# Université Lille 1 Sciences et Technologies 

 École Doctorale Sciences Pour l'Ingénieur Lille Nord-de-France
## THEXS

pour obtenir le titre de

## Docteur de l'Université Lille 1

Displine : MATHÉMATIQUES
Présentée par
Ying CHEN

## ENSEMBLES DE BIFURCATION DES POLYNÔMES MIXTES ET POLYĖDRES DE NEWTON

Bifurcation values of mixed polynomials and Newton polyhedra

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## Remerciements

Je tiens tout d'abord à remercier mon directeur de thèse, Prof. Mihai Tibăr pour avoir dirigé ce travail et pour m'avoir aidé à aller jusqu'au bout. Il a su me proposer des sujets intéressants qui m'ont permis d'étudier la Théorie de Singularité dans différents directions. J'ai beaucoup apprécié son enthousiasme et sa clairvoyance tout au long de ma thèse.
Ce fut un grand honneur pour moi que Messieurs Mutsuo Oka et Krzysztof Kurdyka aient accepté de rapporter cette thèse. Je les remercie de l'intérêt qu'ils ont porté à mon travail, ainsi que les autres membres du jury: Arnaud Bodin, Masaharu Ishikawa, Patrick Popescu-Pampu, Kiyoshi Takeuchi, qu'ils viennent de loin ou de près.
J'exprime toute ma reconnaissance aux nombreuses personnes qui m'ont soutenu et conseillé pendant ces années, Prof. Changgui Zhang pour ses encouragements et son écoute patient. Je remercie Renato Dias et Raimundo dos Santos qui m'ont donné beaucoup d'aide sur ce travail. Je n'oublierai pas de remercier aussi les membres du laboratoire Paul Painlevé et le personnel administratif pour leur aide précieuse. Durant ces années, j'ai également profité d'un environnement de travail très agréable. Je tiens à remercier tous mes collègue et mes amis pour les bons moments passés ensemble. Je pense en particulier à Qidi, TianWen, Chang, Zuqi, JianWei, WenYuan, Jorge, Saja, Merian, Faten, Ming Fu, Yi Liang, Xian Shi, Qi Zhang, etc.
Je garde une place toute particuliére pour ma famille. Je remercie mes parents proches et éloignés qui m'ont suivi et encouragé de loin. J'addresse enfin toute mon affection à Shuying qui m'a beaucoup soutenu.

## Résumé

Milnor a observé que pour les germes des application réelles, la fibration de Milnor sur la sphère n'existe pas en qénéral même avec la condition de singularité isolée, contrairement au cas des fonctions holomorphes. Récemment, Oka a introduit la terminologie de "fonction mixte" qui est une fonction polynomiale $\mathbb{C}^{n} \rightarrow \mathbb{C}$ en $\mathbf{z}$ et $\overline{\mathbf{z}}$, donc une application polynomiale réelle $\mathbb{R}^{2 n} \rightarrow \mathbb{R}^{2}$. Dans cette thèse, on s'intéresse aux propriétés des polynômes mixtes en comparaison avec les résultats connus pour le cas holomorphe. Le travail consiste en quatre parties.
Dans la première partie, on met l'accent sur l'étude des conditions de régularité à l'infini. On définit une conditon de régularité à l'infini plus générale que cette introduite par Rabier et par Kurdyka, Orro et Simon. On montre un théorème de fibration globale qui implique que l'ensemble de bifurcation pour un polynôme mixte est inclus dans un ensemble semi-algébrique fermé de dimension réelle inférieure ou égale à un.
Dans la deuxième partie, on définit le polyèdre de Newton à l'infini pour un polynôme mixte. On distingue deux notions de non-dégénérescence à l'infini qui sont équivalentes dans le cas holomorphe, une est apelée Newton non-dégénéré et l'autre est apelée Newton fortement non-dégénéré). Il s'avère que la condition de non-dégénéré est une condition semi-algébrique ouverte, mais que la condition de fortement non-dégénéré n'est pas dense ni connexe.
Dans la troisième partie, on généralise un théorème de Néméthi et Zaharia pour donner une approximation de l'ensemble de bifurcation dans le cas mixte. On prouve un théorème de stabilité pour la monodromie dans une famille de polynômes mixtes fortement non-dégénérés en supposant l'invariance des polyèdres de Newton. On obtient aussi l'analogue global des théorèmes de Néméthi et Zaharia et d'Oka sur l'existence de la fibration de Milnor à l'infini. Dans le cas local, on étudie plus en détail les polynômes mixtes polaires quasi-homogènes avec singularités non-isolées, par rapport à la condition de Thom.
Dans la dernière partie, on introduit une nouvelle définition de non-dégénéré cette fois-ci pour une application polynomiale mixte qui est plus générale que la définition de non dégénéré introduite par Bivià-Ausina dans le cas réel. En exploitant un résultat de Kurdyka, Orro et Simon, on trouve une extension du théorème de Bivià-Ausina en rapport avec la conjecture Jacobienne.
Mots Clés: Valeurs de bifurcation, polynôme mixte, polyèdre de Newton, nondégénéré, fibration de Milnor, conditions de régularité

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## CHAPTER 1

Introduction

The objective of this thesis is to study the global and local fibrations produced by a mixed polynomial map. The concept of Milnor fibration was proposed by John Milnor in his well-known Princeton lecture notes [Mil68], where he studied the local fibrations of complex holomorphic function germs $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow(\mathbb{C}, 0)$. At the end of this book, Milnor also explained the study to a real analytic map germ $g:\left(\mathbb{R}^{n}, 0\right) \rightarrow\left(\mathbb{R}^{p}, 0\right)$ with isolated singularity and pointed out the difference with the holomorphic case. This topic has been extended by many authors and provided some new viewpoints to other domains.
For a complex polynomial function $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$, it is well known that there is a locally trivial fibration $f_{\mid}: \mathbb{C}^{n} \backslash f^{-1}(\Lambda) \rightarrow \mathbb{C} \backslash \Lambda$ over the complement of some finite subset $\Lambda \subset \mathbb{C}$, see e.g. [Var72], [Ver76]. The minimal such $\Lambda$ is called the set of bifurcation values, or the set of atypical values, and shall be denoted by $B(f)$. The first approach to investigate this global fibration is due to Broughton [Bro, Bro88], who proved that under some regularity condition (called tame condition), the fiber is homotopic equivalent to a bouquet of spheres. This foundational work opened the way to further studies of regularity condition at infinity and the topology of Milnor fibration at infinity, see for example [ST95], [Par95], [HT97], [Tib99], [Tib07], etc. In two variables, Durfee [Dur98] and Tibăr [Tib99] give several characterizations of atypical values at infinity and show that all of them are equivalent, since only isolated singularities at infinity can occur in this setting. As determining the atypical values at infinity of a complex polynomial in higher dimensions is still an open problem, one looks for some significant finite set $A \supset B(f)$ which bounds $B(f)$ reasonably well. For instance, in case of non-convenient but still Newton non-degenerate polynomials, Némethi and Zaharia [NZ90] found an interesting approximation $A \supset B(f)$ in terms of certain faces of the support of $f$. This provides a large class of polynomials for which we control rather well the bifurcation locus. This point of view will be adopted in our work. In contrast to a complex polynomial function germ, the existence of Milnor fibration on the sphere for a real analytic map germ needs to be discussed with reference to some explicit classes of maps. Recently, Oka introduced the terminology "mixed function" $f(\mathbf{z}, \overline{\mathbf{z}}): \mathbb{C}^{n} \rightarrow \mathbb{C}$ of variables $\mathbf{z}$ and $\overline{\mathbf{z}}$ which is actually a real analytic map $f: \mathbb{R}^{2 n} \rightarrow \mathbb{R}^{2}$. Under certain non-degeneracy condition with respect to the local Newton boundary, he proved the existence and equivalence between Milnor tube fibration and Milnor fibration on the sphere for some classes of mixed function
germs (see [Oka10a] for more details). It is an interesting question whether we can get some global results. In the holomorphic case, this idea goes back at least as far as [NZ92] where Némethi and Zaharia studied the global Milnor fibration for semitame polynomials.
In this thesis, we investigate some regularity conditions at infinity for mixed polynomials. The study of regularity condition at infinity for real polynomial functions started with the work by Tibăr ([Tib99]) and for complex polynomial maps with the work of Gaffney [Gaf99] who generalized Malgrange Condition for holomorphic polynomial maps, and is receiving a great deal of attention. In 2000, Kurdyka, Orro and Simon [KOS00] introduced generalized critical values for semi-algebraic maps and improved Gaffney's result to a more general setting. In particular, it was shown in [Tib07] that in the setting of real or complex analytic functions, various regularity conditions could be compared. For instance, $\rho$-regularity condition implies the Malgrange condition. It is natural to try to interpret the regularity conditions in the setting of mixed polynomials. So we will characterize $\rho$-regularity condition and compare it with the regularity condition of [Rab97] and [KOS00]. This also leads us to a global fibration theorem for mixed polynomials. We will also focus on establishing certain notions related to Newton polyhedron at infinity and non-degeneracy conditions in this setting.

This thesis is organized as follows. In Chapter 2 and Chapter 6, we recall some of the standard facts on local Milnor fibrations and study the Milnor fibrations with non-isolated mixed singular locus. In Chapter 3, we will be concerned with the regularity at infinity of mixed polynomials. In Chapter 4 and 5, we proceed with the study of global behavior of non-degenerate mixed polynomials. In Chapter 7, we treat the case of non-degenerate analytic maps and apply this to the real Jacobian problem.

In what follows we shall briefly outline our main results.

Local fibrations of germs (Chapter 2, Chapter 6). In Chapter 2, we review some results for holomorphic polynomial germs and mixed function germs in the local case. This is the starting point of this thesis and also the preparation for Chapter 6. We also recall some basic definitions for mixed polynomials, for e.g. polar weighted homogeneous mixed polynomials and radial weighted homogeneous mixed polynomials. In Chapter 6, we study the existence of Milnor fibration for mixed function germs with non-isolated singular locus. After proving a mixed version of Łojasiewicz's inequality, we give an example to show that Thom regularity does not hold in general like in holomorphic case (see [HL73]). We generalize a result by Oka [Oka08] and Cisneros-Molina [CM08], which asserts that one has a local trivial fibration for polar weighted homogeneous mixed polynomials without the assumption of radial weighted homogeneity (see Theorem 6.3.4).

The proof of this result uses two facts which we prove. One is that $0 \in \mathbb{C}$ is an isolated critical value. The other is that in a neighborhood of $0 \in \mathbb{C}^{n}$, we have a Milnor's type transverse condition (see Definition 6.3.1). In [Mas10], Massey proved a fibration theorem for E -analytic map germs. Our Example 6.3.5 is not $£$-analytic but polar weighted homogeneous. Moreover, our theorem provides a new explicit class of mixed polynomial germs which have equivalent Milnor fibrations. We also emphasize that in general, a polar weighted homogeneous mixed polynomial germ may have non-isolated mixed singular locus.

Regularity at infinity of mixed polynomials (Chapter 3). In various contexts, regularity conditions at infinity are taken into account for trivializing the fibrations at infinity. The importance of studying these regularity conditions lies on the two following points: on one hand one, can use them to detect more easily the bifurcation locus; on the other hand, different regularity conditions yield different intrinsic geometric ingredients.
In Chapter 3, we introduce the non $\rho$-regular set of a mixed polynomial $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ which is the critical locus of the map $(f, \rho)$, where $\rho: \mathbb{R}^{2 n} \rightarrow \mathbb{R}_{\geq 0}$ is the Euclidean distance function. In order to prove a global fibration theorem like [KOS00, Theorem 3.1], we introduce the set of asymptotic non $\rho$-regular values and denote it by $S(f)$ (see Definition 3.2.4). By an extension of the Curve Selection Lemma at infinity, it is shown that $S(f)$ and $f(\operatorname{Sing} f) \cup S(f)$ are closed semi-algebraic sets. Our main result is the following:

Theorem 1.0.1 (Theorem 3.1.8).
Let $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ be a mixed polynomial. Then the restriction:

$$
f_{l}: \mathbb{C}^{n} \backslash f^{-1}(f(\operatorname{Sing} f) \cup S(f)) \rightarrow \mathbb{C} \backslash f(\text { Sing } f) \cup S(f)
$$

is a locally trivial $C^{\infty}$ fibration over each connected component of $\mathbb{C} \backslash(f(\operatorname{Sing} f) \cup$ $S(f))$. In particular $B(f) \subset f(\operatorname{Sing} f) \cup S(f)$.

In order to prove this theorem, we first take $c \notin f(\operatorname{Sing} f) \cup S(f)$. Since $\mathbb{C} \backslash$ $(f(\operatorname{Sing} f) \cup S(f))$ is an open set, there is a closed disk $D \subset \mathbb{C} \backslash f(\operatorname{Sing} f) \cup S(f)$ centered at $c$. The second step is to show that out of a sufficiently large sphere $B_{R_{0}}^{2 n}$, there is a trivial fibration over $D$. Since we do not know whether the restriction of $f$ on $f^{-1}(D) \backslash B_{R_{0}}^{2 n}$ is proper, we consider instead the restriction of the map $(f, \rho)$. Applying Ehresmann's theorem to this restriction and composing the map ( $f, \rho$ ) with the projection $\pi: D \times\left[R_{0}, \infty[\rightarrow D\right.$, we obtain the trivial fibration desired. In [KOS00], it was shown that $B(f) \subset f(\operatorname{Sing} f) \cup K_{\infty}(f)$ where $K_{\infty}(f)$ is the set of asymptotic critical values (see [KOS00, p. 76] for the definition of $K_{\infty}(f)$ ). To say that our theorem provides a better approximation than that of [KOS00], we show that $S(f) \subset K_{\infty}(f)$. Combining with $[K O S 00$, Theorem 3.1], we get that
the dimension of $S(f)$ is strictly less than 2 . On the other hand, the examples of Păunescu and Zaharia in [PZ97] assert that this inclusion is strict. Furthermore, by a modification of these examples in the case of mixed polynomials, we construct mixed polynomials with $S(f)=\emptyset$ and $\operatorname{dim} K_{\infty}(f)=1$.

Non-degeneracy at infinity and Milnor fibrations (Chapter 4, Chapter 5). In Chapter 4, inspired by Oka's construction for mixed functions, we consider the radial Newton polyhedron for a mixed polynomial $f$ with respect to the radial degree for every monomial. This allows us to set up the notion of Newton boundary at infinity as that for holomorphic polynomials. We denote the Newton polyhedron of $f$ by $\Gamma_{0}(f)$ and the Newton boundary at infinity by $\Gamma^{+}(f)$. Note that in the holomorphic case, every vertex of a Newton polyhedron represents a monomial. While for mixed polynomials, every vertex of a Newton polyhedron represents the family of monomials which have the same radial degree.
We also define non-degeneracy conditions at infinity called Newton non-degeneracy and Newton strong non-degeneracy (see Definition 4.2.2). For holomorphic polynomials, strong non-degeneracy is equivalent to non-degeneracy, while it turns out that these two non-degeneracy conditions are not equivalent for mixed polynomials. Consequently, this gives rise to the notions of bad face and strictly bad face (see Definition 4.2.3). Moreover, we prove that strong non-degeneracy condition is neither dense nor connected but still an open condition, while in the holomorphic setting, it follows from [Kus76, Oka79] that strong non-degeneracy condition is a "Zariski open" condition.
Along our construction, we obtain a real counterpart of Némethi and Zaharia's main result [NZ90, Theorem 2] as follows:

Theorem 1.0.2 (Theorem 4.1.3).
Let $f: \mathbb{R}^{2 n} \rightarrow \mathbb{R}^{2}$ be a mixed polynomial which depends effectively on all the variables and let $f(0)=0$. If $f$ is Newton non-degenerate at infinity then:
(a) $S(f) \subset\{0\} \cup \bigcup_{\Delta \in \mathcal{B}} f_{\Delta}\left(\operatorname{Sing} f_{\Delta} \cap \mathbb{C}^{* n}\right)$.
(b) If $f$ is moreover Newton strongly non-degenerate at infinity then $f(\operatorname{Sing} f)$ and $S(f)$ are bounded.
where $\mathfrak{B}$ is the set of bad faces of the support $\operatorname{supp}(f)$.
The proof involves some valuations at infinity coming from Curve Selection Lemma. It is worth pointing out that the second statement yields a global monodromy fibration at infinity for strongly non-degenerate mixed polynomials. As a consequence of the above theorem, for a non-degenerate and convenient mixed polynomial $f$, it follows that $S(f)=\emptyset$. Note that a special case for holomorphic polynomials was
proved in [Kus76, Bro88]. The section 5 of Chapter 4 is devoted to the study of the monodromy of mixed polynomials. In [NZ92], the authors proved that two holomorphic non-degenerate and convenient polynomials with the same Newton boundary at infinity have the same monodromy at infinity. Recently, Pham [Pha08] improved this result by dropping the convenient condition. In order to extend their results for mixed polynomials, we consider a family of strongly non-degenerate mixed polynomials with the same Newton boundary at infinity and prove that:

Theorem 1.0.3 (Theorem 4.5.2).
Let $F_{s}(\mathbf{z}, \overline{\mathbf{z}}):=F(\mathbf{z}, \overline{\mathbf{z}}, s): \mathbb{R}^{2 n} \rightarrow \mathbb{R}^{2}$ be a family of Newton strongly non-degenerate polynomials depending analytically on a parameter $s$, where $s \in[0,1]$. If the Newton boundary $\Gamma^{+}\left(F_{s}\right)$ is constant in this family, then the monodromy at infinity is stable.

The proof of this theorem is based on the analogues of [Pha08, Lemmas 3.2-3.5] and Ehresmann's theorem. The main application of this theorem is to detect the topology of generic fiber for a strongly non-degenerate mixed polynomial. A corollary of this theorem is that if two Newton strongly non-degenerate mixed polynomials with the same Newton boundary at infinity and their restrictions to the boundaries at infinity are both holomorphic (or both anti-holomorphic), then their monodromies at infinity are isotopic. This gives a slight generalization of [NZ92, Pha08].
For the sake of deriving different global behaviors to holomorphic polynomials, some relevant counterexamples are indicated at the end of Chapter 4. For instance, we construct the example 4.6.7 to deduce that [NZ90, Proposition 6] is not adapted to mixed polynomials.

Chapter 5 is devoted to investigate the existence of Milnor fibration $\frac{f}{|f|}$ at infinity for some classes of mixed polynomials. If one makes the definition of semitame mixed polynomial in the same manner as [NZ92], then the example 5.3.3 presents that even if under the condition $S(f)=\emptyset$, the Milnor fibration on the sphere at infinity does not exist in general. Due to Oka's construction for non-degenerate mixed functions, it is our hope to treat this problem under certain non-degeneracy condition. We prove an analogue of Theorem 4.1.3, which gives an approximation of asymptotic non $\rho$-regular values of $\frac{f}{|f|}$. Our main theorem states that:

Theorem 1.0.4 (Theorem 5.1.4).
If $f$ is a Newton strongly non-degenerate mixed polynomial, then $\exists \delta_{0}>0$ and $R_{0}>0$ sufficient large such that for any $\delta \geq \delta_{0}$ and $R>R_{0}$

$$
\frac{f}{|f|}: S_{R}^{2 n-1} \backslash f^{-1}\left(D_{\delta}\right) \longrightarrow S^{1}
$$

is a locally trivial fibration for $R \geq R_{0}$ and is equivalent to the global fibration

$$
f_{\mid}: f^{-1}\left(S_{\delta}^{1}\right) \rightarrow S_{\delta}^{1} .
$$

The proof is done by developing the technics of Oka for mixed function germs. In our situation, it is also required to apply Theorem 4.1.3 for constructing a vector field to which the associated flow gives the above equivalence. In [NZ92], Némethi and Zaharia also observed a similar type of equivalent fibrations for semitame holomorphic polynomials. Since under the assumption of semitameness, the set $S(f)$ is contained in 0 , one can take a small disk centered at the origin and consider the restriction of $\frac{f}{|f|}$ to a big enough sphere away from the fibers over this small disk. As a special case of the above theorem, we prove the following corollary which can be regarded as a global version of [Oka10a, Theorem 29, 33, 36]:

Corollary 1.0.5 (Corollary 5.1.5).
If $f$ is a Newton strongly non-degenerate convenient mixed polynomial, then there exists $R_{0}>0$ sufficient large such that for all $R \geq R_{0}$ the Milnor fibration at infinity

$$
\left.\frac{f}{|f|} \right\rvert\,: S_{R}^{2 n-1} \backslash K \longrightarrow S^{1}
$$

exists and is equivalent to the global fibration

$$
f_{l}: f^{-1}\left(S_{\delta}^{1}\right) \rightarrow S_{\delta}^{1}
$$

where $\delta>0$ is sufficient large.
Note that in these two assertions, strong non-degeneracy condition can not be replaced by non-degeneracy condition.

Generalization of non-degeneracy (Chapter 7). Chapter 7 is based on the work of Chapter 4 and provides an extension of our results for non-degenerate mixed polynomials to non-degenerate mixed polynomial maps. We have worked out this problem together with Renato Dias. He develops in his thesis the notion of nondegeneracy for real polynomial maps. The Newton non-degeneracy condition for a polynomial map was first introduced by Khovanskii in [Kho77]. Since we wish to estimate the bifurcation locus by some significant faces, instead of Khovanskii's definition, we make another characterizations of non-degeneracy at infinity for real and mixed polynomial maps (see Definition 7.2.1 for more details). Our construction is based on the study of mixed non-degenerate polynomials.
Let $F=\left(f_{1}, \ldots, f_{k}\right): \mathbb{C}^{n} \rightarrow \mathbb{C}^{k}$ be a mixed polynomial map. Recall the notation $K_{\infty}(F)$ for the set of asymptotic critical values of $F$. Our main result states that:

Theorem 1.0.6 (Theorem 7.1.1).
If $F$ is non-degenerate at infinity with $F(0)=0$ and depends effectively on all the variables, then:

$$
K_{\infty}(F) \subset \mathbb{C}^{k} \backslash \mathbb{C}^{* k} \cup \underset{\Delta \in \mathfrak{B}(F)}{\cup} F_{\Delta}\left(\operatorname{Sing} F_{\Delta} \cap \mathbb{C}^{* n}\right)
$$

where $\mathfrak{B}(F)$ is the set of bad faces of $F$.
The proof is similar in spirit to that of non-degenerate mixed polynomials. To avoid dealing with $\rho$-regularity, we use KOS-regularity which is more convenient. In particular, we have:

Corollary 1.0.7 (Corollary 7.3.3).
Suppose that $F$ is non-degenerate at infinity and that $f_{i}$ is convenient, for all $i=$ $1, \ldots, k$. Then $K_{\infty}(F)=\emptyset$.

One particular application of this corollary is in the study of the real Jacobian problem. Using corollary above, we obtain an application to the real Jacobian problem from the non-degeneracy viewpoint.

Theorem 1.0.8 (Theorem 7.4.8).
Let $F: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ be a mixed polynomial map such that $J(F)(\mathbf{x}) \neq 0$, for all $\mathbf{x} \in \mathbb{C}^{n}$ If $F$ is non-degenerate at infinity (under Definition 7.2.1) and if $f_{i}$ is convenient, for all $i=1, \ldots, n$, then $F$ is a homeomorphism.

In [Aus07], Bivià-Ausina made another definition of non-degeneracy at infinity for real polynomials. Under his definition of non-degeneracy, he found that for a nondegenerate and convenient real polynomial map $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ if the Jacobian matrix $J(F) \neq 0$, then $F$ is a global diffeomorphism. In the real setting, our definition of non-degeneracy turns out to be equivalent to Bivià-Ausina's definition when $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ and $f_{i}$ is convenient for all $i=1, \ldots, n$. The interest of the above theorem lies in the fact that for a convenient mixed polynomial, if one consider it as a real polynomial map, it might not be convenient. We also observe that in general our definition of non-degeneracy is weaker than Bivià-Ausina's definition, namely our class of non-degenerate polynomials is larger than his (see for instance Example 7.4.11).

## Chapter 2

## Local fibrations for germs of functions

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### 2.1 Introduction

In this chapter, our purpose is to give a brief summary of the singularity theory for mixed functions. The study of fibrations for germs of functions stems from Milnor's classical fibration theorem. Even he considered only isolated singularities, however his viewpoints sheds some new lights on the study of topological properties for algebraic singularities. For the convenience, we restate his result without proof. Consider a holomorphic function germ

$$
f:\left(\mathbb{C}^{n}, 0\right) \rightarrow(\mathbb{C}, 0)
$$

with an isolated singular point at 0 . Let $V=f^{-1}(0)$ and $K_{\varepsilon}=V \cap S_{\varepsilon}$ the link of singularity at 0 , where $S_{\varepsilon}$ is the real $(2 n-1)$ sphere centered at 0 with the radial $\varepsilon$ small enough. In the book [Mil68], Milnor's classical fibration theorem says that we have two types of locally trivial fibrations and we call these fibrations Milnor fibrations. The first Milnor fibration is the following:

$$
\begin{equation*}
\varphi:=\frac{f}{|f|}: S_{\varepsilon}^{2 n-1} \backslash K_{\varepsilon} \longrightarrow S^{1} \tag{2.1.1}
\end{equation*}
$$

and the second Milnor fibration is:

$$
\begin{equation*}
f: f^{-1}\left(S_{\delta}\right) \cap B_{\varepsilon}^{2 n} \rightarrow S_{\delta} \tag{2.1.2}
\end{equation*}
$$

which is obtained from the restriction of Milnor tube fibration:

$$
f: f^{-1}\left(D_{\delta}^{*}\right) \cap B_{\varepsilon}^{2 n} \rightarrow D_{\delta}^{*}
$$

where $B_{\varepsilon}^{2 n}$ is an open ball centered at 0 with the radial $\varepsilon$ small enough and $D_{\delta}^{*}$ is a punctured disk with the radial $\delta \ll \varepsilon$. In the first fibration (2.1.1), the closure of the fiber $\varphi^{-1}\left(e^{i \theta}\right)$ is a compact $(2 n-2)$ real manifold with boundary equal to $K_{\varepsilon}$ and the link $K_{\varepsilon}$ is $(n-2)$-connected. In the second fibration (2.1.2), the closure of the fiber $f^{-1}(y) \cap B_{\varepsilon}^{2 n}$ is also a compact $(2 n-2)$ real manifold with boundary equal to $f^{-1}(y) \cap S_{\varepsilon}^{2 n-1}$. Moreover, it was shown that these two Milnor fibrations are equivalent and the boundaries of the fibers are isotopic.

In higher dimensions, Milnor's fibration theorems were later generalized in [Ham71] for complete intersections, and also for holomorphic functions defined on complex varieties with an isolated singularity. More generally, since holomorphic functions have Thom $a_{f}$-property [HL73, Hir77], Lê showed that:

Theorem 2.1.1 [Lêrry] Let $(X, 0)$ be the germ of a complex analytic variety in $\mathbb{C}^{n}$ and $f:(X, 0) \rightarrow(\mathbb{C}, 0)$ holomorphic. Then for any open ball $B_{\varepsilon}$ centered at 0 with sufficiently small radius $\varepsilon$, there exists $0<\delta \ll \varepsilon$ such that:

$$
f: f^{-1}\left(S_{\delta}\right) \cap B_{\varepsilon} \rightarrow S_{\delta}
$$

is a locally trivial fibration which does not depend on the choice of $\varepsilon$ and $\delta$.
We call this the Milnor-Le fibration of $f$.
In the book [Mil68], Milnor also considered the isolated singularities for real analytic germs. His main result is:

Theorem 2.1.2 [Mil68] Let $f:\left(\mathbb{R}^{n}, 0\right) \rightarrow\left(\mathbb{R}^{m}, 0\right)$ a real analytic germ. Suppose $f$ verifies Milnor condition, namely $f$ is a submersion on a punctured neighborhood of 0 . Then the complement of an open tubular neighborhood of $K_{\varepsilon}=V \cap S_{\varepsilon}^{n-1}$ is the total space of a smooth fiber bundle over the sphere $S_{\delta}^{m-1}$. Each fiber $F$ is a smooth compact $(n-m)$-dimensional manifold bounded by a copy of $K_{\varepsilon}$.

The proof was done by the same method as in the complex setting. However he gave an example $f\left(x_{1}, x_{2}\right)=\left(x_{1}, x_{1}^{2}+x_{2}\left(x_{1}^{2}+x_{2}^{2}\right)\right)$ which shows that under this hypothesis, the first Milnor fibration (2.1.1) does not exist. It is worth pointing out that the entire complement $S_{\varepsilon}^{n-1} \backslash K_{\varepsilon}$ also fibers over $S^{m-1}$ and each fiber is the interior of a compact manifold with boundary equal to $K_{\varepsilon}$. We say that a real mapping germ $f$ satisfies the strong Milnor fibration if and only if the two fibrations (2.1.1) and (2.1.2) exist. In [Jac89], considering a real analytic map $f=(g, h)$, Jacquemard proposed the notion of Jacquemard condition and proved the strong Milnor fibration for germs which verify this condition. Later, Ruas and Dos Santos improved the result in [Jac89]. They proved the strong Milnor
fibration with Bekka's (c)-regularity and Jacquemard condition is equivalent to $(w)$-regularity. For higher dimensions, in [AT10], Tibăr and Dos Santos considered the open book structure with non isolated singularities and gave a criterion of the strong Milnor condition under the assumption $\operatorname{Sing} f \cap f^{-1}(0) \subset\{0\}$. Another approach is using the strong Łojasiewicz inequality introduced by Massey [Mas10]. He proved a fibration theorem for Ł -analytic mappings. Therefore, it is of interest to know more explicit classes of real mappings germs which have Milnor fibrations and good topological properties.

The structure of the chapter is as follows. In Section 2, we review some of the
standard facts of mixed functions in the locally setting. Due to Oka's observation of mixed singularity, we can compute these singularities explicitly for any mixed polynomial. In addition, we recall two types of mixed homogeneous polynomials and the properties of these polynomials. In Section 3, we state the outline of Oka's construction for non-degenerate mixed functions, which provides another view to study the real analytic mapping germs. This is also the starting point of our work.

### 2.2 Mixed singularity and homogeneous polynomials

In a recent series of papers [Oka08, Oka10a, Oka09, Oka10b, Oka11], Oka has studied some subclasses of mixed polynomial germs $\left(\mathbb{C}^{n}, 0\right) \rightarrow(\mathbb{C}, 0)$. The terminology "mixed polynomial" was introduced by M.Oka, but the concept also appears in the work by $A^{\prime}$ Campo [A'C73]. In [PS08], the authors studied the fibered multilink and singularities $f \bar{g}$ which can be also considered as a subclass of mixed polynomial germs. The basic idea of these constructions is using the singularity theory of holomorphic function germs to study some subclasses of mixed polynomial germs.

Let $f:=(g, h): \mathbb{R}^{2 n} \rightarrow \mathbb{R}^{2}$ be a polynomial application, where $g\left(x_{1}, \ldots, y_{n}\right)$ and $h\left(x_{1}, \ldots, y_{n}\right)$ are real polynomials. By writing $\mathbf{z}=\mathbf{x}+i \mathbf{y} \in \mathbb{C}^{n}$, where $z_{k}=x_{k}+i y_{k}$ for $k=1,2 \ldots n$, we get a polynomial function $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ in variables $\mathbf{z}$ and $\overline{\mathbf{z}}$, namely $f(\mathbf{z}, \overline{\mathbf{z}}):=g\left(\frac{\mathbf{z}+\overline{\mathbf{z}}}{2}, \frac{\mathbf{z}-\overline{\mathbf{z}}}{2 i}\right)+i h\left(\frac{\mathbf{z}+\overline{\mathbf{z}}}{2}, \frac{\mathbf{z}-\overline{\mathbf{z}}}{2 i}\right)$, and reciprocally for a polynomial function $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ in variables $\mathbf{z}$ and $\overline{\mathbf{z}}$, we can consider it as a polynomial application $(\operatorname{Re} f, \operatorname{Im} f)$. Then $f$ is called a mixed polynomial. We write $f$ as follows:

$$
\begin{equation*}
f(\mathbf{z}, \overline{\mathbf{z}})=\sum_{\nu, \mu} c_{v, \mu} \mathbf{z}^{\nu} \overline{\mathbf{Z}}^{\mu} \tag{2.2.1}
\end{equation*}
$$

where $c_{v, \mu} \neq 0, \mathbf{z}^{\nu}:=z_{1}^{v_{1}} \cdots z_{n}^{v_{n}}$ and $\overline{\mathbf{z}}^{\mu}:=\bar{z}_{1}^{\mu_{1}} \cdots z_{n}^{\mu_{n}}$ for n-tuples $v=\left(v_{1}, \ldots, v_{n}\right)$, $\mu=\left(\mu_{1}, \ldots, \mu_{n}\right) \in \mathbb{N}^{n}$. In the sequel, given a mixed polynomial $f$, we consider $f$ as in the form of equation (2.2.1).

For a mixed polynomial $f$, we use the notation:

$$
\mathrm{d} f:=\left(\frac{\partial f}{\partial z_{1}}, \cdots, \frac{\partial f}{\partial z_{n}}\right), \overline{\mathrm{d}} f:=\left(\frac{\partial f}{\partial \bar{z}_{1}}, \cdots, \frac{\partial f}{\partial \bar{z}_{n}}\right)
$$

Definition 2.2.1 We call $w$ a mixed singularity of $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$, if $w$ is a critical point of the mapping $f:=(g, h): \mathbb{R}^{2 n} \rightarrow \mathbb{R}^{2}$.

From the above definition, if a fiber of $f$ is mixed and non singular, then it has real codimension 2. Note that a mixed singular point of $f$ is not always a singular point of its fiber. For example, every point of $S^{2 n-1}=\left\{\left|z_{1}\right|^{2}+\cdots+\left|z_{n}\right|^{2}=1\right\}$ is a mixed singularity of $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ where $f(\mathbf{z}, \overline{\mathbf{z}})=\|\mathbf{z}\|^{2}$. By abuse of notation, we continue to denote the set of mixed singularities for a mixed polynomial $f$ by $\operatorname{Sing} f$.

The next proposition give us a straight way to calculate the locus of mixed singularity.

Proposition 2.2.2 [Oka08, Proposition 1] Let $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ be a mixed polynomial. Then $w \in \mathbb{C}^{n}$ is a mixed singularity of $f$ if and only if there exists a complex number $\lambda$ with $|\lambda|=1$ such that $\overline{\mathrm{d} f}=\lambda \overline{\mathrm{d}} f$.

In particular, it is easily seen from above proposition that the notion of mixed singularity coincides with the definition of singularity for a holomorphic function.

EXAMPLE 2.2.3 Let $f: \mathbb{C}^{2} \rightarrow \mathbb{C}, f\left(z_{1}, z_{2}\right)=z_{1}+z_{2}+\bar{z}_{1}^{2}+\bar{z}_{2}^{2}$. We have $\overline{\mathrm{d} f}=(1,1)$ and $\overline{\mathrm{d}} f=\left(2 \bar{z}_{1}, 2 \bar{z}_{2}\right)$. Suppose $\lambda \in \mathbb{C}$ with $|\lambda|=1$. The solutions of the equation $\overline{\mathrm{d} f}=\lambda \overline{\mathrm{d}} f$ are $\left\{z_{1}=z_{2}=\frac{\lambda}{2}\right\}$. By Proposition 2.2.2, hence $f(\operatorname{Sing} f)=\left\{\lambda+\frac{1}{2 \lambda^{2}}\right\}$ which is a simple closed curve with three cusps $-\frac{3}{4}+i \frac{3 \sqrt{3}}{4},-\frac{3}{4}-i \frac{3 \sqrt{3}}{4}$ and $\frac{3}{2}$. (See the following picture).

In the mixed setting, there are two definitions of homogeneous polynomials.
Definition 2.2.4 A mixed polynomial $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ is called radial weighted homogeneous if there exist $n$ integers $q_{1}, \ldots, q_{n}$ with $\operatorname{gcd}\left(q_{1}, \ldots, q_{n}\right)=1$ and a positive integer $m_{r}$ such that $\sum_{j=1}^{n} q_{j}\left(v_{j}+\mu_{j}\right)=m_{r}$ for every n-tuples $\nu$ and $\mu$. We call $\left(q_{1}, \ldots, q_{n}\right)$ the radial weight of $f$ and $m_{r}$ the radial degree of $f$. More precisely, $f$ is radial weighted homogeneous of type $\left(q_{1}, \ldots, q_{n} ; m_{r}\right)$ if and only if it verifies the following equation for all $t \in \mathbb{R}^{*}=\mathbb{R} \backslash\{0\}$ :

$$
f(t \circ \mathbf{z})=\left(t^{q_{1}} z_{1}, \ldots, t^{q_{n}} z_{n}, t^{q_{1}} \bar{z}_{1}, \ldots, t^{q_{n}} \bar{z}_{n}\right)=t^{m_{r}} f(\mathbf{z}, \overline{\mathbf{z}})
$$

From Definition 2.2.4, we see that if $f:=(g, h): \mathbb{R}^{2 n} \rightarrow \mathbb{R}^{2}$ is a radially weighted homogeneous mixed polynomial, then $g$ and $h$ are real weighted homogeneous polynomials with the same weights and degrees as $f$.


Figure 2.1: Figure of Critical values

Definition 2.2.5 A mixed polynomial $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ is called polar weighted homogeneous if there exist $n$ integers $p_{1}, \ldots, p_{n}$ with $\operatorname{gcd}\left(p_{1}, \ldots, p_{n}\right)=1$ and a positive integer $m_{p}$ such that $\sum_{j=1}^{n} p_{j}\left(v_{j}-\mu_{j}\right)=m_{p}$ for every n-tuples $\nu$ and $\mu$. We call $\left(p_{1}, \ldots, p_{n}\right)$ the polar weight of $f$ and $m_{r}$ the polar degree of $f$. More precisely, $f$ is polar weighted homogeneous of type $\left(p_{1}, \ldots, p_{n} ; m_{p}\right)$ if and only if it verifies the following equation for all $\lambda \in S^{1}$ :

$$
f(\lambda \circ \mathbf{z})=f\left(\lambda^{p_{1}} z_{1}, \ldots, \lambda^{p_{n}} z_{n}, \lambda^{-p_{1}} \bar{z}_{1}, \ldots, \lambda^{-p_{n}} \bar{z}_{n}\right)=\lambda^{m_{p}} f(\mathbf{z}, \overline{\mathbf{z}})
$$

EXAMPLE 2.2.6 Let $f, g: \mathbb{C}^{2} \rightarrow \mathbb{C}, f(x, y)=|x|^{2}+|y|^{2}$ and $g(x, y)=x^{2}+x^{4} \bar{y}^{2}+y^{2}$. We see that $f$ is a radial weighted homogeneous polynomial of radial weight $(1,1)$ and degree 2 , but $f$ is not polar weighted homogeneous. $g$ is a polar weighted homogeneous polynomial of polar weight $(1,1)$ and degree 2 , but $g$ is not radial weighted homogeneous.

REmARK 2.2.7 If $f$ is a holomorphic weighted polynomial, then $f$ is radial and polar weighted homogeneous. Moreover, the radial weight is equal to the polar weight and the radial degree is equal to the polar degree. Conversely, if $f$ is a radial and polar weighted homogeneous mixed polynomial with the same weights, then $f$ is a holomorphic weighted polynomial.

The following theorem implies that $f(\operatorname{Sing} f) \subset\{0\}$. Let $V(f)=f^{-1}(0)$ and
$K_{\varepsilon}=S_{\varepsilon}^{2 n-1} \cap V(f)$, where $S_{\varepsilon}^{2 n-1}$ is a real $(2 n-1)$ dimensional sphere with radius $\varepsilon$.

Theorem 2.2.8 [RSV02] If $f$ is a radial and polar weighted homogeneous mixed polynomial of radial weight type $\left(q_{1}, \ldots, q_{n} ; m_{r}\right)$ and polar weight type of $\left(p_{1}, \ldots, p_{r} ; m_{p}\right)$, not necessarily $m_{r}=m_{p}$, then

$$
f: \mathbb{C}^{n} \backslash V(f) \rightarrow \mathbb{C} \backslash\{0\}
$$

is a locally trivial fibration. In particular, the geometric monodromy maph:F $\rightarrow F$ is given by $h(\mathbf{z})=\left(z_{1} \exp \left(\frac{2 p_{1} \pi i}{m_{p}}\right), \ldots, z_{n} \exp \left(\frac{2 p_{n} \pi i}{m_{p}}\right)\right)$, where $F=f^{-1}(1)$.
From the definitions of radial and polar weighted homogeneous mixed polynomials, the above theorem can be shown by using $\mathbb{R}^{*} \times S^{1}$-action. A similar result have been obtained independently by Oka, see [Oka08, Proposition 2].

Proposition 2.2.9 [CM08, Oka08] Let $f$ be a radial and polar weighted homogeneous mixed polynomial. Then the map

$$
\varphi:=\frac{f}{|f|}:\left(S_{\varepsilon}^{2 n-1} \backslash K_{\varepsilon}\right) \rightarrow S^{1}
$$

is a locally trivial fibration for any $\varepsilon>0$, which is equivalent to the locally trivial fibration

$$
f: f^{-1}\left(S^{1}\right) \rightarrow S^{1}
$$

In fact, the local triviality of $\varphi$ can be obtained directly by using polar action as follow (see [Oka10a, subsection 5.4]):

$$
\begin{align*}
\psi & \left.: \varphi^{-1}(\theta) \times\right] \theta-\pi, \theta+\pi\left[\rightarrow \varphi^{-1}(] \theta-\pi, \theta+\pi[),\right.  \tag{2.2.2}\\
\psi(\mathbf{z}, \theta+\eta) & :=\left(z_{1} \exp \left(\frac{2 p_{1} \eta i}{m_{p}}\right), \ldots, z_{n} \exp \left(\frac{2 p_{n} \eta i}{m_{p}}\right)\right) .
\end{align*}
$$

### 2.3 Newton boundary and mixed non-degenerate function germs

In this section, we recall some definitions and results by Oka [Oka10a] in the local setting. We adopt the conventions that $\mathbb{C}^{*}=\mathbb{C} \backslash\{0\}, \mathbb{C}^{* n}=(\mathbb{C} \backslash\{0\})^{n}$ and $\mathbb{R}^{+n}=\left(\mathbb{R}_{\geq 0}\right)^{n}$.

Consider a mixed analytic function $f: \mathbb{C}^{n} \rightarrow \mathbb{C}, f(\mathbf{z}, \overline{\mathbf{z}})=\sum_{\nu, \mu} c_{v, \mu} \mathbf{z}^{\nu} \overline{\mathbf{z}}^{\mu}$ with $f(0)=0$. The radial Newton polyhedron $\Gamma^{0}(f)$ at the origin is defined by the convex hull of $\bigcup_{c, \mu \neq 0}(v+\mu)+\mathbb{R}^{+n}$. Let us mention that each point of the polyhedron correspond finitely many monomials with the same radial degree. The Newton
boundary at the origin denoted by $\Gamma_{+}(f)$, is the union of the compact faces of $\Gamma^{0}(f)$. Given a vector $\mathbf{P}=\left(p_{1}, \ldots, p_{n}\right)$ with $p_{i} \geq 0$ for all $i$, there exists a face denoted by $\triangle_{\mathbf{P}}$ such that the linear function $l_{\mathbf{P}}:=\sum_{i=1}^{n} p_{i}\left(v_{i}+\mu_{i}\right)$ takes its minimal value $d_{\mathbf{P}}$ on $\Gamma_{+}(f)$. The face function $f_{\triangle_{\mathbf{P}}}$ is defined by $f_{\triangle_{\mathbf{P}}}(\mathbf{z}, \overline{\mathbf{z}})=\sum_{v+\mu \in \triangle_{\mathbf{P}}} c_{v, \mu} \mathbf{z}^{\nu} \overline{\mathbf{Z}}^{\mu}$. In fact, $f_{\triangle_{\mathbf{P}}}$ is a radial weighted mixed homogeneous polynomial with radial degree $d_{\mathbf{P}}$ and weights $\left(p_{1}, \ldots, p_{n}\right)$. We say $f$ is convenient if $\Gamma_{+}(f)$ intersects each coordinate axis.

Definition 2.3.1 Let $\mathbf{P}=\left(p_{1}, \ldots, p_{n}\right)$ with $p_{i}>0$ for all $i$, we say $f$ is nondegenerate for $\mathbf{P}$, if $\operatorname{Sing} f_{\triangle_{\mathbf{P}}} \cap f_{\triangle_{\mathbf{P}}}^{-1}(0) \cap \mathbb{C}^{* n}=\emptyset$. In particular, if Sing $f_{\triangle_{\mathbf{P}}} \cap \mathbb{C}^{* n}=$ $\emptyset$, we say $f$ is strongly non-degenerate for $\mathbf{P}$. A mixed function is called nondegenerate (resp. strongly non-degenerate) if $f$ is non-degenerate (resp. strongly non-degenerate) for any strictly positive weight vector $\mathbf{P}$.

REmark 2.3.2 Since $\Gamma_{+}(f)$ is consist of finitely many faces which coincide with the set of strictly positive weight vectors, the non-degenerate condition could be explicitly justified on every face of $\Gamma_{+}(f)$.

EXAMPLE 2.3.3 ${ }^{1}$ Let $f(\mathbf{z}, \overline{\mathbf{z}})=\sum_{i=1}^{n} z_{i}^{a_{i}+b_{i}} \bar{z}_{i}^{b_{i}}, \quad a_{i}$ and $b_{i}$ are positive integers for $i=1, \ldots, n$. The Newton polyhedron $\Gamma^{0}(f)$ is the convex hull of $\underset{1 \leq i \leq n}{\cup}(0, \ldots, \underbrace{a_{i}+2 b_{i}}_{i \text { th }}, \ldots, 0)+\mathbb{R}^{+n}$ and the Newton boundary at the origin $\Gamma_{+}(f)$ is simply the convex hull of $\underset{1 \leq i \leq n}{\cup}(0, \ldots, \underbrace{a_{i}+2 b_{i}}_{i \text { th }}, \ldots, 0)$. Since $a_{i}>0$, by Definition 2.3.1, we see that $f$ is strongly non-degenerate. If we suppose that for some $i$, we have $a_{i}=0$, then $f$ is non-degenerate but not strongly po non-degenerate.
Oka proved that for a strongly non-degenerate convenient mixed function $f$, the origin is an isolated mixed singularity. Moreover, he proved following fibration theorem:

Theorem 2.3.4 [Oka10a, Theorem 29, 33, 36] Assume that $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow(\mathbb{C}, 0)$ is a strongly non-degenerate convenient mixed function. There exist positive numbers $r_{0}, \delta_{0}$ and $\delta \ll \delta_{0}$, such that for any $r \leq r_{0}$, we have: $f: f^{-1}\left(D_{\delta}^{*}\right) \cap B_{r}^{2 n} \rightarrow D_{\delta}^{*}$ is locally trivial fibrations and the topological isomorphism class does not depend on the choice of $r$ and $\delta_{0}$. Moreover, $\varphi:=\frac{f}{|f|}: S_{r}^{2 n-1} \backslash K_{r} \longrightarrow S^{1}$ is also a locally trivial fibration which is equivalent to the fibration $f: f^{-1}\left(S_{\delta}\right) \cap B_{r}^{2 n} \rightarrow S_{\delta}$.

Let us give a brief sketch of Oka's proof. At first, it was shown that for any $\eta \neq 0$ $\|\eta\| \ll \delta_{0}$, the fiber $f^{-1}(\eta)$ has no mixed singularities inside the ball $B_{r_{0}}^{2 n}$. Therefore, Milnor tube fibration of $f$ exits. On the other hand, by proving that $\varphi$ has no critical

[^0]points on $S_{r}^{2 n-1} \backslash K_{r}$, he constructed a vector field on $S_{r}^{2 n-1} \backslash K_{r}$ and showed the local triviality by the integration. Finally, motivated by Milnor's technique of inflating the empty tube $f^{-1}\left(S_{\delta}\right) \cap B_{r}^{2 n}$ to $S_{r}^{2 n-1} \backslash K_{r}$, he proved the equivalence of the two fibrations. However the proof is much more complicated compare with the case of holomorphic functions. Since in his situation, the two normal vectors of the fibers are not always perpendicular. He needed a more delicate argument to control the vector field. We will also use this construction at infinity in chapter 5 .

In Section 7 of [Oka10a], Oka introduced another definition of non-degeneracy to deal with the case for non convenient mixed functions.

In general, if we consider a positive vector $\mathbf{P}=\left(p_{1}, \ldots, p_{n}\right)$, where $I(\mathbf{P})=$ $\left\{i \mid p_{i}=0\right\}$ and $J(\mathbf{P})=\left\{j \mid p_{j}>0\right\}$. The face function $f_{\triangle_{\mathbf{P}}}(\mathbf{z}, \overline{\mathbf{z}})$ is in fact a mixed polynomial in variables $z_{j}$ for $j \in J(\mathbf{P})$ with the other $z_{i}$ constant for $i \in I(\mathbf{P})$. Thus this vector defines a family of mixed polynomial functions in $\mathbf{z}_{J\{\mathbf{P}\}}$ with coefficient in $\mathbb{C}\left\{\mathbf{z}_{I(\mathbf{P})}, \overline{\mathbf{z}}_{I(\mathbf{P})}\right\}$. For abbreviation, we write $\mathbb{C}^{* J(\mathbf{P})}\left(\mathbf{w}_{I(\mathbf{P})}\right)$ instead of $\left\{\mathbf{z} \in \mathbb{C}^{* n} \mid \mathbf{z}_{I(\mathbf{P})}=\mathbf{w}_{I(\mathbf{P})}\right\} \cong \mathbb{C}^{* J(\mathbf{P})}$.

Definition 2.3.5 We say $f$ is super strongly non-degenerate if the following condition satisfied:

For any positive vector $\mathbf{P}$,
(a). If $d_{\mathbf{P}}=0$, then $f_{\triangle_{\mathbf{P}}}(\mathbf{z}, \overline{\mathbf{z}}) \in \mathbb{C}\left\{\mathbf{z}_{I(\mathbf{P})}, \overline{\mathbf{z}}_{I(\mathbf{P})}\right\}$;
(b). If $d_{\mathbf{P}}>0$, for any $\mathbf{w}_{I(\mathbf{P})} \in \mathbb{C}^{* I(\mathbf{P})}, f_{\triangle_{\mathbf{P}}}: \mathbb{C}^{* J(\mathbf{p})}\left(\mathbf{w}_{I(\mathbf{P})}\right) \rightarrow \mathbb{C}^{*}$ has no critical points.

An immediate consequence is that every convenient strongly non-degenerate mixed function is super strongly non-degenerate. The interest of the above definition is that it allows one to establish the following fibration theorem for non isolated singularities.

Theorem 2.3.6 [Oka10a, Theorem 52] Assume that $f$ is a super strongly nondegenerate mixed function. Then there exists $r_{0}>0$, such that for any $r$ with $0<$ $r \leq r_{0}$, and a sufficiently small number $\delta$ (compared with $r$ ) we have two equivalent fibrations:

$$
\begin{align*}
& f: f^{-1}\left(S_{\delta_{0}}\right) \cap B_{r}^{2 n} \rightarrow S_{\delta_{0}}  \tag{2.3.1}\\
& \varphi=\frac{f}{|f|}: S_{r}^{2 n-1} \backslash K_{r} \longrightarrow S^{1} \tag{2.3.2}
\end{align*}
$$

where $K_{r}=f^{-1}(0) \cap S_{r}^{2 n-1}$.
EXAMPLE 2.3.7 Let $f: \mathbb{C}^{2} \rightarrow \mathbb{C}, f(x, y)=\left(2 x^{2}+\|x\|^{2}\right) y$. The Newton boundary at the origin $\Gamma_{+}(f)$ is a single point $(2,1)$ and $f$ is strongly non-degenerate but
not convenient. Taking any vector $\mathbf{P}=\left(p_{1}, 0\right)$ with $p_{1}>0$, we have $f_{\triangle_{\mathbf{P}}}(x, w)=$ $\left(2 x^{2}+\|x\|^{2}\right) w$ for $w \in \mathbb{C}^{*}$ fixed. Then $\frac{\partial f_{\Delta_{\mathbf{P}}}(x, w)}{\partial x}=(4 x+\bar{x}) w$ and $\frac{\partial f_{\Delta_{\mathbf{P}}}(x, w)}{\partial \bar{x}}=x w$. By triangular inequality, we have:

$$
\left\|\frac{\partial f_{\triangle_{\mathbf{P}}}(x, w)}{\partial x}\right\|=\|(4 x+\bar{x}) w\| \geq 3\|w x\|>\left\|\frac{\partial f_{\triangle_{\mathbf{P}}}(x, w)}{\partial \bar{x}}\right\|
$$

where $(x, w) \in \mathbb{C}^{* 2}$. From Proposition 2.2.2, $f_{\triangle_{\mathbf{P}}}(x, w)$ is strongly non-degenerate. On the other hand, taking any vector $\mathbf{Q}=\left(0, q_{1}\right)$ with $q_{1}>0$, we have $f_{\triangle_{\mathbf{Q}}}(w, y)=\left(2 w^{2}+\|w\|^{2}\right) y$ for $w \in \mathbb{C}^{*}$ fixed. We check once that $f_{\triangle_{\mathbf{Q}}}(w, y)$ is strongly non-degenerate. By Definition 2.3.5, we conclude that $f$ is super strongly non-degenerate. Note also that the mixed singular locus is the whole $y$-axis which is non isolated. According to the previous theorem for non convenient mixed functions, there exist two equivalent Milnor fibrations.

## Chapter 3

## Global behavior of mixed polynomials

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### 3.1 Introduction

In this chapter, we study the regularity conditions at infinity for mixed polynomials. We begin with some results known for holomorphic polynomials. For a complex polynomial function $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$, it is well known that there is a $C^{\infty}$ locally trivial fibration $f_{i}: \mathbb{C}^{n} \backslash f^{-1}(\Lambda) \rightarrow \mathbb{C} \backslash \Lambda$ over the complement of some finite subset $\Lambda \subset \mathbb{C}$, see e.g. [Var72], [Ver76]. The minimal such $\Lambda$ is called the set of bifurcation values, or the set of atypical values, and shall be denoted by $B(f)$. It is not difficult to see the inclusion $f(\operatorname{Sing} f) \subset B(f)$ can be strict, like the following famous example given by Broughton cf. [Bro88]:

Example 3.1.1 $f: \mathbb{C}^{2} \rightarrow \mathbb{C}, f(x, y)=x^{2} y+x$. We have $\operatorname{Sing} f=\emptyset$. For $c \neq 0$, the fiber $f^{-1}(c)=\left\{y=\frac{c-x}{x^{2}}\right\}$ and $f^{-1}(0)=\{x(x y+1)=0\}$. Therefore $f^{-1}(c)$ is homeomorphic to $\mathbb{C}^{*}:=\mathbb{C} \backslash\{0\}$, whereas $f^{-1}(0)$ is homeomorphic to the disjoint union $\mathbb{C} \sqcup \mathbb{C}^{*}$. This implies that $0 \in B(f)$ and $f(\operatorname{Sing} f) \subset B(f)$ is strict.

Let us recall certain cases for which $f$ has no atypical values at infinity. We define $\operatorname{grad} f(z)=\left(\overline{\frac{\partial f}{\partial z_{1}}(z)}, \ldots, \overline{\frac{\partial f}{\partial z_{n}}(z)}\right)$. In [Bro88], Broughton considered the following class of holomorphic polynomials.

Definition 3.1.2 [Bro88, Definition 3.1] A holomorphic polynomial $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ is called a "tame polynomial" if there is a compact neighborhood $U$ of the critical points of $F$ such that $\|\operatorname{grad} f(z)\|$ is bounded away from 0 on $\mathbb{C}^{m} \backslash U$.

It was shown that holomorphic convenient polynomials with non-degenerate Newton principal part at infinity are tame and that "tame condition" is an open and dense condition. Since tame polynomials have only isolated singularities, for any $c \in \mathbb{C}$, let $p_{1}, \ldots, p_{k}$ be the critical points of $F$ lying on $F^{-1}(c)$ and set

$$
\mu^{c}=\mu^{c}(F)=\sum_{i=1}^{n} \mu_{p_{i}}(F), \quad \mu=\mu(F)=\sum_{c \in \mathbb{C}}(F)
$$

where $\mu_{p_{i}}(F)$ is the Milnor number of $F$ at $p_{i}$. We call $\mu^{c}$ the fibre Milnor number and $\mu$ the total Milnor number. The most important property of tame polynomials is the following:

Theorem 3.1.3 [Bro88, Theorem 1.2] Let $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ be a tame polynomial and $\mu, \mu^{c}, c \in \mathbb{C}$ be the total and fibre Milnor numbers of $F$ respectively. Then for any $c \in \mathbb{C}, f^{-1}(c)$ has the homotopy type of a bouquet of $\mu-\mu^{c}$ sphere of dimension $n$.

The above theorem provides that the topology of the fiber depends only on the singularities of $F$. Later, more general classes without atypical values at infinity like "M-tame", "cohomologically tame" were studied in [N8́6], [N8́8], [NS99], [Sab99].

Another problem in the study of global behavior of holomorphic polynomials is how to determine the atypical values at infinity.

For $n=2$, one has several equivalent characterizations of the atypical values at infinity, see e.g. [Dur98], [HL84], [Par95], [Tib99]. In higher dimension the problem is still open, and one can look for some significant set $A \supset B(f)$ which bounds $B(f)$ reasonably well.

For instance in [ N 86 ] and [ N 88 ], Némethi proved that $B(f) \subset \Lambda(f)$, where
$\Lambda(f):=\left\{c \in \mathbb{C}\right.$; there exists a sequence $\left\{z^{k}\right\}_{k} \subset \mathbb{C}^{n}$ such that

$$
\lim _{k \rightarrow \infty} \operatorname{grad} f\left(z^{k}\right)=0 \text { and } \lim _{k \rightarrow \infty}\left(f\left(z^{k}\right)-\left\langle z^{k}, \operatorname{grad} f\left(z^{k}\right)\right\rangle=c\right\} .
$$

When $\Lambda(f)$ is empty, we call $f$ is quasitame.
In [NZ90], the authors defined the Milnor set of $f$, namely

$$
M(f):=\left\{z \in \mathbb{C}^{n} ; \text { there exists } \lambda \in \mathbb{C}^{n} \text { such that } \operatorname{grad} f(z)=\lambda z\right\} .
$$

They gave an explicit set $S(f)$ by:

$$
\begin{aligned}
S(f): & =\left\{c \in \mathbb{C} \text {; there exists a sequence }\left\{z^{k}\right\}_{k} \subset M(f)\right. \text { such that } \\
& \left.\lim _{k \rightarrow \infty}\left\|z^{k}\right\|=\infty \text { and } \lim _{k \rightarrow \infty} f\left(z^{k}\right)=c\right\} .
\end{aligned}
$$

They proved that $B(f) \subset f(\operatorname{Sing} f) \cup S(f)$ and showed that $f(\operatorname{Sing} f) \cup S(f) \subset$ $\Lambda(f)$ is a better approximation for $B(f)$. When $S(f)$ is empty, $f$ is called M-tame, see e.g. [NZ90, NZ92, NS99].

The above definition is later generalized by Tibăr cf. [Tib99] called $\rho$-regularity which is concerned with the transversality of the fibers and the distance function $\rho$. He proved that $\rho$-regularity condition implies the topological triviality at infinity.

Let us turn to the cases of real polynomial maps. We may also define the bifurcation locus as follows:

Definition 3.1.4 For a polynomial map $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{p}, n>p$, the bifurcation locus $B(F)$ is the minimal set such that $F$ is a $\mathbb{C}^{\infty}$ locally trivial fibration over each connected component of $\mathbb{R}^{p} \backslash B(F)$.

One has therefore imagined various ways to approximate $B(F)$, essentially through the use of regularity conditions at infinity. For holomorphic polynomials one has the Malgrange regularity condition, mentioned by Pham and used in many papers, see e.g. [Par95, ST95]. This is known to be more general than "tame" [Bro88] or "quasi-tame" [N8́8]. It was extended to real maps by Kurdyka, Orro and Simon. These authors defined in [KOS00] the set of generalized critical values $K(F)=$ $F(\operatorname{Sing} F) \cup K_{\infty}(F)$ of a differentiable semi-algebraic map $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{k}$, where

$$
\begin{aligned}
K_{\infty}(F):= & \left\{c \in \mathbb{R}^{k} \mid \exists\left\{x_{\ell}\right\}_{l} \subset \mathbb{R}^{n},\left\|x_{\ell}\right\| \rightarrow \infty\right. \\
& \left.F\left(x_{\ell}\right) \rightarrow c \text { and }\left\|x_{\ell}\right\| \nu\left(\mathrm{d} F\left(x_{\ell}\right)\right) \rightarrow 0\right\}
\end{aligned}
$$

is the set of asymptotic critical values of $F$. In this definition they use the following distance function:

$$
\begin{equation*}
\nu(A):=\inf _{\|\varphi\|=1}\left\|A^{*} \varphi\right\| \tag{3.1.1}
\end{equation*}
$$

for $A \in L\left(\mathbb{R}^{n}, \mathbb{R}^{k}\right)$ and $A^{*}$ stands for the transpose of $A$. In particular for a holomorphic polynomial, one has $\nu(\mathrm{d} f(x))=\|\operatorname{grad} f(x)\|$. In the sequel, we call their condition KOS-regularity. The main result of [KOS00] is the following:

Theorem 3.1.5 [KOS00, Theorem 3.1]
Let $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{k}$ be a $C^{1}$ semi-algebraic map. Then $K(F)$ is a closed semi-algebraic set of dimension strictly less than $k$.

Moreover, if $F$ is of class $C^{2}$, then $F: \mathbb{R}^{n} \backslash F^{-1}(K(F)) \rightarrow \mathbb{R}^{k} \backslash K(F)$ is a locally trivial fibration over each connected component of $\mathbb{R}^{k} \backslash K(F)$. In particular, the set $B(F)$ of bifurcation values of $F$ is included in $K(F)$.

Remark 3.1.6 In the above theorem, the existence of the fibration is based on a result proved by Rabier [Rab97]. More generally, he introduced the notion of strong submersion and using the norm of $\nu$, he proved the fibration theorem for Finsler manifolds.

REmARK 3.1.7 In particular, for $k=1$, KOS-regularity can be interpreted as a Malgrange's condition. When $F: \mathbb{C}^{n} \rightarrow \mathbb{C}^{k}$ a complex polynomial mapping,
T.Gaffney [Gaf99] generalized Malgrange's condition. Even if his definition of $\nu$ is not exactly the same as definition of [KOS00], the two definitions are equivalent.

In the mixed case, we generalize $\rho$-regularity condition and define the set of asymptotic $\rho$-non-regular values of a mixed polynomial $f$ which is denoted by $S(f)$.(For Definition of $S(f)$, we refer to 3.2.4.) Since a mixed polynomial $F: \mathbb{C}^{n} \rightarrow \mathbb{C}$ can be considered as a real polynomial map from $\mathbb{R}^{2 n}$ to $\mathbb{R}^{2}$ obviously we can deal with the approximation of $B(F)$ by KOS-regularity condition. It is to be expected that one can compare these two regularity conditions. More precisely, the question is what is the difference between $S(f)$ and $K_{\infty}(f)$. This work is intended as an attempt to motivate not only the study of global behavior of mixed polynomials but also for real polynomial maps. Using a generalized version of Curve selection lemma, we make a preliminary observation to the structure of $S(f)$ which is turned out to be a closed semi-algebraic set. Moreover, $f(\operatorname{Sing} f) \cup S(f)$ is also a closed semialgebraic set. With an interpretation of the norm used to define KOS-regularity for mixed polynomials, this enable us to compare the two regularity conditions via some inequalities. Consequently, it is shown that $S(f) \subset K_{\infty}(f)$ and there exist the examples such that this inclusion is strict. (See Remark 3.3.2 and example 3.3.4) Namely, $\rho$-regularity condition is strictly stronger than KOS-regularity condition. Now, we state our main result of this chapter:

## Theorem 3.1.8 Fibration Theorem

Let $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ be a mixed polynomial. Then the restriction:

$$
f_{l}: \mathbb{C}^{n} \backslash f^{-1}(f(\operatorname{Sing} f) \cup S(f)) \rightarrow \mathbb{C} \backslash f(\operatorname{Sing} f) \cup S(f)
$$

is a locally trivial $C^{\infty}$ fibration over each connected component of $\mathbb{C} \backslash(f(\operatorname{Sing} f) \cup$ $S(f))$. In particular $B(f) \subset f(\operatorname{Sing} f) \cup S(f)$.

Remark 3.1.9 In the setting of mixed functions, our Theorem 3.1.8 extends [KOS00, Theorem 3.1]. Since we have $S(f) \subsetneq K_{\infty}(f)$, we get a sharper approximation of the bifurcation set $B(f)$. While our proof 3.3 does not explicitly bound the dimension of $S(f)$, it follows from the preceding inclusion and from Theorem 3.1.5 that $S(f)$ has real dimension less than 2.

The structure of the chapter is as follows. In Section 2, we reformulate the definition of Milnor set according to the terminology in [NZ90] which we call $\rho$-non-regular set and introduce asymptotic $\rho$-non-regular values for mixed polynomials. In Section 3, we compare our regularity condition with KOS-regularity and prove a version of fibration theorem which gives a better approximation of $B(f)$. In the end of this chapter, we will construct a mixed join polynomial type of Păunescu and Zaharia's example which implies that $K_{\infty}(f)$ and $S(f)$ can have different dimensions.

### 3.2 Preliminaries

In order to describe $\rho$-regularity in the mixed setting, the following lemma gives an explicit formula for the Milnor set.

Lemma 3.2.1 Let $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ be a mixed polynomial. The intersection of the fibre $f^{-1}(f(\mathbf{z}, \overline{\mathbf{z}}))$ with the sphere $S_{r}^{2 n-1}$ of radius $r=\|\mathbf{z}\|$ is not transversal at $\mathbf{z} \in \mathbb{C}^{n} \backslash\{0\}$ if and only if there exist $\mu \in \mathbb{C}^{*}, \lambda \in \mathbb{R}$ such that:

$$
\lambda \mathbf{z}=\mu \overline{\mathrm{d} f}(\mathbf{z}, \overline{\mathbf{z}})+\bar{\mu} \overline{\mathrm{d}} f(\mathbf{z}, \overline{\mathbf{z}}) .
$$

Proof. Let us write $f$ as the map:

$$
f: \mathbb{C}^{n}=\mathbb{R}^{2 n} \longrightarrow \mathbb{R}^{2}, f\left(z_{1}, \ldots, z_{n}\right)=(\operatorname{Re} f, \operatorname{Im} f)
$$

where $z_{k}=x_{k}+i y_{k}=\left(x_{k}, y_{k}\right)$, for $k=1, \ldots, n$, and let us denote $\mathbf{v}:=$ $\left(x_{1}, y_{1}, \ldots, x_{n}, y_{n}\right)$.

If $f^{-1}(f(\mathbf{z}, \overline{\mathbf{z}}))$ does not intersect transversely the sphere $S_{r}^{2 n-1}$ at $\mathbf{z}$, then there exist $\alpha, \beta, \gamma \in \mathbb{R},|\alpha|+|\beta|+|\gamma| \neq 0$, such that

$$
\begin{equation*}
\gamma \mathbf{v}=\alpha \mathrm{d} \operatorname{Re} f(\mathbf{v})+\beta \operatorname{dIm} f(\mathbf{v}) . \tag{3.2.1}
\end{equation*}
$$

Since $\operatorname{Re} f=\frac{f+\bar{f}}{2}, \operatorname{Im} f=\frac{f-\bar{f}}{2 i}$ and $\frac{\partial f}{\partial x_{k}}=\frac{\partial f}{\partial z_{k}}+\frac{\partial f}{\partial \bar{z}_{k}}, \frac{\partial f}{\partial y_{k}}=i \frac{\partial f}{\partial z_{k}}-i \frac{\partial f}{\partial \bar{z}_{k}}, k=1, \ldots, n$, we get:

$$
\begin{align*}
& \gamma x_{k}=\frac{\alpha}{2}\left(\frac{\partial f}{\partial z_{k}}+\frac{\partial f}{\partial \bar{z}_{k}}+\frac{\partial \bar{f}}{\partial z_{k}}+\frac{\partial \bar{f}}{\partial \bar{z}_{k}}\right)+\frac{\beta}{2 i}\left(\frac{\partial f}{\partial z_{k}}+\frac{\partial f}{\partial \bar{z}_{k}}-\frac{\partial \bar{f}}{\partial z_{k}}-\frac{\partial \bar{f}}{\partial \bar{z}_{k}}\right)  \tag{3.2.2}\\
& \gamma y_{k}=\frac{\alpha i}{2}\left(\frac{\partial f}{\partial z_{k}}-\frac{\partial f}{\partial \bar{z}_{k}}+\frac{\partial \bar{f}}{\partial z_{k}}-\frac{\partial \bar{f}}{\partial \bar{z}_{k}}\right)+\frac{\beta}{2}\left(\frac{\partial f}{\partial z_{k}}-\frac{\partial f}{\partial \bar{z}_{k}}-\frac{\partial \bar{f}}{\partial z_{k}}+\frac{\partial \bar{f}}{\partial \bar{z}_{k}}\right) . \tag{3.2.3}
\end{align*}
$$

Therefore:

$$
\begin{equation*}
\gamma z_{k}=(\alpha+\beta i) \frac{\partial \bar{f}}{\partial \bar{z}_{k}}+(\alpha-\beta i) \frac{\partial f}{\partial \bar{z}_{k}} \tag{3.2.4}
\end{equation*}
$$

for every $k \in\{1, \ldots, n\}$. We get our claim by taking $\lambda=\gamma$ and $\mu=\alpha+\beta i$.

REmARk 3.2.2 The singular locus Sing $f$ of a mixed polynomial $f$ is by definition the set of critical points of $f$ as a real-valued map. From Lemma 3.2.1, by taking $\lambda=0$ and dividing by $\mu$, we obtain [Oka08, Proposition 1].

Definition 3.2.3 The $\rho$-non-regular set of a mixed polynomial $f$ is

$$
M(f)=\left\{\mathbf{z} \in \mathbb{C}^{n} \mid \exists \lambda \in \mathbb{R} \text { and } \mu \in \mathbb{C}^{*}, \text { such that } \lambda \mathbf{z}=\mu \overline{\mathrm{d} f}(\mathbf{z}, \overline{\mathbf{z}})+\bar{\mu} \overline{\mathrm{d}} f(\mathbf{z}, \overline{\mathbf{z}})\right\}
$$

Lemma 3.2.1 gives the geometric interpretation of $M(f)$ as the critical locus of the map $(f, \rho)$, where $\rho: \mathbb{R}^{2 n} \rightarrow \mathbb{R}_{\geq 0}$ is the Euclidean distance function. From this interpretation, $M(f)$ is a real algebraic subset of $\mathbb{C}^{n}$ and this fact will be used in the following. Like in the holomorphic setting [NZ90], one may define:

Definition 3.2.4 The set of asymptotic $\rho$-non-regular values of a mixed polynomial $f$ is

$$
S(f)=\left\{c \in \mathbb{C} \mid \exists\left\{\mathbf{z}_{k}\right\}_{k \in \mathbb{N}} \subset M(f), \lim _{k \rightarrow \infty}\left\|\mathbf{z}_{k}\right\|=\infty \text { and } \lim _{k \rightarrow \infty} f\left(\mathbf{z}_{k}, \overline{\mathbf{z}_{k}}\right)=c\right\} .
$$

A value $c \notin S(f)$ will be called an asymptotic $\rho$-regular value.
In order to investigate the properties of $S(f)$ we need a version of the curve selection lemma at infinity. Milnor [Mil68, Lemma 3.1] has proved this lemma at points of the closure of a semi-algebraic set. Némethi and Zaharia [NZ90], [NZ92], proved how to extend the result at infinity at some fibre of a holomorphic polynomial function. We give here a more general statement including the case where the value of $|f|$ tends to infinity. Let us denote $+\infty$ simply by $\infty$.

## Lemma 3.2.5 Curve Selection Lemma at infinity

Let $U \subseteq \mathbb{R}^{n}$ be a semi-analytic set. Let $g: \mathbb{R}^{n} \longrightarrow \mathbb{R}$ be a polynomial function. If there is $\left\{\mathbf{x}_{k}\right\}_{k \in \mathbb{N}} \subset U$ such that $\lim _{k \rightarrow \infty}\left\|\mathbf{x}_{k}\right\|=\infty$ and $\lim _{k \rightarrow \infty} g\left(\mathbf{x}_{k}\right)=c$, where $c \in \mathbb{R}$, $c=\infty$ or $c=-\infty$, then there exist a real analytic path $x(t) \in U$ and $\mathbf{x}(t)=$ $\mathbf{x}_{\mathbf{0}} t^{\alpha}+\mathrm{x}_{\mathbf{1}} t^{\alpha+1}+$ h.o.t. defined on some small enough interval $] 0, \varepsilon\left[\right.$, such that $\mathbf{x}_{\mathbf{0}} \neq 0$, $\alpha<0, \alpha \in \mathbb{Z}$, and $\lim _{t \rightarrow 0} g(x(t))=c$.

Proof. Our proof starts with the following observations.
Let $A \subset \mathbb{R}^{m}$ and $B \subset \mathbb{R}^{n}$ be semi-algebraic sets, then we have:
(a) If $f: A \longrightarrow B$ is a polynomial map, then it is a semi-algebraic map.
(b) If $f: A \longrightarrow B$ is a regular rational map, then it is semi-algebraic map.
(c) If $f: A \longrightarrow \mathbb{R}$ is a semi-algebraic function, then $\|f\|$ is semi-algebraic function.
(d) If $f: A \longrightarrow \mathbb{R}$ is a semi-algebraic function and $f \geq 0$ on $A$, then $\sqrt{f}$ is a semi-algebraic function.

For the proofs of the above observation we refer to [BCR98] for more details. Next, we proceed to the proof of the lemma. When $c$ is finite, the lemma was proved by Némethi and Zaharia in [NZ92]. Thus it is sufficient to prove the case for $c=\infty$. Considering the embedding of $\mathbb{R}^{n}$ into $\mathbb{R}^{n} \times \mathbb{R}^{2}$ given by the following map:

$$
\varphi: \mathbf{x}=\left(x^{1}, \ldots, x^{n}\right) \mapsto\left(\frac{x^{1}}{\sqrt{1+\|\mathbf{x}\|^{2}}}, \ldots, \frac{x^{n}}{\sqrt{1+\|\mathbf{x}\|^{2}}}, \frac{1}{\sqrt{1+\|\mathbf{x}\|^{2}}}, h_{1}(\mathbf{x}), h_{2}(\mathbf{x})\right)
$$

Where $h_{1}(\mathbf{x})=\frac{1}{1+|f(\mathbf{x})|}$ and $h_{2}(\mathbf{x})=\frac{f(x)}{1+|f(\mathbf{x})|}$. By the above observations, it follows that $\varphi$ is semi-algebraic. In addition, by Tarski-Seidenberg theorem, $V=\varphi(U)$ is semi-algebraic. Since $\left(\frac{x^{1}}{\sqrt{1+\|\mathbf{x}\|^{2}}}, \ldots, \frac{1}{\sqrt{1+\|\mathbf{x}\|^{2}}}\right) \in S^{n}, \lim _{k \rightarrow \infty}\left\|\mathbf{x}_{k}\right\|=\infty$ and $\lim _{k \rightarrow \infty} f\left(\mathbf{x}_{k}\right)=$ $\infty$, we can suppose that the sequence $\left\{\varphi\left(\mathbf{x}_{k}\right)\right\}_{k \in \mathbb{N}}$ is sub-convergent to some point $\left(\mathbf{c}_{0}, 0,1\right) \in S^{n} \times[0,1] \times[0,1]$. Thus we can apply the Curve Selection Lemma (see [Mil68]) for the point $\left(\mathbf{c}_{0}, 0,1\right) \in \bar{V}$, then we obtain a real analytic path $\hat{\mathbf{x}}(t)$ in $\bar{V}$ which tends to $\left(\mathbf{c}_{0}, 0,1\right) \in \bar{V}$ when $t \rightarrow 0$, and $\hat{\mathbf{x}}(t) \in V$ for $\left.t \in\right] 0, \varepsilon[$. On the other hand we have the following analytic isomorphism $\psi$ between $\mathbb{R}^{n}$ and $S^{n} \cap\left\{\mathbf{x} \in \mathbb{R}^{n+1} \mid x_{n+1}>0\right\}:$

$$
\psi:\left(x^{1}, \ldots, x^{n}\right) \mapsto\left(\frac{x^{1}}{\sqrt{1+\|\mathbf{x}\|^{2}}}, \ldots, \frac{x^{n}}{\sqrt{1+\|\mathbf{x}\|^{2}}}, \frac{1}{\sqrt{1+\|\mathbf{x}\|^{2}}}\right) .
$$

Considering the pre-image of $\hat{x_{i}}(t)$ for all $i$, where $1 \leq i \leq n+1$ and $\left.t \in\right] 0, \varepsilon[$, we therefore get a real analytic path $\mathbf{x}(t)$ as desired.
Now we turn to the following structure result of $S(f)$.
Proposition 3.2.6 If $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ is a mixed polynomial, then $S(f)$ and $f(\operatorname{Sing} f) \cup$ $S(f)$ are closed semi-algebraic sets.

Proof. $S(f)$ may be presented as the projection of a semi-algebraic set. Indeed, consider the embedding of $\mathbb{C}^{n}$ into $\mathbb{C}^{n+1} \times \mathbb{C}$ given by the semi-algebraic map:

$$
\varphi:\left(z_{1}, \ldots, z_{n}\right) \mapsto\left(\frac{z_{1}}{\sqrt{1+\|\mathbf{z}\|^{2}}}, \ldots, \frac{z_{n}}{\sqrt{1+\|\mathbf{z}\|^{2}}}, \frac{1}{\sqrt{1+\|\mathbf{z}\|^{2}}}, f(\mathbf{z}, \overline{\mathbf{z}})\right)
$$

Then $U_{1}:=\overline{\varphi(M(f))} \cap\left\{\left(x_{1}, \ldots, x_{n+1}, c\right) \in \mathbb{C}^{n+1} \times \mathbb{C} \mid x_{n+1}=0\right\}$ is a semi-algebraic set and $S(f)=\pi\left(U_{1}\right)$, where $\pi: \mathbb{C}^{n+1} \times \mathbb{C} \rightarrow \mathbb{C}$ is the projection. Therefore $S(f)$ is semi-algebraic, by the Tarski-Seidenberg theorem.

Let now $c \in \overline{S(f)}$. There exists a sequence $\left\{c_{i}\right\}_{i} \subset S(f)$ such that $\lim _{i \rightarrow \infty} c_{i}=c$. For any $i$, we have by definition a sequence $\left\{\mathbf{z}_{i, n}\right\}_{n} \subset M(f)$ such that $\lim _{n}\left\|\mathbf{z}_{i, n}\right\|=\infty$ and $\lim _{n \rightarrow \infty} f\left(\mathbf{z}_{i, n}, \overline{\mathbf{z}}_{i, n}\right)=c_{i}$. Take a sequence $\left\{r_{i}\right\}_{i} \subset \mathbb{R}_{+}$such that $\lim _{i \rightarrow \infty} r_{i}=\infty$. For each $i$ there exists $n(i) \in \mathbb{N}$ such that $\mathbf{z}_{i, n}>r_{i}$ implies $\left|f\left(\mathbf{z}_{i, n}, \overline{\mathbf{z}}_{i, n}\right)-c_{i}\right|<\frac{1}{r_{i}}$, $\forall n \geqslant n(i)$. Setting $\mathbf{z}_{k}:=\mathbf{z}_{k, n(k)}$ we get a sequence $\left\{\mathbf{z}_{k}\right\}_{k}$ such that $\lim _{k \rightarrow \infty}\left\|\mathbf{z}_{k}\right\|=\infty$ and $\lim _{k \rightarrow \infty} f\left(\mathbf{z}_{k}, \overline{\mathbf{z}}_{k}\right)=c$, which proves that $c \in S(f)$.

Let now $a \in \overline{f(\operatorname{Sing} f)} \cup \overline{S(f)}$. Since we have proved that $S(f)$ is closed, we may assume that $a \in \overline{f(\operatorname{Sing} f)}$. Then there exists a sequence $\left\{\mathbf{z}_{n}\right\}_{j \in \mathbb{N}} \subset \operatorname{Sing} f$, such that $\lim _{j \rightarrow \infty} f\left(\mathbf{z}_{j}, \overline{\mathbf{z}}_{j}\right)=a$. If $\left\{\mathbf{z}_{j}\right\}_{j \in \mathbb{N}}$ is not bounded, then we may choose a subsequence $\left\{\mathbf{z}_{j_{k}}\right\}_{k \in \mathbb{N}}$ such that $\lim _{k \rightarrow \infty}\left\|\mathbf{z}_{j_{k}}\right\|=\infty$ and $\lim _{k \rightarrow \infty} f\left(\mathbf{z}_{j_{k}}, \overline{\mathbf{z}}_{j_{k}}\right)=a$. Since $\operatorname{Sing} f \subset M(f)$, it follows that $a \in S(f)$, see also Remark 4.4.2. In the other case, if $\left\{\mathbf{z}_{j}\right\}_{j \in \mathbb{N}}$ is bounded, then we may choose a subsequence $\left\{\mathbf{z}_{j_{k}}\right\}_{k \in \mathbb{N}}$ such that
$\lim _{k \rightarrow \infty} \mathbf{z}_{j_{k}}=\mathbf{z}_{0}$ and $\lim _{k \rightarrow \infty} f\left(\mathbf{z}_{j_{k}}, \overline{\mathbf{z}}_{j_{k}}\right)=a$. Since $\operatorname{Sing} f$ is a closed algebraic set, this implies $\mathbf{z}_{0} \in \operatorname{Sing} f$, so $a=f\left(\mathbf{z}_{0}, \overline{\mathbf{z}}_{0}\right) \in f(\operatorname{Sing} f)$.

### 3.3 The fibration theorem

By the next two results we prove that $S(f)$ contains the atypical values due to the asymptotical behavior and that $S(f)$ is contained in $K_{\infty}(f)$.

Proposition 3.3.1 Let $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ be a mixed polynomial. Then $S(f) \subset K_{\infty}(f)$.
Proof. Let $(g, h)$ be the corresponding real map of the mixed polynomial $f$ and denote $\nu(\mathbf{x}):=\nu(\mathrm{d}(g, h)(\mathbf{x}))$ where $\mathbf{x}=\left(x_{1}, y_{1}, \ldots, x_{n}, y_{n}\right)$. Let us first claim that:

$$
\begin{equation*}
\nu(\mathbf{x})=\inf _{\mu \in S_{1}^{1}}\|\mu \overline{\mathrm{~d} f}(\mathbf{z}, \overline{\mathbf{z}})+\bar{\mu} \overline{\mathrm{d}} f(\mathbf{z}, \overline{\mathbf{z}})\| . \tag{3.3.1}
\end{equation*}
$$

By the definition (3.1.1) of $\nu(\mathbf{x})$, we have $\nu(\mathbf{x})=\inf _{(a, b) \in S_{1}^{1}}\|a \mathrm{~d} g(\mathbf{x})+b \mathrm{~d} h(\mathbf{x})\|$. But the proof of Lemma 3.2.1 shows the equality: $\|a \mathrm{~d} g(\mathbf{x})+b \mathrm{~d} h(\mathbf{x})\|=\| \mu \overline{\mathrm{d} f}(\mathbf{z}, \overline{\mathbf{z}})+$ $\bar{\mu} \overline{\mathrm{d}} f(\mathbf{z}, \overline{\mathbf{z}}) \|$ for $\mu=a+i b \in S_{1}^{1}$. Our claim is proved.

Let then $c \in S(f)$. By Definition 3.2.4 and Lemma 3.2.5, there exist real analytic paths, $\mathbf{z}(t)$ in $M(f), \lambda(t)$ in $\mathbb{R}$ and $\mu(t)$ in $\mathbb{C}^{*}$, defined on a small enough interval $] 0, \varepsilon\left[\right.$, such that $\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty$ and $\lim _{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=c$ and that:

$$
\begin{equation*}
\lambda(t) \mathbf{z}(t)=\mu(t) \overline{\mathrm{d} f}(\mathbf{z}(t), \overline{\mathbf{z}}(t))+\bar{\mu}(t) \overline{\mathrm{d}} f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) . \tag{3.3.2}
\end{equation*}
$$

Let us assume that $\lambda(t) \not \equiv 0$. Dividing (3.3.2) by $\|\mu(t)\|$ yields:

$$
\begin{equation*}
\lambda_{0}(t) \mathbf{z}(t)=\mu_{0}(t) \overline{\mathrm{d} f}(\mathbf{z}(t), \overline{\mathbf{z}}(t))+\bar{\mu}_{0}(t) \overline{\mathrm{d}} f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \tag{3.3.3}
\end{equation*}
$$

where $\lambda_{0}(t):=\frac{\lambda(t)}{\|\mu(t)\|}$ and $\mu_{0}(t):=\frac{\mu(t)}{\|\mu(t)\|} ;$ therefore $\beta:=\operatorname{ord}_{t}\left(\mu_{0}(t)\right)=0$.
Since $\lim _{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=c$, we have $\alpha:=\operatorname{ord}_{\mathrm{t}} \frac{\mathrm{d}}{\mathrm{d} t} f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \geq 0$. Then the following computation:

$$
\begin{aligned}
\bar{\mu}_{0}(t) \frac{\mathrm{d}}{\mathrm{~d} t} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))+\mu_{0}(t) \frac{\mathrm{d}}{\mathrm{~d} t} \bar{f}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) & =\left\langle\mu_{0}(t) \overline{\mathrm{d} f}(\mathbf{z}(t), \overline{\mathbf{z}}(t))+\bar{\mu}_{0}(t) \overline{\mathrm{d}} f(\mathbf{z}(t), \overline{\mathbf{z}}(t)), \frac{\mathrm{d}}{\mathrm{~d}} \mathbf{z}(t)\right\rangle \\
& +\left\langle\frac{\mathrm{d}}{\mathrm{~d} t} \mathbf{z}(t), \mu_{0}(t) \mathrm{d} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))+\bar{\mu}_{0}(t) \overline{\mathrm{d}} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))\right\rangle \\
& \text { by }\left(\stackrel{3.33)}{=} \lambda_{0}(t)\left(\left\langle\mathbf{z}(\mathrm{t}), \frac{\mathrm{d}}{\mathrm{dt}} \mathbf{z}(\mathrm{t})\right\rangle+\left\langle\frac{\mathrm{d}}{\mathrm{dt}} \mathbf{z}(\mathrm{t}), \mathbf{z}(\mathrm{t})\right\rangle\right)\right. \\
& =\lambda_{0}(t) \frac{\mathrm{d}}{\mathrm{~d} t}\|\mathbf{z}(t)\|^{2}
\end{aligned}
$$

implies that $\operatorname{ord}_{t}\left(\lambda_{0}(t) \frac{\mathrm{d}}{\mathrm{d} t}\|\mathbf{z}(t)\|^{2}\right) \geq \alpha+\beta \geq 0$. But since $\operatorname{ord}_{\mathrm{t}}(\mathbf{z}(t))<0$, this implies that $\lim _{t \rightarrow 0}\left|\lambda_{0}(t)\right|\|\mathbf{z}(t)\|^{2}=0$. Note that this limit holds true for $\lambda(t) \equiv 0$ too.

From the last limit, by using (3.3.3), we get:

$$
\begin{equation*}
\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|\left\|\mu_{0}(t) \overline{\mathrm{d} f}(\mathbf{z}(t), \overline{\mathbf{z}}(t))+\bar{\mu}_{0}(t) \overline{\mathrm{d}} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))\right\|=0 \tag{3.3.4}
\end{equation*}
$$

which, by (3.3.1), implies $\lim _{t \rightarrow 0}\|\mathbf{x}(t)\|\|\nu(\mathbf{x}(t))\|=0$, showing that $c \in K_{\infty}(f)$.

Remark 3.3.2 The above inclusion is strict in general. This holds already in the holomorphic setting; to prove it, we may use the examples constructed by Păunescu and Zaharia in [PZ97], as follows. Let $f_{n, q}: \mathbb{C}^{3} \rightarrow \mathbb{C}, f_{n, q}(x, y, z):=$ $x-3 x^{2 n+1} y^{2 q}+2 x^{3 n+1} y^{3 q}+y z$, where $n, q \in \mathbb{N} \backslash\{0\}$. These polynomials are $\rho$ regular at infinity and therefore we have $S\left(f_{n . q}\right)=\emptyset$. It was also shown in [PZ97] that $f_{n, q}$ satisfies Malgrange's condition for any $t \in \mathbb{C}$ if and only if $n \leq q$. Therefore, in case $n>q$, we have $\emptyset=S\left(f_{n . q}\right) \subsetneq K_{\infty}\left(f_{n . q}\right)=\neq \emptyset$.

Next paragraph is devoted to the proof of our main result.

Proof of Fibration theorem 3.1.8 Let $c \notin f(\operatorname{Sing} f) \cup S(f)$. Then there is a closed disk $D$ centered at $c$ such that $D \subset \mathbb{C} \backslash f(\operatorname{Sing} f) \cup S(f)$, since the latter is an open set by Proposition 3.2.6. Let us first observe that there exists $R_{0} \gg 0$ such that $M(f) \cap f^{-1}(D) \backslash B_{R_{0}}^{2 n}=\emptyset$. Indeed, if this were not true, then there would exist a sequence $\left\{\mathbf{z}_{k}\right\}_{k \in \mathbb{N}} \subset f^{-1}(D) \cap M(f)$ such that $\lim _{k \rightarrow \infty}\left\|\mathbf{z}_{k}\right\|=\infty$. Since $D$ is compact, there is a sub-sequence $\left\{\mathbf{z}_{k_{i}}\right\}_{i \in \mathbb{N}} \subset M(f)$ and $c_{0} \in D$ such that $\lim _{i \rightarrow \infty}\left\|\mathbf{z}_{k_{i}}\right\|=\infty$ and $\lim _{i \rightarrow \infty} f\left(\mathbf{z}_{k_{i}}\right)=c_{0}$, which contradicts $D \subset \mathbb{C} \backslash S(f)$.

We claim that the map:

$$
\begin{equation*}
f_{l}: f^{-1}(D) \backslash B_{R_{0}}^{2 n} \rightarrow D \tag{3.3.5}
\end{equation*}
$$

is a trivial fibration on the manifold with boundary $\left(f^{-1}(D) \backslash B_{R_{0}}^{2 n}, f^{-1}(D) \cap S_{R}^{2 n-1}\right)$, for any $R \geq R_{0}$. Indeed, this is a submersion by hypothesis but it is not proper, so one cannot apply Ehresmann's theorem directly. Instead, we consider the map $(f, \rho): f^{-1}(D) \backslash B_{R_{0}}^{2 n} \rightarrow D \times\left[R_{0}, \infty[\right.$. As a direct consequence of its definition, this is a proper map. It is moreover a submersion since $\operatorname{Sing}(f, \rho) \cap f^{-1}(D) \backslash B_{R_{0}}^{2 n}=\emptyset$ by the above remark concerning the set $M(f)$, which is nothing else but $\operatorname{Sing}(f, \rho)$. We then apply to $(f, \rho)$ Ehresmann's theorem to conclude that it is a locally trivial, hence a trivial fibration over $D \times\left[R_{0}, \infty\left[\right.\right.$. Take now the projection $\pi: D \times\left[R_{0}, \infty[\rightarrow\right.$ $D$ which is a trivial fibration by definition and observe that our map (3.3.5) is the composition $\pi \circ(f, \rho)$ of two trivial fibrations, hence a trivial fibration too.

Next observe that, since $D \cap f(\operatorname{Sing} f)=\emptyset$, the restriction:

$$
\begin{equation*}
f_{\mid}: f^{-1}(D) \cap \bar{B}_{R_{0}}^{2 n} \rightarrow D \tag{3.3.6}
\end{equation*}
$$

is a proper submersion on the manifold with boundary $\left(f^{-1}(D) \cap \bar{B}_{R_{0}}^{2 n}, f^{-1}(D) \cap\right.$ $S_{R_{0}}^{2 n-1}$ ) and therefore a locally trivial fibration by Ehresmann's theorem, hence a trivial fibration over $D$.

Finally we glue the two trivial fibrations (3.3.6) and (3.3.5) by using an isotopy
and the trivial fibration from the following commuting diagram, for some $R>R_{0}$ :

where $\hat{F}$ denotes the fibre of the trivial fibration $f_{\mid}: S_{R} \cap f^{-1}(D) \rightarrow D$ and does not depend on the radius $R \geq R_{0}$.

EXAMPLE 3.3.3 Let $f: \mathbb{C}^{2} \rightarrow \mathbb{C}, f(x, y)=x(1+\bar{x} y)$. We have $\overline{\mathrm{d} f}=\left(1+x \bar{y},|x|^{2}\right)$ and $\overline{\mathrm{d}} f=(x y, 0)$. By the definition of $\operatorname{Sing} f$, there does not exist $\lambda \in S_{1}^{1}$ such that the equation $\overline{\mathrm{d} f}=\lambda \overline{\mathrm{d}} f$ has a solution. Hence $f(\operatorname{Sing} f)=\emptyset$. We proceed to calculate $M(f) \backslash \operatorname{Sing} f$ which is the solution of the following system:

$$
\begin{aligned}
& x=\mu(1+x \bar{y})+\bar{\mu} x y \\
& y=\mu|x|^{2}
\end{aligned}
$$

where $\mu \in \mathbb{C}^{*}$. We conclude $M(f)=\left\{(x, y) \in \mathbb{C}^{2} \mid x\left(|x|^{2}-2|y|^{2}\right)-y=0\right.$ and $x y \neq$ $0\}$.

If we take any sequence $\left\{\left(x_{k}, y_{k}\right)\right\}_{k \in \mathbb{N}}$ in $M(f)$ with $x_{k} \rightarrow \infty$, then we must have $y_{k} \rightarrow \infty$ and therefore, $f\left(x_{k}, y_{k}\right) \rightarrow \infty$.

Let us assume that $y_{k} \rightarrow \infty$. Then we must have $x_{k} \rightarrow 0$ and $2 x_{k} \bar{y}_{k} \rightarrow-1$. This implies that $f\left(x_{k}, y_{k}\right) \rightarrow 0$. We therefore get $S(f)=\{0\}$.

Now, let $c \in K_{\infty}(f)$ and by KOS-regularity we suppose that there exists a sequence $\left\{\left(x_{k}, y_{k}\right)\right\}_{k \in \mathbb{N}}$ such that $\left|\left(x_{k}, y_{k}\right)\right| \rightarrow \infty$ and:

$$
\begin{gather*}
f\left(x_{k}, y_{k}\right) \rightarrow c \\
\left|\left(x_{k}, y_{k}\right)\right|\left(\min _{\mu \in S_{1}^{1}}\|\mu \overline{\mathrm{~d} f}+\overline{\mu \mathrm{d}} f\|\right) \rightarrow 0 \tag{3.3.8}
\end{gather*}
$$

It follows from Equation 3.3.8 that $x_{k} \rightarrow 0, y_{k} \rightarrow \infty$ and $\left|x_{k}^{2} y_{k}\right| \rightarrow 0$. On the other hand, since $S(f) \subset K_{\infty}(f)$, we therefore conclude $K_{\infty}(f)=S(f)=\{0\}$.

In fact, we also have $B(f)=\{0\}$. Since $f^{-1}(0)=\left\{(x, y) \in \mathbb{C}^{2} \mid x=0\right.$ or $\bar{y}=$ $\left.-\frac{1}{x}\right\}$, the fiber $f^{-1}(0) \cong \mathbb{C} \sqcup \mathbb{C}^{*}$. For any $\varepsilon \neq 0$, we get $f^{-1}(\varepsilon)=\left\{(x, y) \in \mathbb{C}^{2} \mid\right.$ $\left.y=\frac{\varepsilon-x}{|x|^{2}}\right\}$ which is homeomorphic to $\mathbb{C}^{*}$. In consequence, there does not exist the locally trivial fibration over any neghibourhood of value $0 \in \mathbb{C}$. Finally, we get $K_{\infty}(f)=S(f)=B(f)=\{0\}$.

Next we consider a mixed polynomial type of Păunescu and Zaharia's example, which shows that $S(f)=\emptyset$ and $\operatorname{dim} K_{\infty}(f)=1$.

EXAMPLE 3.3.4 Let $f: \mathbb{C}^{4} \rightarrow \mathbb{C}, f(x, y, z, u)=x-3 x^{5} y^{2}+2 x^{7} y^{3}+y z+u \bar{u}$. We have:

$$
\begin{aligned}
& \frac{\partial f}{\partial x}=1-15 x^{4} y^{2}+14 x^{6} y^{3} \\
& \frac{\partial f}{\partial y}=-6 x^{5} y+6 x^{7} y^{2}+z \\
& \frac{\partial f}{\partial z}=y \\
& \frac{\partial f}{\partial u}=\bar{u} \\
& \frac{\partial f}{\partial \bar{u}}=u .
\end{aligned}
$$

The singular locus of $f$ consists of the points which satisfy the equation $\overline{\mathrm{d} f}=\lambda \overline{\mathrm{d}} f$ for some $\lambda \in S_{1}^{1}$. Explicitly, the system is written as follows:

$$
\begin{aligned}
1-15 x^{4} y^{2}+14 x^{6} y^{3} & =0 \\
-6 x^{5} y+6 x^{7} y^{2}+z & =0 \\
y & =0 \\
u & =\lambda u .
\end{aligned}
$$

Since the condition $y=0$ does not verify the first equation of the above system, this gives $\operatorname{Sing} f=\emptyset$.

We proceed to show that $S(f)=\emptyset$ and $K_{\infty}(f)=\left\{(a, 0) \in \mathbb{R}^{2} \mid a \geq 0\right\}$.
First, according to the defintion of KOS-regularity, for any point $\mathbf{a}=$ $(x, y, z, u) \in \mathbb{C}^{4}$, we have:

$$
\begin{align*}
\nu(\mathbf{a}) & =\min _{\mu \in S_{1}^{1}}\|\mu \overline{\mathrm{~d} f}+\bar{\mu} \overline{\mathrm{d}} f\| \\
& =\min _{\mu \in S_{1}^{1}}\left\|\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z},(\mu+\bar{\mu}) u\right)\right\| \tag{3.3.9}
\end{align*}
$$

From (3.3.9), in order to attain the minimum, it follows that $\mu= \pm i$ and therefore $\nu(\mathbf{a})=\left\|\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, 0\right)\right\|$.

By the computations of [PZ97], it yields that for $g(x, y, z)=x-3 x^{5} y^{2}+2 x^{7} y^{3}+$ $y z$, the sets $K_{\infty}(g)=\{0\}$ and $S(g)=\emptyset$.

On one hand, we fix a sequence $\left\{x_{k}, y_{k}, z_{k}\right\}_{k \in \mathbb{N}}$ which satisfies KOS-regularity for $g$ and $g\left(x_{k}, y_{k}, z_{k}\right) \rightarrow 0$. On the other hand, for any $u \in \mathbb{C}$, we choose a sequence $\left\{u_{k}\right\}_{k \in \mathbb{N}}$ such that $u_{k} \rightarrow u$. Therefore taking the sequence $\left\{\mathbf{a}_{k}=\left(x_{k}, y_{k}, z_{k}, u_{k}\right)\right\}_{k \in \mathbb{N}}$, we conclude $\left\{(a, 0) \in \mathbb{R}^{2} \mid a \geq 0\right\} \subset K_{\infty}(f)$.

Conversely, if $c \in K_{\infty}(f)$, then there exists a real analytic path $\mathbf{a}(t)=$ $(x(t), y(t), z(t), u(t))$ such that:

$$
\begin{align*}
& \lim _{t \rightarrow 0}\|\mathbf{a}(t)\|=\infty  \tag{3.3.10}\\
& \lim _{t \rightarrow 0} f(\mathbf{a}(t), \overline{\mathbf{a}}(t))=c  \tag{3.3.11}\\
& \lim _{t \rightarrow 0}\|\mathbf{a}(t)\|\|\nu(\mathbf{a}(t))\|=0 . \tag{3.3.12}
\end{align*}
$$

From (3.3.12), we have $\|\mathbf{a}(t)\|\|y(t)\| \rightarrow 0$ and $\left\|x^{2}(t) y(t)\right\| \rightarrow b$, where $b$ is a root of the following equation:

$$
\begin{equation*}
14 b^{3}-15 b^{2}+1=0 \tag{3.3.13}
\end{equation*}
$$

Hence by (3.3.10), we conclude $\operatorname{ord}(x(t))=A<0, \operatorname{ord}(y(t))=-2 A>0$.
If $b \neq 1$, then from (3.3.12), it follows that $\left\|-6 x^{3}(t)\left(b+b^{2}\right)+z(t)\right\| \rightarrow 0$. Consequently, we must have $\operatorname{ord}(z(t))=3 A$ and $\operatorname{ord}(z(t))+\operatorname{ord}(y(t))=A<0$ which contradicts $\|\mathbf{a}(t)\|\|y(t)\| \rightarrow 0$.

We may now assume that $b=1$. Since the roots of (3.3.13) are $1, \tau_{1}=\frac{1-\sqrt{57}}{28}$ and $\tau_{2}=\frac{1+\sqrt{57}}{28}$, we may write (3.3.12) as follows:

$$
\lim _{t \rightarrow 0}\|\mathbf{a}(t)\|\left\|\left(1-x^{2}(t) y(t)\right)\left(\tau_{1}-x^{2}(t) y(t)\right)\left(\tau_{2}-x^{2}(t) y(t)\right)\right\|=0 .
$$

This gives $\left\|x(t)\left(1-x^{2}(t) y(t)\right)\right\| \rightarrow 0$. To deal with the limit of $f(\mathbf{a}(t), \overline{\mathbf{a}}(t))$, we divide this limit in three parts. The first part is: $x(t)-3 x^{5}(t) y^{2}(t)+2 x^{7}(t) y^{3}(t)=$ $x(t)\left(1-x^{2}(t) y(t)\right)^{2}\left(x^{2}(t) y(t)+\frac{1}{2}\right)$. When $t \rightarrow 0$, we have:

$$
\lim _{t \rightarrow 0}\left\|x(t)-3 x^{5}(t) y^{2}(t)+2 x^{7}(t) y^{3}(t)\right\|=0
$$

The second part is $y(t) z(t)$ and $\lim _{t \rightarrow 0} y(t) z(t)=0$ which is a consequence of $\|\mathbf{a}(t)\|\|y(t)\| \rightarrow 0$. The last part is $\|u(t)\|^{2}$ and by (3.3.11), its limit is $c$. On accounts of the above arguments, we get $K_{\infty}(f)=\left\{(a, 0) \in \mathbb{R}^{2} \mid a \geq 0\right\}$.

We now proceed to show $S(f)=\emptyset$. Suppose $c \in S(f)$. Then there exists a real analytic path $\mathbf{a}(t)=(x(t), y(t), z(t), u(t)) \subset M(f)$ such that:

$$
\begin{aligned}
& \lim _{t \rightarrow 0}\|\mathbf{a}(t)\|=\infty \\
& \lim _{t \rightarrow 0} f(\mathbf{a}(t), \overline{\mathbf{a}}(t))=c
\end{aligned}
$$

By (3.3.4) of the proof of our fibration theorem, we conclude that the path $\mathbf{a}(t)$ satisfies (3.3.12). This implies:

$$
\begin{aligned}
\lim _{t \rightarrow 0}\|(x(t), y(t), z(t))\| & =\infty \\
\lim _{t \rightarrow 0} g(x(t), y(t), z(t)) & =0 .
\end{aligned}
$$

Since $\mathbf{a}(t) \subset M(f)$, by Definition of $M(f)$, we get $(x(t), y(t), z(t)) \subset M(g)$. From the above two limits, we obtain $S(g)=\{0\}$ which is in contradiction with $S(g)=\emptyset$.

## Chapter 4

## Newton non-degenerate polynomials

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### 4.1 Introduction

The purpose of this chapter is devoted to give an approximation of the bifurcation set $B(f)$ for a class of mixed polynomials via the Newton boundary at infinity. In the holomorphic case, Kushnirenko [Kus76] had first introduced the Newton boundary of holomorphic germs and polynomials. Lately, Némethi and Zaharia studied the Newton non-degenerate polynomials with respect to the Newton boundary at infinity. In order to estimate the set of bifurcation values $B(f)$, they introduced the notion of "bad faces". In [NZ90], it was proved the following

Theorem 4.1.1 [NZ90, Theorem 2] Suppose that $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ is a non convenient, Newton non-degenerate polynomial and $f(0)=0$. Then

$$
\begin{equation*}
B(f) \subset f(\operatorname{Sing} f) \cup\{0\} \cup \underset{\Delta \in \mathfrak{B}}{\cup} f_{\triangle}\left(\operatorname{Sing} f_{\triangle} \cap \mathbb{C}^{* n}\right) \tag{4.1.1}
\end{equation*}
$$

where $\mathfrak{B}$ is the set of "bad faces" of the support $\operatorname{supp}(f)$.
In the convenient case, it was shown that $B(f)=f(\operatorname{Sing} f)$, see [Bro88, Kus76]. When $n=2$, the above inclusion turns out to be an equality [NZ90, Proposition 6]. By using some technics of toric resolution, Zaharia showed that

Theorem 4.1.2 [Zah96, Proposition 5.3] Let $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ be a not convenient, Newton non-degenerate polynomial. Suppose moreover for all the bad faces $\triangle \in \mathfrak{B}$, we have $\operatorname{dim} \triangle=n-1$. Then

$$
f(\operatorname{Sing} f) \cup \underset{\Delta \in \mathfrak{B}}{\cup}\left(f_{\triangle}\left(\operatorname{Sing} f_{\triangle} \cap \mathbb{C}^{* n}\right) \backslash\{0\}\right) \subset B(f)
$$

Therefore it is of interest to know whether we can develop this technic to estimate the bifurcation set $B(f)$ for some special classes of mixed polynomials. Inspired by Oka's work of non-degenerate mixed functions, we consider the Newton boundary of a mixed polynomial at infinity and define a Newton non-degeneracy condition at infinity. We get the following effective estimation of $\left.S_{( } f\right)$ in the mixed case, which is also a generalization of [NZ90, Theorem 2].

Theorem 4.1.3 Let $f: \mathbb{R}^{2 n} \rightarrow \mathbb{R}^{2}$ be a mixed polynomial which depends effectively on all the variables and let $f(0)=0$. If $f$ is Newton non-degenerate at infinity then:
(a) $S(f) \subset\{0\} \cup \bigcup_{\Delta \in \mathcal{B}} f_{\Delta}\left(\operatorname{Sing} f_{\Delta} \cap \mathbb{C}^{* n}\right)$.
(b) If $f$ is moreover Newton strongly non-degenerate at infinity then $f(\operatorname{Sing} f)$ and $S(f)$ are bounded.
where $\mathfrak{B}$ is the set of bad faces of the support $\operatorname{supp}(f)$.
Remark 4.1.4 If $f$ satisfies the conditions of Theorem 4.1.3 except for $f(0)=0$, then we replace $f$ by $h=f-f(0)$ and apply to it Theorem 4.1.3. Since $\overline{\mathrm{d} f}(\mathbf{z}, \overline{\mathbf{z}})=$ $\overline{\mathrm{d} h}(\mathbf{z}, \overline{\mathbf{z}})$ and $\overline{\mathrm{d}} f(\mathbf{z}, \overline{\mathbf{z}})=\overline{\mathrm{d}} h(\mathbf{z}, \overline{\mathbf{z}})$, we get $M(f)=M(h)$ and $c \in S(f) \Leftrightarrow c-f(0) \in$ $S(h)$.

Unlike in the holomorphic case for two complex variables, we will construct an example which shows that the value zero is not contained in $B(f)$. We also consider a mixed polynomial type of King's example which demonstrates that for two complex variables, even if $S(f) \neq \emptyset$, we may also have $B(f)=\emptyset$. Our another purpose of this chapter is to discuss the stability of the monodromy at infinity for some families of mixed polynomials. In [NZ92], it was shown that for two non-degenerate and convenient polynomials which has the same Newton boundary at infinity, the monodromy at infinity is equivalent. This result was later
extended by Pham [Pha08] to non-convenient and non-degenerate polynomial. In our mixed setting, the technique developed for the proof of Theorem 4.1.3 enables us to pursue the extension of these results for families of mixed polynomials, along the pattern of [NZ92] and [Pha08, Lemmas 3.2-3.5]. Let us point out the difference is that in the holomorphic case the Newton non-degeneracy is a Zariski open dense and connected condition, hence there exists a family of Newton non-degenerate polynomials with the same Newton boundary at infinity joining any two such polynomials. However, in the mixed case, we will show in Remark 4.3.2 that the Newton strong non-degeneracy condition at infinity is neither dense, nor connected, but it is still an open condition (see §4.3). Therefore, in order to obtain a stability theorem in the mixed case, one has to work with a given family of mixed polynomials.

The organization of this chapter is as follows. In Section 2, we first define some notions concerned with the Newton polyhedron at infinity. Then we introduce Newton non-degeneracy condition at infinity and the stronger one. In Section 3, we prove that non-degeneracy condition is an open condition. The first two subsections of Section 4 is devoted to the proof of our main Theorem 4.1.3. In the end of Section 4 we indicate some corollaries which extend the results known in the holomorphic case. The Section 5 is motivate to our investigation of the stability of the monodromy for some families of mixed polynomials. As a consequence, we give a slight generalization of the main result in [Pha08]. In the final section of this chapter, we illustrate our conclusions with some explicit examples which provide the detailed expositions of the differences to holomorphic polynomials.

### 4.2 Newton non-degeneracy at infinity

To introduce our main theorem 4.1.3, we begin with some necessary notions. Let $f$ be a mixed polynomial:

Definition 4.2.1 We call $\operatorname{supp}(f)=\left\{\nu+\mu \in \mathbb{N}^{n} \mid c_{\nu, \mu} \neq 0\right\}$ the support of $f$. We say that $f$ is convenient if the intersection of $\operatorname{supp}(f)$ with each coordinate axis is non-empty. We denote by $\overline{\operatorname{supp}(f)}$ the convex hull of the set $\operatorname{supp}(f) \backslash\{0\}$. The Newton polyhedron of a mixed polynomial $f$, denoted by $\Gamma_{0}(f)$, is the convex hull of the set $\{0\} \cup \operatorname{supp}(f)$. The Newton boundary at infinity, denoted by $\Gamma^{+}(f)$, is the union of the faces of the polyhedron $\Gamma_{0}(f)$ which do not contain the origin. By "face" we mean face of any dimension.

Definition 4.2.2 For any face $\Delta$ of $\overline{\operatorname{supp}(f)}$, we denote the restriction of $f$ to $\Delta \cap \operatorname{supp}(f)$ by $f_{\Delta}:=\sum_{\nu+\mu \in \Delta \cap \operatorname{supp}(f)} c_{\nu, \mu} \mathbf{z}^{\nu} \overline{\mathbf{z}}^{\mu}$. The mixed polynomial $f$ is called Newton non-degenerate at infinity if $\operatorname{Sing} f_{\Delta} \cap f_{\Delta}^{-1}(0) \cap \mathbb{C}^{* n}=\emptyset$, for any face $\Delta$
of $\Gamma^{+}(f)$. Following Oka's terminology [Oka10a], we say that $f$ is Newton strongly non-degenerate at infinity if Sing $f_{\Delta} \cap \mathbb{C}^{* n}=\emptyset$ for any face $\Delta$ of $\Gamma^{+}(f)$.

The later condition is stronger and in general not equivalent to the former, but they coincide in the holomorphic setting since $f_{\Delta}$ is quasi-homogeneous of non-zero degree.

Before giving the proof in $\S 4.4$, we need to define the ingredients and prove several preliminary facts. We consider a mixed polynomial $f: \mathbb{C}^{n} \rightarrow \mathbb{C}, f \not \equiv 0$.

Definition 4.2.3 A face $\Delta \subseteq \overline{\operatorname{supp}(f)}$ is called bad whenever:
(i) there exists a hyperplane $H \subset \mathbb{R}^{n}$ with equation $a_{1} x_{1}+\cdots+a_{n} x_{n}=0$ (where $x_{1}, \ldots, x_{n}$ are the coordinates of $\mathbb{R}^{n}$ ) such that:
(a) there exist $i$ and $j$ with $a_{i}<0$ and $a_{j}>0$,
(b) $H \cap \overline{\operatorname{supp}(f)}=\Delta$.

Let $\mathfrak{B}$ denote the set of bad faces of $\overline{\operatorname{supp}(f)}$. A face $\Delta \in \mathfrak{B}$ is called strictly bad if it satisfies in addition the following condition:
(ii) the affine subspace of the same dimension spanned by $\Delta$ contains the origin.

Remark 4.2.4 In our Theorem 4.1.3 we use the above definition for "bad faces". For holomorphic mappings, the set $\mathfrak{B}$ of bad faces used in the main formula (4.1.1) of [NZ90] corresponds to our definition of "strictly bad faces".

Let us observe that not all bad faces are strictly bad. Nevertheless, our Theorem 4.1.3(a) reduces in case of complex polynomials to precisely the statement (4.1.1) of [NZ90]. If $\Delta$ is a bad face which is not strictly bad, then it follows from the definitions that $\Delta$ is a face of $\Gamma^{+}(f)$. If we assume that $f$ is non-degenerate at infinity, then $f_{\Delta}$ is non-degenerate at infinity. If $f_{\Delta}$ is moreover holomorphic, then it follows that $f_{\Delta}$ is strongly non-degenerate at infinity. Indeed, there exists a hyperplane $V$ not passing through 0 and such that $V \cap \overline{\operatorname{supp}(f)}=\Delta$, thus $f_{\Delta}$ is also weighted homogeneous of degree $\neq 0$ and therefore Sing $f_{\Delta} \subset\left\{f_{\Delta}=0\right\}$. This shows in particular that in case of holomorphic $f$, the bad faces which are not strictly bad do not contribute with nonzero values in the formula of our Theorem 4.1.3(a), hence indeed only the strictly bad faces may play a role.

The following lemma will be used in the proof of our theorem.
Lemma 4.2.5 Let $l_{\mathbf{p}}(x)=\sum_{i=1}^{n} p_{i} x_{i}$ be a linear function such that $p=\min _{1 \leq i \leq n}\left\{p_{i}\right\}<$ 0 . We consider the restriction of $l_{\mathbf{p}}(x)$ to $\overline{\operatorname{supp}(f)}$ and denote by $\Delta_{\mathbf{p}}$ the unique maximal face of $\overline{\operatorname{supp}(f)}$ (with respect to the inclusion of faces) where $l_{\mathbf{p}}(x)$ takes its minimal value $d_{\mathbf{p}}$. Let $d_{\mathbf{p}} \leq 0$.
(a) If $d_{\mathbf{p}}<0$, then $\Delta_{\mathbf{p}}$ is a face of $\Gamma^{+}(f)$.
(b) If $d_{\mathbf{p}}=0$, then either $\Delta_{\mathbf{p}}$ is a face of $\Gamma^{+}(f)$ or $\Delta_{\mathbf{p}}$ satisfies condition (ii) of Definition 4.2.3.

Proof. Let us first remark that from Definition 4.2.1 we have $\Gamma_{0}(f)=$ cone $_{0}\left(\Gamma^{+}(f)\right)$, where cone ${ }_{0}(A)$ denotes the compact cone over the set $A$ with vertex the origin. For each face $\Delta$ of $\Gamma_{0}(f)$ we have that either $\Delta$ is a face of $\Gamma^{+}(f)$ or $\Delta \ni 0$ and in this case we have $\Delta=\operatorname{cone}_{0}(\Delta \cap \overline{\operatorname{supp}(f)})=\operatorname{cone}_{0}\left(\Delta \cap \Gamma^{+}(f)\right)$.

Next, considering the restriction of $l_{\mathbf{p}}(x)$ to $\Gamma_{0}(f)$, we denote by $\Delta_{1}$ the maximal face of $\Gamma_{0}(f)$ where $l_{\mathbf{p}}(x)$ takes its minimal value $d$. Note that $l_{\mathbf{p}}(x)$ can not attain its minimal value $d$ at interior points of $\Gamma_{0}(f)$. Since $\Gamma^{+}(f) \subset \overline{\operatorname{supp}(f)} \subset \Gamma_{0}(f)$, we have $d \leq d_{\mathbf{p}}$.
(a). If $d_{\mathbf{p}}<0$ then it follows by our initial remark that $\Delta_{1}$ is a face of $\Gamma^{+}(f)$, since otherwise we have $0 \in \Delta_{1}$ and $d=0$. We therefore get $\Delta_{\mathrm{p}}=\Delta_{1} \subset \Gamma^{+}(f)$ and $d=d_{\mathbf{p}}$.
(b). If $d_{\mathbf{p}}=0$ and $\Delta_{1}$ is not a face of $\Gamma^{+}(f)$, then by the same initial remark we have $\Delta_{1} \ni 0$ and therefore $d=0$. Since $\Delta_{1}$ is the maximal face of $\Gamma_{0}(f)$ where $l_{\mathbf{p}}(x)$ takes its minimal value $d$, we get $\Delta_{\mathrm{p}} \subset \Delta_{1} \subset H$, where $H$ denotes the hyperplane $\left\{x \in \mathbb{R}^{n} \mid l_{\mathbf{p}}(x)=0\right\}$. We then have $\Delta_{\mathbf{p}}=\overline{\operatorname{supp}(f)} \cap H, \Delta_{1}=\Gamma_{0}(f) \cap H$, and therefore $\Delta_{\mathbf{p}}=\Delta_{1} \cap \overline{\operatorname{supp}(f)}$. Let us assume that $\Delta_{\mathbf{p}}$ does not verify condition (ii) of Definition 4.2.3, namely that we have $\operatorname{dim} \operatorname{cone}_{0}\left(\Delta_{p}\right)>\operatorname{dim} \Delta_{p}$. This implies that $\Delta_{p}$ does not contain any interior point of $\operatorname{cone}_{0}\left(\Delta_{p}\right)$. By the initial remark, $\Delta_{1}=\operatorname{cone}_{0}\left(\Delta_{1} \cap \Gamma^{+}(f)\right)=\operatorname{cone}_{0}\left(\Delta_{\mathbf{p}}\right)$. Then $\Delta_{\mathbf{p}}$ is a face of $\Gamma^{+}(f)$, which contradicts our assumption.

Let $I \subset\{1, \ldots, n\}$. We shall use the following notations:
$\mathbb{C}^{I}=\left\{\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{C}^{n} \mid z_{j}=0, j \notin I\right\}$, and similarly $\mathbb{R}_{\geq 0}^{I}, \mathbb{C}^{* I}:=\mathbb{C}^{I} \cap \mathbb{C}^{* n}$, $f^{I}:=f_{\mid \mathbb{C}^{I}}$.

From Definition 4.2.1, the faces of $f^{I}$ are among the faces of $f$, so we have the following:

Remark 4.2.6 Let $f$ be a mixed Newton (strongly) non-degenerate polynomial. If $I \subset\{1,2, \ldots, n\}$ is such that $f^{I}$ is not identically zero then:
(a) $f^{I}$ is a mixed Newton (strongly) non-degenerate polynomial.
(b) $\Gamma^{+}\left(f^{I}\right)=\Gamma^{+}(f) \cap \mathbb{R}_{\geq 0}^{I}$.

We shall use the following fact for the restriction of $f$ to its bad faces.
Remark 4.2.7 If a mixed polynomial $f$ is Newton (strongly) non-degenerate at infinity then, for any bad face $\Delta \subset \overline{\operatorname{supp}(f)}, f_{\Delta}$ is Newton (strongly) non-degenerate at infinity. Indeed, any face $\Delta^{\prime}$ of $\Gamma^{+}\left(f_{\Delta}\right)$ is also a subface of $\Delta$, hence a subface of
$\Gamma^{+}(f)$. The Newton (strong) non-degeneracy of $f$ implies that the restriction $f_{\Delta}$ is also Newton (strongly) non-degenerate at infinity.

### 4.3 Newton non-degeneracy is an open condition

For a fixed polyhedron $\Gamma$ which is the Newton boundary at infinity of some mixed polynomial, we may define the subset $U_{\Gamma}^{s}:=\left\{\left[c_{1}, c_{2}, \ldots, c_{m}\right] \in \mathbb{P}_{\mathbb{C}}^{m-1} \mid f_{c}(\mathbf{z}, \overline{\mathbf{z}})=\right.$ $f(\mathbf{z}, \overline{\mathbf{z}}, c)=\sum_{j=1}^{m} c_{j} \mathbf{Z}^{\mu_{j}} \overline{\mathbf{Z}}^{\nu_{j}}$ is a Newton strongly non-degenerate mixed polynomial and $\left.\Gamma^{+}\left(f_{c}\right)=\Gamma\right\}$. Similarly we define the set $U_{\Gamma} \supset U_{\Gamma}^{s}$ by just dropping the word "strongly" in the above definition. Then:

Proposition 4.3.1 The subsets $U_{\Gamma} \subset \mathbb{P}^{m-1}$ and $U_{\Gamma}^{s} \subset U_{\Gamma}$ of Newton nondegenerate and, respectively, strongly non-degenerate mixed polynomials, with fixed Newton boundary $\Gamma$ at infinity, are semi-algebraic open sets.

Proof. Let us show that $U_{\Gamma}^{s}$ is open and semi-algebraic. The idea of this proof took its inspiration from Oka's alternate proof in the holomorphic setting [Oka79, Appendix]. For every face $\Delta \subset \Gamma$ we define:

$$
\begin{aligned}
V(\Delta) & :=\left\{(\mathbf{z}, c) \in \mathbb{C}^{n} \times \mathbb{P}^{m-1} \mid \exists \lambda \in S_{1}^{1}, \overline{\mathrm{~d} f_{\Delta}}(\mathbf{z}, \overline{\mathbf{z}}, c)=\lambda \overline{\mathrm{d}} f_{\Delta}(\mathbf{z}, \overline{\mathbf{z}}, c)\right\}, \\
V(\Delta)^{*} & :=V(\Delta) \cap\left\{(\mathbf{z}, c) \in \mathbb{C}^{n} \times \mathbb{P}^{m-1} \mid z_{1} z_{2} \ldots z_{n} \neq 0\right\}
\end{aligned}
$$

Note that $V(\Delta)$ is closed and that $\overline{V(\Delta)^{*}}=V(\Delta)$. Let us consider the union $V^{*}=\cup_{\Delta \subset \Gamma} V(\Delta)^{*}$ and the projection $\pi: \mathbb{C}^{n} \times \mathbb{P}^{m-1} \rightarrow \mathbb{P}^{m-1}$. Showing that $U_{\Gamma}^{s}$ is an open set means to prove that its complement $W=\pi\left(V^{*}\right)$ is a closed set. One observes that $W$ is a semi-algebraic set, since it is the projection of a semi-algebraic set.

Let $c_{0} \in \bar{W}$. By Curve Selection Lemma, there exists a face $\Delta_{0}$ of $\Gamma$ and a real analytic path $(\mathbf{z}(t), c(t)) \subset V\left(\Delta_{0}\right)^{*}$ defined on a small enough interval $] 0, \varepsilon[$ such that $\lim _{t \rightarrow 0} c(t)=c_{0}$ and either $\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty$ or $\lim _{t \rightarrow 0} \mathbf{z}(t)=\mathbf{z}_{0} \in V\left(\Delta_{0}\right)$.

Let then $z_{i}(t)=a_{i} t^{p_{i}}+$ h.o.t. for $1 \leq i \leq n$ where $a_{i} \neq 0, p_{i} \in \mathbb{Z}$ and $\lambda(t)=$ $\lambda_{0}+\lambda_{1} t+$ h.o.t., where $\lambda_{0} \in S_{1}^{1}$. Let $\mathbf{a}:=\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{C}^{* n}, \mathbf{P}:=\left(p_{1}, \ldots, p_{n}\right) \in \mathbb{Z}^{n}$ and consider the linear function $l_{\mathbf{P}}=\sum_{i=1}^{n} p_{i} x_{i}$ defined on $\Delta_{0}$. Let $\Delta_{1}$ be the maximal face of $\Delta_{0}$ where $l_{\mathbf{P}}$ takes its minimal value, say this value is $d_{\mathbf{P}}$. We have:

$$
\overline{\frac{\partial f_{\Delta_{1}}}{\partial z_{i}}}(\mathbf{a}, \overline{\mathbf{a}}, c(t)) t^{d_{\mathbf{p}}-p_{i}}+\text { h.o.t. }=\lambda_{0} \frac{\partial f_{\Delta_{1}}}{\partial \bar{z}_{i}}(\mathbf{a}, \overline{\mathbf{a}}, c(t)) t^{d_{\mathbf{p}}-p_{i}}+\text { h.o.t. }
$$

By taking the limit $c(t) \rightarrow c_{0}$ and focusing on the first terms of the expansions:

$$
\overline{\mathrm{d} f_{\Delta_{1}}}\left(\mathbf{a}, \overline{\mathbf{a}}, c_{0}\right)=\lambda_{0} \overline{\mathrm{~d}} f_{\Delta_{1}}\left(\mathbf{a}, \overline{\mathbf{a}}, c_{0}\right)
$$

we get $\left(\mathbf{a}, c_{0}\right) \in V\left(\Delta_{1}\right)^{*} \subset V^{*}$, since $\mathbf{a} \in \mathbb{C}^{* n}$, thus $c_{0} \in W$, which concludes the proof that $W=\bar{W}$.

If in the definition of $V(\Delta)$ we add the supplementary equation $f_{\Delta}=0$, then the same proof works for $U_{\Gamma}$ instead of $U_{\Gamma}^{s}$.

REMARK 4.3.2 In the holomorphic setting one has "Zariski-open" instead of "open" and such a result was proved by Kushnirenko [Kus76] as a consequence of the Bertini-Sard theorem and of the fact that "strong non-degeneracy" is equivalent to "non-degeneracy". Nevertheless in the real setting this proof does not apply and, in general, one does not have neither the connectedness, nor the density. For instance, in following Example 4.3.3 the inequality for strong non-degeneracy condition at infinity describes a homogeneous open set in $\mathbb{C}^{3}$ which is not dense and not connected. Note also that $\operatorname{supp}(f)$ is a single point.

Example 4.3.3 Consider the mixed polynomial:

$$
f(z, \bar{z})=a z^{2}+b z \bar{z}+c \bar{z}^{2}
$$

where $a, b, c \in \mathbb{C}$ are three parameters.
Then we have the following conclusions:
(a) If $\left(|a|^{2}-|c|^{2}\right)^{2} \leq|\bar{a} b-c \bar{b}|^{2}$, then $f$ is strongly degenerate. Otherwise, $f$ is strongly non-degenerate at infinity.
(b) $M(f)=\mathbb{C}$, and $S(f)=\emptyset$ if and only if the equation $a x^{2}+b x+c=0$ has no solution on $S_{1}^{1}$.

In fact, $f$ is strongly degenerate if and only if the following equation has at least one solution on $\mathbb{C}^{*}$ :

$$
\begin{equation*}
2 \bar{a} \bar{z}+\bar{b} z=\lambda(2 c \bar{z}+b z), \quad \text { where } \lambda \in S_{1}^{1} . \tag{4.3.1}
\end{equation*}
$$

Here note that 0 is always a solution of the above equation, therefore $0 \in \operatorname{Sing} f$. Since Equation 4.3.1 is radial homogeneous, to show the strong degeneracy it is sufficient to find a solution of (4.3.1) on the unit disc. Therefore assume that $z \in S_{1}^{1}$. Then equation (4.3.1) is equivalent to the following by taking the norm:

$$
\begin{equation*}
(a \bar{b}-\bar{c} b) z^{2}+2\left(|a|^{2}-|c|^{2}\right)|z|^{2}+(\bar{a} b-c \bar{b}) \bar{z}^{2}=0 \tag{4.3.2}
\end{equation*}
$$

Let us denote $z^{2}$ by $x$. Multiplying by $z^{2}$ in (4.3.2), we get:

$$
\begin{equation*}
(a \bar{b}-\bar{c} b) x^{2}+2\left(|a|^{2}-|c|^{2}\right) x+(\bar{a} b-c \bar{b})=0 . \tag{4.3.3}
\end{equation*}
$$

Then $f$ is strongly degenerate if and only if (4.3.3) has at least a solution on $S_{1}^{1}$. We first observe that if $a \bar{b}-\bar{c} b=0$ and $|a|=|c|$, then any $x \in S_{1}^{1}$ is a solution of (4.3.3), which implies that $\operatorname{Sing} f=\mathbb{C}$. If $b=0$ and $|a| \neq|c|$, then (4.3.3) has no solution on $S_{1}^{1}$, which implies that $\operatorname{Sing} f=\{0\}$.

If $a \bar{b}-\bar{c} b \neq 0$, then (4.3.3) is a quadratic equation and the discriminant $\triangle=$ $4\left(|a|^{2}-|c|^{2}\right)^{2}-4|\bar{a} b-c \bar{b}|^{2}$. We have the following three cases:
(I) If $\triangle<0$, then the roots $x=\frac{|c|^{2}-|a|^{2} \pm i \sqrt{|\bar{a} b-c \bar{b}|^{2}-\left(|a|^{2}-|c|^{2}\right)^{2}}}{a \bar{b}-\bar{c} b}$ are contained in $S_{1}^{1}$. We conclude that $\operatorname{Sing} f$ consists of four lines and therefore $f$ is strongly degenerate.
(II) If $\triangle=0$, then the roots $x=\frac{|c|^{2}-|a|^{2}}{a \bar{b}-\bar{c} b}$ are contained in $S_{1}^{1}$. We conclude that $\operatorname{Sing} f$ consists of two lines and therefore $f$ is strongly degenerate.
(III) If $\Delta>0$, then the roots $x=\frac{|c|^{2}-|a|^{2} \pm \sqrt{\left(|a|^{2}-|c|^{2}\right)^{2}-|\bar{a} b-c \bar{b}|^{2}}}{a \bar{b}-\bar{c} b}$ are not contained in $S_{1}^{1}$. We conclude that $\operatorname{Sing} f$ consists of the origin and therefore $f$ is strongly non-degenerate at infinity.

For $z \in M(f)$, by definition, we have:

$$
\begin{equation*}
\lambda z=\mu(2 \bar{a} \bar{z}+\bar{b} z)+\bar{\mu}(2 c \bar{z}+b z) \tag{4.3.4}
\end{equation*}
$$

where $\lambda \in \mathbb{R}, \mu \in \mathbb{C}^{*}$. We observe that $z=0$ is a solution of (4.3.4). Let us suppose $z \neq 0$ and divide (4.3.4) by $z$. Then we obtain:

$$
\begin{equation*}
\frac{\mu(2 \overline{a z}+\bar{b} z)+\bar{\mu}(2 c \bar{z}+b z)}{z}=\frac{\overline{\mu(2 \overline{a z}+\bar{b} z)+\bar{\mu}(2 c \bar{z}+b z)}}{\bar{z}} . \tag{4.3.5}
\end{equation*}
$$

By a simplification, (4.3.4) gives:

$$
\begin{equation*}
\mu\left(\bar{a} \overline{z^{2}}-\bar{c} z^{2}\right)=\bar{\mu}\left(a z^{2}-c \bar{z}^{2}\right) . \tag{4.3.6}
\end{equation*}
$$

We deduce that for any $z \in \mathbb{C}^{n}$, there always exists some $\mu \in \mathbb{C}^{*}$ such that (4.3.6) holds. Thus $M(f)=\mathbb{C}$. Consider any real analytic path $\mathbf{z}(t)$ defined on a small enough interval $] 0, \varepsilon[$ such that:

$$
z(t)=a_{1} t^{p_{1}}+\text { h.o.t., where } a_{1} \in \mathbb{C}^{*}, p_{1}<0 .
$$

We have $f(z(t), \bar{z}(t))=\left(a a_{1}^{2}+b a_{1} \bar{a}_{1}+c \bar{a}_{1}^{2}\right) t^{2 p_{1}}+$ h.o.t.
If $a x^{2}+b x+c=0$ has no solution on $S_{1}^{1}$, then $\left(a a_{1}^{2}+b a_{1} \bar{a}_{1}+c \bar{a}_{1}^{2}\right)$ can not be equal to zero, otherwise $\frac{a}{|a|^{2}}$ is the root of $a x^{2}+b x+c=0$, which contradicts our assumption. Therefore $\operatorname{ord}_{\mathrm{t}} f(z(t), \bar{z}(t))=2 p_{1}<0$ and in consequence, $S(f)=\emptyset$.

If $S(f)=\emptyset$ and $a x^{2}+b x+c=0$ has a solution $x_{0}$ on $S_{1}^{1}$, then we take a real analytic path $z(t)=\frac{x_{0}}{t}$ which gives $0 \in S(f)$. Therefore we get a contradiction with $S(f)=\emptyset$. Combining the above arguments, our second claim is proved.

### 4.4 Proof of Main theorem and some consequences

### 4.4.1 Proof of Theorem 4.1.3(a).

Let $c \in S(f)$. By the definition of $S(f)$ and Curve selection Lemma 3.2.5, there exist real analytic paths, $\mathbf{z}(t)$ in $M(f), \lambda(t)$ in $\mathbb{R}$ and $\mu(t)$ in $\mathbb{C}^{*}$, defined on a small
enough interval $] 0, \varepsilon\left[\right.$, such that $\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty$ and $\lim _{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=c$ and that:

$$
\begin{equation*}
\lambda(t) \mathbf{z}(t)=\mu(t) \overline{\mathrm{d} f}(\mathbf{z}(t), \overline{\mathbf{z}}(t))+\bar{\mu}(t) \overline{\mathrm{d}} f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) . \tag{4.4.1}
\end{equation*}
$$

Consider the expansion of $f(\mathbf{z}(t), \overline{\mathbf{z}}(t))$. We have two situations, either:

$$
\begin{equation*}
f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \equiv c \tag{4.4.2}
\end{equation*}
$$

or

$$
\begin{equation*}
f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=c+b t^{\delta}+\text { h.o.t., } \quad \text { where } c, b \in \mathbb{C}, b \neq 0, \delta \in \mathbb{N}^{*} . \tag{4.4.3}
\end{equation*}
$$

Let $I=\left\{i \mid z_{i}(t) \not \equiv 0\right\}$, observe that $I \neq \emptyset$ since $\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty$, and write:

$$
\begin{equation*}
z_{i}(t)=a_{i} t^{p_{i}}+\text { h.o.t., } \quad \text { where } \quad a_{i} \neq 0, p_{i} \in \mathbb{Z}, i \in I . \tag{4.4.4}
\end{equation*}
$$

By eventually transposing the coordinates, we may assume that $I=\{1, \ldots, m\}$ and that $p=p_{1} \leq p_{2} \leq \cdots \leq p_{m}$. Since $\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty$, this implies $p=\min _{j \in I}\left\{p_{j}\right\}<0$. We denote $\left.\mathbf{a}=\left(a_{1}, \ldots, a_{m}\right) \in \mathbb{C}^{* I}, \underline{\mathbf{p}=\left(p_{1}\right.}, \ldots, p_{m}\right) \in \mathbb{Z}^{m}$ and consider the linear function $l_{\mathbf{p}}=\sum_{i=1}^{m} p_{i} x_{i}$ defined on $\overline{\operatorname{supp}\left(f^{I}\right)}$.

Let us observe that since $f(0)=0$, if $c \neq 0$, then $\overline{\operatorname{supp}\left(f^{I}\right)}$ is not empty in both situations (4.4.2) and (4.4.3). Let then $\Delta$ be the maximal face of $\overline{\operatorname{supp}\left(f^{I}\right)}$ where $l_{\mathbf{p}}$ takes its minimal value, say $d_{\mathbf{p}}$. We have:

$$
\begin{equation*}
f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=f^{I}(\mathbf{z}(t), \overline{\mathbf{z}}(t))=f_{\Delta}^{I}(\mathbf{a}, \overline{\mathbf{a}}) t^{d_{\mathbf{p}}}+\text { h.o.t. } \tag{4.4.5}
\end{equation*}
$$

where $d_{\mathbf{p}} \leq \operatorname{ord}_{\mathrm{t}}(f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=0$.
In the following we keep the assumption ${ }^{1} c \neq 0$. For $i \in I$ we have the equalities: $\frac{\partial f}{\partial z_{i}}(\mathbf{z}(t), \overline{\mathbf{z}}(t))=\frac{\partial f^{I}}{\partial z_{i}}(\mathbf{z}(t), \overline{\mathbf{z}}(t))$ and $\frac{\partial f}{\partial \bar{z}_{i}}(\mathbf{z}(t), \overline{\mathbf{z}}(t))=\frac{\partial f^{I}}{\partial \bar{z}_{i}}(\mathbf{z}(t), \overline{\mathbf{z}}(t))$. Then we may write:

$$
\begin{align*}
\frac{\partial f}{\partial z_{i}}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) & =\frac{\partial f_{\Delta}^{I}}{\partial z_{i}}(\mathbf{a}, \overline{\mathbf{a}}) t^{d_{\mathbf{p}}-p_{i}}+\text { h.o.t. }  \tag{4.4.6}\\
\frac{\partial f}{\partial \bar{z}_{i}}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) & =\frac{\partial f_{\Delta}^{I}}{\partial \bar{z}_{i}}(\mathbf{a}, \overline{\mathbf{a}}) t^{d_{\mathbf{p}}-p_{i}}+\text { h.o.t. }
\end{align*}
$$

Consider the expansion of $\lambda(t)$, in case $\lambda(t) \not \equiv 0$, and that of $\mu(t)$ :

$$
\begin{array}{ll}
\lambda(t)=\lambda_{0} t^{\gamma}+\text { h.o.t., } & \text { where } \lambda_{0} \in \mathbb{R}^{*}, \gamma \in \mathbb{Z} \\
\mu(t)=\mu_{0} t^{l}+\text { h.o.t., } & \text { where } \mu_{0} \neq 0, l \in \mathbb{Z}
\end{array}
$$

[^1]Using all the expansions we get from (4.4.1), for any $i \in I$ :

$$
\left(\mu_{0} \frac{\overline{\partial f_{\Delta}^{I}}}{\partial z_{i}}(\mathbf{a}, \overline{\mathbf{a}})+\overline{\mu_{0}} \frac{\partial f_{\Delta}^{I}}{\partial \bar{z}_{i}}(\mathbf{a}, \overline{\mathbf{a}})\right) t^{d_{\mathbf{p}}-p_{i}+l}+\text { h.o.t. }=\lambda_{0} a_{i} t^{p_{i}+\gamma}+\text { h.o.t. }
$$

Since $\lambda_{0} a_{i} \neq 0$, comparing the orders of the two sides in the above formula, we obtain:

$$
\mu_{0} \frac{\overline{\partial f_{\Delta}^{I}}}{\partial z_{i}}(\mathbf{a}, \overline{\mathbf{a}})+\overline{\mu_{0}} \frac{\partial f_{\Delta}^{I}}{\partial \bar{z}_{i}}(\mathbf{a}, \overline{\mathbf{a}})= \begin{cases}\lambda_{0} a_{i}, & \text { if } d_{\mathbf{p}}-p_{i}+l=p_{i}+\gamma  \tag{4.4.7}\\ 0, & \text { if } d_{\mathbf{p}}-p_{i}+l<p_{i}+\gamma\end{cases}
$$

Let $J=\left\{j \in I \mid d_{\mathbf{p}}-p_{j}+l=p_{j}+\gamma\right\}$. If we suppose that $J \neq \emptyset$, then $J=\{j \in$ $\left.I \mid p_{j}=p=\min _{j \in I}\left\{p_{j}\right\}<0\right\}$. In the situation (4.4.3) we have $\frac{\mathrm{d} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t}=b \delta t^{\delta-1}+$ h.o.t and on the other hand:

$$
\begin{align*}
\frac{\mathrm{d} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t}= & \sum_{i=1}^{m}\left(\frac{\partial f}{\partial z_{i}} \cdot \frac{\partial z_{i}}{\partial t}+\frac{\partial f}{\partial \overline{z_{i}}} \cdot \frac{\partial \overline{z_{i}}}{\partial t}\right)=\sum_{i=1}^{m}\left(\frac{\partial f^{I}}{\partial z_{i}} \cdot \frac{\partial z_{i}}{\partial t}+\frac{\partial f^{I}}{\partial \overline{z_{i}}} \cdot \frac{\partial \overline{z_{i}}}{\partial t}\right) \\
& =\left[\left\langle\mathbf{p a}, \overline{\mathrm{d} f_{\Delta}^{I}}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle+\left\langle\mathbf{p} \overline{\mathbf{a}}, \overline{\mathrm{d}} f_{\Delta}^{I}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle\right] t^{d_{\mathbf{p}}-1}+\text { h.o.t. } \tag{4.4.8}
\end{align*}
$$

where pa $=\left(p_{1} a_{1}, \ldots, p_{m} a_{m}\right)$. Comparing the orders of the two expansions of $\frac{\mathrm{d} f(\mathbf{z}(t), \mathbf{z}(t))}{\mathrm{d} t}$ and using the inequality $d_{\mathbf{p}}<\delta$ implied by $c \neq 0$ (see after (4.4.5)), we find:

$$
\begin{equation*}
\left\langle\mathbf{p a}, \overline{\mathrm{d} f_{\Delta}^{I}}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle+\left\langle\mathbf{p} \overline{\mathbf{a}}, \overline{\overline{\mathrm{d}} f_{\Delta}^{I}(\mathbf{a}, \overline{\mathbf{a}})}\right\rangle=0 . \tag{4.4.9}
\end{equation*}
$$

Let us observe here that the proof of formula (4.4.9) holds under the more general condition $d_{\mathbf{p}}<\delta$.

Let now consider the situation (4.4.2). In this case the formula (4.4.9) is true more directly, since $\frac{\mathrm{d} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t}=0$ and after comparing this to (4.4.8).

Next, multiplying (4.4.9) by $\bar{\mu}_{0}$ and taking the real part, we get:

$$
\begin{gather*}
\operatorname{Re}\left\langle\mathbf{p a}, \mu_{0} \overline{\mathrm{~d} f_{\Delta}^{I}}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle+\operatorname{Re}\left\langle\mathbf{p} \overline{\mathbf{a}}, \mu_{0} \overline{\overline{\mathrm{~d}} f_{\Delta}^{I}(\mathbf{a}, \overline{\mathbf{a}})}\right\rangle  \tag{4.4.10}\\
=\operatorname{Re}\left\langle\mathbf{p a}, \mu_{0} \overline{\mathrm{~d} f_{\Delta}^{I}}(\mathbf{a}, \overline{\mathbf{a}})+\bar{\mu}_{0} \overline{\mathrm{~d}} f_{\Delta}^{I}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle=0 .
\end{gather*}
$$

On the other hand, from (4.4.7), we have:

$$
\operatorname{Re}\left\langle\mathbf{p a}, \mu_{0} \overline{\mathrm{~d} f_{\Delta}^{I}}(\mathbf{a}, \overline{\mathbf{a}})+\bar{\mu}_{0} \overline{\mathrm{~d}} f_{\Delta}^{I}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle=\sum_{i \in J} \lambda_{0} p\left\|a_{j}\right\|^{2}
$$

which is different from zero since $\lambda_{0} \neq 0, p<0$ and $a_{j} \neq 0$. This contradicts formula (4.4.10). We have therefore proved that $J=\emptyset$.

From (4.4.7) we obtain:

$$
\begin{equation*}
\mu_{0} \overline{\mathrm{~d} f_{\Delta}^{I}}(\mathbf{a}, \overline{\mathbf{a}})+\bar{\mu}_{0} \overline{\mathrm{~d}} f_{\Delta}^{I}(\mathbf{a}, \overline{\mathbf{a}})=0 \tag{4.4.11}
\end{equation*}
$$

Let us observe that in case $\lambda(t) \equiv 0$ we have $J=\emptyset$ and therefore we get directly (4.4.11).

What (4.4.11) tells us is that a is a singularity of $f_{\Delta}^{I}$. Set now $\mathbf{A}=(\mathbf{a}, 1,1, \ldots, 1)$ with the $i^{\text {th }}$ coordinate $z_{i}=1$ for $i \notin I$. Since $\Delta \subset \overline{\operatorname{supp}\left(f^{I}\right)}$, the restriction $f_{\Delta}$ does not depend on the variables $z_{m+1}, \ldots, z_{n}$ or their conjugates. Thus for any $i \in\{1,2, \ldots, n\}$, we have $\frac{\partial f_{\Delta}}{\partial \bar{z}_{i}}(\mathbf{z}(t), \overline{\mathbf{z}}(t))=\frac{\partial f_{\Delta}^{I}}{\partial \bar{z}_{i}}(\mathbf{z}(t), \overline{\mathbf{z}}(t))$ and $\frac{\partial f_{\Delta}}{\partial z_{i}}(\mathbf{z}(t), \overline{\mathbf{z}}(t))=$ $\frac{\partial f_{\Delta}^{I}}{\partial z_{i}}(\mathbf{z}(t), \overline{\mathbf{z}}(t))$. By replacing $f_{\Delta}^{I}$ with $f_{\Delta}$ in (4.4.11), we get that $\mathbf{A} \in \mathbb{C}^{* n}$ is a singularity of $f_{\Delta}$.
We may now apply Lemma 4.2 .5 to $d_{\mathbf{p}}$ and $\Delta$. We have the following two cases:
(I). If $d_{\mathbf{p}}<0$, then, by Lemma 4.2.5(a), $\Delta$ is a face of $\Gamma^{+}\left(f^{I}\right)$. Since $\mathbf{A} \in \mathbb{C}^{* n}$ is a singularity of $f_{\Delta}$ and since we have $f_{\Delta}(\mathbf{A}, \overline{\mathbf{A}})=0$ by (4.4.5) for $d_{\mathbf{p}}<0$, this contradicts the Newton non-degeneracy of $f$ (Definition 4.2.2) assumed in the statement of Theorem 4.1.3.
(II). Let $d_{\mathbf{p}}=0$. Then $c=f_{\Delta}^{I}(\mathbf{a}, \overline{\mathbf{a}})=f_{\Delta}(\mathbf{A}, \overline{\mathbf{A}}) \in f_{\Delta}\left(\operatorname{Sing} f_{\Delta} \cap \mathbb{C}^{* n}\right)$. By Lemma 4.2.5(b), $\Delta$ is either a face of $\Gamma^{+}\left(f^{I}\right)$ or satisfies the condition (ii) of Definition 4.2.3. Note that these two conditions are exclusive, which fact follows immediately from the definitions. Let us show that $\Delta$ is a bad face of $\overline{\operatorname{supp}(f)}$.
Let $d$ denote the minimal value of the restriction of $l_{\mathbf{p}}$ to $\overline{\operatorname{supp}(f)}$. Since $\overline{\operatorname{supp}\left(f^{I}\right)}=\overline{\operatorname{supp}(f)} \cap \mathbb{R}_{>0}^{I}$, we have $d \leq d_{\mathbf{p}}=0$. Let $H$ be the hyperplane defined by the equation $\underline{\sum_{i=1}^{\bar{m}} p_{i} x_{i}+q \sum_{i=m+1}^{n} x_{i}=0 \text {, where } q>-d+1>0 \text {. Then, }}$ for any $\left(x_{1}, \ldots, x_{n}\right) \in \overline{\operatorname{supp}(f)} \backslash \overline{\operatorname{supp}\left(f^{I}\right)}$, the value of $\sum_{i=1}^{m} p_{i} x_{i}+q \sum_{i=m+1}^{n} x_{i}$ is positive. We therefore get $\Delta=\overline{\operatorname{supp}\left(f^{I}\right)} \cap H=\overline{\operatorname{supp}(f)} \cap H$.

If $\Delta$ does not satisfy condition (i)(a) of Definition 4.2.3, then we have $m=n$ and $p_{i} \leq 0$ for all $1 \leq i \leq n$. Since by hypothesis $f$ depends effectively on all variables, in particular on the variable $z_{1}$, the value $d_{\mathbf{p}}$ must be negative, which is a contradiction to the above original assumption.

This ends our proof.
REmark 4.4.1 The equality (4.4.11) is the key of the above proof of Theorem 4.1.3(a). If $c=0$, then we have two cases in situation (4.4.3):
(1) If $d_{\mathbf{p}}=\operatorname{ord}_{t}(f(\mathbf{z}(t), \overline{\mathbf{z}}(t))$, then formula (4.4.11) might be not true.
(2) If $d_{\mathbf{p}}<\operatorname{ord}_{t}(f(\mathbf{z}(t), \overline{\mathbf{z}}(t))$, then we get the same proof of formula (4.4.11) as in Proof of (a) (see the remark after formula (4.4.9)).

REmARK 4.4.2 Let $\Sigma^{\infty}:=\left\{c \in \mathbb{C} \mid f^{-1}(c) \cap M(f)\right.$ is not bounded $\}$. Under the hypotheses of Theorem 4.1.3, the above proof also shows that if $c \in \Sigma^{\infty}$ and $c \neq 0$ then $c$ is a critical value of $f_{\Delta}$, for some bad face $\Delta$. Indeed, if the path $\mathbf{z}(t) \subset$
$M(f) \cap f^{-1}(c)$ is not bounded, then it must be included in the singular locus Sing $f^{-1}(c)$ since the fibre $f^{-1}(c)$ is an algebraic set. (An alternate argument may be extracted from the last part of the proof of Proposition 3.3.1). This shows the inclusion $\Sigma^{\infty} \subset S(f) \cap f(\operatorname{Sing} f)$. By Theorem 4.1.3(a) we then have $\Sigma^{\infty} \backslash\{0\} \subset$ $\bigcup_{\Delta \in \mathfrak{B}} f_{\Delta}\left(\operatorname{Sing} f_{\Delta}\right)$.

### 4.4.2 Proof of Theorem 4.1.3(b).

By absurd, let us suppose $f(\operatorname{Sing} f)$ is not bounded. Since $\operatorname{Sing} f$ is a semi-algebraic set, by Curve Selection Lemma 3.2.5, there exists a real analytic path $\mathbf{z}(t) \subset \operatorname{Sing} f$ defined on a small enough interval $] 0, \varepsilon[$ such that:

$$
\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty, \text { and } \lim _{t \rightarrow 0}|f(\mathbf{z}(t), \overline{\mathbf{z}}(t))|=\infty
$$

We follow the proof of (a). Since $\mathbf{z}(t) \subset \operatorname{Sing} f$, we have $\lambda(t) \equiv 0$ in (4.4.1) and therefore we obtain (4.4.11) directly, as remarked after it. From $\lim _{t \rightarrow 0}|f(\mathbf{z}(t), \overline{\mathbf{z}}(t))|=$ $\infty$ it follows that $d_{\mathbf{p}} \leq \operatorname{ord}_{\mathrm{t}}(f(\mathbf{z}(t), \overline{\mathbf{z}}(t))<0$. We are in the situation of (I) from the proof of Theorem 4.1.3(a) but without being able to insure the equality $f_{\Delta}(\mathbf{A}, \overline{\mathbf{A}})=$ 0 . That is why we need here the Newton strong non-degeneracy in order to get a contradiction.

To prove that $f_{\Delta}\left(\operatorname{Sing} f_{\Delta}\right)$ is bounded, for any bad face $\Delta \subset \overline{\operatorname{supp}(f)}$, we use Remark 4.2.7 and the above proof for $f_{\Delta}$ in place of $f$.

Since $\overline{\operatorname{supp}(f)}$ has finitely many faces and since, by Theorem 4.1.3(a), we have the inclusion $S(f) \subset\{0\} \cup \underset{\Delta \in \mathfrak{B}}{\cup} f_{\Delta}\left(\operatorname{Sing} f_{\Delta}\right)$, it follows that $S(f)$ is bounded.

### 4.4.3 Some consequences

We get some sharper statements for significant particular classes of non-degenerate mixed polynomials. The following result extends the one for holomorphic polynomials proved in [Kus76, Bro88].

Corollary 4.4.3 If $f$ is a mixed Newton non-degenerate and convenient polynomial, then $S(f)=\emptyset$.

Proof. Under the same notations and definitions as in the proof of Theorem 4.1.3(a), since $l_{\mathbf{p}}(x)=\sum_{i=1}^{m} p_{i} x_{i}$ has at least a coefficient $p_{j}<0$ for some $j$ and the intersection of $\operatorname{supp}(f)$ with each positive coordinate axis is non-empty, the value of $l_{\mathbf{p}}(x)$ at a point of the intersection of $\operatorname{supp}(f)$ with the $j$-axis is negative. This implies that the minimal value $d_{\mathbf{p}}$ is negative. By Lemma 4.2.5(a), $\Delta$ is a face of $\Gamma^{+}(f)$.

### 4.5. Families of mixed polynomials and stability of the monodromy at infinity

Since we have here $d_{\mathbf{p}}<\operatorname{ord}_{t}(f(\mathbf{z}(t), \overline{\mathbf{z}}(t))$, by using Remark 4.4.1, we get formula (4.4.11) and a singularity $\mathbf{A} \in \mathbb{C}^{* n}$ of $f_{\Delta}$ with $f_{\Delta}(\mathbf{A})=0$ as in (I) above. This contradicts the Newton non-degeneracy of $f$.

Corollary 4.4.4 Let $f$ be a mixed polynomial, radial weighted-homogeneous and Newton strongly non-degenerate at infinity. Then:
(a) $\operatorname{Sing} f \cap \mathbb{C}^{* n}=\emptyset$,
(b) $S(f) \cup f(\operatorname{Sing} f) \subset\{0\}$.

Proof. Since $f$ is radial weighted-homogeneous, let's say of degree $m$, we have $f(0)=0$ and $\overline{\operatorname{supp}(f)}$ is contained in a single hyperplane which does not pass through the origin. Therefore the Newton boundary $\Gamma^{+}(f)$ has a single maximal face and its strong non-degeneracy implies $\operatorname{Sing} f \cap \mathbb{C}^{* n}=\emptyset$. Since $\overline{\operatorname{supp}(f)}$ has no bad face and since by Theorem 4.1.3(a) we have $S(f) \subset\{0\} \cup \underset{\Delta \in \mathfrak{B}}{\cup} f_{\Delta}\left(\operatorname{Sing} f_{\Delta}\right)$, it follows that $S(f) \subset\{0\}$.

By absurd, let us suppose that $c \in f(\operatorname{Sing} f) \cap \mathbb{C}^{*}$. For any $\mathbf{z} \in \operatorname{Sing} f$ such that $f(\mathbf{z}, \overline{\mathbf{z}})=c$, there exists $\lambda \in S_{1}^{1}$ such that $\overline{\mathrm{d} f}(\mathbf{z}, \overline{\mathbf{z}})=\lambda \overline{\mathrm{d}} f(\mathbf{z}, \overline{\mathbf{z}})$. Multiplying by $t^{m-q_{i}}$ the equalities $\frac{\overline{\partial f}}{\partial z_{i}}(\mathbf{z}, \overline{\mathbf{z}})=\lambda \frac{\partial f}{\partial \bar{z}_{i}}(\mathbf{z}, \overline{\mathbf{z}})$ for $i=1,2 \ldots, n$, and using that $f$ is radial weighted-homogeneous, we get $\overline{\mathrm{d} f}(t \circ \mathbf{z}, t \circ \overline{\mathbf{z}})=\lambda \overline{\mathrm{d}} f(t \circ \mathbf{z}, t \circ \overline{\mathbf{z}})$. This implies that $t \circ \mathbf{z} \in \operatorname{Sing} f$ and $t^{m} c \in f(\operatorname{Sing} f)$, therefore $f(\operatorname{Sing} f)$ is not bounded, which contradicts Theorem 4.1.3(b). This proves that $f(\operatorname{Sing} f) \subset\{0\}$.

### 4.5 Families of mixed polynomials and stability of the monodromy at infinity

As a consequence of Theorems 3.1.8 and 4.1.3(b), the class of Newton strongly non-degenerate polynomials $f$ has the property that $B(f)$ is bounded. One has the following general definition.

## Definition 4.5.1 (Monodromy at infinity)

Let $f: \mathbb{R}^{2 n} \rightarrow \mathbb{R}^{2}$ be a real polynomial map and assume that the bifurcation set $B(f)$ is bounded. Let $\delta_{0}>0$ such that $B(f)$ is included in the open disk $D_{\delta_{0}}$ of radius $\delta_{0}$ centered at $0 \in \mathbb{C}$. We call monodromy (fibration) at infinity the fibration:

$$
f_{l}: f^{-1}\left(S_{\delta}^{1}\right) \rightarrow S_{\delta}^{1} .
$$

over some circle $S_{\delta}^{1}$ of radius $\delta$ which, by the Fibration Theorem 3.1.8, exists and is independent of $\delta \geq \delta_{0}$.

We then prove the following result:

## Theorem 4.5.2 Stability of monodromy at infinity

Let $F_{s}(\mathbf{z}, \overline{\mathbf{z}}):=F(\mathbf{z}, \overline{\mathbf{z}}, s): \mathbb{R}^{2 n} \rightarrow \mathbb{R}^{2}$ be a family of Newton strongly non-degenerate polynomials depending analytically on a parameter $s$, where $s \in[0,1]$. If the Newton boundary $\Gamma^{+}\left(F_{s}\right)$ is constant in this family, then the monodromy at infinity is stable ${ }^{2}$.

For the proof of the theorem, we need some preliminaries. Let $F_{s}$ stand for $F(\mathbf{z}, \overline{\mathbf{z}}, s)$, let $F(\operatorname{Sing} F):=\underset{s \in[0,1]}{\bigcup} F_{s}\left(\operatorname{Sing} F_{s}\right), S(F):=\underset{s \in[0,1]}{\bigcup} S\left(F_{s}\right)$. We also consider the restriction $F_{s, \Delta}$ of $F_{s}$ to some face $\Delta$ of $\overline{\operatorname{supp} F_{s}}$ and write $F_{\Delta}(\mathbf{z}, \overline{\mathbf{z}}, s):=F_{s, \Delta}$.

Proposition 4.5.3 Under the assumption of Theorem 4.5.2, the set $F(\operatorname{Sing} F) \cup$ $S(F)$ is bounded.

Proof. If $F(\operatorname{Sing} F)$ were not bounded then, by Curve Selection Lemma 3.2.5, there exist analytic paths $\mathbf{z}(t) \in \mathbb{C}^{n}, \lambda(t) \in S_{1}^{1}$ and $s(t) \in[0,1]$ defined on a small enough interval $] 0, \varepsilon[$ such that

$$
\begin{align*}
& \lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty, \lim _{t \rightarrow 0} F(\mathbf{z}(t), \overline{\mathbf{z}}(t), s(t))=\infty,  \tag{4.5.1}\\
& \lim _{t \rightarrow 0} s(t)=s_{0}, \overline{\mathrm{~d} F_{s(t)}}(\mathbf{z}(t), \overline{\mathbf{z}}(t))=\lambda(t) \overline{\mathrm{d}} F_{s(t)}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) . \tag{4.5.2}
\end{align*}
$$

We may then apply the proof of Theorem 4.1.3(b) and find a face $\Delta$ of $\overline{\operatorname{supp}\left(F_{s(t)}^{I}\right)}$, which by assumption is independent of $s$, such that $F_{s(t), \Delta}^{I}$ has a singularity in $\mathbb{C}^{* n}$. By using Remark 4.2.6, this contradicts the Newton strong nondegeneracy.

To show that $S(F)$ is bounded, we proceed as follows. By Theorem 4.1.3(a), one has the inclusion $S(F) \subset \cup_{s \in[0,1]}\left\{F_{s}(0)\right\} \cup_{s \in[0,1]} \cup_{\Delta \in \mathfrak{B}_{s}} F_{s, \Delta}\left(\operatorname{Sing} F_{s, \Delta} \cap \mathbb{C}^{* n}\right)$ where $\mathfrak{B}_{s}$ is the set of bad faces of $\overline{\operatorname{supp}\left(F_{s}\right)}$ for $s \in[0,1]$. We have that $\cup_{s}\left\{F_{s}(0)\right\}$ is bounded by the continuity with respect to $s$, and that $\left\{\mathfrak{B}_{s}\right\}_{s \in[0,1]}$ is a finite set since $\Gamma^{+}\left(F_{s}\right)$ is independent of $s$. If $S(F)$ were not bounded, then we may assume that $F_{\Delta_{0}(s)}\left(\operatorname{Sing} F_{\Delta_{0}(s)} \cap \mathbb{C}^{* n}\right)$ is not bounded as $s \rightarrow s_{0}$, for some bad face $\Delta_{0}(s)$ which is actually independent of $s$ in some small enough interval $] s_{0}-\varepsilon, s_{0}+\varepsilon[$. Since $\Gamma^{+}\left(F_{s}\right)$ is independent of $s$ and since $\Gamma^{+}\left(F_{s, \Delta_{0}}\right) \subset \Gamma^{+}\left(F_{s}\right)$, we get that $\Gamma^{+}\left(F_{s, \Delta_{0}}\right)$ is independent of $s$ within a neighborhood of $s_{0}$. We may then apply the above proof to $F_{\Delta_{0}}$ in place of $F$.

Proposition 4.5.4 Under the assumption of Theorem 4.5.2, there exists $r_{0}>0$ such that, for any $r \geq r_{0}$, there exists $R_{0}(r) \gg 1$ such that one has the transversality $f_{s}^{-1}(c) \pitchfork S_{R}^{2 n-1}, \forall c \in S_{r}^{1}, \forall R \geq R_{0}(r)$ and $\forall s \in[0,1]$.

[^2]Proof. The above Proposition 4.5.3 implies that there exists $r_{0}>0$ independent on $s \in[0,1]$ such that the following inclusion holds:

$$
\begin{equation*}
F(\operatorname{Sing} F) \cup_{s \in[0,1]}\left\{F_{s}(0)\right\} \cup_{s \in[0,1]} \underset{\Delta \in \mathfrak{B}_{s}}{\cup} F_{s, \Delta}\left(\operatorname{Sing} F_{s, \Delta} \cap \mathbb{C}^{* n}\right) \subset \stackrel{\circ}{D}_{r_{0}} \tag{4.5.3}
\end{equation*}
$$

If the above assertion were not true, then by Curve Selection Lemma 3.2.5 there exist analytic paths $\mathbf{z}(t) \subset \mathbb{C}^{n}, \lambda(t) \subset \mathbb{R}, \mu(t) \subset \mathbb{C}^{*}$ and $s(t) \subset[0,1]$ such that:

$$
\begin{align*}
& \lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty, \lim _{t \rightarrow 0} F(\mathbf{z}(t), \overline{\mathbf{z}}(t), s(t))=c \in S_{r}^{1}  \tag{4.5.4}\\
& \lim _{t \rightarrow 0} s(t)=s_{0}, \lambda(t) \mathbf{z}(t)=\mu(t) \overline{\mathrm{d} F}(\mathbf{z}(t), \overline{\mathbf{z}}(t), s(t))+\bar{\mu}(t) \overline{\mathrm{d}} F(\mathbf{z}(t), \overline{\mathbf{z}}(t), s(t)) \tag{4.5.5}
\end{align*}
$$

By a similar analysis as in the proof of Theorem 4.1.3(a) one finds a singular point $\mathbf{A} \in \mathbb{C}^{* n}$ of $F_{\Delta}$ where $\Delta$ is either a face of $\Gamma^{+}\left(F_{s}\right)$ or a bad face of $\overline{\operatorname{supp}\left(F_{s}\right)}$. This contradicts (4.5.3).

### 4.5.1 Proof of Theorem 4.5.2

By the above two propositions, for $r \geq r_{0}$, the global monodromy fibration $F_{s \mid}$ : $F_{s}^{-1}\left(S_{r}^{1}\right) \rightarrow S_{r}^{1}$ is diffeomorphic to the fibration

$$
\begin{equation*}
F_{s \mid}: F^{-1}\left(S_{r}^{1}\right) \cap B_{R} \rightarrow S_{r}^{1} \tag{4.5.6}
\end{equation*}
$$

for all $R \geq R_{0}(r)$ and all $s \in[0,1]$.
Consider the map $\tilde{F}: \mathbb{C}^{n} \times I \rightarrow \mathbb{C} \times I,(\mathbf{z}, s) \mapsto\left(F_{s}(\mathbf{z}, \overline{\mathbf{z}}), s\right)$, where $I:=[0,1]$.
The above proposition show that the restriction $\tilde{F}_{\mid}: \tilde{F}^{-1}\left(S_{r}^{1} \times I\right) \cap\left(B_{R} \times I\right) \rightarrow$ $S_{r}^{1} \times I$ is a proper submersion on the couple of manifolds $\left(\tilde{F}^{r}\left(S_{r}^{1} \times I\right) \cap\left(B_{R} \times\right.\right.$ I), $\left.\tilde{F}^{-1}\left(S_{r}^{1} \times I\right) \cap\left(\partial B_{R} \times I\right)\right)$. Then Ehresmann's theorem tells that the fibrations (4.5.6) are isotopic for varying $s$.

Theorem 4.5.2 appears to be useful in finding the topology of the non atypical fibres for Newton strongly non-degenerate mixed polynomial. As another consequence, one may extend the range of applicability of the stability theorems in [NZ92, Theorem 17] and [Pha08, Theorem 1.1], as follows:

Corollary 4.5.5 If $f$ and $g$ are two Newton strongly non-degenerate mixed polynomials, such that $\Gamma^{+}(f)=\Gamma^{+}(g)$ and that their restrictions to the boundaries at infinity $f_{\Gamma^{+}}$and $g_{\Gamma^{+}}$are both holomorphic (or both anti-holomorphic), then the monodromies at infinity of $f$ and of $g$ are isotopic.

Proof. In the holomorphic setting, the Newton strong non-degeneracy condition at infinity is the same as Newton non-degenerate and is a Zariski open and connected condition. This holds for anti-holomorphic instead of holomorphic. This
fact allows us to connect $f$ to $g$ by a family of Newton strongly non-degenerate mixed polynomials. For instance, one may do as follows. First, one applies [Pha08, Theorem 1.1] to the restrictions $f_{\Gamma^{+}}$and $g_{\Gamma^{+}}$. Next, we write $f=f_{\Gamma^{+}}+h$ and observe that the family of mixed polynomials $F_{s}:=f_{\Gamma^{+}}+(1-s) h$ satisfies the hypotheses of our Theorem 4.5.2 and connects $f$ to $f_{\Gamma^{+}}$, hence the monodromy is stable in this family. A similar construction for $g$ completes the picture and ends our proof.

### 4.6 Some useful examples

EXAMPLE 4.6.1 Let $f: \mathbb{R}^{4} \rightarrow \mathbb{R}^{2}, f=z_{1} z_{2}+\bar{z}_{1}^{2} \bar{z}_{2}^{2}$. This is a Newton strongly non-degenerate mixed polynomial, where $\Gamma^{+}(f)=(2,2)$ and $\overline{\operatorname{supp}(f)}$ consists of just one face $\Delta$ which is a bad face. The solutions of $\overline{\mathrm{d} f}(\mathbf{z}, \overline{\mathbf{z}})=\lambda \overline{\mathrm{d}} f(\mathbf{z}, \overline{\mathbf{z}})$ for $\lambda \in S_{1}^{1}$ are $\left\{z_{1} z_{2}=\frac{1}{2 \bar{\lambda}}\right\} \cup\left\{z_{1}=z_{2}=0\right\}$. Thus we obtain $f\left(\operatorname{Sing} f_{\Delta}\right)=f(\operatorname{Sing} f)=$ $\{0\} \cup\left\{\left.\frac{1}{2 \bar{\lambda}}+\frac{1}{4 \lambda^{2}} \right\rvert\, \lambda \in S_{1}^{1}\right\}$. Taking $z_{1} z_{2}=\frac{1}{2 \bar{\lambda}}$ with $z_{1} \rightarrow 0$ and $z_{2} \rightarrow \infty$, by straightforward computations we get $f(\operatorname{Sing} f) \backslash\{0\} \subset S(f)$ and $\{0\} \notin S(f)$. On the other hand, for $\left\{\mathbf{z}^{k}\right\}_{k \in \mathbb{N}} \subset M(f) \backslash \operatorname{Sing} f$ such that $\lim _{k \rightarrow \infty}\left\|\mathbf{z}^{k}\right\|=\infty$, we get $\left|f\left(\mathbf{z}^{k}\right)\right| \rightarrow \infty$. This shows that $S(f) \backslash f(\operatorname{Sing}(f))=\emptyset$. Moreover, it yields that the inclusion of Theorem 4.1.3(a) may be strict.
In order to investigate how does the topology of fibers change, let us denote $D$ the domain bounded by the bifurcation set $B(f)=f(\operatorname{Sing} f)=\{0\} \cup\left\{\left.\frac{1}{2 \bar{\lambda}}+\frac{1}{4 \lambda^{2}} \right\rvert\, \lambda \in S_{1}^{1}\right\}$. On one hand, since $f=z_{1} z_{2}+\bar{z}_{1}^{2} \bar{z}_{2}^{2}$ is strongly non-degenerate at infinity and the polyhedron at infinity $\Gamma^{+}(f)$ is a single point $(2,2)$, by Theorem 4.5.2, the fibers at infinity of $f$ is isotopic to the one of $g=\bar{z}_{1}^{2} \bar{z}_{2}^{2}$. Hence the fiber out of $D$ can be calculated via the fiber of $g$ which is homeomorphic to $\mathbb{C}^{*} \sqcup \mathbb{C}^{*}$. On the other hand, for any $\lambda \in S_{1}^{1}$, the topology of the fiber $f=\frac{1}{2 \bar{\lambda}}+\frac{1}{4 \lambda^{2}}$ depends on $\lambda$. Let $z_{1} z_{2}=a+i b$ and $\frac{1}{2 \bar{\lambda}}+\frac{1}{4 \lambda^{2}}=c+i d$, where $a, b, c, d$ are real numbers. If $d=0$, then $c=-\frac{1}{4}$ or $\frac{3}{4}$. For $f=-\frac{1}{4}$, the solutions are $\left\{z_{1} z_{2}=\frac{1}{2} \pm i\right\} \cup\left\{z_{1} z_{2}=-\frac{1}{2}\right\}$. For $f=\frac{3}{4}$, the solutions are $\left\{z_{1} z_{2}=\frac{1}{2}\right\} \cup\left\{z_{1} z_{2}=-\frac{3}{2}\right\}$. If $d \neq 0$, then we get:

$$
4 a^{4}-(3+4 c) a^{2}+a(1+4 c)-c-d^{2}=0
$$

where $b=\frac{d}{1-2 a}$. The question is reduced to determine how many different real roots of the above equation, where $a$ is regarded as the variable. From the following Figure of Solutions, we see that in this case, there are two different real roots. (In the Figure, we take the argument of $\lambda$ as the vertical coordinate.) Now, we proceed to calculate the generic fiber inside of $D$. For example, taking $f=\frac{1}{5}$, we have the solutions $\left\{z_{1} z_{2}=\frac{-\sqrt{5} \pm 3}{2 \sqrt{5}}\right\} \cup\left\{z_{1} z_{2}=\frac{1}{2} \pm i \sqrt{\frac{11}{20}}\right\}$ and therefore the generic fiber inside of $D$ is homeomorphic to $\mathbb{C}^{*} \sqcup \mathbb{C}^{*} \sqcup \mathbb{C}^{*} \sqcup \mathbb{C}^{*}$. Finally, the fiber $f=0$ is
$\left\{z_{1}^{3} z_{2}^{3}=-1\right\} \cup\left\{z_{1} z_{2}=0\right\}$ which is homeomorphic to

$$
\mathbb{C}^{*} \sqcup \mathbb{C}^{*} \sqcup \mathbb{C}^{*} \sqcup\left\{\mathbb{C}^{2} \backslash \mathbb{C}^{* 2}\right\}
$$



Figure 4.1: Figure of $S(f)$


Figure 4.2: Figure of Solution
Our next example will show that in Theorem 4.1.3(a), the set of bad faces can not be replaced to the set of strictly bad faces.

EXAMPLE 4.6.2 Let $f: \mathbb{R}^{4} \rightarrow \mathbb{R}^{2}, f=\left|z_{1}\right|^{2}\left(z_{2}^{2}+2 z_{2} \bar{z}_{2}+1\right)$, which is Newton non-degenerate at infinity but not strongly non-degenerate at infinity. There is only one bad face of $f$ which is not a strictly bad face and the restriction to this face is $f_{\triangle_{1}}=\left|z_{1}\right|^{2}\left(z_{2}^{2}+2 z_{2} \bar{z}_{2}\right)$. We shall prove that $f(\operatorname{Sing} f)=\{0\} \cup \mathbb{R}_{+}, S(f)=$ $f_{\triangle_{1}}\left(\operatorname{Sing} f_{\triangle_{1}} \cap \mathbb{C}^{* 2}\right) \cup\{0\}$ and $B(f)=f(\operatorname{Sing} f) \cup S(f)$.
We begin by proving the non-degeneracy of $f$. In fact, there are three faces of
$\overline{\operatorname{supp}(f)}$ which are contained in $\Gamma^{+}(f)$. Let us write the restrictions of $f$ to these three faces: $f_{\triangle_{1}}=\left|z_{1}\right|^{2}\left(z_{2}^{2}+2 z_{2} \bar{z}_{2}\right), f_{\triangle_{2}}=\left|z_{1}\right|^{2}$ and $f_{\triangle_{3}}=f$. It is easy to see that $\left\{f_{\triangle_{i}}=0\right\} \cap \mathbb{C}^{* 2}=\emptyset$ and therefore $f$ is non-degenerate at infinity. On the other hand, since $f_{\triangle_{2}}=\left|z_{1}\right|^{2}$ is not strongly non-degenerate at infinity, the strong non-degeneracy fails for $f$.
Next, let us show that $f(\operatorname{Sing} f)=\{0\} \cup \mathbb{R}_{+}$. We have:

$$
\begin{aligned}
& \frac{\partial f}{\partial z_{1}}=\bar{z}_{1}\left(z_{2}^{2}+2 z_{2} \bar{z}_{2}+1\right) \\
& \frac{\partial f}{\partial z_{2}}=2\left|z_{1}\right|^{2}\left(z_{2}+\bar{z}_{2}\right) \\
& \frac{\partial f}{\partial \bar{z}_{1}}=z_{1}\left(z_{2}^{2}+2 z_{2} \bar{z}_{2}+1\right) \\
& \frac{\partial f}{\partial \bar{z}_{2}}=2\left|z_{1}\right|^{2} z_{2} .
\end{aligned}
$$

By the definition of mixed singularity, $\operatorname{Sing} f$ is the solutions of the following system:

$$
\begin{align*}
z_{1}\left(\bar{z}_{2}^{2}+2 z_{2} \bar{z}_{2}+1\right) & =\lambda z_{1}\left(z_{2}^{2}+2 z_{2} \bar{z}_{2}+1\right)  \tag{4.6.1}\\
2\left|z_{1}\right|^{2}\left(z_{2}+\bar{z}_{2}\right) & =\lambda 2\left|z_{1}\right|^{2} z_{2} \tag{4.6.2}
\end{align*}
$$

where $\lambda \in S_{1}^{1}$. We first conclude $\left\{z_{1}=0, z_{2} \in \mathbb{C}\right\} \cup\left\{z_{2}=0, z_{1} \in \mathbb{C}\right\} \subset \operatorname{Sing} f$. This gives $\{0\} \cup \mathbb{R}_{+} \subset f(\operatorname{Sing} f)$. Let us suppose $z_{1} z_{2} \neq 0$. From (4.6.2), we have $\lambda=\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ and $z_{2}+\bar{z}_{2}=\lambda z_{2}$. Applying these equalities in (4.6.1), we deduce $\lambda=1$ which is impossible. Thus $f(\operatorname{Sing} f)=\{0\} \cup \mathbb{R}_{+}$and by taking the analytic curve contained in $\left\{z_{1}=0, z_{2} \in \mathbb{C}\right\} \subset \operatorname{Sing} f \subset M(f)$, it follows that $\{0\} \in S(f)$. Finally, we shall prove $f_{\triangle_{1}}\left(\operatorname{Sing} f_{\Delta_{1}} \cap \mathbb{C}^{* 2}\right) \subset B(f)$. The singular locus of $f_{\triangle_{1}}$ is defined by (4.6.2) and the following equation:

$$
z_{1}\left(\bar{z}_{2}^{2}+2 z_{2} \bar{z}_{2}\right)=\lambda z_{1}\left(z_{2}^{2}+2 z_{2} \bar{z}_{2}\right)
$$

Suppose $z_{1} z_{2} \neq 0$, we have:

$$
\text { Sing } f_{\triangle_{1}} \cap \mathbb{C}^{* 2}=\left\{\left(z_{1}, z_{2}\right) \in \mathbb{C}^{* 2} \mid z_{2}+\bar{z}_{2}=\lambda z_{2}, \text { where } \lambda=\frac{1}{2} \pm \frac{\sqrt{3}}{2} \mathrm{i}\right\}
$$

Thus $f_{\triangle_{1}}\left(\operatorname{Sing} f_{\triangle_{1}} \cap \mathbb{C}^{* 2}\right)=\left(\frac{3}{2} \pm \frac{\sqrt{3}}{2} i\right) t$, where $t>0$. Now fix $t=t_{0}>0$, we consider a neighborhood $U$ of $\left(\frac{3}{2}+\frac{\sqrt{3}}{2} i\right) t_{0}$. Assume that $a+i b \in U \backslash f_{\triangle_{1}}\left(\operatorname{Sing} f_{\triangle_{1}} \cap \mathbb{C}^{* 2}\right)$, $\left|z_{1}\right|^{2}=c \neq 0$ and $z_{2}=x+i y$ where $(a, b) \in \mathbb{R}^{* 2}$ and $(x, y) \in \mathbb{R}^{2}$. From $f=a+i b$, it follows that:

$$
\begin{aligned}
c\left(3 x^{2}+y^{2}+1\right) & =a \\
-2 c x y & =b
\end{aligned}
$$

We therefore conclude from above equations that

$$
\begin{equation*}
3 b x^{2}+2 a x y+b\left(y^{2}+1\right)=0 . \tag{4.6.3}
\end{equation*}
$$

The discriminant of (4.6.3) is $\triangle=4 a^{2} y^{2}-12 b^{2}\left(y^{2}+1\right)$. Since there is no solution of (4.6.3) if and only if $a^{2}<3 b^{2}$, this implies that there exist no neighborhoods of $\left(\frac{3}{2}+\frac{\sqrt{3}}{2} i\right) t_{0}$ such that the restriction of $f$ is a locally trivial fibration. When $t_{0}$ varies, with the same procedure as above, we observe that there exist no neighborhoods of $\left(\frac{3}{2}-\frac{\sqrt{3}}{2} i\right) t$ such that the restriction of $f$ is a locally trivial fibration. This gives $f_{\triangle_{1}}\left(\operatorname{Sing} f_{\triangle_{1}} \cap \mathbb{C}^{* 2}\right) \subset B(f)$. On the other hand by Theorem 4.1.3(a), we have $S(f) \subset f_{\triangle_{1}}\left(\operatorname{Sing} f_{\triangle_{1}} \cap \mathbb{C}^{* 2}\right) \cup\{0\}$. Consequently, $B(f)=f(\operatorname{Sing} f) \cup S(f)=$ $f_{\Delta_{1}}\left(\operatorname{Sing} f_{\triangle_{1}} \cap \mathbb{C}^{* 2}\right) \cup\{0\} \cup \mathbb{R}_{+}$.

### 4.6.1 Family of twisted Brieskorn mixed polynomials

Let us first recall a join theorem proved by Cisneros-Molina:
Theorem 4.6.3 [CM08, Theorem 4.1] Let $g: \mathbb{C}^{n} \rightarrow \mathbb{C}$ and $h: \mathbb{C}^{m} \rightarrow \mathbb{C}$ be radial and polar weighted homogeneous mixed polynomials. Consider the polynomial on $\mathbb{C}^{n} \times \mathbb{C}^{m}$ defined by

$$
f(\mathbf{z}, \mathbf{w})=g(\mathbf{z}, \overline{\mathbf{z}})+h(\mathbf{w}, \overline{\mathbf{w}})
$$

which is also radial and polar weighted homogeneous polynomials. Let

$$
\begin{aligned}
X & =f^{-1}(1) \subseteq \mathbb{C}^{n} \times \mathbb{C}^{m} \\
Y & =g^{-1}(1) \subseteq \mathbb{C}^{n} \\
Z & =h^{-1}(1) \subseteq \mathbb{C}^{m}
\end{aligned}
$$

Then there is a homotopy equivalence from $X$ to the join $Y * Z$ which is compatible with the monodromy maps and their join.

Example 4.6.4 Consider the mixed Brieskorn polynomial $f: \mathbb{C}^{n} \rightarrow \mathbb{C}, f=$ $\sum_{i=1}^{n} z_{i}^{a_{i}+b_{i}} \bar{z}_{i}^{b_{i}}$. Since every monomial $z_{i}^{a_{i}+b_{i}} \bar{z}_{i}^{b_{i}}$ is radial and polar weighted homogeneous and $\sum_{i=1}^{n} z_{i}^{a_{i}+b_{i}} \bar{z}_{i}^{b_{i}}$ is also radial and polar weighted homogeneous, then by the above join theorem, the generic fiber of $f$ is homotopic equivalent to a bouquet $\vee S^{n-1}$ of spheres of real dimension $(n-1)$ and the number of spheres in the wedge is $\left(a_{1}-1\right)\left(a_{2}-1\right) \cdots\left(a_{n}-1\right)$.

Our next example shows that with a small deformation, the monodromy of a twisted Brieskorn mixed polynomial does not change.

Example 4.6.5 Let us consider a family of twisted Brieskorn mixed polynomials: $F_{s}(\mathbf{z}, \overline{\mathbf{z}})=\sum_{i=1}^{n} z_{i}^{a_{i}+b_{i}} \bar{z}_{i}^{b_{i}}+s \sum_{i=1}^{n} z_{i}^{a_{i}+2 b_{i}}$, where $a_{i}, b_{i} \in \mathbb{N}^{+}$for $1 \leq i \leq n$ and
$0 \leq s<\min _{1 \leq i \leq n} \frac{a_{i}}{a_{i}+2 b_{i}}$. Note that the restriction of $F_{s}$ to every face of the Newton boundary $\Gamma^{+}\left(F_{s}\right)$ is a sum of monomials $z_{i}^{a_{i}+b_{i}} \bar{z}_{i}^{b_{i}}+s z_{i}^{a_{i}+2 b_{i}}$ and:

$$
\begin{align*}
& \frac{\partial F_{s}}{\partial z_{i}}=\left(a_{i}+b_{i}\right) z_{i}^{a_{i}+b_{i}-1} \bar{z}_{i}^{b_{i}}+s\left(a_{i}+2 b_{i}\right) z_{i}^{a_{i}+2 b_{i}-1}  \tag{4.6.4}\\
& \frac{\partial F_{s}}{\partial \bar{z}_{i}}=b_{i} z_{i}^{a_{i}+b_{i}} \bar{z}_{i}^{b_{i}-1} . \tag{4.6.5}
\end{align*}
$$

When $z_{i} \neq 0$, by $0 \leq s<\min _{1 \leq i \leq n} \frac{a_{i}}{a_{i}+2 b_{i}}$ and triangle inequality, we get:

$$
\left|\frac{\partial F_{s}}{\partial z_{i}}\right| \geq\left|a_{i}+b_{i}-s\left(a_{i}+2 b_{i}\right)\right|\left|z_{i}\right|^{a_{i}+2 b_{i}-1}>b_{i}\left|z_{i}\right|^{a_{i}+2 b_{i}-1}=\left|\frac{\partial F_{s}}{\partial \bar{z}_{i}}\right| .
$$

It turns out that $F_{s}(\mathbf{z}, \overline{\mathbf{z}})$ is a family of Newton strongly non-degenerate polynomials. Thus, by Theorem 4.5.2, the monodromy at infinity of $F_{s}$ is isotopic to that of $F_{0}(\mathbf{z}, \overline{\mathbf{z}})=\sum_{i=1}^{n} z_{i}^{a_{i}+b_{i}} \bar{z}_{i}^{b_{i}}$. Using join theorem 4.6.3, the monodromy of $F_{0}$ at infinity is equivalent to $f=\sum_{i=1}^{n} z_{i}^{a_{i}}$.

### 4.6.2 King's example in the mixed case

In this subsection, we consider the following mixed version of King's example ([TZ99]) which shows that for a mixed polynomial with two variables, the inclusion $B(f) \subset f(\operatorname{Sing} f) \cup S(f)$ is strict. In the holomorphic case with two variables, $B(f)=f(\operatorname{Sing} f) \cup S(f)=f(\operatorname{Sing} f) \cup K_{\infty}(f)$, see e.g. [Par95, Dur98]

Example 4.6.6 Consider the mixed polynomial $f: \mathbb{C}^{2} \rightarrow \mathbb{C}$,

$$
f(x, y)=y\left(2|x|^{4}|y|^{4}-9|x|^{2}|y|^{2}+18\right)
$$

Let us show that $\operatorname{Sing} f=\emptyset, S(f)=\{0\}$ and $B(f)=\emptyset$.
We first prove that $f$ is strongly non-degenerate at infinity. There are three faces contained in $\Gamma^{+}(f)$ and the restrictions of $f$ to these faces are: $f_{\triangle_{1}}=18 y, f_{\triangle_{2}}=$ $2|x|^{4}|y|^{4} y$ and $f_{\triangle_{3}}=f$. Since $\frac{\partial f_{\Delta_{1}}}{\partial y}=18$ and $\frac{\partial f_{\Delta_{1}}}{\partial \bar{y}}=0$, we have $\operatorname{Sing} f_{\triangle_{1}}=\emptyset$. For $f_{\Delta_{2}}$, we have $\frac{\partial f_{\Delta_{2}}}{\partial y}=6|x|^{4}|y|^{4}$ and $\frac{\partial f_{\Delta_{2}}}{\partial \bar{y}}=4|x|^{4} y^{3} \bar{y}$. When $(x, y) \in \mathbb{C}^{* 2}$, the inequality $\left|\frac{\partial f_{\Delta_{2}}}{\partial y}\right|>\left|\frac{\partial f_{\Delta_{2}}}{\partial \bar{y}}\right|$ gives $\operatorname{Sing} f_{\triangle_{2}} \cap \mathbb{C}^{* 2}=\emptyset$. To deduce the strong non-degeneracy, it remains to prove $\operatorname{Sing} f \cap \mathbb{C}^{* 2}=\emptyset$. We have:

$$
\begin{aligned}
& \frac{\partial f}{\partial y}=6|x|^{4}|y|^{4}-9|x|^{2}|y|^{2}+18 \\
& \frac{\partial f}{\partial \bar{y}}=4|x|^{4} y^{3} \bar{y}-9|x|^{2} y^{2} \\
& \frac{\partial f}{\partial x}=y\left(4|y|^{4}|x|^{2} \bar{x}-9|y|^{2} \bar{x}\right) \\
& \frac{\partial f}{\partial \bar{x}}=y\left(4|y|^{4}|x|^{2} x-9|y|^{2} x\right) .
\end{aligned}
$$

By the definition of mixed singularity, $\operatorname{Sing} f$ consists of the solutions of the following system:

$$
\begin{align*}
6|x|^{4}|y|^{4}-9|x|^{2}|y|^{2}+18 & =\lambda\left(4|x|^{4} y^{3} \bar{y}-9|x|^{2} y^{2}\right)  \tag{4.6.6}\\
\bar{y}\left(4|y|^{4}|x|^{2} x-9|y|^{2} x\right) & =\lambda y\left(4|y|^{4}|x|^{2} x-9|y|^{2} x\right) \tag{4.6.7}
\end{align*}
$$

where $\lambda \in S_{1}^{1}$. From (4.6.6), we see $(x, y) \in \mathbb{C}^{* 2}$. Assume that $|x||y| \neq \frac{3}{2}$, then from (4.6.7), we get $y=\overline{\lambda y}$. Hence (4.6.7) is equivalent to the following:

$$
6|x|^{4}|y|^{4}-9|x|^{2}|y|^{2}+18=4|x|^{4}|y|^{4}-9|x|^{2}|y|^{2}
$$

We check at once that there is no solution of the above equation.
Now we suppose that $|x||y|=\frac{3}{2}$. Taking the norm on the two sides of (4.6.6), we get:

$$
6|x|^{4}|y|^{4}-9|x|^{2}|y|^{2}+18=0
$$

Since our assumption $|x||y|=\frac{3}{2}$ does not satisfy this equation, we have $\operatorname{Sing} f=\emptyset$. Consequently, $f$ is strongly non-degenerate at infinity.
Using Theorem 4.1.3(a), since $f$ does not have any strictly bad face, we get $S(f) \subset$ $\{0\}$. In order to show $S(f)=\{0\}$, we assume that $(x, y) \in M(f) \cap \mathbb{R}^{* 2}$. By the definition of $M(f)$, we deduce the following equation:

$$
10 x^{4} y^{4}-27 x^{2} y^{2}+18=8 x^{2} y^{6}-18 y^{4}
$$

Consider the sequences $\left\{x_{k}\right\}_{k \in \mathbb{N}}$ and $\left\{y_{k}\right\}_{k \in \mathbb{N}}$ which satisfy the above equation with $x_{k} y_{k} \rightarrow \frac{3}{2}$ and $y_{k} \rightarrow 0$. It follows that $f\left(x_{k}, y_{k}\right) \rightarrow 0$, and therefore $S(f)=\{0\}$.
Let us denote $g(x, y)=2|x|^{4}|y|^{4}-9|x|^{2}|y|^{2}+18$. The map $F: \mathbb{C}^{2} \rightarrow \mathbb{C}^{2}, F(x, y)=$ $\left(f(x, y), \frac{x}{g(x, y)}\right)$ is a diffeomorphism and $F^{-1}=F$. Thus $f$ is a trivial $C^{\infty}$ fibration and $B(f)=\emptyset$.

### 4.6.3 A counterexample with two variables

The following Example 4.6 .7 shows that Némethi and Zaharia's proposition [NZ90, Proposition 6] $B(f)=f(\operatorname{Sing} f) \cup\{0\} \underset{\Delta \in \mathfrak{B}}{\cup} f_{\triangle}\left(\operatorname{Sing} f_{\triangle} \cap \mathbb{C}^{* 2}\right)$ for $\mathbb{C}^{2} \rightarrow \mathbb{C}$ does not hold in the mixed case, and the inclusion of Theorem 4.1.3(a) can be strict.

EXAMPLE 4.6.7 Let $f\left(z_{1}, z_{2}\right)=z_{1}\left(1+\left|z_{2}\right|^{2}+z_{1} z_{2}^{4}\right)$.

Let us first prove that $\operatorname{Sing} f=\emptyset$ and the strong non-degeneracy of $f$. We have:

$$
\begin{aligned}
& \frac{\partial f}{\partial z_{1}}=1+\left|z_{2}\right|^{2}+2 z_{1} z_{2}^{4} \\
& \frac{\partial f}{\partial z_{2}}=z_{1} \bar{z}_{2}+4 z_{1}^{2} z_{2}^{3} \\
& \frac{\partial f}{\partial \bar{z}_{1}}=0 \\
& \frac{\partial f}{\partial \bar{z}_{2}}=z_{1} z_{2}
\end{aligned}
$$

By the definition of mixed singularity, $\operatorname{Sing} f$ consists of the solutions of the following system:

$$
\begin{align*}
1+\left|z_{2}\right|^{2}+2 z_{1} z_{2}^{4} & =0  \tag{4.6.8}\\
z_{1} \bar{z}_{2}+4 z_{1}^{2} z_{2}^{3} & =\lambda \overline{z_{1} z_{2}} \tag{4.6.9}
\end{align*}
$$

where $\lambda \in S_{1}^{1}$. Multiplying by $z_{2}$ in (4.6.9) and $2 z_{1}$ in (4.6.8), we have:

$$
z_{1}\left(-2-\left|z_{2}\right|^{2}\right)=\lambda \bar{z}_{1}\left|z_{2}\right|^{2} .
$$

If $z_{1}=0$, then there is no solution for (4.6.8). If $z_{1} \neq 0$, then from the above formula, we get $2+\left|z_{2}\right|^{2}=\left|z_{2}\right|^{2}$ which implies Sing $f=\emptyset$. In fact, $\Gamma^{+}(f)$ consists of three faces and the restrictions of $f$ are respectively $z_{1}, z_{1}\left(1+z_{1} z_{2}^{4}\right)$ and $z_{1}^{2} z_{2}^{4}$. We check at once that $f$ is strongly non-degenerate at infinity.
Next, let us show that $K_{\infty}(f)=S(f)=\bigcup_{\Delta \in \mathfrak{B}} f_{\triangle}\left(\operatorname{Sing} f_{\triangle} \cap \mathbb{C}^{* 2}\right)=\left\{c \in \mathbb{C}| | c \left\lvert\,=\frac{1}{4}\right.\right\}$. There is only one strictly bad face $\Delta$ of $\overline{\operatorname{supp}(f)}$, and the restriction to this face is $f_{\triangle}=z_{1}\left(\left|z_{2}\right|^{2}+z_{1} z_{2}^{4}\right)$. By the definition of mixed singularity, $\operatorname{Sing} f_{\triangle}$ is the set

$$
\left\{\left.\left(z_{1}, z_{2}\right) \in \mathbb{C}^{2}| | z_{2}\right|^{2}+2 z_{1} z_{2}^{4}=0, z_{1} \bar{z}_{2}+4 z_{1}^{2} z_{2}^{3}=\lambda \overline{z_{1} z_{2}} \text {, where } \lambda \in S_{1}^{1}\right\}
$$

Therefore $\operatorname{Sing} f_{\triangle} \cap \mathbb{C}^{* 2}=\left\{\left(z_{1}, z_{2}\right) \in \mathbb{C}^{2} \left\lvert\, z_{1}=\frac{\bar{z}_{2}}{2 z_{2}^{z}}\right., z_{1}+\lambda \bar{z}_{1}=0\right.$, where $\left.\lambda \in S_{1}^{1}\right\}$. In consequence, we have $f_{\triangle}\left(\operatorname{Sing} f_{\triangle} \cap \mathbb{C}^{* 2}\right)=\left\{c \in \mathbb{C}| | c \left\lvert\,=\frac{1}{4}\right.\right\}$.
In the following, we will show $0 \notin K_{\infty}(f)$. By absurd, let us assume that $0 \in K_{\infty}(f)$. Using Curve selection lemma and KOS-regularity, suppose that there exist analytic curves $z_{1}(t)$ and $z_{2}(t)$ defined on a small enough interval $] 0, \varepsilon[$ such that:

$$
\begin{align*}
\lim _{t \rightarrow 0}\left(\left|z_{1}(t)\right|^{2}+\left|z_{2}(t)\right|^{2}\right)^{\frac{1}{2}} & =\infty  \tag{4.6.10}\\
\lim _{t \rightarrow 0} z_{1}(t)\left(1+\left|z_{2}(t)\right|^{2}+z_{1}(t) z_{2}^{4}(t)\right) & =0  \tag{4.6.11}\\
\lim _{t \rightarrow 0}\left(\left|z_{1}(t)\right|+\left|z_{2}(t)\right|\right)\left(1+\left|z_{2}(t)\right|^{2}+2 z_{1}(t) z_{2}^{4}(t)\right) & =0 \tag{4.6.12}
\end{align*}
$$

If $z_{1}(t) \equiv 0$ or $z_{2}(t) \equiv 0$, then from (4.6.12) we have $\lim _{t \rightarrow 0}\left(\left|z_{1}(t)\right|^{2}+\left|z_{2}(t)\right|^{2}\right)^{\frac{1}{2}}=0$, which contradicts (4.6.10). Hence, we suppose that $z_{1}(t)=a_{1} t^{\alpha}+$ h.o.t. and $z_{2}(t)=$ $b_{1} t^{\beta}+$ h.o.t. where $a_{1} b_{1} \neq 0$ and $\min (\alpha, \beta)<0$. We obtain:
(1). If $\alpha \leq 0$ or $\beta \leq 0$, then we have $z_{1}(t)\left(1+\left|z_{2}(t)\right|^{2}+z_{1}(t) z_{2}^{4}(t)\right) \rightarrow \infty$, which gives a contradiction with (4.6.11).
(2). If $\alpha<0$ and $\beta>0$, then by (4.6.11), we have $\alpha=2 \alpha+4 \beta$ and $a_{1}+a_{1}^{2} b_{1}^{4}=0$. It follows from (4.6.12) that $1+2 a_{1} b_{1}^{4}=0$. On the other hand, there is no solution of the equations $a_{1}+a_{1}^{2} b_{1}^{4}=1+2 a_{1} b_{1}^{4}=0$.
(3). If $\alpha>0$ and $\beta<0$, then by (4.6.11), we have $\alpha+2 \beta \geq 0$. When $\alpha+2 \beta=0$, it follows from (4.6.11) and (4.6.12) that $a_{1}\left(|b|^{2}+a_{1} b_{1}^{4}\right)=0$ and $\left|b_{1}\right|^{2}+2 a_{1} b_{1}^{4}=0$. This gives $b_{1}=0$, which contradicts our assumption. When $\alpha+2 \beta>0$, we have $2 \beta<\alpha+4 \beta$ and therefore $\lim _{t \rightarrow 0}\left(\left|z_{1}(t)\right|+\left|z_{2}(t)\right|\right)\left(1+\left|z_{2}(t)\right|^{2}+2 z_{1}(t) z_{2}^{4}(t)\right) \rightarrow \infty$, contrary to (4.6.12).

With the above method, we have actually proved $K_{\infty}(f) \subset\left\{c \in \mathbb{C}| | c \left\lvert\,=\frac{1}{4}\right.\right\}$. It remains to show the inverse inclusion. Choosing $c \in S_{\frac{1}{4}}$, we set two analytic paths, $z_{1}(t)$ and $z_{2}(t)$ such that $z_{1}(t)=a_{1} t^{2}+t^{8}+$ h.o.t and $z_{2}(t)=b_{1} t^{-1}+c_{1} t+t^{8}+$ h.o.t, where $a_{1}^{2} b_{1}^{4}=-c, b_{1} \bar{c}_{1}=-1$ and $2 a_{1} b_{1}^{3}+\bar{b}_{1}=0$. From this, we conclude $c \in K_{\infty}(f)$ and therefore $K_{\infty}(f)=S(f)=\left\{c \in \mathbb{C}| | c \left\lvert\,=\frac{1}{4}\right.\right\}$.
Finally, let us show $B(f)=K_{\infty}(f)$. By Theorem 4.5.2, for any $c \in \mathbb{C}$ with $\|c\|>\frac{1}{4}$, the fiber $f^{-1}(c) \simeq g^{-1}(c) \simeq{ }_{4} S^{1}$ where $g\left(z_{1}, z_{2}\right)=z_{1}\left(1+z_{1} z_{2}^{4}\right)$, here the equivalence " $\simeq$ " means homotopic equivalence. There is another straight way to see $f^{-1}(c) \simeq$ $\underset{4}{\vee} S^{1}$. The equation $z_{1}\left(1+\left|z_{2}\right|^{2}+z_{1} z_{2}^{4}\right)=c$ has the solutions $\{(c, 0)\}$ and

$$
z_{1}=\frac{-\left(1+\left|z_{2}\right|^{2}\right) \pm \sqrt{\left(1+\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}}}{2 z_{2}^{4}}
$$

where $(c, 0)$ is in the closure of the set $z_{1}=\frac{-\left(1+\left|z_{2}\right|^{2}\right)+\sqrt{\left(1+\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}}}{2 z_{2}^{4}}$. Since

$$
z_{1}=\frac{-\left(1+\left|z_{2}\right|^{2}\right)+\sqrt{\left(1+\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}}}{2 z_{2}^{4}}=\frac{2 c}{\left(1+\left|z_{2}\right|^{2}\right)+\sqrt{\left(1+\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}}}
$$

when $z_{2} \rightarrow 0$, we have $z_{1} \rightarrow c$. Now, we proceed to discuss the topological properties of the fibers.
(1). If $|c|>\frac{1}{4}$, then $\left(1+\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}=0$ has four distinct solutions. The two solution sets are glued at these points, which is homotopic to $\underset{4}{\vee} S^{1}$.
(2). If $|c| \leq \frac{1}{4}$, then $\left(1+\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}=0$ has no solutions. The two solution sets are disjoint, which is homeomorphic to $\mathbb{C} \sqcup \mathbb{C}^{*}$. In particular, for $c=0$, we have $f^{-1}(0)=\left\{z_{1}=0\right\} \cup\left\{z_{1}=-\frac{1+\left|z_{2}\right|^{2}}{z_{2}^{4}}\right\}$.

Hence for every $c \in S_{\frac{1}{4}}$, there exist no neighborhoodss of $c$ such that the restriction of $f$ is a locally trivial fibration and, in consequence, $B(f)=S(f)=\left\{c \in \mathbb{C}| | c \left\lvert\,=\frac{1}{4}\right.\right\}$.

REMARK 4.6.8 In general, by applying the above method to the family of mixed polynomials $f_{t}\left(z_{1}, z_{2}\right)=z_{1}\left(1+t\left|z_{2}\right|^{2}+z_{1} z_{2}^{4}\right)$ for $t \in[-1,1]$, we have the following observations:
(a) If $t=0$, then $B\left(f_{0}\right)=S\left(f_{0}\right)=\{0\}$ and $f_{0}\left(\operatorname{Sing} f_{0}\right)=\emptyset$.
(b) If $0<t \leq 1$, then $B\left(f_{t}\right)=S\left(f_{t}\right)=\left\{\lambda \in \mathbb{C} \left\lvert\,\|\lambda\|=\frac{t^{2}}{4}\right.\right\}$ and $f_{t}\left(\operatorname{Sing} f_{t}\right)=\emptyset$.
(c) If $-1 \leq t<0$, then $f_{t}\left(\operatorname{Sing} f_{t}\right)=\{0\}, S\left(f_{t}\right)=\left\{\lambda \in \mathbb{C} \left\lvert\,\|\lambda\|=\frac{t^{2}}{4}\right.\right\}$ and $B\left(f_{t}\right)=f_{t}\left(\operatorname{Sing} f_{t}\right) \cup S\left(f_{t}\right)$.

Given $t \in\left[-1,0\left[\right.\right.$ and every $c \neq 0$, the solutions of $f_{t}^{-1}(c)$ are $(c, 0)$ and:

$$
z_{1}=\frac{-\left(1+t\left|z_{2}\right|^{2}\right) \pm \sqrt{\left(1+t\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}}}{2 z_{2}^{4}}
$$

where $(c, 0)$ is in the closure of the set $z_{1}=\frac{-\left(1+t\left|z_{2}\right|^{2}\right)+\sqrt{\left(1+t\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}}}{2 z_{2}^{4}}$. Since

$$
z_{1}=\frac{-\left(1+t\left|z_{2}\right|^{2}\right)+\sqrt{\left(1+t\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}}}{2 z_{2}^{4}}=\frac{2 c}{\left(1+t\left|z_{2}\right|^{2}\right)+\sqrt{\left(1+t\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}}}
$$

, when $z_{2} \rightarrow 0$, we have $z_{1} \rightarrow c$.
Let us denote the graph of $z_{1}=\frac{-\left(1+t\left|z_{2}\right|^{2}\right)+\sqrt{\left(1+t\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}}}{2 z_{2}^{4}}$ (resp. $\left.\frac{-\left(1+t\left|z_{2}\right|^{2}\right)-\sqrt{\left(1+t\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}}}{2 z_{2}^{4}}\right)$ by $C_{1}$ (resp. $C_{2}$ ). We see that $\bar{C}_{1} \simeq \mathbb{C}$ and $C_{2} \simeq \mathbb{C}^{*}$. Next, we will ${ }^{2}$ describe the topological properties of the graph.
I. If $|c|>\frac{t^{2}}{4}$, then from $\left(1+t\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}=0$, we have $\left|z_{2}\right|^{2}=\frac{1}{2 \sqrt{|c|}-t}$ and therefore $z_{2}^{4}=-\frac{|c|}{c(2 \sqrt{|c|}-t)^{2}}$. This gives four distinct solutions of this equation. Thus the two graphs $C_{1}$ and $C_{2}$ are glued at these points, which is homotopic to $\underset{4}{\vee} S^{1}$.
II. If $|c|<\frac{t^{2}}{4}$, then from $\left(1+t\left|z_{2}\right|^{2}\right)^{2}+4 c z_{2}^{4}=0$, we have either $\left|z_{2}\right|^{2}=\frac{1}{2 \sqrt{|c|}-t}$ or $\left|z_{2}\right|^{2}=\frac{1}{-2 \sqrt{|c|}-t}$. Therefore $z_{2}^{4}=-\frac{|c|}{c(2 \sqrt{|c|}-t)^{2}}$ or $z_{2}^{4}=-\frac{|c|}{c(2 \sqrt{|c|}+t)^{2}}$. This gives eight solutions of the discriminant. Consequently, the two graphs $C_{1}$ and $C_{2}$ are glued at these points, which is homotopic to $\underset{8}{\vee} S^{1}$.

Combining with the above arguments, for every $c \in S_{\frac{t^{2}}{4}}$, we conclude that there is no neighborhood $D$ of $c$ such that $f_{t}$ is a fibration over $D$. This shows $S_{\frac{t^{2}}{4}} \subset B\left(f_{t}\right)$. On the other hand, with the similar analysis as that of $f\left(z_{1}, z_{2}\right)=z_{1}\left(1+\left|z_{2}\right|^{2}+z_{1} z_{2}^{4}\right)$, we get $S\left(f_{t}\right)=\left\{\lambda \in \mathbb{C} \left\lvert\,\|\lambda\|=\frac{t^{2}}{4}\right.\right\}$. Consequently, we have $B\left(f_{t}\right)=f_{t}\left(\operatorname{Sing} f_{t}\right) \cup S\left(f_{t}\right)$. Now we deduce that for a family of strongly non-degenerate mixed polynomial with the same Newton boundary at infinity, $\operatorname{dim} B(f), \operatorname{dim} f(\operatorname{Sing} f)$ and $\operatorname{dim} S(f)$ could be non-constant.

## Chapter 5

## Milnor fibration

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### 5.1 Introduction

The aim of this chapter is to investigate some classes of mixed polynomials which have the Milnor fibration $\frac{f}{|f|}$ at infinity. We first review some of the standard facts on holomorphic polynomials. In general, for a holomorphic polynomial $f$, the Milnor fibration $\frac{f}{|f|}$ at infinity does not always exist. If $f$ is a convenient polynomials with non degenerate Newton principal part at infinity, then by using Milnor's proof in the local case [Mil68], one can prove that there exists a Milnor fibration at infinity

$$
\varphi:=\frac{f}{|f|}: S_{R} \backslash f^{-1}(0) \rightarrow S^{1}
$$

in a sufficiently large sphere, which is equivalent to the Milnor fibration $f$ : $f^{-1}\left(S_{r}^{1}\right) \rightarrow S_{r}^{1}$ for $r$ sufficiently large. In [NZ92], the authors considered a special class of holomorphic polynomials called "semitame":

Definition 5.1.1 Let $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ be a holomorphic polynomial, we say $f$ is semitame if the set of asymptotic $\rho$-non regular values $S(f) \subset\{0\}$

They proved that:
Theorem 5.1.2 [NZ92, Theorem 6] If $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ is a semitame polynomial, then $\exists R_{0}>0$ for $R \geq R_{0}$ sufficiently large, the map:

$$
\frac{f}{|f|}: S_{R} \backslash f^{-1}(0) \rightarrow S^{1}
$$

is a locally trivial fibration which does not depend on the choice of $R$. It is called Milnor fibration at infinity.

Even if the above fibration could be not equivalent to the usual one $f: f^{-1}\left(S_{r}^{1}\right) \rightarrow S_{r}^{1}$ for $r$ sufficiently large (see for instance Example 3.1.1), one still have:

Theorem 5.1.3 [NZ92, Theorem 11] Let $f$ be a semitame polynomial. For a small disk $D_{\delta}$ centered at $0 \in \mathbb{C}$, the restriction map

$$
\frac{f}{|f|}: S_{R} \backslash f^{-1}\left(D_{\delta}\right) \rightarrow S^{1}
$$

where $R$ sufficiently large with respect to $\delta$ is a locally trivial fibration which is equivalent to the Milnor fibration $f: f^{-1}\left(S_{r}^{1}\right) \rightarrow S_{r}^{1}$ for $r$ sufficiently large.

For mixed polynomials, one can also ask under which condition does the Milnor fibration $\frac{f}{|f|}$ at infinity exist? At least, the example 5.3 .3 shows that the condition $S(f) \subset\{0\}$ is not sufficient to insure the existence of the Milnor fibration $\frac{f}{|f|}$ at infinity. Therefore we get another approach of this question by using nondegeneracy condition at infinity. Consider a mixed polynomial $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$. We first define the $\rho$-regularity for $\frac{f}{|f|}$. Then we prove an analogue of Theorem 4.1.3, which gives an approximation of asymptotic $\rho$-non-regular values of $\frac{f}{|f|}$. Inspired by Oka's construction in the local case, we prove that:

Theorem 5.1.4 If $f$ is a Newton strongly non-degenerate mixed polynomial, then $\exists \delta_{0}>0$ and $R_{0}>0$ sufficient large such that for any $\delta \geq \delta_{0}$ and $R>R_{0}$

$$
\frac{f}{|f|}: S_{R}^{2 n-1} \backslash f^{-1}\left(D_{\delta}\right) \longrightarrow S^{1}
$$

is a locally trivial fibration for $R \geq R_{0}$ and is equivalent to the global fibration

$$
f_{l}: f^{-1}\left(S_{\delta}^{1}\right) \rightarrow S_{\delta}^{1} .
$$

Note that in this theorem the strong non-degeneracy can not be replaced to nondegeneracy. (See Remark 5.3.4)
Combining with the approximation established for the atypical values of $\frac{f}{|f|}$, we get the following global version of Oka's Theorem 2.3.4:

Corollary 5.1.5 If $f$ is a Newton strongly non-degenerate convenient mixed polynomial, then there exists $R_{0}>0$ sufficient large such that for all $R \geq R_{0}$ the Milnor fibration at infinity

$$
\frac{f}{|f|}: S_{R}^{2 n-1} \backslash K \longrightarrow S^{1}
$$

exists and is equivalent to the global fibration

$$
f_{l}: f^{-1}\left(S_{\delta}^{1}\right) \rightarrow S_{\delta}^{1}
$$

where $\delta>0$ is sufficient large.

It is worth pointing out that the advantage of using strong non-degeneracy lies in the fact that $f(\operatorname{Sing} f)$ and $S(f)$ are bounded. Hence it is possible to consider some more general classes of mixed polynomials which have the Milnor fibrations $\frac{f}{|f|}$ at infinity, but we will not develop this point in this chapter.

The structure of the chapter is as follows. Section 2 is devoted to the study of asymptotic $\rho$-non-regular values of $\frac{f}{|f|}$. We first set up notation and terminology. Then we proceed to the study of non-degeneracy condition at infinity. Finally we derive an estimation of bifurcation locus $B(\varphi)$ for strongly non-degenerate mixed polynomials. In Section 3, we will look more closely at the Milnor fibrations $\frac{f}{|f|}$ at infinity. We indicate an equivalence of the fibrations at infinity (Theorem 5.1.4) which is similar as that of [NZ92, Theorem 6]. As a consequence, we prove Corollary 5.1.5 which extends the result of holomorphic case to mixed case. At the end of this chapter, we give two examples with computations on the critical values and asymptotic $\rho$-non-regular values.

### 5.2 Approximation of atypical values of $\frac{f}{|f|}$

For a mixed polynomial $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$, we denote by $\varphi$ the function $\frac{f}{|f|}: \mathbb{C}^{n} \backslash V(f) \rightarrow$ $S^{1}$ where $V(f)=f^{-1}(0)$. To simplify notation, we continue to write $\mathrm{d} \varphi$ and $\overline{\mathrm{d}} \varphi$ specifically for the partial derivatives of the variables $\mathbf{z}$ and $\overline{\mathbf{z}}$.

Lemma 5.2.1 For $\mathbf{z} \in \mathbb{C}^{n} \backslash V(f)$, the fibre $\varphi^{-1}(\varphi(\mathbf{z}, \overline{\mathbf{z}}))$ does not intersect transversely the sphere $S_{\|\mathbf{z}\|}^{2 n-1}$ at $\mathbf{z} \in \mathbb{C}^{n}$, if and only if there exists $\lambda \in \mathbb{R}$, such that

$$
\begin{equation*}
\lambda \mathbf{z}=i \bar{f} \overline{\mathrm{~d}} f(\mathbf{z}, \overline{\mathbf{z}})-i f \overline{\mathrm{~d} f}(\mathbf{z}, \overline{\mathbf{z}}) . \tag{5.2.1}
\end{equation*}
$$

In particular, $\operatorname{Sing} \varphi=\left\{\mathbf{z} \in \mathbb{C}^{n} \backslash V(f) \mid \bar{f} \overline{\mathrm{~d}} f(\mathbf{z}, \overline{\mathbf{z}})=f \overline{\mathrm{~d} f}(\mathbf{z}, \overline{\mathbf{z}})\right\}$.
Proof. Observe first $\varphi=-\operatorname{Re}(i \log f)$. By Lemma 3.2.1, the non-transversality of the fiber $\varphi^{-1}(\varphi(\mathbf{z}, \overline{\mathbf{z}}))$ and the sphere $S_{\|\mathbf{z}\|}^{2 n-1}$ implies:

$$
\begin{equation*}
\gamma \mathbf{z}=\mu \overline{\mathrm{d} \varphi}(\mathbf{z}, \overline{\mathbf{z}})+\bar{\mu} \overline{\mathrm{d}} \varphi(\mathbf{z}, \overline{\mathbf{z}}), \tag{5.2.2}
\end{equation*}
$$

for some $\gamma \in \mathbb{R}$ and $\mu \in \mathbb{R}^{*}$. By definition of $\overline{\mathrm{d} \varphi}$ and $\overline{\mathrm{d}} \varphi$, we have:

$$
\begin{aligned}
& \overline{\mathrm{d}} \varphi(\mathbf{z}, \overline{\mathbf{z}})=-\overline{\mathrm{d}} \operatorname{Re}(i \log f)=i \frac{\overline{\mathrm{~d}} f(\mathbf{z}, \overline{\mathbf{z}})}{\bar{f}} \\
& \overline{\mathrm{~d} \varphi}(\mathbf{z}, \overline{\mathbf{z}})=-\overline{\mathrm{dRe}(\mathrm{i} \log \mathrm{f})}=-i \frac{\overline{\mathrm{~d} f}(\mathbf{z}, \overline{\mathbf{z}})}{f} .
\end{aligned}
$$

Multipling the two sides of (5.2.2) by $|f|^{2}$, we conclude (5.2.1), where $\lambda=\frac{\gamma}{\mu}|f|^{2} \in \mathbb{R}$. In particular, taking $\lambda=0$ in (5.2.1), we obtain $\operatorname{Sing} \varphi$.

Combining with the above lemma, we are led to define $\rho$-regularity for $\varphi$.
Definition 5.2.2 We call $\rho$-non-regular locus of $\varphi$ the semi-algebraic set:

$$
M(\varphi)=\left\{\mathbf{z} \in \mathbb{C}^{n} \backslash V(f) \mid \exists \lambda \in \mathbb{R}, \text { such that } \lambda \mathbf{z}=i \overline{f \bar{d}} f(\mathbf{z}, \overline{\mathbf{z}})-i f \overline{\mathrm{~d} f}(\mathbf{z}, \overline{\mathbf{z}})\right\} .
$$

and we call asymptotic $\rho$-non-regular values of $\frac{f}{|f|}$ the set:

$$
S(\varphi)=\left\{c \in S^{1} \mid \exists\left\{\mathbf{z}_{k}\right\}_{k \in \mathbb{N}} \subset M(\varphi), \lim _{k \rightarrow \infty}\left\|\mathbf{z}_{k}\right\|=\infty \text { and } \lim _{k \rightarrow \infty} \varphi\left(\mathbf{z}_{k}, \overline{\mathbf{z}_{k}}\right)=c\right\}
$$

The above definition enables us to obtain the following structure result of $S(\varphi)$.
Lemma 5.2.3 $S(\varphi)$ is semi-algebraic and $M(\varphi) \subset M(f) \backslash V(f)$.
Proof. The inclusion of $M(\varphi) \subset M(f) \backslash V(f)$ follows from the definitions 3.2.3 and 5.2.2. Since $M(\varphi)$ is a semi-algebraic set, we now proceed analogously to the proof of Proposition 3.2.6 and we see that $S(\varphi)$ is semi-algebraic.

Our next proposition shows that under some homogeneous condition, $\operatorname{Sing} \varphi$ could be equal to $M(\varphi)$.

Proposition 5.2.4 If $f$ is a mixed radial weighted homogeneous polynomial and not constant, then $\operatorname{Sing} \varphi=\operatorname{Sing} f \backslash V(f)=M(\varphi)$.

Proof. Let us denote the radial weights of $f$ by $q_{1}, \cdots, q_{n}$ and the radial degree of $f$ by $m_{r}$, where $q_{1}, \cdots, q_{n} \in \mathbb{Z}$ and $m_{r} \neq 0$. By definition, we have $\operatorname{Sing} \varphi \subset$ $\operatorname{Sing} f \backslash V(f)$ and $\operatorname{Sing} \varphi \subset M(\varphi)$. To prove the equality, let a $\in \operatorname{Sing} f$ and $f(\mathbf{a}, \overline{\mathbf{a}}) \neq 0$. Therefore $\exists \lambda \in S^{1}$ such that for $1 \leq i \leq n$ :

$$
\begin{equation*}
\overline{\frac{\partial f}{\partial z_{i}}}(\mathbf{a}, \overline{\mathbf{a}})=\lambda \frac{\partial f}{\partial \bar{z}_{i}}(\mathbf{a}, \overline{\mathbf{a}}) . \tag{5.2.3}
\end{equation*}
$$

Since $f$ is radial weighted homogeneous, by Euler's lemma, we have:

$$
\begin{equation*}
\sum_{i=1}^{n} q_{i} a_{i} \frac{\partial f}{\partial z_{i}}(\mathbf{a}, \overline{\mathbf{a}})+\sum_{i=1}^{n} q_{i} \bar{a}_{i} \frac{\partial f}{\partial \bar{z}_{i}}(\mathbf{a}, \overline{\mathbf{a}})=m_{r} f(\mathbf{a}, \overline{\mathbf{a}}) . \tag{5.2.4}
\end{equation*}
$$

Let $A=\sum_{i=1}^{n} q_{i} a_{i} \frac{\partial f}{\partial z_{i}}(\mathbf{a}, \overline{\mathbf{a}})$ and $B=\sum_{i=1}^{n} q_{i} \bar{a}_{i} \frac{\partial f}{\partial \bar{z}_{i}}(\mathbf{a}, \overline{\mathbf{a}})$. Multiplying (5.2.3) by $q_{i} \bar{a}_{i}$, we obtain:

$$
\begin{equation*}
\bar{A}=\lambda B \tag{5.2.5}
\end{equation*}
$$

which implies $A \bar{A}=B \bar{B}$ since $\lambda \in S^{1}$. From (5.2.4), (5.2.5) and $f(\mathbf{a}, \overline{\mathbf{a}}) \neq 0$, we therefore get $A B \neq 0$. Consequently,

$$
\begin{equation*}
\frac{\overline{f(\mathbf{a}, \overline{\mathbf{a}})}}{f(\mathbf{a}, \overline{\mathbf{a}})}=\frac{\bar{A}+\bar{B}}{A+B}=\frac{B \bar{A}+B \bar{B}}{B(A+B)}=\frac{B \bar{A}+A \bar{A}}{B(A+B)}=\lambda \tag{5.2.6}
\end{equation*}
$$

which proves that $\mathbf{a} \in \operatorname{Sing} \varphi$ from 5.2.3. Thus, we have $\operatorname{Sing} \varphi=\operatorname{Sing} f \backslash V(f)$. Using Euler vector field as in the proof of [ACT12, Proposition 3.1], we have $M(\varphi) \subset$ Sing $f \backslash V(f)$. This finishes the proof.
For simplicity of notation, we write $\varphi_{\Delta}:=\frac{f_{\Delta}}{\left|f_{\Delta}\right|}$ for the restriction of $\frac{f}{|f|}$, where $\triangle$ is a face of $\overline{\operatorname{supp}(f)}$.

Theorem 5.2.5 If $f$ is Newton strongly non-degenerate at infinity for any face of $\overline{\operatorname{supp}(f)}$, then $M(\varphi)$ is bounded and $S(\varphi)=\emptyset$.

Proof. Assume that $M(\varphi)$ is not bounded, then by Lemma 3.2.5, there exists $\mathbf{z}(t)$ of $M(\varphi)$ a real analytic path defined on a small enough interval $] 0, \varepsilon[$ such that

$$
\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty
$$

Since $\mathbf{z}(t) \subset M(\varphi)$, there exists a real analytic curve $\lambda(t)$, such that for $t \in] 0, \varepsilon[$ we have:

$$
\begin{equation*}
\lambda(t) \mathbf{z}(t)=i \bar{f} \overline{\mathrm{~d}} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))-i f \overline{\mathrm{~d} f}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) . \tag{5.2.7}
\end{equation*}
$$

Suppose here $\lambda(t) \not \equiv 0$ and let $I=\left\{i \mid z_{i}(t) \not \equiv 0\right\}$. Then $I \neq \emptyset$ since $\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty$. Assuming that $I=\{1, \ldots, m\}$, we write the expansions of $f(\mathbf{z}(t), \overline{\mathbf{z}}(t)), \mathbf{z}(t)$ and $\lambda(t)$ explicitly as follows:

$$
\begin{align*}
z_{i}(t) & =a_{i} t^{p_{i}}+\text { h.o.t., } \\
f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) & =\left\{\begin{array}{lc}
b t^{\delta}+\text { h.o.t., } & \text { where } a_{i} \neq 0, p_{i} \in \mathbb{Z}, 1 \leq i \leq m . \\
c+b t^{\delta}+\text { h.o.t. } & \text { where } b \in \mathbb{C}^{*}, \delta \neq 0,
\end{array} \quad \text { if } \lim _{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=0 \text { or } \infty .\right. \tag{5.2.9}
\end{align*}
$$

$$
\begin{equation*}
\lambda(t)=\lambda_{0} t^{\gamma}+\text { h.o.t., } \quad \text { where } \lambda_{0} \in \mathbb{R}^{*}, \gamma \in \mathbb{Z}, \lambda(t) \in \mathbb{R} \text {. } \tag{5.2.10}
\end{equation*}
$$

Set $\mathbf{a}=\left(a_{1}, \ldots, a_{m}\right) \in \mathbb{C}^{* I}, \mathbf{P}=\left(p_{1}, \ldots, p_{m}\right) \in \mathbb{R}^{m}$ and consider the linear function $l_{\mathbf{P}}=\sum_{i=1}^{m} p_{i} x_{i}$ defined on $\overline{\operatorname{supp}\left(f^{I}\right)}$. Let $\triangle$ be the maximal face of $\overline{\operatorname{supp}\left(f^{I}\right)}$ where $l_{\mathbf{P}}$ takes its minimal value, say this value is $d_{\mathbf{P}}$. We have:

$$
\begin{equation*}
f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=f_{\Delta}^{I}(\mathbf{a}, \overline{\mathbf{a}}) t^{d_{\mathbf{P}}}+\text { h.o.t. } \tag{5.2.11}
\end{equation*}
$$

Let us discuss the following two cases:
(I). If $\lim _{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=0$ or $\infty$, we get $d_{\mathbf{P}} \leq \delta$. Since $\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty$, this implies $p:=\min _{j \in I}\left\{p_{j}\right\}<0$. Now using (5.2.8)-(5.2.11) in (5.2.7), we get:

$$
i \bar{b} \frac{\partial f_{\triangle}^{I}}{\partial \bar{z}_{i}}(\mathbf{a}, \overline{\mathbf{a}})-i b \frac{\overline{\partial f_{\triangle}^{I}}}{\partial z_{i}}(\mathbf{a}, \overline{\mathbf{a}})= \begin{cases}\lambda_{0} a_{i}, & \text { if } d_{\mathbf{P}}-p_{i}+\delta=p_{i}+\gamma  \tag{5.2.12}\\ 0, & \text { if } d_{\mathbf{P}}-p_{i}+\delta<p_{i}+\gamma\end{cases}
$$

Let $J=\left\{j \mid d_{\mathbf{P}}-p_{j}+\delta=p_{j}+\gamma\right\}$. We suppose $J \neq \emptyset$ which gives $J=\left\{j \mid p_{j}=\right.$ $\left.p=\min _{1 \leq j \leq m}\left\{p_{j}\right\}<0\right\}$. Consider the derivative of $f(\mathbf{z}(t), \overline{\mathbf{z}}(t))$ with respect to $t$. On one hand, we have:

$$
\begin{equation*}
\frac{\mathrm{d} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t}=b \delta t^{\delta-1}+\text { h.o.t. } \tag{5.2.13}
\end{equation*}
$$

On the other hand, we have:

$$
\begin{align*}
\frac{\mathrm{d} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t} & =\sum_{i=1}^{m}\left(\frac{\partial f}{\partial z_{i}} \cdot \frac{\partial z_{i}}{\partial t}+\frac{\partial f}{\partial \bar{z}_{i}} \cdot \frac{\partial \overline{z_{i}}}{\partial t}\right)  \tag{5.2.14}\\
& =\left[\left\langle\mathbf{P a}, \overline{\mathrm{d} f_{\triangle}^{I}}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle+\left\langle\mathbf{P} \overline{\mathbf{a}}, \overline{\overline{\mathrm{d}} f_{\triangle}^{I}(\mathbf{a}, \overline{\mathbf{a}})}\right\rangle\right] t^{d_{\mathbf{P}}-1}+\text { h.o.t. }
\end{align*}
$$

where $\mathbf{P a}=\left(p_{1} a_{1}, \ldots, p_{m} a_{m}\right)$. From (5.2.12), we obtain:

$$
\begin{equation*}
\operatorname{Re}\left\langle\mathbf{P a}, i \bar{b} \overline{\mathrm{~d}} f_{\triangle}^{I}(\mathbf{a}, \overline{\mathbf{a}})-i b \overline{\mathrm{~d} f_{\triangle}^{I}}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle=\sum_{i \in J} \lambda_{0} p\left\|a_{j}\right\|^{2} \neq 0 . \tag{5.2.15}
\end{equation*}
$$

If $d_{\mathbf{P}}<\delta$, then comparing the orders of the expansions (5.2.13) and (5.2.14) with respect to $t$, we have $\left\langle\mathbf{P a}, \overline{\mathrm{d} f_{\triangle}^{I}}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle+\left\langle\mathbf{P} \overline{\mathbf{a}}, \overline{\overline{\mathrm{d}} f_{\triangle}^{I}(\mathbf{a}, \overline{\mathbf{a}})}\right\rangle=0$ and $f_{\triangle}^{I}(\mathbf{a}, \overline{\mathbf{a}})=0$. Multiplying (5.2.14) by $i \bar{b}$ and comparing the real parts of the equality, we obtain a contradiction with (5.2.15).

If $d_{\mathbf{P}}=\delta$, then by (5.2.13), we have $\operatorname{Re}\left\langle\mathbf{P a}, i \bar{b} \overline{\mathrm{~d}} f_{\Delta}^{I}(\mathbf{a}, \overline{\mathbf{a}})-i b \overline{\mathrm{~d} f_{\triangle}^{I}}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle=$ $\operatorname{Re}\left(i|b|^{2} \delta\right)=0$, which contradicts (5.2.15). It follows that $J=\emptyset$. Hence $\mathbf{a} \in \mathbb{C}^{* I}$ is a singularity of $f_{\Delta}^{I}$ and $f_{\Delta}^{I}(\mathbf{a}, \overline{\mathbf{a}})=b$. By Remark 4.2.6, this is contrary to the strong non-degeneracy of $f^{I}$.
(II). If $\lim _{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=c \in \mathbb{C}^{*}$, comparing the orders of the expansions (5.2.10) and (5.2.12) with respect to $t$, we have $d_{\mathbf{P}}<\delta$. Now using (5.2.9)-(5.2.12) in (5.2.8), we get:

$$
i \bar{c} \frac{\partial f_{\triangle}^{I}}{\partial \bar{z}_{i}}(\mathbf{a}, \overline{\mathbf{a}})-i c \frac{\overline{\partial f_{\triangle}^{I}}}{\partial z_{i}}(\mathbf{a}, \overline{\mathbf{a}})= \begin{cases}\lambda_{0} a_{i}, & \text { if } d_{\mathbf{P}}-p_{i}=p_{i}+\gamma .  \tag{5.2.16}\\ 0, & \text { if } d_{\mathbf{P}}-p_{i}<p_{i}+\gamma .\end{cases}
$$

Let $J=\left\{j \mid d_{\mathbf{P}}-p_{j}=p_{j}+\gamma\right\}$. We suppose $J \neq \emptyset$ which implies $J=\left\{j \mid p_{j}=\right.$ $\left.p=\min _{1 \leq j \leq m}\left\{p_{j}\right\}<0\right\}$. We derive $f(\mathbf{z}(t), \overline{\mathbf{z}}(t))$ with respect to $t$. On one hand, we get (5.2.13). On the other hand, we have (5.2.14). From (5.2.16), we obtain:

$$
\begin{equation*}
\operatorname{Re}\left\langle\mathbf{P a}, i \bar{c} \overline{\mathrm{~d}} f_{\triangle}^{I}(\mathbf{a}, \overline{\mathbf{a}})-i c \overline{\mathrm{~d} f_{\triangle}^{I}}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle=\sum_{i \in J} \lambda_{0} p\left\|a_{j}\right\|^{2} \neq 0 . \tag{5.2.17}
\end{equation*}
$$

Since $d_{\mathbf{P}}<\delta$, comparing the orders of the expansions (5.2.13) and (5.2.14), we have $\left\langle\mathbf{P a}, \overline{\mathrm{d} f_{\triangle}^{I}}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle+\left\langle\mathbf{P} \overline{\mathbf{a}}, \overline{\overline{\mathrm{d}} f_{\triangle}^{I}(\mathbf{a}, \overline{\mathbf{a}})}\right\rangle=0$. Multiplying (5.2.14) by $i \bar{c}$ and comparing the real parts, we obtain a contradiction with (5.2.17). It follows that $J=\emptyset$. Hence $\mathbf{a} \in \mathbb{C}^{* I}$ is a singularity of $f_{\Delta}^{I}$ and $f_{\Delta}^{I}(\mathbf{a}, \overline{\mathbf{a}})=0$. By Remark 4.2.6 this is contrary to the non-degeneracy of $f^{I}$.

In general, if we do not assume the strong non-degeneracy of $f$ and let $\mathbf{A}=(\mathbf{a}, 1,1, \ldots, 1)$ with the $i^{\text {th }}$ coordinate $z_{i}=1$ for $i \notin I$, then we have the following conclusion:
(a). if $d_{\mathbf{P}}<\delta$, then $\mathbf{A}$ is a singularity of $V\left(f_{\triangle}\right)$.
(b). if $d_{\mathbf{P}}=\delta$, then $\mathbf{A} \in \operatorname{Sing} \varphi=\operatorname{Sing} f_{\triangle} \backslash V\left(f_{\triangle}\right)$ by Proposition 5.2.4.

When $\lambda(t) \equiv 0$, by comparing the orders with respect to $t$ in (5.2.7), we have:

$$
\begin{cases}i \bar{\partial} \frac{\partial f_{\Delta}^{I}}{\partial \bar{z}_{i}}(\mathbf{a}, \overline{\mathbf{a}})-i b \frac{\frac{\overline{\partial f_{\Delta}^{I}}}{\partial z_{i}}}{}(\mathbf{a}, \overline{\mathbf{a}})=0, & \text { if } \lim _{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=0 \text { or } \infty .  \tag{5.2.18}\\ i \bar{c} \frac{\partial f_{\Delta}^{I}}{\partial \bar{z}_{i}}(\mathbf{a}, \overline{\mathbf{a}})-i c \frac{\overline{\partial f_{\Delta}^{I}}}{\partial z_{i}}(\mathbf{a}, \overline{\mathbf{a}})=0, & \text { if } \lim _{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=c \in \mathbb{C}^{*} .\end{cases}
$$

It follows that $\mathbf{a} \in \mathbb{C}^{* I}$ is a singularity of $f_{\triangle}^{I}$. By Remark 4.2.6 this is contrary to the non-degeneracy of $f^{I}$.

Hence $M(\varphi)$ is bounded and $S(\varphi)=\emptyset$.
We now proceed to formulate the analogue of Theorem 4.1.3. Recall the notation $\mathfrak{S B}$ the union of strictly bad faces of $\overline{\operatorname{supp}(f)}$.

Theorem 5.2.6 Let $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ be a mixed polynomial. Suppose that $f$ is Newton strongly non-degenerate polynomial which depends effectively on all the variables. Let $f(0)=0$ and $0 \notin S(f)$. Then:

$$
S(\varphi) \subset \bigcup_{\Delta \in \mathfrak{S} \mathfrak{B}} \varphi_{\Delta}\left(\operatorname{Sing} \varphi_{\Delta} \bigcap \mathbb{C}^{* n}\right)
$$

Proof. We use the same notations as in the proof of Theorem 5.2.5. For any $c \in S(\varphi)$, by Lemma 3.2.5, there exists $\mathbf{z}(t)$ of $M(\varphi)$ a real analytic path defined on a small enough interval $] 0, \varepsilon[$ such that

$$
\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty, \text { and } \lim _{t \rightarrow 0} \varphi(\mathbf{z}(t), \overline{\mathbf{z}}(t))=c_{0}
$$

where either $f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=b t^{\delta}+$ h.o.t. and $d_{\mathbf{P}} \leq \delta<0, c_{0}=\frac{b}{|b|}$, or $f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=c+$ $b t^{\delta}+$ h.o.t. and $d_{\mathbf{P}} \leq 0, c \in \mathbb{C}^{*}, c_{0}=\frac{c}{|c|}$. Consider $\lambda(t) \not \equiv 0$, if $\operatorname{ord}_{\mathrm{t}}(f(\mathbf{z}(t), \overline{\mathbf{z}}(t)))<0$, we are in case (I) as in the proof of Theorem 5.2.5, then we get that $\mathbf{a} \in \mathbb{C}^{* I}$ is a singularity of $f_{\triangle}^{I}$. If $\operatorname{ord}_{\mathrm{t}}(f(\mathbf{z}(t), \overline{\mathbf{z}}(t)))=0$, we are in case (II) as in the proof of Theorem 5.2.5, then $\mathbf{a} \in \mathbb{C}^{* I}$ is a singularity of $f_{\triangle}^{I}$.

Set $A=(\mathbf{a}, 1,1, \ldots, 1)$ with the $i^{\text {th }}$ coordinate $z_{i}=1$ for $i \notin I$. By recalling the definition of Newton boundary at infinity for mixed polynomial, we have the following two cases:
(I). If $d_{\mathbf{P}}<0$, then, by Lemma 4.2.5, we conclude that $\triangle$ is a face of $\Gamma^{+}\left(f^{I}\right)$. On the other hand since a is a singularity of $f_{\Delta}^{I}$, by Remark 4.2.6 this contradicts the Newton strong non degeneracy of $f^{I}$.
(II). If $d_{\mathbf{P}}=\delta=0$, then, from Lemma 4.2.5, it follows that either $\triangle$ is a face of $\Gamma^{+}\left(f^{I}\right)$ or $\triangle$ satisfies condition (ii) of Definition 4.2.3. Assume first $\triangle$ is a face of $\Gamma^{+}\left(f^{I}\right)$, then we get the same contradiction as that in (I). Thus $\triangle$ verifies condition (ii) of Definition 4.2.3. We proceed to show that $\triangle$ is strictly bad face of $\overline{\operatorname{supp}(f)}$. Let us denote by $d$ the minimal value of the restriction of $l_{\mathbf{P}}$ to $\overline{\operatorname{supp}(f)}$. Since $\overline{\operatorname{supp}\left(f^{I}\right)}=\overline{\operatorname{supp}(f)} \cap \mathbb{R}_{+}^{I}$, we have $d \leq d_{\mathbf{P}}=0$. Let $H$ be the hyperplane of the equation $\sum_{i=1}^{m} p_{i} x_{i}+q \sum_{i=m+1}^{n} x_{i}=0$, where $q>-d+1>0$. Hence for any $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in \overline{\operatorname{supp}(f)} \backslash \overline{\operatorname{supp}\left(f^{I}\right)}$, the value of $\sum_{i=1}^{m} p_{i} x_{i}+q \sum_{i=m+1}^{n} x_{i}$ is positive. We therefore get $\triangle=\overline{\operatorname{supp}\left(f^{I}\right)} \cap H=\overline{\operatorname{supp}(f)} \cap H$. On the other hand, note that $p_{1}=p=\min _{1 \leq i \leq m}\left\{p_{i}\right\}<0$ and $q>0$. If $\triangle$ does not satisfy condition (i)(a) of Definition 4.2.3, then we have $m=n$ and $p_{i} \leq 0$ for all $1 \leq i \leq n$. It follows that $f$ can not depend on $z_{1}$ otherwise $d_{\mathbf{P}}$ will be negative. This contradicts the effectiveness of $f$. Hence we conclude that $\triangle$ is a strictly bad face of $\overline{\operatorname{supp}(f)}$. Since $d_{\mathbf{P}}=0$, we obtain $c=f_{\Delta}^{I}(\mathbf{a}, \overline{\mathbf{a}})=f_{\triangle}(\mathbf{A}, \overline{\mathbf{A}}) \neq 0$. By $\mathbf{A} \in \operatorname{Sing} \varphi_{\triangle}$ and Proposition 5.2.4, we get $c_{0} \in \varphi_{\Delta}\left(\operatorname{Sing} \varphi_{\Delta}\right)$.

When $\lambda(t) \equiv 0$, it follows that $\mathbf{a} \in \mathbb{C}^{* I}$ is a singularity of $f_{\Delta}^{I}$ from (5.2.18). In the same manner as above reasoning, we get the desired conclusion.

Remark 5.2.7 In particular, if a mixed polynomial $f$ is Newton strongly nondegenerate at infinity and convenient, then by Corollary 4.4.3, we have $S(f)=\emptyset$. Combining this conclusion with the above theorem, we get $S(\varphi)=\emptyset$ since $\mathfrak{S B}=\emptyset$.

### 5.3 Fibration at infinity

Recall that for a strongly non-degenerate polynomial $f$, we have the monodromy fibration:

$$
f_{l}: f^{-1}\left(S_{\delta}^{1}\right) \rightarrow S_{\delta}^{1} .
$$

over some circle $S_{\delta}^{1}$ of radius $\delta$ which is sufficiently large. We define two vectors on $\mathbb{C}^{n} \backslash V(f)$ :

$$
\begin{aligned}
v_{1}(\mathbf{z}, \overline{\mathbf{z}}) & =\overline{\mathrm{d} \log f}(\mathbf{z}, \overline{\mathbf{z}})+\overline{\mathrm{d}} \log f(\mathbf{z}, \overline{\mathbf{z}}) \\
v_{2}(\mathbf{z}, \overline{\mathbf{z}}) & =i(\overline{\mathrm{~d} \log f(\mathbf{z}, \overline{\mathbf{z}})-\overline{\mathrm{d}} \log f(\mathbf{z}, \overline{\mathbf{z}})) .}
\end{aligned}
$$

which have the following geometrical meanings: $v_{1}(\mathbf{z}, \overline{\mathbf{z}})$ is the normal vector of $\log |f|$ and $v_{2}(\mathbf{z}, \overline{\mathbf{z}})$ is the normal vector of $-i \log \frac{f}{|f|}$. In order to prove the main theorem, we shall first prove the following proposition.

Proposition 5.3.1 Under the same assumption as in Theorem 5.1.4, there exists $\delta_{2}>0$ sufficient large, such that for any $\mathbf{z}$ of $\left\{\mathbf{z} \in \mathbb{C}^{n}| | f(\mathbf{z}, \overline{\mathbf{z}}) \mid \geq \delta_{2}\right\}$ the there vectors

$$
\mathbf{z}, \quad v_{1}(\mathbf{z}, \overline{\mathbf{z}}), \quad v_{2}(\mathbf{z}, \overline{\mathbf{z}})
$$

are either linearly independent over $\mathbb{R}$ or they are linearly dependent over $\mathbb{R}$ with the following relation

$$
\mathbf{z}=a v_{1}(\mathbf{z}, \overline{\mathbf{z}})+b v_{2}(\mathbf{z}, \overline{\mathbf{z}})
$$

where $a>0$.
Proof. Since $f$ is strongly non-degenerate at infinity, by Theorem 4.1.3, $f(\operatorname{Sing} f) \cup$ $S(f)$ is bounded. Let us suppose that $f(\operatorname{Sing} f) \cup S(f) \subset D_{\delta_{1}}$. For $|f(\mathbf{z}, \overline{\mathbf{z}})|$ sufficient large we shall prove either $\mathbf{z}, v_{1}(\mathbf{z}, \overline{\mathbf{z}}), v_{2}(\mathbf{z}, \overline{\mathbf{z}})$ are linearly independent over $\mathbb{R}$ or $\mathbf{z}=a v_{1}(\mathbf{z}, \overline{\mathbf{z}})+b v_{2}(\mathbf{z}, \overline{\mathbf{z}})$ where $a b \neq 0$. Assume that $\mathbf{z}$ and $v_{2}(\mathbf{z}, \overline{\mathbf{z}})$ are linearly dependent over $\mathbb{R}$. By Lemma 3.2.5, there exist two analytic paths $\mathbf{z}(t) \subset \mathbb{C}^{n}$ and $\lambda(t) \subset \mathbb{R}$ defined on a small enough interval $] 0, \varepsilon[$ such that

$$
\begin{gather*}
\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty, \lim _{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=\infty .  \tag{5.3.1}\\
i(\overline{\mathrm{~d} \log f}-\overline{\mathrm{d}} \log f)(\mathbf{z}(t), \overline{\mathbf{z}}(t))=\lambda(t) \mathbf{z}(t) . \tag{5.3.2}
\end{gather*}
$$

By Lemma 5.2.3, we have $\mathbf{z}(t) \subset M(f)$. Thus $\lim _{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=\infty$ contradicts our condition $f(\operatorname{Sing} f) \cup S(f) \subset D_{\delta_{1}}$. For $|f(\mathbf{z}, \overline{\mathbf{z}})|$ sufficiently large, we have actually proved that $\mathbf{z}$ and $v_{2}(\mathbf{z}, \overline{\mathbf{z}})$ are linearly independent over $\mathbb{R}$. Since $\overline{\mathrm{d} \log f}+\overline{\mathrm{d}} \log f=$ $\frac{1}{|f|^{2}}(f \overline{\mathrm{~d} f}+\overline{f \mathrm{~d}} f)$, a slightly change in the proof of the above linearly independence shows that for $|f(\mathbf{z}, \overline{\mathbf{z}})|$ sufficient large, $\mathbf{z}$ and $v_{1}(\mathbf{z}, \overline{\mathbf{z}})$ are also linearly independent over $\mathbb{R}$. If $v_{1}(\mathbf{z}, \overline{\mathbf{z}}), v_{2}(\mathbf{z}, \overline{\mathbf{z}})$ are linearly dependent over $\mathbb{R}$, we have $\mathbf{z} \in \operatorname{Sing} f \backslash V(f)$. Hence $f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \subset f(\operatorname{Sing} f)$. This contradicts the boundness of $f(\operatorname{Sing} f)$. It follows that $v_{1}(\mathbf{z}, \overline{\mathbf{z}}), v_{2}(\mathbf{z}, \overline{\mathbf{z}})$ are linearly independent over $\mathbb{R}$ for $|f(\mathbf{z}, \overline{\mathbf{z}})|$ sufficient large. We are reduce to proving the proposition for $a>0$. In the remainder of the proof, we assume $a<0$. By Lemma 3.2.5, there exist the analytic curves $\mathbf{z}(t) \in \mathbb{C}^{n}$,
$a(t)<0$ and $b(t) \in \mathbb{R}$ defined on a small enough interval $] 0, \varepsilon[$ such that

$$
\begin{align*}
\lim _{t \rightarrow 0}\|\mathbf{z}(t)\| & =\infty, \lim _{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=\infty .  \tag{5.3.3}\\
\mathbf{z}(t) & =a(t) v_{1}(\mathbf{z}, \overline{\mathbf{z}})(t)+b(t) v_{2}(\mathbf{z}, \overline{\mathbf{z}})(t) . \tag{5.3.4}
\end{align*}
$$

Let $I=\left\{i \mid z_{i}(t) \not \equiv 0\right\}$. Without loss of generality we can assume $I=\{1, \ldots, m\}$, then we have:

$$
\begin{array}{rlrl}
z_{i}(t) & =a_{i} t^{p_{i}}+\text { h.o.t., } & \quad \text { where } a_{j} \neq 0, p_{i} \in \mathbb{Z}, i \in I . \\
f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) & =b t^{q}+\text { h.o.t., } & & \text { where } b \in \mathbb{C}^{*}, q \in \mathbb{Z}, q<0 \\
a(t) & =\lambda_{0} t^{v_{0}}+\text { h.o.t., } & & \text { where } \lambda_{0} \in \mathbb{R}, v_{0} \in \mathbb{Z} \\
b(t) & =\beta_{0} t^{v_{0}}+\text { h.o.t., } & & \text { where } \beta_{0} \in \mathbb{R}, v_{0} \in \mathbb{Z}
\end{array}
$$

where $\left|\lambda_{0}\right|+\left|\beta_{0}\right| \neq 0$. If $\lambda_{0} \in \mathbb{R}^{*}$, then, by our assumption $a(t)<0$, we have $\lambda_{0}<0$. To shorten notation, we write $\mathbf{a}=\left(a_{1}, \ldots, a_{m}\right) \in \mathbb{C}^{* I}, \mathbf{P}=\left(p_{1}, \ldots, p_{m}\right) \in \mathbb{R}^{m}$ and consider the linear function $l_{\mathbf{P}}=\sum_{i=1}^{m} p_{i} x_{i}$ defined on $\overline{\operatorname{supp}\left(f^{1}\right)}$. Let $\triangle$ be the maximal face of $\operatorname{supp}\left(f^{I}\right)$ where $l_{\mathbf{P}}$ takes its minimal value, say this value is $d_{\mathbf{P}}$. We have $d_{\mathbf{P}} \leq \operatorname{ord}_{\mathrm{t}}(f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=q<0$. By the above expansions, we get from (5.3.4):
$\lambda_{0}\left(\frac{\frac{\overline{\partial f_{\triangle}^{I}}}{\partial z_{i}}}{\bar{b}} \overline{\mathbf{a}, \overline{\mathbf{a}})}+\frac{\frac{\partial f_{\Delta}^{I}}{\partial \bar{z}_{i}}(\mathbf{a}, \overline{\mathbf{a}})}{b}\right)+i \beta_{0}\left(\frac{\frac{\overline{\partial f_{\Delta}^{I}}}{\partial z_{i}}(\mathbf{a}, \overline{\mathbf{a}})}{\bar{b}}-\frac{\frac{\partial f_{\triangle}^{I}}{\partial \bar{z}_{i}}(\mathbf{a}, \overline{\mathbf{a}})}{b}\right)= \begin{cases}a_{i}, & \text { if } d_{\mathbf{P}}-p_{i}-q+v_{0}=p_{i} . \\ 0, & \text { if } d_{\mathbf{P}}-p_{i}-q+v_{0}<p_{i} .\end{cases}$

Let $J=\left\{j \in I \mid d_{\mathbf{P}}-p_{j}-q+v_{0}=p_{j}\right\}$. We observe $J=\left\{j \in I \mid p_{j}=p=\right.$ $\left.\min _{j \in I}\left\{p_{j}\right\}<0\right\}$. If $J=\emptyset$, then from (5.3.5), we have $\mathbf{a} \in \operatorname{Sing} f_{\Delta}^{I}$. Since $d_{\mathbf{P}}<0$, by Lemma 4.2.5, we conclude that $\triangle$ is a face of $\Gamma^{+}\left(f^{I}\right)$. This contradicts the Newton strongly non degeneracy of $f^{I}$. Hence $J \neq \emptyset$. To deduce the contradiction, consider the following expansion:

$$
\begin{array}{r}
\frac{\lambda_{0}+i \beta_{0}}{\bar{b}} \frac{\mathrm{~d} \bar{f}(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t}+\frac{\lambda_{0}-i \beta_{0}}{b} \frac{\mathrm{~d} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t} \\
=2 \lambda_{0} q t^{q-1}+\text { h.o.t. }
\end{array}
$$

We also have:

$$
\frac{\mathrm{d} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t}=\left[\left\langle\mathbf{P a}, \overline{\mathrm{d} f_{\triangle}^{I}}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle+\left\langle\mathbf{P} \overline{\mathbf{a}}, \overline{\overline{\mathrm{d}} f_{\triangle}^{I}(\mathbf{a}, \overline{\mathbf{a}})}\right\rangle\right] t^{d_{\mathbf{P}}-1}+\text { h.o.t. }
$$

By (5.3.5), we obtain:

$$
\begin{array}{r}
\frac{\lambda_{0}+i \beta_{0}}{\bar{b}} \frac{\mathrm{~d} \bar{f}(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t}+\frac{\lambda_{0}-i \beta_{0}}{b} \frac{\mathrm{~d} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t} \\
=\left(2 \sum_{j \in J} p\left\|a_{j}\right\|^{2}\right) t^{d_{\mathbf{P}}-1}+\text { h.o.t. }
\end{array}
$$

Since $d_{\mathbf{P}} \leq q$, comparing the two expansions of $\frac{\lambda_{0}+i \beta_{0}}{\bar{b}} \frac{\mathrm{~d} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t}+\frac{\lambda_{0}-i \beta_{0}}{b} \frac{\mathrm{~d} f(\mathbf{z}(t), \mathbf{z}(t))}{\mathrm{d} t}$, It follows that $d_{\mathbf{P}}=q$ and $\lambda_{0}=\frac{\sum_{j \in J} p\left\|a_{j}\right\|^{2}}{q}>0$ from $p<0$ and $q<0$. This contradicts $\lambda_{0}<0$.

REmark 5.3.2 In the holomorphic setting, the parallel results of this proposition are [Mil68, Lemma 4.4] and [NZ90, Lemma 4 and Lemma 5].

Proof of Theorem 5.1.4 The proof is done as in the case of a holomorphic polynomial. The strong non-degeneracy of $f$ yields a global fibration:

$$
f_{l}: f^{-1}\left(S_{\delta}^{1}\right) \rightarrow S_{\delta}^{1}
$$

where $\delta>0$ is sufficiently large. Since $S_{\delta}^{1}$ is compact and $f(\operatorname{Sing} f) \cup S(f)$ is bounded, there exists $R_{0}>0$ sufficiently large such that all the fibers intersect $S_{R}$ transversely for any $R \geq R_{0}$. We therefore get the restriction

$$
f_{l}: f^{-1}\left(S_{\delta}^{1}\right) \cap B_{R} \rightarrow S_{\delta}^{1}
$$

which is equivalent to the global fibration. By Proposition 5.3.1, there exists a non-zero vector field $\omega$ on $N=\left\{\mathbf{z} \in B_{R}| | f(\mathbf{z}, \overline{\mathbf{z}}) \mid \geq \delta\right\}$ such that

$$
\begin{cases}\operatorname{Re}\left\langle w(\mathbf{z}), v_{2}(\mathbf{z}, \overline{\mathbf{z}})\right\rangle & =0 \\ \operatorname{Re}\left\langle w(\mathbf{z}), v_{1}(\mathbf{z}, \overline{\mathbf{z}})\right\rangle & >0 \\ \operatorname{Re}\langle w(\mathbf{z}), \mathbf{z}\rangle>0 & \end{cases}
$$

Along the integral curve $\gamma\left(t, \mathbf{z}_{0}\right)$ of $w$ with $\gamma\left(0, \mathbf{z}_{0}\right)=\mathbf{z}_{0} \in N$, the argument of $f\left(\gamma\left(t, \mathbf{z}_{0}\right), \overline{\gamma\left(t, \mathbf{z}_{0}\right)}\right)$ is constant and $\left|f\left(\gamma\left(t, \mathbf{z}_{0}\right), \overline{\gamma\left(t, \mathbf{z}_{0}\right)}\right)\right|,\left\|\gamma\left(t, \mathbf{z}_{0}\right)\right\|$ are monotone increasing. Thus for every $\mathbf{z}_{0} \in N$, there exists a unique $h\left(\mathbf{z}_{0}\right) \in S_{R}^{2 n-1} \backslash f^{-1}\left(D_{\delta}\right)$ and $t_{0} \in \mathbb{R}_{+}$such that $\left\|\gamma\left(t_{0}, h\left(\mathbf{z}_{0}\right)\right)\right\|=R$. Consequently, there is an isomorphism $\phi: f^{-1}\left(S_{\delta}^{1}\right) \cap B_{R} \rightarrow S_{R}^{2 n-1} \backslash f^{-1}\left(D_{\delta}\right)$. We therefore get $\frac{f}{|f|}: S_{R}^{2 n-1} \backslash f^{-1}\left(D_{\delta}\right) \longrightarrow S^{1}$ a locally trivial fibration which is equivalent to the fibration $f_{\mid}: f^{-1}\left(S_{\delta}^{1}\right) \cap B_{R} \rightarrow S_{\delta}^{1}$. So $\frac{f}{|f|}: S_{R}^{2 n-1} \backslash f^{-1}\left(D_{\delta}\right) \longrightarrow S^{1}$ is also equivalent to the global one. This completes our proof.

We next turn to prove the analogue of Oka's Theorem 2.3.4 in the global setting
Proof of Corollary 5.1.5 From Remark 5.2.7, it follows that $S(\varphi)=\emptyset$ and $M(\varphi)$ is bounded. Thus we have $\frac{f}{|f|}: S_{R}^{2 n-1} \backslash K \longrightarrow S^{1}$ is a locally trivial fibration. Note that the proof of Theorem 5.1.4 yields that this fibration is equivalent to the global fibration:

$$
f_{l}: f^{-1}\left(S_{\delta}^{1}\right) \rightarrow S_{\delta}^{1}
$$

where $\delta>0$ is sufficient large.
We now proceed to show some examples to illustrate our theorem.
Example 5.3.3 [Oka10a, Example 5 IV] Consider a mixed polynomial

$$
f(\mathbf{z}, \overline{\mathbf{z}})=\frac{1}{4} z_{1}^{2}-\frac{1}{4} \bar{z}_{1}^{2}+z_{1} \bar{z}_{1}-(1+i)\left(z_{1}+z_{2}\right)\left(\bar{z}_{1}+\bar{z}_{2}\right) .
$$

Then we have:
(a) $f$ is not Newton strongly non-degenerate at infinity and $S(f)=\emptyset$.
(b) $\operatorname{Sing} f=\left\{\mathbf{z} \in \mathbb{C}^{2} \mid z_{1}=0, z_{2} \in \mathbb{C}\right\} \cup\left\{\mathbf{z} \in \mathbb{C}^{2} \mid z_{1}+z_{2}=0, z_{1}-i \bar{z}_{1}=0\right\} \cup\{\mathbf{z} \in$ $\left.\mathbb{C}^{2} \mid z_{1}+z_{2}=0, z_{1}+i \bar{z}_{1}=0\right\}$
(c) $M(\varphi)$ is not bounded and $S(\varphi)=\left\{-\frac{1+i}{\sqrt{2}}, \frac{2 \pm i}{\sqrt{5}}\right\}$.

To see that $f$ is not Newton strongly non-degenerate at infinity, it is sufficient to observe that the restriction $f_{\triangle}=-(1+i) z_{2} \bar{z}_{2}$ does not satisfy the strong nondegeneracy. Since in this example, the faces of the Newton boundary at infinity are the same as the compact faces of the Newton boundary at the origin, by Oka's argument, we conclude that $f$ is Newton non-degenerate at infinity.

For any real analytic path $\mathbf{z}(t)$ defined on a small enough interval $] 0, \varepsilon[$ such that:

$$
\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty
$$

Our next claim is that $\lim _{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=\infty$, which implies $S(f)=\emptyset$.
Let $z_{i}(t)=a_{i} t^{p_{i}}+$ h.o.t., where $a_{i} \in \mathbb{C}^{*}, p_{i} \in \mathbb{Z}$ for $i=1,2$ and $\min \left\{p_{1}, p_{2}\right\}<0$. We divide the question into three cases:
(I). If $p_{2}<p_{1}$, then by the expansion of $z_{i}(t)$, we get:

$$
f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=-(1+i)\left|a_{2}\right|^{2} t^{2 p_{2}}+\text { h.o.t. }
$$

where $p_{2}<0$ and $-(1+i)\left|a_{2}\right|^{2} \neq 0$. When $t \rightarrow 0$, it follows that $f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \rightarrow$ $\infty$.
(II). If $p_{2}=p_{1}$, then by the expansion of $z_{i}(t)$, we get:

$$
f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=\left(\frac{1}{4} a_{1}^{2}-\frac{1}{4} \bar{a}_{1}^{2}+\left|a_{1}\right|^{2}-(1+i)\left|a_{1}+a_{2}\right|^{2}\right) t^{2 p_{1}}+\text { h.o.t. }
$$

where $p_{1}<0$ and $\frac{1}{4} a_{1}^{2}-\frac{1}{4} \bar{a}_{1}^{2}+\left|a_{1}\right|^{2}-(1+i)\left|a_{1}+a_{2}\right|^{2} \neq 0$. When $t \rightarrow 0$, it follows that $f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \rightarrow \infty$.
(III). If $p_{1}<p_{2}$, then by the expansion of $z_{i}(t)$, we get:

$$
f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=\left(\frac{1}{4} a_{1}^{2}-\frac{1}{4} \bar{a}_{1}^{2}+i\left|a_{1}\right|^{2}\right) t^{2 p_{1}}+\text { h.o.t. }
$$

where $p_{1}<0$ and $\frac{1}{4} a_{1}^{2}-\frac{1}{4} \bar{a}_{1}^{2}+i\left|a_{1}\right|^{2} \neq 0$. When $t \rightarrow 0$, it follows that $f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \rightarrow \infty$.
On account of the above discussion, our claim is proved. The task is now to show the observations (b) and (c). By Proposition 2.2.2, the singular locus of $f$ is the solutions of the following system:

$$
\begin{aligned}
\frac{1}{2} \bar{z}_{1}+z_{1}-(1-i)\left(z_{1}+z_{2}\right) & =\lambda\left(-\frac{1}{2} \bar{z}_{1}+z_{1}-(1+i)\left(z_{1}+z_{2}\right)\right. \\
(i-1)\left(z_{1}+z_{2}\right) & =-\lambda(1+i)\left(z_{1}+z_{2}\right)
\end{aligned}
$$

where $\lambda \in S_{1}^{1}$. By an easy computation, we conclude the singular locus of $f$ is: $\left\{\mathbf{z} \in \mathbb{C}^{2} \mid z_{1}=0, z_{2} \in \mathbb{C}^{*}\right\},\left\{\mathbf{z} \in \mathbb{C}^{2} \mid z_{1}+z_{2}=0, z_{1}-i \bar{z}_{1}=0\right\}$ and $\left\{\mathbf{z} \in \mathbb{C}^{2} \mid\right.$ $\left.z_{1}+z_{2}=0, z_{1}+i \bar{z}_{1}=0\right\}$. Since $f$ is radial weighted homogeneous, by Proposition 5.2.4 we have $\operatorname{Sing} \varphi=\operatorname{Sing} f \backslash V(f)=M(\varphi)$. It is easily seen that $\operatorname{Sing} f \backslash V(f)$ is not bounded. Choosing a real analytic path $\mathbf{z}(t) \subset \operatorname{Sing} \varphi$ defined on a small enough interval $] 0, \varepsilon\left[\right.$, we have $S(\varphi)=\left\{-\frac{1+i}{\sqrt{2}}, \frac{2 \pm i}{\sqrt{5}}\right\}$. This finishes the proof.

Remark 5.3.4 The above example is due to Oka. In the holomorphic case, Némethi and Zaharia proved the existence of the Milnor fibration at infinity for semitame polynomials in [NZ90]. The definition of semitame is $S(f) \subset\{0\}$. But this example shows that in the mixed case, the condition $S(f) \subset\{0\}$ fails to insure the existence of the Milnor fibration $\frac{f}{|f|}$ at infinity. We also observe that the Newton strong nondegeneracy condition at infinity of Theorem 5.1.4 can not be replaced by Newton non-degeneracy condition at infinity.

EXAMPLE 5.3.5 consider the following mixed Brieskorn polynomial:

$$
f(\mathbf{z}, \overline{\mathbf{z}})=\sum_{i=1}^{n} z_{i}^{a_{i}} \bar{z}_{i}^{b_{i}},
$$

where $a_{i}$ and $b_{i}$ are positive integers for any $i$. We have:
(a) If $b_{i}=a_{i}$ for every $i$, then $f$ is not strongly non-degenerate at infinity. $\operatorname{Sing} f=$ $\mathbb{C}^{n}, f(\operatorname{Sing} f)=\mathbb{R}^{+} \cup\{0\}, S(f)=\emptyset, \operatorname{Sing} \varphi=M(\varphi)=\mathbb{C}^{n} \backslash 0$ and $S(\varphi)=\{1\}$.
(b) If $b_{i} \neq a_{i}$ for every $i$, then $f$ is strongly non-degenerate at infinity and convenient. $\operatorname{Sing} f=\{0\}, f(\operatorname{Sing} f)=\{0\}, S(f)=\emptyset, \operatorname{Sing} \varphi=M(\varphi)=\emptyset$ and $S(\varphi)=\emptyset$.
(c) If there exists some $i$ such that $a_{i}=b_{i}$, then $f$ is not strongly non-degenerate at infinity. We denote $I=\left\{i \mid a_{i}=b_{i}\right\}$. Then $\operatorname{Sing} f=\mathbb{C}^{I} \times\{0\}, f(\operatorname{Sing} f)=$ $\mathbb{R}^{+} \cup\{0\}, S(f)=\emptyset, \operatorname{Sing} \varphi=M(\varphi)=\mathbb{C}^{I} \times\{0\} \backslash 0$ and $S(\varphi)=\{1\}$.

For the first case, $f(\mathbf{z}, \overline{\mathbf{z}})=\left\|z_{1}\right\|^{2 a_{1}}+\cdots+\left\|z_{n}\right\|^{2 a_{n}}$. Consider the restrictions of $f$ to the faces which represent the monomial. We check at once the strong nondegeneracy fails for these faces while $f$ is non-degenerate at infinity. Since $\overline{\mathrm{d} f}=\overline{\mathrm{d}} f$ for any $z \in \mathbb{C}^{n}$, by Proposition 2.2.2, we conclude that $\operatorname{Sing} f=\mathbb{C}^{n}$ and $f(\operatorname{Sing} f)=$ $\mathbb{R}^{+} \cup\{0\}$. If $\|\mathbf{z}\| \rightarrow \infty$, then by Cauchy inequality, it is easily seen that $f(\mathbf{z}, \overline{\mathbf{z}}) \rightarrow \infty$. Thus $S(f)=\emptyset$. Since $f=\|f\|$ which is radial weighted homogeneous and $\overline{\mathrm{d} f}=\overline{\mathrm{d}} f$ for any $z \in \mathbb{C}^{n}$, by Definition of $\operatorname{Sing} \varphi$ and $M(\varphi)$, we deduce $M(\varphi)=\operatorname{Sing} \varphi=\mathbb{C}^{n} \backslash 0$ and therefore $S(\varphi)=\{1\}$.

For the second case, on one hand since $b_{i} \neq a_{i}$ for every $i$, for every monomial $z_{i}^{a_{i}} \bar{z}_{i}^{b_{i}}$, the strong non-degeneracy is verified; on the other hand, every face of the Newton boundary at infinity is a mixed join type polynomial consisted of these monomials. We conclude that $f(\mathbf{z}, \overline{\mathbf{z}})$ is strongly non degenerate. By Proposition 2.2.2, we have $\operatorname{Sing} f=\{0\}$ and $f(\operatorname{Sing} f)=\{0\}$. According to Corollary 4.4.3 and Remark 5.2.7, it follows that $S(f)=\emptyset$ and $S(\varphi)=\emptyset$. In order to compute $\operatorname{Sing} \varphi$ and $M(\varphi)$, we first observe that in this case, $f$ is radial weighted homogeneous and the radial degree of $f$ is not equal to zero. By Proposition 5.2.4, we therefore get $\operatorname{Sing} \varphi=M(\varphi)=\operatorname{Sing} f \backslash V(f)=\emptyset$.

For the third case, since there exists some $i$ such that $a_{i}=b_{i}$, the restriction $f_{\triangle}=\left\|z_{i}\right\|^{2 a_{i}}$ does not verify the strong non-degeneracy condition at infinity, where $\triangle$ is a face of $\Gamma^{+}(f)$. Therefore $f$ is not strongly non-degenerate at infinity. Suppose $S(f) \neq \emptyset$. Let $c \in S(f)$, by Curve Selection Lemma and the equality (3.3.3) in the proof of Proposition 3.3.1, there exist two real analytic paths $\mathbf{z}(t) \in M(f)$ and $\mu(t) \in S_{1}^{1}$ defined on a small enough interval $] 0, \varepsilon[$, such that:

$$
\begin{gather*}
\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty, \text { and } \lim _{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))=c, \\
\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|\|\mu(t) \overline{\mathrm{d} f}(\mathbf{z}(t), \overline{\mathbf{z}}(t))+\bar{\mu}(t) \overline{\mathrm{d}} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))\|=0 . \tag{5.3.6}
\end{gather*}
$$

We write explicitly the expansions:

$$
\begin{aligned}
& z_{i}(t)=\alpha_{i} t^{p_{i}}+\text { h.o.t., } \quad \text { where } a_{i} \neq 0, p_{i} \in \mathbb{Z} \\
& \mu(t)=\mu_{0}+\text { h.o.t., } \quad \text { where } \mu_{0} \in S^{1} .
\end{aligned}
$$

If $\left\|z_{j}(t)\right\| \rightarrow \infty$ for some $j$, then we have $p_{j}<0$. It follows from (5.3.6) that:

$$
\lim _{t \rightarrow 0}\left\|z_{j}(t)\right\|\|\mu(t) \overline{\mathrm{d} f}(\mathbf{z}(t), \overline{\mathbf{z}}(t))+\bar{\mu}(t) \overline{\mathrm{d}} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))\|=0
$$

By the expansions of $z_{j}(t)$ and $\mu(t)$, we get:

$$
\left(\mu_{0} a_{j} \bar{\alpha}_{j}^{a_{j}-1} \alpha_{j}^{b_{j}}+\bar{\mu}_{0} b_{j} \alpha_{j}^{a_{j}} \bar{\alpha}_{j}^{b_{j}-1}\right) t^{a_{j} p_{j}+b_{j} p_{j}}+\text { h.o.t. } \rightarrow 0
$$

where $t \rightarrow 0$. Since $\left(a_{j}+b_{j}\right) p_{j}<0$, we have:

$$
\mu_{0} a_{j} \bar{\alpha}_{j}^{a_{j}-1} \alpha_{j}^{b_{j}}=-\bar{\mu}_{0} b_{j} \alpha_{j}^{a_{j}} \bar{\alpha}_{j}^{b_{j}-1} .
$$

Taking the modules in the above equality, we get $a_{j}=b_{j}$. Let $I=\left\{i \mid a_{i}=b_{i}\right\}$. On account of the above arguments, we see that if $\left\|z_{j}(t)\right\| \rightarrow \infty$, then $j \in I$. We write $f(\mathbf{z}, \overline{\mathbf{z}})=\sum_{i \notin I} z_{i}^{a_{i}} \bar{z}_{i}^{b_{i}}+\sum_{i \in I}\left\|z_{i}\right\|^{2 a_{i}}$ and conclude that $f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \rightarrow \infty$ since $I \neq \emptyset$. This is contrary to $f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \rightarrow c$. Thus $S(f)=\emptyset$.

By Proposition 2.2.2, we get $\operatorname{Sing} f=\mathbb{C}^{I} \times\{0\}$ and $f(\operatorname{Sing} f)=\mathbb{R}^{+} \cup\{0\}$.
Since $f$ is radial weighted homogeneous, using Proposition 5.2.4, we have $M(\varphi)=\operatorname{Sing} \varphi=\operatorname{Sing} f \backslash V(f)=\mathbb{C}^{I} \times\{0\} \backslash 0$. Therefore the restriction of $f$ to $M(\varphi)$ is $\|f\|$. Consequently, $S(\varphi)=\{1\}$.

## Chapter 6

# Remarks on local fibration for non isolated singular locus 

## Contents

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### 6.1 Introduction

In this chapter, we focus on the local fibration for mixed polynomials. In [HL73], Hamm and Lê, by using the complex analytic Łojasiewicz inequality, showed the existence of Thom stratification for holomorphic function germs. However, Lê noticed that the analogous result for complete intersections with non isolated singularities could not hold. Consider the map germ $F:\left(\mathbb{C}^{3}, 0\right) \rightarrow\left(\mathbb{C}^{2}, 0\right)$ with $F(x, y, z)=\left(y, z^{2}-x y^{2}\right)$ which has the fiber over 0 , a double line $l_{x}=\{y=z=0\}$. For the pair $\left(\mathbb{C}^{3}-l_{x}, l_{x}\right)$, Thom property fails at all points along $l_{x}$. It turns out that for a real map germ, one needs more hypothesis to guarantee the existence of Thom property. In the following, we recall some definitions and results in [Mas10].

Let $U$ denote an open set in $\mathbb{R}^{n}$ and $p$ denote a point in $U$. For a real analytic application $f:=(g, h): U \rightarrow \mathbb{R}^{2}$, Massey introduced the following Łojasiewicz inequality.

Definition 6.1.1 We say that $f$ satisfies the strong Łojasiewicz inequality at $p$ or $f$ is Ł-analytic at $p$ if there exists an open neighborhood $U_{p}$ of $p$ in $U$, and $\exists C, \theta_{p} \in \mathbb{R}^{+}$ with $0<\theta_{p}<1$, for all $x \in U_{p}$, we have:

$$
\begin{equation*}
\|f(x)-f(p)\|^{\theta_{p}} \leq c \min _{|(a, b)|=1}\|a \nabla g(x)+b \nabla h(x)\| \tag{6.1.1}
\end{equation*}
$$

where $\nabla g(x)$ and $\nabla h(x)$ are the gradients of $g$ and $h$.

REmark 6.1.2 Note that in the holomorphic case, the above inequality turns out to be the classical Łojasiewicz inequality. More generally, if we consider this equality in the mixed setting, by using equation 3.3.1, the rightside of the inequality 6.1.1 can be written in the form: $\inf _{\mu \in S^{1}}\|\mu \overline{\mathrm{~d} f}(\mathbf{z}, \overline{\mathbf{z}})+\bar{\mu} \overline{\mathrm{d}} f(\mathbf{z}, \overline{\mathbf{z}})\|$.
For Ł-analytic maps, Massey showed that:
Lemma 6.1.3 Suppose that $f$ is E-analytic at 0 and $f(0)=0$. Then there exists an open neighborhood $U \subset \mathbb{R}^{n}$ of 0 and a Whitney stratification $W$ of $U \cap V(f)$ such that, for all $W_{\alpha} \in W$, the pair $\left(U \backslash V(f), W_{\alpha}\right)$ satisfies Thom $a_{f}$-property.

Let us recall here the definition of Thom $a_{f}$-property.
Definition 6.1.4 The pair $\left(U \backslash V(f), W_{\alpha}\right)$ satisfies Thom $a_{f}$-property, if we have that for any sequence $\left\{p_{j}\right\} \in U \backslash V(f)$ such that $p_{j} \rightarrow p \in W_{\alpha}$ and the sequence of tangent spaces $T_{p_{j}}\left(f^{-1}\left(f\left(p_{j}\right)\right) \cap(U \backslash V(f))\right)$ has a limit $T$, then $T$ contains the tangent space of $W_{\alpha}$ at $p$.

As a consequence of the above lemma, one has:
Theorem 6.1.5 [Mas10, Main result] Suppose $f(0)=0$ and $f$ is not locally constant near the origin. If $f$ is E -analytic at 0 , then there exists $\varepsilon_{0}>0$, for all $0<\delta \ll \varepsilon \leq \varepsilon_{0}$, we have:

$$
f: f^{-1}\left(S_{\delta}\right) \cap \overline{B_{\varepsilon}^{n}} \rightarrow S_{\delta}
$$

is a proper, stratified submersion. So $f: f^{-1}\left(S_{\delta}\right) \cap \overline{B_{\varepsilon}^{n}} \rightarrow S_{\delta}$ and $f: f^{-1}\left(S_{\delta}\right) \cap B_{\varepsilon}^{n} \rightarrow$ $S_{\delta}$ are locally trivial fibration. Moreover, the the topological isomorphism class does not depend on the choice of $\varepsilon$ and $\delta$ small enough.

In general, for a mixed polynomial germ $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow(\mathbb{C}, 0)$ with isolated critical value at the origin, we see from Example 6.3.6 that $f$ does not have Thom property at the origin. The main diffculty of investigating the existence of the Milnor-Lê fibration inside an open ball, is that on one hand, the germ can have non-isolated singularities; on the other hand, in a small enough neighborhood of 0 , we don't know if the map is locally surjective. As we have seen the proposition 2.2.9 in the Chapter 2, the local trivialization can be directly constructed from polar and radial actions. Note that from Equation (2.2.2) of Chapter 2, by simply using the polar action, there always exsits a local trivial fibration on the sphere for mixed polar weighted homogeneous polynomials (the fibration $\frac{f}{|f|}:\left(S_{\varepsilon}^{2 n-1} \backslash K_{\varepsilon}\right) \rightarrow S_{1}^{1}$, where $K_{\varepsilon}$ is the link on the sphere with the radius $\varepsilon$ sufficiently small). One may ask for a polar weighted homogeneous polynomial whether there exists the Milnor-Lê fibration inside an open ball without assuming the radial homogeneity property. Our Theorem 6.3.4 gives a positive answer to this question which is an extension of Proposition 2.2.9 in the local case. It is also shown in the example 6.3 .5 that
the $£$-analyticity fails at 0 , but our Theorem 6.3 .4 provides the local fibrations. It would be desirable to know in this case whether we have Thom property for polar weighted homogeneous polynomial germ, but we have not been able to do this.

This chapter is organized as follows. In Section 2, we generalize a type of Parusiński's inequality [Par99] which holds for all the mixed polynomials. In Section 3, we will look more closely at polar weighted homogeneous polynomials and give the proof of Theorem 6.3.4. Moreover, we consider two examples. One is used to illustrate Theorem 6.3.4. The other shows that in the case of mixed function germs with non-isolated singular locus, the Thom property does not hold in general.

## 6.2 Łojasiewicz inequalities for mixed functions

Theorem 6.2.1 Let $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow(\mathbb{C}, 0), g:\left(\mathbb{C}^{n}, 0\right) \rightarrow(\mathbb{C}, 0)$ be two germs of mixed analytic functions. Then there exists a real constant $M$ such that for $p \in$ $g^{-1}(0)$, and sufficiently close to the origin,

$$
\begin{equation*}
|f(p)| \leq M|p| \inf _{\mu \in \mathbb{C}}(|\mu \mathrm{d} g+\mathrm{d} f|+|\mu \overline{\mathrm{d}} g+\overline{\mathrm{d}} f|) \tag{6.2.1}
\end{equation*}
$$

Proof. By absurd, we suppose that this is not the case. Then, by the curve selection lemma, there exists real analytic curves $\mathbf{z}(t)$ and $\mu(t)$ defined on a small enough interval $[0, \varepsilon[$, such that $\mathbf{z}(0)=0, g(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \equiv 0$ and

$$
\operatorname{ord}_{t}|f(\mathbf{z}(t), \overline{\mathbf{z}}(t))|<\operatorname{ord}_{t}(|\mu \mathrm{~d} g+\mathrm{d} f|+|\mu \overline{\mathrm{d}} g+\overline{\mathrm{d}} f|)
$$

where we consider the order of the expansions at $t=0$. Since $\mathbf{z}(0)=0$ and $f((\mathbf{z}(0), \overline{\mathbf{z}}(0))=0$, we have:

$$
\begin{aligned}
\operatorname{ord}_{t}|\mathbf{z}(t)| & =\operatorname{ord}_{t}\left|\frac{\mathrm{~d} \mathbf{z}(t)}{\mathrm{d} t}\right|+1 \\
\operatorname{ord}_{t} f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) & =\operatorname{ord}_{t} \frac{\mathrm{~d} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t}+1
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\frac{\mathrm{d} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t} & =\langle\mathrm{d} f, \mathrm{~d} \mathbf{z}\rangle+\langle\overline{\mathrm{d}} f, \overline{\mathrm{~d}} \mathbf{z}\rangle \\
& =\langle\mu \mathrm{d} g+\mathrm{d} f, \mathrm{~d} \mathbf{z}\rangle+\langle\mu \overline{\mathrm{d}} g+\overline{\mathrm{d}} f, \overline{\mathrm{~d}} \mathbf{z}\rangle
\end{aligned}
$$

By the cauchy inequality, we get:

$$
\frac{\mathrm{d} f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d} t} \leq\left|\frac{\mathrm{d} \mathbf{z}(t)}{\mathrm{d} t}\right|(|\mu \mathrm{d} g+\mathrm{d} f|+|\mu \overline{\mathrm{d}} g+\overline{\mathrm{d}} f|)
$$

Then from the above equalities of orders, it follows that:

$$
\operatorname{ord}_{t}|f(\mathbf{z}(t), \overline{\mathbf{z}}(t))| \geq \operatorname{ord}_{t}(|\mu \mathrm{~d} g+\mathrm{d} f|+|\mu \overline{\mathrm{d}} g+\overline{\mathrm{d}} f|)
$$

This gives a contradiction
Remark 6.2.2 In particular, let $g \equiv 0$. From the above theorem 6.2.1, there exists a constant $0<\theta<1$ and a constant $M>0$ such that:

$$
\begin{equation*}
|f|^{\theta} \leq M(|\mathrm{~d} f|+|\overline{\mathrm{d}} f|) \tag{6.2.2}
\end{equation*}
$$

which can be understood as a version of Łojasiewicz inequality for the mixed case.

### 6.3 Fibration for non-isolated singular germs

By Definition 3.2.3, we consider a submersion condition defined as follows.
Definition 6.3.1 Let $f:\left(\mathbb{R}^{n}, 0\right) \rightarrow\left(\mathbb{R}^{p}, 0\right), n>p$, be an analytic mapping germ. We say that $f$ satisfies condition $(*)$ at 0 , if and only if:

$$
\begin{equation*}
\overline{M(f) \backslash V(f)} \cap V(f)=\{0\} . \tag{6.3.1}
\end{equation*}
$$

Remark 6.3.2 The important point to note here is that if a real analytic mapping germ $f$ satisfies Thom $a_{f}$ stratification for $V(f)$, then $f$ has the condition (*) (see for instance [Lê77], [Mas10, Theorem 5.7]). In [Mas10], Massey calls it "Milnor condition (b)".

To show our Theorem 6.3.4, we begin by proving the following proposition.
Proposition 6.3.3 If $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow(\mathbb{C}, 0)$ is a mixed polar weighted homogeneous polynomial germ not locally constant and the polar degree is not equal to zero, then:
(a) $f$ has isolated critical value 0 .
(b) $f$ satisfies condition (*).

Proof. Throughout the proof, we use the notation as in Definition 2.2.5. Let us first show conclusion (a) of the proposition. To obtain a contradiction, we suppose that 0 is not an isolated critical value. Then there exists a curve $f(t) \subset f(\operatorname{Sing} f)$ such that $f(0)=0$ and $t \in[0,1]$. Therefore the image $|f(t)|$ is an interval $[a, b]$ for $t \in[0,1]$, since $f$ is not locally constant. Let $c \in f(t)$ and $f\left(\mathbf{z}_{0}, \overline{\mathbf{z}}_{0}\right)=c$ for $z_{0} \in \operatorname{Sing} f$. Then we have:

$$
\begin{equation*}
\overline{\frac{\partial f}{\partial z_{i}}}\left(\mathbf{z}_{0}, \overline{\mathbf{z}}_{0}\right)=\eta \frac{\partial f}{\partial \bar{z}_{i}}\left(\mathbf{z}_{0}, \overline{\mathbf{z}}_{0}\right) \tag{6.3.2}
\end{equation*}
$$

where $\eta \in S^{1}$ and $i=1,2 \ldots, n$. By assumption that the polar degree is not zero, for any $d$ with $|d|=|c|$, there is a $\lambda \in S^{1}$ such that $d=\lambda^{m_{p}} c$. We set $\left(\mathbf{w}_{0}, \overline{\mathbf{w}}_{0}\right)=\left(\lambda \circ \mathbf{z}_{0}\right)=\left(\lambda^{p_{1}} z_{1}, \ldots \lambda^{p_{n}} z_{n}, \lambda^{-p_{1}} \bar{z}_{1}, \ldots, \lambda^{-p_{n}} \bar{z}_{n}\right)$ for $\mathbf{w}=\left(w_{1}, \ldots, w_{n}\right)$. Since $f$ is polar weighted homogeneous, we get $f\left(\mathbf{w}_{0}, \overline{\mathbf{w}}_{0}\right)=d$ and

$$
\begin{align*}
& \frac{\overline{\partial f}}{\partial w_{i}}\left(\mathbf{w}_{0}, \overline{\mathbf{w}}_{0}\right)=\lambda^{p_{i}-m_{p}} \frac{\overline{\partial f}}{\partial z_{i}}\left(\mathbf{z}_{0}, \overline{\mathbf{z}}_{0}\right)  \tag{6.3.3}\\
& \frac{\partial f}{\partial \bar{w}_{i}}\left(\mathbf{w}_{0}, \overline{\mathbf{w}}_{0}\right)=\lambda^{m_{p}+p_{i}} \frac{\partial f}{\partial \bar{z}_{i}}\left(\mathbf{z}_{0}, \overline{\mathbf{z}}_{0}\right) \tag{6.3.4}
\end{align*}
$$

for $i=1,2 \ldots n$. From (6.3.2), (6.3.3) and (6.3.4), we thus get:

$$
\overline{\frac{\partial f}{\partial w_{i}}}\left(\mathbf{w}_{0}, \overline{\mathbf{w}}_{0}\right)=\eta \lambda^{-2 m_{p}} \frac{\partial f}{\partial \bar{w}_{i}}\left(\mathbf{w}_{0}, \overline{\mathbf{w}}_{0}\right)
$$

which implies that $\mathbf{w}_{0} \in \operatorname{Sing} f$ by Proposition 2.2.2. It follows that $S_{|c|} \subset f(\operatorname{Sing} f)$ for every $c \in[a, b]$. We thus get a contradiction with Sard theorem.
To show that $f$ satisfies condition $(*)$, we shall first prove that the image of $f$ contains a small disk at $0 \in \mathbb{C}$. Since $f$ is not locally constant and $f(0)=0$, by the Curve selection lemma, the image contains a curve $l$ which intersects the circles $S_{\eta}$ for any sufficiently small radius $\eta$. Let $a \in l \cap S_{\eta}$ and $\mathbf{z} \in f^{-1}(a)$. Since $f(\lambda \circ \mathbf{z})=\lambda^{m_{p}} f(\mathbf{z}, \overline{\mathbf{z}})$, we have $\lambda^{m_{p}} a$ is also contained in the image of $f$ for any $\lambda \in S_{1}$. Therefore the image of $f$ contains a small disk $D_{\delta}$ for some small enough $\delta>0$. Take now the restriction of $f$ to some small enough sphere $S_{\varepsilon}^{n-1}$. Its image must contain a non-constant curve germ at 0 . On one hand the polar action preserves the sphere; on the other hand, taking $\mathbf{z} \in\{M(f) \backslash V(f)\} \cap S_{\varepsilon}^{n-1}$, we have:

$$
\begin{equation*}
\gamma z_{i}=\mu \frac{\overline{\partial f}}{\partial z_{i}}(\mathbf{z}, \overline{\mathbf{z}})+\bar{\mu} \frac{\partial f}{\partial \bar{z}_{i}}(\mathbf{z}, \overline{\mathbf{z}}) \tag{6.3.5}
\end{equation*}
$$

where $\gamma \in \mathbb{R}$ and $\mu \in \mathbb{C}$. For any $c$ with $|c|=|f(\mathbf{z}, \overline{\mathbf{z}})|$, let $\lambda \in S^{1}$ be the number uniquely determined by the equality $c=\lambda^{m_{p}} f(\mathbf{z}, \overline{\mathbf{z}})$.
We set $(\mathbf{w}, \overline{\mathbf{w}})=(\lambda \circ \mathbf{z})=\left(\lambda^{p_{1}} z_{1}, \ldots \lambda^{p_{n}} z_{n}, \lambda^{-p_{1}} \bar{z}_{1}, \ldots, \lambda^{-p_{n}} \bar{z}_{n}\right)$ and conclude that:

$$
\begin{equation*}
\gamma w_{i}=\mu_{0} \frac{\overline{\partial f}}{\partial w_{i}}(\mathbf{w}, \overline{\mathbf{w}})+\overline{\mu_{0}} \frac{\partial f}{\partial \bar{w}_{i}}(\mathbf{w}, \overline{\mathbf{w}}) \tag{6.3.6}
\end{equation*}
$$

where $\mu_{0}=\lambda^{-m_{p}} \mu$. Therefore the fiber $f^{-1}(c)$ does not intersect transversally with $S_{\varepsilon}^{n-1}$. Reciprocally, if $\alpha$ is a regular value of the restriction, then $\lambda^{m_{p}} \alpha$ is also regular, for any $\lambda \in S_{1}$. by the same argument as in proof of conclusion (a), the image of the restriction must contain a small disk $D_{\delta_{0}}$. Since the regular values of $f_{\mid S_{\varepsilon}^{n-1}}$ are a dense semi-analytic set, we conclude that all the values of $D_{\delta_{0}} \backslash\{0\}$ are regular. Hence $f$ satisfies condition ( $*$ ).

Due to above proposition, we get the following fibration theorem without the assumption of radial homogeneity.

Theorem 6.3.4 [ACT12, Theorem 1.4] Let $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow(\mathbb{C}, 0)$ be a polar weighted homogeneous mixed polynomial germ not locally constant. Assume that the polar degree is not equal to zero, then there exists $\varepsilon_{0}>0$, for all $0<\delta \ll \varepsilon \leq \varepsilon_{0}$, such that:

$$
f: f^{-1}\left(S_{\delta}\right) \cap \overline{B_{\varepsilon}^{2 n}} \rightarrow S_{\delta}
$$

is a locally trivial fibration which is equivalent to

$$
\varphi:=\frac{f}{|f|}:\left(S_{\varepsilon}^{2 n-1} \backslash K_{\varepsilon}\right) \rightarrow S_{1}^{1},
$$

the fibration on the sphere.
Proof. From the above proposition 6.3.3 and [Mas10, Theorem 4.4], there exists $\varepsilon_{0}>0$, for all $0<\delta \ll \varepsilon \leq \varepsilon_{0}$, such that:

$$
f: f^{-1}\left(S_{\delta}\right) \cap \overline{B_{\varepsilon}^{2 n}} \rightarrow S_{\delta}
$$

is a locally trivial fibration. On the other hand, recall Definition 5.2.2 for $M(\varphi)$. In order to prove that $M(\varphi) \cap B_{\varepsilon_{0}}^{2 n}=\emptyset$, we apply to the mapping $\varphi$ the same reasoning used in the proof of condition $(*)$ for Proposition 6.3.3, since the polar action yields $\varphi(\lambda \circ \mathbf{z})=\lambda^{m_{p}} \varphi(\mathbf{z}, \overline{\mathbf{z}})$ and preserves the spheres centred at the origin. Now, by [ACT12, Theorem 1.3], we conclude that the fibration

$$
\varphi:=\frac{f}{|f|}:\left(S_{\varepsilon}^{2 n-1} \backslash K_{\varepsilon}\right) \rightarrow S_{1}^{1}
$$

is equivalent to the previous Milnor-Lê fibration inside the open ball.
Next, we consider an example which is not Ł-analytic at the origin, but satisfies our theorem 6.3.4.

EXAMPLE 6.3.5 Let $f: \mathbb{C}^{3} \rightarrow \mathbb{C}, f=\bar{x} y^{2}+x \bar{z}^{2}$ which is a radial homogeneous and polar weighted homogeneous mixed polynomial. We first claim that $f$ is not Ł-analytic at zero. We set $\mu=1$ and the curve $(x(t), \underline{y(t)}, z(t))=\left(i t^{\alpha}, t^{\beta}, i t^{\beta}\right)$ where $\alpha, \beta \in \mathbb{N}, t \in \mathbb{R}$. On one hand, since $v(f)=\min _{\|\mu\|=1}\|\mu \overline{\mathrm{~d} f}+\bar{\mu} \overline{\mathrm{d}} f\|$, we have

$$
v(f) \leq\|\overline{\mathrm{d} f}+\overline{\mathrm{d}} f\|=2 \sqrt{2}|t|^{\alpha+\beta}
$$

On the other hand, if $f$ is E -analytic at 0 , then there exist $C>0$ and $0<\theta<1$ such that:

$$
2^{\theta}|t|^{\theta(\alpha+2 \beta)}=|f|^{\theta} \leq C v(f) \leq 2 \sqrt{2} C|t|^{\alpha+\beta}
$$

When $t \rightarrow 0$, it follows that $\theta(\alpha+2 \beta) \geq \alpha+\beta$, for all $\alpha, \beta \in \mathbb{N}$. Suppose that $\alpha=n$ and $\beta=1$. We therefore get $\theta \geq \frac{n+1}{n+2}$. This inequality is true for all $n \in \mathbb{N}$. Thus we conclude $\theta=1$ which is impossible. This completes the proof of our claim. Since $f$ is a polar weighted homogeneous mixed polynomial, the conclusion of our theorem holds.

For a mixed polynomial germ with isolated critical value at the origin, our next example demonstrates that in Remark 6.2.2, the Łojasiewicz inequality (6.2.2) cannot insure the Thom property.

Example 6.3.6 Consider the mixed polynomial germ $f:\left(\mathbb{C}^{2}, 0\right) \rightarrow(\mathbb{C}, 0)$,

$$
f\left(z_{1}, z_{2}\right)=\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}+2 z_{1} z_{2}
$$

Let us show that $f(\operatorname{Sing} f)=\{0\}$ but $f$ does not have Thom $a_{f}$-property at 0 .
At first, if $\left(z_{1}, z_{2}\right) \in \operatorname{Sing} f$, then there exists $\lambda \in S^{1}$ such that:

$$
\begin{align*}
& z_{1}+2 \bar{z}_{2}=\lambda z_{1}  \tag{6.3.7}\\
& z_{2}+2 \bar{z}_{1}=\lambda z_{2} \tag{6.3.8}
\end{align*}
$$

Suppose that $\left(z_{1}, z_{2}\right) \neq(0,0)$. From (6.3.7) and (6.3.8), we get $|\lambda-1|=2$ and therefore $\lambda=-1$.

Thus the solutions of (6.3.7) and (6.3.8) are $\left\{\left(z_{1}, z_{2}\right) \in \mathbb{C}^{2} \mid z_{1}=-\bar{z}_{2}\right\}$. Therefore $f$ has non isolated singularities and $\operatorname{dim}(\operatorname{Sing} f)=2$. Since $V(f)=f^{-1}(0)=$ $\left\{\left(z_{1}, z_{2}\right) \in \mathbb{C}^{2} \mid z_{1}=-\bar{z}_{2}\right\}$, we have $\operatorname{Sing} f=V(f)$ which implies that 0 is an isolated critical value of $f$.

By Definition of $M(f)$, if $\left(z_{1}, z_{2}\right) \in M(f) \backslash \operatorname{Sing} f$, then there exists $\mu \in \mathbb{C}^{*}$ such that:

$$
\begin{align*}
& z_{1}=\mu\left(z_{1}+2 \bar{z}_{2}\right)+\bar{\mu} z_{1}  \tag{6.3.9}\\
& z_{2}=\mu\left(z_{2}+2 \bar{z}_{1}\right)+\bar{\mu} z_{2} \tag{6.3.10}
\end{align*}
$$

From (6.3.9) and (6.3.10), we get $\left|z_{1}\right|=\left|z_{2}\right|$. Thus $V(f)$ is contained in $\overline{M(f) \backslash V(f)} \cap V(f)$, namely that there is $\varepsilon_{0}>0$ sufficiently small and for any $0<\varepsilon \leq \varepsilon_{0}$ there exists $\mathbf{z}_{\varepsilon} \in S_{\varepsilon}$ such that the fiber $f^{-1}\left(f\left(\mathbf{z}_{\varepsilon}\right)\right)$ and $S_{\varepsilon}$ do not intersect transversally. This yields that $f$ does not have Thom $a_{f}$-property at 0 , by Remark 6.3.2.

Let us give a direct computation to show that Thom $a_{f}$-property fails at 0 . Consider $z_{1}=x+i y, z_{2}=u+i v$ and $f=(\operatorname{Re} f, \operatorname{Im} f), f:\left(\mathbb{R}^{4}, 0\right) \rightarrow\left(\mathbb{R}^{2}, 0\right)$. We take a sequence $\left(x_{k}, y_{k}, u_{k}, v_{k}\right) \subset M(f) \cap\left(S_{\rho} \times S_{\rho}\right) \backslash V(f)$ with $0<\rho \ll 1$ fixed. Let us write the sequence in form of polar coordinate $\left(\rho \cos \theta_{k}, \rho \sin \theta_{k}, \rho \cos \beta_{k}, \rho \sin \beta_{k}\right)$ and $\theta_{k}+\beta_{k} \neq \pi, 3 \pi$. Without loss of generality, we suppose that $\left(x_{k}, y_{k}, u_{k}, v_{k}\right)$ converges to $p_{0} \in S_{\rho} \times S_{\rho} \cap V(f)$ with $\theta_{k} \rightarrow \theta_{0} \in[0, \pi]$ and $\beta_{k} \rightarrow \pi-\theta_{0}$. In addition, we assume that $p_{0}$ is belong to a two dimensional stratum $W$ of $V(f)$.

Consider the two normal vectors $N_{k}^{1}=\left(x_{k}+u_{k}, y_{k}-v_{k}, x_{k}+u_{k}, v_{k}-y_{k}\right)$ and $N_{k}^{2}=\left(v_{k}, u_{k}, y_{k}, x_{k}\right)$ of the fiber $f^{-1}\left(f\left(\mathbf{z}_{\varepsilon}\right)\right)$. Let $b_{k}=\frac{2}{\pi-\theta_{k}-\beta_{k}}$ and $a_{k} \in \mathbb{R}$ such that $\left\|a_{k} N_{k}^{1}+b_{k} N_{k}^{2}\right\|^{2}=1$. We therefore have:

$$
\begin{equation*}
4 a_{k}^{2} \rho^{2}\left(1+\cos \left(\theta_{k}+\beta_{k}\right)\right)+4 a_{k} b_{k} \rho^{2} \sin \left(\theta_{k}+\beta_{k}\right)+2 b_{k}^{2} \rho^{2}-1=0 \tag{6.3.11}
\end{equation*}
$$

The discriminant of (6.3.11) is:

$$
\triangle=16 \rho^{2}\left(1+\cos \left(\theta_{k}+\beta_{k}\right)\right)\left(1-2 b_{k}^{2} \rho^{2} \cos ^{2} \frac{\theta_{k}+\beta_{k}}{2}\right)
$$

On one hand, when $\left(\pi-\theta_{k}-\beta_{k}\right)$ is sufficiently small, we have $\left|\sin \frac{1}{b_{k}}\right| \leq \frac{1}{b_{k}}$. On the other hand, we have $2 b_{k}^{2} \rho^{2} \cos ^{2} \frac{\theta_{k}+\beta_{k}}{2} \leq 2 \rho^{2}$ and $1+\cos \left(\theta_{k}+\beta_{k}\right)>0$. It follows that $\triangle>0$ from $0<\rho \ll 1$ which shows the existence of $a_{k}$.

If $f$ has Thom $a_{f}$-property at 0 , then the limit of the sequence $\left\{a_{k} N_{k}^{1}+b_{k} N_{k}^{2}\right\}$ should be orthogonal to tangent space $T_{p_{0}} W$. Recall that $V(f)$ is the hyperplan $(x, y, z, u) \in \mathbb{R}^{4} \mid x+u=0, y-u=0$. Therefore taking two basis $T_{1}=(1,0,-1,0)$ and $T_{2}=(0,1,0,1)$ of $T_{p_{0}} V$, we have:

$$
\begin{align*}
& \lim _{k \rightarrow \infty}\left\langle a_{k} N_{k}^{1}+b_{k} N_{k}^{2}, T_{1}\right\rangle=\lim _{k \rightarrow \infty} b_{k}\left(\sin \beta_{k}-\sin \theta_{k}\right)=2 \cos \theta_{0}  \tag{6.3.12}\\
& \lim _{k \rightarrow \infty}\left\langle a_{k} N_{k}^{1}+b_{k} N_{k}^{2}, T_{2}\right\rangle=\lim _{k \rightarrow \infty} b_{k}\left(\cos \beta_{k}+\cos \theta_{k}\right)=2 \sin \theta_{0} \tag{6.3.13}
\end{align*}
$$

From the above equalities, we conclude that at least one of the above limits is not equal to 0 . Since $W$ is a 2 -dimensional stratum of $V(f)$ and also a submanifold of $V(f)$, this forces $T_{p_{0}} W=T_{p_{0}} V$. Note that we have actually proved the limit of the sequence $\left\{a_{k} N_{k}^{1}+b_{k} N_{k}^{2}\right\}$ cannot be orthogonal to the tangent space $T_{p_{0}} W$. This completes the proof.

REmARK 6.3.7 In the above example, $f$ is not locally surjective since $\operatorname{Re} f \geq 0$. From the conclusion of [Mas10, Corollary 4.7] and Remark 6.3.2, we observe that if $f$ is not locally constant with $\operatorname{Sing} f \subseteq V(f)$ and $f$ has Thom $a_{f}$-property at 0 , then $f$ must be locally surjective.

## Chapter 7

## Newton non-degeneracy for maps

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### 7.1 Introduction

The purpose of this chapter is to investigate Newton non-degeneracy condition in the global setting for a polynomial map. Due to Khovanskii's non-degeneracy definition in [Kho77], the study of Newton polyhedron involves many branches in the algebraic geometry, especially in the singularity theory. However plenty of the results are related to the local case, when we look at polynomial maps. The work of reference on Newton polyhedron in the singularity theory can be found in [Oka97]. In this chapter, we will define a Newton non-degeneracy condition at infinity for a mixed polynomial map. The particular interest we have in mind lies in the estimation of the set of global bifurcation values.

Let $F=\left(f_{1}, \ldots, f_{k}\right): \mathbb{C}^{n} \rightarrow \mathbb{C}^{k}$ be a mixed polynomial map for $n \geq k$. Recall the notation $K_{\infty}(F)$ for the set of asymptotic critical values of $F$. From now on we adopt the non-degeneracy notion of ours until further notice (see Definition 7.2.1). Our main result states that:

Theorem 7.1.1 If $F$ is non-degenerate at infinity with $F(0)=0$ and depends effectively on all the variables, then:

$$
K_{\infty}(F) \subset \mathbb{C}^{k} \backslash \mathbb{C}^{* k} \cup \underset{\Delta \in \mathfrak{B}(F)}{\cup} F_{\Delta}\left(\operatorname{Sing} F_{\Delta} \cap \mathbb{C}^{* n}\right)
$$

where $\mathfrak{B}(F)$ is the set of bad faces of $F$.

Remark 7.1.2 For the notion of bad faces of a polynomial map, we refer to Definition 7.2.5. In Section 2 of this chapter, we will see that when $k=1$, our nondegeneracy definition agrees with Definition 4.2.2. By Proposition 3.3.1, we have $S(f) \subset K_{\infty}(f)$ for a mixed polynomial $f$. Consequently, our Theorem 4.1.3(a) is a special case of Theorem 7.1.1.

The proof of the above theorem is similar in spirit to that we used in Section 4.4. One can see that it is more convenient to work with $K_{\infty}(F)$ than $S(F)$, since we don't need to deal with $\rho$-regularity in this general setting.
In a recent work [Aus07], Bivià-Ausina made a different definition of non-degeneracy condition at infinity of a real polynomial map. He was only concerned with setting up an estimation of Eojasiewicz exponent at infinity via the Newton polyhedron. According to his construction, he proved:

Theorem 7.1.3 [Aus07, Corollary 3.10] Let $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be a polynomial map such that the determinant of the Jacobian matrix $J(F) \neq 0$, for all $x \in \mathbb{R}^{n}$. Suppose that $F$ is non-degenerate at infinity and that $F_{i}$ is convenient, for all $i=1,2, \ldots, n$. Then $F$ is bijective.

The key argument he used to prove the above corollary is that the positiveness of Łojasiewicz exponent at infinity implies that $F$ is proper. By an explicit calculation, he concluded that Pinchuk's example [Pin94] is "degenerate at infinity" by his definition. In this chapter, we will give another approach to this problem in the mixed setting. One can see the advantage of using our definition lies in the fact
that our result is based on a necessary and sufficient condition for the properness of a polynomial map. The chapter is organized as follows. In Section 2, we define non-degeneracy condition for mixed polynomial maps. Then we deduce several properties on the non-degeneracy for the restrictions to the faces. In Section 3, we begin with the proof of Theorem 7.1.1, and then we conclude some sharpened results under other assumptions. In Section 4, we briefly introduce Bivià-Ausina's non-degeneracy condition and compare his definition with ours. We will indicate under some hypothesis our definition is weaker than his.
The following notation will be used throughout this chapter:
$\mathbb{C}^{L}=\left\{\mathbf{z}=\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{C}^{n} \mid z_{i}=0\right.$, for all $\left.i \notin L\right\}$.
$\mathbb{C}^{* L}=\left\{\mathbf{z}=\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{C}^{n} \mid x_{i}=0 \Longleftrightarrow i \notin L\right\}$.
$F^{L}:=F_{\mid \mathbb{C}^{L}}$ the restriction of $F$ on $\mathbb{C}^{L}$. In particular, $f_{i}^{L}$ is the restriction of $f_{i}$ on $\mathbb{C}^{L}$.
$\mathrm{d} F$ : the Jacobian matrix of $F$ and $\frac{\partial F}{\partial x_{i}}:=\left(\frac{\partial f_{1}}{\partial x_{i}}, \ldots, \frac{\partial f_{k}}{\partial x_{i}}\right)$, for $i=1,2, \ldots, n$. For a mixed polynomial map $F=\left(f_{1}, \ldots, f_{m}\right): \mathbb{C}^{n} \rightarrow \mathbb{C}^{m}$, we denote the Jacobian matrix of the variable $\left(\bar{z}_{1}, \ldots, \bar{z}_{n}\right)$ by $\overline{\mathrm{d}} F:=\left(\frac{\partial f_{i}}{\partial \bar{z}_{j}}\right)$.
$\operatorname{Sing} F$ : the set of singularities of $F$.

### 7.2 Newton polyhedron and non-degeneracy condition at infinity

In this section, we follow the notations and definitions used in Section 4.2. Consider a mixed polynomial map $F=\left(f_{1}, \ldots, f_{k}\right): \mathbb{C}^{n} \rightarrow \mathbb{C}^{k}$ for $n \geq k$. For a vector $\mathbf{P}=\left(p_{1}, \ldots, p_{n}\right) \in \mathbb{Z}^{n}$, let $p=\min _{1 \leq i \leq n} p_{i}, J=\left\{j \mid p_{j}=p\right\}$ and suppose that $p<0$.
 us denote by $\triangle_{\mathbf{p}}^{i}$ the unique maximal face of $\operatorname{supp}\left(f_{i}\right)$ (Here "maximal face" is considered with respect to the inclusion of faces) where $l_{\mathrm{p}}(x)$ takes its minimal value $d_{\mathbf{p}}^{i}$, for $1 \leq i \leq k$. Taking any index set $I \subset\{1,2 \ldots, k\}$, we set $F_{\triangle_{\mathbf{P}}^{I}}=\left(f_{\triangle_{\mathbf{P}}^{i}}\right)_{i \in I}$, where every component is the restriction of $f_{i}$ to the face $\triangle_{\mathbf{p}}^{i}$ for all $i \in I$. If $I=\{1,2 \ldots, k\}$, we will write $F_{\Delta_{\mathbf{P}}}$ for $\left(f_{\triangle_{\mathrm{P}}^{1}}, \ldots, f_{\triangle_{\mathrm{P}}}\right)$ when no confusion can arise. Recall the notation $\Gamma^{+}$for the Newton boundary at infinity. We define:

$$
\begin{equation*}
\mathrm{N}_{\mathbf{P}}:=\left\{j \in\{1, \ldots, k\} \mid \triangle_{\mathbf{P}}^{j} \text { is a face of } \Gamma^{+}\left(f_{j}\right) \text { and } d_{\mathbf{p}}^{j}<0\right\} . \tag{7.2.1}
\end{equation*}
$$

We can now formulate our non-degeneracy notion as follows:
Definition 7.2.1 (Newton non-degeneracy for maps) We say that a mixed polynomial map $F: \mathbb{C}^{n} \rightarrow \mathbb{C}^{k}$ is Newton non-degenerate at infinity resp. Newton strongly non-degenerate at infinity if for any vector $\mathbf{P} \in \mathbb{Z}^{n} \backslash\{0\}$ such that $\mathrm{N}_{\mathbf{P}} \neq \emptyset$, the following condition is satisfied:
$(*) \quad \operatorname{Sing} F_{\triangle_{\mathbf{P}}} \cap\left\{\mathbf{x} \in \mathbb{C}^{n} \mid f_{\triangle_{\mathbf{P}}^{j}}(\mathbf{x})=0, \forall j \in \mathrm{~N}_{\mathbf{P}}\right\} \cap\left(\mathbb{C}^{*}\right)^{n}=\emptyset$.
respectively
$(* *) \quad \operatorname{Sing} F_{\triangle_{\mathbf{P}}} \cap\left(\mathbb{C}^{*}\right)^{n}=\emptyset$.
From now on, we call Newton non-degeneracy or Newton strong non-degeneracy simply non-degeneracy or strong non-degeneracy. Let us mention three remarks after the above definition.

Remark 7.2.2 If $k=1$, then Condition (*) agrees with Definition 4.2.2. Therefore Definition 7.2.1 extends Definition 4.2 .2 to $k \geq 1$ in the mixed setting.

Remark 7.2.3 Let $F=\left(f_{1}, \ldots, f_{k}\right): \mathbb{C}^{n} \rightarrow \mathbb{C}^{k}$ be a mixed polynomial map. We observe that in Definitions 7.2.1, fixed a vector $\mathbf{P} \in \mathbb{Z}^{n}$, the non-degeneracy and the strong non-degeneracy conditions depend only of the map $F_{\Delta_{\mathbf{P}}}=\left(f_{\triangle_{\mathbf{P}}^{1}}, \ldots, f_{\triangle_{\mathbf{P}}^{k}}\right)$. On the other hand, since each $f_{i}$ is a polynomial function, the faces of $\operatorname{supp}\left(f_{i}\right)$ are finite and the cardinality of the set $\left\{F_{\triangle_{\mathbf{P}}} \mid \mathbf{P} \in \mathbb{Z}^{n} \backslash 0\right\}$ is finite. Therefore the non-degeneracy of $F$ or the strong non-degeneracy of $F$ is provided by a finite family of conditions.

REmark 7.2.4 Since for every $i$, the faces of $f_{i}^{L}$ are among the faces of $f_{i}$, if $F$ is non-degenerate at infinity (resp. strongly non-degenerate at infinity) and $F^{L} \not \equiv 0$, then $F^{L}$ is also non-degenerate at infinity (resp. strongly non-degenerate at infinity).

Recall the definition of bad faces in Section 4.2. Our new definition of bad faces for a polynomial map is:

Definition 7.2.5 (Bad face) Let $F=\left(f_{1}, \ldots, f_{k}\right): \mathbb{C}^{n} \rightarrow \mathbb{C}^{k}$ be a mixed polynomial map and let $\mathbf{P} \in \mathbb{Z}^{n} \backslash 0$. If $\triangle_{\mathbf{P}}^{i}$ is a bad (resp. strictly bad) face of $f_{i}$ for all $i$ by Definition 4.2.3, we say that $\triangle_{\mathbf{P}}=\triangle_{\mathbf{P}}^{1} \times \triangle_{\mathbf{P}}^{2} \times \cdots \times \triangle_{\mathbf{P}}^{k}$ is a bad (resp. strictly bad) face of $F$. We denote by $\mathfrak{B}(F)$ the set of the bad faces of $F$ and by $\mathcal{S} \mathfrak{B}(F)$ the set of the strictly bad faces of $F$.

In the above definition, we still use the word "bad face" for a polynomial map. Actually the vector $\mathbf{P}$ plays a role in the definition, since there is no polyhedron at infinity attached to the map $F$.
As every face of $\Gamma^{+}\left(f_{\triangle_{\mathbf{P}}^{i}}\right)$ is also a subface of $\Gamma^{+}\left(f_{i}\right)$, we have:
REmark 7.2.6 If $F$ is non-degenerate at infinity (resp. strongly non-degenerate at infinity) and $\triangle_{\mathbf{P}}$ is a bad face of $F$, then $F_{\Delta_{\mathbf{P}}}$ is also non-degenerate at infinity (resp. strongly non-degenerate at infinity).

In the statement of Theorem 4.1.3, we suppose that the mixed polynomial $f$ depends effectively on all the variables. For a polynomial map $F$, we make the following definition:

Definition 7.2.7 (Effectiveness for maps) We say that $F$ depends effectively on all the variables, if for every variable $z_{i}$ there exists some $j(i)$ such that $f_{j(i)}$ depends effectively on $z_{i}$.

### 7.3 Proof of Theorem 7.1.1 and some consequences

In this section, we will prove Theorem 7.1.1 and show some consequences of this theorem. By recalling the formula (3.1.1) used to define the distance function for KOS-regularity, we first remark here:

REmARK 7.3.1 Let $F=\left(f_{1}, \ldots, f_{k}\right): \mathbb{C}^{n} \rightarrow \mathbb{C}^{k}$ be a mixed polynomial map. Assume that $z_{j}=x_{j}+i y_{j}, f_{l}=g_{l}+i h_{l}$ and $\mu_{j}=a_{j}+i b_{j}$ for all $1 \leq j \leq n$ and $1 \leq l \leq k$. We regard $F$ as a polynomial map $F: \mathbb{R}^{2 n} \rightarrow \mathbb{R}^{2 k}$. By Proof of Lemma 3.2.1, we have:

$$
\left(a_{j}+i b_{j}\right) \frac{\partial \bar{f}_{l}}{\partial \bar{z}_{j}}+\left(a_{j}-i b_{j}\right) \frac{\partial f_{l}}{\partial \bar{z}_{j}}=a_{j} \frac{\partial g_{l}}{\partial x_{j}}+b_{j} \frac{\partial h_{l}}{\partial x_{j}}+i\left(a_{j} \frac{\partial g_{l}}{\partial y_{j}}+b_{j} \frac{\partial h_{l}}{\partial y_{j}}\right) .
$$

By the definition of distance function for KOS-regularity, we therefore get:

$$
\nu(\mathrm{d} F(\mathbf{z}))=\inf \left\|\sum_{i=1}^{k}\left(\mu_{i} \overline{\mathrm{~d} f}_{i}(\mathbf{z}, \overline{\mathbf{z}})+\overline{\mu_{i}} \overline{\mathrm{~d}} f_{i}(\mathbf{z}, \overline{\mathbf{z}})\right)\right\|
$$

where $\mu_{i} \in \mathbb{C}$ and $\sum_{i=1}^{k}\left|\mu_{i}\right|^{2}=1$. In particular, the singular locus of $F$ consists of the points such that $\nu(\mathrm{d} F(\mathbf{z}))=0$. (see also Proof of proposition 3.3.1)

### 7.3.1 Proof of Theorem 7.1.1

Let $c=\left(c_{1}, \ldots, c_{k}\right) \in K_{\infty}(F) \cap \mathbb{C}^{* k}$. By Curve selection lemma, there exists an analytic path $\mathbf{z}(t)=\left(z_{1}(t), \ldots, z_{n}(t)\right)$ defined on a small enough interval $] 0, \varepsilon[$, such that $\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty, \lim _{t \rightarrow 0} F(\mathbf{z}(t), \overline{\mathbf{z}}(t))=c$. By definition of KOS-regularity, we have:

$$
\begin{equation*}
\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|\|\nu(\mathrm{d} F(\mathbf{z}(t)))\|=0 \tag{7.3.1}
\end{equation*}
$$

From above remark for all $1 \leq i \leq n$, we get:
$\lim _{t \rightarrow 0}\left\|z_{i}(t)\right\|\left\|\mu_{1}(t) \frac{\overline{\partial f_{1}}}{\partial z_{i}}(\mathbf{z}(t), \overline{\mathbf{z}}(t))+\overline{\mu_{1}}(t) \frac{\partial f_{1}}{\partial \bar{z}_{i}}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \cdots+\overline{\mu_{k}}(t) \frac{\partial f_{k}}{\partial \bar{z}_{i}}(\mathbf{z}(t), \overline{\mathbf{z}}(t))\right\|=0$
where $\mu_{j}(t) \in \mathbb{C}$ and $\sum_{j=1}^{k}\left|\mu_{j}(t)\right|^{2}=1$. Note that the left side of (7.3.2) is less than $\|\mathbf{z}(t)\|\|\nu(\mathrm{d} F(\mathbf{z}(t)))\|$. Let $L=\left\{l \in\{1, \ldots, n\} \mid z_{l}(t) \not \equiv 0\right\}$. Observe that $L \neq \emptyset$ since $\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty$, and write:

$$
\begin{equation*}
z_{l}(t)=z_{l} t^{p_{l}}+\text { h.o.t., } \quad \text { where } z_{l} \in \mathbb{C}^{*}, p_{l} \in \mathbb{Z}, \forall l \in L . \tag{7.3.3}
\end{equation*}
$$

Consider the expansion of $F(\mathbf{z}(t), \overline{\mathbf{z}}(t))$ for all $i=1, \ldots, k$, we have either:

$$
f_{i}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \equiv c_{i}
$$

or

$$
\begin{equation*}
f_{i}(\mathbf{z}(t), \overline{\mathbf{z}}(t))=c_{i}+\text { h.o.t., } \quad \text { where } \quad c_{i} \in \mathbb{C}^{*} . \tag{7.3.4}
\end{equation*}
$$

By eventually transposing the coordinates, we may assume that $L=\{1, \ldots, m\}$ and that $p=p_{1} \leq p_{2} \leq \cdots \leq p_{m}$. Since $\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty$, this implies $p=$ $\min _{i \in L}\left\{p_{i}\right\}<0$. We denote $J=\left\{j \in L \mid p_{j}=p\right\}, \mathbf{z}_{0}=\left(z_{1}, \ldots, z_{m}\right) \in \mathbb{C}^{* L}, \mathbf{p}=$ $\left(p_{1}, \ldots, p_{m}, g, \ldots, g\right) \in \mathbb{Z}^{n}$ with $g>0$ big enough and consider the linear function $l_{\mathbf{p}}=\sum_{l=1}^{m} p_{l} x_{l}+\sum_{l=m+1}^{n} g x_{l}$ defined on $\overline{\operatorname{supp}\left(f_{i}\right)}$. Since $g>0$ big enough, the minimal values of $l_{\mathbf{p}}$ are attained on the faces of $\operatorname{supp}\left(f_{i}^{L}\right)$. Let $\triangle_{\mathbf{p}}^{i}$ be the maximal face of $\overline{\operatorname{supp}\left(f_{i}^{L}\right)}$ where $l_{\mathbf{p}}$ takes its minimal value, say $d_{\mathbf{p}}^{i}$. Therefore $f_{\triangle_{\mathbf{p}}^{i}}^{L}=f_{\triangle_{\mathbf{p}}^{i}}$ and we have:

$$
\begin{equation*}
f_{i}(\mathbf{z}(t), \overline{\mathbf{z}}(t))=f_{i}^{L}(\mathbf{z}(t), \overline{\mathbf{z}}(t))=f_{\triangle_{\mathbf{p}}^{i}}^{L}\left(\mathbf{z}_{0}, \overline{\mathbf{z}}_{0}\right) t^{t_{\mathbf{p}}^{i}}+\text { h.o.t. } \tag{7.3.5}
\end{equation*}
$$

where $d_{\mathbf{p}}^{i} \leq 0$ for all $i=1, \ldots, k$, since $\lim _{t \rightarrow 0} F(\mathbf{z}(t), \overline{\mathbf{z}}(t))=c \in \mathbb{C}^{* k}$. We write:

$$
\begin{equation*}
\mu_{i}(t)=\mu_{i} t^{q_{i}}+\text { h.o.t., } \quad \text { where } \mu_{i} \in \mathbb{C}^{*} \text { and } q_{i} \geq 0 . \tag{7.3.6}
\end{equation*}
$$

If $\mu_{i} \equiv 0$, we put $q_{i}=\infty$ in (7.3.6). Let $I=$ $\left\{i \in\{1, \ldots, k\} \mid q_{i}+d_{\mathbf{p}}^{i}=\min _{1 \leq i \leq k}\left(q_{i}+d_{\mathbf{p}}^{i}\right)\right\}$. As $\sum_{i=1}^{k}\left\|\mu_{i}(t)\right\|^{2}=1$, we have $\min _{1 \leq i \leq k} q_{i}=0$. Hence $I \neq \emptyset$ and $\mu_{i}(t) \not \equiv 0$ for $i \in I$. We conclude therefore $q_{i}+d_{\mathbf{p}}^{i} \leq 0$, $\forall i \in I$. Then for any $l \in L$, from (7.3.2), we have:

$$
\sum_{i \in I}\left(\mu_{i} z_{l} \frac{\overline{\partial f_{\triangle_{\mathbf{p}}^{i}}^{L}}}{\partial z_{l}}\left(\mathbf{z}_{0}, \overline{\mathbf{z}}_{0}\right)+\bar{\mu}_{i} z_{l} \frac{\partial f_{\triangle_{\mathrm{p}}^{i}}^{L}}{\partial \bar{z}_{l}}\right)\left(\mathbf{z}_{0}, \overline{\mathbf{z}}_{0}\right) t^{q_{i}+d_{\mathbf{p}}^{i}}+\text { h.o.t. } \rightarrow 0 .
$$

Comparing the orders of the two sides in the above formula, since $\forall l \in L, z_{l} \neq 0$, we obtain:

$$
\begin{equation*}
\sum_{i \in I}\left(\mu_{i} \frac{\overline{\partial f_{\triangle_{\mathrm{p}}^{i}}^{L}}}{\partial z_{l}}\left(\mathbf{z}_{0}, \overline{\mathbf{z}}_{0}\right)+\bar{\mu}_{i} \frac{\partial f_{\triangle_{\mathrm{p}}^{i}}^{L}}{\partial \bar{z}_{l}}\left(\mathbf{z}_{0}, \overline{\mathbf{z}}_{0}\right)\right)=0 \tag{7.3.7}
\end{equation*}
$$

Let $\mathbf{z}_{1}=\left(\mathbf{z}_{0}, 1, \ldots, 1\right)$, here we use the construction like in the proof of Theorem 4.1.3(a). It follows from (7.3.7) and Remark 7.3.1 that $\mathbf{z}_{1} \in \operatorname{Sing}\left(F_{\triangle_{\mathbf{P}}}(\mathbf{z}, \overline{\mathbf{z}})\right) \cap \mathbb{C}^{* n}$. Recall the index sets defined in (7.2.1). For every $j \in\{1, \ldots, k\}$ such that $d_{\mathbf{p}}^{j}<0$, it is shown by Lemma 4.2 .5 that the face $\triangle_{\mathbf{p}}^{j}$ is a face of $\Gamma^{+}\left(f_{j}\right)$ and hence $j \in \mathrm{~N}_{\mathbf{P}}$. On the other hand, from (7.3.4) and (7.3.5), we must have $f_{\triangle_{\mathbf{p}}^{j}}\left(\mathbf{z}_{1}, \overline{\mathbf{z}}_{1}\right)=0$ since $d_{\mathbf{p}}^{j}<0$. Therefore $\mathbf{z}_{1} \in \operatorname{Sing} F_{\triangle_{\mathbf{P}}} \cap\left\{\mathbf{z} \in \mathbb{C}^{n} \mid f_{\triangle_{\mathbf{P}}^{j}}(\mathbf{z}, \overline{\mathbf{z}})=0, \forall j \in \mathrm{~N}_{\mathbf{P}}\right\} \cap\left(\mathbb{C}^{*}\right)^{n}$ which contradicts the non degeneracy of $F$ by Condition (*) of Definition 7.2.1.
Otherwise, we may assume that $d_{\mathbf{p}}^{i}=0$ for all $i=1,2, \ldots, k$. We will denote the minimal value of the restriction of $l_{\mathbf{p}}$ to $\overline{\operatorname{supp}\left(f_{i}\right)}$ simply by $d_{i}$ when no confusion can arise. Since $\overline{\operatorname{supp}\left(f_{i}^{L}\right)}=\overline{\operatorname{supp}\left(f_{i}\right)} \cap \mathbb{R}_{\geq 0}^{L}$, we have $d_{i} \leq d_{\mathbf{p}}^{i}=0$. Choose the hyperplane defined by the equation $\sum_{i=1}^{m} p_{i} v_{i}+g \sum_{i=m+1}^{n} v_{i}=0$, where $g>\max _{1 \leq i \leq k}\left(-d_{i}+1\right)>0$. Thus for any $\left(v_{1}, \ldots, v_{n}\right) \in \overline{\operatorname{supp}\left(f_{i}\right)} \backslash \overline{\operatorname{supp}\left(f_{i}^{L}\right)}$, the value of $\sum_{i=1}^{m} v_{i} p_{i}+g \sum_{i=m+1}^{n} v_{i}$ is positive. In consequence, we deduce that $\triangle_{\mathbf{p}}^{i}=\overline{\operatorname{supp}\left(f_{i}^{L}\right)} \cap H=\overline{\operatorname{supp}\left(f_{i}\right)} \cap H$ for all $i=1,2 \ldots, k$. On the contrary, suppose that $\triangle_{\mathbf{p}}^{1}$ does not satisfy condition (i)(a) of Definition 4.2.3. Consequently, $m=n$ and $p_{l} \leq 0$ for all $l=1,2 \ldots, n$. By our hypothesis of effectiveness for $F$, there exists some $j$, such that $f_{j}$ depends effectively on the variable $z_{1}$, which implies that $d_{\mathbf{p}}^{j}<0$ since $p_{1}=p<0$ and there exist some $\left(v_{1}, \ldots, v_{n}\right) \in \overline{\operatorname{supp}\left(f_{j}\right)}$ such that $v_{1}>0$. This contradicts our assumption $d_{\mathbf{p}}^{j}=0$. From Definition 7.2.5, we conclude that $\triangle_{\mathbf{P}}$ is a bad face of $F$. On the other hand since $d_{\mathbf{p}}^{i}=0$ for all $i=1, \ldots, k$, we have $c=F_{\triangle_{\mathbf{p}}}\left(\mathbf{z}_{0}, \overline{\mathbf{z}}_{0}\right) \in$ $F_{\triangle_{\mathbf{P}}}\left(\operatorname{Sing}\left(F_{\triangle_{\mathbf{P}}}(\mathbf{z}, \overline{\mathbf{z}})\right) \cap \mathbb{C}^{* n}\right)$. This completes the proof.

### 7.3.2 Some consequences

Let us prove two important consequences of Theorem 7.1.1.
Proposition 7.3.2 If $F$ is strongly non-degenerate at infinity, then $F(\operatorname{Sing} F) \cap \mathbb{C}^{* k}$ and $K_{\infty}(F) \cap \mathbb{C}^{* k}$ are bounded.

Proof. By Theorem 7.1.1, we have $K_{\infty}(F) \subset \mathbb{C}^{k} \backslash \mathbb{C}^{* k} \cup \underset{\Delta \in \mathfrak{B}(F)}{\cup} F_{\triangle}\left(\operatorname{Sing} F_{\triangle} \cap \mathbb{C}^{* n}\right)$, where $\mathfrak{B}(F)$ is the set of bad faces of $F$. By Remark 7.2.6, it follows that $F_{\triangle}$ is strongly non-degenerate at infinity for every face $\triangle \in \mathfrak{B}(F)$. If we have proved that $F(\operatorname{Sing} F) \cap \mathbb{C}^{* k}$ is bounded, then so is $F_{\Delta}\left(\operatorname{Sing} F_{\Delta}\right) \cap \mathbb{C}^{* k}$. Since the cardinality of the bad faces is finite for a mixed polynomial map, it is sufficient to show that $F(\operatorname{Sing} F) \cap \mathbb{C}^{* k}$ is bounded. Suppose that $F(\operatorname{Sing} F)$ is not bounded. By Curve selection lemma, there exists a real analytic path $\mathbf{z}(t) \subset \operatorname{Sing} F$ defined on a small enough interval $] 0, \varepsilon[$ such that:

$$
\lim _{t \rightarrow 0}\|\mathbf{z}(t)\|=\infty, \text { and } \lim _{t \rightarrow 0}\|F(\mathbf{z}(t), \overline{\mathbf{z}}(t))\|=\infty
$$

We apply the same notations and arguments as in Proof of Theorem 7.1.1. According to our assumption $\lim _{t \rightarrow 0}\|F(\mathbf{z}(t), \overline{\mathbf{z}}(t))\|=\infty$, there exists some $i$ such that $d_{\mathbf{p}}^{i}<0$. It is shown by Lemma 4.2.5 that $\triangle_{\mathbf{p}}^{i}$ is a face of $\Gamma^{+}\left(f_{i}\right)$. Therefore $\mathrm{N}_{\mathbf{P}} \neq \emptyset$. On the other hand Since $\mathbf{z}(t) \subset \operatorname{Sing} F$, by comparing the orders of the expansion of the equation for $\operatorname{Sing} F$, we get directly (7.3.7) and $\mathbf{z}_{1} \in \operatorname{Sing}\left(F_{\triangle_{\mathbf{P}}}(\mathbf{z}, \overline{\mathbf{z}})\right) \cap \mathbb{C}^{* n}$. This contradicts the strong non-degeneracy of $F$ by Condition ( $* *$ ) of Definition 7.2.1. Therefore $F(\operatorname{Sing} F) \cap \mathbb{C}^{* k}$ and $K_{\infty}(F) \cap \mathbb{C}^{* k}$ are bounded.
Recall the definition of convenient polynomials. Our next result is an extension of Corollary 4.4.3.

Corollary 7.3.3 Suppose that $F$ is non-degenerate at infinity and that $f_{i}$ is convenient, for all $i=1, \ldots, k$. Then $K_{\infty}(F)=\emptyset$.

Proof. We apply the same notations and arguments as in the proof of Theorem 7.1.1. The only difference is that we may assume that $c_{i}$ is not necessarily equal to 0 , for every $i$. Since $f_{i}$ is convenient, we have $d_{\mathbf{P}}^{i}<0 \leq \operatorname{ord}_{t}\left(f_{i}(\mathbf{x}(t))\right.$ for every $i$. By Lemma 4.2.5, every face $\triangle_{\mathbf{p}}^{i}$ is a face of $\Gamma^{+}\left(f_{i}\right)$. Therefore $\mathrm{N}_{\mathbf{P}}=\{1,2 \ldots, k\}$. On the other hand, by same reasoning applied in the proof of Theorem 7.1.1, we get $\mathbf{z}_{1} \in \operatorname{Sing}\left(F_{\triangle_{\mathbf{P}}}\right) \cap \mathbb{C}^{* n}$. In order to obtain a contradiction with Condition ( $*$ ), it remains to show that $f_{\triangle_{\mathbf{p}}^{i}}\left(\mathbf{z}_{1}, \overline{\mathbf{z}}_{1}\right)=0$ for all $i$. This is due to the fact that $d_{\mathbf{P}}^{i}<$ $\operatorname{ord}_{t}\left(f_{i}(\mathbf{z}(t), \overline{\mathbf{z}}(t))\right.$. Thus $\mathbf{z}_{1} \in \operatorname{Sing} F_{\triangle_{\mathbf{P}}} \cap\left\{\mathbf{z} \in \mathbb{C}^{n} \mid f_{\triangle_{\mathbf{P}}^{j}}(\mathbf{z}, \overline{\mathbf{z}})=0, \forall j \in \mathrm{~N}_{\mathbf{P}}\right\} \cap\left(\mathbb{C}^{*}\right)^{n}$ which contradicts the non degeneracy of $F$ by Condition (*) of Definition 7.2.1. This ends our proof.
Next, let us consider an example of strongly non-degenerate mixed polynomial map.

EXAMPLE 7.3.4 Let $F: \mathbb{C}^{3} \rightarrow \mathbb{C}^{2}, F\left(z_{1}, z_{2}, z_{3}\right)=\left(z_{1}+z_{2}^{2}, \bar{z}_{1} z_{3}\right)$. We begin with the computation of $\operatorname{Sing} F$ and $F(\operatorname{Sing} F)$. The Jacobian matrix of $F$ are:

$$
\begin{aligned}
\mathrm{d} F\left(z_{1}, z_{2}, z_{3}\right) & =\left(\begin{array}{ccc}
1 & 2 z_{2} & 0 \\
0 & 0 & \bar{z}_{1}
\end{array}\right) \\
\overline{\mathrm{d}} F\left(z_{1}, z_{2}, z_{3}\right) & =\left(\begin{array}{ccc}
0 & 0 & 0 \\
z_{3} & 0 & 0
\end{array}\right) .
\end{aligned}
$$

By Remark7.3.1, the singular locus $\operatorname{Sing} F$ consists of the points such that:

$$
\begin{align*}
\mu_{1}+\bar{\mu}_{2} z_{3} & =0  \tag{7.3.8}\\
2 \mu_{1} \bar{z}_{2} & =0  \tag{7.3.9}\\
\mu_{2} z_{1} & =0 \tag{7.3.10}
\end{align*}
$$

where $\mu_{1}, \mu_{2} \in \mathbb{C}$ and $\left|\mu_{1}\right|+\left|\mu_{2}\right| \neq 0$. From (7.3.8) and (7.3.10), we get $z_{1}=0$.
If $\mu_{1}=0$ and $\mu_{2} \neq 0$, then the solutions of the equations are:

$$
\left\{\left(z_{1}, z_{2}, z_{3}\right) \in \mathbb{C}^{3} \mid z_{1}=z_{3}=0\right\}
$$

If $\mu_{1} \mu_{2} \neq 0$, then the solutions of the above equations are:

$$
\left\{\left(z_{1}, z_{2}, z_{3}\right) \in \mathbb{C}^{3} \mid z_{1}=z_{2}=0\right\}
$$

Therefore $\quad \operatorname{Sing} F \quad=\quad\left\{\left(z_{1}, z_{2}, z_{3}\right) \in \mathbb{C}^{3} \mid z_{1}=z_{2}=0\right\}$
$\left\{\left(z_{1}, z_{2}, z_{3}\right) \in \mathbb{C}^{3} \mid z_{1}=z_{3}=0\right\}$ and $F(\operatorname{Sing} F)=\left\{(a, b) \in \mathbb{C}^{2} \mid b=0\right\}$.
On the other hand, let us show that $F$ is strongly non-degenerate at infinity. The possible restrictions of $F$ to the faces are $F_{\Delta_{1}}=\left(z_{1}, \bar{z}_{1} z_{3}\right), F_{\Delta_{2}}=\left(z_{2}^{2}, \bar{z}_{1} z_{3}\right)$ and $F_{\Delta_{3}}=F$. From the above computation, we see that $\operatorname{Sing} F_{\Delta_{3}} \cap \mathbb{C}^{* 3}=\emptyset$.

For the restriction $F_{\Delta_{1}}=\left(z_{1}, \bar{z}_{1} z_{3}\right)$, the Jacobian matrix are:

$$
\begin{aligned}
& \mathrm{d} F_{\triangle_{1}}\left(z_{1}, z_{2}, z_{3}\right)=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & \bar{z}_{1}
\end{array}\right) \\
& \overline{\mathrm{d}} F_{\triangle_{1}}\left(z_{1}, z_{2}, z_{3}\right)=\left(\begin{array}{ccc}
0 & 0 & 0 \\
z_{3} & 0 & 0
\end{array}\right) .
\end{aligned}
$$

By an easy computation, we have $\operatorname{Sing} F_{\Delta_{1}}=\left\{\left(z_{1}, z_{2}, z_{3}\right) \in \mathbb{C}^{3} \mid z_{1}=0\right\}$. Consequently, $\operatorname{Sing} F_{\triangle_{3}} \cap \mathbb{C}^{* 3}=\emptyset$.

For the restriction $F_{\Delta_{2}}=\left(z_{2}^{2}, \bar{z}_{1} z_{3}\right)$, the Jacobian matrix are:

$$
\begin{aligned}
& \mathrm{d} F_{\Delta_{2}}\left(z_{1}, z_{2}, z_{3}\right)=\left(\begin{array}{ccc}
0 & 2 z_{2} & 0 \\
0 & 0 & \bar{z}_{1}
\end{array}\right) \\
& \overline{\mathrm{d}} F_{\triangle_{2}}\left(z_{1}, z_{2}, z_{3}\right)=\left(\begin{array}{ccc}
0 & 0 & 0 \\
z_{3} & 0 & 0
\end{array}\right) .
\end{aligned}
$$

By an easy computation, we have $\operatorname{Sing} F_{\Delta_{2}}=\left\{\left(z_{1}, z_{2}, z_{3}\right) \in \mathbb{C}^{3} \mid z_{1}=z_{3}=0\right\} \cup$ $\left\{\left(z_{1}, z_{2}, z_{3}\right) \in \mathbb{C}^{3} \mid z_{2}=0\right\}$. Consequently, $\operatorname{Sing} F_{\Delta_{2}} \cap \mathbb{C}^{* 3}=\emptyset$.

On account of the above arguments, we conclude that $F$ is strongly nondegenerate at infinity. In order to calculate $K_{\infty}(F)$, by Remark 7.3.1, we have:

$$
\begin{equation*}
\nu(\mathrm{d} F)=\min \|\left(\mu_{1}+\bar{\mu}_{2} z_{3}, 2 \mu_{1} \bar{z}_{2}, \mu_{2} \bar{z}_{1} \|\right. \tag{7.3.11}
\end{equation*}
$$

where $\left(\mu_{1}, \mu_{2}\right) \in \mathbb{C}^{2}$ and $\left|\mu_{1}\right|^{2}+\left|\mu_{2}\right|^{2}=1$.
Let $(a, b) \in K_{\infty}(F)$. Then by Curve selection lemma, there exist the curves $z(t)=\left(z_{1}(t), z_{2}(t), z_{3}(t)\right)$ and $\left(\mu_{1}(t), \mu_{2}(t)\right) \in S_{1}^{3}$ such that:

$$
\begin{align*}
\lim _{t \rightarrow 0}\left\|\left(z_{1}(t), z_{2}(t), z_{3}(t)\right)\right\| & =\infty  \tag{7.3.12}\\
\lim _{t \rightarrow 0} F\left(z_{1}(t), z_{2}(t), z_{3}(t)\right) & =(a, b)  \tag{7.3.13}\\
\lim _{t \rightarrow 0}\|z(t)\| \nu(\mathrm{d} F)(t) & =0 \tag{7.3.14}
\end{align*}
$$

If $\operatorname{ord}_{\mathrm{t}}\left(\mu_{1}(\mathrm{t})\right)=\operatorname{ord}_{\mathrm{t}}\left(\mu_{2}(\mathrm{t})\right)=0$, then from (7.3.11) and (7.3.14), we must have $\lim _{t \rightarrow 0} z_{1}(t)=\lim _{t \rightarrow 0} z_{2}(t)=0$. This implies that $\lim _{t \rightarrow 0} z_{3}(t)=\infty$ by (7.3.14) which contradicts $\lim _{t \rightarrow 0}\left|\mu_{1}+\bar{\mu}_{2} z_{3}\right|=0$.

If $\operatorname{ord}_{\mathrm{t}}\left(\mu_{1}(\mathrm{t})\right)>0$ and $\mu_{2}(t) \rightarrow 1$, then from (7.3.11) and (7.3.14), we must have $\lim _{t \rightarrow 0} z_{1}(t)=0$. Using (7.3.12) and (7.3.13), we conclude that $\lim _{t \rightarrow 0} z_{3}(t)=\infty$ which contradicts $\lim _{t \rightarrow 0}\left|\mu_{1}+\bar{\mu}_{2} z_{3}\right|=0$.

If $\operatorname{ord}_{\mathrm{t}}\left(\mu_{2}(\mathrm{t})\right)>0$ and $\mu_{1}(t) \rightarrow 1$, then from (7.3.11) and (7.3.14), we must have $\lim _{t \rightarrow 0} z_{2}(t)=0$. Using (7.3.12) and (7.3.13), we conclude that $\lim _{t \rightarrow 0} z_{3}(t)=\infty$ and $\lim _{t \rightarrow 0} z_{1}(t)=0$.

Therefore $K_{\infty}(F) \subset\{0\} \times \mathbb{C}$. Now, let us show that $K_{\infty}(F)=\{0\} \times \mathbb{C}$. In fact for any point $(0, c) \in\{0\} \times \mathbb{C}$, we choose the curves $z(t)=\left(\bar{c} t, 0, \frac{1}{t}\right)$ and $\left(\mu_{1}(t), \mu_{2}(t)\right)=\left(-\frac{1}{\sqrt{t^{2}+1}}, \frac{t}{\sqrt{t^{2}+1}}\right)$ where $t \in \mathbb{R}$ is sufficiently small. Then we get $\mu_{1}(t)+\bar{\mu}_{2}(t) z_{3}(t) \equiv 0$ and $\mu_{2}(t) \bar{z}_{1}(t)=\frac{\bar{c} t^{2}}{\sqrt{t^{2}+1}}$. Since $\operatorname{ord}_{t}\|z(t)\|=-1$ and $\operatorname{ord}_{\mathrm{t}}\left\|\left(\mu_{1}+\bar{\mu}_{2} z_{3}, 2 \mu_{1} \bar{z}_{2}, \mu_{2} \bar{z}_{1}\right)\right\|=2$, we have therefore $\lim _{t \rightarrow 0}\|z(t)\| \nu(\mathrm{d} F)(t)=0$. On the other hand, $\lim _{t \rightarrow 0} F\left(z_{1}(t), z_{2}(t), z_{3}(t)\right)=\lim _{t \rightarrow 0}(\bar{c} t, c)=(0, c)$. Hence $(0, c) \in K_{\infty}(F)$ which gives $\{0\} \times \mathbb{C} \subset K_{\infty}(F)$. Therefore $K_{\infty}(F)=\{0\} \times \mathbb{C}$.

### 7.4 Non-degeneracy and global diffeomorphism

In this section, we first expose some basic definitions and facts concerning Łojasiewicz exponents at infinity which is used to prove Theorem 7.1.3.

Definition 7.4.1 [Aus07, Definition 2.1] Let $F=\left(f_{1}, \ldots, f_{k}\right): \mathbb{R}^{n} \rightarrow \mathbb{R}^{k}$ be a polynomial map. Define the set $E_{\infty}(F)$ formed by those $\alpha \in \mathbb{R}$ satisfying the following inequalities with positive constants $C$ and $r$

$$
\|\mathbf{x}\|^{\alpha} \leq C \sup _{i}\left|f_{i}(\mathbf{x})\right|,
$$

where $\mathbf{x} \in \mathbb{R}^{n}$ such that $\|\mathbf{x}\| \geq B$. If $E_{\infty}(F) \neq$ emptyset, we denote by $\mathcal{L}_{\infty}(F)$ the supremum of $E_{\infty}(F) \neq \emptyset$ and call it the Eojasiewicz exponent at infinity of $F$.

Remark 7.4.2 When $E_{\infty}(F)=\emptyset$, we set $\mathcal{L}_{\infty}(F)=-\infty$. It was shown that the set $E_{\infty}(F)$ is upper bounded when it is non-empty by [Aus07, Proposition 2.4].

Remark 7.4.3 Note that $F$ is proper if and only if

$$
\begin{equation*}
\lim _{\|\mathbf{x}\| \rightarrow+\infty}\|F(\mathbf{x})\|=+\infty \tag{7.4.1}
\end{equation*}
$$

It follows immediately by Definition 7.4.1 that the positiveness of $\mathcal{L}_{\infty}(F)$ implies the properness of $F$.

Now, let us turn to the definition of non-degeneracy at infinity for real polynomial maps. With the similar construction like mixed polynomial maps (see Definition 7.2.1), one can still use Condition (*) to define the non-degeneracy condition at infinty for real polynomial maps and obtain the parallel results for the real case (see [Dia]). In the sequel, it will cause no confusion if we say a real polynomial map is non-degenerate under Definition 7.2.1. In the following, we first state BiviàAusina's non-degeneracy condition:

Definition 7.4.4 [Aus07, Definition 3.5] Let $\mathbf{P}=\left(p_{1}, \ldots, p_{n}\right) \in \mathbb{Z}^{n}$ such that $p=\min _{1 \leq i \leq n} p_{i}<0$. We say that $F$ is non-degenerate at infinity if the following condition is satisfied for any $\mathbf{P}$

$$
\begin{equation*}
\left\{\mathbf{x} \in\left(\mathbb{R}^{*}\right)^{n} \mid f_{\triangle_{\mathbf{P}}^{j}}(\mathbf{x})=0, \text { for all } j=1, \ldots, k\right\}=\emptyset \tag{7.4.2}
\end{equation*}
$$

REmark 7.4.5 In our construction, we used the minimal value of the linear function $l_{\mathbf{p}}(x)=\sum_{i=1}^{n} p_{i} x_{i}$ defined on $\overline{\operatorname{supp}\left(f_{i}\right)}$, since by curve selection lemma, we consider the analytic curves for $t \rightarrow 0$; While in [Aus07], the author used the maximal value of the linear function $l_{\mathbf{p}}(x)=\sum_{i=1}^{n} p_{i} x_{i}$ defined on $\overline{\operatorname{supp}\left(f_{i}\right)}$ where the vector $\mathbf{P}=\left(p_{1}, \ldots, p_{n}\right) \in \mathbb{Z}^{n}$ such that $p=\max _{1 \leq i \leq n} p_{i}>0$ since he considered the analytic curves for $t \rightarrow \infty$. Therefore the above definition 7.4.4 is indeed equivalent to Bivià-Ausina's non-degeneracy condition.

Consider a real polynomial map $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$. We denote by $J_{F}$ the set of points at which $F$ is not proper (see [Jel99] for more details). Obviously, by the definition of $K_{\infty}(F)$, we have the inclusion $K_{\infty}(F) \subset J_{F}$. The next two theorem will lead us to formulate our main result in this section. The first theorem is due to Hadamard, which implies the sufficient and necessary condition on the global homeomorphism for a $C^{1}$ map.

Theorem 7.4.6 [Ess00, Theorem 8, p.240] Let $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be a $C^{1}$ map. Then $F$ is a homeomorphism if and only if $F$ is a local homeomorphism and $F$ is proper.

The interest of the next theorem is that it allows us to study the properness of a polynomial map under KOS-regularity condition.

Theorem 7.4.7 [KOS00, Proposition 3.1 and Theorem 3.4] Let $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{k}$, $k \leq n$ be a $C^{1}$ semialgebraic map. Assume that the set of regular points of $F$ is dense and that $F^{-1}(y)$ is compact for any $y \in \mathbb{R}^{k} \backslash F(\operatorname{Sing} F)$, then $K_{\infty}(F)=J_{F}$. In particular, if $k=n$, the assumption for density of the regular points is automatically verified, and $K_{\infty}(F)=J_{F}$.

Let us denote by $J(F)(\mathbf{z})$ the determinant of the Jacobian matrix of $F$ at point $\mathbf{z}$, where $F: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ is a mixed polynomial map. Note that here the Jacobian matrix of $F$ is the one that we regard $F$ as a polynomial map $\mathbb{R}^{2 n} \rightarrow \mathbb{R}^{2 n}$. According to above two theorems, we have the following formulation of Jacobian problem:

Theorem 7.4.8 Let $F: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ be a mixed polynomial map such that $J(F)(\mathbf{z}) \neq$ 0 , for all $\mathbf{z} \in \mathbb{C}^{n}$. If $F$ is non-degenerate at infinity under Definition 7.2.1 and if $f_{i}$ is convenient for all $i=1, \ldots, n$, then $F$ is a homeomorphism.

Proof. Since $F$ is non-degenerate at infinity and $f_{i}$ is convenient for all $i=$ $1, \ldots, k$, by Corollary 7.3.3, we have $K_{\infty}(F)=\emptyset$. Therefore we get the conclusion from Theorem 7.4.6 and 7.4.7.

Let us compare the definitions 7.2 .1 and 7.4 .4 of non-degeneracy condition in the real setting.

Proposition 7.4.9 Suppose that $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{k}, k \leq n$ is a polynomial map and that $f_{i}$ is convenient, for all $i=1, \ldots, k$. If $F$ is non-degenerate at infinity under Definition 7.4.4, then it is also non-degenerate infinity under Definition 7.2.1.

Proof. We apply the notations as in Section 7.2. Let us fix a vector $\mathbf{P}=$ $\left(p_{1}, \ldots, p_{n}\right) \in \mathbb{Z}^{n}$ and assume that $p=\min _{1 \leq i \leq n} p_{i}<0$. Since $f_{i}$ is convenient for all $i=1, \ldots, k$, the minimal value $d_{\mathbf{p}}^{i}$ of $l_{\mathbf{p}}(x)$ must be strictly negative on $\overline{\operatorname{supp}\left(f_{i}\right)}$. (This argument was also used in the proof of Corollary 7.3.3) Therefore by Lemma 4.2.5, the face $\triangle_{\mathbf{P}}^{i}$ is contained in $\Gamma^{+}\left(f_{i}\right)$ for all $i=1, \ldots, k$. As $d_{\mathbf{p}}^{i}<0$ we have $\mathrm{N}_{\mathbf{P}}=\{1,2 \ldots, k\}$. If $F$ is non-degenerate at infinity under Definition 7.4.4, then $F$ is non-degenerate at infinity under Definition 7.2.1, by Condition (*).

REMARK 7.4.10 When $k=n$ and $f_{i}$ is a real convenient polynomial function for all $i=1, \ldots, n$, the two definitions are equivalent. In fact, assume that $F$ is degenerate at infinity under Definition 7.4.4 but non-degenerate under Definition 7.2.1. Then
there exists $\mathbf{x} \in\left(\mathbb{R}^{*}\right)^{n}$ and a vector $\mathbf{P}=\left(p_{1}, \ldots, p_{n}\right) \in \mathbb{Z}^{n}$ with $\min _{1 \leq i \leq n} p_{i}<0$ such that $f_{\triangle_{\mathrm{P}}^{i}}(\mathbf{x})=0$, for every $i$. Using the same argument as that of the above proof, we conclude that $\triangle_{\mathbf{P}}^{i}$ is a face of $\Gamma^{+}\left(f_{i}\right)$ for all $i=1, \ldots, k$. On the other hand, by Euler's identity, we have the inner product $\left\langle\mathrm{d} f_{\triangle_{\mathbf{P}}^{i}}, \mathbf{P x}\right\rangle=d_{\mathbf{p}}^{i} f_{\triangle_{\mathbf{P}}^{i}}=0$, where $\mathbf{P} \mathbf{x}=\left(p_{1} x_{1}, \ldots, p_{n} x_{n}\right) \neq \mathbf{0}$. It follows from $\langle\mathrm{d} F, \mathbf{P} \mathbf{x}\rangle=0$, that $\mathbf{x} \in \operatorname{Sing}\left(F_{\triangle_{\mathbf{P}}}\right)$, which implies $\mathbf{x} \in \operatorname{Sing}\left(F_{\triangle_{\mathbf{P}}}\right) \cap\left\{\mathbf{x} \in \mathbb{R}^{n} \mid f_{\triangle_{\mathbf{P}}^{j}}(\mathbf{x})=0, \forall j \in \mathrm{~N}_{\mathbf{P}}\right\} \cap\left(\mathbb{R}^{*}\right)^{n}$. This contradicts our non-degeneracy assumption of $F$ by Definition 7.2.1. Then from proposition 7.4.9, we observe that our definition 7.2 .1 is equivalent to Definition 7.4.4. We also note that Bivià-Ausina was only concerned with the real case of Theorem 7.4.8 in [Aus07], so our Theorem 7.4.8 is still more general. We also refer to the next example 7.4 .11 which shows that our non-degeneracy condition is strictly weaker than Bivià-Ausina's definition.

Our next example is a real non-degenerate polynomial map which is degenerate at infinity under Definition 7.4.4

Example 7.4.11 Consider $F: \mathbb{R}^{3} \rightarrow \mathbb{R}^{2}, F(x, y, z)=\left(x+y, y^{2}-z^{2}\right)$. The Jacobian matrix of $F$ at the point $(x, y, z)$ is:

$$
\mathrm{d} F(x, y, z)=\left(\begin{array}{ccc}
1 & 1 & 0 \\
0 & 2 y & -2 z
\end{array}\right) .
$$

Therefore we get $\operatorname{Sing} F=\left\{(x, y, z) \in \mathbb{R}^{3} \mid y=z=0\right\}$ and $F(\operatorname{Sing} F)=$ $\{(c, 0) \mid c \in \mathbb{R}\}$. Let us show that $F$ is non-degenerate at infinity in the sense of Definition 7.2.1. Let $\mathbf{P}=\left(p_{1}, p_{2}, p_{3}\right) \in \mathbb{Z}^{3}$ such that $p=\min _{1 \leq i \leq 3} p_{i}<0$. Then we get following situations:
(a) If $p=p_{3}<p_{2}$, the possible restrictions of $F$ to the faces are $F_{\Delta_{1}}=\left(x,-z^{2}\right)$, $F_{\Delta_{2}}=\left(y,-z^{2}\right)$ and $F_{\Delta_{3}}=\left(x+y,-z^{2}\right)$. Since for every restriction $F_{\Delta_{i}}$, the rank of the Jacobian matrix $\mathrm{d} F_{\Delta_{i}}$ is not full if and only if $z=0$, where $i=$ $1,2,3$. Therefore the strong non-degeneracy condition at infinity is satisfied in this case.
(b) If $p=p_{2}<p_{3}$, the possible restrictions of $F$ to the faces are $F_{\triangle_{4}}=\left(y, y^{2}\right)$, and $F_{\Delta_{5}}=\left(x+y, y^{2}\right)$. For the restriction $F_{\Delta_{5}}$, the singular locus is $\{(x, y, z) \mid y=0\}$ which implies that $\operatorname{Sing} F_{\Delta_{5}} \cap \mathbb{R}^{* 3}=\emptyset$. For the restriction $F_{\triangle_{4}}$, we have $\operatorname{Sing} F_{\triangle_{4}}=\mathbb{R}^{3}$ but $\operatorname{Sing} F_{\Delta_{4}} \cap\left(F_{\Delta_{4}}=0\right) \cap \mathbb{R}^{* 3}=\emptyset$. Therefore in this case, $F$ is non-degenerate at infinity but not strongly non-degenerate at infinity.
(c) If $p=p_{2}=p_{3}$, the possible restrictions of $F$ to the faces are $F_{\triangle_{6}}=\left(x+y, y^{2}-\right.$ $z^{2}$ ), and $F_{\Delta_{7}}=\left(y, y^{2}-z^{2}\right)$. Since $F_{\Delta_{6}}=F$, from the above computation, we
see that $\operatorname{Sing} F_{\Delta_{6}} \cap \mathbb{R}^{* 3}=\emptyset$. For the restriction $F_{\Delta_{7}}$, the Jacobian matrix at the point $(x, y, z)$ is:

$$
\mathrm{d} F_{\triangle_{7}}(x, y, z)=\left(\begin{array}{ccc}
0 & 1 & 0 \\
0 & 2 y & -2 z
\end{array}\right)
$$

which implies that $\operatorname{Sing} F_{\Delta_{7}}=\{(x, y, z) \mid z=0\}$. Hence $\operatorname{Sing} F_{\Delta_{7}} \cap \mathbb{R}^{* 3}=\emptyset$. In this case, $F$ is strongly non-degenerate at infinity.

On the other hand, when $F_{\Delta_{7}}=0$, we get $x=-y=z$ or $x=-y=-z$. Therefore $F$ does not satisfy the non-degeneracy condition in the sense of Definition 7.4.4. Now, we proceed to calculate $K_{\infty}(F)$. By the definition of the norm used to define KOS-regularity, we have:

$$
\nu(\mathrm{d} F)=\min \|(a, a+2 b y,-2 b z)\|
$$

where $(a, b) \in \mathbb{R}^{2}$ and $a^{2}+b^{2}=1$. Let $\left(c_{1}, c_{2}\right) \in K_{\infty}(F)$. Then by Curve selection lemma, there exist the curves $\varphi(t)=(x(t), y(t), z(t))$ and $(a(t), b(t))$ such that:

$$
\begin{align*}
\lim _{t \rightarrow 0}\|(x(t), y(t), z(t))\| & =\infty  \tag{7.4.3}\\
\lim _{t \rightarrow 0} F(x(t), y(t), z(t)) & =\left(c_{1}, c_{2}\right)  \tag{7.4.4}\\
\lim _{t \rightarrow 0}\|\varphi(t)\|\|\nu(\mathrm{d} F)\| & =0 \tag{7.4.5}
\end{align*}
$$

where $a^{2}(t)+b^{2}(t)=1$. Since $\nu(\mathrm{d} F) \rightarrow 0$, we must have $\lim _{t \rightarrow 0} a(t)=0$, $\lim _{t \rightarrow 0} y(t)=0$ and $\lim _{t \rightarrow 0} z(t)=0$. It follows that $\lim _{t \rightarrow 0}\|x(t)\|=\infty$ from (7.4.3). This is in contradiction with (7.4.4), since $c_{1}$ is finite. Hence $K_{\infty}(F)=\emptyset$.

For any critical value $(c, 0)$, we have:
$F^{-1}((c, 0))=\left\{(x, y, z) \in \mathbb{R}^{3} \mid x=c-y, z=y\right\} \cup\left\{(x, y, z) \in \mathbb{R}^{3} \mid x=c-y, z=-y\right\}$ which is the union of two lines and these two lines intersect at the point $(c, 0,0)$.

If we fix $\varepsilon<0$ sufficiently small, then regular fiber is

$$
F^{-1}((c, \varepsilon))=\left\{(x, y, z) \in \mathbb{R}^{3} \mid x=c-y, z= \pm \sqrt{y^{2}-\varepsilon}\right\} .
$$

Therefore the regular fiber is a hyperbolic curve and moreover, the asymptotes of this hyperbolic curve are $F^{-1}((c, 0))$.

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[^0]:    ${ }^{1}$ terminology used by Oka [Oka08]

[^1]:    ${ }^{1}$ For the case $c=0$, we refer to Remark 4.4.1.

[^2]:    ${ }^{2}$ Here, "stable" means that the monodromy fibrations at infinity are equivalent in this family.

