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Immeubles à Angles Droits et Modules Combinatoires au Bord

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devant le jury composé de :

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Immeubles à angles droits et modules combinatoires au bord

Antoine Clais

Il restait encore une cinquième combinaison.
Dieu s'en est servi pour achever le dessin de l'univers.
Platon, Timé

Résumé

L'objet de cette thèse est d'étudier la géométrie des immeubles à angles droits. Ces espaces, définis par J. Tits sont des espaces singuliers qui peuvent être vus comme des généralisations des arbres en dimension supérieure.

La thèse est divisée en deux parties. Dans la première partie, nous décrivons comment la notion de résidus parallèles permet de comprendre l'action d'un groupe sur un immeuble. En corollaire nous retrouvons que dans un groupe de Coxeter et dans un produit graphé les intersections de sous-groupes paraboliques sont paraboliques.

Dans la seconde partie, nous abordons la structure quasi-conforme du bord des immeubles hyperboliques à angles droits. En particulier, nous trouvons des exemples d'immeubles de dimension 3 et 4 dont le bord vérifie la propriété combinatoire de Loewner. Cette propriété est une version faible de la propriété de Loewner. Cette partie est motivée par le fait que, depuis G.D. Mostow, la structure quasi-conforme au bord a mené à plusieurs résultats de rigidités dans les espaces hyperboliques. Dans le cas des immeubles de dimension 2, M. Bourdon et H. Pajot ont prouvé la rigidité des quasi-isométries en utilisant la propriété de Loewner au bord.

Abstract

The object of this thesis is to study the geometry of right-angled buildings. These spaces, defined by J. Tits, are singular spaces that can be seen as trees of higher dimension.

The thesis is divided in two parts. In the first part, we describe how the notion of parallel residues allows to understand the action of a group on the building. As a corollary we recover that in Coxeter groups and in graph products intersections of parabolic subgroups are parabolic.

In the second part, we discuss the quasiconformal structure of boundaries of right-angled hyperbolic buildings thanks to combinatorial tools. In particular, we exhibit some examples of buildings of dimension 3 and 4 whose boundary satisfy the combinatorial Loewner property.This property is a weak version of the Loewner property. This part is motivated by the fact that the quasiconformal structure of the boundary led to many results of rigidity in hyperbolic spaces since G.D. Mostow. In the case of buildings of dimension 2, a lot of work has been done by M. Bourdon and H. Pajot. In particular, the Loewner property on the boundary permitted them to prove the quasi-isometry rigidity for some buildings of dimension 2.

Remerciements

Je tiens à adresser mes plus vifs remerciements à Marc Bourdon qui pendant mes années de thèse, et avant pendant mes années de master, a été un professeur disponible, patient et impliqué. Je lui dois de m'avoir ouvert l'esprit à des mondes qui nourriront longtemps mon imagination : la géométrie hyperbolique et les immeubles.

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Je suis spécialement reconnaissant à María-Dolores Parrilla Ayuso pour m'avoir aidé à réaliser les figures en 3 dimensions qui illustrent cette thèse.

Enfin, je ne saurais trop remercier mes parents qui m'ont accompagné pendant les trois années de mon doctorat et surtout qui m'ont soutenu pendant les années qui précédèrent. Sans ce soutien, il est certain que ce travail n'aurait pas vu le jour.

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Chapitre 0

Introduction

0.0.1 Point de départ

L'étude des modules de courbes dans les espaces métriques compacts a pour origine la théorie classique des applications quasi-conformes dans les espaces euclidiens (cf. [Väi71] ou [Vuo88]). L'objectif de cette théorie est de décrire la régularité des applications quasi-conformes dans $\mathbb{R}^d$ et de trouver des invariants pour ces applications. La notion d'espace de Loewner a été introduite par J. Heinonen et P. Koskela (cf. [HK98] ou [Hei01]) pour décrire les espace métriques mesurés dont les applications quasi-conformes ont un comportement aussi régulier que les applications quasi-conformes dans les espace euclidiens.

Par ailleurs, on sait depuis G.D. Mostow que la structure quasi-conforme du bord d'un espace hyperbolique contrôle la géométrie de l'espace. Cette idée se généralise aux espaces hyperboliques au sens de Gromov. Quand la structure d'espace de Loewner apparaît au bord d'un espace hyperbolique on peut montrer des résultats de rigidité dans l'espace (cf. [Haï09b] pour un survol de ce type de résultats). Pour montrer la rigidité d'un espace hyperbolique on cherche à montrer que les quasi-isométries de l'espace sont données par les homéomorphismes quasi-symétriques du bord. Les espaces de Loewner permettent cette identification car dans ces espaces les classes d'applications quasi-symétriques, quasi-Moebius, et quasi-conformes sont égales.

A comparer avec un texte qui occupe toute la largeur de la page, comme celui-ci. Cependant, il est très difficile de montrer que le bord d'un espace hyperbolique vérifie la propriété de Loewner. Pour ce faire il est nécessaire de connaître une mesure au bord qui soit optimale pour la *dimension conforme*. Cet invariant conforme a été introduit par P. Pansu dans [Pan89]. Trouver une telle mesure et même calculer cette dimension sont des problèmes très difficiles qui ne sont résolus que pour peu d'exemples à l'heure actuelle.

Un exemple particulièrement intéressant pour nous est le travail réalisé par M. Bourdon et H. Pajot dans les immeubles fuchsiens. Ils ont montré que les bords de ces immeubles étaient des espaces de Loewner et ont ensuite utilisé cette structure pour prouver la rigidité des quasi-isométries de ces immeubles (cf. [BP00]).

Les immeubles sont des espaces singuliers introduits par J. Tits pour étudier les groupes de Lie exceptionnels. De nos jours, les immeubles constituent un domaine d'étude à part entière en théorie géométrique des groupes. Parmi les immeubles, ceux à angles droits ont été classifiés par F. Haglund et F. Paulin dans [HP03]. Ces immeubles sont munis d'une structure de mur et d'une action de groupe simplement transitive sur les chambres, ce qui en fait des objet singuliers très réguliers. Les immeubles fuchsiens étudiés par M. Bourdon et H. Pajot sont les immeubles hyperboliques de dimension 2 au sens que leurs appartements sont isométriques à $\mathbb{H}^2$. Au regard du résultat de M. Bourdon et H. Pajot on peut se demander :

Question 0.0.1. *Les immeubles hyperboliques à angles droits de dimension supérieure sont-ils rigides ? Quelles-sont les propriétés quasi-conformes de leurs bords ?*

La géométrie des immeubles hyperboliques à angles droits de dimension supérieure est proche de celles des immeubles fuchsiens. Cela autorise à penser que ces questions doivent avoir des réponses intéressantes. Cependant les méthodes utilisées au bord des immeubles fuchsiens sont très spécifiques à la dimension 2. Ces questions ne sont donc pas simples. Dans cette thèse on utilise les *modules combinatoires* pour fournir une première approche de la structure quasi-conforme des immeubles hyperboliques à angles droits de dimension supérieure.

Un célèbre problème de rigidité étudié à l'aide des propriétés quasi-conforme du bord est la conjecture suivante dû à J.W. Cannon.

Conjecture 0.0.2 ([CS98, Conjecture 5.1.])**.** *Si Γ est un groupe hyperbolique et que son bord est homéomorphe à $\mathbb{S}^2$, alors Γ agit géométriquement sur $\mathbb{H}^3$.*

Cette conjecture implique notamment la conjecture d'hyperbolisation des variétés de dimension 3 de Thurston. Bien que la conjecture de Thurston soit devenu un théorème de G. Perelman, la conjecture de Cannon reste une question intéressante puisqu'elle est indépendante logiquement de celle de Thurston.

Les modules combinatoires ont été introduits par J.W. Cannon dans [Can94] et par M. Bonk et B. Kleiner dans [BK02] au cours de l'étude de la structure quasi-conforme des sphères de dimension 2 en vu de résoudre la conjecture et par P. Pansu dans un contexte plus général dans [Pan89]. Les modules combinatoires conduisent à définir une version faible de la propriété de Loewner : la *propriété de Loewner combinatoire* (CLP). Ces modules combinatoires permettent aussi de caractériser la dimension conforme à l'aide d'un exposant critique au bord.

Récemment, M. Bourdon et B. Kleiner (cf. [BK13]) ont trouvé des exemples de bords de groupes de Coxeter qui vérifient la CLP mais dont on ne sait pas encore s'ils vérifient la propriété de Loewner. Ils ont, de plus, utilisé la CLP pour donner une nouvelle preuve de la conjecture de Cannon pour les groupes de Coxeter. Certaines des méthodes qu'ils ont utilisées dans les groupes de Coxeter sont transposables dans les immeubles à angles droits. Cela fournit une motivation pour explorer le bord des immeubles hyperboliques à angles droits à l'aide de modules combinatoires.

0.0.2 Principaux résultats

Dans le premier chapitre de cette thèse, on présente les immeubles dans un cadre général et on s'intéresse en particulier à la notion de résidus parallèle dans les immeubles. Les résidus parallèles sont les résidus obtenus par projection de deux résidus l'un sur l'autre. Le principal résultat de ce chapitre est le suivant.

Théorème 1 (Théorème 1.2.14)**.** *Soit Δ un immeuble et G un groupe d'automorphismes de Δ. Supposons que pour toute paire de résidus parallèles P, P' on ait $\mathrm{Stab}_G(P) = \mathrm{Stab}_G(P')$. Alors pour toute paire de résidus R et Q on a :*

$$\mathrm{Stab}_G(R) \cap \mathrm{Stab}_G(Q) = \mathrm{Stab}_G(\mathrm{proj}_R(Q)) = \mathrm{Stab}_G(\mathrm{proj}_Q(R)).$$

En conséquence de ce théorème on obtient deux preuves parallèles du résultat suivant concernant les sous-groupes paraboliques dans les groupes de Coxeter et dans les produits graphés. Les sous-groupes paraboliques sont les conjugués des sous-groupes de générateurs. Dans le cas des groupes de Coxeter ce résultat classique est dû à J. Tits. Dans le cas des produits graphés il a été montré récemment dans [AM13] dans un cadre plus général par des méthodes combinatoires.

Théorème 2 (Théorèmes 1.3.2 et 1.5.3)**.** *Soit G un groupe de Coxeter ou un produit graphé. Dans G les intersections de sous-groupes paraboliques sont paraboliques.*

Dans le second chapitre, on se restreint à l'étude des immeubles hyperboliques à angles droits. On utilise les modules combinatoires pour explorer la structure quasi-conforme au bord de ces immeubles. Grâce à des méthodes venant de [BK13], on peut contrôler les modules combinatoires au bord par les courbes contenues dans les *ensembles limites paraboliques* (cf. Partie 2.5). Ensuite, on introduit un *module à poids* aux bords des appartements. Cela permet de contrôler les modules aux bords des immeubles par des modules aux bords des appartements (cf. Partie 2.7). Pour des exemples bien choisis, le bord des appartements a beaucoup de symétries, ce qui fournit un fort contrôle des modules. En particulier, on utilise ces symétries pour trouver des exemples d'immeubles hyperboliques de dimension 3 et 4 dont les bords vérifient la CLP.

Théorème 3 (Corolaire 2.9.3)**.** *Soit D le dodécaèdre régulier à angles droits dans $\mathbb{H}^3$ ou le* 120*-cell régulier dans $\mathbb{H}^4$. Soit W_D le groupe engendré par les réflexions hyperboliques par rapport au faces de D. Pour $q \geq 3$, notons Δ l'immeuble à angles droits d'épaisseur constante égale à q et de groupe de Coxeter W_D. Alors $\partial\Delta$ vérifie la CLP.*

En complément de ce résultat nous donnons aussi dans le Théorème 2.8.1 une caractérisation de la dimension conforme du bord de l'immeuble par un exposant critique calculé au bord d'un appartement.

0.0.3 Terminology and notations

All along this thesis, we will use the following conventions. The identity element in a group will always be designated by e. For a set E, the *cardinality* of E is designated by $\#E$. A *proper* subset F of E is a subset $F \subsetneq E$.

If $\mathcal{G}$ is a graph then $\mathcal{G}^{(0)}$ is the *set of vertices* of $\mathcal{G}$ and $\mathcal{G}^{(1)}$ is the *set of edges* of $\mathcal{G}$. For $v, w \in \mathcal{G}^{(0)}$, we write $v \sim w$ if there exists an edge in $\mathcal{G}$ whose extremities are v and w. If $V \subset \mathcal{G}^{(0)}$, the *full subgraph* generated by V is the graph $\mathcal{G}_V$ such that $\mathcal{G}_V^{(0)} = V$ and an edge lies between to vertices v, w if and only if there exists an edge between v and w in $\mathcal{G}$. A full subgraph is called a *circuit* if it is a cyclic graph C_n for $n \geq 3$. A graph is called a *complete graph* if for any pair of distinct vertices v,w there exists an edge between v and w.

A *curve* in a compact metric space (Z, d) is a continuous map $\eta : [0,1] \longrightarrow Z$. Usually, we identify a curve with its image. If η is a curve in Z, then $\mathcal{U}_\epsilon(\eta)$ denotes the ϵ-neighborhood of η for the C^0-topology. This means that a curve $\eta' \in \mathcal{U}_\epsilon(\eta)$ if and only if there exists $s : t \in [0,1] \longrightarrow [0,1]$ a parametrization of η such that for any $t \in [0,1]$ one has $d(\eta(s(t)), \eta'(t)) < \epsilon$.

In a metric space Z, if $A \subset Z$ then $N_r(A)$ is the *r-neighborhood* of A. The *closure* of A is designated by $\overline{A}$ and the *interior* of A by $\mathrm{Int}(A)$. If $B = B(x, R)$ is an open ball and $\lambda \in \mathbb{R}$ then λB is the ball of radius λR and of center x. A ball of radius R is called an *R-ball*. The closed ball of center x and radius R is designated by $\overline{B(x, R)}$.

A *geodesic line* (resp. *ray*) in a metric space (Z, d) is an isometry from $(\mathbb{R}, |\cdot - \cdot|)$ (resp. $([0, +\infty), |\cdot - \cdot|)$ to (Z, d). The real hyperbolic space (resp. Euclidean space) of dimension d is denoted $\mathbb{H}^d$ (resp. $\mathbb{E}^d$).

Chapter 1

Groups acting on buildings

Introduction

At the starting point of this thesis, we studied the geometry of right-angled buildings. The geometry of these buildings can be understood thanks to a wall structure and a simply transitive group action. Coxeter systems come also with a wall structure and a simply transitive group action. This provides some similarities in the discussions about right-angled buildings and Coxeter systems. Regarding these similarities, it seemed relevant to discuss these two cases in parallel. This is what we do in this chapter.

The main tools used in this chapter are the parallel residues in the Davis complex of the building. Along with a wall structure, the notion of parallel residues makes it possible to recover that intersections of parabolic subgroups are parabolic in Coxeter groups and in graph products (see Theorems 1.3.2 and 1.5.3).

The principal result about parallel residues (Theorem 1.2.14) and the geometric method used to prove Theorems 1.3.2 and 1.5.3 are new. Yet these last two theorems were already known. Theorem 1.3.2 is due to J. Tits and Theorem 1.5.3 has been proved recently in a more general case thanks to combinatorial arguments (see [AM13, Proposition 3.4.]). Besides this chapter is enriched with facts about residues. Some of these were already proved in slightly different contexts than the one we use. For the sake of completeness we chose to expose here the proofs in the setting of the Davis complex that we will use all along this thesis. In particular, some of these results in right-angled buildings (Proposition 1.4.15 and Theorem 1.4.17) will find their use in Chapter 2.

Organization of the chapter

In Section 1.1, we start by reminding basic facts about chamber systems and Coxeter groups. In particular we insist on the geometric realization of the Coxeter groups. In Section 1.2, we remind the abstract definition of buildings and then we discuss the notion of parallel residues in buildings. Parallel residues will help us to describe intersections of stabilizers of residues. In Sections 1.3 and 1.5, we follow a blue print that gives us as corollaries that intersections of parabolic subgroups are parabolic in Coxeter groups and in graph products. Note that essentially, the proofs of Theorems 1.3.2 and 1.5.3 including the intermediate results are the same. In Section 1.4, we discuss the geometric structure of right-angled buildings that will be used in Section 1.5 but also in Chapter 2.

1.1 Chamber and Coxeter systems

Chamber systems provide an abstract context in which J. Tits defined buildings. Coxeter groups are chambers systems that are used as block pattern to construct buildings.

In this section we start by reminding basic facts about chamber and Coxeter systems. Then we describe the geometric realization of Coxeter groups due to M.W. Davis. Eventually, we discuss walls and residues in the Davis complex.

For details concerning the notions reminded in this section, we refer to [Tit74], [Ron89], or [AB08]. Concerning the Davis realization, we can refer to [Dav08, Chapter 8] or to [Mei96] for an example of the Davis construction along with suggestive pictures.

Here $S = \{s_1, \ldots, s_n\}$ is a fixed finite set.

1.1.1 Chamber system

Following the definition of J. Tits, a *chamber system* X over S is a set endowed with a family of partitions indexed by S. The elements of X are called *chambers*.

Hereafter X is a chamber system over S. For $s \in S$, two chambers $c, c' \in X$ are said to be *s-adjacent* if they belong to the same subset of X in the partition associated with s. Then we write $c \sim_s c'$. Usually, omitting the type of adjacency we refer to *adjacent* chambers and we write $c \sim c'$. Note that any chamber is adjacent to itself.

A *morphism* $f : X \longrightarrow X'$ between two chamber systems X, X' over S is a map that preserves the adjacency relations. A bijection of X that preserves the adjacency relations is called an *automorphism* and we designate by $\mathrm{Aut}(X)$ the *group of automorphisms* of X. A *subsystem of chamber* Y of X is a subset $Y \subset X$ such that the inclusion map is a morphism of chamber systems.

We call *gallery*, a finite sequence $\{c_k\}_{k=1,\ldots,\ell}$ of chambers such that $c_k \sim c_{k+1}$ for $k = 1, \ldots, \ell - 1$. The galleries induce a *metric on* X.

Definition 1.1.1. *The* distance between two chambers x *and* y *is the length of the shortest gallery connecting* x *to* y*.*

We use the notation $d_c(\cdot, \cdot)$ for this metric over X. A shortest gallery between two chambers is called *minimal*.

Let $I \subset S$. A subset C of X is said to be *I-connected* if for any pair of chambers $c, c' \in C$ there exists a gallery $c = c_1 \sim \cdots \sim c_\ell = c'$ such that for any $k = 1, \ldots, \ell - 1$, the chambers c_k and c_{k+1} are i_k-adjacent for some $i_k \in I$.

Definition 1.1.2. *The I-connected components are called the* I-residues *or the* residues of type I. *The cardinality of I is called the* rank *of the residues of type I. The residues of rank* 1 *are called* panels.

The following notion of convexity is used in chamber systems.

Definition 1.1.3. *A subset C of X is called* convex *if every minimal gallery whose extremities belong to C is entirely contained in C.*

The convexity is stable by intersection and for $A \subset X$, the *convex hull* of A is the smallest convex subset containing A. In particular, convex subsets of X are subsystems. The following example is crucial because it will be used to equip Coxeter groups and graph products with structures of chamber systems (see Definition 1.1.7 and Theorem 1.4.4).

Example 1.1.4. *Let G be a group, B a subgroup and $\{H_i\}_{i\in I}$ a family of subgroups of G containing B. The set of left cosets of H_i/B defines a partition of G/B. We denote by $C(G, B, \{H_i\}_{i\in I})$ this chamber system over I. This chamber system comes with a natural action of G. The group G acts by automorphisms and transitively on the set of chambers.*

1.1.2 Coxeter systems

A *Coxeter matrix over S* is a symmetric matrix $M = \{m_{r,s}\}_{r,s\in S}$ whose entries are elements of $\mathbb{N} \cup \{\infty\}$ such that $m_{s,s} = 1$ for any $s \in S$ and $\{m_{r,s}\} \geq 2$ for any $r, s \in S$ distinct. Let M be a Coxeter matrix. The *Coxeter group* of type M is the group given by the following presentation

$$W = \langle s \in S | (rs)^{m_{r,s}} = 1 \text{ for any } r, s \in S \rangle .$$

We call *special subgroup* a subgroup of W of the form

$$W_I = \langle s \in I | (rs)^{m_{r,s}} = 1 \text{ for any } r, s \in I \rangle \text{ with } I \subset S.$$

Definition 1.1.5. *We call* parabolic subgroup *a subgroup of W of the form wW_Iw^{-1} where $w \in W$ and $I \subset S$. An involution of the form wsw^{-1} for $w \in W$ and $s \in S$ is called a* reflection.

Example 1.1.6. *Let $\mathbb{X}^d = \mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$. A* Coxeter polytope *is a convex polytope of $\mathbb{X}^d$ such that any dihedral angle is of the form $\frac{\pi}{k}$ with k not necessarily constant. Let D be a Coxeter polytope and let $\sigma_1, \ldots, \sigma_n$ be the codimension 1 faces of D. We set $M = \{m_{i,j}\}_{i,j=1,\ldots,n}$ the matrix defined by $m_{i,i} = 1$, if σ_i and σ_j do not meet in a codimension 2 face $m_{i,j} = \infty$, and if σ_i and σ_j meet in a codimension 2 face $\frac{\pi}{m_{i,j}}$ is the dihedral angle between σ_i and σ_j.*

Then a theorem of H. Poincaré (see [GP01, Theorem 1.2.]) says that the reflection group of $\mathbb{X}^d$ generated by the codimension 1 faces of D is a discrete subgroup of $\mathrm{Isom}(\mathbb{X}^d)$ *and is isomorphic to the Coxeter group of type M.*

On Figure 1.1 are represented the Coxeter systems associated with the following Coxeter matrices

$$M_1 = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix} \text{ and } M_3 = \begin{pmatrix} 1 & 4 & \infty & \infty & 4 \\ 4 & 1 & 4 & \infty & \infty \\ \infty & 4 & 1 & 4 & \infty \\ \infty & \infty & 4 & 1 & 4 \\ 4 & \infty & \infty & 4 & 1 \end{pmatrix}.$$

Definition 1.1.7. *With the notation introduced in Example 1.1.4, the* Coxeter system *associated with W is the chamber system over S given by $C(W, \{e\}, \{W_{\{s\}}\}_{s\in S})$. We use the notation (W,S) to designate this chamber system.*

The chambers of (W,S) are the elements of W and two distinct chambers $w, w' \in W$ are s-adjacent if and only if $w = w's$. For $I \subset S$, notice that for any I-residue R in (W,S) there exists $w \in W$ such that, as a set $R = wW_I$. Again W is a group of automorphisms of (W,S) that acts transitively on the set of chambers.

Hereafter (W,S) is a fixed Coxeter system.

Example 1.1.8. *In the case of Example 1.1.6, the chamber system associated with W is realized geometrically by the tilling of $\mathbb{X}^d$ by copies of the polytope D (see Figure 1.1). Two chambers are adjacent in (W,S) if and only if the corresponding copies of D in $\mathbb{X}^d$ share a codimension 1 face.*

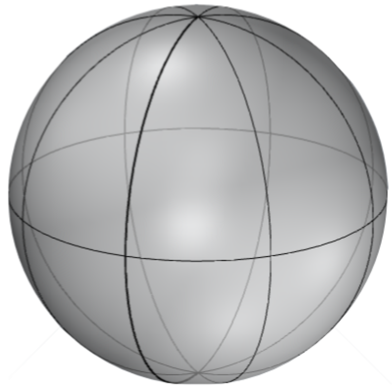
1.1.3 The Davis chamber of (W,S)

The geometric realization of M.W. Davis can be seen as a generalization to all the Coxeter groups of the Example 1.1.8. The first step to describe the geometric realization of W is to construct its *Davis chamber*. We remind that $S = \{s_1, \ldots, s_n\}$ is a set of generators of W. Let $\mathcal{S}_{\neq S}$ be the set of subsets of S different from S. We denote by

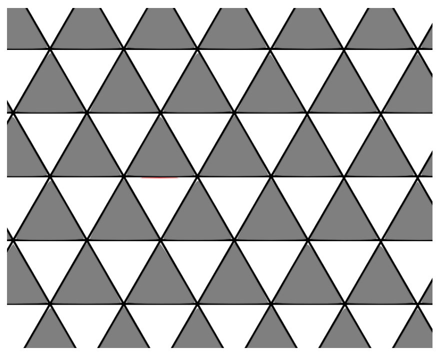
$$\mathcal{S}_f \text{ the set of subsets } F \subsetneq S \text{ such that } W_F \text{ is finite.}$$

Following [Dav08, Appendix A], a poset admits a geometric realization which is a simplicial complex. This complex is such that the inclusion relations between cells represent the partial order. We denote by D the *Davis chamber* which is the geometric realization of the poset $\mathcal{S}_f$. In the following we give details of this construction. An example of this construction is given by Figure 1.2.

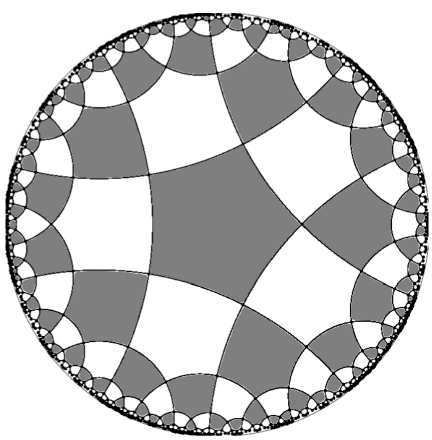
Let Δ^{n-1} be the standard $(n-1)$-simplex and label the codimension 1 faces of Δ^{n-1} with distinct elements of S. Then σ a codimension k face of Δ^{n-1} is associated with a *type i.e* a subset $I \subset S$ of cardinality k. In this setting, we write σ_I for the *face of type* I. Equivalently, we can say that each vertex of the barycentric subdivision of Δ^{n-1} is associated with a subset of S. Adding the fact that the empty set is associated with the barycenter of the whole simplex, we get a bijection between the vertices of the barycentric



(a) The spherical Coxeter system of M_1



(b) The Euclidean Coxeter system of M_2



(c) The hyperbolic Coxeter system of M_3

Figure 1.1: Examples of geometric realizations of Coxeter systems

subdivision and $\mathcal{S}_{\neq S}$. Hence a vertex in the barycentric subdivision is designated by $(s_i)_{i\in K}$ for $K \subset \{1,\ldots,n\}$. Using this identification, let $\mathcal{T}$ be the subgraph of the 1-skeleton of the barycentric subdivision of Δ^{n-1} obtained as follows:

- $\mathcal{T}^{(0)} = \mathcal{S}_{\neq S}$,
- the vertices $(s_i)_{i\in I}$ and $(s_j)_{j\in J}$, with $\#J \geq \#I$, are adjacent if and only if $I \subset J$ and $\#I = \#J - 1$.

In the following definition, for $k \geq 1$ we call a k-cube, a CW-complex that is isomorphic, as a cellular complex, to the Euclidean k-cube $[0,1]^k$. In particular, it is not necessary to equip these cubes with a metric for the purpose of this chapter.

Definition 1.1.9. *The* 1*-skeleton of the Davis chamber* $D^{(1)}$ *is the full subgraph of* $\mathcal{T}$ *generated by the elements of* $\mathcal{S}_f$. *The Davis chamber is obtained from* $D^{(1)}$ *by attaching a* k*-cube inside any full subgraph generated by* 2^k *vertices that is the* 1*-skeleton of a* k*-cube.*

By construction, $D \subset \Delta^{n-1}$. We call *maximal faces* of D the subsets of the form $\sigma \cap D$ where σ is a codimension 1 face of Δ^{n-1}. Likewise, for $I \subset S$, the *face of D of type I* is $D \cap \sigma_I$. Note that that the faces of D are made of branching of cubes of various dimensions.

Example 1.1.10. *In the case of Example 1.1.6, the Davis chamber is combinatorially identified with the Coxeter polytope. Then if we equip D with the appropriate metric (Euclidean, spherical, or hyperbolic) we recover the Coxeter polytope.*

1.1.4 The Davis complex associated with W

For $x \in D$, if I is the type of the face containing x in its interior, we set $W_x := W_I$. To the interior points of D we associate the trivial group $W_\emptyset$.

Now we can define the *Davis complex*: $\Sigma(W,S) = D \times W/\sim$ with

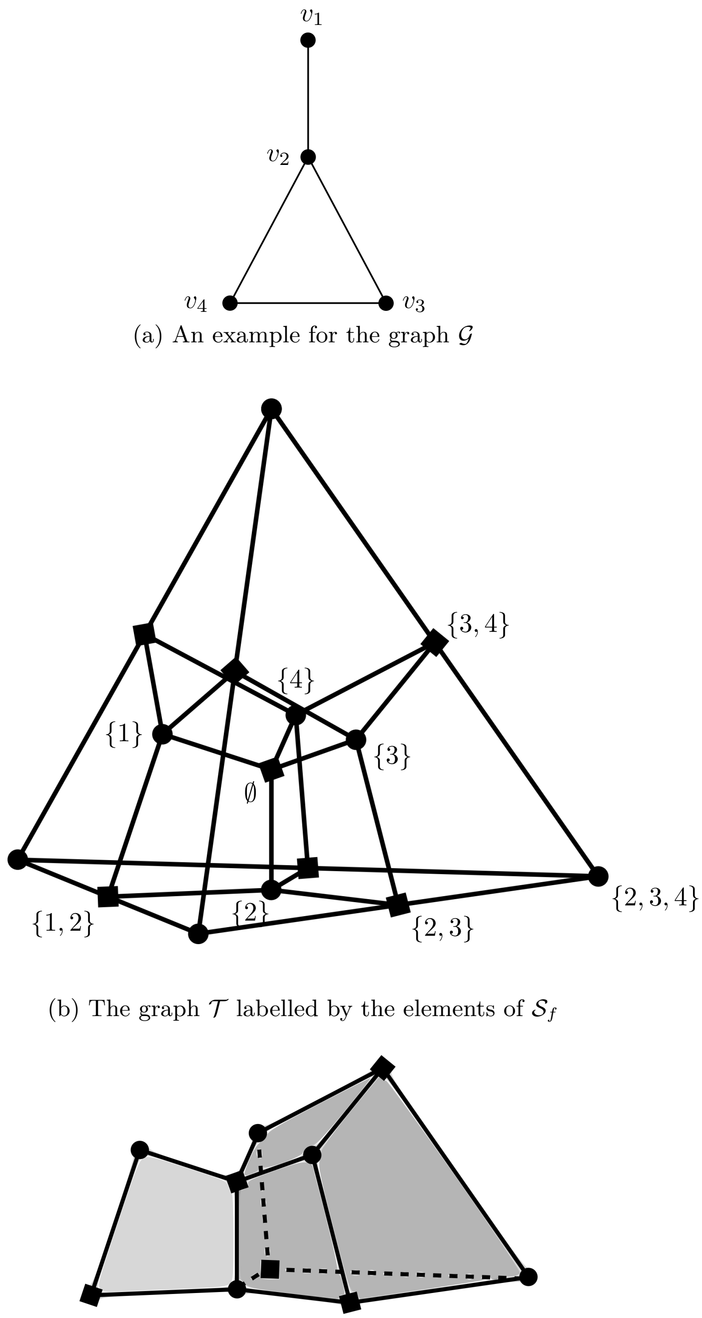
$$(x,w) \sim (y,w') \text{ if and only if } x = y \text{ and } w^{-1}w' \in W_x.$$

We call *chamber of* $\Sigma(W,S)$ a subset of $\Sigma(W,S)$ of the form $[D \times \{w\}]$ with $w \in W$. Two chambers of $\Sigma(W,S)$ are adjacent if and only if they share a maximal face in $\Sigma(W,S)$. For a subset $E \subset \Sigma(W,S)$ we designate by $\mathrm{Ch}(E)$ the set of chambers contained in E.

It appears that $\Sigma(W,S)$ endowed with this structure of chamber system is isomorphic to (W,S). Thanks to this identification, it makes sense to write (W,S) for the set of the chambers in $\Sigma(W,S)$.

In particular, the left action of W on itself induces an action on $\Sigma(W,S)$. For $g \in W$ and $[(x,w)] \in W$, we set $g[(x,w)] := [(x,gw)]$. This action is simply transitive and isometric on (W,S) equipped with $d_c(\cdot,\cdot)$.

Example 1.1.11. *In the case of Example 1.1.6, if we equip D with the appropriate metric we recover that the Davis complex is realized by the tilling of $\mathbb{X}^d$ by D (see Figure 1.1).*



(a) An example for the graph $\mathcal{G}$

(b) The graph $\mathcal{T}$ labelled by the elements of $\mathcal{S}_f$

(c) The Davis chamber resulting from $\mathcal{G}$

Figure 1.2: Example of the construction of a Davis chamber

1.1.5 Walls and residues in the Davis complex

We call *base chamber* of $\Sigma(W,S)$, denoted by x_0, the chamber $[D \times \{e\}]$. For $w \in W$, as $[D \times \{w\}]$ is the image of x_0 under w, we designate this chamber by wx_0. Here we present some classical tools used to describe the structure of (W,S).

Definition 1.1.12.

1. *We call* wall *in $\Sigma(W,S)$ the proper subcomplex stabilized by a reflection $wsw^{-1} \in W$. We denote by $\mathcal{M}(W,S)$ the set of walls of $\Sigma(W,S)$.*

2. *Let M be a wall associated with a reflection $r \in W$. For a chamber $c \in (W,S)$ we say that M is* along *c if $r(c)$ is adjacent to c.*

Proposition 1.1.13. *[Tit74, Corollary 2.8.] Let $M \in \mathcal{M}(W,S)$ and $r \in W$ be the reflection stabilizing M. Then $\Sigma(W,S)\backslash M$ consists of two disjoint connected components. We call* half-spaces *bounded by M the closure in Σ of these components. We designate by $H_0(M)$ and $H_1(M)$ these half-spaces, with the convention that $x_0 \subset H_0(M)$.*

In this setting, $H_0(M)$ and $H_1(M)$ are such that

- $\mathrm{Ch}(H_0(M)) = \{x \in (W,S) : d_c(x_0,x) < d_c(x_0,rx)\}$,
- $\mathrm{Ch}(H_1(M)) = \{x \in (W,S) : d_c(rx_0,x) < d_c(rx_0,rx)\}$.

Moreover r permutes $H_0(M)$ and $H_1(M)$.

By symmetry of the relation induced by the connected components, we obtain that for $i = 1,2$ and any $y \in \mathrm{Ch}(H_i(M))$,

$$\mathrm{Ch}(H_i(M)) = \{x \in \mathrm{Ch}(\Sigma) : d_c(y,x) < d_c(y,rx)\}.$$

In the following, we write $\mathcal{H}(W,S)$ for the set of all the half-spaces of (W,S). In the literature, half-spaces are also called *roots*.

Definition 1.1.14. *Let $M \in \mathcal{M}(W,S)$ and $E,F \subset \Sigma(W,S)$.*

i) We say that M crosses *E if $E\backslash M$ has several connected components.*

ii) We say that M separates *E* and *F if their interior are entirely contained in the two distinct connected components of $\Sigma(W,S)\backslash M$.*

As we see with the following results, walls and half-spaces can be used to describe the metric structure of (W,S).

Theorem 1.1.15 ([Tit74, Theorem 2.19.])**.** *A subset $C \subset (W,S)$ is convex if and only if it is the set of chambers of an intersection of half-spaces.*

Thanks to this proposition we notice that residues are convex in (W, S).

Proposition 1.1.16 ([Tit74, Proposition 2.22.])**.** *Let $x_1, x_2 \in (W, S)$. If $d_c(\cdot, \cdot)$ denotes the metric on the chambers, then*

$$d_c(x_1, x_2) = \#\{M \in \mathcal{M}(W, S) : M \text{ seperates } x_1 \text{ and } x_2\}.$$

Now we discuss the notion of residues in the geometric realization. For $I \subset S$ and $w \in W$, the subset $wW_I x_0 \subset \Sigma(W, S)$ is the union of the chambers of the I-residue containing wx_0. For simplicity, in the following we also call a subset of the form $wW_I x_0 \subset \Sigma(W, S)$ a *residue*. Notice that a reflection relative to a wall that crosses $wW_I x_0$ is of the form $wgsg^{-1}w^{-1}$ with $s \in I$ and $g \in W_I$. Along with the definitions of the action and the residues we obtain the following fact.

Fact 1.1.17. *Let $R = wW_I x_0$ be a residue. Then*

- *R is stabilized by the reflections relative to the walls that cross it,*
- $\mathrm{Stab}_W(R) = wW_I w^{-1}$ *is generated by these reflections,*
- *the type I is given by the types of the maximal faces of wx_0 contained in a wall that crosses R.*

The following result gives a converse to the last fact. The proof is the same as the proof of [BK13, Theorem 5.5.].

Theorem 1.1.18. *Let $R \subset \Sigma(W, S)$ be the union of a convex set of chambers and let P_R denote the group generated by the reflections relative to the walls that cross R. If P_R stabilizes R, then R is a residue in $\Sigma(W, S)$.*

Proof. Up to a translation on R and a conjugation on P_R, we assume that $x_0 \subset R$. Let $C = \mathrm{Ch}(R)$. We start by proving that P_R acts freely and transitively on C. The action of W is free thus the action of P_R is free. For $x \in C$, by convexity of C, there exists a gallery

$$x_0 \sim x_1 \sim \cdots \sim x_\ell = x$$

of distinct chambers in C. Let M_i be the wall along x_{i-1} and x_i. We set $s_i \in S$ the type of the adjacency relation between x_{i-1} and x_i. Then

$$s_1 x_0 = x_1, \ s_1 s_2 x_0 = x_2, \ \ldots, \ s_1 \ldots s_\ell x_0 = x.$$

We notice that $s_1 \ldots s_{i-1} s_i (s_1 \ldots s_{i-1})^{-1}$ is the reflection relative to M_i. Hence x may be obtained from x_0 by successive reflections relative to the walls M_i. These walls cross R, thus the action is transitive. This proves that $R = P_R x_0$ and $\mathrm{Stab}_W(R) = P_R$. It remains to prove that P_R is of the form W_I for a certain $I \subset S$.

We set $I \subset S$ the types of the maximal faces of x_0 contained in a wall that crosses R and we identify P_R with W_I. The inclusion $W_I < P_R$ comes from the definitions of P_R and I. We proceed by induction on $d_c(x_0, wx_0) = \ell$ to check that every element w of P_R is a product of elements of W_I. If $\ell = 0$, there is nothing to say. If $\ell > 0$ we choose $w = s_1 \ldots s_\ell$ such that $d_c(x_0, wx_0) = \ell$. By convexity, $s_1 x_0 \in C$ so $s_1 \in W_I$, because s_1 is a reflection relative to a wall that crosses R along x_0. Then $d_c(x_0, s_1^{-1} w x_0) = \ell - 1$ and $s_1^{-1} w \in P_R$. The induction assumption allows us to conclude. □

Remark 1.1.19. *G. Moussong proved that $\Sigma(W, S)$ equipped with a natural piecewise Euclidean metric is CAT(0) (see [Mou88] or [Dav08, Chapter 12]).*

1.2 Buildings

Buildings are singular spaces defined by J. Tits. At first sight, we can understand them as higher dimensional trees. In well chosen examples, the geometry of the residues, helps to understand the action of a group.

First, in this section, we give basic definitions and properties about buildings. Then we see how residues behave under projection maps. Eventually, we discuss the notion of parallel residues. In particular, we discuss how these residues behave under the action of an automorphism group.

Again, we refer to [Tit74], [Ron89], or [AB08] for details concerning the reminders of this section.

Hereafter (W, S) is a fixed Coxeter system.

1.2.1 Definition and general properties

Definition 1.2.1 ([Tit74, Definition 3.1.]). *A chamber system Δ over S is a* building *of type* (W, S) *if it admits a maximal family $\mathcal{A}p(\Delta)$ of subsystems isomorphic to (W, S), called apartments, such that*

- *any two chambers lie in a common apartment,*
- *for any pair of apartments A and B, there exists an isomorphism from A to B fixing $A \cap B$.*

For $A \in \mathcal{A}p(\Delta)$ we write $\mathcal{H}(A)$ for the set of half-spaces and $\mathcal{M}(A)$ for the set of walls of A.

A straightforward application of this definition is the existence of retraction maps of the building over apartments.

Definition 1.2.2. *Let $x \in \Delta$ and $A \in \mathcal{A}p(\Delta)$. Assume that x is contained in A. We call* retraction onto A centered x *the map $\pi_{A,x} : \Delta \longrightarrow A$ defined by the following property.*

For $c \in \Delta$, there exists a chamber $\pi_{A,x}(c) \in A$ such that for any apartment A' containing x and c, for any isomorphism $f : A \longrightarrow A'$ that fixes $A \cap A'$, then $f(c) = \pi_{A,x}(c)$

Example 1.2.3. *i) Any infinite tree without leaf is a building of type (W, S) with $W = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $S = \{(1,0),(0,1)\}$.*

ii) The flags of subspaces in a vector space $\mathbb{F}^{n+1}$ where $\mathbb{F}$ is a finite field is a building of spherical type (see [Ron89, Chapter 1]).

iii) For $n \geq 5$ a group of the form

$$\Gamma = \left\langle s_i \text{ for } i = \dots n | s_i^3 = 1 \text{ and } s_i s_{i+1} = s_{i+1} s_i \text{ for } i \in \mathbb{Z}/n\mathbb{Z} \right\rangle$$

is acting geometrically on a right-angled Fuchsian building (see Subsection 1.4.1)

Hereafter, Δ is a fixed building of type (W, S). The building Δ is called a *thin* (resp. *thick*) building if any panel contains exactly two (resp. at least three) chambers. Note that thin buildings are Coxeter systems.

Proposition 1.2.4 ([Tit74, Proposition 3.18.])**.** *Let x and y be two chambers of Δ. Then the convex hull of $\{x, y\}$ in Δ is the convex hull of $\{x, y\}$ in any apartment containing x and y.*

From the previous proposition, J. Tits constructs projection maps on the residues.

Proposition 1.2.5 ([Tit74, Proposition 3.19.3.])**.** *Let R be a residue and x be a chamber in Δ. There exists a unique chamber $\mathrm{proj}_R(x) \in R$ such that $d_c(x, \mathrm{proj}_R(x)) = \mathrm{dist}(x, R)$. Moreover, for any chamber y in R there exists a minimal gallery from x to y passing through $\mathrm{proj}_R(x)$.*

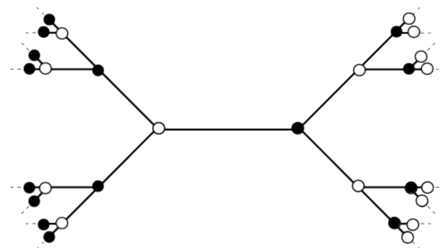
The following proposition is a straightforward consequence of Proposition 1.2.5. To prove it, we adapt word by word the proof of [Tit74, Prop 2.31.] for Coxeter systems to the case of buildings.

Proposition 1.2.6. *Let x and y be two adjacent chambers and R be a residue of Δ. Then $\mathrm{proj}_R(x)$ and $\mathrm{proj}_R(y)$ are adjacent.*

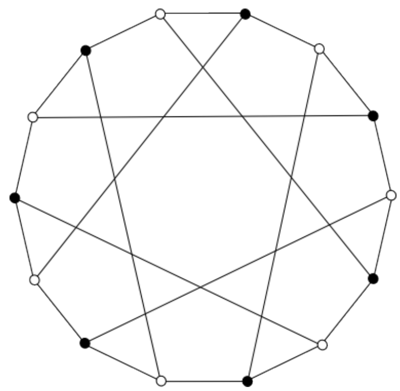
Proof. We may assume that $d_c(y, \mathrm{proj}_R(y)) \leq d_c(x, \mathrm{proj}_R(x))$. Then, according to Proposition 1.2.5, there exists a minimal gallery from x to $\mathrm{proj}_R(y)$ passing through $\mathrm{proj}_R(x)$. In particular, $d_c(x, \mathrm{proj}_R(y)) = d_c(x, \mathrm{proj}_R(x)) + d_c(\mathrm{proj}_R(x), \mathrm{proj}_R(y))$. Then, as x and y are adjacent $d_c(x, \mathrm{proj}_R(y)) - d_c(y, \mathrm{proj}_R(y)) \leq 1$. Eventually

$$\begin{aligned} d_c(\mathrm{proj}_R(x), \mathrm{proj}_R(y)) =& d_c(x, \mathrm{proj}_R(y)) - d_c(x, \mathrm{proj}_R(x)), \\ \leq& d_c(x, \mathrm{proj}_R(y)) - d_c(y, \mathrm{proj}_R(y)) \leq 1. \end{aligned}$$

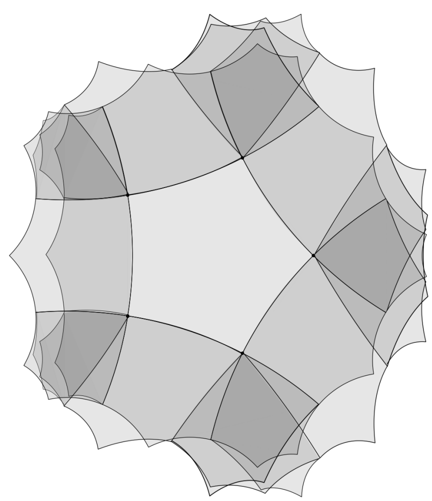
□



(a) The infinite tree of constant valency 3 (see Example 1.2.3.*i*))



(b) The building of $\mathbb{F}_2^3$ (see Example 1.2.3.*ii*))



(c) The neighborhood of a chamber in a Fuchsian building of constant thickness 2 (see Example 1.2.3.*iii*))

Figure 1.3: Examples of buildings

Proposition 1.2.7. *Let x and y be two adjacent chambers and R be a residue of Δ such that $\mathrm{proj}_R(x)$ and $\mathrm{proj}_R(y)$ are distinct. Then $d_c(x, \mathrm{proj}_R(x)) = d_c(y, \mathrm{proj}_R(y))$.*

Proof. Assume for instance that $d_c(y, \mathrm{proj}_R(y)) \leq d_c(x, \mathrm{proj}_R(x)) - 1$. By triangular inequality, it comes that $d_c(x, \mathrm{proj}_R(y)) \leq d_c(y, \mathrm{proj}_R(y)) + 1$. Combining the two inequalities we obtain $d_c(x, \mathrm{proj}_R(y)) \leq d_c(x, \mathrm{proj}_R(x))$. Therefore $\mathrm{proj}_R(x) = \mathrm{proj}_R(y)$ which contradict our hypothesis. □

Proposition 1.2.8. *Let x and y be two adjacent chambers and R be a residue of Δ such that $\mathrm{proj}_R(x)$ and $\mathrm{proj}_R(y)$ are distinct. Then there exists an apartment A containing x, y, $\mathrm{proj}_R(x)$, and $\mathrm{proj}_R(y)$. Besides, in A, the wall M that separates x and y is the wall that separates $\mathrm{proj}_R(x)$ and $\mathrm{proj}_R(y)$.*

Proof. On the one hand, according to Proposition 1.2.5, there exists a minimal gallery from x to $\mathrm{proj}_R(y)$ passing through $\mathrm{proj}_R(x)$. This gallery is of length $d_c(x, \mathrm{proj}_R(x)) + 1$. On the other hand, there exists a gallery of length $d_c(y, \mathrm{proj}_R(y)) + 1$ from x to $\mathrm{proj}_R(y)$ passing through y. Thanks to Proposition 1.2.7, $d_c(y, \mathrm{proj}_R(y)) = d_c(x, \mathrm{proj}_R(x))$, thus this second gallery is minimal. Hence, according to Proposition 1.2.4, an apartment A containing x and $\mathrm{proj}_R(y)$ also contains y and $\mathrm{proj}_R(x)$.

Let M be the wall in A between $\mathrm{proj}_R(x)$ and $\mathrm{proj}_R(y)$. Assume, for instance, that x, y, and $\mathrm{proj}_R(x)$ are in the same connected component of $A \backslash M$. According to Proposition 1.2.5, there exists a minimal gallery from y to $\mathrm{proj}_R(x)$ passing through $\mathrm{proj}_R(y)$. This gallery crosses twice M which reveals a contradiction. □

1.2.2 Projections of residues

Here we prove that the image of a residue under a projection map is a residue.

Proposition 1.2.9. *Let R and Q be two distinct residues in Δ. If $R' = \mathrm{proj}_R(Q)$ and $Q' = \mathrm{proj}_Q(R)$, then the maps $\mathrm{proj}_{R|Q'} : Q' \longrightarrow R'$ and $\mathrm{proj}_{Q|R'} : R' \longrightarrow Q'$ are reciprocal bijections. Moreover, $d_c(x, \mathrm{proj}_Q(x)) = d_c(y, \mathrm{proj}_Q(y))$ for any $x, y \in R'$.*

Proof. Let $x \in Q'$ and set $y = \mathrm{proj}_R(x) \in R'$. To prove the first statement, we must check that $x = \mathrm{proj}_Q(y)$. Let $z \in R$ be such that $x = \mathrm{proj}_Q(z)$. According to Proposition 1.2.5, there exists a minimal gallery of length $\mathrm{dist}(z, Q)$ and of the form

$$x \sim \cdots \sim y \sim \cdots \sim z.$$

Now consider the sub gallery $y \sim \cdots \sim z$ and a minimal gallery $\mathrm{proj}_Q(y) \sim \cdots \sim y$. Then the gallery

$$\mathrm{proj}_Q(y) \sim \cdots \sim y \sim \cdots \sim z$$

is of length at most $\mathrm{dist}(z, Q)$. Eventually, $x = \mathrm{proj}_Q(y)$.

Let x_1 and x_2 be two distinct adjacent chambers in R'. According to the first point, $\mathrm{proj}_Q(x_1)$ and $\mathrm{proj}_Q(x_2)$ are two adjacent and distinct chambers. Then with Proposition 1.2.7, these chambers are such that $d_c(x_1, \mathrm{proj}_Q(x_1)) = d_c(x_2, \mathrm{proj}_Q(x_2))$. Eventually, by Proposition 1.2.6, $\mathrm{proj}_R(Q)$ is connected and the proof is complete. □

Proposition 1.2.10. *Assume that Δ is a thin building. Let R and Q be two residues in Δ. Then $\mathrm{proj}_R(Q)$ is a residue in Δ.*

Proof. First we notice that, thanks to Proposition 1.2.6, $\mathrm{proj}_R(Q)$ is a non-empty connected set of chambers. If $\mathrm{proj}_R(Q)$ is made of a single chamber, then it is a $\emptyset$-residue. Otherwise, thanks to Theorem 1.1.18, if $\mathrm{proj}_R(Q)$ is stabilized by any reflection relative to a wall that crosses it then it is a residue.

Let M be a wall that crosses $\mathrm{proj}_R(Q)$ and $r \in W$ be the reflection relative to this wall. Let x' and y' be two distinct adjacent chambers in $\mathrm{proj}_R(Q)$ separated by the wall M. Let $x = \mathrm{proj}_R(x')$ and $y = \mathrm{proj}_R(y')$. Then with Propositions 1.2.6 and 1.2.9, it comes that x and y are adjacent distinct in Q and such that $x' = \mathrm{proj}_R(x)$ and $y' = \mathrm{proj}_R(y)$. Moreover, according to the second statement in Proposition 1.2.8, x and y are adjacent along M. In particular, M crosses both R and Q. Hence, according to Fact 1.1.17, r stabilizes R and Q thus $\mathrm{proj}_{R|Q}(\cdot)$ is equivariant by r. Then $\mathrm{proj}_R(Q)$ is stabilized by r and the proof is completed. □

For a I-residue R in a building Δ and an apartment A, the subset $R \cap A \subset A$ is either a I-residue in A, or does not contain any chamber . This remark motivates the following notations.

Notation. *Let R be a residue in Δ and $A \in \mathcal{A}p(\Delta)$ such that $\mathrm{Ch}(R \cap A) \neq \emptyset$. Then we write $R_A := R \cap A$. Naturally any residue of A is of the form R_A for R a residue in Δ. Then, if we write $\mathrm{proj}_{R_A}(\cdot)$ the projection maps in A there is no confusion possible with the projection maps in Δ that we write $\mathrm{proj}_R(\cdot)$.*

With these notations and by convexity, the map $\mathrm{proj}_{R_A}(\cdot)$ coincides with the restriction to A of the map $\mathrm{proj}_R(\cdot)$. Now we can extend the previous proposition from the apartments to Δ.

Proposition 1.2.11. *Let R and Q be two residues in Δ. Then $\mathrm{proj}_R(Q)$ is a residue in Δ.*

Proof. For the purpose of this proof we write $R' = \mathrm{proj}_R(Q)$ and $Q' = \mathrm{proj}_Q(R)$. Besides I_1 and I_2 are respectively the types of R and Q. Again, thanks to Proposition 1.2.6, R' is a non-empty connected set of chambers. Fix $x \in Q$ and let $A \in \mathcal{A}p(\Delta)$ be such that both x and $\mathrm{proj}_R(x)$ are in A. The residues R_A and Q_A are of type I_1 and I_2 in A. Let I be the type of the residue $\mathrm{proj}_{R_A}(Q_A)$ given by Proposition 1.2.10 applied in A. Note that I is a subset of I_1.

Now we verify that the type I is invariant under a change of apartment in the following sense. Consider $B \in \mathcal{A}p(\Delta)$ such that both x and $\mathrm{proj}_R(x)$ are in B. Then let $f : A \longrightarrow B$ be the isomorphism that fixes $A \cap B$. This map preserves the types. Hence $f(R_A)$ (resp. $f(Q_A)$) is the residue of type I_1 (resp. I_2) in B that contains $\mathrm{proj}_R(x)$ (resp. x). We get

$$f(R_A) = R_B \text{ and } f(Q_A) = Q_B.$$

Likewise, $f(\mathrm{proj}_{R_A}(Q_A))$ is the I-residue in B containing $\mathrm{proj}_R(x)$. Yet, as f is an isometry for the metric over the chambers $f(\mathrm{proj}_{R_A}(Q_A)) = \mathrm{proj}_{f(R_A)}(f(Q_A))$. Therefore we proved that $\mathrm{proj}_{R_B}(Q_B)$ is the I-residue in B containing $\mathrm{proj}_R(x)$.

We set C the I-residue in Δ containing $\mathrm{proj}_R(x)$ and we check that $C = R'$. Let $y \in C$. As $I \subset I_1$ then $y \in R$. In particular, there exists a minimal gallery from x to y passing through $\mathrm{proj}_R(x)$. Hence there exists $B \in \mathcal{A}p(\Delta)$ containing x, $\mathrm{proj}_R(x)$, and y. In B, $\mathrm{proj}_{R_B}(Q_B)$ is the I residue containing $\mathrm{proj}_R(x)$. Thus $y \in \mathrm{proj}_{R_B}(Q_B)$ and there exists $z \in Q_B$ such that $y = \mathrm{proj}_{R_B}(z)$. Therefore $C \subset R'$.

Pick $y \in R'$ and let $y' \in Q'$ be such that $y' = \mathrm{proj}_Q(y)$ and $y = \mathrm{proj}_R(y')$. On the one hand, there exists a minimal gallery from x to y passing through $\mathrm{proj}_R(x)$. On the other hand, there exists a minimal gallery from y to x passing through y'. Therefore, thanks to Proposition 1.2.4, an apartment B that contains x and y also contains $\mathrm{proj}_R(x)$ and y'. Then, the residue $\mathrm{proj}_{R_B}(Q_B)$ is the I-residue of B containing $\mathrm{proj}_R(x)$. As $y' \in Q_B$ we obtain that there exists a gallery of type I in B from $\mathrm{proj}_R(x)$ to y. Hence $y \in C$ and $R' \subset C$.

□

1.2.3 A theorem about parallel residues

In this subsection, Δ is a building. The following definition is due to B. Mühlherr.

Definition 1.2.12. *Let R and Q be two distinct residues in Δ. We say that R is* parallel *to Q if*

$$\mathrm{proj}_R(Q) = R \textit{ and } \mathrm{proj}_Q(R) = Q.$$

It appears that to any pair of residues is canonically associated a pair of parallel residues.

Proposition 1.2.13. *For R and Q two residues in Δ, let $Q' = \mathrm{proj}_Q(R)$ and $R' = \mathrm{proj}_R(Q)$. Then R' and Q' are parallel residues.*

Proof. By symmetry, it is enough to check that $\mathrm{proj}_{Q'}(R') = Q'$. On the one hand, the inclusion $\mathrm{proj}_{Q'}(R') \subset Q'$ is trivial. On the other hand, let $x \in Q'$, by Proposition 1.2.9, there exists a unique $y \in R'$ such that $x = \mathrm{proj}_Q(y)$. As $Q' \subset Q$ we obtain $\mathrm{dist}(y, Q) \leq \mathrm{dist}(y, Q')$. Yet on the one hand, by definition of the projection map $d_c(y, x) = \mathrm{dist}(y, Q)$. On the other hand, $x \in Q'$ and $\mathrm{dist}(y, Q') \leq d_c(y, x)$. Hence $d_c(y, x) = \mathrm{dist}(y, Q')$ and $x = \mathrm{proj}_{Q'}(y)$.

□

In the following we see that parallel residues are convenient tools to describe intersections of stabilizers of residues under the action of an automorphism group.

Theorem 1.2.14. *Let G be a group of automorphisms of Δ. Suppose that $\mathrm{Stab}_G(P) = \mathrm{Stab}_G(P')$ for any pair P, P' of parallel residues. Then for any pair of residues R, Q one has*

$$\mathrm{Stab}_G(R) \cap \mathrm{Stab}_G(Q) = \mathrm{Stab}_G(\mathrm{proj}_R(Q)) = \mathrm{Stab}_G(\mathrm{proj}_Q(R)).$$

Proof. By symmetry, it is sufficient to prove, that

$$\mathrm{Stab}_G(R) \cap \mathrm{Stab}_G(Q) = \mathrm{Stab}_G(\mathrm{proj}_Q(R)).$$

Let $g \in \mathrm{Stab}_G(R) \cap \mathrm{Stab}_G(Q)$. As g is an isometry of Δ that stabilizes both R and Q, the map $\mathrm{proj}_{Q|R}(\cdot)$ is equivariant by g. Then $g(\mathrm{proj}_Q(R)) = \mathrm{proj}_Q(R)$ and

$$\mathrm{Stab}_G(R) \cap \mathrm{Stab}_G(Q) < \mathrm{Stab}_G(\mathrm{proj}_Q(R)).$$

Let $g \in \mathrm{Stab}_G(\mathrm{proj}_Q(R))$. As g preserves the types, if Q is a I-residue then $g(Q)$ is also a I-residue. In particular, Q and $g(Q)$ are two I-residues containing $\mathrm{proj}_Q(R)$. Therefore $g(Q) = Q$. Thanks to Proposition 1.2.13, $\mathrm{proj}_Q(R)$ is a residue parallel to $\mathrm{proj}_R(Q)$. Then by assumption $g \in \mathrm{Stab}_G(\mathrm{proj}_R(Q))$. Now we can use the previous argument to prove that $g(R) = R$ and

$$\mathrm{Stab}_G(\mathrm{proj}_Q(R)) < \mathrm{Stab}_G(R) \cap \mathrm{Stab}_G(Q).$$

□

1.3 Parallel residues in Coxeter systems

In Coxeter systems, parallel residues can be used to prove that intersections of parabolic subgroups are parabolic. In this section, $\Sigma(W, S)$ is the Davis complex of a Coxeter system (W, S).

Proposition 1.3.1. *Let R and Q be two parallel residues in $\Sigma(W, S)$. Then*

$$\mathrm{Stab}_W(R) = \mathrm{Stab}_W(Q).$$

Proof. If R and Q contain only one chamber then

$$\mathrm{Stab}_W(R) = \mathrm{Stab}_W(Q) = \{e\}.$$

Otherwise, let M be a wall that crosses R. Let x and y be two distinct adjacent chambers of R along M. According to Proposition 1.2.9, the chambers $\mathrm{proj}_Q(x)$ and $\mathrm{proj}_Q(y)$ are distinct. Then thanks to the second statement of Proposition 1.2.8, M also separates $\mathrm{proj}_Q(x)$ and $\mathrm{proj}_Q(y)$. In particular, M crosses Q. By symmetry of the argument, we can also prove that any wall that crosses Q crosses R. Eventually, according to Fact 1.1.17 $\mathrm{Stab}_W(R) = \mathrm{Stab}_W(Q)$. □

Now we apply Theorem 1.2.14 to the action of W on $\Sigma(W, S)$. We get the following theorem that is a classical result about parabolic subgroups in Coxeter groups (see [Dav08, Lemma 5.3.6.]).

Theorem 1.3.2. *Let $w \in W$, and $I, J \subset S$. If we set $P = W_I \cap wW_J w^{-1}$, then there exist $w_I \in W_I$ and $w_J \in W_J$ such that*

$$P = w_I^{-1} W_K w_I,$$

where $K = I \cap w_I^{-1} w w_J J (w_I^{-1} w w_J)^{-1} \subset S$.

Proof. Let $R = W_I x_0$ and $Q = wW_J x_0$. We set $w_I \in W_I$ and $w_J \in W_J$ such that

$$\mathrm{dist}(w_I x_0, Q) = \mathrm{dist}(R, Q) \text{ and } d_c(w_I x_0, w w_J x_0) = \mathrm{dist}(R, Q).$$

Up to conjugate P by w_I^{-1} and to change $w_I^{-1} w w_J$ for w we are in the situation where $P = W_I \cap wW_J w^{-1}$ with $d_c(x_0, wx_0) = \mathrm{dist}(R, Q)$. In this setting, K designates $I \cap wJw^{-1}$ *i.e.*

$$K = \{s \in I : s = wrw^{-1} \text{ for some } r \in J\}.$$

The inclusion $W_K < P$ is obviously true and we prove now the converse.

We know that $\mathrm{Stab}_W(R) = W_I$ and $\mathrm{Stab}_W(Q) = wW_J w^{-1}$. Then, according to Proposition 1.3.1 and Theorem 1.2.14,

$$P = \mathrm{Stab}_W(R) \cap \mathrm{Stab}_W(Q) = \mathrm{Stab}_W(\mathrm{proj}_R(Q)).$$

Thanks to Fact 1.1.17, $\mathrm{Stab}_W(\mathrm{proj}_R(Q)) = W_{\overline{K}}$ where $\overline{K}$ designates the set of types of the walls that cross $\mathrm{proj}_R(Q)$ along x_0.

It is now enough to prove that $\overline{K} \subset K$. Let $s \in \overline{K}$ and M be the wall that crosses $\mathrm{proj}_R(Q)$ along x_0 and sx_0. According to Proposition 1.2.8, M crosses $\mathrm{proj}_Q(R)$ along $\mathrm{proj}_Q(x_0)$ and $\mathrm{proj}_Q(sx_0)$. This situation is illustrated by Figure 1.4.

Hence $s(\mathrm{proj}_Q(x_0)) = \mathrm{proj}_Q(sx_0)$. Besides, at the beginning of the proof we chose $w \in W$ such that $\mathrm{proj}_Q(x_0) = wx_0$. Moreover, $\mathrm{proj}_Q(x_0)$ and $\mathrm{proj}_Q(sx_0)$ are adjacent in a J-residue. Thus, with $\mathrm{proj}_Q(x_0) = wx_0$ there exists $r \in J$ such that $wrw^{-1}(\mathrm{proj}_Q(x_0)) = \mathrm{proj}_Q(sx_0)$. Eventually we obtain

$$wx_0 = \mathrm{proj}_Q(x_0) = s^{-1}(\mathrm{proj}_Q(sx_0)) = s^{-1} wrx_0.$$

Therefore $sw = wr$ and $\overline{K} \subset K$.

□

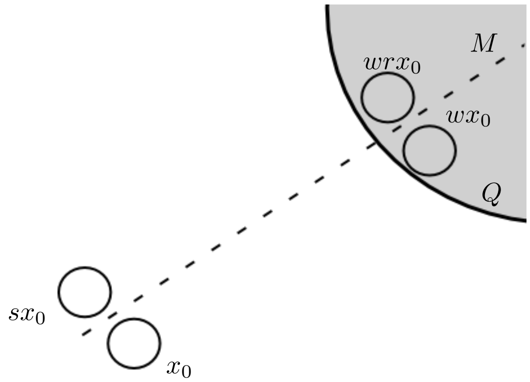


Figure 1.4

1.4 Geometry of right-angled buildings

Right-angled buildings are convenient buildings to work with as they are equipped with a wall structure and a simply transitive group action. This permits to apply to them several methods coming from Coxeter systems.

In this section, we remind basic facts about right-angled buildings and graph products. Then, we discuss the wall structure of the Davis complex as we did in Coxeter systems. In particular, we prove Proposition 1.4.15 and Theorem 1.4.17, that is an analogous of Theorem 1.1.18. These results will be used Chapter 2.

The geometry of right-angled buildings in general and of parallel residues in particular are discussed in [Cap14]. Some of the result of this section are contained in this article and appear here for the sake of completeness.

1.4.1 Graph products and right-angled buildings

Let $\mathcal{G}$ denote a *finite simplicial graph i.e* $\mathcal{G}^{(0)}$ is finite, each edge has two different vertices, and no edge is double. We denote by $\mathcal{G}^{(0)} = \{v_1, \dots, v_n\}$ the vertices of $\mathcal{G}$. If for $i \neq j$, the corresponding vertices v_i, v_j are connected by an edge, we write $v_i \sim v_j$. A cyclic group $G_i = \langle s_i \rangle$ of order $q_i \in \{2, 3, \dots\} \cup \{\infty\}$ is associated with each $v_i \in \mathcal{G}^{(0)}$ and we set $S = \{s_1, \dots, s_n\}$. In this section we assume that $n \geq 2$ and that $\mathcal{G}$ has at least one edge.

Definition 1.4.1. *The* graph product *given by* $(\mathcal{G}, \{G_i\}_{i=1,\dots,n})$ *is the group defined by the following presentation*

$$\Gamma = \langle s_i \in S | s_i^{q_i} = 1, s_i s_j = s_j s_i \text{ if } v_i \sim v_j \rangle .$$

Example 1.4.2. *If the graph $\mathcal{G}$ is fixed and the orders $\{q_i\}_{i=1,\ldots,n}$ go from 2 to $+\infty$, graph products are groups between* right-angled Coxeter groups *(see [Dav08]) and* right-angled Artin groups *(see [Cha07]). If the integers $\{q_i\}_{i=1,\ldots,n}$ are fixed and we add edges to the graph starting from a graph with no edge, those groups are groups between free products and direct products of cyclic groups.*

From nowon, Γ is a fixed graph product given by a pair $(\mathcal{G}, \{G_i\}_{i=1,\ldots,n})$. By analogy with Definition 1.1.5, we define parabolic subgroups in Γ.

Definition 1.4.3. *The subgroup of Γ generated by a subset $I \subset S$ is denoted by Γ_I and a subgroup of the form $g\Gamma_I g^{-1}$, with $g \in \Gamma$, is called a* parabolic subgroup.

Let W be the graph product defined by the pair $(\mathcal{G}, \{\mathbb{Z}/2\mathbb{Z}\}_{i=1,\ldots,n})$. This graph product is isomorphic to the right-angled Coxeter group of type $M = \{m_{i,j}\}_{i,j=1,\ldots,n}$ defined by : $m_{i,j} = 2$ if $v_i \sim v_j$ and $m_{i,j} = \infty$ if $v_i \nsim v_j$.

All along this section, W denotes this Coxeter group canonically associated with Γ and (W, S) is the Coxeter system associated with W.

Theorem 1.4.4 ([Dav98, Theorem 5.1.])**.** *Let Δ be the chamber system $C(\Gamma, \{e\}, \{\Gamma_{\{s\}}\}_{s\in S})$ (see Example 1.1.4). Then Δ is a building of type (W, S).*

Hereafter, Δ denotes the right-angled building associated with Γ by the previous theorem. In the following, we describe the Davis complex associated with this building. This geometric realization will help us to prove the analogous of Proposition 1.1.16 and Theorem 1.1.18 in Δ.

1.4.2 The Davis complex associated with Γ

Now we introduce the Davis complex associated with Γ. This complex is analogous to the Davis complex of a Coxeter system. Again we refer to [Mei96] for an example along with suggestive pictures.

Let D be the Davis chamber associated with W as in Subsection 1.1.3. Again a face of D is associated with a type $I \subset S$. For $x \in D$, if I is the type of the face containing x in its interior, we set $\Gamma_x := \Gamma_I$. To the interior points of D we associate the trivial group $\Gamma_\emptyset$.

Now we can define the *Davis complex* : $\Sigma = D \times \Gamma / \sim$ with

$$(x, g) \sim (y, g') \text{ if and only if } x = y \text{ and } g^{-1}g' \in \Gamma_x.$$

We study the building Δ through it geometric realization Σ and we briefly remind what this mean.

A chamber of Σ is a subset of the form $[D \times \{g\}]$ with $g \in \Gamma$. Two chambers are adjacent if and only if they share a maximal face. For a subset $E \subset \Sigma$ we designate by $\mathrm{Ch}(E)$ the set of chambers contained in E. Equipped with this chamber system structure, Σ is isomorphic to Δ. In particular, the set of apartments in Σ is designated by $\mathcal{A}p(\Sigma)$.

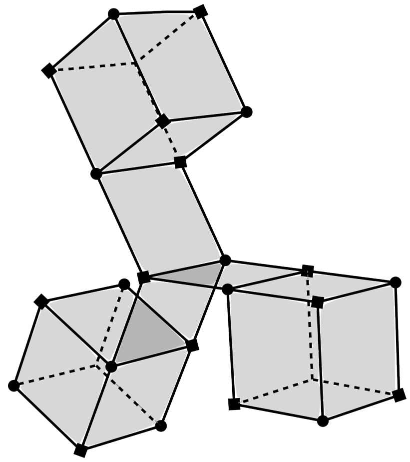


Figure 1.5: A panel of type $\{1,2\}$ and of thickness 2 given by Figure 1.2

Then the left action of Γ on itself induces an action on Σ. For $\gamma \in \Gamma$ and $[(x,g)] \in \Sigma$ we set $\gamma[(x,g)] := [(x,\gamma g)]$. Moreover this action induces a simply transitive action of Γ on $\mathrm{Ch}(\Sigma)$. Naturally this action is also isometric for $d_c(\cdot,\cdot)$.

1.4.3 Building-walls and residues in the Davis complex

We call *base chamber* of Σ, denoted by x_0, the chamber $[D \times \{e\}]$. For $g \in \Gamma$, as $[D \times \{g\}]$ is the image of x_0 under g, we designate this chamber by gx_0. Here we present some basic tools used to describe the structure of Σ. In particular, we extend to Σ some definitions and properties that have been used in Coxeter systems.

The notion of walls in a Coxeter system extends to right-angled buildings.

Definition 1.4.5.

1. *We call* building-wall *in Σ the proper subcomplex M stabilized by a non-trivial isometry $r = gs^{\alpha}g^{-1}$ with $g \in \Gamma$, $s \in S$, $\alpha \in \mathbb{Z}$ and $s^{\alpha} \neq e$. The isometry r is called a* rotation around *M. We denote by $\mathcal{M}(\Sigma)$ the set of building-walls of Σ.*

2. *Let M be a building-wall associated with a rotation $r \in \Gamma$. For $x \in \mathrm{Ch}(\Sigma)$ we say that M is* along *x if $r(x)$ is adjacent to x.*

In [Cap14], a building-wall is called a *wall-residue*. Indeed, a building-wall is isomorphic to the residue generated by the building-walls orthogonal to it.

Note that with the notations of the definition, M is the building-wall fixed by any rotation $gs^{\alpha'}g^{-1}$ with $s^{\alpha'} \neq e$. Besides, because of the right-angles, a building-wall is associated with a type. Which is not true for walls in a generic Coxeter system.

We say that the building-wall M is non-trivial if it contains more than one point. A non-trivial building-wall M may be equipped with a building structure. Indeed, if s_i is the type of M, associated with $v_i \in \mathcal{G}^{(0)}$, we write $I = \{j : v_j \sim v_i, v_j \neq v_i\}$ and $V = \{v_j \in \mathcal{G}^{(0)} : j \in I\}$. Then if $\mathcal{G}_V$ is the full subgraph generated by V, we can check that, M is isomorphic to the geometric realization of the graph product $(\mathcal{G}_V, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i \in I})$. The Davis chamber of this geometric realization is the face of type s_i of D. Moreover building-walls also divide Σ in isomorphic connected components. In the case of infinite dimension 2 buildings, the building-walls are trees and thus they have been called *trees-walls* by M. Bourdon and H. Pajot in [BP00]. These explain our terminology.

Example 1.4.6. *Let Γ be the graph product given by Example 1.2.3.iii) and let Σ be its Davis complex. Let Γ' be the graph product associated with the pair $(\mathcal{G}, \{\mathbb{Z}/3\mathbb{Z}\}_{i=1,\dots,12})$ where $\mathcal{G}$ is the dual graph of the dodecahedron and let Σ' be its Davis complex.*

Then the building-walls in Σ are infinite trees of constant valency 3. In Σ' the building-walls are isomorphic as chamber systems to Σ.

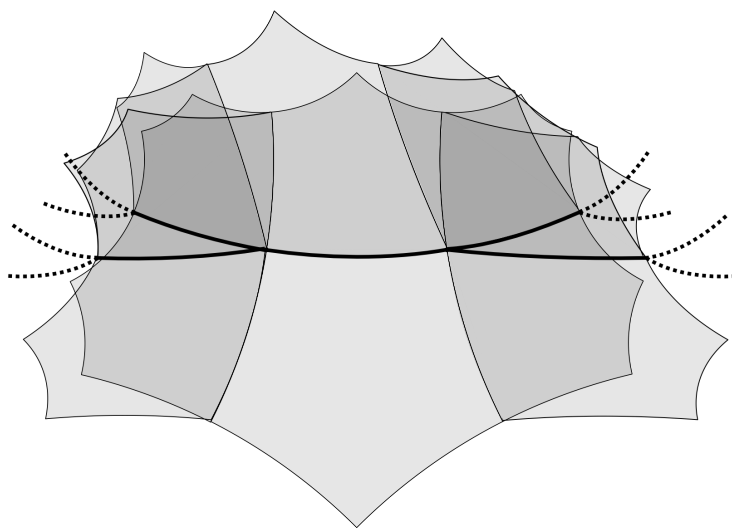


Figure 1.6: In Fuchsian buildings building-walls are trees.

Definition 1.4.7. *Let M be a building-wall of type s and let $r \in \Gamma$ be a rotation around M. We call* dial of building bounded by M *the closure in Σ of the connected components of $\Sigma \backslash M$.*

In [Cap14], a dial of building is called a *wing*. This definition is similar to Definition 1.1.12 and imply an analogous of Proposition 1.1.13.

Fact 1.4.8. *Let M be a building-wall of type s. Assume that s is of finite order q. Then $\Sigma\backslash M$ consists of q connected components. We designate by $D_0(M), D_1(M), \ldots, D_{q-1}(M)$ these dials of building, with the convention that $x_0 \subset D_0(M)$. In this setting, for any $i = 1, \ldots q-1$, if $y \in \mathrm{Ch}(D_i(M))$ then*

$$\mathrm{Ch}(D_i(M)) = \{x \in (W,S) : d_c(y,x) < d_c(y,rx)\}.$$

Eventually r permutes $D_0(M), D_1(M), \ldots, D_{q-1}(M)$.

For a building-wall associated with a type $s \in S$ of infinite order, the analogous property holds.

In the following, we write $\mathcal{D}(\Sigma)$ for the set of all the dials of building of Σ. Naturally, if we consider thin right-angled buildings, the definitions of building-wall and dial of building coincide with the definition of wall and half-space. It is also important to notice that the structure of building-walls in Σ comes from the wall structures in the apartments.

Fact 1.4.9. *For any $M \in \mathcal{M}(\Sigma)$ (resp. $D \in \mathcal{D}(\Sigma)$), any $A \in \mathcal{A}p(\Sigma)$, and any retraction $\pi : \Sigma \longrightarrow A$, there exists a wall m (resp. a half-space H) in A such that M (resp. $\mathrm{Int}(D)$) is one of the connected component of $\pi^{-1}(m)$ (resp. $\pi^{-1}(\mathrm{Int}(H))$).*

The following terminology is frequently used to describe the building-walls relatively to the chambers.

Definition 1.4.10. *Let $M \in \mathcal{M}(\Sigma)$ and $E, F \subset \Sigma$.*

i) We say that M crosses *E if $E\backslash M$ has several connected components.*

ii) We say that M the building-wall separates *E* and *F if their interior are entirely contained in two distinct connected components of $\Sigma\backslash M$.*

Now we have enough to prove the analogous of Proposition 1.1.16.

Proposition 1.4.11. *Let x_1 and x_2 be two chambers. If we denote $d_c(\cdot,\cdot)$ the metric on the chamber system, then*

$$d_c(x_1, x_2) = \#\{M \in \mathcal{M}(\Sigma) : M \text{ seperates } x_1 \text{ and } x_2\}.$$

Proof. This proposition is straightforward consequence of Propositions 1.1.16, 1.2.4, and of Fact 1.4.9. □

In a right-angled building it appears that two distinct building-walls are either parallel or orthogonal. This explains the following notations.

Notation. *Let M and M' be two distinct building-walls.*

i) if $M \cap M' \neq \emptyset$ we write $M \perp M'$ and we say that M is orthogonal to M', *ii) if $M \cap M' = \emptyset$ we write $M \parallel M'$ and we say that M is parallel to M'.*

If $M \perp M'$, then $D \cap D'$ contains a chamber for any pair D, D' of dials of building bounded by M and M'. If $M \parallel M'$ then there exists D bounded by M and D' bounded M' such that $M' \subset D$ and $M \subset D'$.

In right-angled buildings, we can define projection maps not only on residues but also on dials of building. This will be useful in Section 2.6 to understand the metric on the boundary in the hyperbolic case.

Lemma 1.4.12. *Let M and M' be two building-walls such that $M \perp M'$. Let $r \in \Gamma$ be a rotation around M and D' be a dial of building bounded by M' then $r(D') = D'$.*

Proof. Up to a translation on the dials and a conjugation on the rotations we can assume that M and M' are along x_0 and $x_0 \in D'$. If s is a rotation around M', we can write

$$\mathrm{Ch}(D') = \{x \in \mathrm{Ch}(\Sigma) : d_c(x_0, x) < d_c(x_0, sx)\}.$$

Hence

$$\begin{aligned} r(\mathrm{Ch}(D')) =& \{rx \in \mathrm{Ch}(\Sigma) : d_c(x_0, x) < d_c(x_0, sx)\} \\ =& \{x \in \mathrm{Ch}(\Sigma) : d_c(x_0, r^{-1}x) < d_c(x_0, sr^{-1}x)\}. \end{aligned}$$

By assumption $rs = sr$, thus $r(\mathrm{Ch}(D')) = \{x \in \mathrm{Ch}(\Sigma) : d_c(rx_0, x) < d_c(rx_0, sx)\}$. Besides, $d_c(x_0, rx_0) = 1$ and $d_c(x_0, srx_0) = 2$ so $rx_0 \in \mathrm{Ch}(D')$ and with Fact 1.4.8 we obtain $\mathrm{Ch}(D') = r(\mathrm{Ch}(D'))$.

□

Proposition 1.4.13. *Let D be a residue or a dial of building and $C = \mathrm{Ch}(D)$. Then for any $x \in \mathrm{Ch}(\Sigma)$ there exists a unique chamber $\mathrm{proj}_C(x) \in C$ such that*

$$d_c(x, \mathrm{proj}_C(x)) = \mathrm{dist}(x, C).$$

Moreover, for any chamber $y \in C$ there exists a minimal gallery from x to y passing through $\mathrm{proj}_C(x)$.

Proof. If D is a residue, then we refer to Proposition 1.2.5. If D is a dial of building, let $y \in C$ be such that $d_c(x, y) = \mathrm{dist}(x, C)$. Then for $z \in C$ we set $x = x_1 \sim x_2 \sim \cdots \sim y$ and $y = x_\ell \sim \cdots \sim z = x_k$ two minimal galleries. Assume that the gallery

$$x = x_1 \sim x_2 \sim \cdots \sim y = x_\ell \sim \cdots \sim z = x_k$$

is not minimal. Then there exists a building-wall M and two indices i, j with $1 \leq i < \ell$ and $\ell \leq j < k$ such that

- M separates x_i and x_{i+1},

- M separates x_j and x_{j+1}.

Now consider $r \in \Gamma$ the rotation around M such that $rx_{i+1} = x_i$. According to Proposition 1.4.12, $r(D) = D$. As a consequence, the gallery

$$x \sim \cdots \sim x_i \sim rx_{i+2} \cdots \sim rx_\ell = ry$$

connects x to C and is of length $\mathrm{dist}(x, C) - 1$, which is a contradiction.

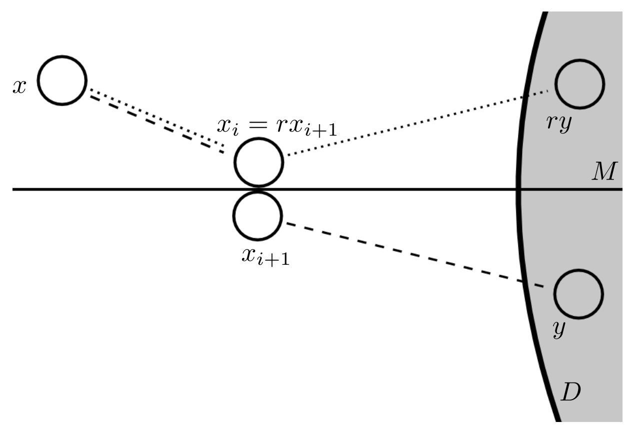


Figure 1.7

We proved that for any $z \in C$, there exists a minimal gallery from x to z passing through y. This proves in particular that y is unique and the proof is achieved.

□

For simplicity, $\mathrm{proj}_D(\cdot)$ designate $\mathrm{proj}_{\mathrm{Ch}(D)}(\cdot)$ for $D \in \mathcal{D}(\Sigma)$. Note that in non-right-angled Coxeter systems, it is not possible to define a unique projection on a half-space. In particular, half-spaces are stabilized by the reflections relative to the walls that cross it if and only if the angles are right.

With the following lemma we can see that the projection on the dials of building are orthogonal relatively to the building-wall structure.

Lemma 1.4.14. *Let D and D' be two dials of building such that $\mathrm{Ch}(D \cap D') \neq \emptyset$. If $x \in \mathrm{Ch}(D)$ then $\mathrm{proj}_{D'}(x) \in \mathrm{Ch}(D \cap D')$.*

Proof. As we know that $\mathrm{proj}_{D'}(x) \subset D'$, we check that $\mathrm{proj}_{D'}(x) \subset D$. The assumption $\mathrm{Ch}(D \cap D') \neq \emptyset$ means that three cases are possible. First, if $D' \subseteq D$ then $\mathrm{proj}_{D'}(x) \subset D' \subset D$. Then if $D \subseteq D'$, as $x \subset D$ we get $\mathrm{proj}_{D'}(x) = x$ and $\mathrm{proj}_{D'}(x) \subset D$.

Let M and M' be the building-walls that bound D and D'. The last cases is realized when $M \perp M'$. In this case consider a minimal gallery $x \sim \cdots \sim \mathrm{proj}_{D'}(x)$. Assume that

$\text{proj}_{D'}(x) \notin D$. Then the previous gallery crosses M and we can write that there exists a minimal gallery of the form

$$x \sim \cdots \sim x_i \sim x_{i+1} = rx_i \sim x_{i+2} \sim \cdots \sim \text{proj}_{D'}(x)$$

where $r \in \Gamma$ is rotation around M. Then with Lemma 1.4.12 we obtain that

$$x \sim \cdots \sim x_i \sim r^{-1}x_{i+2} \sim \cdots \sim r^{-1}\text{proj}_{D'}(x)$$

is a gallery between x and D' of length $d_c(x, \text{proj}_{D'}(x)) - 1$. Which is a contradiction.

□

Applying inductively the projection maps on dials of building, we define projection maps on finite intersections of dials of building.

Proposition 1.4.15. *Let $D_1, \dots, D_k \in \mathcal{D}(\Sigma)$ and $C = \text{Ch}(D_1 \cap \cdots \cap D_k)$. Assume that $C \neq \emptyset$. Then for any $x \in \text{Ch}(\Sigma)$ there exists a unique chamber $\text{proj}_C(x) \in C$ such that*

$$d_c(x, \text{proj}_C(x)) = \text{dist}(x, C).$$

Moreover, for any chamber $y \in C$ there exists a minimal gallery from x to y passing through $\text{proj}_C(x)$. Eventually $\text{proj}_C(x) = \text{proj}_{D_k} \circ \cdots \circ \text{proj}_{D_1}(x)$.

Proof. First, according to Lemma 1.4.14, we can assume, up to a subfamily that $x \notin D_i$ for each $i = 1, \dots, k$. Now we set

- $C_1 = \text{Ch}(D_1)$ and $C_i = C_{i-1} \cap \text{Ch}(D_i)$ for any $i = 2, \dots, k$,
- $x_1 = \text{proj}_{D_1}(x)$ and $x_i = \text{proj}_{D_i}(x_{i-1})$ for any $i = 2, \dots, k$.

By induction on i we prove the following property:

> *$x_i \in C_i$ and is the unique chamber of C_i such that $d_c(x, x_i) = \text{dist}(x, C_i)$. Moreover, for any chamber $y \in C_i$ there exists a minimal gallery from x to y passing through x_i.*

If $i = 1$ the property holds by Proposition 1.4.13. Let $i > 1$ and assume that the property holds at rank i. In particular, for $j = 1, \dots, i$ one has $x_i \in \text{Ch}(D_j)$ and $\text{Ch}(D_j) \cap \text{Ch}(D_{i+1}) \neq \emptyset$. Therefore, with Lemma 1.4.14, $x_{i+1} \in \text{Ch}(D_1) \cap \cdots \cap \text{Ch}(D_i) \cap \text{Ch}(D_{i+1}) = C_{i+1}$.

Then consider $y \in C_i$. There exists a minimal gallery from x to y passing through x_i. This means also that if $x \sim \cdots \sim x_i$ is a minimal gallery, then for any minimal gallery $x_i \sim \cdots \sim y$ contained in C_i the gallery

$$x \sim \cdots \sim x_i \sim \cdots \sim y$$

is minimal. As a consequence, for any subset $C' \subset C_i$ if $y' \in C'$ is such that $d_c(x_i, y') = \mathrm{dist}(x_i, C')$ then
$$d_c(x, y') = \mathrm{dist}(x, C').$$

Thanks to Proposition 1.4.13, x_{i+1} is the unique chamber in C_{i+1} such that

$$d_c(x_i, x_{i+1}) = \mathrm{dist}(x_i, C_{i+1}).$$

Thus x_{i+1} is the unique chamber in C_{i+1} such that $d_c(x, x_{i+1}) = \mathrm{dist}(x, C_{i+1})$. Moreover, for any $y \in C_{i+1}$ there exists a minimal gallery from

$$x_i \sim \cdots \sim x_{i+1} \sim \cdots \sim y$$

that is entirely contained in C_i. Hence if the gallery $x \sim \cdots \sim x_i$ is minimal, the gallery

$$x \sim \cdots \sim x_i \sim \cdots \sim x_{i+1} \sim \cdots \sim y$$

is also minimal.

□

For simplicity, if $D_1, \ldots, D_k$ is a collection of dials of building such that $C = D_1 \cap \cdots \cap D_k$ contains a chamber, we write $\mathrm{proj}_C(\cdot)$ instead of $\mathrm{proj}_{\mathrm{Ch}(C)}(\cdot)$.

Notice that it is not always possible to define a projection on a convex set of chambers. For instance, if Σ is a thick building there exist pairs of adjacent chambers x and y with $d_c(x_0, x) = d_c(x_0, y)$. This is because there is no analogous of Proposition 1.1.15 in Σ *i.e* convex sets are not always intersections of dials of building.

1.4.4 Geometric characterization of parabolic subgroups

Now we discuss the notion of residues in Σ as it is done at the end of Subsection 1.1.2.

Notation. *For $I \subset S$ and $g \in \Gamma$, let $g\Sigma_I$ denote the union of the chambers of the I-residue containing gx_0.*

Notice that $g\Sigma_I = g\Gamma_I x_0$ and $\mathrm{Ch}(g\Sigma_I) = g\Gamma_I$. For simplicity, in the following we also call a subset $g\Sigma_I \subset \Sigma$ a *residue*. Notice that a rotation around a building-wall that crosses $g\Sigma_I$ is of the form $g\gamma s^\alpha \gamma^{-1} g^{-1}$ with $s \in S$, $s^\alpha \neq e$ and $\gamma \in \Gamma_I$. Along with the definitions of the action and the residues we obtain the following fact.

Fact 1.4.16. *Let $R = g\Sigma_I$ be a residue. Then*

- *R is stabilized by the rotations around the building-walls that cross it,*
- $\mathrm{Stab}_\Gamma(R) = g\Gamma_I g^{-1}$ *is generated by these rotations,*
- *the type I is given by the type of the building-walls that cross R along gx_0.*

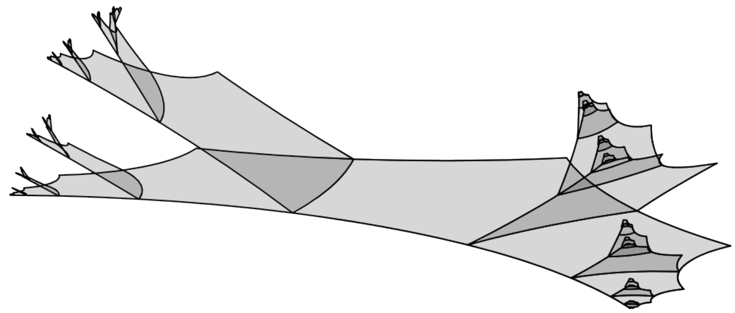


Figure 1.8: A residue of rank 2 in a Fuchsian building

The following result gives a converse to the previous fact. It is the analogous of Theorem 1.1.18.

Theorem 1.4.17. *Let $R \subset \Sigma$ be the union of a convex set of chambers and let P_R denote the group generated by the rotations around the building-walls that cross R. If P_R stabilizes R, then R is a residue in Σ.*

Proof. Up to a translation on R and a conjugation on P_R, we assume that $x_0 \subset R$. Let $C = \mathrm{Ch}(R)$. We start by proving that P_R acts freely and transitively on C. The action of Γ is free thus the action of P_R is free. For $x \in C$, by convexity of C, there exists a gallery

$$x_0 \sim x_1 \sim \cdots \sim x_\ell = x$$

of distinct chambers in C. Let M_i, of type $s_i \in S$, be the building-wall along x_{i-1} and x_i. Then

$$s_1^{\alpha_1} x_0 = x_1,\ s_1^{\alpha_1} s_2^{\alpha_2} x_0 = x_2,\ \ldots,\ s_1^{\alpha_1} \ldots s_\ell^{\alpha_\ell} x_0 = x,$$

for some exponent $\alpha_i \in \mathbb{Z}$.

We notice that $s_1^{\alpha_1} \ldots s_{i-1}^{\alpha_{i-1}} s_i (s_1^{\alpha_1} \ldots s_{i-1}^{\alpha_{i-1}})^{-1}$ is a rotation around M_i. Therefore x may be obtained from x_0 by successive around the building-walls M_i. These building-walls cross R, thus the action is transitive. This proves that $R = P_R x_0$ and $\mathrm{Stab}_\Gamma(R) = P_R$. It remains to prove that P_R is of the form Γ_I for a certain $I \subset S$.

We set $I \subset S$ the types of the building-walls that cross R along x_0 and we identify P_R with Γ_I. The inclusion $\Gamma_I < P_R$ comes from the definitions of P_R and I. We proceed by induction on $d_c(x_0, gx_0) = \ell$ to check that every element g of P_R is a product of elements of Γ_I. If $\ell = 0$, there is nothing to say. If $\ell > 0$ we choose $g = s_1^{\alpha_1} \ldots s_\ell^{\alpha_\ell}$ such that $d_c(x_0, gx_0) = \ell$. By convexity, $s_1^{\alpha_1} x_0 \in C$ so $s_1 \in \Gamma_I$, because s_1 is a rotation around a building-wall that crosses R along x_0. Then $d_c(x_0, s_1^{-\alpha_1} g x_0) = \ell - 1$ and $s_1^{-\alpha_1} g \in P_R$. The induction assumption permits us to conclude. □

In particular, this last theorem is used in Subsection 2.5.1 where we discuss the boundaries of the residues in the hyperbolic case.

1.5 Parallel residues in right-angled buildings

Here we apply the same ideas as in Section 1.3. We start by proving the analogous of Propositions 1.2.8 and 1.3.1. Then we conclude that the analogous of Theorem 1.3.2 holds.

In this section, Γ is a fixed graph product associated with the pair $(\mathcal{G}, \{G_i\}_{i=1,\dots n})$ as in Definition 1.4.1. We designate by Δ and Σ respectively the right-angled building and the Davis complex associated with Γ.

Proposition 1.5.1. *Let x and y be two adjacent chambers in Σ. Assume that R is a residue such that $\operatorname{proj}_R(x)$ and $\operatorname{proj}_R(y)$ are distinct. Then the building-wall M that separates x and y is the building-wall that separates $\operatorname{proj}_R(x)$ and $\operatorname{proj}_R(y)$.*

Proof. According to Proposition 1.2.8, there exists $A \in \mathcal{A}p(\Sigma)$ containing x, y, $\operatorname{proj}_R(x)$, and $\operatorname{proj}_R(y)$. Moreover in A, x and y (resp. $\operatorname{proj}_R(x)$ and $\operatorname{proj}_R(y)$) are separated by a common wall m. Thanks to Fact 1.4.9, we obtain that $m = M \cap A$. Thus M separates $\operatorname{proj}_R(x)$ and $\operatorname{proj}_R(y)$. □

Proposition 1.5.2. *Let R and Q be two parallel residues in Σ. Then*

$$\operatorname{Stab}_\Gamma(R) = \operatorname{Stab}_\Gamma(Q).$$

Proof. With Fact 1.4.16, to identify the stabilizers it is enough to identify the building-walls that crosses R and Q. Let M be a building-wall that crosses R. Let x and y be two distinct adjacent chambers of R along M. According to Proposition 1.2.9, the chambers $\operatorname{proj}_Q(x)$ and $\operatorname{proj}_Q(y)$ are distinct. Then thanks to Proposition 1.5.1, M also separates $\operatorname{proj}_Q(x)$ and $\operatorname{proj}_Q(y)$. In particular, M crosses Q. By symmetry of the argument, we can also prove that any wall that crosses Q crosses R. Eventually, according to Fact 1.4.16 $\operatorname{Stab}_W(R) = \operatorname{Stab}_W(Q)$. □

Now we prove that intersections of parabolic subgroups are parabolic subgroups.

Theorem 1.5.3. *Let $g \in \Gamma$, and $I, J \subset S$. If we set $P = \Gamma_I \cap g\Gamma_J g^{-1}$, then there exist $\gamma_I \in \Gamma_I$ and $\gamma_J \in \Gamma_J$ such that*

$$P = \gamma_I^{-1} W_K \gamma_I,$$

where $K = I \cap \gamma_I^{-1} g\gamma_J J(\gamma_I^{-1} g\gamma_J)^{-1} \subset S$.

Proof. Let $R = \Sigma_I$ and $Q = g\Sigma_I$ be the two residues stabilized Γ_I and $g\Gamma_J g^{-1}$. We set $\gamma_I \in \Gamma_I$ and $\gamma_J \in \Gamma_J$ such that

$$\operatorname{dist}(\gamma_I x_0, Q) = \operatorname{dist}(R, Q) \text{ and } d_c(\gamma_I x_0, g\gamma_J x_0) = \operatorname{dist}(R, Q).$$

Up to conjugate P by γ_I^{-1} and to change $\gamma_I^{-1} g\gamma_J$ for g we are in the situation where $P = \Gamma_I \cap g\Gamma_J g^{-1}$ with $d(x_0, gx_0) = \operatorname{dist}(R, Q)$. In this setting K designates $I \cap gJg^{-1}$ *i.e.*, thanks to the right angles $K = \{s \in I \cap J : sg = gs\}$. The inclusion $\Gamma_K < P$ is obviously true and we prove now the converse.

We know that $\mathrm{Stab}_\Gamma(R) = \Gamma_I$ and $\mathrm{Stab}_\Gamma(Q) = g\Gamma_J g^{-1}$. Then, according to Theorem 1.2.14,

$$P = \mathrm{Stab}_\Gamma(R) \cap \mathrm{Stab}_\Gamma(Q) = \mathrm{Stab}_\Gamma(\mathrm{proj}_R(Q)).$$

According to Fact 1.4.16, $\mathrm{Stab}_\Gamma(\mathrm{proj}_R(Q)) = \Gamma_{\overline{K}}$ where $\overline{K}$ designates the set of types of the walls that cross $\mathrm{proj}_R(Q)$ along x_0. It is now enough to prove that $\overline{K} \subset K$. Let $s \in \overline{K}$ and M be the building-wall that crosses $\mathrm{proj}_Q(R)$ along x_0 and sx_0. According to Proposition 1.5.1, M crosses $\mathrm{proj}_Q(R)$ along $\mathrm{proj}_Q(x_0)$ and $\mathrm{proj}_Q(sx_0)$. This situation is illustrated by Figure 1.9.

Hence $s(\mathrm{proj}_Q(x_0)) = \mathrm{proj}_Q(sx_0)$. Besides, at the beginning of the proof we chose $g \in \Gamma$ such that $\mathrm{proj}_Q(x_0) = gx_0$. Moreover, $\mathrm{proj}_Q(x_0)$ and $\mathrm{proj}_Q(sx_0)$ are adjacent along M of type s. Therefore, with $\mathrm{proj}_Q(x_0) = gx_0$, we get $gsg^{-1}(\mathrm{proj}_Q(x_0)) = \mathrm{proj}_Q(sx_0)$. Eventually we obtain

$$gx_0 = \mathrm{proj}_Q(x_0) = s^{-1}(\mathrm{proj}_Q(sx_0)) = s^{-1}gsx_0.$$

Hence $sg = gs$ and $\overline{K} \subset K$.

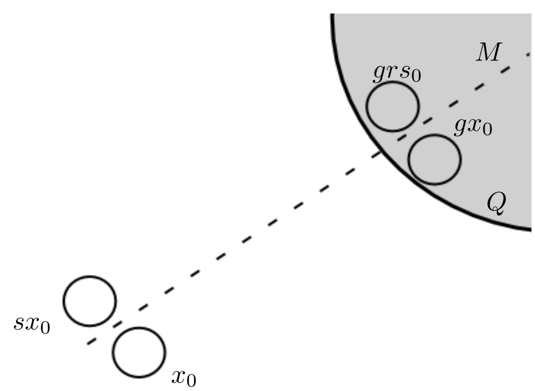


Figure 1.9

□

Remark 1.5.4. *A classification of F. Haglund and F. Paulin states that the construction presented in Subsection 1.4.1 describes all the right-angled buildings in the following sense.*

Theorem 1.5.5 ([HP03, Proposition 5.1.])**.** *Let Γ be the graph product given by the pair $(\mathcal{G}, \{G_i\}_{i=1,\dots,n})$ as in Definition 1.4.1. Let Δ be the building of type (W, S) associated with Γ. Assume that Δ' is a building of type (W, S) such that for any $s_i \in S$ the $\{s_i\}$-residues of Δ' are of cardinality $\#G_i$. Then Δ and Δ' are isomorphic.*

Remark 1.5.6. *Equipped with a natural piecewise Euclidean polyhedral metric, the Davis complex* Σ *of a graph product is CAT(0) (see [Dav98]).*

Remark 1.5.7. *If* Γ *is a graph product as in Definition 1.4.1, J. Meier gave a sufficient and necessary condition on the pair* $(\mathcal{G}, \{G_i\}_{i=1,\dots,n})$ *for* Γ *to be Gromov hyperbolic.*

To state this theorem we need to introduce two notations. For $v \in \mathcal{G}^{(0)}$ *we write* $\mathcal{L}_v$ *for the full subgraph of* $\mathcal{G}$ *generated by the vertices adjacent to* v *(which does not include* v*). Then* $\mathcal{FG}$ *is the full subgraph of* $\mathcal{G}$ *generated by the vertices associated with a finite group.*

Theorem 1.5.8 ([Mei96, Main Theorem])**.** *The group* Γ *is hyperbolic, if and only if the three following conditions are satisfied:*

i) The full subgraph generated by the vertices in $\mathcal{G} \backslash \mathcal{FG}$ *has no edge.*

ii) If v *is a vertex in* $\mathcal{G} \backslash \mathcal{FG}$*, then* $\mathcal{L}_v$ *is a complete graph.*

iii) Every circuit in $\mathcal{FG}$ *of length four contains a chord.*

Remark 1.5.9. *In [AM13], a proof of Theorem 1.5.3 is given in a more general case that includes graph products associated with a graph* $\mathcal{G}$ *with infinitely many vertices.*

Chapter 2

Combinatorial modulus on boundary of right-angled hyperbolic buildings

Introduction

The quasiconformal structure of boundaries of dimension 2 right-angled hyperbolic buildings were the object of exhaustive works by M. Bourdon and H. Pajot. Their results led them to prove the quasi-isometry rigidity for dimension 2 right-angled hyperbolic buildings. Since then, one can ask whether such a strong rigidity result is valid for some hyperbolic buildings of higher dimension. Yet, this is a hard question as the methods of M. Bourdon and H. Pajot are strongly specific to the buildings of dimension 2.

Recently, M. Bourdon and B. Kleiner have investigated the quasiconformal structure of boundaries of hyperbolic Coxeter groups thanks to combinatorial modulus of curves. These tools are weaker than the one using the classical modulus introduced by J. Heinonen and P. Koskela in metric measured spaces. Nevertheless, it appears that these tools can be used for a first approach of the quasiconformal structure of boundaries of some hyperbolic buildings of higher dimension. In particular, in this chapter we exhibit some hyperbolic buildings of dimension 3 and 4 whose boundary satisfy the *Combinatorial Loewner Property* (CLP).

Indeed, in some well chosen examples of higher dimensional hyperbolic buildings it is possible to proceed as follow. First, we can reduce the problem of the modulus of all the curves of the boundary to a discussion about the modulus of the curves contained in the boundary of some parabolic subgroups. Then the combinatorial modulus allow us to see the boundary of a building as the product of the boundary of an apartment by a discrete set. As a result the whole discussion is projected to the boundary of an apartment. Then if the boundary of the apartment has enough symmetry we obtain the CLP.

The main new result of this chapter is Theorem 2.9.1. This theorem provides the first hyperbolic buildings of higher dimension to be known for satisfying such a quasiconformal property (see Corollary 2.9.3). Besides, some partial results (for instance Theorem 2.7.13 and Corollary 2.7.16) are valid in a larger context. Yet, a work need to be done to use them to prove the CLP for other examples or to approach effectively the conformal dimension of the boundary. Moreover, we add to this chapter a discussion about the topology and the metric of the boundary. In particular, we describe how they can be understood thanks to the wall structure of the building. The results of Section 2.5, that lead to Theorem 2.5.13, are directly inspired by the the work of M. Bourdon and B. Kleiner. Their ideas in Coxeter groups apply without much subtleties to the case of right-angled hyperbolic buildings.

Organization of the chapter

In Section 2.1, we introduce the combinatorial modulus of curves in the general setting of compact metric spaces. Then in Section 2.2, we restrict to the case of boundaries of hyperbolic spaces.

After these reminders, we give the main steps and ideas of the proof of Theorem 2.9.1 in Section 2.3. This section is essentially a summary of this chapter.

Then, in Section 2.4, we restrict the setting of Subsection 1.4.1 to the case of locally finite right-angled hyperbolic buildings. Besides, we give basic geometric properties satisfied by the boundary of these buildings.

The key notion of parabolic limit sets is introduced in Section 2.5 where we study the modulus of curves in parabolic limit sets. This section is based on, the ideas used in Coxeter groups in [BK13, Section 5 and 6]. In particular, Theorem 2.5.13 is the first major step in direction of the proof of Theorem 2.9.1. As a consequence of this theorem, we obtain a first application to the CLP (Theorem 2.5.14).

In Section 2.6, we describe the combinatorial metric on the boundary of the group thanks to the geometry of the building. This metric is convenient to use to compute combinatorial modulus. Then, in Section 2.7 we discuss how the modulus in the boundary of an apartment, may be related to a modulus in the boundary of the building. In particular, Theorem 2.7.9 is the second major step necessary to prove Theorem 2.9.1. We use this theorem to prove Theorem 2.8.1 that relates the conformal dimension of the boundary of the building to a critical exponent computed in the boundary of an apartment.

In Section 2.8, we add the assumption of constant thickness of the buildings which specifies the results of the preceding section. In particular, we find that the conformal dimension of the boundary of the building is equal to a critical exponent computed in the boundary of an apartment (see Theorem 2.8.1). Eventually in Section 2.9, we gather these tools to obtain examples of right-angled-buildings of dimension 3 and 4 whose boundary satisfies the CLP (see Corollary 2.9.3).

2.1 Combinatorial modulus and the CLP

The combinatorial modulus are tools developed to compute modulus of curves in a metric space without a natural measure. The idea is to approximate the metric space, with a sequence of thinner and thinner approximations. Then with these approximations we can construct discrete measures and compute combinatorial modulus. Finally, for well chosen examples we can check that this sequence of modulus has a good asymptotic behavior.

In this first section, we present the general theory of combinatorial modulus in compact metric spaces. We also remind basic definitions and facts about abstract Loewner spaces as they inspired the theory of combinatorial modulus. Most of this section is a reminder of [BK13, Section 2] to which we refer for details.

In this section (Z, d) denotes a compact metric space.

2.1.1 General properties of combinatorial modulus of curves

For $k \geq 0$ and $\kappa > 1$, a *κ-approximation of Z on scale k* is a finite covering G_k by open subsets such that for any $v \in G_k$ there exists $z_v \in v$ satisfying the following properties:

- $\forall v \in G_k$ one has $B(z_v, \kappa^{-1}2^{-k}) \subset v \subset B(z_v, \kappa 2^{-k})$,
- $\forall v, w \in G_k$ with $v \neq w$ one has $B(z_v, \kappa^{-1}2^{-k}) \cap B(z_w, \kappa^{-1}2^{-k}) = \emptyset$.

A sequence $\{G_k\}_{k\geq 0}$ is called a *κ-approximation of Z*.

Example 2.1.1. *For $k \geq 0$, a 2^{-k}-*separated subset *of Z is a subset E such that $d(z, z') \geq 2^{-k}$ for any $z, z' \in E_k$. As Z is compact any 2^{-k}-separated subset of Z is finite. Let E_k be a 2^{-k}-separated subset of Z of maximal cardinality. Then E_k satisfies the following property:*

$$\text{for any } x \in Z, \text{ there exists } z \in E_k \text{ such that } d(x, z) \leq 2^{-k}.$$

The set $\{B(z, 2^{-k})\}_{z\in E_k}$ defines a 2-approximation at scale k of Z.

Now we fix the approximation $\{G_k\}_{k\geq 0}$. We construct a discrete measure based on each G_k for $k \geq 0$. Let $\rho : G_k \longrightarrow \mathbb{R}^+$ be a positive function and γ be a curve in Z. The *ρ-length* of γ is

$$L_\rho(\gamma) = \sum_{\gamma\cap v\neq\emptyset} \rho(v).$$

For $p \geq 1$, the *p-mass* of ρ is

$$M_p(\rho) = \sum_{v\in G_k} \rho(v)^p.$$

Until the end of this subsection $p \geq 1$ is fixed. Let $\mathcal{F}$ be a non-empty set of curves in Z. We say that the function ρ is *$\mathcal{F}$-admissible* if $L_\rho(\gamma) \geq 1$ for any curve $\gamma \in \mathcal{F}$.

Definition 2.1.2. *The* G_k-combinatorial p-modulus of $\mathcal{F}$ *is*

$$\mathrm{Mod}_p(\mathcal{F}, G_k) = \inf\{M_p(\rho)\}$$

where the infimum is taken over the set of $\mathcal{F}$*-admissible functions and with the convention* $\mathrm{Mod}_p(\emptyset, G_k) = 0$.

The following equality is an alternative definition of the modulus:

$$\mathrm{Mod}_p(\mathcal{F}, G_k) = \inf_\rho \frac{M_p(\rho)}{L_\rho(\mathcal{F})^p},$$

where the infimum is taken over the set of positive functions on G_k and with $L_\rho(\mathcal{F}) = \inf_{\gamma\in\mathcal{F}} L_\rho(\gamma)$.

The next proposition allows us to see the G_k-combinatorial p-modulus as a weak outer measure on the set of curves of Z. Usually, for an outer measure the subadditivity must hold over countable sets. This is useful to get intuition on these tools.

Proposition 2.1.3 ([BK13, Proposition 2.1.]).

1. *Let* $\mathcal{F}$ *be a family of curves and* $\mathcal{F}' \subset \mathcal{F}$*. Then*

$$\mathrm{Mod}_p(\mathcal{F}', G_k) \leq \mathrm{Mod}_p(\mathcal{F}, G_k).$$

2. *Let* $\mathcal{F}_1, \ldots, \mathcal{F}_n$ *be families of curves. Then*

$$\mathrm{Mod}_p(\bigcup_{i=1}^n \mathcal{F}_i, G_k) \leq \sum_{i=1}^n \mathrm{Mod}_p(\mathcal{F}_i, G_k).$$

A function $\rho : G_k \longrightarrow \mathbb{R}^+$ is called a *minimal function* for a set of curves $\mathcal{F}$ if $\mathrm{Mod}_p(\mathcal{F}, G_k) = M_p(\rho)$. As we only compute finite sums, minimal functions always exist. Along with a convexity argument, this also provides an elementary control of the modulus as follow. For $\mathcal{F}$ a non-empty set of curves in Z and $k \geq 0$

$$\frac{1}{(\#G_k)^{p-1}} \leq \mathrm{Mod}_p(\mathcal{F}, G_k) \leq \#G_k.$$

In the sequel of this chapter we mainly discuss the curves of Z of diameter larger than a fixed constant. For these curves the following basic property is useful.

Proposition 2.1.4. *Let* $\mathcal{F}$ *be a non-empty set of curves in* Z*. Assume that there exists* $d > 0$ *such that* $\mathrm{diam}\,\gamma \geq d$ *for any* $\gamma \in \mathcal{F}$*. Then for any* $\epsilon > 0$*, there exists* $k_0 \geq 0$ *such that for any* $k \geq k_0$*, there exists a minimal function* $\rho : G_k \longrightarrow \mathbb{R}^+$ *such that* $\rho(v) \leq \epsilon$ *for any* $v \in G_k$*.*

Proof. Let $\gamma \in \mathcal{F}$. We remind that κ denotes the multiplicative constant of the approximation $\{G_k\}_{k\geq 0}$. For $k \geq \frac{\log(\kappa/d)}{\log 2}$, as diam $\gamma > d$ the following inequality holds

$$\#\{v \in G_k : v \cap \gamma \neq \emptyset\} \geq \frac{d}{\kappa 2^{-k}}.$$

Hence the constant function : $v \in G_k \longrightarrow \frac{\kappa}{d}2^{-k} \in \mathbb{R}^+$ is $\mathcal{F}$-admissible. This finishes the proof. $\square$

A metric space Z is called *doubling* if there exists a uniform constant N, called the *doubling constant*, such that each ball B of radius r is covered by N balls of radius $r/2$. In doubling spaces, the G_k-combinatorial p-modulus does not depend, up to a multiplicative constant, on the choice of the approximation.

Proposition 2.1.5 ([BK13, Proposition 2.2.]). *Let (Z, d) be a compact doubling metric space. For each $p \geq 1$, if G_k and G'_k are respectively κ and κ'-approximations, there exists $D = D(\kappa, \kappa')$ such that for any $k \geq 0$*

$$D^{-1} \cdot \mathrm{Mod}_p(\mathcal{F}, G_k) \leq \mathrm{Mod}_p(\mathcal{F}, G'_k) \leq D \cdot \mathrm{Mod}_p(\mathcal{F}, G_k).$$

Usually, we work with $p \geq 1$ fixed and with approximately self-similar spaces (see Section 2.2). As these spaces are doubling, now we refer to the *combinatorial modulus on scale k*, omitting p and the approximation.

2.1.2 Combinatorial Loewner property

In this subsection, we assume that (Z, d) is a compact arcwise connected doubling metric space. Let $\kappa > 1$ and let $\{G_k\}_{k\geq 0}$ denote a κ-approximation of Z. Moreover we fix $p \geq 1$.

A compact and connected subset $A \subset Z$ is called a *continuum*. Moreover, if A contains more than one point, A is called a *non-degenerate* continuum. The relative distance between two disjoint non-degenerate continua $A, B \subset Z$ is

$$\Delta(A, B) = \frac{\mathrm{dist}(A, B)}{\min\{\mathrm{diam}\, A, \mathrm{diam}\, B\}}.$$

If A and B are two such continua, $\mathcal{F}(A, B)$ denotes the set of curves in Z joining A and B and we write $\mathrm{Mod}_p(A, B, G_k) := \mathrm{Mod}_p(\mathcal{F}(A, B), G_k)$.

Definition 2.1.6. *Let $p > 1$. We say that Z satisfies the* Combinatorial p-Loewner Property *(CLP) if there exist two increasing functions ϕ and ψ on $(0, +\infty)$ with $\lim_{t\to 0} \psi(t) = 0$, such that*

i) for any pair of disjoint non-degenerate continua A and B in Z and for all $k \geq 0$ with $2^{-k} \leq \min\{\mathrm{diam}\, A, \mathrm{diam}\, B\}$ one has:

$$\phi(\Delta(A, B)^{-1}) \leq \mathrm{Mod}_p(A, B, G_k),$$

ii) for any pair of open balls B_1, B_2 in Z, with same center and $B_1 \subset B_2$, and for all $k \geq 0$ with $2^{-k} \leq \operatorname{diam} B_1$ one has:

$$\operatorname{Mod}_p(\overline{B_1}, Z \backslash B_2, G_k) \leq \psi(\Delta(\overline{B_1}, Z \backslash B_2)^{-1}).$$

As we assume that Z is doubling, thanks to Proposition 2.1.5, the CLP is independent of the choice of the approximation. As we noticed, the modulus on scale k is an outer measure (in a weak sense) over the set of curves in Z. With the previous remarks we can say intuitively that

> *A metric space satisfies the CLP if, the farer are the continua, the smaller is this amount of curves joining them.*

We present examples and properties about the CLP in Subsection 2.1.4.

2.1.3 Loewner spaces

Now we define the notion of Loewner space. This notion introduced in [HK98] has inspired the definition of the CLP. Moreover, the proof of many basic properties of combinatorial modulus are directly inspired by the classical theory of modulus (see [BK13]).

Now we consider (X, d, μ) a metric measured space. For simplicity, we assume that X is compact and *Q-Ahlfors-regular* (Q-AR or AR) for $Q > 1$. This means that there exists a constant $C > 1$ such that for any $0 < R \leq \operatorname{diam} X$ and any R-ball $B \subset X$ one has

$$C^{-1} \cdot R^Q \leq \mu(B) \leq C \cdot R^Q.$$

Note that under this assumption the measure μ is comparable to the Hausdorff measure $\mathcal{H}_d$.

Let $\mathcal{F}$ be a set of curves in X. A measurable function $f : X \longrightarrow \mathbb{R}^+$ is said to be $\mathcal{F}$-admissible if for any rectifiable curve $\gamma \in \mathcal{F}$

$$\int_{\gamma(t)} f(\gamma(t))dt \geq 1.$$

Note that the notion of admissibility does not use the measure on X but only the structure of metric space.

Definition 2.1.7. *The Q-*modulus *of $\mathcal{F}$ is*

$$\operatorname{Mod}_Q(\mathcal{F}) = \inf \Big\{ \int_X f^Q d\mu \Big\}$$

where the infimum is taken over the set of $\mathcal{F}$-admissible functions and with the convention that $\operatorname{Mod}_Q(\mathcal{F}) = 0$ if $\mathcal{F}$ does not contain rectifiable curves.

As before, if A and B are two disjoint non-degenerate continua, $\mathcal{F}(A,B)$ denotes the set of curves in X joining A and B. Moreover, we write $\mathrm{Mod}_Q(A,B) := \mathrm{Mod}_Q(\mathcal{F}(A,B))$. In the literature about quasiconformal maps the pair (A,B) is called a *condenser* and the modulus (with respect to the Lebesgue measure) $\mathrm{Mod}_Q(A,B)$ the *capacity* of (A,B) (see [Vuo88]).

In X, the classical modulus are comparable to the combinatorial modulus in the following sense.

Proposition 2.1.8 ([Haï09a, Prop B.2]). *Assume that X is equipped with an approximation $\{G_k\}_{k\geq 0}$. For $d_0 > 0$, let $\mathcal{F}_0$ be the set of curves in X of diameter larger than d_0. For k large enough one has*

$$\mathrm{Mod}_Q(\mathcal{F}_0, G_k) \asymp \mathrm{Mod}_Q(\mathcal{F}_0)$$

if $\mathrm{Mod}_Q(\mathcal{F}_0) > 0$ and $\mathrm{Mod}_Q(\mathcal{F}_0)$ goes to 0 when k goes to infinity otherwise. Besides for any pair A, B of non-degenerate disjoint continua and for k large enough one has

$$\mathrm{Mod}_Q(A,B,G_k) \asymp \mathrm{Mod}_Q(A,B)$$

if $\mathrm{Mod}_Q(A,B) > 0$ and $\mathrm{Mod}_Q(A,B)$ goes to 0 when k goes to infinity otherwise.

Note that this connection between combinatorial and classical modulus is only valid for the dimension Q.

Now we can define Loewner spaces.

Definition 2.1.9. *We say that (X,d,μ) is a Q-*Loewner space *if there exists an increasing function $\phi : (0,+\infty) \longrightarrow (0,+\infty)$ such that for any pair of non-degenerate disjoint continua A and B in X one has:*

$$\phi(\Delta(A,B)^{-1}) \leq \mathrm{Mod}_Q(\mathcal{F}(A,B)).$$

We also say that X satisfies the *Loewner property* or the *classical Loewner property* to avoid the confusion with the CLP.

The control of the modulus from above is not required in this definition because it is automatically provided by the structure of Q-AR space.

Theorem 2.1.10 ([HK98, Lemma 3.14.]). *There exists $C > 0$ a constant such that the following property holds. Let A and B be two non-degenerate disjoint continua. Let $0 < 2r < R$ and $x \in X$ be such that $A \subset \overline{B(x,r)}$ and $B \subset x\backslash B(x,R)$. Then*

$$\mathrm{Mod}_Q(A,B) \leq C\Big(\log\frac{R}{r}\Big)^{1-Q}.$$

As a consequence there exists an increasing function ψ on $(0,+\infty)$ with $\lim_{t\to 0}\psi(t) = 0$, such that for any pair of disjoint non-degenerate continua A and B

$$\mathrm{Mod}_Q(A,B) \leq \psi(\Delta^{-1}(A,B)).$$

More precisely, there exist some constants $K, C > 0$ such that $\psi(t) = K\Big(\log(\frac{1}{t} + C)\Big)^{1-Q}$ for any $t > 0$.

When X is a Loewner space, the asymptotic behavior of ϕ is described in [HK98, Theorem 3.6.]. For t small enough $\phi(t) \approx \log \frac{1}{t}$, for t large enough $\phi(t) \approx (\log t)^{1-Q}$.

As we will see in the sequel an essential difference between the combinatorial and the classical modulus property is the importance of the dimension Q in the discussions about classical modulus.

2.1.4 First properties and examples

In this section Z is a compact arcwise connected doubling metric space and X is a compact Q-AR metric space ($Q > 1$). First we remind a theorem and a conjecture that compare the CLP and the classical Loewner property.

Theorem 2.1.11 ([BK13, Theorem 2.6.])**.** *If X is a compact Q-AR and Loewner metric space, then X satisfies the combinatorial Q-Loewner property.*

The next conjecture is a motivation for looking to boundaries of groups that satisfy the CLP.

Conjecture 2.1.12 ([Kle06, Conjecture 7.5.])**.** *Assume that Z is quasi-Moebius homeomorphic to the boundary of a hyperbolic group. If Z satisfies the CLP then it is quasi-Moebius homeomorphic to a Loewner space.*

As announced we want to find and use the CLP on boundaries of hyperbolic groups. Quasi-isometries between hyperbolic spaces extends in quasi-Moebius homeomorphisms between the boundaries, so it is fundamental to know how these properties behave under quasi-Moebius maps. These maps have been introduced by J. Vaïsälä in [Väi85]. We remind that in a metric space (Z, d) the *cross-ratio* of four distinct points $a, b, c, d \in Z$ is

$$[a : b : c : d] = \frac{d(a,b)}{d(a,c)} \cdot \frac{d(c,d)}{d(b,d)}.$$

For Z, Z' two metric spaces, an homeomorphism $f : Z \longrightarrow Z'$ is *quasi-Moebius* if there exists an homeomorphism $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that for any quadruple of distinct points $a, b, c, d \in Z$

$$[f(a) : f(b) : f(c) : f(d)] \leq \phi([a : b : c : d]).$$

If f is quasi-Moebius, as $[a : c : b : d] = [a : b : c : d]^{-1}$, $f^{-1} : Z' \longrightarrow Z$ is also quasi-Moebius.

Theorem 2.1.13 ([BK13, Theorem 2.6.])**.** *If Z' is quasi-Moebius homeomorphic to a compact space Z satisfying the CLP, then Z' also satisfies the CLP (with the same exponent).*

The Loewner property does not behave so well under quasi-Moebius maps. In particular, it is perturbed by a change of dimension whereas the CLP is not.

Theorem 2.1.14 ([Tys98])**.** *Let X and X' be respectively Q-Loewner and Q'-AR compact metric spaces. Assume that X' is quasi-Moebius homeomorphic to X. Then X' is a Loewner space if and only if $Q = Q'$.*

If we apply to X a snowflake transformation $f_\epsilon : (X, d) \longrightarrow (X, d^\epsilon)$, $0 < \epsilon < 1$ then $\dim_{\mathcal{H}}(X, d^\epsilon) = \frac{1}{\epsilon} \dim_{\mathcal{H}}(Z, d)$. Such a transformation is a quasi-Moebius homeomorphism and along with the previous theorem we get the following fact.

Fact 2.1.15. *The Loewner property is not invariant by quasi-Moebius homeomorphisms.*

Yet quasi-Moebius maps are the appropriate homeomorphisms to discuss between Loewner spaces.

Theorem 2.1.16 ([HK98])**.** *Let X and X' be two compact Q-regular Loewner spaces and let $f : X \longrightarrow X'$ be a homeomorphism. The following are equivalent*

1. *f is quasi-Moebius,*
2. *there exists $C > 1$ such that*

$$C^{-1} \cdot \mathrm{Mod}_Q(\mathcal{F}) \leq \mathrm{Mod}_Q(f(\mathcal{F})) \leq C \cdot \mathrm{Mod}_Q(\mathcal{F})$$

for any set of curves $\mathcal{F}$ in X.

The next proposition gives examples of spaces that do not satisfy the CLP.

Proposition 2.1.17 ([HK98] or [BK13, Theorem 2.6.])**.** *Assume that Z satisfies the CLP then it has no local cut point,* i.e *no connected open subset is disconnected by removing a point.*

Along with the theorem of Bowditch (see [Bow98]) this proposition says that the boundary of a one-ended hyperbolic group which splits along a two-ended subgroup does not satisfy the CLP.

The first examples of spaces that satisfy the CLP are provided by Theorem 2.1.11 and by the examples of formerly known Loewner spaces. The next examples are provided by [BK13].

Example 2.1.18.

1. *The following spaces are Loewner spaces*

   i) the Euclidean space $\mathbb{R}^d$ for $d \geq 2$, this result is due to C. Loewner for $d \geq 3$ (see [Loe59]), *ii) any Riemannian compact manifold modeled by* $\mathbb{R}^d$ *for* $d \geq 2$ *(see [HK98]),*

iii) boundaries of right-angled Fuchsian buildings (see [BP99]).

2. The following spaces verify the CLP (see [BK13])

i) the Sierpiński carpet and the n-dimensional Menger sponge embedded in the Euclidean space,

ii) boundaries of Coxeter groups of various type: simplex groups, some prism groups, some highly symmetric groups and some groups with planar boundary.

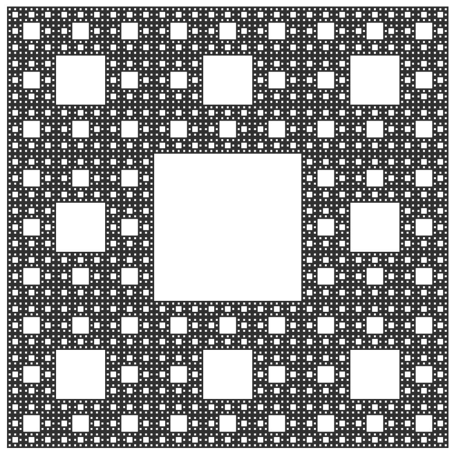


Figure 2.1: The Sierpiński carpet in $\mathbb{E}^2$ satisfies the CLP

For Examples 2.1.18.2, we do not know if they are Loewner spaces. This provides a first kind of interesting questions.

Question 2.1.19. *Is one of the Examples 2.1.18.2. quasi-Moebius homeomorphic to a Loewner space?*

Among these cases the one of the Sierpiński carpet is the first that should be discussed as it should be the easiest one.

Note that Example 2.1.18.1.*ii*) provides many examples of hyperbolic groups whose boundaries are Loewner spaces. Indeed consider a group Γ that is acting geometrically (see Subsection 2.2.1) on the standard hyperbolic space $\mathbb{H}^d$ for $d \geq 3$. Then $\partial\Gamma$ is quasi-Moebius equivalent to $\mathbb{S}^{d-1}$ equipped with the standard spherical metric. Hence with Example 2.1.18.1.*ii*), $\partial\Gamma$ is a Loewner space.

Actually the Loewner property and the CLP have been interesting subject of discussion on boundaries of groups to approach Cannon's conjecture (see Conjecture 0.0.2) that is a reverse problem. By a theorem of D. Sullivan in [Sul82], Cannon's conjecture is equivalent to the following in which the quasiconformal structure of the boundary is the main point.

Conjecture 2.1.20 ([BK02, Conjecture 1.3.]). *If Γ is a hyperbolic group and $\partial\Gamma$ is homeomorphic to $\mathbb{S}^2$, then it is quasi-Moebius homeomorphic to the standard 2-sphere.*

The notion of Loewner space is then interesting to find on boundaries of groups as quasi-Moebius maps are the good morphisms to consider between Loewner spaces. If Conjecture 2.1.12 is true, the CLP would provide many interesting examples of Loewner spaces. This motivates this second question.

Question 2.1.21. *Can we find new examples of compact metric spaces satisfying the CLP?*

In this chapter, we find examples of boundaries of right-angled buildings of dimension 3 and 4 that satisfy the CLP. The discussion about these buildings is suggested by Examples 2.1.18.1.*iii*) and 2.1.18.2.*ii*). The strategy applied is to lift some tools used in [BK13] to study the Coxeter groups thanks to some ideas used in [BP00] to study buildings of dimension 2.

The examples of buildings that we obtain, have been studied by J. Dymara and D. Osajda who described the topology of the boundary.

Theorem 2.1.22 ([DO07]). *Let Δ be a right-angled thick building whose associated Coxeter group is a cocompact reflection group in $\mathbb{H}^d$. Then $\partial\Delta$ is homeomorphic to the universal $(d-1)$-dimensional Menger space μ^{d-1}.*

2.2 Combinatorial modulus of curves on boundaries of hyperbolic groups

Boundaries of hyperbolic groups are naturally endowed with metric structures that satisfy a property of self-similarity. This permits to rescale the sets of curves of the boundary with a controlled modulus. Which is very useful to prove the CLP.

In this section, we present how the boundary of a hyperbolic group can be seen as approximately self-similar spaces. Then, we remind the connection between the combinatorial modulus and the conformal dimension of the boundary. Eventually, we give a sufficient condition for the boundary to satisfy the CLP.

Most of this section is a reminder of [BK13, Section 3 and 4] to which we refer for details.

2.2.1 Boundaries of hyperbolic groups and approximately self-similar spaces

For details concerning hyperbolic groups and spaces, we refer to [CDP90], [GdlH90] or [KB02]. Here (X, d) is a proper geodesic metric space. We say that a finitely generated group Γ *acts geometrically* on X, if:

- $\Gamma < \mathrm{Isom}(X)$,

- Γ acts cocompactly,
- Γ acts properly discontinuously.

We say that X is *hyperbolic* (in the sense of Gromov) if there exists a constant $\delta > 0$ such that for every geodesic triangle $[x, y] \cup [y, z] \cup [z, x] \subset X$ and every $p \in [x, y]$, one has

$$\mathrm{dist}(p, [y, z] \cup [z, x]) \leq \delta.$$

A finitely generated group that acts geometrically on a hyperbolic space X is called a *hyperbolic group.* In this case, the Cayley graph of a hyperbolic group is a hyperbolic space.

From nowon, X is a hyperbolic space with a fixed base point x_0 and Γ is a hyperbolic group acting geometrically on X. Let ∂X be the quotient space defined by the set of half-geodesics and by the equivalence relation:

> $\gamma, \gamma' : [0, +\infty) \longrightarrow X$ are equivalent if and only if there exists $K > 0$ such that $d(\gamma(t), \gamma'(t)) \leq K$ for any $t \in [0, +\infty)$.

Thanks to the hyperbolicity condition, we can restrict to the set of half-geodesics starting from x_0. We can equip ∂X and $X \cup \partial X$ with topologies which make them compact sets. In this setting X, is dense in $X \cup \partial X$ and ∂X is called the *boundary of* X. Moreover we can equip ∂X with a family of *visual metric.* A metric $\delta(\cdot, \cdot)$ is visual if there exist two constants $A \geq 1$ and $\alpha > 0$ such that for all $\xi, \xi' \in \partial X$:

$$A^{-1} e^{-\alpha \ell} \leq \delta(\xi, \xi') \leq A\, e^{-\alpha \ell},$$

where ℓ is the distance from x_0 to a geodesic line (ξ, ξ'). In such a situation we also write

$$\delta(\xi, \xi') \asymp e^{-\alpha \ell}.$$

The action of Γ on X extends naturally on $(\partial X, \delta)$ and elements of Γ are uniform quasi-Moebius homeomorphisms of the boundary. Moreover, if $\partial \Gamma$ is also equipped with a visual metric, the homeomorphism $\partial \Gamma \longrightarrow \partial X$ induced by the orbit map $g \in \Gamma \longrightarrow g x_0 \in X$ is quasi-Moebius.

The following definition is a generalization of the classical notion of self-similar space.

Definition 2.2.1. *A compact metric space* (Z, d) *is called* approximately self-similar *if there exists a constant* $L \geq 1$ *such that for every* r*-ball* B *with* $0 < r < \mathrm{diam}\, Z$*, there exists an open subset* $U \subset Z$ *which is* L*-bi-Lipschitz homeomorphic to the rescaled ball* $(B, \frac{1}{r} d)$*.*

The property that follows shows that the notion of approximately self-similar space fits with the metric structure of the boundary of a hyperbolic group and with the action of the group on its boundary.

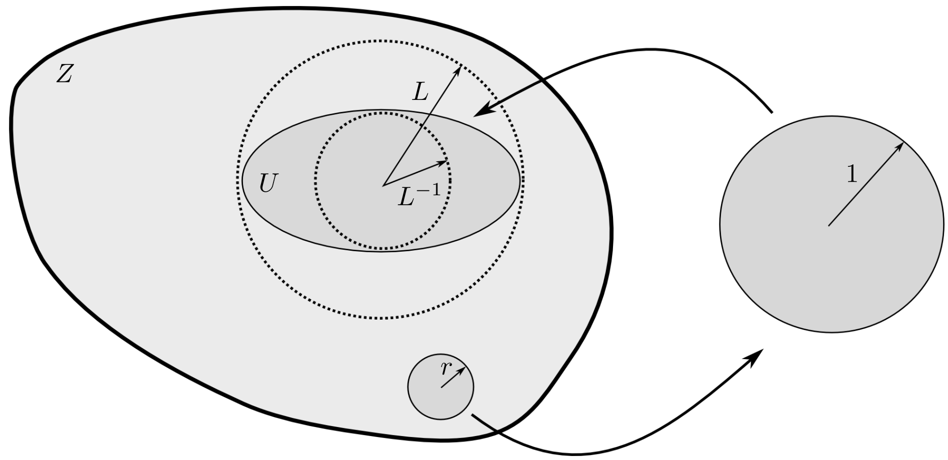


Figure 2.2

Definition 2.2.2. *Let* Γ *be a hyperbolic group. A metric* d *on* $\partial\Gamma$ *is called a* self-similar metric *if there exists a hyperbolic space* X *on which* Γ *acts geometrically, such that* d *is the preimage of a visual metric on* ∂X *by the canonical quasi-Moebius homeomorphism* $\partial\Gamma \longrightarrow \partial X$.

Proposition 2.2.3 ([BK13, Proposition 3.3.])**.** *The space* $\partial\Gamma$ *equipped with a self-similar metric is doubling and is an approximately self-similar space, the partial bi-Lipschitz maps being restrictions of group elements. Moreover,* Γ *acts on* $(\partial\Gamma, d)$ *by (non-uniformly) bi-Lipschitz homeomorphisms.*

2.2.2 Combinatorial modulus and conformal dimension

Here we present the connection between combinatorial modulus and the conformal dimension in approximately self-similar spaces.

Let Z be an arcwise connected approximately self-similar metric space. In practice Z will be the boundary of a hyperbolic space. Let $\{G_k\}_{k\geq 0}$ be a κ-approximation of Z and d_0 be a small constant compared with diam Z and with the constant of self-similarity.

Let $\mathcal{F}_0$ denote the family of curves in Z of diameter larger than d_0. In [BK13], it is proved that the properties of the combinatorial modulus are contained in the asymptotic behavior of $\mathrm{Mod}_p(\mathcal{F}_0, G_k)$. This point is explained in Subsection 2.2.3. Here we write $M_k := \mathrm{Mod}_p(\mathcal{F}_0, G_k)$.

The following proposition allows to define a critical exponent that is related to the conformal dimension of Z.

Proposition 2.2.4. *There exists* $p_0 \geq 1$ *such that for* $p \geq p_0$ *the modulus* M_k *goes to zero as* k *goes to infinity.*

Proof. We assume that $\{G_k\}_{k\geq 0}$ is a κ-approximation of Z. Then we write $K > 0$ the cardinal of a covering of Z by balls of radius κ. Then, by the doubling condition, we write N' the number of balls of radius $\kappa^{-1} \cdot \frac{1}{2}$ that cover a ball of radius κ. By induction we obtain

$$\#G_k \leq K \cdot N'^k \text{ for any } k \geq 1.$$

Moreover, as we saw in the proof of Proposition 2.1.4, there exists a constant $K' > 0$ such that the constant function $\rho : G_k \longrightarrow K' \cdot 2^{-k}$ is $\mathcal{F}_0$-admissible.

As a consequence

$$M_k \leq C \cdot (\frac{N'}{2^p})^k,$$

where C is a positive constant. Thus, for p large enough, M_k goes to zero. □

According to this proposition the next definition makes sense.

Definition 2.2.5. *The* critical exponent Q *associated with the curve family $\mathcal{F}_0$ is defined as follows*

$$Q = \inf\{p \in [1, +\infty) : \lim_{k\to+\infty} \mathrm{Mod}_p(\mathcal{F}_0, G_k) = 0\}.$$

We notice that for $k \geq 0$ fixed the function : $p \longmapsto \mathrm{Mod}_p(\mathcal{F}_0, G_k)$ is non-increasing. Hence $\mathrm{Mod}_p(\mathcal{F}_0, G_k)$ goes to zero for $p > Q$.

This critical exponent is related to the conformal dimension, which provides another motivation to study combinatorial modulus. The conformal dimension has been introduced by P. Pansu in [Pan89]. It is a key point in the conformal structure of the boundary of a hyperbolic group. In particular, it is invariant under quasi-Moebius maps.

In the following, $\mathcal{H}_d(\cdot)$ and $\dim_{\mathcal{H}}(Z, d)$ respectively denote the Hausdorff measure and the Hausdorff dimension of Z equipped with d. The *Ahlfors-regular conformal gauge* of (Z, d) is defined as follow

$$\mathcal{J}_c(Z, d) := \{(Z', \delta) \text{ AR and quasi-moebius homeomorphic to } (Z, d)\}.$$

Definition 2.2.6. *Let (Z, d) be a compact metric space. The* Ahlfors-regular conformal dimension *of (Z, d) is*

$$\mathrm{Confdim}(Z, d) := \inf\{\dim_{\mathcal{H}}(Z', \delta) : (Z', \delta) \in \mathcal{J}_c(Z, d)\}.$$

For simplicity, in the rest of the chapter we will refer to the *conformal dimension* for the Ahlfors-regular conformal dimension.

In [KK], S. Keith and B. Kleiner proved that the combinatorial modulus are related to the conformal dimension. The proof of the following theorem may be found in [Car11].

Theorem 2.2.7 ([KK] or [Car11, Theorem 4.5.])**.** *The critical exponent Q (see Definition 2.2.5) is equal to* $\mathrm{Confdim}(Z, d)$.

The definition of the conformal dimension, along with basic topology give the following inequalities:

$$\dim_T(Z) \leq \mathrm{Confdim}(Z,d) \leq \dim_{\mathcal{H}}(Z,d),$$

where $\dim_T(Z)$ is the topological dimension of Z.

The following theorem due to J. Tyson makes a connection between the conformal dimension and the Loewner property.

Theorem 2.2.8 ([MT10, Corollary 4.2.2.]). *If X is a Q-regular and Q-Loewner space, then* $\mathrm{Confdim}(X) = Q$.

Example 2.2.9. *It has been proved independently by B. Kleiner and in [KL04] that the Euclidean metric on the Sierpiński carpet does not realize the conformal dimension. As a consequence the Sierpiński carpet equipped with this metric does not satisfy the Loewner property. Yet it satisfies the CLP (see Example 2.1.18).*

Again, Cannon's conjecture has been an important motivation to discuss the conformal dimension of the boundary of a hyperbolic group. In particular in [BK05] it is proved that Conjecture 0.0.2 is equivalent to the following.

Conjecture 2.2.10. *If Γ is a hyperbolic group and $\partial\Gamma$ is homeomorphic to $\mathbb{S}^2$, then* $\mathrm{Confdim}(\partial\Gamma)$ *is attained by a metric in $\mathcal{J}_c(\partial\Gamma)$.*

2.2.3 How to prove the CLP

Now we give the sufficient condition that will be used in this chapter to exhibit some examples of groups with a boundary that satisfy the CLP.

Let Z be an arcwise connected approximately self-similar metric space and let $\{G_k\}_{k\geq 0}$ be a κ-approximation of Z.

The approximately self-similar structure and Proposition 2.1.5, allow to deform any set of small curves to a set of large curves with a control of the modulus. This is what we prove in the following proposition.

Proposition 2.2.11. *Let B be a ball in $\partial\Gamma$ such that* $\operatorname{diam} B < 1$. *Let $g \in \Gamma$ be the local L-bi-Lipschitz homeomorphism that rescales B (given by Definition 2.2.3). Let $\mathcal{F}$ be a set of curves contained in λB for $\lambda < 1$. Then there exist $\ell \in \mathbb{N}$, and $D > 1$ depending only on L and on the doubling constant of $\partial\Gamma$ such that the following property holds.*

If $k \geq 0$ is large enough so that

$$\{v \in G_k : \gamma \cap v \neq \emptyset \text{ for some } \gamma \in \mathcal{F}\} \subset \{v \in G_k : v \subset B\},$$

then

$$D^{-1} \cdot \mathrm{Mod}_p(g\mathcal{F}, G_k) \leq \mathrm{Mod}_p(\mathcal{F}, G_{k+\ell}) \leq D \cdot \mathrm{Mod}_p(g\mathcal{F}, G_k),$$

where $g\mathcal{F} = \{g\gamma : \gamma \in \mathcal{F}\}$.

Proof. Let $k \geq 0$ be large enough so that, if $\gamma \cap v \neq \emptyset$ with $\gamma \in \mathcal{F}$ and $v \in G_k$, then $v \subset B$. Let $d = \operatorname{diam} B$ and let $\ell \in \mathbb{N}$ denote the integer verifying $2^{-(\ell+1)} < d \leq 2^{-\ell}$. Let $v \in G_{k+\ell}$ such that $v \subset B$ and assume

$$B(\xi, \kappa^{-1}2^{-(k+\ell)}) \subset v \subset B(\xi, \kappa 2^{-(k+\ell)}).$$

Then

$$B(g\xi, (L\kappa)^{-1}2^{-k}) \subset gv \subset B(g\xi, 2L\kappa 2^{-k}).$$

We write $G'_k \cap gB = \{gv : v \in G_{k+\ell}, v \subset B\}$. Then $G'_k \cap gB$ is a κ'-approximation of gB on scale k, with κ' depending only on κ and L. As the curves of $\mathcal{F}$ are strictly contained in B we obtain the following equality

$$\operatorname{Mod}_p(\mathcal{F}, G_{k+\ell}) = \operatorname{Mod}_p(g\mathcal{F}, G'_k \cap gB).$$

Thanks to Proposition 2.1.5, there exists $D > 1$ such that

$$D^{-1} \cdot \operatorname{Mod}_p(g\mathcal{F}, G_k) \leq \operatorname{Mod}_p(g\mathcal{F}, G'_k \cap gB) \leq D \cdot \operatorname{Mod}_p(g\mathcal{F}, G_k),$$

and the proposition follows. □

Now we fix $d_0 > 0$ a small constant compared with diam Z and with the constant of self-similarity. More precisely it must be small enough so that any non-constant curve in Z may be rescaled to a curve of diameter larger than d_0 by self-similarity. In practice, Z will be the boundary of a hyperbolic group and d_0 will depend on the hyperbolicity constant.

Let $\mathcal{F}_0$ denote the family of curves in Z of diameter larger than d_0. This rescaling property explains that the properties of the combinatorial modulus are contained in the asymptotic behavior of $\operatorname{Mod}_p(\mathcal{F}_0, G_k)$. Again, we use the letter Q to designate the critical exponent of Definition 2.2.5.

Proposition 2.2.12 ([BK13, Proposition 4.5.]). *Let Z be an arcwise connected approximately self-similar metric space. Let $\{G_k\}_{k\geq 0}$ be a κ-approximation of Z and d_0 be a small constant compared with* diam Z *and with the constant of self-similarity. Let $\mathcal{F}_0$ denote the family of curves in Z of diameter larger than d_0.*

For $p = 1$, we assume that $\operatorname{Mod}_p(\mathcal{F}_0, G_k)$ *is unbounded. For $p \geq 1$, we assume that for every non-constant curve $\eta \subset Z$ and every $\epsilon > 0$, there exists $C = C(p, \eta, \epsilon)$ such that for every $k \in \mathbb{N}$:*

$$\operatorname{Mod}_p(\mathcal{F}_0, G_k) \leq C \cdot \operatorname{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

Suppose furthermore when p belongs to a compact subset of $[1, +\infty)$ the constant C may be chosen independent of p. Then Z satisfies the combinatorial Q-Loewner property.

2.3 Steps of the proof of Theorem 2.9.1

Before going into details about boundaries of right-angled buildings, we give a sketch of proof of the main theorem of this chapter. In this section, D is the right-angled dodecahedron in $\mathbb{H}^3$ or the right-angled 120-cell in $\mathbb{H}^4$. We write W_D for the hyperbolic reflection group generated by the faces of D. The main theorem of this chapter may be stated as follows.

Theorem 2.3.1 (Theorem 2.9.1)**.** *Let $q \geq 3$ and let Γ be the a graph product of constant thickness $q-1$ and of Coxeter group W_D. Then $\partial\Gamma$ equipped with a visual metric satisfies the CLP.*

As announced, we will verify that $\partial\Gamma$ satisfies the hypothesis of Proposition 2.2.12. To prove that $\mathrm{Mod}_1(\mathcal{F}_0, G_k)$ is unbounded, it is enough to see that for every $N \in \mathbb{N}$ there exist N disjoint curves in $\partial\Gamma$. Indeed, this implies that for $k \geq 0$ large enough $\mathrm{Mod}_1(\mathcal{F}_0, G_k) > N$.

To follow curves to control the modulus. For $p > 1$, we want to prove that the curves of $\partial\Gamma$ satisfy a property of the following form.

> (P) : For $\epsilon > 0$, there exists a finite set F of bi-Lipschitz maps $f : \partial\Gamma \longrightarrow \partial\Gamma$ such that for any curve $\gamma \in \mathcal{F}_0$ and any curve η in $\partial\Gamma$, the set $\bigcup_{f\in F} f(\gamma)$ contains a curve that belongs to $\mathcal{U}_\epsilon(\eta)$.

Where $\mathcal{U}_\epsilon(\eta)$ denotes the ϵ-neighborhood of η for the C^0 distance (see Subsection 0.0.3 for details). Intuitively, (P) holds if from any curve γ we can *follow* any other curve η thanks to bi-Lipschitz maps. The following computation shows that the property (P) implies the desired inequality.

Proposition 2.3.2. *If $\mathrm{Mod}_1(\mathcal{F}_0, G_k)$ is unbounded, then property (P) implies the CLP.*

Proof. Let η be a curve in $\partial\Gamma$ and $\epsilon > 0$. Fix ρ a $\mathcal{U}_\epsilon(\eta)$-admissible function. The inequality required by the hypothesis of Proposition 2.2.12 is obtained if we find a constant $K > 0$ independent of the scale k and a $\mathcal{F}_0$-admissible function ρ' such that $M_p(\rho') \leq K \cdot M_p(\rho)$.

Let F be the set of bi-Lipschitz maps given by the property (P). We assume, without loss of generality that F contains F^{-1}. We define the function $\rho' : G_k \longrightarrow \mathbb{R}^+$ by:

$$(*) \qquad \rho'(v) = \sum_{f\in F} \sum_{fw\cap v\neq\emptyset} \rho(w).$$

Let $\gamma \in \mathcal{F}_0$ and $\theta \subset \bigcup\limits_{f\in F} f\gamma$ such that $\theta \in \mathcal{U}_\epsilon(\eta)$. Then

$$L_{\rho'}(\gamma) = \sum_{f\in F} \sum_{v\cap\gamma\neq\emptyset} \sum_{fw\cap v\neq\emptyset} \rho(w) \geq \sum_{f\in F} \sum_{w\cap f\gamma\neq\emptyset} \rho(w).$$

Yet

$$L_\rho(\theta) \leq \sum_{f\in F} L_\rho(f\gamma) = \sum_{f\in F} \sum_{w\cap f\gamma\neq\emptyset} \rho(w).$$

Hence $L_{\rho'}(\gamma) \geq L_\rho(\theta)$ and ρ' is $\mathcal{F}_0$-admissible.

Then the number of terms in the right-hand side of the definition $(*)$ is bounded by a constant N depending on $\#F$, the bi-Lipschitz constants of the elements of F, and the doubling constant of $\partial\Gamma$. Therefore by convexity

$$\begin{aligned} M_p(\rho') = &\sum_{v\in G_k} \Big(\sum_{f\in F} \sum_{fw\cap v\neq\emptyset} \rho(w)\Big)^p, \\ \leq &N^{p-1} \cdot \sum_{v\in G_k} \sum_{f\in F} \sum_{w\cap fv\neq\emptyset} \rho(w)^p \leq N^p \cdot \#F \cdot M_p(\rho). \end{aligned}$$

□

Note that this idea of following curves may be used to obtain an inequality between any pair of sets of curves.

The issue of parabolic limit sets. As Γ acts on $\partial\Gamma$ by bi-Lipschitz homeomorphisms, it is natural to intend to follow curves thanks to elements of Γ. Yet, in right-angled buildings some curves may be contained in parabolic limit sets (boundaries of residues). As we will see in Example 2.5.9, these curves are an obstacle to prove the property (P) with the elements of Γ.

To solve this problem we start by showing that $\mathrm{Mod}_p(\mathcal{F}_0, G_k)$ is determined by the combinatorial modulus of the sets of curves of the form $\mathcal{F}_{\delta,r}(\partial P)$ as defined at the beginning Subsection 2.5.2. This is what is done at the beginning of the proof of Theorem 2.7.13. As a first approximation, it is enough to see $\mathcal{F}_{\delta,r}(\partial P)$ as the set of curves contained in a parabolic limit set ∂P.

Following curves inside parabolic limit sets. Then inside the parabolic limit set ∂P it is possible to follow curves. An analogous of property (P) inside the parabolic limit sets is proved in Proposition 2.5.12. From this property we can obtain Theorem 2.5.13. This theorem is the first major step toward the proof. Essentially it says that the combinatorial modulus of $\mathcal{F}_{\delta,r}(\partial P)$ is controlled by any curve contained in ∂P.

Controlling the modulus in $\partial\Gamma$ by the modulus in the boundary of an apartment. The second major step in the proof is to use the building structure to reduce the problem in $\partial\Gamma$ to a problem in the boundary of an apartment *i.e* in ∂W_D. This is what allows Theorem 2.7.9. Essentially, this theorem says that the modulus of a curve in $\partial\Gamma$ is controlled by a weighted modulus defined in the boundary of an apartment. The idea used to prove this is that, from the point of view of the modulus, $\partial\Gamma$ can be seen as the direct product of the boundary of an apartment by a finite set whose cardinality depends on the scale.

Conclusion of the proof thanks to the symmetries of D. Now, thanks to Theorems 2.5.13 and 2.7.9, we arrive at a point where the modulus of $\mathcal{F}_{\delta,r}(\partial P)$, and thus of $\mathcal{F}_0$, is controlled by some modulus of the parabolic limit sets of W_D. The idea we use to conclude is that the symmetries of D extends to the boundary of W_D. Along with the elements of the groups, these symmetries permit us to follow curves in ∂W_D. As a consequence we obtain a strong control of the modulus of the parabolic limit sets in ∂W_D and we can complete the proof.

2.4 Locally finite right-angled hyperbolic buildings

As we said before, the aim of this chapter is to discuss combinatorial modulus on boundaries of hyperbolic buildings. Here we set up the context about hyperbolic buildings that will be used until the end of this chapter. In particular, we discuss the geometry of locally finite right-angled hyperbolic buildings. This short section prolongs naturally the discussion about right-angled hyperbolic buildings in Section 1.4.

2.4.1 Setting and assumptions

Here we fix $\mathcal{G}$ a finite simplicial graph. We write $\mathcal{G}^{(0)} = \{v_1, \ldots, v_n\}$ and to each vertex v_i we attach a finite cyclic group $\langle s_i \rangle = \mathbb{Z}/q_i\mathbb{Z}$ with $q_i \geq 2$. According to Definition 1.4.1, if $S = \{s_1, \ldots, s_n\}$ the graph product Γ given by $(\mathcal{G}, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\ldots,n})$ admits the following presentation

$$\Gamma = \langle s_i \in S | s_i^{q_i} = 1, s_i s_j = s_j s_i \text{ if } v_i \sim v_j \rangle .$$

Now we assume first that Γ is infinite. This happens if and only if there exists two distinct vertices v_i, v_j such that $v_i \nsim v_j$. Then, we assume that Γ is hyperbolic. A criterion of M. Gromov allows J. Meier to prove that Γ is a hyperbolic group if and only if in $\mathcal{G}$ any circuit of length 4 contains a chord (see [Mei96]). Eventually, we assume that $\partial\Gamma$ is arcwise connected. In [DM02], a necessary and sufficient condition on $\mathcal{G}$ is given for $\partial\Gamma$ to be arcwise connected (see Theorem 2.5.17 in this chapter).

We use the same notations as in Section 1.4 to designate the geometrical objects associated with Γ:

- (W, S) is the Coxeter system,
- Δ is the building of type (W, S),
- Σ is the Davis realization of Δ.

Under our assumptions, Δ equipped with the chamber distance $d_c(\cdot, \cdot)$ is a hyperbolic metric space.

2.4.2 Geodesic metrics on Σ

A natural geodesic metric on Σ is obtained as follow. We designate by D the Davis chamber of Γ. We remind that D is obtained from $D^{(1)}$ by attaching a k-cube inside any full subgraph generated by 2^k vertices that is the 1-skeleton of a k-cube. Now, for any k, we equip each k-cube of D with the Euclidean metric of the $[0,1]^k$.

The polyhedral metric $d(\cdot,\cdot)$ induced on Σ by this construction is geodesic and complete. Moreover, any automorphism of Δ is an isometry of (Σ, d). In particular, Γ acts geometrically on (Σ, d). Thus (Σ, d) is a hyperbolic metric space.

In (Σ, d) the building-walls are convex and connected subsets. More precisely, let $M \in \mathcal{M}(\Sigma)$ of type s and let $x \in \mathrm{Ch}(\Sigma)$ such that M is along x. Then M coincides with the geodesic extensions of all the segments contained in the maximal face of type s of x.

Yet in the case where W is a reflection group of the hyperbolic space $\mathbb{H}^d$ it seems more natural to equip D with the hyperbolic metric. Then D is isometric to the Coxeter polytop provided by W. We designate by $d'(\cdot,\cdot)$ the piecewise hyperbolic metric on Σ induced by this construction. This metric satisfies the same properties stated above (geodesic, complete, hyperbolic and a geometric action of Γ.

The two metrics d and d' are quasi-isometric. As our purpose is to study $\partial\Gamma$, essentially it makes no difference to consider (Σ, d) or (Σ, d'). Yet, the arguments presented in Sections 2.5, 2.6, 2.7, and 2.8 hold in the generic case so we consider (Σ, d) in these sections. For Section 2.9 it will be more convenient to consider (Σ, d').

2.4.3 Boundary of the building

Here we describe basic properties of $\partial\Gamma$. In the sequel, we use the geometric action of Γ on (Σ, d) to identify $\partial\Gamma$ and $\partial\Sigma$. Now we remind that in a hyperbolic group G if $H < G$ there are two possible cases:

- either $\partial H \simeq \partial G$,
- or $\mathrm{Int}(\partial H)$ is empty in ∂G.

Now consider M a building-wall of type s. Under the identification $\partial\Sigma \simeq \partial\Gamma$ we get $\partial M \simeq \partial \mathrm{Stab}_\Gamma(M)$. Therefore, thanks to the previous reminder, to describe ∂M we essentially need to distinguish two cases.

In the first case, s commutes with any generator $r \in S$. In the Davis complex this means that all the other building-walls are orthogonal to M. Then $\mathrm{Stab}_\Gamma(M) = \Gamma$. In this case the boundary of M as well as the boundary of any dial of building bounded by M is $\partial\Gamma$.

In the second case, there exists $r \in S$ that does not commute with s. In the Davis complex this means that there exists a building-wall M' parallel to M. This implies that $\partial M \subsetneq \partial\Gamma$. In this case, $\partial\Gamma \backslash \partial M$ is the disjoint union $\mathrm{Int}(\partial D_0(M)) \sqcup \cdots \sqcup \mathrm{Int}(\partial D_{q-1}(M))$ where

$D_0(M), \ldots, D_{q-1}(M)$ are the dials of building bounded by M. Naturally a rotation around M extends to the boundary as an homeomorphism that permutes $\partial D_0(M), \ldots, \partial D_{q-1}(M)$ and fixes ∂M. Moreover $\mathrm{Int}(\partial M) = \emptyset$, $\mathrm{Int}(\partial D_i(M)) \neq \emptyset$, and the topological boundary of $\partial D_i(M)$ in $\partial\Gamma$ is ∂M for any $i = 1, \ldots, q-1$.

We summarize the last two paragraphs by the following fact.

Fact 2.4.1. *Let D be a dial of building bounded by the building-wall M. Then*

- *either $\partial M = \partial D = \partial\Gamma$,*
- *or the topological boundary of ∂D in $\partial\Gamma$ is ∂M. In this case,* $\mathrm{Int}(\partial D) \neq \emptyset$ *and* $\mathrm{Int}(\partial M) = \emptyset$.

Besides, the hyperbolicity condition gives the following fact describing the asymptotic position of two distinct building-walls.

Fact 2.4.2. *Let M and M' be two distinct building-walls. If $M \parallel M'$ then $\partial M \cap \partial M' = \emptyset$.*

From this fact we can construct many parallel building-walls.

Lemma 2.4.3. *Let $M_1, \ldots, M_k$ be a collection of building-walls such that any M_i admits a parallel building-wall. Assume that $M_i \perp M_j$ for any $i \neq j$. Then there exists $M \in \mathcal{M}(\Sigma)$ such that $M \parallel M_i$ for any i.*

Proof. We prove the proposition by induction on k. For $k = 1$ there is nothing to prove. For $k \geq 1$, pick $M_1, \ldots, M_k$ a collection of building-walls verifying the hypothesis of the lemma. Assume that there exists M a building-wall such that $M \parallel M_1, \ldots, M \parallel M_k$. Let $M_{k+1} \in \mathcal{M}(\Sigma)$ be such that $M_{k+1} \perp M_1, \ldots, M_{k+1} \perp M_k$.

If M is parallel to M_{k+1} there is nothing more to say. Now we assume $M \perp M_{k+1}$ and we pick a wall M' parallel to M_{k+1}. If M' is parallel to $M_1, \ldots, M_k$ there is nothing more to say. Now we assume that there exists $1 \leq i \leq k$ such that, up to a reordering

$$M' \perp M_1, \ldots, M' \perp M_i, M' \parallel M_{i+1}, \ldots, M' \parallel M_k, M' \parallel M_{k+1}.$$

First we consider the case $M' \perp M$. With

- $M \perp M_{k+1}$, $M_{k+1} \perp M_1$, $M_1 \perp M'$,
- and $M' \parallel M_{k+1}$, $M \parallel M_1$,

we obtain that the building-walls M', M, M_{k+1}, and M_1 form a right-angled rectangle. Which is a contradiction with the hyperbolicity of Σ.

Secondly we consider the case $M' \parallel M$. Let $r' \in \Gamma$ be a rotation around M'. Then $r'(M)$ is such that

$$r'(M) \parallel M_1, \ldots, r'(M) \parallel M_i.$$

Indeed for $1 \leq j \leq i$, as $M \parallel M_j$ it comes that $r'(M) \parallel r'(M_j)$. Besides $M' \perp M_j$ and according to Lemma 1.4.12, $r'(M_j) = M_j$. Then $r'(M) \parallel M_j$.

Moreover $r'(M)$ is such that

$$r'(M) \parallel M_{i+1}, \ldots, r'(M) \parallel M_{k+1}.$$

Indeed $M_{i+1} \cap \cdots \cap M_{k+1} \neq \emptyset$. As $M' \parallel M_j$ for $i+1 \leq j \leq k+1$, this means that the building-walls $M_{i+1}, \ldots, M_{k+1}$ are entirely contained in the same connected component of $\Sigma \backslash M'$. Let C be this connected component. As $M \parallel M'$ and $M \cap M_{k+1} \neq \emptyset$ it comes that M is also contained in C. Thus $r'(M)$ is not contained in C and $r'(M) \parallel M_j$ for $i+1 \leq j \leq k+1$. □

In [BK13, Proposition 5.2.], it is proved that the boundary of half-spaces in a hyperbolic Coxeter group generates that visual topology. In the case of right-angled building the analogous statement holds.

Fact 2.4.4. *The topology generated by $\{\partial D : D \in \mathcal{D}(\Sigma)\}$ is equivalent to the topology induced by a visual metric on $\partial\Gamma$.*

Eventually, consider an apartment A containing the base chamber x_0 and the retraction map $\pi_{A,x_0} : \Sigma \longrightarrow A$. This retraction maps any geodesic ray of Σ starting from a based point $p_0 \in x_0$ to a geodesic ray in A starting from p_0. Hence π_{A,x_0} extends naturally to the boundaries and we keep the notation $\pi_{A,x_0} : \partial\Sigma \longrightarrow \partial A$ for this extension.

Remark 2.4.5. *In [DM02], M.W. Davis and J. Meier described how properties of connectedness of $\partial\Gamma$ are encoded in the combinatorial structure of $\mathcal{G}$. We use a corollary of their result in Subsection 2.5.2.*

2.5 Curves in connected parabolic limit sets

As we will see with Example 2.5.9, parabolic limit sets (*i.e* boundaries of residues) play a key role in the proof of the CLP in boundaries of graph product.

In this section, we use the ideas of [BK13, Section 5 and 6] to prove Theorem 2.5.13 that is a first major step to prove the main result of this chapter (Theorem 2.9.1). The idea of this theorem is to control the modulus of the curves of a parabolic limit set by the neighborhood of a single curve. Then we apply this theorem to recover a result about boundaries of right-angled Fuchsian buildings.

We use the notations and the setting of Section 2.4. In particular Γ is a fixed graph product given by the pair $(\mathcal{G}, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\ldots,n})$. We identify the building Δ with its Davis complex Σ equipped with the piecewise Euclidean metric. The base chamber is x_0. We assume that Γ and Σ are hyperbolic and that $\partial\Gamma$ is connected. The metric on $\mathrm{Ch}(\Sigma)$ is denoted $d_c(\cdot,\cdot)$. Moreover, in this section we equip $\partial\Gamma$ with a self-similar metric that comes from the action of Γ on Σ.

2.5.1 Parabolic limit sets in $\partial\Gamma$

Here we give basic properties of boundaries of parabolic subgroups. At the end of this subsection we will see that these subsets of the boundary are an issue to prove the CLP.

Definition 2.5.1. *Let $P = g\Gamma_I g^{-1}$ be a parabolic subgroup of Γ. If the limit set of P in $\partial\Gamma$ is non-empty, we call it a* parabolic limit set*. If moreover $\partial P \neq \partial\Gamma$ the parabolic limit set is called a* proper *parabolic limit set.*

Equivalently we could have said that a subset $F \subset \partial\Gamma$ is a parabolic limit set if there exists a residue $g\Sigma_I$ such that F is equal to $\partial(g\Sigma_I)$ under the canonical homeomorphism between $\partial\Gamma$ and $\partial\Sigma$. In the following we will frequently use this point of view about parabolic limit sets.

The following convex hull of a subset of the boundary will be used in the sequel of this section.

Definition 2.5.2. *Let F be a subset of $\partial\Gamma$ containing more than one point and such that $\overline{F} \neq \partial\Gamma$. Let*

$$\mathcal{D}^c(F) = \{D \in \mathcal{D}(\Sigma) : F \subset \partial\Gamma \backslash \partial D\}.$$

Then we call convex hull of F *in* Σ

$$\mathrm{Conv}(F) = \Sigma \backslash \cup_{D \in \mathcal{D}^c(F)} D.$$

If $\overline{F} = \partial\Gamma$ then we set $\mathrm{Conv}(F) = \Sigma$.

We extend the notion of minimal gallery to infinite galleries.

Definition 2.5.3. *An infinite gallery $x_0 \sim x_1 \sim \cdots$ (resp. a bi-infinite gallery $\cdots \sim x_{-1} \sim x_0 \sim x_1 \sim \cdots$) is* minimal *if for any $k \in \mathbb{N}$ (resp. $k \in \mathbb{Z}$) and $\ell \in \mathbb{N}$ the gallery $x_k \sim \cdots \sim x_{k+\ell}$ is minimal.*

Naturally, $\mathrm{Ch}(\Sigma\backslash D)$ is a convex set of chambers for any $D \in \mathcal{D}(\Sigma)$ in the sense of Definition 1.1.3. Thus if F contains at least two points, $\mathrm{Ch}(\mathrm{Conv}(F))$ is a convex set of chambers. We can see $\mathrm{Conv}(F)$ as the maximal convex whose boundary is inside $\overline{F}$. Indeed, let C be the union of a convex set of chambers with $\partial C \subset \overline{F}$, then $C \subset \mathrm{Conv}(F)$.

In particular, $\mathrm{Conv}(F)$ contains any bi-infinite minimal gallery whose end points are in F. Besides note that $\partial\mathrm{Conv}(F) = \overline{F}$.

Example 2.5.4. *Let ∂P be a parabolic limit set and assume that $P = \Gamma_I$. Then we can verify that $\mathrm{Conv}(\partial P) = \Sigma_J$ where $J = \{s_j \in S : s_j s_i = s_i s_j$ for any $s_i \in I\}$. In particular, if $M \in \mathcal{M}(\Sigma)$ then $\mathrm{Conv}(\partial M)$ is the union of all the chambers along M.*

In the following definition $\overline{\Sigma} = \Sigma \cup \partial\Sigma$ and if M is a building-wall $\overline{M} = M \cup \partial M$.

Definition 2.5.5.

i) Let F be a subset of $\partial\Sigma$. We say that a building-wall M cuts *F if there exist two distinct indices i and j such that F meets both $\partial D_i(M)$ and $\partial D_j(M)$.*

ii) If $E_1 \subset \partial\Gamma$ and $E_2 \subset \Sigma$ (resp. $E_2 \subset \partial\Gamma$) we say that the building-wall separates E_1 and E_2 *if they are entirely contained in two distinct connected components of $\overline{\Sigma}\backslash\overline{M}$.*

The proof of the following fact is identical to the proof of [BK13, Lemma 5.7].

Fact 2.5.6. *Let F be a subset of $\partial\Sigma$. The building-wall M cuts F if and only if M crosses* $\mathrm{Conv}(F)$ *(see Definition 1.4.10).*

Along with Example 2.5.4, this fact implies that the characterization of the residues thanks to rotations (Theorem 1.4.17) extends to the boundary into a characterization of the parabolic limit sets.

Corollary 2.5.7. *Let F be a subset of $\partial\Sigma$ containing at least two distinct points and P_F denotes the group generated by the rotations around the building-walls that cut F. If P_F stabilizes F, then F is a parabolic limit set.*

This characterization gives the following corollary concerning the connectedness of the parabolic limit sets.

Corollary 2.5.8. *Let ∂P be a parabolic limit set. Then any connected component F of ∂P in $\partial\Gamma$ is a parabolic limit set.*

Proof. Let M be a building-wall that cuts F. As M cuts ∂P a rotation $r \in \Gamma$ around M stabilize ∂P so in particular it sends a connected component of ∂P on another connected component of ∂P. With $r(M \cap F) = M \cap F$ we deduce that $r(F) = F$ and so F is a parabolic limit set.

□

Eventually the next example is crucial in the sense that it illustrates the issue of the parabolic limit sets to prove the CLP.

Example 2.5.9. *Let $M \in \mathcal{M}(\Sigma)$ be a building-wall of type s along the base chamber x_0. Let $P = \mathrm{Stab}_\Gamma(M)$. The group P is the parabolic subgroup that is generated by the generators $r \in S\backslash\{s\}$ such that $rs = sr$. Moreover, as we reminded in Subsection 2.4.3, $\partial P \simeq \partial M$. Now we assume that ∂P is a proper parabolic limit set and we pick $g \in \Gamma$. Then we verify that*

- *either $g\partial P = \partial P$,*
- *or $\partial P \cap g\partial P = \emptyset$.*

Indeed if $M' \perp M$ then M and M' are of distinct types. As M and gM are of the same types it comes that $M \cap gM \neq \emptyset$ implies $M = gM$ and $g\partial P = \partial P$. Then thanks to Fact 2.4.2, if $M \cap gM = \emptyset$ we obtain $\partial P \cap g\partial P = \emptyset$.

Eventually the set $\cup_{g\in\Gamma} g\partial M$ is made of countably many disjoint copies of ∂M. In the introduction we reminded that an efficient way to prove the CLP is to follow curves thanks to bi-Lipschitz maps (see Section 2.3). As Γ acts by bi-Lipschitz homeomorphisms on its boundary, the first idea is to use Γ to follow curves. Yet if a non-constant curve η is contained in ∂M we cannot hope to follow the curves of $\partial\Gamma$ thanks to η and Γ.

Figure 2.3: Example 2.5.9 on the boundary of a thin building

2.5.2 Modulus of curves in connected parabolic limit set

Here we apply the ideas of [BK13, Section 5 and 6] to Γ. In this subsection, as in Subsection 2.2.2, d_0 denotes a small constant compared with diam $\partial\Gamma$ and with the constant of approximate self-similarity. Then $\mathcal{F}_0$ is the set of curves of diameter larger than d_0. Here we prove that, from the point of view of the modulus, curves in a parabolic limit set are all the same (see consequences of Theorem 2.5.13).

Until the end of this chapter, we use the following notations:

Notation. *Let ∂P be a connected parabolic limit set in $\partial\Gamma$. For $\delta, r > 0$, let $\mathcal{F}_{\delta,r}(\partial P)$ denote the set of curves in $\partial\Gamma$ such that:*

- $\operatorname{diam}\gamma \geq d_0$,
- $\gamma \subset N_\delta(\partial P)$,
- $\gamma \not\subseteq N_r(\partial Q)$ *for any connected parabolic limit set* $\partial Q \subsetneq \partial P$.

As we saw in Example 2.5.9, the curves contained in a parabolic limit set ∂P are an issue to follow other curves. Nevertheless, here we prove that these curves can be used to follow the curves in ∂P (Proposition 2.5.12). Then we deduce from this property a control of the modulus of the curves in parabolic limit sets (Theorem 2.5.13). To this purpose we use the following notion.

Definition 2.5.10. *Let $L \geq 0$ and I a non-empty subset of S. A curve γ in $\partial\Gamma$ is called a (L, I)-*curve *if*

- $x_0 \subset \mathrm{Conv}(\gamma)$,
- *for all $s \in I$, there exists a panel σ_s of type s inside* $\mathrm{Conv}(\gamma)$ *with* $\mathrm{dist}(x_0, \sigma_s) \leq L$.

As we see, curves in parabolic limit sets are (L, I)-curves.

Proposition 2.5.11. *Let $I \subset S$ and $P = h\Gamma_I h^{-1}$. Then for all $r > 0$, there exist $L > 0$ and $\delta > 0$ such that if $x_0 \subset \mathrm{Conv}(\gamma)$ and $\gamma \in \mathcal{F}_{\delta,r}(\partial P)$, then γ is a (L, I)-curve.*

Proof. We fix $r > 0$ and we assume that for every integer $n \geq 1$, there exists a curve γ_n such that:

- $x_0 \subset \mathrm{Conv}(\gamma_n)$,
- $\gamma_n \in \mathcal{F}_{1/n,r}(\partial P)$,
- γ is not a (n, I)-curve.

For $n \geq 1$, we designate the ball of center x_0 and of radius n for the distance over the chambers by

$$B_c(x_0, n) = \{x \in \mathrm{Ch}(\Sigma) : d_c(x_0, x) \leq n\}.$$

For simplicity we also designate by $B_c(x_0, n)$ the union of its chambers. Up to a subsequence we can suppose that for a fixed $s \in I$, there is no panel of type s in $B_c(x_0, n) \cap \mathrm{Conv}(\gamma_n)$ for $n \geq 1$.

We want to reveal a contradiction thanks to the sequences γ_n and $\mathrm{Conv}(\gamma_n)$. According to [Mun75, p. 281] the set of non-degenerate continua in a compact space is a compact set for the Hausdorff distance. Therefore, up to a subsequence, we can suppose that γ_n tends to a non-degenerate continuum $\mathcal{L} \subset \partial P$.

As $x_0 \subset \mathrm{Conv}(\gamma_n)$, using a diagonal argument we can also assume that, up to a subsequence, $\mathrm{Conv}(\gamma_{n+k}) \cap B_c(x_0, n)$ is non-empty and constant for $k \geq 0$. Then we denote $C := \bigcup_{n \geq 1} \mathrm{Conv}(\gamma_n) \cap B_c(x_0, n)$.

With $\gamma_n \subset \partial\mathrm{Conv}(\gamma_n)$, it comes that $\mathcal{L} \subset \partial C$. As C does not contain any panel of type s, $\mathcal{L}$ is contained in the limit set of a parabolic subgroup of the form $g\Gamma_J g^{-1}$ with $s \notin J$. Then, by Theorem 1.5.3, intersections of parabolic subgroups are parabolic subgroups. Therefore $\mathcal{L}$ is contained in the limit set of a proper parabolic subgroup of P. Let ∂Q be the connected component of this parabolic limit set that contains $\mathcal{L}$. Thanks to Corollary 2.5.8, ∂Q is a parabolic limit set. Now we see that for $n \geq 0$ large enough $\gamma_n \subset N_r(\partial Q)$ which is a contradiction with $\gamma_n \in \mathcal{F}_{1/n,r}(\partial P)$.

□

An interesting feature of (L, I)-curves is that these curves are crossed by building-walls of type in I. Which means that from a (L, I)-curve, we can follow curves using rotations around building-walls of type in I.

Proposition 2.5.12. *Let $\epsilon > 0$, $L > 0$ and I be a non-empty subset of S. For $P = h\Gamma_I h^{-1}$, let η denote a curve contained in ∂P. Then there exists a finite subset $F \subset \Gamma$ such that for any (L, I)-curve γ the set $\bigcup_{g \in F} g\gamma$ contains a curve that belongs to $\mathcal{U}_\epsilon(\eta)$.*

Proof. We divide the proof in four steps. In this proof M_s denotes the building-wall of type $s \in S$ along x_0.

i) First, we can suppose without loss of generality that $P = \Gamma_I$. Indeed, as $h \in \Gamma$ is a bi-Lipschitz homeomorphism of $(\partial\Gamma, d)$, then if the property holds for Γ_I it holds for $h\Gamma_I h^{-1}$.

ii) Now we prove that the following property holds.

The set $\bigcup_{g \in F_L} g\gamma$ contains a curve passing through ∂M_s for every $s \in I$.

Where $F_L = \{g \in \Gamma : |g| \leq L\}$ and $|g| = d_c(x_0, gx_0)$.

As γ is a (L, I)-curve, there exist $s \in I$, $\alpha \in \mathbb{Z}$ and $g \in F_L$ such that gx_0 and $gs^\alpha x_0$ belongs to $\text{Conv}(\gamma)$. Let $x_0 \sim g_1 x_0 \sim \cdots \sim g_\ell x_0$ be a gallery contained in $\text{Conv}(\gamma)$ with $g_{\ell-1} = gs^\alpha$ and $g_\ell = g$. For any $i = 0, \ldots, \ell-1$, let M_i denote the building-wall separating $g_i x_0$ and $g_{i+1} x_0$. In particular, ∂M_i cuts γ for any $i = 0, \ldots, \ell-1$. This means that if M_i is of type s_i, then $\gamma \cap g_i s_i^{\alpha_i} g_i^{-1}\gamma \neq \emptyset$ for any $\alpha_i \in \mathbb{Z}$. In particular, if α_i is such that $g_{i+1} = g_i s_i^{-\alpha_i}$ then $\gamma \cap g_i g_{i+1}^{-1}\gamma \neq \emptyset$ for any $i = 0, \ldots, \ell-1$. Hence the set $\gamma \cup g_1^{-1}\gamma \cup \cdots \cup g_\ell^{-1}\gamma$ is arcwise connected and $g_\ell^{-1}\gamma$ intersects ∂M_s. Thus the property is verified.

iii) Let $\Sigma_I \subset \Sigma$ be the residue associated with Γ_I. We remind that this means $\Sigma_I = \Gamma_I x_0$. Let $D_1, \ldots, D_k$ be a collection of dials of building bounded by the building-walls $M_1, \ldots, M_k$. Then we assume that each D_i intersects Σ_I properly (*i.e.* $\Sigma_I \cap D_i \neq \emptyset$ and $\Sigma_I \cap D_i \neq \Sigma_I$). In particular, this means that the building-walls $M_1, \ldots, M_k$ have their types contained in I.

Now we prove that the following property holds.

There exists a finite subset $F_0 \subset \Gamma$ such that for every (L, I)-curve γ the set $\bigcup_{g \in F_0} g\gamma$ contains a curve passing through $\partial D_1, \ldots, \partial D_k$.

For $i = 1, \ldots, k$ pick $h_i \in \Gamma_I$ such that M_i is along $h_i x_0 \in \Sigma_I$. In particular, for any i, we can write $M_i = h_i(M_s)$ where $s \in I$ is the type of M_i. Let $g_1 x_0 = h_1 x_0 \sim g_2 x_0 \sim \cdots \sim g_\ell x_0 = h_k x_0$ be a gallery in Σ_I passing through $h_1 x_0, \ldots, h_k x_0$ in this order.

Applying the second step of the proof, there exists a curve θ in $\bigcup_{g \in F_L} g\gamma$ such that θ crosses every ∂M_s for $s \in I$. Therefore the set $\bigcup_{i=1,\ldots,\ell} g_i\theta$ meets any $g_i(M_s)$ for any $i = 1, \ldots, k$ and any $s \in I$. In particular, it meets any $h_i(M_s)$ and intersects every $\partial D_1, \ldots, \partial D_k$.

We set $F_0 = \{g_i g \in \Gamma : |g| \leq L, 1 \leq i \leq \ell\}$, and it is now enough to check that $\bigcup_{g \in F_0} g\gamma$ is arcwise connected. For any $i = 1, \ldots, \ell-1$ let $s_i \in I$ and $\alpha_i \in \mathbb{Z}$ be such that

$g_{i+1} = g_i s_i^{\alpha_i}$. Then $g_{i+1}\theta = (g_i s_i^{\alpha_i} g_i^{-1}) g_i \theta$. As θ intersects any ∂M_{s_i} then $g_i\theta \cap g_i(\partial M_{s_i}) \neq \emptyset$ and this intersection is fixed by $g_i s_i^{\alpha_i} g_i^{-1}$. Thus $g_i\theta \cap g_{i+1}\theta \neq \emptyset$.

iv) With Fact 2.4.4, we can choose, $D'_1, \dots, D'_{k+1}$ a collection of dials of building such that the union of their boundaries is a neighborhood of η contained in the $\epsilon/2$ neighborhood of η. We also assume that η enters in the boundaries of the $D'_1, \dots, D'_{k+1}$ in this order. For any $i = 1, \dots, k+1$, let r_i denote the rotation around the building-wall associated with D'_i. Let $D_1, \dots, D_k$ be a collection of dials of building such that $\partial D_i \subset \partial D'_i \cap \partial D'_{i+1}$. Applying the previous step of the proof, there exists a finite set $F_0 \subset \Gamma$ such that for every (L, I)-curve γ the set $\bigcup_{g \in F_0}$ contains a curve passing through each $\partial D_1, \dots, \partial D_k$.

If for some $i = 1, \dots, k+1$ the curve η leaves $\partial D'_i$ then $\theta \bigcup_{\alpha \in \mathbb{Z}} r_i^{\alpha}\theta$ contains a curve that does not leave $\partial D'_i$. Eventually we set $F = \{r_i^{\alpha} g : \alpha \in \mathbb{Z} \text{ and } g \in F_0\}$ and F satisfies the desired property. □

We use the two previous propositions to control $\mathrm{Mod}_p(\mathcal{F}_{\delta,r}(\partial P))$.

Theorem 2.5.13. *There exists an increasing function $\delta_0 : (0, +\infty) \longrightarrow (0, +\infty)$ verifying the following property. Let $p \geq 1$, let $\eta \in \mathcal{F}_0$, and let ∂P be the smallest parabolic limit set containing η. Let $r > 0$ be small enough so that $\eta \nsubseteq \overline{N_r}(\partial Q)$ for any connected parabolic limit set $\partial Q \subsetneq \partial P$. Let $\delta < \delta_0(r)$ and let $\epsilon > 0$ be small enough so that $\mathcal{U}_\epsilon(\eta) \subset \mathcal{F}_{\delta,r}(\partial P)$. Then there exists a constant $C = C(d_0, p, \eta, r, \epsilon)$ such that*

$$\mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k) \leq \mathrm{Mod}_p(\mathcal{F}_{\delta,r}(\partial P), G_k) \leq C \cdot \mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

Furthermore when p belongs to a compact subset of $[1, +\infty)$ the constant C may be chosen independent of p.

Before the proof, we need to explain the statement of the theorem. Indeed, it is not clear that for $\epsilon > 0$ small enough $\mathcal{U}_\epsilon(\eta) \subset \mathcal{F}_{\delta,r}(\partial P)$. In particular, if $\mathrm{diam}\,\eta = d_0$ then $\mathcal{U}_\epsilon(\eta)$ may not be made of curves of diameter larger than d_0. Nevertheless thanks to Proposition 2.2.11, we can rescale $\mathcal{U}_\epsilon(\eta)$ to a set of larger curves with a control of the modulus. Hence we can say that, up to rescaling, for $\epsilon > 0$ small enough, $\mathcal{U}_\epsilon(\eta) \subset \mathcal{F}_{\delta,r}(\partial P)$. This gives sense to the statement of the theorem and the left-hand side inequality is now trivial by Proposition 2.1.3 (1).

Proof of 2.5.13. Let $P = h\Gamma_I h^{-1}$, let η be a curve in ∂P, $r > 0$ and $\epsilon > 0$ as in the hypothesis of the theorem.

To prove the right-hand side inequality, thanks to Proposition 2.2.11, we can assume without loss of generality that if $\gamma \in \mathcal{F}_0$ then $x_0 \subset \mathrm{Conv}(\gamma)$. Indeed, there exists an upper bound N depending on d_0 such that if $\gamma \in \mathcal{F}_0$ then $\mathrm{dist}(x_0, \mathrm{Conv}(\gamma)) \leq N$. So there exists only a finite set E of elements of Γ such that if $g \in E$, there exists $\gamma \in \mathcal{F}_0$ with $\mathrm{dist}(x_0, \mathrm{Conv}(\gamma)) = d_c(x_0, gx_0)$.

With this assumption we can apply Proposition 2.5.11 and set $L > 0$ and $\delta > 0$ such that the curves of $\mathcal{F}_{\delta,r}(\partial P)$ are (L, I)-curves. Let $F \subset \Gamma$ be the finite set given by Proposition 2.5.12 and let $\rho : G_k \longrightarrow \mathbb{R}^+$ be a $\mathcal{U}_\epsilon(\eta)$-admissible function. We define $\rho' : G_k \longrightarrow \mathbb{R}^+$ by:

$$(*) \qquad \rho'(v) = \sum_{g \in F} \sum_{w \cap gv \neq \emptyset} \rho(w).$$

Let $\gamma \in \mathcal{F}_{\delta,r}(\partial P)$ and $\theta \subset \bigcup_{g \in F} g\gamma$ such that $\theta \in \mathcal{U}_\epsilon(\eta)$. Then

$$L_{\rho'}(\gamma) = \sum_{g \in F} \sum_{v \cap \gamma \neq \emptyset} \sum_{w \cap gv \neq \emptyset} \rho(w) \geq \sum_{g \in F} \sum_{w \cap g\gamma \neq \emptyset} \rho(w).$$

Yet

$$L_\rho(\theta) \leq \sum_{g \in F} L_\rho(g\gamma) = \sum_{g \in F} \sum_{w \cap g\gamma \neq \emptyset} \rho(w).$$

Thus $L_{\rho'}(\gamma) \geq L_\rho(\theta)$ and ρ' is $\mathcal{F}_{\delta,r}(\partial P)$-admissible.

Then the number or terms in the right-hand side of the definition $(*)$ is bounded by a constant N depending on $\#F$, the bi-Lipschitz constants of the elements of F, and the doubling constant of $\partial\Gamma$. Therefore by convexity

$$\begin{aligned} M_p(\rho') = & \sum_{v \in G_k} \Big(\sum_{g \in F} \sum_{w \cap gv \neq \emptyset} \rho(w) \Big)^p, \\ & \leq N^{p-1} \cdot \sum_{v \in G_k} \sum_{g \in F} \sum_{w \cap gv \neq \emptyset} \rho(w)^p \leq N^p \cdot \#F \cdot \sum_{w \in G_k} \rho(w)^p. \end{aligned}$$

Which proves the inequality.

□

As a straightforward application, we notice that under the assumptions of the theorem, $\mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k)$ does not depend, up to multiplicative constants, on the choice of η and ϵ. Indeed, for $r > 0$ fixed and $\delta < \delta_0(r)$, if $\eta, \eta' \subset \partial P$ and $\epsilon, \epsilon' > 0$ such that the hypothesis of the theorem are satisfied. Then there exist $C = C(\eta, r, \epsilon)$ and $C' = C'(\eta', r, \epsilon')$ such that

$$C^{-1} \cdot \mathrm{Mod}_p(\mathcal{U}_{\epsilon'}(\eta'), G_k) \leq \mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k) \leq C' \cdot \mathrm{Mod}_p(\mathcal{U}_{\epsilon'}(\eta'), G_k).$$

Of course, if $\eta = \eta'$ and $\epsilon' < \epsilon$ we can choose $C = 1$.

Another consequence of the theorem is that if the boundary of a graph product does not contain connected parabolic limit sets, then it satisfies the CLP.

Theorem 2.5.14. *Let Γ be a thick hyperbolic graph product such that $\partial\Gamma$ is connected and any proper parabolic limit set is disconnected. Then $\partial\Gamma$ equipped with a visual metric satisfies the CLP.*

Proof. We check the hypothesis of Proposition 2.2.12. To prove that $\mathrm{Mod}_1(\mathcal{F}_0, G_k)$ is unbounded, it is enough to check that for every $N \in \mathbb{N}$ there exist N disjoint curves of diameter larger than d_0 in $\partial\Gamma$. Indeed, this implies that for $k \geq 0$ large enough

$\mathrm{Mod}_1(\mathcal{F}_0, G_k) > N$. Now, as we assume that the building associated with Γ is thick, for every $N \in \mathbb{N}$ there exist N apartments with disjoint boundaries that intersects in a compact domain inside the building. This assure the existence of N disjoint curves of diameter larger than d_0.

Let η be a non-constant curve in $\partial\Gamma$. Up to a change of scale, by Proposition 2.2.11, we can assume $\eta \in \mathcal{F}_0$. Then as $\partial\Gamma$ is the only parabolic limit set containing η, it is enough to apply Theorem 2.5.13 to verify the second hypothesis of Proposition 2.2.12.

□

In particular, we can apply this result to the case of right-angled Fuchsian buildings. In the following, we call *right-angled Fuchsian building* a building associated with a graph product $(C_n, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\dots,n})$ where C_n is the cyclic graph with $n \geq 5$ vertices and $q_1, \dots, q_n$ is a family of integers larger or equal than 3.

Corollary 2.5.15. *For $n \geq 5$, let C_n be the cyclic graph with n vertices and let $q_1, \dots, q_n$ be a family of integers larger or equal than 3. Let Γ be the graph product given by the pair $(C_n, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\dots,n})$. Then $\partial\Gamma$ equipped with a visual metric satisfies the CLP.*

This result was known since boundaries of right-angled Fuchsian buildings are Loewner spaces (see [BP00, Proposition 2.3.4.]). Yet, here we give a direct proof of this result.

Besides, we can prove that these thick graph products are the only one to satisfy the hypothesis of Theorem 2.5.14. To verify this we need to introduce the following simplicial complex.

Definition 2.5.16. *Let $\Gamma = (\mathcal{G}, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\dots,n})$ be a graph product, the* nerve of Γ *is the simplicial complex $L = L(\mathcal{G})$ such that:*

- *the 1-skeleton of L is $\mathcal{G}$,*
- *k vertices of $\mathcal{G}$ span a $(k-1)$-simplex in L a corresponding parabolic subgroup in Γ is finite.*

The following theorem is a special case of [DM02, Corollary 5.14.].

Theorem 2.5.17. *The boundary of Γ is connected if and only if $L \backslash \sigma$ is connected for any simplex $\sigma \subset L$.*

Now we can prove the following.

Proposition 2.5.18. *Let $\Gamma = (\mathcal{G}, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\dots,n})$ be a hyperbolic graph product. Assume that $\partial\Gamma$ is connected and that any proper parabolic limit set ∂P is disconnected, then the building associated with Γ is a right-angled Fuchsian building.*

Proof. Here we only need to prove that $\mathcal{G}$ contains a circuit of length $n \geq 5$. According to Corollary 2.5.8, if any proper parabolic limit set in $\partial\Gamma$ is disconnected then any proper parabolic limit set in $\partial\Gamma$ is discrete. Moreover, $\partial\Gamma$ contains at least one proper parabolic limit set of the form $\partial\Gamma_I$ with $\#I = n-1$ otherwise $\partial\Gamma = \emptyset$. The subgroup Γ_I is a graph product associated with the graph $\mathcal{G}_I$. This graph is obtained from $\mathcal{G}$ to which we remove a vertex p and all the edges adjacent to p. Then if L_I is the nerve associated with Γ_I, we get L_I from L to which we remove the interior of any simplex containing p.

Now, thanks to Theorem 2.5.17, we know that there exists a simplex $\sigma \subset L_I$ such that $L_I \backslash \sigma$ is disconnected. Let C_1 and C_2 be two connected components of $L_I \backslash \sigma$. Up to a subsimplex, we can assume that any vertex of σ is connected to C_1 or to C_2 by an edge. Yet, if we consider the simplex σ in L, it comes that $L \backslash \sigma$ is connected because $\partial\Gamma$ is connected. Therefore there exist two edges in L attaching p respectively to C_1 and to C_2.

We set $V = \{v_1, \ldots, v_k\}$ the vertices of σ that are not connected to p by an edge and $V' = \{v'_1, \ldots, v'_{k'}\}$ the rest of the vertices of σ. At this point, we assume by contradiction, that $\mathcal{G}$ contains no circuit of length $n \geq 4$. We can check that under this assumption the following situations does not occur

i) V' is empty,

ii) there exists $v \in V$ such that v is adjacent to both C_1 and C_2,

iii) there exist $v, w \in V$ such that v is adjacent to C_1 and w is adjacent to C_2.

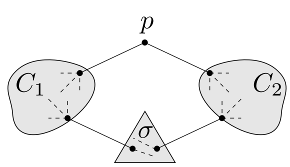


Figure 2.4: Forbidden situation *i*)

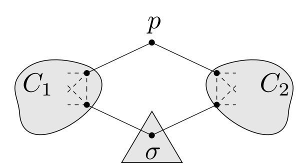


Figure 2.5: Forbidden situation *ii*)

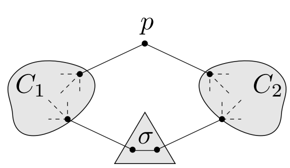


Figure 2.6: Forbidden situation *iii*)

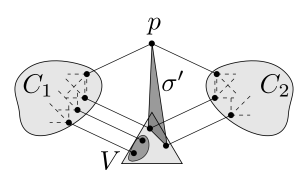


Figure 2.7: Resulting situation

Hence, for instance, the vertices in V are all adjacent to C_1. As a consequence, if σ' designates the simplex in L spanned by $V' \cup \{p\}$ then $L \backslash \sigma'$ is not connected. Which is not possible because $\partial\Gamma$ is connected.

Therefore $\mathcal{G}$ contains a circuit of length $n \geq 4$, but as Γ is hyperbolic it contains no circuit of length 4. This conclude the proof.

□

2.6 Combinatorial metric on boundaries of right-angled hyperbolic buildings

As we saw in Chapter 1, the geometric structure of a right-angled building is contained in the wall structure. Here we describe how, in the hyperbolic case, the geometry of the boundary is determined by the boundaries of the walls.

In this section, we first start by discussing boundaries of intersections of dials of building. Afterwards, we describe a combinatorial and self-similar metric on $\partial\Gamma$ in terms of dials of building. Then, we build an approximation of $\partial\Gamma$ that will be convenient to use in Section 2.7.

Here we use the notations and the setting of Section 2.4 and 2.5. In particular, Γ is a fixed graph product given by $(\mathcal{G}, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\dots,n})$ and acting on the building Σ. The base chamber is x_0, and W is the right-angled Coxeter group associated with Γ. We assume that Γ is hyperbolic and $\partial\Gamma$ is connected.

2.6.1 Shadows on $\partial\Gamma$

The following tools are used in the sequel of this chapter to describe the topology and the metric on $\partial\Gamma$. We remind that the boundary of Γ is canonically identified with the boundary of Σ.

Definition 2.6.1. *Let $x \in \mathrm{Ch}(\Sigma)$. We call* cone of chambers of base x *and we denote $C_x \subset \Sigma$, the union of the set of chambers $y \in \mathrm{Ch}(\Sigma)$ such that there exists a minimal gallery from x_0 to y passing through x.*

Thanks to projection maps, we characterize the cones in terms of dials of building.

Proposition 2.6.2. *Let $D_1, \dots, D_k \in \mathcal{D}(\Sigma)$ and $C = D_1 \cap \dots \cap D_k$. Assume that C contains a chamber and that $x_0 \nsubseteq D_i$ for $i = 1, \dots, k$. If we set $x = \mathrm{proj}_C(x_0)$ then $C_x = C$.*

Proof. According to Definition 2.6.1 and to Proposition 1.4.15, $C \subset C_x$. Now let $y \in \mathrm{Ch}(C_x)$ and for $i = 1, \dots, k$ let M_i be the building-wall that bounds D_i. If $x_0 \sim \dots \sim x \sim \dots \sim y$ is a minimal gallery, then the subgallery $x \sim \dots \sim y$ does not cross any building-wall M_i and $y \subset C$. □

Reciprocally cones are intersections of dials of building.

Proposition 2.6.3. *Let $x \in \mathrm{Ch}(\Sigma)$ and let $D_1, \dots, D_k$ denote the family of dials of building such that for any $i = 1, \dots, k$*

$$x_0 \nsubseteq D_i \text{ and } x \subset D_i.$$

Then $C_x = D_1 \cap \dots \cap D_k$.

Proof. Let $C = D_1 \cap \cdots \cap D_k$. According to Proposition 2.6.2, it is enough to prove that $\mathrm{proj}_C(x_0) = x$. If we write $x' = \mathrm{proj}_C(x_0)$, with Proposition 2.6.2, $C = C_{x'}$. Hence there exists a minimal gallery

$$x_0 \sim \cdots \sim x' \sim \cdots \sim x.$$

Now assume that $x' \neq x$, this means that there exists a building-wall M that separates x and x'. As the gallery is minimal, the dial of building D bounded by M that contains x does not contain x' and x_0. Thus D is one of the $D_1, \ldots, D_k$ and $x' \nsubseteq C$ which is a contradiction. □

In particular, it makes sense to consider projection maps over cones of chambers with $\mathrm{proj}_{C_x}(x_0) = x$. Thanks to the previous proposition, we obtain the following fact that describes how dials of building intersect.

Fact 2.6.4. *Let $D_1, \ldots, D_k$ be a family of distinct dials of building such that $D_i \nsubseteq D_j$ for any $i \neq j$. Assume that $x_0 \nsubseteq D_i$ for any i. For any i, set M_i the building-wall that bounds D_i and let $C = D_1 \cap \cdots \cap D_k$. Then exactly one of these assertion holds.*

- *There exists i, j such that $M_i \parallel M_j$ and $C = \emptyset$.*
- *For any i, j, $M_i \cap M_j \neq \emptyset$ and there exists $i \neq j$ such that $M_i = M_j$. In this case C is contained in M_i.*
- *$M_i \perp M_j$ for any $i \neq j$. In this case C is a cone.*

This fact, up to a translation and up to a subfamily, describes how a finite family of dials intersects. The following lemma specifies the case when the intersection is a cone.

Lemma 2.6.5. *Let $D_1, \ldots, D_k$ be a family of distinct dials of building bounded by the building-walls $M_1, \ldots, M_k$. Let $C = D_1 \cap \cdots \cap D_k$. Assume that $x_0 \nsubseteq D_i$ for any i and that for any $i \neq j$ $M_i \perp M_j$. Then any M_i is along* $\mathrm{proj}_C(x_0)$.

Proof. First we remark that if $k = 1$ the property is trivial. According to Lemma 1.4.14, $\mathrm{proj}_{D_1}(x_0) \notin \mathrm{Ch}(D_2) \cup \cdots \cup \mathrm{Ch}(D_k)$. Applying this lemma $k-2$ times we obtain that

$$\mathrm{proj}_{D_{k-1}} \circ \cdots \circ \mathrm{proj}_{D_1}(x_0) \notin \mathrm{Ch}(D_k).$$

Hence $\mathrm{proj}_C(x_0) = \mathrm{proj}_{D_k}(\mathrm{proj}_{D_{k-1}}(\circ) \cdots \circ \mathrm{proj}_{D_1}(x_0))$ is along M_k. Changing the order of the family of building-walls, the same argument applies to $M_1, \ldots, M_{k-1}$ and the prove is finished. □

Eventually we obtain the following characterization of cones.

Proposition 2.6.6. *Let $x \in \mathrm{Ch}(\Sigma)$ and let $D_1, \ldots, D_k$ be the family of dials of building bounded by $M_1, \ldots, M_k$ such that for any $i = 1, \ldots, k$*

$$x_0 \nsubseteq D_i, \ x \subset D_i, \ \textit{and} \ \ M_i \textit{ is along } x.$$

Then $C_x = D_1 \cap \cdots \cap D_k$.

Proof. Let $D'_1, \dots, D'_\ell$ be the family of dials of building such that $x_0 \not\subseteq D_i$ and $x \subset D_i$ for any $i = 1, \dots, \ell$. Then

$$\{D_1, \dots, D_k\} \subseteq \{D'_1, \dots, D'_\ell\}.$$

According to Proposition 2.6.3, $C_x = D'_1 \cap \cdots \cap D'_\ell$, thus $C_x \subset D_1 \cap \cdots \cap D_k$. For any i, let M'_i be the wall that bounds D'_i. Up to a subfamily, we can assume that $C_x = D'_1 \cap \cdots \cap D'_\ell$ and for any $i \neq j$, $M'_i \perp M'_j$. In this case, according to Lemma 2.6.5, any building-wall M'_i is along x. Indeed $\mathrm{proj}_{C_x}(x_0) = x$. Eventually, we get

$$\{D'_1, \dots, D'_\ell\} \subseteq \{D_1, \dots, D_k\}$$

and $D_1 \cap \cdots \cap D_k \subset C_x$.

□

In the sequel of this chapter, we use boundaries of cones as a base of the topology of $\partial\Gamma$ and to construct approximations.

Proposition 2.6.7. *Let $x \in \mathrm{Ch}(\Sigma)$ and C_x be the cone based at x. Then ∂C_x is of non-empty interior in $\partial\Gamma$.*

Proof. According to Proposition 2.6.3 and Fact 2.6.4, we can write

$$\partial C_x = \partial D_1 \cap \cdots \cap \partial D_k,$$

where if $D_1, \dots, D_k$ is a collection of dials of building bounded by the building-walls $M_1, \dots, M_k$ then $M_i \perp M_j$ for any $i \neq j$. By the rotations around $M_1, \dots, M_k$, all the connected components of $\Sigma \backslash (M_1 \cup \cdots \cup M_k)$ are isomorphic. Hence, thanks to Lemma 2.4.3, there exists $M \in \mathcal{M}(\Sigma)$ such that

$$M \parallel M_i \text{ for any } i = 1, \dots, k \text{ and } M \subset D_1 \cap \cdots \cap D_k.$$

In particular, there exists $D \in \mathcal{D}(\Sigma)$ bounded by M such that $D \subset D_1 \cap \cdots \cap D_k$. As ∂D is of non-empty interior, we obtain that ∂C_x is of non-empty interior. □

Definition/Notation 2.6.8. *Let $x \in \mathrm{Ch}(\Sigma)$ and C_x be the corresponding cone of chambers. We call* shadow of x *the boundary of C_x in $\partial\Gamma$ and we write $v_x = \partial C_x$.*

2.6.2 Combinatorial metric on $\partial\Gamma$

Until now we have been considering on $\partial\Gamma$ the visual metric coming from the geometric action of Γ on Σ. Now we use infinite minimal galleries to describe a combinatorial metric on $\partial\Gamma$ that will be more convenient to use in the sequel.

Let $\mathcal{DG}(\Sigma)$ denote the *dual graph* of Σ. This graph is defined by:

- The set of vertices $\mathcal{DG}(\Sigma)^{(0)}$ is given by $\mathrm{Ch}(\Sigma)$ the set of chambers in Σ. If $v \in \mathcal{DG}(\Sigma)^{(0)}$ then c_v denotes the corresponding chamber in $\mathrm{Ch}(\Sigma)$.

- There exists an edge between two vertices v_1 and v_2 if and only if c_{v_1} is adjacent to c_{v_2} in Σ.
- Each edge is isometric to the segment $[0, 1]$.

Naturally, $\mathcal{DG}(\Sigma)$ is a proper geodesic and hyperbolic space. It is quasi-isometric to Σ and the action of Γ on $\mathcal{DG}(\Sigma)$ is geometric. Therefore we identify

$$\partial\mathcal{DG}(\Sigma) \simeq \partial\Gamma.$$

Example 2.6.9. *If for any $i = 1, \dots, n$, $q_i = 2$ or 3 then $\mathcal{DG}(\Sigma)$ is identified with $\mathrm{Cay}(\Gamma)$ the Cayley graph of Γ. Otherwise, if you consider a generator $s \in S$ of Γ of order $q \geq 4$ then in $\mathcal{DG}(\Sigma)$ the full sub-graph generated by the vertices associated with $e, s, \dots, s^{q-1}$ is a complete graph. In $\mathrm{Cay}(\Gamma)$ the full sub-graph generated by the vertices associated with $e, s, \dots, s^{q-1}$ is a cyclic graph of length q. Nevertheless $\mathcal{DG}(\Sigma)$ and $\mathrm{Cay}(\Gamma)$ are always quasi-isometric.*

With Definition 2.5.3, infinite minimal galleries are identified with geodesic rays in $\mathcal{DG}(\Sigma)$ starting from a vertex. Therefore we can write $\partial\Gamma$ as a quotient of the set of infinite minimal galleries as follow

$$\partial\Gamma \simeq \{x_0 \sim x_1 \sim \dots \sim x_i \sim \dots : x_i \in \mathrm{Ch}(\Sigma)\}/\mathcal{R}$$

where the equivalence relation $\mathcal{R}$ is defined by

$[x_0 \sim x_1 \sim \cdots] = [y_0 = x_0 \sim y_1 \sim \cdots]$ if and only if there exists $K > 0$ such that $d_c(x_i, y_i) < K$ for all $i \in \mathbb{N}$.

Example 2.6.10. *Here again we consider only minimal galleries. Let $x \in \mathrm{Ch}(\Sigma)$ with $d_c(x_0, x) = k \geq 1$. Then we can describe the shadow v_x as follow*

$$v_x \simeq \{x_0 \sim x_1 \sim \dots \sim x_i \sim \dots : x_i \in \mathrm{Ch}(\Sigma) \textit{ and } x_k = x\}/\mathcal{R}.$$

Likewise, if ∂P is a parabolic limit set associated with the residue $g\Sigma_I$. Let $x := \mathrm{proj}_{g\Sigma_I}(x_0)$ and assume that $d_c(x_0, x) = k \geq 1$. Then we can describe ∂P as follow

$$\partial P \simeq \{x_0 \sim x_1 \sim \cdots : x_k = x \textit{ and } x_{k+i} \sim_{s_i} x_{k+i+1} \textit{ with } s_i \in I \textit{ for any } i \geq 0\}/\mathcal{R}.$$

Now we use the following notation.

Notation. *If $x_0 \sim x_1 \sim \cdots$ is a minimal infinite gallery that goes asymptotically to $\xi \in \partial\Gamma$, then we write*

$$\xi = [x_0 \sim x_1 \sim \cdots].$$

Definition 2.6.11. *Let ξ, ξ' be two distinct points in $\partial\Gamma$, let $\{\xi|\xi'\}_{x_0}$ denote the largest integer ℓ such that there exist two infinite minimal galleries representing ξ and ξ'*

$$\xi = [x_0 \sim x_1 \sim \cdots \sim x_i \sim \cdots] \text{ and } \xi' = [x_0 \sim x'_1 \sim \cdots \sim x'_i \sim \cdots]$$

with

$$x_i = x'_i \text{ for } i \leq \ell \text{ and } x_{\ell+1} \neq x'_{\ell+1}.$$

In terms of shadows, $\{\xi|\xi'\}_{x_0}$ is the largest integer such that there exists a shadow v_x, with $d_c(x_0, x) = \{\xi|\xi'\}_{x_0}$, that contains both ξ and ξ'. The following proposition gives a characterization of this quantity in terms of building-walls. We remind that $D_0(M)$ designates the dial of building bounded by M and containing x_0.

Proposition 2.6.12. *Let ξ, ξ' be two distinct points in $\partial\Gamma$. Then*

$$\{\xi|\xi'\}_{x_0} = \#\{M \in \mathcal{M}(\Sigma) : \text{ there exists } \alpha \neq 0 \text{ s.t. } \{\xi, \xi'\} \subset \partial D_\alpha(M)\}.$$

Proof. Let $M_1, \ldots, M_k$ be the set of building-walls such that there exists $\alpha \neq 0$ with $\{\xi, \xi'\} \subset \partial D_\alpha(M)$. Let $\ell = \{\xi|\xi'\}_{x_0}$. We prove that $k = \ell$.

For each M_i, pick D_i such that $\{\xi, \xi'\} \subset \partial D_i$ and $x_0 \nsubseteq D_i$. We set $C = D_1 \cap \cdots \cap D_k$. As the building-walls are distinct and $\partial C \neq \emptyset$ it follows from Fact 2.6.4 that C is a cone. Let $x = \text{proj}_C(x_0)$. As $\{\xi, \xi'\} \subset \partial C$, there exists an infinite minimal gallery starting from x_0 going asymptotically to ξ (resp. ξ') passing through x. Eventually we obtain $\ell \geq d_c(x_0, x) \geq k$.

Consider $x_0 \sim x_1 \sim \cdots \sim x_i \sim \cdots$ (resp. $x_0 \sim x'_1 \sim \cdots \sim x'_i \sim \cdots$) a minimal infinite gallery representing ξ (resp. ξ') in $\partial\Gamma$. Assume that

$$x_i = x'_i \text{ for } i \leq \ell \text{ and } x_{\ell+1} \neq x'_{\ell+1}.$$

For any $i = 1, \ldots, \ell$ let D'_i be the dial of building such that $x_{i-1} \nsubseteq D'_i$ and $x_i \subset D'_i$. By minimality of the galleries, we get that $\{\xi, \xi'\} \subset \partial D'_i$ for any index i. Therefore $\ell \leq k$ and the proof is finished.

□

In the following, we prove that $\{\cdot|\cdot\}_{x_0}$ coincides with a Gromov product in $\partial\Gamma$ and thus controls a visual metric on $\partial\Gamma$.

Proposition 2.6.13. *Let ξ, ξ' be two distinct points in $\partial\Gamma$. Then there exists a bi-infinite minimal gallery between ξ and ξ' that lies at a distance smaller than $\{\xi|\xi'\}_{x_0} + 1$ of x_0.*

Proof. Let $\ell = \{\xi|\xi'\}_{x_0}$ and assume that $\xi = [x_0 \sim x_1 \sim \cdots \sim x_i \sim \cdots]$ and $\xi' = [x_0 \sim x'_1 \sim \cdots \sim x'_i \sim \cdots]$ with

$$x_i = x'_i \text{ for } i \leq \ell \text{ and } x_{\ell+1} \neq x'_{\ell+1}.$$

We consider two cases. Either $x_{\ell+1}$ is adjacent to $x'_{\ell+1}$, or $x_{\ell+1}$ is not adjacent to $x'_{\ell+1}$. In the first case, the bi-infinite gallery

$$\cdots \sim x_{\ell+2} \sim x_{\ell+1} \sim x'_{\ell+1} \sim x'_{\ell+2} \sim \cdots$$

is minimal. Indeed, thanks to Proposition 2.6.12, it only crosses once the building-walls that separate ξ and ξ'. In the second case, we apply the same reasoning to the bi-infinite gallery

$$\cdots \sim x_{\ell+2} \sim x_{\ell+1} \sim x_\ell \sim x'_{\ell+1} \sim x'_{\ell+2} \sim \cdots .$$

Eventually $\{\xi|\xi'\}_{x_0}$ or $\{\xi|\xi'\}_{x_0} + 1$ is the distance between x_0 and a bi-infinite minimal gallery between ξ and ξ'. □

Notation. *Let $d(\cdot,\cdot)$ be the self-similar metric on $\partial\Gamma$ coming from the geometric action of Γ on $\mathcal{DG}(\Sigma)$(see Definition 2.2.2).*

With Proposition 2.6.13, there exist two constants $A \geq 1$ and $\alpha > 0$ such that for any $\xi, \xi' \in \partial\Gamma$:

$$A^{-1}e^{-\alpha\{\xi|\xi'\}_{x_0}} \leq d(\xi,\xi') \leq A\ e^{-\alpha\{\xi|\xi'\}_{x_0}}.$$

In the sequel we also write

$$d(\xi,\xi') \asymp e^{-\alpha\{\xi|\xi'\}_{x_0}}.$$

Which means that, $d(\xi,\xi')$ is, up to a multiplicative constant, equal to $e^{-\alpha\{\xi|\xi'\}_{x_0}}$. An application of this description of this visual metric on $\partial\Gamma$ is the following proposition.

Proposition 2.6.14. *For every $\epsilon > 0$, there exists only a finite set of parabolic limit sets of diameter larger than ϵ.*

Proof. Let ∂P be a parabolic limit set. Let $g'\Sigma_I$ be a residue in Σ such that $\partial P \simeq \partial(g'\Sigma_I)$. According to Proposition 1.2.5, there exists a unique chamber $x \subset g'\Sigma_I$ such that for every chamber $y \subset g'\Sigma_I$ there exists a minimal gallery from x_0 to y passing through x. Let $g \in \Gamma$ such that $x = gx_0$. Then the diameter of ∂P is controlled by $e^{-\alpha|g|}$ with $|g| = d_c(x_0, gx_0)$. As there exists only a finite number of $g \in \Gamma$ such that $|g|$ is smaller than a fixed constant, the proposition is proved.

□

2.6.3 Shadows and balls of the boundary

Here we discuss how shadows control the balls of $\partial\Gamma$.

Lemma 2.6.15. *Let $x \in \mathrm{Ch}(\Sigma)$ and C_x be the cone of chambers in Σ based at x. Then there exist $g \in \Gamma$ and $C \subset \Sigma$ of the form $C = D_0(M_1) \cap \cdots \cap D_0(M_k)$ with*

- *for any i, j distinct: $M_i \perp M_j$,*

- *for any i: M_i is along x_0,*

such that $g(C) = C_x$.

Proof. Let $D_1, \dots, D_k$ designate the family of dials of building bounded by $M_1, \dots, M_k$ such that for any $i = 1, \dots, k$

$$x_0 \nsubseteq D_i,\ x \subset D_i, \text{ and } M_i \text{ is along } x.$$

According to Proposition 2.6.6, $C_x = D_1 \cap \dots \cap D_k$. Now if we choose $g \in \Gamma$ such that $gx_0 = x$ and set $C = g^{-1}(C_x)$, we obtain the desired property.

□

Proposition 2.6.16. *There exists $\lambda > 1$ such that for any $x \in \mathrm{Ch}(\Sigma)$ with $d_c(x_0, x) = k$ there exists $z \in v_x$ with*

$$B(z, \lambda^{-1}e^{-\alpha k}) \subset v_x \subset B(z, \lambda e^{-\alpha k}).$$

Proof. To prove the right-hand side inclusion it is enough to notice that $\operatorname{diam} v_x \leq Ae^{-\alpha k}$ where A and α are the visual constants. Let C_x be the cone based at x. Let $C = D_0(M_1) \cap \dots \cap D_0(M_k)$ and $g \in \Gamma$ such that $g(C) = C_x$, provided by Lemma 2.6.15. Now we remind that g^{-1} is a bi-Lipschitz homeomorphism. Restricted to v_x, it essentially rescales the metric by a factor $e^{\alpha k}$. Now, according to Proposition 2.6.7, there exist $r > 0$ and $z \in \partial C$ such that $B(z, r) \subset \partial C$. As there is only a finite number of possible C, this achieves the proof. □

Adapting the proof of [BK13, Proposition 5.2.], to the case of right-angled hyperbolic buildings, we obtain the following proposition.

Proposition 2.6.17. *There exists $\lambda \geq 1$ depending only on the geometry of Σ, such that for every $\xi \in \partial\Gamma$ and every $0 < r \leq \operatorname{diam} \partial\Gamma$ there exists $D \in \mathcal{D}(\Sigma)$ associated with such that:*

$$B(\xi, \lambda^{-1}r) \subset \partial D \subset B(\xi, \lambda r).$$

2.6.4 Approximation of $\partial\Gamma$ with shadows

Let $x \in \mathrm{Ch}(\Sigma)$ and v_x be the associated shadow as in Definition 2.6.8. Thanks to Proposition 2.6.13, if $d_c(x_0, x) = k$ then $\operatorname{diam} v_x \asymp e^{-\alpha k}$. We use this property to build an approximation of $\partial\Gamma$ made of shadows.

For an integer $k \geq 0$ we set

$$S_k = \{x \in \mathrm{Ch}(\Sigma) : d_c(x_0, x) = k\}.$$

The set $\{v_x : x \in S_k\}$ is a finite covering of $\partial\Gamma$. Now let S'_k be a subset of S_k such that $\{v_x : x \in S'_k\}$ defines a minimal covering of $\partial\Gamma$. This means that for every $x \in S'_k$ there exists $z \in v_x$ such that $z \notin v_y$ for any $y \in S'_k \backslash \{x\}$. Finally we set

$$G_k = \{v_x : x \in S'_k\}$$

and, in the following, we prove that the sequence $\{G_k\}_{k\geq 0}$ defines an approximation of $\partial\Gamma$.

Proposition 2.6.18. *For $k \geq 0$, let S'_k be the set of chambers previously defined and G_k be the minimal covering of $\partial\Gamma$ associated with S'_k. There exists $\kappa > 1$ such that for any $x \in S'_k$, there exists $\xi_x \in v_x$ such that:*

- *$\forall x \in S'_k$: $B(\xi_x, \kappa^{-1}e^{-\alpha k}) \subset v_x \subset B(\xi_x, \kappa e^{-\alpha k})$,*
- *$\forall x, y \in S'_k$ with $x \neq y$: $B(\xi_x, \kappa^{-1}e^{-\alpha k}) \cap B(\xi_y, \kappa^{-1}e^{-\alpha k}) = \emptyset$.*

This property is enough to construct an approximation of $\partial\Gamma$. Indeed the visual constant α can be chosen such that $1/2 \leq e^{-\alpha} < 1$. In this case we can extract from $\{G_k\}_{k\geq 0}$ a subsequence that is an approximation of $\partial\Gamma$ as defined in Subsection 2.1.1.

Proof of Proposition 2.6.18. Let $x \in S'_k$, and let $\xi_x \in v_x$. Up to multiplicative constant, $\operatorname{diam} v_x$ is $e^{-\alpha k}$, hence there exists $\kappa > 1$ such that for all $x \in S'_k$: $v_x \subset B(\xi_x, \kappa e^{-\alpha k})$.

We remind that the hyperbolicity, provides a constant $N \geq 1$, depending only on the hyperbolicity parameter, such that for $x, x' \in \mathrm{Ch}(\Sigma)$ with $d_c(x_0 x) = d_c(x_0, x')$ if $d_c(x, x') \geq N$ then $v_x \cap v_{x'} = \emptyset$.

For any $x \in S'_k$, we pick $z_x \in v_x$ such that $z_x \notin v_y$ for any $y \in S'_k \backslash \{x\}$. Let $x, y \in S'_k$, $x \neq y$ and let $c \in \mathrm{Ch}(\Sigma)$ be such that $d_c(x_0, c) = \{z_x | z_y\}_{x_0}$ and $\{z_x, z_y\} \subset v_c$. In this setting we can write that z_x and z_y are represented by infinite minimal galleries of the form:

- $z_x = [x_0 \sim x_1 \sim \cdots \sim x_i \sim \cdots \sim x_k \sim x_{k+1} \sim \cdots]$
- $z_y = [x_0 \sim y_1 \sim \cdots \sim y_i \sim \cdots \sim y_k \sim y_{k+1} \sim \cdots]$

with

- $x_i = c$ and $y_i = c$ for one $i \in \{1, \ldots, k-1\}$,
- $x_k = x$ and $y_k = y$.

Then, as we saw in the proof of Proposition 2.6.13, one of the following galleries:

- $\cdots \sim x_{k+1} \sim x_k \sim \cdots \sim x_{i+1} \sim c \sim y_{i+1} \sim \cdots \sim y_k \sim y_{k+1} \sim \cdots$
- $\cdots \sim x_{k+1} \sim x_k \sim \cdots \sim x_{i+1} \sim y_{i+1} \sim \cdots \sim y_k \sim y_{k+1} \sim \cdots$

is a bi-infinite minimal gallery from z_x to z_y. In particular

$$d_c(x_{k+N}, y_{k+N}) > N$$

and the corresponding shadows do not intersect:

$$v_{x_{k+N}} \cap v_{y_{k+N}} = \emptyset.$$

Now according to Proposition 2.6.16 there exist $\xi_x \in v_{x_{k+N}}$ and $\xi_y \in v_{y_{k+N}}$ such that

$$B(\xi_x, \lambda^{-1} e^{-\alpha(k+N)}) \subset v_{x_{k+N}} \text{ and } B(\xi_y, \lambda^{-1} e^{-\alpha(k+N)}) \subset v_{y_{k+N}}.$$

With $v_{x_{k+N}} \cap v_{y_{k+N}} = \emptyset$, $v_{x_{k+N}} \subset v_x$, and $v_{y_{k+N}} \subset v_y$ we obtain the desired property.

□

2.7 Modulus in the boundary of a building and in the boundary of an apartment

The boundary of an apartment is, in a well chosen case, much easier to understand than the boundary of the building. This is why we want to compare the modulus in the boundary of the building with some modulus in the boundary of an apartment.

In this section, we start by defining a convenient approximations on $\partial\Gamma$ and on the boundaries of the apartments thanks to shadows and retraction maps. Afterwards, we introduce the weighted modulus on the boundary of an apartment. Then, we prove Theorem 2.7.9. This theorem is, after Theorem 2.5.13, the second major step to prove the main result of this chapter (Theorem 2.9.1). Theorem 2.7.9 states that weighted modulus are comparable to the modulus in $\partial\Gamma$. Eventually, using the ideas of Subsection 2.2.2, we reveal a connection between the conformal dimension of $\partial\Gamma$ and a critical exponent computed in the boundary of an apartment.

We use the notations and the setting of Section 2.4, 2.5 and 2.6. In particular $p \geq 1$ is fixed constant. We fix Γ the graph product associated with the pair $(\mathcal{G}, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\dots,n})$. The self-similar metric $d(\cdot,\cdot)$ on $\partial\Gamma$ is defined as in Subsection 2.6.2. The visual exponent of $d(\cdot,\cdot)$ is α. As in Section 2.2, d_0 denotes a small constant compared with diam $\partial\Gamma$ and with the constant of approximate self-similarity. Then $\mathcal{F}_0$ is the set of curves of diameter larger than d_0.

2.7.1 Notations and conventions in ∂A and in $\partial\Gamma$

In the sequel of this chapter we fix an apartment A containing the base chamber x_0. Then we try to connect the geometry and the modulus in ∂A and in $\partial\Gamma$. Naturally we will use in ∂A and in $\partial\Gamma$ the same concepts, this is why we summarize some of the notations used in the following to avoid confusion. Fist we write

$$\mathcal{A}p_0(\Sigma) = \{B \in \mathcal{A}p(\Sigma) : x_0 \subset B\}.$$

For simplicity, π denotes the retraction $\pi_{A,x_0} : \Sigma \longrightarrow A$. We also denote by π the extension of the retraction on the boundary. The notations $d(\cdot,\cdot)$ and α are also used to describe the metric on ∂B for any $B \in \mathcal{A}p_0(\Sigma)$.

An apartment is a thin building, so we can use in ∂A the tools presented in Subsections 2.6.1 and 2.6.2. First, we define on ∂A a combinatorial self-similar metric as in Subsection 2.6.2. Yet $x_0 \subset A$, so for $\xi, \xi' \in \partial A$, the quantity $\{\xi|\xi'\}_{x_0}$ is the same if we compute it in A or in Σ. Hence, if we choose the same visual exponents for the visual metric in $\partial\Gamma$ and the visual metric in ∂A, then the metrics coincide up to a multiplicative constant. For this reason we use the same notation in ∂A and in $\partial\Gamma$ for the metric $d(\cdot,\cdot)$ and for the visual constants α and A.

Eventually, it makes sense to talk about cones of chambers in A and shadows ∂A. In ∂A the results of 2.6.1 holds.

Notation.

- $\mathcal{D}(\Sigma)$ *(resp.* $\mathcal{H}(A)$*) designates the set of dials of building in* Σ *(resp. half-spaces in* A*),*
- $\mathcal{M}(\Sigma)$ *(resp.* $\mathcal{M}(A)$*) designates the set of building-walls in* Σ *(resp. walls in* A*),*
- *for* $\xi \in \partial A$ *and* $r > 0$ *we write* $B(\xi, r) \subset \partial\Gamma$ *(resp.* $B^A(\xi, r) \subset \partial A$*) for the open ball of radius* r *and center* ξ*),*
- *for* $x \in \mathrm{Ch}(A)$ *we write* C_x *(resp.* C_x^A*) for the cone of chambers based at* x *in* Σ *(resp. in* A*),*
- *for* $x \in \mathrm{Ch}(A)$ *we write* v_x *(resp.* w_x*) for the shadow of* x *in* $\partial\Gamma$ *(resp.* ∂A*).*

Usually we will use the following conventions.

- v (resp. w) designates an open subset of $\partial\Gamma$ (resp. of ∂A),
- ∂P (resp. ∂Q) designates a parabolic limit set in $\partial\Gamma$ (resp. in ∂A),
- D (resp. H) designates a dial of building in Σ (resp. a half-space in A).

2.7.2 Choice of approximations

The following lemma says that shadows have a nice behavior under retraction maps.

Lemma 2.7.1. *Let* $A \in \mathcal{A}p_0(\Sigma)$ *and let* $x \in \mathrm{Ch}(\Sigma)$ *and* v_x *be the associated shadow in* $\partial\Gamma$ *as defined in Definition 2.6.8. Then*

- *either* $x \notin \mathrm{Ch}(A)$ *and* $\mathrm{Int}(v_x) \cap \partial A = \emptyset$*,*
- *or* $x \in \mathrm{Ch}(A)$ *and* $v_x \cap \partial A$ *is a shadow in* ∂A*.*

In the second case $v_x \cap \partial A = \pi(v_x)$.

Proof. Let C_x be the cone based at x. If $\mathrm{Int}(v_x) \cap \partial A \neq \emptyset$ then there exists a chamber c in $A \cap C_x$. By convexity, a minimal gallery from x_0 to c that passes through x is included in A and $x \subset A$. Therefore $v_x \cap \partial A$ is the shadow in ∂A associated with x. □

We fix $\{G_k^A\}_{k\geq 0}$ an approximation of ∂A based on shadows as constructed in Subsection 2.6.4.

Notation. *For* $k \geq 0$ *we set*

$$G_k := \{v_y \subset \partial\Gamma : \pi(v_y) \in G_k^A\}.$$

We remind that we chose the same visual exponents for the metrics in $\partial\Gamma$ and ∂A. Moreover for $v, v' \in G_k$, we observe that $\mathrm{Int}(v \cap v') \neq \emptyset$ if and only if $\pi(v) \cap \pi(v') \neq \emptyset$ and $\pi(v) \neq \pi(v')$. As a consequence and with Lemma 2.7.1 we get the following fact.

Fact 2.7.2. *There exists* $\kappa > 1$ *such that* $\{G_k^A\}_{k\geq 0}$ *and* $\{G_k\}_{k\geq 0}$ *are* κ*-approximations. Moreover, for any* $w \in G_k^A$ *there exists a unique* $\widetilde{w} \in G_k$ *such that* $\widetilde{w} \cap \partial A \neq \emptyset$ *and* $\pi(\widetilde{w}) = w$.

Hereafter, $\{G_k\}_{k\geq 0}$ designates the approximation of $\partial\Gamma$ obtained from $\{G_k^A\}_{k\geq 0}$ thanks to the previous fact. This approximation of $\partial\Gamma$ is canonically associated with $\{G_k^A\}_{k\geq 0}$ in the following sense: from $\{G_k\}_{k\geq 0}$ we can equip any $B \in \mathcal{A}p_0(\Sigma)$ with an approximation isometric to $\{G_k^A\}_{k\geq 0}$. Indeed if $B \in \mathcal{A}p_0(\Sigma)$, for $k \geq 0$ we set

$$G_k^B := \{w = \partial B \cap v : v \in G_k\}.$$

Now let $B \in \mathcal{A}p_0(\Sigma)$ and $f : B \longrightarrow A$ be the type preserving isometry that fixes x_0. The map f is realized by the restriction to B of the retraction π and we get the following fact.

Fact 2.7.3. $G_k^A = \{f(v)\}_{v \in G_k^B}$.

Now that an approximation $\{G_k\}_{k\geq 0}$ is fixed the results we will obtain on the combinatorial modulus in $\partial\Gamma$ will be valid, up to multiplicative constants, for any approximation thanks to Proposition 2.1.5.

2.7.3 Weighted modulus in ∂A

On scale $k \geq 0$, to compare the modulus in the building with the modulus in the apartment means to compare the cardinality of G_k with the cardinality of G_k^A. If the building is thick these quantities differ by an exponential factor in k. This is the reason why we attach a weight to the elements of G_k^A.

Definition 2.7.4. *Let* $w \in G_k^A$*, we set* $q(w) = \#\{v \in G_k : \pi(v) = w\}$.

Let $k \geq 0$ and let $\mathcal{F}^A$ be a set of curves contained in ∂A. As in Subsection 2.1.1, a positive function $\rho : G_k^A \longrightarrow \mathbb{R}^+$ is said to be $\mathcal{F}^A$-admissible if for any $\gamma \in \mathcal{F}^A$

$$\sum_{\gamma \cap w \neq \emptyset} \rho(w) \geq 1.$$

The *weighted p-mass* of ρ in ∂A is

$$WM_p^A(\rho) = \sum_{w \in G_k^A} q(w)\rho(w)^p.$$

Definition 2.7.5. *Let $k \geq 0$ and let $\mathcal{F}^A$ be a set of curves contained in ∂A, we define the* weighted G_k^A-combinatorial p-modulus of $\mathcal{F}^A$ *by*

$$\text{Mod}_p^A(\mathcal{F}^A, G_k^A) := \inf\{WM_p^A(\rho)\}.$$

Where the infimum is taken over the set of $\mathcal{F}^A$-admissible functions and with the convention $\text{Mod}_p^A(\emptyset, G_k^A) = 0$*. For simplicity, we usually use the terminology* weighted modulus*.*

We can check that, Proposition 2.1.3 holds for weighted modulus as well and the proof is identical to the one for the usual combinatorial modulus.

Yet, this definition of the weighted modulus is strongly depending on the choice we have made for the approximation. In particular, it does not permit to compute the weighted modulus relatively to a generic approximation of ∂A. As a consequence, an analogous to Proposition 2.1.5 would make no sense here. This is a huge restriction on the use we can do of the weighted modulus. Indeed, this proposition is essential to prove Proposition 2.2.11 or Theorem 2.5.13 for the usual combinatorial modulus.

Moreover, as we are interested in the modulus in $\partial\Gamma$ we can use weighted modulus computed in a precise approximation and deduce, up to multiplicative constant, generalities about modulus in $\partial\Gamma$.

The weights are given by the types of the building-walls crossed by a minimal galleries.

Proposition 2.7.6. *Let $w \in G_k^A$ be such that w is a shadow $w = w_x$ for $x \in \text{Ch}(A)$. Let $x_0 \sim_{s_1} x_1 \sim_{s_2} \cdots \sim_{s_{k-1}} x_{k-1} \sim_{s_k} x$ be a minimal gallery where $s_1, \ldots, s_k$ is the family of types of the walls crossed by this gallery. If $q_1, \ldots, q_k$ are the orders of these generators of Γ, then*

$$q(w) = \prod_{i=1,\ldots,k} q_i - 1.$$

Proof. Let $w \in G_k^A$ and $x \in \text{Ch}(A)$ be such that $w = w_x$ in ∂A. Then we observe that $\{v \subset \partial\Gamma : \pi(v) = w\} = \{v_y \subset \partial\Gamma : \pi(y) = x\}$. As a consequence, we obtain $q(w) = \#\pi^{-1}(x)$.

Now consider the gallery $x_0 \sim_{s_1} x_1 \sim_{s_2} \cdots \sim_{s_{k-1}} x_{k-1} \sim_{s_k} x$ given in the statement of proposition. As π preserves the types, $y \in \text{Ch}(\Sigma)$ is in $\pi^{-1}(x)$ if and only if there exists a minimal gallery from x_0 to y in Σ of the form $x_0 \sim_{s_1} y_1 \sim_{s_2} \cdots \sim_{s_{k-1}} y_{k-1} \sim_{s_k} y$. Eventually, we obtain $q(w) = \prod_{i=1,\ldots,k} q_i - 1$.

□

Thanks to the choices we have made, the weighted modulus is invariant up to a change of apartment in the following sense. For $B \in \mathcal{A}p_0(\Sigma)$ consider the approximation G_k^B given by Fact 2.7.3. To any element $w \in G_k^B$ we attach a weight and define a *weighted* G_k^B*-combinatorial p-modulus* as it is done in ∂A. Now let $f : B \longrightarrow A$ be a type preserving isometry that fixes x_0 and denote $f : \partial B \longrightarrow \partial A$ the extension of this map to the boundary. The map f is realized by the restriction of the retraction π to B. Thus f preserves the weights. Then the following fact is a straightforward consequence of Fact 2.7.3.

Fact 2.7.7. *Let $B \in \mathcal{A}p_0(\Sigma)$. Then for any $k \geq 0$ and any set of curves $\mathcal{F}^B$ contained in ∂B one has*

$$\mathrm{Mod}_p^B(\mathcal{F}^B, G_k^B) = \mathrm{Mod}_p^A(f(\mathcal{F}^B), G_k^A).$$

Note that, for any $k \geq 0$ and any $w \in G_k^A$ one has

$$1 \leq q(w) \leq (q-1)^k \text{ with } q := \max\{q_1, \ldots, q_n\}.$$

Therefore for any set of curves $\mathcal{F}^A$ contained in ∂A, the following inequalities come directly from the definition

$$\mathrm{mod}_p^A(\mathcal{F}^A, G_k^A) \leq \mathrm{Mod}_p^A(\mathcal{F}^A, G_k^A) \leq (q-1)^k \mathrm{mod}_p^A(\mathcal{F}^A, G_k^A).$$

Where the modulus in small letters designates the usual *modulus computed in ∂A*. In particular if Γ is of constant thickness $q - 1 \geq 2$ then

$$\mathrm{Mod}_p^A(\mathcal{F}^A, G_k^A) = (q-1)^k \mathrm{mod}_p^A(\mathcal{F}^A, G_k^A).$$

As a consequence, at fixed scale $k \geq 0$, the weighted modulus depends only on the boundary of an apartment. We will discuss this particular case in Sections 2.8 and 2.9.

The following proposition is a major motivation of the definition of the weighted modulus.

Proposition 2.7.8. *Let $\mathcal{F}$ be a set of curves in $\partial\Gamma$ and let $\mathcal{F}^A$ be a set of curves in ∂A such that $\pi(\mathcal{F}) \subset \mathcal{F}^A$. Then*

$$\mathrm{Mod}_p(\mathcal{F}, G_k) \leq \mathrm{Mod}_p^A(\mathcal{F}^A, G_k^A).$$

Proof. Let ρ_A be a $\mathcal{F}^A$-admissible function. We set $\rho : G_k \longrightarrow \mathbb{R}_+$ defined by

$$\rho(v) = \rho_A \circ \pi(v).$$

If $\gamma \in \mathcal{F}$, let $\gamma_A := \pi \circ \gamma$. Then, as $\gamma_A \in \mathcal{F}^A$

$$L_\rho(\gamma) = \sum_{v \cap \gamma \neq \emptyset} \rho_A \circ \pi(v) \geq \sum_{w \cap \gamma_A \neq \emptyset} \rho_A(w) \geq 1,$$

thus ρ is $\mathcal{F}$-admissible. Furthermore, one has:

$$M_p(\rho) = \sum_{v \in G_k} \rho_A \circ \pi(v)^p = \sum_{w \in G_k^A} q(w) \cdot \rho_A(w)^p = WM_p^A(\rho_A).$$

With the first point it follows that $\mathrm{Mod}_p(\mathcal{F}, G_k) \leq \mathrm{Mod}_p^A(\mathcal{F}^A, G_k^A)$. □

2.7.4 Modulus in $\partial\Gamma$ compared with weighted modulus in ∂A

We remind that the apartment $A \in \mathcal{A}p_0(\Sigma)$ is fixed. Yet, thanks to Fact 2.7.7 the following result holds for any apartment containing x_0.

Here we keep considering the approximations G_k and G_k^A defined in the begin of Subsection 2.7.2. We remind that if η is a non-constant curve of $\partial\Gamma$, the notation $\mathcal{U}_\epsilon(\eta)$ designates the ϵ-neighborhood of η relative to the C^0 topology. If η is a non-constant curve contained in ∂A, we use the notation

$$\mathcal{U}_\epsilon^A(\eta) := \{\gamma \in \mathcal{U}_\epsilon(\eta) : \gamma \subset \partial A\}.$$

The next theorem proves that in this case, the modulus of $\mathcal{U}_\epsilon(\eta)$ in the boundary of the building is controlled by the weighted modulus of $\mathcal{U}_\epsilon^A(\eta)$ in the boundary of the apartment. It is a key point to prove the main results of this chapter (Theorem 2.9.1).

Theorem 2.7.9. *Let $p \geq 1$, let $\eta \in \mathcal{F}_0$ and assume $\eta \subset \partial A$. For $\epsilon > 0$ small enough so that the hypothesis of Theorem 2.5.13 hold in $\partial\Gamma$, there exists a positive constant $C = C(d_0, p, \eta, \epsilon)$ independent of k such that for $k \geq 0$ large enough*

$$\mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k) \leq \mathrm{Mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A) \leq C \cdot \mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

Furthermore, when p belongs to a compact subset of $[1, +\infty)$ the constant C may be chosen independent of p.

For the rest of the subsection, we fix $\eta \in \mathcal{F}_0$ and $\epsilon > 0$ as in the hypothesis of Theorem 2.5.13. Moreover, we assume $\eta \subset \partial A$. To prove the theorem we need to introduce the following notations:

- Aut_Σ is the full group of type preserving isometries of Σ.
- For $K < \mathrm{Aut}_\Sigma$, if $x \subset \Sigma$ (resp. $v \subset \partial\Gamma$) then $K.x$ (resp. $K.v$) designate the orbit of x (resp. v) under K.

- For $n \geq 0$, $B_n \subset \mathrm{Ch}(\Sigma)$ is the ball of center x_0 and of radius n for the distance over the chambers $d_c(\cdot,\cdot)$.
- For $n \geq 0$, $K_n < \mathrm{Aut}_\Sigma$ is the fixed point stabilizer of B_n under the action of Aut_Σ.
- $\mathcal{F}_n := \{g\gamma \subset \partial\Gamma : g \in K_n \text{ and } \gamma \in \mathcal{U}_\epsilon^A(\eta)\}$.

The main step to prove the theorem is to show that $\mathcal{F}_n$ is an intermediate set of curves between $\mathcal{U}_\epsilon^A(\eta)$ and $\mathcal{U}_\epsilon(\eta)$. This is what is done in Lemma 2.7.11. Before proving this, we need to discuss the action of K_n on the chambers. The next lemma makes use of the ideas of [Cap14, Lemma 3.5 and Proposition 8.1].

Lemma 2.7.10. *There exists an integer $N > 0$ depending on n and verifying the following property. Let $x \in \mathrm{Ch}(\Sigma)$, set $d_c(x_0, x) = k$ and assume $k > n$. Let*

$$x_0 \sim_{s_1} x_1 \sim_{s_2} \cdots \sim_{s_n} x_n \sim_{s_{n+1}} \cdots \sim_{s_{k-1}} x_{k-1} \sim_{s_k} x$$

be a minimal gallery where $s_1, \ldots, s_k$ is the family of types of the building-walls crossed by this gallery. Then

$$\frac{1}{(q-1)^N} \cdot \prod_{i=n+1}^{k} q_i - 1 \leq \#K_n.x \leq \prod_{i=n+1}^{k} q_i - 1,$$

where $q := \max\{q_1, \ldots, q_n\}$.

Proof. As K_n preserves the types and fixes $x_0, \ldots, x_n$ it comes that

$$\#K_n.x \leq \prod_{i=n+1}^{k} q_i - 1.$$

Now for $D \in \mathcal{D}(\Sigma)$ we write $U(D)$ the fix point stabilizer of D under the action of Aut_Σ and we set

$$U(n) = \langle U(D) | B_n \subset \mathrm{Ch}(D) \rangle .$$

Naturally $U(n) < K_n$ and

$$\#K_n.x \geq \#U(n).x.$$

Now if we write M_i the building-wall between x_i and x_{i+1}, we observe that the orbit of x_{i+1} under $U(D_0(M_i))$ has $q_i - 1$ elements. Indeed, $U(D_0(M_i))$ acts as the full group of permutations on the set $\{D_1(M_i), \ldots, D_{q_i-1}(M_i)\}$.

Yet $U(D_0(M_i)) < U(n)$ if and only if $B_n \subset \mathrm{Ch}(D_0(M_i))$. Otherwise M_i crosses B_n, because $x_0 \in \mathrm{Ch}(D_0(M_i))$. As a consequence, if we set N the number of building-walls that cross B_n we obtain

$$\#U(n).x \geq \frac{1}{(q-1)^N} \cdot \prod_{i=n+1}^{k} q_i - 1.$$

This achieves the proof.

□

Now we can prove the main lemma.

Lemma 2.7.11. *Let $p \geq 1$. For $n \geq 0$ large enough, there exist two positive constants C_1, C_2 depending on d_0, p, η, ϵ, n, and independent of k such that for $k > n$:*

$$\mathrm{Mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A) \leq C_1 \cdot \mathrm{Mod}_p(\mathcal{F}_n, G_k) \leq C_2 \cdot \mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

Furthermore, when p belongs to a compact subset of $[1, +\infty)$ the constants may be chosen independent of p.

Proof. First we prove the right-hand side inequality. According to Proposition 2.6.13, for any $g \in K_n$ and any $\xi \in \partial\Gamma$, $d(\xi, g\xi) \asymp e^{-\alpha n}$. Then for $n \geq 0$ large enough, by triangular inequality, $\mathcal{F}_n \subset \mathcal{U}_{2\epsilon}(\eta)$. As a consequence of Theorem 2.5.13, $\mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k)$ does not depend up to a multiplicative constant on ϵ. Hence, with Proposition 2.1.3 (1), there exists $C = C(p, d_0, \epsilon, \eta)$ such that

$$\mathrm{Mod}_p(\mathcal{F}_n, G_k) \leq C \cdot \mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

Now we fix an integer $n \geq 0$ large enough so that the first point of the proof holds. We use the notation $K := K_n$ for simplicity. Moreover we assume that $k > n$. Let $\rho : G_k \longrightarrow \mathbb{R}^+$ be a minimal $\mathcal{F}_n$-admissible function and set $\rho_A : G_k^A \longrightarrow \mathbb{R}^+$ the function defined by:

$$\rho_A(w) = \int_K \rho(g\widetilde{w}) d\mu(g),$$

where μ denotes the Haar probability measure over K and where the function : $w \in G_k^A \longrightarrow \widetilde{w} \in G_k$ is given by Fact 2.7.2. Let $w \in G_k^A$ and let $x \in \mathrm{Ch}(\Sigma)$ be such that $v_x = \widetilde{w}$, then $d_c(x_0, x) = k$. As in Proposition 2.7.6, let

$$x_0 \sim_{s_1} x_1 \sim_{s_2} \cdots \sim_{s_n} x_n \sim_{s_{n+1}} \cdots \sim_{s_{k-1}} x_{k-1} \sim_{s_k} x$$

be a minimal gallery where $s_1, \ldots, s_k$ is the family of types of the building-walls crossed by this gallery. Then we set

$$q(w, n) = \prod_{i=n+1,\ldots,k} q_i - 1.$$

We notice that for any $g \in K$ the translated $g\widetilde{w} = gv_x$ is the shadow v_{gx}. In particular, this means that $\#K.\widetilde{w} = \#K.x$. Then according to Lemma 2.7.10

$$(*) \qquad \frac{q(w, n)}{(q-1)^N} \leq \#K.\widetilde{w} \leq q(w, n),$$

where $q := \max\{q_1, \dots, q_n\}$ and N is the number of building-walls crossing B_n.

As a consequence we can write

$$\rho_A(w) = \frac{1}{\#K.\widetilde{w}} \cdot \sum_{v \in K.\widetilde{w}} \rho(v),$$

and we prove the second inequality of the proposition.

Let $\gamma \in \mathcal{U}_\epsilon^A(\eta)$:

$$\begin{aligned} L_{\rho_A}(\gamma) &= \sum_{w \cap \gamma \neq \emptyset} \int_K \rho(g\widetilde{w}) d\mu(g), \\ &= \int_K \sum_{w \cap \gamma \neq \emptyset} \rho(g\widetilde{w}) d\mu(g) = \int_K \sum_{v \cap g(\gamma) \neq \emptyset} \rho(v) d\mu(g). \end{aligned}$$

Yet $g(\gamma) \in \mathcal{F}_n$, thus $\sum_{v \cap g(\gamma) \neq \emptyset} \rho(v) \geq 1$ and ρ_A is $\mathcal{F}_A$-admissible.

Then, thanks to Jensen's inequality, for $p \geq 1$ one has:

$$WM_p^A(\rho_A) \leq \sum_{w \in G_k^A} q(w) \int_K \rho(g\widetilde{w})^p d\mu(g) = \sum_{w \in G_k^A} \frac{q(w)}{\#K.\widetilde{w}} \cdot \sum_{v \in K.\widetilde{w}} \rho(v)^p.$$

Hence with $(*)$ we obtain

$$WM_p^A(\rho_A) \leq \sum_{w \in G_k^A} (q-1)^N \cdot \frac{q(w)}{q(w,n)} \cdot \sum_{v \in K.\widetilde{w}} \rho(v)^p \leq (q-1)^{n+N} M_p(\rho).$$

Eventually we get:

$$\mathrm{Mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A) \leq (q-1)^{n+N} \mathrm{Mod}_p(\mathcal{F}_n, G_k).$$

□

Proof of Theorem 2.7.9. As $\pi(\mathcal{U}_\epsilon(\eta)) \subset \mathcal{U}_\epsilon^A(\eta)$, Proposition 2.7.8 and Lemma 2.7.11 prove the theorem.

□

2.7.5 Consequences

Here we keep considering the approximations G_k and G_k^A defined in Subsection 2.7.2. For η a non-constant curve in ∂A, ∂Q a parabolic limit set in ∂A, and $\delta, r, \epsilon > 0$, we use the following notations:

- $\mathcal{F}_0^A = \{\gamma \in \mathcal{F}_0 : \gamma \subset \partial A\}$,

- $\mathcal{F}_{\delta,r}^{A}(\partial Q)$ is the subset of $\mathcal{F}_0^A$ of the curves γ such that:
  - $\gamma \subset N_\delta(\partial Q)$,
  - $\gamma \nsubseteq N_r(\partial Q')$ for any connected parabolic limit set $\partial Q' \subsetneq \partial Q$,
- $\delta_0(\cdot)$ refers to the increasing function define in Theorem 2.5.13.

We remind that the apartment $A \in \mathcal{A}p_0(\Sigma)$ is fixed. Yet, thanks to Fact 2.7.7, the following results holds for any apartment containing x_0.

Lemma 2.7.12. *Let $p \geq 1$ and $A \in \mathcal{A}p_0(\Sigma)$. Let ∂P be a parabolic limit set in $\partial\Gamma$ and assume that $x_0 \subset \mathrm{Conv}(\partial P)$. Let γ be a non-constant curve in $\partial Q = \partial P \cap \partial A$ such that ∂Q is the smallest parabolic limit set of ∂A containing γ. Let $r > 0$ be small enough so that $\gamma \nsubseteq \overline{N_r}(\partial Q')$ for any connected parabolic limit set $\partial Q' \subsetneq \partial Q$. Let $\delta < \delta_0(r)$ and $\epsilon > 0$ be small enough so that $\mathcal{U}_\epsilon^A(\gamma) \subset \mathcal{F}_{\delta,r}^A(\partial Q)$. Then there exists a constant $C = C(d_0, p, \gamma, r, \epsilon)$ such that*

$$\mathrm{Mod}_p(\mathcal{F}_{\delta,r}(\partial P), G_k) \leq C \cdot \mathrm{Mod}_p(\mathcal{U}_\epsilon(\gamma), G_k) \leq C \cdot \mathrm{Mod}_p^A(\mathcal{F}_{\delta,r}^A(\partial Q), G_k^A).$$

In particular

$$\mathrm{Mod}_p(\mathcal{F}_{\delta,r}(\partial P), G_k) \leq C \cdot \mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A).$$

Furthermore, when p belongs to a compact subset of $[1, +\infty)$ the constant C may be chosen independent of p.

Before the proof, we do the same remark as we did at Theorem 2.5.13. Up to rescaling, the assumption *for $\epsilon > 0$ small enough $\mathcal{U}_\epsilon^A(\gamma) \subset \mathcal{F}_{\delta,r}^A(\partial Q)$* makes sense.

Proof. With the assumption on ϵ and Proposition 2.1.3(1) we obtain:

$$\mathrm{Mod}_p^A(\mathcal{U}_\epsilon^A(\gamma), G_k^A) \leq \mathrm{Mod}_p^A(\mathcal{F}_{\delta,r}^A(\partial Q), G_k^A) \leq \mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A).$$

As $\pi(\, \mathcal{U}_\epsilon(\gamma)) \subset \mathcal{U}_\epsilon^A(\gamma)$, with Proposition 2.7.8 one has

$$\mathrm{Mod}_p(\mathcal{U}_\epsilon(\gamma), G_k) \leq \mathrm{Mod}_p^A(\mathcal{U}_\epsilon^A(\gamma), G_k^A).$$

Finally thanks to Theorem 2.5.13 there exists $C = C(d_0, p, \gamma, \epsilon, r)$ such that:

$$\mathrm{Mod}_p(\mathcal{F}_{\delta,r}(\partial P), G_k) \leq C \cdot \mathrm{Mod}_p(\mathcal{U}_\epsilon(\gamma), G_k).$$

□

Now we have enough to prove the following theorem that is used in the proof of Theorem 2.9.1.

Theorem 2.7.13. *For any $p \geq 1$, there exists a constant $D = D(p, d_0)$ such that for $k \geq 0$*

$$\mathrm{Mod}_p(\mathcal{F}_0, G_k) \leq D \cdot \mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A).$$

Proof. First, as it is done in [BK13, as a remark of Corollary 6.2.] in boundaries of Coxeter groups, we observe that $\mathcal{F}_0$ splits in a finite disjoint union

$$\mathcal{F}_0 = \mathcal{F}_{\delta_1, r_1}(\partial P_1) \sqcup \cdots \sqcup \mathcal{F}_{\delta_N, r_N}(\partial P_N)$$

with $\delta_i < \delta_0(r_i)$. To see this we remind that for $\delta > 0$ small enough compared with d_0, according to Proposition 2.6.14, there exists only a finite number of parabolic limit sets ∂P such that $\mathcal{F}_{\delta,r}(\partial P) \neq \emptyset$. Then we call the *height* of a parabolic limit set ∂P the maximal length of a sequence of parabolic limit sets included in ∂P of the form:

$$\partial Q_0 \subsetneq \partial Q_1 \subsetneq \cdots \subsetneq \partial Q_k = \partial P.$$

If the parameters δ_k and r_k are given for the parabolic limit sets of height k, then we set $r_{k+1} = \delta_k$ and $\delta_{k+1} < \delta_0(\delta_k)$. Starting with δ_0 small enough, we obtain the desired decomposition by induction on the height.

Let ∂P be one of the parabolic limit sets involved in the previous decomposition of $\mathcal{F}_0$ and $\delta, r > 0$ be the corresponding constants. Applying the same argument as in the begin of the proof of Theorem 2.5.13, we can assume that $x_0 \subset \mathrm{Conv}(\partial P)$. Pick $B \in \mathcal{A}p_0(\Sigma)$ such that $\partial B \cap \partial P \neq \emptyset$. With C the constant provided by the Lemma 2.7.12 we get

$$\mathrm{Mod}_p(\mathcal{F}_{\delta,r}(\partial P), G_k) \leq C \cdot \mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A).$$

Moreover, with Fact 2.7.7, the weighted modulus on the right-hand side of the inequality is independent of ∂P. Eventually, with the Proposition 2.1.3 (2), there exists a constant $D = D(p, d_0)$ such that

$$\mathrm{Mod}_p(\mathcal{F}_0, G_k) \leq D \cdot \mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A).$$

□

Note that for the moment we cannot prove a converse inequality between the modulus. Indeed, in the proof of the Lemma 2.7.12 the use of Theorem 2.5.13 in $\partial\Gamma$ is a key point. As we said before, we cannot prove an analogous of Theorem 2.5.13 for the weighted modulus.

Nevertheless, we can define a critical exponent in connection with the weighted modulus as it is done in the reminders of Subsection 2.2.2. Then Theorem 2.7.13 helps us to understand this new critical exponent.

Proposition 2.7.14. *There exists $p_0 \geq 1$ such that for $p \geq p_0$ the weighted modulus $\mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A)$ goes to zero as k goes to infinity.*

Proof. This proof is the same as the proof of Proposition 2.2.4. We remind that κ is the constant of the approximations $\{G_k\}_{k\geq 0}$ and $\{G_k^A\}_{k\geq 0}$. Then we write $K > 0$ the cardinal of a covering of ∂A by balls of radius κ. Then, by the doubling condition, we write N' the number of balls of radius $\kappa^{-1} \cdot \frac{1}{2}$ that cover a ball of radius κ. By induction we obtain

$$\# G_k^A \leq K \cdot N'^k \text{ for any } k \geq 1.$$

Moreover, as we saw in the proof of Proposition 2.1.4, there exists a constant $K' > 0$ such that the constant function $\rho : G_k \longrightarrow K' \cdot 2^{-k}$ is $\mathcal{F}_0^A$-admissible.

As a consequence

$$\mathrm{mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq C \cdot \Big(\frac{N'}{2^p}\Big)^k,$$

where C is a positive constant. Then we obtain

$$\mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq (q-1)^k \cdot \mathrm{mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq C \cdot \Big(\frac{(q-1)N'}{2^p}\Big)^k,$$

Thus, for p large enough, $\mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A)$ goes to zero. □

It is now natural to define a critical exponent for the weighted modulus in the apartment.

Definition 2.7.15. *The* critical exponent Q_W *of the weighted modulus in ∂A is defined as follow*

$$Q_W = \inf\{p \in [1, +\infty) : \lim_{k\to+\infty} \mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A) = 0\}.$$

To avoid confusion, we use the following notations

- Q for the critical exponent associated with the usual modulus $\mathrm{Mod}_p(\cdot, G_k)$ in $\partial\Gamma$,
- Q_A for the critical exponent associated with the usual modulus $\mathrm{mod}_p^A(\cdot, G_k^A)$ in ∂A,
- Q_W for the critical exponent associated with the weighted modulus $\mathrm{Mod}_p^A(\cdot, G_k^A)$ in ∂A.

We remind that Q and Q_A are respectively the conformal dimension of $\partial\Gamma$ and of $\partial A \simeq \partial W$ (see Theorem 2.2.7). The inequalities between the different modulus provide the next corollary.

Corollary 2.7.16. *The following inequalities hold*

$$Q_A \leq Q \leq Q_W.$$

Proof. With Proposition 2.1.3 (1) and Theorem 2.7.13, one has

$$\mathrm{mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq \mathrm{Mod}_p(\mathcal{F}_0, G_k) \leq D \cdot \mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A).$$

The inequalities between the critical exponents follow. □

2.8 Application to buildings of constant thickness

Here we use the notations and the setting of the previous section. In particular, the self-similar metric on $\partial\Gamma$ is $d(\cdot,\cdot)$. We fix d_0 a small constant compared with diam $\partial\Gamma$ and with the constant of approximate self-similarity. Then $\mathcal{F}_0$ is the set of curves of diameter larger than d_0. The notation $\delta_0(\cdot)$ still refers to the increasing function define in Theorem 2.5.13.

As before we fix an apartment $A \in \mathcal{A}p_0(\Sigma)$ and $\mathcal{F}_0^A$ is the set of curves in ∂A of diameter larger than d_0.

We assume that Γ is of constant thickness $q-1 \geq 2$. This means that Γ is the graph product given by the pair $(\mathcal{G}, \{\mathbb{Z}/q\mathbb{Z}\}_{i=1,\dots,n})$. As before $\{G_k^A\}_{k\geq 0}$ and $\{G_k\}_{k\geq 0}$ are the approximations of ∂A and $\partial\Gamma$ provided by Fact 2.7.2. We already noticed that, with the constant thickness assumption, we obtain for $k \geq 0$ and $\mathcal{F}^A$ a set of curves contained in ∂A

$$\mathrm{Mod}_p^A(\mathcal{F}^A, G_k^A) = q^k \mathrm{mod}_p^A(\mathcal{F}^A, G_k^A),$$

where the modulus in small letters designates the usual modulus computed in ∂A. In particular, this means that from Theorem 2.5.13 applied to $\mathrm{mod}_p^A(\cdot, G_k^A)$ we obtain analogous inequalities for $\mathrm{Mod}_p^A(\cdot, G_k^A)$.

Along with the results of Subsection 2.7.5, this leads to control $\mathrm{Mod}_p(\mathcal{F}_0, G_k)$ by $\mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A)$.

Theorem 2.8.1. *For any $p \geq 1$, there exists a constant $D = D(p, d_0)$ such that for $k \geq 0$*

$$D^{-1} \cdot \mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq \mathrm{Mod}_p(\mathcal{F}_0, G_k) \leq D \cdot \mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A).$$

In particular $Q_W = Q$.

Proof. The right-hand side inequality is given by Theorem 2.7.13. The proof is almost the same for the left-hand side inequality. Indeed, $\mathcal{F}_0^A$ admits a decomposition analogous to the decomposition used in the beginning of the proof of Theorem 2.7.13. Hence, with Proposition 2.1.3 (2), it is sufficient to prove that for any parabolic limit set $\partial Q \subset \partial A$ and any $\delta, r > 0$ with $\delta < \delta_0(r)$, there exists a constant $C = C(p, d_0, \partial Q, \delta, r)$ such that

$$\mathrm{Mod}_p^A(\mathcal{F}_{\delta,r}^A(\partial Q), G_k^A) \leq C \cdot \mathrm{Mod}_p(\mathcal{F}_0, G_k).$$

To this purpose, pick η a non-constant curve in ∂Q and $\epsilon > 0$ such that the hypothesis of Theorem 2.5.13 in ∂A and of Theorem 2.7.9 are satisfied. Then there exist two constants K and K' independent of k such that

$$\mathrm{Mod}_p^A(\mathcal{F}_{\delta,r}^A(\partial Q), G_k^A) \leq K \cdot \mathrm{Mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A) \leq K' \cdot \mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

Eventually, with the hypothesis on η and ϵ, one has, up to rescaling, $\mathcal{U}_\epsilon(\eta) \subset \mathcal{F}_0$. This achieves the proof.

The equality of the critical exponents is a straightforward consequence of the inequalities between the modulus. □

Remark 2.8.2. *In the case where Σ is a right-angled Fuchsian building of constant thickness, M. Bourdon gave the explicit value of the conformal dimension of $\partial\Gamma$.*

Theorem 2.8.3 ([Bou97])**.** *Let Γ be the graph product associated with a pair $(C_n, \{\mathbb{Z}/q\mathbb{Z}\}_{i=1,\dots,n})$ where C_n is a cyclic graph of length $n \geq 5$ and $q \geq 2$, then*

$$\mathrm{Confdim}(\partial\Gamma) = 1 + \frac{\log(q-1)}{\mathrm{Arg}\cosh\frac{n-2}{2}}.$$

2.9 Dimension 3 and 4 right-angled buildings with boundary satisfying the CLP

In a well chosen case, the symmetries of the Davis chamber, that extend to the boundary of an apartment, provide a strong control of the weighted modulus. This lead to the proof of the main theorem of this chapter.

Here we still assume that Γ is of constant thickness $q-1 \geq 2$. As usual, W is the Coxeter group, associated with Γ. As before $\{G_k^A\}_{k\geq 0}$ and $\{G_k\}_{k\geq 0}$ are the approximations of ∂A and $\partial\Gamma$ provided by Fact 2.7.2.

In this subsection, we assume that W is the group generated by the reflections along the faces of a compact right-angled polytope $D \subset \mathbb{H}^d$. Now, under some assumptions on the regularity of D, we prove that $\partial\Gamma$ satisfies the CLP.

Theorem 2.9.1. *Let Γ be a graph product of constant thickness $q-1 \geq 2$. Assume that W is the group generated by the reflections along the faces of a compact right-angled polytope $D \subset \mathbb{H}^d$ and let $R_{ef}(D)$ be the finite group of the hyperbolic reflections that preserve D. Moreover, assume that the quotient of D by $R_{ef}(D)$ is a simplex in $\mathbb{H}^d$. Then $\partial\Gamma$ verifies the CLP.*

Now we assume that the hypotheses of the previous theorem hold and we use the following notations.

Notation.

- *T is the hyperbolic simplex in $\mathbb{H}^d$ isometric to $D/R_{ef}(D)$.*
- *W_T is the hyperbolic reflection group generated by the reflections along the codimension 1 faces of T.*

We notice that W is a finite index subgroup of W_T. Indeed, W is a subgroup of W_T and both are acting discretely on $\mathbb{H}^d$ with finite co-volume. Then W_T acts by polyhedral isometries on an apartment of Σ. Indeed, a reflection along a face of T either preserves D, or is a reflection along a face of D. In particular, it preserves the tilling of $\mathbb{H}^d$ by D.

Thanks to the results of the previous section and of the constant thickness, we essentially need to study the usual combinatorial modulus in the apartment to prove the theorem.

Lemma 2.9.2. *Let $p \geq 1$ and let $A \in \mathcal{A}p_0(\Sigma)$. Let η be a non-constant curve in ∂A. There exists a constant $C = C(p, \eta, \epsilon)$ such that*

$$\mathrm{mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq C \cdot \mathrm{mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A).$$

Furthermore, when p belongs to a compact subset of $[1, +\infty)$ the constant C may be chosen independent of p.

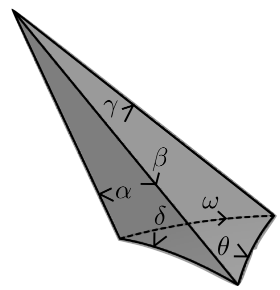


Figure 2.8: If D is a dodecahedron, T is the hyperbolic tetrahedron with dihedral angles $\alpha = \pi/5, \beta = \pi/3, \gamma = \delta = \omega = \pi/2$ and $\theta = \pi/4$,

Proof. To prove this lemma, we use the fact that ∂W_T is identified with ∂A and that ∂W_T contains no proper parabolic limit set. The group W_T acts geometrically on A, so the combinatorial visual metric on ∂A defines a self-similar metric d_{W_T} on ∂W_T. Then, a κ-approximation $\{G_k^A\}_{k \geq 0}$ induces a κ-approximation on ∂W_T with same modulus.

Now, a proper parabolic limit set ∂P in ∂A, is not a parabolic limit set in ∂W_T. Indeed, in W_T all the proper parabolic subgroups are finite. In particular, for any non-constant curve $\eta \subset \partial W_T$, the smallest parabolic subset containing η is ∂W_T. Thus, according to [BK13, Corollary 6.2.] in ∂W_T (which is the equivalent on boundaries of Coxeter groups of Theorem 2.5.13), we get that for every $\epsilon > 0$, there exists $C = C(p, \eta, \epsilon)$ such that

$$\mathrm{mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq C \cdot \mathrm{mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A).$$

□

Proof of Theorem 2.9.1. We check that the hypothesis of Proposition 2.2.12 are satisfied. To prove that $\mathrm{Mod}_1(\mathcal{F}_0, G_k)$ we do the same as at the beginning of the proof of Theorem 2.5.14.

Now we set $p \geq 1$, η a non-constant curve in $\partial\Gamma$, and $\epsilon > 0$. Thanks to Proposition 2.2.11, we can assume that $\eta \in \mathcal{F}_0$.

Then we can assume that there exists an apartment A containing η. Indeed, let ∂P be the smallest parabolic limit set containing η and let η' be a non-constant curve in $\partial A \cap \partial P$

such that ∂P is the smallest parabolic limit set of ∂A containing η'. Then, as a consequence of Theorem 2.5.13, the modulus of $\mathcal{U}_\epsilon(\eta)$ and $\mathcal{U}_{\epsilon'}(\eta')$ are essentially the same.

Using the same argument as in the beginning of the proof of Theorem 2.5.13, we can also assume that $x_0 \in \mathrm{Ch}(A)$.

Then, because of the constant thickness, the inequality of the Lemma 2.9.2 becomes

$$\mathrm{Mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq C \cdot \mathrm{Mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A).$$

Eventually, it is enough to apply Theorem 2.7.13 to the left-hand term and Theorem 2.7.9 to the right-hand term to complete the proof. □

Corollary 2.9.3. *Let q be a positive integer $q \geq 3$. Let Σ be a building of constant thickness q. Assume that the Coxeter group of Σ is the reflection group of the right-angled dodecahedron in $\mathbb{H}^3$ or the reflection group of the right-angled 120-cells in $\mathbb{H}^4$, then $\partial\Sigma$ verifies the CLP.*

Remark 2.9.4. *The hyperbolic* 120*-cell has been described by H.S.M. Coxeter in [Cox73] (see also [Dav08, Appendix B.2.]). It has been used by M.W. Davis to build a compact hyperbolic* 4*-manifold in [Dav85].*

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