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Λ-MODULES AND HOLOMORPHIC LIE ALGEBROIDS

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Abstract

The thesis is concerned with the construction and the study of moduli spaces of holomorphic Lie algebroid connections. It provides a classification of sheaves of almost polynomial filtered algebras on a smooth projective complex variety in terms of holomorphic Lie algebroids and their cohomology classes. This permits to build moduli spaces of holomorphic Lie agebroid connections via Simpson's formalism of Lambda-modules. Furthermore, the deformation theory of such connections is studied, and the germ of their moduli spaces in the rank two case is computed when the base variety is a curve.

Résumé

La thèse est consacrée à la construction et à l'étude des espaces de modules des connexions holomorphes algébroïdes de Lie. On commence par une classification des faisceaux d'algèbres filtrées quasi-polynômiales sur une variété complexe lisse projective en termes d'algébroïdes de Lie holomorphes et de leurs classes de cohomologie. Cela permet de construire les espaces de modules de connexions holomorphes algébroïdes de Lie par le formalisme des Lambda-modules de Simpson. Par ailleurs, on étudie la théorie des déformations de telles connexions, et on calcule le germe de leur espace de modules dans le cas de rang deux, lorsque la variété de base est une courbe.

To my family

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Chapter 1

Introduction

The moduli spaces of coherent sheaves over an algebraic variety X are geometric objects parametrizing the isomorphism classes of coherent $\mathscr{O}_X$-modules. Their study is one of the leading themes of contemporary algebraic geometry. There are various precise definitions for these spaces. Some of these introduce a notion of *(semi)stability* of coherent sheaves such that one can construct moduli spaces of (semi)stable coherent sheaves by using GIT techniques. As a consequence, one obtains moduli spaces that are quasi projective schemes.

Besides moduli spaces of sheaves, one is interested in moduli spaces of sheaves with some additional structure, as *flat connections* or *Higgs fields*. A connection on a coherent sheaf $\mathcal{E}$ is a sheaf map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_X$ satisfying the Leibniz rule $\nabla(fs) = f\nabla s + s \otimes \mathrm{d}f$ for any $f \in \mathscr{O}_X$ and $s \in \mathcal{E}$. This is equivalent to have a linear map $\mathcal{T}_X \to \mathrm{Der}_{\mathscr{O}_X}\mathcal{E}$. A connection is said *flat* or *integrable* when this map is a Lie algebra morphism. A Higgs field on a coherent sheaf $\mathcal{E}$ is a $\mathscr{O}_X$-linear morphism $\phi : \mathcal{E} \to \mathcal{E} \otimes \Omega_X$ satisfying the integrability condition $\phi \wedge \phi = 0$. For X a complex curve of genus greater then 1, it was shown in [15] that an open part of the moduli space of Higgs sheaves can be identified with the cotangent bundle of the moduli space of vector bundles on X, and that the induced symplectic structure is an algebraically complete integrable system. These constructions have then been generalized in many directions, and nowadays moduli spaces of Higgs sheaves are important in many branches of algebraic geometry and mathematical physics (for example geometric Langlands program, non abelian Hodge theory). Moduli spaces of flat connections are much studied too, since they play a fundamental role in the non abelian Hodge correspondence and they are diffeomorphic to the moduli spaces of representations of the fundamental group of X (see [35]).

In [34], Simpson develops a uniform approach to construct moduli spaces of sheaves with structures as this kind: namely, let X be a smooth algebraic variety and Λ a sheaf of filtered algebras satisfying some axioms that we will state in Section 3.2. Then Simpson shows how to construct, for each numerical polynomial P, a quasi projective scheme $\mathcal{M}_\Lambda(P)$ parametrizing semistable Λ-modules. By varying the algebra Λ, one obtains moduli spaces for a large class of objects. For example, for $\Lambda = \mathscr{O}_X$, semistable Λ-modules are just semistable coherent sheaves, while for $\Lambda = \mathscr{D}_X$, the sheaf of differential operators of X, Λ-modules are flat connections, and for $\Lambda = \mathrm{Sym}_{\mathscr{O}_X} \mathcal{T}_X$ semistable Λ-modules are semistable Higgs sheaves. Moreover, via this theory one can construct other interesting spaces. For example, when Λ is the one parameter deformation of the symmetric algebra $\mathrm{Sym}_{\mathscr{O}_X}$ into $\mathscr{D}_X$, the corresponding moduli space $\mathcal{M}_\Lambda(P)$ parametrizes semistable λ-connections, and realizes the twistor space of the hyperkähler structure of the moduli space of Higgs bundles (see [36]).

In this thesis, the moduli spaces of Λ-modules are the main object of study. In particular, we give them another interpretation, proving that they are in fact moduli spaces of holomorphic Lie algebroid connections.

Among the axioms that the sheaves of algebras Λ have to satisfy, there is the *almost commutativity*. We say that a filtered k-algebra A is almost commutative when its associated graded algebra is commutative. It is well known that the first graded piece $\mathrm{Gr}_1 A$ of an almost commutative filtered k-algebra carries naturally a structure of (k, R)Lie-Rinehart algebra, for $R = A_{(0)}$. A (k, R)-Lie-Rinehart algebra L is a k-Lie algebra carrying an R-module structure and acting on R by derivations in a compatible way. Lie-Rinehart algebras are the algebraic analogues of Lie algebroids, and we will see that the almost commutativity of Λ induces naturally a holomorphic Lie algebroid structure on $\mathrm{Gr}_1\Lambda$.

Lie algebroids are a generalization of both Lie algebras and the tangent bundle of a manifold: on one side they are the infinitesimal objects associated to Lie groupoids, just as Lie algebras are the infinitesimal objects associated to Lie groups; on the other side, they are naturally vector bundles over a manifold with a morphism to the tangent bundle of the manifold, and this allows to generalize many constructions of differential geometry.

Holomorphic Lie algebroids are the generalization of smooth Lie algebroids to the holomorphic case. One constructs their theory in a similar way to the theory of complex manifolds, and many important features of the latter generalize to holomorphic Lie algebroids. In particular, the cohomological theory of a holomorphic Lie algebroid $\mathcal{L}$ is very rich: the cohomology groups $H^i(\mathcal{L}; \mathbb{C})$, defined as the hypercohomology of a suitable complex, carry a natural filtration $F^p H^i(\mathcal{L}; \mathbb{C})$ similar to the Hodge filtration of the cohomology of a complex variety. Moreover, there are generalization of the holomorphic De Rham theorem, Dolbeault theorem, etc. (see for example [20] and [7]).

In particular, one can generalize to holomorphic Lie algebroids the notion of a holomorphic connection: for a holomorphic Lie algebroid $\mathcal{L}$, a holomorphic $\mathcal{L}$-connection on a coherent $\mathscr{O}_X$-module $\mathcal{E}$ is a sheaf map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L}^*$. This is equivalent to having a linear map $\mathcal{L} \to \mathrm{Der}_{\mathscr{O}_X} \mathcal{E}$, and one says that the $\mathcal{L}$-connection is *flat* or *integrable* when it is a morphism of Lie algebras.

One of the main results of this thesis is a generalization of [37] leading to a classification of sheaves of filtered algebras Λ: first to any Λ we naturally associate the short exact sequence

$$0 \to \mathscr{O}_X \to \Lambda_{(1)} \to \mathrm{Gr}_1 \Lambda \to 0$$

that, since Λ is almost commutative, is an extension of holomorphic Lie algebroids. Then we generalize Sridharan's construction of twisted enveloping algebra to associate to any of such extensions a sheaf of filtered algebras. Finally we show that these two constructions are inverse to each other. Thus, since holomorphic Lie algebroids extensions are parametrized by $F^1 H^2(\mathcal{L}; \mathbb{C})$, we obtain

Theorem 1. *There is a one to one correspondence between*

- *pairs (Λ, Ξ), where Λ is a sheaf of filtered algebras satisfying the axioms we state in Section 3.2 and Ξ an isomorphism of graded algebras between $Gr\Lambda$ and the symmetric algebra over the first graded piece $Gr_1\Lambda$;*
- *holomorphic Lie algebroid extensions where the first term equal $\mathscr{O}_X$;*
- *pairs $(\mathcal{L}, \Sigma)$, where $\mathcal{L}$ is a holomorphic Lie algebroid and Σ is a cohomology class in $F^1 H^2(\mathcal{L}, \mathbb{C})$.*

Then we see that for a sheaf of algebras Λ corresponding to the pair $(\mathcal{L}, \Sigma)$, a Λ-module structure on a coherent $\mathscr{O}_X$-module $\mathcal{E}$ is equivalent to a collection of holomorphic $\mathcal{L}$-connections on $\mathcal{E}$ satisfying an integrability condition involving the cohomology class Σ.

As a consequence, we deduce from the theory of Λ-modules a construction of moduli spaces for semistable flat $\mathcal{L}$-connections for any holomorphic Lie algebroid $\mathcal{L}$. We obtain an interesting application of this to generalized complex geometry [13]: a generalized complex structure on a smooth manifold M is defined in terms of a maximal isotropic subspace $L \subseteq T_M \oplus T_M^*$, where the latter is equipped with the Courant algebroid structure. Since L is maximal isotropic, the Courant bracket restricts to a Lie algebroid structure. When M is a complex manifold and Π is a Poisson bivector on M, there is a natural way to associate to Π a generalized complex structure L_Π on M, and it turns out that actually L_Π is induced by a holomorphic Lie algebroid $\mathcal{L}_\Pi$ (see [20]). For L a generalized complex structure, flat L-connections on vector bundles are called *generalized holomorphic bundles*. So we see that our construction provides moduli spaces for semistable generalized holomorphic bundles, with respect to generalized complex structures associated to holomorphic Poisson bivectors.

In the latest chapters of the thesis, we adress problems of deformation theory for Lie algebroid connections. The theory of deformations of coherent sheaves is well estabilished: the first order deformations of a coherent sheaf $\mathcal{E}$ over a smooth projective variety X are naturally parametrized by the group $\mathrm{Ext}^1_{\mathscr{O}_X}(\mathcal{E},\mathcal{E})$, while obstructions to integrate a first order deformation to higher orders lie in $\mathrm{Ext}^2_{\mathscr{O}_X}(\mathcal{E},\mathcal{E})$ (cf. [6], [14]). This is a manifestation of the general theory of deformation functors, that associates to a deformation problem a differential graded Lie algebra (DGLA) $L^\bullet$, and says that the tangent space to the deformation functor is $H^1(L^\bullet)$, while an obstruction theory is given by $H^2(L^\bullet)$ and the Maurer-Cartan equation (cf. [29]).

In [26] these results are generalized to the case of (integrable) connections with poles along a divisor, and we will further generalize these results to $\mathcal{L}$-connections for $\mathcal{L}$ any holomorphic Lie algebroid on X. In particular, we will find that the DGLA associated to the deformation functor of the $\mathcal{L}$-connection $(\mathcal{E},\nabla)$ is the total complex computing the hypercohomology of the complex

$$\mathscr{E}nd\mathcal{E} \to \mathscr{E}nd\mathcal{E}\otimes\Omega_{\mathcal{L}} \to \mathscr{E}nd\mathcal{E}\otimes\Omega^2_{\mathcal{L}} \to \dots ,$$

where the differential is given by the $\mathcal{L}$-connection induced by ∇ on $\mathscr{E}nd\mathcal{E}$.

We will finally use this result and Luna's slice theorem [24] to study the local structure of the moduli spaces in a simple case: for X a smooth projective curve and $\mathcal{L} = \mathcal{T}_X$ the canonical Lie algebroid, we will try to understand the structure of the moduli space $\mathcal{M}_X(2,0)$ around the "most degenerate point" $z_0 = (\mathscr{O}_X^{\oplus 2}, \mathrm{d})$. We will give explicit equations for $K_2//H$, where K_2 is $\mathrm{obs}_2^{-1}(0)$, the variety defined by the degree 2 homogeneous part of the Kuranishi map of of z_0. If the higher degree terms of the Kuranishi map vanish, as in the genus 1 case, or the deformation functor is rigid as in [21], this is isomorphic to the germ of the moduli space $\mathcal{M}_X(2,0)$ at z_0.

In particular, we show that $K_2//H$ is a symplectic reduction, so that there is a symplectic form on its smooth locus. Moreover we will see that, for C of genus $g \geq 2$, the germ of $K_2//H$ at a singular point different from zero is the germ of an affine cone over the Segre embedding of the incidence divisor in $\mathbb{P}^N \times \mathbb{P}^N$ for $N = 2g-3$, and that the singularities of $K_2//H$ are symplectic, i. e. for any resolution of singularities the symplectic form lifts to a globally defined 2-form.

The second chapter of the thesis concerns with the theory of Lie algebroids. The first 2 sections are a quick review of basic definitions about

Lie algebras and Lie-Rinehart algebras; in particular, we focus on the definitions of the Chevalley-Eilenberg and Lie-Rinehart cohomology, and recall the correspondence between the isomorphism classes of abelian extensions

$$0 \to M \to L' \to L \to 0$$

of Lie algebras (resp. Lie-Rinehart algebras) and the second cohomology group $H^2_{CE}(L; M)$ (resp. $H^2_{LR}(L; M)$). These are classical results, and one can refer to [39] and [17] for a systematic treatment. Section 2.3, is an introduction to the general theory of smooth Lie algebroids, Lie algebroid connections and the L-characteristic ring of a vector bundle, L being a smooth Lie algebroid. The exposition follows mainly [28] and [11]. In Section 2.4 we introduce complex and holomorphic Lie algebroids, and develop their theory: first (Subsection 2.4.1) we introduce matched pairs of Lie algebroids and construct the canonical complex Lie algebroid associated to a holomorphic Lie algebroid. We follow [20], though this construction had previously been introduced in [23]. Then (Subsection 2.4.2) we present various results about the cohomology of a holomorphic Lie algebroid: first we recall from [20] and [7] the generalizations to holomorphic Lie algebroids of the holomorphic De Rham theorem and Dolbeault's theorems, and then study the Čech-DeRham spectral sequence of a holomorphic Lie algebroid. In particular, we show a mild degeneration of this spectral sequence, that gives a description of $F^1H^2(\mathcal{L}; \mathbb{C})$ in terms of Čech cocycles. Finally (Subsection 2.4.3) we present a generalization of Atiyah's theory on holomorphic connections to a Lie algebroid setting: for a holomorphic Lie algebroid $\mathcal{L}$ and its associated canonical complex Lie algebroid $\mathcal{L}_h$, first we study some relations between holomorphic $\mathcal{L}$-connections and smooth $\mathcal{L}_h$-connections; then we define the $\mathcal{L}$-Atiyah classs of a holomorphic vector bundle $\mathcal{E}$ and show that it generates the $\mathcal{L}_h$-characteristic ring of $\mathcal{E}$. In particular, this gives a necessary criterion for the existence of holomorphic $\mathcal{L}$-connections on $\mathcal{E}$. Finally we study the situation where $\mathcal{E}$ is not locally free: this is a situation that does not occur in the theory of usual connections, and we are able to generalize these results to the case where $\mathcal{E}$ is torsion free.

In Chapter 3 we will show Theorem 1. In Section 3.1 we will present the main ideas, that come from the work of Sridharan [37]: he classifies the isomorphism classes of pairs (A, Ξ), with A a filtered k-algebra with the 0th piece isomorphic to k and Ξ an isomorphism of the associated graded algebra with the symmetric algebra over the first graded piece $\mathrm{Gr}_1 A$, in terms of pairs (L, Σ), with L a k-Lie algebra and $\Sigma \in H^2_{CE}(L, k)$. In the second part of the section we generalize this result, letting the 0th graded piece of A to be any commutative k-algebra R: in this case, $A_{(1)}$ and $\mathrm{Gr}_1 A$ acquire a natural structure of (k, R)-Lie-Rinehart algebra, and the construction of Sridharan readily generalizes to this case. In particular, for any a (k, R)-Lie-Rinehart algebra L and 2-cocycle F in the Lie-Rinehart cohomology of L with values in R, we construct an F-twisted Rinehart-enveloping algebra $\tilde{U}_F(L)$, which is a filtered k-algebra with graded object isomorphic to the symmetric algebra $\mathrm{Sym}_R L$.

In Section 3.2 we sheafify these reults: first we classify the isomorphism classes of extensions of a holomorphic Lie algebroid $\mathcal{L}$ by $\mathscr{O}_X$, and show that these are in a one to one correspondence with $F^1H^2(\mathcal{L};\mathbb{C})$, the first filtration of the second cohomology group of $\mathcal{L}$. Then we generalize the construction of the twisted enveloping algebra to Lie algebroids: we will see that if we have a cohomology class $\Sigma \in F^1H^2(\mathcal{L};\mathbb{C})$ and a representative $\sigma = (Q_\alpha, \phi_{\alpha\beta})$ with respect to a sufficiently good open covering $\mathfrak{U} = \{U_\alpha\}_\alpha$, we can glue the sheafifications of the algebras $\tilde{U}_{Q_\alpha}(\mathcal{L}(U_\alpha))$ using the $\phi_{\alpha\beta}$, to define a sheaf of filtered $\mathbb{C}_X$-algebras.

We then conclude the chapter with some examples of this correspondence.

Chapter 4 gives applications of this result to moduli spaces of Λ-modules: first we recall from [34] the construction of moduli spaces for Λ-modules. Then we see that, for the sheaf Λ of filtered algebras associated to the Lie algebroid $\mathcal{L}$ and the class Σ, the Λ-modules structures on a coherent sheaf $\mathcal{E}$ are equivalent to a collection of holomorphic $\mathcal{L}$-connections satisfying a gluing condition depending on Σ. In particular, for $\Sigma \in F^2H^2(\mathcal{L};\mathbb{C})$, a Λ-module structure corresponds to a globally defined holomorphic $\mathcal{L}$-connection ∇ and we show, using the results of section 1.9, that the moduli spaces are empty unless $\Sigma = 0$, and when this happens the corresponding moduli space parametrize semistable holomorphic $\mathcal{L}$-connections. The ideas on the correspondence between Λ-module structures and Lie algebroid connection are already contained in [34]: there Simpson introduces triples (H, δ, γ) that are equivalent to holomorphic Lie algebroid structures on H^*. Then, the algebra that he constructs from such a triple correspond, in our notations, to the algebra Λ associated to the Lie algebroid H^* and the class $\Sigma = 0$. Our construction completes this correspondence and embeds it in the more natural setting of holomorphic Lie algebroid connections. In the last subsections we show examples of moduli spaces that one can construct with this theory: we see that flat connections, Higgs bundles, co-Higgs bundles are holomorphic $\mathcal{L}$-connections for $\mathcal{L} = \mathcal{T}_X, (\mathcal{T}_X)_0, (\Omega_X)_0$. Finally, using some results of [20] we show how with this techniques one can construct moduli spaces of semistable generalized holomorphic vector bundles for a generalized complex structure associated to a holomorphic Poisson bivector Π.

In Chapter 5 we study deformations of $\mathcal{L}$-connections: first we recall some background theory on deformation functors from [29] and [14], while in Section 5.2 we study the deformation functor associated to a flat $\mathcal{L}$-connection. We will see that there is not much difference from the case of usual integrable connections, that has been studied in [26]. In the last section we first study the tangent space to a subscheme of $\mathfrak{Quot}(\Lambda_{(1)} \otimes V(-N), P)$ parametrizing flat $\mathcal{L}$-connections then, following [31], we will present an application of Luna's slice theorem to study the local structure of the moduli spaces of integrable $\mathcal{L}$-connections.

In Chapter 6 we study the local structure of the moduli spaces of flat ($\mathcal{T}_X$-)connections in a particular case: for X a smooth projective curve and $z_0 = (\mathscr{O}_X^{\oplus 2}, \mathrm{d})$ the "most degenerate point", the tangent space and the first obstruction map have an easy description in terms of linear data. In Section 6.1 we review some theory of invariants for the groups $\mathrm{O}(V)$ and $\mathrm{SO}(V)$ acting on the direct sums of the vector space V, while in section 6.2 we use this result to find equations for $K_2//H$, the quotient of the versal space of second order deformations of z_0 by the stabilizer of z_0. This is an approximation of the germ of of the moduli space $\mathcal{M}_X(2,0)$ at z_0, that when the deformation functor is rigid is actually isomorphic to it.

In the last chapter, we present some ideas about possible developments of this work. First we present a proposal for an extension of Theorem 1 which would involve the whole cohomology group $H^2(\mathcal{L};\mathbb{C})$ instead of its first filtration. This requires a stacky generalization of holomorphic Lie algebroid extensions, and the corresponding algebras Λ have to be generalized to some Lie algebroid version of twisted differential operators, similarly to [5]. In the second section we present some links of Theorem 1 to deformation quantization problems; connections of this with the arguments of the previous section are very strong. Finally, we propose a generalization of the moduli space of λ-modules using the theory of deformations of Lie algebroids [8].

Chapter 2

Lie algebroids

Lie algebroids are geometric objects which generalize both of the concepts of a Lie algebra and the tangent bundle of a manifold. One of the ideal principles in the theory of Lie algebroids is that one can generalize to a Lie algebroid setting classical construction of differential geometry, by replacing the vector fields with sections of the Lie algebroid, differential forms with sections of the exterior power of its dual and the exterior differential with the Lie algebroid differential. This pilosophy will be clear later with the examples we present.

Before to develop the theory of Lie algebroids that we will need, we review some known facts about extensions of Lie algebras and Lie-Rinehart algebras, that we need to recall because they will be central in our further constructions.

2.1 Extensions of Lie algebras

Let k be a commutative unital ring.

Definition 1. *A* Lie algebra *over k is a k-module L equipped with an operation $[\cdot,\cdot] : L \times L \to L$ satisfying:*

- $[\cdot,\cdot]$ *is k-bilinear;*
- $[\cdot,\cdot]$ *is antisymmetric;*
- $[\cdot,\cdot]$ *satisfies the Jacobi identity, that is*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

for any $x, y, z \in L$.

For a k-module M, a Lie algebra morphism $\rho : L \to \mathrm{End}_k(M)$ (where the latter is given the Lie algebra structure of commutation: $[A, B] = A \circ$

$B - B \circ A$) is called *representation* of L, and M is said to have the structure of L-module.

Now, we are going to assume that L is a projective k-module. For the constructions of this section, this is not really necessary: with some more effort one has the same results for any k-Lie algebra. Anyway, in next sections we will work with free k-algebras, and the proof in this setting are much more intuitive, and we prefer this to a broader generality. We refer to [39] for the general constructions.

Let L be a projective k-Lie algebra, and M an L-module. Define the cochain groups:

$$C^n(L; M, \rho) = \mathrm{Hom}_k(\bigwedge_k^p L, M)$$

and the coboundary operator

$$\begin{aligned}(\delta\theta)(x_1 \dots x_{p+1}) &= \sum_{i<j} (-1)^{i+j} \theta([x_i, x_j], x_1, \dots, \hat{x_i}, \hat{x_j}, \dots, x_{p+1}) + \\ &+ \sum_i \rho(x_i) \theta(x_1, \dots, \hat{x_i}, \dots x_{p+1})\end{aligned}$$

for $\theta \in C^p(L, M)$ and $x_i \in L$. We have that $\delta^2 = 0$, and we call the the corresponding cohomology groups $H^p_{CE}(L; M, \rho)$ *Chevalley-Eilenberg cohomology groups.*

Let L be a projective k-Lie algebra and M a k-module. An *abelian extension of Lie algebras* (or an extension of L by the k-module M) is a short exact sequence of k-Lie algebras

$$0 \to M \to L' \to L \to 0$$

where M is equipped with the abelian (i. e. trivial) Lie algebra structure. A morphism between two extensions L', L'' of L by M is a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & L' \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & L'' \arrow[r] & L \arrow[r] & 0 \end{tikzcd}$$

By the 5-lemma, every morphism of extensions of L by M is an isomorphism.

Remark that any extension of L by M induces on M an L-module structure: let

$$0 \to M \xrightarrow{\iota} L' \xrightarrow{\pi} L \to 0$$

be an extension of L by M, and for $x \in L$ and $m \in M$ set $\rho(x)(m) = [y, \iota(m)]_{L'}$, where y is an element of L' mapped to x. The above expression takes value in M because $\pi([y,\iota(m)]) = [\pi(y),\pi(\iota(m))] = 0$, and is well defined because for another elements y' of L' mapped to x, one has $y'-y = \iota(n)$, and $[n,m]=0$ because M is abelian.

The following theorem is classical:

Theorem 2. *Let k be a commutative ring, L a projective k-Lie algebra, M a k-module and ρ a representation of L on M.*

Then the set of equivalence classes of extensions of L by M inducing ρ as L-module structure on M is in a one to one correspondence with the second cohomology group $H^2_{CE}(L;M,\rho)$.

Proof. Let F' be a closed 2-cochain in $C^2(L;M,\rho)$. Define on $M\oplus L$ the k-bilinear antisymmetric bracket $[\cdot,\cdot]_{L'}$ by

$$[m+x,n+y]_{L'} = (\rho(x)(n)-\rho(y)(m)+F'(x,y)) + [x,y] \ .$$

Since F' is a cocycle, we have that $[\cdot,\cdot]_{L'}$ satisfies the Jacobi identity, so we have a k-Lie algebra structure L' on $M\oplus L$. Moreover by construction the standard injection $M\hookrightarrow M\oplus L$ and projection $M\oplus L\to L$ are Lie algebra morphisms, so we have that

$$o\to M\to L'\to L\to 0$$

defines an extension of L by M.

Let F'' be another 2-cocycle cohomologous to F', with $F''-F' = \delta\phi$, and construct the extension L'' of L by M as before. Define the endomorphism $\tilde{\phi}$ of $M\oplus L$ by

$$\tilde{\phi}(m+x) = (m+\phi(x)) + x \ .$$

This is a actually a morphism of k-Lie algebras $\tilde{\phi}: L'\to L''$, so we see that cohomologous cochains define the same equivalence class of extension of L by M.

Let

$$0\to M\to L'\to L\to 0$$

be an extension of L by M. Since L is projective, this sequence splits. Choose a splitting $\zeta': L\to L'$. This yields an isomorhpism of k-modules $\hat{\zeta}': L'\to M\oplus L$. According to this, we obtain a k-Lie algebra bracket on $M\oplus L$, that we write

$$[m+x,n+y]' = [m,n]' + [x,n]' - [y,m]' + [x,y]'$$

Since M is abelian $[m,n]'=0$, and since the L-module structure on M coincide with ρ, we have $[x,m]' = \rho(x)(m)$. Finally, write $[x,y]' = F'(x,y) + G(x,y)$ with $F'(x,y)\in M$ and $G(x,y)\in L$. Since the map $L'\to L$ is a Lie algebra morphism, $G(x,y)$ coincide with the Lie bracket in L. Further, F' is an antisymmetric, k-bilinear map from L to M, so it is a 2-cochain of L with values in M. Moreover, one can check that, since the bracket on L' satisfies the Jacobi identiy, it is closed w. r. t. the Chevalley-Eilenberg differential, so it F' is a 2-cocycle, and the Lie algebra L' is obtained from F' as above.

If ζ'' is another splitting, the difference $\phi = \zeta'' - \zeta'$ is a k-linear map from L to M, so it is a 1-cochain. The composition of the isomorphisms $\hat{\zeta}^{-1}$ and $\hat{\zeta}'$ is the automorphism of $M \oplus L$ given by $m + x \to m + \phi(x) + x$. Let F', F'' be the 2-cocycle associated to ζ', ζ'' respectively. It is then clear that $F'' - F' = \delta\phi$, so the proof is complete. □

2.2 Extensions of Lie-Rinehart algebras

We now present a generalization of the results of the previous section to Lie-Rinehart algebras, that are Lie algebras endowed with additional structures making them an algebraic analogue of the algebra of vector fields over a manifold. See for example [17] for more details.

Let k be a commutative ring, and R a commutative k-algebra.

Definition 2. *A (k, R)-Lie-Rinehart algebra is a k-Lie algebra L with the following structures:*

- *an R-module structure on L, i. e. an associative algebra momorphism $R \to End_k(L)$;*
- *an L-action by derivations on R, i. e. a Lie algebra morphism $\sharp : L \to Der_k(R)$;*
- *this actions are compatible, i. e. satisfy*

$$[x, ay] = a[x, y] + \sharp(x)(a)y$$

for any $x, y \in L$ and $a \in R$.

Morphisms between two (k, R)-Lie-Rinehart algebras are morphisms as k-Lie algebras that are compatible with the additional actions.

If X is a real manifold and T_X is is tangent bundle, then $\Gamma(T_X)$ the vector space of smooth sections of T_X is a real Lie algebra, with the bracket given by the commutator of vector fields. Moreover this has a structure of a $(\mathbb{R}, C^\infty(X))$-Lie-Rinehart algebra, where $\sharp(V)(f)$ is given by the derivative of f along V, for any vector field V and a smooth function f.

Remark that if L is a (k, R)-Lie-Rinehart algebra and L acts trivially on R, then L is actually an R-Lie algebra, since the compatibility condition implies that the bracket is R-linear.

Let L be a (k, R)-Lie-Rinehart algebra, and M an R-module.

Definition 3. *An L-connection on M is a map $\nabla : L \otimes_k M \to M$ satisfying*

- $\nabla((ax) \otimes m) = a\nabla(x \otimes m)$,
- $\nabla(x \otimes (am)) = a\nabla(x \otimes m) + \sharp(x)(a)m$

for any $a \in R$, $x \in L$ and $m \in M$.

Remark that an L-connection on the R-module M induces a map $L \to \mathrm{End}_k(M)$, that in general is not a Lie algebra morphism. When it is a Lie algebra morphism, we will say that the connection is *flat*, and M is said to be an L-module.

If $L = \Gamma(T_X)$ is the $(\mathbb{R}, C^\infty(X))$-Lie-Rinehart of the vector fields of a manifold and $M = \Gamma(E)$ is the $C^\infty(M)$-module of the smooth sections of a vector bundle E, we see that the notion of a $\Gamma(T_X)$-connection on M is equivalent to the usual notion of connection on the vector bundle E.

As before, from now on assume that L is a (k, R)-Lie-Rinehart algebra wich is projective as an R-module. Let (M, ∇) be an L-module, and define the cochain groups

$$C^p(R, L; M, \nabla) = \mathrm{Hom}_R(\bigwedge_R^p L, M)$$

and the coboundary operator

$$\begin{array}{rcl}(\delta\theta)(x_1 \wedge \ldots \wedge x_{p+1}) & = & \sum_{i<j}(-1)^{i+j}\theta([x_i, x_j] \wedge x_1 \wedge \ldots \hat{x_i}, \hat{x_j} \ldots \wedge x_{p+1}) \\ & & +\sum_i(-1)^i \nabla_{x_i}\theta(x_1 \wedge \ldots \hat{x_i} \ldots \wedge x_{p+1})\end{array}$$

for $\theta \in C^p(R, L; M, \nabla)$ and $x_i \in L$. Denote the associated cohomology groups by $H^p_{LR}(R, L; M, \nabla)$, and call them *Lie-Rinehart cohomology groups.*

Remark that the coboundary operator δ is R-linear if and only if L acts trivially on R. When this happens, we have isomorphisms between the Lie-Rinehart cohomology of L and its Chevalley-Eilenberg cohomology as an R-Lie algebra.

When $L = \Gamma(T_X)$ is the $(\mathbb{R}, C^\infty(X))$-Lie-Rinehart algebra of the vector fields of a real manifold X, the Lie-Rinehart cohomology of $\Gamma(T_X)$ coincides with the De Rham cohomology of X.

Let L be a (k, R)-Lie-Rinehart algebra and M an R-module. An extension of L by M is a short exact sequence

$$0 \to M \to L' \to L \to 0$$

in the category of (k, R)-Lie-Rinehart algebras, where M is equipped with the trivial Lie algebra structure. As before, a morphism of two extensions of Lie-Rinehart algebras is a morphism of short exact sequences with the identity corresponding to M and L.

An extension of (k, R)-Lie-Rinehart algebras is in particular an extension of k-Lie algebras. So whenever we have an extension of L by M, this induces on M an L-module structure.

We have the following generalization of Theorem 2:

Theorem 3. *Let k be a commutative ring, R a commutative unital k-algebra, L a (k, R)-Lie-Rinehart algebra wich is projective as R-module, M an R-module and ∇ an L-module structure on M.*

Then the equivalence classes of extensions of (k, R)-Lie-Rinehart algebras of L by M are in one to one correspondence with $H^2_{LR}(R, L; M, \nabla)$ the second Lie-Rinehart cohomology group of L with values in M

Proof. Let L' be a (k, R)-Lie-Rinehart algebra extension of L by M. In particular, this is a k-Lie algebra extension, so, as in in the second part of Theorem 2, for any R-linear splitting $\zeta' : L \to L'$ we obtain a 2-Chevalley-Eilenberg cocycle F'. Recall that F is a k-linear antisymmetric map $F' : L \wedge_k L \to M$. This is a 2-Lie-Rinehart cocycle if and only if F' is R-bilinear. But this is true since $[m+x, f(n+y)]_{L'} = f[m+x, n+y]_{L'} + \sharp(x)(f) \cdot (n+y)$.

On the other hand, a 2-Lie-Rinehart cocycle F' is in particular a 2-Chevalley-Eilenberg cocycle, so, as in the first part of Theorem 2, we can define a k-Lie algebra structure $[\cdot, \cdot]_{L'}$ on $M \oplus L$. It is then easy to see that since F' is R-bilinear, the bracket $[\cdot, \cdot]_{L'}$, together with the anchor $\sharp_{L'} = \sharp \circ p_L$, define a (k, R)-Lie-Rinehart algebra extension of L by M. □

2.3 Smooth Lie algebroids

Let M be a smooth manifold, and T_M its tangent bundle. For any smooth vector bundle E on M we will denote by $\Gamma(E)$ the space of smooth sections of E.

Definition 4. *A real* Lie algebroid *on M is a triple $(L, \sharp, [\cdot, \cdot])$ such that:*

- *L is a vector bundle on M;*
- *$[\cdot, \cdot]$ is a $\mathbb{R}$-Lie algebra structure on $\Gamma(L)$;*
- *$\sharp : L \to T_M$ is a vector bundle morphism, called the* anchor*, that induces an $\mathbb{R}$-Lie algebra morphism on global sections and satisfies the following Leibniz rule:*

$$[u, fv] = f[u, v] + \sharp(u)(f)v$$

for any $u, v \in \Gamma(L)$ and $f \in C^\infty(M)$.

A morphism between two Lie algebroids L and L' over the same manifold M is a vector bundle morphism $L \to L'$ that commutes with the anchors and such that the induced morphism on global sections is a Lie algebra morphism.

Clearly, a real Lie algebra is equivalent to a real Lie algebroid defined over a point. The tangent bundle has a *canonical* structure of a Lie algebroid, with anchor equal to the identity and the bracket equal the ommutator of vector fields.

If $(L, \sharp, [\cdot,\cdot])$ is a real Lie algebroid, we see that the space $\Gamma(L)$ of smooth sections of L is naturally endowed with a $(\mathbb{R}, C^\infty(M))$-Lie-Rinehart structure: functions act on sections of L by multiplication, a section s of L acts on functions via the derivation along the vector field $\sharp(s)$, and the Lie bracket is just $[\cdot,\cdot]$.

As differential forms on a manifold are sections of the exterior powers of the cotangent bundle, we call *k-L-forms* the sections of the vector bundle $\bigwedge^k L^*$; we denote the corresponding sheaf of sections by $\mathcal{A}_L^k$, and by A_L^k the vector space of global sections of $\bigwedge^k L^*$.

Consider the map $(f,u) \mapsto \mathrm{d}_L f(u) := \sharp(u)(f)$ for $f \in C^\infty(M)$ and $u \in \Gamma(L)$. It is linear w. r. t. u, so it defines a map $\mathrm{d}_L : C^\infty(M) \to A_L^1$. It satisfies the Leibniz rule $\mathrm{d}_L(fg) = f\mathrm{d}_L g + g\mathrm{d}_L f$, so it is a derivation, that we will call the *exterior differential* of L. Remark that it is obtained simply by composing the usual exterior differential with the dual of the anchor $\sharp^* : T_M^* \to L^*$.

It extends to a derivation $\mathrm{d}_L : A_L^p \to A_L^{p+1}$ via the formula

$$\begin{array}{rcl}(\mathrm{d}_L\theta)(u_1,\dots,u_{p+1}) & = & \sum_i (-1)^{i+1} a(u_i)(\theta(u_1,\dots,\hat{u}_i,\dots,u_{p+1})) + \\ & + & \sum_{i<j}(-1)^{i+j}\theta(\{u_i,u_j\},u_1,\dots,\hat{u}_i,\hat{u}_j,\dots u_{p+1})\end{array}$$

for $\theta \in A_L^p$ and $u_1,\dots,u_{p+1} \in \Gamma(L)$.

One can check that $\mathrm{d}_L^2 = 0$, so the pair $(A_L^\bullet, \mathrm{d}_L)$ forms a complex. We define the *Lie algebroid cohomology* of L to be the cohomology of this complex, and denote it by $H^p(L, \sharp, [\cdot,\cdot]; \mathbb{R})$ (or simply by $H^p(L;\mathbb{R})$).

Remark that the definition of the Lie algebroid cohomology for a Lie algebroid L coincides with the definition of the Lie-Rinehart cohomology for the $(\mathbb{R}, C^\infty(M))$-Lie-Rinehart algebra $\Gamma(L)$ with coeffficients in the trivial $\Gamma(L)$-module $\mathbb{R}$.

The construction of the Lie algebroid cohomology is contravariantly functorial: a morphism of Lie algebroids $\psi : L \to L'$ induces a morphism of complexes $\psi^* : A_{L'}^\bullet \to A_L^\bullet$, yielding pull-back morphisms in cohomology $H^p(L', \mathbb{R}) \to H^p(L, \mathbb{R})$. In particular, we can always see the anchor of a Lie algebroid L as a Lie algebroid morphism between L and T_M equipped with the canonical Lie algebroid structure, so, since the Lie algebroid cohomology of the canonical Lie algebroid is just the De Rham cohomology of M, we always have morphisms $H^p_{DR}(M,\mathbb{R}) \to H^p(L,\mathbb{R})$.

2.3.1 Basic examples

As noted previously, the tangent bundle T_M has a canonical structure of Lie algebroid. Similarly, if $\mathscr{F} \subseteq T_M$ is any subbundle of T_M that satisfies the involutivity condition $[\mathscr{F},\mathscr{F}] \subseteq \mathscr{F}$, we can define a natural Lie algebroid structure over it, taking the inclusion as anchor and the restriction of the commutator of vector fields to $\Gamma(\mathscr{F})$ as bracket. Remark that in this case $\mathscr{F}$, by Frobenius theorem, defines a foliation of M.

Vice versa, if L is a Lie algebroid with injective anchor, it is a subbundle of T_M, and since the anchor has to be a morphism of Lie algebras we see that it defines an integrable foliation.

Let K be a vector bundle such that each fiber K_x is equipped with a $\mathbb{R}$-Lie algebra structure $[\cdot,\cdot]_x$. Assume that the coefficients defining the bracket w. r. t. any local frame of K vary smoothly. This is a so called *bundle of Lie algebras*. Then, for any open subset $U \subseteq M$ such that there exists a frame of K over U, the $\mathbb{R}$-Lie algebra structures $[\cdot,\cdot]_x$ glue together to define a $C^\infty(U)$-Lie algebra structure on $\Gamma(K_{|U})$, and these glue to define a $C^\infty(M)$-Lie algebra structure on $\Gamma(K)$. So we can define a Lie algebroid structure on K with anchor equal to 0 and the bracket just defined.

Vice versa, any Lie algebroid $(L,\sharp,[\cdot,\cdot])$ with $\sharp = 0$ defines naturally a structure of bundle of Lie algebras on L.

In particular, we can equip any vector bundle with the trivial Lie algebroid structure, since we can put the 0 Lie algebra structure on each of its fibers.

Let $(L,a,\{\cdot,\cdot\})$ be a Lie algebroid. The image of the anchor $\mathscr{F} = \mathrm{Im}\,(\sharp)$ defines a subsheaf of T_M involutive under the commutator of vector fields, so it defines a (not necessarily regular) foliation on M, that is usually called the *foliation of the Lie algebroid*. On the other hand, the kernel of the anchor $K = \mathrm{Ker}\,(\sharp)$ is clearly invariant under the bracket of L.

When the anchor is regular, i. e. the dimension of kernel and image of $\sharp_x$ are constant as x varies, both $\mathscr{F}$ and K are vector bundles over M, and they inherit a natural Lie algebroid structure from L: $\mathscr{F}$ with injective anchor, i. e. a foliation, and K with anchor equal 0, i. e. a bundle of Lie algebras. Moreover they form an exact sequence of Lie algebroids

$$0 \to K \to L \to \mathscr{F} \to 0 \ .$$

Another situation where Lie alebroids arise naturally is Poisson geometry:

let M be a smooth manifold and $\Pi \in \Gamma(\bigwedge^2 T_M)$ a Poisson bivector. Then we can define a Lie algebroid structure on T_M^* with

- anchor $\sharp : T_M^* \to T_M$ given by the contraction with Π,
- bracket defined by the formula

$$[\alpha, \beta] = \mathrm{d}\langle \Pi, \alpha \wedge \beta \rangle - \mathscr{L}_{\sharp(\beta)}\alpha + \mathscr{L}_{\sharp(\alpha)}\beta$$

  for any $\alpha, \beta \in \Gamma(T_X^*)$, where $\mathscr{L}_V$ is the Lie derivative along the vector field V.

2.3.2 Lie algebroid connections and characteristic classes

For the whole subsection, fix a Lie algebroid $(L, \sharp, [\cdot, \cdot])$ on a manifold M.

Let E be a vector bundle over M. There are many equivalent notion of a connection on E. One of these is given in terms of an operator

$$\nabla : \Gamma(E) \to \Gamma(E) \otimes A_X^1$$

satisfying the Leibniz rule $\nabla(fs) = f\nabla s + s \otimes \mathrm{d}f$ for any $f \in C^\infty(M)$ and $s \in \Gamma(E)$. Similarly, we give the following:

Definition 5. *An* L-connection *on a vector bundle* E *is a map* $\nabla : \Gamma(E) \to \Gamma(E) \otimes A_L^1$ *satisfying the Leibniz rule*

$$\nabla(fs) = f\nabla s + s \otimes d_L f$$

for any $f \in \mathbb{C}^\infty(M)$ *and* $s \in \Gamma(E)$.

Remark that any (T_M-)connection on E induces an L-connection: if $\nabla : \Gamma(E) \to \Gamma(E) \otimes A_X^1$ is a connection, composing it with $\mathbf{1}_E \otimes \sharp^* : E \otimes T_M^* \to E \otimes L^*$ we obtain a L-connection. In particular, this shows that on any vector bundle E there always exist L-connections.

One can extend an L-connection to higher degree E-valued L-forms: set $A_L^k(E) = \Gamma(\bigwedge^k L^* \otimes E)$, the space of E-valued global k-L-forms, and define $\nabla : A_L^k(E) \to A_L^{k+1}(E)$, by

$$\nabla(\eta \otimes s) = \mathrm{d}_L \eta \otimes s + (-1)^k \eta \wedge \nabla s$$

for any $\eta \in A_L^k$ and $s \in \Gamma(E)$.

As for usual connections, an L-connection ∇ on a vector bundle E induces an L-connection ∇^* on E^* via the formula

$$\langle \nabla^* \sigma, s \rangle + \langle \sigma, \nabla s \rangle = \mathrm{d}_L \langle \sigma, s \rangle \ ,$$

while L-connections ∇_E, ∇_F on two vector bundles E, F induce, in an appropriate way, an L-connection $\nabla_{E \oplus F} = \nabla_E \oplus \nabla_F$ on $E \oplus F$ and an L-connection $\nabla_{E \otimes F} = \nabla_E \otimes \mathbf{1}_F + \mathbf{1}_E \otimes \nabla_F$ on $E \otimes F$.

Define the *curvature* of ∇ as

$$F_\nabla = \nabla \circ \nabla : A_L^0(E) \to A_L^2(E).$$

One can easily check that this is a $C^\infty(M)$-linear mapping, so it defines an element of $A_L^2(\operatorname{End} E)$. Moreover one can easily express it in terms of the covariant derivative associated to ∇:

$$F_\nabla(u,v)(e) = [\nabla_u, \nabla_v](e) - \nabla_{\{u,v\}} e \ , \qquad u, v \in \Gamma(L),\ e \in \Gamma(E) \ ,$$

where $\nabla_w : E \to E$ denotes the 1st order differential operator $e \to \langle \nabla e, w \rangle$.

We have the analogue of Bianchi identity for Lie algebroids (cf. [11]):

Proposition 4. *Let ∇ be an L-connection on the vector bundle E, and F_∇ its curvature. Denote by $\tilde{\nabla}$ the L-connection induced by ∇ on $\operatorname{End} E$.*

Then $\tilde{\nabla} F_\nabla = 0$.

We will say that an L-connection on the vector bundle E is *flat* when its curvature vanishes. When this happens, the map $\Gamma(L) \to \operatorname{Der}(E)$ given by $w \to \nabla_w$ is a morphism of $\mathbb{R}$-Lie algebras. We shall sometime use the terminology *representation of* L or L*-module* to mean a flat L-connection on a vector bundle E.

If (E, ∇) is a L-module we have that $(A_L^\bullet(E), \nabla)$ is a complex, so we can define the cohomology groups of L with values in E, and denote them by $H^k(L; E, \nabla)$.

Remark that if we consider $\Gamma(L)$ the $(\mathbb{R}, C^\infty(M))$-Lie-Rinehart algebra associated to L, and $M = \Gamma(E)$ is the $C^\infty(M)$-module of sections of the vector bundle E, then an L-connection on E is equivalent to a $\Gamma(L)$-connection on M, and the cohomologies $H^k(L; E, \nabla)$ and $H^k_{LR}(C^\infty(M), \Gamma(L); \Gamma(E), \nabla)$ coincide.

Let us recall the construction of the charateristic ring of a vector bundle: let E be a rank r complex vector bundle on M. Consider the complex general linear group $\mathrm{GL}(r, \mathbb{C})$ and its Lie algebra $\mathfrak{gl}_r$; the group $\mathrm{GL}(r, \mathbb{C})$ acts on $\mathfrak{gl}_r$ via the adjoint representation, so it acts naturally on $\mathfrak{gl}_r^{\oplus k}$ for any positive integer k. Consider $I^k(\mathrm{GL}(r, \mathbb{C}))$ the set of k-multilinear maps $P : \mathfrak{gl}_r \times \ldots \times \mathfrak{gl}_r \to \mathbb{C}$ invariant w. r. t. the adjoint representation of $\mathrm{GL}(r, \mathbb{C})$.

Let ∇ be an L-connection on E and $F_\nabla \in A_L^2(\operatorname{End} E)$ its curvature. For any $P \in I^k(\mathrm{GL}(r, \mathbb{C}))$ define $\lambda_\nabla(P) \in A_L^{2k}$ via

$$\lambda_\nabla(P)(u_1, \ldots, u_{2k}) = \sum_{\sigma \in \mathfrak{S}_{2k}} (-1)^\sigma P(F_\nabla(u_{\sigma(1), \sigma(2)}), \ldots, F_\nabla(u_{\sigma(2k-1)}, u_{\sigma(2k)}))$$

for any $u_1, \ldots, u_{2k} \in \Gamma(L)$. To give a precise meaning to the right hand side of the equation, one has to work a little. Let $U \subseteq M$ be an open set such that over it a trivialization τ of E is given. Then $\tau^*(F_\nabla)_{|U} \in A^{2k}_L(\mathfrak{gl}_r)$, so over any such U we have $\tau^* F_\nabla(u,v)) \in \mathfrak{gl}_r$, and we can apply P to it. Now, if τ' is another trivialization of E, the matrix of 2-L-forms $\tau'^* F_\nabla$ is obtained from $\tau^* F_\nabla$ by conjugation. So, since P is $\mathrm{Ad}_{\mathrm{GL}(r,\mathbb{C})}$-invariant the right hand side is well defined.

Now, from [11] we have:

Lemma 5. *1. The $2k$-L-forms $\lambda_\nabla(P)$ are d_L-closed.*

2. Their cohomology class in $H^{2k}(L,\mathbb{C})$ does not depend on the choice of the L-connection ∇.

Via this lemma, for any vector bundle E we have a morphism

$$\lambda : I^\bullet(\mathrm{GL}(r,\mathbb{C})) \to H^\bullet(L,\mathbb{C}) \ ,$$

that we will call the *L-Chern-Weil homomorphism.* Define the L-characteristic ring of E to be the image of this morphism, and denote it by $\mathcal{R}^\bullet_L(E)$.

Remark that actually we have not defined anything new: since the curvature of any L-connection computes the L-characteristic ring of E, we can choose a L-connection ∇ induced by a T_M-connection ∇^{T_M}; then, since $F_\nabla = \sharp^* F_{\nabla^{T_M}}$, we obtain the commutative diagram:

$$\begin{tikzcd} I^\bullet(\mathrm{GL}_r) \arrow[r] \arrow[dr] & H^\bullet_{DR}(X,\mathbb{C}) \arrow[d] \\ & H^\bullet(L,\mathbb{C}) \end{tikzcd}$$

where the upper arrow is the usual Chern-Weil morphism. So we see that the L-characteristic ring of E is just the pull-back in cohomology of its usual characteristic ring. The construction we gave of $\mathcal{R}_L(E)$ provides another way to compute this ring, and later we will exploit this fact.

2.4 Holomorphic Lie algebroids

As real Lie algebroids generalize the tangent bundle of a real manifold, complex and holomorphic Lie algebroids generalize the tangent bundle of a complex manifold.

Let X be a complex manifold, T_X its tangent bundle and $\mathcal{T}_X$ the associated sheaf of holomorphic vector fields. As a general notation, if E is a complex vector bundle, we denote by $\mathcal{A}^0(E)$ the associated sheaf of smooth sections. When it is equipped with a holomorphic structure we will denote by the corresponding calligraphic letter $\mathcal{E}$ the coherent $\mathscr{O}_X$-module of its holomorphic sections, and we may use also $\mathcal{A}^0(\mathcal{E})$ the to denote $\mathcal{A}^0(E)$. Recall that we denote by $\mathcal{A}^k_X$ the sheaf of smooth k-forms on X, while when X

is a complex manifold we denote by Ω_X^k the sheaf of holomorphic k-forms of X.

Definition 6. *A* holomorphic Lie algebroid *is a triple* $(\mathcal{L}, \sharp, [\cdot,\cdot])$ *with*

- $\mathcal{L}$ *a coherent locally free* $\mathscr{O}_X$*-module;*
- $[\cdot,\cdot]$ *a* $\mathbb{C}_X$*-Lie algebra structure on* $\mathcal{L}$ *(i.e. for each open* U *of* X *the bracket is a* $\mathbb{C}$*-Lie algebra structure on* $\mathcal{L}(U)$ *compatible with the restriction morphisms);*
- *the anchor* $\sharp : \mathcal{L} \to \mathcal{T}_X$ *is a morphism of* $\mathscr{O}_X$*-modules which is also a morphism of sheaves of* $\mathbb{C}_X$*-Lie algebras, satisfying the Leibniz rule*

  $$[u, av] = a[u, v] + \sharp(u)(a)v$$

  for any local sections $a \in \mathscr{O}_X$ *and* $u, v \in \mathcal{L}$*.*

We will call holomorphic k-$\mathcal{L}$-forms the sections of $\Omega_{\mathcal{L}}^k = \bigwedge^k \mathcal{L}^*$, and, similarly to the smooth case, we have an exterior differential $\mathrm{d}_{\mathcal{L}} : \Omega_{\mathcal{L}}^k \to \Omega_{\mathcal{L}}^{k+1}$.

The *underlying real Lie algebroid* $\mathcal{L}_{\mathbb{R}}$ of the holomorphic Lie algebroid $\mathcal{L}$ is the real Lie algebroid obtained forgetting the holomorphic structures: if L is the smooth bundle underlying $\mathcal{L}$, we have that $\sharp$ induces a real anchor $\sharp_{\mathbb{R}} : L \to T_X$, and that the bracket naturally extends in a unique way to smooth sections of L.

Moreover, we can associate to $\mathcal{L}$ two complex Lie algebroids $\mathcal{L}^{1,0}$ and $\mathcal{L}^{0,1}$: consider the complexification $L_{\mathbb{C}} = L \otimes \mathbb{C}$ of the smooth Lie algebroid underlying $\mathcal{L}$. The complex structures on the fibers of L gives a bundle map $J : L_{\mathbb{C}} \to L_{\mathbb{C}}$ whose square is $-\mathbf{1}$. So we have a splitting of L_C in eingenbundles $L^{1,0}$ and $L^{0,1}$ according to the eigenvalues $\pm i$ of J. Since the anchor $\sharp_{\mathbb{R}} : L \to T_X$ comes from a holomorphic Lie algebroid, it is easy to see that $L^{1,0}$ and $L^{0,1}$ are complex Lie subalgebroids of $L_{\mathbb{C}}$, that we will denote by $\mathcal{L}^{1,0}$ and $\mathcal{L}^{0,1}$ respectively.

Let $(L, \sharp, \{\cdot,\cdot\})$ be a real Lie algebroid over X.

Definition 7. *An* almost complex structure *on* $(L, \sharp, \{\cdot,\cdot\})$ *is a vector bundle endomorphism* $J_L : L \to L$ *such that* $J_L^2 = -\mathbf{1}_L$ *and commuting with the canonical almost complex structure* J_X *on* T_X*, i.e.* $J_X \circ \sharp = \sharp \circ J_L$*.*

An almost complex structure J_L on a Lie algebroid L induces a splitting $L \otimes \mathbb{C} = L^{1,0} \oplus L^{0,1}$ according to the eigenvalues $\pm i$ of J_L. This induces a splitting on L^* and its exterior power:

$$\bigwedge^k (L^* \otimes \mathbb{C}) = \bigoplus_{p+q=k} \bigwedge^p (L^*)^{1,0} \otimes \bigwedge^q (L^*)^{0,1}.$$

We denote by $\mathcal{A}_L^{p,q}$ the sheaf of local sections of $\bigwedge^p(L^*)^{1,0} \otimes \bigwedge^q(L^*)^{0,1}$ and call them (p,q)-L-forms. Accordingly to this splitting, one has the splitting of the exterior differential $\mathrm{d}_L = \partial'_L + \partial_L + \bar\partial_L + \partial''_L$, where

$$\partial'_L : \mathcal{A}_L^{p,q} \to \mathcal{A}_L^{p+2,q-1}, \qquad \partial_L : \mathcal{A}_L^{p,q} \to \mathcal{A}_L^{p+1,q}$$
$$\bar\partial_L : \mathcal{A}_L^{p,q} \to \mathcal{A}_L^{p,q+1}, \qquad \partial''_L : \mathcal{A}_L^{p,q} \to \mathcal{A}_L^{p-1,q+2}.$$

We will say that an almost complex structure J_L on a Lie algebroid $(L, \sharp, [\cdot,\cdot])$ is *integrable* if there exists a holomorphic bundle structure $\mathcal{L}$ on L such that $\mathcal{L}^{1,0} = L^{1,0}$.

Clearly, when an almost complex structure on a real Lie algebroid $(L, \sharp, [\cdot,\cdot])$ is integrable, the operators $\partial'_L, \partial''_L$ clearly vanish, and the differentials $\mathrm{d}_L, \partial_L, \mathrm{d}_{\mathcal{L}}$ coincide on the subspaces $\Omega^k_{\mathcal{L}} \subseteq \mathcal{A}^k_{L,\mathbb{C}}$.

2.4.1 Algebraic Lie algebroids

If X is a smooth algebraic variety over an algebraically closed field k of characteristic 0, we have can define an *algebraic Lie algebroid* as a triple $(\mathcal{L}, \sharp, [\cdot,\cdot])$ with $\mathcal{L}$ a coherent $\mathscr{O}_X$-module, $\sharp : \mathcal{L} \to \mathcal{T}_X$ a morphism of $\mathscr{O}_X$-modules ($\mathcal{T}_X$ being the tangent sheaf of X) and $[\cdot,\cdot]$ a k-Lie algebra structure on $\mathcal{L}$ that satisfies the usual Leibniz rule with respect to the anchor $\sharp$.

If X is projective and defined over $\mathbb{C}$, by the GAGA principle the notions of algebraic and holomorphic Lie algebroid over X coincide. In the following chapters, we will often work on smooth projective algebraic varieties over $\mathbb{C}$, and we will often switch from the algebraic to the holomorphic setting without mentioning. In particular, all the statements where we will work on complex algebraic varieties with holomorphic Lie algebroids are to be understood in this way.

2.4.2 Matched pairs of Lie algebroids and the canonical complex Lie algebroid

We say that two real (or complex) Lie algebroids $(L_i, \sharp_i, [\cdot,\cdot]_i)$, $i = 1, 2$, form a *matched pair* if we are given a L_1-module structure on L_2 and a L_2-module structure on L_1 (that we both denote by ∇) satisfying the following equations:

$$\begin{aligned}
[\sharp_1(u_1), \sharp_2(u_2)] &= -\sharp_1(\nabla_{u_2} u_1) + \sharp_2(\nabla_{u_1} u_2)\ , \\
\nabla_{u_1}([u_2, v_2]_2) &= [\nabla_{u_1} u_2, v_2]_2 + [u_2, \nabla_{u_1} v_2]_2 + \nabla_{\nabla_{v_2} u_1} u_2 - \nabla_{\nabla_{u_2} u_1} v_2\ , \\
\nabla_{u_2}([u_1, v_1]_1) &= [\nabla_{u_2} u_1, v_1]_1 + [u_1, \nabla_{u_2} v_1]_1 + \nabla_{\nabla_{v_1} u_2} u_1 - \nabla_{\nabla_{u_1} u_2} v_1\ .
\end{aligned}$$

Consider the groups $K^{p,q} = A^p_{L_1} \otimes A^q_{L_2}$. The L_i-module structures on $L_{\hat\imath}$ (where $\hat\imath = 2$ for $i = 1$ and $\hat i = 1$ for $i = 2$) induces an L_i-module structure on $A^p_{L_{\hat\imath}}$, so we have two differentials $\mathrm{d}_{L_1} : K^{p,q} \to K^{p+1,q}$ and $\mathrm{d}_{L_2} : K^{p,q} \to$

$K^{p,q+1}$ induced by these module structures. The three equations above are equivalent to $\mathrm{d}_{L_1}\mathrm{d}_{L_2} = (-1)^p \mathrm{d}_{L_2}\mathrm{d}_{L_1}$, i.e. the triple $(K^{\bullet,\bullet}, \mathrm{d}_{L_1}, \mathrm{d}_{L_2})$ is a double complex.

If (L_1, L_2) is a matched pair of Lie algebroids, define the following Lie algebroid $L_1 \bowtie L_2$:

- the underlying vector bundle is $L_1 \oplus L_2$;
- the anchor is the sum: $\sharp = \sharp_1 + \sharp_2$;
- the bracket is defined by

$$\begin{aligned} [u_1 + u_2, v_1 + v_2] &= ([u_1, v_1]_1 + \nabla_{v_1}(u_2) - \nabla_{v_2}(u_1)) + \\ &+ ([u_2, v_2]_2 + \nabla_{u_1}(v_2) - \nabla_{u_2}(v_1)) \end{aligned}$$

  for any $u_i, v_i \in \Gamma(L_i)$, $i = 1, 2$.

Proposition 6 (see [20]). *The Lie algebroid cohomology of $L_1 \bowtie L_2$ is the cohomology of the total complex associated to $K^{\bullet,\bullet}$.*

Proof. This follows directly from the fact that, because of the previous compatibility equations, the exterior differential of the Lie algebroid $L_1 \bowtie L_2$ is $\mathrm{d}_{L_1 \bowtie L_2} = \mathrm{d}_{L_1} + (-1)^p \mathrm{d}_{L_2}$. □

We recall the following result of [20]:

Theorem 7. *Let $\mathcal{L}$ be a holomorphic Lie algebroid, and $\mathcal{L}^{1,0}$ the associated complex Lie algebroid.*

Then the pair $(\mathcal{L}^{1,0}, T_X^{0,1})$ is naturally a matched pair of complex Lie algebroids.

The $T_X^{0,1}$-module structure on $\mathcal{L}^{1,0}$ is given by the holomorphic structure of $\mathcal{L}$, while the $\mathcal{L}^{1,0}$-module structure on $T_X^{0,1}$ is given by

$$\nabla_u(V) = \mathrm{pr}^{0,1}([\sharp(u), V])$$

for $u \in \Gamma(\mathcal{L}^{1,0})$ and $V \in \Gamma(T_X^{0,1})$, and where $\mathrm{pr}^{0,1}$ denotes the projection from the complexified tangent space $T_{X,\mathbb{C}}$ to $T_X^{0,1}$.

$\mathcal{L}^{1,0} \bowtie T_X^{0,1}$ carries naturally an almost complex structure, and we call it the *canonical complex Lie algebroid* associated to the holomorphic Lie algebroid $\mathcal{L}$, and denote it by $\mathcal{L}_h$. Remark that we have a natural morphism of complex Lie algebroids

$$\mathcal{L}_{\mathbb{C}} = \mathcal{L}_{\mathbb{R}} \otimes \mathbb{C} \to \mathcal{L}_h.$$

2.4.3 Cohomology of holomorphic Lie algebroids

We now study some cohomological properties of a holomorphic Lie algebroid $(\mathcal{L}, \sharp, [\cdot,\cdot])$. In particular, we present some results from [20] and [7] generalizing to holomorphic Lie algebroids the theorems of De Rham and Dolbeault.

Let $\mathcal{L}$ be a holomorphic Lie algebroid. Consider the complex of coherent sheaves over X

$$0 \longrightarrow \mathscr{O}_X = \Omega^0_{\mathcal{L}} \xrightarrow{\mathrm{d}_{\mathcal{L}}} \Omega^1_{\mathcal{L}} \xrightarrow{\mathrm{d}_{\mathcal{L}}} \cdots \xrightarrow{\mathrm{d}_{\mathcal{L}}} \Omega^n_{\mathcal{L}} \xrightarrow{\mathrm{d}_{\mathcal{L}}} 0 \ ,$$

and define the *holomorphic Lie algebroid cohomology* of $\mathcal{L}$ to be the hypercohomology of this complex:

$$H^p(\mathcal{L};\mathbb{C}) = \mathbb{H}^p(X;\Omega^\bullet_{\mathcal{L}}, \mathrm{d}_{\mathcal{L}}) \ .$$

Let $\mathcal{L}_h = \mathcal{L}^{1,0} \bowtie T^{0,1}_X$ be the canonical complex Lie algebroid associated to $\mathcal{L}$. Then we have:

Theorem 8. *Let $\mathcal{L}$ be a holomorphic Lie algebroid. Then we have the following isomorphisms:*

1. *(holomorphic De Rham) $H^p(\mathcal{L};\mathbb{C}) \cong H^p(\mathcal{L}_h;\mathbb{C})$;*
2. *(Dolbeault)*
$$H^q(X, \Omega^p_{\mathcal{L}}) \cong H^q(A^{p,\bullet}_{\mathcal{L}_h}, \bar\partial_{\mathcal{L}_h}).$$

Proof. Both the statement are a direct consequence of the holomorphic Poincaré Lemma: consider the double complex

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & \mathcal{A}^{p+1,q}_{\mathcal{L}_h} \arrow[u, "\partial_{\mathcal{L}_h}"] \arrow[r, "\bar\partial_{\mathcal{L}_h}"] & \mathcal{A}^{p+1,q+1}_{\mathcal{L}_h} \arrow[u, "\partial_{\mathcal{L}_h}"] \arrow[r, "\bar\partial_{\mathcal{L}_h}"] & \cdots \\
\cdots \arrow[r] & \mathcal{A}^{p,q}_{\mathcal{L}_h} \arrow[u, "\partial_{\mathcal{L}_h}"] \arrow[r, "\bar\partial_{\mathcal{L}_h}"] & \mathcal{A}^{p,q+1}_{\mathcal{L}_h} \arrow[u, "\partial_{\mathcal{L}_h}"] \arrow[r, "\bar\partial_{\mathcal{L}_h}"] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

associated to $\mathcal{L}_h$. As we have seen previously, the associated total complex computes the (smooth) Lie algebroid cohomology of $\mathcal{L}_h$ with values in $\mathbb{C}$. On the other hand, by the holomorphic Poincaré Lemma the rows of the double complex are exact, so the total complex computes the hypercohomology of the complex of the kernels $\mathcal{K}^\bullet = \mathrm{Ker}\,(\mathcal{A}^{\bullet,0}_{\mathcal{L}_h} \to \mathcal{A}^{\bullet,1}_{\mathcal{L}_h})$. But $\mathcal{K}^p = \Omega^p_{\mathcal{L}}$, so (1) is proven.

(2) follows by the same argument, considering each row separately. $\square$

The second of these statements can be readily generalized to the case of coefficients in a holomorphic vector bundle: let $\mathcal{E}$ be a holomorphic vector bundle. Then we can define the $\mathcal{E}$-valued holomorphic k-$\mathcal{L}$-forms as the sections of

$$\Omega^k_{\mathcal{L}}(\mathcal{E}) = \Omega^k_{\mathcal{L}} \otimes_{\mathscr{O}_X} \mathcal{E} \ ,$$

and the $\mathcal{E}$-valued (p,q)-$\mathcal{L}_h$-forms as the sections of

$$\mathcal{A}^{p,q}_{\mathcal{L}_h}(\mathcal{E}) = \mathcal{E} \otimes_{\mathscr{O}_X} \mathcal{A}^{p,q}_{\mathcal{L}_h} \ .$$

Now, $\bar\partial_{\mathcal{L}_h}(f) = 0$ for $f \in \mathscr{O}_X$, so $\bar\partial_{\mathcal{L}_h}$ is an $\mathscr{O}_X$-linear homomorphism $\mathcal{A}^{p,q}_{\mathcal{L}_h} \to \mathcal{A}^{p,q+1}_{\mathcal{L}_h}$; then we can extend it to $\mathcal{E}$-valued forms as

$$\bar\partial_{\mathcal{E}} = \mathbf{1}_{\mathcal{E}} \otimes_{\mathscr{O}_X} \bar\partial_{\mathcal{L}_h} : \mathcal{A}^{p,q}_{\mathcal{L}_h}(\mathcal{E}) \to \mathcal{A}^{p,q+1}_{\mathcal{L}_h}(\mathcal{E}) \ .$$

Consider the global sections $A^{p,q}_{\mathcal{L}_h}(\mathcal{E}) = \Gamma(\mathcal{A}^{p,q}_{\mathcal{L}_h}(\mathcal{E}))$; since $\bar\partial^2_{\mathcal{E}} = 0$ we have a complex of vector spaces $(A^{p,\bullet}_{\mathcal{L}_h}, \bar\partial_{\mathcal{E}})$ whose q-th cohomology group we denote by $H^{p,q}(\mathcal{L}_h, \mathcal{E})$.

Theorem 9. *Let $\mathcal{L}$ be a holomorphic Lie algebroid, $\mathcal{L}_h$ its canonical complex Lie algebroid, and $\mathcal{E}$ a holomorphic vector bundle.*

Then there are isomorphisms

$$H^{p,q}(\mathcal{L}_h, \mathcal{E}) \cong H^q(X, \Omega^p_{\mathcal{L}}(\mathcal{E})) \ .$$

Proof. Since $(\mathcal{A}^{p,\bullet}_{\mathcal{L}_h}, \bar\partial_{\mathcal{L}_h})$ is a fine resolution of $\Omega^p_{\mathcal{L}}$, tensoring it by a locally free sheaf $\mathcal{E}$ we still obtain a fine resolution

$$0 \to \Omega^p_{\mathcal{L}}(\mathcal{E}) \to \mathcal{A}^{p,\bullet}_{\mathcal{L}_h}(\mathcal{E}) \ .$$

The thesis then follows. □

We now study closely the Čech-DeRham double complex of a holomorphic Lie algebroid $\mathcal{L}$ to have a better description of some of its cohomology groups (in particular $H^2(\mathcal{L}, \mathbb{C})$).

Let $\mathfrak{U} = \{U_\alpha\}$ be a sufficiently fine open covering of X, such that we have an isomorphism between sheaf and Čech cohomology over it.

Consider the double complex

$$K^{p,q}_{\mathcal{L}} = \check{C}^q(\mathfrak{U}, \Omega^p_{\mathcal{L}}) \ ,$$

with differentials $\mathrm{d}_{\mathcal{L}}, \check\delta$; its associated total complex $(T^\bullet_{\mathcal{L}}, \delta)$ computes the hypercohomology of $\Omega^\bullet_{\mathcal{L}}$, so it computes $H^k(\mathcal{L}, \mathbb{C})$.

The filtration by columns of the total complex

$$F^r T^k_{\mathcal{L}} = \bigoplus_{p+q=k,\ q \geq r} K^{p,q}_{\mathcal{L}}$$

induces a filtration in the Lie algebroid cohomology (the bête filtration):

$$F^p H^k(\mathcal{L}, \mathbb{C}) = \operatorname{Im}(H^k(F^p T^\bullet_{\mathcal{L}}) \to H^k(\mathcal{L}, \mathbb{C})).$$

The associated spectral sequence has E_1 and E_2 terms given by

$$E_1^{p,q} = H^q(X, \Omega^p_{\mathcal{L}}) \qquad E_2^{p,q} = H^p(H^q(X, \Omega^\bullet_{\mathcal{L}}), \mathrm{d}_{\mathcal{L}}).$$

In the classical case, when $\mathcal{L} = \mathcal{T}_X$ and X is a smooth projective variety, the Hodge decomposition implies that $E_1^{p,q} = E_\infty^{p,q}$ and that the differential d_1 is zero. Let us rewrite this fact as:

Lemma 10. *Let X be a smooth projective variety, and $\phi_{i_0,\dots,i_q}$ a closed Čech q-cochain of Ω^p_X.*

Then the Čech q-cochain $d\phi_{i_0,\dots,i_q}$ is Čech-exact, i.e. $d\phi = \check{\delta}\tau$ for some $\tau \in \check{C}^{q-1}(\mathfrak{U}, \Omega^{p+1}_X)$.

Now, for a general holomorphic Lie algebroid $\mathcal{L}$ there is no analogue of Hodge decomposition, so we do not have degeneration of the spectral sequence at the first step. Anyway, we can use this lemma to find a mild degeneration of the spectral sequence:

Lemma 11. *Restricted to $p = 0$, the differential d_1 of the spectral sequence $d_1 : E_1^{0,q} \to E_1^{1,q}$ is zero.*

Proof. Remark that on functions $\mathrm{d}_{\mathcal{L}}$ coincides with the composition of the exterior differential d with the dual of the anchor, i.e. $\mathrm{d}_{\mathcal{L}}(f) = \sharp^*(\mathrm{d}f)$ for any $f \in \mathscr{O}_X$.

The differential d_1 coincide with $\mathrm{d}_{\mathcal{L}}$. More precisely, an element $\xi \in E_1^{0,q}$ is a q-Čech cocycle of the sheaf $\mathscr{O}_X$, so $\xi = \{\xi_{i_0,\dots,i_q}\}$, and $d_1\xi = \{\mathrm{d}_{\mathcal{L}}\xi_{i_0,\dots,i_q}\}$.

By previous lemma $\{\mathrm{d}\xi_{i_0,\dots,i_q}\} = \check{\delta}\tau$, so $d_1\xi = \check{\delta}(\sharp^*\tau)$, so its class is zero. □

In particular, we have the following corollary:

Corollary 12. *We have an isomorphism*

$$F^1 H^2(\mathcal{L}, \mathbb{C}) \cong H^2(F^1 T^\bullet_{\mathcal{L}}) \ .$$

Proof. We have the exact sequence

$$0 \to \frac{N}{F^1 B^2_{\mathcal{L}}} \to H^2(F^1 T^\bullet_{\mathcal{L}}) \to F^1 H^2(\mathcal{L}, \mathbb{C}) \to 0$$

where by $F^r B^p_{\mathcal{L}}$ we denote the p-coboundaries of the complex $F^r T^\bullet_{\mathcal{L}}$, while N is the set of 2-coboundaries of $T^\bullet_{\mathcal{L}}$ living in $F^1 T^2_{\mathcal{L}}$, i. e.

$$N = \left\{ \delta x \mid x \in T^1_{\mathcal{L}} \text{ and } \delta x \in F^1 T^2 \right\} \ .$$

We need to show that $N = F^1 B^2_{\mathcal{L}}$.

Let $\delta x \in N$. We have $x = x^{1,0} + x^{0,1}$ according to $T^1_{\mathcal{L}} = K^{1,0}_{\mathcal{L}} \oplus K^{0,1}_{\mathcal{L}}$. We already have $\delta x^{1,0} \in F^1 B^2_{\mathcal{L}}$, we need to show that $\delta x^{0,1} \in F^1 B^2_{\mathcal{L}}$. This happens if and only if $\mathrm{d}_{\mathcal{L}} x^{0,1} = \check{\delta} y^{1,0}$ for some $y^{1,0} \in K^{1,0}_{\mathcal{L}}$. But $x \in N$ if and only if $\check{\delta} x^{0,1} = 0$, so we can apply the previous lemma and obtain 1-forms $\omega_\alpha \in \Omega_X(U_\alpha)$ such that $\mathrm{d}x^{0,1} = \check{\delta}\omega_\alpha$. So, $\mathrm{d}_{\mathcal{L}} x^{0,1} = \check{\delta}(\sharp^*\omega_\alpha)$, since $\sharp^*\omega_\alpha \in K^{0,1}_{\mathcal{L}}$. □

This corollary and the following computations will be useful in the next sections: remark that the elements of $H^2(F^1 T^\bullet_{\mathcal{L}})$ are represented by closed elements of $F^1 T^2_{\mathcal{L}} = K^{2,0} \oplus K^{1,1}$, that is, pairs $(Q_\alpha, \phi_{\alpha\beta})$ satisfying the equations

$$\begin{gathered} (\check{\delta}\phi)_{\alpha\beta\gamma} = 0, \\ \mathrm{d}_{\mathcal{L}}\phi_{\alpha\beta} = (\check{\delta}Q)_{\alpha\beta}, \\ \mathrm{d}_{\mathcal{L}} Q_\alpha = 0; \end{gathered} \tag{2.1}$$

while coboundaries of $F^1 T^2_{\mathcal{L}}$ are of the form $(\mathrm{d}_{\mathcal{L}}\eta_\alpha, (\check{\delta}\eta)_{\alpha\beta})$ for $\eta_\alpha \in K^{1,0}$.

Hence we have a natural projection

$$F^1 H^2(\mathcal{L};\mathbb{C}) \to H^1(X, \Omega^1_{\mathcal{L}}) \ , \qquad [(Q,\phi)] \to [\phi] \ ,$$

well defined by the fact that if $(Q,\phi) = \delta\eta$ then $\phi = \check{\delta}\eta$.

One can check that the constructions so far are controvariantly functorial: any holomorphic Lie algebroid morphism $\Psi : \mathcal{L} \to \mathcal{L}'$ yields pull back morphisms $H^p(\mathcal{L}',\mathbb{C}) \to H^p(\mathcal{L},\mathbb{C})$ and $F^p H^k(\mathcal{L}',\mathbb{C}) \to F^p H^k(\mathcal{L},\mathbb{C})$.

Moreover these are compatible with the holomorphic De Rham and Dolbeault theorems: Ψ induces a morphism of the associated canonical complex algebroids $\Psi_h : \mathcal{L}_h \to \mathcal{L}'_h$, that gives a pull-back in cohomology $\Psi^* : H^p(\mathcal{L}'_h;\mathbb{C}) \to H^p(\mathcal{L}_h;\mathbb{C})$ and the following diagrams commute:

$$\begin{tikzcd} H^p(\mathcal{L}';\mathbb{C}) \arrow[r, "\cong"] \arrow[d, "\Psi^*"] & H^p(\mathcal{L}'_h;\mathbb{C}) \arrow[d, "\Psi^*_h"] \\ H^p(\mathcal{L};\mathbb{C}) \arrow[r, "\cong"] & H^p(\mathcal{L}_h;\mathbb{C}) \end{tikzcd} \quad , \quad \begin{tikzcd} H^q(X;\Omega^p_{\mathcal{L}'}(\mathcal{E})) \arrow[r, "\cong"] \arrow[d, "\Psi^*"] & H^{p,q}(\mathcal{L}'_h;\mathcal{E}) \arrow[d, "\Psi^*_h"] \\ H^q(X;\Omega^p_{\mathcal{L}}(\mathcal{E})) \arrow[r, "\cong"] & H^{p,q}(\mathcal{L}_h;\mathcal{E}) \end{tikzcd} .$$

2.4.4 Holomorphic $\mathcal{L}$-connections

In this subsection we introduce holomorphic $\mathcal{L}$-connections on holomorphic vector bundles for $\mathcal{L}$ a holomorphic Lie algebroid. While there always exist smooth L-connections on smooth bundles, we will see that, similarly to what happens with usual connections, the existence of holomorphic $\mathcal{L}$-connections has topological obstructions.

Let $\mathcal{L}$ be a holomorphic Lie algebroid over a smooth complex manifold X, and $\mathcal{E}$ a holomorphic vector bundle on X. Similarly to the smooth case, we have:

Definition 8. *A holomorphic $\mathcal{L}$-connection on $\mathcal{E}$ is a map of sheaves $\nabla : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L}^*$ satisfying the Leibniz rule $\nabla(fe) = f\nabla e + e \otimes d_{\mathcal{L}} f$ for any $f \in \mathscr{O}_X$ and $e \in \mathcal{E}$.*

Define the curvature $F_\nabla \in H^0(X, \mathscr{E}nd(\mathcal{E}) \otimes \Omega^2_{\mathcal{L}})$ of a holomorphic $\mathcal{L}$-connection ∇ in the same way as in the smooth case.

Let $\mathcal{L}$ be a holomorphic Lie algebroid and consider the associated Lie algebroid $\mathcal{L}_h = \mathcal{L}^{1,0} \bowtie T^{0,1}_X$. Let E be a smooth vector bundle and ∇ a $\mathcal{L}_h$-connection on it. Since $\mathcal{L}_h = \mathcal{L}^{1,0} \oplus T^{0,1}_X$ as a vector bundle, ∇ splits in two operators

$$\nabla' : \Gamma(E) \to \Gamma(E) \otimes A^1_{\mathcal{L}^{1,0}} \ , \qquad \nabla'' : \Gamma(E) \to \Gamma(E) \otimes A^{0,1}_X \ ,$$

satisfying the Leibniz rules

$$\nabla'(fs) = f\nabla'(s) + \sharp_{\mathcal{L}^{1,0}}(f) \otimes s \ , \qquad \nabla''(fs) = f\nabla'' s + \bar\partial f \otimes s \ ,$$

for $f \in C^\infty(X)$ and $s \in \Gamma(E)$.

It is well known that giving a holomorphic structure on a smooth vector bundle E is equivalent to giving an operator $\bar\partial_E : \Gamma(E) \to \Gamma(E) \otimes A^{0,1}_X$ satisfying the latter Leibniz rule and such that $\bar\partial_E^2 = 0$. So we see that the ∇'' piece of an $\mathcal{L}_h$-connection defines an operator of this kind. We call a $\mathcal{L}_h$-connection *holomorphing* ∇ if $(\nabla'')^2 = 0$, so that it induces a holomorphic structure on E.

If D is a holomorphic $\mathcal{L}$-connection on the holomorphic vector bundle $\mathcal{E}$, we can define a $\mathcal{L}_h$-connection ∇ on E, the smooth vector bundle underlying $\mathcal{E}$, as follows: set $\nabla'' = \bar\partial_{\mathcal{E}}$, the operator defining the holomorphic structure of $\mathcal{E}$, $\nabla' = D \otimes_{\mathscr{O}_X} \mathcal{A}^0_X$, and $\nabla = \nabla' + \nabla''$. Since ∇' is defined from a holomorphic connection, we see that

$$\nabla'\nabla'' + \nabla''\nabla' = 0 \ . \tag{2.2}$$

Vice versa, if ∇ is a holomorphing $\mathcal{L}_h$-connection satisfying equation (2.2), it is easy to see that there exists a holomorphic $\mathcal{L}$-connection inducing it as described above.

Finally consider, for an $\mathcal{L}_h$-connection ∇ on the smooth vector bundle E, the flatness condition $\nabla^2 = 0$: it implies the following 3 conditions:

- on $\bigwedge^2 (T^{0,1}_X)^*$, we have $(\nabla'')^2 = 0$, that is, ∇ is a holomorphing $\mathcal{L}_h$-connection on E;
- on $(\mathcal{L}^{1,0})^* \otimes (T^{0,1}_X)^*$, we have equation (2.2), that is, ∇' is induced by a holomorphic $\mathcal{L}$-connection $D : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L}^*$;
- on $\bigwedge^2 \mathcal{L}^*$, the condition is $(\nabla')^2 = 0$, that is equivalent to D to be a flat holomorphic $\mathcal{L}$-connection.

Summing up, we have the following:

Proposition 13. *Let $\mathcal{L}$ be a holomorphic Lie algebroid over X and E a smooth vector bundle on X.*

Then

1. *a $\mathcal{L}_h$-connection ∇ on E with $F^{0,2}_\nabla = 0$ induces a holomorphic structure on E;*

2. *a $\mathcal{L}_h$-connection ∇ on E with $F^{1,1}_\nabla + F^{0,2}_\nabla = 0$ is equivalent to a holomorphic structure $\mathcal{E}$ on E and a holomorphic $\mathcal{L}$-connection on $\mathcal{E}$;*

3. *a flat $\mathcal{L}_h$-connection on E is equivalent to a holomorphic structure $\mathcal{E}$ on it and a flat holomorphic $\mathcal{L}$-connection.*

We now study the problem of existence of holomorphic $\mathcal{L}$-connections over a holomorphic vector bundle $\mathcal{E}$.

For $\mathcal{L} = \mathcal{T}_X$ the problem is well known: let $\mathcal{J}^1_X(\mathcal{E})$ be the bundle of holomorphic first order operators on $\mathcal{E}$ with scalar symbol. It admits naturally a Lie algebroid structure, whose anchor is the symbol and whose bracket is the commutator of differential operators. It is usually called the *Atiyah Lie algebroid* of $\mathcal{E}$. There is a natural short exact sequence

$$0 \to \mathscr{E}nd(\mathcal{E}) \to \mathcal{J}^1_X(\mathcal{E}) \to \mathcal{T}_X \to 0 \ . \tag{2.3}$$

A holomorphic ($\mathcal{T}_X$-)connection on $\mathcal{E}$ is equivalent to a splitting of this exact sequence, so there exists a holomorphic connection on $\mathcal{E}$ if and only if the extension class

$$\mathfrak{a}(\mathcal{E}) \in \mathrm{Ext}^1(\mathcal{T}_X, \mathscr{E}nd(\mathcal{E})) = H^1(X, \mathscr{E}nd(\mathcal{E}) \otimes \Omega_X)$$

is zero. It is a theorem of Atiyah (cf. [2]) that the class $\mathfrak{a}(\mathcal{E})$ generates the characteristic ring of $\mathcal{E}$, so one has:

Theorem 14. *Let $\mathcal{E}$ be a holomorphic vector bundle over a smooth projective variety X.*

Then $\mathcal{R}(\mathcal{E}) = 0$ is a necessary condition for the existence of a holomorphic connection on $\mathcal{E}$.

Let us recall how one can prove this: fix a Hermitian metric on $\mathcal{E}$. Let ∇ be the Hermitian connection on $\mathcal{E}$; by definition it is the unique connection compatible with both the metric and the holomorphic structure of $\mathcal{E}$. Its curvature F is of type $(1,1)$. The cohomology class $[F] \in H^{1,1}(X, \mathscr{E}nd(\mathcal{E}))$ does not depend on the choice of the metric, and via the generalized Dolbeault isomorphism

$$H^{1,1}(X, \mathscr{E}nd(\mathcal{E})) \cong H^1(X, \mathscr{E}nd(\mathcal{E}) \otimes \Omega_X) \ ,$$

$[F]$ corresponds to the class $\mathfrak{a}(\mathcal{E})$. The theorem then follows since, by definition, $\mathcal{R}(\mathcal{E})$ is generated by F.

Let now $\mathcal{L}$ be a holomorphic Lie algebroid, $\mathcal{L}_h = \mathcal{L}^{1,0} \bowtie T_X^{0,1}$, and $\mathcal{E}$ a holomorphic vector bundle.

Let

$$\mathfrak{a}_{\mathcal{L}}(\mathcal{E}) = \sharp^*(\mathfrak{a}(\mathcal{E})) \in \operatorname{Ext}^1(\mathcal{L}, \mathscr{E}nd(\mathcal{E})) \cong H^1(X, \mathscr{E}nd(\mathcal{E}) \otimes \Omega_{\mathcal{L}})$$

be the pullback via the anchor $\sharp$ of the Atiyah class of $\mathcal{E}$. We have the corresponding diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E}nd(\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{J}^1_{\mathcal{L}}(\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{E}nd(\mathcal{E}) \arrow[r] & \mathcal{J}^1_X(\mathcal{E}) \arrow[r] & \mathcal{T}_X \arrow[r] & 0.
\end{tikzcd}$$

An $\mathcal{L}$-connection on $\mathcal{E}$ is equivalent to a splitting of the upper row, so it exists if and only if $\mathfrak{a}_{\mathcal{L}}(\mathcal{E}) = 0$.

We have the following:

Proposition 15. *Let $\mathcal{L}$ be a holomorphic Lie algebroid and $\mathcal{E}$ a holomorphic vector bundle over a smooth projective variety X.*

Then the class $\mathfrak{a}_{\mathcal{L}}(\mathcal{E})$ generates the $\mathcal{L}_h$-characteristic ring $\mathcal{R}_{\mathcal{L}_h}(\mathcal{E})$.

Proof. By the generalization of Dolbeault Theorem given in Theorem 9 we have a commutative diagram

$$\begin{tikzcd}
H^1(X, \mathscr{E}nd(\mathcal{E}) \otimes \Omega_X) \arrow[r] \arrow[d, "\cong"] & H^1(X, \mathscr{E}nd(\mathcal{E}) \otimes \Omega_{\mathcal{L}}) \arrow[d, "\cong"] \\
H^{1,1}(X, \mathscr{E}nd(\mathcal{E})) \arrow[r] & H^{1,1}(\mathcal{L}_h, \mathscr{E}nd(\mathcal{E})) .
\end{tikzcd}$$

Consider the images of $\mathfrak{a}(\mathcal{E}), \mathfrak{a}_{\mathcal{L}}(\mathcal{E})$, living in the lower row.

In general there are not inclusions of $H^{k,k}(\mathcal{L}_h, \mathbb{C})$ in $H^{2k}(\mathcal{L}_h, \mathbb{C})$, so Atiyah's argument does not apply straightforwardly. To obtain the thesis, we need, for each invariant polynomial P, the existence of a $\mathrm{d}_{\mathcal{L}_h}$-closed representative of the class $P(\mathfrak{a}_{\mathcal{L}}(\mathcal{E})) \in H^{k,k}(\mathcal{L}_h, \mathbb{C})$. But $P(\mathfrak{a}_{\mathcal{L}}(\mathcal{E}))$ is the pullback of $P(\mathfrak{a}(\mathcal{E}))$, and, since X is a smooth projective variety, we can choose a d-closed representative of $P(\mathfrak{a}(\mathcal{E}))$, whose pullback is a $\mathrm{d}_{\mathcal{L}}$-closed representative of $P(\mathfrak{a}_{\mathcal{L}}(\mathcal{E}))$. □

It is well known that if a coherent $\mathscr{O}_X$-module $\mathcal{E}$ admits a smooth T_X-connection then it is locally free. This is no more true in the Lie algebroid case: if $\mathcal{L}$ is a holomorphic Lie algebroid and $\mathcal{G}$ the associated holomorphic foliation, $\mathcal{E}$ a coherent $\mathscr{O}_X$-module and ∇ a holomorphic $\mathcal{L}$-connection on $\mathcal{E}$, then we can only say that $\mathcal{E}_{|G}$ is locally free for any leaf G of $\mathcal{G}$ (see [11]). We want to generalize Atiyah's construction to the case when $\mathcal{E}$ is not locally free, and we will show the following:

Theorem 16. *Let $\mathcal{L}$ be a holomorphic Lie algebroid over a smooth projective variety X, and $\mathcal{E}$ a torsion free $\mathscr{O}_X$-module.*

Then if there exists a holomorphic $\mathcal{L}$-connection on $\mathcal{E}$ we have that $\mathcal{R}_{\mathcal{L}_h}(\mathcal{E}) = 0$.

We now follow [25] to define the $\mathcal{L}$-Atiyah class for general coherent $\mathscr{O}_X$-modules: consider the following sheaf $\mathcal{J}^1_{\mathcal{L}} = \Omega_{\mathcal{L}} \oplus \mathscr{O}_X$ and equip it with the unital associative $\mathscr{O}_X$-algebra structure given by the product

$$(\alpha, a)(\beta, b) = (a\beta + b\alpha, ab)$$

for $\alpha, \beta \in \Omega_{\mathcal{L}}$ and $a, b \in \mathscr{O}_X$.

$\Omega_{\mathcal{L}}$ is a sheaf of ideals of $\mathcal{J}^1_{\mathcal{L}}$ of square 0, and we have the exact sequence

$$0 \to \Omega_{\mathcal{L}} \to \mathcal{J}^1_{\mathcal{L}} \to \mathscr{O}_X \to 0$$

of $\mathscr{O}_X$-algebras; moreover this sequence splits via the morphism of $\mathscr{O}_X$-algebras $a \to (0, a)$.

For $\mathcal{E}$ a coherent $\mathscr{O}_X$-module, over the sheaf $\mathcal{J}^1_{\mathcal{L}}(\mathcal{E}) = \Omega_{\mathcal{L}} \otimes \mathcal{E} \oplus \mathcal{E}$ define the following left and right $\mathcal{J}^1_{\mathcal{L}}$-module structures:

$$(\alpha, a)(\beta \otimes e, f) = (\alpha \otimes f + a\beta \otimes e + \mathrm{d}_{\mathcal{L}} a \otimes f, af)$$

$$(\beta \otimes e, f)(\alpha, a) = (a\beta \otimes e, af)$$

Via the previous splitting, the $\mathcal{J}^1_{\mathcal{L}}$-module structures induce a left and a right $\mathscr{O}_X$-module structure on $\mathcal{J}^1_{\mathcal{L}}(\mathcal{E})$: the right $\mathscr{O}_X$-module structure coincides with the $\mathscr{O}_X$-module structure of the direct sum, while the left $\mathscr{O}_X$-module structure is given by:

$$a(\beta \otimes e, f) = (a\beta \otimes e + \mathrm{d}_{\mathcal{L}} a \otimes f, af)\ . \tag{2.4}$$

We have the exact sequence

$$0 \to \Omega_{\mathcal{L}} \otimes \mathcal{E} \to \mathcal{J}^1_{\mathcal{L}}(\mathcal{E}) \to \mathcal{E} \to 0 \tag{2.5}$$

of both left and right $\mathscr{O}_X$ and $\mathcal{J}^1_{\mathcal{L}}$-modules. This sequence is always split as right $\mathscr{O}_X$ and $\mathcal{J}^1_{\mathcal{L}}$-modules, while we have

Proposition 17. *The above sequence splits as left $\mathscr{O}_X$-modules if and only if there exists a holomorphic $\mathcal{L}$-connection on $\mathcal{E}$.*

Proof. Write a splitting $\zeta : \mathcal{E} \to \mathcal{J}^1_{\mathcal{L}}(\mathcal{E})$ as

$$\zeta(e) = (\nabla(e), e)$$

with, a priori, $\nabla(e) : \mathcal{E} \to \mathcal{E} \otimes \Omega_{\mathcal{L}}$ a sheaf map. Then it is easy to see that by equation (2.4) $\zeta(ae) = a\zeta(e)$ if and only if ∇ is a holomorphic $\mathcal{L}$-connection. □

We define $\mathfrak{a}_{\mathcal{L}}(\mathcal{E})$, the $\mathcal{L}$-Atiyah class of the coherent $\mathscr{O}_X$-module $\mathcal{E}$, as the class of the sequence (2.5) seen as an extension of left $\mathscr{O}_X$-modules in $\mathrm{Ext}^1_{\mathscr{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega_{\mathcal{L}})$. Clearly we have that $\mathfrak{a}_{\mathcal{L}}(\mathcal{E}) = 0$ if and only if there exist holomorphic $\mathcal{L}$-connections on $\mathcal{E}$. Remark that when $\mathcal{E}$ is locally free this definition coincide with the previous one, since $\mathrm{Ext}^1_{\mathscr{O}_X}(\mathcal{E}, \Omega_{\mathcal{L}} \otimes \mathcal{E}) \cong \mathrm{Ext}^1_{\mathscr{O}_X}(\mathcal{T}_X, \mathscr{E}nd\, \mathcal{E})$.

The next step in generalizing Atiyah's construction is the following:

Proposition 18. *Let $\mathcal{L}$ a holomorphic Lie algebroid over X and $\mathcal{E}$ a holomorphic Lie algebroid.*

Then the $\mathcal{L}$-Atiyah class of $\mathcal{E}$ generates $\mathcal{R}_{\mathcal{L}}(\mathcal{E})$.

Proof. One can check that when $\mathcal{L} = \mathcal{T}_X$ this definition coincides with the usual definition of the Atiyah class of a coherent sheaf (see for example [18], chapter 10). In particular, the proposition is well known in the case $\mathcal{L} = \mathcal{T}_X$. We can then apply the same arguments of Proposition 15 and obtain the assertion. □

To conclude the proof of Theorem 16 it remains to show that one can compute the $\mathcal{L}_h$-characteristic ring of $\mathcal{E}$ (that, recall, is just the pull-back of the usual characteristic ring) by means of a holomorphic $\mathcal{L}$-connection on it. This is shown in the following:

Lemma 19. *Let $\mathcal{L}$ be a holomorphic Lie algebroid, $\mathcal{L}_h = \mathcal{L}^{1,0} \bowtie T^{0,1}_X$, $\mathcal{E}$ a torsion free $\mathscr{O}_X$-module on X and ∇ a holomorphic $\mathcal{L}$-connection on it.*

Then F_∇, the curvature of ∇, generates $\mathcal{R}_{\mathcal{L}_h}(\mathcal{E})$.

Proof. We show how one can obtain the first $\mathcal{L}_h$-Chern class of $\mathcal{E}$ from F_∇, for the other $\mathcal{L}_h$-characteristic classes it works similarly.

Let $U \subseteq X$ be the open subset where $\mathcal{E}$ is locally free. Since $\mathcal{E}$ is torsion free, its complement has codimension at least 2. Consider $\det(\mathcal{E})$, the determinant line bundle of $\mathcal{E}$: on the open U where it is locally free we have

$$\det(\mathcal{E})_{|U} \cong \bigwedge^r \mathcal{E}_{|U} \ ,$$

where r is the rank of $\mathcal{E}$. Since $c_{1,\mathcal{L}_h}(\mathcal{E})$ is the pull back of $c_1(\mathcal{E})$, we have $c_{1,\mathcal{L}_h}(\mathcal{E}) = c_{1,\mathcal{L}_h}(\det(\mathcal{E}))$.

Over U, ∇ induces a holomorphic $\mathcal{L}$-connection $\tilde{\nabla}$ on $\det \mathcal{E}$, defined as the rth exterior power of ∇. Now, since the complement of U has codimension at least 2, we can extend $\tilde{\nabla}$ to the whole X by Hartogs lemma: for any $s \in \det(\mathcal{E})_{|U}$, $\sigma \in (\det(\mathcal{E})_{|U})^*$ and $u \in \mathcal{L}$, the function

$$f_{s,\sigma,u} = \langle \sigma, \tilde{\nabla}_u(s) \rangle$$

is holomorphic on U, so it extends uniquely to X. So we can define $\tilde{\nabla}_u(s) = f_{s,u} \in \det(\mathcal{E})$, since $\sigma \mapsto f_{s,\sigma,u}$ is a $\mathscr{O}_X$-linear map.

Let $\tilde{F}$ be the curvature of $\tilde{\nabla}$. Since over U we have $\tilde{F}_{|U} = \mathrm{Trace}(F_\nabla)_{|U}$, by the same extension argument this equality holds on the whole X.

Since $\det(\mathcal{E})$ is locally free, $c_{1,\mathcal{L}_h}(\det(\mathcal{E})) = [\tilde{F}]$ holds, and since $\tilde{F} = \mathrm{Trace}(F_\nabla)$ we finally have the equality $c_{1,\mathcal{L}_h}(\mathcal{E}) = [\mathrm{Trace}(F_\nabla)]$.

□

2.4.5 Example: logarithmic connections

Let X be a smooth projective variety and D an effective normal crossing divisor on it. Consider the sheaf $\Omega_X(\log D)$ of meromorphic 1-forms with logarithmic poles along D. This is the locally free $\mathscr{O}_X$-module locally generated by $\frac{\mathrm{d}x_1}{x_1}, \dots, \frac{\mathrm{d}x_t}{x_t}, \mathrm{d}x_{t+1}, \dots, \mathrm{d}x_n$, where $x_1, \dots, x_n$ are local coordinates on X such that D has equation $x_1 \cdots x_t = 0$ in this coordinates. Remark that we have a natural inclusion $\Omega_X \hookrightarrow \Omega_X(\log D)$, and the quotient is isomorphic to the structure sheaf of $\tilde{D}$, the normalization of D. The exterior differential extends naturally to $\mathrm{d} : \Omega_X^p(\log D) \to \Omega_X^{p+1}(\log D)$, where $\Omega_X^p(\log D) = \bigwedge^p \Omega_X(\log D)$.

Consider $\mathcal{T}_X(\log D)$, the dual of $\Omega_X(\log D)$. Dualizing the sequence

$$0 \to \Omega_X \to \Omega_X(\log D) \to \mathscr{O}_{\tilde{D}} \to 0$$

we obtain an inclusion $\mathcal{T}_X(\log D) \hookrightarrow \mathcal{T}_X$. This subsheaf is locally free and closed under the commutator of vector fields, so it inherits a holomorphic Lie algebroid structure.

The holomorphic Lie algebroid cohomology $H^p(\mathcal{T}_X(\log D), \mathbb{C})$ is equal, just by definition, to the hypercohomology of the logarithmic deRham complex $\mathbb{H}^p(X, \Omega_X^\bullet(\log D))$, that is well known to be isomorphic to the deRham cohomology $H^p_{DR}(U, \mathbb{C})$ of the open $U = X \setminus D$.

Starting from important works of Deligne, holomorphic integrable connections with logarithmic poles along a divisor D have been extensively studied, see for example [9], [10]. From our point of view, an integrable connection with logarithmic poles along D on a coherent sheaf $\mathcal{E}$ is just a flat $\mathcal{T}_X(\log D)$-connection on $\mathcal{E}$.

In the Appendix B of [10], it is shown how one can compute the Chern classes of a holomorphic vector bundle in terms of a connection with logarithmic poles anong D. In particular, it is shown:

Theorem 20. *Let X be a smooth projective variety and D an effective normal crossing divisor on X, $\mathcal{E}$ a coherent $\mathscr{O}_X$-module and ∇ a $T_X(\log D)$-connection on $\mathcal{E}$. Let $D = \sum a_i D_i$ with D_i irreducible, and $[D_i]$ the class of D_i in $H^2(X, \mathbb{C})$.*

Then the p-th Chern class of $\mathcal{E}$ is a complex linear combination of $[D_{i_1}]^{k_1} \cdots [D_{i_t}]^{k_t}$ with $\sum k_\alpha = p$.

Now we can easily see that this theorem implies Theorem 16 when $\mathcal{L} = \mathcal{T}_X(\log D)$: since

$$H^p((\mathcal{T}_X(\log D))_h, \mathbb{C}) \cong H^p_{hol}(\mathcal{T}_X(\log D)) \cong H^p_{DR}(U, \mathbb{C}) \ ,$$

the pullbacks of $[D_i]$ to the Lie algebroid cohomology vanish. So if $\mathcal{E}$ admits a holomorphic connection with logarithmic poles along D, the pullback of its Chern classes to $H^p(\mathcal{T}_X(\log D); \mathbb{C})$ vanish, i. e. its its $\mathcal{T}_X(\log D)$-Chern classes vanish.

Chapter 3

(Sheaves of)Filtered algebras and their classification

In this chapter we present subsequent generalizations of Sridharan's construction of the twisted enveloping algebra associated to an almost polynomial filtered algebra.

In [37], the author classifies the almost polynomial k-algebras with the 0th piece isomorphic to k by means of a Lie algebra structure on the first graded piece and a class in the second cohomology of this Lie algebra with trivial coefficients. First we recall this construction, then generalize it to the case when the 0th filtration is any k-algebra and finally use this result to classify sheaves of almost polynomial filtered algebras on a smooth projective variety over $\mathbb{C}$ by means of holomorphic Lie algebroids and their cohomology classes.

3.1 Classification of almost polynomial filtered algebras

Let k be a commutative ring. A *filtered k-algebra* is a unital associative k-algebra A equipped with an increasing filtration of k-submodules $A_{(i)} \subseteq A_{(i+1)} \subseteq \ldots \subseteq A$ for $i \geq 0$, such that $A_{(i)}A_{(j)} \subseteq A_{(i+j)}$.

Define the associated graded modules $\mathrm{Gr}_i A = A_{(i)}/A_{(i-1)}$, and let $[a]_i$ denote the class in $\mathrm{Gr}_i A$ of $a \in A_{(i)}$. Define on $\mathrm{Gr}_\bullet A = \bigoplus_i \mathrm{Gr}_i A$ a product by $[a]_i[b]_j = [ab]_{i+j}$, for any $a \in A_{(i)}$, $b \in A_{(j)}$. This gives $\mathrm{Gr}_\bullet A$ the structure of a k-algebra, that we call *associated graded algebra.*

A *graded k-algebra* is a k-algebra A with k-submodules A_i such that $A = \bigoplus_i A_i$ and the product satisfies $A_iA_j \subseteq A_{i+j}$. Any graded algebra is naturally a filtered algebra defining the filtration $A_{(i)} = \bigoplus_{j\leq i} A_j$. Remark that if A is a graded algebra, the graded algebra associated to its filtration is naturally isomorphic to A itself.

Remark that the filtered condition on the product implies that $A_{(0)}$ is a k-subalgebra of A, and each of the $A_{(i)}$ carries an $A_{(0)}$-bimodule structure. Moreover since the algebra is unital and the product is k-bilinear, k is a subalgebra of $A_{(0)}$ and is contained in the center of A.

We will say that the filtered algebra A is *almost polynomial* when the associated graded algebra is isomorphic to the symmetric algebra over $A_{(0)}$ of the first graded piece, i. e. $\mathrm{Gr}_\bullet A \cong \mathrm{Sym}_{A_{(0)}} \mathrm{Gr}_1 A$.

In particular, the graded algebra of an almost polynomial filtered algebra is commutative. This implies that for any $a \in A_{(i)}$ and $b \in A_{(j)}$ their commutator $ab-ba$ belongs to $A_{(i+j-1)}$, because $[ab-ba]_{i+j} = [a]_i[b_j]-[b]_j[a]_i$ and the latter is 0 because of the commutativity condition.

Proposition 21. *Let A be a filtered k-algebra such that the associated graded algebra is commutative.*

Then $R = A_{(0)}$ is a commutative k-algebra, and $A_{(1)}$ and $Gr_1 A$ carry a natural structure of (k, R)-Lie-Rinehart algebra.

Proof. Since $\mathrm{Gr}_0 A = A_{(0)}$, clearly R is commutative.

Because of the previous observation, the commutator of two elements of $A_{(1)}$ belongs to $A_{(1)}$. One can check that the bracket $[a, b] = ab - ba$ for $a, b \in A_{(1)}$ defines a k-Lie algebra structure on $A_{(1)}$.

For $x, y \in \mathrm{Gr}_1 A$, for any two representative $a, b \in A_{(1)}$ define

$$[x, y] = [\, [a, b]\,]_1 \ ;$$

one can check that since $R = A_{(0)}$ is commutative this definition does not depend on the representatives, and that it gives a k-Lie bracket on $\mathrm{Gr}_1 A$.

The anchor $\sharp : A_{(1)} \to \mathrm{Der}_k(R)$ is given by the commutator $\sharp(a)(f) = [a, f] = af - fa$ for $f \in R$ and $a \in A_{(1)}$. The Jacobi identity assurres that $\sharp$ is a morphism of Lie algebras. Finally,

$$[a, fb] = a(fb) - fba = f(ab - ba) + (af - fa)b = f[a, b] + \sharp(a)(f)b$$

so the bracket on $A_{(1)}$ forms with $\sharp$ a (k, R)-Lie-Rinehart algebra structure on $A_{(1)}$.

Since R is commutative, we have that $\sharp(a) = \sharp(a + f)$ for any $a \in A_{(1)}$ and $f \in R$, so $\sharp$ factors through the quotient, and induces a map $\sharp : \mathrm{Gr}_1 A \to \mathrm{Der}_k(R)$. The same calculation of before shows that this is an anchor for the Lie bracket on $\mathrm{Gr}_1 A$, and so that we have a (k, R)-Lie-Rinehart structure on $\mathrm{Gr}_1 A$. □

3.1.1 The subalgebra $A_{(0)}$ isomorphic to k

Now we proceed to the classification of almost polynomial filtered k-algebras with $A_{(0)} = k$. Remark that in this case the structures of the previous proposition are reduced to a k-Lie algebra structure.

Let A be an almost polynomial filtered k-algebra with $A_{(0)} = k$. We can associate to it the exact sequence of k-Lie algebras:

$$0 \to k \to A_{(1)} \to \mathrm{Gr}_1 A \to 0\ ;$$

where $A_{(0)}$ is equiped with the abelian structure. This is an extension of $\mathrm{Gr}_1 A$ by k, as studied in section 2.1, and k has the trivial $\mathrm{Gr}_1 A$-module structure since in A we have $a\mu - \mu a = 0$ for any $a \in A_{(1)}$ and $\mu \in k$.

The main idea of Sridharan is that from an extension of $\mathrm{Gr}_1 A$ by k equipped with the trivial $\mathrm{Gr}_1 A$-module structure it is possible to construct an almost polynomial filtered algebra, and show that this operation is an inverse to the previous one. To have a precise correspondence, fix a polynomial k-algebra S and consider the category of pairs (A, Ξ), where A is an almost polynomial filtered k-algebra with $A_{(0)} = k$ and $\Xi : \mathrm{Gr}_\bullet A \to S$ is an isomorphism of graded algebras. Morphisms between two pairs (A, Ξ) and (A', Ξ') are morphisms of filtered algebras $A \to A'$ such that the induce graded morphism $\mathrm{Gr}_\bullet A \to \mathrm{Gr}_\bullet A'$ commutes with the isomorphisms Ξ, Ξ'.

We have (cf. [37]):

Theorem 22. *Let k be a commutative ring and V a free k-module.*
Then there is a one to one correspondence between

1. *isomorphism classes of pairs (A, Ξ), where A is a filtered k-algebra with $A_{(0)} = k$ and $\Xi : Gr_\bullet A \to Sym_k^\bullet V$ is an isomorphism of graded algebras;*

2. *pairs (L, Σ), where L is a k-Lie algebra structure on V and $\Sigma \in H^2_{CE}(L; k)$ (k endowed with the trivial L-module structure).*

Proof. Let (A, Ξ) be as in (1). Let L be the k-Lie algebra structure on $\mathrm{Gr}_1 A$ given by Proposition 21. Consider the extension of k-Lie algebras

$$0 \to k \to A_{(1)} \to \mathrm{Gr}_1 \Lambda \to 0\ .$$

The isomorphism $\Xi_1 : \mathrm{Gr}_1 A \to V$ equip V with the k-Lie algebra structure of $\mathrm{Gr}_1 A$, while the extension gives a class $\Sigma \in H^2_{CE}(L; k)$ by Theorem 2. So we obtain a pair (L, Σ) as in (2).

The converse follows from the following construction: let (L, Σ) be as in (2), and F be a representative of Σ. In the full tensor algebra $T_k^\bullet L$ consider the ideal I_F generated by elements of the form

$$x \otimes y - y \otimes x - [x, y] - F(x, y)$$

and define $U_F(L)$ as the quotient $T_k^\bullet L / I_F$. This is a twist on the universal enveloping algebra of L, and we will call it *F-Sridharan enveloping algebra* of L. It inherits a filtration from the grading of $T_k^\bullet L$. It is easy to see that

$\mathrm{Gr}_\bullet U_F(L)$ is commutative, and that the quotient map $T_k^\bullet L \to U_F(L)$ induces a homomorphism $T_k^\bullet L \to \mathrm{Gr}_\bullet U_F(L)$. So there is a natural homomorphism $\Xi_F : \mathrm{Sym}_k^\bullet L \to \mathrm{Gr}_\bullet U_F(L)$, that in the next proposition we show to be an isomorphism.

For another representative F' of Σ, with $F' - F = \delta\eta$, define the following endomorphism of $T_k^\bullet L$: for $f \in k$, $x \in L$ let

$$\Upsilon_\eta : f \mapsto f \qquad \Upsilon_\eta : x \mapsto x + \eta(x)$$

and extend it to the full tensor algebra requiring it to be an algebra homomorphism. This is clearly a surjective endomorphism, since $x = \Upsilon_\eta(x - \eta(x))$ for any $x \in L$, and L generates $T_k^\bullet L$. So Υ_η determines a surjective morphism of algebras $T_k^\bullet L \to U_F(L)$. The following calculation shows that the ideal $I_{F'}$ is in the kernel of Υ_η, and so this induces an isomorphism between $U_{F'}(L)$ and $U_F(L)$, concluding the proof:

$$\begin{aligned} & \Upsilon_\eta(x \otimes y - y \otimes x - [x,y] - F'(x,y)) \ = \\ = \ & [(x + \eta(x)) \otimes (y + \eta(y)) - (y + \eta(y)) \otimes (x + \eta(x)) + \\ & -[x,y] - \eta([x,y]) - F(x,y)] = \\ = \ & [x \otimes y - y \otimes x - [x,y] - \eta([x,y]) - F(x,y)] = 0 \ , \end{aligned}$$

where the last equality follows from the definition of the coboundary:

$$(\delta\eta)(x,y) = -\eta(x,y) \ .$$

□

In the proof of the theorem, we used the following fact:

Proposition 23. *The morphism of graded algebras $\Xi_F : Sym_k^\bullet L \to Gr_\bullet U_F(L)$, constructed in the second part of the previous proof, is an isomorphism.*

Proof. Let L_F be the k-Lie algebra defined as the central extension associated to the 2-cocycle F:

$$0 \to k \to L_F \to L \to 0 \ ;$$

let e_0 denote the generator of k in L_F, and consider the universal enveloping algebra $U_0(L_F)$.

There is a natural algebras morphism $U_0(L_F) \to U_F(L)$ given by

$$(f_1 e_0 + x_1) \otimes \ldots \otimes (f_l e_0 + x_l) \mapsto \sum_{I \subseteq \{1,\ldots l\}} \prod_{i \in I} f_i \bigotimes_{j \in \{1,\ldots,l\} \setminus I} x_j \ ,$$

wich is clearly surjective, and whose kernel is the ideal generated by $e_0 - 1$. So we have the algebra isomorphism $U_0(L_F)/(e_0 - 1) \cong U_F(L)$.

Now, since L is k-free, let $\{e_i\}_{i \in I}$ be a basis of L; assume that I is ordered, that $0 \notin I$ and extend the order of I to $I \cup \{0\}$ by $0 < i$ for any $i \in I$. So

$\{e_i\}_{i\in I\cup\{0\}}$ is a basis of L_F. Now we can apply the Poincaré - Witt theorem to the algebra $U_0(L_F)$, and obtain that the images of $\{e_{\alpha_1}\otimes\ldots\otimes e_{\alpha_l}\}$ in $\mathrm{Gr}_\bullet U_0(L_F)$ form a basis of the latter, where $\alpha_1\leq\alpha_2\leq\ldots\leq\alpha_l\in I\cup\{0\}$ run over all non-decreasing sequence of indices of finite length. Since this is a basis of the symmetric algebra $\mathrm{Sym}_k^\bullet L_F$, this gives an isomorphism $\mathrm{Gr}_\bullet U_0(L_F)\cong\mathrm{Sym}_k^\bullet L_F$.

Now the proof is complete, since we have the diagram

$$\begin{tikzcd}
\mathrm{Sym}_k^\bullet L_F/(e_0-1) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{Gr}_\bullet U_0(L_F)/(e_0-1) \arrow[d, "\cong"] \\
\mathrm{Sym}_k^\bullet L \arrow[r, "\Xi_F"] & \mathrm{Gr}_\bullet U_F(L)\ .
\end{tikzcd}$$

□

3.1.2 The case of arbitrary $A_{(0)}$

We now let $A_{(0)}$ to be any commutative k-algebra, and generalize Sridharan's constructions.

As before, we assocciate to each almost polynomial filtered k-algebra A the exact sequence

$$0\to R\to A_{(1)}\to\mathrm{Gr}_1 A\to 0\ .$$

This is an exact sequence of (k,R)-Lie-Rinehart algebras, with R abelian and endowed with the $\mathrm{Gr}_1 A$-module structure given by the anchor.

Then we will show that from such an extension it is possible to construct an almost polynomial filtered k-algebra, and that this construction is the inverse to the one above.

Theorem 24. *Let k be a commutative ring, R be an associative k-algebra and V a free R-module.*

Then there is a one-to-one correspondence between:

1. *isomorphism classes of pairs (A,Ξ), where A is a filtered k-algebra with $A_{(0)}=R$ and $\Xi: Gr_\bullet A\to Sym_R^\bullet V$ is an isomorphism of graded algebras;*

2. *pairs (L,Σ), where L is a (k,R)-Lie-Rinehart algebra structure on V and $\Sigma\in H^2_{LR}(L;R,\sharp)$, where $\sharp$ is the L-module structure on R associated to the anchor of L.*

From a pair as in (1) one obtains a pair as in (2) simply repeating the argument of Theorem 22 and using Theorem 3 to associate a class in $H^2_{LR}(L;R,\sharp)$ to the extension of (k,R)-Lie-Rinehart algebras.

Now let (L,Σ) be a pair as in (2), and let $F:\bigwedge_R^2 L\to R$ be a representative of Σ.

The construction of the algebra $U_F(L)$ has to be modified to take account of the (k, R)-Lie-Rinehart algebra structure of L: set

$$\tilde{T}^0 L = R \ , \qquad \tilde{T}^1 L = L \otimes_k R \ ,$$

and equip $\tilde{T}^1 L$ with the following R-bimodule structure:

$$f(x \otimes g)h = (fx) \otimes (gh) \ ,$$

for $x \in L$ and $f, g, h \in R$. Define $\tilde{T}^i L$ to be the ith R-tensor power of $\tilde{T}^1 L$, where on $\tilde{T}^1 L \otimes_R \tilde{T}^1 L$ we equip the first factor with the right R-module structure, and the second factor with the left R-module structure. On $\tilde{T}^\bullet L = \bigoplus_{i \geq 0} \tilde{T}^i L$ the tensor $\otimes_R$ defines a k-bilinear graded product.

Let I_F be the two-sided ideal of $\tilde{T}^\bullet L$ generated by elements of the form

$$fx \otimes_k 1 - x \otimes_k f - \sharp(x)(f) \ ,$$

$$(x \otimes_k 1) \otimes_R (y \otimes_k 1) - (y \otimes_k 1) \otimes_R (x \otimes_k 1) - [x, y] \otimes_k 1 - F(x, y) \ ,$$

for $x, y \in L$ and $f \in R$.

Define the algebra $\tilde{U}_F(L)$ to be the quotient of $\tilde{T}^\bullet L$ by I_F. Now we see some properties of the algebra $\tilde{U}_F(L)$:

Lemma 25. *1. $\tilde{U}_F(L)$ is a filtered algebra with $(\tilde{U}_F(L))_{(0)} = R$;*

2. the first graded piece $Gr_1 \tilde{U}_F(L)$ is isomorphic, as a left R-module, to L;

3. the left and right R-module structures on the graded objects $Gr_i \tilde{U}_F(L)$ coincide;

4. the graded algebra $Gr_\bullet \tilde{U}_F(L)$ is commutative;

5. there is a natural morphism of graded algebras $\Xi_F : Sym^\bullet_R L \to Gr_\bullet \tilde{U}_F(L)$;

6. for $F = 0$, the algebra $\tilde{U}_0(L)$ is isomorphic to the universal enveloping algebra $V(R, L)$ of the (k, R)-Lie-Rinehart algebra L, as defined in [32].

Proof. (1) is clear from the construction of $\tilde{U}_F(L)$, since $\tilde{T}^\bullet_k L$ has a natural grading and the ideal I_F does not contain any element of R.

To see (2), let $\eta \in (\tilde{U}_F(L))_{(1)}$. Then η is the class of an element of the form $x \otimes_k f + g$ with $x \in L$ and $f, g \in R$. We may assume that $f = 1$, since $fx \otimes 1 - x \otimes f - \sharp(x)(f)$ is in the ideal I_F. But $g \in (\tilde{U}_F(L))_{(0)}$, so the map $x \to [x \otimes_k 1]_1$ is an isomorphism between L and $\mathrm{Gr}_1 \tilde{U}_F(L)$.

(3) and (4) follow from the fact that the generators of I_F are of the form "commutator of two elements" + "lower degree terms", so for $\xi \in (\tilde{U}_F(L))_i$ and $\eta \in (\tilde{U}_F(L))_j$ we have that $f\xi - \xi f \in (\tilde{U}_F(L))_{i+j-1}$.

To prove (5), since by (4) the associated graded algebra is commutative, it suffices to construct a morphism of graded algebras $T_R^\bullet L \to \mathrm{Gr}_\bullet \tilde{U}_F(L)$. This is given by

$$x_1 \otimes_R \ldots \otimes_R x_t \to [(x_1 \otimes_k 1) \otimes_r \ldots \otimes_R (x_t \otimes_k 1)]_t$$

for $x_i \in L$, where on the righ hand side, the bracket $[\cdot]_t$ denotes the class modulo $(\tilde{U}_F(L))_{t-1}$ of the class modulo the ideal I_F of the element of $\tilde{T}^t L$. To show that this map is well defined one has to check that

$$f[(x_1 \otimes_k 1) \otimes_R \ldots \otimes_R (x_t \otimes_k 1)]_t = [(x_1 \otimes_k 1) \otimes_R \ldots \otimes_R (x_i \otimes f) \otimes_R \ldots \otimes_R (x_t \otimes_k 1)]_n$$

for any $f \in R$. This is true by the same argument as before, since $fx \otimes_k 1 - x \otimes_k f - \sharp(x)(f)$ is in the ideal I_F, and the last summand has degree lower than the others.

For (6), recall how to define $V(R, L)$: we can equip the R-module $R \oplus L$ with the k-Lie bracket given by

$$[(f, x), (g, y)] = (\sharp(x)(g) - \sharp(y)(f), [x, y])$$

and consider the universal enveloping algebra $U_0^k(R \oplus L)$ of $R \oplus L$ over k. Let $U^\dagger$ be the subalgebra generated by the canonical image of $R \oplus L$ in $U_0^k(R \oplus L)$, and P the ideal in $U^\dagger$ generated by elements of the form $(f, 0) \otimes_k (g, x) - (fg, fx)$ for $f, g \in R$ and $x \in L$. Set

$$V(R, L) = \frac{U^\dagger}{P} \ .$$

To show that $\tilde{U}_0(L)$ is isomorphic to $V(R, L)$, consider the algebra morphism from $\tilde{T}^\bullet L$ to $V(R, L)$ defined by

$$\begin{array}{rccl} \Psi : & f & \mapsto & [(f, 0)] \ , \\ \Psi : & x \otimes_k 1 & \mapsto & [(0, x)] \ , \end{array}$$

for $f \in R = \tilde{T}^0 L$ and $x \otimes_k 1 \in \tilde{T}^1 L$, where the bracket $[\cdot]$ on the right hand side denotes the class in $V(R, L)$ of an element of $T_k^\bullet(R \oplus L)$, and extended to $\tilde{T}^\bullet L$ by $\Psi(\xi \otimes_R \eta) = \Psi(\xi) \cdot \Psi(\eta)$.

The map Ψ is surjective: elements of the form $[(f, x)]$ are in its image, since

$$[(f, x)] = [(f, 0) + (x, 0)] = \Psi(f) + \Psi(x \otimes_k 1) \ ,$$

and if $A, B \in T_k^\bullet(R \oplus L)$ are such that $[A], [B]$ are in the image of Ψ, then the class of their tensor product $[A \otimes_k B]$ is in the image of Ψ too, since

$$[A \otimes_k B] = [A][B] = \Psi(\xi)\Psi(\eta) = \Psi(\xi \otimes_R \eta) \ ,$$

so Ψ is onto, since $V(R, L)$ is generated by elements of $R \oplus L$.

It remains to show that the ideal I_0 is in the kernel of Ψ:

$$\begin{array}{ll} \Psi(fx \otimes_k 1 - x \otimes_k f - \sharp(x)(f)) & = \\ = \Psi(fx \otimes 1) - \Psi((x \otimes_k 1) \otimes_R f) - \Psi(\sharp(x)(f)) & = \\ = [(0, fx)] - [(0, x) \otimes_k (f, 0)] - [(\sharp(x)(f), 0)] & = \\ = [(0, fx)] - [(f, 0) \otimes_k (0, x) - [(f, 0), (0, x)]] - [(\sharp(x)(f), 0)] & = \\ = [(0, fx)] - [(0, fx)] + [(\sharp(x)(f), 0)] - [(\sharp(x)(f), 0)] & = 0 \ , \end{array}$$

and

$$\begin{array}{ll} \Psi((x \otimes_k 1) \otimes_R (y \otimes_k 1) - (y \otimes_k 1) \otimes_R (x \otimes_k 1) - [x, y] \otimes_k 1) & = \\ = [(0, x) \otimes_k (0, y) - (0, y) \otimes_k (0, x) - (0, [x, y])] & = 0 \end{array}$$

□

Proposition 26. *The morphism Ξ_F constructed above is an isomorphism.*

Proof. Let L_F be the (k, R)-Lie-Rinehart algebras extension defined by the class F

$$0 \to R \to L_F \to L \to 0$$

and consider its Rinehart enveloping algebra $V(R, L_F)$. Denoting by e_0 the generator of R in L_F, we have $V(R, L_F)/(e_0 - 1) \cong \tilde{U}_F(L)$.

Now, a Poincaré - Witt theorem for $V(R, L_F)$ is proven in [32], so we have $\mathrm{Sym}_R^\bullet L_F \cong \mathrm{Gr}_\bullet V(R, L_F)$. The thesis then follows by the same argument of Proposition 23. □

To conclude the proof of Theorem 24, we need to see what happens when one changes representative of Σ.

Lemma 27. *Let F, F' be two representatives of $\Sigma \in H^2_{LR}(L; R)$, with $F' - F = \delta\eta$ for $\eta \in Hom_R(L, R)$.*

Then there exists an isomorphism of filtered algebras $\Upsilon_\eta : \tilde{U}_{F'} L \to \tilde{U}_F L$ such that the induced morphism of graded algebras commutes with the isomorphisms $\Xi_{F'}$ and Ξ_F.

Proof. We construct this isomorphism as we did in Theorem 22: consider the endomorphism of $\tilde{T}^\bullet L$ given by

$$f \mapsto f \quad \text{for } f \in R = \tilde{T}^0 L$$

$$x \otimes 1 \mapsto x \otimes 1 + \eta(x) \quad \text{for } x \otimes 1 \in L \otimes_k R = \tilde{T}^1 L$$

and extended naturally to a morphism of algebras. Clearly Υ_η is surjective, so it induces a surjective morphism of algebras $\tilde{T}^\bullet L \to \tilde{U}_F L$. To conclude the proof, one just needs to check that $I_{F'}$ is in the kernel of Υ_α, which is a straightforward calculation. □

3.2 Classification of sheaves of almost polynomial filtered algebras

In this section, by sheafifying the arguments of the previous section, we classify the sheaves of rings Λ satisfying the axioms of Simpson's paper [34].

Let X be a smooth algebraic variety over $\mathbb{C}$. We denote by $\mathcal{T}_X$ its tangent sheaf. By a *sheaf of filtered algebras on X* we shall mean what Simpson calls "almost polynomial sheaf of rings of differential operators", that is, a sheaf of $\mathbb{C}_X$-algebras Λ over X ($\mathbb{C}_X$ being the constant sheaf on X with group $\mathbb{C}$) with a filtration of subsheaves of abelian subgroups $\Lambda_{(i)} \subseteq \Lambda_{(i+1)} \subseteq \ldots \subseteq \Lambda$ for $i \in \mathbb{Z}_{\geq 0}$, satisfying the following axioms:

1. $\Lambda_0 = \mathscr{O}_X$, $\Lambda_{(i)}\Lambda_{(j)} \subseteq \Lambda_{(i+j)}$ and $\Lambda = \bigcup \Lambda_{(i)}$;
2. the left and right $\mathscr{O}_X$-module structures on $\mathrm{Gr}_i\Lambda = \Lambda_{(i)}/\Lambda_{(i-1)}$ coincide;
3. the graded $\mathscr{O}_X$-modules $\mathrm{Gr}_i\Lambda$ are coherent and locally free;
4. the graded algebra $\mathrm{Gr}_\bullet\Lambda$ is isomorphic to the symmetric algebra over the first graded summand $\mathrm{Gr}_1\Lambda$.

By a *splitting* of Λ we mean a left $\mathscr{O}_X$-module morphism $\zeta : \mathrm{Gr}_1\Lambda \to \Lambda_{(1)}$ that splits the exact sequence

$$0 \to \mathscr{O}_X \to \Lambda_{(1)} \to \mathrm{Gr}_1\Lambda \to 0 \ . \tag{3.1}$$

For each open subset $U \subseteq X$, we have that $\Lambda(U)$ is a filtered $\mathbb{C}$-algebra with $\Lambda(U)_{(0)} = \mathscr{O}_X(U)$, and $\mathrm{Gr}_\bullet\Lambda(U) \cong \mathrm{Sym}^\bullet_{\mathscr{O}_X(U)}\mathrm{Gr}_1\Lambda(U)$, so is an object that we studied in the previous section. In particular, $\mathrm{Gr}_1\Lambda(U)$ carries a $(\mathbb{C}, \mathscr{O}_X(U))$-Lie-Rinehart algebra structure. These structures glue in a natural way, and give $\mathrm{Gr}_1\Lambda$ a holomorphic Lie algebroid structure:

Proposition 28. *The coherent $\mathscr{O}_X$-modules $\Lambda_{(1)}$ and $Gr_1\Lambda$ carry a natural holomorphic Lie algebroid structure.*

Proof.

- For each $x \in \Lambda_{(1)}$, consider the map

$$\sharp_\Lambda(x) : f \to [x, f] = xf - fx \ .$$

Since the graded object is commutative, this map takes values in $\Lambda_{(1)}$. Moreover it satisfies the Leibniz rule, so $\sharp_\Lambda(x)$ is a derivation of $\mathscr{O}_X$. This yield a map $\sharp_\Lambda : \Lambda_{(1)} \to \mathcal{T}_X$.

- For $x \in \Lambda_{(1)}$ and $g \in \mathscr{O}_X$, one has $\sharp_\Lambda(x+g) = \sharp_\Lambda(x)$, since

$$[x+g, f] = (x+g)f - f(x+g) = xf - fx + fg - gf = xf - fx \ ,$$

so that $\sharp_\Lambda$ factors through the quotient, and we obtain the anchor $\sharp_{\mathrm{G}} : \mathrm{Gr}_1\Lambda \to \mathcal{T}_X$.

- Since the graded object is commutative, for $x, y \in \Lambda_{(1)}$ one has $[x, y] = xy - yx \in \Lambda_{(1)}$. It is easy to check that this bracket satisfies the Jacobi identity and the Leibniz rule w. r. t. the anchor a_Λ, so that $(\Lambda_{(1)}, \sharp_\Lambda, [\cdot,\cdot])$ is a holomorphic Lie algebroid.
- For each $u, v \in \mathrm{Gr}_1\Lambda$, define

$$[u, v]_{\mathrm{G}} = [x, y] \bmod \mathscr{O}_X$$

where $x, y \in \Lambda_{(1)}$ are representatives of u and v respectively. It is standard to check that this definition does not depend on the choice of x and y in their classes, and that the bracket $[\cdot,\cdot]_G$ satisfies the Jacobi identity and the Leibniz rule w. r. t. the anchor a_{G}.

□

We will call $(\mathrm{Gr}_1\Lambda, a_{\mathrm{G}}, [\cdot,\cdot]_{\mathrm{G}})$ the Lie algebroid associated to Λ.

It follows straightforwardly from the definition that the projection $\Lambda_{(1)} \to \mathrm{Gr}_1\Lambda$ is a Lie algebroid map, so we can look at (3.1) as an exact sequence of Lie algebroids, where $\mathscr{O}_X$ is given the trivial Lie algebroid structure. Hence $\Lambda_{(1)}$ is an extension of holomorphic Lie algebroids of $\mathrm{Gr}_1\Lambda$ by $\mathscr{O}_X$.

We will classify such extensions in the next section.

3.2.1 Lie algebroid extensions

Let $\mathcal{L}$ be a holomorphic Lie algebroid, and

$$0 \to \mathscr{O}_X \to \mathcal{L}' \to \mathcal{L} \to 0$$

an abelian extension of holomorphic Lie algebroids.

First assume that there exists a global splitting $\zeta : \mathcal{L} \to \mathcal{L}'$ of the sequence considered as a sequence of left $\mathscr{O}_X$-modules. This gives an isomorphism of left $\mathscr{O}_X$-modules $\hat{\zeta} : \mathcal{L}' \to \mathscr{O}_X \oplus \mathcal{L}$, and let $[\cdot,\cdot]_\zeta$ be the bracket on $\mathscr{O}_X \oplus \mathcal{L}$ obtained via this isomorphism. So we can write

$$[f + u, g + v]_\zeta = [u, v]_{\mathcal{L}} + Q(u, v) + a_{\mathcal{L}}(u)(g) - a_{\mathcal{L}}(v)(f)$$

with $Q(u, v) \in \mathscr{O}_X$. Obviously, Q is antisymmetric and $\mathscr{O}_X$-bilinear, so it is a holomorphic 2-$\mathcal{L}$-form; moreover, one can check that the Jacobi identity for $[\cdot,\cdot]_\zeta$ implies that $\mathrm{d}_{\mathcal{L}}Q = 0$.

We have the following:

Lemma 29. *Let ζ_1, ζ_2 be two global left $\mathscr{O}_X$-module splittings of*

$$0 \to \mathscr{O}_X \to \mathcal{L}' \to \mathcal{L} \to 0\ ,$$

and let $\psi = \zeta_2 - \zeta_1 \in Hom(\mathcal{L}, \mathscr{O}_X) = H^0(X, \Omega_{\mathcal{L}})$.

Let Q_1, Q_2 be the closed holomorphic 2-$\mathcal{L}$-forms associated to the short exact sequence via ζ_1, ζ_2 respectively.

Then $Q_2 - Q_1 = d_{\mathcal{L}}\psi$.

Proof. This works in the same way as in the case of Lie-Rinehart algebras: on $\mathscr{O}_X \oplus \mathcal{L}$ we have brackets $[\cdot,\cdot]_{\zeta_i}$, for $i = 1, 2$, induced by ζ_i. For $\xi, \eta \in \mathcal{L}'$ we have

$$\zeta_1(\xi) = (f, u) \ , \qquad \zeta_1(\eta) = (g, v) \ ,$$

$$\zeta_1([\xi, \eta]) = (\sharp(u)(g) - \sharp(v)(f) + Q_1(u, v), [u, v]) \ ,$$

$$\zeta_2(\xi) = (f + \psi(u), u) \ , \qquad \zeta_2(\eta) = (g + \psi(v), v) \ ,$$

$$\zeta_2([\xi, \eta]) = (\sharp(u)(g + \psi(v)) - \sharp(v)(f + \psi(u)) + Q_1(u, v) + \psi([u, v]), [u, v])$$

with $f, g \in \mathscr{O}_X$ and $u, v \in \mathcal{L}$. On the other hand

$$\zeta_2([\xi, \eta]) = (\sharp(u)(g) - \sharp(v)(f) + Q_2(u, v), [u, v]) \ ,$$

so that we obtain $(Q_2 - Q_1)(u, v) = \psi([u, v]) = (\mathrm{d}_{\mathcal{L}}\psi)(u, v)$. □

Remember the notation and results of Section 2.4.3:

$$\frac{H^0(X, \Omega^2_{\mathcal{L}})_{\text{closed}}}{\mathrm{d}_{\mathcal{L}}(H^0(X, \Omega_{\mathcal{L}}))} = E_2^{2,0} \cong F^2H^2(\mathcal{L}, \mathbb{C}) \ ,$$

Then the previous discussion and lemma imply:

Corollary 30. *The isomorphism classes of holomorphic Lie algebroid extensions*

$$0 \to \mathscr{O}_X \to \mathcal{L}' \to \mathcal{L} \to 0$$

which are split as sequences of left $\mathscr{O}_X$-modules are in one to one correspondence with the elements of $F^2H^2(\mathcal{L}, \mathbb{C})$.

Now we will see what happens if the extension does not split as a sequence of $\mathscr{O}_X$-modules: let $\Phi \in \mathrm{Ext}^1(\mathcal{L}, \mathscr{O}_X) \cong H^1(X, \Omega_{\mathcal{L}})$ be the class of the extension of left $\mathscr{O}_X$-modules

$$0 \to \mathscr{O}_X \to \mathcal{L}' \to \mathcal{L} \to 0 \ .$$

This sequence admits a global splitting if and only if $\Phi = 0$. For a sufficiently good open covering $\mathfrak{U} = \{U_\alpha\}$ of X we can represent Φ by a closed 1-Čech-cocycle $\phi_{\alpha\beta}$, and choose local splittings

$$\zeta_\alpha : \mathcal{L}_{|U_\alpha} \to \mathcal{L}'_{|U_\alpha}$$

satisfying $\zeta_\beta - \zeta_\alpha = \phi_{\alpha\beta}$. Then we can do the previous construction on each U_α and obtain a closed holomorphic 2-$\mathcal{L}$-form $Q_\alpha \in H^0(U_\alpha, \Omega^2_{\mathcal{L}})$. Because of Lemma 29, these satisfy $Q_\beta - Q_\alpha = \mathrm{d}_{\mathcal{L}}\phi_{\alpha\beta}$ on the double overlaps $U_{\alpha\beta}$. So the pair $(Q_\alpha, \phi_{\alpha\beta})$ is a closed element of $F^1T^2_{\mathcal{L}}$.

Now let $\phi'_{\alpha\beta}$ be another representative of Φ; then $\phi'_{\alpha\beta} - \phi_{\alpha\beta} = (\check{\delta}\eta)_{\alpha\beta}$ for some $\eta \in \check{C}^1(\mathfrak{U}, \Omega^1_{\mathcal{L}})$. Let ζ'_α be local splittings over U_α satisfying $\zeta'_\beta - \zeta'_\alpha =$

$\phi'_{\alpha\beta} = \phi_{\alpha\beta} + (\check{\delta}\eta)_{\alpha\beta}$, and $Q'_\alpha \in H^0(U_\alpha, \Omega^2_{\mathcal{L}})$ the closed 2-$\mathcal{L}$-forms associated to the splittings ζ'_α. Then $\check{\delta}(Q' - Q)_{\alpha\beta} = \mathrm{d}_{\mathcal{L}}(\check{\delta}\eta)_{\alpha\beta}$, which means, since $\check{\delta}$ and $\mathrm{d}_{\mathcal{L}}$ commute, that the local 2-$\mathcal{L}$-forms $Q'_\alpha - Q_\alpha - \mathrm{d}_{\mathcal{L}}\eta_\alpha$ glue to a global 2-$\mathcal{L}$-form G. So

$$(Q'_\alpha, \phi'_{\alpha\beta}) - (Q_\alpha, \phi_{\alpha\beta}) = \delta(\eta_\alpha + G_{|U_\alpha}),$$

Hence the cohomology class of $(Q_\alpha, \phi_{\alpha\beta})$ in $H^2(F^1T^\bullet) = F^1H^2(\mathcal{L}, \mathbb{C})$ is independent of the choices we made. Summing up, we have shown:

Theorem 31. *Let $\mathcal{L}$ be a holomorphic Lie algebroid.*

The isomorphism classes of abelian extensions of $\mathcal{L}$ by $\mathscr{O}_X$ are in a one to one correspondence with the elements of the cohomology group $F^1H^2(\mathcal{L}, \mathbb{C})$.

Remark that through this correspondence, the map

$$F^1H^2(\mathcal{L}, \mathbb{C}) \to H^1(X, \Omega_{\mathcal{L}})$$

described at the end of Section 2.4.3 associates to an extension of Lie algebroids its class as an extension of $\mathscr{O}_X$-modules, while Corollary 30 describes the fiber of this map over 0.

3.2.2 Sheafifying the twisted enveloping algebras

Now we sheafify the construction of twisted enveloping algebras of the previous section, to give analogous results for Lie algebroids: we will see that the datum of an extension of $\mathcal{L}$ by $\mathscr{O}_X$ as holomorphic Lie algebroids is equivalent to a pair (Λ, Ξ) with Λ a sheaf of filtered algebras over X and $\Xi : \mathrm{Gr}\Lambda \to \mathrm{Sym}^\bullet\mathcal{L}$ an isomorphism of sheaves of graded algebras.

Let

$$0 \to \mathscr{O}_X \to \mathcal{L}' \to \mathcal{L} \to 0$$

be an abelian extension of holomorphic Lie algebroids, with $\mathscr{O}_X$ endowed with the $\mathcal{L}$-module structure given by the anchor, and let $\sigma = (Q_\alpha, \phi_{\alpha\beta})$ be a representative of the class $\Sigma \in F^1H^2(\mathcal{L}, \mathbb{C})$ associated to this extension.

Consider the $\mathscr{O}_X$-bimodule structure on $\mathcal{L} \otimes_{\mathbb{C}_X} \mathscr{O}_X$ given by

$$f(u \otimes_{\mathbb{C}_X} g)h = (fu) \otimes_{\mathbb{C}_X} (gh)$$

and let $\tilde{T}^\bullet\mathcal{L}$ its $\mathscr{O}_X$-tensor algebra. Let $\tilde{T}^\bullet_\alpha\mathcal{L}$ and $\tilde{T}^\bullet_{\alpha\beta}$ be its restriction to U_α, $U_{\alpha\beta}$ respectively. In $\tilde{T}^\bullet_\alpha\mathcal{L}$ consider the ideal I_{Q_α} generated by sections of the form

$$fu \otimes_{\mathbb{C}_{U_\alpha}} 1 - u \otimes_{\mathbb{C}_{U_\alpha}} f - \sharp(u)(f)\ ,$$

$$(u \otimes_{\mathbb{C}_{U_\alpha}} 1) \otimes_{\mathscr{O}_{U_\alpha}} (v \otimes_{\mathbb{C}_{U_\alpha}} 1) - (v \otimes_{\mathbb{C}_{U_\alpha}} 1) \otimes_{\mathscr{O}_{U_\alpha}} (u \otimes_{\mathbb{C}_{U_\alpha}} 1) - [u, v] \otimes_{\mathbb{C}_{U_\alpha}} - Q_\alpha(u, v)$$

for $f \in \mathscr{O}_{U_\alpha}$ and $u, v \in \mathcal{L}_{|U_\alpha}$, and let $\tilde{\mathcal{U}}_{\sigma,\alpha}\mathcal{L}$ be the quotient.

Now, on double overlaps $U_{\alpha\beta}$ define the map

$$\tilde{T}^{\bullet}_{\alpha\beta}\mathcal{L} \to \tilde{T}^{\bullet}_{\alpha\beta}\mathcal{L}$$

by

$$f \to f \qquad f \in \mathscr{O}_{U_{\alpha\beta}} \ ,$$

$$u \to u + \phi_{\alpha\beta} \qquad u \in \mathcal{L}_{|U_{\alpha\beta}} \ .$$

One can check that this map descends to an isomorphism of sheaves of algebras

$$g_{\alpha\beta} : (\tilde{\mathcal{U}}_{\sigma,\alpha}\mathcal{L})_{|U_{\alpha\beta}} \to (\tilde{\mathcal{U}}_{\sigma,\beta}\mathcal{L})_{|U_{\alpha\beta}}.$$

Since $\phi_{\alpha\beta}$ is $\check{\delta}$-closed, we have $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \mathbf{1}$ on the triple intersections, so we can glue the local sheaves $\tilde{\mathcal{U}}_{\sigma,\alpha}\mathcal{L}$ via the isomorphisms $g_{\alpha\beta}$, and get a sheaf of algebras $\tilde{\mathcal{U}}_{\sigma}(\mathcal{L})$ on X.

Now, it is easy to generalize Lemma 25 to the following:

Lemma 32. *1. $\tilde{\mathcal{U}}_{\sigma}\mathcal{L}$ is a filtered algebra with $(\tilde{U}_F(L))_{(0)} = \mathscr{O}_X$;*

2. the first graded piece $Gr_1\tilde{\mathcal{U}}_{\sigma}\mathcal{L}$ is isomorphic, as a left $\mathscr{O}_X$-module, to $\mathcal{L}$;

3. the left and right $\mathscr{O}_X$-module structures on the graded objects $Gr_i\tilde{\mathcal{U}}_{\sigma}\mathcal{L}$ coincide;

4. the graded algebra $Gr_{\bullet}\tilde{\mathcal{U}}_{\sigma}\mathcal{L}$ is commutative;

5. there is a natural morphism of sheaves of graded algebras $\Xi_{\sigma} : Sym^{\bullet}_{\mathscr{O}_X}\mathcal{L} \to Gr_{\bullet}\tilde{\mathcal{U}}_{\sigma}\mathcal{L}$.

Proposition 33. *The map Ξ_{σ} defined in the last point of the previous lemma is an isomorphism.*

Proof. It suffices to show that locally Ξ_{σ} is an isomorphism. On each U_{α}, by construction, we have that each $\tilde{\mathcal{U}}_{\sigma,\alpha}\mathcal{L}$ is the sheafification of the $\mathscr{O}_X(U_{\alpha})$-algebra $\tilde{U}_{Q_{\alpha}}(\mathcal{L}(U_{\alpha}))$.

So

$$\Xi_{\sigma|U_{\alpha}} : \mathrm{Sym}^{\bullet}_{\mathscr{O}_{U_{\alpha}}}\mathcal{L}_{|U_{\alpha}} \to \tilde{\mathcal{U}}_{\sigma,\alpha}\mathcal{L}$$

is the sheafification of

$$\Xi_{Q_{\alpha}} : \mathrm{Sym}^{\bullet}_{\mathscr{O}_X(U_{\alpha})}\mathcal{L}(U_{\alpha}) \to \tilde{U}_{Q_{\alpha}}(\mathcal{L}(U_{\alpha})) \ ,$$

and so is an isomorphism by Proposition 26. □

If σ' is another representative of Σ and $\sigma' - \sigma = \delta(\eta)$, it is possible to construct from η an isomorphism

$$\Upsilon_\eta : \mathcal{U}_{\sigma'}(\mathcal{L}) \to \mathcal{U}_\sigma(\mathcal{L})$$

commuting with the isomorphisms $\Xi_{\sigma'}$ and Ξ_σ: let $\sigma' = (Q'_\alpha, \phi'_{\alpha\beta})$ and $\sigma = (Q_\alpha, \phi_{\alpha\beta})$; then $Q'_\alpha - Q_\alpha = \mathrm{d}_\mathcal{L}\eta_\alpha$ and $\phi'_{\alpha\beta} - \phi_{\alpha\beta} = -(\check{\delta}\eta)_{\alpha\beta} = \eta_\alpha - \eta_\beta$. Over each U_α define $\Upsilon_{\eta,\alpha}$ as the sheafification of the Υ_{η_α} as defined in Lemma 27. These are isomorphisms of $\tilde{\mathcal{U}}_{\sigma',\alpha}\mathcal{L}$ with $\tilde{\mathcal{U}}_{\sigma,\alpha}\mathcal{L}$. For these to define an isomorphism of the sheaf $\tilde{\mathcal{U}}_{\sigma'}\mathcal{L}$ with $\tilde{\mathcal{U}}_\sigma\mathcal{L}$ we need the following diagram to commute on the double overlaps $U_{\alpha\beta}$:

$$\begin{tikzcd}
\tilde{\mathcal{U}}_{\sigma',\alpha}\mathcal{L} \arrow[r, "\Upsilon_{\eta,\alpha}"] \arrow[d, "g'_{\alpha\beta}"] & \tilde{\mathcal{U}}_{\sigma,\alpha}\mathcal{L} \arrow[d, "g_{\alpha\beta}"] \\
\tilde{\mathcal{U}}_{\sigma',\beta}\mathcal{L} \arrow[r, "\Upsilon_{\eta,\beta}"'] & \tilde{\mathcal{U}}_{\sigma,\beta}\mathcal{L}
\end{tikzcd} .$$

This is true because

$$g_{\alpha\beta}(\Upsilon_{\eta,\alpha}(u \otimes 1)) = g_{\alpha\beta}(u \otimes 1 + \eta_\alpha(u)) = u \otimes 1 + \phi_{\alpha\beta}(u) + \eta_\alpha(u) ,$$

$$\Upsilon_{\eta,\beta}(g'_{\alpha\beta}(u \otimes 1)) = \Upsilon_{\eta,\beta}(u \otimes 1 + \phi'_{\alpha\beta}(u)) = u \otimes 1 + \eta_\beta(u) + \phi'_{\alpha\beta}(u)$$

and $\phi_{\alpha\beta} + \eta_\alpha = \eta_\beta + \phi'_{\alpha\beta}$.

Summing up, we have shown:

Theorem 34. *Let X be a smooth projective variety and $\mathcal{L}$ a holomorphic Lie algebroid on X.*

Then there is a 1-to-1 correspondence between

- *holomorphic Lie algebroid extensions of $\mathcal{L}$ by $\mathscr{O}_X$;*
- *elements of the vector space $F^1H^2(\mathcal{L}, \mathbb{C})$;*
- *isomorphism classes of pairs (Λ, Ξ), where Λ is a sheaf of filtered algebras on X and $\Xi : Gr_\bullet\Lambda \to Sym^\bullet_{\mathscr{O}_X}\mathcal{L}$ an isomorphism of sheaves of graded algebras.*

In particular, if $\mathcal{L}$ is a holomorphic Lie algebroid and $\Sigma \in F^1H^2(\mathcal{L}, \mathbb{C})$, we will denote the associated sheaf of filtered algebras by $\Lambda_{\mathcal{L},\Sigma}$.

Moreover, this construction is functorial: if $\mathcal{L}, \mathcal{L}'$ are two holomorphic Lie algebroids and $\Psi : \mathcal{L} \to \mathcal{L}'$ is a Lie algebroid morphism, we have an induced pull back morphism between the cohomologies $\Psi^* : H^p(\mathcal{L}', \mathbb{C}) \to H^p(\mathcal{L}, \mathbb{C})$ preserving the filtration.

Then it is easy to show the following:

Lemma 35. *Let $\Psi : \mathcal{L} \to \mathcal{L}'$ be a morphism of holomorphic Lie algebroids and $\Sigma \in H^2(\mathcal{L}', \mathbb{C})$. Then Ψ extends to a morphism of sheaves of filtered algebras*

$$\Psi : \Lambda_{\mathcal{L},\Psi^*\Sigma} \longrightarrow \Lambda_{\mathcal{L}',\Sigma}.$$

3.2.3 Examples: algebras associated to the canonical Lie algebroid

Let X be a smooth projective variety over $\mathbb{C}$, and consider $\mathcal{L} = \mathcal{T}_X$ the holomorphic tangent bundle with the canonical Lie algebroid structure. Remark that since X is smooth projective, we have the Hodge decomposition, that implies

$$F^1 H^2(\mathcal{T}_X, \mathbb{C}) = F^1 H^2_{DR}(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \ .$$

If $\Sigma = 0$, then $\Lambda_{\mathcal{T}_X,0}$ is the sheaf of algebras of holomorphic differential operators $\mathscr{D}_X$.

For $\Sigma = [(0, Q)]$ we can describe $\Lambda_{\mathcal{T}_X,[(0,Q)]}$ in terms of local coordinates as follows. Let $x^1, \dots, x^n$ be local holomorphic coordinates of X and ∂_{x^i} the corresponding frame of $\mathcal{T}_X$. Let Q_{ij} be such that $Q = \sum_{i,j} Q_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j$. Then the commutator of elements in $\Lambda_{\mathcal{T}_X,[(0,Q)]}$ is determined by:

$$\begin{aligned} &[x^i, x^j] = 0 \ , \\ &\left[x^i, \partial_{x^j}\right] = \delta^i_j \ , \\ &\left[\partial_{x_i}, \partial_{x_j}\right] = Q_{ij} \ . \end{aligned}$$

This is the operator algebra corresponding to a magnetic monopole of charge Q.

Another case that has an explicit description is when $[\phi] \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. Let $\mathcal{M}$ be a holomorphic line bundle on X given by transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$; this defines a class in $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ represented, through Dolbeault isomorphism, by a cocycle $\phi_{\alpha\beta} = g^{-1}_{\alpha\beta} \mathrm{d} g_{\alpha\beta} \in H^1(X, \Omega_X)$. For $Q = 0$, the class $[(\phi, 0)]$ defines the Atiyah Lie algebroid of $\mathcal{M}$:

$$0 \to \mathscr{O}_X \to \mathcal{J}^1_X(\mathcal{M}) \to \mathcal{T}_X \to 0 \ ,$$

so we have $\Lambda_{\mathcal{T}_X,[(\phi,0)]} \cong \mathscr{D}(\mathcal{M})$, the algebra of differential operators on the line bundle $\mathcal{M}$. One can then think of $\Lambda_{\mathcal{T}_X,[(\phi_{\alpha\beta},Q)]}$ as the operator algebra of a twisted magnetic monopole with charge Q and twisting line bundle $\mathcal{M}$.

3.2.4 Other examples

Remark that when $\Sigma = 0$, the corresponding algebra $\Lambda_{\mathcal{L},0}$ are the algebras that arise from a triple (H, δ, γ) as in paragraph XX of [34]: actually the definition of that triples is equivalent to define a holomorphic Lie algebroid, with anchor induced by δ and bracket induced by γ.

Let now $\mathcal{L} = \mathcal{K}$ be a holomorphic bundle of Lie algebras. Then $\mathcal{K}(U)$ is actually an $\mathscr{O}_X(U)$-Lie algebra for each open $U \subseteq X$, and $H^p(\Omega^\bullet_{\mathcal{K}}(U), \mathrm{d}_{\mathcal{K}})$ coincides with the Chevalley-Eilenberg cohomology $H^p_{CE}(\mathcal{K}(U), \mathscr{O}_X(U))$.

For $\Sigma = 0$, the associated sheaf of filtered algebras $\Lambda_{\mathcal{K},0}$ is the sheaf of universal enveloping algebras of $\mathcal{K}$: indeed, we have that $\Lambda_{\mathcal{K},0}(U)$ is the universal enveloping algebra of the $\mathscr{O}_X(U)$-Lie algebra $\mathcal{K}(U)$ for each open $U \subseteq X$.

For a general $\Sigma = [(Q_\alpha, \phi_{\alpha\beta})]$, we have that $\Lambda_{\mathcal{K},\Sigma}(U_\alpha)$ is the Sridharan enveloping algebra of the $\mathscr{O}_X(U_\alpha)$-Lie algebra $\mathcal{K}(U_\alpha)$ associated to the class $Q_\alpha \in \Omega^2_{\mathcal{K}}(U_\alpha) = H^2_{CE}(\mathcal{K}(U_\alpha), \mathscr{O}_X(U_\alpha))$. $(\Lambda_{\mathcal{K},\Sigma})_{|U_\alpha}$ is the sheafification of this algebra. In a similar way as in the proof of Theorem 34, the $\phi_{\alpha\beta}$ give isomorphisms on the double overlaps satisfying the gluing condition.

For any regular integrable holomorphic foliation $\mathscr{F} \subseteq \mathcal{T}_X$, we have a holomorphic Lie algebroid structure induced by the canonical one on $\mathcal{T}_X$.

If $\Sigma = 0$, then $\Lambda_{\mathscr{F},0}$ is isomorphic to the algebra $\mathscr{D}_{\mathscr{F}}$ of differential operators along the foliation. For more general Σ we obtain operator algebras of (twisted) monopoles propagating along the foliation.

Chapter 4

Λ-modules

In this chapter we provide an application of Theorem 34 to moduli spaces: first we quicly review the construction of the moduli spaces of Λ-modules of [34], then see that this provides moduli spaces for semistable flat $\mathcal{L}$-connections for any holomorphic Lie algebroid $\mathcal{L}$.

4.1 The moduli spaces of semistable Λ-modules

Let X be a smooth projective variety over $\mathbb{C}$, $\mathscr{O}_X(1)$ an ample line bundle and Λ a sheaf of filtered algebras. Let $\mathcal{E}$ be a coherent sheaf on X.

Definition 9. *A Λ-module structure on $\mathcal{E}$ is a $\mathscr{O}_X$-morphism $\mu : \Lambda \otimes \mathcal{E} \to \mathcal{E}$ satisfying the usual module axioms and such that the $\mathscr{O}_X$-module structure on $\mathcal{E}$ induced by $\mathscr{O}_X \to \Lambda$ coincides with the original one.*

We will say that a Λ-module $(\mathcal{E}, \mu)$ is *(semi)stable* if $\mathcal{E}$ is torsion free and for any subsheaf $\mathcal{F} \subseteq \mathcal{E}$ invariant under μ (i.e. such that $\mu(\Lambda \otimes \mathcal{F}) \subseteq \mathcal{F}$), one has $p(\mathcal{F}) < p(\mathcal{E})$ (resp. $p(\mathcal{F}) \leq p(\mathcal{E})$), where $p(\mathcal{E})$ is the reduced Hilbert polynomial of $\mathcal{E}$, defined as the ratio $P(\mathcal{E})/rnk(\mathcal{E})$, where $P(\mathcal{E})(n) = \chi(\mathcal{E} \otimes \mathscr{O}_X(n))$ is the Hilbert polynomial of $\mathcal{E}$, and $rnk(\mathcal{E})$ its rank.

Remark that if $(\mathcal{E}, \mu)$ is a Λ-module and $\mathcal{F} \subseteq \mathcal{E}$ is a μ-invariant subsheaf, then μ induces a Λ-module structure on both $\mathcal{F}$ and $\mathcal{E}/\mathcal{F}$.

If $(\mathcal{E}, \mu)$ is a semistable Λ-module, there exist filtrations $\mathcal{E}_i \subseteq \mathcal{E}_{i-1}$ of μ-invariant subsheaves of $\mathcal{E}$ such that each $\mathrm{gr}_i\mathcal{E} = \mathcal{E}_i/\mathcal{E}_{i+1}$ is a stable Λ-module with Hilbert polynomial equal to the Hilbert polynomial of $\mathcal{E}$. In general, such filtrations (that we call *Jordan filtration* of the semistable Λ-module $(\mathcal{E}, \mu)$ are not unique, but the graded object $\mathrm{gr}\mathcal{E} = \bigoplus \mathrm{gr}_i\mathcal{E}$ associated to different Jordan filtrations of the same semistable Λ-module are isomorphic. In general, different semistable Λ-modules may have graded objects isomorphic, so this induce an equivalence relation on semistable Λ-modules. We say that two Λ-modules are Jordan equivalent when their graded objects are isomorphic.

For any scheme S, let p_S, p_X denote the projections from the product $S \times X$ to the factors, and Λ_S be the pullback $p_X^* \Lambda$. Consider the contravariant functor $\underline{\mathcal{M}_\Lambda}(P)$: **Schemes** $\to$ **Sets** such that

- to each scheme S associates the set of isomorphism classes of Λ_S-modules $(\mathcal{F}, \mu)$ such that $\mathcal{F}$ is flat over S and each fiber $(\mathcal{F}_s, \mu_s)$ is a semistable Λ-module and has Hilbert polynomial P. Two such families $\mathcal{F}, \mathcal{F}'$ are isomorphic if there exist a line bundle $\mathcal{M}$ on S such that $\mathcal{F}' = \mathcal{F} \otimes p_S^* \mathcal{M}$ and $\mu' = \mu \otimes \mathbf{1}_{p_S^* \mathcal{M}}$;
- to each morphism of schemes $\psi : T \to S$ associates the pull back: if $\mathcal{F}$ is a Λ_S-module flat over S, then $\psi^* \mathcal{F}$ is a Λ_T-module flat over T.

In [34] is shown the following:

Theorem 36. *Let X be a smooth projective variety, Λ a sheaf of filtered algebras on X and P a numerical polynomial.*

Then there exists a quasi projective variety $\mathcal{M}_\Lambda(P)$ that universally corepresents the functor $\underline{\mathcal{M}_\Lambda}(P)$. The closed points of $\mathcal{M}_\Lambda(P)$ are in a one to one correpondence with Jordan equivalence classes of semistable Λ-modules over X with Hilbert polynomial equal to P.

The scheme $\mathcal{M}_\Lambda(P)$ is obtained as a GIT quotient of a locally closed subscheme of a $\mathfrak{Quot}$-scheme by the action of a special linear group: let $\mathscr{F}$ be the family of coherent sheaves $\mathcal{E}$ over X with Hilbert polynomial P that admits a Λ-module structure μ such that the pair $(\mathcal{E}, \mu)$ is semistable. It can be shown (see [34], Corollary 3.5) that $\mathscr{F}$ is a bounded family, so that there exists an integer N such that, letting $V = \mathbb{C}^{P(N)}$, for each $\mathcal{E} \in \mathscr{F}$ there exist a surjective map $q : V(-N) \to \mathcal{E}$, where by $V(-N)$ we denote $V \otimes_{\mathbb{C}} \mathscr{O}_X(-N)$. In particular, since $\mathscr{O}_X \hookrightarrow \Lambda_{(1)}$, for each $\mathcal{E} \in \mathscr{F}$ there exist a surjective map

$$\Lambda_{(1)} \otimes V(-N) \to \mathcal{E} \ .$$

Now consider the scheme $\mathfrak{Quot}(\Lambda_{(1)} \otimes V(-N), P)$.

Theorem 37 (cf. [34], Theorem 3.8)**.** *There exist a locally closed subscheme $Q \subseteq \mathfrak{Quot}(\Lambda_{(1)} \otimes V(-N), P)$ parametrizing triples $(\mathcal{E}, \mu, \alpha)$, with $(\mathcal{E}, \mu)$ a semistable Λ-module and $\alpha : V \to H^0(X; \mathcal{E}(N))$ an isomorphism of vector spaces.*

The moduli space $\mathcal{M}_\Lambda(P)$ is the quotient $Q//\mathrm{SL}(V)$. We review the construction of Q, since later we will study its tangent space. First of all consider the open subscheme $Q_1 \subseteq \mathfrak{Quot}(\Lambda_{(1)} \otimes V(-N), P)$ consisting of quotients $q : \Lambda_{(1)} \otimes V(-N) \to \mathcal{E}$ such that $\tilde{q}$, the composition $V(-N) \hookrightarrow \Lambda_{(1)} \otimes V(-N) \to \mathcal{E}$ is surjective. If we let $\mathcal{K}_1$ be the kernel of q and $\mathcal{K}_0$ be the kernel of $\tilde{q}$, we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}_0 \arrow[r] \arrow[d] & V(-N) \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K}_1 \arrow[r] & \Lambda_{(1)} \otimes V(-N) \arrow[r] & \mathcal{E} \arrow[r] & 0
\end{tikzcd}$$

Now, tensoring the first row of this diagram by $\Lambda_{(1)}$, we obtain the map $\Lambda_{(1)} \otimes \mathcal{K}_0 \to \Lambda_{(1)} \otimes V(-N) \to \mathcal{E}$. Define Q_2 to be the closed subscheme of Q_1 consisting of the quotients q such that this map vanishes. For a point q in Q_2 we have the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \Lambda_{(1)} \otimes \mathcal{K}_0 \arrow[r, equal] \arrow[d] & \Lambda_{(1)} \otimes \mathcal{K}_0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{K}_1 \arrow[r] \arrow[d] & \Lambda_{(1)} \otimes V(-N) \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{W}_1 \arrow[r] \arrow[d] & \Lambda_{(1)} \otimes \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{4.1}$$

Denote by μ_1 the morphism $\Lambda_{(1)} \otimes \mathcal{E} \to \mathcal{E}$ on the bottom row.

For any positive integer j, let $\mathcal{B}_j$ be the kernel of the product $(\Lambda_{(1)})^{\otimes j} \to \Lambda_{(j)}$. Let $Q_{3,j} \subseteq Q_2$ the closed subscheme of points q such that the composition $\mathcal{B}_j \otimes \mathcal{E} \to (\Lambda_{(1)})^{\otimes j} \to \mathcal{E}$ vanishes, where the last map is obtained by iterating μ_1 j times. Let Q_3 denote the intersection of all $Q_{3,j}$'s, that is a closed subscheme of Q_2 since the latter is Noetherian. For $q \in Q_3$, the map μ_1 extends to define a Λ-module structure μ on $\mathcal{E}$.

Finally, the open subscheme $Q \subseteq Q_3$ of quotients $q : \Lambda_{(1)} \otimes V(-N) \to \mathcal{E}$ such that $\mathcal{E}$ is semistable as Λ-module, is the scheme parametrizing the triples $(\mathcal{E}, \nabla, \alpha)$ as in the statement of the theorem.

4.2 Λ-modules and $\mathcal{L}$-connections

The ideas underlying this subsection are already contained in the section "The split almost polynomial case" of [34]: Simpson's definition of triple (H, δ, γ) is equivalent to define a holomorphic Lie algebroid structure on H^*, and we remark that Simpson's Lemma 2.13 is equivalent to our Proposition 38 for $Q = 0$.

Let $\mathcal{L}$ be a holomorphic Lie algebroid, $\Sigma \in F^2H^2(\mathcal{L},\mathbb{C})$ and $\Lambda = \Lambda_{\mathcal{L},\Sigma}$. Since we have taken Σ in the second filtration piece, there exist left $\mathscr{O}_X$-module splittings of the sequence

$$0 \to \mathscr{O}_X \to \Lambda_{(1)} \to \mathcal{L} \to 0\ .$$

Choose a splitting ζ, which provides a representative $Q \in H^0(X,\Omega^2_{\mathcal{L}})_{\mathrm{closed}}$ of Σ.

To a Λ-module structure μ on $\mathcal{E}$, we canonically associate a sheaf map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L}^*$, defined by

$$\langle \nabla e, v\rangle = \mu(\zeta(v) \otimes e)$$

for all $v \in \mathcal{L}$. This map satisfies the $\mathrm{d}_{\mathcal{L}}$-Leibniz rule $\nabla(fe) = f\nabla e + e \otimes \mathrm{d}_{\mathcal{L}} f$ for any $f \in \mathscr{O}_X$ and $e \in \mathcal{E}$, so it is a holomorphic $\mathcal{L}$-connection on $\mathcal{E}$.

Vice versa, if we have a holomorphic $\mathcal{L}$-connection ∇ on $\mathcal{E}$, we can define a morphism $\mu_1 : \Lambda_{(1)} \otimes \mathcal{E} \to \mathcal{E}$ by

$$\mu_1((f + \zeta(u)) \otimes e) = fe + \langle \nabla e, v\rangle.$$

This morphism can be extended to a Λ-module structure if and only if

$$\mu_1(a \otimes \mu_1(b \otimes e)) - \mu_1(b \otimes \mu_1(a \otimes e)) = \mu_1([a,b] \otimes e)$$

for any $a, b \in \Lambda_{(1)}$, $e \in \mathcal{E}$. Since $[a,b] = \zeta([[a],[b]]) + Q([a],[b])$, we see that this condition is satisfied if and only if

$$\langle \nabla e, [u,v]\rangle + Q(u,v)e - \langle \nabla(\langle \nabla e, u\rangle), v\rangle + \langle \nabla(\langle \nabla e, v\rangle), u\rangle = 0$$

for any $u, v \in \mathcal{L}$ and $e \in \mathcal{E}$, that is, if and only if the curvature of ∇ satisfies

$$F_\nabla = Q \cdot \mathbf{1}_{\mathcal{E}}\ .$$

So we have:

Proposition 38. *Let $\mathcal{L}$ be a holomorphic Lie algebroid, $Q \in H^0(X,\Omega^2_{\mathcal{L}})$ a closed 2-$\mathcal{L}$-form, and $\mathcal{E}$ a coherent sheaf on X.*

Then giving a $\Lambda_{\mathcal{L},[(0,Q)]}$-module structure μ on $\mathcal{E}$ is equivalent to giving a holomorphic $\mathcal{L}$-connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes L^*$ such that $F_\nabla = Q\mathbf{1}_{\mathcal{E}}$.

By virtue of this proposition and Theorem 36, there exist quasi projective moduli schemes $M_{\mathcal{L},Q}(P)$ that are coarse moduli spaces for semistable pairs $(\mathcal{E},\nabla)$ with $\mathcal{E}$ a torsion free $\mathscr{O}_X$-module with Hilbert polynomial P, ∇ a holomorphic $\mathcal{L}$-connection on $\mathcal{E}$ satisfying $F_\nabla = Q \cdot \mathbf{1}_{\mathcal{E}}$, and semistable means that for any subsheaf $\mathcal{F} \subseteq \mathcal{E}$ with $\nabla(\mathcal{F}) \subseteq \mathcal{F} \otimes \Omega_{\mathcal{L}}$ one has $p(\mathcal{F}) \leq p(\mathcal{E})$.

Now, if Q, Q' are two cohomologous closed 2-$\mathcal{L}$-forms, for any $\eta \in \Omega^1_{\mathcal{L}}$ such that $Q' - Q = \mathrm{d}_{\mathcal{L}}\eta$ we can construct an isomorphism of algebras $\Upsilon_\eta : \Lambda_{\mathcal{L},[(0,Q)]} \to \Lambda_{\mathcal{L},[(0,Q')]}$. This induces an isomorphism of the moduli spaces $M_{\mathcal{L},Q}(P) \to M_{\mathcal{L},Q'}(P)$. Moreover, we have:

Proposition 39. *If Q is not cohomologous to 0, the moduli spaces $M_{\mathcal{L},Q}(P)$ are empty for any polynomial P.*

Proof. On one side, by Theorem 16, if $\mathcal{E}$ is a torsion free $\mathscr{O}_X$-module and ∇ is a holomorphic $\mathcal{L}$-connection on it, its $\mathcal{L}_h$-characteristic ring is 0. So the cohomology class of the trace of F_∇ is zero.

On the other side we have

$$\mathrm{Trace}(Q \cdot \mathbf{1}_{\mathcal{E}}) = rnk(\mathcal{E}) \cdot Q \ ,$$

so if Q is not cohomologous to zero we have a contradiction.

□

Summing up, we have:

Corollary 40. *Let $\mathcal{L}$ be a holomorphic Lie algebroid over a smooth projective variety X, $Q \in H^0(X, \Omega^2_{\mathcal{L}})_{closed}$ and P a numerical polynomial.*

Then for Q not cohomologous to 0 the moduli spaces $M_{\mathcal{L},Q}(P)$ are empty, while for Q cohomologous to 0 the moduli spaces $M_{\mathcal{L},Q}(P)$ parametrizes semistable flat holomorphic $\mathcal{L}$-connections with Hilbert polynomial P.

For a general $\Sigma \in F^1H^2(\mathcal{L}, \mathbb{C})$, we do not have a global splitting of the 1st-order sequence. But, if $(Q_\alpha, \phi_{\alpha\beta})$ is a representative of Σ, we can choose local splittings ζ_α over U_α such that $\zeta_\beta - \zeta_\alpha = \phi_{\alpha\beta}$ on the overlaps, and repeat the previous argument to find that a $\Lambda_{\mathcal{L},[(Q_\alpha,\phi_{\alpha\beta})]}$-module structure on a coherent sheaf $\mathcal{E}$ is equivalent to a collection of holomorphic $\mathcal{L}$-connections ∇_α on $\mathcal{E}_{|U_\alpha}$ such that

- $F_{\nabla_\alpha} = Q_\alpha \mathbf{1}_{\mathcal{E}|U_\alpha}$,
- $\nabla_\beta - \nabla_\alpha = \mathbf{1}_{\mathcal{E}} \otimes \phi_{\alpha\beta}$ over the double intersections $U_{\alpha\beta}$.

Any morphism of holomorphic Lie algebroids $\Psi : \mathcal{L} \to \mathcal{L}'$ induces by Lemma 35 a morphism of sheaves of filtered algebras $\Psi : \Lambda_{\mathcal{L},0} \to \Lambda_{\mathcal{L}',0}$.

Let $\mathcal{E}$ be a coherent sheaf on X and μ' a $\Lambda_{\mathcal{L}',0}$-module structure on it. Then $\mu = \mu' \circ \Psi$ is a $\Lambda_{\mathcal{L},0}$-module structure on $\mathcal{E}$. If ∇ (resp. ∇') is the flat $\mathcal{L}$ (resp. $\mathcal{L}'$)-connection associated to μ (resp. μ'), then $\nabla = (\mathbf{1}_{\mathcal{E}} \otimes \Psi^*) \circ \nabla'$.

Lemma 41. *Let $\Psi : \mathcal{L} \to \mathcal{L}'$ be a morphism of holomorphic Lie algebroids, and denote by $\mathcal{C}$ its cokernel. Let $\mathcal{E}$ be a coherent $\mathscr{O}_X$-module and ∇ a $\mathcal{L}$-connection on it.*

If there exist a $\mathcal{L}'$-connection ∇' inducing ∇ by the previous construction, then the set of all $\mathcal{L}'$-connection inducing ∇ is parametrized by $Hom(\mathcal{E}, \mathcal{E} \otimes \mathcal{C}^*)$.

Proof. The kernel of Ψ^* is $\mathcal{C}^*$, so the sequence

$$0 \to \mathcal{E} \otimes \mathcal{C}^* \to \mathcal{E} \otimes \Omega_{\mathcal{L}'} \to \mathcal{E} \otimes \Omega_{\mathcal{L}}$$

is exact. So for any $\xi \in \mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{C}^*)$ the composition $(\mathbf{1}_{\mathcal{E}} \otimes \Psi^*) \circ \xi$ is zero. Then, if ∇' is a $\mathcal{L}'$-connection inducing ∇, so is $\nabla' + \xi$.

Vice versa, if ∇'_1, ∇'_2 are two $\mathcal{L}'$-connection inducing ∇ their difference is a $\mathscr{O}_X$-linear map taking values in $\mathcal{E} \otimes \mathcal{C}^*$. □

If $\mathcal{F}$ is a ∇'-invariant subsheaf of $\mathcal{E}$, then it is clearly ∇-invariant too. So if $(\mathcal{E}, \nabla)$ is a semistable $\mathcal{L}$-connection, then $(\mathcal{E}, \nabla')$ is a semistable $\mathcal{L}'$-connection. In particular, this implies the following:

Proposition 42. *Let $\Psi : \mathcal{L} \to \mathcal{L}'$ be a morphism of holomorphic Lie algebroids, $(\mathcal{E}, \nabla')$ a $\mathcal{L}'$-connection such that $(\mathcal{E}, (\mathbf{1} \otimes \Psi^*) \circ \nabla')$ is a semistable $\mathcal{L}$-connection. Denote by Z, Z' the irreducible components of the moduli spaces $\mathcal{M}_{\mathcal{L}}(P), \mathcal{M}_{\mathcal{L}'}(P)$ to wich respectively $(\mathcal{E}, (\mathbf{1}_{\mathcal{E}} \otimes \Psi^*) \circ \nabla')$ and $(\mathcal{E}, \nabla')$ belong.*

Then there is a rational map

$$Z' \dashrightarrow Z$$

whose fibers are linear spaces.

Proof. We need to show that the set of semistable $\mathcal{L}'$-connections inducing semistable $\mathcal{L}$-connections is open in $\mathcal{M}_{\mathcal{L}'}(P)$.

Let S be a scheme, and consider a family of semistable $\mathcal{L}'$-connections parametrized by S, that is a $(p_X^*\mathcal{L}')$-connection $(\mathcal{G}, D')$ on $S \times X$ flat over S such that for any closed point $s \in S$ the fiber $(\mathcal{G}, D')_{s \times X}$ is semistable. Composing D' with $(\mathbf{1}_{\mathcal{G}} \otimes p_X^* \Psi^*)$ we obtain a $(p_X^*\mathcal{L})$-connection $(\mathcal{G}, D)$. Now, the set of points of S such that $(\mathcal{G}, D)_{s \times X}$ is a semistable $\mathcal{L}$-connection is an open in S, so the assertion is proved. □

This proposition generalize the usual construction of the "forgetful" map from the moduli space of semistable flat connections (resp. Higgs sheaves) to the moduli space of coherent $\mathscr{O}_X$-modules, that can be used to study the structure of the moduli spaces of flat connections (resp. Higgs sheaves) (see for example [22]).

In particular, for any holomorphic Lie algebroid $\mathcal{L}$ the anchor is a morphism of holomorphic Lie algebroids to $\mathcal{T}_X$, so the irreducible components of $M_{DR}(P)$ have rational morphisms to appropriate irreducible components of $M_{\mathcal{L}}(P)$.

4.3 Examples

If the Lie algebroid is the canonical one $(\mathcal{T}_X, \mathbf{1}, [\cdot, \cdot])$, then $\mathcal{T}_X$-connections on a sheaf $\mathcal{E}$ are just usual connections. So $M_{\mathcal{T}_X}(P)$ is the moduli space of semistable flat connections with Hilbert polynomial P, that usually is denoted by $M_{DR}(P)$.

If $(\mathcal{K}, 0, \{,\})$ is a holomorphic bundle of Lie algebras, then a $\mathcal{K}$-connection ∇ on a sheaf $\mathcal{E}$ is an $\mathscr{O}_X$-linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \mathcal{K}^*$, because, since the anchor is zero, so is the restriction of $\mathrm{d}_{\mathcal{K}}$ to functions. So ∇ may be seen as a section of $\mathrm{End}(\mathcal{E}) \otimes \mathcal{L}^*$. Remark that, since the anchor is 0, the bracket is $\mathscr{O}_X$-bilinear, hence we can see it as a section $\Theta \in H^0(\bigwedge^2 \mathcal{L}^* \otimes \mathcal{L})$.

The curvature of a $\mathcal{K}$-connection in this case is given by

$$F_\nabla = \nabla \wedge \nabla + \langle \Theta, \nabla \rangle \in H^0(\mathrm{End}(\mathcal{E}) \otimes \bigwedge^2 \mathcal{K}^*) \ .$$

A particular case of this is when $\mathcal{K} = \mathcal{T}_X$ equipped with the trivial Lie algebroid bundle structure. In this case a flat $(\mathcal{T}_X, 0, 0)$-connection on $\mathcal{E}$ is an $\mathscr{O}_X$-linear map $\phi : \mathcal{E} \to \mathcal{E} \otimes \Omega_X$ satisfying $\phi \wedge \phi = 0$, i.e. it is a *Higgs field* on $\mathcal{E}$.

Another interesting case is when $\mathcal{K} = \Omega_X$ with the trivial structure: similarly to the above, a flat $(\Omega_X, 0, 0)$-connection on $\mathcal{E}$ is a $\mathscr{O}_X$-linear map $\phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_X$ satisfying $\phi \wedge \phi = 0$, that is, a *co-Higgs field* as introduced by Hitchin in [16]. This is a particular case of the construction described in the next subsection.

4.3.1 Holomorphic Poisson structures and generalized complex geometry

Let X be a smooth projective variety and $\Pi \in H^0(X, \bigwedge^2 \mathcal{T}_X)$ a Poisson bivector. As is Section 2.3.1, it defines a holomorphic Lie algebroid structure on Ω_X that we denote by $(\Omega_X)_\Pi$.

According to [13] and [20], this Lie algebroid defines a generalized complex structure on X. It can be described as follows. Recall that a generalized complex structure on a smooth manifold M is defined by the $(+i)$-eigenbundle L of an endomorphism $\mathbb{J}$ of $(T_M \oplus T_M^*) \otimes \mathbb{C}$ satisfying $\mathbb{J}^2 = -1$ and $\mathbb{J}^* = -\mathbb{J}$. Since L is an isotropic subbundle of the Courant algebroid $T_M \oplus T_M^*$, the resriction of the Courant bracket of $T_M \oplus T_M^*$ to L defines a real Lie algebroid structure on it.

If E is a vector bundle on M, a *L-generalized holomorphic structure* on E is a flat L-connection on E.

When X is a holomorphic Poisson manifold with holomorphic Poisson bivector Π, we can define the following endomorphism of $T_X \oplus T_X^*$:

$$\mathbb{J}_{4\Pi} = \begin{pmatrix} -J & 4\sharp_I \\ 0 & J^* \end{pmatrix}$$

where J is the almost complex structure on X and $\sharp_I$ is the morphism $T_X^* \to T_X$ associated to the bivector Π_I, where $\Pi = \Pi_R + i\Pi_I$ for $\Pi_R, \Pi_I \in \Gamma(\bigwedge^2 T_X)$. It is easy to see that $\mathbb{J}_{4\Pi}$ defines a generalized complex structure on X, that we call $L_{4\Pi}$.

Remark that the elements of $L_{4\Pi}$ are of the form $(V + i4\sharp_I\xi, \xi)$ with $V \in T_X^{0,1}$ and $\xi \in T_X^{*1,0}$, which gives an isomorphism $L_{4\Pi} \cong T_X^{0,1} \oplus T_X^{*1,0}$. Moreover we have $L_{4\Pi}^* \cong T_X^{*0,1} \oplus T_X^{1,0}$, and the Lie algebroid differential on functions is $d_{L_{4\Pi}} f = \bar{\partial} f + \sharp_I(\partial f)$.

The following theorem is proved in [20]:

Theorem 43. *The Lie algebroid $L_{4\Pi}$ is isomorphic to $(\Omega_X)_\Pi \bowtie T_X^{0,1}$.*

By Proposition 13, flat $L_{4\Pi}$-connections on a smooth bundle E are equivalent to a holomorphic structure $\mathcal{E}$ on E and a flat $(\Omega_X)_\Pi$-connection on $\mathcal{E}$. For the latter we have constructed moduli spaces of semistable objects, so we have the following:

Corollary 44. *Let X be a smooth complex projective variety, and $\Pi \in H^0(X, \bigwedge^2 \mathcal{T}_X)$ an algebraic Poisson bivector on X inducing a generalized complex structure $L_{4\Pi}$ on X.*

Then for any numerical polynomial P, there exists a quasi-projective scheme $M_\Pi(P)$ parametrizing semistable $L_{4\Pi}$-generalized holomorphic vector bundles with Hilbert polynomial P.

Chapter 5

Deformation theory of $\mathcal{L}$-connections

In this chapter we adress some questions of deformation theory for integrable $\mathcal{L}$-connections: we find the DGLA associated to the deformations of an integrable $\mathcal{L}$-connection $(\mathcal{E}, \nabla)$, and give an application of Luna's slice theorem to study the local structure of the moduli spaces of integrable $\mathcal{L}$-connection.

5.1 Generalities on deformation functors

Following [14] and [29] we recall some facts about deformation theory.

Let k be an algebraically closed field of characteristi zero, and consider $\mathbf{Art}_k$, the category of local Artin k-algebras with residue field equal to k. For a local Artin k-algebra A, we denote by $\mathfrak{m}_A$ its (unique) maxiaml ideal.

Remark that there exist fibered products in $\mathbf{Art}_k$: if $B \to A$ and $C \to A$ are morphisms in $\mathbf{Art}_k$, then $B \times_A C$ is the k-algebra whose elements are pairs (b, c) such that b, c have the same image in A, and the multiplication is defined componentwise.

A *small extension* of artinian rings is a short exact sequence

$$0 \to I \to B \xrightarrow{\phi} A \to 0 \ ,$$

where ϕ is a morphism of local artinian rings, I is its kernel satisfying $I\mathfrak{m}_B = 0$. Remark that this implies that $I^2 = 0$, and so I inherits a natural structure of A-module. Moreover I is a finite dimensional vector space over $B/\mathfrak{m}_B = k$.

Now consider covariant functors $\mathbf{Art}_k \to \mathbf{Sets}$ on Artin rings. A particular class of these is given by $h_R = \mathrm{Hom}_k(R, \cdot)$ for $R \in \hat{\mathbf{Art}}_k$, where $\hat{\mathbf{Art}}_k$ denotes the category of local complete Noetherian k-algebras with residue field k. A functor of Artin rings F is said *pro-representable* if it is isomorphic to h_R for some $R \in \hat{\mathbf{Art}}_k$.

A functor of Artin rings F is said *homogeneous* if the natural map $F(B\times_A C) \to F(B) \times_{F(A)} F(C)$ is a bijection whenever $B \to A$ is surjective.

For the functors h_R, the maps $h_R(B \times_C A) \to h_R(B) \times_{h_R(A)} h_R(C)$ are always isomorphisms, so homogeneity is a necessary condition for a functor of Artin rings to be prorepresentable.

Now we are going to relax these hypothese to define a larger class of functors of Artin rings, that usually arise in deformation problems:

Definition 10. *A* deformation functor *is a functor*

$$F : \mathbf{Art}_k \to \mathbf{Sets}$$

such that $F(k) = \{pt.\}$ and for any two morphisms $B \to A$, $C \to A$ in **Art**, *the natural map $\eta : F(B \times_A C) \to F(B) \times_{F(A)} F(C)$ satisfies:*

- *η is surjective whenever $B \to A$ is surjective;*
- *if $A = k$, η is an isomorphism.*

We call the set $F(\mathbb{C}[\epsilon]/\epsilon^2)$ *tangent space* to F, and denote it by T^1_F. The second condition in the previous definition allows us to define a $\mathbb{C}$-vector space structure on the tangent space of a deformation functor: any $\lambda \in k$ defines the endomorphism of $k[\epsilon]/(\epsilon^2)$ given by the multiplication by λ; this induces a morphism $\lambda : T^1_F \to T^1_F$, that gives the structure of multiplication by scalat. Moreover, by the second axiom

$$T^1_F \times T^1_F \cong F(k[\epsilon_1]/(\epsilon_1^2) \times_k k[\epsilon_2]/(\epsilon_2^2)) \ ;$$

now, the morphism $k[\epsilon_1]/(\epsilon_1^2)\times_k k[\epsilon_2]/(\epsilon_2^2) \to k[\epsilon]/(\epsilon^2)$ given by $a+b\epsilon_1+c\epsilon_2 \mapsto a + (b + c)\epsilon$ defines the sum.

A functor of Artin rings $F : \mathbf{Art}_k \to \mathbf{Sets}$ can be extended to a functor $\hat{F} : \hat{\mathbf{Art}}_k \to \mathbf{Sets}$ by $\hat{F}(R) = \lim_{\leftarrow n} F(R/\mathfrak{m}_R^n)$. Now, to give an element $\xi \in \hat{F}(R)$ is equivalent to define a morphism of functors $\Theta : h_R \to F$: if we are given $h_R \to F$, we can complete it to a morphism of functors $\hat{h_R} \to \hat{F}$, and ξ is the image to the identity in $\hat{h_R}(R)$. If we are given $\xi \in \hat{F}(R)$, for any $A \in \mathbf{Art}_k$ there exists a positive integer n such that $\mathrm{Hom}(R, A) = \mathrm{Hom}(R/\mathfrak{m}_R^n, A)$; let $\xi_n \in F(R/\mathfrak{m}_R^n)$ be the image of ξ via the natural map $F(R) \to F(R/\mathfrak{m}_R^n)$. Then we define $\Theta_A : h_R(A) \to F(A)$ to be the map $f \mapsto F(f)(\xi_n)$.

So to a prorepresentable functor of Artin rings F we associate a pair (R, ξ), with $R \in \hat{\mathbf{Art}}_k$ and $\xi \in \hat{F}(R)$ inducing an isomorphism of functors $h_R \to F$. We call (R, ξ) a *universal family* for the functor F. If there exist a universal family for a functor F, then this is unique up to unique isomorphism.

We call *versal family* for a functor F a pair (R, ξ) with $R \in \hat{\mathbf{Art}}_k$ and $\xi \in \hat{F}(R)$, such that the associated morphism $h_R \to F$ is surjective and such that for any surjection $B \to A$ in $\mathbf{Art}_k$, the corresponding morphism $\mathrm{Hom}(R, B) \to \mathrm{Hom}(R, A) \times_{F(A)} F(B)$ is surjective. If moreover the morphism between the tangent spaces $T^1_{h_R} \to T^1_F$ is bijective, we call (R, ξ) a *miniversal family.*

If (R, ξ) and (R', ξ') are (mini)versal families of the same functor F, then one can construct an isomorphism between the two (that is an isomorphism of the algebras $R \to R'$ that induces a commuting diagram with the morphisms to F), but in general this isomorphism is not unique.

It is a theorem of Schlessinger [33] that a deformation functor with finite dimensional tangent space admits a miniversal family (R, ξ).

Let X be a Notherian scheme over k, and consider the functor of points $\underline{X} : \mathbf{Schemes} \to \mathbf{Sets}$ that associates to any scheme S the set of morphisms of schemes $\mathrm{Mor}(S, X)$. For any closed point $x \in X$, we have a functor of Artin rings $\mathfrak{Def}_x$ that to an Artin algebra $A \in \mathbf{Art}_k$ associates the morphism from $\mathrm{SPEC}A$ to X that send the closed point of $\mathrm{SPEC}A$ to x. Clearly this functor is pro-represented by the local complete Noetherian algebra $\hat{\mathscr{O}}_{X,x}$.

Actually, we can do the same construction for any contravariant functor $F : \mathbf{Schemes}/k \to \mathbf{Sets}$: given $x \in F(\mathrm{SPEC}\, k)$, we can define $\mathfrak{Def}_x : \mathbf{Art} \to \mathbf{Sets}$ as the functor that associates to A the subset $\mathfrak{Def}_x(A) \subseteq F(\mathrm{SPEC}\, A)$ of elements η that are mapped to x by the map $F(\mathrm{SPEC}\, A) \to F(\mathrm{SPEC}\, k)$. Clearly, if F is represented by a scheme X, the two deformation functors coincide, so we see that a necessary condition for a functor to be representable is that the functors $\mathfrak{Def}_x$ are prorepresentable, for any $x \in F(\mathrm{SPEC}\, k)$. The converse is not true, since there are functors of schemes that are not representable but whose associated deformation functors are prorepresentable.

The tangent of the deformation functor $\mathfrak{Def}_x$ associated toa closed point of a scheme X coincide with the fiber of the tangent sheaf over x: $T^1_{\mathfrak{Def}_x} = T_{X,x} \otimes_{\mathscr{O}_{X,x}} k(x)$.

An *obstruction theory* for a deformation functor D is a pair (W, obs), where W is a finite dimensional vector space and obs is a functor associating for any small extension in $\mathbf{Art}_k$

$$e : \quad 0 \to I \to B \to A \to 0$$

a map $\mathrm{obs}(e) : D(A) \to W \otimes I$ such that:

- if for $x \in D(A)$ there exists a $y \in D(B)$ suhc that $y \mapsto x$ via the natural map $D(B) \to D(A)$, then $\mathrm{obs}(x) = 0$;
- the assignment $e \mapsto \mathrm{obs}(e)$ is functorial in the following sense: for any morpihsm $\phi : e \to e'$ of small extensions, we have the commutativity

of the diagram

$$\begin{tikzcd} D(A) \arrow[r, "\mathrm{obs}(e)"] \arrow[d, "\phi"] & W \otimes I \arrow[d, "\mathbf{1}_W \otimes \phi"] \\ D(A') \arrow[r, "\mathrm{obs}(e')"] & W \otimes I' \end{tikzcd}$$

Remark that with this definition $(0,0)$ is always an obstruction theory for any deformation functor. An obstruction theory is said *complete* when $\mathrm{obs}(e)(x) = 0$ if and only if there exist a lifting of x to B. A deformation functor D is said *smooth* when $(0,0)$ is a complete obstruction theory for D, i. e. when $D(B) \to D(A)$ is surjective for every surjective morphism $B \to A$.

One can check that the prorepresentable functors h_R are smooth if and only if R is the algebra of formal power series $k[[x_1, \ldots, x_n]]$. From this follows that the functors $\mathfrak{Def}_x$ are smooth if and only if x is a smooth point of X.

5.2 Deformation of $\mathcal{L}$-connections

Fix a smooth complex projective variety X and $\mathcal{L}$ a holomorphic Lie algebroid on X. We study deformation theory for flat holomorphic $\mathcal{L}$-connections. For $A \in \mathbf{Art}_{\mathbb{C}}$, let $X_A = \mathrm{Spec}\, A \times X$ and $p_X : X_A \to X$ the projection. Let $X_0 = \{pt\} \times X$, with pt the closed point of $\mathrm{Spec}\, A$; we have $x \cong X_0 \hookrightarrow X_A$. As $\mathbb{C}$-vector spaces, we have $A = \mathbb{C} \oplus \mathfrak{m}_A$, and $\mathfrak{m}_A$ is finite dimensional. Then $\mathscr{O}_{X_A} = \mathscr{O}_X \otimes A = \mathscr{O}_X \oplus (\mathscr{O}_X \otimes \mathfrak{m}_A)$.

Let $(\mathcal{E}, \nabla)$ be a flat $\mathcal{L}$-connection with $\mathcal{E}$ a vector bundle of rank r, and consider the functor of Artin rings $\mathfrak{Def}_{(\mathcal{E},\nabla)}$ that associates to any $A \in \mathbf{Art}_{\mathbb{C}}$ the set of isomorphism classes of flat $p_X^*\mathcal{L}$-connections $(\mathcal{E}', \nabla')$ on X_A such that the restriction of $(\mathcal{E}', \nabla')$ to $\mathfrak{m}_A \times X$ is isomorphic to $(\mathcal{E}, \nabla)$.

We know that if $(\mathcal{E}, \nabla)$ is stable the functor $\mathfrak{Def}_{(\mathcal{E},\nabla)}$ is prorepresentable, since the functor for stable $\mathcal{L}$-connection is representable. More generally, from [6] we know that $\mathfrak{Def}_{(\mathcal{E},\nabla)}$ is prorepresentable for $(\mathcal{E}, \nabla)$ simple.

Let $(\mathcal{E}, \nabla)$ be a flat $\mathcal{L}$-connection. Fix an open cover $\mathfrak{U} = \{U_\alpha\}$ of X and trivializations $\psi_\alpha : U_\alpha \times \mathbb{C}^r \to \mathcal{E}_{|U_\alpha}$ of $\mathcal{E}$ over U_α, and let $e^{(\alpha)} = (e_1^{(\alpha)}, \ldots, e_r^{(\alpha)})$ be the corresponding frame of $\mathcal{E}$ over U_α. Let $g_{\alpha\beta} \in \mathscr{O}_X(U_{\alpha\beta}) \times \mathrm{GL}_r$ be transition functions for $\mathcal{E}$, defined by $g_{\alpha\beta} = \psi_\beta^{-1} \circ \psi_\alpha$.

Let $A \in \mathbf{Art}_{\mathbb{C}}$ and $(\mathcal{E}', \nabla')$ be a deformation of $(\mathcal{E}, \nabla)$ over X_A. If $g'_{\alpha\beta} \in \mathscr{O}_{X_A}(U_{A,\alpha\beta})$ are transition functions for $\mathcal{E}'$ relatively to the open cover $U_{A,\alpha} = \mathrm{Spec}\ A \times U_\alpha$, we can write $g'_{\alpha\beta} = g_{\alpha\beta} + g^A_{\alpha\beta}$, with $g^A_{\alpha\beta} \in \mathscr{O}_X(U_{\alpha\beta}) \otimes \mathfrak{m}_A \otimes \mathfrak{gl}_r$. Since $\mathcal{E}'$ is a vector bundle, the following equations have to be satisfied:

$$g'_{\alpha\beta}g'_{\beta\alpha} = \mathbf{1} \ , \quad g'_{\alpha\beta}g'_{\beta\gamma}g'_{\gamma\alpha} = \mathbf{1} \ . \tag{5.1}$$

Relatively to the frames $e^{(\alpha)}$, let $G^j_{\alpha,i} \in \Omega_{\mathcal{L}}(U_\alpha)$ be such that $\nabla e^{(\alpha)}_i = \sum_j G^j_{\alpha,i} e^{(\alpha)}_j$. Then we can see the matrix G_α as an element of $\Omega_{\mathcal{L}}(U_\alpha) \otimes \mathfrak{gl}_r$, and write formally $\nabla = \mathrm{d}_{\mathcal{L}} + G_\alpha$. The gauge transformation is the same as for a usual connection, namely on the double overlaps $U_{\alpha\beta}$ we have

$$G_\beta = g_{\beta\alpha}G_\alpha g_{\alpha\beta} + g_{\beta\alpha}\mathrm{d}_{\mathcal{L}} g_{\alpha\beta} \ . \tag{5.2}$$

Simliarly, for ∇' we have $\nabla' = \mathrm{d}_{\mathcal{L}} + G'_\alpha$, and since it is a deformation of ∇, we can write $G'_\alpha = G_\alpha + G^A_\alpha$.

Finally recall that the integrability equation for ∇ translates, in terms of the local 1-$\mathcal{L}$-form of connection G_α, as

$$G_\alpha \wedge G_\alpha + \mathrm{d}_{\mathcal{L}} G_\alpha = 0 \ . \tag{5.3}$$

5.2.1 First order deformations

First order deformations of $(\mathcal{E}, \nabla)$ are deformations $(\mathcal{E}', \nabla')$ over X_A for $A = \mathbb{C}[\epsilon]/(\epsilon^2)$. In this case $\mathfrak{m}_A = \mathbb{C} \cdot \epsilon$, so it is one dimensional.

Then in this case

$$g'_{\alpha\beta} = g_{\alpha\beta} + \epsilon g^1_{\alpha\beta} \ , \quad G'_\alpha = G_\alpha + \epsilon G^1_\alpha$$

with $g^1_{\alpha\beta} \in \mathscr{O}_X(U_{\alpha\beta}) \times \mathfrak{gl}_r$ and $G^1_\alpha \in \Omega_{\mathcal{L}}(U_\alpha) \otimes \mathfrak{gl}_r$.

Define sections $c_{\alpha\beta}$ of $\mathscr{E}nd\mathcal{E}$ over $U_{\alpha\beta}$ as follows: for any endomorphism θ of $\mathcal{E}$ we denote by $(\theta)^{(\alpha)}$ the matrix associated to θ w. r. t. the frame $e^{(\alpha)}_i$, i. e. $\theta(e^{(\alpha)}_i) = \sum_j (\theta^{(\alpha)})^j_i e^{(\alpha)}_j$; then $c_{\alpha\beta}$ is defined by $(c_{\alpha\beta})^{(\alpha)} = g^1_{\alpha\beta} g_{\beta\alpha}$ (so that, respectively, we have $(c_{\alpha\beta})^{(\beta)} = g_{\beta\alpha} g^1_{\alpha\beta}$). We look at the collection $\{c_{\alpha\beta}\}$ as a 1-Čech cocycle of $\mathscr{E}nd\mathcal{E}$.

The first of the equations (5.1) implies that $g^1_{\beta\alpha} = -g_{\beta\alpha} g^1_{\alpha\beta} g_{\beta\alpha}$; in terms of $c_{\alpha\beta}$ this gives

$$(c_{\beta\alpha})^{(\alpha)} = g_{\alpha\beta} g^1_{\beta\alpha} = -g^1_{\alpha\beta} g_{\beta\alpha} = -(c_{\alpha\beta})^{(\alpha)}$$

i.e. $c_{\beta\alpha} = -c_{\alpha\beta}$.

The second of the equations (5.1) implies

$$g^1_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} + g_{\alpha\beta} g^1_{\beta\gamma} g_{\gamma\alpha} + g_{\alpha\beta} g_{\beta\gamma} g^1_{\gamma\alpha} \ .$$

Now,

$$\begin{aligned} g^1_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} &= g^1_{\alpha\beta} g_{\beta\alpha} = (c_{\alpha\beta})^{(\alpha)} \ , \\ g_{\alpha\beta} g^1_{\beta\gamma} g_{\gamma\alpha} &= g_{\alpha\beta} g^1_{\beta\gamma} g_{\gamma\beta} g_{\beta\alpha} = (c_{\beta\gamma})^{(\alpha)} \ , \\ g_{\alpha\beta} g_{\beta\gamma} g^1_{\gamma\alpha} &= g_{\alpha\gamma} g^1_{\gamma\alpha} = (c_{\gamma\alpha})^{(\alpha)} \ , \end{aligned}$$

so that $c_{\alpha\beta} + c_{\beta\gamma} + c_{\gamma\alpha} = 0$, i. e. $c_{\alpha\beta}$ is a closed Čech 1-cocycle.

Similarly to the definition of $c_{\alpha\beta}$, let C_α be the section over U_α of $\mathscr{E}nd\mathcal{E} \otimes \Omega_{\mathcal{L}}$ defined by $(C_\alpha)^{(\alpha)} = G^1_\alpha$.

The equation (5.2) implies that

$$\begin{aligned} G^1_\beta = \ & g^1_{\beta\alpha} G_\alpha g_{\alpha\beta} + g_{\beta\alpha} G^1_\alpha g_{\alpha\beta} + g_{\beta\alpha} G_\alpha g^1_{\alpha\beta} + \\ & + g^1_{\beta\alpha} \mathrm{d}_{\mathcal{L}} g_{\alpha\beta} + g_{\beta\alpha} \mathrm{d}_{\mathcal{L}} g^1_{\alpha\beta} \ , \end{aligned} \tag{5.4}$$

that we can rewrite as

$$\begin{aligned} (C_\beta - C_\alpha)^{(\alpha)} &= g_{\alpha\beta} g^1_{\beta\alpha} G_\alpha + G_\alpha g^1_{\alpha\beta} g_{\beta\alpha} + \\ & \quad + g_{\alpha\beta} g^1_{\beta\alpha} (\mathrm{d}_{\mathcal{L}} g_{\alpha\beta}) g_{\beta\alpha} + (\mathrm{d}_{\mathcal{L}} g^1_{\alpha\beta}) g_{\beta\alpha} = \\ &= (c_{\beta\alpha})^{(\alpha)} (C_\alpha)^{(\alpha)} + (C_\alpha)^{(\alpha)} (c_{\alpha\beta})^{(\alpha)} + (\mathrm{d}_{\mathcal{L}} c_{\alpha\beta})^{(\alpha)} \end{aligned}$$

where we used

$$\begin{aligned} (\mathrm{d}_{\mathcal{L}} c_{\alpha\beta})^{(\alpha)} = \mathrm{d}_{\mathcal{L}} (g^1_{\alpha\beta} g_{\beta\alpha}) &= g^1_{\alpha\beta} \mathrm{d}_{\mathcal{L}} g_{\beta\alpha} + (\mathrm{d}_{\mathcal{L}} g^1_{\alpha\beta}) g_{\beta\alpha} = \\ &= g^1_{\alpha\beta} g_{\beta\alpha} (\mathrm{d}_{\mathcal{L}} g_{\alpha\beta}) g_{\beta\alpha} + (\mathrm{d}_{\mathcal{L}} g^1_{\alpha\beta}) g_{\beta\alpha} \ . \end{aligned}$$

So, the Čech cocycle C satisfies

$$(\check{\delta} C)_{\alpha\beta} = [G_\alpha, c_{\alpha\beta}] + \mathrm{d}_{\mathcal{L}} c_{\alpha\beta} \ . \tag{5.5}$$

Finally, from the integrability condition (5.3) the cocycles C_α have to satisfy

$$C_\alpha \wedge G_\alpha + G_\alpha \wedge C_\alpha + \mathrm{d}_{\mathcal{L}} C_\alpha = 0 \ . \tag{5.6}$$

So we have:

Proposition 45. *Cochains $c \in \check{C}^1(\mathfrak{U}, \mathscr{E}nd\mathcal{E})$ and $C \in \check{C}^0(\mathfrak{U}, \mathscr{E}nd\mathcal{E} \otimes \Omega_{\mathcal{L}})$ define a first order deformation of the flat $\mathcal{L}$-connection $(\mathcal{E}, \nabla)$ given if and only if the following are satisfied:*

$$(\check{\delta} c)_{\alpha\beta\gamma} = 0 \ , \tag{5.7}$$

$$(\check{\delta} C)_{\alpha\beta} = [A_\alpha, c_{\alpha\beta}] + d_{\mathcal{L}} c_{\alpha\beta} \ , \tag{5.8}$$

$$d_{\mathcal{L}} C_\alpha + G_\alpha \wedge C_\alpha + C_\alpha \wedge G_\alpha = 0 \ . \tag{5.9}$$

We can rewrite these equations in a more invariant form: let $\tilde{\nabla}$ be the $\mathcal{L}$-connection on $\mathscr{E}nd\mathcal{E}$ induced by ∇. Recall that it satisfies $\tilde{\nabla}\phi = \nabla \circ \phi - \phi \circ \nabla$ as endomorphism of $\mathcal{E}$. If $e_i^{(\alpha)}$ is a local frame of $\mathcal{E}$, the matrix associated to this endomorphism is given by $(\tilde{\nabla}\phi)^{(\alpha)} = [A_\alpha, \phi^{(\alpha)}] + \mathrm{d}_{\mathcal{L}} \phi^{(\alpha)}$. So we can rewrite the equations (5.7), (5.8) and (5.9) as

$$\check{\delta} c = 0 \quad \check{\delta} C = \tilde{\nabla} c \ , \quad \tilde{\nabla} C = 0 \ .$$

Now, since ∇ is flat, $\tilde{\nabla}$ is flat too, so we have the complex of holomorphic bundles

$$\mathcal{C}^\bullet \ : \ \mathscr{E}nd\mathcal{E} \to \mathscr{E}nd\mathcal{E} \otimes \Omega_{\mathcal{L}} \to \mathscr{E}nd\mathcal{E} \otimes \Omega^2_{\mathcal{L}} \to \dots \ . \tag{5.10}$$

Consider its hypercohomology groups $\mathbb{H}^p(X;\mathcal{C}^\bullet)$. If $\mathfrak{U}$ is a sufficiently nice open cover of X, they are computed as the cohomology of the total complex associated to the double complex

$$K^{p,q} = \check{C}^p(\mathfrak{U}, \mathscr{E}nd\mathcal{E} \otimes \Omega^q_{\mathcal{L}}) \ ,$$

with differentials $\check{\delta} : K^{p,q} \to K^{p+1,q}$ and $\tilde{\nabla} : K^{p,q} \to K^{p,q+1}$. In particular, elements of $\mathbb{H}^1(X;\mathcal{C}^\bullet)$ are cohomology class of pairs $(s_{\alpha\beta}, t_\alpha)$ that are closed, i. e. satisfy the equations

$$\check{\delta}s = 0, \quad \tilde{\nabla}s = \check{\delta}t, \quad \tilde{\nabla}t = 0.$$

So we have:

Proposition 46. *To any* 1*st order deformation* $(\mathcal{E}', \nabla')$ *of a flat* $\mathcal{L}$*-connection* $(\mathcal{E}, \nabla)$ *we can associate a hypercohomology class* $[(c, C)] \in \mathbb{H}^1(X;\mathcal{C}^\bullet)$*, where* $\mathcal{C}^\bullet$ *is the complex defined by* (5.10)*.*

Recall that two first order deformation $(\mathcal{E}', \nabla'), (\mathcal{E}'', \nabla'')$ of $(\mathcal{E}, \nabla)$ are equivalent when there exist an isomorphism of flat $\mathcal{L}$-connections $\Theta : (\mathcal{E}', \nabla') \to (\mathcal{E}'', \nabla'')$ such that its restriction to $\mathcal{E}'_{|X_0}$ is equal to the identity of $(\mathcal{E}, \nabla)$.

If the deformations are given by the local data

$$g'_{\alpha\beta} = g_{\alpha\beta} + \epsilon g^1_{\alpha\beta}, \quad G'_\alpha = G_\alpha + \epsilon G^1_\alpha$$
$$g''_{\alpha\beta} = g_{\alpha\beta} + \epsilon \tilde{g}^1_{\alpha\beta}, \quad G''_\alpha = G_\alpha + \epsilon \tilde{G}^1_\alpha,$$

they are equivalent if there exist functions $\xi_\alpha : U_\alpha \to \mathrm{GL}_r$ such that on $U_{\alpha\beta}$

$$g''_{\alpha\beta}(g'_{\alpha\beta})^{-1} = \xi_\beta \xi_\alpha^{-1} \tag{5.11}$$

and on U_α

$$G''_\alpha = \xi_\alpha^{-1} G'_\alpha \xi_\alpha + \xi_\alpha^{-1} \mathrm{d}_{\mathcal{L}} \xi_\alpha \ . \tag{5.12}$$

Writing $\xi_\alpha = \mathbf{1} + \epsilon\eta_\alpha$, we see that equation (5.11) becomes

$$g_{\alpha\beta} g^1_{\beta\alpha} + \tilde{g}^1_{\alpha\beta} g_{\beta\alpha} = \eta_\beta - \eta_\alpha \ .$$

Defining the 1-Čech cochains $c_{\alpha\beta}$, $\tilde{c}_{\alpha\beta}$ as above, we see that this is equivalent to

$$\tilde{c}_{\alpha\beta} - c_{\alpha\beta} = (\check{\delta}\eta)_{\alpha\beta} \ .$$

Similarly, equation (5.12) is equivalent to

$$\tilde{G}^1_\alpha = G^1_\alpha - \eta_\alpha G_\alpha + G_\alpha \eta_\alpha + \mathrm{d}_{\mathcal{L}} \eta_\alpha \ .$$

Defining the 0-Čech cocycles C, $\tilde{C}$ as above, this implies

$$\tilde{C}_\alpha - C_\alpha = [G_\alpha, \eta_\alpha] + \mathrm{d}_{\mathcal{L}} \eta_\alpha = \tilde{\nabla}\eta_\alpha \ .$$

These equations imply that if $(\mathcal{E}', \nabla')$ and $(\mathcal{E}'', \nabla'')$ are two equivalent deformations of $(\mathcal{E}, \nabla)$, then the associated cocycles (c, C), $(\tilde{c}, \tilde{C})$ differ by $(\check{\delta}\eta, \tilde{\nabla}\eta) = \delta(\eta)$. So we have shown

Theorem 47. *Let X be a smooth projective variety, $\mathcal{L}$ a holomorphic Lie algebroid and $(\mathcal{E}, \nabla)$ a flat $\mathcal{L}$-connection, with $\mathcal{E}$ locally free.*

Then the equivalence classes of first order deformations of $(\mathcal{E}, \nabla)$ are in one to one correspondence with the first hypercohomology group $\mathbb{H}^1(X; \mathscr{E}nd\mathcal{E} \otimes \Omega^\bullet_{\mathcal{L}})$.

In particular, the tangent space of the deformation functor $\mathfrak{Def}_{(\mathcal{E},\nabla)}$ is $\mathbb{H}^1(X; \mathscr{E}nd\mathcal{E} \otimes \Omega^\bullet_{\mathcal{L}})$.

5.2.2 Cup product in hypercohomology

Recall the construction of cup product in cohomology: if E and F are two sheaves over the same variety X, and $\mathfrak{U}$ is any open cover of X, we have a map

$$\smile : \check{C}^p(\mathfrak{U}, E) \otimes \check{C}(\mathfrak{U}, F) \to \check{C}^{p+q}(\mathfrak{U}, E \otimes F)$$

given by

$$(\xi \smile \eta)_{i_0,\dots,i_{p+q}} = (-1)^q \xi_{i_0,\dots,i_p} \otimes \eta_{i_p,\dots,i_{p+q}} \ .$$

for $\xi \in \check{C}^p(\mathfrak{U}, E)$ and $\eta \in \check{C}^q(\mathfrak{U}, F)$. The Čech differential satisfies a graded Leibniz rule $\check{\delta}(\xi \smile \eta) = \check{\delta}\xi \smile \eta + (-1)^p \xi \smile \check{\delta}\eta$, so $\smile$ defines a map in cohomology

$$\smile : \check{H}^p(\mathfrak{U}; E) \otimes H^q(\mathfrak{U}; F) \to \check{H}^{p+q}(\mathfrak{U}; E \otimes F) \ .$$

In the case $F = E$ and E has a ring structure, we can compose $\smile$ with the product of E, and we have a ring structure over the cohomology $\check{H}^\bullet(\mathfrak{U}; E)$.

If we replace E and F by complexes of sheaves we have a similar construction: let $(E^\bullet, d_E)$ and $(F^\bullet, d_F)$ be two complexes of sheaves over X. We can build the complex $(E \otimes F)^\bullet$ given by $(E \otimes F)^k = \bigoplus_{p+q=k} E^p \otimes F^q$ and differential $d_{E \otimes F} = d_E \otimes \mathbf{1}_F + \mathbf{1}_E \otimes d_F$. Define the map

$$\smile : \check{C}^p(\mathfrak{U}; E^r) \otimes \check{C}^q(\mathfrak{U}; F^s) \to \check{C}^{p+q}(\mathfrak{U}; (E \otimes F)^{r+s})$$

by

$$(\xi \smile \eta)_{i_0,\dots,i_{p+q}} = (-1)^{qr} \xi_{i_0,\dots,i_p} \otimes \eta_{i_p,\dots,i_{p+q}}$$

This induces a map at the hypercohomology level

$$\smile : \mathbb{H}^i(X; E^\bullet) \otimes \mathbb{H}^j(X; F^\bullet) \to \mathbb{H}^{i+j}(X; (E \otimes F)^\bullet).$$

In the case $E^\bullet = F^\bullet$ has a graded ring structure, i. e. there are maps

$$E^r \otimes E^s \to E^{r+s}$$

making $E^\bullet$ a graded ring, we can compose $\smile$ with the product of $E^\bullet$, and obtain maps

$$\mathbb{H}^i(X; E^\bullet) \otimes \mathbb{H}^j(X; E^\bullet) \to \mathbb{H}^{i+j}(X; E^\bullet) \ .$$

For $\mathcal{L}$ a Lie algberoid over a smooth projective variety X, and $\mathcal{E}$ a coherent $\mathscr{O}_X$-module, the complex $\mathcal{C}^\bullet = \mathscr{E}nd\mathcal{E} \otimes \Omega^\bullet_{\mathcal{L}}$ has a natural graded ring structure, given by

$$(\theta \otimes \mu)(\sigma \otimes \nu) = (\theta \circ \sigma) \otimes (\mu \wedge \nu) \ .$$

As before, this induces a cup product in hypercohomology:

$$\smile : \mathbb{H}^i(X; \mathcal{C}^\bullet) \otimes \mathbb{H}^j(X; \mathcal{C}^\bullet) \to \mathbb{H}^{i+j}(X; \mathcal{C}^\bullet) \ .$$

Since we are going to use it soon, we write explicitly the $\smile$-product for elements of the first hypercohomology group of $\mathcal{C}^\bullet$. Let $(a, A), (b, B)$ be two cocycles representing two elements $[a, A][b, B] \in \mathbb{H}^1(X; \mathcal{C}^\bullet)$. Let $(a, A) \smile (b, B) = (f, F, \Phi)$. Then

$$\begin{aligned} f_{\alpha\beta\gamma} &= a_{\alpha\beta} \circ b_{\beta\gamma} \ , \\ F_{\alpha\beta} &= A_\alpha \circ b_{\alpha\beta} - a_{\alpha\beta} \circ B_\beta \ , \\ \Phi_\alpha &= A_\alpha \wedge B_\alpha \ . \end{aligned}$$

5.2.3 First obstruction

Let $(\mathcal{E}', \nabla')$ be a deformation of $(\mathcal{E}, \nabla)$ over X_A for $A = \mathbb{C}[\epsilon]/(\epsilon^3)$. We can write the local data of $(\mathcal{E}', \nabla')$ as

$$g'_{\alpha\beta} = g_{\alpha\beta} + \epsilon g^1_{\alpha\beta} + \epsilon^2 g^2_{\alpha\beta} \ ,$$

$$G'_\alpha = G_\alpha + \epsilon G^1_\alpha + \epsilon^2 G^2_\alpha$$

for $g^i_{\alpha\beta} \in \mathscr{O}_X(U_{\alpha\beta}) \otimes \mathfrak{gl}_r$ and $G^i \in \Omega_{\mathcal{L}}(U_\alpha) \otimes \mathfrak{gl}_r$.

As before, for $i = 1, 2$ introduce $c^i_{\alpha\beta}$, the 1-Čech cochains of $\mathscr{E}nd\mathcal{E}$ given by $(c^1_{\alpha\beta})^{(\alpha)} = g^i_{\alpha\beta} g_{\beta\alpha}$, and C^i_α, the 0-Čech cochains of $\mathscr{E}nd\mathcal{E} \otimes \Omega_{\mathcal{L}}$ given by $(C^i_\alpha)^{(\alpha)} = G^i_\alpha$.

By the previous subsection we know that $\delta(c^1, C^1) = (\check{\delta} c^1, \check{\delta} C^1 - \tilde{\nabla} c^1, \tilde{\nabla} C^1 = 0$.

Now, the term in ϵ^2 of the first of the equations (5.1) is

$$g^2_{\alpha\beta} g_{\beta\alpha} + g^1_{\alpha\beta} g^1_{\beta\alpha} + g_{\alpha\beta} g^2_{\beta\alpha} = 0 \ ,$$

that gives the equation

$$c^2_{\beta\alpha} = -c^2_{\alpha\beta} - c^1_{\alpha\beta} c^1_{\beta\alpha} \ . \tag{5.13}$$

The term in ϵ^2 of equation (5.1) is

$$\begin{aligned} 0 \ &= \ g^2_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} + g_{\alpha\beta} g^2_{\beta\gamma} g_{\gamma\alpha} + g_{\alpha\beta} g_{\beta\gamma} g^2_{\gamma\alpha} + \\ &+ \ g^1_{\alpha\beta} g^1_{\beta\gamma} g_{\gamma\alpha} + g^1_{\alpha\beta} g_{\beta\gamma} g^1_{\gamma\alpha} + g_{\alpha\beta} g^1_{\beta\gamma} g^1_{\gamma\alpha} \ . \end{aligned} \tag{5.14}$$

A quick computation shows that the right hand side of the equation is the matrix w. r. t. the frame $e^{(\alpha)}$ of the endomorphism

$$c^2_{\alpha\beta} + c^2_{\beta\gamma} + c^2_{\gamma\alpha} + c^1_{\alpha\beta}c^1_{\beta\gamma} + c^1_{\beta\gamma}c^1_{\gamma\alpha} + c^1_{\gamma\alpha}c^1_{\alpha\beta} \ .$$

Since $(\check{\delta}c^2)_{\alpha\beta\gamma} = c^2_{\alpha\beta} + c^2_{\beta\gamma} - c^2_{\alpha\gamma}$, using the equation (5.13) we see that equation (5.14) implies

$$(\check{\delta}c^2)_{\alpha\beta\gamma} = -c^1_{\alpha\beta}c^1_{\beta\gamma} - (c^1_{\alpha\beta} + c^1_{\beta\gamma})c^1_{\gamma\alpha} + c^1_{\alpha\gamma}c^1_{\gamma\alpha}$$

and since $\check{\delta}c^1 = 0$ we obtain

$$(\check{\delta}c^2)_{\alpha\beta\gamma} = (c^1 \smile c^1)_{\alpha\beta\gamma} \ .$$

Remark that the left hand side of the equation is the $(2,0)$ part of $\delta(c^2, C^2)$, while the right hand side is equal to the $(2,0)$-part of the cup product $(c^1, C^1) \smile (c^1, C^1)$.

The term in ϵ^2 of the equation (5.2) is

$$\begin{array}{rcl} G^2_\beta &=& g_{\beta\alpha}G^2_\alpha g_{\alpha\beta} + g^2_{\beta\alpha}G_\alpha g_{\alpha\beta} + g_{\beta\alpha}G_\alpha g^2_{\alpha\beta} + g^1_{\beta\alpha}G^1_\alpha g_{\alpha\beta} + g^1_{\beta\alpha}G_\alpha g^1_{\alpha\beta} + \\ &+& g_{\beta\alpha}G^1_\alpha g^1_{\alpha\beta} + g^2_{\beta\alpha}\mathrm{d}_{\mathcal{L}}g_{\alpha\beta} + g_{\beta\alpha}\mathrm{d}_{\mathcal{L}}g^2_{\alpha\beta} + g^1_{\beta\alpha}\mathrm{d}_{\mathcal{L}}g^1_{\alpha\beta} \ . \end{array}$$

Adjoining both terms of the equation by $g_{\alpha\beta}$, we see that this is the equation of the matrices w. r. t. the frame $e^{(\alpha)}$ of the cochains

$$\begin{array}{rcl} C^2_\beta - C^2_\alpha &=& c^2_{\beta\alpha}G_\alpha + G_\alpha c^2_{\alpha\beta} + c^1_{\beta\alpha}C^1_\alpha + c^1_{\beta\alpha}G_\alpha c^1_{\alpha\beta} + \\ &+& C^1_\alpha c^1_{\alpha\beta} + \mathrm{d}_{\mathcal{L}}c^2_{\alpha\beta} - c^1_{\alpha\beta}\mathrm{d}_{\mathcal{L}}c^1_{\alpha\beta} \end{array} \tag{5.15}$$

where we have used

$$\begin{array}{rl} & (\mathrm{d}_{\mathcal{L}}c^2_{\alpha\beta} - c^1_{\alpha\beta}\mathrm{d}_{\mathcal{L}}c^1_{\alpha\beta})^{(\alpha)} = \\ = & \mathrm{d}_{\mathcal{L}}g^2_{\alpha\beta}g_{\beta\alpha} + (-g^2_{\alpha\beta}g_{\beta\alpha} + g^1_{\alpha\beta}g_{\beta\alpha}g^1_{\alpha\beta}g_{\beta\alpha})(\mathrm{d}_{\mathcal{L}}g_{\alpha\beta})g_{\beta\alpha} + \\ & -g^1_{\alpha\beta}g_{\beta\alpha}(\mathrm{d}_{\mathcal{L}}g^1_{\alpha\beta})g_{\beta\alpha} = \\ = & \mathrm{d}_{\mathcal{L}}g^2_{\alpha\beta}g_{\beta\alpha} + g_{\alpha\beta}g^2_{\beta\alpha}(\mathrm{d}_{\mathcal{L}}g_{\alpha\beta})g_{\beta\alpha} + g_{\alpha\beta}g^1_{\beta\alpha}(\mathrm{d}_{\mathcal{L}}g^1_{\alpha\beta})g_{\beta\alpha} \ . \end{array}$$

Now, using equation (5.13) and recalling the definition of $\tilde{\nabla}$, we can rewrite equation (5.15) as

$$(\check{\delta}C^2)_{\alpha\beta} - \tilde{\nabla}c^2_{\alpha\beta} = -c^1_{\alpha\beta}\tilde{\nabla}c^1_{\alpha\beta} + [C^1_\alpha, c^1_{\alpha\beta}] \ .$$

Finally, since $\tilde{\nabla}c^1_{\alpha\beta} = (\check{\delta}C^1)_{\alpha\beta}$, this is equivalent to

$$(\check{\delta}C^2)_{\alpha\beta} - \tilde{\nabla}c^2_{\alpha\beta} = C^1_\alpha c^1_{\alpha\beta} - c^1_{\alpha\beta}C^1_\beta$$

where the left hand side of the equation is the $(1,1)$-part of $\delta(c^2, C^2)$, while the right hand side is equal to the $(1,1)$-part of the cup product $(c^1, C^1) \smile (c^1, C^1)$.

Finally the ϵ^2 term of the equation (5.3) is

$$0 = G^2_\alpha \wedge G_\alpha + G^1_\alpha \wedge G^1_\alpha + G_\alpha \wedge G^2_\alpha + \mathrm{d}_{\mathcal{L}} G^2_\alpha \ .$$

that we rewrite as

$$\tilde{\nabla} C^2_\alpha = C^1_\alpha \wedge C^1_\alpha \ .$$

Since the left hand side of the equation is the $(0,2)$-part of $\delta(c^2, C^2)$ and the right hand side is the $(0,2)$-part of the cup product $(c^1, C^1) \smile (c^1, C^1)$, we have shown

Proposition 48. *Let $(\mathcal{E}', \nabla')$ be a second order deformation of the flat $\mathcal{L}$-connection $(\mathcal{E}, \nabla)$ as above.*

Then $\delta(a^2, \mathcal{A}^2) = (a^1, \mathcal{A}^1) \smile (a^1, \mathcal{A}^1)$.

We can interpret this result in terms of obstruction for the deformation functor $\mathfrak{Def}_{(\mathcal{E},\nabla)}$: consider the small extension in $\mathbf{Art}_{\mathbb{C}}$

$$e_{1,2} \ : \ 0 \to \mathbb{C} \to \mathbb{C}[\epsilon]/(\epsilon^3) \to \mathbb{C}[\epsilon]/(\epsilon^2) \to 0 \ .$$

A k-th order deformation of $(\mathcal{E}, \nabla)$ is an element of $\mathfrak{Def}_{(\mathcal{E},\nabla)}(\mathbb{C}[\epsilon]/(\epsilon^{k+1}))$.

Let $(\mathcal{E}', \nabla')$ be a first order deformation given by the cohomology class $[c^1, C^1] \in \mathbb{H}^1(X; \mathcal{C}^\bullet)$, and consider the cup product $[c^1, C^1] \smile [c^1, C^1] \in \mathbb{H}^2(X; \mathcal{C}^\bullet)$. This is zero if and only if there exist a 1-cochain (c^2, C^2) satisfying $\delta(c^2, C^2) = (c^1, C^1) \smile (c^1, C^1)$, that is if and only if there exists a lift of $(\mathcal{E}', \nabla')$ to a second order deformation.

So

$$\mathrm{obs}_2 : \mathbb{H}^1(X; \mathcal{C}^\bullet) \to \mathbb{H}^2(X; \mathcal{C}^\bullet) \ , \quad \mathrm{obs}_2 : [c^1, C^1] \mapsto [c^1, C^1] \smile [c^1, C^1]$$

is an obstruction map for the extension $e_{1,2}$.

5.2.4 General obstruction and miniversal deformation space

Consider $T = \mathbb{H}^1(X; \mathcal{C}^\bullet)$ the tangent space to the deformation functor. Fix a basis $\tau_1, \dots, \tau_N$ and let $t_1, \dots, t_N$ be the dual coordinates. The k-th infinitesimal neighbourhood of T is, by definition, the spectrum of

$$\mathbb{C}[t_1, \dots, t_N]/(t_1, \dots, t_N)^{k+1} \ ,$$

and we denote it by $T_{(k)}$. We can construct a versal first order deformation $(\mathcal{E}^{(1)}, \nabla^{(1)})$ of $(\mathcal{E}_0, \nabla_0)$ on $T_{(1)} \times X$ as follows: let $(c^1_{\alpha\beta,i}, C^1_{\alpha,i})$ be cocycles representing the classes τ_i, and $g^1_{\alpha\beta,i} = (c^1_{\alpha\beta,i})^{(\alpha)} g_{\beta\alpha}$, $G^1_{\alpha,i} = (C^1_{\alpha,i})^{(\alpha)}$. Then define the $\mathcal{L}$-connection $(\mathcal{E}', \nabla')$ on $T_{(1)} \times X$ by local data

$$g^{(1)}_{\alpha\beta} = g_{\alpha\beta} + \sum_i g^1_{\alpha\beta,i} t_i$$

$$G_\alpha^{(1)} = G_\alpha + \sum_i G^1_{\alpha,i} \ .$$

The family $(\mathcal{E}^{(1)}, \nabla^{(1)}$ is a versal first order deformation of $(\mathcal{E}_0, \nabla_0)$ in the following sense: by the results of Section 5.2.1, any deformation over $\mathrm{Spec}\mathbb{C}[\epsilon/(\epsilon^2)$ is uniquely determined by a vector in T, that is equivalent to a morphism $\mathrm{Spec}\mathbb{C}[\epsilon]/(\epsilon^2) \to T_{(1)}$.

Now, let $\rho_{ij} = \tau_i \smile \tau_j$ and $f_2 = \sum_{i,j} \rho_{ij} t_i t_j \in \mathbb{H}^2(X; \mathcal{C}^\bullet) \otimes \mathbb{C}[t_1, \ldots, t_N]$. Let I_2 be the ideal of $\mathbb{C}[t_1, \ldots, t_N]$ generated by the image of the adjoint $f_2^* : \mathbb{H}^2(X; \mathcal{C}^\bullet) \to \mathbb{C}[t_1, \ldots, t_N]$. By the results of Section 5.2.3 we see that for $\tau \in T$ such that $f_2(\tau) = 0$ there exists second order deformations of $(\mathcal{E}_0, \nabla_0)$ integrating τ. Let $K_2 \subseteq T_{(2)}$ be defined by the ideal I_2. Then on $K_2 \times X$ there exists a flat $p_X^*\mathcal{L}$-connection $(\mathcal{E}^{(2)}, \nabla^{(2)})$ extending $(\mathcal{E}^{(1)}, \nabla^{(1)})_{|K_2 \cap T_{(1)}}$, which is a versal second order deformation of $(\mathcal{E}_0, \nabla_0)$.

Now, by induction on $l \geq 2$ we can construct closed subvarieties $K_l \subseteq T_{(l)}$ such that there exist a versal l-th order deformation of $(\mathcal{E}, \nabla)$ on $K_l \times X$.

Assume we have a versal $(l-1)$-th order defosrmation $(\mathcal{E}^{(l-1)}, \nabla^{(l-1)})$ over $K_{l-1} \times X$. Let I_{l-1} be the ideal in $\mathbb{C}[t_1, \ldots, t_N]/(t_1, \ldots, t_N)^l$ defining K_{l-1}, and let $g_{\alpha\beta}^{(l-1)} = \sum_{k \leq l-1} g^k_{\alpha\beta}$, $C_\alpha^{(l-1)} = \sum_{k \leq l-1} C^k_\alpha$ be the local data defining $(\mathcal{E}^{(l)}, \nabla^{(l)})$.

Let $F^{2,0}_{\alpha\beta\gamma,l}, F^{1,1}_{\alpha\beta,l}, F^{0,2}_{\alpha,l}$ be the homogeneous parts of degree l of, respectively, the following expressions:

$$g_{\alpha\beta}^{(l-1)} g_{\beta\gamma}^{(l-1)} g_{\gamma\alpha}^{(l-1)} - 1 \quad \in \mathrm{GL}_r \times \mathscr{O}_X(U_{\alpha\beta\gamma}) \ , \tag{5.16}$$

$$G_\beta^{(l-1)} - g_{\beta\alpha}^{(l-1)} G_\alpha^{(l-1)} g_{\alpha\beta}^{(l-1)} - g_{\beta\alpha}^{(l-1)} \mathrm{d}_\mathcal{L} g_{\alpha\beta}^{(l-1)} \quad \in \mathfrak{gl}_r \otimes \Omega_\mathcal{L}(U_\alpha\beta) \ , \tag{5.17}$$

$$G_\alpha^{(l-1)} \wedge G_\alpha^{(l-1)} + \mathrm{d}_\mathcal{L} G_\alpha^{(l-1)} \quad \in \mathfrak{gl}_r \otimes \Omega^2_\mathcal{L}(U_\alpha) \ . \tag{5.18}$$

Lemma 49 (see [26]). *The triple $(F^{2,0}, F^{1,1}, F^{0,2})$ defines a cochain of the total complex associated to the double complex $K^{p,q} = \check{p}(\mathscr{E}nd\mathcal{E} \otimes \Omega^q_\mathcal{L}) \otimes \mathbb{C}[t_1, \ldots, t_N]$.*

Moreover, this cochain is closed modulo I_{l-1}.

Let f_l be the cohomology class of the triple $(F^{2,0}, F^{1,1}, F^{0,2})$ in $\mathbb{H}^2(X; \mathcal{C}^\bullet) \otimes \mathbb{C}[t_1, \ldots, t_N]/I_{l-1}$, and I_l be the ideal generated by I_{l-1} and the image of the adjoint of f_l. Then it is clear that, as a cocycle on $\mathbb{C}[t_1, \ldots, t_N]/I_l$, $(F^{2,0}, F^{1,1}, F^{0,2})$ is exact, so we can choose $(c^l_{\alpha\beta}, C^l_\alpha)$ whose coboundary is $(F^{2,0}, F^{1,1}, F^{0,2})$. So we can define $g^l_{\alpha\beta} = (c^l_{\alpha\beta})^{(\alpha)} g_{\alpha\beta}$ and $G^l_\alpha = (C^l_\alpha)^{(\alpha)}$ that provide an integration of $(g_{\alpha\beta}^{(l-1)}, G_\alpha^{(l-1)})$ to the l-th order. Now define K_l to be the subvariety of $T_{(l)}$ defined by I_l, and the induction step is proved.

Consider the formal power series ring $\mathbb{C}[[t_1, \ldots, t_N]]$, and let I be the ideal generated by $f = \sum_{k=2}^\infty f_k$, and K the analytic space defined by $\mathbb{C}[[t_1, \ldots, t_N]]/I$. The families $(\mathcal{E}^{(l)}, \nabla^{(l)})$ on $K_l \times X$ define a family $(\mathcal{E}^\infty, \nabla^\infty)$

on $K \times X$. This family is versal by construction, and since $K_1 = T$ it is miniversal.

Summing up, we have:

Theorem 50. *Let X be a smooth projective complex variety, $\mathcal{L}$ a holomorphic Lie algebroid and $(\mathcal{E}, \nabla)$ a flat holomorphic $\mathcal{L}$-connection. Associate to this the complex of sheaves $\mathcal{C}^\bullet = (\mathscr{E}nd\mathcal{E}, \tilde{\nabla})$, and let $T = \mathbb{H}^1(X; \mathcal{C}^\bullet)$, $W = \mathbb{H}^2(X; \mathcal{C}^\bullet)$.*

Then there exist $f_k \in W \otimes \mathbb{C}[T]$ homogeneous of degree k such that the following holds. Let $f = \sum_{k=2}^{\infty} f_k \in W \otimes \mathbb{C}[[T]]$; since W is finite dimensional this is equivalent to a map $f^* : W^* \to \mathbb{C}[[T]]$. Let I be the ideal in $\mathbb{C}[[T]]$ generated by the image of f^*. Then K, the spectrum of $\mathbb{C}[[T]]/I$, has a natural versal family $(\mathcal{E}, \nabla)$ making it a miniversal deformation space of $(\mathcal{E}, \nabla)$.

5.3 Versal deformations spaces and Luna's slices

5.3.1 The tangent to Q

Recall from Section 4.1 that Q denotes the locally closed subscheme of $\mathfrak{Quot}(\Lambda_{(1)} \otimes V(-N), P)$ parametrizing triples $(\mathcal{E}, \mu, \alpha)$, with $(\mathcal{E}, \mu)$ a semistable Λ-module with Hilbert polynomial P and α an isomorphism between V and $H^0(X; \mathcal{E}(N))$.

Now we want to compute the tangent space $T_q Q$ at a closed point $q \in Q$. It is well known that if q represents the quotient $0 \to \mathcal{K}_1 \to \Lambda_{(1)} \otimes V(-N) \to \mathcal{E} \to 0$, the tangent space to $\mathfrak{Quot}(\Lambda_{(1)} \otimes V(-N), P)$ at q is isomorphic to $\mathrm{Hom}_{\mathscr{O}_X}(\mathcal{K}_1, \mathcal{E})$. Q_1 is an open subscheme of the $\mathfrak{Quot}$-scheme, so $T_q Q_1 \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathcal{K}_1, \mathcal{E})$.

Now consider the following: for $q \in Q$, define $q_i : \Lambda_{(i)} \otimes V(-N) \to \mathcal{E}$ by

$$q_i(\lambda \otimes s) = \mu(\lambda \otimes \tilde{q}(s)) \ .$$

From the diagram (4.1) we see that $q_{t|\Lambda_{(1)} \otimes V(-N)}$ coincides with q. Moreover, by the definition of Q_3, these maps are compatible, i. e. for $i < j$ we have the diagrams

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & \mathcal{K}_i \arrow[r] \arrow[d] & \Lambda_{(i)} \otimes V(-N) \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{K}_j \arrow[r] & \Lambda_{(j)} \otimes V(-N) \arrow[r] & \mathcal{E} \arrow[r] & 0 \end{tikzcd}$$

Let $q_t : \Lambda \otimes V(-N) \to \mathcal{E}$ denote the quotient obtained as the limit of the q_i. q_t is a Λ-module morphism, so its kernel $\mathcal{K}_t$ has a natural Λ-module structure.

Lemma 51. *$Hom_\Lambda(\mathcal{K}_t, \mathcal{E})$ naturally injects in $Hom_{\mathscr{O}_X}(\mathcal{K}_1, \mathcal{E})$.*

Proof. The inclusion $\mathcal{K}_1 \hookrightarrow \mathcal{K}_t$ gives a morphism

$$\mathrm{Hom}_\Lambda(\mathcal{K}_t, \mathcal{E}) \to \mathrm{Hom}_{\mathscr{O}_X}(\mathcal{K}_1, \mathcal{E}) \ .$$

We have to check that this is injective, i. e. that a Λ-module morphism $\phi : \mathcal{K}_t \to \mathcal{E}$ vanishing on $\mathcal{K}_1$ is zero.

Let $\mathcal{K}_i$ be the intersection of $\mathcal{K}_t \cap \Lambda_{(i)} \otimes V(-N)$. We show that if we know that ϕ vanish on $\mathcal{K}_i$, then it vanish on $\mathcal{K}_{i+1}$, and by induction we obtain the thesis.

So, assume that the Λ-module morphism $\phi : \mathcal{K}_t \to \mathcal{E}$ vanish on $\mathcal{K}_{i-1}$ for some $i \geq 0$. An element in $\mathcal{K}_i$ is of the form $x_1 \cdots x_i \otimes s$. Let $\sigma \in V(-N)$ be such that $\tilde{q}(\sigma) = q_t(x_2 \cdots x_i \otimes s)$. Since $0 = q_t(x_1 \cdots x_i \otimes s) = \mu_1(x_1 \otimes \sigma)$, we have $x_1 \otimes \sigma \in \mathcal{K}_1$, so that $\phi(x_1 \otimes \sigma) = 0$. Now, $\sigma - x_2 \cdots x_i \otimes s \in \mathcal{K}_t$, so

$$\phi(x_1 \cdots x_i \otimes s) = \phi(x_1 \otimes \sigma - x_1 \cdots x_i \otimes s) = \mu_1(x_1 \otimes \phi(\sigma - x_2 \cdots x_i \otimes s)) \ .$$

Since $\sigma - x_2 \cdots x_i \otimes s \in \mathcal{K}_{i-1}$ we can conclude. □

Proposition 52. *The tangent space T_qQ is naturally isomorphic to the vector space $Hom_\Lambda(\mathcal{K}_t, \mathcal{E})$.*

Proof. Let $\phi_1 \in \mathrm{Hom}(\mathcal{K}_1, \mathcal{E}) = T_qQ_1$. We want to show that ϕ_1 is extendable to a Λ-module morphism $\phi : \mathcal{K}_t \to \mathcal{E}$ if and only if $\phi_1 \in T_qQ$.

Since $q \in Q_1$, $\mathcal{K}_0 \to \mathcal{K}_1$ induces a morphism

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathcal{K}_1, \mathcal{E}) \to \mathrm{Hom}_{\mathscr{O}_X}(\mathcal{K}_0, \mathcal{E}) \ .$$

We will denote by $\tilde{\phi}$ the image of $\phi_1 \in \mathrm{Hom}_{\mathscr{O}_X}(\mathcal{K}_1, \mathcal{E})$ via this map.

The condition Q_2 is easy to translate for tangent vectors: $\phi_1 \in T_qQ_2$ if and only if the following diagram commutes

$$\begin{tikzcd}
 & \Lambda_{(1)} \otimes \mathcal{K}_1 \arrow[r, "\mathbf{1}\otimes\phi_1"] & \Lambda_{(1)} \otimes \mathcal{E} \arrow[dd, "\mu_1"] \\
\Lambda_{(1)} \otimes \mathcal{K}_0 \arrow[ur] \arrow[dr] & & \\
 & \mathcal{K}_1 \arrow[r, "\phi_1"] & \mathcal{E}
\end{tikzcd}$$

This is equivalent to say that for $s \in \mathcal{K}_0$ and $x \in \Lambda_{(1)}$ we have $\phi(x \otimes s) = \mu(x \otimes \tilde{\phi}(s))$.

To understand the condition Q_3, consider the following: let $\mathcal{W}_j$ denote the kernel of the morphism $(\Lambda_{(1)})^{\otimes j} \otimes \mathcal{E} \to \mathcal{E}$ obtained applying j times μ_1.

We have the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{A}_j \arrow[r] \arrow[d] & \mathcal{B}_j \otimes V(-N) \arrow[r] \arrow[d] & \mathcal{W}_{j-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\Lambda_{(1)})^{\otimes j-1} \otimes \mathcal{K}_1 \arrow[r] \arrow[d] & (\Lambda_{(1)})^{\otimes j} \otimes V(-N) \arrow[r] \arrow[d] & (\Lambda_{(1)})^{\otimes j-1} \otimes \mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K}_j \arrow[r] \arrow[d] & \Lambda_{(j)} \otimes V(-N) \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

for an appropriate $\mathcal{A}_j$. It turns out that $\phi_1 \in T_q Q_{3j}$ if and only if the composition of the morphism

$$\mathbf{1} \otimes \tilde{\phi} : (\Lambda_{(1)})^{\otimes j-1} \otimes \mathcal{K}_1 \to (\Lambda_{(1)})^{\otimes j-1} \otimes \mathcal{E}$$

with $(\Lambda_{(1)})^{\otimes j-1} \otimes \mathcal{E} \to \mathcal{E}$ vanish on $\mathcal{A}_j$.

Now let $\phi_1 \in \mathrm{Hom}_{\mathscr{O}_X}(\mathcal{K}_1, \mathcal{E})$ be a tangent vector in $T_q Q_3$. Then we can extend ϕ_1 to $\phi_j : \mathcal{K}_j \to \mathcal{E}$ as follows: for $\eta \in \mathcal{K}_j$ there exist $\xi \in \mathcal{K}_1$ and $x_1, \dots, x_{j-1} \in \Lambda_{(1)}$ such that $x_1 \otimes \cdots \otimes x_{j-1} \otimes \xi$ is mapped to η. Then set $\phi(\eta) = \mu(x_1 \cdots x_{j-1} \otimes \phi_1(\xi))$. This defines a Λ-module morphism from $\mathcal{K}_t$ to $\mathcal{E}$, and the claim is proven. $\square$

Lemma 53. *Let $(\mathcal{E}, \nabla), (\mathcal{F}, \nabla)$ be two flat $\mathcal{L}$-connection, and $\mu_{\mathcal{E}}, \mu_{\mathcal{F}}$ the associated Λ module structure. Let $\mathcal{C}^\bullet$ be the complex $\mathscr{H}om(\mathcal{E}, \mathcal{F}) \otimes \Omega^\bullet_{\mathcal{L}}$ with differential $\tilde{\nabla}$, the flat $\mathcal{L}$-connection on $\mathscr{H}om_{\mathscr{O}_X}(\mathcal{E}, \mathcal{F})$ induced by the $\mathcal{L}$-connections on $\mathcal{E}, \mathcal{F}$.*

Then there are natural isomorphisms $\mathbb{H}^i(X; \mathcal{C}^\bullet) \cong Ext^i_\Lambda(\mathcal{E}, \mathcal{F})$.

Proof. We show this for $i = 1$.

An element in $\mathrm{Ext}^1_\Lambda(\mathcal{E}, \mathcal{F})$ represents an equivalence class of an extension of Λ-modules

$$0 \to \mathcal{F} \to \mathcal{Z} \to \mathcal{E} \to 0 \ . \tag{5.19}$$

In particular, this is an extension of $\mathscr{O}_X$-modules, so we have an associated class $x \in \mathrm{Ext}^1_{\mathscr{O}_X}(\mathcal{E}, \mathcal{F})$.

Fix a good open cover $\mathfrak{U}$ of X, and let $x_{\alpha\beta} \in \check{C}^1(\mathfrak{U}, \mathscr{H}om_{\mathscr{O}_X}(\mathcal{E}, \mathcal{F})$ be a representative of x. Recall that $x_{\alpha\beta}$ are obtained by choosing local $\mathscr{O}_X$-splittings of (5.19) giving isomorphisms $\zeta_\alpha : \mathcal{Z}_{|U_\alpha} \cong (\mathcal{F} \oplus \mathcal{E})_{|U_\alpha}$, and $x_{\alpha\beta}$ are such that on $U_{\alpha\beta}$ we have

$$\zeta_\beta \circ \zeta_\alpha^{-1} = \begin{pmatrix} \mathbf{1}_{\mathcal{F}} & 0 \\ x_{\alpha\beta} & \mathbf{1}_{\mathcal{E}} \end{pmatrix}$$

as endomorphism of $(\mathcal{F} \oplus \mathcal{E})_{|U_{\alpha\beta}}$.

Now consider the Λ-module structure $\mu_{\mathcal{Z}}$ of $\mathcal{Z}_{|U_\alpha}$: via ζ_α, since (5.19) is an exact sequence of Λ-modules we have

$$\mu_{\mathcal{Z}}(\lambda \otimes (f, e)) = (\lambda f + Y_\alpha(\lambda, e), \lambda e)$$

for any $\lambda \in \Lambda$, $e \in \mathcal{E}$ and $f \in \mathcal{F}$. Remark that Y_α is $\mathscr{O}_X$-linear w. r. t. both arguments, and satisfies

$$Y_\alpha(\lambda\lambda', e) = \lambda Y_\alpha(\lambda', e) + Y_\alpha(\lambda, \mu_{\mathcal{E}}(\lambda' \otimes e)) \ . \tag{5.20}$$

Now, suppose a splitting of $\Lambda_{(1)} \to \mathcal{L}$ is fixed, and define $y_\alpha : \mathcal{E}_{|U_\alpha} \to (\mathcal{F} \otimes \Omega_{\mathcal{L}})_{|U_\alpha}$ via $\langle y_\alpha(e), v\rangle = Y_\alpha(v, e)$ for any $e \in \mathcal{E}_{|U_\alpha}$ and $v \in \mathcal{L}_{|U_\alpha}$ (where on the right hand side of the equation we see it in Λ via the splitting). Now, equation (5.20) implies that $\tilde{\nabla} y_\alpha = 0$, while by the compatibility of the $\mathscr{O}_X$and Λ-module structures of $\mathcal{Z}$ follows that on the double overlaps we have $(\check{\delta} y)_{\alpha\beta} = \tilde{\nabla} x_{\alpha\beta}$, so the pair $(x_{\alpha\beta}, y_\alpha)$ defines a cohomology class in $\mathbb{H}^1(X; \mathcal{C}^\bullet)$.

It is the easy to verify that the converse holds, i. e. that if $(x_{\alpha\beta}, y_\alpha)$ is a representative of a cohomology class in $\mathbb{H}^1(X; \mathcal{C}^\bullet)$, then y_α defines a Λ-module structure on the $\mathscr{O}_X$-modules extension defined by $x_{\alpha\beta}$, and that equivalence classes of extensions correspond to the same cohomology class. □

5.3.2 An application of Luna's slices theorem

Let k be an algebraically closed field of characteristic 0 and X a k-scheme. Let G be a k-algebraic group acting on X. Let $\sigma : G \times X \to X$ denote the action morphism. For any point $x \in X$ let $O(x)$ be the orbit of x, that is the image of $\sigma_{|G\times\{x\}}$.

A *good quotient* of X by the action of G is a pair (Y, ϕ) with Y a scheme and $\varphi : X \to Y$ a G-equivariant morphism (where Y has the trivial G-action) such that

- φ is affine, surjective and open;
- the morphism $\mathscr{O}_Y \to (\varphi_* \mathscr{O}_X)^G$ is an isomorphism;
- for any $Z \subseteq X$ closed and G-invariant, the image $\varphi(Z)$ is closed in Y, and for any two $Z_1, Z_2 \subseteq X$ closed, G-invariant and disjoint, the images $\varphi(Z_1)$ and $\varphi(Z_2)$ are disjoint.

Given an action of an algebraic group G on a scheme X, a good quotient may not exist. When it does it is unique up to isomorphism, and we will denote it by $X//G$.

Let $H \subseteq G$ be an algebraic subgroup, and let H act on a scheme Y. Define the H-action on $G \times Y$ by $h(g, y) = (gh^{-1}, h \cdot x)$. Then there exists a good quotient $(G \times Y)//H$ such that the corresponding map $\varphi : G \times Y \to (G \times Y)//H$ is a principal bundle with fiber H. Denote this by $G \times^H Y$: it has a natural action of G (on the left).

Let X be a scheme on which an algebraic group G acts. Let x be a closed point in X, $O(x)$ its orbit under the G-action and $G_x \subseteq G$ the stabilizer of x. A *normal slice* of $O(x)$ at x is an affine subscheme $S \subseteq X$ such that

- $x \in S$ and S is invariant for the action of G_x;
- the natural morphism $G \times^{G_x} S \to X$ has an open image and is étale over its image.

Luna's slice theorem (cf. [24]):

Theorem 54. *Let X be a scheme, G a reductive algebraic group acting on X admitting a good quotient $X//G$. Moreover, let $x \in X$ be a closed point such that its orbit $O(x)$ is closed, and G_x its stabilizer.*

Then there exists a normal slice S to $O(x)$ at x, and the G_x-action on S admits a good quotient $S//G_x$. Moreover, there is a natural morphism of good quotients $S//G_x \to X//G$, whose image is affine and open in $X//G$, and which is étale over the image. Furthermore, the following diagram is commutative:

$$\begin{tikzcd} S \times^{G_x} G \arrow[r] \arrow[d] & X \arrow[d] \\ S//G_x \arrow[r] & X//G \end{tikzcd} .$$

Proposition 55 (cf. [31], Proposition 1.2.3)**.** *Let $(\mathcal{E}, \nabla)$ be a semistable $\mathcal{L}$-connection, and $z \in Q_{(j)}$ representing a triple $(\mathcal{F}, \mu, \alpha)$, with $\mathcal{F} = \mathcal{E}$, μ the Λ-module structure corresponding to the $\mathcal{L}$-connection ∇ and α an isomorphism as before. Assume that the $SL(V)$-orbit of z is closed, so that there is a normal slice S of $O(z)$ at z, and let $\mathcal{T}_{|S}$ be the pull-back of the tautological bundle to S.*

Then $(S, z, \mathcal{T})$ is a miniversal family for $(\mathcal{E}, \nabla)$.

Proof. The fact that $\mathcal{T}$ is a versal family for $(\mathcal{E}, \nabla)$ follows directly from the fact that $\mathcal{T}$ is the restriction of the tautological bundle of the $\mathfrak{Quot}$ scheme. So it remains to show the morphism between the tangent spaces $\tilde{\kappa} : T_z S \to \mathbb{H}^1(X; \mathcal{C}^\bullet)$ is an isomorphism. By the universality of $\mathcal{T}$ we already know that κ is surjective, so it suffice to show that it is injective.

The tangent to the slice $T_z S$ is isomorphic to the quotient $T_z Q / T_z O(z)$. Let $\kappa : T_z Q \to \mathbb{H}^1(X; \mathcal{C}^\bullet) = \mathrm{Ext}^1_\Lambda(\mathcal{E}, \mathcal{E})$ be the map between the tangent spaces associated to the tautological bundle $\mathcal{T}_Q$ on Q. We have to show that $T_z O(z) = \mathrm{Ker}\,(\kappa)$.

The map κ is given by the connection morphism obtained by applying $\mathrm{Hom}_\Lambda(\bullet, \mathcal{E})$ to the exact sequence

$$0 \to \mathcal{K}_t \to \Lambda \otimes V(-N) \to \mathcal{E} \to 0 \ .$$

So it suffices to show that $T_z O(z)$ is equal to the image of

$$\mathrm{Hom}_\Lambda(\Lambda \otimes V((-N), \mathcal{E}) \to \mathrm{Hom}_\Lambda(\mathcal{K}_t, \mathcal{E}) \ .$$

Now, a point $z' \in O(z)$ is obtained as the composition $(\mathbf{1}_\Lambda \otimes M) \circ z$, where $M : V(-N) \to V(-N)$ is the isomorphism associated to a $\tilde{M} \in \mathrm{SL}(V)$, so the tangent to $O(z)$ is naturally identified with the image of the composition

$$\mathrm{Hom}_{\mathscr{O}_X}(V(-N), V(-N)) \cong \mathrm{Hom}_\Lambda(\Lambda \otimes V(-N), \Lambda \otimes V(-N)) \xrightarrow{\beta} \\ \xrightarrow{\beta} \mathrm{Hom}_\Lambda(\Lambda \otimes V(-N), \mathcal{E}) \to \mathrm{Hom}_\Lambda(\mathcal{K}_t, \mathcal{E}) \ ,$$

so it suffices to show that β is surjective, but this follows directly from $V \hookrightarrow H^0(X; \Lambda \otimes V)$ and $V \cong H^0(X; \mathcal{E}(N))$. $\square$

Chapter 6

The singularity of $\mathcal{M}_X(2,0)$ at $(\mathcal{E}_0, \nabla_0)$

Let X be a smooth projective curve, $\mathcal{L} = \mathcal{T}_X$ and $(\mathcal{E}_0, \nabla_0) = (\mathscr{O}_X^{\oplus 2}, \mathrm{d})$. We describe the tangent space at $(\mathcal{E}_0, \nabla_0)$ of $\mathcal{M}_X(2,0)$ and the first obstruction in terms of linear data, and find equations for the quotient $K_2//H$.

6.1 Some invariant theory

Let V be a complex vector space of finite dimension d, and $\langle \cdot, \cdot \rangle$ a non degenerate symmetric bilinear form on V. Denote by $\mathrm{O}(V)$ the orthogonal group of V associated to this bilinear form, that is the group of k-linear endomorphisms $B \in \mathrm{End}(V)$ such that $\langle BX, BY \rangle = \langle X, Y \rangle$ for any $X, Y \in V$. The special orthogonal group $\mathrm{SO}(V)$ is the subgroup of $\mathrm{O}(V)$ consisting of endomorphisms whose determinant is 1.

For any positive integer n, both $\mathrm{O}(V)$ and $\mathrm{SO}(V)$ act on $V^{\oplus n}$ by $B \cdot (X_1, \dots, X_n) = (BX_1, \dots BX_n)$. We study the invariant theory of these actions.

The algebra of functions on $V^{\oplus n}$ is the polynomial algebra $\mathbb{C}[V^{\oplus n}]$ in $d \cdot n$ variables. The functions

$$T_{i,j} = \langle X_i, X_j \rangle \qquad \forall i, j = 1, \dots, n$$

are clearly $\mathrm{O}(V)$-invariant. It is known (see Appendix F of [12]) that any $\mathrm{O}(V)$-invatiant function on $V^{\oplus n}$ can be written as a linear combination of products of the $T_{i,j}$'s. We have:

Theorem 56. *The algebra $(\mathbb{C}[V^{\oplus n}])^{O(V)}$ is isomorphic to $\mathbb{C}[T_{i,j}]/I$, where I is the ideal generated by*

$$T_{i,j} - T_{j,i} \tag{6.1}$$

$$\det(M_{i_0,\dots,i_d;j_0,\dots,j_d}) = 0 \tag{6.2}$$

for $i, j, 1_0, \ldots, i_d, j_0, \ldots, j_d \in \{1, \ldots, n\}$, *where.* $M_{i_0,\ldots,i_d;j_0,\ldots,j_d}$ *is the* $(d+1)\times(d+1)$ *matrix obtained by choosing the rows* $i_0, \ldots, i_d$ *and the columns* $j_0, \ldots, j_d$ *of* $(T_{i,j})$.

The relations $T_{j,i} = T_{i,j}$ come from the fact that the biliear form is symmetric, while the relations $\det(M_{i_0,\ldots,i_d;j_0,\ldots,j_d}) = 0$ are due to the fact that since V has dimension d, any $d+1$ vectors of V are linearly dependent. Clearly the equations (6.2) are non trivial only if $n > d$.

So, for $n \leq d$ the quotient $V^{\oplus n}//\mathrm{O}(V)$ is the spectrum of

$$\mathbb{C}[T_{i,j}]/(T_{i,j} - T_{j,i}) \cong \mathbb{C}[T_{i,j}]_{i\leq j} \ ,$$

i. e. it is a linear space of dimension $n(n+1)/2$.

For $n > d$, the quotient $V^{\oplus n}//\mathrm{O}(V) = \mathrm{Spec}\mathbb{C}[V^{\oplus n}]^{\mathrm{O}(V)}$ is the spectrum of the algebra $\mathbb{C}[T_{i,j}]/I$ where I is the ideal generated by $T_{i,j} - T_{j,i}$ and $\det(M_{i_0,\ldots,i_d;j_0,\ldots,j_d})$. From this we see that $V^{\oplus n}//\mathrm{O}(V)$ is a determinantal variety in the space of symmetric maps. We denote by $W(k,m)$ the variety of symmetric maps of an m-dimensional complex vector space of rank at most k; then $V^{\oplus n}//\mathrm{O}(V) \cong W(d,n)$ These have been studied for instance in [38], where the following theorem is proven:

Theorem 57. *The singular locus of* $W(k,m)$ *coincide with* $W(k-1,m)$.

The blow up of $W(k,m)$ *along* $W(k-1,m)$ *is naturally isomorphic to* $S(K_k)$, *where* K_k *is the tautological bundle of the Grassmannian* $\mathfrak{Gr}(k,m)$, *and for any vector bundle* $\mathcal{F}$ *we denote by* $S(\mathcal{F})$ *the dual of the subbundle of* $\mathscr{H}om(\mathcal{F}^*, \mathcal{F})$ *consisting of symmetric maps.*

In particular, we see that $W(k,m)$ has dimension $k(m-k) + k(k+1)/2$, and that it is smooth in codimension $m-k+1$

Now we go over to the case of $\mathrm{SO}(V)$. The $T_{i,j}$ are invariant for $\mathrm{SO}(V)$ too. Moreover we have another class of $\mathrm{SO}(V)$-invariant functions, namely:

$$S_{i_1,\ldots,i_d} = \det(X_{i_1}|\cdots|X_{i_d}) \ ,$$

for any $i_1, \ldots i_d \in \{1, \ldots, n\}$, where the vertical bar means juxtapposition of vectors. Any $\mathrm{SO}(V)$-invariant function on $V^{\oplus n}$ is a linear combination of products of $T_{i,j}$ and $S_{i_1,\ldots,i_d}$ (cf. [12]). These have to satisfy the relations $T_{j,i} = T_{i,j}$ and (6.2). Moreover, $S_{i_1,\ldots,i_d} = (-1)^\sigma S_{i_{\sigma(1)},\ldots,i_{\sigma(d)}}$ for any σ permutation of $\{1, \ldots, d\}$, while polarizing the Gram relation

$$\det(X_1|\cdots|X_d)^2 = \begin{pmatrix} \langle X_1, X_1\rangle & \cdots & \langle X_1, X_d\rangle \\ \vdots\rangle & \ddots & \vdots \\ \langle X_d, X_1\rangle & \cdots & \langle X_d, X_d\rangle \end{pmatrix}$$

we obtain

$$R_{i_1,\dots,i_d;j_1,\dots,j_d} = S_{i_1,\dots,i_d} S_{j_1,\dots,j_d} - \begin{vmatrix} T_{i_1,j_1} & \cdots & T_{i_1,j_d} \\ \vdots & \ddots & \vdots \\ T_{i_d,j_1} & \cdots & T_{i_d,j_d} \end{vmatrix} = 0 \ . \tag{6.3}$$

So we have:

Theorem 58. *The algebra $\mathbb{C}[V^{\oplus n}]^{SO(V)}$ is isomorphic to $\mathbb{C}[T_{i,j}, S_{i_1,\dots,i_d}]/J$, where J is the ideal generated by*

$$T_{i,j} - T_{j,i} \ , \tag{6.4}$$

$$\det(M_{i_1,\dots,i_d;j_0,\dots,j_d}) \ , \tag{6.5}$$

$$R_{i_1,\dots,i_d;j_1,\dots,j_d} \ . \tag{6.6}$$

Moreover, $\mathrm{SO}(V)$ is a subgroup of $\mathrm{O}(V)$ of index 2, so the quotient $V^{\oplus n}//\mathrm{SO}(V)$ has a natural double covering φ to $V^{\oplus n}//\mathrm{O}(V)$.

Lemma 59. *The ramification locus of $\varphi : V^{\oplus n}//SO(V) \to V^{\oplus n}//O(V)$ is the set of $SO(V)$-equivalence classes of n-tuples of vectors $(X_1, \dots, X_n)$ such that Rank$(X_1|\cdots|X_n) < d$.*

Proof. The ramification locus is the set of equivalence classes of n-tuples $(X_1, \dots, X_n)$ whose $\mathrm{SO}(V)$-orbit coincides with the $\mathrm{O}(V)$-orbit. So we have to look for elements of $V^{\oplus n}$ having this property.

Recall that $\mathrm{O}(V)$ acts tranitively on the d-tuples of orthonormal vectors, and that $\mathrm{SO}(V)$ acts transitively on the d-tuples of orthonormal vectors having a fixed orientation.

Assume for the moment that $n < d$, and let $(X_1, \dots, x_n)$ be a n-tuple of elements of V. For any $A \in \mathrm{O}(V)$ we have to find a $B \in \mathrm{SO}(V)$ such that $A(X_1, \dots, X_n) = B(X_1, \dots, X_n)$. Consider W the linear subspace of V span of $X_1, \dots, X_n$, and let AW be its image via A, that is the span of $AX_1, \dots, AX_n$. Since $n < d$, we can complete $\{x_i\}$ and $\{Ax_i\}$ to bases e and f of V having the same orienatation. Since A is an isometry of W in AW, by Witt's theorem there exists an isometry B sending e to f, and since these have the same orientation $B \in \mathrm{SO}(V)$.

For general n, we can repeat the argument above whenever we can complete the basis of W to a basis of V having the freedom to choose the orientation of the base, and this is possible if and only if the rank of $(X_1|\cdots|X_n)$ is less then d. □

From this lemma, we see that the ramification locus of ϕ in $V^{\oplus n}//\mathrm{O}(V)$ coincide with its singularity locus. In particular, $V^{\oplus n}//\mathrm{SO}(V)$ is smooth outside its ramification locus, has dimension $k(m-k) + k(k+1)/2$ and is smooth in codimension $m - k + 1$.

6.2 The germ of the moduli space of connections on a curve at the most degenerate point

Let X be a genus g smooth projective complex curve. Consider $\mathcal{M}(2,0)$, the moduli space of semistable flat connections of rank 2 and Chern class equal to 0. Let $\mathcal{E}_0 = \mathscr{O}_X^{\oplus 2}$ be the trivial rank 2 bundle, and $\nabla_0 = d$ the trivial flat connection on it. Clearly $(\mathcal{E}_0, \nabla_0) \in \mathcal{M}(2,0)$, and this is a strictly semistable point. We study the singularity of $\mathcal{M}(2,0)$ at $(\mathcal{E}_0, \nabla_0)$, i. e. we look for equations of the germ of $\mathcal{M}(2,0)$ around $(\mathcal{E}_0, \nabla_0)$.

By Proposition 55, we know that the germ of $\mathcal{M}(2,0)$ at $(\mathcal{E}_0, \nabla_0)$ is isomorphic to the germ of the quotient of K, a miniversal deformation of $(\mathcal{E}_0, \nabla_0)$, by $H = \mathrm{Stab}(\mathcal{E}_0, \nabla_0) = \mathrm{PGL}_2$. Recall that $K = \mathrm{obs}^{-1}(0)$ for $\mathrm{obs} : \mathbb{H}^1(X; \mathcal{C}^\bullet) \to \mathbb{H}^2(X; \mathcal{C}^\bullet)$ the Kuranishi map constructed in Theorem 50.

Actually, in the following we will find equations for the quotient of $K_2//H$, K_2 being the variety defined by obs_2, the degree 2 homogeneous part of obs. In the case when X is a curve of genus 1, the terms of degree > 2 of obs vanish, so the germ of $K_2//H$ is actually the germ of the moduli space $\mathcal{M}(2,0)$ at $(\mathcal{E}_0, \nabla_0)$ (cf. [27]). In the higher genus case, we expect that one can generalize the argument of deformation to the normal cone of [21] to conclude that $K_2//H$ is the germ of $\mathcal{M}_X(2,0)$.

Let $\gamma_1, \dots, \gamma_g$ be generators of $H^1(X; \mathscr{O}_X)$ and $\zeta_1, \dots, \zeta_g$ the dual generators of $H^0(X; \Omega_X)$. With respect to a good open cover $\mathfrak{U}$ of X, we represent γ_i by 1-Čech cocylces $\{\gamma_{\alpha\beta,i}\}$ and ζ_i by 0-Čech cocycles $\{\zeta_{\alpha,i}\}$. Since ζ_i's are dual to γ_i's, the cohomology class of the 1-Čech cocyle $e_{\alpha\beta,i,j} = \gamma_{\alpha\beta,i}\zeta_{\beta,j}$ equals ω for $i = j$ and 0 for $i \neq j$, where ω is the generator of $H^1(X; \Omega_X)$.

Consider a g-tuple of 0-Čech cochains ϕ_i, $i = 1, \dots, g$ such that $\check{\delta}\phi_i = \mathrm{d}\gamma_i$. Then the map $H^1(X; \mathscr{O}_X) \oplus H^0(X; \Omega_X) \to \mathbb{H}^(X; \Omega_X^\bullet)$ given by

$$(\sum_i a_i\gamma_i, \sum_i b_i\zeta_i) \mapsto (\sum_i a_i\gamma_i, \sum_i b_i\zeta_i - \sum_i a_i\phi_i)$$

is an isomorphism of vector spaces. Since $\mathbb{H}^1(X; \Omega_X^\bullet) \cong H^1(X; \mathbb{C})$, g-tuples ϕ_i as above exists. Via this isomorphism, the cup product $\smile: \mathbb{H}^1(X; \Omega_X^\bullet) \times \mathbb{H}^1(X; \Omega_X^\bullet) \to \mathbb{H}^2(X; \Omega_X^\bullet)$ induces a product on the De Rham cohomology $H^1(X; \mathbb{C}) \times H^1(X; \mathbb{C}) \to H^2(X; \mathbb{C})$.

Lemma 60. *There exist ϕ_i, 0-Čech cocycles of Ω_X, such that the induced product on De Rham cohomology coincides with the cup product induced by the wedge product of differential forms.*

Proof. We need to show that there exist ϕ_i's such that the class in $\mathbb{H}^2(X; \Omega_X^\bullet)$ of $e_{i,j} = (\gamma_i, \phi_i) \smile (\gamma_j, \phi_j)$ is zero.

Since X is a curve, $\gamma_i \smile \gamma_j = \check{\delta} g_{i,j}$ for some $g_{i,j} \in \check{C}^1(\mathfrak{U}, \mathscr{O}_X)$. Let ϕ_i be any g-tuple of 0-Čech cocycles such that $\check{\delta}\phi_i = \mathrm{d}\gamma_i$. Let Ψ be the isomorphism $\Psi : \mathbb{H}^2(X;\Omega_X^\bullet) \to H^1(X;\Omega_X)$. This is given by $\Psi(e_{i,j}) = [e_{i,j}^{1,1} - \mathrm{d}g_{i,j}]$. Let $\lambda_{i,j} \in \mathbb{C}$ be such that $\Psi(e_{i,j}) = \lambda_{i,j}\omega$.

Let $\tilde{\phi}_i = \phi_i - \sum_j \lambda_{i,j}\zeta_j$, and $\tilde{e}_{i,j}$ be the class in $\mathbb{H}^2(X;\Omega_X^\bullet)$ defined as $e_{i,j}$'s replacing the ϕ_i's with the $\tilde{\phi}_i$'s. Then $\Phi(\tilde{e}_{i,j}) = 0$. □

Consider the trivial flat connection of rank 2 $(\mathcal{E}_0, \nabla_0)$. Since X is a curve, $\mathcal{C} = [\mathscr{E}nd\mathcal{E} \to \mathscr{E}nd\mathcal{E} \otimes \Omega_X]$, so we can compute its hypercohomology via the exact sequence

$$0 \longrightarrow \mathbb{H}^0(X,\mathcal{C}^\bullet) \longrightarrow H^0(X,\mathscr{E}nd\mathcal{E}_0) \xrightarrow{d_1} H^0(X,\mathscr{E}nd\mathcal{E}_0 \otimes \Omega_X) \longrightarrow$$

$$\longrightarrow \mathbb{H}^1(X,\mathcal{C}^\bullet) \longrightarrow H^1(X,\mathscr{E}nd\mathcal{E}_0) \xrightarrow{d_2} H^1(X,\mathscr{E}nd\mathcal{E}_0 \otimes \Omega_X) \longrightarrow$$

$$\longrightarrow \mathbb{H}^2(X,\mathcal{C}^\bullet) \longrightarrow H^2(X,\mathscr{E}nd\mathcal{E}_0) = 0\ .$$

By Serre's duality $H^0(X;\mathscr{E}nd\mathcal{E}_0 \otimes \Omega_x) \cong H^1(X;\mathscr{E}nd\mathcal{E}_0)^*$ and $H^1(X;\mathscr{E}nd\mathcal{E}_0 \otimes \Omega_X) \cong H^0(X;\mathscr{E}nd\mathcal{E}_0)^*$. Moreover, since $\mathcal{E}_0$ is trivial, $\mathscr{E}nd\mathcal{E}_0 = \mathfrak{gl}_2 \otimes_{\mathbb{C}} \mathscr{O}_X$. So

$$H^0(X;\mathscr{E}nd\mathcal{E}_0) \cong \mathfrak{gl}_2 \otimes H^0(X;\mathscr{O}_X) \cong \mathfrak{gl}_2^{\oplus g}$$

and

$$H^1(X;\mathscr{E}nd\mathcal{E}_0) \cong \mathfrak{gl}_2 \otimes H^1(X;\mathscr{O}_X) \cong \mathfrak{gl}_2^{\oplus g}.$$

Now, the map d_1 above is just $\tilde{\nabla}_0$, the connection on $\mathscr{E}nd\mathcal{E}_0$ induced by $\nabla_0 = \mathrm{d}$. So, since $H^0(X;\mathscr{E}nd\mathcal{E}_0) = \mathfrak{gl}_2$, it is zero.

The map d_2 is the adjoint of d_1 via the Serre isomorphism, so it is zero too.

So we have isomorphisms

$$\mathbb{H}^1(X,\mathcal{C}^\bullet) = H^0(X,\mathscr{E}nd\mathcal{E} \otimes \Omega_X^1) \oplus H^1(X,\mathscr{E}nd\mathcal{E}) \cong \mathfrak{gl}_2^{\oplus g} \oplus \mathfrak{gl}_2^{\oplus g}\ ,$$

$$\mathbb{H}^2(X,\mathcal{C}^\bullet) = H^1(X,\mathscr{E}nd\mathcal{E} \otimes \Omega_X^1) \cong \mathfrak{gl}_2\ .$$

Fix an open cover $\mathfrak{U}$ of X, and for $i = 1,\ldots,g$, let $\phi_i \in \check{C}(\mathfrak{U},\Omega_X)$ be a cocycle such that $\check{\delta}\phi_i = \mathrm{d}\gamma_i$. Choose ϕ_i as in Lemma 60. These give an isomorphism $\mathfrak{gl}_2^{\oplus g} \oplus \mathfrak{gl}_2^{\oplus g} \to \mathbb{H}^1(X;\mathcal{C}^\bullet)$ via the formula

$$(A_i, B_i) \mapsto (A_i\gamma_i, B_i\zeta_i - A_i\phi_i)\ .$$

Via this isomorphisms, the map obs_2 is given by

$$\mathrm{obs}_2(A_1,\ldots A_n, B_1,\ldots B_n) = \sum_i [A_i, B_i]\ ,$$

where the bracket is the commutator of matrices and $A_i, B_i \in \mathfrak{gl}_2$.

Remark that the target is a traceless element. Moreover, remark that $[A+\lambda\mathbf{1}, B+\mu\mathbf{1}] = [A,B]$ for any $\lambda, \mu \in \mathbb{C}$. Via the isomorphism $\mathfrak{gl}_2 \cong \mathbb{C}\oplus\mathfrak{sl}_2$ given by $A \mapsto (\mathrm{Trace}(A), A - \frac{1}{2}\mathrm{Trace}(A)\mathbf{1})$, this implies that obs_2 factors as

$$\begin{tikzcd} \mathfrak{gl}_2^{\oplus 2g} \arrow[r, "\mathrm{obs}_2"] \arrow[d] & \mathfrak{sl}_2 \\ \mathfrak{sl}_2^{\oplus 2g} \arrow[ur, "\tilde{\mathrm{obs}}_2"'] & \end{tikzcd} .$$

So $K_2 \cong \mathbb{C}^{2g} \times \tilde{K}$, where $\tilde{K}$ is the preimage of 0 via $\tilde{\mathrm{obs}}_2$.

Now consider $W = \mathfrak{sl}_2^{\oplus 2g}$. The following 2-form

$$\omega(A_1, \ldots, A_{2g}; B_1, \ldots, B_{2g}) = \sum_{i=1}^{n} \mathrm{Trace}(A_i B_{n+i}) - \mathrm{Trace}(A_{n+i} B_i) \ .$$

is symplectic.

Let PGL_2 act on W by symultaneous conjugation: for $M \in \mathrm{PGL}_2$ and $A_i \in \mathfrak{sl}_2$ define

$$M(A_1, \ldots, A_{2g}) = (MA_1M^{-1}, \ldots, MA_{2g}M^{-1}) \ .$$

Since the trace is invariant for conjugation, this action is symplectic. The corresponding moment map is given by

$$F : W \to \mathfrak{sl}_2$$

$$F(A_1, \ldots, A_{2g}) = \sum_{i=1}^{n} [A_i, A_{g+i}] \ ,$$

that coincides with $\tilde{\mathrm{obs}}_2$.

Summing up, we have shown

Proposition 61. *The quotient $K_2//H$ is isomorphic to the symplectic reduction of $V = \mathfrak{sl}_2^{\oplus 2g}$ with the symplectic form*

$$\omega(A_1, \ldots, A_{2g}; B_1, \ldots, B_{2g}) = \sum_{i=1}^{g} \mathit{Trace}(A_i B_{g+i}) - \mathit{Trace}(A_{g+i} B_i)$$

by the PGL_2 action.

For $A \in \mathfrak{sl}_2$, write

$$A = \begin{pmatrix} a^{(0)} & a^{(1)} \\ a^{(2)} & -a^{(0)} \end{pmatrix}$$

with $a^{(k)} \in \mathbb{C}$. This realizes an isomorphism of vector spaces between $\mathfrak{sl}_2$ and $\mathbb{C}^3$. In this coordinates, the determinant $\det(A) = -(a^{(0)})^2 - a^{(1)}a^{(2)}$ gives a quadratic form on $\mathbb{C}^3$. The associated matrix is

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}$$

that is non-degenerate. The associated bilinear form is $\langle A, B\rangle = -\frac{1}{2}\mathrm{Trace}(AB)$. On $\mathbb{C}^3$ equipped with a non degenerate quadratic form there exists a unique antisymmetric operation $[\cdot,\cdot]$ such that $\langle [X,Y], Z\rangle = \det(X|Y|Z)$. If X, Y are the vectors of $\mathbb{C}^3$ corresponding to $A, B \in \mathfrak{sl}_2$, we have $[X,Y] = \frac{1}{4}[A,B]$, where the bracket on the right hand side is the commutator of the matrices.

Now consider the action of PGL_2 on $\mathfrak{sl}_2$: it is linear and preserves the determinant. So, if we denote by X_A the vector of $\mathbb{C}^3$ corersponding to A, for any $M \in \mathrm{PGL}_2$ there is an $N \in \mathrm{O}(3)$ such that $MAM^{-1} = NX_A$, where the orthogonal group is taken w. r. t. the matrix D.

Moreover, since an orthogonal matrix N is in $N \in \mathrm{SO}(3)$ if and only if $\det(X|Y|Z) = \det(NX|NY|NZ)$, the action of PGL_2 corresponds to the action of the special orthogonal group, because $[MAM^{-1}, MBM^{-1}] = M[A,B]M^{-1}$.

So we have the isomorphism of vector spaces with group action between $(\mathfrak{sl}_2, \mathrm{PGL}_2)$ and $(\mathbb{C}^3, \mathrm{SO}(3))$.

In particular, W with the PGL_2 action coincides with the vector space $V = (\mathbb{C}^3)^{\oplus 2n}$ with the $\mathrm{SO}(3)$-action given by simultaneous multiplication, studied in Section 6.1.

Recall that the $\mathrm{SO}(3)$-invariant functions on V are, for $(X_1, \ldots, X_{2g}) \in V$, $T_{i,j} = \langle X_i, X_j\rangle$ for $i,j = 1,\ldots,2g$ and $S_{i,j,k} = \det(X_i|X_j|X_k)$ for $i,j,k = 1,\ldots,2g$, subject to relations $T_{i,j} = T_{j,i}$, (6.2) and (6.3).

Writing $X_i = (x_i^{(0)}, x_i^{(1)}, x_i^{(k)})^t$ with $x_i^{(k)} \in \mathbb{C}$, the moment map is given by

$$F(A_1,\ldots,A_{2n}) = \begin{pmatrix} \sum_{i=1}^n (x_i^{(1)}x_{n+i}^{(2)} - x_i^{(2)}x_{n+i}^{(1)}) \\ \sum_{i=1}^n (x_i^{(2)}x_{n+i}^{(0)} - x_i^{(0)}x_{n+i}^{(2)}) \\ \sum_{i=1}^n (x_i^{(0)}x_{n+i}^{(1)} - x_i^{(1)}x_{n+i}^{(0)}) \end{pmatrix}. \tag{6.7}$$

The differential of F is given by

$$\mathrm{d}F = \begin{pmatrix} 0 & 0 & x_{n+i}^{(2)} & -x_i^{(2)} & -x_{n+i}^{(1)} & x_i^{(1)} \\ -x_{n+i}^{(2)} & x_i^{(2)} & 0 & 0 & x_{n+i}^{(0)} & -x_{n+i}^{(0)} \\ x_{n+i}^{(1)} & -x_i^{(1)} & -x_{n+i}^{(0)} & x_i^{(0)} & 0 & 0 \end{pmatrix}$$

where the elements of the $(2k+1)i$-th column are the derivatives w. r. t. $x_i^{(k)}$, and those of the $(2k+2)i$-th are derivatives w. r. t. $x_{n+i}^{(k)}$, for $k =$

$0,1,2$. From this we see that the rank of $\mathrm{d}F$ is zero only at $(X_1,\ldots,X_{2n}) = (0,\ldots,0)$, that it can never be 1, that it is 2 exactly at points $(X_1,\ldots,X_{2n})$ such that the three vectors $(x_i^{(k)})_{i=1,\ldots,2n}$ for $k=0,1,2$ span a 1-dimensional vector space, while the rank of $\mathrm{d}F$ is 3 in all the other points. In particular, F is smooth in the open subset of V consisting of $(X_1,\ldots,X_{2g})$ such that the matrix $(X_1|\cdots|X_{2g})$ has rank 3.

This yields a stratification of $\tilde{K} = \tilde{K}_3 \sqcup \tilde{K}_2 \sqcup \tilde{K}_0$, where $\tilde{K}_3$ are the smooth points of $\tilde{K}$, and $\tilde{K}_0$ is a single point. This stratification descends to the stratification $Z_3 \sqcup Z_2 \sqcup Z_0$ of the quotient $Z = \tilde{K}//\mathrm{SO}(3)$.

Proposition 62. *The singularities of Z are symplectic, i. e. for any resolution of the singularities $\tilde{Z} \to Z$ the symplectic form defined on the smooth locus of Z by the symplectic reduction is extendable to a globally defined 2-form.*

Proof. Recall that the natural $2:1$ cover $\varphi : V//\mathrm{SO}(3) \to V//\mathrm{O}(3)$ is ramified in the locus $\mathcal{R}$, wich is exactly the singularity locus of the quotient, that is the set of vectors $(X_1,\ldots,X_{2g})$ whose associated matrix has rank lesser than 3, and outside the ramification locus both $V//\mathrm{SO}(3)$ and $V//\mathrm{O}(3)$ are smooth (cf. Lemma 59).

In particular, F is smooth outside $\mathcal{R}$, so the quotient $Z = F^{-1}(0)//\mathrm{SO}(3)$ is smooth in Z_3. Now, the dimension of Z is the dimension of V minus 3 by the equation $F=0$ and minus 3 by the dimsension of $\mathrm{SO}(3)$, so it is $6g-6$. The dimension of Z_{sing} is $4g-4$, so Z is smooth in codimension $2g-2$, that is greater or equal than 4 for $g \geq 2$. So by Proposition 1.4 in [4] we can directly conclude. □

Now we add the equations $F(X_1,\ldots,X_{2g}) = 0$, wich wee have to express in terms of the invariant coordinates $T_{i,j}, S_{i,j,k}$. First we use the relation $\langle F(X), Y\rangle = 0$ for any $Y \in \mathbb{C}^3$; for $Y = X_l$ this gives

$$0 = \sum_{i=1}^{g} \langle [X_i, X_{g+i}], X_l \rangle$$

that, with the formula $\langle [X,Y],Z\rangle = \det(X|Y|Z)$, brings us to the following linear equations:

$$\sum_{i=1}^{g} S_{i,g+i,l} = 0 \tag{6.8}$$

for any $l = 1,\ldots,2g$. The smallest possible degree of the relations involving $T_{i,j}$ only is 2. These are obtained as before, for $Y = [X_k, X_l]$ we have

$$0 = \sum_{i,j=1}^{g} \langle [X_i, X_{g+i}], [X_k, X_l] \rangle$$

that, with the formula $\langle [X,Y],[Z,U]\rangle = \langle X,Z\rangle\langle Y,U\rangle - \langle X,U\rangle\langle Y,Z\rangle$, leads to

$$\sum_{i,j=1}^{g} T_{i,k}T_{g+i.l} - T_{i,l}T_{g+i,k} = 0 \quad \forall k,l = 1,\ldots,2g. \tag{6.9}$$

This brings to a complete set of equations for Z:

Proposition 63. *The quotient $Z = F^{-1}(0)//SO(V)$ is isomorphic to the spectrum of $\mathbb{C}[T_{i,j}, S_{a,b,c}]_{i,j,a,b,c=1,\ldots,2g}/I$ where I is the ideal generated by*

$$T_{i,j} - T_{j,i} \tag{6.10}$$

$$S_{a,b,c} + S_{b,a,c}\ , \quad S_{a,b,c} + S_{a,c,b} \tag{6.11}$$

$$\begin{vmatrix} T_{i_1,j_1} & T_{i_1,j_2} & T_{i_1,j_3} & T_{i_1,j_4} \\ T_{i_2,j_1} & T_{i_2,j_2} & T_{i_2,j_3} & T_{i_2,j_4} \\ T_{i_3,j_1} & T_{i_3,j_2} & T_{i_3,j_3} & T_{i_3,j_4} \\ T_{i_4,j_1} & T_{i_4,j_2} & T_{i_4,j_3} & T_{i_4,j_4} \end{vmatrix} \tag{6.12}$$

$$S_{a_1,b_1,c_1}S_{a_2,b_2,c_2} - \begin{vmatrix} T_{a_1,b_1} & T_{a_1,b_2} & T_{a_1,b_3} \\ T_{a_2,b_1} & T_{a_2,b_2} & T_{a_2,b_3} \\ T_{a_3,b_1} & T_{a_3,b_2} & T_{a_3,b_3} \end{vmatrix} \tag{6.13}$$

$$\sum_{i=1}^{g} S_{i,g+i,j} \tag{6.14}$$

$$\sum_{i,j=1}^{g} T_{i,k}T_{g+i,l} - T_{i,l}T_{g+i,k} \tag{6.15}$$

We can give an explicit description of these equations for low genera: for $g = 1$ the problem has been studied in [27], where the author finds that the germ of $\mathcal{M}(2,0)$ at $(\mathcal{E}_0, \nabla_0)$ is isomorphic to $\mathbb{C}^2 \times Q$ at 0, where Q is the quadric in $\mathbb{C}^3$ defined by the equation $z_0^2 = z_1z_2$.

For $g = 2$ the problem ireduces to the one examined in [21]: first of all remark that in this case equation (6.14) is simply $S_{1,3,i} + S_{2,4,i} = 0$ for $i = 1,\ldots,4$. This with equation (6.11) implies that all $S_{i,j,k}$ vanish. So equation (6.13) reduces to $\mathrm{Rank}(T_{i,j}) \leq 2$, while equation (6.15) reduces to three quadratic equations.

6.2.1 Singularities in Z_2

In this subsection we determine the type of singularities of Z along the stratum Z_2.

Let U be the open set of $F^{-1}(0)$ where $x_1^{(0)} \neq 0$. On this open U for $k = 1,2$, and for $i = 2,\ldots,g,g+2,\ldots 2g$ set

$$\mu_k = x_1^{(k)}/x_1^{(0)}\ , \tag{6.16}$$

$$x_i^{(1)} = \mu_1 x_i^{(0)} + y_i\ , \tag{6.17}$$

$$x_i^{(2)} = \mu_2 x_i^{(0)} + z_i\ . \tag{6.18}$$

Let $\mathcal{U}$ be the open set of $\mathbb{C}^{2g+2}$ where the first coordinate is different from zero. Then

Lemma 64. *The open set $U \subseteq F^{-1}(0)$ is naturally isomorphic to $\mathcal{U} \times Y$, where Y is the hypersurface of $\mathbb{C}^{4g-4}$ defined by the equation $G(y,z) = 0$, where y_i, z_i for $i = 2, \ldots, g, g+2, \ldots, 2g$ are coordinates on $\mathbb{C}^{4g-4}$ and*

$$G(y,z) = \sum_{i=2}^{g} y_i z_{g+i} - y_{g+i} z_i \ .$$

Proof. We prove this showing that $x_i^{(0)}, \mu_k, y_j, z_j$ for $i = 1, \ldots, 2g$, $k = 1, 2$ and $j = 2, \ldots, g, g+2, \ldots, 2g$, with y, z satisfying $G(y,z) = 0$, are coordinates for V.

Remark that $x_{g+1}^{(1)}, x_{g+1}^{(2)}$ can be recovered in terms of μ_k, y_i, z_i from the second and third equations for the coefficient of $F(A) = 0$:

$$\begin{aligned} x_{g+1}^{(1)} &= \tfrac{1}{x_1^{(0)}}(-x_{g+1}^{(0)} x_1^{(1)} - \textstyle\sum_{i=2}^{g} x_i^{(0)} a_{g+i}^{(1)} - x_i^{(1)} x_{g+i}^{(0)}) = \\ &= \tfrac{1}{x_1^{(0)}}(-\mu_1 x_{g+1}^{(0)} x_1^{(0)} - \textstyle\sum x_i^{(0)} y_{g+i} - x_{g+i}^{(0)} y_i) \end{aligned}$$

and similarly

$$x_{g+1}^{(2)} = \frac{1}{x_1^{(0)}}(-\mu_2 x_{g+1}^{(0)} a_1^{(0)} - \sum x_i^{(0)} z_{g+i} - x_{g+i}^{(0)} z_i)$$

while the first of these equations becomes

$$\sum_{i=2}^{g} (y_i z_{g+i} - y_{g+i} z_i) = 0$$

that is exactly $G(y,z) = 0$.

This, together with equations (6.16), proves the lemma. $\square$

Remark that $\mathcal{U} \times \{0\}$ coincide with $\tilde{K}_2 \cap U$, while if we fix $\xi \in \mathcal{U}$, then $\{\xi\} \times Y$ povides a slice of $\tilde{K}$ passing through ξ and "orthogonal" to $\tilde{K}_2$.

So by Luna's slice theorem we have that $Y//\mathrm{Stab}(\xi)$ is isomorphic to the germ of Z near the class of ξ.

Take for simplicity $\xi = \xi_0 = (a_i^{(0)}, \mu_1, \mu_2)$ with $(a_i^{(0)}) = (1, 0, \ldots, 0)$ and $\mu_1 = \mu_2 = 0$, for the calculation in the general case is only notationally different: if μ_i are different from 0, we can change coordinates in such a way to diagonalize A_1, so that we can assume that μ_i vanish; if $a_i^{(0)} \neq 0$ for some $i > 1$, we can repeat the argument that follows in the same way, since all matrices A_i are proportional to A_1.

The stabilizer of ξ_0 via the PGL_2-action is isomorphic to $\mathbb{C}^*$, acting by

$$\lambda A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} A \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}^{-1}$$

The action of $\mathbb{C}^*$ on Y is given by

$$\lambda(y_2,\ldots,y_2g;z_2,\ldots z_{2g}) = (\lambda^2 y_2,\ldots,\lambda^2 y_{2g};\lambda^{-2}z_2,\ldots,\lambda^{-2}z_{2g}) \ .$$

From this we see that $u_{ij} = y_i z_j$ for $i,j = 2,\ldots,g,g+2,\ldots,2g$ is a basis of $\mathbb{C}^*$-invariant functions on Y, subject to the relations

$$u_{i_1 j_1} u_{i_2 j_2} = u_{i_1 j_2} u_{i_2 j_1} \ .$$

These are the equations of the Segre embedding

$$\mathbb{P}^{2g-3} \times \mathbb{P}^{2g-3} \hookrightarrow \mathbb{P}^{(2g-2)^2-1} \ ,$$

so that the quotient of $\mathrm{SPAN}(y_i,z_i) = \mathbb{C}^{4g-2}$ by this $\mathbb{C}^*$-action is isomorphic to the affine cone over $\mathbb{P}^{2g-3} \times \mathbb{P}^{2g-3}$ embedded in the affine cone of $\mathbb{P}^{(2g-2)^2-1}$.

In these coordinates, the equation $G = 0$ becomes

$$\sum_{i=2}^{g} (u_{i,g+i} - u_{g+i,i})$$

that determines a hyperplane in the cone of $\mathbb{P}^{(2g-2)^2-1}$.

The polynomial G is a homogeneous non-degenerate quadric. We associate to it the matrix $\mathfrak{G}$ given by $G(y,z) = y^t \mathfrak{G} z$. We have

$$\mathfrak{G} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \ .$$

This defines on $\mathbb{C}^{2g-2}$ a non-degenerate bilinear form, that yields an isomorphism $(\mathbb{C}^{2g-2})^* \cong \mathbb{C}^{2g-2}$. So we have the identification $\mathbb{C}^{4g-4} \cong \mathbb{C}^{2g-2} \oplus (\mathbb{C}^{2g-2})^*$.

Via this identification, we see that the equation defining the hyperplane, is actually the incidence divisor

$$\{(y,\eta) \in \mathbb{C}^{2g-2} \oplus (\mathbb{C}^{2g-2})^* \mid \eta(y) = 0\} \ ,$$

so the variety $Y/\mathbb{C}^*$ is isomorphic to the affine cone over the Segre embedding over this incidence variety.

So we have shown that

Proposition 65. *The germ of Z at a point $\xi \in Z_2$ is isomorphic to the germ of the affine cone over the Segre embedding $\mathbb{P}^{2g-3} \times (\mathbb{P}^{2g-3})^* \hookrightarrow \mathbb{P}^{(2g-2)^2-1}$ restricted to the incidence divisor $D = \{(y,\eta) | \eta(y) = 0\}$.*

Chapter 7

Further developments

7.1 Generalization to linear stacks

As far as Theorem 34 is concerned, it looks somewhat unnatural to have a cohomology class in the first filtered piece $F^1H^2(\mathcal{L}, \mathbb{C})$ of the cohomology of $\mathcal{L}$, so a natural question is in which way one should generalize the sheaf of filtered algebras Λ (and the corresponding holomorphic Lie algebroid extensions) to have a correspondence with the full cohomology group $H^2(\mathcal{L}, \mathbb{C})$.

Relatively to a nice open cover $\mathfrak{U} = \{U_\alpha\}$, we can represent a cohomology class in $H^2(\mathcal{L}, \mathbb{C})$ by a triple $(Q_\alpha, \phi_{\alpha\beta}, f_{\alpha\beta\gamma})$ with $Q_\alpha \in \Omega^2_{\mathcal{L}}(U_\alpha)$, $\phi_{\alpha\beta} \in \Omega_{\mathcal{L}}(U_{\alpha\beta})$ and $f_{\alpha\beta\gamma} \in \mathscr{O}_X(U_{\alpha\beta\gamma})$.

Over each U_α it is possible to construct the sheaf of twisted enveloping algebras $\tilde{\mathcal{U}}_{Q_\alpha}\mathcal{L}$, and with $\phi_{\alpha\beta}$ on the double overlaps $U_{\alpha\beta}$ one can construct isomorphisms

$$g_{\alpha\beta} : (\tilde{\mathcal{U}}_{Q_\beta}\mathcal{L})_{|U_{\alpha\beta}} \to (\tilde{\mathcal{U}}_{Q_\alpha}\mathcal{L}_{|U_{\alpha\beta}})$$

as in Section 3.2. The problem is that on the triple intersections $U_{\alpha\beta\gamma}$ these isomorphisms do not satisfy a cocycle relation, but

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = F_{\alpha\beta\gamma}$$

where $F_{\alpha\beta\gamma}$ is constructed from the $f_{\alpha\beta\gamma}$, and precisely satisfies:

$$F_{\alpha\beta\gamma}(f) = f \qquad F_{\alpha\beta\gamma}(x) = x + \mathrm{d}_{\mathcal{L}} f_{\alpha\beta\gamma}(x)$$

for $f \in \mathscr{O}_X$ and $x \in \mathcal{L}$. Since $\check{\delta} f = 0$, on the four-fold intersections the isomorphisms $F_{\alpha\beta\gamma}$ satisfy a cocycle condition.

This situation is typically a stacky one. To have a correspondence similar to the one of Theorem 34, one can try to generalize [5], and introduce, for each open subset $U \subseteq X$, the category $\mathbf{Ext}^{\mathscr{O}_X}_{\mathcal{L}}(U)$ of extensions of Lie algebroids defined over U, whose objects are holomorphic Lie algebroid extensions

$$0 \to \mathscr{O}_{X|U} \to \mathcal{L}' \to \mathcal{L}_{|U} \to 0$$

over U, and the morphisms between two objects are morphisms of Lie algebroid extensions. Since each morphism of Lie algebroids extensions is invertible, $\mathbf{Ext}_{\mathcal{L}}^{\mathscr{O}_X}(U)$ is a groupoid for each U. So we can see $\mathbf{Ext}_{\mathcal{L}}^{\mathscr{O}_X}$ as a controvariant functor from the category of open subsets of X to the category of groupoids.

One can show the following:

Proposition 66 ([1]). $\mathbf{Ext}_{\mathcal{L}}^{\mathscr{O}_X}$ *is a stack.*

Then one should try to generalize Theorem 31, and show that $\mathbf{Ext}_{\mathcal{L}}^{\mathscr{O}_X}(X)$ is in a one to one correspondence with $H^2(\mathcal{L};\mathbb{C})$, and define the enveloping algebra for elements of $\mathbf{Ext}_{\mathcal{L}}^{\mathscr{O}_X}(X)$. The resulting object of this operation will be some kind of generalization of rings of twisted differential operators to the Lie algebroid case.

7.2 Links with deformation quantization

The main question deformation quantization theory asks is: given a manifold (or a variety) and a Poisson structure ϖ on it, whether it is possible to integrate it to a formal $*$-product whose first term equals ϖ? (cf. [3])

Kontsevich's formality theorem gives a complete positive answer to this question in the smooth case, while in the algebraic case there are obstructions for the integration (see [19]).

Our construction of the enveloping algebra for holomorphic Lie algebroids gives an explicit construction of a star product in some cases: let $\mathcal{L}$ be a holomorphic Lie algebroid and $Q \in H^0(X;\Omega^2_{\mathcal{L}})$. Then we can define the Poisson structure on the total space of L^*, the dual of the vector bundle underlying $\mathcal{L}$, as follows:set

$$\varpi(f,x) = -\sharp(x)(f)\ , \quad \varpi(x,y) = [x,y] + Q(x,y)$$

for $f \in \mathscr{O}_X$ and $x, y \in \mathcal{L}$ and extend it to $\mathrm{Sym}^\bullet_{\mathscr{O}_X}\mathcal{L}$ by requiring it to be a biderivation. So ϖ is a Poisson structure on $\mathrm{Sym}^\bullet_{\mathscr{O}_X}\mathcal{L}$, that we see as the sheaf of holomorphic functions on L^* that are polyomial along the fibers.

Then for $\Sigma = [Q] \in F^2H^2(\mathcal{L};\mathbb{C})$, the algebra $\Lambda_{\mathcal{L},\Sigma}$ is a deformation of $\mathrm{Sym}^\bullet_{\mathscr{O}_X}\mathcal{L}$, so gives a $*$-product, whose first term is exactly ϖ. This is an algebraic analogue of the result obtained in [30].

Hopefully also the algebras $\Lambda_{\mathcal{L},\Sigma}$ for $\Sigma \in F^1H^2(\mathcal{L};\mathbb{C})$ (or even in $H^2(\mathcal{L};\mathbb{C})$ when properly defines) admit an interpretation in terms of deformation quantization. It is an interesting question for further works.

7.3 Generalization of λ-connections

Let $\mathcal{E}$ be a coherent $\mathscr{O}_X$-module. For $\lambda \in \mathbb{C}$, define a (integrable) λ-connection on $\mathcal{E}$ to be a map of sheaves

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_X$$

satisfying the Leibniz rule $\nabla(fs) = f\nabla s + \lambda \mathrm{d}f \otimes s$ for any $f \in \mathscr{O}_X$ and $s \in \mathcal{E}$. Clearly, for $\lambda \neq 0$ this is equivalent to a usual (integrable) connection, while for $\lambda = 0$ this is just a $\mathscr{O}_X$-linear map (Higgs field).

One can construct the moduli space of integrable λ-connections using the formalism of Λ-modules, for Λ a one parameter deformation of $\mathrm{Sym}^\bullet \mathcal{T}_X$ into $\mathscr{D}_X$. This construction is particularly interesting because the moduli space of λ-connections is a realization of the twistor space of the hyperkähler structure of the moduli space of Higgs bundles (cf. [36]).

We expect that the theory of Lie algebroids can be used to generalize this construction for a better understanding of the moduli spaces of Higgs bundles and flat connections.

In [8] it is studied (in the smooth case) deformation theory for Lie algebroids. In particular, it is shown that, if we fix a vector bundle L, the space of Lie algebroid structure that L admits is a subspace of the sections of a bundle of differential operators associated to L. In the holomorphic case, this space is finite dimensional.

In the particular case of $L = T_X$, the Lie algebroid structure in L that can be deformed to the canonical one are all of the form $(T_X, \Theta, [\cdot,\cdot]_\Theta)$, where Θ is an endomorphism of T_X and the bracket is given by

$$[U, V]_\Theta = [U, \Theta(V)] + [\Theta(U), V] - \Theta([U, V])$$

and satisfies the Jacobi identity.

If it is possible to define a "universal" algebra Λ on the space parametrizing the Lie algebroid structures on $\mathcal{T}_X$, it should be interesting to study the corresponding moduli space of Λ-modules, that, just as the moduli space of λ-connections does, should encode important informations on the moduli space of Higgs bundles.

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