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Théorie de l'homotopie des algèbres sur un PROP

THÈSE

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par

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Avant-propos

Cette version préliminaire de ma thèse se compose de quatre parties :

Chapitre 0. une introduction aux thèmes étudiés dans la thèse, suivie d'un résumé des travaux présentés dans les parties suivantes,

Chapitre 1. une version longue (48 pages) de l'article *The homotopy theory of bialgebras over pairs of operads*, 27 pages, 2013 (soumis),

Chapitre 2. l'article *Classifying spaces and moduli spaces of algebras over a prop*, 41 pages, 2012 (soumis),

Chapitre 3. l'article *Simplicial localization of homotopy generalized bialgebras*, 12 pages, 2013.

La première partie est en français alors que les trois chapitres suivants sont en anglais. Chaque partie est suivie de sa propre bibliographie.

Avertissement

Les articles constituant les chapitres 1 à 3 ont été intégré sans modifications. Chaque article comprend notamment une introduction propre rappelant les définitions et résultats qui y sont utilisés. Ces trois chapitres peuvent donc être lus indépendamment les uns des autres.

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Résumé

Le but de cette thèse est de mettre en place une théorie d'homotopie générale pour les catégories de bigèbres différentielles graduées.

Une première partie est consacrée au cas des catégories de bigèbres définies par un couple d'opérades en distribution. Les bigèbres classiques, les bigèbres de Lie, les bigèbres de Poisson fournissent des exemples de telles structures de bigèbres. Le résultat principal de cette partie montre que la catégorie des bigèbres associée à un couple d'opérades en distribution hérite d'une structure de catégorie de modèles.

La notion de PROP donne un cadre pour étudier des structures de bigèbres générales, impliquant des opérations à plusieurs entrées et plusieurs sorties comme générateurs de la structure, par opposition aux opérades en distribution qui ne permettent de coder que des opérations à une seule entrée ou à une seule sortie seulement. Les PROPs forment une catégorie, dans laquelle on peut définir une notion d'objet cofibrant avec de bonnes propriétés homotopiques.

La seconde partie de la thèse est consacrée à la théorie homotopique des bigèbres sur un PROP. Le résultat principal de la thèse est que les catégories de bigèbres associées à des PROPs cofibrants faiblement équivalents ont des catégories homotopiques équivalentes. En fait, on prouve un théorème plus précis qui donne une équivalence au niveau des localisations simpliciales des catégories.

Notre théorème entraine que la catégorie des bigèbres associée à une résolution cofibrante d'un PROP donné P définit une notion de bigèbre à homotopie près sur P indépendante du choix de la résolution, et permet de donner un sens à des problèmes de réalisation homotopiques dans ce cadre.

Mots clés : PROP, bigèbre, algèbre homotopique, espace classifiant.

Abstract

The purpose of this thesis is to set up a general homotopy theory for categories of differential graded bialgebras.

A first part is devoted to the case of bialgebras defined by a pair of operads in distribution. Classical bialgebras, Lie bialgebras and Poisson bialgebras provide examples of such bialgebra structures. The main result of this part asserts that the category of bialgebras associated to a pair of operads in distribution inherits a model category structure.

The notion of a PROP provides a setting for the study of general bialgebras structures, involving operations with multiple inputs and multiple outputs as generators of the structure, in contrast to operads in distribution which only encode operations with either one single input or one single output. PROPs form a category, in which one can define a notion of cofibrant object with good homotopical properties.

The second part of the thesis is devoted to the homotopy theory of bialgebras over a PROP. The main result of the thesis asserts that the categories of bialgebras associated to weakly equivalent cofibrant props have equivalent homotopy categories. We actually prove a more precise theorem asserting that this equivalence holds at the level of a simplicial localization of the categories.

Our theorem implies that the category of bialgebras associated to a cofibrant resolution of a given PROP P defines a notion of bialgebra up to homotopy over P independent of the choice of the resolution, and enables us to give a sense to homotopical realization problems in this setting.

Keywords : PROP, bialgebra, homotopical algebra, classifying space.

Chapitre 0

Présentation des résultats

La notion de PROP remonte aux travaux de Mac Lane [31] en algèbre, dont le but était d'étudier les propriétés de la construction bar itérée sur les algèbres de Hopf. Il s'agit d'un acronyme pour PROduct and Permutation. Les PROPs firent leur apparition en topologie quelques années plus tard sous le nom de catégories d'opérateurs de forme standard dans les travaux de Boardman et Vogt [5] et [6]. Les cas particuliers de PROPs considérés dans leurs travaux sont engendrés par des opérations avec de multiples entrées mais une seule sortie. May [34] axiomatisa la structure définie par ces opérations génératrices et lui donna le nom d'opérade. Les opérades et PROPs sont notamment intéressants à étudier pour leur représentations, i.e leurs algèbres. En effet, leurs structures combinatoires permettent de décrire respectivement diverses structures d'algèbres et de bigèbres, en un sens plus large que les algèbres associatives et bigèbres classiques. La première application d'une telle idée fut le principe de reconnaissance de May identifiant les espaces de lacets itérés comme des algèbres sur l'opérade des petits disques.

Dans les années 80 et 90, de multiples découvertes apportèrent un nouvel éclairage sur l'importance de travailler avec de telles structures. Concernant les opérades, plusieurs papiers fondateurs devinrent les piliers de travaux fondamentaux sur des sujets variés. Le formalisme opéradique s'est par conséquent étendu à divers domaines des Mathématiques : algèbre, topologie, théorie de la déformation, théorie des catégories, géométries différentielle, algébrique et énumérative, physique mathématique, informatique théorique... Concernant les PROPs, les travaux de Drinfeld sur les groupes quantiques [9] et [10] mirent en évidence l'importance de structures algébriques à plusieurs entrées et sorties différentes des bigèbres commutatives et cocommutatives usuelles. Le formalisme des props joue un rôle fondamental dans la quantification des bigèbres de Lie par Etingof et Kazdhan [15] ainsi que l'approche de Merkulov [36], et plus généralement dans l'étude des foncteurs de quantification comme l'ont montré Enriquez et Etingof [14] En topologie, l'homologie de certains espaces topologiques s'est révélée contenir différentes structures de bigèbres. Ainsi, à la fin des années 90, la naissance de la topologie des cordes avec les travaux de Chas et Sullivan [7] mit en évidence une structure de bigèbre de Lie involutive sur l'homologie (équivariante) des espaces de lacets libres sur les variétés [8]. L'homologie de tout espace à dualité de Poincaré (en particulier les variétés compactes orientées) possède quant à elle une structure de bigèbre de Frobenius, qui s'avère provenir d'une structure plus fine au niveau des chaines d'après les travaux de Lambrechts et Stanley [28]. On peut aussi noter l'importance des PROPs pour définir de manière compacte différentes notions de théories de champs comme les théories de champs quantiques topologiques [27] (bigèbres de Frobenius) ou les théories de champs conformes (algèbres sur les chaines du PROP de Segal). Plus récemment, les opérations naturelles sur l'homologie de Hochschild d'une algèbre sur un PROP avec multiplication ont fait l'objet des travaux de Wahl dans [42], recouvrant en particulier les opérations obtenues dans le cas des théories de champs et permettant d'exhiber des familles non triviales d'opérations supérieures en topologie des cordes. Notons finalement que les props ont été utilisés par Merkulov pour décrire des structures géométriques locales en géométrie différentielle formelle [35].

On définit les structures algébriques avec lesquels on souhaite travailler en se fixant une opérade ou un PROP P. Il est alors naturel de les organiser en une catégorie, la catégorie des algèbres sur P ou des P-algèbres, puis d'étudier les propriétés de tels objets au sein de cette catégorie. On peut s'intéresser notamment à leurs déformations et à la construction d'une notion d'homotopie. L'algèbre homotopique développée dans le cadre des catégories de modèles de Quillen [37] fournit une panoplie d'outils effectifs afin de mener à bien une telle étude. La construction d'une catégorie de modèles des algèbres sur une opérade remonte au travaux de Hinich dans le cadre différentiel gradué [20] avant d'être généralisée aux catégories de modèles monoïdales symétriques. Les props forment quant à eux des objets combinatoires a priori gigantesques comparés aux opérades : ils font intervenir des graphes à plusieurs entrées et sorties, de genre quelconque et non connexes, alors que les opérades se restreignent à des arbres. Cette "explosion combinatoire" des props a laissé ouverte l'étude de leurs algèbres dans un contexte homotopique, notamment à cause de l'absence d'un foncteur algèbre libre dans les catégories non cartésiennes (par exemple les complexes de chaines). Notre travail dans cette thèse a été d'apporter des réponses sur différents points d'une telle étude. Ce premier chapitre introduit brièvement dans une première section quelques notions et résultats fondamentaux sur les opérades, props et leurs algèbre. La deuxième section est dédiée à la théorie de l'homotopie des théories d'homotopie, ou des $(\infty, 1)$-catégories, dont les catégories de modèles forment un cas particulier. Les trois sections suivantes résument les résultats et points clés des trois articles.

0.1 Introduction aux opérades, PROPs et leurs algèbres

L'objet de cette section, ainsi que de la suivante, est de rappeler le cadre de travail dans lequel s'énoncent les résultats originaux des chapitres suivants. On travaille dans la catégorie $Ch_{\mathbb{K}}$ des complexes de chaines sur un corps $\mathbb{K}$, néanmoins beaucoup de résultats énoncés dans cette section ont leur analogue dans des catégories de modèles monoïdales symétriques plus générales. On renvoie le lecteur aux ouvrages [16], [30] et [33] pour plus de détails et une exploration en profondeur de la théorie et de ses applications.

0.1.1 Opérades

Opérades et leurs algèbres

Rappelons tout d'abord qu'un Σ-objet est une collection $M = \{M(n)\}_{n\in\mathbb{N}}$ de complexes de chaines telle que chaque $M(n)$ forme un $\mathbb{K}[\Sigma_n]$-module à droite, i.e est muni d'une action à droite du groupe Σ_n des permutations de $\{1, ..., n\}$. A tout Σ-objet M on peut associer un foncteur dit de Schur

$$M : Ch_{\mathbb{K}} \to Ch_{\mathbb{K}}$$

défini par

$$M(X) = \bigoplus_n (M(n) \otimes X^{\otimes n})_{\Sigma_n}$$

pour tout complexe de chaines X. Ici $(M(n) \otimes X^{\otimes n})_{\Sigma_n}$ est l'espace des coinvariants de $M(n) \otimes X^{\otimes n}$ sous l'action diagonale de Σ_n. En caractéristique zéro, l'espace des coinvariants est isomorphe à l'espace des invariants $(M(n) \otimes X^{\otimes n})^{\Sigma_n}$.

Définition 0.1.2. *Une opérade est un Σ-objet tel que le foncteur de Schur associé forme un monoïde dans la catégorie des endofoncteurs de $Ch_{\mathbb{K}}$ pour la composition des foncteurs. Il s'agit donc d'un cas particulier de monade.*

On peut voir une opérade comme une collection $P = \{P(n)\}$ d'espaces $P(n)$ d'opérations à n entrées et une sortie, munie d'une unité $1 \in P(1)$ et de produits de composition de ces opérations

$$\circ : P(n) \otimes P(k_1) \otimes ... \otimes P(k_n) \to P(k_1 + ... + k_n).$$

Ces produits de composition vérifient des diagrammes d'associativité, d'unitarité et de compatibilité avec les actions des groupes symétriques. Graphiquement, si l'on représente les opérations par des arbres étiquetés, ces produits de compositions correspondent à greffer des arbres entre eux, en particulier greffer la racine d'un arbre à l'une des entrées d'un autre arbre. On appelle n l'arité d'une opération de $P(n)$.

Exemple 0.1.3. *Soit X un complexe de chaines, l'opérade des endomorphismes de X est notée End_X et est définie par*

$$End_X(n) = Hom_{\mathbb{K}}(X^{\otimes n}, X)$$

où $Hom_{\mathbb{K}}(-,-)$ est le bifoncteur hom interne des complexes de chaines (i.e l'espace des morphismes de complexes de chaines enrichi dans les complexes de chaines). Les produits de composition correspondent ici aux compositions partielles usuelles des applications multilinéaires, et les groupes symétriques agissent par permutation des variables à la source.

L'opérade des endomorphismes permet d'encoder de manière concise la notion d'algèbre sur une opérade :

Définition 0.1.4. *Soit P une opérade. Une algèbre sur P, ou P-algèbre, est la donnée d'un complexe X et d'un morphisme d'opérades $P \to End_X$.*

Il s'agit donc d'un complexe de chaines muni d'une famille d'applications équivariantes $P(n) \otimes X^{\otimes n} \to X$ associatives par rapport aux produits de composition de P.

Il existe par exemple une opérade Ass telle que les Ass-algèbres sont les algèbres associatives différentielles graduées, une opérade Com telle que les Com-algèbres sont les algèbres associatives commutatives, une opérade Lie telle que les Lie-algèbres sont les algèbres de Lie, etc. De plus, pour toute opérade P et tout complexe X, l'objet $P(X)$ est la P-algèbre libre sur X, de sorte que le foncteur de Schur associé à P est le foncteur P-algèbre libre

$$P : Ch_{\mathbb{K}} \to Ch_{\mathbb{K}}^P$$

où $Ch_{\mathbb{K}}^P$ est la catégorie des P-algèbres.

De manière duale, on peut définir une coopérade comme un Σ-objet dont le foncteur de Schur est muni d'une structure de comonoïde pour la composition (i.e forme une comonade), ainsi qu'une notion de cogèbre. On reviendra sur cette dernière ultérieurement.

Théorie de l'homotopie des opérades

On rappelle que $Ch_{\mathbb{K}}$ forme une catégorie de modèles cofibrement engendrée dont les fibrations sont les surjections degré par degré, les cofibration les injections degré par degré et les équivalences faibles les quasi-isomorphismes.

Théorème 0.1.5. *(Hinich [20]) Les opérades dans $Ch_{\mathbb{K}}$ forment une catégorie de modèles cofibrement engendrée, dont les fibrations et équivalences faibles sont déterminées arité par arité.*

On peut donc produire des résolutions cofibrantes afin d'étudier les déformations des opérades et de leurs algèbres. Il existe même des résolutions explicites. Une résolution cofibrante fonctorielle applicable à toute opérade est la résolution bar-cobar. La construction bar est un foncteur $\mathcal{B}$ des opérades vers les coopérades et la construction cobar Ω est un foncteur défini dans l'autre direction, les deux formant une adjonction. L'évaluation de la counité de cette adjonction sur une opérade P est la résolution bar-cobar $\Omega\mathcal{B}(P) \xrightarrow{\sim} P$. La construction cobar produit une opérade quasi-libre, i.e de la forme $(\mathcal{F}(M), \partial)$ où $\mathcal{F}$ est le foncteur opérade libre sur les Σ-objets et ∂ une différentielle. Les opérades quasi-libres forment des objects cofibrants dans la catégorie de modèles des opérades.

Une telle résolution est toutefois peu utilisable concrètement car combinatoirement énorme (il s'agit grosso modo d'un foncteur colibre composé avec un foncteur libre, donc d'arbres indexés par des arbres). La théorie de la dualité de Koszul développée par Ginzburg et Kapranov [19] permet d'obtenir pour certaines opérades une résolution cofibrante beaucoup plus petite. L'idée est que l'on peut associer à toute opérade P dite quadratique une coopérade $P^{¡}$ appelée sa duale de Koszul, ainsi qu'un morphisme d'opérades $\Omega(P^{¡}) \to P$, de source cofibrante. La duale de Koszul d'une opérade est bien plus petite que sa construction bar. On peut dire approximativement qu'il s'agit de l'homologie de la construction bar par rapport à une différentielle que l'on ne précisera pas ici. L'opérade P est de Koszul si et seulement si ce morphisme est une équivalence faible. De plus, lorsque la différentielle interne de P est nulle (on rappelle que P est définie dans les complexes de chaines donc a priori munie d'une différentielle), la résolution $\Omega(P^{¡}) \to P$ est un modèle minimal de P :

Définition 0.1.6. *L'opérade libre $\mathcal{F}(M)$ admet une décomposition*

$$\mathcal{F}(M) = \bigoplus_n \mathcal{F}_{(n)}(M)$$

où $\mathcal{F}_{(n)}(M)$ est engendré par les arbres à n sommets décorés par des éléments de M. Une opérade quasi-libre $(\mathcal{F}(M), \partial)$ est minimale si la différentielle vérifie

$$\partial(M) \subset \bigoplus_{n\geq 2} \mathcal{F}_{(n)}(M).$$

Cette notion analogue aux modèles minimaux en homotopie rationnelle a été introduite par Markl dans [32]. Ici M est l'espace des indécomposables, les générateurs de l'opérade. La différentielle d'un modèle minimal envoie les générateurs dans les éléments décomposables, i.e les opérations s'écrivant comme des compositions opéradiques d'autres opérations. Un modèle minimal d'une opérade est unique à isomorphisme près, on parle donc "du" modèle minimal.

Les résolutions cofibrantes d'opérades permettent de définir la notion de P-algèbre à homotopie près :

Définition 0.1.7. *(1) Soit P une opérade. Une P-algèbre à homotopie près est une algèbre sur une résolution cofibrante de P.*

(2) Soit P une opérade de Koszul. Une P-algèbre fortement à homotopie près est une $\Omega(P^{¡})$-algèbre.

Intuitivement, sur une algèbre à homotopie près, les relations définissant l'algèbre "stricte" sont vérifiées seulement à un système cohérent d'homotopies supérieures près. On voit que la notion d'algèbre fortement à homotopie près est bien définie par unicité du modèle minimal. La question d'une cohérence, au moins homotopique, de la notion d'algèbre à homotopie près est abordée dans la section suivante.

Théorie de l'homotopie des algèbres sur une opérade

La structure de modèles des opérades est héritée par théorème de transfert le long de l'adjonction entre le foncteur opérade libre et les Σ-objets. Les Σ-objets forment en effet eux-mêmes une catégorie de modèles cofibrement engendrée. On peut donc définir une opérade Σ-cofibrante comme une opérade dont le Σ-objet sous-jacent est cofibrant. Cela nous permet d'énoncer le théorème suivant dû à Hinich [20] :

Théorème 0.1.8. *(Hinich [20]) Soit P une opérade Σ-cofibrante. La catégorie des P-algèbres forme une catégorie de modèles cofibrement engendrée dont les fibrations et équivalences faibles sont définies par le foncteur oubli $U : Ch_{\mathbb{K}}^P \rightarrow Ch_{\mathbb{K}}$. Lorsque le corps $\mathbb{K}$ est de caractéristique zéro, ce résultat est valable pour une opérade quelconque.*

Ce résultat s'obtient lui aussi par transfert le long de l'adjonction entre foncteur libre et foncteur oubli.

Revenons à la notion d'algèbre à homotopie près. Les opérades cofibrantes étant en particulier Σ-cofibrantes, le théorème précédent s'applique à elles. De plus, tout morphisme d'opérades $\varphi : P \rightarrow Q$ induit une adjonction $\varphi^* : Ch_{\mathbb{K}}^Q \leftrightarrows Ch_{\mathbb{K}}^P : \varphi_!$. Ces observations permettent d'obtenir le résultat suivant :

Théorème 0.1.9. *(Rezk [38]) Soit $\varphi : P \xrightarrow{\sim} Q$ une équivalence faible d'opérades Σ-cofibrantes. L'adjonction*

$$\varphi^* : Ch_{\mathbb{K}}^Q \leftrightarrows Ch_{\mathbb{K}}^P : \varphi_!$$

forme une équivalence de Quillen.

Une adjonction entre deux catégories de modèles est une équivalence de Quillen lorsque l'on peut dériver (au sens de Quillen) les foncteurs adjoints, et que les foncteurs dérivés forment une équivalence des catégories homotopiques associées. En particulier, la catégorie homotopique des P-algèbres à homotopie près ne dépend pas du choix d'une résolution cofibrante de P.

Lois de distribution et bigèbres généralisées

Bien que les bigèbres soient a priori codées par des PROPs, on peut dans certains cas utiliser des couples d'opérades pour obtenir une notion équivalente (on peut reconstruire le PROP en question à partir des deux opérades). Pour cela on utilise la notion de loi de distribution. Une loi de distribution entre une monade P et une comonade C est une transformation naturelle $P \circ C \rightarrow C \circ P$ vérifiant des relations de compatibilité avec la structure de monoïde de P et celle de comonoïde de Q. Soient P une opérade et Q^* la coopérade obtenue à partir d'une opérade Q par dualisation linéaire des $Q(n)$. On doit ici supposer les $Q(n)$ de dimensions finies afin d'avoir une structure coopéradique bien définie. Les foncteurs de Schur associés forment respectivement une monade et une comonade. Une loi de distribution opéradique entre P et Q^* est une certaine famille d'applications telle que les foncteurs de Schurs associés sont reliés par une loi de distribution au sens monadique. Pour la définition explicite et plus de détails on renvoie le lecteur à l'article de Fox et Markl sur le sujet [18]. Cette compatibilité permet de définir alternativement une (P,Q)-bigèbre comme une P-algèbre dans les Q-cogèbres ou une Q-cogèbre dans les P-algèbres.

De nombreux types de bigèbres, appelées bigèbres généralisées, sont en fait codées par un couple d'opérades en distribution. Des exemples notables sont les bigèbres associatives-coassociatives, les bigèbres infinitésimales, les bigèbres de Lie. Pour plus d'exemples on peut lire la liste proposée dans la monographie de Loday [29]. Néanmoins, il existe des bigèbres ne rentrant pas dans le cadre des lois de distribution opéradiques, deux exemples notoires étant les bigèbres de Frobenius (où l'on a en fait deux règles de réécriture simultanées) codant les théories de champs quantiques topologiques et les algèbres sur les chaines du PROP de Segal codant les théories de champ conformes.

0.1.10 PROPs

PROPs et leurs algèbres

La définition originelle de MacLane [31] est la suivante :

Définition 0.1.11. *Un PROP dans une catégorie monoïdale symétrique $\mathcal{C}$ est une petite catégorie monoïdale symétrique P, catégorie enrichie dans $\mathcal{C}$, dont les objets sont les entiers naturels ($ob(P) = ob(\mathbb{N})$) et le produit tensoriel est donné par l'addition au niveau des objets ($m \otimes n = m + n$).*

On peut définir un PROP dans un formalisme plus "opéradique". Un Σ-biobjet est une collection $M = \{M(m,n)\}_{m,n\in\mathbb{N}}$ d'objets de $\mathcal{C}$ telle que chaque $M(m,n)$ est muni d'une action à droite de Σ_m et d'une action à gauche de Σ_n commutant entre elles. Un PROP est alors un Σ-biobjet P muni de produits de composition verticaux

$$\circ_v : P(k,n) \otimes P(m,k) \rightarrow P(m,n),$$

de produits de composition horizontaux

$$\circ_h : P(m_1,n_1) \otimes P(m_2,n_2) \rightarrow P(m_1+n_1, m_2+n_2)$$

et d'unités

$$1_{\mathcal{C}} \rightarrow P(n,n)$$

vérifiant des axiomes de compatibilité issus de ceux d'une catégorie monoïdale symétrique. Une description détaillée est donné dans [14] dans le cas des props dans les modules sur un anneau. On travaille ici dans le cas $\mathcal{C} = Ch_{\mathbb{K}}$.

Le PROP des endomorphismes d'un complexe X est noté End_X et est défini par $End_X(m,n) = Hom_{\mathbb{K}}(X^{\otimes m}, X^{\otimes n})$ où $Hom_{\mathbb{K}}(-,-)$ est le hom interne des complexes de chaines. Les groupes symétriques agissent par permutation des variables à la source et au but. Les produits de composition verticaux sont les compositions usuelles d'applications, et les produits de compositions horizontaux sont les produits tensoriels d'applications. On peut alors définir la notion d'algèbre sur un PROP P comme la donnée d'un complexe X muni d'un morphisme de props $P \rightarrow End_X$. Les algèbres sur un PROP comprennent un vaste éventail d'exemples, puisque l'on autorise des opérations le long de tout type de graphe dirigé. Cela inclut donc en particulier les bigèbres généralisées de [29] et les structures associées aux diverses variantes de théories de champs topologiques.

Remarque 0.1.12. *Il existe des versions colorées des opérades et PROPs, i.e les entrées et sorties des opérations sont indexées par un ensemble fixé de couleurs, et la composition verticale de deux opérations est autorisée uniquement si les couleurs des sorties de la première opération coïncident avec celles des entrées de la seconde. On a aussi une notion d'algèbre sur un PROP coloré pour laquelle on renvoie le lecteur à l'article de Johnson et Yau [23]. Par exemple, à toute petite catégorie $\mathcal{I}$ et à toute opérade P on peut associer une opérade $ob(\mathcal{I})$-colorée dont les algèbres sont les $\mathcal{I}$-diagrammes de P-algèbres.*

Théorie de l'homotopie des props

De manière analogue au cas opéradique, il existe un foncteur PROP libre sur les Σ-biobjets, i.e un adjoint à gauche du foncteur oubli. Ces derniers forment une catégorie de modèles cofibrement engendrée dont les fibrations et équivalences faibles sont déterminées composante par composante. On aimerait par conséquent transférer cette structure sur la catégorie des props, le long de l'adjonction entre le foncteur PROP libre et le foncteur oubli. Néanmoins, en caractéristique quelconque il y a des obstructions à l'existence d'une telle structure comme l'a expliqué Fresse dans [17]. Il faut alors se restreindre au props P vérifiant $P(0,0) = \mathbb{K}$ et $P(0,n) = 0$ pour $n \neq 0$, appelés props à entrées non vides, ou dualement aux props à sorties non-vides. On obtient plus précisément le résultat suivant dû à Fresse [17] :

Théorème 0.1.13. *(Fresse [17]) Les props à entrées non vides, respectivement à sorties non vides, dans les complexes de chaines sur un corps forment une catégorie de semi-modèles cofibrement engendrée. Lorsque le corps est de caractéristique nulle, on a une structure de catégorie de modèles cofibrement engendrée sur tous les props.*

Une catégorie de semi-modèles est une catégorie vérifiant les mêmes axiomes que ceux d'une catégorie de modèles à l'exception des axiomes de factorisation et de relèvement, qui s'appliquent uniquement aux morphismes de source cofibrante. La structure de semi-modèles est suffisante pour définir une relation d'homotopie et construire la catégorie homotopique comme dans le cas des catégories de modèles.

Remarque 0.1.14. *Les props colorés forment aussi une catégorie de modèles lorsque $\mathbb{K}$ est de caractéristique nulle [23].*

On peut à présent se demander comment construire des résolutions cofibrantes explicites dans les props. La question est plus difficile que dans le cas des opérades, et n'a pas de réponse en toute généralité. Néanmoins, on peut y répondre en travaillant avec des versions restreintes des props appelées les propérades. Une propérade est un monoïde dans la catégorie des Σ-biobjets pour un produit monoïdal appelé produit de composition connexe, introduit par Vallette dans [41]. Intuitivement, une propérade décrit des espaces d'opérations le long de graphes à plusieurs entrées et à plusieurs sorties, de genres quelconques, mais connexes. Cette restriction aux graphes connexes permet non seulement de définir les propérades comme des monoïdes mais aussi de développer une théorie analogue à celle des opérades. Les propérades forment une catégorie de modèles cofibrement engendrée, il existe une résolution bar-cobar analogue au cas opéradique, et on peut construire une théorie de la dualité de Koszul des propérades [41]. De manière identique au cas opéradique, on définit les notions de propérade quasi-libre et de modèle minimal.

Soit P un PROP. On peut définir une P-algèbre à homotopie près comme une algèbre sur une résolution cofibrante de P. On obtient en particulier les versions à homotopie près des bigèbres généralisées, des bigèbres de Frobenius et des théories de champs. Néanmoins, la cohérence homotopique d'une telle définition ne peut résulter d'une méthode analogue au cas opéradique, comme on l'explique ci-dessous.

Difficultés concernant les algèbres sur un PROP

Dans le cas des catégories cartésiennes (les catégories dans lesquelles le produit catégorique coïncide avec le produit monoïdal), le fait que le produit catégorique coïncide avec le produit tensoriel permet de définir un foncteur P-algèbre libre : on peut scinder toute opération $X^{\otimes m} \rightarrow X^{\otimes n}$ en un n-uplet d'opérations $X^{\otimes m} \rightarrow X$ par propriété universelle du produit. On a alors une adjonction permettant de transférer depuis $Ch_{\mathbb{K}}$ une structure de catégorie de modèles cofibrement engendrée sur $Ch_{\mathbb{K}}^P$ lorsque P est cofibrant. Ce résultat est démontré dans [23] et reste valide dans le cas coloré. Les ensembles et ensembles simpliciaux sont des exemples de telles catégories. Mais dès lors que l'on travaille dans une catégorie non cartésienne comme notre catégorie favorite, celle des complexes de chaines, une telle construction n'est plus possible et l'on ne connait pas de structure de modèles sur les P-algèbres. Une telle structure n'existe souvent tout simplement pas car il peut ne pas y avoir de produits ou de coproduits. Ainsi, les bigèbres généralisées à homotopie près ne forment pas une catégorie de modèles, et la cohérence homotopique d'une telle notion ne peut s'obtenir par équivalence de Quillen comme dans le cas opéradique. La section suivante décrit en particulier une autre manière de capturer de l'information de nature homotopique dans une catégorie munie d'une classe d'équivalences faibles.

0.2 Théories d'homotopie et $(\infty, 1)$-catégories

Les constructions qui suivent sont a priori effectuées dans le contexte des petites catégories, afin d'éviter des difficultés liées à la théorie des ensembles comme le fait d'obtenir des catégories qui ne sont plus localement petites. Toutefois en pratique on veut souvent appliquer ces résultats à des catégories larges. Nous adoptons donc le point de vue usuel dans la littérature, qui est d'ajouter à la théorie des ensembles l'axiome des univers de Grothendieck : pour tout ensemble il existe un univers dans lequel cet ensemble est un élément. En particulier il existe un univers U dans lequel les catégories qui nous intéressent sont U-petites.

On appelle catégorie relative une paire (C, W) où C est une catégorie et W une sous-catégorie de C telle que $ob(W) = ob(C)$. La catégorie W est appelée catégorie des équivalences faibles de C. Le but de section est d'expliquer, au moins de manière informelle, trois moyens d'obtenir une théorie de l'homotopie des théories d'homotopie. Le premier est la notion de catégorie simpliciale, fondamentale dans les travaux de Dwyer et Kan [11], [12] et [13], qui fournit la notion la plus intuitive de théorie d'homotopie ou de $(\infty, 1)$-catégorie. Un exemple important est la localisation simpliciale qui construit une théorie d'homotopie à partir de toute catégorie relative. Ensuite on montre que la notion d'espace classifiant encode une partie intéressante de l'information homotopique. La dernière sous-section exhibe deux autres modèles des $(\infty, 1)$-catégories fortement reliés entre eux, à savoir la catégorie de modèles des espaces de Segal complets de Rezk [39] et celle obtenue directement sur les catégories relatives dans un travail récent de Barwick et Kan [1].

Catégories simpliciales et localisation à la Dwyer-Kan

Une catégorie simpliciale est une catégorie enrichie dans la catégorie des ensembles simpliciaux. On note $SCat$ la catégorie des catégories simpliciales. On peut munir $SCat$ d'une notion d'équivalence introduite par Dwyer et Kan :

Définition 0.2.1. *Soient M et N deux catégories simpliciales. Elles sont équivalentes au sens de Dwyer-Kan s'il existe un foncteur $F : M \to N$ induisant des équivalences faibles d'ensembles simpliciaux*

$$Mor_M(X, Y) \xrightarrow{\sim} Mor_N(FX, FY)$$

pour tous objets X et Y de M, et tel que

$$\pi_0 F : \pi_0 M \to \pi_0 N$$

est une équivalence de catégories.

Bergner a démontré dans [3] que l'on peut munir $SCat$ d'une structure de catégorie de modèles :

Théorème 0.2.2. *(Bergner [3]) La catégorie $SCat$ possède une structure de catégorie de modèles dont les équivalences faibles sont les équivalences de Dwyer-Kan.*

Pour voir le lien avec les théories d'homotopie, on rappelle quelques propriétés de la localisation simpliciale au sens de Dwyer et Kan. Ils ont en fait introduit deux types de localisation simpliciales, la localisation simpliciale "classique" et la localisation hamac. On renvoie le lecteur à [11] et [12] pour des définitions détaillées. Brièvement, la localisation simpliciale usuelle d'une catégorie relative (C, W)

consiste à prendre la résolution simpliciale libre sur C et à la localiser formellement (au sens de Gabriel-Zisman) par rapport à la résolution simpliciale libre sur W. On note $L(C,W)$ la catégorie simpliciale ainsi obtenue. On a :

Proposition 0.2.1. *(Dwyer, Kan [12]) Pour toute catégorie relative (C,W), il y a une équivalence de catégories*

$$\pi_0 L(C,W) \cong C[W^{-1}]$$

où $C[W^{-1}]$ est la catégorie homotopique de (C,W), obtenue en inversant formellement les flèches de W.

L'idée de la localisation simpliciale est de capturer toutes les informations sur les homotopies supérieures dans une catégorie relative, qui sont perdues lorsqu'on passe à la catégorie homotopique. Dans ces structures homotopiques supérieures on peut citer les mapping spaces simpliciaux, et les limites et colimites homotopiques dans les catégories de modèles. Il s'agit dont de la théorie de l'homotopie de (C,W). La localisation hamac est une alternative à la localisation simpliciale usuelle, avec l'avantage d'une description explicite des espaces de morphismes en termes de diagrammes particuliers appelés hammacs. Les deux localisations sont équivalentes [12].

Inversement, on peut démontrer que toute catégorie simpliciale est Dwyer-Kan équivalente à la localisation simpliciale d'une certaine catégorie relative. Ainsi, la structure de catégorie de modèles de $SCat$ constitue bien une théorie de l'homotopie des théories d'homotopie. On parle aussi de modèle pour les $(\infty,1)$-catégories, car les espaces de morphismes sont enrichis de sorte que toutes les n-flèches sont inversibles "à une $(n+1)$-flèche près" pour $n > 1$ (on pourrait dire alternativement que ce sont des catégories enrichies dans les ∞-groupoïdes).

Le lien avec les catégories de modèles de Quillen est le suivant. Les composantes connexes de la localisation simpliciale d'une catégorie de modèles correspondent à la catégorie homotopique associée à cette catégorie de modèle, et toute équivalence de Quillen induit une équivalence de Dwyer-Kan des localisations simpliciales.

Espaces classifiants

L'espace classifiant est un objet clé de notre travail :

Définition 0.2.3. *Soit (C,W) une catégorie relative. L'espace classifiant de (C,W) est le nerf simplicial $\mathcal{N}W$ de W.*

Dwyer et Kan ont prouvé dans [13] que cet espace renferme les informations suivantes :

Théorème 0.2.4. *(Dwyer, Kan [13]) Soit (C,W) une catégorie relative. Alors il y a une équivalence d'homotopie*

$$\mathcal{N}W \sim \coprod_{[X]} LW(X,X)$$

où $\mathcal{N}$ est le foncteur nerf simplicial, $[X]$ parcourt les classes d'équivalences faibles des objets de C et $L(-)$ est le foncteur de localisation simpliciale. Lorsque (C,W) provient d'une catégorie de modèles on a de plus

$$\mathcal{N}W \sim \coprod_{[X]} Bhaut(X)$$

où $Bhaut(X)$ est l'espace classifiant du monoïde simplicial $haut(X)$ des auto-équivalences faibles d'une résolution fibrante-cofibrante de X.

Cela signifie que l'espace classifiant $\mathcal{N}W$ encode les informations sur les types d'homotopie des objets et leurs symétries internes, i.e leurs automorphismes homotopiques. Dwyer et Kan ont aussi démontré que le type d'homotopie de l'espace classifiant est invariant par équivalences de Quillen dans le cas d'une catégorie de modèles.

Espaces de Segal complets et catégories relatives

Bien que les catégories simpliciales forment le modèle le plus intuitif pour les théories d'homotopie, les équivalences de Dwyer-Kan sont difficiles à identifier. Il est donc naturel de chercher à produire un modèle alternatif des $(\infty, 1)$-catégories avec des équivalences faibles plus aisément manipulables. Cet objectif a été atteint dans [39]. On se contente ici de donner une petite idée de la construction ainsi que quelques propriétés. Brièvement, la catégorie de modèles des espaces de Segal complets est une certaine localisation de Bousfield de la structure de modèles de Reedy des ensembles bisimpliciaux. La structure de Reedy sur les ensembles bisimpliciaux a pour équivalences faibles les morphismes formant des équivalences faibles en chaque dimension simpliciale. La localisation de Bousfield rajoute aux équivalences de Reedy des flèches supplémentaires à inverser, tout en préservant une structure de catégorie de modèles.

Notons CSS la catégorie de modèles ainsi obtenue. Les objets fibrants de CSS sont les espaces de Segal complets. A tout espace de Segal complet on peut associer la théorie d'homotopie d'une certaine catégorie relative. Ils satisfont de plus le principe de reconnaissance suivant [39] :

Théorème 0.2.5. *(Rezk [39]) Les équivalences faibles entre espaces de Segal complets sont exactement les équivalences de Reedy, et correspondent aux équivalences de Dwyer-Kan entre les théories d'homotopie associées.*

Ainsi, la catégorie CSS est une théorie de l'homotopie des théories d'homotopie, dans laquelle les équivalences de Reedy entre objets fibrants correspondent équivalences de Dwyer-Kan.

Un exemple important d'espace de Segal complet est le diagramme de classification d'une catégorie de modèles :

Définition 0.2.6. *Soit (C, W) une catégorie relative. Son diagramme de classification, noté $N(C, W)$, est l'ensemble bisimplicial $N(C, W) : \Delta^{op} \to sSets$ défini par*

$$N(C, W)([n]) = \mathcal{N}we(C^{[n]})$$

où $[n]$ est la catégorie $\{0 \to 1 \to ... \to n\}$, le foncteur $\mathcal{N}$ est le nerf simplicial et $we(C^{[n]})$ la sous-catégorie des équivalences faibles de la catégorie de diagrammes $C^{[n]}$, i.e les transformations naturelles sont des morphismes de W en chaque composante.

L'ensemble simplicial $N(C, W)([0]) = \mathcal{N}W$ est l'espace classifiant de (C, W).

Proposition 0.2.2. *(Rezk [39]) Si (C, W) provient d'une catégorie de modèles alors $N(C, W)$ est un espace de Segal complet.*

Récemment, Barwick et Kan ont construit dans [1] une théorie de l'homotopie des théories d'homotopie directement sur les catégories relatives. Notons $RelCat$ la catégorie des catégories relatives et $ssSets$ la catégorie des ensembles bisimpliciaux. Leur théorème est le suivant :

Théorème 0.2.7. *(Barwick, Kan [1]) Il existe une adjonction*

$$K_\xi : ssSets \leftrightarrows RelCat : N_\xi$$

qui relève toute localisation de Bousfield de la structure de modèles de Reedy sur $ssSets$ en une structure de modèles cofibrement engendrée propre à gauche sur $RelCat$ qui lui est Quillen équivalente. Dans cette catégorie de modèles, un morphisme est une équivalence faible ou une fibration si et seulement si son image par N_ξ l'est dans $ssSets$.

En particulier, la structure de modèles CSS est transférée sur $RelCat$. Barwick et Kan ont aussi démontré que deux catégories relatives sont faiblement équivalentes si et seulement si leurs localisations simpliciales sont Dwyer-Kan équivalentes, i.e le foncteur de localisation $L : RelCat \to SCat$ induit une équivalence des catégories homotopiques [2].

0.3 Résultats du chapitre 1

Le but de cet article est de munir les catégories de bigèbres codées par des couples d'opérades en distribution d'une structure de catégorie de modèles cofibrement engendrée. La démonstration se déroule en deux étapes. La première étape consiste à munir les cogèbres sur une opérade d'une structure de catégorie de modèles cofibrement engendrée. Il existe deux analogues de ce résultat dans la littérature, qui toutefois imposent des restrictions sur l'opérade que l'on ne souhaite pas ici et ne fournissent pas d'engendrement cofibrant. Cet engendrement cofibrant est crucial pour la seconde étape de la démonstration. Le fait que la loi de distribution relève l'adjonction entre complexes de chaines et algèbres en une adjonction entre cogèbres et bigèbres nous permet d'effectuer un transfert de structure de modèles cofibrement engendrée.

Dans la version étendue de l'article proposée dans cette thèse, en plus de la démonstration esquissée ci-dessus on a rédigé des rappels sur la construction de l'argument du petit objet, crucial pour construire et transférer des structures de modèles cofibrement engendrées. Une section est aussi consacrée à la construction détaillée de la structure de modèles des algèbres sur un opérade, afin de se familiariser avec les arguments utilisés dans le cas classique.

0.3.1 Cogèbres sur une opérade

Précisons nos hypothèses de travail. Soit P une opérade dans $Vect_{\mathbb{K}}$ (la catégorie des $\mathbb{K}$-espaces vectoriels). Premièrement, il faut que la dualisation linéaire de P forme une coopérade, donc on supposera les $P(n)$ de dimension finie. Ensuite, on aimerait que le foncteur de Schur associé à P^* soit le foncteur P-cogèbre colibre, i.e que la notion de P-cogèbre soit équivalente à la notion de cogèbre sur la coopérade $P^* = \{P(n)^*\}$ vue comme une comonade (voir [18] pour la notion de cogèbre sur une comonade). Pour cela, on a besoin d'hypothèses restrictives non seulement sur P mais aussi sur la catégorie dans laquelle on définit les P-algèbres. En effet, produire des objets colibres dans une catégorie nécessite de considérer des produits infinis, or le produit tensoriel ne commute en général pas avec les produits infinis. Ainsi, il nous faut rajouter une condition de conilpotence. Celle choisie ici est de définir nos cogèbres dans la sous-catégorie $Ch_{\mathbb{K}}^+$ des complexes de chaines sur un corps $\mathbb{K}$ nuls en degré 0. Concernant P, on va supposer que $P(0) = 0$ et $P(1) = \mathbb{K}$. Ce sont les hypothèses de travail usuelles avec les algèbres non unitaires et cogèbres non counitaires : la condition $P(0) = 0$ signifie l'absence d'unité (ou de counité), et la condition $P(1) = \mathbb{K}$ signifie qu'il n'y a que l'opération identité en arité 1.

On remarquera que P est définie dans $Vect_{\mathbb{K}}$ et les P-cogèbres dans $Ch_{\mathbb{K}}^+$. Cela ne pose aucun problème, la deuxième catégorie étant tensorisée sur la première avec le produit tensoriel usuel des complexes de chaines (en considérant $Vect_{\mathbb{K}}$ comme la sous-catégorie de $Ch_{\mathbb{K}}$ des complexes concentrés en degré 0).

On peut à présent énoncer le résultat suivant :

Théorème 0.3.2. *Soit P une opérade dans $Vect_{\mathbb{K}}$ telle que $P(0) = 0$, $P(1) = \mathbb{K}$ et les $P(n)$ sont de dimension finie. Alors on a une adjonction*

$$U :^P Ch_{\mathbb{K}}^+ \leftrightarrows Ch_{\mathbb{K}}^+ : P^*$$

entre le foncteur de Schur P^* et le foncteur oubli. Ici $^P Ch_{\mathbb{K}}^+$ désigne la catégorie des P-cogèbres dans $Ch_{\mathbb{K}}^+$.

On va ensuite utiliser cette adjonction pour transférer une structure de catégorie de modèles cofibrement engendrée sur les P-cogèbres. Pour des raisons techniques, on est obligé de supposer que $car(\mathbb{K}) = 0$. On prouve précisément le théorème suivant :

Théorème 0.3.3. *Supposons que $car(\mathbb{K}) = 0$. La catégorie des P-cogèbres $^P Ch_{\mathbb{K}}^+$ hérite d'une structure de catégorie de modèles cofibrement engendrée telle qu'un morphisme f de $^P Ch_{\mathbb{K}}^+$ est*

(i) une équivalence faible si $U(f)$ est une équivalence faible dans $Ch_{\mathbb{K}}^+$;

(ii) une cofibration si $U(f)$ est une cofibration dans $Ch_{\mathbb{K}}^+$;

(iii) une fibration si f a la propriété de relèvement à droite par rapport aux cofibrations acycliques.

L'engendrement cofibrant est quant à lui défini de la manière suivante :

Proposition 0.3.1. *Un morphisme* $p : X \to Y$ *de* ${}^{P}Ch_{\mathbb{K}}^{+}$ *est*

(i) une fibration si et seulement si il a la propriété de relèvement à droite par rapport aux injections acycliques $A \hookrightarrow B$ *où* B *admet une base homogène dénombrable ;*

(ii) une fibration si et seulement si il a la propriété de relèvement à droite par rapport aux injections $A \hookrightarrow B$ *où* B *est de dimension finie (i.e admet une base finie de générateurs en tant qu'espace vectoriel gradué).*

Une construction essentielle à la preuve est la coopérade coenveloppante d'une P-cogèbre. Soit A une P-cogèbre. On considère le Σ-objet $P^*[A]$ défini par

$$P^*[A](n) = \bigoplus_{r \geq 1} P(n+r)^* \otimes_{\Sigma_r} A^{\otimes r}.$$

La structure de P-cogèbre $A \to P^*(A)$ sur A et le coproduit $P^* \to P^* \circ P^*$ de la coopérade P^* induisent respectivement les morphismes de Σ-objets $d_0, d_1 : P^*[A] \rightrightarrows P^*[P^*[A]]$. La counité de P^* induit un morphisme $s_0 : P^*[P^*[A]] \to P^*[A]$. La coopérade coenveloppante est alors définie comme l'égalisateur coréflexif

$$U_{P^*}(A) = ker(d_0 - d_1) \longrightarrow P^*[A] \underset{d_1}{\overset{d_0}{\rightrightarrows}} P^*[P^*(A)] \quad (s_0 : P^*[P^*(A)] \to P^*[A])$$

dans les Σ-objets avec la structure de coopérade induite par celle de $P^*[A]$. La notion de coopérade coenveloppante permet de démontrer le résultat suivant :

Proposition 0.3.2. *Soient* A *une* P*-cogèbre et* C *un complexe de chaines. Si* $H_*(C) = 0$ *alors la projection canonique* $A \times P^*(C) \to A$ *est une équivalence faible dans* ${}^{P}Ch_{\mathbb{K}}^{+}$.

Les principales difficultés résident dans l'engendrement cofibrant, ainsi que les preuves des axiomes de relèvement (MC4) et de factorisation (MC5) des catégories de modèles. La proposition ci-dessus permet notamment de prouver MC4. Afin de prouver MC5, on utilise une version raffinée de l'argument usuel du petit objet : contrairement aux cas classiques, on ne peut pas considérer de petitesse générale mais la petitesse uniquement par rapport aux systèmes d'injections.

0.3.4 Bigèbres sur un couple d'opérades en distribution

Soient P et Q deux opérades dans $Vect_{\mathbb{K}}$ munies d'une loi de distribution. Supposons que Q vérifie $Q(0) = 0$, $Q(1) = 1$ et les $Q(n)$ sont de dimension finie. Dans ce cas, l'adjonction

$$P : Ch_{\mathbb{K}}^{+} \rightleftarrows_{P} Ch_{\mathbb{K}}^{+} : U$$

se relève en une adjonction

$$P : {}^{Q}Ch_{\mathbb{K}}^{+} \rightleftarrows_{P}^{Q} Ch_{\mathbb{K}}^{+} : U.$$

De même l'adjonction

$$U : {}^{Q}Ch_{\mathbb{K}}^{+} \rightleftarrows Ch_{\mathbb{K}}^{+} : Q^*$$

se relève en une adjonction

$$U : {}_{P}^{Q}Ch_{\mathbb{K}}^{+} \rightleftarrows_{P} Ch_{\mathbb{K}}^{+} : Q^*.$$

On obtient alors :

Théorème 0.3.5. *La catégorie* ${}_{P}^{Q}Ch_{\mathbb{K}}^{+}$ *des* (P, Q)*-bigèbres hérite d'une structure de catégorie de modèles cofibrement engendrée de sorte qu'un morphisme* f *de* ${}_{P}^{Q}Ch_{\mathbb{K}}^{+}$ *est*

(i) une équivalence faible si $U(f)$ *est une équivalence faible dans* ${}^{Q}Ch_{\mathbb{K}}^{+}$ *;*

(ii) une fibration si $U(f)$ *est une fibration dans* ${}^{Q}Ch_{\mathbb{K}}^{+}$ *;*

(iii) une cofibration si f *a la propriété de relèvement à gauche par rapport aux fibrations acycliques.*

L'engendrement cofibrant est obtenu de la manière suivante. Notons $\{j : A \hookrightarrow B\}$ la famille des cofibrations génératrices de ${}^{Q}Ch_{\mathbb{K}}^{+}$ et $\{i : A \hookrightarrow B\}$ la famille des cofibrations génératrices acycliques. Alors les $P(j)$ forment les cofibrations génératrices de ${}^{Q}_{P}Ch_{\mathbb{K}}^{+}$ et les $P(i)$ forment les cofibrations acycliques génératrices :

Proposition 0.3.3. *Soit f un morphisme de ${}^{Q}_{P}Ch_{\mathbb{K}}^{+}$. Alors*
(i) f est une fibration si et seulement si f a la propriété de relèvement à droite par rapport aux $P(i)$, où $i : A \hookrightarrow B$ est une injection acyclique de Q-cogèbres telle que B admet une base homogène dénombrable ;
(ii) f est une fibration acyclique si et seulement si f a la propriété de relèvement à droite par rapport aux $P(j)$, où $j : A \hookrightarrow B$ est une injection de Q-cogèbres telles que B est de dimension finie.

La preuve de l'engendrement cofibrant est analogue à celle utilisée dans le cas des algèbres sur une opérade. Pour les axiomes de factorisation MC5, on doit encore une fois considérer la petitesse par rapport aux systèmes d'injections afin d'appliquer l'argument du petit objet.

0.4 Résultats du chapitre 2

Dans cette section, on désigne par $Ch_{\mathbb{K}}$ la catégorie des complexes de chaines $\mathbb{Z}$-gradués sur un corps de caractéristique quelconque. Le premier objectif principal de cet article est un résultat d'invariance homotopique de l'espace classifiant d'une catégorie d'algèbres sur un PROP cofibrant. Un tel résultat ne pouvant s'obtenir de la même manière que dans le cas opéradique (qui utilise une équivalence de Quillen), on emploie une méthode complètement différente. Le deuxième objectif est une application de ce résultat. Il s'agit de la généralisation du théorème principal de [38], prouvé initialement pour les opérades simpliciales, aux props différentiels gradués codant des algèbres dans une catégorie de modèles monoïdale symétrique tensorisée sur $Ch_{\mathbb{K}}$. De plus, ces résultats sont aussi valables pour les props colorés si l'on suppose que $car(\mathbb{K}) = 0$.

0.4.1 Invariance homotopique de l'espace classifiant

Afin de comprendre dans quel cadre on travaille, on rappelle la définition suivante :

Définition 0.4.2. *Soit $\mathcal{C}$ une catégorie monoïdale symétrique. Une catégorie monoïdale symétrique sur $\mathcal{C}$ est une catégorie monoïdale symétrique $(\mathcal{E}, \otimes, 1_{\mathcal{E}})$ munie d'un produit tensoriel externe $\otimes_e : \mathcal{C} \times \mathcal{E} \to \mathcal{E}$ satisfaisant les conditions naturelles suivantes :*
(1) $\forall X \in \mathcal{E}, 1_{\mathcal{C}} \otimes X \cong X$,
(2) $\forall X \in \mathcal{E}, \forall C, D \in \mathcal{C}, (C \otimes D) \otimes X \cong C \otimes (D \otimes X)$,
(3) $\forall C \in \mathcal{C}, \forall X, Y \in \mathcal{E}, C \otimes (X \otimes Y) \cong (C \otimes X) \otimes Y \cong X \otimes (C \otimes Y)$.

Exemple 0.4.3. *Soit I une petite catégorie ; les I-diagrammes dans une catégorie monoïdale symétrique $\mathcal{C}$ forment une catégorie monoïdale symétrique sur $\mathcal{C}$. Le produit tensoriel interne est le produit tensoriel point par point, et le produit tensoriel externe entre un objet X de $\mathcal{C}$ et un diagramme $\{D_i\}$ est le produit tensoriel point par point entre le diagramme constant sur X et $\{D_i\}$.*

On rappelle aussi brièvement qu'une catégorie de modèles monoïdale symétrique est une catégorie monoïdale symétrique munie d'une structure de catégorie modèles d'une manière compatible avec le produit tensoriel (voir [22]). Cette compatibilité s'exprime notamment via l'axiome du pushout-produit, et signifie que le produit tensoriel est un bifoncteur de Quillen que l'on peut dériver pour obtenir une structure monoïdale sur la catégorie homotopique. La catégorie $Ch_{\mathbb{K}}$ est notre principal exemple de catégorie de modèles monoïdale symétrique cofibrement engendrée.

Pour un PROP P dans une catégorie monoïdale symétrique $\mathcal{C}$, on peut définir une algèbre dans une catégorie monoïdale symétrique sur $\mathcal{C}$ de la manière suivante :

Définition 0.4.4. *Soit $\mathcal{E}$ une catégorie monoïdale symétrique sur $\mathcal{C}$.*
(1) Le PROP des endomorphismes de $X \in \mathcal{E}$ est donné par $End_X(m,n) = Hom_{\mathcal{E}}(X^{\otimes m}, X^{\otimes n})$ où $Hom_{\mathcal{E}}(-,-) : \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{C}$ est le bifoncteur hom externe de $\mathcal{E}$.

(2) Soit P un PROP dans $\mathcal{C}$. Une P-algèbre dans $\mathcal{E}$ est un objet $X \in \mathcal{E}$ muni d'un morphisme de props $P \to End_X$.

On peut à présent énoncer le théorème d'invariance homotopique :

Théorème 0.4.5. *Soit $\mathcal{E}$ une catégorie de modèles monoïdale symétrique cofibrement engendrée sur $Ch_{\mathbb{K}}$. Soit $\varphi : P \xrightarrow{\sim} Q$ une équivalence faible entre deux props cofibrants définis dans $Ch_{\mathbb{K}}$. Ce morphisme φ induit un foncteur $\varphi^* : w(\mathcal{E}^c)^Q \to w(\mathcal{E}^c)^P$ qui lui-même induit une équivalence faible d'ensembles simpliciaux $\mathcal{N}\varphi^* : \mathcal{N}w(\mathcal{E}^c)^Q \to \mathcal{N}w(\mathcal{E}^c)^P$.*

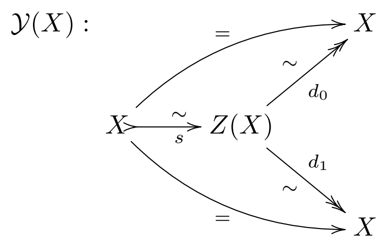
Ici $\mathcal{E}^c$ désigne la sous-catégorie de $\mathcal{E}$ des objets cofibrants et $w(\mathcal{E}^c)^P$ la sous-catégorie de $(\mathcal{E}^c)^P$ des morphismes de P-algèbres qui sont des équivalences faibles dans $\mathcal{E}$. En particulier, si P est un PROP quelconque, ce théorème affirme que le type d'homotopie de l'espace classifiant des P-algèbres à homotopie près ne dépend pas du choix d'une résolution cofibrante de P.

La démonstration de ce théorème repose sur des résultats intermédiaires, notamment celui-ci :

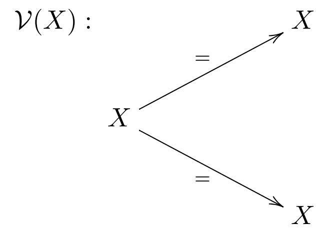
Proposition 0.4.1. *Soit P un PROP cofibrant. Il existe un objet en chemin fonctoriel dans la catégorie des P-algèbres.*

Voici une esquisse de la démonstration dans le cas $\mathcal{E} = \mathcal{C} = Ch_{\mathbb{K}}$:

Démonstration. On considère premièrement l'objet en chemin fonctoriel associé à tout X dans $Ch_{\mathbb{K}}$

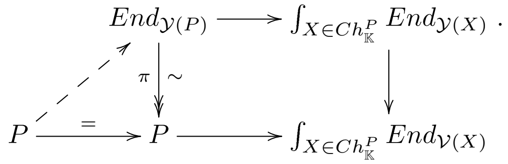


et son sous-diagramme $\mathcal{Z}(X) = \{X \overset{\sim}{\twoheadleftarrow} Z(X) \overset{\sim}{\twoheadrightarrow} X\}$. On prouve que la P-action naturelle existant sur le diagramme



s'étend en une P-action naturelle sur $\mathcal{Y}(X)$. Pour cela on construit des props d'endomorphismes "P-modifiés" obtenus en remplaçant les opérations $X^{\otimes m} \to X^{\otimes n}$ dans le PROP des endomorphismes d'un diagramme donné par des opérations de $P(m,n)$. On note $End_{\mathcal{Y}(P)}$, $End_{\mathcal{Z}(P)}$ et $End_{\mathcal{V}(P)}$ les versions P-modifiées. Il faut ensuite vérifier que chaque construction produit un PROP agissant naturellement sur le PROP des endomorphismes du diagramme associé.

On utilise les colimites de la forme $\int_{X \in Ch_{\mathbb{K}}^P}(-)$ (i.e les "fin", voir [31]) pour obtenir une version fonctorielle de nos props d'endomorphismes de diagrammes. Pour tout $X \in Ch_{\mathbb{K}}^P$, on a $End_{\mathcal{V}(X)} \cong End_X$ donc le morphisme $P \to End_X$ induit trivialement un morphisme $P \to \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{V}(X)}$. On construit un diagramme



On définit le PROP des endomorphismes P-modifié $End_{\mathcal{Y}(P)}$ avec un forme similaire à celle de $End_{\mathcal{Y}(X)}$ afin d'obtenir le morphisme de props $End_{\mathcal{Y}(P)} \to \int_{X\in Ch_{\mathbb{K}}^P} End_{\mathcal{Y}(X)}$ induit par les morphismes $P \to End_X$, $X \in Ch_{\mathbb{K}}^P$.

Ensuite on prouve que π est une fibration acyclique de props, d'où le relèvement souhaité $P \to \int_{X\in Ch_{\mathbb{K}}^P} End_{\mathcal{Y}(X)}$ par l'axiome MC4 des catégories de modèles (puisque P est supposé cofibrant). □

0.4.6 Les espaces de modules comme fibres homotopiques

L'idée d'un espace de modules de structures algébriques d'un type donné sur un objet est de prendre l'ensemble des structures algébriques de ce type sur un objet et d'enrichir cet ensemble simplicialement. Les points de l'ensemble simplicial obtenu sont exactement les structures algébriques. On formalise ce principe dans la définition suivante :

Définition 0.4.7. *Soit $\mathcal{E}$ une catégorie de modèles monoïdale symétrique sur $\mathcal{C}$ et P un PROP à entrées non-vides dans $\mathcal{C}$. Soit X un objet de $\mathcal{E}$. L'espace de modules des structures de P algèbres sur X est l'ensemble simplicial défini par*

$$P\{X\} = Mor_{\mathcal{P}_0}(P \otimes \Delta^\bullet, End_X)$$

où $P \otimes \Delta^\bullet$ est une résolution cosimpliciale de P et $\mathcal{P}_0$ la catégorie des props à entrées non-vides. On obtient donc un foncteur

$$\begin{aligned} \mathcal{P}_0 &\to sSet \\ P &\mapsto P\{X\}. \end{aligned}$$

Les résolutions cosimpliciales existant dans toute catégorie de modèles [13], cet espace est toujours défini. Notons que lorsque P est cofibrant, il suffit de munir P d'un encadrement cosimplicial, car un encadrement cosimplicial d'un objet cofibrant est une résolution cosimpliciale de cet objet (voir [21] pour la notion d'encadrement cosimplicial et ses propriétés). Rappelons que lorsque la catégorie de base $\mathcal{C}$ est celle des complexes de chaines sur un corps de caractéristique nulle, l'hypothèse des entrées non vides n'est plus nécessaire.

Supposons à présent que P est cofibrant. Lorsque $\mathcal{E}$ satisfait des axiomes supplémentaires appelés "limit monoid axioms", et que X est un objet fibrant-cofibrant de $\mathcal{E}$, alors l'espace de modules $P\{X\}$ est un complexe de Kan dont le type d'homotopie est invariant par équivalences faibles de props cofibrants. En particulier, la catégorie $Ch_{\mathbb{K}}$ vérifie de tels axiomes.

On utilise le théorème d'invariance homotopique de l'espace classifiant pour prouver le résultat suivant, qui généralise un théorème central de la thèse de Rezk [38] :

Théorème 0.4.8. *Soit $\mathcal{E}$ une catégorie de modèles monoïdale symétrique sur $Ch_{\mathbb{K}}$ cofibrement engendrée et satisfaisant les "limit monoid axioms". Soient P un PROP cofibrant dans $Ch_{\mathbb{K}}$ et X un objet fibrant-cofibrant de $\mathcal{E}$. Notons $\mathcal{E}^{cf}$ la sous-catégorie de $\mathcal{E}$ des objets fibrants et cofibrants. Alors le carré commutatif*

$$\begin{array}{ccc} P\{X\} & \longrightarrow & \mathcal{N}(w(\mathcal{E}^{cf})^P) \\ \downarrow & & \downarrow \\ \{X\} & \longrightarrow & \mathcal{N}(w\mathcal{E}^{cf}) \end{array}$$

est un pullback homotopique d'ensembles simpliciaux.

Remarque 0.4.9. *Ce résultat implique une décomposition de l'espace de modules*

$$P\{X\} \sim \coprod_{[X]} Lw((\mathcal{E}^{cf})^P)(X,X)$$

où $[X]$ parcourt les classes d'équivalences faibles de P-algèbres ayant X comme objet sous-jacent. Notons finalement que, l'espace de modules formant un ensemble, c'est aussi le cas des automorphismes homotopiques $Lw((\mathcal{E}^{cf})^P)(X,X)$, chose qui n'était pas assurée au départ (a priori la localisation d'une catégorie large donne une catégorie qui n'est localement petite que dans un certain univers de Grothendieck).

0.5 Résultats du chapitre 3

On rappelle la problématique posée à la fin de la section 1.1. Soit P un PROP dans $Ch_{\mathbb{K}}$. On a défini une P-algèbre à homotopie près comme une algèbre sur une résolution cofibrante de P. Soit $\mathcal{E}$ une catégorie de modèle monoïdale symétrique sur $Ch_{\mathbb{K}}$ cofibrement engendrée. Notons P_∞ une résolution cofibrante de P et $\mathcal{E}^{P_\infty}$ la catégorie des P_∞-algèbres dans $\mathcal{E}$. Une condition naturelle de cohérence pour une telle définition serait que la théorie de l'homotopie des P-algèbres à homotopie près ne dépende pas du choix d'une résolution cofibrante de P. Contrairement au cas opéradique nous n'avons pas de structure de catégorie de modèles sur les algèbres sur un PROP cofibrant. Nous pouvons néanmoins définir la théorie de l'homotopie de $\mathcal{E}^{P_\infty}$ comme la localisation simpliciale de $\mathcal{E}^{P_\infty}$ par rapport à la sous-catégorie $w\mathcal{E}^{P_\infty}$ des morphismes de P_∞-algèbres qui sont des équivalences faibles de $\mathcal{E}$. La question qui se pose est donc l'invariance de $L(\mathcal{E}^{P_\infty}, w\mathcal{E}^{P_\infty})$ par équivalences faibles de props cofibrants. Nous y apportons une réponse positive, en se restreignant à la sous-catégorie des objets cofibrants :

Théorème 0.5.1. *Une équivalence faible $P \xrightarrow{\sim} Q$ entre deux props (colorés) cofibrants induit un zigzag d'équivalences de Dwyer-Kan de localisations simpliciales*

$$L((\mathcal{E}^c)^Q, w(\mathcal{E}^c)^Q) \cong L((\mathcal{E}^c)^P, w(\mathcal{E}^c)^P).$$

Dans le cas où tous les objets de $\mathcal{E}$ sont cofibrants, on a donc un résultat invariance pour la théorie de l'homotopie de toutes les algèbres à homotopie près. C'est le cas par exemple lorsque $\mathcal{E} = Ch_{\mathbb{K}}$.

La démonstration repose sur deux résultats intermédiaires. Le point clé est une amélioration du théorème d'invariance de l'espace classifiant :

Théorème 0.5.2. *Une équivalence faible de props cofibrants $\varphi : P \xrightarrow{\sim} Q$ induit une équivalence faible des diagrammes de classification*

$$N((\mathcal{E}^c)^Q, w(\mathcal{E}^c)^Q) \xrightarrow{\sim} N((\mathcal{E}^c)^P, w(\mathcal{E}^c)^P).$$

A présent, on aimerait en déduire le théorème principal en affirmant que cette équivalence de Reedy induit une équivalence faible des localisations simpliciales. Or, les catégories relatives considérées ici ne sont pas des catégories de modèles, donc leurs diagrammes de classification ne sont a priori pas des espaces de Segal complets. La question est par conséquent de savoir comment relier le diagramme de classification d'une catégorie relative quelconque avec sa localisation simpliciale. La réponse est la suivante :

Proposition 0.5.1. *(Cisinksi, Schommer-Pries) Soit (C, W) une catégorie relative. Alors les ensembles bisimpliciaux $(t^! \circ \tilde{N} \circ L)(C, W)$ et $N(C, W)$ sont faiblement équivalents dans CSS.*

Les arguments de la preuve sont dus de deux manières différentes à Cisinski et Schommer-Pries. Nous avons rédigé dans le détail l'approche de Schommer-Pries. Celle-ci réside dans les points suivants. Premièrement, il est prouvé dans [1] que le foncteur N est naturellement Reedy équivalent au foncteur N_Ξ du théorème 0.2.7. Deuxièmement, le foncteur L induit une équivalence de catégories homotopiques [2] et les foncteurs $t^!$ et $\tilde{N}$ sont des foncteurs de Quillen à droite dans les paires de Quillen

$$t_! : CSS \leftrightarrows QCat : t^!$$

et

$$J : QCat \leftrightarrows SCat : \tilde{N}$$

(voir les travaux de Joyal et Tierney [25], et de Joyal [26]). Ici $QCat$ désigne la catégorie des quasi-catégories de Joyal [24]. Notons RN_Ξ et $R(t^! \circ \tilde{N} \circ L)$ les foncteurs dérivés à droite de N_Ξ et $(t^! \circ \tilde{N} \circ L)$. Ce sont deux équivalences de catégories $Ho(RelCat) \xrightarrow{\sim} Ho(CSS)$. On voit que ce sont les mêmes en utilisant un théorème de Toen sur les automorphismes homotopiques d'une catégorie de modèles des $(\infty, 1)$-catégories [40].

On déduit le théorème 0.5.1 du théorème 0.5.2 combiné à la proposition 0.5.3.

Ce résultat d'invariance s'applique en particulier aux bigèbres généralisées à homotopie près. Il est aussi valable pour les props colorés lorsqu'on travaille sur un corps de caractéristique nulle. Dans ce cas, il fournit un résultat d'invariance homotopique des catégories de diagrammes de différentes sortes d'algèbres à homotopie près.

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Chapitre 1

The homotopy theory of bialgebras over pairs of operads

We endow the category of bialgebras over a pair of operads in distribution with a cofibrantly generated model category structure. We work in the category of chain complexes over a field of characteristic zero. We split our construction in two steps. In the first step, we equip coalgebras over an operad with a cofibrantly generated model category structure. In the second one we use the adjunction between bialgebras and coalgebras via the free algebra functor. This result allows us to do classical homotopical algebra in various categories such as associative bialgebras, Lie bialgebras or Poisson bialgebras in chain complexes.

Introduction

The goal of this paper is to define a model category structure for the categories of bialgebras governed by operads in distribution. The work of Drinfeld on quantum groups (see [2] and [3]) has initiated the study of bialgebra structures where the product and the coproduct belong to various types of algebras. Besides the classical Hopf algebras, examples include their non-commutative non-cocommutative variant, Lie bialgebras and Poisson bialgebras. Applications ranges from knot theory, in topology, to integrable systems in mathematical physics. The theory of operads in distribution, introduced by Fox and Markl in [5], provides a convenient generalization of the classical categories of bialgebras defined by products and coproducts in distribution. The general idea is that there is an operad encoding the operations (where we have several inputs and a single output) and another operad encoding the cooperations (a single input and several outputs). The distributive law then formalizes the interplay between these operads, i.e the compatibilities between operations and cooperations. We refer the reader to [16] for a detailed survey providing many examples of these generalized bialgebras. One may then wonder how to transpose homotopical algebra methods in this setting, as it has been done successfully for algebras over operads. For this aim, this paper construct a closed model category structure for this kind of bialgebras. We expect interesting homotopical outcomes in the various fields of mathematics where such bialgebraic structures appear.

The existence of a cofibrantly generated model category structure on algebras over a suitable operad is a classical result, see [12]. When working over a field of characteristic zero, such a structure exists for any operad. Let $\mathbb{K}$ be a field of characteristic zero and $Ch_{\mathbb{K}}$ the category of non-negatively graded chain complexes over $\mathbb{K}$. Let $Ch_{\mathbb{K}}^+$ be the full subcategory of $Ch_{\mathbb{K}}$ of positively graded chain complexes. Let P be an operad. We assume to simplify that the operad P belongs to the category of $\mathbb{K}$-modules, and we consider the category of coalgebras over P in $Ch_{\mathbb{K}}^+$. We denote this category by ${}^PCh_{\mathbb{K}}^+$. Our results can be generalized to other situations, but then we have to require extra conilpotence conditions, which are automatically fulfilled in our setting (see 1.1.19). We suppose moreover that $P(0) = 0$, $P(1) = \mathbb{K}$ and that the $P(n)$ are of finite dimension. In a first step we establish the existence of a model category structure for coalgebras over P:

Theorem 1.0.2. *The category of P-coalgebras ${}^{P}Ch_{\mathbb{K}}^{+}$ inherits a cofibrantly generated model category structure such that a morphism f of ${}^{P}Ch_{\mathbb{K}}^{+}$ is*

(i) a weak equivalence if $U(f)$ is a weak equivalence in $Ch_{\mathbb{K}}^{+}$;

(ii) a cofibration if $U(f)$ is a cofibration in $Ch_{\mathbb{K}}^{+}$;

(iii) a fibration if f has the right lifting property with respect to acyclic cofibrations.

Note that an analoguous result has been proven in [23] in the context of unbounded chain complexes. We follow another simpler approach. We do not address the same level of generality, but we obtain a stronger result. To be more precise, in constrast with [23], we obtain a cofibrantly generated structure. These generating cofibrations are crucial to transfer the model structure on bialgebras. Moreover, we do not need the hypothesis considered in [23] about the underlying operad (see [23], condition 4.3). Our method is close to the ideas of [10]. Such a result also appears in [1], but for coalgebras over a quasi-free cooperad. We prove this theorem via the following steps. First, we prove two crucial results. The first is the structure of the cofree coalgebra over an operad. The second one is based on the construction, for any P-coalgebra A, of its enveloping cooperad. It expresses the coproduct of A with a cofree coalgebra in terms of the evaluation of the associated enveloping cooperad functor. Axioms MC2 and MC3 are obvious. Axioms MC1 is proved in an analogue way than in the case of algebras. The main difficulty lies in the proofs of MC4 and MC5. For this aim, we use proofs inspired from that of [10] and adapted to our operadic setting. In order to produce the desired factorization axioms, our trick here is to use a slightly modified version of the usual small object argument. We use smallness with respect to injections systems.

We denote by ${}_{P}^{Q}Ch_{\mathbb{K}}^{+}$ the category of (P, Q)-bialgebras in $Ch_{\mathbb{K}}^{+}$, where P encodes the operations and Q the cooperations. We use the adjunction

$$U : {}_{P}^{Q} Ch_{\mathbb{K}}^{+} \rightleftarrows {}_{P} Ch_{\mathbb{K}}^{+} : Q^{*}$$

betweent the forgetful functor and the cofree P-coalgebra functor. The model category structure on (P, Q)-bialgebras is then given by our main theorem:

Theorem 1.0.3. *The category of (P, Q)-bialgebras ${}_{P}^{Q}Ch_{\mathbb{K}}^{+}$ inherits a cofibrantly generated model category structure such that a morphism f of ${}_{P}^{Q}Ch_{\mathbb{K}}^{+}$ is*

(i) a weak equivalence if $U(f)$ is a weak equivalence in ${}^{Q}Ch_{\mathbb{K}}^{+}$ (i.e a weak equivalence in $Ch_{\mathbb{K}}^{+}$ by definition of the model structure on ${}^{Q}Ch_{\mathbb{K}}^{+}$);

(ii) a fibration if $U(f)$ is a fibration in ${}^{Q}Ch_{\mathbb{K}}^{+}$;

(iii) a cofibration if f has the left lifting property with respect to acyclic fibrations.

The main difficulty is the proof of MC5. We use mainly the small object argument for smallness with respect to injections systems, combined with a result about cofibrations in algebras over an operad due to Hinich [12]. We cannot use the usual simplifying hypothesis of smallness with respect to all morphisms.

1.1 Preliminary notions

In this section, we first recall some notions and facts about Σ-modules, operads and algebras over operads. Then we review the interplay between monads and comonads by means of distributive laws and make the link with operads. It leads us to the crucial definition of bialgebras over a pair of operads in distribution. Finally, we recall a classical tool of homotopical algebra, namely the small object argument, aimed to produce factorizations in model categories. The material of this section is taken from [17], [5] and [14].

1.1.1 Operads and their algebras

For simplicity, definitions of this subsection are given in the category of vector spaces $Vect_{\mathbb{K}}$, where $\mathbb{K}$ is a field of characteristic zero. They extend to the categories of chain complexes which we consider in the next section.

Σ-modules, Schur functors and operads

Let us start with Σ-modules and their associated Schur functors:

Definition 1.1.1. (1) A Σ-module is a family $M = \{M(n)\}_{n\in\mathbb{N}}$ of right $\mathbb{K}[\Sigma_n]$-modules $M(n)$, where Σ_n is the symmetric group of permutations of $\{1,...,n\}$. It is connected if $M(0) = 0$, simply connected if moreover $M(1) = 0$. It is of finite type if for every $n \in \mathbb{N}$, the vector space $M(n)$ is of finite dimension. For any element $x \in M(n)$, the integer n is the arity of x.

(2) A morphism of Σ-modules $f : M \to N$ is a family of Σ_n-equivariant maps $f_n : M(n) \to N(n)$, i.e maps of Σ_n-modules. When all the f_n are injective, M is a sub-Σ-module of N.

To each Σ-module corresponds a Schur functor:

Definition 1.1.2. Let M be a Σ-module, its Schur functor $M : Vect_{\mathbb{K}} \to Vect_{\mathbb{K}}$ is defined by

$$M(V) = \bigoplus_{n\in\mathbb{N}} M(n) \otimes_{\Sigma_n} V^{\otimes n}$$

where Σ_n acts on $V^{\otimes n}$ by permuting variables: for any $\sigma \in \Sigma_n$ and $(v_1, ..., v_n) \in V^{\otimes n}$, $\sigma.(v_1, ..., v_n) = (v_{\sigma^{-1}(1)}, ..., v_{\sigma^{-1}(n)})$. Then $M(n) \otimes_{\Sigma_n} V^{\otimes n}$ is the space of coinvariants under the diagonal action of Σ_n, that is the action of Σ_n on the left term and the right term.

Example 1.1.3. The Schur functor of $I = (0, \mathbb{K}, 0, ..., 0, ...)$ is the identity functor.

Now let us give two definitions of an operad. The first is a concise "monoidal" definition in terms of Schur functor:

Definition 1.1.4. (1) Consider the quasicategory of endofunctors $Vect_{\mathbb{K}} \to Vect_{\mathbb{K}}$ endowed with the functor composition product. An operad is a Σ-module $P = \{P(n)\}_{n\in\mathbb{N}}$ whose Schur functor forms a monoid for the composition product. It means that there are two natural transformations, the composition $\gamma : P \circ P \to P$ and the unit $\iota : I \to P$ satisfying the usual monoid axioms:

-*associativity*:

$$\begin{array}{ccc} P \circ P \circ P & \xrightarrow{P\gamma} & P \circ P \\ {\scriptstyle \gamma P}\downarrow & & \downarrow{\scriptstyle \gamma} \\ P \circ P & \xrightarrow[\gamma]{} & P \end{array}$$

-*unitarity*:

$$\begin{array}{ccccc} I \circ P & \xrightarrow{\iota P} & P \circ P & \xleftarrow{P\gamma} & P \circ I \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle \gamma} & \swarrow{\scriptstyle =} & \\ & & P & & \end{array}$$

The Schur functor P is also called a monad on $Vect_{\mathbb{K}}$.

(2) A morphism of operads $f : P \to Q$ is a natural transformation of Schur functors commuting with the monoid structures.

The second one is the classical definition due to Peter May. One can show that these two definitions coincide (see [17] for a proof):

Definition 1.1.5. (1) An operad is a Σ-module $P = \{P(n)\}_{n\in\mathbb{N}}$ endowed with $\mathbb{K}$-linear applications called the operadic compositions

$$\gamma(k_1,...,k_n) : P(n)\otimes P(k_1)\otimes ...\otimes P(k_n) \to P(k_1+...+k_n)$$

for $n \geq 1$ and $k_1,...,k_n \geq 0$, and a unit application $\eta : \mathbb{K} \to P(1)$. These applications satisfy the following axioms:

associativity. Let $n \geq 1$ and $m_1,...,m_n,k_1,...,k_m$ natural integers, where $m = m_1+...+m_n$. Let us denote $g_s = m_1+...+m_{s-1}$ and $h_s = k_{g_s+1}+...+k_{g_s+m_s}$ for $1 \leq s \leq n$. Then the following diagram commutes:

$$\begin{array}{ccc}
(P(n)\otimes\bigotimes_{s=1}^n P(m_s))\otimes\bigotimes_{r=1}^m P(k_r) & \xrightarrow{\gamma(m_1,...,m_n)\otimes id} & P(m)\bigotimes_{r=1}^m P(k_r) \\
\downarrow{\scriptstyle permutations} & & \downarrow{\scriptstyle \gamma(k_1,...,k_m)} \\
P(n)\otimes\bigotimes_{s=1}^n(P(m_s)\otimes\bigotimes_{q=1}^{m_s}P(k_{g_s+q})) & \xrightarrow[\gamma(h_1,...,h_n)\circ id\otimes(\bigotimes_{s=1}^n\gamma(k_{g_s+1},...,k_{g_s+m_s}))]{} & P(k_1+...+k_m)
\end{array}$$

equivariance. Let $n\geq 1$, $k_1,...,k_n$ be natural integers and $\sigma\in\Sigma_n$, $\tau_1\in\Sigma_{k_1},...,\tau_n\in\Sigma_{k_n}$ be permutations. Let us note $\sigma(k_1,...,k_n)\in\Sigma_{k_1+...+k_n}$ the permutation permuting the blocks $(1,...,k_1),...,(k_{n-1}+1,...,k_n)$ as σ permutes the elements of $\{1,...,n\}$. Denote also by $\tau_1\oplus...\oplus\tau_n\in\Sigma_{k_1+...+k_n}$ the blockwise sum of the permutations τ_i, that is if $x\in\{k_{i-1}+1,...,k_i\}$ then $(\tau_1\oplus...\oplus\tau_n) = \tau_i(x)$. Then the following diagrams commute:

$$\begin{array}{ccc}
P(n)\otimes P(i_1)\otimes...\otimes P(i_n) & \xrightarrow{\sigma\otimes\sigma^{-1}} & P(n)\otimes P(i_{\sigma(1)})\otimes...\otimes P(i_{\sigma(n)}) \\
\downarrow{\scriptstyle \gamma(i_1,...,i_n)} & & \downarrow{\scriptstyle \gamma(i_{\sigma(1)},...,i_{\sigma(n)})} \\
P(i_1+...+i_n) & \xrightarrow[\sigma(i_{\sigma(1)},...,i_{\sigma(n)})]{} & P(i_{\sigma(1)}+...+i_{\sigma(n)})
\end{array}$$

$$\begin{array}{ccc}
P(n)\otimes P(i_1)\otimes...\otimes P(i_n) & \xrightarrow{id\otimes\tau_1\otimes...\otimes\tau_n} & P(n)\otimes P(i_{\sigma(1)})\otimes...\otimes P(i_{\sigma(n)}) \\
\downarrow{\scriptstyle \gamma(i_1,...,i_n)} & & \downarrow{\scriptstyle \gamma(i_{\sigma(1)},...,i_{\sigma(n)})} \\
P(i_1+...+i_n) & \xrightarrow[\tau_1\oplus...\oplus\tau_n]{} & P(i_1+...+i_n)
\end{array}$$

unitarity. For every $n\geq 1$ the following diagrams commute :

$$\begin{array}{ccc}
\mathbb{K}\otimes P(n) & \xrightarrow{\cong} & P(n) \\
\downarrow{\scriptstyle \eta\otimes id} & \nearrow{\scriptstyle \gamma(n)} & \\
P(1)\otimes P(n) & &
\end{array}$$

$$\begin{array}{ccc}
P(n)\otimes\mathbb{K}^{\otimes n} & \xrightarrow{\cong} & P(n) \\
\downarrow{\scriptstyle id\otimes\eta^{\otimes n}} & \nearrow{\scriptstyle \gamma(1,...,1)} & \\
P(n)\otimes P(1)^{\otimes n} & &
\end{array}$$

(2) A morphism of operads $f : P\to Q$ is a family $\{f(n) : P(n)\to Q(n)\}_{n\in\mathbb{N}}$ of Σ_n-equivariant $\mathbb{K}$-linear maps commuting with operadic compositions and preserving units.

Remark 1.1.6. When $\mathbb{K}$ is an infinite field, we have a fully faithful embedding of the category of Σ-modules in the category of endofunctors of $Vect_{\mathbb{K}}$. However, this is no more true if we suppose $\mathbb{K}$ to be finite. We refer the reader to [18] about this fact.

We can also define suboperads and operadic ideals:

Definition 1.1.7. Let $P = \{P(n)\}_{n\geq 0}$ and $R = \{R(n)\}_{n\geq 0}$ be two operads. The operad R is a suboperad of P if for every $n \geq 0$, the space $R(n)$ is a sub-$\mathbb{K}[\Sigma_n]$-module of $P(n)$ and if all operadic compositions of R are restrictions of that of P.

Definition 1.1.8. An ideal in the operad P is a collection $\mathcal{I} = \{\mathcal{I}(n)\}_{n\geq 0}$ of Σ_n-invariant subspaces $\mathcal{I}(n) \subset P(n)$ such that $\gamma(f, g_1, ..., g_n) \in \mathcal{I}(k_1 + ... + k_n)$ if $f \in \mathcal{I}(n)$ or $g_i \in \mathcal{I}(k_i)$ for some $1 \geq i \geq n$.

Algebras over operads

Operads are aimed to parametrize various kind of algebraic structures: associative, commutative, Poisson or Lie algebras for instance. This leads us to the general notion of algebra over an operad. The operads we carry about in this paper are algebraic operads, but the reader should note that the first examples of operads where topological operads, namely the little disks operads, introduced in homotopy theory in the 60s in order to understand the structure of iterated loop spaces. We can formulate two alternative definitions of an algebra over an operad:

Definition 1.1.9. (1) Let P be an operad. A P-algebra is a vector space A endowed with a linear application $\gamma_A : P(A) \rightarrow A$ such that the following diagrams commute

$$\begin{array}{ccc} (P\circ P)(A) & \xrightarrow{P(\gamma_A)} & P(A) \\ {\scriptstyle \gamma(A)}\downarrow & & \downarrow{\scriptstyle \gamma_A} \\ P(A) & \xrightarrow[\gamma_A]{} & A \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\iota(A)} & P(A) \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle \gamma_A} \\ & & A \end{array}$$

i.e A is an algebra (or a module) over the monad P.

(2) A morphism of P-algebras $f : A \rightarrow B$ is a linear application such that the following diagram commutes :

$$\begin{array}{ccc} P(B) & \xrightarrow{\gamma_A} & A \\ {\scriptstyle P(f)}\downarrow & & \downarrow{\scriptstyle f} \\ P(B) & \xrightarrow[\gamma_B]{} & B \end{array}$$

Definition 1.1.10. Let P be an operad. A P-algebra is a vector space A equiped with linear applications $\alpha_n : P(n) \otimes A^{\otimes n} \rightarrow A$ for $n \in \mathbb{N}$ satisfying the following axioms:

associativity. Let $n \geq 1$ and $k_1, ..., k_n$ be natural integers, then the following diagram commutes:

$$\begin{array}{ccc} (P(n)\otimes \bigotimes_{s=1}^n P(k_s))\otimes \bigotimes_{s=1}^n A^{\otimes k_s} & \xrightarrow{\gamma(k_1,\ldots,k_n)\otimes id} & P(A) \\ {\scriptstyle permutation}\downarrow & & \downarrow{\scriptstyle \alpha_{k_1+\ldots+k_n}} \\ P(n)\otimes \bigotimes_{s=1}^n (P(k_s)\otimes A^{\otimes k_s}) & \xrightarrow[\alpha_n\circ(id\otimes \bigotimes_{s=1}^n \alpha_{k_s})]{} & A \end{array}$$

equivariance. For every $n \geq 1$ and $\sigma \in \Sigma_n$ the following diagram commutes:

$$\begin{array}{ccc} P(n)\otimes A^{\otimes n} & \xrightarrow{\sigma\otimes\sigma^{-1}} & P(n)\otimes A^{\otimes n} \\ & \searrow^{\alpha_n} \quad {}^{\alpha_n}\swarrow & \\ & A & \end{array}$$

unitarity. For every $n \geq 1$ the following diagram commutes:

$$\begin{array}{ccc} \mathbb{K}\otimes A & \xrightarrow{\cong} & A\ . \\ {\scriptstyle \eta\otimes id}\downarrow & \nearrow_{\alpha_1} & \\ P(1)\otimes A & & \end{array}$$

We will denote ${}_PVect_{\mathbb{K}}$ the category of P-algebras in vector spaces and ${}_PCh_{\mathbb{K}}$ the category of P-algebras in non-negatively graded chain complexes. Let us give some fundamental examples of operads.

Example 1.1.11. Let $As : Vect_{\mathbb{K}} \to Vect_{\mathbb{K}}$ the functor defined by

$$As(V) = \bigoplus_{n\geq 1} V^{\otimes n}.$$

As a Σ-module we have $As(n) = \mathbb{K}[\Sigma_n]$ for $n \geq 1$ and $As(0) = 0$. The composition product γ of As is given by the composition of non commutative polynomials. Every non-unitary associative algebra is an algebra over the operad As. If A is an associative algebra and γ_A the application of definition 1.1.9, then the component $\gamma_A(2)$ of γ_A in arity 2 determines the associative product on A. We can parametrize unitary associative algebras by slightly modifying As and defining uAs by $uAs(0) = \mathbb{K}$ and $uAs(n) = \mathbb{K}[\Sigma_n]$ for $n \geq 1$. The unit of A is then given by the component $\mathbb{K} \to A$ of γ_A in arity 0.

Example 1.1.12. Let $Com : Vect_{\mathbb{K}} \to Vect_{\mathbb{K}}$ be the functor defined by

$$Com(V) = \bigoplus_{n\geq 1} (V^{\otimes n})_{\Sigma_n}$$

where $(V^{\otimes n})_{\Sigma_n}$ is the quotient of $V^{\otimes n}$ by the left action of Σ_n. As a Σ-module, we have $Com(0) = 0$ and $Com(n) = \mathbb{K}$ for $n \geq 1$. The composition product γ of Com is given by the composition of polynomials. The operad Com parametrizes non-unitary associative and commutative algebras in the way we explained before for As.

Example 1.1.13. The operad Lie is a more involved example. There is a Schur functor $Lie : Vect_{\mathbb{K}} \to Vect_{\mathbb{K}}$ sending a vector space V to the free Lie algebra $Lie(V) \subset T(V)$, where $T(V) = As(V)$ is the tensor algebra of V. This is the subspace generated by V under the commutator $[v, w] = vw - wv$. One can show that there is an operad structure on Lie induced by that of As. One can also show that $Lie(V)$ is the space of primitive elements in the bialgebra structure of $T(V)$. The explicit description of $Lie(n)$ for every n is more complicated than the previous examples, and we refer to [17] for more details.

Example 1.1.14. To any vector space V we can associate its endomorphism operad End_V defined by $End_V(n) = Hom_{\mathbb{K}}(V^{\otimes n}, V)$. The right action of Σ_n on $End_V(n)$ is induced by its left action on $V^{\otimes n}$. The operadic compositions $\gamma(k_1, ..., k_n)$ are given by partial compositions of morphisms:

$$\gamma(f; f_1, ..., f_n)(v_1, ...v_{k_1+...+k_n}) = f(f_1(v_1, ..., v_{k_1}), ..., f_n(v_{k_1+...+k_{n-1}+1}, ...v_{k_1+...+k_n}))$$

where $f \in Hom_{\mathbb{K}}(V^{\otimes n}, V)$ and $f_i \in Hom_{\mathbb{K}}(V^{\otimes k_i}, V)$.

The endomorphism operad allows us to give a third definition of algebras over operads equivalent to definitions 1.9 and 1.10:

Definition 1.1.15. Let P be an operad. A P-algebra is the data of a vector space A and an operad morphism $P \to End_A$.

It is well known that the tensor algebra and the symmetric algebra constructions are respectively the free associative algebra functor and the free commutative algebra functor. There exists a notion of free algebra functor in the operadic setting (which coincides respectively with the tensor algebra for the associative operad and the symmetric algebra for the commutative operad):

Definition 1.1.16. Let V be a vector space. In the category of P-algebras, a P-algebra $\mathcal{F}(V)$ endowed with a linear map $i : V \rightarrow \mathcal{F}(V)$ is the free P-algebra on V if it satisfies the following universal property: for every P-algebra A and every linear application $f : V \rightarrow A$, there exists a unique factorization

$$\begin{array}{ccc} V & \xrightarrow{i} & \mathcal{F}(V) \\ & \searrow^{f} & \downarrow \overline{f} \\ & & A \end{array}$$

where $\overline{f}$ is a morphism of P-algebras. A free algebra is unique up to isomorphism. The functor $\mathcal{F}$: $Vect_{\mathbb{K}} \rightarrow_P Vect_{\mathbb{K}}$ is called the free P-algebra functor and is by definition the left adjoint of the forgetful functor $U :_P Vect_{\mathbb{K}} \rightarrow Vect_{\mathbb{K}}$.

For every vector space V, we can equip $P(V)$ with a P-algebra structure by setting $\gamma_{P(V)} = \gamma(V)$: $P(P(V)) \rightarrow P(V)$.

Proposition 1.1.17. *(see [17], proposition 5.2.6) The P-algebra $(P(V), \gamma(V))$ equiped with the map $\iota(V) : I(V) = V \rightarrow P(V)$ is the free P-algebra on V.*

Coalgebras and cooperads

Let us introduce first the coendomorphism operad :

Definition 1.1.18. To any vector space V we can associate its coendomorphism operad $coEnd_V$ defined by $coEnd_V(n) = Hom_{\mathbb{K}}(V, V^{\otimes n})$. The right action of Σ_n on $coEnd_V(n)$ is induced by its right action on $V^{\otimes n}$. The operadic compositions $\gamma(k_1, ..., k_n)$ are given by partial compositions of morphisms:

$$\gamma(f; f_1, ..., f_n) = (f_1 \circ f^1) \otimes ... \otimes (f_n \circ f^n) \in Hom_{\mathbb{K}}(V, V^{\otimes k_1 + ... + k_n})$$

where $f \in Hom_{\mathbb{K}}(V, V^{\otimes n})$ and $f_i \in Hom_{\mathbb{K}}(V, V^{\otimes k_i})$. The map f^i is the i^{th} component of f in $V^{\otimes n}$.

Now we can introduce two equivalent definitions of coalgebras over an operad:

Definition 1.1.19. (1) Let P be an operad. A P-coalgebra is a vector space C equiped with linear applications $\rho_n : P(n) \otimes C \rightarrow C^{\otimes n}$ for every $n \geq 0$. These maps are Σ_n-equivariant and associative with respect to the operadic compositions, i.e the following diagram commutes for every $n, k_1, ..., k_n \in \mathbb{N}$:

$$\begin{array}{ccc} P(n) \otimes \bigotimes_{s=1}^n P(k_s) \otimes C & \xrightarrow{\gamma(k_1,...,k_n) \otimes id} & P(k_1 + ... + k_n) \otimes C \\ \downarrow {\scriptstyle (id \otimes ... \otimes id \otimes \rho_n) \circ permutation} & & \downarrow {\scriptstyle \rho_{k_1 + ... + k_n}} \\ \bigotimes_{s=1}^n P(k_s) \otimes C^{\otimes n} & \xrightarrow[(\bigotimes_{s=1}^n \rho_{k_s}) \circ permutation]{} & C^{\otimes (k_1 + ... + k_n)} \end{array}$$

If $\mathbb{K}$ is a field of characteristic zero and the $P(n)$ are finite dimensional, then it is equivalent to define applications $\overline{\rho}_n : C \rightarrow P(n)^* \otimes_{\Sigma_n} C^{\otimes n}$.

(2) A P-coalgebra is a vector space C equiped with an operad morphism $P \rightarrow coEnd_C$.

To see that (1) is equivalent to (2), we basically use that each $p \in P(n)$ gives rise to a cooperation $p^* : C \rightarrow C^{\otimes n}$. The coalgebra C is usually said to be conilpotent if for each $c \in C$, there exists $N \in \mathbb{N}$ so that $p^*(c) = 0$ when we have $p \in P(n)$ with $n > N$.

We can also define the dual notion of operads, namely the cooperads:

Definition 1.1.20. Let $C = \{C(n)\}_{n\in\mathbb{K}}$ be a Σ-module such that $C(0) = 0$. It is a cooperad if the associated Schur functor is a comonoid in the endofunctors of $Vect_{\mathbb{K}}$. This means that there exist two natural transformations, the counit $\eta : C \to I$ and the decomposition coproduct $\Delta : C \to C \circ C$ satisfying the following axioms:

coassociativity.

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\circ C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle C\Delta} \\ C\circ C & \xrightarrow[\Delta C]{} & C\circ C\circ C \end{array}$$

counitarity.

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{=} & \downarrow{\scriptstyle\Delta} & \searrow^{=} & \\ I\circ C & \xleftarrow[\eta C]{} & C\circ C & \xrightarrow[C\eta]{} & C\circ I \end{array}.$$

Such a structure is called a comonad on $Vect_{\mathbb{K}}$. We also suppose that there exists an element $id \in C(1)$ such that $\eta(id) = 1 \in I(1) = \mathbb{K}$, called the identity cooperation.

There is a notion of coalgebra over a cooperad:

Definition 1.1.21. Let C be a cooperad. A C-coalgebra is a vector space X equiped with a linear application $\rho : X \to C(X)$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\rho} & C(X) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\Delta\rho} \\ C(X) & \xrightarrow[C(\rho)]{} & C(C(X)) \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{\rho} & C(X) \\ & \searrow_{=} & \downarrow{\scriptstyle\eta} \\ & & X \end{array}.$$

We can go from operads to cooperads and vice-versa by dualization. Indeed, if C is a cooperad, then the Σ-module P defined by $P(n) = C(n)^* = Hom_{\mathbb{K}}(C(n), \mathbb{K})$ form an operad. Conversely, suppose that $\mathbb{K}$ is of characteristic zero and P is an operad such that each $P(n)$ is finite dimensional. Then the $P(n)^*$ form a cooperad. The additional hypotheses are needed because you have to use, for finite dimensional vector spaces V and W, the isomorphism $(V \otimes W)^* \cong V^* \otimes W^*$ to define properly the decomposition product.

Now suppose that P is an operad such that the $P(n)$ are finite dimensional, $P(0) = 0$ and $P(1) = \mathbb{K}$. Then we have a cofree conilpotent P-coalgebra functor, which is by definition the right adjoint to the forgetful functor and is given by the comonad associated to the cooperad P^* (theorem 2.3.2). The conilpotence condition is automatically fulfilled when we deal with operads in $Vect_{\mathbb{K}}$ and coalgebras in $Ch^+_{\mathbb{K}}$, because these hypotheses ensure that the morphisms $P(n)\otimes C_d \to (C^{\otimes n})_d$ are zero for $n > d$. In the next sections we will use these assumptions and just say P-coalgebra to refer to a conilpotent P-coalgebra. Under these assumptions, we also get as a corollary of theorem 2.3.2 an equivalence between the notion of coalgebra over the operad P (Definition 1.1.19) and the notion of coalgebra over the cooperad P^* (Definition 1.1.21).

1.1.2 Monads, comonads and distributive laws

In certain cases, bialgebras can be parametrized by a pair of operads in the following way: one operad encodes the operations, the other encodes the cooperations, such that the concerned bialgebra forms an algebra over the first operad and a coalgebra over the second operad. The compatibility relations between operations and cooperations are formalized by the notion of distributive law between the two operads. The purpose of this subsection is to explain these notions, starting in the more general context of monads and comonads.

Definition 1.1.22. Let $\mathcal{C}$ be a category. (a) A monad $\mathcal{T} = (\mathcal{T}, \gamma, \iota)$ in $\mathcal{C}$ is a functor $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ equiped with two natural transformations $\gamma : \mathcal{T} \circ \mathcal{T} \to \mathcal{T}$ and $\iota : \mathcal{I} \to \mathcal{T}$ (where $\mathcal{I}$ is the identity functor) satisfying the usual monoid axioms:

-*associativity*:

$$\begin{array}{ccc} \mathcal{T}\circ\mathcal{T}\circ\mathcal{T} & \xrightarrow{\mathcal{T}\gamma} & \mathcal{T}\circ\mathcal{T} \\ {\scriptstyle \gamma\mathcal{T}}\downarrow & & \downarrow{\scriptstyle \gamma} \\ \mathcal{T}\circ\mathcal{T} & \xrightarrow[\gamma]{} & \mathcal{T} \end{array}$$

-*unitarity*:

$$\begin{array}{ccccc} \mathcal{I}\circ\mathcal{T} & \xrightarrow{\iota\mathcal{T}} & \mathcal{T}\circ\mathcal{T} & \xleftarrow{\mathcal{T}\gamma} & \mathcal{T}\circ\mathcal{I} \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle \gamma} & \swarrow{\scriptstyle =} & \\ & & \mathcal{T} & & \end{array}.$$

(b) A $\mathcal{T}$-algebra (A, α) is an object A of $\mathcal{C}$ equiped with a morphism $\alpha : \mathcal{T}(A) \to A$ such that the following diagrams commute:

$$\begin{array}{ccc} (\mathcal{T}\circ\mathcal{T})(A) & \xrightarrow{\mathcal{T}(\alpha)} & \mathcal{T}(A) \\ {\scriptstyle \gamma(A)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathcal{T}(A) & \xrightarrow[\alpha]{} & A \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\iota(A)} & \mathcal{T}(A) \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle \alpha} \\ & & A \end{array}.$$

(c) A morphism of $\mathcal{T}$-algebras $(A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ of $\mathcal{C}$ such that the following diagram commute:

$$\begin{array}{ccc} (\mathcal{T}(A) & \xrightarrow{\alpha} & A \\ {\scriptstyle \mathcal{T}(f)}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{T}(B) & \xrightarrow[\beta]{} & B \end{array}$$

We denote $\mathcal{T} - Alg$ the category of $\mathcal{T}$-algebras.

One immediately sees that it corresponds to definitions 1.4 and 1.9, such that operads and their algebras are special cases of monads and their algebras. In a dual way, we can define comonads and coalgebras over comonads (see [5] for instance):

Definition 1.1.23. (a) A comonad $\mathcal{S} = (\mathcal{S}, \delta, \epsilon)$ in $\mathcal{C}$ is a functor $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ equiped with two natural transformations $\delta : \mathcal{S} \to \mathcal{S} \circ \mathcal{S}$ and $\epsilon : \mathcal{S} \to \mathcal{I}$ satisfying the usual comonoid axioms:

-*coassociativity*:

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\delta} & \mathcal{S}\circ\mathcal{S} \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \mathcal{S}\delta} \\ \mathcal{S}\circ\mathcal{S} & \xrightarrow[\delta\mathcal{S}]{} & \mathcal{S}\circ\mathcal{S}\circ\mathcal{S} \end{array}$$

-*counitarity*:

$$\begin{array}{ccccc} & & \mathcal{S} & & \\ & \overset{=}{\swarrow} & \downarrow \delta & \overset{=}{\searrow} & \\ \mathcal{I}\circ\mathcal{S} & \xleftarrow[\epsilon\mathcal{S}]{} & \mathcal{S}\circ\mathcal{S} & \xrightarrow[\mathcal{S}\epsilon]{} & \mathcal{S}\circ\mathcal{I} \end{array}.$$

(b) A $\mathcal{S}$-coalgebra (C,c) is an object C of $\mathcal{C}$ equiped with a morphism $c : C \to \mathcal{S}(C)$ such that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{c} & \mathcal{S}(C) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \delta(c)} \\ \mathcal{S}(C) & \xrightarrow[\mathcal{S}(c)]{} & \mathcal{S}(\mathcal{S}(C)) \end{array}$$

$$\begin{array}{ccc} C & \xrightarrow{c} & \mathcal{S}(C) \\ & \overset{=}{\searrow} & \downarrow{\scriptstyle \epsilon(c)} \\ & & C \end{array}.$$

(c) A morphism of $\mathcal{S}$-coalgebras $(C,c) \to (D,d)$ is a morphism $f : C \to D$ of $\mathcal{C}$ such that the following diagram commute:

$$\begin{array}{ccc} (C & \xrightarrow{c} & \mathcal{S}(C) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{S}(f)} \\ D & \xrightarrow[d]{} & \mathcal{S}(D) \end{array}$$

We denote $\mathcal{S} - Coalg$ the category of $\mathcal{S}$-coalgebras.

Again, we see that cooperads and their coalgebras are special cases of comonads and their coalgebras. Now, suppose we have in our category $\mathcal{C}$ a monad $(\mathcal{T},\gamma,\iota)$ and a comonad $(\mathcal{S},\delta,\epsilon)$. We would like to make $\mathcal{T}$ and $\mathcal{S}$ compatible, that is to define $\mathcal{S}$-coalgebras in $\mathcal{T}$-algebras or conversely $\mathcal{T}$-algebras in $\mathcal{S}$-coalgebras. This compatibility is formalized by the notion of mixed distributive law:

Definition 1.1.24. A mixed distributive law $\lambda : \mathcal{T}\mathcal{S} \to \mathcal{S}\mathcal{T}$ between $\mathcal{T}$ and $\mathcal{S}$ is a natural transformation satisfying the following conditions:

(i)$\Lambda \circ \gamma\mathcal{S} = \mathcal{S}\gamma \circ \Lambda : \mathcal{T}\mathcal{T}\mathcal{S} \to \mathcal{S}\mathcal{T}$

(ii)$\delta\mathcal{T} \circ \Lambda = \Lambda \circ \mathcal{T}\delta : \mathcal{T}\mathcal{S} \to \mathcal{S}\mathcal{S}\mathcal{T}$

(iii)$\lambda \circ \iota\mathcal{S} = \mathcal{S}\iota : \mathcal{S} \to \mathcal{S}\mathcal{T}$

(iv)$\epsilon\mathcal{T} \circ \lambda = \mathcal{T}\epsilon : \mathcal{T}\mathcal{S} \to \mathcal{T}$

where the $\Lambda : \mathcal{T}^m\mathcal{S}^n \to \mathcal{S}^n\mathcal{T}^m$, for natural numbers m and n, are the natural transformations obtained by iterating λ. For instance, for $m = 2$ and $n = 3$ we have

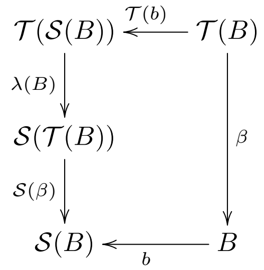
$$\mathcal{T}^2\mathcal{S}^3 \xrightarrow{\mathcal{T}\lambda\mathcal{S}^2} \mathcal{T}\mathcal{S}\mathcal{T}\mathcal{S}^2 \xrightarrow{\lambda^2\mathcal{S}} \mathcal{S}\mathcal{T}\mathcal{S}\mathcal{T}\mathcal{S} \xrightarrow{\mathcal{S}\lambda^2} \mathcal{S}^2\mathcal{T}\mathcal{S}\mathcal{T} \xrightarrow{\mathcal{S}^2\lambda\mathcal{T}} \mathcal{S}^3\mathcal{T}^2$$

This conditions allow us to lift $\mathcal{T}$ as an endofunctor of $\mathcal{S} - Coalg$ and $\mathcal{S}$ as an endofunctor of $\mathcal{T} - Alg$ (see remark 1.1.26).

Finally we introduce the notion of bialgebra over a pair (monad,comonad) endowed with a mixed distributive law:

Definition 1.1.25. (a) Given a monad $\mathcal{T}$, a comonad $\mathcal{S}$ and a mixed distributive law $\lambda : \mathcal{T}\mathcal{S} \to \mathcal{S}\mathcal{T}$, a $(\mathcal{T},\mathcal{S})$-bialgebra (B,β,b) is an object B of $\mathcal{C}$ equiped with two morphisms $\beta : \mathcal{T}(B) \to B$ and $b : B \to \mathcal{S}(B)$ defining respectively a $\mathcal{T}$-algebra structure and a $\mathcal{S}$-coalgebra structure. Furthermore, the maps β

and b satisfy a compatibility condition expressed through the commutativity of the following diagram:



(b) A morphism of $(\mathcal{T},\mathcal{S})$-bialgebras is a morphism of $\mathcal{C}$ which is both a morphism of $\mathcal{T}$-algebras and a morphism of $\mathcal{S}$-coalgebras.

The category of $(\mathcal{T},\mathcal{S})$-bialgebras is denoted $(\mathcal{T},\mathcal{S})-Bialg$.

Remark 1.1.26. The application $\mathcal{S}(\beta)\circ\lambda(B)$ endows $\mathcal{S}(B)$ with a $\mathcal{T}$-algebra structure, and the application $\lambda(B)\circ\mathcal{T}(b)$ endows $\mathcal{T}(B)$ with a $\mathcal{S}$-coalgebra structure. Moreover, given these two structures, the compatibility diagram of definition 1.1.25 shows that β is a morphism of $\mathcal{S}$-coalgebras and b a morphism $\mathcal{T}$-algebras. The $(\mathcal{T},\mathcal{S})$-bialgebras can therefore be considered as $\mathcal{S}$-coalgebras in $\mathcal{T}-Alg$ or as $\mathcal{T}$-algebras in $\mathcal{S}-Coalg$.

In the particular case of operads, the mixed distributive laws can be defined by explicit formulae:

Definition 1.1.27. Let P and Q be two operads. A mixed distributive law between P and Q is a family of applications $\{M(m,n)\}_{m,n\geq 1}$ where

$$M(m,n):P(m)\otimes Q(n)\to\bigoplus(Q(t_1)\otimes...\otimes Q(t_m))\otimes_{\Sigma_{t_1}\times...\times\Sigma_{t_m}}\mathbb{K}(\Sigma_N)\otimes_{\Sigma_{s_1}\times...\times\Sigma_{s_n}}(P(s_1)\otimes...\otimes P(s_n))$$

where the direct sum is indexed by every $N\geq 1$ and $t_1+...+t_m=s_1+...+s_n=N$. Moreover, these applications have to be compatible with operadic compositions and symmetric group actions at the inputs and the outputs. The detailed axioms can be found in [5].

Theorem 1.1.28. *(cf. [5]) Let P and Q be two operads endowed with a mixed distributive law. Then the monad $P(-)$ and the comonad $Q^*(-)$ inherits a distributive law in the sense of definition 1.1.24, induced by the mixed distributive law of definition 1.1.27.*

We define a (P,Q)-bialgebra as a bialgebra over these monads in distribution in the sense of definition 1.1.25. Suppose that $\mathbb{K}$ is of characteristic zero and that every $Q(n)$ is finite dimensional. Then we know that the notions of Q-coalgebras and Q^*-coalgebras coincide. A (P,Q)-bialgebra is thus equiped with a P-algebra structure, a Q-coalgebra structure and compatibilities with respect to the distributive law. The operadic distributive law as defined in definition 1.1.27 formalizes the interplay between algebraic operations and coalgebraic cooperations of the bialgebra.

Let us finally note that if B is a (P,Q)-coalgebra, then, as a corollary of Theorem 1.1.28 and Remark 1.1.26, the free P-algebra $P(B)$ has a natural structure of Q-coalgebra and the cofree Q-coalgebra $Q^*(B)$ has a natural structure of P-algebra.

1.1.3 Model categories and the small object argument

Model categories were introduced in [22] as an effective approach to do localization with respect to a particular class of morphisms called the weak equivalences. The original motivation was to transpose ideas related to homotopies, fibrations and cofibrations in topological spaces in a more general and categorical framework. Model categories are therefore the natural setting to do homotopical algebra. This means that they encode well defined notions of cylinder objects and path objects, homotopy classes, non-abelian cohomology theories and non abelian functor derivation (Quillen's derived functors). We will just recall here some basic facts about model categories, cofibrantly generated model categories and the small object argument. We will not review the construction of the associated homotopy category and its properties. We refer the reader to the classical reference [22], but also to [4] for a well-written and detailed account on basis of model categories and their homotopy theories, as well as [14] to push the analysis further.

Definition 1.1.29. A (closed) model category is a category $\mathcal{M}$ with the data of three classes of morphisms: the weak equivalences $\xrightarrow{\sim}$, the cofibrations $\rightarrowtail$ and the fibrations $\twoheadrightarrow$. Each of theses classes is stable by composition and contains the identity morphisms. A morphism which is both a cofibration and a weak equivalence is called an acyclic cofibration, and a morphism which is both a fibration and a weak equivalence is called an acyclic fibration. The following axioms hold:

MC1. Small limits and colimits exist in $\mathcal{M}$ (*completeness axiom*).

MC2. Weak equivalences satisfy the "two-out-of-three" property: if f and g are two composable morphisms such that two among f, g and $f \circ g$ are weak equivalences, then so is the third (*two-out-of-three axiom*).

MC3. If f is a retract of g and g belongs to one of the three aforementioned classes, then so does f (*retract axiom*).

MC4. Given a commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

a lifting exists in the two following situations :

(i) i is cofibration and p an acyclic fibration. One says that cofibrations have the left lifting property with respect to acyclic fibrations.

(ii)i is an acyclic cofibration and p a fibration. One says that fibrations have the right lifting property with respect to acyclic cofibrations. These are the *lifting axioms*.

MC5. Every morphism f admits the two following factorizations:

(i)$f = p \circ i$ where p is a fibration and i an acyclic cofibration.

(ii)$f = p \circ i$ where p is an acyclic fibration and i a cofibration.

These are the *factorization axioms.*

An object is cofibrant if its initial morphism is a cofibration, and fibrant if its final morphism is a fibration. Cofibrations and fibrations satisfy the following properties:

Proposition 1.1.30. *(cf. [4], proposition 3.13) Let $\mathcal{M}$ be a model category.*

(i)The cofibrations of $\mathcal{M}$ are the morphisms which have the left lifting property with respect to the acyclic fibrations.

(ii) The acyclic cofibrations of $\mathcal{M}$ are the morphisms which have the left lifting property with respect to the fibrations.

(iii)The fibrations of $\mathcal{M}$ are the morphisms which have the right lifting property with respect to the acyclic cofibrations.

(iv)The acyclic fibrations of $\mathcal{M}$ are the morphisms which have the right lifting property with respect to the cofibrations.

Remark 1.1.31. According to this proposition, it is sufficient to define the weak equivalences and the cofibrations to get the fibrations, or to define the weak equivalences and the fibrations to get the cofibrations.

Proposition 1.1.32. *(cf. [4], proposition 3.14) Let $\mathcal{M}$ be a model category.*

(i)The class of cofibrations is stable by cobase change.

(ii)The class of acyclic cofibrations is stable by cobase change.

(iii)The class of fibrations is stable by base change.

(iv)The class of acyclic fibrations is stable by base change.

In the most common cases (for instance chain complexes and simplicial sets), the core of the model category structure consists of the class of weak equivalences and two sets of generators for the cofibrations and acyclic cofibrations. Any (acyclic) cofibration is obtained by retracts and pushouts of these generators, and (acyclic) fibrations are obtained by their right lifting property. Such a model category is said to be cofibrantly generated. Moreover, when one has found the sets of generators, there is a classical construction called the small object argument which produces the factorization axioms needed to satisfy axiom MC5. Let us define more precisely these notions. We start with the small object argument, which is a general and useful way to produce factorizations with lifting properties with respect to a given class of morphisms. We just sum up the construction given in [14] without detailing the process. We refer the reader for section 2.1.1 of [14] for recollections about ordinals, cardinals and transfinite composition.

Suppose that $\mathcal{C}$ is a category admitting small colimits. Let λ be an ordinal. A λ-sequence is a colimit preserving functor $B : \lambda \rightarrow \mathcal{C}$, written as

$$B(0) \rightarrow B(1) \rightarrow ... \rightarrow B(\beta) \rightarrow ...$$

Now let us fix an object A of $\mathcal{C}$, a collection $\mathcal{D}$ of morphisms of $\mathcal{C}$ and a cardinal κ.

Definition 1.1.33. (1) The object A is κ-small with respect to $\mathcal{D}$ if for all κ-filtered ordinals λ and all λ-sequences

$$B(0) \rightarrow B(1) \rightarrow ... \rightarrow B(\beta) \rightarrow ...$$

such that each map $B(\beta) \rightarrow B(\beta+1)$ is in $\mathcal{D}$ for $\beta + 1 < \lambda$, the canonical induced map

$$colim_{\beta<\lambda} Mor_{\mathcal{C}}(A, B(\beta)) \rightarrow Mor_{\mathcal{C}}(A, colim_{\beta<\lambda} B(\beta))$$

is a bijection.

(2) The object A is small if it is κ-small with respect to all morphisms of $\mathcal{C}$ for a certain cardinal κ.

A fundamental example of small object is the following:

Lemma 1.1.34. *([14], Lemma 2.3.2) Every chain complex over a ring is small.*

Let κ be a cardinal. Let $\mathcal{F} = \{f_i : A_i \rightarrow B_i\}_{i\in I}$ be a set of morphisms of $\mathcal{C}$. We consider a morphism $g : X \rightarrow Y$ of $\mathcal{C}$ for which we want to produce a factorization $X \rightarrow X' \rightarrow Y$, such that $X' \rightarrow Y$ has the right lifting property with respect to the morphisms of $\mathcal{F}$. There is a recursive construction providing the following commutative diagram:

$$\begin{array}{ccccccccc} X & \xrightarrow{i_1} & G^1(\mathcal{F},g) & \xrightarrow{i_2} & ... & \xrightarrow{i_\beta} & G^\beta(\mathcal{F},g) & \xrightarrow{i_{\beta+1}} & ... \\ {\scriptstyle g}\downarrow & & {\scriptstyle g_1}\downarrow & & & & {\scriptstyle g_\beta}\downarrow & & \\ Y & = & Y & = & ... & = & Y & = & ... \end{array}$$

where the upper line is a λ-sequence for a certain κ-filtered ordinal λ. In this recursive procedure, each i_β is obtained by a pushout of the form

$$\begin{array}{ccc} \bigoplus_\alpha A_\alpha & \longrightarrow & G^{\beta-1}(\mathcal{F},g) \\ {\scriptstyle \bigoplus_\alpha f_\alpha}\downarrow & & \downarrow{\scriptstyle i_\beta} \\ \bigoplus_\alpha B_\alpha & \longrightarrow & G^\beta(\mathcal{F},g) \end{array}$$

where the f_α are morphisms of $\mathcal{F}$. The category $\mathcal{C}$ is supposed to admit small colimits, so we can consider the infinite composite $i_\infty : X \rightarrow G^\infty(\mathcal{F},g)$ of the sequence of maps

$$X \xrightarrow{i_1} G^1(\mathcal{F},p) \xrightarrow{i_2} ... \xrightarrow{i_\beta} G^\beta(\mathcal{F},p) \xrightarrow{i_{\beta+1}} ... \longrightarrow G^\infty(\mathcal{F},p)$$

where $G^\infty(\mathcal{F},p) = colim_{\beta<\lambda} G^\beta(\mathcal{F},p)$ is the colimit of this system. A morphism like $i_\infty : X \rightarrow G^\infty(\mathcal{F},g)$, that is obtained by a (possibly transfinite) composition of pushouts of maps of $\mathcal{F}$, is called a relative $\mathcal{F}$-cell complex. By universal property of the colimit, the morphism g has a factorization $g = g_\infty \circ i_\infty$ where $g_\infty : G^\infty(\mathcal{F},g) \rightarrow Y$.

Theorem 1.1.35. *(cf. [14], Theorem 2.1.14) In the preceding situation, suppose that for every $i \in I$, the object A_i is κ-small in $\mathcal{C}$ with respect to relative $\mathcal{F}$-cell complexes. Then the morphism p_∞ has the right lifting property with respect to the morphisms of $\mathcal{F}$.*

Now we can define the notion of cofibrantly generated model category:

Definition 1.1.36. The model category $\mathcal{M}$ is said to be cofibrantly generated if there exists two sets of morphisms $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{M}$ such that:
(i) The domains of the morphisms of $\mathcal{I}$ are small.
(ii) The domains of the morphisms of $\mathcal{J}$ are small.
(iii) The class of fibrations is the class of morphisms having the right lifting property with respect to the morphisms of $\mathcal{J}$.
(iv) The class of acyclic fibrations is the class of morphisms having the right lifting property with respect to the morphisms of $\mathcal{I}$.
One says that $\mathcal{I}$ is the set of generating cofibrations and $\mathcal{J}$ the set of generating acyclic cofibrations.

A usual way to construct a cofibrantly generated model category is the following: start by defining these two sets of generators. Then use the small object argument in order to obtain for any morphism f a factorization $f = f_\infty \circ i_\infty$ such that f_∞ has the right lifting property with respect to $\mathcal{I}$ (resp. $\mathcal{J}$). Define the acyclic fibrations (resp. fibrations) as the morphisms having the right lifting property with respect to the morphisms of $\mathcal{I}$ (resp. $\mathcal{J}$). The application f_∞ forms therefore an acyclic fibration (resp. a fibration). Afterwards, define cofibrations as relative $\mathcal{I}$-cell complexes and acyclic cofibrations as relative $\mathcal{J}$-cell complexes. In particular, this implies that i_∞ is a cofibration (resp. acyclic cofibration), and thus the factorization axioms MC5 hold.

In the remaining sections of our paper, in order to deal with applications of the form of i_∞ we will need the two following lemmas:

Lemma 1.1.37. *(see [13]) Let us consider a pushout of the form*

$$\begin{array}{ccc} K & \xrightarrow{f} & K' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ L & \xrightarrow[g]{} & L' \end{array}$$

in a category $\mathcal{C}$ admitting small colimits. Suppose that i has the left lifting property with respect to a given family $\mathcal{F}$ of morphisms of $\mathcal{C}$. Then j has also the left lifting property with respect to $\mathcal{F}$. Another way to state this result is to say that the left lifting property with respect to a given family of morphisms is invariant under cobase change.

Lemma 1.1.38. *(see [13]) Let us consider a λ-sequence*

$$G^0 \xrightarrow{i_1} G^1 \xrightarrow{i_2} ... \xrightarrow{i_\beta} G^\beta \xrightarrow{i_{\beta+1}} ... \longrightarrow colim_{\beta<\lambda} G^\beta = G^\infty \ .$$

Let us note $i_\infty : G^0 \to G^\infty$ the transfinite composite of the i_β. If for every ordinal $\beta < \lambda$, the morphism i_β has the left lifting property with respect to a given family $\mathcal{F}$ of morphisms of $\mathcal{C}$, then so does i_∞.

It is time now to give a concrete example of model category. Of course, topological spaces provide the initial example from which the theory of model categories arised. However, the example we will use to illustrate these notions is that of chain complexes. This choice is motivated by two reasons. Firstly, this will be the base category for the remaining part of our paper. Secondly, the model category structures of algebras and coalgebras over operads will be transferred from this one via adjunctions.

Theorem 1.1.39. *(cf. [4], theorem 7.2) The category $Ch_{\mathbb{K}}$ of non-negatively graded chain complexes over a field $\mathbb{K}$ forms a cofibrantly generated model category such that a morphism f of $Ch_{\mathbb{K}}$ is*
(i) a weak equivalence if for every $n \geq 0$, the induced map $H_n(f)$ in homology is an isomorphism.
(ii) a fibration if for every $n > 0$, the map f_n is surjective.
(iii) a cofibration if for every $n \geq 0$, the map f_n is injective.

For $n \geq 1$, the chain complex D^n is defined by

$$D^n_k = \begin{cases} 0 & k \neq n, n-1 \\ \mathbb{K}b_{n-1} & k = n-1 \\ \mathbb{K}e_n & k = n \end{cases}$$

with $deg(b_{n-1}) = n-1$, $deg(e_n) = n$, and a differential δ satisfying $\delta(e_n) = b_{n-1}$. The chain complex S^n is defined by

$$S^n_k = \begin{cases} 0 & k \neq n \\ \mathbb{K}b_n & k = n \end{cases}$$

with $deg(b_n) = n$. We have for every $n \geq 1$ an obvious inclusion $j_n : S^{n-1} \to D^n$ which is the identity on $\mathbb{K}b_{n-1}$. The sets of generating cofibrations and generating acyclic cofibrations are given by the following proposition:

Proposition 1.1.40. *(cf. [4], proposition 7.19) A morphism f of $Ch_{\mathbb{K}}$ is*

(i) a fibration if and only if for every $n \geq 1$, it has the right lifting property with respect to the inclusions $i_n : 0 \to D^n$.

(ii)an acyclic fibration if and only if for every $n \geq 1$, it has the right lifting property with respect to the inclusions $j_n : S^{n-1} \to D^n$.

To conclude this section, let us note that the commonly used small object argument, for instance to prove theorem 1.1.39 but also for various other examples (like algebras over operads), is Proposition 7.17 in [4]. That is, use the simplifying assumption of smallness with respect to all morphisms. We will need a more refined version for coalgebras an operad.

1.2 The model category of algebras over an operad

The most general statement about model categories of algebras over operads holds in any cofibrantly generated symmetric monoidal model category. The category of Σ-modules inherits a model structure as a category of diagrams over a cofibrantly generated model category. Algebras over a Σ-cofibrant operad (i.e the underlying Σ-module is cofibrant in the model category of Σ-modules) form a cofibrantly generated semi-model category, see [9] theorem 12.3.A. The semi-model structure is a weakened model structure in which the lifting and factorization axioms can be applied only on morphisms with cofibrant domains.

We fix an operad P in the category of non-negatively graded chain complexes $Ch_{\mathbb{K}}$ over a field $\mathcal{K}$ of characteristic zero. In this case we construct a full model category structure. Moreover, the proof of theorem 1.3.1 illustrates fundamental methods for the remaining part of this paper. The central theorem of this section is the following:

Theorem 1.2.1. *The category of P-algebras ${}_PCh_{\mathbb{K}}$ inherits a cofibrantly generated model category structure such that a morphism f of ${}_PCh_{\mathbb{K}}$ is*

(i)a weak equivalence if $U(f)$ is a weak equivalence in $Ch_{\mathbb{K}}$, where U is the forgetful functor;

(ii)a fibration if $U(f)$ is a fibration in $Ch_{\mathbb{K}}$;

(iii)a cofibration if it has the left lifting property with respect to acyclic fibrations.

We can also say that cofibrations are relative cell complexes with respect to the generating cofibrations.

We will make the generating cofibrations and generating acyclic cofibrations explicit in 2.3. The three classes defined above contain identity and are clearly stable by composition. Thus it remains to prove the MC axioms. Axioms MC2 and MC3 are clear and easily proved as in the case of chain complexes (theorem 1.1.39). Axiom MC4(i) is obvious by definition of the cofibrations.

Actually, what we have obtained is a transfer of cofibrantly generated model category structure via the adjunction $P : Ch_{\mathbb{K}} \rightleftarrows_P Ch_{\mathbb{K}} : U$. The forgetful functor creates fibrations and weak equivalences. The free P-algebra functor P preserves generating (acyclic) cofibrations as we will see in 2.3, by definition of the generating (acyclic) cofibrations of ${}_PCh_{\mathbb{K}}$. Moreover, it preserves colimits as a left adjoint (it is a general property of adjunctions, see [19] for instance). Thus it preserves all (acyclic) cofibrations, which are relative cell complexes with respect to the generating (acyclic) cofibrations. Such a pair of functors is called a Quillen adjunction, and induces an adjunction at the level of the associated homotopy categories.

1.2.1 Small limits and colimits

The small limits

The forgetful functor creates the small limits in ${}_PCh_{\mathbb{K}}$. Indeed, let us consider a diagram $\{A_i\}_{i\in I}$ of P-algebras. We obtain a diagram $\{U(A_i)\}_{i\in I}$ in $Ch_{\mathbb{K}}$ via the forgetful functor U. The category $Ch_{\mathbb{K}}$ admits small limits so $lim_iU(A_i)$ exists.

The structure of P-algebra on A_i is the data of $\mathbb{K}$-linear maps $\alpha_n^i : P(n)\otimes A_i^{\otimes n}\rightarrow A_i$ satisfying the appropriate properties of associativity, equivariance and unitarity. The limit $lim_iU(A_i)$ is equiped with projections $\pi_i : lim_iU(A_i)\rightarrow U(A_i)$, thus we get a linear map

$$P(n)\otimes(lim_iU(A_i))^{\otimes n}\stackrel{id\otimes\pi_i^{\otimes n}}{\rightarrow}P(n)\otimes U(A_i)^{\otimes n}\stackrel{\alpha_n^i}{\rightarrow}U(A_i)$$

which factorizes via $lim_iU(A_i)$ by universal property of the limit, hence a commutative square for every i and n

$$\begin{array}{ccc} P(n)\otimes U(A_i)^{\otimes n} & \xrightarrow{\alpha_n^i} & U(A_i) \\ \uparrow{\scriptstyle id\otimes\pi_i^{\otimes n}} & & \uparrow{\scriptstyle \pi_i} \\ P(n)\otimes(lim_iU(A_i))^{\otimes n} & \xrightarrow[\alpha_n^{\infty}]{} & lim_iU(A_i) \end{array}.$$

The α_n^{∞} endow $lim_iU(A_i)$ with a structure of P-algebra. The structure morphism $\gamma_{A_i} : P(A_i)\rightarrow A_i$ is the sum of the α_n^i in each arity n, so the following diagram commutes for every i:

$$\begin{array}{ccc} P(A_i) & \xrightarrow{\gamma_{A_i}} & A_i \\ \uparrow{\scriptstyle P(\pi_i)} & & \uparrow{\scriptstyle \pi_i} \\ P(lim_iU(A_i)) & \xrightarrow[\gamma_{\infty}]{} & lim_iA_i \end{array}$$

where γ_{∞} is the sum of the α_n^{∞} in each arity n and constitutes the structure morphism of P-algebra of lim_iA_i. It proves that the π_i are morphisms of P-algebras. We conclude that $lim_iU(A_i)$ endowed with the P-algebra structure defined by the α_n^{∞} is the limit of $\{A_i\}$ in ${}_PCh_{\mathbb{K}}$.

The small colimits

First recall the definition of a reflexive coequalizer:

Definition 1.2.2. Let $\mathcal{C}$ be a category and $f,g : A\rightarrow B$ two arrows $\mathcal{C}$. A coequalizer of (f,g) is an arrow $u : B\rightarrow E$ such that
(i) $u\circ f = u\circ g$
(ii) if $h : B\rightarrow C$ satisfies $h\circ f = h\circ g$, then h admits a unique factorization $h = h'\circ u$.

Definition 1.2.3. A pair of morphisms $f,g : A\rightarrow B$ is reflexive if there exists a morphism $s : B\rightarrow A$ such that $f\circ s = g\circ s = id_B$. The coequalizer of a reflexive pair is called a reflexive coequalizer.

The reflexive coequalizers allow us to build all the small colimits. This arises from the following theorem:

Theorem 1.2.4. *(see [19]) Let $\mathcal{C}$ be a category. If $\mathcal{C}$ contains the reflexive coequalizers of every pairs of arrows and all small coproducts, then $\mathcal{C}$ contains all small colimits.*

Proof. Let $\{X_i\}_{i\in I}$ be a diagram in $\mathcal{C}$, i.e a functor $X : I \rightarrow \mathcal{C}$ where I is a small category. We consider the following pair of morphisms:

$$\bigvee_{u:i\rightarrow j\in Mor(I)} X_i \underset{d_1}{\overset{d_0}{\rightrightarrows}} \bigvee_{k\in ob(I)} X_k$$

where

$$d_0 : (X_i, u : i \rightarrow j) \mapsto (X_i, i) \hookrightarrow \bigvee_k X_k$$

and

$$d_0 : (X_i, u : i \rightarrow j) \mapsto (X_j, j) \hookrightarrow_{-} u^* \bigvee_k X_k.$$

We define $s_0 : \bigvee_{k\in ob(I)} X_k \rightarrow \bigvee_{u:i\rightarrow j\in Mor(I)} X_i$ by $s_0(X_k) = (X_k, id_k : k \rightarrow k)$. We see clearly that $d_0 \circ s_0 = d_1 \circ s_0 = id_{\bigvee_{k\in ob(I)} X_k}$. By hypothesis, there exists a reflexive coequalizer for the reflexive pair (d_0, d_1):

$$\bigvee_{u:i\rightarrow j\in Mor(I)} X_i \underset{d_1}{\overset{d_0}{\rightrightarrows}} \bigvee_{k\in ob(I)} X_k \xrightarrow{\pi} Y.$$

It remains to show that this coequalizer Y satisfies the universal property of a colimit of $\{X_i\}_{i\in I}$. Let $\{f_i : X_i \rightarrow Y\}_{i\in I}$ a familiy of morphisms of $\mathcal{C}$. The f_i admit a unique factorization via $\bigvee_{k\in ob(I)} X_k$ through a map $f : \bigvee_{k\in ob(I)} X_k \rightarrow Y$. The fact that $f \circ d_0 = f \circ d_1$ is equivalent to the fact that the f_i commute with the arrows of the diagram $\{X_i\}_{i\in I}$. Indeed, it means that for every $u : i \rightarrow j \in Mor(I)$, $(f \circ d_0) |_(X_i, u) = (f \circ d_1) |_(X_i, u)$, i.e that $f_i = f_j \circ u^*$. In this case, the morphism f admit a unique factorization via the reflexive coequalizer Y of (d_0, d_1), so the f_i admits a unique factorization via Y. We conclude that $Y = colim_I X_i$. ☐

Now we verify that ${}_P Ch_{\mathbb{K}}$ contains the reflexive coequalizers of every pairs of morphisms and all small coproducts, which conclude our proof of the existence of small colimits and thus our proof of axiom MC1.

Lemma 1.2.5. *Let us consider a reflexive pair $(d_0, d_1 : A \rightrightarrows B, s_0 : B \rightarrow A)$ in ${}_P Ch_{\mathbb{K}}$. Then $coker(d_0 - d_1)$ has a structure of P-algebra and forms the reflexive coequalizer of (d_0, d_1) in ${}_P Ch_{\mathbb{K}}$.*

Proof. We recall briefly that an ideal of the P-algebra B is a sub-chain complex $I \subset U(B)$ (where U is the forgetful functor) such that for every $\mu \in P(n), r_1, ..., r_{n-1} \in B, x \in I$, $\mu(r_i, ..., r_{n-1}) \in I$. Let us show that $im(d_0 - d_1)$ is an ideal of B. Let $\mu \in P(n), b_1, ..., b_{n-1} \in B, a \in A$. Then

$$\begin{cases} \mu(b_1, ..., b_{n-1}, d_0(a)) = \mu((d_0 \circ s_0)(b_1), ..., (d_0 \circ s_0)(b_{n-1}), d_0(a)) \\ d_0(\mu(s_0(b_1), ..., s_0(b_{n-1}), a)) \\ (d_0 - d_1)(\mu(s_0(b_1), ..., s_0(b_{n-1}), a)) + d_1(\mu(s_0(b_1), ..., s_0(b_{n-1}), a)) \\ (d_0 - d_1)(\mu(s_0(b_1), ..., s_0(b_{n-1}), a)) + \mu(b_1, ..., b_{n-1}, d_1(a)). \end{cases}$$

In this series of equalities, we use that $d_0 \circ s_0 = d_1 \circ s_0 = id_B$ and that these are morphisms of P-algebras, therefore they commute with the operations. We deduce that

$$\mu(b_1, ..., b_{n-1}, (d_0 - d_1)(a)) = (d_0 - d_1)(\mu(s_0(b_1), ..., s_0(b_{n-1}), a)) \in im(d_0 - d_1).$$

Then we can equip $coker(d_0 - d_1) = B/im(d_0 - d_1)$ with the structure of P-algebra induced by that of B via the projection $\pi : B \rightarrow B/im(d_0 - d_1)$:

$$\begin{array}{ccc} P(n) \otimes B^{\otimes n} & \xrightarrow{\alpha_n^B} & B \\ \downarrow{\scriptstyle id\otimes\pi^{\otimes n}} & & \downarrow{\scriptstyle \pi} \\ P(n) \otimes coker(d_0 - d_1)^{\otimes n} & \xrightarrow[\alpha_n^{coker}]{} & coker(d_0 - d_1) \end{array}$$

where α_n^B and α_n^{coker} denote the P-algebras structures respectively of B and $coker(d_0 - d_1)$. We deduce the following commutative square:

$$\begin{array}{ccc} P(B) & \xrightarrow{\gamma_B} & B \\ {\scriptstyle P(\pi)}\downarrow & & \downarrow{\scriptstyle \pi} \\ P(coker(d_0-d_1)) & \xrightarrow[\gamma_{coker}]{} & coker(d_0-d_1) \end{array}$$

so π is a morphism of P-algebras. Furthermore, the space $coker(d_0 - d_1)$ is the reflexive coequalizer of (d_0, d_1) in $Ch_{\mathbb{K}}$. We conclude that it is their coequalizer in ${}_PCh_{\mathbb{K}}$. □

Lemma 1.2.6. *Let $\{R_i\}_{i\in I}$ be a set of P-algebras. We set*

$$d_0 = P(\bigoplus \gamma_{R_i}) : P(\bigoplus P(R_i)) \to P(\bigoplus R_i)$$

and

$$d_1 = \gamma(\bigoplus R_i) \circ P(+_i P(i_{R_i})) : P(\bigoplus P(R_i)) \hookrightarrow P(P(\bigoplus_{R_i})) \to P(\bigoplus R_i)$$

where γ is the composition product of the monad (P, γ, ι) and the $i_{R_j} : R_j \to \bigoplus R_i$ are inclusions. Then $\bigvee R_i = coker(d_0 - d_1)$ is the coproduct of the R_i in ${}_PCh_{\mathbb{K}}$.

Proof. We detail the proof in the case of two P-algebras R and S. The method is the same in the general case. Let us consider $s_0 = P(\iota(R) \oplus \iota(S)) : P(R \oplus S) \to P(P(R) \oplus P(S))$. Then

$$d_0 \circ s_0 = P(\gamma_R \oplus \gamma_S) \circ P(\iota(R) \oplus \iota(S)) = P((\gamma_R \circ \iota(R)) \oplus (\gamma_S \circ \iota(S))) = P(id_{R\oplus S}) = id_{P(R\oplus S)}$$

by definition of γ_R, γ_S and the functoriality of P. We also get

$$d_1 \circ s_0 = \gamma(R \oplus S) \circ P(P(i_R), P(i_S)) \circ P(\iota(R) \oplus \iota(S)) = id_{P(R\oplus S)}$$

by unitality of ι. According to the preceding lemma, the space $coker(d_0, d_1)$ is the reflexive coequalizer of (d_0, d_1) in ${}_PCh_{\mathbb{K}}$. Now let X be a P-algebra. Two linear maps $u : R \to X$, $v : S \to X$ induce a unique application $u + v : R \oplus S \to X$, which admits a unique factorization through a morphism of P-algebras $\varphi_{(u,v)} : P(R \oplus S) \to X$ by universal property of the free P-algebra. It remains to prove that $\varphi_{(u,v)}$ factorizes via $coker(d_0 - d_1)$ if and only if u and v are morphisms of P-algebras. The map $\varphi_{(u,v)}$ factorizes via $coker(d_0 - d_1)$ if and only if $\varphi_{(u,v)} \circ d_0 = \varphi_{(u,v)} \circ d_1$, or

$$\varphi_{(u,v)\circ d_0} = \varphi_{(u,v)} \circ d_1 \quad \Leftrightarrow \quad \begin{cases} \varphi_{(u,v)\circ d_0} \mid_{P(R)} = \varphi_{(u,v)} \circ d_1 \mid_{P(R)} \\ \varphi_{(u,v)\circ d_0} \mid_{P(S)} = \varphi_{(u,v)} \circ d_1 \mid_{P(S)} \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} u \circ \gamma_R = \gamma_X \circ P(u) \\ v \circ \gamma_S = \gamma_X \circ P(v) \end{cases}$$

which is by definition equivalent to the fact that u and v are morphims of P-algebras. □

1.2.2 Enveloping operad

Let A be a P-algebra. We can associate to it a particular operad called the enveloping operad of A. First, for every natural integers r and n, we consider the group injection $\Sigma_r \hookrightarrow \Sigma_{n+r}$ defined as follows: every permutation of Σ_r can be extended to a permutation of Σ_{n+r} by fixing the elements $\{r+1, ..., n+r\}$. We then consider the Σ-module $P[A]$ given by

$$P[A](n) = \bigoplus_{r=1}^{\infty} P(n + r) \otimes_{\Sigma_r} A^{\otimes r}.$$

We keep the same notation to denote the associated Schur functor. We need the following lemma:

Lemma 1.2.7. *Let A be a P-algebra. For every chain complex C of $Ch_{\mathbb{K}}$, we have $P[A](C) \cong P(A \oplus C)$.*

Proof. Let us note

$$Sh_{p,q} = \{\sigma \in \Sigma_{p+q} \mid \sigma(1) < ... < \sigma(p), \sigma(p+1) < ... < \sigma(p+q)\}$$

the set of (p,q)-shuffles, then

$$P(n) \otimes_{\Sigma_n} (A \oplus C)^{\otimes n} = \left(\bigoplus_{Sh_{p,q}, p+q=n} P(n) \otimes A^{\otimes p} \otimes C^{\otimes q} \right)_{\Sigma_n} = \bigoplus_{p+q=n} P(p+q) \otimes_{\Sigma_p \times \Sigma_q} (A^{\otimes p} \otimes C^{\otimes q})$$

hence

$$\begin{aligned} P(A \oplus C) &= \bigoplus_{n \geq 0} P(n) \otimes_{\Sigma_n} (A \oplus C)^{\otimes n} \\ &= \bigoplus_n \bigoplus_{p+q=n} P(p+q) \otimes_{\Sigma_p \times \Sigma_q} (A^{\otimes p} \otimes C^{\otimes q}) \\ &= \bigoplus_n \bigoplus_{p+q=n} (P(p+q) \otimes_{\Sigma_p} A^{\otimes p}) \otimes_{\Sigma_q} C^{\otimes q}) \\ &= \bigoplus_q P[A](q) \otimes_q C^{\otimes q} \\ &= P[A](C). \end{aligned}$$

□

The P-algebra structure morphism $\gamma_A : P(A) \rightarrow A$ induces a morphism of Σ-modules $d_0 : P[P(A)] \rightarrow P[A]$ defined by

$$d_0(n) = \bigoplus_{r=1}^{\infty} (id \otimes \gamma_A^{\otimes r}) : \bigoplus_{r=1}^{\infty} P(n+r) \otimes_{\Sigma_r} P(A)^{\otimes r} \rightarrow \bigoplus_{r=1}^{\infty} P(n+r) \otimes_{\Sigma_r} A^{\otimes r}.$$

The operadic composition product $\gamma : P \circ P \rightarrow P$ induces another morphism of Σ-modules $d_1 : P[P(A)] \rightarrow P[A]$: for every chain complex C of $Ch_{\mathbb{K}}$, there is a map

$$P[P(A)](C) \cong P(P(A) \oplus C) \stackrel{P(P(i_A), i_C)}{\rightarrow} P(P(A \oplus C)) \stackrel{\gamma(A \oplus C)}{\rightarrow} P(A \oplus C) \cong P[A](C)$$

where $i_A : A \hookrightarrow A \oplus C$ and $i_C : C \hookrightarrow P(A \oplus C)$. We know that we have a faithful embedding of the category of Σ-modules in the category of endofunctors (the one which associates to every Σ-module its Schur functor, see remark 1.6). Therefore the morphism of Schur functors above corresponds to a unique morphism of Σ-modules $d_1 : P[P(A)] \rightarrow P[A]$.

The unit $\iota : I \rightarrow P$ induces a morphism of Σ-modules $s_0 : P[A] \rightarrow P[P(A)]$ with $s_0(0) = P(\iota(A))$, obtained by the following morphism of Schur functors: for every chain complex C, define

$$P[A](C) \cong P(A \oplus C) \stackrel{P(\iota(A \oplus C))}{\rightarrow} P(P(A \oplus C)) \stackrel{P(P(pr_A), \pi \circ P(pr_C))}{\rightarrow} P(P(A) \oplus C) \cong P[P(A)](C)$$

where π is the projection on the component of arity 1 and $pr_A : A \oplus C \rightarrow A$, $pr_C : A \circ C \rightarrow C$ the obvious projections. Thus we finally get a reflexive pair (d_0, d_1) of morphisms of Σ-modules induced by a reflexive pair of morphisms of Schur functors. The enveloping operad of A is then the reflexive coequalizer $U_P(A) = coker(d_0 - d_1)$ in the Σ-modules, endowed with the operad structure induced by that of $P[A]$.

Now we want to prove that for every chain complex C, there is an isomorphism $U_P(A)(C) \cong A \vee P(C)$ where $\vee$ is the coproduct in ${}_P Ch_{\mathbb{K}}$. We need the following expression of such a coproduct:

Lemma 1.2.8. *Let A be a P-algebra and C be a chain complex of $Ch_{\mathbb{K}}$. The following coequalizer defines the coproduct $A \vee P(C)$ in the P-algebras:*

$$P(P(A) \oplus C) \underset{d_1}{\overset{d_0}{\rightrightarrows}} P(A \oplus C) \longrightarrow coker(d_0 - d_1) = A \vee P(C) \qquad (s_0 : P(A \oplus C) \to P(P(A) \oplus C))$$

where $d_0 \mid_A = \gamma_A$, $d_0 \mid_C = id_C$, $d_1 \mid_A = \gamma(A)$, $d_1 \mid_C = id_C$, $s_0 \mid_A = \iota(A)$, $s_0 \mid_C = id_C$.

Proof. We clearly have $d_0 \circ s_0 = d_1 \circ s_0 = id$ hence a reflexive pair in ${}_PCh_{\mathbb{K}}$. The cokernel $coker(d_0 - d_1)$ is thus the reflexive coequalizer of (d_0, d_1) in ${}_PCh_{\mathbb{K}}$ according to lemma 1.2.6. Let X be a P-algebra, $u : A \to X$ a morphism of P-algebras and $v : C \to X$ a linear map. These two maps induce an application $(u, v) : A \oplus C \to X$ hence an application $\varphi_{(u,v)} : P(A \oplus C) \to X$ by universal property of the free P-algebra. The proof ends by noting that $\varphi_{(u,v)}$ admits a unique factorization through $coker(d_0 - d_1)$. □

The reflexive coequalizer defining the enveloping operad induces a reflexive coequalizer in P-algebras

$$P[P(A)](C) \underset{d_1}{\overset{d_0}{\rightrightarrows}} P[A] \longrightarrow U_P(A)(C) \qquad (s_0 : P[A] \to P[P(A)](C))$$

where $P[A](C) \cong P(A \oplus C)$, $P[P(A)](C) \cong P(P(A) \oplus C)$ and d_0, d_1, s_0 turn out to be the morphisms of the lemma above. By unicity of the colimit, we have proved the following result:

Proposition 1.2.9. *Let A be a P-algebra and C a chain complex of $Ch_{\mathbb{K}}$, then*

$$U_P(A)(C) \cong A \vee P(C).$$

1.2.3 Generating (acyclic) cofibrations, proofs of MC4 and MC5

Generating (acyclic) cofibrations

The generating (acyclic) cofibrations are, as expected, the images of the generating (acyclic) cofibrations of $Ch_{\mathbb{K}}$ under the free P-algebra functor P. Recall that the $j_n : S^{n-1} \hookrightarrow D^n$ and the $i_n : 0 \hookrightarrow D^n$ are respectively the generating cofibrations and the generating acyclic cofibrations of $Ch_{\mathbb{K}}$.

Proposition 1.2.10. *Let $f : A \to B$ be a morphism of P-algebras.*

(i)It is a fibration if and only if it has the right lifting property with respect to the $P(i_n)$ for every $n \geq 1$, i.e the $P(i_n)$ are the generating acyclic cofibrations.

(ii)It is an acyclic fibration if and only if it has the right lifting property with respect to the $P(j_n)$ for every $n \geq 1$, i.e the $P(j_n)$ are the generating cofibrations.

Proof. The proof relies on standard adjunction arguments used to transfer cofibrantly generated model structures. Part (ii) can be proved in the same way than part (i), so we only give the details for part (i). Suppose that $f : A \to B$ is a fibration and consider a commutative square

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow{\scriptstyle P(i_n)} & & \downarrow{\scriptstyle f} \\ P(D^n) & \underset{v}{\longrightarrow} & B \end{array}$$

in ${}_PCh_{\mathbb{K}}$. Via the forgetful functor we obtain in $Ch_{\mathbb{B}}$ a commutative square

$$\begin{array}{ccc} 0 & \longrightarrow & U(A) \\ \downarrow{\scriptstyle (U\circ P)(i_n)} & & \downarrow{\scriptstyle U(f)} \\ (U \circ P)(D^n) & \underset{U(v)}{\longrightarrow} & U(B) \end{array} .$$

The unit $\eta : id_{Ch_{\mathbb{K}}} \rightarrow U \circ P$ associated to the adjunction between P and U provides a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & & \longrightarrow & U(A) \\ \downarrow{\scriptstyle i_n} & & \nearrow & & \downarrow{\scriptstyle U(f)} \\ D^n & \xrightarrow{\eta(D^n)} & (U\circ P)(D^n) & \xrightarrow{U(v)} & U(B) \end{array} .$$

A lifting $\hat{v} : D^n \rightarrow U(A)$ exits in this diagram, given that $U(f)$ is a fibration and has therefore the right lifting property with respect to i_n. By applying P we obtain a new commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & & \longrightarrow & (P\circ U)(A) \\ \downarrow{\scriptstyle P(i_n)} & & \overset{P(\hat{v})}{\nearrow} & & \downarrow{\scriptstyle (P\circ U)(f)} \\ P(D^n) & \xrightarrow{P(\eta(D^n))} & (P\circ U\circ P)(D^n) & \xrightarrow{(P\circ U)(v)} & (P\circ U)(B) \end{array}$$

in ${}_P Ch_{\mathbb{K}}$. The counity $\epsilon : P \circ U \rightarrow id_{{}_P Ch_{\mathbb{K}}}$ associated to the adjunction gives rise to the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & & \longrightarrow & (P\circ U)(A) & \xrightarrow{\epsilon(A)} & A \\ \downarrow{\scriptstyle P(i_n)} & & \overset{P(\hat{v})}{\nearrow} & & \downarrow{\scriptstyle (P\circ U)(f)} & & \downarrow{\scriptstyle f} \\ P(D^n) & \xrightarrow{P(\eta(D^n))} & (P\circ U\circ P)(D^n) & \xrightarrow{(P\circ U)(v)} & (P\circ U)(B) & \xrightarrow{\epsilon(B)} & B \end{array} .$$

Moreover, the following diagrams commute:

$$\begin{array}{ccc} (P\circ U)(P(D^n)) & \xrightarrow{(P\circ U)(v)} & (P\circ U)(B) \\ \downarrow{\scriptstyle \epsilon(P(D^n))} & & \downarrow{\scriptstyle \epsilon(B)} \\ P(D^n) & \xrightarrow{v} & B \end{array}$$

by naturality of ϵ, and

$$\begin{array}{ccc} P & \xrightarrow{P\eta} & P\circ U\circ P \\ & \searrow\!\!\searrow & \downarrow{\scriptstyle \epsilon P} \\ & & P \end{array}$$

which is a property associated to any adjunction, see [19] for more details. We conclude that $\epsilon(B) \circ (P \circ U)(v) \circ P(\eta(D^n)) = v \circ (\epsilon P \circ P\eta)(D^n) = v \circ id_{P(D^n)} = v$. Thus $\epsilon(A) \circ P(\hat{v}) : P(D^n) \rightarrow A$ is the desired lifting.

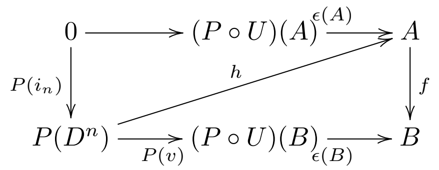
We have to prove the other direction of the equivalence. Let us suppose that f has the right lifting property with respect to the $P(i_n)$ and consider the commutative square

$$\begin{array}{ccc} 0 & \longrightarrow & U(A) \\ \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle U(f)} \\ D^n & \xrightarrow{v} & U(B) \end{array} .$$

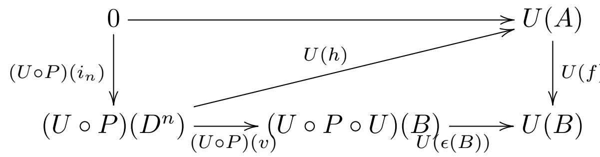
By applying P we obtain

$$\begin{array}{ccc} 0 & \longrightarrow & (P\circ U)(A) \\ \downarrow{\scriptstyle P(i_n)} & & \downarrow{\scriptstyle (P\circ U)(f)} \\ P(D^n) & \xrightarrow{P(v)} & (P\circ U)(B) \end{array} ,$$

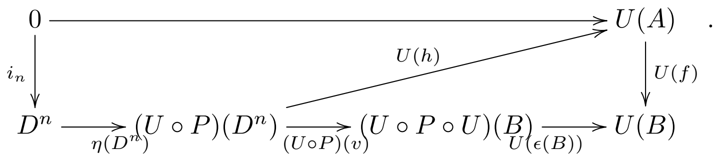
hence via the counity ϵ of the adjunction



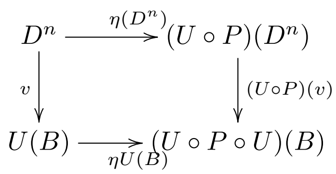
where h exists by hypothesis about f. We apply U:



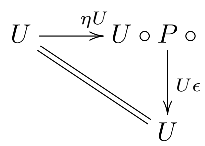
hence via the unity η of the adjunction:



Morevoer the following diagrams commute:



by naturality of η, and



which is a property associated to any adjunction. We deduce that $U(\epsilon(B)) \circ (U \circ P)(v) \circ \eta(D^n) = (U\epsilon \circ \eta U)(B) \circ v = id_{U(B)} \circ v = v$. Therefore $U(h) \circ \eta(D^n) : D^n \to U(A)$ is the desired lifting: the morphism $U(f)$ forms a fibration in $Ch_{\mathbb{K}}$, which implies by definition that f forms a fibration in ${}_P Ch_{\mathbb{K}}$. □

Axioms MC4 and MC5

MC5 (i). In order to apply the small object argument to the $P(j_n)$ and consequently obtain MC5 (i), we need the following lemma:

Lemma 1.2.11. *Let C be a chain complex of $Ch_{\mathbb{K}}$. If C is sequentially small in $Ch_{\mathbb{K}}$, then $P(C)$ is sequentially small in ${}_P Ch_{\mathbb{K}}$.*

Proof. Let us suppose that C is sequentially small in $Ch_{\mathbb{K}}$, and let $F : \mathbb{N} \to_P Ch_{\mathbb{K}}$ be a functor. For every $n \in \mathbb{N}$,

$$Hom_{{}_P Ch_{\mathbb{K}}}(P(C), F(n)) \cong Hom_{Ch_{\mathbb{K}}}(C, (U \circ F)(n))$$

hence

$$\begin{aligned} colim_n Hom_{{}_PCh_{\mathbb{K}}}(P(C), F(n)) &\cong colim_n Hom_{Ch_{\mathbb{K}}}(C, (U \circ F)(n)) \\ &\cong Hom_{Ch_{\mathbb{K}}}(C, colim_n (U \circ F)(n)) \end{aligned}$$

because $U \circ F : \mathbb{N} \to Ch_{\mathbb{K}}$ and C is sequentially small. We can equip $colim_n(U \circ F)(n)$ with a structure of P-algebra, such that with this structure it forms the colimit of the $F(n)$ in ${}_PCh_{\mathbb{K}}$. Indeed, we have $colim_n(U \circ F)(n) = \{[a], a \in F(n)\}/\sim$ where $a \sim b$ (i.e $[a] = [b]$), $a \in F(n), b \in F(m)$, $n \leq m$, if the application $F(n) \to F(m)$ in the sequential system sends a to b. Let $[a_1], ..., [a_r] \in colim_n(U \circ F)(n)$ such that $a_1 \in F(n_1), ..., a_r \in F(n_r)$. We consider $F(n)$ for a given $n \geq max(n_1, ..., n_r)$ and we set, for $\mu \in P(n)$, $\mu([a_1], ..., [a_r]) = \mu(a'_1, ..., a'_r)$ where $a'_1, ..., a'_r$ are representing elements of $[a_1], ..., [a_r]$ in $F(n)$. We then obtain a P-algebra structure on $colim_n(U \circ F)(n)$ (one says that the forgetful functor creates the sequential colimits). We can finally write

$$\begin{aligned} colim_n Hom_{{}_PCh_{\mathbb{K}}}(P(C), F(n)) &\cong Hom_{Ch_{\mathbb{K}}}(C, U(colim_n F(n))) \\ &\cong Hom_{{}_PCh_{\mathbb{K}}}(P(C), colim_n F(n)). \end{aligned}$$

□

The S^{n-1} are sequentially small in $Ch_{\mathbb{K}}$, so the $P(S^{n-1})$ are sequentially small in ${}_PCh_{\mathbb{K}}$. We can then apply the small object argument to a given morphism $f : A \to B$ of ${}_PCh_{\mathbb{K}}$ and the family of morphisms $\mathcal{F} = \{P(j_n)\}_{n \geq 1}$. We obtain a factorization $f = p_\infty \circ i_\infty$ where $i_\infty : A \to G^\infty(\mathcal{F}, f)$, $p_\infty : G^\infty(\mathcal{F}, f) \to B$ and p_∞ has the right lifting property with respect to the $P(j_n)$. According to proposition 1.2.10, the morphism p_∞ is an acyclic fibration. According to lemmas 1.37 and 1.38, the morphism i_∞ has the right lifting property and forms therefore a cofibration. We have the desired factorization.

MC5 (ii). In order to prove MC5 (ii), we need two general results about Σ-modules:

Proposition 1.2.12. *Let M be a Σ-module and C a chain complex. If $H_*(C) = 0$ then $H_*(M(C)) = H_*(M(0))$.*

Proof. Recall that we work over a field $\mathbb{K}$ of characteristic 0. We use the norm map $N : M(n) \otimes_{\Sigma_n} C^{\otimes n} \to M(n) \otimes C^{\otimes n}$ defined by

$$N(c \otimes v_1 \otimes ... \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma.c \otimes v_{\sigma(1)} \otimes ... \otimes v_{\sigma(n)}.$$

If we denote $p : M(n) \otimes C^{\otimes n} \to M(n) \otimes_{\Sigma_n} C^{\otimes n}$ the projection, then

$$\begin{aligned} (p \circ N)(c \otimes v_1 \otimes ... \otimes v_n) &= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} p(\sigma.c \otimes v_{\sigma(1)} \otimes ... \otimes v_{\sigma(n)}) \\ &= \frac{1}{n!} \mid \Sigma_n \mid c \otimes v_1 \otimes ... \otimes v_n \\ &= c \otimes v_1 \otimes ... \otimes v_n \end{aligned}$$

so $p \circ N = id$. Therefore $M(n) \otimes_{\Sigma_n} C^{\otimes n}$ is a retract of $M(n) \otimes C^{\otimes n}$. For $n \geq 1$, the Künneth formula gives us for every $k \geq 0$

$$H_k(M(n) \otimes C^{\otimes n}) = \bigoplus_{p+q=k} H_p(M(n) \otimes C) \otimes H_q(C^{\otimes n-1}).$$

This is equal to 0 for $n > 1$ because the fact that $H_*(C) = 0$ implies recursively that $H_*(C^{\otimes n}) = 0$ by the Künneth formula. This is also equal to 0 for $n = 1$ because the fact that $H_k(C) = 0$ implies that $H_k(M(1) \otimes C) = 0$. For $n = 0$, we have $H_k(M(0))$. We conclude that $H_k(M(C)) = H_k(M(0))$. □

Lemma 1.2.13. *Let M be a Σ-module. An injection $A \hookrightarrow B$ in $Ch_{\mathbb{K}}$ induces an injection $M(A) \hookrightarrow M(B)$.*

Proof. We know that $M(n)\otimes_{\Sigma_n}A^{\otimes n}$ is a retract of $M(n)\otimes A^{\otimes n}$ by using the projection and the norm maps. Moreover, the tensor product over a field preserves the injections so if $A\hookrightarrow B$ is an injection, then we obtain for every $n\geq 1$ the injections $u_n:M(n)\otimes A^{\otimes n}\hookrightarrow M(n)\otimes B^{\otimes n}$. We deduce injections $\tilde{u_n}=p\circ u_n\circ N:M(n)\otimes_{\Sigma_n}A^{\otimes n}\hookrightarrow M(n)\otimes_{\Sigma_n}B^{\otimes n}$, hence an injection $\bigoplus_n\tilde{u_n}:M(A)\hookrightarrow M(B)$. □

Now we can start the proof of MC5 (ii). Recall that the family of generating acyclic cofibrations of ${}_PCh_{\mathbb{K}}$ is given by $\mathcal{F}=\{P(i_n):P(0)\to P(D^n)\}_{n\geq 1}$. Let $f:X\to Y$ be a morphism of P-algebras. For every $k>0$, the construction of $G^k(\mathcal{F},f)$ (see section 1.3 about the small object argument) follows from a pushout of the form:

$$\begin{array}{ccc} \bigvee P(0) & \longrightarrow & G^{k-1}(\mathcal{F},f) \\ {\scriptstyle \bigvee P(i_n)}\downarrow & & \downarrow{\scriptstyle i_k} \\ \bigvee P(D^n) & \longrightarrow & G^k(\mathcal{F},f) \end{array}.$$

According to lemma 1.1.37, the left lifting property with respect to a given family of morphisms is invariant under cobase change. The coproduct $\bigvee P(i_n)$ has the left lifting property with respect to fibrations, so i_k has the same property, in particular i_k has the left lifting property with respect to acyclic fibrations. According to lemma 1.1.38, the transfinite composite i_∞ of the i_k inherits such a property and forms therefore a cofibration. The small object argument provides a factorization $f=p_\infty\circ i_\infty$ where $p_\infty:G^\infty(\mathcal{F},f)\to Y$ has the right lifting property with respect to generating acyclic cofibrations and forms therefore a fibration. We have seen that i_∞ is a cofibration. It remains to prove that it is acyclic. For this aim, we use the following classical lemma:

Lemma 1.2.14. *Let $\{C_n\}$ be a sequential direct system of chain complexes, then there is an isomorphism $colim_nH_*(C_n)\cong H_*(colim_nC_n)$.*

Suppose that all the i_k are weak equivalences, i.e induce isomorphisms in homology. If we apply the homology functor H_* to the sequential direct system of the i_k, we obtain a sequential direct system in which every arrow is an isomorphism, so the transfinite composite of these arrows is also an isomorphism. By composing it with the isomorphism of the lemma above we obtain the isomorphism $H_*(i_\infty)$, i.e i_∞ is acyclic. Therefore we just have to prove that the i_k are acyclic.

The chain complex 0 is the initial object of $Ch_{\mathbb{K}}$, so via the adjunction $P:Ch_{\mathbb{K}}\rightleftarrows_P Ch_{\mathbb{K}}:U$ the object $P(0)$ is initial in ${}_PCh_{\mathbb{K}}$. The coproduct of any object A with the initial object is isomorphic to A. Another general categorical fact is that in any category endowed with an initial object I and admitting the coproduct of two objects A and B, this coproduct $A\vee B$ corresponds to the pushout

$$\begin{array}{ccc} I & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & A\vee B \end{array}.$$

We deduce from these facts that $\bigvee P(0)\cong P(0)$, $G^{k-1}(\mathcal{F},f)\cong G^{k-1}(\mathcal{F},f)\vee P(0)$ and therefore $G^k(\mathcal{F},f)=G^{k-1}(\mathcal{F},f)\vee(\bigvee P(D^n))$. Furthermore, $\bigvee P(D^n)\cong P(\bigoplus D^n)$ and proposition 1.2.9 implies that

$$G^{k-1}(\mathcal{F},f)\vee P(0)\cong U_P(G^{k-1}(\mathcal{F},f))(0)$$

and

$$G^k(\mathcal{F},f)\cong G^{k-1}(\mathcal{F},f)\vee P(\bigoplus D^n)\cong U_P(G^{k-1}(\mathcal{F},f))(\bigoplus D^n).$$

We obtain consequently the following pushout:

$$\begin{array}{ccc} P(0) & \longrightarrow & U_P(G^{k-1}(\mathcal{F},f))(0) \\ \downarrow & & \downarrow{\scriptstyle i_k} \\ P(\bigoplus D^n) & \longrightarrow & U_P(G^{k-1}(\mathcal{F},f))(\bigoplus D^n) \end{array}.$$

Lemma 1.2.13 applied to the Σ-module $U_P(G^{k-1}(\mathcal{F}, f))$ implies that i_k is an injection. Given that $H_*(\bigoplus D^n) = 0$, proposition 1.2.12 implies that $H_*U_P(G^{k-1}(\mathcal{F}, f))(\bigoplus D^n) \cong U_P(G^{k-1}(\mathcal{F}, f))(0)$ and i_k is acyclic. We conclude that i_∞ is acyclic, which achieve our proof of MC5 (ii).

MC4 (i). Obvious by definition of the cofibrations.

MC4 (ii). We have to use axiom MC5 (ii), which will be proved below. Let $f : X \to Y$ be an acyclic cofibration, according to MC5 (ii) f admits a factorization $f = p_\infty \circ i_\infty$ where $i_\infty : X \to G^\infty(\mathcal{F}, f)$ is an acyclic cofibration and $p_\infty : G^\infty(\mathcal{F}, f) \to Y$ a fibration. The morphisms f and i_∞ are weak equivalences, therefore p_∞ is also a weak equivalence according to MC2. We obtain a commutative square

$$\begin{array}{ccc} X & \xrightarrow{i_\infty} & G^\infty(\mathcal{F}, f) \\ {\scriptstyle f}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p_\infty} \\ Y & = & Y \end{array}$$

where a lifting h exists because f is a cofibration and has thus the left lifting property with respect to acyclic fibrations. The morphism f is therefore a retract of i_∞ via the retraction diagram

$$\begin{array}{ccccc} X & = & X & = & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle i_\infty}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow[h]{} & G^\infty(\mathcal{F}, f) & \xrightarrow[p_\infty]{} & Y \end{array}.$$

The $P(0) \to P(D^n)$ have the left lifting property with respect to fibrations, so by lemma 1.1.37 the maps i_k inherit this property, and so does i_∞ by lemma 1.1.38. Now consider a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[\beta]{} & B \end{array}$$

where p is a fibration. Combined with the retraction diagram it gives rise to the diagram

$$\begin{array}{ccccccc} X & = & X & = & X & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\downarrow & & {\scriptstyle i_\infty}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[h]{} & G^\infty(\mathcal{F}, f) & \xrightarrow[p_\infty]{} & Y & \xrightarrow[\beta]{} & B \end{array}.$$

In the square

$$\begin{array}{ccc} X & \longrightarrow & A \\ {\scriptstyle i_\infty}\downarrow & & \downarrow{\scriptstyle p} \\ G^\infty(\mathcal{F}, f) & \xrightarrow[\beta\circ p_\infty]{} & B \end{array}$$

there exists a lifting $\tilde{h}$. Now let us define $\hat{h} = \tilde{h} \circ h$. Then

$$\hat{h} \circ f = \tilde{h} \circ h \circ f = \tilde{h} \circ i_\infty = \alpha$$

and

$$p \circ \hat{h} = p \circ \tilde{h} \circ h = \beta \circ p_\infty \circ h = \beta$$

so h is the desired lifting : the acyclic cofibration f has the left lifting property with respect to fibrations.

1.3 The model category of coalgebras over an operad

We work in the full subcategory $Ch_{\mathbb{K}}^+$ of $Ch_{\mathbb{K}}$ whose objects are the chain complexes C such that $C_0 = 0$, i.e the positively graded chain complexes. The category $Ch_{\mathbb{K}}^+$ is actually a model subcategory of $Ch_{\mathbb{K}}$. We suppose that P is an operad in $Vect_{\mathbb{K}}$ such that the $P(n)$ are finite dimensional, $P(0) = 0$ and $P(1) = \mathbb{K}$. Note that the commonly used operads satisty this hypothesis, for instance As (for the associative algebras), Com (for the commutative associative algebras), Lie (for the Lie algebras), $Pois$ (for the Poisson algebras). There are two difficulties appearing here. Firstly, our operad is not defined exactly in the same category as our algebras. Secondly, the category $Ch_{\mathbb{K}}^+$ inherits the symmetric monoidal structure of $Ch_{\mathbb{K}}$ but not the unit (which is $\mathbb{K}$ concentrated in degree 0). However, $Vect_{\mathbb{K}}$ acts on $Ch_{\mathbb{K}}^+$ via the usual tensor product of chain complexes, when we identify $Vect_{\mathbb{K}}$ with the subcategory of $Ch_{\mathbb{K}}$ consisting in complexes concentrated in degree 0 . The convenient notion to deal with such situations is the one of symmetric monoidal category over a base category. We refer the reader to [9], chapter 1, for a precise definition and the associated properties. In our case, we work in the reduced symmetric monoidal category $Ch_{\mathbb{K}}^+$ over the base $Vect_{\mathbb{K}}$ (see also [9], 1.1.17). As shown in [9], all the usual definitions and properties of operads and their algebras hold in the reduced setting. The situation is analogous for cooperads and their coalgebras. The model category structure on coalgebras is given by the following theorem:

Theorem 1.3.1. *The category of P-coalgebras ${}^P Ch_{\mathbb{K}}^+$ inherits a cofibrantly generated model category structure such that a morphism f of ${}^P Ch_{\mathbb{K}}^+$ is*
(i) a weak equivalence if $U(f)$ is a weak equivalence in $Ch_{\mathbb{K}}^+$;
(ii) a cofibration if $U(f)$ is a cofibration in $Ch_{\mathbb{K}}^+$;
(iii) a fibration if f has the right lifting property with respect to acyclic cofibrations.

The three class of morphisms defined in this theorem are clearly stable by composition and contain the identity maps. Axioms MC2 and MC3 are clear, and MC4 (ii) is obvious by definition of the fibrations. It remains to prove axioms MC1, MC4 (i) and MC5. We first need a description of the cofree P-coalgebra functor, and the notion of enveloping cooperad.

1.3.1 Cofree coalgebra over an operad

There exists a cofree P-algebra functor $P^* : Ch_{\mathbb{K}}^+ \rightarrow^P Ch_{\mathbb{K}}^+$, which is by definition the right adjoint to the forgetful functor and is given by the following theorem. In this theorem, we start with definition 1.1.19 when we consider the category ${}^P Ch_{\mathbb{K}}^+$.

Theorem 1.3.2. *Let V be an object of $Ch_{\mathbb{K}}^+$. Then*

$$P^*(V) = \bigoplus_{r=1}^{\infty} P(r)^* \otimes_{\Sigma_r} V^{\otimes r}$$

inherits a P-coalgebra structure and forms the cofree P-coalgebra.

The result of this theorem actually provides the equivalence between the category given by definition 1.1.19 and the category of coalgebras over the cooperad P^* as defined in definition 1.1.21. In our verifications we implicitely establishes that the coalgebras of ${}^P Ch_{\mathbb{K}}^+$ automatically satisfy the conilpotence condition of definition 1.1.19.

For the needs of the proof we give the following definition:

Definition 1.3.3. Let I be a finite set of cardinal k, then we define $P(I)$ by $P(I) = Bij(\underline{k}, I) \otimes_{\Sigma_k} P(k)$ where $\underline{k} = 1, ..., k$. The elements of $P(I)$ of the form $u \otimes \mu$, $u \in Bij(\underline{k}, I)$ and $\mu \in P(k)$, satisfy $u \otimes \sigma.\mu = (u \circ \sigma) \otimes \mu$ for any permutation $\sigma \in \Sigma_k$.

The classical operadic compositions

$$\gamma(n_1, ..., n_r) : P(r) \otimes P(n_1) \otimes ... \otimes P(n_r) \rightarrow P(n_1 + ... + n_r)$$

extend to these objects:

$$\gamma(I_1,...,I_r): P(r)\otimes P(I_1)\otimes ...\otimes P(I_r)\rightarrow P(I_1\amalg ...\amalg I_r)$$

is defined by

$$\gamma(I_1,...,I_r)(\mu\otimes(q_1,v_1)\otimes ...\otimes(q_r,v_r)) = (\mu(q_1,...,q_r),v_1\amalg ...\amalg v_r)$$

where $q_i\in P(n_i)$, the map $v_i:\underline{n_i}\rightarrow I_i$ is a bijection, and $\mu(q_1,...,q_r)=\gamma(n_1,...,n_r)(\mu,q_1,...,q_r)$. We have a bijection $\underline{n_1}\amalg ...\amalg\underline{n_r}\cong\underline{n_1+...+n_r}$ by renumbering the elements, so $v_1\amalg ...\amalg v_r:\underline{n_1+...+n_r}\rightarrow I_1\amalg ...\amalg I_r$ is a well defined bijection.

Proof. We want to equip $P^*(V)$ with a P-coalgebra structure, i.e linear applications $\rho_r: P(r)\otimes P^*(V)\rightarrow P^*(V)^{\otimes r}$ with the adequate properties. Let us first compute explicitely $P(r)\otimes P^*(V)$ and $P^*(V)^{\otimes r}$:

$$P(r)\otimes P^*(V)=\bigoplus_{n=1}^{\infty}P(r)\otimes P(n)^*\otimes_{\Sigma_n}V^{\otimes n}.$$

In arity n, we have

$$(P^*(V)^{\otimes r})_n=\bigoplus_{i_1+...+i_r=n}P^*(V)_{i_1}\otimes ...\otimes P^*(V)_{i_r}$$

so

$$\begin{aligned} P^*(V)^{\otimes r} &= \bigoplus_n\bigoplus_{i_1+...+i_r=n}P^*(V)_{i_1}\otimes ...\otimes P^*(V)_{i_r}\\ &= \bigoplus_n\bigoplus_{I_1\amalg ...\amalg I_r=\underline{n}}(P(I_1)^*\otimes_{\Sigma_{I_1}}V^{\otimes I_1})\otimes ...\otimes(P(I_n)^*\otimes_{\Sigma_{I_n}}V^{\otimes I_n})\end{aligned}$$

. We want now to define maps

$$diag_{r,n}: P(r)\otimes P(n)^*\otimes V^{\otimes n}\rightarrow\bigoplus_{I_1\amalg ...\amalg I_r=\underline{n}}(P(I_1)^*\otimes_{\Sigma_{I_1}}V^{\otimes I_1})\otimes ...\otimes(P(I_n)^*\otimes_{\Sigma_{I_n}}V^{\otimes I_n}).$$

For $I_1\amalg ...\amalg I_r=\underline{n}$, the map

$$\gamma(I_1,...,I_r): P(r)\otimes P(I_1)\otimes ...\otimes P(I_r)\rightarrow P(I_1\amalg ...\amalg I_r)=P(\underline{n})\cong P(n)$$

induces a map $\overline{\gamma}(I_1,...,I_r): P(r)\otimes P(n)^*\rightarrow P(I_1)^*\otimes ...\otimes P(I_r)^*$. Indeed, the $P(I_k), k=1,...,r$ are finite dimensional so we can use the following sequence of isomorphisms:

$$\begin{aligned} Hom_{\mathbb{K}}(P(r)\otimes P(I_1)\otimes ...\otimes P(I_r),P(n)) &\cong Hom_{\mathbb{K}}(P(r)\otimes P(I_1)\otimes ...\otimes P(I_r),Hom_{\mathbb{K}}(P(n)^*,\mathbb{K}))\\ &\cong Hom_{\mathbb{K}}(P(r)\otimes P(n)^*\otimes P(I_1)\otimes ...\otimes P(I_r),\mathbb{K})\\ &\cong Hom_{\mathbb{K}}(P(r)\otimes P(n)^*,Hom_{\mathbb{K}}(P(I_1)\otimes ...\otimes P(I_r),\mathbb{K}))\\ &\cong Hom_{\mathbb{K}}(P(r)\otimes P(n)^*,P(I_1)^*\otimes ...\otimes P(I_r)^*).\end{aligned}$$

We define

$$diag_{r,n}=\sum_{I_1\amalg ...\amalg I_r=\underline{n}}\overline{\gamma}(I_1,...,I_r)\otimes id$$

which is defined on $P(r)\otimes P(n)^*\otimes_{\Sigma_n}V^{\otimes n}$ and take its values in

$$\begin{aligned} &\textstyle\bigoplus_{I_1\amalg ...\amalg I_r=\underline{n}}(P(I_1)^*\otimes ...\otimes P(I_r)^*)\otimes_{\Sigma_n}V^{\otimes n}\\ \cong\ &\textstyle\bigoplus_{I_1\amalg ...\amalg I_r=\underline{n}}(P(I_1)\otimes_{\Sigma_{I_1}}V^{\otimes I_1})\otimes ...\otimes(P(I_r)\otimes_{Sigma_{I_r}}V^{\otimes I_r}).\end{aligned}$$

The hypothesis $P(0)=0$ ensures that only a finite number of $\overline{\gamma}(I_1,...,I_r)$ such that $I_1\amalg ...\amalg I_r=\underline{n}$ are non zero, so the application $diag_{r,n}$ is well defined. Then we can set

$$\rho_r=\bigoplus_n diag_{r,n}: P(r)\otimes P^*(V)\rightarrow P^*(V)^{\otimes r}.$$

By construction, the ρ_n make the appropriate diagram commute and are Σ_n-equivariant, so they equip $P^*(V)$ with a structure of P-coalgebra.

Now we have to prove that $P^*(V)$ is cofree. It means that for every morphism $f : C \to V$ of $Ch_{\mathbb{K}}$ where C is a P-coalgebra and V any chain complex, there exists a unique factorization

$$\begin{array}{ccc} C & \xrightarrow{f} & V \\ {\scriptstyle \tilde{f}}\downarrow & \nearrow_{\pi} & \\ P^*(V) & & \end{array}$$

where $\tilde{f}$ is a morphism of P-coalgebras and $\pi : \bigoplus_{r=1}^{\infty} P(r)^* \otimes_{\Sigma_r} V^{\otimes r} \to P(1)^* \otimes V \cong V$ (recall that $P(1) = \mathbb{K}$) is the projection on the component of arity 1.

The structure of P-coalgebra on C is given by morphisms $\rho_r : C \to P(r)^* \otimes_{\Sigma_r} V^{\otimes r}$, hence in degree n the maps $(\rho_r)_n : C_n \to (P(r)^* \otimes_{\Sigma_r} V^{\otimes r})_n$. We have $C_0 = 0$ so $(C^{\otimes r})_n = \bigoplus_{i_1+...+i_r=n} C_{i_1} \otimes ... \otimes C_{i_r} = 0$ if $r > n$. We deduce that for a fixed degree n, only a finite number of $(\rho_r)_n$ are non zero. We can then set

$$\phi_n = \sum_r (\rho_r)_n : C_n \to \bigoplus_{r=1}^{\infty} (P(r)^* \otimes_{\Sigma_r} C^{\otimes r})_n$$

hence

$$\phi_C = \bigoplus_n \phi_n : C \to P^*(C).$$

Consequently we set $\tilde{f} = P^*(f) \circ \phi_C : C \to P^*(V)$. Given that π is the projection on the component of arity 1, in order to show that $\pi \circ \tilde{f} = f$ we look after what it gives in arity 1:

$$C \overset{\phi|_{r=1}}{\to} P(1)^* \otimes C \overset{id \otimes f}{\to} P(1)^* \otimes V \overset{\pi|_{r=1}}{\to} V.$$

We obtain f as expected. It remains to prove the unicity of $\tilde{f}$. For every n we have a commutative square

$$\begin{array}{ccc} P^*(V) & \xrightarrow{\rho_n^{P^*(V)}} & P(n)^* \otimes P^*(V)^{\otimes n} \\ {\scriptstyle projection}\downarrow & & \downarrow{\scriptstyle id \otimes \pi^{\otimes n}} \\ P(n)^* \otimes_{\Sigma_n} V^{\otimes n} & \xrightarrow[N]{} & P(n)^* \otimes V^{\otimes n} \end{array}$$

where N is the norm map. The map $\tilde{f}$ is a morphism P-coalgebras, so we have for every n another commutative square

$$\begin{array}{ccc} C & \xrightarrow{\tilde{f}} & P^*(V) \\ {\scriptstyle \rho_n^C}\downarrow & & \downarrow{\scriptstyle \rho_n^{P^*(V)}} \\ P(n)^* \otimes C^{\otimes n} & \xrightarrow[id \otimes \tilde{f}^{\otimes n}]{} & P(n)^* \otimes P^*(V)^{\otimes n} \end{array}.$$

Combining these two squares we obtain a new commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{\tilde{f}} & P^*(V) & \xrightarrow{projection} & P(n)^* \otimes_{\Sigma_n} V^{\otimes n} \\ {\scriptstyle \rho_n^C}\downarrow & & \downarrow{\scriptstyle \rho_n^{P^*(V)}} & & \downarrow{\scriptstyle N} \\ P(n)^* \otimes C^{\otimes n} & \xrightarrow[id \otimes \tilde{f}^{\otimes n}]{} & P(n)^* \otimes P^*(V)^{\otimes n} & \xrightarrow[id \otimes \pi^{\otimes n}]{} & P(n)^* \otimes V^{\otimes n} \end{array}.$$

We have $(id \otimes \pi^{\otimes n}) \circ (id \otimes \tilde{f}^{\otimes n}) = id \otimes f^{\otimes n}$, so $\tilde{f}$ is determined in a unique way by f in each arity n. Indeed, according to the commutative diagram above, we have $N \circ (\tilde{f})_n = id \otimes f^{\otimes n}$. Recall that $p \circ N = id$, where $p : P(n) \otimes V^{\otimes n} \to P(n) \otimes_{\Sigma_n} V^{\otimes n}$ is the projection. It implies that $(\tilde{f})_n = p \circ (id \otimes f^{\otimes n})$. $\square$

1.3.2 Proof of MC1

The forgetful functor creates the small colimits. The proof of this fact is exactly the same as the proof of the existence of small limits in the P-algebras case. To prove the existence of small limits in ${}^PCh_{\mathbb{K}}^+$, we use the following categorical result:

Theorem 1.3.4. *(see [19]) Let $\mathcal{C}$ be a category. If $\mathcal{C}$ admits the coreflexive equalizers of every pair of arrows and all small coproducts, then $\mathcal{C}$ admits all the small limits.*

Now let us prove that ${}^PCh_{\mathbb{K}}^+$ admits the coreflexive equalizers and the small products.

Lemma 1.3.5. *Let $d^0, d^1 : A \rightarrow B$ be two morphisms in ${}^PCh_{\mathbb{K}}^+$ and $s^0 : B \rightarrow A$ a morphism of $Ch_{\mathbb{K}}^+$ such that $s^0 \circ d^0 = s^0 \circ d^1 = id_A$. Then $ker(d^0 - d^1)$ is the coreflexive equalizer of (d^0, d^1) in ${}^PCh_{\mathbb{K}}^+$.*

Proof. The subspace $ker(d^0 - d^1) \subset A$ is the coreflexive equalizer of (d^0, d^1) in $Ch_{\mathbb{K}}^+$. Moreover, it is a sub-P-coalgebra of A and the inclusion is obviously a P-coalgebras morphism. Indeed, let $\alpha \in A$ such that $d^0(\alpha) = d^1(\alpha)$, i.e $\alpha \in ker(d^0 - d^1)$. We want to prove that $ker(d^0 - d^1)$ is stable under the cooperations of A. That is, for every cooperation $p^* : A \rightarrow A^{\otimes n}$ associated to $p \in P(n)$, the image $p^*(\alpha)$ actually lies in $ker(d^0 - d^1)^{\otimes n}$. We have

$$ker(d^0 - d^1)^{\otimes n} = \bigcap_i A \otimes ... \otimes ker(d^0 - d^1) \otimes ... \otimes A.$$

Let $p^* : A \rightarrow A^{\otimes n}$ be a cooperation associated to $p \in P(n)$. Then the following equalities hold:

$$\begin{aligned} id \otimes ... \otimes d^0 \otimes ... \otimes id \circ p^*(\alpha) &= (s^0 \circ d^0) \otimes ... \otimes d^0 \otimes ... \otimes (s^0 \circ d^0) \circ p^*(\alpha) \\ &= s^0 \otimes ... \otimes id \otimes ... \otimes s^0 \circ (d^0)^{\otimes n} \circ p^*(\alpha) \\ &= s^0 \otimes ... \otimes id \otimes ... \otimes s^0 \circ p^* \circ d^0(\alpha) \\ &= s^0 \otimes ... \otimes id \otimes ... \otimes s^0 \circ p^* \circ d^1(\alpha) \\ &= id \otimes ... \otimes d^1 \otimes ... \otimes id \circ p^*(\alpha). \end{aligned}$$

The first line holds because of the equality $s^0 \circ d^0 = id$ satisfied by hypothesis. The third line comes from the fact that d^0 is a P-coalgebras morphism. The fourth line follows from our assumption that $\alpha \in ker(d^0 - d^1)$. The last line is obtained by following the preceding arguments in the converse direction. According to our decomposition of $ker(d^0 - d^1)^{\otimes n}$, it precisely means that $p^*(\alpha) \in ker(d^0 - d^1)^{\otimes n}$, which concludes the proof. □

Lemma 1.3.6. *Let $\{R_i\}_{i \in I}$ be a set of P-coalgebras. Let us set*

$$d^0 = P^*(\bigoplus \rho_{R_i}) : P^*(\bigoplus R_i) \rightarrow P^*(\bigoplus P^*(R_i))$$

and

$$d^1 = \pi \circ \Delta(\bigoplus R_i) : P^*(\bigoplus R_i) \rightarrow P^*(\bigoplus P^*(R_i))$$

where $\pi : P^*(P^*(\bigoplus R_i)) \rightarrow P^*(\bigoplus P^*(R_i))$ is the canonical projection and Δ the comultiplication of the comonad (P^*, Δ, ϵ). Then $\times R_i = ker(d^0 - d^1)$ is the product of the R_i in ${}^PCh_{\mathbb{K}}^+$.

Proof. We prove the lemma in the case of two P-coalgebras R and S. The proof is the same in the general case. Let us set

$$s^0 = P^*(\epsilon(R) \oplus \epsilon(S)) : P^*(P^*(R) \oplus P^*(S)) \rightarrow P^*(R \oplus S),$$

then $s^0 \circ d^0 = s^0 \circ d^1 = id$. According to Lemma 1.3.5, the space $ker(d^0 - d^1)$ is the coreflexive equalizer of (d^0, d^1) in ${}^PCh_{\mathbb{K}}^+$. Let X be a P-coalgebra. Two linear maps $u : X \rightarrow R$ and $v : X \rightarrow S$ induce a map $(u, v) : X \rightarrow R \oplus S$. This map admits a unique factorization through $P^*(R \oplus S)$ to give a P-coalgebras morphism $\varphi_{(u,v)} : X \rightarrow P^*(R \oplus S)$ by the universal property of the cofree P^*-coalgebra. This morphism admits a unique factorization through $ker(d^0 - d^1)$ if and only if u and v are morphisms of P-coalgebras. By unicity of the limit this concludes our proof, since $ker(d^0 - d^1)$ satisfies the same universal property than $R \times S$. □

Now we introduce a crucial construction, the enveloping cooperad, which provides a handable expression of the coproduct of a P-coalgebra with a cofree P-coalgebra. This cooperad plays a key role in the proof of axiom MC5 (i).

1.3.3 Enveloping cooperad

Let A be a P-coalgebra. We want to construct a particular cooperad associated to A and called the enveloping cooperad of A. This is a "dual version" of the enveloping operad of [7]. We need the following classical result :

Proposition 1.3.7. *(see [17], and Proposition 1.2.5 of [8]) When $\mathbb{K}$ is an infinite field, we have a fully faithful embedding of the category of Σ-modules in the category of endofunctors of $Vect_{\mathbb{K}}$, which associates to any Σ-module its Schur functor.*

We consider the Σ-module $P^*[A]$ defined by

$$P^*[A](n) = \bigoplus_{r=1}^{\infty} P(n+r)^* \otimes_{\Sigma_r} A^{\otimes r}.$$

We need the following lemma:

Lemma 1.3.8. *Let A be a P-coalgebra. For every chain complex C of $Ch_{\mathbb{K}}^+$ we have $P^*[A](C) \cong P^*(A \oplus C)$.*

Proof. We use the equality

$$P(n)^* \otimes_{\Sigma_n} (A \oplus C)^{\otimes n} = \bigoplus_{p+q=n} P(p+q)^* \otimes_{\Sigma_p \times \Sigma_q} (A^{\otimes p} \otimes C^{\otimes q})$$

to get

$$\begin{aligned}
P^*(A \oplus C) &= \bigoplus_{n\geq 0} P(n)^* \otimes_{\Sigma} (A \oplus C)^{\otimes n} \\
&= \bigoplus_{n} \bigoplus_{p+q=n} P(p+q)^* \otimes_{\Sigma_p \times \Sigma_q} (A^{\otimes p} \otimes C^{\otimes q}) \\
&= \bigoplus_{n} \bigoplus_{p+q=n} (P(p+q)^* \otimes_{\Sigma_p} A^{\otimes p}) \otimes_{\Sigma_q} C^{\otimes q}) \\
&= \bigoplus_{q} P^*[A](q) \otimes_{\Sigma_q} C^{\otimes q} \\
&= P^*[A](C).
\end{aligned}$$

□

This lemma allows us to equip $P^*[A]$ with a cooperad structure induced by the one of P^*. The cooperad coproduct $P^*[A] \to P^*[A] \circ P^*[A]$ comes from a comonad coproduct $P^*[A](-) \to P^*[A](-) \circ P^*[A](-)$ on the Schur functor $P^*[A](-)$ associated to $P^*[A]$. This comonad coproduct is a natural map defined for every chain complex C by

$$\begin{aligned}
P^*[A](C) \cong P^*(A \oplus C) &\overset{\Delta(A\oplus C)}{\to} P^*(P^*(A \oplus C)) \\
&\overset{P^*(proj\circ\epsilon, id)}{\to} P^*(A \oplus P^*(A \oplus C)) \cong (P^*[A] \circ P^*[A])(C)
\end{aligned}$$

where Δ is the coproduct of P^*, ϵ its counit and $proj$ the projection on the first component.

The P-coalgebra structure morphism $\rho_A : A \to P^*(A)$ of A induces a Σ-modules morphism

$$d^0 : P^*[A] \to P^*[P^*(A)],$$

given by

$$d^0(n) = \bigoplus_{r=1}^{\infty} id \otimes \rho_A^{\otimes r} : \bigoplus_{r=1}^{\infty} P(n+r)^* \otimes_{\Sigma_r} A^{\otimes r} \to \bigoplus_{r=1}^{\infty} P(n+r)^* \otimes_{\Sigma_r} P^*(A)^{\otimes r}.$$

It comes from a natural map $d^0(-)$ defined for every chain complex C by

$$d^0(C) : P^*[A](C) \cong P^*(A \oplus C) \overset{P^*(\rho_A \oplus id)}{\to} P^*(P^*(A) \oplus C) \cong P^*[P^*[A]](C)$$

where $\rho_A : A \to P^*(A)$ is the map defining the P^*-coalgebra structure of A. The natural map $d^0(-)$ is by construction a morphism of comonads, so d^0 forms a morphism of cooperads.

The coproduct $\Delta : P^* \to P^* \circ P^*$ associated to the comonad (P^*, Δ, ϵ) induces another morphism of Σ-modules

$$d^1 : P^*[A] \to P^*[P^*(A)]$$

defined in the following way: we have a natural map $d^1(-)$ defined for every chain complex C by

$$\begin{aligned} d^1(C) : P^*[A](C) \cong P^*(A \oplus C) & \overset{\Delta(A \oplus C)}{\to} P^*(P^*(A \oplus C)) \\ & \overset{P^*(P^*(pr_A), \pi \circ P^*(pr_C))}{\to} P^*(P^*(A) \oplus C) \\ & \cong P^*[P^*(A)](C) \end{aligned}$$

where $P^*[A](-)$ is the Schur functor associated to $P^*[A]$ and the map π is the projection on the component of arity 1. According to Proposition 1.3.7, any natural transformation between two Schur functors determines a unique morphism between their corresponding Σ-modules. Thus there is a unique morphism of Σ-modules $d^1 : P^*[A] \to P^*[P^*(A)]$ associated to $d^1(-)$. The natural map $d^1(-)$ is by construction a morphism of comonads, so d^1 forms a morphism of cooperads.

The counit $\epsilon : P^* \to Id$ induces a morphism of Σ-modules

$$s^0 : P^*[P^*(A)] \to P^*[A]$$

defined in the following way: we have a natural map $s^0(-)$ defined for every chain complex C by

$$\begin{aligned} s^0(C) : P^*[P^*(A)](C) \cong P^*(P^*(A) \oplus C) & \overset{P^*(P^*(i_A), i_C)}{\to} P^*(P^*(A \oplus C)) \\ & \overset{P^*(\epsilon(A \oplus C))}{\to} P^*(A \oplus C) \cong P^*[A](C) \end{aligned}$$

where $i_A : A \to A \oplus C$ and $i_C : C \to A \oplus C$, hence the unique associated morphism of Σ-modules $s^0 : P^*[P^*(A)] \to P^*[A]$. Note that a priori s^0 is not a morphism of cooperads. We finally obtain a coreflexive pair (d^0, d^1) of morphisms of Σ-modules induced by the associated reflexive pair of morphisms of Schur functors. The enveloping cooperad of A is the coreflexive equalizer

$$U_{P^*}(A) = ker(d^0 - d^1) \longrightarrow P^*[A] \underset{d^1}{\overset{d^0}{\rightrightarrows}} P^*[P^*(A)] \quad (s^0 : P^*[P^*(A)] \to P^*[A])$$

in Σ-modules, where d^0 and d^1 are cooperad morphisms. The fact that such a coreflexive equalizer inherits a cooperad structure from the one of $P^*[A]$ follows from arguments similar to those of the proof of Lemma 1.3.5. The cooperad $U_{P^*}(A)$ is coaugmented over P (by dualizing the proof of Lemma 1.2.4 in [7]). One can even go further and prove that the category of $U_{P^*}(A)$-coalgebras is equivalent to the category of P-coalgebras over A (this is the dual statement of Theorem 1.2.5 in [7]).

Now we want to prove that for every P-coalgebra A and every chain complex C, we have an isomorphism of P-coalgebras $U_{P^*}(A)(C) \cong A \times P^*(C)$ where $\times$ is the product in ${}^P Ch_{\mathbb{K}}^+$. For this aim we need the following lemma:

Lemma 1.3.9. *Let A be a P-coalgebra and C be a chain complex. The following equalizer defines the product $A \times P^*(C)$ in the category of P-coalgebras:*

$$A \times P^*(C) = ker(d^0 - d^1) \longrightarrow P^*(A \oplus C) \underset{d^1}{\overset{d^0}{\rightrightarrows}} P^*(P^*(A) \oplus C) \quad (s^0 : P^*(P^*(A) \oplus C) \to P^*(A \oplus C))$$

where $d^0 |_A = \rho_A$, $d^0 |_C = id_C$, $d^1 |_A = \Delta(A)$, $d_l |_C = id_C$, $s^0 |_A = \epsilon(A)$, $s^0 |_C = id_C$.

Proof. We clearly have $s^0 \circ d^0 = s^0 \circ d^1 = id$ so (d^0, d^1) is a reflexive pair in ${}^P Ch_{\mathbb{K}}^+$. The space $ker(d^0 - d^1)$ is the coreflexive equalizer of (d^0, d^1) in $Ch_{\mathbb{K}}^+$ and is a sub-P-coalgebra of $P^*(A \oplus C)$, so it is the coreflexive equalizer of (d^0, d^1) in ${}^P Ch_{\mathbb{K}}^+$. Let X be a P-coalgebra, $u : X \to A$ a morphism of P-coalgebras and $v : X \to C$ a linear map. They induce a map $(u, v) : X \to A \oplus C$, hence a morphism of P-coalgebras $\varphi_{(u,v)} : X \to P^*(A \oplus C)$ obtained by the universal property of the cofree P-coalgebra. The proof ends by seeing that $\varphi_{(u,v)}$ admits a unique factorization through $ker(d^0 - d^1)$. □

The coreflexive equalizer in Σ-modules defining the enveloping cooperad induces a coreflexive equalizer in P-coalgebras

$$U_{P^*}(A)(C) \longrightarrow P^*[A](C) \underset{d^1}{\overset{d^0}{\rightrightarrows}} P^*[P^*(A)](C), \quad s^0 : P^*[P^*(A)](C) \to P^*[A](C)$$

where $P^*[A](C) \cong P^*(A \oplus C)$, $P^*[P^*(A)](C) \cong P^*(P^*(A) \oplus C)$ and d^0, d^1, s^0 turn out to be the morphisms of the lemma above. By unicity of the limit, we have proved the following result:

Proposition 1.3.10. *Let A be a P-coalgebra and C be a chain complex, then there is an isomorphism of P-coalgebras*

$$U_{P^*}(A)(C) \cong A \times P^*(C).$$

We also need the following general result about Σ-modules:

Proposition 1.3.11. *Let M be a Σ-module and C a chain complex. If $H_*(C) = 0$ then $H_*(M(C)) = H_*(M(0))$.*

Proof. Recall that we work over a field $\mathbb{K}$ of characteristic 0. We use the norm map $N : M(n) \otimes_{\Sigma_n} C^{\otimes n} \to M(n) \otimes C^{\otimes n}$ defined by

$$N(c \otimes v_1 \otimes ... \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma.c \otimes v_{\sigma(1)} \otimes ... \otimes v_{\sigma(n)}.$$

If we denote $p : M(n) \otimes C^{\otimes n} \to M(n) \otimes_{\Sigma_n} C^{\otimes n}$ the projection, then

$$\begin{aligned} (p \circ N)(c \otimes v_1 \otimes ... \otimes v_n) &= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} p(\sigma.c \otimes v_{\sigma(1)} \otimes ... \otimes v_{\sigma(n)}) \\ &= \frac{1}{n!} \mid \Sigma_n \mid c \otimes v_1 \otimes ... \otimes v_n \\ &= c \otimes v_1 \otimes ... \otimes v_n \end{aligned}$$

so $p \circ N = id$. Therefore $M(n) \otimes_{\Sigma_n} C^{\otimes n}$ is a retract of $M(n) \otimes C^{\otimes n}$. For $n \geq 1$, the Künneth formula gives us for every $k \geq 0$

$$H_k(M(n) \otimes C^{\otimes n}) = \bigoplus_{p+q=k} H_p(M(n) \otimes C) \otimes H_q(C^{\otimes n-1}).$$

This is equal to 0 for $n > 1$ because the fact that $H_*(C) = 0$ implies recursively that $H_*(C^{\otimes n}) = 0$ by the Künneth formula. This is also equal to 0 for $n = 1$ because the fact that $H_k(C) = 0$ implies that $H_k(M(1) \otimes C) = 0$. For $n = 0$, we have $H_k(M(0))$. We conclude that $H_k(M(C)) = H_k(M(0))$. □

We finally reach the crucial result of this section:

Corollary 1.3.12. Let A be a P-coalgebra and C be a chain complex. If $H_*(C) = 0$ then the canonical projection $A \times P^*(C) \to A$ is a weak equivalence in ${}^P Ch_{\mathbb{K}}^+$.

Proof. According to Proposition 1.3.10, we have $U_{P^*}(A)(C) \cong A \times P^*(C)$. We can apply Proposition 1.3.11 to the Σ-module $U_{P^*}(A)$ since $H_*(C) = 0$ by hypothesis, so

$$H_*(A \times P^*(C)) = H_*(U_{P^*}(A)(C)) = H_*(U_{P^*}(A)(0)).$$

It remains to prove that $H_*(U_{P^*}(A)(0)) = H_*(A)$. For this aim we show that $U_{P^*}(A)(0) \cong A$. It comes from a categorical result: in any category with a final object and admitting products, the product of any object A with the final object is isomorphic to A. We apply this fact to $U_{P^*}(A)(0) \cong A \times P^*(0)$. Indeed, the chain complex 0 is final in $Ch_{\mathbb{K}}^+$ so $P^*(0)$ is final in ${}^P Ch_{\mathbb{K}}^+$. □

1.3.4 Generating (acyclic) cofibrations, proofs of MC4 and MC5

Before specifying the families of generating cofibrations and generating acyclic cofibrations, we prove axioms MC4 (i) and MC5 (i). The cofibrantly generated structure will then be used to prove MC5 (ii) by means of a small object argument, slightly different from the preceding one since we will use smallness with respect to injections systems. The plan and some arguments parallel those of [10]. However, they work in cocommutative differential graded coalgebras. Some care is necessary in our more general setting. This is the reason why we give full details in proofs, in order to see where we can readily follow [10] and where our modifications (for instance the notion of enveloping cooperad) step in.

MC5 (i). We first need a preliminary lemma:

Lemma 1.3.13. *Every chain complex X of $Ch_{\mathbb{K}}^+$ can be embedded in a chain complex V satisfying $H_*(V) = 0$.*

This lemma helps us to prove the following result:

Proposition 1.3.14. *(i) Let C be a P-coalgebra and V be a chain complex such that $H_*(V) = 0$. Then the projection $C \times P^*(V) \to C$ is an acyclic fibration with the right lifting property with respect to all cofibrations.*

(ii) Every P-coalgebras morphism $f : D \to C$ admits a factorization

$$D \stackrel{j}{\to} X \stackrel{q}{\to} C$$

where j is a cofibration and q an acyclic fibration with the right lifting property with respect to all cofibrations (in particular we obtain axiom MC5 (i)).

Proof. (i) According to Corollary 1.3.12, the map $C \times P^*(V) \to C$ is a weak equivalence so it remains to prove that it has the right lifting property with respect to all cofibrations (which implies in particular that it is a fibration). Let us consider the following commutative square in ${}^P Ch_{\mathbb{K}}$:

$$\begin{array}{ccc} A & \xrightarrow{a} & C \times P^*(V) \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \xrightarrow[b]{} & C \end{array}$$

where i is a cofibration. A lifting in this square is equivalent to a lifting in each of the two squares

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle i}\downarrow & & \| \\ B & \longrightarrow & C \end{array}$$

and

$$\begin{array}{ccc} A & \longrightarrow & P^*(V) \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \longrightarrow & 0 \end{array} .$$

In the first square this is obvious, just take the bottom map $B \to C$ as a lifting. In the second square, via the adjunction $U : {}^{P}Ch_{\mathbb{K}}^{+} \rightleftarrows Ch_{\mathbb{K}}^{+} : P^*$, the lifting problem is equivalent to a lifting problem in the following square of $Ch_{\mathbb{K}}^{+}$:

$$\begin{array}{ccc} U(A) & \longrightarrow & V \\ {\scriptstyle U(i)}\downarrow & & \downarrow \\ U(B) & \longrightarrow & 0 \end{array} .$$

The map $V \to 0$ is degreewise surjective so it is a fibration of $Ch_{\mathbb{K}}^{+}$, which is acyclic because $H_*(V) = 0$. The map i is a cofibration, so $U(i)$ is a cofibration by definition and has therefore the left lifting property with respect to acyclic fibrations.

(ii) According to Lemma 1.3.13, there exists an injection $i : U(D) \hookrightarrow V$ in $Ch_{\mathbb{K}}^{+}$ where V is such that $H_*(V) = 0$. Let us set $X = C \times P^*(V)$, $q : X \to C$ the projection and

$$j = (f, \tilde{i}) : D \to C \times P^*(V)$$

where $\tilde{i} : D \to P^*(V)$ is the factorization of i by universal property of the cofree P-coalgebra. We have $q \circ j = f$. According to (i), the map q is an acyclic fibration with the right lifting property with respect to all cofibrations. It remains to prove that j is a cofibration. Let us consider the composite

$$D \stackrel{j}{\to} C \times P^*(V) \stackrel{pr_2}{\to} P^*(V) \stackrel{\pi}{\to} V$$

where pr_2 is the projection on the second component and π the projection associated to the cofree P-coalgebra on V. We have $\pi \circ pr_2 \circ j = \pi \circ \tilde{i} = i$ by definition of $\tilde{i}$. The map i is injective so j is also injective, which implies that $U(j)$ is a cofibration in $Ch_{\mathbb{K}}^{+}$. By definition it means that j is a cofibration in ${}^{P}Ch_{\mathbb{K}}^{+}$. □

MC4 (i). Let $p : X \to Y$ be an acyclic fibration, let us consider the commutative square

$$\begin{array}{ccc} C & \stackrel{a}{\longrightarrow} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ D & \underset{b}{\longrightarrow} & Y \end{array}$$

where i is a cofibration. According to Proposition 1.3.14, the map p admits a factorization $p = q \circ j$ where $j : X \to T$ is a cofibration and $q : T \to Y$ an acyclic fibration with the right lifting property with respect to all cofibrations. Axiom MC2 implies that j is a weak equivalence. Let us consider the commutative square

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\ T & \underset{q}{\longrightarrow} & Y \end{array} .$$

According to axiom MC4 (ii), there exists a lifting $r : T \to X$ in this square and p is consequently a retract of q via the following retraction diagram:

$$\begin{array}{ccccc} X & \stackrel{j}{\longrightarrow} & T & \stackrel{r}{\longrightarrow} & X \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & = & Y & = & Y \end{array} .$$

The fact that p is a retract of q implies that p inherits from q the property of right lifting property with respect to cofibrations. Indeed, we get the commutative diagram

$$\begin{array}{ccccccc} C & \xrightarrow{a} & X & \xrightarrow{j} & T & \xrightarrow{r} & X \\ {\scriptstyle i}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ D & \xrightarrow[b]{} & Y & = & Y & = & Y \end{array} .$$

In the square

$$\begin{array}{ccc} C & \xrightarrow{j\circ a} & T \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle q} \\ D & \xrightarrow[b]{} & Y \end{array}$$

there exists a lifting h by property of q. Now let us define $\tilde{h} = r \circ h$. Then

$$\tilde{h} \circ i = r \circ h \circ i = r \circ j \circ a = a$$

and

$$p \circ \tilde{h} = p \circ r \circ h = q \circ h = b,$$

so $\tilde{h}$ is the desired lifting. This concludes the proof of MC4 (i).

Generating (acyclic) cofibrations We first need two preliminary lemmas:

Lemma 1.3.15. *Let C be a P-coalgebra. For every homogeneous element $x \in C$ there exists a sub-P-coalgebra $D \subset C$ of finite dimension such that $x \in D$ and $D_k = 0$ for every $k > deg(x)$.*

Proof. Let $x \in C_n$ be an homogeneous element of degree n and $p \in P(m)$. The element p gives rise to an operation $p^* : C \to C^{\otimes m}$. We have the formula

$$\begin{aligned} p^*(x) &= \sum_{i_1+...+i_m=n} (\sum x_{i_1} \otimes ... \otimes x_{i_m}) \\ &= \sum_{(x)} x_{(1)} \otimes ... \otimes x_{(m)} \end{aligned}$$

where the second line is written in Sweedler's notation.

We do a recursive reasoning on the degree n of x. For $n = 0$, we have $x = 0$ since $C_0 = 0$ and 0 is the trivial sub-P-coalgebra. Now suppose the lemma true for every $k < n$. Let $x \in C_n$ and $p^*(x)$ as above, for a certain element $p \in P(m)$. By hypothesis, there exist sub-P-coalgebras of finite dimension $D_{(1)}(p), ..., D_{(m)}(p)$ such that for every $1 \leq j \leq m$, we have $x_{(j)} \in D_{(j)}(p)$ and $(D_{(j)}(p))_l = 0$ for $l > deg(x_{(j)})$. We set

$$D(p) = \sum D_{(1)}(p) + ... + \sum D_{(m)}(p).$$

A finite sum of sub-P-coalgebras is stable under the operations of the P-coalgebra structure of C and thus form also a sub-P-coalgebra of C. To build our sub-P-coalgebra D, we need to include x and all its images under the operations induced by P. We need also to include the image dx of x under the differential of C. For this aim, we set

$$D = \mathbb{K}.x \oplus D(dx) \oplus \sum_p D(p)$$

where $D(dx)$ is the sub-P-coalgebra containing dx given by our hypothesis (which exists since $deg(dx) < n$), and the sum $\sum_p$ ranges over fixed bases of each $P(m)$ for every m. This sum is actually finite. Indeed, each $P(m)$ is of finite dimension, and when $m > deg(x)$ we have $p^*(x) = 0$ for degree reasons. Consequently, the space D is a sub-P-coalgebra of C containing x. It is of finite dimension as a finite sum of finite dimensional spaces. Moreover, since for every $1 \leq j \leq m$ we have $deg(x) > deg(x_{(j)})$, this construction implies that $D_l = 0$ if $l > deg(x)$. ☐

Lemma 1.3.16. *Let $j : C \to D$ be an acyclic cofibration and $x \in D$ a homogeneous element. Then there exists a sub-P-coalgebra $B \subseteq D$ such that:*
(i) $x \in B$;
(ii) B has a countable homogeneous basis;
(iii) the injection $C \cap B \hookrightarrow B$ is an acyclic cofibration in ${}^P Ch_{\mathbb{K}}^+$ (we denote also by C the image of C under j, since j is injective and thus $j(C) \cong C$).

Proof. We want to define recursively sub-P-coalgebras

$$B(1) \subseteq B(2) \subseteq ... \subseteq D$$

such that $x \in B(1)$, each $B(n)$ is finite dimensional and the induced map

$$\frac{B(n-1)}{C \cap B(n-1)} \to \frac{B(n)}{C \cap B(n)}$$

is zero in homology. This map is well defined, since we do the quotient by an intersection of two sub-P-coalgebras which is still a sub-P-coalgebra. This property will be essential to prove the acyclicity of the injection the injection $C \cap B \hookrightarrow B$.

The P-coalgebra $B(1)$ is given by Lemma 1.3.15. Now suppose that for some integer $n \geq 1$ the coalgebra $B(n-1)$ has been well constructed. The space $B(n-1)$ is of finite dimension, so we can choose a finite set of homogeneous elements $z_i \in B(n-1)$ giving cycles $\overline{z_i} \in \frac{B(n-1)}{C \cap B(n-1)}$, such that the homology classes of the $\overline{z_i}$ span $H_*(\frac{B(n-1)}{C \cap B(n-1)})$. The existence of an acyclic cofibration $j : C \to D$ implies that $H_*(D/C) = 0$. Consequently, there exists elements $z'_i \in D$ such that $\delta(z'_i) = z_i mod C$. For every i, Lemma 1.3.15 provides us a finite dimensional sub-P-coalgebra $A(z'_i) \subseteq D$ containing z'_i. We can then define

$$B(n) = B(n-1) + \sum_i A(z'_i).$$

The sub-P-coalgebra $B(n)$ is of finite dimension because it is the sum of finite dimensional sub-p-coalgebras. We have $\delta(z'_i) - z_i \in C$ and $z_i, z'_i \in B(n)$, so $\delta(z'_i) - z_i \in C \cap B(n)$, hence the induced map in homology

$$H_*(\frac{B(n-1)}{C \cap B(n-1)}) \to H_*(\frac{B(n)}{C \cap B(n)})$$

is zero because it sends the homology classes of the $\overline{z_i}$ to 0.

Let us define $B = \bigcup B(n)$ and prove that $C \cap B \hookrightarrow B$ is an acyclic cofibration. Firs it is injective so it is a cofibration. To prove its acyclicity, let us consider the following short exact sequence:

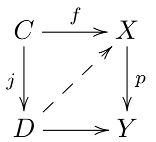
$$0 \to B \cap C \hookrightarrow B \to \frac{B}{C \cap B} \to 0.$$

It is sufficient to consider the long exact sequence induced by this sequence in homology and to prove that $H_*(\frac{B}{C \cap B}) = 0$. Let $z \in B$ such that $\partial(\overline{z}) = 0$ in $\frac{B}{C \cap B}$, where ∂ is the differential of $\frac{B}{C \cap B}$. We have $\partial(z) \in B \cap C = \bigcup B(n) \cap C$ and $B(1) \subseteq ... \subseteq D$ so there exists an integer n such that $z \in B(n-1)$ and $\partial(z) \in B(n-1) \cap C$. It implies that $[\overline{z}] \in H_*(\frac{B(n-1)}{C \cap B(n-1)})$, where $[\overline{z}]$ is the homology class of $\overline{z}$. Thus $[\overline{z}] = 0$ in $H_*(\frac{B(n)}{C \cap B(n)})$, since the map $H_*(\frac{B(n-1)}{C \cap B(n-1)}) \to H_*(\frac{B(n)}{C \cap B(n)})$ is zero in homology. We deduce that $z = \partial(b) + B(n) \cap C$ for a certain $b \in B(n)$, so $\overline{z} = \partial(\overline{b})$ in $\frac{B}{B \cap C}$ (the projection $x \mapsto \overline{x}$ commutes with the differentials). Finally, it means that every cycle of $\frac{B}{B \cap C}$ is a boundary, i.e that $H_*(\frac{B}{C \cap B}) = 0$. To conclude, the complex B is a colimit over $\mathbb{N}$ of finite dimensional complexes and thus has a homogeneous countable basis. □

Now we can give a characterization of generating cofibrations and generating acyclic cofibrations.

Proposition 1.3.17. *A morphism $p : X \to Y$ of ${}^P Ch_{\mathbb{K}}^+$ is*
(i) a fibration if and only if it has the right lifting property with respect to the acyclic cofibrations $A \hookrightarrow B$ where B has a countable homogeneous basis;
(ii) an acyclic fibration if and only if it has the right lifting property with respect to the cofibrations $A \hookrightarrow B$ where B is of finite dimension.

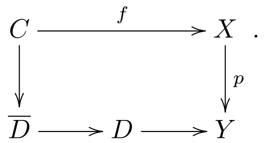
Proof. (i) One of the two implications is obvious. Indeed, if p is a fibration then it has the right lifting property with respect to acyclic cofibrations by definition. Conversely, suppose that p has the right lifting property with respect to the acyclic cofibrations $A \hookrightarrow B$ where B has a countable homogeneous basis. We consider the following lifting problem:



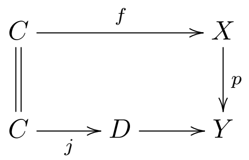
where j is an acyclic cofibration. Let us define Ω as the set of pairs $(\overline{D}, g)$, where $\overline{D}$ fits in the composite of two acyclic cofibrations

$$C \hookrightarrow \overline{D} \hookrightarrow D$$

such that this composite is equal to j. The map $g : \overline{D} \to X$ is a lifting in

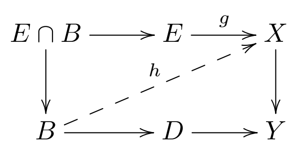


The collection Ω is really a set: it is the union of hom sets $Mor_{{}_PCh_{\mathbb{K}}^+}(\overline{D}, X)$ indexed by a subset of the set of subcomplexes $\overline{D}$ of D. Recall that cofibrations are injective P-coalgebras morphisms. We endow Ω with a partial order defined by $(\overline{D_1}, g_1) \leq (\overline{D_2}, g_2)$ if $\overline{D_1} \subseteq \overline{D_2}$ and $g_2 \mid_{\overline{D_1}} = g_1$. The commutative square



admits f as an obvious lifting, so $(C, f) \in \Omega$ and thus Ω is not empty. Moreover, any totally ordered subset of Ω admits an upper bound, just take the sum of its elements. We can therefore apply Zorn lemma. Let $(E, g) \in \Omega$ be a maximal element. We know that E is injected in D by definition, and we want to prove that D is injected in E in order to obtain $E = D$.

Let $x \in D$ be a homogeneous element. According to Lemma 1.3.16 applied to the acyclic cofibration $E \hookrightarrow D$, there exists a sub-P-coalgebra $B \subseteq D$ with a countable homogeneous basis such that $x \in B$ and $E \cap B \hookrightarrow B$ is an acyclic cofibration. The lifting problem



admits a solution h by hypothesis about p. We therefore extend g into a map $\tilde{g} : E + B \to X$ such that $\tilde{g} \mid_E = g$, $\tilde{g} \mid_B = h$. According to the diagram above, we have $h \mid_{E\cap B} = g \mid_{E\cap B}$ so $\tilde{g}$ is well defined.The short exact sequences

$$0 \to E \cap B \to B \to \frac{B}{E \cap B} \to 0$$

and

$$0 \to E \to E + B \to \frac{E + B}{B} \to 0$$

induce long exact sequences in homology

$$... \to H_{n+1}(\frac{B}{E \cap B}) \to H_n(E \cap B) \to H_n(B) \to H_n(\frac{B}{E \cap B}) \to ...$$

and

$$... \to H_{n+1}(\frac{E+B}{E}) \to H_n(E) \to H_n(E+B) \to H_n(\frac{E+B}{E}) \to ...$$

But $E \cap B \hookrightarrow B$ induces an isomorphism in homology so in the first exact sequence $H_*(\frac{B}{E\cap B}) = 0$. Furthermore, the isomorphism $\frac{B}{E\cap B} \cong \frac{E+B}{E}$ implies that $H_*(\frac{E+B}{E}) = 0$. Accordingly, the map $E \hookrightarrow E+B$ in the second exact sequence induces an isomorphism in homology, i.e $E \hookrightarrow E+B$ is an acyclic cofibration. It means that $(E+B, \tilde{g}) \in \Omega$, and by definition of $\tilde{g}$ the inequality $(E,g) \leq (E+B,\tilde{g})$ holds in Ω. Given that (E,g) is supposed to be maximal, we conclude that $E = E+B$, hence $x \in E$ and $E = D$. The map g is the desired lifting, and the map p is a fibration.

(ii) If p is an acyclic fibration, then p has the right lifting property with respect to cofibrations according to axiom MC4 (i). Conversely, let us suppose that p has the right lifting property with respect to cofibrations $A \hookrightarrow B$ where B is finite dimensional. The proof is similar to that of (i) with a slight change in the definition of Ω. Indeed, we consider the lifting problem

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\ D & \longrightarrow & Y \end{array}$$

where j is a cofibration. We define Ω as the set of pairs $(\overline{D}, g)$ where $\overline{D}$ fits in a composite of cofibrations $C \hookrightarrow \overline{D} \hookrightarrow D$ such that this composite is equal to j. We define the same partial order on Ω as in (i), and Ω is clearly not empty since $(C,f) \in \Omega$. The set Ω is inductive so we can apply Zorn's lemma. Let (E,g) be a maximal element of Ω, as before E is injected in D and we want to prove that D is injected in E. Let $x \in D$ be a homogeneous element, then there exists a finite dimensional sub-P-coalgebra $B \subseteq D$ containing x (this property is an adaptation to P-coalgebras of Proposition 1.5 in [10]). The map p has the right lifting property with respect to $E \cap B \hookrightarrow B$ by hypothesis (since $E \cap B \hookrightarrow B$ is an injection of P-coalgebras, thus a cofibration, and B is of finite dimension), so the method of (i) works here. We extend g to $\tilde{g} : E+B \to X$, we have $(E+B, \tilde{g}) \in \Omega$ and $(E,g) \leq (E+B,\tilde{g})$. The maximality of (E,g) implies that $E = E+B$ and $g : E = D \to X$ is the desired lifting. ☐

MC5 (ii). We need here to use a refined version of the usual small object argument. We use this expression to emphasize the fact that we will not use the usual simplifying hypothesis of smallness with respect to any sequences of morphisms. We will consider smallness only with respect to injections systems. Moreover, we will have to consider colimits running over a certain ordinal bigger than $\mathbb{N}$.

We use the notations of section 1.1.3. In the case where $\mathcal{C} = {}^P Ch_{\mathbb{K}}^+$ and $\mathcal{D}$ is the collection of injections of P-coalgebras, the colimit $colim_{\beta<\lambda} B(\beta)$ is a union $\bigcup_{\beta<\lambda} B(\beta)$. We say that a P-coalgebra A is small with respect to direct systems of injections if the map

$$\bigcup_{\beta<\lambda} Mor_{{}^P Ch_{\mathbb{K}}^+}(A, B(\beta)) \to Mor_{{}^P Ch_{\mathbb{K}}^+}(A, \bigcup_{\beta<\lambda} B(\beta))$$

is a bijection. Consider a morphism f of $\mathcal{C}$ and a family of morphisms $\mathcal{F} = \{f_i : A_i \to B_i\}_{i\in I}$ such that the A_i are small with respect to injections systems. If we can prove that the i_β obtained in the construction of the $G^\beta(\mathcal{F}, f)$ (see section 1.1.3 for the notation) are injections, then we can use this refined version of the small object argument to obtain a factorization $f = f_\infty \circ i_\infty$ where f_∞ has the right lifting property with respect to the morphisms of $\mathcal{F}$ and i_∞ is an injection (the injectivity passes to the transfinite composite). This is the argument we are going to use to prove axiom MC5 (ii) in ${}^P Ch_{\mathbb{K}}^+$.

Recall that the generating acyclic cofibrations of ${}^P Ch_{\mathbb{K}}^+$ are the acyclic injections $j_i : A_i \hookrightarrow B_i$ of P-coalgebras such that the B_i have countable homogeneous bases. In order to apply the refined small object argument explained above, we need the following lemma:

Lemma 1.3.18. *Let C be a object of ${}^P Ch_{\mathbb{K}}^+$ and κ a cardinal. If $U(C)$ is κ-small with respect to injections systems, then so is C in ${}^P Ch_{\mathbb{K}}^+$.*

Proof. Let us consider a κ-filtered ordinal λ and a λ-sequence of injections

$$... \hookrightarrow B(\beta) \hookrightarrow B(\beta+1) \hookrightarrow ...$$

of P-coalgebras, and let $f : C \to \bigcup_{\beta<\lambda} B(\beta)$ be a morphism of P-coalgebras. The chain complex $U(C)$ is κ-small with respect to injections systems, so there exists an ordinal $\beta < \lambda$ such that we have a unique factorization in $Ch_{\mathbb{K}}^+$

$$f : C \stackrel{\tilde{f}}{\to} B(\beta) \hookrightarrow \bigcup_{\beta<\lambda} B(\beta).$$

The map f is a morphism of P-coalgebras and so is $B(\beta) \hookrightarrow \bigcup_{\beta<\lambda} B(\beta)$. We deduce that $\tilde{f}$ is a morphism of P-coalgebras by the following argument. For any cooperations $p_C^* : C \to C^{\otimes n}$, $p_\beta^* : B(\beta) \to B(\beta)^{\otimes n}$ and $p_\lambda^* : \bigcup_{\beta<\lambda} B(\beta) \to (\bigcup_{\beta<\lambda} B(\beta))^{\otimes n}$ associated to an element $p \in P^*(n)$, we consider the diagram

$$\begin{array}{ccccc} C & \xrightarrow{\tilde{f}} & B(\beta) & \xhookrightarrow{i_\beta} & \bigcup_{\beta<\lambda} B(\beta) \\ \downarrow{\scriptstyle p_C^*} & & \downarrow{\scriptstyle p_\beta^*} & & \downarrow{\scriptstyle p_\lambda^*} \\ C^{\otimes n} & \xrightarrow[\tilde{f}^{\otimes n}]{} & B(\beta)^{\otimes n} & \xhookrightarrow[i_\beta^{\otimes n}]{} & (\bigcup_{\beta<\lambda} B(\beta))^{\otimes n} \end{array}.$$

The external rectangle commutes because f is a morphism of P-coalgebras. The right square commutes because i_β is a morphism of P-coalgebras as a transfinite composite of morphisms of P-coalgebras. We deduce that

$$i_\beta^{\otimes n} \circ \tilde{f}^{\otimes n} \circ p_C^* = i_\beta^{\otimes n} \circ p_\beta^* \circ \tilde{f}.$$

By injectivity of $i_\beta^{\otimes n}$ we get the commutativity of the left square

$$\tilde{f}^{\otimes n} \circ p_C^* = p_\beta^* \circ \tilde{f},$$

so $\tilde{f}$ is a morphism of P-coalgebras. We have the desired factorization in ${}^P Ch_{\mathbb{K}}^+$. □

Let us consider the family of generating acyclic cofibrations $\mathcal{F} = \{j_i : A_i \hookrightarrow B_i\}_{i\in I}$. According to Lemma 2.3.2 of [14], the $U(A_i)$ are κ-small for a certain cardinal κ. In particular, they are κ-small with respect to injections systems. Lemma 1.3.18 implies that the A_i are κ-small with respect to injection systems. Now, let $f : X \to Y$ be a morphism of P-coalgebras and λ a κ-filtered ordinal. Recall that the construction of $G^\beta(\mathcal{F}, f)$, $\beta < \lambda$, is given by a pushout

$$\begin{array}{ccc} \bigvee_i A_i & \longrightarrow & G^{\beta-1}(\mathcal{F}, f) \\ \downarrow{\scriptstyle \bigvee_i j_i} & & \downarrow{\scriptstyle i_\beta} \\ \bigvee_i B_i & \longrightarrow & G^\beta(\mathcal{F}, f) \end{array}.$$

The forgetful functor creates the small colimits, so we obtain the same pushout in $Ch_{\mathbb{K}}^+$ by forgetting P-coalgebras structures. By definition of cofibrations and weak equivalences in $Ch_{\mathbb{K}}^+$, given that $\bigvee_i j_i$ is an acyclic cofibration, the map $U(\bigvee_i j_i)$ is an acyclic cofibration in $Ch_{\mathbb{K}}^+$. In any model category, acyclic cofibrations are stable by pushouts, so the $U(i_\beta)$ are acyclic cofibrations. By definition, it means that the i_β are acyclic cofibrations, i.e in our case acyclic injections of P-coalgebras. We use our refined version of the small object argument to obtain a factorization $f = f_\infty \circ i_\infty$. Injectivity and acyclicity are two properties which passes to the transfinite composite i_∞, so i_∞ is an acyclic cofibration of ${}^P Ch_{\mathbb{K}}^+$. Moreover, the map f_∞ has by construction the right lifting property with respect to the generating acyclic cofibrations and forms consequently a fibration. Our proof is now complete.

Remark 1.3.19. This method provides us another way to prove MC5 (i), by using this time the family of generating cofibrations.

1.4 The model category of bialgebras over a pair of operads in distribution

Let P be an operad in $Vect_{\mathbb{K}}$. Let Q be an operad in $Vect_{\mathbb{K}}$ such that $Q(0) = 0$, $Q(1) = \mathbb{K}$ and the $Q(n)$ are of finite dimension for every $n \in \mathbb{K}$. We suppose that there exists a mixed distributive law between P and Q (see Definition 1.1.27). In the following, the operad P will encode the operations of our bialgebras and the operad Q will encode the cooperations.

Recall that there exists a cofibrantly generated model category structure on the category ${}_PCh_{\mathbb{K}}$ of P-algebras:

Theorem 1.4.1. *(see [12] or [9]) The category of P-algebras ${}_PCh_{\mathbb{K}}$ inherits a cofibrantly generated model category structure such that a morphism f of ${}_PCh_{\mathbb{K}}$ is*
(i)a weak equivalence if $U(f)$ is a weak equivalence in $Ch_{\mathbb{K}}$, where U is the forgetful functor;
(ii)a fibration if $U(f)$ is a fibration in $Ch_{\mathbb{K}}$;
(iii)a cofibration if it has the left lifting property with respect to acyclic fibrations.

We can also say that cofibrations are relative cell complexes with respect to the generating cofibrations, where the generating cofibrations and generating acyclic cofibrations are, as expected, the images of the generating (acyclic) cofibrations of $Ch_{\mathbb{K}}$ under the free P-algebra functor P. Recall that the $j_n : S^{n-1} \hookrightarrow D^n$ and the $i_n : 0 \hookrightarrow D^n$ are respectively the generating cofibrations and the generating acyclic cofibrations of $Ch_{\mathbb{K}}$.

Actually, this structure exists via a transfer of cofibrantly generated model category structure via the adjunction $P : Ch_{\mathbb{K}} \rightleftarrows {}_PCh_{\mathbb{K}} : U$ (see [12] and [9]). The forgetful functor creates fibrations and weak equivalences. The free P-algebra functor P preserves generating (acyclic) cofibrations by definition of the generating (acyclic) cofibrations of ${}_PCh_{\mathbb{K}}$. Moreover, it preserves colimits as a left adjoint (it is a general property of adjunctions, see [19] for instance). Thus it preserves all (acyclic) cofibrations, which are relative cell complexes with respect to the generating (acyclic) cofibrations. Such a pair of functors is called a Quillen adjunction, and induces an adjunction at the level of the associated homotopy categories. According to Theorem 1.1.28, we can lift this free P-algebra functor to the category of Q-coalgebras, so the adjunction

$$P : Ch_{\mathbb{K}}^+ \rightleftarrows {}_PCh_{\mathbb{K}}^+ : U$$

becomes an adjunction

$$P : {}^QCh_{\mathbb{K}}^+ \rightleftarrows {}^Q_PCh_{\mathbb{K}}^+ : U.$$

Similarly, the adjunction

$$U : {}^QCh_{\mathbb{K}}^+ \rightleftarrows Ch_{\mathbb{K}}^+ : Q^*$$

becomes an adjunction

$$U : {}^Q_PCh_{\mathbb{K}}^+ \rightleftarrows {}_PCh_{\mathbb{K}}^+ : Q^*.$$

The model category structure on (P, Q)-bialgebras is then given by the following theorem:

Theorem 1.4.2. *The category ${}^Q_PCh_{\mathbb{K}}^+$ inherits a cofibrantly generated model category structure such that a morphism f of ${}^Q_PCh_{\mathbb{K}}^+$ is*
(i) a weak equivalence if $U(f)$ is a weak equivalence in ${}^QCh_{\mathbb{K}}^+$ (i.e a weak equivalence in $Ch_{\mathbb{K}}^+$ by definition of the model structure on ${}^QCh_{\mathbb{K}}^+$);
(ii) a fibration if $U(f)$ is a fibration in ${}^QCh_{\mathbb{K}}^+$;
(iii) a cofibration if f has the left lifting property with respect to acyclic fibrations.

It is clear that this three classes of morphisms are stable by composition and contain the identity morphisms. Axioms MC2 and MC3 are clear, axiom MC4 (i) is obvious by definition of the cofibrations. It remains to prove axioms MC1, MC4 (ii) and MC5.

MC1. The forgetful functor $U : {}^Q_PCh_{\mathbb{K}}^+ \rightarrow {}^QCh_{\mathbb{K}}^+$ creates small limits. The proof is the same than in the case of P-algebras. The forgetful functor $U : {}^Q_PCh_{\mathbb{K}}^+ \rightarrow {}_PCh_{\mathbb{K}}^+$ creates the small colimits. The proof is the same as in the case of P-coalgebras, see section 1.3.2.

Generating (acyclic) cofibrations. The treatment is similar to the case of P-algebras. Let us note $\{j : A \hookrightarrow B\}$ the family of generating cofibrations and $\{i : A \hookrightarrow B\}$ the family of generating acyclic cofibrations. Then the $P(j)$ form the generating cofibrations of ${}^Q_P Ch^+_{\mathbb{K}}$ and the $P(i)$ form the generating acyclic cofibrations:

Proposition 1.4.3. *Let f be a morphism of ${}^Q_P Ch^+_{\mathbb{K}}$. Then*

(i) f is a fibration if and only if it has the right lifting property with respect to the $P(i)$, where $i : A \hookrightarrow B$ an acyclic injection of Q-coalgebras such that B has a countable homogeneous basis;

(ii) f is an acyclic fibration if and only if it has the right lifting property with respect to the $P(j)$, where $j : A \hookrightarrow B$ is an injection of Q-coalgebras such that B is finite dimensional.

Proof. This is a standard argument using only properties of the unit and the counit of the adjunction as well as lifting properties. This is the same for instance than the one giving the generating (acyclic) cofibrations of the category of P-algebras. It is part of more general results giving conditions to transfer a cofibrantly generated model structure via an adjunction, see for instance theorem 11.3.2 in [13]. □

MC4 (ii). If MC5 (ii) is proved, then MC4 (ii) follows from MC4 (i) and MC5 (ii) by the same arguments as in the proof of MC4 (i) for coalgebras over an operad (see section 2). Indeed, any acyclic cofibration f admits a factorization $f = p \circ i$, where p is a fibration and i an acyclic cofibration. The map i has the left lifting property with respect to fibrations because it is constructed via the small object argument (see the proof of MC5), and f is a retract of i so it inherits this lifting property.

MC5. The main difficulty here is to prove axiom MC5. Let f be a morphism of ${}^Q_P Ch^+_{\mathbb{K}}$. Let us note $\mathcal{F} = \{P(j_i), j_i : A_i \hookrightarrow B_i\}_{i\in I}$ the family of generating cofibrations. Recall that the A_i are κ-small with respect to injections systems for a certain cardinal κ. Actually, since the generating cofibrations have finite dimensional targets, they are small with respect to sequences indexed by $\mathbb{N}$. However, we write the proof without specifying κ, because after we use the same method to prove MC5 for generating acyclic cofibrations, whose domains are small for a non countable cardinal.

We use the following Lemma to prove that the $P(A_i)$ are also κ-small with respect to injections systems:

Lemma 1.4.4. *Let C be an object of ${}^Q Ch^+_{\mathbb{K}}$. If C is κ-small in ${}^Q Ch^+_{\mathbb{K}}$ with respect to injections systems, then $P(C)$ is κ-small in ${}^Q_P Ch^+_{\mathbb{K}}$ with respect to injections systems.*

Proof. Let us suppose that C is κ-small with respect to injections systems in ${}^Q Ch^+_{\mathbb{K}}$. Let λ be a κ-filtered ordinal and let $F : \lambda \rightarrow {}^Q_P Ch^+_{\mathbb{K}}$ be a λ-sequence whose arrows are injections.. For every $\beta < \lambda$,

$$Hom_{{}^Q_P Ch^+_{\mathbb{K}}}(P(C), F(\beta)) \cong Hom_{{}^Q Ch^+_{\mathbb{K}}}(C, (U \circ F)(\beta))$$

hence

$$\begin{aligned} colim_{\beta<\lambda} Hom_{{}^Q_P Ch^+_{\mathbb{K}}}(P(C), F(\beta)) &\cong colim_{\beta<\lambda} Hom_{{}^Q Ch^+_{\mathbb{K}}}(C, (U \circ F)(\beta)) \\ &\cong Hom_{{}^Q Ch^+_{\mathbb{K}}}(C, colim_{\beta<\lambda}(U \circ F)(\beta)) \end{aligned}$$

because $U \circ F : \lambda \rightarrow {}^Q Ch^+_{\mathbb{K}}$ and C is κ-small with respect to injections systems. We can equip $colim_{\beta<\lambda}(U \circ F)(\beta)$ with a structure of P-algebra, such that with this structure it forms $colim_{\beta<\lambda} F(\beta)$ in ${}^Q_P Ch^+_{\mathbb{K}}$. Indeed, we have

$$colim_{\beta<\lambda}(U \circ F)(\beta) = \{[a], a \in F(\beta)\}/\sim$$

where $a \sim b$ (i.e $[a] = [b]$), $a \in F(\beta), b \in F(\beta')$, $\beta \leq \beta'$, if the application $F(\beta) \rightarrow F(\beta')$ in the λ-sequence sends a to b. Let $[a_1], ..., [a_r] \in colim_{\beta<\lambda}(U \circ F)(\beta)$ such that $a_1 \in F(\beta_1), ..., a_r \in F(\beta_r)$. We consider $F(\beta)$ for a given ordinal $\beta \geq max(\beta_1, ..., \beta_r)$ and we set, for $\mu \in P(\beta)$, $\mu([a_1], ..., [a_r]) = \mu(a'_1, ..., a'_r)$ where $a'_1, ..., a'_r$ are representing elements of $[a_1], ..., [a_r]$ in $F(\beta)$. We then obtain a P-algebra structure on $colim_{\beta<\lambda}(U \circ F)(\beta)$ (one says that the forgetful functor creates the sequential colimits). Moreover, this P-algebra structure is compatible with the Q-coalgebras structures of the $F(\beta)$, since it is defined via their

P-algebra structures. It is therefore compatible with the Q-coalgebra structure of $colim_{\beta<\lambda}(U\circ F)(\beta)$, such that we obtain the colimit in ${}_P^QCh_{\mathbb{K}}^+$. We can finally write

$$\begin{array}{rcl} colim_{\beta<\lambda}Hom_{{}_P^QCh_{\mathbb{K}}^+}(P(C),F(\beta)) & \cong & Hom_{{}^QCh_{\mathbb{K}}^+}(C,U(colim_{\beta<\lambda}F(\beta))) \\ & \cong & Hom_{{}_P^QCh_{\mathbb{K}}^+}(P(C),colim_{\beta<\lambda}F(\lambda)). \end{array}$$

□

We want to apply the small object argument to obtain a factorization $f=f_\infty\circ i_\infty$ of f. Recall that, given a κ-filtered ordinal λ, for every $\beta<\lambda$, the space $G^\beta(\mathcal{F},f)$ is obtained by a pushout

$$\begin{array}{ccc} \bigvee_i P(A_i) & \longrightarrow & G^{\beta-1}(\mathcal{F},f) \\ {\scriptstyle \bigvee_i P(j_i)}\downarrow & & \downarrow{\scriptstyle i_\beta} \\ \bigvee_i P(B_i) & \longrightarrow & G^\beta(\mathcal{F},f) \end{array}.$$

The forgetful functor $U:{}_P^QCh_{\mathbb{K}}^+\rightarrow {}_PCh_{\mathbb{K}}^+$ creates small colimits, so we obtain the same pushout diagram in ${}_PCh_{\mathbb{K}}^+$ by forgetting the Q-coalgebras structures. The j_i are cofibrations of ${}^QCh_{\mathbb{K}}^+$, so the underlying chain complexes morphisms are cofibrations of $Ch_{\mathbb{K}}^+$. Thus, via the adjunction $P:Ch_{\mathbb{K}}^+\rightleftarrows {}_PCh_{\mathbb{K}}^+:U$, the $P(j_i)$ are cofibrations of ${}_PCh_{\mathbb{K}}^+$ and so are $\bigvee_i P(j_i)$. In any model category, cofibrations are stable by pushouts, so the i_β are cofibrations of ${}_PCh_{\mathbb{K}}^+$. By definition of cofibrations in ${}_PCh_{\mathbb{K}}^+$, we can apply Lemma 1.1.38 to i_∞ to deduce that i_∞ forms a cofibration in ${}_PCh_{\mathbb{K}}^+$. We now use the following proposition:

Proposition 1.4.5. *An (acyclic) cofibration of ${}_PCh_{\mathbb{K}}^+$ forms an (acyclic) cofibration in $Ch_{\mathbb{K}}^+$.*

Proof. See section 4.6.3 in [12] (Note that for a base field of characteristic zero, every operad is Σ-split in the sense defined by Hinich). □

The maps i_β (and thus i_∞) forms therefore cofibrations in $Ch_{\mathbb{K}}^+$, i.e injections. This is crucial to apply our version of the small object argument, since the $P(A_i)$ are κ-small only with respect to injections systems. Finally, i_∞ forms a cofibration in ${}_P^QCh_{\mathbb{K}}^+$. The map f_∞ has the right lifting property with respect to the generating cofibrations and forms thus an acyclic fibration. Axiom MC5 (i) is proved.

The method to prove MC5 (ii) is the same up to two minor changes: we consider the family of generating acyclic cofibrations, and use the stability of acyclic cofibrations under pushouts.

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Chapitre 2

Classifying spaces and moduli spaces of algebras over a prop

The purpose of this article is two-fold. First we show that a weak equivalence between cofibrant props induces a weak equivalence between the associated classifying spaces of algebras. This statement generalizes to the prop setting a homotopy invariance result which is well known in the case of algebras over operads. The absence of model category structure on algebras over a prop leads us to introduce new methods to overcome this difficulty.
We also explain how our result can be extended to algebras over colored props in any symmetric monoidal model category tensored over chain complexes.
Then we provide a generalization of a theorem of Charles Rezk in the setting of algebras over a (colored) prop: we introduce the notion of moduli space of algebra structures over a prop, and prove that under certain conditions such a moduli space is the homotopy fiber of a map between classifying spaces.

Introduction

The notion of a prop has been introduced by MacLane ([17]) in algebra. The name prop is actually an acronyme for "product and permutation". Briefly, a prop P is a double sequence of objects $P(m,n)$ whose elements represent operations with m inputs and n outputs.

Certain categories of algebras, like associative, Poisson or Lie algebras, have a structure which is fully determined by operations with a single output. These categories are associated to props P of a certain form, where operations in components $P(m,1)$ generate the prop. Boardman and Vogt coined the name categories of operators of standard form to refer to props of this particular form [1]. Peter May introduced the axioms of operads to deal with the components $P(m,1)$ which define the core of such prop structures [20]. The work of these authors was initially motivated by the theory of iterated loop spaces, in topology (see [2] and [20]). Operads have now proved to be a powerful device to handle a variety of algebraic structures occurring in many branches of mathematics.

However, if one wants to deal with bialgebras it becomes necessary to use general props instead of operads. Important examples appeared especially in mathematical physics and string topology : the Frobenius bialgebras (whose category is equivalent to the category of two-dimensional topological quantum field theories), the topological conformal field theories (which are algebras over the chain Segal prop), or the Lie bialgebras introduced by Drinfeld in quantization theory are categories of bialgebras associated to props.

The purpose of this article is to set up a theory for the homotopical study of algebras over a (possibly colored) prop. In a seminal series of papers at the beginning of the 80's, Dwyer and Kan investigated the simplicial localization of categories. They proved that simplicial localization gives a good device to capture secondary homology structures usually defined in the framework of Quillen's model categories ([5]). An important homotopy invariant of a model category is its classification space, defined as the nerve of its subcategory of weak equivalences. The interest of such a classifiying space has been shown in the work of Dwyer and Kan [5]. More precisely they prove the following result:

Theorem 2.0.6. *(Dwyer-Kan) Let M be a category, W a class of morphisms of M and wM the subcategory of M defined by $ob(wM) = ob(M)$ and $mor(wM) = W$. Then one has a homotopy equivalence*

$$\mathcal{N}wM \sim \coprod_{[X]} LwM(X,X)$$

where $\mathcal{N}$ is the simplicial nerve functor, $[X]$ ranges over the weak equivalence classes of the objects of M and $L(-)$ is the simplicial localization functor. When M is a model category, one has moreover

$$\mathcal{N}wM \sim \coprod_{[X]} Bhaut(X)$$

where $Bhaut(X)$ is the classifying space of the simplicial monoid $haut(X)$ of self weak equivalences on a fibrant-cofibrant resolution of X.

It means that the classification space $\mathcal{N}wM$ encode informations about the homotopy types of the objects and their internal symmetries, i.e their homotopy automorphisms. They also proved that such classification space is homotopy invariant under Quillen equivalences of model categories.

The algebras over an operad in a model category form themselves, under suitable assumptions, a model category. A consequence of usual results about model categories is that the classification space of such a category is homotopy invariant up to the weak homotopy type of the underlying operad. Unfortunately, there is no model category structure on the algebras over a prop in general. We can not handle our motivating examples of bialgebras occurring in mathematical physics and string topology by using this approach, and we aim to overcome this difficulty.

The basic problem is to compare categories of algebras over a prop. In order to bypass difficulties due to the absence of model structure on these algebras, our overall strategy is to stay at the prop level as far as possible, and to use factorization and lifting properties in the model category of props. The structure of an algebra over a prop P can be encoded by a prop morphism $P \to End_A$, where End_A is the endomorphism prop associated to A. One can construct a version of endomorphism props modeling P-algebras structures on diagrams. We can notably use these diagram endomorphism props to define path objects in the category of P-algebras. But we need an analogue of this device for a variable P-algebra A, not a fixed object. The idea is to perform such a construction on the abstract prop P itself before moving to endomorphisms props. Combining this P-modified prop feature with lifting and factorization techniques, we endow the category of P-algebras with functorial path objects.

Consequently, the first main outcome of our study is the following homotopy invariance theorem. Let $Ch_{\mathbb{K}}$ be the category of $\mathbb{Z}$-graded chain complexes over a field $\mathbb{K}$ of characteristic zero. Let $(Ch_{\mathbb{K}})^P$ be the category of algebras associated to a prop P in this category, and $w(Ch_{\mathbb{K}})^P$ its subcategory obtained by restricting to morphisms which are weak equivalences in $Ch_{\mathbb{K}}$. Our result reads:

Theorem 2.0.7. *Let $\varphi : P \xrightarrow{\sim} Q$ be a weak equivalence between two cofibrant props. The map φ gives rise to a functor $\varphi^* : w(Ch_{\mathbb{K}})^Q \to w(Ch_{\mathbb{K}})^P$ which induces a weak equivalence of simplicial sets $\mathcal{N}\varphi^* : \mathcal{N}w(Ch_{\mathbb{K}})^Q \xrightarrow{\sim} \mathcal{N}w(Ch_{\mathbb{K}})^P$.*

We can withdraw the hypothesis about the characteristic of $\mathbb{K}$ if we suppose that P is a prop with non-empty inputs or outputs (see definition 2.1.12 and theorem 2.1.13). We explain in 2.2.7 how to extend theorem 2.0.2 to the case of a category tensored over $Ch_{\mathbb{K}}$. In section 2.2.4, we also briefly show that the proof of theorem 2.0.2 extends readily to the colored props context if we suppose $\mathbb{K}$ to be of characteristic zero (this hypothesis is needed to put a model category structure on colored props in Ch_K, see the work of Johnson and Yau[15]). Recall that examples includes cofibrant resolutions of the props encoding associative bialgebras, Lie bialgebras, Frobenius bialgebras or Poisson bialgebras for instance. Algebras over a cofibrant resolution of a given prop P are called homotopy P-algebras. Theorem 2.1.3 implies that the classification space does not depend on the choice of the cofibrant resolution, and thus provides a well defined homotopy invariant.

Rezk considers in his thesis [23] the moduli spaces $\mathcal{A}\{X\}$ of algebras structures over an operad $\mathcal{A}$, which are simplicial sets whose 0-simplexes are operad morphisms $\mathcal{A} \rightarrow End_X$ representing all the $\mathcal{A}$-algebra structures on a given object X. The first main result of his thesis, proved in the case of simplicial sets and simplicial modules, is that under certain conditions such a moduli space is the homotopy fiber of a map between classification spaces. Combining our theorem 2.0.2 with the technical proposition 2.3.1 proved in 2.3.2, we obtain a broad generalization of this result:

Theorem 2.0.8. *(Generalization of [23], theorem 1.1.5, in the case of props). Let P be a cofibrant prop defined in $Ch_{\mathbb{K}}$ and X a fibrant and cofibrant object of $Ch_{\mathbb{K}}$. Then the commutative square*

$$\begin{array}{ccc} P\{X\} & \longrightarrow & \mathcal{N}(w(Ch_{\mathbb{K}})^P) \\ \downarrow & & \downarrow \\ \{X\} & \longrightarrow & \mathcal{N}(wCh_{\mathbb{K}}) \end{array}$$

is a homotopy pullback of simplicial sets.

As previously, we can extend this result to colored props and categories tensored over $Ch_{\mathbb{K}}$.

Remark 2.0.9. This result implies that

$$P\{X\} \sim \coprod_{[X]} Lw(Ch_{\mathbb{K}})^P)(X,X)$$

where $[X]$ ranges over the weak equivalence classes of P-algebras having X as underlying object.

Remark 2.0.10. We do not address the case of simplicial sets. However, theorem 1.4 in [15] endows the algebras over a colored prop in simplicial sets with a model category structure. Moreover, the free algebra functor exists in this case. Therefore one can transpose the methods used in the operadic setting to obtain a simplicial version of theorem 2.0.2. Theorem 2.0.3 in simplicial sets can be proved by following step by step Rezk's original proof. We also conjecture that our results have a version in simplicial modules which follows from arguments similar to ours.

We would like to emphasize some links with two other objects encoding algebraic structures and their deformations. Theorem 2.0.3 asserts that we can use a function space of props, the moduli space $P\{X\}$, to determine classification spaces of categories of algebras over props $\mathcal{N}(w(Ch_{\mathbb{K}})^P)$. The homotopy groups of this moduli space, in turn, can be approached by means of a Bousfield-Kan type spectral sequence. The E_2 page of this spectral sequence is identified with the cohomology of certain deformation complexes. These complexes have been studied in [7], [19] and [21]. These papers prove the existence of an L_∞ structure on such complexes which generalizes the *intrinsic Lie bracket* of Schlessinger and Stasheff [24]. We aim to apply this spectral sequence technique and provide new results about the deformation theory of bialgebras in an ongoing work. To complete this outlook, let us point out that Ciocan-Fontanine and Kapranov used a similar approach to that of Rezk in [3] to define a derived moduli space of algebras structures in the formalism of dg schemes. This construction remains valid for props and formalizes, in the context of algebraic geometry, the idea that the deformation complexes form the tangent space of the moduli space $P\{X\}$.

Organization: the overall setting is reviewed in section 1. We recall some definitions about symmetric monoidal categories over a base category and axioms of monoidal model categories. Then we introduce the precise definition of props and algebras over a prop. We conclude these recollections with a fundamental result, the existence of a model structure on the category of props.

The heart of this paper consists of sections 2.2 and 2.3, devoted to the proofs of theorem 2.0.2 and proposition 2.3.1. The proof of theorem 2.0.2 is quite long and has been consequently divided in several steps. Subsection 2.2.1 gives a sketch of our main arguments. In 2.2.3, 2.2.4 and 2.2.5 we define particular props called P-modified endomorphism props, which allow us to build a functorial path object in P-algebras. In 2.2.6 we give a proof of theorem 2.0.2. To conclude section 2 we generalize theorem 2.0.2 to categories tensored over $Ch_{\mathbb{K}}$. In 2.3.1, we prove that under suitable assumptions a diagram category inherits a monoidal model category structure from the base category. The transfer of model structure is a well known result, but the compatibility with the symmetric monoidal structure over the base category does not seem to appear in the literature. This general result allows us to give a proof of proposition 2.3.1 in 2.3.2. Then we quickly explain in section 2.2.4 the extension of our arguments to colored props.

Finally, we show in section 2.5 how theorem 2.0.2 and proposition 2.3.1 fit in an adaptation of Rezk's proof of theorem 1.1.5 in [23]. We thus obtain theorem 2.0.3. This adaptation need some preliminary results given in 2.5.1 and 2.5.2. The last remark of 2.5.3 shows how to recover theorem A of Fresse in [8] as a consequence of theorem 2.0.3.

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2.1 Recollections and general results

2.1.1 Symmetric monoidal categories over a base category

Definition 2.1.1. Let $\mathcal{C}$ be a symmetric monoidal category. A *symmetric monoidal category over* $\mathcal{C}$ is a symmetric monoidal category $(\mathcal{E},\otimes,1_{\mathcal{E}})$ endowed with an external tensor product $\otimes : \mathcal{C}\times\mathcal{E}\rightarrow\mathcal{E}$ satisfying the following natural unit, associativity and symmetry constraints:

(1) $\forall X\in\mathcal{E}, 1_{\mathcal{C}}\otimes X\cong X$,

(2) $\forall X\in\mathcal{E},\forall C,D\in\mathcal{C},(C\otimes D)\otimes X\cong C\otimes(D\otimes X)$,

(3) $\forall C\in\mathcal{C},\forall X,Y\in\mathcal{E},C\otimes(X\otimes Y)\cong(C\otimes X)\otimes Y\cong X\otimes(C\otimes Y)$.

We will tacitely assume throughout the paper that all small limits and small colimits exist in $\mathcal{C}$ and that the internal tensor product of $\mathcal{C}$ preserves colimits in each variable. We assume the same hypotheses for $\mathcal{E}$, and suppose moreover that the external tensor product of $\mathcal{E}$ also preserves colimits in each variable. This last condition implies the existence of an external hom bifunctor $Hom_{\mathcal{E}}(-,-):\mathcal{E}^{op}\times\mathcal{E}\rightarrow\mathcal{C}$ satisfying an adjunction relation

$$\forall C\in\mathcal{C},\forall X,Y\in\mathcal{E}, Mor_{\mathcal{E}}(C\otimes X,Y)\cong Mor_{\mathcal{C}}(C,Hom_{\mathcal{E}}(X,Y))$$

(so $\mathcal{E}$ is naturally an enriched category over $\mathcal{C}$).

Examples.

(1) The differential graded $\mathbb{K}$-modules (where $\mathbb{K}$ is a commutative ring) form a symmetric monoidal category over the $\mathbb{K}$-modules. This is the main category used in this paper.

(2) Any symmetric monoidal category $\mathcal{C}$ with small coproducts forms a symmetric monoidal category over Set (the category of sets) with an external tensor product defined by

$$\begin{aligned}\otimes : Set\times\mathcal{C} &\rightarrow \mathcal{C}\\ (S,C) &\mapsto \bigoplus_{s\in S} C.\end{aligned}$$

(3) Let I be a small category ; the I-diagrams in a symmetric monoidal category $\mathcal{C}$ form a symmetric monoidal category over $\mathcal{C}$. The external hom $Hom_{\mathcal{C}^I}(-,-) : \mathcal{C}^I \times \mathcal{C}^I \to \mathcal{C}$ is given by

$$Hom_{\mathcal{C}^I}(X,Y) = \int_{i\in I} Hom_{\mathcal{C}}(X_i, Y_i).$$

This example will be useful especially in section 3.

Remark 2.1.2. If $\mathcal{E}$ is a symmetric monoidal category over $\mathcal{D}$ and $\mathcal{D}$ a symmetric monoidal category over $\mathcal{C}$, then $\mathcal{E}$ is a symmetric monoidal category over $\mathcal{C}$.

Proposition 2.1.3. *Let $F : \mathcal{D} \rightleftarrows \mathcal{E} : G$ be an adjunction between two symmetric monoidal categories over $\mathcal{C}$. If F preserves the external tensor product then F and G satisfy an enriched adjunction relation*

$$Hom_{\mathcal{E}}(F(X), Y) \cong Hom_{\mathcal{D}}(X, G(Y))$$

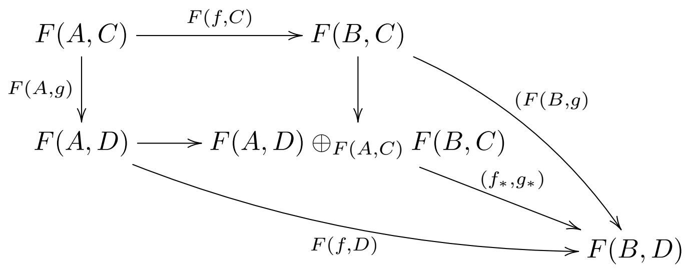
at the level of the external hom bifunctors (see proposition 1.1.16 in Fresse's lecture notes [9] for the proof).

We now deal with symmetric monoidal categories equipped with a model structure. We assume that the reader is familiar with the basics of model categories. We refer to the well written paper of Dwyer and Spalinski [4] for a complete and accessible introduction, and to Hirschhorn [12] and Hovey [14] for a comprehensive treatment. We just recall the axioms of symmetric monoidal model categories formalizing the interplay between the tensor and the model structure.

Definition 2.1.4. Let $\mathcal{C}$ be a category with colimits and $F : \mathcal{A}\times\mathcal{B} \to \mathcal{C}$ a bifunctor. The *pushout-product* of two morphisms $f : A \to B \in \mathcal{A}$ and $g : C \to D \in \mathcal{B}$ is the morphism

$$(f_*, g_*) : F(A,D) \oplus_{F(A,C)} F(B,C) \to F(B,D)$$

given by the commutative diagram:



Definition 2.1.5. (1) A *symmetric monoidal model category* is a symmetric monoidal category $\mathcal{C}$ equipped with a model category structure such that the following axioms hold:

MM0. The unit object $1_{\mathcal{C}}$ of $\mathcal{C}$ is cofibrant.

MM1. The pushout-product $(i_*, j_*) : A \otimes D \oplus_{A\otimes C} B \otimes C \to B \otimes D$ of cofibrations $i : A \rightarrowtail B$ and $j : C \rightarrowtail D$ is a cofibration which is also acyclic as soon as i or j is so.

(2) Suppose that $\mathcal{C}$ is a symmetric monoidal model category. A symmetric monoidal category $\mathcal{E}$ over $\mathcal{C}$ is a symmetric monoidal model category over $\mathcal{C}$ if the axiom MM0 holds and the axiom MM1 holds for both the internal and external tensor products of $\mathcal{E}$.

Example: the category $Ch_{\mathbb{K}}$ of chain complexes over a field $\mathbb{K}$ is our main working example of symmetric monoidal model category.

Lemma 2.1.6. *In a symmetric monoidal model category $\mathcal{E}$ over $\mathcal{C}$ the axiom MM1 for the external tensor product is equivalent to the following one:*

MM1'. *The morphism*

$$(i^*, p_*) : Hom_{\mathcal{E}}(B, X) \to Hom_{\mathcal{E}}(A, X) \times_{Hom_{\mathcal{E}}(A,Y)} Hom_{\mathcal{E}}(B, Y)$$

induced by a cofibration $i : A \rightarrowtail B$ and a fibration $p : X \twoheadrightarrow Y$ is a fibration in $\mathcal{C}$ which is also acyclic as soon as i or p is so (cf. lemma 4.2.2 in [14]).

In good cases, the fact that the internal tensor product of $\mathcal{E}$ preserves colimits in each variable implies the existence of an internal hom bifunctor. The axiom MM1 for the internal tensor product is in the same way equivalent to a "dual" axiom MM1'.

To conclude this section, we define additional axioms (introduced in [8]) that we will need to prove proposition 2.3.1.

Definition 2.1.7. We say that a symmetric monoidal category $\mathcal{C}$ satisfies the *limit monoid axioms* when the following properties hold:

LM1 (final monoid axiom). Let $*$ be the terminal object of $\mathcal{C}$, the natural morphism $* \otimes * \to *$ is an isomorphism.

LM2 (cartesian monoid axiom). For every fibration of the form

$$(f, g) : S \twoheadrightarrow X \times_T Y$$

and every cofibrant-fibrant object Z, the morphism

$$(f \otimes Z, g \otimes Z) : S \otimes Z \twoheadrightarrow X \otimes Z \times_{T \otimes Z} Y \otimes Z$$

is also a fibration.

Example: the category $Ch_{\mathbb{K}}$ of chain complexes over a field $\mathbb{K}$ is an example of category satisfying these axioms. It will be again our main example of such a category in our paper.

Proposition 2.1.8. *(cf. [8], proposition 6.7) The following properties hold in any symmetric monoidal category satisfying the limit monoid axioms:*

(1) If X is cofibrant-fibrant, then $X^{\otimes n}$ is fibrant for every $n \in \mathbb{N}$.

(2) If $p : X \twoheadrightarrow Y$ is a fibration and Y is cofibrant-fibrant, then $p^{\otimes n} : X^{\otimes n} \twoheadrightarrow Y^{\otimes n}$ is also a fibration.

(3) If $p : Z \twoheadrightarrow X \times Y$ is a fibration and X, Y are cofibrant-fibrant objects, then

$$p^{\otimes n} : Z^{\otimes n} \twoheadrightarrow X^{\otimes n} \times Y^{\otimes n}$$

is also a fibration.

2.1.2 On Σ-bimodules, props and algebras over a prop

Let $\mathcal{C}$ be a symmetric monoidal category admitting all small limits and small colimits, whose tensor product preserves colimits and endowed with an internal hom bifunctor. Let $\mathbb{B}$ be the category having the pairs $(m, n) \in \mathbb{N}^2$ as objects together with morphisms sets such that:

$$Mor_{\mathbb{B}}((m, n), (p, q)) = \begin{cases} \Sigma_m^{op} \times \Sigma_n, & \text{if } (p, q) = (m, n), \\ \emptyset & \text{otherwise.} \end{cases}$$

The Σ-biobjects in $\mathcal{C}$ are the $\mathbb{B}$-diagrams in $\mathcal{C}$. So a Σ-biobject is a double sequence $\{M(m, n) \in \mathcal{C}\}_{(m,n)\in\mathbb{N}^2}$ where each $M(m, n)$ is equipped with a right action of Σ_m and a left action of Σ_n commuting with each other. Let $\mathbb{A}$ be the discrete category of pairs $(m,n) \in \mathbb{N}^2$. We have an obvious forgetful functor $\phi^* : \mathcal{C}^{\mathbb{B}} \to \mathcal{C}^{\mathbb{A}}$. This functor has a left adjoint $\phi_! : \mathcal{C}^{\mathbb{A}} \to \mathcal{C}^{\mathbb{B}}$ defined on objects by

$$\forall M \in \mathcal{C}^{\mathbb{A}}, \forall (m,n) \in \mathbb{N}^2, \phi_! M(m,n) = 1_{\mathcal{C}}[\Sigma_n \times \Sigma_m^{op}] \otimes M(m,n)$$
$$\cong \bigoplus_{\Sigma_n \times \Sigma_m^{op}} M(m,n).$$

Definition 2.1.9. (1) Let $\mathcal{C}$ be a symmetric monoidal category. A *prop* in $\mathcal{C}$ is a symmetric monoidal category P, enriched over $\mathcal{C}$, with $\mathbb{N}$ as object set and the tensor product given by $m \otimes n = m + n$ on objects. Let us unwrap this definition. Firstly we see that a prop is a Σ-biobject. Indeed, the group Σ_m acts on $m = 1 + ... + 1 = 1^{\otimes m}$ and the group ${\Sigma_n}^{op}$ acts on $n = 1 + ... + 1 = 1^{\otimes n}$ by permuting the variables at the morphisms level. A prop is endowed with horizontal products

$$\circ_h : P(m_1, n_1) \otimes P(m_2, n_2) \to P(m_1 + m_2, n_1 + n_2)$$

which are defined by the tensor product of homomorphisms, since $P(m_1 \otimes m_2, n_1 \otimes n_2) = P(m_1 + m_2, n_1 + n_2)$ by definition of the tensor product on objects. It also admits vertical composition products

$$\circ_v : P(k,n) \otimes P(m,k) \to P(m,n)$$

corresponding to the composition of homomorphisms, and units $1 \to P(n,n)$ corresponding to identity morphisms of the objects $n \in \mathbb{N}$ in P. These operations satisfy relations coming from the axioms of symmetric monoidal categories. We refer the reader to Enriquez and Etingof [6] for an explicit description of props in the context of modules over a ring. A morphism of props is a symmetric monoidal functor. It is equivalently a collection of equivariant maps compatible with the composition products at the source and at the target. We denote by $\mathcal{P}$ the category of props.

Another construction of props is given in [15]: props are defined there as $\boxtimes_h$-monoids in the $\boxtimes_v$-monoids of colored Σ-biobjects, where $\boxtimes_h$ and $\boxtimes_v$ denote respectively a horizontal composition product and a vertical composition product.

Appendix A of [8] provides a construction of the free prop on a Σ-biobject. The free prop functor is left adjoint to the forgetful functor:

$$F : \mathcal{C}^{\mathbb{B}} \rightleftarrows \mathcal{P} : U.$$

Definition 2.1.10. (1) To any object X of $\mathcal{C}$ we can associate an *endomorphism prop* End_X defined by

$$End_X(m,n) = Hom_{\mathcal{C}}(X^{\otimes m}, X^{\otimes n}).$$

The actions of the symmetric groups are the permutations of the input variables and of the output variables, the horizontal product is the tensor product of homomorphisms and the vertical composition product is the composition of homomorphisms. The units $1_{\mathcal{C}} \to Hom_{\mathcal{C}}(X^{\otimes n}, X^{\otimes n})$ represent $id_{X^{\otimes n}}$.

(2) An *algebra over a prop* P, or P-algebra, is an object $X \in \mathcal{C}$ equipped with a prop morphism $P \to End_X$.

We can also define a P-algebra in a symmetric monoidal category over $\mathcal{C}$:

Definition 2.1.11. Let $\mathcal{E}$ be a symmetric monoidal category over $\mathcal{C}$.

(1) The endomorphism prop of $X \in \mathcal{E}$ is given by $End_X(m,n) = Hom_{\mathcal{E}}(X^{\otimes m}, X^{\otimes n})$ where $Hom_{\mathcal{E}}(-,-)$ is the external hom bifunctor of $\mathcal{E}$.

(2) Let P be a prop in $\mathcal{C}$. A P-algebra in $\mathcal{E}$ is an object $X \in \mathcal{E}$ equipped with a prop morphism $P \to End_X$.

We unwrap the definition in the case of a diagram category over $\mathcal{E}$: let $\{X_i\}_{i\in I}$ be a I-diagram in $\mathcal{E}$, then

$$End_{\{X_i\}_{i\in I}} = \int_{i\in I} Hom_{\mathcal{E}}(X_i^{\otimes m}, X_i^{\otimes n}).$$

This end can equivalently be defined as a coreflexive equalizer

$$End_{\{X_i\}}(m,n) \longrightarrow \prod_{i\in I} Hom_{\mathcal{E}}(X_i^{\otimes m}, X_i^{\otimes n}) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \prod_{u:i\to j\in mor(I)} Hom_{\mathcal{E}}(X_i^{\otimes m}, X_j^{\otimes n}), \quad s_0 : \prod_{u:i\to j\in mor(I)} Hom_{\mathcal{E}}(X_i^{\otimes m}, X_j^{\otimes n}) \to \prod_{i\in I} Hom_{\mathcal{E}}(X_i^{\otimes m}, X_i^{\otimes n})$$

where d_0 is the product of the maps

$$u_* : Hom_{\mathcal{E}}(X_i^{\otimes m}, X_i^{\otimes n}) \to Hom_{\mathcal{E}}(X_i^{\otimes m}, X_j^{\otimes n})$$

induced by the morphisms $u : i \to j$ of I and d_1 is the product of the maps

$$u^* : Hom_{\mathcal{E}}(X_j^{\otimes m}, X_j^{\otimes n}) \to Hom_{\mathcal{E}}(X_i^{\otimes m}, X_j^{\otimes n})$$

The section s_0 is the projection on the factors associated to the identities $id : i \to i$.

This construction is functorial in I: given a J-diagram $\{X_j\}_{j\in J}$, every functor $\alpha : I \to J$ gives rise to a prop morphism $\alpha^* : End_{\{X_j\}_{j\in J}} \to End_{\{X_{\alpha(i)}\}_{i\in I}}$.

2.1.3 The semi-model category of props

Suppose that $\mathcal{C}$ is a cofibrantly generated symmetric monoidal model category. The category of Σ-biobjects $\mathcal{C}^{\mathbb{B}}$ is a diagram category over $\mathcal{C}$, so it inherits a cofibrantly generated model category structure. The weak equivalences and fibrations are defined componentwise. The generating (acyclic) cofibrations are given by $i \otimes \phi_! G_{(m,n)}$, where $(m,n) \in \mathbb{N}^2$ and i ranges over the generating (acyclic) cofibrations of $\mathcal{C}$. Here $\otimes$ is the external tensor product of $\mathcal{C}^{\mathbb{B}}$ and $G_{(m,n)}$ is the double sequence defined by

$$G_{(m,n)}(p,q) = \begin{cases} 1_{\mathcal{C}}, & \text{if } (p,q) = (m,n), \\ 0 & \text{otherwise.} \end{cases}$$

We can also see this result as a transfer of cofibrantly generated model category structure via the adjunction $\phi_! : \mathcal{C}^{\mathbb{A}} \rightleftarrows \mathcal{C}^{\mathbb{B}} : \phi^*$ (this is a propic version of proposition 11.4.A in [9]). The question is to know if the adjunction $F : \mathcal{C}^{\mathbb{B}} \rightleftarrows \mathcal{P} : U$ transfer this model category structure to the props. In the general case it works only with the subcategory $\mathcal{P}_0$ of props with non-empty inputs or outputs and does not give a full model category structure. We give the precise statement in theorem 2.1.13.

Definition 2.1.12. A Σ-biobject M has *non-empty inputs* if it satisfies

$$M(0,n) = \begin{cases} 1_{\mathcal{C}}, & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We define in a symmetric way a Σ-biobject with *non-empty outputs*. The category of Σ-biobjects with non-empty inputs is noted $\mathcal{C}_0^{\mathbb{B}}$.

The composite adjunction

$$\mathcal{C}^{\mathbb{A}} \rightleftarrows \mathcal{C}^{\mathbb{B}} \rightleftarrows \mathcal{P}$$

restricts to an adjunction

$$\mathcal{C}_0^{\mathbb{A}} \rightleftarrows \mathcal{C}_0^{\mathbb{B}} \rightleftarrows \mathcal{P}_0.$$

We define the weak equivalences (respectively fibrations) in $\mathcal{P}_0$ componentwise, i.e their images by the forgetful functor $U : \mathcal{P}_0 \to \mathcal{C}_0^{\mathbb{A}}$ are weak equivalences (respectively fibrations) in $\mathcal{C}_0^{\mathbb{A}}$. We define the generating (acyclic) cofibrations as the images under the free prop functor of the generating (acyclic) cofibrations of $\mathcal{C}_0^{\mathbb{B}}$. We have the following result:

Theorem 2.1.13. *(cf. [8], theorem 4.9) Let $\mathcal{C}$ be a cofibrantly generated symmetric monoidal model category. The category $\mathcal{P}_0$ of props with non-empty inputs (or outputs) equipped with the classes of weak equivalences, fibrations and cofibrations of 1.3 forms a semi-model category. Moreover the forgetful functor $U : \mathcal{P}_0 \to \mathcal{C}_0^{\mathbb{A}}$ preserves cofibrations with cofibrant domain.*

A semi-model category structure is a slightly weakened version of model category structure: the lifting axioms work only with cofibrations with cofibrant domain, and the factorization axioms work only on a map with cofibrant domain (see the relevant section of [8]). The notion of a semi-model category is sufficient to do homotopy theory. In certain categories we recover a full model structure on the whole category of props:

Theorem 2.1.14. *(cf. [8], theorem 5.5) If the base category $\mathcal{C}$ is the category of dg-modules over a ring $\mathbb{K}$ such that $\mathbb{Q} \subset \mathbb{K}$, simplicial modules over a ring, simplicial sets or topological spaces, then the definition of theorem 2.1.13 provides $\mathcal{P}$ with a model category structure.*

2.2 Proof of theorem 2.0.2

The purpose of this section is to establish theorem 2.0.2. We give the details of our arguments in the case $\mathcal{E} = \mathcal{C} = Ch_{\mathbb{K}}$ (the $\mathbb{Z}$-graded chain complexes over a field $\mathbb{K}$ of characteristic zero). We explain briefly afterwards the generalization of these arguments when $\mathcal{E}$ is a cofibrantly generated symmetric monoidal model category over $Ch_{\mathbb{K}}$.

2.2.1 Statement of the result and outline of the proof

In the work of Dwyer-Kan [5], the classification space of a model category $\mathcal{M}$ is the simplicial set $\mathcal{N}(w\mathcal{M})$ where $\mathcal{N}$ is the simplicial nerve functor and $w\mathcal{M}$ is the subcategory of weak equivalences of $\mathcal{M}$. In the case of $\mathcal{E}^P$ (the P-algebras in $\mathcal{E}$ for a prop P defined in $\mathcal{C}$) we use the expression of classification space to refer to the simplicial set $\mathcal{N}w(\mathcal{E}^{cf})^P$, where $w(\mathcal{E}^{cf})^P$ is the subcategory of P-algebra morphisms whose underlying morphisms in $\mathcal{E}$ are weak equivalences between fibrant-cofibrant objects. In the operadic context, algebras over operads satisfy the following useful property: a weak equivalence between two cofibrant operads induces a weak equivalence between their associated classification spaces of algebras. The proof of this result works in two steps: firstly, one show the existence of an adjunction between the two categories of algebras induced by the operads morphism, then one prove that this adjunction forms actually a Quillen equivalence. Such a method fails in the prop setting: there is no free algebra functor, and accordingly a model structure does not exist on the category of algebras over a prop, except in some particular cases such as simplicial sets (see [15]). So the difficult part is to deal with this absence of model structure to get a similar result for algebras over props. Therefore, our method is entirely different from this one. The crux of our proof is given by the following statement:

Theorem 2.2.1. *Let P be a cofibrant prop. The mappings $\mathcal{N}\varphi^*, \mathcal{N}\psi^* : \mathcal{N}w(\mathcal{E}^{cf})^P \rightrightarrows \mathcal{N}w(\mathcal{E}^{cf})^P$ associated to homotopic prop morphisms $\varphi, \psi : P \rightrightarrows P$ are homotopic in sSet.*

Let us outline the main steps of the proof of theorem 2.2.1 in the case $\mathcal{E} = \mathcal{C} = Ch_{\mathbb{K}}$. The idea is to construct a zigzag of natural transformations $\varphi^* \xleftarrow{\sim} Z \xrightarrow{\sim} \psi^*$, where Z is a functorial path object in $Ch_{\mathbb{K}}^P$. We proceed as follows. We use functional notations $\mathcal{Y}(X)$, $\mathcal{Z}(X)$ and $\mathcal{V}(X)$ to refer to diagrams functorially associated to an object X which, in our constructions, ranges within (some subcategory of) Ch_K. We first consider the functorial path object diagram associated to any X in $Ch_{\mathbb{K}}$

$$\mathcal{Y}(X): \quad \xymatrix{ & & X \, . \\ X \ar@/^2pc/[urr]^{=} \ar@/_2pc/[drr]_{=} \ar@{>->}[r]^{\sim}_{s} & Z(X) \ar@{->>}[ur]^{\sim}_{d_0} \ar@{->>}[dr]^{d_1}_{\sim} & \\ & & X }$$

and its subdiagram $\mathcal{Z}(X) = \{X_0 \overset{\sim}{\twoheadleftarrow} Z(X) \overset{\sim}{\twoheadrightarrow} X_1\}$. We prove that the natural P-action existing on the diagram

$$\mathcal{V}(X): \quad \xymatrix{ & X \\ X \ar[ur]_{=} \ar[dr]^{=} & \\ & X }$$

extends to a natural P-action on $\mathcal{Y}(X)$. For this aim, we consider "P-modified endomorphism props", which are built by replacing all the operations $X^{\otimes m} \rightarrow X^{\otimes n}$ in the endomorphism prop of a given diagram by operations of $P(m,n)$. We use notations $End_{\mathcal{Y}(P)}$, $End_{\mathcal{Z}(P)}$ and $End_{\mathcal{V}(P)}$ to refer to these P-modified endomorphism props. We verify that these constructions give rise to props acting naturally on the endomorphism prop of the associated diagram. We use these P-modified endomorphism props to give a P-action on the zigzag of endofunctors $Id \overset{\sim}{\twoheadleftarrow} Z \overset{\sim}{\twoheadrightarrow} Id$. We check that we retrieve the action given by φ and ψ on the extremity of this zigzag. We thus have a zigzag connecting φ^* and ψ^* and yielding the desired homotopy between $\mathcal{N}\varphi^*$ and $\mathcal{N}\psi^*$.

Our argument line is divided in two steps. For every $X \in Ch_{\mathbb{K}}^P$, we have $End_{\mathcal{V}(X)} \cong End_X$ so the morphism $P \rightarrow End_X$ trivially induces a morphism $P \rightarrow \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{V}(X)}$. We use ends to obtain a functorial version of our endomorphism props of diagrams (we refer to MacLane [16] for the definition of ends and coends). In our first step we build a diagram

$$\xymatrix{ & End_{\mathcal{Y}(P)} \ar[r] \ar[d]_{\pi}^{\sim} & \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{Y}(X)} \ar[d] \\ P \ar@{-->}[ur] \ar[r]^{=} & P \ar[r] & \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{V}(X)} }$$

In $Ch_{\mathbb{K}}$, the endomorphism prop $End_{\mathcal{Y}(X)}$ is built via the two following pullbacks:

$$\xymatrix{ End_{\mathcal{Y}(X)} \ar[r] \ar[d] & End_{\mathcal{Z}(X)} \ar[d]^{s^* \circ pr} \\ End_X \ar[r]_{s_*} & Hom_{X,Z(X)} }$$

and

$$\xymatrix{ End_{\mathcal{Z}(X)} \ar[r] \ar[d] & End_{X_0} \times End_{X_1} \ar[d]^{d_0^* \times d_1^*} \\ End_{Z(X)} \ar[r]_{(d_0,d_1)_*} & Hom_{Z(X),X_0} \times Hom_{Z(X),X_1} }$$

where s_* and $(d_0,d_1)_*$ are maps induced by the composition by s and (d_0,d_1), and s^*, d_0^*, d_1^* are maps induced by the precomposition by s, d_0 and d_1 . The projection $pr : End_{\mathcal{Z}(X)} \rightarrow End_{Z(X)}$ is induced by the inclusion of diagrams $\{Z(X)\} \hookrightarrow \{X_0 \overset{\sim}{\twoheadleftarrow} Z(X) \overset{\sim}{\twoheadrightarrow} X_1\}$ (see [8], section 8). The idea is to define a P-modified endomorphism prop $End_{\mathcal{Y}(P)}$ with a form similar to that of $End_{\mathcal{Y}(X)}$, in order to get the prop morphism $End_{\mathcal{Y}(P)} \rightarrow \int_{X\in Ch_{\mathbb{K}}^P} End_{\mathcal{Y}(X)}$ induced by the morphisms $P \rightarrow End_X$, $X \in Ch_{\mathbb{K}}^P$. For this aim we use two pullbacks similar to those above with P-modified endomorphisms props and Σ-biobjects replacing the usual ones.

In our second step, we show that π is an acyclic fibration in $\mathcal{P}$ in order to obtain the desired lifting $P \rightarrow \int_{X\in Ch_{\mathbb{K}}^P} End_{\mathcal{Y}(X)}$, which respects the P-algebra structures on the diagrams $\mathcal{V}(X)$ for every $X \in Ch_{\mathbb{K}}^P$. It endows the category of P-algebras with a functorial path object. Finally, we prove theorem 2.2.1 in section 2.6, by using lifting properties in the category of props and providing the desired zigzag of natural transformations $\varphi^* \overset{\sim}{\leftarrow} Z \overset{\sim}{\rightarrow} \psi^*$. Then we show how to deduce theorem 2.0.2.

Remark 2.2.2. We can also wonder about the homotopy invariance of the classification space up to Quillen equivalences. Let P be a prop in $\mathcal{E}_1$. Let $F : \mathcal{E}_1 \rightleftarrows \mathcal{E}_2 : G$ be a symmetric monoidal adjunction. The prop $F(P)$ is defined by applying the functor F entrywise to P: the fact that F is symmetric monoidal ensures the preservation of the composition products of P, giving to $F(P)$ a prop structure. Lemma 7.1 of [15] says that the adjoint pair (F,G) induces an adjunction $\overline{F} : \mathcal{E}_1^P \rightleftarrows \mathcal{E}_2^{F(P)} : \overline{G}$. Now suppose that (F,G) forms a Quillen adjunction. By Brown's lemma, the functor F preserves weak equivalences between cofibrant objects and the functor G preserves weak equivalences between fibrant objects. If all the objects of $\mathcal{E}_1$ and $\mathcal{E}_2$ are fibrant and cofibrant, then the adjoint pair $(\overline{F},\overline{G})$ restricts to an adjunction $\overline{F} : w(\mathcal{E}_1)^P \rightleftarrows w(\mathcal{E}_2)^{F(P)} : \overline{G}$ and thus gives rise to a homotopy equivalence $\mathcal{N}w(\mathcal{E}_1)^P \sim \mathcal{N}w(\mathcal{E}_2)^{F(P)}$.

2.2.2 The path object $Z(X) = Z \otimes X$

Recall that in the model category structure of $Ch_{\mathbb{K}}$, the fibrations are the degreewise surjections, the cofibrations the degreewise injections and the weak equivalences the morphisms inducing isomorphisms in homology. The category $Ch_{\mathbb{K}}$ has moreover the simplifying feature that the product and the coproduct coincide (it is an abelian category). Let Z be the chain complex defined by

$$Z = \mathbb{K}\rho_0 \oplus \mathbb{K}\rho_1 \oplus \mathbb{K}\sigma_0 \oplus \mathbb{K}\sigma_1 \oplus \mathbb{K}\tau.$$

The elements τ, ρ_0 and ρ_1 are three generators of degree 0 and σ_0, σ_1 two generators of degree -1. The differential d_Z is defined by $d_Z(\sigma_0) = d_Z(\sigma_1) = 0$, $d_Z(\tau) = 0$, $d_Z(\rho_0) = \sigma_0$ and $d_Z(\rho_1) = \sigma_1$.

Lemma 2.2.3. *The chain complex $Z \otimes X$ defines a path object on X in $Ch_{\mathbb{K}}$, fitting in a factorization $X \overset{\sim}{\rightarrowtail}_s Z \otimes X \twoheadrightarrow_{(d_0,d_1)} X \oplus X$ of the diagonal $\Delta = (id_X, id_X) : X \rightarrow X \oplus X$ such that s is an acyclic cofibration and (d_0,d_1) a fibration.*

Proof. Let $s : X \rightarrow Z \otimes X$ be the map defined by $s(x) = \tau \otimes x$. Given the differential of Z, the map s is clearly an injective morphism of $Ch_{\mathbb{K}}$, i.e a cofibration. We can also write $Z \otimes X \cong (\tilde{Z} \otimes X) \oplus X$ where

$$\tilde{Z} = \mathbb{K}\rho_0 \oplus \mathbb{K}\rho_1 \oplus \mathbb{K}\sigma_0 \oplus \mathbb{K}\sigma_1$$

is an acyclic complex. The acyclicity of $\tilde{Z}$ implies that s is an acyclic cofibration. We now define a map $(d_0,d_1) : Z \otimes X \twoheadrightarrow X \oplus X$ such that $(d_0,d_1) \circ s = (id_X, id_X)$ and (d_0,d_1) is a fibration. The map d_0 is determined for every $x \in X$ by $d_0(\tau \otimes x) = x$ and $d_0(\sigma_0 \otimes x) = d_0(\sigma_1 \otimes x) = d_0(\rho_0 \otimes x) = d_0(\rho_1 \otimes x) = 0$. The map d_1 is determined for every $x \in X$ by $d_1(\rho_0 \otimes x) = x$, $d_1(\tau \otimes x) = x$, $d_1(\sigma_0 \otimes x) = d_1(\sigma_1 \otimes x) = d_1(\rho_1 \otimes x) = 0$. The map (d_0,d_1) is clearly a surjective chain complexes morphism, i.e a fibration, and satisfies the equality $(d_0,d_1) \circ s = (id_X, id_X)$. ☐

The two advantages of this path object on X are its writing under the form of a tensor product with X and its decomposition in a direct sum of X with an acyclic complex.

2.2.3 The prop $End_{Z(P)}$

Consider the endomorphism prop of $Z(X)$:

$$\begin{aligned} End_{Z(X)}(m,n) &= Hom_{Ch_{\mathbb{K}}}(Z(X)^{\otimes m}, Z(X)^{\otimes n}) \\ &\cong Hom_{Ch_{\mathbb{K}}}(Z^{\otimes m}\otimes X^{\otimes m}, Z^{\otimes n}\otimes X^{\otimes n}) \\ &\cong (Z^{\otimes m})^{*}\otimes Z^{\otimes n}\otimes End_X(m,n). \end{aligned}$$

We define a P-modified endomorphism prop such that

$$\begin{aligned} End_{Z(P)}(m,n) &= (Z^{\otimes m})^{*}\otimes Z^{\otimes n}\otimes P(m,n) \\ &= \bigoplus t_1^*\otimes ...\otimes t_m^*\otimes t_1\otimes ...\otimes t_n\otimes P(m,n), \end{aligned}$$

where $t_i\in\{\rho_0,\rho_1,\sigma_0,\sigma_1,\tau\}$, together with the following structure maps:

-Vertical composition product. Let

$$\alpha\in t_1^*\otimes ...\otimes t_k^*\otimes t_1\otimes ...\otimes t_n\otimes P(k,n)$$

and

$$\beta\in u_1^*\otimes ...\otimes u_m^*\otimes u_1\otimes ...\otimes u_k\otimes P(m,k).$$

We set

$$\alpha\circ_v\beta=\begin{cases}\alpha\circ_v^P\beta & \text{if } (u_1,...,u_k)=(t_1,...,t_k),\\ 0 & \text{otherwise,}\end{cases}$$

where $\circ_v^P$ is the vertical composition product of P.

-Horizontal product. Let

$$\alpha\in t_1^*\otimes ...\otimes t_{m_1}^*\otimes t_1\otimes ...\otimes t_{n_1}\otimes P(m_1,n_1)$$

and

$$\beta\in u_1^*\otimes ...\otimes u_{m_2}^*\otimes u_1\otimes ...\otimes u_{n_2}\otimes P(m_2,n_2).$$

We set

$$\begin{aligned} \alpha\circ_h\beta &= t_1^*\otimes ...\otimes t_{m_1}^*\otimes u_1^*\otimes ...\otimes u_{m_2}^* \\ &\quad \otimes t_1\otimes ...\otimes t_{n_1}\otimes u_1\otimes ...\otimes u_{n_2}\otimes(\alpha\mid_{P(m_1,n_1)}\circ_h^P\beta\mid_{P(m_2,n_2)}) \\ &\in t_1^*\otimes ...\otimes t_{m_1}^*\otimes u_1^*\otimes ...\otimes u_{m_2}^* \\ &\quad \otimes t_1\otimes ...\otimes t_{n_1}\otimes u_1\otimes ...\otimes u_{n_2}\otimes P(m_1+n_1,m_2+n_2), \end{aligned}$$

where $\circ_h^P$ is the horizontal product of P.

-Actions of the symmetric groups. Let $\alpha=t_1^*\otimes ...\otimes t_m^*\otimes t_1\otimes ...\otimes t_n\otimes\alpha_P\in End_{Z(P)}(m,n)$ with $\alpha_P\in P(m,n)$. The action of a permutation $\sigma\in\Sigma_m$ on the right of this prop element is given by $\alpha.\sigma=t_{\sigma(1)}^*\otimes ...\otimes t_{\sigma(m)}^*\otimes t_1\otimes ...\otimes t_n\otimes\alpha_P.\sigma$. The action of a permutation $\tau\in\Sigma_n$ on the left of this prop element is given by $\tau.\alpha=t_1^*\otimes ...\otimes t_m^*\otimes t_{\tau^{-1}(1)}\otimes ...\otimes t_{\tau^{-1}(n)}\otimes\tau.\alpha_P$.

Let $X\in Ch_{\mathbb{K}}^P$ be a P-algebra. From the definition of $End_{Z(P)}(m,n)$, we easily see that the prop morphism $P\rightarrow End_X$ induces a prop morphism

$$End_{Z(P)}\rightarrow\int_{X\in Ch_{\mathbb{K}}^P}End_{Z(X)}.$$

2.2.4 The prop $End_{\mathcal{Z}(P)}$

The pullback defining $End_{\mathcal{Z}(X)}$ and its explicit maps

For every $(m,n)\in\mathbb{N}^2$, we have a pullback

$$\begin{array}{ccc} End_{\mathcal{Z}(X)}(m,n) & \longrightarrow & End_{X_0}(m,n)\oplus End_{X_1}(m,n) \\ \downarrow & & \downarrow{\scriptstyle (d_0^{\otimes m})^*\oplus(d_1^{\otimes m})^*} \\ End_{Z(X)}(m,n) & \xrightarrow[(d_0^{\otimes n},d_1^{\otimes n})_*]{} & Hom_{Z(X),X_0}(m,n)\oplus Hom_{Z(X),X_1}(m,n) \end{array}.$$

For every $X\in Ch_{\mathbb{K}}^P$ and $(m,n)\in\mathbb{N}^2$ we have the isomorphisms

$$\begin{aligned} Hom_{X,Z(X)}(m,n) &= Hom_{Ch_{\mathbb{K}}}(X^{\otimes m},Z(X)^{\otimes n}) \\ &\cong Hom_{Ch_{\mathbb{K}}}(X^{\otimes m},Z^{\otimes n}\otimes X^{\otimes n}) \\ &\cong Z^{\otimes n}\otimes End_X(m,n) \end{aligned}$$

and

$$\begin{aligned} Hom_{Z(X),X_i}(m,n) &= Hom_{Ch_{\mathbb{K}}}(Z(X)^{\otimes m},X^{\otimes n}) \\ &\cong Hom_{Ch_{\mathbb{K}}}(Z^{\otimes m}\otimes X^{\otimes m},X^{\otimes n}) \\ &\cong (Z^{\otimes m})^*\otimes End_{X_i}(m,n). \end{aligned}$$

Applying these isomorphisms, we get a pullback

$$\begin{array}{ccc} End_{\mathcal{Z}(X)}(m,n) & \longrightarrow & End_{X_0}(m,n)\oplus End_{X_1}(m,n) \\ \downarrow & & \downarrow{\scriptstyle \overline{(d_0^{\otimes m})^*}\oplus\overline{(d_1^{\otimes m})^*}} \\ (Z^{\otimes m})^*\otimes Z^{\otimes n}\otimes End_X(m,n) & \xrightarrow[\overline{(d_0^{\otimes n},d_1^{\otimes n})_*}]{} & (Z^{\otimes m})^*\otimes End_{X_0}(m,n)\oplus(Z^{\otimes m})^*\otimes End_{X_1}(m,n) \end{array}.$$

We have to make explicit the maps $\overline{(d_0^{\otimes n},d_1^{\otimes n})_*}$ and $\overline{(d_0^{\otimes m})^*}\oplus\overline{(d_1^{\otimes m})^*}$ and replace $End_{X_0}(m,n)$, $End_{X_1}(m,n)$ and $End_X(m,n)$ by $P_0(m,n)$, $P_1(m,n)$ and $P(m,n)$ to obtain a P-modified endomorphism prop $\{End_{\mathcal{Z}(P)}(m,n)\}_{(m,n)\in\mathbb{N}^2}$ acting naturally on $End_{\mathcal{Z}(X)}(m,n)$, $X\in Ch_{\mathbb{K}}^P$. Then we apply the same method to build a P-modified endomorphism prop $End_{\mathcal{Y}(P)}$ acting naturally on $End_{\mathcal{Y}(X)}$, $X\in Ch_{\mathbb{K}}^P$.

Lemma 2.2.4. *Let $\{\underline{z}_i\}_{i\in I}$ be a basis of $Z^{\otimes m}$. The map*

$$\overline{(d_1^{\otimes m})^*}:End_X(m,n)\rightarrow(Z^{\otimes m})^*\otimes End_X(m,n)$$

is defined by the formula

$$\overline{(d_1^{\otimes m})^*}(\xi)=\sum_{j\in J}(\underline{z}_j^*\otimes\xi)=(\sum_{j\in J}\underline{z}_j^*)\otimes\xi,$$

where J is the subset of I such that $d_1^{\otimes m}(\underline{z}_j\otimes\underline{x})=\underline{x}$ for $\underline{x}\in X^{\otimes m}$ and $j\in J$.

Proof. First we give an explicit inverse to the well known isomorphism

$$\begin{aligned} \lambda:U^*\otimes Hom_{Ch_{\mathbb{K}}}(V,V') &\stackrel{\cong}{\rightarrow} Hom_{Ch_{\mathbb{K}}}(U\otimes V,V') \\ \varphi\otimes f &\mapsto [u\otimes v\mapsto\varphi(u).f(v)] \end{aligned}$$

where U is supposed to be of finite dimension. Let $\{u_i\}_i \in I$ be a basis of U. We have $\lambda = \sum_{i\in I} \lambda_i$ where

$$\begin{aligned}\lambda_i : \mathbb{K}u_i^* \otimes Hom_{Ch_{\mathbb{K}}}(V,V') &\to Hom_{Ch_{\mathbb{K}}}(\mathbb{K}u_i \otimes V, V')\\ u_i^* \otimes f &\mapsto u_i^*.f : u_i \otimes v \mapsto u_i^*(u_i).f(v) = f(v)\end{aligned}$$

so

$$\begin{aligned}\lambda^{-1} : Hom_{Ch_{\mathbb{K}}}(U\otimes V, V') &\to U^* \otimes Hom_{Ch_{\mathbb{K}}}(V,V')\\ f &\mapsto \sum_{i\in I}(u_i^* \otimes f \mid_{\mathbb{K}u_i\otimes V}).\end{aligned}$$

Let $\sigma : Z^{\otimes m} \otimes X^{\otimes m} \to (Z\otimes X)^{\otimes m}$ be the map permuting the variables. Recall that the map d_1 is determined for every $x \in X$ by $d_1(\rho_0 \otimes x) = x$, $d_1(\tau\otimes x) = x$, $d_1(\sigma_0 \otimes x) = d_1(\sigma_1\otimes x) = d_1(\rho_1 \otimes x) = 0$. The map

$$\overline{(d_1^{\otimes m})^*} : Hom_{Ch_{\mathbb{K}}}(X^{\otimes m}, X^{\otimes n}) \to Hom_{Ch_{\mathbb{K}}}(Z^{\otimes m}\otimes X^{\otimes m}, X^{\otimes n}) \xrightarrow{\cong} (Z^{\otimes m})^* \otimes Hom_{Ch_{\mathbb{K}}}(X^{\otimes m}, X^{\otimes n})$$

is defined by

$$\xi \mapsto \xi \circ d_1^{\otimes m}\circ\sigma \mapsto \sum_{i\in I}(\underline{z}_i^* \otimes (\xi\circ d_1^{\otimes m}\circ \sigma)\mid_{\mathbb{K}\underline{z}_i\otimes V}).$$

We obtain finally

$$\begin{aligned}\overline{(d_1^{\otimes m})^*} : End_X(m,n) &\to (Z^{\otimes m})^* \otimes End_X(m,n)\\ \xi &\mapsto \sum_{j\in J}(\underline{z}_j^*\otimes\xi) = (\sum_{j\in J}\underline{z}_j^*)\otimes\xi\end{aligned}$$

where J is the subset of I such that $d_1^{\otimes m}(\underline{z}_j \otimes \underline{x}) = \underline{x}$ for $\underline{x} \in X^{\otimes m}$ and $j \in J$. If $j \notin J$ then $d_1^{\otimes m}\mid_{\mathbb{K}\underline{z}_j\otimes X^{\otimes m}} = 0$. □

Recall that the map $d_0 : Z\otimes X \to X$ is defined for every $x \in X$ by $d_0(\tau\otimes x) = x$ and $d_0(\sigma_0\otimes x) = d_0(\sigma_1 \otimes x) = d_0(\rho_0\otimes x) = d_0(\rho_1\otimes x) = 0$. As previously, the map $\overline{(d_0^{\otimes m})^*}$ has a form similar to that of $\overline{(d_1^{\otimes m})^*}$, and we have determined $\overline{(d_0^{\otimes m})^*} \oplus \overline{(d_1^{\otimes m})^*}$.

Lemma 2.2.5. *The map* $\overline{(d_0^{\otimes n}, d_1^{\otimes n})_*}$ *is determined by*

$$\overline{(d_0^{\otimes n}, d_1^{\otimes n})_*} : \underline{z}_j^* \otimes \underline{z}_i' \otimes \xi \mapsto \sum_{k\in I}(\underline{z}_k^* \otimes ((d_0^{\otimes n}, d_1^{\otimes n})\circ \underline{z}_j^*(-).\underline{z}_i'\otimes\xi)\mid_{\mathbb{K}\underline{z}_k\otimes X^{\otimes m}}).$$

Proof. Let $\{\underline{z}_i'\}_{i\in I'}$ be the basis of $Z^{\otimes n}$. We have the isomorphism

$$\begin{aligned}(Z^{\otimes m})^*\otimes Z^{\otimes n}\otimes Hom_{Ch_{\mathbb{K}}}(X^{\otimes m}, X^{\otimes n}) &\to Hom_{Ch_{\mathbb{K}}}(Z^{\otimes m}\otimes X^{\otimes m}, Z^{\otimes n}\otimes X^{\otimes n})\\ \underline{z}_j^*\otimes\underline{z}_i'\otimes\xi &\mapsto \underline{z}_j^*(-).\underline{z}_i'\otimes\xi\end{aligned}$$

that we compose with

$$\begin{aligned}(d_0^{\otimes n}, d_1^{\otimes n}) : Z^{\otimes n}\otimes X^{\otimes n} &\to X_0^{\otimes n}\oplus X_1^{\otimes n}\\ \underline{z}_j\otimes x &\mapsto \begin{cases} x\oplus x & \text{if } j\in J',\\ x\oplus 0 \textit{ or } 0\oplus x & \text{otherwise,}\end{cases}\end{aligned}$$

where J' is the subset of I such that $d_0\mid_{\mathbb{K}\underline{z}_j\otimes X^{\otimes n}} \neq 0$ and $d_1\mid_{\mathbb{K}\underline{z}_j\otimes X^{\otimes n}}\neq 0$ for $j\in J'$. Finally we compose with the isomorphism

$$\begin{aligned}Hom_{Ch_{\mathbb{K}}}(Z^{\otimes m}\otimes X^{\otimes m}, X_0^{\otimes n}\oplus X_1^{\otimes n}) &\xrightarrow{\cong} (Z^{\otimes m})^*\otimes Hom_{Ch_{\mathbb{K}}}(X^{\otimes m}, X_0^{\otimes n}\oplus X_1^{\otimes n})\\ f &\mapsto \sum_{i\in I}(\underline{z}_i^*\otimes f\mid_{Kz_i\otimes X^{\otimes m}})\end{aligned}$$

and get the map

$$\overline{(d_0^{\otimes n}, d_1^{\otimes n})_*} : \underline{z}_j^* \otimes \underline{z}_i' \otimes \xi \mapsto \sum_{k \in I} (\underline{z}_k^* \otimes ((d_0^{\otimes n}, d_1^{\otimes n}) \circ \underline{z}_j^*(-).\underline{z}_i' \otimes \xi) \mid_{\mathbb{K}\underline{z}_k \otimes X^{\otimes m}}).$$

□

The associated P-modified prop

The key observation is that these two maps $\overline{(d_0^{\otimes m})^*} \oplus \overline{(d_1^{\otimes m})^*}$ and $\overline{(d_0^{\otimes n}, d_1^{\otimes n})_*}$, fixing the prop structure on $End_{\mathcal{Z}(X)}(m,n)$ in function of those of $(Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes End_X(m,n)$ and $End_{X_0}(m,n) \oplus End_{X_1}(m,n)$, do not modify the operations $\xi \in End_X(m,n)$ themselves. Therefore, we replace $End_{X_0}(m,n)$, $End_{X_1}(m,n)$ and $End_X(m,n)$ by $P_0(m,n)$, $P_1(m,n)$ and $P(m,n)$ to get this new pullback

$$\begin{array}{ccc}
End_{\mathcal{Z}(P)}(m,n) & \longrightarrow & P_0(m,n) \oplus P_1(m,n) \\
\downarrow & & \downarrow {\scriptstyle \overline{(d_0^{\otimes m})^*} \oplus \overline{(d_1^{\otimes m})^*}} \\
(Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes P(m,n) & \xrightarrow[\overline{(d_0^{\otimes n}, d_1^{\otimes n})_*}]{} & (Z^{\otimes m})^* \otimes P_0(m,n) \oplus (Z^{\otimes m})^* \otimes P_1(m,n)
\end{array}.$$

The explicit formulae of the applications defining this pullback, given by lemmas 2.4 and 2.5, show that these replacements do not break the prop structure transfer. Thus we get the desired P-modified endomorphism prop $End_{\mathcal{Z}(P)}$ having the same shape as that of $End_{\mathcal{Z}(X)}$ and thus acting naturally on the associated diagram of P-algebras:

$$End_{\mathcal{Z}(P)} \to \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{Z}(X)}.$$

2.2.5 The prop $End_{\mathcal{Y}(P)}$ and the functorial path object in P-algebras

Now let us define $End_{\mathcal{Y}(P)}$. For every $(m,n) \in \mathbb{N}^2$, the pullback

$$\begin{array}{ccc}
End_{\mathcal{Y}(X)}(m,n) & \longrightarrow & End_{\mathcal{Z}(X)}(m,n) \\
\downarrow & & \downarrow {\scriptstyle (s^{\otimes m})^* \circ pr} \\
End_X(m,n) & \xrightarrow[(s^{\otimes n})_*]{} & Hom_{X,Z(X)}(m,n)
\end{array}$$

induces via the isomorphims explained at the beginning of 3.3 and 3.4.1 a pullback

$$\begin{array}{ccc}
End_{\mathcal{Y}(X)}(m,n) & \longrightarrow & End_{\mathcal{Z}(X)}(m,n) \\
\downarrow & & \downarrow {\scriptstyle \overline{(s^{\otimes m})^* \circ pr}} \\
End_X(m,n) & \xrightarrow[\overline{(s^{\otimes n})_*}]{} & Z^{\otimes n} \otimes End_X(m,n)
\end{array}.$$

In the same manner as before, given that $s : X \to Z \otimes X$ sends every $x \in X$ to $\tau \otimes x$, the map $\overline{(s^{\otimes m})^*}$ is of the form

$$\begin{aligned}
(Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes End_X(m,n) &\to Z^{\otimes n} \otimes End_X(m,n) \\
\underline{z}_j^* \otimes \underline{z}_i' \otimes \xi &\mapsto \begin{cases} \underline{z}_i' \otimes \xi & \text{if } j \in K, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

where K is a certain subset of I and $\overline{(s^{\otimes n})_*}$ is of the form

$$\begin{aligned} End_X(m,n) &\to Z^{\otimes n} \otimes End_X(m,n) \\ \xi &\mapsto \sum_{i \in K'} \underline{z}'_i \otimes \xi \end{aligned}$$

where K' is a certain subset of I'. These two maps $\overline{(s^{\otimes m})^* \circ pr}$ and $\overline{(s^{\otimes n})_*}$, fixing the prop structure on $End_{\mathcal{Y}(X)}(m,n)$ in function of those of $End_{\mathcal{Z}(X)}(m,n)$ and $End_X(m,n)$, do not modify the operations $\xi \in End_X(m,n)$ themselves. Therefore, we replace $End_X(m,n)$ by $P(m,n)$ and $End_{\mathcal{Z}(X)}(m,n)$ by $End_{\mathcal{Z}(P)}(m,n)$ to get this new pullback

$$\begin{array}{ccc} End_{\mathcal{Y}(P)}(m,n) & \longrightarrow & End_{\mathcal{Z}(P)}(m,n) \\ \downarrow & & \downarrow{\scriptstyle \overline{(s^{\otimes m})^* \circ pr}} \\ P(m,n) & \xrightarrow[\overline{(s^{\otimes n})_*}]{} & Z^{\otimes n} \otimes P(m,n) \end{array} .$$

The explicit formulae of the applications defining this pullback show that these replacements do not break the prop structure transfer. Thus we get the desired P-modified endomorphism prop $End_{\mathcal{Y}(P)}$ having the same shape as that of $End_{\mathcal{Y}(X)}$ and thus acting naturally on the associated diagram of P-algebras:

$$End_{\mathcal{Y}(P)} \to \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{Y}(X)}.$$

We finally obtain the following lemma:

Lemma 2.2.6. *There is a commutative diagram of props*

$$\begin{array}{ccccc} & & End_{\mathcal{Y}(P)} & \longrightarrow & \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{Y}(X)} \\ & & {\scriptstyle \pi}\downarrow & & \downarrow \\ P & \xrightarrow{=} & P & \longrightarrow & \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{V}(X)} \end{array}$$

Now we want to prove that the morphism $P \to \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{V}(X)}$ lifts to a morphism $P \to \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{Y}(X)}$:

Lemma 2.2.7. *The map π is an acyclic fibration in the category of props.*

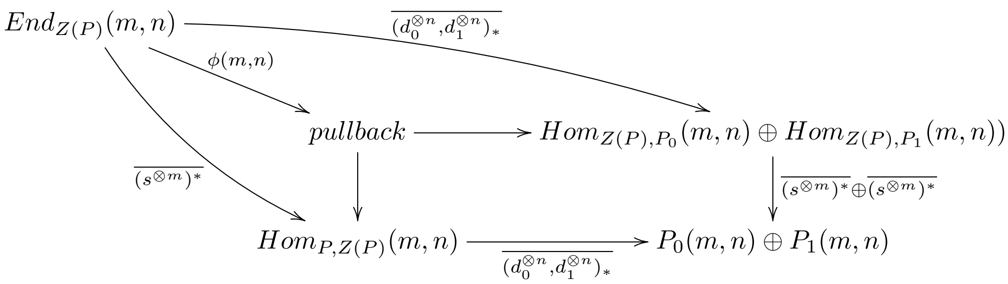
Proof. According to the model category structure on $\mathcal{P}$, it is sufficient to prove that for every $(m,n) \in \mathbb{N}^2$, $\pi(m,n)$ is an acyclic fibration of chain complexes. The map $\pi(m,n)$ is given by the base extension

$$\pi(m,n) = P(m,n) \underset{Hom_{P,Z(P)}(m,n)}{\times} \phi(m,n) \underset{Hom_{Z(P),P_0}(m,n) \oplus Hom_{Z(P),P_1}(m,n)}{\times} (P_0(m,n) \oplus P_1(m,n))$$

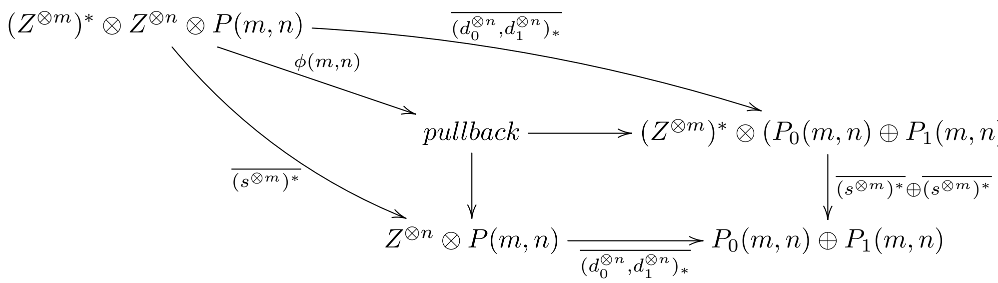
where

$$\phi(m,n) : End_{Z(P)}(m,n) \to Hom_{P,Z(P)}(m,n) \underset{P_0(m,n) \oplus P_1(m,n)}{\times} (Hom_{Z(P),P_0}(m,n) \oplus Hom_{Z(P),P_1}(m,n))$$

comes from the diagram



i.e



.

We have an isomorphism

$$\begin{aligned} P_0(m,n) \oplus P_1(m,n) &\stackrel{\cong}{\to} (\mathbb{K}p_0 \oplus \mathbb{K}p_1) \otimes P(m,n) \\ p \oplus p' &\mapsto p_0 \otimes p + p_1 \otimes p \end{aligned}$$

where p_0 and p_1 are two generators of degree 0. The previous computations give

$$\begin{aligned} \overline{(d_0^{\otimes n}, d_1^{\otimes n})_*} : Z^{\otimes n} \otimes P(m,n) &\to (\mathbb{K}p_0 \oplus \mathbb{K}p_1) \otimes P(m,n) \\ \underline{z}'_i \otimes p &\mapsto \begin{cases} (p_0 \oplus p_1) \otimes p & \text{if } i \in J', \\ p_0 \otimes p \text{ } or \text{ } p_1 \otimes p & \text{otherwise,} \end{cases} \end{aligned}$$

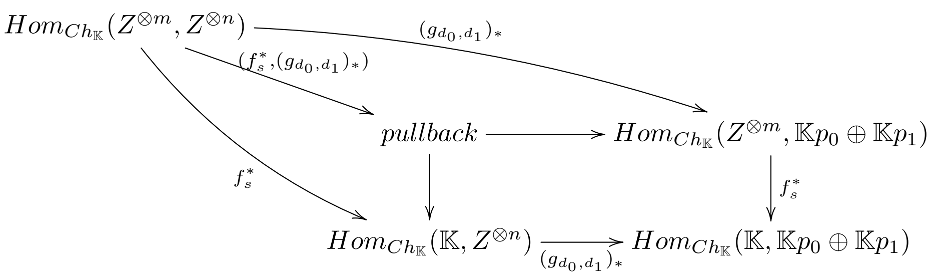
and the map

$$\overline{(s^{\otimes m})^*} \oplus \overline{(s^{\otimes m})^*} : (Z^{\otimes m})^* \otimes (\mathbb{K}p_0 \oplus \mathbb{K}p_1) \otimes P(m,n) \to (\mathbb{K}p_0 \oplus \mathbb{K}p_1) \otimes P(m,n)$$

is defined by

$$\underline{z}_j^* \otimes (\lambda p_0 \oplus \mu p_1) \otimes p \mapsto \begin{cases} (\lambda p_0 \oplus \mu p_1) \otimes p \text{ } or \text{ } \lambda p_0 \otimes p \text{ } or \text{ } \mu p_1 \otimes p, & \text{if } j \in K, \\ 0 = 0 \otimes p & \text{otherwise.} \end{cases}$$

We have similar results for the two maps starting from $(Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes P(m,n)$. We deduce that the previous diagram is the image under the functor $- \otimes P(m,n)$ of the dual pushout-product



modulo the isomorphisms

$$Z^{\otimes n} \cong Hom_{Ch_{\mathbb{K}}}(\mathbb{K}, Z^{\otimes n}),$$

$$(Z^{\otimes m})^* \otimes Z^{\otimes n} \cong Hom_{Ch_{\mathbb{K}}}(Z^{\otimes m}, Z^{\otimes n}),$$

$$(Z^{\otimes m})^* \otimes (\mathbb{K}p_0 \oplus \mathbb{K}p_1) \cong Hom_{Ch_{\mathbb{K}}}(Z^{\otimes m}, \mathbb{K}p_0 \oplus \mathbb{K}p_1)$$

and

$$\mathbb{K}p_0 \oplus \mathbb{K}p_1 \cong Hom_{Ch_{\mathbb{K}}}(\mathbb{K}, \mathbb{K}p_0 \oplus \mathbb{K}p_1).$$

The map $g_{d_0,d_1} : Z^{\otimes n} \to Kp_0 \oplus Kp_1$ is surjective so it is a fibration of chain complexes. Recall that we have a decomposition of Z into $Z = \widetilde{Z} \oplus \mathbb{K}\tau$ where $\widetilde{Z}$ is acyclic, which implies a decomposition of $Z^{\otimes m}$ of the form $Z^{\otimes m} \cong S_m \oplus \mathbb{K}(\tau^{\otimes n})$ where S_m is acyclic because it is a sum of tensor products containing $\widetilde{Z}$. The map f_s is an injection sending $\mathbb{K}$ on $\mathbb{K}(\tau^{\otimes n})$ so it is a cofibration, and S_m is acyclic so f_s is an acyclic cofibration. Applying the axiom MM1' in $Ch_{\mathbb{K}}$ we conclude that $(f_s^*, (g_{d_0,d_1})_*)$ is an acyclic fibration. Therefore $\phi(m,n) = (f_s^*, (g_{d_0,d_1})_*) \otimes id_{P(m,n)}$ is an acyclic fibration, and so is $\pi(m,n)$, given that the class of acyclic fibrations is stable by base extension. □

We have proved the following result:

Proposition 2.2.8. *There is a prop morphism $P \to \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{Y}(X)}$, and consequently a functorial path object $Z : (Ch_{\mathbb{K}})^P \to (Ch_{\mathbb{K}})^P$ in the category of cofibrant-fibrant P-algebras $(Ch_{\mathbb{K}})^P$.*

2.2.6 Proof of the final result

Consider now the square of inclusions of diagrams

$$\begin{array}{ccc} \mathcal{T}(X) & \xhookrightarrow{t} & \mathcal{V}(X) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ \mathcal{Z}(X) & \xhookrightarrow[w]{} & \mathcal{Y}(X) \end{array}$$

where $\mathcal{V}(X)$, $\mathcal{Z}(X)$ and $\mathcal{Y}(X)$ are the diagrams defined previously and $\mathcal{T}(X)$ is the diagram $\{X_0, X_1\}$ consisting of two copies of X and no arrows between them. This square of inclusions induces the following commutative square of endomorphisms props

$$\begin{array}{ccc} End_{\mathcal{Y}(X)} & \xrightarrow{w_*} & End_{\mathcal{Z}(X)} \\ {\scriptstyle v_*}\downarrow & & \downarrow{\scriptstyle u^*} \\ End_{\mathcal{V}(X)} & \xrightarrow[t^*]{} & End_{\mathcal{T}(X)} \end{array}$$

where u^*, v^*, t^* and w^* are the maps induced by the inclusions of the associated diagrams P-algebras. We have a commutative diagram of P-modified endomorphism props reflecting this square

$$\begin{array}{ccc} End_{\mathcal{Y}(P)} & \xrightarrow{w_*} & End_{\mathcal{Z}(P)} \\ {\scriptstyle v_*}\downarrow & & \downarrow{\scriptstyle u^*} \\ End_{\mathcal{V}(P)} = P & \xrightarrow[t^*]{} & End_{\mathcal{T}(P)} = P_0 \times P_1 \end{array}$$

where v^* is the acyclic fibration π of lemma 2.2.7 and u^* is a fibration because it is clearly surjective in each biarity (recall that the surjective morphisms are the fibrations of $Ch_{\mathbb{K}}$ and that the fibrations of $\mathcal{P}$ are determined componentwise). Now we can use this commutative square to prove the final result:

Theorem 2.2.9. *Let P be a cofibrant prop and $\varphi, \psi : P \to P$ two homotopic prop morphisms, then there exists a diagram of functors*

$$\varphi^* \overset{\sim}{\twoheadleftarrow} Z \overset{\sim}{\twoheadrightarrow} \psi^*$$

where Z is the path object functor defined in proposition 2.2.8 and the natural transformations are pointwise acyclic fibrations.

Proof. We consider a cylinder object of P fitting in a diagram of the form:

$$P \vee P \xrightarrow[(d_0,d_1)]{} \widetilde{P} \overset{\sim}{\underset{s_0}{\twoheadrightarrow}} P$$

The components d_0 and d_1 of the morphism (d_0, d_1) are acyclic cofibrations because P is cofibrant by assumption(see lemma 4.4 in [4]) and s_0 an acyclic fibration. The fact that φ and ψ are homotopic implies the existence of a lifting in

$$\begin{array}{ccc} P \vee P & \xrightarrow{(\varphi,\psi)} & P \\ {\scriptstyle (d_0,d_1)}\downarrow & \overset{h}{\nearrow} & \downarrow \\ \widetilde{P} & \longrightarrow & 0 \end{array}$$

We produce the lifting

$$\begin{array}{ccc} I & \longrightarrow & End_{\mathcal{Y}(P)} \\ \downarrow & \overset{k}{\nearrow} & \sim\downarrow{\scriptstyle v^*} \\ P & \xrightarrow[\varphi]{} & P \end{array}$$

(by the axiom MC4 of model categories, see [4]) and form $(\varphi \circ s_0, h) : \widetilde{P} \to P_0 \times P_1$ in order to get the following commutative diagram:

$$\begin{array}{ccccc} P & \xrightarrow{k} & End_{\mathcal{Y}(P)} & \xrightarrow{w^*} & End_{\mathcal{Z}(P)} \\ {\scriptstyle d_0}\downarrow\sim & & \overset{l}{\nearrow} & & \downarrow{\scriptstyle u^*} \\ \widetilde{P} & & \xrightarrow[(\varphi\circ s_0,h)]{} & & P_0 \times P_1 \end{array}$$

We have $(\varphi \circ s_0, h) \circ d_0 = (\varphi \circ s_0 \circ d_0, h \circ d_0) = (\varphi, \varphi)$ and $u^* \circ w^* \circ k = t^* \circ v^* \circ k = t^* \circ \varphi = (\varphi, \varphi)$ so this diagram is indeed commutative and there exists a lifting (axiom MC4) $l : \widetilde{P} \to End_{\mathcal{Z}(P)}$. Then we form $l \circ d_1 : P \to End_{\mathcal{Z}(P)}$ and observe that $u^* \circ l \circ d_1 = (\varphi \circ s_0, h) \circ d_1 = (\varphi \circ s_0 \circ d_1, h \circ d_1) = (\varphi, \psi)$, i.e we obtain the following diagram:

$$\begin{array}{ccccc} & & End_{\mathcal{Z}(P)} & \longrightarrow & \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{Z}(X)} \\ & {\scriptstyle l\circ d_1}\nearrow & \downarrow{\scriptstyle u^*} & & \downarrow \\ P & \xrightarrow[(\varphi,\psi)]{} & P_0 \times P_1 & \longrightarrow & \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{T}(X)} \end{array}$$

and consequently a diagram of functors $\varphi^* \overset{\sim}{\twoheadleftarrow} Z \overset{\sim}{\twoheadrightarrow} \psi^*$. The functorial path object Z on $Ch_{\mathbb{K}}$ preserves weak equivalences and restrict to an endofunctor of $wCh_{\mathbb{K}}$, so the associated functorial path object Z on $Ch_{\mathbb{K}}^P$ do the same. Moreover, the natural transformations are weak equivalences in each component, so this diagram restricts to the desired diagram of endofunctors of $wCh_{\mathbb{K}}^P$. $\square$

Now we can conclude the proof of theorem 2.0.2 in the case $\mathcal{E} = Ch_K$:

Theorem 2.2.10. *Let $Ch_{\mathbb{K}}$ be the category of $\mathbb{Z}$-graded chain complexes over a field $\mathbb{K}$ of characteristic zero. Let $\varphi : P \xrightarrow{\sim} Q$ be a weak equivalence between two cofibrant props. The map φ gives rise to a functor $\varphi^* : w(Ch_{\mathbb{K}})^Q \to w(Ch_{\mathbb{K}})^P$ which induces a weak equivalence of simplicial sets $\mathcal{N}\varphi^* : \mathcal{N}w(Ch_{\mathbb{K}})^Q \xrightarrow{\sim} \mathcal{N}w(Ch_{\mathbb{K}})^P$.*

Proof. Recall that $\mathcal{P}$ is the category of props in Ch_K. Let us suppose first that $\varphi : P \overset{\sim}{\rightarrowtail} Q$ is an acyclic cofibration between two cofibrants props of $\mathcal{P}$. All objects in $Ch_{\mathbb{K}}$ are fibrant, so by definition of the model category structure on $\mathcal{P}$ the prop P is fibrant and thus we have the following lifting

$$\begin{array}{ccc} P & \xrightarrow{=} & P \\ {\scriptstyle\varphi}\downarrow\sim & \nearrow_{s} & \downarrow \\ Q & \longrightarrow & pt \end{array}$$

where $s : Q \xrightarrow{\sim} P$ satisfies

$$\begin{cases} s \circ \varphi = id_P \\ \varphi \circ s \sim id_Q \end{cases}$$

(the relation $\sim$ is the homotopy relation for the model category structure of $\mathcal{P}$). These maps induce functors $\varphi^* : (w\mathcal{E}^{cf})^Q \to (w\mathcal{E}^{cf})^P$ and $s^* : (w\mathcal{E}^{cf})^P \to (w\mathcal{E}^{cf})^Q$. Applying the simplicial nerve functor, we obtain

$$\begin{cases} \mathcal{N}(s \circ \varphi)^* = \mathcal{N}\varphi^* \circ \mathcal{N}s^* = id_{(w\mathcal{E}^{cf})^P} \\ \mathcal{N}(\varphi \circ s)^* = \mathcal{N}s^* \circ \mathcal{N}\varphi^* \sim id_{(w\mathcal{E}^{cf})^Q} \end{cases}$$

so $\mathcal{N}\varphi^*$ is a homotopy equivalence in $sSet$, which implies that it is a weak equivalence of simplicial sets . The functor

$$\begin{aligned} \mathcal{P} &\to sSet \\ P &\mapsto \mathcal{N}w(\mathcal{E}^{cf})^P \end{aligned}$$

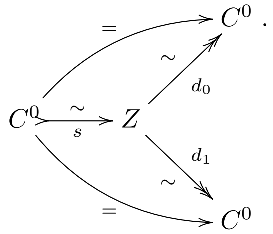
is defined between two model categories, and maps the acyclic cofibrations between cofibrants objects to weak equivalences, so it preserves weak equivalences between cofibrant objects according to Brown's lemma. □

2.2.7 The general case of a category $\mathcal{E}$ tensored over $Ch_{\mathbb{K}}$

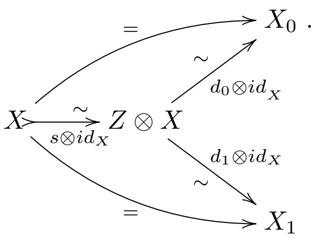
To complete our results we explain how the proof of theorem 2.2.9 extends to a category $\mathcal{E}$ tensored over $Ch_{\mathbb{K}}$.

Theorem 2.2.11. *Let $\mathcal{E}$ be a cofibrantly generated symmetric monoidal model category over $Ch_{\mathbb{K}}$. Let $\varphi : P \xrightarrow{\sim} Q$ be a weak equivalence between two cofibrant props defined in $Ch_{\mathbb{K}}$. This morphism φ gives rise to a functor $\varphi^* : w(\mathcal{E}^c)^Q \to w(\mathcal{E}^c)^P$ which induces a weak equivalence of simplicial sets $\mathcal{N}\varphi^* : \mathcal{N}w(\mathcal{E}^c)^Q \to \mathcal{N}w(\mathcal{E}^c)^P$.*

Proof. The chain complex Z defined previously is itself the path object on C^0, so we have the commutative diagram



Given that C^0 is the unit of $Ch_{\mathbb{K}}$, for any $X \in \mathcal{E}$ we have $C^0 \otimes X \cong X$, thus by applying the functor $- \otimes X$ we get the commutative diagram



The axiom MM1 for the external tensor product $\otimes$ implies that if X is cofibrant, then the functor $- \otimes X$ preserves acyclic cofibrations of $Ch_{\mathbb{K}}$ (all the objects of $Ch_{\mathbb{K}}$ are cofibrant) and thus, by Brown's lemma, it preserves the weak equivalences. Therefore $s \otimes id_X$ is still an acyclic cofibration and $d_0 \otimes id_X$, $d_1 \otimes id_X$ are weak equivalences. Moreover, given the properties of $\otimes$ and the fact that endomorphism props in $Ch_{\mathbb{K}}$ for objects of $\mathcal{E}$ are defined with the external hom bifunctor $Hom_{\mathcal{E}}(-,-)$ of $\mathcal{E}$, we have the following isomorphisms:

$$\begin{aligned} End_{Z \otimes X}(m,n) &= Hom_{\mathcal{E}}((Z \otimes X)^{\otimes m}, (Z \otimes X)^{\otimes n}) \\ &\cong Hom_{\mathcal{E}}(Z^{\otimes m} \otimes X^{\otimes m}, Z^{\otimes n} \otimes X^{\otimes n}) \\ &\cong (Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes End_X(m,n) \end{aligned}$$

$$\begin{aligned} Hom_{X, Z \otimes X}(m,n) &= Hom_{\mathcal{E}}(X^{\otimes m}, (Z \otimes X)^{\otimes n}) \\ &\cong Hom_{\mathcal{E}}(X^{\otimes m}, Z^{\otimes n} \otimes X^{\otimes n}) \\ &\cong Z^{\otimes n} \otimes End_X(m,n), \end{aligned}$$

and

$$\begin{aligned} Hom_{Z \otimes X, X_i}(m,n) &= Hom_{\mathcal{E}}((Z \otimes X)^{\otimes m}, X^{\otimes n}) \\ &\cong Hom_{\mathcal{E}}(Z^{\otimes m} \otimes X^{\otimes m}, X^{\otimes n}) \\ &\cong (Z^{\otimes m})^* \otimes End_{X_i}(m,n). \end{aligned}$$

The proofs of 3.3, 3.4 and 3.5 extend without changes to the case of a category $\mathcal{E}$ tensored over $Ch_{\mathbb{K}}$: we still work in $Ch_{\mathbb{K}}$, and as before the operations associated to $s \otimes id_X$, $d_0 \otimes id_X$ and $d_1 \otimes id_X$ in the pullbacks do not transform the elements of $End_X(m,n)$ themselves, so that the replacement of $End_X(m,n)$ by $P(m,n)$ does not break the transfert of prop structure in these pullbacks. We obtain a diagram of endofunctors $\varphi^* \overset{\sim}{\leftarrow} Z \overset{\sim}{\rightarrow} \psi^*$ of $(\mathcal{E}^c)^P$ where the natural transformations are weak equivalences in each component, so this diagram restricts to the desired diagram of endofunctors of $w(\mathcal{E}^c)^P$. The theorem 2.0.2 is proved in the general case. $\square$

2.3 The subcategory of acyclic fibrations

The goal of this section is to show that the classification space $\mathcal{N}w(\mathcal{E}^{cf})^P$ is weakly equivalent to $\mathcal{N}fw(\mathcal{E}^{cf})^P$, that is the nerve of the subcategory of acyclic fibrations. It works in the broader context of a category $\mathcal{E}$ tensored over any symmetric monoidal cofibrantly generated model category $\mathcal{C}$. The following result is a key point in the proof of theorem 2.0.3:

Proposition 2.3.1. *Let $\mathcal{E}$ be a symmetric monoidal cofibrantly generated model category over $\mathcal{C}$ satisfying the limit monoid axioms. Let P be a cofibrant prop with non-empty inputs (or outputs) defined in $\mathcal{C}$. The inclusion of categories $i : fw(\mathcal{E}^{cf})^P \hookrightarrow w(\mathcal{E}^{cf})^P$ gives rise to a weak equivalence of simplicial sets $\mathcal{N}fw(\mathcal{E}^{cf})^P \xrightarrow{\sim} \mathcal{N}w(\mathcal{E}^{cf})^P$.*

For this aim, we will have to deal in section 3.2 with the lifting of P-algebras structures in a certain diagram category. Therefore we need a compatibility between tensor and model structures on diagram categories, which we check in section 3.1.

2.3.1 The monoidal model structure of a diagram category

Let us start by recalling the model category structure on a diagram category:

Definition 2.3.2. Let $\mathcal{M}$ be a cocomplete category.

(1) Let X be an object of $\mathcal{M}$ and S a set. We set $S \otimes X = \amalg_S X$.

(2) Let I be a small category and X an object of $\mathcal{M}$. We consider an I-diagram $F^i_X : I \to \mathcal{M}$ defined by $F^i_X = Mor_I(i,-) \otimes X$, i.e for every $j \in I$, $F^i_X(j) = \amalg_{Mor_I(i,j)} X$.

(3) Let $\mathcal{K}$ be a set of morphisms of $\mathcal{M}$. We denote by $F^I_{\mathcal{K}}$ the set of morphisms in $\mathcal{M}^I$ of the form $Mor_I(i,-) \otimes f : F^i_X \to F^i_Y$ where $f : A \to B \in \mathcal{K}$ and $i \in I$.

Theorem 2.3.3. *(cf. [12], theorem 11.6.1) Let I be a small category. Let $\mathcal{M}$ be a cofibrantly generated model category, with C as set of generating cofibrations and C_a as set of generating acyclic cofibrations. The diagram category $\mathcal{M}^I$ is endowed with a cofibrantly generated model category structure such that:*

(1) $f : X \xrightarrow{\sim} Y$ is a weak equivalence (respectively a fibration) of $\mathcal{M}^I$ if and only if for every $i \in I$ the morphism $f(i) : X(i) \xrightarrow{\sim} Y(i)$ is a weak equivalence (respectively a fibration) of $\mathcal{M}$;

(2) The set of generating cofibrations (respectively acyclic generating cofibrations) of $\mathcal{M}^I$ is F^I_C (respectively $F^I_{C_a}$).

Consequently, a cofibration is a rectract or a transfinite composition of pushouts of elements of F^I_C.

Proposition 2.3.4. *(cf. [12], proposition 11.6.3) If $f : M \to N$ is a cofibration of I-diagrams, then $f(i) : M(i) \to N(i)$ is a cofibration in $\mathcal{M}$ for each $i \in I$.*

We now assume that $\mathcal{M}$ is a cofibrantly generated symmetric monoidal model category. The category $\mathcal{M}^I$ inherits the structure of a symmetric monoidal category over $\mathcal{M}$:

(1) The internal tensor product $\otimes : \mathcal{M}^I \times \mathcal{M}^I \to \mathcal{M}^I$ is defined pointwise: $\forall i \in I, \forall X, Y \in \mathcal{M}^I, (X \otimes Y)(i) = X(i) \otimes Y(i)$.

(2) We have a constant diagram functor

$$\begin{aligned} C : \mathcal{M} &\to \mathcal{M}^I \\ X &\mapsto C_X \end{aligned}$$

where $C_X(i) = X$, $C_X(i \to j) = id_X$, and for $f : X \to Y \in \mathcal{M}$, C_f is defined by $C_f(i) = f$. The external tensor product $\otimes : \mathcal{M} \times \mathcal{M}^I \to \mathcal{M}^I$ is given by $X \otimes F = C_X \otimes F$ for $X \in \mathcal{M}$, $F \in \mathcal{M}^I$.

(3) The external hom $Hom_{\mathcal{M}^I}(-,-) : \mathcal{M}^I \times \mathcal{M}^I \rightarrow \mathcal{M}$ is given by

$$Hom_{\mathcal{M}^I}(X,Y) = \int_{i\in I} Hom_{\mathcal{M}}(X(i),Y(i))$$

where $Hom_{\mathcal{M}}(-,-)$ is the internal hom of $\mathcal{M}$.

According to [14], definition 4.2.6, there is a weaker version of the axiom MM0 of definition 2.1.5 which is sufficient to recover a full symmetric monoidal model category structure. Let $\mathcal{C}$ be a symmetric monoidal category. We call this weaker axiom MM0':

MM0'. Let $1_{\mathcal{C}}$ be the unit object of $\mathcal{C}$. Let $Q1_{\mathcal{M}} \overset{\sim}{\twoheadrightarrow} 1_{\mathcal{C}}$ be a cofibrant replacement of $1_{\mathcal{C}}$ obtained by using the functorial factorization axioms of the model structure to factor $0 \rightarrow 1_{\mathcal{C}}$ into a cofibration followed by an acyclic fibration. Then the natural map $Q1_{\mathcal{C}} \otimes X \xrightarrow{\sim} 1_{\mathcal{C}} \otimes X$ is a weak equivalence for every cofibrant object X. This condition is automatic if $1_{\mathcal{C}}$ is cofibrant.

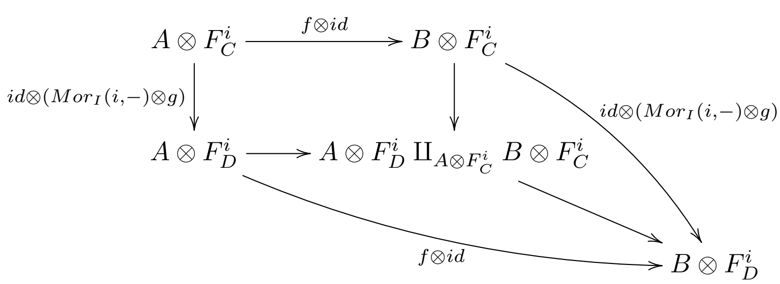
We prove that $\mathcal{M}^I$ with the model structure of theorem 2.3.3 satisfies the axiom MM0' and the axiom MM1 for the external tensor product. The axiom MM1 for the internal tensor product of $\mathcal{M}^I$ fails if we do not impose an extra assumption: we have to suppose that I admits finite coproducts.

Lemma 2.3.5. *The unit $C_{1_{\mathcal{M}}}$ of the symmetric monoidal category $\mathcal{M}^I$ satisfies axiom MM0'.*

Proof. Let $QC_{1_{\mathcal{M}}} \overset{\sim}{\twoheadrightarrow} C_{1_{\mathcal{M}}}$ be the acyclic fibration associated to the canonical cofibrant resolution $QC_{1_{\mathcal{M}}}$ of $C_{1_{\mathcal{M}}}$ in the model structure of $\mathcal{M}^I$, obtained by using the factorization axioms. Let $F \in \mathcal{M}^I$ be any cofibrant object. Then, according to [12], proposition 11.6.3, for every $i \in I$ the objects $F(i)$ and $QC_{1_{\mathcal{M}}}(i)$ are cofibrant in $\mathcal{M}$. Recall now that acyclic fibrations are defined pointwise in the model structure of $\mathcal{M}^I$. It implies that for every $i \in I$, the map $QC_{1_{\mathcal{M}}}(i) \overset{\sim}{\twoheadrightarrow} C_{1_{\mathcal{M}}}(i) = 1_{\mathcal{M}}$ is an acyclic fibration, thus $QC_{1_{\mathcal{M}}}(i)$ is a cofibrant resolution of $1_{\mathcal{M}}$ in $\mathcal{M}$. Now we apply MM0 in $\mathcal{M}$ to affirm that $QC_{1_{\mathcal{M}}}(i) \otimes F(i) \xrightarrow{\sim} 1_{\mathcal{M}} \otimes F(i)$ is a weak equivalence in $\mathcal{M}$ for every $i \in I$, since $F(i)$ is cofibrant. Given that weak equivalences in $\mathcal{M}^I$ are defined pointwise, we conclude that $QC_{1_{\mathcal{M}}} \otimes F \xrightarrow{\sim} C_{1_{\mathcal{M}}} \otimes F$ is a weak equivalence in $\mathcal{M}^I$, i.e axiom MM0' is satisfied. ☐

Lemma 2.3.6. *The axiom MM1 holds in $\mathcal{M}^I$ for the external tensor product.*

Proof. Recall that C and C_a denote respectively the sets of generating cofibrations and generating acyclic cofibrations of $\mathcal{M}$. The sets F_C^I and $F_{C_a}^I$ denote the generating cofibrations and generating acyclic cofibrations of $\mathcal{M}^I$. According to corollary 4.2.5 in [14], it is sufficient to check the axiom MM1 for C and F_C^I, respectively C and $F_{C_a}^I$, respectively C_a and F_C^I. We will just explain this verification for C and F_C^I, because the two other cases work in the same way. Let $f : A \rightarrowtail B$ and $g : C \rightarrowtail D$ be two generating cofibrations of $\mathcal{M}$. Let $Mor_I(i,-) \otimes g$ be a generating cofibration of $\mathcal{M}^I$ induced by g. We form their pushout-product for the external tensor product:



We take the evaluation of this diagram of diagrams in $j \in I$:

$$\begin{array}{ccc} A\otimes \amalg_{Mor_I(i,j)}C & \longrightarrow & B\otimes \amalg_{Mor_I(i,j)}C \\ \downarrow & & \downarrow \\ A\otimes \amalg_{Mor_I(i,j)}D & \longrightarrow & A\otimes \amalg_{Mor_I(i,j)}D\bigvee_{A\otimes \amalg_{Mor_I(i,j)}C} B\otimes \amalg_{Mor_I(i,j)}C \\ & & \qquad\qquad \searrow \\ & & \qquad\qquad\qquad B\otimes \amalg_{Mor_I(i,j)}D \end{array}$$

The tensor product $\otimes$ commutes with colimits so

$$A\otimes \amalg_{Mor_I(i,j)}D \bigvee_{A\otimes \amalg_{Mor_I(i,j)}C} B\otimes \amalg_{Mor_I(i,j)}C = \amalg_{Mor_I(i,j)}(A\otimes D\bigvee_{A\otimes C} B\otimes C)$$

and

$$B\otimes \amalg_{Mor_I(i,j)}D = \amalg_{Mor_I(i,j)}(B\otimes D).$$

Thus

$$(f_*,(Mor_I(i,-)\otimes g)_*)(j) = \amalg_{Mor_I(i,j)}(f_*,g_*),$$

for every $j\in J$, i.e

$$(f_*,(Mor_I(i,-)\otimes g)_*) = Mor_I(i,-)\otimes (f_*,g_*).$$

The morphism (f_*,g_*) is a cofibration because axiom MM1 holds in $\mathcal{M}$ for the internal tensor product, and the functor $Mor_I(i,-)\otimes -$ preserves cofibrations, so $(f_*,(Mor_I(i,-)\otimes g)_*)$ is a cofibration in $\mathcal{M}^I$. □

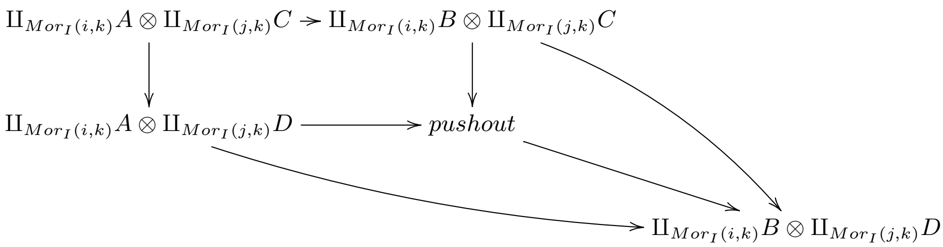
Remark 2.3.7. Lemma 2.3.6 implies axiom MM1', the "dual pushout-product" axiom, for the external hom $Hom_{\mathcal{M}^I}(-,-)$.

Lemma 2.3.8. *If I admits finite coproducts, then the axiom MM1 holds in $\mathcal{M}^I$ for the internal tensor product.*

Proof. It is sufficient to check axiom MM1 in two cases: two morphisms of F^I_C , and two morphisms belonging respectively to F^I_C and $F^I_{C_a}$ (cf. [14], corollary 4.2.5). We just prove the first case, given that the second case can be treated in the same way. Let $Mor_I(i,-)\otimes f$ and $Mor_I(j,-)\otimes g$ be two generating cofibrations of $\mathcal{M}^I$ obtained from the two generating cofibrations $f:A\rightarrowtail B$ and $g:C\rightarrowtail D$ of $\mathcal{M}$. We form their pushout-product for the internal tensor product:

$$\begin{array}{ccc} F^i_A\otimes F^j_C & \longrightarrow & F^i_B\otimes F^j_C \\ \downarrow & & \downarrow \\ F^i_A\otimes F^j_D & \longrightarrow & F^i_A\otimes F^j_D \amalg_{F^i_A\otimes F^i_C} F^i_B\otimes F^j_C \\ & & \qquad\qquad \searrow \\ & & \qquad\qquad\qquad F^i_B\otimes F^i_D \end{array}$$

We take the evaluation of this diagram of diagrams in $k\in I$:



We have

$$pushout = \amalg_{Mor_I(i,k)\times Mor_I(j,k)}(A\otimes D\bigvee_{A\otimes C} B\otimes C)$$
$$= \amalg_{Mor_I(i\vee j,k)}(A\otimes D\bigvee_{A\otimes C} B\otimes C)$$

and

$$\amalg_{Mor_I(i,k)}B\otimes\amalg_{Mor_I(j,k)}D = \amalg_{Mor_I(i\vee j,k)}(B\otimes D)$$

so $((Mor_I(i,-)\otimes f)_*, (Mor_I(j,-)\otimes g)_*) = Mor_I(i\vee j,-)\otimes(f_*,g_*)$. The morphism (f_*,g_*) is a cofibration because the axiom MM1 holds in $\mathcal{M}$ for the internal tensor product, and the functor $Mor_I(i\vee j,-)\otimes -$ preserves cofibrations, so $((Mor_I(i,-)\otimes f)_*, (Mor_I(j,-)\otimes g)_*)$ is a cofibration in $\mathcal{M}^I$. □

From lemmas 3.5, 3.6 and 3.8 we conclude that:

Proposition 2.3.9. *Let $\mathcal{M}$ be a cofibrantly generated symmetric monoidal model category and I a small category. If I admits finite coproducts, then $\mathcal{M}^I$ forms a cofibrantly generated symmetric monoidal model category over $\mathcal{M}$.*

Before going to the heart of the matter, we warn the reader about the following subtlety: proposition 2.3.9 is available for a definition of symmetric monoidal model category with axioms MM0' and MM1. However, theorem 2.1.13 has been proven for a base category sastisfying MM0. This is the reason why we precise that, in proposition 2.3.1, the definition of symmetric monoidal category we use is definition 2.1.5 with MM0. Axiom MM0' is just used for the category of diagrams $\mathcal{M}^I$, to get proposition 2.3.9. Actually, we do not need axiom MM0' in the proof of proposition 2.3.1, but only the pushout-product axioms of certain diagram categories.

Let us point out the fact that we can form the endomorphism prop of a diagram of diagrams. Transposing the construction proposed in the section 3.4.4 of [9] in the prop context, we get:

Definition 2.3.10. Let I, J be two small categories and $F: I\rightarrow\mathcal{M}^J$ a functor. We can define an *endomorphism prop* End_F in $\mathcal{M}$ by

$$End_F(m,n) = \int_{i\in I} Hom_{\mathcal{M}^J}(F(i)^{\otimes m}, F(i)^{\otimes n})$$

where $Hom_{\mathcal{M}^J}(-,-)$ is the external hom of $\mathcal{M}^J$. The inner bifunctor of this formula is the endomorphism prop of the diagram $F(i)$ in $\mathcal{M}$.

Proposition 2.3.11. *Let P be a prop in $\mathcal{M}$. The props morphisms $P\rightarrow End_F$ are in bijection with the functorial P-actions on the objects $F(i)\in\mathcal{M}^J$, $i\in I$, such that $i\mapsto F(i)$ defines a functor $I\rightarrow(\mathcal{M}^J)^P$.*

2.3.2 Proof of proposition 2.3.1

We are now in position to begin the proof of proposition 2.3.1. Let $\mathcal{E}$ be a symmetric monoidal cofibrantly generated model category over $\mathcal{C}$ satisfying the limit monoid axioms. Let P be a cofibrant prop, defined in $\mathcal{C}$, with non-empty inputs (or outputs) and let $i : fw(\mathcal{E}^c)^P \hookrightarrow w(\mathcal{E}^c)^P$ be the inclusion of categories. The overall strategy is to show that for every $X \in (\mathcal{E}^{cf})^P$, the category $\mathcal{N}(X \downarrow i)$ is contractible and to apply Quillen's theorem A (cf. Quillen [22]). Let $fw(X \downarrow (\mathcal{E}^{cf})^P)$ denote the category whose objects are morphisms $X \to Y$ and morphisms are commutative triangles

$$\begin{array}{ccc} & & Y \\ & \nearrow & \Big\downarrow{\scriptstyle\sim} \\ X & & \\ & \searrow & \\ & & Y' \end{array}$$

where $Y \overset{\sim}{\twoheadrightarrow} Y'$ is an acyclic fibration. By unraveling definitions, we see that $(X \downarrow i)$ is the full subcategory of $fw(X \downarrow (\mathcal{E}^{cf})^P)$ formed by the objects weakly equivalent to X. We will use the short notation $\mathcal{K} = (X \downarrow (\mathcal{E}^{cf})^P)$, and we set $\mathcal{L} = (X \downarrow \mathcal{E}^{cf})$ for the image of $\mathcal{K}$ under the forgetful functor. We also consider $\mathcal{K}' = (X \downarrow i) = fw(X \downarrow w(\mathcal{E}^{cf})^P)$. The category $\mathcal{K}$ admits an initial object $X \overset{=}{\to} X$, and $\mathcal{K}'$ is a full subcategory of $fw\mathcal{K}$ including $X \overset{=}{\to} X$. The category of subdivisions of a simplicial set is the poset of its non degenerate simplices, where the partial order is given by the faces: a morphism in this category is a face map between two non degenerate simplices. Recall the following standard result about the simplicial nerve:

Lemma 2.3.12. *Let I, J be two small categories. Every simplicial map $\varphi : \mathcal{N}I \to \mathcal{N}J$ is induced by a functor $F : I \to J$, i.e the simplicial nerve functor $\mathcal{N} : Cat \to sSet$ is full (Cat is the category of small categories).*

Proof. The map φ defines F on the objects and morphisms, and we use the fact that φ commutes with faces and degeneracies to obtain the functoriality of F. □

For the proof of proposition 2.3.1 we will need to apply proposition 2.3.9 to the category of diagrams $\mathcal{L}^{sdK}$, where sdK denotes the category of subdivisions of a simplicial set K. The idea is to use the model category structure of the props and the endomorphisms props of definition 2.3.10, in the case $\mathcal{M} = \mathcal{L}$ and $J = sdK$, to lift a P-algebra structure on a certain diagram in $\mathcal{L}^{sdK}$. We determine a coproduct of two non degenerate simplices for the poset structure of sdK in the case of a simplicial complex K. If two simplices α and β in sdK have no common face, then for every $\gamma \in sdK$, we have $Mor_{sdK}(\alpha, \gamma) \times Mor_{sdK}(\beta, \gamma) = \emptyset$ because at least one of these two sets of morphisms is empty, and in this case the pushout obtained in the proof of lemma 2.3.8 is empty. We have only to define a coproduct for two non degenerate simplices which have at least one common face. This is the subject of the following lemma:

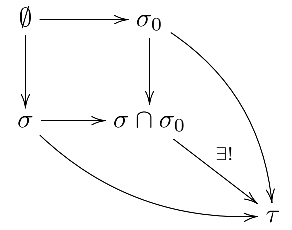
Lemma 2.3.13. *We can define the coproduct of two non-degenerate simplices having at least one common face in the subdivision category of any simplicial complex.*

Proof. Recall that a simplicial complex is a particular case of simplicial set consisting of a finite collection of simplices K such that

(i) if $\sigma \in K$ and τ is a face of σ then $\tau \in K$;

(ii) for any $\sigma, \sigma_0 \in K$, the intersection $\sigma \cap \sigma_0$ is either the empty set or a common face of σ and σ_0.

Let σ and σ_0 be two simplices of K having at least one common face. We then have $\sigma \cap \sigma_0 \neq \emptyset$, which implies that $\sigma \cap \sigma_0$ is a common face of σ and σ_0 (condition (ii)). We define their coproduct by $\sigma \vee \sigma_0 = \sigma \cap \sigma_0$. We easily check that this $\sigma \cap \sigma_0$ defines the coproduct of σ and σ_0: $\sigma \cap \sigma_0$ is unique and satisfy the universal property of the coproduct



□

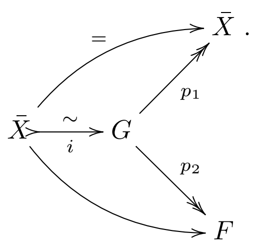
We want to prove that for every $n \in \mathbb{N}$, $\pi_n(|\,\mathcal{N}\mathcal{K}'\,|) = [S^n, |\,\mathcal{N}\mathcal{K}'\,|] = *$, where $|-|$ is the geometric realization functor and $[-,-]$ the homotopy classes. We consider, for every $n \in \mathbb{N}$, the simplicial complex $\partial\Delta^{n+1}$ as simplicial model of S^n. The simplicial complex $\partial\Delta^{n+1}$ is the boundary of Δ^{n+1}, i.e the simplicial complex obtained by withdrawing the $(n+1)$-simplexe of Δ^{n+1}. We prove in full generality the following result :

Proposition 2.3.14. *For any simplicial complex K_n, any simplicial map $\varphi : \mathcal{N}sdK_n \to \mathcal{N}\mathcal{K}'$ is null up to homotopy.*

Proof. We adapt the argument line of Rezk (lemma 4.2.5 in [23]). The main difference with his proof lies in the fact that we deal with the absence of model category structure on the category of P-algebras (recall that the lemma 4.2.5 in [23] involves model category devices). Therefore we use proposition 2.3.9, proposition 2.3.11 and lemma 2.3.13 to apply lifting techniques from [8] and get the desired P-algebras structures on our diagrams. According to lemma 2.3.12, it is sufficient to prove that any functor $F : sdK_n \to \mathcal{K}'$ induces a map between the simplicial nerves null up to homotopy. Let F be such a functor. In order to use the factorization axioms of a model category, we work temporarily in $\mathcal{L}^{sdK_n}$ and note also F the composite functor $U \circ F$ taking values in $U(\mathcal{K}') \subset \mathcal{L}$ and thus belonging to $\mathcal{L}^{sdK_n}$. The initial diagram $\bar{X}$ is the constant diagram in $X \overset{=}{\to} X$. To simplify, we will abreviate the notation of a morphism $X \to Y$, regarded as an object of $\mathcal{L}$, to the single target object Y, assuming that any such given Y comes together with a canonical morphism $X \to Y$. We similarly abreviate the notation of a morphism between $X \to Y$ and $X \to Y'$, which is a commutative triangle, to the morphism $Y \to Y'$ between the targets. The functor $F \times \bar{X}$ is defined on the objects by $(F \times \bar{X})(k) = F(k) \times X$ and on the morphisms by $(F \times \bar{X})(k_1 \overset{\phi}{\to} k_2) = F(k_1) \times X \overset{F(\phi)\times id_X}{\to} F(k_2) \times X$. For every $\phi : k_1 \to k_2 \in sdK_n$, $F(\phi)$ is an acyclic fibration so $F(\phi) \times id_X$ too, thus $F \times \bar{X}$ takes actually its values in $fw\mathcal{L}$. Recall that $\mathcal{L}$ inherits a cofibrantly generated model category structure, according to Hirschhorn [13]. Therefore $\mathcal{L}^{sdK_n}$ is endowed with a model category structure, so that we can give a decomposition of the unique map $\bar{X} \to F \times \bar{X}$ in an acyclic cofibration followed by a fibration:

$$\bar{X} \overset{\sim}{\rightarrowtail} G \twoheadrightarrow F \times \bar{X}.$$

It gives us a diagram $\mathcal{Y}$



The map $(p_1, p_2) : G \twoheadrightarrow F \times \bar{X}$ is a fibration so for every $k \in sdK_n$, $(p_1, p_2)(k)$ is a fibration. Moreover, $F(k)$ and X are fibrant so $p_1(k)$ and $p_2(k)$ are acyclic fibrations (cf [4], lemma 4.4). By definition of the weak equivalences and fibrations in a diagram category, we conclude that p_1 and p_2 are acyclic fibrations in $U(\mathcal{M})^{sdK_n}$. Now, considering again F and $\bar{X}$ as functors $sdK_n \to \mathcal{K}' \subset \mathcal{K}$, we want to put a P-algebra structure on $\mathcal{Y}$ which preserves the P-algebra structures existing on F and $\bar{X}$. To be more explicit, we want to produce a lifting

$$\begin{array}{ccc} 0 & \longrightarrow & End_{\mathcal{Y}} \\ \downarrow & \nearrow & \downarrow \sim \\ P & \longrightarrow & End_{\mathcal{V}} \end{array}$$

where $\mathcal{V}$ is the subdiagram of $\mathcal{Y}$ constisting of the two external arrows obtained by withdrawing G, i, p_1 and p_2 from $\mathcal{Y}$. The idea is to show that we have an acyclic fibration $End_{\mathcal{Y}} \stackrel{\sim}{\twoheadrightarrow} End_{\mathcal{V}}$ and to use the cofibrancy of P to produce the lifting. The category $\mathcal{L}$ forms a cofibrantly generated symmetric monoidal model category over $\mathcal{C}$ satisfying the limit monoid axioms, as does $\mathcal{E}$. We examine the case of $\mathcal{L}^{sdK_n}$. We know that it possesses a cofibrantly generated model category structure. It is a symmetric monoidal category over $\mathcal{E}$, which is a symmetric monoidal category over $\mathcal{C}$, so $\mathcal{L}^{sdK_n}$ is also a symmetric monoidal category over $\mathcal{C}$. Moreover, $\mathcal{L}^{sdK_n}$ satisfies the properties described in proposition 2.1.8, given that these properties are satisfied in $\mathcal{L}$ and that the fibrations of $\mathcal{L}^{sdK_n}$ are defined pointwise. Lemmas 3.5 and 3.6 ensure that the axioms MM0 and MM1 for the external tensor product hold in $\mathcal{L}^{sdK_n}$. The fact that K_n is a simplicial complex and lemma 2.3.13 allows us to apply lemma 2.3.8, i.e to check the axiom MM1 for the internal tensor product in $\mathcal{L}^{sdK_n}$. We have now verified all the necessary conditions to apply the proof of lemma 8.3 in [8] to obtain the desired acyclic fibration.

Finally, we want the diagram of functors $F \stackrel{\sim}{\twoheadleftarrow} G \stackrel{\sim}{\twoheadrightarrow} \bar{X}$ to live in $\mathcal{K}'^{sdK_n}$. We have already checked that F and $\bar{X}$ take their values in $\mathcal{K}'$. The maps p_1 and p_2 are natural transformations in $\mathcal{K}'^{sdK_n}$. So it remains to prove that G takes its values in $\mathcal{K}'$. For this, we refer the reader to the proof of lemma 4.2.5 in [23]. This diagram of functors induces simplicial homotopies $\mathcal{N}F \sim \mathcal{N}G \sim 0$, that is what we expected. □

Recall that $\mid \mathcal{N}sdK_n \mid \simeq \mid K_n \mid$. It includes especially the case $K_n = \partial\Delta^{n+1}$. We deduce that the nerve $\mathcal{N}\mathcal{K}' = \mathcal{N}(X \downarrow i)$ is contractible for every $X \in (\mathcal{E}^{cf})^P$, so $i : fw(\mathcal{E}^c)^P \hookrightarrow w(\mathcal{E}^c)^P$ gives rise to a weak equivalence of nerves according to Quillen's theorem A. It concludes the proof of proposition 2.3.1.

2.4 Extension of the results in the colored prop setting

Definition 2.4.1. Let C be a non-empty set, called the *set of colors*, and $\mathcal{C}$ a symmetric monoidal category.

(1) A *C-colored Σ-biobject* M is a double sequence of objects $\{M(m,n) \in \mathcal{E}\}_{(m,n)\in\mathbb{N}^2}$ where each $M(m,n)$ admits commuting left Σ_m action and right Σ_n action as well as a decomposition

$$M(m,n) = colim_{c_i,d_i\in C} M(c_1,...,c_m;d_1,...,d_n)$$

compatible with these actions. The objects $M(c_1,...,c_m;d_1,...,d_n)$ should be thought as spaces of operations with colors $c_1,...,c_m$ indexing the m inputs and colors $d_1,...,d_n$ indexing the n outputs.

(2) A *C-colored prop* P is a C-colored Σ-biobject endowed with a horizontal composition

$$\circ_h : P(c_{11},...,c_{1m_1};d_{11},...,d_{1n_1}) \otimes ... \otimes P(c_{k1},...,c_{km_k};d_{k1},...,d_{kn_1}) \to$$
$$P(c_{11},...,c_{km_k};d_{k1},...,d_{kn_k}) \subseteq P(m_1+...+m_k, n_1+...+n_k)$$

and a vertical composition

$$\circ_v : P(c_1, ..., c_k; d_1, ..., d_n) \otimes P(a_1, ..., a_m; b_1, ..., b_k) \to P(a_1, ..., a_m; d_1, ..., d_n) \subseteq P(m, n)$$

which is equal to zero unless $b_i = c_i$ for $1 \leq i \leq k$. These two compositions satisfy associativity axioms (we refer the reader to [15] for details).

Definition 2.4.2. (1) Let $\{X_c\}_C$ be a collection of objects of $\mathcal{E}$. The *C-colored endomorphism prop* $End_{\{X_c\}_C}$ is defined by

$$End_{\{X_c\}_C}(c_1, ..., c_m; d_1, ..., d_n) = Hom_{\mathcal{E}}(X_{c_1} \otimes ... \otimes X_{c_m}, X_{d_1} \otimes ... \otimes X_{d_n})$$

with a horizontal composition given by the tensor product of homomorphisms and a vertical composition given by the composition of homomorphisms with matching colors.

(2) Let P be a C-colored prop. A *P-algebra* is the data of a collection of objects $\{X_c\}_C$ and a C-colored prop morphism $P \to End_{\{X_c\}_C}$.

Example 2.4.3. Let I be a small category, P a prop in $\mathcal{C}$. We can build an $ob(I)$-colored prop P_I such that the P_I-algebras are the I-diagrams of P-algebras in $\mathcal{E}$ in the same way as that of [18].

To endow the category of colored props with a model category structure, the cofibrantly generated symmetric monoidal model structure on $\mathcal{C}$ is not sufficient. We have to suppose moreover that the domains of the generating cofibrations and acyclic generating cofibrations are small (cf [12], 10.4.1), that is to say the model structure is strongly cofibrantly generated:

Theorem 2.4.4. *(cf. [15], theorem 1.1) Let C be a non-empty set. Let $\mathcal{C}$ be a strongly cofibrantly generated symmetric monoidal model category with a symmetric monoidal fibrant replacement functor, and either:*

(1) a cofibrant unit and a cocommutative interval, or

(2) functorial path data.

Then the category $\mathcal{P}_C$ of C-colored props in $\mathcal{C}$ forms a strongly cofibrantly generated model category with fibrations and weak equivalences defined componentwise in $\mathcal{C}$.

This theorem works especially with the categories of simplicial sets, simplicial modules over a commutative ring and chain complexes over a characteristic 0 ring (our main category in this paper).

This model structure is similar to that of 1-colored props, and we can define C-colored endomorphism props of morphisms (see [15], section 4) and more generally of any kind of diagram, so the lifting properties used in the two previous sections works in the C-colored case. Morevoer, in the proof of theorem 2.0.2, the replacement of the operations $X^{\otimes m} \to X^{\otimes n}$ by $P(m, n)$ can be done using a C-colored prop P instead of a 1-colored one without changing anything to the proof, therefore we finally get the C-colored version of theorem 2.0.2 and proposition 2.3.1. We do not have to change anything to theorem 2.0.2, given that $Ch_{\mathbb{K}}$ satisfies the hypotheses of theorem 2.4.4. Concerning proposition 2.3.1, we just have to suppose that $\mathcal{C}$ verifies the additional hypotheses of theorem 2.4.4.

2.5 Application: the moduli space of P-algebra structures as a homotopy fiber

2.5.1 Moduli spaces of algebra structures over a prop

A moduli space of algebra structures over a prop P, on a given object X of $\mathcal{E}$, is a simplicial set whose points are the prop morphisms $P \to End_X$. Such a moduli space can be more generally defined on diagrams of $\mathcal{E}$. We then deal with endomorphism props of diagrams. To construct properly such a simplicial set and give its first fundamental properties, we have to recall some results about the cosimplicial resolutions in a model category. For the sake of brevity and clarity, we refer the reader to the chapter 16 in [12] for a complete treatment of the notions of simplicial resolutions, cosimplicial resolutions and Reedy model categories.

Definition 2.5.1. Let $\mathcal{M}$ be a model category and let X be an object of $\mathcal{M}$.

(1) A *cosimplicial resolution* of X is a cofibrant approximation to the constant cosimplicial object cc_*X in the Reedy model category structure on cosimplicial objects $\mathcal{M}^\Delta$ of $\mathcal{M}$.

(2) A *simplicial resolution* of X is a fibrant approximation to the constant simplicial object cs_*X in the Reedy model category structure on simplicial objects $\mathcal{M}^{\Delta^{op}}$ of $\mathcal{M}$.

Definition 2.5.2. Let $\mathcal{M}$ be a model category and let X be an object of $\mathcal{M}$.

(1) A *cosimplicial frame* on X is a cosimplicial object $\tilde{X}$ in $\mathcal{X}$, together with a weak equivalence $\tilde{X} \to cc_*X$ in the Reedy model category structure of $\mathcal{M}^\Delta$. It has to satisfy the two following properties : the induced map $\tilde{X}^0 \to X$ is an isomorphism, and if X is cofibrant in $\mathcal{M}$ then $\tilde{X}$ is cofibrant in $\mathcal{M}^\Delta$.

(2) A *simplicial frame* on X is a simplicial object $\tilde{X}$ in $\mathcal{X}$, together with a weak equivalence $cs_*X \to \tilde{X}$ in the Reedy model category structure of $\mathcal{M}^\Delta$. It has to satisfy the two following properties : the induced map $X \to \tilde{X}^0$ is an isomorphism, and if X is fibrant in $\mathcal{M}$ then $\tilde{X}$ is fibrant in $\mathcal{M}^{\Delta^{op}}$.

Proposition 2.5.3. *(cf [12], proposition 16.1.9) Let $\mathcal{M}$ be a model category. There exists functorial simplicial resolutions and functorial cosimplicial resolutions in $\mathcal{M}$.*

Proposition 2.5.4. *(cf. [12], corollaries 16.5.3 and 16.5.4) Let $\mathcal{M}$ be a model category and C a cosimplicial resolution in $\mathcal{M}$.*

(1) If X is a fibrant object of $\mathcal{M}$, then $Mor_{\mathcal{M}}(C,X)$ is a fibrant simplicial set.

(2) If $p : X \twoheadrightarrow Y$ is a fibration in $\mathcal{M}$, then $p_* : Mor_{\mathcal{M}}(C,X) \twoheadrightarrow Mor_{\mathcal{M}}(C,Y)$ is a fibration of simplicial sets, acyclic if p is so.

(3) If $p : X \xrightarrow{\sim} Y$ is a weak equivalence of fibrant objects in $\mathcal{M}$, then $p_* : Mor_{\mathcal{M}}(C,X) \twoheadrightarrow Mor_{\mathcal{M}}(C,Y)$ is a weak equivalence of fibrant simplicial sets.

Proposition 2.5.5. *(cf. [12], proposition 16.6.3) Let X be an object of $\mathcal{M}$.*

(1) If X is cofibrant then every cosimplicial frame of X is a cosimplicial resolution of X.

(2) If X is fibrant then every simplicial frame of X is a simplicial resolution of X.

Definition 2.5.6. Let $\mathcal{E}$ be a symmetric monoidal model category over $\mathcal{C}$ and P a cofibrant prop with non-empty inputs in $\mathcal{C}$. Let I be a small category and $\{X_i\}_{i\in I}$ a I-diagram in $\mathcal{E}$. The *moduli space of P-algebra structures* on $\{X_i\}_{i\in I}$ is the simplicial set defined by

$$P\{X_i\}_{i\in I} = Mor_{\mathcal{P}_0}(P\otimes\Delta[-], End_{\{X_i\}_{i\in I}}).$$

We get a functor

$$\begin{aligned} \mathcal{P}_0 &\to sSet \\ P &\mapsto P\{X_i\}_{i\in I}. \end{aligned}$$

We can already get two interesting properties of these moduli spaces:

Lemma 2.5.7. *Suppose moreover that $\mathcal{E}$ satisfies the limit monoid axioms. If X is a fibrant and cofibrant object of $\mathcal{E}$, then $P\{X\}$ is a fibrant simplicial set.*

Proof. If X is fibrant and cofibrant, we can show by arguments based on the pushout-product axiom and the limit monoid axiom LM2 that End_X is a fibrant prop (cf lemma 7.2 in [8]), so $P\{X\}$ is fibrant according to proposition 2.5.4. □

Lemma 2.5.8. *Suppose that $\mathcal{E}$ satisfies the limit monoid axioms. Let X be a fibrant and cofibrant object of $\mathcal{E}$. Every weak equivalence of cofibrant props $P \xrightarrow{\sim} Q$ gives rise to a weak equivalence of fibrant simplicial sets (which is a homotopy equivalence, since every object is automatically cofibrant in the model category structure of simplicial sets) $Q\{X\} \xrightarrow{\sim} P\{X\}$.*

Proof. Let $\varphi : P \to Q$ be a weak equivalence of cofibrant props. According to proposition 16.1.24 of [12], the map φ induces a Reedy weak equivalence of cosimplicial resolutions $P \otimes \Delta[-] \xrightarrow{\sim} Q \otimes \Delta[-]$. The object X is fibrant and cofibrant so End_X is fibrant, and we conclude by corollary 16.5.5 of [12] that $P \otimes \Delta[-] \xrightarrow{\sim} Q \otimes \Delta[-]$ induces $Q\{X\} \xrightarrow{\sim} P\{X\}$. □

2.5.2 Moduli spaces of algebra structures on fibrations

We start by recalling lemma 7.2 of [8]. Let $f : A \to B$ be a morphism of $\mathcal{E}$, we have a pullback

$$\begin{array}{ccc} End_{\{A \to^f B\}} & \xrightarrow{d_0} & End_B \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle f^*} \\ End_A & \xrightarrow[f_*]{} & Hom_{AB} \end{array}$$

where Hom_{AB} is defined by $Hom_{AB}(m,n) = Hom_{\mathcal{E}}(A^{\otimes m}, B^{\otimes n})$.

Lemma 2.5.9. *(cf. [8], lemma 7.2) Suppose that A and B are fibrant and cofibrant. Then End_A and End_B are fibrant props. Moreover:*

(1) If f is a (acyclic) fibration then so is d_0.

(2) If f is a cofibration, then d_1 is a fibration. If f is also acyclic then d_1 is an acyclic fibration and d_0 a weak equivalence.

Remark 2.5.10. It is a generalization in the prop context of propositions 4.1.7 and 4.1.8 of [23].

Lemma 2.5.11. *Let $X_n \twoheadrightarrow ... \twoheadrightarrow X_1 \twoheadrightarrow X_0$ be a chain of fibrations in $\mathcal{E}^{cf}$ (the full subcategory of $\mathcal{E}$ consisting of objects which are both fibrant and cofibrant). For every $0 \leq k \leq n-1$, the map d_0 in the pullback*

$$\begin{array}{ccc} End_{\{X_n \twoheadrightarrow ... \twoheadrightarrow X_0\}} & \xrightarrow{d_0} & End_{\{X_k \twoheadrightarrow ... \twoheadrightarrow X_0\}} \\ {\scriptstyle d_1}\downarrow & & \downarrow \\ End_{\{X_n \twoheadrightarrow ... \twoheadrightarrow X_{k+1}\}} & \longrightarrow & Hom_{X_{k+1}X_k} \end{array}$$

is a fibration . Moreover, if the fibrations in the chain $X_n \twoheadrightarrow ... \twoheadrightarrow X_1 \twoheadrightarrow X_0$ are acyclic then so is d_0.

Proof. We prove this lemma by induction. For $n = 1$ it is lemma 2.5.9. Now suppose that our lemma is true for a given integer $n \geq 1$. Let $X_{n+1} \twoheadrightarrow ... \twoheadrightarrow X_1 \twoheadrightarrow X_0$ be a chain of fibrations in $\mathcal{E}^{cf}$. We distinguish two cases:

-the case $k = n$: we have the pullback

$$\begin{array}{ccc} End_{\{X_{n+1} \twoheadrightarrow ... \twoheadrightarrow X_0\}} & \xrightarrow{d_0} & End_{\{X_n \twoheadrightarrow ... \twoheadrightarrow X_0\}} \\ \downarrow{\scriptstyle d_1} & & \downarrow \\ End_{X_{n+1}} & \xrightarrow[f_*]{} & Hom_{X_{n+1}X_n} \end{array}$$

where $f : X_{n+1} \twoheadrightarrow X_n$. The fact that f is a fibration implies that f_* is a fibration, so d_0 is a fibration because of the stability of fibrations under pullback, and the acyclicity of f implies the acyclicity of d_0. The detailed proof of these statements is done in the proof of lemma 7.2 in [8].

-the case $0 \leq k \leq n-1$: $d_0 = End_{\{X_{n+1} \twoheadrightarrow ... \twoheadrightarrow X_0\}} \rightarrow End_{\{X_n \twoheadrightarrow ... \twoheadrightarrow X_0\}} \rightarrow End_{\{X_k \twoheadrightarrow ... \twoheadrightarrow X_0\}}$ is the composite of an map satisfying the induction hypothesis with the map of the case $k = n$, so the conclusion follows. □

Remark 2.5.12. This lemma is the generalization in the prop context of proposition 4.1.9 of [23].

We deduce from lemmas 5.9 and 5.11 the following properties of our moduli spaces:

Proposition 2.5.13. *Let $f : X \rightarrow Y$ be a morphism of $\mathcal{E}^{cf}$ and P a cofibrant prop in $\mathcal{C}$. The pullback of lemma 2.5.9 gives rise to the following diagram of simplicial sets:*

$$P\{X\} \overset{(d_1)_*}{\leftarrow} P\{f\} \overset{(d_0)_*}{\rightarrow} P\{Y\}$$

(1) If f is a cofibration then $(d_1)_*$ is a fibration. Moreover, if f is acyclic then $(d_0)_*$ and $(d_1)_*$ are weak equivalences.

(2) If f is a fibration then $(d_0)_*$ is a fibration. Moreover, if f is acyclic then $(d_0)_*$ and $(d_1)_*$ are weak equivalences.

Proof. (1) If f is a cofibration then d_1 is a fibration. So $(d_1)_*$ is a fibration of simplicial sets according to proposition 2.5.4. If f is acyclic, then d_0 and d_1 are weak equivalences. The objects X and Y are fibrant and cofibrant so End_X and End_Y are fibrant props, which implies that $End_{\{f\}}$ is also fibrant. We deduce from this and proposition 2.5.4 that $(d_0)_*$ and $(d_1)_*$ are weak equivalences.

(2) The proof is the same as in the previous case. □

By induction we can also prove the following proposition:

Proposition 2.5.14. *Let $X_n \overset{\sim}{\twoheadrightarrow} ... \overset{\sim}{\twoheadrightarrow} X_1 \overset{\sim}{\twoheadrightarrow} X_0$ be a chain of acyclic fibrations in $\mathcal{E}^{cf}$ and P a cofibrant prop in $\mathcal{C}$. For every $0 \leq k \leq n-1$, the map $(d_0)_*$ is an acyclic fibration and $(d_1)_*$ a weak equivalence in the diagram below:*

$$P\{X_n \overset{\sim}{\twoheadrightarrow} ... \overset{\sim}{\twoheadrightarrow} X_{k+1}\} \overset{(d_1)_*}{\leftarrow} P\{X_n \overset{\sim}{\twoheadrightarrow} ... \overset{\sim}{\twoheadrightarrow} X_0\} \overset{(d_0)_*}{\rightarrow} P\{X_k \overset{\sim}{\twoheadrightarrow} ... \overset{\sim}{\twoheadrightarrow} X_1\}.$$

Remark 2.5.15. Propositions 5.13 and 5.14 are generalizations in the prop context of propositions 4.1.11, 4.1.12 and 4.1.13 in [23].

2.5.3 Proof of theorem 2.0.3

We have now all the key results to generalize Rezk's theorem to algebras over props and colored props. The remaining arguments are the same as that of Rezk, so we will not repeat it with all details but essentially show how our theorem 2.0.2 and proposition 2.3.1 fit in the proof.

Let $\mathcal{P}$ a cofibrant prop, and $\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[-]\otimes P}$ the bisimplicial set defined by $(\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[-]\otimes P})_{m,n} = (\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[n]\otimes P})_m$. The prop P is cofibrant, thus so is $\Delta[n]\otimes P$ for every $n \geq 0$. According to proposition 2.3.1, we have a weak équivalence induced by an inclusion of categories

$$\mathcal{N}fw(\mathcal{E}^{cf})^{\Delta[n]\otimes P} \xrightarrow{\sim} \mathcal{N}w(\mathcal{E}^{cf})^{\Delta[n]\otimes P}$$

Morevoer, for every $n, n' \geq 0$, $\Delta[n] \to \Delta[n']$ induces a weak equivalence of cofibrant props $\Delta[n] \otimes P \to \Delta[n'] \otimes P$ and thereby a weak equivalence of simplicial sets

$$\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[n']\otimes P} \xrightarrow{\sim} \mathcal{N}w(\mathcal{E}^{cf})^{\Delta[n]\otimes P}$$

according to theorem 2.0.2. We obtain a zigzag of weak equivalences

$$diag\mathcal{N}fw(\mathcal{E}^{cf})^{\Delta[-]\otimes P} \xrightarrow{\sim} diag\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[-]\otimes P} \xleftarrow{\sim} \mathcal{N}w(\mathcal{E}^{cf})^{P}$$

We use an adaptation of a slightly modified version of Quillen's theorem B (cf. [22]), namely lemma 4.2.2 in [23], in order to determine the homotopy fiber of the map $diag\mathcal{N}fw(\mathcal{E}^{cf})^{\Delta[-]\otimes P} \to \mathcal{N}fw(\mathcal{E}^{cf})$. To prove that our map verifies the hypotheses of this lemma we use the propositions of section 5.2 exactly in the same way as Rezk in the operadic case. Then we check that $diag(U \downarrow X) \simeq P\{X\}$ where $U : fw(\mathcal{E}^{cf})^{\Delta[-]\otimes P} \to fw\mathcal{E}^{cf}$ is the forgetful functor (by using again the propositions of section 5.2) and finally we get the following diagram:

$$\begin{array}{ccccccc}
P\{X\} & \longrightarrow & diag\mathcal{N}fw(\mathcal{E}^{cf})^{\Delta[-]\otimes P} & \xrightarrow{\sim} & diag\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[-]\otimes P} & \xleftarrow{\sim} & \mathcal{N}w(\mathcal{E}^{cf})^{P} \\
\downarrow & & \downarrow & & & & \downarrow \\
pt & \longrightarrow & \mathcal{N}(fw\mathcal{E}^{cf}) & & \xrightarrow{\sim} & & \mathcal{N}(w\mathcal{E}^{cf})
\end{array}$$

The proof of theorem 2.0.3 is complete.

Given the model category structure on the colored props in $Ch_{\mathbb{K}}$, the construction of moduli spaces from cosimplicial framings makes sense. We can obtain the colored prop version of theorem 2.0.3 by replacing the cofibrant prop P by a colored cofibrant prop (According to the section 4 of our paper, theorem 2.0.2 and proposition 2.3.1 have their equivalent in the colored prop setting, and so do the propositions of section 5.2 according to [15], section 4):

Theorem 2.5.16. *(generalization of [23], theorem 1.1.5, in the case of colored props) Let $\mathcal{E}$ be a cofibrantly generated symmetric monoidal model category over $Ch_{\mathbb{K}}$ satisfying the limit monoid axioms. Let P be a cofibrant C-colored prop in $Ch_{\mathbb{K}}$, where C is a non-empty set, and $\{X_c\}_{c\in C}$ a collection of objects of $\mathcal{E}^{cf}$. Then the commutative square*

$$\begin{array}{ccc}
P\{X_c\}_{c\in C} & \longrightarrow & \mathcal{N}(w(\mathcal{E}^{cf})^{P}) \\
\downarrow & & \downarrow \\
\{X_c\}_{c\in C} & \longrightarrow & \mathcal{N}(w\mathcal{E}^{cf})
\end{array}$$

is a homotopy pullback of simplicial sets.

Remark 2.5.17. Note that we can recover the transfer theorem of bialgebras structures obtained in [8] (theorem A) and its colored version as a consequence of theorem 2.5.16. Indeed, let P be a cofibrant prop in $Ch_{\mathbb{K}}$. Let $X \stackrel{\sim}{\to} Y$ be a morphism of $\mathcal{E}^{cf}$ such that Y is endowed with a P-algebra structure. We have a homotopy pullback of simplicial sets

$$\begin{array}{ccc} P\{X\} & \xrightarrow{p} & \mathcal{N}(w(\mathcal{E}^{cf})^P) \\ \downarrow & & \downarrow \mathcal{N}U \\ \{X\} & \longrightarrow & \mathcal{N}(w\mathcal{E}^{cf}) \end{array}$$

which induces an exact sequence of pointed sets

$$\pi_0 P\{X\} \to \pi_0\mathcal{N}(w(\mathcal{E}^{cf})^P) \to \pi_0\mathcal{N}(w\mathcal{E}^{cf}).$$

The base point of the set $\pi_0\mathcal{N}(w\mathcal{E}^{cf})$ is the weak equivalence class of X, denoted by $[X]$. The weak equivalence $X \stackrel{\sim}{\to} Y$ in $\mathcal{E}^{cf}$ implies that we have the equality $[Y] = [X]$ and thus $\pi_0\mathcal{N}U([Y]_P) = [X]$, where $[Y]_P$ is the weak equivalence class of Y in $(\mathcal{E}^{cf})^P$. The exactness of the sequence above implies that $\pi_0 p(P\{X\}) = (\pi_0\mathcal{N}U)^{-1}([X])$ so $[Y]_P \in \pi_0 p(P\{X\})$. It means that there exists a P-algebra structure on X such that we have a zigzag of P-algebras morphisms

$$X \stackrel{\sim}{\leftarrow} ... \stackrel{\sim}{\to} Y$$

which are weak equivalences of $\mathcal{E}^{cf}$.

Remark 2.5.18. Theorem 2.5.16 applies especially in the following case. Let I be a small category, P a prop in $Ch_{\mathbb{K}}$, and P_I the $ob(I)$-colored prop such that the P_I-algebras are the I-diagrams of P-algebras in $Ch_{\mathbb{K}}$ (see example 37). Let us take a cofibrant replacement of P_I, namely $(P_I)_\infty$, then we have the following homotopy pullback of simplicial sets for any collection $\{X_i\}$ of chain complexes indexed by $ob(I)$:

$$\begin{array}{ccc} (P_I)_\infty\{X_i\} & \longrightarrow & \mathcal{N}(w(Ch_{\mathbb{K}})^P) \\ \downarrow & & \downarrow \\ pt & \longrightarrow & \mathcal{N}(wCh_{\mathbb{K}}) \end{array}$$

(recall that every chain complex over a field of characteristic zero is fibrant and cofibrant, so there is no need of writing this hypothesis here). Moreover, given that a weak equivalence between two cofibrant props induces a weak equivalences between the associated classification spaces, we can take any cofibrant replacement of P_I to get the same classification space up to weak equivalence.

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Chapitre 3

Simplicial localization of homotopy generalized bialgebras

The goal of this paper is to show that weakly equivalent cofibrant PROPs describe the same homotopy theory of bialgebras. Precisely, we prove that two weakly equivalent cofibrant PROPs encode two Dwyer-Kan equivalent simplicial localizations of algebras. In particular, it gives a homotopical coherence to the definition of homotopy algebras over a prop as algebras over a cofibrant resolution of this prop. This result relies on a homotopy invariance theorem of the classifying space obtained by the author in a previous work, and the properties of different models of $(\infty,1)$-categories.

3.1 Introduction

Props are combinatorial tools introduced by MacLane in [22] in order to parametrize operations with multiple inputs and outputs. Later, the work of Drinfeld on quantum groups ([5],[6]) shed a new light on the importance of algebraic structures with coproducts. These generalized bialgebras (see [19] for a wide range of examples) and even more general ones are encoded by props. Under suitable conditions on the ambient category, the category of props admits a model category structure [12] allowing one to perform usual homotopical algebra. Let P be a prop. We define a P-algebra up to homotopy, or homotopy P-algebra, as an algebra over a cofibrant resolution of P. It means that the relations defining the algebraic structure are satisfied only up to homotopies and higher coherent homotopies. This definition relies on the choice of a cofibrant resolution, so a natural request is to ask for the homotopy theory of homotopy algebras to not depend on this choice. In the operadic setting, homotopy algebras and their properties have been the object of an extensive study, see for instance [20]. It is a classical result that the model categories of algebras over two weakly equivalent cofibrant operads are Quillen equivalent. But algebras over a cofibrant prop does not form a model category in general, so in this case one has to find another way to prove the homotopy invariance of the homotopy theory.

Let (C,W) be a pair of categories such that W is a subcategory of C called its subcategory of weak equivalences. Such a pair is called a relative category [1]. Its classification space is defined as the simplicial nerve of W. It encompasses the datas of weak equivalence classes of the objects of C as well as their homotopy automorphisms (in the sense of simplicial localization) [9]. A Quillen equivalence of model categories induces a weak equivalence at the classification spaces level. In the framework of algebras over a prop we cannot use such a result. However, we managed in [27] to get round this difficulty and prove that the homotopy type of the classification space of algebras over a cofibrant prop is invariant under weak equivalences of cofibrant props. So the informations encoded by this space do not depend on the choice of a cofibrant resolution. However, this space does not encapsulate the whole homotopy theory of the relative category. It does not contain for instance informations about the homotopy of morphisms between two different objects. Consequently, we provide a definitive answer for the whole homotopy theory. We prove that algebras over two weakly equivalent cofibrant props describe the same homotopy theory with its higher homotopy datas, namely the same simplicial localization. We work with props in the category $Ch_{\mathbb{K}}$ of chain complexes over a field $\mathbb{K}$. Let $\mathcal{E}$ be a symmetric monoidal model category over $Ch_{\mathbb{K}}$ and $(\mathcal{E}^c)$ its subcategory of cofibrant objects. Let $(\mathcal{E}^c)^P$ denote the category of algebras over a dg prop P, and $w(\mathcal{E}^c)^P$ its subcategory of morphisms of P-algebras which are weak equivalences in $\mathcal{E}$. Our main result in this paper reads:

Theorem 3.1.1. *A weak equivalence $P \xrightarrow{\sim} Q$ between two cofibrant (colored) props induces a zigzag of Dwyer-Kan equivalences of simplicial localizations*

$$L((\mathcal{E}^c)^Q, w(\mathcal{E}^c)^Q) \cong L((\mathcal{E}^c)^P, w(\mathcal{E}^c)^P).$$

Thus the homotopy theory (at both basic and higher level) of homotopy P-algebras does not depend on the choice of a cofibrant resolution of P. It works in particular for $\mathcal{E} = Ch_{\mathbb{K}}$, whose objects are all cofibrant.

We combine results of [27] about the homotopy invariance of the classification space with ∞-category methods to get the proof of this theorem.

Organization. Section 1 consists in brief recollections about symmetric monoidal model categories over a base category and props and algebras over a prop in this setting. Section 2 is the heart of this paper. We recall the notion of classification diagram introduced by Rezk [25]. We then show how our result of [27] can be improved into a homotopy invariance of the classification diagram in the category of bisimplicial sets. Then we give the link between the classification diagram of a relative category and its simplicial localization with proposition 3.3.7. This result relies on several properties of different models of $(\infty, 1)$-categories, which are deferred to the appendix section. Finally we explain how to combine it with the homotopy invariance of the classification diagram in order to prove theorem 3.1.1. Section 3 is the appendix section, devoted to recollections about different models of $(\infty, 1)$-categories and the proof of proposition 3.3.7. We provide the argument line of [23] with all details.

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3.2 Recollections on props, algebras and homotopy

We will need categories tensored over chain complexes and formalize this notion as follows:

Definition 3.2.1. Let $\mathcal{C}$ be a symmetric monoidal category. A symmetric monoidal category over $\mathcal{C}$ is a symmetric monoidal category $(\mathcal{E}, \otimes, 1_{\mathcal{E}})$ endowed with an external tensor product $\otimes : \mathcal{C} \times \mathcal{E} \to \mathcal{E}$ satisfying the following natural unit, associativity and symmetry constraints:
(1) $\forall X \in \mathcal{E}, 1_{\mathcal{C}} \otimes X \cong X$,
(2) $\forall X \in \mathcal{E}, \forall C, D \in \mathcal{C}, (C \otimes D) \otimes X \cong C \otimes (D \otimes X)$,
(3) $\forall C \in \mathcal{C}, \forall X, Y \in \mathcal{E}, C \otimes (X \otimes Y) \cong (C \otimes X) \otimes Y \cong X \otimes (C \otimes Y)$.

From now we work in the case $\mathcal{C} = Ch_{\mathbb{K}}$, the category of chain complexes over a field $\mathbb{K}$. We assume that all small limits and small colimits exist in $\mathcal{E}$, and suppose moreover that the external tensor product of $\mathcal{E}$ also preserves colimits in each variable. This last condition implies the existence of an external hom bifunctor $Hom_{\mathcal{E}}(-,-) : \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{C}$ satisfying an adjunction relation

$$\forall C \in \mathcal{C}, \forall X, Y \in \mathcal{E}, Mor_{\mathcal{E}}(C \otimes X, Y) \cong Mor_{\mathcal{C}}(C, Hom_{\mathcal{E}}(X, Y))$$

(so $\mathcal{E}$ is naturally an enriched category over $\mathcal{C}$).

When working with symmetric monoidal categories equiped with a model structure, one can wonder how the tensor product behaves with respect to the model structure. The precise question is to know under which conditions the homotopy category inherits a monoidal structure. The answer lies in the following axioms:

Definition 3.2.2. (1) A symmetric monoidal model category is a symmetric monoidal category $\mathcal{C}$ endowed with a model category structure such that the following axioms hold:

MM0. The unit object $1_{\mathcal{C}}$ of $\mathcal{C}$ is cofibrant.

MM1. The pushout-product $(i_*, j_*) : X \otimes T \oplus_{X \otimes Z} Y \otimes Z \to Y \otimes T$ of cofibrations $i : X \rightarrowtail Y$ and $j : Z \rightarrowtail T$ is a cofibration which is also acyclic as soon as i or j is so.

(2) Suppose that $\mathcal{C}$ is a symmetric monoidal model category. A symmetric monoidal category $\mathcal{E}$ over $\mathcal{C}$ is a symmetric monoidal model category over $\mathcal{C}$ if the axiom MM0 holds and the axiom MM1 holds for both the internal and external tensor products of $\mathcal{E}$.

Axiom MM0 ensures the existence of a unit for the monoidal structure of the homotopy category. Axiom MM1 provides the necessary assumptions to make the tensor product a Quillen bifunctor, and thus obtain the monoidal product on the homotopy category by deriving the tensor product. Let us note that axiom MM0 as we enunciate it can be weakened. We refer the reader to [14] for more details about monoidal model categories. The category $Ch_{\mathbb{K}}$ of chain complexes over a field $\mathbb{K}$ is our main working example of symmetric monoidal model category.

A differential graded (dg) Σ-biobject is a sequence $M = \{M(m,n)\}_{m,n \in \mathbb{N}}$ of chain complexes such that each $M(m,n)$ is endowed with a left action of the symmetric group Σ_m and a right action of the symmetric group Σ_n commuting with the left one. We can see the $M(m,n)$ as spaces of operations with m inputs and n outputs, and the action of the symmetric groups as permutations of the inputs and the outputs. We call (m,n) the biarity of such an operation. Composing operations of two Σ-biobjects M and N amounts then to consider 2-levelled directed graphs (with no loops) with the first level indexed by operations of M and the second level by operations of N. Vertical composition by grafting and horizontal composition by concatenation allows one to define props.

Definition 3.2.3. A dg prop is a symmetric monoidal category P, enriched over $Ch_{\mathbb{K}}$, with $\mathbb{N}$ as object set and the tensor product given by $m \otimes n = m + n$ on objects.

Equivalently, a prop is a Σ-biobject equiped with horizontal products

$$\circ_h : P(m_1, n_1) \otimes P(m_2, n_2) \to P(m_1 + m_2, n_1 + n_2),$$

vertical composition products

$$\circ_v : P(k,n) \otimes P(m,k) \to P(m,n)$$

and units $1 \to P(n,n)$ corresponding to identity morphisms of the objects $n \in \mathbb{N}$ in P. These operations satisfy relations coming from the axioms of symmetric monoidal categories. We refer the reader to Enriquez and Etingof [10] for an explicit description of props in the context of modules over a ring. We denote by $\mathcal{P}$ the category of props.

There exists a free prop functor $\mathcal{P}$ forming an adjunction

$$\mathcal{F} : \Sigma \rightleftarrows \mathcal{P} : U$$

with the forgetful functor U between the category of props and the category of Σ-biobjects. There is an explicit construction of the free prop for which we refer the reader to [12].

For a prop P in $Ch_{\mathbb{K}}$, we can define the notion of P-algebra in a symmetric monoidal category over $Ch_{\mathbb{K}}$:

Definition 3.2.4. Let $\mathcal{E}$ be a symmetric monoidal category over $Ch_{\mathbb{K}}$.

(1) The endomorphism prop of $X \in \mathcal{E}$ is given by $End_X(m,n) = Hom_{\mathcal{E}}(X^{\otimes m}, X^{\otimes n})$ where $Hom_{\mathcal{E}}(-,-)$ is the external hom bifunctor of $\mathcal{E}$.

(2) Let P be a prop in $Ch_{\mathbb{K}}$. A P-algebra in $\mathcal{E}$ is an object $X \in \mathcal{E}$ equipped with a prop morphism $P \to End_X$.

Now we are interested in doing homotopical algebra with props and their algebras. Concerning props, we have to distinguish two cases. In the case where $\mathbb{K}$ is of characteristic zero, the adjunction $F : \Sigma \rightleftarrows \mathcal{P} : U$ transfer the cofibrantly generated model category structure on Σ-biobjects to the whole category of props:

Theorem 3.2.5. *(Fresse [12], theorem 5.5) If the base category is the category of dg-modules over a ring $\mathbb{K}$ such that $\mathbb{Q} \subset \mathbb{K}$, then the adjunction between the free prop functor and the forgetful functor provides $\mathcal{P}$ with a cofibrantly generated model category structure. The weak equivalences and fibrations are the componentwise quasi-isomorphisms and surjections, and the generating (acyclic) cofibrations are the images under the free prop functor of the generating (acyclic) cofibrations of Σ.*

In the general case it works only with the subcategory $\mathcal{P}_0$ of props with non-empty inputs or outputs and does not give a full model category structure, but a semi-model category structure:

Definition 3.2.6. A prop P has non-empty inputs if it satisfies

$$P(0,n) = \begin{cases} \mathbb{K}, & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We define in a symmetric way a prop with non-empty outputs.

We refer the reader to [12] for more details about this result. The semi-model structure is close enough to a full model structure to define the homotopy category and do similarly homotopical algebra. In the remaining part of the paper, we will work in a field of any characteristic, and assume tacitely that if the characteristic is positive then the considered props satisfy the non-emptyness property. Let us note that most of props encountered, especially the ones encoding generalized bialgebras [19], are of this form.

Concerning algebras over a dg prop, the situation is far more involved than in the operadic case. In general, there is no model structure on such a category, there is even no limits or colimits (and no free algebra functor). However, there are other ways to recover informations of a homotopical nature about algebras over a prop. If C is a category equiped with a subcategory wC satisfying $ob(wC) = ob(C)$, called the subcategory of weak equivalences, then one can define its classification space as the simplicial nerve of wC. We denote it by $\mathcal{N}wC$. The classification space has been introduced in [9]. Dwyer and Kan proved that such a space encode the homotopy types of the objects of C as well as their homotopy automorphisms (when one does not work with a model category, this notion is still meaningful by using the simplicial localization, see [9]). In [27], the author proved the following homotopy invariance result of the classification space of algebras over weakly equivalent cofibrant props:

Theorem 3.2.7. *(Yalin [27], see chapter 2 theorem 2.2.11 in this thesis) Let $\mathcal{E}$ be a cofibrantly generated symmetric monoidal model category over $Ch_{\mathbb{K}}$. Let $\varphi : P \xrightarrow{\sim} Q$ be a weak equivalence between two cofibrant props defined in $Ch_{\mathbb{K}}$. This morphism φ gives rise to a functor $\varphi^* : w(\mathcal{E}^c)^Q \to w(\mathcal{E}^c)^P$ which induces a weak equivalence of simplicial sets $\mathcal{N}\varphi^* : \mathcal{N}w(\mathcal{E}^c)^Q \to \mathcal{N}w(\mathcal{E}^c)^P$.*

If we suppose that $\mathbb{K}$ is of characteristic zero, then this result can be extended to colored props, i.e props equiped with a set of colors labeling inputs and outputs of the operations (see [11]).

However, this classification space do not recover all the homotopical information. In order to get the full homotopy theory of a category with weak equivalences, one has to consider its simplicial localization. This is the topic of the next section.

3.3 Simplicial localization of algebras over cofibrant props

Let us note $sSets$ the category of simplicial sets.

Definition 3.3.1. Let (C, W) be a relative category. Its classification diagram, denoted $N(C, W)$, is the bisimplicial set $N(C, W) : \Delta^{op} \to sSets$ defined by

$$N(C, W)([n]) = \mathcal{N}we(C^{[n]})$$

where $\mathcal{N}$ is the simplicial nerve functor and $we(C^{[n]})$ the subcategory of weak equivalences of the diagram category $C^{[n]}$, i.e the natural transformations are morphisms of W in each component.

The simplicial set $N(C,W)([0]) = \mathcal{N}W$ is the classification space of (C,W).

Let $\mathcal{E}$ be a cofibrantly generated symmetric monoidal model category over $Ch_{\mathbb{K}}$. Let I be a small category and $\mathcal{E}^I$ the category of I-diagrams in $\mathcal{E}$. There exists two Quillen equivalent cofibrantly generated model categories on $\mathcal{E}^I$: the projective model structure $(\mathcal{E}^I)_{proj}$ with pointwise weak equivalences and pointwise fibrations, and the injective model structure $(\mathcal{E}^I)_{inj}$ with pointwise weak equivalences and cofibrations. We refer the reader to proposition A.2.8.2 of [21] for a proof of this result. Since the two have the same weak equivalences, their classification spaces are also the same.

Lemma 3.3.2. *The model category $(\mathcal{E}^I)_{inj}$ is a cofibrantly generated symmetric monoidal model category for the pointwise tensor product.*

Proof. The pushout-product is a pointwise pushout-product since colimits are created pointwise, and (acyclic) cofibrations are the pointwise (acyclic) cofibrations so one has just to apply axiom MM1 of $\mathcal{E}$. ☐

Lemma 3.3.3. *The category $(\mathcal{E}^I)_{inj}$ is a symmetric monoidal model category over $\mathcal{E}$ for the external tensor product defined for every $E \in \mathcal{E}$ and $F \in \mathcal{E}^I$ by*

$$\forall i \in I, (E \otimes_e F)(i) = E \otimes F(i).$$

Proof. Let f be a cofibration of $\mathcal{E}$ and ϕ be a cofibration of $(\mathcal{E}^I)_{inj}$. We form the pushout-product $f\Box\phi$. By definition of the external tensor product, for every $i \in I$ the map $(f\Box\phi)(i) = f\Box\phi(i)$ is a pushout-product of f and $\phi(i)$ in $\mathcal{E}$. We apply axiom MM1 in $\mathcal{E}$ and use the fact that cofibrations are defined pointwise in $(\mathcal{E}^I)_{inj}$, so $f\Box\phi$ is a cofibration. If one of these two maps is acyclic then $f\Box\phi$ is acyclic by the same argument. ☐

We would like the category $(\mathcal{E}^I)_{inj}$ to be a symmetric monoidal model category over $Ch_{\mathbb{K}}$. This follows from the following general fact:

Lemma 3.3.4. *Let $\mathcal{E}$ be a symmetric monoidal model category over $\mathcal{C}$ and $\mathcal{D}$ be a symmetric monoidal model category over $\mathcal{E}$. Then $\mathcal{D}$ is a symmetric monoidal model category over $\mathcal{C}$.*

Proof. We have a symmetric monoidal functor $\eta_{\mathcal{E}} : \mathcal{C} \to \mathcal{E}$ defined by $\eta_{\mathcal{E}}(C) = C \otimes_{\mathcal{C}}^{\mathcal{E}} 1_{\mathcal{E}}$ where $\otimes_{\mathcal{C}}^{\mathcal{E}}$ is the external tensor product of $\mathcal{E}$ over $\mathcal{C}$ and $1_{\mathcal{E}}$ is the unit of $\mathcal{E}$. According to axiom MM0 in $\mathcal{E}$, the unit $1_{\mathcal{E}}$ is cofibrant, so by applying axiom MM1 for the external tensor product we see that $\eta_{\mathcal{E}}$ preserves cofibrations and acyclic cofibrations.

Now, let $\otimes_{\mathcal{E}}^{\mathcal{D}}$ denote the external tensor product of $\mathcal{D}$ over $\mathcal{E}$. We can define an external tensor product of $\mathcal{D}$ over $\mathcal{C}$ by

$$C \otimes_{\mathcal{C}}^{\mathcal{D}} D = \eta_{\mathcal{E}}(C) \otimes_{\mathcal{E}}^{\mathcal{D}} D$$

where $C \in \mathcal{C}$ and $D \in \mathcal{D}$. It satisfies all the recquired axioms of an external tensor product since $\otimes_{\mathcal{C}}^{\mathcal{E}}$ is an external tensor product and $eta_{\mathcal{E}}$ is a symmetric monoidal functor. The external tensor product $\otimes_{\mathcal{E}}^{\mathcal{D}}$ satisfies MM1 and $\eta_{\mathcal{E}}$ preserves cofibrations and acyclic cofibrations, so $\otimes_{\mathcal{C}}^{\mathcal{D}}$ satisfies also MM1. ☐

In the injective model structure, cofibrant diagrams are diagrams of cofibrant objects of $\mathcal{E}$, that is $((\mathcal{E}^I)_{inj})^c = ((\mathcal{E}^c)^I)_{inj}$. We now apply theorem 3.2.7 to $(\mathcal{E}^I)_{inj}$ as a category tensored over $Ch_{\mathbb{K}}$ and deduce:

Proposition 3.3.5. *A weak equivalence of cofibrant dg props $\varphi : P \xrightarrow{\sim} Q$ induces a weak equivalence of classification spaces*

$$\mathcal{N}w((\mathcal{E}^c)^Q)^I = \mathcal{N}w((\mathcal{E}^c)^I)^Q \xrightarrow{\sim} \mathcal{N}w((\mathcal{E}^c)^I)^P = \mathcal{N}w((\mathcal{E}^c)^P)^I.$$

We then deduce from theorem 3.2.7:

Theorem 3.3.6. *A weak equivalence of cofibrant dg props $\varphi : P \xrightarrow{\sim} Q$ induces a Reedy weak equivalence of classification diagrams*

$$N((\mathcal{E}^c)^Q, w(\mathcal{E}^c)^Q) \xrightarrow{\sim} N((\mathcal{E}^c)^P, w(\mathcal{E}^c)^P).$$

The weak equivalences of the model category of complete Segal spaces CSS [25] contains the Reedy weak equivalences (CSS is a Bousfield localization of the standard Reedy model structure on bisimplicial sets), so the map above is a weak equivalence in CSS. The link with the simplicial localization is given by:

Proposition 3.3.7. *(Cisinski, Schommer-Pries) Let* (C, W) *be a relative category. Then the bisimplicial sets* $(t^! \circ \tilde{N} \circ L^H)(C, W)$ *and* $N_\xi(C, W)$ *are weakly equivalent in CSS.*

Here N_ξ is a functor introduced in [1], which is pointwise Reedy equivalent to N. The three functors of the composite $(t^! \circ \tilde{N} \circ L^H)$ are recalled in the appendix, and their composite is a right Quillen functor from relative categories to complete Segal spaces. The proof of this proposition is postponed to the appendix section 4.

We now argue as follows to get the result of thm 1.1. According to lemma 5.4 of [1] and theorem 3.3.6 of this paper, we have a zigzag of weak equivalences of bisimplicial sets

$$N_\xi((\mathcal{E}^c)^Q, w(\mathcal{E}^c)^Q) \overset{\sim}{\leftarrow} N((\mathcal{E}^c)^Q, w(\mathcal{E}^c)^Q) \overset{\sim}{\rightarrow} N((\mathcal{E}^c)^P, w(\mathcal{E}^c)^P) \overset{\sim}{\rightarrow} N_\xi((\mathcal{E}^c)^P, w(\mathcal{E}^c)^P).$$

According to proposition 3.3.7, we then obtain in the homotopy category of complete Segal spaces $Ho(CSS)$ the isomorphism

$$\begin{aligned} R(t^! \circ \tilde{N} \circ L^H)((\mathcal{E}^c)^Q, w(\mathcal{E}^c)^Q) &= RN_\xi((\mathcal{E}^c)^Q, w(\mathcal{E}^c)^Q) \\ &\cong RN_\xi((\mathcal{E}^c)^P, w(\mathcal{E}^c)^P) \\ &= R(t^! \circ \tilde{N} \circ L^H)((\mathcal{E}^c)^Q, w(\mathcal{E}^c)^Q). \end{aligned}$$

where $R(-)$ stands for the right derived Quillen functor. Since $R(t^! \circ \tilde{N} \circ L^H)$ is an equivalence of categories, the relative categories $((\mathcal{E}^c)^Q, w(\mathcal{E}^c)^Q)$ and $((\mathcal{E}^c)^P, w(\mathcal{E}^c)^P)$ are isomorphic in $Ho(RelCat)$, i.e connected by a zigzag of weak equivalences of $RelCat$

$$((\mathcal{E}^c)^Q, w(\mathcal{E}^c)^Q) \overset{\sim}{\leftarrow} . \overset{\sim}{\rightarrow} ... \overset{\sim}{\rightarrow} ((\mathcal{E}^c)^P, w(\mathcal{E}^c)^P).$$

According to [2], these weak equivalences induce Dwyer-Kan equivalences of the associated simplicial localizations

$$L^H((\mathcal{E}^c)^Q, w(\mathcal{E}^c)^Q) \overset{\sim}{\leftarrow} . \overset{\sim}{\rightarrow} ... \overset{\sim}{\rightarrow} L^H((\mathcal{E}^c)^P, w(\mathcal{E}^c)^P).$$

This is theorem 3.1.1.

Consequence.

Recall that according to theorem 1.1 of [15], theorem 3.2.5 can be extended to the colored case when working over a field of characteristic zero. Let P be a differential graded colored prop over a field of characteristic zero. The homotopy theory of homotopy P-algebras does not depend, up to Dwyer-Kan equivalence, on the choice of a cofibrant resolution of P. This result works also for 1-colored dg props (i.e the props of definition 3.2.3) over a field of any characteristic when considering props with non-empty inputs.

Examples.

Examples in the 1-colored case include the list of generalized bialgebras described in [19], for instance Hopf algebras, non-commutative Hopf algebras, Lie bialgebras, Frobenius bialgebras, etc. Typical examples of colored props are the one encoding diagrams of algebras (see [11]), so this is an invariance result in particular for homotopy diagrams of various kinds of homotopy algebras.

3.4 Appendix: models for $(\infty, 1)$-categories

We refer the reader to [1] for the more recent model of $(\infty, 1)$-categories based on relative categories, and to the survey [4] for more informations and references about the previous models. Here we will focus on four models: complete Segal spaces ([25]), simplicial categories ([7], [8],[9], [3]), quasi-categories ([16],[17],[18]) and relative categories ([1],[2]). We will not enter in the details of these theories but just recall informally the definitions of these objects and some Quillen equivalences linking their model categories.

Set theoretic warning. These constructions are a priori established in the setting of small categories, in order to avoid set theoretic problems like simplicial proper classes instead of sets or categories which are not locally small. However in practice one often wants to apply these results to large categories. We adopt therefore the usual position in the litterature, that is assuming Grothendieck axiom of universes: for every set there exists a universe in which this set is an element. Thus there exists a universe U in which the categories we will use are U-small.

Simplicial categories.

To be more precise, Dwyer and Kan originally defined simplicial categories as simplicial objects in the category of (small) categories. However, the definition we use there and which is the good one to model $(\infty, 1)$-categories [3] is that a simplicial category is a category enriched over simplicial sets. The former definition of course implies the later. We note $SCat$ the category of simplicial categories.

There exists functorial cosimplicial resolutions and simplicial resolutions in any model category ([9],[13]), so model categories provide examples of simplicial categories. One recovers the morphisms of the homotopy category between a cofibrant resolution and a fibrant resolution by taking the set of connected components of the corresponding simplicial mapping space. Another more general example is the simplicial localization developed by Dwyer and Kan [7]. Let (C, W) be a pair of categories such that W is a subcategory of C containing all the objects of W. We call W the category of weak equivalences of C. Such a pair is called a relative category in [1]. To any such pair Dwyer and Kan associates a simplicial category $L(C, W)$ called its simplicial localization. They developed also another simplicial localization, the hammock localization $L^H(C, W)$ [8]. The two are actually equivalent, and each one has its own advantages. The hammock localization has an explicit description of the simplicial mapping spaces. By taking the sets of connected components of the mapping spaces, we get $\pi_0 L(C, W) \cong C[W^{-1}]$ where $C[W^{-1}]$ is the localization of C with respect to W (the homotopy category of (C, W)).

Now let us define Dwyer-Kan equivalences:

Definition 3.4.1. Let C and D be two simplicial categories. A functor $f : C \to D$ is a Dwyer-Kan equivalence if it induces weak equivalences of simplicial sets $Map_C(X, Y) \xrightarrow{\sim} Map_D(FX, FY)$ for every $X, Y \in C$ and an equivalence of categories $\pi_0 C \xrightarrow{\sim} \pi_0 D$.

In particular, every Quillen equivalence of model categories gives rise to a Dwyer-Kan equivalence of their simplicial localizations.

Theorem 3.4.2. *(Bergner [3]) There exists a model category structure on the category of (small) simplicial categories with the Dwyer-Kan equivalences as weak equivalences.*

One can prove that actually every simplicial category is Dwyer-Kan equivalent to the simplicial localization of a certain category with respect to a subcategory of weak equivalences. Thus we can really speak of simplicial categories as models for the homotopy theories, and their model structure as the homotopy theory of the homotopy theories.

Complete Segal spaces. Although simplicial categories are the most intuitive model for $(\infty, 1)$-categories, Dwyer-Kan equivalences are difficult to detect. A nice model with weak equivalences easier to handle has been developed by Rezk in [25], namely the category of complete Segal spaces. We note CSS this category. It has a model structure defined by a certain left Bousfield localization of the standard Reedey model structure on bisimplicial sets. The fibrant objects of CSS are precisely the complete Segal spaces. They satisfy the following interesting recognition property:

Theorem 3.4.3. *(Rezk [25]) The Reedy weak equivalences between two complete Segal spaces are precisely the Dwyer-Kan equivalences between their associated homotopy theories.*

An important instance of complete Segal space is the classification diagram of a model category, which will be defined in defition 4.1. Thus a Reedy weak equivalence of classification diagrams of model categories corresponds to an equivalence of their homotopy categories.

Quasi-categories. Quasi-categories as models for homotopy theories have been extensively studied by Joyal. They are a particular sort of simplicial sets:

Definition 3.4.4. A simplicial set is said to be a quasi-category if every inner horn $\Lambda^k[n] \to X$ (i.e a horn for $0 < k < n$) has a filler $\Delta[n] \to X$.

Examples include Kan complexes and nerves of categories. We do not recall the construction of the model structure, since it will only serves us as an intermediate category between Quillen functors. We note $QCat$ the model category of quasi-categories. Two Quillen pairs which will be of crucial interest for us are the following. Firstly, there is a Quillen pair between complete Segal spaces and quasi-categories [17]:

Theorem 3.4.5. *(Joyal, Tierney [17]) The adjoint pair*

$$t_! : CSS \leftrightarrows QCat : t^!$$

is a Quillen equivalence.

The functor $t^!$ is defined on every quasi-category X by

$$t^!(X)_{m,n} = Mor_{sSets}(\Delta[m] \times \Delta'[n], X)$$

where $\Delta'[n]$ stands for the nerve of the groupoid freely generated by the category $[n]$. Secondly, there is a Quillen pair between simplicial categories and quasi-categories [18]. Given a simplicial category X and the simplicial resolution $C_*[n]$ of the category $[n] = \{0 \to ... \to n\}$, the coherent nerve of X is defined by

$$\tilde{N}(X)_n = Mor_{SCat}(C_*[n], X).$$

It forms a right Quillen functor:

Theorem 3.4.6. *(Joyal [18]) The adjoint pair*

$$J : QCat \leftrightarrows SCat : \tilde{N}$$

is a Quillen equivalence.

Relative categories. We have already defined relative categories. We note $RelCat$ the category of relative categories. According to [1], there is an adjunction between the category of bisimplicial sets and the category of relative categories which lifts any Bousfield localization of the Reedy model structure of bisimplicial sets into a model structure on $RelCat$. In the particular case of the Bousfield localization defining the complete Segal spaces one obtains a Quillen equivalent homotopy theory of the homotopy theories [1]:

Theorem 3.4.7. *(Barwick, Kan [1]) There exists an adjunction*

$$K_\xi : sSets^{\Delta^{op}} \leftrightarrows RelCat : N_\xi$$

which lifts every left Bousfield localization of the Reedy model structure of $sSets^{\Delta^{op}}$ to a Quillen equivalent cofibrantly generated left proper model structure on $RelCat$. In this model structure, a map is weak equivalence or a fibration if and only if its image under N_ξ is so in $sSets^{\Delta^{op}}$.

The functor N_ξ is defined for every relative category (C,W) by

$$N_\xi(C,W)_{m,n} = Mor_{RelCat}(\xi(\hat{m} \times \check{n}), (C,W)).$$

We refer the reader to [1] for the definition of the relative category $\xi(\hat{m} \times \check{n})$. Another crucial feature of $RelCat$ is its behaviour with respect to the simplicial localization functor [2]:

Theorem 3.4.8. *(Barwick, Kan [2])*

(1) The simplicial localization functor induces an equivalence of homotopy categories between $Ho(RelCat)$ and $Ho(SCat)$.

(2) A morphism of relative categories is a weak equivalence if and only if it induces a Dwyer-Kan equivalence of the associated simplicial localizations.

Proof of proposition 3.3.7.

We want to prove the following result:

Proposition 3.4.9. *(Cisinski, Schommer-Pries) Let (C,W) be a relative category. Then the bisimplicial sets $(t^! \circ \tilde{N} \circ L^H)(C,W)$ and $N_\xi(C,W)$ are weakly equivalent in CSS.*

We quote the arguments given in [23] with all the details. As explained in this appendix, the functors $t^!$, $\tilde{N}$, L^H and N_ξ are functors inducing equivalences of homotopy categories. The result we want to prove is equivalent to say that the functors $(t^! \circ \tilde{N} \circ L^H)$ and N_ξ gives rise to the same equivalence of homotopy categories

$$R(t^! \circ \tilde{N} \circ L^H) = RN_\xi : Ho(RelCat) \stackrel{\sim}{\to} Ho(CSS)$$

where the notation R stands for the right derived Quillen functor. Since Quillen equivalences induce Dwyer-Kan equivalences of the simplicial localizations, we want to compare equivalences of simplicial categories $L(RelCat) \stackrel{\sim}{\to} L(CSS)$.

In [26], Toen proved that

$$Aut(LCSS) = \{Id, Inv_{CSS}\} \cong \mathbb{Z}/2\mathbb{Z}$$

where Id is the identity functor and Inv_{CSS} is an involutive functor. This involutive functor associates to any bisimplicial set $X : \Delta^{op} \to sSets$ the precomposition $X \circ \sigma^{op} : \Delta^{op} \to sSets$ by an automorphism $\sigma : \Delta \to \Delta$ of Δ which preserves the objects $[n]$ and permutes the two morphisms $[0] \rightrightarrows [1]$. Actually there is more: such a charaterization of the homotopy automorphisms is available for any homotopy theory of $(\infty,1)$-categories satisfying Toen's axioms in [26]. In particular, according to [1] (section 6.5) this works for $RelCat$ and we get

$$Aut(LRelCat) = \{Id, Inv_{RelCat}\}$$

where the involutive functor Inv_{RelCat} sends a relative category to its opposite category.

These results imply that the functors $R(t^! \circ \tilde{N} \circ L^H)$ and RN_ξ are related by three possible equalities:
(i) $R(t^! \circ \tilde{N} \circ L^H) = RN_\xi$;
(ii) $R(t^! \circ \tilde{N} \circ L^H) = Inv_{CSS} \circ RN_\xi$;
(iii) $R(t^! \circ \tilde{N} \circ L^H) = RN_\xi \circ Inv_{RelCat}$.

To determine which equality is the correct one amounts then to compare $(t^! \circ \tilde{N} \circ L^H)$ and N_ξ on the interval category (two objects and one non trivial arrow). Recall from the appendix that these two functors are defined by the following formulae:

$$(t^! \circ \tilde{N} \circ L^H)(C,W)_{m,n} = Mor_{sSets}(\Delta[m] \times \Delta'[n], Mor_{SCat}(C_*[\bullet], L^H(C,W)))$$

and

$$N_\xi(C,W)_{m,n} = Mor_{RelCat}(\xi(\hat{m} \times \check{n}), (C,W)).$$

Since neither functors invert the interval subcategory, the good relation is (i), which concludes the proof.

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Résumé

Le but de cette thèse est de mettre en place une théorie d'homotopie générale pour les catégories de bigèbres différentielles graduées.

Une première partie est consacrée au cas des catégories de bigèbres définies par un couple d'opérades en distribution. Les bigèbres classiques, les bigèbres de Lie, les bigèbres de Poisson fournissent des exemples de telles structures de bigèbres. Le résultat principal de cette partie montre que la catégorie des bigèbres associée à un couple d'opérades en distribution hérite d'une structure de catégorie de modèles.

La notion de PROP donne un cadre pour étudier des structures de bigèbres générales, impliquant des opérations à plusieurs entrées et plusieurs sorties comme générateurs de la structure, par opposition aux opérades en distribution qui ne permettent de coder que des opérations à une seule entrée ou à une seule sortie seulement. Les PROPs forment une catégorie, dans laquelle on peut définir une notion d'objet cofibrant avec de bonnes propriétés homotopiques.

La seconde partie de la thèse est consacrée à la théorie homotopique des bigèbres sur un PROP. Le résultat principal de la thèse est que les catégories de bigèbres associées à des PROPs cofibrants faiblement équivalents ont des catégories homotopiques équivalentes. En fait, on prouve un théorème plus précis qui donne une équivalence au niveau des localisations simpliciales des catégories.

Notre théorème entraine que la catégorie des bigèbres associée à une résolution cofibrante d'un PROP donné P définit une notion de bigèbre à homotopie près sur P indépendante du choix de la résolution, et permet de donner un sens à des problèmes de réalisation homotopiques dans ce cadre.

Mots-clés: PROP, bigèbre, algèbre homotopique, espace classifiant.

Abstract

The purpose of this thesis is to set up a general homotopy theory for categories of differential graded bialgebras.

A first part is devoted to the case of bialgebras defined by a pair of operads in distribution. Classical bialgebras, Lie bialgebras and Poisson bialgebras provide examples of such bialgebra structures. The main result of this part asserts that the category of bialgebras associated to a pair of operads in distribution inherits a model category structure.

The notion of a PROP provides a setting for the study of general bialgebras structures, involving operations with multiple inputs and multiple outputs as generators of the structure, in contrast to operads in distribution which only encode operations with either one single input or one single output. PROPs form a category, in which one can define a notion of cofibrant object with good homotopical properties.

The second part of the thesis is devoted to the homotopy theory of bialgebras over a PROP. The main result of the thesis asserts that the categories of bialgebras associated to weakly equivalent cofibrant props have equivalent homotopy categories. We actually prove a more precise theorem asserting that this equivalence holds at the level of a simplicial localization of the categories.

Our theorem implies that the category of bialgebras associated to a cofibrant resolution of a given PROP P defines a notion of bialgebra up to homotopy over P independent of the choice of the resolution, and enables us to give a sense to homotopical realization problems in this setting.

Keywords: Operads, algebras, Koszul duality, homology theories, obstruction theories.