Université des Sciences et Technologies de Lille
École Doctorale Sciences Pour l'Ingénieur Lille
Nord-de-France

Numéro d'ordre:
40583

T H E S E

pour obtenir le titre de

Docteur en Sciences

Mention : MATHÉMATIQUES APPLIQUÉES

Présentée pour être soutenue par

Qidi PENG

Inférence statistique pour des processus multifractionnaires cachés dans un cadre de modèles à volatilité stochastique

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Date de soutenance : lundi 21 novembre 2011

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Remerciements

Je souhaite tout d'abord exprimer ma profonde gratitude à Antoine Ayache, mon directeur de thèse. Il a su me proposer un sujet motivant qui a avivé mon goût pour la recherche ; ses qualités scientifiques et humaines, son suivi attentif ainsi que le soutien qu'il m'a toujours apporté, m'ont permis de mener à bien cette thèse.

Je suis très honoré que Donatas Surgailis, qui est l'un des spécialistes internationaux les plus reconnus de mon domaine de recherche, ait accepté de rapporter sur ma thèse. Je le remercie vivement.

L'article d'Arnaud Gloter et Marc Hoffmann "stochastic volatility and fractional Brownian motion" [35] (ainsi que sa version plus longue [36]), a été le point de départ de ma thèse et m'a apporté de nombreuses idées ; c'est pour moi un grand honneur qu'ils fassent, tous les deux, partie de mon jury et qu'Arnaud Gloter ait accepté d'être rapporteur. Je leur adresse, à tous les deux, mes profonds remerciements.

Je suis très honoré que Stéphane Jaffard, malgré son emploi du temps très chargé et ses nombreuses responsabilités, ait accepté d'être membre de mon jury. L'intérêt qu'il accorde à ma thèse me touche et je le remercie vivement.

Certains travaux de Jean-Marc Bardet et ses co-auteurs [11, 12, 13, 14] m'ont considérablement aidé au cours de ma thèse ; c'est pour moi un grand honneur qu'il soit dans mon jury. Je lui adresse mes profonds remerciements.

Je suis très honoré que Viet Chi Tran et Ciprian Tudor fassent partie de mon jury. Je suis touché par l'intérêt qu'ils accordent à ma thèse et je les remercie vivement.

J'exprime toute ma reconnaissance à mes anciens enseignants et plus particulièrement à Azzouz Dermoune, Youri Davydov, Nikolay Tzvetkov et Marie-Claude Viano, qui par la qualité de leurs cours, ont considérablement contribué à me donner envie de faire une thèse. Je suis aussi reconnaissant à Charles Suquet, mon ancien tuteur de monitorat, pour l'aide qu'il m'a apporté.

J'exprime ma profonde gratitude à Changgui Zhang et Yimin Xiao pour tous les précieux conseils qu'ils m'ont donnés.

Un grand merci, aux personnes que j'ai connues à l'Université Lille 1 et plus particulièrement à Yin Chen, David Coupier, Julien Hamonier, Philippe Heinrich, Radu Stoica et Tianwen Wei.

Enfin, je pense beaucoup à mes parents en Chine, à mes proches et à mes amis, qui m'ont beaucoup soutenu tout au long de cette thèse.

Inférence statistique pour des processus multifractionnaires cachés dans un cadre de modèles à volatilité stochastique

Résumé

L'exemple paradigmatique d'un processus stochastique multifractionnaire est le mouvement brownien multifractionnaire (mbm). Ce processus gaussien de nature fractale admet des trajectoires continues nulle part dérivables et étend de façon naturelle le célèbre mouvement brownien fractionnaire (mbf). Le mbf a été introduit depuis longtemps par Kolmogorov et il a ensuite été "popularisé" par Mandelbrot ; dans plusieurs travaux remarquables, ce dernier auteur a notamment insisté sur la grande importance de ce modèle dans divers domaines applicatifs.

Le mbm, quant à lui, a été introduit, depuis plus de quinze ans, par Benassi, Jaffard, Lévy Véhel, Peltier et Roux. Grossièrement parlant, il est obtenu en remplaçant le paramètre constant de Hurst du mbf, par une fonction $H(t)$ qui dépend de façon régulière du temps t. Ainsi, contrairement au mbf, les accroissements du mbm sont non stationnaires et la rugosité locale de ses trajectoires (mesurée habituellement par l'exposant de Hölder ponctuel) peut évoluer significativement au cours du temps ; en fait, à chaque instant t, l'exposant de Hölder ponctuel du mbm vaut $H(t)$. Notons que cette dernière propriété, rend ce processus plus flexible que le mbf ; grace à elle, le mbm est maintenant devenu un modèle utile en traitement du signal et de l'image ainsi que dans d'autres domaines tels que la finance.

Depuis plus d'une décennie, plusieurs auteurs se sont intéressés à des problèmes d'inférence statistique liés au mbm et à d'autres processus/champs multifractionnaires ; leurs motivations comportent à la fois des aspects applicatifs et théoriques. Parmi les plus importants, figure le problème de l'estimation de $H(t)$, l'exposant de Hölder ponctuel en un instant arbitraire t. Dans ce type de problématique, la méthode des variations quadratiques généralisées, initialement introduite par Istas et Lang dans un cadre de processus à accroissements stationnaires, joue souvent un rôle crucial. Cette méthode permet de construire des estimateurs asymptotiquement normaux à partir de moyennes quadratiques d'accroissements généralisés d'un processus observé sur une grille.

A notre connaissance, dans la littérature statistique qui concerne le mbm, jusqu'à présent, il a été supposé que, l'observation sur une grille des valeurs exactes de ce processus est disponible ; cependant une telle hypothèse ne semble pas toujours réaliste. L'objectif principal de la thèse est d'étudier des problèmes d'inférence statistique liés au mbm, lorsque seulement une version corrompue de ce dernier est observable sur une grille régulière. Cette version corrompue est donnée par une classe de modèles à volatilité stochastique dont la définition s'inspire de certains travaux antérieurs de Gloter et Hoffmann ; signalons enfin que la formule d'Itô permet de ramener ce cadre statistique au cadre classique : "signal+bruit".

Statistical inference for hidden multifractional processes in a setting of stochastic volatility models

Abstract

The paradigmatic example of a multifractional stochastic process is multifractional Brownian motion (mBm). This fractal Gaussian process with continuous nowhere differentiable trajectories is a natural extension of the well-known fractional Brownian motion (fBm). FBm was introduced a long time ago by Kolmogorov and later it has been made "popular" by Mandelbrot; in several outstanding works, the latter author has emphasized the fact that this model is of a great importance in various applied areas.

Regarding mBm, it was introduced, more than fifteen years ago, by Benassi, Jaffard, Lévy Véhel, Peltier and Roux. Roughly speaking, it is obtained by replacing the constant Hurst parameter of fBm by a smooth function $H(t)$ which depends on the time variable t. Therefore, in contrast with fBm, the increments of mBm are non stationary and the local roughness of its trajectories (usually measured through the pointwise Hölder exponent) is allowed to significantly evolve over time; in fact, at each time t, the pointwise Hölder exponent of mBm is equal to $H(t)$. It is worth noticing that the latter property makes this process more flexible than fBm; thanks to it, mBm has now become a useful model in the area of signal and image processing, as well as in other areas such as finance.

Since at least one decade, several authors have been interested in statistical inference problems connected with mBm and other multifractional processes/fields; their motivations have both applied and theoretical aspects. Among those problems, an important one is the estimation of $H(t)$, the pointwise Hölder exponent at an arbitrary time t. In the solutions of such issues, the generalized quadratic variation method, which was first introduced by Istas and Lang in a setting of stationary increments processes, usually plays a crucial role. This method allows to construct asymptotically normal estimators starting from quadratic means of generalized increments of a process observed on a grid.

So far, to our knowledge, in the statistical literature concerning mBm, it has been assumed that, the observation of the true values of this process on a grid, is available; yet, such an assumption does not always seem to be realistic. The main goal of the thesis is to study statistical inference problems related to mBm, when only a corrupted version of it, can be observed on a regular grid. This corrupted version is given by a class of stochastic volatility models whose definition is inspired by some Gloter and Hoffmann's earlier works; last, notice that thanks to Itô formula this statistical setting can be viewed as the classical setting: "signal+noise".

Contents

Chapter 1
Introduction en français

Le mouvement brownien multifractionnaire (mbm), ici noté par $\{X(t)\}_{t\in[0,1]}$, est l'exemple paradigmatique d'un processus gaussien multifractionnaire ; il a été introduit dans [51, 18]. Le mbm est une extension naturelle du mouvement brownien fractionnaire (mbf) de paramètre de Hurst α [31, 53], ici noté par $\{B_\alpha(t)\}_{t\in[0,1]}$; en effet, il est obtenu en remplaçant le paramètre constant α par un paramètre fonctionnel $H(\cdot)$, qui dépend de façon suffisamment régulière de t. Ainsi, contrairement au mbf, les accroissements du mbm sont non stationnaires et la rugosité locale de ses trajectoires (mesurée habituellement par l'exposant de Hölder ponctuel) peut évoluer considérablement au cours du temps ; en fait à chaque instant t, l'exposant de Hölder ponctuel du mbm vaut $H(t)$. Cette dernière propriété, rend ce processus plus flexible que le mbf ; grâce à elle le mbm est devenu un modèle d'une utilité considérable en traitement du signal et dans d'autres domaines tels que la finance (voir par exemple [2, 19, 20, 21, 41, 45]).

Depuis plus d'une décennie, plusieurs auteurs se sont intéressés à des problèmes d'inférence statistique liés au mbm ou encore à d'autres processus ou champs multifractionnaires (voir par exemple [10, 12, 14, 13, 15, 17, 16, 19, 20, 21, 22, 23][1]) ; parmi les plus importants, figure l'estimation de $H(t)$, l'exposant de Hölder ponctuel en un instant arbitraire t. Dans de tels problèmes, la méthode des variations quadratiques généralisées, qui a d'abord été introduite par Istas et Lang dans un cadre de processus à accroissements stationnaires [42], joue un rôle crucial.

A notre connaissance, dans la littérature statistique qui concerne le mbm, jusqu'à présent, il a été supposé que, l'observation des valeurs exactes de $\{X(t)\}_{t\in[0,1]}$ sur une grille, est disponible ; cependant, une telle hypothèse ne semble pas toujours réaliste. L'objectif principal de la thèse est d'étudier des problèmes d'inférence statistique liés au mbm, lorsque seulement une version corrompue de ce dernier peut être observée sur une grille régulière. Plus précisément, dans la thèse, on suppose que l'on observe un échantillon $\big\{Z(0), Z(1/n), \ldots, Z(n/n)\big\}$ du processus $\{Z(t)\}_{t\in[0,1]}$ défini par,

$$Z(t) = z_0 + \int_0^t \Phi\big(X(s)\big)\, \mathrm{d}W(s), \tag{$*$}$$

où :

- $\{W(s)\}_{s\in[0,1]}$ est un mouvement brownien standard indépendant du mbm $\{X(s)\}_{s\in[0,1]}$,
- Φ est une fonction déterministe inconnue de classe C^2, à croissance lente (au plus polynômiale) à l'infini ainsi que ses deux dérivées ;

[1] Cette liste est loin d'être exhaustive.

en outre, à l'exception du chapitre numéro 4, on suppose toujours que le paramètre $H(\cdot)$ du mbm caché $\{X(s)\}_{s\in[0,1]}$, est une fonction deux fois continûment dérivable ; cette dernière hypothèse nous permet d'obtenir de "bonnes estimations" des corrélations entre les accroissements généralisés du mbm.

Nous appelons le processus $\{Z(t)\}_{t\in[0,1]}$ défini par $(*)$, *modèle à volatilité stochastique multifractionnaire*, parce qu'on peut voir ce processus comme la modélisation du logarithme du prix d'un actif financier qui est une solution de l'équation différentielle stochastique :

$$dZ_t = \sigma_t \, dW_t,$$

où la volatilité $\sigma_t = \Phi\big(X(t)\big)$ est gouvernée par le mbm $\{X(t)\}_{t\in[0,1]}$. De tels modèles à volatilité stochastique possèdent les trois propriétés suivantes, qui leur permettent de reproduire certaines caractéristiques observées empiriquement dans nombre de séries financières [19, 20, 21] :

- les accroissements de $\{Z(t)\}_{t\in[0,1]}$ sont non corrélés, cela résulte de l'hypothèse d'indépendance des processus $\{X(s)\}_{s\in[0,1]}$ et $\{W(s)\}_{s\in[0,1]}$;
- en revanche, les carrés des accroissements de $\{Z(t)\}_{t\in[0,1]}$ sont significativement corrélés, et les intensités de leurs corrélations peuvent changer au cours du temps permettant ainsi d'avoir des périodes avec de la longue dépendance qui alternent avec d'autres périodes avec de la faible dépendance ;
- la rugosité locale de la volatilité (autrement dit son exposant de Hölder ponctuel) peut être calibrée au moyen du paramètre fonctionnel $H(\cdot)$ du mbm et peut donc évoluer au cours du temps.

Il convient de noter que *des modèles à volatilité stochastique fractionnaire* de la forme $(*)$, où le mbm $\{X(s)\}_{s\in[0,1]}$ est remplacé par un mbf $\{B_\alpha(s)\}_{s\in[0,1]}$, ont été étudiés précédemment par Gloter et Hoffmann [35, 36]. Une idée importante, déjà utilisée par ces deux auteurs, est que les valeurs moyennes inconnues,

$$\overline{Y}_{i,N_n} = N_n \int_{i/N_n}^{(i+1)/N_n} \big(\Phi(X(s))\big)^2 \, ds \quad i = 0, \dots, N_n - 1,$$

où $N_n = [n^\beta]$ (ici $[\cdot]$ désigne la fonction partie entière et $\beta \in]0,1[$ un paramètre fixé), peuvent être estimées à partir des observations disponibles, c'est-à-dire de $\big\{Z(0), Z(1/n), \dots, Z(n/n)\big\}$; cette importante idée qui repose notamment sur la formule d'Itô, se traduit, de façon plus précise (voir le Lemme 7.4.2), par

$$\widehat{Y}_{i,N_n,n} = \overline{Y}_{i,N_n} + \mathscr{E}_{i,N_n,n} \quad i = 0, \dots, N_n - 1, \tag{$**$}$$

où:

- les $\widehat{Y}_{i,N_n,n}$ sont les valeurs approchées des $\overline{Y}_{i,N_n}$, ces valeurs approchées sont obtenus aux moyen de variations quadratiques normalisées de certaines des observations $Z(i/n)$;

- les $\mathscr{E}_{i,N_n,n}$ désignent les termes d'erreur, il est à noter que ces variables aléatoires possèdent des propriétés "sympathiques", entre autres :
  - (i) conditionnellement à la tribu $\mathcal{G}_X = \sigma(X(s), 0 \leq s \leq 1)$, leurs lois sont indépendantes entre elles,
  - (ii) on dispose d'un contrôle de leur vitesse de convergence vers 0 lorsque N_n tend vers l'infini, au moyen de leurs moments conditionnels $|\mathbb{E}(\mathscr{E}_{i,N_n,n}|\mathcal{G}_X)|$, $\mathbb{E}(\mathscr{E}^2_{i,N_n,n}|\mathcal{G}_X)$ et $\mathbb{E}(\mathscr{E}^4_{i,N_n,n}|\mathcal{G}_X)$, plus précisément, il existe une variable aléatoire C dont les moments de tout ordre sont finis, telle que, presque sûrement, pour tout n assez grand, l'on a,
$$|\mathbb{E}(\mathscr{E}_{i,N_n,n}|\mathcal{G}_X)| \leq CN_n^{-(\beta^{-1}-1)},$$
$$\mathbb{E}(\mathscr{E}^2_{i,N_n,n}|\mathcal{G}_X) \leq CN_n^{-(\beta^{-1}-1)}$$
et
$$\mathbb{E}(\mathscr{E}^4_{i,N_n,n}|\mathcal{G}_X) \leq CN_n^{-2(\beta^{-1}-1)}.$$

Ainsi la formulation $(**)$ permet de se ramener à un cadre statistique classique de la forme : "signal+bruit".

Il convient aussi de souligner que la formule de Taylor-Lagrange permet de montrer que les accroissements généralisés des $\overline{Y}_{i,N_n}$ sont intimement liés aux accroissements généralisés des valeurs moyennes $\overline{X}_{i,N_n}$, $i = 0, \ldots, N_n - 1$, du processus mbm $\{X(s)\}_{s\in[0,1]}$ lui-même ; ces dernières valeurs sont définies par,
$$\overline{X}_{i,N_n} = N_n \int_{i/N_n}^{(i+1)/N_n} X(s)\, ds.$$

Tout comme Gloter et Hoffmann [35, 36], pour faire de l'inférence statistique dans le cadre des modèles du type $(*)$ ou encore $(**)$, on adoptera bien souvent des méthodes qui consistent essentiellement en les trois étapes suivantes :

- on construit d'abord un estimateur au moyen des $\overline{X}_{i,N_n}$;
- on se ramène ensuite à un estimateur qui dépend des $\overline{Y}_{i,N_n}$;
- enfin, dans ce dernier estimateur les $\overline{Y}_{i,N_n}$ sont remplacées par leurs valeurs approchées $\widehat{Y}_{i,N_n,n}$.

Nous allons maintenant décrire le contenu de chacun des chapitres de la thèse, nous nous limiterons ici à une description succincte qui permet d'avoir une vue d'ensemble de la thèse ; signalons au passage, qu'au début de chaque chapitre, une introduction détaillée présentera plus précisément, les problématiques qui y seront étudiées.

Rappelons qu'Istas et Lang (voir [42]) ont construit des estimateurs asymptotiquement normaux de l'exposant de Hölder uniforme [2] d'un processus à accroissements stationnaires $\{S(t)\}_{t\in[0,1]}$ appartenant à une large classe ; leurs estimateurs sont obtenus par la méthode des variations quadratiques généralisées,

[2] Cet exposant permet de mesurer la régularité de Hölder globale d'un processus sur un intervalle.

à partir des valeurs exactes de $\{S(t)\}_{t\in[0,1]}$ sur une grille régulière, c'est-à-dire à partir de $\{S(0), S(1/N), \ldots, S(N/N)\}$. Dans le premier chapitre de la thèse (chapitre numéro 3), il est essentiellement établi que les principaux résultats d'Istas et Lang restent valables lorsque $\{S(0), S(1/N), \ldots, S(N/N)\}$ est remplacé par $\{\overline{S}_{0,N}, \ldots, \overline{S}_{N-1,N}\}$, c'est-à-dire que l'on observe les valeurs moyennes de $\{S(t)\}_{t\in[0,1]}$, sur une grille régulière, au lieu de ses valeurs exactes. Les motivations sont les suivantes : (a) lorsque le processus $\{S(t)\}_{t\in[0,1]}$ est erratique (il fluctue beaucoup au cours du temps), il semble plus réaliste de dire que l'on observe sur une grille ses valeurs moyennes plutôt que ses valeurs exactes ; (b) comme on l'a déjà souligné, les valeurs moyennes jouent un rôle important dans le cadre de l'étude statistique des modèles du type $(*)$ ou encore $(**)$. Signalons enfin qu'une version de ce chapitre a déjà été publiée dans le journal "Statistics and Probability Letters".

Avant de se lancer dans une étude statistique des modèles du type $(*)$, il faudrait au préalable avoir une représentation intuitive de ces modèles. Pour ce faire, il convient d'étudier la régularité de Hölder, globale et locale, de leurs trajectoires et il convient également de simuler ces dernières ; tels sont les objectifs du second chapitre de la thèse (chapitre numéro 4). Il convient de souligner que les principaux résultats de ce chapitre sont valables, non seulement dans le cas où $\{X(s)\}_{s\in[0,1]}$ est le mbm, mais aussi, dans le cadre beaucoup plus général, où $\{X(s)\}_{s\in[0,1]}$ est un processus gaussien arbitraire dont les trajectoires vérifient une condition de Hölder uniforme d'ordre arbitraire $\alpha > 1/2$. Dans ce chapitre, d'abord, l'on montre qu'avec probabilité 1, les trajectoires de $\{Z(t)\}_{t\in[0,1]}$ appartiennent à tous les espaces de Hölder $C^\gamma([0,1])$ où $\gamma < 1/2$ et que, l'exposant de Hölder ponctuel de $\{Z(t)\}_{t\in[0,1]}$ vaut $1/2$ en tout point de $]0,1[$, presque sûrement. Ensuite, afin de disposer d'une méthode de simulation efficace de ces trajectoires, une représentation en série aléatoire de $\{Z(t)\}_{t\in[0,1]}$, via la base de Haar, est introduite ; la principale idée qui permet de l'obtenir, consiste, à décomposer, pour tous $t \in [0,1]$ et ω fixés, la fonction $s \mapsto \Phi\big(X(s,\omega)\big)\mathbb{1}_{[0,t]}(s)$, dans la base de Haar de l'espace $L^2([0,1])$, puis à utiliser la propriété d'isométrie de l'intégrale stochastique dans $(*)$. Enfin, il est montré qu'avec probabilité 1, la série est convergente dans tous les espaces $C^\gamma([0,1])$ où $\gamma < 1/2$, de plus, une fine estimation de sa vitesse de convergence, mesurée au moyen de la norme $\|\cdot\|_{C^\gamma([0,1])}$, est donnée.

Nous avons déjà souligné au début de cette introduction que l'exposant de Hölder ponctuel du mbm caché $\{X(s)\}_{s\in[0,1]}$ peut évoluer considérablement au cours du temps, paradoxalement, l'exposant de Hölder ponctuel de sa version corrompue $\{Z(t)\}_{t\in[0,1]}$ reste constant (comme on l'a vu dans le paragraphe précédent) ; cela signifie *qu'il y a une perte considérable d'information lorsque l'on observe les valeurs de* $\{Z(t)\}_{t\in[0,1]}$ *sur une grille, au lieu de celles de* $\{X(s)\}_{s\in[0,1]}$. L'objectif principal des deux derniers chapitres de la thèse est de montrer que, malgré cette perte d'information, il est encore possible de construire des estimateurs consistants de certains indices pertinents liés à $\{X(s)\}_{s\in[0,1]}$ et/ou à la fonction inconnue Φ.

Dans le troisième chapitre de la thèse (chapitre numéro 5), on suppose que le paramètre fonctionnel $H(\cdot)$ du mbm est connu et prend ses valeurs dans l'intervalle ouvert $(1/2, 1)$, h désigne une fonction arbitraire connue de classe C^1 à croissance lente à l'infini ainsi que sa dérivée, enfin $(\mu_N)_N$ et $(\nu_N)_N$ sont deux suites arbitraires vérifiant pour tout N, $0 \le \mu_N < \nu_N \le 1$ et $\lim_{N\to+\infty} N(\nu_N - \mu_N) = +\infty$. Un premier résultat de ce chapitre, montre qu'en construisant, au moyen des $\{\overline{X}_{i,N} : i \in \mathbb{N} \text{ et } i/N \in [\mu_N, \nu_N]\}$, la variation quadratique généralisée pondérée par la fonction h, on obtient un estimateur de l'intégrale :

$$\frac{1}{\nu_N - \mu_N} \int_{\mu_N}^{\nu_N} h(X(s))\, ds.$$

Un deuxième résultat, qui généralise le premier résultat, montre qu'en construisant, au moyen des $\{\overline{Y}_{i,N} : i \in \mathbb{N} \text{ et } i/N \in [\mu_N, \nu_N]\}$, $\overline{V}(h; \mu_N, \nu_N)$ la variation quadratique généralisée pondérée par la fonction h, on obtient un estimateur de l'intégrale :

$$\frac{1}{\nu_N - \mu_N} \int_{\mu_N}^{\nu_N} \big(f'(X(s))\big)^2 h(Y(s))\, ds.$$

Un troisième résultat montre que lorsque dans $\overline{V}(h; \mu_N, \nu_N)$, les $\overline{Y}_{i,N}$ sont remplacés par les $\widehat{Y}_{i,N_n,n}$, quitte à rajouter à $\overline{V}(h; \mu_N, \nu_N)$ un terme de correction, on continue à avoir un estimateur noté par $\widehat{V}(h; \mu_{N_n}, \nu_{N_n})$ qui converge vers la même intégrale. Signalons que ces trois estimateurs convergent dans $L^1(\Omega)$, à la vitesse $\big(N(\nu_N - \mu_N)\big)^{-1/2}$. Enfin, un dernier résultat, montre que dans le cas d'un modèle à volatilité stochastique multifractionnaire linéaire (i.e. $\Phi(x) = \theta x$ pour tout réel x), sous certaines conditions, un estimateur $\widehat{\theta}_n^2$ de θ^2, peut être obtenu à partir de $\widehat{V}(1; \mu_{N_n}, \nu_{N_n})$; de plus $\widehat{\theta}_n^2$ converge en probabilité, au moins à la vitesse

$$n^{-\left(4 \max_{s\in[\mu_{N_n}, \nu_{N_n}]} H(s)+2\right)^{-1}} (\nu_{N_n} - \mu_{N_n})^{-1/2}.$$

Signalons qu'une version de ce chapitre a été publiée dans un ouvrage chez "Springer" (http://www.springer.com/mathematics/probability/book/978-3-642-22367-9).

Dans le quatrième chapitre de la thèse (chapitre numéro 6), on désigne par $(v(N))_{N\ge 3}$ une suite arbitraire de réels strictement positifs qui tend vers 0 plus vite que $(\log N)^{-1}$ et moins vite que $(\log N)^2/N$, on fixe t_0 un point arbitraire de $]0,1[$ et pour tout N assez grand, on pose

$$\nu_N(t_0) = \big\{i \in \mathbb{N} : |t_0 - i/N| \le v(N)\big\}.$$

Un premier résultat montre que la variation quadratique généralisée construite au moyen de $\{\overline{X}_{i,N} : i \in \nu_N(t_0)\}$, permet d'obtenir un estimateur de $H(t_0)$ qui converge en probabilité ; de plus lorsqu'on impose la condition :

$$\sum_{N=3}^{+\infty} \big(Nv(N)\big)^{-2} < \infty, \tag{$\mathcal{C}$}$$

l'estimateur converge alors presque sûrement ; par ailleurs, lorsqu'on impose la condition :

$$v(N) = o\left(N^{-1/3}(\log N)^{-2/3}\right),$$

l'estimateur devient alors asymptotiquement normal. Un deuxième résultat montre que la variation quadratique généralisée construite au moyen de $\{\overline{Y}_{i,N} : i \in \nu_N(t_0)\}$, permet d'obtenir un estimateur de $H(t_0)$ qui converge en probabilité ; de plus lorsqu'on impose la condition $(\mathcal{C})$ l'estimateur converge alors presque sûrement ; par ailleurs, lorsqu'on impose les conditions $H(t_0) \in]1/2, 1[$ et

$$v(N) = o\left(N^{-1/2}(\log N)^{\eta}\right),$$

où $\eta > 1/2$ est arbitraire, l'estimateur devient alors asymptotiquement normal. Enfin, un troisième résultat montre que pour certaines valeurs du paramètre β, le deuxième résultat reste vrai lorsque les $\overline{Y}_{i,N}$ sont remplacées par les $\widehat{Y}_{i,N_n,n}$.

Chapter 2

Introduction

The paradigmatic example of a multifractional Gaussian process is multifractional Brownian motion (mBm), we denote by $\{X(t)\}_{t\in[0,1]}$. MBm was introduced in [51, 18]. It is a natural extension of the well-known fractional Brownian motion (fBm) [31, 53] of Hurst parameter α, we denote by $\{B_\alpha(t)\}_{s\in[0,1]}$; roughly speaking, it is obtained by replacing the constant parameter α of the latter Gaussian process by a smooth enough function $H(\cdot)$, depending on the time variable t. Therefore, in contrast with fBm, the increments of mBm are non stationary and the local roughness of its trajectories (usually measured through the pointwise Hölder exponent) is allowed to evolve considerably over time, since at each t the pointwise Hölder exponent of mBm equals $H(t)$. It is worth noticing that the latter property makes mBm to be a useful model in the area of signal processing and other areas such as finance (see for example [2, 19, 20, 21, 41, 45]).

Since at least one decade, several authors have been interested in statistical inference problems connected with mBm and related processes (see for example [10, 12, 14, 13, 15, 17, 16, 19, 20, 21, 22, 23][1]); an important one of them is the estimation of $H(t)$, the pointwise Hölder exponent of mBm at an arbitrary time t. In such problems, the generalized quadratic variation method, which was first introduced by Istas and Lang in a setting of stationary increments processes [42], plays a key role.

So far, to our knowledge, in the statistical literature concerning mBm, it has been assumed that the observation, of the true values of $\{X(t)\}_{t\in[0,1]}$ on a grid, is available; yet, such an assumption does not always seem to be realistic. The main goal of the thesis is to study some statistical inference problems related to mBm, when only a corrupted version of it over a regular grid, can be observed. More precisely, in the thesis, it is assumed that one observes a sample $\big\{Z(0), Z(1/n), \dots, Z(n/n)\big\}$ of the process $\{Z(t)\}_{t\in[0,1]}$ defined as:

$$Z(t) = z_0 + \int_0^t \Phi\big(X(s)\big)\, \mathrm{d}W(s), \qquad (*)$$

where:

- $\{W(s)\}_{s\in[0,1]}$ is a standard Brownian motion independent on the mBm $\{X(s)\}_{s\in[0,1]}$;
- Φ is an unknown deterministic C^2 function having slow increase (at most polynomial) at infinity, as well as its two derivatives;

[1]This list is not exhaustive.

also, except in the chapter number 4, it is always assumed that the parameter $H(\cdot)$ of the hidden mBm $\{X(s)\}_{s\in[0,1]}$, is a two times continuously differentiable function; actually we need this assumption in order to get "nice estimates" of the correlations between the generalized increments of mBm.

We call the process $\{Z(t)\}_{t\in[0,1]}$ defined by $(*)$ *a multifractional stochastic volatility model*, since it can be viewed as a model for the logarithm of the price of a financial asset which is a solution of the stochastic differential equation:

$$dZ_t = \sigma_t \, dW_t,$$

where the volatility $\sigma_t = \Phi\big(X(t)\big)$ is governed by the mBm $\{X(t)\}_{t\in[0,1]}$. Such stochastic volatility models have the following three properties which allow them to reproduce some features empirically observed in financial series [19, 20, 21]:

- increments of $\{Z(t)\}_{t\in[0,1]}$ are uncorrelated, this is a consequence of the independence of the processes $\{X(s)\}_{s\in[0,1]}$ and $\{W(s)\}_{s\in[0,1]}$;
- on the other hand, the square of increments of $\{Z(t)\}_{t\in[0,1]}$ are significantly correlated and intensities of correlations can change over time in such a way which allows to have some periods with long range dependance and other periods with short range dependance;
- the volatility local roughness (its pointwise Hölder exponent) can be prescribed via the functional parameter $H(\cdot)$ of mBm and thus can evolve over time.

It is worth mentioning that *fractional stochastic volatility models* of the form $(*)$ where the mBm $\{X(s)\}_{s\in[0,1]}$ is replaced by a fBm $\{B_\alpha(s)\}_{s\in[0,1]}$, have been previously studied by Gloter and Hoffmann [35, 36]. An important idea, already used by the latter two authors, is that the unknown local average values,

$$\overline{Y}_{i,N_n} = N \int_{i/N_n}^{(i+1)/N_n} \big(\Phi(X(s))\big)^2 \, ds \quad i = 0, \ldots, N_n - 1,$$

where $N_n = [n^\beta]$ (here $[\cdot]$ denotes the integer part function and $\beta \in (0,1)$ a fixed parameter), can be estimated starting from the available observations, namely $\big\{Z(0), Z(1/n), \ldots, Z(n/n)\big\}$; this important idea basically relies on Itô formula. More precisely (see Lemma 7.4.2), this idea can be expressed as,

$$\widehat{Y}_{i,N_n,n} = \overline{Y}_{i,N_n} + \mathscr{E}_{i,N_n,n} \quad i = 0, \ldots, N_n - 1, \qquad (**)$$

where:

- the $\widehat{Y}_{i,N_n,n}$'s are the approximate values of the $\overline{Y}_{i,N_n}$'s, these approximate values can be obtained through normalized quadratic variations of some of the observations $Z(i/n)$;
- the $\mathscr{E}_{i,N_n,n}$'s denote the error terms, observe that these random variables have nice properties:

(i) conditional on the σ-algebra $\mathcal{G}_X = \sigma(X(s), 0 \leq s \leq 1)$, their distributions are independents,

(ii) their rate of convergence to 0 when N_n goes to infinity, can be controled through their conditional moments $|\mathbb{E}(\mathscr{E}_{i,N_n,n}|\mathcal{G}_X)|$, $\mathbb{E}(\mathscr{E}^2_{i,N_n,n}|\mathcal{G}_X)$ and $\mathbb{E}(\mathscr{E}^4_{i,N_n,n}|\mathcal{G}_X)$, more precisely, there is a random variable C of finite moment of any order, such that one has almost surely, for all n big enough,

$$|\mathbb{E}(\mathscr{E}_{i,N_n,n}|\mathcal{G}_X)| \leq C N_n^{-(\beta^{-1}-1)},$$

$$\mathbb{E}(\mathscr{E}^2_{i,N_n,n}|\mathcal{G}_X) \leq C N_n^{-(\beta^{-1}-1)}$$

and

$$\mathbb{E}(\mathscr{E}^4_{i,N_n,n}|\mathcal{G}_X) \leq C N_n^{-2(\beta^{-1}-1)}.$$

Thus the formulation $(**)$ allows to recover a classical statistical framework of the form: "signal+noise".

Also, it is worth to notice that Taylor-Lagrange formula allows to show that the generalized increments of the $\overline{Y}_{i,N}$'s are closely connected with the generalized increments of the local average values $\overline{X}_{i,N_n}$, $i = 0, \ldots, N_n - 1$, of the process $\{X(s)\}_{s\in[0,1]}$ itself, defined as,

$$\overline{X}_{i,N_n} = N_n \int_{i/N_n}^{(i+1)/N_n} X(s)\, ds.$$

Similarly to Gloter and Hoffmann [35, 36], in order to make statistical inference in the setting of models of type $(*)$ or $(**)$, we will very often use methods which mainly consist in the following three steps:

- we build an estimator starting from the $\overline{X}_{i,N_n}$'s;
- then we obtain from it an estimator which depends on the $\overline{Y}_{i,N_n}$'s;
- finally, we replace in the latter estimator the $\overline{Y}_{i,N_n}$'s by their approximate values $\widehat{Y}_{i,N_n,n}$'s.

Let us know describe the content of each chapter in the thesis. We limit ourselves here to a brief description that provides an overview of the entire thesis; note in passing that at the beginning of each chapter, a detailed introduction describes more explicitly, the issues that will be studied.

Recall that, Istas and Lang (see [42]) have built asymptotically normal estimators of the uniform Hölder exponent[2] of a stationary increments Gaussian process $\{S(t)\}_{t\in[0,1]}$ belonging to a wide class; their estimators are obtained by the generalized quadratic variation method, starting from the observation of the true values of $\{S(t)\}_{t\in[0,1]}$ over a regular grid, namely $\{S(0), S(1/N), \ldots, S(N/N)\}$. In the first chapter of the thesis (chapter number 3), basically, it is shown that the main results

[2] This exponent provides a measure of the global Hölder regularity of a process over an interval.

of Istas and Lang remain valid when $\{S(0), S(1/N), \ldots, S(N/N)\}$ is replaced by $\{\overline{S}_{0,N}, \ldots, \overline{S}_{N-1,N}\}$, that is one observes the local average values of $\{S(t)\}_{t\in[0,1]}$, over a regular grid, instead of its true values. The motivations are the following: (a) when the process $\{S(t)\}_{t\in[0,1]}$ is erratic (that is it fluctuates a lot over time), it seems to be more realistic to say that one observes, over a regular grid, its local average values instead of its true values; (b) as we have mentioned before, local average values play an important role in the statistical study of models of the type $(*)$ or $(**)$. At last, note that a version of this chapter has already been published in the journal "Statistics and Probability Letters".

Before making a statistical study of models of the type $(*)$, it seems to be useful to have an intuitive representation of these models. To this end, the study of the global and the local Hölder regularity of their trajectories as well as their simulation can be of some help; here are the main goals of the second chapter of the thesis (chapter number 4). It is worth noticing that the main results of the chapter, are valid, not only in the case where $\{X(s)\}_{s\in[0,1]}$ is mBm, but also in the much more general setting in which $\{X(s)\}_{s\in[0,1]}$ is an arbitrary Gaussian process whose trajectories satisfy a uniform Hölder condition of any arbitrary order $\alpha > 1/2$. In this chapter, first, it is shown that, with probability 1, the trajectories of $\{Z(t)\}_{t\in[0,1]}$ belong to any Hölder space $C^\gamma([0,1])$ with $\gamma < 1/2$ and that, the pointwise Hölder exponent of $\{Z(t)\}_{t\in[0,1]}$ at any point in $(0,1)$, equals almost surely to $1/2$. Then, in order to have an efficient simulation method of these trajectories, a random series representation of $\{Z(t)\}_{t\in[0,1]}$, via the Haar basis, is introduced; the main idea which allows to obtain it, consists in expanding for all fixed $t \in [0,1]$ and ω, the function $s \mapsto \Phi(X(s,\omega))\mathbb{1}_{[0,t]}(s)$ in the Haar basis of the Lebesgue Hilbert space $L^2([0,1])$ and then in using the isometry property of the stochastic integral in $(*)$. At last, it is shown that, with probability 1, the series is convergent in all the spaces $C^\gamma([0,1])$ with $\gamma < 1/2$ and a sharp estimation, of its convergence rate, in the sense of the norm $\|\cdot\|_{C^\gamma([0,1])}$, is given.

We have already mentioned that the pointwise Hölder exponent of the hidden mBm $\{X(s)\}_{s\in[0,1]}$ may evolve considerably over time, while that of its corrupted version $\{Z(t)\}_{t\in[0,1]}$ remains constant; this means that *there is a considerable loss of information when one observes the values of* $\{Z(t)\}_{t\in[0,1]}$ *over a grid, instead of those of* $\{X(s)\}_{s\in[0,1]}$. The main goal of the last two chapters of the thesis is to show that, in spite of this loss of information, it is still possible to construct consistent estimators of some relevant indices related to $\{X(s)\}_{s\in[0,1]}$ and/or to the unknown function Φ.

In the third chapter of the thesis (chapter number 5), one assumes that the functional parameter $H(\cdot)$ of mBm, is known and takes its values in the open interval $(1/2,1)$, h denotes an arbitrary known C^1 function having slow increase at infinity as well as its derivative, at last $(\mu_N)_N$ and $(\nu_N)_N$ are two arbitrary sequences satisfying for each N, $0 \le \mu_N < \nu_N \le 1$ and $\lim_{N\to+\infty} N(\nu_N - \mu_N) = +\infty$. A first result of the chapter shows that, when one constructs starting from $\{\overline{X}_{i,N} : i \in \mathbb{N} \text{ and } i/N \in [\mu_N, \nu_N]\}$, the generalized quadratic variation weighted by the function h, one gets an estimator of the integral:

$$\frac{1}{\nu_N-\mu_N}\int_{\mu_N}^{\nu_N} h(X(s))\,ds.$$

A second result, which generalizes the first result, shows that when one constructs starting from $\big\{\overline{Y}_{i,N}\,:\,i\in\mathbb{N}\text{ and } i/N\in[\mu_N,\nu_N]\big\}$, $\overline{V}(h;\mu_N,\nu_N)$ the generalized quadratic variation weighted by the function h, one gets an estimator of the integral:

$$\frac{1}{\nu_N-\mu_N}\int_{\mu_N}^{\nu_N} \big(f'(X(s))\big)^2 h(Y(s))\,ds.$$

A third result shows that when in the expression giving $\overline{V}(h;\mu_N,\nu_N)$, the $\overline{Y}_{i,N}$'s are replaced by the $\widehat{Y}_{i,N_n,n}$'s and one adds a correction term, we still have an estimator, denoted by $\widehat{V}(h;\mu_{N_n},\nu_{N_n})$, which converges to the same integral. Notice that these three estimators converge in $L^1(\Omega)$ at the rate $\big(N(\nu_N-\mu_N)\big)^{-1/2}$. At last, a fourth result, shows that in the case of a linear multifractional stochastic volatility model (i.e. $\Phi(x)=\theta x$ for all real x), under some conditions, an estimator $\widehat{\theta}_n^2$ of θ^2, can be obtained starting from $\widehat{V}(1;\mu_{N_n},\nu_{N_n})$; moreover $\widehat{\theta}_n^2$ converges in probability, at least at the rate,

$$n^{-\left(4\max_{s\in[\mu_{N_n},\nu_{N_n}]}H(s)+2\right)^{-1}}(\nu_{N_n}-\mu_{N_n})^{-1/2}.$$

At last notice that a version of this chapter, has been published in a book edited by "Springer" (http://www.springer.com/mathematics/probability/book/978-3-642-22367-9).

In the fourth chapter of the thesis (chapter number 6), we denote by $(v(N))_{N\geq 3}$ an arbitrary sequence of strictly positive real numbers, which converges to 0 more quickly than $(\log N)^{-1}$ and less quickly than $(\log N)^2/N$, we fix t_0 an arbitrary point of $(0,1)$ and for N large enough, we set

$$\nu_N(t_0)=\big\{i\in\mathbb{N}\,:\,|t_0-i/N|\leq v(N)\big\}.$$

A first result shows that the generalized quadratic variation corresponding to $\big\{\overline{X}_{i,N}\,:\,i\in\nu_N(t_0)\big\}$, allows to obtain an estimator of $H(t_0)$ which converges in probability; moreover, under the additional condition:

$$\sum_{N=3}^{+\infty}\big(Nv(N)\big)^{-2}<\infty, \tag{$\mathcal{C}$}$$

the estimator converges almost surely; on the other hand, when one imposes the condition:

$$v(N)=o\left(N^{-1/3}(\log N)^{-2/3}\right),$$

the estimator becomes asymptotically normal. A second result shows that the generalized quadratic variation corresponding to $\big\{\overline{Y}_{i,N}\,:\,i\in\nu_N(t_0)\big\}$, allows to obtain an estimator of $H(t_0)$ which converges in probability; moreover, under the additional condition $(\mathcal{C})$ the estimator converges almost surely; on the other hand, when one imposes the conditions $H(t_0) \in (1/2, 1)$ and

$$v(N) = o\left(N^{-1/2}(\log N)^{\eta}\right),$$

where $\eta > 1/2$ is arbitrary, the estimator becomes asymptotically normal. At last a third result shows that for some well-chosen values of the parameter β, the second result remains true when the $\overline{Y}_{i,N}$'s are replaced by the $\widehat{Y}_{i,N_n,n}$'s.

Chapter 3

Uniform Hölder exponent of a stationary increments Gaussian process: estimation starting from average values

Contents

3.1 Introduction

Since several years, there has been a considerable interest in the statistical estimation of some indices related to Hölder regularity of sample paths of stochastic processes, we refer e.g. to [10, 16, 11, 12, 14, 22, 37, 42]; as for instance the uniform Hölder exponent over a compact interval. Let us recall the definition of the latter exponent. Denote by $\{X(t)\}_{t\in\mathbb{R}}$ a real-valued stochastic process and by K a fixed compact interval in $\mathbb{R}$; one says that a sample path $X(\cdot,\omega): t \longmapsto X(t,\omega)$ belongs to the Hölder space $C^\gamma(K)$ where $\gamma \in \mathbb{R}_+ \setminus \mathbb{Z}_+$, if the following two conditions (i) and (ii) are satisfied:

(i) $X(\cdot,\omega)$ is a $[\gamma]$-times continuously differentiable function over K, where $[\cdot]$ denotes the integer part function;

(ii) there is a constant $c = c(K,\omega) > 0$ such that for all $s_1, s_2 \in K$,

$$\left|X^{([\gamma])}(s_1,\omega) - X^{([\gamma])}(s_2,\omega)\right| \le c|s_1 - s_2|^{\gamma-[\gamma]}, \tag{3.1.1}$$

where the function $X^{([\gamma])}(\cdot,\omega)$ denotes the derivative of order $[\gamma]$ of the function $X(\cdot,\omega)$, with the convention that $X^{(0)}(\cdot,\omega) = X(\cdot,\omega)$.

The uniform Hölder exponent of $X(\cdot,\omega)$ over K is defined as,

$$h_X(K,\omega) := \sup\left\{\gamma \in \mathbb{R}_+ \setminus \mathbb{Z}_+ : \quad X(\cdot,\omega) \in C^\gamma(K)\right\}.$$

From now on, we suppose that $\{X(t)\}_{t\in\mathbb{R}}$ is a centered stationary increments Gaussian process satisfying almost surely $X(0)=0$; thus the distribution of $\{X(t)\}_{t\in\mathbb{R}}$ is completely determined by its variogram, namely the even function v defined for all $t\in\mathbb{R}$, as,

$$v(t)=2^{-1}\mathbb{E}\left(X(t)\right)^2; \tag{3.1.2}$$

moreover it follows from zero-one law that there is a deterministic quantity $H=H_X(K)\in[0,+\infty]$ such that one has, for almost all ω, $h_X(K,\omega)=H_X(K)$. There is no restriction to assume that $K=[0,1]$. In their seminal article [42], by using the notion of generalized quadratic variation, Istas and Lang have constructed, under some assumptions, asymptotically normal estimators of H, starting from the observation of $\{X(i\delta_N)\}_{i=0,\ldots,[\delta_N]^{-1}-1}$, the true values of X over a regular grid. However, in the setting of some applications, for example when one has to model a quite fluctuating signal, it sounds to be more realistic to say that one observes average values of the process X, namely $\{\delta_N^{-1}\int_{i\delta_N}^{(i+1)\delta_N}X(s)\,\mathrm{d}s\}_{i=0,\ldots,[\delta_N]^{-1}-1}$, rather than a true discretized trajectory of X. The goal of this chapter is to construct a strongly consistent and asymptotically normal estimator of H starting from such data. From now on, for the sake of simplicity, we assume that the discretization mesh $\delta_N=1/N$ (N being an integer big enough) and we set

$$\left\{\overline{X}_{i,N}\right\}_{i=0,\ldots,N-1}:=\left\{N\int_{i/N}^{(i+1)/N}X(s)\,\mathrm{d}s\right\}_{i=0,\ldots,N-1}. \tag{3.1.3}$$

Our results as well as their proofs are inspired by [42], however new difficulties appear in our setting; they are mainly due to the fact that

$$\mathrm{Cov}\Big(\sum_{k=0}^{p}a_k\overline{X}_{i+k,N},\sum_{k'=0}^{p}a_{k'}\overline{X}_{j+k',N}\Big),$$

is more difficult to estimate, than

$$\mathrm{Cov}\Big(\sum_{k=0}^{p}a_kX\big(\frac{i+k}{N}\big),\sum_{k'=0}^{p}a_{k'}X\big(\frac{j+k'}{N}\big)\Big),$$

here $p\geq 1$ is an integer and $a=(a_0,\ldots,a_p)$ denotes a finite sequence of real numbers satisfying Assumption (3.2.6).

3.2 Statement of the main results

Let us first precisely present the assumptions we need for obtaining our main results. Note in passing that these assumptions are nearly similar to some fundamental hypotheses in [42].

(A1) Assumptions on the variogram function v: We assume that there exists a finite nonnegative integer d such that v is $2d$-times continuously differentiable on $[-2,2]$ and v is not $2(d+1)$-times continuously differentiable on this interval. We denote by $v^{(2d)}$ the derivative of v of order $2d$, with the convention that $v^{(0)} = v$. Also we assume that there are two real numbers $c \neq 0$ and $0 < s_0 < 2$ such that for all $t \in [-2, 2]$, one has

$$v^{(2d)}(t) = v^{(2d)}(0) + c|t|^{s_0} + r(t), \tag{3.2.1}$$

where the remainder r satisfies the following two properties:

- For $|t|$ small enough, one has,

$$r(t) = o(|t|^{s_0}). \tag{3.2.2}$$

- There are two real numbers $c > 0$, $w > s_0$ and an integer $q > w + 1/2$ such that the remainder r is q-times continuously differentiable on $[-2, 2] \setminus \{0\}$ and for all $t \in [-2, 2] \setminus \{0\}$, one has

$$|r^{(q)}(t)| \leq c|t|^{w-q}. \tag{3.2.3}$$

□

The integers d and q are supposed to be known; in fact the unknown parameter we want to estimate, starting from the data (3.1.3), is s_0. It is worth noticing that Assumption (A1) implies (see for instance [40] or [26]) that the uniform Hölder exponent H satisfies,

$$H = d + \frac{s_0}{2}. \tag{3.2.4}$$

Though, this assumption might seem to be a bit technical, it is satisfied (see [42]) by fractional Brownian motion (i.e. $v(t) = c|t|^{2\alpha}$ where $c > 0$ is a constant and $\alpha \in (0, 1)$ the Hurst parameter) and other, more or less, classical classes of stationary increments Gaussian processes (for example when $v(t) = 1 - exp(-|t|^{\beta})$, where $\beta \in (0, 2)$ is a parameter; observe that in this case $d = 0$ and $s_0 = \beta$, moreover standard computation show that one can take $q = 3$).

For any integer $N \geq p + 1$, the generalized increments of the average values $\overline{X}_{i,N}$, $i = 0, \ldots, N-1$ of the process X are defined as,

$$\left\{\Delta_a \overline{X}_{i,N}\right\}_{i=0,\ldots,N-p-1} = \left\{\sum_{k=0}^{p} a_k \overline{X}_{i+k,N}\right\}_{i=0,\ldots,N-p-1}, \tag{3.2.5}$$

where $a = (a_0, \ldots, a_p) \in \mathbb{R}^{p+1}$ is an arbitrary finite fixed sequence with $M(a) \geq d + q/2$ vanishing moments, that is:

$$\sum_{k=0}^{p} k^l a_k = 0, \text{ for all } l = 0, \ldots, M(a) - 1 \text{ and } \sum_{k=0}^{p} k^{M(a)} a_k \neq 0 \tag{3.2.6}$$

The integer $M(a)$ is called the order of the generalized increment and one always has $p \geq M(a)$. For example, $a^{(1)} = (1, -1)$ is of order 1 and $a^{(2)} = (1, -2, 1)$ is of order 2.

Note in passing that (3.2.6) implies that for all $l \in \{0, \dots, 2M(a)-1\}$,

$$\sum_{k=0}^{p}\sum_{k'=0}^{p} a_k a_{k'} (k-k')^l = 0. \tag{3.2.7}$$

The important idea of replacing usual 1-order increments by generalized increments has been initially introduced by Istas and Lang [42]. The main advantage in doing so, is that the statistical estimator of H, defined through generalized quadratic variation, is asymptotically normal whatever the value of H might be. Basically, this asymptotic normality comes from the fact that generalized increments are less correlated than usual increments.

In fact, we need to impose to the sequence a an additional assumption.

(A2) Assumption related to the generalized increments: For all $\nu \in (0,2)$, one has

$$R\big(0,1,2d,(\cdot)^\nu\big) \neq 0, \tag{3.2.8}$$

where $R\big(0,1,0,(\cdot)^\nu\big)$ is defined by (3.4.5) and $R\big(0,1,2d,(\cdot)^\nu\big)$ is defined by (3.4.6) when $d \geq 1$. □

It is worth noticing that standard computations allow to show that: the sequence $a^{(2)} = (1,-2,1)$ has 2 vanishing moments (i.e. $M(a^{(2)}) = 2$) and satisfies Assumption (A2) when $d = 0$; the sequence $a^{(3)} = (1,-3,3,-1)$ has 3 vanishing moments (i.e. $M(a^{(3)}) = 3$) and satisfies Assumption (A2) when $d \in \{0,1\}$.

Now we are in position to state the two main results of this chapter.

Theorem 3.2.1 *Let us denote by*

$$\widehat{H}_N = \frac{1}{2}\Big(1 + \log_2\big(\frac{V_N}{V_{2N}}\big)\Big), \tag{3.2.9}$$

where V_N is the generalized quadratic variation defined as,

$$V_N = \sum_{i=0}^{N-p-1} \big(\Delta_a \overline{X}_{i,N}\big)^2. \tag{3.2.10}$$

Then, under Assumptions (A1) and (A2), when $N \to +\infty$, $\widehat{H}_N$ converges almost surely to H,

Theorem 3.2.2 *Under the same assumptions as in Theorem 3.2.1 and the additional assumption that*

$$r(t) = o(|t|^{s_0+1/2}), \tag{3.2.11}$$

a Central Limit Theorem holds, namely $N^{1/2}(\widehat{H}_N - H)$ converges in law to a centered Gaussian random variable.

3.3 Some simulations

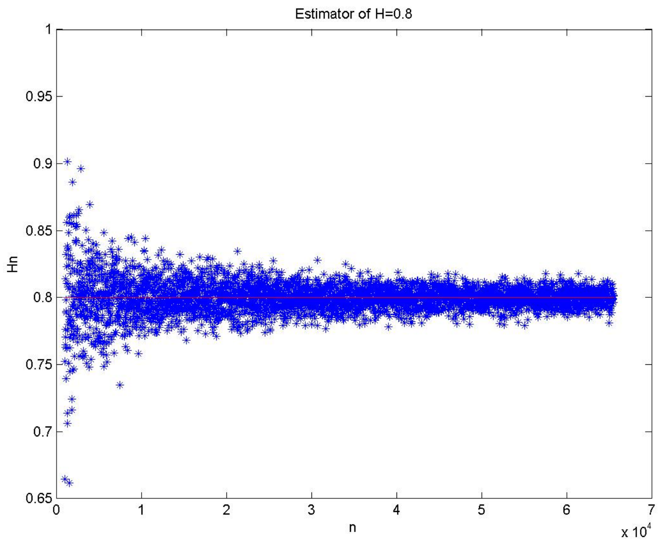


Figure 3.1: Simulation of the convergence of $\widehat{H}_N$, for a fBm of Hurst parameter 0.8

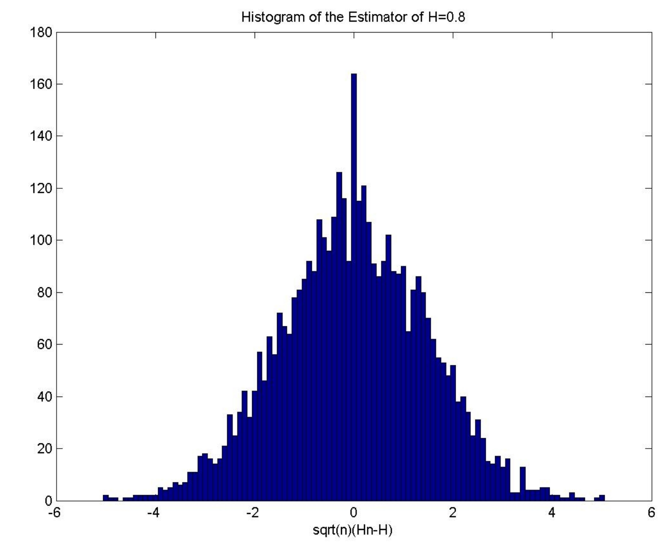


Figure 3.2: Histogram of $N^{1/2}(\widehat{H}_N - 0.8)$

3.4 Proof of the main results

First it is convenient to notice that the stationarity of the increments of the process X, implies that for all $N \geq p+1$ and $i \in \{0, \ldots, N-p-1\}$, one has

$$\sigma_{a,N}^2 := \mathrm{Var}(\Delta_a \overline{X}_{0,N}) = \mathrm{Var}(\Delta_a \overline{X}_{i,N}). \tag{3.4.1}$$

The proof of Theorem 3.2.1 mainly relies on the following proposition.

Proposition 3.4.1 *Under Assumptions (A1) and (A2), there exist two constants $c_1 > 0$ and $c_2 > 0$, such that the following two equalities hold for all $N \geq p+1$:*

$$\sigma_{a,N}^2 = c_1 N^{-2H} + o(N^{-2H}), \tag{3.4.2}$$

and

$$s_N^2 := Var\Big(\sum_{i=0}^{N-p-1} \frac{(\Delta_a \overline{X}_{i,N})^2}{\sigma_{a,N}^2}\Big) = c_2 N + o(N). \tag{3.4.3}$$

Now, let us focus on the proof of Proposition 3.4.1. In order to show that the latter proposition holds, we need several preliminary results. The following lemma (whose proof has been omitted since it is more or less similar to that of Lemma 1 in [35]) gives a nice expression of $\mathrm{Cov}\big(\Delta_a \overline{X}_{i,N}, \Delta_a \overline{X}_{j,N}\big)$, in terms of the variogram function v.

Lemma 3.4.2 *For all integer $N \geq p+1$ and all $i, j \in \{0, \ldots, N-p-1\}$, the following equality holds:*

$$\mathbb{E}\left(\Delta_a \overline{X}_{i,N} \Delta_a \overline{X}_{j,N}\right) = -N^2 \sum_{0 \leq k,l \leq p} a_k a_l \Big(\int_0^{\frac{1}{N}} \int_0^{\frac{1}{N}} v\Big(\frac{|i-j|}{N} + s - s' + \frac{k-l}{N}\Big)\, \mathrm{d}s'\, \mathrm{d}s \Big). \tag{3.4.4}$$

Next we will use (3.4.4) and (3.2.1) for estimating $\mathbb{E}\left(\Delta_a \overline{X}_{i,N} \Delta_a \overline{X}_{j,N}\right)$; to this end, we need to introduce some notations. Let h and g be two real-valued Borel functions defined on the real line, let $N \geq p+1$ and $u \geq 1$ be two integers and let $x \in \mathbb{R}$, we set

$$R(x, N, 0, h) = \sum_{0 \leq k,l \leq p} a_k a_l \int_0^{\frac{1}{N}} \int_0^{\frac{1}{N}} h\big(x + s - s' + \frac{k-l}{N}\big)\, \mathrm{d}s'\, \mathrm{d}s, \tag{3.4.5}$$

and

$$\begin{aligned} R(x, N, u, g) \;&=\; \sum_{0 \leq k,l \leq p} a_k a_l (k-l)^u \int_0^{\frac{1}{N}} \int_0^{\frac{1}{N}} \int_0^1 \frac{(1-\eta)^{u-1}}{(u-1)!} \\ &\quad \times g\big(x + s - s' + \frac{(k-l)\eta}{N}\big)\, \mathrm{d}\eta\, \mathrm{d}s'\, \mathrm{d}s. \end{aligned} \tag{3.4.6}$$

Of course we assume that h, g, N, u and x have been chosen in such a way that all the integrals in (3.4.5) and (3.4.6) are well-defined and finite. Observe that in view of Lemma 3.4.2, one has

$$\mathbb{E}\left(\Delta_a\overline{X}_{i,N}\Delta_a\overline{X}_{j,N}\right) = -N^2R\Big(\frac{|i-j|}{N}, N, 0, v\Big). \tag{3.4.7}$$

In the sequel we set $m=|i-j|$. Let us now give a nice property of R.

Lemma 3.4.3 *Let $x\in\mathbb{R}$ and u be an integer such that $0\le u\le 2M(a)$. Assume that h is u-times continuously differentiable on the interval $[x-(p+1)/N, x+(p+1)/N]$. Then, for all integer u', satisfying $0\le u'\le u$, $R(x,N,u',h^{(u')})$ is well-defined and one has*

$$R(x,N,u',h^{(u')}) = N^{-(u-u')}R(x,N,u,h^{(u)}). \tag{3.4.8}$$

Proof of Lemma 3.4.3: First observe that, Lemma 3.4.3 clearly holds when $u=0$, so from now on we assume that $u\ge 1$. Next observe that for all $(s,s')\in[0,1/N]^2$ and all $(k,l)\in\{0,\dots,p\}^2$, h is a C^u function on the compact interval of extremities $x+s-s'$ and $x+s-s'+(k-l)/N$. By applying Taylor formula to h on this interval, one has

$$h\Big(x+s-s'+\frac{k-l}{N}\Big) = \sum_{m=0}^{u-1}\frac{h^{(m)}(x+s-s')}{m!}\Big(\frac{k-l}{N}\Big)^m + \int_0^1\frac{(1-\eta)^{u-1}}{(u-1)!}h^{(u)}\Big(x+s-s'+\frac{(k-l)\eta}{N}\Big)\,\mathrm{d}\eta\times\Big(\frac{k-l}{N}\Big)^u.$$

Then using (3.4.5), (3.2.7) and (3.4.6), it follows that

$$R(x,N,0,h) = N^{-u}R(x,N,u,h^{(u)}). \tag{3.4.9}$$

By replacing in (3.4.9) u by u', one also has

$$R(x,N,0,h) = N^{-u'}R(x,N,u',h^{(u')}). \tag{3.4.10}$$

Finally combining (3.4.9) with (3.4.10) one obtains (3.4.8). □

The following remark is a consequence of (3.4.8), (3.4.5), (3.4.6), (3.2.1) and (3.2.7).

Remark 3.4.1 *For all integer $N\ge p+1$ and for all $m\in\{0,\dots,N-p-1\}$, one has*

$$R\big(\frac{m}{N},N,0,v\big) = N^{-2d}\Big(cR\big(\frac{m}{N},N,2d,|\cdot|^{s_0}\big)+R\big(\frac{m}{N},N,2d,r\big)\Big). \tag{3.4.11}$$

Now our goal will be to estimate $R\big(m/N,N,2d,|\cdot|^{s_0}\big)$ and $R\big(m/N,N,2d,r\big)$.

The following lemma can be obtained by setting in the integrals in (3.4.5) and (3.4.6), $(y,y')=(Ns,Ns')$, and then by using the fact that $|\cdot|^{s_0}$ is a homogeneous function of degree s_0.

Lemma 3.4.4 *For all integers $N \geq p+1$, $m \in \{0, \dots, N-p-1\}$ and $u \geq 0$, one has*

$$R\big(\frac{m}{N}, N, u, |\cdot|^{s_0}\big) = N^{-2-s_0} R\big(m, 1, u, |\cdot|^{s_0}\big). \tag{3.4.12}$$

Lemma 3.4.5 *There is a constant $c > 0$, only depending on d, q and a, such that one has for all $m \in \{p+2, \dots, N-p-1\}$,*

$$\big|R\big(m, 1, 2d, |\cdot|^{s_0}\big)\big| \leq c(m-p-1)^{s_0-q}. \tag{3.4.13}$$

Proof of Lemma 3.4.5: First notice that for all $s, s' \in [0,1]$, $m \in \{p+2, \dots, N-p-1\}$, $k, l \in \{0, \dots, p\}$ and $\eta \in [0,1]$, one has

$$m + s - s' + (k-l)\eta \geq m - p - 1 = 1 > 0. \tag{3.4.14}$$

Then by using the definition of R ((3.4.5) and (3.4.6)) as well as (5.4.1), one gets

$$R\big(m, 1, 2d, |\cdot|^{s_0}\big) = R\big(m, 1, 2d, (\cdot)^{s_0}\big). \tag{3.4.15}$$

Moreover, denoting by $\big((\cdot)^{s_0}\big)^{(q)}$ the derivative of order q of the function $z \mapsto z^{s_0}$, it follows from Lemma 3.4.3 that

$$\begin{aligned} R\big(m, 1, 2d, (\cdot)^{s_0}\big) &= R\big(m, 1, 2d+q, \big((\cdot)^{s_0}\big)^{(q)}\big) \\ &= c_1 R\big(m, 1, 2d+q, (\cdot)^{s_0-q}\big), \end{aligned} \tag{3.4.16}$$

where c_1 is a constant only depending on s_0 and q. Since $2d+q \geq 1$, again by using (3.4.6), one has

$$\begin{aligned} R\big(m, 1, 2d+q, (\cdot)^{s_0-q}\big) &= \sum_{0 \leq k,l \leq p} a_k a_l (k-l)^{2d+q} \int_0^1 \frac{(1-\eta)^{2d+q-1}}{(2d+q-1)!} \\ &\times \int_0^1\int_0^1 \big(m + s - s' + (k-l)\eta\big)^{s_0-q} \,\mathrm{d}s' \,\mathrm{d}s \,\mathrm{d}\eta. \end{aligned} \tag{3.4.17}$$

Then, it results from (3.4.17), the triangle inequality, (5.4.1) and the inequality $s_0 - q < 0$, that

$$\big|R\big(m, 1, u+q, (\cdot)^{s_0-q}\big)\big| \leq c_2 (m-p-1)^{s_0-q}, \tag{3.4.18}$$

where c_2 is a constant only depending on a, d and q. Finally, putting together (3.4.15), (3.4.16) and (3.4.18) one obtains the lemma.□

Lemma 3.4.6 *For all $N \geq p+1$ and $u \in \mathbb{Z}_+$, let us set*

$$r^*_{u,N} = \max_{0 \leq m \leq p+1} \big|R\big(\frac{m}{N}, N, u, r\big)\big|. \tag{3.4.19}$$

Then one has, when N is big enough

$$r^*_{u,N} = o(N^{-s_0-2}). \tag{3.4.20}$$

Proof of Lemma 3.4.6: We will only show that the lemma holds in the case where $u=0$, since it can be proved similarly in the case where $u\ge 1$. First observe that, by using the triangle inequality, one has that, for all integer $0\le m\le p+1$, all $s,s'\in[0,1/N]$ and $k,l\in\{0,\dots,p\}$,

$$\Big|\frac{m}{N}+s-s'+\frac{k-l}{N}\Big|\le\frac{m}{N}+|s-s'|+\frac{|k-l|}{N}\le\frac{2p+2}{N}. \tag{3.4.21}$$

Moreover, the assumption $r(x)=o(|x|^{s_0})$ implies that, there is a sequence $\{e_N\}_{N\ge p+1}$ of positive real-numbers converging to 0, such that for all real number x satisfying $|x|\le(2p+2)/N\le 2$, one has

$$\big|r(x)\big|\le e_N|x|^{s_0}. \tag{3.4.22}$$

It follows from (3.4.5), (3.4.21) and (3.4.22) that

$$\begin{aligned}
&\big|R\big(\frac{m}{N},N,0,r\big)\big|\\
&=\Big|\sum_{0\le k,l\le p}a_ka_l\int_0^{\frac{1}{N}}\int_0^{\frac{1}{N}}r\big(\frac{m}{N}+s-s'+\frac{k-l}{N}\big)\,\mathrm{d}s\,\mathrm{d}s'\Big|\\
&\le e_N\sum_{0\le k,l\le p}|a_ka_l|\int_0^{\frac{1}{N}}\int_0^{\frac{1}{N}}\big(\frac{m}{N}+|s-s'|+\frac{|k-l|}{N}\big)^{s_0}\,\mathrm{d}s\,\mathrm{d}s'\\
&=ce_NN^{-s_0-2},
\end{aligned} \tag{3.4.23}$$

where $c>0$ is a constant only depending on a. □

Lemma 3.4.7 *For all $N\ge p+1$, there is a constant $c>0$, only depending on a, r and d, such that one has for each $m\in\{p+2,\dots,N-p-1\}$,*

$$\big|R\big(\frac{m}{N},N,2d,r\big)\big|\le cN^{-w-2}(m-p-1)^{w-q}. \tag{3.4.24}$$

Proof of Lemma 3.4.7: First observe that, in view of Assumption (A1), r is q-times continuously differentiable on the interval $[m/N-(p+1)/N,m/N+(p+1)/N]\subset[1/N,1]$. Therefore we are allowed to use Lemma 3.4.3 and we obtain that,

$$R\big(\frac{m}{N},N,2d,r\big)=N^{-q}R\big(\frac{m}{N},N,2d+q,r^{(q)}\big). \tag{3.4.25}$$

Moreover (3.4.6) implies that,

$$\begin{aligned}
R\big(\frac{m}{N},N,2d+q,r^{(q)}\big)\;=\;&\sum_{0\le k,l\le p}a_ka_l(k-l)^{2d+q}\int_0^{\frac{1}{N}}\int_0^{\frac{1}{N}}\int_0^1\frac{(1-\eta)^{2d+q-1}}{(2d+q-1)!}\\
&\times r^{(q)}\big(\frac{m}{N}+s-s'+\frac{(k-l)\eta}{N}\big)\,\mathrm{d}\eta\,\mathrm{d}s'\,\mathrm{d}s.
\end{aligned} \tag{3.4.26}$$

Then, it follows from (3.4.26), the triangle inequality, (3.2.3) and the inequality $w-q<-1/2<0$ that

$$\begin{aligned}\big|R(\frac{m}{N},N,u,r^{(q)})\big| &\le c_1\sum_{0\le k,l\le p}|a_ka_l||k-l|^{2d+q}\\ &\quad\times\int_0^{\frac{1}{N}}\int_0^{\frac{1}{N}}\int_0^1\frac{\big|m/N+s-s'+(k-l)\eta/N\big|^{w-q}}{(2d+q-1)!}\,\mathrm{d}\eta\,\mathrm{d}s'\,\mathrm{d}s\\ &\le c_2N^{-w+q-2}(m-p-1)^{w-q}, \end{aligned}\qquad (3.4.27)$$

where $c_1>0$ is the constant c in (3.2.3) and $c_2>0$ is a constant only depending on a, r and d. Finally, putting together (3.4.25), (3.4.26) and (3.4.27) we get the lemma. □

Let us now recall a useful result concerning centered, 2-D, Gaussian random vectors whose proof is given in Appendix (see the proof of Lemma 5.3.4).

Lemma 3.4.8 *Let (Z,Z') be a centered, 2-D, Gaussian random vector and let us assume that the variances of Z and Z' are equal to ν. Then*

$$\mathbb{E}\left((Z^2-\nu)(Z'^2-\nu)\right)=2\big(Cov(Z,Z')\big)^2. \qquad (3.4.28)$$

Now we are in position to prove Proposition 3.4.1.

Proof of Proposition 3.4.1: First observe that it follows from (3.4.7) and (3.4.11) that for all integers $N\ge p+1$ and $i,j\in\{0,\dots,N-p-1\}$, one has

$$\mathbb{E}\left(\Delta_a\overline{X}_{i,N}\Delta_a\overline{X}_{j,N}\right)=-N^{2-2d}\Big(cR\big(\frac{|i-j|}{N},N,2d,|\cdot|^{s_0}\big)+R\big(\frac{|i-j|}{N},N,2d,r\big)\Big). \qquad (3.4.29)$$

Taking $i=j$ in (3.4.29) and using (3.4.1), we get

$$\sigma_{a,N}^2=\mathrm{Var}\big(\Delta_a\overline{X}_{i,N}\big)=-N^{2-2d}\big(cR(0,N,2d,|\cdot|^{s_0})+R(0,N,2d,r)\big). \qquad (3.4.30)$$

Then (3.4.2) results from (3.4.30), Lemma 3.4.4, (3.2.4), (3.2.8) and Lemma 3.4.6.

Let us now prove that Relation (3.4.3) holds. We denote by $\rho_N(|i-j|)$ the correlation coefficient between $\Delta_a\overline{X}_{i,N}$ and $\Delta_a\overline{X}_{j,N}$, i.e.

$$\rho_N(|i-j|)=\frac{\mathbb{E}\left(\Delta_a\overline{X}_{i,N}\Delta_a\overline{X}_{j,N}\right)}{\sigma_{a,N}^2}. \qquad (3.4.31)$$

One has,

$$\begin{aligned} s_N^2 &:= \mathrm{Var}\Big(\sum_{i=0}^{N-p-1}\frac{(\Delta_a\overline{X}_{i,N})^2}{\sigma_{a,N}^2}\Big)\\ &= \mathrm{Var}\Big(\sum_{i=0}^{N-p-1}\big(\frac{(\Delta_a\overline{X}_{i,N})^2}{\sigma_{a,N}^2}-1\big)\Big)\\ &= \sum_{i=0}^{N-p-1}\sum_{j=0}^{N-p-1}\mathbb{E}\Big(\big(\frac{(\Delta_a\overline{X}_{i,N})^2}{\sigma_{a,N}^2}-1\big)\big(\frac{(\Delta_a\overline{X}_{j,N})^2}{\sigma_{a,N}^2}-1\big)\Big). \end{aligned} \qquad (3.4.32)$$

Then, it follows from (3.4.32), Lemma 3.4.8 and (3.4.31), that

$$\begin{aligned} s_N^2 &= 2\sum_{i=0}^{N-p-1}\sum_{j=0}^{N-p-1}\rho_N^2(|i-j|) \\ &= 2\sum_{|j|\le N-p-1}(N-p-|j|)\rho_N^2(|j|). \end{aligned} \quad (3.4.33)$$

From now on we split s_N^2 into two parts according to the values of j:

$$s_N^2 = F_N + G_N, \quad (3.4.34)$$

where

$$F_N = 2\sum_{|j|\le p+1}(N-p-|j|)\rho_N^2(|j|), \quad (3.4.35)$$

and

$$G_N = 2\sum_{p+2\le|j|\le N-p-1}(N-p-|j|)\rho_N^2(|j|). \quad (3.4.36)$$

It remains to show that there exist non-vanishing constants c_1 and c_2, such that

$$F_N = c_1 N + o(N), \quad (3.4.37)$$

and

$$G_N = c_2 N + o(N). \quad (3.4.38)$$

By using (3.4.29) and Lemma 3.4.4, one gets

$$\rho_N(|j|) = \frac{cR(|j|,1,2d,|\cdot|^{s_0}) + N^{2+s_0}R(|j|/N,N,2d,r)}{cR(0,1,2d,|\cdot|^{s_0}) + N^{2+s_0}R(0,N,2d,r)}. \quad (3.4.39)$$

Thanks to Lemma 3.4.6, the two terms $N^{2+s_0}R(|j|/N,N,2d,r)$ (for $|j|\le p+1$) and $N^{2+s_0}R(0,N,2d,r)$ converge to 0. This fact together with (3.4.39) implies

$$\lim_{N\to+\infty}\max_{j\in\mathbb{Z},|j|\le p+1}\left|\rho_N(|j|) - C(|j|)\right| = 0, \quad (3.4.40)$$

where for all $0\le|j|\le p+1$,

$$C(|j|) = \frac{R(|j|,1,2d,|\cdot|^{s_0})}{R(0,1,2d,|\cdot|^{s_0})}.$$

Then, by using (3.4.35) and (3.4.40), one obtains that

$$\begin{aligned} \frac{F_N - 2\sum_{|j|\le p+1}(C(|j|))^2 N}{N} &= \frac{-2\sum_{|j|\le p+1}(p+|j|)C(|j|)^2}{N} \\ &\quad + \frac{2\sum_{|j|\le p+1}(N-p-|j|)(\rho_N^2(|j|) - (C(|j|))^2)}{N} \\ &\xrightarrow[N\to+\infty]{} 0, \end{aligned}$$

which proves that (3.4.37) holds; observe that (3.2.8) implies that

$$c_1 := 2 \sum_{|j|\le p+1} (C(|j|))^2 \neq 0.$$

Let us now prove that (3.4.38) is satisfied. For the sake of simplicity, from now on, for all $|j| \in \{p+2,\dots,N-p-1\}$, we set

$$L(|j|) = R(|j|, 1, 2d, (\cdot)^{s_0}). \tag{3.4.41}$$

First, we will show that there is a non-vanishing constant c_3 such that

$$\sum_{p+2\le |j|\le N-p-1} \Big|\rho_N^2(|j|) - c_3 L^2(|j|)\Big| \xrightarrow[N\to+\infty]{} 0. \tag{3.4.42}$$

By using (3.4.31), (3.4.29) and the fact that for $|j| \ge p+1$, one has,

$$\rho_N(|j|) = \frac{-N^{2-2d}}{\sigma_{a,N}^2}\Big(cR\big(\frac{|j|}{N}, N, 2d, (\cdot)^{s_0}\big) + R\big(\frac{|j|}{N}, N, 2d, r\big)\Big). \tag{3.4.43}$$

It follows from (3.4.43), (3.2.4), (3.4.12) and (3.4.41) that,

$$\rho_N(|j|) = -cL(|j|)\frac{N^{-2H}}{\sigma_{a,N}^2} - \frac{N^{2-2d}}{\sigma_{a,N}^2} R\big(\frac{|j|}{N}, N, 2d, r\big). \tag{3.4.44}$$

Let $c_1 > 0$ be the constant introduced in (3.4.2), then by using (3.4.2), the following relation holds:

$$\big(\frac{c}{c_1}\big)L(|j|) = cL(|j|)\frac{N^{-2H}}{\sigma_{a,N}^2} + cL(|j|)M(N), \tag{3.4.45}$$

where

$$M(N) = \frac{1}{c_1} - \frac{N^{-2H}}{\sigma_{a,N}^2} = \frac{\sigma_{a,N}^2 - c_1 N^{-2H}}{c_1\sigma_{a,N}^2} = o(1). \tag{3.4.46}$$

By using the inequality $|x^2-y^2| = |(x+y)^2 - 2(x+y)y| \le (x+y)^2 + 2|(x+y)y|$ for all real numbers x and y, (3.4.44) and (3.4.45), one gets

$$\begin{aligned}
\big|\rho_N^2(|j|) - \big(\frac{c}{c_1}\big)^2 L^2(j)\big| &\le \big|\rho_N(|j|) + \big(\frac{c}{c_1}\big)L(j)\big|^2 + 2\Big|\rho_N(|j|) + \big(\frac{c}{c_1}\big)L(|j|)\Big|\Big|\big(\frac{c}{c_1}\big)L(|j|)\Big| \\
&= \Big|\frac{N^{2-2d}}{\sigma_{a,N}^2} R\big(\frac{|j|}{N}, N, 2d, r\big) - cL(|j|)M(N)\Big|^2 \\
&\quad +2\Big|\frac{N^{2-2d}}{\sigma_{a,N}^2} R\big(\frac{|j|}{N}, N, 2d, r\big) - cL(|j|)M(N)\Big|\Big|\big(\frac{c}{c_1}\big)L(|j|)\Big|.
\end{aligned} \tag{3.4.47}$$

It follows from (3.4.47), (3.4.2), (3.2.4), Lemma 3.4.7 and Lemma 3.4.5 that there exists a constant $c_4 > 0$, non depending on j and N, such that,

$$\begin{aligned}
&\big|\rho_N^2(|j|) - (\frac{c}{c_1})^2 L^2(j)\big| \\
&\le c_4\Big(\big(N^{s_0-w}(|j|-p-1)^{w-q} + |M(N)|(|j|-p-1)^{s_0-q}\big)^2 \\
&\quad +\big(N^{s_0-w}(|j|-p-1)^{w-q} + |M(N)|(|j|-p-1)^{s_0-q}\big)(|j|-p-1)^{s_0-q}\Big) \\
&\le 2c_4\Big(N^{2(s_0-w)}(|j|-p-1)^{2(w-q)} + (M(N))^2(|j|-p-1)^{2(s_0-q)} \\
&\quad +\big(N^{s_0-w} + |M(N)|\big)(|j|-p-1)^{w+s_0-2q}\Big).
\end{aligned} \tag{3.4.48}$$

Next, setting $c_3 = c^2/c_1^2$, it follows from (3.4.48), (3.4.46) and the inequalities $s_0 - w < 0$, $s_0 - q < -1/2$, $w - q < -1/2$, that (3.4.42) is satisfied.

Now we are in position to show that (3.4.38) holds. On one hand, (3.4.43) and (3.4.42) imply

$$\begin{aligned}
&\frac{\big|G_N - 2c_3\sum_{p+2\le|j|\le N-p-1}(N-p-|j|)L^2(|j|)\big|}{N} \\
&\le 2\sum_{p+2\le|j|\le N-p-1}\frac{(N-p-|j|)}{N}\big|\rho_N^2(|j|) - c_3L^2(|j|)\big| \\
&\le 4\sum_{p+2\le|j|\le N-p-1}\big|\rho_N^2(j) - c_3L^2(|j|)\big| \\
&\xrightarrow[N\to+\infty]{} 0.
\end{aligned} \tag{3.4.49}$$

On the other hand, standard computations allow to show that the sequence

$$\Big(\frac{\sum_{p+2\le|j|\le N-p-1}(N-p-|j|)L^2(|j|)}{N}\Big)_{N\ge p+1}$$

is increasing and bounded. Therefore this sequence converges to finite positive limit denoted by c_5. Then setting $c_2 = 2c_3c_5$, (3.4.49) implies that (3.4.38) is satisfied. Finally, (3.4.3) results from (3.4.34), (3.4.37) and (3.4.38). □

The previous proposition will play a crucial role in the proof of Theorem 3.2.1; the proof of this theorem will also make use of the following lemma, which, roughly speaking, means that, almost surely, the generalized quadratic variation V_N behaves like cN^{1-2H} when N goes to infinity.

Lemma 3.4.9 *There exists a constant $c > 0$ such that*

$$W_N := \frac{V_N}{cN^{1-2H}} - 1 \xrightarrow[N\to+\infty]{a.s.} 0.$$

Lemma 3.4.9 is a straightforward consequence of (3.4.2) and the following result.

Lemma 3.4.10 *The generalized quadratic variation V_N satisfies*

$$\frac{V_N}{N\sigma_{a,N}^2}-1\xrightarrow[N\to+\infty]{a.s.}0,$$

where $\sigma_{a,N}$ has been defined in (3.4.1).

Proof of Lemma 3.4.10: First, notice that, similarly to the end of the proof of Proposition 2 in [16], we can show that

$$\mathbb{E}\Big(\sum_{i=0}^{N-p-1}(\Delta_a\overline{X}_{i,N})^2-(N-p)\sigma_{a,N}^2\Big)^4\leq c_1\Big(\mathrm{Var}\Big(\sum_{i=0}^{N-p-1}(\Delta_a\overline{X}_{i,N})^2\Big)\Big)^2, \quad (3.4.50)$$

where c_1 is constant. On the other hand, it follows from (3.4.3) that,

$$\mathrm{Var}\Big(\sum_{i=0}^{N-p-1}(\Delta_a\overline{X}_{i,N})^2\Big)=s_N^2\sigma_{a,N}^4. \quad (3.4.51)$$

Then, by using (3.4.50) and (3.4.51), we get

$$\mathbb{E}\Big(\sum_{i=0}^{N-p-1}(\Delta_a\overline{X}_{i,N})^2-(N-p)\sigma_{a,N}^2\Big)^4\leq c_1s_N^4\sigma_{a,N}^8, \quad (3.4.52)$$

Next combining Markov inequality with (3.4.52) and (3.4.2), we obtain that, there exists a constant $c_2>0$ such that for all real $\eta>0$ and integer $N\geq p+1$, one has

$$\begin{aligned}&\mathbb{P}\Big(\Big|\frac{\sum_{i=0}^{N-p-1}(\Delta_a\overline{X}_{i,N})^2}{(N-p)\sigma_{a,N}^2}-1\Big|>\eta\Big)\\&\leq((N-p)\sigma_{a,N}^2)^{-4}\eta^{-4}\mathbb{E}\Big(\sum_{i=0}^{N-p-1}(\Delta_a\overline{X}_{i,N})^2-(N-p)\sigma_{a,N}^2\Big)^4\\&\leq c\eta^{-4}N^{-2}.\end{aligned} \quad (3.4.53)$$

Next, in view of (3.4.53), applying Borel-Cantelli Lemma we get that

$$\frac{\sum_{i=0}^{N-p-1}(\Delta_a\overline{X}_{i,N})^2}{(N-p)\sigma_{a,N}^2}\xrightarrow[N\to+\infty]{a.s}1. \quad (3.4.54)$$

Finally (3.2.10) and (3.4.54) imply that the lemma holds. □

Now, we are in position to prove Theorem 3.2.1.

Proof of Theorem 3.2.1: Let $c_1=c$ and W_N be respectively the constant and the random variable which have been introduced in Lemma 3.4.9. Therefore, one has

$$V_N=c_1N^{1-2H}(W_N+1).$$

Thus, it follows from (3.2.9) that

$$\begin{aligned}\widehat{H}_N &= \frac{1}{2}\Big(1+\log_2\Big(\frac{V_N}{V_{2N}}\Big)\Big)\\ &= \frac{1}{2}\Big(1+\log_2\Big(\frac{c_1N^{1-2H}(W_N+1)}{c_1(2N)^{1-2H}(W_{2N}+1)}\Big)\Big)\\ &= H+\frac{1}{2}\Big(\log_2(W_N+1)-\log_2(W_{2N}+1)\Big).\end{aligned}$$

Finally applying Lemma 3.4.9 one obtains the theorem.□

From now on, our goal will be to show that Theorem 3.2.2 holds. The proof of this theorem mainly relies on the following lemma, which, roughly speaking, means that a Central Limit Theorem holds for the generalized quadratic variation V_N.

Lemma 3.4.11 *For any $N \geq p+1$ let us set,*

$$\tau_N^2 := Var\left(\frac{\sum_{i=0}^{N-p-1}(\Delta_a\overline{X}_{i,N})^2}{(N-p)(\sigma_{a,N})^2}\right) = \frac{s_N^2}{(N-p)^2}. \tag{3.4.55}$$

Then, there is a constant $c>0$ such that one has

$$\tau_N = cN^{-1/2}+o(N^{-1/2}). \tag{3.4.56}$$

Moreover,

$$\frac{1}{\tau_N}\Big(\frac{\sum_{i=0}^{N-p-1}(\Delta_a\overline{X}_{i,N})^2}{(N-p)\sigma_{a,N}^2}-1\Big)\xrightarrow[N\to+\infty]{\mathrm{d}}\mathcal{N}(0,1). \tag{3.4.57}$$

The proof of Lemma 3.4.11 mainly relies on the following two lemmas.

Lemma 3.4.12 (Csörgo and Révész [27]) *Consider the sequence of random variables $\{S_N\}_{N\geq p+1}$ defined by $S_N = \sum_{j=0}^{N-p-1}\lambda_{j,N}(\varepsilon_{j,N}^2-1)$, where for all $N \geq p+1$, $\{\varepsilon_{j,N}\}_{j=0,\dots,N-p-1}$ is a finite sequence of i.i.d. standard Gaussian random variables and $\{\lambda_{j,N}\}_{j=0,\dots,N-p-1}$ a finite sequence of positive real numbers. Let $\lambda_N = \max_{j=0,\dots,N-p-1}\lambda_{j,N}$, if $\lambda_N = o((Var(S_N))^{1/2})$, then*

$$\frac{S_N}{(Var(S_N))^{1/2}}\xrightarrow[N\to+\infty]{\mathrm{d}}\mathcal{N}(0,1).$$

This lemma is in fact a consequence of Lindeberg-Féller Central Limit Theorem.

Lemma 3.4.13 (Luenberger [46], Ch. 6.2, P. 194) *For all integer $n \geq 1$, let $C=(C_{ij})_{n\times n}$ be a symmetric positive definite matrix and let λ be its largest eigenvalue. Then one has,*

$$\lambda \leq \max_{1\leq i\leq n}\sum_{j=1}^n |C_{ij}|.$$

Proof of Lemma 3.4.11: First observe that (3.4.56) easily follows from (3.4.3) and the second equality in (3.4.55). From now on our goal will be to show that (3.4.57) holds. Let $M = (M_{ij})_{(N-p)\times(N-p)}$ be the covariance matrix of the centered Gaussian vector $\{\Delta_a \overline{X}_{i,N}\}_{i=0,\dots,N-p-1}$, i.e.

$$M_{ij} = \mathbb{E}\left(\Delta_a \overline{X}_{i,N} \Delta_a \overline{X}_{j,N}\right). \tag{3.4.58}$$

Denote by $\{\widetilde{\lambda}_{i,N}\}_{i=0,\dots,N-p-1}$ the finite sequence of the eigenvalues of M. Then, let P be the orthogonal matrix such that

$$D := \mathrm{Diag}(\widetilde{\lambda}_{i,N}) = P'MP. \tag{3.4.59}$$

For every $i \in \{0,\dots,N-p-1\}$, we set

$$\varepsilon_{i,N} = (\widetilde{\lambda}_{i,N})^{-1/2} \sum_{k=0}^{N-p-1} P_{ki} \Delta_a \overline{X}_{k,N}. \tag{3.4.60}$$

Observe that the $\varepsilon_{i,N}$, $i = 0,\dots,N-p-1$, are standard independent Gaussian random variables. By using the fact that P is an orthogonal matrix, as well as the fact that $\mathrm{Tr}(M) = \sum_{i=0}^{N-p-1} \widetilde{\lambda}_{i,N}$, one has,

$$\sum_{i=0}^{N-p-1} (\Delta_a \overline{X}_{i,N})^2 - (N-p)\sigma_{a,N}^2 = \sum_{i=0}^{N-p-1} \widetilde{\lambda}_{i,N}(\varepsilon_{i,N}^2 - 1) \tag{3.4.61}$$

and, as a consequence,

$$S_N := \frac{\sum_{i=0}^{N-p-1} (\Delta_a \overline{X}_{i,N})^2}{(N-p)\sigma_{a,N}^2} - 1 = \sum_{i=0}^{N-p-1} \lambda_{i,N}(\varepsilon_{i,N}^2 - 1), \tag{3.4.62}$$

where

$$\lambda_{i,N} := \frac{\widetilde{\lambda}_{i,N}}{(N-p)\sigma_{a,N}^2}. \tag{3.4.63}$$

Moreover, it follows from Lemma 3.4.13, (7.1.5) and (3.4.29) that,

$$\begin{aligned}
\widetilde{\lambda}_N &:= \max_{0\le j\le N-p-1} \widetilde{\lambda}_{j,N} \le \max_{0\le i\le N-p-1} \sum_{k=0}^{N-p-1} |M_{ik}| \qquad (3.4.64)\\
&= \max_{0\le i\le N-p-1} N^{2-2d} \sum_{k=0}^{N-p-1} \left(c_1 \left| R\Big(\frac{|i-k|}{N}, N, 2d, |\cdot|^{s_0}\Big)\right| + \left| R\Big(\frac{|i-k|}{N}, N, 2d, r\Big)\right| \right),
\end{aligned}$$

where $c_1 > 0$ is a constant non depending on i, k and N. Let us now derive a convenient upper bound for the latter maximum . On one hand, combining Lemma 3.4.4

with Lemma 3.4.5 and the inequality $s_0 < w$, one gets,

$$\begin{aligned}\max_{0\le i\le N-p-1}\sum_{k=0}^{N-p-1}\Big|R\big(\frac{|i-k|}{N},N,2d,|\cdot|^{s_0}\big)\Big| &\le 2N^{-s_0-2}\sum_{m=0}^{N-p-1}\Big|R\big(m,1,2d,|\cdot|^{s_0}\big)\Big|\\ &\le 2N^{-s_0-2}\Big(c_2+c_3\sum_{m=p+2}^{N-p-1}(m-p-1)^{s_0-q}\Big)\\ &\le c_4N^{-s_0-2}\sum_{m=p+2}^{N-p-1}(m-p-1)^{w-q},\end{aligned} \tag{3.4.65}$$

where $c_2 = (p+1)\max_{0\le m\le p+1}\big|R\big(m,1,2d,|\cdot|^{s_0}\big)\big|$ and c_3, c_4 are two positive constants non depending on N. On the other hand, Lemma 3.4.6 and Lemma 3.4.7 imply that

$$\begin{aligned}\max_{0\le i\le N-p-1}\sum_{k=0}^{N-p-1}\Big|R\big(\frac{|i-k|}{N},N,2d,r\big)\Big| &\le 2\sum_{m=0}^{N-p-1}\Big|R\big(\frac{m}{N},N,2d,r\big)\Big|\\ &\le N^{-s_0-2}\Big(c_5+c_6\sum_{m=p+2}^{N-p-1}(m-p-1)^{w-q}\Big)\\ &\le c_7N^{-s_0-2}\sum_{m=p+2}^{N-p-1}(m-p-1)^{w-q},\end{aligned} \tag{3.4.66}$$

where $c_5 = (p+1)\sup_{N\ge p+1}N^{s_0+2}r^*_{2d,N} < \infty$ and c_6, c_7 are two positive constants non depending on N. Next, putting together (3.4.64), (3.4.65), (3.4.66), (3.2.4) and the fact that $\sum_{m=p+2}^{N-p-1}(m-p-1)^{w-q} = \mathcal{O}\big(\log(N)+N^{1+w-q}\big)$ one obtains that,

$$\widetilde{\lambda}_N = \mathcal{O}\big(\log(N)N^{-2H}+N^{1+w-q-2H}\big). \tag{3.4.67}$$

Next it follows from (3.4.63), (3.4.2), (3.4.67) and the inequality $w-q<-1/2$, that

$$\lambda_N := \max_{0\le j\le N-p-1}\lambda_{j,N} = o(N^{-1/2}). \tag{3.4.68}$$

Thus, combining (3.4.56) with (3.4.68) one obtains that,

$$\lambda_N = o(\tau_N). \tag{3.4.69}$$

Finally, in view of (3.4.62), the first equality in (3.4.55) and (3.4.69), we are allowed to apply Lemma 3.4.12 to the sequence $\{S_N\}_{N\ge p+1}$, and thus we obtain (3.4.57). □

Lemma 3.4.14 *Let V_N be the generalized quadratic variation defined in (3.2.10) and let τ_N be the quantity defined in (3.4.55). Then, under the additional assumption (3.2.11), there is $c>0$ a constant non depending on N, such that*

$$\frac{1}{\tau_N}\Big(\frac{V_N}{cN^{1-2H}}-1\Big)\xrightarrow[N\to+\infty]{\mathrm{d}}\mathcal{N}(0,1). \tag{3.4.70}$$

The proof of Lemma 3.4.14 mainly relies on Lemma 3.4.11 as well as on the following classical lemma.

Lemma 3.4.15 *Let (Z_N) be a sequence of random variables which converges in distribution to a standard Gaussian random variable. Let (a_N) and (b_N) be two arbitrary sequences of real numbers satisfying $\lim_{N\to+\infty} a_N = 1$ and $\lim_{N\to+\infty} b_N = 0$. Then, the sequence of random variables $(a_N Z_N + b_N)$ converges in distribution to a standard Gaussian random variable.*

Proof of Lemma 3.4.14: First observe that a careful inspection of the proof of Relation (3.4.2) shows that under the additional assumption (3.2.11), there is a constant $c_1 > 0$, non depending on N, such that one has

$$\sigma_{a,N}^2 = c_1 N^{-2H} + o(N^{-2H-1/2}).$$

Therefore, there exists a constant $c > 0$, such that one has

$$(N-p)\sigma_{a,N}^2 = cN^{1-2H} + o(N^{1/2-2H}),$$

and, as a consequence,

$$a_N := \frac{(N-p)\sigma_{a,N}^2}{cN^{1-2H}} = 1 + o(N^{-1/2}). \tag{3.4.71}$$

Next, let us set,

$$Z_N := \frac{1}{\tau_N}\Big(\frac{V_N}{(N-p)\sigma_{a,N}^2} - 1\Big). \tag{3.4.72}$$

It follows from (3.4.71) and (3.4.72) that

$$\frac{1}{\tau_N}\Big(\frac{V_N}{cN^{1-2H}} - 1\Big) = a_N Z_N + b_N, \tag{3.4.73}$$

where

$$b_N = \tau_N^{-1}\big(a_N - 1\big). \tag{3.4.74}$$

Observe that (3.4.56), (3.4.71) and (3.4.74) imply that $\lim_{N\to+\infty} b_N = 0$. Also, observe that (3.4.71) entails that $\lim_{N\to+\infty} a_N = 1$. Finally, Lemma 3.4.14 results from (3.2.10), Lemma 3.4.11 and Lemma 3.4.15. □

At last let us give the proof of Theorem 3.2.2.

Proof of Theorem 3.2.2: This Theorem can be obtained by using Lemma 3.4.14, the δ-method (see for instance Theorem 3.3.11 in [28]) and Relation (3.4.56). □

Chapter 4

Hölder regularity and series representation of a stochastic volatility model $\{Z(t)\}_{t\in[0,1]}$

Contents

4.1 Introduction

In this chapter we consider the class of stochastic volatility models in which the price process $\{Z(t)\}_{t\in[0,1]}$ is defined for each $t\in[0,1]$, as,

$$Z(t)=\int_0^t \Phi(X(s))\,\mathrm{d}W(s), \tag{4.1.1}$$

where

- $\{X(s)\}_{s\in[0,1]}$ is a centered Gaussian process whose trajectories are, with probability 1, Hölder continuous functions of order $\alpha>1/2$ i.e. there is a positive random variable C such that one has for almost all ω and for all $s_1,s_2\in[0,1]$,

  $$\big|X(s_1,\omega)-X(s_2,\omega)\big|\le C(\omega)|s_1-s_2|^{\alpha}. \tag{4.1.2}$$

  Observe that (4.1.2) holds when $\{X(s)\}_{s\in[0,1]}$ is a multifractional Brownian motion whose functional parameter $H(\cdot)$ is a β-Hölder function with values in $(1/2,\min\{1,\beta\})$ (see [18, 51, 8]). Also, it holds for the stationary increments processes studied in the previous chapter, when $d\ge 1$ or when $s_0\in(1,2)$.

- Φ is a deterministic function belonging to $C^1_{pol}(\mathbb{R})$ (see (5.1.3)), the space of continuously differentiable functions over the real line with a slow decrease at infinity as well as their derivatives.

- $\{W(s)\}_{s\in[0,1]}$ denotes a standard Brownian motion which is independent on $\{X(s)\}_{s\in[0,1]}$.

In section 4.2 we show that there exists a modification of $\{Z(t)\}_{t\in[0,1]}$, also denoted by $\{Z(t)\}_{t\in[0,1]}$, whose trajectories belong, with probability 1, to any Hölder space $C^\gamma([0,1])$ of order $\gamma < 1/2$; on the other hand, under the assumption that Φ vanishes only on a Lebesgue negligible set, we show that the pointwise Hölder exponent of $\{Z(t)\}_{t\in[0,1]}$, at any point $t_0 \in (0,1)$, is almost surely equal to $1/2$; observe that the latter result implies that, with probability 1, the trajectories of $\{Z(t)\}_{t\in[0,1]}$ fail to belong to $C^\gamma([0,1])$ when $\gamma > 1/2$. In section 4.3, by using the Haar basis, we introduce a random series representation of $\{Z(t)\}_{t\in[0,1]}$, for which the convergence holds, almost surely in $C^\gamma([0,1])$, for all $\gamma < 1/2$. Thanks to the latter nice representation of $\{Z(t)\}_{t\in[0,1]}$, we give in section 4.4 an algorithm which allows to simulate this process.

4.2 Hölder regularity of $\{Z(t)\}_{t\in[0,1]}$

Let us first recall the definition of a Hölder space of order $\gamma \in [0,1]$.

Definition 4.2.1 *For any $\gamma \in [0,1]$, the Hölder space $C^\gamma([0,1])$ is defined as the Banach space of the continuous real valued functions u which satisfy,*

$$\sup_{0\le t_1<t_2\le 1} \frac{|u(t_1)-u(t_2)|}{|t_1-t_2|^\gamma} < +\infty.$$

It is equipped with the norm,

$$\|u\|_{C^\gamma([0,1])} = \|u\|_\infty + \sup_{0\le t_1<t_2\le 1} \frac{|u(t_1)-u(t_2)|}{|t_1-t_2|^\gamma}, \tag{4.2.1}$$

where $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$.

Observe that $C^0([0,1])$ reduces to $C([0,1])$ the usual Banach space of real valued continuous functions over $[0,1]$. The main goal of this section is to prove the following two theorems.

Theorem 4.2.1 *Let $\{Z(t)\}_{t\in[0,1]}$ be the stochastic process defined in (4.1.1), then there exists a modification of $\{Z(t)\}_{t\in[0,1]}$, also denoted by $\{Z(t)\}_{t\in[0,1]}$, such that, with probability 1, for all $\gamma < 1/2$, the trajectories of $\{Z(t)\}_{t\in[0,1]}$ belong to the Hölder space $C^\gamma([0,1])$.*

Recall that when a trajectory $t \mapsto Y(t,\omega)$ of an arbitrary stochastic process $\{Y(t)\}_{t\in[0,1]}$ is a continuous and non-differentiable at a point $t_0 \in [0,1]$, then $\rho_Y(t_0,\omega)$, its pointwise Hölder exponent at t_0, is defined as,

$$\rho_Y(t_0,\omega) = \sup\left\{\rho \in [0,1] \,:\, \limsup_{h\to 0} \frac{Y(t_0+h,\omega)-Y(t_0,\omega)}{|h|^\rho} = 0\right\}. \tag{4.2.2}$$

Theorem 4.2.2 *Let $\{Z(t)\}_{t\in[0,1]}$ be the modification of the process $\{Z(t)\}_{t\in[0,1]}$ introduced in Theorem 4.2.1 and let ρ_Z be the corresponding pointwise Hölder exponent; then, under the assumption that Φ vanishes only on a Lebesgue negligible set, one has for all $t_0 \in (0,1)$,*

$$\mathbb{P}\big(\rho_Z(t_0) = 1/2\big) = 1. \tag{4.2.3}$$

Observe that a straightforward consequence of Theorem 4.2.1 is that, one has for all $t_0 \in (0,1)$,

$$\mathbb{P}\big(\rho_Z(t_0) \geq 1/2\big) = 1. \tag{4.2.4}$$

Therefore, for obtaining Theorem 4.2.2, it is sufficient to prove that Theorem 4.2.1 holds and that one has for all $t_0 \in (0,1)$,

$$\mathbb{P}\big(\rho_Z(t_0) \leq 1/2\big) = 1. \tag{4.2.5}$$

The proof of Theorem 4.2.1 mainly relies on the following two lemmas.

Lemma 4.2.3 (Kolmogorov, Centsov) *(see for instance [43]) Let $T > 0$ be an arbitrary and fixed real number and $\{M(t)\}_{t\in[0,T]}$ be a stochastic process verifying: for all $s_1, s_2 \in [0,T]$,*

$$\mathbb{E}|M(s_1) - M(s_2)|^\tau \leq c|s_1 - s_2|^{1+\beta},$$

where $c, \tau, \beta > 0$ are constants. Then this process has a modification $\{\tilde{M}(t)\}_{t\in[0,T]}$ whose trajectories are with probability 1, γ-Hölderian functions, for any $\gamma \in [0, \beta/\tau)$; in other words,

$$\mathbb{P}\Big(\sup_{s_1,s_2\in[0,T]} \frac{|\tilde{M}(s_1) - \tilde{M}(s_2)|}{|s_1 - s_2|^\gamma} < +\infty\Big) = 1.$$

Lemma 4.2.4 *Conditional on $\mathcal{G}_X = \sigma(X(s), 0 \leq s \leq 1)$, $\{Z(t)\}_{t\in[0,1]}$ is a centered Gaussian process and its covariance is given, for all $t_1, t_2 \in [0,1]$, by,*

$$\mathbb{E}\left(Z(t_1)Z(t_2)|\mathcal{G}_X\right) = \int_0^{\min(t_1,t_2)} |\Phi(X(s))|^2 \, \mathrm{d}s.$$

Proof of Lemma 4.2.4: This lemma follows from the fact that $\{X(s)\}_{s\in[0,1]}$ is independent on $\{W(s)\}_{s\in[0,1]}$as well as from the isometry property of Wiener integral. $\square$

Now we are in position to prove Theorem 4.2.1.

Proof of Theorem 4.2.1: We can write, for any $\tau > 0$ and for each $t_1, t_2 \in (0,1]$,

$$\mathbb{E}\,|Z(t_1) - Z(t_2)|^\tau = \mathbb{E}\Big\{\mathbb{E}\left(|Z(t_1) - Z(t_2)|^\tau|\mathcal{G}_X\right)\Big\}. \tag{4.2.6}$$

Then, it follows from (4.2.6), Lemma 4.2.4 and the equivalence of Gaussian moments, that

$$\mathbb{E}\,|Z(t_1) - Z(t_2)|^\tau = c_1\mathbb{E}\left\{\left(\mathbb{E}\left(|Z(t_1) - Z(t_2)|^2|\mathcal{G}_X\right)\right)^{\tau/2}\right\}, \tag{4.2.7}$$

where $c_1 > 0$ is a constant depending only on τ. Again by using Lemma 4.2.4, we get

$$\begin{aligned}
&\mathbb{E}\left(|Z(t_1)-Z(t_2)|^2|\mathcal{G}_X\right)\\
&=\mathbb{E}\left(|Z(t_1)|^2|\mathcal{G}_X\right)+\mathbb{E}\left(|Z(t_2)|^2|\mathcal{G}_X\right)-2\mathbb{E}\left(Z(t_1)Z(t_2)|\mathcal{G}_X\right)\\
&=\int_0^{t_1}\Phi^2(X(s))\,\mathrm{d}s+\int_0^{t_2}\Phi^2(X(s))\,\mathrm{d}s-2\int_0^{\min(t_1,t_2)}\Phi^2(X(s))\,\mathrm{d}s\\
&=\int_{\min(t_1,t_2)}^{\max(t_1,t_2)}|\Phi(X(s))|^2\,\mathrm{d}s. \qquad (4.2.8)
\end{aligned}$$

One sets $\|X\|_\infty=\sup_{t\in[0,1]}|X(t)|$; observe that $\|X\|_\infty<+\infty$ almost surely, since the trajectories of $\{X(s)\}_{s\in[0,1]}$ are continuous with probability 1. As a consequence (see [44]), for all real $p>0$,

$$\mathbb{E}\left(\|X\|_\infty^p\right)<+\infty. \qquad (4.2.9)$$

One clearly has, for any $s\in[0,1]$,

$$|\Phi(X(s))|\le C_1:=\sup_{x\in[-\|X\|_\infty,\|X\|_\infty]}|\Phi(x)|; \qquad (4.2.10)$$

moreover by using the fact that $\Phi\in C^1_{pol}(\mathbb{R})$ as well as (4.2.9), one can show that for all real $p>0$,

$$\mathbb{E}\left(C_1^p\right)<\infty \qquad (4.2.11)$$

all the moments of the positive random variable C_1 are finite. Therefore, (4.2.8) yields

$$\mathbb{E}\left(|Z(t_1)-Z(t_2)|^2|\mathcal{G}_X\right)\le C_1^2|t_1-t_2|. \qquad (4.2.12)$$

It results from (4.2.7) and (4.2.12) that

$$\mathbb{E}\,|Z(t_1)-Z(t_2)|^\tau\le c_1\big(\mathbb{E}\,|C_1|^\tau\big)|t_1-t_2|^{\tau/2}. \qquad (4.2.13)$$

Finally, in view of (4.2.13), by using Lemma 4.2.3, in which one takes $\beta=\tau/2-1$ with τ big enough, we obtain Theorem 4.2.1. □

To prove that Relation (4.2.5) holds we need the following lemma.

Lemma 4.2.5 *Denote by $(h_n)_n$ an arbitrary sequence of non vanishing real numbers which converges to 0. For all $\varepsilon>0$ and $t_0\in(0,1)$, conditional on $\mathcal{G}_X$, the random variable*

$$\frac{Z(t_0+h_n)-Z(t_0)}{|h_n|^{1/2+\varepsilon}}$$

has a centered Gaussian distribution whose variance is equal to

$$|h_n|^{-1-2\varepsilon}\int_{\min(t_0,t_0+h_n)}^{\max(t_0,t_0+h_n)}|\Phi(X(s))|^2\,\mathrm{d}s;$$

moreover the latter variance tends to $+\infty$ when $h_n\to0$.

Proof of Lemma 4.2.5: First it follows from Lemma 4.2.4 and (4.2.8) that, conditional on $\mathcal{G}_X$,

$$\frac{Z(t_0+h_n)-Z(t_0)}{|h_n|^{1/2+\varepsilon}}\sim\mathcal{N}\Big(0,|h_n|^{-1-2\varepsilon}\int_{\min(t_0,t_0+h_n)}^{\max(t_0,t_0+h_n)}|\Phi(X(s))|^2\,\mathrm{d}s\Big).$$

Moreover, the Mean Value Theorem, implies that, that there is $\tilde{s}_n\in\big(\min(t_0,t_0+h_n),\max(t_0,t_0+h_n)\big)$ such that,

$$\int_{\min(t_0,t_0+h_n)}^{\max(t_0,t_0+h_n)}|\Phi(X(s))|^2\,\mathrm{d}s=|h_n||\Phi(X(\tilde{s}_n))|^2; \tag{4.2.14}$$

which in turn entails that

$$|h_n|^{-1}\int_{\min(t,t+h_n)}^{\max(t,t+h_n)}|\Phi(X(s))|^2\,\mathrm{d}s\xrightarrow[n\to+\infty]{a.s.}|\Phi(X(t_0))|^2. \tag{4.2.15}$$

Observe that the fact that $X(t)$ is non degenerate Gaussian random variable and the assumption that Φ vanishes only on a set of Lebesgue measure zero, imply that,

$$|\Phi(X(t_0))|^2>0,\text{a.s.}. \tag{4.2.16}$$

Finally it follows from (4.2.15) and (4.2.16) that

$$|h_n|^{-1-2\varepsilon}\int_{\min(t_0,t_0+h_n)}^{\max(t_0,t_0+h_n)}|\Phi(X(s))|^2\,\mathrm{d}s\xrightarrow[n\to+\infty]{a.s.}+\infty.$$

□

Now, we are in position to prove Relation (4.2.5).

Proof of Relation (4.2.5): our proof is inspired from that of Proposition 2.4 in [9]. It consists in showing that for all $\varepsilon>0$, there exists a sequence $(\tau_k)_{k\in\mathbb{N}}$ of non vanishing real numbers which converges to 0 and satisfies

$$\frac{|Z(t_0+\tau_k)-Z(t_0)|}{|\tau_k|^{1/2+\varepsilon}}\xrightarrow[k\to+\infty]{a.s.}+\infty. \tag{4.2.17}$$

To this end, it is sufficient to prove that there exists a sequence $(h_n)_n$ of non vanishing real numbers which converges to 0 and satisfies

$$\frac{|h_n|^{1/2+\varepsilon}}{|Z(t_0+h_n)-Z(t_0)|}\xrightarrow[n\to+\infty]{\mathbb{P}}0. \tag{4.2.18}$$

Indeed, assuming that (4.2.18) holds, then one can extract from $(h_n)_n$ a subsequence denoted by $(\tau_k)_{k\in\mathbb{N}}$ such that one has (4.2.17). Let us now prove (4.2.18). Denote by $(h_n)_n$ an arbitrary sequence of non vanishing real numbers which converges to 0 and set

$$\sigma^2(t_0+h_n,t_0)=|h_n|^{-1-2\varepsilon}\mathbb{E}\left(|Z(t_0+h_n)-Z(t_0)|^2\Big|\mathcal{G}_X\right)=|h_n|^{-1-2\varepsilon}\int_{\min(t_0,t_0+h_n)}^{\max(t_0,t_0+h_n)}|\Phi(X(s))|^2\,\mathrm{d}s.$$

Observe that in view of Lemma 4.2.5,

$$\sigma^2(t_0+h_n,t_0)\xrightarrow[k\to+\infty]{a.s.}+\infty. \tag{4.2.19}$$

One has, for all $\eta>0$,

$$\begin{aligned}\mathbb{P}\Big(\frac{|h_n|^{1/2+\varepsilon}}{|Z(t_0+h_n)-Z(t_0)|}\le\eta\Big) &= \mathbb{E}\Big(\mathbb{1}_{\big(\frac{|h_n|^{1/2+\varepsilon}}{|Z(t_0+h_n)-Z(t_0)|}\le\eta\big)}\Big)\\ &= \mathbb{E}\Big(\mathbb{E}\Big(\mathbb{1}_{\big(\frac{|h_n|^{1/2+\varepsilon}}{|Z(t_0+h_n)-Z(t_0)|}\le\eta\big)}\Big|\mathcal{G}_X\Big)\Big)\\ &= \mathbb{E}\Big(\mathbb{E}\Big(\mathbb{1}_{\big(\frac{|Z(t_0+h_n)-Z(t_0)|}{|h_n|^{1/2+\varepsilon}}\ge 1/\eta\big)}\Big|\mathcal{G}_X\Big)\Big)\end{aligned} \tag{4.2.20}$$

On the other hand Lemma 4.2.5 entails that, a.s.,

$$\mathbb{E}\Big(\mathbb{1}_{\big(\frac{|Z(t_0+h_n)-Z(t_0)|}{|h_n|^{1/2+\varepsilon}}\ge 1/\eta\big)}\Big|\mathcal{G}_X\Big)=\sqrt{\frac{2}{\pi}}\int_{\frac{1}{\eta\sigma(t_0+h_n,t_0)}}^{+\infty}e^{-x^2/2}\,\mathrm{d}x. \tag{4.2.21}$$

Combining (4.2.21) with (4.2.19) one obtains that,

$$\mathbb{E}\Big(\mathbb{1}_{\big(\frac{|Z(t_0+h_n)-Z(t_0)|}{|h_n|^{1/2+\varepsilon}}\ge 1/\eta\big)}\Big|\mathcal{G}_X\Big)\xrightarrow[n\to+\infty]{a.s.}\sqrt{\frac{2}{\pi}}\int_0^{+\infty}e^{-x^2/2}\,\mathrm{d}x=1. \tag{4.2.22}$$

Next, in view of (4.2.20) and (4.2.22), by using the dominated convergence theorem, it follows that (4.2.18) holds. □

4.3 Random series representation of $\{Z(t)\}_{t\in[0,1]}$ via the Haar basis

In order to state the main result of this section, we need to introduce some notations.

- We denote by $L^2([0,1])$ the usual Lebesgue Hilbert of the square integrable real valued deterministic functions over $[0,1]$.
- The Haar orthonormal basis of $L^2([0,1])$, is the collection of the functions $\big\{\varphi_{0,0},\psi_{j,k}:j\in\mathbb{N}\text{ and }k\in\{0,\dots,2^j-1\}\big\}$, defined as

$$\varphi_{0,0}=\mathbb{1}_{[0,1)} \tag{4.3.1}$$

  and for all $j\ge0$, $0\le k\le 2^j-1$,

$$\psi_{j,k}=2^{j/2}(\mathbb{1}_{[\frac{k}{2^j},\frac{2k+1}{2^{j+1}})}-\mathbb{1}_{[\frac{2k+1}{2^{j+1}},\frac{k+1}{2^j})}). \tag{4.3.2}$$

- We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the underlying probability space, that is the probability space on which the processes $\{X(s)\}_{s\in[0,1]}$, $\{W(s)\}_{s\in[0,1]}$ and $\{Z(t)\}_{t\in[0,1]}$ are defined; moreover, for the sake of simplicity, we assume that this space has been chosen in such a way that, *for all* $\omega \in \Omega$ Relation (4.1.2) holds and $Z(\cdot,\omega) \in C^{\gamma}([0,1])$ for all $\gamma \in [0,1/2)$. Also, we denote by $\big\{\delta_{0,0}, \lambda_{j,k} : j \in \mathbb{N} \text{ and } k \in \{0,\ldots,2^j-1\}\big\}$ the sequence of standard independent Gaussian random variables defined, on this space as,

$$\delta_{0,0} = \int_0^1 \varphi_{0,0}(s)\,\mathrm{d}W(s) = W(1), \tag{4.3.3}$$

and for all $j \in \mathbb{N}$ and all $k \in \{0,\ldots,2^j-1\}$,

$$\lambda_{j,k} = \int_0^1 \psi_{j,k}(s)\,\mathrm{d}W(s) = -2^{j/2}\Big(W(\frac{k+1}{2^j}) - 2W(\frac{2k+1}{2^{j+1}}) + W(\frac{k}{2^j})\Big). \tag{4.3.4}$$

Observe that, similarly to Lemma 2 in [8], one can show that there exist $\Omega^* \subseteq \Omega$ an event of probability 1, and a positive random variable C_* of finite moment of any order, such that one has for all $\omega \in \Omega^*$, all $j \in \mathbb{N}$ and all $k \in \{0,\ldots,2^j-1\}$,

$$|\lambda_{j,k}(\omega)| \le C_*(\omega)\sqrt{1+j}. \tag{4.3.5}$$

- $\{K(t,s)\}_{(t,s)\in[0,1]^2}$ is the stochastic field defined for all $(t,s,\omega) \in [0,1]^2 \times \Omega$ as,

$$K(t,s,\omega) := \Phi(X(s,\omega))\mathbb{1}_{[0,t]}(s); \tag{4.3.6}$$

thus, the process $\{Z(t)\}_{t\in[0,1]}$ defined in (4.1.1) can be expressed as,

$$Z(t) = \int_0^1 K(t,s)\,\mathrm{d}W(s). \tag{4.3.7}$$

Observe that (4.2.10), (4.1.2) and the fact $\Phi \in C^1_{pol}(\mathbb{R})$, imply that for all $\omega \in \Omega$,

$$\sup_{(t,s)\in[0,1]^2} \big|K(t,s,\omega)\big| = \sup_{s\in[0,1]} \big|\Phi(X(s,\omega))\big| \le C_1(\omega) < +\infty; \tag{4.3.8}$$

- We denote by $\{b_{0,0}(t)\}_{t\in[0,1]}$ the stochastic process defined for all $(t,\omega) \in [0,1]\times\Omega$, as,

$$b_{0,0}(t,\omega) = \int_0^1 K(t,s,\omega)\varphi_{0,0}(s)\,\mathrm{d}s; \tag{4.3.9}$$

moreover, for all $j \in \mathbb{N}$ and $k \in \{0,\ldots,2^j-1\}$, we denote by $\{a_{j,k}(t)\}_{t\in[0,1]}$ the stochastic process defined for all $(t,\omega) \in [0,1]\times\Omega$, as,

$$a_{j,k}(t,\omega) = \int_0^1 K(t,s,\omega)\psi_{j,k}(s)\,\mathrm{d}s. \tag{4.3.10}$$

Observe that it follows from (4.3.9), (4.3.10), (4.3.8), (4.3.1) and (4.3.2), that for all $\omega \in \Omega$ and for all $t_1, t_2 \in [0,1]$,

$$\big|b_{0,0}(t_1,\omega) - b_{0,0}(t_2,\omega)\big| \le C_1(\omega)|t_1 - t_2| \tag{4.3.11}$$

and for all $j \in \mathbb{N}$ and $k \in \{0, \dots, 2^j - 1\}$,

$$\big|a_{j,k}(t_1,\omega) - a_{j,k}(t_2,\omega)\big| \le C_1(\omega)2^{j/2}|t_1 - t_2|. \tag{4.3.12}$$

Now we are in position to state the main result of this section:

Theorem 4.3.1 *Let $\gamma \in [0, 1/2)$ be arbitrary and fixed. For all $J \in \mathbb{N}$ and $(t,\omega) \in [0,1] \times \Omega$, one sets,*

$$Z_J(t,\omega) = b_{0,0}(t,\omega)\delta_{0,0}(\omega) + \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} a_{j,k}(t,\omega)\lambda_{j,k}(\omega). \tag{4.3.13}$$

In view of (4.3.11) and (4.3.12) the trajectories of the process $\{Z_J(t)\}_{t\in[0,1]}$ belong to the Hölder space $C^\gamma([0,1])$ since they are in fact Lipschitz functions; moreover, there exist Ω_2^* an event of probability 1 and a positive random variable D of finite moment of any order, such that one has for all $\omega \in \Omega_2^*$ and $J \in \mathbb{N}$,

$$\big\|Z(\cdot,\omega) - Z_J(\cdot,\omega)\big\|_{C^\gamma([0,1])} \le D(\omega)2^{-J\min(1/2-\gamma,\alpha-1/2)}\sqrt{1+J}, \tag{4.3.14}$$

where $\|\cdot\|_{C^\gamma([0,1])}$ is the usual norm on $C^\gamma([0,1])$ (see Definition 4.2.1) and where α has been introduced in (4.1.2).

In order to prove the latter theorem we need two lemmas. The following lemma is a weak version of Theorem 4.3.1.

Lemma 4.3.2 *We use the same notations as in Theorem 4.3.1. Let $t \in [0,1]$ be arbitrary and fixed. When J goes to $+\infty$, the random variable $Z_J(t)$ converge to the random variable $Z(t)$ in the Hilbert space $L^2(\Omega)$, namely, one has,*

$$\lim_{J\to+\infty} E\big(|Z_J(t) - Z(t)|^2\big) = 0. \tag{4.3.15}$$

As a straightforward consequence, there exist $\Omega_{1,t}^* \subseteq \Omega^*$ an event of probability 1 and a subsequence $n \mapsto J_n$ (a priori depending on t) such that for all $\omega \in \Omega_{1,t}^*$, one has,

$$\lim_{n\to+\infty} Z_{J_n}(t,\omega) = Z(t,\omega). \tag{4.3.16}$$

Proof of Lemma 4.3.2: Let K be as in (4.3.6). Observe that in view of (4.3.8), for all fixed $(t,\omega) \in [0,1] \times \Omega$, the function $K(t,\cdot,\omega) : s \mapsto K(t,s,\omega)$ belongs to $L^2([0,1]; \mathrm{d}s)$. By expanding the latter function in the Haar basis, one obtains that,

$$K(t,\cdot,\omega) = b_{0,0}(t,\omega)\varphi_{0,0}(\cdot) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} a_{j,k}(t,\omega)\psi_{j,k}(\cdot), \tag{4.3.17}$$

where $b_{0,0}(t,\omega)$ and $a_{j,k}(t,\omega)$ have been defined respectively in (4.3.9) and (4.3.10). A priori the series in (4.3.17) is convergent in the $L^2([0,1]; \mathrm{d}s)$ norm, namely,

$$\lim_{J\to+\infty}\int_0^1 \Big|K(t,s,\omega)-b_{0,0}(t,\omega)\varphi_{0,0}(s)-\sum_{j=0}^{J}\sum_{k=0}^{2^j-1}a_{j,k}(t,\omega)\psi_{j,k}(s)\Big|^2\,\mathrm{d}s=0. \quad (4.3.18)$$

Let us show that it is also convergent in the $L^2\big([0,1]\times\Omega; \mathrm{d}s\otimes\mathbb{P}\big)$, that is,

$$\lim_{J\to+\infty} E\left(\int_0^1 \Big|K(t,s)-b_{0,0}(t)\varphi_{0,0}(s)-\sum_{j=0}^{J}\sum_{k=0}^{2^j-1}a_{j,k}(t)\psi_{j,k}(s)\Big|^2\,\mathrm{d}s\right)=0. \quad (4.3.19)$$

To derive (4.3.19), we will use the dominated convergence theorem. It follows from Parseval formula and from (4.3.8) that for all $J\in\mathbb{N}$,

$$\int_0^1 \Big|K(t,s)-b_{0,0}(t)\varphi_{0,0}(s)+\sum_{j=0}^{J}\sum_{k=0}^{2^j-1}a_{j,k}(t)\psi_{j,k}(s)\Big|^2\,\mathrm{d}s\le\int_0^1\big|K(t,s)\big|^2\,\mathrm{d}s\le C_1^2 \quad (4.3.20)$$

Next, in view of (4.3.18), (4.3.8) and $\mathbb{E}\,(C_1^2)<+\infty$ we are allowed to use the dominated convergence theorem and thus we obtain (4.3.19). Finally, it follows from the latter relation, the isometry property of the stochastic integral in (4.3.7), (4.3.3), (4.3.4) and (4.3.13), that (4.3.15) holds. □

The following lemma provides sharp estimates for $a_{j,k}(t)$.

Lemma 4.3.3 *There is a positive random variable A of finite moment of any order, such that for all $\omega\in\Omega$, $t\in[0,1]$, $j\in\mathbb{N}$ and $k\in\{0,\dots,2^j-1\}$,*

(i) when $t\ge(k+1)/2^j$, one has

$$\big|a_{j,k}(t,\omega)\big|\le A(\omega)2^{-j(\alpha+1/2)}, \quad (4.3.21)$$

where α has been introduced in (4.1.2);

(ii) when $k/2^j<t<(k+1)/2^j$, one has

$$\big|a_{j,k}(t,\omega)\big|\le A(\omega)2^{j/2}\big(t-k/2^j\big); \quad (4.3.22)$$

(iii) when $t\le k/2^j$, one has,

$$a_{j,k}(t,\omega)=0 \quad (4.3.23)$$

Proof of Lemma 4.3.3: First observe that (4.3.23) easily results from (4.3.2), (4.3.6) and (4.3.10). Let us now prove that (4.3.21) holds, so we assume that $t\ge(k+1)/2^j$. For all $(y,\omega)\in[0,1]\times\Omega$, we set

$$X_\Phi^{(-1)}(y,\omega)=\int_0^y\Phi(X(s,\omega))\,\mathrm{d}s. \quad (4.3.24)$$

Then, using (4.3.2), (4.3.6), (4.3.10) and (4.3.24), it follows that $a_{j,k}(t,\omega)$ can be expressed as an increment of order 2 of the function $2^{j/2}X_{\Phi}^{(-1)}(\cdot,\omega)$: namely, when $t\ge (k+1)/2^j$, one has,

$$\begin{aligned} &a_{j,k}(t,\omega) \\ &= 2^{j/2}\Big(X_{\Phi}^{(-1)}\big(\frac{2k+1}{2^{j+1}},\omega\big)-X_{\Phi}^{(-1)}\big(\frac{k}{2^j},\omega\big)\Big)-2^{j/2}\Big(X_{\Phi}^{(-1)}\big(\frac{k+1}{2^j},\omega\big)-X_{\Phi}^{(-1)}\big(\frac{2k+1}{2^{j+1}},\omega\big)\Big). \end{aligned} \tag{4.3.25}$$

Applying the Mean Value Theorem to the function

$$y\mapsto X_{\Phi}^{(-1)}\big(\frac{k+1}{2^j}-y,\omega\big)-X_{\Phi}^{(-1)}\big(\frac{2k+1}{2^{j+1}}-y,\omega\big)$$

on the interval $[0,2^{-j-1}]$, it follows from (4.3.25) and (4.3.24), that there exists $z\in(0,2^{-j-1})$ such that

$$a_{j,k}(t,\omega)=2^{-j/2-1}\left(\Phi\Big(X\Big(\frac{2k+1}{2^{j+1}}-z,\omega\Big)\Big)-\Phi\Big(X\Big(\frac{k+1}{2^j}-z,\omega\Big)\Big)\right). \tag{4.3.26}$$

Next, applying the Mean Value Theorem to the function $x\mapsto\Phi(x)$, on the compact interval whose endpoints are $X\Big(\frac{2k+1}{2^{j+1}}-z,\omega\Big)$ and $X\Big(\frac{k+1}{2^j}-z,\omega\Big)$, one obtains, in view of (4.3.26), that,

$$\big|a_{j,k}(t,\omega)\big|\le C_2(\omega)2^{-j/2-1}\left|X\Big(\frac{2k+1}{2^{j+1}}-z,\omega\Big)-X\Big(\frac{k+1}{2^j}-z,\omega\Big)\right|, \tag{4.3.27}$$

where

$$C_2(\omega):=\sup\Big\{\big|\Phi'(x)\big|\ :\ |x|\le\|X\|_\infty(\omega)\Big\}.$$

Observe that (4.2.9) and the fact that $\Phi\in C^1_{pol}(\mathbb{R})$, imply that the positive random variable C_2 is of finite moment of any order. Next, combining (4.3.27) with (4.1.2), one gets (4.3.21). Let us now show that (4.3.22) holds. It follows from (4.3.2), (4.3.6) and (4.3.10) that

$$\big|a_{j,k}(t,\omega)\big|\le 2^{j/2}\int_{k/2^j}^t\big|\Phi(X(s))\big|\,\mathrm{d}s\le C_1(\omega)2^{j/2}\big(t-k/2^j\big),$$

where the random variable C_1 has been defined in (4.2.10); thus we obtain (4.3.22). □

Lemma 4.3.4 *Let δ be an arbitrary fixed strictly positive real number. There is a constant $c>0$, only depending on δ, such that for all $x,y\in(0,1]$, satisfying $x\le y$, one has*

$$x^\delta\sqrt{1+\log_2(x^{-1})}\le c\,y^\delta\sqrt{1+\log_2(y^{-1})}$$

Proof of Lemma 4.3.4: The derivative over $(0,1]$ of the (strictly) positive function $s\mapsto s^\delta\sqrt{1+\log_2(s^{-1})}$ equals to

$$s^{\delta-1}\left(1-\frac{\log s}{\log 2}\right)^{-1/2}\left(\delta-\delta\frac{\log s}{\log s}-\frac{1}{2\log 2}\right).$$

Therefore there exists $\bar{s}_\delta \in (0,1]$ such that the function $s \mapsto s^\delta \sqrt{1+\log_2(s^{-1})}$ is increasing on $(0,\bar{s}_\delta]$ and decreasing on $[\bar{s}_\delta\, 1]$. Thus, setting

$$c := \frac{\max_{s\in(0,1]}\left\{s^\delta \sqrt{1+\log_2(s^{-1})}\right\}}{\min_{s\in[\bar{s}_\delta,1]}\left\{s^\delta \sqrt{1+\log_2(s^{-1})}\right\}},$$

one obtains the lemma. □

Now we are in position to prove Theorem 4.3.1.

Proof of Theorem 4.3.1: First notice that it is sufficient to show that for all $\omega \in \Omega^*$ (recall that Ω^* is the event of probability 1 on which (4.3.5) is satisfied), $J \in \mathbb{N}$ and $Q \in \mathbb{N}$ one has,

$$\big\| Z_{J+Q}(\cdot,\omega) - Z_J(\cdot,\omega)\big\|_{C^\gamma([0,1])} \le D(\omega) 2^{-J\min(1/2-\gamma,\alpha-1/2)}\sqrt{1+J}. \tag{4.3.28}$$

Indeed, (4.3.28) implies that $\big(Z_J(\cdot,\omega)\big)_{J\in\mathbb{N}}$ is a Cauchy sequence in the Banach $C^\gamma([0,1])$ and, as consequence, it converges to some limit denoted by $\widetilde{Z}(\cdot,\omega)$ in this space. Next let Ω_2^* be the event of probability 1 defined as

$$\Omega_2^* = \bigcap_{q\in[0,1]\cap\mathbb{Q}} \Omega_{1,q}^*,$$

where $\Omega_{1,q}^*$ is the event $\Omega_{1,t}^*$ introduced in Lemma 4.3.2 when $t = q$. Thus it follows from the latter lemma, that for each $\omega \in \Omega_2^*$, one has $Z(\cdot,\omega) = \widetilde{Z}(\cdot,\omega)$. Then, letting in (4.3.28) Q goes to $+\infty$, one gets (4.3.14).

From now on, our goal will be to show that (4.3.28) holds. To this end, in view of (4.2.1), it is sufficient to prove that there exist two positive random variables D_1 and D_2 such that, one has for all $\omega \in \Omega^*$, $J \in \mathbb{N}$ and $Q \in \mathbb{N}\setminus\{0\}$,

$$\begin{aligned}&\sup_{0\le t_1<t_2\le 1} \frac{\big|Z_{J+Q}(t_1,\omega) - Z_J(t_1,\omega) - Z_{J+Q}(t_2,\omega) + Z_J(t_2,\omega)\big|}{|t_1-t_2|^\gamma}\\ &\le D_1(\omega)2^{-J\min(1/2-\gamma,\alpha-1/2)}\sqrt{1+J}\end{aligned} \tag{4.3.29}$$

and

$$\sup_{0\le t\le 1}\big|Z_{J+Q}(t,\omega) - Z_J(t,\omega)\big| \le D_2(\omega)2^{-J\min(1/2-\gamma,\alpha-1/2)}\sqrt{1+J}. \tag{4.3.30}$$

We will only show that (4.3.29) is satisfied, since (4.3.30) can be obtained in the same way. Let $t_1 < t_2$ be two arbitrary and fixed real numbers belonging to the interval $[0,1]$. Observe that in view of (4.3.13), one has

$$\begin{aligned}&|Z_{J+Q}(t_1,\omega) - Z_J(t_1,\omega) - Z_{J+Q}(t_2,\omega) + Z_J(t_2,\omega)|\\ &= \Big|\sum_{j=J+1}^{J+Q}\sum_{k=0}^{2^j-1}\big(a_{j,k}(t_1,\omega) - a_{j,k}(t_2,\omega)\big)\lambda_{j,k}(\omega)\Big|\\ &\le \sum_{j=J+1}^{+\infty}\Big|\sum_{k=0}^{2^j-1}\big(a_{j,k}(t_1,\omega) - a_{j,k}(t_2,\omega)\big)\lambda_{j,k}(\omega)\Big|.\end{aligned} \tag{4.3.31}$$

Let us give appropriate bounds for

$$U_j(t_1,t_2,\omega) := \Big|\sum_{k=0}^{2^j-1}\big(a_{j,k}(t_1,\omega)-a_{j,k}(t_2,\omega)\big)\lambda_{j,k}(\omega)\Big| \tag{4.3.32}$$

and derive form them (4.3.29). We denote by $j_0\in\mathbb{N}$ the unique integer such that

$$2^{-j_0-1} < |t_1-t_2| \le 2^{-j_0}. \tag{4.3.33}$$

Also for all $t\in[0,1]$ and $j\in\mathbb{N}$, we denote by $\tilde{k}(j,t)$ the unique integer belonging to $\{0,\dots,2^j-1\}$ such that

$$\frac{\tilde{k}(j,t)}{2^j} \le t < \frac{\tilde{k}(j,t)+1}{2^j}, \tag{4.3.34}$$

with the convention that

$$\tilde{k}(j,1) = 2^j-1. \tag{4.3.35}$$

Observe that when $t\in[0,1)$,

$$\tilde{k}(j,t) = \big[2^j t\big], \tag{4.3.36}$$

$[\cdot]$ being the integer part function. Also, observe that (4.3.32), (4.3.23) and (4.3.25) imply that

$$U_j(t_1,t_2,\omega) = \Big|\sum_{k=\tilde{k}(j,t_1)}^{\tilde{k}(j,t_2)}\big(a_{j,k}(t_1,\omega)-a_{j,k}(t_2,\omega)\big)\lambda_{j,k}(\omega)\Big|. \tag{4.3.37}$$

Let us now study two cases $j\le j_0$ and $j>j_0$. First assume that $j\le j_0$, then (4.3.33) implies that $2^{-j}\ge|t_1-t_2|$ and, as a consequence that $\tilde{k}(j,t_2)\in\big\{\tilde{k}(j,t_1),\tilde{k}(j,t_1)+1\big\}$. When, $\tilde{k}(j,t_2)=\tilde{k}(j,t_1)$, it follows from (4.3.37) and (4.3.5) and (4.3.12) that

$$\begin{aligned} U_j(t_1,t_2,\omega) &= \big|a_{j,\tilde{k}(j,t_1)}(t_1,\omega)-a_{j,\tilde{k}(j,t_1)}(t_2,\omega)\big|\big|\lambda_{j,\tilde{k}(j,t_1)}(\omega)\big| \\ &\le G_1(\omega)|t_1-t_2|2^{j/2}\sqrt{1+j}, \end{aligned}$$

where $G_1(\omega):=C_*(\omega)C_1(\omega)$. When, $\tilde{k}(j,t_2)=\tilde{k}(j,t_1)+1$, putting together (4.3.37), (4.3.5), (4.3.12), (4.3.22) and the fact that $(\tilde{k}(j,t_1)+1)/2^j\in[t_1,t_2]$, one obtains that,

$$\begin{aligned} &U_j(t_1,t_2,\omega)\\ &\le \big|a_{j,\tilde{k}(j,t_1)}(t_1,\omega)-a_{j,\tilde{k}(j,t_1)}(t_2,\omega)\big|\big|\lambda_{j,\tilde{k}(j,t_1)}(\omega)\big| + \big|a_{j,\tilde{k}(j,t_1)+1}(t_2,\omega)\big|\big|\lambda_{j,\tilde{k}(j,t_1)+1}(\omega)\big| \\ &\le G_1(\omega)|t_1-t_2|2^{j/2}\sqrt{1+j} + G_2(\omega)\big(t_2-(\tilde{k}(j,t_1)+1)/2^j\big)|t_1-t_2|2^{j/2}\sqrt{1+j} \\ &\le G_3(\omega)|t_1-t_2|2^{j/2}\sqrt{1+j}, \end{aligned}$$

where $G_2(\omega):=A(\omega)C_*(\omega)$ and $G_3(\omega):=G_1(\omega)+G_2(\omega)$. Thus, we have shown that there is a positive random variable G_4, non depending on γ, j_0, t_1 and t_2, such that for all $j\le j_0$, one has,

$$\frac{U_j(t_1,t_2,\omega)}{|t_1-t_2|^\gamma} \le G_4(\omega)|t_1-t_2|^{1-\gamma}\,2^{j/2}\sqrt{1+j}.$$

Therefore, in view of (4.3.33), for each integer J satisfying $0 \le J < j_0$, one has,

$$\begin{aligned}\sum_{j=J+1}^{j_0} \frac{U_j(t_1,t_2,\omega)}{|t_1-t_2|^\gamma} &\le G_4(\omega)|t_1-t_2|^{1-\gamma} \sum_{j=J+1}^{j_0} 2^{j/2}\sqrt{1+j}\\ &\le \sqrt{2}\big(\sqrt{2}-1\big)^{-1} G_4(\omega)|t_1-t_2|^{1-\gamma}\, 2^{j_0/2}\sqrt{1+j_0}\\ &\le \sqrt{2}\big(\sqrt{2}-1\big)^{-1} G_4(\omega)|t_1-t_2|^{1/2-\gamma}\sqrt{1+\log_2(|t_1-t_2|^{-1})}.\end{aligned} \tag{4.3.38}$$

Next, it follows from (4.3.38), the inequality $2^{-J} > 2^{-j_0} \ge |t_1-t_2|$ (see (4.3.33)) and Lemma 4.3.4 (in which one takes $\delta = 1/2-\gamma$, $x = |t_1-t_2|$ and $y = 2^{-J}$), that

$$\sum_{j=J+1}^{j_0} \frac{U_j(t_1,t_2,\omega)}{|t_1-t_2|^\gamma} \le G_5(\omega) 2^{-J(1/2-\gamma)}\sqrt{1+J}, \tag{4.3.39}$$

where

$$G_5(\omega) := c\sqrt{2}\big(\sqrt{2}-1\big)^{-1} G_4(\omega),$$

c being the constant introduced in Lemma 4.3.4. Let us now study the case where $j > j_0$. Observe that in this case, in view of Relations (4.3.33) and (4.3.34), one necessarily has $\tilde{k}(j,t_1) < \tilde{k}(j,t_2)$ and

$$2^j|t_1-t_2| > 1. \tag{4.3.40}$$

Also observe that, using (4.3.35), (4.3.36) and (4.3.40), one obtain that

$$\tilde{k}(j,t_2) - \tilde{k}(j,t_1) < 2^j t_2 - 2^j t_1 + 1 < 2^{j+1}|t_1-t_2|. \tag{4.3.41}$$

It follows from (4.3.32), (4.3.25), (4.3.34), Lemma 4.3.3, (4.3.41), (4.3.5) and (4.3.12), that

$$\begin{aligned}U_j(t_1,t_2,\omega) &\le \big|a_{j,\tilde{k}(j,t_1)}(t_1,\omega)\big|\big|\lambda_{j,\tilde{k}(j,t_1)}(\omega)\big| + \sum_{k=\tilde{k}(j,t_1)}^{\tilde{k}(j,t_2)-1} \big|a_{j,k}(t_2,\omega)\big|\big|\lambda_{j,k}(\omega)\big|\\ &\quad + \big|a_{j,\tilde{k}(j,t_2)}(t_2,\omega)\big|\big|\lambda_{j,\tilde{k}(j,t_2)}(\omega)\big|\\ &\le 2G_2(\omega)\Big(2^{-j/2}\sqrt{1+j} + |t_1-t_2|\,2^{-j(\alpha-1/2)}\sqrt{1+j}\Big).\end{aligned} \tag{4.3.42}$$

Thus (4.3.42) and (4.3.33) imply that for all $j > j_0$, one has,

$$\begin{aligned}\frac{U_j(t_1,t_2,\omega)}{|t_1-t_2|^\gamma} &\le 2G_2(\omega)\Big(2^{-j(1/2-\gamma)}\sqrt{1+j} + |t_1-t_2|^{1-\gamma}\,2^{-j(\alpha-1/2)}\sqrt{1+j}\Big)\\ &\le G_6(\omega)2^{-j\min(1/2-\gamma,\alpha-1/2)}\sqrt{1+j}.\end{aligned} \tag{4.3.43}$$

where $G_6(\omega) := 4G_2(\omega)$. Next, observe that one has for $J \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{j=J+1}^{+\infty} 2^{-j\min(1/2-\gamma,\alpha-1/2)}\sqrt{1+j}\\
&= 2^{-J\min(1/2-\gamma,\alpha-1/2)}\sqrt{1+J}\sum_{j=J+1}^{+\infty} 2^{-(j-J)\min(1/2-\gamma,\alpha-1/2)}\sqrt{\frac{1+j}{1+J}}\\
&\le c_1 2^{-J\min(1/2-\gamma,\alpha-1/2)}\sqrt{1+J}, \qquad (4.3.44)
\end{aligned}$$

where the constant $c_1 := \sum_{l=0}^{+\infty} 2^{-l\min(1/2-\gamma,\alpha-1/2)}\sqrt{1+l} < +\infty$. Thus combining (4.3.43) with (4.3.44), it follows that for all $J \ge j_0$, one has,

$$\sum_{j=J+1}^{+\infty} \frac{U_j(t_1,t_2,\omega)}{|t_1-t_2|^{\gamma}} \le G_7(\omega)2^{-J\min(1/2-\gamma,\alpha-1/2)}\sqrt{1+J}, \qquad (4.3.45)$$

where $G_7(\omega) := c_1G_6(\omega)$. Let now set now show that, for all $J \in \mathbb{N}$, one has,

$$\sum_{j=J+1}^{+\infty} \frac{U_j(t_1,t_2,\omega)}{|t_1-t_2|^{\gamma}} \le D_1(\omega)2^{-J\min(1/2-\gamma,\alpha-1/2)}\sqrt{1+J}, \qquad (4.3.46)$$

where $D_1(\omega) := (1+c)G_7(\omega) + G_5(\omega)$, c is the constant introduced in Lemma 4.3.4 and $G_5(\omega)$ has been introduced in (4.3.39). It is clear that (4.3.45) implies that (4.3.46) holds when $J \ge j_0$, so from now on, we assume that $j_0 \ge 1$ and that J is an arbitrary nonnegative integer satisfying $J < j_0$. It follows from (4.3.45) that

$$\sum_{j=j_0+1}^{+\infty} \frac{U_j(t_1,t_2,\omega)}{|t_1-t_2|^{\gamma}} \le G_7(\omega)2^{-j_0\min(1/2-\gamma,\alpha-1/2)}\sqrt{1+j_0}.$$

Then using Lemma 4.3.4 (in which one takes $\delta = \min(1/2-\gamma,\alpha-1/2)$, $x = 2^{-j_0}$ and $y = 2^{-J}$), one obtains that

$$\sum_{j=j_0+1}^{+\infty} \frac{U_j(t_1,t_2,\omega)}{|t_1-t_2|^{\gamma}} \le cG_7(\omega)2^{-J\min(1/2-\gamma,\alpha-1/2)}\sqrt{1+J}. \qquad (4.3.47)$$

Next combining (4.3.47) with (4.3.39), it follows that (4.3.46) holds in the case where $J < j_0$. Finally (4.3.31), (4.3.32) and (4.3.46) imply that (4.3.29) is satisfied. □

4.4 Simulation of $\{Z(t)\}_{t\in[0,1]}$ when $\{X(t)\}_{t\in[0,1]}$ is the mBm

Our algorithm for simulating $\{Z(t)\}_{t\in[0,1]}$ mainly relies on Theorem 4.3.1 which allows to approximate $\{Z(t)\}_{t\in[0,1]}$, for J large enough, by the process $\{Z_J(t)\}_{t\in[0,1]}$ defined in (4.3.13) as a finite of sum. First we will give an expression for the latter process which makes it rather easy to simulate, to this end, we need to introduce some notations.

- For all fixed integers $J \geq 0$ and $l \in \{0,\ldots,2^J-1\}$, the function $\varphi_{J,l}$ is defined as,
$$\varphi_{J,l} = 2^{J/2}\mathbb{1}_{[\frac{l}{2^J},\frac{l+1}{2^J})}. \tag{4.4.1}$$
- For all fixed integer $J \geq 0$, we denote by $\{\delta_{J,l} : l \in \{0,\ldots,2^J-1\}\}$ the finite sequence of independent standard Gaussian random variables defined as,
$$\delta_{J,l} := \int_0^1 \varphi_{J,l}(s)\,\mathrm{d}W(s) = 2^{J/2}\Big(W(\frac{l+1}{2^J}) - W(\frac{l}{2^J})\Big). \tag{4.4.2}$$
- For all fixed integers $J \in \mathbb{N}$ and $l \in \{0,\ldots,2^J-1\}$, the stochastic process $\{b_{J,l}(t)\}_{t\in[0,1]}$ is defined for all $(t,\omega) \in [0,1]\times\Omega$, as,
$$b_{J,l}(t,\omega) = \int_0^1 K(t,s,\omega)\varphi_{J,l}(s)\,\mathrm{d}s, \tag{4.4.3}$$
where $K(t,s,\omega)$ has been introduced in (4.3.6)

The following proposition provides a nice expression for the process $\{Z_J(t)\}_{t\in[0,1]}$ in Theorem 4.3.1.

Proposition 4.4.1 *For all fixed integer $J \geq 0$, let $\{Z_J(t)\}_{t\in[0,1]}$ be the stochastic process introduced in (4.3.13). Then one has for all $t \in [0,1]$, almost surely,*
$$Z_J(t) = \sum_{l=0}^{2^J-1} b_{J,l}(t)\delta_{J,l}. \tag{4.4.4}$$

Observe that, Relation (4.4.4), also holds almost surely for all $J \in \mathbb{N}$ and $t \in [0,1]$, since the trajectories of the processes $\{Z_J(t)\}_{t\in[0,1]}$ and $\Big\{\sum_{l=0}^{2^J-1} b_{J,l}(t)\delta_{J,l}\Big\}_{t\in[0,1]}$, are with probability 1, continuous functions.

Proof of Proposition 4.4.1: The proof mainly relies on the notion of multiresolution analysis (see e.g. [49, 29, 60]). Let V_J be the finite dimensional subspace of the Hilbert space $L^2([0,1])$ defined as,
$$V_J := \mathrm{Span}\big\{\varphi_{0,0},\, \psi_{j,k} :\; j \in \{0,\ldots,J\} \text{ and } k \in \{0,\ldots,2^j-1\}\big\}.$$
Relation (4.3.17) implies that for every fixed $(t,\omega) \in [0,1]\times\Omega$, the function
$$K_J(t,\cdot,\omega) := b_{0,0}(t,\omega)\varphi_{0,0}(\cdot) + \sum_{j=0}^{J}\sum_{k=0}^{2^j-1} a_{j,k}(t,\omega)\psi_{j,k}(\cdot), \tag{4.4.5}$$
can be viewed as the orthogonal projection of $K(t,\cdot,\omega)$ on the space V_J. On the other hand, it is known that (see e.g. [49, 29, 60]),
$$\{\varphi_{J,l},\, l \in \{0,\ldots,2^J-1\}\},$$

is an orthonormal basis of V_J. Therefore, one has

$$K_J(t,\cdot,\omega)=\sum_{l=0}^{2^J-1}b_{J,l}(t,\omega)\varphi_{J,l}(\cdot). \tag{4.4.6}$$

Thus, using the definition of the stochastic integral $\int_0^1(\cdot)\,dW$, (4.4.5), (4.3.4), (4.4.6) and (4.4.2), one obtains (4.4.4). □

Now, we are ready to describe the main step of our algorithm for simulating $\{Z(t)\}_{t\in[0,1]}$.

Main steps of our algorithm for simulating $\{Z(t)\}_{t\in[0,1]}$:

(1) We take J large enough and we simulate the finite sequence

$$\{\delta_{J,l}:\, l\in\{0,\dots,2^J-1\}\}$$

of the standard independent Gaussian random variables defined in (4.4.2).

(2) By using the efficient methods described in [15], we simulate

$$\Big\{X(0),X\Big(\frac{1}{2^{2J}}\Big),\dots,X\Big(\frac{2^{2J}-1}{2^{2J}}\Big)\Big\},$$

where $\{X(t)\}_{t\in[0,1]}$ denotes a multifractional Brownian motion.

(3) Noticing that, for all $l\in\{0,\dots,2^J-1\}$, and $m\in\{l+1,\dots,2^J\}$, Relations (4.4.1), (4.4.3) and (4.3.6) imply that,

$$b_{J,l}\Big(\frac{m}{2^J}\Big)=2^{J/2}\int_{\frac{l}{2^J}}^{\frac{l+1}{2^J}}\phi(X(s))\,\mathrm{d}s,$$

we approximate the latter integral, by the Riemann sum

$$\widehat{b}_{J,l}\Big(\frac{m}{2^J}\Big):=2^{-3J/2}\sum_{q=0}^{2^J-1}\phi\left(X\Big(\frac{l}{2^J}+\frac{q}{2^{2J}}\Big)\right). \tag{4.4.7}$$

On the other hand, observe that (4.4.1), (4.4.3) and (4.3.6) entail that for each $m\in\{0,\dots,l\}$,

$$b_{J,l}\Big(\frac{m}{2^J}\Big)=0$$

(4) Thus, in view of (4.4.4), for all $m\in\{1,\dots,2^J\}$, we approximate $Z_J\big(\frac{m}{2^J}\big)$ by

$$\widehat{Z}_J\Big(\frac{m}{2^J}\Big):=\sum_{l=0}^{m-1}\widehat{b}_{J,l}\Big(\frac{m}{2^J}\Big)\delta_{J,l}. \tag{4.4.8}$$

Then we simulate

$$\Big\{\widehat{Z}_J\Big(\frac{1}{2^J}\Big),\dots,\widehat{Z}_J\Big(\frac{2^J-1}{2^J}\Big),\widehat{Z}_J(1)\Big\},$$

by using the fact that

$$\widehat{Z}_J\Big(\frac{1}{2^J}\Big) = \widehat{b}_{J,0}\Big(\frac{1}{2^J}\Big)\delta_{J,0} \tag{4.4.9}$$

and the induction relation, for all $m \in \{2,\dots,2^J\}$,

$$\widehat{Z}_J\Big(\frac{m}{2^J}\Big) = \widehat{Z}_J\Big(\frac{m-1}{2^J}\Big) + \widehat{b}_{J,m-1}\Big(\frac{m}{2^J}\Big)\delta_{J,m-1}. \tag{4.4.10}$$

Observe that (4.4.9) and (4.4.10) easily result from (4.4.7) and (4.4.8).

(5) Finally, by interpolating the 2^J+1 points

$$(0,0)\,;\ \Big(\frac{1}{2^J},\widehat{Z}_J\Big(\frac{1}{2^J}\Big)\Big)\ ;\ \dots\ ;\ \Big(\frac{2^J-1}{2^J},\widehat{Z}_J\Big(\frac{2^J-1}{2^J}\Big)\Big)\ ;\ \big(1,\widehat{Z}_J(1)\big)$$

we obtain a stochastic process $\big\{\widehat{Z}_J(t)\big\}_{t\in[0,1]}$ which, in view of Theorem 4.3.1, satisfies the following property: for all fixed $\gamma\in[0,1/2)$, there exists a random variable D' such that one has, almost surely, for all $J\in\mathbb{N}$,

$$\big\|Z-\widehat{Z}_J\big\|_{C^\gamma([0,1])} \le D' 2^{-J\min(1/2-\gamma,H_*-1/2)}\sqrt{1+J},$$

where $\|\cdot\|_{C^\gamma([0,1])}$ is the usual norm on the Hölder space $C^\gamma([0,1])$ and where $H_* = \min_{s\in[0,1]} H(s)$, recall that $H(\cdot)$ denotes the functional parameter of the mBm $\{X(s)\}_{s\in[0,1]}$.

□

To test our algorithm, we have taken $H(s) = 0.6+0.2s$ for all $s\in[0,1]$, then we have simulated the corresponding mBm $\{X(s)\}_{s\in[0,1]}$ as well as $\{Z(t)\}_{t\in[0,1]}$ in the following two cases:

(1) $\Phi(x) = 0.5+0.5x$ for all real x (see Figure 1 below),

(2) $\Phi(x) = \sin(x)$ for all real x (see Figure 2 below).

Our simulations tend to confirm the fact that the pointwise Hölder exponent of $\{Z(t)\}_{t\in(0,1]}$ does not change from one place to another and is equal to 1/2 (see Theorem 4.2.2).

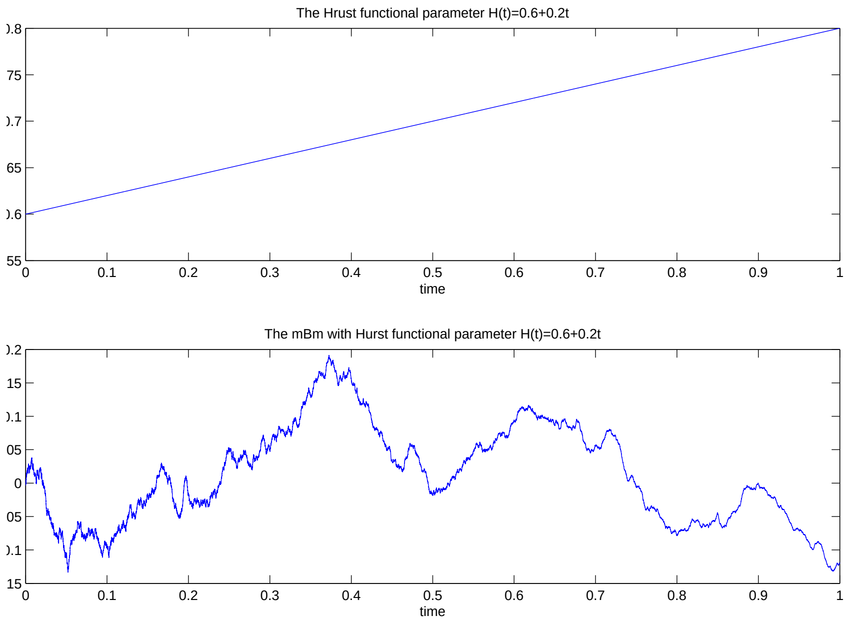


Figure 1

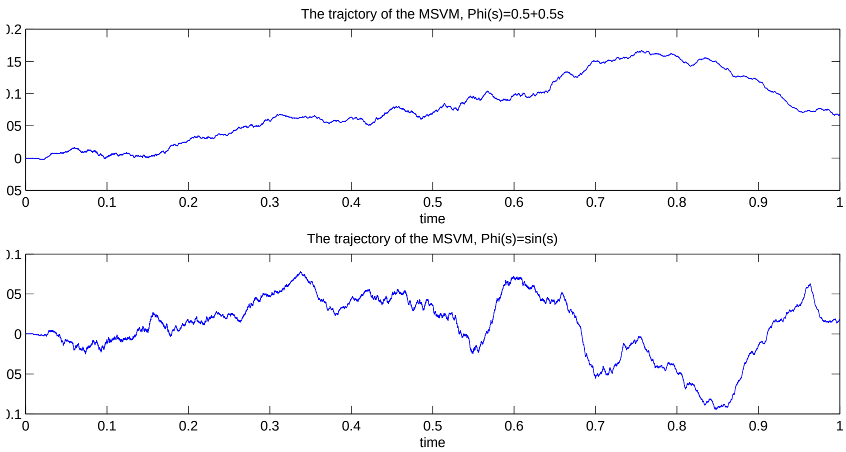


Figure 2

Chapter 5

Multifractional stochastic volatility models: statistical inference related to the hidden volatility

Contents

5.1 Introduction

Stochastic volatility models are extensions of the well-known Black and Scholes model. Hull and White [39] and other authors in mathematical finance (see for instance [54] and [50]) introduced them in the eighties in order to account the volatility effects of exogenous arrivals of information. The results of this chapter are inspired by a work of Gloter and Hoffmann [35, 36] which concerns statistical inference in a parametric stochastic volatility model driven by a fractional Brownian motion (fBm for short). Namely, the model considered in [35, 36] can be expressed as:

$$\begin{cases} Z(t) = z_0 + \int_0^t \sigma(s)\,\mathrm{d}W(s) \\ \sigma(s) = \sigma_0 + \Phi(\theta, B_\alpha(s)), \end{cases} \tag{5.1.1}$$

where:

- $Z(t)$ denotes logarithm of the price of the underlying asset, the original price z_0 is supposed to be deterministic and known.
- $\{W(s)\}_{s\in[0,1]}$ denotes a standard Brownian motion (Bm for short).

- $\{\sigma(s)\}_{s\in[0,1]}$ denotes the volatility process (σ_0 is real-valued and known); the deterministic function $x \mapsto \Phi(\theta, x)$, through which it is defined, is known up to a real-valued parameter θ. For the sake of convenience, one sets for every $x \in \mathbb{R}$,
$$f(x) := (\sigma_0 + \Phi(\theta, x))^2 \tag{5.1.2}$$
and throughout this chapter one assumes that the function f belongs to $C^2_{pol}(\mathbb{R})$. Observe that for every integer $l \geq 0$, $C^l_{pol}(\mathbb{R})$ denotes the vector space of l-times continuously differentiable functions over the real line, which slowly increase at infinity as well as their derivative of any order, more formally,
$$C^l_{pol}(\mathbb{R}) := \Big\{ h \in C^l(\mathbb{R}) :\ \exists\, c, K > 0\,, \forall\, x \in \mathbb{R}, \quad \sum_{k=0}^{l} |h^{(k)}(x)| \leq c\big(1 + |x|^K\big) \Big\}. \tag{5.1.3}$$

- $\{B_\alpha(s)\}_{s\in[0,1]}$ denotes a fractional Brownian motion (fBm for short) with Hurst parameter α (see e.g. [30, 31, 53]), which is assumed to be independent on the Bm $\{W(s)\}_{s\in[0,1]}$; one makes the latter independence assumption for the stochastic integral $\int_0^t \sigma(s)\,\mathrm{d}W(s)$ to be well-defined. Note that the idea of replacing the Bm governing the volatility by a fBm is due to Comte and Renault (see [24, 25]), who have proposed to do so in order to account some long memory effects.

In order to clearly explain the main goal of this chapter, we need to briefly present some of the main results obtained in [35, 36]. In the latter articles, it is assumed that one observes a discretized trajectory of the process $\{Z(t)\}_{t\in[0,1]}$, namely the high frequency data $Z(j/n)$, $j = 0, \ldots, n$. Also, it is assumed that the fBm B_α governing the volatility is hidden; however one knows the value of its Hurst parameter α, moreover $\alpha \in (1/2, 1)$. Though, the hypothesis that the Hurst parameter is known may seem to be restrictive from a practical point of view, it has already been made by other authors (see for example [59]), in some settings more or less related to that of the model (5.1.1). Under additional technical assumptions, we will not give here for the sake of simplicity, Gloter and Hoffmann [35, 36] have obtained the following results (i) and (ii):

(i) By using the notion of generalized quadratic variation, one can construct estimators of integrated functional of the volatility of the form: $\int_0^1 f'(B_\alpha(s))^2 h(\sigma^2(s))\,\mathrm{d}s$, where f' is the derivative of f and $h \in C^1_{pol}(\mathbb{R})$ is arbitrary and fixed. Note that the problem of the estimation of such quantities is of some importance in its own right, since more or less similar integrals appear in some option pricing formulas (see for instance [39]).

(ii) Thanks to the result (i), it is possible to build a minimax optimal estimator of the unknown parameter θ. Also, it is worth noticing that, it has been shown in [35, 36] that the minimax rate of convergence for estimating θ, in the setting of the model (5.1.1), is not the usual rate $n^{-1/2}$ but the slower rate $n^{-1/(4\alpha+2)}$, which deteriorates when the Hurst parameter α increases; basically, the reason for this unusual phenomenon is that the volatility is hidden and the Brownian motion W makes the approximation of the volatility more noisy.

Let us now present the main motivation behind the introduction of multifractional stochastic volatility models. To this end, first we need to introduce the notion of pointwise Hölder exponent. Let $\{X(s)\}_{s\in[0,1]}$ be a stochastic process whose trajectories are with probability 1, continuous and nowhere differentiable functions (this is the case of fBm and of multifractional Brownian motion which will soon be introduced), $\rho_X(t)$ the pointwise Hölder exponent of the process $\{X(s)\}_{s\in[0,1]}$ at an arbitrary time t, is defined as,

$$\rho_X(t) = \sup\Big\{\rho\in[0,1] \,:\, \limsup_{\tau\to 0}\frac{X(t+\tau)-X(t)}{|\tau|^\rho} = 0\Big\}.$$

The quantity of $\rho_X(t)$ provides a measure of $\{X(s)\}_{s\in[0,1]}$ roughness (i.e. of the maximum of the fluctuations amplitudes of $\{X(s)\}_{s\in[0,1]}$) in a neighborhood of t; the smaller $\rho_X(t)$ is the rougher (i.e. the more fluctuating) is $\{X(s)\}_{s\in[0,1]}$ in t neighborhood. In [1] numerical evidences have shown that for a better understanding of stock price dynamics, it is important to analyze volatility local roughness. With this respect, fractional stochastic volatility model has a serious limitation: its volatility local roughness cannot evolve over time; more precisely, when $\Phi(\theta,\cdot)$ is a continuously differentiable function with a nowhere vanishing derivative, then one has, almost surely, at any time t, $\rho_\sigma(t)=\alpha$, where α is the constant Hurst parameter of the fBm $\{B_\alpha(s)\}_{s\in[0,1]}$ and $\rho_\sigma(t)$ the pointwise Hölder exponent at t, of the volatility process $\{\sigma(s)\}_{s\in[0,1]}$ defined in (5.1.1). The latter limitation is due to the fact that the local roughness of $\{B_\alpha(s)\}_{s\in[0,1]}$ itself cannot change from time to time, namely one has almost surely, for all t, $\rho_{B_\alpha}(t)=\alpha$ (see e.g. [47]). In order to overcome this drawback, we propose to replace in (5.1.1), the fBm $\{B_\alpha(s)\}_{s\in[0,1]}$ by a multifractional Brownian motion (mBm for short), denoted in all the sequel by $\{X(s)\}_{s\in[0,1]}$, which is independent on $\{W(s)\}_{s\in[0,1]}$. Thus, we obtain a new stochastic volatility model we call multifractional stochastic volatility model. Its precise definition is the following:

$$\begin{cases} Z(t) = z_0 + \int_0^t \sigma(s)\,\mathrm{d}W(s) \\ \sigma(s) = \sigma_0 + \Phi(\theta, X(s)), \end{cases} \tag{5.1.4}$$

where z_0, σ_0, θ, $\Phi(\theta,\cdot)$ and $\{W(s)\}_{s\in[0,1]}$ satisfy the same assumptions as before; note that the stochastic integral $\int_0^t \sigma(s)\,\mathrm{d}W(s)$ is well-defined since the mBm $\{X(s)\}_{s\in[0,1]}$ is assumed to be independent on the Bm $\{W(s)\}_{s\in[0,1]}$. In order to clearly explain the reason why multifractional stochastic volatility model allows to overcome the limitation of fractional volatility model we have already pointed out, we need to make some brief recalls concerning mBm. The latter non stationary increments centered Gaussian process was introduced independently in [18] and [51], in order to avoid some drawbacks coming from the fact that the Hurst parameter of fBm cannot evolve with time. The mBm $\{X(s)\}_{s\in[0,1]}$ can be obtained by substituting to the constant Hurst parameter α in the harmonizable representation of fBm:

$$B_\alpha(s) = \int_{\mathbb{R}} \frac{e^{is\xi}-1}{|\xi|^{\alpha+1/2}}\,\mathrm{d}\widehat{B}(\xi), \tag{5.1.5}$$

a function $H(\cdot)$ depending continuously on time and with values in $(0,1)$. The process $\{X(s)\}_{s\in[0,1]}$ can therefore be expressed as,

$$X(s) := \int_{\mathbb{R}} \frac{e^{is\xi}-1}{|\xi|^{H(s)+1/2}}\,\mathrm{d}\widehat{B}(\xi). \tag{5.1.6}$$

Throughout this chapter not only we assume that $H(\cdot)$ is continuous but also that it is a C^2-function; actually we need to impose this condition in order to be able to estimate the correlations between the generalized increments of local averages of mBm (see Proposition 5.3.6). Moreover, to obtain Lemma 5.3.2, we need to assume that $H(\cdot)$ is with values in $(1/2,1)$.

For the sake of clarity, notice that $\mathrm{d}\widehat{B}$ is defined as the unique complex-valued stochastic Wiener measure which satisfies for all $f\in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} f(s)\,\mathrm{d}B(s) = \int_{\mathbb{R}} \widehat{f}(\xi)\,\mathrm{d}\widehat{B}(\xi), \tag{5.1.7}$$

where $\{B(s)\}_{s\in\mathbb{R}}$ denotes a real-valued Wiener process and $\widehat{f}$ the Fourier transform of f (with the convention that the Fourier transform of $g\in L^1(\mathbb{R})$ is defined for real ξ as $\widehat{g}(\xi)=\int_{\mathbb{R}} e^{-is\xi}g(s)\,\mathrm{d}s$). Observe that it follows from (5.1.7) that, one has up to a negligible deterministic smooth real-valued multiplicative function (see [23, 57]), for all $s\in[0,1]$,

$$\int_{\mathbb{R}} \Big\{|s+x|^{H(s)-1/2} - |x|^{H(s)-1/2}\Big\}\,\mathrm{d}B(x) = \int_{\mathbb{R}} \frac{e^{is\xi}-1}{|\xi|^{H(s)+1/2}}\,\mathrm{d}\widehat{B}(\xi),$$

which implies that the process $\{X(s)\}_{s\in[0,1]}$ is real-valued.

Since several years, there is an increasing interest in the study of mBm and related processes (see for instance [4, 2, 3, 5, 8, 6, 19, 20, 21, 32, 33, 34, 58, 56, 55, 57]). The usefulness of such processes as models in financial frame has been emphasized by several authors (see for example [19, 20, 21, 47, 48]). Generally speaking, mBm offers a larger spectrum of applications than fBm, mainly because its local roughness can be prescribed via its functional parameter $H(\cdot)$ and thus is allowed to change with time; more precisely, one has almost surely, for all t, $\rho_X(t)=H(t)$, where $\rho_X(t)$ denotes the pointwise Hölder exponent at t of the mBm $\{X(s)\}_{s\in[0,1]}$. It is worth noticing that the latter result, in turn, implies that in the model (5.1.1) the volatility local roughness can evolve with time, namely when $\Phi(\theta,\cdot)$ is a continuously differentiable function with a nowhere vanishing derivative, then one has, almost surely, at any time t, $\rho_\sigma(t)=H(t)$, where $\rho_\sigma(t)$ is the pointwise Hölder exponent at t, of the volatility process $\{\sigma(s)\}_{s\in[0,1]}$ defined in (5.1.1).

Having given the main motivation behind multifractional stochastic volatility models, let us now clearly explain the goal of chapter. Our aim is to study to which extent it is possible to extend to the setting of these new models Gloter and Hoffmann results (i) and (ii) stated above. Basically, we use some techniques which are reminiscent to those in [35, 36]; however new difficulties appear in our multifractional setting. These new difficulties are essentially due to the fact that local properties of mBm change from one time to another.

Throughout this chapter we assume that the functional parameter $H(\cdot)$ of the mBm $\{X(s)\}_{s\in[0,1]}$ is known. We show that the result (i) can be stated in a more general form and can be extended to multifractional stochastic volatility models. The challenging problem of extending the result (ii) to these models remains open; the major difficulty in it, consists in precisely determining the minimax rate of convergence for estimating θ. Yet, in the linear case, that is for a model of the form:

$$\begin{cases} Z(t) = z_0 + \int_0^t \sigma(s)\,\mathrm{d}W(s) \\ \sigma(s) = \sigma_0 + \theta X(s), \end{cases} \tag{5.1.8}$$

assuming that there exists $t_0 \in (0,1)$ such that $H(t_0) = \min_{t\in[0,1]} H(t)$, we give a partial solution to this problem; namely, we show that by localizing Gloter and Hoffmann estimator in a well-chosen neighborhood of t_0, it is possible to obtain an estimator of θ^2 whose rate of convergence can be bounded in probability by $n^{-1/(4H(t_0)+2)}\big(\log n\big)^{1/4}$.

5.2 Statement of the main results

Let us consider an integrated functional of the volatility of the form:

$$\int_0^1 \big(f'(X(s))\big)^2 h(Y(s))\,\mathrm{d}s, \tag{5.2.1}$$

where, $\{X(s)\}_{s\in[0,1]}$ denotes the mBm, $\{Y(s)\}_{s\in[0,1]}$ is the process defined as

$$Y(s) := f(X(s)) := \sigma^2(s) := \big(\sigma_0 + \Phi(\theta, X(s))\big)^2, \tag{5.2.2}$$

and h an arbitrary function of $C^1_{pol}(\mathbb{R})$. An important difficulty in the problem of the nonparametric estimation of the integral (5.2.1) comes from the fact that the process $\{Y(s)\}_{s\in[0,1]}$ is hidden; as we have mentioned before, we only observe the sample $(Z(0), Z(1/n), \ldots, Z(1))$, where $\{Z(t)\}_{t\in[0,1]}$ is the process defined in (5.1.4). Let us first explain how to overcome this difficulty. $\overline{Y}_{i,N}$, $i = 0, \ldots, N-1$, the local average values of the process $\{Y(s)\}_{s\in[0,1]}$ over a grid $\{0, 1/N, \ldots, 1\}$, $N \geq 1$ being an arbitrary integer, are defined, for all $i = 0, \ldots, N-1$, as

$$\overline{Y}_{i,N} := N \int_{\frac{i}{N}}^{\frac{i+1}{N}} Y(s)\,\mathrm{d}s. \tag{5.2.3}$$

Let us now assume that N is a well chosen integer depending on n (this choice will be made more precise in the statements of Theorems 5.2.2 and 5.2.3 given below), such an N is denoted by N_n; moreover, we set

$$m_n := [n/N_n] \text{ and for every } i = 0, \ldots, N_n,\ j_i := [in/N_n], \tag{5.2.4}$$

with the convention that $[\cdot]$ is the integer part function. The key idea to overcome the difficulty, we have already pointed out, consists in using the fact that for n big enough, $\overline{Y}_{i,N_n}$ can be approximated by

$$\widehat{Y}_{i,N_n,n} := N_n \sum_{k=0}^{j_{i+1}-j_i-1} \left(Z\left(\frac{j_i+k+1}{n}\right) - Z\left(\frac{j_i+k}{n}\right)\right)^2. \tag{5.2.5}$$

The rigorous proof of the latter approximation result relies on Itô formula, it is given in Appendix (see Lemma 7.4.2). This is why we will only give here a short heuristic proof. Using (5.2.5), (5.1.4), the fact that n is big enough, and (5.2.2), one has

$$\begin{aligned}
\widehat{Y}_{i,N_n,n} &= N_n \sum_{k=0}^{j_{i+1}-j_i-1} \left(\int_{\frac{j_i+k}{n}}^{\frac{j_i+k+1}{n}} \sigma(s)\,\mathrm{d}W(s)\right)^2 \\
&\approx N_n \sum_{k=0}^{j_{i+1}-j_i-1} \left(\int_{\frac{j_i+k}{n}}^{\frac{j_i+k+1}{n}} \mathrm{d}W(s)\right)^2 \sigma^2\left(\frac{j_i+k}{n}\right) \\
&= N_n \sum_{k=0}^{j_{i+1}-j_i-1} \left(W\left(\frac{j_i+k+1}{n}\right) - W\left(\frac{j_i+k}{n}\right)\right)^2 Y\left(\frac{j_i+k}{n}\right) \\
&\approx N_n \left(n^{-1} \sum_{k=0}^{j_{i+1}-j_i-1} Y\left(\frac{j_i+k}{n}\right)\right),
\end{aligned} \tag{5.2.6}$$

where the latter approximation follows from the fact that $\left(W\left(\frac{j_i+k+1}{n}\right) - W\left(\frac{j_i+k}{n}\right)\right)^2$, $k = 0, \ldots, j_{i+1} - j_i - 1$ are i.i.d random variables whose expectation equals n^{-1}. Then noticing that $n^{-1}\sum_{k=0}^{j_{i+1}-j_i-1} Y\left(\frac{j_i+k}{n}\right)$ is a Riemann sum which, in view of (5.2.4), converges to the integral $\int_{\frac{i}{N_n}}^{\frac{i+1}{N_n}} Y(s)\,\mathrm{d}s$; it follows from (5.2.3) and (5.2.6) that

$$\widehat{Y}_{i,N_n,n} \approx \overline{Y}_{i,N_n}.$$

The main goal of Section 5.3 is to construct estimators of the integrated functional of the volatility

$$V(h; \mu_N, \nu_N) := \frac{1}{\nu_N - \mu_N} \int_{\mu_N}^{\nu_N} \left(f'(X(s))\right)^2 h(Y(s))\,\mathrm{d}s, \tag{5.2.7}$$

where $(\mu_N)_N$ and $(\nu_N)_N$ are two arbitrary sequences satisfying:

(i) for every N, $0 \leq \mu_N < \nu_N \leq 1$,

(ii) $\lim_{N\to+\infty} N(\nu_N - \mu_N) = +\infty$.

Observe that when we take for every N, $\mu_N = 0$ and $\nu_N = 1$, then the integral in (5.2.7) is equal to the integral in (5.2.1).

In order to be able to state the main two results of Section 5.3, one needs to introduce some additional notations. Throughout this chapter one denotes by $a = (a_0, \dots, a_p)$ a finite sequence of $p+1$ arbitrary fixed real numbers whose $M(a)$ first moments vanish i.e. one has

$$\sum_{k=0}^{p} k^l a_k = 0, \text{ for all } l = 0, \dots, M(a) - 1 \text{ and } \sum_{k=0}^{p} k^{M(a)} a_k \neq 0. \tag{5.2.8}$$

One always assumes that $M(a) \geq 3$ (observe that one has necessarily $p > M(a)$). For each integer $N \geq p+1$ and any $i = 0, \dots, N-p-1$, $\Delta_a \overline{Y}_{i,N}$ is the generalized increment of local average values of Y, defined as

$$\Delta_a \overline{Y}_{i,N} := \sum_{k=0}^{p} a_k \overline{Y}_{i+k,N} \tag{5.2.9}$$

and $\Delta_a \overline{X}_{i,N}$ is the generalized increment of local average values of mBm X, defined as

$$\Delta_a \overline{X}_{i,N} := \sum_{k=0}^{p} a_k \overline{X}_{i+k,N}, \tag{5.2.10}$$

where

$$\overline{X}_{i,N} := N \int_{\frac{i}{N}}^{\frac{i+1}{N}} X(s)\, \mathrm{d}s. \tag{5.2.11}$$

At last, for each integer n big enough and any $i = 0, \dots, N_n - p - 1$, one denotes by $\Delta_a \widehat{Y}_{i,N_n,n}$ the generalized increment defined as

$$\Delta_a \widehat{Y}_{i,N_n,n} := \sum_{k=0}^{p} a_k \widehat{Y}_{i+k,N_n,n}. \tag{5.2.12}$$

One is now in position to state the two main results of Section 5.3. The following theorem provides an estimator of the integrated functional of the volatility $V(h; \mu_N, \nu_N)$ starting from $\overline{Y}_{i,N}$, $\mu_N \leq i/N \leq \nu_N$, the local average values of the process $\{Y(s)\}_{s\in[0,1]}$ over the grid $\{0, 1/N, \dots, 1\} \cap [\mu_N, \nu_N]$. It also provides an upper bound of the rate of convergence.

Theorem 5.2.1 *For every integer $N \geq p+1$ and for every function $h \in C^1_{pol}(\mathbb{R})$ one sets*

$$\overline{V}(h; \mu_N, \nu_N) := \frac{1}{N(\nu_N - \mu_N)} \sum_{i \in \mathcal{J}(\mu_N, \nu_N)} \frac{(\Delta_a \overline{Y}_{i,N})^2}{C(i/N) N^{-2H(i/N)}} h(\overline{Y}_{i,N}), \tag{5.2.13}$$

where:

- *$\mathcal{J}(\mu_N,\nu_N)$ denotes the set of indices,*

$$\mathcal{J}(\mu_N,\nu_N)=\big\{i\in\{0,\dots,N-p-1\}:\ \mu_N\le i/N\le\nu_N\big\};\tag{5.2.14}$$

- *for all $s\in[0,1]$,*

$$C(s):=\int_{\mathbb{R}}\frac{|e^{i\eta}-1|^2|\sum_{k=0}^p a_k e^{ik\eta}|^2}{|\eta|^{2H(s)+3}}\,d\eta.\tag{5.2.15}$$

Then there exists a constant $c>0$, such that one has for each integer $N\ge p+1$,

$$\mathbb{E}\Big\{\big|\overline{V}(h;\mu_N,\nu_N)-V(h;\mu_N,\nu_N)\big|\Big\}\le c\Big(N(\nu_N-\mu_N)\Big)^{-1/2}.\tag{5.2.16}$$

Recall that the integrated functional of the volatility $V(h;\mu_N,\nu_N)$ has been defined in (5.2.7).

In view of the previous theorem, in order to construct an estimator of $V(h;\mu_{N_n},\nu_{N_n})$ starting from the observed data $Z(j/n)$, $j=0,\dots,n$, a natural idea consists in replacing in (5.2.13), the $\overline{Y}_{i,N_n}$'s by their approximations $\widehat{Y}_{i,N_n,n}$. However (this has already been noticed in [35, 36] in the case where X is the fBm, $\mu_{N_n}=0$ and $\nu_{N_n}=1$),

$$\frac{1}{N_n(\nu_{N_n}-\mu_{N_n})}\sum_{i\in\mathcal{J}(\mu_{N_n},\nu_{N_n})}\frac{(\Delta_a\widehat{Y}_{i,N_n,n})^2}{C(i/N_n)N_n^{-2H(i/N_n)}}h(\widehat{Y}_{i,N_n,n})-V(h;\mu_{N_N},\nu_{N_n})$$

does not converge to zero in the $L^1(\Omega)$ norm; one needs therefore to add the correction term:

$$-\frac{1}{N_n(\nu_{N_n}-\mu_{N_n})}\sum_{i\in\mathcal{J}(\mu_{N_n},\nu_{N_n})}\frac{2\|a\|_2^2(\widehat{Y}_{i,N_n,n})^2}{C(i/N_n)N_n^{-2H(i/N_n)}m_n}h(\widehat{Y}_{i,N_n,n}),$$

where $\|a\|_2=\sqrt{\sum_{k=0}^p a_k^2}$ denotes the Euclidian norm of a. More precisely, the following theorem holds.

Theorem 5.2.2 *For every integer n big enough and $h\in C^1_{pol}(\mathbb{R})$, one sets*

$$\widehat{V}(h;\mu_{N_n},\nu_{N_n}):=\frac{1}{N_n(\nu_{N_n}-\mu_{N_n})}\sum_{i\in\mathcal{J}(\mu_{N_n},\nu_{N_n})}\left(\frac{(\Delta_a\widehat{Y}_{i,N_n,n})^2}{C(i/N_n)N_n^{-2H(i/N_n)}}-\frac{2\|a\|_2^2(\widehat{Y}_{i,N_n,n})^2}{C(i/N_n)N_n^{-2H(i/N_n)}m_n}\right)h(\widehat{Y}_{i,N_n,n}),\tag{5.2.17}$$

where m_n is as in (5.2.4). Then assuming that

$$\sup_n m_n^{-1}N_n^{2\max_{s\in[\mu_{N_n},\nu_{N_n}]}H(s)}<+\infty,\tag{5.2.18}$$

it follows that there exists a constant $c>0$, such that one has for all n big enough,

$$\mathbb{E}\Big\{\Big|\widehat{V}(h;\mu_{N_n},\nu_{N_n})-V(h;\mu_{N_n},\nu_{N_n})\Big|\Big\}\le c\Big(N_n(\nu_{N_n}-\mu_{N_n})\Big)^{-1/2}.\tag{5.2.19}$$

Remark 5.2.1 *When the mBm X is a fBm with Hurst parameter $\alpha \in (1/2, 1)$, one can take in Theorems 5.2.1 and 5.2.2, $H(\cdot) = \alpha$, $\mu_{N_n} = 0$ and $\nu_{N_n} = 1$; then one recovers Theorem 3 in [36] and Proposition 1 in [35].*

Let us now turn to Section 5.4. The goal of this section is to construct an estimator of θ^2 in the setting of a linear stochastic volatility model driven by a mBm, that is a model of the type (5.1.8); and also to give an evaluation of the rate of convergence of this estimator in terms of $\min_{t\in[0,1]} H(t)$. Notice that, in Section 5.4, we assume that $\theta \neq 0$ and there exists $t_0 \in (0,1)$ such that

$$H(t_0) = \min_{t\in[0,1]} H(t). \tag{5.2.20}$$

In order to be able to state the main result of this section, we need to introduce some additional notations. For n big enough, we set,

$$\mathscr{E}_{N_n}^{min}(t_0) = t_0 - \frac{1}{\sqrt{\log(N_n)}}, \tag{5.2.21}$$

$$\mathscr{E}_{N_n}^{max}(t_0) = t_0 + \frac{1}{\sqrt{\log(N_n)}} \tag{5.2.22}$$

and

$$\mathcal{V}_{N_n,t_0} := \mathscr{J}\big(\mathscr{E}_{N_n}^{min}(t_0), \mathscr{E}_{N_n}^{max}(t_0)\big) = \Big\{ i \in \{0,\dots,N_n - p - 1\} \; : \; \Big|t_0 - \frac{i}{N_n}\Big| \leq \frac{1}{\sqrt{\log N_n}} \Big\}, \tag{5.2.23}$$

where $\mathscr{J}$ has been introduced in (5.2.14).

Let $(a_n)_n$ and $(b_n)_n$ be two arbitrary sequences of positive real numbers. The notation $a_n \asymp b_n$ means there exist two constants $0 < c_1 \leq c_2$, such that for all n, one has $c_1 a_n \leq b_n \leq c_2 a_n$.

We are now in position to state the main result of Section 5.4.

Theorem 5.2.3 *Consider a linear stochastic volatility model driven by a mBm. For n big enough, let,*

$$\widehat{\theta}^2_{n,t_0} = \frac{\widehat{V}(1; \mathscr{E}_{N_n}^{min}(t_0), \mathscr{E}_{N_n}^{max}(t_0))}{4\big(2(\log(N_n))^{-1/2} N_n\big)^{-1} \sum_{i\in\mathcal{V}_{N_n,t_0}} \widehat{Y}_{i,N_n,n}}, \tag{5.2.24}$$

where $\widehat{V}$ has been introduced in (5.2.17). Assume that

$$N_n \asymp n^{1/(2H(t_0)+1)}. \tag{5.2.25}$$

Then the sequence of random variables

$$\Big(n^{1/(4H(t_0)+2)} (\log n)^{-1/4} (\widehat{\theta}^2_{n,t_0} - \theta^2)\Big)_n,$$

is bounded in probability i.e. one has

$$\lim_{\lambda\to+\infty} \limsup_{n\to+\infty} \mathbb{P}\Big\{ n^{1/(4H(t_0)+2)} (\log n)^{-1/4} |\widehat{\theta}^2_{n,t_0} - \theta^2| > \lambda \Big\} = 0. \tag{5.2.26}$$

Remark 5.2.2 *Theorem 5.2.3 is an extension of Proposition 2 in [35].*

In fact, Theorem 5.2.3 is a straightforward consequence of the following result.

Theorem 5.2.4 *Consider a linear stochastic volatility model driven by a mBm. For n big enough, we set,*

$$\widehat{\theta}_n^2(\mu_{N_n}, \nu_{N_n}) = \frac{\widehat{V}(1; \mu_{N_n}, \nu_{N_n})}{4(N_n(\nu_{N_n} - \mu_{N_n}))^{-1} \sum_{i \in \mathcal{J}(\mu_{N_n}, \nu_{N_n})} \widehat{Y}_{i,N_n,n}}. \tag{5.2.27}$$

Assume that $(\mu_{N_n})_n$ *and* $(\nu_{N_n})_n$ *are two convergent sequences and also that*

$$N_n \asymp n^{1/(2 \max_{s \in [\mu_{N_n}, \nu_{N_n}]} H(s)+1)}. \tag{5.2.28}$$

Then the sequence of random variables

$$\Big(n^{1/(4 \max_{s \in [\mu_{N_n}, \nu_{N_n}]} H(s)+2)} (\nu_{N_n} - \mu_{N_n})^{1/2} \Big(\widehat{\theta}_n^2(\mu_{N_n}, \nu_{N_n}) - \theta^2 \Big) \Big)_n,$$

is bounded in probability.

5.3 Estimation of integrated functionals of the volatility

5.3.1 Proof of Theorem 5.2.1 when Y is the mBm

The goal of this subsection is to show that Theorem 5.2.1 holds in the particular case where the process Y (see (5.2.2)) is the mBm X itself i.e. the function f is equal to the identity. Namely, we will prove the following theorem.

Theorem 5.3.1 *For every integer* $N \geq p+1$ *and every function* $h \in C^1_{pol}(\mathbb{R})$ *one sets*

$$Q(h; \mu_N, \nu_N) := \frac{1}{\nu_N - \mu_N} \int_{\mu_N}^{\nu_N} h(X(s))\, \mathrm{d}s \tag{5.3.1}$$

and

$$\overline{Q}(h; \mu_N, \nu_N) := \frac{1}{N(\nu_N - \mu_N)} \sum_{i \in \mathcal{J}(\mu_N, \nu_N)} \frac{(\Delta_a \overline{X}_{i,N})^2}{C(i/N) N^{-2H(i/N)}} h(\overline{X}_{i,N}). \tag{5.3.2}$$

Then there is a constant $c > 0$, *such that one has for each* $N \geq p+1$,

$$\mathbb{E}\Big\{ \big| \overline{Q}(h; \mu_N, \nu_N) - Q(h; \mu_N, \nu_N) \big| \Big\} \leq c \big(N(\nu_N - \mu_N) \big)^{-1/2}. \tag{5.3.3}$$

Remark 5.3.1 *This theorem generalizes Proposition 1 in [36].*

Let us explain the main intuitive ideas which lead to the estimator $\overline{Q}(h; \mu_N, \nu_N)$.

- First one approximates the integral $(\nu_N - \mu_N)^{-1} \int_{\mu_N}^{\nu_N} h(X(s))\, \mathrm{d}s$ by the Riemann sum

$$\frac{1}{N(\nu_N - \mu_N)} \sum_{i \in \mathcal{J}(\mu_N, \nu_N)} h\big(X(\frac{i}{N})\big);$$

- then one approximates the latter quantity by

$$\frac{1}{N(\nu_N-\mu_N)}\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\frac{(\Delta_a\overline{X}_{i,N})^2}{\mathrm{Var}(\Delta_a\overline{X}_{i,N})}h\big(X(\frac{i}{N})\big);$$

- finally, one approximates the latter quantity by $\overline{Q}(h;\mu_N,\nu_N)$ since $X(i/N)\simeq\overline{X}_{i,N}$ and $\mathrm{Var}(\Delta_a\overline{X}_{i,N})\simeq C(i/N)N^{-2H(i/N)}$.

Upper bounds of the L^1-norms of the successive approximation errors are given in the following lemma.

Lemma 5.3.2 *Let $h\in C^1_{pol}(\mathbb{R})$, then there exist four constants $c_1,c_2,c_3,c_4>0$, such that the following inequalities hold for every $N\geq p+1$.*

(i)

$$\begin{aligned} E_1 &:= \mathbb{E}\Big\{\Big|\frac{1}{N(\nu_N-\mu_N)}\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}h\big(X(\frac{i}{N})\big)-\frac{1}{\nu_N-\mu_N}\int_{\mu_N}^{\nu_N}h(X(s))\,\mathrm{d}s\Big|\Big\} \\ &\leq c_1\big(N(\nu_N-\mu_N)\big)^{-1/2}; \end{aligned} \tag{5.3.4}$$

(ii)

$$\begin{aligned} E_2 &:= \mathbb{E}\Big\{\Big|\frac{1}{N(\nu_N-\mu_N)}\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\Big\{\frac{(\Delta_a\overline{X}_{i,N})^2}{Var(\Delta_a\overline{X}_{i,N})}-1\Big\}h\big(X(\frac{i}{N})\big)\Big|\Big\} \\ &\leq c_2(N(\nu_N-\mu_N))^{-1/2}; \end{aligned} \tag{5.3.5}$$

(iii)

$$\begin{aligned} E_3 &:= \mathbb{E}\Big\{\Big|\frac{1}{N(\nu_N-\mu_N)}\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\frac{(\Delta_a\overline{X}_{i,N})^2}{Var(\Delta_a\overline{X}_{i,N})}\Big(h(\overline{X}_{i,N})-h\big(X(\frac{i}{N})\big)\Big)\Big|\Big\} \\ &\leq c_3(N(\nu_N-\mu_N))^{-1/2}; \end{aligned} \tag{5.3.6}$$

(iv)

$$E_4:=\mathbb{E}\Big\{\Big|\overline{Q}(h;\mu_N,\nu_N)-\widetilde{Q}(h;\mu_N,\nu_N)\Big|\Big\}\leq c_4\big(N(\nu_N-\mu_N)\big)^{-1/2}, \tag{5.3.7}$$

where

$$\widetilde{Q}(h;\mu_N,\nu_N):=\frac{1}{N(\nu_N-\mu_N)}\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\frac{(\Delta_a\overline{X}_{i,N})^2}{Var(\Delta_a\overline{X}_{i,N})}h(\overline{X}_{i,N}). \tag{5.3.8}$$

Proof of Theorem 5.3.1: This theorem is a straightforward consequence of Lemma 5.3.2 and the triangle inequality. Indeed, one has that

$$\mathbb{E}\left\{\left|\widetilde{Q}(h;\mu_N,\nu_N)-Q(h;\mu_N,\nu_N)\right|\right\}\le E_1+E_2+E_3+E_4\le c\big(N(\nu_N-\mu_N)\big)^{-1/2}.$$

where $c=c_1+c_2+c_3+c_4>0$ is a constant. □

Now we are going to explain how to get Lemma 5.3.2. Since the proofs of Parts (i), (iii) and (iv) of the lemma are less difficult than that of Part (ii), we will first give them.

Proof of Lemma 5.3.2 (i)**:** Denote by $J_N^{min}=\min\mathcal{J}(\mu_N,\nu_N)$ and $J_N^{max}=\max\mathcal{J}(\mu_N,\nu_N)$, thus, in view of (5.2.14), one has,

$$\begin{aligned}&\int_{\mu_N}^{\nu_N}h(X(s))\,\mathrm{d}s\\&=\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\int_{i/N}^{(i+1)/N}h(X(s))\,\mathrm{d}s+\int_{\mu_N}^{J_N^{min}/N}h(X(s))\,\mathrm{d}s+\int_{J_N^{max}/N}^{\nu_N}h(X(s))\,\mathrm{d}s.\end{aligned}\tag{5.3.9}$$

Next using (5.3.9) and the triangle inequality, one gets,

$$\begin{aligned}&\Big|N^{-1}\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}h\big(X(\frac{i}{N})\big)-\int_{\mu_N}^{\nu_N}h(X(s))\,\mathrm{d}s\Big|\\&=\Big|\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\int_{i/N}^{(i+1)/N}h\big(X(\frac{i}{N})\big)\,\mathrm{d}s-\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\int_{i/N}^{(i+1)/N}h(X(s))\,\mathrm{d}s\\&\quad-\int_{\mu_N}^{J_N^{min}/N}h(X(s))\,\mathrm{d}s-\int_{J_N^{max}/N}^{\nu_N}h(X(s))\,\mathrm{d}s\Big|\\&\le\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\int_{i/N}^{(i+1)/N}\Big|h\big(X(\frac{i}{N})\big)-h(X(s))\Big|\,\mathrm{d}s\\&\quad+\int_{\mu_N}^{J_N^{min}/N}|h(X(s))|\,\mathrm{d}s+\int_{J_N^{max}/N}^{\nu_N}|h(X(s))|\,\mathrm{d}s.\end{aligned}\tag{5.3.10}$$

In view of (5.3.10), in order to prove that Lemma 5.3.2 (i) holds, it remains to show that there are two positive constants c_1,c_2 such that for all $N\ge p+1$,

$$E\Big\{\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\int_{i/N}^{(i+1)/N}\Big|h\big(X(\frac{i}{N})\big)-h(X(s))\Big|\,\mathrm{d}s\Big\}\le c_1N^{-1/2}(\nu_N-\mu_N)^{1/2};\tag{5.3.11}$$

and

$$E\Big\{\int_{\mu_N}^{J_N^{min}/N}|h(X(s))|\,\mathrm{d}s+\int_{J_N^{max}/N}^{\nu_N}|h(X(s))|\,\mathrm{d}s\Big\}\le c_2N^{-1/2}(\nu_N-\mu_N)^{1/2}.\tag{5.3.12}$$

Let us first prove that (5.3.11) is satisfied. Using the Taylor expansion of h of order 1 around $X(i/N)$ with integral remainder, one has,

$$h(X(s)) = h\big(X(\frac{i}{N})\big) + \big(X(\frac{i}{N}) - X(s)\big)\int_0^1 h'\Big(X(\frac{i}{N}) + \theta\big(X(s) - X(\frac{i}{N})\big)\Big)\,\mathrm{d}\theta. \tag{5.3.13}$$

We denote by $\|X\|_\infty$ the random variable defined as $\|X\|_\infty = \sup_{s\in[0,1]}|X(s)|$. Since $\{X(s)\}_{s\in[0,1]}$ is a Gaussian process with continuous trajectories, by applying Dudley's Theorem and Borell's inequality (more precisely, with the same arguments for the proof of $\mathbb{E}\,(e^{\tilde{V}}) < +\infty$ on Page 1445 − 1446 of [52], see also [44]), one can show that

$$\mathbb{E}\,(e^{\|X\|_\infty}) < +\infty. \tag{5.3.14}$$

Recall that for all fixed nonnegative real l, there is a constant $c > 0$ such that every nonnegative real x, one has,

$$x^l \le ce^x. \tag{5.3.15}$$

It follows from (5.3.14) and (5.3.15) that,

$$\mathbb{E}\,(\|X\|_\infty^l) \le c\mathbb{E}\,(e^{\|X\|_\infty}) < +\infty. \tag{5.3.16}$$

Next, using (5.3.13) and the fact that $h \in C^1_{pol}(\mathbb{R})$, one obtains

$$\begin{aligned}\Big|h(X(s)) - h\big(X(\frac{i}{N})\big)\Big| &\le \Big|X(\frac{i}{N}) - X(s)\Big| \sup_{t\in[-\|X\|_\infty,\|X\|_\infty]}|h'(t)|\\ &\le c_3\Big|X(\frac{i}{N}) - X(s)\Big|(1 + \|X\|_\infty^K),\end{aligned} \tag{5.3.17}$$

where c_3, K are the two constants appearing in (5.1.3). It follows from (5.3.17) and Cauchy-Schwarz inequality that

$$\begin{aligned}E\Big|h(X(s)) - h\big(X(\frac{i}{N})\big)\Big| &\le c_3\Big(E\Big|X(\frac{i}{N}) - X(s)\Big|^2\Big)^{1/2}\Big(E(1 + \|X\|_\infty^K)^2\Big)^{1/2}\\ &= c_4\Big(E\Big|X(\frac{i}{N}) - X(s)\Big|^2\Big)^{1/2},\end{aligned} \tag{5.3.18}$$

where $c_4 = c_3\Big(E(1 + \|X\|_\infty^K)^2\Big)^{1/2} < +\infty$ (thanks to (5.3.16)) is a positive constant which does not depend on N. Moreover, Lemma 2.12 in [7], the fact that $s \in [i/N, (i+1)/N]$ and the assumption that $H(\cdot)$ is with values in $(1/2, 1)$, yield that there is a constant $c_5 > 0$ such that

$$E\Big\{\big|X(\frac{i}{N}) - X(s)\big|^2\Big\} \le c_5\big|\frac{i}{N} - s\big|^{2H(i/N)} \le c_5 N^{-2H(i/N)} \le c_5 N^{-1}. \tag{5.3.19}$$

Thus, combining 5.3.18) with (5.3.19), one gets that,

$$E\Big|h(X(s)) - h\big(X(\frac{i}{N})\big)\Big| \le c_4(c_5)^{1/2}N^{-1/2} \tag{5.3.20}$$

Next, it follows from (5.3.11), Fubini Theorem and (5.3.20) that

$$
\begin{aligned}
& E\Big\{ \sum_{i\in\mathcal{J}(\mu_N,\nu_N)} \int_{i/N}^{(i+1)/N} \Big| h(X(\frac{i}{N})) - h(X(s)) \Big| \, \mathrm{d}s \Big\} \\
& = \sum_{i\in\mathcal{J}(\mu_N,\nu_N)} \int_{i/N}^{(i+1)/N} E\Big\{ \Big| h(X(\frac{i}{N})) - h(X(s)) \Big| \Big\} \, \mathrm{d}s \\
& \le c_4 (c_5)^{1/2} N^{-3/2} \mathrm{Card}(\mathcal{J}(\mu_N,\nu_N)). \qquad (5.3.21)
\end{aligned}
$$

Moreover, the assumption that $\lim_{N\to+\infty} N(\nu_N - \mu_N) = +\infty$ implies that, one has for all N big enough,

$$
\mathrm{Card}(\mathcal{J}(\mu_N,\nu_N)) \le [N\nu_N] + 1 - [N\mu_N] + 1 \le 2N(\nu_N - \mu_N). \qquad (5.3.22)
$$

Next, (5.3.21), (5.3.22) and the fact that $0 < \nu_N - \mu_N \le 1$ lead to

$$
\begin{aligned}
& E\Big\{ \sum_{i\in\mathcal{J}(\mu_N,\nu_N)} \int_{i/N}^{(i+1)/N} \Big| h(X(\frac{i}{N})) - h(X(s)) \Big| \, \mathrm{d}s \Big\} \\
& \le 2c_4 (c_5)^{1/2} N^{-1/2} (\nu_N - \mu_N) \\
& \le c_6 N^{-1/2} (\nu_N - \mu_N)^{1/2}, \qquad (5.3.23)
\end{aligned}
$$

where $c_6 = 2c_4(c_5)^{1/2}$; which proves that (5.3.11) holds.

Let us now show that (5.3.12) is satisfied. Observe that for all $N \ge p+1$,

$$
J_N^{min} \in \Big\{ [N\mu_N], [N\mu_N] + 1 \Big\} \qquad (5.3.24)
$$

and

$$
J_N^{max} = \min\Big([N\nu_N], N - p - 1 \Big). \qquad (5.3.25)
$$

(5.3.24) and (5.3.25) imply that

$$
\begin{aligned}
\Big| \frac{J_N^{min}}{N} - \mu_N \Big| & \le \max\Big\{ \Big| \frac{[N\mu_N]}{N} - \mu_N \Big|, \Big| \frac{[N\mu_N]+1}{N} - \mu_N \Big| \Big\} \\
& \le \max\Big\{ \mu_N - \frac{N\mu_N - 1}{N}, \frac{N\mu_N + 1}{N} - \mu_N \Big\} \\
& = N^{-1}; \qquad (5.3.26)
\end{aligned}
$$

and when $[N\nu_N] \le N - p - 1$,

$$
\begin{aligned}
\Big| \frac{J_N^{max}}{N} - \nu_N \Big| & = \Big| \frac{[N\nu_N]}{N} - \nu_N \Big| \\
& \le \nu_N - \frac{N\nu_N - 1}{N} \\
& = N^{-1}; \qquad (5.3.27)
\end{aligned}
$$

when $N-p-1<[N\nu_N]$, one has $N-p-1<N\nu_N\le N$, i.e. $0<|N-p-1-N\nu_N|\le p+1$, thus

$$\begin{aligned}\Big|\frac{J_N^{max}}{N}-\nu_N\Big| &= \Big|\frac{N-p-1}{N}-\nu_N\Big|\\ &= \frac{|N-p-1-N\nu_N|}{N}\\ &\le (p+1)N^{-1}. \end{aligned} \tag{5.3.28}$$

Combining (5.3.27) with (5.3.28), we get

$$\Big|\frac{J_N^{max}}{N}-\nu_N\Big|\le (p+1)N^{-1}. \tag{5.3.29}$$

It follows from the fact that $h\in C^1_{pol}(\mathbb{R})$, (5.3.26) and (5.3.29) that

$$\begin{aligned}&E\Big\{\int_{\mu_N}^{J_N^{min}/N}|h(X(s))|\,\mathrm{d}s+\int_{J_N^{max}/N}^{\nu_N}|h(X(s))|\,\mathrm{d}s\Big\}\\ &\le c_3\Big(\Big|\frac{J_N^{min}}{N}-\mu_N\Big|+\Big|\frac{J_N^{max}}{N}-\nu_N\Big|\Big)\mathbb{E}\,(1+\|X\|_\infty^K)\\ &\le c_7N^{-1}. \end{aligned} \tag{5.3.30}$$

where $c_3>0$ is the constant in (5.3.17) and $c_7=(p+2)c_3\mathbb{E}\,(1+\|X\|_\infty^K)$. Next observe that the assumption $\lim_{N\to+\infty}N(\nu_N-\mu_N)=+\infty$ implies that $N^{-1}\le N^{-1/2}(\nu_N-\mu_N)^{1/2}$ for N big enough. Putting together the latter inequality and (5.3.30), we prove that (5.3.12) holds. Combining (5.3.10), (5.3.11) and (5.3.12), we get Lemma 5.3.2 (i). $\square$

Let us now show that Lemma 5.3.2 (iii) holds.

Proof of Lemma 5.3.2 (iii)**:** By using the triangle inequality and Cauchy-Schwarz inequality, one obtains

$$\begin{aligned}&E\Big|\frac{1}{N(\nu_N-\mu_N)}\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\frac{(\Delta_a\overline{X}_{i,N})^2}{\mathrm{Var}(\Delta_a\overline{X}_{i,N})}\Big(h(\overline{X}_{i,N})-h\big(X(\frac{i}{N})\big)\Big)\Big|\\ &\le\frac{1}{N(\nu_N-\mu_N)}\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\frac{\big(\mathbb{E}\,\big((\Delta_a\overline{X}_{i,N})^4\big)\big)^{1/2}}{\mathrm{Var}(\Delta_a\overline{X}_{i,N})}\Big(\mathbb{E}\,\Big|h(\overline{X}_{i,N})-h\big(X(\frac{i}{N})\big)\Big|^2\Big)^{1/2}. \end{aligned} \tag{5.3.31}$$

On one hand, the equivalence of Gaussian moments, implies that there is constant $c_1>0$, non depending on N and i, such that

$$\mathbb{E}\,\big((\Delta_a\overline{X}_{i,N})^4\big)=c_1\big(\mathrm{Var}(\Delta_a\overline{X}_{i,N})\big)^2; \tag{5.3.32}$$

On the other hand, similarly to (5.3.20), one can show that there exists a constant $c_2>0$, non depending on N and i, such that

$$\mathbb{E}\,\Big|h(\overline{X}_{i,N})-h\big(X(\frac{i}{N})\big)\Big|^2\le c_2N^{-1}. \tag{5.3.33}$$

Finally, Lemma 5.3.2 (*iii*) results from (5.3.31), (5.3.32), (5.3.33), (5.3.22) and the fact that $0 < \nu_N - \mu_N \leq 1$:

$$
\begin{aligned}
&E\Big|\frac{1}{N(\nu_N-\mu_N)}\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\frac{(\Delta_a\overline{X}_{i,N})^2}{\mathrm{Var}(\Delta_a\overline{X}_{i,N})}\Big(h(\overline{X}_{i,N})-h\big(X(\frac{i}{N})\big)\Big)\Big|\\
&\leq (c_1c_2)^{1/2}(N(\nu_N-\mu_N))^{-1}\mathrm{Card}(\mathcal{J}(\mu_N,\nu_N))N^{-1/2}\\
&\leq 2(c_1c_2)^{1/2}N^{-1/2}\\
&\leq c_3(N(\nu_N-\mu_N))^{-1/2}, \qquad (5.3.34)
\end{aligned}
$$

where $c_3 = 2(c_1c_2)^{1/2}$. □

In order to prove Part (*iv*) of Lemma 5.3.2, we need the following lemma whose proof is given in Appendix.

Lemma 5.3.3 *There is a constant $c > 0$ such that for every $i \in \{0, \dots, N-p-1\}$ one has,*

$$
\Big|Var(\Delta_a\overline{X}_{i,N}) - C(i/N)N^{-2H(i/N)}\Big| \leq c\log(N)N^{-1-2H(i/N)}. \qquad (5.3.35)
$$

Proof of Lemma 5.3.2 (*iv*)**:** By applying Lemma 5.3.3, it follows that there is a constant $c_1 > 0$ such that, for all $N \geq p+1$ and all $i \in \{0, \dots, N-p-1\}$,

$$
\begin{aligned}
\Big|\frac{1}{C(i/N)N^{-2H(i/N)}}-\frac{1}{\mathrm{Var}(\Delta_a\overline{X}_{i,N})}\Big| &= \frac{|\mathrm{Var}(\Delta_a\overline{X}_{i,N})-C(i/N)N^{-2H(i/N)}|}{C(i/N)N^{-2H(i/N)}\mathrm{Var}(\Delta_a\overline{X}_{i,N})}\\
&\leq \frac{c_1\log(N)N^{-1}}{C(i/N)\mathrm{Var}(\Delta_a\overline{X}_{i,N})}. \qquad (5.3.36)
\end{aligned}
$$

By using (5.3.2), (5.3.8), the triangle inequality, (5.3.36) and Cauchy-Schwarz inequality, one has

$$
\begin{aligned}
&\mathbb{E}\left\{\Big|\overline{Q}(h;\mu_N,\nu_N)-\widetilde{Q}(h;\mu_N,\nu_N)\Big|\right\}\\
&\leq \frac{1}{N(\nu_N-\mu_N)}\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\mathbb{E}\left(\frac{|\mathrm{Var}(\Delta_a\overline{X}_{i,N})-C(i/N)N^{-2H(i/N)}|}{C(i/N)N^{-2H(i/N)}\mathrm{Var}(\Delta_a\overline{X}_{i,N})}(\Delta_a\overline{X}_{i,N})^2h(\overline{X}_{i,N})\right)\\
&\leq c_1(\min_{s\in[0,1]}C(s))^{-1}(\log(N)N^{-1})(N(\nu_N-\mu_N))^{-1}\\
&\quad\times\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\left(\frac{\mathbb{E}\,(\Delta_a\overline{X}_{i,N})^4}{\big(\mathrm{Var}(\Delta_a\overline{X}_{i,N})\big)^2}\right)^{1/2}\Big(\mathbb{E}\,|h(\overline{X}_{i,N})|^2\Big)^{1/2}. \qquad (5.3.37)
\end{aligned}
$$

Notice that (5.2.15) implies that $0 < \min_{s\in[0,1]} C(s)$ and consequently that $(\min_{s\in[0,1]} C(s))^{-1} < +\infty$. It follows from (5.3.37), (5.3.32), the fact that $h \in$

$C^1_{pol}(\mathbb{R})$ and (5.3.22) that

$$\begin{aligned}
&\mathbb{E}\left\{\left|\overline{Q}(h;\mu_N,\nu_N)-\widetilde{Q}(h;\mu_N,\nu_N)\right|\right\}\\
&\le c_1(c_2)^{1/2}(\min_{s\in[0,1]}C(s))^{-1}(\log(N)N^{-1})(N(\nu_N-\mu_N))^{-1}\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\Big(\mathbb{E}\,|h(\overline{X}_{i,N})|^2\Big)^{1/2}\\
&\le 2c_1(c_2c_3)^{1/2}(\min_{s\in[0,1]}C(s))^{-1}(\log(N)N^{-1})\Big(\mathbb{E}\,(1+\|X\|_\infty^K)^2\Big)^{1/2}\\
&\le c_4\log(N)N^{-1}, \qquad (5.3.38)
\end{aligned}$$

where

$$c_4=2c_1(c_2c_3)^{1/2}(\min_{s\in[0,1]}C(s))^{-1}\Big(\mathbb{E}\,(1+\|X\|_\infty^K)^2\Big)^{1/2},$$

with c_2,K being the two constants introduced in (5.1.3) and c_3 being the constant in (5.3.32). Finally Lemma 5.3.2 (iv) results from (5.3.38) and the fact that $\log(N)N^{-1/2}\le(\nu_N-\mu_N)^{-1/2}$ for all $N\ge p+1$. □

Let us now focus on the proof of Part (ii) of Lemma 5.3.2; this proof relies on some technics which are more or less similar to those in [35, 36]. First we need to give some preliminary results. The following lemma is a more or less classical result on centered 2-dimensional Gaussian vectors; its proof is given in Appendix.

Lemma 5.3.4 *Let (Z,Z') be a 2-D centered Gaussian random vector and assume that the variances of Z and Z' are both equal to the same quantity denoted by v. Then, one has,*

$$\mathbb{E}\Big\{(Z^2-v)(Z'^2-v)\Big\}=2\Big(Cov(Z,Z')\Big)^2. \qquad (5.3.39)$$

Lemma 5.3.5 *For every $N\ge p+1$, let $\rho_N:[\mu_N,\nu_N]\to\mathbb{R}$ be an arbitrary deterministic bounded function and let $\Sigma_N(\rho_N)$ be the quantity defined as,*

$$\Sigma_N(\rho_N)=\sum_{j\in\mathcal{J}(\mu_N,\nu_N)}\Big\{\frac{(\Delta_a\overline{X}_{j,N})^2}{Var(\Delta_a\overline{X}_{j,N})}-1\Big\}\rho_N(j/N), \qquad (5.3.40)$$

where $\mathcal{J}(\mu_N,\nu_N)$ is the set introduced in (5.2.14). Then the inequality,

$$\mathbb{E}\Big\{\big(\Sigma_N(\rho_N)\big)^2\Big\}\le c\|\rho_N\|_\infty^2N(\nu_N-\mu_N),$$

where $\|\rho_N\|_\infty:=\sup_{x\in[\mu_N,\nu_N]}|\rho_N(x)|$ and $c>0$ is a constant non depending on N, holds for every $N\ge p+1$.

In order to prove Lemma 5.3.5, we need the following proposition which concerns the estimation of the correlation between the generalized increments $\Delta_a\overline{X}_{j,N}$ and $\Delta_a\overline{X}_{j',N}$ of the mBm $\{X(s)\}_{s\in[0,1]}$; we refer to Appendix for its proof.

Proposition 5.3.6 *Assume that $H(\cdot) \in C^2([0,1])$ and $M(a) \geq 3$ (recall that $M(a)$ is the number of the vanishing moments of the sequence a). Then there is a constant $c > 0$ such that one has for any integer $N \geq p+1$ and each $j, j' \in \{0, \ldots, N-p-1\}$,*

$$\left| Corr\left\{\Delta_a \overline{X}_{j,N}, \Delta_a \overline{X}_{j',N}\right\}\right| \leq c\left(\frac{1}{1+|j-j'|}\right), \tag{5.3.41}$$

where

$$Corr\left\{\Delta_a \overline{X}_{j,N}, \Delta_a \overline{X}_{j',N}\right\} := \mathbb{E}\left\{\frac{\Delta_a \overline{X}_{j,N}}{\sqrt{Var\left\{\Delta_a \overline{X}_{j,N}\right\}}}\frac{\Delta_a \overline{X}_{j',N}}{\sqrt{Var\left\{\Delta_a \overline{X}_{j',N}\right\}}}\right\}.$$

Proof of Lemma 5.3.5: In view of (5.3.40), one clearly has that

$$\begin{aligned}
&\mathbb{E}\left\{(\Sigma_N(\rho_N))^2\right\} \\
&= \sum_{j,j' \in \mathcal{J}(\mu_N,\nu_N)} \rho_N(j/N)\rho_N(j'/N)\mathbb{E}\left\{\Big(\frac{(\Delta_a \overline{X}_{j,N})^2}{\mathrm{Var}(\Delta_a \overline{X}_{j,N})} - 1\Big)\Big(\frac{(\Delta_a \overline{X}_{j',N})^2}{\mathrm{Var}(\Delta_a \overline{X}_{j',N})} - 1\Big)\right\}.
\end{aligned}$$

Next it follows from Lemma 5.3.4, Proposition 5.3.6 and (5.3.22), that

$$\begin{aligned}
&\mathbb{E}\left\{(\Sigma_N(\rho_N))^2\right\} = 2\sum_{j,j' \in \mathcal{J}(\mu_N,\nu_N)} \rho_N(j/N)\rho_N(j'/N)\Big(\mathrm{Corr}(\Delta_a \overline{X}_{j,N}, \Delta_a \overline{X}_{j',N})\Big)^2 \\
&\leq 2c_1\|\rho_N\|_\infty^2 \sum_{j,j' \in \mathcal{J}(\mu_N,\nu_N)} \Big(\frac{1}{1+|j-j'|}\Big)^2 \\
&\leq 4c_1\|\rho_N\|_\infty^2 \sum_{j \in \mathcal{J}(\mu_N,\nu_N)} \sum_{l=-\infty}^{\infty} \Big(\frac{1}{1+|l|}\Big)^2 \\
&= 4c_1\|\rho_N\|_\infty^2 \mathrm{Card}(\mathcal{J}(\mu_N,\nu_N)) \sum_{l=-\infty}^{\infty} \Big(\frac{1}{1+|l|}\Big)^2 \\
&\leq c_2\|\rho_N\|_\infty^2 N(\nu_N - \mu_N),
\end{aligned} \tag{5.3.42}$$

where c_1 is the constant introduced in (5.3.41) and $c_2 = 8c_1\sum_{l=-\infty}^{\infty}(1+|l|)^{-2}$ are two constants which do not depend on N. □

Lemma 5.3.7 *For every $N \geq p+1$, let $\rho_N : [\mu_N, \nu_N] \to \mathbb{R}$ be an arbitrary bounded deterministic function that vanishes outside a dyadic interval of the form $[k2^{-j_0}\mathcal{L}_N + \mu_N, k'2^{-j_0}\mathcal{L}_N + \mu_N]$, where $\mathcal{L}_N := \nu_N - \mu_N$ and where the integers j_0, k and k' are arbitrary and satisfy $j_0 \geq 1$ and $0 \leq k < k' \leq 2^{j_0}$. Then there exists a constant $c > 0$ which does not depend on N, k, k' and j_0, such that for all integers $N \geq p+1$, and j_1 satisfying $2^{j_0} \leq 2^{j_1} \leq N\mathcal{L}_N < 2^{j_1+1}$, one has*

$$\mathbb{E}\left\{(\Sigma_N(\rho_N))^2\right\} \leq c\|\rho_N\|_\infty^2 (k'-k)2^{j_1-j_0}. \tag{5.3.43}$$

Proof of Lemma 5.3.7: Let $\mathcal{I}(k,k',j_0,N)$ be the set of indices defined as,

$$\mathcal{I}(k,k',j_0,N)=\Big\{i\in\mathcal{J}(\mu_N,\nu_N):i\in\big[(k2^{-j_0}\mathcal{L}_N+\mu_N)N,(k'2^{-j_0}\mathcal{L}_N+\mu_N)N\big]\Big\}.$$

One has,

$$\mathrm{Card}\big(\mathcal{I}(k,k',j_0,N)\big)\le N(k'-k)2^{-j_0}\mathcal{L}_N+1\le 4(k'-k)2^{j_1-j_0}, \tag{5.3.44}$$

where the last inequality follows from the fact that $N\mathcal{L}_N\le 2^{j_1+1}$. Using the method which allowed us to obtain (5.3.42) and replacing $\mathcal{J}(\mu_N,\nu_N)$ by $\mathcal{I}(k,k',j_0,N)$, one can show that,

$$\begin{aligned}\mathbb{E}\left\{(\Sigma_N(\rho_N))^2\right\}&=2\sum_{j,j'\in\mathcal{I}(k,k',j_0,N)}\rho_N(j/N)\rho_N(j'/N)\Big(\mathrm{Corr}(\Delta_a\overline{X}_{j,N},\Delta_a\overline{X}_{j',N})\Big)^2\\&\le c\|\rho_N\|_\infty^2(k'-k)2^{j_1-j_0},\end{aligned} \tag{5.3.45}$$

where $c>0$ is a constant which does not depend on k,k',j_0,N.

We are now in position to prove part (ii) of Lemma 5.3.2.

Proof of Part (ii) of Lemma 5.3.2: Let $N\ge p+1$ be fixed. Observe that, with probability 1, the function $t\mapsto h(X(t))$ belongs to $C\big([\mu_N,\nu_N]\big)$, the space of the continuous functions over $[\mu_N,\nu_N]$. By expanding it, in the Schauder basis of this space, one obtains that

$$h(X(t))=\lambda_0\phi_0(t)+\lambda_1\phi_1(t)+\sum_{j=0}^{\infty}\sum_{k=1}^{2^j}\lambda_{j,k}\phi_{j,k}(t), \tag{5.3.46}$$

where $\lambda_0=h(X(\mu_N))$, $\lambda_1=h(X(\nu_N))$, $\phi_0(t)=(\nu_N-t)\mathcal{L}_N^{-1}$, $\phi_1(t)=(t-\mu_N)\mathcal{L}_N^{-1}$, with $\mathcal{L}_N=\nu_N-\mu_N$,

$$\lambda_{j,k}=2^{-\frac{j}{2}}\Big\{2h\Big(X\Big(\frac{2k-1}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big)-h\Big(X\Big(\frac{2k}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big)-h\Big(X\Big(\frac{2k-2}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big)\Big\},$$

and

$$\begin{aligned}&\phi_{j,k}(t)\\&=2^{3j/2}\mathcal{L}_N^{-1}\int_{\mu_N}^t\Big(\mathbb{1}_{[\frac{(2k-2)\mathcal{L}_N}{2^{j+1}}+\mu_N,\frac{(2k-1)\mathcal{L}_N}{2^{j+1}}+\mu_N]}(s)-\mathbb{1}_{[\frac{(2k-1)\mathcal{L}_N}{2^{j+1}}+\mu_N,\frac{2k\mathcal{L}_N}{2^{j+1}}+\mu_N]}(s)\Big)\,ds.\end{aligned} \tag{5.3.47}$$

Observe that the series in (5.3.46) is, with probability 1, uniformly convergent in $[\mu_N,\nu_N]$. Now let us show that there is a constant $c>0$ such that

$$\mathbb{E}\left\{\lambda_0^2+\lambda_1^2\right\}\le c, \tag{5.3.48}$$

and

$$\mathbb{E}\left\{\lambda_{j,k}^2\right\}\le c2^{-j(1+2\min_{s\in[\mu_N,\nu_N]}H(s))},\ \text{for every } j,k. \tag{5.3.49}$$

By using the fact that $h \in C^1_{pol}(\mathbb{R})$ as well as the fact that all the moments of the random variable $\|X\|_\infty := \sup_{s\in[0,1]} |X(s)|$ are finite, one gets,

$$\mathbb{E}\,|\lambda_0|^2 = \mathbb{E}\Big(h(X(\mu_N))\Big)^2 \le \mathbb{E}\Big(c(1+\|X\|_\infty)^K\Big)^2 < +\infty. \tag{5.3.50}$$

Similarly, one can show that

$$\mathbb{E}\,|\lambda_1|^2 < +\infty. \tag{5.3.51}$$

Combining (5.3.50) and (5.3.51), one obtains (5.3.48). Let us now show that (5.3.49) holds. Using, the expression of $\lambda_{j,k}$, the inequality $(a+b)^2 \le 2a^2+2b^2$ for all $a,b \in \mathbb{R}$ and the triangle inequality one gets,

$$\begin{aligned}
&\mathbb{E}\,|\lambda_{j,k}|^2\\
&= \mathbb{E}\,\Big|2^{-j/2}\Big\{2h\Big(X\Big(\frac{2k-1}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big) - h\Big(X\Big(\frac{2k}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big)\\
&\quad -h\Big(X\Big(\frac{2k-2}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big)\Big\}\Big|^2\\
&\le 2^{-j}\Big\{2\mathbb{E}\,\Big|h\Big(X\Big(\frac{2k-1}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big) - h\Big(X\Big(\frac{2k}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big)\Big|^2\\
&\quad +2\mathbb{E}\,\Big|h\Big(X\Big(\frac{2k-1}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big) - h\Big(X\Big(\frac{2k-2}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big)\Big|^2\Big\}.
\end{aligned} \tag{5.3.52}$$

Thus, in view of (5.3.52), for proving (5.3.49), it suffices to show that there is a constant $c > 0$ (which does not depend on N, j and k) such that one has,

$$\mathbb{E}\,\Big|h\Big(X\Big(\frac{2k-1}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big) - h\Big(X\Big(\frac{2k}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big)\Big|^2 \le c2^{-2j\min_{s\in[\mu_N,\nu_N]}H(s)} \tag{5.3.53}$$

and

$$\mathbb{E}\,\Big|h\Big(X\Big(\frac{2k-1}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big) - h\Big(X\Big(\frac{2k-2}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big)\Big|^2 \le c2^{-2j\min_{s\in[\mu_N,\nu_N]}H(s)} \tag{5.3.54}$$

We will only prove that (5.3.53) holds since (5.3.54) can be obtained in the same way. By using the fact that $h \in C^1_{pol}(\mathbb{R})$, the Mean Value Theorem and Cauchy-Schwarz inequality, one has

$$\begin{aligned}
&\mathbb{E}\,\Big|h\Big(X\Big(\frac{2k-1}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big) - h\Big(X\Big(\frac{2k}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big)\Big|^2\\
&\le \mathbb{E}\,\Big(\Big(\sup_{s\in[-\|X\|_\infty,\|X\|_\infty]}|h'(s)|\Big)\Big|X\Big(\frac{2k-1}{2^{j+1}}\mathcal{L}_N+\mu_N\Big) - X\Big(\frac{2k}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big|\Big)^2\\
&\le \Big(\mathbb{E}\,\Big(c(1+\|X\|_\infty)^K\Big)^4\Big)^{1/2}\Big(\mathbb{E}\,\Big|X\Big(\frac{2k-1}{2^{j+1}}\mathcal{L}_N+\mu_N\Big) - X\Big(\frac{2k}{2^{j+1}}\mathcal{L}_N+\mu_N\Big)\Big|^4\Big)^{1/2}.
\end{aligned} \tag{5.3.55}$$

On the other hand, standard computations (see e.g. [8]) allow to show that, there is a constant $c > 0$ (non depending on N) such that for all $t, t' \in [\mu_N, \nu_N]$, one has

$$\mathbb{E}\,|X(t) - X(t')|^2 \le c|t-t'|^{2\min_{s\in[\mu_N,\nu_N]}H(s)}.$$

Then the latter inequality, the equivalence of the Gaussian moments and (5.3.55) imply that (5.3.53) holds; recall that (5.3.54) can be obtained in the same way. Next combining (5.3.52) with (5.3.53) and (5.3.54), one gets (5.3.49).

Now observe that (5.3.46) (5.3.40) entail that

$$\sum_{i\in\mathcal{J}(\mu_N,\nu_N)}\Big\{\frac{(\Delta_a\overline{X}_{i,N})^2}{\mathrm{Var}(\Delta_a\overline{X}_{i,N})}-1\Big\}h(X(i/N))$$
$$=\lambda_0\Sigma_N(\phi_0)+\lambda_1\Sigma_N(\phi_1)+\sum_{j=0}^{\infty}\sum_{k=1}^{2^j}\lambda_{j,k}\Sigma_N(\phi_{j,k}). \tag{5.3.56}$$

Also observe that by using the triangle inequality, Cauchy-Schwarz inequality, Lemma 5.3.5, (5.3.48) and the fact that $\|\phi_0\|_\infty=\|\phi_1\|_\infty=1$, one gets that there is a constant $c>0$ such that for all N,

$$\begin{aligned}
&\mathbb{E}\big|\lambda_0\Sigma_N(\phi_0)+\lambda_1\Sigma_N(\phi_1)\big|\\
&\le\mathbb{E}\big|\lambda_0\Sigma_N(\phi_0)\big|+\mathbb{E}\big|\lambda_1\Sigma_N(\phi_1)\big|\\
&\le\Big(\mathbb{E}\big|\lambda_0\big|^2\Big)^{1/2}\Big(\mathbb{E}\big|\Sigma_N(\phi_0)\big|^2\Big)^{1/2}+\Big(\mathbb{E}\big|\lambda_1\big|^2\Big)^{1/2}\Big(\mathbb{E}\big|\Sigma_N(\phi_1)\big|^2\Big)^{1/2}\\
&\le c\big(N(\nu_N-\mu_N)\big)^{1/2}.
\end{aligned}\tag{5.3.57}$$

Thus in view of (5.3.56) and (5.3.57), in order to finish our proof, it remains to show that there exists a constant $c>0$ such that for all $N\ge p+1$, one has,

$$\mathbb{E}\Big|\sum_{j=0}^{\infty}\sum_{k=1}^{2^j}\lambda_{j,k}\Sigma_N(\phi_{j,k})\Big|\le c(N(\nu_N-\mu_N))^{1/2}. \tag{5.3.58}$$

Observe that, in view of the assumption that $\lim_{N\to+\infty}N\mathcal{L}_N=+\infty$, one has for all N big enough, $N\mathcal{L}_N\ge 2$. Let $j_1\ge 1$ be the unique integer such that

$$2^{j_1}\le N\mathcal{L}_N<2^{j_1+1}.$$

It follows from the triangle inequality, Cauchy-Schwarz inequality, (5.3.49), the fact that for all j,k,

$$\mathrm{supp}\,\phi_{j,k}\subseteq\Big[\frac{k-1}{2^j}\mathcal{L}_N+\mu_N,\frac{k}{2^j}\mathcal{L}_N+\mu_N\Big], \tag{5.3.59}$$

and Lemma 5.3.7 in which one takes $j_0 = j$, that

$$\begin{aligned}
&\mathbb{E}\Big|\sum_{j=0}^{j_1}\sum_{k=1}^{2^j}\lambda_{j,k}\Sigma_N(\phi_{j,k})\Big| \\
&\le \sum_{j=0}^{j_1}\sum_{k=1}^{2^j}\Big(\mathbb{E}\,|\lambda_{j,k}|^2\Big)^{1/2}\Big(\mathbb{E}\,|\Sigma_N(\phi_{j,k})|^2\Big)^{1/2} \\
&\le c\sum_{j=0}^{j_1}\sum_{k=1}^{2^j}2^{-j(1+2\min_{s\in[\mu_N,\nu_N]}H(s))/2}\Big(\|\phi_{j,k}\|_\infty^2(k-(k-1))2^{j_1-j}\Big)^{1/2} \\
&= c\sum_{j=0}^{j_1}\sum_{k=1}^{2^j}2^{-j(1+\min_{s\in[\mu_N,\nu_N]}H(s))+j_1/2}\|\phi_{j,k}\|_\infty,
\end{aligned} \tag{5.3.60}$$

where $c>0$ is a constant. Then using (5.3.60), the fact that $\|\phi_{j,k}\|_\infty \le 2^{j/2}$, the inequalities $\min_{s\in[\mu_N,\nu_N]}H(s) \ge \min_{s\in[0,1]}H(s) > 1/2$ and $2^{j_1} \le N\mathcal{L}_N$, one gets that

$$\begin{aligned}
&\mathbb{E}\Big|\sum_{j=0}^{j_1}\sum_{k=1}^{2^j}\lambda_{j,k}\Sigma_N(\phi_{j,k})\Big| \\
&\le c\sum_{j=0}^{j_1}\sum_{k=1}^{2^j}2^{-j(1/2+\min_{s\in[\mu_N,\nu_N]}H(s))+j_1/2} \\
&\le c2^{j_1/2}\sum_{j=0}^{+\infty}2^{j(1/2-\min_{s\in[\mu_N,\nu_N]}H(s))} \\
&\le c_1(N(\nu_N-\mu_N))^{1/2},
\end{aligned} \tag{5.3.61}$$

where $c_1 = c\sum_{j=0}^{+\infty}2^{j(1/2-\min_{s\in[0,1]}H(s))} < +\infty$. Let us now show that there is a constant $c_2>0$ non depending on j_1 and N, such that

$$\mathbb{E}\Big|\sum_{j=j_1+1}^{\infty}\sum_{k=1}^{2^j}\lambda_{j,k}\Sigma_N(\phi_{j,k})\Big| \le c_2(N(\nu_N-\mu_N))^{1/2}. \tag{5.3.62}$$

First observe that, for every fixed (j,k), (5.3.59) and the inequalities $2^{-j} \le 2^{-j_1-1} < (N\mathcal{L}_N)^{-1}$ imply that there is at most one index $i \in \mathcal{J}(\mu_N,\nu_N)$ such that $\phi_{j,k}(i/N) \ne 0$. Therefore one has

$$\mathbb{E}\,|\Sigma_N(\phi_{j,k})|^2 \le \|\phi_{j,k}\|_\infty^2\,\mathbb{E}\Big(\frac{(\Delta_a\overline{X}_{i,N})^2}{\mathrm{Var}(\Delta_a\overline{X}_{i,N})}-1\Big)^2. \tag{5.3.63}$$

Then noticing that $\|\phi_{j,k}\|_\infty \le 2^{j/2}$ and that $\mathbb{E}\left(\frac{(\Delta_a \overline{X}_{i,N})^2}{\mathrm{Var}(\Delta_a \overline{X}_{i,N})} - 1\right)^2 = \mathbb{E}\left(Z^2-1\right)^2$, where Z is standard Gaussian random variable. It follows (5.3.63), that

$$\mathbb{E}\left|\Sigma_N(\phi_{j,k})\right|^2 \le c2^j, \tag{5.3.64}$$

where $c > 0$ is a constant non depending on N, j and k. Now, for every fixed $j \ge j_1 + 1$, let us denote by $\mathcal{K}_{j,N}$ the set of indices $k \in \{1, \dots, 2^j\}$ defined as

$$\mathcal{K}_{j,N} = \Big\{k \in \{1, \dots, 2^j\} : \exists\, i \in \mathcal{J}(\mu_N, \nu_N) \quad \text{such that} \quad \phi_{j,k}(i/N) \ne 0\Big\}.$$

Observe that when $k \notin \mathcal{K}_{j,N}$ then for every $i \in \mathcal{J}(\mu_N, \nu_N)$, one has $\phi_{j,k}(i/N) = 0$ and as a consequence,

$$\Sigma_N(\phi_{j,k}) = 0. \tag{5.3.65}$$

On the other hand, by using (5.3.59) and the fact that for all k and k' satisfying $k \ne k'$, one has

$$\Big(\frac{k-1}{2^j}\mathcal{L}_N + \mu_N, \frac{k}{2^j}\mathcal{L}_N + \mu_N\Big) \cap \Big(\frac{k'-1}{2^j}\mathcal{L}_N + \mu_N, \frac{k'}{2^j}\mathcal{L}_N + \mu_N\Big) = \emptyset,$$

it follows that

$$\mathrm{Card}(\mathcal{K}_{j,N}) \le \mathrm{Card}\mathcal{J}(\mu_N, \nu_N) \le 2N(\nu_N - \mu_N). \tag{5.3.66}$$

Next it follows from, (5.3.65), the triangle inequality, Cauchy-Schwarz inequality, (5.3.49), (5.3.64), (5.3.66), and the inequalities $\min_{s\in[0,1]} H(s) > 1/2$, $2^{-j_1} < 2(N\mathcal{L}_N)^{-1}$, that

$$\begin{aligned}
&\mathbb{E}\Big|\sum_{j=j_1+1}^{+\infty}\sum_{k=1}^{2^j}\lambda_{j,k}\Sigma_N(\phi_{j,k})\Big| \\
&\le \sum_{j=j_1+1}^{+\infty}\sum_{k\in\mathcal{K}_{j,N}}\Big(\mathbb{E}\left|\lambda_{j,k}\right|^2\Big)^{1/2}\Big(\mathbb{E}\left|\Sigma_N(\phi_{j,k})\right|^2\Big)^{1/2} \\
&\le c\sum_{j=j_1+1}^{+\infty}\sum_{k\in\mathcal{K}_{j,N}} 2^{-j(1+2\min_{s\in[\mu_N,\nu_N]} H(s))/2}2^{j/2} \\
&\le 2cN(\nu_N-\mu_N)\sum_{j=j_1}^{+\infty}2^{-j\min_{s\in[0,1]} H(s)} \\
&= 2cN(\nu_N-\mu_N)\frac{2^{-j_1\min_{s\in[0,1]} H(s)}}{1-2^{-\min_{s\in[0,1]} H(s)}} \\
&\le c'\big(N\mathcal{L}_N\big)^{1-\min_{s\in[0,1]} H(s)} \\
&\le c'(N(\nu_N-\mu_N))^{1/2},
\end{aligned} \tag{5.3.67}$$

where the constant $c' = 4c\big(1 - 2^{-\min_{s\in[0,1]} H(s)}\big)^{-1}$ and thus we obtain (5.3.62). Moreover combining (5.3.62) with (5.3.61) one gets (5.3.58).

Finally, Part (ii) of Lemma 5.3.2 results from (5.3.56), (5.3.57) and (5.3.58).
□

5.3.2 Proof of Theorem 5.2.1 when Y is arbitrary

We need the following lemmas.

Lemma 5.3.8 *For all $N \geq p+1$ and $j \in \{0, \dots, N-p-1\}$, one sets*

$$e_{j,N} = \Delta_a \overline{Y}_{j,N} - f'(\overline{X}_{j,N}) \Delta_a \overline{X}_{j,N} \text{ and } e'_{j,N} = h(\overline{Y}_{j,N}) - h(f(\overline{X}_{j,N})).$$

Then for all real $l \geq 1$ there exists a constant $c = c(l) > 0$, such that the inequalities,

$$\mathbb{E}\left\{|e_{j,N}|^l\right\} \leq cN^{-2H(j/N)l} \tag{5.3.68}$$

and

$$\mathbb{E}\left\{|e'_{j,N}|^l\right\} \leq cN^{-H(j/N)l}, \tag{5.3.69}$$

hold for every $N \geq p+1$ and $j \in \{0, \dots, N-p-1\}$.

Proof of Lemma 5.3.8: First we prove that (5.3.68) is satisfied. By using (5.2.9), (5.2.3) and the fact that $\sum_{k=0}^{p} a_k = 0$, we get

$$\Delta_a \overline{Y}_{j,N} = \sum_{k=0}^{p} a_k N \int_{(j+k)/N}^{(j+k+1)/N} \big(f(X(s)) - f(\overline{X}_{j,N})\big)\, \mathrm{d}s. \tag{5.3.70}$$

Moreover, a second order Taylor expansion of f around $\overline{X}_{j,N}$ allows to obtain that,

$$\begin{aligned} f(X(s)) - f(\overline{X}_{j,N}) &= (X(s) - \overline{X}_{j,N}) f'(\overline{X}_{j,N}) \\ &+ (X(s) - \overline{X}_{j,N})^2 \int_0^1 (1-\vartheta) f^{(2)}(\overline{X}_{j,N} + \vartheta(X(s) - \overline{X}_{j,N}))\, \mathrm{d}\vartheta. \end{aligned} \tag{5.3.71}$$

Next, (5.3.70) and (5.3.71) imply that

$$e_{j,N} = N \sum_{k=0}^{p} a_k \int_{(j+k)/N}^{(j+k+1)/N} (X(s) - \overline{X}_{j,N})^2 \int_0^1 f^{(2)}(\overline{X}_{j,N} + \vartheta(X(s) - \overline{X}_{j,N}))\, \mathrm{d}\vartheta\, \mathrm{d}s. \tag{5.3.72}$$

Then, using the fact that $x \mapsto x^l$ is a convex function, Hölder inequality, Fubini Theorem and Cauchy-Schwarz inequality, it follows from (5.3.72), that for all real

$l \geq 1$,

$$\begin{aligned}
&\mathbb{E}\,|e_{j,N}|^l \\
&\leq N^l(p+1)^{l-1}\sum_{k=0}^{p}|a_k|^l N^{-l+1} \\
&\quad\times\mathbb{E}\left(\int_{(j+k)/N}^{(j+k+1)/N}|X(s)-\overline{X}_{j,N}|^{2l}\Big|\int_0^1 f^{(2)}(\overline{X}_{j,N}+\vartheta(X(s)-\overline{X}_{j,N}))\,\mathrm{d}\vartheta\Big|^l\,\mathrm{d}s\right) \\
&= N(p+1)^{l-1}\sum_{k=0}^{p}|a_k|^l \\
&\quad\times\int_{(j+k)/N}^{(j+k+1)/N}\mathbb{E}\left(|X(s)-\overline{X}_{j,N}|^{2l}\Big|\int_0^1 f^{(2)}(\overline{X}_{j,N}+\vartheta(X(s)-\overline{X}_{j,N}))\,\mathrm{d}\vartheta\Big|^l\right)\mathrm{d}s \\
&\leq N(p+1)^{l-1}\sum_{k=0}^{p}|a_k|^l\int_{(j+k)/N}^{(j+k+1)/N}\left(\mathbb{E}\Big|X(s)-\overline{X}_{j,N}\Big|^{4l}\right)^{1/2} \\
&\quad\times\left(\mathbb{E}\Big|\int_0^1 f^{(2)}(\overline{X}_{j,N}+\vartheta(X(s)-\overline{X}_{j,N}))\,\mathrm{d}\vartheta\Big|^{2l}\right)^{1/2}\mathrm{d}s. \qquad (5.3.73)
\end{aligned}$$

Moreover, using (5.2.11), Hölder inequality and Fubini Theorem, one has that,

$$\begin{aligned}
\mathbb{E}\Big|X(s)-\overline{X}_{j,N}\Big|^{4l} &\leq \mathbb{E}\left(N\int_{j/N}^{(j+1)/N}|X(s)-X(t)|\,\mathrm{d}t\right)^{4l} \\
&\leq N\int_{j/N}^{(j+1)/N}\mathbb{E}\,|X(s)-X(t)|^{4l}\,\mathrm{d}t. \qquad (5.3.74)
\end{aligned}$$

Since for any $s,t \in [0,1]$, $X(s)-X(t)$ is a Gaussian random variable, then it follows from the equivalence of Gaussian moments, there is a constant $c_1(l) > 0$ only depending on l, such that

$$\mathbb{E}\,|X(s)-X(t)|^{4l} = c_1(l)\Big(\mathbb{E}\,|X(s)-X(t)|^2\Big)^{2l}. \qquad (5.3.75)$$

Also, observe that with the similarly to (5.3.19) and by using the fact H is a C^2 function, one can show that two constants c_1, c_2 (non depending j, j, k, s and t) such that, for all $s \in [(j+k)/N, (j+k+1)/N]$ and $t \in [j/N, (j+1)/N]$, one has,

$$\mathbb{E}\,|X(s)-X(t)|^2 \leq c_1 N^{-2\min_{s\in[j/N,(j+p+1)/N]}H(s)} \leq c_2 N^{-2H(j/N)}. \qquad (5.3.76)$$

Next putting together (5.3.74), (5.3.75) and (5.3.76), it follows that,

$$\begin{aligned}
&\mathbb{E}\Big|X(s)-\overline{X}_{j,N}\Big|^{4l} \\
&\leq c_1(l)N\int_{j/N}^{(j+1)/N}\Big|\mathbb{E}\,(X(s)-X(t))^2\Big|^{2l}\,\mathrm{d}t \\
&\leq c_2(l)N^{-4lH(j/N)}, \qquad (5.3.77)
\end{aligned}$$

where $c_2(l) = c_1(l)c_2^{2l}$ is a constant only depending on l.

On the other hand, the fact that $f \in C^2_{pol}(\mathbb{R})$, the triangle inequality, the inequality $|X(s)| \leq \|X\|_\infty$, and the inequality $|\overline{X}_{j,N}| \leq \|X\|_\infty$, imply that,

$$\begin{aligned}
&\mathbb{E}\Big|\int_0^1 f^{(2)}(\overline{X}_{j,N} + \vartheta(X(s) - \overline{X}_{j,N}))\,\mathrm{d}\vartheta\Big|^{2l} \\
&\leq \mathbb{E}\Big(c(1 + 3^K\|X\|_\infty^K)\Big)^{2l} \\
&= c_3(l), \qquad (5.3.78)
\end{aligned}$$

where c, K are the constants introduced in (5.1.3) and $c_3(l) > 0$ is a constant only depending on l, K. Then (5.3.68) results from (5.3.73), (5.3.77) and (5.3.78). The inequality (5.3.69) can be proved in a rather similar way. □

Lemma 5.3.9 *For all $N \geq p+1$, one sets*

$$e_N^{(1)} = \frac{1}{N(\nu_N - \mu_N)} \sum_{j \in \mathcal{J}(\mu_N,\nu_N)} \frac{\big(e_{j,N}^2 + 2\Delta_a\overline{X}_{j,N} f'(\overline{X}_{j,N})e_{j,N}\big)}{C(j/N)N^{-2H(j/N)}} h(\overline{Y}_{j,N})$$

and

$$e_N^{(2)} = \frac{1}{N(\nu_N - \mu_N)} \sum_{j \in \mathcal{J}(\mu_N,\nu_N)} \frac{\big(\Delta_a\overline{X}_{j,N} f'(\overline{X}_{j,N})\big)^2}{C(j/N)N^{-2H(j/N)}} e'_{j,N}.$$

Then there is a constant $c > 0$, such that the inequality

$$\mathbb{E}\{|e_N^{(1)}|\} + \mathbb{E}\{|e_N^{(2)}|\} \leq cN^{-1/2},$$

holds for every $N \geq p+1$.

Proof of Lemma 5.3.9: The lemma can be obtained by using Lemma 5.3.8, Lemma 7.2.2, the fact that $h \in C^1_{pol}(\mathbb{R})$ and the fact that $f \in C^2_{pol}(\mathbb{R})$. □

Lemma 5.3.10 *For every function $h \in C^1_{pol}(\mathbb{R})$ one has*

$$V(h; \mu_N, \nu_N) = Q((f')^2 \times h \circ f; \mu_N, \nu_N).$$

Moreover, for each $N \geq p+1$ one has

$$\overline{V}(h; \mu_N, \nu_N) = \overline{Q}((f')^2 \times h \circ f; \mu_N, \nu_N) + e_N^{(1)} + e_N^{(2)}.$$

Proof of Lemma 5.3.10: The lemma can be obtained just by using the definitions of $V(h; \mu_N, \nu_N)$, $Q((f')^2 \times h \circ f; \mu_N, \nu_N)$, $\overline{V}(h; \mu_N, \nu_N)$, $\overline{Q}((f')^2 \times h \circ f; \mu_N, \nu_N)$ and standard computations. □

We are now in position to prove Theorem 5.2.1.

Proof of Theorem 5.2.1: We use Lemmas 5.3.8, 5.3.9 and 5.3.10 as well as Theorem 5.3.1 and we follow the same lines as in the proof of Theorem 3 in [36]. □

5.3.3 Proof of Theorem 5.2.2

Theorem 5.2.2 is a straightforward consequence of Theorem 5.2.1 and the following proposition which allows to control the $L^1(\Omega)$ norm of the error one makes when one replaces the estimator $\overline{V}(h;\mu_{N_n},\nu_{N_n})$ by the estimator $\widehat{V}(h;\mu_{N_n},\nu_{N_n})$.

Proposition 5.3.11 *For any n big enough one sets*

$$v(N_n,m_n)=(N_n^{-1/2}+m_n^{-1/2})(1+m_n^{-1}N_n^{2\max_{s\in[\mu_{N_n},\nu_{N_n}]}H(s)}).$$

Recall that m_n has been defined in (5.2.4). Let us assume that N_n is chosen such that $m_n^{-1}N_n^{2\max_{s\in[\mu_N,\nu_N]}H(s)}$ remains bounded when n goes to infinity. Then, for all $h\in C^1_{pol}(\mathbb{R})$, there exists a constant $c>0$ such that for any n big enough, one has

$$\mathbb{E}\left\{\left|\widehat{V}(h;\mu_{N_n},\nu_{N_n})-\overline{V}(h;\mu_{N_n},\nu_{N_n})\right|\right\}\le cv(N_n,m_n)=\mathcal{O}(N_n^{-1/2}) \tag{5.3.79}$$

The proof of this proposition is given in Appendix.□

5.4 Estimation of the unknown parameter in the linear case

Lemma 5.4.1 *Assume that $(\mu_{N_n})_n$ and $(\nu_{N_n})_n$ are two convergent sequences. Then, when n goes to infinity, the random variable,*

$$T_{N_n}:=(\nu_{N_n}-\mu_{N_n})^{-1}\int_{\mu_{N_n}}^{\nu_{N_n}}Y(s)\,\mathrm{d}s,$$

almost surely converges to an almost surely strictly positive random variable T.

Lemma 5.4.2 *Assume that (5.2.18) holds. Then the sequence*

$$\Big(\frac{1}{N_n(\nu_{N_n}-\mu_{N_n})}\sum_{i\in\mathcal{J}(\mu_{N_n},\nu_{N_n})}\widehat{Y}_{i,N_n,n}-\frac{1}{\nu_{N_n}-\mu_{N_n}}\int_{\mu_{N_n}}^{\nu_{N_n}}Y(s)\,\mathrm{d}s\Big)_n$$

converges to 0 in $L^1(\Omega,\mathscr{F},\mathbb{P})$ with the rate $(N_n(\nu_{N_n}-\mu_{N_n}))^{-1/2}$ when n goes to infinity. Note that in this lemma we do not necessarily suppose that Φ is of the form $\Phi(x,\theta)=\theta x$.

Proof of Lemma 5.4.2: It follows from (5.2.3) that for any n big enough one has

$$\mathbb{E}\Big|(N_n(\nu_{N_n}-\mu_{N_n}))^{-1}\sum_{i\in\mathcal{J}(\mu_{N_n},\nu_{N_n})}\widehat{Y}_{i,N_n,n}-(\nu_{N_n}-\mu_{N_n})^{-1}\int_{\mu_{N_n}}^{\nu_{N_n}}Y(s)\,\mathrm{d}s\Big|$$
$$\le M_n+\mathcal{O}\big((N_n(\nu_{N_n}-\mu_{N_n}))^{-1}\big), \tag{5.4.1}$$

where

$$M_n:=\frac{1}{N_n(\nu_{N_n}-\mu_{N_n})}\sum_{i\in\mathcal{J}(\mu_{N_n},\nu_{N_n})}\mathbb{E}\left|\widehat{Y}_{i,N_n,n}-\overline{Y}_{i,N_n}\right|. \tag{5.4.2}$$

Next Cauchy-Schwarz inequality, Part (ii) of Lemma 7.4.2, (5.2.18), the inequalities $0 \le \nu_{N_n} - \mu_{N_n} \le 1$ and the fact that $H(\cdot)$ is with values in $(1/2, 1)$, imply that for all n big enough,

$$\begin{aligned} M_n &\le \frac{1}{N_n(\nu_{N_n} - \mu_{N_n})} \sum_{i \in \mathcal{J}(\mu_{N_n}, \nu_{N_n})} \sqrt{\mathbb{E}\,(|\widehat{Y}_{i,N_n,n} - \overline{Y}_{i,N_n}|^2)} \\ &= \mathcal{O}(m_n^{-1/2}) = \mathcal{O}\big(N_n^{-\max_{s\in[\mu_N,\nu_N]} H(s)}\big) = \mathcal{O}\big((N_n(\nu_{N_n} - \mu_{N_n}))^{-1/2}\big). \end{aligned} \tag{5.4.3}$$

Finally putting together (5.4.1), (5.4.2) and (5.4.3) one obtains the lemma. □

We are now in position to prove Theorem 5.2.4.

Proof of Theorem 5.2.4: Let us set

$$T_{N_n} = (\nu_{N_n} - \mu_{N_n})^{-1} \int_{\mu_{N_n}}^{\nu_{N_n}} Y(s)\, \mathrm{d}s, \tag{5.4.4}$$

$$\overline{T}_n = (N_n(\nu_{N_n} - \mu_{N_n}))^{-1} \sum_{i \in \mathcal{J}(\mu_{N_n}, \nu_{N_n})} \widehat{Y}_{i,N_n,n}, \tag{5.4.5}$$

$$u_n = \widehat{V}(1; \mu_{N_n}, \nu_{N_n}) - V(1; \mu_{N_n}, \nu_{N_n}), \tag{5.4.6}$$

and

$$v_n = \overline{T}_n - T_{N_n}. \tag{5.4.7}$$

Moreover, observe that (5.1.2), (5.2.7), the fact that $\Phi(x, \theta) = \theta x$, and (5.4.4) imply that

$$\widehat{V}(1; \mu_{N_n}, \nu_{N_n}) = 4\theta^2 (\nu_{N_n} - \mu_{N_n})^{-1} \int_{\mu_{N_n}}^{\nu_{N_n}} Y(s)\, \mathrm{d}s = 4\theta^2 T_{N_n}. \tag{5.4.8}$$

Then it follows from (5.2.27), (5.4.4), (5.4.5), (5.4.6), (5.4.7) and (5.4.8) that

$$\widehat{\theta}_n^2(\mu_{N_n}, \nu_{N_n}) - \theta^2 = \frac{u_n - 4\theta^2 v_n}{4T_{N_n} + 4v_n}. \tag{5.4.9}$$

Therefore, one has for any real $\lambda > 1$ and any integer n big enough

$$\begin{aligned} &\mathbb{P}\Big((N_n(\nu_{N_n} - \mu_{N_n}))^{1/2} |\widehat{\theta}_n^2(\mu_{N_n}, \nu_{N_n})) - \theta| > \lambda\Big) \\ &\le \mathbb{P}\left(\left\{\frac{(N_n(\nu_{N_n} - \mu_{N_n}))^{1/2}|u_n - 4\theta^2 v_n|}{4T_{N_n} + 4v_n} > \lambda\right\} \cap \Big\{T_{N_n} \ge \lambda^{-1/2}\Big\} \cap \Big\{|v_n| \le 4^{-1}\lambda^{-1/2}\Big\}\right) \\ &\qquad + \mathbb{P}\Big(T_{N_n} < \lambda^{-1/2}\Big) + \mathbb{P}\Big(|v_n| > 4^{-1}\lambda^{-1/2}\Big) \\ &\le \mathbb{P}\Big((N_n(\nu_{N_n} - \mu_{N_n}))^{1/2}|u_n - 4\theta^2 v_n| > 3\lambda^{3/2}\Big) \\ &\quad + \mathbb{P}\Big(T_{N_n} < \lambda^{-1/2}\Big) + \mathbb{P}\Big(|v_n| > 4^{-1}\lambda^{-1/2}\Big). \end{aligned}$$

Next the latter inequality, (5.4.4), (5.4.5), (5.4.6), (5.4.7), Theorem 5.2.2, Lemma 5.4.2 and Markov inequality, imply that there is a constant $c > 0$ such that for all real $\lambda > 1$, one has

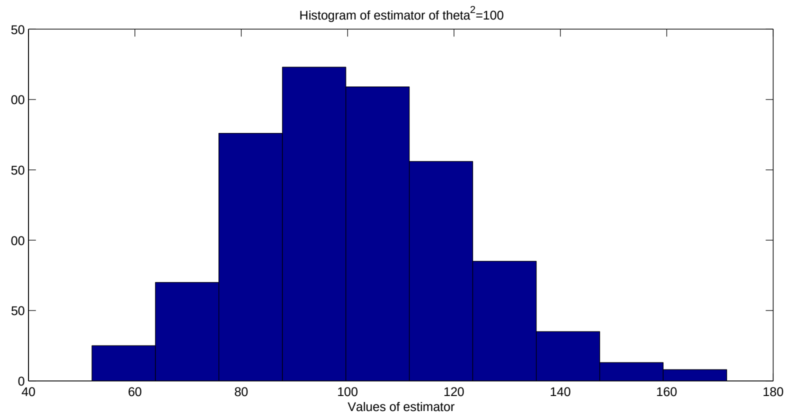
$$\limsup_{n\to+\infty} \mathbb{P}\Big((N_n(\nu_{N_n} - \mu_{N_n}))^{1/2} \Big| \widehat{\theta}_n^2(\mu_{N_n}, \nu_{N_n})) - \theta^2 \Big| > \lambda \Big) \\ \leq 3^{-1} c \lambda^{-3/2} + \mathbb{P}\Big(T < \lambda^{-1/2} \Big), \tag{5.4.10}$$

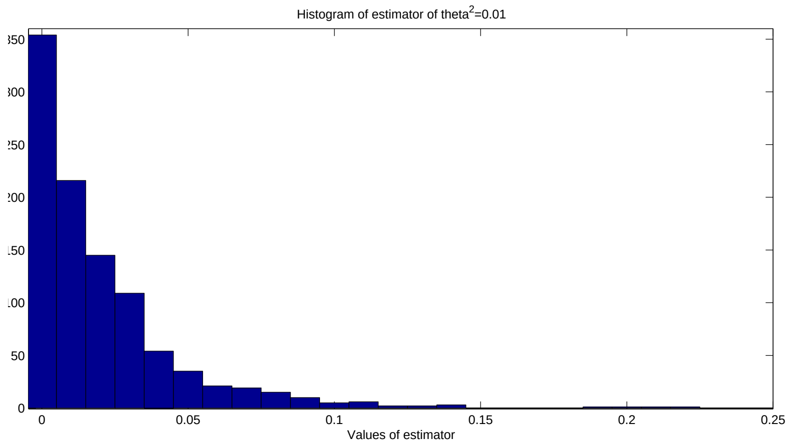
where $T := \lim_{n\to+\infty} T_{N_n}$. Thus, it follows from (5.2.28), (5.4.10), (5.4.4) and Lemma 5.4.1, that Theorem 5.2.4 holds. □

5.5 Histograms of the estimated values

We have tested our estimator of θ^2 on simulated data and our numerical results are summarized in the following two histograms.

To obtain the first histogram, we have proceeded as follows: we assume that $\theta = 10$ and $H(s) = (s - 0.5)^2 + 0.6$ for all $s \in [0, 1]$, then we simulate 1000 discretized trajectories of the process $\{Z(t)\}_{t\in[0,1]}$, finally we apply our estimator to each trajectory which gives us 1000 estimations of θ^2. The second histogram has been obtained by using a similar method; $H(\cdot)$ is defined in the same way, yet in this case we assume that $\theta = 0.1$.


Histogram of estimator of theta²=100
50
00
50
00
50
0
40
60
80
100
120
140
160
180
Values of estimator


Histogram of estimator of theta²=0.01
0
0.05
0.1
0.15
0.2
0.25
Values of estimator

Chapter 6

Multifractional stochastic volatility models: estimation of hidden pointwise Hölder exponents

Contents

6.1 Introduction

In this chapter we consider the class of multifractional stochastic volatility models. We are interested in constructing an estimator of the pointwise Hölder exponent of the hidden volatility starting from an observation of the discretized trajectory of the model. Notice that, the local Hölder regularity of a continuous nowhere differentiable process at each point can be measured by its pointwise Hölder exponent. Let us recall the definition of this exponent; denote by $X = \{X(t)\}_{t\in[0,1]}$ a stochastic process with continuous and nowhere differentiable trajectories; the pointwise Hölder exponent of X, is the stochastic process denoted by $\{\rho_X(t)\}_{t\in[0,1]}$ and defined as

$$\rho_X(t) := \sup\Big\{\rho : \limsup_{h\to 0} \frac{|X(t+h)-X(t)|}{|h|^\rho} = 0\Big\}.$$

It measures the local smoothness of X: the larger is $\rho_X(t)$, the more regular is the process X in a neighborhood of the point t. Since several years, a number of authors have been interested in the statistical problem of the estimation of $\rho_X(t)$ starting from the observation of a discretized trajectory of the process X (see for example [10, 17, 13, 22, 19, 21]). However, it does not always seem to be realistic to assume that such an observation is available, but only a corrupted version of it; therefore a natural question one can address is that whether it is still possible to estimate $\rho_X(t)$. To our knowledge, only a few number of articles in the literature deal with this problem and it has been studied only in a setting which basically remains to be that of Gaussian stationary increments processes; typically when the hidden process X is a fractional Brownian motion (fBm for short) [37, 52]. In such a setting the hidden pointwise Hölder exponent of X has a rather simple structure since it cannot evolve with time; recall that for a fBm $\{B_\alpha(t)\}_{t\in[0,1]}$ with Hurst parameter $\alpha \in (0,1)$, one has with probability 1, for each $t \in [0,1]$, $\rho_{B_\alpha}(t) = \alpha$. The goal of this chapter, is to study the statistical problem of the estimation of a hidden pointwise Hölder exponent in a new setting where this exponent has a rather complex structure since it is allowed to evolve over time. More precisely we assume that the corresponding hidden process X is a multifractional Brownian motion (mBm for short). As we have already pointed out in the previous chapter, mBm $\{X(t)\}_{t\in[0,1]}$ with functional parameter $H(\cdot)$ is an extension of fBm. It is more flexible than fBm since its local Hölder regularity can change from one time to another. More precisely, under the assumption that $H(\cdot)$ is a β-Hölder function on $[0,1]$ with values in $(0,\beta)$, it has been shown in [51, 18, 6] that almost surely for all $t \in [0,1]$,

$$\rho_X(t) = H(t).$$

Let us now describe our statistical setting. Similarly to the previous chapter, we consider the multifractional stochastic volatility model $\{Z(t)\}_{t\in[0,1]}$, defined for each $t \in [0,1]$, as:

$$Z(t) = z_0 + \int_0^t \Phi(X(s))\,\mathrm{d}W(s), \tag{6.1.1}$$

where:

- in practice, $\{Z(t)\}_{t\in[0,1]}$ denotes the logarithm of the price of the underlying asset ($z_0 \in \mathbb{R}$ is deterministic and known);
- $\{W(s)\}_{s\in[0,1]}$ is a standard Brownian motion;
- $\Phi : \mathbb{R} \to \mathbb{R}$ is an unknown deterministic function. For all $x \in \mathbb{R}$, we set

$$f(x) = (\Phi(x))^2. \tag{6.1.2}$$

  We assume that $f \in C^2_{pol}(\mathbb{R})$ (see (5.1.3)). This means that f is two times continuously differentiable on the real line and there exist two constants $c, K > 0$ (which a priori depend on f) such that for all $x \in \mathbb{R}$,

$$\sum_{k=0}^{2} |f^{(k)}(x)| \le c\left(1 + |x|^K\right).$$

  Moreover, We assume that f', the first derivative of f, vanishes only on a Lebesgue negligible set, that is $\mathcal{L}\big(\{x \in \mathbb{R} : f'(x) = 0\}\big) = 0$, $\mathcal{L}$ being the Lebesgue measure on $\mathbb{R}$.

- $\{X(s)\}_{s\in[0,1]}$ denotes a multifractional Brownian motion (mBm) of functional parameter $H(\cdot)$. We assume that $\{X(s)\}_{s\in[0,1]}$ is independent on the Brownian motion $\{W(s)\}_{s\in[0,1]}$. We assume furthermore that $H(\cdot)$ is two times continuously differentiable on $[0,1]$ i.e. $H(\cdot)\in C^2([0,1])$. Let us also point out that, unless we mention the contrary, the function $H(\cdot)$ is allowed to take its values on the whole open interval $(0,1)$.

It is worth noticing that, *there is a considerable loss of information when one observes* $\{Z(t)\}_{t\in[0,1]}$ *instead of* $\{X(t)\}_{t\in[0,1]}$*: indeed, when* $H(\cdot)$ *is with values in* $(1/2,1)$*, contrarily to* X*, the pointwise Hölder exponent of* Z *remains constant since it is, at each time, almost surely equal to* $1/2$ *(see Theorem 4.2.2)*. Yet we will show that, in spite of this considerable loss of information, it is still possible, starting from the observation of a discretized trajectory of Z, to estimate $H(t_0)$, the pointwise Hölder exponent of X, at each time $t_0\in(0,1)$ [1].

Before ending this introduction, let us recall that stochastic volatility models of the type (6.1.1), have been already considered by Gloter and Hoffmann [36, 35] in the case where X is fBm, and also by Rosenbaum [52] in a more general case where X can nicely be expressed in terms of a Gaussian stationary increment process. In [36, 35], Gloter and Hoffmann were not interested in the problem of the estimation of pointwise Hölder exponents, which they suppose to be known. In [52], Rosenbaum was interested in the same problem as us; he constructed wavelet estimators of hidden pointwise Hölder exponents which converge in probability at the optimal minimax rate. Here, we have preferred to adopt a rather different estimation strategy from him, in order to be able to construct, under some conditions, a strongly consistent and asymptotically normal estimator.

6.2 Statement of the main results

We suppose to have observed the sample,

$$\Big\{Z(0),\, Z\Big(\frac{1}{2n}\Big),\, Z\Big(\frac{2}{2n}\Big),\ldots,\, Z\Big(\frac{2n-1}{2n}\Big),\, Z(1)\Big\},$$

where n denotes an integer large enough and where Z is defined by (6.1.1). Our goal is to propose a method allowing to estimate, starting from the latter observation, the pointwise Hölder exponent $H(t_0)$ of the hidden mBm $\{X(s)\}_{s\in[0,1]}$ at an arbitrary fixed point $t_0\in(0,1)$. To this end, we use a localized generalized quadratic variations method which is reminiscent to that of the previous chapter. Before stating our main results, we need to briefly fix some notations which will extensively be used in all the sequel.

- As usual, $a=(a_0,\ldots,a_p)$ is an arbitrary finite fixed sequence of $\mathbb{R}^{p+1}$ with

[1] Observe that, we restrict to the open interval $(0,1)$, in order to avoid the border effect.

$M(a)$ vanishing moments, that is,

$$\sum_{k=0}^{p} k^l a_k = 0, \text{for all } l = 0, \dots, M(a) - 1 \text{ and } \sum_{k=0}^{p} k^{M(a)} a_k \neq 0. \tag{6.2.1}$$

As a consequence, one has for any $l \in \{0, \dots, 2M(a) - 1\}$,

$$\sum_{k=0}^{p} \sum_{k'=0}^{p} a_k a_{k'} (k - k')^l = 0. \tag{6.2.2}$$

Observe that one always has $p \geq M(a)$. Throughout this chapter, we assume that $M(a) \geq 2$.

- For all integer $N \geq p + 1$, we denote by $\nu_N(t_0)$ the set of indices defined as,

$$\nu_N(t_0) = \Big\{ i \in \{0, \dots, N - p - 1\} : \ |i/N - t_0| \leq v(N) \Big\}, \tag{6.2.3}$$

  where $v(\cdot)$ is an arbitrary fixed function defined on $[3, +\infty)$, with values in $(0, 1]$ which satisfies for each integer $N \geq 3$, $v(N) \geq N^{-1}$ and $\lim_{N \in \mathbb{N}, N \to +\infty} N v(N) = +\infty$.

- For all integer $N \geq p + 1$ we set

$$N_{t_0} = card(\nu_N(t_0)); \tag{6.2.4}$$

  observe that

$$N_{t_0} \in \big\{ [2Nv(N)], [2Nv(N)] + 1 \big\}, \tag{6.2.5}$$

  where $[\cdot]$ is the integer part function.

- We denote by $\{Y(s)\}_{s \in [0,1]}$ the process defined for each $s \in [0, 1]$, as,

$$Y(s) = f(X(s)); \tag{6.2.6}$$

  recall that $\{X(s)\}_{s \in [0,1]}$ is the hidden mBm and the deterministic function f has been introduced in (6.1.2).

- For all integer $N \geq p+1$ and $i \in \{0, \dots, N-p-1\}$ the generalized increments $\Delta_a \overline{Y}_{i,N}$ and $\Delta_a \overline{X}_{i,N}$ are defined as,

$$\Delta_a \overline{Y}_{i,N} := \sum_{k=0}^{p} a_k \overline{Y}_{i+k,N} \tag{6.2.7}$$

  and

$$\Delta_a \overline{X}_{i,N} := \sum_{k=0}^{p} a_k \overline{X}_{i+k,N}. \tag{6.2.8}$$

Recall that $\overline{Y}_{i+k,N}$ and $\overline{X}_{i+k,N}$ are respectively the average values of the processes Y and X, on the interval $[(i+k)/N, (i+k+1)/N]$, that is,

$$\overline{Y}_{i+k,N} := N \int_{(i+k)/N}^{(i+k+1)/N} Y(s)\,\mathrm{d}s \tag{6.2.9}$$

and

$$\overline{X}_{i+k,N} := N \int_{(i+k)/N}^{(i+k+1)/N} X(s)\,\mathrm{d}s. \tag{6.2.10}$$

- The $\overline{Y}_{i,N}$'s are hidden; yet, in view of Lemma 7.4.2 (see also the previous chapter), when N is of the form,

$$N_n = [n^{\beta}], \tag{6.2.11}$$

  $\beta \in (0,1)$ being a fixed parameter, $\overline{Y}_{i,N_n}$ can be approximated by $\widehat{Y}_{i,n}$ defined as,

$$\widehat{Y}_{i,n} := N_n \sum_{k=0}^{j_{i+1}-j_i-1} \Big(Z\big((j_i+k+1)/n\big) - Z\big((j_i+k)/n\big)\Big)^2,$$

  where $j_i := [in/N_n]$.

- At last, for all n big enough and all $i \in \{0, \dots, N_n - p - 1\}$ the generalized increment $\Delta_a \widehat{Y}_{i,n}$ is defined as,

$$\Delta_a \widehat{Y}_{i,n} = \sum_{k=0}^{p} a_k \widehat{Y}_{i+k,n}.$$

The main results of this chapter are the following three theorems.

Theorem 6.2.1 *Assume that v satisfies the following 3 conditions:*

(i) $\sum_{N\in\mathbb{N},\, N\geq 3} \big(v(N)N\big)^{-2} < +\infty$*;*

(ii) for all integer N big enough, one has,

$$v(N) = o\Big(\big(\log N\big)^{-1}\Big),$$

which means that $\log(N)v(N) \xrightarrow[N\to+\infty]{} 0$*;*

(iii) there exists $\beta \in (0, 1/6]$ and $c > 0$ such that $\lim_{n\to+\infty} \frac{v(N_{2n})}{v(N_n)} = c$, recall that N_n has been defined in (6.2.11).

For all integer n large enough, one sets,

$$V_n(t_0) = \sum_{i\in\nu_{N_n}(t_0)} \Big(\Delta_a \widehat{Y}_{i,n}\Big)^2, \tag{6.2.12}$$

and

$$\widehat{H}_{n,t_0,\beta} = \frac{1}{2}\Big(1+\beta^{-1}\log_2(c)+\beta^{-1}\log_2\Big(\frac{V_n(t_0)}{V_{2n}(t_0)}\Big)\Big), \tag{6.2.13}$$

Then, one has,

$$\widehat{H}_{n,t_0,\beta} \xrightarrow[n\to+\infty]{a.s.} H(t_0).$$

Theorem 6.2.2 *Assume that v satisfies condition (ii), as well as condition (iii) for some $\beta \in (0,1/3)$. Also assume that $\widehat{H}_{n,t_0,\beta}$ is as in (6.2.13). Then, one has,*

$$\widehat{H}_{n,t_0,\beta} \xrightarrow[n\to+\infty]{\mathbb{P}} H(t_0),$$

where the sumbol $\xrightarrow{\mathbb{P}}$ means that the convergence holds in probability.

Theorem 6.2.3 *Assume that $H(t_0) \in (1/2,1)$; also assume that v satisfies condition (iii) for some $\beta \in (0,1/4)$, and the following condition:*

(iv) one has

$$\log N = o\left(\big(Nv(N)\big)^{1/2}\right),$$

morever there exists $\eta \geq 0$, such that,

$$v(N) = \mathcal{O}\left(N^{-1/2}\big(\log(N)\big)^{\eta}\right);$$

the notation $v(N) = \mathcal{O}\left(N^{-1/2}\big(\log(N)\big)^{\eta}\right)$ means that there is a constant $c > 0$, such that for all integer $N \geq 3$, one has $v(N) \leq cN^{-1/2}\big(\log(N)\big)^{\eta}$.

Let $\widehat{H}_{n,t_0,\beta}$ be as in (6.2.13). Then,

$$\big(v(N_n)N_n\big)^{1/2}\left(\widehat{H}_{n,t_0,\beta} - H(t_0)\right),$$

converges in distribution to a centered Gaussian variable, when n goes to infinity.

Before ending this section, observe that for all fixed parameter $\alpha > 1$, the function $x \mapsto (\log(x))^{-\alpha}$ is a natural example of a function v satisfying conditions (i), (ii) and (iii) (for any $\beta \in (0,1)$). Another natural example is given by the function $x \mapsto x^{-\gamma}$, where $\gamma \in (0,1/2)$ is a fixed parameter; also notice that the latter function satisfies (ii), (iii) (for any $\beta \in (0,1)$) and (iv), when $\gamma \in [1/2,1)$. Last but not least, let us point out that, for all fixed parameter $\eta > 1/2$, the function $x \mapsto x^{-1/2}\big(\log(x)\big)^{\eta}$, satisfies (i), (ii), (iii) (for any $\beta \in (0,1)$) and (iv).

6.3 Estimation when the $\overline{Y}_{i,N}$'s are known

The main goal of this section is to construct, starting from the $\overline{Y}_{i,N}$'s (see (6.2.9)), a consistent estimator of $H(t_0)$, the pointwise Hölder exponent of the mBm $\{X(s)\}_{s\in[0,1]}$ at an arbitrary point $t_0 \in (0,1)$. The main results of this section are the following two theorems.

Theorem 6.3.1 *Assume that v satisfies conditions (i) and (ii) in Theorem 6.2.1, and also that there is $c>0$ such that $\lim_{N\in\mathbb{N},\, N\to+\infty}\frac{v(2N)}{v(N)}=c$. Set*

$$V_{N,1}(t_0)=\sum_{i\in\nu_N(t_0)}\Big(\Delta_a\overline{Y}_{i,N}\Big)^2, \tag{6.3.1}$$

and

$$\widehat{H}^{(1)}_{N,t_0}=\frac{1}{2}\Big(1+\log_2(c)+\log_2\Big(\frac{V_{N,1}(t_0)}{V_{2N,1}(t_0)}\Big)\Big). \tag{6.3.2}$$

Then

$$\widehat{H}^{(1)}_{N,t_0}\xrightarrow[N\to+\infty]{a.s.}H(t_0).$$

Theorem 6.3.2 *Assume that v satisfies condition (ii) in Theorem 6.2.1, and also that there is $c>0$ such that $\lim_{N\in\mathbb{N},\, N\to+\infty}\frac{v(2N)}{v(N)}=c$. Let $\widehat{H}^{(1)}_{N,t_0}$ be as in (6.3.2). Then*

$$\widehat{H}^{(1)}_{N,t_0}\xrightarrow[N\to+\infty]{\mathbb{P}}H(t_0).$$

The proof of Theorem 6.3.1 mainly relies on the following proposition.

Proposition 6.3.3 *Let $\widetilde{C}_a:(0,1)\to(0,+\infty)$ be the function defined for all $\alpha\in(0,1)$ as,*

$$\widetilde{C}_a(\alpha)=2\int_{\mathbb{R}}\frac{(1-\cos\eta)\big|\sum_{k=0}^p a_k e^{ik\eta}\big|^2}{|\eta|^{2\alpha+3}}\,d\eta. \tag{6.3.3}$$

Then, assuming that v satisfies conditions (i) and (ii) in Theorem 6.2.1, one has,

$$\frac{\sum_{i\in\nu_N(t_0)}\Big(\Delta_a\overline{X}_{i,N}\Big)^2}{2\widetilde{C}_a(H(t_0))v(N)N^{1-2H(t_0)}}\xrightarrow[N\to+\infty]{a.s.}1.$$

Observe that, in view of (5.2.15) and (6.3.3), one has, for all $s\in[0,1]$,

$$\widetilde{C}_a(H(s))=C(s).$$

The proof of Theorem 6.3.2 mainly relies on the following proposition.

Proposition 6.3.4 *Let $\widetilde{C}_a$ be as in (6.3.3). Then, assuming that v satisfies condition (ii) in Theorem 6.2.1, one has,*

$$\frac{\sum_{i\in\nu_N(t_0)}\Big(\Delta_a\overline{X}_{i,N}\Big)^2}{2\widetilde{C}_a(H(t_0))v(N)N^{1-2H(t_0)}}\xrightarrow[N\to+\infty]{\mathbb{P}}1.$$

In order to prove Propositions 6.3.3 and 6.3.4, we need several preliminary results.

Lemma 6.3.5 *There is a constant $c>0$ such that for all integer $N\geq p+1$, one has*

$$\mathbb{E}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2-\sum_{i\in\nu_N(t_0)}Var(\Delta_a\overline{X}_{i,N})\Big)^4\leq c\Big(Var\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2\Big)\Big)^2. \tag{6.3.4}$$

Proof of Lemma 6.3.5. Let $M = (M_{ij})_{N_{t_0}\times N_{t_0}}$ denote the covariance matrix of the centered Gaussian vector $\{\Delta_a\overline{X}_{i,N}\}_{i\in\nu_N(t_0)}$, i.e.

$$M_{ij} = \mathbb{E}\left(\Delta_a\overline{X}_{i,N}\Delta_a\overline{X}_{j,N}\right). \tag{6.3.5}$$

The sequence of the eigenvalues of M is denoted by $\{\widetilde{\lambda}_{i,N}\}_{i\in\nu_N(t_0)}$; observe that they are strictly positive real numbers. Let P be the orthogonal matrix such that

$$D := \mathrm{Diag}(\widetilde{\lambda}_{i,N}) = P'MP. \tag{6.3.6}$$

For all $i\in\nu_N(t_0)$, we set,

$$\varepsilon_{i,N} = (\widetilde{\lambda}_{i,N})^{-1/2}\sum_{k\in\nu_N(t_0)} P_{ki}\Delta_a\overline{X}_{k,N}. \tag{6.3.7}$$

Then $\{\varepsilon_{i,N}\}_{i\in\nu_N(t_0)}$ is a sequence of independent standard Gaussian variables; indeed on has,

$$\mathbb{E}\left(\varepsilon_{i,N}\right) = (\widetilde{\lambda}_{i,N})^{-1/2}\sum_{k\in\nu_N(t_0)} P_{ki}\mathbb{E}\left(\Delta_a\overline{X}_{k,N}\right) = 0, \tag{6.3.8}$$

$$\begin{aligned}
\mathrm{Var}(\varepsilon_{i,N}) &= (\widetilde{\lambda}_{i,N})^{-1}\sum_{k,k'\in\nu_N(t_0)} P_{ki}P_{k'i}\mathbb{E}\left(\Delta_a\overline{X}_{k,N}\Delta_a\overline{X}_{k',N}\right) = 0\\
&= (\widetilde{\lambda}_{i,N})^{-1}\sum_{k,k'\in\nu_N(t_0)} P_{ki}P_{k'i}M_{kk'}\\
&= (\widetilde{\lambda}_{i,N})^{-1}D_{ii}\\
&= 1,
\end{aligned} \tag{6.3.9}$$

and for any $i\neq j$,

$$\begin{aligned}
\mathbb{E}\left(\varepsilon_{i,N}\varepsilon_{j,N}\right) &= (\widetilde{\lambda}_{i,N}\widetilde{\lambda}_{j,N})^{-1/2}\sum_{k,k'\in\nu_N(t_0)} P_{ki}P_{k'j}M_{kk'}\\
&= (\widetilde{\lambda}_{i,N}\widetilde{\lambda}_{j,N})^{-1/2}D_{ij}\\
&= 0.
\end{aligned} \tag{6.3.10}$$

It follows from (6.3.6) and (6.3.7) that,

$$\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2 - \sum_{i\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{X}_{i,N}) = \sum_{i\in\nu_N(t_0)}\widetilde{\lambda}_{i,N}(\varepsilon_{i,N}^2-1). \tag{6.3.11}$$

Let us denote by ε an arbitrary standard Gaussian random variable. Using (6.3.11) as well as the fact that $\{\varepsilon_{i,N}\}_{i\in\nu_N(t_0)}$ is a sequence of independent standard Gaussian variables, one obtains that

$$\begin{aligned}
&\mathbb{E}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2-\sum_{i\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{X}_{i,N})\Big)^4\\
&=\mathbb{E}\Big(\sum_{i\in\nu_N(t_0)}\lambda_{i,N}(\varepsilon_{i,N}^2-1)\Big)^4\\
&=\sum_{i_1,i_2,i_3,i_4\in\nu_N(t_0)}\lambda_{i_1,N}\lambda_{i_2,N}\lambda_{i_3,N}\lambda_{i_4,N}\mathbb{E}\left((\varepsilon_{i_1,N}^2-1)(\varepsilon_{i_2,N}^2-1)(\varepsilon_{i_3,N}^2-1)(\varepsilon_{i_4,N}^2-1)\right)\\
&=\sum_{i\in\nu_N(t_0)}\lambda_{i,N}^4\mathbb{E}\,(\varepsilon_{i,N}^2-1)^4+\sum_{i,j\in\nu_N(t_0),i\neq j}\lambda_{i,N}^2\lambda_{j,N}^2\mathbb{E}\left((\varepsilon_{i,N}^2-1)^2(\varepsilon_{j,N}^2-1)^2\right)\\
&=\sum_{i\in\nu_N(t_0)}\lambda_{i,N}^4\mathbb{E}\,(\varepsilon^2-1)^4+\sum_{i,j\in\nu_N(t_0),i\neq j}\lambda_{i,N}^2\lambda_{j,N}^2\Big(\mathbb{E}\,(\varepsilon_{i,N}^2-1)^2\mathbb{E}\,(\varepsilon_{j,N}^2-1)^2\Big)\\
&=\frac{\mathbb{E}\,(\varepsilon^2-1)^4}{\Big(\mathbb{E}\,(\varepsilon^2-1)^2\Big)^2}\sum_{i\in\nu_N(t_0)}\lambda_{i,N}^4\Big(\mathbb{E}\,(\varepsilon_{i,N}^2-1)^2\Big)^2\\
&\quad+\sum_{i,j\in\nu_N(t_0),i\neq j}\lambda_{i,N}^2\lambda_{j,N}^2\Big(\mathbb{E}\,(\varepsilon_{i,N}^2-1)^2\mathbb{E}\,(\varepsilon_{j,N}^2-1)^2\Big)\\
&\le c\Big(\sum_{i\in\nu_N(t_0)}\lambda_{i,N}^2\mathbb{E}\,(\varepsilon_{i,N}^2-1)^2\Big)^2\\
&=c\Big(\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2\Big)\Big)^2,
\end{aligned}$$

where

$$c=\frac{\mathbb{E}\,(\varepsilon^2-1)^4}{\Big(\mathbb{E}\,(\varepsilon^2-1)^2\Big)^2}+1.$$

Thus we obtain (6.3.4). Observe that to derive the previous inequality, we have used the fact that for any $i_1,i_2,i_3,i_4\in\nu_N(t_0)$, if $i_1=i_2=i_3=i_4$,

$$\mathbb{E}\left((\varepsilon_{i_1,N}^2-1)(\varepsilon_{i_2,N}^2-1)(\varepsilon_{i_3,N}^2-1)(\varepsilon_{i_4,N}^2-1)\right)=\mathbb{E}\,(\varepsilon_{i_1,N}^2-1)^4;$$

if there are $k,k'\in\{1,2,3,4\}$, $k\neq k'$ such that two indices of $i_1,i_2,i_3,i_4\in\nu_N(t_0)$ equal to i_k and the two others equal to $i_{k'}$,

$$\mathbb{E}\left((\varepsilon_{i_1,N}^2-1)(\varepsilon_{i_2,N}^2-1)(\varepsilon_{i_3,N}^2-1)(\varepsilon_{i_4,N}^2-1)\right)=\mathbb{E}\left((\varepsilon_{i_k,N}^2-1)^2(\varepsilon_{i_k',N}^2-1)^2\right);$$

if else,

$$\mathbb{E}\left((\varepsilon_{i_1,N}^2-1)(\varepsilon_{i_2,N}^2-1)(\varepsilon_{i_3,N}^2-1)(\varepsilon_{i_4,N}^2-1)\right)=0.$$

□

The following lemma is a more or less classical result whose proof is given in Appendix (see in Appendix the proof of Lemma 5.3.4).

Lemma 6.3.6 *Let (Z_1, Z_2) be an arbitrary centered 2-D Gaussian vector such that $Var(Z_1) = Var(Z_2) = \tau$. Then one has,*

$$\mathbb{E}\Big((Z_1 Z_2)^2 - \tau^2\Big) = 2(Cov(Z_1, Z_2))^2.$$

A careful inspection of the proof of Proposition 5.3.6 (see Appendix), shows that the following lemma holds even in the case where $M(a) = 2$.

Lemma 6.3.7 *Assume that v satisfies condition (ii) in Theorem 6.2.1 and $M(a) \geq 2$, then there is a constant $c > 0$ such that for all $N \geq p+1$ and for all $i, j \in \nu_N(t_0)$, one has,*

$$\big|Cov(\Delta_a \overline{X}_{i,N}, \Delta_a \overline{X}_{j,N})\big| \leq c\Big(\frac{N^{-H(i/N)-H(j/N)}}{1+|i-j|}\Big).$$

Lemma 6.3.8 *Assume that v satisfies conditions (i) and (ii) in Theorem 6.2.1. Then, we have*

$$\frac{\sum_{i\in\nu_N(t_0)}(\Delta_a \overline{X}_{i,N})^2}{\sum_{i\in\nu_N(t_0)} Var(\Delta_a \overline{X}_{i,N})} \xrightarrow[N\to+\infty]{a.s.} 1.$$

Proof of Lemma 6.3.8. First observe that by using the fact that

$$\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a \overline{X}_{i,N})^2\Big) = \sum_{i,j\in\nu_N(t_0)} \mathrm{Cov}\Big((\Delta_a \overline{X}_{i,N})^2, (\Delta_a \overline{X}_{j,N})^2\Big)$$

and by using Lemma 6.3.6 (in which we take $Z_1 = \frac{\Delta_a \overline{X}_{i,N}}{\sqrt{\Delta_a \overline{X}_{i,N}}}$, $Z_2 = \frac{\Delta_a \overline{X}_{j,N}}{\sqrt{\Delta_a \overline{X}_{j,N}}}$ and $\tau = 1$), we get

$$\begin{aligned}
&\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a \overline{X}_{i,N})^2\Big)\\
&= \sum_{i,j\in\nu_N(t_0)}\Big(\mathbb{E}\Big(\Delta_a \overline{X}_{i,N}\Delta_a \overline{X}_{j,N}\Big)^2 - \mathbb{E}\,(\Delta_a \overline{X}_{i,N})^2\mathbb{E}\,(\Delta_a \overline{X}_{j,N})^2\Big)\\
&= \sum_{i,j\in\nu_N(t_0)} \mathrm{Var}(\Delta_a \overline{X}_{i,N})\mathrm{Var}(\Delta_a \overline{X}_{j,N})\Big(\mathbb{E}\Big(\frac{\Delta_a \overline{X}_{i,N}\Delta_a \overline{X}_{j,N}}{\sqrt{\mathrm{Var}(\Delta_a \overline{X}_{i,N})\mathrm{Var}(\Delta_a \overline{X}_{j,N})}}\Big)^2 - 1\Big)\\
&= 2\sum_{i,j\in\nu_N(t_0)} \mathrm{Var}(\Delta_a \overline{X}_{i,N})\mathrm{Var}(\Delta_a \overline{X}_{j,N})\Big(\mathrm{Cov}\Big(\frac{\Delta_a \overline{X}_{i,N}}{\sqrt{\Delta_a \overline{X}_{i,N}}}, \frac{\Delta_a \overline{X}_{j,N}}{\sqrt{\Delta_a \overline{X}_{j,N}}}\Big)\Big)^2\\
&= 2\sum_{i,j\in\nu_N(t_0)}\Big(\mathrm{Cov}(\Delta_a \overline{X}_{i,N}, \Delta_a \overline{X}_{j,N})\Big)^2, \qquad (6.3.12)
\end{aligned}$$

Then it follows from (6.3.12) and Lemma 6.3.7 that there is a constant $c_1 > 0$ such that for all $N \geq p+1$,

$$\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a \overline{X}_{i,N})^2\Big) \leq c_1 \sum_{i,j\in\nu_N(t_0)} \frac{N^{-2(H(i/N)+H(j/N))}}{(1+|i-j|)^2}. \qquad (6.3.13)$$

On the other hand, since $H \in C^2([0,1])$, there is a constant $c_2 > 0$ such that for any $s, s' \in [0,1]$,

$$|H(s) - H(s')| \leq c_2 |s - s'|. \tag{6.3.14}$$

Thus, it follows from (6.2.3) that for all $N \geq p+1$ and $i \in \nu_N(t_0)$,

$$N^{H(t_0)-H(i/N)} \leq N^{|H(t_0)-H(i/N)|} \leq N^{c_2|t_0-i/N|} \leq N^{c_2 v(N)} \leq c_3, \tag{6.3.15}$$

where $c_3 = \sup_{N\in\mathbb{N},\, N\geq 3} e^{c_2 v(N)\log N}$ is finite thanks to condition (ii) (see Theorem 6.2.1). Also (6.3.14) and (6.2.3) imply that

$$N^{H(t_0)-H(i/N)} \geq N^{-|H(t_0)-H(i/N)|} \geq N^{-c_2 v(N)} \geq c_3', \tag{6.3.16}$$

where $c_3' = \inf_{N\in\mathbb{N},\, N\geq 3} e^{-c_2 v(N)\log N}$ is strictly positive thanks to condition (ii) (see Theorem 6.2.1). It follows from (6.3.13), (6.3.15), (6.2.4), (6.2.5) that for all $N \geq p+1$,

$$\begin{aligned}
\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)} (\Delta_a \overline{X}_{i,N})^2\Big) &\leq c_4 N^{-4H(t_0)} \sum_{i,j\in\nu_N(t_0)} \big(1+|i-j|\big)^{-2} \\
&\leq c_5 N^{-4H(t_0)} N_{t_0} \\
&\leq c_6 N^{1-4H(t_0)} v(N),
\end{aligned} \tag{6.3.17}$$

where the constants c_4, $c_5 = 2c_4 \sum_{l=1}^{+\infty} l^{-2}$ and c_6 do not depend on N. Thus using Markov's inequality, (6.3.4) and (6.3.17), one obtains that, for any $\eta > 0$,

$$\begin{aligned}
&\mathbb{P}\Big(\Big|\frac{\sum_{i\in\nu_N(t_0)} \Delta_a(\overline{X}_{i,N})^2}{\sum_{i\in\nu_N(t_0)} \mathrm{Var}(\Delta_a \overline{X}_{i,N})} - 1\Big| > \eta\Big) \\
&\leq \Big(\sum_{i\in\nu_N(t_0)} \mathrm{Var}(\Delta_a \overline{X}_{i,N})\Big)^{-4} \eta^{-4} \mathbb{E}\Big(\sum_{i\in\nu_N(t_0)} (\Delta_a \overline{X}_{i,N})^2 - \sum_{i\in\nu_N(t_0)} \mathrm{Var}(\Delta_a \overline{X}_{i,N})\Big)^4 \\
&\leq c_6'\Big(\sum_{i\in\nu_N(t_0)} \mathrm{Var}(\Delta_a \overline{X}_{i,N})\Big)^{-4} \eta^{-4} (v(N))^2 N^{2-8H(t_0)}.
\end{aligned} \tag{6.3.18}$$

On the other hand, in view of Lemma 7.2.2 in Appendix, there is constant $c_7 > 0$ such that one has for all $N \geq p+1$ and all $i \in \{0, \ldots, N-p-1\}$,

$$\mathrm{Var}(\Delta_a \overline{X}_{i,N}) \geq c_7 N^{-2H(i/N)}. \tag{6.3.19}$$

Thus, combining (6.3.19) with (6.3.16), it follows that there exists a constant $c_8 > 0$ such that one has for all N big enough at $i \in \nu_N(t_0)$,

$$\mathrm{Var}(\Delta_a \overline{X}_{i,N}) \geq c_8 N^{-2H(t_0)}. \tag{6.3.20}$$

Relations (6.3.18) and (6.3.20) imply that, for all N big enough,

$$\begin{aligned}
&\mathbb{P}\Big(\Big|\frac{\sum_{i\in\nu_N(t_0)} (\Delta_a \overline{X}_{i,N})^2}{\sum_{i\in\nu_N(t_0)} \mathrm{Var}(\Delta_a \overline{X}_{i,N})} - 1\Big| > \eta\Big) \\
&\leq c_2 c_6 (c_8)^{-4} \eta^{-4} (v(N))^{-2} N^{-2}.
\end{aligned} \tag{6.3.21}$$

Then using condition (i) (see Theorem 6.2.1), one obtains that

$$\sum_{N=p+1}^{+\infty} \mathbb{P}\Big(\Big|\frac{\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}{\sum_{i\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{X}_{i,N})}-1\Big|>\eta\Big)<+\infty.$$

Therefore by Borel-Cantelli's Lemma, we get

$$\frac{\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}{\sum_{i\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{X}_{i,N})}\xrightarrow[N\to+\infty]{a.s.}1.$$

Lemma 6.3.8 has been proved. □

Lemma 6.3.9 *Assume that v satisfies condition (ii) in Theorem 6.2.1. Then, we have*

$$\frac{\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}{\sum_{i\in\nu_N(t_0)}Var(\Delta_a\overline{X}_{i,N})}\xrightarrow[N\to+\infty]{\mathbb{P}}1.$$

Proof of Lemma 6.3.9. The lemma follows from (6.3.21) and the assumption that $\lim_{N\to+\infty}Nv(N)=+\infty$. □

Lemma 6.3.10 *Let $\{B_{H(t_0)}(s)\}_{s\in[0,1]}$ be the fBm with Hurst parameter $H(t_0)$. For any integer $N\geq p+1$ and any $j\in\{0,\dots,N-p-1\}$, define the following generalized increment:*

$$\Delta_a\overline{B}_{j,N}^{H(t_0)}=\sum_{k=0}^{p}a_k\overline{B}_{j+k,N}^{H(t_0)}=N\sum_{k=0}^{p}a_k\int_{(j+k)/N}^{(j+k+1)/N}B_{H(t_0)}(s)\,\mathrm{d}s. \tag{6.3.22}$$

Then one has,

$$Var(\Delta_a\overline{B}_{j,N}^{H(t_0)})=\widetilde{C}_a(H(t_0))N^{-2H(t_0)}, \tag{6.3.23}$$

where $\widetilde{C}_a$ is the function introduced in (6.3.3).

Proof of Lemma 6.3.10. It follows from (6.3.22) and Fubini Theorem, that

$$\begin{aligned}&\mathrm{Var}(\Delta_a\overline{B}_{j,N}^{H(t_0)}) \qquad (6.3.24)\\&=N^2\int_0^{1/N}\int_0^{1/N}\mathbb{E}\left\{\Big(\sum_{k=0}^{p}a_kB_{H(t_0)}(\frac{j+k}{N}+s)\Big)\Big(\sum_{k'=0}^{p}a_kB_{H(t_0)}(\frac{j+k'}{N}+s')\Big)\right\}dsds'.\end{aligned}$$

Recall that, for each $s\in[0,1]$,

$$B_{H(t_0)}(s)=\int_{\mathbb{R}}\frac{e^{is\xi}-1}{|\xi|^{H(t_0)+1/2}}\,\mathrm{d}\widehat{B}(\xi), \tag{6.3.25}$$

where $\mathrm{d}\widehat{B}$ is the Fourier transformation of the white noise $\mathrm{d}W$. Relations (6.3.24), (6.3.25), (6.2.1), the isometry property of the integral $\int_{\mathbb{R}}(\cdot)\,d\widehat{B}$ and Fubini Theorem, imply that

$$\mathrm{Var}(\Delta_a\overline{B}_{j,N}^{H(t_0)})=N^2\int_{\mathbb{R}}\frac{\big|\sum_{k=0}^{p}a_ke^{ik\xi/N}\big|^2}{|\xi|^{2H(t_0)+1}}\Big|\int_0^{1/N}e^{is\xi}\,\mathrm{d}s\Big|^2\,\mathrm{d}\xi. \tag{6.3.26}$$

Moreover, for all $\xi \in \mathbb{R} \setminus \{0\}$,

$$\Big| \int_0^{1/N} e^{is\xi}\, \mathrm{d}s \Big|^2 = \Big| \frac{e^{i\xi/N} - 1}{i\xi} \Big|^2 = 4 \frac{\sin^2(\xi/2N)}{|\xi|^2} = 2 \frac{\big(1 - \cos(\xi/N)\big)}{|\xi|^2}. \tag{6.3.27}$$

Finally setting in (6.3.26) $\eta = \xi/N$ and using (6.3.27), one obtains the lemma. □

Lemma 6.3.11 *Assume that v satisfies condition (ii) in Theorem 6.2.1. Then, there exists a constant $c > 0$ such that for all N big enough, one has,*

$$\max_{j \in \nu_N(t_0)} \Big\{ E \Big| \Delta_a \overline{X}_{j,N} - \Delta_a \overline{B}_{j,N}^{H(t_0)} \Big|^2 \Big\} \leq c\big(v(N) \log(N)\big)^2 N^{-2H(t_0)}. \tag{6.3.28}$$

Proof of Lemma 6.3.11. Using, triangle inequality, one has,

$$\|\Delta_a \overline{X}_{j,N} - \Delta_a \overline{B}_{j,N}^{H(t_0)}\|_{L^2} \leq \|\Delta_a \overline{X}_{j,N} - \Delta_a \overline{B}_{j,N}^{H(j/N)}\|_{L^2} + \|\Delta_a \overline{B}_{j,N}^{H(j/N)} - \Delta_a \overline{B}_{j,N}^{H(t_0)}\|_{L^2}, \tag{6.3.29}$$

where

$$\Delta_a \overline{B}_{j,N}^{H(j/N)} = \sum_{k=0}^{p} a_k \overline{B}_{j+k,N}^{H(j/N)} = N \sum_{k=0}^{p} a_k \int_{(j+k)/N}^{(j+k+1)/N} B_{H(j/N)}(s)\, \mathrm{d}s. \tag{6.3.30}$$

First, let us bound $\|\Delta_a \overline{X}_{j,N} - \Delta_a \overline{B}_{j,N}^{H(j/N)}\|_{L^2}^2 = \mathbb{E} \Big| \Delta_a \overline{X}_{j,N} - \Delta_a \overline{B}_{j,N}^{H(j/N)} \Big|^2$. Recall that, one has for each $s \in [0,1]$,

$$X(s) = \int_{\mathbb{R}} \frac{e^{is\xi} - 1}{|\xi|^{H(s)+1/2}}\, \mathrm{d}\widehat{B}(\xi) \tag{6.3.31}$$

and

$$B_{H(j/N)}(s) = \int_{\mathbb{R}} \frac{e^{is\xi} - 1}{|\xi|^{H(j/N)+1/2}}\, \mathrm{d}\widehat{B}(\xi), \tag{6.3.32}$$

where $\mathrm{d}\widehat{B}$ is the Fourier transformation of the white noise $\mathrm{d}W$. Relations (6.3.31) and (6.3.32) imply that,

$$\begin{aligned}
&\Big| \Delta_a \overline{X}_{j,N} - \Delta_a \overline{B}_{j,N}^{H(j/N)} \Big| \\
&= \Big| N \int_0^{1/N} \sum_{k=0}^{p} a_k \Big(X(s + \frac{j+k}{N}) - B_{H(j/N)}(s + \frac{j+k}{N}) \Big)\, \mathrm{d}s \Big| \\
&= N \Big| \int_0^{1/N} \sum_{k=0}^{p} a_k \int_{\mathbb{R}} \Big(\frac{e^{i(s+(j+k)/N)\xi} - 1}{|\xi|^{H(s+(j+k)/N)+1/2}} - \frac{e^{i(s+(j+k)/N)\xi} - 1}{|\xi|^{H(j/N)+1/2}} \Big)\, \mathrm{d}\widehat{B}(\xi)\, \mathrm{d}s \Big| \\
&= N \Big| \int_0^{1/N} \int_{\mathbb{R}} \Big(\sum_{k=0}^{p} a_k (e^{i(s+(j+k)/N)\xi} - 1) |\xi|^{-1/2} \big(g(s + (j+k)/N, \xi) - g(j/N, \xi) \big) \Big)\, \mathrm{d}\widehat{B}(\xi)\, \mathrm{d}s \Big|,
\end{aligned} \tag{6.3.33}$$

where g is the function defined for each $x \in [0,1]$ and $\xi \in \mathbb{R} \setminus \{0\}$, as,

$$g(x, \xi) = |\xi|^{-H(x)}. \tag{6.3.34}$$

Observe that

$$\partial_x g(x,\xi) = -H'(x)|\xi|^{-H(x)}\log|\xi| = -H'(x)g(x,\xi)\log|\xi| \tag{6.3.35}$$

and

$$\partial_x^2 g(x,\xi) = \Big(\big(H'(x)\big)^2\log|\xi| - H''(x)\Big)g(x,\xi)\log|\xi|. \tag{6.3.36}$$

It is worth noticing that (6.3.34), (6.3.35), (6.3.36) and the fact that

$$\sup_{x\in[0,1]}\big(|H'(x)|+|H''(x)|\big)<\infty, \tag{6.3.37}$$

imply that there exists a constant $c_{11}>0$ such that for all $m\in\{0,1,2\}$, $x\in[0,1]$ and $\xi\in\mathbb{R}\setminus\{0\}$, one has,

$$\big|\partial_x^m g(x,\xi)\big| \le c_{11}\big(|\xi|^{-H_*}+|\xi|^{-H^*}\big)\big(1+\big|\log|\xi|\big|^m\big). \tag{6.3.38}$$

where $H_* = \min_{s\in[0,1]}H(s)$ and $H^* = \max_{s\in[0,1]}H(s)$. It follows from (6.3.33), Cauchy-Schwarz inequality and the isometry property of the integral $\int_{\mathbb{R}}(\cdot)\,\mathrm{d}\widehat{B}$ that,

$$\begin{aligned}&\mathbb{E}\Big|\Delta_a\overline{X}_{j,N}-\Delta_a\overline{B}^{H(j/N)}_{j,N}\Big|^2 \\ &\le N\int_0^{1/N}\int_{\mathbb{R}}\Big|\sum_{k=0}^p a_k(e^{i(s+(j+k)/N)\xi}-1)|\xi|^{-1/2}\big(g(s+(j+k)/N,\xi)-g(j/N,\xi)\big)\Big|^2\mathrm{d}\xi\,\mathrm{d}s.\end{aligned} \tag{6.3.39}$$

Moreover, using for each fixed $\xi\in\mathbb{R}\setminus\{0\}$, a Taylor expansion of order 2 of $g(\cdot,\xi)$ at j/N, one has, for all $x\in[0,1]$,

$$\begin{aligned}g(x,\xi) &= g(j/N,\xi)+(x-j/N)\partial_x g(j/N,\xi) \\ &\quad +(x-j/N)^2\int_0^1(1-\theta)\partial_x^2 g(j/N+\theta(x-j/N),\xi)\,\mathrm{d}\theta.\end{aligned} \tag{6.3.40}$$

Combining (6.3.39) with (6.3.40) one gets that

$$\mathbb{E}\Big|\Delta_a\overline{X}_{j,N}-\Delta_a\overline{B}^{H(j/N)}_{j,N}\Big|^2 \le 2U_{j,N}+2V_{j,N}, \tag{6.3.41}$$

where

$$U_{j,N} = N\int_0^{1/N}\int_{\mathbb{R}}\Big|\sum_{k=0}^p a_k(e^{i(s+(j+k)/N)\xi}-1)|\xi|^{-1/2}(s+k/N)\partial_x g(j/N,\xi)\Big|^2\mathrm{d}\xi\,\mathrm{d}s \tag{6.3.42}$$

and

$$\begin{aligned}V_{j,N} &= N\int_0^{1/N}\int_{\mathbb{R}}\Big|\sum_{k=0}^p a_k(e^{i(s+(j+k)/N)\xi}-1)|\xi|^{-1/2}(s+k/N)^2 \\ &\qquad\times\int_0^1(1-\theta)\partial_x^2 g(j/N+\theta(x-j/N),\xi)\,\mathrm{d}\theta\Big|^2\mathrm{d}\xi\,\mathrm{d}s.\end{aligned} \tag{6.3.43}$$

Let us now give a suitable bound for $U_{j,N}$. Relations (6.3.42), (6.2.1) and (6.3.35) entail that

$$\begin{aligned} U_{j,N} &\le 2N\int_0^{1/N} s^2 \int_{\mathbb{R}} \frac{\big|A(\xi/N)\big|^2\big(H'(j/N)\big)^2\big(\log|\xi|\big)^2}{|\xi|^{2H(j/N)+1}}\,\mathrm{d}\xi\,\mathrm{d}s \\ &\quad +2N^{-1}\int_0^{1/N}\int_{\mathbb{R}} \frac{\big|A'(\xi/N)\big|^2\big(H'(j/N)\big)^2\big(\log|\xi|\big)^2}{|\xi|^{2H(j/N)+1}}\,\mathrm{d}\xi\,\mathrm{d}s \\ &\le 2N^{-2}\int_{\mathbb{R}} \frac{\big(\big|A(\xi/N)\big|^2+\big|A'(\xi/N)\big|^2\big)\big(H'(j/N)\big)^2\big(\log|\xi|\big)^2}{|\xi|^{2H(j/N)+1}}\,d\xi, \end{aligned} \tag{6.3.44}$$

where A is the trigonometric polynomial defined for $\eta\in\mathbb{R}$ as,

$$A(\eta)=\sum_{k=0}^{p} a_k e^{ik\eta} \tag{6.3.45}$$

and A' is its derivative. Observe that all the integrals in (6.3.44) are finite since

$$\big|A(\eta)\big| = 0\big(\min\{1,|\eta|\}\big) \text{ and } \big|A'(\eta)\big| = 0\big(\min\{1,|\eta|\}\big); \tag{6.3.46}$$

relation (6.3.46) is in fact a consequence of (6.2.1). Setting in (6.3.44) $\eta=\xi/N$ and using (6.3.34), (6.3.35), (6.3.37), (6.3.38) and (6.3.15), it follows that, for all N big enough and $j\in\nu_N(t_0)$,

$$\begin{aligned} U_{j,N} &\le 4\big(H'(j/N)\big)^2 N^{-2-2H(j/N)}\big(\log N\big)^2\int_{\mathbb{R}}\big(\big|A(\eta)\big|^2+\big|A'(\eta)\big|^2\big)\big(g(j/N,\eta)\big)^2\,d\eta \\ &\quad +4N^{-2-2H(j/N)}\int_{\mathbb{R}}\big(\big|A(\eta)\big|^2+\big|A'(\eta)\big|^2\big)\big(\partial_x g(j/N,\eta)\big)^2\,d\eta \\ &\le c_{12}N^{-2-2H(t_0)}\big(\log N\big)^2, \end{aligned} \tag{6.3.47}$$

where $c_{12}>0$ is a constant non depending on N and j. Let us now give a suitable bound for $V_{j,N}$. Relation (6.3.43), the triangle inequality, the inequality $\big(\sum_{k=0}^p b_k\big)^2 \le (p+1)\sum_{k=0}^p b_k^2$ for all are reals $b_0,\dots,b_p$ and Relation (6.3.38) imply that,

$$\begin{aligned} V_{j,N} &\le (p+1)N\sum_{k=0}^{p}|a_k|^2\int_0^{1/N}(s+k/N)^4\int_{\mathbb{R}}\big|e^{i(s+(j+k)/N)\xi}-1\big|^2|\xi|^{-1} \\ &\qquad\qquad \times\Big|\int_0^1(1-\theta)\partial_x^2 g(j/N+\theta(x-j/N),\xi)\,\mathrm{d}\theta\Big|^2\,\mathrm{d}\xi\,\mathrm{d}s \\ &\le c_{11}^2(p+1)^5N^{-3}\sum_{k=0}^{p}|a_k|^2\int_0^{1/N}\int_{\mathbb{R}}\big|e^{i(s+(j+k)/N)\xi}-1\big|^2|\xi|^{-1} \\ &\qquad\qquad \times\big(|\xi|^{-H_*}+|\xi|^{-H^*}\big)^2\big(1+\big|\log|\xi|\big|^2\big)^2\,\mathrm{d}\xi\,\mathrm{d}s; \end{aligned} \tag{6.3.48}$$

moreover, one has,

$$c_{13}=\sup_{x\in[0,1]}\int_{\mathbb{R}}\big|e^{ix\xi}-1\big|^2|\xi|^{-1}\big(|\xi|^{-H_*}+|\xi|^{-H^*}\big)^2\big(1+\big|\log|\xi|\big|^2\big)^2\,\mathrm{d}\xi<\infty, \tag{6.3.49}$$

since

$$x \mapsto \int_{\mathbb{R}} \big|e^{ix\xi}-1\big|^2 |\xi|^{-1}\big(|\xi|^{-H_*}+|\xi|^{-H^*}\big)^2\big(1+\big|\log|\xi|\big|^2\big)^2\,\mathrm{d}\xi,$$

is a continuous function on the compact interval $[0,1]$. Next, combining (6.3.48) with (6.3.49), one obtains that for all N big enough and $j\in\nu_N(t_0)$,

$$V_{j,N} \le c_{14}N^{-4}, \tag{6.3.50}$$

where $c_{14}>0$ is a constant non depending on N and j. Next, it follows from (6.3.41), (6.3.47) and (6.3.50) that, for all N big enough and $j\in\nu_N(t_0)$,

$$\mathbb{E}\left|\Delta_a\overline{X}_{j,N}-\Delta_a\overline{B}^{H(j/N)}_{j,N}\right|^2 \le c_{15}N^{-2-2H(t_0)}\big(\log N\big)^2, \tag{6.3.51}$$

where $c_{15}>0$ is a constant non depending on N and j. Now let us bound $\|\Delta_a\overline{B}^{H(j/N)}_{j,N}-\Delta_a\overline{B}^{H(t_0)}_{j,N}\|^2_{L^2}=\mathbb{E}\left|\Delta_a\overline{B}^{H(j/N)}_{j,N}-\Delta_a\overline{B}^{H(t_0)}_{j,N}\right|^2$. Relations (6.3.32) and (6.3.25) imply that for all N big enough and $j\in\nu_N(t_0)$, one has,

$$\begin{aligned}&\left|\Delta_a\overline{B}^{H(j/N)}_{j,N}-\Delta_a\overline{B}^{H(t_0)}_{j,N}\right| \\ &= N\Big|\int_0^{1/N}\int_{\mathbb{R}}\Big(\sum_{k=0}^{p}a_k\big(e^{i(s+(j+k)/N)\xi}-1\big)\Big)|\xi|^{-1/2}\big(g(j/N,\xi)-g(t_0,\xi)\big)\,\mathrm{d}\widehat{B}(\xi)\,\mathrm{d}s\Big|,\end{aligned} \tag{6.3.52}$$

where g is the function introduced in (6.3.34). It follows from (6.3.52), Cauchy-Schwarz inequality , the isometry property of the stochastic integral $\int_{\mathbb{R}}(\cdot)\,d\widehat{B}$, (6.2.1) and (6.3.45) that,

$$\begin{aligned}&\mathbb{E}\left|\Delta_a\overline{B}^{H(j/N)}_{j,N}-\Delta_a\overline{B}^{H(t_0)}_{j,N}\right|^2 \\ &\le N\int_0^{1/N}\int_{\mathbb{R}}\big|A(\xi/N)\big|^2|\xi|^{-1}\big|g(j/N,\xi)-g(t_0,\xi)\big|^2\,\mathrm{d}\xi\,\mathrm{d}s \\ &= N\int_0^{1/N}\int_{\mathbb{R}}\big|A(\eta)\big|^2|\eta|^{-1}\big|g(j/N,N\eta)-g(t_0,N\eta)\big|^2\,\mathrm{d}\xi\,\mathrm{d}s,\end{aligned} \tag{6.3.53}$$

where the last equality results from the change of variable $\eta=\xi/N$. Next, using, for each fixed N and $\eta\in\mathbb{R}\setminus\{0\}$, a Taylor expansion of $g(\cdot,N\eta)$ of order 1 on t_0, it follows that,

$$\begin{aligned}\big|g(j/N,N\eta)-g(t_0,N\xi)\big| &= |j/N-t_0|\Big|\int_0^1\partial_x g(t_0+\theta(j/N-t_0),N\eta)\,\mathrm{d}\theta\Big| \\ &\le v(N)\Big|\int_0^1\partial_x g(t_0+\theta(j/N-t_0),N\eta)\,\mathrm{d}\theta\Big|,\end{aligned} \tag{6.3.54}$$

where the last inequality results from (6.2.3). Now let us bound

$$\Big|\int_0^1\partial_x g(t_0+\theta(j/N-t_0),N\eta)\,\mathrm{d}\theta\Big|.$$

First, observe that (6.3.35) and (6.3.34) imply that,

$$\begin{aligned}
&\int_0^1 \partial_x g(t_0+\theta(j/N-t_0),N\eta)\,\mathrm{d}\theta \\
&= \int_0^1 (\log|N\eta|)(|N\eta|^{-H(t_0+\theta(j/N-t_0))})(-H'(t_0+\theta(j/N-t_0)))\,\mathrm{d}\theta \\
&= \int_0^1 (\log(N)+\log(|\eta|))N^{-H(t_0+\theta(j/N-t_0))}|\eta|^{-H(t_0+\theta(j/N-t_0))}(-H'(t_0+\theta(j/N-t_0)))\,\mathrm{d}\theta \\
&= \int_0^1 N^{-H(t_0+\theta(j/N-t_0))}\Big(\log(N)(-H'(t_0+\theta(j/N-t_0)))g(t_0+\theta(j/N-t_0),\eta) \\
&\quad +\partial_x g(t_0+\theta(j/N-t_0),\eta)\Big)\,\mathrm{d}\theta. \qquad (6.3.55)
\end{aligned}$$

Next observe that, similarly to (6.3.15), one can show that there is a constant $c_{16}>0$, non depending on N, j and t_0, such that

$$\begin{aligned}
N^{-H(t_0+\theta(j/N-t_0))} &= N^{-H(t_0)}N^{H(t_0)-H(t_0+\theta(j/N-t_0))} \\
&\le c_{16}N^{-H(t_0)}. \qquad (6.3.56)
\end{aligned}$$

It follows from (6.3.55), (6.3.56), (6.3.37) and (6.3.38), that for all $N\ge p+1$,

$$\begin{aligned}
&\Big|\int_0^1 \partial_x g(t_0+\theta(j/N-t_0),N\eta)\,\mathrm{d}\theta\Big| \\
&\le c_{17}\log(N)N^{-H(t_0)}\big(|\eta|^{-H_*}+|\eta|^{-H^*}\big)\big(1+\big|\log|\eta|\big|\big), \qquad (6.3.57)
\end{aligned}$$

where $c_{17}>0$ is a constant non depending on N, j and t_0. Thus, (6.3.53), (6.3.54) and (6.3.57) imply that, for all N big enough, $j\in\nu_N(t_0)$, $x\in[(j+k)/N,(j+k+1)/N]$ and $\eta\in\mathbb{R}\setminus\{0\}$,

$$\mathbb{E}\Big|\Delta_a\overline{B}_{j,N}^{H(j/N)}-\Delta_a\overline{B}_{j,N}^{H(t_0)}\Big|^2 \le c_{18}\big(v(N)\log(N)\big)^2N^{-2H(t_0)}, \qquad (6.3.58)$$

where the finite constant

$$c_{18}=c_{17}^2\int_{\mathbb{R}}\big|A(\eta)\big|^2|\eta|^{-1}\big(|\eta|^{-H_*}+|\eta|^{-H^*}\big)^2\big(1+\big|\log|\eta|\big|\big)^2\,d\eta.$$

Next, putting together (6.3.29), (6.3.51), (6.3.58) and the inequality $v(N)\ge N^{-1}$, one gets (6.3.28). □

The following remark is a straightforward consequence of Lemma 6.3.10, Lemma 6.3.11 and condition (ii) in Theorem 6.2.1.

Remark 6.3.1 *Assume that v satisfies condition (ii) in Theorem 6.2.1, then there exist two constants $0<c'\le c$ such that for all N big enough and all $j\in\nu_N(t_0)$, one has,*

$$c'N^{-2H(t_0)}\le Var(\Delta_a\overline{X}_{j,N})\le cN^{-2H(t_0)}. \qquad (6.3.59)$$

Lemma 6.3.12 *Assume that v satisfies condition (ii) in Theorem 6.2.1. Then, there is a constant $c>0$, such that for all N big enough, one has,*

$$\max_{j\in\nu_N(t_0)}\Big|\frac{Var(\Delta_a\overline{X}_{j,N})}{Var(\Delta_a\overline{B}_{j,N}^{H(t_0)})}-1\Big|\le c\log(N)v(N). \tag{6.3.60}$$

Proof of Lemma 6.3.12. One has,

$$\begin{aligned}
&\left|\frac{\mathrm{Var}(\Delta_a\overline{X}_{j,N})}{\mathrm{Var}(\Delta_a\overline{B}_{j,N}^{H(t_0)})}-1\right|=\left|\frac{\|\Delta_a\overline{X}_{j,N}\|_{L^2}^2}{\|\Delta_a\overline{B}_{j,N}^{H(t_0)}\|_{L^2}^2}-1\right|\\
&=\left|\frac{\|\Delta_a\overline{X}_{j,N}\|_{L^2}^2-\|\Delta_a\overline{B}_{j,N}^{H(t_0)}\|_{L^2}^2}{\|\Delta_a\overline{B}_{j,N}^{H(t_0)}\|_{L^2}^2}\right|\\
&=\frac{\big|\|\Delta_a\overline{X}_{j,N}\|_{L^2}-\|\Delta_a\overline{B}_{j,N}^{H(t_0)}\|_{L^2}\big|\big(\|\Delta_a\overline{X}_{j,N}\|_{L^2}+\|\Delta_a\overline{B}_{j,N}^{H(t_0)}\|_{L^2}\big)}{\|\Delta_a\overline{B}_{j,N}^{H(t_0)}\|_{L^2}^2}\\
&\le\frac{\|\Delta_a\overline{X}_{j,N}-\Delta_a\overline{B}_{i,N}^{H(t_0)}\|_{L^2}\big(\|\Delta_a\overline{X}_{j,N}\|_{L^2}+\|\Delta_a\overline{B}_{j,N}^{H(t_0)}\|_{L^2}\big)}{\|\Delta_a\overline{B}_{j,N}^{H(t_0)}\|_{L^2}^2}.
\end{aligned}$$

Then the lemma follows from Lemma 6.3.10, Lemma 6.3.11 and Remark 6.3.1. □

Lemma 6.3.13 *Assume that v satisfies condition (ii) in Theorem 6.2.1 and let c be the constant introduced in Lemma 6.3.12. Then, for all N big enough, one has,*

$$\Big|\frac{\sum_{j\in\nu_N(t_0)}Var(\Delta_a\overline{X}_{j,N})}{\sum_{j\in\nu_N(t_0)}Var(\Delta_a\overline{B}_{j,N}^{H(t_0)})}-1\Big|\le c\log(N)v(N). \tag{6.3.61}$$

Proof of Lemma 6.3.13. Using the triangle inequality and (6.3.60) one has,

$$\begin{aligned}
&\Big|\frac{\sum_{j\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{X}_{j,N})}{\sum_{j\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{B}_{j,N}^{H(t_0)})}-1\Big|\\
&=\Big|\frac{\sum_{j\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{X}_{j,N})-\sum_{j\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{B}_{j,N}^{H(t_0)})}{\sum_{j\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{B}_{j,N}^{H(t_0)})}\Big|\\
&\le\Big(\sum_{j\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{B}_{j,N}^{H(t_0)})\Big)^{-1}\sum_{j\in\nu_N(t_0)}\big|\mathrm{Var}(\Delta_a\overline{X}_{j,N})-\mathrm{Var}(\Delta_a\overline{B}_{j,N}^{H(t_0)})\big|\\
&=\Big(\sum_{j\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{B}_{j,N}^{H(t_0)})\Big)^{-1}\sum_{j\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{B}_{j,N}^{H(t_0)})\Big|\frac{\mathrm{Var}(\Delta_a\overline{X}_{j,N})}{\mathrm{Var}(\Delta_a\overline{B}_{j,N}^{H(t_0)})}-1\Big|\\
&\le c\log(N)v(N).
\end{aligned}$$

□

Now we are in position to show that Propositions 6.3.3 and 6.3.4 hold; we will only give the proof of Proposition 6.3.3, since that of Proposition 6.3.4 is quite similar.

Proof of Proposition 6.3.3. One has,

$$\frac{\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}{v(N)N^{1-2H(t_0)}} = \tag{6.3.62}$$
$$\Big(\frac{\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}{\sum_{i\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{X}_{i,N})}\Big)\Big(\frac{\sum_{i\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{X}_{i,N})}{\sum_{i\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{B}_{i,N}^{H(t_0)})}\Big)\Big(\frac{\sum_{i\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{B}_{i,N}^{H(t_0)})}{v(N)N^{1-2H(t_0)}}\Big).$$

Moreover, Lemma 6.3.10 and (6.2.4) imply that,

$$\sum_{i\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{B}_{i,N}^{H(t_0)}) = N_{t_0}\widetilde{C}_a(H(t_0))N^{-2H(t_0)}, \tag{6.3.63}$$

and (6.2.5) entails that,

$$\lim_{N\to+\infty}\frac{N_{t_0}}{2Nv(N)} = 1, \tag{6.3.64}$$

Thus, combining (6.3.63) with (6.3.64), one gets,

$$\frac{\sum_{i\in\nu_N(t_0)}\mathrm{Var}(\Delta_a\overline{B}_{i,N}^{H(t_0)})}{v(N)N^{1-2H(t_0)}}\xrightarrow[N\to+\infty]{} 2\widetilde{C}_a(H(t_0)), \tag{6.3.65}$$

Finally, Proposition 6.3.3 results from (6.3.62), Lemma 6.3.8, Lemma 6.3.13 and (6.3.65).□

The following remark easily results from Proposition 6.3.4 and from Lemma 6.3.14 below.

Remark 6.3.2 *Let $\widetilde{C}_a$ be as in (6.3.3). Then, assuming that v satisfies condition (ii) in Theorem 6.2.1, one has,*

$$\frac{2\widetilde{C}_a(H(t_0))v(N)N^{1-2H(t_0)}}{\sum_{i\in\nu_N(t_0)}\Big(\Delta_a\overline{X}_{i,N}\Big)^2}\xrightarrow[N\to+\infty]{\mathbb{P}} 1.$$

Lemma 6.3.14 *Let $(B_l)_l$ be a sequence of random variables with values in $(0,+\infty)$ which converges in probability to a deterministic quantity $b\in(0,+\infty)$. Let g be a deterministic real-valued function defined on $(0,+\infty)$ which is continuous at b. Then, the sequence of random variables $\big(g(B_l)\big)_l$ converges in probability to $g(b)$.*

Lemma 6.3.14 is a well-known result whose proof is straightforward, this is why it has been omitted.

Lemma 6.3.15 *For any integer $N\geq p+1$, one set*

$$V_{N,2}(t_0) = \Big(f'(X(t_0))\Big)^2\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2 \tag{6.3.66}$$

and

$$V_{N,3}(t_0) = \sum_{i\in\nu_N(t_0)} \Big(f'(X(i/N))\Big)^2 (\Delta_a \overline{X}_{i,N})^2. \tag{6.3.67}$$

Then, for all arbitrarily small $\varepsilon > 0$, there is an almost surely finite and strictly positive random variable $C > 0$, such that one has, almost surely, for all $N \geq p+1$,

$$\Big|\frac{V_{N,3}(t_0)}{V_{N,2}(t_0)} - 1\Big| \leq C\big(v(N)\big)^{H(t_0)-\varepsilon}. \tag{6.3.68}$$

Proof of Lemma 6.3.15. It is clear that

$$V_{N,3}(t_0) = V_{N,2}(t_0) + \sum_{i\in\nu_N(t_0)} \Big(\Big(f'(X(i/N))\Big)^2 - \Big(f'(X(t_0))\Big)^2\Big)(\Delta_a \overline{X}_{i,N})^2. \tag{6.3.69}$$

Since $f \in C^2_{pol}(\mathbb{R})$, there exist $c_1, K > 0$ such that, for all $x \in \mathbb{R}$,

$$\sum_{l=0}^{2} |f^{(l)}(x)| \leq c_1(1+|x|^K). \tag{6.3.70}$$

Using the Mean Value Theorem and the latter inequality, one obtains, almost surely, that,

$$\begin{aligned} \Big|\big(f'(X(i/N))\big)^2 - \big(f'(X(t_0))\big)^2\Big| &= |f'(X(i/N)) + f'(X(t_0))||f'(X(i/N)) - f'(X(t_0))| \\ &\leq C_2 \sup_{u\in[-\|X\|_\infty,\|X\|_\infty]} |f^{(2)}(u)||X(i/N) - X(t_0)| \\ &\leq C_3|X(i/N) - X(t_0)|, \end{aligned} \tag{6.3.71}$$

where $C_2 = 2c_1(1+\|X\|_\infty^K)$, $C_3 = 2c_1^2(1+\|X\|_\infty^K)^2$ and $\|X\|_\infty = \max_{s\in[0,1]}|X(s)|$; observe that the positive random variables C_2 and C_3 are of finite moment of any order. Let us now state an important result [2] concerning the path behavior of the mBm $\{X(s)\}_{s\in[0,1]}$; namely, for all fixed arbitrarily small $\varepsilon > 0$, there is a positive random C_4 of finite moment of any order, only depending on ε, $H_* = \min_{s\in[0,1]} H(s)$ and $H^* = \max_{s\in[0,1]} H(s)$, such that one has almost surely, for all $s, s' \in [0,1]$,

$$|X(s) - X(s')| \leq C_4|s-s'|^{\max\{H(s),H(s')\}-\varepsilon}. \tag{6.3.72}$$

It follows from (6.3.71), (6.3.72) and the inequalities $|i/N - t_0| \leq v(N) \leq 1$ that, almost surely,

$$|\big(f'(X(i/N))\big)^2 - \big(f'(X(t_0))\big)^2| \leq C_5(v(N))^{H(t_0)-\varepsilon}, \tag{6.3.73}$$

[2] This result can be derived from [6], its precise proof is given in [38].

where $C_5 = C_3 C_4$. Finally, (6.3.66), (6.3.67), (6.3.69) and (6.3.73), imply that

$$\begin{aligned}\Big|\frac{V_{N,3}(t_0)}{V_{N,2}(t_0)}-1\Big| &= \left|\frac{\sum_{i\in\nu_N(t_0)}\big(f'(X(i/N))\big)^2(\Delta_a\overline{X}_{i,N})^2}{\big(f'(X(t_0))\big)^2\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}-1\right|\\ &\le \frac{\sum_{i\in\nu_N(t_0)}\Big|\big(f'(X(i/N))\big)^2-\big(f'(X(t_0))\big)^2\Big|(\Delta_a\overline{X}_{i,N})^2}{\big(f'(X(t_0))\big)^2\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}\\ &\le \frac{C(v(N))^{H(t_0)-\varepsilon}\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}{\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}\\ &= C\big(v(N)\big)^{H(t_0)-\varepsilon},\end{aligned}$$

where $C = C_5\big(f'(X(t_0))\big)^{-2}$ is an almost surely finite and strictly positive random variable, since, C_5 is almost surely finite and strictly positive, f' does not vanish except on a Lebesgue negligible set and $X(t_0)$ is a non degenerate Gaussian random variable. □

Lemma 6.3.16 *Assume that v satisfies conditions (i) and (ii) in Theorem 6.2.1. Then, for all arbitrarily small $\varepsilon > 0$, there is an almost surely finite and strictly positive random variable C, such that, one has almost surely for all $N \geq p+1$,*

$$\Big|\frac{V_{N,1}(t_0)}{V_{N,3}(t_0)}-1\Big| \le CN^{-(H(t_0)-\varepsilon)}. \tag{6.3.74}$$

Recall that $V_{N,1}(t_0)$ has been introduced in (6.3.1) and $V_{N,3}(t_0)$ in (6.3.67).

Proof of Lemma 6.3.16. In view of (6.3.1), (6.2.7), (6.2.9) and (6.2.6), $V_{N,1}(t_0)$ can be expressed as,

$$V_{N,1}(t_0) = \sum_{i\in\nu_N(t_0)}\Big(N\sum_{k=0}^{p}a_k\int_{(i+k)/N}^{(i+k+1)/N}f(X(s))\,\mathrm{d}s\Big)^2. \tag{6.3.75}$$

A second order Taylor expansion of $f(x)$ on $X(i/N)$ with integral remainder gives

$$\begin{aligned}f(X(s)) &= f(X(i/N)) + f'(X(i/N))(X(s)-X(i/N))\\ &+ (X(s)-X(i/N))^2\int_0^1(1-\eta)f^{(2)}(X(i/N+\eta(X(s)-X(i/N))))\,\mathrm{d}\eta.\end{aligned} \tag{6.3.76}$$

It follows from (6.3.75), (6.3.76) and (6.2.1), that

$$\begin{aligned}
V_{N,1}(t_0) &= N^2 \sum_{i\in\nu_N(t_0)} \Big(\sum_{k=0}^p a_k \int_{(i+k)/N}^{(i+k+1)/N} \Big(f(X(i/N)) + f'(X(i/N))(X(s)-X(i/N)) \\
&+(X(s)-X(i/N))^2 \int_0^1 (1-\eta) f^{(2)}(X(i/N+\eta(X(s)-X(i/N))))\,\mathrm{d}\eta\Big)\,\mathrm{d}s\Big)^2 \\
&= \sum_{i\in\nu_N(t_0)} \Big(f'(X(i/N))\Delta_a \overline{X}_{i,N} + N\sum_{k=0}^p a_k \int_{(i+k)/N}^{(i+k+1)/N} (X(s)-X(i/N))^2 \\
&\times \int_0^1 (1-\eta) f^{(2)}(X(i/N+\eta(X(s)-X(i/N))))\,\mathrm{d}\eta\,\mathrm{d}s\Big)^2 \\
&= V_{N,3}(t_0) + 2\sum_{i\in\nu_N(t_0)} f'(X(i/N))(\Delta_a\overline{X}_{i,N})e_{i,N} + \sum_{i\in\nu_N(t_0)} e_{i,N}^2, \qquad (6.3.77)
\end{aligned}$$

where

$$e_{i,N} = N\sum_{k=0}^p a_k \int_{(i+k)/N}^{(i+k+1)/N} (X(s)-X(i/N))^2 \int_0^1 (1-\eta) f^{(2)}(X(i/N+\eta(X(s)-X(i/N))))\,\mathrm{d}\eta\,\mathrm{d}s. \qquad (6.3.78)$$

Next observe that (6.3.72) and (6.3.15) imply that, for all fixed arbitrarily small $\varepsilon > 0$, there is a positive random variable C_1 of finite moment of any order such that one has, almost surely, for each $i \in \nu_N(t_0)$ and for each $s \in [(i+k)/N, (i+k+1)/N]$,

$$|X(i/N) - X(s)| \le C_1 N^{-(H(t_0)-\varepsilon)}. \qquad (6.3.79)$$

Moreover (6.3.70) and the fact that the random variable $\sup_{x\in[0,1]} |X(x)|$ is with finite moment of each order, entail that,

$$\int_0^1 (1-\eta)\big|f^{(2)}(X(i/N+\eta(X(s)-X(i/N))))\big|\,\mathrm{d}\eta \le C_2, \qquad (6.3.80)$$

where C_2 is a positive random variable of finite moment of any order, non depending on N, i and s. Next putting together, (6.3.78), (6.3.79) and (6.3.80), it follows that, there exists a positive random variable C_3 of finite moment of each order, such that almost surely, for all $N \ge p+1$ and $i \in \nu_N(t_0)$,

$$|e_{i,N}| \le C_3 N^{-2(H(t_0)-\varepsilon)}, \qquad (6.3.81)$$

On the other hand, Cauchy-Schwarz inequality and (6.3.67) imply that

$$\begin{aligned}
&\Big|\sum_{i\in\nu_N(t_0)} f'(X(i/N))\Delta_a\overline{X}_{i,N} e_{i,N}\Big| \\
&\le \Big(\sum_{i\in\nu_N(t_0)} (f'(X(i/N))\Delta_a\overline{X}_{i,N})^2\Big)^{1/2}\Big(\sum_{i\in\nu_N(t_0)} e_{i,N}^2\Big)^{1/2} \\
&= \big(V_{N,3}(t_0)\big)^{1/2}\Big(\sum_{i\in\nu_N(t_0)} e_{i,N}^2\Big)^{1/2}. \qquad (6.3.82)
\end{aligned}$$

Putting together (6.3.77), (6.3.82), (6.3.81), (6.3.66), the fact that $(f'(X(t_0)))^{-2}$ is an almost surely finite random variable, (6.2.4), (6.2.5), Proposition 6.3.3 and Lemma 6.3.15, one obtains that, almost surely for all $N \geq p+1$,

$$
\begin{aligned}
\frac{|V_{N,1}(t_0) - V_{N,3}(t_0)|}{V_{N,3}(t_0)} &\leq 2\frac{\big(\sum_{i\in\nu_N(t_0)} e_{i,N}^2\big)^{1/2}}{\big(V_{N,3}(t_0)\big)^{1/2}} + \frac{\sum_{i\in\nu_N(t_0)} e_{i,N}^2}{V_{N,3}(t_0)}\\
&= \frac{2\big(\sum_{i\in\nu_N(t_0)} e_{i,N}^2\big)^{1/2}}{\big((f'(X(t_0)))^2 v(N) N^{1-2H(t_0)}\big)^{1/2}} \times \frac{\big((f'(X(t_0)))^2 v(N) N^{1-2H(t_0)}\big)^{1/2}}{\big(V_{N,2}(t_0)\big)^{1/2}} \times \Big(\frac{V_{N,2}(t_0)}{V_{N,3}(t_0)}\Big)^{1/2}\\
&\quad + \frac{\big(\sum_{i\in\nu_N} e_{i,N}^2\big)}{(f'(X(t_0)))^2 v(N) N^{1-2H(t_0)}} \times \frac{(f'(X(t_0)))^2 v(N) N^{1-2H(t_0)}}{V_{N,2}(t_0)} \times \frac{V_{N,2}(t_0)}{V_{N,3}(t_0)}\\
&\leq C N^{-(H(t_0)-4\varepsilon)},
\end{aligned}
\tag{6.3.83}
$$

where C is a positive and almost surely finite random variable non depending on N. □

In order to be able to give a weaker version of Lemma 6.3.16 which will be useful in the proof of Theorem 6.3.2, one needs to make some recalls on the notion of boundedness in probability.

Definition 6.3.1 *(a) One says that a sequence $(U_n)_n$ of real-valued random variables is bounded in probability if and only if one has,*

$$
\lim_{\eta\in\mathbb{R}_+,\,\eta\to+\infty} \limsup_{n\in\mathbb{N}, n\to+\infty} \mathbb{P}\big(|U_n| > \eta\big) = 0. \tag{6.3.84}
$$

(b) Let $(x_n)_n$ be a sequence of strictly positive reals and let $(V_n)_n$ be a sequence of positive random variables. The notation:

$$
V_n = \mathcal{O}_{\mathbb{P}}(x_n),
$$

means that the sequence $\big(x_n^{-1} V_n\big)_n$ is bounded in probability.

Remark 6.3.3 *(a) Let $(U_n)_n$ be a sequence of real-valued random variables which converges in probability to some real-valued random variable U, then the sequence $(U_n)_n$ is bounded in probability.*

(b) Let $(K_n)_n$ and $(L_n)_n$ be two sequences of real-valued random variables which are bounded in probability, then the sequence $(K_n L_n)_n$ is bounded in probability.

(c) Let $(K_n)_n$ and $(L_n)_n$ be two sequences of real-valued random variables. Assume $(K_n)_n$ converges in probability to 0 and that $(L_n)_n$ is bounded in probability, then the sequence of random variables $(K_n L_n)_n$ converges in probability to 0.

(d) Let $(U_n)_n$ be a sequence of real-valued random variables which is bounded in probability, let V be a real-valued random variable; then the sequence of random variables $\big(VU_n\big)_n$ is bounded in probability.

(e) *Let $(U_n)_n$ be a sequence of real-valued random variables which converges in probability to some real valued random variable U, let V be a real-valued random variable; then the sequence of random variables $\big(VU_n\big)_n$ converges in probability to the random VU .*

Remark 6.3.3 will play a crucial role in the sequel; the results in it are known, yet we will give their proofs for the sake of completness.

Proof of Remark 6.3.3. Let us first show that Part (a) holds. One has for all strictly positive real $\eta > 0$ and all n,

$$\mathbb{P}\big(|U_n| > \eta\big) \leq \mathbb{P}\big(|U_n - U| > \eta/2\big) + \mathbb{P}\big(|U| > \eta/2\big).$$

Therefore

$$\limsup_{n\to+\infty} \mathbb{P}\big(|U_n| > \eta\big) \leq \mathbb{P}\big(|U| > \eta/2\big),$$

then, using the fact that $\lim_{\eta\to+\infty} \mathbb{P}\big(|U| > \eta/2\big) = 0$, it follows that

$$\lim_{\eta\to+\infty} \limsup_{n\to+\infty} \mathbb{P}\big(|U_n| > \eta\big) = 0.$$

Part (b) easily follows from the inequality, for all real $\eta > 0$ and integer n,

$$\mathbb{P}\big(|K_n||L_n| > \eta\big) \leq \mathbb{P}\big(|K_n| > \eta^{1/2}\big) + \mathbb{P}\big(|L_n| > \eta^{1/2}\big).$$

Let us now prove that Part (c) holds. As previously, η denotes an arbitrary strictly positive real number. One has for all n,

$$\begin{aligned} \mathbb{P}\big(|K_n||L_n| > \eta\big) &\leq \mathbb{P}\left(\big\{|K_n||L_n| > \eta\big\} \cap \big\{|K_n| \leq \eta^2\big\}\right) + \mathbb{P}\big(|K_n| > \eta^2\big) \\ &\leq \mathbb{P}\big(|L_n| > \eta^{-1}\big) + \mathbb{P}\big(|K_n| > \eta^2\big). \end{aligned} \tag{6.3.85}$$

On the other hand, in view of (6.3.84), for all fixed arbitrarily $\varepsilon > 0$, there exists $\eta_0 \in (0,1)$ such that for $\eta \in (0, \eta_0]$, one has,

$$\limsup_{n\to+\infty} \mathbb{P}\big(|L_n| > \eta^{-1}\big) \leq \varepsilon. \tag{6.3.86}$$

Next, Putting together, (6.3.85), (6.3.86) and the fact that

$$\lim_{n\to+\infty} \mathbb{P}\big(|K_n| > \eta^2\big) = 0,$$

one obtains that for all $\eta \in (0, \eta_0]$,

$$\limsup_{n\to+\infty} \mathbb{P}\big(|K_n||L_n| > \eta\big) \leq \varepsilon. \tag{6.3.87}$$

Notice that (6.3.87) remains valid in the case where $\eta > \eta_0$, since one has in this case $\mathbb{P}\big(|K_n||L_n| > \eta\big) \leq \mathbb{P}\big(|K_n||L_n| > \eta_0\big)$ for all n; therefore (6.3.87) implies that for all real $\eta > 0$,

$$\lim_{n\to+\infty} \mathbb{P}\big(|K_n||L_n| > \eta\big) = 0,$$

which proves that Part (c) holds. Finally, observe that Part (d) easily follows from (b), and that Part (e) easily results from Part (c). □

Now we are in position to give the weaker version of Lemma 6.3.16.

Lemma 6.3.17 *Assume that v satisfies conditions (ii) in Theorem 6.2.1. Then, one has, for all arbitrarily small $\varepsilon > 0$,*

$$\Big|\frac{V_{N,1}(t_0)}{V_{N,3}(t_0)} - 1\Big| = \mathcal{O}_{\mathbb{P}}\big(N^{-(H(t_0)-\varepsilon)}\big). \tag{6.3.88}$$

Recall that $V_{N,1}(t_0)$ has been introduced in (6.3.1) and $V_{N,3}(t_0)$ in (6.3.67).

Proof of Lemma 6.3.17. The proof is quite similar to that of Lemma 6.3.16 except that in the arguments leading to (6.3.83),

$$\frac{(f'(X(t_0)))^2 v(N) N^{1-2H(t_0)}}{V_{N,2}(t_0)} = \mathcal{O}_{a.s.}(1),$$

has to be replaced by

$$\frac{(f'(X(t_0)))^2 v(N) N^{1-2H(t_0)}}{V_{N,2}(t_0)} = \mathcal{O}_{\mathbb{P}}(1); \tag{6.3.89}$$

thus we obtain the following weaker version of Relation (6.3.83):

$$\Big|\frac{V_{N,1}(t_0)}{V_{N,3}(t_0)} - 1\Big| = \mathcal{O}_{\mathbb{P}}\big(N^{-(H(t_0)-4\varepsilon)}\big).$$

Before ending our proof, let us notice (6.3.89) is straightforward consequence of Remark 6.3.2 and of (6.3.66). □

The following remark is a straightforward consequence of Proposition 6.3.3, Lemma 6.3.15 and Lemma 6.3.16

Remark 6.3.4 *Assume that v satisfies conditions (i) and (ii) in Theorem 6.2.1, then one has,*

$$\frac{V_{N,1}(t_0)}{2\widetilde{C}_a(H(t_0))(f'(X(t_0)))^2 v(N) N^{1-2H(t_0)}} \xrightarrow[N\to+\infty]{a.s.} 1. \tag{6.3.90}$$

The following remark is a straightforward consequence of Proposition 6.3.4, Lemma 6.3.15 and Lemma 6.3.17

Remark 6.3.5 *Assume that v satisfies condition (ii) in Theorem 6.2.1, then one has,*

$$\frac{V_{N,1}(t_0)}{2\widetilde{C}_a(H(t_0))(f'(X(t_0)))^2 v(N) N^{1-2H(t_0)}} \xrightarrow[N\to+\infty]{\mathbb{P}} 1. \tag{6.3.91}$$

Now we are in position to prove Theorem 6.3.1 and Theorem 6.3.2.

Proof of Theorem 6.3.1. Let $g:(0,+\infty)\to\mathbb{R}$, be the continuous function defined for all $x\in(0,+\infty)$, as:

$$g(x) = \frac{1}{2}\big(1+\log_2(c)+\log_2(x)\big), \tag{6.3.92}$$

where $c>0$ is such that,

$$c=\lim_{N\in\mathbb{N},N\to+\infty}\frac{v(2N)}{v(N)}. \tag{6.3.93}$$

Observe that, in view of (6.3.2), one has for all $N\geq p+1$,

$$\widehat{H}^{(1)}_{N,t_0}=g\left(\frac{V_{N,1}(t_0)}{V_{2N,1}(t_0)}\right). \tag{6.3.94}$$

On the other hand (6.3.90) and (6.3.93) imply that,

$$\frac{V_{N,1}(t_0)}{V_{2N,1}(t_0)}\xrightarrow[N\to+\infty]{a.s.}c^{-1}2^{2H(t_0)-1}. \tag{6.3.95}$$

Thus combining (6.3.94) with (6.3.95), one obtains that,

$$\widehat{H}^{(1)}_{N,t_0}\xrightarrow[N\to+\infty]{a.s.}g\big(c^{-1}2^{2H(t_0)-1}\big)=H(t_0).$$

□

Proof of Theorem 6.3.2. Observe that (6.3.91) and (6.3.93) imply that,

$$\frac{V_{N,1}(t_0)}{V_{2N,1}(t_0)}\xrightarrow[N\to+\infty]{\mathbb{P}}c^{-1}2^{2H(t_0)-1}. \tag{6.3.96}$$

Thus, combining (6.3.94) with (6.3.96), one obtains in view of Lemma 6.3.14,

$$\widehat{H}^{(1)}_{N,t_0}\xrightarrow[N\to+\infty]{\mathbb{P}}g\big(c^{-1}2^{2H(t_0)-1}\big)=H(t_0).$$

□

6.4 Estimation when the $\overline{Y}_{i,N}$'s are unknown

The main goal of this section is to show that Theorem 6.2.1 and Theorem 6.2.2 hold. The proofs respectively rely on Proposition 6.4.1 and Proposition 6.4.2 below.

Proposition 6.4.1 *For all integer n, let N_n be the integer (depending on a parameter $\beta\in(0,1)$) which was defined in (6.2.11). Then, assuming that v satisfies conditions (i) and (ii) in Theorem 6.2.1, for all $\beta<\frac{1}{4H(t_0)+2}$, one has,*

$$\frac{V_n(t_0)}{2\widetilde{C}_a(H(t_0))\big(f'(X(t_0))\big)^2v(N_n)N_n^{1-2H(t_0)}}\xrightarrow[n\to+\infty]{a.s.}1. \tag{6.4.1}$$

where $V_n(t_0)$ was introduced in (6.2.12).

Proof of Proposition 6.4.1. First notice that $V_{N_n,1}(t_0)$ is defined by (6.3.1) in which N is replaced by N_n. Let $\eta>0$ be arbitrary and fixed, it follows from the Markov's inequality that, for all n big enough,

$$\begin{aligned}&\mathbb{P}\left(\big(v(N_n)N_n^{1-2H(t_0)}\big)^{-1/2}\big||V_n(t_0)|^{1/2}-|V_{N_n,1}(t_0)|^{1/2}\big|>\eta\right)\\&\leq\eta^{-4}\big(v(N_n)\big)^{-2}N_n^{-2+4H(t_0)}\mathbb{E}\,\big||V_n(t_0)|^{1/2}-|V_{N_n,1}(t_0)|^{1/2}\big|^4;\end{aligned} \tag{6.4.2}$$

moreover, by using (6.2.12), (6.3.1), the triangle inequality, the inequality $\big(\sum_{k=0}^{q} b_k\big)^2 \le (q+1)\sum_{k=0}^{q} b_k^2$ for all integer $q \ge 0$ and all reals $b_0, \dots, b_q$, (6.2.4) and (6.2.5), we get,

$$
\begin{aligned}
&\big||V_n(t_0)|^{1/2} - |V_{N_n,1}(t_0)|^{1/2}\big|^4 \le \Big(\sum_{i\in\nu_{N_n}(t_0)} \Big(\sum_{k=0}^{p} a_k \mathscr{E}_{i+k,n}\Big)^2\Big)^2 \\
&\le (p+1)^2\Big(\sum_{i\in\nu_{N_n}(t_0)} \Big(\sum_{k=0}^{p} a_k^2 \mathscr{E}_{i+k,n}^2\Big)\Big)^2 \\
&\le 4(p+1)^2 N_n v(N_n) \sum_{i\in\nu_{N_n}(t_0)} \Big(\sum_{k=0}^{p} a_k^2 \mathscr{E}_{i+k,n}^2\Big)^2 \\
&\le 4(p+1)^3 N_n v(N_n) \sum_{i\in\nu_{N_n}(t_0)} \sum_{k=0}^{p} a_k^4 \mathscr{E}_{i+k,n}^4,
\end{aligned} \tag{6.4.3}
$$

where we have set $\mathscr{E}_{i+k,n} = \widehat{Y}_{i+k,n} - \overline{Y}_{i+k,N_n}$. Next, (6.2.11) and Part (ii) of Lemma 7.4.2 (notice that $m_n = [n/N_n]$) imply that there are two constants $c_1 > 0$ and $c_2 > 0$ such that, for all n big enough and all $i \in \{0, \dots, N_n\}$, one has,

$$
\mathbb{E}\,(\mathscr{E}_{i,n}^4) \le c_1\big([n/N_n]\big)^{-2} \le c_2 n^{-2(1-\beta)}, \tag{6.4.4}
$$

where $[\cdot]$ denotes the integer part function. Next, it follows from (6.4.2), (6.4.3), (6.4.4), (6.2.4), (6.2.5) and (6.2.11) that, there is a constant $c_3 > 0$, non depending on n and η, such that one has for each n big enough,

$$
\begin{aligned}
&\mathbb{P}\Big(\big(v(N_n)N_n^{1-2H(t_0)}\big)^{-1/2}\big||V_n(t_0)|^{1/2} - |V_{N_n,1}(t_0)|^{1/2}\big| > \eta\Big) \\
&\le 4\eta^{-4}(p+1)^3 (v(N_n))^{-1} N_n^{-1+4H(t_0)} \sum_{i\in\nu_{N_n}(t_0)} \sum_{k=0}^{p} a_k^4 \mathbb{E}\,(\mathscr{E}_{i+k,n}^4) \\
&\le c_3\eta^{-4} N_n^{4H(t_0)} n^{-2(1-\beta)} \le c_3\eta^{-4} n^{(4H(t_0)+2)\beta-2}.
\end{aligned} \tag{6.4.5}
$$

Next, using the assumption that $(4H(t_0)+2)\beta < 1$, it follows from (6.4.5) that,

$$
\sum_n \mathbb{P}\Big(\big(v(N_n)N_n^{1-2H(t_0)}\big)^{-1/2}\big||V_n(t_0)|^{1/2} - |V_{N_n,1}(t_0)|^{1/2}\big| > \eta\Big) < \infty;
$$

therefore, Borel-Cantelli's Lemma entails that

$$
\big(v(N_n)N_n^{1-2H(t_0)}\big)^{-1/2}\big||V_n(t_0)|^{1/2} - |V_{N_n,1}(t_0)|^{1/2}\big| \xrightarrow[n\to+\infty]{a.s.} 0. \tag{6.4.6}
$$

On the other hand, one has,

$$\Big|\frac{|V_n(t_0)|^{1/2}}{|V_{N_n,1}(t_0)|^{1/2}}-1\Big| = \frac{\big||V_n(t_0)|^{1/2}-|V_{N_n,1}(t_0)|^{1/2}\big|}{|V_{N_n,1}(t_0)|^{1/2}}$$
$$= \frac{\big||V_n(t_0)|^{1/2}-|V_{N_n,1}(t_0)|^{1/2}\big|}{\big(2\widetilde{C}_a(H(t_0))\big)^{1/2}|f'(X(t_0))|\big(v(N_n)\big)^{1/2}N_n^{1/2-H(t_0)}}$$
$$\times\frac{\big(2\widetilde{C}_a(H(t_0))\big)^{1/2}|f'(X(t_0))|\big(v(N_n)\big)^{1/2}N_n^{1/2-H(t_0)}}{|V_{N_n,1}(t_0)|^{1/2}}. \tag{6.4.7}$$

Finally, putting together, (6.4.7), (6.4.6) and (6.3.90), one obtains

$$\frac{V_n(t_0)}{V_{N_n,1}(t_0)} \xrightarrow[n\to+\infty]{a.s.} 1;$$

then using Remark 6.3.4 and Lemma 6.4.1, one gets the lemma. □

Proposition 6.4.2 *For all integer n, let N_n be the integer (depending on a parameter $\beta\in(0,1)$) which was defined in (6.2.11). Assume that v satisfies condition (ii) in Theorem 6.2.1, then for all $0<\beta\le\frac{1}{2H(t_0)+1}$, the sequence*

$$\left(\frac{V_n(t_0)}{2\widetilde{C}_a(H(t_0))\big(f'(X(t_0))\big)^2 v(N_n)N_n^{1-2H(t_0)}}\right)_n,$$

is bounded in probability. Moreover, when $0<\beta<\frac{1}{2H(t_0)+1}$, then one has,

$$\frac{V_n(t_0)}{2\widetilde{C}_a(H(t_0))\big(f'(X(t_0))\big)^2 v(N_n)N_n^{1-2H(t_0)}} \xrightarrow[n\to+\infty]{\mathbb{P}} 1. \tag{6.4.8}$$

where the symbol "$\xrightarrow{\mathbb{P}} 1$" means that the convergence to 1 holds in probability.

Proof of Proposition 6.4.2: Relation (6.4.5) implies that the sequence of random variables

$$\Big(\big(v(N_n)N_n^{1-2H(t_0)}\big)^{-1/2}\big||V_n(t_0)|^{1/2}-|V_{N_n,1}(t_0)|^{1/2}\big|\Big)_n,$$

is bounded in probability when $\beta\le\frac{1}{2H(t_0)+1}$ and that it converges in probability to 0 when $\beta<\frac{1}{2H(t_0)+1}$. Next, it follows Parts (d) and (e) of Remark 6.3.3, in which one takes

$$U_n=\big(v(N_n)N_n^{1-2H(t_0)}\big)^{-1/2}\big||V_n(t_0)|^{1/2}-|V_{N_n,1}(t_0)|^{1/2}\big|,$$

and

$$V=\Big(\big(2\widetilde{C}_a(H(t_0))\big)^{1/2}|f'(X(t_0))|\Big)^{-1},$$

that the sequence of random variables

$$\left(\frac{\big||V_n(t_0)|^{1/2}-|V_{N_n,1}(t_0)|^{1/2}\big|}{\big(2\widetilde{C}_a(H(t_0))\big)^{1/2}\big|f'(X(t_0))\big|\big(v(N_n)\big)^{1/2}N_n^{1/2-H(t_0)}}\right)_n$$

is bounded in probability when $\beta \le \frac{1}{2H(t_0)+1}$ and that it converges in probability to 0 when $\beta < \frac{1}{2H(t_0)+1}$. On the other hand observe that (6.3.91) and Lemma 6.3.14 (in which one takes $g(x)=x^{-1}$), imply that the sequence of random variables,

$$\left(\frac{\big(2\widetilde{C}_a(H(t_0))\big)^{1/2}\big|f'(X(t_0))\big|\big(v(N_n)\big)^{1/2}N_n^{1/2-H(t_0)}}{|V_{N_n,1}(t_0)|^{1/2}}\right)_n$$

is bounded in probability. Combining the latter two facts with (6.4.7) and Parts (b) and (c) of Remark 6.3.3, it follows that the sequence of random variables,

$$\left(\Big|\frac{|V_n(t_0)|^{1/2}}{|V_{N_n,1}(t_0)|^{1/2}}-1\Big|\right)_n,$$

is bounded in probability when $\beta \le \frac{1}{2H(t_0)+1}$ and that it converges in probability to 0 when $\beta < \frac{1}{2H(t_0)+1}$. This implies that the sequence of random variables,

$$\left(\Big|\frac{|V_n(t_0)|^{1/2}}{|V_{N_n,1}(t_0)|^{1/2}}+1\Big|\right)_n,$$

is bounded in probability when $\beta \le \frac{1}{2H(t_0)+1}$. Next combining the latter two facts with the equality:

$$\Big|\frac{V_n(t_0)}{V_{N_n,1}(t_0)}-1\Big| = \Big|\frac{|V_n(t_0)|^{1/2}}{|V_{N_n,1}(t_0)|^{1/2}}-1\Big| \times \Big|\frac{|V_n(t_0)|^{1/2}}{|V_{N_n,1}(t_0)|^{1/2}}+1\Big|,$$

and with Parts (b) and (c) of Remark 6.3.3, it follows that the sequence of random variables,

$$\left(\frac{V_n(t_0)}{V_{N_n,1}(t_0)}-1\right)_n,$$

is bounded in probability when $\beta \le \frac{1}{2H(t_0)+1}$ and that it converges in probability to 0 when $\beta < \frac{1}{2H(t_0)+1}$. Finally, putting together the latter fact, the equality

$$\begin{aligned}
&\frac{V_n(t_0)}{2\widetilde{C}_a(H(t_0))\big(f'(X(t_0))\big)^2 v(N_n)N_n^{1-2H(t_0)}}-1\\
&=\left(\frac{V_n(t_0)}{V_{N_n,1}(t_0)}-1\right)\times\frac{V_{N_n,1}(t_0)}{2\widetilde{C}_a(H(t_0))\big(f'(X(t_0))\big)^2 v(N_n)N_n^{1-2H(t_0)}}\\
&\qquad\qquad+\frac{V_{N_n,1}(t_0)}{2\widetilde{C}_a(H(t_0))\big(f'(X(t_0))\big)^2 v(N_n)N_n^{1-2H(t_0)}}-1,
\end{aligned}$$

Relation (6.3.91), and Parts (b) and (c) of Remark 6.3.3, one obtains the proposition. □

Now, we are in position to show that Theorem 6.2.1 and Theorem 6.2.2 hold.
Proof of Theorem 6.2.1. First observe that, in view of (6.2.11), one has,

$$\lim_{n\in\mathbb{N},n\to+\infty}\frac{N_n^{1-2H(t_0)}}{N_{2n}^{1-2H(t_0)}}=2^{(2H(t_0)-1)\beta}. \tag{6.4.9}$$

Let $c>0$ be such that,

$$c=\lim_{n\in\mathbb{N},n\to+\infty}\frac{v(N_{2n})}{V(N_n)}. \tag{6.4.10}$$

Observe that the fact that $H(t_0)\in(0,1)$ and $\beta\in(0,1/6]$, imply that $\beta<\frac{1}{4H(t_0)+2}$. Therefore, we are allowed to use Proposition 6.4.1. Putting together, the latter proposition, (6.4.9) and (6.4.10), one gets

$$\frac{V_n(t_0)}{V_{2n}(t_0)}\xrightarrow[n\to+\infty]{a.s.}c^{-1}2^{(2H(t_0)-1)\beta}.$$

The rest of the proof follows the same lines as that of Theorem 6.3.1. □
Proof of Theorem 6.2.2. The proof relies on Proposition 6.4.2 and Lemma 6.3.14; it is similar to that of Theorem 6.2.1. □

6.5 Asymptotic normality of the estimators

The main goal of this section is to show that Theorem 6.2.3 holds, to this end we need several preliminary results. The following proposition provides a Central Limit Theorem for the generalized quadratic variation of mBm, $\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2$.

Proposition 6.5.1 *Assume that v satisfies the following condition:*

(ii") for all integer N big enough, one has,

$$v(N)=o\Big((\log N)^{-2}\Big)\ \textit{and}\ \log N=o\Big(\big(v(N)N\big)^{1/2}\Big).$$

Then, there is a constant $c>0$, such that,

$$c\big(Nv(N)\big)^{1/2}\Big(\frac{\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}{\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{X}_{i,N})^2}-1\Big)\xrightarrow[n\to+\infty]{\mathrm{d}}\mathcal{N}(0,1),$$

where the symbol $\xrightarrow{\mathrm{d}}\mathcal{N}(0,1)$ means: converge in distribution to a standard Gaussian random variable.

The proof of Proposition 6.5.1 relies on the following three lemmas as well as the following remark.

Lemma 6.5.2 *(see [27]) Consider the sequence of random variables* $(S_N)_N$ *defined by*

$$S_N = \sum_{j\in\nu_N(t_0)} \lambda_{j,N}(\varepsilon_{j,N}^2 - 1), \tag{6.5.1}$$

where $\{\varepsilon_{j,N}\}_{j\in\nu_N(t_0)}$ *are i.i.d. centred standard Gaussian random variables and* $\{\lambda_{j,N}\}_{j\in\nu_N(t_0)}$ *is a sequence of positive reals. Let* $\lambda_N = \max_{j\in\nu_N(t_0)} \lambda_{j,N}$*, if* $\lambda_N = o\big((Var(S_N))^{1/2}\big)$*, then*

$$\frac{S_N}{\big(Var(S_N)\big)^{1/2}} \xrightarrow[n\to+\infty]{\mathbb{d}} \mathcal{N}(0,1).$$

Lemma 6.5.3 *(see [46]) For all integer* $q \geq 1$*, let* $C = (C_{ij})_{q\times q}$ *be a positively defined symmetric matrix and* λ *its largest eigenvalue, then*

$$\lambda \leq \max_{1\leq i\leq q} \sum_{j=1}^{q} |C_{ij}|.$$

The following remark is a classical result, this is why we will not give its proof.

Remark 6.5.1 *Let* $(X_n)_n$ *be a sequence of real-valued random variable such that* $X_n \xrightarrow[n\to+\infty]{\mathbb{d}} \mathcal{N}(0,1)$*. Let* $(Y_n)_n$ *be a sequence of real-valued random variable satisfying* $Y_n \xrightarrow[n\to+\infty]{\mathbb{P}} 0$*. Let* $(c_n)_n$ *be a sequence of real numbers such that,* $c_n \xrightarrow[n\to+\infty]{} 1$*. Then, one has,* $c_nX_n + Y_n \xrightarrow[n\to+\infty]{\mathbb{d}} \mathcal{N}(0,1)$*.*

Lemma 6.5.4 *For all integer* $N \geq p+1$*, let us set*

$$\sigma_N^2 = Var\left(\frac{\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}{\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{X}_{i,N})^2} - 1\right) = \frac{Var\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2\Big)}{\Big(\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{X}_{i,N})^2\Big)^2}. \tag{6.5.2}$$

Then, assuming that, v *satisfies condition* (ii'') *in Proposition 6.5.1, it follows that,*

$$Nv(N)\sigma_N^2 \xrightarrow[n\to+\infty]{} d^2, \tag{6.5.3}$$

where d *is a strictly positive real number.*

Proof of Lemma 6.5.4. First, observe that using (6.5.2), one has,

$$\begin{aligned}\sigma_N^2 &= \frac{\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2\Big)}{\Big(\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{X}_{i,N})^2\Big)^2}\\
&= \left(\frac{\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{B}_{i,N}^{H(t_0)})^2\Big)}{\Big(\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{B}_{i,N}^{H(t_0)})^2\Big)^2}\right.\\
&\qquad\left.+\frac{\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2\Big)-\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{B}_{i,N}^{H(t_0)})^2\Big)}{\Big(\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{B}_{i,N}^{H(t_0)})^2\Big)^2}\right)\\
&\quad\times\left(\frac{\Big(\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{B}_{i,N}^{H(t_0)})^2\Big)^2}{\Big(\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{X}_{i,N})^2\Big)^2}\right).\end{aligned}\tag{6.5.4}$$

Also observe that, similarly to (3.4.3), one can show that there is a constant $c_1>0$ such that

$$\frac{\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{B}_{i,N}^{H(t_0)})^2\Big)}{(Nv(N))^{-1}\Big(\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{B}_{i,N}^{H(t_0)})^2\Big)^2}\xrightarrow[n\to+\infty]{}c_1.\tag{6.5.5}$$

Moreover (6.3.28) and (6.3.23) imply that,

$$\frac{\Big(\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{B}_{i,N}^{H(t_0)})^2\Big)^2}{\Big(\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{X}_{i,N})^2\Big)^2}\xrightarrow[n\to+\infty]{}1.\tag{6.5.6}$$

Thus, in view of (6.5.4), (6.5.5) and (6.5.6), in order to show that (6.5.2) holds, it is sufficient to prove that,

$$\frac{\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2\Big)-\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{B}_{i,N}^{H(t_0)})^2\Big)}{(Nv(N))^{-1}\Big(\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{B}_{i,N}^{H(t_0)})^2\Big)^2}\xrightarrow[N\to+\infty]{}0.\tag{6.5.7}$$

Similarly to (6.3.12), one has that,

$$\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{B}_{i,N}^{H(t_0)})^2\Big)=2\sum_{i,j\in\nu_N(t_0)}\Big(\mathrm{Cov}(\Delta_a\overline{B}_{i,N}^{H(t_0)},\Delta_a\overline{B}_{j,N}^{H(t_0)})\Big)^2.\tag{6.5.8}$$

Using (6.3.12), (6.5.8) and the triangle inequality, it follows that,

$$\begin{aligned}&\Big|\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2\Big)-\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{B}_{i,N}^{H(t_0)})^2\Big)\Big|\\
&\le 2\sum_{i,j\in\nu_N(t_0)}\Big|\mathrm{Cov}\,(\Delta_a\overline{X}_{i,N},\Delta_a\overline{X}_{j,N})-\mathrm{Cov}\Big(\Delta_a\overline{B}_{i,N}^{H(t_0)},\Delta_a\overline{B}_{j,N}^{H(t_0)}\Big)\Big|\\
&\qquad\times\Big(\Big|\mathrm{Cov}\,(\Delta_a\overline{X}_{i,N},\Delta_a\overline{X}_{j,N})\Big|+\Big|\mathrm{Cov}\Big(\Delta_a\overline{B}_{i,N}^{H(t_0)},\Delta_a\overline{B}_{j,N}^{H(t_0)}\Big)\Big|\Big).\end{aligned}\tag{6.5.9}$$

Next, observe that, the triangle inequality, Cauchy-Schwarz inequality, (6.3.28), (6.3.59) and (6.3.23), imply that, for all $N \geq p+1$, and $i, j \in \nu_N(t_0)$, one has,

$$\begin{aligned}
&\Big|\mathrm{Cov}\left(\Delta_a \overline{X}_{i,N}, \Delta_a \overline{X}_{j,N}\right) - \mathrm{Cov}\Big(\Delta_a \overline{B}^{H(t_0)}_{i,N}, \Delta_a \overline{B}^{H(t_0)}_{j,N}\Big)\Big| \\
&\leq \Big|\mathrm{Cov}\Big(\Delta_a \overline{X}_{i,N} - \Delta_a \overline{B}^{H(t_0)}_{i,N}, \Delta_a \overline{X}_{j,N}\Big)\Big| + \Big|\mathrm{Cov}\Big(\Delta_a \overline{B}^{H(t_0)}_{i,N}, \Delta_a \overline{X}_{j,N} - \Delta_a \overline{B}^{H(t_0)}_{j,N}\Big)\Big| \\
&\leq \big\|\Delta_a \overline{X}_{i,N} - \Delta_a \overline{B}^{H(t_0)}_{i,N}\big\|_{L^2}\big\|\Delta_a \overline{X}_{j,N}\big\|_{L^2} + \big\|\Delta_a \overline{B}^{H(t_0)}_{i,N}\big\|_{L^2}\big\|\Delta_a \overline{X}_{j,N} - \Delta_a \overline{B}^{H(t_0)}_{j,N}\big\|_{L^2} \\
&\leq c_2\big(v(N)\log N\big)N^{-2H(t_0)}, \qquad (6.5.10)
\end{aligned}$$

where c_2 is a constant non depending on N, i and j. On the other hand, Lemma 6.3.7 and (6.3.15) entail that for all $N \geq p+1$ and all fixed $i \in \nu_N(t_0)$, one has

$$\begin{aligned}
&\sum_{j\in\nu_N(t_0)} \big|\mathrm{Cov}\left(\Delta_a \overline{X}_{i,N}, \Delta_a \overline{X}_{j,N}\right)\big| \leq c_3 N^{-2H(t_0)} \sum_{j\in\nu_N(t_0)} \big(1+|i-j|\big)^{-1} \\
&\leq 2c_3 N^{-2H(t_0)} \sum_{l=1}^{N-p} l^{-1} \\
&\leq c_4 N^{-2H(t_0)}\big(\log N\big), \qquad (6.5.11)
\end{aligned}$$

where c_3 and c_4 are two constants non depending on N and i. Moreover, similarly to (6.5.11), one can show that, for all $N \geq p+1$ and all fixed $i \in \nu_N(t_0)$, one has

$$\sum_{j\in\nu_N(t_0)} \Big|\mathrm{Cov}\Big(\Delta_a \overline{B}^{H(t_0)}_{i,N}, \Delta_a \overline{B}^{H(t_0)}_{j,N}\Big)\Big| \leq c_4 N^{-2H(t_0)}\big(\log N\big), \qquad (6.5.12)$$

where c_5 is a constant non depending on N and i. Next putting together, (6.2.4), (6.2.5) and (6.5.9) to (6.5.12), one obtains that, for all $N \geq p+1$,

$$\Big|\mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)} (\Delta_a \overline{X}_{i,N})^2\Big) - \mathrm{Var}\Big(\sum_{i\in\nu_N(t_0)} (\Delta_a \overline{B}^{H(t_0)}_{i,N})^2\Big)\Big| \leq c_6\big(v(N)\log N\big)^2 N^{1-4H(t_0)}, \qquad (6.5.13)$$

where c_6 is a constant non depending N. On the other hand, (6.2.4), (6.2.5) and (6.3.23) imply that, for all $N \geq p+1$,

$$(Nv(N))^{-1}\Big(\sum_{i\in\nu_N(t_0)} \mathbb{E}\,(\Delta_a \overline{B}^{H(t_0)}_{i,N})^2\Big)^2 \geq c_7 v(N) N^{1-4H(t_0)}, \qquad (6.5.14)$$

where $c_7 > 0$ is a constant non depending on N. Finally, putting together, (6.5.13), (6.5.14) and condition (ii'') in Proposition 6.5.1, one gets (6.5.7). □

Now we are in position to prove Proposition 6.5.1.

Proof of Proposition 6.5.1. First observe that, in view of (6.3.11),

$$S_N := \frac{\sum_{i\in\nu_N(t_0)} (\Delta_a \overline{X}_{i,N})^2}{\sum_{i\in\nu_N(t_0)} \mathbb{E}\,(\Delta_a \overline{X}_{i,N})^2} - 1, \qquad (6.5.15)$$

can be expressed in the from (6.5.1), where, for all j,

$$\lambda_{j,N} = \frac{\widetilde{\lambda}_{j,N}}{\sum_{i\in\nu_N(t_0)} \mathbb{E}\,(\Delta_a \overline{X}_{i,N})^2}, \tag{6.5.16}$$

the $\widetilde{\lambda}_{j,N}$'s being the eigenvalues of the covariance matrix M of the centered Gaussian vector $\big(\Delta_a \overline{X}_{i,N}\big)_{i\in\nu_N(t_0)}$. We denote by $\widetilde{\lambda}_N$ the biggest eigenvalue, then it follows from Lemma 6.5.3 and (6.5.11), that,

$$\widetilde{\lambda}_N \le \max_{i\in\nu_N(t_0)} \sum_{k\in\nu_N(t_0)} \big|\mathrm{Cov}(\Delta_a \overline{X}_{i,N}, \Delta_a \overline{X}_{i,N})\big| = \mathcal{O}\big(N^{-2H(t_0)}\log N\big). \tag{6.5.17}$$

Thus, denoting λ_N the biggest of the $\lambda_{N,j}$'s, one has, in view of (6.5.16), (6.5.17) and (6.3.59),

$$\lambda_N = \mathcal{O}\big((Nv(N))^{-1}\log N\big). \tag{6.5.18}$$

On the other hand, Lemma 6.5.4 and (6.5.15) imply that

$$N^{-1/2}(v(N))^{-1/2} = \mathcal{O}\big((\mathrm{Var}(S_N))^{1/2}\big). \tag{6.5.19}$$

Relations (6.5.18) and (6.5.19) imply that,

$$\lambda_N = \mathcal{O}\big((\mathrm{Var}(S_N))^{1/2}(Nv(N))^{-1/2}\log N\big). \tag{6.5.20}$$

Finally, putting together (6.5.20) and condition (ii'') in Proposition 6.5.1, it follows that

$$\lambda_N = o\big(\mathrm{Var}(S_N))^{1/2}\big);$$

therefore Lemma 6.5.2, Lemma 6.5.4 and Remark 6.5.1 entail that the proposition holds. □

Let us now give two important consequences of Proposition 6.5.1.

Proposition 6.5.5 *Assume that v satisfies the following condition:*

(ii"') for all integer N big enough, one has,

$$v(N) = o\Big(N^{-1/3}(\log N)^{-2/3}\Big) \text{ and } \log N = o\Big(\big(v(N)N\big)^{1/2}\Big).$$

Then, there is a constant $c > 0$, such that,

$$c\big(Nv(N)\big)^{1/2}\Big(\frac{\sum_{i\in\nu_N(t_0)}(\Delta_a \overline{X}_{i,N})^2}{2\widetilde{C}_a(H(t_0))v(N)N^{1-2H(t_0)}} - 1\Big) \xrightarrow[n\to+\infty]{\mathrm{d}} \mathcal{N}(0,1),$$

Proof of Proposition 6.5.5. For all $N \ge p+1$, let us set,

$$X_N = c\big(Nv(N)\big)^{1/2}\Big(\frac{\sum_{i\in\nu_N(t_0)}(\Delta_a \overline{X}_{i,N})^2}{\sum_{i\in\nu_N(t_0)} \mathbb{E}\,(\Delta_a \overline{X}_{i,N})^2} - 1\Big)$$

and

$$Y_N = c\big(Nv(N)\big)^{1/2}\Big(\frac{\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{X}_{i,N})^2}{2\widetilde{C}_a(H(t_0))v(N)N^{1-2H(t_0)}}-1\Big)\frac{\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}{\sum_{i\in\nu_N(t_0)}\mathbb{E}\,(\Delta_a\overline{X}_{i,N})^2},$$

where c is the constant introduced in Proposition 6.5.1. Then, one has,

$$c\big(Nv(N)\big)^{1/2}\Big(\frac{\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}{2\widetilde{C}_a(H(t_0))v(N)N^{1-2H(t_0)}}-1\Big)=X_N+Y_N;$$

Observe that, Proposition 6.5.1 implies that,

$$X_N \xrightarrow[n\to+\infty]{\mathrm{d}} \mathcal{N}(0,1).$$

Also observe that, one has in view of Lemma 6.3.9, (6.3.61), (6.3.23), (6.2.4), (6.2.5) and condition (ii''') in Proposition 6.5.5, that,

$$Y_N \xrightarrow[n\to+\infty]{\mathbb{P}} 0.$$

Therefore Proposition 6.5.5 results from Remark 6.5.1. □

Proposition 6.5.6 *Assume that $H(t_0)\in(1/2,1)$ and that v satisfies condition (iv) in Theorem 6.2.3. Then there is a constant $c>0$, such that,*

$$c\big(Nv(N)\big)^{1/2}\Big(\frac{V_{N,1}(t_0)}{2\widetilde{C}_a(H(t_0))|f'(X(t_0))|^2v(N)N^{1-2H(t_0)}}-1\Big)\xrightarrow[n\to+\infty]{\mathrm{d}}\mathcal{N}(0,1);$$

recall that $V_{N,1}(t_0)$ has been introduced in (6.3.1).

Proof of Proposition 6.5.6. Let c be the constant introduced in Proposition 6.5.5. Observe that one has,

$$\begin{aligned}
&c\big(Nv(N)\big)^{1/2}\Big(\frac{V_{N,1}(t_0)}{2\widetilde{C}_a(H(t_0))|f'(X(t_0))|^2v(N)N^{1-2H(t_0)}}-1\Big)\\
&=c\big(Nv(N)\big)^{1/2}\Big(\frac{\sum_{i\in\nu_N(t_0)}(\Delta_a\overline{X}_{i,N})^2}{2\widetilde{C}_a(H(t_0))v(N)N^{1-2H(t_0)}}-1\Big)\\
&\quad+cN^{-(1-4H(t_0))/2}v(N)^{-1/2}\frac{\big(V_{N,3}(t_0)-V_{N,2}(t_0)\big)}{2\widetilde{C}_a(H(t_0))|f'(X(t_0))|^2}\\
&\quad+cN^{-(1-4H(t_0))/2}v(N)^{-1/2}\frac{\big(V_{N,1}(t_0)-V_{N,3}(t_0)\big)}{2\widetilde{C}_a(H(t_0))|f'(X(t_0))|^2}
\end{aligned}\tag{6.5.21}$$

Recall that $V_{N,2}(t_0)$ and $V_{N,3}(t_0)$ have respectively been introduced in (6.3.66) and (6.3.67). In view of (6.5.21), Proposition 6.5.5 and Remark 6.5.1, in order to prove that Proposition 6.5.5 holds, it is sufficient to show that,

$$N^{-(1-4H(t_0))/2}v(N)^{-1/2}\big|V_{N,2}(t_0)-V_{N,3}(t_0)\big|\xrightarrow[n\to+\infty]{\mathbb{P}}0\tag{6.5.22}$$

and

$$N^{-(1-4H(t_0))/2} v(N)^{-1/2} \big| V_{N,3}(t_0) - V_{N,1}(t_0) \big| \xrightarrow[n \to +\infty]{\mathbb{P}} 0. \tag{6.5.23}$$

In the sequel ε denotes a fixed strictly positive real number such that,

$$H(t_0) - \varepsilon > 1/2; \tag{6.5.24}$$

observe that such an ε exists since the function $H(t_0) \in (1/2, 1)$. Let us now prove that (6.5.22) is satisfied. By using (6.3.68), (6.3.66) and Proposition 6.3.4, we get,

$$\begin{aligned}
& N^{-(1-4H(t_0))/2} v(N)^{-1/2} \big| V_{N,2}(t_0) - V_{N,3}(t_0) \big| \\
&= N^{-(1-4H(t_0))/2} v(N)^{-1/2} \left| \frac{V_{N,3}(t_0)}{V_{N,2}(t_0)} - 1 \right| \big| V_{N,2}(t_0) \big| \\
&= \mathcal{O}_{\mathbb{P}} \Big(N^{(-1+4H(t_0))/2} v(N)^{-1/2} v(N)^{H(t_0)-\varepsilon} v(N) N^{1-2H(t_0)} \Big) \\
&= \mathcal{O}_{\mathbb{P}} \Big(N^{1/2} v(N)^{H(t_0)+1/2-\varepsilon} \Big)
\end{aligned} \tag{6.5.25}$$

Next combining (6.5.25) with (6.5.24) and condition (iv) in Theorem 6.2.3, one obtains (6.5.22).

Let us now prove that (6.5.23) is satisfied. It follows from the triangle inequality, (6.5.25), (6.3.88), (6.3.68), (6.3.66), and Proposition 6.3.4, that

$$\begin{aligned}
& N^{-(1-4H(t_0))/2} v(N)^{-1/2} \big| V_{N,3}(t_0) - V_{N,1}(t_0) \big| \\
&= N^{-(1-4H(t_0))/2} v(N)^{-1/2} \big| V_{N,3}(t_0) \big| \left| \frac{V_{N,1}(t_0)}{V_{N,3}(t_0)} - 1 \right| \\
&\leq N^{-(1-4H(t_0))/2} v(N)^{-1/2} \Bigg(\big| V_{N,3}(t_0) - V_{N,2}(t_0) \big| \left| \frac{V_{N,1}(t_0)}{V_{N,3}(t_0)} - 1 \right| \\
&\qquad\qquad + \big| V_{N,2}(t_0) \big| \left| \frac{V_{N,1}(t_0)}{V_{N,3}(t_0)} - 1 \right| \Bigg) \\
&= \mathcal{O}_{\mathbb{P}} \Big(N^{1/2} v(N)^{H(t_0)+1/2-\varepsilon} N^{-H(t_0)+\varepsilon} \\
&\qquad\qquad + N^{-(1-4H(t_0))/2} v(N)^{-1/2} v(N) N^{1-2H(t_0)} N^{-H(t_0)+\varepsilon} \Big) \\
&= \mathcal{O}_{\mathbb{P}} \Big(N^{1/2-H(t_0)+\varepsilon} v(N)^{1/2} \Big).
\end{aligned} \tag{6.5.26}$$

Finally (6.5.23) results from (6.5.24) and (6.5.26). □

Lemma 6.5.7 *Assume that v satisfies condition (ii) in Theorem 6.2.1. Also, assume that for every integer n, the integer N_n is of the form (6.2.11), where the parameter β is such that,*

$$0 < \beta \leq \frac{1}{4}. \tag{6.5.27}$$

Then, one has,

$$\big(N_n v(N_n) \big)^{1/2} \Big(\frac{\big| V_n(t_0) - V_{N_n,1}(t_0) \big|}{2 \widetilde{C}_a(H(t_0)) |f'(X(t_0))|^2 v(N_n) N_n^{1-2H(t_0)}} \Big) = \mathcal{O}_{\mathbb{P}} \Big(\big(v(N_n) \big)^{1/2} \Big), \tag{6.5.28}$$

Notice that $V_n(t_0)$ has been introduced in (6.2.12) and $V_{N_n,1}(t_0)$ is defined by the formula (6.3.1), in which N has been replaced by N_n.

Proof of Lemma 6.5.7. First observe that similarly to (6.4.5), there is a constant $c_1 > 0$, non depending on n and η, such that one has for all real $\eta > 0$ and for each integer n big enough,

$$\mathbb{P}\Big(n^{1/8}\big(v(N_n)N_n^{1-2H(t_0)}\big)^{-1/2}\big||V_n(t_0)|^{1/2} - |V_{N_n,1}(t_0)|^{1/2}\big| > \eta\Big) \\ \leq c_1\eta^{-4}n^{(4H(t_0)+2)\beta-3/2}. \tag{6.5.29}$$

Next, (6.2.11), (6.5.29) and (6.5.27) imply that,

$$\big(N_n v(N_n)\big)^{1/2}\Big(\frac{\big|\big(V_n(t_0)\big)^{1/2} - \big(V_{N_n,1}(t_0)\big)^{1/2}\big|}{2\widetilde{C}_a(H(t_0))|f'(X(t_0))|^2 v(N_n)N_n^{1-2H(t_0)}}\Big) \\ = \mathcal{O}_{\mathbb{P}}\left(N_n^{H(t_0)-\frac{1}{8\beta}}\right). \tag{6.5.30}$$

On the other hand, Proposition 6.4.2 implies that,

$$\big(V_n(t_0)\big)^{1/2} = \mathcal{O}_{\mathbb{P}}\left(\big(v(N_n)\big)^{1/2}N_n^{1/2-H(t_0)}\right), \tag{6.5.31}$$

and Remark 6.3.5 entails that,

$$\big(V_{N_n,1}(t_0)\big)^{1/2} = \mathcal{O}_{\mathbb{P}}\left(\big(v(N_n)\big)^{1/2}N_n^{1/2-H(t_0)}\right). \tag{6.5.32}$$

Finally, putting together (6.5.30), (6.5.31), (6.5.32), and the equality

$$\big|V_n(t_0) - V_{N_n,1}(t_0)\big| = \Big|\big(V_n(t_0)\big)^{1/2} - \big(V_{N_n,1}(t_0)\big)^{1/2}\Big|\Big|\big(V_n(t_0)\big)^{1/2} + \big(V_{N_n,1}(t_0)\big)^{1/2}\Big|,$$

one obtains (6.5.28). □

Proposition 6.5.8 *Assume $H(t_0) \in (1/2, 1)$ and that v satisfies condition (iv) in Theorem 6.2.3. Also, assume that for every integer n, the integer N_n is of the form (6.2.11), where the parameter β is such that, $0 < \beta \leq \frac{1}{4}$. Then there is a constant $c > 0$, such that,*

$$c\big(N_n v(N_n)\big)^{1/2}\Big(\frac{V_n(t_0)}{2\widetilde{C}_a(H(t_0))|f'(X(t_0))|^2 v(N_n)N_n^{1-2H(t_0)}} - 1\Big) \xrightarrow[n\to+\infty]{\mathbb{d}} \mathcal{N}(0,1).$$

Notice that $V_n(t_0)$ has been introduced in (6.2.12)

Proof of Proposition 6.5.8. The proposition easily follows from Proposition 6.5.6, Remark 6.5.1 and Lemma 6.5.7. □

Now, we are in position to prove Theorem 6.2.3.

Proof of Theorem 6.2.3. By using Proposition 6.5.8 and the δ-method one can get the theorem. □

CHAPTER 7

Appendix

7.1 Proof of Lemma 5.3.4

Lemma 7.1.1 *Let (Z, Z') be a 2-D centered Gaussian random vector and assume that the variances of Z and Z' are both equal to the same quantity denoted by v. Then, one has,*

$$\mathbb{E}\Big\{(Z^2 - v)(Z'^2 - v)\Big\} = 2\Big(Cov(Z, Z')\Big)^2. \tag{7.1.1}$$

Proof: For the sake of simplicity we denote by $u = \mathrm{Corr}(Z, Z')$, then M the correlation of the centered random vector (Z, Z') can be expressed as, $M = \begin{pmatrix} 1 & u \\ u & 1 \end{pmatrix}$. Let us study the following two cases: $\det(M) = 0$ and $\det(M) \neq 0$.

Case 1: $\det(M) = 0$ i.e Z and Z' are colinear. Observe that in this case one has $u = \pm 1$. Then the latter equality and the fact that $\mathrm{Var}(Z) = \mathrm{Var}(Z') = v$, imply that,

$$Z^2 = Z'^2. \tag{7.1.2}$$

It follows from (7.1.2) and the equality $\mathbb{E}\,(Z/\sqrt{v})^4 = 3$ that

$$\begin{aligned} E\big\{(Z^2 - v)(Z'^2 - v)\big\} &= E\big\{(Z^2 - v)^2\big\} \\ &= E\{Z^4 - 2Z^2 v + v^2\} \\ &= E\{Z^4\} - 2E\{Z^2\}v + v^2 \\ &= 3v^2 - 2v^2 + v^2 \\ &= 2v^2. \end{aligned} \tag{7.1.3}$$

On the other hand (7.1.2), implies,

$$2\big(E\{ZZ'\}\big)^2 = 2\big(E\{Z^2\}\big)^2 = 2v^2. \tag{7.1.4}$$

Thus combining (7.1.3) with (7.1.4), one can show that the lemma holds in this case.

Case 2: $\det(M) \neq 0$. Standard computations show that

$$M = P \begin{pmatrix} 1+u & 0 \\ 0 & 1-u \end{pmatrix} P, \tag{7.1.5}$$

where $P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. We set

$$A = P \begin{pmatrix} \sqrt{1+u} & 0 \\ 0 & \sqrt{1-u} \end{pmatrix}. \tag{7.1.6}$$

Let ε and ε' be two independent standard Gaussian variables, then the centered Gaussian random vector

$$A\begin{pmatrix}\varepsilon\\ \varepsilon'\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}(1+u)^{1/2}\varepsilon+(1-u)^{1/2}\varepsilon'\\ (1+u)^{1/2}\varepsilon-(1-u)^{1/2}\varepsilon'\end{pmatrix}$$

has the same distribution as centered Gaussian random vector $\left(\frac{Z}{\sqrt{v}},\frac{Z'}{\sqrt{v}}\right)$. Therefore,

$$\begin{aligned}
&E\big\{(ZZ')^2\big\}\\
&=\frac{v^2}{4}E\Big\{\Big((1+u)^{1/2}\varepsilon+(1-u)^{1/2}\varepsilon'\Big)^2\Big((1+u)^{1/2}\varepsilon-(1-u)^{1/2}\varepsilon'\Big)^2\Big\}\\
&=\frac{v^2}{4}E\Big\{\Big((1+u)\varepsilon^2-(1-u)\varepsilon'^2\Big)^2\Big\}\\
&=\frac{v^2}{4}E\big\{(1+u)^2\varepsilon^4-2(1-u^2)\varepsilon^2\varepsilon'^2+(1-u)^2\varepsilon'^4\big\}\\
&=\frac{v^2}{4}\left((1+u)^2E\{\varepsilon^4\}-2(1-u^2)E\{\varepsilon^2\varepsilon'^2\}+(1-u)^2E\{\varepsilon'^4\}\right)\\
&=(2u^2+1)v^2 \qquad (7.1.7)
\end{aligned}$$

and

$$\begin{aligned}
&2\Big(E\big\{(ZZ')\big\}\Big)^2\\
&=2\cdot\frac{v^2}{4}\bigg(E\Big\{\Big((1+u)^{1/2}\varepsilon+(1-u)^{1/2}\varepsilon'\Big)\Big((1+u)^{1/2}\varepsilon-(1-u)^{1/2}\varepsilon'\Big)\Big\}\bigg)^2\\
&=\frac{v^2}{2}\Big(E\big\{(1+u)\varepsilon^2-(1-u)\varepsilon'^2\big\}\Big)^2\\
&=\frac{v^2}{2}\Big((1+u)E\{\varepsilon^2\}-(1-u)E\{\varepsilon'^2\}\Big)^2\\
&=\frac{v^2}{2}\Big((1+u)-(1-u)\Big)^2\\
&=2u^2v^2 \qquad (7.1.8)
\end{aligned}$$

Observe that (7.1.7) entails that,

$$E\big\{(Z^2-v)(Z'^2-v)\big\}=E\big\{(ZZ')^2-Z^2v-Z'^2v+v^2\big\}=2u^2v^2. \qquad (7.1.9)$$

It follows from (7.1.9) and (7.1.8) that

$$E\big\{(Z^2-v)(Z'^2-v)\big\}=2\big(E\{ZZ'\}\big)^2,$$

which proves that the lemma also holds in this case. □

7.2 Proof of Proposition 5.3.6

Proposition 7.2.1 *Assume that $H(\cdot) \in C^2([0,1])$ and $M(a) \geq 3$ (recall that $M(a)$ is the number of the vanishing moments of the sequence a). Then there is a constant $c > 0$ such that one has for any integer $N \geq p+1$ and each $j, j' \in \{0, \ldots, N-p-1\}$,*

$$\left| Corr\left\{\Delta_a \overline{X}_{j,N}, \Delta_a \overline{X}_{j',N}\right\}\right| \leq c \left(\frac{1}{1+|j-j'|}\right),$$

where

$$Corr\left\{\Delta_a \overline{X}_{j,N}, \Delta_a \overline{X}_{j',N}\right\} := \mathbb{E}\left\{\frac{\Delta_a \overline{X}_{j,N}}{\sqrt{Var\left\{\Delta_a \overline{X}_{j,N}\right\}}}\frac{\Delta_a \overline{X}_{j',N}}{\sqrt{Var\left\{\Delta_a \overline{X}_{j',N}\right\}}}\right\}.$$

This proposition is a straightforward consequence of Lemmas 7.2.2 and 7.2.3, given below.

Lemma 7.2.2 *There are two constants $0 < c_1 \leq c_2$ such that one has for any integer $N \geq p+1$ and $j \in \{0, \ldots, N-p-1\}$,*

$$c_1 N^{-2H(j/N)} \leq Var\left\{\Delta_a \overline{X}_{j,N}\right\} \leq c_2 N^{-2H(j/N)}. \tag{7.2.1}$$

Proof of Lemma 7.2.2 : The lemma can be obtained by using Lemma 7.3.1 given in the next section. Indeed, Lemma 7.3.1 can be equivalently expressed as follows: there exists a constant $c > 0$ such that for all $N \geq p+1$ and all $j \in \{0, \ldots, N-p-1\}$, one has

$$\mathrm{Var}\left\{\Delta_a \overline{X}_{j,N}\right\} \begin{array}{l} \geq -c\log(N)N^{-2H(j/N)-1} + C(j/N)N^{-2H(j/N)} \\ \leq c\log(N)N^{-2H(j/N)-1} + C(j/N)N^{-2H(j/N)}. \end{array}$$

By using the fact that for all $s \in [0,1]$,

$$C(s) = \int_{\mathbb{R}} \frac{|e^{i\eta}-1|^2|\sum_{k=0}^p a_k e^{ik\eta}|^2}{|\eta|^{2H(s)+3}}\,\mathrm{d}\eta,$$

we get for any real $s \in [0,1]$,

$$C(s) \geq c_1 = \int_{|\eta|\leq 1} \frac{|e^{i\eta}-1||\sum_{k=0}^p a_k e^{ik\eta}|}{|\eta|^{2\min_{s\in[0,1]} H(s)+3}}\,\mathrm{d}\eta + \int_{|\eta|>1} \frac{|e^{i\eta}-1||\sum_{k=0}^p a_k e^{ik\eta}|}{|\eta|^{2\max_{s\in[0,1]} H(s)+3}}\,\mathrm{d}\eta > 0,$$

and

$$C(s) \leq c_2 = \int_{|\eta|\leq 1} \frac{|e^{i\eta}-1||\sum_{k=0}^p a_k e^{ik\eta}|}{|\eta|^{2\max_{s\in[0,1]} H(s)+3}}\,\mathrm{d}\eta + \int_{|\eta|>1} \frac{|e^{i\eta}-1||\sum_{k=0}^p a_k e^{ik\eta}|}{|\eta|^{2\min_{s\in[0,1]} H(s)+3}}\,\mathrm{d}\eta > 0.$$

it yields

$$(c_1 - c\log(N)/N)N^{-2H(j/N)} \leq \mathrm{Var}\left\{\Delta_a \overline{X}_{j,N}\right\} \leq (c_2 + c\log(N)/N)N^{-2H(j/N)}.$$

Finally (7.2.1), easily follows from the latter two inequalities since $\lim_{N\to+\infty} \log(N)/N = 0$. $\square$

Lemma 7.2.3 *Under the same conditions as in Proposition 7.2.1, there is a constant $c > 0$ such that one has for any integer $N \geq p+1$ and each $j, j' \in \{0, \ldots, N-p-1\}$,*

$$\left| \mathbb{E}\left\{ \Delta_a \overline{X}_{j,N} \Delta_a \overline{X}_{j',N} \right\} \right| \leq \frac{cN^{-H(j/N)-H(j'/N)}}{1+|j-j'|}. \tag{7.2.2}$$

Proof of Lemma 7.2.3 : In order to conveniently express the quantity $\mathbb{E}\left\{ \Delta_a \overline{X}_{j,N} \Delta_a \overline{X}_{j',N} \right\}$ let us introduce the function g defined for all reals $\xi \neq 0$ and $x \in [0,1]$ as

$$g(x,\xi) := |\xi|^{-H(x)-1/2}. \tag{7.2.3}$$

Recall that for all $j \in \{0, \ldots, N-p-1\}$,

$$\Delta_a \overline{X}_{j,N} = N \sum_{k=0}^{p} a_k \int_{j/N}^{(j+1)/N} X(s+k/N)\, \mathrm{d}s,$$

and for any $s, s' \in [0,1]$, the covariance between $X(s)$ and $X(s')$ can be expressed as

$$\mathbb{E}\left(X(s)X(s') \right) = \int_{\mathbb{R}} \frac{(e^{is\xi}-1)(e^{-is'\xi}-1)}{|\xi|^{H(s)+H(s')+1}}\, \mathrm{d}\xi.$$

Then it follows from Fubini Theorem that

$$\begin{aligned}
&\mathbb{E}\left\{ \Delta_a \overline{X}_{j,N} \Delta_a \overline{X}_{j',N} \right\} \\
&= \mathbb{E}\left\{ N^2 \sum_{k=0}^{p} \sum_{k'=0}^{p} a_k a_{k'} \int_{\frac{j}{N}}^{\frac{j+1}{N}} \int_{\frac{j'}{N}}^{\frac{j'+1}{N}} X(s+k/N) X(s'+k'/N)\, \mathrm{d}s'\, \mathrm{d}s \right\} \\
&= N^2 \int_{\frac{j}{N}}^{\frac{j+1}{N}} \int_{\frac{j'}{N}}^{\frac{j'+1}{N}} \mathbb{E}\left\{ \Big(\sum_{k=0}^{p} a_k X(s+k/N) \Big) \Big(\sum_{k'=0}^{p} a_{k'} X(s'+k'/N) \Big)\, \mathrm{d}s'\, \mathrm{d}s \right\} \\
&= N^2 \int_{\frac{j}{N}}^{\frac{j+1}{N}} \int_{\frac{j'}{N}}^{\frac{j'+1}{N}} \int_{\mathbb{R}} \Big(\sum_{k=0}^{p} a_k (e^{i(s+k/N)\xi}-1) g(s+k/N,\xi) \Big) \\
&\quad \times \Big(\sum_{k'=0}^{p} a_{k'} (e^{-i(s'+k'/N)\xi}-1) g(s'+k'/N,\xi) \Big)\, \mathrm{d}\xi\, \mathrm{d}s'\, \mathrm{d}s.
\end{aligned} \tag{7.2.4}$$

Now let us fix $\xi \in \mathbb{R} \setminus \{0\}$ and $s \in \left[\frac{j+k}{N}, \frac{j+k+1}{N} \right]$. By applying the Taylor formula of order 2 with integral remainder to the function $x \mapsto g(x,\xi)$ around j/N, one gets that

$$g(s,\xi) = A_0 + (s-j/N)A_1 + (s-j/N)^2 A_2 \tag{7.2.5}$$

where

$$A_0 := A_0(j/N,\xi) = g(j/N,\xi) = |\xi|^{-H(j/N)-1/2}, \tag{7.2.6}$$

$$A_1 := A_1(j/N,\xi) = \partial_x g(j/N,\xi) = -H'(j/N)|\xi|^{-H(j/N)-1/2} \log|\xi| \tag{7.2.7}$$

and

$$A_2 := A_2(j/N, s, \xi) = \int_0^1 (1-\theta)\partial_x^2 g(j/N + \theta(s - j/N), \xi)\, \mathrm{d}\theta. \tag{7.2.8}$$

Using the same argument as before, one has for any fixed $\xi \in \mathbb{R} \setminus \{0\}$ and $s' \in \left[\frac{j'+k'}{N}, \frac{j'+k'+1}{N}\right]$,

$$g(s', \xi) = A'_0 + (s' - j'/N)A'_1 + (s' - j'/N)^2 A'_2 \tag{7.2.9}$$

where A'_0, A'_1 and A'_2 are defined similarly to A_0, A_1 and A_2 but by replacing j/N by j'/N and s by s'. It follows from (7.2.4), (7.2.5) and (7.2.9) that

$$\mathbb{E}\left\{\Delta_a \overline{X}_{j,N} \Delta_a \overline{X}_{j',N}\right\} = \sum_{0 \le l', l \le 2} \mathcal{I}_{l,l'}(j, j', N) \tag{7.2.10}$$

where for any $l, l' \in \{0, 1, 2\}$,

$$\begin{aligned} \mathcal{I}_{l,l'}(j,j',N) \;=\; & N^2 \int_{\frac{j}{N}}^{\frac{j+1}{N}} \int_{\frac{j'}{N}}^{\frac{j'+1}{N}} \int_{\mathbb{R}} \Big(\sum_{k=0}^{p} a_k (e^{i(s+k/N)\xi} - 1)(s + k/N - j/N)^l A_l \Big) \\ & \times \Big(\sum_{k'=0}^{p} a_{k'} (e^{-i(s'+k'/N)\xi} - 1)(s' + k'/N - j'/N)^{l'} A'_{l'} \Big)\, \mathrm{d}\xi\, \mathrm{d}s'\, \mathrm{d}s. \end{aligned} \tag{7.2.11}$$

Then Lemma 7.2.3 can easily be obtained by using (7.2.10) and Lemma 7.2.4 below, which allows to conveniently bound the $I_{l,l'}$'s. □

Lemma 7.2.4 *There exist four constants $c_1, c_2, c_3, c_4 > 0$, such that for any $N \ge p+1$, any $j, j' \in \{0, \dots, N-p-1\}$, one has:*

- $$|\mathcal{I}_{0,0}(j,j',N)| \le \frac{c_1 N^{-H(j/N)-H(j'/N)}}{1 + |j - j'|}; \tag{7.2.12}$$

- *for all $l, l' \in \{0, 1\}$ satisfying $l + l' = 1$,*

$$|\mathcal{I}_{l,l'}(j,j',N)| \le \frac{c_2 (\log N) N^{-H(j/N)-H(j'/N)-1}}{1 + |j - j'|}; \tag{7.2.13}$$

- *for all $l, l' \in \{0, 1, 2\}$ satisfying $l + l' = 2$,*

$$|\mathcal{I}_{l,l}(j,j',N)| \le c_3 (\log N)^2 N^{-H(j/N)-H(j'/N)-2}; \tag{7.2.14}$$

- *for all $l, l' \in \{1, 2\}$ satisfying $3 \le l + l' \le 4$,*

$$|\mathcal{I}_{l,l'}(j,j',N)| \le c_4 N^{-3}. \tag{7.2.15}$$

Let us stress that the main tool for proving Lemma 7.2.4 is integration by parts. Let us first state some technical lemmas which will play key rôles in the proof.

Lemma 7.2.5 *For any real $x \in \mathbb{R}$, one has $|e^{ix} - 1| \leq 2\min(1, |x|)$.*

Proof of Lemma 7.2.5: Observe that, for all $x \in \mathbb{R}$,

$$|e^{ix} - 1| = |e^{ix/2} - e^{-ix/2}| = 2|\sin(x/2)|.$$

Then combining the latter equality with the inequality: for all $x \in \mathbb{R}$,

$$|\sin(x)| \leq \min(1, |x|), \tag{7.2.16}$$

we get Lemma 7.2.5. □

Lemma 7.2.6 *There exists a constant $c > 0$, such that for any $x \in \mathbb{R}$, any integer $N \geq p+1$, any $j, j' \in \{0, \ldots, N-p-1\}$ and $m \in \{0, 1\}$, one has*

$$|r_{j,j'}^{(m)}(x)| \leq c \min\left(1, |x|^{2-m}\right), \tag{7.2.17}$$

where $r_{j,j'}(x) = e^{-i\delta(j,j')x}|e^{ix} - 1|^2 = 4e^{-i\delta(j,j')x}|\sin(x/2)|^2$ and $r_{j,j'}^{(m)}$ is the derivative of order m of $r_{j,j'}$; we assume that $\delta(j,j') = 1$ if $j \geq j'$ and $\delta(j,j') = -1$ else.

Proof of Lemma 7.2.6: When $m = 0$, (7.2.17) is a straightforward consequence of Lemma 7.2.5. When $m = 1$, by using the triangle inequality and the fact that for all integers $j, j' \in \{0, \ldots, N-p-1\}$ one has $|\delta(j,j')| = 1$, it follows that for all $x \in \mathbb{R}$ and all integers $N \geq p+1$, $j, j' \in \{0, \ldots, N-p-1\}$,

$$\begin{aligned} |r_{j,j'}^{(1)}(x)| &= \Big| -4i\delta(j,j')e^{-i\delta(j,j')x}|\sin(x/2)|^2 + 4e^{-i\delta(j,j')x}\sin(x/2)\cos(x/2)\Big| \\ &= \Big| -4i\delta(j,j')e^{-i\delta(j,j')x}|\sin(x/2)|^2 + 2e^{-i\delta(j,j')x}\sin(x)\Big| \\ &\leq 4|\sin(x/2)|^2 + 2|\sin(x)|. \end{aligned} \tag{7.2.18}$$

Finally combining (7.2.18) with (7.2.16), one gets (7.2.17). □

Lemma 7.2.7 *Let h be the trigonometric polynomial defined for every real x as $h(x) = \sum_{k=0}^{p} a_k e^{ikx}$. Then there is a constant $c > 0$ such that for all $m \in \{0, \ldots, M(a)\}$ and real x, one has*

$$|h^{(m)}(x)| \leq c \min\left(1, |x|^{M(a)-m}\right). \tag{7.2.19}$$

Proof of Lemma 7.2.7: Let us set $c = \sum_{k=0}^{p}(1+|k|)^{M(a)}|a_k|$. First observe that for all $m \in \mathbb{N}$ and all $x \in \mathbb{R}$, one has

$$h^{(m)}(x) = i^m \sum_{k=0}^{p} k^m a_k e^{ikx}, \tag{7.2.20}$$

which entails that for all $m \in \{0, \dots, M(a)\}$ and all $x \in \mathbb{R}$, one has

$$|h^{(m)}(x)| \le c. \tag{7.2.21}$$

This proves that (7.2.19) is satisfied when $m = M(a)$. On the hand (5.2.8) and (7.2.20) imply that $h(0) = h^{(1)}(0) = \dots h^{(M(a)-1)}(0) = 0$. Thus, for all fixed $m \in \{0, \dots, M(a)-1\}$, applying the Taylor formula of order $M(a)-m$ with an integral remainder to the function $h^{(m)}$ on the interval of endpoints 0 and x, one gets that

$$h^{(m)}(x) = \frac{x^{M(a)-m}}{(M(a)-m-1)!} \int_0^1 (1-\theta)^{M(a)-m-1} h^{(M(a))}(\theta x)\, d\theta,$$

which implies that

$$|h^{(m)}(x)| \le c|x|^{M(a)-m}. \tag{7.2.22}$$

Finally combining (7.2.21) with (7.2.22) one obtains the lemma. □

The proof of the following lemma has been omitted since it is given in [5] (see the proof of Lemma 2.1 in this article):

Lemma 7.2.8 *The function g introduced in (7.2.3) satisfies the following two properties.*

(i) There is a constant $c_1 > 0$ such that for all $m \in \{0,1\}$, $x \in [0,1]$ and $\xi \in \mathbb{R} \setminus \{0\}$, one has

$$|\partial_\xi^m g(x,\xi)| \le c_1 \max\left(|\xi|^{-\max_{s\in[0,1]} H(s)-m-1/2}, |\xi|^{-\min_{s\in[0,1]} H(s)-m-1/2}\right). \tag{7.2.23}$$

(ii) For any arbitrarily small real $\varepsilon > 0$, there exists a constant $c_2 > 0$ such that for all $n \in \{0,1,2\}$, $m \in \{0,1,2\}$, $x \in [0,1]$ and $\xi \in \mathbb{R} \setminus \{0\}$, one has

$$|\partial_x^n \partial_\xi^m g(x,\xi)| \le c_2 \max\left(|\xi|^{-\max_{s\in[0,1]} H(s)-\varepsilon-m-1/2}, |\xi|^{-\min_{s\in[0,1]} H(s)+\varepsilon-m-1/2}\right). \tag{7.2.24}$$

We are now ready to prove Lemma 7.2.4.

Proof of Lemma 7.2.4: Let us first prove that (7.2.12) holds. Fix $N \ge p+1$ and $j, j' \in \{0, \dots, N-p-1\}$, by using (7.2.11), the fact that $\sum_{k=0}^p a_k = 0$, Fubini Theorem and the equality: for all $\xi \in \mathbb{R} \setminus \{0\}$,

$$\int_{\frac{j}{N}}^{\frac{j+1}{N}} \int_{\frac{j'}{N}}^{\frac{j'+1}{N}} e^{i(s-s')\xi}\, ds'\, ds = \xi^{-2} e^{i((j-j')/N)\xi} |e^{i\xi/N} - 1|^2,$$

we obtain

$$\begin{aligned}
&\mathcal{I}_{0,0}(j,j',N)\\
&= N^2\int_{\frac{j}{N}}^{\frac{j+1}{N}}\int_{\frac{j'}{N}}^{\frac{j'+1}{N}}\int_{\mathbb{R}} e^{i(s-s')\xi}\Big|\sum_{k=0}^{p}a_k e^{i(k/N)\xi}\Big|^2 A_0(j/N,\xi)A_0(j'/N,\xi)\,\mathrm{d}\xi\,\mathrm{d}s'\,\mathrm{d}s\\
&= N^2\int_{\mathbb{R}}\Big(\int_{\frac{j}{N}}^{\frac{j+1}{N}}\int_{\frac{j'}{N}}^{\frac{j'+1}{N}} e^{i(s-s')\xi}\,\mathrm{d}s'\,\mathrm{d}s\Big)\Big|\sum_{k=0}^{p}a_k e^{i(k/N)\xi}\Big|^2 A_0(j/N,\xi)A_0(j'/N,\xi)\,\mathrm{d}\xi\\
&= N^2\int_{\mathbb{R}} e^{(i(j-j')/N)\xi}\frac{|e^{i\xi/N}-1|^2|\sum_{k=0}^{p}a_k e^{i(k/N)\xi}|^2}{\xi^2}A_0(j/N,\xi)A_0(j'/N,\xi)\,\mathrm{d}\xi.
\end{aligned} \tag{7.2.25}$$

Next by setting $\eta=\xi/N$ in the last integral and by using (7.2.6), we get

$$\mathcal{I}_{0,0}(j,j',N) = N^{-H(j/N)-H(j'/N)}\int_{\mathbb{R}} e^{i(\delta(j,j')+(j-j'))\eta}L_{j,j',N}(\eta)\,\mathrm{d}\eta,$$

with

$$L_{j,j',N}(\eta) := \begin{cases} \eta^{-2}r_{j,j'}(\eta)|h(\eta)|^2A_0(j/N,\eta)A_0(j'/N,\eta) & \text{when } \eta\neq 0,\\ 0 & \text{else,}\end{cases} \tag{7.2.26}$$

where we use the same notations as in Lemmas 7.2.6 and 7.2.7. Next, using the triangle inequality as well as Lemmas 7.2.5, 7.2.7 and 7.2.8 (Part (i)), one obtains that there is a constant $c_1>0$ such that for all $\eta\in\mathbb{R}\setminus\{0\}$,

$$|L_{j,j',N}(\eta)| \le \begin{cases} c_1|\eta|^{2M(a)-2\max_{s\in[0,1]}H(s)-1} & \text{if } 0<|\eta|<1\\ c_1|\eta|^{-2\min_{s\in[0,1]}H(s)-3} & \text{if } |\eta|\ge 1,\end{cases} \tag{7.2.27}$$

Let us now bound bound $L'_{j,j',N}(\eta)$, to this end it is convenient to set

$$w_{j,j'}(\eta) := \begin{cases} \eta^{-2}r_{j,j'}(\eta)|h(\eta)|^2 & \text{when } \eta\neq 0,\\ 0 & \text{else,}\end{cases} \tag{7.2.28}$$

It follows from Lemmas 7.2.6 and 7.2.7, that there exists a constant $c_2>0$, non depending on j, j' and η, such that for all $\eta\in\mathbb{R}\setminus\{0\}$, one has,

$$|w_{j,j'}(\eta)| \le c_2|\eta|^{-2}\min\big(1,|\eta|^{2(M(a)+1)}\big). \tag{7.2.29}$$

Observe that, in view of (7.2.26) and (7.2.28), one has, for all $\eta\in\mathbb{R}\setminus\{0\}$,

$$L_{j,j',N}(\eta) = w_{j,j'}(\eta)A_0(j/N,\eta)A_0(j'/N,\eta)$$

and, as a consequence,

$$\begin{aligned} L'_{j,j',N}(\eta) &= w'_{j,j'}(\eta)A_0(j/N,\eta)A_0(j'/N,\eta)+w_{j,j'}(\eta)\partial_\eta A_0(j/N,\eta)A_0(j'/N,\eta)\\ &\quad +w_{j,j'}(\eta)A_0(j/N,\eta)(\partial_\eta A_0)(j'/N,\eta). \end{aligned} \tag{7.2.30}$$

It follows from (7.2.28) that for each $\eta \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} w'_{j,j'}(\eta) &= \Big(-2\frac{r_{j,j'}(\eta)}{\eta} + r'_{j,j'}(\eta)\Big)\frac{|h(\eta)|^2}{\eta^2} \\ &\quad + \frac{r_{j,j'}(\eta)}{\eta^2}\big(h'(\eta)h(-\eta) - h(\eta)h'(-\eta)\big) \end{aligned} \tag{7.2.31}$$

Then (7.2.31), the triangle inequality, Lemma 7.2.6 and Lemma 7.2.7 imply that there is a constant $c_3 > 0$, non depending on j, j' and η, such that, for each $\eta \in \mathbb{R} \setminus \{0\}$,

$$|w'_{j,j'}(\eta)| \le c_3|\eta|^{-2}\min\big(1, |\eta|^{2M(a)+1}\big). \tag{7.2.32}$$

Also, observe that by using (7.2.30) and the triangle inequality, one has,

$$\begin{aligned} &|L'_{j,j',N}(\eta)| \\ &\le |w'_{j,j'}(\eta)||A_0(j/N,\eta)||A_0(j'/N,\eta)| + |w_{j,j'}(\eta)||\partial_\eta A_0(j/N,\eta)||A_0(j'/N,\eta)| \\ &\quad + |w_{j,j'}(\eta)||A_0(j/N,\eta)||\partial_\eta A_0(j'/N,\eta)|. \end{aligned} \tag{7.2.33}$$

Next (7.2.33), (7.2.29), (7.2.32), (7.2.6) and (7.2.23) allow to prove that,

$$|L'_{j,j',N}(\eta)| \le \begin{cases} c_4|\eta|^{2M(a)-2\max_{s\in[0,1]}H(s)-2} & \text{if } 0 < |\eta| < 1 \\ c_4|\eta|^{-2\min_{s\in[0,1]}H(s)-3} & \text{if } |\eta| \ge 1, \end{cases} \tag{7.2.34}$$

where $c_4 > 0$ is a constant not depending on N, j, j' and η. It is clear $L_{j,j',N}$ is a C^1-function over $\mathbb{R} \setminus \{0\}$ and this is also the case over $\mathbb{R}$ since (7.2.27), (7.2.34) and the fact that $M(a) \ge 3$, imply that $\lim_{\eta\to 0} L_{j,j',N}(\eta) = \lim_{\eta\to 0} L'_{j,j',N}(\eta) = 0$. Next integrating by parts we obtain that

$$\begin{aligned} \mathcal{I}_{0,0}(j,j',N) &= N^{-H(j/N)-H(j'/N)}\bigg(\Big[\frac{e^{i(\delta(j,j')+(j-j'))\eta}}{i(\delta(j,j')+(j-j'))}L_{j,j',N}(\eta)\Big]_{-\infty}^{+\infty}\bigg) \\ &\quad - \frac{1}{i(\delta(j,j')+(j-j'))}\int_{\mathbb{R}} e^{i(\delta(j,j')+(j-j'))\eta}L'_{j,j',N}(\eta)\,\mathrm{d}\eta \\ &= \frac{i}{(\delta(j,j')+(j-j'))}\int_{\mathbb{R}} e^{i(\delta(j,j')+(j-j'))\eta}L'_{j,j',N}(\eta)\,\mathrm{d}\eta. \end{aligned} \tag{7.2.35}$$

Observe that (7.2.27) implies that $\Big[\frac{e^{i(\delta(j,j')+(j-j'))\eta}}{i(\delta(j,j')+(j-j'))}L_{j,j',N}(\eta)\Big]_{-\infty}^{+\infty} = 0$ and (7.2.34) that $\int_{\mathbb{R}}|L'_{j,j',N}(\eta)|\,\mathrm{d}\eta < \infty$, which means that the integral in the right hand side of (7.2.35) exists. Finally, it follows from (7.2.35) that (7.2.12) holds.

Before pursuing our proof, observe that for all l, l', j and j', one has

$$\mathcal{I}_{l,l'}(j,j',N) = \mathcal{I}_{l',l}(j',j,N);$$

so we can assume in all the sequel that $l \le l'$. Now our goal will be to prove that (7.2.13) holds. It suffices to show that this inequality is satisfied when $l = 0$ and

$l'=1$. By setting in (7.2.11) $l=0$, $l'=1$, $\xi=N\eta$ and by using the equalities $\sum_{k=0}^p a_k=0$ and $\sum_{k=0}^p ka_k=0$, one obtains that

$$\begin{aligned}
&\mathcal{I}_{0,1}(j,j',N)\\
&=N^3\int_{\frac{j}{N}}^{\frac{j+1}{N}}\int_{\frac{j'}{N}}^{\frac{j'+1}{N}}\int_{\mathbb{R}}\Big(\sum_{k=0}^p a_k e^{i(k+sN)\eta}\Big)\Big(\sum_{k'=0}^p a_{k'}e^{-i(k'+s'N)\eta}(s'+k'/N-j'/N)\Big)\\
&\quad\times A_0(j/N,N\eta)A_1(j'/N,N\eta)\,\mathrm{d}\eta\,\mathrm{d}s'\,\mathrm{d}s\\
&=N^3\int_{\frac{j}{N}}^{\frac{j+1}{N}}\int_{\frac{j'}{N}}^{\frac{j'+1}{N}}\int_{\mathbb{R}}e^{i(s-s')N\eta}\Big(\sum_{k=0}^p a_k e^{ik\eta}\Big)\Big(\sum_{k'=0}^p a_{k'}e^{-ik'\eta}(s'+k'/N-j'/N)\Big)\\
&\quad\times A_0(j/N,N\eta)A_1(j'/N,N\eta)\,\mathrm{d}\eta\,\mathrm{d}s'\,\mathrm{d}s\\
&=B_1(j,j',N)+B_2(j,j',N), \qquad (7.2.36)
\end{aligned}$$

where

$$\begin{aligned}
B_1(j,j',N)=N^3&\int_{\frac{j}{N}}^{\frac{j+1}{N}}\int_{\frac{j'}{N}}^{\frac{j'+1}{N}}\int_{\mathbb{R}}e^{i(s-s')N\eta}(s'-j'/N)\Big|\sum_{k=0}^p a_k e^{ik\eta}\Big|^2\\
&\times A_0(j/N,N\eta)A_1(j'/N,N\eta)\,\mathrm{d}\eta\,\mathrm{d}s'\,\mathrm{d}s, \qquad (7.2.37)
\end{aligned}$$

and

$$\begin{aligned}
B_2(j,j',N)=N^2&\int_{\frac{j}{N}}^{\frac{j+1}{N}}\int_{\frac{j'}{N}}^{\frac{j'+1}{N}}\int_{\mathbb{R}}e^{i(s-s')N\eta}\Big(\sum_{k=0}^p\sum_{k'=0}^p k'a_k a_{k'}e^{i(k-k')\eta}\Big)\\
&\times A_0(j/N,N\eta)A_1(j'/N,N\eta)\,\mathrm{d}\eta\,\mathrm{d}s'\,\mathrm{d}s. \qquad (7.2.38)
\end{aligned}$$

Now we are going to show that there are two constants $c_5,c_6>0$, non depending on N, j and j', such that

$$|B_1(j,j',N)|\le\frac{c_5(\log N)N^{-H(j/N)-H(j'/N)-1}}{1+|j-j'|}, \qquad (7.2.39)$$

and

$$|B_2(j,j',N)|\le\frac{c_6(\log N)N^{-H(j/N)-H(j'/N)-1}}{1+|j-j'|}. \qquad (7.2.40)$$

Let us first prove that (7.2.39) holds. By using Fubini Theorem and the following equality: for all $\eta\in\mathbb{R}\setminus\{0\}$,

$$\int_{\frac{j}{N}}^{\frac{j+1}{N}}\int_{\frac{j'}{N}}^{\frac{j'+1}{N}}e^{i(s-s')N\eta}(s'-j'/N)\,\mathrm{d}s'\,\mathrm{d}s=\frac{e^{i(j-j')\eta}}{N^2\eta^2}\Big(\frac{1-e^{-i\eta}}{N}+\frac{|e^{i\eta}-1|^2}{iN\eta}\Big),$$

we can write

$$\begin{aligned}B_1(j,j',N) &= N^3\int_{\mathbb{R}}\Big(\int_{\frac{j}{N}}^{\frac{j+1}{N}}\int_{\frac{j'}{N}}^{\frac{j'+1}{N}} e^{i(s-s')N\eta}(s'-j'/N)\,\mathrm{d}s'\,\mathrm{d}s\Big)\Big|\sum_{k=0}^{p}a_ke^{ik\eta}\Big|^2\\&\quad\times A_0(j/N,N\eta)A_1(j'/N,N\eta)\,\mathrm{d}\eta\\&= N^3\int_{\mathbb{R}}\frac{e^{i(j-j')\eta}}{N^2\eta^2}\Big(\frac{1-e^{-i\eta}}{N}+\frac{|e^{i\eta}-1|^2}{iN\eta}\Big)\\&\quad\times\Big|\sum_{k=0}^{p}a_ke^{ik\eta}\Big|^2A_0(j/N,N\eta)A_1(j'/N,N\eta)\,\mathrm{d}\eta\\&= \int_{\mathbb{R}}e^{i(j-j')\eta}\Big(\frac{1-e^{-i\eta}}{\eta^2}+\frac{|e^{i\eta}-1|^2}{i\eta^3}\Big)\\&\quad\times\Big|\sum_{k=0}^{p}a_ke^{ik\eta}\Big|^2A_0(j/N,N\eta)A_1(j'/N,N\eta)\,\mathrm{d}\eta.\end{aligned}\tag{7.2.41}$$

Observe that (7.2.7) implies that

$$A_1(j'/N,N\eta) = -H'(j'/N)A_0(j'/N,N\eta)(\log N+\log|\eta|).\tag{7.2.42}$$

Then the latter equality and (7.2.41) entail that $B_1(j,j',N)$ can be expressed as:

$$\begin{aligned}B_1(j,j',N) &= -H'(j'/N)\int_{\mathbb{R}}e^{i(j-j')\eta}\Big(\frac{1-e^{-i\eta}}{\eta^2}+\frac{|e^{i\eta}-1|^2}{i\eta^3}\Big)\\&\quad\times\Big|\sum_{k=0}^{p}a_ke^{ik\eta}\Big|^2(\log N+\log|\eta|)A_0(j/N,N\eta)A_0(j'/N,N\eta)\,\mathrm{d}\eta\\&= B_{1,1}(j,j',N)+B_{1,2}(j,j',N),\end{aligned}\tag{7.2.43}$$

where

$$\begin{aligned}B_{1,1}(j,j',N) &= -H'(j'/N)\log N\int_{\mathbb{R}}e^{i(j-j')\eta}\Big(\frac{1-e^{-i\eta}}{\eta^2}+\frac{|e^{i\eta}-1|^2}{i\eta^3}\Big)\\&\quad\times\Big|\sum_{k=0}^{p}a_ke^{ik\eta}\Big|^2A_0(j/N,N\eta)A_0(j'/N,N\eta)\,\mathrm{d}\eta\end{aligned}\tag{7.2.44}$$

and

$$\begin{aligned}B_{1,2}(j,j',N) &= -H'(j'/N)\int_{\mathbb{R}}e^{i(j-j')\eta}\Big(\frac{1-e^{-i\eta}}{\eta^2}+\frac{|e^{i\eta}-1|^2}{i\eta^3}\Big)\\&\quad\times\Big|\sum_{k=0}^{p}a_ke^{ik\eta}\Big|^2\log|\eta|A_0(j/N,N\eta)A_0(j'/N,N\eta)\,\mathrm{d}\eta.\end{aligned}\tag{7.2.45}$$

Next using (7.2.6), (7.2.44) and (7.2.45), it follows that

$$\begin{aligned}&B_{1,1}(j,j',N)\\&= -H'(j'/N)(\log N)N^{-H(j/N)-H(j'/N)-1}\int_{\mathbb{R}}e^{i(\delta(j,j')+(j-j'))\eta}K_{j,j',N}(\eta)\,\mathrm{d}\eta,\end{aligned}\tag{7.2.46}$$

with

$$K_{j,j',N}(\eta) := \begin{cases} e^{-i\delta(j,j')\eta}\Big(\frac{1-e^{-i\eta}}{\eta^2}+\frac{|e^{i\eta}-1|^2}{i\eta^3}\Big)\Big|\sum_{k=0}^{p} a_k e^{ik\eta}\Big|^2 A_0(j/N,\eta)A_0(j'/N,\eta) & \text{when } \eta\neq 0,\\ 0 & \text{else,}\end{cases}$$

and

$$B_{1,2}(j,j',N) = -H'(j'/N)N^{-H(j/N)-H(j'/N)-1}\int_{\mathbb{R}} e^{i(\delta(j,j')+(j-j'))\eta}M_{j,j',N}(\eta)\,\mathrm{d}\eta, \tag{7.2.47}$$

with

$$M_{j,j',N}(\eta) := \begin{cases} (\log|\eta|)K_{j,j',N}(\eta) & \text{when } \eta\neq 0,\\ 0 & \text{else,}\end{cases} \tag{7.2.48}$$

Observe that by using the fact that the function φ defined as $\varphi(0) := 2^{-1}$ and for all $\eta\in\mathbb{R}\setminus\{0\}$, as,

$$\varphi(\eta) := \frac{1-e^{-i\eta}}{\eta^2}+\frac{|e^{i\eta}-1|^2}{i\eta^3}, \tag{7.2.49}$$

is an entire function, one has that,

$$\sup_{|\eta|\le 1}\big(|\varphi(\eta)|+|\varphi'(\eta)|\big) < \infty; \tag{7.2.50}$$

moreover standard computations allow to show that, for all η such that $|\eta|>1$, one has,

$$|\varphi(\eta)|+|\varphi'(\eta)| = \mathcal{O}\big(|\eta|^{-2}\big). \tag{7.2.51}$$

Then, using (7.2.49), (7.2.50), (7.2.51) and a method similar to that which allowed to obtain (7.2.27) and (7.2.34), we can show that there are two constants $c_7, c_8 > 0$ such that

$$|K_{j,j',N}(\eta)| \le \begin{cases} c_7|\eta|^{2M(a)-2\max_{s\in[0,1]}H(s)-1} & \text{if } 0<|\eta|<1,\\ c_7|\eta|^{-2\min_{s\in[0,1]}H(s)-3} & \text{if } |\eta|\ge 1,\end{cases} \tag{7.2.52}$$

and

$$|K'_{j,j',N}(\eta)| \le \begin{cases} c_8|\eta|^{2M(a)-2\max_{s\in[0,1]}H(s)-2} & \text{if } 0<|\eta|<1,\\ c_8|\eta|^{-2\min_{s\in[0,1]}H(s)-3} & \text{if } |\eta|\ge 1.\end{cases} \tag{7.2.53}$$

Moreover, in view of (7.2.48), Relations (7.2.52) and (7.2.53) entail that there are two constants $c_9, c_{10} > 0$ such that

$$|M_{j,j',N}(\eta)| \le \begin{cases} c_9(\log|\eta|)|\eta|^{2M(a)-2\max_{s\in[0,1]}H(s)-1} & \text{if } 0<|\eta|<1,\\ c_9(\log|\eta|)|\eta|^{-2\min_{s\in[0,1]}H(s)-3} & \text{if } |\eta|\ge 1;\end{cases} \tag{7.2.54}$$

and

$$|M'_{j,j',N}(\eta)| \le \begin{cases} c_{10}(\log|\eta|)|\eta|^{2M(a)-2\max_{s\in[0,1]}H(s)-2} & \text{if } 0<|\eta|<1, \\ c_{10}(\log|\eta|)|\eta|^{-2\min_{s\in[0,1]}H(s)-3} & \text{if } |\eta|\ge 1; \end{cases} \tag{7.2.55}$$

Il follows from (7.2.52), (7.2.53), (7.2.54), (7.2.55) and the inequality $M(a)\ge 3$, that

$$\lim_{\eta\to 0} K_{j,j',N}(\eta) = \lim_{\eta\to 0} K'_{j,j',N}(\eta) = \lim_{\eta\to 0} M_{j,j',N}(\eta) = \lim_{\eta\to 0} M'_{j,j',N}(\eta) = 0.$$

Thus, in view of (7.2.47) and (7.2.48), $K_{j,j',N}$ and $M_{j,j',N}$ are C^1 functions over $\mathbb{R}$. Then, similarly to (7.2.35), integrating by parts in (7.2.46) and (7.2.47) and using Relations (7.2.52) to (7.2.55) as well as the fact H' is bounded, allow to show that there are two constants $c_{11}, c_{12} > 0$ such that

$$|B_{1,1}(j,j',N)| \le \frac{c_{11}(\log N)N^{-H(j/N)-H(j'/N)-1}}{1+|j-j'|} \tag{7.2.56}$$

and

$$|B_{1,2}(j,j',N)| \le \frac{c_{12}N^{-H(j/N)-H(j'/N)-1}}{1+|j-j'|}. \tag{7.2.57}$$

Next putting together (7.2.43), (7.2.56) and (7.2.57), one obtains (7.2.39).

Now let us show that (7.2.40) holds. In view of (7.2.38), and by using Fubini Theorem, $B_2(j,j',N)$ can be expressed as follows:

$$\begin{aligned} B_2(j,j',N) &= N^2 \int_{\mathbb{R}} \Big(\int_{\frac{j}{N}}^{\frac{j+1}{N}} \int_{\frac{j'}{N}}^{\frac{j'+1}{N}} e^{i(s-s')N\eta}\, \mathrm{d}s'\, \mathrm{d}s \Big) \Big(\sum_{k=0}^{p} \sum_{k'=0}^{p} k' a_k a_{k'} e^{i(k-k')\eta} \Big) \\ &\quad \times A_0(j/N, N\eta) A_1(j'/N, N\eta)\, \mathrm{d}\eta \\ &= N^2 \int_{\mathbb{R}} \frac{e^{i(j-j')\eta}}{N^2\eta^2} \Big(|e^{i\eta}-1|^2 \Big) \\ &\quad \times \Big(\sum_{k=0}^{p} \sum_{k'=0}^{p} k' a_k a_{k'} e^{i(k-k')\eta} \Big) A_0(j/N, N\eta) A_1(j'/N, N\eta)\, \mathrm{d}\eta \\ &= \int_{\mathbb{R}} e^{i(j-j')\eta} \Big(\frac{|e^{i\eta}-1|^2}{|\eta|^2} \Big) \\ &\quad \times \Big(\sum_{k=0}^{p} \sum_{k'=0}^{p} k' a_k a_{k'} e^{i(k-k')\eta} \Big) A_0(j/N, N\eta) A_1(j'/N, N\eta)\, \mathrm{d}\eta. \end{aligned}$$

Then putting together, the latter equality, (7.2.42) and (7.2.6), it follows that,

$$B_2(j,j',N) = B_{2,1}(j,j',N) + B_{2,2}(j,j',N), \tag{7.2.58}$$

where

$$B_{2,1}(j,j',N) = -H'(j'/N)(\log N)$$
$$\times \int_{\mathbb{R}} e^{i(j-j')\eta}\Big(\frac{|e^{i\eta}-1|^2}{|\eta|^2}\Big)\Big(\sum_{k=0}^{p}\sum_{k'=0}^{p} k' a_k a_{k'} e^{i(k-k')\eta}\Big) A_0(j/N,N\eta)A_0(j'/N,N\eta)\,\mathrm{d}\eta$$
$$= -H'(j'/N)(\log N)N^{-H(j/N)-H(j'/N)-1}\int_{\mathbb{R}} e^{i(\delta(j,j')+(j-j'))\eta} S_{j,j',N}(\eta)\,\mathrm{d}\eta, \tag{7.2.59}$$

with

$$S_{j,j',N}(\eta) \tag{7.2.60}$$
$$:= \begin{cases} e^{-i\delta(j,j')\eta}\Big(\frac{|e^{i\eta}-1|^2}{|\eta|^2}\Big)\Big(\sum_{k=0}^{p}\sum_{k'=0}^{p} k' a_k a_{k'} e^{i(k-k')\eta}\Big)A_0(j/N,\eta)A_0(j'/N,\eta) & \text{if } \eta\neq 0,\\ 0 & \text{else,}\end{cases}$$

and

$$B_{2,2}(j,j',N) = -H'(j'/N)$$
$$\times \int_{\mathbb{R}} e^{i(j-j')\eta}\Big(\frac{|e^{i\eta}-1|^2}{|\eta|^2}\Big)\Big(\sum_{k=0}^{p}\sum_{k'=0}^{p} k' a_k a_{k'} e^{i(k-k')\eta}\Big)(\log|\eta|) A_0(j/N,N\eta)A_0(j'/N,N\eta)\,\mathrm{d}\eta$$
$$= -H'(j'/N)N^{-H(j/N)-H(j'/N)-1}\int_{\mathbb{R}} e^{i(\delta(j,j')+(j-j'))\eta} T_{j,j',N}(\eta)\,\mathrm{d}\eta, \tag{7.2.61}$$

with

$$T_{j,j',N}(\eta) := \begin{cases} (\log|\eta|)S_{j,j',N}(\eta) & \text{when } \eta\neq 0,\\ 0 & \text{else.}\end{cases} \tag{7.2.62}$$

Then noticing that

$$\sum_{k'=0}^{p} k' a_k a_{k'} e^{i(k-k')\eta} = -ih(\eta)h'(-\eta),$$

where h is the trigonometric polynomial introduced in Lemma 7.2.7, and using a method similar to that which allowed to obtain (7.2.27) and (7.2.34), one can show that there are two constants $c_{13}, c_{14} > 0$ such that

$$|S_{j,j',N}(\eta)| \le c_{13}\frac{|e^{i\eta}-1|^2}{\eta^2}\min\big(1,|\eta|^{2M(a)-1}\big)$$
$$\times\max\big(|\eta|^{-2\max_{s\in[0,1]}H(s)-1}, |\eta|^{-2\min_{s\in[0,1]}H(s)-1}\big)$$
$$\le \begin{cases} c_{13}|\eta|^{2M(a)-2\max_{s\in[0,1]}H(s)-2} & \text{if } 0<|\eta|<1,\\ c_{13}|\eta|^{-2\min_{s\in[0,1]}H(s)-3} & \text{if } |\eta|\ge 1,\end{cases} \tag{7.2.63}$$

and

$$|S'_{j,j',N}(\eta)| \le \begin{cases} c_{14}|\eta|^{2M(a)-2\max_{s\in[0,1]}H(s)-3} & \text{if } 0<|\eta|<1,\\ c_{14}|\eta|^{-2\min_{s\in[0,1]}H(s)-3} & \text{if } |\eta|\ge 1.\end{cases} \tag{7.2.64}$$

Moreover in view of (7.2.62), Relations (7.2.63) and (7.2.64), entail that there are two constants $c_{15}, c_{16} > 0$ such that,

$$|T_{j,j',N}(\eta)| \le \begin{cases} c_{15}(\log|\eta|)|\eta|^{2M(a)-2\max_{s\in[0,1]} H(s)-2} & \text{if } 0<|\eta|<1, \\ c_{16}(\log|\eta|)|\eta|^{-2\min_{s\in[0,1]} H(s)-3} & \text{if } |\eta|\ge 1, \end{cases} \tag{7.2.65}$$

and

$$|T'_{j,j',N}(\eta)| \le \begin{cases} c_{16}(\log|\eta|)|\eta|^{2M(a)-2\max_{s\in[0,1]} H(s)-3} & \text{if } 0<|\eta|<1, \\ c_{16}(\log|\eta|)|\eta|^{-2\min_{s\in[0,1]} H(s)-3} & \text{if } |\eta|\ge 1. \end{cases} \tag{7.2.66}$$

Next putting together (7.2.63), (7.2.64), (7.2.65), (7.2.66), the inequality $M(a) \ge 3$ and the fact that H' is bounded, it follows that,

$$\lim_{\eta\to 0} S_{j,j',N}(\eta) = \lim_{\eta\to 0} S'_{j,j',N}(\eta) = \lim_{\eta\to 0} T_{j,j',N}(\eta) = \lim_{\eta\to 0} T'_{j,j',N}(\eta) = 0,$$

which, in view of (7.2.60) and (7.2.62), implies that $S_{j,j',N}$ and $T_{j,j',N}$ are C^1 functions over $\mathbb{R}$. Then, similarly to (7.2.35), integrating by parts in (7.2.59) and (7.2.61) and using Relations (7.2.63) to (7.2.66) as well as the fact that H' is bounded, allow to show that there are two constants $c_{17}, c_{18} > 0$ such that

$$|B_{2,1}(j,j',N)| \le \frac{c_{17}(\log N)N^{-H(j/N)-H(j'/N)-1}}{1+|j-j'|} \tag{7.2.67}$$

and

$$|B_{2,2}(j,j',N)| \le \frac{c_{18}N^{-H(j/N)-H(j'/N)-1}}{1+|j-j'|}. \tag{7.2.68}$$

Thus (7.2.40) results from (7.2.58), (7.2.67) and (7.2.68). Finally (7.2.13) follows from (7.2.36), (7.2.39) and (7.2.40).

Now let us prove that (7.2.14) is satisfied. By setting in (7.2.11) $l = l' = 1$, $\xi = N\eta$ and by using the fact that $\sum_{k=0}^p a_k = \sum_{k=0}^p ka_k = 0$ one obtains that

$$\begin{aligned} \mathcal{I}_{1,1}(j,j',N) &= N^3 \int_{\frac{j}{N}}^{\frac{j+1}{N}} \int_{\frac{j'}{N}}^{\frac{j'+1}{N}} \int_{\mathbb{R}} \Big(\sum_{k=0}^p a_k e^{i(sN+k)\eta}(s+k/N-j/N) \Big) \\ &\quad \times \Big(\sum_{k'=0}^p a_{k'} e^{-i(s'N+k')\eta}(s'+k'/N-j'/N) \Big) \\ &\quad \times A_1(j/N,N\eta)A_1(j'/N,N\eta)\,\mathrm{d}\eta\,\mathrm{d}s'\,\mathrm{d}s. \end{aligned} \tag{7.2.69}$$

Observe that by using (7.2.7) and Part (*ii*)) Lemma 7.2.8, one can show that for every arbitrarily small $\varepsilon > 0$, there is a constant $c_{19} > 0$ such that for all $N \ge p+1$, $j \in \{0, \dots N-p-1\}$ and $\eta \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} &|A_1(j/N,N\eta)| \\ &\le c_{19}(\log(N))N^{-H(j/N)-1/2} \max\Big(|\eta|^{-\max_{s\in[0,1]} H(s)-\varepsilon-1/2}, |\eta|^{-\min_{s\in[0,1]} H(s)+\varepsilon-1/2}\Big). \end{aligned} \tag{7.2.70}$$

On the other hand, by using (7.2.19), one can show that there is a constant $c_{20} > 0$, such that for all $N \geq p+1$, $j \in \{0, \dots, p+1\}$, $s \in [\frac{j}{N}, \frac{j+1}{N}]$ and $\eta \in \mathbb{R}$, one has

$$\Big|\sum_{k=0}^{p} a_k e^{i(sN+k)\eta}(s + k/N - j/N)\Big| \leq c_{20} N^{-1} \min\big(1, |\eta|^{M(a)-1}\big). \tag{7.2.71}$$

Next (7.2.69), (7.2.71) and (7.2.70), imply that that

$$|\mathcal{I}_{1,1}(j, j', N)| \leq c_{21} N^{-H(j/N)-H(j'/N)-2} (\log N)^2,$$

where the constant

$$c_{21} = (c_{19}c_{20})^2 \int_{\mathbb{R}} \min\big(1, |\eta|^{2M(a)-2}\big) \max\big(|\eta|^{-2\max_{s\in[0,1]} H(s)-2\varepsilon-1}, |\eta|^{-2\min_{s\in[0,1]} H(s)+2\varepsilon-1}\big)\, d\eta,$$

is finite since $M(a) \geq 3$. Thus we have proved that (7.2.14) holds when $l = l' = 1$.

Let us now prove that (7.2.14) is satisfied when $l = 0$ and $l' = 2$. By setting in (7.2.11), $l = 0$, $l' = 2$ and $\eta = \xi/N$, one obtains

$$\begin{aligned}
\mathcal{I}_{0,2}(j, j', N) &= N^3 \int_{\frac{j}{N}}^{\frac{j+1}{N}} \int_{\frac{j'}{N}}^{\frac{j'+1}{N}} \int_{\mathbb{R}} e^{isN} \Big(\sum_{k=0}^{p} a_k e^{ik\eta}\Big) A_0(j/N, N\eta) \\
&\quad \times \Big(\sum_{k'=0}^{p} a_{k'} (e^{-i(s'N+k')\eta} - 1)(s' + k'/N - j'/N)^2 \\
&\quad \times A_2(j'/N, s' + k'/N, N\eta)\Big)\, d\eta\, ds'\, ds.
\end{aligned} \tag{7.2.72}$$

Standard computations allow to obtain that for all $N \geq p+1$, $j \in \{0, \dots, N-p-1\}$, $s \in [\frac{j}{N}, \frac{j+1}{N}]$, $\theta \in [0, 1]$, $\eta \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}
&\partial_x^2 g(j/N + \theta(s - j/N), N\eta) \\
&= N^{-H(j/N+\theta(s-j/N))-1/2} \Big((H'(j/N + \theta(s - j/N)))^2 (\log N)^2 A_0(j/N + \theta(s - j/N), \eta) \\
&\quad + 2(-H'(j/N + \theta(s - j/N)))(\log N) A_1(j/N + \theta(s - j/N), \eta) + \partial_x^2 g(j/N + \theta(s - j/N), \eta) \\
&\quad - (H^{(2)}(j/N + \theta(s - j/N)))(\log N) A_0(j/N + \theta(s - j/N), \eta) \Big).
\end{aligned} \tag{7.2.73}$$

Next, by using (7.2.8), (7.2.3), the triangle inequality, (7.2.73), the fact that for any $m \in \{0, 1, 2\}$ the function $H^{(m)}(\cdot)$ is bounded, (7.2.6), (7.2.7), Lemma 7.2.8 and the inequalities $N^{-H(j/N+\theta(s-j/N))} \leq e^{\|H'\|_\infty} N^{-H(j/N)}$ and $\log N \geq \log 3 \geq 1$, one can show that for every arbitrarily small $\varepsilon > 0$, there is a constant $c_{22} > 0$ such that for all $N \geq p+1$, $j' \in \{0, \dots, N-p-1\}$, $k' \in \{0, \dots, p\}$, $s' \in [\frac{j'}{N}, \frac{j'+1}{N}]$, $\eta \in \mathbb{R} \setminus \{0\}$, one has,

$$\begin{aligned}
&|A_2(j'/N, s' + k'/N, N\eta)| \\
&\leq c_{22} (\log N)^2 N^{-H(j'/N)-1/2} \max\Big(|\eta|^{-\max_{s\in[0,1]} H(s)-\varepsilon-1/2}, |\eta|^{-\min_{s\in[0,1]} H(s)+\varepsilon-1/2}\Big).
\end{aligned} \tag{7.2.74}$$

Next it follows from (7.2.72), (7.2.74), the inequality $(s'+k'/N-j'/N)^2 \le (p+1)^2/N^2$, Lemma 7.2.5, (7.2.6), (7.2.23) and (7.2.19) that

$$|\mathcal{I}_{0,2}(j,j',N)| \le c_{23} N^{-H(j/N)-H(j'/N)-2}(\log N)^2,$$

where the constant

$$\begin{aligned} c_{23} &= c_{24}c_{22}\int_{\mathbb{R}} \min\Big(1,|\eta|^{M(a)}\Big)\max\Big(|\eta|^{-2\max_{s\in[0,1]}H(s)-2\varepsilon-1},|\eta|^{-2\min_{s\in[0,1]}H(s)+2\varepsilon-1}\Big)\,\mathrm{d}\eta \\ &< +\infty, \end{aligned}$$

with c_{24} being the product of $2\Big((p+1)^2\sum_{k'=0}^{p}|a_{k'}|\Big)$ and the constants in the right hand side of (7.2.19) and (7.2.23). Thus we have shown that (7.2.14) holds when $l=0$ and $l'=2$.

Now our goal will be to prove that (7.2.15) holds. First we will prove that this inequality is satisfied when $l=1$ and $l'=2$. Setting in (7.2.11), $l=1$ and $l'=2$, one obtains that

$$\begin{aligned} \mathcal{I}_{1,2}(j,j',N) &= N^2\int_{\frac{j}{N}}^{\frac{j+1}{N}}\int_{\frac{j'}{N}}^{\frac{j'+1}{N}}\int_{\mathbb{R}}\Big(\sum_{k=0}^{p}a_k(e^{i(s+k/N)\xi}-1)(s+k/N-j/N)A_1(j/N,\xi)\Big) \\ &\quad\times\Big(\sum_{k'=0}^{p}a_{k'}(e^{-i(s'+k'/N)\xi}-1)(s'+k'/N-j'/N)^2 \\ &\quad\times A_2(j'/N,s'+k'/N,\xi)\Big)\,\mathrm{d}\xi\,\mathrm{d}s'\,\mathrm{d}s. \end{aligned} \tag{7.2.75}$$

Observe that it follows from (7.2.7) and (7.2.24) that for all arbitrarily small $\varepsilon>0$, there exists a constant $c_{25}>0$ such that for all $N\ge p+1$, $j\in\{0,\dots,N-p-1\}$ and $\xi\in\mathbb{R}\setminus\{0\}$, one has

$$|A_1(j/N,\xi)| \le c_{25}\max\Big(|\xi|^{-\max_{s\in[0,1]}H(s)-\varepsilon-1/2},|\xi|^{-\min_{s\in[0,1]}H(s)+\varepsilon-1/2}\Big). \tag{7.2.76}$$

Also note that (7.2.8) and (7.2.24) imply that for all arbitrarily small $\varepsilon>0$, there is a constant $c_{26}>0$ such that for all $N\ge p+1$, $j'\in\{0,\dots,N-p-1\}$, $k'\in\{0,\dots,p\}$, $s'\in[\frac{j'}{N},\frac{j'+1}{N}]$ and $\xi\in\mathbb{R}\setminus\{0\}$ one has

$$|A_2(j'/N,s'+k'/N,\xi)| \le c_{26}\max\Big(|\xi|^{-\max_{s\in[0,1]}H(s)-\varepsilon-1/2},|\xi|^{-\min_{s\in[0,1]}H(s)+\varepsilon-1/2}\Big). \tag{7.2.77}$$

It follows from (7.2.75), Lemma 7.2.5, (7.2.76), (7.2.77), and the inequalities $|s+k/N|\le 1$, $|s'+k'/N|\le 1$, $|s+k/N-j/N|\le (p+1)/N$ and $|s'+k'/N-j'/N|^2\le (p+1)^2/N^2$ that

$$|\mathcal{I}_{1,2}(j,j',N)| \le c_{27}N^{-3},$$

where the constant

$$\begin{aligned} c_{27} &= c_{26}c_{25}\Big(2(p+1)^{3/2}\sum_{k=0}^{p}|a_k|\Big)^2 \\ &\quad\times\int_{\mathbb{R}}\min\Big(1,|\xi|^2\Big)\max\Big(|\xi|^{-2\max_{s\in[0,1]}H(s)-2\varepsilon-1},|\xi|^{-2\min_{s\in[0,1]}H(s)+2\varepsilon-1}\Big)\,\mathrm{d}\xi \\ &< +\infty. \end{aligned}$$

Finally let us prove that the inequality (7.2.15) is satisfied when $l = l' = 2$. Setting in (7.2.11) $l = l' = 2$ one obtains that

$$\begin{aligned}
&\mathcal{I}_{2,2}(j,j',N)\\
&= N^2 \int_{\frac{j}{N}}^{\frac{j+1}{N}} \int_{\frac{j'}{N}}^{\frac{j'+1}{N}} \int_{\mathbb{R}} \Big(\sum_{k=0}^{p} a_k (e^{i(s+k/N)\xi} - 1)(s+k/N-j/N)^2 A_2(j/N, s+k/N, \xi) \Big)\\
&\quad \times \Big(\sum_{k'=0}^{p} a_{k'} (e^{i(s'+k'/N)\xi} - 1)(s'+k'/N-j'/N)^2 A_2(j'/N, s'+k'/N, \xi) \Big) \,\mathrm{d}\xi\, \mathrm{d}s'\, \mathrm{d}s
\end{aligned} \tag{7.2.78}$$

Then it follows from (7.2.78), Lemma 7.2.5, (7.2.77) and the inequalities $|s+k/N| \leq 1$, $|s'+k'/N| \leq 1$, $(s+k/N-j/N)^2 \leq (p+1)^2/N^2$ and $(s'+k'/N-j'/N)^2 \leq (p+1)^2/N^2$ that

$$|\mathcal{I}_{2,2}(j,j',N)| \leq c_{28} N^{-4},$$

where the constant

$$\begin{aligned}
c_{28} &= \Big(2c_{26}(p+1)^2 \sum_{k=0}^{p} |a_k| \Big)^2\\
&\quad \times \int_{\mathbb{R}} \min\Big(1, |\xi|^2\Big) \max\Big(|\xi|^{-2\max_{s\in[0,1]} H(s) - 2\varepsilon - 1}, |\xi|^{-2\min_{s\in[0,1]} H(s) + 2\varepsilon - 1} \Big) \,\mathrm{d}\xi\\
&< +\infty.
\end{aligned}$$

Finally Lemma 7.2.4 holds. □

7.3 Proof of Lemma 5.3.3

Lemma 7.3.1 *There is a constant $c > 0$ such that for every $i \in \{0, \ldots, N-p-1\}$ one has,*

$$\Big| Var(\Delta_a \overline{X}_{i,N}) - C(i/N) N^{-2H(i/N)} \Big| \leq c \log(N) N^{-1-2H(i/N)}.$$

Recall that

$$C(s) := \int_{\mathbb{R}} \frac{|e^{i\eta} - 1| |\sum_{k=0}^{p} a_k e^{ik\eta}|}{|\eta|^{2H(s)+3}} \,\mathrm{d}\eta.$$

Proof of Lemma 7.3.1: Observe that (7.2.25), (5.2.15), (7.2.6) and the change of variables $\eta = \xi/N$ show that for all $j \in \{0, \ldots, N-p-1\}$,

$$\begin{aligned}
\mathcal{I}_{0,0}(j,j,N) &= N^2 \int_{\mathbb{R}} \frac{|e^{i\xi/N} - 1|^2 |\sum_{k=0}^{p} a_k e^{i(k/N)\xi}|^2}{\xi^2} \big(A_0(j/N,\xi)\big)^2 \,\mathrm{d}\xi\\
&= N^2 \int_{\mathbb{R}} \frac{|e^{i\xi/N} - 1|^2 |\sum_{k=0}^{p} a_k e^{i(k/N)\xi}|^2}{|\xi|^{2H(j/N)+3}} \,\mathrm{d}\xi\\
&= N^{-2H(j/N)} \int_{\mathbb{R}} \frac{|e^{i\eta} - 1|^2 |\sum_{k=0}^{p} a_k e^{ik\eta}|^2}{|\eta|^{2H(j/N)+3}} \,\mathrm{d}\eta\\
&= C(j/N) N^{-2H(j/N)}.
\end{aligned} \tag{7.3.1}$$

Then it follows from (7.3.1), (7.2.10), the triangle inequality, Relations (7.2.13) to (7.2.15), and the inequality $\max_{s\in[0,1]} H(s) < 1$, that there exists a constant $c > 0$ such that for all $N \geq p+1$ and $j \in \{0,\dots,N-p-1\}$,

$$\begin{aligned}\left|\mathrm{Var}(\Delta_a \overline{X}_{j,N}) - C(j/N)N^{-2H(j/N)}\right| &= \left|\mathrm{Var}(\Delta_a \overline{X}_{j,N}) - \mathcal{I}_{0,0}(j,j,N)\right| \\ &= \left|\sum_{0\le l,l'\le 2,(l,l')\neq(0,0)} \mathcal{I}_{l,l'}(j,j,N)\right| \\ &\le \sum_{0\le l,l'\le 2,(l,l')\neq(0,0)} \left|\mathcal{I}_{l,l'}(j,j,N)\right| \\ &\le c\log(N)N^{-2H(j/N)-1}.\end{aligned}$$

Thus Lemma 5.3.3 has been proved. □

7.4 Proof of Proposition 5.3.11

Proposition 7.4.1 *For any n big enough one sets*

$$v(N_n,m_n) = \left(N_n^{-1/2} + m_n^{-1/2}\right)\left(1 + m_n^{-1} N_n^{2\max_{s\in[\mu_{N_n},\nu_{N_n}]} H(s)}\right). \qquad (7.4.1)$$

Recall that m_n has been defined in (5.2.4). Let us assume that N_n is chosen such that $m_n^{-1} N_n^{2\max_{s\in[\mu_{N_n},\nu_{N_n}]} H(s)}$ remains bounded when n goes to infinity. Then for all $h \in C^1_{pol}(\mathbb{R})$, there exists a constant $c > 0$ such that for any n big enough one has

$$\mathbb{E}\left\{\left|\widehat{V}(h;\mu_{N_n},\nu_{N_n}) - \overline{V}(h;\mu_{N_n},\nu_{N_n})\right|\right\} \le cv(N_n,m_n) = \mathcal{O}(N_n^{-1/2}) \qquad (7.4.2)$$

In order to prove of Proposition 7.4.1, we need the following quite useful lemma which is a natural extension of Lemma 7 in ([35]) (see also ([36])).

Lemma 7.4.2 *For all $i = 0,\dots,N_n-1$, we set*

$$\mathscr{E}_{i,N_n,n} = \widehat{Y}_{i,N_n,n} - \overline{Y}_{i,N_n}. \qquad (7.4.3)$$

Moreover, we denote by $\mathcal{G}_X$ the σ-field generated by $\{X(s)\}_{s\in[0,1]}$, that is,

$$\mathcal{G}_X = \sigma(X(s), 0 \le s \le 1).$$

Then:

(i) *Conditional on $\mathcal{G}_X$, the random variables $\{\mathscr{E}_{i,N_n,n}\}_{i\in\{0,\dots,N_n-1\}}$ are independent.*

(ii) *There exist a positive random variable $C > 0$ of finite moment of any order, and a positive constant c, such that, for all n big enough and $i \in \{0, \dots, N_n - 1\}$, one has,*

$$|\mathbb{E}(\mathscr{E}_{i,N_n,n}|\mathcal{G}_X)| \le C m_n^{-1}, \quad |\mathbb{E}(\mathscr{E}_{i,N_n,n})| \le c m_n^{-1}, \tag{7.4.4}$$
$$\mathbb{E}(\mathscr{E}_{i,N_n,n}^2|\mathcal{G}_X) \le C m_n^{-1}, \quad \mathbb{E}(\mathscr{E}_{i,N_n,n}^2) \le c m_n^{-1}, \tag{7.4.5}$$
$$\mathbb{E}(\mathscr{E}_{i,N_n,n}^4|\mathcal{G}_X) \le C m_n^{-2}, \quad \mathbb{E}(\mathscr{E}_{i,N_n,n}^4) \le c m_n^{-2}. \tag{7.4.6}$$

(iii) *Conditional on $\mathcal{G}_X$, the second moment of $\mathscr{E}_{i,N_n,n}$ has the expansion,*

$$\mathbb{E}(\mathscr{E}_{i,N_n,n}^2|\mathcal{G}_X) = 2m_n^{-1}(Y_{i/N_n})^2 + \alpha_{i,N_n,n}, \tag{7.4.7}$$

where for all real $l \ge 1$, there exists a constant $c(l)$ only depending on l, such that,

$$\mathbb{E}\{|\alpha_{i,N_n,n}|^l\} \le c(l)\{m_n^{-l} N_n^{-lH(i/N_n)} + m_n^{-2l}\}. \tag{7.4.8}$$

(iv) *For all real $l \ge 1$, there exists a constant $c(l)$ only depending on l, such that,*

$$\mathbb{E}\{|\widehat{Y}_{i,N_n,n}|^l\} \le c(l).$$

Proof of Lemma 7.4.2: In order to show that Part (i) holds, we will prove that, almost surely for all $i \in \{0, \dots, N_n - 1\}$,

$$\mathscr{E}_{i,N_n,n} = \tilde{\mathscr{E}}_{i,N_n,n} + N_n \int_{j_i/n}^{i/N_n} Y(s)\,\mathrm{d}s - N_n \int_{(i+1)/N_n}^{j_{i+1}/n} Y(s)\,\mathrm{d}s, \tag{7.4.9}$$

where

$$\tilde{\mathscr{E}}_{i,N_n,n} := 2N_n \sum_{k=0}^{j_{i+1}-j_i-1} \int_{(j_i+k)/n}^{(j_i+k+1)/n} (Z(s) - Z((j_i+k)/n))\sigma(s)\,\mathrm{d}W(s). \tag{7.4.10}$$

Observe that, assuming that (7.4.9) and (7.4.10) are satisfied are satisfied then on can easily obtain Part (i) of the lemma. Indeed, on one hand, conditional on $\mathcal{G}_X$, $\{\tilde{\mathscr{E}}_{i,N_n,n}\}_{i\in\{0,\dots,N_n-1\}}$ is a sequence of independent random variables because the intervals $[j_i/n, j_{i+1}/n)$, $i \in \{0, \dots, N_n - 1\}$ are disjoint; on the other hand,

$$\Big\{N_n \int_{j_i/n}^{i/N_n} Y(s)\,\mathrm{d}s - N_n \int_{j_{i+1}/n}^{(i+1)/N_n} Y(s)\,\mathrm{d}s\Big\}_{i\in\{0,\dots,N_n-1\}}$$

are $\mathcal{G}_X$-measurable.

Relation (7.4.9) will result from Itô formula, we are now going to recall. Let $\{P_t\}_{t\in[0,1]}$ be an arbitrary centered real-valued process defined for all $t \in [0,1]$, as,

$$P_t = P_0 + \int_0^t J_s\,\mathrm{d}W(s),$$

where:

- P_0 is a bounded random variable;
- $\{J_s\}_{s\in[0,1]}$ is a (Lebesgue) mesurable stochastic process, which is (stochastically) independent on the Brownian motion $\{W(s)\}_{s\in[0,1]}$, and which satisfies,

$$\mathbb{E}\left(\int_0^1 J_s^2\,\mathrm{d}s\right) < +\infty.$$

Then the following Itô formula: a.s.

$$F(P_t) = F(P_0) + \int_0^t F'(P_s)J_s\,\mathrm{d}W(s) + \frac{1}{2}\int_0^t F^{(2)}(P_s)J_s^2\,\mathrm{d}s, \tag{7.4.11}$$

holds, for all real-valued function F which is two times continuously on the real line, almost surely.

Now we set, in (7.4.11), $P_0 = 0$, for each real x, $F(x) = x^2$, for all $i \in \{0,\dots,N_n-1\}$, $k \in \{0,\dots,j_{i+1}-j_i-1\}$ and $t \in [0,1]$,

$$P_t = Z_t^{i,k} := \int_0^t \sigma(s)\mathbf{1}_{[(j_i+k)/n,+\infty)}(s)\,\mathrm{d}W(s),$$

Thus, we obtain that,

$$\left(Z_t^{i,k}\right)^2 = 2\int_0^t Z_s^{i,k}\sigma(s)\mathbf{1}_{[(j_i+k)/n,+\infty)}(s)\,\mathrm{d}W(s) + \int_0^t (\sigma(s))^2\mathbf{1}_{[(j_i+k)/n,+\infty)}(s)\,\mathrm{d}s. \tag{7.4.12}$$

Then, observing that

$$Z((j_i+k+1)/n) - Z((j_i+k)/n) = Z^{i,k}_{(j_i+k+1)/n}, \tag{7.4.13}$$

it follows from (7.4.12) and the fact that $Y(s) = (\sigma(s))^2$, that,

$$\begin{aligned}&\big(Z((j_i+k+1)/n) - Z((j_i+k)/n)\big)^2\\ &= \left(Z^{i,k}_{(j_i+k+1)/n}\right)^2\\ &= 2\int_{(j_i+k)/n}^{(j_i+k+1)/n}\Big(Z(s) - Z((j_i+k)/n)\Big)\sigma(s)\,\mathrm{d}W(s) + \int_{(j_i+k)/n}^{(j_i+k+1)/n} Y(s)\,\mathrm{d}s.\end{aligned} \tag{7.4.14}$$

Then, (5.2.5) and (7.4.14) entail that

$$\widehat{Y}_{i,N_n,n} = \tilde{\mathscr{E}}_{i,N_n,n} + N_n\int_{j_i/n}^{j_{i+1}/n} Y(s)\,\mathrm{d}s, \tag{7.4.15}$$

where the error term $\tilde{\mathscr{E}}_{i,N_n,n}$ has been defined in (7.4.10). Relation (7.4.15) implies that $\mathscr{E}_{i,N_n,n} := \widehat{Y}_{i,N_n,n} - \overline{Y}_{i,N_n}$ can be expressed as:

$$\mathscr{E}_{i,N_n,n} = \tilde{\mathscr{E}}_{i,N_n,n} + \Big(N_n\int_{j_i/n}^{j_{i+1}/n} Y(s)\,\mathrm{d}s - N_n\int_{i/N_n}^{(i+1)/N_n} Y(s)\,\mathrm{d}s\Big), \tag{7.4.16}$$

which means that (7.4.9) is satisfied.

Let us now show that Part (ii) of the lemma holds. By using, (7.4.15), (5.2.5), the fact that $\{Y(s)\}_{s\in[0,1]}$ is $\mathcal{G}_X$ measurable, (5.1.4), and the equality $Y(s)=(\sigma(s))^2$ (see (5.2.2)), we get

$$\begin{aligned}\mathbb{E}\,(\tilde{\mathscr{E}}_{i,N_n,n}|\mathcal{G}_X) &= \mathbb{E}\,(\widehat{Y}_{i,N_n,n}|\mathcal{G}_X)-N_n\int_{j_i/n}^{j_{i+1}/n}Y(s)\,\mathrm{d}s\\ &= N_n\sum_{k=0}^{j_{i+1}-j_i-1}\mathbb{E}\left(\Big|\int_{(j_i+k)/n}^{(j_i+k+1)/n}\sigma(s)\,dW(s)\Big|^2\Big|\mathcal{G}_X\right)\\ &= N_n\sum_{k=0}^{j_{i+1}-j_i-1}\int_{(j_i+k)/n}^{(j_i+k+1)/n}Y(s)\,\mathrm{d}s-N_n\int_{j_i/n}^{j_{i+1}/n}Y(s)\,\mathrm{d}s\\ &= 0 \end{aligned}\tag{7.4.17}$$

and

$$\mathbb{E}\left(N_n\int_{j_i/n}^{i/N_n}Y(s)\,\mathrm{d}s-N_n\int_{j_{i+1}/n}^{(i+1)/N_n}Y(s)\,\mathrm{d}s\Big|\mathcal{G}_X\right)=N_n\int_{j_i/n}^{i/N_n}Y(s)\,\mathrm{d}s-N_n\int_{j_{i+1}/n}^{(i+1)/N_n}Y(s)\,\mathrm{d}s.\tag{7.4.18}$$

Then, it follows from (7.4.9), (7.4.17), (7.4.18), the triangle inequality, that,

$$\begin{aligned}&\Big|\mathbb{E}\,(\mathscr{E}_{i,N_n,n}|\mathcal{G}_X)\Big|\\ &=\Big|\mathbb{E}\,(\tilde{\mathscr{E}}_{i,N_n,n}|\mathcal{G}_X)+\mathbb{E}\left(N_n\int_{j_i/n}^{i/N_n}Y(s)\,\mathrm{d}s-N_n\int_{j_{i+1}/n}^{(i+1)/N_n}Y(s)\,\mathrm{d}s\Big|\mathcal{G}_X\right)\Big|\\ &=\Big|N_n\int_{j_i/n}^{i/N_n}Y(s)\,\mathrm{d}s-N_n\int_{j_{i+1}/n}^{(i+1)/N_n}Y(s)\,\mathrm{d}s\Big|\\ &\le N_n\int_{j_i/n}^{i/N_n}|Y(s)|\,\mathrm{d}s+N_n\int_{j_{i+1}/n}^{(i+1)/N_n}|Y(s)|\,\mathrm{d}s.\end{aligned}\tag{7.4.19}$$

Moreover, in view of the $Y(s)=f(X(s))$ where $f\in C^2_{pol}(\mathbb{R})$, one has,

$$\sup_{s\in[0,1]}|Y(s)|\le c_1(1+\|X\|_\infty^K),\tag{7.4.20}$$

where $\|X\|_\infty=\sup_{s\in[0,1]}|X(s)|$ and $c_1,K>0$ are two constants only depending on f. Then, (7.4.19), (7.4.20) and the fact that the random variable $\|X\|_\infty$ is $\mathcal{G}_X$ mesurable, imply that,

$$\begin{aligned}&\Big|\mathbb{E}\,(\mathscr{E}_{i,N_n,n}|\mathcal{G}_X)\Big|\\ &\le c_1(1+\|X\|_\infty^K)N_n\Big((\frac{i}{N_n}-\frac{j_i}{n})+(\frac{i+1}{N_n}-\frac{j_{i+1}}{n})\Big).\end{aligned}\tag{7.4.21}$$

Observe that for each $i \in \{0, \dots, N_n - p - 1\}$,

$$\begin{aligned} N_n(\frac{i}{N_n} - \frac{j_i}{n}) &= N_n(\frac{i}{N_n} - \frac{[in/N_n]}{n}) \\ &\leq N_n(\frac{i}{N_n} - \frac{in/N_n - 1}{n}) \\ &= \frac{N_n}{n} \leq m_n^{-1}. \end{aligned} \tag{7.4.22}$$

It follows from (7.4.21) and (7.4.22) that

$$\left| \mathbb{E}\left(\mathscr{E}_{i,N_n,n} | \mathcal{G}_X\right) \right| \leq C_1 m_n^{-1}, \tag{7.4.23}$$

where $C_1 = 2c_1(1 + \|X\|_\infty^K)$ is a positive random variable of finite moment of any order. Moreover, (7.4.23) implies

$$\left|\mathbb{E}\left(\mathscr{E}_{i,N_n,n}\right)\right| = \left|\mathbb{E}\left(\mathbb{E}\left(\mathscr{E}_{i,N_n,n}|\mathcal{G}_X\right)\right)\right| \leq \mathbb{E}\left(\left|\mathbb{E}\left(\mathscr{E}_{i,N_n,n}|\mathcal{G}_X\right)\right|\right) \leq c_1 m_n^{-1}, \tag{7.4.24}$$

where $c_1 = \mathbb{E}(C_1)$. Thus (7.4.4) has been proved.
Now, we are going to prove that (7.4.5) holds. First observe that, by using (7.4.19), (7.4.21) and (7.4.22), we have for all real $l > 0$,

$$\left| N_n \int_{j_i/n}^{i/N_n} Y(s)\,\mathrm{d}s - N_n \int_{j_{i+1}/n}^{(i+1)/N_n} Y(s)\,\mathrm{d}s \right|^l \leq C_1^l m_n^{-l}. \tag{7.4.25}$$

Now we need to show the following inequality:

$$\mathbb{E}\left(\tilde{\mathscr{E}}_{i,N_n,n}^2 | \mathcal{G}_X\right) \leq 4c_1^2\big(1 + \|X\|_\infty^K\big)^2 m_n^{-1}. \tag{7.4.26}$$

In order to prove (7.4.26), recall that for all $t \in [0,1]$,

$$Z(t) = z_0 + \int_0^t \sigma(s)\,\mathrm{d}W(s).$$

Putting the above definition of Z and (7.4.10) together, it yields

$$\tilde{\mathscr{E}}_{i,N_n,n} = 2N_n \sum_{k=0}^{j_{i+1}-j_i-1} \int_{(j_i+k)/n}^{(j_i+k+1)/n} \Big(\int_{(j_i+k)/n}^{t} \sigma(s)\,\mathrm{d}W(s)\Big)\sigma(t)\,\mathrm{d}W(t). \tag{7.4.27}$$

Then conditional on $\mathcal{G}_X$, we have

$$\begin{aligned} \mathbb{E}\left(\tilde{\mathscr{E}}_{i,N_n,n}^2 | \mathcal{G}_X\right) &= 4N_n^2 \sum_{k=0}^{j_{i+1}-j_i-1} \int_{(j_i+k)/n}^{(j_i+k+1)/n} \Big(\int_{(j_i+k)/n}^{t} \sigma^2(s)\,\mathrm{d}s\Big)\sigma^2(t)\,\mathrm{d}t \\ &+8N_n^2 \mathbb{E}\Big(\sum_{0 \leq k < k' \leq j_{i+1}-j_i-1} \Big(\Big(\int_{(j_i+k)/n}^{(j_i+k+1)/n} \Big(\int_{(j_i+k)/n}^{t} \sigma(s)\,\mathrm{d}W(s)\Big)\sigma(t)\,\mathrm{d}W(t)\Big) \\ &\times\Big(\int_{(j_i+k')/n}^{(j_i+k'+1)/n} \Big(\int_{(j_i+k')/n}^{t} \sigma(s)\,\mathrm{d}W(s)\Big)\sigma(t)\,\mathrm{d}W(t)\Big)\Big| \mathcal{G}_X\Big). \end{aligned} \tag{7.4.28}$$

Observe that, the second term in the right side of equality (7.4.28) is almost surely equal to 0, since, conditional on $\mathcal{G}_X$, the random variables

$$\Big\{\int_{(j_i+k)/n}^{(j_i+k+1)/n}\Big(\int_{(j_i+k)/n}^{t}\sigma(s)\,\mathrm{d}W(s)\Big)\sigma(t)\,\mathrm{d}W(t)\Big\}_{k=0,\dots,j_{i+1}-j_i-1}$$

are centered and independent. Therefore, we get

$$\begin{aligned}\mathbb{E}\,(\tilde{\mathscr{E}}^2_{i,N_n,n}|\mathcal{G}_X) &= 4N_n^2\sum_{k=0}^{j_{i+1}-j_i-1}\int_{(j_i+k)/n}^{(j_i+k+1)/n}\Big(\int_{(j_i+k)/n}^{t}\sigma^2(s)\,\mathrm{d}s\Big)\sigma^2(t)\,\mathrm{d}t\\ &= 4N_n^2\sum_{k=0}^{j_{i+1}-j_i-1}\int_{(j_i+k)/n}^{(j_i+k+1)/n}\Big(\int_{(j_i+k)/n}^{t}Y(s)\,\mathrm{d}s\Big)Y(t)\,\mathrm{d}t.\end{aligned} \tag{7.4.29}$$

It follows from (7.4.29) and (7.4.20) that

$$\begin{aligned}\mathbb{E}\,(\tilde{\mathscr{E}}^2_{i,N_n,n}|\mathcal{G}_X) &\le 4c_1^2\big(1+\|X\|_\infty^K\big)^2N_n^2\sum_{k=0}^{j_{i+1}-j_i-1}\int_{(j_i+k)/n}^{(j_i+k+1)/n}\Big(\int_{(j_i+k)/n}^{t}1\,\mathrm{d}s\Big)\,\mathrm{d}t\\ &= 2c_1^2\big(1+\|X\|_\infty^K\big)^2N_n^2n^{-2}(j_{i+1}-j_i).\end{aligned} \tag{7.4.30}$$

Next combining (7.4.30) with the inequalities $j_{i+1}-j_i\le m_n+1\le 2m_n$ and $N_nn^{-1}\le m_n^{-1}$, one gets, (7.4.26). Next, setting,

$$A_{i,N_n,n}=N_n\int_{j_i/n}^{j_{i+1}/n}Y(s)\,\mathrm{d}s-N_n\int_{i/N_n}^{(i+1)/N_n}Y(s)\,\mathrm{d}s, \tag{7.4.31}$$

and using (7.4.9), the inequality for any reals $x,y\in\mathbb{R}$, $(x+y)^2\le 2x^2+2y^2$, (7.4.26) and (7.4.25), it follows that,

$$\begin{aligned}\mathbb{E}\,(\mathscr{E}^2_{i,N_n,n}|\mathcal{G}_X) &= \mathbb{E}\Big((\tilde{\mathscr{E}}_{i,N_n,n}+A_{i,N_n,n})^2\Big|\mathcal{G}_X\Big)\\ &\le 2\mathbb{E}\,(\tilde{\mathscr{E}}^2_{i,N_n,n}|\mathcal{G}_X)+2\mathbb{E}\,(A^2_{i,N_n,n}|\mathcal{G}_X)\\ &\le 8c_1^2(1+\|X\|_\infty^K)^2m_n^{-1}+2c_1^2(1+\|X\|_\infty^K)^2m_n^{-2}\\ &\le C_2m_n^{-1},\end{aligned} \tag{7.4.32}$$

where $C_2=10c_1^2(1+\|X\|_\infty^K)^2$ is a positive random variable of finite moment of any order. Moreover, (7.4.32) implies that

$$\mathbb{E}\,(\mathscr{E}^2_{i,N_n,n})=\mathbb{E}\,(\mathbb{E}\,(\mathscr{E}^2_{i,N_n,n}|\mathcal{G}_X))\le c_2m_n^{-1}, \tag{7.4.33}$$

where $c_2=\mathbb{E}\,(C_2)$. Thus, we have shown that (7.4.5) is satisfied.

Let us now show that (7.4.6) holds. The classical Wiener's calculus (see for example, [49], P. 202) shows that, there exists a constant $c_3>0$ such that for all n

big enough, $i \in \mathcal{J}(\mu_{N_n}, \nu_{N_n})$ and $k \in \{0, \dots, j_{i+1} - j_i - 1\}$, one has,

$$\mathbb{E}\Big(\Big(\int_{(j_i+k)/n}^{(j_i+k+1)/n}\Big(\int_{(j_i+k)/n}^{t}\sigma(s)\,\mathrm{d}W(s)\Big)\sigma(t)\,\mathrm{d}W(t)\Big)^4\Big|\mathcal{G}_X\Big) \tag{7.4.34}$$

$$\le c_3\Big(\mathbb{E}\Big(\Big(\int_{(j_i+k)/n}^{(j_i+k+1)/n}\Big(\int_{(j_i+k)/n}^{t}\sigma(s)\,\mathrm{d}W(s)\Big)\sigma(t)\,\mathrm{d}W(t)\Big)^2\Big|\mathcal{G}_X\Big)\Big)^2$$

$$= c_3\Big(\int_{(j_i+k)/n}^{(j_i+k+1)/n}\int_{(j_i+k)/n}^{t} Y(s)Y(t)\,\mathrm{d}s\,\mathrm{d}t\Big)^2. \tag{7.4.35}$$

Next, it follows from (7.4.29) and (7.4.20), (7.4.34) that,

$$\mathbb{E}\Big(\Big(\int_{(j_i+k)/n}^{(j_i+k+1)/n}\Big(\int_{(j_i+k)/n}^{t}\sigma(s)\,\mathrm{d}W(s)\Big)\sigma(t)\,\mathrm{d}W(t)\Big)^4\Big|\mathcal{G}_X\Big) \le 4^{-1}c_3c_1^4(1+\|X\|_\infty^K)^4 n^{-4}. \tag{7.4.36}$$

Next (7.4.27), the equality that for all independent centered real-valued random variables $X_1, \dots, X_n$,

$$\mathbb{E}\Big(\big(\sum_{k=1}^n X_k\big)^4\Big) = \sum_{k=1}^n \mathbb{E}(X_k^4) + \sum_{k=1}^n\sum_{k'=1}^n \mathbb{E}(X_k^2)\mathbb{E}(X_{k'}^2),$$

(7.4.36), (7.4.29) and (7.4.30), imply that,

$$\begin{aligned}
&\mathbb{E}(\tilde{\mathscr{E}}_{i,N_n,n}^4|\mathcal{G}_X)\\
&= 16N_n^4\sum_{k=0}^{j_{i+1}-j_i-1}\mathbb{E}\Big(\Big(\int_{(j_i+k)/n}^{(j_i+k+1)/n}\Big(\int_{(j_i+k)/n}^{t}\sigma^2(s)\,\mathrm{d}W(s)\Big)\sigma^2(t)\,\mathrm{d}W(t)\Big)^4\Big|\mathcal{G}_X\Big)\\
&\quad +16N_n^4\sum_{k=0}^{j_{i+1}-j_i-1}\sum_{k'=0}^{j_{i+1}-j_i-1}\mathbb{E}\Big(\Big(\int_{(j_i+k)/n}^{(j_i+k+1)/n}\Big(\int_{(j_i+k)/n}^{t}\sigma^2(s)\,\mathrm{d}W(s)\Big)\sigma^2(t)\,\mathrm{d}W(t)\Big)^2\Big|\mathcal{G}_X\Big)\\
&\quad \times\mathbb{E}\Big(\Big(\int_{(j_i+k')/n}^{(j_i+k'+1)/n}\Big(\int_{(j_i+k')/n}^{t}\sigma^2(s)\,\mathrm{d}W(s)\Big)\sigma^2(t)\,\mathrm{d}W(t)\Big)^2\Big|\mathcal{G}_X\Big)\\
&\le 4c_3c_1^4\big(1+\|X\|_\infty^K\big)^4 n^{-4}N_n^4(j_{i+1}-j_i) + 4c_3c_1^4(1+\|X\|_\infty^K)^4 n^{-4}N_n^4(j_{i+1}-j_i)^2\\
&\le 8c_3c_1^4\big(1+\|X\|_\infty^K\big)^4 m_n^{-2}.
\end{aligned} \tag{7.4.37}$$

Next, using (7.4.9), (7.4.31), the inequality for any reals $x, y \in \mathbb{R}$, $(x+y)^4 \le 8x^4 + 8y^4$, (7.4.37) and (7.4.25), one can show that,

$$\begin{aligned}
\mathbb{E}(\mathscr{E}_{i,N_n,n}^4|\mathcal{G}_X) &= \mathbb{E}\Big((\tilde{\mathscr{E}}_{i,N_n,n} + A_{i,N_n,n})^4\Big|\mathcal{G}_X\Big)\\
&\le 8\mathbb{E}(\tilde{\mathscr{E}}_{i,N_n,n}^4|\mathcal{G}_X) + 8\mathbb{E}(A_{i,N_n,n}^4|\mathcal{G}_X)\\
&\le 8c_3c_1^4(1+\|X\|_\infty^K)^4 m_n^{-2} + 8c_1^4(1+\|X\|_\infty^K)^4 m_n^{-4}\\
&\le C_3 m_n^{-2},
\end{aligned} \tag{7.4.38}$$

where $C_3 = 8(c_3+1)c_1^4(1+\|X\|_\infty^K)^4$ is a random variable of finite moment of any order. Moreover, (7.4.38) entails that,

$$\mathbb{E}\,(\mathscr{E}_{i,N_n,n}^4) = \mathbb{E}\Big(\mathbb{E}\,(\mathscr{E}_{i,N_n,n}^4|\mathcal{G}_X)\Big) \leq c_3 m_n^{-2}, \tag{7.4.39}$$

where $c_3 = \mathbb{E}\,(C_3)$. Thus, we have shown that (7.4.6) is satisfied.

Let us now prove that Part *(iii)* of the lemma holds. By using (7.4.29), we obtain

$$\begin{aligned}
\mathbb{E}\,(\tilde{\mathscr{E}}_{i,N_n,n}^2|\mathcal{G}_X) &= 4N_n^2\sum_{k=0}^{j_{i+1}-j_i-1}\int_{(j_i+k)/n}^{(j_i+k+1)/n}\Big(\int_{(j_i+k)/n}^{t} Y(s)\,\mathrm{d}s\Big)Y(t)\,\mathrm{d}t\\
&= 4N_n^2\sum_{k=0}^{j_{i+1}-j_i-1}\int_{(j_i+k)/n}^{(j_i+k+1)/n}\int_{(j_i+k)/n}^{t}\Big(Y(\frac{i}{N_n})\Big)^2\,\mathrm{d}s\,\mathrm{d}t+\tilde{\alpha}_{i,N_n,n}\\
&= 2N_n^2 n^{-2}(j_{i+1}-j_i)\Big(Y(\frac{i}{N_n})\Big)^2+\tilde{\alpha}_{i,N_n,n},
\end{aligned} \tag{7.4.40}$$

where

$$\tilde{\alpha}_{i,N_n,n} = 4N_n^2\sum_{k=0}^{j_{i+1}-j_i-1}\int_{(j_i+k)/n}^{(j_i+k+1)/n}\int_{(j_i+k)/n}^{t}\Big(Y(s)Y(t)-\Big(Y(\frac{i}{N_n})\Big)^2\Big)\,\mathrm{d}s\,\mathrm{d}t. \tag{7.4.41}$$

Thus we can write

$$\mathbb{E}\,(\tilde{\mathscr{E}}_{i,N_n,n}^2|\mathcal{G}_X) = 2m_n^{-1}\Big(Y(\frac{i}{N_n})\Big)^2+\alpha_{i,N_n,n}, \tag{7.4.42}$$

where

$$\alpha_{i,N_n,n} = \tilde{\alpha}_{i,N_n,n}+B_{i,N_n,n}, \tag{7.4.43}$$

with

$$B_{i,N_n,n} = 2\Big(N_n^2 n^{-2}(j_{i+1}-j_i)-m_n^{-1}\Big)\Big(Y(\frac{i}{N_n})\Big)^2. \tag{7.4.44}$$

Thus, in order to obtain Part *(iii)* of the lemma, it is sufficient to show that for all real $l \geq 1$, there is a constant $c(l) > 0$ only depending on l such that,

$$\mathbb{E}\,|B_{i,N_n,n}|^l \leq c(l)m_n^{-2l}, \tag{7.4.45}$$

and

$$\mathbb{E}\,|\tilde{\alpha}_{i,N_n,n}|^l \leq c(l)m_n^{-l}N_n^{-lH(i/N_n)}. \tag{7.4.46}$$

Let us first prove that (7.4.45) holds. The facts that $n/N_n \in (m_n, m_n+1)$ and $j_{i+1}-j_i \in \{m_n, m_n+1\}$ imply

$$N_n^2 n^{-2}(j_{i+1}-j_i)-m_n^{-1} \in \Big[\frac{m_n}{(m_n+1)^2}-m_n^{-1}, \frac{m_n+1}{m_n^2}-m_n^{-1}\Big] = \Big[\frac{-2(m_n+1)}{m_n(m_n+1)^2}, m_n^{-2}\Big].$$

Thus there exists a constant $c_4 > 0$ such that

$$|N_n^2 n^{-2}(j_{i+1}-j_i)-m_n^{-1}| \leq c_4 m_n^{-2}. \tag{7.4.47}$$

Putting together (7.4.44), (7.4.47) and (7.4.20), we get

$$\mathbb{E}\,|B_{i,N_n,n}|^l \le c_{1,l} m_n^{-2l}, \tag{7.4.48}$$

where $c_{1,l} = 2^l c_4^l c_1^{2l} \mathbb{E}\left((1+\|X\|_\infty^K)^{2l}\right)$ is a finite constant only depending on l; thus we have shown that (7.4.45) is satisfied.

Let us now prove that (7.4.46) holds. By using (7.4.41), the fact that that $x \mapsto x^l$ is a convex function, Hölder inequality, and the inequality $|j_{i+1} - j_i| \le m_n + 1$, we obtain

$$\begin{aligned}
|\tilde{\alpha}_{i,N_n,n}|^l &\le \left|4N_n^2 \sum_{k=0}^{j_{i+1}-j_i-1} \int_{(j_i+k)/n}^{(j_i+k+1)/n} \int_{(j_i+k)/n}^{t} \Big(Y(s)Y(t) - (Y(\frac{i}{N_n}))^2\Big)\,\mathrm{d}s\,\mathrm{d}t\right|^l \\
&\le 4^l N_n^{2l} |j_{i+1}-j_i|^{l-1} \sum_{k=0}^{j_{i+1}-j_i-1} n^{-(l-1)} \int_{(j_i+k)/n}^{(j_i+k+1)/n} \left|t - \frac{j_i+k}{n}\right|^{l-1} \\
&\quad \times \int_{(j_i+k)/n}^{t} \left|Y(s)Y(t) - (Y(i/N_n))^2\right|^l \mathrm{d}s\,\mathrm{d}t \\
&\le 4^l N_n^{2l} (m_n+1)^{l-1} n^{-2(l-1)} \\
&\quad \times \sum_{k=0}^{j_{i+1}-j_i-1} \int_{(j_i+k)/n}^{(j_i+k+1)/n} \int_{(j_i+k)/n}^{t} \left|Y(s)Y(t) - (Y(i/N_n))^2\right|^l \mathrm{d}s\,\mathrm{d}t.
\end{aligned} \tag{7.4.49}$$

Next it follows from (7.4.49) and Fubini Theorem that,

$$\mathbb{E}\,|\tilde{\alpha}_{i,N_n,n}|^l \le 4^l N_n^{2l} (m_n+1)^{l-1} n^{-2(l-1)} \sum_{k=0}^{j_{i+1}-j_i-1} \int_{(j_i+k)/n}^{(j_i+k+1)/n} \int_{(j_i+k)/n}^{t} \mathbb{E}\left|Y(s)Y(t) - (Y(i/N_n))^2\right|^l \mathrm{d}s\,\mathrm{d}t. \tag{7.4.50}$$

Let us now prove that there is a constant $c_{3,l} > 0$, only depending only on l, such that,

$$\mathbb{E}\left|Y(s)Y(t) - (Y(i/N_n))^2\right|^l \le c_{3,l} N_n^{-lH(i/N_n)}. \tag{7.4.51}$$

Using equality: for all reals x, y, z, one has,

$$xy - z^2 = z(x-z) + z(y-z) + (x-z)(y-z),$$

one gets

$$\begin{aligned}
&\left|Y(s)Y(t) - (Y(i/N_n))^2\right| \\
&= \left|Y(i/N_n)\Big((Y(s) - Y(i/N_n)) + (Y(t) - Y(i/N_n))\Big) + (Y(s) - Y(i/N_n))(Y(t) - Y(i/N_n))\right|.
\end{aligned} \tag{7.4.52}$$

It follows from (7.4.52), the triangle inequality, the equality $Y(u) = f(X(u))$ for all real u (see (5.2.2), the Mean Value Theorem and the fact that $f \in C^2_{pol}(\mathbb{R})$, that

$$
\begin{aligned}
&\Big|Y(s)Y(t) - (Y(i/N_n))^2\Big| \\
&\le \Big|Y(i/N_n)(Y(s) - Y(i/N_n))\Big| + \Big|Y(i/N_n)(Y(t) - Y(i/N_n))\Big| \\
&\quad + \Big|(Y(s) - Y(i/N_n))(Y(t) - Y(i/N_n))\Big| \\
&\le |Y(i/N_n)| \sup_{u\in[-\|X\|_\infty,\|X\|_\infty]} |f'(u)| \Big(\Big|X(s) - X(i/N_n)\Big| + \Big|X(t) - X(i/N_n)\Big|\Big) \\
&\quad + \Big(\sup_{u\in[-\|X\|_\infty,\|X\|_\infty]} |f'(u)|\Big)^2 \Big|X(s) - X(i/N_n)\Big|\Big|X(t) - X(i/N_n)\Big| \\
&\le c_1^2(1 + \|X\|_\infty^K)^2 \bigg(\Big|X(s) - X(i/N_n)\Big| + \Big|X(t) - X(i/N_n)\Big| \\
&\quad + \Big|X(s) - X(i/N_n)\Big|\Big|X(t) - X(i/N_n)\Big|\bigg) \\
&\le c_1^2\big(1 + \|X\|_\infty^K\big)^2 \Big(\Big|X(s) - X(i/N_n)\Big| + \Big|X(t) - X(i/N_n)\Big|\Big)^2 \\
&\le 2c_1^2\big(1 + \|X\|_\infty^K\big)^2 \Big(\Big|X(s) - X(i/N_n)\Big|^2 + \Big|X(t) - X(i/N_n)\Big|^2\Big) \\
&= C_4\Big(\Big|X(s) - X(i/N_n)\Big|^2 + \Big|X(t) - X(i/N_n)\Big|^2\Big), \qquad (7.4.53)
\end{aligned}
$$

where $C_4 = 2c_1^2\big(1 + \|X\|_\infty^K\big)^2$ is a random variable of finite moment of any order. Next, Observe that

$$\Big|\frac{j_i}{n} - \frac{i}{N_n}\Big| = \frac{i}{N_n} - \frac{[in/N_n]}{n} \le \frac{i}{N_n} - \frac{in/N_n - 1}{n} = \frac{1}{n} \le \frac{1}{N_n}$$

and

$$\Big|\frac{j_{i+1}}{n} - \frac{i}{N_n}\Big| = \frac{[(i+1)n/N_n]}{n} - \frac{i}{N_n} \le \frac{(i+1)n/N_n - 1}{n} - \frac{i}{N_n} = \frac{1}{N_n}.$$

Then, Lemma 2.12 in [7] and the equivalence of Gaussian moments, imply that there exists a constant $c_{4,l} > 0$, only depending on l, such that for all n, i and $s \in [\frac{j_i}{n}, \frac{j_{i+1}}{n}]$, one has,

$$\mathbb{E}\Big|X(s) - X(i/N_n)\Big|^{2l} \le c_{4,l}|s - i/N_n|^{2lH(i/N_n)} \le c_{4,l}N_n^{-2lH(i/N_n)}. \qquad (7.4.54)$$

It follows from (7.4.53), the fact that $x \mapsto x^l$ is a convex function, Cauchy-Schwarz inequality and (7.4.54), that

$$
\begin{aligned}
&\mathbb{E}\Big|Y(s)Y(t) - (Y(i/N_n))^2\Big|^l \\
&\le 2^{l-1}\Big(E(C_4^l)\Big)\Big(\Big(\mathbb{E}\Big|X(s) - X(i/N_n)\Big|^{2l}\Big)^{1/2} + \Big(\mathbb{E}\Big|X(t) - X(i/N_n)\Big|^{2l}\Big)^{1/2}\Big) \\
&\le c_{3,l}N_n^{-lH(i/N_n)}, \qquad (7.4.55)
\end{aligned}
$$

Thus we have shown that (7.4.51) holds. Next, it results from (7.4.50) and (7.4.51) that, for all n big enough,

$$\begin{aligned}
&\mathbb{E}\,|\tilde{\alpha}_{i,N_n,n}|^l \\
&\le 4^l N_n^{2l}(m_n+1)^{l-1}n^{-2(l-1)}(j_{i+1}-j_i)n^{-2}\mathbb{E}\left|Y(s)Y(t)-(Y(i/N_n))^2\right|^l \\
&\le 4^l c_{3,l} N_n^{2l} n^{-2l}(m_n+1)^l N_n^{-lH(i/N_n)} \\
&\le c_{5,l} m_n^{-l} N_n^{-lH(i/N_n)}, \qquad (7.4.56)
\end{aligned}$$

where $c_{5,l} = 8^l c_{3,l}$. This shows that (7.4.46) is satisfied.

Next, using (7.4.43), the fact that $x \mapsto x^l$ is a convex function, (7.4.45) and (7.4.46), we get

$$\begin{aligned}
\mathbb{E}\,|\alpha_{i,N_n,n}|^l &\le 2^{l-1}\mathbb{E}\,|\tilde{\alpha}_{i,N_n,n}|^l + 2^{l-1}\mathbb{E}\,|B_{i,N_n,n}|^l \\
&\le c(l)\Big(m_n^{-l}N_n^{-lH(i/N_n)} + m_n^{-2l}\Big), \qquad (7.4.57)
\end{aligned}$$

which proves that Part (iii) of the lemma is satisfied.

Let us now show that Part (iv) of the lemma holds. First, recall that (see (7.4.15), one has,

$$\widehat{Y}_{i,N_n,n} = \tilde{\mathscr{E}}_{i,N_n,n} + N_n \int_{j_i/n}^{j_{i+1}/n} Y(s)\,\mathrm{d}s. \qquad (7.4.58)$$

Observe that using a method, similar to that which allowed to show (7.4.6) holds, one can prove that for all real $l \ge 1$, for all n big enough,

$$\mathbb{E}\,(|\tilde{\mathscr{E}}_{i,N_n,n}|^l) \le c_{6,l} m_n^{-l/2} \le c_{6,l}, \qquad (7.4.59)$$

where $c_{6,l} > 0$ is a constant only depending on l. On the other hand, Relation (7.4.20) and the fact that $N_n n^{-1}(j_{i+1}-j_i)$ can be bounded independently on i and n, imply that,

$$\mathbb{E}\left|N_n \int_{j_i/n}^{j_{i+1}/n} Y(s)\right|^l \le c_{7,l}, \qquad (7.4.60)$$

where $c_{6,7} > 0$ is a constant only depending on l. Finally Part (iv) of the lemma results from (7.4.58), the fact that $x \mapsto x^l$ is a convex function, (7.4.59) and (7.4.60). □

Now we are in position to show that Proposition 5.2.2 holds. Our proof is inspired by that of Proposition 1 in [35] (see also [36]).

Proof of Proposition 5.2.2: For all $i = 0, \ldots, N_n - p - 1$, one sets,

$$\Delta_a \mathscr{E}_{i,N_n,n} := \sum_{k=0}^{p} a_k \mathscr{E}_{i+k,N_n,n}. \qquad (7.4.61)$$

Thus, in view of (5.2.9), (5.2.12) and (7.4.3), one has,

$$\Delta_a \mathscr{E}_{i,N_n,n} = \Delta_a \widehat{Y}_{i,N_n,n} - \Delta_a \overline{Y}_{i,N_n} \qquad (7.4.62)$$

Next oberve that, (5.2.17), (5.2.13) and (7.4.61) imply that,

$$\widehat{V}(h;\mu_{N_n},\nu_{N_n}) - \overline{V}_N(h;\mu_N,\nu_N) = \sum_{r=1}^{4} B_{n,r}(h), \tag{7.4.63}$$

where

$$\begin{aligned}
B_{n,1}(h) &= N_n^{-1} \sum_{i\in\mathcal{J}(\mu_{N_n},\nu_{N_n})} (C(i/N_n))^{-1} N_n^{2H(i/N_n)} \\
&\quad \times\Big\{ \big(\Delta_a \mathscr{E}_{i,N_n,n}\big)^2 - 2\|a\|_2^2 m_n^{-1} \big(Y_{i/N_n}\big)^2 \Big\} h(\overline{Y}_{i,N_n}) &(7.4.64)\\
B_{n,2}(h) &= N_n^{-1} \sum_{i\in\mathcal{J}(\mu_{N_n},\nu_{N_n})} (C(i/N_n))^{-1} N_n^{2H(i/N_n)} \\
&\quad \times\Big\{ \big(\Delta_a \widehat{Y}_{i,N_n,n}\big)^2 - 2\|a\|_2^2 m_n^{-1} \big(\widehat{Y}_{i,N_n,n}\big)^2 \Big\} \big\{ h(\widehat{Y}_{i,N_n,n}) - h(\overline{Y}_{i,N_n}) \big\} \\
& &(7.4.65)\\
B_{n,3}(h) &= N_n^{-1} \sum_{\in\mathcal{J}(\mu_{N_n},\nu_{N_n})} (C(i/N_n))^{-1} N_n^{2H(i/N_n)} \\
&\quad \times 2\|a\|_2^2 m_n^{-1} \Big\{ \big(Y_{i/N_n}\big)^2 - \big(\widehat{Y}_{i,N_n,n}\big)^2 \Big\} h(\overline{Y}_{i,N_n}) &(7.4.66)\\
B_{n,4}(h) &= N_n^{-1} \sum_{i\in\mathcal{J}(\mu_{N_n},\nu_{N_n})} (C(i/N_n))^{-1} N_n^{2H(i/N_n)} \\
&\quad \times 2(\Delta_a \mathscr{E}_{i,N_n,n})\big(\Delta_a \overline{Y}_{i,N_n}\big) h(\overline{Y}_{i,N_n}\big) &(7.4.67)
\end{aligned}$$

Thus, in order to prove the proposition, it is sufficient to show that, for $r = 1, 2, 3, 4$, one has

$$\mathbb{E}\left\{|B_{n,r}(h)|\right\} \le cv(N_n, m_n). \tag{7.4.68}$$

We will only prove that (7.4.68) is satisfied in the case where $r = 1$ since the other cases can be treated in a rather similar way. We set,

$$b_i = \Big\{ \big(\Delta_a \mathscr{E}_{i,N_n,n}\big)^2 - 2\|a\|_2^2 m_n^{-1} \big(Y_{i/N_n}\big)^2 \Big\} h(\overline{Y}_{i,N_n}) \tag{7.4.69}$$

Observe that in view of Part (i) of Lemma 7.4.2, conditional on $\mathcal{G}_X$, the random variables b_i and b_j are independent when $|i-j| \ge p+1$. Using the latter fact as well as (7.4.64),

$$\begin{aligned}
&\mathbb{E}\left((B_{n,1}(h))^2|\mathcal{G}_X\right) \\
&= N_n^{-2} \sum_{i,j\in\mathcal{J}(\mu_{N_n},\nu_{N_n}),|i-j|\le p} \big(C(i/N_n)C(j/N_n)\big)^{-1} N_n^{2(H(i/N_n)+H(j/N_n))} \mathbb{E}\left(b_i b_j|\mathcal{G}_X\right) \\
&\quad + N_n^{-2} \sum_{i,j\in\mathcal{J}(\mu_{N_n},\nu_{N_n}),|i-j|\ge p+1} \big(C(i/N_n)C(j/N_n)\big)^{-1} N_n^{2(H(i/N_n)+H(j/N_n))} \mathbb{E}\left(b_i|\mathcal{G}_X\right)\mathbb{E}\left(b_j|\mathcal{G}_X\right).
\end{aligned} \tag{7.4.70}$$

Next, using (7.4.20), the fact $h \in C^1_{pol}(\mathbb{R})$, Cauchy-Schwarz inequality and Part (ii) of Lemma 7.4.2, it follows that there is a constant $c_1 > 0$, non depending on n, i and j, such that,

$$\mathbb{E}\left\{|b_i b_j|\right\} \le c_1 m_n^{-2}. \tag{7.4.71}$$

On the other hand, observe that (5.2.15) implies that

$$c_2 := \sup_{s\in[0,1]} (C(s))^{-1} < \infty. \tag{7.4.72}$$

Next combining (7.4.71) with (7.4.72), it follows that, there exists a constant $c_3 > 0$, such that, for all n big enough,

$$\begin{aligned}&\mathbb{E}\left\{\sum_{i,j\in\mathcal{J}(\mu_{N_n},\nu_{N_n})1,|i-j|\le p}\big(C(i/N_n)C(j/N_n)\big)^{-1}N_n^{2(H(i/N_n)+H(j/N_n))}\mathbb{E}\left(b_ib_j|\mathcal{G}_X\right)\right\}\\&\le c_3 N_n^{4\max_{s\in[\mu_{N_n},\nu_{N_n}]}H(s)+1}m_n^{-2}\\&\le c_3N_n^2\big(v(N_n,m_n)\big)^2,\end{aligned} \tag{7.4.73}$$

where the last inequality results from (7.4.1). Let us now study the second term in the right hand side of (7.4.70). To this end, first we need to show that, that there is a constant $c_4 > 0$ such that for all n big enough and all i, one has,

$$\mathbb{E}\left\{\big(\mathbb{E}\left(b_i|\mathcal{G}_X\right)\big)^2\right\} \le c_4\Big(m_n^{-2}N_n^{-2H(i/N_n)} + m_n^{-4}\Big) \tag{7.4.74}$$

Using (7.4.69), the fact the process $\{Y(s)\}_{s\in[0,1]}$ is $\mathcal{G}_X$) measurable and Part (iii) of Lemma 7.4.2, one has that,

$$\begin{aligned}&\mathbb{E}\left(b_i|\mathcal{G}_X\right)\\&=\Big\{\mathbb{E}\left(\left(\Delta_a\mathscr{E}_{i,N_n,n}\right)^2|\mathcal{G}_X\right)-2\|a\|_2^2m_n^{-1}\big(Y_{i/N_n}\big)^2\Big\}h(\overline{Y}_{i,N_n})\\&=\Big\{2\sum_{k=0}^p a_k^2m_n^{-1}\big(Y_{(i+k)/N_n}\big)^2-2\|a\|_2^2m_n^{-1}\big(Y_{i/N_n}\big)^2+\sum_{k=0}^p a_k^2\alpha_{i+k,N_n,n}+\beta_{i,N_n,n}\Big\}h(\overline{Y}_{i,N_n})\end{aligned} \tag{7.4.75}$$

where

$$\beta_{i,N_n,n} = 2\sum_{0\le k<l\le p} a_ka_l\mathbb{E}\left(\mathscr{E}_{i+k,N_n,n}\mathscr{E}_{i+l,N_n,n}|\mathcal{G}_X\right). \tag{7.4.76}$$

Next observe that, using the fact that conditional on $\mathcal{G}_X$, the random variables $\mathscr{E}_{i+k,N_n,n}$ and $\mathscr{E}_{i+l,N_n,n}$ in (7.4.76) are independent (this results from Part (i) of Lemma 7.4.2)) as well as the first inequality in Relation (7.4.4), it follows (7.4.76) that, almost surely,

$$\begin{aligned}|\beta_{i,N_n,n}| &\le 2\sum_{0\le k<l\le p}|a_ka_l|\,\left|\mathbb{E}\left(\mathscr{E}_{i+k,N_n,n}\mathscr{E}_{i+l,N_n,n}|\mathcal{G}_X\right)\right|\\&\le 2\sum_{0\le k<l\le p}|a_ka_l|,\left|\mathbb{E}\left(\mathscr{E}_{i+k,N_n,n}\mathcal{G}_X\right)\right|\,\left|\mathbb{E}\left(\mathscr{E}_{j+l,N_n,n}\mathcal{G}_X\right)\right|\\&\le C_5m_n^{-2},\end{aligned} \tag{7.4.77}$$

where C_5 is a random variable of finite moment of any order, which does not depend on n and i. Also, observe that similarly to (7.4.51), one can show that for all real

$l \geq 1$, there is a constant $c_{6,l} > 0$, only depending on l, such that, for all n big enough, $i \in \{0, \ldots, N_n - p - 1\}$ and $k \in \{0, \ldots, p\}$, one has,

$$\mathbb{E}\left|\big(Y_{(i+k)/N_n}\big)^2 - \big(Y_{i/N_n}\big)^2\right|^l \leq c_{6,l} N^{-lH(i/N_n)},$$

which in turn implies that

$$\mathbb{E}\left|2\sum_{k=0}^{p} a_k^2 m_n^{-1}\big(Y_{(i+k)/N_n}\big)^2 - 2\|a\|_2^2 m_n^{-1}\big(Y_{i/N_n}\big)^2\right|^l \leq c_{7,l} m_n^{-1} N^{-lH(i/N_n)}, \quad (7.4.78)$$

where $c_{7,l}$ is a constant only depending on l. Next by using (7.4.75), (7.4.78), (7.4.8), (7.4.77), Cauchy-Schwarz inequality, the fact $h \in C^1_{pol}(\mathbb{R})$ and (7.4.20), one obtains (7.4.74). Next, it follows from (7.4.72), (7.4.74) and Cauchy-Schwarz inequality, the assumption that

$$\sup_n m_n^{-1} N_n^{2\max_{s\in[\mu_{N_n},\nu_{N_n}]} H(s)} < \infty,$$

the fact that H is with values in $(1/2, 1)$, that

$$\begin{aligned}
&\mathbb{E}\left\{\Big|\sum_{i,j\in\mathcal{J}(\mu_{N_n},\nu_{N_n}),|i-j|\geq p+1} \big(C(i/N_n)C(j/N_n)\big)^{-1} N_n^{2(H(i/N_n)+H(j/N_n))}\mathbb{E}\left(b_i|\mathcal{G}_X\right)\mathbb{E}\left(b_j|\mathcal{G}_X\right)\Big|\right\}\\
&\leq c_8 N_n^{4\max_{s\in[\mu_{N_n},\nu_{N_n}]} H(s)+2}\left(m_n^{-2} N_n^{-2\min_{s\in[\mu_{N_n},\nu_{N_n}]} H(s)} + m_n^{-4}\right)\\
&\leq c_9 N_n^2\left(N_n^{-2\min_{s\in[\mu_{N_n},\nu_{N_n}]} H(s)} + m_n^{-2}\right)\\
&\leq c_9 N_n^2\left(N_n^{-1} + m_n^{-2}\right)\\
&\leq c_9 N_n^2\big(v(N_n, m_n)\big)^2, \qquad (7.4.79)
\end{aligned}$$

where c_8, c_9 are two constants non depending on n and where the last inequality results from (7.4.1). Next using (7.4.70), (7.4.73), (7.4.79) and Cauchy-Schwarz inequality, it follows that, (7.4.68) is satisfied when $r = 1$.

□

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