$	$\searrow$	$\searrow$	$\nearrow$
$q \uparrow$	$\searrow$	$\searrow$	$\nearrow$
$T \uparrow$	$\nearrow$	$\nearrow$	$\searrow$

Table 2.1: Influence of parameters for causal and anti-causal estimators $D^{(n)}_{\kappa,\mu,\beta T,q}x(\beta T\xi_q + t_0)$.

$\hat{D}^{(n)}_{\kappa,\mu,T,q}x(t_0)$	Truncated term error	Noise error contribution
$\kappa \uparrow$	$\searrow$	$\nearrow$
$\mu \uparrow$	$\searrow$	$\nearrow$
$q \uparrow$	$\searrow$	$\nearrow$
$T \uparrow$	$\nearrow$	$\searrow$

Table 2.2: Influence of parameters for central estimators $\hat{D}^{(n)}_{\kappa,\mu,T,q}x(t_0)$.

2.6.1 Numerical tests for central Jacobi estimators

Let $x^\delta(t_i) = x(t_i) + c\varpi(t_i)$ be a generated noise data with an equidistant sampling period $T_s = 10^{-3}$ where $c > 0$. The noise $c\varpi(x_i)$ is simulated from a zero-mean white Gaussian *iid* sequence. By using the well-known three-sigma rule, we can assume that the noise level δ for $c\varpi$ is equal to $3c$. In this subsection, we use central Jacobi estimator given by (1.169) to estimate the derivatives of x. According to Corollary 1.6.32, we set $\kappa = \mu$ and choose the truncation order q to be an even integer. Moreover, according to Table 2.2 the associated truncated term error is decreasing with respect to κ and q, the associated noise error contribution is increasing with respect to κ and q. In order to reduce the truncated term error and to avoid a large noise error contribution, we set $q = 4$ and $\kappa = 5$. The noise error decreases with respect to T and the truncated term error increases with respect to T. In the following examples, we are going to choose an appropriate value for T by using the knowledge of function x.

We use the trapezoidal rule to approximate the integral in central estimators with $2m + 1$ values. The estimated derivatives of x at the point $t_i \in I = [-2, 2]$ are calculated from the noise data $x^\delta(t_j)$ with $t_j \in [-t_i - T, t_i + T]$, where $T = mT_s$ and $2m+1$ is the number of sampling data used to calculate our estimation inside the sliding integration windows. When all the parameters are chosen, the values of $\hat{Q}_{\kappa,n,q}$ can be calculated explicitly by off-line work with the $\mathcal{O}(n^2)$ complexity. Hence, the central Jacobi estimators can be written like a discrete convolution product of these pre-calculated coefficients. Thus, we only need $2m+1$ multiplications and $2m$ additions to calculate each estimation.

The numerical integration method has an approximation error. Thus, the total error for our estimators can be bounded by

$$\begin{aligned}\left|I^{x^\delta,m}_{\hat{Q}_{\kappa,n,q}} - x^{(n)}(t_i)\right| &\leq \left|I^{x^\delta,m}_{\hat{Q}_{\kappa,n,q}} - I^{x,m}_{\hat{Q}_{\kappa,n,q}}\right| + \left|I^{x,m}_{\hat{Q}_{\kappa,n,q}} - \hat{D}^{(n)}_{\kappa,T,q}x(t_i)\right| + \left|\hat{D}^{(n)}_{\kappa,T,q}x(t_i) - x^{(n)}(t_i)\right| \\ &\leq B_{noise} + B_{num} + B_{bias} = B_{total},\end{aligned}$$

where $I^{x,m}_{\hat{Q}_{\kappa,n,q}}$ (resp. $I^{x^\delta,m}_{\hat{Q}_{\kappa,n,q}}$) is the numerical approximation of $\hat{D}^{(n)}_{\kappa,T,q}x(t_i)$ (resp. $\hat{D}^{(n)}_{\kappa,T,q}x^\delta(t_i)$) with the trapezoidal rule and B_{num} is the well-known error bound for the numerical integration error [Ralston 1965]:

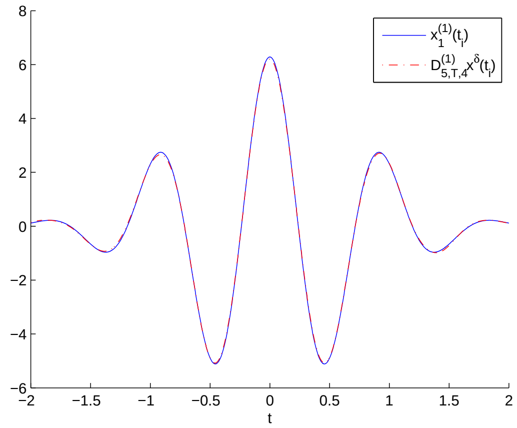
$$\left|\hat{D}^{(n)}_{\kappa,T,q}x(t_i) - I^{x,m}_{\hat{Q}_{\kappa,n,q}}\right| \leq \frac{2^3}{12(2m)^2} \sup_{\tau\in[-1,1]} \left|\hat{Q}_{\kappa,n,q}(\tau)\, x(t_i+T\tau)\right|^2 = B_{num}. \tag{2.52}$$

According to Proposition 2.3.48 and (1.166), we take $B_{noise} = E_{\kappa,\kappa,n,q,\xi}\frac{\delta}{T^n}$ with $\xi = \frac{1}{2}$. According to Corollary 1.6.32, we take $B_{bais} = M_{q+n+2}\hat{C}_{\kappa,n,q}T^{q+2}$. We are going to set the value of m such that B_{total} reaches its minimum and consequently the total errors in the following two examples can be minimized. For this, we need to calculate some values of $x^{(j)}$ with $j = 0,\cdots,n+q+2$. According to Remark 2, we calculate the value of M_{n+2+q} in the interval $[-2-\frac{T}{n+q+3}, 2+\frac{T}{n+q+3}]$. However, in practice, the function x is unknown.

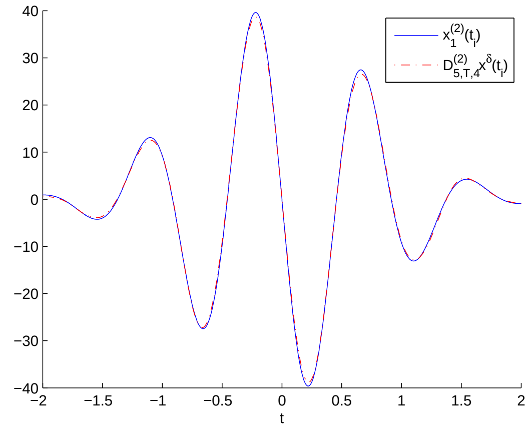
Example 1. We choose $x_1(t) = \sin(2\pi t)e^{-t^2}$ as the exact function. The numerical results are shown in Figure 6.2, where the noise level δ is equal to 0.15. The solid lines represent the exact derivative values of $x_1^{(n)}$ for $n = 1,2,3,4$ and the dash-dotted lines represent the estimated derivative values $\hat{D}^{(n)}_{\kappa,T,q}x_1^\delta(t_i)$. Moreover, we give in Table 2.3 the total error values $\max_{t_i\in[2,2]}\left|\hat{D}^{(n)}_{\kappa,T,q}x_1^\delta(t_i) - x_1^{(n)}(t_i)\right|$ for the following noise levels: $\delta = 0.15$ and $\delta = 0.015$. We can see also the total error values produced with a larger sampling period $T_s' = 10T_s = 10^{-2}$.

Table 2.3: $\max_{t_i\in[2,2]}\left|\hat{D}^{(n)}_{5,T,4}x_1^\delta(t_i) - x_1^{(n)}(t_i)\right|$.

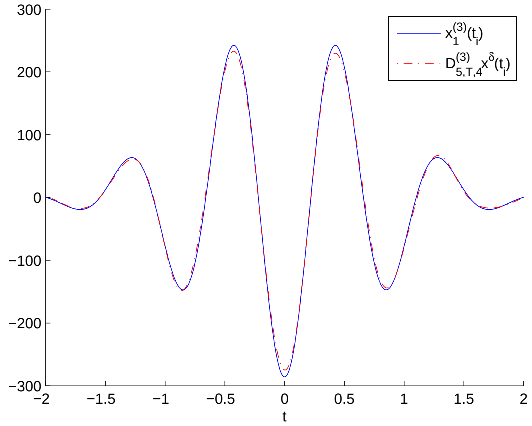
δ	$n=1$ (m)	$n=2$ (m)	$n=3$ (m)	$n=4$ (m)
0.15	$9.45e-002$ (591)	1.1 (698)	$1.258e+001$ (777)	$1.278e+002$ (850)
0.015	$1.85e-002$ (425)	$2.951e-001$ (523)	3.888 (601)	$4.588e+001$ (675)
0.015 ($T_s' = 0.01$)	$4.06e-002$ (47)	$5.645e-001$ (55)	7.359 (62)	$9.686e+001$ (69)



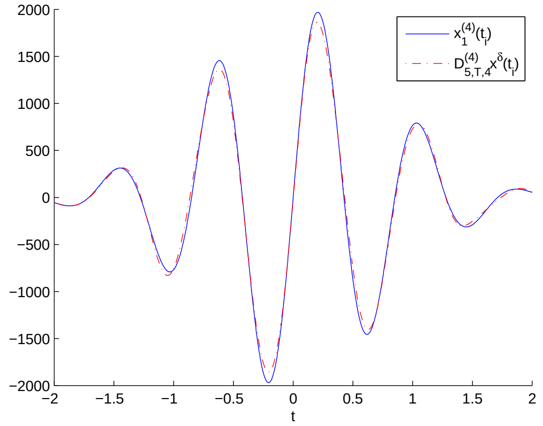
(a) $n = 1$, $\kappa = 5$, $q = 4$ and $T = 591T_s$.



(b) $n = 2$, $\kappa = 5$, $q = 4$ and $T = 698T_s$.



(c) $n = 3$, $\kappa = 5$, $q = 4$ and $T = 777T_s$.



(d) $n = 4$, $\kappa = 5$, $q = 4$ and $T = 850T_s$.

Figure 2.8: The exact values of $x_1^{(n)}(t_i)$ and the estimated values $\hat{D}_{\kappa,T,q}^{(n)} x_1^{\delta}(t_i)$ for $\delta = 0.15$.

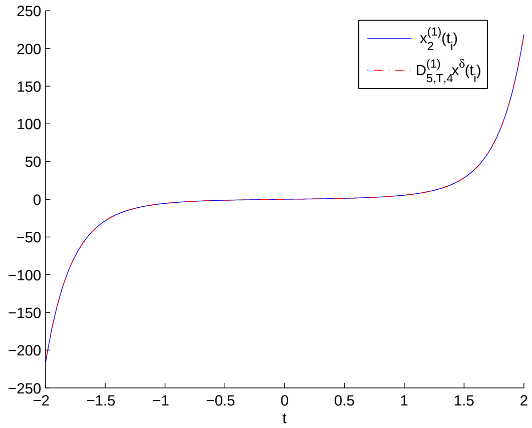
Example 2. When $x_2(t) = e^{t^2}$, we give our numerical results in Figure 6.1 with the noise level $\delta = 0.15$, where the corresponding errors are given in Figure 2.10. In Table 2.4, we also give the total error values $\max_{t_i \in [2,2]} \left| \hat{D}^{(n)}_{\kappa,T,q} x_2^\delta(t_i) - x_2^{(n)}(t_i) \right|$ for $\delta = 0.15$ and $\delta = 0.015$, where the total error values are produced with T_s and a larger sampling period $T_s' = 10^{-2}$.

Table 2.4: $\max_{x_i \in [2,2]} \left| \hat{D}^{(n)}_{5,T,4} x_2^\delta(t_i) - x_2^{(n)}(t_i) \right|$.

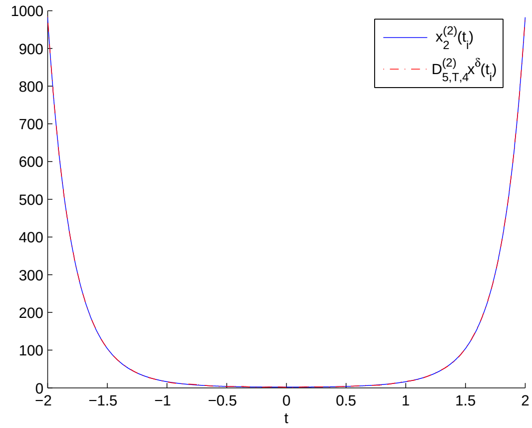
δ	$n = 1$ (m)	$n = 2$ (m)	$n = 3$ (m)	$n = 4$ (m)
0.15	$1.42e-001$ (442)	2.152 (549)	$2.982e+001$ (643)	$3.756e+002$ (733)
0.015	$2.22e-002$ (346)	$4.435e-001$ (428)	5.973 (510)	$8.769e+001$ (595)
0.015 ($T_s' = 0.01$)	$3.404e-001$ (54)	3.425 (61)	$3.638e+001$ (68)	$5.235e+002$ (79)

We can see in Figure 2.10 that the maximum of the total error for each estimation (solid line) is produced nearby the extremities where the truncated term error plus the numerical error (dash line) are much larger than the noise error. The noise error (dash-dotted line) is much larger elsewhere. This is due to the fact that the total error bound B_{total} is calculated globally in the interval $[-T-2, 2+T]$. The value of m with which B_{total} reaches its minimum is used for all the estimations $\hat{D}^{(n)}_{\kappa,T,q} x_2^\delta(t_i)$ with $t_i \in [-2, 2]$. This value is only appropriate for the estimations nearby the extremities, but not for the others. In fact, when the truncated term error and the numerical integration error decrease, we should increase the value of m so as to reduce the noise errors.

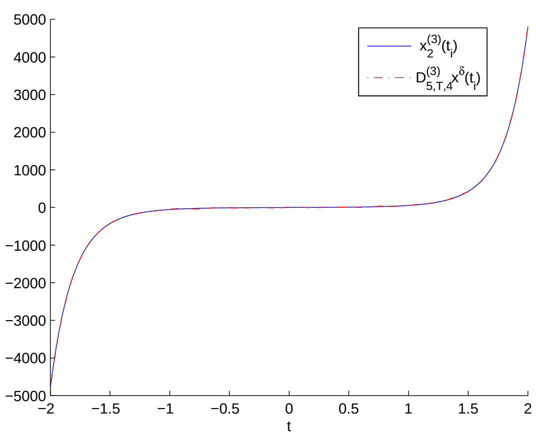
In order to improve our estimations, we can locally choose the value of $m = m_i$, *i.e.* we search the value m_i which minimizes B_{total} on $[-T_i + t_i, t_i + T_i]$ where $T_i = m_i T_s$. We can see in Figure 2.11 the errors for these improved estimations $\hat{D}^{(n)}_{\kappa,T_i,q} x_2^\delta(t_i)$. The different values of m_i are also given in Figure 2.11. The corresponding error bounds are given in Figure 2.12. We can observe that the proposed error bounds are correct but not optimal. However, the parameters' influence to these error bounds can help us to know the tendency of errors so as to choose parameters for our estimations. On the one hand, the chosen parameters may not be optimal, but as we have seen in our examples, they give good estimations. On the other hand, the optimal parameters q_{op}, κ_{op} and m_{op} with which the total error bound reaches its minimum may not give the best estimation. That is why we only use these error bounds to choose the value of m.



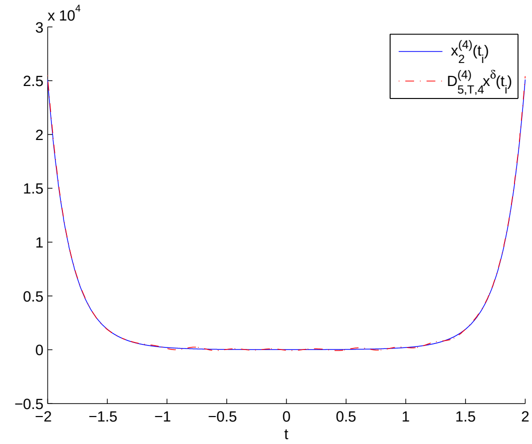
(a) $n = 1$, $\kappa = 5$, $q = 4$ and $T = 442T_s$.



(b) $n = 2$, $\kappa = 5$, $q = 4$ and $T = 549T_s$.

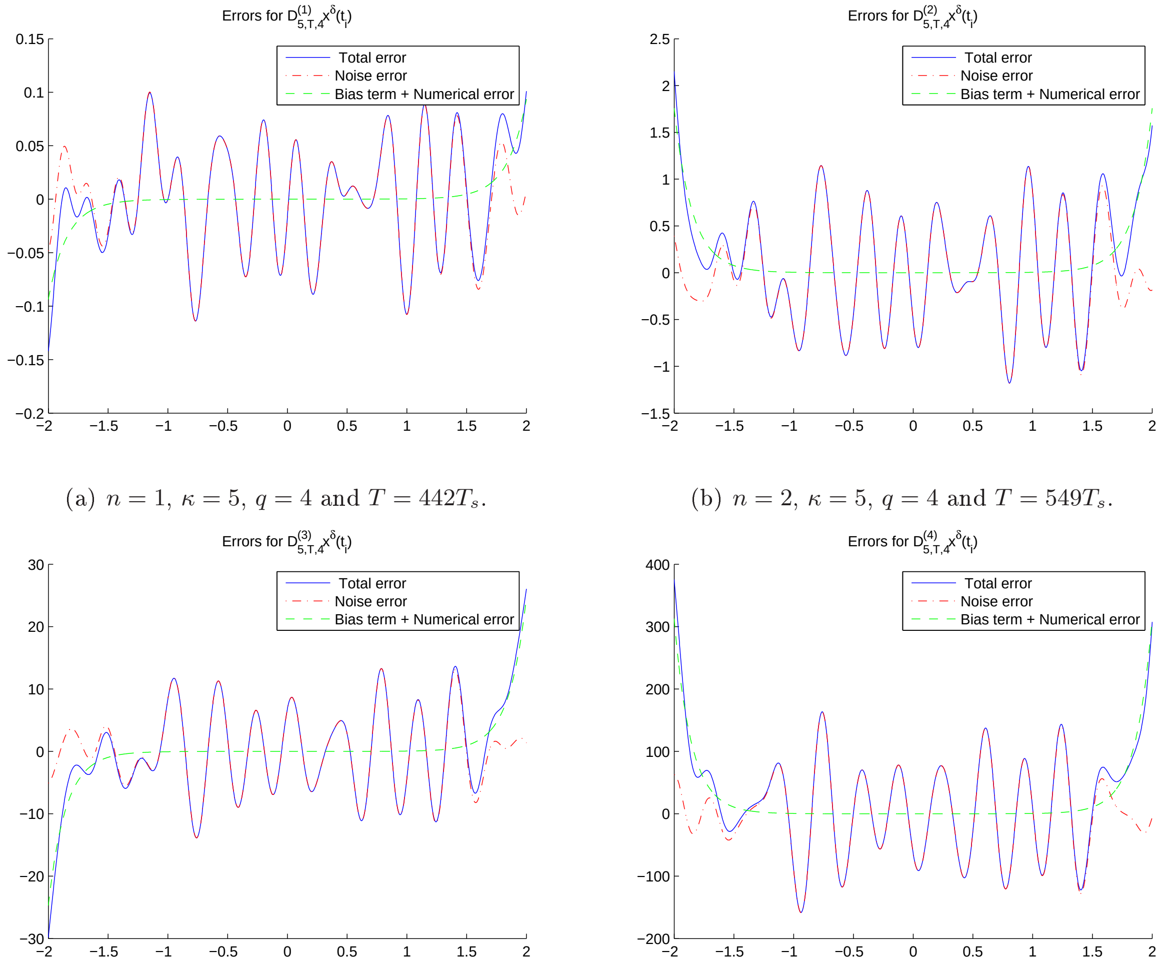


(c) $n = 3$, $\kappa = 5$, $q = 4$ and $T = 643T_s$.



(d) $n = 4$, $\kappa = 5$, $q = 4$ and $T = 733T_s$.

Figure 2.9: The exact values of $x_2^{(n)}(t_i)$ and the estimated values $\hat{D}^{(n)}_{\kappa,T,q} x_2^{\delta}(t_i)$ for $\delta = 0.15$.



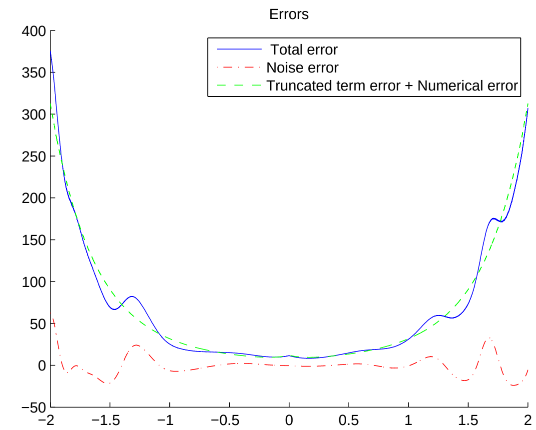
(a) $n = 1$, $\kappa = 5$, $q = 4$ and $T = 442T_s$.

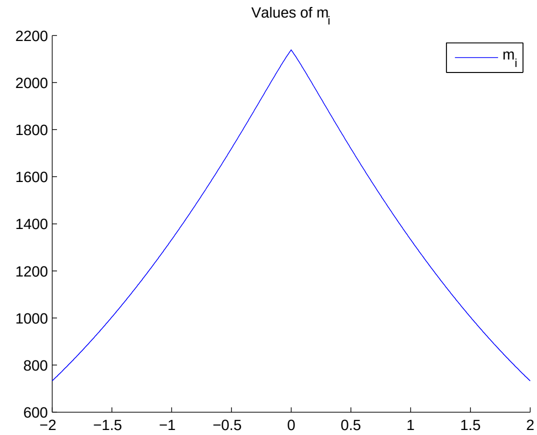
(b) $n = 2$, $\kappa = 5$, $q = 4$ and $T = 549T_s$.

(c) $n = 3$, $\kappa = 5$, $q = 4$ and $T = 643T_s$.

(d) $n = 4$, $\kappa = 5$, $q = 4$ and $T = 733T_s$.

Figure 2.10: The estimation errors for the estimated values $\hat{D}^{(n)}_{\kappa,T,q}x^\delta_2(t_i)$ for $\delta = 0.15$.

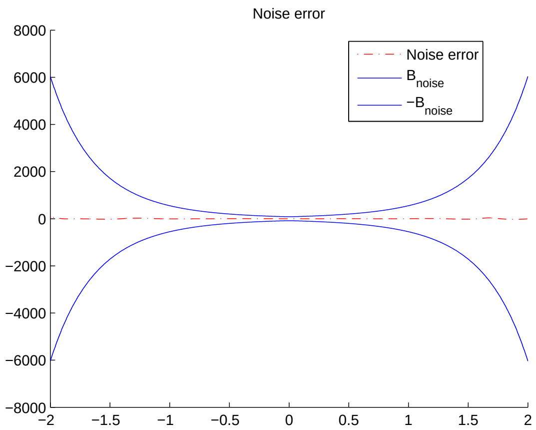


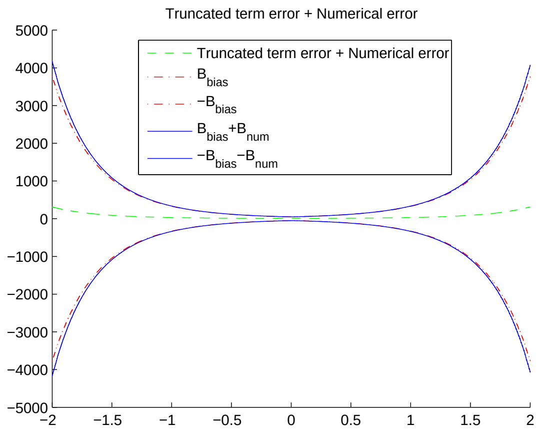


(a) The estimation errors for $\hat{D}^{(n)}_{5,T_i,4}x^{\delta}_2(t_i)$ with varying values of T_i for $\delta = 0.15$.

(b) Values of m_i

Figure 2.11: Errors for improved estimations with varying values of T_i.

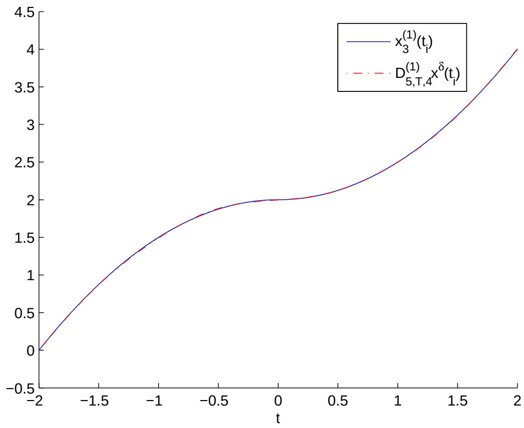




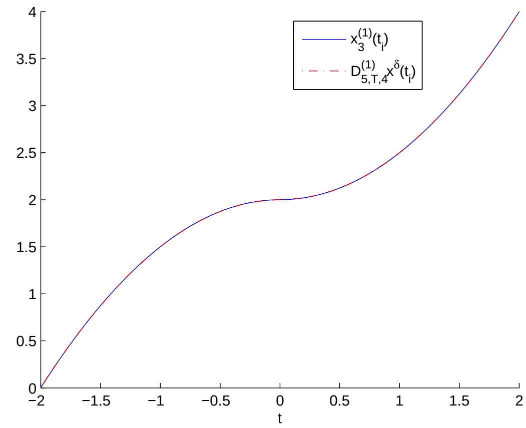
(a) Noise error with its error bounds

(b) Bias term error + numerical integration error with the error bounds

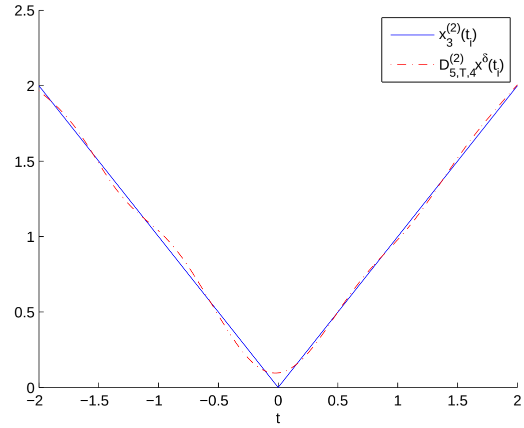
Figure 2.12: The estimation errors and their corresponding error bounds for $\hat{D}^{(n)}_{5,T_i,4}x^{\delta}_2(t_i)$ with varying values of T_i for $\delta = 0.15$.



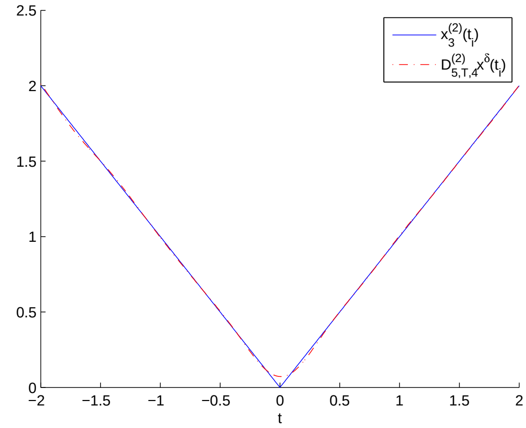
(a) $\delta = 0.15$, $n = 1$, $\kappa = 5$, $q = 4$ and $T = 1700T_s$.



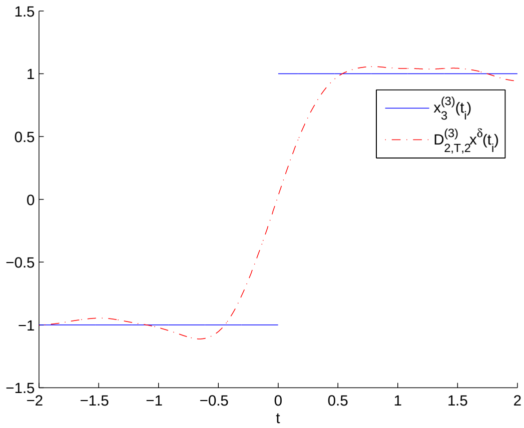
(b) $\delta = 0.015$, $n = 1$, $\kappa = 5$, $q = 4$ and $T = 1200T_s$.



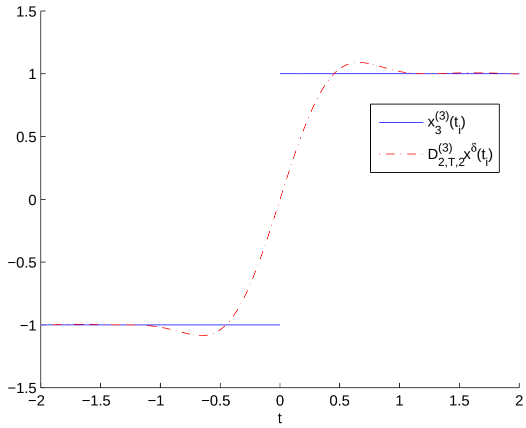
(c) $\delta = 0.15$, $n = 2$, $\kappa = 5$, $q = 4$ and $T = 1700T_s$.



(d) $\delta = 0.015$, $n = 2$, $\kappa = 5$, $q = 4$ and $T = 1200T_s$.



(e) $\delta = 0.15$, $n = 3$, $\kappa = 2$, $q = 2$ and $T = 1700T_s$.



(f) $\delta = 0.015$, $n = 3$, $\kappa = 2$, $q = 2$ and $T = 1500T_s$.

Figure 2.13: The exact values of $x_3^{(n)}(t_i)$ and the estimated values $\hat{D}^{(n)}_{\kappa,T,q}x_3^\delta(t_i)$.

Example 3. Let us consider the following function

$$x_3(t) = \begin{cases} -\frac{1}{6}t^3 + 2t, & \text{if } t \leq 0, \\ \frac{1}{6}t^3 + 2t, & \text{if } t > 0, \end{cases}$$

which is C^2 on $I = [-2, 2]$. The second derivative of x_3 is equal to $|\cdot|$. Consequently, $x_3^{(3)}$ does not exist at $t = 0$. If $n \geq 1$, then this function does not satisfy the condition $x \in C^{n+2+q}(I)$ of Corollary 1.6.32. The numerical results are shown in Figure 2.13, where the sampling period is $T_s = 10^{-3}$ and the noise level δ is equal to 0.15 and 0.015 respectively. The solid lines represent the exact derivative values of $x_3^{(n)}$ for $n = 1, 2, 3$ and the dash-dotted lines represent the estimated derivative values $\hat{D}^{(n)}_{\kappa,T,q} x_3^{\delta}(t_i)$. For the estimations of $x^{(1)}$ and $x^{(2)}$, we set $\kappa = 5$ and $q = 4$. When we estimate $x^{(3)}$, the noise error increases. Hence, we need to reduce the values of κ and q to $\kappa = 2$ and $q = 2$. In Table 2.5, we give also the total error values $\max_{t_i \in [2,2]} \left| \hat{D}^{(n)}_{\kappa,T,q} x_3^{\delta}(t_i) - x_3^{(n)}(t_i) \right|$ for $n = 1, 2$ and $\delta = 0.015, 0.15$.

Table 2.5: $\max_{t_i \in [2,2]} \left| \hat{D}^{(n)}_{5,T,4} x_3^{\delta}(t_i) - x_3^{(n)}(t_i) \right|$.

δ	$n = 1$ (m)	$n = 2$ (m)
0.15	$9.7e-003$ (1700)	$9.65e-002$ (1700)
0.015	$4.7e-003$ (1200)	$7.23e-002$ (1200)

2.6.2 Numerical tests for causal Jacobi estimators

In this subsection, we give some numerical results for the causal minimal Jacobi estimator $D^{(n)}_{\kappa,\mu,-T} x(t_0)$ given in (1.29). Let $n = 1$ in (1.29), then we have

$$D^{(1)}_{\kappa,\mu,-T} x^{\delta}(t_0) = \int_0^1 p_{\kappa,\mu,T}(\tau)\ x^{\delta}(t_0 - T\tau)\ d\tau, \tag{2.53}$$

where

$$p_{\kappa,\mu,T}(\tau) = -\frac{1}{T\mathrm{B}(\kappa+2, \mu+2)} \left((\mu+\kappa+2)\tau - (\kappa+1)\right)(1-\tau)^{\mu}\tau^{\kappa} \tag{2.54}$$

with $\kappa, \mu \in]-1, +\infty[$ and $T \in D_{t_0}$. Observing that

$$\dot{w}_{\mu+1,\kappa+1}(\tau) = -\left((\mu+\kappa+2)\tau - (\kappa+1)\right)(1-\tau)^{\mu}\tau^{\kappa}, \tag{2.55}$$

then by applying integration by parts, we obtain

$$\begin{aligned} D^{(1)}_{\kappa,\mu,-T} x(t_0) &= \frac{1}{T\mathrm{B}(\kappa+2, \mu+2)} \int_0^1 \dot{w}_{\mu+1,\kappa+1}(\tau)\ x(t_0 - T\tau)\ d\tau, \\ &= \frac{1}{\mathrm{B}(\kappa+2, \mu+2)} \int_0^1 w_{\mu+1,\kappa+1}(\tau)\, \dot{x}(t_0 - T\tau)\, d\tau. \end{aligned} \tag{2.56}$$

By using the well known Taylor's formula, we have for any $T \in D_{t_0}$ there exist $\hat{\theta}_{\tau,t_0} \in]t_0 - T\tau, t_0[$ such that

$$\forall \tau \in [0, 1],\ \dot{x}(t_0 - T\tau) = \dot{x}(t_0) - T\tau x^{(2)}(t_0) + \frac{(-T\tau_i)^2}{2} x^{(3)}(\hat{\theta}_{\tau,t_0}). \tag{2.57}$$

Then, by using (2.57) in (2.56) we get

$$\begin{aligned} D^{(1)}_{\kappa,\mu,-T}x(t_0) =& \frac{1}{\mathrm{B}(\kappa+2,\mu+2)}\int_0^1 w_{\mu+1,\kappa+1}(\tau)\left(\dot{x}(t_0) - T\tau x^{(2)}(t_0) + \frac{(-T\tau)^2}{2}x^{(3)}(\hat{\theta}_{\tau,t_0})\right)d\tau \\ =& \dot{x}(t_0) - T\frac{\kappa+2}{\kappa+\mu+4}x^{(2)}(t_0) + \frac{T^2}{2\mathrm{B}(\kappa+2,\mu+2)}\int_0^1 w_{\mu+1,\kappa+1}(\tau)\tau^2 x^{(3)}(\hat{\theta}_{\tau,t_0})\,d\tau. \end{aligned} \tag{2.58}$$

Hence, the truncated term error for $D^{(1)}_{\kappa,\mu,-T}x(t_0)$ is given as follows

$$D^{(1)}_{\kappa,\mu,-T}x(t_0) - \dot{x}(t_0) = -T\frac{\kappa+2}{\kappa+\mu+4}x^{(2)}(t_0) + \frac{T^2}{2\mathrm{B}(\kappa+2,\mu+2)}\int_0^1 w_{\mu+1,\kappa+1}(\tau)\tau^2 x^{(3)}(\hat{\theta}_{\tau,t_0})\,d\tau. \tag{2.59}$$

From now on, let us take the trapezoidal rule to approximate the integral given in (2.53)

$$I^{x^\delta,m}_{p_{\kappa,\mu,T}} := \frac{1}{m}\sum_{i=0}^{m} w_i\, p_{\kappa,\mu,T}(\tau_i)\; x^\delta(t_0 - T\tau_i), \tag{2.60}$$

where $w_0 = w_m = \frac{1}{2}$, $w_i = 1$ for $i = 1, \cdots, m-1$, and $\tau_i = \frac{i}{m}$ for $i = 0, \cdots, m$. If the value of κ (resp. μ) is negative, then we set $w_0 = 0$ (resp. $w_m = 0$) so as to avoid the singular value at $\tau_0 = 0$ (resp. $\tau_m = 1$). We denote by $I^{\tau^j,m}_{p_{\kappa,\mu,T}} := \sum_{i=0}^{m}\frac{w_i}{m}\frac{(-T\tau_i)^j}{j!}p_{\kappa,\mu,T}(\tau_i)$ for $j = 0,1,2$. Then, we can give the following proposition.

Proposition 2.6.59 *Let $I^{x^\delta,m}_{p_{\kappa,\mu,T}}$ be the minimal Jacobi estimator given by (2.60) in discrete case. If we have the following conditions:*

(E_1) : *x is a solution of the following equation*

$$\forall t \in I,\ \ddot{x}(t) + c\,x(t) = \varepsilon(t), \tag{2.61}$$

where $c \in \mathbb{R}^*$, and there exists $M_\varepsilon \in \mathbb{R}_+$ such that $\sup_{t\in I}|\varepsilon(t)| \leq M_\varepsilon$,

(E_2) : $c = -\frac{I^{\tau^2,m}_{p_{\kappa,\mu,T}}}{I^{1,m}_{p_{\kappa,\mu,T}}}$ *with $I^{1,m}_{p_{\kappa,\mu,T}}$ being not negligible,*

(E_3) : $I^{\tau,m}_{p_{\kappa,\mu,T}} \approx 1$,

then the estimation error for $I^{x^\delta,m}_{p_{\kappa,\mu,T}}$ is given by

$$I^{x^\delta,m}_{p_{\kappa,\mu,T}} - \dot{x}(t_0) \approx \varepsilon(t_0) I^{\tau^2,m}_{p_{\kappa,\mu,T}} + e_{R_2,m} + e_{\varpi,m}, \tag{2.62}$$

where $e_{\varpi,m} := \sum_{i=0}^{m}\frac{w_i}{m}p_{\kappa,\mu,T}(\tau_i)\varpi(t_0 - T\tau_i)$ *and* $e_{R_2,m} := \sum_{i=0}^{m}\frac{w_i}{m}\frac{(-T\tau_i)^3}{6}p_{\kappa,\mu,T}(\tau_i)x^{(3)}(\theta_{\tau_i,t_0})$ *with* $\theta_{\tau_i,t_0} \in]t_0 - T\tau_i, t_0[$.

Proof. By using the well known Taylor's formula, we have for any $T \in D_{t_0}$ there exist $\theta_{\tau_i,t_0} \in]t_0 - T\tau_i, t_0[$ such that

$$\forall \tau_i \in [0,1],\ x(t_0 - T\tau_i) = \sum_{j=0}^{2} \frac{(-T\tau_i)^j}{j!} x^{(j)}(t_0) + \frac{(-T\tau_i)^3}{3!} x^{(3)}(\theta_{\tau_i,t_0}). \tag{2.63}$$

Then, by taking (2.63) in (2.60) we get

$$I_{p_{\kappa,\mu,T}}^{x^\delta,m} = x(t_0) I_{p_{\kappa,\mu,T}}^{1,m} + \dot{x}(t_0) I_{p_{\kappa,\mu,T}}^{\tau,m} + x^{(2)}(t_0) I_{p_{\kappa,\mu,T}}^{\tau^2,m} + e_{R_2,m} + e_{\varpi,m}, \tag{2.64}$$

where $x^{(2)}(t_0) I_{p_{\kappa,\mu,T}}^{\tau^2,m} + e_{R_2,m}$ corresponds to the truncated error for $D_{\kappa,\mu,-T}^{(1)} x(t_0)$ in discrete case. Hence, the estimation error can be given by

$$I_{p_{\kappa,\mu,T}}^{x^\delta,m} - \dot{x}(t_0) = x(t_0) I_{p_{\kappa,\mu,T}}^{1,m} + \dot{x}(t_0)\left(I_{p_{\kappa,\mu,T}}^{\tau,m} - 1\right) + x^{(2)}(t_0) I_{p_{\kappa,\mu,T}}^{\tau^2,m} + e_{R_2,m} + e_{\varpi,m}. \tag{2.65}$$

Finally, by considering the conditions $(E_1) - (E_3)$ this proof can be completed. □

Recall that $I_{p_{\kappa,\mu,T}}^{\tau^j,m}$ for $j = 0,1,2$ is the numerical approximated value for the integral value $\int_0^1 p_{\kappa,\mu,T}(\tau) \frac{(-T\tau)^j}{j!} d\tau$ which is given in the following lemma.

Lemma 2.6.60 *Let $p_{\kappa,\mu,T}$ be the function defined by (2.54) and $i \in \mathbb{N}$, then we have*

$$\int_0^1 p_{\kappa,\mu,T}(\tau) \frac{(-T\tau)^{1+i}}{(1+i)!} d\tau = \frac{(-T)^i}{i!} \frac{\Gamma(2+i+\kappa)}{\Gamma(2+\kappa)} \frac{\Gamma(4+\mu+\kappa)}{\Gamma(4+\mu+\kappa+i)}. \tag{2.66}$$

Proof. By using (2.55) we obtain

$$\int_0^1 p_{\kappa,\mu,T}(\tau) \frac{(-T\tau)^{1+i}}{(1+i)!} d\tau = \frac{T^i}{\mathrm{B}(\kappa+2,\mu+2)} \int_0^1 \dot{w}_{\mu+1,\kappa+1}(\tau) \frac{(-\tau)^{1+i}}{(1+i)!} d\tau. \tag{2.67}$$

By applying integration by parts and using the classical Beta function, we get

$$\begin{aligned} \int_0^1 p_{\kappa,\mu,T}(\tau) \frac{(-T\tau)^{1+i}}{(1+i)!} d\tau &= \frac{T^i}{\mathrm{B}(\kappa+2,\mu+2)} \int_0^1 w_{\mu+1,\kappa+1}(\tau) \frac{(-\tau)^i}{i!} d\tau \\ &= \frac{\mathrm{B}(\kappa+2+i,\mu+2)}{\mathrm{B}(\kappa+2,\mu+2)} \frac{(-T)^i}{i!} \\ &= \frac{\Gamma(2+i+\kappa)\Gamma(4+\mu+\kappa)}{\Gamma(2+\kappa)\Gamma(4+\mu+\kappa+i)} \frac{(-T)^i}{i!}. \end{aligned} \tag{2.68}$$

□

In particular, by using (2.66) with $i = 0$ and 1, we get

$$\int_0^1 -T\tau p_{\kappa,\mu,T}(\tau) d\tau = 1 \text{ and } \int_0^1 p_{\kappa,\mu,T}(\tau) \frac{(-T\tau)^2}{2} d\tau = -T\frac{\kappa+2}{\mu+\kappa+4}.$$

Moreover, it is easy to get that $\int_0^1 p_{\kappa,\mu,T}(\tau) d\tau = 0$. Consequently, the condition (E_2) corresponds that the numerical error for $I_{p_{\kappa,\mu,T}}^{1,m}$ is not negligible, and the condition (E_3) corresponds that the numerical error for $I_{p_{\kappa,\mu,T}}^{\tau,m}$ is negligible. If we assume that the numerical errors for $I_{p_{\kappa,\mu,T}}^{\tau^2,m}$ and $e_{R_2,m}$ are negligible, then we can give the following corollary.

Corollary 2.6.61 *Let us take the same assumptions given in Proposition 2.6.59. Moreover, we assume that*

(E_4) : *there exists* $M_3 \in \mathbb{R}_+^*$ *such that* $\left\|x^{(3)}\right\|_\infty \leq M_3$

(E_5) : $I_{p_{\kappa,\mu,T}}^{\tau^2,m} \approx -T\frac{\kappa+2}{\mu+\kappa+4}$,

(E_6) : $e_{R_2,m} \approx \int_0^1 \frac{(-T\tau)^3}{6} p_{\kappa,\mu,T}(\tau) x^{(3)}(\theta_{\tau,t_0})\, d\tau$ *where* $\theta_{\tau,t_0} \in]t_0 - T\tau, t_0[$.

Then the estimation error for $I_{p_{\kappa,\mu,T}}^{x^\delta,m}$ *is bounded by*

$$\left\|I_{p_{\kappa,\mu,T}}^{x^\delta,m} - \dot{x}(t_0)\right\|_\infty \leq T\xi_0 M_\varepsilon + \frac{T^2\xi_0}{2}\frac{\kappa+3}{\kappa+\mu+5}M_3 + M_\varpi, \tag{2.69}$$

where $\xi_0 = \frac{\kappa+2}{\mu+\kappa+4}$, M_ϖ *is an error bound for the noise error contribution* $e_{\varpi,m}$.

Proof. By observing (2.59) and (2.64), we can obtain that

$$\int_0^1 \frac{(-T\tau)^3}{6} p_{\kappa,\mu,T}(\tau) x^{(3)}(\theta_{\tau,t_0})\, d\tau = \frac{T^2}{2\mathrm{B}(\kappa+2,\mu+2)}\int_0^1 w_{\mu+1,\kappa+1}(\tau)\tau^2 x^{(3)}(\hat{\theta}_{\tau,t_0})\, d\tau, \tag{2.70}$$

where the truncated term error $\int_0^1 \frac{(-T\tau)^3}{6} p_{\kappa,\mu,T}(\tau) x^{(3)}(\theta_{\tau,t_0})\, d\tau$ is approximated by $e_{R_2,m}$ in (2.64). Then, this proof can be easily completed by using Proposition 2.6.59. □

We are going to give some numerical examples where we take a class of functions satisfying the conditions (E_1) and (E_4). We set $\kappa < 0$ and $\mu \geq 0$ such that the conditions (E_2), (E_3), (E_5) and (E_6) meet.

Let $x^\delta(t_i) = x(t_i) + c\varpi(t_i)$ be a noisy generated data with an equidistant sampling period $T_s = \frac{1}{2000}$. The noise $c\varpi(x_i)$ is simulated from a zero-mean white Gaussian *iid* sequence. In the following examples, the constant c is set such that the signal-to-noise ratios ([Haykin 2002]) $SNR = 10\log_{10}\left(\frac{\sum|x^\delta(t_i)|^2}{\sum|c\varpi(t_i)|^2}\right)$ is equal to 30dB. By comparing with the minimal estimator $D_{\kappa,\mu,-T}^{(1)} x^\delta(t_0)$ given by (1.29) with $\kappa = \mu = 0$, we show the improvement of $D_{\kappa,\mu,-T}^{(1)} x^\delta(t_0)$ with $\mu = 0$ and $\kappa < 0$.

The error bounds of the truncated term error and the noise error contribution for $D_{\kappa=0,\mu=0,-T}^{(1)} x^\delta(t_0)$ are given by (1.49) and (2.51) respectively. Similarly to (2.52), an error bound of the numerical error for $D_{\kappa=0,\mu=0,-T}^{(1)} x^\delta(t_0)$ can be given. Hence, the value of the parameter m ($T = mT_s$) is chosen such that the total error bound arrives its minimum.

The values of κ and T for $D_{\kappa<0,\mu=0,-T}^{(1)} x^\delta(t_0)$ are set by considering the two following criterions

- the noise error bound for $D_{\kappa<0,\mu=0,-T}^{(1)} x^\delta(t_0)$ is equal to the one for $D_{\kappa=0,\mu=0,-T}^{(1)} x^\delta(t_0)$,
- condition (E_2) meets.

Hence, we compare $D_{\kappa=0,\mu=0,-T}^{(1)} x^\delta(t_0)$ and $D_{\kappa<0,\mu=0,-T}^{(1)} x^\delta(t_0)$ by giving the same noise error contributions.

In the following examples, the dotted lines refer to the estimations of $D_{\kappa=0,\mu=0,-T}^{(1)} x^\delta(t_0)$ and the dash-dotted lines refer to the estimations of $D_{\kappa<0,\mu=0,-T}^{(1)} x^\delta(t_0)$.

Example 1. We choose $x_1^\delta(t_i) = \sin(t_i) + c\varpi(t_i)$ with $I = 4\pi$. Hence, x_1 satisfies conditions (E_1), (E_4) and c is equal to 0.0225. By some calculations, we obtain $T_2 = 707T_e$ for $D^{(1)}_{\kappa=0,\mu=0,-T_2} x_1^\delta(t_0)$ and $T_1 = 820T_e$, $\kappa = -0.458$ for $D^{(1)}_{\kappa<0,\mu=0,-T_1} x_1^\delta(t_0)$. With these parameters' values, we can verify that conditions (E_3), (E_5) and (E_6) meet. Hence, we can apply Corollary 2.6.61 to obtain error bounds for $D^{(1)}_{\kappa<0,\mu=0,-T_1} x_1^\delta(t_0)$. The obtained estimations and corresponding error bounds are shown in Figure 2.14. We can see that with the same level of noise error contributions, the estimator $D^{(1)}_{\kappa<0,\mu=0,-T_1} x_1^\delta(t_0)$ produces a smaller truncated term error than $D^{(1)}_{\kappa=0,\mu=0,-T_1} x_1^\delta(t_0)$. Consequently, we obtain a delay-free estimation.

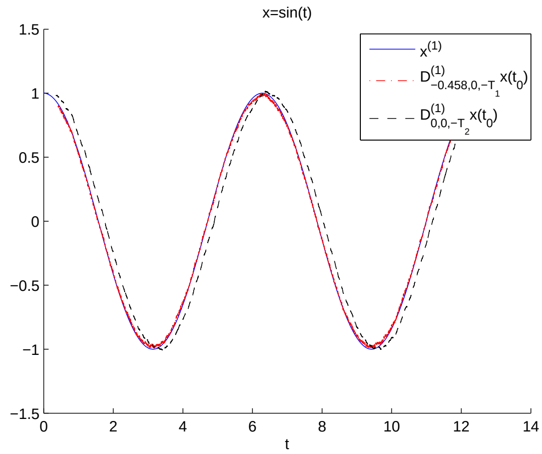
Example 2. We choose $x_2^\delta(t_i) = t_i \sin(t_i) + c\varpi(t_i)$ with $I = 14$ and $c = 0.18$. Hence, x_2 satisfies conditions (E_1) and (E_4) with $x_2^{(2)}(t_i) + x_2(t_i) = 2\cos(t_i)$. By some calculations, we obtain $T_2 = 639T_e$ for $D^{(1)}_{\kappa=0,\mu=0,-T_2} x_2^\delta(t_0)$ and $T_1 = 705T_e$, $\kappa = -0.4$ for $D^{(1)}_{\kappa<0,\mu=0,-T_1} x_2^\delta(t_0)$. The obtained estimations and corresponding error bounds are shown in Figure 2.15.

Example 3. We choose $x_3^\delta(t_i) = \exp(\frac{-t}{1.2})\sin(6t_i + \pi) + c\varpi(t_i)$ with $I = 5$ and $c = 0.0075$. Hence, x_3 satisfies conditions (E_1) and (E_4) with

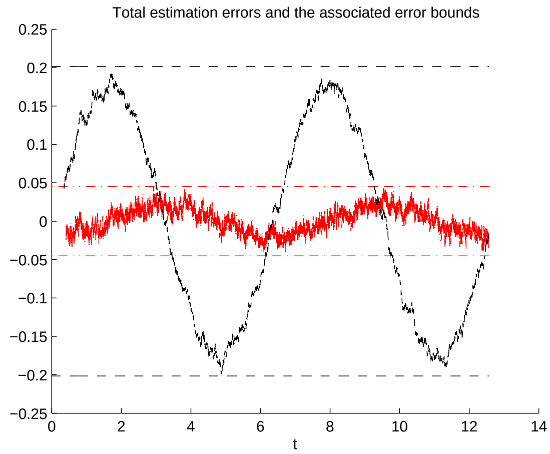
$$x_3^{(2)}(t_i) + \frac{1271}{36} x_3(t_i) = -10\exp(\frac{-t_i}{1.2})\cos(6t_i + \pi), \tag{2.71}$$

$$x_3^{(3)}(t_i) = \frac{19315}{216}\exp(\frac{-t_i}{1.2})\sin(6t_i + \pi) - 203.5\exp(\frac{-t_i}{1.2})\cos(6t_i + \pi). \tag{2.72}$$

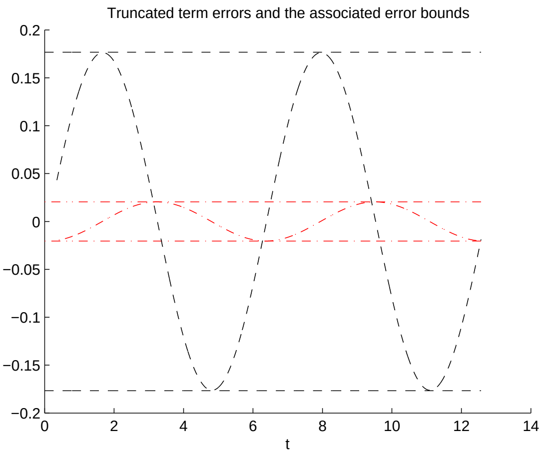
By some calculations, we obtain $T_2 = 194T_e$ for $D^{(1)}_{\kappa=0,\mu=0,-T_2} x_3^\delta(t_0)$ and $T_1 = 210T_e$, $\kappa = -0.4771$ for $D^{(1)}_{\kappa<0,\mu=0,-T_1} x_3^\delta(t_0)$. The obtained estimations and corresponding error bounds are shown in Figure 2.16. We can see in Figure 2.16(c) that the truncated term error for $D^{(1)}_{\kappa<0,\mu=0,-T_1} x_3^\delta(t_0)$ is much smaller than the corresponding error bounds obtained by Corollary 2.6.61. This is because the truncated error part $\varepsilon(t_i) I^{\tau^2,m}_{p_{\kappa,\mu,T}}$ given by (2.62) with $\varepsilon(t_i) = -10\exp(\frac{-t_i}{1.2})\cos(6t_i + \pi)$ reduces the truncated error part $e_{R_2,m}$ obtained by $x_3^{(3)}$. Consequently, we obtain a delay-free estimation.



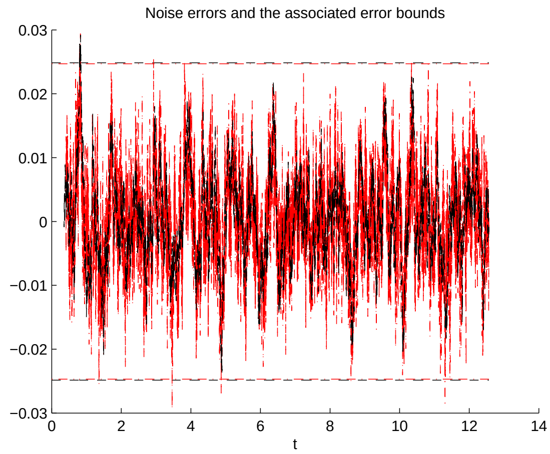
(a) Estimations.



(b) Associated estimation errors and corresponding error bounds.

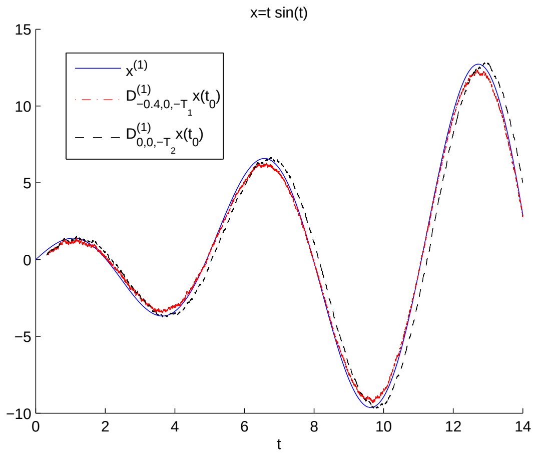


(c) Associated truncated term errors and corresponding error bounds.

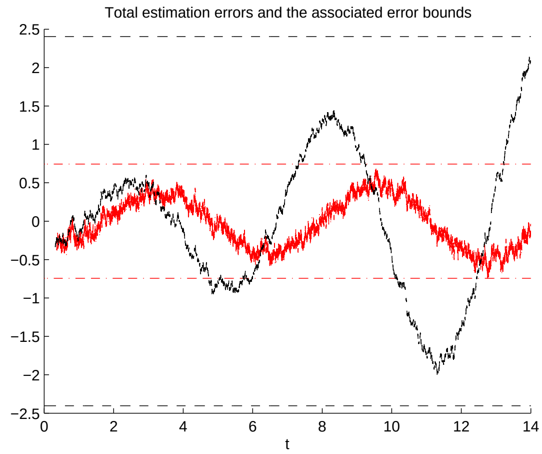


(d) Associated noise errors and corresponding error bounds.

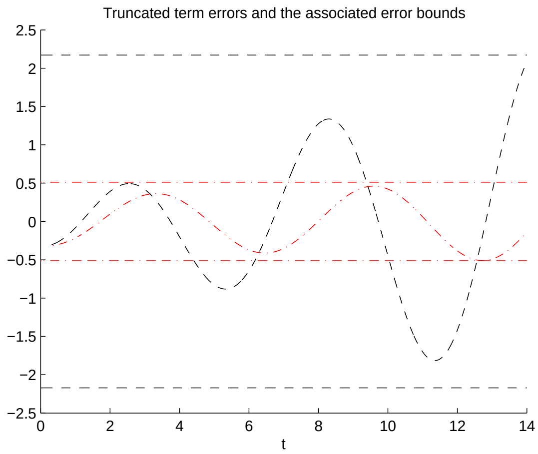
Figure 2.14: Comparison between the minimal estimator $D^{(1)}_{0,0,-T_2}x_1^\delta(t_0)$ and $D^{(1)}_{\kappa,0,-T_1}x_1^\delta(t_0)$ with $T_2 = 707T_e$, $T_1 = 820T_e$ and $\kappa = -0.458$.



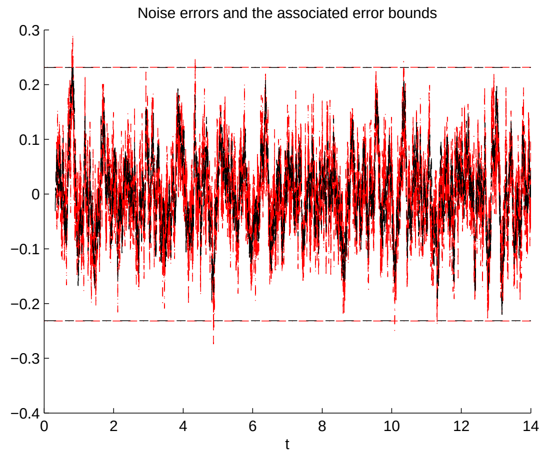
(a) Estimations.



(b) Associated estimation errors and corresponding error bounds.

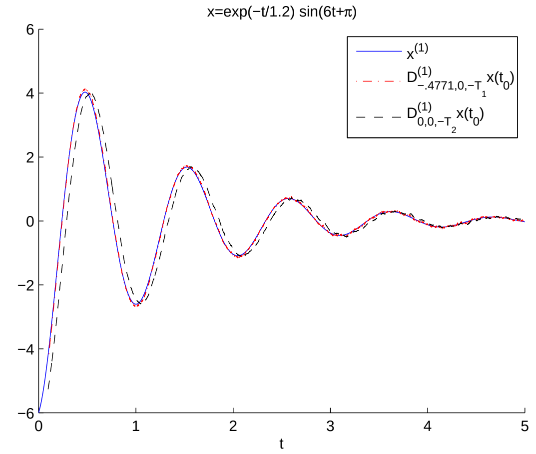


(c) Associated truncated term errors and corresponding error bounds.

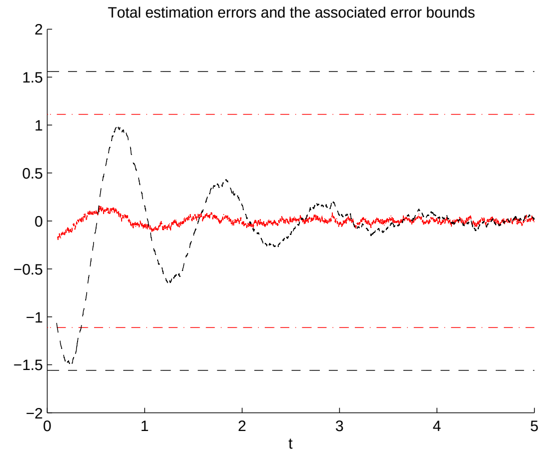


(d) Associated noise errors and corresponding error bounds.

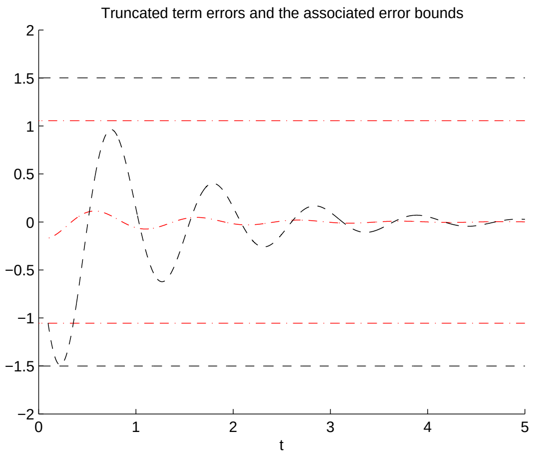
Figure 2.15: Comparison between the minimal estimator $D^{(1)}_{0,0,-T_2}x_2^\delta(t_0)$ and $D^{(1)}_{\kappa,0,-T_1}x_2^\delta(t_0)$ with $T_2 = 639T_e$, $T_1 = 705T_e$ and $\kappa = -0.4$.



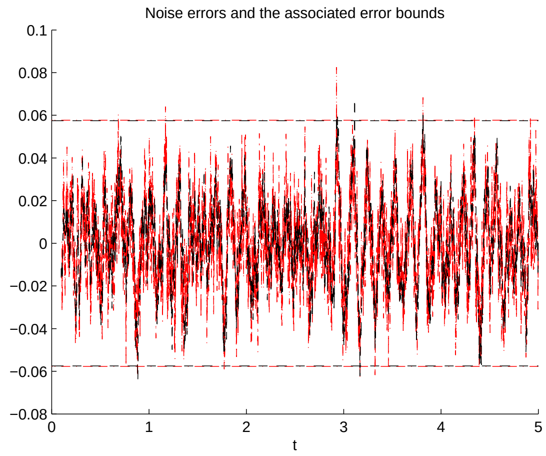
(a) Estimations.



(b) Associated estimation errors and corresponding error bounds.



(c) Associated truncated term errors and corresponding error bounds.



(d) Associated noise errors and corresponding error bounds.

Figure 2.16: Comparison between the minimal estimator $D^{(1)}_{0,0,-T_1} x^{\delta}_3(t_0)$ and $D^{(1)}_{\kappa,0,-T_2} x^{\delta}_3(t_0)$ with $T_1 = 194$, $T_2 = 210$ and $\kappa = -0.4771$.

2.7 Conclusion

In this chapter, we have studied the noise error contribution for Jacobi estimators. We have respectively given some error bounds for the noise error contributions due to a bounded integrable noise, a non-independent stochastic process and an independent stochastic process. These error bounds gave us a guide for choosing parameters so as to reduce noise error contribution. We recall in Table 2.6 the results obtained of the parameters' influence on the variance of the noise error contributions. In the numerical simulations, we have shown how to choose parameters for central Jacobi estimators. If the smooth function x satisfied a differential equation, then by taking negative value for κ in minimal causal Jacobi estimators we obtained some "delay-free" estimations in discrete case. Let us remark that by using affine Jacobi estimators we can also obtain some delay-free estimations without taking negative value for κ. However, we should take much more points in each sliding integration window so as to reduce the associated noise error contributions.

Estimator	Type of noise	Figure
causal (anti-causal) minimal estimators	Wiener or Poisson process	Figure 2.1
causal (anti-causal) affine estimators with $q = 1$	Wiener or Poisson process	Figure 2.2
causal (anti-causal) estimators with $\kappa, \mu \geq 0$	*iid* random variables	Figure 2.3
central estimators with $\kappa, \mu \geq 0$	*iid* random variables	Figure 2.4
causal (anti-causal) minimal estimators with $\kappa, \mu < 0$	*iid* random variables	Figure 2.6
causal (anti-causal) affine estimators with $\kappa, \mu < 0$	*iid* random variables	Figure 2.7

Table 2.6: Parameters' influence on the noise error contributions

Chapter 3

Application to non linear observation

Contents

3.1 Introduction

3.1.1 Context and motivations

Physical processes are often represented by the models described in the following form (explicit state representation):

$$\begin{cases} \dot{x} = f(x(t), u(t)) \\ y = h(x(t)) \end{cases} \tag{3.1}$$

where

- $x \in \mathcal{X}$ denotes the state vector with $\mathcal{X}$ being a differentiable manifold open subset of $\mathbb{R}^n$,
- $u \in \mathcal{U}$ denotes the vector of known inputs with $\mathcal{U} \subset \mathbb{R}^m$ being a set of admissible input,
- $y \in \mathcal{Y}$ denotes the vector of measured outputs with $\mathcal{Y}$ being an open set of $\mathbb{R}^p$.

Functions f and h are in general assumed to be $\mathcal{C}^\infty$, and inputs functions $u(\cdot)$ to be locally essentially bounded and measurable (in sense of Lebesgue).

In general, it is clear that one can not use as many sensors as signals of interest required to characterize the behavior of the system (for cost reasons, technological constraints, etc...), and the size of vector output is lower than the one of state vector. Most of the time this implies that for a given time t, the state $x(t)$ can not be algebraically deduced from the measured output y (observed at the time t). However, the need for information on the state is motivated by various purposes: modeling (identification), monitoring (fault detection), or driving (control) the system (Cf. Figure 3.1). For this, we can see [Besançon 2007]. Consequently, the problem of reconstruction of state or observer is one of the most essential part of a general control problem.

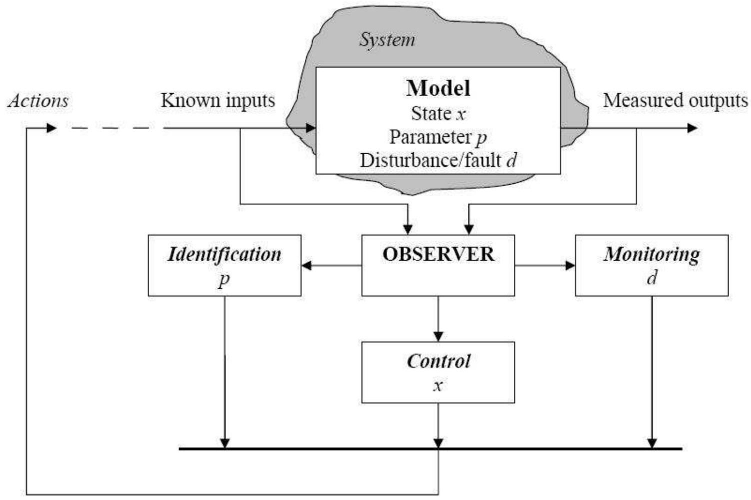


Figure 3.1: Observer: the essential part of control system.

3.1.2 Observer problem

An observer can be achieved if the system is observable. This means that it is possible to reconstruct the initial state from the information on its inputs u and output y during a finite time interval $[t_i, t_f]$.

The purpose of an observer is precisely to provide an estimate of the current value of the state as a function of the input and output of system.

The observer design is often based on the idea of "feedback". More precisely, on the one hand, if the initial value $x(0)$ is known, then the estimated value $x(t)$ can be simply obtained by integrating the system (3.1) from $x(0)$. On the other hand, if the initial value $\tilde{x}(0)$ is unknown, then we can try to correct on-line the integration of $\tilde{x}(t)$ from some erroneous initial value $\tilde{x}(0)$, depending on the measurable error $h(\tilde{x}(t)) - y(t)$. That is to say we look for an estimate $\tilde{x}$ of x as the solution of a system:

$$\dot{\tilde{x}} = f(\tilde{x}(t), u(t)) + k(t, h(\tilde{x}(t)) - y(t)) \text{ with } k(t, 0) = 0. \tag{3.2}$$

In most cases, the auxiliary system (3.2) is defined as an observer for the system (3.1).

3.2 Observability in a non linear context

The purpose of this section is to discuss some conditions required on the system for possible solutions to the above mentioned observer problem. Such conditions correspond to what are usually called *observability* conditions. For this, we recall some results of observability for a nonlinear system within a differential geometric framework mainly due to Hermann and Krener [Hermann 1977] (see also Isidori [Isidori 1989], Nijmeijer and van der Schaft [Nijmeijer 1990]) and a differential algebraic framework due to Diop and Fliess [Diop 1991a, Diop 1991b] (see also [Barbot 2007]), respectively.

3.2.1 Review of observability within a geometric framework

In order to design an observer, one must be able to recover the information on the state x via the output measured y from the initial time, and more particularly to recover the corresponding initial value of the state. This means that observability is characterized by the fact that from an output measurement, one must be able to distinguish between various initial states. The observability is indeed defined from the notion of indiscernibility which is given in the following definition.

Definition 4 *Indiscernibility [Besançon 2007] A sate $x(0)$ is indistinguishable from another sate $x'(0)$ for the system (3.1) if*

$$\forall u \in \mathcal{U}, \forall t \geq 0, \;\; y(t, 0, x(0), u) = y(t, 0, x'(0), u), \tag{3.3}$$

where $y(t, 0, x(0), u) = h(x(t, 0, x(0), u))$ (resp. $y(t, 0, x'(0), u) = h(x(t, 0, x'(0), u))$) is the output of (3.1) for the input u and the initial state $x(0)$ (resp. $x'(0)$).

This notion of indiscernibility of two initial states permits us to give the definition of observability.

Definition 5 *(Observability) [Besançon 2007] The system (3.1) is called observable at x if there is no indistinguishable state from x in $\mathcal{X}$. The system (3.1) is observable if it is observable for all $x \in \mathcal{X}$.*

The previous definition is too general for practical use, since one might be mainly interested in distinguishing states from their neighbors. Let us consider for instance the case of the following system:

$$\begin{cases} \dot{x} = u \\ y = \cos(x). \end{cases}$$

Clearly, y cannot help to distinguish between x_0 and $x_0 + 2k\pi$, and thus the system is not observable. However, it is yet clear that y allows to distinguish states of $]-\frac{\pi}{2}, \frac{\pi}{2}[$. This brings to consider a weaker notion of observability.

Definition 6 *(Local observability) The system (3.1) is locally observable at x if there exists a neighborhood $\mathcal{V}(x)$ of x such that there is no indistinguishable state from x in $\mathcal{V}(x)$. The system (3.1) is locally observable if it is locally observable for all $x \in \mathcal{X}$.*

Notice that two states may be indistinguishable for some input whereas they can be distinguished for other inputs. Let us consider for instance the case of the following system:

$$\begin{cases} \dot{x}_1 = x_2 u \\ \dot{x}_2 = 0 \\ y = x_1. \end{cases}$$

It is clear that for the null input u one cannot distinguish the two states x and x' such that $x_1 = x'_1$ and $x_2 \neq x'_2$. However, this system is observable for any $u \neq 0$. For example, if $u \equiv 1$, then we obtain an observable linear system. Hence, in order to prevent this situation, we give the following definition of observability.

Definition 7 *(Weak observability) The system (3.1) is weakly observable at x if there exists a neighborhood $\mathcal{V}(x)$ of x such that $\mathcal{I}_u(x) \cap \mathcal{V}(x) = x$, where $\mathcal{I}_u(x)$ denotes the set of states indistinguishable from x_0 with the input u. The system (3.1) is weakly observable if it is weakly observable for all $x \in \mathcal{X}$.*

If the system (3.1) is locally observable and weakly observable, then it is locally weakly observable. This notion is of more interest in practice, and also presents the advantage of admitting some 'rank condition' characterization. Such a condition relies on the notion of observation space roughly corresponding to the space of all observable states.

Definition 8 *(Observation space) [Besançon 2007] The observation space for the system (3.1) is defined as the smallest real vector space (denoted by $\mathcal{O}(h)$) of $\mathcal{C}^\infty$ functions containing the components of h and closed under Lie derivation along $f_u := f(.,u)$ for any constant $u \in \mathbb{R}^m$ (namely such that for any $\phi \in \mathcal{O}(h)$, $L_{f_u}\phi \in \mathcal{O}(h)$, where $L_{f_u}\phi = \frac{\partial \phi}{\partial x} f(x,u)$).*

Definition 9 *(Observability rank condition) [Besançon 2007] The system (3.1) is said to satisfy the observability rank condition if:*

$$\forall x \in \mathcal{X}, \quad \dim d\mathcal{O}(h)|_x = n, \tag{3.4}$$

where $d\mathcal{O}(h)|_x = \operatorname{span}\{d\phi(x); \phi \in \mathcal{O}(h)\}$ is called the codistribution of observability.

Then, we give the following theorem.

Theorem 3.2.62 *[Hermann 1977] The system (3.1) satisfying the observability rank condition is locally weakly observable. Conversely, a system (3.1) locally weakly observable satisfies the observability rank condition in an open dense subset of $\mathcal{X}$.*

Examples: Let us consider the following non linear system

$$\begin{cases} \dot{x}_1 &= x_2 + x_1 x_2 \\ \dot{x}_2 &= -x_1 x_2 + u \\ y &= x_1. \end{cases} \tag{3.5}$$

In this system, we have $h(x) = x_1$ and $f(x) = (x_2 + x_1 x_2, -x_1 x_2 + u)^T$. Then, we get $L_f h = x_2 + x_1 x_2$, $dh = dx_1$ and $dL_f h = dx_2 + x_1 dx_2 + dx_1 x_2$. Hence, we have $d\mathcal{O}(h) = \text{span}\{dx_1, dx_2\}$ with $\dim d\mathcal{O}(h) = 2$. This system is locally weakly observable.

3.2.2 Review of observability within an algebraic framework

Diop and Fliess introduced in [Diop 1991a, Diop 1991b] a new approach of nonlinear observability based on differential algebra. With respect to the differential geometric theory presented in Subsection 3.2.1, it has among other features the possibility of defining observability for systems represented by an arbitrary set of algebra-differential equations.

Let us recall some useful notations.

Definition 10 *(Differential ring and differential field) [Diop 1991a] A differential ring* R *is a commutative ring with* 1*, which is equipped with a derivation, i.e., a mapping* $\frac{d}{dt} = \dot{} : \mathrm{R} \longrightarrow \mathrm{R}$ *such that*

$$\forall a, b \in \mathrm{R}, \quad \frac{d}{dt}(a + b) = \dot{a} + \dot{b}, \tag{3.6}$$

$$\forall a, b \in \mathrm{R}, \quad \frac{d}{dt}(ab) = \dot{a}b + a\dot{b}. \tag{3.7}$$

A differential field is a differential ring which is a field.

Definition 11 *(Differential field extension) [Diop 1991a] A differential field extension* L/K *is a field extension* L/K *such that the derivation of* K *is the restriction to* K *of the derivation of* L.

We denote the differential field generated by K and a subset S of L by K⟨S⟩.

Definition 12 *(Differentially* K*-algebraic) [Diop 1991a] An element* $z \in \mathrm{L}$ *is said to be differentially algebraic over* K*, or differentially* K*-algebraic if, and only if, it satisfies an algebraic differential equation over* K.

This definition means that there exists a non-zero polynomial p over K in $v + 1$ indeterminates such that $p(z, \dot{z}, \ldots, z^{(v)}) = 0$.

Definition 13 *(Deferentially* K*-algebraically dependent) [Diop 1991a] A set* $\xi = \{\xi_i; i \in \mathrm{I}\}$ *of element in* L *is said to be deferentially* K*-algebraically dependent if, and only if, there exists* $\xi_{i_0} \neq 0$ *which is differentially algebraic over* $\mathrm{K}\langle\bar{\xi}\rangle$ *with* $\bar{\xi} = \{\xi_i; i \in \mathrm{I}, i \neq i_0\}$.

Let k be a given differential ground field. Denote by k⟨u⟩ the differential field generated by k and a finite set $\mathrm{u} = (\mathrm{u}_1, \ldots, \mathrm{u}_m)$ of differential quantities. The set u plays the role of control variables or input, which may be assumed to be independent. This means that u is differentially k-algebraically independent.

Definition 14 *(Dynamic) [Fliess 1989] A dynamic is a finitely generated differential algebraic extension* $\mathcal{D}/\mathrm{k}\langle\mathrm{u}\rangle$.

This means that any element of $\mathcal{D}$ satisfies an algebraic differential equation with coefficients which are rational functions over k in the components of u and a finite number of their derivatives. As output variables can be viewed as sensors on the dynamics, we formally define an *output* as a finite set $y = (y_l, \ldots, y_p) \in \mathcal{D}$.

Theorem 3.2.63 *[Diop 1991b] Choose a subset $z = \{z_i; i \in \mathrm{I}\}$ of $\mathcal{D}$ in a dynamics $\mathcal{D}/\mathrm{k}\langle \mathrm{u}\rangle$. An element ξ in $\mathcal{D}$ is said to be observable with respect to z if it is algebraic over $\mathrm{k}\langle z\rangle$.*

This result intuitively means that ξ can be expressed as an algebraic function of the components of z and a finite number of their derivatives. A subset S of $\mathcal{D}$ is said to be observable with respect to z if, and only if, any element of S is so.

In the usual definition of observability, one takes for z the set $\{u, y\}$ of input and output variables and for S the set $\mathcal{X}$ of state variables. A state x is said to be observable if, and only if, it is observable with respect to $\{u, y\}$. Indeed, a nonlinear system is observable if, and only if, any state variable is a differential function of the control and output variables, *i.e.*, a function of those variables and their derivatives up to some finite order.

Remark 7 *This algebraic approach of nonlinear observability can be used for systems represented by an arbitrary set of algebra-differential equations. While the geometric approach can be only used for the systems defined in (3.1). It is shown that the system defined in (3.5) is observable by using the geometric approach. Since $x_1 = y$ and $x_2 = \frac{\dot{x}_1}{1+x_1}$, this system is also observable by using the algebraic approach. Moreover, let us consider the ball and beam system which is described by (3.8). Since (3.8) is not given in the form of (3.1), we are going to show that this system is observable by using the algebraic approach.*

3.3 Case study: comparison between some observers and our numerical differentiation techniques

3.3.1 The Ball and Beam system

The ball and beam system is one of the most enduring popular and important laboratory models for teaching control systems engineering. This system is widely used because it is very simple to understand. It has a very important property: it is open loop unstable.

The system can be shown in Figure 3.2. A steel ball rolling on the top of a long beam which can be tilted about its center axis by applying a control. The position of the ball on the beam and the angle of the beam can be measured by using sensors.

The dynamics of the ball rolling on the beam can be described as follows:

$$\begin{cases} (mr^2 + J)\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\cos(\theta) &= u \\ m\ddot{r} + mg\sin(\theta) - mr\dot{\theta}^2 &= 0, \end{cases} \tag{3.8}$$

where m is the mass of the ball, J is the length of the beam, g is the gravitational constant, r is the position of the ball on the beam, θ is the beam angle and u is a control.

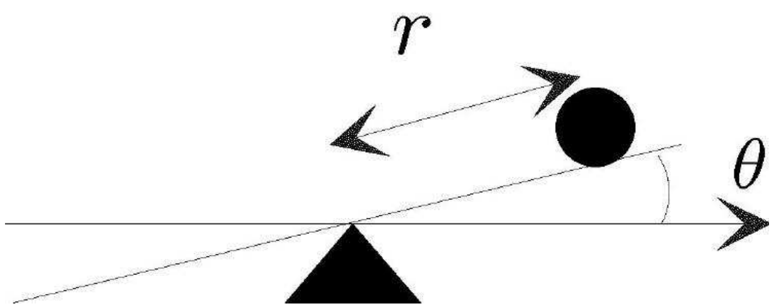


Figure 3.2: Ball and beam system.

The state vector of the ball and beam system is $(r, \dot{r}, \theta, \dot{\theta})^T$. Assume that the output vector is $(y_1, y_2)^T = (r, \theta)^T$, then we get

$$\begin{cases} r &= y_1 \\ \dot{r} &= -\frac{1}{2my_1\dot{y}_2}\left((my_1^2 + J)\ddot{y}_2 + mgy_1\cos(y_2) - u\right) \\ \theta &= y_2 \\ \dot{\theta} &= \left(\frac{\ddot{y}_1 + g\sin(y_2)}{y_1}\right)^{\frac{1}{2}}. \end{cases} \tag{3.9}$$

Hence, this system is observable by using the algebraic approach.

From now on, by taking $u = (mr^2 + J)v + 2mr\dot{r}\dot{\theta} + mgr\cos(\theta)$, the ball and beam system is simplified as follows:

$$\begin{cases} \ddot{\theta} &= v \\ \ddot{r} + g\sin(\theta) - r\dot{\theta}^2 &= 0, \end{cases} \tag{3.10}$$

where $v \in \mathbb{R}$.

Linearization of this equation about the beam angle, $\theta = 0$, gives us the following linear approximation of the system:

$$\begin{cases} \ddot{\theta} &= v \\ \ddot{r} &= -g\theta. \end{cases} \tag{3.11}$$

In order to stabilize the system, we take $v = \frac{k_3}{g}r^{(3)} + \frac{k_2}{g}r^{(2)} + \frac{k_1}{g}\dot{r} + \frac{k_0}{g}r$ where

$$\begin{cases} k_0 &= w_{n_1}^2 w_{n_2}^2 \\ k_1 &= 2\xi(w_{n_1}w_{n_2}^2 + w_{n_2}w_{n_1}^2) \\ k_2 &= w_{n_1}^2 + w_{n_2}^2 + 4\xi^2 w_{n_1}w_{n_2} \\ k_3 &= 2\xi(w_{n_1} + w_{n_2}), \end{cases}$$

with $\xi = 0.7$, $w_{n_1} = 1/\xi$ and $w_{n_2} = 3w_{n_1}$.

By denoting the state vector $(r, \dot{r}, \theta, \dot{\theta})^T$ by $z = (z_1, z_2, z_3, z_4)^T$ and considering the output vector, the system in (3.11) can be written as follows:

$$\begin{cases} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= v \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= -g\sin(z_1) - z_2 z_3^2 \\ y_1 &= z_1 \\ y_2 &= z_3. \end{cases} \tag{3.12}$$

Hence, (3.12) is in the form of (3.1) with $h(z) = (z_1, z_3)^T$. Then, we get $dh(z) = (dz_1, dz_3)^T$, $L_f h = (z_2, z_4)^T$ and $dL_f h = (dz_2, dz_4)^T$. Hence, we have $d\mathcal{O}(h) = \text{span}\{dz_1, dz_2, dz_3, dz_4\}$ with $\dim d\mathcal{O}(h) = 4$. This system is locally weakly observable.

Since the measurements of r and θ are noisy and their first order derivatives $\dot{r}$ and $\dot{\theta}$ are unknown, we use in the following subsections the high-gain observer, the high-order sliding modes differentiation and our Jacobi estimators to estimate r, θ, $\dot{r}$ and $\dot{\theta}$.

3.3.2 High-gain observer

During the past few years, high-gain observers played an important role in the design of nonlinear output feedback control of nonlinear systems. They are mainly used to estimate the derivatives of the output. In this subsection, we use the high-gain observer to estimate r, θ, $\dot{r}$ and $\dot{\theta}$.

By using (3.12), we get

$$\begin{cases} \dot{R} &= AR + Bv \\ y_1 &= r, \end{cases} \tag{3.13}$$

and

$$\begin{cases} \dot{\Theta} &= A\Theta + \Phi(\Theta, R) \\ y_2 &= \theta, \end{cases} \tag{3.14}$$

where $R = \begin{pmatrix} r \\ \dot{r} \end{pmatrix}$, $\Theta = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\Phi(\Theta, R) = \begin{pmatrix} 0 \\ -g\sin(\theta) - r\dot{\theta}^2 \end{pmatrix}$.

Hence, by using high-gain observer we get

$$\begin{cases} \dot{\widehat{R}} &= A\widehat{R} - K_r(\widehat{y}_1 - y_1) + Bv \\ \widehat{y}_1 &= r_{est}, \end{cases} \tag{3.15}$$

and

$$\begin{cases} \dot{\widehat{\Theta}} &= A\widehat{\Theta} - K_\theta(\widehat{y}_2 - y_2) + \Phi(\widehat{\Theta}, \widehat{R}) \\ \widehat{y}_2 &= \theta_{est}, \end{cases} \tag{3.16}$$

where $\widehat{R} = \begin{pmatrix} r_{est} \\ \dot{r}_{est} \end{pmatrix}$, $\widehat{\Theta} = \begin{pmatrix} \theta_{est} \\ \dot{\theta}_{est} \end{pmatrix}$, $K_r = \begin{pmatrix} k_{r1} \\ k_{r2} \end{pmatrix}$ and $K_\theta = \begin{pmatrix} k_{\theta 1} \\ k_{\theta 2} \end{pmatrix}$.

The high-gains are given by $K_r = S_\infty^{-1}(\alpha_r)C$ and $K_\theta = S_\infty^{-1}(\alpha_\theta)C$, where $S_\infty^{-1}(\alpha)$ is the unique solution of the matrix equation:

$$\left\{ \begin{array}{rcl} \alpha S_\infty(\alpha) + A^T S_\infty(\alpha) + S_\infty(\alpha)A & = & CC^T \\ S_\infty(\alpha) & = & S_\infty^T(\alpha), \end{array} \right. \tag{3.17}$$

where $C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Consequently, we get $K_r = \begin{pmatrix} 2\alpha_r \\ \alpha_r^2 \end{pmatrix}$ and $K_\theta = \begin{pmatrix} 2\alpha_\theta \\ \alpha_\theta^2 \end{pmatrix}$ with $\alpha_r \in \mathbb{R}$ and $\alpha_\theta \in \mathbb{R}$.

3.3.3 High-order sliding modes differentiator

The high-order sliding modes differentiator described in [Levant 2003] can be expressed in a dynamic form as follows:

$$\left\{ \begin{array}{rcl} \dot{\alpha}_0 & = & -\lambda_n M^{\frac{1}{n+1}} |\alpha_0 - x|^{\frac{n}{n+1}} \operatorname{sign}(\alpha_0 - x) + \alpha_1 \\ \dot{\alpha}_1 & = & -\lambda_{n-1} M^{\frac{1}{n}} |\alpha_1 - \dot{\alpha}_0|^{\frac{n-1}{n}} \operatorname{sign}(\alpha_1 - \dot{\alpha}_0) + \alpha_2 \\ \vdots & & \\ \dot{\alpha}_{n-1} & = & -\lambda_1 M^{\frac{1}{2}} |\alpha_{n-1} - \dot{\alpha}_{n-2}|^{\frac{1}{2}} \operatorname{sign}(\alpha_{n-1} - \dot{\alpha}_{n-2}) + \alpha_n \\ \dot{\alpha}_n & = & -\lambda_0 M \operatorname{sign}(\alpha_n - \dot{\alpha}_{n-1}), \end{array} \right. \tag{3.18}$$

where $x \in \mathcal{C}^{n+1}$ with $n \in \mathbb{N}$, the derivatives of which we want to estimate. Then, it was shown in [Levant 2003] that, if the gains of λ_i are chosen properly, then the differentiator converges in a finite time T, *i.e.*, $\alpha_i(t) = x^{(i)}(t)$, for all $t \geq T$ and $i = 0, 1, \ldots, n$. M is a constant such that $\|x^{(n+1)}\|_\infty \leq M$. For the case when $n = 5$, the gains could be chosen as $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 8$ and $\lambda_5 = 12$.

In order to estimate r and $\dot{r}$ (resp. θ and $\dot{\theta}$), we apply the high-order sliding modes differentiator to (3.15) (resp. (3.16)) with $n = 2$ and $x = y_1$ (resp. $x = y_2$).

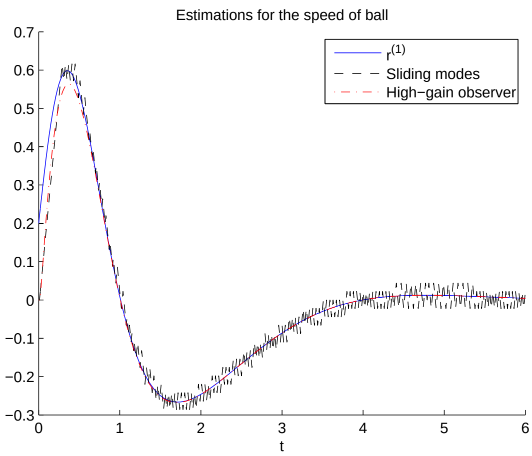
3.3.4 Numerical comparisons

In the subsection, by observing the state vector of the ball and beam system we give some comparisons among high-gain observer, sliding modes differentiator and Jacobi estimators. When we use high-gain observer, we have two parameters to set: α_r and α_θ. If we set these parameters to be large, then the estimation errors in noise-free case are small. However, large values for these parameters can produce large noise errors in noisy case. In the following estimations, we take $\alpha_r = 10$ and $\alpha_\theta = 30$ so as to produce small estimation errors in noise-free case. By simulating the ball and beam system we obtain that $r^{(2)} \leq 2$ and $\theta^{(2)} \leq 6.7$. These values are used for sliding modes differentiators. However, these values are usually unknown. Finally, we apply the causal Jacobi estimator $D^{(n)}_{\kappa,\mu,-T,1} x(-T\xi + t_0)$ given in Corollary 1.3.15 and the central Jacobi estimator $\hat{D}^{(n)}_{\kappa,\mu,T,q} x(t_0)$ given in (1.166). According to the previous study of the parameters' influence on the estimation errors for the Jacobi estimators, we take $\kappa = \mu = 0$, $q = 2$ and ξ as the smaller root of the Jacobi polynomial $P_2^{(n,n)}$ in each following estimations. Hence, the only parameter to be set is the length of each sliding window. However, we need different value of T to estimate each state, *i.e.*, we have four different values of T to set. When we estimate each state, we take the same value of T for causal and central Jacobi estimators.

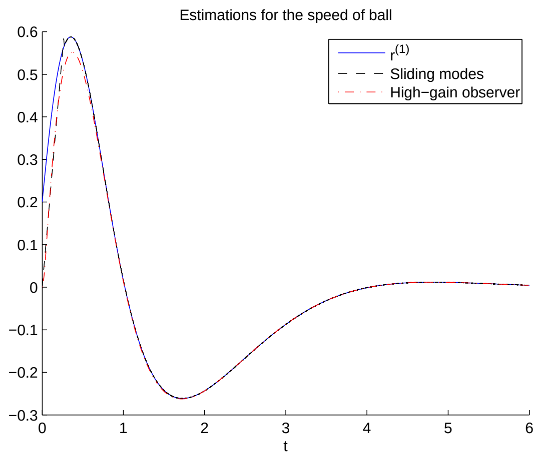
Firstly, we compare high-gain observer to sliding modes differentiator without considering noises. Then, we obtain that the sliding modes differentiator is not robust to "large" sampling period, especially when they are used to estimate the speed of ball and the first order derivative of beam angle. We can see this result in Figure 3.3 and Figure 3.4. When the sampling period is set to $0.01s$, there are large estimation errors for the sliding modes differentiator, which can be reduced by decreasing the sampling period to $0.0001s$.

Secondly, we compare these two observers in noisy case. We add a white gaussian noise to the measurements of ball position and beam angle. The obtained SNR are equal to 24.5dB and 23.6dB respectively (see Figure 3.5). Then, we can see in Figure 3.6-3.9 the obtained estimations and associated absolute estimation errors. Especially, it is shown in Figure 3.6 that there is a time-delay for the estimation obtained by sliding modes modes differentiator. This time-delay is due to the noise. Moreover, we can observe in Figure 3.8 and Figure 3.9 that the time of convergence for high-gain observer is smaller than the one for sliding modes modes differentiator. Hence, it is shown that high-gain observer is more appreciate than sliding modes modes differentiator for the ball and beam system.

Thirdly, we compare high-gain observer to Jacobi estimators by taking the same noisy measurements given in Figure 3.5. The obtained estimations and associated absolute estimation errors are given in Figure 3.10-3.13. The time-delay values for causal Jacobi estimators can be calculated. Hence, by shifting these causal Jacobi estimators we get smaller estimation errors than the ones for high-gain observer. Moreover, it is shown that central Jacobi estimators are better than high-gain observer for off-line estimations.

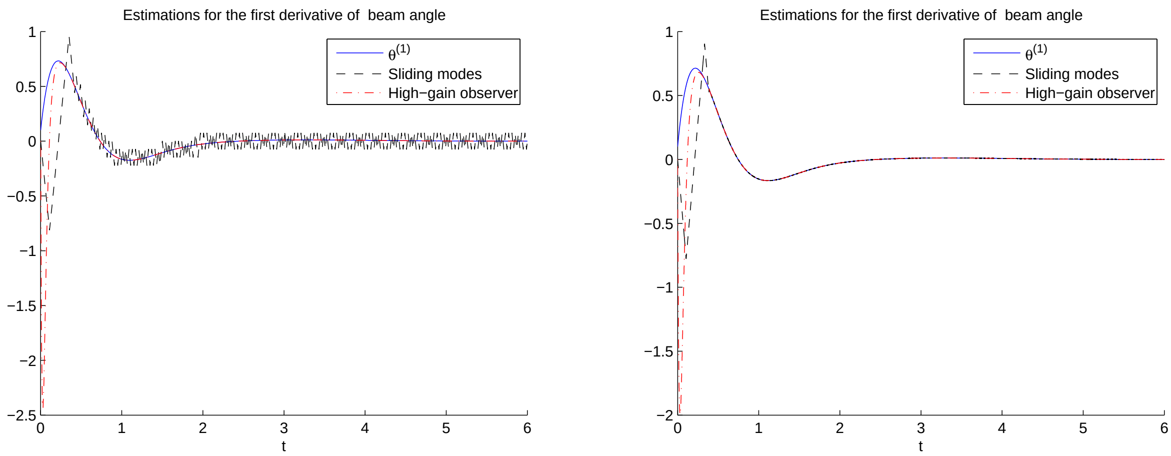


(a) Comparison of estimations with $T_s = 0.01$.



(b) Comparison of estimations with $T_s = 0.0001$.

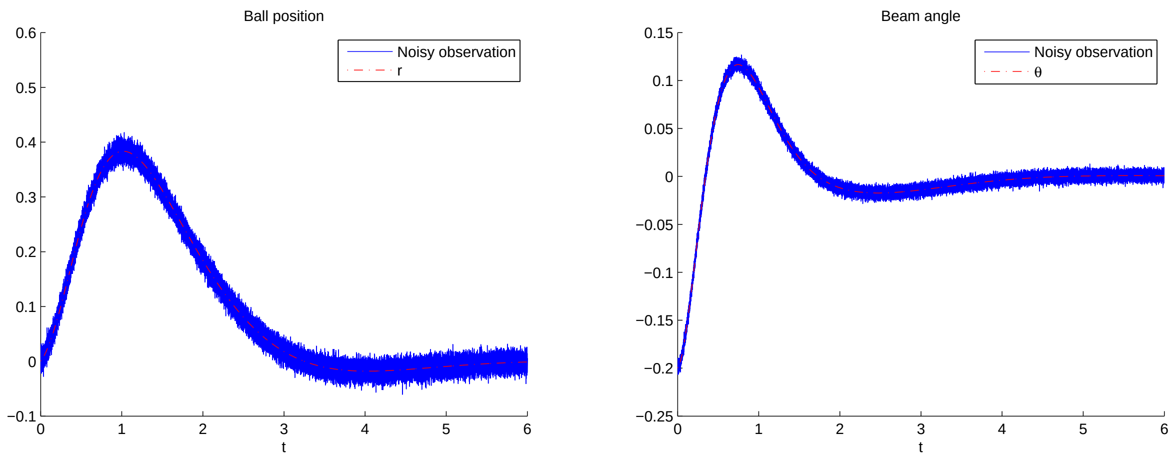
Figure 3.3: Estimations for the ball speed obtained by high-gain observer with $\alpha_r = 10$ and sliding modes with $M = 2$.



(a) Comparison of estimations with $T_s = 0.01$.

(b) Comparison of estimations with $T_s = 0.0001$.

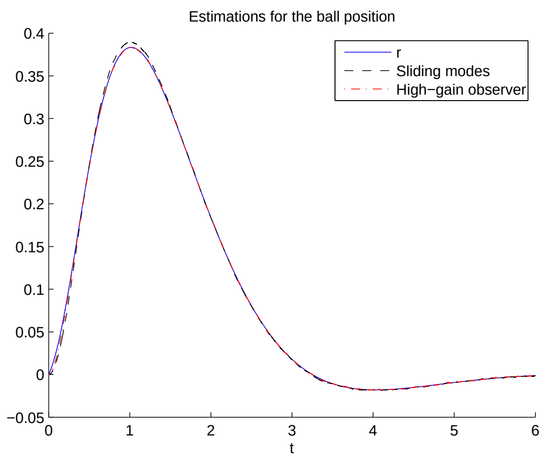
Figure 3.4: Estimations for the first order derivative of beam angle obtained by high-gain observer with $\alpha_\theta = 30$ and sliding modes with $M = 6.7$.



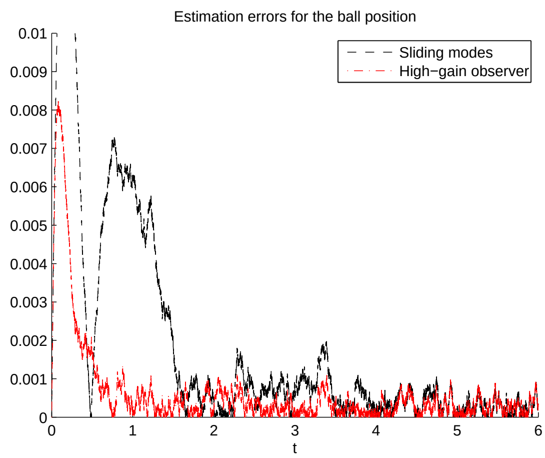
(a) Noisy observation of the ball position with $SNR =$ 24.5dB.

(b) Noisy observation of the beam angle with $SNR =$ 23.6dB.

Figure 3.5: Noisy observations obtained with a sampling period $T_s = 0.0001s$.

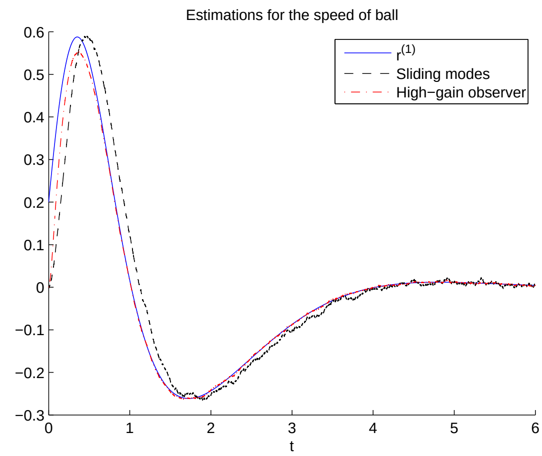


(a) Comparison of estimations.

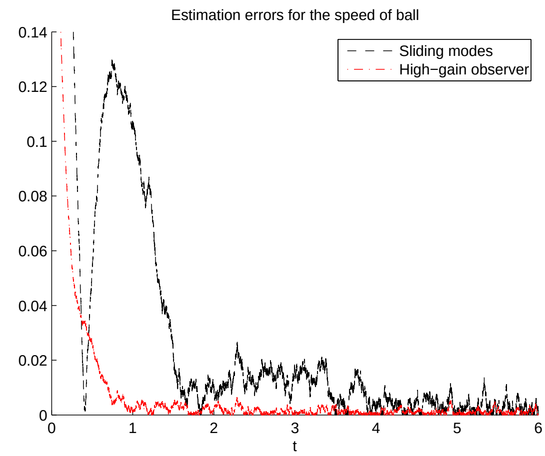


(b) Comparison of absolute estimation errors.

Figure 3.6: Estimations for the ball position obtained by high-gain observer with $\alpha_r = 10$ and sliding modes with $M = 2$ where the sampling period is $T_s = 0.0001s$.

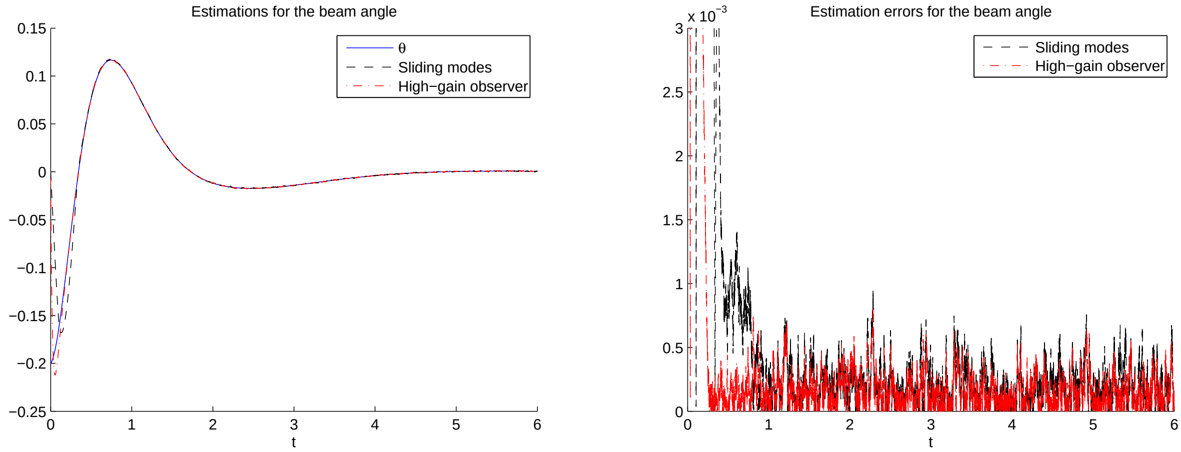


(a) Comparison of estimations.



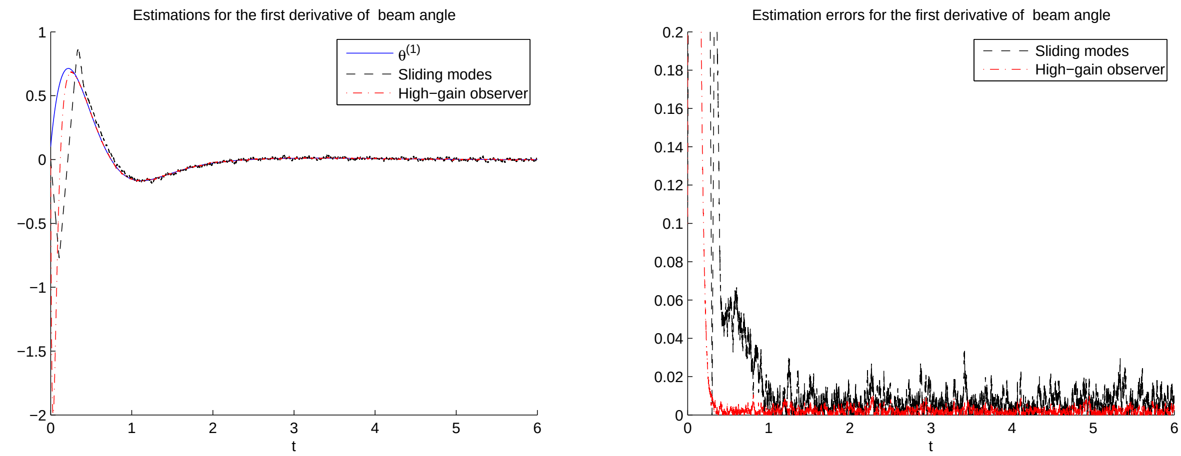
(b) Comparison of absolute estimation errors.

Figure 3.7: Estimations for the ball speed obtained by high-gain observer with $\alpha_r = 10$ and sliding modes with $M = 2$ where the sampling period is $T_s = 0.0001s$.



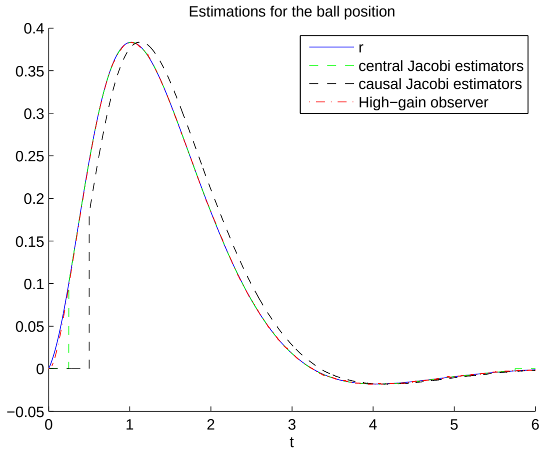
(a) Comparison of estimations. (b) Comparison of absolute estimation errors.

Figure 3.8: Estimations for the beam angle by high-gain observer with $\alpha_\theta = 30$ and sliding modes with $M = 6.7$ where the sampling period is $T_s = 0.0001s$.

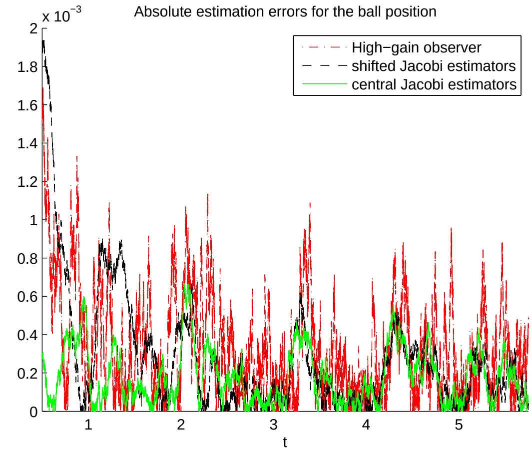


(a) Comparison of estimations. (b) Comparison of absolute estimation errors.

Figure 3.9: Estimations for the first order derivative of beam angle by high-gain observer with $\alpha_\theta = 30$ and sliding modes $M = 6.7$ where the sampling period is $T_s = 0.0001s$.

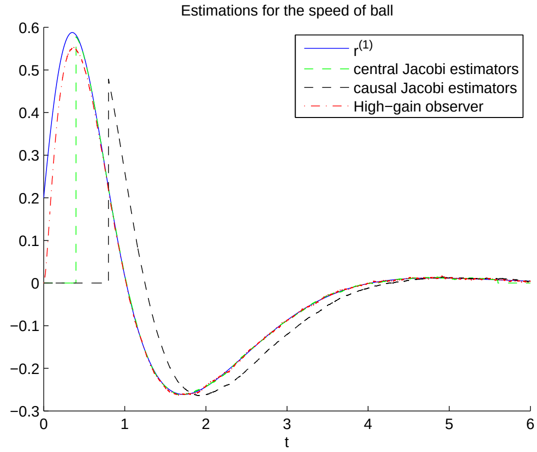


(a) Comparison of estimations.

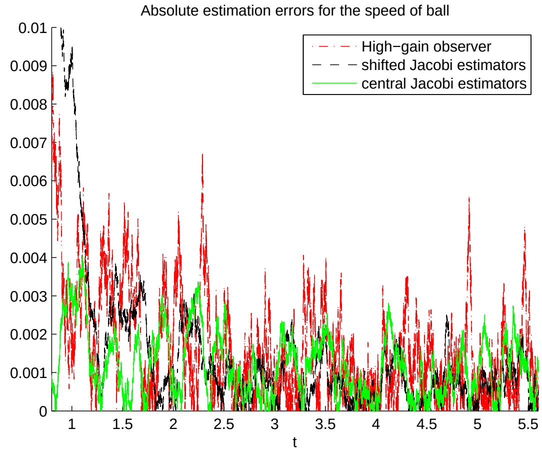


(b) Comparison of absolute estimation errors.

Figure 3.10: Estimations for the ball position obtained by high-gain observer with $\alpha_r = 10$ and Jacobi estimators with $T = 5000Ts$ where the sampling period is $T_s = 0.0001s$.

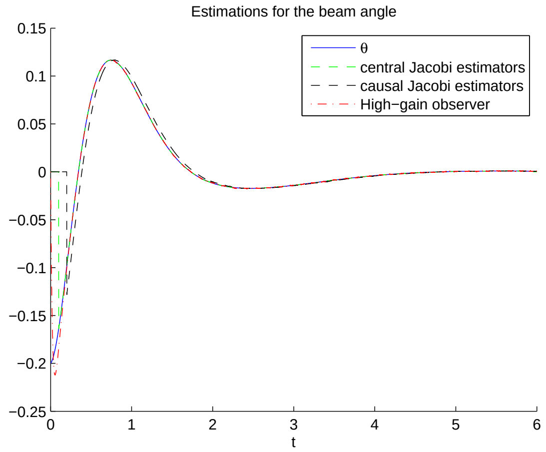


(a) Comparison of estimations.

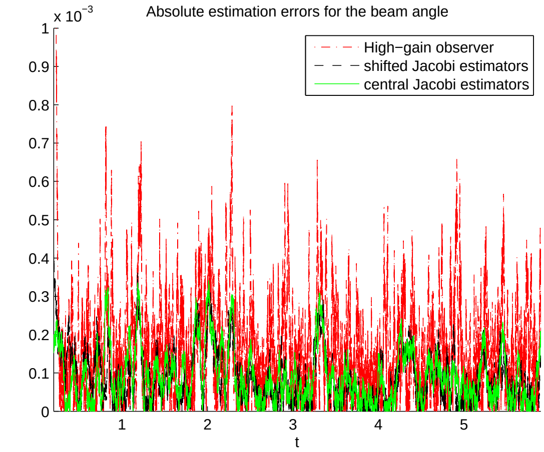


(b) Comparison of absolute estimation errors.

Figure 3.11: Estimations for the ball speed obtained by high-gain observer with $\alpha_r = 10$ and Jacobi estimators with $T = 8000Ts$ where the sampling period is $T_s = 0.0001s$.

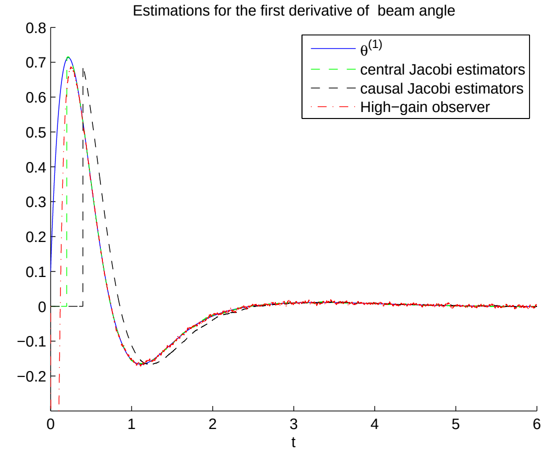


(a) Comparison of estimations.

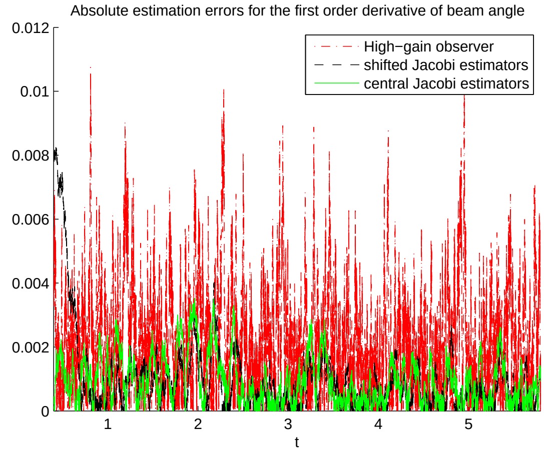


(b) Comparison of absolute estimation errors.

Figure 3.12: Estimations for the beam angle by high-gain observer with $\alpha_\theta = 30$ and Jacobi estimators with $T = 2000Ts$ where the sampling period is $T_s = 0.0001s$.



(a) Comparison of estimations.



(b) Comparison of absolute estimation errors.

Figure 3.13: Estimations for the first order derivative of beam angle by high-gain observer with $\alpha_\theta = 30$ and Jacobi estimators with $T = 4000Ts$ where the sampling period is $T_s = 0.0001s$.

3.4 Conclusion

In this chapter, we recall some results of observability for a nonlinear system within a differential geometric framework and a differential algebraic framework. By taking the ball and beam system, we compare Jacobi estimators to high-gain observer and high order sliding modes differentiator. By calculating the estimation errors, we have obtained that high-gain observer is more appreciate than sliding modes observer for the ball and beam system, and Jacobi estimators are better than high-gain observer for off-line estimations and for on-line estimations by admitting a time-delay. According to the previous simulations, we get in the Table 3.1 and Table 3.2 their comparison by considering other different criterions.

Observer	Convergence time	Number of parameters to be set
Jacobi estimators	the length of sliding integration window	4 (one for each state estimation)
High-gain observer	unknown	2
Sliding modes differentiator	unknown	2

Table 3.1: Comparison by considering different criterions

Observer	Time-delay	Robustness to noise and sampling period
Jacobi estimators	known	good
High-gain observer	unknown (small)	good
Sliding modes differentiator	unknown	bad

Table 3.2: Comparison by considering different criterions

Part II

Sinusoidal parameters estimation

Chapter 4

Frequency, amplitude and phase estimations

Contents

4.1 Introduction

The problem of reliably estimating the defining parameters in a sinusoidal signal, namely: the amplitude, the frequency and the phase, from noisy measurements, has drawn considerable attention in the last decade among signal processing researchers and applied mathematicians. Several interesting applications of this problem are described in science and engineering, such as

- control theory [Fedele 2009a], [Becedas 2009], [Pereira 2009],
- intelligent instrumentation of power systems [Yang 2001], [Karimi 2004], [Wu 2005],
- signal processing [Liu 2001], [Klapuri 2003], [So 2006],
- biomedical engineering [Östlund 2004],
- global position systems [Hackman 2006].

Because of its importance, many schemes and many solutions which offer different approaches to the problem have been suggested in the literature (see [Stoica 1993], [Roy 1989], [Bittanti 2000], [Mojiri 2004], [Cheng 2006],[Zhang 2006],[Fu 2007],[Li 2009]).

The key problem is to find a method that improves speed of convergence, accuracy, noise rejection, etc. In [Trapero 2007a, Trapero 2007b, Sira-Ramírez 2006, Trapero 2008], recent algebraic parametric techniques are devoted to estimate the frequency, amplitude and phase of time-invariant amplitude noisy biased sinusoidal signals. It is an on-line, robust, continuous time identification method capable of estimating the unknown parameters. The obtained estimators are given by exact formulae in terms of iterated integrals of the noisy observation signals. The calculations of the unknown parameters are performed in a fraction of the sinusoid signal period, independently of all initial conditions. Very recently, it is shown in [Coluccio 2008, Fedele 2009b, Fedele 2010] that the unknown frequency can be also estimated by using modulating functions method [Shinbrot 1957, Rao 1976, Rao 1983, Pearson 1985, Jordan 1986, Jordan 1990, Preising 1993, Co 1997, Ungarala 2000] which has similar advantages to the algebraic parametric techniques especially concerning the robustness of the estimation to corrupting noises.

The aim of this chapter is to use the algebraic parametric techniques and the modulating functions method to give some estimators for the frequency, amplitude and phase of noisy sinusoidal signals the amplitude of which are time-invariant or not. This chapter is organized as follows. In Section 4.2, we give the estimators for the unknown parameters by using the algebraic parametric techniques via some differential operators in the operational domain. In Section 4.3, by providing an extended frequency estimator we show the link between the algebraic parametric techniques and modulating functions method. Then, we estimate the amplitudes and phases by using modulating functions method.

4.2 Algebraic parametric techniques

Let $y = x + \varpi$ be a noisy observation on a finite time interval $I \subset \mathbb{R}_+$ of a real valued signal x, where ϖ is an additive corrupting noise and

$$\forall t \in I,\ x(t) = (A_0 + A_1 t)\,\sin(\omega t + \phi) \tag{4.1}$$

with $A_0 \in \mathbb{R}_+^*$, $A_1 \in \mathbb{R}$, $\omega \in \mathbb{R}_+^*$ and $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$. In this section, by using algebraic parametric techniques we estimate the parameters ω, A_0 and ϕ from the noisy observation y. The estimations are given in two cases where the amplitude of x is time-invariant ($A_1 = 0$) and time-varying ($A_1 \neq 0$) respectively.

Let us denote by $D_T := \{T; T \in \mathbb{R}_+^*, [0,T] \subset I\}$, $c_{\mu,\kappa} = \frac{(-1)^\kappa}{\Gamma(\mu+1)}$, $W_{\mu,\kappa}(t) = (T-t)^\mu t^\kappa$ for any $t \in [0,T]$ with $\mu, \kappa \in]-1, +\infty[$, $T \in D_T$, and recall the expression of $w_{\mu,\kappa}^{(n)}$ given by (7.22) in Appendix with $n \in \mathbb{N}$. Hence, $w_{\mu,\kappa}$ is the normalized form of $W_{\mu,\kappa}$.

4.2.1 Time-invariant amplitude case

In this subsection, we assume that $A_1 = 0$. Then, x is a sinusoidal signal with time-invariant amplitude, which is a solution of the harmonic oscillator equation

$$\forall\, t \in I, \quad \ddot{x}(t) + \omega^2 x(t) = 0. \tag{4.2}$$

We are going to estimate ω by applying the algebraic parametric techniques to the previous equation. Let us give the following lemma.

Lemma 4.2.64 *[Liu 2011d] Let f be a $\mathcal{C}^n$-continuous function ($n \in \mathbb{N}$) defined on I and $\Pi^n_{k,\mu}$ be the integral annihilator defined by (1.20). Then, the inverse of the Laplace transform of $\Pi^n_{k,\mu}\hat{f}$, where $\hat{f}$ is the laplace transform of f with the Laplace variable s, is given by*

$$\mathcal{L}^{-1}\left\{\Pi^n_{k,\mu}\hat{f}(s)\right\}(T) = T^{n+1+\mu+k}c_{\mu+n,k}\int_0^1 w^{(n)}_{\mu+n,k+n}(\tau)\, f(T\tau)d\tau, \tag{4.3}$$

where $T \in D_T$.

Proof. By substituting $\hat{x}_n$ by $\hat{f}$ and $x^{(n)}_n(\beta\tau + t_0)$ by $f(\tau)$ in (1.24), we get

$$\mathcal{L}^{-1}\left\{\Pi^n_{k,\mu}\left(\hat{f}\right)\right\}(T) = \frac{(-1)^{n+k}}{\Gamma(n+1+\mu)}\int_0^T (T-\tau)^{n+\mu}\tau^{n+k}f^{(n)}(\tau)\,d\tau. \tag{4.4}$$

Then, this proof can be completed by applying a change of variable $\tau \to T\tau$ and n times integrations by parts in (4.4). □

Proposition 4.2.65 *[Liu 2008] Let $k \in \mathbb{N}$, $-1 < \mu \in \mathbb{R}$ and $T \in D_T$ such that $\int_0^1 w_{\mu+2,k+2}(\tau)\,x(T\tau)d\tau \neq 0$, then the parameter ω is estimated from the noisy observation y by*

$$\tilde{\omega} = \frac{1}{T}\left(-\frac{\int_0^1 \ddot{w}_{\mu+2,k+2}(\tau)\,y(T\tau)\,d\tau}{\int_0^1 w_{\mu+2,k+2}(\tau)\,y(T\tau)\,d\tau}\right)^{\frac{1}{2}}. \tag{4.5}$$

Proof. By applying the Laplace transform to (4.2), we get

$$s^2\hat{x}(s) - sx_0 - \dot{x}_0 + \omega^2\hat{x}(s) = 0, \tag{4.6}$$

where s is the Laplace variable. Let us take $k+2\,(k \in \mathbb{N})$ times derivations to both sides of (4.6) with respect to s so as to annihilate the initial conditions x_0 and $\dot{x}_0$. Then, we multiply the resulting equation by $s^{-3-\mu}$ so as to apply (7.13) given in the Appendix to obtain Riemann-Liouville integrals when returning back into the time domain. Thus, by applying these operations to (4.6) we get

$$\Pi^2_{k,\mu}\hat{x}(s) + \omega^2\Pi^0_{k+2,\mu+2}\hat{x}(s) = 0. \tag{4.7}$$

Let us apply the inverse Laplace transform to (4.7). Then by using Lemma 4.2.64, we obtain

$$T^{k+\mu+3}c_{\mu+2,k+2}\int_0^1 \left(\ddot{w}_{\mu+2,k+2}(\tau) + \omega^2T^2w_{\mu+2,k+2}(\tau)\right)x(T\tau)d\tau = 0. \tag{4.8}$$

Assume that $\int_0^1 w_{\mu+2,k+2}(\tau)\,x(T\tau)d\tau \neq 0$, then the frequency ω is calculated by

$$\omega = \frac{1}{T}\left(-\frac{\int_0^1 \ddot{w}_{\mu+2,k+2}(\tau)\,x(T\tau)\,d\tau}{\int_0^1 w_{\mu+2,k+2}(\tau)\,x(T\tau)\,d\tau}\right)^{\frac{1}{2}}. \tag{4.9}$$

Finally, an estimator for ω can be obtained by substituting x by y in (4.9). □

Let us recall that the parameter k defined in the integral annihilator $\Pi^n_{k,\mu}$ is extended in Subsection 1.2.2 with $k \in]-1,+\infty[$ for minimal Jacobi estimators. We can also extend the parameter k to $]-1,+\infty[$ for the frequency estimator given in Proposition 4.2.65. Moreover, the formula of the frequency estimator still has a sense with this extension.

By observing that $x_0 = x(0) = A_0 \sin\phi$ and $\dot{x}_0 = \dot{x}(0) = A_0\omega\cos\phi$, if $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, then we have

$$A_0 = \left(x_0^2 + \frac{\dot{x}_0^2}{\omega^2}\right)^{\frac{1}{2}}, \ \phi = \arctan\left(\omega\frac{x_0}{\dot{x}_0}\right). \tag{4.10}$$

Hence, we should estimate $\dot{x}_0$ and x_0 in order to obtain estimators for A_0 and ϕ.

Proposition 4.2.66 *[Liu 2008] Let $-1 < \mu \in \mathbb{R}$ and $T \in D_T$, then the parameters A_0 and ϕ are estimated from the noisy observation y and the estimated value $\tilde{\omega}$ defined by (4.5) as follows*

$$\begin{aligned}\tilde{A}_0 &= \left(\tilde{x}_0^2 + \frac{\tilde{\dot{x}}_0^2}{\tilde{\omega}^2}\right)^{\frac{1}{2}},\\ \tilde{\phi} &= \arctan\left(\tilde{\omega}\frac{\tilde{x}_0}{\tilde{\dot{x}}_0}\right),\end{aligned} \tag{4.11}$$

where $\tilde{x}_0 = \int_0^1 P_0^{\tilde{\omega}}(\tau)\, y(T\tau)\, d\tau$ and $\tilde{\dot{x}}_0 = \frac{1}{T}\int_0^1 P_1^{\tilde{\omega}}(\tau)\, y(T\tau)\, d\tau$ with

$P_0^{\tilde{\omega}}(\tau) = 2(\mu+2)w_{\mu+1,0}(\tau) - (\mu+1)(\mu+2)w_{\mu,1}(\tau) - (\tilde{\omega}T)^2 w_{\mu+2,1}(\tau),$

$P_1^{\tilde{\omega}}(\tau) = (\mu+1)(\mu+2)(\mu+3)w_{\mu,1}(\tau) - (\mu+2)(\mu+3)w_{\mu+1,0}(\tau) + (\tilde{\omega}T)^2(\mu+3)w_{\mu+2,1}(\tau) + (\tilde{\omega}T)^2 w_{\mu+3,0}(\tau).$

Proof. In order to calculate x_0, we take the derivative of both sides of (4.6) with respect to s

$$s^2\hat{x}'(s) + 2s\hat{x}(s) - x_0 + \omega^2\hat{x}'(s) = 0. \tag{4.12}$$

Then, by multiplying both sides of (4.12) by $s^{-\mu-3}$ with $-1 < \mu \in \mathbb{R}$, we get

$$\frac{1}{s^{\mu+1}}\hat{x}'(s) + \frac{2}{s^{\mu+2}}\hat{x}(s) - \frac{1}{s^{\mu+3}}x_0 + \frac{\omega^2}{s^{\mu+3}}\hat{x}'(s) = 0. \tag{4.13}$$

Let us express (4.13) in the time domain and denote by T as the length of the window used for estimation. Since $-1 < \mu \in \mathbb{R}$, by applying (7.11) and (7.13) given in the Appendix we get the following Riemann-Liouville integral

$$\frac{T^{\mu+2}}{\Gamma(\mu+3)}x_0 = \int_0^T \left(c_{\mu,1}W_{\mu,1}(\tau) + 2c_{\mu+1,0}W_{\mu+1,0}(\tau) + \omega^2 c_{\mu+2,1}W_{\mu+2,1}(\tau)\right) x(\tau)\, d\tau.$$

Hence, by substituting τ by $T\tau$, x by y and taking the estimation of ω given in Proposition 4.2.65 we obtain an estimate for x_0

$$\tilde{x}_0 = \int_0^1 P_0^{\tilde{\omega}}(\tau)\, y(T\tau)\, d\tau. \tag{4.14}$$

In order to estimate $\dot{x}_0$, we multiply both sides of (4.6) by s^{-1}

$$s\hat{x}(s) - x_0 - \frac{1}{s}\dot{x}_0 + \frac{\omega^2}{s}\hat{x}(s) = 0. \tag{4.15}$$

By taking the derivative of both sides of (4.15) with respect to s, we can annihilate the term x_0

$$s\hat{x}'(s) + \hat{x}(s) + \frac{1}{s^2}\dot{x}_0 + \frac{\omega^2}{s}\hat{x}'(s) - \frac{\omega^2}{s^2}\hat{x}(s) = 0. \tag{4.16}$$

Then, by multiplying by $s^{-\mu-2}$ with $-1 < \mu \in \mathbb{R}$ we get

$$\frac{1}{s^{\mu+1}}\hat{x}'(s) + \frac{1}{s^{\mu+2}}\hat{x}(s) + \frac{1}{s^{4+\mu}}\dot{x}_0 + \frac{\omega^2}{s^{3+\mu}}\hat{x}'(s) - \frac{\omega^2}{s^{4+\mu}}\hat{x}(s) = 0. \tag{4.17}$$

By applying (7.11) and (7.13) given in the Appendix to the last equation and denoting by T as the length of the window used for estimation, we get in the time domain

$$-\frac{T^{\mu+3}}{\Gamma(\mu+4)}\dot{x}_0 = \int_0^T \left(c_{\mu,1}W_{\mu,1}(\tau) + c_{\mu+1,0}W_{\mu+1,0}(\tau) + \omega^2 c_{\mu+2,1}W_{\mu+2,1}(\tau) - \omega^2 c_{\mu+3,0}W_{\mu+3,0}(\tau)\right) x(\tau)\, d\tau.$$

By applying a change of variable, substituting x by y and taking the estimation for ω, we can estimate $\dot{x}_0$ by

$$\tilde{\dot{x}}_0 = \frac{1}{T}\int_0^1 P_1^{\tilde{\omega}}(\tau)\, y(T\tau)\, d\tau.$$

Finally, we get estimations for A_0 and ϕ from the relations (4.10) and by using the estimations of x_0, $\dot{x}_0$ and ω. □

Let us remark that in the previous proof, we have applied the two following differential operators $\frac{1}{s^{\mu+3}} \cdot \frac{d}{ds}$ and $\frac{1}{s^{\mu+2}} \cdot \frac{d}{ds} \cdot \frac{1}{s}$ with $-1 < \mu \in \mathbb{R}$.

4.2.2 Time-varying amplitude case

In this subsection, we assume that $A_1 \in \mathbb{R}^*$. Then x is a sinusoidal signal with time-varying amplitude, which is a solution of the harmonic oscillator equation

$$\forall\, t \in I, \quad x^{(4)}(t) + 2\omega^2\ddot{x}(t) + \omega^4 x(t) = 0. \tag{4.18}$$

Similarly to Proposition 4.2.65, we are going to estimate ω by using the above equation.

Proposition 4.2.67 *[Liu 2011d] Let $k \in \mathbb{N}$, $-1 < \mu \in \mathbb{R}$ and $T \in D_T$ such that $\int_0^1 w_{\mu+4,k+4}(\tau)\, x(T\tau)d\tau \neq 0$, then the parameter ω is estimated from the noisy observation y by*

$$\tilde{\omega} = \begin{cases} \left(\frac{-B_y - \sqrt{B_y^2 - A_y C_y}}{A_y}\right)^{\frac{1}{2}}, & \text{if } \Delta \geq 0, \\ \left(\frac{-B_y + \sqrt{B_y^2 - A_y C_y}}{A_y}\right)^{\frac{1}{2}}, & \text{else}, \end{cases} \tag{4.19}$$

where $\Delta = A_1 \int_0^1 \dot{w}_{\mu+4,k+4}(\tau)\, \sin(\omega T\tau + \phi)d\tau$, $A_y = T^4 \int_0^1 w_{\mu+4,k+4}(\tau)\, y(T\tau)d\tau$, $B_y = T^2 \int_0^1 \ddot{w}_{\mu+4,k+4}(\tau)\, y(T\tau)d\tau$, $C_y = \int_0^1 w^{(4)}_{\mu+4,k+4}(\tau)\, y(T\tau)d\tau$.

Proof. By applying the Laplace transform to (4.18), we get

$$s^4\hat{x}(s) + 2\omega^2 s^2\hat{x}(s) + \omega^4\hat{x}(s) = s^3x_0 + s^2\dot{x}_0 + (2\omega^2x_0 + \ddot{x}_0)s + (2\omega^2\dot{x}_0 + x_0^{(3)}). \tag{4.20}$$

Let us apply $k+4$ ($k \in \mathbb{N}$) times derivations to both sides of (4.20) with respect to s and by multiplying the resulting equation by $s^{-5-\mu}$ with $-1 < \mu \in \mathbb{R}$, we get

$$\Pi^4_{k,\mu}\hat{x}(s) + 2\omega^2\Pi^2_{k+2,\mu+2}\hat{x}(s) + \omega^4\Pi^0_{k+4,\mu+4}\hat{x}(s) = 0. \tag{4.21}$$

Let us apply the inverse Laplace transform to (4.21). Then by using Lemma 4.2.64, we obtain

$$\int_0^1 w^{(4)}_{\mu+4,k+4}(\tau)x(T\tau) + 2(\omega T)^2\ddot{w}_{\mu+4,k+4}(\tau)x(T\tau)\ d\tau + (\omega T)^4\int_0^1 w_{\mu+4,k+4}(\tau)\,x(T\tau)d\tau = 0. \tag{4.22}$$

According to (7.22) given in the Appendix, we have $w^{(i)}_{\mu+4,k+4}(0) = w^{(i)}_{\mu+4,k+4}(1)$ for $i = 0,\dots,3$. Then by applying integration by parts, we get

$$\int_0^1 \left(\omega^4 x(T\tau) + 2\omega^2 x^{(2)}(T\tau)\ d\tau + x^{(4)}(T\tau)\right) w_{\mu+4,k+4}(\tau)d\tau = 0. \tag{4.23}$$

Assume that $\int_0^1 w_{\mu+4,k+4}(\tau)\,x(T\tau)d\tau \neq 0$, then ω^2 is obtained by

$$\omega^2 = \frac{-\hat{B}_x \pm \sqrt{\hat{B}_x^2 - \hat{A}_x\hat{C}_x}}{\hat{A}_x}, \tag{4.24}$$

where $\hat{A}_x = \int_0^1 w_{\mu+4,k+4}(\tau)\,x(T\tau)d\tau$, $\hat{B}_x = \int_0^1 w_{\mu+4,k+4}(\tau)x^{(2)}(T\tau)\ d\tau$, $\hat{C}_x = \int_0^1 w_{\mu+4,k+4}(\tau)x^{(4)}(T\tau)d\tau$.

Since $x^{(4)}(T\tau) + 2\omega^2 x^{(2)}(T\tau) + \omega^4 x(T\tau) = 0$ for any $\tau \in [0,1]$, we get

$$\hat{B}_x^2 - \hat{A}_x\hat{C}_x = \left(\int_0^1 w_{\mu+4,k+4}(\tau)x^{(2)}(T\tau)\ d\tau + \int_0^1 \omega^2 w_{\mu+4,k+4}(\tau)x(T\tau)\ d\tau\right)^2.$$

Observe that $x^{(2)}(T\tau)+(\omega T)^2x(T\tau) = 2\omega A_1T^2\cos(\omega T\tau+\phi)$ for any $\tau \in [0,1]$, and $\omega A_1\int_0^1 w_{\mu+4,k+4}(\tau)\ \cos(\omega Tt+\phi)d\tau = -\frac{A_1}{T}\int_0^1 \dot{w}_{\mu+4,k+4}(\tau)\ \sin(\omega T\tau+\phi)d\tau$. Hence, we obtain that

$$\omega = \begin{cases} \left(\frac{-B_x-\sqrt{B_x^2-A_xC_x}}{A_x}\right)^{\frac{1}{2}}, & \text{if } \Delta \geq 0, \\ \left(\frac{-B_x+\sqrt{B_x^2-A_xC_x}}{A_x}\right)^{\frac{1}{2}}, & \text{else}, \end{cases} \tag{4.25}$$

where $\Delta = A_1\int_0^1 \dot{w}_{\mu+4,k+4}(\tau)\ \sin(\omega T\tau+\phi)d\tau$. Finally, this proof can be completed by applying integration by parts and substituting x by y in the last equation. □

Observe that

$$\begin{cases} x_0 = x(0) & = A_0\sin\phi, \\ \dot{x}_0 = \dot{x}(0) & = \omega A_0\cos\phi + A_1\sin\phi, \\ x_0^{(2)} = x^{(2)}(0) & = -\omega^2A_0\sin\phi + 2\omega A_1\cos\phi, \\ x_0^{(3)} = x^{(3)}(0) & = -\omega^3A_0\cos\phi - 3\omega^2A_1\sin\phi, \end{cases} \tag{4.26}$$

then we have the fowling linear system

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 1 & 0 \\ -\omega^2 & 0 & 0 & 2\omega \\ 0 & -\omega^3 & -3\omega^2 & 0 \end{pmatrix} \begin{pmatrix} A_0 \sin\phi \\ A_0 \cos\phi \\ A_1 \sin\phi \\ A_1 \cos\phi \end{pmatrix} = \begin{pmatrix} x_0 \\ \dot{x}_0 \\ x_0^{(2)} \\ x_0^{(3)} \end{pmatrix}. \tag{4.27}$$

By solving the previous linear system, we obtain

$$\begin{cases} A_0 \sin\phi & = x_0, \\ A_0 \cos\phi & = \frac{1}{2\omega^3}\left(x_0^{(3)} + 3\omega^2 \dot{x}_0\right), \\ A_1 \sin\phi & = -\frac{1}{2\omega^2}\left(x_0^{(3)} + \omega^2 \dot{x}_0\right), \\ A_1 \cos\phi & = \frac{1}{2\omega}\left(\omega^2 x_0 + x_0^{(2)}\right). \end{cases} \tag{4.28}$$

If $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, then we have

$$\begin{aligned} A_0 &= \left(x_0^2 + \frac{\left(x_0^{(3)} + 3\omega^2 \dot{x}_0\right)^2}{4\omega^6}\right)^{\frac{1}{2}}, \\ A_1 &= \left(\frac{\left(x_0^{(3)} + \omega^2 \dot{x}_0\right)^2}{4\omega^4} + \frac{\left(\omega^2 x_0 + x_0^{(2)}\right)^2}{4\omega^2}\right)^{\frac{1}{2}}, \\ \phi &= \arctan\left(\frac{2\omega^3 x_0}{x_0^{(3)} + 3\omega^2 \dot{x}_0}\right). \end{aligned} \tag{4.29}$$

Hence, similarly to Proposition 4.2.66 we are going to estimate x_0, $\dot{x}_0$, $x_0^{(2)}$ and $x_0^{(3)}$ so as to obtain estimations of parameters A_0, A_1 and ϕ.

Proposition 4.2.68 *Let $-1 < \mu \in \mathbb{R}$ and $T \in D_T$, then the parameters A_0, A_1 and ϕ are estimated from the noisy observation y and the estimated value $\tilde{\omega}$ defined by (4.19) as follows*

$$\begin{aligned} \tilde{A}_0 &= \left(\tilde{x}_0^2 + \frac{\left(\tilde{x}_0^{(3)} + 3\tilde{\omega}^2 \tilde{\dot{x}}_0\right)^2}{4\tilde{\omega}^6}\right)^{\frac{1}{2}}, \\ \tilde{A}_1 &= \left(\frac{\left(\tilde{x}_0^{(3)} + \tilde{\omega}^2 \tilde{\dot{x}}_0\right)^2}{4\tilde{\omega}^4} + \frac{\left(\tilde{\omega}^2 \tilde{x}_0 + \tilde{x}_0^{(2)}\right)^2}{4\tilde{\omega}^2}\right)^{\frac{1}{2}}, \\ \tilde{\phi} &= \arctan\left(\frac{2\tilde{\omega}^3 x_0}{\tilde{x}_0^{(3)} + 3\tilde{\omega}^2 \tilde{\dot{x}}_0}\right), \end{aligned} \tag{4.30}$$

where

$$\tilde{x}_0 = \int_0^1 P_2^{\tilde{\omega}}(\tau)\, y(T\tau)\, d\tau, \qquad \tilde{x}_0^{(2)} = \frac{1}{T^3}\int_0^1 P_4^{\tilde{\omega}}(\tau)\, y(T\tau)\, d\tau - 2\tilde{\omega}^2\tilde{x}_0,$$
$$\tilde{\dot{x}}_0 = \frac{1}{T}\int_0^1 P_3^{\tilde{\omega}}(\tau)\, y(T\tau)\, d\tau, \quad \tilde{x}_0^{(3)} = \frac{1}{T^3}\int_0^1 P_5^{\tilde{\omega}}(\tau)\, y(T\tau)\, d\tau - 2\tilde{\omega}^2\tilde{\dot{x}}_0$$

with

$$\begin{aligned}
P_2^{\tilde{\omega}}(\tau) =& \frac{\Gamma(\mu+5)}{6}\sum_{i=0}^{3}\binom{3}{i}\frac{4!\, c_{\mu+i,3-i}}{(4-i)!}w_{\mu+i,3-i}(\tau) \\
&+ \frac{2(\tilde{\omega}T)^2\Gamma(\mu+5)}{3}\sum_{i=0}^{2}\binom{3}{i}\frac{c_{\mu+i+2,3-i}}{(2-i)!}w_{\mu+i+2,3-i}(\tau) + (\tilde{\omega}T)^4 c_{\mu+4,3}w_{\mu+4,3}(\tau), \\
P_3^{\tilde{\omega}}(\tau) =& -\frac{\Gamma(\mu+6)}{2}\left(c_{\mu,3}w_{\mu,3}(\tau) + 11c_{\mu+1,2}w_{\mu+1,2}(\tau) + 28c_{\mu+2,1}w_{\mu+2,1}(\tau) + 12c_{\mu+3,0}w_{\mu+3,0}(\tau)\right) \\
&- \Gamma(\mu+6)(\tilde{\omega}T)^2\left(c_{\mu+2,3}w_{\mu+2,3}(\tau) + 5c_{\mu+3,2}w_{\mu+3,2}(\tau) + 2c_{\mu+4,1}w_{\mu+4,1}(\tau) - 2c_{\mu+5,0}w_{\mu+5,0}(\tau)\right) \\
&- \frac{\Gamma(\mu+6)(\tilde{\omega}T)^4}{2}\left(c_{\mu+4,3}w_{\mu+4,3}(\tau) - c_{\mu+5,2}w_{\mu+5,2}(\tau)\right), \\
P_4^{\tilde{\omega}}(\tau) =& \frac{\Gamma(\mu+7)}{2}\left(c_{\mu,3}w_{\mu,3}(\tau) + 10c_{\mu+1,2}w_{\mu+1,2}(\tau) + 22c_{\mu+2,1}w_{\mu+2,1}(\tau) + 8c_{\mu+3,0}w_{\mu+3,0}(\tau)\right) \\
&+ \Gamma(\mu+7)(\tilde{\omega}T)^2\left(c_{\mu+2,3}w_{\mu+2,3}(\tau) + 3c_{\mu+3,2}w_{\mu+3,2}(\tau)\right) \\
&+ \frac{\Gamma(\mu+7)(\tilde{\omega}T)^4}{2}\left(c_{\mu+4,3}w_{\mu+4,3}(\tau) - 2c_{\mu+5,2}w_{\mu+5,2}(\tau) + 2c_{\mu+6,1}w_{\mu+6,1}(\tau)\right), \\
P_5^{\tilde{\omega}}(\tau) =& -\frac{\Gamma(\mu+8)}{6}\left(\sum_{i=0}^{3}\binom{3}{i}\frac{3!\, c_{\mu+i,3-i}}{(3-i)!}w_{\mu+i,3-i}(\tau) + 2(\tilde{\omega}T)^2\sum_{i=0}^{1}\binom{3}{i}c_{\mu+i+2,3-i}w_{\mu+i+2,3-i}(\tau)\right) \\
&- \frac{\Gamma(\mu+8)}{6}(\tilde{\omega}T)^4\sum_{i=0}^{3}\binom{3}{i}(-1)^i i!\, c_{\mu+4+i,3-i}w_{\mu+4+i,3-i}(\tau).
\end{aligned}$$

Proof. In order to estimate x_0, we apply the following operator $\Pi_1 = \frac{1}{s^{\mu+5}} \cdot \frac{d^3}{ds^3}$ to (4.20) with $-1 < \mu \in \mathbb{R}$, which annihilates each terms containing $x_0^{(i)}$ for $i = 1, 2, 3$. Then, by using the Leibniz formula, we get

$$\sum_{i=0}^{3}\binom{3}{i}\frac{4!}{(4-i)!}\frac{1}{s^{\mu+1+i}}\hat{x}^{(3-i)}(s) + 2\omega^2\sum_{i=0}^{2}\binom{3}{i}\frac{2!}{(2-i)!}\frac{1}{s^{\mu+3+i}}\hat{x}^{(3-i)}(s) + \frac{\omega^4}{s^{\mu+5}}\hat{x}^{(3)}(s) = \frac{6}{s^{\mu+5}}x_0. \quad (4.31)$$

Let us apply (7.11) and (7.13) given in the Appendix so as to express (4.31) in the time domain and denote by T the length of the window used for estimation

$$\begin{aligned}
x_0 =& \frac{\Gamma(\mu+5)}{6T^{\mu+4}}\int_0^T \sum_{i=0}^{3}\binom{3}{i}\frac{4!\, c_{\mu+i,3-i}}{(4-i)!}W_{\mu+i,3-i}(\tau)\, x(\tau)\, d\tau \\
&+ \frac{2\omega^2\Gamma(\mu+5)}{3T^{\mu+4}}\int_0^T \sum_{i=0}^{2}\binom{3}{i}\frac{c_{\mu+i+2,3-i}}{(2-i)!}W_{\mu+i+2,3-i}(\tau)\, x(\tau) + \omega^4 c_{\mu+4,3}W_{\mu+4,3}(\tau)\, x(\tau)\, d\tau.
\end{aligned}$$

Hence, by substituting τ by $T\tau$, x by y and taking the estimation of ω given in Proposition 4.2.67 we obtain an estimate for x_0.

In order to compute an estimate for $\dot{x}_0$, we apply the following operator $\Pi_2 = \frac{1}{s^{\mu+4}} \cdot \frac{d}{ds} \cdot \frac{1}{s} \cdot \frac{d^2}{ds^2}$ to (4.20). Then we get

$$\begin{aligned}
\Pi_2 \left\{s^4\hat{x}(s)\right\} &= \frac{1}{s^{\mu+1}}\hat{x}^{(3)}(s) + \frac{11}{s^{\mu+2}}\hat{x}^{(2)}(s) + \frac{28}{s^{\mu+3}}\hat{x}'(s) + \frac{12}{s^{\mu+4}}\hat{x}(s),\\
\Pi_2 \left\{2\omega^2 s^2\hat{x}(s)\right\} &= \frac{2\omega^2}{s^{\mu+3}}\hat{x}^{(3)}(s) + \frac{10\omega^2}{s^{\mu+4}}\hat{x}^{(2)}(s) + \frac{4\omega^2}{s^{\mu+5}}\hat{x}'(s) - \frac{4\omega^2}{s^{\mu+6}}\hat{x}(s),\\
\Pi_2 \left\{\omega^4\hat{x}(s)\right\} &= \frac{\omega^4}{s^{\mu+5}}\hat{x}^{(3)}(s) - \frac{\omega^4}{s^{\mu+6}}\hat{x}^{(2)}(s),\\
\Pi_2 \left\{s^2\dot{x}_0\right\} &= -\frac{2}{s^{\mu+6}}\dot{x}_0.
\end{aligned} \tag{4.32}$$

By applying (7.13) given in the Appendix to (4.32), we get

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\Pi_2 s^4\hat{x}(s)\right\}(T)\\
=&\int_0^T \left(c_{\mu,3}W_{\mu,3}(\tau) + 11c_{\mu+1,2}W_{\mu+1,2}(\tau) + 28c_{\mu+2,1}W_{\mu+2,1}(\tau) + 12c_{\mu+3,0}W_{\mu+3,0}(\tau)\right)x(\tau)\,d\tau,\\
&\mathcal{L}^{-1}\left\{\Pi_2 2\omega^2 s^2\hat{x}(s)\right\}(T)\\
=&2\omega^2\int_0^T \left(c_{\mu+2,3}W_{\mu+2,3}(\tau) + 5c_{\mu+3,2}W_{\mu+3,2}(\tau) + 2c_{\mu+4,1}W_{\mu+4,1}(\tau) - 2c_{\mu+5,0}W_{\mu+5,0}(\tau)\right)x(\tau)\,d\tau,\\
&\mathcal{L}^{-1}\left\{\Pi_2\omega^4\hat{x}(s)\right\}(T) = \omega^4\int_0^T \left(c_{\mu+4,3}W_{\mu+4,3}(\tau) - c_{\mu+5,2}W_{\mu+5,2}(\tau)\right)x(\tau)\,d\tau,\\
&\mathcal{L}^{-1}\left\{\Pi_2 s^2\dot{x}_0\right\}(T) = -\frac{2T^{\mu+5}}{\Gamma(\mu+6)}\dot{x}_0.
\end{aligned}$$

Thus, by applying a change of variable, substituting x by y and using the estimation of ω, we get an estimate of $\dot{x}_0$.

In order to estimate $x_0^{(2)}$, we apply the following operator $\Pi_3 = \frac{1}{s^{\mu+4}} \cdot \frac{d^2}{ds^2} \cdot \frac{1}{s} \cdot \frac{d}{ds}$ to (4.20) with $-1 < \mu \in \mathbb{R}$. Then we get

$$\begin{aligned}
\Pi_3 \left\{s^4\hat{x}(s)\right\} &= \frac{1}{s^{\mu+1}}\hat{x}^{(3)}(s) + \frac{10}{s^{\mu+2}}\hat{x}^{(2)}(s) + \frac{22}{s^{\mu+3}}\hat{x}'(s) + \frac{8}{s^{\mu+4}}\hat{x}(s),\\
\Pi_3 \left\{2\omega^2 s^2\hat{x}(s)\right\} &= \frac{2\omega^2}{s^{\mu+3}}\hat{x}^{(3)}(s) + \frac{6\omega^2}{s^{\mu+4}}\hat{x}^{(2)}(s),\\
\Pi_3 \left\{\omega^4\hat{x}(s)\right\} &= \frac{\omega^4}{s^{\mu+5}}\hat{x}^{(3)}(s) - \frac{2\omega^4}{s^{\mu+6}}\hat{x}^{(2)}(s) + \frac{2\omega^4}{s^{\mu+7}}\hat{x}'(s),\\
\Pi_3 \left\{(2\omega^2 x_0 + x_0^{(2)})s\right\} &= \frac{2}{s^{\mu+7}}(2\omega^2 x_0 + x_0^{(2)}).
\end{aligned} \tag{4.33}$$

By applying (7.13) given in the Appendix to (4.33), we get

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\Pi_3 s^4\hat{x}(s)\right\}(T)\\
=&\int_0^T \left(c_{\mu,3}W_{\mu,3}(\tau)+10c_{\mu+1,2}W_{\mu+1,2}(\tau)+22c_{\mu+2,1}W_{\mu+2,1}(\tau)+8c_{\mu+3,0}W_{\mu+3,0}(\tau)\right)x(\tau)\,d\tau,\\
&\mathcal{L}^{-1}\left\{\Pi_3 2\omega^2 s^2\hat{x}(s)\right\}(T)=2\omega^2\int_0^T \left(c_{\mu+2,3}W_{\mu+2,3}(\tau)+3c_{\mu+3,2}W_{\mu+3,2}(\tau)\right)x(\tau)\,d\tau,\\
&\mathcal{L}^{-1}\left\{\Pi_3 \omega^4\hat{x}(s)\right\}(T)=\omega^4\int_0^T \left(c_{\mu+4,3}W_{\mu+4,3}(\tau)-2c_{\mu+5,2}W_{\mu+5,2}(\tau)+2c_{\mu+6,1}W_{\mu+6,1}(\tau)\right)x(\tau)\,d\tau,\\
&\mathcal{L}^{-1}\left\{\Pi_3(2\omega^2x_0+x_0^{(2)})s\right\}(T)=\frac{2T^{\mu+6}}{\Gamma(\mu+7)}(2\omega^2x_0+x_0^{(2)}).
\end{aligned}$$

Thus, by applying a change of variable, substituting x by y and using the estimation of ω, we get an estimate of $x_0^{(2)}$.

In order to estimate $x_0^{(3)}$, we apply the following operator $\Pi_4=\frac{1}{s^{\mu+4}}\cdot\frac{d^3}{ds^3}\cdot\frac{1}{s}$ to (4.20) with $-1<\mu\in\mathbb{R}$. Then, by using the Leibniz formula, we get

$$\begin{aligned}
&-\frac{6}{s^{\mu+8}}(2\omega^2\dot{x}_0+x_0^{(3)})\\
=&\sum_{i=0}^{3}\binom{3}{i}\frac{3!}{(3-i)!}\frac{1}{s^{\mu+1+i}}\hat{x}^{(3-i)}(s)+2\omega^2\sum_{i=0}^{1}\binom{3}{i}\frac{1}{s^{\mu+3+i}}\hat{x}^{(3-i)}(s)+\omega^4\sum_{i=0}^{3}\binom{3}{i}\frac{(-1)^i i!}{s^{\mu+5+i}}\hat{x}^{(3-i)}(s).
\end{aligned}\tag{4.34}$$

Then, by applying (7.11) and (7.13) given in the Appendix we get

$$\begin{aligned}
2\omega^2\dot{x}_0+x_0^{(3)}=&-\frac{\Gamma(\mu+8)}{6T^{\mu+7}}\int_0^T\sum_{i=0}^{3}\binom{3}{i}\frac{3!\,c_{\mu+i,3-i}}{(3-i)!}W_{\mu+i,3-i}(\tau)\,x(\tau)\,d\tau\\
&-\frac{\Gamma(\mu+8)}{3T^{\mu+7}}\omega^2\int_0^T\sum_{i=0}^{1}\binom{3}{i}c_{\mu+i+2,3-i}W_{\mu+i+2,3-i}(\tau)\,x(\tau)\,d\tau\\
&-\frac{\Gamma(\mu+8)}{6T^{\mu+7}}\omega^4\int_0^T\sum_{i=0}^{3}\binom{3}{i}(-1)^i i!c_{\mu+4+i,3-i}W_{\mu+4+i,3-i}(\tau)\,x(\tau)\,d\tau.
\end{aligned}$$

Hence, by applying a change of variable $\tau\to T\tau$, substituting x by y and using the obtained estimations of ω and $\dot{x}_0$ we obtain an estimate for $x_0^{(3)}$.

Finally, we get estimations for A_0 and ϕ from the relations (4.29) and the estimations of x_0, $\dot{x}_0$, $x_0^{(2)}$, $x_0^{(3)}$ and ω. □

We can observe that all the obtained estimators using the algebraic parametric techniques are expressed as iterated time integrals of the noisy observation y. Note that the noisy function y may not be integrable in these integrals. Hence, the expressions of our estimators are only formal.

4.3 Modulating functions method

The identification procedure, based on modulating functions, was pioneered by Shinbrot [Shinbrot 1957] in the 1957. Essentially, the use of modulating functions allows to transform a differential expression, involving input-output signals on a specified time interval, into a sequence of algebraic equations. Moreover, the modulating functions method annihilates the effects of initial conditions and allows the direct use of noisy data signals [Co 1997]. These features make the modulating functions method desirable for use in several real processes. In more recent years, many authors have focused on the choice of modulating functions type including Walsh functions [Rao 1983], Hermite functions [Jordan 1986, Jordan 1990], Fourier modulating functions [Pearson 1985, Co 1997, Ungarala 2000] and spline-type functions [Coluccio 2008, Fedele 2009b, Fedele 2010, Rao 1976].

The modulating functions method can be used to estimate parameters directly from any differential equation possessing the following structure:

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{i=0}^{m} b_i u^{(i)}(t), \; n \geq m, \tag{4.35}$$

where y and u are the output and input signals respectively, and $\{a_i, b_i\}$ are the unknown system parameters. Without loss of generality, let us assume that $a_0 = 1$. A function $g_K \in \mathcal{C}^K$, defined on a finite time interval $[0, T]$, which satisfies the following terminal conditions

$$g_K^{(i)}(0) = g_K^{(i)}(T) = 0, \; \forall i = 0, 1, \ldots, K-1, \tag{4.36}$$

is called a *modulating function* [Preising 1993]. A function $f \in \mathcal{L}^1([0, T])$ is modulated by taking the inner product with a modulating function g_K

$$\langle f, g_K \rangle = \int_0^T f(t) g_K(t) dt. \tag{4.37}$$

The terminal constraints of (4.36) essentially make the boundary conditions of the function f irrelevant after modulations. Moreover, they make possible the transfer of the differentiation operation from the function f on to the modulating function g_K (as when dealing with distribution and test functions):

$$\left\langle f^{(i)}, g_K \right\rangle = (-1)^i \left\langle f, g_K^{(i)} \right\rangle, \; i = 0, 1, \ldots, K-1. \tag{4.38}$$

Hence, we do not need to approximate time derivatives from noisy measurement data. The modulating function procedure starts by multiplying (4.35) with the modulating function and integrating over the interval $[0, T]$:

$$\sum_{i=0}^{n} a_i \int_0^T g_K(\tau) y^{(i)}(\tau) d\tau = \sum_{i=0}^{m} b_i \int_0^T g_K(\tau) u^{(i)}(\tau) d\tau. \tag{4.39}$$

By integrating by parts and using property (4.38), (4.39) becomes

$$\sum_{i=0}^{n} (-1)^i a_i \int_0^T g_K^{(i)}(\tau) y(\tau) d\tau = \sum_{i=0}^{m} (-1)^i b_i \int_0^T g_K^{(i)}(\tau) u(\tau) d\tau. \tag{4.40}$$

In order to determine all parameters $\{a_i, b_i\}$, at least the same number of linearly independent algebraic equations similar to (4.40) must be generated. The proposed approach gains advantages from the low-pass filtering property of modulating functions integrals and gives explicit formulae for the parameters.

Let us recall that the frequency estimator given in Proposition 4.2.65 is obtained by applying algebraic parametric techniques with the following annihilator

$$\Pi_{k,\mu}^n = \frac{1}{s^{n+1+\mu}} \cdot \frac{d^{n+k}}{ds^{n+k}} \cdot s^n, \tag{4.41}$$

where $k \in \mathbb{N}$ and $-1 < \mu \in \mathbb{R}$. The differentiation operation $\frac{d^{n+k}}{ds^{n+k}}$ is used to annihilate initial conditions, and the multiplication $\frac{1}{s^{n+1+\mu}}$ is used to obtain an integral. In fact, we can generalize this annihilator by taking modulating functions so as to obtain an extended frequency estimator.

Lemma 4.3.69 *Let f be a $\mathcal{C}^n$-continuous function ($n \in \mathbb{N}$) defined on I and Π_g^n be defined as follows*

$$\Pi_g^n = \hat{g}(s) \cdot s^n, \tag{4.42}$$

where $\hat{g}$ is the Laplace transform of a $\mathcal{C}^n$-continuous function g satisfying $g^{(i)}(T) = g^{(i)}(0) = 0$ for $i = 0, \cdots, n-1$ with $T \in D_T$. Then, the inverse Laplace transform of $\Pi_g^n \hat{f}$, where $\hat{f}$ is the laplace transformation of f, is given by

$$\mathcal{L}^{-1}\left\{\Pi_g^n \hat{f}\right\}(T) = \int_0^T g^{(n)}(T-\tau)\, f(\tau) d\tau. \tag{4.43}$$

Proof. By applying (7.9) given in the Appendix, we get

$$\mathcal{L}\left\{\int_0^T g^{(n)}(\tau)\, f(T-\tau) d\tau\right\} = \hat{f}(s) \cdot \mathcal{L}\left\{g^{(n)}(\tau)\right\}.$$

Since $g^{(i)}(0) = 0$ for $i = 0, \cdots, n-1$, by applying (7.8) given in the Appendix, we obtain

$$\hat{f}(s) \cdot \mathcal{L}\left\{g^{(n)}(\tau)\right\} = s^n \hat{g}(s) \cdot \hat{f}(s) = \Pi_g^n \hat{f}(s).$$

Then, this proof can be completed by applying the inverse Laplace transform and a change of variable $\tau \to T - \tau$. □

The conditions $g^{(i)}(T) = 0$ for $i = 0, \cdots, n-1$ are used to annihilate the initial conditions.

Recall the following equation given in (4.6)

$$s^2 \hat{x}(s) - s x_0 - \dot{x}_0 + \omega^2 \hat{x}(s) = 0. \tag{4.44}$$

Then, by multiplying the Laplace transform of a modulating function $g \in \mathcal{C}^2$ satisfying $g(0) = g(T) = \dot{g}(0) = \dot{g}(T) = 0$, we get

$$s^2 \hat{g}(s) \hat{x}(s) - s \hat{g}(s) x_0 - \hat{g}(s) \dot{x}_0 + \hat{g}(s) \omega^2 \hat{x}(s) = 0. \tag{4.45}$$

By applying the inverse Laplace transform, we get $\mathcal{L}^{-1}\{s\hat{g}(s)x_0\}(T) = x_0 \dot{g}(T) = 0$ and $\mathcal{L}^{-1}\{\hat{g}(s)\dot{x}_0\} = \dot{x}_0 g(T) = 0$. Hence, by applying Lemma 4.3.69, we get

$$\omega = \left(-\frac{\int_0^T g^{(2)}(T-\tau)\, x(\tau) d\tau}{\int_0^T g(T-\tau)\, x(\tau) d\tau}\right)^{\frac{1}{2}}. \tag{4.46}$$

Consequently, the frequency estimator given in Proposition 4.2.65 can also be obtained by using modulating functions. The frequency estimator given in Proposition 4.2.67 can be obtained similarly. In this section, by using modulating functions method we are going to estimate the parameters ω, A_0 and ϕ in the time-invariant amplitude case and in the time-varying amplitude case.

4.3.1 Time-invariant amplitude case

In this subsection, we assume $A_1 = 0$. Then, x satisfies the equation given by (4.2). Thus, we can give the following proposition.

Proposition 4.3.70 *Let g be a function belonging to $\mathcal{C}^2([0,1])$ which satisfies the following condition $g(0) = g(1) = \dot{g}(0) = \dot{g}(1) = 0$. Assume that there exists $T \in D_T$ such that $\int_0^1 g(\tau)\, x(T\tau)\, d\tau \neq 0$. Then, the frequency ω is estimated by*

$$\tilde{\omega} = \frac{1}{T}\left(-\frac{\int_0^1 \ddot{g}(\tau)\, y(T\tau) d\tau}{\int_0^1 g(\tau)\, y(T\tau) d\tau}\right)^{\frac{1}{2}}.$$

Proof. By substituting t by $T\tau$ in (4.2) with $\tau \in [0,1]$ and $T \in D_T$, we get $\ddot{x}(T\tau) + \omega^2\, x(T\tau) = 0$ for any $\tau \in [0,1]$. Since g is integrable on $[0,1]$, we get

$$\int_0^1 g(\tau)\left(\ddot{x}(T\tau) + \omega^2\, x(T\tau)\right)\, d\tau = 0.$$

As $\int_0^1 g(\tau)\, x(T\tau)\, d\tau \neq 0$, we have

$$\omega^2 = -\frac{\int_0^1 g(\tau)\, \ddot{x}(T\tau)\, d\tau}{\int_0^1 g(\tau)\, x(T\tau)\, d\tau}. \tag{4.47}$$

By applying two times integrations by parts and using $g(0) = g(1) = \dot{g}(0) = \dot{g}(1) = 0$, we obtain

$$\omega^2 = -\frac{1}{T^2}\frac{\int_0^1 \ddot{g}(\tau)\, x(T\tau)\, d\tau}{\int_0^1 g(\tau)\, x(T\tau)\, d\tau}. \tag{4.48}$$

Then, an estimation of ω is obtained by substituting x by y in (4.48). □

Let us take an expansion of x

$$x(T\tau) = A_0 \cos\phi \sin\omega T\tau + A_0 \sin\phi \cos\omega T\tau, \tag{4.49}$$

where $\tau \in [0,1]$, $T \in D_T$. Then, by using the modulating functions method used in the previous proposition, we can calculate A_0 and ϕ from (4.49). Hence, we give the following proposition.

Proposition 4.3.71 *[Liu 2008] Let g_i for $i = 1,2$ be two continuous functions defined on $[0,1]$. Then, for any $\phi \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, the parameters A_0 and ϕ are estimated by*

$$\begin{aligned}\tilde{A}_0 &= \left\{\left(\frac{G_4^{\tilde{\omega}}\, I_{g_1}^y - G_2^{\tilde{\omega}}\, I_{g_2}^y}{L_{\tilde{\omega}}}\right)^2 + \left(\frac{G_1^{\tilde{\omega}}\, I_{g_2}^y - G_3^{\tilde{\omega}}\, I_{g_1}^y}{L_{\tilde{\omega}}}\right)^2\right\}^{1/2},\\ \tilde{\phi} &= \arctan\left(\frac{G_1^{\tilde{\omega}}\, I_{g_2}^y - G_3^{\tilde{\omega}}\, I_{g_1}^y}{G_4^{\tilde{\omega}}\, I_{g_1}^y - G_2^{\tilde{\omega}}\, I_{g_2}^y}\right),\end{aligned} \tag{4.50}$$

where $T \in D_T$, $\tilde{\omega}$ is the estimate of ω given by Proposition 4.3.70, and

$$\begin{aligned} I_{g_1}^y &= \int_0^1 g_1(\tau)\, y(T\tau)\, d\tau, & I_{g_2}^y &= \int_0^1 g_2(\tau)\, y(T\tau)\, d\tau, \\ G_1^{\tilde{\omega}} &= \int_0^1 g_1(\tau) \sin(\tilde{\omega} T\tau) d\tau, & G_2^{\tilde{\omega}} &= \int_0^1 g_1(\tau) \cos(\tilde{\omega} T\tau)\, d\tau, \\ G_3^{\tilde{\omega}} &= \int_0^1 g_2(\tau) \sin(\tilde{\omega} T\tau)\, d\tau, & G_4^{\tilde{\omega}} &= \int_0^1 g_2(\tau) \cos(\tilde{\omega} T\tau)\, d\tau. \end{aligned}$$

Moreover, we assume that $L_\omega = G_1^\omega G_4^\omega - G_2^\omega G_3^\omega \neq 0$ and $G_4^\omega I_{g_1}^x - G_2^\omega I_{g_2}^x \neq 0$.

Proof. By multiplying both sides of (4.49) by the continuous function g_1 (resp. g_2) and integrating between 0 and 1, we obtain

$$\begin{aligned} I_{g_1}^x &= A_0 \cos\phi\; G_1^\omega + A_0 \sin\phi\; G_2^\omega \\ I_{g_2}^x &= A_0 \cos\phi\; G_3^\omega + A_0 \sin\phi\; G_4^\omega. \end{aligned}$$

It yields a linear system

$$\begin{pmatrix} G_1^\omega & G_2^\omega \\ G_3^\omega & G_4^\omega \end{pmatrix} \begin{pmatrix} A_0 \cos\phi \\ A_0 \sin\phi \end{pmatrix} = \begin{pmatrix} I_{g_1}^x \\ I_{g_2}^x \end{pmatrix}.$$

Assume that $L_\omega = G_1^\omega G_4^\omega - G_2^\omega G_3^\omega \neq 0$, then by solving the system we get

$$\begin{aligned} A_0 \cos\phi &= \frac{G_4^\omega I_{g_1}^x - G_2^\omega I_{g_2}^x}{L_\omega}, \\ A_0 \sin\phi &= \frac{G_1^\omega I_{g_2}^x\, d\tau - G_3^\omega I_{g_1}^x}{L_\omega}. \end{aligned} \tag{4.51}$$

Then, by assuming that $G_4^\omega I_{g_1}^x - G_2^\omega I_{g_2}^x \neq 0$ the parameters A_0 and ϕ are given by (4.50) by using trigonometric relations. The proof can be completed by substituting x by y and ω by $\tilde{\omega}$ respectively.

□

Let us remark that the calculation of $G_1^{\tilde{\omega}}$ is obtained by the following way: once the function g_1 is given, the integral G_1^ω can be formally calculated where ω is an unknown parameter. Then, we substitute ω by $\tilde{\omega}$ in the obtained integral value. The calculations of $G_i^{\tilde{\omega}}$ for $i = 2, 3, 4$ can be given similarly.

4.3.2 Time-varying amplitude case

In this subsection, we assume that $A_1 \in \mathbb{R}^*$. Then, x satisfies the equation given by (4.18). Thus, similarly to Proposition 4.2.67 we can estimate the frequency by using the modulating functions method.

Proposition 4.3.72 *[Liu 2011d] Let f be a function belonging to $\mathcal{C}^4([0,1])$ which satisfies the following conditions $f^{(i)}(0) = f^{(i)}(1)$ for $i = 0, \ldots, 3$. Assume that $\int_0^1 f(\tau)\, x(T\tau) d\tau \neq 0$ with $T \in D_T$, then the parameter ω is estimated from the noisy observation y by*

$$\tilde{\omega} = \begin{cases} \left(\frac{-B_y - \sqrt{B_y^2 - A_y C_y}}{A_y} \right)^{\frac{1}{2}}, & \text{if } \Delta \geq 0, \\ \left(\frac{-B_y + \sqrt{B_y^2 - A_y C_y}}{A_y} \right)^{\frac{1}{2}}, & \text{else}, \end{cases} \tag{4.52}$$

where $\Delta = A_1 \int_0^1 \dot{w}_{\mu+4,k+4}(\tau)\, \sin(\omega T\tau + \phi)d\tau,\ A_y = T^4 \int_0^1 f(\tau)\, y(T\tau)d\tau,\ B_y = T^2 \int_0^1 \ddot{f}(\tau)\, y(T\tau)d\tau,$ $C_y = \int_0^1 f^{(4)}(\tau)\, y(T\tau)d\tau$.

Proof. Recall that $x^{(4)}(T\tau) + 2\omega^2 x^{(2)}(T\tau) + \omega^4 x(T\tau) = 0$ for any $\tau \in [0,1]$. As f is continuous on $[0,1]$, the we have

$$\int_0^1 f(\tau)x^{(4)}(T\tau)d\tau + 2\omega^2 \int_0^1 f(\tau)x^{(2)}(T\tau)d\tau + \omega^4 \int_0^1 f(\tau)x(T\tau)d\tau = 0.$$

Then, this proof can be completed similarly to the one of Proposition 4.2.67. □

Let us take an expansion of x

$$x(T\tau) = A_0 \cos\phi\, \sin(\omega T\tau) + A_0 \sin\phi\, \cos(\omega T\tau) + A_1 \cos\phi\, T\tau \sin(\omega T\tau) + A_1 \sin\phi\, T\tau \cos(\omega T\tau), \quad (4.53)$$

where $\tau \in [0,1]$, $T \in D_T$. Then, similarly to Proposition 4.3.71 we can estimate A_0 and ϕ by using the modulating functions method in the following proposition.

Proposition 4.3.73 *[Liu 2011d] Let f_i for $i = 1,\ldots,4$ be four continuous functions defined on $[0,1]$. Assume that there exists $T \in D_T$ such that the determinant of the matrix $M_\omega = (M^\omega_{i,j})_{1\le i,j\le 4}$ is different to zero, where for $i = 1,\ldots,4$*

$$M^\omega_{i,1} = \int_0^1 f_i(\tau)\, \sin(\omega T\tau)d\tau, \qquad M^\omega_{i,2} = \int_0^1 f_i(\tau)\, \cos(\omega T\tau)d\tau,$$
$$M^\omega_{i,3} = \int_0^1 f_i(\tau)\, T\tau\, \sin(\omega T\tau)d\tau, \quad M^\omega_{i,4} = \int_0^1 f_i(\tau)\, T\tau\, \cos(\omega T\tau)d\tau.$$

Then, for any $\phi \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ the parameters A_0, A_1 and ϕ are estimated by

$$\begin{aligned} \widetilde{A_i} &= \left(\left(\widetilde{A_i \cos\phi}\right)^2 + \left(\widetilde{A_i \sin\phi}\right)^2 \right)^{1/2}, \\ \tilde{\phi} &= \arctan\left(\frac{\widetilde{A_0 \sin\phi}}{\widetilde{A_0 \cos\phi}} \right), \end{aligned} \quad (4.54)$$

where the estimates of $A_i \cos\phi$ and $A_i \sin\phi$ for $i = 0,1$ are obtained by solving the following linear system

$$M_{\tilde{\omega}} \begin{pmatrix} \widetilde{A_0 \cos\phi} \\ \widetilde{A_0 \sin\phi} \\ \widetilde{A_1 \cos\phi} \\ \widetilde{A_1 \sin\phi} \end{pmatrix} = \begin{pmatrix} I^y_{f_1} \\ I^y_{f_2} \\ I^y_{f_3} \\ I^y_{f_4} \end{pmatrix}, \quad (4.55)$$

where $I^y_{f_i} = \int_0^1 f_i(\tau)\, y(T\tau)d\tau$ for $i = 1,\ldots,4$, and $\tilde{\omega}$ is the estimate of ω given by Proposition 4.3.72.

Proof. By multiplying both sides of (4.53) by the continuous function f_i for $i = 1,\ldots,4$ and by integrating between 0 and 1, we obtain

$$I^x_{f_i} = A_0 \cos\phi M^\omega_{i,1} + A_0 \sin\phi M^\omega_{i,2} + A_1 \cos\phi M^\omega_{i,3} + A_1 \sin\phi M^\omega_{i,4}. \quad (4.56)$$

Then, it yields the following linear system

$$M_\omega \begin{pmatrix} A_0 \cos\phi \\ A_0 \sin\phi \\ A_1 \cos\phi \\ A_1 \sin\phi \end{pmatrix} = \begin{pmatrix} I^x_{f_1} \\ I^x_{f_2} \\ I^x_{f_3} \\ I^x_{f_4} \end{pmatrix}.$$

Since $\det(M_\omega) \neq 0$, by solving the previous system, we obtain $A_i \cos\phi$ and $A_i \sin\phi$. Finally, the proof can be completed by substituting x by y in the obtained formulae of $A_i \cos\phi$ and $A_i \sin\phi$ for $i = 0, 1$.
□

Remark 8 *The estimations given by Proposition 4.3.71 and Proposition 4.3.73 are obtained by solving a system linear which depends on modulating functions. Hence, in order to obtain stable estimations we should choose the modulating functions which make the system to be equalized.*

If we choose function $w^{(n)}_{\mu+n,\kappa+n}$ with $n \in \mathbb{N}$, $\mu, \kappa \in]-1, +\infty[$ as the previous modulating functions. Then, the principle of such estimators is connected with that of orthogonal projection. An interpretation in terms of least squares follows from [Mboup 2009a].

4.4 Conclusion

In this chapter, by using the algebraic parametric techniques and the modulating functions method we have given some estimators for the frequency, amplitude and phase of noisy sinusoidal signals the amplitude of which are time-invariant or not. Moreover, we have shown the connection between these two methods.

Let us remark that whatever the algebraic identification techniques and the modulating functions method the obtained amplitude and phase estimators were given by solving a linear system. Similarly, by providing some necessary equations these two methods can also be used to estimate the unknown amplitudes and phases of noisy multi-sinusoidal signal with known frequency.

In the following chapter, we give some noise error bounds to study the choice of parameters κ, μ, n and the length of integration window T for our estimators.

Chapter 5

Error analysis for estimators of sinusoidal signal

Contents

5.1 Introduction

Let us recall that the algebraic parametric techniques exhibit good robustness properties with respect to corrupting noises, without the need of knowing their statistical properties. A weakness of these methods is a lack of any precise error analysis, when they are implemented in practice. In this chapter, such error analysis is performed for the estimators given in Chapter 4.

Let us denote by q the involved functions in the integrals of our estimators in Chapter 4. Then, we denote by

$$I_q^y := \int_0^1 q(\tau)\, y(T\tau) d\tau. \tag{5.1}$$

Since $y = x + \varpi$, we get

$$I_q^y = I_q^x + I_q^{\varpi}, \tag{5.2}$$

where I_q^{ϖ} is the noise error (ϖ being the noise)

$$I_q^{\varpi} := \int_0^1 q(\tau)\,\varpi(T\tau)d\tau. \tag{5.3}$$

Thus the integral I_q^y is only corrupted by the noise error contributions I_q^{ϖ} that will be denoted by e_q^{ϖ} in the following.

From now on, assume that $y(t_i) = x(t_i) + \varpi(t_i)$ ($t_i \in I$) is a noisy measurement of x in discrete case with an equidistant sampling period T_s, then we need to use a numerical integration method to approximate the integrals used in the previous estimators. Let $\tau_i = \frac{i}{m}$ and $a_i > 0$ for $i = 0, \dots, m$ with $m = \frac{T}{T_s} \in \mathbb{N}^*$ (except for $a_0 \geq 0$ and $a_m \geq 0$) be respectively the abscissas and the weights for a given numerical integration method. Weight a_0 (resp. a_m) is set to zero in order to avoid the infinite values at $\tau = 0$ when $-1 < \kappa < 0$ (resp. $\tau = 1$ when $-1 < \mu < 0$) [Lyness 1994]. Hence, I_q^y can be approximated by

$$I_q^{y,m} := \sum_{i=0}^{m} \frac{a_i}{m}\, q(\tau_i)\, y(T\tau_i). \tag{5.4}$$

By writing $y(t_i) = x(t_i) + \varpi(t_i)$, we get

$$I_q^{y,m} = I_q^{x,m} + e_q^{\varpi,m}, \tag{5.5}$$

where

$$e_q^{\varpi,m} := \sum_{i=0}^{m} \frac{a_i}{m}\, q(\tau_i)\, \varpi(T\tau_i). \tag{5.6}$$

Thus the integral

$$I_q^y = I_q^{x,m} + (I_q^x - I_q^{x,m}) + e_q^{\varpi,m} \tag{5.7}$$

is corrupted in the discrete case by two sources of errors:

- the numerical error $I_q^x - I_q^{x,m}$,
- the noise error contributions $e_q^{\varpi,m}$.

Consequently, the estimation error for each previously obtained estimator is due to these two sources of errors. Hence, by reducing the errors for the integral I_q^y we can reduce the total error for our estimators. To do so, we give some noise error bounds. Let us recall that these estimators depend on the parameters κ, μ, n and T. These error bounds permit us to know precisely the influence of these parameters on our estimators. Hence, we can choose the optimal parameters so as to get "good" estimations.

This chapter is organized as follows. In Section 5.2, we roughly show the parameters' influence on the numerical error. In Section 5.3, we consider bounded and integral noise. In particular, we roughly study the parameters' influence on the noise error which is due to a biased sinusoidal perturbation. In Section 5.4, we consider two classes of stochastic processes noise. Firstly, we provide some noise error bounds for the corresponding noise error contributions. Secondly, we show the influence of the sampling period onto these noise error contributions. In Section 5.5, by using the error bounds for noise error contributions obtained in the previous subsection, we give some error bounds for the estimators obtained in the time-invariant amplitude case.

5.2 Analysis for the numerical error

We apply the trapezoidal rule as the numerical integration method in (5.4). Let $h(\tau) := q(\tau)x(T\tau)$ for $\tau \in [0,1]$ be a function which is at least $2l+2$ ($l \in \mathbb{N}$) times continuously differentiable on $[0,1]$, then by using the Euler-Maclaurin formula [Atkinson 1989] (p. 285) we have

$$I_q^x - I_q^{x,m} = -\sum_{j=1}^{l} \frac{B_{2j}}{(2j)!\, m^{2j}} \left(h^{(2j-1)}(1) - h^{(2j-1)}(0)\right) - \frac{B_{2l+2}}{(2l+2)!\, m^{2l+2}} h^{(2l+2)}(\xi), \tag{5.8}$$

for some $\xi \in [0,1]$. The coefficients B_j are the Bernoulli numbers [Abramowitz 1965] (p. 804) which are equal to the value of the Bernoulli polynomial $B_n(t)$ at $t = 0$. Let us recall that the Bernoulli polynomials are defined as follows

1. $\forall t \in \mathbb{R},\ B_0(t) = 1$,
2. $\forall n \in \mathbb{N},\ B'_{n+1}(t) = (n+1)B_n(t)$,
3. $\forall n \in \mathbb{N}^*,\ \int_0^1 B_n(t)dt = 1$.

It is clear that if the value of T is set then by decreasing the sampling period T_s we can increase the value of m. Hence, the numerical error $I_q^x - I_q^{x,m}$ given in (5.8) can be reduced. Then, we roughly show the other parameters' influence on this numerical error.

On the one hand, we denote the ceil function by $\lceil\cdot\rceil$. Then, we assume that $q \equiv w_{\mu+n,\kappa+n}^{(n)}$ (such that $q^{(i)}(0) = q^{(i)}(1) = 0$ for $i = 0, 1, \cdots, \lceil\min(\kappa,\mu)\rceil - 1$) for the modulating functions method, where $\kappa, \mu \in]-1, +\infty[$ and $n \in \mathbb{N}$ such that $q \in \mathcal{C}^{2l+2}([0,1])$. If $\lceil\min(\kappa,\mu)\rceil = 2l$, then by applying the Rodrigues formula (given in (7.22)) in (5.8) we obtain that $h^{(2j-1)}(1) = h^{(2j-1)}(0)$. It yields

$$I_q^x - I_q^{x,m} = -\frac{B_{2l+2}}{(2l+2)!m^{2l+2}} h^{(2l+2)}(\xi), \tag{5.9}$$

where

$$h^{(2l+2)}(\xi) = \sum_{i=0}^{2l+2} \binom{2l+2}{i} (-1)^{n+i}(n+i)!w_{\mu-i,\kappa-i}(\xi)P_{n+i}^{(\mu-i,\kappa-i)}(\xi)(mT_s)^{2l+2-i}x^{(2l+2-i)}(T\xi).$$

Denoting by $M_i = \|x^{(i)}\|_\infty$, we obtain

$$\left|I_q^x - I_q^{x,m}\right| \leq \frac{B_{2l+2}}{(2l+2)!m^{2l+2}} \sum_{i=0}^{2l+2} M_{2l+2-i}(mT_s)^{2l+2-i} \binom{2l+2}{i} \left|w_{\mu-i,\kappa-i}(\xi)P_{n+i}^{(\mu-i,\kappa-i)}(\xi)\right|. \tag{5.10}$$

Since the value of l depends on the minimum value between κ and μ and large values of κ and μ (resp. of l) can get a large value for $P_{n+i}^{(\mu-i,\kappa-i)}(\xi)$ (resp. for $x^{(2l+2-i)}(T\xi)$), a natural idea is to increase appropriately the values of κ and μ such that the numerical error can be negligible even for a small value of m (large value of T_s). The value of $P_{n+i}^{(\mu-i,\kappa-i)}(\xi)$ can increase with respect to the value of n, hence we set $n=0$ by default.

On the other hand, the estimators for A_0 and ϕ given by the algebraic parametric technique in Proposition 4.2.66 and Proposition 4.2.68 depend only on the parameter μ for a given value T. The integrals in these estimators contain the function $w_{\mu+j,0}$ with $j \in \mathbb{N}^*$. Then, by taking $l=0$ in (5.8), we obtain

$$I_{w_{\mu+j,0}}^{x} - I_{w_{\mu+j,0}}^{x,m} = -\frac{B_2}{m^2}\sum_{i=0}^{2}\frac{(mT_s)^{2-i}}{i!(2-i)!}x^{(2-i)}(T\xi)\frac{\Gamma(\mu+1+j)}{\Gamma(\mu+1+j-i)}(1-\xi)^{\mu+j-i}. \tag{5.11}$$

Thus, if the value of j is large and the value of m is not large enough (T_s is not small enough), then the numerical error can not be negligible. Moreover, if the value of μ increases, then according to (5.11) the numerical error can be increased also. Hence, the estimators obtained by the algebraic parametric techniques in Proposition 4.2.68 can produce larger numerical errors than the ones given in Proposition 4.2.66. Moreover, the estimators obtained by using the modulating functions method can produce smaller numerical errors than the ones obtained by using the algebraic parametric techniques. Finally, we recall in Table 5.1 the influence of parameters on the numerical error. $a \downarrow$ and $b \nearrow$ (resp. $b \searrow$) refer that by increasing (resp. decreasing) the value of b, we can reduce the value of a.

Method	Numerical error	T_s (m)	μ	n
Modulation functions method	$\downarrow$	$\searrow$ $(\nearrow)$ (T being set)	$\min(\kappa,\mu) \nearrow$	$\searrow$
Algebraic parametric technique	$\downarrow$	$\searrow$ $(\nearrow)$ (T being set)	$\searrow$	

Table 5.1: The influence of parameters on the numerical error.

5.3 Analysis for an integrable noises

In this section, we assume that the noise ϖ is a bounded with a noise level $\hat{\delta}$ and integrable function on I, which is given (2.6). Then, the associated noise error contribution $e_q^{\varpi,m}$ given in (5.6) is bounded by

$$\left|e_q^{\varpi,m}\right| \leq M_q^{max}, \tag{5.12}$$

where $M_q^{max} := \hat{\delta}\sum_{i=0}^{m}\frac{a_i}{m}\,|q(\tau_i)|$.

In the following subsection, we study the noise error due to a biased sinusoidal perturbation.

5.3.1 Analysis for a sinusoidal perturbation

There are many applications where the sinusoidal signal is corrupted by another sinusoidal perturbation of higher frequency. Let us assume in the subsection that $\varpi(t_i) = A_\varpi \sin(\omega_\varpi t_i)$ for $t_i \in I$ with $A_\varpi \in \mathbb{R}_+$ and $\omega_\varpi \in \mathbb{R}_+$. If $\omega_\varpi \gg \omega$, then according to [Fliess 2006] ϖ is a noise understood as a high frequency perturbations. If there exists a small integer k such that $\omega_\varpi = k\omega$, we consider ϖ as a *low frequency* *sinusoidal perturbation*. We study the parameters' influence on such sinusoidal perturbation error contribution.

On the one hand, we assume that $q \equiv w_{\mu+n,\kappa+n}^{(n)}$ with $\kappa, \mu \in]-1, +\infty[$ and $n \in \mathbb{N}$ for the modulating functions method. If $l+1 = \lceil \min(\kappa, \mu) \rceil$, then by taking the Rodrigues formula, we get $w_{\mu+n,\kappa+n}^{(n+i)}(0) = w_{\mu+n,\kappa+n}^{(n+i)}(1) = 0$ for $i = 0, \ldots, l$. Hence, by applying integration by parts, we obtain

$$\int_0^1 w_{\mu+n,\kappa+n}^{(n)}(\tau) \sin(\omega_\varpi T\tau)\, d\tau = \frac{(-1)^{n+l}(n+l)!}{(\omega_\varpi T)^l} \int_0^1 w_{\mu-l,\kappa-l}(\tau) P_{n+l}^{(\mu-l,\kappa-l)}(\tau) \sin^{(l)}(\omega_\varpi T\tau)\, d\tau. \quad (5.13)$$

Hence, by increasing the value of T we can reduce this sinusoidal perturbation error. Moreover, if $\omega_\varpi \gg \omega$ and the numerical error for I_q^ϖ is negligible, then this high frequency sinusoidal noise error can be negligible. Otherwise, by increasing the values of κ and μ, the power of $\frac{1}{\omega_\varpi T}$ becomes larger and larger. Recall that by increasing appropriately the values for κ and μ, the numerical error for $I_q^{\varpi_0}$ can become negligible. Thus, if $\frac{1}{\omega_\varpi T} < 1$, then the low frequency sinusoidal perturbation error can be also negligible.

On the other hand, as is shown in the previous section, the integrals in the estimators for A_0 and ϕ obtained by the algebraic parametric techniques contain the functions $w_{\mu+j,0}$ with $j \in \mathbb{N}^*$. Then, by taking integration by parts we obtain

$$A_\varpi \int_0^1 w_{\mu+j,0}(\tau) \sin(\omega_\varpi T\tau)\, d\tau = -\frac{A_\varpi}{\omega_\varpi T} - (\mu+j)\frac{A_\varpi}{\omega_\varpi T} \int_0^1 w_{\mu+j-1,0}(\tau) \cos(\omega_\varpi T\tau)\, d\tau. \quad (5.14)$$

Hence, similarly to the modulating functions method, the sinusoidal perturbation error can be reduced by increasing the value of T or by taking a high frequency ω_ϖ. However, if the frequency ω_ϖ is not high enough, then this sinusoidal perturbation error can not be negligible. Moreover, if the value of μ increases, then according to (5.14) this error can be increased also. We recall in Table 5.2 the influence of parameters on the sinusoidal perturbation error. $a \downarrow$ and $b \nearrow$ (resp. $b \searrow$) refer that by increasing (resp. decreasing) the value of b, we can reduce the value of a.

Method	Sinusoidal perturbation	T (m)	μ	n
Modulation functions method	$\downarrow$	$\nearrow$ (T_s being set)	$\min(\kappa, \mu) \nearrow$	$\searrow$
Algebraic parametric technique	$\downarrow$	$\nearrow$ (T_s being set)	$\searrow$	

Table 5.2: The influence of parameters on the sinusoidal perturbation error.

Finally, we assume that the sinusoidal perturbation ϖ is corrupted by a structured perturbation where $\varrho(t_i) = \sum_{j=0}^{n-1} \nu_j\, t_i^j$. Then, according to Lemma 2.3.47, the estimators obtained by using the modulating functions method with $q \equiv w_{\mu+n,\kappa+n}^{(n)}$ can annihilate this structured perturbation.

5.4 Analysis for a stochastic processes noise error

In this section, we study the noise in the framework of stochastic process. By using the Bienaymé-Chebyshev inequality, we give some appropriate error bounds for the noise error contributions. Then, we show the influence of the sampling period on these noise errors.

5.4.1 Non-independent cases

In this subsection, we assume that the noise ϖ satisfies the condition (C_1) defined in Subsection 2.4.1. Then, by applying Lemma 2.4.49 we obtain that the noise error $e_q^{\varpi,m}$ defined in (5.6) converges in mean square to e_q^{ϖ}.

Similarly to (2.15), by calculating the mean value and variance of the noise error contribution $e_q^{\varpi,m}$ we obtain an error bound for $e_q^{\varpi,m}$:

$$\left|e_q^{\varpi,m}\right| \overset{p_r}{<} M_q^{max}, \tag{5.15}$$

where $M_q^{max} = \max(|M_q^l|, |M_q^h|)$ with $M_q^l = E[e_q^{\varpi,m}] - \gamma\sqrt{Var[e_q^{\varpi,m}]}$ and $M_q^h = E[e_q^{\varpi,m}] + \gamma\sqrt{Var[e_q^{\varpi,m}]}$. In particular, similarly to Subsection 2.4.2, we can give error bounds when the noise is a Wiener process or a Poisson process.

Now, we assume that the noise $\{\varpi(\tau), \tau \geq 0\}$ is a continuous parameter stochastic process satisfying conditions (C_1) to (C_3) defined in Section 2.4. Let us choose function $w_{\mu+n,\kappa+n}^{(n)}$ with $n \in \mathbb{N}$, $\mu, \kappa \in]-1, +\infty[$ as the modulating functions used in Section 4.3. Then, similarly to Theorem 2.4.51 the mean value and variance of the noise error e_q^{ϖ} for the estimators obtained by using modulating functions methods are equal to 0. Hence, similarly to Theorem 2.4.56, we can obtain that $e_q^{\varpi,m}$ converges in mean square to 0 when $T_s \to 0$.

5.4.2 Independent cases

In this subsection, we assume that the noise ϖ satisfies the condition $(C_4) - (C_6)$ defined in Subsection 2.5. Then, similarly to the previous subsection, by calculating the mean value and variance of the noise error contribution $e_q^{\varpi,m}$, we can give error bound M_q^{max} for $e_q^{\varpi,m}$.

In particular, if ϖ is a white Gaussian noise, then according to the three-sigma rule, we have

$$M_q^l \overset{p_r}{\leq} e_q^{\varpi,m} \overset{p_r}{\leq} M_q^h, \tag{5.16}$$

where $M_q^l = E[e_q^{\varpi,m}] - \gamma\sqrt{Var[e_q^{\varpi,m}]}$ and $M_q^h = E[e_q^{\varpi,m}] + \gamma\sqrt{Var[e_q^{\varpi,m}]}$ with $p_1 = 68.26\%$, $p_2 = 95.44\%$ and $p_3 = 99.73\%$. In this case, we have

$$\left|e_q^{\varpi,m}\right| \overset{p_r}{\leq} M_q^{max}, \tag{5.17}$$

where $M_q^{max} = \max(|M_q^l|, |M_q^h|)$.

Now, let us assume that $q \in \mathcal{L}^2(I)$. Then, similarly to Theorem 2.5.58, we obtain that $e_q^{\varpi,m}$ converges in mean square to $\int_0^1 q(\tau)\, E[\varpi(T\tau)]\, d\tau$ when $T_s \to 0$. Moreover, if $E[\varpi(\tau)] = \sum_{i=0}^{n-1} \bar{\nu}_i\, \tau^i$ with $\bar{\nu}_i \in \mathbb{R}$, then by taking $q \equiv w_{\mu+n,\kappa+n}^{(n)}$ with $n \in \mathbb{N}$, $\mu, \kappa \in]-\frac{1}{2}, +\infty[$, the noise error e_q^{ϖ} for the estimators obtained by using modulating functions methods converges in mean square to 0 when $T_s \to 0$.

Remark 9 *According to the proof of Theorem 2.5.58, the variance of the noise error contribution* $e_q^{\varpi,m}$ *is bounded by the term* $U\frac{a(m)}{m}\sum_{i=0}^{m}\frac{a_i}{m}\, q^2(\tau_i)$ *which tends to zero when* $m \to +\infty$. *As* $\sum_{i=0}^{m}\frac{a_i}{m}\, q^2(\tau_i)$ *tends* *to the integral value* $\int_0^1 q^2(\tau)\,d\tau$ *when* $m \to +\infty$*, the convergence rate of the term* $U\frac{a(m)}{m}\sum_{i=0}^{m}\frac{a_i}{m}q^2(\tau_i)$ *depends on the value of* $\int_0^1 q^2(\tau)\,d\tau$*. Thus, we can choose the function* q *which minimizes the integral value* $\int_0^1 q^2(\tau)\,d\tau$ *so as to minimize the variance* $Var[e_q^{\varpi,m}]$*. If* $q \equiv w_{\mu,\kappa}$*, then we have* $\int_0^1 q^2(\tau)\,d\tau = \mathrm{B}(2\mu+1, 2\kappa+1)$ *where* $\mathrm{B}(\cdot,\cdot)$ *is the classical beta function. Since* $\mathrm{B}(2\mu+1, 2\kappa+1)$ *increases with respect to* μ *and* κ*, we can reduce* $Var[e_q^{\varpi,m}]$ *by decreasing the values of* μ *and* κ*. Consequently, we can reduce the noise errors contributions for our estimators by decreasing the values of* μ *and* κ*.*

5.5 Some error bounds for estimators

In this section, by using the noise error bound M_q^{max} obtained in the previous sections we give some error bounds for the estimators obtained in the time-invariant amplitude case. These error bounds permit us to know precisely the influence of these parameters on our estimators.

5.5.1 Some error bounds in the unknown ω case

We obtain in Proposition 4.3.70 an estimator for the frequency ω in the time-invariant amplitude case, which is more general to the one obtained in Proposition 4.2.65. We give an error bound for this estimation in the following proposition.

Proposition 5.5.74 *Let* $\tilde{\omega}$ *be the estimation given in Proposition 4.3.70 for* ω*. Then the total error for* $\tilde{\omega}^2$ *is bounded as follows*

$$\left|\tilde{\omega}^2 - \omega^2\right| \leq M_{\ddot{g},g},$$

where

$$\begin{aligned} M_{\ddot{g},g} &= \left|\omega_m^2 - \omega^2\right| + \frac{1}{T^2}\frac{M_{g_x}^{max}}{\left||I_g^{x,m}| - M_g^{max}\right| |I_g^{x,m}|}, \\ \omega_m^2 &= \frac{1}{T^2}\frac{I_{\ddot{g}}^{x,m}}{I_g^{x,m}}, \\ g_x &= \ddot{g} I_g^{x,m} - g I_{\ddot{g}}^{x,m}, \end{aligned}$$

$I_{\ddot{g}}^{x,m}$ *and* $I_g^{x,m}$ *are given by (5.4) with* $y = x$*,* $q = \ddot{g}$ *and* $q = g$ *respectively,* $M_{g_x}^{max}$ *and* M_g^{max} *are given with* $q = g_x$ *and* $q = g$ *respectively. Moreover, if* $\omega^2 > M_{\ddot{g},g}$*, then the total error for* $\tilde{\omega}$ *is bounded as follows*

$$|\tilde{\omega} - \omega| \leq B_{\ddot{g},g} = \frac{M_{\ddot{g},g}}{2\left(\omega^2 - M_{\ddot{g},g}\right)^{\frac{1}{2}}}. \tag{5.18}$$

Proof. We have

$$\begin{aligned}\left|\tilde{\omega}^2 - \omega_m^2\right| &= \frac{1}{T^2}\left|\frac{I_{\ddot{g}}^{x,m} + e_{\ddot{g}}^{\varpi,m}}{I_g^{x,m} + e_g^{\varpi,m}} - \frac{I_{\ddot{g}}^{x,m}}{I_g^{x,m}}\right| \\ &= \frac{1}{T^2}\left|\frac{e_{\ddot{g}}^{\varpi,m} I_g^{x,m} - e_g^{\varpi,m} I_{\ddot{g}}^{x,m}}{(I_g^{x,m} + e_g^{\varpi,m}) I_g^{x,m}}\right| \\ &\leq \frac{1}{T^2}\frac{\left|e_{\ddot{g}}^{\varpi,m} I_g^{x,m} - e_g^{\varpi,m} I_{\ddot{g}}^{x,m}\right|}{\left||I_g^{x,m}| - |e_g^{\varpi,m}|\right| |I_g^{x,m}|}.\end{aligned}$$

Denote the noise error $e_{\ddot{g}}^{\varpi,m} I_g^{x,m} - e_g^{\varpi,m} I_{\ddot{g}}^{x,m}$ by $e_{g_x}^{\varpi,m}$ with $g_x = \ddot{g} I_g^{x,m} - g I_{\ddot{g}}^{x,m}$. Then, by using noise error bound $M_{g_x}^{max}$ (resp. M_g^{max}) for $e_{g_x}^{\varpi,m}$ (resp. $e_g^{\varpi,m}$) we get

$$\begin{aligned}\left|\tilde{\omega}^2 - \omega^2\right| &\leq \left|\tilde{\omega}^2 - \omega_m^2\right| + \left|\omega^2 - \omega_m^2\right| \\ &\leq \frac{1}{T^2}\frac{M_{g_x}^{max}}{\left||I_g^{x,m}| - M_g^{max}\right| |I_g^{x,m}|} + \left|\omega^2 - \omega_m^2\right| = M_{\ddot{g},g}.\end{aligned} \tag{5.19}$$

Then, by applying the mean value theorem, we get

$$|\tilde{\omega} - \omega| \leq \frac{1}{2}\left|\tilde{\omega}^2 - \omega^2\right| \sup_{\xi \in [\min(\tilde{\omega}^2,\omega^2),\max(\tilde{\omega}^2,\omega^2)]} \xi^{\frac{-1}{2}}$$

Observe that $\omega^2 - M_{\ddot{g},g} \leq \omega^2 - \left|\tilde{\omega}^2 - \omega^2\right| \leq \min(\tilde{\omega}^2, \omega^2) \leq \max(\tilde{\omega}^2, \omega^2) \leq \omega^2 + \left|\tilde{\omega}^2 - \omega^2\right| \leq \omega^2 + M_{\ddot{g},g}$, hence if $\omega^2 > M_{\ddot{g},g}$ then this proof can be easily completed. □

We use the estimations of x_0 and $\dot{x}_0$ in Proposition 4.2.66 to estimate A_0 and ϕ. In these estimations, we need an estimation value for the frequency ω. Hence, we can give error bounds for these estimations by using the error bound given in the previous proposition. Such that by studying these error bounds we can choose the optimal parameters for x_0 and $\dot{x}_0$ so as to estimate A_0 and ϕ.

Proposition 5.5.75 *Let $\tilde{x}_0$ be the estimation given in Proposition 4.2.66 for x_0. Then the total error for $\tilde{x}_0$ is bounded as follows*

$$|\tilde{x}_0 - x_0| \leq M_{P_0^{\omega,x}}, \tag{5.20}$$

where

$$\begin{aligned}M_{P_0^{\omega,x}} &= |x_{0m} - x_0| + M_{\ddot{g},g} T^2 I_{w_{\mu+2,1}}^{x,m} + M_{P_0^{\omega,x}}^{max}, \\ x_{0m} &= I_{P_0^{\omega}}^{x,m}, \\ P_0^{\omega,x}(\tau) &= |p_0(\tau)| + (\omega^2 + M_{\ddot{g},g}) T^2 w_{\mu+2,1}(\tau),\end{aligned}$$

$$p_0(\tau) = 2(\mu+2) w_{\mu+1,0}(\tau) - (\mu+1)(\mu+2) w_{\mu,1}(\tau), \tag{5.21}$$

P_0^{ω} is given in Proposition 4.2.66, $I_{P_0^{\omega}}^{x,m}$ and $I_{w_{\mu+2,1}}^{x,m}$ are given by (5.4) with $y = x$,$q = P_0^{\omega}$ and $q = w_{\mu+2,1}$ respectively, $M_{P_0^{\omega,x}}^{max}$ is given with $q = P_0^{\omega,x}$, $M_{\ddot{g},g}$ is given by (5.19).

Proof. Let Denote $P_0^{\tilde{\omega}}(\tau) = p_0(\tau) - (\tilde{\omega}T)^2 w_{\mu+2,1}(\tau)$ with $p_0(\tau)$ being given by (5.21). Then, we get

$$\tilde{x}_0 - x_{0m} = (\omega^2 - \tilde{\omega}^2)T^2 I^{x,m}_{w_{\mu+2,1}} + e^{\varpi,m}_{P_0^{\tilde{\omega}}}. \tag{5.22}$$

Observe that

$$\begin{aligned} \left|P_0^{\tilde{\omega}}(\tau)\right| &\leq |p_0(\tau)| + (\tilde{\omega}T)^2 w_{\mu+2,1}(\tau) \\ &\leq |p_0(\tau)| + (\omega^2 + M_{\ddot{y},g})T^2 w_{\mu+2,1}(\tau). \end{aligned}$$

Denote $|p_0(\tau)| + (\omega^2 + M_{\ddot{y},g})T^2 w_{\mu+2,1}(\tau)$ by $P_0^{\omega,x}(\tau)$, then by using (5.22) we get

$$|\tilde{x}_0 - x_{0m}| \leq M_{\ddot{y},g} T^2 I^{x,m}_{w_{\mu+2,1}} + M^{max}_{P_0^{\omega,x}}.$$

Then this proof can be easily completed. □

Similarly, we give the following proposition.

Proposition 5.5.76 *Let $\tilde{\dot{x}}_0$ be the estimation given in Proposition 4.2.66 for $\dot{x}_0$. Then the total error for $\tilde{\dot{x}}_0$ is bounded as follows*

$$\left|\tilde{\dot{x}}_0 - \dot{x}_0\right| \leq M_{P_1^{\omega,x}}, \tag{5.23}$$

where

$$\begin{aligned} M_{P_1^{\omega,x}} &= |\dot{x}_{0m} - \dot{x}_0| + \frac{1}{T}\left(M_{\ddot{y},g} T^2 I^{x,m}_{w^\mu} + M^{max}_{P_1^{\omega,x}}\right), \\ w^\mu(\tau) &= (\mu+3)w_{\mu+2,1}(\tau) + w_{\mu+3,0}(\tau), \\ \dot{x}_{0m} &= \frac{1}{T} I^{x,m}_{P_1^{\omega}}, \\ P_1^{\omega,x} &= |p_1(\tau)| + (\omega^2 + M_{\ddot{y},g})T^2((\mu+3)w_{\mu+2,1}(\tau) + w_{\mu+3,0}(\tau)), \\ p_1(\tau) &= (\mu+1)(\mu+2)(\mu+3)w_{\mu,1}(\tau) - (\mu+2)(\mu+3)w_{\mu+1,0}(\tau), \end{aligned}$$

$I^{x,m}_{P_1^{\omega}}$ and $I^{x,m}_{w^\mu}$ are given by (5.4) with $y = x$, $q = P_1^{\omega}$ and $q = w^\mu$ respectively, $M^{max}_{P_1^{\omega,x}}$ is given with $q = P_1^{\omega,x}$, $M_{\ddot{y},g}$ is given by (5.19).

By observing the estimators obtained in Proposition 4.3.71 by the modulating functions method, we can find out that these estimators are not linear with respect to $\tilde{\omega}^2$. Hence, it is better to know the value of ω so as to give error bounds for these estimators.

5.5.2 Some error bounds in the known ω case

In this subsection, we assume that the frequency ω is known. With this assumption, we can give directly some error bounds for the estimators of A_0 and ϕ obtained in Proposition 4.2.66 and in Proposition 4.3.71.

Proposition 5.5.77 *Let $\tilde{A}_0$ and $\tilde{\phi}$ be the estimations obtained in Proposition 4.2.66 for A_0 and ϕ respectively. Then the total error for $\tilde{A}_0^2$ is bounded as follows*

$$\left|\tilde{A}_0^2 - A_0^2\right| \leq M_{P_2^{\omega,x}},$$

where

$$\begin{aligned} M_{P_2^{\omega,x}} &= \left|A_{0m}^2 - A_0^2\right| + M_{P_2^{\omega,x}}^{max} + \left(M_{P_0^{\omega}}^{max}\right)^2 + \frac{1}{(T\omega)^2}\left(M_{P_1^{\omega}}^{max}\right)^2, \\ A_{0m}^2 &= x_{0m}^2 + \frac{\dot{x_0}_m^2}{\omega^2}, \\ P_2^{\omega,x} &= 2I_{P_0^{\omega}}^{x,m} P_0^{\omega} + \frac{2}{(T\omega)^2} I_{P_1^{\omega}}^{x,m} P_1^{\omega}, \end{aligned}$$

P_i^{ω} for $i = 0,1$ is given by Proposition 4.2.66, x_{0m} and $\dot{x}_{0m}$ are given by Proposition 5.5.75 and Proposition 5.5.76 respectively, $M_{P_i^{\omega,x}}^{max}$ is given with $q = P_i^{\omega,x}$ for $i = 2,3$. Moreover, if $A_0^2 > M_{P_2^{\omega,x}}$, then the total error for $\tilde{A}_0$ is bounded as follows

$$\left|\tilde{A}_0 - A_0\right| \leq B_{P_2^{\omega,x}} = \frac{M_{P_2^{\omega,x}}}{2\left(A_0^2 - M_{P_2^{\omega,x}}\right)^{\frac{1}{2}}}.$$

The total error for $\widetilde{\tan(\phi)}$ is bounded as follows

$$\left|\widetilde{\tan(\phi)} - \tan(\phi)\right| \leq M_{P_3^{\omega,x}}, \tag{5.24}$$

where

$$\begin{aligned} M_{P_3^{\omega,x}} &= \left|\tan(\phi_m) - \tan(\phi)\right| + \frac{\omega T M_{P_3^{\omega,x}}^{max}}{|I_{P_1^{\omega}}^{x,m}| \left||I_{P_1^{\omega}}^{x,m}| - M_{P_1^{\omega}}^{max}\right|}, \\ \tan(\phi_m) &= \omega \frac{x_{0m}}{\dot{x}_{0m}}, \\ P_3^{\omega,x} &= I_{P_1^{\omega}}^{x,m} P_0^{\omega} - I_{P_0^{\omega}}^{x,m} P_1^{\omega}. \end{aligned}$$

Moreover, if $\tan(\phi) > M_{P_3^{\omega,x}}$, then the total error for $\tilde{\phi}$ is bounded as follows

$$\left|\tilde{\phi} - \phi\right| \leq B_{P_3^{\omega,x}} = \frac{M_{P_3^{\omega,x}}}{1 + \left(\tan(\phi) - M_{P_3^{\omega,x}}\right)^2}. \tag{5.25}$$

Proof. Observe that

$$\begin{aligned} \tilde{A}_0^2 - A_{0m}^2 &= \tilde{x}_0^2 + \frac{\tilde{\dot{x}}_0^2}{\omega^2} - x_{0m}^2 - \frac{\dot{x_0}_m^2}{\omega^2} \\ &= 2I_{P_0^{\omega}}^{x,m} e_{P_0^{\omega}}^{\varpi,m} + \frac{2}{(T\omega)^2} I_{P_1^{\omega}}^{x,m} e_{P_1^{\omega}}^{\varpi,m} + \left(e_{P_0^{\omega}}^{\varpi,m}\right)^2 + \frac{1}{(T\omega)^2}\left(e_{P_1^{\omega}}^{\varpi,m}\right)^2. \end{aligned} \tag{5.26}$$

Denote $2I_{P_0^{\omega}}^{x,m} e_{P_0^{\omega}}^{\varpi,m} + \frac{2}{(T\omega)^2} I_{P_1^{\omega}}^{x,m} e_{P_1^{\omega}}^{\varpi,m}$ by $e_{P_2^{\omega,x}}^{\varpi,m}$ with $P_2^{\omega,x} = 2I_{P_0^{\omega}}^{x,m} P_0^{\omega} + \frac{2}{(T\omega)^2} I_{P_1^{\omega}}^{x,m} P_1^{\omega}$. Then we obtain

$$\left|\tilde{A_0}^2 - A_{0m}^2\right| \leq M_{P_2^{\omega,x}}^{max} + \left(M_{P_0^{\omega}}^{max}\right)^2 + \frac{1}{(T\omega)^2}\left(M_{P_1^{\omega}}^{max}\right)^2. \tag{5.27}$$

Then by applying the mean value theorem the error bound for $\tilde{A}_0$ is easily obtained. Similarly to Proposition 5.5.74, the total error for $\tilde{\phi}$ can be easily bounded. Then this proof is completed. □

Similarly, we give in the following proposition two error bounds for the estimators obtained by the modulating functions method.

Proposition 5.5.78 *Let $\tilde{A}_0$ and $\tilde{\phi}$ be the estimations obtained in Proposition 4.3.71 for A_0 and ϕ respectively. Then, the total error for $\tilde{A}_0^2$ is bounded as follows*

$$\left|\tilde{A}_0^2 - A_0^2\right| \leq M_{g^\omega},$$

where

$$\begin{aligned} M_{g^\omega} &= \left|A_{0m}^2 - A_0^2\right| + \frac{1}{L_\omega^2}\left(M_{g^\varpi}^{max} + \left(M_{g_o}^{max}\right)^2 + \left(M_{g_e}^{max}\right)^2\right), \\ A_{0m}^2 &= \frac{(I_{g_e}^{x,m})^2}{L_\omega^2} + \frac{(I_{g_o}^{x,m})^2}{L_\omega^2}, \\ g_e &= G_4^\omega g_1 - G_2^\omega g_2, \\ g_o &= G_1^\omega g_2 - G_3^\omega g_1, \\ g^\varpi &= 2I_{g_e}^{x,m}\, g_e + 2I_{g_o}^{x,m}\, g_o, \end{aligned}$$

$I_{g_i}^{x,m}$ is given by (5.4) with $y = x$ and $q = g_i$ for $i = 1, 2, e, o$, $M_{g^\varpi}^{max}$ and $M_{g_j}^{max}$ are given with $q = g^\varpi$ and $q = g_j$ for $j = e, o, \varpi$ respectively. Moreover, if $A_0^2 > M_{g^\omega}$, then the total error for $\tilde{A}_0$ is bounded as follows

$$\left|\tilde{A}_0 - A_0\right| \leq B_{g^\omega} = \frac{M_{g^\omega}}{2\left(A_0^2 - M_{g^\omega}\right)^{\frac{1}{2}}}.$$

The total error for $\widetilde{\tan(\phi)}$ is bounded as follows

$$\left|\widetilde{\tan(\phi)} - \tan(\phi)\right| \leq M_{g_\omega}, \tag{5.28}$$

where

$$\begin{aligned} M_{g_\omega} &= \left|\tan(\phi_m) - \tan(\phi)\right| + \frac{M_{g_\omega}^{max}}{|I_{g_e}^{x,m}|\left||I_{g_e}^{x,m}| - M_{g_e}^{max}\right|}, \\ \tan(\phi_m) &= \frac{I_{g_o}^{x,m}}{I_{g_e}^{x,m}}, \\ g_\omega &= I_{g_e}^{x,m} g_o - I_{g_o}^{x,m} g_e. \end{aligned}$$

Moreover, if $\tan(\phi) > M_{g_\omega}$, then, the total error for $\tilde{\phi}$ is bounded as follows

$$\left|\tilde{\phi} - \phi\right| \leq B_{g_\omega} = \frac{M_{g_\omega}}{1 + \left(\tan(\phi) - M_{g_\omega}\right)^2}. \tag{5.29}$$

Proof. Denote $g_e = G_4^\omega g_1 - G_2^\omega g_2$ and $g_o = G_1^\omega g_2 - G_3^\omega g_1$, then we get

$$\left(\tilde{A}_0^2 - A_{0m}^2\right) L_\omega^2 = 2I_{g_e}^{x,m}\, e_{g_e}^{\varpi,m} + \left(e_{g_e}^{\varpi,m}\right)^2 + 2I_{g_o}^{x,m}\, e_{g_o}^{\varpi,m} + \left(e_{g_o}^{\varpi,m}\right)^2. \tag{5.30}$$

Denote $2I_{g_e}^{x,m}\, e_{g_e}^{\varpi,m} + 2I_{g_o}^{x,m}\, e_{g_o}^{\varpi,m}$ by $e_{g^\varpi}^{\varpi,m}$ with $g^\varpi = 2I_{g_e}^{x,m}\, g_e + 2I_{g_o}^{x,m}\, g_o$, then we get

$$\left|\tilde{A}_0^2 - A_{0m}^2\right| \leq \frac{1}{L_\omega^2}\left(M_{g^\varpi}^{max} + \left(M_{g_o}^{max}\right)^2 + \left(M_{g_e}^{max}\right)^2\right). \tag{5.31}$$

Then the total error for $\tilde{A}_0$ is easily bounded. Similarly to Proposition 5.5.74, the total error for $\tilde{\phi}$ can be easily bounded. Then this proof is completed. □

In particular, if ϖ is a zero-mean white Gaussian noise satisfying the condition (C_4) defined in Subsection 2.5, then the noise error $e^{\varpi,m}_{P^{\omega}_0}$ obtained in (5.26) is also a white Gaussian noise. Hence, we get that the random variable $\frac{(e^{\varpi,m}_{P^{\omega}_0})^2}{Var\left[e^{\varpi,m}_{P^{\omega}_0}\right]}$ follows the χ^2 distribution with the probability density function $f(t) = \frac{\sqrt{2}}{2\Gamma(\frac{1}{2})} t^{-\frac{1}{2}} e^{-\frac{t}{2}}$. Consequently, we can get a more precise error bound for $(e^{\varpi,m}_{P^{\omega}_0})^2$ than $(M^{max}_{P^{\omega}_0})^2$. For example, we can obtain

$$(e^{\varpi,m}_{P^{\omega}_0})^2 \overset{95.6\%}{\leq} 3Var\left[e^{\varpi,m}_{P^{\omega}_0}\right]. \tag{5.32}$$

Similarly, we can get such error bounds for $e^{\varpi,m}_{P^{\omega}_1}$ obtained in (5.26) and $(e^{\varpi,m}_{g_j})^2$ for $j = o, e$ obtained in (5.30).

5.6 Conclusion

In this chapter, our estimators were given in discrete case. We have roughly shown the parameters' influence on the numerical error and on the noise error due to a biased sinusoidal perturbation. Then, we have considered the two classes of stochastic processes noise which have been studied in Chapter 2 for our Jacobi derivative estimators. Hence, we could give some error bounds for the estimators obtained in the time-invariant amplitude case so as to study precisely the parameters' influence. In future work, the analysis for colored noises will be done.

Let us mention that some similar error bounds can also be given for the obtained estimators in the time-varying amplitude case. However, the disadvantage of these error bounds is that the signal x is assumed to be known. The error bounds obtained in this section are only used to let us know better the parameters' influence on our estimators. Hence, instead of giving error bounds, we use the knowledge of the parameters' influence studied previously to choose parameters for the estimators obtained in the time-varying amplitude. We are going to show it in numerical implementations.

Finally, we recall all the obtained error bounds in Table 5.3. In the following chapter, before giving some numerical examples we will apply these error bounds to choose the parameters for our estimators.

Proposition	Estimator	Error bound	Needed condition
5.5.74	$\tilde{\omega}^2$	$M_{\ddot{g},g}$	
5.5.74	$\tilde{\omega}$	$B_{\ddot{g},g}$	$\omega^2 > M_{\ddot{g},g}$
5.5.75	$\tilde{x}_0$	$M_{P_0^{\omega,x}}$	
5.5.76	$\tilde{\dot{x}}_0$	$M_{P_1^{\omega,x}}$	
5.5.77	$\tilde{A}_0^2$	$M_{P_2^{\omega,x}}$	ω is known
5.5.77	$\tilde{A}_0$	$B_{P_2^{\omega,x}}$	ω is known and $A_0^2 > M_{P_2^{\omega,x}}$
5.5.77	$\widetilde{\tan(\phi)}$	$M_{P_3^{\omega,x}}$	ω is known
5.5.77	$\tilde{\phi}$	$B_{P_3^{\omega,x}}$	ω is known and $\tan(\phi) > M_{P_3^{\omega,x}}$
5.5.78	$\tilde{A}_0^2$	M_{g^ω}	ω is known
5.5.78	$\tilde{A}_0$	B_{g^ω}	ω is known and $A_0^2 > M_{g^\omega}$
5.5.78	$\widetilde{\tan(\phi)}$	M_{g_ω}	ω is known
5.5.78	$\tilde{\phi}$	B_{g_ω}	ω is known and $\tan(\phi) > M_{g_\omega}$

Table 5.3: Error bounds

Chapter 6

Numerical implementation of estimators for sinusoidal signal

Contents

6.1 Introduction

In this chapter, we are going to show the efficiency and stability of ours estimators. We choose function $w^{(n)}_{\mu+n,\kappa+n}$ with $n \in \mathbb{N}$, $\mu, \kappa \in]-1, +\infty[$ as the modulating functions. In Section 6.2, Section 6.3 and Section 6.4 we respectively explain how to apply the frequency, amplitude and phase estimators by using a sliding integration window in our identification procedure. In Section 6.5, we consider a time-invariant amplitude sinusoidal signal corrupted by a zero-mean white gaussian noise. Then, we use the error bounds given by (5.17) and (5.32) in the error bounds obtained in Proposition 5.5.77 and Proposition 5.5.78 to study the choice of parameters for our estimators. This helps us to globally select parameters for our estimators. In Section 6.6, some numerical examples are given in the time-invariant amplitude and time-varying amplitude cases, where the noises are white gaussian noises with zero-mean or not and low frequency sinusoidal perturbations respectively. We give also two examples where the sampling period is large so as to compare the estimators obtained by using the algebraic parametric techniques to the ones obtained by using the modulating functions method.

6.2 Recursive algorithms for the frequency estimators

In our identification procedure, we use a sliding integration window. Hence, there can be a singular value in the frequency estimators. Inspired by [Coluccio 2008, Fedele 2009b, Fedele 2010], we are going to use the weighted least square criterion to improve our estimators. Moreover, similarly to [Coluccio 2008, Fedele 2009b, Fedele 2010], two recursive algorithms for these frequency estimators are given.

6.2.1 Time-invariant amplitude case

Let us recall that the estimation of ω at the instant t_i is obtained by applying Proposition 4.3.70 as follows

$$\forall t_i \in I, \quad \tilde{\omega}(t_i) = \frac{1}{T}\left(-\frac{\int_0^1 \ddot{g}(\tau)\, y(T\tau + t_i)d\tau}{\int_0^1 g(\tau)\, y(T\tau + t_i)d\tau}\right)^{\frac{1}{2}}, \; i = 0, 1, \ldots. \tag{6.1}$$

Then, in the discrete case we have

$$\forall t_i \in I, \quad \tilde{\omega}(t_i) = \frac{1}{T}\left(-\frac{I_{\ddot{g}}^{y_{t_i},m}}{I_g^{y_{t_i},m}}\right)^{\frac{1}{2}}, \; i = 0, 1, \ldots, \tag{6.2}$$

where $y_{t_i} \equiv y(T \cdot + t_i)$. Note that if $I_g^{y_{t_i},m} = 0$, then there is a singular value in (6.2). If we denote by $\theta_i = T^2\tilde{\omega}^2(t_i)$, then we can apply the following weighted least square criterion so as to improve the estimation of ω

$$\min_{\theta_i \in \mathbb{R}} J(\theta_i) = \frac{1}{2}\sum_{j=0}^{i} \nu^{i+1-j}\left(I_{\ddot{g}}^{y_{t_j},m} + I_g^{y_{t_j},m}\theta_i\right)^2, \; i = 0, 1, \ldots, \tag{6.3}$$

where the parameter ν is within the interval $(0, 1]$ and represents a forgetting factor to exponentially discard the "old" data in the recursive schema.

The value of θ_i, which minimizes the criterion (6.3), is obtained by seeking the value which cancels $\frac{\partial J(\theta_i)}{\partial \theta_i}$. Hence, we get

$$\theta_i = -\frac{\sum_{j=0}^{i} \nu^{i+1-j} I_{\ddot{g}}^{y_{t_j},m} I_g^{y_{t_j},m}}{\sum_{j=0}^{i} \nu^{i+1-j}\left(I_g^{y_{t_j},m}\right)^2}. \tag{6.4}$$

Then, similarly to [Coluccio 2008, Fedele 2009b, Fedele 2010], we obtain the following recursive algorithm for (6.4)

$$\theta_{i+1} = \frac{\nu}{\alpha_{i+1}}\left(\alpha_i\theta_i - \beta_{i+1}\right), \; i = 0, 1, \ldots, \tag{6.5}$$

where $\alpha_i = \sum_{j=0}^{i} \nu^{i+1-j}\left(I_g^{y_{t_j},m}\right)^2$ and $\beta_i = I_{\ddot{g}}^{y_{t_i},m} I_g^{y_{t_i},m}$. Moreover, α_{i+1} can be recursively calculated as follows

$$\alpha_{i+1} = \nu\left(\alpha_i + \left(I_g^{y_{t_{i+1}},m}\right)^2\right). \tag{6.6}$$

6.2.2 Time-varying amplitude case

The estimation of ω at t_i in the time-varying amplitude case was obtained in Proposition 4.3.72. In the discrete case it becomes

$$\forall t_i \in I, \quad \tilde{\omega}^2(t_i) = -\frac{B_{y_{t_i}}}{2A_{y_{t_i}}} + \frac{\Delta_{y_{t_i}}}{2A_{y_{t_i}}}, \; i = 0, 1, \ldots, \tag{6.7}$$

where $A_{y_{t_i}} = T^4 I_{f}^{y_{t_i},m}$, $B_{y_{t_i}} = 2T^2 I_{\ddot{f}}^{y_{t_i},m}$, $C_{y_{t_i}} = I_{f^{(4)}}^{y_{t_i},m}$ with $y_{t_i} \equiv y(T \cdot + t_i)$ and

$$\Delta_{y_{t_i}} = \begin{cases} -\sqrt{B_{y_{t_i}}^2 - 4A_{y_{t_i}} C_{y_{t_i}}}, & \text{if } A_1 \int_0^1 \dot{w}_{\mu+4,k+4}(\tau)\, \sin(\omega T\tau + t_i + \phi) d\tau \geq 0, \\ \sqrt{B_{y_{t_i}}^2 - 4A_{y_{t_i}} C_{y_{t_i}}}, & \text{else.} \end{cases} \tag{6.8}$$

Similarly to the previous subsection, we apply the criterion (6.3) to $\frac{B_{y_{t_i}}}{2A_{y_{t_i}}}$ and $\frac{\Delta_{y_{t_i}}}{2A_{y_{t_i}}}$ so as to improve the estimation. Denote $\theta(D_{y_{t_i}}) = \frac{D_{y_{t_i}}}{2A_{y_{t_i}}}$ where $D_{y_{t_i}} = B_{y_{t_i}}$ or $D_{y_{t_i}} = \Delta_{y_{t_i}}$, then similarly to (6.5) we get the following recursive algorithm

$$\theta(D_{y_{t_{i+1}}}) = \frac{\nu}{\alpha(A_{y_{t_{i+1}}})} \Big(\alpha(A_{y_{t_i}}) \theta(D_{y_{t_i}}) + 2A_{y_{t_{i+1}}} D_{y_{t_{i+1}}} \Big), \; i = 0, 1, \ldots, \tag{6.9}$$

where $\alpha(A_{y_{t_i}}) = 4 \sum_{j=0}^{i} \nu^{i+1-j} (A_{y_{t_i}})^2$.

6.3 Causal formulae for the amplitude estimators

Let us remark that the formula given in (6.2) is in fact an anti-causal formula, where we use integration window $[t_i, t_i + T]$ to estimate the frequency value at t_i. In the identification procedure, the estimate value is given at instant $t_i + T$. This induces a time-delay of value greater than T. If the frequency is time-invariant, then this is not a problem. However, it is not the case for the time-varying amplitude estimators.

Let $I_q^{y_{t_i},m} = \sum_{j=0}^{m} \frac{a_j}{m} q(\tau_j)\, y(T\tau_j + t_i)$ be the approximated integral value in the amplitude estimators in the identification procedure, where $y_{t_i} \equiv y(T \cdot + t_i)$. In order to avoid a time-delay, we use the following causal formula to estimate the amplitude value at t_i

$$I_q^{y,m}(t_i) = \sum_{j=0}^{m} \frac{a_j}{m} q(\tau_j)\, y(t_i - T\tau_j). \tag{6.10}$$

6.4 Algorithms for the phase estimators

Since we use a sliding integration window $[t_i, t_i + T]$ in the identification procedure of phase, the estimated phase value at t_i is equal to $\tilde{\phi}(t_i) = \omega t_i + \tilde{\phi}$, where $\tilde{\phi}$ is the estimate for the initial phase value at t_0. Hence, we get $\tilde{\phi} = \tilde{\phi}(t_i) - \omega t_i$. Moreover, since we use the function $\arctan(\cdot)$ in the phase estimators, we have $\tilde{\phi} \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. Hence, we take $\tilde{\phi}_0 \equiv \tilde{\phi}(\mod \pi)$ as the estimate of the initial phase value.

6.5 Analysis of parameters' choice for the estimators of A_0 and ϕ

In this section, let us assume that $y(t_i) = x(t_i) + c\varpi(t_i)$ is a generated noise data of x in the interval $[0, 2\pi]$ where $x(t_i) = A_0 \sin(\omega t_i + \phi)$ with $A_0 = 1$, $\omega = 1$, $\phi = 0.25$ and noise $c\varpi(x_i)$ is simulated from a zero-mean white Gaussian *iid* sequence with $c = 0.05 \in \mathbb{R}_+$. The sampling period is set to $T_s = \frac{2\pi}{100\omega}$, *i.e.* there are 100 data per period.

We are going to do some analysis for the choice of parameters in the estimators of A_0 given by Proposition 4.2.66 (resp. Proposition 4.3.71) by using the error bounds obtained in Proposition 5.5.77 (resp. Proposition 5.5.78). For the estimations of A_0 at each time t_i, we use the moving integration window $[t_i, t_i + T]$ with $T = T_s m$. Hence, these estimations depend on the instant t_i. We denote them by $\tilde{A}_0(t_i)$.

Firstly, we consider the estimator of A_0 given in Proposition 4.2.66. Thus, the error bound $B_{P_{2\omega,x}}$ obtained in Proposition 5.5.77 depends on the estimation parameters μ, m and the instant t_i. We denote it by $B_{P_{2\omega,x}}(t_0; \mu, m)$. Then, we can see the variation of $B_{P_{2\omega,x}}(t_0; \mu, m)$ in Fig. 6.1 for $\mu = 0, 0.2, \ldots, 10$ and $m = 30, 21, \ldots, 130$ at t_0, where the minimum of $B_{P_{2\omega,x}}(t_0; \cdot, \cdot)$ is equal to 0.0211 at $\mu = 1$ and $m = 46$. Thus, the optimal parameters for the estimation of A_0 at t_0 are $\mu = 1$ and $m = 46$. Similarly, we can find the optimal parameters for the estimations of A_0 at the other instants t_i $(i \neq 0)$. The minimal value of $B_{P_{2\omega,x}}(t_i; \cdot, \cdot)$ at each t_i and the corresponding optimal parameters' values m_{op}, μ_{op} are shown in Fig. 6.2. We can observe that the minimal values of $B_{P_{2\omega,x}}(t_i; \cdot, \cdot)$ are near to 0.02, and the corresponding optimal values for μ_{op} are between 0 and 1. However, the change of the corresponding optimal values for m_{op} is large (from 37 to 105). In order to choose appropriate parameters' values for the estimations of A_0 at each t_i, we set the value of μ to 0 and 1 respectively, and vary the values of t_i and m. As the curves in Fig. 6.2 are π-periodic, the values of $B_{P_{2\omega,x}}(t_i; \mu, m)$ are shown in Fig. 6.3 with $t_i = 0, T_s, \ldots, \pi - T_s$ and $m = 30, \ldots, 100$. Hence, if $\mu = 0$ (resp. $\mu = 1$) then we can choose $m = 48$ (resp. $m = 65$) so as to get minimal values for $B_{P_{2\omega,x}}(t_i; \mu, \cdot)$.

Secondly, we consider the estimator of A_0 given in Proposition 4.3.71 where the modulating functions are set to $g_i \equiv w^{(n_i)}_{\kappa_i+n_i,\mu_i+n_i}$ for $i = 1, 2$. According to Remark 8, in order to obtain stable estimations we set $n = n_1 = n_2$, $\kappa_2 = \mu_1 = \mu$ and $\mu_2 = \kappa_1 = \kappa$. Moreover, we set $\mu = \kappa + 1$. Thus, the error bound B_{g^ω} obtained in Proposition 5.5.78 depends on the estimation parameters κ, m, n and the instant t_i. We denote it by $B_{g^\omega}(t_i; \kappa, m, n)$. By calculating the variation of $B_{g^\omega}(t_i; \kappa, m, n)$, we can observe that for different instant t_i the minimum of $B_{g^\omega}(t_i; \kappa, m, n)$ holds with different values of κ and m. Moreover, the influence of n on $B_{g^\omega}(t_i; \kappa, m, n)$ is more important than the one of κ. We can see in Fig. 6.5 the different values of $B_{g^\omega}(t_i; \kappa = 1, m, n)$ for $t_i = 0, T_s, \ldots, \pi - T_s$ and $n = 0, 1, 2, 3$. Consequently, we can conclude that by increasing the value of n, we can get smaller and smaller values of $B_{g^\omega}(t_i; \kappa = 1, m, n)$. These values are smaller than the minimal value of $B_{P_{2\omega,x}}(t_i; \mu, \cdot)$ obtained previously. However, we need larger and larger value of m.

Finally, we consider the error bound $B_{P_{3\omega,x}}$ (resp. B_{g_ω}) obtained in Proposition 5.5.77 (resp. Proposition 5.5.78) for the estimator of ϕ obtained in Proposition 4.2.66 (resp. Proposition 4.3.71). Since we use a moving integration window, the estimations of ϕ depend on the instant t_i. We have $\tilde{\phi}(t_i) = \omega t_i + \tilde{\phi}$ at t_i. Let us denote the error bounds by $B_{P_{3\omega,x}}(t_i; \mu, m)$ and $B_{g_\omega}(t_i; \kappa, m, n)$ respectively. Similarly to the previous analysis, we show the variations of $B_{P_{3\omega,x}}(t_i; \mu, m)$ and $B_{g_\omega}(t_i; \kappa, m, n)$. The values of $B_{P_{2\omega,x}}(t_i; \mu, m)$ are shown in Fig. 6.4 with $t_i = 0, T_s, \ldots, 19T_s$ and $m = 30, \ldots, 100$ such that we can ensure the condition $\tan(\phi(t_i)) > M_{P_3^{\omega,x}}$. Hence, if $\mu = 0$ (resp. $\mu = 1$) then we can choose $m = 48$ (resp. $m = 65$) so as to get minimal values for $B_{P_3\omega,x}(t_i; \mu, \cdot)$. Moreover, we can see in Fig. 6.6 the different values of $B_{g_\omega}(t_i; \kappa = 1, m, n)$ for $t_i = 0, T_s, \dots, 19T_s$ and $n = 0, 1, 2, 3$. We give in Table 6.1 and Table 6.2 the obtained parameters for amplitude and phase parameters.

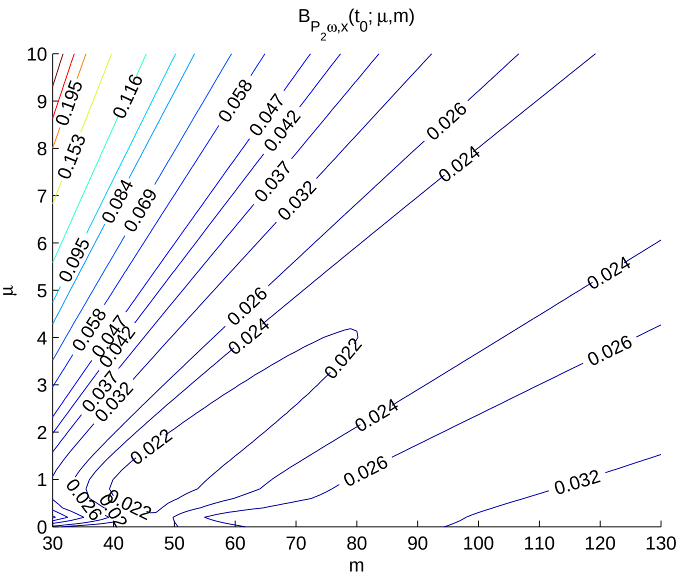
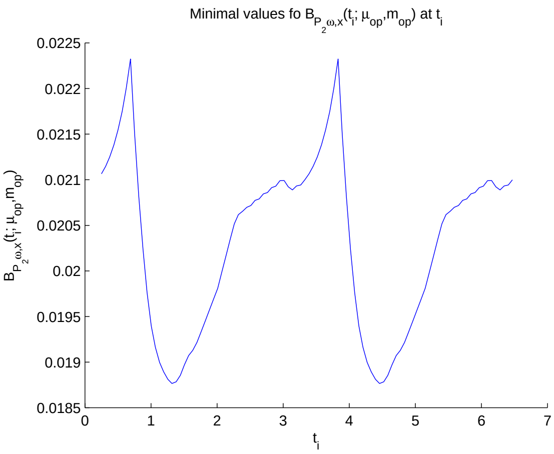
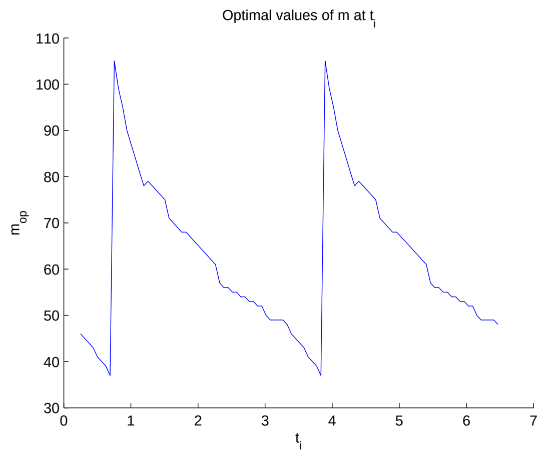


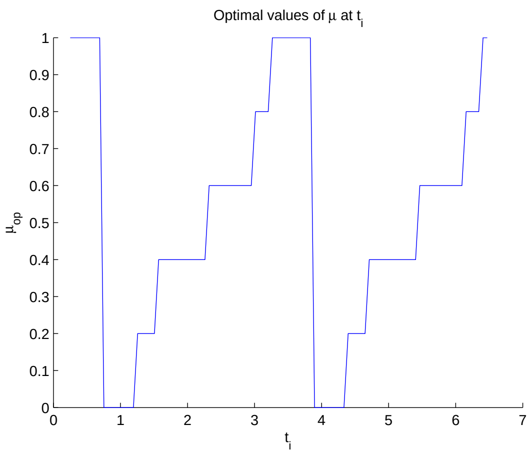
Figure 6.1: $B_{P_2\omega,x}(t_0; \mu, m)$ with $\mu = 0, 0.2, \dots, 10$ and $m = 30, 31, \dots, 130$.



(a) Optimal values of $B_{P_2\omega,x}(t_i;\cdot,\cdot)$.

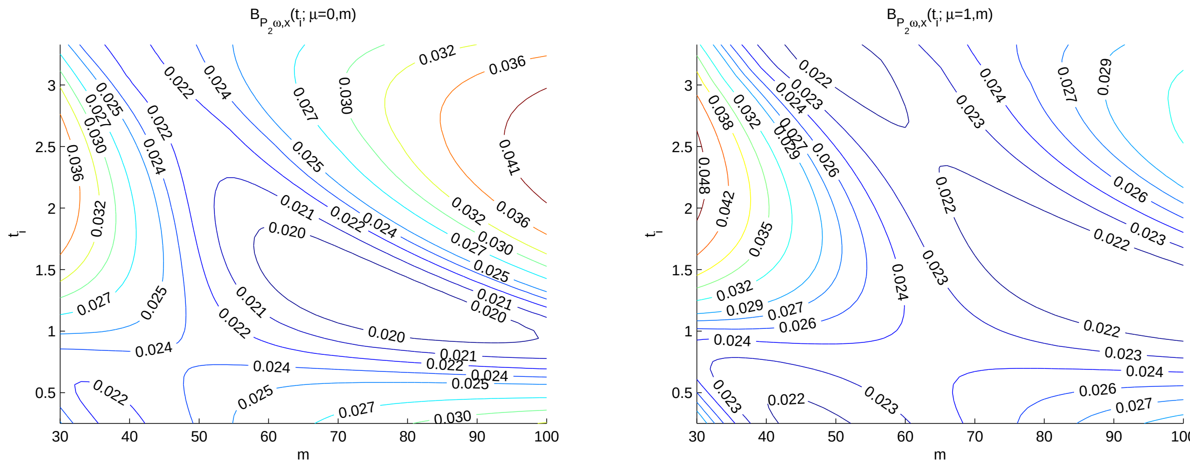


(b) Corresponding optimal parameter' values m_{op}.



(c) Corresponding optimal parameter' values μ_{op}.

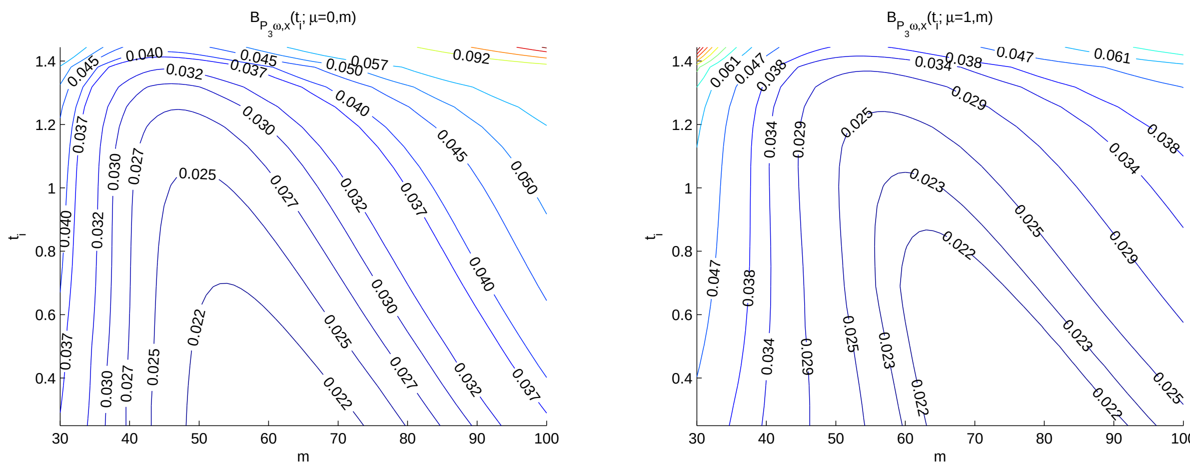
Figure 6.2: Optimal values of error bound and corresponding optimal parameter' values



(a) Different values of $B_{P_2\omega,x}(t_i;\mu = 0, m)$ for $t_i = 0, T_s, \ldots, \pi - T_s$ and $m = 30, \ldots, 100$.

(b) Different values of $B_{P_2\omega,x}(t_i;\mu = 1, m)$ for $t_i = 0, T_s, \ldots, \pi - T_s$ and $m = 30, \ldots, 100$.

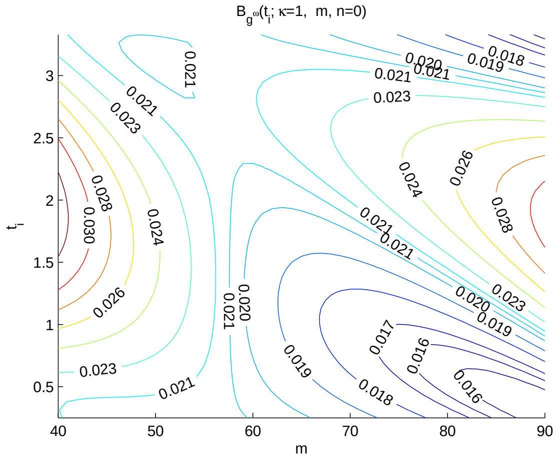
Figure 6.3: Different values of $B_{P_2\omega,x}(t_i;\mu, m)$ for $t_i = 0, T_s, \ldots, \pi - T_s$ and $m = 30, \ldots, 100$ with $\mu = 0, 1$.



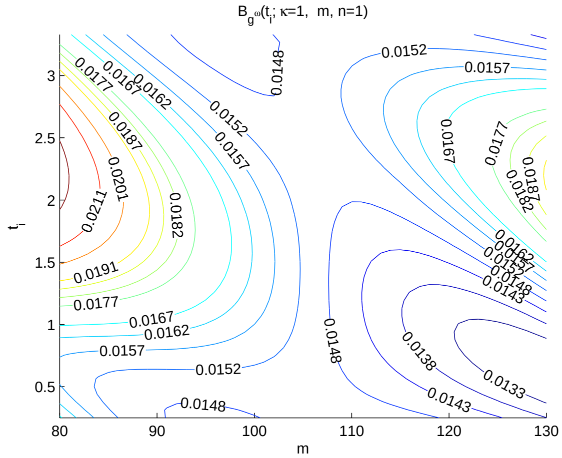
(a) Different values of $B_{P_3\omega,x}(t_i;\mu = 0, m)$ for $t_i = 0, T_s, \ldots, 19T_s$ and $m = 30, \ldots, 100$.

(b) Different values of $B_{P_3\omega,x}(t_i;\mu = 1, m)$ for $t_i = 0, T_s, \ldots, 19T_s$ and $m = 30, \ldots, 100$.

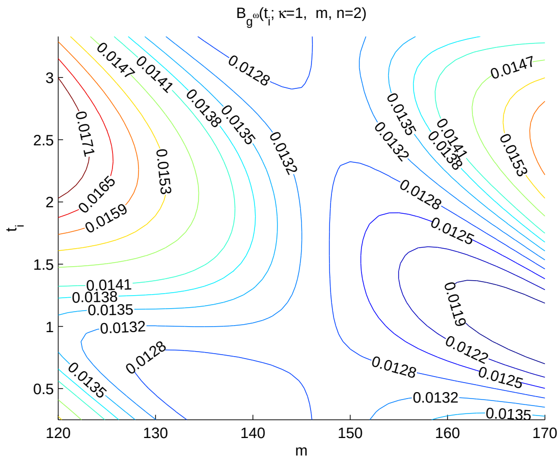
Figure 6.4: Different values of $B_{P_3\omega,x}(t_i;\mu, m)$ for $t_i = 0, T_s, \ldots, 19T_s$ and $m = 30, \ldots, 100$ with $\mu = 0, 1$.



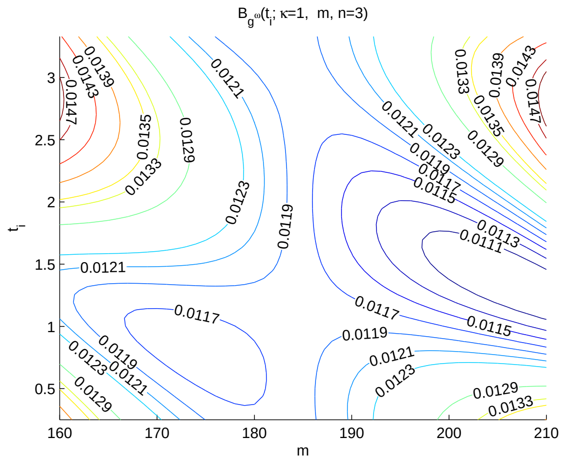
(a) Different values of $B_{g^\omega}(t_i;\kappa=1,m,n=0)$ for $t_i = 0, T_s, \ldots, \pi - T_s$ and $m = 40, \ldots, 90$.



(b) Different values of $B_{g^\omega}(t_i;\kappa=1,m,n=1)$ for $t_i = 0, T_s, \ldots, \pi - T_s$ and $m = 80, \ldots, 130$.

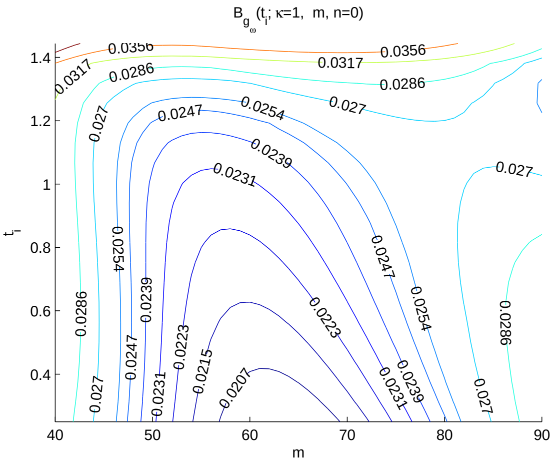


(c) Different values of $B_{g^\omega}(t_i;\kappa=1,m,n=2)$ for $t_i = 0, T_s, \ldots, \pi - T_s$ and $m = 120, \ldots, 170$.

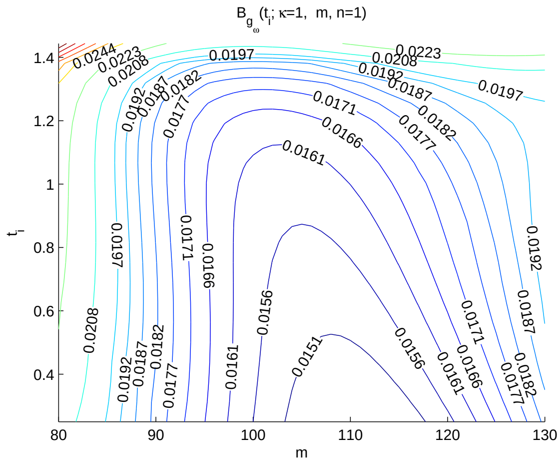


(d) Different values of $B_{g^\omega}(t_i;\kappa=1,m,n=3)$ for $t_i = 0, T_s, \ldots, \pi - T_s$ and $m = 160, \ldots, 210$.

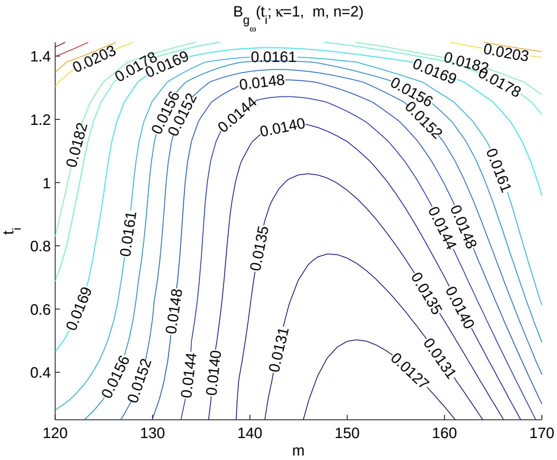
Figure 6.5: Different values of $B_{g^\omega}(t_i;\kappa=1,m,n)$ for $t_i = 0, T_s, \ldots, \pi - T_s$ and $n = 0, 1, 2, 3$.



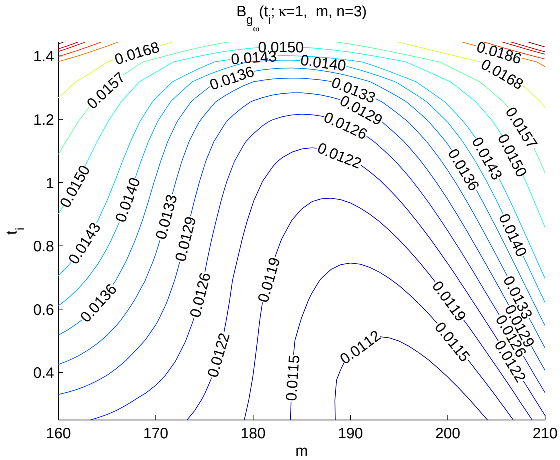
(a) Different values of $B_{g_\omega}(t_i; \kappa = 1, m, n = 0)$ for $t_i = 0, T_s, \ldots, 19T_s$ and $m = 40, \ldots, 90$.



(b) Different values of $B_{g_\omega}(t_i; \kappa = 1, m, n = 1)$ for $t_i = 0, T_s, \ldots, 19T_s$ and $m = 80, \ldots, 130$.



(c) Different values of $B_{g_\omega}(t_i; \kappa = 1, m, n = 2)$ for $t_i = 0, T_s, \ldots, 19T_s$ and $m = 120, \ldots, 170$.



(d) Different values of $B_{g_\omega}(t_i; \kappa = 1, m, n = 3)$ for $t_i = 0, T_s, \ldots, 19T_s$ and $m = 160, \ldots, 210$.

Figure 6.6: Different values of $B_{g_\omega}(t_i; \kappa = 1, m, n)$ for $t_i = 0, T_s, \ldots, 19T_s$ and $n = 0, 1, 2, 3$.

Method	Error bound	μ	n	m	Minimum
Algebraic parametric	$B_{P_{2\omega,x}}(t_i;\mu,m)$	0		48	$0.022-0.024$
	$B_{P_{2\omega,x}}(t_i;\mu,m)$	1		65	$0.022-0.024$
Modulating functions	$B_{g^\omega}(t_i;\kappa,m,n)$	$\mu_1=\kappa_2=2,\ \mu_2=\kappa_1=1$	0	58	0.021
	$B_{g^\omega}(t_i;\kappa,m,n)$	$\mu_1=\kappa_2=2,\ \mu_2=\kappa_1=1$	1	105	$0.0148-0.0152$
	$B_{g^\omega}(t_i;\kappa,m,n)$	$\mu_1=\kappa_2=2,\ \mu_2=\kappa_1=1$	2	146	$0.0128-0.0132$
	$B_{g^\omega}(t_i;\kappa,m,n)$	$\mu_1=\kappa_2=2,\ \mu_2=\kappa_1=1$	3	184	$0.0118-0.0119$

Table 6.1: Parameters for amplitude estimators.

Method	Error bound	μ	n	m
Algebraic parametric	$B_{P_{2\omega,x}}(t_i;\mu,m)$	0		48
	$B_{P_{2\omega,x}}(t_i;\mu,m)$	1		65
Modulating functions	$B_{g^\omega}(t_i;\kappa,m,n)$	$\mu_1=\kappa_2=2,\ \mu_2=\kappa_1=1$	0	58
	$B_{g^\omega}(t_i;\kappa,m,n)$	$\mu_1=\kappa_2=2,\ \mu_2=\kappa_1=1$	1	105
	$B_{g^\omega}(t_i;\kappa,m,n)$	$\mu_1=\kappa_2=2,\ \mu_2=\kappa_1=1$	2	146
	$B_{g^\omega}(t_i;\kappa,m,n)$	$\mu_1=\kappa_2=2,\ \mu_2=\kappa_1=1$	3	184

Table 6.2: Parameters for phase estimators.

6.6 Numerical examples

Example 1

In this example, let us assume that $y(t_i)=x(t_i)+c\varpi(t_i)$ is a generated noise data of x with a sampling period $T_s=\frac{\pi}{50}$ in the interval $[0,30\pi]$ (see Fig. 6.7) where

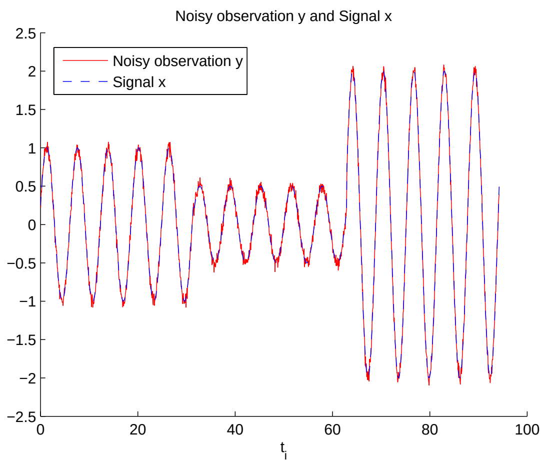
$$x(t_i)=\begin{cases}\sin(t_i+\frac{1}{4}), & \text{if } 0\le t_i\le 10\pi,\\ \frac{1}{2}\sin(t_i+\frac{1}{4}), & \text{if } 10\pi<t_i\le 20\pi,\\ 2\sin(t_i+\frac{1}{4}), & \text{if } 20\pi<t_i\le 30\pi,\end{cases} \tag{6.11}$$

and noise $c\varpi(x_i)$ is simulated from a zero-mean white Gaussian *iid* sequence with $c=0.05$. Hence, the signal-to-noise ratios ([Haykin 2002]) $SNR=10\log_{10}\left(\frac{\sum|y(t_i)|^2}{\sum|c\varpi(t_i)|^2}\right)$ in each interval are equal to 23.5dB, 17.7dB and 26.3dB respectively.

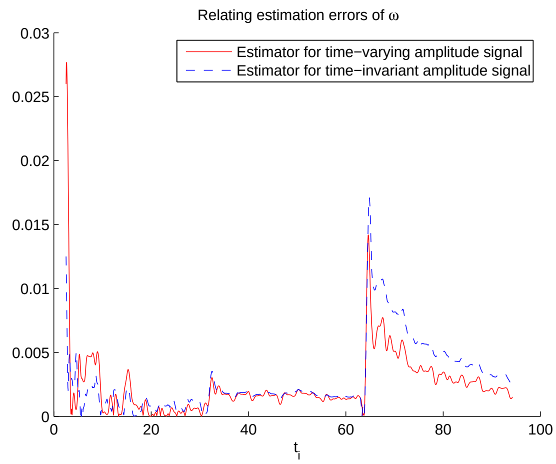
In order to estimate the frequency, the amplitude and the phase, we use the estimators given in Proposition 4.3.70, Proposition 4.2.66 and Proposition 4.3.71 which are obtained in time-invariant amplitude case and the one given in Proposition 4.3.72, Proposition 4.2.68 and Proposition 4.3.73 which are obtained in time-varying amplitude case. We apply the recursive algorithms proposed in Section 6.2 with $\nu=1$ so as to estimate the frequency. The relating estimation errors are shown in Fig. 6.7 where $\kappa=\mu=n=0$, $\nu=1$ and $m=40$. We can observe that these errors are very small. Since we need frequency estimated values in the estimators of amplitude and phase, we use the frequency estimator given in Proposition 4.3.70. As is studied in the previous subsection, we set $\mu=0$, $m=48$

for the estimators for the amplitude and the phase obtained by the algebraic parametric techniques in Proposition 4.2.66, and $\kappa = 1$, $n = 0$, $m = 58$ for the ones obtained by the modulating functions method in Proposition 4.3.71. The so obtained estimations are shown in Fig. 6.8 and Fig. 6.10 with the corresponding relating estimation errors. We can see that the estimations of the amplitude are very good. When $0 \leq t_i \leq 10\pi$, according to Fig 6.3 (resp. Fig 6.5) the relating estimation error for the estimation obtained by using the algebraic parametric techniques (resp. the modulating functions method) is smaller than 0.023 (resp. 0.021) with a probability of 95.6%. However, when the amplitude of the signal changes, there are large errors in the estimations of the phase.

According to Section 5.2 and Remark 9, the estimators obtained by the algebraic parametric techniques in Proposition 4.2.68 can produce larger estimation errors than the ones given in Proposition 4.2.66. Hence, we set a lager value for m as so to reduce estimation errors. Thus, we set $\mu = 0$ and $m = 80$. Finally, according to Remark 8 and Remark 9, we set $f_1 \equiv w_{1,0}$, $f_2 \equiv w_{0,1}$, $f_3 \equiv w_{2,1}$ and $f_4 \equiv w_{1,2}$ for the estimators for the amplitude and the phase obtained by the modulating functions method in Proposition 4.3.73. According to Section 5.2, they can produce larger numerical errors than the ones given in Proposition 4.3.71 with $g_1 \equiv w_{1,2}$, $g_2 \equiv w_{2,1}$. Hence, we set a lager value for m ($m = 80$). The so obtained estimations are shown in Fig. 6.9 and Fig. 6.11 with the corresponding relating estimation errors. We can see that when the amplitude of the signal changes, there are large errors in the estimations. Consequently, if the amplitude of a noisy sinusoidal signal changes quickly, we shall use the estimators given by the algebraic parametric techniques in Proposition 4.2.66 and the ones given by the modulating function method in Proposition 4.3.71 to estimate its amplitude and phase. Moreover, according to the analysis in the previous subsection, if we increase the value of n for the estimators given in Proposition 4.3.71 then the obtained estimation error can be smaller than the ones for the estimations given in Proposition 4.2.66. We recall in Table 6.3 all the parameters used in this example.

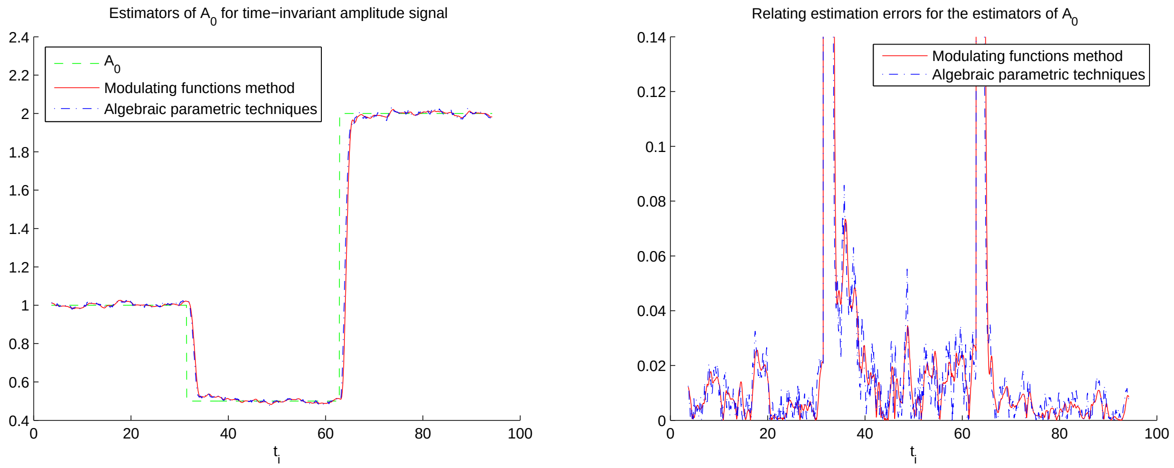


(a) Noisy observation y and the signal x



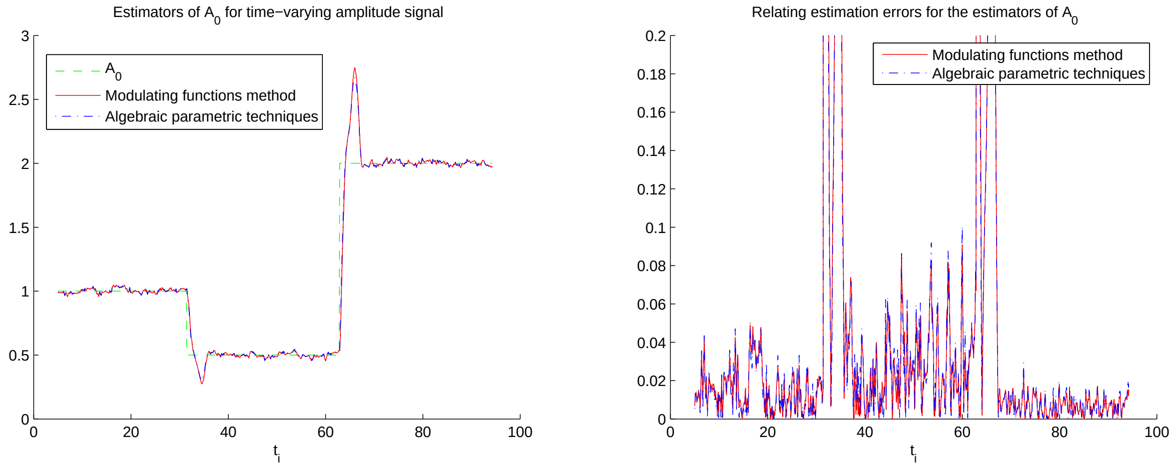
(b) Relating estimation errors of ω

Figure 6.7: Signals and relating estimation errors of ω obtained with $\kappa = \mu = n = 0$, $\nu = 1$ and $m = 40$.



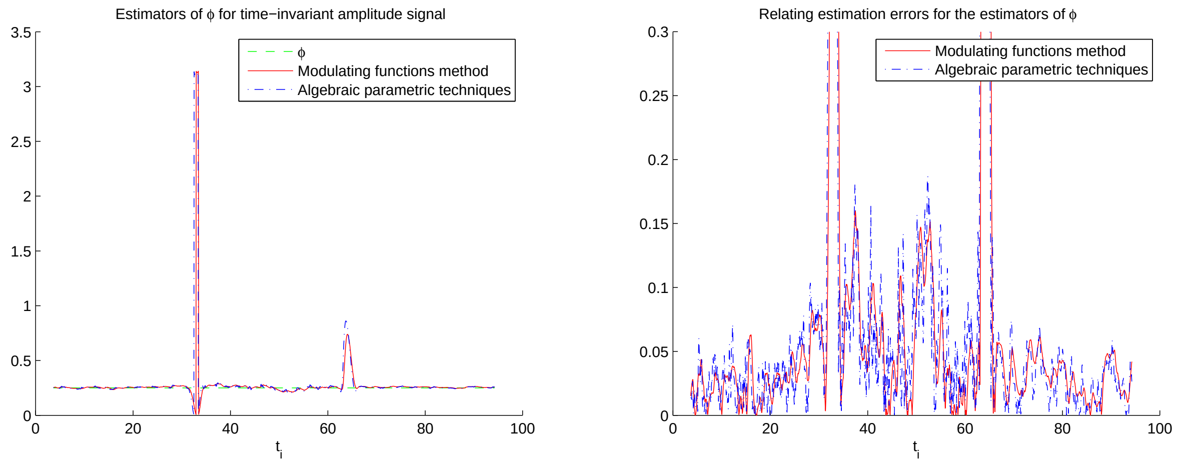
(a) Estimations of amplitude (b) Relating estimation errors of amplitude

Figure 6.8: The amplitude estimation obtained in Proposition 4.2.66 with $\mu = 0$, $m = 48$ and the one obtained in Proposition 4.3.71 with $g_1 \equiv w_{1,2}$, $g_2 \equiv w_{2,1}$ and $m = 58$.



(a) Estimations of amplitude (b) Relating estimation errors of amplitude

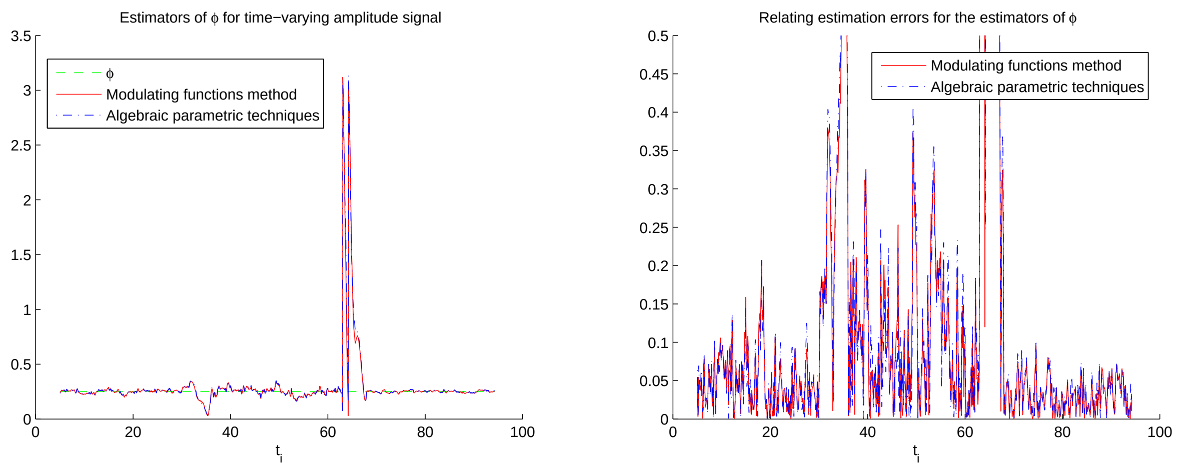
Figure 6.9: The amplitude estimation obtained in Proposition 4.2.68 with $\mu = 0$, $m = 80$ and the one obtained in Proposition 4.3.73 with $f_1 \equiv w_{1,0}$, $f_2 \equiv w_{0,1}$, $f_3 \equiv w_{2,1}$ and $f_4 \equiv w_{1,2}$ and $m = 80$.



(a) Estimations of phase

(b) Relating estimation errors of phase

Figure 6.10: The phase estimation obtained in Proposition 4.2.66 with $\mu = 0$, $m = 48$ and the one obtained in Proposition 4.3.71 with $g_1 \equiv w_{1,2}$, $g_2 \equiv w_{2,1}$ and $m = 58$.



(a) Estimations of phase

(b) Relating estimation errors of phase

Figure 6.11: The phase estimation obtained in Proposition 4.2.68 with $\mu = 0$, $m = 80$ and the one obtained in Proposition 4.3.73 with $f_1 \equiv w_{1,0}$, $f_2 \equiv w_{0,1}$, $f_3 \equiv w_{2,1}$ and $f_4 \equiv w_{1,2}$ and $m = 80$.

Proposition	m ($T_s = \frac{\pi}{50}$)	μ	κ	ν	n
4.3.70	40	0	0	1	0
4.3.72	40	0	0	1	0
4.2.66	48	0			
4.2.68	80	0			
4.3.71	58	$\mu_1 = 1$, $\mu_2 = 2$	$\kappa_1 = 2$, $\kappa_2 = 1$		0
4.3.73	80	$\mu_1 = 1$, $\mu_2 = 0$, $\mu_3 = 2$, $\mu_4 = 1$	$\kappa_1 = 0$, $\kappa_2 = 1$, $\kappa_3 = 1$, $\kappa_4 = 2$		0

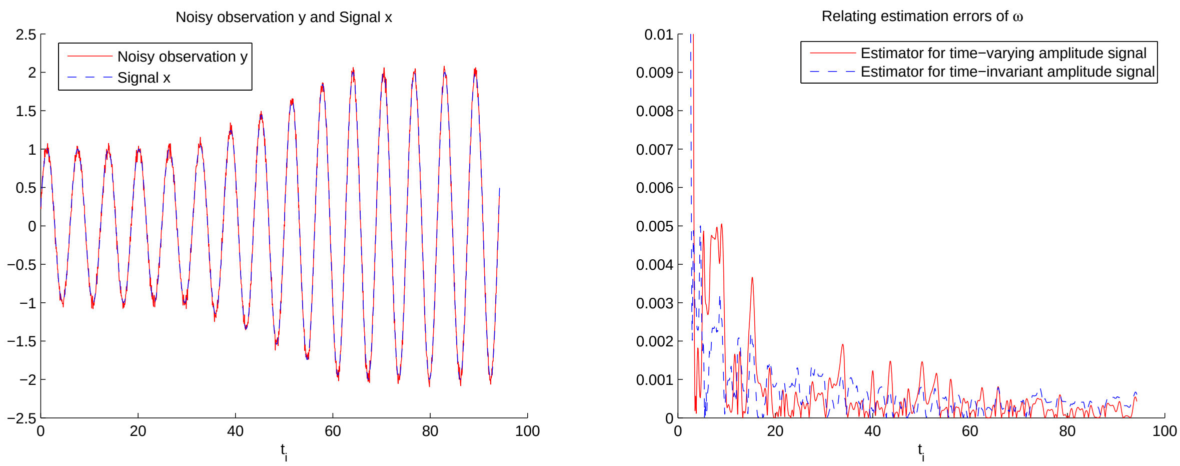
Table 6.3: Parameters used in Example 1.

Example 2

In this example, we change the amplitude of the signal defined in the previous example by taking

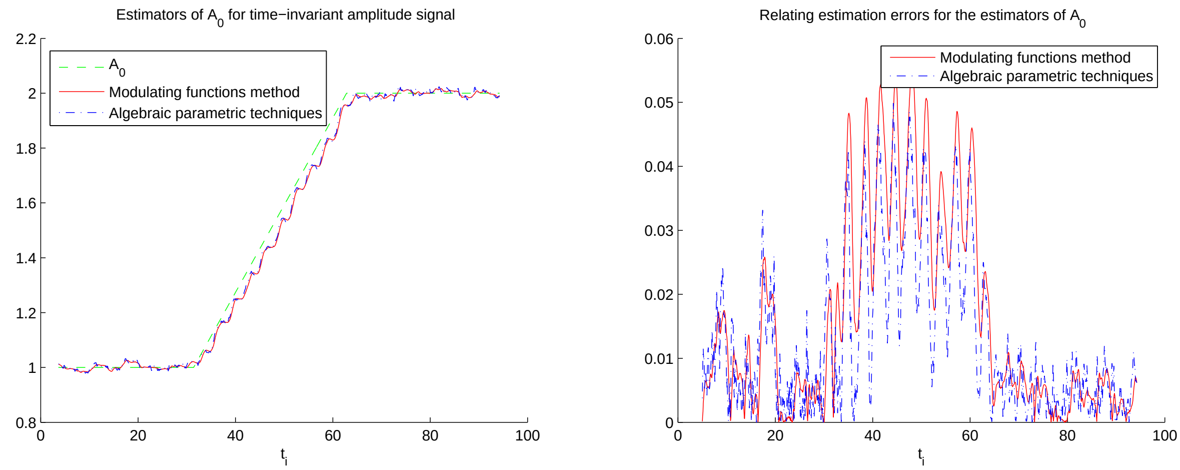
$$x(t_i) = \begin{cases} \sin(t_i + \frac{1}{4}), & \text{if } 0 \leq t_i \leq 10\pi, \\ \frac{t_i}{10}\sin(t_i + \frac{1}{4}), & \text{if } 10\pi < t_i \leq 20\pi, \\ 2\sin(t_i + \frac{1}{4}), & \text{if } 20\pi < t_i \leq 30\pi, \end{cases} \tag{6.12}$$

The signal-to-noise ratios of this signal (see Fig. 6.12) in each interval become equal to 23.5dB, 27.2dB and 28dB respectively. We use the same estimators with the same parameters to the ones used in the previous example to estimate the frequency, the amplitude and the phase. The relating estimation errors are shown in Fig. 6.12. Then, we use the frequency estimator obtained in Proposition 4.3.72 in the estimators of amplitude and phase. The estimations obtained by using the estimators obtained in the time-invariant amplitude case are shown in Fig. 6.13 and Fig. 6.15 with the corresponding relating estimation errors. We can see that when the amplitude of the signal changes slowly, the estimation errors become larger. The estimations obtained by using the estimators given in the time-varying amplitude case are shown in Fig. 6.14 and Fig. 6.16 with the corresponding relating estimation errors. We can see that when the amplitude of the signal changes slowly, the estimations of the amplitude are better than the ones obtained in Fig. 6.13. The noise errors for the estimations of the phase are important such that these estimations of the phase are not better than the ones obtained in Fig. 6.15. However, by decreasing the sampling period, we can reduce these noise error contributions. Consequently, if the amplitude of a noisy sinusoidal signal changes slowly, we shall use the estimators given in Proposition 4.2.68 and Proposition 4.3.73 to estimate its amplitude and phase. We recall in Table 6.4 all the parameters used in this example.



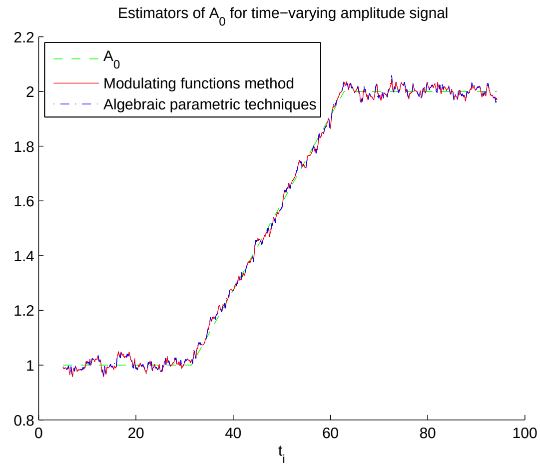
(a) Noisy observation y and the signal x (b) Relating estimation errors of ω

Figure 6.12: Signals and relating estimation errors of ω obtained with $\kappa = \mu = n = 0$, $\nu = 1$ and $m = 40$.

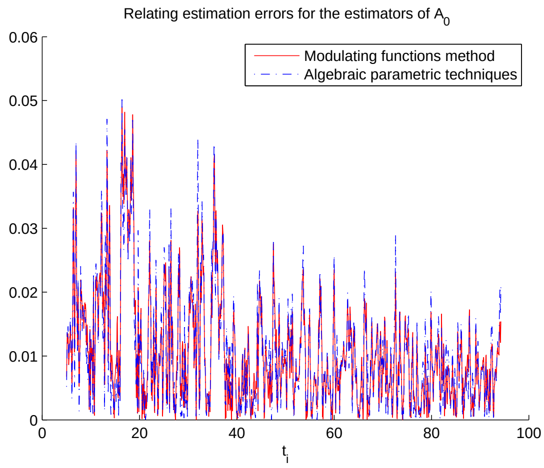


(a) Estimations of amplitude (b) Relating estimation errors of amplitude

Figure 6.13: The amplitude estimation obtained in Proposition 4.2.66 with $\mu = 0$, $m = 48$ and the one obtained in Proposition 4.3.71 with $g_1 \equiv w_{1,2}$, $g_2 \equiv w_{2,1}$ and $m = 58$.

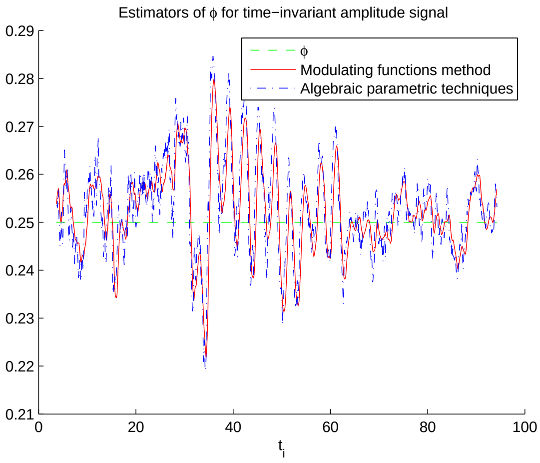


(a) Estimations of amplitude

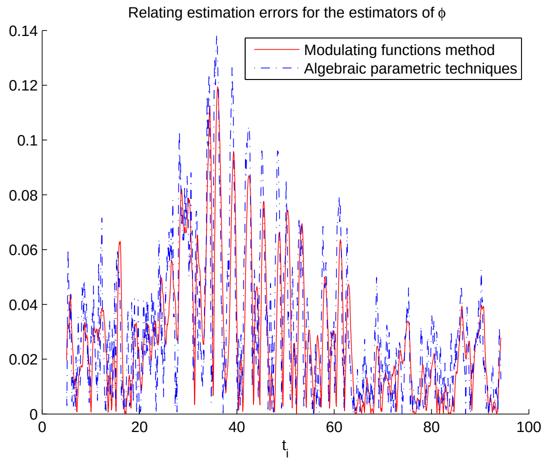


(b) Relating estimation errors of amplitude

Figure 6.14: The amplitude estimation obtained in Proposition 4.2.68 with $\mu = 0$, $m = 80$ and the one obtained in Proposition 4.3.73 with $f_1 \equiv w_{1,0}$, $f_2 \equiv w_{0,1}$, $f_3 \equiv w_{2,1}$ and $f_4 \equiv w_{1,2}$ and $m = 80$.

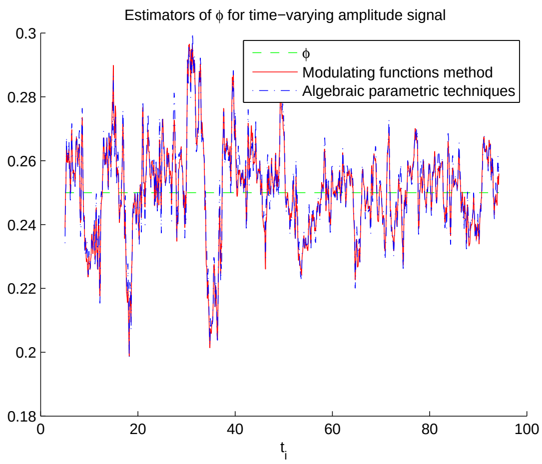


(a) Estimations of phase

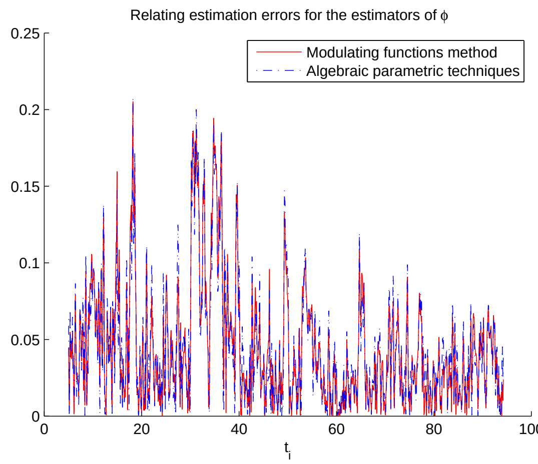


(b) Relating estimation errors of phase

Figure 6.15: The phase estimation obtained in Proposition 4.2.66 with $\mu = 0$, $m = 48$ and the one obtained in Proposition 4.3.71 with $g_1 \equiv w_{1,2}$, $g_2 \equiv w_{2,1}$ and $m = 58$.



(a) Estimations of phase



(b) Relating estimation errors of phase

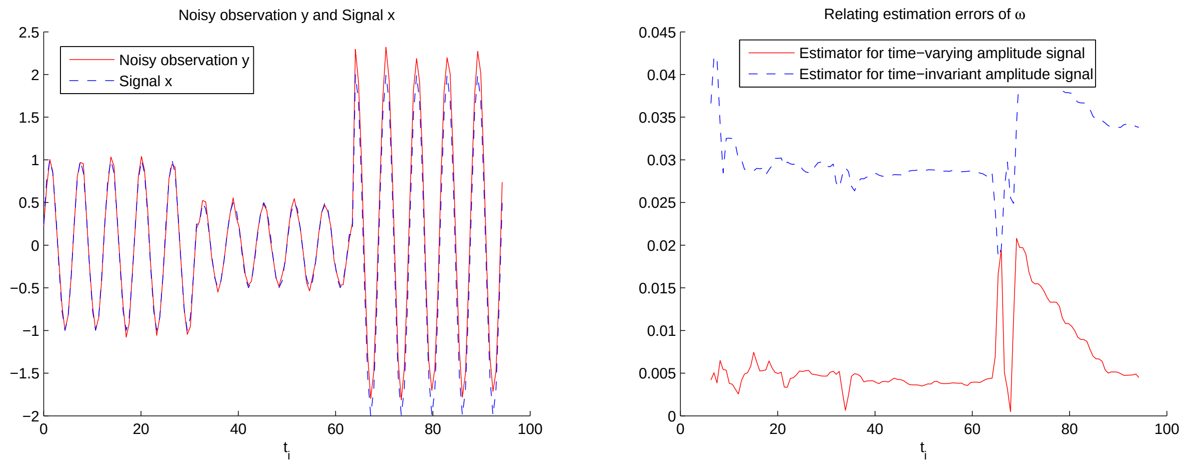
Figure 6.16: The phase estimation obtained in Proposition 4.2.68 with $\mu = 0$, $m = 80$ and the one obtained in Proposition 4.3.73 with $f_1 \equiv w_{1,0}$, $f_2 \equiv w_{0,1}$, $f_3 \equiv w_{2,1}$ and $f_4 \equiv w_{1,2}$ and $m = 80$.

Proposition	m ($T_s = \frac{\pi}{50}$)	μ	κ	ν	n
4.3.70	40	0	0	1	0
4.3.72	40	0	0	1	0
4.2.66	48	0			
4.2.68	80	0			
4.3.71	58	$\mu_1 = 1$, $\mu_2 = 2$	$\kappa_1 = 2$, $\kappa_2 = 1$		0
4.3.73	80	$\mu_1 = 1$, $\mu_2 = 0$, $\mu_3 = 2$, $\mu_4 = 1$	$\kappa_1 = 0$, $\kappa_2 = 1$, $\kappa_3 = 1$, $\kappa_4 = 2$		0

Table 6.4: Parameters used in Example 2.

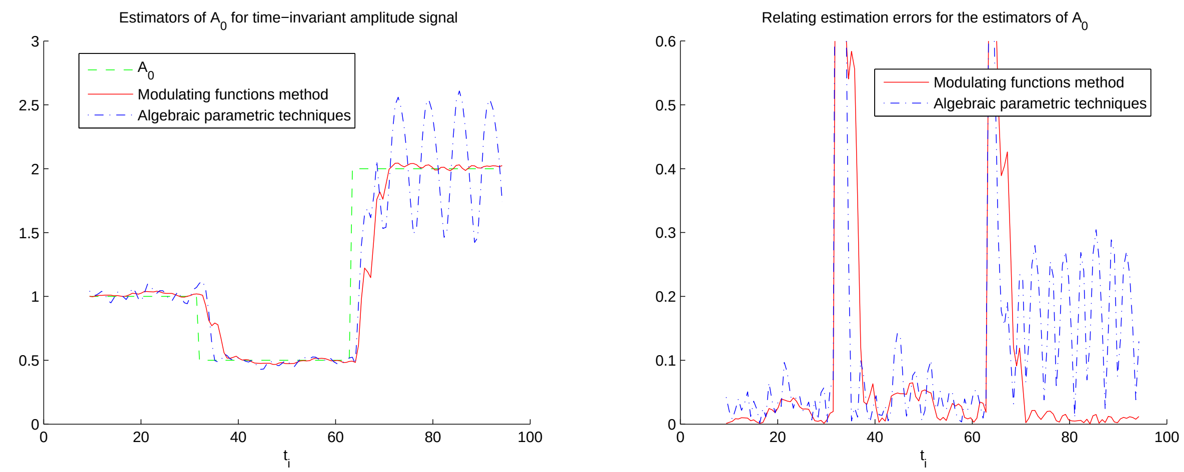
Example 3

In this example, we increase the sampling period of the signal defined in Example 1 to $T_s = \frac{\pi}{5}$. Moreover, we add a bias term perturbation $\xi = 0.25$ in the interval $[20\pi, 30\pi]$. The signal-to-noise ratios of this signal (see Fig. 6.17) in each interval become equal to 23.3dB, 19.4dB and 15.5dB respectively. In order to estimate the frequency, we use the same estimators used in the previous two examples. The relating estimation error are shown in Fig. 6.17 where we take $\kappa = \mu = 0$, $m = 10$ and $n = \nu = 1$. The value of n is set to 1 so as to annihilate the bias term perturbation. Then, we use the frequency estimation obtained by Proposition 4.3.72 in the estimators of amplitude and phase. By doing similar analysis to the one done in Section 6.5, we can get the "optimal" parameters for the estimators given in Proposition 4.2.66 are $\mu = 0$, $m = 7$. Since the sampling period become larger than the one in the previous examples, according to Section 5.2 we shall increase the value of κ and μ for the estimators given in Proposition 4.3.71 so as to reduce the numerical error. Then, when $n = 1$ we find the "optimal" values are $\kappa = 2$ and $m = 12$. The so obtained estimations for the amplitude and the phase are shown in Fig. 6.18 and Fig. 6.20 with the corresponding relating estimation errors. Then, we take $\mu = 0$ and $m = 15$ for the estimators for the amplitude and the phase given in Proposition 4.2.68 and $f_1 \equiv w_{3,2}^{(1)}$, $f_2 \equiv w_{2,3}^{(1)}$, $f_3 \equiv w_{3,4}^{(1)}$, $f_4 \equiv w_{4,3}^{(1)}$ with $m = 15$ for the ones given in Proposition 4.3.73. The obtained estimations are shown in Fig. 6.19 and Fig. 6.21 with the corresponding relating estimation errors. We can observe that the modulating functions method is more robust to the sampling period and to the non zero-mean noise than the algebraic parametric techniques. We recall in Table 6.5 all the parameters used in this example.



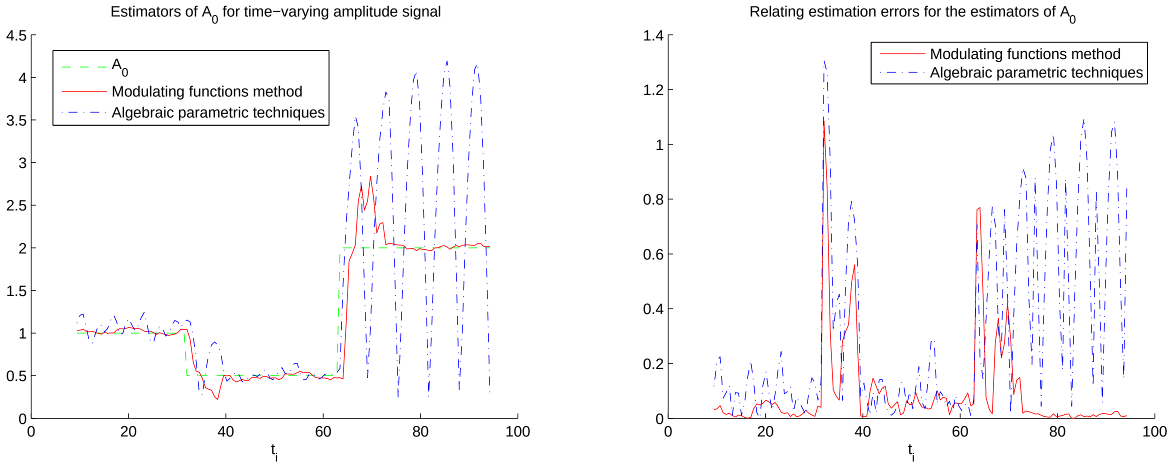
(a) Noisy observation y and the signal x (b) Relating estimation errors of ω

Figure 6.17: Signals and relating estimation errors of ω obtained with $\kappa = \mu = 0$, $n = \nu = 1$ and $m = 10$.



(a) Estimations of amplitude (b) Relating estimation errors of amplitude

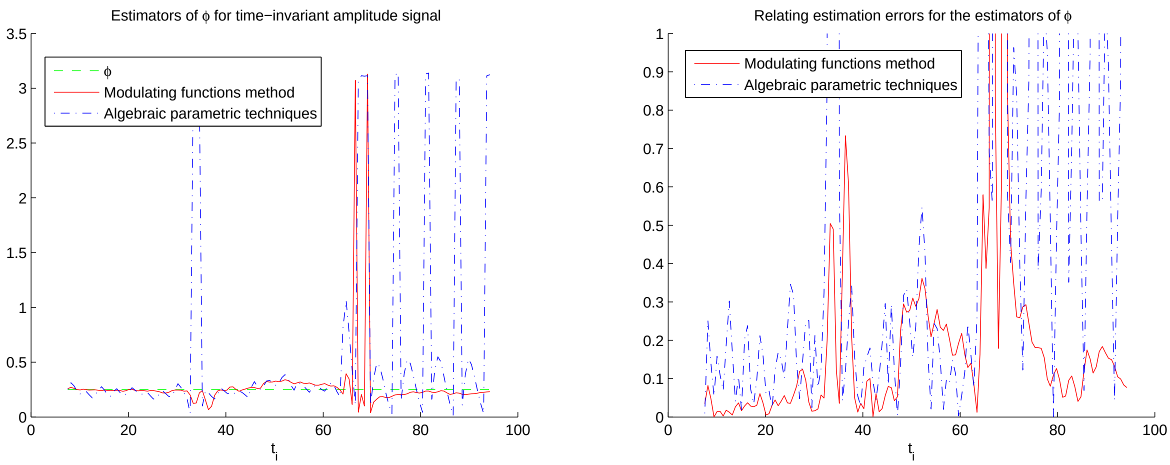
Figure 6.18: The amplitude estimation obtained in Proposition 4.2.66 with $\mu = 0$, $m = 7$ and the one obtained in Proposition 4.3.71 with $g_1 \equiv w_{2,3}^{(1)}$, $g_2 \equiv w_{3,2}^{(1)}$ and $m = 12$.



(a) Estimations of amplitude

(b) Relating estimation errors of amplitude

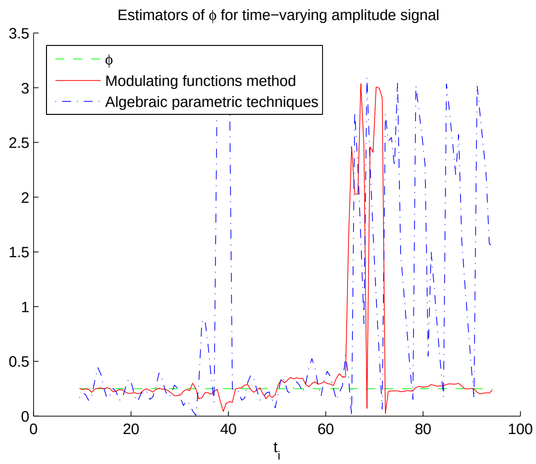
Figure 6.19: The amplitude estimation obtained in Proposition 4.2.68 with $\mu = 0$, $m = 15$ and the one obtained in Proposition 4.3.73 with $f_1 \equiv w_{3,2}^{(1)}$, $f_2 \equiv w_{2,3}^{(1)}$, $f_3 \equiv w_{3,4}^{(1)}$, $f_4 \equiv w_{4,3}^{(1)}$ and $m = 15$.



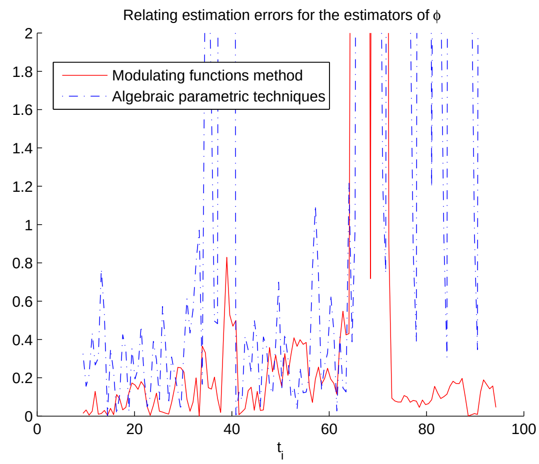
(a) Estimations of phase

(b) Relating estimation errors of phase

Figure 6.20: The phase estimation obtained in Proposition 4.2.66 with $\mu = 0$, $m = 7$ and the one obtained in Proposition 4.3.71 with $g_1 \equiv w_{2,3}^{(1)}$, $g_2 \equiv w_{3,2}^{(1)}$ and $m = 12$.



(a) Estimations of phase



(b) Relating estimation errors of phase

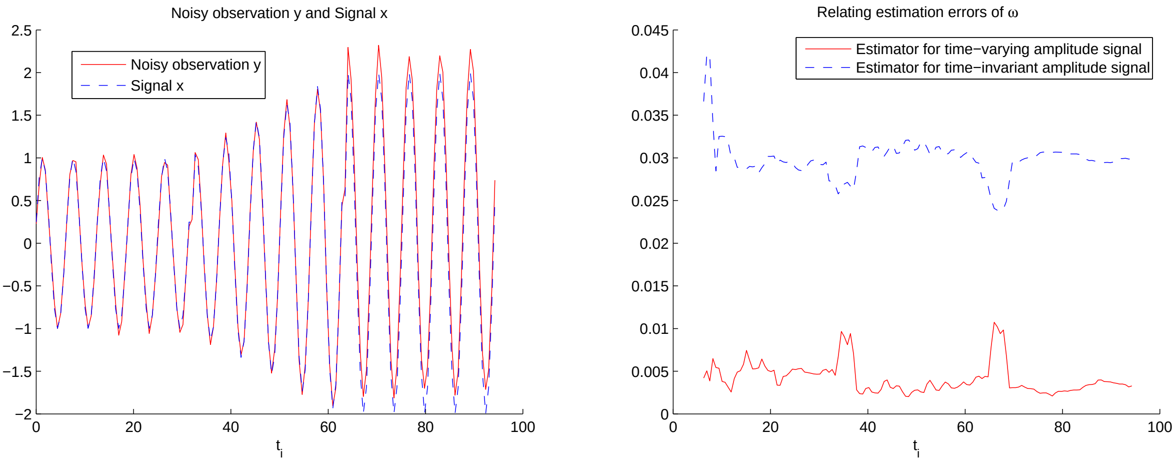
Figure 6.21: The phase estimation obtained in Proposition 4.2.68 with $\mu = 0$, $m = 15$ and the one obtained in Proposition 4.3.73 with $f_1 \equiv w^{(1)}_{3,2}$, $f_2 \equiv w^{(1)}_{2,3}$, $f_3 \equiv w^{(1)}_{3,4}$, $f_4 \equiv w^{(1)}_{4,3}$ and $m = 15$.

Proposition	m ($T_s = \frac{\pi}{5}$)	μ	κ	ν	n
4.3.70	10	0	0	1	1
4.3.72	10	0	0	1	1
4.2.66	7	0			
4.2.68	15	0			
4.3.71	12	$\mu_1 = 2,\ \mu_2 = 3$	$\kappa_1 = 3,\ \kappa_2 = 2$		1
4.3.73	15	$\mu_1 = 3,\ \mu_2 = 2,\ \mu_3 = 3,\ \mu_4 = 4$	$\kappa_1 = 2,\ \kappa_2 = 3,\ \kappa_3 = 4,\ \kappa_4 = 3$		1

Table 6.5: Parameters used in Example 3.

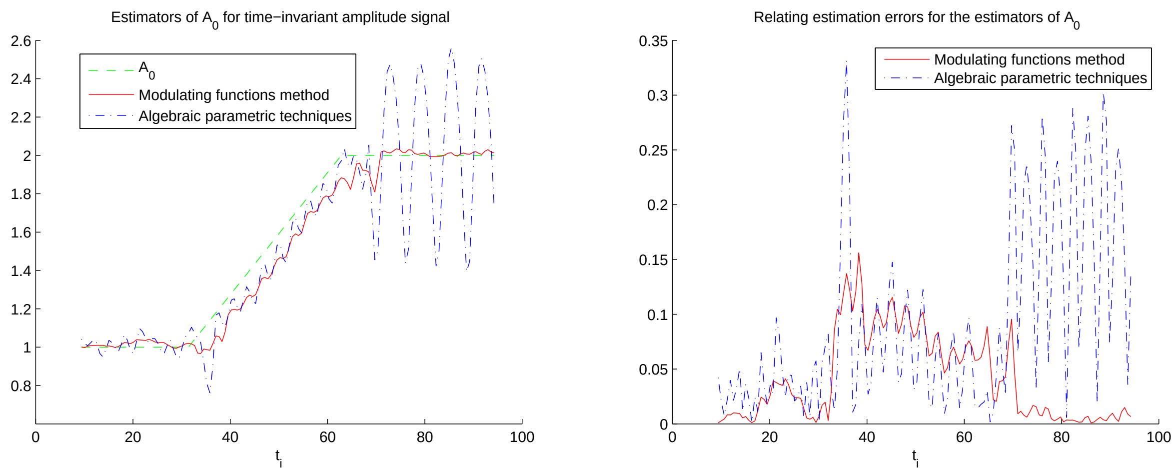
Example 4

In this example, we increase the sampling period of the signal defined in Example 2 to $T_s = \frac{\pi}{5}$, and add a bias term perturbation $\xi = 0.25$ in the interval $[20\pi, 30\pi]$. The signal-to-noise ratios of this signal (see Fig. 6.22) in each interval become equal to 23.3dB, 29dB and 17.2dB respectively. We use the same estimators with the same parameters to the ones used in Example 3 to estimate the frequency, the amplitude and the phase. The relating estimation errors are shown in Fig. 6.22. Then, we can see the estimations of the amplitude and the phase with the corresponding relating estimation errors in Fig.6.23-Fig.6.26 where we use the frequency estimator obtained in Proposition 4.3.72. Consequently, we can observe that the estimator of amplitude given in Proposition 4.3.73 and the estimator of phase given in Proposition 4.3.71 are most appropriate for this signal. We recall in Table 6.6 all the parameters used in this example.



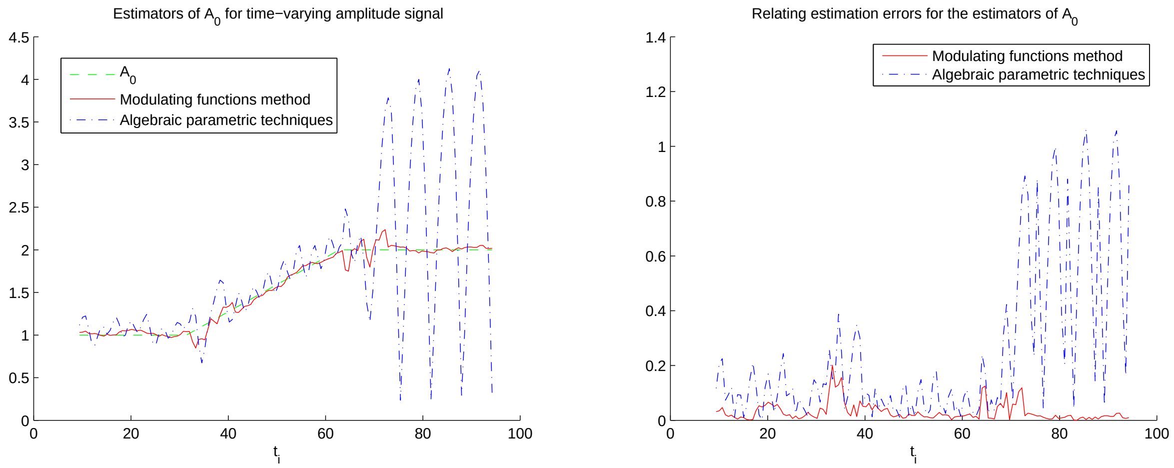
(a) Noisy observation y and the signal x (b) Relating estimation errors of ω

Figure 6.22: Signals and relating estimation errors of ω obtained with $\kappa = \mu = 0$, $n = \nu = 1$ and $m = 10$.



(a) Estimations of amplitude (b) Relating estimation errors of amplitude

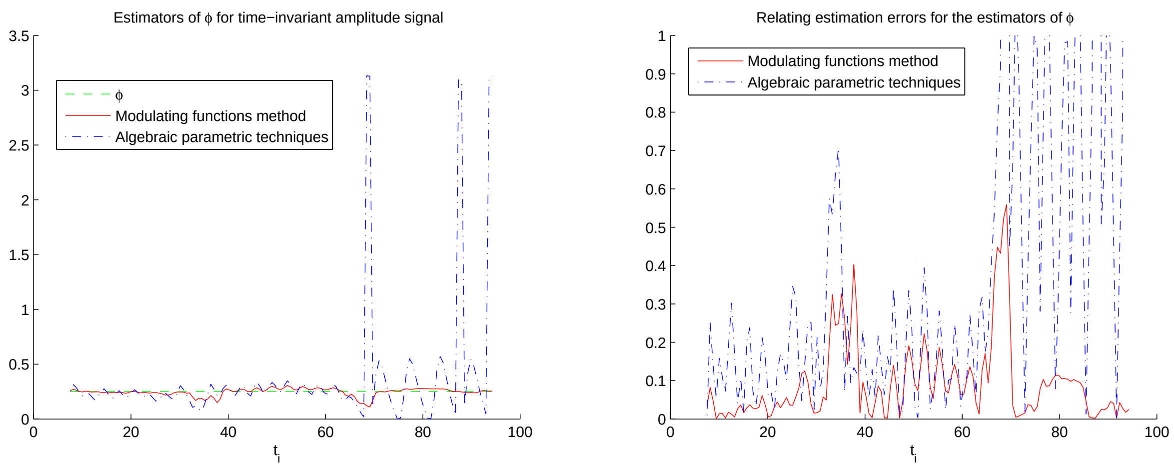
Figure 6.23: The amplitude estimation obtained in Proposition 4.2.66 with $\mu = 0$, $m = 7$ and the one obtained in Proposition 4.3.71 with $g_1 \equiv w^{(1)}_{2,3}$, $g_2 \equiv w^{(1)}_{3,2}$ and $m = 12$.



(a) Estimations of amplitude

(b) Relating estimation errors of amplitude

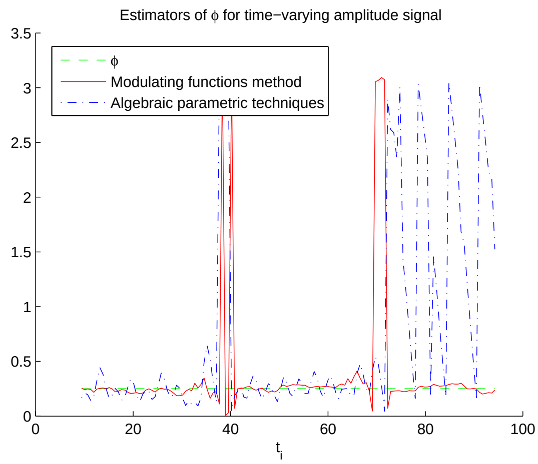
Figure 6.24: The amplitude estimation obtained in Proposition 4.2.68 with $\mu = 0$, $m = 15$ and the one obtained in Proposition 4.3.73 with $f_1 \equiv w_{3,2}^{(1)}$, $f_2 \equiv w_{2,3}^{(1)}$, $f_3 \equiv w_{3,4}^{(1)}$, $f_4 \equiv w_{4,3}^{(1)}$ and $m = 15$.



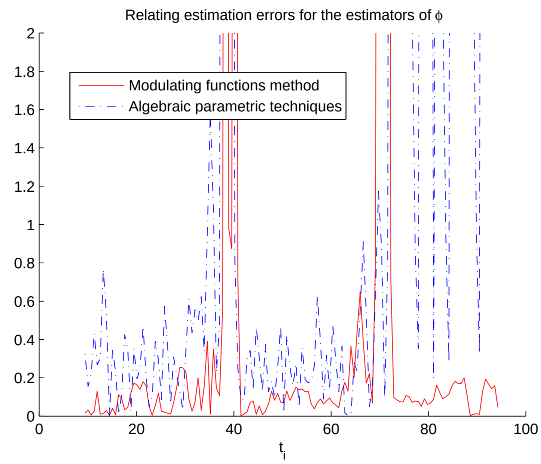
(a) Estimations of phase

(b) Relating estimation errors of phase

Figure 6.25: The phase estimation obtained in Proposition 4.2.66 with $\mu = 0$, $m = 7$ and the one obtained in Proposition 4.3.71 with $g_1 \equiv w_{2,3}^{(1)}$, $g_2 \equiv w_{3,2}^{(1)}$ and $m = 12$.



(a) Estimations of phase



(b) Relating estimation errors of phase

Figure 6.26: The phase estimation obtained in Proposition 4.2.68 with $\mu = 0$, $m = 15$ and the one obtained in Proposition 4.3.73 with $f_1 \equiv w_{3,2}^{(1)}$, $f_2 \equiv w_{2,3}^{(1)}$, $f_3 \equiv w_{3,4}^{(1)}$, $f_4 \equiv w_{4,3}^{(1)}$ and $m = 15$.

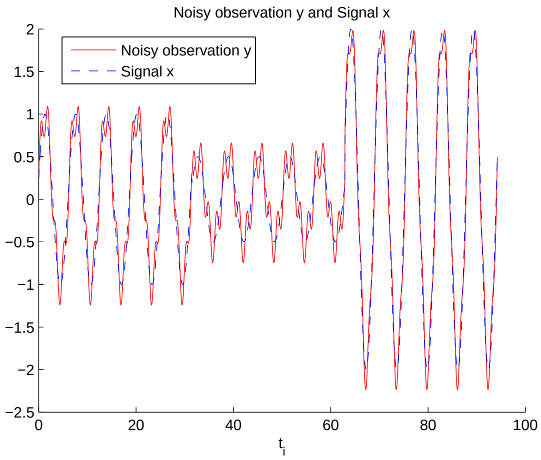
Proposition	m ($T_s = \frac{\pi}{5}$)	μ	κ	ν	n
4.3.70	10	0	0	1	1
4.3.72	10	0	0	1	1
4.2.66	7	0			
4.2.68	15	0			
4.3.71	12	$\mu_1 = 2,\ \mu_2 = 3$	$\kappa_1 = 3,\ \kappa_2 = 2$		1
4.3.73	15	$\mu_1 = 3,\ \mu_2 = 2,\ \mu_3 = 3,\ \mu_4 = 4$	$\kappa_1 = 2,\ \kappa_2 = 3,\ \kappa_3 = 4,\ \kappa_4 = 3$		1

Table 6.6: Parameters used in Example 4.

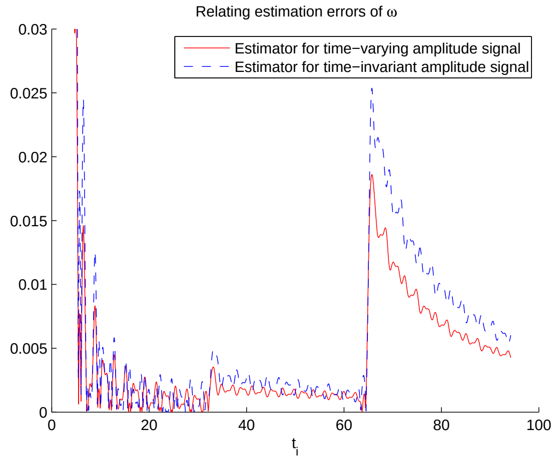
Example 5

In this example, we change the noise of the signal defined in Example 1 by a sinusoidal perpetration $\varpi = 0.25\sin(4t_i)$. The frequency of this sinusoidal perpetration is four times to the one of x. Hence, it can be considered as a low frequency sinusoidal perpetration. The signal-to-noise ratios of this signal (see Fig. 6.27) in each interval become equal to 12.3dB, 7dB and 15.4dB respectively. In order to estimate the frequency, we use the same estimators used in the previous examples. The relating estimation errors are shown in Fig. 6.27 where we take $\kappa = \mu = n = 0$, $m = 60$ and $\nu = 1$. According to Subsection 5.3.1, we can take large values for κ and μ in the estimators given by the modulating functions method so as to reduce this sinusoidal perpetration. Thus, we take $\kappa = 3$ and $n = 0$ for the estimators given in Proposition 4.3.71, and take $f_1 \equiv w_{4,5}$, $f_2 \equiv w_{5,4}$, $f_3 \equiv w_{5,6}$, $f_4 \equiv w_{6,5}$ for the estimators given in Proposition 4.3.73. Then, we take $m = 80$ in the estimators obtained in the time-invariant amplitude case and take $m = 120$ in the estimators obtained in the time-varying amplitude case. Moreover, we take $\mu = 0$ for the estimators obtained by the algebraic parametric techniques. Then, we can see the estimations of the amplitude and the phase with the corresponding relating estimation errors in Fig.6.28-Fig.6.31 where we use the frequency estimator obtained by Proposition

4.3.72. We can observe that the modulating functions method is more robust to the low frequency sinusoidal perpetration than the algebraic parametric techniques. Moreover, the estimators given in Proposition 4.3.71 are most appropriate for this signal. We recall in Table 6.7 all the parameters used in this example.

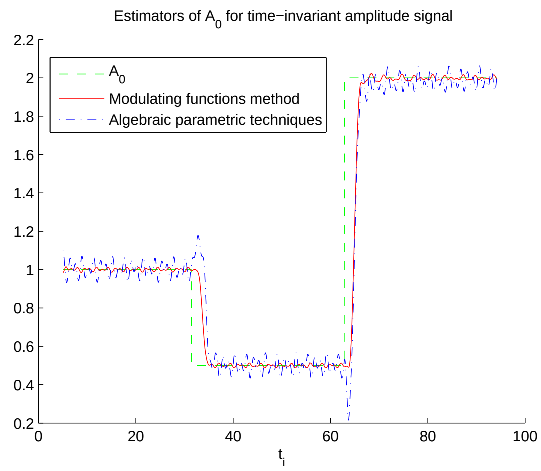


(a) Noisy observation y and the signal x

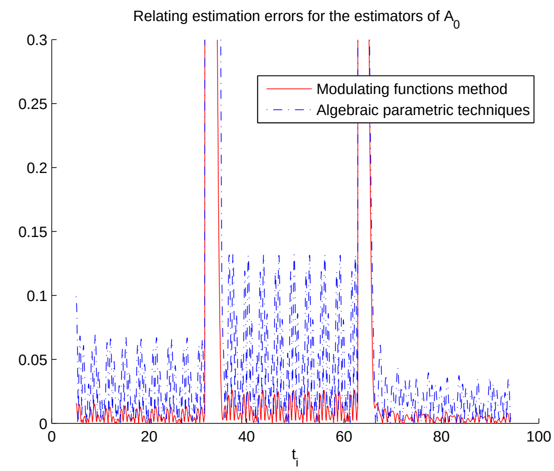


(b) Relating estimation errors of ω

Figure 6.27: Signals and relating estimation errors of ω obtained with $\kappa = \mu = n = 0$, $\nu = 1$ and $m = 60$.

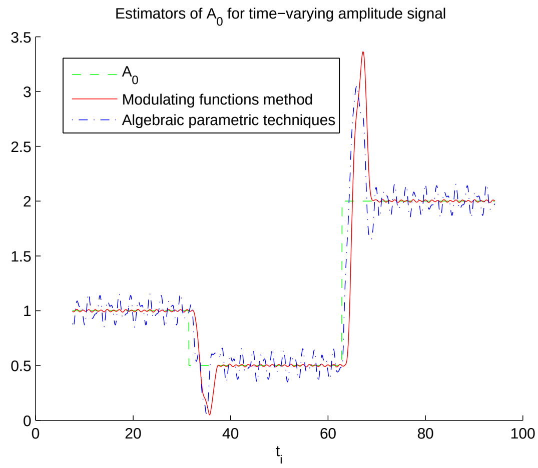


(a) Estimations of amplitude

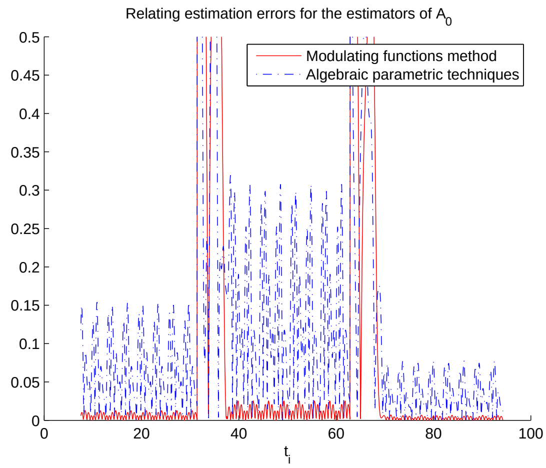


(b) Relating estimation errors of amplitude

Figure 6.28: The amplitude estimation obtained in Proposition 4.2.66 with $\mu = 0$, $m = 80$ and the one obtained in Proposition 4.3.71 with $g_1 \equiv w_{3,4}$, $g_2 \equiv w_{4,3}$ and $m = 80$.

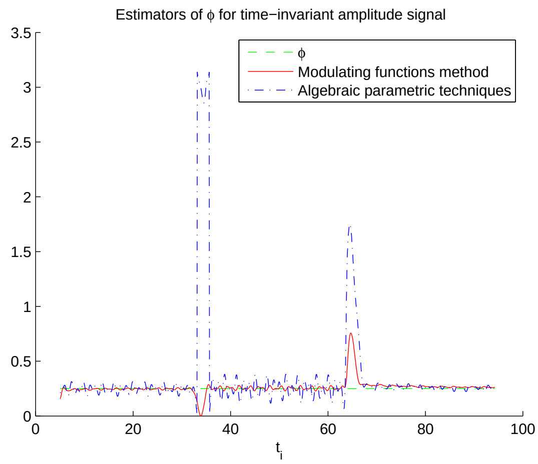


(a) Estimations of amplitude

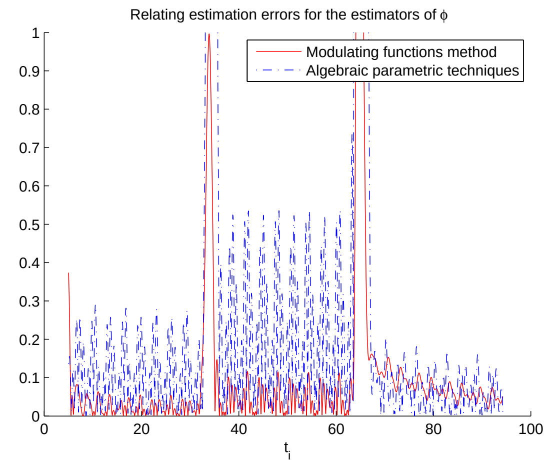


(b) Relating estimation errors of amplitude

Figure 6.29: The amplitude estimation obtained in Proposition 4.2.68 with $\mu = 0$, $m = 120$ and the one obtained in Proposition 4.3.73 with $f_1 \equiv w_{4,5}$, $f_2 \equiv w_{5,4}$, $f_3 \equiv w_{5,6}$, $f_4 \equiv w_{6,5}$ and $m = 120$.

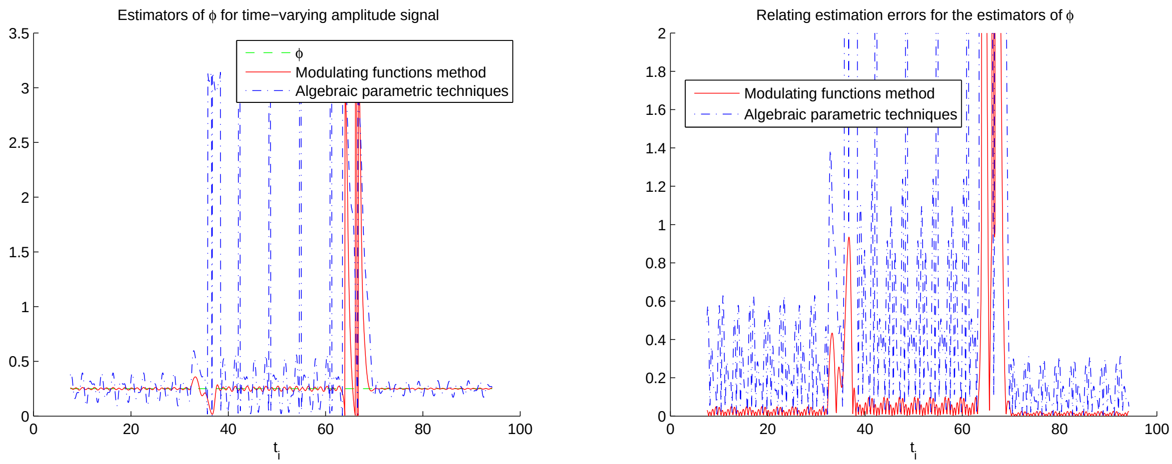


(a) Estimations of phase



(b) Relating estimation errors of phase

Figure 6.30: The phase estimation obtained in Proposition 4.2.66 with $\mu = 0$, $m = 80$ and the one obtained in Proposition 4.3.71 with $g_1 \equiv w_{3,4}$, $g_2 \equiv w_{4,3}$ and $m = 80$.



(a) Estimations of phase (b) Relating estimation errors of phase

Figure 6.31: The phase estimation obtained in Proposition 4.2.68 with $\mu = 0$, $m = 120$ and the one obtained in Proposition 4.3.73 with $f_1 \equiv w_{4,5}$, $f_2 \equiv w_{5,4}$, $f_3 \equiv w_{5,6}$, $f_4 \equiv w_{6,5}$ and $m = 120$.

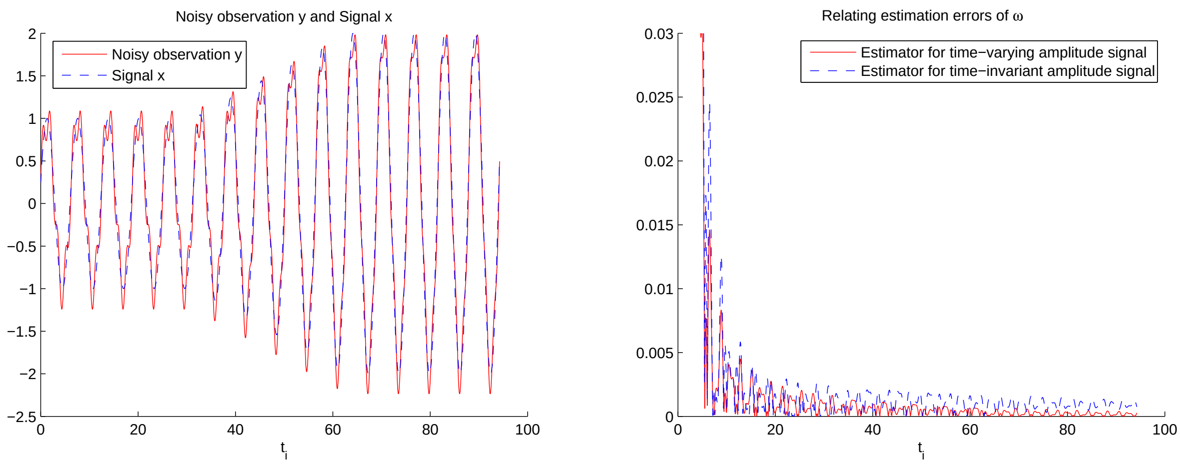
Proposition	m ($T_s = \frac{\pi}{50}$)	μ	κ	ν	n
4.3.70	60	0	0	1	0
4.3.72	60	0	0	1	0
4.2.66	80	0			
4.2.68	120	0			
4.3.71	80	$\mu_1 = 3$, $\mu_2 = 4$	$\kappa_1 = 4$, $\kappa_2 = 3$		0
4.3.73	120	$\mu_1 = 4$, $\mu_2 = 5$, $\mu_3 = 5$, $\mu_4 = 6$	$\kappa_1 = 5$, $\kappa_2 = 4$, $\kappa_3 = 6$, $\kappa_4 = 5$		0

Table 6.7: Parameters used in Example 5.

Example 6

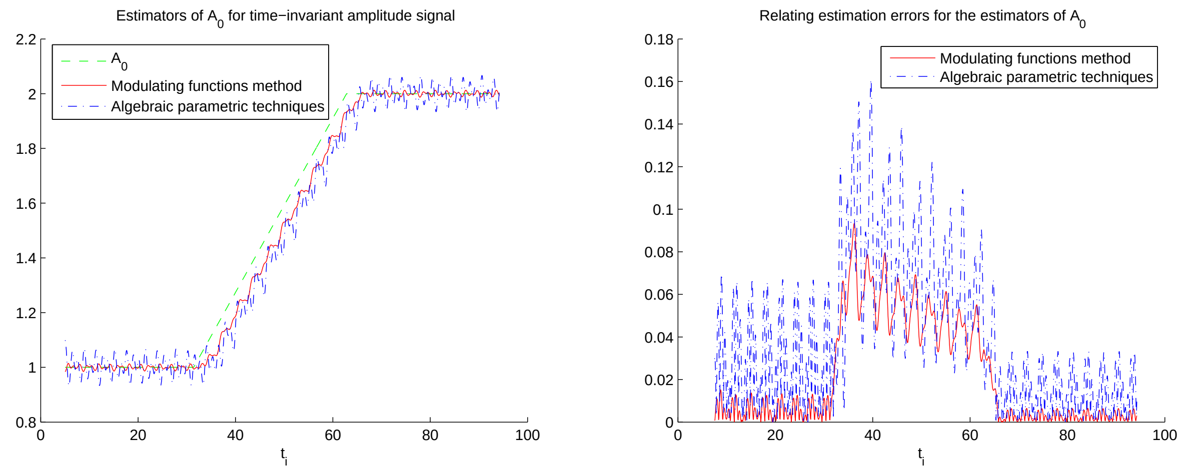
In this example, we change the noise of the signal defined in Example 2 by the same low frequency sinusoidal perpetration defined in Example 5. The signal-to-noise ratios of this signal (see Fig. 6.32) in each interval become equal to 12.3dB, 15.8dB and 17.1dB respectively. We use the same estimators with the same parameters to the ones used in Example 5 to estimate the frequency, the amplitude and the phase. The relating estimation errors are shown in Fig. 6.32. We can see the estimations of the amplitude and the phase with the corresponding relating estimation errors in Fig.6.33-Fig.6.36 where we use the frequency estimator obtained by Proposition 4.3.72. We can observe that the estimators given in Proposition 4.3.73 are most appropriate for this signal. We recall in Table 6.8 all the parameters used in this example.

Let us remark that if the frequency ω is known, then the period of the function $A_0 \sin(\omega \cdot + \phi)$ is equal to $\frac{2\pi}{\omega}$. Then, we set $T_s = \frac{2\pi}{\omega N}$, where $N \in \mathbb{N}^*$ is the number of sampling data per period. By observing the estimators of A_0 and ϕ given by Proposition 4.2.66, Proposition 4.2.68, Proposition 4.3.71 and Proposition 4.3.73, we can find out that they do not depend only on ω but on $T\omega$. Hence, by taking $T\omega = \frac{2\pi m}{N}$ with $T = mT_s$ we can conclude that these estimators do not depend on the value of ω but on $\frac{m}{N}$. Consequently, if we change the value of ω in the previous examples, then by taking the same estimators with the same parameters used previously we can obtain similar estimation results to the ones shown in the the previous examples.



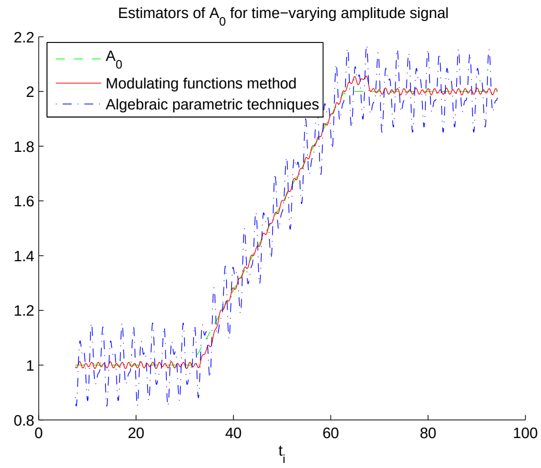
(a) Noisy observation y and the signal x (b) Relating estimation errors of ω

Figure 6.32: Signals and relating estimation errors of ω obtained with $\kappa = \mu = n = 0$, $\nu = 1$ and $m = 60$.

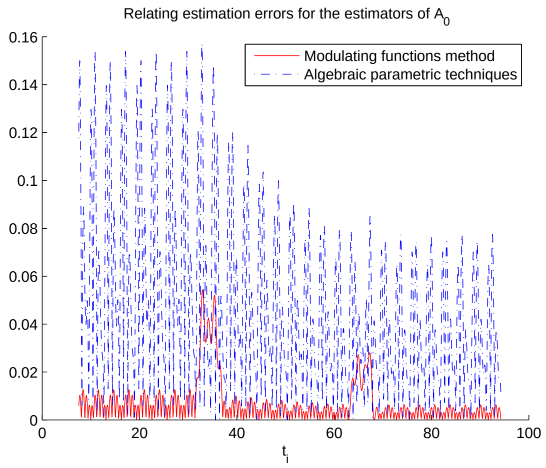


(a) Estimations of amplitude (b) Relating estimation errors of amplitude

Figure 6.33: The amplitude estimation obtained in Proposition 4.2.66 with $\mu = 0$, $m = 80$ and the one obtained in Proposition 4.3.71 with $g_1 \equiv w_{3,4}$, $g_2 \equiv w_{4,3}$ and $m = 80$.

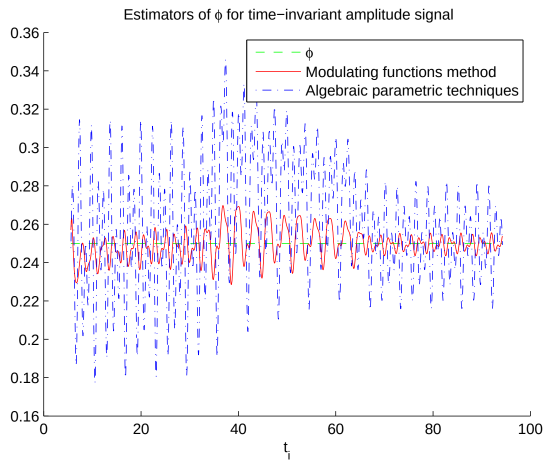


(a) Estimations of amplitude

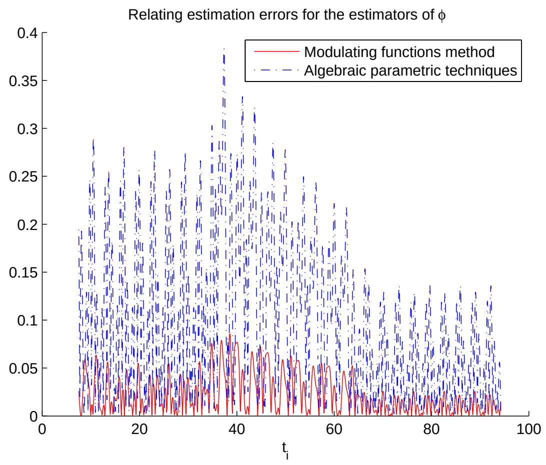


(b) Relating estimation errors of amplitude

Figure 6.34: The amplitude estimation obtained in Proposition 4.2.68 with $\mu = 0$, $m = 120$ and the one obtained in Proposition 4.3.73 with $f_1 \equiv w_{4,5}$, $f_2 \equiv w_{5,4}$, $f_3 \equiv w_{5,6}$, $f_4 \equiv w_{6,5}$ and $m = 120$.

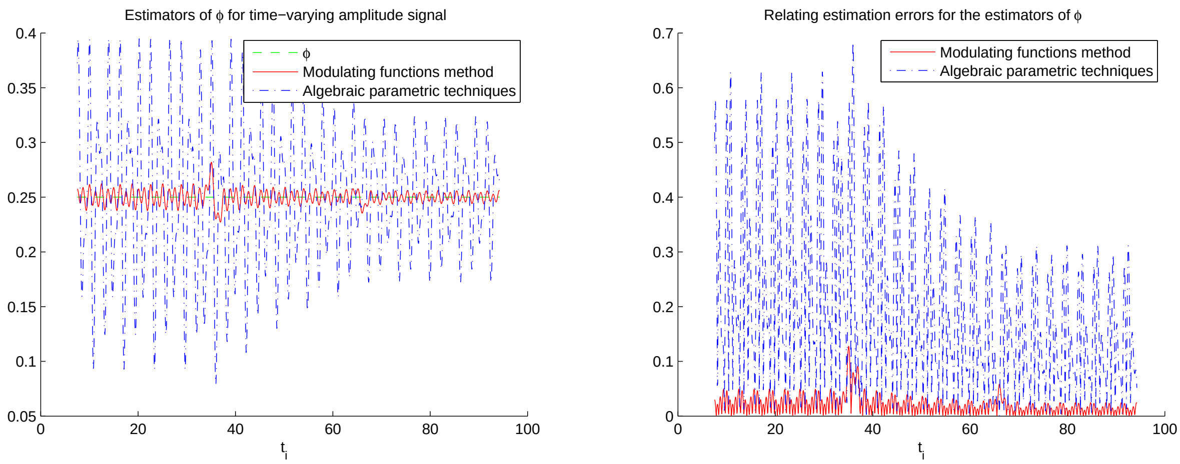


(a) Estimations of phase



(b) Relating estimation errors of phase

Figure 6.35: The phase estimation obtained in Proposition 4.2.66 with $\mu = 0$, $m = 80$ and the one obtained in Proposition 4.3.71 with $g_1 \equiv w_{3,4}$, $g_2 \equiv w_{4,3}$ and $m = 80$.



(a) Estimations of phase

(b) Relating estimation errors of phase

Figure 6.36: The phase estimation obtained in Proposition 4.2.68 with $\mu = 0$, $m = 120$ and the one obtained in Proposition 4.3.73 with $f_1 \equiv w_{4,5}$, $f_2 \equiv w_{5,4}$, $f_3 \equiv w_{5,6}$, $f_4 \equiv w_{6,5}$ and $m = 120$.

Proposition	m ($T_s = \frac{\pi}{50}$)	μ	κ	ν	n
4.3.70	60	0	0	1	0
4.3.72	60	0	0	1	0
4.2.66	80	0			
4.2.68	120	0			
4.3.71	80	$\mu_1 = 3,\ \mu_2 = 4$	$\kappa_1 = 4,\ \kappa_2 = 3$		0
4.3.73	120	$\mu_1 = 4,\ \mu_2 = 5,\ \mu_3 = 5,\ \mu_4 = 6$	$\kappa_1 = 5,\ \kappa_2 = 4,\ \kappa_3 = 6,\ \kappa_4 = 5$		0

Table 6.8: Parameters used in Example 6.

6.7 Conclusion

In this chapter, firstly we respectively explained how to apply the frequency, amplitude and phase estimators obtained in Chapter 4 in our identification procedure. Secondly, by taking a time-invariant amplitude sinusoidal signal corrupted by a zero-mean white gaussian noise we applied the error bounds obtained in Chapter 4 to select parameters for our estimators. Thirdly, some numerical examples have been given in the time-invariant amplitude and time-varying amplitude cases to show the efficiency and stability and to compare different estimators. It is shown that the estimators obtained by using the modulating functions method are more robust to a large sampling period, a biased noise and a sinusoidal perturbation than the ones obtained by using the algebraic parametric techniques. In the following chapter, we give some experimental results by applying the amplitude estimator obtained by using the modulating functions method.

Chapter 7

Applications to the AFM in tapping mode

Contents

7.1 Introduction

In this chapter, we give some experimental results obtained at LNE (Laboratoire National de métrologie et d'Essais) by applying our amplitude estimators previously presented. These results are based on the comparison of our results with respect to a DSP lock-in amplifier. Such device is usually used as an amplitude detector for the atomic force microscopy in tapping mode up to date industrial solution. This chapter is organized as follows. In Subsection 7.1.1, we present the atomic force microscopy in tapping mode. In Subsection 7.1.2, we give the basic principles of a lock-in amplifier. In Subsection 7.2.1, we recall the materials used in our experimental tests. Finally, we give our experimental results in Subsection 7.2.2.

7.1.1 Atomic force microscopy in tapping mode

Atomic force microscopy (AFM) is a very high resolution type of scanning probe microscopy, with demonstrated resolution on the order of fractions of a nanometer, more than 1000 times better than the optical diffraction limit. The precursor to the AFM, the scanning tunneling microscope (STM), was developed by Gerd Binnig and Heinrich Rohrer in the early 1980s, a development that earned them the Nobel Prize for Physics in 1986. Binnig, Quate and Gerber invented the first AFM in 1986.

The AFM was developed to overcome a basic drawback with STM which can only image conducting or semiconducting surfaces. The AFM, however, has the advantage of imaging almost any type of surface, including polymers, ceramics, composites, glass, and biological samples.

The original AFM consisted in a diamond shard attached to a strip of gold foil. The diamond tip contacted the surface directly, with the interatomic van der Waals forces providing the interaction mechanism. Detection of the cantilever's vertical movement was done with a second tip - a STM placed above the cantilever. Today, most AFMs use a laser beam deflection system, introduced by Meyer and Amer, where a laser is reflected from the back of the reflective AFM lever and onto a position-sensitive detector. To avoid that the tip crushes into the sample surface and damages the sample and/or the delicate tip a fast feedback electronic is used to maintain a constant force between the tip and the sample and therefore a resultant constant bending of the cantilever (see Figure 7.1). AFM tips and cantilevers are microfabricated from Si or Si_3N_4. Typical tip radius ranges from 1 to $100nm$.

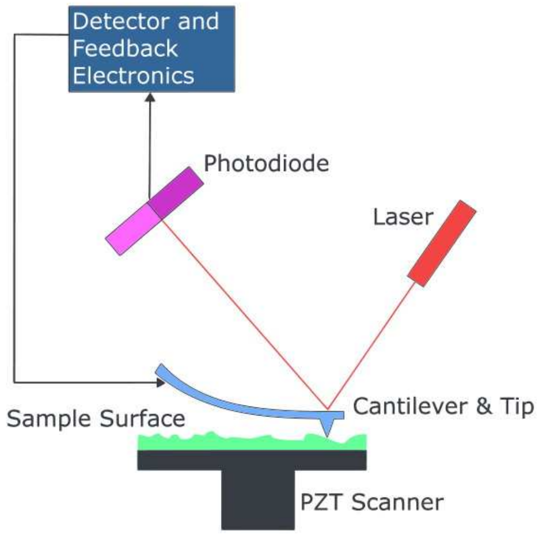


Figure 7.1: Schematic assembly of an AFM.

Because the AFM relies on the forces between the tip and sample, knowing these forces is important for proper imaging. The force is not measured directly, but calculated by measuring the deflection of the cantilever, and knowing its stiffness. Hook's law gives $F = -kz$, where F is the force, k is the stiffness of the cantilever, and z is the distance between the tip of cantilever and the sample (see Figure 7.2).

Since there are many types of tip-surface interactions, different types of interaction maps may be obtained. Furthermore, different types of maps require slightly different tip-surface positioning: sometimes the tip is scanned in contact with the surface, sometimes it is scanned in non-contact, and sometimes it is run in occasional contact (so-called tapping mode), which is considered here.

Tapping mode is a key advance in AFM. This powerful technique allows high resolution topographic imaging of sample surfaces that are easily damaged, loosely hold to their substrate, or difficult to image by other AFM techniques. Tapping mode overcomes problems associated with friction, adhesion,

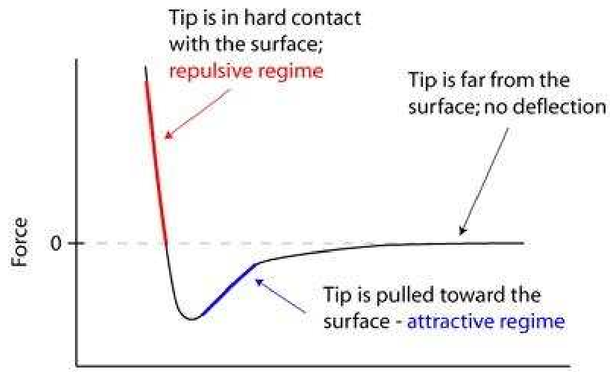


Figure 7.2: Probe Distance from Sample (z distance).

electrostatic forces, and other difficulties that an conventional AFM scanning methods by alternately placing the tip in contact with the surface to provide high resolution and then lifting the tip from the surface to avoid dragging the tip across the surface.

Tapping mode imaging is implemented in ambient air by oscillating the cantilever assembly at or near the cantilever's resonant frequency using a piezoelectric crystal. The piezo motion causes the cantilever to oscillate with a high amplitude (typically greater than $20nm$) when the tip is not in contact with the surface. The oscillating tip is then moved toward the surface until it begins to lightly touch, or tap the surface. During scanning, the vertically oscillating tip alternately contacts the surface and lifts off, generally at a frequency of $50,000$ to $500,000$ cycles per second. As the oscillating cantilever begins to intermittently contact the surface, the cantilever oscillation is necessarily reduced due to energy loss caused by the tip contacting the surface. The reduction in oscillation amplitude is used to identify and measure surface features.

During tapping mode operation, the cantilever oscillation amplitude is maintained constant by a feedback loop. Selection of the optimal oscillation frequency is software-assisted and the force on the sample is automatically set and maintained at the lowest possible level. When the tip passes over a bump in the surface, the cantilever has less room to oscillate and the amplitude of oscillation decreases. Conversely, when the tip passes over a depression, the cantilever has more room to oscillate and the amplitude increases (approaching the maximum free air amplitude). The oscillation amplitude of the tip is measured by a photodiode detector and it is used as an input for a controller. The digital feedback loop then adjusts the tip-sample separation to maintain a constant amplitude and force on the sample.

In the next subsection, we present lock-in amplifier which is an amplitude detector usually used by AFM.

7.1.2 Lock-in Amplifiers

Lock-in amplifiers are used to detect and measure very small Alternating Current (AC) signals. Accurate measurements may be obtained when the signal to be observed has an amplitude to ten thousands times smaller than the one of the noise measurement. All lock-in amplifiers, whether analogue or digital, use a technique known as phase sensitive detection to single out the component of the signal at a specific reference frequency and phase. Noise signals, at frequencies other than the reference frequency, are rejected and do not affect the measurement.

Now, we follow the amplifier with a phase sensitive detector (PSD). Lock-in measurements require a frequency reference. Typically, an experiment is excited at a fixed frequency (from an oscillator or function generator), and the lock-in detects the response from the experiment at the reference frequency. In Figure 7.3, the reference signal is a square wave at frequency ω_r. This might be the sync output from a function generator. If the sine output from the function generator is used to excite the experiment, the response might be the signal waveform shown below. The signal is $A_{sig}\sin(\omega_r t+\theta_{sig})$ where A_{sig} is the signal amplitude, ω_r is the signal frequency, and θ_{sig} is the signal's phase.

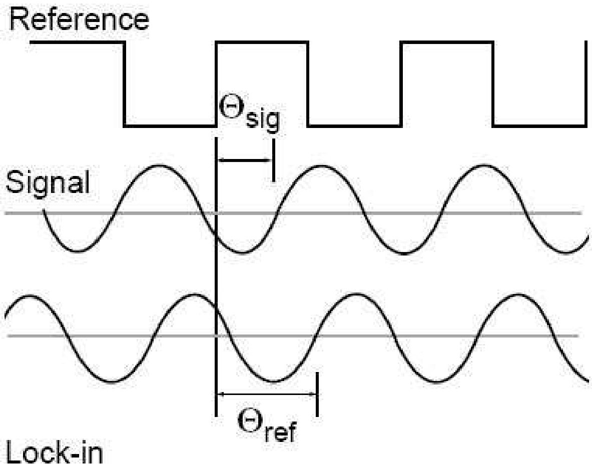


Figure 7.3: Phase Sensitive Detection.

Lock-in amplifiers generate their own internal reference signal usually by a phase-locked-loop locked to the external reference. In Figure 7.3, the external reference, the lock-in's reference, and the signal are all shown. The internal reference is $A_L\sin(\omega_L t+\theta_{ref})$. The lock-in amplifies the signal and then multiplies it by the lock-in reference using a phase sensitive detector or multiplier. The output of the PSD is simply the product of two sine waves:

$$\begin{aligned} A_{psd} &= A_{sig}A_L\sin(\omega_r t+\theta_{sig})\sin(\omega_L t+\theta_{ref}) \\ &= \frac{1}{2}A_{sig}A_L\cos\left((\omega_r-\omega_L)t+\theta_{sig}-\theta_{ref}\right)+\frac{1}{2}A_{sig}A_L\cos\left((\omega_r+\omega_L)t+\theta_{sig}+\theta_{ref}\right). \end{aligned} \tag{7.1}$$

The PSD output is two AC signals, one at the difference frequency $(\omega_r-\omega_L)$ and the other at the sum frequency $(\omega_r+\omega_L)$.

If the PSD output is passed through a low pass filter, the AC signals are removed. What will be left? In the general case, nothing. However, if ω_r equals ω_L, the difference frequency component will be a DC signal. In this case, the filtered PSD output will be:

$$\hat{A}_{psd} = \frac{1}{2}A_{sig}A_L\cos\left(\theta_{sig}-\theta_{ref}\right). \tag{7.2}$$

This is a very nice signal. It is a DC signal proportional to the signal amplitude.

It is important to consider the physical nature of this multiplication and filtering process in different types of lock-ins. In traditional analog lock-ins, the signal and reference are analog voltage signals. The signal and reference are multiplied in an analog multiplier, and the result is filtered with one or more stages of Resistor-Capacitor (RC) filters. In a digital lock-in, such as the $SR830$ or $SR850$, the signal and reference are represented by sequences of numbers. Multiplication and filtering are performed mathematically by a digital signal processing (DSP) chip.

The PSD output is proportional to $A_{sig}\cos\theta$, where $\theta = (\theta_{sig} - \theta_{ref})$. θ is the phase difference between the signal and the lock-in reference oscillator. By adjusting θ_{ref} we can make θ equal to zero. In which case we can measure A_{sig} ($\cos\theta = 1$). Conversely, if θ is $\frac{\pi}{2}$, there will be no output at all. A lock-in with a single PSD is called a single-phase lock-in and its output is $A_{sig}\cos\theta$. This phase dependency can be eliminated by adding a second PSD. If the second PSD multiplies the signal with the reference oscillator shifted by $\frac{\pi}{2}$, i.e. $A_L \sin(\omega_L t + \theta_{ref} + \frac{\pi}{2})$, its low pass filtered output will be:

$$\hat{A}_{psd2} = \frac{1}{2} A_{sig} A_L \sin\left(\theta_{sig} - \theta_{ref}\right). \tag{7.3}$$

Now we have two outputs: one proportional to $\cos\theta$ and the other proportional to $\sin\theta$. If we call the first output X and the second Y, $X = A_{sig}\cos\theta$, $Y = A_{sig}\sin\theta$, these two quantities represent the signal as a vector relative to the lock-in reference oscillator. X is called the "in-phase" component and Y the "quadrature" component. This is because when $\theta = 0$, X measures the signal while Y is zero. By computing the amplitude (A_{sig}) of the signal vector, the phase dependency is removed:

$$A_{sig} = (X^2 + Y^2)^{\frac{1}{2}}. \tag{7.4}$$

A_{sig} does not depend upon the phase between the signal and lock-in reference. A dual-phase lock-in has two PSDs with reference oscillators $\frac{\pi}{2}$ apart, and can measure X, Y and A_{sig} directly. In addition, the phase (θ) between the signal and lock-in is defined as:

$$\theta = \arctan\left(\frac{Y}{X}\right). \tag{7.5}$$

However, the main disadvantage of the lock-in amplifier is the limited speed at which we detect the amplitude. We consider in this chapter the model 7280 DSP Lock-in Amplifier which is an exceptionally versatile instrument with outstanding performance. With direct digital demodulation over an operating frequency extending up to $2.0MHz$, output filter time constants down to $1\mu s$ and a main Analog-to-Digital Converter (ADC) sampling rate of $7.5MHz$ it is ideal for recovering fast changing signals. But unlike some other high frequency lock-ins, it also works in the traditional audio frequency band.

7.2 Comparison of modulating function method and DSP lock-in amplifier

7.2.1 The experiment materials

We have obtained some experimental results at LNE. These experimental results are realized by using the following materials:

- Signal generator;
- PC with ADbasic;
- Real time target (Adwin gold) (see Figure 7.4);
- Signal Recovery 7280 DSP Lock-in Amplifier (see Figure 7.5);
- Oscilloscope (see Figure 7.6).

Figure 7.4: Adwin gold.

Figure 7.5: Signal Recovery 7280 DSP Lock-in Amplifier.

We use a signal generator to generate a sinusoidal signal, the amplitude of which we want to estimate. The code of our estimator is written with ADbasic on PC. ADbasic is the solution for flexible and simple programming of fast data acquisition, open-loop and closed-loop control procedures. The ADbasic programs are executed on the real-time CPU of the ADwin hardware after the occurrence of the generated signal. The ADwin CPU reacts to a new event within microsecond range. The processing of the event such as the calculation of a correction value is done with high-speed so that precise process response times (reaction times) of a few microseconds can be guaranteed. The obtained estimate is shown by an oscilloscope. Simultaneously, the generated signal is also sent to a DSP lock-in amplifier. DSP lock-in amplifier calculates the amplitude of the signal, which is also shown by the oscilloscope. Hence, we can compare the obtained estimates.

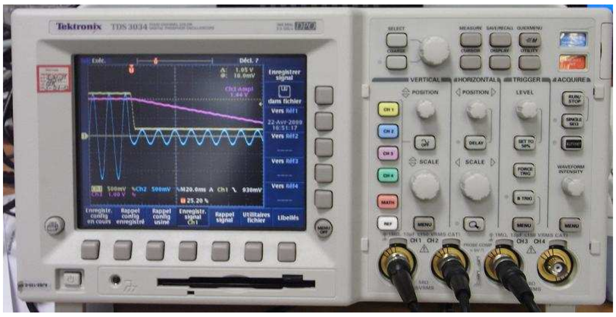


Figure 7.6: Oscilloscope.

7.2.2 Experiment results

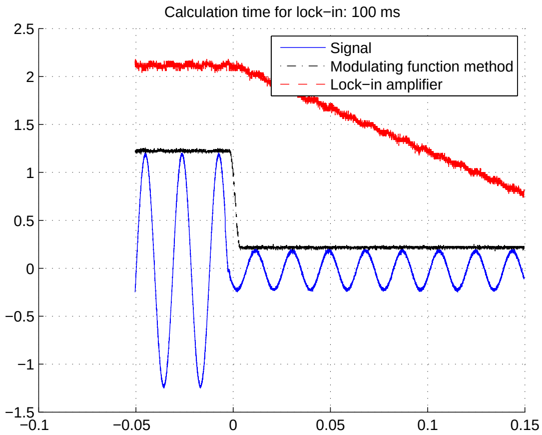
We give in this subsection some numerical experimental tests where we estimate the amplitude of a sinusoidal signal during a fast change of the oscillation conditions. This signal is generated with a frequency of $53Hz$, a fixed sampling period and an amplitude varying from 1.2 to 0.2.

We use the amplitude estimator obtained in Proposition 4.3.71 to estimate the time-varying amplitude where the two continuous functions are $\exp(\cdot)$ and $\exp(2\cdot)$. We apply the trapezes rule in order to approximate the integrals in this estimator with 17 points. The codes are implemented on Real time target (Adwin gold). Moreover, we compare our estimations with the ones given by the Signal Recovery 7280 DSP lock-in amplifier.

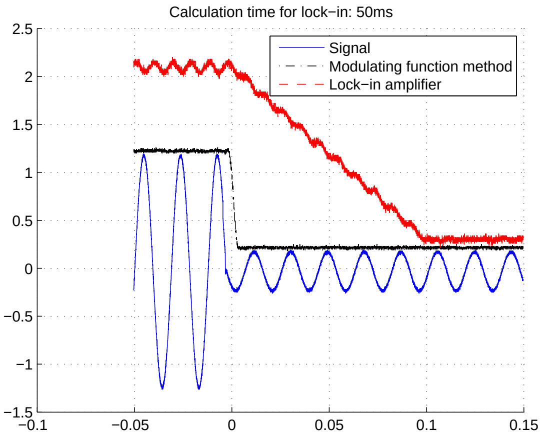
We can see the comparisons of estimations in Figure 7.7 where the sampling period is set to $375\mu s$. Hence the integration window length for our estimator is $6.4ms$. The calculation times for lock-in amplifier are varying from $100ms$ to $2ms$ so as to arrive the same rising time as the one of our estimator. In Figure 7.8, the sampling period is set to $100\mu s$. The integration window length for our estimator is $1.7ms$ (17 points) and the calculation time for lock-in amplifier is $50ms$. We can see the last estimations in Figure 7.9 where the signal is noisy with a which gaussian noise and the the sampling period is set to $750\mu s$. The integration window length for our estimator is $12.8ms$ (17 points) and the calculation time for lock-in amplifier is $50ms$. We show in Table 7.1 the calculations time used for our estimator and DSP lock-in amplifier in the three previous cases. Hence, we can see that when we estimate the amplitude of a time-varying amplitude sinusoidal signal the rise time for our estimator is much smaller than the one for a DSP lock-in amplifier.

	Fig. 7.7	Fig. 7.8	Fig. 7.9
Our estimator	$6.4ms$ (17 points)	$1.7ms$ (17 points)	$12.8ms$ (17 points)
Lock-in amplifier	$50ms$ (3.75×10^4 points)	$50ms$ (3.75×10^4 points)	$50ms$ (3.75×10^4 points)

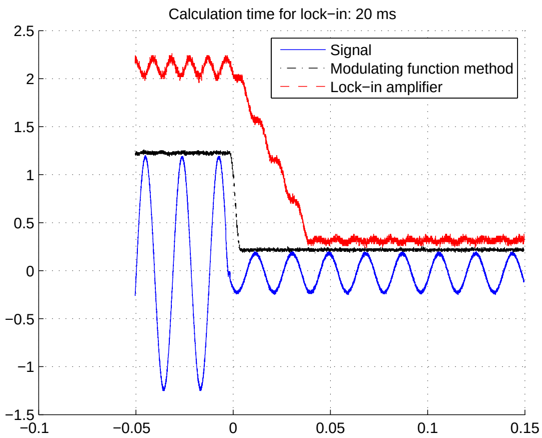
Table 7.1: Comparison of calculations time.



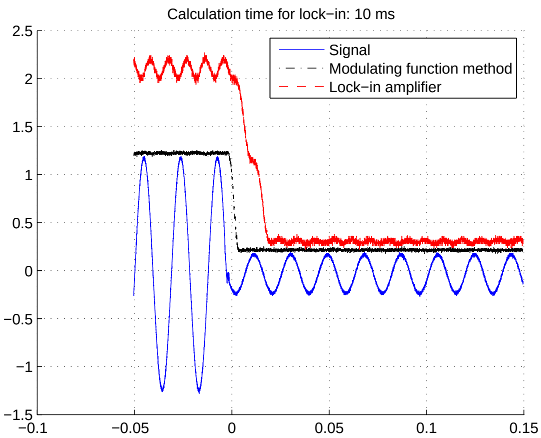
(a) Calculation time for lock-in: $100ms$.



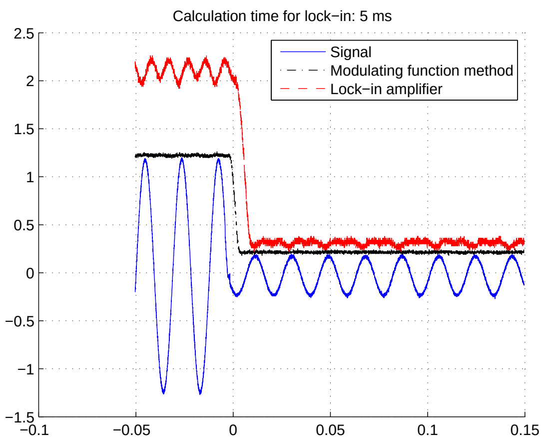
(b) Calculation time for lock-in: $50ms$.



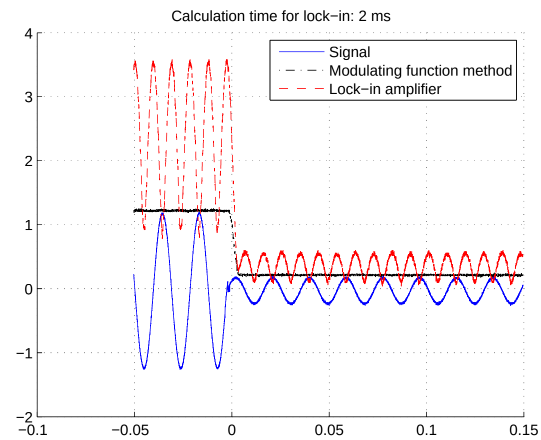
(c) Calculation time for lock-in: $20ms$.



(d) Calculation time for lock-in: $10ms$.



(e) Calculation time for lock-in: $5ms$.



(f) Calculation time for lock-in: $2ms$.

Figure 7.7: Integration window length for RT system is $6.4ms$.

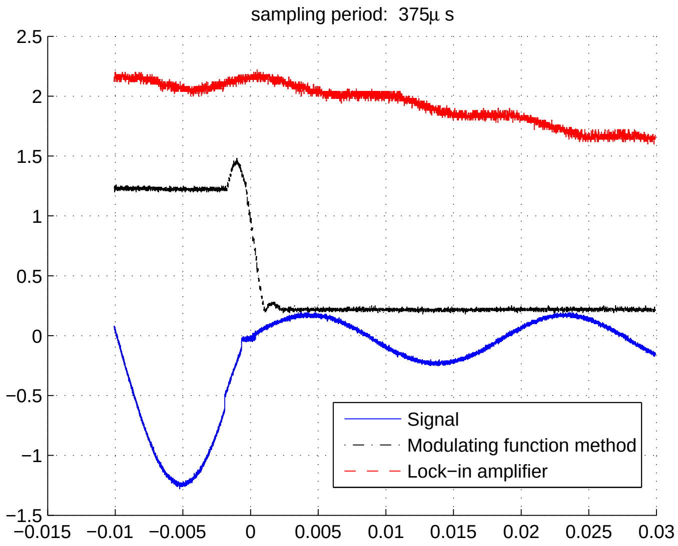


Figure 7.8: Integration window length for RT system is 1.7ms and calculation time for lock-in amplifier is 50ms.

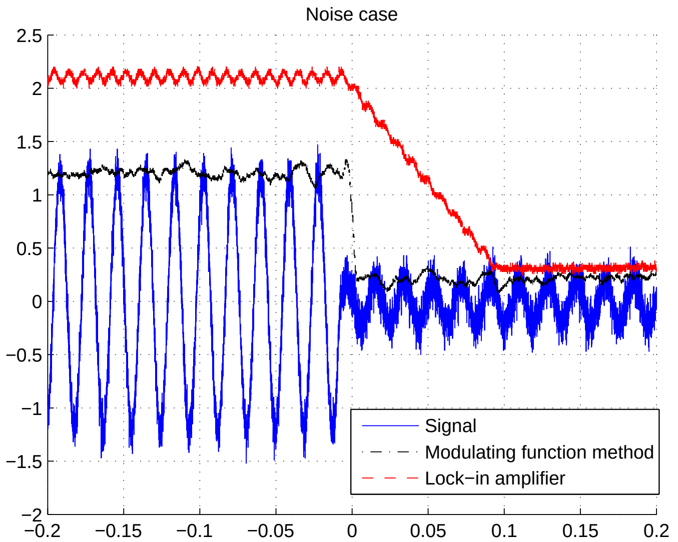


Figure 7.9: Integration window length for RT system is 12.8ms and calculation time for lock-in amplifier is 50ms.

7.3 Conclusion

In this chapter, we have given some experimental results by comparing our amplitude estimator to a DSP lock-in amplifier which is usually used for the AFM in tapping mode. We have shown that when we estimated the amplitude of a time-varying amplitude sinusoidal signal the rise time for our estimator was much smaller than the one for a DSP lock-in amplifier.

In the future, firstly we will consider sinusoidal signals with higher frequency which corresponds to the oscillating tip of an AFM in tapping mode. Secondly, we will use the modulating functions used in Chapter 6 to estimate the amplitude of a noisy signal with smaller estimation error. Thirdly, the frequency value is assumed to be known in our experimental tests, which will be estimated by our frequency estimator.

Conclusions and perspective

Conclusions

This PhD thesis is devoted to extend the derivative estimators and some parameter estimators recently obtained by the algebraic parametric techniques from noisy signals and to analyze the corresponding estimation errors. The whole document is structured around two themes:

- Numerical differentiation in finite time of noisy signals and the application to non linear observation.
- Numerical parameter estimations in finite time for noisy sinusoidal signals and the applications to the atomic force microscope in tapping mode.

Seven main contributions can be noted:

- The derivative estimators introduced by M. Mboup, M. Fliess and C. Join in [Mboup 2009b] were obtained by using the algebraic parametric techniques to the truncated Taylor series expansion. The estimation errors are only due to the truncated terms in continuous noise-free case. These estimators were improved in [Mboup 2009b] by taking the truncated Jacobi orthogonal expansion and by allowing a small time-drift in the derivative estimations. These Jacobi estimators depend on a set of parameters among which the parameters κ and μ come from the Jacobi polynomial's expression. We extend these estimators by letting these two parameters belong to $]-1,\infty[$. It is shown that with this extension we can have smaller values for the truncated term errors, especially for the time-drift. Moreover, we show in some numerical examples that if the function x satisfies the following differential equation $x^{(2)} + cx = \varepsilon$ where $c \in \mathbb{R}$ and ε is a continuous function, then the numerical error due to a negative value for κ allows us to compensate the reduced time-drift for minimal estimators. Differently from [Mboup 2009b], extended affine Jacobi estimations are given, where there is no constraint to the truncated order of the Jacobi orthogonal series expansion and the associated coordinates are given without solving a linear system. Moreover, we show that the Jacobi estimators for the n^{th} order derivative of a smooth function can be obtained by taking n derivations to the zero-order estimators of this function. The corresponding convergence rate for these estimators and the influence of the parameters on the truncated term errors are studied. These corresponding results were published [Liu 2011c, Liu 2011a].
- The Lanczos generalized derivative estimator [Lanczos 1956] is called a method of *differentiation by integration*. We extend this method by introducing central Jacobi estimators. These central

estimators are defined like causal Jacobi estimators. They can be only used for off-line estimations. However, their convergence rate is better than the one of causal Jacobi estimators. These corresponding results were published in [Liu 2011b].

- The study of the convergence rate of the Jacobi estimators is given by considering the Beppo-Levi space. Moreover, the n^{th} order Jacobi estimators are generalized by considering the Beppo-Levi space $\mathcal{H}^n(I)$. Three classes of Richardson-Jacobi estimators are introduced which improve the convergence rate of central minimal Jacobi estimators. Finally, by using the algebraic parametric techniques with a general annihilator we provide a general form of our obtained estimators.

- Non-asymptotic estimators for fractional order derivative estimations are originally introduced by using the algebraic parametric techniques.

- A weakness of the algebraic parametric techniques methods was a lack of any precise error analysis. By considering integrable noises and a large class of stochastic process noises, we provide appreciate error bounds for the Jacobi derivative estimators. These error bounds help us to choose the "optimal" parameters for our estimators. Hence, it is shown that the variance of the noise error can be smaller in the case of negative real parameters κ and μ than it was in [Mboup 2009b] for integer values. When the noise is a stochastic process, the existence of integrals obtained in our estimators are studied in the sense of convergence in mean square. The influence of the sampling period on such noise error is also studied in discrete case. Moreover, it is shown that the Jacobi derivative estimators can cope with a class of noises for which the mean and covariance are polynomials in time (with degree smaller than the order of derivative to be estimated). These results can also be applied to other estimators obtained by the algebraic parametric techniques. These corresponding results were published [Liu 2009, Liu 2011c, Liu 2011a].

- The estimators for the parameters of noisy sinusoidal signals are given by using the algebraic parametric techniques. They can cope with both the cases when there is a step or a sweep in the amplitude. The experimental results show that when we estimate the amplitude of noisy sinusoidal signals the rise time for our estimators is much smaller than the one for the DSP lock-in amplifier. These corresponding results were published [Liu 2008, Liu 2011d].

- The modulating function methods are considered to estimate the parameters of noisy sinusoidal signals with simple calculations. These methods have the similar advantages to the algebraic parametric techniques. Especially, by choosing appreciate modulating functions, the obtained estimators can also cope with a class of noises for which the mean and covariance are polynomials in time. Moreover, it is shown that they are more robust to "large" sampling period and to sinusoidal perturbations with "low" frequency. These corresponding results were published [Liu 2008, Liu 2011d].

Perspectives

Based on the results given by this thesis, several perspectives should be considered:

- The obtained "delay-free" minimal estimator for the first order derivative, the numerical error for which compensates the time-drift, should be improved by introducing some "delay-free" affine Jacobi estimators which produce small noise error contributions. Similar "delay-free" estimators should be obtained for higher order derivative.

- We show how to choose parameters for our estimators. The length of sliding window is chosen by assuming to know the noise level and the smooth signal. Hence, it should give some criterions to choose the length of sliding window. Moreover, the analysis for colored noises will be done.

- The obtained non-asymptotic estimators for the fractional order derivatives will be developed and verified by numerical simulations.

- The applications to non linear observation of Jacobi estimators are given in numerical simulations by comparing to high gain observer and sliding modes differentiator. Applicability of Jacobi estimators in a practical scenario will be verified by comparing to other existing methods.

- In [Trapero 2008], the algebraic parametric techniques are used to estimate the parameters of two sinusoidal signals from their noisy sum. The modulating functions method will be used to estimate these parameters. Furthermore, we will estimate the parameters of a finite number of sinusoidal signals from their noisy sum.

- We estimate the parameters of the noisy sinusoidal signals with time-varying amplitude, where the frequency is assumed to be constant. The obtained estimators do not adapted to the noisy sinusoidal signals with time-varying frequency, especially when there is a frequency sweep. For this problem, we will consider the following signal: $x(t) = A\sin((\omega_0 + \omega t)t + \phi)$ where $A \in \mathbb{R}^*$, $\omega_0 \in \mathbb{R}_+^*$, $\omega_1 \in \mathbb{R}$ and $\phi \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. By finding out a differential equation of x, the frequency ω_0 can be estimated by the algebraic parametric techniques and the modulating functions method. Then, the amplitude and phase can be estimated by the estimators given in this thesis.

- The experimental results are only given by using the amplitude estimators. More experimental tests will be done to compare other parameter estimators to the DSP lock-in amplifier.

Appendix

Laplace transform and Riemann-Liouville integral

In this section, let us recall some useful formulae on the Laplace transform and Riemann-Liouville integral.

Laplace transform (see [Abramowitz 1965] p. 1020) Let f be a function defined on $\mathbb{R}_+$. If it exists $\beta \in \mathbb{R}$ such that $e^{-\beta} f(\cdot) \in \mathcal{L}^1(\mathbb{R}_+)$, then the Laplace transform of f is defined by

$$\hat{f}(s) = \mathcal{L}\{f\}(s) := \int_0^{+\infty} e^{-st} f(t)\, dt, \tag{7.6}$$

where $s \in \mathbb{C}$ et $Re(s) \geq \beta$.

Derivation formulae (see [Abramowitz 1965] p. 1020) By applying derivations and integration by parts, we can get the following formulae

$$\mathcal{L}\left\{t^n f(t)\right\}(s) = (-1)^n \frac{d^n}{ds^n}\left\{\hat{f}(s)\right\}, \tag{7.7}$$

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n \hat{f}(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0). \tag{7.8}$$

Convolution theorem (see [Abramowitz 1965] p. 1020) Let f_1 and f_2 two functions, the Laplace transform of which exist. Then, we have

$$\mathcal{L}\left\{\int_0^t f_1(\tau)\, f_2(t-\tau) d\tau\right\} = \hat{f}_1(s) \cdot \hat{f}_2(s). \tag{7.9}$$

Riemann-Liouville integral (see [Loverro 2004]) The $\alpha \in \mathbb{R}_+$ order Riemann-Liouville integral of a real function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$J^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{7.10}$$

Properties (see [Loverro 2004]) We have for any $\alpha \in \mathbb{R}_+$,

$$\mathcal{L}\left\{t^{\alpha-1}\right\}(s) = \frac{\Gamma(\alpha)}{s^\alpha}, \tag{7.11}$$

$$\mathcal{L}\left\{J^\alpha f(t)\right\}(s) = \frac{1}{s^\alpha} \hat{f}(s). \tag{7.12}$$

Important formula By applying (7.7) and (7.12), we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\frac{d^n\hat{f}(s)}{ds^n}\right\}(t)=\frac{(-1)^n}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^n f(\tau)\,d\tau. \tag{7.13}$$

Jacobi orthogonal polynomial

We recall in this section some useful formulae on the Jacobi orthogonal polynomials.

Definitions ([Abramowitz 1965] p. 775) The n^{th} $(n \geq 0)$ order Jacobi polynomials on $[-1,1]$ are defined as follows

$$\hat{P}_n^{(\mu,\kappa)}(t)=\sum_{j=0}^{n}\binom{n+\mu}{j}\binom{n+\kappa}{n-j}\left(\frac{t-1}{2}\right)^{n-j}\left(\frac{t+1}{2}\right)^{j}, \tag{7.14}$$

where $\mu,\kappa \in]-1,+\infty[$. Let us substituting t by $-1+2\tau$ in (7.14), then the n^{th} order Jacobi polynomials on $[0,1]$ are defined as follows

$$P_n^{(\mu,\kappa)}(\tau)=\sum_{j=0}^{n}\binom{n+\mu}{j}\binom{n+\kappa}{n-j}(\tau-1)^{n-j}\,\tau^{j}. \tag{7.15}$$

Scalar product ([Abramowitz 1965] p. 774) Let us denote $\forall \hat{g}_1,\, \hat{g}_2 \in \mathcal{C}([-1,1])$,

$$\langle \hat{g}_1,\hat{g}_2\rangle_{\mu,\kappa}=\int_{-1}^{1}\hat{w}_{\mu,\kappa}(t)\hat{g}_1(t)\hat{g}_2(t)dt, \tag{7.16}$$

where $\hat{w}_{\mu,\kappa}(t)=(1-t)^\mu(1+t)^\kappa$ is the weight function. We denote its associated norm by $\|\cdot\|_{\mu,\kappa}$ and we have

$$\left\|\hat{P}_n^{(\mu,\kappa)}\right\|_{\mu,\kappa}^2=\frac{2^{\mu+\kappa+1}}{2n+\mu+\kappa+1}\frac{\Gamma(\mu+n+1)\Gamma(\kappa+n+1)}{\Gamma(\mu+\kappa+n+1)\Gamma(n+1)}. \tag{7.17}$$

Let us denote $\forall g_1,\, g_2 \in \mathcal{C}([0,1])$,

$$\langle g_1,g_2\rangle_{\mu,\kappa}=\int_{0}^{1}w_{\mu,\kappa}(\tau)g_1(\tau)g_2(\tau)d\tau, \tag{7.18}$$

where

$$w_{\mu,\kappa}(\tau)=(1-\tau)^\mu\tau^\kappa \tag{7.19}$$

is the associated weight function defined on $[0,1]$, then we have

$$\left\|P_n^{(\mu,\kappa)}\right\|_{\mu,\kappa}^2=\frac{1}{2n+\mu+\kappa+1}\frac{\Gamma(\mu+n+1)\Gamma(\kappa+n+1)}{\Gamma(\mu+\kappa+n+1)\Gamma(n+1)}. \tag{7.20}$$

Rodrigues formulae (see [Szegö 1967] p. 67)

$$\forall t\in[-1,1],\ \frac{d^n}{dt^n}\left\{\hat{w}_{\mu+n,\kappa+n}(t)\right\}=(-1)^n2^n n!\,\hat{P}_n^{(\mu,\kappa)}(t)\hat{w}_{\mu,\kappa}(t), \tag{7.21}$$

$$\forall \tau\in[0,1],\ \frac{d^n}{d\tau^n}\left\{w_{\mu+n,\kappa+n}(\tau)\right\}=(-1)^n n!\,P_n^{(\mu,\kappa)}(\tau)w_{\mu,\kappa}(\tau). \tag{7.22}$$

Orthogonality By applying (7.22) and integration by parts, we get

$$\int_0^1 w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau)\, \tau^m \, d\tau = 0, \;\; 0 \leq m < n, \tag{7.23}$$

$$\int_0^1 w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau)\, \tau^n \, d\tau = \mathrm{B}(\mu + n + 1, \kappa + n + 1), \tag{7.24}$$

where $\mathrm{B}(\cdot,\cdot)$ is the classical Beta function (see [Abramowitz 1965] p. 258).

Derivation relation (see [Szegö 1967] p. 63)

$$\forall \tau \in [0,1], \;\; \frac{d}{d\tau}\{P_n^{(\mu,\kappa)}(\tau)\} = (n + \mu + \kappa + 1) P_{n-1}^{(\mu+1,\kappa+1)}(\tau). \tag{7.25}$$

Parity (see [Szegö 1967] p. 80) Let us set $\kappa = \mu$ for the Jacobi estimators defined on $[-1,1]$, then we have

$$\forall t \in [-1,1], \;\; \hat{P}_n^{(\kappa,\kappa)}(-t) = (-1)^n \hat{P}_n^{(\kappa,\kappa)}(t). \tag{7.26}$$

Recurrence relations (see [Abramowitz 1965] p. 782) $\forall \tau \in [0,1]$,

$$(2n + 2 + \mu + \kappa)\,(1 - \tau)\, P_n^{(\mu+1,\kappa)}(\tau) = (1 + n + \mu) P_n^{(\mu,\kappa)}(\tau) - (n+1) P_{n+1}^{(\mu,\kappa)}(\tau), \tag{7.27}$$

$$(2n + 2 + \mu + \kappa)\, \tau \, P_n^{(\mu,\kappa+1)}(\tau) = (1 + n + \kappa) P_n^{(\mu,\kappa)}(\tau) + (n+1) P_{n+1}^{(\mu,\kappa)}(\tau). \tag{7.28}$$

Résumé étendu en français

Ce mémoire concerne la construction et l'analyse d'estimateurs robustes pour le calcul numérique des dérivés de signaux bruités et des paramètres de signaux sinusoïdaux bruités. Ces estimateurs, originalement introduits par Fliess, Mboup et Sira Ramírez, sont actuellement étudiés au sein de l'équipe projet NON-A de l'INRIA Lille Nord Europe. Pour une classe d'entres eux, nous les obtenons à partir de la réécriture dans le domaine opérationnel de Laplace des équations différentielles linéaires des signaux analysés. Par des manipulations algébriques simples dans l'anneau $\mathbb{R}(s)\left[\frac{d}{ds}\right]$ des polynômes différentiels en $\frac{d}{ds}$ à coefficients rationnels en la variable opérationnelle s, nous montrons que ces estimateurs sont non-asymptotiques et que les estimations numériques obtenues, même en présence de bruits, sont robustes pour un faible nombre d'échantillons des signaux. Nous montrons, de plus, que ces propriétés sont vérifiées pour une large classe de type de bruits. Ces estimateurs exprimés dans le domaine temporel s'écrivent en général *via* des fractions d'intégrales itérées des signaux analysés. Dans la première partie du mémoire, nous étudions des familles d'estimateurs de dérivées obtenus par ces méthodes algébriques. Nous montrons que pour une classe d'entre eux, il est possible de les formuler directement en tronquant une série orthogonale de polynômes de Jacobi. Cette considération nous permet alors d'étendre à $\mathbb{R}$ le domaine de définition des paramètres de ces estimateurs. Nous analysons ensuite l'influence de ces paramètres étendus sur l'erreur de troncature, qui produit un retard d'estimation dans le cas causal, puis sur l'erreur due aux bruits, considérés comme des processus stochastiques, et enfin sur l'erreur numérique de discrétisation des intégrales. Ainsi, nous montrons comment réduire le retard d'estimation et l'effet du aux bruits. Une validation de cette approche est réalisée par la construction d'un observateur non asymptotique de variables d'état d'un système non linéaire. Dans la deuxième partie de ce mémoire, nous construisons par cette approche algébrique des estimateurs des paramètres d'un signal sinusoïdal bruité dont l'amplitude varie avec le temps. Nous montrons que les méthodes classiques de fonctions modulatrices sont un cas particulier de cette approche. Nous étudions ensuite l'influence des paramètres algébriques sur l'erreur d'estimation due au bruit et l'erreur numérique d'intégration. Des majorations de ces erreurs sont données pour une classe d'estimateurs. Finalement, une comparaison entre ces estimateurs et la méthode classique de détection synchrone est réalisée pour démontrer l'efficacité de notre approche sur ce type de signaux.

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