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by

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ON TWO PROBLEMS CONCERNING THE LAURENT-STIELTJES COEFFICIENTS OF DIRICHLET L-SERIES

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Abstract

In this thesis, we give explicit upper bound for the Laurent-Stieltjes constants in the following two cases:

1. The character χ is fixed and the order n goes to infinity.
2. The order n is 0 and the conductor q goes to infinity.

The Laurent-Stieltjes constants $\gamma_n(\chi)$ are the coefficients of the expansion in the Laurent series of the Dirichlet L-series. This thesis is divided to three parts:

In first part of this thesis, we study the asymptotic behavior in n of these constants. In 1985, Matsuoka gave an excellent asymptotic expansion for these constants in the case of the zeta function, We extend the formula of Matsuoka to the Dirichlet L- functions. We give an upper bound for $\left|\gamma_n(\chi)\right|$ when $1 \le q < \frac{\pi}{2}\,\frac{e^{(n+1)/2}}{n+1}$. In the case when $\chi = \chi_0$ and $q = 1$, this leads to a sizeable improvement of Matsuoka's bound and on previous results. By using this result, we also deduce an approximation of the Dirichlet L-functions in the neighborhood of $z = 1$ by a short Taylor polynomial. We provide a direct proof of Matsuoka's formula that avoids complex analysis and, more importantly, the functional equation of $\zeta(z)$ and $L(z,\chi)$.

The second part of this thesis deals more specifically with the first Laurent-Stieltjes coefficient, namely $L(1,\chi)$. In 2001, Ramaré gave an approximate formula for $L(1,\chi)$ depending on Fourier transforms. Thanks to this formula, he obtained the best explicit upper bound for $\left|L(1,\chi)\right|$ to date. Here, we study the most difficult case when $\chi(2) = 1$, and get an improvement. As application, we deduce an explicit upper bound for the class number of any real quadratic field $\mathbb{Q}\left(\sqrt{q}\right)$, improving on a result by Le.

In the last part, we follow the method of Ramaré for giving an upper bound of the first Laurent-Stieltjes coefficient but this time when $\chi(3) = 0$. This result is an improvement on a similar one due to Louboutin.

Résumé (French version)

Dans cette thèse, nous donnons des majorations explicites pour les constantes de Laurent-Stieltjes dans les deux cas suivants :

1. Le caractère χ est fixé et n tend vers l'infini.
2. L'ordre n est égal à 0 et le conducteur q tend vers l'infini.

Les constantes de Laurent-Stieltjes $\gamma_n(\chi)$ sont les coefficients qui interviennent dans le développement en série de Laurent des séries L de Dirichlet. Cette thèse est composée de trois parties :

Dans la première partie de cette thèse, nous étudions le comportement asymptotique en n de ces constantes. En 1985, Matsuoka a donné un développement asymptotique des constantes de Laurent-Stieltjes pour la fonction zêta de Riemann. Ici, nous prolongeons la formule de Matsuoka aux fonctions L de Dirichlet. Nous donnons une majoration de $|\gamma_n(\chi)|$ pour $1 \leq q < \frac{\pi}{2} \frac{e^{(n+1)/2}}{n+1}$. Dans le cas où $\chi = \chi_0$ et $q = 1$, notre résultat donne une réelle amélioration de la majoration de Matsuoka et des résultats précédents. En utilisant cette majoration, nous déduisons aussi une approximation des fonctions L de Dirichlet au voisinage de $z = 1$ par un polynôme de Taylor relativement court. En outre, nous donnons une preuve directe de la formule de Matsuoka en évitant l'analyse complexe et de façon notoire l'équation fonctionnelle de la fonction zêta de Riemann et des fonctions L de Dirichlet.

Dans la deuxième partie de cette thèse, nous nous intéressons au premier coefficient de Laurent-Stieltjes $L(1,\chi)$. En 2001, Ramaré a donné une formule approchée pour $L(1,\chi)$ en utilisant des transformées de Fourier. Grâce à cette formule, Ramaré a obtenu la meilleure majoration explicite pour $|L(1,\chi)|$ à ce jour. Nous étudions le cas le plus difficile lorsque $\chi(2) = 1$ et obtenons une amélioration. Nous en déduisons aussi une majoration explicite pour le nombre de classes pour les corps quadratiques réels $\mathbb{Q}(\sqrt{q})$. Cette majoration est une amélioration d'un résultat de Le.

Dans la dernière partie, nous suivons la méthode de Ramaré pour donner une majoration explicite de $|L(1,\chi)|$ dans le cas où $\chi(3) = 0$, améliorons un résultat de Louboutin.

Keywords: Dirichlet characters, Functional equation, Gauss sums, Gamma function, Matsuoka's formula, Riemann zeta function, Laurent-Stieltjes coefficients, Dirichlet L-functions, Euler's constant, class number, Euler-Maclaurin formula.

Mots clés : Caractère de Dirichlet, équation fonctionnelle, somme de Gauss, fonction Gamma, formule de Matsuoka, fonction zêta de Riemann, coefficients de Laurent-Stieltjes, fonction L de Dirichlet, constante d'Euler, nombre de classes, formule d'Euler-Maclaurin.

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I dedicate this thesis to
my father Mohammed, my mother Aida,
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my brother Jamal
and my love Waseem,
for their constant support and unconditional love.
I love you all.

Science sans conscience n'est que
ruine de l'âme

François RABELAIS

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Sumaia Saad Eddin

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History of the Laurent-Stieltjes constants

The Laurent-Stieltjes constants first appeared in the coefficients of Laurent expansion at the pole of the Riemann zeta function. These constants are named after the Dutch mathematician *Thomas Jan Stieltjes* (1885-1894). The Laurent-Stieltjes constants are sometimes referred to as generalized Euler constants. They was studied by other eminent mathematicians like Charles Hermite, Srinivasa Ramanujan, William E. Briggs and Bruce Berndt $\cdots$ as well as many others.

The Laurent-Stieltjes constants also appear in various areas as Dirichlet L-functions, Hurwitz zeta function, definite integration and hyper-geometric series. Indeed, these constants have many uses in analytic number theory and elsewhere. Among other applications, estimates for γ_n may be used to determine a zero-free region of the zeta function near the real axis in the critical strip $0 \leq \Re(z) \leq 1$, to compute the Riemann and Hurwitz zeta functions in the complex plane, to give new derivations of Binet's formulae for $\log \Gamma(x)$, to evaluate some integrals related to the Barnes multiple gamma functions which has several connections with the sine and cosine integrals.

In this chapter, we present a short historical survey of these constants during the last 150 years.

The Laurent-Stieltjes constants for the zeta function

After Riemann's famous memoir, we known that the function

$$\zeta(z) - \frac{1}{z-1} \tag{1}$$

is entire; i. e., $\zeta(z)$ has an analytic continuation throughout the whole complex plane except for a simple pole at $z = 1$ with residue 1. Close to the singular point $z = 1$ the zeta function can be represented as a Laurent series in $(z-1)$:

$$\zeta(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_n}{n!} (z-1)^n. \tag{2}$$

Thomas Jan Stieltjes was the first to propose the following definition of γ_n

$$\gamma_n = \lim_{T\to\infty}\left(\sum_{m=1}^{T}\frac{(\log m)^n}{m} - \frac{(\log T)^{n+1}}{(n+1)}\right). \tag{3}$$

In an exchange of letters between Stieltjes and Hermite, they have thoroughly discussed the proof of this formula. Ramanujan [45] did not explicitly define γ_n by Eq (3). Instead, he said that γ_n are "the constants of $\sum_{m=1}^{\infty}(\log m)^n/m$". In 1887, Jensen [25] rediscovered this equation. It has been proved also by Verma [55] and Ferguson [18]. In 1955, Briggs & Chowla [10] stated two proofs of Eq (3) and found estimates for these constants. These estimates were improved by Lammel[32]. By using these estimates, he also gave a simple proof of the fact that $\zeta(z)$ has no zeros on $|z-1|\le 1$.

It is interesting to observe that the Eq (3) is slowly convergent and is therefore useless in numerical computations so one is forced to seek some method to speed up convergence.

Different integral or series representations for γ_n were successively obtained by several authors, among them, Kluyver [27] has established an infinite series representation for these constants. In the paper [6], Briggs proved that:

Theorem 0.1 1. *There are infinitely many n for which γ_n is positive, and infinitely many n for which that γ_n is negative.*

2. *There exists a function $\epsilon(n)$ such that $|\epsilon(n)|\le 1$ and*

$$\gamma_n = \left(\frac{n}{2e}\right)^n \epsilon(n). \tag{4}$$

3. *For any $\epsilon > 0$, the inequalities*

$$n^{-\epsilon n} < |\gamma_n| < n^{\epsilon n} \tag{5}$$

hold for infinitely many n.

It is important to point out that Briggs was the first to obtain an upper bound for $|\gamma_n|$.

In 1972, Berndt [4] proved Eq (3) and gave the following bound for $\left|\gamma_n\right|$.

Theorem 0.2 *For $n\ge 1$, we have:*

$$\left|\gamma_n\right| \le \left(3+(-1)^n\right)\frac{(n-1)!}{\pi^n}. \tag{6}$$

From Eq (6), we note that the radius of convergence of Eq (2) is at least π. So Eq (6) is stronger than Eq (4), but Eq (5) is asymptotically stronger than Eq (6).

A detailed discussion of the constants γ_n in terms of the Bernoulli numbers has been provided by many authors, like Liang & Todd [35]. In 1981, Israilov [23] gave some information on the quantities γ_n and proved that

Theorem 0.3 *Let $f_n^{(v)}(x) = d^v\left(x^{-1}\log^n x\right)/dx^v$. Then, for all integral $n \geq 1$ and $k \geq 1$, we have*

$$\gamma_n = \frac{(-1)^n}{n!}\left[\sum_{m=1}^{N}\frac{\log^n m}{m} - \frac{\log^{n+1} N}{n+1} - \frac{\log^n N}{2N} - \sum_{r=1}^{k_0-1}\frac{B_{2r}}{2r}f^{(2r-1)}(N) - \theta\frac{B_{2k_0}}{(2k_0)!}f^{(2k_0-1)(N)}\right]$$

and for $n=0$, we have

$$\gamma_0 = \sum_{m=1}^{N}\frac{1}{m} - \log N - \frac{1}{2N} + \sum_{r=1}^{k-1}\frac{B_{2r}}{2rN^{2r}} + \theta\frac{B_{2k}}{2kN^{2k}},$$

where $0<\theta<1$, the B_j denoting the Bernoulli numbers and where $b_{j,v}$ is defined by the formula

$$b_{j,v} = \sum_{r_1=j}^{v}\frac{1}{r_1}\sum_{r_2=j-2}^{r_1-1}\frac{1}{r_2}\cdots\sum_{r_j=1}^{r_{j-1}-1}\frac{1}{r_j}.$$

Thanks to this theorem, Israilov, in the same his paper, gave the following upper bound for $|\gamma_n|$.

Theorem 0.4 *For $n \geq 2k$ and $k = 1, 2, \cdots$, we have*

$$\left|\frac{\gamma_n}{n!}\right| \leq C(k)(2k)^{-n}, \tag{7}$$

where

$$C(k) = \frac{|B_{2k}|}{2k}\left(1 + \sum_{j=1}^{2k} b_{j,2k}(2k)^j\right).$$

In particular,

$$C(1) = 1/2, \quad C(2) = 7/12, \quad C(3) = 11/3, \quad C(4) \leq 73.3.$$

In 1985, Matsuoka [40] produced an excellent asymptotic expansion for γ_n, and he was also able to simplify his method to derive the explicit form.

Theorem 0.5 *Let m be a sufficiently large integer, and let c be any positive constant. Then, for all integer n with $|n-m| < c\log m$, we have:*

$$\gamma_n = \sqrt{\frac{2n}{\pi}}\frac{e^{G(n)}}{\log n!}\left[\cos\left(F(m) - \frac{\pi}{2}\frac{n-m}{\log m}\right) + \mathcal{O}\left(\frac{\log\log m}{\log m}\right)\right].$$

The constant in the $\mathcal{O}$-term depends only on c and

$$F(n) = -\frac{\pi n}{2\log n} + \mathcal{O}\left(\frac{n\log\log n}{\log^2 n}\right),$$

and where

$$G(n) = -n\log n + n\log\log n + n + o(n), \quad \text{when } n \to \infty.$$

Matsuoka also gave an excellent explicit upper bound for $|\gamma_n|$ for $n \geq 10$.

Theorem 0.6 *For $n \geq 10$, we have*

$$|\gamma_n| \leq 10^{-4} e^{n \log\log n}. \tag{8}$$

In the same paper, Matsuoka proved that:

Theorem 0.7 *For any $\varepsilon > 0$, there exist infinitely many integers n for which all of $\gamma_n, \gamma_{n+1}, \gamma_{n+2}, \cdots, \gamma_{n+\lfloor(2-\varepsilon)\log n\rceil}$ have the same sign, and there exist only finitely many integers n for which all of $\gamma_n, \gamma_{n+1}, \gamma_{n+2}, \cdots, \gamma_{n+\lfloor(2+\varepsilon)\log n\rceil}$ have the same sign. Also, if*

$$f(n) = \left|\{0 \leq k \leq n \,:\, \gamma_k > 0\}\right|, \quad g(n) = \left|\{0 \leq k \leq n \,:\, \gamma_k < 0\}\right|,$$

then, $f(n) = n/2 + o(n)$ and $g(n) = n/2 + o(n)$.

In 1994, Zhang & Williams [57] proved that:

Theorem 0.8 *For $n = 0, 1, \cdots$, we have*

$$\gamma_n = \frac{1}{n+1} \sum_{j=0}^{n+1} \binom{n+1}{j} B_{n+1-j} \left(\log 2\right)^{n-j} \alpha_j,$$

where

$$\alpha_j = \sum_{m=1}^{\infty} \frac{(-1)^m \log^j m}{m}, \ (j \geq 1), \qquad \alpha_0 = \sum_{m=1}^{\infty} \frac{(-1)^m}{m} = -\log 2.$$

Thanks to this theorem, they showed that:

Theorem 0.9 *For $n \geq 1$, we have*

$$\left|\gamma_n\right| \leq \frac{(3 + (-1)^n)\,(2n)!}{(2\pi)^n n^{n+1}}. \tag{9}$$

From this bound, we note that the radius of convergence of Eq (2) is at least $\pi e/2$. So Eq (9) is stronger than Eq (6).

At the beginning of this century, Coffey [11], [12] and [13] has obtained rapidly convergent expressions for γ_n in terms of Bernoulli numbers. In another paper [14], Coffey gave the series representations for γ_0 and γ_1 with an exponential rate of decay. In 2010, Coffey & Knessl [29] have proved a variation of Matsuoka's asymptotic expression for these constants, namely:

Theorem 0.10 *Let $v = v(n)$ be the unique solution of the equation*

$$2\pi \exp[v \tan v] = n \frac{\cos v}{v},$$

in the interval $(0, \pi/2)$*, with* $v \to \pi/2$ *as* $n \to \infty$*. Let* $u = v \tan v$ *with* $u(n) \sim \log n$ *as* $n \to \infty$*. Then we have for* $n \gg 1$,

$$\gamma_n \sim \frac{B}{\sqrt{n}} e^{nA} \cos(an + b),$$

where

$$A = \frac{1}{2}\log(u^2 + v^2) - \frac{u}{u^2 + v^2}, \quad B = \frac{2\sqrt{2\pi}\sqrt{u^2 + v^2}}{[(u+1)^2 + v^2]^{1/4}},$$

and

$$a = \tan^{-1}\left(\frac{v}{u}\right) + \frac{v}{u^2 + v^2}, \quad b = \tan^{-1}\left(\frac{v}{u}\right) - \frac{1}{2}\tan^{-1}\left(\frac{v}{u+1}\right).$$

The $\sim$-sign above is loosely defined in their paper: it holds as long as we stay bounded away from the zeros of the cosine factor. They showed that $A \sim \log\log n$ and $B \sim \frac{\pi}{2}(\log n)^{-1}$ and that this form encapsulates both the basic growth rate $\log\log n$ and the oscillations $\cos[\frac{n(\pi/2)}{\log n}]$. It therefore has implications for the sign changes observed in γ_n with increasing n.

In 2011, Knessl & Coffey [28] have provided hyper-geometric summation representations for γ_1 and γ_2 with summands of the order of $O(n^{-6})$ and $O(n^{-3})$, respectively. In the same year, Adell [1] proved that:

Theorem 0.11 *For any* $n \geq 4$*, we have*

$$|\gamma_n| \leq \left(\frac{n! e^m}{m^{n+1}}\left(\frac{n+1}{m} + 1\right) + \frac{1}{n+1}\right)\log^{n+1}(m+1), \tag{10}$$

where $m = \lfloor n(1 - 1/\log n)\rfloor$ *and* $\lfloor x \rfloor$ *denotes the integer part of* x.

The order of magnitude, as $n \to \infty$ of the upper bound for $|\gamma_n|$ the last inequality above is

$$\exp\left\{n\log\log n - \frac{n}{2\log^2 n}\left(1 + \mathcal{O}\left(\frac{1}{\log n}\right)\right)\right\}.$$

Numerical computations of the constants γ_n some were first carried out by Jensen [25], who calculated the first 9 constants to 9 decimal places. In 1895, Gram[19] published a table of the first 16 constants to 16 decimal places. In 1927, Liang & Todd [35] employed the Euler-Maclaurin summation for calculate the first 20 constants to 15 decimal places, but they did not give error bounds. Using contour integration, Ainsworth & Howell [2] computed the first 2000 of γ_n. In 2002, Kreminski [31] expressed γ_n as a limit of a more rapidly convergent sequence than in Eq (3). He believed that his method produces high-precision estimates of γ_n more quickly, for larger values of n, and to more digits of accuracy than has been attained to date (see Table 4.1). He has computed γ_n with n ranging [0, 3200]. The most extensive calculations to date have been made by Coffey [13], who calculated γ_n with n ranging [0, 35000].

At the moment, the value of γ_n are not known precisely and each author gives different value of γ_n. But although these values are close, they are distinct. We note that, the Euler's constant $\gamma = \gamma_0$ is not known to be irrational or transcendental. The question of its irrationality has challenged mathematicians since Euler and remains a famous unresolved problem. By computing a large number of digits of γ and using continued fraction expansion, it has been shown that if γ is a rational number a/b then the denominator b must have at least 242080 digits.
Even if γ is less famous than the constants π and e, it deserves a great attention since it plays an important role in Analysis (Gamma function, Bessel functions, exponential-integral, ...) and occurs frequently in Number Theory.

The Laurent-Stieltjes constants for the Dirichlet L-functions

For a primitive and non-principal Dirichlet character χ of modulo q, where $q > 1$ and $0 \le n \in \mathbb{Z}$. The function $L(z,\chi)$ has an analytic continuation throughout the whole complex plane. Thus, the Dirichlet L-function can be represented as a Taylor series at $z = 1$:

$$L(z,\chi) = \sum_{n\ge 0} \frac{(-1)^n \gamma_n(\chi)}{n!} (z-1)^n.$$

where $(-1)^n\gamma_n(\chi)$ is simply the value of the n-th derivative of $L(z,\chi)$ at $z = 1$. These constants are well known the *Laurent-Stieltjes constants* . Recall that χ is called even or odd according to whether $\chi(-1) = 1$ or $\chi(-1) = -1$.

In 1837, Dirichlet produced finite expansions for $L(1,\chi)$ in the form:

$$L(1,\chi) = \sum_{n\ge 1} \frac{\chi(n)}{n} = -\frac{2\tau(\chi)}{q} \begin{cases} 2 \sum_{1\le m\le q/2} \bar{\chi}(m) \log\left|\sin \frac{\pi m}{q}\right| & \text{when } \chi(-1) = +1, \\ i\pi \sum_{1\le m\le q/2} \bar{\chi}(m) \left(1 - \frac{2m}{q}\right) & \text{when } \chi(-1) = -1. \end{cases}$$

where $\tau(\chi)$ is the Gaussian sum attached to χ. A similar finite expansion for the first derivative form of $L(z,\chi)$ at $z = 1$ has been obtained by Berger [3], de Séguier [15], Selberg & Chowla [49] in the case when $\chi(-1) = -1$ and by Gut [21] and Deninger [16] when $\chi(-1) = +1$.

$$\gamma_1(\chi) = \begin{cases} \pi i \frac{\tau(\chi)}{q} \left((\gamma + \log 2\pi) B_{1,\overline{\chi}} + \sum_{j=0}^{q-1} \overline{\chi}(j) \log\Gamma(j/q) \right) & \text{if } \chi(-1) = -1, \\ -\frac{\tau(\chi)}{q} \left((\gamma + \log 2\pi) \sum_{j=0}^{q-1} \overline{\chi}(j) \log\left|1 - e^{2\pi i j/q}\right| + \sum_{j=0}^{q-1} \overline{\chi}(j) R(j/q) \right) & \text{if } \chi(-1) = +1. \end{cases}$$

where $R(x)$ is Deninger's function and where $B_{1,\chi}$ is the first generalized Bernoulli number belonging to the character χ; it can be expressed as

$$B_{1,\chi} = \frac{1}{q}\sum_{m=1}^{q} m\chi(m).$$

In 1989, Kanemitsu [26] gave similar results for $\gamma_n(\chi)$ with $n \geq 2$. In the paper [30], Knopfmacher studied the following direct relationship between $\gamma_n(\chi)$ and $\gamma_n(a,q)$.

$$\gamma_n(\chi) = \sum_{a=1}^{q} \chi(a)\gamma_n(a,q),$$

where the constants $\gamma_n(a,q)$ are defined by

$$\gamma_n(a,q) = \lim_{T\to\infty}\left\{\sum_{1\leq m\equiv a \bmod q}^{T} \frac{(\log m)^n}{m} - \frac{(\log T)^{n+1}}{q(n+1)}\right\}.$$

In particular, $\gamma_0(1,1)$ is again the well known *Euler's constant*. The constants $\gamma_n(a,q)$ are called sometimes the *generalized Euler constants or the Laurent-Stieltjes constants for arithmetical progressions.*

The study of $\gamma_n(\chi)$ is closely related to that of $\gamma_n(a,q)$. Many mathematicians tend to study $\gamma_n(a,q)$ instead of $\gamma_n(\chi)$ since it is less complicated to define than the latter. Among them, Kanemitsu [26] , who proved that $\gamma_n(a,q)$ can be expressed in terms of classical functions. In 1992, K. Dilcher[17] derived further properties of $\gamma_n(a,q)$, calculated $\gamma_n(a,q)$ explicitly in many cases and computed many approximate values of $\gamma_n(a,q)$.

The problem of finding an explicit upper bound for $|\gamma_n(\chi)|$ has been addressed by a number of the authors, like Toyoizumi [52] who studied this problem when n is fixed and q goes to infinity. He showed that:

Theorem 0.12 *Assume that q is cube-free. Then for any $\epsilon > 0$, we have*

$$|\gamma_n(\chi)| \leq \left(\frac{1}{(n+1)4^{n+1}}\cdot\frac{L(1+\epsilon,\chi)}{\zeta(1+\epsilon)} + \epsilon\right)\log^{n+1} q, \tag{11}$$

when $q > q_0(\epsilon)$, where $q_0(\epsilon)$ is a constant depending only on ϵ.

From this, he deduced the following corollaries.

Corollary 0.13 *Let q be as above. Then*

$$|\gamma_n(\chi)| \leq \left(\frac{1}{(n+1)4^{n+1}} + o(1)\right)\log^{n+1} q,$$

as $q \to \infty$.

Corollary 0.14 *Let q be as above. Then*

$$|\gamma_n(\chi)| \leq \begin{cases} \left(\frac{1}{3(n+1)4^{n+1}} + o(1)\right)\log^{n+1} q & \text{if } \chi(2) = -1, \\ \left(\frac{1}{2(n+1)4^{n+1}} + o(1)\right)\log^{n+1} q & \text{if } \chi(3) = -1, \\ \left(\frac{1}{6(n+1)4^{n+1}} + o(1)\right)\log^{n+1} q & \text{if } \chi(2) = \chi(3) = -1, \end{cases}$$

as $q \to \infty$.

In another direction and pursuing the groundbreaking result of Matsuoka, Ishikawa [22] studied the asymptotic behaviour of $\gamma_n(\chi)$ as $n \to \infty$. He showed that:

Theorem 0.15 *There exists an $n_0(q)$ such that for all $n \geq n_0(q)$,*

$$|\gamma_n(\chi)| \leq q^{n/\log n - 1/2} \exp\left\{ n \log\log n - \frac{n \log\log n}{\log n} \right\}. \tag{12}$$

We note here that this last result is better than Eq (11) when q is small with respect to n.

Brief history of upper bounds for $\left|L(1,\chi)\right|$

The upper bound for $\left|L(1,\chi)\right|$, i. e., the first coefficient of $\gamma_n(\chi)$, has received considerable attention near the end of the 20th century, mainly because of the importance of this bound in number theory. Several authors have obtained upper bounds for $\left|L(1,\chi)\right|$ via character sums estimates, the functional equation and approximate formulae, or a mix of three.

Let χ be a Dirichlet character $\mod q$ and let N and H be non-negative real numbers with $H \geq 1$. We consider

$$S_\chi(H) = \sum_{N < n \leq N+H} \chi(n).$$

In 1918, Polya-Vinogradov stated the first upper bound for $\left|S_\chi(H)\right|$.

Theorem 0.16 *There is a universal constant c such that for χ a non-principal Dirichlet character* $\mod q$*. We have*

$$\left|S_\chi(H)\right| \leq c\sqrt{q}\log q.$$

This inequality is now called the *Polya-Vinogradov inequality.*

In 1956, Burgess[7] also gave another upper bound for $\left|S_\chi(H)\right|$, he proved that:

Theorem 0.17 *Let χ be a primitive character $\mod q$, where $q > 1$, r is a positive integer and $\epsilon > 0$ is a real number. Let N and H be non-negative real numbers with $H \geq 1$. Then*

$$\left|\sum_{N<n\leq N+H}\chi(n)\right| \ll H^{1-1/r}q^{\frac{r+1}{4r^2}+\epsilon},$$

for $r = 1, 2, 3$ and for any $r \geq 1$ if q is cube-free, the implied constant depending only on ϵ and r.

For $q = p$ is a prime number and for χ quadratic characters. Using Burgess's inequality, Chowla [9] proved that

Theorem 0.18 *Let χ be a real non- principal character modulo p. Then, we have:*

$$\left|L(1,\chi)\right| \leq \left(\tfrac{1}{4}+o(1)\right)\log p.$$

In 1966, Burgess [7] used an argument of Polya-Vinogradov to improve on Chowla's result. He proved the inequality:

Theorem 0.19 *For p sufficiently large, we have*

$$\left|L(1,\chi)\right| \leq 0.246\log p.$$

In the paper [50], Stephens showed that

Theorem 0.20 *For p sufficiently large, we have*

$$\left|L(1,\chi)\right| < \frac{1}{2}\left(1-\frac{1}{\sqrt{e}}+o(1)\right)\log p \approx 0.197\log p.$$

In 1977, Pintz [43] gave an elementary proof of Stephens's result generalizing it for every quadratic characters, whose modulus is not necessarily prime. No analogous improvements over the Burgess bound were known for complex characters χ.

In 1990, Toyoizumi [51] improved on Stephens's result under certain conditions.

Theorem 0.21 *For real non-principal χ, when q is cube-free, for any $\epsilon > 0$, we have*

$$|L(1,\chi)| \leq \left(\frac{L(1+\epsilon,\chi)}{4\zeta(1+\epsilon)}+\epsilon\right)\log q,$$

when $q > q_0(\epsilon)$, where $q_0(\epsilon)$ is a constant depending only on ϵ.

Form this, he deduced the following corollary.

Corollary 0.22 *Let q be as above. Then*

$$\left|L(1,\chi)\right| \leq \begin{cases} \left(\frac{1}{12}+o(1)\right)\log q & \textit{if } \chi(2)=-1, \\ \left(\frac{1}{8}+o(1)\right)\log q & \textit{if } \chi(3)=-1, \\ \left(\frac{1}{6}+o(1)\right)\log q & \textit{if } \chi(5)=-1, \\ \left(\frac{1}{24}+o(1)\right)\log q & \textit{if } \chi(2)=\chi(3)=-1, \\ \left(\frac{1}{18}+o(1)\right)\log q & \textit{if } \chi(2)=\chi(5)=-1, \\ \left(\frac{1}{12}+o(1)\right)\log q & \textit{if } \chi(3)=\chi(5)=-1, \\ \left(\frac{1}{36}+o(1)\right)\log q & \textit{if } \chi(2)=\chi(3)=\chi(5)=-1, \end{cases}$$

as $q \to \infty$.

In 2002, Granville & Soundararajan [20] stated an improvement on Toyoizumi's result. They proved that:

Theorem 0.23 *For a non-principal χ, when q is cube-free, for any $\epsilon > 0$, we have*

$$\left|L(1,\chi)\right| \leq \left(\frac{17}{70}+o(1)\right)\log q.$$

It is important to point out that all these bounds of $\left|L(1,\chi)\right|$ are asymptotical and explicit error terms are not known. So it is useful to have explicit estimates for the Burgess inequality to obtain an explicit upper bound for $\left|L(1,\chi)\right|$. Several authors have worked out explicit estimates for Burgess's theorem, i.e., finding an explicit upper bound for $\left|S_\chi(H)\right|$, among them, Iwaniec & Kowalski [24] who gave, in their analytic number theory book, a sketch of a proof of the following explicit result.

Theorem 0.24 *Let χ be a primitive character* $\bmod p$, *where $p > 1$ is prime. Let r be a positive integer, and let N and H be non-negative real with $H \geq 1$. Then*

$$\left|S_\chi(H)\right| \leq 30 H^{1-1/r} p^{\frac{r+1}{4r^2}} \left(\log p\right)^{1/r}.$$

In his doctoral thesis in 2011, Enrique Treviño [53] improved this result to

Theorem 0.25 *Let p be a prime such that $p \geq 10^7$. Let χ be a non-principal Dirichlet character* $\bmod p$. *Let r be a positive integer, and let N and H be non-negative real with $H \geq 1$. Then*

$$\left|S_\chi(H)\right| \leq 2.71 H^{1-1/r} p^{\frac{r+1}{4r^2}} \left(\log p\right)^{1/r}.$$

In 2006, Booker [5] provided an explicit version of Burgess' inequality. He proved that

r	$\alpha(r)$	$\beta(r)$	r	$\alpha(r)$	$\beta(r)$	r	$\alpha(r)$	$\beta(r)$
2	1.8221	8.9077	7	1.5363	1.0405	12	1.3721	-1.1232
3	1.8000	5.3948	8	1.4921	0.4856	13	1.3512	-1.4323
4	1.7263	3.6658	9	1.4548	0.0085	14	1.3328	-1.7169
5	1.6526	2.5405	10	1.4231	- 0.4106	15	1.3164	-1.9808
6	1.5892	1.7059	11	1.3958	- 0.7848	-	-	-

Table 1: Explicit constants on the Burgess inequality for quadratic characters.

Theorem 0.26 *Let $q > 10^{20}$ be a prime number $\equiv 1 \mod 4$, $r \in \{2, \cdots, 15\}$ and N, H integers with $0 < N, H \leq 2\sqrt{q}$. Let χ be a non-principal quadratic character $\mod p$. Then*

$$\left|S_\chi(H)\right| \leq \alpha(r) d^{\frac{r+1}{4r^2}} \left(\log d + \beta(r)\right)^{\frac{1}{2r}} H^{1-\frac{1}{r}},$$

where $\alpha(r), \beta(r)$ are given by Table 1.

Using this result, Booker computed the class number of 32-digit discriminant.

The best explicit upper bound known for $\left|L(1,\chi)\right|$ are of the form

$$\left|L(1,\chi)\right| \leq \tfrac{1}{2}\log q + C.$$

Concerning C, Louboutin [36] and [37] showed that

Theorem 0.27 *Let χ be a primitive character of conductor q. Then*

$$\left|L(1,\chi)\right| \leq \begin{cases} \frac{1}{2}\log q + 0.009 & \text{if } \chi(-1) = +1, \\ \frac{1}{2}\log q + 0.716 & \text{if } \chi(-1) = -1. \end{cases}$$

Theorem 0.28 *Let χ be a primitive character of even conductor q. Then*

$$\left|L(1,\chi)\right| \leq \begin{cases} \frac{1}{4}\log q + 0.358 & \text{if } \chi(-1) = +1, \\ \frac{1}{4}\log q + 0.704 & \text{if } \chi(-1) = -1. \end{cases}$$

He proved his results by using an integral representations of Dirichlet L-function. In 2001, Ramaré [46] gave new approximate formulae for $L(1,\chi)$ depending on Fourier transforms. He proved that

Theorem 0.29 *Let χ be a primitive Dirichlet character of conductor $q > 1$. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be such that $f(t) = F(t)/t$ is $C^2(\mathbb{R})$ (even at 0), vanishes at $t = \mp\infty$ and its first and second derivatives belong to $L^1(\mathbb{R})$. Assume that further that F is even if χ is odd and that F is odd if χ is even. then for any $\delta > 0$, we have*

$$L(1,\chi) = \sum_{n\geq 1} \frac{(1 - F(\delta n))\,\chi(n)}{n} + \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \geq 1} \overline{\chi}(m) \int_{-\infty}^{+\infty} \frac{F(t)}{t}\, e\big(mt/(\delta q)\big)\, dt.$$

With a proper choice of the function F in the above formula, in fact F an approximation of $\operatorname{sgn}(x)$ in the even case and of 1 in the odd case, Ramaré provided the following upper bound for $|L(1,\chi)|$.

Proposition 0.30 *Let χ be a primitive character of conductor q. Then*

$$|L(1,\chi)| \leq \begin{cases} \frac{1}{2}\log q & \text{if } \chi(-1)=+1, \\ \frac{1}{2}\log q + 0.716 & \text{if } \chi(-1)=-1. \end{cases}$$

He found also

Proposition 0.31 *Let χ be a primitive character of even conductor q. Then*

$$|L(1,\chi)| \leq \begin{cases} \frac{1}{4}\log q + 0.34657 & \text{if } \chi(-1)=+1, \\ \frac{1}{4}\log q + 0.70069 & \text{if } \chi(-1)=-1. \end{cases}$$

In 2002, Louboutin [38] proved that

Theorem 0.32 *Let χ be an even primitive character of conductor q. Then*

$$|L(1,\chi)| \leq \begin{cases} \frac{1}{4}\log q + 0.358 & \text{if } 2|q, \\ \frac{1}{3}\log q + 0.381 & \text{if } 3|q, \\ \frac{1}{6}\log q + 0.421 & \text{if } 2|q \text{ and } 3|q, \\ \frac{1}{6}\log q + \frac{5}{6} & \text{if } \chi(2)=-1, \\ \frac{1}{8}\log q + \frac{6}{8} & \text{if } \chi(2)=0 \text{ and } \chi(3)=-1. \end{cases}$$

This last result for $|L(1,\chi)|$ in the case when $\chi(2)=0$ and $\chi(3)=-1$ is an improvement on the following result due to Le [33], who proved that

Lemma 0.33 *If $\chi(2)=0$ and $\chi(3)=-1$, then*

$$|L(1,\chi)| \leq \frac{1}{8}\log q + \frac{3\log 6+8}{8}.$$

In another paper [39], Louboutin collected his results in the following theorem. This upper bound is valid for any primitive character χ modulo q.

Theorem 0.34 *Let S be a given finite set of pairwise distinct rational primes. Then, for any primitive Dirichlet character χ of conductor $q_\chi > 1$, we have*

$$\left|\left\{\prod_{p\in S}\left(1-\frac{\chi(p)}{p}\right)\right\} L(1,\chi)\right| \leq \frac{1}{2}\left|\left\{\prod_{p\in S}\left(1-\frac{1}{p}\right)\right\}\right| \times \left(\log q_\chi + \kappa_\chi + \omega\log 4 + 2\sum_{p\in S}\frac{\log p}{p-1}\right) + o(1),$$

where

$$\kappa_\chi = \begin{cases} \kappa_{even} = 2+\gamma-\log(4\pi) = 0.046191\cdots & \text{if } \chi(-1)=+1, \\ \kappa_{odd} = 2+\gamma-\log(\pi) = 1.432485\cdots & \text{if } \chi(-1)=-1. \end{cases}$$

where $\omega \geq 0$ is the number of primes $p \in S$ which do not divide q_χ, and where $o(1)$ is an explicit error term which tends rapidly to zero when q_χ goes to infinity. Moreover, if $S=\phi$ or if $S=2$, then this error term $o(1)$ is always less than or equal to zero, and if none of the primes in S divides q_χ then this error term $o(1)$ is less than or equal to zero for q_χ large enough.

In 2004, Ramaré [47] stated a general formula

Theorem 0.35 *Let χ be a primitive Dirichlet character of conductor q and let h be an integer prime to q. Let $F:\mathbb{R}\longrightarrow\mathbb{R}$ be such that $f(t)=F(t)/t$ is $C^2(\mathbb{R})$ (even at 0), vanishes at $t=\mp\infty$ and its first and second derivatives belong to $L^1(\mathbb{R})$. Assume that further that F is even if χ is odd and that F is odd if χ is even. then for any $\delta>0$, we have*

$$\prod_{p|h}\left(1-\frac{\chi(p)}{p}\right)L(1,\chi) = \sum_{\substack{n\geq 1\\(n,h)=1}}\frac{(1-F(\delta n))\,\chi(n)}{n} + \frac{\chi(-h)\tau(\chi)}{qh}\sum_{m\geq 1}c_h(m)\overline{\chi}(m)\int_{-\infty}^{+\infty}\frac{F(t)}{t}\,e\big(mt/(\delta qh)\big)\,dt.$$

Thanks to this formula, Ramaré proved that

Corollary 0.36 *Let χ be a primitive Dirichlet character modulo odd q. Then*

$$\left|\big(1-\chi(2)/2\big)\,L(1,\chi)\right| \leq \tfrac{1}{4}\big(\log q+\kappa(\chi)\big),$$

where $\kappa(\chi)=4\log 2$ if χ is even, and $\kappa(\chi)=5-2\log(3/2)$ otherwise.

We are still very far from proving that:

$$\left|L(1,\chi)\right| \leq c_1\log\log q + c_2,$$

for every $q\geq 10$ and some decent constants c_1 and c_2 though such a bound is expected to hold and indeed holds under the *Generalized Riemann Hypothesis.*

Introduction

The Laurent-Stieltjes constants $\gamma_n(\chi)$ are, up to a trivial coefficient, the coefficients of the Laurent expansion of the usual Dirichlet L-series : when χ is a non-principal, $(-1)^n\gamma_n(\chi)$ is simply the value of the n-th derivative of $L(z,\chi)$ at $z=1$.

The interest in these constants has a long history (started by Stieltjes in 1885) recalled in the previous chapter. Among the applications, let us cite: determining zero-free regions for Dirichlet L-functions near the real axis in the critical strip $0 \le \Re(z) \le 1$, computing the values of the Riemann and Hurwitz zeta functions in the complex plane and studying the class number of the quadratic field, etc. In this thesis, we give explicit upper bounds for the Laurent-Stieltjes constants in the following two cases:

- The character χ is fixed and the order n goes to infinity.
- The order n is 0 and the modulus q goes to infinity.

We note here that there has been numerous trials at geometrical interpretations of these coefficients. Such an approach has been successful for the first coefficient (this is Dirichlet class number formula); the second coefficient has been examined, with, up to now, some results due to Berger, Séguier, Selberg, Chowla & Gut and, more recently, Deninger when χ is quadratic.

The first part of this thesis deals with the asymptotic in the order n. In 1985, the theory made a huge progress via an asymptotic expansion produced by Matsuoka, for these constants in the case of the zeta function. Among other results pertaining to these constants, Matsuoka [40] gave the best upper bound for $\left|\gamma_n\right|$ for $n \ge 10$. He proved that

$$|\gamma_n| \le 10^{-4} e^{n \log\log n}.$$

Many authors have tried to improve on Matsuoka's bound, with few success. Matsuoka's work relied on a formula that is essentially a consequence of Cauchy's Theorem and the functional equation. We extend this formula to Dirichlet L- functions (this is essentially technical); our input is to produce a second explicit formula from this first one that involves only real valued functions. By using this second form and some combinatorial bound, we give the following upper bound for $\left|\gamma_n(\chi)\right|$ for $1 \le q < \frac{\pi}{2}\,\frac{e^{(n+1)/2}}{n+1}$.

Theorem 0.37 *Let χ be a primitive Dirichlet character to modulus q. Then, for every $1 \le q < \frac{\pi}{2}\frac{e^{(n+1)/2}}{n+1}$ and every $n \ge 2$, we have*

$$\frac{|\gamma_n(\chi)|}{n!} \le q^{-1/2}\, C(n,q)\, \min\left(1 + D(n,q), \frac{\pi^2}{6}\right),$$

with

$$C(n,q) = 2\sqrt{2}\exp\left\{-(n+1)\log\theta(n,q) + \theta(n,q)\left(\log\theta(n,q) + \log\frac{2q}{\pi e}\right)\right\},$$

and

$$\theta(n,q) = \frac{n+1}{\log\frac{2q(n+1)}{\pi}} - 1, \quad D(n,q) = 2^{-\theta(n,q)-1}\,\frac{\theta(n,q)+1}{\theta(n,q)-1}.$$

In the case when $\chi = \chi_0$ and $q = 1$, this bound is asymptotically as strong as provided by Matsuoka and numerically better as soon as $n \ge 11$. The short table below offers many comparative values of $|\gamma_n|$. For more values of $|\gamma_n|$. (you can see Table 4.1).

n		10	500	1000
$\lvert\gamma_n\rvert$	Matsuoka	0.41894487	$5.0918232 \cdot 10^{392}$	$2.17241813 \cdot 10^{835}$
	Adell	21622.6705	$9.4804345 \cdot 10^{393}$	$4.4078946 \cdot 10^{834}$
	Our result	0.71412027	$3.82043826 \cdot 10^{277}$	$1.3320458 \cdot 10^{618}$
γ_n	Kreminski	$2.0533281 \cdot 10^{-4}$	$-1.1655052 \cdot 10^{204}$	$-1.5709538 \cdot 10^{486}$
	Coffey & Knessl	0.000210539	$-1.16551 \cdot 10^{204}$	$-1.570953 \cdot 10^{486}$

Table 2: We see that while there is still some room for improvement, our upper bound is much closer to the truth than the previous ones

Remark 1 *This result has been published in the Journal of Number Theory [48].*

Our theorem enables us to approximate $L(z,\chi)$ by a short Taylor polynomial. For instance, we have

Corollary 0.38 *Let χ be a primitive Dirichlet character to modulus q. For $N = 4\log q$ and $q \ge 150$, we have*

$$\left| L(z,\chi) - \sum_{1\le n\le N} \frac{(-1)^n \gamma_n(\chi)}{n!}(z-1)^n \right| \le \frac{32.3}{q^{2.5}},$$

when $|z-1| \le e^{-1}$.

If we replace $4\log q$ by N and increase the latter, the approximation becomes better.

We further provide a direct proof of this second form of Matsuoka's formula, avoiding complex analysis and, more importantly, the functional equation. This is also related to the 1972-work of Liang & Todd [35] where they computed the γ_n by using the Euler-Maclaurin summation formula. They appealed to some recurrence relations for the function $\frac{\log^n x}{x}$ to calculate the derivatives needed in the Bernoulli terms and in the error term. Using this summation but with a different relation recurrence for $\frac{\log^n x}{x}$, we provide a proof of the Matsuoka's formula that avoids any use of the functional equation of $\zeta(z)$ or $L(z,\chi)$.

The second part of this thesis concerns $L(1,\chi)$. In 2001, Ramaré [46] gave an approximate formula for $L(1,\chi)$ depending on Fourier transforms. Thanks to this formula, he obtained the best explicit upper bound for $|L(1,\chi)|$ to date (see Proposition 0.30). In particular, when $\chi(-1)=1$, Ramaré proved that

$$|L(1,\chi)| \leq \tfrac{1}{2}\log q.$$

Using numerical evidence,Ramaré proposed the conjecture

$$\begin{cases} \max\limits_{\chi \text{ even}} \left\{|L(1,\chi)| - \frac{1}{2}\log q\right\} \stackrel{?}{=} -0.32404\cdots, \\ \max\limits_{\chi \text{ odd}} \{|L(1,\chi)| - \frac{1}{2}\log q\} \stackrel{?}{=} 0.51482\cdots. \end{cases}$$

The first one being reached by a character modulo 241 and the second one by a character modulo 311. Recently and using a very refined algorithm, Platt has checked this conjecture for $2 \leq q \leq 2\,000\,000$.

The problem of beating the $\frac{1}{2}\log q$ has since been addressed but the only results obtained have been under the additional hypothesis that $\chi(2)$ is noticeably different from 1 (or if, not $\chi(2)$, then $\chi(p)$ for some small prime p). We study the more difficult case, i.e. when $\chi(2)=1$ and prove that

Theorem 0.39 *Let χ be an even primitive Dirichlet character of conductor $q>1$ and suppose that $\chi(2)=1$. Then, we have*

$$|L(1,\chi)| \leq \tfrac{1}{2}\log q - 0.02012.$$

As an example of application, we deduce an explicit upper bound for the class number for any real quadratic field $\mathbb{Q}(\sqrt{q})$, improving on a result by Le [33]. We prove that

Theorem 0.40 *For every real quadratic field of discriminant $q>1$ and $\chi(2)=1$, we have :*

$$h\left(\mathbb{Q}\left(\sqrt{q}\right)\right) \leq \frac{\sqrt{q}}{2}\left(1 - \frac{1}{25\log q}\right),$$

where $h\left(\mathbb{Q}(\sqrt{q})\right)$ is the class number of $\mathbb{Q}\left(\sqrt{q}\right)$.

B. Oriat [41] has computed the class number of this field when $1 < q < 24572$. Here, we prove this theorem for $q \geq 24572$ and extend it to $q > 226$ via the table of Oriat. Thanks to the precious remarks of F. Pappalardi, we can check that our result is also correct for $1 < q \leq 226$.

We complete this work by a bound concerning the case $\chi(3) = 0$. In 2004 and on using an integral representation of the Dirichlet L-functions, Louboutin[39] gave the following upper bound for $\left|L(1,\chi)\right|$ when 3 divides the conductor q.

$$|L(1,\chi)| \leq \frac{1}{3}\log q + \begin{cases} 0.3816 & \text{when } \chi(-1) = 1, \\ 0.8436 & \text{when } \chi(-1) = -1. \end{cases}$$

In the third part of this thesis, we show how we use the method of Ramaré (Proposition 6.4) to improve on Louboutin's bound. We prove that

Theorem 0.41 *Let χ be a primitive character of conductor $q > 1$ such that $3|q$. Then*

$$|L(1,\chi)| \leq \frac{1}{3}\log q + \begin{cases} 0.368296 & \textit{when } \chi(-1) = 1, \\ 0.838374 & \textit{when } \chi(-1) = -1. \end{cases}$$

We prove our result for $q > 2 \cdot 10^6$. To check that our result is valid for $1 < q \leq 2 \cdot 10^6$, Platt has kindly agreed to run his algorithm from his thesis [44], (which is rigorous and efficient for computing $L(1,\chi)$ for all primitive χ of conductor $2 \leq q \leq 2 \cdot 10^6$).

Introduction (French version)

Les constantes de Laurent-Stieltjes $\gamma_n(\chi)$ sont les coefficients qui interviennent dans le développement en série de Laurent des séries L de Dirichlet : quand χ n'est pas principal, $(-1)^n\gamma_n(\chi)$ est simplement la valeur de la dérivée n-ième de $L(z,\chi)$ en $z=1$.

L'intérêt de ces coefficients a une longue histoire qui a commencé par Stieltjes en 1885. Parmi les applications, rappelons qu'ils permettent de déterminer les régions sans zéro des fonctions L de Dirichlet au voisinage de l'axe réel dans la bande critique $0 \le \Re(z) \le 1$. Ils permettent aussi de calculer les valeurs de la fonction zêta de Riemann et/ou Hurwitz dans le plan complexe, ainsi que le nombre des classes des corps quadratiques.

Dans cette thèse, nous donnons les majorations explicites pour les constantes de Laurent-Stieltjes dans les deux cas suivants :

1. Le caractère χ est fixé et n tend vers l'infini.
2. L'ordre n est égal à 0 et le conducteur q tend vers l'infini.

Nous notons ici qu'il y a eu plusieurs tentatives d'interpréter géométriquement ces coefficients. Avec succès pour le premier coefficient (c'est la formule de Dirichlet du nombre des classes) ; le deuxième coefficient a été étudié par Berger, Séguier, Selberg, Chowla et Gut et plus récemment, Deninger quand χ est quadratique.

Dans la première partie dans cette thèse, nous étudions le comportement asymptotique en n de ces coefficients. En 1985, cette théorie a fait beaucoup de progrès grâce au développement asymptotique de Matsuoka [40] dans le cas de la fonction zêta de Riemann. Parmi les résultats concernant ces coefficients, Matsuoka a donné la meilleure majoration explicite de $\left|\gamma_n\right|$ pour $n \ge 10$. Il a montré que ; Pour $n \ge 10$, nous avons

$$|\gamma_n| \le 10^{-4} e^{n \log\log n}.$$

D'autre auteurs ont essayé avec peu de succès d'améliorer la majoration de Matsuoka. Le travail de Matsuoka repose sur une formule qui est une conséquence du théorème de Cauchy et de l'équation fonctionnelle de la fonction zêta de Riemann. Nous prolongeons cette formule aux fonctions L de Dirichlet ; notre contribution est de produire une seconde formule affinant la première et qui ne contient plus que des termes réels (et non complexes). En utilisant cette formule et de la combinatoire, nous donnons une majoration explicite de $|\gamma_n(\chi)|$ pour $1 \leq q < \frac{\pi}{2}\,\frac{e^{(n+1)/2}}{n+1}$.

Théorème 0.1 *Soit χ un caractère de Dirichlet primitif modulo q. Alors, pour tout $1 \leq q < \frac{\pi}{2}\,\frac{e^{(n+1)/2}}{n+1}$ et tout $n \geq 2$, nous avons*

$$\frac{|\gamma_n(\chi)|}{n!} \leq q^{-1/2}\, C(n,q)\, \min\left(1 + D(n,q), \frac{\pi^2}{6}\right),$$

avec

$$C(n,q) = 2\sqrt{2}\exp\left\{-(n+1)\log\theta(n,q) + \theta(n,q)\left(\log\theta(n,q) + \log\frac{2q}{\pi e}\right)\right\},$$

et

$$\theta(n,q) = \frac{n+1}{\log\frac{2q(n+1)}{\pi}} - 1, \quad D(n,q) = 2^{-\theta(n,q)-1}\,\frac{\theta(n,q)+1}{\theta(n,q)-1}.$$

Dans le cas où $\chi = \chi_0$ et $q = 1$, ce résultat est asymptotiquement aussi forte que prévu par Matsuoka et numériquement mieux dès que $n \geq 11$. Le tableau ci-dessous offre certaines valeurs de $|\gamma_n|$. Pour plus de valeurs de $|\gamma_n|$ (voir Table 4.1).

n		10	500	1000
$\lvert\gamma_n\rvert$	Matsuoka	0.41894487	$5.0918232 \cdot 10^{392}$	$2.17241813 \cdot 10^{835}$
	Adell	21622.6705	$9.4804345 \cdot 10^{393}$	$4.4078946 \cdot 10^{834}$
	Notre Résultat	0.71412027	$3.82043826 \cdot 10^{277}$	$1.3320458 \cdot 10^{618}$
γ_n	Kreminski	$2.0533281 \cdot 10^{-4}$	$-1.1655052 \cdot 10^{204}$	$-1.5709538 \cdot 10^{486}$
	Coffey & Knessl	0.000210539	$-1.16551 \cdot 10^{204}$	$-1.570953 \cdot 10^{486}$

TABLE 3 – Nous voyons que, même s'il y a encore place à l'amélioration, notre majoration est beaucoup plus proche de la vérité que les précédents

Remarque 1 *Ce résultat est publié dans la revue Journal of number theory [48].*

Ce théorème donne aussi une approximation de $L(z,\chi)$ au voisinage de $z = 1$ par un polynôme de Taylor relativement court. Nous montrons que :

Corollaire 0.2 *Soit χ un caractère de Dirichlet primitif modulo q. Pour $N = 4\log q$ et $q \geq 150$, nous avons*

$$\left| L(z,\chi) - \sum_{1\leq n\leq N} \frac{(-1)^n\gamma_n(\chi)}{n!}(z-1)^n \right| \leq \frac{32.3}{q^{2.5}},$$

où $|z-1| \leq e^{-1}$.

Si nous remplaçons $4\log q$ par N et augmentons celui-ci, l'approximation devient meilleure.

En outre, nous donnons aussi une preuve directe de la formule de Matsuoka en évitant l'analyse complexe et plus précisément l'équation fonctionnelle pour $\zeta(z)$ ou $L(z,\chi)$. Ceci a un lien avec le travail de Liang & Todd [35], qui ont calculé γ_n en utilisant la formule sommatoire d'Euler-Maclaurin. Ils ont eu recours à une relation récurrence de la fonction $\frac{\log^n x}{x}$ pour calculer les dérivées nécessaires dans les termes de Bernoulli et dans le terme d'erreur; nous utilisons une expression différente.

Dans la deuxième partie de cette thèse, nous nous intéressons à $L(1,\chi)$. En 2001, Ramaré [46] a donné une formule approchée pour $L(1,\chi)$ en utilisant des transformées de Fourier. Grâce à cette formule, Ramaré a obtenu la meilleure majoration explicite pour $\left|L(1,\chi)\right|$ à ce jour (see Proposition 0.30). En particulier, il a montré que

$$\left|L(1,\chi)\right| \leq \tfrac{1}{2}\log q.$$

lorsque $\chi(-1)=1$. Ramaré a justifié numériquement la question suivante :

$$\begin{cases} \max\limits_{\chi\text{pair}}\left\{\left|L(1,\chi)\right| - \tfrac{1}{2}\log q\right\} \overset{?}{=} -0.32404\cdots, \\ \max\limits_{\chi\text{impair}}\left\{\left|L(1,\chi)\right| - \tfrac{1}{2}\log q\right\} \overset{?}{=} 0.51482\cdots. \end{cases}$$

Le premier étant atteint par un caractère modulo 241 et le second par un caractère modulo 311. Récemment, en utilisant un algorithme très raffiné, Platt a vérifié cette conjecture pour $2 \leq q \leq 2\,000\,000$.

Certains auteurs ont réussi à améliorer la majoration $\frac{1}{2}\log q$, mais sous l'hypothèse supplémentaire que $\chi(2)$ est sensiblement différent de 1 (ou sinon, alors $\chi(p)$ pour un petit nombre premier p). Nous étudions le cas le plus difficile, c'est à dire lorsque $\chi(2)=1$. Nous prouvons que

Théorème 0.3 *Soit χ un caractère de Dirichlet primitif et pair de conducteur $q>1$ et supposons que $\chi(2)=1$. Alors, nous avons :*

$$\left|L(1,\chi)\right| \leq \tfrac{1}{2}\log q - 0.02012.$$

En conséquence, nous déduisons une majoration pour le nombre des classes pour les corps quadratiques réels $\mathbb{Q}\left(\sqrt{q}\right)$. Cette majoration est une amélioration d'un résultat de Le [33]. Nous montrons que

Théorème 0.4 *For tout corps quadratique réel de discriminant $q>1$ et $\chi(2)=1$. Alors, nous avons :*

$$h\left(\mathbb{Q}\left(\sqrt{q}\right)\right) \leq \frac{\sqrt{q}}{2}\left(1 - \frac{1}{25\log q}\right),$$

où $h\left(\mathbb{Q}\left(\sqrt{q}\right)\right)$ est le nombre de classes de $\mathbb{Q}\left(\sqrt{q}\right)$.

Oriat [41] a calculé le nombre des classes de ce corps quand $1 < q < 24572$. Ici, nous montrons ce théorème pour $q \geq 24572$ et l'étendons à $q > 226$ via la table de Oriat. Grâce à un remarque de F. Pappalardi, nous pouvons vérifier que notre résultat est aussi vrai pour $1 < q < 226$.

En 2004, Louboutin [39], en utilisant la représentation intégrale de la fonction L de Dirichlet, a donné une majoration de $\left|L(1,\chi)\right|$ dans le cas où 3 divise le conducteur q.

$$|L(1,\chi)| \leq \frac{1}{3}\log q + \begin{cases} 0.3816 & \text{si } \chi(-1) = 1, \\ 0.8436 & \text{si } \chi(-1) = -1. \end{cases}$$

Dans la troisième partie de cette thèse, nous montrons comment nous utilisons la méthode de Ramaré (Proposition 6.4) pour améliorer le résultat de Louboutin. Nous montrons que :

Théorème 0.5 *Soit χ un caractère de Dirichlet primitif de conducteur $q \geq 2$ tel que $3|q$. Alors :*

$$|L(1,\chi)| \leq \frac{1}{3}\log q + \begin{cases} 0.368296 & \textit{si } \chi(-1) = 1, \\ 0.838374 & \textit{si } \chi(-1) = -1. \end{cases}$$

Nous montrons ce résultat pour $q > 2\,000\,000$. Afin de vérifier que notre résultat est valable pour $1 < q \leq 2 \cdot 10^6$, Platt a gentiment accepté de faire tourner son algorithme, issu de sa thèse [44] (il s'agit d'un algorithme rigoureux et efficace pour calculer $L(1,\chi)$ pour tout χ de conducteur q et $1 < q \leq 2 \cdot 10^6$).

Notation

Throughout this thesis we will have the following notation:

- When x is a real number, $[x]$ is the integer part of x, i.e., the greatest integer less than or equal to x.
- For x is real number, $\{x\}$ is the fractional part of x, i.e., $\{x\} = x - [x]$.
- p always denoted a prime number.
- log refers to the natural logarithm, i.e., the logarithm in base e.
- z denotes a complex variable with $z = \sigma + it$.
- $\tau(\chi)$ denotes the Gauss sum.
- For integers a and b, (a, b) denotes the greatest common divisor of a and b.
- The notation of Vinogradov $f \ll g$ means that $\left|f(t)\right| \leq Cg(t)$ for some constant C independent of the variable t.
- The notation $f \ll_A g$ means that $f \ll g$ with a constant C that may depend on A.
- We write $f(t) = \mathcal{O}\left(g(t)\right)$ to mean that there is a constant C such that $\left|f(t)\right| \leq Cg(t)$ for all values of t under consideration. This is usually called the big Oh notation or Landau O notation.
- The notation $f = \mathcal{O}_A(g(t))$ means that $f(t) = \mathcal{O}(g(t))$ with the constant C that may depend on A.
- We write $f(t) \sim g(t)$ when $\lim f(t)/g(t) = 1$ as t tends to some limit.

- We write $f(t) = o\big(g(t)\big)$ when $\lim f(t)/g(t) = 0$ as t tends to infinity.
- $\gamma_n(\chi)$ denotes the coefficient of Laurent-Stieltjes for the Dirichlet L-functions.
- γ_n denotes the coefficient of Laurent-Stieltjes for the zeta function.
- γ is Euler's constant or Euler-Mascheroni constant, i.e.,

$$\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \log n.$$

- $h\big(\mathbb{Q}(\sqrt{q})\big)$ denotes the class number of the field $\mathbb{Q}\big(\sqrt{q}\big)$.
- $\Lambda(n)$ denotes the von Mangoldt Λ-function, which is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{when } n = p^k, \text{ and } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- $\varphi(n)$ denotes the Euler totient or phi function, which counts the number of positive integers less than or equal to n that are relatively prime to n.

Chapter 1

On the values of the Dirichlet L-functions and their derivatives at $z = 1$

Dirichlet series are extremely useful in tackling problems in number theory as well as in other branches of mathematics. In this chapter, we recall the main definitions and the preliminaries concerning L-series.

1.1 Groups character

Let G be a finite abelian group. A **character** of G is a homomorphism from G to multiplicative group $\mathbb{C}^{\times}$. The image of χ is contained in the group of $|G|$-th roots of unity. For any two characters χ_1 and χ_2 of G, the product character defined by

$$\left(\chi_1\chi_2\right)(n) = \chi_1(n)\chi_2(n)$$

is again a character of G. Then the set of characters forms an abelian group under this multiplication. This group is the **character group** of G and is sometimes denoted as $\widehat{G}$. The identity element of $\widehat{G}$, mapping every element of G to 1, is the **principal (or the trivial) character**, denoted χ_0. The inverse of χ is $\overline{\chi}$ defined by $\overline{\chi}(n) = \overline{\chi(n)}$ for each element $n \in G$.

Lemma 1.1 *Let G be a finite abelian group. Then $\widehat{G} \cong G$.*

To proceed further we quote a well-known result, and develop its consequences on the characters.

Lemma 1.2 *G is a direct product of cyclic groups.*

Lemma 1.3 *We have:*

$$\sum_{n\in G}\chi(n)=\begin{cases}|G| & \text{if } \chi=\chi_0\\ 0 & \text{otherwise}\end{cases}\quad,\quad \sum_{\chi\in\hat{G}}\chi(n)=\begin{cases}|G| & \text{if } n=0\\ 0 & \text{otherwise}\end{cases}$$

et

$$\sum_{n\in G}\chi_1(n)\overline{\chi_2(n)}=\begin{cases}|G| & \text{if } \chi_1=\chi_2\\ 0 & \text{otherwise}\end{cases}$$

$$\sum_{\chi\in\hat{G}}\chi(n)\overline{\chi(m)}=\begin{cases}|G| & \text{if } n=m\\ 0 & \text{otherwise.}\end{cases}$$

1.2 Dirichlet Characters

Let $q\geq 1$ be an integer, and consider the group $G=(\mathbb{Z}/q\mathbb{Z})^{\times}$ consisting of the $\varphi(q)$ residues coprime to q. Let $\chi: G\longrightarrow\mathbb{C}^{\times}$ be a character. We may use χ to also denote the function on $\mathbb{Z}$, defined $\chi(n)=\chi(n \mod q)$ if $(n,q)=1$ and $\chi(n)=0$ if $(n,q)>1$ and the rule $\chi(nm)=\chi(n)\chi(m)$ still holds. Such a function is called a **Dirichlet character** to the modulus q. Observe that χ (the Dirichlet character) has q as a period, and so it descends to give a well-defined function on the additive group $\mathbb{Z}/q\mathbb{Z}$. This we will also denote by χ. Although the triple abuse of notation is perhaps distasteful to some, it rarely results in confusion.

Suppose that $q|q'$, and that χ is a Dirichlet character to the modulus q. Then we may use χ to define an **induced character** χ' by setting $\chi'(n)=\chi(n)$ when $(n,q')=1$. Observe that if q' has prime factors which do not appear in q then the conditions $(n,q)=1$ and $(n,q')=1$ will differ. We say that a character χ to the modulus q is **primitive** if it is not induced from a character to a smaller modulus. Much of the theory that we will develop really only works properly for primitive characters. If χ is a Dirichlet character then the smallest divisor q_1 of qs for which χ is induced from a Dirichlet character to the modulus q_1 is called the **conductor** of χ.

Lemma 1.4 *Suppose that χ is a Dirichlet character to the modulus q, and that q_1 is a factor of q with $q_1<q$. Suppose that q_1 is a pseudo-period of χ, by which we mean that whenever $(n,q)=(n',q)=1$ and $n\equiv n'(\mod q_1)$ we have $\chi(n)=\chi(n')$. Then χ is induced from a Dirichlet character to the modulus q_1.*

Definition 1.5 *A character χ is called **real** if $\chi(n)$ is a real number, for all $n\in\mathbb{Z}$. A character that takes complex values is called **complex**, and the character whose value at n is the complex conjugate of $\chi(n)$ is called the **complex conjugate** of χ and is denoted $\overline{\chi}$.*

1.2.1 The Gauss Sums

Definition 1.6 *If χ is a character modulo q one calls Gauss sum $\tau(\chi)$ the value*

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e(a/q)$$

where $e(a/q) = \exp(2\pi i a/q)$.

Proposition 1.7 *If χ is primitive modulo q, then we have:*

$$\left|\tau(\chi)\right| = \sqrt{q}, \tag{1.1}$$

and

$$\tau(\chi)\tau(\overline{\chi}) = \chi(-1) q. \tag{1.2}$$

Proposition 1.8 *Let q be a positive integer. If $q = 1$ or $q = 2$ then every Dirichlet character χ modulo q satisfies $\chi(-1) = 1$. If $q > 2$ then the number of Dirichlet characters modulo q is even, half of them satisfying $\chi(-1) = 1$ and the other half satisfying $\chi(-1) = -1$.*

1.3 Some Dirichlet L-Series

Our next task is to recall some basic results about the zeta function and Dirichlet L-functions. Let's begin by reviewing the zeta function.

1.3.1 The Riemann zeta function

First recall the **Riemann's zeta function** is defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \tag{1.3}$$

for $z = \sigma + it \in \mathbb{C}$ with $\sigma > 1$. Then $\zeta(z)$ can be meromorphically continued to a function of the whole complex plane with only a simple pole at $z = 1$, which is a unique pole and for which the residue is 1. Euler observed that

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p} \frac{1}{1 - p^{-z}}$$

The series is absolutely convergent in the half plane $\sigma > 1$ and uniformly for $\sigma > 1 + \delta$ for $\delta > 0$. The fact that $\zeta(z)$ has a pole at $z = 1$ implies there are infinitely many prime numbers. This is the first hint that the analytic behaviour of $\zeta(z)$ can encodes deep arithmetic information. One important feature of $\zeta(z)$ is its functional equation relating the values at z to the values at $1-z$, which can be written

$$\zeta(1-z) = 2\,(2\pi)^{-z}\cos(\pi z/2)\,\Gamma(z)\zeta(z). \tag{1.4}$$

Here Γ is called the Gamma function. This function is analytic in the half-plane $\Re(z)>0$. Since neither ζ nor Γ have poles to the right of $z=1$, the simple poles of $\Gamma(z/2)$ for $z\in\{-2,-4,\cdots\}$ is the simple zeros of ζ. These are the **trivial zeros** of ζ. Moreover, it has infinitely many zeros with real part in $[0,1]$, they are called **non-trivial**. We call the region $0\le\Re(z)\le 1$ the **critical strip** and the line of symmetry $\Re(z)=1/2$ the **critical line**. The very deep **Riemann hypothesis** asserts that all non-trivial zeroes of ζ actually lie on the critical line, but it has not been yet either proved or disproved.

1.3.2 The Dirichlet L-functions

For any Dirichlet character χ modulo q, let

$$L(z,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^z}. \tag{1.5}$$

For $\sigma>1$, the series (1.5) converges absolutely. Since χ is multiplicative, again one an Euler product expansion have the Euler product representations respectively

$$L(z,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^z} = \prod_p (1-\chi(p)p^{-z})^{-1}.$$

As with $\zeta(z)$, one can show $L(z,\chi)$ has meromorphic continuation to $\mathbb{C}$, but this time it will in fact be entire whenever χ is a non-principal character. It is important to point out that the analytic nature of Dirichlet L-functions is different for principal and non principal characters.

Theorem 1.9 *Let χ be a principal Dirichlet character modulo $q\in\mathbb{N}$. Then the function $L(z,\chi_0)$ admits an analytic continuation to the half plane $\sigma>0$. Furthermore, $L(1,\chi_0)$ is analytic in $\sigma>0$ except for a simple pole at $z=1$ with residue $\varphi(q)/q$.*

Theorem 1.10 *Let χ be a non-principal Dirichlet character modulo q. Then the function $L(z,\chi)$ converges in $\sigma>0$ and uniformly in any compact subset, in particular, $L(z,\chi)$ is analytic in $\sigma>0$.*

In the special case where Dirichlet character is principal modulo 1, (this means $\chi_0(n)=1$ for all integers n), we notice that the Dirichlet L-functions $L(z,\chi_0)$ for $\sigma>1$

is the Riemann zeta function, and also for larger moduli q the Dirichlet L-functions to principal characters have a similar analytic behaviour as the zeta function. For χ_0 a principal character modulo q and $\sigma > 1$, the above formula of $L(z,\chi)$ can be written in the form

$$L(z,\chi_0) = \zeta(z)\prod_{p|q}(1-p^{-z}).$$

A fundamental property of the Dirichlet L-functions that is key to any deeper study of these functions is the functional equation.

If χ is a primitive Dirichlet character modulo q, the completed Dirichlet L-functions is

$$\psi(z,\chi) = \left(\frac{q}{\pi}\right)^{\frac{z+\kappa}{2}} \Gamma\left(\frac{z+\kappa}{2}\right) L(z,\chi) \tag{1.6}$$

where $\kappa = \frac{1}{2}\big(1-\chi(-1)\big)$ and $\kappa = 0$ if χ is even and $\kappa = 1$ if χ is odd. Then the functional equation leads

$$\psi(z,\chi) = \varepsilon(\chi)\psi(1-z,\overline{\chi}) \tag{1.7}$$

$$\varepsilon(\chi) = i^{-\kappa}\frac{\tau(\chi)}{\sqrt{q}}.$$

Note here the function equation does not relate $L(z,\chi)$ with $L(1-z,\chi)$ in general. But if χ is real valued, which is equivalent to χ^2 is trivial, then this functional equation relates $L(z,\chi)$ with $L(1-z,\chi)$.

From (1.6) and (1.7), the functional equation of $L(z,\chi)$ is given by the following relations

1. When χ is even, we have:

$$L(1-z) = \frac{2\tau(\chi)}{q}\left(\frac{q}{2\pi}\right)^z \cos(\pi z/2)\Gamma(z)L(z,\overline{\chi}). \tag{1.8}$$

2. When χ is odd, we have:

$$L(1-z) = \frac{2i}{\tau(\chi)}\left(\frac{q}{2\pi}\right)^z \sin(\pi z/2)\Gamma(z)L(z,\overline{\chi}). \tag{1.9}$$

The following are some properties of these functions.

Theorem 1.11 *Suppose that $\sigma > 1$. Then $\zeta(z) \neq 0$. Furthermore, $L(z,\chi) \neq 0$ for every Dirichlet character χ modulo q.*

Theorem 1.12 *Suppose that $\sigma > 1$. Then*

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{n=1}^{\infty} \Lambda(n)n^{-z}.$$

Furthermore, for every Dirichlet character χ modulo q, we have

$$-\frac{L'(z,\chi)}{L(z,\chi)}=\sum_{n=1}^{\infty}\chi(n)\Lambda(n)n^{-z},$$

where

$$\Lambda(n)=\begin{cases}\log p & \text{if } n=p^k,\ k\geq 1\\ 0 & \text{otherwise.}\end{cases}$$

is the von Mangoldt Λ-function.

Theorem 1.13 *For any character χ modulo q, we have $L(1,\chi)\neq 0$.*

Lemma 1.14 *For $z=\sigma+it$ with $|t|\geq 2$, we have:*

$$\zeta(z)=\mathcal{O}(|t|^{\tau(\sigma)}\log|t|),$$

where:

$$\tau(\sigma)=\begin{cases}\frac{1}{2}-\sigma & \text{if } \sigma\leq 0,\\ \frac{1}{2} & \text{if } 0\leq\sigma\leq\frac{1}{2},\\ 1-\sigma & \text{if } \frac{1}{2}\leq\sigma\leq 1,\\ 0 & \text{if } \sigma\geq 1.\end{cases}$$

Proof. A proof of this lemma can be found in [56]. ⋄⋄⋄

Lemma 1.15 *Let χ be a non-principal Dirichlet character modulo q. Then we have*

$$|L(z,\chi)|\leq\log\big(q(2+|t|)\big)+\mathcal{O}(1)$$

where $z=\sigma+it$ with $\sigma\geq 1$.

Proof. Recall that

$$L(z,\chi)=\sum_{m\geq 1}\frac{\chi(m)}{m^z}=z\int_1^{\infty}\frac{\sum_{m\leq u}\chi(m)}{u^{z+1}}\,du,$$

Putting $\sum_{m\leq u}\chi(m)=B(u)$. Thanks to Abel's summation formula, we get

$$\sum_{m\leq x}\frac{\chi(m)}{m^z}=\frac{B(x)}{x^z}+z\int_1^{x}\frac{B(u)}{u^{z+1}}\,du.$$

It follows that

$$\begin{aligned}L(z,\chi)-\sum_{m\leq x}\frac{\chi(m)}{m^z} &= z\int_1^{\infty}\frac{B(u)}{u^{z+1}}\,du-\frac{B(x)}{x^z}-z\int_1^{x}\frac{B(u)}{u^{z+1}}\,du\\ &= -\frac{B(x)}{x^z}+z\int_x^{\infty}\frac{B(u)}{u^{z+1}}\,du.\end{aligned}$$

Then

$$\begin{aligned} L(z,\chi) &= \sum_{m\le x}\chi(m)m^{-z} - \frac{B(x)}{x^z} + z\int_x^\infty \frac{B(u)}{u^{z+1}}\,du \\ &= \sum_{m\le x}\chi(m)m^{-z} + z\int_x^\infty \frac{B(u)-B(x)}{u^{z+1}}\,du \\ &= \sum_{m\le x}\chi(m)m^{-z} + z\int_x^\infty \frac{\sum_{x<m\le u}\chi(m)}{u^{z+1}}\,du. \end{aligned}$$

Taking $x = q(2+|t|)$ with $\Im(z) = t$, we obtain

$$|L(z,\chi)| \le \log\big(q(2+|t|)\big) + \mathcal{O}(1).$$

◇◇◇

1.3.3 The Polya-Vinogradov inequality

The Polya-Vinogradov inequality states that characters can not be constant on a long sequence of consecutive integers:

Theorem 1.16 *Let χ be a non-principal Dirichlet character modulo q. Then we have*

$$\left|\sum_{1\le n\le N}\chi(n)\right| \le 2\sqrt{q}\log q \qquad \forall N \ge 1.$$

Theorem 1.17 *Let χ be a non-principal Dirichlet character modulo q and let q is a prime number. Then we have*

$$\left|\sum_{1\le n\le N}\chi(n)\right| \le \sqrt{q}\log q \qquad \forall N \ge 1.$$

For more information about the estimates for character sums, you can see "History of the Laurent-Stieltjes constants".

1.4 The Dirichlet class number formula

Let d be square free, and let $q, h(K)$ denote the discriminant and the class number of the quadratic field $K = \mathbb{Q}\left(\sqrt{d}\right)$ respectively. Then

$$q = \begin{cases} d & \text{if } d \equiv 1 \mod 4, \\ 4d & \text{if } d \equiv 2,3 \mod 4. \end{cases}$$

and the Dirichlet character modulo q given by

$$\chi_q(n) = \left(\frac{q}{n}\right),$$

where $\left(\frac{q}{n}\right)$ is the Kronecker symbol. Here, we distinguish two cases:

- If $d < 0$, it means that K is imaginary quadratic, the Dirichlet class number formula says
$$L(1,\chi_q) = \frac{2\pi h(K)}{\omega\sqrt{|q|}}$$
where ω is the number of roots of unity in K.
- If $d > 0$, it means that K is real quadratic, the Dirichlet class number formula says
$$L(1,\chi_q) = \frac{2h(K)\log\epsilon}{\sqrt{q}}$$
where ϵ is the fundamental unit.

Lemma 1.18 *For any square free $d \in \mathbb{N}$, we have*

$$\epsilon^2 > \begin{cases} d-3 & \text{if } d = a^2 \pm 4,\ a \in \mathbb{N}, \\ 4d-3 & \text{otherwise.} \end{cases}$$

Moreover, if d is a prime with $d \equiv 3 \mod 4$, then

$$\epsilon > \begin{cases} 2d-3 & \text{if } d = a^2 \pm 2,\ a \in \mathbb{N}, \\ 18d-3 & \text{otherwise,} \end{cases}$$

where ϵ is the fundamental unit.

Proof. See [33, Lemma 4]. ◇◇◇

The following result is immediate consequence of this previous Lemma.

Lemma 1.19 *For $q \geq 5$, we have*

$$\epsilon \geq \frac{\sqrt{q-4}+\sqrt{q}}{2},$$

where ϵ is the fundamental unit.

1.5 Some explicit expressions for $\gamma_n(\chi)$

It is well known n-th derivative for the Dirichlet L-functions is given by the following relation

$$L^{(n)}(z,\chi) = (-1)^n \sum_{k=1}^{\infty} \chi(k) \frac{\log^n k}{k^z}.$$

In 1978, Knopfmacher [30] proved that

Proposition 1.20 *For $n \geq 0$ we have*

$$\gamma_n(\chi) = \sum_{r=1}^{m} \chi(r)\gamma_n(r,m),$$

where the coefficients $\gamma_n(a,q)$ are the *Laurent-Stieltjes constants for arithmetical progressions.* In particular, $\gamma_0(1,1)$ is again the well known Euler constant. We recall that $(-1)^n\gamma_n(\chi)$ is the value of the n-th derivative of $L(z,\chi)$ at $z=1$.

For $n=0$, this was shown by Lehmer [34]. who gave the connection between $L(1,\chi)$ and the class number of quadratic fields. For $n=1$, Berger [3], de Séguier [15], Selberg & Chowla [49] gave an explicit expression of $\gamma_1(\chi)$ in the case when $\chi(-1) = -1$ and by Gut [21] and Deninger [16] gave an explicit expression of $\gamma_1(\chi)$in the case when $\chi(-1)=+1$.

$$\gamma_1(\chi) = \begin{cases} \pi i \dfrac{\tau(\chi)}{q}\left(\left(\gamma + \log 2\pi\right) B_{1,\overline{\chi}} + \displaystyle\sum_{j=0}^{q-1} \overline{\chi}(j) \log\Gamma(j/q)\right) & \text{if } \chi(-1) = -1, \\ -\dfrac{\tau(\chi)}{q}\left(\left(\gamma + \log 2\pi\right) \displaystyle\sum_{j=0}^{q-1} \overline{\chi}(j) \log\left|1 - e^{2\pi i j/q}\right| + \displaystyle\sum_{j=0}^{q-1} \overline{\chi}(j) R(j/q)\right) & \text{if } \chi(-1) = +1. \end{cases}$$

where $R(x)$ is Deninger's function and $B_{1,\chi}$ is the first generalized Bernoulli number belonging to the character χ; it can be expressed as

$$B_{1,\chi} = \frac{1}{q} \sum_{m=1}^{q} m\chi(m).$$

In 1989, Kanemitsu [26] gave similar results for $\gamma_n(\chi)$ with $n \geq 2$.

Part I

Explicit upper bounds for the Stieltjes constants

Chapter 2

The formulae of Matsuoka

In 1985, Matsuoka [40] produced an excellent asymptotic expansion for the Laurent-Stieltjes constants in the case of the zeta function, and he was also able to simplify his method to derive the explicit form. Matsuoka gave the best upper bound for $|\gamma_n|$ for $n \geq 10$. He proved that

$$|\gamma_n| \leq 10^{-4}\, e^{n \log\log n}.$$

Matsuoka's work relied on a formula that is essentially a consequence of Cauchy's Theorem and the functional equation.

In this chapter, we give an analytic proof of the Matsuoka formula for the zeta function. We extend this formula to Dirichlet L-functions.

2.1 Preliminary lemma

Lemma 2.1 *For $\frac{1}{2} < a < n + \frac{1}{2}$ and $y > 0$, we have:*

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)}{z^{n+1}}\,dz = \int_{a-i\infty}^{a+i\infty} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)}{z^{n+1}}\,dz,$$

and

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)}{z^{n+1}}\,dz = \int_{a-i\infty}^{a+i\infty} \frac{2y^{-z}\sin(\pi z/2)\,\Gamma(z)}{z^{n+1}}\,dz.$$

Proof. Putting

$$F_n(y,z) = \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)}{z^{n+1}}.$$

We consider a rectangle C in the z plane with vertices at the points $1/2 \pm iT$ and $a \pm iT$, where $T > 0$ and $a > 1/2$.

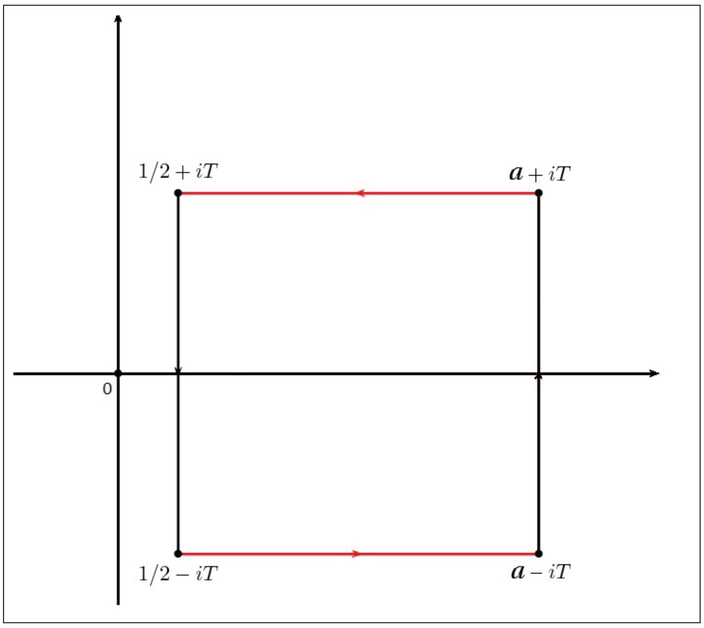

Figure 2.1: The rectangle C in the complex z plane.

By Cauchy's theorem the integral of $F_n(y,z)$ around this rectangle is zero (see Figure 2.1). We get

$$\left(\int_{1/2-iT}^{1/2+iT} + \int_{a+iT}^{a-iT} + \int_{1/2+iT}^{a+iT} + \int_{a-iT}^{1/2-iT}\right) F_n(y,z)\,dz = 0.$$

For the third integral above, we notice that:

$$\left|\int_{1/2+iT}^{a+iT} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)}{z^{n+1}}\,dz\right| \leq \int_{1/2+iT}^{a+iT} \left|\frac{2y^{-z}\Gamma(z)}{z^{n+1}}\,dz\right|$$

Thanks to Stirling's formula for Gamma function,

$$\Gamma(z) = \sqrt{2\pi}e^{-z}z^{z-1/2}\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right) \quad \text{when } |\arg(z)| < \pi,$$

and the fact that $\log(\sigma+it) = \log\left(\sqrt{\sigma^2+t^2}\right) + i\arg(\sigma+it)$. We get

$$\Gamma(\sigma+it) =$$
$$\sqrt{2\pi}e^{(\sigma-1/2)\log\sqrt{\sigma^2+t^2} - t\arg(a+it) + i\left((\sigma-1/2)\arg(\sigma+it)+t\log\sqrt{\sigma^2+t^2}\right)}\, e^{-\sigma-it}\left(1+\mathcal{O}\left(\frac{1}{\sigma+it}\right)\right).$$

Then, we find that:

$$\begin{aligned} |\Gamma(\sigma+it)| &\ll \sqrt{2\pi}e^{(\sigma-1/2)\log\sqrt{\sigma^2+t^2} - t\arg(\sigma+it) - \sigma} \\ &\ll \sqrt{2\pi}\left(\sigma^2+t^2\right)^{\frac{\sigma-1/2}{2}}. \end{aligned}$$

Hence, we obtain

$$\left|\int_{1/2+iT}^{a+iT} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)}{z^{n+1}}\,dz\right| \ll \sqrt{2\pi}\int_{1/2}^{a} \frac{2y^{-\sigma}\left(\sigma^2+T^2\right)^{\frac{\sigma-1/2}{2}}}{(\sigma^2+T^2)^{\frac{n+1}{2}}}\,|d\sigma|$$

Here, we notice that the last integral tend to 0 under the condition $n > a-1/2$.
A similar argument, we find that the integral over the segment $[1/2-iT, a-iT]$ tend to 0 as $T \longrightarrow +\infty$. Hence, we get

$$\int_{1/2-iT}^{1/2+iT} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)}{z^{n+1}}\,dz = \int_{a-iT}^{a+iT} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)}{z^{n+1}}\,dz,$$

which completes the proof of the first equation. A similar argument, we can prove the second equation in this lemma. ⋄⋄⋄

Lemma 2.2 *For* $1 < a < n+\frac{1}{2}$ *and* $|\Im(z)| \geq 2$, *we have:*

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)\zeta(z)}{z^{n+1}}\,dz = \int_{a-i\infty}^{a+i\infty} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)\zeta(z)}{z^{n+1}}\,dz,$$

and

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)\zeta(z)}{z^{n+1}}\,dz = \int_{a-i\infty}^{a+i\infty} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)L(z,\overline{\chi})}{z^{n+1}}\,dz.$$

Proof. Putting

$$G_n(y,z) = \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)\zeta(z)}{z^{n+1}}.$$

We consider a rectangle C in the z plane with vertices at the points $1/2 \pm iT$ and $a \pm iT$, where $T \geq 2$ is a sufficiently larger number and $a > 1$ (see Figure 2.1). By Cauchy's theorem the integral of $F_n(y,z)$ around this rectangle is zero. We get

$$\left(\int_{1/2-iT}^{1/2+iT} + \int_{a+iT}^{a-iT} + \int_{1/2+iT}^{a+iT} + \int_{a-iT}^{1/2-iT}\right) G_n(y,z)\,dz = 0.$$

For the third integral above, we notice that:

$$\left|\int_{1/2+iT}^{a+iT} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)\zeta(z)}{z^{n+1}}\,dz\right| \leq \int_{1/2+iT}^{1+iT} \left|\frac{2y^{-z}\Gamma(z)\zeta(z)}{z^{n+1}}\,dz\right| + \int_{1+iT}^{a+iT} \left|\frac{2y^{-z}\Gamma(z)\zeta(z)}{z^{n+1}}\,dz\right|$$

Using Lemma 1.14, we get

$$\left|\int_{1/2+iT}^{a+iT} G_n(y,z)dz\right| \leq \int_{1/2+iT}^{1+iT} \left|\frac{2y^{-z}\Gamma(z)}{z^{n+1}}\right| \cdot |t|^{1-\sigma}\log|t|\,dt + \int_{1+iT}^{a+iT} \left|\frac{2y^{-z}\Gamma(z)}{z^{n+1}}\right| \cdot \log|t|\,dt.$$

A similar argument to that in the proof of Lemma 2.1 shows that

$$\left|\int_{1/2+iT}^{a+iT} G_n(y,z)\,dz\right| \ll$$

$$\sqrt{2\pi}\int_{1/2}^{1} \frac{y^{-\sigma}\log(\sigma^2+T^2)}{\left(\sigma^2+T^2\right)^{\frac{n+1}{2}}}\,|d\sigma| + \sqrt{2\pi}\int_{1+iT}^{a+iT} \frac{y^{-\sigma}\left(\sigma^2+T^2\right)^{\frac{\sigma-1/2}{2}}\log(\sigma^2+T^2)}{\left(\sigma^2+T^2\right)^{\frac{n+1}{2}}}\,|d\sigma|.$$

Here, we notice that the first integral on the right-hand side above tend to 0 when $T\to+\infty$ and the second integral tend to 0 under the condition $n > a-1/2$.
A similar argument, we find that the integral over the segment $[1/2-iT, a-iT]$ tend to 0 as $T\to+\infty$. Hence, we get

$$\int_{1/2-iT}^{1/2+iT} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)\zeta(z)}{z^{n+1}}\,dz = \int_{a-iT}^{a+iT} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)\zeta(z)}{z^{n+1}}\,dz,$$

which completes the proof of the first equation. A similar argument, using Lemma 1.15 for the function $L(z,\chi)$, we can prove the second equation in this lemma. ⋄⋄⋄

2.2 Matsuoka's formula for the zeta function

Proposition 2.3 *For all real number $1 < a < n+1/2$, we have:*

$$\frac{\gamma_n}{n!} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2(a+it)^{-n-1}\,(2\pi)^{-a-it}\cos\frac{\pi}{2}(a+it)\,\Gamma(a+it)\,\zeta(a+it)\,dt,$$

where $z=\sigma+it$.

Proof. Recall that

$$\zeta(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n(z-1)^n.$$

By Caucy's formula, we can write

$$\gamma_n = \frac{(-1)^n n!}{2\pi i}\int_D \frac{\zeta(z)}{(z-1)^{n+1}}\,dz.$$

where D is the rectangular path passing through the vertices $1/2+iT$, $1/2-iT$, $A-iT$ and $A+iT$, in the anticlockwise direction, where T is a sufficiently larger number and $A\geq 2$ is large and fixed for a moment.

Let us now divide D into the line segments D_1, D_2, D_3 and D_4 joining $1/2+iT$, $1/2-iT$, $A-iT$, $A+iT$ and $1/2+iT$, as in Figure 2.2. Then, we have

$$\gamma_n = \frac{(-1)^n n!}{2\pi i}\left(\int_{D_1}+\int_{D_2}+\int_{D_3}+\int_{D_4}\right)\frac{\zeta(z)}{(z-1)^{n+1}}\,dz.$$

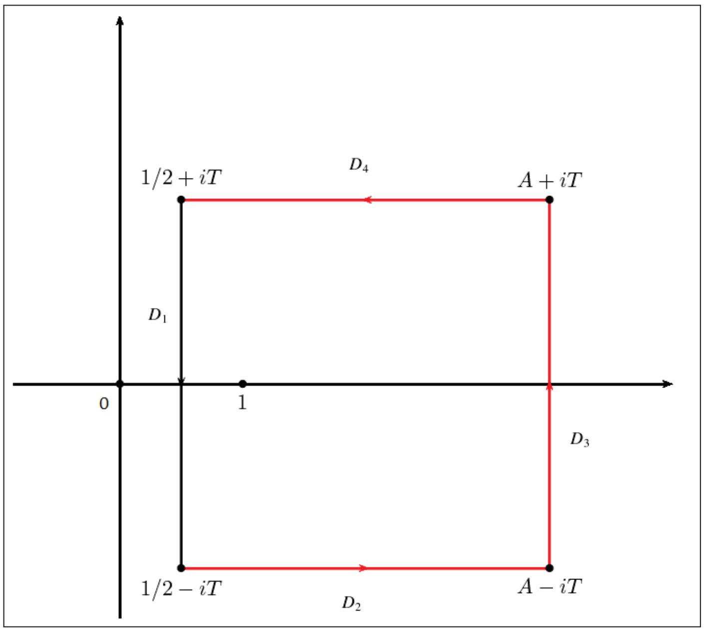

Figure 2.2: The contour of integration D enclosing 1 in the complex plane.

By Lemma 1.14, the integral over D_2 is bounded by

$$\begin{aligned}\int_{D_2} &= \int_{\frac{1}{2}-iT}^{A-iT} \frac{\zeta(z)}{(z-1)^{n+1}}\,dz,\\ &= \mathcal{O}\left(\int_{\frac{1}{2}}^{1} T^{-n-\sigma}\log|T|\,d\sigma + \int_{1}^{A} T^{-n-1}\log|T|\,d\sigma\right),\\ &= \mathcal{O}\left(|T|^{-n}\log|T|\left(\int_{\frac{1}{2}}^{1} e^{-\sigma\log|T|}\,d\sigma + |T|^{-1}(A-1)\right)\right),\\ &= \mathcal{O}\left(|T|^{-n-1}\log|T|\left(\frac{|T|^{1/2}-1}{\log|T|} + (A-1)\right)\right).\end{aligned}$$

If we allow $T\to\infty$, we obtain the limit of the integral over D_2 is 0. A similar argument, we find that the integral over D_4 tends to 0 as $T\to\infty$. While the modulus of the integral over D_3 is bounded by

$$\left|\int_{D_3}\right| = \left|\int_{A-iT}^{A+iT} \frac{\zeta(z)}{(z-1)^{n+1}}\,dz\right| \le \int_{A-iT}^{A+iT} \left|\frac{\zeta(z)\,dz}{(z-1)^{n+1}}\right|.$$

Since

$$|\zeta(z)| \le \sum_{m\ge1}\frac{1}{m^A} \le \sum_{m\ge1}\frac{1}{m^2} = \frac{\pi^2}{6},$$

then we have

$$\left|\int_{D_3}\right| \le \int_{A-iT}^{A+iT} \left|\frac{\zeta(z)\,dz}{(z-1)^{n+1}}\right| \le \frac{\pi^2}{6}\int_{A-iT}^{A+iT} \frac{|dz|}{|z-1|^{n+1}}.$$

We recall that $z=\sigma+it$ and A is fixed for a moment. If we let $T\to\infty$ to get

$$\left|\int_{D_3}\right| \le \frac{\pi^2}{6}\int_{-\infty}^{+\infty} \frac{|dt|}{\left((1-A)^2+t^2\right)^{\frac{n+1}{2}}}.$$

By making a simple change of variable $(A-1)u=t$, we obtain that

$$\left|\int_{D_3}\right| \le \frac{\pi^2}{6(A-1)^n}\int_{-\infty}^{+\infty} \frac{|du|}{\left(1+u^2\right)^{\frac{n+1}{2}}} \le \frac{\pi^2}{6(A-1)^n}\int_{-\infty}^{+\infty} \frac{|du|}{\left(1+u^2\right)} = \frac{\pi^3}{6(A-1)^n}.$$

If we now let $A\to\infty$, it follows that the integral over D_3 tends to 0. Therefore we get

$$\gamma_n = \frac{(-1)^{n+1}n!}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\zeta(z)}{(z-1)^{n+1}}\,dz.$$

Now, change z to $1-z$ to get

$$\gamma_n = \frac{n!}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\zeta(1-z)}{z^{n+1}}\,dz.$$

Here we use the functional equation of $\zeta(1-z)$ given by Eq (1.4). Then we have

$$\gamma_n = \frac{n!}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{2(2\pi)^{-z}\cos(\frac{\pi z}{2})\Gamma(z)\zeta(z)}{z^{n+1}}\,dz.$$

To complete the proof, using Lemma 2.2, it suffices to shift the line of integration to $\sigma=a$ where $1<a<n+1/2$. We conclude that

$$\frac{\gamma_n}{n!} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2(a+it)^{-n-1}\ (2\pi)^{-a-it}\cos\frac{\pi}{2}(a+it)\ \Gamma(a+it)\ \zeta(a+it)\,dt.$$

◇◇◇

2.3 Matsuoka's formula for the Dirichlet L functions

We now state and prove the following Matsuoka formula for Dirichlet L-functions. Recall that $(-1)^n\gamma_n(\chi)$ denotes the value of the n-th derivative of $L(z,\chi)$ at $z=1$ when χ is non-principal Dirichlet character modulo $q>1$. Then, we have

Proposition 2.4 *Let χ be a non-principal even primitive Dirichlet character of modulus q. Then for any real parameter $1<a<n+1/2$, we have :*

$$\frac{\gamma_n(\chi)}{n!}=\frac{\tau(\chi)}{2\pi i q}\int_{a-i\infty}^{a+i\infty}\left(\frac{2\pi}{q}\right)^{-z}\frac{2\,\cos(\pi z/2)\,\Gamma(z)\,L(z,\overline{\chi})}{z^{n+1}}\,dz.$$

where $\tau(\chi)$ is again the Gaussian sum attached to χ.

Proposition 2.5 *Let χ be a non-principal odd primitive Dirichlet character of modulus q. Then for any real parameter $1<a<n+1/2$, we have :*

$$\frac{\gamma_n(\chi)}{n!}=\frac{-\tau(\chi)}{2\pi q}\int_{a-i\infty}^{a+i\infty}\left(\frac{2\pi}{q}\right)^{-z}\frac{2\,\sin(\pi z/2)\,\Gamma(z)\,L(z,\overline{\chi})}{z^{n+1}}\,dz.$$

The idea of the proof is exactly the same as in Proposition 2.3.

Proof. Recall that

$$L(z,\chi)=\sum_{n=0}^{\infty}\frac{(-1)^n\gamma_n(\chi)}{n!}(z-1)^n.$$

By Cauchy's integral formula, we can write

$$\frac{\gamma_n(\chi)}{n!}=\frac{-1}{2\pi i}\int_D\frac{L(z,\chi)}{(1-z)^{n+1}}\,dz,$$

where the path of integration D is taken as in Proposition 2.3 (see also Figure (2.2)). Then we have

$$\frac{\gamma_n(\chi)}{n!}=\frac{-1}{2\pi i}\left(\int_{D_1}+\int_{D_2}+\int_{D_3}+\int_{D_4}\right)\frac{L(z,\chi)}{(1-z)^{n+1}}\,dz.$$

For the integral over D_2, we write

$$\int_{D_2}\frac{L(z,\chi)}{(1-z)^{n+1}}\,dz=\int_{1/2-iT}^{A-iT}\frac{L(z,\chi)}{(1-z)^{n+1}}\,dz=\int_{1/2}^{A}\frac{L(\sigma-iT,\chi)}{\left((1-\sigma)^2+T^2\right)^{\frac{n+1}{2}}}\,d\sigma$$

Using Lemma 1.15 and letting $A\to\infty$, we find that

$$\left|\int_{D_2}\right|\le\log\big(q(2+|T|)\big)\int_{1/2}^{\infty}\frac{|d\sigma|}{\left((1-\sigma)^2+T^2\right)^{\frac{n+1}{2}}}+\mathcal{O}\left(\int_{1/2}^{\infty}\frac{d\sigma}{\left((1-\sigma)^2+T^2\right)^{\frac{n+1}{2}}}\right).$$

We can change of the variable $\sigma - 1 = uT$ to get

$$\left|\int_{D_2}\right| \leq \frac{\log\left(q(2+|T|)\right)}{T^n} \int_{-\infty}^{1/(2T)} \frac{|du|}{\left(1+u^2\right)^{\frac{n+1}{2}}} + \mathcal{O}\left(\frac{1}{T^n}\int_{-\infty}^{1/(2T)} \frac{du}{\left(1+u^2\right)^{\frac{n+1}{2}}}\right).$$

To compute the above integral, we divide it into two parts

$$\begin{aligned} \int_{-\infty}^{1/(2T)} \frac{|du|}{\left(1+u^2\right)^{\frac{n+1}{2}}} &= \int_{-\infty}^{0} \frac{|du|}{\left(1+u^2\right)^{\frac{n+1}{2}}} + \int_{0}^{1/(2T)} \frac{|du|}{\left(1+u^2\right)^{\frac{n+1}{2}}} \\ &\leq \int_{0}^{\infty} \frac{|du|}{1+u^2} + \int_{0}^{1/(2T)} \frac{du}{1+u^2} = \frac{\pi}{2} + \arctan\left(\frac{1}{2T}\right). \end{aligned}$$

If we now let $T \to \infty$, it follows that the integral over D_2 tends to zero. Similar reasoning, we show that the integral over D_4 tends to 0. While the modulus of the integral over D_3 is bounded by

$$\left|\int_{D_3}\right| \leq \int_{A-i\infty}^{A+i\infty} \left|\frac{L(z,\chi)\,dz}{(1-z)^{n+1}}\right|.$$

Taking A is fixed for a moment and using Polya-Vinogradov theorem 1.16, we get

$$\left|\int_{D_3}\right| \leq 2\sqrt{q}\log q \int_{-\infty}^{+\infty} \frac{|dt|}{\left((A-1)^2+t^2\right)^{\frac{n+1}{2}}}.$$

By making a simple change of variable $(A-1)u = t$, we obtain that

$$\left|\int_{D_3}\right| \leq \frac{2\sqrt{q}\log q}{(A-1)^n} \int_{-\infty}^{+\infty} \frac{|du|}{\left(1+u^2\right)^{\frac{n+1}{2}}} \leq \frac{4\sqrt{q}\log q}{(A-1)^n} \int_{0}^{+\infty} \frac{|du|}{\left(1+u^2\right)} \leq \frac{2\pi\sqrt{q}\log q}{(A-1)^n}$$

If we let $A \to \infty$, it follows that the integral over D_3 tends to 0. Therefore we infer that:

$$\frac{\gamma_n(\chi)}{n!} = \frac{\tau(\chi)}{2\pi q i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{L(z,\chi)}{(1-z)^{n+1}}\,dz, \qquad \text{when } \chi(-1) = 1,$$

and

$$\frac{\gamma_n(\chi)}{n!} = \frac{i}{2\pi\tau(\overline{\chi})} \int_{1/2-i\infty}^{1/2+i\infty} \frac{L(z,\chi)}{(1-z)^{n+1}}\,dz, \qquad \text{when } \chi(-1) = -1.$$

Now, change z to $1-z$ and use the functional equation of $L(z,\chi)$ given by Eq (1.8) and Eq (1.9) to get

$$\frac{\gamma_n(\chi)}{n!} = \frac{\tau(\chi)}{2\pi q i}\int_{1/2-i\infty}^{1/2+i\infty}\left(\frac{2\pi}{q}\right)^{-z}\frac{2\cos(\pi z/2)\,\Gamma(z)\ L(z,\overline{\chi})}{z^{n+1}}\,dz, \qquad \text{when}\chi(-1)=1,$$

and

$$\frac{\gamma_n(\chi)}{n!} = \frac{i}{2\pi\tau(\overline{\chi})}\int_{1/2-i\infty}^{1/2+i\infty}\left(\frac{2\pi}{q}\right)^{-z}\frac{2\sin\left(\frac{\pi}{2}z\right)\Gamma(z)\ L(z,\overline{\chi})}{z^{n+1}}\,dz \qquad \text{when}\chi(-1)=-1.$$

To complete the proof, using Lemma 2.2, it therefore suffices to shift the line of integration to $\sigma = a$ where $1 < a < n+1/2$. Applying Eq (1.2) to the last equality above, we conclude that

$$\frac{\gamma_n(\chi)}{n!} = \frac{\tau(\chi)}{2\pi q i}\int_{a-i\infty}^{a+i\infty}\left(\frac{2\pi}{q}\right)^{-z}\frac{2\cos(\pi z/2)\,\Gamma(z)\ L(z,\overline{\chi})}{z^{n+1}}\,dz \qquad \text{when } \chi(-1)=1,$$

and

$$\frac{\gamma_n(\chi)}{n!} = \frac{1}{2\pi i}\frac{\tau(\chi)}{iq}\int_{a-i\infty}^{a+i\infty}\left(\frac{2\pi}{q}\right)^{-z}\frac{2\sin\left(\frac{\pi}{2}z\right)\Gamma(z)\ L(z,\overline{\chi})}{z^{n+1}}\,dz \qquad \text{when } \chi(-1)=-1.$$

This completes the proof. ⋄⋄⋄

Chapter 3

The special functions $\mathscr{F}_n(y)$ and $\tilde{\mathscr{F}}_n(y)$

In this chapter, we first use the Matsuoka formulae to define two complex functions $\mathscr{F}_n(y)$ and $\tilde{\mathscr{F}}_n(y)$ for any $y > 0$. Next, we attempt to give the real expressions of $\mathscr{F}_n$(y) and $\tilde{\mathscr{F}}_n(y)$. In the last section, we study the behaviour of these functions.

3.1 Complex expression of $\mathscr{F}_n(y)$ and $\tilde{\mathscr{F}}_n(y)$

We define the following two complex functions $\mathscr{F}_n$ and $\tilde{\mathscr{F}}_n$.

Definition 3.1 *For $y > 0$, we define:*

$$\mathscr{F}_n(y) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{2y^{-z}\cos(\pi z/2)\,\Gamma(z)}{z^{n+1}}\,dz \tag{3.1}$$

and

$$\tilde{\mathscr{F}}_n(y) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{2y^{-z}\sin(\pi z/2)\,\Gamma(z)}{z^{n+1}}\,dz. \tag{3.2}$$

These functions will play the central role in the proof of the following two results which are similar in nature.

Theorem 3.2 *Let χ be a non-principal primitive Dirichlet character of modulo $q > 1$. For $n \geq 1$, we have*

$$\frac{\gamma_n(\chi)}{n!} = \begin{cases} \dfrac{\tau(\chi)}{q}\displaystyle\sum_{m\geq 1}\overline{\chi}(m)\mathscr{F}_n\left(\frac{2\pi m}{q}\right), & \text{when } \chi(-1) = 1,\\ \dfrac{\tau(\chi)}{iq}\displaystyle\sum_{m\geq 1}\overline{\chi}(m)\tilde{\mathscr{F}}_n\left(\frac{2\pi m}{q}\right), & \text{when } \chi(-1) = -1. \end{cases}$$

where $\mathscr{F}_n$ and $\tilde{\mathscr{F}}_n$ are defined by Eq (3.1) *and Eq* (3.2), *respectively .*

Proof. For $1 < a < n + \frac{1}{2}$, we recall that

$$\frac{\gamma_n(\chi)}{n!} = \frac{\tau(\chi)}{2\pi q i} \int_{a-i\infty}^{a+i\infty} \left(\frac{2\pi}{q}\right)^{-z} \frac{2\cos(\pi z/2)\,\Gamma(z)\, L(z,\overline{\chi})}{z^{n+1}}\, dz \qquad \text{when } \chi(-1) = 1,$$

and that

$$\frac{\gamma_n(\chi)}{n!} = \frac{-\tau(\chi)}{2\pi q} \int_{a-i\infty}^{a+i\infty} \left(\frac{2\pi}{q}\right)^{-z} \frac{2\sin\left(\frac{\pi}{2}z\right)\Gamma(z)\, L(z,\overline{\chi})}{z^{n+1}}\, dz \qquad \text{when } \chi(-1) = -1.$$

Since

$$L(z,\chi) = \sum_{m\geq 1} \frac{\chi(m)}{m^z}$$

is uniformly convergent for $\Re(z) > 0$, we may interchange summation and integration above. This yields

$$\frac{\gamma_n(\chi)}{n!} = \frac{\tau(\chi)}{2\pi q i} \sum_{m\geq 1} \overline{\chi}(m) \int_{a-i\infty}^{a+i\infty} \left(\frac{2\pi m}{q}\right)^{-z} \frac{2\cos(\pi z/2)\,\Gamma(z)}{z^{n+1}}\, dz \qquad \text{when } \chi(-1) = 1,$$

and that

$$\frac{\gamma_n(\chi)}{n!} = \frac{-\tau(\chi)}{2\pi q} \sum_{m\geq 1} \overline{\chi}(m) \int_{a-i\infty}^{a+i\infty} \left(\frac{2\pi m}{q}\right)^{-z} \frac{2\sin\left(\frac{\pi}{2}z\right)\Gamma(z)}{z^{n+1}}\, dz \qquad \text{when } \chi(-1) = -1.$$

From Lemma 2.1, we found that

$$\int_{a-i\infty}^{a+i\infty} \left(\frac{2\pi m}{q}\right)^{-z} \frac{2\sin\left(\frac{\pi}{2}z\right)\Gamma(z)}{z^{n+1}}\, dz = \int_{1/2-i\infty}^{1/2+i\infty} \left(\frac{2\pi m}{q}\right)^{-z} \frac{2\sin\left(\frac{\pi}{2}z\right)\Gamma(z)}{z^{n+1}}\, dz,$$

and

$$\int_{a-i\infty}^{a+i\infty} \left(\frac{2\pi m}{q}\right)^{-z} \frac{2\cos\left(\frac{\pi}{2}z\right)\Gamma(z)}{z^{n+1}}\, dz = \int_{1/2-i\infty}^{1/2+i\infty} \left(\frac{2\pi m}{q}\right)^{-z} \frac{2\cos\left(\frac{\pi}{2}z\right)\Gamma(z)}{z^{n+1}}\, dz.$$

This completes the proof. ◇◇◇

The case $q = 1$ being of the special importance, we state the following corollary:

Corollary 3.3 *For $n \geq 1$, we have :*

$$\frac{\gamma_n}{n!} = \sum_{m\geq 1} \mathscr{F}_n(2\pi m), \tag{3.3}$$

where $\mathscr{F}_n$ is defined by Eq (3.1) *above.*

Proof. The proof of this result is a similar as to that in the proof of Theorem 3.2 and under the condition $\Re(z) > 1$. ◇◇◇

These expressions for $\mathscr{F}_n$ and $\tilde{\mathscr{F}}_n$ are not efficient for bounding explicitly these functions. In order to bound them, we will have recourse to the following formulae which avoid complex analysis.

3.2 Real expression of $\mathscr{F}_n(y)$ and $\tilde{\mathscr{F}}_n(y)$

Theorem 3.4 *For any positive real number y, we have:*

$$\mathscr{F}_n(y) = \frac{2}{n!}\int_y^\infty \left(\log\frac{u}{y}\right)^n \frac{\cos u}{u}\,du,$$

and

$$\tilde{\mathscr{F}}_n(y) = \frac{2}{n!}\int_y^\infty \left(\log\frac{u}{y}\right)^n \frac{\sin u}{u}\,du.$$

3.2.1 Lemmas for Theorem 3.4

In this section, we prove some technical results on integral for use later in the proof of Theorem 3.4.

Lemma 3.5 *When $0 < \Re(z) < 1$, we have:*

$$\cos(\pi z/2)\Gamma(z) = \int_0^\infty u^{z-1}\cos u\,du,$$

and

$$\sin(\pi z/2)\Gamma(z) = \int_0^\infty u^{z-1}\sin u\,du.$$

Proof. A proof of this lemma can be found in [42]. ⋄⋄⋄

Lemma 3.6 *For every $x > 0$ and $n \geq 1$, we have :*

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{x^{-z}}{z^{n+1}}\,dz = \begin{cases} \frac{(-1)^n}{n!}\log^n x, & \text{when } x \leq 1,\\ 0, & \text{when } x > 1.\end{cases}$$

Proof. First and foremost, the integral over an infinite path is the limit of the integral on a finite path:

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{x^{-z}}{z^{n+1}}\,dz = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{x^{-z}}{z^{n+1}}\,dz. \tag{3.4}$$

When $x > 1$, we consider a rectangle D in the z plane with vertices at the points $1/2 \pm iT$ and $A \pm iT$, where $T > 0$ and $A > 1/2$ going to infinity.

By Cauchy's theorem the integral of x^{-z}/z^{n+1} around this rectangle is zero (see Figure 3.1). we get

$$\left(\int_{1/2-iT}^{1/2+iT} + \int_{A+iT}^{A-iT} + \int_{1/2+iT}^{A+iT} + \int_{A-iT}^{1/2-iT}\right)\frac{x^{-z}}{z^{n+1}}\,dz = 0.$$

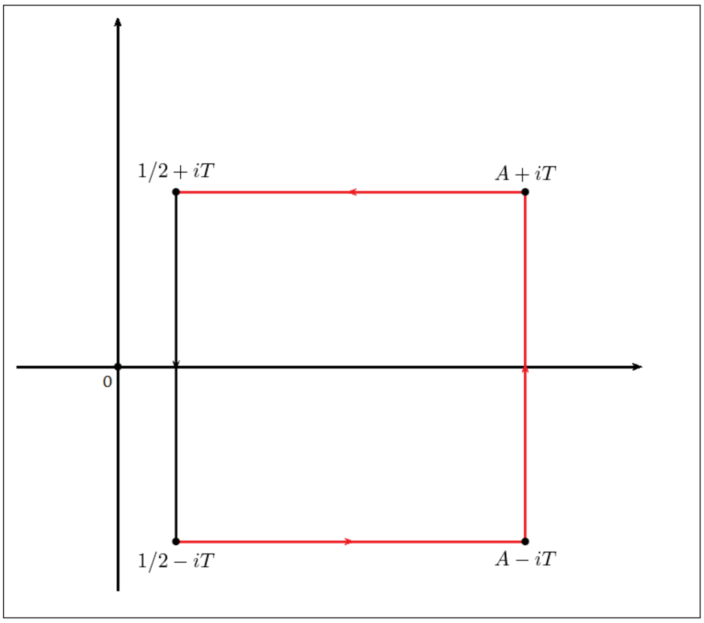

Figure 3.1: The rectangle D in the complex z plane.

The second integral dwindles to zero when A increases. Both integrals on the horizontal segments are bounded by $\frac{x^{-1/2}A}{T^{n+1}}$. This yields

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{x^{-z}}{z^{n+1}}\,dz = \mathscr{O}\left(\frac{x^{-1/2}A}{T^{n+1}}\right).$$

If we now let $T \to \infty$ in the last equality, we find that the limit in Eq (3.4) is equal 0. Next, for $x \leq 1$, we remove this time the line of integration to the far left, i.e we consider the integral of x^{-z}/z^{n+1} taken around a rectangle with vertices of the form $1/2 \pm iT$ and $-A \pm iT$, where $T > 0$ and $A > 1/2$ going to infinity. In this case, the integral around this rectangle is equal to $\frac{(-1)^n}{n!}\log^n x$, since the pole of order n at $z = 0$ lies inside the contour. By the same reasoning used in the case $x > 1$, we find that the limit in Eq (3.4) is equal to $\frac{(-1)^n}{n!}\log^n x$, when $x \leq 1$. ◇◇◇

3.2.2 Proof of Theorem 3.4

From Lemma 3.5, we get

$$\cos(\pi z/2)\Gamma(z) = \int_0^\infty \cos(u)u^z\,\frac{du}{u}.$$

An integration by parts easily gives the following approximation of the above:

$$y^{-z}\cos(\pi z/2)\Gamma(z) = \int_0^U (y/u)^{-z}\cos u \frac{du}{u} + \mathscr{O}\left(y^{-\Re(z)}U^{\Re(z)-1}\right)$$

We first note that

$$\mathscr{F}_n(y) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{2y^{-z}\cos(\frac{\pi z}{2})\,\Gamma(z)}{z^{n+1}}\,dz,$$

and we use the above representation of the integrand. We get

$$\mathscr{F}_n(y) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\int_0^U 2\frac{(y/u)^{-z}}{z^{n+1}}\frac{\cos u}{u}\,du\,dz + \mathscr{O}\left(y^{-1/2}U^{-1/2}\right)$$

Thanks to Lemma 3.6, we find that

$$\mathscr{F}_n(y) = \frac{2}{n!}\int_y^\infty \left(\log\frac{u}{y}\right)^n \frac{\cos u}{u}\,du,$$

and the proof of the first equality in Theorem 3.2 is complete. Then the proof of the second equality is similar to the previous proof.

3.3 Behaviour of $\mathscr{F}_n(y)$

In this section, we hope to deepen our knowledge on the analytic behaviour of the special function $\mathscr{F}_n(y)$. We recall that

$$\mathscr{F}_n(y) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{2y^{-z}\cos(\pi z/2)\Gamma(z)}{z^{n+1}}. \tag{3.5}$$

This function clearly does not carry any arithmetic any more and we should be able to grasp its behaviour quite fully. It turns out that achieving such an understanding is more difficult expected and we prefer to spend a full section on proving the following result, and continue our study of this function in the next chapter.

Lemma 3.7 *Let $\mathscr{F}_n(y)$ be a function defined by Eq (3.5). For $y > 0$, we have:*

$$\mathscr{F}_n(y) = \sum_{i=0}^{n+1}\sum_{j=0}^{n-i+1}(-1)^{i+1}\frac{(n+1)!\,(\pi/2)^j}{i!\,j!(n-i-j+1)!}(\log y)^i\,\Gamma^{(n-i-j+1)}(1) + \sum_{d\geq 1}\frac{(-1)^{n+d+1}y^{2d}}{(2d)!(2d)^{n+1}},$$

where $\Gamma^{(k)}(1)$ is the value of k-th derivative of the Gamma function at $z = 1$.

Proof. We see that we can send the line of integration to the far left provided we compute the contribution of the poles. The first candidates, $z = 0$ is a pole of order $n+2$, so we have access to the residue via the computation of the derivative at $z = 0$ of

$$\frac{y^{-z}\ \cos(\frac{\pi}{2}z)\Gamma(z)}{z^{n+1}} = \frac{y^{-z}\ \cos(\frac{\pi}{2}z)\Gamma(z+1)}{z^{n+2}}.$$

We find that

$$\begin{aligned}
\mathscr{R}es\big(\mathscr{F}_n(y),0\big) &= \lim_{z\to 0}\frac{d^{n+1}}{dz^{n+1}}\Big(y^{-z}\cos(\frac{\pi}{2}z)\Gamma(z+1)\Big)\\
&= \lim_{z\to 0}\left(\sum_{i=0}^{n+1}\binom{n+1}{i}\big(y^{-z}\big)^{(i)}\sum_{j=0}^{n+1-i}\binom{n+1-i}{j}\cos^{(j)}\left(\frac{\pi z}{2}\right)\ \Gamma^{(n+1-i-j)}(z+1)\right)\\
&= \sum_{i=0}^{n+1}\sum_{j=0}^{n-i+1}(-1)^{i+1}\frac{(n+1)!}{i!j!(n-i-j+1)!}\log^i(y)\left(\frac{\pi}{2}\right)^j\ \Gamma^{(n-i-j+)}(1),
\end{aligned}$$

where

$$\lim_{z\to 0}\left(\frac{d^j}{dz^j}\cos\left(\frac{\pi}{2}z\right)\right) = \begin{cases} -(\pi/2)^j & \text{when } j \text{ even,}\\ \ \ 0 & \text{when } j \text{ odd.}\end{cases}$$

Next we are to take care of the contribution of the poles at $z = -m$ for $m \geq 1$. To do so we use the complement formula and write

$$\begin{aligned}
\frac{y^{-z}\cos(\pi z/2)\Gamma(z)}{z^{n+1}} &= \frac{\pi y^{-z}\ \cos(\frac{\pi}{2}z)}{z^{n+1}\sin(\pi z)\Gamma(1-z)}\\
&= \frac{\pi y^{-z}}{2\sin(\pi z/2)z^{n+1}\Gamma(1-z)}
\end{aligned}$$

which shows we have a pole only if $m = 2d$ and that its residue is then

$$\frac{(-1)^{n+1+d}y^{2d}}{(2d)^{n+1}(2d)!}.$$

This completes the proof. ⋄⋄⋄

Chapter 4

Proof of Theorem 0.37

Our task in this chapter is to prove our **main result** (Theorem 0.37) in this part. The proof of this theorem depends on two technical results. The first result is basically Matsuoka's formula. Recall that:

$$\mathscr{F}_n(y) = \frac{2}{n!}\int_y^\infty \left(\log\frac{u}{y}\right)^n \frac{\cos u}{u}\,du,$$

and

$$\tilde{\mathscr{F}}_n(y) = \frac{2}{n!}\int_y^\infty \left(\log\frac{u}{y}\right)^n \frac{\sin u}{u}\,du.$$

These integrals do not converge very fast, but this is corrected in Lemma 4.2 and 4.3. In the following section we collect some preliminary results which we will use to improve on these expressions.

4.1 Calculating the derivatives

Lemma 4.1 *The derivatives of the function* $f_n(u) = \frac{(\log u)^n}{u}$ *are given by*

$$\frac{d^k f_n(u)}{du^k} = \sum_{\ell=0}^{k} \sum_{1\le m_1<m_2<\cdots<m_\ell\le k} \frac{(-1)^{k-\ell}k!}{m_0\cdots m_\ell}\,\frac{n!}{(n-\ell)!}\,\big(\log u\big)^{n-\ell}\,\frac{1}{u^{k+1}} \tag{4.1}$$

where $m_0 = 1$ *and* $0 \le k \le n$.

Proof. We prove this formula by induction on k. When $k = 0$, Eq (4.1) is clearly valid. We prove that it is also valid on $k = 1$. If S the right-hand side of Eq (4.1) when $k = 1$ and

$$S = -(\log u)^n \frac{1}{u^2} + n(\log u)^{n-1}\frac{1}{u^2}.$$

Then $S = \frac{d}{du} f_n(u)$ and so Eq (4.1) is valid for $k = 1$.
Now, we assume that Eq (4.1) is valid for any fixed and non-negative integer k. Then we have to prove that Eq (4.1) is also valid for $k+1$, i.e.,

$$\frac{d^{k+1} f_n(u)}{du^{k+1}} = \sum_{\ell=0}^{k+1} \sum_{1\le m_1<m_2<\cdots<m_\ell\le k+1} \frac{(-1)^{k-\ell+1}(k+1)!}{m_0\cdots m_\ell} \frac{n!}{(n-\ell)!} \Big(\log u\Big)^{n-\ell} \frac{1}{u^{k+2}}. \quad (4.2)$$

By induction hypothesis, we get

$$\frac{d}{du}\Big(\frac{d^k f_n}{du^k}\Big) = \sum_{\ell=0}^{k} \sum_{1\le m_1<m_2<\cdots<m_\ell\le k} \frac{(-1)^{k-\ell} k!}{m_0\cdots m_\ell} \frac{n!}{(n-\ell-1)!} \Big(\log u\Big)^{n-\ell-1} \frac{1}{u^{k+2}}$$
$$+ \sum_{\ell=0}^{k} \sum_{1\le m_1<m_2<\cdots<m_\ell\le k} \frac{(-1)^{k-\ell+1}(k+1)!}{m_0\cdots m_\ell} \frac{n!}{(n-\ell)!} \Big(\log u\Big)^{n-\ell} \frac{1}{u^{k+2}}.$$

By making a simple change of variable $\ell+1 = \ell'$ in the first sum above and multiplying the resulting equality by $k+1 = m_{\ell'}$. We obtain

$$\frac{d}{du}\Big(\frac{d^k f_n(u)}{du^k}\Big) =$$
$$\sum_{\ell'=1}^{k+1} \sum_{1\le m_1<m_2<\cdots<m_{\ell'-1}\le k} \frac{(-1)^{k-\ell'+1}(k+1)!}{m_0\cdots m_{\ell'-1} m_{\ell'}} \frac{n!}{(n-\ell')!} \Big(\log u\Big)^{n-\ell'} \frac{1}{u^{k+2}}$$
$$+ \sum_{\ell=0}^{k} \sum_{1\le m_1<m_2<\cdots<m_\ell\le k} \frac{(-1)^{k-\ell+1}(k+1)!}{m_0\cdots m_\ell} \frac{n!}{(n-\ell)!} \Big(\log u\Big)^{n-\ell} \frac{1}{u^{k+2}}.$$

Noting that the last term (for $\ell' = k+1$) in the first sum on ℓ' gives us the last term of Eq (4.2) for $k+1$, which equal to

$$(n-1)\cdots(n-k)(\log u)^{n-k+1} \frac{1}{u^{k+2}},$$

and that the first term (for $\ell = 0$) in the second sum on ℓ gives the first term of Eq (4.2) for $k+1$, which equal to

$$(-1)^{k+1}(k+1)!\,(\log u)^n \frac{1}{u^{k+2}}.$$

On the other hand, we have

$$\sum_{1\le m_1<m_2<\cdots<m_\ell\le k+1} \frac{1}{m_0\cdots m_\ell} =$$
$$\sum_{1\le m_1<m_2<\cdots<m_{\ell-1}\le k} \frac{1}{m_0\cdots m_{\ell-1}(k+1)} + \sum_{1\le m_1<m_2<\cdots<m_\ell\le k} \frac{1}{m_0\cdots m_\ell}.$$

We conclude from the above that the claimed formula is also valid for $k+1$. Then it is valid for all values of $k \ge 0$. The lemma is proved. ⋄⋄⋄

Lemma 4.2 *For all $y > 0$, we have*

$$\mathscr{F}_n(y) = \frac{2}{n!}\frac{1}{y^k}\int_1^\infty P_{n,k}(\log u)\frac{S_k(yu)}{u^{k+1}}\,du,$$

where

$$P_{n,k}(\log u) = \sum_{\ell=0}^{k}\sum_{1\le m_1<m_2<\cdots<m_\ell\le k}(-1)^{k-\ell}\frac{k!}{m_0\cdots m_\ell}\,\frac{n!}{(n-\ell)!}(\log u)^{n-\ell},$$

with $m_0 = 1$, and

$$S_k(u) = \begin{cases} i^k\cos u & \text{if } k\equiv 0,2 \pmod 4,\\ i^{k+1}\sin u & \text{if } k\equiv \pm 1 \pmod 4.\end{cases}$$

Proof. Recall that

$$\mathscr{F}_n(y) = \frac{2}{n!}\int_y^\infty \Big(\log\frac{u}{y}\Big)^n\frac{\cos u}{u}\,du.$$

By making a simple change of variable $u/y = t$, we get

$$\mathscr{F}_n(y) = \frac{2}{n!}\int_1^\infty \log^n t\,\frac{\cos(yt)}{t}\,dt.$$

By using integration by parts and using Lemma 4.1, we conclude immediately the proof of this lemma. ⋄⋄⋄

Lemma 4.3 *For all $y > 0$, we have :*

$$\tilde{\mathscr{F}}_n(y) = \frac{2}{n!}\frac{1}{y^k}\int_1^\infty P_{n,k}(\log u)\frac{\tilde{S}_k(yu)}{u^{k+1}}\,du.$$

where

$$\tilde{S}_k(u) = \begin{cases} i^k\sin u & \text{if } k\equiv 0,2 \pmod 4,\\ i^{1-k}\cos u & \text{if } k\equiv \pm 1 \pmod 4.\end{cases}$$

and $P_{n,k}(\log u)$ is given by Lemma 4.2.

Proof. The proof of this lemma is similar to that in the proof of Lemma 4.2. ⋄⋄⋄

4.2 Recurrence relation for $\log^n x/x$

Lemma 4.4 *For any integer number $k \geq 1$, we have*

$$P_{n,k}(X) = P'_{n,k-1}(X) - kP_{n,k-1}(X), \tag{4.3}$$

where $P_{n,0}(X) = X^n$.

Proof. We prove this lemma by induction on k. In order to show that Eq (4.3) is valid for $k = 1$, we start with

$$\begin{aligned}\mathscr{F}_n(y) &= \frac{2}{n!}\int_1^\infty \log^n u \frac{\cos(yu)}{u}\, du,\\ &= \frac{2}{n!}\int_1^\infty P_{n,0}(\log u)\frac{S_0(yu)}{u}\, du,\end{aligned}$$

where $S_k(u)$ are defined in Lemma 4.2. By using integration by parts, we get

$$\begin{aligned}\mathscr{F}_n(y) &= \frac{2}{n!}\frac{1}{y}\int_1^\infty \left(\frac{P'_{n,0}(\log u)}{u^2} - \frac{P_{n,0}(\log u)}{u^2}\right) S_1(yu)\, du\\ &= \frac{2}{n!}\frac{1}{y}\int_1^\infty P_{n,1}(\log u)\frac{S_1(yu)}{u^2}\, du.\end{aligned}$$

Then

$$P_{n,1}(\log u) = P'_{n,0}(\log u) - P_{n,0}(\log u)$$

and so Eq (4.3) is valid for $k = 1$. Now, we assume that Eq (4.3) is valid for any fixed and non-negative integer k. Then we have to prove that it is also valid for $k+1$. By induction hypothesis, we have

$$\mathscr{F}_n(y) = \frac{2}{n!}\frac{1}{y^k}\int_1^\infty P_{n,k}(\log u)\frac{S_k(yu)}{u^{k+1}}\, du.$$

By using integration by parts, and selecting $v = P_{n,k}(\log u)\, u^{-k-1}$ and $dw = S_k(yu)\, du$, we obtain

$$\begin{aligned}\mathscr{F}_n(y) &= \frac{2}{n!}\frac{1}{y^{k+1}}\int_1^\infty \Big(P'_{n,k}(\log u) - (k+1)P_{n,k}(\log u)\Big)\frac{S_{k+1}(yu)}{u^{k+2}}\, du,\\ &= \frac{2}{n!}\frac{1}{y^{k+1}}\int_1^\infty P_{n,k+1}(\log u)\frac{S_{k+1}(yu)}{u^{k+2}}\, du,\end{aligned}$$

which gives us

$$P_{n,k+1}(\log u) = P'_{n,k}(\log u) - kP_{n,k}(\log u).$$

We conclude from the above that Eq (4.3) is valid for $k+1$. Then it is valid for all $k \geq 1$. This completes the proof. ◇◇◇

The second technical result will be provided in the following section (by Theorem 4.11). This theorem will play a crucial role in the proof of our main theorem. Now, we shall state some definitions and Preliminary lemmas without proof (The proof of these results can be found in the paper [8]).

4.3 Generalized harmonic numbers $H(k,\ell)$

4.3.1 Some definitions

Definition 4.5 *Let $s(k,\ell)$ be denote the Stirling numbers of the first kind defined by*

$$(m)_k := \prod_{r=1}^{k}(m-r+1) = \sum_{\ell=0}^{k} s(k,\ell)m^{\ell}.$$

Then we define the generalized harmonic numbers by

$$H(k,\ell) := \frac{c(k+1,\ell+1)}{k!},$$

where denote $c(k,\ell) = |s(k,\ell)|$, i. e., $c(k,\ell)$ is the unsigned Stirling number of the first kind which counts the permutations of k elements that are the product of ℓ disjoint cycles.

Definition 4.6 *A polynomial $P(m_1, m_2, \cdots, m_k)$ in the variables $m_1, m_2, \cdots, m_k$ is called a symmetric polynomial or a symmetric function if it is invariant under all possible permutations of the variables $m_1, m_2, \cdots, m_k$. Especially, important symmetric polynomials that will be considered in this chapter are the elementary symmetric polynomial $\sigma_\ell^{(k)}$ for integer k, ℓ with $k \geq \ell \geq 0$ is defined by:*

$$\sigma_\ell^{(k)}(m_1, m_2, \cdots, m_k) = \sum_{1\leq r_1<r_2<\cdots<r_\ell\leq k} m_{r_1} m_{r_2} \cdots m_{r_\ell}.$$

Definition 4.7 *The generating function $E(t)$ for $\sigma_\ell^{(k)}$ is given by*

$$E(t) = \prod_{i=1}^{k}(1+m_i t) = \sum_{\ell=0}^{k} \sigma_\ell^{(k)}(m_1, m_2, \cdots, m_k)\ t^{\ell}.$$

The unsigned Stirling numbers of the first kind $c(k,\ell)$ is related to $\sigma_\ell^{(k)}$ by

$$c(k,\ell) = \sigma_{k-\ell}^{(k-1)}(1,2,\cdots,k-1).$$

4.3.2 Preliminary lemmas

Lemma 4.8 *For all nonzero real numbers $m_1, m_2, \cdots, m_k$ and for all non-negative integer i, we have*

$$\sigma_i^{(k)}\left(\frac{1}{m_1}, \frac{1}{m_2}, \cdots, \frac{1}{m_k}\right) = \frac{1}{m_1 m_2 \cdots m_k}\sigma_{k-i}^{(k)}(m_1, m_2, \cdots, m_k).$$

As a direct consequence of this lemma, we see that for specific positive integer n, the special case $m_i = i$ for each $i = 1, 2, \cdots, k$ yields:

$$\sigma_i^{(k)}\left(1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{k}\right) = \frac{1}{k!}\,\sigma_{k-i}^{(k)}(1, 2, \cdots, k).$$

Lemma 4.9 *For integers k, ℓ with $k \geq \ell \geq 2$, we have*

$$c(k,\ell) = (k-1)!\,\sigma_{\ell-1}^{(k-1)}\left(1, \frac{1}{2}, \cdots, \frac{1}{k-1}\right).$$

Lemma 4.10 *The generalized harmonic numbers $H(k,\ell)$ satisfy*

$$H(k,\ell) = \sum_{1 \leq m_1 < m_2 < \cdots < m_\ell \leq k} \frac{1}{m_0 \cdots m_\ell}.$$

Then we deduce the following principal result concerning $H(k,\ell)$.

Theorem 4.11 *For $k \geq 1$, we have:*

$$\sum_{\ell=0}^{k} \sum_{1 \leq m_1 < m_2 < \cdots < m_\ell \leq k} \frac{k^\ell}{m_0 \cdots m_\ell} = \binom{2k}{k}.$$

The final lemma needed is the following:

Lemma 4.12 *For all $k \geq 1$, we have :*

$$\frac{(2k)!}{k!} \leq \sqrt{2}\left(\frac{4k}{e}\right)^k. \tag{4.4}$$

Proof. We prove this lemma by using the fact that

$$\left(\frac{k}{e}\right)^k \sqrt{2\pi k}\, e^{1/(12k+1)} \leq k! \leq \left(\frac{k}{e}\right)^k \sqrt{2\pi k}\, e^{1/(12k)},$$

which gives us

$$\frac{(2k)!}{k!} \leq \sqrt{2}\left(\frac{4k}{e}\right)^k e^{\frac{1}{24k} - \frac{1}{12k+1}} \leq \sqrt{2}\left(\frac{4k}{e}\right)^k,$$

as required. ◇◇◇

We are now ready to prove our main theorem.

4.4 Proof of Theorem 0.37

Recall that our function is

$$\mathscr{F}_n(y) = \frac{2}{n!}\int_y^{\infty}\left(\log\frac{u}{y}\right)^n \frac{\cos u}{u}\,du.$$

By Lemma 4.1 and an integration by parts, we obtain

$$\mathscr{F}_n(y) = \frac{2}{n!}\sum_{\ell=0}^{k}\sum_{1\le m_1<m_2<\cdots<m_\ell\le k}\frac{(-1)^{k-\ell}k!}{m_0\cdots m_\ell}\,\frac{n!}{(n-\ell)!}\int_y^{\infty}\left(\log\frac{u}{y}\right)^{n-\ell}\frac{S_k(u)}{u^{k+1}}\,du,$$

where

$$S_k(u) = \begin{cases} i^k\cos u & \text{if } k\equiv 0,2 \pmod 4,\\ i^{k+1}\sin u & \text{if } k\equiv \pm 1 \pmod 4,\end{cases}$$

and where $k\le n$. It follows that

$$|\mathscr{F}_n(y)| \le 2\sum_{\ell=0}^{k}\sum_{1\le m_1<m_2<\cdots<m_\ell\le k}\frac{k!}{m_0\cdots m_\ell}\,\frac{1}{(n-\ell)!}\int_y^{\infty}\left(\log\frac{u}{y}\right)^{n-\ell}\frac{1}{u^{k+1}}\,du.$$

After a change of variable in the integral $\frac{u}{y}=t$, we get

$$|\mathscr{F}_n(y)| \le \frac{2}{y^k}\sum_{\ell=0}^{k}\sum_{1\le m_1<m_2<\cdots<m_\ell\le k}\frac{k!}{m_0\cdots m_\ell}\,\frac{1}{(n-\ell)!}\int_1^{\infty}(\log t)^{n-\ell}\frac{1}{t^{k+1}}\,dt.$$

Since

$$\int_1^{\infty}(\log t)^{n-\ell}\frac{dt}{t^{k+1}} = \frac{(n-\ell)!}{k^{n-\ell+1}},$$

we get

$$|\mathscr{F}_n(y)| \le 2\;k!\;\frac{y^{-k}}{k^{n+1}}\sum_{\ell=0}^{k}\sum_{1\le m_1<m_2<\cdots<m_\ell\le k}\frac{k^\ell}{m_0\cdots m_\ell}.$$

Using Theorem 4.11 yields

$$|\mathscr{F}_n(y)| \le 2\;k!\;\frac{y^{-k}}{k^{n+1}}\binom{2k}{k}.$$

Using Lemma 4.12, we get

$$|\mathscr{F}_n(y)| \le 2\sqrt{2}\,\frac{y^{-k}}{k^{n+1}}\left(\frac{4k}{e}\right)^k.$$

This inequality may be written as

$$|\mathscr{F}_n(y)| \leq 2\sqrt{2}y^{-k} \, \exp\left\{-(n+1)\log k + k\log k + (-1+\log 4)k\right\}. \tag{4.5}$$

Replace y by $2\pi m/q$ and sum both sides of Eq (4.5) from $m=1$ to ∞. We find that

$$\sum_{m\geq 1}\left|\mathscr{F}_n\left(\frac{2\pi m}{q}\right)\right| \leq C(n,k,q)\sum_{m\geq 1}\frac{1}{m^k} \tag{4.6}$$

where

$$C(n,k,q) = 2\sqrt{2}\exp\left\{-(n+1)\log k + k\log k + k\log q + (-1-\log 2\pi + \log 4)k\right\}.$$

Now, let us determine a proper choice an admissible value of k. To do so, we put

$$\frac{d}{dk}C(n,k,q) = C'(n,k,q).$$

We see that the derivative of $C(n,k,q)$ with respect to k verifies

$$\begin{aligned} \frac{C\prime(n,k,q)}{C(n,k,q)} &= -\frac{(n+1)}{k} + \log k + \log q - \log 2\pi + \log 4 \\ &= -\frac{(n+1)}{k} + \log\left(\frac{2qk}{\pi}\right). \end{aligned}$$

The quantity $C(n,k,q)$ is maximal when

$$\frac{(n+1)}{k} = \log\left(\frac{2qk}{\pi}\right).$$

We multiply both sides of this equation by $2qk/\pi$ and put $K = 2qk/\pi$, to get

$$\frac{2q}{\pi}(n+1) = K\log K.$$

By setting $\frac{2q}{\pi}(n+1) = x$, we find that

$$f(K) = K\log K = x.$$

This equation has a unique solution K_0 given by

$$K_0 = \frac{x}{W(x)},$$

where W denotes the Lambert W-function. In addition, K_0 corresponds to a parameter k_0 given by

$$k_0 = \frac{n+1}{W(2q(n+1)/\pi)}.$$

The above analysis shows that $C(n,k,q)$ decreases when $k \le k_0$ and increases afterwards. Letting K_1 denote non-negative real number given by

$$K_1 = \frac{x}{\log x},$$

which corresponds to a parameter k_1 given by

$$k_1 = \frac{(n+1)}{\log \frac{2q(n+1)}{\pi}}.$$

We stress that k_1 is not necessarily an integer. For $x > e$ (this condition is satisfied when $q > 1$. We have to take $n \ge 4$ when $q = 1$), we have

$$f\left(\frac{x}{\log x}\right) = \frac{x}{\log x} \log \frac{x}{\log x} < x,$$

it follows that $k_1 \le k_0$. Then, we get

$$1 \le \left\lfloor \frac{n+1}{\log \frac{2q(n+1)}{\pi}} \right\rfloor \le k_0.$$

Since $C' < 0$ for any integer number less than k_0 and putting

$$k_2 = \frac{n+1}{\log \frac{2q(n+1)}{\pi}} - 1. \tag{4.7}$$

We conclude that

$$C(k_0,n,q) \le C(k_1,n,q) \le C(\lfloor k_1 \rfloor, n, q) \le C(k_2,n,q),$$

when $k_2 > 1$, i.e. when

$$q < \frac{\pi}{2} \frac{e^{(n+1)/2}}{n+1}.$$

We now return to Eq (4.6) and replaces k_2 by $\left(\frac{n+1}{\log \frac{2q(n+1)}{\pi}} - 1\right)$ to deduce that

$$\sum_{m \ge 1} \left| \mathscr{F}_n\left(\frac{2\pi m}{q}\right) \right| \le C(n,q) \sum_{m \ge 1} m^{-\frac{n+1}{\log\left(\frac{2q(n+1)}{\pi}\right)} + 1},$$

where

$$C(n,q) = 2\sqrt{2} \exp\left\{ -(n+1)\log\left(\frac{n+1}{\log \frac{2q(n+1)}{\pi}} - 1\right) + \left(\frac{(n+1)}{\log \frac{2q(n+1)}{\pi}} - 1\right)\left(\log\left(\frac{n+1}{\log \frac{2q(n+1)}{\pi}} - 1\right) + \log \frac{2q}{\pi e}\right)\right\}.$$

For the last sum of this inequality, we write

$$\sum_{m\geq 1}\frac{1}{m^{k_2}}\leq 1+\frac{1}{2^{k_2}}+\int_2^{\infty}\frac{dt}{t^{k_2}}\leq 1+\frac{1}{2^{k_2}}\left(1+\frac{2}{k_2-1}\right)\leq 1+\frac{1}{2^{k_2}}\left(\frac{k_2+1}{k_2-1}\right).$$

Replacing k_2 by $\left(\frac{n+1}{\log\frac{2q(n+1)}{\pi}}-1\right)$, we find that

$$\sum_{m\geq 1}\left|\mathscr{F}_n\left(\frac{2\pi m}{q}\right)\right|\leq C(n,q)\ (1+D(n,q))$$

where

$$D(n,q)=2^{-\frac{n+1}{\log\frac{2q(n+1)}{\pi}}}\left(\frac{n+1}{n-2\log\frac{2q(n+1)}{\pi}+1}\right).$$

Since $k\geq k_2>1$, we have $k\geq 2$. By Eq (4.6), we find that

$$\sum_{m\geq 1}\left|\mathscr{F}_n\left(\frac{2\pi m}{q}\right)\right|\leq C(n,q)\ \frac{\pi^2}{6}$$

Therefore, we conclude that:

$$\sum_{m\geq 1}\left|\mathscr{F}_n\left(\frac{2\pi m}{q}\right)\right|\leq C(n,q)\min\left(1+D(n,q),\frac{\pi^2}{6}\right).$$

From Theorem 3.2 and the fact that $|\tau(\chi)|=\sqrt{q}$, Theorem now readily follows.

	$\|\gamma_n\|$			γ_n	
n	**Matsuoka**	**Adell**	**Our result**	**Kreminski**	**Coffey, Knessl**
2	-	-	0.070245078183852	-	-
3	-	-	0.067719699218508	-	0.00190188
4	-	62.66226985729476870185	0.079906995076743	-	0.00231644
5	-	253.25511937574449246600	0.101671067958902	-	0.000812965
10	0.418944879802952047	21622.67050207189979471	0.714120274361758	$2.0533281 \cdot 10^{-4}$	0.000210539
20	338890.9439428998678	8108016580.566186017960	340.4220669628279	$4.6634356 \cdot 10^{-4}$	0.000471981
30	889109155106.7848077	13920583854580676.03702	858791.0355985352	$3.5577288 \cdot 10^{-3}$	0.00359535
40	4740002165959979407.4	535443396840771326697735.2	5972285271.236439	$2.4873155 \cdot 10^{-1}$	0.251108
100	$2.111389883 \cdot 10^{62}$	$6.131935463 \cdot 10^{65}$	$3.3048214111 \cdot 10^{37}$	$-4.2534015 \cdot 10^{17}$	$-4.2534 \cdot 10^{17}$
150	$9.636492256 \cdot 10^{100}$	$1.201425843 \cdot 10^{104}$	$7.3910426765 \cdot 10^{63}$	$8.0288537 \cdot 10^{35}$	$8.02885 \cdot 10^{35}$
200	$6.723474349 \cdot 10^{140}$	$4.028139634 \cdot 10^{143}$	$7.4716821273 \cdot 10^{91}$	$-6.9746497 \cdot 10^{55}$	$-6.97465 \cdot 10^{55}$
250	$3.262125160 \cdot 10^{181}$	$1.005492856 \cdot 10^{184}$	$1.1136521652 \cdot 10^{121}$	$3.0592128 \cdot 10^{79}$	$3.05921 \cdot 10^{79}$
300	$7.061327217 \cdot 10^{222}$	$1.171684058 \cdot 10^{2251}$	$1.3071668576 \cdot 10^{151}$	$-5.5567282 \cdot 10^{102}$	$-5.55673 \cdot 10^{102}$
400	$1.030854890 \cdot 10^{307}$	$5.475491836 \cdot 10^{308}$	$2.0198164612 \cdot 10^{213}$	$-1.7616421 \cdot 10^{152}$	$-1.76164 \cdot 10^{152}$
500	$5.091823280 \cdot 10^{392}$	$9.480434509 \cdot 10^{393}$	$3.8204382632 \cdot 10^{277}$	$-1.1655052 \cdot 10^{204}$	$-1.16551 \cdot 10^{204}$
600	$3.827762436 \cdot 10^{479}$	$2.665372229 \cdot 10^{480}$	$2.9158215571 \cdot 10^{343}$	$3.5627462 \cdot 10^{257}$	$3.56275 \cdot 10^{257}$
700	$2.624506258 \cdot 10^{567}$	$7.173560533 \cdot 10^{567}$	$4.4254777108 \cdot 10^{410}$	$-3.5494521 \cdot 10^{312}$	$-3.549452 \cdot 10^{312}$
800	$1.150665897 \cdot 10^{656}$	$1.282553972 \cdot 10^{656}$	$8.1932108605 \cdot 10^{478}$	$4.9135405 \cdot 10^{369}$	$4.9135405 \cdot 10^{369}$
1000	$2.172418132 \cdot 10^{835}$	$4.407894620 \cdot 10^{834}$	$1.3320458753 \cdot 10^{618}$	$-1.5709538 \cdot 10^{486}$	$-1.570953 \cdot 10^{486}$
1200	$6.041294880 \cdot 10^{1016}$	$2.443852795 \cdot 10^{1015}$	$1.7273464190 \cdot 10^{760}$	$8.6840299 \cdot 10^{605}$	$8.6840303 \cdot 10^{605}$
1400	$9.747968498 \cdot 10^{1199}$	$8.441567295 \cdot 10^{1197}$	$5.0329835559 \cdot 10^{904}$	$-4.0972873 \cdot 10^{728}$	$-4.097289 \cdot 10^{728}$

Table 4.1: This table compares our upper bound for $|\gamma_n|$ with the ones given by Matsuoka and Adell and also with the exact value computed by Kreminski, Coffey and Knessl.

4.5 An application

Our theorem enable us to approximate $L(z,\chi)$ by a short Taylor polynomial. For instance, we have

Corollary 4.13 *Let χ be a primitive Dirichlet character to modulus q. For $N = 4\log q$ and $q \geq 150$, we have*

$$\left| L(z,\chi) - \sum_{n\leq N} \frac{(-1)^n \gamma_n(\chi)}{n!}(z-1)^n \right| \leq \frac{32.3}{q^{2.5}},$$

where $|z-1| \leq e^{-1}$.

Proof. From Theorem 0.37, we recall that

$$\frac{|\gamma_n(\chi)|}{n!} \leq q^{-1/2}\, C(n,q) \min\left(1 + D(n,q), \frac{\pi^2}{6}\right),$$

with

$$C(n,q) = 2\sqrt{2} \exp\left\{ -(n+1)\log\theta(n,q) + \theta(n,q)\left(\log\theta(n,q) + \log\frac{2q}{\pi e}\right)\right\},$$

and

$$\theta(n,q) = \frac{n+1}{\log\frac{2q(n+1)}{\pi}} - 1, \quad D(n,q) = 2^{-\theta(n,q)-1}\, \frac{\theta(n,q)+1}{\theta(n,q)-1}.$$

For $n+1 \geq 4\log q$, we note that the function $\theta(n,q)$ is non-decreasing function of n, it follows that the function $D(n,q)$ is decreasing function of θ. For $n+1 \geq 4\log q$ and $q \geq 150$ we find that

$$\theta(n,q) \geq \frac{4\log q}{\log\left(\frac{8q\log q}{\pi}\right)} - 1 \geq 1.65 \quad \text{and} \quad D(n,q) \leq 0.65.$$

On the other hand, we have

$$\log\theta(n,q) + \log\frac{2q}{\pi e} \leq \log\left(\frac{\frac{2q(n+1)}{\pi e}}{\log\left(\frac{2q(n+1)}{\pi}\right)}\right).$$

Putting $H = 2q(n+1)/\pi$, we obtain that

$$\theta(n,q)\left(\log\theta(n,q) + \log\frac{2q}{\pi e}\right) \leq \frac{n+1}{\log H}\log\left(\frac{H/e}{\log H}\right).$$

For $H \geq 1.45$, we infer that

$$\theta(n,q)\left(\log\theta(n,q)+\log\frac{2q}{\pi e}\right)\leq n+1,$$

it follows that

$$C(n,q)\leq 2\sqrt{2}\exp\left\{-(n+1)\log\theta(n,q)+(n+1)\right\}=2\sqrt{2}\left(\frac{e}{\theta(n,q)}\right)^{n+1}.$$

For $n+1\geq N$, we have $\theta(n,q)\geq\theta(N,q)$. Then

$$\frac{|\gamma_n(\chi)|}{n!}\leq 3.3\frac{\sqrt{2}}{\sqrt{q}}\left(\frac{e}{\theta(N,q)}\right)^{n+1}.$$

Now, we recall that

$$L(z,\chi)=\sum_{n\geq 1}\frac{(-1)^n\gamma_n(\chi)}{n!}(z-1)^n.$$

Let $\varepsilon>0$ such that $|z-1|\leq\varepsilon$. Then, for $n+1\geq N=4\log q$, we find that

$$\begin{aligned}\left|L(z,\chi)-\sum_{n\leq N-2}\frac{(-1)^n\gamma_n(\chi)}{n!}(z-1)^{n+1}\right| &\leq \sum_{n\geq N-1}\frac{\left|\gamma_n(\chi)\right|}{n!}|z-1|^n\\ &\leq 3.3\frac{\sqrt{2}}{\varepsilon\sqrt{q}}\sum_{n\geq N-1}\left(\frac{e\varepsilon}{\theta(N,q)}\right)^{n+1}\\ &\leq 3.3\frac{\sqrt{2}}{\varepsilon\sqrt{q}}\left(\frac{e\varepsilon}{\theta(N,q)}\right)^{N}\frac{1}{1-\frac{\varepsilon e}{\theta(N,q)}}.\end{aligned}$$

Taking $\varepsilon=e^{-1}$, we get

$$\left|L(z,\chi)-\sum_{n\leq N-2}\frac{(-1)^n\gamma_n(\chi)}{n!}(z-1)^{n+1}\right|\leq 3.3\frac{e\sqrt{2}}{\sqrt{q}}\cdot\frac{1}{q^{4\log\left(\frac{4\log q}{\log(8q\log q/\pi)}-1\right)}}\cdot\frac{1}{1-\frac{1}{1.65}}.$$

For $q\geq 150$, we conclude that:

$$\left|L(z,\chi)-\sum_{n\leq N-2}\frac{(-1)^n\gamma_n(\chi)}{n!}(z-1)^{n+1}\right|\leq\frac{32.3}{q^{2.5}}.$$

◇◇◇

Remark 2 *If we replace* $4\log q$ *by* N *and increase the latter, the approximation becomes better.*

Chapter 5

Matsuoka's formula via Euler-Maclaurin summation formula.

In this chapter, we prove Theorem 3.2 without using the functional equation of $\zeta(z)$ and $L(z,\chi)$. To do so, we begin with a slight modification of Eq when $1 \le a \le q$

$$\gamma_n(a,q) = \lim_{T\to\infty}\left\{\sum_{t=0}^{T}\frac{\log^n(a+qt)}{a+qt} - \frac{\log^{n+1}(a+qT)}{q(n+1)}\right\}.$$

Recall that the Euler-Maclaurin summation formula can be written as :

$$\sum_{t=0}^{T} h_n(a+qt) = \frac{1}{q}\int_a^{a+qT} h_n(t)\,dt + \frac{1}{2}\left(h_n(a)+h_n(a+qT)\right) + \sum_{r=1}^{k}\frac{B_{2r}}{(2r)!}\,q^{2r-1}\left(h_n^{(2r-1)}(a+qT) - h_n^{(2r-1)}(a)\right) + \frac{q^{2k}}{(2k+1)!}\int_a^{a+qT} B_{2k+1}\left(\frac{t-a}{q}\right)\,h_n^{(2k+1)}(t)\,dt,$$

where $B_{k+1}(t)$ are the Bernoulli polynomials and B_k are the Bernoulli numbers, such that $B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}\cdots$ and that $h_n^{(r)}(t)$ being the r-th derivative of the function $h_n(t) = \log^n t/t$. We use this summation in the following way. On taking $k = 1$, we get

$$\sum_{t=0}^{T}\frac{\log^n(a+qt)}{a+qt} - \frac{\log^{n+1}(a+qT)}{q(n+1)} = -\frac{\log^{n+1}a}{q(n+1)} + \frac{\log^n a}{2a} + \frac{\log^n(a+qT)}{2(a+qT)} + \frac{q}{12}\left(n\frac{\log^{n-1}(a+qT)}{(a+qT)^2} - \frac{\log^n(a+qT)}{(a+qT)^2}\right) - \frac{q}{12}\left(n\frac{\log^{n-1}a}{a^2} - \frac{\log^n a}{a^2}\right) + \frac{q^2}{6}\left(\int_1^{a+qT} B_3\left(\frac{t-a}{q}\right)\,h_n^{(3)}(t)\,dt - \int_1^{a} B_3\left(\frac{t-a}{q}\right)\,h_n^{(3)}(t)\,dt\right).$$

Letting $T \to \infty$ gives us

$$\gamma_n(a,q) = -\frac{\log^{n+1} a}{q(n+1)} + \frac{\log^n a}{2a} - \frac{q}{12}\left(n\frac{\log^{n-1} a}{a^2} - \frac{\log^n a}{a^2}\right) + \frac{q^2}{6}\int_1^\infty B_3\left(\frac{t-a}{q}\right)\, h_n^{(3)}(t)\,dt - \frac{q^2}{6}\int_1^a B_3\left(\frac{t-a}{q}\right)\, h_n^{(3)}(t)\,dt$$

Using integration by parts the last integral above, selecting $dv = h_n^{(3)}(t)\,dt$ and $w = B_3(\frac{t-a}{q})$. We obtain

$$\gamma_n(a,q) = -\frac{\log^{n+1} a}{q(n+1)} + \frac{\log^n a}{2a} - \frac{q}{12}\left(n\frac{\log^{n-1} a}{a^2} - \frac{\log^n a}{a^2}\right) + \frac{q^2}{6}\int_1^\infty B_3\left(\frac{t-a}{q}\right)\, h_n^{(3)}(t)\,dt + \frac{q}{2}\int_1^a B_2\left(\frac{t-a}{q}\right)\, h_n^{(2)}(t)\,dt$$

We integrate by parts one more time to get

$$\gamma_n(a,q) = -\frac{\log^{n+1} a}{q(n+1)} + \frac{\log^n a}{2a} + \frac{q^2}{6}\int_1^\infty B_3\left(\frac{t-a}{q}\right)\, h_n^{(3)}(t)\,dt - \int_1^a B_1\left(\frac{t-a}{q}\right)\, h_n^{(1)}(t)\,dt.$$

Since $B_1(t) = t - 1/2$ and $h_n^{(1)}(t) = (n\log^{n-1} t - \log^n t)/t^2$, then

$$\int_1^a B_1\left(\frac{t-a}{q}\right)\, h_n^{(1)}(t)\,dt = \frac{\log^{n+1} a}{q(n+1)} - \frac{\log^n a}{2a}.$$

It follows that

$$\gamma_n(a,q) = \frac{q^2}{6}\int_1^\infty B_3\left(\frac{t-a}{q}\right) h_n^{(3)}(t)\,dt \tag{5.1}$$

where For, $0 \le t \le 1$ we recall that :

$$B_{2k+1}(t) = \frac{(-1)^{k-1}2(2k+1)!}{(2\pi)^{2k+1}}\sum_{m=1}^{\infty}\frac{\sin(2\pi m t)}{m^{2k+1}}$$

and that

$$\sin(u-a) = \sin u \cos a - \cos u \sin a.$$

Thus, the right-hand side of Eq (5.1) may be written as

$$\begin{aligned}\gamma_n(a,q) &= \frac{q^2\cdot 2\cdot 3!}{6(2\pi)^3}\int_1^\infty \sum_{m\geq 1}\frac{\cos\left(\frac{2\pi ma}{q}\right)\sin\left(\frac{2\pi mt}{q}\right)}{m^3}\,h_n^{(3)}(t)\,dt \\ &\qquad - \frac{q^2\cdot 2\cdot 3!}{6(2\pi)^3}\int_1^\infty \sum_{m\geq 1}\frac{\sin\left(\frac{2\pi ma}{q}\right)\cos\left(\frac{2\pi mt}{q}\right)}{m^3}\,h_n^{(3)}(t)\,dt \\ &= \frac{2}{q}\sum_{m\geq 1}\frac{\cos\left(\frac{2\pi ma}{q}\right)}{\left(\frac{2\pi m}{q}\right)^3}\int_1^\infty \sin\left(\frac{2\pi mt}{q}\right)\,h_n^{(3)}(t)\,dt \\ &\qquad - \frac{2}{q}\sum_{m\geq 1}\frac{\sin\left(\frac{2\pi ma}{q}\right)}{\left(\frac{2\pi m}{q}\right)^3}\int_1^\infty \cos\left(\frac{2\pi mt}{q}\right)\,h_n^{(3)}(t)\,dt.\end{aligned}$$

It follows that

$$\gamma_n(a,q) = \frac{1}{q}\sum_{m\geq 1}\left\{\cos\left(\frac{2\pi ma}{q}\right)n!\mathscr{F}_n\left(\frac{2\pi m}{q}\right) + \sin\left(\frac{2\pi ma}{q}\right)n!\tilde{\mathscr{F}}_n\left(\frac{2\pi m}{q}\right)\right\}$$

where $\mathscr{F}_n$ and $\tilde{\mathscr{F}}_n$ are defined by Lemma and (with $k=3$). Now notice that:

$$\sum_{a=1}^{q}\chi(a)\cos\left(\frac{2\pi ma}{q}\right) = \bar{\chi}(m)\frac{\tau(\chi)+\chi(-1)\tau(\chi)}{2},$$

and

$$\sum_{a=1}^{q}\chi(a)\sin\left(\frac{2\pi ma}{q}\right) = \bar{\chi}(m)\frac{\tau(\chi)-\chi(-1)\tau(\chi)}{2i}.$$

Recalling Eq , we get

$$\begin{aligned}\frac{\gamma(\chi)}{n!} &= \frac{\tau(\chi)+\chi(-1)\tau(\chi)}{2q}\sum_{m\geq 1}\bar{\chi}(m)\mathscr{F}_n\left(\frac{2\pi m}{q}\right) \\ &\qquad + \frac{\tau(\chi)-\chi(-1)\tau(\chi)}{2iq}\sum_{m\geq 1}\bar{\chi}(m)\tilde{\mathscr{F}}_n\left(\frac{2\pi m}{q}\right).\end{aligned}$$

In the case χ even, we see that this equation equal to

$$\frac{\gamma_n(\chi)}{n!} = \frac{\tau(\chi)}{q}\sum_{m\geq 1}\bar{\chi}(m)\mathscr{F}_n\left(\frac{2\pi m}{q}\right),$$

In the case χ odd, we find that

$$\frac{\gamma_n(\chi)}{n!} = \frac{\tau(\chi)}{iq}\sum_{m\geq 1}\bar{\chi}(m)\tilde{\mathscr{F}}_n\left(\frac{2\pi m}{q}\right),$$

which completes the proof.

Part II

Explicit upper bounds for $|L(1,\chi)|$ when χ is even and $\chi(2)=1$

Chapter 6

Proof structure

The main goal of this part is proving Theorem 0.39, which give us an upper bound for $\left|L(1,\chi)\right|$ in the most difficult case when $\chi(2)=1$. This upper bound of $\left|L(1,\chi)\right|$ has an equally profound impact on number theory. In particular, in studying the class numbers of algebraic extensions. To do so, we need to prove the following theorem.

Theorem 6.1 *Let χ be a primitive Dirichlet character of conductor $q>1$. Let $F:\mathbb{R}\longrightarrow\mathbb{R}$ be such that $F(t)/t$ is in $C^2(\mathbb{R})$ (also at 0), vanishes at $t=\pm\infty$ and its first and second derivatives belong to $L^1(\mathbb{R})$. Assume further that F is even if χ is odd and that F is odd if χ is even. We define*

$$G_F(u)=\sum_{\ell\geq 0}\frac{1-F(2^\ell u)}{2^\ell}\chi(2)^\ell. \tag{6.1}$$

Suppose that $G_F(u)$ and the derivative of $(G_F(u)-2)/u$ are positive for $u_0\leq u\leq 1$, (u_0 approaches to 0) and that $|1-F(t)|\leq c_0/t^2$, $|F^{'}(t)|\leq c_1/t^2$, $|F(t)|\leq c_2$, $|F(t)|\leq c_3 t$ and $|G_F(t)|\leq 2$ for all $t>0$. Then for any $\delta>0$, we have

$$\sum_{\substack{m\geq 1\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}\leq -\log\delta+b_F+\frac{c_3\theta}{2\log 2}\left(-\log\theta+\log\frac{c_2}{c_3}+\log\log 2+\log 2+2\right)$$
$$+\log(1-\delta)+\frac{(4c_0+7c_1)\delta}{21}+\frac{(12c_0+14c_1)\delta^2}{21}+\frac{\delta^2}{6(1-4\delta)^2}+\frac{\delta^4}{30(1-4\delta)^4}.$$

where θ is u_0 or δ according to whether $\delta\leq u_0$ or $\delta>u_0$. The constant b_F is given by

$$b_F=\frac{1}{2}\int_0^1\frac{G_F(t)-2}{t}\,dt+\frac{1}{2}\int_1^\infty\frac{|G_F(t)|}{t}\,dt+\gamma+\log 2.$$

where γ denote Euler's constant and the constants c_i, $i\in\{1,2,3\}$ are chosen according to the function F.

The proof of this theorem is longer and complex, its requires much of the arsenal of tools and tricks. For this reason, we devote this chapter to state all these tools. We begin by collecting some important results proven in the paper [46].

6.1 Results of Ramaré

In 2001, Ramaré gave the following approximate formula of $L(1,\chi)$ by using Fourier's transforms.

Theorem 6.2 *Let χ be a primitive Dirichlet character of conductor $q>1$. Let $F:\mathbb{R}\longrightarrow\mathbb{R}$ be such that $f(t)=F(t)/t$ is $C^2(\mathbb{R})$ (even at 0), vanishes at $t=\mp\infty$ and its first and second derivatives belong to $L^1(\mathbb{R})$. Assume that further that F is even if χ is odd and that F is odd if χ is even. then for any $\delta>0$, we have*

$$L(1,\chi)=\sum_{n\geq 1}\frac{(1-F(\delta n))\,\chi(n)}{n}+\frac{\chi(-1)\tau(\chi)}{q}\sum_{m\geq 1}\overline{\chi}(m)\int_{-\infty}^{+\infty}\frac{F(t)}{t}\,e\big(mt/(\delta q)\big)\,dt$$

With a proper choice of the function F in the above formula, Ramaré provided the following two formulae of $L(1,\chi)$.

Proposition 6.3 *Set*

$$F_1(t)=\frac{\sin(\pi t)}{\pi}\left(\log 4+\sum_{n\geq 1}(-1)^n\left(\frac{2n}{t^2-n^2}+\frac{2}{n}\right)\right),$$
$$F_2(t)=1-\frac{\sin(\pi t)}{\pi t}.$$

Let χ Dirichlet character of conductor $q>1$. Then, we have

$$L(1,\chi)=\begin{cases}\displaystyle\sum_{n\geq 1}\frac{(1-F_1(\delta n))\,\chi(n)}{n}-\frac{2\tau(\chi)}{q}\sum_{1\leq m\leq \delta q/2}\overline{\chi}(m)\log\left|\sin\frac{\pi m}{\delta q}\right| & si\ \chi(-1)=1,\\ \displaystyle\sum_{n\geq 1}\frac{(1-F_2(\delta n))\,\chi(n)}{n}-\frac{i\pi\tau(\chi)}{q}\sum_{1\leq m\leq \delta q/2}\overline{\chi}(m)\left(1-\frac{2m}{\delta q}\right) & si\ \chi(-1)=-1.\end{cases}$$

Proposition 6.4 *Set*

$$F_3(t)=\left(\frac{\sin(\pi t)}{\pi}\right)^2\left(\frac{2}{t}+\sum_{m\in\mathbb{Z}}\frac{\mathrm{sgn}(m)}{(t-m)^2}\right),$$
$$j(t)=2\int_{|t|}^{1}(\pi(1-u)\cot(\pi u)+1)\,du,$$
$$F_4(t)=1-\left(\frac{\sin(\pi t)}{\pi t}\right)^2.$$

Let χ be a primitive Dirichlet character of conductor $q>1$. Then, we have :

$$L(1,\chi)=\begin{cases}\sum_{n\geq 1}\frac{(1-F_3(\delta n))\chi(n)}{n}-\frac{\tau(\chi)}{q}\sum_{1\leq m\leq \delta q}\overline{\chi}(m)j\left(\frac{m}{\delta q}\right) & \text{if } \chi(-1)=1,\\ \sum_{n\geq 1}\frac{(1-F_4(\delta n))\chi(n)}{n}+\frac{i\pi\tau(\chi)}{q}\sum_{1\leq m\leq \delta q}\overline{\chi}(m)\left(1-\frac{m}{\delta q}\right)^2 & \text{if } \chi(-1)=-1.\end{cases}$$

Taking δ to be around $1/\sqrt{q}$ in the first proposition, Ramaré obtained the following explicit upper bounds for $|L(1,\chi)|$

$$\left|L(1,\chi)\right|\leq\begin{cases}\frac{1}{2}\log q+0.006 & \text{si } \chi(-1)=+1,\\ \frac{1}{2}\log q+0.9 & \text{si } \chi(-1)=-1.\end{cases}$$

By using the second proposition. He obtain the best upper bound for $|L(1,\chi)|$ to date.

$$\left|L(1,\chi)\right|\leq\begin{cases}\frac{1}{2}\log q & \text{si } \chi(-1)=+1,\\ \frac{1}{2}\log q+0.7082 & \text{si } \chi(-1)=-1.\end{cases}$$

To understand the difference between these two results, it is instructive to compare the function F_1 with the function F_3 for even characters, and F_2 with F_4 for odd character. In the next section, we will explain these differences.

Lemma 6.5 *We have*

$$-\sum_{1\leq m\leq \delta q/2}\log\left|\sin\frac{\pi m}{\delta q}\right|\leq\frac{q\delta}{2}\log 2.$$

6.2 Comparing F_1 to F_3

In the late 1930's, A. Beurling observed that the entire function

$$F_1(t)=\frac{\sin(\pi t)}{\pi}\left(\log 4+\sum_{n\geq 1}(-1)^n\left(\frac{2n}{t^2-n^2}+\frac{2}{n}\right)\right)$$

is the best approximation of exponential type π of $f(t)=\text{sgn}(t)$ with

$$\int_{-\infty}^{\infty}\left|F_1(t)-\text{sgn}(t)\right|dt=1.$$

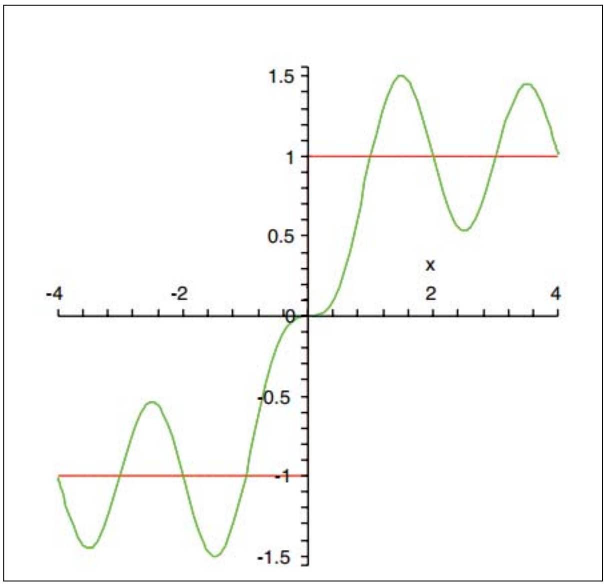

Figure 6.1: Function F_1

The function

$$F_3(t) = \left(\frac{\sin(\pi t)}{\pi}\right)^2 \left(\frac{2}{t} + \sum_{n\in\mathbb{Z}} \frac{\mathrm{sgn}(n)}{(t-n)^2}\right)$$

interpolates $\mathrm{sgn}(t)$ at the integers and its derivative at the non zero integers, i.e., $F_3(n) = \mathrm{sgn}(n)$ and $F_3'(n) = \mathrm{sgn}'(n) = 0$, but it is not a majorizing function of $\mathrm{sgn}(x)$. with

$$\int_{-\infty}^{\infty} \{F_3(t) - \mathrm{sgn}(t)\}\, dt = 1.$$

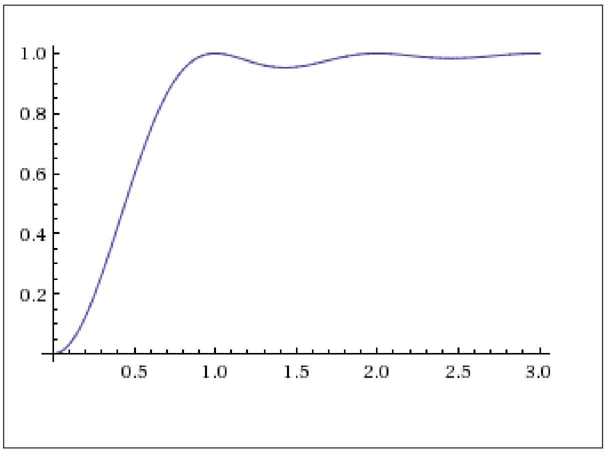

Figure 6.2: Function F_3

We note that the function $1-F_1$ is non-positive on $\mathbb{R}_+$, while that $1-F_3$ is positive on $\mathbb{R}_+$. This shows that the part of the loss of the first upper bound for $\left|L(1,\chi)\right|$ due to study $\sum_n |1-F_1(\delta n)|/n$ instead of $\sum_n (1-F_1(\delta n))/n$. Ramaré eliminated this loss by taking F_3. A similar discussion, we can see the difference between the function F_2 with F_4. To improve the upper bound for $\left|L(1,\chi)\right|$ due to Ramaré, we will study $\sum_n (1-F_1(\delta n))/n$ directly. To do so, we will need to observe the behaviour of the functions F_1 and then G_{F_1}. The following two sections state the most important results of these functions.

6.3 Study of F_1

The fundamental property of the function F_1 is given in the following lemma.

Lemma 6.6 *The function* $F_1(t)$ *satisfies*

$$0 \le \operatorname{sgn}(\sin \pi t)\left\{\operatorname{sgn}(t) - F_1(t)\right\} \le \left|\frac{\sin \pi t}{\pi t}\right| \frac{1}{1+|t|},$$

Proof. See [54]. ◇◇◇

A simple consequence of this lemma is the following result.

Lemma 6.7 *For* t *real number, we have*

$$0 \le \operatorname{sgn}(\sin(\pi t))\,(1-F_1(t)) \le |1/(\pi t)|, \qquad \left|F'(t)\right| \ll t^{-2}.$$

Proof. See [46, Lemma 12]. ◇◇◇

Lemma 6.8 *For* $t \ge 0$*, we have*

$$|F_1(t)| \le (\log 4)\, t.$$

Proof. We recall that

$$\frac{F_1(t)}{t} = -2\int_{-1/2}^{1/2} \log|\sin(\pi u)|\, e(-tu)\, du,$$

which gives

$$\frac{F_1(t)}{t} = -2\int_{-1/2}^{1/2} \log|\sin(\pi u)| \cos(2\pi t u)\, du + 2i\int_{-1/2}^{1/2} \log|\sin(\pi u)| \sin(2\pi t u)\, du. \quad (6.2)$$

We note that the second integral in the right-hand side of Eq (6.2) is equal to 0 because the function $\log|\sin(\pi u)|\sin(2\pi t u)$ is odd. Hence

$$\frac{F_1(t)}{t} = -4\int_{0}^{1/2} \log|\sin(\pi u)| \cos(2\pi t u)\, du,$$

it follows that

$$\frac{|F_1(t)|}{t} \leq 4\left|\int_0^{1/2} \log|\sin(\pi u)|\cos(2\pi t u)\,du\right| \leq 4\int_0^{1/2} \log|\sin(\pi u)|\,du = \log 4.$$

This completes the proof. ⋄⋄⋄

6.4 Study of G_{F_1}

In this section, we collect the most important results of G_{F_1}.

Lemma 6.9 *For $u \geq 0$, we have $\left|G_{F_1}(u)\right| \leq 2$.*

Proof. From Lemma 6.6, we get

$$\left|1 - F_1(2^\ell u)\right| \leq \frac{|\sin(2^\ell \pi u)|}{2^\ell \pi u}\frac{1}{1+2^\ell u} \leq 1$$

it follows that:

$$\left|G_{F_1}(u)\right| \leq \sum_{\ell\geq 0} \frac{\left|1 - F_1(2^\ell u)\right|}{2^\ell} \leq \sum_{\ell \geq 0}\frac{1}{2^\ell} = 2.$$

This completes the proof. ⋄⋄⋄

Lemma 6.10 *For $10^{-5} \leq u \leq 1$, we have $G_{F_1}(u)$ is positive.*

Proof. Recall that :

$$G_{F_1}(u) = \sum_{\ell\geq 0}\frac{1 - F_1(2^\ell u)}{2^\ell} = 2 - \sum_{\ell\geq 0}\frac{F_1(2^\ell u)}{2^\ell}$$

where

$$\begin{aligned}\frac{F_1(t)}{t} &= -2\int_{-1/2}^{1/2} \log|\sin(\pi v)|\,e(tv)\,dv = -4\int_0^{1/2}\log|\sin(\pi v)|\cos(2\pi t v)\,dv \\ &= \frac{2}{t}\int_0^{1/2}\cot(\pi v)\sin(2\pi t v)\,dv.\end{aligned}$$

Then, we find that

$$G_{F_1}(u) = 2 - 2\sum_{\ell\geq 0}\int_0^{1/2}\cot(\pi v)\frac{\sin(2^{\ell+1}\pi u v)}{2^\ell}\,dv, \tag{6.3}$$

and that

$$\frac{d}{du}\big(G_{F_1}(u)\big) = -4\pi\sum_{\ell\geq 0}\int_0^{1/2} v\cot(\pi v)\cos(2^{\ell+1}\pi u v)\,dv.$$

Using integration by parts we get

$$\frac{d}{du}\big(G_{F_1}(u)\big) = \frac{2}{u}\int_0^{1/2}\left(\cot(\pi v)-\frac{\pi v}{\sin^2(\pi v)}\right)\sum_{\ell\geq 0}\frac{\sin(2^{\ell+1}\pi u v)}{2^\ell}\,dv.$$

For $L \geq 0$, we write this sum as follows:

$$\frac{d}{du}\big(G_{F_1}(u)\big) = \frac{2}{u}\int_0^{1/2}\psi(v)\sum_{\ell\leq L}\frac{\sin(2^{\ell+1}\pi u v)}{2^\ell}\,dv + \frac{2}{u}\int_0^{1/2}\psi(v)\sum_{\ell\geq L+1}\frac{\sin(2^{\ell+1}\pi u v)}{2^\ell}\,dv \tag{6.4}$$

where $\psi(v) = \cot(\pi v) - (\pi v)/\sin^2(\pi v)$ is negative function. Suppose that:

$$S_L(u) = \int_0^{1/2}\psi(v)\sum_{\ell\leq L}\frac{\sin(2^{\ell+1}\pi u v)}{2^\ell}\,dv, \qquad R_L(u) = \int_0^{1/2}\psi(v)\sum_{\ell\geq L+1}\frac{\sin(2^{\ell+1}\pi u v)}{2^\ell}\,dv.$$

Then, we get

$$\left|\int_0^{1/2}\psi(v)\sum_{\ell\geq L+1}\frac{\sin(2^{\ell+1}\pi u v)}{2^\ell}\,dv\right| \leq \sum_{\ell\geq L+1}\int_0^{1/2^{\ell+1}}\big|\psi(v)\big|2\pi u v\,dv + \sum_{\ell\geq L+1}\int_{1/2^{\ell+1}}^{1/2}\frac{\big|\psi(v)\big|}{2^\ell}\,dv.$$

Since $|\sin(2\pi v) - 2\pi v| \leq (2\pi v)^3/3!$ and $\sin(\pi v) \geq 2v$, for all $0 \leq v \leq 1/2$, we get $|\psi(v)| \leq \pi^3 v/6$. It follows that

$$\begin{aligned}\left|\int_0^{1/2}\psi(v)\sum_{\ell\geq L+1}\frac{\sin(2^{\ell+1}\pi u v)}{2^\ell}\,dv\right| &\leq \frac{\pi^4 u}{9\cdot 8}\sum_{\ell\geq L+1}\frac{1}{2^{3\ell}} + \frac{\pi^3}{6\cdot 8}\sum_{\ell\geq L+1}\left(\frac{1}{2^\ell}-\frac{1}{2^{3\ell}}\right)\\ &\leq \frac{\pi^4 u}{9\cdot 8\cdot 7\cdot 2^{3L}} + \frac{\pi^3}{6\cdot 8}\left(\frac{1}{2^L}-\frac{1}{7\cdot 2^{3L}}\right).\end{aligned}$$

Therefore

$$|R_L(u)| \leq \frac{\pi^4 u}{504\cdot 2^{3L}} + \frac{\pi^3}{48}\left(\frac{1}{2^L}-\frac{1}{7\cdot 2^{3L}}\right).$$

For $10^{-5} \leq u \leq 1$ and $L = 15$, the maxima of $S_L(u) + |R_L(u)|$ numerically seems non-increasing and GP/PARI needs at most 10 second to prove it is ≤ -0.0001353. Since $G_{F_1}(1) = 0$, it follows that G_{F_1} is positive. This completes the proof. ⋄⋄⋄

Lemma 6.11 *For* $10^{-5} \leq u \leq 1$, *we have*

$$\frac{d}{du}\left(\frac{G_{F_1}(u)-2}{u}\right) \geq 0$$

Proof. Recall that :

$$G_{F_1}(u) = \sum_{\ell\geq 0} \frac{1-F_1(2^\ell u)}{2^\ell},$$

where

$$\frac{F_1(t)}{t} = -2\int_{-\frac{1}{2}}^{\frac{1}{2}} \log|\sin(\pi v)|\, e(-tv)\, dv = \frac{-1}{it}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cot(\pi v) e(-tv)\, dv.$$

Then, we get

$$\begin{aligned} \frac{d}{du}\left(\frac{G_{F_1}(u)-2}{u}\right) = -\frac{d}{du}\left(\sum_{\ell\geq 0}\frac{F_1(2^\ell u)}{2^\ell u}\right) &= -\sum_{\ell\geq 0}\frac{d}{du}\left(\frac{-1}{i2^\ell u}\int_{-\frac{1}{2}}^{\frac{1}{2}}\cot(\pi v)e(-2^\ell uv)\,dv\right)\\ &= \frac{1}{i2^\ell}\sum_{\ell\geq 0}\frac{d}{du}\left(\frac{1}{u}\int_{-\frac{1}{2}}^{\frac{1}{2}}\cot(\pi v)e(-2^\ell uv)\,dv\right), \end{aligned}$$

which gives us

$$\frac{d}{du}\left(\frac{G_{F_1}(u)-2}{u}\right) = -\sum_{\ell\geq 0}\frac{1}{i2^\ell u^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\cot(\pi v)e(-2^\ell uv)\,dv - \sum_{\ell\geq 0}\frac{2^{\ell+1}\pi i}{2^\ell iu}\int_{-\frac{1}{2}}^{\frac{1}{2}} v\cot(\pi v)e(-2^\ell uv)\,dv.$$

It follows that

$$\frac{d}{du}\left(\frac{G_{F_1}(u)-2}{u}\right) = -\frac{1}{iu^2}\sum_{\ell\geq 0}\frac{1}{2^\ell}\int_{-\frac{1}{2}}^{\frac{1}{2}}\cot(\pi v)e(-2^\ell uv)\,dv - \frac{2\pi}{u}\sum_{\ell\geq 0}\int_{-\frac{1}{2}}^{\frac{1}{2}} v\cot(\pi v)e(-2^\ell uv)\,dv.$$

Since $e(-2^\ell uv) = \cos(2^{\ell+1}\pi uv) - i\sin(2^{\ell+1}\pi uv)$ and the functions $\cot(\pi v)\cos(2^{\ell+1}\pi uv)$ and $v\cot(\pi v)\sin(2^{\ell+1}\pi uv)$ are odd, it follows that

$$\frac{d}{du}\left(\frac{G_{F_1}(u)-2}{u}\right) = \frac{2}{u^2}\sum_{\ell\geq 0}\frac{1}{2^\ell}\int_0^{\frac{1}{2}}\cot(\pi v)\sin(2^{\ell+1}\pi uv)\,dv - \frac{4\pi}{u}\sum_{\ell\geq 0}\int_0^{\frac{1}{2}} v\cot(\pi v)\cos(2^{\ell+1}\pi uv)\,dv.$$

Using integration by parts for the last integral above, we obtain that:

$$\begin{aligned} -\sum_{\ell\geq 0}\frac{d}{du}\left(\frac{F_1(2^\ell u)}{2^\ell u}\right) &= \frac{4}{u^2}\sum_{\ell\geq 0}\frac{1}{2^\ell}\int_0^{\frac{1}{2}}\left(\cot(\pi v) - \frac{\pi v}{2\sin^2(\pi v)}\right)\sin(2^{\ell+1}\pi uv)\,dv\\ &= \frac{4}{u^2}\sum_{\ell\geq 0}\frac{1}{2^\ell}\int_0^{\frac{1}{2}}\varphi(v)\sin(2^{\ell+1}\pi uv)\,dv\\ &= \frac{4}{u^2}\int_0^{\frac{1}{2}}\sum_{\ell\geq 0}\frac{\sin(2^{\ell+1}\pi uv)}{2^\ell}\varphi(v)\,dv. \end{aligned}$$

Therefore

$$\frac{d}{du}\left(\frac{G_{F_1}(u)-2}{u}\right)=-\sum_{\ell\geq 0}\frac{d}{du}\left(\frac{F_1(2^\ell u)}{2^\ell u}\right)=\frac{4}{u^2}\int_0^{\frac12}\sum_{\ell\geq 0}\frac{\sin(2^{\ell+1}\pi uv)}{2^\ell}\varphi(v)\,dv \tag{6.5}$$

where $\varphi(v)=\cot(\pi v)-\pi v/(2\sin^2(\pi v))$. Then to prove that this derivative is positive, we divide this sum into two sums as the following:

$$\sum_{\ell\geq 0}\frac{\sin(2^{\ell+1}\pi uv)}{2^\ell}=\sum_{0\leq \ell\leq L}\frac{\sin(2^{\ell+1}\pi uv)}{2^\ell}+\sum_{\ell\geq L+1}\frac{\sin(2^{\ell+1}\pi uv)}{2^\ell}.$$

Thus, Eq (6.5) can be written as follows

$$\frac{d}{du}\left(\frac{G_{F_1}(u)-2}{u}\right)=$$
$$\frac{4}{u^2}\int_0^{1/2}\varphi(v)\sum_{0\leq \ell\leq L}\frac{\sin(2^{\ell+1}\pi uv)}{2^\ell}\,dv+\frac{4}{u^2}\int_0^{1/2}\varphi(v)\sum_{\ell\geq L+1}\frac{\sin(2^{\ell+1}\pi uv)}{2^\ell}\,dv. \tag{6.6}$$

Suppose that

$$\tilde{S}_L(u)=\int_0^{1/2}\varphi(v)\sum_{0\leq \ell\leq L}\frac{\sin(2^{\ell+1}\pi uv)}{2^\ell}\,dv,\qquad \tilde{R}_L(u)=\int_0^{1/2}\varphi(v)\sum_{\ell\geq L+1}\frac{\sin(2^{\ell+1}\pi uv)}{2^\ell}\,dv.$$

Then, we get

$$\begin{aligned}\left|\tilde{R}_L(u)\right| &= \left|\int_0^{1/2}\varphi(v)\sum_{\ell\geq L+1}\frac{\sin(2^{\ell+1}\pi uv)}{2^\ell}\,dv\right|\\ &\leq \sum_{\ell\geq L+1}\int_0^{1/2^{\ell+1}}\left|\varphi(v)\right|2\pi uv\,dv+\sum_{\ell\geq L+1}\int_{1/2^{\ell+1}}^{1/2}\frac{\left|\varphi(v)\right|}{2^\ell}\,dv.\end{aligned}$$

Since $|\sin(2\pi v)-\pi v|\leq \pi v$ and $\sin(\pi v)\geq 2v$, for all $0\leq v\leq 1/2$, we get $|\varphi(v)|\leq \pi/(8v)$. It follows that

$$\begin{aligned}\left|\int_0^{1/2}\varphi(v)\sum_{\ell\geq L+1}\frac{\sin(2^{\ell+1}\pi uv)}{2^\ell}\,dv\right| &\leq \frac{\pi^2 u}{8}\sum_{\ell\geq L+1}\frac{1}{2^\ell}+\frac{\pi\log 2}{8}\sum_{\ell\geq L+1}\frac{\ell}{2^\ell}\\ &\leq \frac{\pi^2 u}{2^{L+3}}+\frac{\pi(L+2)\log 2}{2^{L+3}}.\end{aligned}$$

Therefore

$$\left|\tilde{R}_L(u)\right|\leq \frac{\pi^2 u}{2^{L+3}}+\frac{\pi(L+2)\log 2}{2^{L+3}}.$$

For $10^{-5}\leq u\leq 1$ and $L=21$, the minima of $\tilde{S}_L(u)+\left|\tilde{R}_L(u)\right|$ numerically seems increasing and GP/PARI needs at most 10 second to prove it is ≤ 0.0000019. Then, the derivative of $\left(G_{F_1}-2\right)/u$ is positive. This completes the proof. ⋄⋄⋄

Lemma 6.12 *Let b_{F_1} be the function given by the following equation:*

$$b_{F_1} = \frac{1}{2}\int_0^1 \frac{G_{F_1}(t)-2}{t}\,dt + \frac{1}{2}\int_1^\infty \frac{|G_{F_1}(t)|}{t}\,dt + \gamma + \log 2. \tag{6.7}$$

Then

$$b_{F_1} \le -0.66266.$$

Proof. Recall that

$$F_1(t) = \frac{\sin(\pi t)}{\pi}\left(\log 4 + \sum_{n\ge 1}(-1)^n\left(\frac{2n}{t^2-n^2}+\frac{2}{n}\right)\right),$$

it follows that

$$F_1(t) = \frac{\sin(\pi t)}{\pi}\left\{\log 4 + \sum_{n\ge 1}(-1)^n\left(\frac{1}{t-n}+\frac{1}{n}\right) - \sum_{n\ge 1}(-1)^n\left(\frac{1}{t+n}-\frac{1}{n}\right)\right\}.$$

Thanks to the formula (2.2) is showed in [54], when $F=1$, we write

$$\frac{\pi}{\sin(\pi t)} = \frac{1}{t} + \sum_{n\ge 1}(-1)^n\left(\frac{1}{t-n}+\frac{1}{n}\right) + \sum_{n\ge 1}(-1)^n\left(\frac{1}{t+n}-\frac{1}{n}\right),$$

it follows that

$$\begin{aligned} F_1(t) &= \frac{\sin(\pi t)}{\pi}\left\{\log 4 + \frac{\pi}{\sin(\pi t)} - \frac{1}{t} - 2\sum_{n\ge 1}(-1)^n\left(\frac{1}{t+n}-\frac{1}{n}\right)\right\} \\ &= 1 + \frac{\sin(\pi t)}{\pi}\left\{\log 4 - \frac{1}{t} - 2\sum_{n\ge 1}(-1)^n\left(\frac{-t}{n(t+n)}\right)\right\} \end{aligned}$$

We therefore conclude from this last expression that

$$G_{F_1}(u) = -\sum_{\ell\ge 0}\frac{\sin(\pi 2^\ell u)}{2^\ell \pi}\left\{\log 4 - \frac{1}{2^\ell u} + 2\sum_{n\ge 1}(-1)^n\left(\frac{2^\ell u}{n(2^\ell u+n)}\right)\right\}. \tag{6.8}$$

Pour $T=1000$ and $L=50$, we find that:

$$\left|-\frac{1}{2}\int_0^1 \sum_{l\ge L+1}\frac{F_1(2^l t)}{2^l t}\,dt + \frac{1}{2}\int_1^T \sum_{l\ge L+1}\frac{1-F_1(2^l t)}{2^l t}\,dt + \frac{1}{2}\int_T^\infty \sum_{l\ge 0}\frac{1-F_1(2^l t)}{2^l t}\,dt\right| \le 0.00022$$

Now, we use the following simple GP-PARI code:

```
{F1(t) =
   1+sin(Pi*t)/Pi*(log(4)-1/t+2*t*sumalt(n=1,cos(Pi*n)/n/(t+n)))
   }
```



```
{G(x, borne=50) =
 sum(l=0, borne, (1-F1(2^l*x))/2^l)
}

{GG(x, borne=50) =
   sum(l=0, borne,-F1(2^l*x)/2^l)
   }

{bF1( borne=50, bornex=1000) =
    Euler+log(2)
   +1/2*intnum(x=0, 1, GG(x, borne)/x)
     +1/2*intnum(x=1, bornex, abs(G(x, borne))/x)
   }

 default(realprecision,200)
```

We conclude that $b_{F_1} \le -0.66266$, which completes the proof. ◇◇◇

Lemma 6.13 *Under the same hypotheses of Theorem 6.1, we have*

$$\max_{T\le u}\left|\frac{d}{du}\frac{G_F(u)}{u}\right| \le \frac{8c_0}{7T^4}+\frac{4c_1}{3T^3}.$$

where the constants c_0 and c_1 are chosen According to the function F.

Proof. Notice that

$$\frac{d}{du}\left(\frac{G_F(u)}{u}\right)=\frac{G_F'(u)}{u}-\frac{G_F(u)}{u^2},$$

where $G_F(u)$ is defined by Eq (6.1). Then we find that

$$\left|\frac{d}{du}\frac{G_F(u)}{u}\right| \le \frac{|G'_F(u)|}{u}+\frac{|G_F(u)|}{u^2}.$$

From our hypotheses, it follows that the series $\sum_{\ell\ge0}\frac{1-F(2^\ell u)}{2^\ell u^2}$ converges normally as well as the series $\sum_{\ell\ge0}\frac{(1-F(2^\ell u))'}{2^\ell u}$ is uniform for $u\in[\lambda,\infty[$ for any $\lambda>0$. We infer that

$$\frac{G'_F(u)}{u}=\sum_{\ell\ge0}\frac{\left(1-F(2^\ell u)\right)'}{2^\ell u}=-\sum_{\ell\ge0}\frac{F'(2^\ell u)}{u},$$

and that

$$\left|\frac{G'_F(u)}{u}\right| \le \sum_{\ell\ge0}\frac{\left|F'(2^\ell u)\right|}{u} \le \sum_{\ell\ge0}\frac{c_1}{2^{2\ell}u^3}=\frac{4c_1}{3u^3}$$

$$\frac{|G_F(u)|}{u^2} \le \sum_{\ell\ge0}\frac{|1-F(2^\ell u)|}{2^\ell u^2} \le \sum_{\ell\ge0}\frac{c_0}{2^{3\ell}u^4}=\frac{8c_0}{7u^4}.$$

Hence

$$\max_{T\le u}\left|\frac{d}{du}\frac{G_F(u)}{u}\right| \le \frac{8c_0}{7T^4}+\frac{4c_1}{3T^3}.$$

This completes the proof. ◇◇◇

Lemma 6.14 *Under the same hypotheses of Theorem 6.1. we have*

$$\int_0^{u_0}\left|\frac{G_F(u)-2}{u}\right| du \le \frac{c_3 u_0}{\log 2}\left(-\log u_0+\log\frac{c_2}{c_3}+\log\log 2+\log 2+2\right),$$

where c_2 and c_3 are positive numbers to be chosen later.

Proof. Recall that

$$\frac{G_F(u)-2}{u} = -\sum_{\ell\ge 0}\frac{F(2^\ell u)}{2^\ell u}$$

For $L\ge 0$, we write the sum in right-hand side above as follows :

$$\sum_{\ell\ge 0}\frac{F(2^\ell u)}{2^\ell} = \sum_{\ell\le L}\frac{F(2^\ell u)}{2^\ell}+\sum_{\ell\ge L+1}\frac{F(2^\ell u)}{2^\ell}.$$

Since $|F(t)|\le c_2$ and $|F(t)|\le c_3 t$, we get

$$\begin{aligned}\left|\sum_{\ell\ge 0}\frac{F(2^\ell u)}{2^\ell}\right| &\le \sum_{\ell\le L}\frac{\left|F(2^\ell u)\right|}{2^\ell}+\sum_{\ell\ge L+1}\frac{\left|F(2^\ell u)\right|}{2^\ell}\\ &\le c_3(L+1)u+\frac{c_2}{2^L}.\end{aligned}$$

The best value of L is given by

$$2^L = \frac{c_2\log 2}{c_3 u}.$$

Then, we get

$$\left|\sum_{\ell\ge 0}\frac{F(2^\ell u)}{2^\ell}\right| \le \frac{c_3 u}{\log 2}\left(1+\log\frac{c_2}{c_3}+\log\log 2+\log 2-\log u\right).$$

This yields

$$\int_0^{u_0}\left|\frac{G_F(u)-2}{u}\right| du \le \frac{c_3 u_0}{\log 2}\left(-\log u_0+\log\frac{c_2}{c_3}+\log\log 2+\log 2+2\right),$$

which completes the proof. ◇◇◇

Now, we record the following simple lemmas, but useful results to complete our proof of Theorem 6.1.

6.5 Preliminary lemmas

Lemma 6.15 *For any integer and even number $M \geq 1$, we have*

$$\sum_{\substack{m\leq M\\(m,2)=1}} \frac{1}{m} = \frac{1}{2}\left(\log M+\gamma+\log 2\right)+\frac{1}{12M^2}+\frac{2\Theta_M}{15M^4}$$

where $\Theta_M \in [-1, 1/8]$.

Proof. It is well known that

$$\sum_{m\leq M} \frac{1}{m} - \log M = \gamma + \frac{1}{2M} - \frac{1}{12M^2} + \frac{\theta_M}{60M^4} \tag{6.9}$$

where γ is the Euler constant and $\theta_M \in [0,1]$. Then the sum over all positive even integers less than M is

$$\sum_{\substack{m\leq M\\m=2k}} \frac{1}{m} - \frac{1}{2}\log M = \sum_{k\leq M/2} \frac{1}{2k} - \frac{1}{2}\log M = \frac{1}{2}\left(\sum_{k\leq M/2}\frac{1}{k} - \log\frac{M}{2}\right) - \frac{1}{2}\log 2$$

Apply Eq (6.9) to the last sum above, to obtain

$$\begin{aligned}\sum_{\substack{m\leq M\\m=2k}} \frac{1}{m} - \frac{1}{2}\log M &= \frac{1}{2}\left(\gamma+\frac{1}{M}-\frac{1}{3M^2}+\frac{2^4\theta^*_{M/2}}{60M^4}\right)-\frac{1}{2}\log 2\\ &= \frac{1}{2}\gamma+\frac{1}{2M}-\frac{1}{6M^2}+\frac{2^3\theta^*_{M/2}}{60M^4}-\frac{1}{2}\log 2,\end{aligned}$$

where $\theta^*_{M/2} \in [0,1]$. Therefore

$$\sum_{\substack{m\leq M\\m=2k}} \frac{1}{m} - \frac{1}{2}\log M = \frac{1}{2}\gamma+\frac{1}{2M}-\frac{1}{6M^2}+\frac{2^3\theta^*_{M/2}}{60M^4}-\frac{1}{2}\log 2. \tag{6.10}$$

On the other hand, we have

$$\sum_{m\leq M}\frac{1}{m} = \sum_{\substack{m\leq M\\m=2k}}\frac{1}{m} + \sum_{\substack{m\leq M\\m=2k+1}}\frac{1}{m}.$$

Apply Eq (6.9) and Eq (6.10), to obtain

$$\begin{aligned}\sum_{\substack{m\leq M\\m=2k+1}}\frac{1}{m} &= \frac{1}{2}\left(\log M+\gamma+\log 2\right)+\frac{1}{12M^2}+\frac{\theta_M-2^3\theta^*_{M/2}}{60M^4}\\ &= \frac{1}{2}\left(\log M+\gamma+\log 2\right)+\frac{1}{12M^2}+\frac{8\Theta_M}{60M^4},\end{aligned}$$

where $\Theta_M = \theta_M/8 - \theta^*_{M/2} \in [-1, 1/8]$. This completes the proof. ◇◇◇

Lemma 6.16 *For $M \geq 1$, we have*

$$\sum_{\substack{m \geq M \\ (m,2)=1}} \frac{1}{m^3} = \frac{1}{4M^2} + \mathcal{O}^*\left(\frac{1}{2M^3}\right)$$

and

$$\sum_{\substack{m \geq M \\ (m,2)=1}} \frac{1}{m^4} = \frac{1}{6M^3} + \mathcal{O}^*\left(\frac{1}{2M^4}\right)$$

Proof. Notice that

$$\begin{aligned}
\sum_{\substack{m \geq M \\ (m,2)=1}} \frac{1}{m^3} &= 3 \sum_{\substack{m \geq M \\ (m,2)=1}} \int_m^\infty \frac{dt}{t^4} = 3 \int_M^\infty \sum_{\substack{M \leq m \leq t \\ (m,2)=1}} \frac{dt}{t^4} \\
&= \frac{3}{2} \int_M^\infty \frac{[t]-[M]}{t^4}\, dt \\
&= \frac{3}{2}\left(\int_M^\infty \frac{dt}{t^3} - \int_M^\infty \frac{\{t\}}{t^4}\, dt - M \int_M^\infty \frac{dt}{t^4} + \{M\} \int_M^\infty \frac{dt}{t^4}\right)
\end{aligned}$$

As usual, $\{x\}$ and $[x]$ denote The fractional and integer part of real x. Since $0 \leq \{x\} < 1$, then we obtain that:

$$0 \leq \int_M^\infty \frac{\{t\}}{t^4}\, dt \leq \frac{1}{3M^3}$$

and that

$$0 \leq \int_M^\infty \frac{\{M\}}{t^4}\, dt \leq \frac{1}{3M^3}.$$

This yields

$$\left|\int_M^\infty \frac{\{M\}-\{t\}}{t^4}\, dt\right| \leq \frac{1}{3M^3}.$$

We conclude that

$$\sum_{\substack{m \geq M \\ (m,2)=1}} \frac{1}{m^3} = \frac{1}{4M^2} + \mathcal{O}^*\left(\frac{1}{2M^3}\right).$$

By a similar argument, we prove the second equality of this Lemma. ◇◇◇

Chapter 7

Proofs

After Chapter 6, we are ready to prove our main Theorem 6.1.

7.1 Proof of Theorem 6.1

First, writing n in the form $2^\ell m$ where m is odd integer and under suitable condition $\chi(2)=1$. We may write

$$\sum_{n\geq 1}\chi(n)\frac{1-F(\delta n)}{n}=\sum_{\substack{m\geq 1\\(m,2)=1}}\frac{\chi(m)}{m}\sum_{\ell\geq 0}\frac{1-F(\delta 2^\ell m)}{2^\ell}, \tag{7.1}$$

where the function F satisfies our hypotheses. Recall that:

$$\sum_{\ell\geq 0}\frac{1-F(\delta 2^\ell m)}{2^\ell}=G_F(\delta m).$$

Then, we can write Eq (7.1) in the form:

$$\sum_{n\geq 1}\chi(n)\frac{1-F(\delta n)}{n}=\sum_{\substack{m\geq 1\\(m,2)=1}}\frac{\chi(m)}{m}G_F(\delta m)$$

Now, we compare this sum to an integral. Here we need to consider two cases of $[\delta^{-1}]$.

1. The first case when $[\delta^{-1}]$ is odd. We take $M=[\delta^{-1}]$ to get

$$\delta\sum_{\substack{m\geq M\\(m,2)=1}}\frac{|G_F(\delta m)|}{\delta m}=\frac{1}{2}\int_{\delta M}^{\infty}\frac{|G_F(t)|}{t}\,dt-\frac{1}{2}\sum_{\substack{m\geq M\\(m,2)=1}}\int_{\delta m}^{\delta(m+2)}\left(\frac{|G_F(t)|}{t}-\frac{|G_F(\delta m)|}{\delta m}\right)dt. \tag{7.2}$$

Concerning the inner integral on the far right-hand side above, we notice that:

$$\left|\frac{|G_F(\delta m)|}{\delta m} - \frac{|G_F(t)|}{t}\right| \le (t-\delta m)\max_{\delta m \le u}\left|\frac{d}{du}\frac{G_F(u)}{u}\right|.$$

Applying Lemma 6.13 and using the identity

$$\int_{\delta m}^{\delta(m+2)} (t-\delta m)\,dt = 2\delta^2, \tag{7.3}$$

we get

$$\begin{aligned}\sum_{\substack{m\ge M\\(m,2)=1}} \int_{\delta m}^{\delta(m+2)} \left|\frac{|G_F(t)|}{t} - \frac{|G_F(\delta m)|}{\delta m}\right| dt &\le \sum_{\substack{m\ge M\\(m,2)=1}} \int_{\delta m}^{\delta(m+2)} \left(\frac{8c_0}{7\delta^4 m^4} + \frac{4c_1}{3\delta^3 m^3}\right)(t-\delta m)\,dt \\ &\le \frac{16c_0}{7\delta^2}\sum_{\substack{m\ge M\\(m,2)=1}} \frac{1}{m^4} + \frac{8c_1}{3\delta}\sum_{\substack{m\ge M\\(m,2)=1}} \frac{1}{m^3}.\end{aligned}$$

Applying Lemma 6.16 to the sum on the right-hand side above, we find that:

$$\frac{1}{2}\sum_{\substack{m\ge M\\(m,2)=1}} \int_{\delta m}^{\delta(m+2)} \left|\frac{|G_F(t)|}{t} - \frac{|G_F(\delta m)|}{\delta m}\right| dt \le \frac{8c_0}{7\delta^2}\left(\frac{1}{6M^3} + \frac{1}{2M^4}\right) + \frac{4c_1}{3\delta}\left(\frac{1}{4M^2} + \frac{1}{2M^3}\right).$$

Recall that $M = [\delta^{-1}]$, Eq (7.2) becomes:

$$\sum_{\substack{m\ge M\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \le \frac{1}{2}\int_1^{\infty} \frac{|G_F(t)|}{t}\,dt + \frac{(4c_0+7c_1)\delta}{21} + \frac{(12c_0+14c_1)\delta^2}{21}. \tag{7.4}$$

Now let $u_0 \le \delta m \le 1$ for the remaining m's, we notice that $G_F(\delta m) \ge 0$. Here, we distinguish two cases:

⋆ The first case when $u_0/\delta < 1$. Then we write

$$\sum_{\substack{1\le m\le M-2\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} = \sum_{\substack{1\le m\le M-2\\(m,2)=1}} \frac{G_F(\delta m)-2}{m} + \sum_{\substack{1\le m\le M-2\\(m,2)=1}} \frac{2}{m}$$

Applying Lemma 6.15 on the last sum in the right-hand side above, we find that:

$$\begin{aligned}\sum_{\substack{1\le m\le M-2\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} &= \sum_{\substack{1\le m\le M-2\\(m,2)=1}} \frac{G_F(\delta m)-2}{m} \\ &\quad + \log(M-1) + \gamma + \log 2 + \frac{1}{6(M-1)^2} + \frac{1}{30(M-1)^4}\end{aligned} \tag{7.5}$$

Now we write the sum in the right-hand side of Eq (7.5) as follows:

$$\sum_{\substack{1\le m\le M-2\\(m,2)=1}}\frac{G_F(\delta m)-2}{m}=\frac{1}{2}\int_{\delta}^{\delta M}\frac{G_F(t)-2}{t}\,dt$$
$$-\frac{1}{2}\sum_{\substack{1\le m\le M-2\\(m,2)=1}}\int_{\delta m}^{\delta(m+2)}\left(\frac{G_F(t)-2}{t}-\frac{G_F(\delta m)-2}{\delta m}\right)dt.$$

Since the derivative of $(G_F(u)-2)/u$ is positive on $u_0\le u\le 1$ and $u_0<\delta$. Then the derivative $(G_F(u)-2)/u$ is also positive on $\delta\le u\le 1$, it follows that the last sum above is positive. Hence, Eq (7.5) can be written as follows:

$$\sum_{\substack{1\le m\le M-2\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}\le\frac{1}{2}\int_{\delta}^{\delta M}\frac{G_F(t)-2}{t}\,dt$$
$$+\log(M-1)+\gamma+\log(2)+\frac{1}{6(M-1)^2}+\frac{1}{30(M-1)^4}$$

Recalling that $M=[\delta^{-1}]$, we find that

$$\sum_{\substack{1\le m\le M-2\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}\le\frac{1}{2}\int_{\delta}^{1}\frac{G_F(t)-2}{t}\,dt$$
$$-\log\delta+\log(1-\delta)+\gamma+\log 2+\frac{\delta^2}{6(1-2\delta)^2}+\frac{\delta^4}{30(1-2\delta)^4}\tag{7.6}$$

From (7.5) and (7.6), we get

$$\sum_{\substack{m\ge 1\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}\le\frac{1}{2}\int_{\delta}^{1}\frac{G_F(t)-2}{t}\,dt+\frac{1}{2}\int_{1}^{\infty}\frac{|G_F(t)|}{t}\,dt-\log\delta+\log(1-\delta)$$
$$+\gamma+\log 2+\frac{(4c_0+7c_1)\delta}{21}+\frac{(12c_0+14c_1)\delta^2}{21}+\frac{\delta^2}{6(1-2\delta)^2}+\frac{\delta^4}{30(1-2\delta)^4}\tag{7.7}$$

Notice that

$$\int_{\delta}^{1}\frac{G_F(t)-2}{t}\,dt=\int_{0}^{1}\frac{G_F(t)-2}{t}\,dt-\int_{0}^{\delta}\frac{G_F(t)-2}{t}\,dt.$$

Applying Lemma 6.14 to the last integral above. We write Eq (7.7) as follows

$$\sum_{\substack{m\ge 1\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}\le-\log\delta+b_F+\frac{c_3\delta}{2\log 2}\left(-\log\delta+\log\frac{c_2}{c_3}+\log\log 2+\log 2+2\right)$$
$$+\log(1-\delta)+\frac{(4c_0+7c_1)\delta}{21}+\frac{(12c_0+14c_1)\delta^2}{21}+\frac{\delta^2}{6(1-2\delta)^2}+\frac{\delta^4}{30(1-2\delta)^4},\tag{7.8}$$

and where the constant b_F depends on only F and given by the following equality:

$$b_F = \frac{1}{2}\int_0^1 \frac{G_F(t)-2}{t}\,dt + +\frac{1}{2}\int_1^\infty \frac{|G_F(t)|}{t}\,dt + \gamma + \log 2,$$

and the constants c_i, $i \in \{0,1,2,3\}$ are chosen according to the function F.

⋆ The second case when $u_0/\delta \geq 1$. We have:

$$\sum_{\substack{1\leq m\leq M-2\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \leq \sum_{\substack{1\leq m<[u_0/\delta]\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} + \sum_{\substack{[u_0/\delta]\leq m\leq M-2\\(m,2)=1}} \frac{G_F(\delta m)}{m}.$$

Under the condition $|G_F(u)| \leq 2$, we obtain that

$$\sum_{\substack{1\leq m\leq M-2\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \leq \sum_{\substack{1\leq m\leq M-2\\(m,2)=1}} \frac{2}{m} + \sum_{\substack{[u_0/\delta]\leq m\leq M-2\\(m,2)=1}} \frac{G_F(\delta m)-2}{m}$$

Applying Lemma 6.15 on the last sum in the right-hand side above, we infer

$$\sum_{\substack{1\leq m\leq M-2\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \leq \sum_{\substack{[u_0/\delta]\leq m\leq M-2\\(m,2)=1}} \frac{G_F(\delta m)-2}{m} + \log(M-1) + \gamma + \log 2 + \frac{1}{6(M-1)^2} + \frac{1}{30(M-1)^4} \tag{7.9}$$

Now we write the sum in the right-hand side of Eq (7.9) as follows:

$$\sum_{\substack{[u_0/\delta]\leq m\leq M-2\\(m,2)=1}} \frac{G_F(\delta m)-2}{m} = \frac{1}{2}\int_{\delta[u_0/\delta]}^{\delta M} \frac{G_F(t)-2}{t}\,dt - \frac{1}{2}\sum_{\substack{[u_0/\delta]\leq m\leq M-2\\(m,2)=1}} \int_{\delta m}^{\delta(m+2)} \left(\frac{G_F(t)-2}{t} - \frac{G_F(\delta m)-2}{\delta m}\right) dt.$$

Since the derivative of $(G_F(u)-2)/u$ is positive on the interval $u_0 \leq u \leq 1$, it follows that the last sum above is positive. Then, Eq (7.9) can be written as follows:

$$\sum_{\substack{1\leq m\leq M-2\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \leq \frac{1}{2}\int_{\delta[u_0/\delta]}^{\delta M} \frac{G_F(t)-2}{t}\,dt + \log(M-1) + \gamma + \log(2) + \frac{1}{6(M-1)^2} + \frac{1}{30(M-1)^4}$$

Recalling that $M = [\delta^{-1}]$, we find that:

$$\sum_{\substack{1\le m\le M-2\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \le \frac{1}{2}\int_{\delta[u_0/\delta]}^{1} \frac{G_F(t)-2}{t}\,dt - \log\delta + \log(1-\delta) + \gamma + \log 2 + \frac{\delta^2}{6(1-2\delta)^2} + \frac{\delta^4}{30(1-2\delta)^4} \tag{7.10}$$

From (7.4) and (7.10), we get

$$\sum_{\substack{m\ge1\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \le \frac{1}{2}\int_{\delta[u_0/\delta]}^{1} \frac{G_F(t)-2}{t}\,dt + \frac{1}{2}\int_{1}^{\infty} \frac{|G_F(t)|}{t}\,dt - \log\delta + \log(1-\delta) + \gamma + \log 2 + \frac{(4c_0+7c_1)\delta}{21} + \frac{(12c_0+14c_1)\delta^2}{21} + \frac{\delta^2}{6(1-2\delta)^2} + \frac{\delta^4}{30(1-2\delta)^4}.$$

But, the first integral above is

$$\frac{1}{2}\int_{\delta[u_0/\delta]}^{1} \frac{G_F(t)-2}{t}\,dt = \frac{1}{2}\int_{0}^{1} \frac{G_F(t)-2}{t}\,dt - \frac{1}{2}\int_{0}^{\delta[u_0/\delta]} \frac{G_F(t)-2}{t}\,dt.$$

Thanks to Lemma 6.14, we conclude that:

$$\sum_{\substack{m\ge1\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \le -\log\delta + b_F + \frac{c_3 u_0}{2\log 2}\left(-\log u_0 + \log\frac{c_2}{c_3} + \log\log 2 + \log 2 + 2\right) + \log(1-\delta) + \frac{(4c_0+7c_1)\delta}{21} + \frac{(12c_0+14c_1)\delta^2}{21} + \frac{\delta^2}{6(1-2\delta)^2} + \frac{\delta^4}{30(1-2\delta)^4} \tag{7.11}$$

Form Eq (7.8) and Eq (7.11), we have:

$$\sum_{\substack{m\ge1\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \le -\log\delta + b_F + \frac{c_3 u_0}{2\log 2}\left(-\log u_0 + \log\frac{c_2}{c_3} + \log\log 2 + \log 2 + 2\right) + \log(1-\delta) + \frac{(4c_0+7c_1)\delta}{21} + \frac{(12c_0+14c_1)\delta^2}{21} + \frac{\delta^2}{6(1-2\delta)^2} + \frac{\delta^4}{30(1-2\delta)^4} \tag{7.12}$$

2. The second case, when $[\delta^{-1}]$ is even. We take $M = [\delta^{-1} - 1]$ to get

$$\delta \sum_{\substack{m\ge M\\(m,2)=1}} \frac{|G_F(\delta m)|}{\delta m} = \frac{1}{2}\int_{\delta M}^{\infty} \frac{|G_F(t)|}{t}\,dt - \frac{1}{2}\sum_{\substack{m\ge M\\(m,2)=1}} \int_{\delta m}^{\delta(m+2)} \left(\frac{|G_F(t)|}{t} - \frac{|G_F(\delta m)|}{\delta m}\right)dt. \tag{7.13}$$

Similar arguments apply to the first case (when $[\delta^{-1}]$ is odd), we find that

$$\frac{1}{2}\sum_{\substack{m\geq M\\(m,2)=1}}\int_{\delta m}^{\delta(m+2)}\left|\frac{|G_F(t)|}{t}-\frac{|G_F(\delta m)|}{\delta m}\right|dt\leq\frac{8c_0}{7\delta^2}\left(\frac{1}{6M^3}+\frac{1}{2M^4}\right)+\frac{4c_1}{3\delta}\left(\frac{1}{4M^2}+\frac{1}{2M^3}\right).$$

By substituting this inequality in Eq (7.13), we obtain that

$$\sum_{\substack{m\geq M\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}\leq\frac{1}{2}\int_{\delta M}^{\infty}\frac{|G_F(t)|}{t}\,dt+\frac{8c_0}{7\delta^2}\left(\frac{1}{6M^3}+\frac{1}{2M^4}\right)+\frac{4c_1}{3\delta}\left(\frac{1}{4M^2}+\frac{1}{2M^3}\right)$$

Recall that $M=[\delta^{-1}-1]$, we write

$$\sum_{\substack{m\geq M\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}\leq\frac{1}{2}\int_{\delta[\delta^{-1}-1]}^{\infty}\frac{|G_F(t)|}{t}\,dt+\frac{(4c_0+7c_1)\delta}{21(1-2\delta)^2}+\frac{(12c_0+14c_1)\delta^2}{21(1-2\delta)^3}. \quad (7.14)$$

Let $u_0\leq\delta(m+3)\leq 1$, for the remaining m's, we notice that $G_F(\delta m)\geq 0$. Here, we distinguish two cases:

⋆ The first case when $u_0/\delta\geq 1$. Then, we can write:

$$\sum_{\substack{1\leq m\leq M-2\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}=\sum_{\substack{1\leq m<[u_0/\delta]\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}+\sum_{\substack{[u_0/\delta]\leq m\leq M-2\\(m,2)=1}}\frac{G_F(\delta m)}{m}.$$

By similar arguments as in the first case (when $[\delta^{-1}]$ is odd), we find that:

$$\sum_{\substack{1\leq m\leq M-2\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}\leq\sum_{\substack{[u_0/\delta]\leq m\leq M-2\\(m,2)=1}}\frac{G_F(\delta m)-2}{m}+\sum_{\substack{1\leq m\leq M-2\\(m,2)=1}}\frac{2}{m}$$

Applying Lemma 6.15 on the last sum in the right-hand side above, we find that:

$$\sum_{\substack{1\leq m\leq M-2\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}\leq\sum_{\substack{[u_0/\delta]\leq m\leq M-2\\(m,2)=1}}\frac{G_F(\delta m)-2}{m}$$
$$+\log(M-2)+\gamma+\log 2+\frac{1}{6(M-2)^2}+\frac{1}{30(M-2)^4}, \quad (7.15)$$

Now we write the sum in the right-hand side above as follows:

$$\sum_{\substack{[u_0/\delta]\leq m\leq M-2\\(m,2)=1}}\frac{G_F(\delta m)-2}{m}=\frac{1}{2}\int_{\delta[u_0/\delta]}^{\delta M}\frac{G_F(t)-2}{t}\,dt$$
$$-\frac{1}{2}\sum_{\substack{[u_0/\delta]\leq m\leq M-2\\(m,2)=1}}\int_{\delta m}^{\delta(m+2)}\left(\frac{G_F(t)-2}{t}-\frac{G_F(\delta m)-2}{\delta m}\right)dt.$$

By a similar reasoning as in the case $[\delta^{-1}]$ is odd, we write Eq (7.15) as follows:

$$\sum_{\substack{1\le m\le M-2\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}\le\frac{1}{2}\int_{\delta[u_0/\delta]}^{\delta M}\frac{G_F(t)-2}{t}\,dt + \log(M-2)+\gamma+\log 2+\frac{1}{6(M-2)^2}+\frac{1}{30(M-2)^4}.$$

Recalling that $M=[\delta^{-1}-1]$, we get:

$$\sum_{\substack{1\le m\le M-2\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}\le\frac{1}{2}\int_{\delta[u_0/\delta]}^{\delta[\delta^{-1}-1]}\frac{G_F(t)-2}{t}\,dt -\log\delta+\log(1-3\delta)+\gamma+\log 2+\frac{\delta^2}{6(1-4\delta)^2}+\frac{\delta^4}{30(1-4\delta)^4}. \quad (7.16)$$

Notice that :

$$\int_{\delta[u_0/\delta]}^{\delta[\delta^{-1}-1]}\frac{G_F(t)-2}{t}\,dt=\int_{\delta[u_0/\delta]}^{1}\frac{G_F(t)-2}{t}\,dt-\int_{\delta[\delta^{-1}-1]}^{1}\frac{G_F(t)-2}{t}\,dt.$$

and

$$\int_{\delta[\delta^{-1}-1]}^{\infty}\frac{|G_F(t)|}{t}\,dt=\int_{1}^{\infty}\frac{|G_F(t)|}{t}\,dt+\int_{\delta[\delta^{-1}-1]}^{1}\frac{|G_F(t)|}{t}\,dt.$$

Since the function $G_F(u)$ is positive for $u_0\le u\le 1$ and $\delta\le u_0$, we infer that:

$$\int_{\delta[\delta^{-1}-1]}^{1}\frac{|G_F(t)|}{t}\,dt-\int_{\delta[\delta^{-1}-1]}^{1}\frac{G_F(t)-2}{t}\,dt=2\int_{1-\delta}^{1}\frac{dt}{t}\le -2\log(1-2\delta) \quad (7.17)$$

From Eq (7.14), Eq (7.16) and Eq (7.17) together, we get

$$\sum_{\substack{m\ge 1\\(m,2)=1}}\frac{|G_F(\delta m)|}{m}\le\frac{1}{2}\int_{\delta[u_0/\delta]}^{1}\frac{G_F(t)-2}{t}\,dt+\frac{1}{2}\int_{1}^{\infty}\frac{|G_F(t)|}{t}\,dt-\log\delta+\log(1-\delta) +\gamma+\log 2+\frac{(4c_0+7c_1)\delta}{21(1-2\delta)^2}+\frac{(12c_0+14c_1)\delta^2}{21(1-2\delta)^3}+\frac{\delta^2}{6(1-4\delta)^2}+\frac{\delta^4}{30(1-4\delta)^4}$$

Now, we write the first integral in the right-hand side above as follows:

$$\int_{\delta[u_0/\delta]}^{1}\frac{G_F(t)-2}{t}\,dt=\int_{0}^{1}\frac{G_F(t)-2}{t}\,dt-\int_{0}^{\delta[u_0/\delta]}\frac{G_F(t)-2}{t}\,dt$$

Using Lemma 6.14, we conclude that:

$$\sum_{\substack{m\geq 1\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \leq -\log\delta + b_F + \frac{c_3 u_0}{2\log 2}\left(-\log u_0 + \log\frac{c_2}{c_3} + \log\log 2 + \log 2 + 2\right)$$
$$+\log(1-\delta) + \frac{(4c_0+7c_1)\delta}{21(1-2\delta)^2} + \frac{(12c_0+14c_1)\delta^2}{21(1-2\delta)^3} + \frac{\delta^2}{6(1-4\delta)^2} + \frac{\delta^4}{30(1-4\delta)^4} \tag{7.18}$$

⋆ The second case when $u_0/\delta < 1$. Then, we can write

$$\sum_{\substack{1\leq m\leq M-2\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} = \sum_{\substack{1\leq m\leq M-2\\(m,2)=1}} \frac{G_F(\delta m)-2}{m} + \sum_{\substack{1\leq m\leq M-2\\(m,2)=1}} \frac{2}{m}.$$

By a similar arguments as in the first case when $[\delta^{-1}]$ is odd, we have

$$\sum_{\substack{1\leq m\leq M-2\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \leq \frac{1}{2}\int_{\delta}^{\delta M} \frac{G_F(t)-2}{t}\,dt$$
$$+\log(M-2) + \gamma + \log(2) + \frac{1}{6(M-2)^2} + \frac{1}{30(M-2)^4}$$

Recalling that $M = [\delta^{-1}-1]$, we find that:

$$\sum_{\substack{1\leq m\leq M-2\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \leq \frac{1}{2}\int_{\delta}^{\delta[\delta^{-1}-1]} \frac{G_F(t)-2}{t}\,dt - \log\delta + \log(1-3\delta)$$
$$+\gamma + \log 2 + \frac{\delta^2}{6(1-4\delta)^2} + \frac{\delta^4}{30(1-4\delta)^4} \tag{7.19}$$

Notice that

$$\int_{\delta}^{\delta[\delta^{-1}-1]} \frac{G_F(t)-2}{t}\,dt = \int_{\delta}^{1} \frac{G_F(t)-2}{t}\,dt - \int_{\delta[\delta^{-1}-1]}^{1} \frac{G_F(t)-2}{t}\,dt.$$

and

$$\int_{\delta[\delta^{-1}-1]}^{\infty} \frac{|G_F(t)|}{t}\,dt = \int_{\delta[\delta^{-1}-1]}^{1} \frac{|G_F(t)|}{t}\,dt + \int_{1}^{\infty} \frac{|G_F(t)|}{t}\,dt.$$

Since the function $G_F(u)$ is positive for $u_0 \leq u \leq 1$ and $u_0 \leq \delta$, we infer that:

$$\int_{\delta[\delta^{-1}-1]}^{1} \frac{|G_F(t)|}{t}\,dt - \int_{\delta[\delta^{-1}-1]}^{1} \frac{G_F(t)-2}{t}\,dt \leq -2\log(1-2\delta) \tag{7.20}$$

From Eq (7.14), Eq (7.19) and Eq (7.20) we get

$$\sum_{\substack{m\geq 1\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \leq \frac{1}{2}\int_\delta^1 \frac{G_F(t)-2}{t}\,dt + \frac{1}{2}\int_1^\infty \frac{|G_F(t)|}{t}\,dt - \log\delta + \log(1-\delta)$$
$$+\gamma+\log 2+\frac{\delta^2}{6(1-3\delta)^2}+\frac{\delta^4}{30(1-3\delta)^4}+\frac{(4c_0+7c_1)\delta}{21(1-2\delta)^2}+\frac{(12c_0+14c_1)\delta^2}{21(1-2\delta)^3}. \quad (7.21)$$

Now, we write the first integral in the right-hand side above as follows:

$$\int_\delta^1 \frac{G_F(t)-2}{t}\,dt = \int_0^1 \frac{G_F(t)-2}{t}\,dt - \int_0^\delta \frac{G_F(t)-2}{t}\,dt.$$

Using Lemma 6.14, we conclude that:

$$\sum_{\substack{m\geq 1\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \leq -\log\delta + b_F + \frac{c_3\delta}{2\log 2}\left(-\log\delta+\log\frac{c_2}{c_3}+\log\log 2+\log 2+2\right)$$
$$+\log(1-\delta)+\frac{(4c_0+7c_1)\delta}{21(1-2\delta)^2}+\frac{(12c_0+14c_1)\delta^2}{21(1-2\delta)^3}+\frac{\delta^2}{6(1-4\delta)^2}+\frac{\delta^4}{30(1-4\delta)^4} \quad (7.22)$$

From all previous cases, we find that

$$\sum_{\substack{m\geq 1\\(m,2)=1}} \frac{|G_F(\delta m)|}{m} \leq -\log\delta + b_F + \frac{c_3\theta}{2\log 2}\left(-\log\theta+\log\frac{c_2}{c_3}+\log\log 2+\log 2+2\right)$$
$$+\log(1-\delta)+\frac{(4c_0+7c_1)\delta}{21}+\frac{(12c_0+14c_1)\delta^2}{21}+\frac{\delta^2}{6(1-4\delta)^2}+\frac{\delta^4}{30(1-4\delta)^4}.$$

where θ is u_0 or δ according as to whether $\delta/u_0 \leq 1$ or $\delta/u_0 > 1$. This complete the proof.

7.2 Proof of Theorem 0.39

First, recall that

$$L(1,\chi) = \sum_{n\geq 1} \frac{\chi(n)\,(1-F_1(\delta n))}{n} - \frac{2\tau(\chi)}{q}\sum_{1\leq k\leq \delta q/2} \overline{\chi}(k)\log\left|\sin\frac{\pi k}{\delta q}\right|$$

where F_1 is given by Proposition 6.3. Setting $n = 2^\ell m$ with m is odd, we find that:

$$L(1,\chi) = \sum_{\substack{m\geq 1\\(m,2)=1}} \chi(m)\frac{G_{F_1}(\delta m)}{m} - \frac{2\tau(\chi)}{q}\sum_{1\leq k\leq \delta q/2} \overline{\chi}(k)\log\left|\sin\frac{\pi k}{\delta q}\right| \quad (7.23)$$

From Lemmas 6.9, 6.10 and 6.11, we find that the hypotheses of Theorem 6.1 are satisfied when $u_0 = 10^{-5}$. Then, we can to apply this theorem on the first sum in Eq (7.23), and using Lemma 6.5 for the second sum in Eq (7.23). We obtain that

$$|L(1,\chi)| \leq -\log\delta + b_{F_1} + \delta\sqrt{q}\log 2 + \frac{c_3\theta}{2\log 2}\left(-\log\theta + \log\frac{c_2}{c_3} + \log\log 2 + \log 2 + 2\right) + \log(1-\delta) + \frac{(4c_0+7c_1)\delta}{21} + \frac{(12c_0+14c_1)\delta^2}{21} + \frac{\delta^2}{6(1-4\delta)^2} + \frac{\delta^4}{30(1-4\delta)^4}.$$

where θ is 10^{-5} or δ according as to whether $\delta \leq 10^{-5}$ or $\delta > 10^{-5}$.

Thanks to Lemmas 6.7, 6.8 and 6.12, we find that $c_0 = 1$, $c_1 = 1/\pi$, $c_2 = 2$, $c_3 = \log 4$ and $b_{F_1} \leq -0.66266$. Hence, we can write

$$|L(1,\chi)| \leq -\log\delta - 0.66266 + \delta\sqrt{q}\log 2 + \theta\left(-\log\theta + \log 2 + 2\right) + \log(1-\delta) + \frac{(4\pi+7)\delta}{21\pi} + \frac{(12\pi+14)\delta^2}{21\pi} + \frac{\delta^2}{6(1-4\delta)^2} + \frac{\delta^4}{30(1-4\delta)^4}.$$

The best value for δ is given by $\delta\sqrt{q}\log 2 = 1$. We obtain that

$$|L(1,\chi)| \leq \tfrac{1}{2}\log q + \log\log 2 - 0.66266 + 1 + C(q),$$

where the constant $C(q)$ depends only on q

$$C(q) = \theta\left(-\log\theta + \log 2 + 2\right) + \log\left(1 - \frac{1}{\sqrt{q}\log 2}\right) + \frac{(4\pi+7)}{21\pi\sqrt{q}\log 2} + \frac{(12\pi+14)}{21\pi q\log^2 2} + \frac{1}{6(\sqrt{q}\log 2 - 4)^2} + \frac{1}{30(\sqrt{q}\log 2 - 4)^4}.$$

Here, we distinguish the following cases :

- For $2 \leq q \leq 2\cdot 10^6$. In 2001, Ramaré computed $L(1,\chi)$ for all primitive χ of conductor $q \leq 4500$ by using the version 2.0.11 of the GP- calculator of the PARI system. With slight modifications, we give, in Appendix A, the computer script used for computing the values of $L(1,\chi)$ when $\chi(2) = 1$, $\chi(-1) = 1$ and $2 \leq q \leq 2\cdot 10^5$. In his doctoral thesis in 2011, Platt [44] gave an algorithm more effective for rigorously computing Dirichlet L-functions. He was able to improve on his method for computing $L(1,\chi)$ for all primitive characters of the conductor q less than $2\cdot 10^6$ and to check the conjectures of Ramaré for $2 \leq q \leq 2\cdot 10^6$. Here is the picture obtained for the first 20000 values of $|L(1,\chi)| - \frac{1}{2}\log q$.
- For $2\cdot 10^6 \leq q \leq 2\cdot 10^{10}$, we find that $\theta = 1/(\sqrt{q}\log 2)$. It follows that

$$C(q) + \log\log 2 - 0.66266 + 1 \leq -0.02012.$$

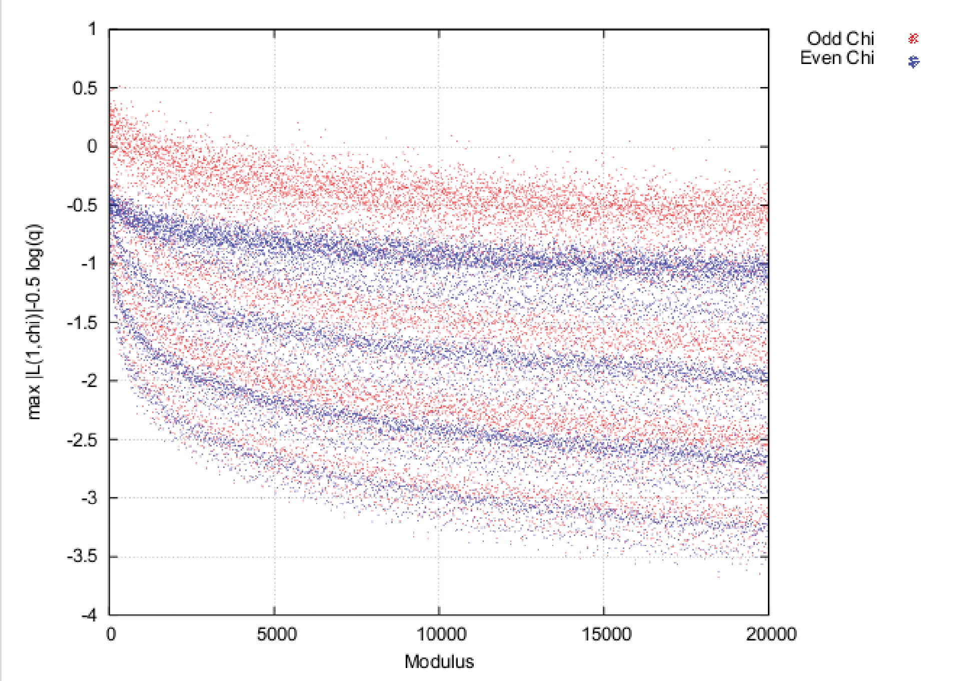

Figure 7.1: The maximum of $|L(1,\chi)| - \frac{1}{2}\log q$.

- For $q \geq 2 \cdot 10^{10}$, we find that $\theta \leq 1.02014 \cdot 10^{-5}$. It follows that

$$C(q) + \log\log 2 - 0.66266 + 1 \leq -0.02012.$$

Therefore, we conclude that:

$$|L(1,\chi)| \leq \tfrac{1}{2}\log q - 0.02012.$$

This completes the proof.

Remark 3 *this result is the best for the function F_1 for the moment, but can be optimized through applying this method on a different function, where the efficiency of the method will be much clear.*

7.3 Proof of Theorem 0.40

Recall that the Dirichlet class number formula is given by

$$h\big(\mathbb{Q}(\sqrt{q})\big) = \frac{\sqrt{q}}{2\log \varepsilon_q} L\big(1, \chi_q\big)$$

where χ_q is the even primitive real character modulo q and ε_q is the fundamental unit. Thanks to Theorem 0.39, we find that:

$$\begin{aligned} \frac{2}{\sqrt{q}} h\left(\mathbb{Q}\left(\sqrt{q}\right)\right) &\leq \frac{\log q - 0.04038}{2\log\left(\frac{1}{2}\sqrt{q-4} + \frac{1}{2}\sqrt{q}\right)} \\ &\leq \frac{\log q - 0.04038}{2\log\sqrt{q} + 2\log\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4}{q}}\right)}. \end{aligned}$$

For $q \geq 4$, we notice that

$$\log\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4}{q}}\right) = \log\left(1 - \frac{2}{q\left(1+\sqrt{1-\frac{4}{q}}\right)}\right) \geq \log\left(1 - \frac{2}{q}\right).$$

We get

$$\frac{2}{\sqrt{q}} h\left(\mathbb{Q}\left(\sqrt{q}\right)\right) \leq \frac{\log q - 0.04038}{\log q + 2\log\left(1-\frac{2}{q}\right)}.$$

Since $\log(1-X) \geq -2X$, when $X \leq \frac{1}{2}$. It follows that:

$$\begin{aligned} \frac{2}{\sqrt{q}} h\left(\mathbb{Q}\left(\sqrt{q}\right)\right) &\leq 1 - \frac{2\log\left(1-\frac{2}{q}\right) + 0.04038}{\log q + 2\log\left(1-\frac{2}{q}\right)} \\ &\leq 1 - \frac{0.04038 - \frac{8}{q}}{\log q} \end{aligned}$$

B. Oriat [41] has computed the class number of this field when $1 < q < 24572$. For $q \geq 24572$, we conclude that

$$\frac{2}{\sqrt{q}} h\left(\mathbb{Q}\left(\sqrt{q}\right)\right) \leq 1 - \frac{1}{25\log q}.$$

We extend it to $q > 226$ via the table of Oriat. Thanks to the precious remarks of F. Pappalardi, and the following simple GP-PARI code sent by him. We can check that our result is also correct for $1 < q \leq 226$

```
for(q=2,226,
if(q==quaddisc(q),U=quadclassunit(q)[1];
if(U>sqrt(q)/2*(1-1/25/log(q)),
print(q" "U),),))
```

This completes the proof.

Part III

Explicit upper bounds for $\left|L(1,\chi)\right|$ when $\chi(3)=0$

Chapter 8

Explicit upper bounds for $\left|L(1,\chi)\right|$ when $\chi(3)=0$

8.1 Introduction

In the papers [38] and [39], Louboutin, by using integral representations of Dirichlet L functions, obtained the following upper bound for $\left|L(1,\chi)\right|$ when 3 divides the conductor q.

$$|L(1,\chi)| \le \frac{1}{3}\log q + \begin{cases} 0.3816 & \text{when } \chi(-1)=1, \\ 0.8436 & \text{when } \chi(-1)=-1. \end{cases}$$

In 2001, Ramaré [46] gave another method, completely different to that of Louboutin, (see Proposition 6.4). In this chapter, we show how this method improves on Louboutin's bound. Specifically, we show :

Theorem 8.1 *Let χ be a primitive character of conductor $q>1$ such that $3|q$. Then*

$$|L(1,\chi)| \le \frac{1}{3}\log q + \begin{cases} 0.368296 & \textit{when } \chi(-1)=1, \\ 0.838374 & \textit{when } \chi(-1)=-1. \end{cases}$$

We prove this result for the conductor q more than $2\cdot 10^6$. Numerically, Ramaré computed $L(1,\chi)$, in 2001, for all primitive χ of conductor $q \le 4500$ by using the version 2.0.11 of the GP- calculator of the PARI system. Platt has kindly agreed to run his algorithm from his thesis [44](which is rigorous and efficient for computing $L(1,\chi)$ for all primitive χ of conductor $2\le q\le 2\cdot 10^6$).

8.2 Results of Ramaré

The proof of the following results can be found in [46].

Lemma 8.2 *Let k and $\beta>0$ be two real numbers. Let f be a continuous, convex and non-increasing L^1-function on $]k-\theta, k+\beta]$. We have*

$$f(k) \le \frac{1}{\theta}\int_{k-\theta}^{k} f(t)\,dt - \theta\frac{f(k)-f(k+\beta)}{2\beta}.$$

Lemma 8.3 *Set*

$$j(t) = 2\int_{|t|}^{1} \left(\pi(1-u)\cot(\pi u)+1\right)\,du.$$

Then, we have

$$\int_0^1 j(t)\,dt = 1, \tag{8.1}$$

and

$$-2\log|t| - 2\left(\log(2\pi)-1\right) \le j(t) \le -2\log|t|. \tag{8.2}$$

Lemma 8.4 *For $\delta\in]0,1]$, we have*

$$\sum_{n\ge1}\frac{1-F_3(\delta n)}{n} = -\log\delta - 1 + \delta.$$

where F_3 is defined in Proposition 6.4.

Lemma 8.5 *For $\delta>0$, we have*

$$\sum_{n\ge1}\frac{1-F_4(\delta n)}{n} = -\log\delta + \frac{3}{2} - \log(2\pi) + 2\int_0^1 (1-t)\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right|\,dt.$$

where F_4 is defined in Proposition 6.4.

Lemma 8.6 *For $0<\delta\le 1/2$, we have*

$$2\int_0^1 (1-t)\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right|\,dt \le \frac{\pi^3\delta^2}{12}. \tag{8.3}$$

8.3 Some auxiliary lemmas

Lemma 8.7 *Let $\alpha > 0$ and a be 1 or 2. Let g be a continuous, convex, non-negative and non-increasing L^1-function on $]0,1]$. Then*

$$\sum_{\substack{a\le n\le \alpha\\ n=3k+a}} g\left(\frac{n}{\alpha}\right) \le \frac{1}{3}\int_0^{\alpha} g\left(\frac{u}{\alpha}\right)du + \frac{1}{2}g(1) + \frac{2-a}{2}g\left(\frac{a}{\alpha}\right) + \frac{a}{6}g\left(\frac{2a}{\alpha}\right) - \frac{1}{2}g\left(\frac{a+3}{\alpha}\right)$$

Proof. Let $3K+a$ be the largest integer less than or equal to α, then we have

$$\sum_{\substack{a\le n\le 3K+a\\ n=3k+a}} g\left(\frac{n}{\alpha}\right) = g\left(\frac{a}{\alpha}\right) + \sum_{\substack{a+3\le n\le 3K+a\\ n=3k+a}} g\left(\frac{n}{\alpha}\right). \tag{8.4}$$

We write the sum on the right-hand side of this equality as

$$\sum_{\substack{a+3\le n\le 3K+a\\ n=3k+a}} g\left(\frac{n}{\alpha}\right) = \sum_{1\le k\le K} g\left(\frac{3k+a}{\alpha}\right).$$

Using Lemma 8.2 with $\theta=\beta=1$, we get

$$\begin{aligned}\sum_{1\le k\le K-1} g\left(\frac{3k+a}{\alpha}\right) &\le \sum_{1\le k\le K-1}\int_{k-1}^{k} g\left(\frac{3t+a}{\alpha}\right)dt - \sum_{1\le k\le K-1}\frac{g\left(\frac{3k+a}{\alpha}\right)-g\left(\frac{3k+a+3}{\alpha}\right)}{2},\\ &\le \int_0^{K-1} g\left(\frac{3t+a}{\alpha}\right)dt - \frac{g\left(\frac{a+3}{\alpha}\right)-g\left(\frac{3K+a}{\alpha}\right)}{2}.\end{aligned}$$

By making the simple change of variable $3t+a=u$ in the last integral above, we obtain that

$$\sum_{1\le k\le K-1} g\left(\frac{3k+a}{\alpha}\right) \le \frac{1}{3}\int_a^{3K+a-3} g\left(\frac{u}{\alpha}\right)du - \frac{g\left(\frac{a+3}{\alpha}\right)-g\left(\frac{3K+a}{\alpha}\right)}{2} \tag{8.5}$$

We again use Lemma 8.2 but this time with $\theta=3$ and $\beta=\alpha-(3K+a)$. When $\beta=0$ the proof is complete. Otherwise $\beta\le 3$ and we get

$$g\left(\frac{3K+a}{\alpha}\right) \le \frac{1}{3}\int_{3K+a-3}^{3K+a} g\left(\frac{u}{\alpha}\right)du - \frac{3\left(g\left(\frac{3K+a}{\alpha}\right)-g(1)\right)}{2\beta}$$

Substituting this in Eq (8.5) with regard to $\beta\le 3$, we find that

$$\sum_{1\le k\le K} g\left(\frac{3k+a}{\alpha}\right) \le \frac{1}{3}\int_a^{3K+a} g\left(\frac{u}{\alpha}\right)du - \frac{g(\frac{a+3}{\alpha})-g(1)}{2}$$

Then Eq (8.4) becomes

$$\sum_{\substack{a\le n\le 3K+a\\ n=3k+a}} g\left(\frac{n}{\alpha}\right) \le \frac{1}{3}\int_a^{3K+a} g\left(\frac{u}{\alpha}\right)du - \frac{g(\frac{a+3}{\alpha})-g(1)}{2} + g\left(\frac{a}{\alpha}\right). \tag{8.6}$$

Now, we apply Lemma 8.2 to $g(a/\alpha)$ with $\theta=\beta=a$, to get

$$g\left(\frac{a}{\alpha}\right) \le \frac{1}{a}\int_0^a g\left(\frac{u}{\alpha}\right)du - \frac{g(a/\alpha)-g(2a/\alpha)}{2},$$

it follows that

$$\frac{3}{2}g\left(\frac{a}{\alpha}\right) \le \frac{1}{a}\int_0^a g\left(\frac{u}{\alpha}\right)du + \frac{g(2a/\alpha)}{2}.$$

Multiplying this inequality with $a/3$, we find that

$$\frac{a}{2}g\left(\frac{a}{\alpha}\right) \le \frac{1}{3}\int_0^a g\left(\frac{u}{\alpha}\right)du + \frac{a}{6}g\left(\frac{2a}{\alpha}\right).$$

Here, we have to distinguish two cases.

– The first case when $a=1$ and write

$$\frac{g(1/\alpha)}{2} \le \frac{1}{3}\int_0^1 g\left(\frac{u}{\alpha}\right)du + \frac{g(2/\alpha)}{6}.$$

Then, Eq (8.6) becomes

$$\sum_{\substack{1\le n\le 3K+1\\ n=3k+a}} g\left(\frac{n}{\alpha}\right) \le \frac{1}{3}\int_0^{3K+1} g\left(\frac{u}{\alpha}\right)du + \frac{g(1)}{2} + \frac{g(1/\alpha)}{2} + \frac{g(2/\alpha)}{6} - \frac{g(4/\alpha)}{2}. \tag{8.7}$$

– The second case when $a=2$, we rewrite

$$g\left(\frac{2}{\alpha}\right) \le \frac{1}{3}\int_0^2 g\left(\frac{u}{\alpha}\right)du + \frac{g(4/\alpha)}{3}.$$

Then, Eq (8.6) becomes

$$\sum_{\substack{2\le n\le 3K+2\\ n=3k+a}} g\left(\frac{n}{\alpha}\right) \le \frac{1}{3}\int_0^{3K+2} g\left(\frac{u}{\alpha}\right)du + \frac{g(1)}{2} + \frac{g(4/\alpha)}{3} - \frac{g(5/\alpha)}{2} \tag{8.8}$$

From Eq (8.7) and Eq (8.8), we conclude that

$$\sum_{\substack{a\le n\le \alpha\\ n=3k+a}} g\left(\frac{n}{\alpha}\right) \le \frac{1}{3}\int_0^{3K+a} g\left(\frac{u}{\alpha}\right)du + \frac{1}{2}g(1) + \frac{2-a}{2}g\left(\frac{a}{\alpha}\right) + \frac{a}{6}g\left(\frac{2a}{\alpha}\right) - \frac{1}{2}g\left(\frac{a+3}{\alpha}\right).$$

We complete the integral from $(3K+a)$ to α by using the non-negativity of g and get

$$\sum_{\substack{a\le n\le 3K+a\\ n=3k+a}} g\left(\frac{n}{\alpha}\right) \le \frac{1}{3}\int_0^{\alpha} g\left(\frac{u}{\alpha}\right)du + \frac{1}{2}g(1) + \frac{2-a}{2}g\left(\frac{a}{\alpha}\right) + \frac{a}{6}g\left(\frac{2a}{\alpha}\right) - \frac{1}{2}g\left(\frac{a+3}{\alpha}\right),$$

this completes the proof. ◇◇◇

Lemma 8.8 *For* $0 < \delta \le 1/2$, *we have :*

$$\int_0^1 (1-t)\left(\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right| - \frac{1}{3}\log\left|\frac{3\pi\delta t}{\sin(3\pi\delta t)}\right|\right) dt \le \frac{\pi^3\delta^2}{36} + \frac{\pi^2\delta^2}{27}$$

Proof. We begin with

$$\log\left|\frac{3\pi\delta t}{\sin(3\pi\delta t)}\right| = \log 3 + \log(\pi\delta t) - \log|\sin(3\pi\delta t)|.$$

Since

$$\sin(3x) = -4\sin^3 x + 3\sin x,$$

it follows that

$$\begin{aligned}\log\left|\frac{3\pi\delta t}{\sin(3\pi\delta t)}\right| &= \log 3 + \log(\pi\delta t) - \log|\sin(\pi\delta t)| - \log\left|4\sin^2(\pi\delta t) - 3\right| \\ &= \log\frac{\pi\delta t}{|\sin(\pi\delta t)|} - \log\left|\frac{4}{3}\sin^2(\pi\delta t) - 1\right|.\end{aligned}$$

Then

$$\int_0^1 (1-t)\left(\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right| - \frac{1}{3}\log\left|\frac{3\pi\delta t}{\sin(3\pi\delta t)}\right|\right) dt = \\ \frac{2}{3}\int_0^1 (1-t)\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right| dt + \frac{1}{3}\int_0^1 (1-t)\log\left|\frac{4}{3}\sin^2(\pi\delta t) - 1\right| dt. \quad (8.9)$$

Since $\log x \le x - 1$ and $\sin x \le x$ if $0 \le x \le \pi/2$, we see that

$$\begin{aligned}\log\left|\frac{4}{3}\sin^2(\pi\delta t) - 1\right| &\le \left|\frac{4}{3}\sin^2(\pi\delta t) - 1\right| - 1 \\ &\le \frac{4}{3}\sin^2(\pi\delta t) \le \frac{4}{3}(\pi\delta t)^2\end{aligned}$$

Hence, we can write Eq (8.9) as

$$\int_0^1 (1-t)\left(\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right| - \frac{1}{3}\log\left|\frac{3\pi\delta t}{\sin(3\pi\delta t)}\right|\right) dt \le \\ \frac{2}{3}\int_0^1 (1-t)\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right| dt + \frac{4(\pi\delta)^2}{9}\int_0^1 (1-t)t^2\, dt.$$

Using Lemma 8.6, we infer

$$\int_0^1 (1-t)\left(\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right| - \frac{1}{3}\log\left|\frac{3\pi\delta t}{\sin(3\pi\delta t)}\right|\right) dt \le \frac{\pi^3\delta^2}{36} + \frac{\pi^2\delta^2}{27}.$$

This completes the proof. ◇◇◇

8.4 Proof of Theorem 8.1

We are now ready to prove our upper bound for $|L(1,\chi)|$ when 3 divides the conductor q. We break the proof into two cases:

– In the case of even characters, we have:

$$\sum_{\substack{n\geq 1\\(n,3)=1}} \frac{1-F_3(\delta n)}{n} = \sum_{n\geq 1} \frac{1-F_3(\delta n)}{n} - \sum_{n\geq 1} \frac{1-F_3(3\delta n)}{3n}$$

where F_3 is defined by Proposition 6.4. Thanks to Lemma 8.4, we can write the first sums in the right-hand side above as

$$\sum_{n\geq 1} \frac{1-F_3(\delta n)}{n} = -\log\delta - 1 + \delta.$$

and

$$\sum_{n\geq 1} \frac{1-F_3(3\delta n)}{n} = -\log(3\delta) - 1 + 3\delta.$$

Again using Proposition 6.4 and recalling $0<\delta\leq 1$, we get

$$|L(1,\chi)| \leq -\frac{2}{3}\log\delta + \frac{1}{3}\log 3 - \frac{2}{3} + \frac{1}{\sqrt{q}} \sum_{\substack{1\leq m\leq \delta q\\(m,3)=1}} j\left(\frac{m}{\delta q}\right) \tag{8.10}$$

where $j(t)$ is defined in Lemma 8.3. Now, we apply Lemma 8.7 and Eq (8.1) of the sum of $j(t)$ given in Eq (8.10) to obtain

$$\begin{aligned}\sum_{\substack{1\leq m\leq \delta q\\(m,3)=1}} j\left(\frac{m}{\delta q}\right) &= \sum_{\substack{1\leq m\leq \delta q\\m=3k+1}} j\left(\frac{m}{\delta q}\right) + \sum_{\substack{2\leq m\leq \delta q\\m=3k+2}} j\left(\frac{m}{\delta q}\right)\\ &\leq \frac{2\delta q}{3} + j(1) + \frac{1}{2} j\left(\frac{1}{\delta q}\right) + \frac{1}{6} j\left(\frac{2}{\delta q}\right) - \frac{1}{6} j\left(\frac{4}{\delta q}\right) - \frac{1}{2} j\left(\frac{5}{\delta q}\right),\end{aligned}$$

Using Lemma 8.3 again, we find that

$$\sum_{\substack{1\leq m\leq \delta q\\(m,3)=1}} j\left(\frac{m}{\delta q}\right) \leq \frac{2\delta q}{3} + \frac{5}{3}\log 2 + \log 5 + \frac{4}{3}\left(\log\pi - 1\right).$$

Then, Eq (8.10) becomes simply

$$|L(1,\chi)| \leq -\frac{2}{3}\log\delta + \frac{1}{3}\log 3 - \frac{2}{3} + \frac{2\delta\sqrt{q}}{3} + \frac{1}{\sqrt{q}}\left(\frac{5}{3}\log 2 + \log 5 + \frac{4}{3}\log\pi - \frac{4}{3}\right)$$

The best possible choice for δ is $1/\sqrt{q}$. This yields

$$\left|L(1,\chi)\right| \le \frac{1}{3}\log q + \frac{1}{3}\log 3 + \frac{1}{\sqrt{q}}\left(\frac{5}{3}\log 2 + \log 5 + \frac{4}{3}\log\pi - \frac{4}{3}\right), \tag{8.11}$$

where $\frac{1}{3}\log 3 = 0.3662\cdots$, with an error term depending only on q. Setting

$$C_{\text{even}}(q) = \frac{1}{3}\log 3 + \frac{1}{\sqrt{q}}\left(\frac{5}{3}\log 2 + \log 5 + \frac{4}{3}\log\pi - \frac{4}{3}\right), \tag{8.12}$$

it follows that

$$\left|L(1,\chi)\right| \le \tfrac{1}{3}\log q + C_{\text{even}}(q). \tag{8.13}$$

– In the case of odd characters, we write again:

$$\sum_{\substack{n\ge 1\\(n,3)=1}} \frac{1-F_4(\delta n)}{n} = \sum_{n\ge 1}\frac{1-F_4(\delta n)}{n} - \sum_{n\ge 1}\frac{1-F_4(3\delta n)}{3n}$$

where F_4 is defined in Proposition 6.4. Thanks to Lemma 8.5, we get

$$\sum_{\substack{n\ge 1\\(n,3)=1}} \frac{1-F_4(\delta n)}{n} = -\log\delta - \log(2\pi) + \frac{3}{2} + 2\int_0^1 (1-t)\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right| dt$$
$$+\frac{1}{3}\log\delta + \frac{1}{3}\log(2\pi) - \frac{1}{2} + \frac{1}{3}\log 3 - \frac{2}{3}\int_0^1 (1-t)\log\left|\frac{3\pi\delta t}{\sin(3\pi\delta t)}\right| dt.$$

It follows that

$$\sum_{\substack{n\ge 1\\(n,3)=1}} \frac{1-F_4(\delta n)}{n} = -\frac{2}{3}\log\delta - \frac{2}{3}\log(2\pi) + 1 + \frac{1}{3}\log 3$$
$$+2\int_0^1 (1-t)\left(\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right| - \frac{1}{3}\log\left|\frac{3\pi\delta t}{\sin(3\pi\delta t)}\right|\right) dt.$$

Now, we use Proposition 6.4 and Lemma 8.8 to obtain that

$$\left|L(1,\chi)\right| \le -\frac{2}{3}\log\delta - \frac{2}{3}\log(2\pi) + 1 + \frac{1}{3}\log 3 + \frac{\pi^3\delta^2}{18} + \frac{2\pi^2\delta^2}{27} + \frac{\pi}{\sqrt{q}}\sum_{\substack{1\le m\le\delta q\\(m,3)=1}}\left(\frac{m}{\delta q}-1\right)^2.$$

For the last sum above, we use Lemma 8.7 to find that

$$\begin{aligned}\sum_{\substack{1\le m\le\delta q\\(m,3)=1}}\left(\frac{m}{\delta q}-1\right)^2 &= \sum_{\substack{m=3k+1\\1\le m\le\delta q}}\left(\frac{m}{\delta q}-1\right)^2 + \sum_{\substack{m=3k+2\\2\le m\le\delta q}}\left(\frac{m}{\delta q}-1\right)^2\\ &\le \frac{2\delta q}{9} - \frac{14}{\delta^2 q^2} + \frac{14}{3\delta q}.\end{aligned}$$

Then, Eq (8.7) becomes

$$\left|L(1,\chi)\right| \le -\frac{2}{3}\log\delta - \frac{2}{3}\log(2\pi)+1+\frac{1}{3}\log 3+\frac{\pi^3\delta^2}{18}+\frac{2\pi^2\delta^2}{27}+\frac{2\pi\delta\sqrt{q}}{9}+\frac{\pi}{\delta q\sqrt{q}}\left(\frac{14}{3}-\frac{14}{\delta q}\right).$$

The choice $\delta = 3/\left(\pi\sqrt{q}\right)$ yields

$$\left|L(1,\chi)\right| \le \frac{1}{3}\log q - \frac{1}{3}\log(12)+\frac{5}{3}+\frac{1}{q}\left(\frac{\pi}{2}+\frac{2}{3}+\frac{14\pi^2}{9}-\frac{14\pi^3}{9\sqrt{q}}\right) \tag{8.14}$$

with $\frac{5}{3}-\frac{1}{3}\log(12) = 0.8383\cdots$. Setting

$$C_{\text{odd}}(q) = \frac{5}{3}-\frac{1}{3}\log(12)+\frac{1}{q}\left(\frac{\pi}{2}+\frac{2}{3}+\frac{14\pi^2}{9}-\frac{14\pi^3}{9\sqrt{q}}\right) \tag{8.15}$$

it follows that

$$\left|L(1,\chi)\right| \le \frac{1}{3}\log q + C_{\text{odd}}(q). \tag{8.16}$$

We list below the values of $C_{\text{even}}(q)$ and $C_{\text{odd}}(q)$ for different values of q. Thus,

$q \ge$	10^4	10^5	10^6	$2\cdot 10^6$	∞
$C_{\text{even}} \le$	0.395781	0.375558	0.369162	0.368296	0.366205
$C_{\text{odd}} \le$	0.840076	0.838539	0.838382	0.838374	0.838365

Table 8.1: The values of $C_{\text{even}}(q)$ and $C_{\text{odd}}(q)$.

for $q \ge 2\cdot 10^6$, we have proved that

$$|L(1,\chi)| \le \tfrac{1}{3}\log q + \begin{cases} 0.368296 & \text{when } \chi(-1)=1, \\ \\ 0.838374 & \text{when } \chi(-1)=-1. \end{cases}$$

It is important to point out that Ramaré computed $L(1,\chi)$ for all primitive χ of conductor ≤ 4500 by using the version 2.0.11 of the GP-calculator of the PARI system. Platt has kindly agreed to run his algorithm from his thesis [44] (which is rigorous and efficient for computing $L(1,\chi)$ for all primitive χ of conductor $q \le 2\cdot 10^6$). Here are the pictures obtained for the first 100 000 values of $|L(1,\chi)| - \frac{1}{2}\log q$ when $\chi(3)=0$. This completes the proof.

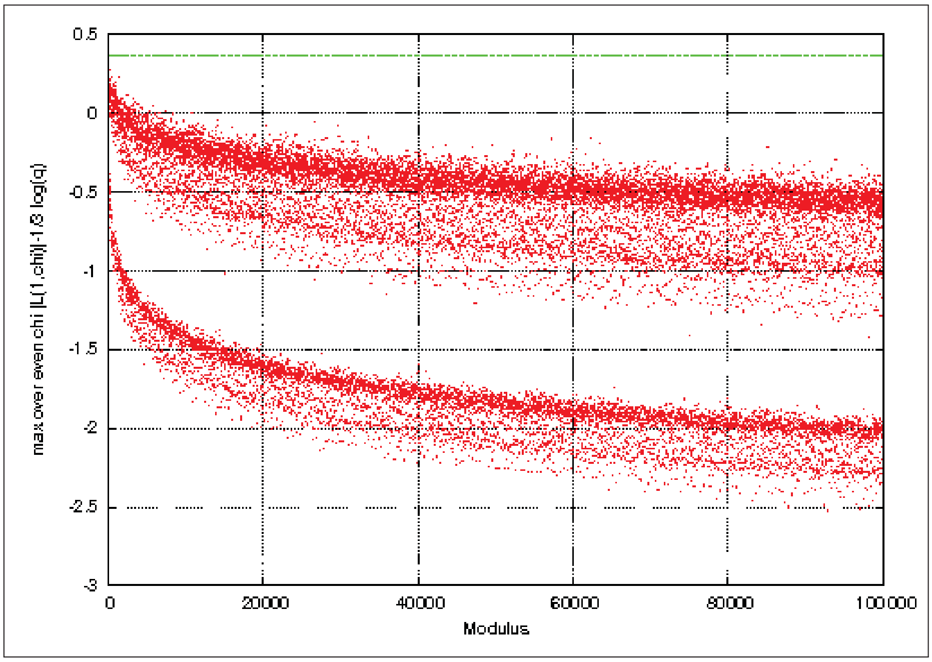

Figure 8.1: The maximum of $|L(1,\chi)| - \frac{1}{3}\log q$ when $\chi(-1) = 1$.

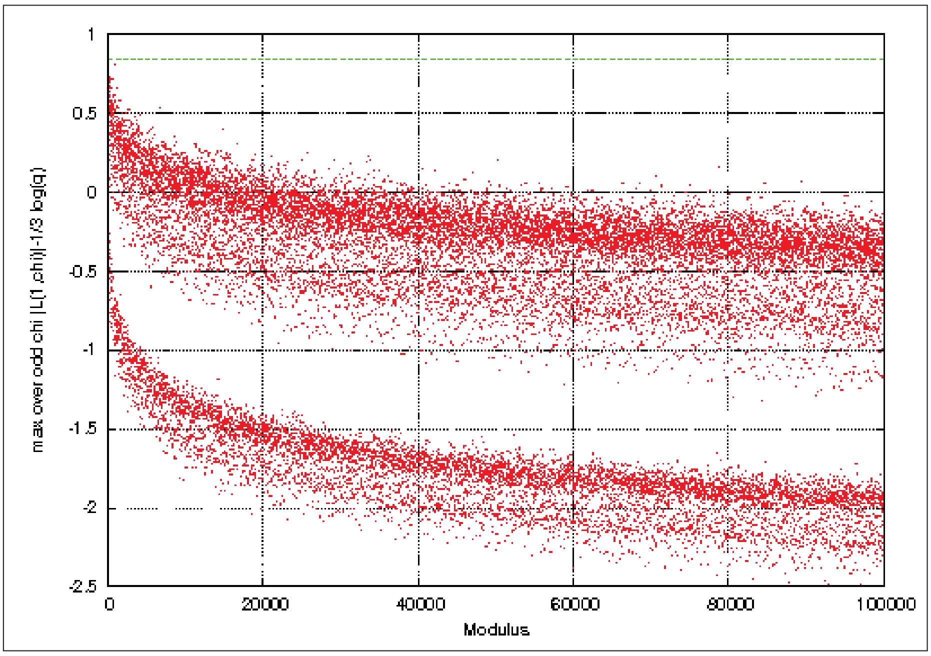

Figure 8.2: The maximum of $|L(1,\chi)| - \frac{1}{3}\log q$ when $\chi(-1) = -1$.

Annexes

Appendix A

Computer programs

In this Appendix, we give the computer code written to do some of the computations in the thesis. All of the code was written for the GP-calculator of the PARI system; version 2.3.4. In the following sections, we go into more details regarding the code created for the second and the third parts in this thesis. The first part in this thesis did not need any special computer programming.

A.1 Structure of computer code for computing $L(1,\chi)$

In this section, we detail the computer script used for computing the value of $L(1,\chi)$ for all primitive χ of conductor q. We use the exact formula of $L(1,\chi)$ which is given by the following form

$$L(1,\chi)=\sum_{n\geq 1}\frac{\chi(n)}{n}=-\frac{2\tau(\chi)}{q}\begin{cases}2\sum_{1\leq m\leq q/2}\bar{\chi}(m)\log\left|\sin\frac{\pi m}{q}\right| & \text{when } \chi(-1)=+1,\\ i\pi\sum_{1\leq m\leq q/2}\bar{\chi}(m)\left(1-\frac{2m}{q}\right) & \text{when } \chi(-1)=-1.\end{cases} \tag{A.1}$$

To use this formula, we need generate the characters, and hence to find generators of the multiplicative groups modulo p^α for each p^α occurring in the prime decomposition of q and to build a table of logarithms for integers $\leq q/2$ and prime to q. With a set of generators at hand, it is very easy to build the characters and to recognize which ones are primitive.

A.1.1 Initializations

We define the data structure used in our script.

- `zqcarac = [1,1,1,1,1];`
  A vector whose coordinates are 5-components vectors:

  $$[q, \varphi(q), \text{case}, \omega(q), \text{number of the primitive characters}]$$

  For the third component vector above, we distinguishe three cases: the first case, when 4 divides q and 8 does not divide q, the second case when 8 divides q, the third case otherwise.
- `qstruct=[[1,1,1,Mod(1,1)]];`
  A vector whose coordinates are 4-components vectors: $[[p,\alpha,\varphi(p^\alpha),\text{primroot}]]$.
  In the third case above, we have
  [1] p a prime number such that p dividing **zqcarac[1]**.
  [2] α is a power of p.
  [3] $\varphi(p^\alpha)$ is the Euler's phi function of p^α.
  [4] primroot means a primitive root modulo p^α.
  In the first case when 4 divides q and 8 does not divide q, we have
  [1] 2
  [2] 2
  [3] 1
  [4] Mod(3, 4)
  In the second case when $8|q$, we have
  [1] 2
  [2] α such that 2^α divides strictly **zqcarac[1]**.
  [3] $2^{\alpha-2}$
  [4] [Mod$(3, 2^\alpha)$, Mod$(5, 2^\alpha)$]
- `zqvalodd=[1];`
  `zqvaleven=[1];`

  It means that there are two vectors of length $[q/2]$ such that the scalar product with a vector of the values of χ gives $L(1,\chi)\cdot\sqrt{q}/\tau(\chi)$.
- `zqlog=[1];`
  A vector of length $[q/2]$ containing 0, when the index m, it is appeared in Eq (A.1), is not prime to q, and containing logarithm of this index otherwise.
- `sup_zqsmalllog=1;`
  `zqsmalllog=[1];`
  `A vector of length sup_zqsmalllog containing logarithm of the index m.`
  `It is faster to compute if sup_zqsmalllog is large`

For simplicity, we define

```
{Initzqdata(zqdata)=
   zqdata = vector(9);
   zqdata[1] = [1,1,1,1,1];
```

```
   zqdata[2] = [[1,1,1,Mod(1,1)]];
   zqdata[3] = 2*Pi;
   zqdata[4] = 2*I*Pi;
   zqdata[5] = [1];
   zqdata[6] = [1];
   zqdata[7] = [1];
   zqdata[8] = 1;
   zqdata[9] = [1];
   return(zqdata);
}
```

The following function build **zqsmalllog**.

```
{Build_zqsmalllog(bound,small=0, zqdata)=
   local(vec_orders, n, compteur,k, zqcarac, zqstruct, TwoPi, TwoIPi,
 zqvalodd, zqvaleven, zqlog, sup_zqsmalllog, zqsmalllog );
   zqcarac = zqdata[1];
   zqstruct = zqdata[2];
   TwoPi = zqdata[3];
   TwoIPi = zqdata[4];
   zqvalodd = zqdata[5];
   zqvaleven = zqdata[6];
   zqlog = zqdata[7];
   sup_zqsmalllog = zqdata[8];
   zqsmalllog = zqdata[9];
   sup_zqsmalllog=bound;
   zqsmalllog=vector(sup_zqsmalllog,n,0);
   vec_orders=vector(zqcarac[4],k,if((k==1)&&(zqcarac[3]==2),[0,0],0));
   for(n=small+1,sup_zqsmalllog,
      if(gcd(n,zqcarac[1])==1,
 for(k=2,zqcarac[4],
    vec_orders[k]=znlog(n,zqstruct[k][4]));
 if(zqcarac[3]==1,
    vec_orders[1]=if(n%4==1,0,1),
    if(zqcarac[3]==2,
       vec_orders[1][1]=if(n%4==1,0,1);
       vec_orders[1][2]=
       znlog((-1)^vec_orders[1][1]*n,Mod(5,zqstruct[1][1]^zqstruct[1][2])),
  \\ Case 3:
       vec_orders[1]=znlog(n,zqstruct[1][4])));
 zqsmalllog[n]=vec_orders
       ,));
```



```
   zqdata[9] = zqsmalllog;
   zqdata[8] = sup_zqsmalllog;
   return(zqdata);
}
```

Here is finally the initialization step used in all the functions of this program.

```
{Get_zqstruct(q, zqdata)=
   local(decomp,k, zqcarac, zqstruct, TwoPi, TwoIPi, zqvalodd,
 zqvaleven, zqlog, sup_zqsmalllog, zqsmalllog);
   zqcarac = zqdata[1];
   zqstruct = zqdata[2];
   TwoPi = zqdata[3];
   TwoIPi = zqdata[4];
   zqvalodd = zqdata[5];
   zqvaleven = zqdata[6];
   zqlog = zqdata[7];
   sup_zqsmalllog = zqdata[8];
   zqsmalllog = zqdata[9];
   if((q%2==0)&&(q%4<>0),q=q/2,);
   decomp=factor(q);
   zqcarac=[q,0,0,length(decomp~),0];
   if(decomp[1,1]==2,
      if(decomp[1,2]==2,zqcarac[3]=1,zqcarac[3]=2),zqcarac[3]=3);
   zqcarac[2]=prod(k=1,zqcarac[4],(decomp[k,1]-1)
    *decomp[k,1]^(decomp[k,2]-1));
   zqcarac[5]=prod(k=1,zqcarac[4],if(decomp[k,2]==1,decomp[k,1]-2,
     decomp[k,1]^(decomp[k,2]-2)*(decomp[k,1]-1)^2));
   zqdata[1] = zqcarac;
   zqstruct=vector(zqcarac[4],k,
   [decomp[k,1],decomp[k,2],
    if((k==1)&&(zqcarac[3]<>3),2^(decomp[k,2]-2),
       (decomp[k,1]-1)*decomp[k,1]^(decomp[k,2]-1)),
    if((k==1)&&(zqcarac[3]==2),[Mod(-1,2^decomp[k,2]),
Mod(5,2^decomp[k,2])],
       znprimroot(decomp[k,1]^decomp[k,2]))]);
   zqdata[2] = zqstruct;
   zqdata[7] = Build_zqlog(zqdata);
   zqvalodd=vector(floor(q/2),n,if(gcd(n,q)==1,Pi*(1-2*n/q)/sqrt(q),0));
   zqdata[5] = zqvalodd;
   zqvaleven = vector(floor(q/2),n,if(gcd(n,q)==1,-2*log(sin(Pi*n/q))
                                                           /sqrt(q),0));
```



```
   zqdata[6] = zqvaleven;
   return(zqdata);
   \\ 0 is set where n is not prime to zqcarac[1]=q ;
   \\ otherwise, exp(TwoIPi*Chi(vec_chi)[n]/zqcarac[2]) is the value of this
   \\ character in n.
}
```

A.1.2 Ranging the field $(\mathbb{Z}/q\mathbb{Z})^\times$

Now, we establishe a mean of ranging over all the elements. The first code gives the first logarithm, the second code gives the next logarithm.

```
{Initial_vec(zqdata)=
   local(zqcarac);
   if(zqcarac[3]==2,return(concat([[1,1]],vector(zqcarac[4]-1,k,1))),
      return(vector(zqcarac[4],k,1)))
}

{Next_elt(vec_orders,  zqdata)=
   local(k,l, zqcarac, zqstruct, TwoPi, TwoIPi, zqvalodd,
 zqvaleven, zqlog, sup_zqsmalllog, zqsmalllog);
   zqcarac = zqdata[1];
   zqstruct = zqdata[2];
   TwoPi = zqdata[3];
   TwoIPi = zqdata[4];
   zqvalodd = zqdata[5];
   zqvaleven = zqdata[6];
   zqlog = zqdata[7];
   sup_zqsmalllog = zqdata[8];
   zqsmalllog = zqdata[9];
   k=zqcarac[4];
   while((k>1)&&(vec_orders[k]==zqstruct[k][3]),k-=1);
   if(k==1,
      if(zqcarac[3]==1,
   \\ Case 1:
 if(vec_orders[1]==1,vec_orders[1]=2,),
 if(zqcarac[3]==2,
   \\ Case 2:
    if(vec_orders[1][2]==zqstruct[1][3],
       if(vec_orders[1][1]==2,,vec_orders[1][1]=2;vec_orders[1][2]=1),
       vec_orders[1][2]+=1),
```



```
   \\ Case 3:
    vec_orders[1]+=1)),
 \\ k<>1:
      vec_orders[k]+=1);
   for(l=k+1,zqcarac[4],vec_orders[l]=1);
   vec_orders
}
```

Now, we build the vector **zqlog** by the following code:

```
{Build_zqlog(zqdata)=
   local(small,vec_orders, n, compteur, k, zqcarac, zqstruct,
   TwoPi, TwoIPi, zqvalodd, zqvaleven, zqlog, sup_zqsmalllog, zqsmalllog);
   zqcarac = zqdata[1];
   zqstruct = zqdata[2];
   TwoPi = zqdata[3];
   TwoIPi = zqdata[4];
   zqvalodd = zqdata[5];
   zqvaleven = zqdata[6];
   zqlog = zqdata[7];
   sup_zqsmalllog = zqdata[8];
   zqsmalllog = zqdata[9];
   zqlog=vector(floor(zqcarac[1]/2),k,0);
   vec_orders=Initial_vec(zqdata);
   compteur=1;
   while(compteur<=zqcarac[2]/2,
 n=Up(vec_orders, zqdata);
 if(n<zqcarac[1]/2, \\ It cannot be equal.
    small=1,
    n=zqcarac[1]-n;
    small=0);
 zqlog[n]=vector(zqcarac[4],k,
 if(k<2,0,
    if(small,vec_orders[k],(vec_orders[k]+zqstruct[k][3]/2)
       %zqstruct[k][3])));
 if(zqcarac[3]==1,
    \\ Case 1:
    zqlog[n][1]=if(small,vec_orders[1]%2,(1-vec_orders[1])%2),
    if(zqcarac[3]==2,
     \\ Case 2:
       zqlog[n][1]=if(small,vec_orders[1],[3-vec_orders[1][1],
```



```
   vec_orders[1][2]]),
     \\ Case 3:
       zqlog[n][1]=if(small,vec_orders[1],(vec_orders[1]+zqstruct[1][3]/2)
      %zqstruct[1][3])));
 vec_orders=Next_elt(vec_orders, zqdata);
 compteur+=1);
   zqdata[7] = zqlog;
   return(zqlog);
}
```

A.1.3 Ranging the characters

We also establish a mean of ranging over all the characters. The first code gives the first character, the second code gives the next character.

```
{Initial_Chi(zqdata)=
   local(zqcarac);
   zqcarac = zqdata[1];
   if(zqcarac[3]==2,return(concat([[1,1]],vector(zqcarac[4]-1,k,1))),
      return(vector(zqcarac[4],k,1)))
}

{Next_chi(vec_chi, zqdata)=
   local(k,l, zqcarac, zqstruct, TwoPi, TwoIPi, zqvalodd,
 zqvaleven, zqlog, sup_zqsmalllog, zqsmalllog);
   zqcarac = zqdata[1];
   zqstruct = zqdata[2];
   TwoPi = zqdata[3];
   TwoIPi = zqdata[4];
   zqvalodd = zqdata[5];
   zqvaleven = zqdata[6];
   zqlog = zqdata[7];
   sup_zqsmalllog = zqdata[8];
   zqsmalllog = zqdata[9];
   k=zqcarac[4];
   while((k>1)&&(vec_chi[k]==zqstruct[k][3]-1),k-=1);
   if(k==1,
      if(zqcarac[3]==1,
 if(vec_chi[1]==1,vec_chi[1]=2,),
 if(zqcarac[3]==2,
    if(vec_chi[1][2]==zqstruct[1][3]-1,
```



```
        if(vec_chi[1][1]==2,,vec_chi[1][1]=2;vec_chi[1][2]=1),
        vec_chi[1][2]+=2),
    \\ k=1, but we're in case 3:
     vec_chi[1]+=1;
     if(vec_chi[1]%zqstruct[1][1],,vec_chi[1]+=1))),
  \\ k<>1:
       vec_chi[k]+=1;
       if(vec_chi[k]%zqstruct[k][1],,vec_chi[k]+=1));
  \\re-initialise the next entries:
    for(l=k+1,zqcarac[4],vec_chi[l]=1);
    return(vec_chi);
}
```

A.1.4 Let us use the above !

We build here the code to detect whether a Dirichlet character is even or odd :

```
{Is_even(vec_chi, zqdata)=
  \\ Utiliser Get_zqstruct avant.
   local(k,res,l, zqcarac);
   zqcarac = zqdata[1];
  \\ Utiliser Get_zqstruct avant.
   if(zqcarac[3]==1,res=vec_chi[1];k=2,
      if(zqcarac[3]==2,res=vec_chi[1][1];k=2,res=0;k=1));
   if(sum(l=k,zqcarac[4],vec_chi[l],res)%2,return(0),return(1));
}
```

The following function returns the **log*zqcarac[2]** of the value of χ to **vec-orders**

```
{Evaluate(vec_chi,vec_orders, zqdata)=
  local(val,k, zqcarac, zqstruct, TwoPi, TwoIPi, zqvalodd,
zqvaleven, zqlog, sup_zqsmalllog, zqsmalllog);
   zqcarac = zqdata[1];
   zqstruct = zqdata[2];
   TwoPi = zqdata[3];
   TwoIPi = zqdata[4];
   zqvalodd = zqdata[5];
   zqvaleven = zqdata[6];
   zqlog = zqdata[7];
   sup_zqsmalllog = zqdata[8];
   zqsmalllog = zqdata[9];
   if(type(vec_orders)=="t_INT",val=0,
```

```
      val=sum(k=2,zqcarac[4],
      vec_chi[k]*vec_orders[k]*zqcarac[2]/zqstruct[k][3],Mod(0,zqcarac[2]));
      if(zqcarac[3]==1,
      \\ Case 1:
 val=lift(val+vec_chi[1]*vec_orders[1]*zqcarac[2]/2),
 if(zqcarac[3]==2,
      \\ Case 2:
    val=lift(val+vec_chi[1][1]*vec_orders[1][1]*zqcarac[2]/2
     +vec_chi[1][2]*vec_orders[1][2]*zqcarac[2]/zqstruct[1][3]),
      \\ Case 3:
    val=lift(val+vec_chi[1]*vec_orders[1]*zqcarac[2]/zqstruct[1][3]))));
   return(val);
}

{Evaluate(vec_chi,vec_orders, zqdata)=
  local(val,k, zqcarac, zqstruct, TwoPi, TwoIPi, zqvalodd,
zqvaleven, zqlog, sup_zqsmalllog, zqsmalllog);
   zqcarac = zqdata[1];
   zqstruct = zqdata[2];
   TwoPi = zqdata[3];
   TwoIPi = zqdata[4];
   zqvalodd = zqdata[5];
   zqvaleven = zqdata[6];
   zqlog = zqdata[7];
   sup_zqsmalllog = zqdata[8];
   zqsmalllog = zqdata[9];
   if(type(vec_orders)=="t_INT",val=0,
   \\ We calculate the value of the character on n.
      val=sum(k=2,zqcarac[4],
      vec_chi[k]*vec_orders[k]*zqcarac[2]/zqstruct[k][3],Mod(0,zqcarac[2]));
      if(zqcarac[3]==1,
      \\ Case 1:
 val=lift(val+vec_chi[1]*vec_orders[1]*zqcarac[2]/2),
 if(zqcarac[3]==2,
      \\ Case 2:
    val=lift(val+vec_chi[1][1]*vec_orders[1][1]*zqcarac[2]/2
     +vec_chi[1][2]*vec_orders[1][2]*zqcarac[2]/zqstruct[1][3]),
      \\ Case 3:
    val=lift(val+vec_chi[1]*vec_orders[1]*zqcarac[2]/zqstruct[1][3]))));
   return(val);
}
```

Thanks to the following code, we check that the modulus of the Gauss sums divided by $\sqrt{q}$ is equal to 1, and reduce execution time is half.

```
{Easy_LOne(vec_chi, zqdata)=
   local(n,beta, zqcarac, zqstruct, TwoPi, TwoIPi, zqvalodd,
 zqvaleven, zqlog, sup_zqsmalllog, zqsmalllog);
   zqcarac = zqdata[1];
   zqstruct = zqdata[2];
   TwoPi = zqdata[3];
   TwoIPi = zqdata[4];
   zqvalodd = zqdata[5];
   zqvaleven = zqdata[6];
   zqlog = zqdata[7];
   sup_zqsmalllog = zqdata[8];
   zqsmalllog = zqdata[9];
   beta = -TwoIPi/zqcarac[2];
   if(Is_even(vec_chi, zqdata),
      return(vector(floor(zqcarac[1]/2),n,exp(beta*Evaluate(vec_chi,
      zqlog[n], zqdata))) *zqvaleven~),
      return(vector(floor(zqcarac[1]/2),n,exp(beta*Evaluate(vec_chi,
      zqlog[n], zqdata))) *zqvalodd~))
}
```

The following function gives the value of χ on n.

```
{Chi_Of(vec_chi,n, zqdata)=
   local(zqcarac, zqstruct, TwoPi, TwoIPi, zqvalodd,
 zqvaleven, zqlog, sup_zqsmalllog, zqsmalllog);
   zqcarac = zqdata[1];
   zqstruct = zqdata[2];
   TwoPi = zqdata[3];
   TwoIPi = zqdata[4];
   zqvalodd = zqdata[5];
   zqvaleven = zqdata[6];
   zqlog = zqdata[7];
   sup_zqsmalllog = zqdata[8];
   zqsmalllog = zqdata[9];
   n=n%zqcarac[1];
   if(n==0,1,  \\ <-- somehow unusual ...
      if(n<=zqcarac[1]/2,
```



```
 if(type(zqlog[n])=="t_INT",
    return(0),
    return(exp(TwoIPi/zqcarac[2]*Evaluate(vec_chi,zqlog[n])))),
 if(type(zqlog[zqcarac[1]-n])=="t_INT",
    return(0),
    if(Is_even(vec_chi, zqdata),
       return(exp(TwoIPi/zqcarac[2]*Evaluate(vec_chi,zqlog[zqcarac [1]-n],
     zqdata))),
       return(-exp(TwoIPi/zqcarac[2]*Evaluate(vec_chi,zqlog[zqcarac[1]-n],
      zqdata)))))))
}
```

A.2 Computer code for computing $L(1,\chi)$ when χ is even and $\chi(2) = 1$

We now have all the ingredients to be able to list the main code for even Dirichlet character and under the condition $\chi(2) = 1$.

```
{Going_alone_even_special(q, ratio=1/2)=
  local(vec_chi,compteur,k,store_even_special,maintermeven,exists_special=0,
zqdata, zqcarac);
   zqdata = Initzqdata(zqdata);
   zqdata = Get_zqstruct(q, zqdata);
   zqcarac = zqdata[1];
   maintermeven=ratio*log(q);
   compteur = 1;
   store_even_special = -10;
   vec_chi = Initial_Chi(zqdata);
   while(compteur <= zqcarac[5], \\Only primitive characters are considered.
 if(Is_even(vec_chi)&&(Chi_Of(vec_chi, 2, zqdata)==1), exists_special=1;
       store_even_special=max(store_even_special,
      abs(Easy_LOne(vec_chi, zqdata))-maintermeven),);
 vec_chi=Next_chi(vec_chi, zqdata);
 compteur+=1);
   if(exists_special==1,
      write(WhereToWrite,
 concat(["q = ",zqcarac[1]," Pairs (chi(2) = 1) : ",store_even_special])),);
}
```

```
{Test(borneinf,bornesup)=
```



```
   local(q);
   write(WhereToWrite,"ratio = 1/2");
   for(q=borneinf,bornesup,
      if((q%2==0)&&(q%4<>0),,print1(q,"...");Going_alone_even_special(q);
 print("done.")));
}
```

The short table below offers for some moduli the maxima of $\left|L(1,\chi)\right| - \frac{1}{2}\log q$ for even χ and $\chi(2) = 1$. Results have been rounded up at the tenth decimal place.

q	χ even
17	-0.4005218381
997	-0.5763130274
1009	-0.3799133660
11089	-0.7162145960
56251	-1.6402531903
87481	-0.8807228987
117049	-0.8890177956
173209	-1.1231276690
223609	-1.1052272316
243721	-1.2973247347

Table A.1: Some moduli the maxima of $\left|L(1,\chi)\right| - \frac{1}{2}\log q$ for χ is even and $\chi(2) = 1$.

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