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Structures périphériques des groupes relativement hyperboliques

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A ma mère et mon père!

Structures périphériques des groupes relativement hyperboliques

Résumé: L'objectif principal de cette thèse est d'étudier les structures périphériques des groupes relativement hyperboliques. En contraste avec l'hyperbolicité ordinaire, l'hyperbolicité relative est définie par rapport à une famille finie de sous-groupes, appelée structure périphérique. Dans cette thèse, on introduit et caractérise une classe de structures paraboliques étendues pour des groupes relativement hyperboliques.

La thèse met également l'accent sur l'étude des sous-groupes relativement quasiconvexes, qui jouent un rôle important en théorie des groupes relativement hyperboliques. Grâce à la flexibilité des structures périphériques, la quasiconvexité relative d'un sous-groupe est caractérisée par rapport aux structures paraboliques étendues. En outre, les sous-groupes relativement quasiconvexes sont étudiés par des méthodes dynamiques en terme des groupes de convergence. Ceci nous conduit à obtenir un théorème décrivant l'intersection des ensembles limites pour une paire de sous-groupes relativement quasiconvexes ; et donner des preuves dynamiques de plusieurs résultats bien connus sur les sous-groupes relativement quasiconvexes.

De plus, on obtient plusieurs résultats dans les groupes kleiniens sur le lien entre les ensembles d'axes et la commensurabilité de deux groupes kleiniens.

Un résultat de la thèse d'intérêt indépendant montre qu'un sous-groupe séparable a la propriété d'empilement borné. Ceci implique que cette propriété est vraie pour tout sous-groupe d'un groupe polycyclique, répondant à une question de Hruska-Wise.

Mots Clés: groupes relativement hyperboliques, structures périphériques, le bord de Floyd, sous-groupes dynamiquement quasiconvexes, ensembles limites

Peripheral structures of relatively hyperbolic groups

Abstract: The main objective of this thesis is to study peripheral structures of relatively hyperbolic groups. In contrast with hyperbolicity, relative hyperbolicity is defined with respect to a finite collection of subgroups, which is referred to as a peripheral structure. In the thesis, we introduce and characterize a class of peripheral structures: parabolically extended structures for relatively hyperbolic groups.

The thesis also focuses on the study of relatively quasiconvex subgroups, which play an important role in the theory of relatively hyperbolic groups. With the flexibility of peripheral structures, relative quasiconvexity of a subgroup is characterized with respect to parabolically extended structures. Moreover, relatively quasiconvex subgroups are studied using dynamical methods in terms of convergence group actions. This leads us to obtain a limit set intersection theorem for a pair of relatively quasiconvex subgroups, and give dynamical proofs of several well-known results on relatively quasiconvex subgroups.

In Kleinian groups, we prove several results on the relationship between the axes sets and commensurability of two Kleinian groups.

A result of independent interest in the thesis is that a separable subgroup has the bounded packing property. This implies that the property is true for each subgroup of a polycyclic group, answering a question of Hruska-Wise.
Keywords: relatively hyperbolic groups, peripheral structures, Floyd boundary, dynamically quasiconvex subgroups, limit sets

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Chapitre I

Introduction

L'objectif de cette thèse est d'étudier des propriétés algébriques et géométriques des groupes relativement hyperboliques. Généralisant le concept de groupe hyperbolique, celui de *groupe relativement hyperbolique* a été proposé par M. Gromov [Gr87] dans sa monographie en 1987. La classe des groupes relativement hyperboliques est avérée être assez large et comprend de nombreuses classes de groupes, par example, groupes kleiniens géométriquement finis, groupes hyperboliques, groupes limites [Da03], groupes des espaces CAT(0) ayant appartements isolés [HK05], et bien d'autres.

Depuis leur apparition dans [Gr87], les groupes relativement hyperboliques ont été étudiés par de nombreux mathématiciens au cours des vingt dernières années. En 1994, B. Farb [Fa98] a introduit un espace combinatoire, *un graphe de Cayley suspendu*, pour étudier la géométrie intrinsèque des groupes relativement hyperboliques. Plus tard, B. Bowditch [Bo99b] a rendue populaire la notion d'hyperbolicité relative de Gromov, en montrant l'équivalence entre la définition de Farb et celle de Gromov. Ces travaux fondamentaux ont permis des études plus avancées sur les groupes relativement hyperboliques dans les années 2000, voir par exemple Tukia [Tu98], Dahmani [Da03], Osin [Os06b] et Gerasimov [Ge09].

Récemment, Hruska [Hr10] a donné un survol sur la théorie de groupes relativement hyperboliques; l'une de ses conclusions affirme que l'hyperbolicité relative peut être étendue dans le cadre des groupes dénombrables. Ce point de vue sera adopté dans notre travail: la plupart des raisonnements restent valables pour des groupes dénombrables relativement hyperboliques.

Contrairement à l'hyperbolicité ordinaire, l'hyperbolicité relative doit être étudiée au moyen d'une famille de sous-groupes, appelée *structure périphérique*. Par conséquent, les propriétés algébriques et géométriques des groupes, qui peuvent être fournies par l'hyperbolicité relative, varient en fonction des structures périphériques considérées. Il est donc intéressant de se poser la question de savoir quel genre de structures périphériques on peut munir à un groupe relativement hyperbolique, et cela constitue le but principal pour la présente thèse.

La thèse met également l'accent sur l'étude des *sous-groupes relativement quasiconvexes*, lesquels constituent une classe naturelle de sous-groupes des groupes relativement hyperboliques. L'étude de tels sous-groupes a été initiée dans les travaux de Dahmani [Da03] et d'Osin [Os06b], puis poursuivie par d'autres auteurs. En particulier, comme le montrent [Hr10], [GP09a] et [GP11], des différentes définitions de la quasiconvexité relative sont équivalentes dans les relativement hyperboliques.

La quasiconvexité relative d'un sous-groupe d'un groupe relativement hyperboliques dépend de la structure périphérique mise sur le groupe en question; autrement dit, ce sous-groupe peut être ou non relativement quasiconvexe si l'on modifie la structure périphérique du groupe initial. Ainsi nous allons caractériser, pour un sous-groupe donné, la façon dont varie sa quasiconvexité relative en fonction des différentes structures périphériques envisagées. L'étude de ces sous-groupes fait intervenir des méthodes dynamiques en termes des groupes de convergence. Cela nous permet de décrire l'intersection des ensembles limites associés à toute paire de sous-groupes relativement quasiconvexes. Par ailleurs, nous donnerons des preuves dynamiques de plusieurs résultats bien connus sur les sous-groupes relativement quasiconvexes.

Dans ce qui suit, nous allons revisiter brièvement les trois approches de l'hyperbolicité relative, avant de donner un aperçu sur nos principaux résultats.

I.1 Approches aux groupes relativement hyperboliques

Les groupes relativement hyperboliques font objet d'étude chez différents auteurs et avec divers points de vue (cf. [Fa98], [Bo99b], [Tu98], [Os06b], [DS05] et [Ge09]). Sans une tentative d'exhaustivité, nous ne parlons que des approches qui seront utilisées dans la thèse. Comme le montre Hruska [Hr10], toutes ces approches conduisent à des définitions équivalentes sur l'hyperbolicité relative pour les groupes dénombrables.

Soit G un groupe dénombrable avec une famille finie de sous-groupes $\mathbb{H} = \{H_i\}_{i\in I}$. Cette famille $\mathbb{H}$ est appelée *structure périphérique* de G.

I.1.1 Modèle de Gromov

Dans la définition initiale de l'hyperbolicité relative, Gromov considère une action proprement discontinue d'un groupe G sur un espace métrique propre δ-hyperbolique (X, ρ). Alors G est *relativement hyperbolique* si le quotient de X par l'action G est quasi-isométrique à la réunion d'un nombre fini de demi-droites $[0, \infty[$ dans laquelle les points à l'origine sont identifiés.

Suite à ses travaux sur la finitude géométrique (cf. [Bo93], [Bo95]), Bowditch a revu la définition de Gromov dans [Bo99b]. Son approche est basée sur l'étude dynamique de l'action de G à l'infini, *le bord de Gromov* ∂X de X. En effet, l'action induite de G sur ∂X est une action d'un groupe de convergence, cf. [Fr95], [Tu98] et [Bo99a].

Dans ce cadre, notons $\mathbb{H}$ un ensemble de représentants des classes de conjugaison de sous-groupes paraboliques maximaux.

Soit Π l'ensemble des points paraboliques de G, et $\mathbb{U} = \{B_p : p \in \Pi\}$ une collection d'horoboules ouvertes, où B_p désigne la horoboule centrée en p. Par définition, $\mathbb{U}$ est dite *G-équivariante* si $B_{gp} = gB_p$ pour tout $g \in G$, $p \in \Pi$; elle sera dite *r-séparée* pour un certain $r > 0$ si $\rho(B_p, B_q) > r$ pour tout distincts $p, q \in \Pi$. Posons enfin $Y(\mathbb{U}) = X \setminus \bigcup_{p\in\Pi} B_p$.

Définition (Définition de Gromov)**.** La paire $(G, \mathbb{H})$ est dite *relativement hyperbolique* s'il existe une collection G-équivariante r-séparée d'horoboules ouvertes $\mathbb{U}$ pour un (ou tout) $r > 0$, telle que le quotient de $Y(\mathbb{U})$ par l'action de G est compact. On dit aussi que G est relativement hyperbolique par rapport à $\mathbb{H}$.

Notons que si $(G, \mathbb{H})$ est relativement hyperbolique, alors l'action de G sur ∂X est géométriquement finie. Dans ce cas, il en résulte que $\mathbb{H}$ est fini, d'après Tukia [Tu98].

I.1.2 Modèle de Farb

Dans sa thèse, Farb [Fa98] a présenté un graphe de Cayley suspendu pour étudier la géométrie des groupes relativement hyperboliques.

Etant donnée une structure périphérique $\mathbb{H}$, on suppose que G est *finiment engendré par rapport à* $\mathbb{H}$. C'est à dire, il existe une partie finie X de G telle que $X \cup (\cup_{H \in \mathbb{H}} H)$ constitue une famille génératrice de G. Désignons par $\mathcal{H}$ l'ensemble alphabet $\bigsqcup_{H \in \mathbb{H}} (H \setminus \{1\})$.

Le *graphe de Cayley suspendu* de G est obtenu à partir du graphe de Cayley $\mathscr{G}(G, X)$ en adjoignant à ce dernier: (i) un *sommet de cône* v_{gH} pour chaque classe à gauche $gH \in \bigcup_{H \in \mathbb{H}} G/H$ et (ii) des arêtes de longueur $1/2$ associés à chaque classe gH, entre le sommet de cône v_{gH} et chacun des éléments de gH. Les classes gH à gauche sont appelées *classes périphériques*.

Grâce aux sommets de cone qu'on vient de rajouter, l'action de G devient cocompacte sur le graphe ainsi obtenu, dit le graphe de Cayley suspendu. Ce point de vue est également employé par Bowditch [Bo99b] pour formuler l'hyperbolicité relative en termes des actions des groupes.

Lorsque l'on ignore le rôle des sommets de cône, un graphe de Cayley suspendu devient alors une copie isométrique du graphe $\mathscr{G}(G, X \cup \mathcal{H})$ de Cayley de G par rapport à $X \sqcup \mathcal{H}$. On dit que le graphe $\mathscr{G}(G, X \cup \mathcal{H})$ est un *graphe de Cayley relatif* de G par rapport à $\mathbb{H}$. Le graphe de Cayley relatif est particulièrement adapté à une étude combinatoire des groupes relativement hyperboliques, comme le démontrent les travaux d'Osin (cf. [Os06b], [Os10], [Os07]). Dans ce qui suit, on va considérer directement les graphes de Cayley relatifs.

En outre, Farb [Fa98] a introduit une propriété de *pénétration bornée des classes à gauche*(BCP) sur la géométrie des graphes de Cayley relatifs, afin de décrire la propriété de poursuite de deux géodésiques par rapport à des classes périphériques. C'est à dire, étant données deux géodésiques reliant les mêmes extrémités, si une géodésique traverse une classe périphérique assez long, alors il faut que l'autre entre et sorte de la classe même périphérique d'une façon uniformément proche.

Définition (Définition de Farb)**.** On dit que $(G, \mathbb{H})$ est *relativement hyperbolique* si le graphe de Caylay $\mathscr{G}(G, X \cup \mathcal{H})$ est hyperbolique et la paire $(G, \mathbb{H})$ satisfait la propriété de BCP.

Soit d une métrique propre et invariante à gauche sur le groupe dénombrable G.

Une telle métrique propre existe, compte tenu du fait qu'un groupe dénombrable peut être prolongé dans un groupe engendré par deux éléments.

Par analogie avec la notion d'un ensemble convexe dans un espace métrique, la notion suivante de la quasiconvexité des sous-groupes est introduite par Osin [Os06b] pour les groupes relativement hyperboliques.

Définition (Sous-groupes relativement quasiconvexes)**.** Supposons que $(G, \mathbb{H})$ est relativement hyperbolique. Un sous-groupe Γ de G est appelé *relativement σ-quasiconvexe* par rapport à $\mathbb{H}$ s'il existe une constante $\sigma = \sigma(d) > 0$ telle que toute géodésique de $\mathscr{G}(G, X \cup \mathcal{H})$ d'extrémités dans Γ est contenue dans le σ-voisinage de Γ relatif à la métrique d.

Dans le chapitre 1, les groupes relativement hyperboliques sont étudiés en utilisant cette approche combinatoire.

I.1.3 L'approche dynamique

L'approche dynamique se place dans le contexte des actions de groupes de convergence. Un groupe de convergence sur la sphère S^n est d'abord introduit par Gehring-Martin [GM87], comme une généralisation topologique d'un groupe kleinien. Plus tard, la théorie de groupes de convergence a été développée dans un cadre assez général, voir Bowditch [Bo99a], Freden [Fr95] et Tukia [Tu94].

Soit M un espace compact métrisable. Une *action de convergence d'un groupe G* sur M est une action telle que G agit de façon proprement discontinue sur l'espace $\Theta^3(M)$ de triples distincts.

Une grande partie des groupes de convergence provient des actions sur les bords de Gromov, induites par des actions proprement discontinues sur des espaces δ-hyperboliques. Une autre classe intéressante de groupes de convergence sont des groupes de type fini avec un bord de Floyd non-trivial (cf. Karlsson [Ka03]).

D'une façon analogue à ce qu'on fait en théorie des groupes kleiniens, on peut considérer l'*ensemble limite* $\Lambda(G)$ de G comme l'ensemble des points d'accumulation de G-orbites dans M. En particulier, il existe deux types de points limites qui méritent une attention particulière: *points paraboliques bornés* et *points coniques*. En termes de points limites, l'hyperbolicité relative peut être formulée comme suit.

Définition (Groupes géométriquement finis)**.** Un groupe G de convergence sur M est dit *géométriquement fini* si tout point de M est conique ou parabolique borné.

Lorsque $\mathbb{H}$ est un ensemble de représentants des classes de conjugaison de sous-groupes paraboliques maximaux, la paire $(G, \mathbb{H})$ est dite *relativement hyperbolique.*

Dans la définition précédente, l'espace compact M est appelé *bord de Bowditch* de $(G, \mathbb{H})$. Lorsqu'on considère des différentes actions géométriquement finies de G, le bord de Bowditch de $(G, \mathbb{H})$ est souvent désigné par $T_{\mathbb{H}}$ pour une $\mathbb{H}$ donnée. Dans ce cas, la notation $\Lambda_{\mathbb{H}}(\Gamma)$ indique l'ensemble limite d'un sous-groupe $\Gamma \subset G$ par rapport à $G \curvearrowright T_{\mathbb{H}}$.

On dit que l'action d'un groupe G sur M est *2-cocompacte* si l'action induite de G sur l'espace $\Theta^2(M)$ de sous-ensembles de cardinal deux est cocompacte. Dans [Ge09], Gerasimov a montré que les groupes de convergence admettant une action 2-cocompacte sont géométriquement finis (même sans supposer que M soit métrisable). La réciproque est obtenue dans le travail de Tukia [Tu98]. Par conséquent, la donnée d'un groupe de convergence admettant une action 2-cocompacte équivaut à celle d'un groupe hyperbolique relative, ce qui donne lieu à une autre caractérisation dynamique de l'hyperbolicité relative.

Dans [Bo99b], une notion de la quasiconvexité dynamique est introduite pour un groupe de convergence; elle est équivalente à celle de la quasiconvexité géométrique dans le cas des groupes hyperboliques.

Définition (Sous-groupes dynamiquement quasiconvexes)**.** Un sous-groupe H d'un groupe de convergence G est dit *dynamiquement quasiconvexe* si, étant donnés des sous-ensembles disjoints fermés K et L de M, l'ensemble

$$\{gH \in G/H : g\Lambda(H) \cap K \neq \emptyset \text{ et } g\Lambda(H) \cap L \neq \emptyset\}$$

est fini.

Dans [GP09a] et [GP11], Gerasimov-Potyagailo ont montré que les sous-groupes dynamiquement quasiconvexes sont exactement les sous-groupes relativement quasiconvexes dans les groupes relativement hyperboliques. Cela permet une étude dynamique pour les sous-groupes relativement quasiconvexes; voir Chapitre 2.

I.2 Nos principaux résultats

Pour cette partie, sauf mention le contraire, on considère des groupes dénombrable relativement hyperboliques et leurs sous-groupes dénombrables relativement quasiconvexes. Dans ce cas, les sous-groupes périphériques ne sont pas nécessairement de type fini.

I.2.1 Structures paraboliques étendues (Chapitre 1)

Dans différents contextes, un groupe infini peut être relativement hyperbolique par rapport à des différentes structures périphériques. Un exemple typique est le groupe modulaire $\mathrm{PSL}(2,\mathbb{Z})$, qui a été bien étudié dans de nombreux domaines. Ce groupe, comme le groupe fondamental d'une surface de Riemann de volume fini avec un cusp, est relativement hyperbolique par rapport à un sous-groupe $\mathbb{Z}$. D'autre part, $\mathrm{PSL}(2,\mathbb{Z})$ peut être décomposé comme un produit libre de deux groupes cycliques finis. Ainsi, il est hyperbolique au sens de Gromov.

D'ailleurs, en tant que groupe relativement hyperbolique, $\mathrm{PSL}(2,\mathbb{Z})$ a un bord de Bowditch associé—le cercle unité S^1. Pourtant, le bord de Gromov de $\mathrm{PSL}(2,\mathbb{Z})$, en tant que groupe hyperbolique, est l'ensemble de Cantor totalement disconnexe. Il est bien connu qu'il existe une application équivariante de l'espace de Cantor dans le cercle unité.

Une autre motivation est le fait que des structures périphériques des groupes relativement hyperboliques peuvent être raffinées. C'est à dire, si G est relativement hyperbolique par rapport à un sous-groupe Γ et que Γ est encore relativement hyperbolique par rapport à un sous-groupe H, alors G est relativement hyperbolique par rapport à H. cf. Drutu-Sapir [DS05, Corollary 1.14].

L'exemple du groupe modulaire nous suggère à définir, dans le chapitre 1, une classe de structures paraboliques étendues pour les groupes relativement hyperboliques.

Soit G un groupe dénombrable avec une collection finie de sous-groupes $\mathbb{H} = \{H_i\}_{i \in I}$. Cette collection $\mathbb{H}$ est appelée *structure périphérique* de G. Soit $\mathbb{P} = \{P_j\}_{j \in J}$ une autre structure périphérique, telle que pour chaque $i \in I$, il existe $j \in J$ avec $H_i \subset P_j$; on dira que $\mathbb{P}$ est une *structure périphérique étendue* pour $(G, \mathbb{H})$. En outre, si $(G, \mathbb{P})$ est relativement hyperbolique, alors $\mathbb{P}$ est appelée *structure parabolique étendue* pour $(G, \mathbb{H})$.

Notre principal résultat est de donner une caractérisation d'une structure périphérique étendue. Définissons $\mathbb{H}_P = \{H_i : H_i \subset P, i \in I\}$ pour un $P \in \mathbb{P}$. Rappelons qu'un sous-groupe $\Gamma \subset G$ est *faiblement malnormale* si $\Gamma \cap g\Gamma g^{-1}$ is fini pour tout $g \in G \setminus \Gamma$.

Théorème (Théorème 1.1.1). *Supposons que $(G, \mathbb{H})$ relativement hyperbolique et $\mathbb{P}$ une structure périphérique étendue pour $(G, \mathbb{H})$. Alors $(G, \mathbb{P})$ est relativement hyperbolique si, et seulement si, chaque $P \in \mathbb{P}$ satisfait les propriétés suivantes:*

(P1). P est relativement quasiconvexe par rapport à $\mathbb{H}$,

(P2). P est faiblement malnormale, et

(P3). $P \cap gHg^{-1}$ est fini pour tout $g \in G$ et $H \in \mathbb{H} \setminus \mathbb{H}_P$.

Remarque. Les résultats suivants conus sont des cas particuliers du théorème cidessus:

1. lorsque G est hyperbolique, voir Gersten [Ge96] et Bowditch [Bo99a];
2. pour chaque $P \in \mathbb{P}$, soit $P \in \mathbb{H}$ soit $\mathbb{H}_P$ est le singleton contenant le sousgroupe trivial uniquement, voir Osin [Os06b].

Afin de démontrer ce théorème, on modifie des chemins dans le graphe de Cayley relatif $\mathscr{G}(G, X \cup \mathcal{P})$ et définit leurs *relèvements* dans $\mathscr{G}(G, X \cup \mathcal{H})$. Sous les conditions (P1)–(P3), plusieurs propriétés des chemins originaux sont préservées par l'opération du relèvement. En particulier, le relèvement d'une quasigéodésique sans retour reste encore une quasigéodésique sans retour. Cette propriété nous conduit à démontrer l'hyperbolicité relative de $(G, \mathbb{P})$ à l'aide de la définition de Farb. Plus précisément, nous avons d'abord relever des quasigéodésiques du $\mathscr{G}(G, X \cup \mathcal{P})$ à $\mathscr{G}(G, X \cup \mathcal{H})$, et puis appliquer la définition de Farb à la paire $(G, \mathbb{H})$ pour vérifier l'hyperbolicité du $\mathscr{G}(G, X \cup \mathcal{P})$ et la propriété BCP pour la paire $(G, \mathbb{P})$.

L'étude du relèvement de chemins consiste à se servir du prolongement quasiisométrique d'un sous-groupe relativement quasiconvexe dans le groupe ambiant relativement hyperbolique. Ce prolongement quasi-isométrique est construit explicitement de telle façon qu'une quasigéodésique sans retour soit envoyée sur une quasigéodésique sans retour. En plus, cette propriété nous permet de donner une nouvelle preuve du théorème de Hruska [Hr10] affirmant qu'un sous-groupe relativement quasiconvexe est relativement hyperbolique.

Afin de comprendre des structures paraboliques étendues, les bords de Bowditch des groupes relativement hyperboliques sont également étudiés d'un point de vue dynamique. Ceci est basé sur le théorème suivant, dû à Gerasimov [Ge10], qui montre que le bord de Floyd est un bord universel pour un groupe relativement hyperbolique.

Soit $\partial_f G$ le bord de Floyd par rapport à une fonction f pour un groupe G de type fini; voir Floyd [Fl80] pour la définition correspondante.

Théorème (Théorème de l'application de Floyd [Ge10])**.** *Soit G un groupe de type fini. S'il admet une action 2-cocompacte sur un espace compact T contenant au moins 3 points, alors il existe une application G-équivariante $\phi : \partial_f G \to T$, avec $f(n) = \alpha^n$ pour $\alpha \in]0, 1[$ suffisamment proche de 1. De plus $\Lambda(G) = \phi(\partial_f G)$.*

En utilisant le théorème de l'application de Floyd, il est facile de voir que les bords de Bowditch associés aux structures paraboliques étendues ne sont que les quotients équivariants des bords attachés aux structures originales. En outre, le noyau d'une application de Floyd est décrite par Gerasimov-Potyagailo [GP09b], [GP10]. La Proposition suivante découle directement du Théorème A dans [GP09b].

Dans l'énoncé suivant, on désigne par $T_{\mathbb{H}}$ le bord de Bowditch du groupe relativement hyperbolique $(G, \mathbb{H})$ et, $\Lambda_{\mathbb{H}}(G_p)$ l'ensemble limite du stabilisateur G_p de $p \in T_{\mathbb{P}}$ par rapport à $G \curvearrowright T_{\mathbb{H}}$.

Proposition (Lemmes 1.4.15 & 1.4.16)**.** *Supposons que $(G, \mathbb{H})$ est relativement hyperbolique. Soit $\mathbb{P}$ une structure parabolique étendue pour $(G, \mathbb{H})$. Alors il existe une application G-équivariante surjective $\varphi : T_{\mathbb{H}} \to T_{\mathbb{P}}$ telle que φ^{-1} est injective sur les points coniques de $G \curvearrowright T_{\mathbb{H}}$. En outre,*

$$\varphi^{-1}(p) = \Lambda_{\mathbb{H}}(G_p)$$

pour tout point parabolique $p \in T_{\mathbb{P}}$.

En plus, nous montrons le résultat suivant: si un groupe G est de type fini et relativement hyperbolique et que l'action de G sur son bord de Floyd est géométriquement finie, alors les structures périphérique étendues sont les seules structures possibles pour que G reste un groupe relativement hyperbolique.

D'autre part, il existe en effet des groupes relativement hyperboliques qui ne sont pas géométriquement finis sur leurs bords de Floyd. Cela est indiqué dans le résultat suivant, qui s'obtient en utilisant un résultat de Behrstock-Drutu-Mosher [BDM09, Proposition 6.3] et [GP09b, Theorem A].

Théorème (Théorème 1.4.23)**.** *Un groupe inaccessible de Dunwoody [Du91] ne peut pas agir sur son bord de Floyd de manière à ce que l'action soit géométriquement finie.*

Notons que le groupe de Dunwoody agit de façon géométriquement finie sur son bord de Bowditch.

I.2.2 Sous-groupes relativement quasiconvexes (Chapitres 1 & 2)

Grâce à la flexibilité de structures périphériques, il est intéressant d'étudier comment varient les propriétés algébriques et géométriques des groupes relativement hyperboliques. Un exemple concernant la quasiconvexité relative d'un sous-groupe est examiné au paragraphe 4, Chapitre 1.

Par des approches dynamiques, on obtient une caractérisation de la quasiconvexité relative par rapport aux structures paraboliques étendues. On remarque qu'un cas particulier est déjà étudié dans Martinez-Pedroza [MP09].

Théorème (Théorème 1.1.3). *Supposons que $(G, \mathbb{H})$ est relativement hyperbolique et que $\mathbb{P}$ est une structure parabolique étendue pour $(G, \mathbb{H})$. Si $\Gamma \subset G$ est relativement quasiconvexe par rapport à $\mathbb{H}$, alors Γ est relativement quasiconvexe par rapport à $\mathbb{P}$.*

Réciproquement, supposons que $\Gamma \subset G$ est relativement quasiconvexe par rapport à $\mathbb{P}$. Alors Γ est relativement quasiconvexe par rapport à $\mathbb{H}$ si, et seulement si, $\Gamma \cap gPg^{-1}$ est relativement quasiconvexe par rapport à $\mathbb{H}$ pour tout $g \in G$ et $P \in \mathbb{P}$.

La première assertion de ce théorème découle du fait suivant: la quasiconvexité dynamique est preservée sous une application équivariante; on complète sa preuve à l'aide de la proposition précédente. Toutefois, la preuve de la seconde assertion est reposée sur une construction d'un domaine compact fondamental pour l'action de Γ sur l'espace $\Theta^2(\Lambda_{\mathbb{H}}(\Gamma))$ des sous-ensembles de cardinal deux. Ceci devient possible grâce à ladite proposition, en relevant un domaine compact fondamental de l'action 2-cocompacte de Γ sur $\Lambda_{\mathbb{P}}(\Gamma)$. On peut alors montrer que Γ opère de faççon 2-cocompacte sur $\Lambda_{\mathbb{H}}(\Gamma)$ et ainsi son action est géométriquement finie sur $\Lambda_{\mathbb{H}}(\Gamma)$.

Le chapitre 2 est consacré à une étude sur les sous-groupes dynamiquement quasiconvexes dans des groupes de convergence, en particulier sur les applications dans des sous-groupes relativement quasiconvexes. Comme le montre Gerasimov-Potyagailo [GP09a], la quasiconvexité relative d'un sous-groupe équivaut à sa quasiconvexite dynamique dans le cas des groupes géométriquement finis.

Nous allons démontrer que les sous-groupes dynamiquement quasiconvexes, même dans des cas plus généraux, possèdent des nombreuses propriétés algébriques avec des sous-groupes relativement quasiconvexes. En particulier, plusieurs résultats bien connus sur des ensembles limites de groupes kleiniens géométriquement finis sont éstablis pour des sous-groupes dynamiquement quasiconvexes. Voir quelques-uns d'entre eux dans le chapitre 2.

Une application particulière est faite sur la propriété de l'intersection des sous-groupes relativement quasiconvexes. Dans le chapitre 2, un théorème décrivant l'intersection des ensembles limites est obtenu dans des groupes relativement hyperboliques, dont certains cas particuliers ont été connus dans [Da03] .

Théorème (Théorème 2.1.1). *Soit H, J deux sous-groupes relativement quasiconvexes d'un groupe G de type fini relativement hyperbolique. Alors,*

$$\Lambda(H) \cap \Lambda(J) = \Lambda(H \cap J) \bigsqcup E$$

où l'ensemble exceptionnel E comprend les points limites isolés dans $\Lambda(H) \cap \Lambda(J)$.

Un corollaire de ce théorème est que l'intersection de la paire de sous-groupes relativement quasiconvexes est encore relativement quasiconvexes. Ce résultat a été demontré à l'aide d'une approche géométrique, voir Hruska [Hr10] et Martinez-Pedroza [MP08] .

Les autres applications de la quasiconvexité dynamique sont fournies par le resultat suivant.

Théorème (Théorème 2.1.5). *Soit H un sous-groupe non-distordu d'un groupe G de type fini avec un bord de Floyd non-trivial. Alors H est dynamiquement quasiconvexe.*

Remarque. L'existence de l'application de Floyd, prouvée dans [Ge10], implique que les groupes non-élémentaires, de type fini et relativement hyperboliques ont un bord de Floyd non-trivial.

Compte tenu du théorème de l'application de Floyd, il s'en suit facilement que des sous-groupes non-distordus sont dynamiquement quasiconvexes dans des groupes relativement hyperboliques. Les sous-groupes non-distordus sont ainsi relativement quasiconvexes, c'est ce qui a été montré par Hruska [Hr10].

I.2.3 Propriétés d'empilement borné de sous-groupes séparables (Chapitre 3)

La propriété d'empilement bornée a été introduite dans Hruska-Wise [HW09] pour un sous-groupe d'un groupe dénombrable. En gros, un sous-groupe d'empilement borné demande une borne supérieure finie sur le nombre de ses classes à gauche qui sont deux à deux proches dans un groupe.

Cette propriété est une généralisation de la notion de largeur d'un sous-groupe dans un groupe hyperbolique (cf. [GMRS98]). En particulier, Hruska-Wise [HW09] a montré que la propriété d'empilement borné implique que la largeur d'un sous-groupe relativement quasiconvexe est finie. Dans le chapitre 4, nous montrons le résultat suivant qui présente lui-même un certain intérêt indépendant.

Théorème (Théorème 3.1.3). *Si H est un sous-groupe séparable d'un groupe G dénombrable, alors H a la propriété d'empilement borné.*

Il est bien connu que chaque sous-groupe d'un groupe polycyclique est séparable. Ainsi, ce théorème donne une réponse positive à une question de Hruska-Wise [HW09], qui demande si chaque sous-groupe d'un groupe virtuellement polycyclique a la propriété d'empilement borné.

I.2.4 Commensurabilités de groupes kleiniens (Chapitre 4)

Le chapitre 5 est basé sur le travail en commun avec Yueping Jiang [YJ10], sur une question posée par J. Anderson.

Soit $\mathrm{Isom}(\mathbf{H}^n)$ le groupe des isométries de l'espace hyperbolique $\mathbf{H}^n$ de dimension n. On désigne par $\mathrm{Ax}(G)$ l'ensemble des axes d'éléments hyperboliques de $G \subset \mathrm{Isom}(\mathbf{H}^n)$.

Question. Si $G_1, G_2 \subset \mathrm{Isom}(\mathbf{H}^n)$ sont de type fini et discrets, est-ce que $\mathrm{Ax}(g_1) = \mathrm{Ax}(G_2)$ implique que G_1 et G_2 sont commensurables ?

La question reçoit une réponse positive quand G_1 et G_2 sont des groupes fuchsiens de type fini [Me90] ou sont des groupes kleiniens arithmétiques [LR98].

Un exemple de Susskind [Su01] montre que la réponse à la question est généralement négative lorsque G_1 et G_2 engendrent un groupe non-discret. Ainsi on est amené à supposer que G_1 et G_2 peuvent être inclus dans un groupe discret; sous cette hypothèse, nous montrons le résultat suivant en cas de dimension 3.

Théorème (Théorème 4.1.3)**.** *Soit G_1, G_2 deux sous-groupes non-élémentaires de type fini d'un groupe kleinien $G \subset \mathrm{Isom}(\mathbf{H}^3)$ et supposons que G est de covolume infini. Alors G_1 et G_2 sont commensurables si, et seulement si,* $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2)$.

Chapitre II
Introduction

The objective of this thesis is to study the algebraic and geometric properties of relatively hyperbolic groups. The concept of a *relatively hyperbolic group* was proposed by M. Gromov in his 1987 monograph [Gr87], as a generalization of the concept of a hyperbolic group. The class of relatively hyperbolic groups has been proven to be wide enough to encompass many naturally occurring groups, such as geometrically finite Kleinian groups, word hyperbolic groups, limit groups [Da03], CAT(0)-groups with isolated flats [HK05] and many others.

In the 20 years since their introduction by Gromov [Gr87], relatively hyperbolic groups have inspired numerous studies. In 1994, B. Farb [Fa98] introduced a combinatorial space, *a coned-off Cayley graph*, to study the intrinsic geometry of relatively hyperbolic groups. Later, B. Bowditch [Bo99b] elaborated on Gromov's definition of relative hyperbolicity, and proved the equivalence of Farb and Gromov's definitions. These foundational works opened the way to further studies of relatively hyperbolic groups in 2000s, see for example, Tukia [Tu98], Dahmani [Da03], Osin [Os06b] and Gerasimov [Ge09].

Recently, Hruska [Hr10] surveyed the theory of relatively hyperbolic groups and concluded that relative hyperbolicity is well-established for countable groups. This point of view has been adopted in our work: most of the analysis are carried out for countable relatively hyperbolic groups.

In contrast to the ordinary hyperbolicity, relative hyperbolicity needs to be discussed with reference to a collection of subgroups. This preferred collection of subgroups is called a *peripheral structure*. Consequently, the algebraic and geometric properties of groups which can be captured by relative hyperbolicity vary with the preferred peripheral structures. So it is an interesting question to consider what kind of peripheral structures we can endow on a given relatively hyperbolic group. This will be the main concern addressed in this thesis.

The thesis also focuses on the study of *relatively quasiconvex subgroups*, which play an important role in the theory of relatively hyperbolic groups. The study of relatively quasiconvex subgroups began with the work of Dahmani [Da03] and Osin [Os06b], and is receiving a great deal of attention. In particular, it is shown in [Hr10], [GP09a] and [GP11] that various definitions of relative quasiconvexity are equivalent in relatively hyperbolic groups.

The relative quasiconvexity of a given subgroup in a relatively hyperbolic group is defined with respect to the peripheral structure of the ambient group; as a consequence, this subgroup may not be relatively quasiconvex with respect to other peripheral structures on the same group. So we will also characterize how the relative quasiconvexity of a given subgroup varies with the given peripheral structures. A dynamical approach in terms of convergence group actions has been taken to study relatively quasiconvex subgroups. This also leads us to a limit set intersection theorem for a pair of relatively quasiconvex subgroups, and gives dynamical proofs of several well-known results on relatively quasiconvex subgroups.

In what follows we shall briefly discuss three approaches to relative hyperbolicity, and then outline our main results.

II.1 Approaches to relative hyperbolicity

Relatively hyperbolic groups have rich structures and have been explored through a number of different approaches (cf.[Fa98], [Bo99b], [Tu98], [Os06b], [DS05], [Ge09]). Without any attempt at completeness, we only discuss the approaches used in this thesis. As shown in Hruska [Hr10], these approaches yield equivalent definitions of relative hyperbolicity for countable groups.

Let G be a countable group with a finite collection of subgroups $\mathbb{H} = \{H_i\}_{i \in I}$. This finite collection is often referred to as the *peripheral structure* of G.

II.1.1 Gromov's Model

In the original definition of relative hyperbolicity, Gromov considered a properly discontinuous action of G on a proper δ-hyperbolic space X. Then G is *relatively hyperbolic in sense of Gromov*, if the quotient of X by the action G is quasi-isometric to a union of finitely many closed half-rays $[0, \infty)$ with the initial endpoints identified.

In [Bo99b], Bowditch gave a rigorous treatment of Gromov's definition, based on his earlier works on geometrical finiteness (cf. [Bo93], [Bo95]). His approach is closely related to the dynamical action of G on the infinity, the *Gromov boundary* ∂X, of X. It is known that the induced action of G on ∂X is a convergence group action, cf. [Fr95], [Tu98] and [Bo99a].

In this setting, let $\mathbb{H}$ be a set of representatives of conjugacy classes of maximal parabolic subgroups.

Let Π be the set of all parabolic points of G, and $\mathbb{U} = \{B_p : p \in \Pi\}$ a collection of open horoballs such that B_p is centered at p. By definition, $\mathbb{U}$ is called *G-equivariant* if $B_{gp} = gB_p$ for all $g \in G, p \in \Pi$; and *r-separated* for some $r > 0$ if $d(B_p, B_q) > r$ for any distinct $p, q \in \Pi$. We write $Y(\mathbb{U}) = X \setminus \bigcup_{p \in \Pi} B_p$.

Definition (Gromov's definition)**.** The pair $(G, \mathbb{H})$ is *relatively hyperbolic* if there is a G-equivariant r-separated collection $\mathbb{U}$ of open horoballs for some (or any) $r > 0$, such that the quotient of $Y(\mathbb{U})$ by the action of G is compact. We also say G is hyperbolic relative to $\mathbb{H}$.

Note that if $(G, \mathbb{H})$ is relatively hyperbolic, then the action of G on ∂X is geometrically finite. So $\mathbb{H}$ is finite by a result of Tukia [Tu98].

II.1.2 Farb's Model

In his thesis, Farb [Fa98] introduced a coned-off Cayley graph to study the geometry of relatively hyperbolic groups.

Given a peripheral structure $\mathbb{H}$, we assume that G is finitely generated with respect to $\mathbb{H}$. That is to say, there exists a finite set X such that $X \cup (\cup_{H \in \mathbb{H}} H)$ generates G. Let $\mathcal{H}$ denote the alphabet set $\bigsqcup_{H \in \mathbb{H}} (H \setminus \{1\})$.

The *coned-off Cayley graph* of G is obtained from the Cayley graph $\mathscr{G}(G, X)$ by adjoining a *cone vertex* v_{gH} for each coset $gH \in \bigcup_{H \in \mathbb{H}} G/H$, and new edges of length $1/2$ from the cone vertex v_{gH} to each element of gH. The cosets gH are called *peripheral* cosets.

Paying attention to cone vertices, we observe that G admits a cocompact group action on its coned-off Cayley graph. This point of view is further explored in Bowditch [Bo99b] to formulate relative hyperbolicity using group actions.

When cone vertices are "removed", a coned-off Cayley graph is in fact an isometric copy of the Cayley graph $\mathscr{G}(G, X \cup \mathcal{H})$ of G with respect to $X \sqcup \mathcal{H}$. We call $\mathscr{G}(G, X \cup \mathcal{H})$ a *relative Cayley graph* of G with respect to $\mathbb{H}$. The relative Cayley graph is particularly suited to a combinatorial study of relatively hyperbolic groups, as demonstrated in a series of works by Osin (cf. [Os06b], [Os10], [Os07]). In the sequel, we will deal directly with relative Cayley graphs.

Additionally, Farb [Fa98] introduced a *Bounded Coset Penetration Property* (or BCP property for short) on the geometry of relative Cayley graphs, describing a fellow traveler property with respect to peripheral cosets. Loosely speaking, given two geodesics with the same endpoints, if one geodesic traverses one peripheral coset long enough, then the other one has to enter and exit the same peripheral coset in a uniformly close way.

Definition (Farb's definition). We say $(G, \mathbb{H})$ is *relatively hyperbolic* if the Caylay graph $\mathscr{G}(G, X \cup \mathcal{H})$ is hyperbolic, and the pair $(G, \mathbb{H})$ satisfies the BCP property.

Let d be a proper left invariant metric on the countable group G. Such a proper metric exists, since a countable group is embedded into a 2-generated group.

Akin to convex sets in metric spaces, the following notion of quasiconvexity of subgroups has been introduced into relatively hyperbolic groups by Osin [Os06b].

Definition (Relatively quasiconvex subgroups). Suppose $(G, \mathbb{H})$ is relatively hyperbolic. A subgroup Γ of G is called *relatively σ-quasiconvex* with respect to $\mathbb{H}$ if there exists a constant $\sigma = \sigma(d) > 0$, such that any geodesic in $\mathscr{G}(G, X \cup \mathcal{H})$ with both endpoints in Γ lies in a σ-neighborhood of Γ with respect to the metric d.

In Chapter 1, the above combinatorial approach is taken to prove the relative hyperbolicity of infinite groups with respect to extended structures.

II.1.3 Dynamical Approach

The dynamical approach takes place in the context of convergence group actions. Convergence group actions on spheres were first introduced by Gehring-Martin

[GM87] as a topological generalization of Kleinian groups. Later, the theory of convergence groups was developed in a rather general setting—see Bowditch [Bo99a], Freden [Fr95] and Tukia [Tu94],

Let M be a compact metrizable space. A *convergence group action* is an action of a group G on M, such that the induced action of G on the space $\Theta^3(M)$ of distinct triples is properly discontinuous.

One rich source of convergence groups is provided by the actions on Gromov boundaries induced from properly discontinuous actions on δ-hyperbolic spaces. Another interesting class of convergence groups are the finitely generated groups with nontrivial Floyd boundary (cf. Karlsson [Ka03]).

Analogous to the theory of Kleinian groups, we can consider the *limit set* $\Lambda(G)$ as the set of accumulation points of G-orbits in M. Two types of limit points deserve special attention: *bounded parabolic points* and *conical points*. In terms of limit points, relative hyperbolicity can be formulated as follows.

Definition (Geometrically finite groups)**.** A convergence group action of G on M is *geometrically finite* if every point of M is conical or bounded parabolic. Then $(G, \mathbb{H})$ is called *relatively hyperbolic*, where $\mathbb{H}$ is a set of representatives of conjugacy classes of maximal parabolic subgroups.

In the definition, the compact metrizable space M is called the *Bowditch boundary* of $(G, \mathbb{H})$. When dealing with different geometrically finite actions of G, the Bowditch boundary of $(G, \mathbb{H})$ is often denoted by $T_{\mathbb{H}}$ for given $\mathbb{H}$. In this case, we denote by $\Lambda_{\mathbb{H}}(\Gamma)$ the limit set of a subgroup $\Gamma \subset G$ with respect to $G \curvearrowright T_{\mathbb{H}}$.

We say a group action of G on M is *2-cocompact* if the induced action of G on the space $\Theta^2(M)$ of subsets of cardinality 2 is cocompact. In [Ge09], Gerasimov proved that 2-cocompact convergence group actions are geometrically finite. The converse statement is implied in the work of Tukia [Tu98]. So a 2-cocompact convergence group action gives another dynamical characterization of relative hyperbolicity.

In [Bo99b], a dynamical notion of quasiconvexity is introduced in a convergence group, and shown to coincide with the geometric quasiconvexity in hyperbolic groups.

Definition (Dynamically quasiconvex subgroups)**.** A subgroup H of a convergence group G is *dynamically quasiconvex* if the following set

$$\{gH \in G/H : g\Lambda(H) \cap K \neq \emptyset \text{ and } g\Lambda(H) \cap L \neq \emptyset\}$$

is finite, whenever K and L are disjoint closed subsets of M.

In [GP09a] and [GP11], Gerasimov-Potyagailo showed that dynamically quasiconvex subgroups are exactly the relatively quasiconvex subgroups in relatively hyperbolic groups. This result allows us to study relatively quasiconvex subgroups via dynamical methods, see Chapter 2.

II.2 Outline of main results

In this section, unless explicitly stated, we consider countable relatively hyperbolic groups and relatively quasiconvex subgroups. In this situation, peripheral subgroups may not be finitely generated.

II.2.1 Parabolically extended peripheral structures (Chapter 1)

In various contexts, an infinite group may be hyperbolic relative to different peripheral structures. A typical example is the well-studied modular group $\mathrm{PSL}(2,\mathbb{Z})$ in many mathematical fields. This group, as the fundamental group of a finite area Riemann surface with one cusp, is relatively hyperbolic to the cusp subgroup $\mathbb{Z}$. On the other hand, $\mathrm{PSL}(2,\mathbb{Z})$ can be decomposed as a free product of two finite cyclic groups. So it is hyperbolic in the sense of Gromov.

Moreover, as a relatively hyperbolic group, $\mathrm{PSL}(2,\mathbb{Z})$ has an associated Bowditch boundary—the circle S^1. As a hyperbolic group, the Gromov boundary of $\mathrm{PSL}(2,\mathbb{Z})$ is the totally disconnected Cantor space. It is well-known that there exists an equivalent map from the Cantor space to the circle.

Another motivation is the fact that peripheral structures of relatively hyperbolic groups are refinable. That is to say, if G is hyperbolic relative to one subgroup Γ, and Γ is hyperbolic relative to a proper subgroup H, then G is hyperbolic relative to H (cf. Drutu-Sapir [DS05, Corollary 1.14]).

In Chapter 1, we define a class of parabolically extended structures for the relatively hyperbolic groups, in order to capture the relations discussed above.

Let G be a countable group with a finite collection of subgroups $\mathbb{H} = \{H_i\}_{i\in I}$. This finite collection is called the *peripheral structure* of G. Let $\mathbb{P} = \{P_j\}_{j\in J}$ be another peripheral structure with the property that, for each $i \in I$, there exists $j \in J$ such that $H_i \subset P_j$. Then $\mathbb{P}$ is called an *extended peripheral structure* for the pair $(G, \mathbb{H})$. Furthermore, if the pair $(G, \mathbb{P})$ is relatively hyperbolic, then we say $\mathbb{P}$ is *parabolically extended* for $(G, \mathbb{H})$

Our main result is to give a characterization of parabolically extended peripheral structure. Let $\mathbb{H}_P = \{H_i : H_i \subset P, i \in I\}$ for a given $P \in \mathbb{P}$. Recall that a subgroup $\Gamma \subset G$ is *weakly malnormal* if $\Gamma \cap g\Gamma g^{-1}$ is finite for any $g \in G \setminus \Gamma$.

Theorem (Theorem 1.1.1)**.** *Suppose $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ is an extended peripheral structure for $(G, \mathbb{H})$. Then $(G, \mathbb{P})$ is relatively hyperbolic if and only if each $P \in \mathbb{P}$ satisfies the following statements:*

(P1). P is relatively quasiconvex with respect to $\mathbb{H}$,

(P2). P is weakly malnormal, and

(P3). $P \cap gHg^{-1}$ is finite for any $g \in G$ and $H \in \mathbb{H} \setminus \mathbb{H}_P$.

Remark. Several special cases of this Theorem were known to Gersten [Ge96] and Bowditch [Bo99b] in the case where G is hyperbolic; and in Osin [Os06b] under the assumption that for each $P \in \mathbb{P}$, either $P \in \mathbb{H}$ or $\mathbb{H}_P$ consists of a trivial subgroup.

In order to prove this Theorem, we modify paths in the relative Cayley graph $\mathscr{G}(G, X \cup \mathcal{P})$ and define their *lifting paths* in $\mathscr{G}(G, X \cup \mathcal{H})$. Assuming Conditions (P1)–(P3), several properties of the original paths are proven to be conserved by the operation of lifting. Particularly, the lifting of a quasigeodesic without backtracking is still a quasigeodesic without backtracking. This property motivates us to prove the relative hyperbolicity of $(G, \mathbb{P})$ via Farb's definition. More precisely, we first lift quasigeodesics in $\mathscr{G}(G, X \cup \mathcal{P})$ to $\mathscr{G}(G, X \cup \mathcal{H})$, and then apply Farb's definition to the pair $(G, \mathbb{H})$ to verify the hyperbolicity of the graph $\mathscr{G}(G, X \cup \mathcal{P})$ and BCP property for the pair $(G, \mathbb{P})$.

The study of lifting paths boils down to a quasi-isometric embedding of a relatively quasiconvex subgroup into the ambient relatively hyperbolic group. Such a quasi-isometric map is constructed explicitly such that a quasigeodesic without backtracking is mapped to a quasigeodesic without backtracking. This property also enables us to give a new proof of Hruska's result [Hr10] that a relatively quasiconvex subgroup is relatively hyperbolic.

From a dynamic point of view, Bowditch boundaries of relatively hyperbolic groups are also studied to order to understand parabolically extended structures. This is based on the theorem below, due to Gerasimov [Ge10], which states that the Floyd boundary is a universal boundary for a relatively hyperbolic group.

The Floyd boundary $\partial_f G$ with respect to a scaling function f is defined in Floyd [Fl80] for a finitely generated group G.

Theorem (Floyd mapping theorem [Ge10]). *Suppose a finitely generated group G admits a 2-cocompact convergence group action on a compactum T containing at least 3 points. Then there exists a G-equivariant map $\phi : \partial_f G \to T$, where $f(n) = \alpha^n$ for some $\alpha \in]0, 1[$ sufficiently close to 1. Furthermore $\Lambda(G) = \phi(\partial_f G)$.*

Using the Floyd mapping theorem, it is easy to see that the Bowditch boundaries associated with the parabolically extended structures are just equivariant quotients of the ones associated with the original structures. Moreover, the kernel of the Floyd map is described in Gerasimov-Potyagailo [GP09b] [GP10]. The Proposition below follows directly from Theorem A in [GP09b].

Recall that $T_{\mathbb{H}}$ denotes the Bowditch boundary of $(G, \mathbb{H})$, and $\Lambda_{\mathbb{H}}(G_p)$ the limit set of the stabilizer G_p of $p \in T_{\mathbb{P}}$ with respect to $G \curvearrowright T_{\mathbb{H}}$.

Proposition (Lemmas 1.4.15 & 1.4.16). *Suppose a finitely generated group G is hyperbolic relative to $\mathbb{H}$. Let $\mathbb{P}$ be a parabolically extended structure for $(G, \mathbb{H})$. Then there exists a G-equivariant surjective map $\varphi : T_{\mathbb{H}} \to T_{\mathbb{P}}$ such that the multivalued inverse map φ^{-1} is injective on conical points of $G \curvearrowright T_{\mathbb{P}}$. Moreover,*

$$\varphi^{-1}(p) = \Lambda_{\mathbb{H}}(G_p)$$

for any parabolic point $p \in T_{\mathbb{P}}$.

Furthermore, we show that if a relatively hyperbolic group G acts geometrically finitely on its Floyd boundary, then extended peripheral structures are the only possible nontrivial ones such that G may be relatively hyperbolic.

On the other hand, there indeed exist relatively hyperbolic groups which are not geometrically finite on Floyd boundaries. This is shown in the next result, which is obtained using a result of Behrstock-Drutu-Mosher [BDM09, Proposition 6.3] and [GP09b, Theorem A].

Theorem (Theorem 1.4.23). *Dunwoody's inaccessible group [Du91] does not act geometrically finitely on its Floyd boundary.*

Note that Dunwoody's inaccessible group does act geometrically finite on its Bowditch boundary.

II.2.2 Relatively quasiconvex subgroups (Chapters 1 & 2)

With the flexibility of the peripheral structures, it is interesting to explore how the algebraic and geometric properties of relatively hyperbolic groups vary. An example concerning the relative quasiconvexity of a given subgroup is examined in Section 4, Chapter 1.

Using dynamical approaches, we give a characterization of relative quasiconvexity with respect to parabolically extended structures. Note that a special case is proven in Martinez-Pedroza [MP09].

Theorem (Theorem 1.1.3). *Suppose $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ is a parabolically extended structure for $(G, \mathbb{H})$. If $\Gamma \subset G$ is relatively quasiconvex with respect to $\mathbb{H}$, then Γ is relatively quasiconvex with respect to $\mathbb{P}$.*

Conversely, suppose that $\Gamma \subset G$ is relatively quasiconvex with respect to $\mathbb{P}$. Then Γ is relatively quasiconvex with respect to $\mathbb{H}$ if and only if $\Gamma \cap gPg^{-1}$ is relatively quasiconvex with respect to $\mathbb{H}$ for any $g \in G$ and $P \in \mathbb{P}$.

The first statement of this Theorem follows from the fact that dynamical quasiconvexity is kept under an equivariant quotient map. The proof is thus completed by the Proposition above. However, the proof of the second statement involves a construction of a compact fundamental domain for the action of Γ on the space $\Theta^2(\Lambda_{\mathbb{H}}(\Gamma))$ of subsets of cardinality 2. This is made possible by using the Proposition above to lift a compact fundamental domain of the 2-cocompact action of Γ on $\Lambda_{\mathbb{P}}(\Gamma)$. Hence we can show that Γ acts 2-cocompactly on $\Lambda_{\mathbb{H}}(\Gamma)$ and thus it acts geometrically finitely on $\Lambda_{\mathbb{H}}(\Gamma)$.

Chapter 2 is devoted to the study of dynamical quasiconvex subgroups in general convergence groups, focusing on applications to relatively quasiconvex subgroups. Recall that Gerasmov-Potyagailo [GP09a][GP11] proved that the relative quasiconvexity of a subgroup is equivalent to its dynamical quasiconvexity in geometrically finite groups.

It is shown that dynamically quasiconvex subgroups, even in the general case, share many algebraic properties with relatively quasiconvex subgroups. In particular, several well-known results on the limit sets of geometrically finite Kleinian groups are derived for dynamically quasiconvex subgroups. See Chapter 2 for details.

One particular application is given to the study of the intersection property of relatively quasiconvex subgroups. In Chapter 2, we prove a limit set intersection theorem for relatively hyperbolic groups, some special cases of which were known in [Da03].

Theorem (Theorem 2.1.1). *Let H, J be two relatively quasiconvex subgroups of a finitely-generated relatively hyperbolic group G. Then*

$$\Lambda(H) \cap \Lambda(J) = \Lambda(H \cap J) \bigsqcup E$$

where the exceptional set E comprises the limit points isolated in $\Lambda(H) \cap \Lambda(J)$.

A corollary of this Theorem is that the intersection of relatively quasiconvex subgroups is relatively quasiconvex. This result was proved using geometrical methods, see Hruska [Hr10] and Martinez-Pedroza [MP08].

Further applications of the dynamical quasiconvexity are provided by the following.

Theorem (Theorem 2.1.5). *If H is an undistorted subgroup of a finitely generated group G with nontrivial Floyd boundary, then H is dynamically quasiconvex.*

Remark. The existence of the Floyd map proved in [Ge10] implies that non-elementary finitely-generated relatively hyperbolic groups have nontrivial Floyd boundary.

Using the Floyd mapping theorem, one easily obtain that undistorted subgroups are dynamically quasiconvex in relatively hyperbolic groups. Hence, undistorted subgroups are relatively quasiconvex, which was first proved by Hruska in [Hr10].

II.2.3 Bounded packing property(Chapter 3)

The bounded packing property was introduced for a subgroup of a countable group in Hruska-Wise [HW09]. Roughly speaking, a subgroup with bounded packing demands a finite upper bound on the number of its left cosets which are pairwise close in the ambient group.

This property is a generalization of the notion of the *width* of a subgroup in a hyperbolic group (cf. [GMRS98]). It is shown in [HW09] that bounded packingness implies that the width of a relatively quasiconvex subgroup is finite. In Chapter 4, we prove the following result, which is interesting in its own right.

Theorem (Theorem 3.1.3). *If H is a separable subgroup of a countable group G, then H has bounded packing in G.*

It is well-known that each subgroup of a polycyclic group is separable. So the Theorem gives a positive answer to the question of Hruska-Wise [HW09], asking whether each subgroup of a virtually polycyclic group has bounded packing.

II.2.4 Commensurability of Kleinian groups (Chapter 4)

Chapter 5 is based on joint work [YJ10] with Yueping Jiang to investigate a question posed by J. Anderson.

Let $\mathrm{Isom}(\mathbf{H}^n)$ be the isometry group of an n-dimensional hyperbolic space $\mathbf{H}^n$. We denote by $\mathrm{Ax}(G)$ the set of axes of hyperbolic elements of $G \subset \mathrm{Isom}(\mathbf{H}^n)$.

Question. [Be04] If $G_1, G_2 \subset \mathrm{Isom}(\mathbf{H}^n)$ are finitely generated and discrete, does $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2)$ imply that G_1 and G_2 are commensurable?

The Question has a positive answer when G_1 and G_2 are finitely generated Fuchsian groups [Me90] and arithmetic Kleinian groups [LR98].

An example of Susskind in [Su01] shows that the Question is answered negatively in the general situation, where G_1 and G_2 together generate a nondiscrete group. So it is interesting to consider the case when G_1 and G_2 lie in a discrete group. In dimension 3, we obtain the following result under certain restrictions.

Theorem (Theorem 4.1.3)**.** *Let G_1, G_2 be two non-elementary finitely generated subgroups of an infinite co-volume Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^3)$. Then G_1 and G_2 are commensurable if and only if $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2)$.*

Chapitre 1

Parabolically extended structures of relatively hyperbolic groups

Abstract. In this chapter, we introduce and characterize a class of parabolically extended structures for relatively hyperbolic groups. A characterization of relative quasiconvexity with respect to parabolically extended structures is obtained using dynamical methods. Some applications are discussed. The class of groups acting geometrically finitely on Floyd boundaries turns out to be easily understood. However, we also show that Dunwoody's inaccessible group does not act geometrically finitely on its Floyd boundary.

Contents

1.1 Introduction

In this chapter, we study peripheral structures of relatively hyperbolic groups. Since introduced by Gromov [Gr87], relative hyperbolicity has several equivalent formulations. If relatively hyperbolic groups are understood as geometrically finite actions(see Bowditch [Bo99b]), peripheral structures can be thought of a set of representatives of the conjugacy classes of maximal parabolic subgroups. On the other hand, following approaches of Farb [Fa98] and Osin [Os06b], a peripheral structure is a preferred collection of subgroups such that the constructed relative Cayley graph satisfies some nice properties. In practice a given group may be hyperbolic relative to different peripheral structures. So it is interesting to understand the relationship between possible peripheral structures that can be endowed on a countable group.

The study of peripheral structures actually stems from hyperbolic groups. A first result of this sort is due to Gersten [Ge96] and Bowditch [Bo99b], who proved that malnormal quasiconvex subgroups of hyperbolic groups yield peripheral structures. In a point of view of relative hyperbolicity, an ordinary hyperbolic group is hyperbolic relative to a trivial subgroup. Later on, in relatively hyperbolic groups, Osin [Os06a] generalized their results and proposed the notion of hyperbolically embedded subgroups. A hyperbolically embedded subgroup can be added into an existing peripheral structure such that the group is also relatively hyperbolic with respect to the enlarged peripheral structure. In this chapter, we shall enlarge peripheral subgroups themselves to get new peripheral structures.

Convention 1. In the remainder of the paper, the term "peripheral structure" will be used in a weaker sense, i.e. it is just a finite collection of subgroups without involving relative hyperbolicity.

Let G be a countable group with a collection of subgroups $\mathbb{H} = \{H_i\}_{i\in I}$. We denote such a pair by $(G, \mathbb{H})$. The collection $\mathbb{H}$ is often referred as a *peripheral structure* of G, and each element of $\mathbb{H}$ a *peripheral subgroup* of G. In this chapter, we always assume that peripheral structures have finite cardinality.

Let $\mathbb{H} = \{H_i\}_{i\in I}$ and $\mathbb{P} = \{P_j\}_{j\in J}$ be two peripheral structures of a countable group G. If for each $i \in I$, there exists $j \in J$ such that $H_i \subset P_j$, then we say $\mathbb{P}$ is an *extended peripheral structure* for the pair $(G, \mathbb{H})$. Let $\mathbb{H}_P = \{H_i : H_i \subset P, i \in I\}$ for a given $P \in \mathbb{P}$.

Furthermore, if the pair $(G, \mathbb{P})$ is relatively hyperbolic, then we say $\mathbb{P}$ is *parabolically extended* for $(G, \mathbb{H})$. Each subgroup $P \in \mathbb{P}$ is said to be *parabolically embedded* into $(G, \mathbb{H})$. Our first result is to give a characterization of parabolically extended peripheral structure. The notation Γ^g denotes the conjugate $g\Gamma g^{-1}$ of a subgroup $\Gamma \subset G$ by an element $g \in G$. Recall that a subgroup $\Gamma \subset G$ is *weakly malnormal* if $\Gamma \cap \Gamma^g$ is finite for any $g \in G \setminus \Gamma$.

Theorem 1.1.1. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ is an extended peripheral structure for $(G, \mathbb{H})$. Then $(G, \mathbb{P})$ is relatively hyperbolic if and only if each $P \in \mathbb{P}$ satisfies the following statements*

(P1). P is relatively quasiconvex with respect to $\mathbb{H}$,

(P2). P is weakly malnormal, and

(P3). $P^g \cap H$ is finite for any $g \in G \setminus P$ and $H \in \mathbb{H} \setminus \mathbb{H}_P$.

In fact, Theorem 1.1.1 follows from a characterization of parabolically embedded subgroups(see Theorem 1.3.10).

Remark 1.1.2. In our terms, a hyperbolically embedded subgroup $\Gamma \subset G$, defined in [Os06a], is parabolically embedded into $(G, \mathbb{H})$ such that $\mathbb{H}_\Gamma$ consists of only one trivial subgroup. In this case, Γ is a hyperbolic group. However, parabolically embedded subgroups may be in general hyperbolic relative to a nontrivial collection of proper subgroups, as stated in Condition (P1).

In relatively hyperbolic groups, we can define a natural class of subgroups named relatively quasiconvex subgroups. In particular a subgroup relatively quasiconvex with respect to one peripheral structure, may not be relatively quasiconvex with respect to others. Our second result is to give a characterization of relative quasi-convexity with respect to parabolically extended structures.

Theorem 1.1.3. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ is a parabolically extended structure for $(G, \mathbb{H})$. If $\Gamma \subset G$ is relatively quasiconvex with respect to $\mathbb{H}$, then Γ is relatively quasiconvex with respect to $\mathbb{P}$.*

Conversely, suppose $\Gamma \subset G$ is relatively quasiconvex with respect to $\mathbb{P}$. Then Γ is relatively quasiconvex with respect to $\mathbb{H}$ if and only if $\Gamma \cap P^g$ is relatively quasiconvex with respect to $\mathbb{H}$ for any $g \in G$ and $P \in \mathbb{P}$.

Remark 1.1.4. Theorem 1.1.3 generalizes the main result of E. Martinez-Pedroza [MP08], where the result is proven under the assumption that for each $P \in \mathbb{P}$, either $P \in \mathbb{H}$ or $\mathbb{H}_P$ consists of a trivial subgroup.

Unlike the proof of Theorem 1.1.1, we give a dynamical proof of Theorem 1.1.3 using the work of Gerasimov [Ge10] and Gerasimov-Potyagailo [GP09b] on Floyd maps.

Using Floyd maps, we can also make some preliminary observations to general peripheral structures of relatively hyperbolic groups. In our study, peripheral structures of groups admitting geometrically finite actions on Floyd boundaries are easily analyzed and shown to be parabolically extended with respect to a canonical one. See Corollary 1.4.21.

It is known that many relatively hyperbolic groups act geometrically finitely on their Floyd boundaries. For instance, geometrically finite Kleinian groups. See List 1.4.4 for more such groups.

However, there indeed exist relatively hyperbolic groups which do not act geometrically finitely on their Floyd boundaries. See Theorem 1.4.23 for the example Dunwoody's inaccessible group. One of our results also shows that for a relatively hyperbolic group, the convergence action on Floyd boundary is largely determined by ones of peripheral subgroups.

Theorem 1.1.5. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic. Then G acts geometrically finitely on its Floyd boundary $\partial_f G$ if and only if each $H \in \mathbb{H}$ acts geometrically finitely on its limit set for the action on $\partial_f G$.*

The structure of the chapter is as follows. In Section 2, we restate Bounded Coset Penetration property for countable relative hyperbolicity and indicate the equivalence of the definitions of countable relative hyperbolicity due to Osin and Farb. We also construct a quasi-isometric map from a relatively quasiconvex subgroup to the ambient group. Our construction leads to a new proof of Hruska's result [Hr10] that relatively quasiconvex subgroups are relatively hyperbolic. In Section 3, we study parabolically extended structures and give the proof of Theorem 1.1.1. In Section 4, we take a dynamical approach to relative hyperbolicity and then prove Theorems 1.1.3 and 1.1.5. Dunwoody's inaccessible group is discussed in this section.

1.2 Preliminaries

1.2.1 Cayley graphs and partial distance functions

Let G be a group with a set $\mathcal{A} \subset G$. Note that the alphabet set $\mathcal{A}$ is assumed to neither be finite and nor generate G. For convenience, we always assume $1 \notin \mathcal{A}$ and $\mathcal{A} = \mathcal{A}^{-1}$.

We define the *Cayley graph* $\mathscr{G}(G, \mathcal{A})$ of a group G with respect to $\mathcal{A}$, as a directed edge-labeled graph with the vertex set $V(\mathscr{G}(G, \mathcal{A})) = G$ and the edge set $E(\mathscr{G}(G, \mathcal{A})) = G \times \mathcal{A}$. An edge $e = [g, a]$ goes from the vertex g to the vertex ga and has the label $\mathbf{Lab}(e) = a$. As usual, we denote the origin and the terminus of the edge e, i.e., the vertices g and ga, by e_- and e_+ respectively. By definition, we set $e^{-1} := [ga, a^{-1}]$.

Let $p = e_1 e_2 \dots e_k$ be a combinatorial path in the Cayley graph $\mathscr{G}(G, \mathcal{A})$, where $e_1, e_2, \dots, e_k \in E(\mathscr{G}(G, \mathcal{A}))$. The length of p is the number of edges in p, i.e. $\ell(p) = k$. We define the label of p as $\mathbf{Lab}(p) = \mathbf{Lab}(e_1)\mathbf{Lab}(e_2)\dots\mathbf{Lab}(e_k)$. The path p^{-1} is defined in a similar way. We also denote by $p_- = (e_1)_-$ and $p_+ = (e_k)_+$ the origin and the terminus of p respectively. A *cycle* p is a path such that $p_- = p_+$.

Definition 1.2.1. (Partial Distance Functions) By assigning the length of each edge in $\mathscr{G}(G, \mathcal{A})$ to be 1, we define a *partial distance function* $d_{\mathcal{A}} : \mathscr{G}(G, \mathcal{A}) \times \mathscr{G}(G, \mathcal{A}) \to [0, \infty]$ as follows. Note that $\mathcal{A}$ is not assumed to generate G and thus $\mathscr{G}(G, \mathcal{A})$ may be disconnected. For $z, w \in \mathscr{G}(G, \mathcal{A})$, if z and w lie in the same path connected component of $\mathscr{G}(G, \mathcal{A})$, we define $d_{\mathcal{A}}(z, w)$ as the length of a shortest path in $\mathscr{G}(G, \mathcal{A})$ between z and w. Otherwise we set $d_{\mathcal{A}}(z, w) = \infty$.

Remark 1.2.2. If $\langle \mathcal{A} \rangle = G$, then the partial distance function $d_{\mathcal{A}}$ actually gives a *word metric* with respect to $\mathcal{A}$ on the Cayley graph $\mathscr{G}(G, \mathcal{A})$. Note that if $g_1, g_2 \in G$ and $g_1^{-1} g_2 \notin \langle \mathcal{A} \rangle$, then $d_{\mathcal{A}}(g_1, g_2) = \infty$. For any element $g \in G$, we define its norm $|g|_{\mathcal{A}} = d_{\mathcal{A}}(1, g)$.

A path p in the Cayley graph $\mathscr{G}(G, \mathcal{A})$ is called (λ, c)-*quasigeodesic* for some $\lambda \geq 1$, $c \geq 0$, if the following inequality holds for any subpath q of p,

$$\ell(p) \leq \lambda d_{\mathcal{A}}(q_-, q_+) + c.$$

We often consider a group G with a collection of subgroups $\mathbb{H} = \{H_i\}_{i \in I}$. Then X is a *relative generating set* for $(G, \mathbb{H})$ if G is generated by the set $(\cup_{i \in I} H_i) \cup X$ in the traditional sense.

Let $\mathcal{H} = \bigsqcup_{i \in I} H_i \setminus \{1\}$. Fixing a relative generating set X for $(G, \mathbb{H})$, the constructed Cayley graph $\mathscr{G}(G, X \cup \mathcal{H})$ is called the *relative Cayley graph* of G with respect to $\mathbb{H}$. We now collect some notions introduced by Osin [Os06b] in relative Cayley graphs.

Definition 1.2.3. Let p, q be paths in $\mathscr{G}(G, X \cup \mathcal{H})$. A subpath s of p is called an *H_i-component*, if s is the maximal subpath of p such that s is labeled by letters from H_i.

Two H_i-components s, t of p, q respectively are called *connected* if there exists a path c in $\mathscr{G}(G, X \cup \mathcal{H})$ such that $c_- = s_-, c_+ = t_-$ and c is labeled by letters from H_i. An H_i-component s of p is *isolated* if no other H_i-component of p is connected to s.

We say a path p *without backtracking* by meaning that all H_i-components of p are isolated. A vertex u of p is *nonphase* if there is an H_i-component s of p such that u is a vertex of s but $u \neq s_-, u \neq s_+$. Other vertices of p are called *phase*.

1.2.2 Relatively hyperbolic groups

In the large part of this chapter, we consider countable relatively hyperbolic groups. In this subsection, we shall recall the definitions of countable relative hyperbolicity in the sense of Osin and Farb, and then indicate their equivalence based on Osin's results in [Os06b].

Let G be a countable group with a finite collection of subgroups $\mathbb{H} = \{H_i\}_{i \in I}$. As the notion of relative generating sets, we can define in a similar fashion the relative presentations and (relative) Dehn functions of G with respect to $\mathbb{H}$. We refer the reader to [Os06b] for precise definitions.

We now give the first definition of relative hyperbolicity due to Osin [Os06b]. Note that the full version of Osin's definition applies to general groups without assuming the finiteness of $\mathbb{H}$.

Definition 1.2.4. (Osin Definition) A countable group G is *hyperbolic relative to* $\mathbb{H}$ *in the sense of Osin* if G is finitely presented with respect to $\mathbb{H}$ and the relative Dehn function of G with respect to $\mathbb{H}$ is linear.

The following lemma plays an important role in Osin's approach [Os06b] to relative hyperbolicity. The finite subset Ω and constant κ below depend on the choice of finite relative presentations of G with respect to $\mathbb{H}$. In our later use of Lemma 1.2.5, when saying there exists κ, Ω such that the inequality (1.1) below holds in $\mathscr{G}(G, X \cup \mathcal{H})$, we have implicitly chosen a finite relative presentation of G with respect to $\mathbb{H}$.

Lemma 1.2.5. *[Os06b, Lemma 2.27] Suppose $(G, \mathbb{H})$ is relatively hyperbolic in the sense of Osin and X is a finite relative generating set for $(G, \mathbb{H})$. Then there exists $\kappa \geq 1$ and a finite subset $\Omega \subset G$ such that the following holds. Let c be a cycle in $\mathscr{G}(G, X \cup \mathcal{H})$ with a set of isolated H_i-components $S = \{s_1, \ldots, s_k\}$ of c for some $i \in I$, Then*

$$\sum_{s \in S} d_{\Omega_i}(s_-, s_+) \leq \kappa \ell(c), \tag{1.1}$$

where $\Omega_i := \Omega \cap H_i$.

Remark 1.2.6. By the definition of d_{Ω_i}, if $d_{\Omega_i}(g,h) < \infty$ for $g,h \in G$, then there exists a path p labeled by letters from Ω_i in this new Cayley graph $\mathscr{G}(G, X \cup \Omega \cup \mathcal{H})$ such that $p_- = g, p_+ = h$.

Using Lemma 1.2.5, the following lemma can be proven exactly as Proposition 3.15 in [Os06b]. The finite set Ω below is given by Lemma 1.2.5.

Lemma 1.2.7. *[Os06b] Suppose $(G, \mathbb{H})$ is relatively hyperbolic in the sense of Osin and X is a finite relative generating set for $(G, \mathbb{H})$. For any $\lambda \geq 1, c \geq 0$, there exists a constant $\epsilon = \epsilon(\lambda, c) > 0$ such that the following holds. Let p, q be (λ, c)-quasigeodesics without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$ such that $p_- = q_-, p_+ = q_+$. Then for any phase vertex u of p(resp. q), there exists a phase vertex v of q(resp.p) such that $d_{X \cup \Omega}(u, v) < \epsilon$.*

The following lemma is well-known in the theory of relatively hyperbolic groups.

Lemma 1.2.8. *[Os06b] Suppose $(G, \mathbb{H})$ is relatively hyperbolic in the sense of Osin. Then the following statements hold for any $g \in G$ and $H_i, H_j \in \mathbb{H}$,*
1) If $H_i^g \cap H_i$ is infinite, then $g \in H_i$,
2) If $i \neq j$, then $H_i^g \cap H_j$ is finite.

In order to formulate the BCP property, we shall put a metric d_G on a group G, which is *proper* if any bounded set is finite, and *left invariant* if $d_G(gx_1, gx_2) = d_G(x_1, x_2)$ for any $g, x_1, x_2 \in G$. For given $g \in G$, we define the norm $|g|_{d_G}$ with respect to d_G to be the distance $d_G(1, g)$.

Let us now recall the following lemma in Hruska-Wise [HW09], which justifies the use of proper left invariant metrics on countable groups.

Lemma 1.2.9. *[HW09] A group is countable if and only if it admits a proper left invariant metric.*

From now on, we assume that $(G, \mathbb{H})$ has a finite relative generating set X.

In terms of proper left invariant metrics, bounded coset penetration property is formulated as follows.

Definition 1.2.10. (Bounded coset penetration) Let d_G be some (any) proper, left invariant metric on G. The pair $(G, \mathbb{H})$ is said to satisfy the *bounded coset penetration property with respect to* d_G (or BCP property with respect to d_G for short) if, for any $\lambda \geq 1$, $c \geq 0$, there exists a constant $a = a(\lambda, c, d_G)$ such that the following conditions hold. Let p, q be (λ, c)-quasigeodesics without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$ such that $p_- = q_-$, $p_+ = q_+$.

1) Suppose that s is an H_i-component of p for some $H_i \in \mathbb{H}$, such that $d_G(s_-, s_+) > a$. Then there exists an H_i-component t of q such that t is connected to s.

2) Suppose that s and t are connected H_i-components of p and q respectively, for some $H_i \in \mathbb{H}$. Then $d_G(s_-, t_-) < a$ and $d_G(s_+, t_+) < a$.

Remark 1.2.11. In [Hr10], Hruska proposed to use the partial distance function d_X (with respect to a finite relative generating set X) instead of d_G in the definition of BCP property, and showed that BCP property is independent of the choice of relative generating sets. However, this is generally not true, due to the following example.

Example 1.2.12. We take a free product $G = L *_F K$ of two finitely generated groups L and K amalgamated over a nontrivial finite group F, which is known to be hyperbolic relative to $\mathbb{H} = \{L, K\}$ in the sense of Farb and Osin. Take a special relative generating set $X = \emptyset$ and construct the relative Cayley graph $\mathscr{G}(G, X \cup \mathcal{H})$. Since $F = L \cap K$ is nontrivial, we take a nontrivial element $f = f_L = f_K \in F$, where f_L and f_K are the corresponding elements in $L \setminus \{1\}$ and $K \setminus \{1\}$ respectively. Thus, there are two different edges p and q with same endpoints 1 and f such that $\mathbf{Lab}(p) = f_L$ and $\mathbf{Lab}(q) = f_K$ in $\mathscr{G}(G, X \cup \mathcal{H})$. Obviously p and q are geodesics and isolated components. Note that $d_X(p_-, p_+) = \infty$. Hence BCP property is not well-defined with respect to d_X.

This example was also known to other researchers in this field, see Remark 2.15 in a latest version of [MP08]. Moreover, the idea using proper metrics to define BCP property also appeared independently in Martinez-Pedroza [MP08]. See Subsection 2.3 in [MP08].

Remark 1.2.13. We remark that Hruska's arguments in [Hr10] remain valid with the new definition 1.2.10 of BCP property. So the main result concerning the equivalence of various definitions of relative hyperbolicity in [Hr10] is still correct. For the convenience of the reader, we will give a direct proof of the equivalence of Osin's and Farb's definitions in the remaining part of this subsection.

The following corollary is immediate by an elementary argument.

Corollary 1.2.14. *BCP property of $(G, \mathbb{H})$ is independent of the choice of left invariant proper metrics.*

In view of Corollary 1.2.14, we shall not mention explicitly proper left invariant metrics when saying the BCP property of $(G, \mathbb{H})$.

With a little abuse of terminology, we also say $(G, \mathbb{H})$ satisfies BCP property with respect to a partial distance function $d_{\mathcal{A}}$ if, for any $\lambda \geq 1$, $c \geq 0$, there exists a constant $a = a(\lambda, c, d_{\mathcal{A}})$ such that the statements of BCP property 1) and 2) are true for $d_{\mathcal{A}}$.

When proving BCP property, we usually do it with respect to some special partial distance function, as stated in the following corollary.

Corollary 1.2.15. *Let $\mathcal{A} \subset G$ be a finite set. If $(G, \mathbb{H})$ satisfies BCP property with respect to $d_{\mathcal{A}}$, then so does $(G, \mathbb{H})$ with respect to any proper left invariant metric.*

The second definition of relative hyperbolicity is due to Farb [Fa98], which will be used in establishing relative hyperbolicity of groups in Section 3.

Definition 1.2.16. (Farb Definition) A countable group G is *hyperbolic relative to* $\mathbb{H}$ *in the sense of Farb* if the Caylay graph $\mathscr{G}(G, X \cup \mathcal{H})$ is hyperbolic and the pair $(G, \mathbb{H})$ satisfies the BCP property.

As observed in Example 1.2.12, BCP property is not well-defined with respect to relative generating sets. But the following lemma states that for a given finite relative generating set, we can always find a finite subset Σ such that $(G, \mathbb{H})$ satisfies BCP property with respect to d_Σ.

Lemma 1.2.17. *Suppose* $(G, \mathbb{H})$ *is relatively hyperbolic in the sense of Osin and* X *is a finite relative generating set for* $(G, \mathbb{H})$*. Then there exists a finite set* $\Sigma \subset G$ *such that then* $(G, \mathbb{H})$ *satisfies BCP property with respect to* d_Σ*.*

Proof. Let Ω be the finite set given by Lemma 1.2.5 for $\mathscr{G}(G, X \cup \mathcal{H})$. We take a new finite relative generating set $\hat{X} := X \cup \Omega$. Using Lemma 1.2.5 again, we obtain a finite set Σ and constant $\mu > 1$ such that the inequality (1.1) holds in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$.

We now verify BCP property 1). Let p, q be (λ, c)-quasigeodesics without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$. Since $\hat{X}$ is finite, the embedding $\mathscr{G}(G, X \cup \mathcal{H}) \hookrightarrow \mathscr{G}(G, \hat{X} \cup \mathcal{H})$ is a quasi-isometry. Regarded as paths in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, p, q are (λ', c')-quasigeodesics without backtracking in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, for some constants $\lambda' \geq 1, c' \geq 0$ depending on $\hat{X}$.

Let $\epsilon = \epsilon(\lambda, c)$ be the constant given by Lemma 1.2.7. Set

$$a = \mu(\lambda' + 1)(2\epsilon + 1) + c'\mu.$$

We claim that a is the desired constant for the BCP property of $(G, \mathbb{H})$. If not, we suppose there exists an H_i-component s of p such that $d_\Sigma(s_-, s_+) > a$ and no H_i-component of q is connected to s.

By Lemma 1.2.7, there exist phase vertices u, v of q such that $d_{X \cup \Omega}(s_-, u) < \epsilon, d_{X \cup \Omega}(s_-, v) < \epsilon$. Thus by regarding p, q as paths in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, there exist paths l and r labeled by letters from Ω such that $l_- = e_-, l_+ = u, r_- = e_+$, and $r_+ = v$. We consider the cycle $c := er[u, v]_q^{-1} l^{-1}$ in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, where $[u, v]_q$ denotes the subpath of q between u and v. Since $[u, v]_q$ is a (λ', c')-quasigeodesic, we compute $\ell(c)$ by the triangle inequality and have

$$\ell(c) \leq (\lambda' + 1)(2\epsilon + 1) + c'.$$

Obviously e is an isolated H_i-component of c. Using Lemma 1.2.5 for the cycle c in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, we have $d_\Sigma(e_-, e_+) < \mu\ell(c) < a$. This is a contradiction.

Therefore, BCP property 1) is verified with respect to d_Σ. BCP property 2) can be proven in a similar way. □

We conclude this subsection with the following theorem which is proven in [Os06b] for finitely generated relatively hyperbolic groups.

Theorem 1.2.18. *The pair* $(G, \mathbb{H})$ *is relatively hyperbolic in the sense of Farb if and only if it is relatively hyperbolic in the sense of Osin.*

Proof. By Corollary 1.2.15, BCP property of $(G, \mathbb{H})$ follows from Lemma 1.2.17. The hyperbolicity of relative Cayley graph $\mathscr{G}(G, X \cup \mathcal{H})$ is proven in [Os06b, Corollary 2.54]. Thus, $(G, \mathbb{H})$ is relatively hyperbolic in the sense of Farb.

The sufficient part is proven in the appendix of Osin [Os06b] for finitely generated relatively hyperbolic groups. We remark that the only argument involved to use word metrics with respect to finite generating sets is in the proof of Lemma 6.12 in [Os06b]. But Osin's argument also works for any proper left invariant metric. Hence, Osin's proof is through for the countable case. □

1.2.3 Relatively quasiconvex subgroups

In this subsection, we shall explicitly describe a quasi-isometric map between a relatively quasiconvex subgroup to the ambient relatively hyperbolic group.

The existence of such a quasi-isometric map is first proven in [Hr10, Theorem 10.1], but whose statement or proof does not tell explicitly how a geodesic is mapped. Using an argument of Short on the geometry of relative Cayley graph, we carry on a more careful analysis to construct the quasi-isometric map explicitly.

As a byproduct in the course of the construction, we are able to produce a new proof of the relative hyperbolicity of relatively quasiconvex subgroups. This was an open problem in [Os06b] and is firstly answered by Hruska [Hr10] using different methods. During the preparation of this thesis, E. Martinez-Pedroza and D. Wise [MPW10] gave another elementary and self-contained proof of this result.

Definition 1.2.19. [Hr10] Suppose $(G, \mathbb{H})$ is relatively hyperbolic and d is some proper left invariant metric on G. A subgroup Γ of G is called *relatively σ-quasiconvex* with respect to $\mathbb{H}$ if there exists a constant $\sigma = \sigma(d) > 0$ such that the following condition holds. Let p be an arbitrary geodesic path in $\mathscr{G}(G, X \cup \mathcal{H})$ such that $p_-, p_+ \in \Gamma$. Then for any vertex $v \in p$, there exists a vertex $w \in \Gamma$ such that $d(u, w) < \sigma$.

Corollary 1.2.20. *[Hr10] Relative quasiconvexity is independent of the choice of proper left invariant metrics.*

In fact, when proving relative quasiconvexity, we usually verify the relative quasiconvexity with respect to some partial distance function, as indicated in the following corollary. See an application of this corollary in the proof of Proposition 1.3.3.

Corollary 1.2.21. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic and Γ is a subgroup of G. Let $\mathcal{A} \subset G$ be a finite set and $d_{\mathcal{A}}$ the partial distance function with respect to $\mathcal{A}$. If there exists a constant $\sigma = \sigma(d_{\mathcal{A}}) > 0$ such that for any geodesic p with endpoints at Γ, the vertex set of p lies in σ-neighborhood of Γ with respect to $d_{\mathcal{A}}$. Then Γ is relatively quasiconvex.*

We are going to construct the quasi-isometric map. The relatively finitely generatedness of Γ in Lemma 1.2.22 is also proved by E. Martinez-Pedroza and D. Wise

[MPW10]. In particular, it also follows from a more general result of Gerasimov-Potyagailo [GP10], which states that 2-cocompact convergence groups are finitely generated relative to a set of maximal parabolic subgroups.

Lemma 1.2.22. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic. Let $\Gamma < G$ be relatively σ-quasiconvex. Then Γ is finitely generated by a finite subset $Y \subset G$ with respect to a finite collection of subgroups*

$$\mathbb{K} = \{H_i^g \cap \Gamma : |g|_d < \sigma, i \in I, \sharp\, H_i^g \cap \Gamma = \infty\}. \tag{1.2}$$

Moreover, X can be chosen such that $Y \subset X$ and there is a Γ-equivariant quasi-isometric map $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \to \mathscr{G}(G, X \cup \mathcal{H})$.

Proof. The argument is inspired by the one of [Os06b, Lemma 4.14].

For any $\gamma \in \Gamma$, we take a geodesic p in $\mathscr{G}(G, X \cup \mathcal{H})$ with endpoints 1 and γ. Suppose the length of p is n. Let $g_0 = 1, g_1, \ldots, g_n = \gamma$ be the consecutive vertices of p. By the definition of relative quasiconvexity, for each vertex g_i of p, there exists an element γ_i in Γ such that $d(g_i, \gamma_i) < \sigma$.

Denote by x_i the element $\gamma_i^{-1} g_i$, and by e_{i+1} the edge of p going from g_i to g_{i+1}. Obviously we have $\gamma_{i+1} = \gamma_i x_i \mathbf{Lab}(e_{i+1}) x_{i+1}^{-1}$.

Set $\kappa = \max\{|x|_d : x \in X\}$. Then κ is finite, as X is finite. Let $Z_0 = \{\gamma \in \Gamma : |\gamma|_d \leq 2\sigma + \kappa\}$ and $Z_{x,y,i} = \{xhy^{-1} : \; h \in H_i\} \cap \Gamma$. Since the metric d is proper, the set $B_\sigma := \{g \in G : |g|_d \leq \sigma\}$ is finite.

For simplifying notations, we define sets

$$\Pi = \{(x, y, i) : x, y \in B_\sigma, i \in I\}$$

and

$$\Xi = \{(x, y, i) : \sharp\, Z_{x,y,i} = \infty, x, y \in B_\sigma, i \in I\}.$$

If e_{i+1} is an edge labeled by a letter from X, then the element $x_i \mathbf{Lab}(e_{i+1}) x_{i+1}^{-1}$ belongs to Z_0. If e_{i+1} is an edge labeled by a letter from H_k, then $x_i \mathbf{Lab}(e_{i+1}) x_{i+1}^{-1}$ belongs to $Z_{x_i, x_{i+1}, k}$. By the construction, we obtain that the subgroup Γ is also generated by the set

$$Z := Z_0 \cup \left(\cup_{(x,y,i) \in \Pi} Z_{x,y,i}\right).$$

For each $(x, y, i) \in \Pi$, if $Z_{x,y,i}$ is nonempty, then we take an element of the form $x h_i y^{-1} \in Z_{x,y,i}$ for some $h_i \in H_i$. Denote by Z_1 the union of all such elements $\bigcup_{(x,y,i) \in \Pi} x h_i y^{-1}$. Note that $Z_1 \subset Z$. Then we have that Γ is generated by the set

$$\hat{Z} := Y \cup \left(\cup_{(z,z,i) \in \Xi} Z_{z,z,i}\right),$$

where $Y := Z_0 \cup Z_1 \cup \left(\bigcup_{(z,z,i) \in \Pi \setminus \Xi} Z_{z,z,i}\right)$. Indeed, for each triple $(x, y, i) \in \Pi$, we have

$$Z_{x,y,i} = Z_{x,x,i} \cdot x h_i y^{-1}, \text{ where } x h_i y^{-1} \in Z_1.$$

On the other direction, it is obvious that $\hat{Z} \subset Z$.

Let $\hat{X} = X \cup Y \cup B_\sigma$. By the above construction, we define a Γ-equivariant map ϕ from $\mathscr{G}(\Gamma, Z)$ to $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$ as follows. For each vertex $\gamma \in V(\mathscr{G}(\Gamma, Z))$, $\phi(\gamma) = \gamma$. For each edge $[\gamma, s] \in E(\mathscr{G}(\Gamma, Z))$, if $s \in Z_0$, then $\phi([\gamma, s]) = [\gamma, s]$; if $s \in Z_{x,y,i}$ for some $(x, y, i) \in \Xi$, then $s = xty^{-1}$ for some $t \in H_i$ and we set $\phi([\gamma, s]) = [\gamma, x][\gamma x, t][\gamma xt, y^{-1}]$.

For any $\gamma_1, \gamma_2 \in V(\mathscr{G}(\Gamma, Z))$, it is easy to see that $d_{\hat{X}\cup\mathcal{H}}(\gamma_1, \gamma_2) < 3d_Z(\gamma_1, \gamma_2)$. For the other direction, we take a geodesic q in $\mathscr{G}(G, X \cup \mathcal{H})$ with endpoints γ_1, γ_2.

Since $\hat{X}$ is finite, there exist constants $\lambda \geq 1, c \geq 0$ depending only on $\hat{X}$, such that the graph embedding $\mathscr{G}(G, X \cup \mathcal{H}) \hookrightarrow \mathscr{G}(G, \hat{X} \cup \mathcal{H})$ is a G-equivariant (λ, c)-quasi-isometry. Thus, q is a (λ, c)-quasigeodesic in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, i.e.

$$d_{X\cup\mathcal{H}}(\gamma_1, \gamma_2) < \lambda d_{\hat{X}\cup\mathcal{H}}(\gamma_1, \gamma_2) + c.$$

Since q is a geodesic in $\mathscr{G}(G, X \cup \mathcal{H})$ ending at Γ, we can apply the above analysis to q and obtain that $d_Z(\gamma_1, \gamma_2) < d_{X\cup\mathcal{H}}(\gamma_1, \gamma_2)$. Then we have

$$d_Z(\gamma_1, \gamma_2) < \lambda d_{\hat{X}\cup\mathcal{H}}(\gamma_1, \gamma_2) + c.$$

Therefore, ϕ is a Γ-equivariant quasi-isometric map.

We now claim the subgraph embedding $\imath : \mathscr{G}(\Gamma, \hat{Z}) \hookrightarrow \mathscr{G}(\Gamma, Z)$ is a Γ-equivariant $(2, 0)$-quasi-isometry. This is due to the following observation: every element of Z can be expressed as a word of $\hat{Z}$ of length at most 2.

Finally, we obtain a Γ-equivariant quasi-isometric map $\iota := \phi \cdot \imath$ from $\mathscr{G}(\Gamma, Y \cup \mathcal{K})$ to $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$. □

Remark 1.2.23. Eliminating redundant entries of $\mathbb{K}$ such that all entries of $\mathbb{K}$ are non-conjugate in Γ, we keep the same notation $\mathbb{K}$ for the reduced collection. It is easy to see the construction of the quasi-isometric map $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \to \mathscr{G}(G, X \cup \mathcal{H})$ works for the reduced $\mathbb{K}$.

In the following of this subsection, we assume the Γ-equivariant quasi-isometric map $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \to \mathscr{G}(G, X \cup \mathcal{H})$ is the one constructed in Lemma 1.2.22. In particular X is the suitable chosen relative generating set such that $Y \subset X$.

Lemma 1.2.24. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic. Let $\Gamma < G$ be relatively σ-quasiconvex. Then the quasi-isometric map $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \to \mathscr{G}(G, X \cup \mathcal{H})$ sends distinct peripheral $\mathbb{K}$-cosets of Γ to a d-distance σ from distinct peripheral $\mathbb{H}$-cosets of G.*

Proof. Taking into account Lemma 1.2.22 and Remark 1.2.23, we suppose all entries of $\mathbb{K}$ are non-conjugate. We continue the notations in the proof of Lemma 1.2.22.

By the construction of ϕ, we can see the map ϕ sends the subset $gZ_{x,x,i}$ to a uniform d-distance σ from the peripheral coset gxH_i of G for each $(x, x, i) \in \Xi$ and $g \in G$. Here σ is the quasiconvex constant associated to Γ. Observe that $\imath : \mathscr{G}(\Gamma, \hat{Z}) \hookrightarrow \mathscr{G}(\Gamma, Z)$ is an embedding. Therefore, we have the quasi-isometric map $\iota = \phi \cdot \imath$ maps each peripheral $\mathbb{K}$-coset to a uniform distance from a peripheral $\mathbb{H}$-coset.

We now show the "injectivity" of ι on $\mathbb{K}$-cosets. Let $\gamma H_i^g \cap \Gamma$, $\gamma' H_{i'}^{g'} \cap \Gamma$ be distinct peripheral $\mathbb{K}$-cosets of Γ, where $\gamma, \gamma' \in \Gamma$ and $H_i^g \cap \Gamma, H_{i'}^{g'} \cap \Gamma \in \mathbb{K}$.

Using Lemma 1.2.8, it is easy to deduce that if $\gamma(H_i^g \cap \Gamma)\gamma^{-1} \cap (H_{i'}^{g'} \cap \Gamma)$ is infinite, then $i = i'$ and $\gamma \in H_i^g \cap \Gamma$.

It is seen from the above discussion that there is a uniform constant $\sigma > 0$, such that $\iota(\gamma H_i^g \cap \Gamma) \subset N_\sigma(\gamma g H_i)$ and $\iota(\gamma' H_{i'}^{g'} \cap \Gamma) \subset N_\sigma(\gamma' g' H_{i'})$. It suffices to show that $\gamma g H_i \neq \gamma' g' H_{i'}$.

Without loss of generality, we assume that $i = i'$. Suppose, to the contrary, that $\gamma g H_i = \gamma' g' H_i$. Then we have $\gamma g = \gamma' g' h$ for some $h \in H_i$. It follows that $\gamma g H_i g^{-1} \gamma^{-1} = \gamma' g' H_i g'^{-1} \gamma'^{-1}$. This implies that $H_i^g \cap \Gamma$ is conjugate to $H_i^{g'} \cap \Gamma$ in Γ, i.e. $H_i^g \cap \Gamma = (H_i^{g'} \cap \Gamma)^{\gamma^{-1}\gamma'}$. Since any two entries of $\mathbb{K}$ are non-conjugate in Γ, we have $H_i^g \cap \Gamma = H_i^{g'} \cap \Gamma$. As a consquence, we have $\gamma^{-1}\gamma' \in H_i^g \cap \Gamma$, as $H_i^g \cap \Gamma \in \mathbb{K}$ is infinite. This is a contradiction, since we assumed $\gamma H_i^g \cap \Gamma \neq \gamma' H_{i'}^{g'} \cap \Gamma$.

Therefore, ι sends distinct peripheral $\mathbb{K}$-cosets of Γ to a uniform distance from distinct peripheral $\mathbb{H}$-cosets of G. □

Before proceeding to prove the relative hyperbolicity of relatively quasiconvex subgroups, we need justify the finite collection $\mathbb{K}$ in (1.2) as a set of representatives of Γ-conjugacy classes of $\hat{\mathbb{K}}$ in (1.3).

Lemma 1.2.25. *[MP09] Suppose $(G, \mathbb{H})$ is relatively hyperbolic. Let $\Gamma < G$ be relatively σ-quasiconvex. Then the following collection of subgroups of Γ*

$$\hat{\mathbb{K}} = \{H_i^g \cap \Gamma : \sharp\, H_i^g \cap \Gamma = \infty, g \in G, i \in I\}. \tag{1.3}$$

consists of finitely many Γ-conjugacy classes. In particular, $\mathbb{K}$ is a set of representatives of Γ-conjugacy classes of $\hat{\mathbb{K}}$.

Proof. This is proven by adapting an argument of Martinez-Pedroza [MP09, Proposition 1.5] with our formulation of BCP property 1.2.10. We refer the reader to [MP09] for the details. □

We are ready to show the relative hyperbolicity of $(\Gamma, \mathbb{K})$. Using notations in the proof of Lemma 1.2.22, we recall that $\mathbb{K} = \{Z_{x,x,i} : (x, x, i) \in \Xi\}$.

Lemma 1.2.26. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic. If $\Gamma < G$ is relatively σ-quasiconvex, then $(\Gamma, \mathbb{K})$ is relatively hyperbolic.*

Proof. Recall that ι is the Γ-equivariant quasi-isometric map from $\mathscr{G}(\Gamma, Y \cup \mathcal{K})$ to $\mathscr{G}(G, X \cup \mathcal{H})$. In particular we assumed $Y \subset X$.

We shall prove the relative hyperbolicity of Γ using Farb's definition. First, it is straightforward to verify that $\mathscr{G}(\Gamma, Y \cup \mathcal{K})$ has the thin-triangle property, using the quasi-isometric map ι and the hyperbolicity of $\mathscr{G}(G, X \cup \mathcal{H})$.

Let d_G be a proper left invariant metric on G. Denote by d_Γ the restriction of d_G on Γ. Obviously d_Γ is a proper left invariant metric on Γ. We are going to verify

BCP property 1) with respect to d_Γ, for the pair $(\Gamma, \mathbb{K})$. The verification of BCP property 2) is similar.

Let $[\gamma, s]$ be an edge of $\mathscr{G}(\Gamma, Y \cup \mathcal{K})$, where $s \in Z_{x,x,i}$ for some $(x, x, i) \in \Xi$. By the construction of ι, $[\gamma, s]$ is mapped by ι to the concatenated path $[\gamma, x][\gamma x, t][\gamma z t, x^{-1}]$, which clearly contains an H_i-component $[\gamma x, t]$. Note that $|x|_d \leq \sigma$. To simplify notations, we reindex $\mathbb{K} = \{K_j\}_{j\in J}$.

Given $\lambda \geq 1$ and $c \geq 0$, we consider two (λ, c)-quasigeodesics p, q without backtracking in $\mathscr{G}(\Gamma, Y \cup \mathcal{K})$ such that $p_- = q_-$, $p_+ = q_+$. By Lemma 1.2.24, as p, q are assumed to have no backtracking, the paths $\hat{p} = \iota(p)$, $\hat{q} = \iota(q)$ in $\mathscr{G}(G, X \cup \mathcal{H})$ also have no backtracking. Moreover, for each H_i-component $\hat{s}$ of $\hat{p}$(resp. $\hat{q}$), there is a K_j-component s of p(resp. q) such that $\hat{s} \subset \iota(s)$.

Note that paths $\hat{p}, \hat{q}$ are (λ',c')-quasigeodesic without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$ for some $\lambda' \geq 1, c' \geq 1$. By BCP property of $(G, \mathbb{H})$, we have the constant $\hat{a} = a(\lambda', c', d_G)$. Set $a = \hat{a} + 2\sigma$, where σ is the quasiconvex constant of Γ. Let s be a K_j-component of p for some $j \in J$. We claim that if $d_\Gamma(s_-, s_+) > a$, then there is a K_j-component t of q connected to s.

By the property of the map ι, there exists an H_i-component $\hat{s}$ of $\hat{p}$ such that the following hold

$$d_G(\hat{s}_-, \iota(s)_-) \leq \sigma, \; d_G(\hat{s}_+, \iota(s)_+) \leq \sigma.$$

Thus, we have $d_G(\hat{s}_-, \hat{s}_+) > \hat{a}$. Using BCP property 1) of $(G, \mathbb{H})$, there exists an H_i-component $\hat{t}$ of $\hat{q}$, that is connected to $\hat{s}$. By the construction of ι, there is a K_k-component t of q for some $k \in J$ such that $\hat{t} \subset \iota(t)$.

Since $\hat{s}$ and $\hat{t}$ are connected as H_i-components, endpoints of $\hat{s}$ and $\hat{t}$ belong to the same H_i-coset. By Lemma 1.2.24, it follows that $k = j$. Furthermore, endpoints of s and t must belong to the same K_j-coset. Hence s and t are connected in $\mathscr{G}(\Gamma, Y \cup \mathcal{K})$. Therefore, it is verified that $(\Gamma, \mathbb{K})$ satisfies BCP property 1). □

1.3 Characterization of parabolically embedded subgroups

Convention 2. Without loss of generality, peripheral structures considered in this section consist of infinite subgroups. It is easy to see that adding or eliminating finite subgroups in peripheral structures still gives relatively hyperbolic groups.

1.3.1 Parabolically embedded subgroups

Let $\mathbb{H} = \{H_i\}_{i\in I}$ and $\mathbb{P} = \{P_j\}_{j\in J}$ be two peripheral structures of a countable group G. Recall that $\mathbb{P}$ is an *extended peripheral structure* for $(G, \mathbb{H})$, if for each $H_i \in \mathbb{H}$, there exists $P_j \in \mathbb{P}$ such that $H_i \subset P_j$. Given $P \in \mathbb{P}$, we define $\mathbb{H}_P = \{H_i : H_i \subset P, i \in I\}$.

Definition 1.3.1. Suppose $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ an extended peripheral structure for $(G, \mathbb{H})$. If $(G, \mathbb{P})$ is relatively hyperbolic, then $\mathbb{P}$ is called a *parabolically extended* structure for $(G, \mathbb{H})$. Moreover, each $P \in \mathbb{P}$ is said to be *parabolically embedded* into $(G, \mathbb{H})$.

In this subsection, we assume that $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ is a parabolically extended structure for $(G, \mathbb{H})$.

Fix a finite relative generating set X for $(G, \mathbb{H})$ and thus $(G, \mathbb{P})$. Since $(G, \mathbb{P})$ is relatively hyperbolic, by Lemma 1.2.5, we obtain a finite subset Ω and $\kappa \geq 1$ such that the inequality (1.1) holds in $\mathscr{G}(G, X \cup \mathcal{P})$.

Due to Lemma 1.2.8 and Convention 2, it is worth to mention that we have $\mathbb{H}_P \cap \mathbb{H}_{P'} = \emptyset$, if P, P' are distinct in $\mathbb{P}$. This implies that each $H \in \mathbb{H}$ belongs to exactly one $P \in \mathbb{P}$.

Since $\mathbb{P}$ is an extended structure for $(G, \mathbb{H})$, then for each $H_i \in \mathbb{H}$, there exists a unique $P_j \in \mathbb{P}$ such that $H_i \hookrightarrow P_j$. By identifying $\mathcal{H} \subset \mathcal{P}$, we regard $\mathscr{G}(G, X \cup \mathcal{H})$ as a subgraph of $\mathscr{G}(G, X \cup \mathcal{P})$.

With a slight abuse of notations, a path p in $\mathscr{G}(G, X \cup \mathcal{H})$ will be often thought of as a path in $\mathscr{G}(G, X \cup \mathcal{P})$. The ambiance will be made clear in the context. The length $\ell(p)$ of a path p should also be understood in the corresponding relative Cayley graphs, but the values are equal by the natural embedding.

Let $\hat{X} = X \cup \Omega$. We first show parabolically embedded subgroups are relatively finitely generated.

Lemma 1.3.2. *Let $\Gamma = P_j \in \mathbb{P}$ be parabolically embedded into G. Then $Y := \hat{X} \cap \Gamma$ is a finite relative generating set for the pair $(\Gamma, \mathbb{H}_\Gamma)$.*

Proof. For any $\gamma \in \Gamma$, we take a geodesic p in $\mathscr{G}(G, X \cup \mathcal{H})$ with endpoints 1 and γ. In $\mathscr{G}(G, X \cup \mathcal{P})$, we can connect p_- and p_+ by an edge e, labeled by some letter from Γ, such that $e_- = p_-$ and $e_+ = p_+$. Then the path $c := pe^{-1}$ is a cycle in $\mathscr{G}(G, X \cup \mathcal{P})$. Without loss of generality, we assume e is a Γ-component of c.

The following two cases are examined separately.

Case 1. If there is no Γ-component of p connected to e in $\mathscr{G}(G, X \cup \mathcal{P})$, then e is an isolated component of c. By Lemma 1.2.5, we have

$$d_{\Omega_j}(e_-, e_+) \leq \kappa \ell(c) \leq \kappa(\ell(p) + 1)$$

where $\Omega_j := \Omega \cap \Gamma$. In particular, there is a path q in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$ labeled by letters from Ω_j, such that $q_- = e_-$, $q_+ = e_+$ and

$$\ell(q) = d_{\Omega_j}(e_-, e_+).$$

Hence, the element γ is a word over the alphabet Ω_j.

Case 2. We suppose that $\{e_1, \ldots, e_i, \ldots, e_n\}$ is the maximal set of Γ-components of p such that each e_i is connected to e. Then p can be decomposed as

$$p = p_1 e_1 \ldots p_i e_i \ldots p_n e_n p_{n+1}. \tag{1.4}$$

Since e_i is a Γ-component of p, each edge of e_i is labeled by an element in Γ. On the other hand, as a subpath of p, e_i has the label $\mathbf{Lab}(e_i)$ which is a word over $X \cup \mathcal{H}$.

Observe that each $H \in \mathbb{H}$ belongs to exactly one $P \in \mathbb{P}$. Thus we obtain that each $\mathbf{Lab}(e_i)$ is a word over $(X \cap \Gamma) \cup \mathbb{H}_\Gamma$

Since the vertex set $\{e_-, (e_1)_-, (e_1)_+, \ldots, (e_n)_-, (e_n)_+, e_+\}$ lies in Γ, we can connect pairs of consequent vertices

$$\{e_-, (e_1)_-\}, \ldots, \{(e_k)_+, (e_{k+1})_-\}, \ldots, \{(e_n)_+, e_+\}$$

by edges $s_0, \ldots, s_k \ldots, s_n$ labeled by letters from Γ respectively. We can get $n+1$ cycles $c_k := p_k s_k^{-1}, 1 \leq k \leq n+1$, such that s_k is an isolated Γ-component of c_k.

As argued in *Case 1* for each cycle c_k, we obtain a path q_k in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$ labeled by letters from Ω_j, such that $(q_k)_- = (e_k)_-$, $(q_k)_+ = (s_k)_+$, $\ell(q_k) = d_{\Omega_j}((s_k)_-, (s_k)_+)$ and the following inequality holds

$$\ell(q_k) \leq \kappa \ell(c_k) \leq \kappa(\ell(p_k) + 1). \tag{1.5}$$

In particular, we obtain a path $\hat{p}$ in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$ as follows

$$\hat{p} := q_1 e_1 \ldots q_i e_i \ldots q_n e_n q_{n+1} \tag{1.6}$$

with same endpoints as p. Note that the label $\mathbf{Lab}(\hat{p})$ is a word over the alphabet $(\hat{X} \cap \Gamma) \cup \mathcal{H}_\Gamma$. Therefore, γ is a word over $(\hat{X} \cap \Gamma) \cup \mathcal{H}_\Gamma$. □

Proposition 1.3.3. *Let $\Gamma = P_j \in \mathbb{P}$ be parabolically embedded into G. Then Γ is relatively quasiconvex with respect to $\mathbb{H}$.*

Proof. Since $\hat{X}$ is a finite relative generating set for $(G, \mathbb{H})$, using Lemma 1.2.5, we obtain a finite set Σ and constant $\mu > 1$ such that the inequality (1.1) holds in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$.

Let p be a geodesic in $\mathscr{G}(G, X \cup \mathcal{H})$ such that $p_-, p_+ \in \Gamma$. By Corollary 1.2.21, it suffices to prove that p lies in a uniform neighborhood of Γ with respect to $d_{\hat{X} \cup \Sigma}$.

By Lemma 1.3.2, we have a finite relative generating set $Y \subset \hat{X}$ for $(\Gamma, \mathbb{H}_\Gamma)$. Then we have $\mathscr{G}(\Gamma, Y \cup \mathcal{H}_\Gamma) \hookrightarrow \mathscr{G}(G, \hat{X} \cup \mathcal{H})$. Let q be a geodesic in $\mathscr{G}(\Gamma, Y \cup \mathcal{H}_\Gamma)$ such that $q_- = p_-, q_+ = p_+$. We claim that q is a quasigeodesic without backtracking in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$. No backtracking of q is obvious. We will show the quasigeodesicity of q.

We apply the same arguments to p, as *Case 2* in the proof of Lemma 1.3.2. Precisely, we decompose p as (1.4) and proceed to obtain the inequality (1.5) and construct a path $\hat{p}$ as in (1.6). Observe that $\hat{p}$ has the same endpoints as p, and $\mathbf{Lab}(\hat{p})$ is a word over the alphabet $Y \cup \mathcal{H}_\Gamma$. As $\hat{p}$ can be regarded as path in $\mathscr{G}(\Gamma, Y \cup \mathcal{H}_\Gamma)$, we obtain

$$\ell(q) \leq \ell(\hat{p}). \tag{1.7}$$

Using the inequality (1.5), we estimate the length of $\hat{p}$ as follows

$$\begin{array}{rl} \ell(\hat{p}) & = \sum_{1 \leq k \leq n+1} \ell(q_k) + \sum_{1 \leq k \leq n} \ell(e_k) \\ & \leq \sum_{1 \leq k \leq n+1} \kappa \ell(p_k) + \sum_{1 \leq k \leq n} \ell(e_k) + (n+1)\kappa \leq 2\kappa \ell(p). \end{array} \tag{1.8}$$

Since $\hat{X}$ is finite, the embedding $\mathscr{G}(G, X \cup \mathcal{H}) \hookrightarrow \mathscr{G}(G, \hat{X} \cup \mathcal{H})$ is a quasi-isometry. Thus there are constants $\lambda \geq 1, c \geq 0$, such that the geodesic p in $\mathscr{G}(G, X \cup \mathcal{H})$ is a (λ, c)-quasigeodesic in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, i.e.:

$$\ell(p) < \lambda d_{\hat{X} \cup \mathcal{H}}(p_-, p_+) + c. \tag{1.9}$$

Combining (1.7), (1.8) and (1.9), we have

$$\ell(q) \leq 2\kappa\lambda d_{\hat{X} \cup \mathcal{H}}(q_-, q_+) + 2\kappa c \tag{1.10}$$

It is easy to see the above estimates (1.7), (1.8) and (1.9) can be applied to arbitrary subpath of q. Thus the same inequality as (1.10) is obtained for arbitrary subpath of q. This proves our claim that q is a $(2\kappa\lambda, 2\kappa c)$-quasigeodesic without backtracking in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$.

As $\kappa \geq 1$, p is a $(2\kappa\lambda, 2\kappa c)$-quasigeodesic in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$. Hence by Lemma 1.2.7, there exists a constant $\epsilon = \epsilon(\kappa, \lambda, c)$ such that, for each vertex $v \in p$, there is a phase vertex $u \in q$ such that $d_{\hat{X} \cup \Sigma}(u, v) \leq \epsilon$.

On the other hand, the vertex set of q lies entirely in Γ. Thus p lies in a ϵ-neighborhood of Γ with respect to $d_{\hat{X} \cup \Sigma}$. Therefore, we have proven the relative quasiconvexity of Γ with respect to $\mathbb{H}$. □

Lemma 1.2.26 and Proposition 1.3.3 together prove the following.

Corollary 1.3.4. *A parabolically embedded subgroup Γ is hyperbolic relative to $\mathbb{H}_\Gamma$.*

1.3.2 Lifting of quasigeodesics

In this subsection, we assume that $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ an extended structure for $(G, \mathbb{H})$. The results established in this subsection will be applied in subsection 3.3 to prove Theorem 1.3.10.

To make our discussion more transparent, we first note the following assumption, on which the notion of a lifting path is defined.

Assumption A. *Each $P_j \in \mathbb{P}$ is relatively quasiconvex with respect to $\mathbb{H}$.*

By Lemma 1.2.22, we assume that each $P_j \in \mathbb{P}$ is finitely generated by a finite set Y_j with respect to $\mathbb{H}_j := \mathbb{H}_{P_j}$. Without loss of generality, we assume X to be a finite relative generating set for $(G, \mathbb{H})$ such that $Y_j \subset X$ for each $j \in J$. So we can identify the relative Cayley graph $\mathscr{G}(P_j, Y_j \cup \mathcal{H}_j)$ of P_j as a subgraph of $\mathscr{G}(G, X \cup \mathcal{H})$. Thus given any path p of $\mathscr{G}(G, X \cup \mathcal{P})$, we can define the *lifting path* of p in $\mathscr{G}(G, X \cup \mathcal{H})$, by replacing each P_j-component of p by a geodesic segment in $\mathscr{G}(P_j, Y_j \cup \mathcal{H}_j)$ with same endpoints.

Precisely, we express the path p in $\mathscr{G}(G, X \cup \mathcal{P})$ in the following form

$$p = s_0 t_0 \ldots s_k t_k \ldots s_n t_n, \tag{1.11}$$

where t_k are P_k-components of p and s_k are labeled by letters from X. It is possible that $P_i = P_j$ for $i \neq j$. We allow s_0 and t_n to be trivial.

Let $\iota_k : \mathscr{G}(P_k, Y_k \cup \mathcal{H}_k) \hookrightarrow \mathscr{G}(G, X \cup \mathcal{H})$ be the graph embedding. For each t_k, we take a geodesic segment $\hat{t}_k$ in $\mathscr{G}(P_k, Y_k \cup \mathcal{H}_k)$ such that $(\hat{t}_k)_- = (t_k)_-$ and $(\hat{t}_k)_+ = (t_k)_+$. Then the following constructed path

$$\hat{p} = s_0 \iota(\hat{t}_0) \ldots s_k \iota(\hat{t}_k) \ldots s_n \iota(\hat{t}_n) \tag{1.12}$$

is the *lifting path* of p in $\mathscr{G}(G, X \cup \mathcal{H})$.

The following two lemmas require only Assumption A above.

Lemma 1.3.5. *Lifting of a path without backtracking in $\mathscr{G}(G, X \cup \mathcal{P})$ has no backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$.*

Proof. We assume the path p and its lifting $\hat{p}$ decompose as (1.11) and (1.12) respectively. By way of contradiction, we assume that, for some $H \in \mathbb{H}$, there exist H-components r_1, r_2 of $\hat{p}$, such that r_1, r_2 are connected. Since s_k are labeled by letters from X, we have $r_1 \subset \iota(\hat{t}_i), r_2 \subset \iota(\hat{t}_j)$ for some $0 \leq i, j \leq n$.

Note that $\hat{t}_k$ is a geodesic in $\mathscr{G}(P_k, Y_k \cup \mathcal{H}_k)$, which has no backtracking. Hence, by Assumption A and Lemma 1.2.24, $\iota(\hat{t}_k)$ has no backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$. It follows that $i \neq j$.

Since p is assumed to have no backtracking, we have that $\iota(\hat{t}_i)$ and $\iota(\hat{t}_j)$ lie entirely in distinct cosets $g_1 P_i$ and $g_2 P_j$ respectively. On the other hand, as r_1 and r_2 are connected H-components, their endpoints lie in the same H-coset. By the assumption that $\mathbb{H}$ consists of infinite subgroups, each $H \in \mathbb{H}$ belongs to exact one subgroup $P \in \mathbb{P}$. Thus it follows that $g_1 P_i$ and $g_2 P_j$ coincide. This leads to a contradiction with non-backtrackingness of p. □

Similar arguments as above allow one to prove the following.

Lemma 1.3.6. *Suppose p, q are two paths in $\mathscr{G}(G, X \cup \mathcal{P})$ such that there are no connected P_j-components of p, q for any $P_j \in \mathbb{P}$. Then for any $H_i \in \mathbb{H}$, their liftings $\hat{p}, \hat{q}$ have no connected H_i-components.*

To deduce our main result Proposition 1.3.8, we make another assumption as follows.

Assumption B. *Let X be a finite relative generate set for $(G, \mathbb{H})$. There exists $\kappa \geq 1$ such that for any cycle o in $\mathscr{G}(G, X \cup \mathcal{P})$ with a set of isolated Γ-components $R = \{r_1, \ldots, r_k\}$, the following holds*

$$\sum_{r \in R} d_{X \cup \mathcal{H}}(r_-, r_+) \leq \kappa \ell(o).$$

Remark 1.3.7. Lemma 1.3.13 below states that Assumption B will be satisfied under the assumptions of Theorem 1.3.10.

Taking this assumption into account, we have the following.

Proposition 1.3.8. *Lifting of a quasigeodesic without backtracking in $\mathscr{G}(G, X \cup \mathcal{P})$ is a quasigeodesic without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$.*

Proof. To simplify our proof, we prove the proposition for the lifting of a geodesic p in $\mathscr{G}(G, X \cup \mathcal{P})$. General cases follow from a quasi-modification of the inequality in (1.13) mentioned below.

We assume the path p and its lifting $\hat{p}$ decompose as (1.11) and (1.12) respectively. Since ι_k is an embedding, we will write $\hat{t}_k$ instead of $\iota(\hat{t}_k)$ for simplicity. We shall show the lifting path $\hat{p} = s_0\hat{t}_0 \dots s_k\hat{t}_k \dots s_n\hat{t}_n$ is a quasigeodesic in $\mathscr{G}(G, X \cup \mathcal{H})$.

By Lemma 1.2.22, we have each $\hat{t}_k$ is a (λ, c)-quasigeodesic in $\mathscr{G}(G, X \cup \mathcal{H})$, where the constants $\lambda \geq 1, c \geq 0$ depend on P_k and X. As $\sharp\, \mathbb{P}$ is finite, λ and c can be made uniform for all $P_k \in \mathbb{P}$.

Let q be a geodesic in $\mathscr{G}(G, X \cup \mathcal{H})$ with same endpoints as $\hat{p}$. Since $\mathscr{G}(G, X \cup \mathcal{H}) \hookrightarrow \mathscr{G}(G, X \cup \mathcal{P})$, it is obvious that

$$\ell(p) \leq \ell(q). \tag{1.13}$$

We consider the cycle $c := pq^{-1}$ in $\mathscr{G}(G, X \cup \mathcal{P})$. For each t_k, we are going to estimate the length of $\hat{t}_k$ in $\mathscr{G}(G, X \cup \mathcal{H})$.

Case 1. The path t_k is isolated in c. By Assumption B, there exists a constant $\kappa \geq 1$ such that

$$d_{X\cup\mathcal{H}}((\hat{t}_k)_-, (\hat{t}_k)_+) \leq \kappa\ell(c) \leq \kappa(\ell(p) + \ell(q)) \leq 2\kappa(\ell(q)). \tag{1.14}$$

Case 2. The path t_k is not isolated in c. Then t_k is connected to some P_k-component of q, as p is assumed to have no backtracking in $\mathscr{G}(G, X \cup \mathcal{P})$. Let e_1 and e_2 be the first and last P_k-components of q connected to t_k. Note that e_1 may coincide with e_2.

Since e_1 and e_2 are connected to t_k, we can take two edges u and v labeled by letters from P_k such that

$$u_- = (t_k)_-, u_+ = (e_1)_-, v_- = (t_k)_+, v_+ = (e_2)_+.$$

Then $c_1 := p_1uq_1^{-1}$ and $c_2 := v^{-1}p_2{q_2}^{-1}$ are two cycles in $\mathscr{G}(G, X \cup \mathcal{P})$. By the choice of P_k-components e_1 and e_2, we deduce that u and v are isolated P_k-components of c_1 and c_2 respectively. By Assumption B, we have the following inequalities

$$d_{X\cup\mathcal{H}}(u_-, u_+) \leq \kappa\ell(c_1) \leq \kappa(\ell(p) + \ell(q) + 1) \leq 2\kappa\ell(q) + \kappa, \tag{1.15}$$

and

$$d_{X\cup\mathcal{H}}(v_-, v_+) \leq \kappa\ell(c_2) \leq \kappa(\ell(p) + \ell(q) + 1) \leq 2\kappa\ell(q) + \kappa. \tag{1.16}$$

Then it follows from (1.15) and (1.16) that

$$\begin{aligned} d_{X\cup\mathcal{H}}((t_k)_-, (t_k)_+) &\leq \ell(q) + d_{X\cup\mathcal{H}}(u_-, u_+) + d_{X\cup\mathcal{H}}(v_-, v_+) \\ &\leq (4\kappa + 1)\ell(q) + 2\kappa. \end{aligned} \tag{1.17}$$

As $\hat{t}_k$ can be regarded as a (λ, c)-quasigeodesic in $\mathscr{G}(G, X \cup \mathcal{H})$, we estimate the length of $\hat{t}_k$ in $\mathscr{G}(G, X \cup \mathcal{H})$ by taking into account (1.14) and (1.17),

$$\begin{aligned} \ell(\hat{t}_k) &\leq \lambda d_{X\cup\mathcal{H}}((t_k)_-, (t_k)_+) + c \\ &\leq \lambda(4\kappa + 1)\ell(q) + 2\lambda\kappa + c. \end{aligned} \tag{1.18}$$

Finally, we have

$$\begin{aligned} \ell(\hat{p}) &= \sum_{0\le k\le n} \ell(s_i) + \sum_{0\le k\le n} \ell(\hat{t}_k) \\ &\le \ell(q) + \ell(q)(\lambda(4\kappa+1)\ell(q) + 2\lambda\kappa + c) \\ &\le \lambda(4\kappa+1)(\ell(q))^2 + (2\lambda\kappa + c + 1)\ell(q). \end{aligned} \tag{1.19}$$

Similarly, we can apply the above estimates to arbitrary subpath of $\hat{p}$ to obtain the same quadratic bound on its length as (1.19). It is well-known that in hyperbolic spaces a sub-exponential path is a quasigeodesic, see Bowditch [Bo99b, Lemma 5.6] for example. Note that $\mathscr{G}(G, X \cup \mathcal{H})$ is hyperbolic. Hence $\hat{p}$ is a quasigeodesic in $\mathscr{G}(G, X \cup \mathcal{H})$. □

Remark 1.3.9. In [MP08], Martinez-Pedroza proves a specical case of Proposition 1.3.8, where for each $P \in \mathbb{P}$, P belongs to $\mathbb{H}$, or P is a hyperbolically embedded subgroup in the sense of Osin [Os06a].

1.3.3 Characterization of parabolically embedded subgroups

Let $\mathbb{H} = \{H_i\}_{i\in I}$ and $\mathbb{K} \subset \mathbb{H}$ be two peripheral structures of a countable group G. We will show the following characterization of parabolically embedded subgroups.

Theorem 1.3.10. *Let G be hyperbolic relative to $\mathbb{H}$. Assume that*
(C0). $\Gamma \subset G$ contains $\mathbb{K} \subset \mathbb{H}$,
(C1). Γ is relatively quasiconvex,
(C2). Γ is weakly malnormal,
(C3). $\Gamma^g \cap H_i$ is finite for any $g \in G$ and $H_i \in \mathbb{H} \setminus \mathbb{K}$.
Then G is hyperbolic relative to $\{\Gamma\} \cup \mathbb{H} \setminus \mathbb{K}$.

Putting in another way, Theorem 1.3.10 implies the following

Corollary 1.3.11. *Under the assumptions of Theorem 1.3.10, Γ is a parabolically embedded subgroup of G with respect to $\mathbb{K}$.*

We now prove Theorem 1.1.1 using Theorem 1.3.10.

Proof of Theorem 1.1.1. For the sufficient part, Condition (P1) follows from Proposition 1.3.3. Since $(G, \mathbb{P})$ is relatively hyperbolic, Conditions (P2) and (P3) are direct consequences of Lemma 1.2.8.

Let $\mathbb{P} = \{P_1, \ldots, P_j, \ldots, P_n\}$. Recall that $\mathbb{H}_{P_j} = \{H_i : H_i \subset P_j; i \in I\}$. Define peripheral structures

$$\mathbb{P}_k = \{P_1, \ldots, P_k\} \cup \left(\mathbb{H} \setminus \cup_{1\le j\le k} \mathbb{H}_{P_j}\right), 0 \le k \le n.$$

Note that $\mathbb{P}_0 = \mathbb{H}$, $\mathbb{P}_n = \mathbb{P}$. By definition, we have $\mathbb{P}_k$ is an extended structure for $(G, \mathbb{P}_{k-1})$ for each $1 \le k \le n$. In particular, Conditions (P1)-(P3) imply that $P_k \subset G$ satisfies Conditions (C0)-(C3) for $(G, \mathbb{P}_{k-1})$. By repeated applications of Theorem 1.3.10, we obtain $\mathbb{P}_k$ is parabolically extended for $(G, \mathbb{P}_{k-1})$. Finally, we prove that $\mathbb{P}$ is parabolically extended for $(G, \mathbb{H})$. □

In what follows, we have all assumptions of Theorem 1.3.10 are satisfied.

Choose a finite relative generating set X for $(G, \mathbb{H})$. Let Ω the finite set obtained by using Lemma 1.2.5 for $\mathscr{G}(G, X \cup \mathcal{H})$. To simplify notations, we denote $\mathbb{P} = \{\Gamma\} \cup \mathbb{H} \setminus \mathbb{K}$.

Since $\Gamma \subset G$ is assumed to satisfy Conditions (C0)–(C3), by Lemma 1.2.22 we have Γ is finitely generated by a subset Y with respect to $\mathbb{K}$. Without loss of generality, we assume $Y \subset X$. So the graph embedding $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \hookrightarrow \mathscr{G}(G, X \cup \mathcal{H})$ is a quasi-isometric map.

Note that $\mathbb{P}$ satisfies Assumption A. So given any path p of $\mathscr{G}(G, X \cup \mathcal{H})$, we can define the lifting path $\hat{p}$ in $\mathscr{G}(G, X \cup \mathcal{P})$ as in Subsection 3.2. So we have exactly Lemmas 1.3.5 and 1.3.6.

Furthermore, by Lemma 1.3.13 below, we have Assumption B satisfied in the current setting. So we have the following result by Proposition 1.3.8.

Proposition 1.3.12. *Under the assumptions of Theorem 1.3.10. Lifting of a quasigeodesic without backtracking in $\mathscr{G}(G, X \cup \mathcal{P})$ is a quasigeodesic without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$.*

The following Lemma 1.3.13 is an analogue of Lemma 1.2.5, without assuming that $(G, \mathbb{P})$ is relatively hyperbolic. Recall that $\mathbb{P} = \{\Gamma\} \cup \mathbb{H} \setminus \mathbb{K}$.

Lemma 1.3.13. *Under the assumptions of Theorem 1.3.10. There exists $\mu \geq 1$ such that for any cycle o in $\mathscr{G}(G, X \cup \mathcal{P})$ with a set of isolated Γ-components $R = \{r_1, \ldots, r_k\}$, the following holds*

$$\sum_{r \in R} d_{X \cup \mathcal{H}}(r_-, r_+) \leq \mu \ell(o).$$

We defer the proof of Lemma 1.3.13 and now finish the proof of Theorem 1.3.10 by using Proposition 1.3.12.

Proof of Theorem 1.3.10. We shall prove the relative hyperbolicity of $(G, \mathbb{P})$ using Farb's definition.

Let pqr be a geodesic triangle in $\mathscr{G}(G, X \cup \mathcal{P})$. We are going to verify the thinness of pqr. Let $\hat{p}, \hat{q}, \hat{r}$ be lifting of p, q, r in $\mathscr{G}(G, X \cup \mathcal{H})$ respectively. Then by Proposition 1.3.12, there exists $\lambda \geq 1, c \geq 0$ such that $\hat{p}\hat{q}\hat{r}$ is a (λ, c)-quasigeodesic triangle in $\mathscr{G}(G, X \cup \mathcal{H})$.

Since $(G, \mathbb{H})$ is relatively hyperbolic, then $\hat{p}\hat{q}\hat{r}$ is ν-thin for the constant $\nu > 0$ depending on λ, c. That is to say, the side $\hat{p}$ belongs to a ν-neighborhood of the union $q \cup r$. Since $\mathscr{G}(G, X \cup \mathcal{H}) \hookrightarrow \mathscr{G}(G, X \cup \mathcal{P})$, we have $d_{X \cup \mathcal{P}}(x, y) \leq d_{X \cup \mathcal{H}}(x, y)$ for $x, y \in G$. By the construction of lifting paths, we have the vertex set of triangle pqr is contained in a 1-neighborhood of the one of triangle $\hat{p}\hat{q}\hat{r}$ in $\mathscr{G}(G, X \cup \mathcal{P})$. Then pqr is $(\nu + 1)$-thin in $\mathscr{G}(G, X \cup \mathcal{P})$.

Given any $\lambda \geq 1, c \geq 0$, we take two (λ, c)-quasigeodesics p, q without backtracking in $\mathscr{G}(G, X \cup \mathcal{P})$ with same endpoints. Let $\hat{p}, \hat{q}$ be lifting of p, q in $\mathscr{G}(G, X \cup \mathcal{H})$ respectively. By Proposition 1.3.12, there exist constants $\lambda' \geq 1, c' \geq 0$, such that $\hat{p}, \hat{q}$ are (λ', c')-quasigeodesic without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$.

Let $\hat{X} = X \cup \Omega$. Using Lemma 1.2.5 again, we obtain a finite set Σ and constant $\mu > 1$ such that the inequality (1.1) holds in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$. Let $\epsilon = \epsilon(\lambda', c')$ the constant given by Lemma 1.2.7.

Suppose s is a Γ-component of p such that no Γ-component of q is connected to s. To verify BCP property 1), it suffices to bound $d_{\hat{X}\cup\Sigma}(s_-, s_+)$ by a uniform constant using Corollary 1.2.15. BCP property 2) can be verified in a similar way.

Since endpoints of s belong to the vertex set of $\hat{p}$, by Lemma 1.2.7, there exists vertices $\hat{u}, \hat{v} \in \hat{q}$ such that

$$d_{\hat{X}}(s_-, \hat{u}) < \epsilon, \; d_{\hat{X}}(s_+, \hat{v}) < \epsilon.$$

If $\hat{u}$ is not a vertex of q, then $\hat{u}$ must belong to a lifting of a Γ-component of q. So we can take a phase vertex $u \in p$ such that $d_{X\cup\mathcal{P}}(u, \hat{u}) \leq 1$. Otherwise, we set $u = \hat{u}$. Similarly, we choose a phase vertex v of q such that $d_{X\cup\mathcal{P}}(v, \hat{v}) \leq 1$. We connect $u, \hat{u}$(resp. $v, \hat{v}$) by a path e_u(resp. e_v), which consists of at most one edge labeled by a letter from Γ. The path e_u is trivial if $u = \hat{u}$.

By regarding p, q as paths in $\mathscr{G}(G, \hat{X}\cup\mathcal{P})$, there exist paths l and r in $\mathscr{G}(G, \hat{X}\cup\mathcal{P})$ labeled by letters from $\hat{X}$, such that $l_- = s_-, l_+ = \hat{u}, r_- = s_+, r_+ = \hat{v}$. Let

$$o = sre_v[u, v]_q^{-1}e_u^{-1}l^{-1}$$

be a cycle in $\mathscr{G}(G, \hat{X} \cup \mathcal{P})$, where $[u, v]_q$ denotes the segment of q between u and v. Since $[u, v]_q$ is a (λ, c)-quasigeodesic in $\mathscr{G}(G, \hat{X} \cup \mathcal{P})$, then by the triangle inequality,

$$\begin{aligned} \ell([u, v]_q) &\leq \lambda d_{X\cup\mathcal{P}}(u, v) + c \\ &\leq \lambda(d_{X\cup\mathcal{P}}(u, \hat{u}) + d_{X\cup\Omega}(\hat{u}, s_-) + 1 + d_{X\cup\mathcal{P}}(s_+, \hat{v}) + d_{X\cup\mathcal{P}}(v, \hat{v})) + c \\ &\leq \lambda(3 + 2\epsilon) + c. \end{aligned}$$

It follows that

$$\begin{aligned} \ell(o) &\leq \ell([u, v]_q) + d_{X\cup\mathcal{P}}(u, \hat{u}) + d_{X\cup\Omega}(\hat{u}, s_-) + 1 + d_{X\cup\mathcal{P}}(s_+, \hat{v}) + d_{X\cup\mathcal{P}}(v, \hat{v}) \\ &\leq (\lambda + 1)(3 + 2\epsilon) + c. \end{aligned}$$

By Lemma 1.3.13, there exists a constant $\mu \geq 1$ such that

$$d_{X\cup\mathcal{H}}(s_-, s_+) \leq \mu\ell(o) \leq \mu(\lambda + 1)(3 + 2\epsilon) + c\mu. \tag{1.20}$$

Let $\hat{s}$ be the lifting of the Γ-component s in $\mathscr{G}(G, X \cup \mathcal{H})$. As a subpath of $\hat{p}$, $\hat{s}$ is a (λ', c')-quasigeodesic in $\mathscr{G}(G, X \cup \mathcal{H})$. Then we have $\ell(\hat{s}) \leq \lambda' d_{X\cup\mathcal{H}}(s_-, s_+) + c'$.

We consider the cycle in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$ as follows

$$\hat{o} := \hat{s}r[\hat{u}, \hat{u}]_{\hat{q}}^{-1}l^{-1},$$

where $[\hat{u}, \hat{v}]_{\hat{q}}$ denotes the subpath of $\hat{q}$ between $\hat{u}$ and $\hat{v}$. As $\hat{q}$ is a (λ', c')-quasigeodesic in $\mathscr{G}(G, X \cup \mathcal{H})$, we have

$$\begin{aligned} \ell([\hat{u}, \hat{v}]_{\hat{q}}) &\leq \lambda' d_{X\cup\mathcal{H}}(\hat{u}, \hat{v}) + c' \\ &\leq \lambda'(d_{X\cup\mathcal{P}}(\hat{u}, s_-) + d_{X\cup\mathcal{H}}(s_-, s_+) + d_{X\cup\mathcal{P}}(s_-, \hat{v})) + c' \\ &\leq \lambda'(2\epsilon + d_{X\cup\mathcal{H}}(s_-, s_+)) + c'. \end{aligned}$$

It follows that

$$\begin{aligned} \ell(\hat{o}) &\leq d_{\hat{X}}(\hat{u}, s_-) + \ell(s) + d_{\hat{X}}(s_+, \hat{v}) + \ell([\hat{u}, \hat{v}]_{\hat{q}}) \\ &< 2\epsilon(\lambda' + 1) + \lambda' d_{X\cup\mathcal{H}}(s_-, s_+) + c'. \end{aligned} \tag{1.21}$$

It is assumed that no Γ-component of q is connected to the Γ-component s of p. By Lemma 1.3.6, we obtain that, for any $H_i \in \mathbb{H}$, no H_i-component of $\hat{s}$ is connected to an H_i-component of $\hat{q}$. Moreover, $\hat{s}$ has no backtracking by Lemma 1.2.24. Hence every H_i-component of $\hat{s}$ is isolated in the cycle $\hat{o}$. Using Lemma 1.2.5 for $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, we have

$$d_{\hat{X}\cup\Sigma}(s_-, s_+) < d_{X\cup\mathcal{H}}(s_-, s_+) \cdot \kappa\ell(\hat{o}).$$

Observe that $d_{X\cup\mathcal{H}}(s_-, s_+)$ and $\ell(\hat{o})$ are upper bounded by uniform constants, as shown in 1.20, 1.21. Thus the distance $d_{\hat{X}\cup\Sigma}(s_-, s_+)$ is also uniformly upper bounded by a constant. Therefore, we have completed the verification of BCP property 1) for $(G, \mathbb{P})$ □

1.3.4 Proof of Lemma 1.3.13

Under the assumptions of Theorem 1.3.10, we now prove Lemma 1.3.13. Our proof is essentially inspired by Osin's arguments in [Os06a]. In particular, we need the following two Lemmas 1.3.14 and 1.3.15 analogous to Lemmas 3.1 and 3.2 in [Os06a] respectively.

Let X be a finite relative generating set for $(G, \mathbb{H})$. Recall that two paths p, q in $\mathscr{G}(G, X \cup \mathcal{H})$ are called *k-connected* for $k \geq 0$, if

$$\max\{d_{X\cup\mathcal{H}}(p_-, q_-), d_{X\cup\mathcal{H}}(p_+, q_+)\} \leq k.$$

Since $(G, \mathbb{H})$ is relatively hyperbolic, we obtain the finite set Ω by using Lemma 1.2.5 for $\mathscr{G}(G, X \cup \mathcal{H})$.

The following lemma requires only the assumption that $(G, \mathbb{H})$ is relatively hyperbolic. It can be proven by combining the proofs of [Os06b, Proposition 3.15] and [Os06a, Lemma 3.1] partially. See the proof in Appendix A.

Lemma 1.3.14. *For any $\lambda \geq 1$, $c \geq 0$, there exists $\alpha_1 = \alpha_1(\lambda, c) > 0$ such that, for any $k \geq 0$, there exists $\alpha_2 = \alpha_2(k, \lambda, c) > 0$ satisfying the following condition. Let p, q be two k-connected (λ, c)-quasigeodesics in $\mathscr{G}(G, X \cup \mathcal{H})$. If p has no backtracking and u is a phase vertex on p such that $\min\{d_{X\cup\mathcal{H}}(u, p_-), d_{X\cup\mathcal{H}}(u, p_+)\} > \alpha_2$. Then there exists a phase vertex v on q such that $d_{X\cup\Omega}(u, v) \leq \alpha_1$.*

Using Lemma 1.3.14, the following lemma, although stated in geometric terms, is a reminiscent of [Os06a, Lemma 3.2] and can be proven along the same line with Conditions (C0)–(C3) on Γ. See the proof in Appendix A.

Lemma 1.3.15. *For any $\lambda \geq 1$, $c \geq 0$, $k > 0$, there exists $L = L(\lambda, c, k) > 0$ such that the following holds. Let p, q be k-connected (λ, c)-quasigeodesics without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$ such that p, q are labeled by letters from $\Gamma \setminus \{1\}$. If $\min\{\ell(p), \ell(q)\} > L$, then p and q as Γ-components are connected in $\mathscr{G}(G, X \cup \mathcal{P})$.*

We define a *geodesic n-polygon* P in a geodesic metric space as a collection of n geodesics $p_1, \ldots, p_n$ such that $(p_i)_+ = (p_{i+1})_-$, where i is taken modulo n. The following lemma follows from the proof of [Ol91, Lemma 25], but is weaker.

Lemma 1.3.16. *[Ol91] There are constants $\beta_1 = \beta_1(\delta) > 0, \beta_2 = \beta_2(\delta) > 0$ such that the following holds for any geodesic n-polygon P in a δ-hyperbolic space. Suppose the set of all sides of P is divided into three subsets R, S and T with length sums Σ_R, Σ_S and Σ_T respectively. If $\Sigma_R > max\{\beta n, 10^3\Sigma_S\}$ for some $\beta \geq \beta_1$. Then there exist distinct sides $p_i \in R, p_j \in R \cup T$ that contain β_2-connected segments of length greater than $10^{-3}\beta$.*

We are now ready to prove Lemma 1.3.13. Its proof uses crucially the quasi-isometric map $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \hookrightarrow \mathscr{G}(G, X \cup \mathcal{H})$.

Proof of Lemma 1.3.13. Without loss of generality, we assume the cycle o in $\mathscr{G}(G, X \cup \mathcal{P})$ can be written as the following form

$$o = r_1 s_1 \ldots r_n s_n$$

such that $\{r_1, \ldots, r_n\}$ is the maximal set of Γ-components of o. Note that s_n is not trivial. We assume that $\{r_{n_1}, \ldots r_{n_k}\}$ is a set of isolated Γ-components of o.

Consider the geodesic $2n$-polygon $a_1 b_1 \ldots a_n b_n$, where a_i and b_i are geodesics in $\mathscr{G}(G, X \cup \mathcal{H})$ with the same endpoints as r_i and s_i respectively. We divide the sides of the 2n-polygon into three disjoint sets. Let $R = \{a_{n_1}, \ldots, a_{n_k}\}$, $S = \{b_1, \ldots, b_n\}$ and $T = \{a_i : a_i \notin R\}$. Set $\Sigma_R = \sum_{i=0}^{n} \ell(a_{n_i})$ and $\Sigma_S = \sum_{i=1}^{k} \ell(b_{n_i})$. Obviously $\Sigma_S \leq \ell(o)$.

Let $\hat{r}_i$ be a geodesic segment in $\mathscr{G}(\Gamma, Y \cup \mathcal{K})$ such that $(\hat{r}_i)_- = (r_i)_-$ and $(\hat{r}_i)_+ = (r_i)_+$. Since the embedding $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \hookrightarrow \mathscr{G}(G, X \cup \mathcal{H})$ is quasi-isometric, then $\hat{r}_i$ is a (λ, c)-quasigeodesic in $\mathscr{G}(G, X \cup \mathcal{H})$ for some $\lambda \geq 1, c \geq 0$. Moreover, $\hat{r}_i$ has no backtracking by Lemma 1.2.24.

Let δ denote the hyperbolicity constant of $\mathscr{G}(G, X \cup \mathcal{H})$. By the stability of quasigeodesics in hyperbolic spaces (see [Gr87] or [GH90]), there exists a constant $\xi = \xi(\delta, \lambda, c)$ such that $\hat{r}_i$ have a uniform Hausdorff ξ-distance from a_i.

Let $\beta_1 = \beta_1(\delta), \beta_2 = \beta_2(\delta)$ be the constants provided by Lemma 1.3.16, $L = L(\lambda, c, \beta_2 + 2\xi)$ the constant provided by Lemma 1.3.15.

It suffices to set $\mu = \max\{\beta_1, 10^3, (L + 2\xi) \cdot 10^3\}$ for showing $\Sigma_R \leq \mu\ell(o)$. Suppose, to the contrary, we have $\Sigma_R > \mu\ell(o)$. This yields

$$\Sigma_R > \mu\ell(o) \geq \max\{\mu\ell(o), 10^3\ell(o) \geq \max\{\mu\ell(o), 10^3\Sigma_S\}.$$

By Lemma 1.3.16, there are distinct sides $a_j \in R$ and $a_k \in R \cup T$, having β_2-connected segments of length at least $\mu \cdot 10^{-3}$. Therefore, there exist $(\beta_2 + 2\xi)$-connected subsegments $q_1 \subset \hat{r}_j, q_2 \subset \hat{r}_k$ such that

$$\min\{\ell(q_1), \ell(q_2)\} \geq \mu \cdot 10^{-3} - 2\xi \geq L.$$

Since q_1, q_2 are (λ, c)-quasigeodesics labeled by letters from Γ, they are connected by Lemma 1.3.15. Thus, r_j and r_k are connected. This is a contradiction, since r_j is an isolated Γ-component of the cycle o. □

1.4 Peripheral structures and Floyd boundary

1.4.1 Convergence groups and dynamical quasiconvexity

Let M be a compact metrizable space. We denote by $\Theta^n M$ the set of subsets of M of cardinality n, equipped with the product topology.

A *convergence group action* is an action of a group G on M such that the induced action of G on the space $\Theta^3 M$ is properly discontinuous. Following Gerasimov [Ge09], a group action of G on M is *2-cocompact* if the quotient space $\Theta^2 M/G$ is compact.

Suppose G has a convergence group action on M. Then M is partitioned into a limit set $\Lambda_M(G)$ and discontinuous domain $M \setminus \Lambda_M(G)$. The *limit set* $\Lambda_M(H)$ of a subgroup $H \subset G$ is the set of limit points, where a *limit point* is an accumulation point of some H-orbit in M. An infinite subgroup $P \subset G$ is a *parabolic subgroup* if the limit set $\Lambda_M(P)$ consists of one point, which is called a *parabolic point*. The stabilizer of a parabolic point is always a (maximal) parabolic group. A parabolic point p with stabilizer $G_p := Stab_G(p)$ is *bounded* if G_p acts cocompactly on $M \setminus \{p\}$. A point $z \in M$ is a *conical* point if there exists a sequence $\{g_i\}$ in G and distinct points $a, b \in M$ such that $g_i(z) \to a$, while for all $q \in M \setminus \{z\}$, we have $g_i(q) \to b$.

Convention 3. For simplicity, we often denote by $G \curvearrowright M$ a convergence group action of G on a compact metrizable M.

Let us begin with the following simple observation.

Lemma 1.4.1. *Suppose a group G admits convergence group actions on compact spaces M and N respectively. If there is a G-equivariant surjective map ϕ from M to N, then for any $H < G$, $\phi(\Lambda_M(H)) = \Lambda_N(H)$.*

Proof. Given $x \in \Lambda_M(H)$, by definition, there exists $z \in M$ and $\{h_n\} \subset H$ such that $h_n(z) \to x$ as $n \to \infty$. Since ϕ is a G-equivariant map, we have $h_n(\phi(z)) = \phi(h_n z) \to \phi(x)$. Then $\phi(x)$ is the limit point of of sequence $\{h_n(\phi(z))\}$ and thus $\phi(x) \in \Lambda_N(H)$.

Conversely, for any $y \in \Lambda_N(H)$, there exists $z \in N$ and $\{h_n\} \subset H$ such that $h_n(z) \to y$. Take $w \in M$ such that $\phi(w) = z$. We have $\phi(h_n w) = h_n \phi(w) = h_n z \to y$. After passage to a subsequence, we assume $x \in \Lambda_M(H)$ to be the limit point of sequence $\{h_n w\}$. Then $\phi(h_n w) \to \phi(x)$ by the continuity of ϕ. It follows that $\phi(x) = y$. Therefore, we obtain $\Lambda_N(H) \subset \phi(\Lambda_M(H))$. □

Remark 1.4.2. Note that in general $\phi^{-1}(\Lambda_N(H)) = \Lambda_M(H)$ is not true. This is readily seen from Lemma 1.4.16 below.

Definition 1.4.3. A subgroup H of a convergence group action $G \curvearrowright M$ is *dynamically quasiconvex* if the following set

$$\{gH \in G/H : g\Lambda_N(H) \cap K \neq \emptyset, g\Lambda_N(H) \cap L \neq \emptyset\}$$

is finite, whenever K and L are disjoint closed subsets of M.

Remark 1.4.4. The notion of dynamical quasiconvexity was introduced by Bowditch [Bo99a] in hyperbolic groups and is proven there to be equivalent to the geometrical quasiconvexity.

In the following lemma, we show that dynamical quasiconvexity is kept under an equivariant quotient.

Lemma 1.4.5. *Suppose a group G admits convergence group actions on compact spaces M and N respectively. Assume, in addition, that there is a G-equivariant surjective map ϕ from M to N. If $H \subset G$ is dynamically quasiconvex with respect to $G \curvearrowright M$, then it is dynamically quasiconvex with respect to $G \curvearrowright N$.*

Proof. Given any disjoint closed subsets K, L of N, we are going to bound the cardinality of the following set

$$\Theta = \{gH \in G/H : g\Lambda_N(H) \cap K \neq \emptyset, g\Lambda_N(H) \cap L \neq \emptyset\}.$$

Let $K' = \phi^{-1}(K)$ and $L' = \phi^{-1}(L)$. Obviously $K' \cap L' = \emptyset$. For each $gH \in \Theta$, we claim $g\Lambda_M(H) \cap K' \neq \emptyset$. Otherwise, we have then

$$\phi(g\Lambda_M(H)) \cap \phi(K') = g\phi(\Lambda_M(H)) \cap K = \emptyset.$$

By Lemma 1.4.1, we have $g\Lambda_N(H) \cap K = \emptyset$. This is a contradiction. Hence $g\Lambda_M(H) \cap K' \neq \emptyset$. Similarly, we have $g\Lambda_M(H) \cap L' \neq \emptyset$.

By the dynamical quasiconvexity of H with respect to $G \curvearrowright M$, we have Θ is a finite set. Thus, H is dynamically quasiconvex with respect to $G \curvearrowright N$. □

Definition 1.4.6. A convergence group action of G on M is *geometrically finite* if every limit point of G in M is either a conical or bounded parabolic.

We now summarize as follows the equivalence of several dynamical formulations of relative hyperbolicity. Theorems 1.4.7 and 1.4.9 shall enable us to translate the results established in previous sections in dynamical terms.

Theorem 1.4.7. *[Bo99b][Ge09][Tu98][Ya04] Suppose a finitely generated group G acts on M as a convergence group action. Let $\mathbb{P}$ be a set of representatives of the conjugacy classes of maximal parabolic subgroups. Then the following statements are equivalent:*
(1) The pair $(G, \mathbb{P})$ is relatively hyperbolic in the sense of Farb,
(2) $G \curvearrowright M$ is geometrically finite,
(3) $G \curvearrowright M$ is a 2-cocompact convergence group action.

Remark 1.4.8. The direction (1) $\Rightarrow$ (2) is due to Bowditch [Bo99b]; (2) $\Rightarrow$ (1) is proved by Yaman [Ya04]; (2) $\Rightarrow$ (3) is implied in the work of Tukia [Tu98, Theorem 1 C]; (3) $\Rightarrow$ (2) is proven in Gerasimov [Ge09] without assuming that G is countable and M metrizable.

In Theorem 1.4.7, the limit set of G with respect to $G \curvearrowright M$ will be referred as *Bowditch boundary* of the relatively hyperbolic group G. We shall often write it as $T_{\mathbb{P}}$, with reference to a particular peripheral structure $\mathbb{P}$. It is shown in [Bo99b] that Bowditch boundary is well-defined up to a G-equivariant homeomorphism.

In different contexts, we can formulate the corresponding notions of relative quasiconvexity, which are proven to be equivalent.

Theorem 1.4.9. *[GP09a] [GP11] [Hr10] Suppose a finitely generated group G acts geometrically finitely on M. Let Γ be a subgroup of G. Then the following statements are equivalent:*
(1) Γ is relatively quasiconvex,
(2) $\Gamma \curvearrowright \Lambda_M(\Gamma)$ is geometrically finite,
(3) Γ is dynamical quasiconvex with respect to $G \curvearrowright M$.

Remark 1.4.10. The equivalence (1) $\Leftrightarrow$ (2) is proved by Hruska [Hr10] for countable relatively hyperbolic groups; (1) $\Leftrightarrow$ (3) is proven in Gerasimov-Potyagailo [GP09a].

Lastly we recall a useful result about peripheral subgroups of finitely generated relatively hyperbolic groups.

Lemma 1.4.11. *[DS05][Os06b][Ge09][Hr10] Suppose G is finitely generated and hyperbolic relative to $\mathbb{H}$. Then each $H \in \mathbb{H}$ is undistorted in G. Moreover H is relatively quasiconvex in any relatively hyperbolic $(G, \mathbb{P})$.*

Remark 1.4.12. The undistortedness of peripheral subgroups are proved by Osin [Os06b], Drutu-Sapir [DS05] and Gerasimov [Ge09], using quite different methods. The last statement is proved by Hruska [Hr10].

1.4.2 Floyd boundary and relative hyperbolicity

In this subsection, we first briefly recall the work of Gerasimov [Ge10] and Gerasimov-Potyagailo [GP09a] on Floyd maps. Based on their results, the Bowditch boundary with respect to a parabolically extended structure is shown as an equivalent quotient, and then the kernel of such an equivariant map is described.

From now on, unless explicitly stated, G is always assumed to be finitely generated by a fixed finite generating set X.

In [Fl80], Floyd introduced a compact boundary for a finitely generated group G. Let f be a suitable chosen function satisfying Conditions (3)–(4) in [GP09b]. We first rescale the length of each edge e of $\mathscr{G}(G, X)$ by $f(n)$, where n is the word distance of the edge e to $1 \in G$. Then we take length metric on $\mathscr{G}(G, X)$ and get the Cauchy completion $\overline{G}_f$ of $\mathscr{G}(G, X)$. The complete metric ρ on $\overline{G}_f$ is called *Floyd metric*. The completion $\overline{G}_f$ is compact, and the remainder $\overline{G}_f \setminus G$ is defined to be the *Floyd boundary* $\partial_f G$ of G with respect to f.

If $\partial_f G$ consists of 0, 1 or 2 points then it is said to be *trivial*. Otherwise, it is uncountable and is called *nontrivial*. If $\partial_f G$ is nontrivial, then G acts on $\partial_f G$ as a convergence group action, by a result of Karlsson [Ka03].

The following Floyd map theorem due to Gerasimov [Ge10] is key to our study of peripheral structures.

Theorem 1.4.13. *[Ge10] Suppose $G \curvearrowright M$ is 2-cocompact and M contains at least 3 points. Then there exists a continuous G-equivariant map $\phi : \partial_f G \to M$, where $f(n) = \alpha^n$ for some $\alpha \in]0, 1[$ sufficiently close to 1. Furthermore $\Lambda(G) = \phi(\partial_f G)$.*

The map ϕ given by Theorem 1.4.13 is called *Floyd map*. According to the discussion in [GP09b], the Floyd map ϕ defines a closed G-invariant equivalent relation $\omega := \{(x, y) : \phi(x) = \phi(y), x, y \in \partial_f G\}$, which induces a *shortcut pseudometric* $\tilde{\rho}$ on $\partial_f G$. This shortcut pseudometric is characterized as the maximal pseudometric, among which vanishes on ω and is less then the Floyd metric ρ. See [GP09b] for more details.

Recall that $T_{\mathbb{H}}$ denotes the Bowditch boundary of $G \curvearrowright M$, where $\mathbb{H}$ is a set of representatives of the conjugacy classes of maximal parabolic subgroups of $G \curvearrowright M$.

Moreover, the push-forward of $\tilde{\rho}$ by ϕ is shown to be a metric on $T_{\mathbb{H}}$ in [Ge10], which is called *shortcut metric* (still denoted by $\tilde{\rho}$). Thus, ϕ is a distance decreasing map from $(\partial_f G, \rho)$ to $(T_{\mathbb{H}}, \tilde{\rho})$:

$$\forall x, y \in \partial_f G : \rho(x, y) \geq \tilde{\rho}(\phi(x), \phi(y)). \tag{1.22}$$

Convention 4. Given a subgroup $J \subset G$, we denote by $\Lambda_f(J)$ and $\Lambda_{\mathbb{H}}(J)$ limit sets with respect to $G \curvearrowright \partial_f G$ and $G \curvearrowright T_{\mathbb{H}}$ respectively.

We now recall the characterization of the "kernel" of Floyd maps given in [GP09b]. Note that a more complete characterization appears in [GP10], but here we do not need that deeper result.

Theorem 1.4.14. *[GP09b] Suppose $G \curvearrowright T$ is 2-cocompact. Let $\phi : \partial_f G \to T$ be a G-equivariant map. Then*

$$\phi^{-1}(p) = \Lambda_f(G_p)$$

for any parabolic point $p \in T$. Moreover, the multivalued inverse map φ^{-1} is injective on conical points of $G \curvearrowright T$.

In the following two lemmas, we shall show the Bowditch boundary with respect to an extended parabolically structure can be described in a nice way.

Lemma 1.4.15. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic. Let $\mathbb{P}$ be a parabolically extended structure for $(G, \mathbb{H})$. Then there exists a G-equivariant surjective map φ such that the following diagram commutes*

$$\begin{array}{ccc} \partial_f G & \xrightarrow{\phi_1} & T_{\mathbb{H}} \\ & \searrow^{\phi_2} & \downarrow \varphi \\ & & T_{\mathbb{P}} \end{array} \tag{1.23}$$

where ϕ_1 and ϕ_2 are Floyd maps given by Theorem 1.4.13. Furthermore, φ is a distance decreasing map with respect to the shortcut metrics $d_{\mathbb{H}}$ and $d_{\mathbb{P}}$.

Proof. The following function φ is well-defined:

$$\forall x \in T_{\mathbb{H}} : \varphi(x) = \phi_2 \phi_1^{-1}(x).$$

It is easy to verify that φ is a G-equivariant continuous map.

We now show the last statement of this lemma. Let ω_1 and ω_2 be G-invariant equivalence relations induced by Floyd map ϕ_1 and ϕ_2 respectively. Observe that $\omega_1 \subset \omega_2$. Thus it follows easily that

$$\forall x, y \in T_{\mathbb{H}} : d_{\mathbb{H}}(x, y) \geq d_{\mathbb{P}}(\varphi(x), \varphi(y)).$$

from the definition of shortcut pseudometrics on $\overline{G}_f$. □

The following lemma follows easily from Theorem 1.4.14 and describes the kernel of the map φ defined in Lemma 1.4.15.

Lemma 1.4.16. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ is a parabolically extended structure for $(G, \mathbb{H})$. Let $\varphi : T_{\mathbb{H}} \to T_{\mathbb{P}}$ be the G-equivariant surjective map provided by Lemma 1.4.15. Then*

$$\varphi^{-1}(p) = \Lambda_{\mathbb{H}}(G_p)$$

for any parabolic point $p \in T_{\mathbb{P}}$. Moreover, the multivalued inverse map φ^{-1} is injective on conical points of $G \curvearrowright T_{\mathbb{P}}$.

Proof. Observe that $\varphi^{-1}(p) = \phi_1 \phi_2^{-1}(p)$ for any $p \in T_{\mathbb{P}}$. Suppose $\varphi(a) = \varphi(b) = p$ for $a, b \in T_{\mathbb{H}}$, i.e. $a, b \in \varphi^{-1}(p)$. If p is conical with respect to $G \curvearrowright T_{\mathbb{P}}$, then $\phi_2^{-1}(p)$ consists of one single point. Thus $a = b$.

If p is bounded parabolic with respect to $G \curvearrowright T_{\mathbb{P}}$, then $\phi_2^{-1}(p) = \Lambda_f(G_p)$ using Theorem 1.4.14. By Lemma 1.4.1, we obtain $\varphi^{-1}(p) = \phi_1(\Lambda_f(G_p)) = \Lambda_{\mathbb{H}}(G_p)$. The proof is complete. □

1.4.3 Proof of Theorem 1.1.3

The proof of Theorem 1.1.3 is divided into the following two propositions. Taking into account Theorem 1.4.9, the first proposition follows immediately from Lemmas 1.4.1 and 1.4.15.

Proposition 1.4.17. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ is a parabolically extended structure for $(G, \mathbb{H})$. If $\Gamma \subset G$ is relatively quasiconvex in G with respect to $\mathbb{H}$, then Γ is relatively quasiconvex in G with respect to $\mathbb{P}$.*

By Theorem 1.4.9, the second statement of Theorem 1.1.3 is restated in the following dynamical terms.

Proposition 1.4.18. *Suppose* $(G, \mathbb{H})$ *is relatively hyperbolic and* $\mathbb{P}$ *is a parabolically extended structure for* $(G, \mathbb{H})$. *Let* $\Gamma \subset G$ *acts geometrically finitely on* $\Lambda_{\mathbb{P}}(\Gamma)$. *Then* Γ *acts geometrically finitely on* $\Lambda_{\mathbb{H}}(\Gamma)$ *if and only if* $\Gamma \cap P_j^g$ *acts geometrically finitely on* $\Lambda_{\mathbb{H}}(\Gamma \cap P_j^g)$ *for any* $j \in J$ *and* $g \in G$.

Proof. $\Rightarrow$: By Lemma 1.3.3, each P_j is relatively quasiconvex with respect to $\mathbb{H}$. It is a well-known fact that the intersection of two relatively quasiconvex subgroups is relatively quasiconvex, see for example, [Hr10] and [MP09]. Hence we have $\Gamma \cap P_j^g$ is relatively quasiconvex with respect to $\mathbb{H}$, and then acts geometrically finitely on its limit set $\Lambda_{\mathbb{H}}(\Gamma \cap P_j^g)$.

$\Leftarrow$: By Lemma 1.4.15, the map $\varphi : T_{\mathbb{H}} \to T_{\mathbb{P}}$ is a distance decreasing function with respect to the induced shortcut metrics $d_{\mathbb{H}}$ and $d_{\mathbb{P}}$.

Since Γ is relatively quasiconvex with respect to $\mathbb{P}$, then the following set

$$\{\Gamma \cap P_j^g : \sharp\, \Gamma \cap P_j^g = \infty, g \in G, j \in J\}$$

contains finitely many Γ-conjugacy classes, say $\{Q_1, \ldots, Q_n\}$. By Theorem 1.4.7, each Q_i acts 2-cocompactly on $\Lambda_{\mathbb{H}}(Q_i)$. We shall show that Γ also acts 2-cocompactly on $\Lambda_{\mathbb{H}}(\Gamma)$.

Since Γ acts 2-cocompactly on $\Lambda_{\mathbb{P}}(\Gamma)$, these exists $\epsilon_0 > 0$ such that for any $(x, y) \in \Theta^2(\Lambda_{\mathbb{P}}(\Gamma))$, there exists $\gamma \in \Gamma$ satisfying $d_{\mathbb{P}}(\gamma x, \gamma y) > \epsilon_0$. Similarly, we have a positive constant $\epsilon_i > 0$ for each $i \in I$ such that for any $(x, y) \in \Theta^2(\Lambda_{\mathbb{H}}(Q_i))$, there exists $\gamma \in Q_i$ satisfying $d_{\mathbb{H}}(\gamma x, \gamma y) > \epsilon_i$.

Let $\epsilon := \min\{\epsilon_0, \min\{\epsilon_i : i \in I\}\}$. We now define a compact $L \subset \Theta^2(\Lambda_{\mathbb{H}}(\Gamma))$ as follows

$$L = \{(x, y) \in \Theta^2(\Lambda_{\mathbb{H}}(\Gamma)) : d_{\mathbb{H}}(x, y) \geq \epsilon\}.$$

Then we claim L is a fundamental domain of Γ on $\Theta^2(\Lambda_{\mathbb{H}}(\Gamma))$.

Given distinct points $p, q \in \Lambda_{\mathbb{H}}(\Gamma)$, we have the following two cases to consider:

Case 1. $\phi(p) \neq \phi(q)$. Then there exists $\gamma_0 \in \Gamma$ such that

$$d_{\mathbb{P}}(\gamma_0(\varphi(p)), \gamma_0(\varphi(q))) = d_{\mathbb{P}}(\varphi(\gamma_0 p), \varphi(\gamma_0 q)) > \epsilon_0 > \epsilon.$$

Since φ is a distance decreasing map, we have $d_{\mathbb{H}}(\gamma_0 p, \gamma_0 q) \geq d_{\mathbb{P}}(\phi(\gamma_0 p), \phi(\gamma_0 q))$. This implies $\gamma_0(p, q) \in L$.

Case 2. $\phi(p) = \phi(q)$. By Lemma 1.4.16, we have the points p, q lie in the limit set $\Lambda_{\mathbb{H}}(Q_i^\gamma)$ for some $1 \leq i \leq n, \gamma \in \Gamma$, i.e. $(\gamma^{-1}(p), \gamma^{-1}(q)) \in \Lambda_{\mathbb{H}}(Q_i)$. Then there exists an element γ_i from Q_i such that $d_{\mathbb{H}}(\gamma_i\gamma^{-1}(p), \gamma_i\gamma^{-1}(q)) > \epsilon_i > \epsilon$. This implies that $\gamma_i\gamma^{-1}(p, q) \in L$.

Combining the above two cases, we showed that Γ acts 2-cocompactly ad thus geometrically finitely on $\Lambda_{\mathbb{H}}(\Gamma)$. □

Remark 1.4.19. Using an argument of [MP08] with Proposition 1.3.8, one is able to obtain the full generality of Theorem 1.1.3 for countable relatively hyperbolic groups. We leave the details to the interested reader.

The proof of Proposition 1.4.18 also produces the following result.

Theorem 1.4.20. *(Theorem 1.1.5) Suppose* $(G, \mathbb{H})$ *is relatively hyperbolic. Then* G *acts geometrically finitely on* $\partial_f G$ *if and only if each* $H \in \mathbb{H}$ *acts geometrically finitely on* $\Lambda_f(H)$*.*

Proof. $\Rightarrow$: Note that each $H \in \mathbb{H}$ is undistorted in G by Lemma 1.4.11. Since G acts geometrically finitely on $\partial_f G$, then by Theorem 1.4.9 each $H \in \mathbb{H}$ acts geometrically finitely on $\Lambda_f(H)$.

$\Leftarrow$: In particular, we use the Floyd map $\phi : \partial_f G \to T_{\mathbb{H}}$ instead of the map φ in the proof of Proposition 1.4.18. Note that F is also a distance decreasing map with respect to ρ and $d_{\mathbb{H}}$. The other arguments are exactly the same as Proposition 1.4.18. □

1.4.4 Some applications

In this subsection, we give some preliminary results on general peripheral structures. The first result roughly states that if a finitely generated group acts geometrically finitely on its Floyd boundary, then every peripheral structure to which it may be hyperbolic relative are parabolically extended for a canonical peripheral structure. This is a direct corollary to Theorem 1.4.13.

Corollary 1.4.21. *Suppose* G *acts geometrically finitely on* $\partial_f G$ *and* $(G, \mathbb{P})$ *is relatively hyperbolic. Then* $\mathbb{P}$ *is parabolically extended for* $(G, \mathbb{H})$*, where* $\mathbb{H}$ *comprises a suitable choice of representatives of the conjugacy classes of maximal parabolic subgroups with respect to* $G \curvearrowright \partial_f G$*, and possibly a trivial subgroup.*

Proof. Let $\phi : \partial_f G \to T_{\mathbb{P}}$ be the Floyd map given by Theorem 1.4.13. Let $\tilde{\mathbb{H}}$ be the collection of maximal parabolic subgroups with respect to $G \curvearrowright \partial_f G$.

Claim 1. *For each* $H \in \tilde{\mathbb{H}}$*, there exists* $g \in G$ *and* $j \in J$ *such that* $H \subset P_j^g$*.*

Proof of Claim 1. As $\Lambda_f(H)$ is a parabolic point, then $\phi(\Lambda_f(H))$ is also fixed by H. Hence $\Lambda_{\mathbb{P}}(H)$ consists of one point or two points. If $\Lambda_{\mathbb{P}}(H)$ is one point, then H contains no hyperbolic elements. By [Tu98, Theorem 3A], the stabilizer of $\Lambda_{\mathbb{P}}(H)$ is a maximal parabolic subgroup for the action $G \curvearrowright T_{\mathbb{P}}$. So the claim is proved in this case.

We now show that $\Lambda_{\mathbb{P}}(H)$ could not consist of two points. Suppose not. Let q be the other point in $\Lambda_{\mathbb{P}}(H)$. Then the preimage $\phi^{-1}(q)$ is H-invariant. Take a point $z \in \phi^{-1}(q)$. As H acts properly discontinuously on $\partial_f G \setminus \{\Lambda_f(H)\}$, then the orbit $H(z)$ should converge to $\Lambda_f(H)$. However, we have $\phi(H(z))$ and $\phi(\Lambda_{\mathbb{P}}(H))$ are distinct points. This contradicts to the continuity of ϕ. □

Let $\mathbb{H}_j$ be a set of representatives of the conjugacy classes of maximal parabolic subgroups with respect to $P_j \curvearrowright \Lambda_f(P_j)$.

Claim 2. *The union* $\mathbb{H} := \cup_{j \in J} \mathbb{H}_j$ *is a set of representatives of* $\tilde{\mathbb{H}}$*.*

Proof of Claim 2. By Claim 1, we have that $\mathbb{H}$ contains at least a set of representatives of the conjugacy classes of $\tilde{\mathbb{H}}$. Moreover, it is easy to verify that no two entries of $\mathbb{H}$ is conjugate in G. The claim is thus proved. □

If there exists a parabolic subgroup $P \in \mathbb{P}$ such that P is a hyperbolic group, then we may add the trivial subgroup into $\mathbb{H}$. Then by the choice of $\mathbb{H}$, we have that P is parabolically extended for $(G, \mathbb{H})$. □

In view of Corollary 1.4.21, Theorem 1.1.3 gives the following corollary, concerning about "universal" relatively quasiconvex subgroups in certain classes of relatively hyperbolic groups.

Corollary 1.4.22. *If G acts geometrically finitely on $\partial_f G$ and $(G, \mathbb{P})$ is relatively hyperbolic. Then relatively quasiconvex subgroups of G with respect to $G \curvearrowright \partial_f G$ are relatively quasiconvex with respect to $(G, \mathbb{P})$.*

Recall that a group H is said *Non-Relatively Hyperbolic* (NRH) if H is not hyperbolic relative to any collection of proper subgroups. By Theorem 1.4.20, we now list some examples of geometrically finite actions on their Floyd boundaries:

(1) Geometrically finite Kleinian groups where maximal parabolic subgroups are virtually abelian.

(2) Hyperbolic groups relative to a collection of unconstricted subgroups. According to [DS05], a group is unconstricted if one of its asymptotic cones has no cut points. By Proposition 4.28 of [OOS09], the Floyd boundary of an unconstricted subgroup is trivial.

(3) Most known hyperbolic groups relative to a collection of NRH subgroups. For example, all NRH subgroups in [AAS07] have trivial Floyd boundaries.

In fact, Olshanskii-Osin-Sapir made the following conjecture on the relationship between NRH groups and their Floyd boundary.

Conjecture A. *[OOS09] If a finitely generated group has non-trivial Floyd boundary, then it is hyperbolic relative to a collection of proper subgroups.*

In [BDM09], Behrstock-Drutu-Mosher studied Dunwoody's inaccessible group J which is constructed in [Du91]. In particular, they proved that there exists no collection $\mathbb{P}$ of NRH proper subgroups such that J is hyperbolic relative to $\mathbb{P}$. Moreover, we have the following observation.

Theorem 1.4.23. *Dunwoody's group J in [Du91] does not act geometrically finitely on its Floyd boundary.*

Proof. By way of contradiction, we suppose $J \curvearrowright \partial_f J$ is geometrically finite. Let $\mathbb{P}$ be a set of representatives of the conjugacy classes of maximal parabolic subgroups with respect to $J \curvearrowright \partial_f J$. Then the Floyd boundary $\partial_f J$ is same as the Bowditch boundary $T_{\mathbb{P}}$. Moreover, the limit set $\Lambda_f(P)$ of each $P \in \mathbb{P}$ consists of only one point.

By Proposition 6.3 in Behrstock-Drutu-Mosher [BDM09], there exists a subgroup $\Gamma \in \mathbb{P}$ such that Γ is hyperbolic relative to a collection of proper subgroups $\mathbb{K} = \{K_j\}_{j \in J}$. By Corollary 1.14 in [DS05], we have that J is hyperbolic relative to $\mathbb{H} := \mathbb{K} \cup (\mathbb{P} \setminus \{\Gamma\})$.

By Theorem 1.4.13, we have a G-equivalent Floyd map $\varphi : T_{\mathbb{P}} = \partial_f J \to T_{\mathbb{H}}$. Note that $\Lambda_f(\Gamma)$ consists of one point. Using Theorem 1.4.14, we will obtain that φ maps $\Lambda_f(\Gamma)$ to different points $\Lambda_{\mathbb{H}}(K_j)$. This gives a contradiction. Hence, the action $J \curvearrowright \partial_f J$ is not geometrically finite □

Remark 1.4.24. Note that a more direct proof (without using [DS05, Corollary 1.14]) follows from [GP10, Theorem C] and Theorem 1.4.13. A consequence of [GP10, Theorem C] says that if $(G, \mathbb{H})$ is relatively hyperbolic, then there is a particular Floyd function f such that for each $H \in \mathbb{H}$, the limit set $\Lambda_f(H)$ is homeomorphic to its Floyd boundary $\partial_f H$. On the other hand, Theorem 1.4.13 implies that a non-elementary relatively hyperbolic group has a non-trivial Floyd boundary. So if J acts geometrically finitely on its Floyd boundary, then Floyd boundary of every $P \in \mathbb{P}$ consists of one point. As above, by [BDM09, Proposition 6.3], there exists $\Gamma \in \mathbb{P}$ acting non-trivially on a compactum, which contradicts to Theorem 1.4.13.

As suggested by Theorem 1.4.23, it seems reasonable to conjecture the following.

Conjecture B. *If a finitely generated group is hyperbolic relative to a collection of NRH proper subgroups, then it acts geometrically finitely on its Floyd boundary.*

As a matter of fact, the converse of Conjecture B is true by Corollary 1.4.21.

Although Conjectures A and B appear to be different claims, they turn out to be equivalent by the following simple arguments.

Conjecture A implies Conjecture B: Suppose Conjecture B is false. Then there exists a relatively hyperbolic group G with respect to a collection $\mathbb{H}$ of NRH proper subgroups such that G does not act geometrically finitely on its Floyd boundary. Then by Theorem 1.1.5, there is a parabolic subgroup $H \in \mathbb{H}$ such that the limit set $\Lambda_f(H)$ is nontrivial and the action $H \curvearrowright \Lambda_f(H)$ is not geometrically finite. By Theorem C in [GP10], $\Lambda_f(H)$ is homeomorphic to the Floyd boundary of H. Therefore, this contradicts to Conjecture A.

Conjecture B implies Conjecture A: Suppose, to the contrary, that there exists a NRH group Γ with non-trivial Floyd boundary. Then we take a free product $G = \Gamma * F_2$, where F_2 is a free group of rank 2. By Conjecture A, G acts geometrically finitely on $\partial_f G$. By Theorem 1.1.5, we have that Γ also acts geometrically finitely on $\Lambda_f(\Gamma)$. Using again Theorem C in [GP10], we obtain that Γ acts geometrically finitely on its non-trivial Floyd boundary. This contradicts to the hypothesis that Γ is a non-relatively hyperbolic group.

Chapitre 2

Limit sets of relatively hyperbolic groups

This chapter is adapted from the paper [Ya10a], which will appear in **Geometriae Dedicata**.

Abstract. In this chapter, we prove a limit set intersection theorem in relatively hyperbolic groups. Our approach is based on a study of dynamical quasiconvexity of relatively quasiconvex subgroups. Using dynamical quasiconvexity, many well-known results on limit sets of geometrically finite Kleinian groups are derived in general convergence groups. We also establish dynamical quasiconvexity of undistorted subgroups in finitely generated groups with nontrivial Floyd boundary.

2.1 Introduction

The purpose of this chapter is to study applications of dynamical quasiconvexity to limit sets of relatively quasiconvex subgroups. The notion of dynamical quasiconvexity is introduced by Bowditch in [Bo99a] and used to characterize a geometric notion of quasiconvexity in word hyperbolic groups. In relatively hyperbolic groups, Gerasimov-Potyagailo [GP09a][GP11] recently showed that dynamically quasiconvex subgroups are exactly the class of relatively quasiconvex subgroups.

In this chapter, we shall show that even in general convergence groups, dynamical quasiconvexity efficiently captures algebraic and geometric properties of subgroups as well. However, the particular interest we have in mind lies in relatively hyperbolic groups, and finitely generated groups with nontrivial Floyd boundary, which is conjectured to be relatively hyperbolic [OOS09].

Let G be a finitely generated group, admitting a *convergence group action* on a compact metric space M. Then the *limit set* $\Lambda(H)$ of a subgroup $H \subset G$ is the set of accumulation points of H-orbits in M. See Section 2 for their precise definitions. Following Anderson [An96], a limit set intersection theorem for convergence groups describes the limit set $\Lambda(H \cap J)$ in terms of $\Lambda(H)$ and $\Lambda(J)$, where H, J are subgroups of a convergence group G. Ideally, we expect such a theorem has the following form

$$\Lambda(H) \cap \Lambda(J) = \Lambda(H \cap J) \cup E$$

where E is an exceptional set consisting of specific parabolic points of $\Lambda(H)$ and $\Lambda(J)$.

Such a limit set intersection theorem has been investigated in several different classes of groups. In 1992, Susskind-Swarup [SS92] showed that the above decomposition of limit sets holds for a pair of geometrically finite Kleinian subgroups. In [An91], [An95] and [An96], using techniques specific to 3 manifolds, Anderson carried out a systematic study of the intersection of two finitely generated subgroups of 3 dimensional Kleinian groups and proved that the limit set intersection theorem holds in this context.

In 1987, Gromov [Gr87] introduced relatively hyperbolic groups as a generalization of many naturally occurred groups, for example, word hyperbolic groups and geometrically finite Kleinian groups and many others. In word hyperbolic groups, the limit set intersection theorem is explained in Gromov [Gr93, Page 164], where the exceptional set E is empty. Our main result is to generalize these limit set intersection theorems in relatively hyperbolic groups as follow.

Theorem 2.1.1. *Let H, J be two relatively quasiconvex subgroups of a relatively hyperbolic group G. Then*

$$\Lambda(H) \cap \Lambda(J) = \Lambda(H \cap J) \bigsqcup E$$

where the exceptional set E consists of parabolic fixed points of $\Lambda(H)$ and $\Lambda(J)$, whose stabilizer subgroups in H and J have finite intersection. Equivalently, the set E consists of the limit points isolated in $\Lambda(H) \cap \Lambda(J)$.

Remark 2.1.2. Two special cases of Theorem 2.1.1 were known in Dahmani [Da03]. The first one is under the assumption that maximal parabolic groups are abelian. The other one is proved for a pair of fully quasiconvex subgroups, where the exceptional set E is empty. By a result of Hruska [Hr10], fully quasiconvex subgroups are relatively quasiconvex .

One corollary to Theorem 2.1.1 is the following well-known result, which is usually proved via geometrical methods by Hruska [Hr10] and, independently, Martinez-Pedroza [MP08].

Corollary 2.1.3. *Let H, J be two relatively quasiconvex subgroups of a relatively hyperbolic group G. Then $H \cap J$ is relatively quasiconvex.*

The proof of Theorem 2.1.1 replies crucially on dynamical quasiconvexity of relatively quasiconvex subgroups. Even in general convergence groups, dynamical quasiconvex subgroups share many nice properties with relatively quasiconvex subgroups. See Section 2 for a few of them.

The following result extends a property of quasiconvex subgroups in word hyperbolic groups proved in Mihalik-Towle [MT94] to dynamically quasiconvex subgroups in general convergence groups.

Theorem 2.1.4. *Let H be dynamically quasiconvex in a convergence group G such that $|\Lambda(H)| \geq 2$. Then for any $g \in G \setminus H$, $gHg^{-1} \subseteq H$ implies that $gHg^{-1} = H$.*

In the final section, we explore dynamical quasiconvexity of subgroups in a special class of convergence groups, i.e. finitely generated groups with nontrivial Floyd boundary. In [Fl80], Floyd boundary was introduced by Floyd to compactify Cayley graphs of finitely generated groups. Later, Karlsson [Ka03] proved that the left multiplication of group elements extends to a convergence group action on Floyd boundary.

Our last result establishes the dynamical quasiconvexity of undistorted subgroups with respect to the convergence action on Floyd boundary.

Theorem 2.1.5. *If H is an undistorted subgroup of a finitely generated group G with nontrivial Floyd boundary, then H is dynamical quasiconvex.*

Remark 2.1.6. The class of groups with nontrivial Floyd boundary includes non-elementary finitely-generated relatively hyperbolic groups [Ge10]. In particular, infinite-ended groups have nontrivial Floyd boundary.

By the Floyd map theorem in [Ge10], there exists an equivariant map from the Floyd boundary of a relatively hyperbolic group to its Bowditch boundary. Then it is easily seen that the dynamical quasiconvexity of a subgroup is kept under an equivalent quotient. See Lemma 1.4.5 for details. This together with Theorem 2.1.5 gives the following result, which was first proved in [Hr10].

Corollary 2.1.7. *Let H be an undistorted subgroup of a relatively hyperbolic group G. Then H is relatively quasiconvex.*

The chapter is organized as follows. In Section 2, we define dynamical quasiconvex subgroups in general convergence groups. Then we deduce several consequences of dynamically quasiconvex subgroups in general convergence groups. In Section 3, we study the intersection of conical limit points and bounded parabolic points of dynamically quasiconvex subgroups, and then conclude with the proof of Theorem 2.1.1. In Section 4, it is shown that a nonparabolic dynamically quasiconvex subgroup cannot contain a proper conjugate of itself. In the final section, we briefly introduce the Floyd boundary for a finitely generated groups and give a proof of Theorem 2.1.5. Moreover, an example is given of dynamically quasiconvex subgroups which are not geometrically finite.

2.2 Preliminary results

Throughout the chapter, we consider a finitely generated group G, and a compact metrizable space M containing at least three points.

A *convergence group action* is an action of a group G on M such that the induced action of G on the space ΘM of distinct unordered triples of points of M is properly discontinuous.

Suppose G has a convergence group action on M. Then M is partitioned into a limit set $\Lambda(G)$ and discontinuous domain $M \setminus \Lambda(G)$. The *limit set* $\Lambda(H)$ of a subgroup $H \subset G$ is the set of limit points, where a *limit point* is an accumulation point of some H-orbit in M. It is well-known that if $|\Lambda(H)| \geq 2$, the limit set $\Lambda(H)$ is also characterized as the minimal H-invariant closed subset in M of cardinality at least two.

An element $g \in G$ is *elliptic* if it has finite order. An element $g \in G$ is *parabolic* if it has infinite order and fixes exactly one point of M. An element $g \in G$ is *loxodromic* if it has infinite order and fixes exactly two points of M. An infinite subgroup $P \subset G$ is a *parabolic subgroup* if it contains no loxodromic element. A parabolic subgroup P has a unique fixed point in M. This point is called a *parabolic point*. The stabilizer of a parabolic point is always a maximal parabolic group. A parabolic point p with stabilizer $G_p := Stab_G(p)$ is *bounded* if G_p acts cocompactly on $M \setminus \{p\}$. A point $z \in M$ is a *conical limit* point if there exists a sequence $\{g_i\}$ in G and distinct points $a, b \in M$ such that $g_i(z) \to a$, while for all $q \in M \setminus \{z\}$ we have $g_i(q) \to b$.

Before discussing relative hyperbolicity, we recall the following well-known result on general convergence groups.

Lemma 2.2.1. *[Tu98, Theorem 3.A] In a convergence group, a conical limit point cannot be parabolic.*

In the literature, various definitions of relative hyperbolicity were proposed, see Farb [Fa98], Bowditch [Bo99b], Osin [Os06b], Drutu-Sapir [DS05] and so on. These different definitions are now proven to be equivalent for countable groups, see Hruska [Hr10] for a complete account, and provide convenient and complement viewpoints to the study of this class of groups. For the sake of the purpose of this chapter, we use the following dynamical formulation of relatively hyperbolic groups.

Definition 2.2.2. A convergence group action of G on M is *geometrically finite* if every limit point of G is either conical or bounded parabolic. Let $\mathbb{P}$ be a set of representatives of the conjugacy classes of maximal parabolic subgroups. Then we say the pair (G, $\mathbb{P}$) is *relatively hyperbolic*. When $\mathbb{P}$ is clear in the context, we just say G is relatively hyperbolic.

Remark 2.2.3. By the work of Drutu-Sapir [DS05], Osin [Os06b] and Gerasimov [Ge09], maximal parabolic subgroups are quasiconvex and finitely generated. So in the definition of relative hyperbolicity, we do not need impose the "finitely generated" condition on maximal parabolic subgroups, as usually do in Bowditch [Bo99b].

The following notion of dynamical convexity is introduced by Bowditch [Bo99a] and proven to be equivalent to the geometric quasiconvexity in word hyperbolic groups.

Definition 2.2.4. A subgroup H of a convergence group G is *dynamically quasi-convex* if the following set

$$\{gH \in G/H : g\Lambda(H) \cap K \neq \emptyset \text{ and } g\Lambda(H) \cap L \neq \emptyset\}$$

is finite, whenever K and L are disjoint closed subsets of M.

Recently, Gerasimov-Potyagailo [GP09a][GP11] proved that dynamical quasiconvexity coincides with relatively quasiconvexity in relatively hyperbolic groups, which gives a positive answer to a question of Osin in his book [Os06b]. We refer the reader to [Os06b] for the definition of relative quasiconvexity.

Theorem 2.2.5. *[GP09a][GP11] Suppose G is relatively hyperbolic. Every subgroup H of G is dynamically quasiconvex if and only if it is relatively quasiconvex.*

Let us first draw some consequences of dynamical quasiconvexity, without assuming G is relatively hyperbolic.

Lemma 2.2.6. *Let H be dynamically quasiconvex in a convergence group G such that $|\Lambda(H)| \ge 2$. Then for any subgroup $H \subset J \subset G$ satisfying $\Lambda(H) = \Lambda(J)$, we have that H is of finite index in J. In particular, J is dynamically quasiconvex.*

Proof. Since $|\Lambda(H)| \ge 2$, we can pick distinct points x and y from $\Lambda(H)$. Since $\Lambda(H) = \Lambda(J)$, we have that each coset of H in J belongs to the following set

$$\{gH \in G/H : g\Lambda(H) \cap \{x\} \neq \emptyset \text{ and } g\Lambda(H) \cap \{y\} \neq \emptyset\}.$$

By the dynamical quasiconvexity of H, the above set is finite. Thus, H is of finite index in J.

In order to prove the dynamical quasiconvexity of J, it suffices to show that the following set

$$\Omega := \{gJ \in G/J : g\Lambda(J) \cap K \neq \emptyset \text{ and } g\Lambda(J) \cap L \neq \emptyset\}$$

is finite, for any given disjoint closed subsets $K, L \subset M$. On the other hand, since H is dynamically quasiconvex, the following set

$$\{gH \in G/H : g\Lambda(H) \cap K \neq \emptyset \text{ and } g\Lambda(H) \cap L \neq \emptyset\}.$$

is finite. Combining with the fact H is of finite index in J, we have that the above set Ω is finite. Therefore, J is dynamically quasiconvex. □

Corollary 2.2.7. *Let H, J be dynamically quasiconvex in a convergence group G such that $\Lambda(H) = \Lambda(J)$. If $|\Lambda(H)| \ge 2$, then H and J are commensurable.*

Proof. Let L be the stabilizer in G of the limit set $\Lambda(H)$. Using Lemma 2.2.6, we have that H, J are both of finite index in L. It thus follows that $H \cap J$ is of finite index in both H and J. □

Recall that the *commensurator* of H in a convergence group G is defined as the subgroup of G, consisting of all $g \in G$ such that $H \cap gHg^{-1}$ has finite index in both H and gHg^{-1}.

Corollary 2.2.8. *Let H be dynamically quasiconvex in a convergence group G such that $|\Lambda(H)| \ge 2$. Then H is of finite index in its commensurator. In particular, H is of finite index in its normalizer.*

Proof. Let L be the commensurator of H in G. Then $H \subset L$. It is obvious that $\Lambda(H) \subset \Lambda(L)$.

It is well-known that the limit set of a subgroup is same as the one of its finite extension. So for each $g \in L$, we have $\Lambda(H) = \Lambda(gHg^{-1}) = g\Lambda(H)$, i.e. L leaves invariant the limit set $\Lambda(H)$. Since the limit set $\Lambda(L)$ is the minimal L-invariant closed subset in M of cardinality at least two, we have $\Lambda(L) = \Lambda(H)$. The conclusion now follows from Lemma 2.2.6. □

Remark 2.2.9. In relatively hyperbolic groups, Corollary 2.2.8 has been proven using different methods in Hruska-Wise [HW09]. We remark the hypothesis $|\Lambda(H)| \geq 2$ is necessary for the above lemma and corollaries, as it is easy to get counterexamples when H is taken as a parabolic subgroup.

2.3 Intersections of limit sets

In this section, we study the intersection of limit sets of dynamically quasiconvex subgroups. The intersection of conical limit points is firstly examined.

Proposition 2.3.1. *Let H be dynamically quasiconvex in a convergence group G. Suppose $J < G$ is infinite and let $z \in \Lambda(H) \cap \Lambda(J)$ be a conical limit point of J. Then $z \in \Lambda(H \cap J)$ is a conical limit point of $H \cap J$.*

Proof. Let $z \in \Lambda(H) \cap \Lambda(J)$ be a conical limit point of J. Then there exists a sequence $\{j_n\}$ in J and distinct points $a, b \in \Lambda(J)$ such that $j_n(z) \to a$, while $j_n(q) \to b$ for all $q \in \Lambda(J) \setminus \{z\}$. By the convergence property of $\{j_n\}$, we also have that $j_n(q) \to b$ for all $q \in M \setminus \{z\}$. In particular, we can choose q to be a limit point in $\Lambda(H) \setminus \{z\}$. Here, we use the fact $|\Lambda(H)| \geq 2$, which follows from Lemma 2.2.1.

Take closed neighborhoods U and V of a and b respectively, such that $U \cap V = \emptyset$. After passage to a subsequence of $\{j_n\}$, we can assume $j_n(z) \in U$ and $j_n(q) \in V$ for all n. This implies that j_nH belongs to the following set for all n,

$$\{gH \in G/H : g\Lambda(H) \cap U \neq \emptyset \text{ and } g\Lambda(H) \cap V \neq \emptyset\}.$$

By the dynamical quasiconvexity of H in G, the above set is finite. Thus, $\{j_nH\}$ is a finite set of cosets. By taking further a subsequence of $\{j_n\}$, we suppose $j_nH = j_1H$ for all n. We can write $j_n = j_1h_n$ for each n, where $h_n \in H$. Then $j_1^{-1}j_n = h_n$ implies that $H \cap J$ is nontrivial and infinite.

It suffices to prove that z is a conical limit point of $H \cap J$. By the convergence property of $\{j_n\}$, it follows that $h_n(z) = j_1^{-1}j_n(z) \to j_1^{-1}(a)$ and $h_n(q) = j_1^{-1}j_n(q) \to j_1^{-1}(b)$ for all $q \in M \setminus \{z\}$. Thus, z is a conical limit point of $H \cap J$. □

Remark 2.3.2. A similar statement of Proposition 2.3.1 in relatively hyperbolic groups appears in the proof of Proposition 3.1.10 in [Da03].

We now study how bounded parabolic points intersect. Compared to that of conical points, the intersection of bounded parabolic points raises some complicated behavior.

Proposition 2.3.3. *Let H, J be infinite subgroups of a convergence group G. If $z \in \Lambda(H) \cap \Lambda(J)$ is a bounded parabolic point of H and J, then z is either a bounded parabolic point of $H \cap J$, or an isolated point in $\Lambda(H) \cap \Lambda(J)$ and does not lie in $\Lambda(H \cap J)$.*

Proof. Since z is a bounded parabolic point of both H and J, there are compact subsets $K \subset \Lambda(H) \setminus z$ and $L \subset \Lambda(J) \setminus \{z\}$, such that $H_z K = \Lambda(H) \setminus \{z\}$ and $J_z L = \Lambda(J) \setminus \{z\}$. Here, H_z and J_z are stabilizers in H and J of z, respectively. Let $P = H_z \cap J_z$.

We claim that there exists a compact subset $C \subset M \setminus \{z\}$ such that $\Lambda(H \cap J) \setminus z \subset PC$.

Note first that $\Lambda(H \cap J) \setminus \{z\} \subset (\Lambda(H) \cap \Lambda(J)) \setminus z = H_z K \cap J_z L$. Therefore, it suffices to show that there exists a compact subset $C \subset M \setminus \{z\}$ such that $H_z K \cap J_z L \subset PC$. Since G is countable, we define the following set

$$\mathcal{A} = \{h_n K \cap j_n L : (h_n, j_n) \in H_z \times J_z, h_n K \cap j_n L \neq \emptyset\}. \tag{2.1}$$

We remark that it is possible that one set hK may have nontrivial intersections with two more sets $j_1 L$ and $j_2 L$, but (h, j_1) and (h, j_2) are counted differently in the set $\mathcal{A}$. Note that $H_z K \cap J_z L \subset \cup \mathcal{A}$.

Define the set $\mathcal{B} = \{j_n^{-1} h_n : j_n^{-1} h_n K \cap L \neq \emptyset,\ (h_n, j_n) \in H_z \times J_z\}$. We now show that $\mathcal{B}$ is finite. Suppose not. By the convergence property, there exists an infinite subsequence $\{j_{n_i}^{-1} h_{n_i}\}$ of $\mathcal{B}$ such that $\{j_{n_i}^{-1} h_{n_i}\}$ converges locally compactly to b on $M \setminus \{a\}$, for some $a, b \in M$. We claim $a = b$. Otherwise, using Lemma 2.5 in Bowditch [Bo99a], we have that $j_{n_i}^{-1} h_{n_i}$ are loxodromic elements for all sufficiently large n_i. But this contradicts to the fact that $\{j_{n_i}^{-1} h_{n_i}\}$ lie in the maximal parabolic subgroup G_z.

Moreover, we have that $a = b = z$, since z is the fixed point of elements $j_n^{-1} h_n$. Note that $K \subset M \setminus \{z\}$ and $L \subset M \setminus \{z\}$ are disjoint compact subsets. Since $j_{n_i}^{-1} h_{n_i} K \cap L \neq \emptyset$, the subsequence $\{j_{n_i}^{-1} h_{n_i}\}$ is a finite set by the convergence property. This is a contradiction. Hence $\mathcal{B}$ is a finite set.

Let $\mathcal{B}$ be a finite set, say $\{j_1^{-1} h_1, ..., j_r^{-1} h_r\}$, for example. Without loss of generality, we first consider the elements in $\{j_n^{-1} h_n\}$ of the form $j_n^{-1} h_n = j_1^{-1} h_1$. Then $j_n j_1^{-1} = h_n h_1^{-1} \in H_z \cap J_z = P$. We write $j_n = p_n j_1$ and $h_n = p_n h_1$ for some $p_n \in P$. So we have $h_n K \cap j_n L = p_n(h_1 K \cap j_1 L)$ for each $j_n^{-1} h_n = j_1^{-1} h_1$.

We can do the rewriting process similarly for other elements in $\{j_n^{-1} h_n\}$, and finally we obtain $H_z K \cap J_z L \subset \cup \mathcal{A} \subset PC$, where C is a compact set defined as $\bigcup_{i=1}^{r} (h_i K \cap j_i L)$. The claim is proved.

Recall that we have proved there exists a compact subset $C \subset M$ such that the following holds

$$\Lambda(H \cap J) \setminus \{z\} \subset (\Lambda(H) \cap \Lambda(J)) \setminus \{z\} \subset PC. \tag{2.2}$$

We now have two cases to consider for finishing the proof of proposition,

P **is finite**. Since the right-hand of (2.2) is a compact set, there exists an open neighborhood of z disjoint with $\Lambda(H) \cap \Lambda(J)$. Thus, z is an isolated point of $\Lambda(H) \cap \Lambda(J)$ and does not lie in $\Lambda(H \cap J)$.

P **is infinite**. P acts cocompactly on $\Lambda(H \cap J) \setminus \{z\}$. Thus, z is a bounded parabolic point of $H \cap J$. □

Summarizing the above results, we can now conclude with the proof of Theorem 2.1.1. Recall that by Theorem 2.2.5, dynamically quasiconvex subgroups coincide with relatively quasiconvex groups in relatively hyperbolic groups.

Proof of Theorem 2.1.1. By (QC-1) definition of relative quasiconvexity in [Hr10], a relatively quasiconvex subgroup acts on its limit set as a geometrically finite convergence action. Then the limit set of a relatively quasiconvex subgroup consists of conical limit points and bounded parabolic points. Therefore, the decomposition of $\Lambda(H) \cap \Lambda(J)$ follows from Propositions 2.3.1 and 2.3.3. □

Remark 2.3.4. In word hyperbolic groups, the exceptional set E is empty since there are no parabolic subgroups. In this case, limit sets of two relatively quasiconvex subgroups intersect at least in two points, once they intersect. But in the relative case, it is possible that their limit sets intersect in only one (necessarily parabolic) point. For example, let $H = \langle z+1 \rangle, J = \langle z+i \rangle$ be two parabolic subgroups of a Fuchsian group G acting on the upper half space $\{z \in \mathbb{C} : \Im(z) > 0\}$. Note that the intersection $H \cap J$ is trivial, but H and J share the same fixed point ∞.

The following corollary follows from the isolatedness of the exceptional set E.

Corollary 2.3.5. *Let H, J be two relatively quasiconvex subgroups of G. If $\Lambda(H) \subset \Lambda(J)$. Then either $\Lambda(H \cap J) = \Lambda(H)$ or H is a parabolic subgroup.*

Proof. Suppose H is not a parabolic subgroup. Then $|\Lambda(H)| \geq 2$. By Theorem 2.1.1, we have $\Lambda(H) = \Lambda(H \cap J) \sqcup E$, where E consists of isolated points in $\Lambda(H)$. It is well-known that limit sets are perfect, if containing at least 3 points. So if $|\Lambda(H)| > 2$, then E is empty.

It suffices to consider the case $|\Lambda(H)| = 2$. In this case, H is a virtually cyclic group. Thus, $\Lambda(H)$ consists of two conical limit points. By Lemma 2.2.1, we have that E is empty. □

2.4 Proper conjugates of dynamically quasiconvex subgroups

Suppose G has a convergence group action on M. According to [GMRS98], a subgroup $H \subset G$ is said to be *maximal* in its limit set if $H = Stab_G(\Lambda(H))$. Recall that Lemma 2.2.6 shows any nonparabolic dynamically quasiconvex subgroup is of finite index in the stabilizer of its limit set.

Lemma 2.4.1. *Let H be dynamically quasiconvex in a convergence group G and suppose H is maximal in its limit set. Then for any $g \in G \setminus H$, $g\Lambda(H) \not\subseteq \Lambda(H)$.*

Proof. If $|\Lambda(H)| = 1$, then H is the maximal parabolic subgroup. In this case, the conclusion is trivial. We now consider the case $|\Lambda(H)| \geq 2$. By way of contradiction, we suppose that $g\Lambda(H) \subsetneq \Lambda(H)$.

Take distinct points x and y from $\Lambda(H)$. Since $g\Lambda(H) \subsetneq \Lambda(H)$, we have $g^n(x) \in \Lambda(H)$ and $g^n(y) \in \Lambda(H)$ for each $n \in \mathbb{N}$. Therefore, we have that the cosets $g^{-n}H$ belong to the following set

$$\{gH \in G/H : g\Lambda(H) \cap \{x\} \neq \emptyset \text{ and } g\Lambda(H) \cap \{y\} \neq \emptyset\}.$$

Since H is dynamically quasiconvex, we have the set $\{g^{-n}H\}$ is finite. Consequently there exist two different integers m and n such that $g^{-m}H = g^{-n}H$, and thus $g^{n-m} \in H$. Then $\Lambda(H) = g^{n-m}\Lambda(H) \subsetneq g\Lambda(H) \subsetneq \Lambda(H)$. Hence $\Lambda(H) = g\Lambda(H)$, which is impossible since H is maximal in its limit set $\Lambda(H)$. □

Remark 2.4.2. Lemma 2.4.1 generalizes Lemma 2.10 in Gitik-Mitra-Rips-Sageev [GMRS98].

We now prove Theorem 2.1.4.

Proof of Theorem 2.1.4. Let g be an element of $G \setminus H$ such that $gHg^{-1} \subsetneq H$. By Lemma 2.4.1, it follows that g belongs to the setwise stabilizer K in G of $\Lambda(H)$. Using Lemma 2.2.6, we obtain that H is of finite index in K. So g^n belongs to H for some n. Thus, we have $H = g^nHg^{-n} \subset gHg^{-1} \subset H$. The proof is complete. □

Remark 2.4.3. The condition that $|\Lambda(H)| \geq 2$ could not be dropped. It is known that there exists a finitely generated group G containing a finitely generated subgroup H such that, for some $g \in G$, $gHg^{-1} \subset H$ but $gHg^{-1} \subsetneq H$. See [WZ95] for an elementary example. We then form a free product $G * F_2$, where F_2 is a free group of rank 2. By the second definition of relative hyperbolicity in [Bo99b], $G * F_2$ is relatively hyperbolic. In particular, G is a maximal parabolic subgroup. But $H \subset G * F_2$ does not satisfy the statement of Theorem 2.1.4 for some $g \in G$.

2.5 Undistorted subgroups of groups with nontrivial Floyd boundary

In this section, we consider a finitely generated group G with a fixed finite generating set S, without assuming relative hyperbolicity of G.

As usual, S is assumed to be symmetric, i.e. $S = S^{-1}$. Then the *Cayley graph* $\Gamma(G, S)$ of G with respect to S, is defined as an oriented graph with vertex set G and edge set $G \times S$. An edge $(g, s) \in G \times S$ goes from g to gs. Note that $\Gamma(G, S)$ is a connected graph, which induces a *word metric* d_S on G by setting the length of each edge to be 1.

Given a rectifiable path p in $\Gamma(G, S)$, we denote by p_-, p_+ the initial and terminal endpoint of p respectively. Let $l(p)$ be the length of p. We say p is a *ϵ-quasigeodesic* for a constant $\epsilon \geq 0$ if, for any subpath q of p, we have $l(q) < \epsilon d_S(q_-, q_+) + \epsilon$. Let $d_S(1, p)$ be the distance from the identity to the path p with respect to d_S.

Recall that a (ϵ-)*quasi-isometric* map $\phi : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is a map such that the following holds

$$\epsilon^{-1} d_X(x, y) - \epsilon \le d_Y(\phi(x), \phi(y)) \le \epsilon d_X(x, y) + \epsilon.$$

Definition 2.5.1. Let $H \subset G$ be a finitely generated subgroup with a finite generating set T. Then H is *undistorted* if the inclusion of (H, d_T) into (G, d_S) is a quasi-isometric map.

Note that the definition of an undistorted subgroup is independent of choices of finite generating sets S, T. Without loss of generality, we assume that $T \subset S$ in the sequel. Then the embedding $\imath : \Gamma(H, T) \hookrightarrow \Gamma(G, S)$ is a quasi-isometric map. In particular, a geodesic in $\Gamma(H, T)$ is naturally embedded as a quasigeodesic in $\Gamma(G, S)$.

We now briefly discuss the construction of Floyd boundary of a finitely generated group. We refer the reader to [Fl80], [Ka03] and [GP09a] for more details.

Let $f : \mathbb{N} \to \mathbb{R}$ be the function $f(n) = n^{-2}$. We rescale the length of each edge e of $\Gamma(G, S)$ by a factor $f(d_S(1, e))$, and then take the Cauchy metric completion $\overline{G}$. Denote by ρ the complete metric on $\overline{G}$. Then *Floyd boundary* $\partial(G)$ is defined as $\overline{G} \setminus G$. With a change of finite generating sets, Floyd boundary is well-defined up to a bi-Lipschitz homeomorphism.

If $\partial(G)$ consists of 0, 1 or 2 points then it is said to be *trivial*. Otherwise, it is uncountable and is called *nontrivial*. The class of groups with nontrivial Floyd boundary includes non-elementary relatively hyperbolic groups [Ge10].

In [Ka03], Karlsson showed that if Floyd boundary is nontrivial, then G acts on $\partial(G)$ as a convergence group action. In what follows, when speaking of limit sets and dynamical quasiconvexity of subgroups in G, we have in mind the convergence action of G on $\partial(G)$.

The following lemma shows that the Floyd length of a far (quasi)geodesic in $\Gamma(G, S)$ is small. The original version was stated in [Ka03] for geodesics, but its proof also works for quasigeodesic in the Cayley graph.

Lemma 2.5.2. *[Ka03] Given $\epsilon > 0$, there is a function Θ_ϵ: $\mathbb{N} \to \mathbb{R}_{\ge 0}$ such that $\Theta_\epsilon(n) \to 0$ as $n \to \infty$ and the following property holds. Let z, w be two points in G and let γ be an ϵ-quasigeodesic between z and w of $\Gamma(G, S)$. Then the following holds*

$$\rho(z, w) \le \Theta_\epsilon(d_S(1, \gamma))$$

Recall that we assume $T \subset S$ and the embedding $\imath : \Gamma(H, T) \hookrightarrow \Gamma(G, S)$ is quasi-isometric. The following lemma roughly says that any two limit points of an undistorted subgroup H can be connected by a geodesic in $\Gamma(H, T)$.

Lemma 2.5.3. *If H is undistorted in G such that $\Lambda(H) \ge 2$, then there exists a constant $\epsilon_0 \ge 0$ such that the following holds. For any two distinct points $p, q \in \Lambda(H)$, there exists an ϵ_0-quasigeodesic γ in $\Gamma(G, S)$ between p and q such that $\gamma \subset \Gamma(H, T)$.*

Proof. Since p and q are distinct limit points of H, there exist two sequences $\{h_n\}$ and $\{h'_n\}$ of H such that $h_n \to p$ and $h'_n \to q$. Let $\delta = \rho(p,q)/3$.

Without loss of generality, we assume for all n, $h_n \in B_\delta(p)$ and $h'_n \in B_\delta(q)$, after passage to subsequences of $\{h_n\}$ and $\{h'_n\}$ respectively. Here, $B_\delta(p)$ and $B_\delta(q)$ denote open metric balls centered at p and q in $\overline{G}$ with radius δ respectively. It then follows by the triangle inequality that $\rho(h_n, h'_n) > d(p,q)/3$ for all n.

Taking geodesics γ_n in the Cayley graph $\Gamma(H,T)$ such that $(\gamma_n)_- = h_n$ and $(\gamma_n)_+ = h'_n$. By the undistortedness of H, there is a positive constant ϵ_0 depending on H, such that any geodesic in $\Gamma(H,T)$ is an ϵ_0-quasigeodesic in $\Gamma(G,S)$. Thus, γ_n are ϵ_0-quasigeodesics in $\Gamma(G,S)$. Observe that the endpoints h_n, h'_n of γ_n have at least a ρ-distance δ in $\overline{G}$.

Let Θ_{ϵ_0} be the function given by Lemma 2.5.2. Since $\Theta_{\epsilon_0}(n) \to 0$ as $n \to \infty$, let R be the maximal integer m such that $\Theta_{\epsilon_0}(m) \geq \delta$. By Lemma 2.5.2, each quasigeodesic γ_n intersects a closed ball B centered at identity with radius R in $\Gamma(G,S)$.

Therefore, using a Cantor diagonal argument based on γ_n, we obtain an ϵ_0-quasigeodesic γ in $\Gamma(G,S)$ between p and q such that the vertex set of γ lies in H. □

Remark 2.5.4. In contrast with hyperbolic groups, two (quasi)geodesics in $\Gamma(G,S)$ with same endpoints may not be uniformly Hausdorff distance bounded. Thus, we could not guarantee that any (quasi)geodesic between p and q satisfies the statement of Lemma 2.5.3.

We are now ready to prove Theorem 2.1.5.

Proof of Theorem 2.1.5. It suffices to establish the conclusion under the assumption $|\Lambda(H)| \geq 2$. We are going to bound the following set

$$\{gH \in G/H : g\Lambda(H) \cap L \neq \emptyset \text{ and } g\Lambda(H) \cap K \neq \emptyset\},$$

whenever K and L are disjoint closed subsets of $\partial(G)$.

Suppose, to the contrary, there exists a sequence of distinct cosets g_nH such that $g_n\Lambda(H) \cap K \neq \emptyset$ and $g_n\Lambda(H) \cap L \neq \emptyset$. Let $p_n \in g_n\Lambda(H) \cap K$ and $q_n \in g_n\Lambda(H) \cap L$. Note that $g_n^{-1}(p_n), g_n^{-1}(q_n) \in \Lambda(H)$. By Lemma 2.5.3, we obtain ϵ_0-quasigeodesics γ_n between $g_n^{-1}(p_n)$ and $g_n^{-1}(q_n)$, such that the vertex set of γ_n lies in H. Hence $g_n(\gamma_n)$ are ϵ_0-quasigeodesics with endpoints $p_n, q_n \in g_n\Lambda(H)$, such that the vertex set of $g_n(\gamma_n)$ lies in the same coset g_nH.

Note that $\{p_n, q_n\} \in K \times L$. Since $K \times L$ is compact in $\partial(G) \times \partial(G)$. there exists a uniform positive constant μ depending on K and L, such that $\rho(p_n, q_n) \geq \mu$ for all n. Let Θ_{ϵ_0} be the function given by Lemma 2.5.2. Since $\Theta_{\epsilon_0}(n) \to 0$ as $n \to \infty$, let R be the maximal integer m such that $\Theta_{\epsilon_0}(m) \geq \mu$.

By Lemma 2.5.2, any ϵ_0-quasigeodesic between p_n and q_n intersects non-trivially with B, where B is the closed ball at the identity with radius R in $\Gamma(G,S)$. Let c_n be an intersection point of $g_n\gamma_n \cap B$. Then we have $d_S(1, c_n) < R$ for every n.

Recall that the vertex set of $g_n(\gamma_n)$ lies in g_nH. Therefore, for each n, there exists $h_n \in H$ such that $d_S(g_nh_n, c_n) < 1$. Then $d_S(1, g_nh_n) < R+1$ for all n. Since S is a finite set, we have the set $\{g_nh_n\}$ is finite. This is a contradiction, as $\{g_nH\}$ is assumed as a sequence of different H-cosets in G. The proof is complete. □

In view of Theorem 2.1.5, the previous Lemma 2.2.6, Corollaries 2.7, 2.8 and Theorem 2.1.4 can be stated in the setting of finitely generated groups with nontrivial Floyd boundary. In favor of applications in group theory, we state the following.

Corollary 2.5.5. *Let H be undistorted in G such that $|\Lambda(H)| \geq 2$. Then H is of finite index in its commensurator. In particular, H is of finite index in its normalizer.*

Corollary 2.5.6. *Let H be undistorted in G such that $|\Lambda(H)| \geq 2$. Then for any $g \in G \setminus H$, $gHg^{-1} \subseteq H$ implies that $gHg^{-1} = H$.*

In relatively hyperbolic groups, the limit set of a relatively quasiconvex subgroup consists of conical points and bounded parabolic points. This fact allows us to complete the limit set intersection theorem 2.1.1 for relatively quasiconvex groups. In general convergence groups it is an interesting question to ask whether dynamically quasiconvex subgroups act geometrically finitely on their limit sets. The following example gives a negative answer to the question.

Example 2.5.7. Dunwoody's inaccessible group J in [Du91] has infinite ends and thus nontrivial Floyd boundary. Since infinitely ended groups are relatively hyperbolic, by using a theorem of Stalling with the second definition of relative hyperbolicity in [Bo99b]. Thus, J is relatively hyperbolic. Let $\mathbb{H}$ be a set of representatives of conjugacy class of maximal parabolic subgroups of J. Since $(J, \mathbb{H})$ is relatively hyperbolic, then each $H \in \mathbb{H}$ is undistorted in J. See the remark 2.2.3.

In Chapter 1, we prove that if $(G, \mathbb{P})$ is relatively hyperbolic, then G acts geometrically finitely on ∂G if and only if each $P \in \mathbb{P}$ acts geometrically finitely on its limit set $\Lambda(P) \subset \partial G$. See Theorem 1.1.5.

However, it is observed in Theorem 1.4.23 that J does not act geometrically finitely on its Floyd boundary ∂J. Therefore, there exists at least one $H \in \mathbb{H}$ such that H does not act geometrically finitely on its limit set $\Lambda(H) \subset \partial J$. Moreover, $\Lambda(H)$ contains infinite limit points, otherwise J would act geometrically finitely on ∂J.

In a word, H is dynamically quasiconvex by Theorem 2.1.5, but does not act geometrically finitely on its limit set in the Floyd boundary ∂J.

CHAPITRE 3

Separable subgroups have bounded packing

This chapter is written based on the paper [Ya10b], which is accepted in **Proc. Amer. Math. Soc.**

Abstract. In this chapter, we prove that separable subgroups have bounded packing in ambient groups. The notion bounded packing was introduced by Hruska and Wise and in particular, our result answers positively a question of theirs, asking whether each subgroup of a virtually polycyclic group has the bounded packing property.

3.1 Introduction

Bounded packing was introduced for a subgroup of a countable group in Hruska-Wise [HW09]. Roughly speaking, this property gives a finite upper bound on the number of left cosets of the subgroup that are pairwise close in G. Precisely,

Definition 3.1.1. Let G be a countable group with a left invariant proper metric d. A subgroup H has *bounded packing* in G (with respect to d) if for each positive constant D, there is a natural number $N = N(G, H, D)$ such that, for any collection $\mathcal{C}$ of N left H-cosets in G, there exist at least two H-cosets $gH, g'H \in \mathcal{C}$ satisfying $d(gH, g'H) > D$.

Remark 3.1.2. Bounded packing of a subgroup is independent of the choice of the left invariant proper metric d. Equivalently, bounded packing says that for each positive constant D, every collection of left H-cosets in G with pairwise distance at most D has a uniform bound $N = N(G, H, D)$ on their cardinality.

This chapter aims to give a proof of the following.

Theorem 3.1.3. *If H is a separable subgroup of a countable group G, then H has bounded packing in G.*

A subgroup H of a group G is *separable* if H is an intersection of finite index subgroups of G. A group is called *subgroup separable* or *LERF* if every finitely generated subgroup is separable. For example, Hall showed that free groups are LERF in [Ha49]. It follows from a theorem of Mal'cev [Ma83] that polycyclic (and in particular finitely generated nilpotent) groups are LERF. A group is called *slender* if every subgroup is finitely generated. Polycyclic groups are also slender by a result of Hirsch [Hi37]. Therefore, we have the following corollary, which gives a positive answer to [HW09, Conjecture 2.14].

Corollary 3.1.4. *Let P be virtually polycyclic. Then each subgroup of P has bounded packing in P.*

Remark 3.1.5. In [Sa10], Jordan Sahattchieve obtained a special case of this Corollary using different methods: any subgroup of (Hirsch) length 1 of a polycyclic group has bounded packing.

3.2 Proof of the Theorem

We define the norm $|g|_d$ of an element $g \in G$ as the distance $d(1, g)$.

Proof of the Theorem. By the definition of bounded packing, it suffices to show, for each positive constant D, that there is a uniform bound on the cardinality of every collection of left H-cosets in G with pairwise distance at most D.

Given such a collection $\mathcal{A}$ satisfying $d(gH, g'H) < D$ for any gH, $g'H \in \mathcal{A}$. Without loss of generality, we can assume H belongs to $\mathcal{A}$, up to a translation of $\mathcal{A}$ by an appropriate element of G. Since $d(H, gH) < D$ for each $gH \in \mathcal{A}$, there exists an element h in H such that $d(1, hgH) < D$. Hence we conclude that the collection $\mathcal{A} \setminus \{H\}$ lies in the finite union of double cosets HgH with $|g|_d < D$ and $g \in G \setminus H$.

Since d is a left invariant proper metric on G, the set $F = \{g \in G \setminus H : |g|_d < D\}$ is finite. Since H is separable in G, we can take a finite index subgroup K of G such that $H < K$ and $F \subset G \setminus K$.

We claim that no two different left H-cosets of $\mathcal{A}$ lie in the same left K-coset. By way of contradiction, we suppose that there are two H-cosets gkH, $gk'H \in \mathcal{A}$ in the same coset gK such that $d(gkH, gk'H) < D$. By a similar argument as above, we get that $k^{-1}k'H$ belongs to a double coset Hg_0H with $|g_0|_d < D$. Moreover, we note that $g_0 \in F$. Since we have $k^{-1}k'H = hg_0H$ for some $h \in H$, it is easy to see that g_0 belongs to K. But by the choice of K, we know that g_0 belongs to $G \setminus K$. This is a contradiction. Our claim is proved.

Since K is of finite index in G, the cardinality of each $\mathcal{A}$ is upper bounded by $[G : K]$. Thus for each D, we have obtained a uniform bound on every $\mathcal{A}$. Hence H has bounded packing in G. □

Chapitre 4

Limit sets and commensurability of Kleinian groups

This chapter is based on the paper [YJ10], joint with Yueping Jiang, published in **Bull. Aust. Math. Soc.** (2010), **82**: 1-9.

Abstract. In this chapter, we obtain several results on the commensurability of two Kleinian groups and their limit sets. We prove that two finitely generated subgroups G_1 and G_2 of an infinite co-volume Kleinian group $G \subset \text{Isom}(\mathbf{H}^3)$ having $\Lambda(G_1) = \Lambda(G_2)$ are commensurable. In particular, it is proved that any finitely generated subgroup H of a Kleinian group $G \subset \text{Isom}(\mathbf{H}^3)$ with $\Lambda(H) = \Lambda(G)$ is of finite index if and only if H is not a virtually fibered subgroup.

4.1 Introduction

Two groups G_1 and G_2 are commensurable if their intersection $G_1 \cap G_2$ is of finite index in both G_1 and G_2. In this chapter, we investigate the following question asked by J. Anderson [Be04]: namely, if $G_1, G_2 \subset \text{Isom}(\mathbf{H}^n)$ are finitely generated and discrete, does $\text{Ax}(G_1) = \text{Ax}(G_2)$ imply that G_1 and G_2 are commensurable? Here we use $\text{Ax}(G)$ to denote the set of axes of the hyperbolic elements of $G \subset \text{Isom}(\mathbf{H}^n)$.

The question has been discussed by several authors. In 1990, G. Mess [Me90] showed that if G_1 and G_2 are non-elementary finitely generated Fuchsian groups having the same nonempty set of simple axes, then G_1 and G_2 are commensurable. Using some technical results on arithmetic Kleinian groups, D. Long and A. Reid [LR98] gave an affirmative answer to this question in the case where G_1 and G_2 are arithmetic Kleinian groups. Note that all the confirmed cases for the question are geometrically finite groups. So it is natural to ask if the question is true with the assumption that G_1 and G_2 are geometrically finite. Recently, P. Susskind [Su01] constructed two geometrically finite Kleinian groups in $\text{Isom}(\mathbf{H}^n)$ (for $n \geq 4$) having the same action on some invariant 2-hyperbolic plane but whose intersection is infinitely generated. So this implies that these two geometrically finite groups are not commensurable although they have the same axes set. But it is worth pointing that these two geometrically finite Kleinian groups generate a non-discrete group. This example suggests that some additional conditions need to be imposed to eliminate such 'bad' groups.

In higher dimensions, we have the following consequence of P. Susskind and G. Swarup's results [SS92] on the limit set of the intersection of two geometrically finite Kleinian groups.

Proposition 4.1.1. *Let G_1 and G_2 be two non-elementary geometrically finite subgroups of a Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^n)(n \geq 3)$. Then G_1 and G_2 are commensurable if and only if the limit sets $\Lambda(G_1)$ and $\Lambda(G_2)$ are equal. In particular, $\Lambda(G_1) = \Lambda(G_2)$ if and only if $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$.*

In several cases, the condition that both subgroups lie in a larger discrete group can be dropped.

Corollary 4.1.2. *Let $G_1, G_2 \subset \mathrm{Isom}(\mathbf{H}^n)$ be two non-elementary geometrically finite Kleinian groups of the second kind leaving no m hyperbolic planes invariant for $m < n-1$. Then G_1 and G_2 are commensurable if and only if $\Lambda(G_1) = \Lambda(G_2)$.*

In dimension 3, we can refine the analysis of the limit sets using Anderson's results [An95] to get the following result in the essence of the recent solution of the Tameness Conjecture (see [Ag04] and [CG06]), which states that all finitely generated Kleinian groups in $\mathrm{Isom}(\mathbf{H}^3)$ are topologically tame.

Theorem 4.1.3. *Let G_1, G_2 be two non-elementary finitely generated subgroups of an infinite co-volume Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^3)$. Then G_1 and G_2 are commensurable if and only if $\Lambda(G_1) = \Lambda(G_2)$. In particular, $\Lambda(G_1) = \Lambda(G_2)$ if and only if $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$*

Similarly, in the following case we are able to remove the ambient discrete group.

Corollary 4.1.4. *Let $G_1, G_2 \subset \mathrm{Isom}(\mathbf{H}^3)$ be two non-elementary finitely generated Kleinian groups of the second kind whose limit sets are not circles. Then G_1 and G_2 are commensurable if and only if $\Lambda(G_1) = \Lambda(G_2)$.*

In fact, under the hypotheses of the above results, the condition of having the same limit sets of two Kleinian subgroups exactly implies having the same axes sets. But Anderson's original formulation of the question is only to suppose that two Kleinian groups have the same axes set. So it is interesting to explore whether there is some essential difference between the limit set and axes set. The following theorem is a result in this direction, suggesting that the 'same axes set' condition is necessary in general for Anderson's question.

Theorem 4.1.5. *Let H be a non-elementary finitely generated subgroup of a Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^3)$. Suppose that $\Lambda(H) = \Lambda(G)$. Then $[G : H]$ is finite if and only if G is not virtually fibered over H. In particular, $[G : H]$ is finite if and only if $\mathrm{Ax}(H) = \mathrm{Ax}(G)$.*

Remark 4.1.6. We remark that the case of H being geometrically finite is proved by P. Susskind and G. Swarup [SS92, Theorem 1]. Theorem 4.1.5 actually proves the special case of Anderson's question where G_1 is a subgroup of G_2.

The chapter is organized as follows. In section 4.2, we collect some results on the limit set of the intersection of two Kleinian groups and prove some useful lemmas for later use. In section 4.3, we prove Theorems 4.1.3 and 4.1.5.

4.2 Preliminaries

Let $\mathbf{B}^n$ denote the closed ball $\mathbf{H}^n \cup \mathbf{S}^{n-1}$, whose boundary $\mathbf{S}^{n-1}$ is identified via stereographic projection with $\overline{\mathbf{R}}^{n-1} = \mathbf{R}^{n-1} \cup \infty$. Let $\mathrm{Isom}(\mathbf{H}^n)$ be the full group of isometries of $\mathbf{H}^n$ and let $G \subset \mathrm{Isom}(\mathbf{H}^n)$ be a *Kleinian* group; that is, a discrete subgroup of $\mathrm{Isom}(\mathbf{H}^n)$. Then G acts discontinuously on $\mathbf{H}^n$ if and only if G is discrete. Furthermore, G acts on $\mathbf{S}^{n-1}$ as a group of conformal homeomorphisms. The *set of discontinuity* $\Omega(G)$ of G is the subset of $\mathbf{S}^{n-1}$ on which G acts discontinuously; the *limit set* $\Lambda(G)$ is the complement of $\Omega(G)$ in $\mathbf{S}^{n-1}$. A Kleinian group is said to be of the *second kind* if $\Omega(G)$ is nonempty otherwise it is said to be of the *first kind*.

The elements of $\mathrm{Isom}(\mathbf{H}^n)$ are classified in terms of their fixed point sets. An element $g \neq \mathrm{id}$ in $\mathrm{Isom}(\mathbf{H}^n)$ is *elliptic* if it has a fixed point in $\mathbf{H}^n$, *parabolic* if it has exactly one fixed point which lies in $\mathbf{S}^{n-1}$, *hyperbolic* if it has exactly two fixed points which lie in $\mathbf{S}^{n-1}$. The unique geodesic joining the two fixed points of the hyperbolic element g, which is invariant under g, is called the *axis* of the hyperbolic element and is denoted by $\mathrm{ax}(g)$. The limit set $\Lambda(G)$ is the closure of the set of fixed points of hyperbolic and parabolic elements of G. A Kleinian group whose limit set contains fewer than three points is called *elementary* and is otherwise called *non-elementary*.

For a non-elementary Kleinian group G, define $\widetilde{C}(G)$ to be the smallest nonempty convex set in $\mathbf{H}^n$ which is invariant under the action of G; this is the *convex hull* of $\Lambda(G)$. The quotient $C(G) = \widetilde{C}(G)/G$ is the *convex core* of $M = \mathbf{H}^n/G$. The group G is *geometrically finite* if the convex core $C(G)$ has finite volume.

By Margulis's lemma, it is known that there is a positive constant ϵ_0 such that for any Kleinian group $G \in \mathrm{Isom}(\mathbf{H}^n)$ and $\epsilon < \epsilon_0$, the part of $\mathbf{H}^n/G$ where the injectivity radius is less than ϵ is a disjoint union of tubular neighbourhoods of closed geodesics, whose lengths are less than 2ϵ, and cusp neighbourhoods. In dimensions 2 and 3, these cusp neighbourhoods can be taken to be disjoint quotients of horoballs by the corresponding parabolic subgroup. This set of disjoint horoballs is called a *precisely invariant system of horoballs* for G. In dimension 3, it is often helpful to identify the infinity boundary $\mathbf{S}^2$ of $\mathbf{H}^3$ with the extended complex plane $\overline{\mathbf{C}}$. In particular, the fixed point of a rank 1 parabolic subgroup J of G is called *doubly cusped* if there are two disjoint circular discs $B_1, B_2 \subset \overline{\mathbf{C}}$ such that $B_1 \cup B_2$ is precisely invariant under J in G. In this case, the parabolic elements of J are also called doubly cusped.

In dimension 3, we call a Kleinian group G *topologically tame* if the manifold $M = \mathbf{H}^3/G$ is homeomorphic to the interior of a compact 3-manifold. Denote by M^c the complement of these cusp neighbourhoods. Using the relative core theorem of McCullough [Mc86], there exists a compact submanifold N of M^c such that the inclusion of N in M^c is a homotopy equivalence, every torus component of $\partial(M^c)$ lies in N, and N meets each annular component of $\partial(M^c)$ in an annulus. Call such an N a *relative compact core* for M. The components of $\partial(N) - \partial(M^c)$ are the *relative boundary components* of N. The *ends* of M^c are in one-to-one correspondence with the components of $M^c - N$. An end E of M^c is *geometrically finite* if it has a neighbourhood disjoint from $C(G)$. Otherwise, E is *geometrically infinite*.

A Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^3)$ is *virtually fibered* over a subgroup H if there are finite index subgroups G^0 of G and H^0 of H such that $\mathbf{H}^3/G^0$ has finite volume and fibers over the circle with the fiber subgroup H^0. Note that H^0 is then a normal subgroup of G^0, and so $\Lambda(H^0) = \Lambda(G^0) = \mathbf{S}^2$.

In order to analyze the geometry of a geometrically infinite Kleinian group, we will use Canary's covering theorem, which generalizes a theorem of Thurston [Th78]. Note that the Tameness Theorem ([Ag04] and [CG06]) states that all finitely generated Kleinian groups in $\mathrm{Isom}(\mathbf{H}^3)$ are topologically tame.

Theorem 4.2.1 ([Ca96, The Covering Theorem])**.** *Let G be a torsion free Kleinian group in $\mathrm{Isom}(\mathbf{H}^3)$ and let H be a non-elementary finitely generated subgroup of G. Let $N = \mathbf{H}^3/G$, let $M = \mathbf{H}^3/H$, and let $p : M \to N$ be the covering map. If M has a geometrically infinite end E, then either G is virtually fibered over H or E has a neighbourhood U such that p is finite-to-one on U.*

Now we list several results on the limit set of the intersection of two Kleinian groups, which describe $\Lambda(G_1 \cap G_2)$ in terms of $\Lambda(G_1)$ and $\Lambda(G_2)$, where G_1 and G_2 are subgroups of a Kleinian group G. Here we only collect the results used in this chapter and state them in an appropriate form for our purpose. See [An96] for a useful survey and the bibliography therein for the results in full details.

Theorem 4.2.2 ([SS92, Theorem 3])**.** *Let G_1, G_2 be two geometrically finite subgroups of Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^n)$. Then $\Lambda(G_1) \cap \Lambda(G_2) = \Lambda(G_1 \cap G_2) \cup P$ where P consists of some parabolic fixed points of G_1 and G_2.*

Proposition 4.2.3 ([SS92, Corollary 1])**.** *Let H be geometrically finite and j be a hyperbolic element with a fixed point in $\Lambda(H)$. If $\langle H, j\rangle$ is discrete, then $j^n \in H$ for some $n > 0$.*

Based on the above results, we get the following lemma characterizing the relationship between limit sets and axes sets.

Lemma 4.2.4. *Let G_1, G_2 be two geometrically finite subgroups of Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^n)$. Then $\Lambda(G_1) = \Lambda(G_2)$ if and only if $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$*

Proof. If G_1 and G_2 are geometrically finite subgroups of a Kleinian group G, then $G_1 \cap G_2$ is again geometrically finite ([SS92, Theorem 4]). It is well known that a hyperbolic element cannot share one fixed point with a parabolic element in a discrete group. Thus, by applying Theorem 4.2.2 to $\Lambda(G_1) \cap \Lambda(G_2)$, we can conclude that any hyperbolic element $h \in G_i$ has at least one fixed point in $\Lambda(G_1 \cap G_2)$ for $i = 1, 2$. Now by Proposition 4.2.3, we have $h^j \in G_1 \cap G_2$ for some large integer $j > 0$. This implies the axis $\mathrm{ax}(h)$ of h belongs to $\mathrm{Ax}(G_1 \cap G_2)$. Therefore, we have $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$. That is to say, P is an empty set.

The other direction is easy to see using the fact that the set of fixed points of hyperbolic elements of G is dense in $\Lambda(G)$. □

In dimension 3, J. Anderson [An95] carried out a more careful analysis on the limit set of the intersection of two topologically tame Kleinian groups. Combined with the recent solution of the Tameness Conjecture ([Ag04] and [CG06]), we have the following theorem.

Theorem 4.2.5 ([An95, Theorem C]). *Let $G \subset \mathrm{Isom}(\mathbf{H}^3)$ be a Kleinian group, and let G_1 and G_2 be non-elementary finitely generated subgroups of G, then $\Lambda(G_1) \cap \Lambda(G_2) = \Lambda(G_1 \cap G_2) \cup P$ where P is empty or consists of some parabolic fixed points of G_1 and G_2.*

Proposition 4.2.6 ([An95, Theorem A]). *Let H be finitely generated Kleinian group and j be a hyperbolic element with a fixed point in $\Lambda(H)$. If $\langle H, j\rangle$ is discrete, then either $\langle H, j\rangle$ is virtually fibered over H or $j^n \in H$ for some $n > 0$.*

Similarly, we obtain the following lemma.

Lemma 4.2.7. *Let G_1 and G_2 be two non-elementary finitely generated subgroups of an infinite co-volume Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^3)$. Then $\Lambda(G_1) = \Lambda(G_2)$ if and only if $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$*

Proof. Observe that our hypothesis 'G is an infinite co-volume Kleinian group' implies that G is not virtually fibered over G_1. Since the intersection of any pair of finitely generated subgroups of a Kleinian group is finitely generated (see [An91]), we see that $G_1 \cap G_2$ is finitely generated. Using Theorem 4.2.5 and Proposition 4.2.6, we can argue exactly as in Lemma 4.2.4 to obtain $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$, if we suppose $\Lambda(G_1) = \Lambda(G_2)$. □

Remark 4.2.8. The condition of G being infinite co-volume cannot be dropped, as will be seen in the proof of Theorem 4.1.5. Namely, the Kleinian group G fibered over H has a different axes set from that of its fiber subgroup H.

In the following two lemmas, we give some useful properties about the same axes sets of two Kleinian groups.

Lemma 4.2.9. *Let G be a non-elementary finitely generated Kleinian group and H a subgroup of finite index in G. Then $\mathrm{Ax}(G) = \mathrm{Ax}(H)$.*

Proof. It is obvious that $\mathrm{Ax}(H) \subset \mathrm{Ax}(G)$. Conversely, since $[G : H]$ is finite, for any hyperbolic element g with axis $\mathrm{ax}(g) \in \mathrm{Ax}(G)$, there are two distinct integers i and j such that $g^i H = g^j H$ and thus $g^{i-j} \in H$. It follows that $\mathrm{ax}(g) \in \mathrm{Ax}(H)$. The proof is complete. □

Remark 4.2.10. In fact, our Theorem 4.1.5 proves that the converse of the above Lemma is also true when H is a finitely generated subgroup of $G \subset \mathrm{Isom}(\mathbf{H}^3)$.

Lemma 4.2.11. *Let G be a non-elementary finitely generated, torsion free Kleinian group and H be a subgroup of G. Suppose that $\mathrm{Ax}(G) = \mathrm{Ax}(H)$. Then for every hyperbolic element $g \in G$, $g^n \in H$ for some $n > 0$.*

Proof. For any hyperbolic element $g \in G$, we can choose a hyperbolic element h from H such that $\mathrm{ax}(g) = \mathrm{ax}(h)$ by the hypothesis $\mathrm{Ax}(G) = \mathrm{Ax}(H)$. It follows that the subgroup $\langle g, h\rangle$ is elementary and torsion free. By the characterization of elementary Kleinian groups it follows that $\langle g, h\rangle$ is actually a cyclic subgroup $\langle f\rangle$ of G. Thus, we can write $g = f^m$ and $h = f^n$ for two appropriate integers m, n. Now we have found the integer n such that $g^n = h^m \in H$, which proves the lemma. □

4.3 Proofs of Theorems

Proof of Proposition 4.1.1. Recall G_1 and G_2 are commensurable if the intersection $G_1 \cap G_2$ is of finite index in both G_1 and G_2. By Lemma 4.2.9, we have $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2)$ and thus $\Lambda(G_1) = \Lambda(G_2)$, if G_1 and G_2 are commensurable. So it remains to prove the converse.

If $\Lambda(G_1) = \Lambda(G_2)$, we have $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$ by Lemma 4.2.4. Therefore, it follows that $\Lambda(G_1) = \Lambda(G_2) = \Lambda(G_1 \cap G_2)$, since the set of fixed points of hyperbolic elements of G is dense in $\Lambda(G)$. Now we can conclude that $G_1 \cap G_2$ is of finite index in both G_1 and G_2, by using Theorem 1 in [SS92] which states that any geometrically finite subgroup H of a Kleinian group G is of finite index in G if $\Lambda(H) = \Lambda(G)$.

The second assertion is just Lemma 4.2.4. This completes the proof. □

Proof of Corollary 4.1.2. It is well known that the stabilizer in $\mathrm{Isom}(\mathbf{H}^n)$ of the limit set of a non-elementary Kleinian group G of the second kind leaving no m hyperbolic planes invariant for $m < n-1$ is itself a Kleinian group. See for example Greenberg [Gr74], where the discreteness of the stabilizer is proved.

Thus, $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2)$ implies that G_1 and G_2 together lie in a common Kleinian group, which is the stabilizer of the common limit set of G_1 and G_2. Thus, Proposition 4.1.1 completes the proof. □

Proof of Theorem 4.1.3. Observe that the fundamental domain of the subgroup H is the union of translates of the fundamental domain of G by left H-coset representatives in G. So the subgroup $\langle G_1, G_2\rangle$ also has infinite co-volume, and we can assume that G is finitely generated by replacing G by $\langle G_1, G_2\rangle$. As the conclusion is easily seen to be unaffected by passage to a finite index subgroup, we may use Selberg's Lemma to pass to a finite index, torsion free subgroup of G. Hence, without loss of generality, we may assume that G is finitely generated and torsion free.

By Proposition 4.1.1, the conclusion is trivial if G is geometrically finite. So we suppose that G is geometrically infinite. Then $M = \mathbf{H}^3/G$ has infinite volume. Let C be a compact core for M. Since M has infinite volume, ∂C contains a surface of genus at least two. Then using Thurston's geometrization theorem for Haken three-manifolds (see [Mo84]), there exists a geometrically finite Kleinian group with non-empty discontinuity domain, which is isomorphic to G.

Now our task is to give an algebraic characterization of the limit set of $G_1 \cap G_2$ in G_i such that the relationships between $\Lambda(G_1 \cap G_2)$ and $\Lambda(G_i)$ can be passed to the ones between target isomorphic groups under the above isomorphism of G. Then the conclusion of Theorem 4.1.3 follows from Proposition 4.1.1. We claim that, for every element $g \in G_1$, there exists an integer k such that $g^k \in G_1 \cap G_2$.

Firstly, by Lemma 4.2.7, we obtain that $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$. So for any hyperbolic element $g \in G_1$, the integer k obtained in Lemma 4.2.11 is such that $g^k \in G_1 \cap G_2$. Now we consider the remaining parabolic elements. Theorem B of [An95] says that if no nontrivial power of a parabolic element $h \in G_1$ lies in $G_1 \cap G_2$, then there exists a doubly cusped parabolic element $f \in G_1 \cap G_2$ with the same fixed point ξ as h. Normalizing their fixed point ξ to ∞, we can suppose that $f(z) = z + 1$ and $h(z) = z + \tau$, where $\mathrm{Im}(\tau) \neq 0$. Since f is doubly cusped in $G_1 \cap G_2$, then $\Lambda(G_1 \cap G_2) \subset \{z : |\mathrm{Im}(z)| < c\}$, for some constant c. But on the other hand, $\Lambda(G_1 \cap G_2)$ is also kept invariant under h, which contradicts the fact that $\Lambda(G_1 \cap G_2)$ is invariant under f. Therefore, the claim is proved for all elements including parabolic elements. A similar claim holds for $G_1 \cap G_2$ and G_2.

Under the isomorphism, using the above claims, we can conclude that the limit set of the (isomorphic) image of $G_1 \cap G_2$ is equal to those of the (isomorphic) images of G_1 and G_2. The proof is complete as a consequence of Proposition 4.1.1. □

Remark 4.3.1. Theorem 4.1.3 can be thought as a geometric version of Lemma 5.4 in [An95], which uses an algebraic assumption on the limit sets of the groups involved.

Proof of Corollary 4.1.4. This is proved similarly to Corollary 4.1.2. □

Proof of Theorem 4.1.5. In view of Lemma 4.2.9, we may assume, without loss of generality, that H is finitely generated and torsion free by using Selberg's Lemma to pass to a finite index, torsion free subgroup of H.

If H is geometrically finite, then the conclusion follows from a result of P. Susskind and G. Swarup [SS92], which states that a non-elementary geometrically finite subgroup sharing the same limit set with the ambient discrete group is of finite index. So next we suppose that H is geometrically infinite. Then there exist finitely many geometrically infinite ends E_i for the manifold $N := \mathbf{H}^3/H$.

We first claim that $\mathrm{Ax}(H) = \mathrm{Ax}(G)$ implies that H cannot be a virtually fibered subgroup of G. Otherwise, by taking finite index subgroups of G and H, we can suppose H is normal in G. Then it follows that every element of the quotient group G/H has finite order by Lemma 4.2.11. Thus, G/H could not be isomorphic to $\mathbf{Z}$. This is a contradiction. So H is not a virtually fibered subgroup of G.

Using the Covering Theorem 4.2.1, we know that, for each geometrically infinite end E_i, there exists a neighbourhood U_i of E_i such that the covering map $\mathcal{P}$: $N \to M := \mathbf{H}^3/G$ is finite to one on U_i.

Now we argue by way of contradiction. Let $\mathcal{Q}_N : \mathbf{H}^3 \to N$ and let $\mathcal{Q}_M : \mathbf{H}^3 \to M$ be the covering maps and notice that $\mathcal{Q}_M = \mathcal{P} \circ \mathcal{Q}_N$. Suppose that $[G : H]$ is infinite. This implies that $\mathcal{P}$ is an infinite covering map. By the definition of a geometrically infinite end, we can take a point z from the neighbourhood U_1 of a geometrically infinite end E_1 such that z also lies in the convex core of N. By lifting the point $\mathcal{P}(z) \in M$ to $\mathbf{H}^3$, it is easy to see that the infinite set $\widetilde{S} := \mathcal{Q}_M^{-1}(\mathcal{P}(z))$ lies in the common convex hull $\widetilde{C}(H) = \widetilde{C}(G) \subset \mathbf{H}^3$, by observing that $\widetilde{C}(G)$ is invariant under G and the preimage $\mathcal{Q}_N^{-1}(z) \subset \widetilde{S}$ lies in $\widetilde{C}(H)$. Since $\mathcal{P}$ is an infinite covering map, the set $S := \mathcal{P}^{-1}(\mathcal{P}(z))$ is infinite. By considering $\mathcal{Q}_M = \mathcal{P} \circ \mathcal{Q}_N$, it follows that $S = \mathcal{Q}_N(\widetilde{S}) \subset N$ and thus S lies in the convex core of N, because $\widetilde{S} \subset \widetilde{C}(H)$.

We claim that we can take a smaller invariant horoball system for H such that infinitely many points of S lie outside all cusp ends of N. Otherwise, we can suppose that infinitely many points of S are contained inside a cusp end E_c of N, since there are only finitely many cusp ends for N. Thus, infinitely many points of $\mathcal{Q}_N^{-1}(S)$ lie in the corresponding horoball B for the end E_c. Normalizing the parabolic fixed point for E_c to ∞ in the upper half space model of $\mathbf{H}^3$, the horoball B at ∞ is precisely invariant under the stabilizer of ∞ in H. On the other hand, we have that infinitely many points of $\mathcal{Q}_N^{-1}(S)$ have the same height, since the covering map $\mathcal{P}$ maps $S \subset N$ to a single point on M, and the horoball B is also precisely invariant under the stabilizer of ∞ in G, which is a Euclidean group preserving the height of points in the horoball B. Then we can take a smaller horoball for E_c such that these infinitely many points of $\mathcal{Q}_N^{-1}(S)$ lie outside the horoball.

Continuing the above process for all cusp ends of N, we can get a new invariant horoball system such that infinitely points of S lie outside these cusp ends of N.

Since S projects to a single point on M, we can conclude that S cannot lie in any compact subset of the convex core of N. Thus, by the above second claim, there exist infinitely many points of S which can only lie in geometrically infinite ends of N. This is a contradiction to the Covering Theorem 4.2.1, which states that the covering map $\mathcal{P}$ restricted to each geometrically infinite end is finite-to-one, if H is not a virtually fibered subgroup of G. □

Annexe A

Proofs of Lemmas 1.3.14 and 1.3.15

In a δ-hyperbolic space, the stability property of quasigeodesics is well-known, cf. [Gr87], [GH90]. This property says that, any two quasigeodesics with the same endpoints have a uniform Hausdorff distance.

Lemma A.0.2. *For any $\delta \geq 0$, $\lambda \geq 1$, $c \geq 0$, there exists $\xi = \xi(\delta, \lambda, c)$ such that for any δ-hyperbolic space, any two (λ, c)-quasigeodesic paths p, q such that $p_- = q_-$, $p_+ = q_+$ are contained in the closed ξ-neighborhoods of each other.*

Recall that two paths p, q in a metric space (X, d_X) are called *k-connected*, if

$$\max\{d_X(p_-, q_-), d_X(p_+, q_+)\} \leq k.$$

Let x, y be two points on the path p. In the following, we denote by $[x, y]_p$ the segment of p between the points x and y.

The next lemma follows directly from Lemma A.0.2 and the thin-triangle property.

Lemma A.0.3. *For any $\delta \geq 0$, $\lambda \geq 1$, $c \geq 0$, there exists $\xi = \xi(\delta, \lambda, c)$ such that the following holds. Suppose that p, q are k-connected (λ, c)-quasigeodesic paths in a δ-hyperbolic space X and u is a point on p such that $min\{d_X(u, p_-), d_X(u, p_+)\} \geq \xi + 2\delta + k$. Then there exists a point v on q such that $d_X(u, v) \leq 2\xi + 2\delta$.*

Following [Os06a, Lemma 3.1] and [Os06b, Proposition 3.15], we prove Lemma 1.3.14 below. At several points of the proof, we will refer the reader to consult [Os06b] for more details.

From now on, we assume $(G, \mathbb{H})$ is relatively hyperbolic and fix a finite relative generating set X for $(G, \mathbb{H})$. Using Lemma 1.2.5, we obtain a finite set Ω and constant κ such that the following inequality holds for a cycle c in $\mathscr{G}(G, X \cup \mathcal{H})$,

$$\sum_{s \in S} d_\Omega(s_-, s_+) \leq \kappa \ell(c), \tag{A.1}$$

where S is a set of isolated H_i-components of c.

Lemma A.0.4 (Lemma 1.3.14)**.** *For any $\lambda \geq 1$, $c \geq 0$, there exists $\alpha_1 = \alpha_1(\lambda, c) > 0$ such that, for any $k \geq 0$, there exists $\alpha_2 = \alpha_2(k, \lambda, c) > 0$ satisfying the following condition.*

Let p, q be two k-connected (λ, c)-quasigeodesics in $\mathscr{G}(G, X \cup \mathcal{H})$. If p has no backtracking and u is a phase vertex on p such that $\min\{d_{X\cup\mathcal{H}}(u, p_-), d_{X\cup\mathcal{H}}(u, p_+)\} > \alpha_2$. Then there exists a phase vertex v on q such that $d_{X\cup\Omega}(u, v) \leq \alpha_1$.

Remark A.0.5. We remark that a result of the same spirit as Lemma A.0.4 was obtained in [Da06, Proposition 1.11].

Proof. Since (λ, c)-quasigeodesics can be modified such that the resulting paths are still (λ, c)-quasigeodesics with the same set of phase vertices. See [Os06b, Lemma 3.12]. Thus it suffices to prove the conclusion by assuming that each vertex of p (and q) is phase.

Since $(G, \mathbb{H})$ is relatively hyperbolic, then the relative Cayley graph $\mathscr{G}(G, X \cup \mathcal{H})$ is a δ-hyperbolic space with respect to the metric $d_{X\cup\mathcal{H}}$.

We define the following constant

$$\alpha_2 = 5\xi + 6\delta + k,$$

where $\xi = \xi(\delta, \lambda, c)$ is the constant given by Lemma A.0.2. For simplicity, we may assume that $2(\xi + \delta)$ is integer.

Suppose u is a phase vertex of p such that $d_{X\cup\mathcal{H}}(p_-, u) \geq \alpha_2$ and $d_{X\cup\mathcal{H}}(p_+, u) \geq \alpha_2$. Then there exists a vertex u_1(resp. u_2) on the segment $[p_-, u]_p$(resp. $[p_+, u]_p$) such that the following holds

$$d_{X\cup\mathcal{H}}(u_i, u) = 4(\xi + \delta), \tag{A.2}$$

for each $i = 1, 2$.

By the choice of α_2, we obtain the following inequality for each $i = 1, 2$,

$$\max\{d_{X\cup\mathcal{H}}(p_-, u_i),\ d_{X\cup\mathcal{H}}(p_+, u_i)\} \geq \xi + 2\delta + k.$$

By Lemma A.0.3, there exist vertices v_1, v_2 on q such that the following holds for each $i = 1, 2$,

$$d_{X\cup\mathcal{H}}(u_i, v_i) \leq 2(\xi + \delta). \tag{A.3}$$

This gives that $[v_1, v_2]_q$ and $[u_1, u_2]_p$ are $2(\xi + \delta)$-connected. Moreover, for each $i \in \{1, 2\}$, we have the following

$$d_{X\cup\mathcal{H}}(u_i, u) = 4(\xi + \delta) \geq \xi + 2\delta + 2(\xi + \delta).$$

Let V be the set of all vertices z on the segment $[v_1, v_2]_q$ of the path q that are closest to u, i.e., for each $v \in V$, we have

$$d_{X\cup\mathcal{H}}(u, z) = \min d_{X\cup\mathcal{H}}(u, v)$$

where v ranges among all vertices of q.

Applying Lemma A.0.3 to $[v_1, v_2]_q$ and $[u_1, u_2]_p$, we obtain that for each $v \in V$,

$$d_{X\cup\mathcal{H}}(u, v) \leq 2(\xi + \delta). \tag{A.4}$$

For each $v \in V$, let $O(v)$ be the set of geodesics in $\mathscr{G}(G, X \cup \mathcal{H})$ connecting u and v. For each $o \in O(v)$, we consider the following combinatorial loops

$$\Box_{(i,v,o)} = [u, u_i]_p o_i [v, v_i]_q{}^{-1} o^{-1}, i = 1, 2,$$

where o_i are fixed geodesics between u_i and v_i in $\mathscr{G}(G, X \cup \mathcal{H})$.

Claim. *For each $v \in V$ and $o \in O(v)$, let s be an H_i-component of o such that $s_+ \neq v$. Then s is isolated in the loop either $\Box_{(1,v,o)}$ or $\Box_{(2,v,o)}$.*

Proof of Claim 1. If the conclusion is false, one easily see that it will produce a backtracking of the path p. This is a contradiction with the assumption. For details, see the proof of Corollary 3.18 in Osin [Os06b]. □

Claim. *Suppose that for each $v \in V$ and $o \in O(v)$, there is an H_i-component s of o such that $s_+ = v$. Then there exists $\bar{v} \in V$ and $\bar{o} \in O(\bar{v})$ such that the H_i-component s of $\bar{o}$ satisfying $s_+ = v$ is isolated in either $\Box_{(1,\bar{v},\bar{o})}$ or $\Box_{(2,\bar{v},\bar{o})}$.*

Proof of Claim 2. This is exactly Corollary 3.20 in Osin [Os06b], whose proof involves an inductive argument to obtain the desired vertex and geodesic. □

We aim to estimate the distance $d_{X\cup\Omega}(u, v)$. By the definition of the set V, it suffices to do it for a particular vertex $\bar{v} \in V$ and geodesic $\bar{o} \in O(\bar{v})$. In the following, we will properly choose $\bar{v}$ and $\bar{o}$.

If the assumption of **Claim 2** is true, we can choose $\bar{v} \in V$ and $\bar{o} \in O(\bar{v})$ such that the H_i-component s of $\bar{o}$ satisfying $s_+ = \bar{v}$ is isolated in one of the two loops $\Box_{(i,\bar{v},\bar{o})}$ (i =1, 2). Otherwise, there exist $\bar{v} \in V$ and $\bar{o} \in O(\bar{v})$ such that no H_i-component s of $\bar{o}$ satisfies $s_+ = v$.

Hence, if s is an H_i-component of $\bar{o}$ such that $s_+ = \bar{v}$, then it is isolated in one of two loops $\Box_{(i,\bar{v},\bar{o})}$ (i =1, 2). By Lemma 1.2.5, we obtain that

$$d_{X\cup\Omega}(s_-, s_+) < \kappa \max_{i=1,2} \ell(\Box_{(i,\bar{v},\bar{o})}) \tag{A.5}$$

For an H_i-component s of $\bar{o}$ satisfying $s_+ \neq v$, we apply **Claim 1** to see that s must be isolated in one of two loops $\Box_{(i,\bar{v},\bar{o})}$ (i =1, 2). Again by Lemma 1.2.5, we have the same inequality

$$d_{X\cup\Omega}(s_-, s_+) < \kappa \max_{i=1,2} \ell(\Box_{(i,\bar{v},\bar{o})}). \tag{A.6}$$

Combining the inequalities (A.2), (A.3) gives the estimate of the length of the segment $[v_1, v_2]_q$,

$$\begin{aligned} \ell([v_1, v_2]_q) &\leq \lambda(d_{X\cup\mathcal{H}}(u_1, v_1) + d_{X\cup\mathcal{H}}(u_1, u_2) + d_{X\cup\mathcal{H}}(u_2, v_2)) + c \\ &\leq 12\lambda(\xi + \delta) + c. \end{aligned}$$

and the length of each cycle $\Box_{(i,\bar{v},\bar{o})}$ is calculated as follows,

$$\begin{aligned} \ell(\Box_{(i,\bar{v},\bar{o})}) &\leq \ell([u, u_i]_p) + d_{X\cup\mathcal{H}}(u_i, v_i) + l([v_1, v_2]_q) + d_{X\cup\mathcal{H}}(u, v) \\ &\leq (16\lambda + 4)(\xi + \delta) + 2c. \end{aligned} \tag{A.7}$$

Finally, we estimate the length of the geodesic $\bar{o}$ using the inequalities (A.5), (A.6) and (A.7),

$$d_{X\cup\Omega}(u, \bar{v}) < \kappa d_{X\cup\mathcal{H}}(u, v) \max_{i=1,2} \ell(\Box_{(i,v,\bar{o})}) < 2\kappa(\xi + \delta)((16\lambda + 4)(\xi + \delta) + 2c).$$

The proof is finished by setting $\alpha_1 = 2\kappa(\xi + \delta)((16\lambda + 4)(\xi + \delta) + 2c)$. □

In what follows, we assume $\mathbb{K} \subset \mathbb{H}$ and the subgroup $\Gamma \subset G$ satisfies the conditions (C0)–(C3) of Theorem 1.3.10. Let $\mathbb{P} = \{\Gamma\} \cup \mathbb{K}$.

By Lemma 1.2.22, Γ is finitely generated by a subset Y with respect to $\mathbb{K}$. Without loss of generality, we may assume that $Y \subset X$. Moreover, the graph imbedding $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \hookrightarrow \mathscr{G}(G, X \cup \mathcal{H})$ is a quasi-isometric map.

Using Lemma A.0.2, the following proof of Lemma 1.3.15 is modeled on the one of Lemma 3.2 in [Os06a]. We present it here for the convenience of the reader.

Lemma A.0.6 (Lemma 1.3.15)**.** *For any $\lambda \geq 1$, $c \geq 0$, $k > 0$, there exists $L = L(\lambda, c, k) > 0$ such that the following holds. Let p, q be k-connected (λ, c)-quasigeodesics without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$ such that p, q are labeled by letters from $\Gamma \setminus \{1\}$. If $\min\{\ell(p), \ell(q)\} > L$, then p and q as Γ-components are connected in $\mathscr{G}(G, X \cup \mathcal{P})$.*

Proof. As in the proof of Lemma A.0.2, we assume that every vertex of p, q are phase. Let $\alpha_1 = \alpha_1(\lambda, c)$ and $\alpha_2 = \alpha_2(k, \lambda, c)$ be the constants given by Lemma A.0.2.

Let $T = \{t \in \langle X \cup \Omega\rangle : |t|_{X \cup \Omega} \leq \alpha_1\}$ and $N = \sharp\, T$. Since $\sharp\,(X \cup \Omega) < \infty$, then $N < \infty$.

By the condition (C2), Γ is almostly . Then there is an integer $M > 0$ such that for $t \in T \cap (G \setminus \Gamma)$, each element $\gamma \in \Gamma \cap \Gamma^t$ has the length strictly less than M with respect to $Y \cup \mathcal{K}$, i.e. $d_{Y \cup \mathcal{K}}(1, \gamma) < M$.

Set

$$L = 2(\lambda\alpha_2 + c) + N(\lambda M + c).$$

Suppose $\min\{\ell(p), \ell(q)\} > L$. Let u be a vertex of p such that $\ell([u, p_-]_p) = \lambda\alpha_2 + c$. As p is a (λ, c)-quasigeodesic, we have $d_{X \cup \Omega}(u, p_-) \geq \alpha_2$.

By the value of L, we continue to choose N vertices, $u_1, u_2, \ldots, u_N$, on the segment $[u, p_+]_p$ of p such that $\ell([u_i, u_{i+1}]_p) = \lambda M + c$, for $1 \leq i < N$. So we have

$$d_{Y \cup \mathcal{K}}(u_i, u_{i+1}) \geq d_{X \cup \mathcal{H}}(u_i, u_{i+1}) \geq (\ell([u_i, u_{i+1}]_p) - c)/\lambda = M. \tag{A.8}$$

Observe that the last vertex u_N satisfies $\ell([u_N, p_+]_p) \geq \lambda\alpha_2 + c$, which gives $d_{X \cup \mathcal{H}}(u_N, p_+) \geq \alpha_2$. Hence we have

$$\min\{d_{X \cup \mathcal{H}}(u_i, p_-), d_{X \cup \mathcal{H}}(u_i, p_+)\} \geq \alpha_2$$

for each u_i. Using Lemma 1.3.14, we obtain a vertex v_i on q such that the following holds

$$d_{X \cup \Omega}(u_i, v_i) \leq \alpha_1. \tag{A.9}$$

Let $\hat{X} = X \cup \Omega$. There exists a path o_i in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$ with the endpoints u_i, v_i such that o_i is labeled by letters from $\hat{X}$, and satisfies $\ell(o_i) = d_{X \cup \Omega}(u_i, v_i)$.

By the definition of N, there exist at least two paths, say o_i, o_j, such that they have the same label $\mathbf{Lab}(o_i) = \mathbf{Lab}(o_i) = t_0$.

Let γ_1, γ_2 be the labels of the segments $[u_i, u_j]_p$, $[v_i, v_j]_q$ respectively. Then the quadrangle $o_i[u_i, u_j]_p o_j^{-1}[v_i, v_j]_q^{-1}$ gives the following equality

$$t_0^{-1}\gamma_1 t_0 = \gamma_2. \tag{A.10}$$

It follows from the inequality (A.9) that $t_0 \in T$. Assume first $t_0 \notin \Gamma$. As $\gamma_1, \gamma_2 \in \Gamma$, we have that $d_{Y\cup\mathcal{K}}(u_i, u_j) = d_{Y\cup\mathcal{K}}(1, \gamma_1) < M$. This contradicts to the inequlaity A.8. Hence we deduce that $t_0 \in \Gamma$. In view of the equality (A.9), we obtain that two segments $[u_i, u_j]_p$ and $[v_i, v_j]_q$ are connected Γ-components in $\mathscr{G}(G, X \cup \mathcal{P})$. So are p, q. □

Bibliographie

[AAS07] J. Anderson, J. Aramayona & K. Shackleton. *An obstruction to the strong relative hyperbolicity of a group.* J. R. Statistical Society, *vol 10*, no. 6, pp. 749–756, 2007. (Cited on page 51.)

[Ag04] I. Agol. *Tameness of hyperbolic 3-manifolds*, 2004. Preprint. (Cited on pages 68, 70 and 71.)

[An91] J. Anderson. *On the finitely generated intersection property for kleinian groups.* Complex Variables Theory Appl., *vol 17*, pp. 111–112, 1991. (Cited on pages 54 and 71.)

[An95] J. Anderson. *Intersections of topologically tame subgroups of kleinian groups.* J. Anal. Math., *vol 65*, pp. 77–94, 1995. (Cited on pages 54, 68, 71 and 73.)

[An96] J. Anderson. *The limit set intersection theorem for finitely generated kleinian groups.* Math. Res. Lett., *vol 3*, pp. 675–692, 1996. (Cited on pages 53, 54 and 70.)

[BDM09] J. Behrstock, C. Drutu & L. Mosher. *Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity.* Math. Annalen, *vol 344*, no. 3, pp. 543–595, 2009. (Cited on pages 7, 17, 51 and 52.)

[Be04] M. Bestvina. *Questions in geometric group theory*, 2004. `http://www.math.utah.edu/~bestvina`. (Cited on pages 19 and 67.)

[Bo93] B. Bowditch. *Geometrical finiteness for hyperbolic groups.* J. Funct. Anal., *vol 113*, pp. 245–317, 1993. (Cited on pages vii, 2 and 12.)

[Bo95] B. Bowditch. *Geometrical finiteness with variable negative curvature.* Duke Math. J., *vol 77*, pp. 229–274, 1995. (Cited on pages 2 and 12.)

[Bo99a] B. Bowditch. *Convergence groups and configuration spaces*, (pp. 23–54). Geometric Group Theory Down Under (J. Cossey, C.F. Miller, W.D. Neumann, M. Shapiro, eds.). de Gruyter, 1999. (Cited on pages 2, 4, 6, 12, 14, 45, 53, 56 and 59.)

[Bo99b] B. Bowditch. *Relatively hyperbolic groups*, 1999. Preprint, Univ. of Southampton. (Cited on pages 1, 2, 3, 5, 11, 12, 13, 14, 15, 22, 39, 45, 46, 56, 61 and 64.)

[Ca96] R. Canary. *A covering theorem for hyperbolic 3-manifolds and its applications.* Topology, *vol 35*, pp. 751–778, 1996. (Cited on page 70.)

[CG06] D. Calegari & D. Gabai. *Shrinkwrapping and the taming of hyperbolic 3-manifolds*. J. Amer. Math. Soc., *vol 19*, pp. 385–446, 2006. (Cited on pages 68, 70 and 71.)

[Da03] F. Dahmani. *Combination of convergence groups*. Geom. Topol., *vol 7*, pp. 933–963, 2003. (Cited on pages 1, 8, 11, 18, 54 and 58.)

[Da06] F. Dahmani. *Accidental parabolics and relatively hyperbolic groups*. Israel J. Math., *vol 153*, pp. 93–127, 2006. (Cited on page 76.)

[DS05] C. Drutu & M. Sapir. *Tree-graded spaces and asymptotic cones of groups*. Topology, *vol 44*, no. 5, pp. 959–1058, 2005. With an appendix by D. Osin and M. Sapir. (Cited on pages 2, 6, 12, 15, 46, 51, 52 and 56.)

[Du91] M. Dunwoody. *An inaccessible group*, vol 1 of *Geometric group theory (volume 181 of London Math. Soc. Lecture Note Ser.)*, (pp. 75–78). Cambridge Univ. Press, Sussex, 1991. (Cited on pages 7, 17, 51 and 64.)

[Fa98] B. Farb. *Relatively hyperbolic groups*. Geom. Funct. Anal., *vol 8*, no. 5, pp. 810–840, 1998. (Cited on pages 1, 2, 3, 11, 12, 13, 22, 28 and 56.)

[Fl80] W. Floyd. *Group completions and limit sets of kleinian groups*. Inventiones Math., *vol 57*, pp. 205–218, 1980. (Cited on pages 7, 16, 46, 55 and 62.)

[Fr95] E. Freden. *Negatively curved groups have the convergence property*. Ann. Acad. Sci. Fenn. Ser. A: Math, *vol 20*, pp. 333–348, 1995. (Cited on pages 2, 4, 12 and 14.)

[Ge96] S. Gersten. *Subgroups of small cancellation groups in dimension 2*. J. London Math. Soc., *vol 54*, pp. 261–283, 1996. (Cited on pages 6, 15 and 22.)

[Ge09] V. Gerasimov. *Expansive convergence groups are relatively hyperbolic*. Geom. Funct. Anal., *vol 19*, pp. 137–169, 2009. (Cited on pages 1, 2, 5, 11, 12, 14, 44, 45, 46 and 56.)

[Ge10] V. Gerasimov. *Floyd maps to the boundaries of relatively hyperbolic groups*, 2010. ArXiv:1001.4482. (Cited on pages 7, 9, 16, 18, 23, 46, 47, 55 and 62.)

[GH90] E. Ghys & P. de la Harpe. *Sur les groupes hyperboliques d'après Mikhael Gromov*. Progress in Math. Birkaüser, 1990. (Cited on pages 43 and 75.)

[GM87] F. Gehring & G. Martin. *Discrete quasiconformal groups i*. Proc. London Math. Soc., *vol 55*, pp. 331–358, 1987. (Cited on pages 4 and 13.)

[GMRS98] R. Gitik, M. Mitra, E. Rips & M. Sageev. *Widths of subgroups*. Trans. Amer. Math. Soc., *vol 350*, pp. 321–329, 1998. (Cited on pages 9, 18, 60 and 61.)

[GP09a] V. Gerasimov & L. Potyagailo. *Dynamical quasiconvexity in relatively hyperbolic groups*, 2009. Preprint. (Cited on pages 1, 5, 8, 11, 14, 17, 46, 53, 56, 57 and 62.)

[GP09b] V. Gerasimov & L. Potyagailo. *Quasi-isometries and floyd boundaries of relatively hyperbolic groups*, 2009. ArXiv:0908.0705. (Cited on pages 7, 16, 17, 23, 46 and 47.)

[GP10] V. Gerasimov & L. Potyagailo. *Horospherical geometry of relatively hyperbolic groups*, 2010. ArXiv:1008.3470. (Cited on pages 7, 16, 30, 47 and 52.)

[GP11] V. Gerasimov & L. Potyagailo. *Quasiconvexity in the relatively hyperbolic groups*, 2011. ArXiv:1103.1211. (Cited on pages 1, 5, 11, 14, 17, 46, 53, 56 and 57.)

[Gr74] L. Greenberg. *Commensurable groups of Moebius transformations*, vol Annals of Mathematics Studies, 79 of *Discontinuous groups and Riemann surfaces (ed. by L. Greenberg)*, (pp. 227–237). Princeton University Press, 1974. (Cited on page 72.)

[Gr87] M. Gromov. *Hyperbolic groups*, vol 1 of *Essays in group theory (S Gersten, editor)*, (pp. 75–263). Springer New York, 1987. (Cited on pages 1, 11, 21, 43, 54 and 75.)

[Gr93] M. Gromov. *Asymptotic invariants of infinite groups*, vol 2 of *Geometric group theory (volume 182 of London Math. Soc. Lecture Note Ser.)*, (pp. 1–295). Cambridge Univ. Press, 1993. (Cited on page 54.)

[Ha49] M. Hall. *Coset representations in free groups.* Trans. Amer. Math. Soc., *vol 67*, pp. 421–432, 1949. (Cited on page 65.)

[Hi37] K. A. Hirsch. *On infinite soluble groups. i.* Proc. London Math. Soc., *vol 44*, no. 2, pp. 53–60, 1937. (Cited on page 65.)

[HK05] G. Hruska & B. Kleiner. *Hadamard spaces with isolated flats.* Geom. Topol., *vol 34*, pp. 1501–1538, 2005. (Cited on pages 1 and 11.)

[Hr10] G. Hruska. *Relative hyperbolicity and relative quasiconvexity for countable groups.* Algebr. Geom. Topol., *vol 10*, pp. 1807–1856, 2010. (Cited on pages 1, 2, 7, 9, 11, 12, 16, 18, 24, 27, 29, 46, 49, 54, 55, 56 and 60.)

[HW09] G. Hruska & D. Wise. *Packing subgroups in relatively hyperbolic groups.* Geom. Topol., *vol 13*, no. 4, pp. 1945–1988, 2009. (Cited on pages 9, 18, 26, 58 and 65.)

[Ka03] A. Karlsson. *Free subgroups of groups with non-trivial floyd boundary.* Comm. Algebra., *vol 31*, pp. 5361–5376, 2003. (Cited on pages 4, 14, 46, 55 and 62.)

[LR98] D. Long & A. Reid. *On fuchsian groups with the same set of axes.* Bull. London Math. Soc., *vol 30*, pp. 533–538, 1998. (Cited on pages 10, 19 and 67.)

[Ma83] A. I. Mal'cev. *On homomorphisms onto finite groups.* American Mathematical Society Translations, *vol 119*, no. 2, pp. 67–79, 1983. (Cited on page 65.)

[Mc86] D. McCullough. *Compact submanifolds of 3-manifolds with boundary.* Quart. J. Math. Oxford, *vol 37*, pp. 299–307, 1986. (Cited on page 69.)

[Me90] G. Mess. *Fuchsian groups with the same simple axes*, 1990. Preprint. (Cited on pages 10, 19 and 67.)

[Mo84] J. Morgan. *On Thurston's uniformization theorem for 3-dimensional manifolds*, (pp. 37–125). The Smith Conjecture(ed. by H. Bass and J. Morgan). Academic Press, 1984. (Cited on page 72.)

[MP08] E. Martinez-Pedroza. *On quasiconvexity and relatively hyperbolic structures on groups*, 2008. ArXiv:0811.2384. (Cited on pages 9, 18, 23, 27, 39, 49 and 54.)

[MP09] E. Martinez-Pedroza. *Combination of quasiconvex subgroups of relatively hyperbolic groups.* Groups, Geometry, and Dynamics, *vol 3*, pp. 317–342, 2009. (Cited on pages 8, 17, 32 and 49.)

[MPW10] E. Martinez-Pedroza & D. Wise. *Relative quasiconvexity using fine hyperbolic graphs*, 2010. ArXiv:1009.3532. (Cited on pages 29 and 30.)

[MT94] M. Mihalik & W. Towle. *Quasiconvex subgroups of negatively curved groups.* Pure and Applied Algebra, *vol 95*, pp. 297–301, 1994. (Cited on page 54.)

[Ol91] A. Olshanskii. *Periodic quotients of hyperbolic groups (in russian).* Mat. Sbornik, *vol 182*, no. 4, pp. 543–567, 1991. English translation in Math. USSR Sbornik 72 (1992), 2, 519-541. (Cited on page 43.)

[OOS09] A. Olshanskii, D. Osin & M. Sapir. *Lacunary hyperbolic groups.* Geom. Topol., *vol 13*, no. 4, pp. 2051–2140, 2009. With an appendix by Michael Kapovich and Bruce Kleiner. (Cited on pages 51 and 53.)

[Os06a] D. Osin. *Elementary subgroups of relatively hyperbolic groups and bounded generation.* Internat. J. Algebra Comput., *vol 16*, no. 1, pp. 99–118, 2006. (Cited on pages 22, 23, 39, 42, 75 and 78.)

[Os06b] D. Osin. *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*, vol 179. Mem. Amer. Math. Soc., 2006. (Cited on pages 1, 2, 3, 4, 6, 11, 12, 13, 15, 22, 25, 26, 28, 29, 30, 42, 46, 56, 75, 76 and 77.)

[Os07] D. Osin. *Peripheral fillings of relatively hyperbolic groups.* Inventiones Math., *vol 167*, no. 2, pp. 295–326, 2007. (Cited on pages 3 and 13.)

[Os10] D. Osin. *Small cancellations over relatively hyperbolic groups and embedding theorems.* Ann. Math., *vol 172*, no. 1, pp. 1–39, 2010. (Cited on pages 3 and 13.)

[Sa10] J. Sahattchieve. *On bounded packing in polycyclic groups*, 2010. ArXiv:1011.5953. (Cited on page 66.)

[SS92] P. Susskind & G. Swarup. *Limit sets of geometrically finite hyperbolic groups.* Amer. J. Math., *vol 114*, pp. 233–250, 1992. (Cited on pages 54, 67, 68, 70, 72 and 73.)

[Su01] P. Susskind. *An infinitely generated intersection of the hyperbolic group.* Proc. Amer. Math. Soc., *vol 129*, pp. 2643–2646, 2001. (Cited on pages 10, 19 and 67.)

[Th78] W. Thurston. *The geometry and topology of 3-manifolds*, 1978. Lecture Notes, Princeton Mathematics Department. (Cited on page 70.)

[Tu94] P. Tukia. *Convergence groups and gromov's metric hyperbolic spaces.* New Zealand J. Math., *vol 23*, pp. 157–187, 1994. (Cited on pages 4 and 14.)

[Tu98] P. Tukia. *Conical limit points and uniform convergence groups.* J. Reine. Angew. Math., *vol 501*, pp. 71–98, 1998. (Cited on pages 1, 2, 3, 5, 11, 12, 14, 45, 50 and 56.)

[WZ95] S. Wang & Q. Zhou. *On the proper conjugation of kleinian groups.* Geom. Dedicata, *vol 56*, pp. 145–154, 1995. (Cited on page 61.)

[Ya04] A. Yaman. *A topological characterisation of relatively hyperbolic groups.* J. reine ang. Math., *vol 566*, pp. 41–89, 2004. (Cited on page 45.)

[Ya10a] W. Yang. *Limit sets of relatively hyperbolic groups*, 2010. To appear in Geom. Dedicata. (Cited on page 53.)

[Ya10b] W. Yang. *Separable subgroups have bounded packing*, 2010. To appear in Proc. Amer. Math. Soc. (Cited on page 65.)

[YJ10] W. Yang & Y. Jiang. *Limit sets and commensurability of kleinian groups.* Bull. Aust. Math. Soc., *vol 82*, pp. 1–9, 2010. (Cited on pages 9, 19 and 67.)