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Holomorphically symplectic varieties with Prym Lagrangian fibrations

PhD thesis by Tommaso Matteini

Members of the Jury:

Supervisors:	Ugo Bruzzo	SISSA
	Dimitri Markushevich	University Lille 1
Referees:	Manfred Lehn	University Mainz
	Kieran O'Grady	University Roma "La Sapienza"
Examiners:	Michel Brion	University Grenoble 1
	Boris Dubrovin	SISSA
	Viacheslav Nikulin	University of Liverpool
	Fabio Perroni	SISSA

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Abstract

This thesis presents a construction of singular holomorphically symplectic varieties as Lagrangian fibrations. They are relative compactified Prym varieties associated to curves on symplectic surfaces with an antisymplectic involution. They are identified with the fixed locus of a symplectic involution on singular moduli spaces of sheaves of dimension 1. An explicit example, giving a singular irreducible symplectic 6-fold without symplectic resolutions, is described for a K3 surface which is the double cover of a cubic surface. In the case of abelian surfaces, a variation of this construction is studied to get irreducible symplectic varieties: relative compactified 0-Prym varieties. A partial classification result is obtained for involutions without fixed points: either the 0-Prym variety is birational to an irreducible symplectic manifold of $K3^{[n]}$-type, or it does not admit symplectic resolutions.

Résumé

Cette thèse présente une construction de variétés holomorphiquement symplectiques singulières comme fibrations lagrangiennes. Celles-ci sont des variétés de Prym compactifiées relatives associées aux courbes sur des surfaces symplectiques avec une involution antisymplectique. Elles sont identifiées au lieu fixe d'une involution symplectique sur des espaces de modules de faisceaux de dimension 1. Un exemple explicite d'une variété symplectique irréductible de dimension 6 singulière et sans résolution symplectique est décrit pour une surface K3 qui est un revêtement double d'une surface cubique. Pour surfaces abéliennes, une variation de la construction est étudiée pour obtenir des variétés symplectiques irréductibles: variétés 0-Prym compacifiées relatives. Un résultat partiel est obtenu pour involutions sans points fixes: soit la variété 0-Prym est birationnelle à une variété symplectique irréductible de $K3^{[n]}$-type, soit elle n'admet pas de résolution symplectique.

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Introduction

The geometry of irreducible symplectic varieties is an important research topic for many different mathematical fields, as algebraic geometry, Riemannian geometry and mathematical physics. One of the reasons why irreducible symplectic manifolds have attracted much attention is that, together with complex tori and Calabi-Yau manifolds, they are building blocks of Kähler manifolds with trivial first Chern class, by the Bogomolov decomposition theorem [12]. Another important (more recent) reason is that the Torelli problem for these manifolds has been solved in the work of Huybrechts [24], Markmann [32] and Verbitsky [65], so that it is known to which extent such a manifold can be recovered from the integral Hodge structure on its second cohomology.

An irreducible symplectic manifold is a natural generalization of a K3 surface to higher (even) dimensions. Very few examples are known, up to deformation equivalence. For a long time, only two families of irreducible symplectic manifolds were known, described by Fujiki [16] (in dimension 4) and by Beauville [8] (in every even dimension).

An important impulse to the research is due to Mukai, who described a symplectic structure on moduli spaces of sheaves on projective symplectic surfaces in [42]. This fact led to the hope that new deformation classes of irreducible symplectic varieties could be found in this context. As stable sheaves are smooth points of the moduli space, the first natural case to study is the one of moduli spaces containing only stable sheaves. It turns out that these moduli spaces are deformation equivalent to Beauville's examples (see [25]). It was reasonable to suggest that some moduli spaces containing strictly semistable sheaves (which usually correspond to singular points - see [25]), upon resolution of singularities, would produce new examples. Indeed, O'Grady described two singular moduli spaces admitting a symplectic resolution which belong to new deformation classes of dimension 6 ([51]) and 10 ([50]). Then Kaledin, Lehn and Sorger proved in [27] that these are isolated examples, in the sense that O'Grady's method may only produce smooth irreducible symplectic varieties in dimension 6 and 10. Finally, Perego and Rapagnetta showed in [53] that the singular moduli spaces admitting the O'Grady desingularization are all in the same two deformation classes.

A particular role in the theory of irreducible symplectic varieties is played

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by Lagrangian fibrations. Indeed, by a result of Matsushita [36], Lagrangian fibrations are the only non-trivial morphisms with connected fibers from an irreducible symplectic manifold to a manifold of smaller positive dimension. By Arnold-Liouville theorem in the projective setting, the smooth fibers of a Lagrangian fibration are abelian varieties, so an irreducible symplectic variety with a Lagrangian fibration can be viewed as a compactification of a family of abelian varieties. All the known examples of irreducible symplectic manifolds deform to Lagrangian fibrations [55], naturally arising in the context of moduli spaces of sheaves on symplectic surfaces. In the case of deformation classes related to K3 surfaces, such a Lagrangian fibration is a relative compactified Jacobian variety of a linear system on a K3 surface, endowed with the (Fitting) support map. In the case of deformation classes related to abelian surfaces, the irreducible symplectic variety is a fiber of the Albanese map of a relative compactified Jacobian variety of a linear system on an abelian surface, endowed with the (Fitting) support map. Hence one might attempt to find new examples of irreducible symplectic varieties in providing new constructions of Lagrangian fibrations as (compactified) families of abelian varieties.

In the theory of abelian varieties, the first natural objects after the Jacobian of curves are Prym varieties related to double covers of curves. Thus it is reasonable to investigate which compactified families of Prym varieties give irreducible symplectic varieties. This problem was suggested by Markushevich in [34] and a first answer was given by Markushevich and Tikhomirov in [35]. Their basic idea is that a natural way to obtain a family of Prym varieties is to consider a K3 surface S admitting an involution τ acting non-trivially on the symplectic form (a so called antisymplectic involution) and take a linear family of curves on S invariant with respect to τ. In this way, we have a relative version of the global involution τ given by its restriction on each invariant curve. So we can consider the relative Prym variety over the locus of smooth τ-invariant curves of a linear system on S. The key-point is that it can be interpreted as a connected component of the fixed locus of a rational involution η preserving the symplectic form on the relative compactified Jacobian $\mathcal{J}$ of the linear system $|C|$ on S. Hence it inherits the symplectic structure of $\mathcal{J}$ and it admits a natural compactification $\mathcal{P}_C$ (which we denote also by $\mathcal{P}$ to simplify the notation) inside $\mathcal{J}$. Moreover the restriction of the support map to $\mathcal{P}$ gives a fibration in Prym varieties over the τ-invariant part $|C|^\tau$ of the linear system. In order to have a symplectic structure on all the relative compactified Prym variety and a Lagrangian fibration $\mathcal{P} \to |C|^\tau$, one has to limit oneself to the situation when η extends to a regular involution on $\mathcal{J}$. It follows that, whenever it is possible to extend η to a regular involution, $\mathcal{J}$ acquires singularities, corresponding to strictly semistable sheaves. Moreover, also $\mathcal{P}$ has singularities, the singular locus of $\mathcal{P}$ being contained in the locus of η-invariant strictly semistable sheaves, because the fixed locus of an involution on a manifold is smooth. Thus the

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problem of the existence of a symplectic resolution arises. Generalizing results of [35] and [4], we prove a sufficient condition for the non-existence of a symplectic resolution of $\mathcal{P}$. This result relies on a study of the local model of $\mathcal{J}$ given by the Kuranishi map. The key point is that, for the simplest non-stable polystable sheaf of dimension 1, the invariant part of the tangent cone to $\mathcal{J}$ under the involution induced by η coincides with the tangent cone to $\mathcal{P}$.

Theorem 3.3.2. *Let (S, τ) be a generic K3 surface with an antisymplectic involution, C a smooth τ-invariant curve. Let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \in \mathcal{P}_C$ be a polystable sheaf, where $\mathrm{supp}(\mathcal{F}_i) =: C_i$ $(i = 1, 2)$ are smooth irreducible τ-invariant curves of genera g_i meeting transversely at 2δ points. Then $(\mathcal{P}_C, [\mathcal{F}])$ is locally analytically equivalent to $(\mathbb{C}^N \times (\mathbb{C}^{2\delta}/\pm 1), 0)$, where $N = 2(g_1 - g_1' + g_2 - g_2')$.*

As this is a $\mathbb{Q}$-factorial and terminal singularity for $\delta \geq 2$, the presence of such an η-invariant polystable sheaf implies that $\mathcal{P}$ has no symplectic resolution.

In [35], an example of a relative compactified Prymian $\mathcal{P}$ of dimension 4 is studied, and it is proven that it is a singular irreducible symplectic variety without symplectic resolutions. Even if, in this case, the construction does not produce a smoothable example, it seems to be promising because it yields a small dimensional singular irreducible symplectic variety. In dimension 4, the only other singular examples were described by Fujiki in [16], as partial desingularizations of quotients of products of two symplectic surfaces. On the contrary, the singular moduli spaces of sheaves on symplectic surfaces without symplectic resolution give examples in dimension at least 12 in the K3 case and 8 in the abelian case. Constructions of new small dimensional examples are important for the development of moduli theory of singular irreducible symplectic varieties. Foundations of this theory were laid by Namikawa [46], where he used the notion of a symplectic singularity introduced by Beauville in [11]. Such a natural generalization of the theory to singular varieties might also provide a new insight into the smooth case.

Following this philosophy, in [38], which is the first part of the original work of the thesis, I consider an example of a relative Prym variety of dimension 6, coming from a K3 surface S which is the double cover of a Del Pezzo surface Y_3 of degree 3, that is a cubic surface. The linear system of curves to which we apply the relative Prym construction is the pullback of the anticanonical linear system on Y_3. So a generic fiber of $\mathcal{P}$ is a Prym variety of a double cover of an elliptic curve by a curve of genus 4, which is naturally endowed with a polarization of type $(1, 1, 2)$. As the singular locus of $\mathcal{P}$ is contained in the locus of η-invariant strictly semistable sheaves, and a strictly semistable sheaf of dimension 1 has non-integral support, we start by determining the non-integral τ-invariant members of the linear system on S. Using the local model of $\mathcal{J}$ given by the Kuranishi map, we observe that the

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invariant part of the tangent cone to $\mathcal{J}$ under the involution induced by η coincides with the tangent cone to $\mathcal{P}$. This permits to describe all the singularities of $\mathcal{P}$. We then check that $\mathcal{P}$ is simply connected and $h^{(2,0)}(\mathcal{P}) = 1$ by representing $S^{[3]}$ as a rational double cover of $\mathcal{P}$. Finally, we find the Euler number of $\mathcal{P}$ by studying the singular fibers of the Lagrangian fibration. In this way, we get the following description of $\mathcal{P}$:

Theorem 4.4.4. *Let (S, τ) be a generic K3 surface with an antisymplectic involution such that $\pi : S \to S/\tau = Y_3$. Let $C \in |\pi^*(-K_{Y_3})|^{\tau}_{S}$.*

Then $\mathcal{P}$ is a singular irreducible symplectic 6-fold without symplectic resolutions and $\chi(\mathcal{P}) = 2283$. Its singular locus Sing($\mathcal{P}$) coincides with the locus of η-invariant strictly semistable sheaves of $\mathcal{J}$, and it is the union of 27 singular K3 surfaces associated to the 27 lines on Y_3. Each K3 surface has 5 A_1-singularities and each singular point is in the intersection of 3 K3 surfaces. A smooth point of Sing($\mathcal{P}$) is a singularity of $\mathcal{P}$ of analytic type $\mathbb{C}^2 \times (\mathbb{C}^4/\pm 1)$. A singular point of Sing($\mathcal{P}$) is a singularity of $\mathcal{P}$ of analytic type $\mathbb{C}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, where the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is given by $\langle(1, 1, -1, -1, -1, -1), (-1, -1, 1, 1, -1, -1)\rangle$.

Markushevich and Tikhomirov's construction is strictly related to the theory of K3 surfaces with an antisymplectic involution. This theory was developed by Nikulin using lattice theory in [47], [48] and [49]. Essentially, there are 75 families: one corresponds to involutions without fixed points, whose quotients are Enriques surfaces; the others to involutions with fixed points, and hence fixed curves, whose quotients are rational surfaces. Thus to approach the problem of the existence of a symplectic resolution of $\mathcal{P}$ in general, it is natural to follow this classification. Arbarello, Saccà and Ferretti in [4] and Saccà in [57] deal with the general case of Enriques surfaces, in which the fibers of $\mathcal{P}$ are principally polarized. They generalize the local description of $\mathcal{P}$ by Markushevich and Tikhomirov [35] at generic singular points. Using classical properties of linear systems on Enriques surfaces (see [14]), they prove that essentially two cases can occur when the curve is primitive. Either a linear system contains only hyperelliptic curves, and $\mathcal{P}$ is birational to a deformation of a Beauville's example. Or this is not the case, and then $\mathcal{P}$ does not admit any symplectic resolution.

The second part of the original work of the thesis consists in extending Markushevich and Tikhomirov's construction to the case of abelian surfaces A admitting an antisymplectic involution τ. Also in this case $\mathcal{P}$ can be identified with a component of the fixed locus of a regular involution η on a singular compactified Jacobian $\mathcal{J}$. A first difference is that $\mathcal{J}$ and $\mathcal{P}$ are not simply connected. Another difference is that the support map of $\mathcal{J}$ has as its image an irreducible component $\{C\}$ of the Hilbert scheme of curves, which is not a projective space as in the K3 case, and when restricted to $\mathcal{P}$ has as its image a component of the τ-invariant part $\{C\}^{\tau}$, which is again not a projective space. In order to get an irreducible symplectic variety, we observe

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that the global Prym involution η induces a regular involution η^0 on a fiber $\mathcal{K}$ of the Albanese map, so one can define the relative 0-Prym variety $\mathcal{P}^0$ as a connected component of the fixed locus of η^0, or equivalently as a fiber of the Albanese map restricted to $\mathcal{P}$. Again, $\mathcal{P}^0$ inherits a symplectic structure, and the restriction of the support map induces a Lagrangian fibration onto a component of $|C|^\tau$. Also in this case, the singular locus of $\mathcal{P}^0$ is contained in the locus of η^0-invariant strictly semistable sheaves of $\mathcal{K}$. Thus the first natural problem to deal with is the existence of symplectic resolutions. Again using the Kuranishi model, we describe the singularities of $\mathcal{P}^0$ corresponding to the simplest strictly semistable sheaves and we find a sufficient condition for the non-existence of symplectic resolutions, similarly to the case of K3 surfaces.

Theorem 5.2.3. *Let (A, τ) be a generic abelian surface with an antisymplectic involution, C a smooth τ-invariant curve. Let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \in \mathcal{P}^0_C$ be a polystable sheaf such that $\mathrm{supp}(\mathcal{F}_i) =: C_i$ $(i = 1, 2)$ are smooth irreducible τ-invariant curves of genera g_i meeting transversely at 2δ points. Then $(\mathcal{P}^0_C, [\mathcal{F}])$ is locally analytically equivalent to $(\mathbb{C}^N \times (\mathbb{C}^{2\delta}/\pm 1), 0)$ with $N = 2(g_1 - g_1' + g_2 - g_2' - 2)$.*

If $\delta \geq 2$, this is a $\mathbb{Q}$-factorial and terminal singularity and $\mathcal{P}$ does not admit any symplectic resolution.

Fujiki in [19] classified complex tori of dimension 2 with finite automorphisms group. In particular, the quotient of an abelian surface by an antisymplectic involution can be either a bielliptic surface if there are no fixed points, or a ruled surface over an elliptic curve otherwise (and so there are fixed curves). In the thesis, I focus on the case of bielliptic surfaces for several reasons. Firstly, the τ-invariant non-integral curves on A correspond to the non-integral curves on the corresponding bielliptic surface Y because the double cover is étale. So using the geometry of divisors on a bielliptic surface, it is easy to describe the η^0-invariant strictly semistable sheaves, where $\mathcal{P}^0$ can acquire singularities. Secondly, the fibers of $\mathcal{P}^0$ may be, in special cases, also principally polarized abelian varieties, which is quite interesting from the theoretical point of view, as this provides a track towards checking or extending conjectures by Sawon [60]. Thirdly, Prym varieties of étale double covers often admit nice descriptions, in particular in the case when the quotient curve is hyperelliptic.

By simple considerations on the lattices of the abelian surfaces, we prove that there are only two types of bielliptic surfaces, arising from generic abelian surfaces with an antisymplectic involution. Remark that the bielliptic surfaces in general were classified by Bagnera and de Franchis in [5].

Theorem 5.3.1. *Let A be a generic abelian surface with an antisymplectic involution τ without fixed points. Then there are only two possibilities:*

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i) $A = E_1 \times E_2$ *and* $\tau = T \times (-1)$*, where* E_1, E_2 *are generic elliptic curves and* T *is a translation by a 2-torsion point of* E_1*;*

ii) $A = E_1 \times E_2/T_1 \times T_2$ *and* $\tau = T \times (-1)$*, where* E_1, E_2 *are generic elliptic curves* T_i *are translations by 2-torsion points on* E_i*,* T *is a translation by a 2-torsion point on* E_1 *such that* $T \neq T_1$*.*

We denote by A, $\bar{A}$ respectively the abelian surfaces from items $i)$, $ii)$ of Theorem 5.3.1, and by Y, $\bar{Y}$ the corresponding bielliptic surfaces. In both cases, the bielliptic surface admits two natural projections p_1 and p_2 onto an elliptic curve and a projective line. The fiber of p_1 is isomorphic to E_2 for Y, to $\bar{E}_2 := E_2/T_2$ for $\bar{Y}$. The fiber of p_2 over the four branch points is isomorphic to $E_1' := E_1/T_1$ for Y, to $\bar{E}_1' := (E_1/T_1)/(-1)$ for $\bar{Y}$. The divisors modulo numerical equivalence are generated by these curves by a result of Serrano [62].

We give a partial answer to the question on the existence of a symplectic resolution of singularities of the relative 0-Prym variety associated to a curve on a generic bielliptic surface, observing that it only depends on the numerical class of the curve. Denote by $\mathcal{P}^0_{a,d}$, $\mathcal{P}^{\sim,0}_{a,d}$ respectively the 0-Prym variety associated to a curve C', $\bar{C}'$ on Y, $\bar{Y}$ of numerical class $aE_1' + dE_2$, $a\bar{E}_1' + d\bar{E}_2$. There are essentially two possibilities for $\mathcal{P}^0$, similar to those for $\mathcal{P}$ in the case of Enriques surfaces. In the case of A, if $C' \cdot E_1' = 1$, then p_2 induces on C' a hyperelliptic double cover. By a result of Mumford [44], the Prym variety of a double cover of a hyperelliptic curve admits a natural decomposition into the product of two Jacobian varieties. In this situation, one of them is an elliptic curve. Considering a relative version of this result, we show that $\mathcal{P}^0_{a,1}$ is birational to a relative Jacobian of a linear system on a K3 surface, which is deformation equivalent to one of Beauville's examples. For A and $\bar{A}$, we show that if $C' \cdot E_1' > 1$, then $\mathcal{P}^0$ contains a polystable sheaf satisfying the hypothesis of Theorem 5.2.3, so $\mathcal{P}^0$ does not admit any symplectic resolution. Summing up, we are stating the following theorem.

Theorem 5.5.12. *Let* E_1, E_2 *be generic elliptic curves,* $A := E_1 \times E_2$ *(*$\bar{A} := E_1 \times E_2/T_1 \times T_2$*),* $\tau := T \times (-1)$*. Let* $Y := A/\tau$ *(*$\bar{Y} := \bar{A}/\tau$*) be the corresponding bielliptic surface. Let* $a, d \in \mathbb{Z}_+$*. Then*

i) $\mathcal{P}^0_{a,1}$ *is birational to an irreducible symplectic manifold of* $K3^{[a-1]}$*-type;*

ii) for $d > 1$ *and* $a > 2$ *with* $(a-1)d \geq 6$*,* $\mathcal{P}^0_{a,d}$ *and* $\mathcal{P}^{\sim,0}_{a,d}$ *are singular symplectic varieties which do not admit any symplectic resolution.*

Structure of the thesis

In Chapter 1 we give basic definitions and results of the theory of irreducible symplectic varieties. In Section 1.1 we introduce the equivalent no-

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tions of irreducible symplectic manifolds and hyperkähler manifolds and describe Beauville's examples, which are the Douady space of 0-dimensional analytic subspaces of length n on a K3 surface and the generalized Kummer variety of an abelian surface. In Section 1.2 we briefly describe the Beauville-Bogomolov form on the second cohomology of hyperkähler manifolds, which is a fundamental tool for the study of moduli spaces of irreducible symplectic manifolds and the Torelli problem for them. In Section 1.3 we introduce Lagrangian fibrations and present a beautiful result by Matsushita. In Section 1.4 we extend the definitions to the singular case, where the Beauville-Bogomolov form still exists.

In Chapter 2 we present the basic definitions and results in the theory of moduli spaces of sheaves on projective symplectic surfaces in the case of 1-dimensional sheaves. In Section 2.1 we review the notions of purity and stability. In Section 2.2 we introduce moduli spaces of sheaves on a projective surface, focusing on the case of sheaves of dimension 1, and we describe the compactified Jacobian of the simplest reducible curve lying on a surface as a fiber of the support map defined on the moduli space. In Section 2.3 we present the local model of the moduli space given by the Kuranishi map, which is a strong tool to describe the singularities and to study the question on the existence of symplectic resolutions. In Section 2.4 we focus on the case of sheaves on a symplectic surface, introducing the symplectic structure on their moduli spaces. Then we describe the Lagrangian fibration in the case of sheaves of dimension 1, which is given by the support map, and we state the main classification result for their deformation classes.

In Chapter 3 we introduce the relative compactified Prym variety associated to linear systems on a K3 surface with an antisymplectic involution. In Section 3.1 we recall the notion of Prym variety. In Section 3.2 we describe the construction of the relative Prym variety $\mathcal{P}$ by Markushevich and Tikhomirov, motivating the choices of polarization and involution. In Section 3.3 we use the Kuranishi model to give a local description of the simplest singularities of $\mathcal{P}$. In Section 3.4 we present Nikulin's classification of K3 surfaces admitting an antisymplectic involution, which provides the list of all the relative Prym varieties constructed in this setting. In Section 3.5 we report briefly on the general results obtained by Arbarello, Saccà and Ferretti, in the case of Enriques surfaces. In Section 3.6 we discuss possible generalizations of the construction.

In Chapter 4 we present some examples of relative Prym varieties coming from double covers of Del Pezzo surfaces. In Section 4.1 we specialize the construction to the case of the anticanonical linear system on a Del Pezzo surface. In Section 4.2 we study the case of a Del Pezzo of degree 1, where $\mathcal{P}$ is a smooth elliptic K3 surface. In Section 4.3 we present the case of a Del Pezzo of degree 2, which is the example considered by Markushevich and Tikhomirov: $\mathcal{P}$ is a singular irreducible symplectic 4-fold without any symplectic resolution. In Section 4.4 we consider the case of a Del Pezzo

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of degree 3, i.e. a cubic surface: $\mathcal{P}$ is a singular irreducible symplectic 6-fold without symplectic resolution. In Subsection 4.4.1 we determine all the singularities of this 6-fold. In Subsection 4.4.2 we describe a double rational cover of $\mathcal{P}$ to deduce that $\mathcal{P}$ is irreducible symplectic. In Subsection 4.4.3 we calculate the Euler number by describing the singular members of the fibration.

In Chapter 5 we adapt Markushevich and Tikhomirov's construction to abelian surfaces with an antisymplectic involution. In Section 5.1 we show that in this case the relative Prym variety is not simply connected, and we define the relative 0-Prym variety $\mathcal{P}^0$ restricting the global Prym involution to a fiber of the Albanese map, in order to have an irreducible symplectic variety. In Section 5.2 we study a local model of this singular variety using the Kuranishi map, and we determine the simplest singularities of $\mathcal{P}^0$. In Section 5.3 we focus on the case of abelian surfaces admitting an antisymplectic involution without fixed points, whose quotient is a bielliptic surface, and we review their classification. In Section 5.4 we study divisors and linear systems on bielliptic surfaces. In Section 5.5 we obtain a partial classification of relative 0-Prym varieties associated to primitive curves on bielliptic surfaces. In the cases under consideration, either the 0-Prym variety is birational to an irreducible symplectic manifold of $K3^{[n]}$-type, or it does not admit any symplectic resolution.

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Chapter 1

Irreducible symplectic varieties

In this chapter we introduce the equivalent notions of irreducible symplectic manifold and hyperkähler manifold, and we briefly present the known examples of deformation classes (Section 1.1). Then we describe their main properties, focusing in particular on the Beauville-Bogomolov form (Section 1.2) and on Lagrangian fibrations (Section 1.3). We then extend the definitions to the singular case (Section 1.4). Even if in the next chapters we consider only projective (singular) irreducible symplectic varieties, we present the general theory in the setting of complex geometry.

1.1 Hyperkähler manifolds

Definition 1.1.1. *An irreducible symplectic manifold X is a compact Kähler manifold such that*

i) X is holomorphically symplectic, i.e. there exists a holomorphic 2-form σ which is closed and non-degenerate;

ii) X is simply connected;

iii) $H^{(2,0)}(X) = \mathbb{C}\sigma$.

The conditions *ii*) and *iii*) express a kind of minimality condition of irreducible symplectic manifolds among the holomorphic symplectic manifolds.

The existence of a symplectic structure implies that the complex dimension is always even.

Moreover, σ induces an alternating homomorphism $\mathcal{T}_X \to \Omega_X$ between the tangent sheaf and the sheaf of 1-forms, which induces a trivialization of the canonical sheaf $\mathcal{K}_X = \mathcal{O}_X$. In particular $c_1(X) = 0$.

The complex-geometric notion of an irreducible symplectic manifold can be translated in a differential-geometric notion of a hyperkähler manifold, because of Yau's proof of Calabi's conjecture [66].

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Definition 1.1.2. *A hyperkähler manifold X is a simply connected compact Kähler manifold endowed with a hyperkähler metric, i.e. a Riemannian metric g and three complex Kähler metrics i, j, k such that $Re(i) = Re(j) = Re(k) = g$ and the three complex structures I, J, K induced by i, j, k satisfy $IJ = K$.*

The importance of irreducible symplectic manifolds consists in the fact that they are building blocks of compact Kähler manifolds with trivial first Chern class [12].

Theorem 1.1.3 (Bogomolov decomposition)**.** *Let X be a compact Kähler manifold with trivial first Chern class. Then there exists a finite unramified cover $\tilde{X} \to X$ such that $\tilde{X}$ decomposes as*

$$\tilde{X} = T \times \prod X_i \times \prod Y_i,$$

where T is a complex torus, X_i are hyperkähler manifolds and Y_i are Calabi-Yau manifolds.

In the lowest dimensional case, the notion of an irreducible symplectic manifold coincides with that of a K3 surface, which, together with complex tori, are the only symplectic surfaces by Kodaira's classification of surfaces.

Definition 1.1.4. *A K3 surface is a complex surface S such that $H^1(\mathcal{O}_S) = 0$ and $\mathcal{K}_S = 0$.*

Example 1.1.5. Some families of projective K3 surfaces are the following:

i) Smooth quartics in $\mathbb{P}^3$;

ii) Smooth complete intersections of bidegree $(2, 3)$ in $\mathbb{P}^4$;

iii) Smooth complete intersections of multidegree $(2, 2, 2)$ in $\mathbb{P}^5$;

iv) Double covers of $\mathbb{P}^2$ branched along a smooth sextic curve;

v) Kummer surfaces, which are the minimal resolutions of singularities of the quotients of abelian surfaces A by the involution (-1), i.e. the blowup of $A/(-1)$ in its 16 simple nodes.

Also in the non-projective case one can define Kummer surfaces.

While many examples of K3 surfaces are known, this is not the case for irreducible symplectic manifolds. For a long time only two families of examples have been known, described by Fujiki [17] in dimension 4 and by Beauville [8] in any even dimension.

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Example 1.1.6. Let S be a projective $K3$ surface. Let $S^{(n)}$ be its n-th symmetric product and $S^{[n]}$ its Hilbert scheme of 0-dimensional subschemes of length n. The Hilbert-Chow morphism

$$S^{[n]} \xrightarrow{HC} S^{(n)}$$
$$\xi \mapsto \sum_{p \in \xi} l(\mathcal{O}_{\xi,p}) \cdot p,$$

is a minimal resolution of singularities by a result of Haiman [21], with the natural symplectic structure induced by S. Since $\pi_1(S^{[n]}) = 0$ and $H^{(2,0)}(S^{[n]}) \cong H^{(2,0)}(S)$, $S^{[n]}$ is an irreducible symplectic manifold. Also in the non-projective case, one can similarly define $S^{[n]}$ as the Douady space parametrizing 0-dimensional analytic subspaces of S of length n.

The resolution of singularities is the blowup along the generalized diagonal:

$$\begin{array}{ccc} Bl_{\Delta}(S^n) & \longrightarrow & S^n \\ \downarrow & & \downarrow \\ S^{[n]} = Bl_{\Delta}(S^{(n)}) & \longrightarrow & S^{(n)}. \end{array}$$

Definition 1.1.7. *An irreducible symplectic manifold X deformation equivalent to $S^{[n]}$ for a K3 surface S is said to be of $K3^{[n]}$-type.*

Example 1.1.8. Let A be an abelian surface. The Hilbert-Chow morphism

$$A^{[n+1]} \xrightarrow{HC} A^{(n+1)}$$

is again a minimal resolution of singularities and the symplectic form on A induces a symplectic form on $A^{[n+1]}$. However $\pi_1(A^{[n+1]}) \neq 0$ and $h^{(2,0)}(A^{[n+1]}) > 1$, hence this manifold is not hyperkähler. But if we consider the summation map

$$A^{(n+1)} \xrightarrow{\Sigma} A$$
$$(p_1, \ldots, p_{n+1}) \to \sum_i p_i,$$

and we set $K_n(A) = HC^{-1} \circ \Sigma^{-1}(0)$, we obtain a new hyperkähler manifold called generalized Kummer variety of A. If $n = 1$, $K_n(A)$ is just the usual Kummer surface (Example 1.1.5 v)). Also in the non-projective case, one can similarly define $K_n(T)$ using the Douady space.

Definition 1.1.9. *An irreducible symplectic manifold X deformation equivalent to $K_n(T)$ for a 2-dimensional torus T is said to be of Kummer-n type.*

Two more examples of irreducible symplectic manifolds are known, discovered by O'Grady ([50] and [51]) in the context of moduli spaces of sheaves on symplectic surfaces (see Chapter 2), in dimension 6 and 10.

Up to deformations, these are all the known examples of irreducible symplectic manifolds.

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1.2 Beauville-Bogomolov form

On a K3 surface S, the intersection form induces a structure of an even unimodular lattice on $H^2(S,\mathbb{Z})$. By Noether's formula, $b_2(S) = 22$. Since a unimodular indefinite even lattice is determined up to isometry by its rank and signature, we have

$$H^2(S,\mathbb{Z}) = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

where U is the hyperbolic lattice and $E_8(-1)$ is the unique unimodular even negative definite lattice of rank 8.

Quite surprisingly, this quadratic form can be generalized to irreducible symplectic manifolds of higher dimension.

Definition 1.2.1. *Let X be an irreducible symplectic manifold of dimension $2n$ and σ its symplectic form normalized by $\int_X (\sigma\overline{\sigma})^n = 1$. There is a natural pairing q'_X on $H^2(X,\mathbb{C})$ defined by*

$$q'_X(\alpha) := n \int_X \alpha^2 (\sigma\overline{\sigma})^{n-1} + (2-2n) \left(\int_X \alpha\sigma^{n-1}\overline{\sigma}^n\right) \left(\int_X \alpha\sigma^n\overline{\sigma}^{n-1}\right).$$

Theorem 1.2.2. *[18] [8] Let X be an irreducible symplectic manifold of dimension $2n$. There exists an integral primitive bilinear symmetric form q_X on $H^2(X,\mathbb{C})$, called Beauville-Bogomolov form, proportional to q'_X, which is nondegenerate of signature $(3, b_2(X) - 3)$, and there exists $c_X \in \mathbb{Q}_{>0}$, called Fujiki constant, such that*

$$\alpha^{2n} = c_X q_X(\alpha)^n.$$

Moreover, q_X and c_X are deformation and birational invariants.

The Beauville-Bogomolov form of the known examples have been determined by Beauville for manifolds of K3$^{[n]}$-type and Kummer-n type in [8], and by Rapagnetta for the O'Grady's examples in [54] and [55].

Example 1.2.3. Let X be an irreducible symplectic manifold of $K3^{[n]}$-type. Then $(H^2(X,\mathbb{Z}), q_X)$ is isomorphic to the lattice

$$U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle 2-2n \rangle,$$

where U is the hyperbolic lattice, $E_8(-1)$ is the unique unimodular even negative definite lattice of rank 8, $\langle -2-2n \rangle$ is $(\mathbb{Z}, q)$ with $q(1) = -2-2n$.

Example 1.2.4. Let X be an irreducible symplectic manifold of Kummer-n type. Then $(H^2(X,\mathbb{Z}), q_X)$ is isomorphic to the lattice

$$U^{\oplus 3} \oplus \langle -2-2n \rangle.$$

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The Beauville-Bogomolov form also gives a useful projectivity criterion.

Proposition 1.2.5. *[22] Let X be an irreducible symplectic manifold.*

Then X is projective if and only if there exists $x \in H^{1,1}(X, \mathbb{Z})$ such that $q_X(x) > 0$.

The Beauville-Bogomolov form is a fundamental tool to study moduli spaces of irreducible symplectic manifolds but in this thesis we do not analyze this beautiful aspect of the theory.

1.3 Lagrangian fibrations

Definition 1.3.1. *A Lagrangian fibration is a proper surjective morphism $f : X \to B$, where X is an irreducible symplectic manifold, B is a Kähler manifold and the generic fiber is a connected Lagrangian submanifold of X.*

Clearly $2\dim(B) = \dim(X)$. By Arnold-Liouville theorem in the compact case, the smooth fibers are complex tori. Moreover, if X is projective, then the smooth fibers are abelian varieties.

Example 1.3.2. In the case of K3 surfaces a Lagrangian fibration is an elliptic pencil. To get higher dimensional examples, one can consider an elliptic pencil $S \to \mathbb{P}^1$ and take the induced morphism $S^{[n]} \to (\mathbb{P}^1)^{[n]} = \mathbb{P}^n$, which is a Lagrangian fibration with a product of elliptic curves as the generic fiber.

A first reason to study these manifolds is that all the known examples of irreducible symplectic manifolds admit a Lagrangian fibration in their defomation class.

Another important reason is that they essentially represent all the possible maps between irreducible symplectic manifolds and manifolds of smaller dimension:

Theorem 1.3.3 (Matsushita)**.** *Let X be an irreducible symplectic manifold of dimension $2n$, B a Kähler manifold of dimension $0 < m < 2n$ and $f : X \to B$ a proper morphism with connected fibers.*

Then B is projective and f is a Lagrangian fibration.

Proof. Since $H^{(2,0)}(X)$ is generated by the symplectic form, $H^{(2,0)}(B) = 0$. Hence B is projective.

If $\alpha \in H^2(B)$, then $\int_X f^*\alpha^{2n} = 0$ because $m < 2n$, so, by the Fujiki relation of Theorem 1.2.2, $q_X(f^*\alpha) = 0$. Let H be an ample class on B and ω the Kähler class on X, then

$$\int_X (f^*H)^m \wedge \omega^{2n-m} > 0.$$

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Using the Fujiki relation, we easily see that the only possibility is $m = n$. Given a generic fiber X_b, to prove that it is Lagrangian it is enough to show that the symplectic form σ satisfies

$$\int_{X_b} \sigma \wedge \bar{\sigma} \wedge \omega^{n-2} = 0.$$

For an ample class H

$$\int_X (f^*H)^n \wedge \sigma \wedge \bar{\sigma} \wedge \omega^{n-2} = H^n \int_{X_b} \sigma \wedge \bar{\sigma} \wedge \omega^{n-2},$$

but the left hand side is zero because $q_X(f^*(H)) = 0$ and $(f^*(H), \sigma) = 0$, where $(,)_X$ is the bilinear form induced by q_X. □

Moreover, Hwang proved the following result in the projective case [26], then generalized by Greb and Lehn [20] in the Kähler case.

Theorem 1.3.4. *Let $f : X \to B$ be a Lagrangian fibration. Then $B = \mathbb{P}^n$.*

In the case of K3 surfaces the existence of an elliptic pencil is equivalent to the existence of a divisor with self-intersection 0. In higher dimension it is a conjecture that the same holds, up to bimeromorphic transformations.

1.4 Singular irreducible symplectic varieties

The notion of an irreducible symplectic manifold can be extended also to singular varieties admitting symplectic singularities in the sense of Beauville [11]. We follow the definition by Namikawa [46].

Definition 1.4.1. *A (possibly singular) symplectic variety is a normal compact Kähler variety X such that its smooth locus X^{sm} admits a non-degenerate holomorphic closed 2-form which extends to a regular 2-form on any desingularization of X. A (possibly singular) irreducible symplectic variety is a symplectic variety X such that $\pi_1(X) = 0$ and $h^{(2,0)}(X) = 1$.*

Definition 1.4.2. *A symplectic resolution $\nu : \tilde{X} \to X$ of a singular symplectic variety X is a resolution of singularities such that the regular 2-form induced on $\tilde{X}$ is also non-degenerate.*

By Proposition 1.1 [16], we have also another characterization of this type of singularities:

Theorem 1.4.3. *Let X be a symplectic variety and $\nu : \tilde{X} \to X$ a resolution. Then ν is a symplectic resolution if and only if ν is a crepant resolution (i.e. $\nu^*(\mathcal{K}_X) = \mathcal{K}_{\tilde{X}}$).*

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In the case of K3 surfaces the symplectic singularities admitting a symplectic resolution are the singularities of type A-D-E (see [6]).

This notion of singular symplectic variety also includes the notion of a V-manifold, i.e. an algebraic variety with at worst finite quotient singularities. Fujiki in [17] also describes examples of irreducible symplectic V-manifolds of dimension 4, which are all, up to deformation equivalence, partial resolutions of finite quotients of the product of two symplectic surfaces. A new construction of symplectic V-manifolds has been suggested by Markushevich and Tikhomirov in [35]. We analyze it in this thesis.

The main feature of these varieties is that, in the projective case, they also admit a Beauville-Bogomolov form (see [37] and [28]), so a theory of moduli spaces of singular irreducible symplectic varieties can be developed.

Theorem 1.4.4. *Let X be a projective singular irreducible symplectic variety of dimension $2n$ with only $\mathbb{Q}$-factorial singularities and a singular locus of codimension ≥ 4. Let $\nu : \tilde{X} \to X$ a resolution. The Beauville-Bogomolov form is the pairing $q_X := q_{\tilde{X}} \circ \nu^*$ on $H^2(X, \mathbb{C})$, where $q_{\tilde{X}}$ is defined as in Definition 1.2.2. Then q_X does not depend on $\tilde{X}$, is non-degenerate of signature $(3, b_2(X) - 3)$ and there exists $c_X \in \mathbb{Q}_{>0}$ such that*

$$\alpha^{2n} = c_X q_X(\alpha)^n.$$

The notion of a Lagrangian fibration can be extended to the singular setting still using Definition 1.3.1.

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Chapter 2

Relative Jacobians on symplectic surfaces

This chapter provides the basic definitions and results in the theory of moduli spaces of sheaves on projective symplectic surfaces in the case when the sheaves have dimension 1. After introducing the notions of purity and stability (Section 2.1), we review the definition of the moduli space of sheaves on a projective surface (Section 2.2). Then we study a local model of the moduli space using the Kuranishi map (Section 2.3), which gives a strong tool to study the type of singularities, and hence also the existence of possible symplectic resolutions. Finally we present the symplectic structure of moduli spaces of sheaves of dimension 1 on symplectic surfaces, we describe their natural Lagrangian fibration, and we state the classification theorem for their deformation classes (Section 2.4). Essentially, the examples of Beauville and O'Grady give all the known deformation classes.

2.1 Semistable sheaves of dimension 1 on a surface

Let X be a smooth projective surface and $\mathcal{F}$ a coherent sheaf on X.

Definition 2.1.1. *The support of $\mathcal{F}$ is the closed set*

$$\mathrm{Supp}(\mathcal{F}) := \{x \in X : \mathcal{F}_x \neq 0\}.$$

The ideal sheaf $I_{\mathcal{F}} := \ker[\mathcal{O}_X \to \mathcal{E}nd(\mathcal{F})]$ defines a subscheme structure on it. Its dimension is called the dimension of $\mathcal{F}$, which is denoted by $\dim(F)$.

Definition 2.1.2. *$\mathcal{F}$ is called pure sheaf of dimension d if $\dim(\mathcal{G}) = d$ for every non-trivial subsheaf $\mathcal{G}$ of $\mathcal{F}$.*

In this work we focus on pure sheaves of dimension 1 on X.

Remark 2.1.3. Equivalently, $\mathcal{F}$ is a pure sheaf of dimension 1 if $\dim(\mathcal{F}) = 1$ and it does not contain any skyscraper sheaf.

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By Proposition 1.1.10 from [25], a pure sheaf $\mathcal{F}$ of dimension 1 admits a minimal locally free resolution of length one:

$$0 \longrightarrow \mathcal{L}_1 \xrightarrow{f} \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0.$$

Definition 2.1.4. *[29] The Fitting support of $\mathcal{F}$ is*

$$\operatorname{supp}(\mathcal{F}) := \ker(det(f)).$$

One can check that it is well defined, i.e. it does not depend on the locally free resolution. Moreover $[\operatorname{supp}(\mathcal{F})] = c_1(\mathcal{F}) \in H^2(X, \mathbb{Z})$. The two notions of support that we have defined do not always agree: if we take a smooth integral curve C on X and consider a first order thickening $C' = 2C$ of it, then for a line bundle $\mathcal{F}$ on C' we have $\operatorname{Supp}(\mathcal{F}) = C$ while $\operatorname{supp}(\mathcal{F}) = C'$.

The notion of a pure sheaf of dimension 1 on X is equivalent to the notion of a pure sheaf of dimension 1 on the Fitting support of the sheaf.

If moreover the Fitting support is a reduced curve with at most nodal singularities, we have the following characterization by Seshadri [63]:

Lemma 2.1.5. *A pure sheaf $\mathcal{F}$ of dimension 1 on C is of type $\mathcal{F} = \bar{\nu}_*(\mathcal{F}')$, where $\mathcal{F}' = \bar{\nu}^*(\mathcal{F})/(tors)$ is an invertible sheaf on a partial normalization $\bar{\nu} : \bar{C} \to C$. $\mathcal{F}$ is not invertible precisely at the nodes where $\bar{\nu}$ is not an isomorphism.*

More precisely, every such $\mathcal{F}$ is obtained by taking a line bundle on the normalization of the support, and by gluing the fibers of the line bundle at the points where $\mathcal{F}$ is locally free. From this description, we get:

Lemma 2.1.6. *Let C be a reduced reducible curve with simple nodes as singularities, and let $C = C_1 \cup C_2$ be a decomposition in two curves with no common components. Set*

$$\mathcal{F}_{C_i} := \mathcal{F}|_{C_i}/(tors). \tag{2.1.1}$$

Let $\Delta_{\mathcal{F}}$ be the subset of $C_1 \cap C_2$ where $\mathcal{F}$ is locally free, and

$$\mathcal{F}^{C_j} := \mathcal{F}_{C_j} \otimes \mathcal{O}_{C_j}(-\Delta_{\mathcal{F}}). \tag{2.1.2}$$

Then

$$0 \to \mathcal{F}^{C_j} \to \mathcal{F} \to \mathcal{F}_{C_i} \to 0 \tag{2.1.3}$$

is a short exact sequence for $i \neq j$.

In order to define a notion of stability, we need to fix an ample divisor H on X. The Hilbert polynomial of such a sheaf with respect to H is

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$$P(\mathcal{F}, m) := \chi(\mathcal{F}(m)) = (c_1(\mathcal{F}) \cdot H)m + \chi(\mathcal{F}),$$

hence the reduced Hilbert polynomial with respect to H is

$$p(\mathcal{F}, m) := m + \frac{\chi(\mathcal{F})}{c_1(\mathcal{F}) \cdot H}.$$

Definition 2.1.7. *The slope of a pure sheaf $\mathcal{F}$ of dimension 1 with respect to a polarization H is*

$$\mu_H(\mathcal{F}) := \frac{\chi(\mathcal{F})}{c_1(\mathcal{F}) \cdot H}.$$

We can then define the notion of μ-stability, which is a particular case of the Gieseker-stability for sheaves of dimension 1.

Definition 2.1.8. *A pure sheaf $\mathcal{F}$ of dimension 1 is called H-(semi)stable if for every non-trivial subsheaf $\mathcal{G}$ of $\mathcal{F}$*

$$\mu_H(\mathcal{G})(\leq)\mu_H(\mathcal{F}).$$

Equivalently, $\mathcal{F}$ is H-(semi)stable if for every proper quotient sheaf $\mathcal{H}$ of $\mathcal{F}$

$$\mu_H(\mathcal{H})(\geq)\mu_H(\mathcal{F}).$$

An H-semistable sheaf which is not H-stable is called strictly H-semistable.

Lemma 2.1.9. *Let $\mathcal{F}$ be a stable sheaf. Then $\operatorname{End}(\mathcal{F}) = \mathbb{C}$, that is $\mathcal{F}$ is a simple sheaf.*

Proof. If f is a non trivial endomorphism of $\mathcal{F}$, then by the definition of stability it has to be an isomorphism. □

From now on, we omit H, except when it is important to specify the polarization.

Since we consider sheaves of dimension 1, the notion of stability on the surface is equivalent to the notion of stability on the curve C given by the Fitting support. Moreover, it reduces to finitely many inequalities [1]:

Lemma 2.1.10. *If $\mathcal{F}$ is a pure sheaf and $C := \operatorname{supp}(\mathcal{F})$ is a reduced curve with at most simple nodes, then it suffices to check the (semi)stability conditions on $\mathcal{F}^D$ (see (2.1.1)) for every subcurve D of C. In particular, if C is integral, $\mathcal{F}$ is stable with respect to any polarization.*

Proof. If $\mathcal{G}$ is a sheaf of dimension 1 and $\mathcal{L}$ is a subsheaf of it such that $\operatorname{supp}(\mathcal{G}/\mathcal{L})$ is finite, then $c_1(\mathcal{G}) = c_1(\mathcal{L})$ and $\chi(\mathcal{G}) = \chi(\mathcal{L}) + \chi(\mathcal{G}/\mathcal{L}) > \chi(\mathcal{L})$. Moreover, given a subcurve D of C, the sheaf $\mathcal{F}^D$ is maximal with respect to the inclusion. Thence the assertion. □

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Definition 2.1.11. *A sheaf $\mathcal{F}$ is called H-polystable if it is isomorphic to a direct sum of H-stable sheaves with slope $\mu_H(\mathcal{F})$.*

We can associate a polystable sheaf to a semistable one, as explained in the following result.

Theorem 2.1.12. *Any H-semistable sheaf $\mathcal{F}$ admits a filtration, called Jordan-Hölder filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_n,$$

such that the quotient $gr^i_H(\mathcal{F}) := \mathcal{F}_i/\mathcal{F}_{i-1}$ is stable with slope $\mu_H(\mathcal{F})$ for every $i = 1, ..., n$. Such a filtration is not unique, but the polystable sheaf

$$gr_H(\mathcal{F}) := \oplus_{i=1}^n gr^i_H(\mathcal{F})$$

is unique up to isomorphisms.

Definition 2.1.13. *Two H-semistable sheaves $\mathcal{F}_1, \mathcal{F}_2$ are called $\mathcal{S}$-equivalent if*

$$gr_H(\mathcal{F}_1) \cong gr_H(\mathcal{F}_2).$$

By Lemma 2.1.10, a strictly semistable $\mathcal{F}$ sheaf has non-integral Fitting support C. If C is reduced and reducible, then $\mathcal{F}$ is $\mathcal{S}$-equivalent to $gr(\mathcal{F})$, which is a direct sum of sheaves supported on subcurves of C.

2.2 Moduli spaces of sheaves of dimension 1

In general, to parametrize sheaves on a complex projective variety, it is necessary to fix their discrete invariants, which are the rank and the Chern classes, and then to consider a good notion of a family. In order to get a projective variety as the moduli space, we need to focus on bounded families. The boundedness is achieved using a notion of stability.

We introduce the notion of Mukai vector as a device keeping the discrete invariants of our sheaves [43].

Definition 2.2.1. *The torsion free part of the even cohomology $H^{2*}(X, \mathbb{Z})_{tf}$ is endowed with a lattice structure given by*

$$(v, w) := \int_X -v_0 w_2 - v_2 w_0 + v_1 w_1,$$

where $v_i \in H^{2i}(X, \mathbb{Z})$ denotes the i-th component of v. It is called Mukai lattice and its elements are said to be Mukai vectors.

The Mukai vector of a sheaf $\mathcal{F}$ on X is

$$v(\mathcal{F}) = ch(\mathcal{F})\sqrt{td(X)} \in H^{2*}(X, \mathbb{Z})_{tf}.$$

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By Riemann-Roch theorem

$$\chi(\mathcal{F},\mathcal{G}) = -(v(\mathcal{F}), v(\mathcal{G})).$$

Remark 2.2.2. A Mukai vector v is a Mukai vector of a sheaf if either $v_0 > 0$, or $v_0 = 0$ and v_1 is effective, or $v_0 = v_1 = 0$ and $v_2 > 0$.

Example 2.2.3. For a 1-dimensional sheaf $\mathcal{F}$ on a symplectic surface X

$$v(\mathcal{F}) = \left(0, c_1(\mathcal{F}), \frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F})\right) = (0, C, 1 - g + d),$$

where $c_1(\mathcal{F}) = [C]$ is a curve of genus g and $\mathcal{F}$ has degree d as a sheaf on C.

Definition 2.2.4. *A family of sheaves of dimension* 1 *over a scheme T with Mukai vector $v = (0, C, k)$, is a T-flat coherent sheaf $\mathcal{E}$ on $X \times T$ such that for all $t \in T$ the fiber $\mathcal{E}_t := \mathcal{E}_{|X\times\{t\}}$ is a semistable sheaf on X with Mukai vector v. Two families $\mathcal{E}_1$ and $\mathcal{E}_2$ over T are said to be equivalent if there exists a line bundle $\mathcal{L}$ on T such that $\mathcal{E}_1 = \mathcal{E}_2 \otimes pr_T^*\mathcal{L}$.*

In the following we prove that no moduli space can separate $\mathcal{F}$ and $gr(\mathcal{F})$, i.e. it does not distinguish among trivial and non-trivial extensions of stable sheaves with the same μ. So the best that one can expect is that the moduli space parametrizes $\mathcal{S}$-equivalence classes, or polystable sheaves.

Lemma 2.2.5. *Let $\mathcal{F}$ be a semistable sheaf. Then there exists a family $\mathcal{E}$ over $\mathbb{A}^1$ such that $\mathcal{E}_t = \mathcal{F}$ for $0 \neq t \in \mathbb{A}^1$ and $\mathcal{E}_0 = gr(\mathcal{F})$.*

Proof. For simplicity, we consider only the case $gr(\mathcal{F}) = \mathcal{F}_1 \oplus \mathcal{F}_2$. Then, setting $p : \mathbb{A}^1 \times X \to X$ the projection and $i : X = \{0\} \times X \to \mathbb{A}^1 \times X$ the inclusion, $\mathcal{E}$ is given by the kernel of $q^*\mathcal{F} \to i_*\mathcal{F}_2$. □

In order to define the notion of a moduli space, we introduce the moduli functor.

Definition 2.2.6. *Given a Mukai vector $v = (0, C, k)$, the corresponding moduli functor is*

$$\begin{aligned} \mathcal{M}_X^H(v) : &(Schemes)^0 \longrightarrow (Sets) \\ &T \mapsto \{\textit{families of sheaves over } T\}/\sim . \end{aligned}$$

Definition 2.2.7. *A scheme Z represents a functor F if F is isomorphic to the functor*

$$\begin{aligned} \underline{Z} : &(Schemes)^0 \to (Sets) \\ &T \mapsto Mor(T, Z). \end{aligned}$$

A scheme Z corepresents a functor F if there is a morphism of functors $F \to \underline{Z}$ such that any other morphism $F \to \underline{Y}$ factors through the morphism induced by $Z \to Y$.

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Definition 2.2.8. *A fine (respectively a coarse) moduli space of H-semistable sheaves with Mukai vector $v = (0, C, k)$ is a projective scheme $M_X^H(v)$ representing (respectively corepresenting) $\mathcal{M}_X^H(v)$.*

By Lemma 2.2.5, if there is a strictly semistable sheaf, then the moduli functor is not representable, i.e. it does not admit a fine moduli space.

Theorem 2.2.9 (Simpson)**.** *[64] There exists a coarse moduli space $M_X^H(v)$ of H-semistable sheaves with Mukai vector $v = (0, C, k)$. Its closed points are in natural bijection with $\mathcal{S}$-equivalence classes. The points corresponding to H-stable sheaves form an open subset $M_X^H(v)^{st}$.*

Using the notion of Fitting support, we can build a natural map on M.

Definition 2.2.10. *Let $\{C\}$ be the irreducible component containing C of the Hilbert scheme of subschemes of X whose class in cohomology is $[C]$. Then we have a natural support morphism*

$$\begin{aligned} \mathrm{supp}: & M_X^H(0, C, 1 - g + d) \to \{C\} \\ & \mathcal{F} \mapsto \mathrm{supp}(\mathcal{F}). \end{aligned}$$

The fibers over the smooth curves in $\{C\}$ are the Jacobians of these curves. For this reason, we also call $M_X^H(0, C, 1 - g + d)$ the relative compactified Jacobian of $\{C\}$, and we denote it by $\mathcal{J}_{X,H}^d(\{C\})$.

This map is well defined because supp behaves nicely in families (while Supp does not).

Lemma 2.2.11. *Let $\mathcal{E}$ be a flat family of sheaves of dimension 1 over T. Then there is a subscheme $R \subset X \times T$ such that $R_t = \mathrm{supp}(\mathcal{E}_t)$ for all $t \in T$.*

Proof. Lemma 1.1.6 [57]. □

By Lemma 2.1.10, the fiber over an integral curve C is $\bar{J}^d$, the compactified Jacobian of rank 1 torsion free sheaves of degree d.

Theorem 2.2.12. *[2] Let C be an integral curve of arithmetic genus g with planar singularities. Then $\bar{J}^d(C)$ is irreducible of dimension g.*

When C is reduced but reducible, there can be several irreducible components and also strictly semistable sheaves.

Lemma 2.2.13. *Let $C = C_1 \cup C_2$ be a curve of arithmetic genus g, where C_i $(i = 1, 2)$ are smooth irreducible curves of genus g_i meeting transversely in $\nu \geq 1$ points. Then*

i) if $\frac{H \cdot C_1}{H \cdot C}(1 - g + d) \notin \mathbb{Z}$, $\bar{J}_H^d(C)$ has ν irreducible components of dimension g, whose generic points parametrize stable sheaves;

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ii) *if* $\frac{H\cdot C_1}{H\cdot C}(1-g+d) \in \mathbb{Z}$, $\bar{J}^d_H(C)$ *has* $\nu - 1$ *irreducible components of dimension* g*, whose generic points parametrize stable sheaves, and the locus of strictly semistable ones is of dimension* $g+1-\nu$*, representing the* $\mathcal{S}$*-equivalence classes* $[\mathcal{F}_1 \oplus \mathcal{F}_2]$ *with* $\mathrm{supp}(\mathcal{F}_i) = C_i$.

Proof. Let $\mathcal{F} \in \bar{J}^d(C)$ and set $\mathcal{F}_i := \mathcal{F}_{C_i}$, $\mathcal{F}^i := \mathcal{F}^{C_i}$, $k_i := H\cdot C_i$, $k := H\cdot C$, $d_i := deg(\mathcal{F}_i)$ and $\epsilon \leq \nu := C_1 \cdot C_2$ the number of points where $\mathcal{F}$ is locally free.

By Lemma 2.1.6, $\chi = \chi_1 + \chi_2 - \epsilon$. Moreover $\chi = 1 - g + d$. Since there are only simple nodes, $g = g_1 + g_2 + \nu - 1$ and we have the following decomposition by Lemma 2.1.5:

$$\bar{J}^d(C) = \bigcup_{\bar{C}} \coprod_{d_1+d_2=d+\bar{\epsilon}} J^{d_1,d_2}(\bar{C}), \tag{2.2.1}$$

where

$$J^{d_1,d_2}(\bar{C}) = \{\mathcal{F} : deg(\mathcal{F}_i) = d_i, \text{ for } i = 1,2\},$$

$\bar{C}$ varies among the partial normalizations of C and $\bar{\epsilon}$ is the number of points where $\bar{C} \to C$ is not smooth.

By Lemma 2.1.10 we get two semistability conditions

$$\frac{\chi}{k}(\leq)\frac{\chi_i}{k_i} \text{ for } i = 1,2$$

hence combining them with (2.1.3) we obtain

$$\frac{k_1}{k}\chi(\leq)\chi_1(\leq)\frac{k_1}{k}\chi + \epsilon. \tag{2.2.2}$$

If $\frac{k_1}{k}\chi \notin \mathbb{Z}$, then χ_1 can assume ν different values satisfying strictly (2.2.2) for locally free sheaves. Since the non-locally free sheaves lie in the closure of the locally free sheaves in the decomposition (2.2.1), there are ν irreducible components of dimension g, whose generic points are stable sheaves. If $\frac{k_1}{k}\chi \in \mathbb{Z}$, then χ_1 can assume $\nu - 1$ different values satisfying strictly (2.2.2) for locally free sheaves. Thus there are $\nu - 1$ irreducible components of dimension g, whose generic points are stable sheaves. Moreover, there are strictly semistable sheaves for $\chi_1 = \frac{k_1}{k}\chi$ by (2.2.2). Each $\mathcal{S}$-equivalence class of strictly semistable sheaves contains a polystable sheaf $\mathcal{F}_1 \oplus \mathcal{F}_2$ with $\mathrm{supp}(\mathcal{F}_i) = C_i$, which is not locally free at all the points of $C_1 \cdot C_2$ because it is a direct sum. Hence $\epsilon = 0$ for such a polystable sheaf, and there is only one stratum of strictly semistable sheaves. □

There is also another natural map on the moduli space.

Definition 2.2.14. *The determinant map is*

$$det : M^H_X(0, C, 1-g+d) \to Pic(X) \tag{2.2.3}$$

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$$\mathcal{F} \mapsto \mathcal{L}_0 \otimes \mathcal{L}_1^*,$$

where

$$0 \longrightarrow \mathcal{L}_1 \xrightarrow{f} \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

is a locally free resolution of $\mathcal{F}$*.*

2.3 Local structure of the moduli space

Let X be a smooth projective surface, H a polarization and C a curve on X. In this section we denote $\mathcal{J}^d_{X,H}(\{C\})$ simply by $\mathcal{J}$. Let $\mathcal{F}$ be a stable sheaf from $\mathcal{J}$, which is simple by Lemma 2.1.9. There is a natural identification

$$T_{[\mathcal{F}]}\mathcal{J} = \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}). \tag{2.3.1}$$

Indeed, an element $\mathcal{G}$ of $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ satisfies a short exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{i} \mathcal{G} \xrightarrow{p} \mathcal{F} \longrightarrow 0\ .$$

If we consider $t := i \circ p \in \mathrm{End}(\mathcal{G})$, it naturally defines an $\mathcal{O}_X[t]/(t^2)$-module, because $\mathcal{G}$ is an $\mathcal{O}_X$-module and $t^2 = 0$. It is also flat over X. Thus we get a family of sheaves over $Spec(\mathbb{C}[t]/(t^2))$ with $\mathcal{F}$ as the central fiber, which is a tangent vector to the deformation functor introduced in Definition 2.2.4.

To study the smoothness of $\mathcal{J}$, we know by a result of Artamkin (Theorem 4.5.3 in [25]) that the obstruction to deform $\mathcal{F}$ on X sits in $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})_0$, the kernel of the trace map.

We briefly recall its definition. If $\mathcal{F}$ is locally free, there is a natural trace map

$$\mathrm{tr} : \mathcal{E}nd(\mathcal{F}) \to \mathcal{O}_X,$$

which induces

$$\mathrm{tr} : \mathrm{Ext}^i(\mathcal{F}, \mathcal{F}) = H^i(X, \mathcal{E}nd(\mathcal{F})) \to H^i(X, \mathcal{O}_X).$$

If $\mathcal{F}$ is not locally free, then the trace map is constructed using a locally free resolution $\mathcal{L}^\bullet$:

$$tr : \mathcal{E}nd(\mathcal{L}^\bullet) \to \mathcal{O}_X$$

gives

$$\mathrm{tr} : \mathrm{Ext}^i(\mathcal{F}, \mathcal{F}) = \mathbb{H}^i(X, \mathcal{E}nd(\mathcal{L}^\bullet)) \to H^i(X, \mathcal{O}_X).$$

It does not depend on the locally free resolution.

If $i = 1$, the trace map is the tangent map to the determinant map (2.2.3):

$$\mathrm{tr} : T_{[\mathcal{F}]}\mathcal{J} = \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to T_{[\mathcal{F}]}\hat{X} = H^1(X, \mathcal{O}_X).$$

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Theorem 2.3.1. *If X is a symplectic surface, then $\mathcal{J}^{st}$ is smooth.*

Proof. By Serre duality $\mathrm{Ext}^2(\mathcal{F},\mathcal{F}) = \mathrm{End}(\mathcal{F})^*$, which is $\mathbb{C}$ if $\mathcal{F}$ is stable. Thus the trace map $\mathrm{tr} : \mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \to H^2(\mathcal{O}_X)$ is an isomorphism, and there is no obtruction. □

The main tool to study the singularities of moduli spaces of sheaves is given by the Kuranishi map (see [30]).

Theorem 2.3.2. *There exists a formal map, called Kuranishi map,*

$$k : \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \to \mathrm{Ext}^2(\mathcal{F},\mathcal{F})_0$$

satisfying the following properties:

i) *k is equivariant with respect to the natural conjugation action of $G := PAut(\mathcal{F})$ on $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$ and $\mathrm{Ext}^2(\mathcal{F},\mathcal{F})$;*

ii) *$k^{-1}(0)$ is a base of the miniversal deformation of $\mathcal{F}$;*

iii) *the expansion of k into a formal series*

$$k = k_2 + k_3 + \ldots$$

starts by a quadratic term

$$k_2(\mathcal{G}) := \frac{1}{2}\mathcal{G} \cup \mathcal{G},$$

where $\cup$ denotes the Yoneda pairing on $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$.

In general, for semistable sheaves Ext^1 is the tangent space to the deformation functor and to the versal deformation. By Luna slice Theorem applied to the GIT construction of the moduli space (see [25]), the Luna slice through a polystable $\mathcal{F}$ is a versal deformation of $\mathcal{F}$ whenever the Quot scheme is smooth over $\mathcal{F}$. Combining this description of the versal deformation with the properties of the Kuranishi map, one obtains the following local description of $\mathcal{J}$.

Theorem 2.3.3. *If the Quot scheme is smooth over $\mathcal{F}$, then $(k^{-1}(0)//G, 0)$ is a local analytic model of $(\mathcal{J}, [\mathcal{F}])$.*

Luckily, it turns out that it suffices to consider the cup product to determine the singularities of $\mathcal{J}$.

Theorem 2.3.4. *[27] If the Quot scheme is smooth over $\mathcal{F}$, then $(k_2^{-1}(0)//G, 0)$ is a local analytic model of $(C_{[\mathcal{F}]}(\mathcal{J}), [\mathcal{F}])$.*

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Thus the local structure of $\mathcal{J}$ at $\mathcal{F}$ is related to the deformation theory of the unique polystable sheaf in the $\mathcal{S}$-equivalence class of $\mathcal{F}$. Focusing on strictly semistable sheaves, to study if they represent smooth or singular points of the moduli space, we consider the tangent cone $C_{[\mathcal{F}]}(\mathcal{J})$.

In the case of symplectic surfaces, strictly semistable sheaves usually correspond to singular points of moduli spaces containing at least one stable sheaf (see [25]). We show this in the case of the simplest (non-stable) polystable sheaf, and we describe the type of singularity.

Lemma 2.3.5. *Let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ be a polystable sheaf of dimension 1 on a symplectic surface X, where $\operatorname{supp}(\mathcal{F}_i) =: C_i$ are smooth irreducible curves of genus g_i meeting transversely at $\nu \geq 2$ points. Then $(\mathcal{J}, [\mathcal{F}])$ is locally analytically equivalent to $(\mathbb{C}^{2g_1+2g_2} \times \hat{Z}, 0)$, where $\hat{Z}$ is the affine cone over a hyperplane section of the Segre embedding $\sigma_{\nu-1,\nu-1} : \mathbb{P}^{\nu-1} \times \mathbb{P}^{\nu-1} \to \mathbb{P}^{\nu^2-1}$.*

Proof. Set

$$U_i := \operatorname{Ext}^1(\mathcal{F}_i, \mathcal{F}_i), W := \operatorname{Ext}^1(\mathcal{F}_1, \mathcal{F}_2), W' := \operatorname{Ext}^1(\mathcal{F}_2, \mathcal{F}_1). \tag{2.3.2}$$

By Serre duality, W' is the dual W^* of W with respect to the pairing $\operatorname{tr} \circ \cup$. As the supports of $\mathcal{F}_1$ and $\mathcal{F}_2$ are transversal, we get $W = \mathbb{C}^\nu$. So

$$\operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) = U_1 \times U_2 \times W \times W^* = \mathbb{C}^{2g_1+2g_2+2\nu} = \mathbb{C}^{2g+2}. \tag{2.3.3}$$

Choose coordinates $x_1, ..., x_\nu$ in W and the dual coordinates $y_1, ..., y_\nu$ in W^*.

By the stability of $\mathcal{F}_i$ we have $Aut(\mathcal{F}) = Aut(\mathcal{F}_1) \times Aut(\mathcal{F}_2) = \mathbb{C}^{*2}$, hence $G = \mathbb{C}^*$. As proven in Lemma 1.4.16 of [50], G acts trivially on $U_1 \times U_2$, while on $W \times W^*$

$$(\lambda_1, \lambda_2) \cdot (\underline{x}, \underline{y}) = (\lambda_1 \lambda_2^{-1} \underline{x}, \lambda_1^{-1} \lambda_2 \underline{y}) \text{ for } (\lambda_1, \lambda_2) \in Aut(\mathcal{F}),$$

so the action of G is

$$\lambda \cdot (\underline{x}, \underline{y}) = (\lambda \underline{x}, \lambda^{-1} \underline{y}), \text{ with } \lambda = \lambda_1/\lambda_2.$$

Hence

$$\operatorname{Ext}^1(\mathcal{F}, \mathcal{F})//G = U_1 \times U_2 \times ((W \times W^*)//G). \tag{2.3.4}$$

The algebra of invariants of the action of G on $\mathbb{P}(W \times W^*)$ is generated by the quadratic monomials

$$u_{ij} = x_i y_j \text{ for } i, j = 1, ..., \nu, \tag{2.3.5}$$

and the generating relations are the quadratic ones

$$u_{ij} u_{kl} = u_{kj} u_{il}. \tag{2.3.6}$$

Hence we get the Segre embedding $\sigma_{\nu-1,\nu-1} : \mathbb{P}^{\nu-1} \times \mathbb{P}^{\nu-1} \to \mathbb{P}^{\nu^2-1}$.

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Thus (2.3.4) becomes

$$\mathrm{Ext}^1(\mathcal{F},\mathcal{F})//G = U_1 \times U_2 \times Im(\widehat{\sigma_{\nu-1,\nu-1}}) \tag{2.3.7}$$

where $Im(\widehat{\sigma_{\nu-1,\nu-1}})$ is the affine cone over the Segre embedding.

To describe M in an analytic neighbourhood of $[\mathcal{F}]$, we can determine $C_{[\mathcal{F}]}(M)$ around $[\mathcal{F}]$, because the tangent cone is analytically equivalent to the singularity by Section 3 [27]. By Theorem 2.3.4, $(k_2^{-1}(0)//G, 0)$ is a local analytic model of $(C_{[\mathcal{F}]}(\mathcal{J}), [\mathcal{F}])$. We thus need to describe k_2.

By Serre duality and by the stability of $\mathcal{F}_i$, we have

$$\mathrm{Ext}^2(\mathcal{F},\mathcal{F}) = \mathrm{Hom}(\mathcal{F},\mathcal{F})^* = \mathrm{Hom}(\mathcal{F}_1,\mathcal{F}_1)^* \times \mathrm{Hom}(\mathcal{F}_2,\mathcal{F}_2)^* = \mathbb{C}^2,$$

so $\mathrm{Ext}^2(\mathcal{F},\mathcal{F})_0 = \mathbb{C}$.

Since the cup product on $U_1 \times U_2$ is trivial, we also have the decomposition

$$k_2^{-1}(0)//G = U_1 \times U_2 \times (k_2^{-1}(0)|_{W\times W^*}//G), \tag{2.3.8}$$

and the singularity depends only on the last term.

On $W \times W^*$ the cup product is bilinear and $k_2^{-1}(0)|_{W\times W^*}//G$ is given by a linear equation in the invariant coordinates u_{ij}, hence it is a hyperplane section of the Segre embedding $Im(\widehat{\sigma_{\nu-1,\nu-1}})$. □

Corollary 2.3.6. *Let X be a symplectic surface and H a polarization on X. Let $\{C\}_X$ contain at least one reducible curve $C_1 \cup C_2$, where C_i are smooth irreducible curves of genus g_i meeting transversely at ν points. If $\frac{H\cdot C_1}{H\cdot C}(1-g+d) \in \mathbb{Z}$, then the singular locus of $\mathcal{J}^d_{X,H}(C)$ contains an irreducible component of dimension $2g_1+2g_2$ with a fibration map*

$$\mathcal{J}^{d_1}_{X,H}(C_1) \times \mathcal{J}^{d_2}_{X,H}(C_2) \to \{C_1\} \times \{C_2\},$$

where $d_1 = \frac{H\cdot C_1}{H\cdot C}(1-g+d) + g_1 - 1$ and $d_2 = d - d_1$.

Proof. A generic strictly semistable sheaf is $\mathcal{S}$-equivalent to a sheaf of type $[\mathcal{F}_1 \oplus \mathcal{F}_2]$, with $\mathrm{supp}(\mathcal{F}_i) = C_i$. By Lemma 2.2.13, the fiber over $C_1 \cup C_2$ has a stratum of strictly semistable sheaves of type $[\mathcal{F}_1 \oplus \mathcal{F}_2]$, which admits a natural fibration on $|C_1| \times |C_2|$. Moreover their degrees d_1, d_2 are determined by the semistability conditions. By Lemma 2.3.5, these polystable sheaves are singular points and the singular locus has dimension $2g_1 + 2g_2$. □

Remark 2.3.7. Of course, there can be several irreducible components of $Sing(\mathcal{J}^d_{X,H}(C))$, for example one for each family of reducible curves satisfying the previous hypothesis.

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2.4 Moduli spaces of sheaves on symplectic surfaces

An interesting aspect of moduli spaces of sheaves on a surface is that usually their geometry reflects the geometry of the surface itself. In the case of a symplectic surface X, we have that they inherit the symplectic structure from X. Moreover, they have the same fundamental group as X.

Theorem 2.4.1 (Mukai). *Let X be a symplectic surface. Then $M_X^H(0, C, d)$ has a symplectic structure, which, by (2.3.1), is given pointwise by*

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \times \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \xrightarrow{\cup} \mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) \xrightarrow{tr} H^2(\mathcal{O}) \xrightarrow{\cdot/\sigma} \mathbb{C}. \quad (2.4.1)$$

The skew-symmetry follows from general properties of the Yoneda product and trace maps. Non-degeneracy is an immediate consequence of Serre duality. The difficulty is to prove the global properties, i.e. this pointwise defined 2-form is holomorphic and closed (see [42]).

To state the general results on moduli spaces of sheaves on a symplectic surface, we introduce the notion of primitivity.

Definition 2.4.2. *A Mukai vector $v = (0, C, 1 - g + d)$ is primitive if it is not divisible by $k \geq 2$.*

Lemma 2.4.3. *If $d = 0$, then v is primitive if and only if v_1 is primitive.*

Proof. Clearly, if v is not primitive, then v_1 is not primitive. Moreover, for $d = 0$, if $v_1 = kC$, then $v = (0, kC, k^2(1 - g)) = k(0, C, k(1 - g))$. □

Definition 2.4.4. *Let $v = (0, C, 1 - g + d)$, where C is a curve of genus g. For every sheaf $\mathcal{F}$ with $v(\mathcal{F}) = v$ and for every subsheaf $\mathcal{G} \subset \mathcal{F}$, we set*

$$D := \chi(\mathcal{G})c_1(\mathcal{F}) - \chi(\mathcal{F})c_1(\mathcal{G}), \quad (2.4.2)$$

and for $D \neq 0$ we define the v-wall associated to D as

$$W^D := D^\perp \cap \mathrm{Amp}(X)_\mathbb{Q}, \quad (2.4.3)$$

where $\mathrm{Amp}(X)_\mathbb{Q}$ is the cone of ample $\mathbb{Q}$-divisors of X. By a result of Yoshioka (Section 1.4 [67]), the number of v-walls is finite.
The v-chambers are the connected components of the complement of the union of all the v-walls. A v-generic polarization H is a divisor not contained in a v-wall.

Essentially, a v-generic polarization forces the Mukai vector of any potentially destabilizing subsheaf to be proportional to v. Hence if the Mukai vector is primitive, there is no strictly semistable sheaf. When $\mathrm{NS}(X) = \mathbb{Z}$, any polarization is v-generic with respect to any Mukai vector v.

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Theorem 2.4.5. *Let S be a K3 surface, $v = (0, C, 1 - g + d)$ a primitive Mukai vector where C is a curve of genus $g \geq 2$ and H a v-generic polarization. Then $M_S^H(v)$ is an irreducible symplectic manifold of $K3^{[g]}$-type, called Beauville-Mukai integrable system. It is the relative compactified Jacobian of the complete linear system $|C|$, and the map $\mathcal{J}^d_{S,H}(|C|) \xrightarrow{\text{supp}} |C| = \mathbb{P}^g$ is a Lagrangian fibration by Matsushita theorem.*

In particular, if $|C|$ contains only integral curves, then $M_S^H(v)$ is smooth by Lemma 2.1.10 and Theorem 2.3.1 and v is primitive.

Example 2.4.6. If $|C|$ is base point free, we can easily check for $d = g$ (and hence for $d = kg$, $k \in \mathbb{Z}$) that it is of K3$^{[g]}$-type. Indeed we can define a birational map

$$S^{[g]} \dashrightarrow \mathcal{J}^g_{S,H}(|C|)$$
$$Z \to [Z] \in J(\bar{C})$$

where $\bar{C}$ is the unique curve in $|C|$ containing Z. $\mathcal{J}^g_{S,H}(|C|)$ is birational to $S^{[g]}$, so by a theorem of Huybrechts [23], they are also deformation equivalent.

Since a theorem by Burns, Hu, Luo [7] states that all the birational maps among hyperkähler 4-folds are composition of Mukai flops, for $g = 2$ we expect to describe more precisely the birational map $\mathcal{J}^2_{S,H}(|C|) \simeq S^{[2]}$. Indeed it is just the Mukai flop obtained blowing up $\mathcal{J}^2_{S,H}(|C|)$ along

$$\{K_{\bar{C}} : \bar{C} \in |C|\} \cong (\mathbb{P}^2)^*,$$

and then blowing down the exceptional locus

$$E = \{(x, l) \in \mathbb{P}^2 \times (\mathbb{P}^2)^* : x \in l\} \cong \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$$

along the other direction (see Example 4.1 [61]).

Beauville-Mukai integrable systems have a central role in hyperkähler geometry because all the known examples coming from K3 surfaces are deformation equivalent to them.

Moreover, it is conjectured that the Beauville-Mukai integrable systems represent all the possible Lagrangian fibrations on irreducible symplectic manifolds that are compactified Jacobians of families of curves. Markushevich proved this for $g = 2$ in [33] (and the same technique works in the case $g = 3$), and Sawon proved it for $g \leq 5$ in [60]:

Theorem 2.4.7. *Let $Y \to \mathbb{P}^g$ be a flat family of integral Gorestein curves of arithmetic genus $g \leq 5$, such that the relative compactified Jacobian $\mathcal{J}_Y \to \mathbb{P}^g$ is an irreducible symplectic manifold with a Lagrangian fibration. Then $\mathcal{J}_Y$ is a Beauville-Mukai integrable system.*

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In the case of an abelian surface, the moduli spaces of sheaves are not simply connected, but it is still possible to get irreducible symplectic varieties taking the fiber of a natural map, as in the case of Kummer surfaces (Example 1.1.5) or generalized Kummer varieties (Example 1.1.8).

Theorem 2.4.8. *[67] Let A be an abelian surface, $v = (0, C, 1 - g + d)$ a primitive Mukai vector, where C is a curve of genus $g \geq 6$, and let H be a v-generic polarization. Let $\mathcal{K}^d_{A,H}(C)$ be a fiber of the Albanese map*

$$\begin{aligned} Alb : &\mathcal{J}^d_{A,H}(C) \to A \times \hat{A} \\ &[\mathcal{F}] \mapsto \left(\sum \widetilde{c}_2(\mathcal{F}), \widetilde{c}_1(\mathcal{F}) \right), \end{aligned} \tag{2.4.4}$$

where $\widetilde{c}_i$ are the Chern classes with values in the Chow ring. Then $\mathcal{K}^d_{A,H}(C)$ is an irreducible symplectic manifold of Kummer-$(g-2)$ type. There is a natural commutative diagram

$$\begin{array}{ccccc} \mathcal{K}^d_{A,H}(C) & \longrightarrow & \mathcal{J}^d_{A,H}(C) & \longrightarrow & A \times \hat{A} \\ \big\downarrow \text{supp} & & \big\downarrow \text{supp} & & \big\downarrow \\ |C| = \mathbb{P}^{g-2} & \longrightarrow & \{C\} & \xrightarrow{det} & \hat{A} \end{array} \tag{2.4.5}$$

On the fibers over the smooth curves, it gives the exact sequence

$$K^d_A(C) \to J^d(C) \to A, \tag{2.4.6}$$

where $J^d(C) \to A$ is the group morphism induced by $C \hookrightarrow A$ via the universal property of the Jacobian, while $K^d_A(C)$ is the complementary abelian variety of A inside $J^d(C)$. Since $C^2 = 2g-2$, C induces a polarization of type $(1, g-1)$ on A. Via (2.4.6) $K^d_A(C)$ inherits a polarization of type $(1, ..., 1, g-1)$. $\mathcal{K}^d_{A,H}(C) \xrightarrow{\text{supp}} |C|$ is a Lagrangian fibration by Matsushita theorem.

It is natural to ask what happens if there are strictly semistable sheaves, i.e. if the moduli space admits singular points.

Theorem 2.4.9. *[68] Let H, H' be two polarizations on a symplectic surface X which lie in the closure of the same v-chamber. Then $\mathcal{J}^d_{X,H}(C)$ and $\mathcal{J}^d_{X,H'}(C)$ are birational (and if X is an abelian surface the same holds for $\mathcal{K}^d_{X,H}(C)$ and $\mathcal{K}^d_{X,H'}(C)$).*

Hence, for a primitive Mukai vector v with $v_0 = 0$ and v_1 effective, a non-v-generic polarization gives a singular irreducible symplectic variety which admits a symplectic resolution. Moreover, each chamber gives the same smooth birational model.

In the case of a non-primitive Mukai vector $v = kv_0$ with v_0 primitive and $k \geq 2$, O'Grady described a symplectic resolution when $k = 2$ and $v_0^2 = 2$.

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He considered two moduli spaces of sheaves of rank 2 in [50] and [51], which give O'Grady's examples of irreducible symplectic manifolds different from Beauville's examples. Lehn and Sorger in [30] proved that, for all the non-primitive Mukai vectors with $k = 2$ and $v_0^2 = 2$, the moduli spaces admit such a symplectic resolution. Then Perego and Rapagnetta showed in [53] that all the vectors with $k = 2$ and $v_0^2 = 2$ give a moduli space in the same deformation class. Kaledin, Lehn and Sorger in [27] showed that there does not exist any symplectic desingularization for $k \geq 2$ and $v_0^2 > 2$ or for $k > 2$ and $v_0^2 = 2$.

We state these result in the case of 1-dimensional sheaves:

Theorem 2.4.10. *Let X be a symplectic surface, v a non-primitive Mukai vector with $v_0 = 0$, v_1 the class of a curve C of genus g and $v_2 \neq 0$, H a polarization on X. Assume $M_S^H(v) = \mathcal{J}^d_{X,H}(C) \neq \emptyset$. Then*

- *i) If g=5, $\mathcal{J}^d_{X,H}(C)$ admits a symplectic resolution. If X is a K3 surface, $\mathcal{J}^d_{X,H}(C)$ is deformation equivalent to the 10-dimensional example by O'Grady. If X is an abelian surface, $\mathcal{K}^d_{X,H}(C)$ is deformation equivalent to the 6-dimensional example by O'Grady.*
- *ii) Otherwise $\mathcal{J}^d_{X,H}(C)$ has $\mathbb{Q}$-factorial and terminal singularities, hence it does not admit any symplectic resolution (and if X is an abelian surface, the same holds for $\mathcal{K}^d_{X,H}(C)$).*

Remark 2.4.11. In Theorem 2.4.10, for $d = 0$ the case *i)* is given by $C = 2C_0$, with C_0 primitive curve. Indeed by Lemma 2.4.3, v is primitive if and only if v_1 is primitive. Moreover, for a multiple of a primitive curve $C = kC_0$, by adjunction formula we get $8 = 2g - 2 = k^2(2g_0 - 2)$, hence the only possibility is $k = 2$.

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Chapter 3

Relative Prym varieties from K3 surfaces

In this chapter we define the relative compactified Prym variety associated to linear systems on K3 surfaces with an antisymplectic involution. We start by reviewing the notion of the Prym variety (Section 3.1). Then we describe the construction of the relative Prym variety (Section 3.2) introduced in the work of Markushevich and Tikhomirov [35], motivating the choice of the polarization and of the involution. We focus in particular on a local description of its simplest singularities, using the Kuranishi model (Section 3.3). After that, we present Nikulin's classification of the K3 surfaces admitting an antisymplectic involution, which gives us a list of all the possible relative Prym varieties to be considered (Section 3.4). We report briefly on the general results obtained by Arbarello, Saccà and Ferretti in [4], in the case of antisymplectic involutions without fixed points (Section 3.5). To conclude we discuss possible generalizations of the construction (Section 3.6).

3.1 Prym variety of a double cover of curves

In the theory of principally polarized abelian varieties (ppav's for short), the first interesting families are the Jacobian varieties of curves, endowed with the theta divisors. For genus $g > 3$, they do not represent a generic ppav, as the dimension of the moduli space of curves is $3g-3$ for $g \geq 2$, while the dimension of ppav's is $\frac{g(g+1)}{2}$. To obtain other families of ppav's, a classical idea is to consider the fixed locus of an involution on a Jacobian variety, and try to get a principal polarization on it.

Definition 3.1.1. *Let* $\tau \circlearrowright C \xrightarrow{\pi} C'$ *be a double cover of smooth curves* C, C' *of genus* g, g' *respectively, with Galois involution* τ*. The Prym variety* $P(C,\tau)$ *is the connected component* $\mathrm{Fix}^0(-\tau^*) \subset J(C)$ *of the fixed locus of* $-\tau^*$ *containing zero. It is the abelian subvariety of* $J(C)$ *of dimension* $g - g'$

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complementary to $\pi^*(J(C'))$. *It inherits a natural polarization from* $J(C)$, *the restriction of* Θ_C.

One can also define $P(C,\tau)$ as the identity component of the kernel of the norm map

$$\begin{aligned} N : & J(C) \to J(C') \\ \sum a_i p_i & \mapsto \sum a_i \pi_*(p_i). \end{aligned} \tag{3.1.1}$$

There is also a simple topological description of the Prym variety.

Theorem 3.1.2. *Let* $\tau \circlearrowright C \xrightarrow{\pi} C'$ *be a double cover of smooth curves. Then* $P(C,\tau) = H^0(\Omega_C)^{-*}/H_1(C,\mathbb{Z})^-$, *where* $H^0(\Omega_C)^-$ *and* $H_1(C,\mathbb{Z})^-$ *are the* (-1)*-eigenspaces of* $H^0(\Omega_C)$ *and* $H_1(C,\mathbb{Z})$.

From the definition of Prym variety, we get that it is trivial if C is hyperelliptic and τ is a hyperelliptic involution. Moreover:

Lemma 3.1.3. *If* C *is a hyperelliptic curve with hyperelliptic involution* ι *and* τ *is another involution on* C, *then* $P(C,\tau) \cong J(\bar{C})$, *where* $\bar{C} = C/(\iota\circ\tau)$.

Proof. The hyperelliptic involution ι on C induces the -1 involution on $J(C)$. Hence $P(C,\tau) = \mathrm{Fix}^0(-\tau^*) = \mu^* J(\bar{C})$, where $\mu : C \to \bar{C}$ is the morphism induced by $\iota \circ \tau$. □

Mumford in [44] determined all the cases where one can obtain a natural principal polarization on $P(C,\tau)$:

Theorem 3.1.4. *If* $\tau \circlearrowright C \xrightarrow{\pi} C'$ *is a double cover of smooth curves, then* $\Theta_C|_{P(C,\tau)} = 2\Theta_P$ *for a principal polarization* Θ_P *if and only if* π *has at most 2 branch points.*

3.2 Relative Prym variety $\mathcal{P}_C$ of $|C|^\tau_S$ on (S,τ)

Let $\tau \circlearrowright S \xrightarrow{\pi} Y$ be a double cover with S a K3 surface, and H a polarization on S.

Let C be a τ-invariant smooth curve on S and $\tau \circlearrowright C \xrightarrow{\pi} C'$ the induced morphism, where C' is a smooth curve on Y.

Let U be a connected component of the subset of smooth τ-invariant curves of $|C|$. Let $\mathcal{J}_S(U) \to U$ be the relative Jacobian of degree 0 of U, which does not depend on the polarization because all the curves are smooth.

We can define naturally a relative version of the Prym variety over U as $\mathcal{P}_S(U) \to U$, which is a fibration in Prym varieties with $P(C,\tau)$ as the fiber over C. Since

$$(-1) = \mathcal{H}om_C(\_,\mathcal{O}_C) = \mathcal{H}om_S(\_,\mathcal{O}_C)$$

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defines fiberwise a regular involution on $\mathcal{J}_S(U)$, we can consider $-\tau^*$ as an involution on $\mathcal{J}_S(U)$. Hence we define

$$\mathcal{P}_S(U) := \mathrm{Fix}(-\tau^*)^0 \subset \mathcal{J}_S(U)$$

as the connected component of the fixed locus of $-\tau^*$ containing the zero section.

The basic idea of Markushevich and Tikhomirov, suggested in [35], is that if $-\tau^*$ is a symplectic involution, i.e. it preserves the symplectic form ($(-\tau^*)(\sigma) = \sigma$), then $\mathcal{P}_S(U)$ inherits a symplectic structure from $\mathcal{J}_S(U)$. If moreover $-\tau^*$ can be extended to a regular involution on all $\mathcal{J}_{S,H}(C)$, then the same construction works for the fixed locus of the extended involution, and it gives a projective symplectic variety.

Lemma 3.2.1. (-1) *is an antisymplectic involution of* $\mathcal{J}_S(U)$*, i.e.* $(-1)^*(\sigma) = -\sigma$*, where* σ *is the symplectic form on* $\mathcal{J}_S(U)$*.*

Proof. Let $C_0 \in U$ and $J := J(C_0)$. Since $\mathrm{supp} : \mathcal{J}_S(U) \to U$ is a Lagrangian fibration, the isomorphism $\mathcal{T}_{\mathcal{J}} \cong \Omega_{\mathcal{J}}$ induced by the symplectic form σ, gives an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}_J & \longrightarrow & \mathcal{T}_{\mathcal{J}|J} & \longrightarrow & \mathcal{N}_{J/\mathcal{J}} & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & \mathcal{N}^*_{J/\mathcal{J}} & \longrightarrow & \Omega_{\mathcal{J}|J} & \longrightarrow & \Omega_J & \longrightarrow & 0. \end{array}$$

In particular, for a point $p \in J$, we have the isomorphism

$$\mathcal{T}_{C_0}|C| \cong (\mathcal{N}_{J/\mathcal{J}})_p \cong (\mathcal{T}_p J)^*. \tag{3.2.1}$$

Since the second isomorphism in (3.2.1) is given by σ and (-1) acts as the identity on $\mathcal{T}_{C_0}|C|$ and as -1 on $\mathcal{T}_p J$, we have that $(-1)^*(\sigma) = -\sigma$. □

Thus in order to get a symplectic involution, we need to determine when τ^* is antisymplectic.

Lemma 3.2.2. *Let* τ *be an involution on a K3 surface* S *and* τ^* *the induced involution on* $\mathcal{J}_S(U)$*. Then* τ^* *is (anti)symplectic if and only if* τ *is (anti)symplectic.*

Proof. In the definition of the symplectic structure (2.4.1), all the identifications are intrinsic except for the last one

$$H^2(\mathcal{F}) \xrightarrow{\cdot/\sigma} \mathbb{C}, \tag{3.2.2}$$

so the action of τ^* on $H^{(2,0)}(\mathcal{J})$ is the same of the action of τ on $H^{(2,0)}(S)$. □

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It remains to understand when $-\tau^*$ can be extended to a regular involution on $\mathcal{J}_{S,H}(C)$.

τ^* naturally extends to an involution on the sheaves supported on τ-invariant curves. The only issue is related to the H-semistability.

Lemma 3.2.3. *Let τ be an involution on a K3 surface S, H a τ-invariant polarization and τ^* the induced involution on $\mathcal{J}_S(U)$. Then τ^* is a regular involution on $\mathcal{J}_{S,H}(C)$.*

Proof. Since by Lemma 2.1.10 a sheaf supported on an irreducible curve is stable with respect to any polarization, τ^* is always defined as a rational involution. Moreover, if $\mathcal{F}$ is an H-(semi)stable sheaf then $\tau^*(\mathcal{F})$ is a $\tau^*(H)$-(semi)stable sheaf. Hence the assertion. □

The involution (-1) is a little more complicated to treat. Indeed, it does not behave well in families, as $\chi(\mathcal{F})$ and $\chi(\mathcal{H}om_S(\mathcal{F},\mathcal{O}_C))$ can jump when $\mathcal{F}$ becomes non-locally-free. Luckily, (-1) admits a natural extension j which commutes with base change in flat families.

Lemma 3.2.4. *Let $\mathcal{F}$ be a pure 1-dimensional sheaf on S such that* $\text{supp}(\mathcal{F})$ *is a curve C. Define*

$$j(\mathcal{F}) := \mathcal{E}xt^1_S(\mathcal{F}, \mathcal{O}_S(-C)). \tag{3.2.3}$$

Then $j(\mathcal{F}) \cong \mathcal{H}om_S(\mathcal{F}, \mathcal{O}_C)$, whenever $\mathcal{F}$ is locally free as $\mathcal{O}_C$-module.

Proof. Let $\{V_i\}$ be an open covering of S such that local isomorphisms $\mathcal{F}|_{V_i} \cong \mathcal{O}_C|_{V_i}$ hold. Applying the functor $\mathcal{H}om_S(\_, \mathcal{O}_S(-C))$ to the short exact sequence

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0,$$

we get, from the definition of $\mathcal{E}xt^1$, the canonical isomorphisms

$$\mathcal{H}om_S(\mathcal{O}_C|_{V_i}, \mathcal{O}_C|_{V_i}) = \mathcal{E}xt^1_S(\mathcal{O}_C|_{V_i}, \mathcal{O}_S(-C)|_{V_i}).$$

We conclude by gluing together these isomorphisms. □

Lemma 3.2.5. *Let $\mathcal{E}$ be a flat family of pure 1-dimensional sheaves on S parametrized by B, and $p : S \times B \to S$ the natural projection. Then $\mathcal{E}xt^1(\mathcal{E}, p^*\mathcal{O}_S)$ is a flat family of pure 1-dimensional sheaves on S parametrized by B, and for every $b \in B$, there is an isomorphism $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{S\times B})_b \cong \mathcal{E}xt^1(\mathcal{E}_b, \mathcal{O}_S)$.*

Proof. Theorem 1.10 [3] □

Lemma 3.2.6. *[4] Let $C = C_1 \cup C_2$ be a curve on S, such that C_i are smooth irreducible τ-invariant curves meeting transversely, and let $\mathcal{F}$ be a sheaf on C. Then, in the notation of Lemma 2.1.6,*

$$(j(\mathcal{F}))_i = j(\mathcal{F}_i \otimes \mathcal{O}(-\Delta_F)) \text{ for } i = 1, 2, \tag{3.2.4}$$

where $\Delta_{\mathcal{F}} \subset C_1 \cap C_2$ is the set of nodes in which $\mathcal{F}$ is locally free.

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Proof. For $i \neq k$ we have the short exact sequence

$$0 \to \mathcal{F}^i \to \mathcal{F} \to \mathcal{F}_k \to 0.$$

By Lemma 2.1.6,

$$\mathcal{F}^i \cong \mathcal{F}_i \otimes \mathcal{O}(-\Delta_{\mathcal{F}}).$$

Applying j, we obtain

$$0 \to j(\mathcal{F}_k) \to j(\mathcal{F}) \xrightarrow{f} j(\mathcal{F}_i \otimes \mathcal{O}(-\Delta_{\mathcal{F}})) \to 0.$$

As $j(\mathcal{F}_i \otimes \mathcal{O}(-\Delta_{\mathcal{F}}))$ is torsion free and supported on C_i, f factors via a surjective map

$$(j(\mathcal{F}))_i \to j(\mathcal{F}_i \otimes \mathcal{O}(-\Delta_{\mathcal{F}})).$$

However, this map is also injective, because it is generically an isomorphism, and the assertion follows. □

Lemma 3.2.7. *j preserves the topological invariants of pure 1-dimensional sheaves with Mukai vector $(0, C, 1-g)$.*

Proof. i) $c_1(\mathcal{F}) = c_1(j(\mathcal{F}))$ for any 1-dimensional sheaf $\mathcal{F}$.
Given a locally free resolution of minimal length of $\mathcal{F}$

$$0 \longrightarrow \mathcal{L}_1 \xrightarrow{f} \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

$c_1(\mathcal{F})$ is the cohomology class of the curve defined by the equation $det(f) = 0$.
Applying $\mathcal{E}xt^1_S(\_, \mathcal{O}_S)$, we get

$$0 \longrightarrow \mathcal{L}_0^* \xrightarrow{f^*} \mathcal{L}_1^* \longrightarrow \mathcal{E}xt^1_S(\mathcal{F}, \mathcal{O}_S) \longrightarrow 0.$$

So $(det(f) = 0)$ and $(det(f^*) = 0)$ define the same subscheme of S. But $c_1(\mathcal{E}xt^1_S(\mathcal{F}, \mathcal{O}_S)) = c_1(j(\mathcal{F}))$, because tensoring by a line bundle does not change the first Chern class of a 1-dimensional sheaf.

ii) $\chi(\mathcal{F}) = \chi(j(\mathcal{F}))$ for any 1-dimensional sheaf $\mathcal{F}$ supported on C.
Since $\mathcal{E}xt^1_S(\mathcal{F}, \mathcal{O}_S(-C)) = \mathcal{F}^* \otimes \mathcal{O}_S(-C) \otimes \mathcal{N}_{C/S}$ and $\mathcal{N}_{C/S} = \mathcal{O}_S(C)|_C = \omega_C$, we get, using Hirzebruch-Riemann-Roch theorem,

$$\chi(j(\mathcal{F})) = -\chi(\mathcal{F} \otimes \mathcal{O}_S(C) \otimes \omega_C^*) = -\chi(\mathcal{F}) - C \cdot c_1(\mathcal{F}). \tag{3.2.5}$$

As $c_1(\mathcal{F}) = C$, we conclude. □

Lemma 3.2.8. *If $H = kC$, then j respects the H-stability.*

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Proof. Applying j to the exact sequence $0 \to \mathcal{G} \to \mathcal{F}$, we obtain $j(\mathcal{F}) \to j(\mathcal{G}) \to 0$. Hence there is a 1-1 correspondence between subsheaves of $\mathcal{F}$ and quotient sheaves of $j(\mathcal{F})$. Moreover, by $i)$ of Lemma 3.2.7 and by (3.2.5),

$$\mu_H(j(\mathcal{G})) := \frac{\chi(j(\mathcal{G}))}{c_1(j(\mathcal{G})) \cdot H} = -\mu_H(\mathcal{G}) - \frac{c_1(\mathcal{G}) \cdot C}{c_1(\mathcal{G}) \cdot H}. \tag{3.2.6}$$

Thus $\mu_H(j(\mathcal{F})) \leq \mu_H(j(\mathcal{G}))$ is equivalent to $\mu_H(\mathcal{G}) \leq \mu_H(\mathcal{F})$ if $H = kC$. □

We remark that for other polarizations, (3.2.6) depends on the subsheaves $\mathcal{G}$, as their supports can intersect with a different proportion H and C. We analyze this aspect in Section 3.5.

We are now ready to introduce the main object studied in this thesis:

Definition 3.2.9. *Let (S, τ) be a K3 surface with an antisymplectic involution. Let C be a smooth irreducible ample τ-invariant curve and $\mathcal{J}_C := \mathcal{J}_{S,C}(C)$. Then $j := \mathcal{E}xt^1_S(\_, \mathcal{O}_S(-C))$ and τ^* are regular antisymplectic involutions on $\mathcal{J}_C$, and*

$$\eta := j \circ \tau^* \circlearrowleft \mathcal{J}_C$$

is a symplectic involution since j and τ^* commute.
The fixed locus $|C|^\tau$ has at most two connected components, both of which are projective spaces. Let $|C|^{\tau,+}$ be one of them, such that its generic member represents an irreducible curve. Denote by $\mathcal{J}_C^+$ the restriction of $\mathcal{J}_C$ over $|C|^{\tau,+}$.

Then the relative Prym variety associated to C is

$$\mathcal{P}_C := \mathrm{Fix}^0(\eta) \subset \mathcal{J}_C,$$

the connected component of the fixed locus containing the zero section of $\mathcal{J}_C^+ \to |C|^{\tau,+}$. It is a symplectic variety endowed with the natural Lagrangian fibration

$$\mathrm{supp} : \mathcal{P}_C \to |C|^{\tau,+}.$$

Remark 3.2.10. When τ has no fixed points, then the irreducible components of $|C|^\tau$ are of the same dimension, and any one of them can be chosen to be $|C|^{\tau,+}$. Denoting the other by $|C|^{\tau,-}$, we obtain two different relative compactified Prymians with their Lagrangian fibration maps $\mathcal{P}_C^\pm \to |C|^{\tau,\pm}$. We will consider only one of them, $\mathcal{P}_C^+$, but all the results we state for it hold as well for $\mathcal{P}_C^-$.

When τ has a fixed curve D, then only one of the components of $|C|^\tau$ is a base point free linear system, and the other component (when it exists) is a linear system with base locus D. In this case our result apply only to the first component, and this is the component we mark by the plus sign.

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3.3 Singularities of $\mathcal{P}_C$

Using the regular version η of $-\tau^*$, it is possible to define the Prym variety of a reducible curve.

Lemma 3.3.1. *Let $C = C_1 \cup C_2$ be a curve of arithmetic genus g, which is a double cover of a curve C' of genus g', where C_i $(i = 1, 2)$ are smooth irreducible τ-invariant curves meeting transversely in $2\delta \geq 2$ points. Then $\bar{P}_C(C, \tau)$ has one irreducible component of dimension $g - g'$, whose generic points parametrize stable sheaves, and the locus of strictly C-semistable sheaves is of dimension $g - g' - \delta$, representing the $\mathcal{S}$-equivalence classes $[\mathcal{F}_1 \oplus \mathcal{F}_2]$ with* $\mathrm{supp}(\mathcal{F}_i) = C_i$.

Proof. We consider only locally free sheaves on C as they are dense in the compactified Jacobian (see [63]). Since $\frac{C \cdot C_1}{C^2}\chi = -g_1 + 1 - \delta \in \mathbb{Z}$ and $C_1 \cdot C_2 = 2\delta$, by (2.2.1) of Lemma 2.2.13 restricted to invertible sheaves on C

$$J^0(C) = \coprod_{d_1 = -\delta, \ldots, \delta} J^{d_1, -d_1}(C),$$

where $J^{\delta,-\delta}(C)$ and $J^{-\delta,\delta}(C)$ are identified. Hence $\bar{J}^0_C(C)$ has $2\delta - 1$ irreducible components of dimension g, whose generic points parametrize stable sheaves, and a locus of dimension $g - g' - 2\delta$ parametrizing strictly semistable sheaves. As C_i are τ-invariant smooth curves, η acts as $-\tau$ on them, so

$$\eta : J^{d_1,-d_1}(C) \to J^{-d_1,d_1}(C). \tag{3.3.1}$$

Thus the fixed locus of η lies only on two strata, $J^{0,0}(C)$ and $J^{\delta,-\delta}(C)$. Consequently, $\bar{P}(C, \tau)$ has only one component of dimension $g - g'$ containing stable sheaves, and a stratum of dimension $g_1 - g'_1 + g_2 - g'_2 = g - \delta$ containing polystable sheaves of type $[\mathcal{F}_1 \oplus \mathcal{F}_2]$ with $\mathrm{supp}(\mathcal{F}_i) = C_i$. □

Let (S, τ) be a generic K3 surface with an antisymplectic involution, C a smooth τ-invariant primitive curve. If $|C|_S^{\tau,+}$ contains a reducible curve $C_1 \cup C_2$ with smooth irreducible τ-invariant curves C_i $(i = 1, 2)$ meeting transversely, then by Lemma 3.3.1, $\mathcal{J}_C$ contains a strictly C-semistable sheaf. In particular this is the case if the linear system $|C'|$ on Y contains a reducible curve $C'_1 \cup C'_2$ such that C'_i $(i = 1, 2)$ are smooth irreducible curves meeting transversely. Hence $\mathcal{J}_C$ is a singular irreducible symplectic variety .

Since C is primitive, by Lemma 2.4.3 and Theorem 2.4.9, the singularities are due to a bad choice of the polarization, and changing it one obtains a symplectic resolution of $\mathcal{J}_C$.

Hence it is reasonable to expect that also $\mathcal{P}_C$ is singular. We check this in the case of the simplest η-invariant polystable non-stable sheaf, and we explicitly describe the type of singularity, following the method used by Arbarello, Saccà and Ferretti in Proposition 4.4 and Proposition 5.1 of [4].

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Theorem 3.3.2. *Let (S,τ) be a generic K3 surface with an antisymplectic involution, C a smooth τ-invariant curve. Let $\mathcal{F}=\mathcal{F}_1\oplus\mathcal{F}_2\in\mathcal{P}_C$ be a polystable sheaf, where* $\mathrm{supp}(\mathcal{F}_i)=: C_i$ *$(i=1,2)$ are smooth irreducible τ-invariant curves of genera g_i meeting transversely at 2δ points. Then $(\mathcal{P}_C,[\mathcal{F}])$ is locally analytically equivalent to $(\mathbb{C}^N\times(\mathbb{C}^{2\delta}/\pm 1),0)$, where $N=2(g_1-g_1'+g_2-g_2')$.*

Proof. At the infinitesimal level, η induces an involution η^* on $\mathrm{Ext}^1_S(\mathcal{F},\mathcal{F})$. By the definition of $\mathcal{P}_C$ and by Theorem 2.3.4, there is a natural sequence of inclusions

$$C_{[\mathcal{F}]}(\mathcal{P}_C)=C_{[\mathcal{F}]}(\mathcal{J}_C^{\eta})\subset C_{[\mathcal{F}]}(\mathcal{J}_C)^{\eta^*}\subset(\mathrm{Ext}^1_S(\mathcal{F},\mathcal{F})//G)^{\eta^*}. \tag{3.3.2}$$

We prove that this is actually a sequence of identities.

We use the same notation as in (2.3.2) of Lemma 2.3.5. By (2.3.7), since C_i is τ-invariant for $i=1,2$, we get

$$(\mathrm{Ext}^1_S(\mathcal{F},\mathcal{F})//G)^{\eta^*}=U_1^{\eta^*}\times U_2^{\eta^*}\times((W\times W^*)//G)^{\eta^*}. \tag{3.3.3}$$

Considering $\mathcal{P}_{C_i}\to|C_i|^{\tau,+}$, we get the natural identification

$$U_i^{\eta^*}=C_{[\mathcal{F}_i]}(\mathcal{P}_{C_i})=T_{[\mathcal{F}_i]}(\mathcal{P}_{C_i}), \tag{3.3.4}$$

because $\mathcal{F}_i$ is a stable and hence smooth point of $\mathcal{P}_{C_i}$. It remains to study the last term of (3.3.3). Let $x_1,...,x_{2\delta}$ be coordinates in W such that $x_{2i}\leftrightarrow x_{2i-1}$ for $i=1,...,\delta$, so that the corresponding points $P_1,...,P_{2\delta}$ of $C_1\cap C_2$ are interchanged in pairs $P_{2i}\leftrightarrow P_{2i-1}$ for $i=1,...,\delta$. Let $y_1,...,y_{2\delta}$ be the dual coordinates in W^*. The involution j exchanges $x_i\leftrightarrow y_i$, thus the action of η^* on $W\times W^*$ is

$$x_{2i}\leftrightarrow y_{2i-1}, y_{2i}\leftrightarrow x_{2i-1}. \tag{3.3.5}$$

Its fixed locus is then

$$x_{2i}=y_{2i-1}, x_{2i-1}=y_{2i}. \tag{3.3.6}$$

Since

$$\eta^*(\lambda\cdot(\underline{x},\underline{y}))=\frac{1}{\lambda}\eta^*(\underline{x},\underline{y}), \tag{3.3.7}$$

η^* induces a well-defined involution on $(W\times W^*)//G$. From (3.3.7), we also see that η^* is not G-invariant, hence its fixed locus on the quotient cannot be described as the quotient of its fixed locus. To characterize it, we observe that in the G-invariant coordinates $u_{ij}=x_iy_j$ (2.3.5), these conditions become

$$u_{2i,2j}=u_{2j-1,2i-1}\text{ for } i,j, \tag{3.3.8}$$

$$u_{2i,2j-1}=u_{2j,2i-1}\text{ and } u_{2i-1,2j}=u_{2j-1,2i}\text{ for } i<j. \tag{3.3.9}$$

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The quadratic relations (2.3.6) in u_{ij}, combined with these ones, give the equations of the image of the Veronese embedding $v_2 : \mathbb{P}^{2\delta-1} \to \mathbb{P}^{\binom{2\delta+1}{2}-1}$. Thus

$$((W \times W^*)//G)^{\eta^*} = \widehat{Im(v_2)} \tag{3.3.10}$$

where $\widehat{Im(v_2)}$ is the affine cone over $Im(v_2)$. Moreover

$$\widehat{Im(v_2)} = \mathbb{C}^{2\delta}/\pm 1. \tag{3.3.11}$$

Indeed, choosing coordinates $w_1, ..., w_{2\delta}$ on $\mathbb{C}^{2\delta}$ with the action of -1 given by $w_i \mapsto -w_i$, the algebra of invariant functions has generators

$$v_{ij} := w_i w_j \text{ for } i \leq j,$$

and relations

$$v_{ij}v_{kl} = v_{kj}v_{il},$$

which describe exactly $\widehat{Im(v_2)}$.

Thus combining (3.3.3), (3.3.10) and (3.3.11), we obtain

$$(\mathrm{Ext}^1_S(\mathcal{F},\mathcal{F})//G)^{\eta^*} = U_1^\tau \times U_2^\tau \times \mathbb{C}^{2\delta}/\pm 1. \tag{3.3.12}$$

Since it is irreducible and it has dimension

$$\begin{aligned}
\dim((\mathrm{Ext}^1_S(\mathcal{F},\mathcal{F})//G)^{\eta^*}) &= \dim U_1^\tau + \dim U_2^\tau + 2\delta = \\
&= 2\dim P(C_1,\tau) + 2\dim P(C_2,\tau) + 2\delta = \\
&= 2(g_1 - g_1') + 2(g_2 - g_2') + 2\delta = \\
&= 2(g - g') = \\
&= \dim C_{[\mathcal{F}]}(\mathcal{P}_C),
\end{aligned}$$

we conclude

$$C_{[\mathcal{F}]}(\mathcal{P}) = U_1^\tau \times U_2^\tau \times (\mathbb{C}^{2\delta}/\pm 1). \tag{3.3.13}$$

As in the case of $\mathcal{J}$, $(C_{[\mathcal{F}]}(\mathcal{P}), [\mathcal{F}])$ is a local analytic model of $(\mathcal{P}, [\mathcal{F}])$ (see Proposition 5.1 [4]), and we conclude the proof. □

Corollary 3.3.3. *Let (S,τ) be a generic K3 surface with an antisymplectic involution, C a smooth τ-invariant curve. Suppose that $|C|_S^{\tau,+}$ contains a reducible curve $C_1 \cup C_2$, where C_i are smooth irreducible τ-invariant curves of genera g_i meeting transversely at $2\delta \geq 2$ points. Then the singular locus of $\mathcal{P}_C$ contains an irreducible component of dimension $2(g_1 - g_1' + g_2 - g_2')$ given by*

$$\mathcal{P}_{C_1} \times \mathcal{P}_{C_2} \to |C_1|^{\tau,+} \times |C_2|^{\tau,+}.$$

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Proof. By Lemma 3.3.1, the fiber of $\mathcal{P}_C \to |C|^{\tau,+}$ over $C_1 \cup C_2$ has an irreducible component of strictly semistable sheaves of type $[\mathcal{F}_1 \oplus \mathcal{F}_2]$, which admits a natural fibration over $|C_1|^{\tau,+} \times |C_2|^{\tau,+}$. By Theorem 3.3.2, these polystable sheaves are singular points, in which the local dimension of the singular locus is $2(g_1 - g_1' + g_2 - g_2')$. □

Unfortunately, the procedure which gives a symplectic resolution of $\mathcal{J}_C$ does not fit to $\mathcal{P}_C$. Indeed changing the polarization we lose the regularity of η, which is only a rational involution in general. Hence the natural problem of understand if $\mathcal{P}_C$ admits a symplectic resolution arises.

Corollary 3.3.4. *Let (S, τ) be a generic K3 surface with an antisymplectic involution, C a smooth τ-invariant curve. Suppose that $|C|^{\tau,+}$ contains a reducible curve $C_1 \cup C_2$, where C_i are smooth irreducible τ-invariant curves of genus g_i meeting transversely at $2\delta \geq 4$ points. Then $\mathcal{P}_C$ does not admit any symplectic desingularization.*

Proof. By Lemma 3.3.1, there exists a polystable sheaf $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ supported on $C_1 \cup C_2$. By Theorem 3.3.2, $(\mathcal{P}_C, \mathcal{F})$ is locally analytically isomorphic to $(\mathbb{C}^N \times (\mathbb{C}^{2\delta}/\pm 1), 0)$. The singularity of $\mathbb{C}^{2\delta}/\pm 1$ is clearly $\mathbb{Q}$-factorial, so it has no small resolutions. Moreover for $\delta \geq 2$, it is also terminal (see [41]), that is the canonical sheaf of any resolution of singularities contains all the exceptional divisors with strictly positive coefficients. Thus, as in Theorem 2.4.10, we conclude that none of the resolutions has trivial canonical class, and hence none of them is symplectic. □

Remark 3.3.5. We point out that if $\delta = 1$, then the singularity of $\mathcal{P}_C$ at a polystable $\mathcal{F}$ as before is $\mathbb{C}^2/\pm 1$, which is an A-D-E singularity of A_1-type. The A-D-E singularities admit a symplectic desingularization.

3.4 K3 surfaces (S, τ) with antisymplectic involutions

Nikulin developed a general theory on K3 surfaces with an antisymplectic involution in [47], [48] and [49]. We remark that, by one of his results in [47], the existence of an antisymplectic involution on a Kähler K3 surface implies that it is projective. So the previous construction works only in the projective setting. We briefly review some facts.

Let (S, τ) be a generic K3 surface with an antisymplectic involution and $Y = S/\tau$ the quotient surface. There are essentially two cases:

- if $\mathrm{Fix}(\tau) = \emptyset$, then Y is smooth and, by the classification of surfaces, Y is an Enriques surface;

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- if $\mathrm{Fix}(\tau) \neq \emptyset$, then Y is again smooth because the differential of τ is non-degenerate at any fixed point, and, by the classification of surfaces, is a rational surface.

We focus on the second case. While in the first case Y is already a minimal model (i.e. it does not contain any (-1)-curve), this is not necessarily true in the second one.
Let Y' be a minimal model of Y (which is not unique because Y is rational). Let B be the branch curve of the double cover and B' its image in Y'. While B is smooth, B' acquires singularities if $Y \neq Y'$. We can then consider the double cover S' of Y' with B' as branch locus. S' is a singular K3 surface with S as symplectic resolution. So S' can have only A-D-E singularities. It is possible to determine the exceptional curves of Y using the Dynkin diagrams of the A-D-E singularities (see Section 4.2 of [31]), but we omit this description. We get the following commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & Y \supset B \\ \downarrow & & \downarrow \\ S' & \longrightarrow & Y' \supset B'. \end{array} \tag{3.4.1}$$

Since $\mathcal{K}_S = \mathcal{O}_S$, by Hurwitz formula $B \in |-2K_Y|$, and analogously $B' \in |-2K_{Y'}|$. Moreover B' must be reduced. The only possibilities are $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4$. Recall that $\mathrm{Pic}(\mathbb{F}_n)$ is generated by the class of a fiber f and the class of a section s_n, and the intersection matrix is $\begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix}$. Moreover $\mathcal{K}_{\mathbb{F}_n} = -2s_n - (n+2)f$. Hence we can easily describe $|-2K_{Y'}|$:

- if $Y' = \mathbb{P}^2$, then B' is a sextic;
- if $Y' = \mathbb{P}^1 \times \mathbb{P}^1$, then B' is a curve of bidegree $(4,4)$;
- if $Y' = \mathbb{F}_2$, then $B' \in |8f + 4s_2|$;
- if $Y' = \mathbb{F}_3$, then $B' = B_1 + s_3$ with $B_1 \in |10f + 3s_3|$;
- if $Y' = \mathbb{F}_4$, then $B' = B_1 + s_4$ with $B_1 \in |12f + 3s_4|$.

Since the Picard number of S is given by the Picard number of S' plus the number of exceptional divisors of the blowups resolving the singularities, we obtain clearly finitely many cases. Moreover, since the minimal model of a ruled surface is not unique, we can obtain the same (S,τ) from different (Y', B').

To obtain a classification, we can study this problem using lattice theory. Let $L_\pm = L_\pm(S,\tau) \subset H^2(S,\mathbb{Z})$ be the lattice of cohomology classes l such that $\tau^* l = \pm l$. Clearly L_+ is the orthogonal complement of L_-. Denoting by r the rank of L_+, L_+ and L_- have signature $(1, r-1)$ and $(2, 20-$

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r) respectively. The discriminant form of $L_\pm$ is the finite quadratic form $(D_{L_\pm}, q_\pm)$, where $L_\pm^\vee$ is the dual lattice of $L_\pm$, $D_{L_\pm} := L_\pm^\vee / L_\pm$ and $q_\pm : D_{L_\pm} \to \mathbb{Q}/2\mathbb{Z}$ is induced by the quadratic form on $L_\pm^\vee$. We have $D_{L_+} \simeq (\mathbb{Z}/2\mathbb{Z})^a$ for some $a \in \mathbb{N}$. The parity δ of q_+ is defined by $\delta = 0$ if $q_+(L_+) \subset \mathbb{Z}$, and $\delta = 1$ otherwise. The triple (r, a, δ) is called the main invariant of (S, τ). By [48], the isometry class of $L_\pm$ is uniquely determined by (r, a, δ).

We can easily relate the main invariant and the pair (Y, B):

Theorem 3.4.1 (Nikulin). *Let (S, τ) be a generic K3 surface with an anti-symplectic involution with main invariant (r, a, δ).*

i) If $(r, a, \delta) = (10, 10, 0)$, then $\mathrm{Fix}(\tau) = \emptyset$.

ii) If $(r, a, \delta) = (10, 8, 0)$, then $\mathrm{Fix}(\tau)$ is a union of two elliptic curves.

iii) Otherwise $\mathrm{Fix}(\tau)$ decomposes as $C \sqcup E_1 \sqcup \cdots \sqcup E_k$ such that C is a genus g curve and $E_1, \cdots, E_k$ are rational curves with

$$g = 11 - \frac{r+a}{2}, \qquad k = \frac{r-a}{2}. \tag{3.4.2}$$

Moreover $\delta = 0$ if and only if the class of $\mathrm{Fix}(\tau)$ is divisible by 2 in $L_+(S, \tau)$.

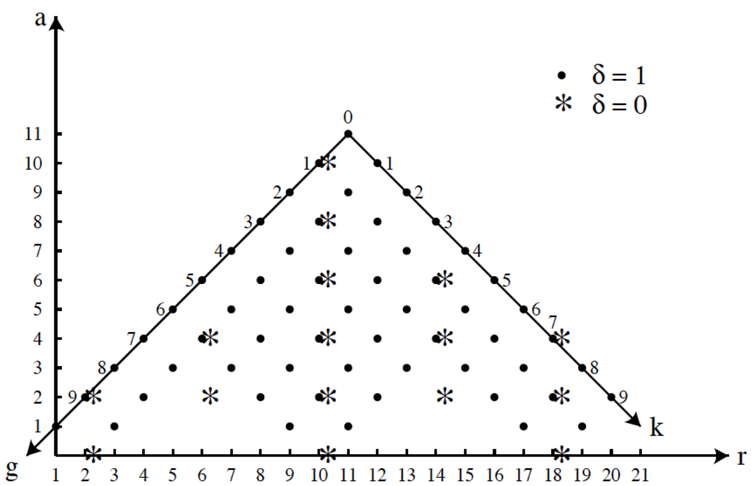

Figure 3.1: Main invariants (r, a, δ)

Theorem 3.4.2 (Nikulin). *The deformation type of (S, τ) is determined by its main invariant (r, a, δ). All the possible main invariants are shown in Figure 3.1.*

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In total there are 75 cases, hence 75 families of relative Prym varieties, associated to primitive curves on each one of (S, τ). In the next section we present the case of Enriques surfaces, following Arbarello, Saccà and Ferretti [4]. In the next chapter we analyze the case of some explicit examples of small dimension when Y is a Del Pezzo surface.

3.5 Relative Prym varieties from Enriques surfaces

Arbarello, Saccà and Ferretti treated the case of an involution without fixed points in [4]. This case is particularly interesting because the fibers of $\mathcal{P}_C$ are principally polarized as the double cover is étale (Theorem 5.5.4). Moreover, the reducible members of $|C|^\tau$ correspond exactly to the reducible members of $|C'|$, again because the double cover is étale. Thus, in order to study the singularities of $\mathcal{P}_C$, it suffices to determine the η-invariant strictly C-semistable sheaves, which are supported on the double covers of the reducible members of $|C'|$.

We briefly sum up the results of [4], from which I took inspiration to obtain the results of Chapter 5.

Let (S, τ) be a generic K3 surface with an antisymplectic involution without fixed points. Let $\tau \circlearrowright S \xrightarrow{\pi} Y$ be the corresponding double cover with Galois involution τ, where Y is a generic Enriques surface.

By Theorem 3.4.1, it has the main invariant $(10, 10, 0)$. Indeed, by Proposition 2.3 of [45], $L_+(S, \tau) = \pi^* H^2(Y, \mathbb{Z})$, and $H^2(Y, \mathbb{Z})$ has rank 10, so we have $r = a = 10$.

In order to obtain a complete classification of $\mathcal{P}_C$, we need to recall classical results about K3 surfaces and Enriques surfaces, due to Saint-Donat [58] and Cossec-Dolgachev [14].

Definition 3.5.1. *A hyperelliptic linear system on a K3 surface S (respectively on an Enriques surface Y) is a linear system $|C|_S$ (respectively $|C|_Y$) such that either $C^2 = 2$ or the associated morphism ϕ_C is of degree 2 onto a rational normal scroll of degree $n-1$ in $\mathbb{P}^n$.*

Proposition 3.5.2. *[14] Let Y be an Enriques surface and C' an irreducible curve on Y with $C'^2 \geq 2$. Then the following facts are equivalents:*

i) *$|C'|$ is hyperelliptic;*

ii) *$|C'|$ has base points;*

iii) *there exists a primitive elliptic curve e_1 such that $e_1 \cdot C' = 1$.*

Suppose furthermore that Y is generic. If one of the above conditions is satisfied, then there exist an integer $n \geq 1$ and a primitive elliptic curve e_2 such that $C' = ne_1 + e_2$ and $e_1 \cdot e_2 = 1$.

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Proposition 3.5.3. *[58] Let S be a K3 surface and C an irreducible curve on S with $C^2 \geq 4$. Then the following facts are equivalent:*

i) $|C|$ is hyperelliptic;

ii) a generic member of $|C|$ is a smooth hyperelliptic curve;

iii) there exists an elliptic pencil $|E_1|$ such that $E_1 \cdot C = 2$.

Suppose furthermore that S is unnodal. If one of the above conditions is satisfied, then there exist an integer $n \geq 1$ and a primitive elliptic curve E_2 such that $C = nE_1 + E_2$ and $E_1 \cdot E_2 = 2$. Moreover the associated morphism ϕ_C is of degree 2 and maps S onto a rational normal scroll of degree $2n$ in $\mathbb{P}^{2n+1}$.

Thus there is a correspondence between hyperelliptic linear systems on an Enriques surface and on the associated K3 surface.

Theorem 3.5.4. *[14] Let Y be a generic Enriques surface and C' an irreducible curve on Y with $C'^2 \geq 2$. Then $|C'|$ contains a member that is the union of two smooth connected curves meeting transversely at $\delta \geq 1$ points. Moreover, there exists such a reducible member with $\delta = 1$ if and only if $|C'|$ is hyperelliptic.*

Corollary 3.5.5. *Let (S, τ) be a generic K3 surface with an antisymplectic involution without fixed points, and C an irreducible curve on S with $C^2 \geq 4$. Then $|C|^\tau$ contains a member that is the union of two smooth connected τ-invariant curves meeting transversely at $2\delta \geq 2$ points. Moreover, $\delta = 1$ if and only if $|C'|$ is hyperelliptic.*

Now we can focus on a generic fiber $P(C, \tau)$. Denoting as usual g and g' the genus of C and of C' respectively, by Riemann-Hurwitz theorem we get $\dim P(C, \tau) = g' - 1$. Moreover on an Enriques surface $\dim |C'| = g' - 1$, as expected. Thus $\mathcal{P}_C$ is a singular symplectic variety of dimension $2(g' - 1)$.

Theorem 3.5.6. *[4] Let Y be a generic Enriques surface and S the corresponding K3 surface. Let C' be a primitive smooth curve of genus $g \geq 2$ on Y and C the corresponding curve on S. Then*

i) if $|C'|$ is hyperelliptic, $\mathcal{P}_C$ is birational to an irreducible symplectic manifold of $K3^{[g-1]}$-type;

ii) if $|C'|$ is not hyperelliptic, $\mathcal{P}_C$ does not admit any symplectic resolution.

The proof of the non hyperelliptic case relies on Corollary 3.5.5, which implies the existence of a reducible curve with two smooth irreducible τ-invariant components meeting transversely in $2\delta \geq 4$ points. Hence we conclude by Theorem 3.3.2.

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In the hyperelliptic case, the key observation is that the -1 involution on $\mathcal{J}_C$ comes from an antisymplectic involution ι on S, by a relative version of Lemma 3.1.3. Thus one can consider the K3 surface $\hat{S}$ obtained by resolution of singularities of $S/(\iota \circ \tau)$, and it is possible to construct a birational map between $\mathcal{P}_C$ and a relative Jacobian on $\hat{S}$, which is an irreducible symplectic variety of $K3^{[g-1]}$-type.

We close the section by reporting another result of [4], which is obtained using techniques of the work of Markushevich and Tikhomirov [35].

Theorem 3.5.7. *Let Y be a generic Enriques surface and S the corresponding K3 surface. Let C' be a primitive smooth curve of genus $g \geq 2$ on Y and C the corresponding curve on S. Then $\mathcal{P}_C$ is simply connected and, when g is odd or $|C|$ is hyperelliptic, $h^{(2,0)}(\mathcal{P}_C) = 1$.*

3.6 Possible generalizations

In Lemma 3.2.3 we proved that τ is regular if H is τ-invariant. In Lemma 3.2.8, we proved that j respects C-stability. For this reason we chose this polarization, which is not v-generic, hence we have a regular involution on a singular variety, and $\mathcal{P}_C$ is also singular. It is natural to ask if one can choose instead of C a v-generic τ-invariant polarization H, in order to obtain a regular symplectic involution on a smooth irreducible symplectic variety and hence a smooth symplectic fixed locus. In Section 3.5 of [4], it is proven that this cannot happen, under a reasonable hypothesis. We adapt their result to our setting.

Theorem 3.6.1. *[4] Let (S, τ) be a generic K3 surface with an antisymplectic involution, C a smooth τ-invariant curve on S and H a τ-invariant polarization on S. Suppose that $|C|_S^{\tau}$ has an element $C_1 \cup C_2$, where C_i are smooth irreducible τ-invariant curves meeting transversely at $2\delta \geq 4$ points. Then j preserves the H-semistability if and only if H is not v-generic.*

Proof. Let $\mathcal{F}$ be an H-stable locally free sheaf supported on $C_1 \cup C_2$. As in the proof of Lemma 2.2.13, set $\mathcal{F}_i := \mathcal{F}_{C_i}$, $k_i := H \cdot C_i$, $k := H \cdot C_i$. Then the H-stability of $\mathcal{F}$ is equivalent to

$$\frac{k_1}{k}\chi < \chi_1 < \frac{k_1}{k}\chi + 2\delta. \tag{3.6.1}$$

Moreover, the H-stability of $j(\mathcal{F})$ is equivalent to

$$\frac{k_1}{k}\chi(j(\mathcal{F})) < \chi(j(\mathcal{F})_1) < \frac{k_1}{k}\chi(j(\mathcal{F})) + 2\delta. \tag{3.6.2}$$

Since $\chi(j(\mathcal{F})) = \chi(\mathcal{F})$ if j is a regular involution, and by Lemma 3.2.6 $\chi(j(\mathcal{F})_1) = \chi(j(\mathcal{F}_1 \otimes \mathcal{O}(-C_1 \cdot C_2))) = -\chi_1 - C_1 \cdot C + C_1 \cdot C_2 = -\chi_1 - 2g_1 + 2,$

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(3.6.2) is equivalent to

$$-\frac{k_1}{k}\chi - 2g_1 + 2 - 2\delta < \chi_1 < -\frac{k_1}{k}\chi - 2g_1 + 2. \tag{3.6.3}$$

Let $a = \left[\frac{k_1}{k}\chi\right]$ and $s = \frac{k_1}{k}\chi - a$. By Lemma 2.2.13, for a v-generic H, $s > 0$. Then (3.6.1) and (3.6.3) become

$$a + 1 \leq \chi_1 \leq a + 2\delta, \qquad -a - 2g_1 + 2 - 2\delta \leq \chi_1 \leq -a - 2g_1 + 2 - 1.$$

Hence they are equivalent if and only if

$$a + 1 = -a - 2g_1 + 2 - 2\delta$$

i.e.

$$2a + 1 = -2g_1 + 2 - 2\delta,$$

absurd. □

Another possible generalization, suggested in the work of Markushevich and Tikhomirov [35], consists in defining η also in the case of the relative Jacobian of degree d over $|C|$. Since there is a canonical identification $\mathcal{J}_C^d(|C|) \cong \mathcal{J}_C^{2g-2+d}(|C|)$ given by the tensor product with the canonical sheaf, there are other $2g - 1$ possible d. The problem in this case is that clearly (-1) (and so j) changes the degree $d \mapsto -d$. Thus for $d = g - 1$, in order to define an involution we can tensor by the canonical sheaf. In this case unfortunately we still get strictly semistable sheaves, because $\frac{\chi}{C^2} = 0$ satisfies $ii)$ of Lemma 2.2.13. For the other values of d, there can be only stable sheaves, but there is no canonical way to define an involution. Analogously to the case $d = g-1$, one can tensor by $\mathcal{O}_S(D)$, for a suitable curve D of degree $2d$, but there is no reason to get a regular involution. One can still hope to find a symplectic variety considering the closure of the fixed locus of such a rational involution. In Section 3 of [35], this situation is analyzed in an example of a relative Prym variety of dimension 4 of odd degree, but the closure of the fixed locus is not symplectic because it contains a rational 3-fold (Remark 5.8 [35]).

Finally, a last natural generalization consists in considering non-primitive curves C on a K3 surface S. Because of Theorem 2.4.10 and Remark 2.4.11, it is quite reasonable to consider only the case of curves of type $2C_0$, where C_0 is a primitive curve of genus 2, because otherwise the relative Jacobian of degree 0 of a non-primitive curve has bad singularities not admitting a symplectic resolution. In this thesis anyway we will focus only on the case of primitive curves.

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Chapter 4

Some relative Prym varieties from Del Pezzo surfaces

In this chapter we describe some examples of relative Prym varieties coming from K3 surfaces which are double covers of Del Pezzo surfaces. We start by specializing the Lagrangian fibration in Prym varieties to the case of the anticanonical linear system on a Del Pezzo surface (Section 4.1). In the case of a Del Pezzo of degree 1, $\mathcal{P}$ is a smooth elliptic K3 surface (Section 4.2). In the case of a Del Pezzo of degree 2, we obtain a singular irreducible symplectic 4-fold without any symplectic resolution (Section 4.3). This is the example considered by Markushevich and Tikhomirov in their fundamental work [35]. In the case of a Del Pezzo of degree 3, $\mathcal{P}$ is a singular irreducible symplectic 6-fold without any symplectic resolution (Section 4.4), which I described in [38]. In particular, we determine all the singularities of this 6-fold (Subsection 4.4.1), we show that it is simply connected and has $h^{(2,0)} = 1$ (Subsection 4.4.2), and we determine its Euler characteristic using its Lagrangian fibration structure (Subsection 4.4.3).

4.1 Looking for small dimensional examples

In the previous chapter, we introduced the relative Prym variety $\mathcal{P}_C$ associated to a τ-invariant curve C on a K3 surface with an antisymplectic involution τ (Definition 3.2.9), which gives an interesting construction of a symplectic variety.

As pointed out in Section 3.3, it inherits singularities from $\mathcal{J}_C$, because of a non-generic choice of the polarization. Moreover, by Corollary 3.3.4, it is reasonable to suspect that in most cases $\mathcal{P}_C$ is a singular variety without symplectic resolution.

In any case, it is interesting to determine small dimensional examples of singular irreducible symplectic varieties via this construction, as there are few known examples in the literature even in the singular case (see [17]

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for 4-folds). Furthermore, the description of new examples can be useful for developing a theory of moduli of singular irreducible symplectic varieties, which is closely related to the Beauville-Bogomolov form (see Theorem 1.4.4), as in the smooth case.

Since the fixed locus of a regular involution on a smooth variety is smooth, the singular points of $\mathcal{P}_C$ are contained in the fixed points of η acting on the singular locus of $\mathcal{J}_C$, i.e. in the locus of η-invariant strictly C-semistable sheaves. By Lemma 2.1.10, it suffices to study the reducible members of $|C|_S^\tau$, and the corresponding strictly semistable sheaves.

In this chapter we analyze some examples coming from K3 surfaces which are double covers of Del Pezzo surfaces. It is natural to consider the linear system of the anticanonical divisor, which contains curves of arithmetic genus 1, hence gives the first non-trivial example (if $g = 0$, then $\mathcal{P}_C = \mathcal{J}_C$).

We denote by Y_d the Del Pezzo surface of degree $1 \leq d \leq 9$, which is the blowup of $\mathbb{P}^2$ in $9 - d$ generic points. Then $-K_{Y_d} = 3H - E_1 - ... - E_{9-d}$, where H is the pullback of the hyperplane section of $\mathbb{P}^2$ and E_i are the exceptional divisors of the blowup. Its linear system is the smallest possible one containing elliptic curves, which is not induced by a linear system on Y_n for some $n > d$. A curve in this linear system is the strict transform in Y of a plane cubic passing through the $9 - d$ base points of the blowup.

As $B \in |-2K_{Y_d}| = |6H - 2E_1 - ... - 2E_{9-d}|$, B is a curve of genus $10 - (9 - d) = 1 + d$, which is the strict transform of a plane sextic with exactly $9-d$ simple nodes, one in each base point of the blowup. By Theorem 3.4.1 and by the classification of Figure 3.1 (there is only one possible value of δ in these cases), we get that the main invariant of the corresponding pair (S, τ) is $(10 - d, 10 - d, 1)$.

Since $\mathrm{Pic}(Y_d) = I_{1,9-d}$, the corresponding generic K3 surface has $\mathrm{Pic}(S) = I_{1,9-d}(2)$, and the dimension of the moduli space (see [36]) of (S, τ) with main invariant $(10 - d, 10 - d, 1)$ is $20 - (10 - d) = 10 + d$.

Moreover $\mathcal{P} \to |-K_{Y_d}| = \mathbb{P}^d$ is a Lagrangian fibration in abelian varieties $\bar{P}(C, \tau)$ of dimension d with polarization of type $(1, ..., 1, 2)$, because τ has $2d$ fixed points (see [44]).

The advantage of working with Del Pezzo surfaces is that the reducible members of a linear system are easy to describe. The disadvantage is that for a non-étale cover, there can be τ-invariant reducible curves which are double covers of irreducible curves.

Lemma 4.1.1. *Let (S, τ) be a K3 surface with an antisymplectic involution, $Y := S/\tau$ and $B \subset Y$ the branch locus of the double cover. Let C be a τ-invariant curve on S, and $\tau \circlearrowright C \longrightarrow C'$ the induced double cover. If C is reducible, then either C' is reducible or C' is totally tangent to B (i.e. C intersects B only at tangency points).*

Proof. Clearly, if C' is reducible, then also C is.
Assume C' irreducible and C reducible. If $C \to C'$ has a simple ramification

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point, corresponding to a point of transversal intersection of C' and B, then C is irreducible for topological reasons. Then C' is necessarily totally tangent to B. □

Remark 4.1.2. Of course, if C' is reducible, then also C is reducible. But if C' is totally tangent to B, C can be irreducible. Hence we can only analyze case by case what happens if C' is totally tangent to B.

4.2 Del Pezzo of degree 1

We start considering Y_1 and (S, τ) with main invariant $(9, 9, 1)$.

Lemma 4.2.1. *Let (S, τ) be a generic K3 surface with an antisymplectic involution such that $\pi : S \to S/\tau = Y_1$. Then $\pi^*(-K_{Y_1})$ represents S as a double cover of $\mathbb{P}^2$ branched along a sextic of type*

$$C_6 : F_6 = x_2^6 + x_2^4 \cdot f_2 + x_2^2 \cdot f_4 + f_6 = 0, \ \textit{with } f_{2i} \in \mathbb{C}[x_0, x_1]_{2i},$$

hence

$$S : y^2 = F_6(x_0, x_1, x_2) \subset \mathbb{P}(3, 1, 1, 1), \ \textit{and } \tau : x_2 \mapsto -x_2. \tag{4.2.1}$$

Proof. Consider (S, τ) as in (4.2.1). The fixed locus of τ is the genus 2 curve $y^2 = f_6(x_0, x_1)$. By Theorem 3.4.1, the main invariant is $(9, 9, 1)$, so $S/\tau = Y_1$. Moreover, this is a generic pair with this main invariant, because counting the dimensions

$$\dim \mathbb{C}[x_0, x_1]_2 + \dim \mathbb{C}[x_0, x_1]_4 + \dim \mathbb{C}[x_0, x_1]_6 - \dim GL(2) = 3+5+7-4 = 11.$$

Since $C^2 = 2(3H - E_1 - ... - E_8)^2 = 2$ implies $g = 2$, $\pi^*(-K_{Y_1})$ expresses S as in (4.2.1) after a suitable choice of coordinates. □

Theorem 4.2.2. *Let (S, τ) be a generic K3 surface with an antisymplectic involution such that $\pi : S \to S/\tau = Y_1$, and let $C \in |\pi^*(-K_{Y_1})|_S^{\tau,+}$. Then $\mathcal{P} \to |C|_S^{\tau,+}$ is an elliptic K3 surface birational to the K3 surface $\hat{S}$ which is the resolution of singularities of $S/(\iota \circ \tau)$, where ι denotes the antisymplectic involution $y \mapsto -y$ on S and the induced involution on $|\pi^*(-K_{Y_1})|_S$.*

Proof. By Lemma 4.2.1, we can assume that S is of type (4.2.1). Then $|\pi^*(-K_{Y_1})|_S^\tau = \mathbb{P}(\langle x_0, x_1 \rangle)$. Moreover $|-K_{Y_1}|_{Y_1}$ corresponds to the pencil of plane cubic curves passing through 8 points in general position. There are no reducible members in $|-K_S|_S$, because there are no singular non-integral plane cubic curves passing through 8 points in general position. By Lemma 4.1.1, there can be reducible members in $|\pi^*(-K_{Y_1})|_S^\tau$ corresponding to double covers of curves totally tangent to B. But if this is the case, such a curve is the union of two smooth curves of genus 1 meeting transversely,

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thus by (a degeneration of) Lemma 2.3.5, a polystable sheaf on it represents a smooth point of $\mathcal{J}$. Hence also a smooth point of $\mathcal{P}$. So $\mathcal{P}$ is a smooth symplectic surface.

The corresponding Prym varieties are principally polarized with the polarization described in Theorem 5.5.4, because there are only 2 branching points.

Since a curve of genus 2 is hyperelliptic, denoting by ι the hyperelliptic involution, by Lemma 3.1.3, for a smooth curve in $|\pi^*(-K_{Y_1})|_S$, we have

$$P(C,\tau) = J(\bar{C}) \text{ with } \bar{C} := C/(\iota \circ \tau). \tag{4.2.2}$$

Moreover the involution ι corresponds to the antisymplectic involution $y \mapsto -y$ on S. Hence $\iota \circ \tau : (y, x_0, x_1, x_2) \mapsto (-y, x_0, x_1, -x_2)$ is a symplectic involution on S with 8 fixed points (6 points given by $y = x_2 = 0$ and 2 points by $x_0 = x_1 = 0$), and the quotient $\bar{S} := S/(\iota \circ \tau)$ is a K3 surface with 8 singular points. Blowing up, we get a smooth K3 surface $\hat{S}$ with an elliptic fibration given by $\hat{C} := \delta'^{-1}\bar{C}$. If we consider the open subset U of smooth curves, the relative version of (4.2.2) on U gives an isomorphism of $\mathcal{P}|_U$ and $\hat{S}|_U$. Thus they are birational.

□

Remark 4.2.3. Note that we have only proved that $\mathcal{P}$ is smooth, not that $\mathcal{J}$ is smooth. If also $\mathcal{J}$ is smooth, then this provides a nice example of a symplectic involution on an irreducible symplectic 4-fold, and $\mathcal{P}$ is the K3 component of its fixed locus, which, as we know from [40], consists of a K3 surface and 28 isolated points.

4.3 Del Pezzo of degree 2

The first interesting example is given by Y_2 and (S, τ) with main invariant $(8, 8, 1)$. This is the case considered by Markushevich and Tikhomirov in [35]. In the following, we describe the singularities of $\mathcal{P}$ using the results of the previous chapter, and then we briefly sum up other results on this example obtained in [35] and [39].

Lemma 4.3.1. *Let (S, τ) be a generic K3 surface with an antisymplectic involution such that $\pi : S \to S/\tau = Y_2$. Then $\pi^*(-K_{Y_2})$ embeds S into $\mathbb{P}^3$ as a quartic of the form*

$$S : x_3^4 + x_3^2 f_2 + f_4 = 0 \subset \mathbb{P}^3, \text{ with } f_{2i} \in \mathbb{C}[x_0, x_1, x_2]_{2i}, \tag{4.3.1}$$

and $\tau : x_3 \mapsto -x_3$.

Proof. Consider (S, τ) as in (4.3.1). The fixed locus of τ is the genus 3 curve $f_4(x_0, x_1, x_2) = 0$. By Theorem 3.4.1, the main invariant is $(8, 8, 1)$,

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so $S/\tau = Y_2$. Moreover, this is a generic pair with this main invariant by dimension count:

$$\dim \mathbb{C}[x_0, x_1, x_2]_2 + \dim \mathbb{C}[x_0, x_1, x_2]_4 - \dim GL(3) = 6 + 15 - 9 = 12.$$

Since $C^2 = 2(3H - E_1 - ... - E_7)^2 = 4$ implies $g = 3$, $\pi^*(-K_{Y_2})$ expresses S as (4.3.1) after a suitable choice of coordinates. □

We determine the non-integral members of $|\pi^*(-K_{Y_2})|_S^{\tau,+}$ using the geometry of $|-K_{Y_2}|$, in a way different from that of Lemma 1.1 of [35].

Lemma 4.3.2. *Let (S, τ) be a generic K3 surface with an antisymplectic involution such that $\pi : S \to S/\tau = Y_2$. Then $|\pi^*(-K_{Y_2})|_S^{\tau,+}$ contains exactly 28 reduced and reducible curves of type $C_1 \cup C_2$, with C_1 and C_2 smooth rational curves meeting transversely in 4 points, and all the other curves are integral.*

Proof. There are $\binom{7}{2} = 21$ reducible plane cubic curves passing through 7 points in general position, given by a conic through 5 points and a line through the remaining 2. Since the two irreducible components meet transversely in 2 points, their preimages on S meet transversely in 4 points. Moreover, in $|-K_{Y_2}| = \mathbb{P}^2$ there is also another type of reducible curves. It corresponds to plane cubic curves with a simple node at one of the base points of the blowup. Indeed, imposing a node at a point P to a plane curve passing through P, we set 2 linear conditions. Thus there are 7 other reducible curves in $|-K_{Y_2}|$, one for each point of the blowup, each one having two smooth rational irreducible components meeting transversely in 2 points (the exceptional divisor and the strict transform of the curve). Again, the corresponding curves on S meet transversely in 4 points.

We claim that these are all the reducible curves in $|\pi^*(-K_{Y_2})|_S^{\tau,+}$. By Lemma 4.1.1, it suffices to exclude the case when a double cover C of a curve C' totally tangent to B is reducible. Such a C has 2 simple nodes. If it is reducible, it has 2 smooth irreducible components of genus 1. But then there is an elliptic curve meeting C in 2 points. By Proposition 3.5.3, this means that all the elements of the linear system are hyperelliptic, which is absurd, since $\pi^*(-K_{Y_2})$ is ample. □

Lemma 4.3.2, combined with Corollary 3.3.4 and Theorem 3.3.2, gives us a complete description of this singular 4-fold:

Corollary 4.3.3. *$\mathcal{P}$ is a singular symplectic variety with 28 singularities of type $\mathbb{C}^4/\pm 1$, which does not admit any symplectic desingularization.*

Moreover, $\mathcal{P}$ can be obtained as a birational transformation of a quotient of $S^{[2]}$, hence we are in the setting of [17].

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Theorem 4.3.4. *Let $\iota_0 : S^{[2]} \to S^{[2]}$ be the involution $\xi \mapsto \langle \xi \rangle \cap S - \xi$, and $\iota_1 := \iota_0 \circ \tau$. Let $M := S^{[2]}/\iota_1$ and M' the symplectic terminalization of M. Then $\mathcal{P}$ is birational to M' via a Mukai flop, hence $\mathcal{P}$ is simply connected and $h^{(2,0)} = 1$, i.e. $\mathcal{P}$ is a singular irreducible symplectic variety.*

Using this birational description, Menet in [39] calculated the Beauville-Bogomolov form of $\mathcal{P}$:

Theorem 4.3.5 (Menet)**.** *Let (S, τ) be a generic K3 surface with an anti-symplectic involution such that $\pi : S \to S/\tau = Y_2$. Let $C \in |\pi^*(-K_{Y_2})|_S^{\tau,+}$ and $\mathcal{P}$ be the corresponding relative Prym. Then $H^2(\mathcal{P}, \mathbb{Z})$ endowed with its Beauville-Bogomolov form q is isomorphic to the lattice*

$$E_8(-1) \oplus U(2)^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}.$$

Determining the singular members in the linear system, Markushevich and Tikhomirov found also the Euler characteristic of $\mathcal{P}$ (see Remark 4.4.23).

4.4 Del Pezzo of degree 3

In this section we analyze the case of Y_3 (which we denote by Y to simplify the notation) and (S, τ) with main invariant $(7, 7, 1)$.

Lemma 4.4.1. *Let (S, τ) be a generic K3 surface with an antisymplectic involution such that $\pi : S \to S/\tau = Y_3$. Then $\pi^*(-K_{Y_3})$ embeds S as the intersection of a quadric 3-fold of the form*

$$Z_2 : F_2 = x_4^2 + f_2 = 0 \text{ with } f_2 \in \mathbb{C}[x_0, x_1, x_2, x_3]_2 \qquad (4.4.1)$$

and a cubic cone with vertex $p_0 = (0, 0, 0, 0, 1)$

$$Z_3 : F_3 = 0 \text{ with } F_3 \in \mathbb{C}[x_0, x_1, x_2, x_3]_3. \qquad (4.4.2)$$

Moreover τ is given by $x_4 \to -x_4$ and the double covering map $\tau \circlearrowright S \xrightarrow{\pi} Y_3$ is the restriction to $S \subset \mathbb{P}^4$ of the projection $\pi : \mathbb{P}^4 \dashrightarrow H_4$ from the point p_0, where H_4 is the hyperplane $x_4 = 0$.

Proof. Consider (S, τ) as in (4.4.1) and (4.4.2). Clearly the projection onto $Z_3 \cap H_4$ gives the map π, as Y_3 is also a cubic surface. The fixed locus of τ is the genus 4 curve given by $B := Z_2 \cap Z_3 \cap H_4$. By Theorem 3.4.1, the main invariant is $(7, 7, 1)$, so $S/\tau = Y_3$. Moreover, this is a generic pair with this main invariant by dimension count:

$$\dim \mathbb{C}[x_0, x_1, x_2, x_3]_2 + \dim \mathbb{C}[x_0, x_1, x_2, x_3]_3 - \dim GL(4) = 10 + 20 - 16 = 14.$$

Since $C^2 = 2(3H - E_1 - ... - E_6)^2 = 6$ implies $g = 4$, $\pi^*(-K_{Y_3})$ embeds S as the intersection of a quadric and a cubic of $\mathbb{P}^4$ as in (4.4.1) and (4.4.2), after a suitable choice of coordinates. □

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Remark 4.4.2. A generic τ-invariant cubic 3-fold is of the form

$$Z_3 : F_3 = x_4^2 \cdot f_1 + f_3 = 0 \text{ with } f_i \in \mathbb{C}[x_0, x_1, x_2, x_3]_i$$

Since we only care about its intersection with a τ-invariant quadric 3-fold (which has equation (4.4.1)), adding $-F_2 \cdot f_1$ to the equation of Z_3, we can bring it to the form (4.4.2).

Corollary 4.4.3. *Let (S, τ) be a generic K3 surface with an antisymplectic involution such that $\pi : S \to S/\tau = Y_3$. Then S admits 27 elliptic fibrations, one for each line on Y.*

Proof. By Lemma 4.4.1, S is given by (4.4.1) and (4.4.2). The double cover of a line of $Y = Z_3 \cap H_4$ is a plane conic on S, and considering the pencil of hyperplanes containing it, we get a pencil of elliptic curves. □

In the following subsections we show the following characterization of $\mathcal{P}$.

Theorem 4.4.4. *Let (S, τ) be a generic K3 surface with an antisymplectic involution such that $\pi : S \to S/\tau = Y_3$. Let $C \in |\pi^*(-K_{Y_3})|_S^{\tau,+}$.*

Then $\mathcal{P}$ is a singular irreducible symplectic 6-fold without symplectic resolutions and $\chi(\mathcal{P}) = 2283$. Its singular locus Sing($\mathcal{P}$) coincides with the locus of η-invariant strictly semistable sheaves of $\mathcal{J}$, and it is the union of 27 singular K3 surfaces associated to the 27 lines on Y_3. Each K3 surface has 5 A_1-singularities and each singular point is in the intersection of 3 K3 surfaces. A smooth point of Sing($\mathcal{P}$) is a singularity of $\mathcal{P}$ of analytic type $\mathbb{C}^2 \times (\mathbb{C}^4/\pm 1)$. A singular point of Sing($\mathcal{P}$) is a singularity of $\mathcal{P}$ of analytic type $\mathbb{C}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, where the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is given by $\langle (1, 1, -1, -1, -1, -1), (-1, -1, 1, 1, -1, -1) \rangle$.

Proof. The description of the singularities follows from Corollary 4.4.9, Theorem 4.4.10 and Corollary 4.4.11 (Subsection 4.4.1). $\mathcal{P}$ is an irreducible symplectic variety by Corollary 4.4.13 and Proposition 4.4.16 (Subsection 4.4.2). The computation of the Euler characteristic is in Theorem 4.4.22 (Subsection 4.4.3). □

4.4.1 Singularities of $\mathcal{P}$

In this subsection we describe the singular locus of $\mathcal{J}^+ := \mathcal{J}|_{|\mathcal{O}_S(1)|^{\tau,+}}$ and $\mathcal{P}$.

Proposition 4.4.5. *Using the natural embedding $Y \to |\mathcal{O}_Y(1)|^*$, the reducible members of $|\mathcal{O}_S(1)|^{\tau,+}$ are parametrized by the 27 lines dual to the 27 lines on Y. They meet in exactly 45 triple points. A point lying on only one line represents $C = C_1 \cup C_2$, where C_1 and C_2 are smooth curves of genus respectively 0 and 1 intersecting in 4 points; moreover $C' = C_1' \cup C_2'$, where C_1' and C_2' are a line and a conic lying on a plane. A point in the intersection*

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of 3 dual lines represents $C = C^1 \cup C^2 \cup C^3$, with C^1, C^2, C^3 smooth rational curves, intersecting each other in 2 points; moreover, $C' = C^{1'} \cup C^{2'} \cup C^{3'}$ is a triangle formed by 3 lines lying on a plane. The triple intersection points represent the curves C with 3 irreducible components that can be obtained as degenerations of curves with 2 irreducible components by fixing one of them and deforming the other 2.

Proof. First we show that all the reducible curves of $|C|^{\tau,+}$ come from reducible curves of $|C'|$. By Lemma 4.1.1, if C' is irreducible and C is reducible, then C' is obtained by intersecting Y with a hyperplane $H \in (\mathbb{P}^3)^*$, totally tangent to B. Since the arithmetic genus of C is 4, C is the union of two smooth curves of genus 1 meeting in 3 points. But $C = S \cap \langle H, P_0 \rangle \subset Z_2 \cap \langle H, P_0 \rangle$, the latter intersection being a quadric in $\mathbb{P}^3$. A smooth genus 1 curve is in the linear system $|\mathcal{O}(2)|$ on a quadric surface, while C belongs to $|\mathcal{O}(3)|$, which is absurd.

We describe then the reducible members C' of $|\mathcal{O}_Y(1)|$. As Z_3 is a cubic cone, it contains 27 planes, each one generated by P_0 and one of the 27 lines on $Y_3 \cap H_4$. If one of these lines is given by $H' \cap Y$, $H = \langle H', P_0 \rangle \in (\mathbb{P}^4)^*$ is a τ-invariant hyperplane containing one of these planes. Then $H \cap Z_3$ is the union of the plane itself with either a quadric surface or other two planes. Restricting to $Z_2 \cap H$, which is smooth for a generic choice of f_2 and hence isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, we obtain a reducible $C \in |\mathcal{O}(1)|^{\tau,+}$, union either of a $(1,1)$-curve C_1 and a $(2,2)$-curve C_2 on $\mathbb{P}^1 \times \mathbb{P}^1$, or of $(1,1)$-curves C^1, C^2, C^3. Moreover $C_1 \cdot C_2 = 4$, while $C^i \cdot C^j = 2$. Since C has 6 ramification points, the only possibilities are $C' = C_1' \cup C_2'$ for two rational curves C_1' and C_2' meeting in 2 points, or $C' = C^{1'} \cup C^{2'} \cup C^{3'}$ with three rational curves $C^{i'}$.

In fact, if C' is reducible, then it is the union either of a conic and a line in Y, or of three lines in Y.

Hence from the dual line of each line in Y, we get a line of reducible elements in $|\mathcal{O}_S(1)|^{\tau,+}$. From the configuration of the 27 lines in a cubic surface, we can deduce the configuration of these dual lines. Their intersection points correspond obviously to the degeneration when C' becomes the union of 3 lines. Considering Y as the blowup in 6 points $p_1, ..., p_6$ of $\mathbb{P}^2$, we call E_i the exceptional divisor of p_i, F_{ij} the pullback of the line $\langle p_i, p_j \rangle$ and G_i the pullback of a conic passing through P_j for all $i \neq j$. Then we get $C' = F_{ij} \cup F_{kl} \cup F_{mn}$ with i, j, k, l, m, n all distinct, or $C' = F_{ij} \cup E_i \cup G_j$. Thus, each line meets other 10 lines in different points forming 5 triangles, and there are in total 45 triangles in the linear system. Since the dual of a triangle in $\mathbb{P}^3$ is given by 3 lines in $\mathbb{P}^3$ meeting in one point, there are 45 points where 3 of the 27 orthogonal lines intersect, and no other intersection points. When C' is given by 3 lines, it can deform to a conic and a line in 3 different ways, by fixing one line and deforming the other 2 to a smooth conic by rotating a plane through the fixed line. The triple points of the dual configuration of lines are either the intersection of $F_{ij}^*, F_{kl}^*, F_{mn}^*$

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with i, j, k, l, m, n all distinct, or the intersection of F^*_{ij}, E^*_i, G^*_j, and there are no other intersections. Hence each dual line meets other 10 dual lines in 5 points, and each intersection point is triple. □

Remark 4.4.6. A geometric description of the reducible curves C' can also be obtained by viewing Y as the blowup of $\mathbb{P}^2$ in $p_1, ..., p_6$. A generic reducible element of $|3H - p_1 - ... - p_6|_{\mathbb{P}^2}$ is the union of a conic and a line passing through the 6 base points of the blowup. They form 21 pencils in $|\mathcal{O}_Y(1)| = |3H - E_1 - ... - E_6|$: 6 $\mathbb{P}^1$'s, denoted by L_i with $i = 1, ..., 6$, are given by a line passing through p_i and the conic passing through the remaining 5 base points; 15 $\mathbb{P}^1$'s, denoted by L_{jk} with $1 \leq j < k \leq 6$, are given by the line spanned by p_j and p_k, and a conic through the remaining 4 points. The remaining 6 $\mathbb{P}^1$'s, parametrizing reducible curves of $|\mathcal{O}_Y(1)|$, correspond to the 6 pencils, denoted by M_i, parametrizing the strict transforms of cubics, singular at p_i, plus the exceptional divisor E_i.

The curves C' with 3 irreducible components correspond to the strict transforms on Y of the unions of three lines passing through the 6 base points of the blowup, and of the unions of a conic through 5 base points with a line through one of the same 5 base points and through the sixth one. These curves correspond to the intersections of the 27 lines. More precisely, $L_i \cdot L_j = 0$; $L_i \cdot L_{jk}$ is 1 if $i \in \{j, k\}$ and 0 otherwise; $L_i \cdot M_j$ is 1 if $i \neq j$ and 0 otherwise; $L_{jk} \cdot L_{hi}$ is 1 if $\#\{h, i, j, k\} = 4$ and 0 otherwise; $L_{jk} \cdot M_i$ is 1 if $i \in \{j, k\}$ and 0 otherwise; $M_i \cdot M_j = 0$. In particular, we obtain again that all the intersection points of the 27 lines are triple points.

Lemma 4.4.7. *All the polystable non-stable sheaves in $\mathcal{J}^+$ are η-invariant singular points. For each curve $C = C_1 \cup C_2$, there is a 1-dimensional family of polystable sheaves of type $\mathcal{O}_{C_1}(-2pt) \oplus \mathcal{F}_2$, with $\mathcal{F}_2 \in J^{-2}(C_2)$. For each curve $C = C^1 \cup C^2 \cup C^3$, there are three 1-dimensional families of polystable sheaves of type $\mathcal{O}_{C_1}(-2pt) \oplus \mathcal{F}_2$, where $\mathcal{F}_2 \in J^{(-1,-1)}(C_2) = \mathbb{C}^*$, C_1 is an irreducible component of C and C_2 is the union of the remaining two irreducible components; these families meet quasi-transversely in one point, represented by $[\mathcal{O}_{C^1}(-2pt) \oplus \mathcal{O}_{C^2}(-2pt) \oplus \mathcal{O}_{C^3}(-2pt)]$.*

Proof. A polystable non-stable sheaf is supported on one of the reducible curves described in Proposition 4.4.5. In both cases, the addendi are elements of Prym varieties of double covers of rational curves, which are the Jacobians of the double covers, and hence the polystable sheaves are so η-invariant. By Lemma 2.3.5, they are singular points of $\mathcal{J}^+$.

The polystable sheaves on $C = C_1 \cup C_2$ are described in Lemma 3.3.1.

If $C = C^1 \cup C^2 \cup C^3$, then it comes from a degeneration of a C_2 into 2 rational curves, which we assume to be C^1 and C^2 (hence in this notation C^3 is identified with C_1). Then we still have the exact sequence $0 \to \mathcal{F}_2 \to \mathcal{F} \to \mathcal{O}_{C_1}(-2pt) \to 0$, with $\mathcal{F}_2$ a C-semistable but not necessarily C-stable sheaf in $J^{-2}(C_2)$ (because it is a limit of stable sheaves on a family of C_2's

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degenerating into two rational curves). Again $\mathcal{F}_2 \in \mathrm{Ext}^1_S(\mathcal{G}_1, \mathcal{G}_2) = \mathbb{C}^{C^1 \cdot C^2} = \mathbb{C}^2$, where $\mathcal{G}_i$ is a pure 1-dimensional sheaf on C^i of degree d^i. So

$$\mu_C(\mathcal{G}_2) \leq \mu_C(\mathcal{F}_2) \leq \mu_C(\mathcal{G}_1),$$

and since

$$\mu_C(\mathcal{G}_i) = \frac{\chi(\mathcal{G}_i)}{C^i \cdot C} = \frac{1 + d^i}{(C^i)^2 + C^i \cdot (C - C^i)} = \frac{1 + d^i}{2},$$

we get

$$\frac{1 + d^2}{2} \leq \frac{-2}{4} \leq \frac{1 + d^1}{2}, \quad i.e. \;\; d^2 \leq -2 \leq d^1.$$

Moreover $\chi(\mathcal{F}_2(n)) = \chi(\mathcal{G}_1(n)) + \chi(\mathcal{G}_2(n))$, i.e. $d^1 + d^2 = -4$.

Thus (d^1, d^2) can be $(-1, -3)$ (and $\mathcal{F}_2$ is C-stable), $(-2, -2)$ (and $\mathcal{F}_2$ is strictly C-semistable) or $(0, -4)$ ($\mathcal{F}_2$ is again strictly C-semistable with a maximal destabilizing subsheaf $\mathcal{G}_1(-C_1 \cdot C_2)$, so that $\mathcal{F}_2$ is considered as an extension given by an element of $\mathrm{Ext}^1_S(\mathcal{G}_2, \mathcal{G}_1)$).

So either $\mathcal{F}$ admits a Jordan-Hölder filtration $0 \to \mathcal{F}_2 \to \mathcal{F}$ and $[\mathcal{F}] = [\mathcal{O}_{C_1}(-2pt) \oplus \mathcal{F}_2]$ with $\mathcal{F}_2 \in J^{(-1,-1)}(C_2) = \mathbb{C}^*$; or $\mathcal{F}$ admits a Jordan-Hölder filtration $0 \to \mathcal{G}_2 \to \mathcal{F}_2 \to \mathcal{F}$, and $[\mathcal{F}] = [\mathcal{O}_{C^1}(-2pt) \oplus \mathcal{O}_{C^2}(-2pt) \oplus \mathcal{O}_{C^3}(-2pt)]$.

The geometric meaning of this degeneration is the following: identifying $\mathcal{F}_2 \in J^{(-1,-1)}(C_2)$ as the gluing of $\mathcal{O}_{C^1}(-p_1)$ and $\mathcal{O}_{C^2}(-p_2)$, where p_i denotes a point on C^i, when p_1 (respectively p_2) tends to one of the two singular points, we obtain the gluing of $\mathcal{O}_{C^1}$ and $\mathcal{O}_{C^2}(-p_1 - p_2)$ in $J^{(0,-2)}(C_2)$ (respectively the gluing of $\mathcal{O}_{C^1}(-p_1 - p_2)$ and $\mathcal{O}_{C^2}$ in $J^{(-2,0)}(C_2)$) in the limit. □

Corollary 4.4.8. *$\mathcal{P}$ has symplectic singularities, i.e. its symplectic form can be extended to a regular form on a resolution of singularities.*

Proof. Since the codimension of the singular locus is at least 4, this is a consequence of Flenner theorem (see [15]). □

Specializing Theorem 3.3.2 and Corollary 3.3.4 to this example, we get:

Corollary 4.4.9. *At a polystable sheaf $\mathcal{F} = \mathcal{O}_{C_1}(-2pt) \oplus \mathcal{F}_2$, $(C_{[\mathcal{F}]}(\mathcal{P}), [\mathcal{F}])$ is locally analytically equivalent to $(\mathbb{C}^2 \times (\mathbb{C}^4/\pm 1), 0)$. Hence it does not admit any symplectic resolution.*

Theorem 4.4.10. *At a polystable sheaf $\mathcal{F} = \mathcal{O}_{C^1}(-2pt) \oplus \mathcal{O}_{C^2}(-2pt) \oplus \mathcal{O}_{C^3}(-2pt)$, $(C_{[\mathcal{F}]}(\mathcal{P}), [\mathcal{F}])$ is locally analytically equivalent to $(\mathbb{C}^6/\mathbb{Z}_2 \times \mathbb{Z}_2, 0)$, where the action on $\mathbb{C}^6$ is given by*

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle (1, 1, -1, -1, -1, -1), (-1, -1, 1, 1, -1, -1) \rangle. \qquad (4.4.3)$$

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Proof. 1) For $i \neq j$ we set

$$W_{ij} := \mathrm{Ext}^1_S(\mathcal{O}_{C^i}(-2pt), \mathcal{O}_{C^j}(-2pt)),\ \ W^*_{ij} := \mathrm{Ext}^1_S(\mathcal{O}_{C^j}(-2pt), \mathcal{O}_{C^i}(-2pt)).$$

Since the supports of $\mathcal{O}_{C^i}(-2)$ and $\mathcal{O}_{C^j}(-2)$ are transversal for $i \neq j$, we get $W_{ij} = \mathbb{C}^{C^i \cdot C^j} = \mathbb{C}^2$. So

$$\mathrm{Ext}^1_S(\mathcal{F}, \mathcal{F}) = W_{12} \times W_{13} \times W_{23} \times W^*_{12} \times W^*_{13} \times W^*_{23} = \mathbb{C}^{12}.$$

Choosing coordinates x^0_{ij}, x^1_{ij} in W_{ij} such that $\tau(x^0_{ij}) = x^1_{ij}$, let y^0_{ij}, y^1_{ij} be the dual ones in W^*_{ij}.

By the stability of $\mathcal{F}_i$, we have $Aut(\mathcal{F}) = \mathbb{C}^{*3}$, hence $G := PAut(\mathcal{F}) = \mathbb{C}^{*2}$, and, setting $\epsilon_1 := \lambda_1/\lambda_2, \epsilon_2 := \lambda_2/\lambda_3$ for $(\lambda_1, \lambda_2, \lambda_3) \in Aut(\mathcal{F})$, its action on $\mathrm{Ext}^1_S(\mathcal{F}, \mathcal{F})$ is

$$(\epsilon_1, \epsilon_2) \cdot (x^k_{ij}, y^k_{ij}) = (\epsilon_1 x^k_{12}, \epsilon_1\epsilon_2 x^k_{13}, \epsilon_2 x^k_{23}, \epsilon_1^{-1} y^k_{12}, \epsilon_1^{-1}\epsilon_2^{-1} y^k_{13}, \epsilon_2^{-1} y^k_{23}). \quad (4.4.4)$$

The algebra of invariants of the action of G on $\mathbb{P}(\mathrm{Ext}^1_S(\mathcal{F}, \mathcal{F}))$ is generated by the 12 quadratic monomials

$$u^{kl}_{ij} := x^k_{ij} y^l_{ij} \qquad i < j, \quad (4.4.5)$$

and by the 16 cubic monomials

$$v^{klm} := x^k_{13} y^l_{12} y^m_{23}, \qquad w^{klm} := y^k_{13} x^l_{12} x^m_{23}. \quad (4.4.6)$$

Its generating relations are the 3 equations in u^{kl}_{ij}

$$u^{00}_{ij} u^{11}_{ij} = u^{01}_{ij} u^{10}_{ij}, \quad (4.4.7)$$

the 18 equations in v^{ijk}, w^{ijk}

$$v^{klm} v^{k'l'm'} = v^{k'lm} v^{kl'm'} = v^{kl'm} v^{k'lm'} = v^{klm'} v^{k'l'm}, \quad (4.4.8)$$

$$w^{klm} w^{k'l'm'} = w^{k'lm} w^{kl'm'} = w^{kl'm} w^{k'lm'} = w^{klm'} w^{k'l'm}, \quad (4.4.9)$$

and the 64 cubic equations

$$v^{klm} w^{k'l'm'} = u^{kk'}_{13} u^{ll'}_{12} u^{mm'}_{23}. \quad (4.4.10)$$

2) The action of j is $x^k_{ij} \leftrightarrow y^k_{ij}$, so that

$$\eta^*(x^k_{ij}, y^k_{ij}) = (y^1_{12}, y^0_{12}, y^1_{13}, y^0_{13}, y^1_{23}, y^0_{23}, x^1_{12}, x^0_{12}, x^1_{13}, x^0_{13}, x^1_{23}, x^0_{23}). \quad (4.4.11)$$

Its fixed locus is then

$$y^1_{12} = x^0_{12}, y^0_{12} = x^1_{12}, y^1_{13} = x^0_{13}, y^0_{13} = x^1_{13}, y^1_{23} = x^0_{23}, y^0_{23} = x^1_{23}. \quad (4.4.12)$$

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Since

$$\eta^*((\epsilon_1,\epsilon_2)\cdot(x^k_{ij},y^k_{ij})) = \left(\frac{1}{\epsilon_1},\frac{1}{\epsilon_2}\right)\eta^*(x^k_{ij},y^k_{ij}),$$

η^* induces a well defined involution on $\mathrm{Ext}^1_S(\mathcal{F},\mathcal{F})//G$. As η^* is not G-invariant, its fixed locus on the quotient cannot be expressed as quotient of its fixed locus. We can describe it using the invariant coordinates $u^{kl}_{ij}, v^{klm}, w^{klm}$. Substituing (4.4.12) in (4.4.5) and (4.4.6), we see that the fixed locus of η^* is

$$u^{00}_{ij} = u^{11}_{ij}, \tag{4.4.13}$$

$$v^{klm} = w^{1-k,1-l,1-m}. \tag{4.4.14}$$

So the function algebra of $(\mathrm{Ext}^1_S(\mathcal{F},\mathcal{F})//G)^{\eta^*}$ has the 9 coordinate functions $u^{00}_{ij}, u^{01}_{ij}, u^{10}_{ij}$ and the 8 coordinate functions v^{klm} as generators. Using (4.4.13) and (4.4.14), the relations (4.4.7) give the 3 equations

$$(u^{00}_{ij})^2 = u^{01}_{ij}u^{10}_{ij}, \tag{4.4.15}$$

(4.4.10) give the 36 equations

$$v^{klm}v^{k'l'm'} = u^{k,1-k'}_{13}u^{l,1-l'}_{12}u^{m,1-m'}_{23}, \tag{4.4.16}$$

while (4.4.8) and (4.4.9) follow from (4.4.15) and (4.4.16).

These equations describe the quotient $\mathbb{C}^6/\mathbb{Z}_2\times\mathbb{Z}_2$. Indeed choosing coordinates $r^0_1, r^1_1, r^0_2, r^1_2, r^0_3, r^1_3$, in which the action is given by 4.4.3, the algebra of invariant functions is generated by the 9 quadratic monomials

$$s^{jk}_i := r^j_i r^k_i$$

and by the 8 cubic monomials

$$t^{ijk} := s^i_1 s^j_2 s^k_3,$$

with the 3 quadratic relations

$$s^{01}_i = s^{00}_i s^{11}_i,$$

and the 36 cubic ones

$$t^{ijk}t^{i'j'k'} = s^{ii'}_1 s^{jj'}_2 s^{kk'}_3.$$

Hence $C_{[\mathcal{F}]}(\mathcal{P}) = (\mathrm{Ext}^1_S(\mathcal{F},\mathcal{F})//G)^{\eta^*} = \mathbb{C}^6/\mathbb{Z}_2\times\mathbb{Z}_2$ by dimension reasons, and we conclude because $(C_{[\mathcal{F}]}(\mathcal{P}),[\mathcal{F}])$ is a local analytic model of $(\mathcal{P},[\mathcal{F}])$ (see Proposition 5.1 [4]). $\square$

By Theorem 1.4.4, $\mathcal{P}$ admits a Beauville-Bogomolov form. The problem of determining it is still open.

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Corollary 4.4.11. *The singular locus of $\mathcal{P}$ is exactly the locus of η-invariant strictly C-semistable sheaves and consists only of quotient singularities. It is a union of 27 singular K3 surfaces, each of which is the $\bar{J}_C^{-2}(C)$ of one of the 27 elliptic pencils on Y associated to the 27 lines on Y. Each K3 surface has 5 A_1-singularities, and each of these singular points is an intersection point of three K3 surfaces, so that the intersection graph of the 27 K3 surfaces coincides with the graph of the dual configuration of lines of Y.*

Proof. The identification of the singular locus with the strictly C-semistable sheaves follows from Corollary 4.4.9 and Theorem 4.4.10.

Consider the support map restricted to the singular locus. It gives an elliptic fibration over the 27 dual lines. Restricting it to the elliptic pencil of a line of Y (Corollary 4.4.3), we get an elliptic fibration which is the relative compactified Jacobian $\bar{J}_C^{-2}(C)$ of this pencil. It can be identified with $\mathcal{J}_{S,C}^{-2}(|C_2|)$, hence it is a K3. Since each one of these 27 elliptic fibrations on S has 5 reducible fibers with two simple nodes, and $\chi(S) = 24$, it has 14 irreducible singular members with a simple node. Hence the elliptic fibration $\bar{J}_C^{-2}(C)$ has exactly 19 irreducible singular fibers and $\chi = 19$, so it is singular. By Lemma 2.3.5, it has 5 A_1-singularities at the points corresponding to the $\mathcal{S}$-equivalence classes of $[\mathcal{O}_{C^1}(-2pt) \oplus \mathcal{O}_{C^2}(-2pt)]$. □

4.4.2 Simple connectedness and irreducibility of $\mathcal{P}$

In this subsection, we prove that $\mathcal{P}$ is simply connected and has $H^{(2,0)}$ generated by the symplectic form, so it is a singular irreducible symplectic 6-fold.

To this aim, we describe a birational model of $\mathcal{P}$ as a quotient of $S^{[3]}$ by an involution.

For a generic $\xi \in S^{[3]}$, $C_\xi := \langle \xi, p_0 \rangle \cap S$ is a generic element of $|\mathcal{O}_S(1)|^{\tau,+}$ and $C'_\xi := \langle \pi(\xi) \rangle \cap Y$ is a generic member of $|\mathcal{O}_Y(1)|$. Indeed, a τ-invariant hyperplane contains p_0, so it is given by p_0 and 3 other points.

Hence generically $\xi \in J^3(C_\xi)$, and $\xi - \tau(\xi) \in P(C_\xi, \tau)$.

We thus obtain the natural map

$$\begin{aligned} \psi :& S^{[3]} \dashrightarrow \mathcal{P} \\ & \xi \mapsto (1-\tau)\xi \end{aligned} \tag{4.4.17}$$

Its indeterminacy locus is generically given by ξ such that $\dim\langle \xi \rangle < 3$, i.e. $\langle \xi \rangle$ is a line. A line meeting S in 3 points, meets also Z_2 in 3 points, so it lies on Z_2. Vice versa, a line on Z_2 clearly meets Z_3 in 3 points, so it also meets S in these 3 points. Hence

$$Indet(\psi) = \{\text{lines in } Z_2\} = \mathbb{P}^3.$$

Let us consider the natural rational involution

$$\begin{aligned} \iota_0 :& S^{[3]} \dashrightarrow S^{[3]} \\ & \xi \mapsto (\langle \xi \rangle \cap S) - \xi. \end{aligned} \tag{4.4.18}$$

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It is antisymplectic as proven in [52], Proposition 4.1. Again

$$Indet(\iota_0) = \{\text{lines in } Z_2\}.$$

If we consider a generic $\xi \in Indet(\iota_0)$, then $\langle \xi, p_0 \rangle \cong \mathbb{P}^2$ meets S in 6 points, respectively ξ and $\tau(\xi)$, and $\tau(\xi) \in Indet(\iota_0)$. So we can extend ι to an involution on the blowup $Bl(S^{[3]})$ of $S^{[3]}$ along $Indet(\psi)$:

$$\iota_1 : Bl(S^{[3]}) \to Bl(S^{[3]}). \tag{4.4.19}$$

τ induces a natural involution on $S^{[3]}$ and on $Bl(S^{[3]})$, which we denote again by τ. Since it comes from a linear involution, it commutes with ι_0 and ι_1. Setting $\iota_2 := \iota_1 \circ \tau$, we get an involution on $Bl(S^{[3]})$, which comes from a rational symplectic involution of $S^{[3]}$.

Lemma 4.4.12. *ψ is a rational double cover with involution ι_2, hence $M := Bl(S^{[3]})/\iota_2$ is birational to $\mathcal{P}$.*

Proof. Let $\xi = \{p_1, p_2, p_3\}$ be generic. We want to determine all the divisors $\xi' = \{p'_1, p'_2, p'_3\}$ on C_ξ such that $\xi - \tau(\xi) \sim \xi' - \tau(\xi')$. Equivalently, setting $\delta := \xi + \tau(\xi')$, we want to determine the solutions of $\delta \sim \tau(\delta)$ for ξ generic.

If $\delta = \tau(\delta)$, then δ is τ-invariant and, modulo the permutations of ξ and of ξ', we have only 3 possibilities:

a) $p'_i = \tau(p_i)$, $i = 1, 2, 3$, then $2\xi \sim 2\tau(\xi)$, hence ξ is non-generic.

b) $p'_1 = \tau(p_1)$, $p'_2 = \tau(p_2)$, $p'_3 = p_3$, then $2(p_1 + p_2) \sim 2(\tau(p_1) + \tau(p_2))$, hence ξ is non-generic.

c) $p'_1 = \tau(p_1)$, $p'_2 = p_2$, $p'_3 = p_3$, then $2p_1 \sim 2\tau(p_1)$, hence ξ is non-generic.

If $\delta \neq \tau(\delta)$, then $\dim |\delta| > 0$. By Riemann-Roch theorem we have $\dim |\delta| = 3 + \dim |K_{C_\xi} - \delta|$, with $\deg K_{C_\xi} = \deg \delta = 6$.
There are 3 subcases:

d) $K_{C_\xi} \sim \delta$, so $\langle \delta \rangle$ is a plane in $\langle C_\xi \rangle \cong \mathbb{P}^3$, and $|\delta| = \mathbb{P}^{3*}$. Then $\tau(\xi')$ is uniquely determined as $\langle \xi \rangle \cap C_\xi - \xi$. So the unique nontrivial solution is $\iota_2(\xi)$.

e) $K_{C_\xi} \neq \delta$ and $|\delta|$ is base point free. Then none of the possible 5-uples of points of δ lies on a plane. Now $|\mathcal{O}_{\mathbb{P}^3}(2)| = \mathbb{P}^9$, and $|\mathcal{O}_{C_\xi}(2)| = |2H_{C_\xi}| \cong \mathbb{P}^8$, since $C_\xi \subset \langle \xi, p_0 \rangle \cap Z_2$. So there exist 6 points $\bar{\delta}$ on C_ξ such that $|\delta|$ consists of the residual intersection $(q \cap C_\xi) - \bar{\delta}$, where $q \in |2H_{C_\xi} - \bar{\delta}| = \mathbb{P}^2$. Moreover, τ acts linearly on $\langle \xi \rangle$, so $q \in |2H_{C_\xi} - \bar{\delta}|$ if and only if $\tau(q) \in |2H_{C_\xi} - \tau(\bar{\delta})|$. As $\delta \sim \tau(\delta)$, the two families coincide. We deduce that $\bar{\delta}$ is τ-invariant, and hence every quadric in $|2H_{C_\xi} - \bar{\delta}|$ is τ-invariant. Thus $\xi = \xi'$, i.e. $\delta = \tau(\delta)$, absurd.

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f) $K_{C_\xi} \neq \delta$, and $|\delta| = \mathbb{P}^2$ has a base point. Then 5 points of δ span a plane Π; assume they are $p_1, p_2, p_3, \tau(p'_1), \tau(p'_2)$. Setting $\bar{p}$ the remaining intersection point of Π with C_ξ, $|\delta|$ is clearly given by $|H_{C_\xi} - \bar{p}| = \mathbb{P}^2$. As $\delta \sim \tau(\delta)$, $\bar{p}$ is τ-invariant. So ξ spans a plane passing through one of the six τ-invariant points of C_ξ, hence ξ is non-generic.

We conclude that the generic fiber of ψ consists of only two points, interchanged by ι_2. □

Corollary 4.4.13. $h^{(2,0)}(\mathcal{P}) = 1$.

Proof. As M admits a rational dominant map onto $\mathcal{P}$ by Lemma 4.4.12, we have $h^{(2,0)}(\mathcal{P}) = h^{(2,0)}(M) = h^{(2,0)}(S^{[3]}) = 1$. □

Lemma 4.4.14. *Fix(ι_2) is the disjoint union of two smooth irreducible 4-folds and 120 isolated points.*

Proof. Obviously, the fixed locus of a biregular involution on a smooth variety is also smooth.

For a generic ι_2-invariant ξ, the plane $\langle\xi\rangle$ is τ-invariant, because the planes are ι_1-invariant. Recalling that Fix(τ)$= H_4 \cup p_0$, either $\langle\xi\rangle \subset H_4$ or $p_0 \in \langle\xi\rangle$ (indeed if there exists $p \in \langle\xi\rangle - H_4$, then $p_0 \in \langle\xi\rangle$).

In the first case $\xi \subset S \cap H_4$ is τ-invariant, so $\iota_1(\xi) = \xi$. Then $\langle\xi\rangle$ is totally tangent to the curve $S \cap H_4$. This imposes 3 conditions in $\mathbb{P}^3$, hence we expect a finite number of such ξ's. These correspond to the odd theta characteristics of the curve, which are exactly $2^3(2^4 - 1) = 120$.

In the second case, we obtain the remaining part of Fix(ι_2) as

$$\Sigma := \{\xi \in S^{[3]} : p_0 \in \langle\xi\rangle\}.$$

To describe it, we consider the natural map $Bl(S^{[3]}) \to \mathbb{G}(2,4)$, implicitly defined before. It is a $\binom{6}{3} = 20$-to-1 covering. The image of Σ is clearly

$$\{\Pi \text{ plane } \subset \mathbb{P}^4 : p_0 \in \Pi\} = \sigma_{1,1} \cong \mathbb{G}(1,3),$$

so Σ is a 20:1 covering of a smooth quadric of $\mathbb{P}^5$. Since $\langle\xi\rangle$ is τ-invariant, $\langle\xi\rangle \cap S = \{\xi, \tau(\xi)\}$. The 12 triples $\{p_i, \tau(p_i), p_j\}$ and $\{p_i, \tau(p_i), \tau(p_j)\}$ sweep a 4-fold $\Sigma_1 \subset S^{[3]}$ which is a double covering of $Y^{[2]}$. The other 8 triples sweep Σ_2, an 8-sheeted covering of $\sigma_{1,1}$. So Σ has 2 disjoint irreducible components, Σ_1 and Σ_2. □

Remark 4.4.15. Considering ι_2 as a rational involution on $S^{[3]}$, it has the same fixed locus, because a line on Z_2 does not lie in H_4 and does not pass through p_0.

Proposition 4.4.16. *$\mathcal{P}$ is simply connected.*

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Proof. As M is a rational double cover of $\mathcal{P}$ by Lemma 4.4.12, M has the same fundamental group as $\mathcal{P}$.

Since there are fixed points, M has the same fundamental group as $Bl(S^{[3]})$, which has the same fundamental group of $S^{[3]}$ because it is a blowup along a smooth locus. □

Remark 4.4.17. Considering the invariant part of the action of η_* on $T_P(S^{[3]})$, it is easy to see that M has singularities of type $\mathbb{C}^4 \times (\mathbb{C}^2/\pm 1)$ on Σ_i and of type $\mathbb{C}^6/\pm 1$ at the isolated points. Since the singularities of M and of $\mathcal{P}$ are different, they are only birational. It is interesting to determine explicitly a birational transformation between them.

Remark 4.4.18. It remains an open question if $\mathcal{P}$ can be expressed as a quotient of another manifold by a regular finite group action.

4.4.3 Euler characteristic of $\mathcal{P}$

To calculate the Euler characteristic of $\mathcal{P}$, we can use the fibration structure. Since the Euler characteristic is additive, i.e. $\chi(X) = \chi(U) + \chi(X - U)$ for any open $U \subset X$, and multiplicative for topologically trivial fibrations, i.e. $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$, we can stratify $|\mathcal{O}_S(1)|^{\tau,+} = |\mathcal{O}_Y(1)| = \mathbb{P}^{3*}$ depending on the fibers. Since $\chi = 0$ for any smooth abelian variety, it is enough to study the locus of singular fibers. They come from the singular curves of the linear system. This subset of $|\mathcal{O}_Y(1)|$ is called discriminant of the fibration, and is denoted by Δ.

Lemma 4.4.19. *The discriminant $\Delta \subset \mathbb{P}^{3*}$ consists of two irreducible components: the dual Y^* of the cubic surface, of degree 12, and the dual B^* of the branch locus of ϕ, of degree 18.*

Proof. C is singular if and only if C' is singular or C' is tangent to B.

The degrees of Y^* and B^* can be easily determined using Schubert calculus in $\mathbb{P}^{3*}$:

$$\deg Y^* = Y^* \cap \sigma_{1,0,0}^2 = Y^* \cap \sigma_{1,1,0} = \{H : l \subset H, H \text{ tangent to } Y\},$$

$$\deg B^* = \{H : l \subset H, H \ \ tangent \ \ to \ \ B\},$$

with l a generic line in $\mathbb{P}^3$. Denoting P, Q, R the intersection points of Y and l, we have a natural map $f : Bl_{P,Q,R}(Y) \to \sigma_{1,1,0} \cong \mathbb{P}^1$ such that $f(p) = \langle p, l \rangle$. The degree of Y^*, which is the number of planes tangent to Y and passing through l, corresponds to the number N of singular fibers of f. Using the good properties of the Euler characteristic, we get

$$\chi(Bl_{P,Q,R}(Y)) = \chi(\mathbb{P}^1 - N \text{ pts})\chi(\text{smooth fiber}) + \chi(N \text{ pts})\chi(\text{singular fiber}),$$

i.e. $N = 12$, because $\chi(Bl_{P,Q,R}(Y)) = \chi(Bl_{9pts}(\mathbb{P}^2)) = \chi(\mathbb{P}^2) + 9 = 12$ and a smooth fiber has χ equal to zero (it is an elliptic curve), while a singular

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fiber has $\chi = 1$ (it is a nodal plane cubic).
To determine $\deg B^*$, we can consider the 6:1 cover $g : B \to \sigma_{1,1,0} \cong \mathbb{P}^1$ such that $g(p) = \langle p, l\rangle$. The degree corresponds to the degree of the branch locus, which is 18 by Riemann-Hurwitz theorem. □

Remark 4.4.20. The degree of the discriminant locus of $\mathcal{P}$ is 30. For irreducible symplectic 6-folds obtained as Beauville-Mukai integrable systems, the degree is 36. General results on the degree of a Lagrangian fibration with Jacobians of integral curves as fibers have been obtained by Sawon in [59].

We focus on the natural stratification of Δ, which will permit us to determine $\chi(\mathcal{P})$.

Theorem 4.4.21. *For a generic S, Δ admits a natural stratification in singular loci (with several irreducible components), corresponding to all the possible singular members of $|\mathcal{O}_S(1)|^{\tau,+}$, as described in the following:*

Dimension 2

a) *C has a simple τ-invariant node, if $\langle C'\rangle \in B^* - Sing(B^* \cup Y^*)$ (i.e. C' is tangent to B);*

b) *C has two simple nodes, interchanged by τ, if $\langle C'\rangle \in Y^* - Sing(B^* \cup Y^*)$ (i.e. C' has a simple node);*

Dimension 1

c) *C has two simple τ-invariant nodes, if $\langle C'\rangle$ lies in the complement of $Sing(Sing(B^* \cup Y^*))$ inside the irreducible component of $Sing(B^*)$ corresponding to C' bitangent to B;*

d) *C has one cusp, if $\langle C'\rangle$ lies in the complement of $Sing(Sing(B^* \cup Y^*))$ inside the irreducible component of $Sing(B^*)$, which corresponds to C' having a point of triple contact with B;*

e) *C has three simple nodes, one fixed by τ and the others interchanged by τ, if $\langle C'\rangle$ lies in the complement of $Sing(Sing(B^* \cup Y^*))$ inside the irreducible component of $B^* \cap Y^*$ corresponding to C' tangent to B and having a simple node;*

f) *C has a tacnode, if $\langle C'\rangle$ lies in the complement of $Sing(Sing(B^* \cup Y^*))$ inside the irreducible component of $B^* \cap Y^*$ corresponding to C' with a simple node on B, in other words C' is cut out by a plane tangent to Y at a point of B;*

g) *C has two simple cusps, interchanged by τ, if $\langle C'\rangle$ lies in the complement of $Sing(Sing(B^* \cup Y^*))$ inside the irreducible component of $Sing(Y^*)$ corresponding to C' with a cusp;*

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h) *C has two irreducible components meeting in four points, interchanged in pairs by τ, if $\langle C'\rangle$ lies in the complement of $Sing(Sing(B^* \cup Y^*))$ inside the irreducible component of $Sing(Y^*)$, which corresponds to C' being a reducible plane cubic that decomposes into a conic and a line;*

<u>*Dimension 0*</u>

i) *C has a cusp and a simple node, if $\langle C'\rangle$ is one of the points of $Sing(Sing(B^*))$ corresponding to C' with a triple contact point with B and another simple tangency point;*

j) *C has a tacnode, if $\langle C'\rangle$ is one of the points of $Sing(Sing(B^*))$ corresponding to C' with a quadruple contact point with B;*

k) *C has three simple τ-invariant nodes, if $\langle C'\rangle$ is one of the points of $Sing(Sing(B^*))$ corresponding to C' with a tritangent to B;*

l) *C has two simple cusps interchanged by τ and a simple τ-invariant node, if C' is one of the points of $B^* \cap Sing(Y^*)$, which corresponds to C' being tangent to B and having a simple cusp outside B;*

m) *C has an A_5-singularity, if $\langle C'\rangle$ is one of the points of $B^* \cap Sing(Y^*)$ corresponding to C' with a simple cusp on B;*

n) *C has two irreducible components meeting in pairs in two points interchanged by τ and a simple node only on one of the two components, if $\langle C'\rangle$ is one of the points of $B^* \cap Sing(Y^*)$, which corresponds to C' being a reducible plane cubic that is the union of a line and a conic tangent to B;*

o) *C has two τ-invariant simple nodes and two simple nodes interchanged by τ, if $\langle C'\rangle$ is one of the points of $Sing(B^*) \cap Y^*$ corresponding to C' bitangent to B and having a simple node outside B;*

p) *C has a tacnode and a simple node, if $\langle C'\rangle$ is one of the points of $Sing(B^*) \cap Y^*$ corresponding to C' tangent to B and having a singular point on B;*

q) *C has a D_4-singularity, if $\langle C'\rangle$ is one of the points of $Sing(B^*) \cap Y^*$ corresponding to C' tangent to B in a singular point;*

r) *C has two simple nodes interchanged by τ and a simple τ-invariant cusp, if $\langle C'\rangle$ is one of the points of $Sing(B^*) \cap Y^*$ corresponding to C' with a triple tangency point on B and a singular point outside B;*

s) *C has three irreducible components meeting in pairs in two points interchanged by τ, if $\langle C'\rangle$ is one of the points of $Sing(Sing(Y^*))$ (i.e. a reducible plane cubic given by three lines).*

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Proof. As in the previous description of the discriminant locus, to describe the singularities of C it is enough to look at C'. If C' has a simple node/cusp outside B, then C inherits two nodes/cusps interchanged by τ. If C' has a double/triple/quadruple tangency point with B, then C inherits a τ-invariant simple node/cusp/tacnode. If C' has a simple node on B, then C has a tacnode, because locally C' has equation $u^2 + v^2 = 0$, hence C is given by $t^2 = u, u^2 + v^2 = 0$, or $t^4 + v^2 = 0$. If C' has a simple node on B and B is tangent to one of the two branches of the curve through it, then C has a D_4-singularity, because locally C' has equation $uv + v^3 = 0$, hence C is given by $t^2 = u, uv + v^3 = 0$, or $(t^2 + v^2)v = 0$. If C' has a simple cusp on B, then C has an A_5-singularity, because locally C' has equation $u^3 + v^2 = 0$, hence C is given by $t^2 = u, u^3 + v^2 = 0$, or $t^6 + v^2 = 0$.

□

Continuing the calculation of $\chi(\mathcal{P})$, we denote by $\Pi_\bullet$ the locus of points such that the condition $\bullet$) of Theorem 4.4.21 holds, and by $\bar{P}_\bullet$ the fiber over a point of $\Pi_\bullet$ (i.e. the compactified Prym variety of a curve from $\Pi_\bullet$). Then

$$\chi(\mathcal{P}) = \chi(\Pi_a)\chi(\bar{P}_a) + ... + \chi(\Pi_s)\chi(\bar{P}_s). \tag{4.4.20}$$

To calculate $\chi(\bar{P}_\bullet)$, we follow the stratification of $\bar{P}_\bullet$ used in [35], based on a description of $\bar{J}(C)$ by Cook in [13]. We recall that $\bar{J}(C)$ admits a stratification in smooth strata whose codimension is equal to the index $i(\mathcal{F})$ of the sheaves $\mathcal{F}$ represented by points of these strata. By Lemma 2.1.5, the normalization map $\nu : \tilde{C} \to C$ factorizes through a partial normalization $\bar{\nu} : \bar{C} \to C$ such that $\bar{\nu}^*(\mathcal{F})/(tors)$ is invertible, and $i(\mathcal{F})$ is the minimum of $length(\bar{\nu}_*(\mathcal{O}_{\bar{C}})/\mathcal{O}_C)$. When C is integral, the index takes values between 0 and $\delta(C) = length(\bar{\nu}_*(\mathcal{O}_{\tilde{C}})/\mathcal{O}_C) = p_a(C) - g(C)$. Each stratum can be described as an extension of $J(\bar{C})$ by an algebraic group. Let $J_i(C)$ be the stratum of codimension i. So $J_0(C) = J(C)$. The map $\mathcal{F} \mapsto \nu^*(\mathcal{F})/(tors)$, restricted to $J_i(C)$, gives a morphism $v_i : J_i(C) \to \mathrm{Pic}^{-i}(\tilde{C})$.

We denote by P_i the stratum $J_i(C) \cap \bar{P}$ induced on $\bar{P}$. So $P_0 = P(C, \tau)$, an algebraic group of dimension 4. Moreover, τ extends to an involution on $\tilde{C}$ corresponding to the double cover $\tilde{C} \to \tilde{C}'$, where $\tilde{C}'$ is the normalization of C'. Each stratum can be obtained as an extension of $P(\tilde{C}, \tau)$ with an algebraic group.

Theorem 4.4.22. $\chi(\mathcal{P}) = 2283$.

Proof. Similarly to Proposition 2.2 of [10], $\chi(P_\bullet)$ corresponds to the number of 0-dimensional strata of $P_\bullet$. Hence it is nonzero only in the cases $k), n), o), s)$, and it suffices to determine the cardinality of $\Pi_\bullet$ and the 0-dimensional strata of $P_\bullet$ in these cases.

k) The number of tritangents to B corresponds to the number of odd theta characteristics. Hence we get $2^3(2^4 - 1) = 120$ points.

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We need to determine the zero-dimensional strata of $\bar{P}$, which is P_3, because C is irreducible and $\delta(C) = 3$. First, we observe that $P(\tilde{C}/C')$ is given by two points. Indeed C has three τ-invariant nodes, p_1, p_2, p_3. By Riemann-Hurwitz, the induced τ on $\tilde{C}$ is base-point-free, so $\tau(p_i') = p_i''$, with $\nu^{-1}(p_i) = \{p_i', p_i''\}$, and τ is a translation by a 2-torsion point $q = [p_1' - p_1''] = [p_1' - p_2''] = [p_3' - p_3''] \in J(\tilde{C}) = \tilde{C}$. So η has 4 fixed points on $\tilde{C}$ (the four solutions of $2p = q$) and $P(\tilde{C}/C')$ consists of two points.

Following Cook [13], the elements of $J_3(C)$ are of the form $\nu_*(\mathcal{L})$, with $\mathcal{L} \in \mathrm{Pic}^{-3}(\tilde{C})$. To determine P_3, we need to describe the action of η on $J_3(C)$:

$$j(\nu_*(\mathcal{L})) = \nu_*((\mathcal{L}^{-1})(-p_1' - p_1'' - p_2' - p_2'' - p_3' - p_3'')),$$

$$\tau(\nu_*(\mathcal{L})) = \nu_*(\tau(\mathcal{L})).$$

Hence $\nu_*(\mathcal{L}) \in P_3$ if and only if $\mathcal{L} \in P(\tilde{C}/C')$, and P_3 consists of two points.

$n)$ The number of reducible curves given by a conic tangent to B and a line on Y does not correspond to the intersection number of $h)$ and $a)$, because the line is not generic. To calculate it, we consider the pencil of planes of $\mathbb{P}^3$ containing a fixed line on Y. Then B intersects the line in 2 points, and a plane of the pencil in the same 2 points plus other 4 points. Thus we get a 4:1 cover $B \to \mathbb{P}^1$, and the degree of the branch locus is 14 by Riemann-Hurwitz. As there are 27 lines on a cubic surface, the number of points of $n)$ is $14 \cdot 27 = 378$.

The zero-dimensional stratum of $\bar{P}$ is P_5, because C has four simple nodes. $P(\tilde{C}/\tilde{C}')$ is a point, because $\tilde{C}$ and $\tilde{C}'$ are rational curves. The elements of $J_5(C)$ are of the form $\nu_*(\mathcal{O}_{\tilde{C}}(d-1) \oplus \mathcal{O}_{\tilde{C}}(-d))$, for d satisfying semistability conditions, i.e. $d = 0, \pm 1, \pm 2$ (± 1 represent the same $\mathcal{S}$-equivalence class, and also ± 2). P_5 thus consists of three points.

$o)$ To determine the number of nodal curves bitangent to B, we calculate it indirectly, determining the degree of the curve (case $c)$) of bitangents to B and the number (case $p)$) of curves tangent to B and having a singular point on B. Indeed, $b) \cap c) = 2p) + o)$, since $b)$ and $c)$ meet transversely and $p)$ has intersection multiplicity 2 because the bitangents of $b)$ can acquire a node in one of the two tangency points.

The degree of $c)$ can be obtained considering a projection of B onto a plane from a generic fixed point: the number of bitangents of B corresponds to the number of bitangents of the image B', which is a plane curve with the same geometric genus and degree, i.e. $g = 4$ and

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$d = 6$. Since the arithmetic genus of a plane sextic is 10, B' has 6 simple nodes. By Plücker formulas, we have

$$g = (d^* - 1)(d^* - 2)/2 - b - f, \qquad d = d^*(d^* - 1) - 2b - 3f,$$

where d^* is the degree of the dual curve of B', b is the number of bitangents, f the number of flexes. So we need to determine d^*. Again by Plücker formulas

$$d^* = d(d - 1) - 2\delta - 3\kappa,$$

where δ is the number of simple nodes and κ the number of cusps. Hence $d^* = 18$ and $b = 90$, so c) has degree 90.

The degree of p) can be obtained considering the curve of the case f). From it we can define

$$D := \{\Pi \cap B - \{p\} : \Pi \text{ tangent to } Y \text{ at } p\}_{p\in B}.$$

It is a 4:1 cover of B with branching p). By Riemann-Hurwitz, to calculate p), it is enough to determine the genus of D. D can be seen as a subvariety of $B \times B \subset \mathbb{P}^3 \times \mathbb{P}^3$: the equation $\sum x_i \partial_i F(\underline{p}) = 0$, with $((x_i), (\underline{p})) \in B \times B$, gives $D + 2\Delta_B$. Setting $f_1 := B \times pt$, $f_2 := pt \times B$, we get that $D \sim 12f_1 + 6f_2 - 2\Delta_B$ numerically, hence $K_D \sim (K_{B\times B} + D)D \sim (18f_1 + 12f_2 - 2\Delta_B)(12f_1 + 6f_2 - 2\Delta_B)$ and using the intersection relations $\Delta_B \cdot f_1 = \Delta_B \cdot f_2 = 0, f_1^2 = f_2^2 = 0, \Delta_B^2 = \deg \mathcal{N}_B = \deg \mathcal{T}_B = 2 - 2g_B = -6$, we have $K_D \sim 132$. So $g(D) = 67$, and by Riemann-Hurwitz the branch locus consists of 108 points.

In conclusion, o) consists of $90 \cdot 12 - 2 \cdot 108 = 864$ points. The zero-dimensional stratum of $\bar{P}$ is P_4, because C is irreducible and $\delta(C) = 4$. $P(\tilde{C}/\tilde{C}')$ is a point, because $\tilde{C}$ and $\tilde{C}'$ are rational curves. Similarly to k), the elements of $J_4(C)$ are of the form $\nu_*(\mathcal{O}_{\tilde{C}}(-4))$, and $P_4 = P(\tilde{C}/\tilde{C}')$ is a point.

s) We have 45 points, which are the intersection points of the orthogonal lines to the 27 lines on Y.

Collecting the previous calculation, we obtain by (4.4.20)

$$\chi(\mathcal{P}) = 120 \cdot 2 + 378 \cdot 3 + 864 \cdot 1 + 45 \cdot 1 = 2283.$$

□

Remark 4.4.23. Many computations of this proof are similar to those of Proposition 4.3 of [35]. In particular, we remark that at point iv) there is a mistake: indeed P_2 consists of 2 points, not 4 (and analogously in P_1 and

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P_0 there are half of the copies of $\mathbb{C}^*$ and of $\mathbb{C}^* \times \mathbb{C}^*$). This follows from the same considerations as in the previous proof for the item k). For this reason, the computation in [35] of the Euler characteristic of the 4-fold from the Del Pezzo of degree 2 described in Section 4.3 should be corrected as follows:

$$\chi(\mathcal{P}) = 28 \cdot 2 + 128 \cdot 1 + 28 \cdot 1 = 212.$$

This computation agrees with the one done by Menet in [39], Proposition 2.40, where he determines the Euler characteristic of the 4-fold relating it to the quotient of a K3 surface by an involution.

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Chapter 5

Relative 0-Prym varieties from abelian surfaces

In this chapter we extend Markushevich and Tikhomirov's contruction to the case of abelian surfaces admitting an antisymplectic involution. In this context, the relative Prym variety is not simply connected. In order to get an irreducible symplectic variety, we define the relative 0-Prym variety by restricting the global Prym involution to a fiber of the Albanese map (Section 5.1). As we obtain again a singular variety, we study a local model of it using the Kuranishi map and we determine its simplest singularities (Section 5.2). We then focus on the case of abelian surfaces admitting an antisymplectic involution without fixed points, which we classify using elementary properties of lattices (Section 5.3). The quotients by such involutions are bielliptic surfaces. In order to characterize the corresponding relative 0-Prym varieties, we study divisors and linear systems on bielliptic surfaces (Section 5.4). We finally obtain our main result on relative 0-Prym varieties associated to curves on bielliptic surfaces (Section 5.5). Essentially, in the cases under consideration, either the 0-Prym variety is birational to an irreducible symplectic manifold of $K3^{[n]}$-type, or it does not admit any symplectic resolution.

5.1 Relative 0-Prym variety

Let (A, τ) be a generic abelian surface with an antisymplectic involution. Analogously to the case of a K3 surface [47], a complex torus of dimension 2 admitting an antisymplectic involution is automatically projective. Let $\tau \circlearrowright A \xrightarrow{\pi} Y$ be the induced double cover. Let C be a τ-invariant smooth curve on A and $\tau \circlearrowright C \xrightarrow{\pi} C'$ the induced morphism, where C' is a smooth curve on Y. Let $\{C\}$ be the irreducible component of the Hilbert scheme of curves on A with the same cohomology class as C. It admits a natural action of A by translations, and its quotient by A is $|C|$. Moreover $\{C\}$ admits a

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natural map onto $\hat{A}$ associating to C_0 the line bundle $\mathcal{O}_A(C_0 - C)$, which has $|C_0|$ as the fiber over C_0. This map trivializes after the isogeny

$$\begin{array}{ccc} A \times |C| & \longrightarrow & \{C\} \\ \downarrow{\scriptstyle p_1} & & \downarrow \\ A & \xrightarrow{(g-1):1} & \hat{A}, \end{array} \tag{5.1.1}$$

where the map between A and $\hat{A}$ is the natural map associating to p the invertible sheaf $\mathcal{O}_A(t_p^* C - C)$, and g is the genus of C.

Let $\mathcal{J}_C := \mathcal{J}_A^C(C)$ be the relative Jacobian variety associated to $\{C\}$. The Albanese map

$$\begin{aligned} Alb : &\mathcal{J}_C \to A \times \hat{A} \\ &[\mathcal{F}] \mapsto \left(\sum \widetilde{c}_2(\mathcal{F}), \widetilde{c}_1(\mathcal{F})\right) \end{aligned}$$

is surjective. Moreover for 1-dimensional sheaves, it admits the following description. The first component $\sum \tilde{c}_2$ restricted to the fiber of the support map over C_0 is the map $-\bar{\iota} : J(C_0) \to A$, where $\bar{\iota}$ is the map induced by $\iota : C_0 \to A$ via the universal property of the Jacobian. The negative sign is due to the fact that $C \to J(C)$ sends p to $\mathcal{O}_C(p)$ and $\tilde{c}_2(\mathcal{O}_C(p)) = C^2 - p$. The second component $\tilde{c}_1$ is given by the determinant map (2.2.3), and is constant on $\mathcal{J}(|C_0|)$ for every $C_0 \in \{C\}$, because it factors through $|C_0| = \mathbb{P}^{g-2} \to A$, and the image of a projective space in an abelian variety is obviously a point.

As the Albanese map is surjective and A is not simply connected, $\mathcal{J}_C$ cannot be simply connected.

Similarly to the case of a K3 surface (Definition 3.2.9), we can define the relative Prym variety $\mathcal{P}_C$ associated to C as a connected component of the fixed locus of a symplectic involution η on $\mathcal{J}_C$. It has a natural fibration structure given by the support map $\mathcal{P}_C \to \{C\}^{\tau,+}$ over one of the two connected components of $\{C\}^\tau$. $\mathcal{P}_C$ inherits a symplectic structure from $\mathcal{J}_C$. The main difference with respect to the case of a K3 surface is that $\mathcal{P}_C$ is not simply connected, essentially because its image under the Albanese map is not simply connected.

Lemma 5.1.1. *Let (A, τ) be an abelian surface with an antisymplectic involution. Then the Albanese map induces a map*

$$\mathcal{P}_C \to \mathrm{Fix}_A^0(L(\tau)) \times \mathrm{Fix}_{\hat{A}}(\tau^*)$$

where $L(\tau)$ is the linear part of τ and $\mathrm{Fix}_A^0(L(\tau))$ is the connected component of $\mathrm{Fix}_A(L(\tau))$ containing zero. In particular, $\mathcal{P}_C$ is not simply connected.

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Proof. Consider the restriction of the Albanese map to $\mathcal{P}_C$. Let C_0 be a smooth curve in $\{C\}^\tau$. By the universal property of the Jacobian, the image of $P(C_0, \tau) = \mathrm{Fix}^0_{J(C_0)}(-\tau)$ is $\mathrm{Fix}_A(L(\tau))$, as the map $\bar{\iota}$ is unique up to translations and $\sum \tilde{c}_2$ is $-\bar{\iota} : J(C_0) \to A$. Since $L(\tau)$ is an antisymplectic involution with fixed points preserving the zero point, the fixed locus is a union of elliptic curve. The image of the first component is the one passing through the origin. Thus $\mathcal{P}_C$ is not simply connected.

Since the support map on $\mathcal{P}_C$ has $\{C\}^{\tau,+}$ as the image, the image of the second component of the Albanese map is $\mathrm{Fix}_{\hat{A}}(\tau)$. If τ has fixed points then $\mathrm{Fix}_{\hat{A}}(\tau) = \mathrm{Pic}^0(Y)$; if not, then the map from $\mathrm{Pic}^0(Y)$ to $\hat{A} = \mathrm{Pic}^0(A)$ is not injective (the kernel is given by the divisor defining the double cover), but $\mathrm{Fix}_{\hat{A}}(\tau)$ still is the image of $\mathrm{Pic}^0(Y)$. □

As in Theorem 2.4.8, in order to get an irreducible symplectic variety, we can consider a fiber $\mathcal{K}_C := \mathcal{K}_{A,C}(C)$ of the Albanese map. By Theorem 2.4.9 and Theorem 2.4.8, if C is primitive of genus $g \geq 6$, then $\mathcal{K}_C$ is birational to an irreducible symplectic manifold of Kummer-$(g-2)$ type. Moreover $\mathcal{K}_C \xrightarrow{\text{supp}} |C|$ is a Lagrangian fibration in abelian varieties with polarization of type $(1, ..., 1, g-1)$. A generic fiber is $K_A(C)$, the abelian subvariety of $J(C)$, complementary to A.

The natural commutative diagram

$$\begin{array}{ccccc} \mathcal{K}_C & \longrightarrow & \mathcal{J}_C & \xrightarrow{Alb} & A \times \hat{A} \\ \downarrow{\scriptstyle \text{supp}} & & \downarrow{\scriptstyle \text{supp}} & & \downarrow \\ |C| = \mathbb{P}^{g-2} & \longrightarrow & \{C\} & \xrightarrow{det} & \hat{A} \end{array} \tag{5.1.2}$$

gives fiberwise, over the locus of smooth curves, the exact sequence

$$K_A(C) \to J(C) \to A, \tag{5.1.3}$$

where $J(C) \to A$ is the group morphism induced by $C \hookrightarrow A$ via the universal property of the Jacobian.

As $\{C\}$ trivializes to $A \times |C|$ after the isogeny (5.1.1), we also have the cartesian square

$$\begin{array}{ccc} A \times \hat{A} \times \mathcal{K}_C & \longrightarrow & \mathcal{J}_C \\ \downarrow{\scriptstyle p_1 \times p_2} & & \downarrow \\ A \times |C| & \longrightarrow & \{C\}. \end{array} \tag{5.1.4}$$

Lemma 5.1.2. *Let (A, τ) be an abelian surface with an antisymplectic involution. Let C be a τ-invariant smooth curve on A. Then $-\tau$ induces an*

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involution on $K_A(C)$, its fixed locus $P^0(C,\tau)$ is an abelian variety of codimension 1 in $P(C,\tau)$ and there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^0(C,\tau) & \longrightarrow & P(C,\tau) & \longrightarrow & \mathrm{Fix}^0_A(L(\tau)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_A(C) & \longrightarrow & J(C) & \longrightarrow & A & \longrightarrow & 0. \end{array} \tag{5.1.5}$$

Moreover, if τ has no fixed points, then $P^0(C,\tau)$ has polarization of type $(1,..,1,k)$, where $2k = C \cdot \mathrm{Fix}^0_A(L(\tau))$.

Proof. As shown in the proof of Lemma 5.1.1, the image of $P(C,\tau)$ in A is $\mathrm{Fix}^0_A(L(\tau))$, which is an elliptic curve. Hence we have the diagram (5.1.5). If τ has no fixed points, then $P(C,\tau)$ is principally polarized by Theorem 3.1.4, and the polarization $C \cdot \mathrm{Fix}^0_A(L(\tau))$ on $\mathrm{Fix}^0_A(L(\tau))$ induces naturally a polarization of type $(1,...,1,k)$ on $P^0(C,\tau)$ by (5.1.5). □

So η induces a symplectic involution on $\mathcal{K}_C$ and *Alb* induces a map on $\mathcal{P}_C$, and we have the following definition.

Definition 5.1.3. *Let (A,τ) be an abelian surface with an antisymplectic involution. Let C be a smooth irreducible ample τ-invariant curve and $\mathcal{K}_C := \mathcal{K}_{A,C}(C)$. Then $j := \mathcal{E}xt^1(\_, \mathcal{O}_S(-C))$ and τ^* are regular antisymplectic involutions on $\mathcal{K}_C$, and $\eta^0 := j \circ \tau^*$ is a symplectic involution. The fixed locus $|C|^\tau$ has at most two connected components, both of which are projective spaces. Let $|C|^{\tau,+}$ be one of them, such that its generic member represents an irreducible curve. Denote by $\mathcal{K}_C^+$ the restriction of $\mathcal{K}_C$ over $|C|^{\tau,+}$.*

The relative 0-Prym variety associated to C is

$$\mathcal{P}^0_C := \mathrm{Fix}^0(\eta^0),$$

the connected component of the fixed locus containing the zero section. It is a symplectic variety endowed with the Lagrangian fibration

$$\mathrm{supp} : \mathcal{P}^0_C \to |C|^{\tau,+}.$$

There is a natural commutative diagram

$$\begin{array}{ccccc} \mathcal{P}^0_C & \longrightarrow & \mathcal{P}_C & \xrightarrow{Alb} & A \times \hat{A} \\ \downarrow{\scriptstyle \mathrm{supp}} & & \downarrow{\scriptstyle \mathrm{supp}} & & \downarrow \\ |C|^{\tau,+} & \longrightarrow & \{C\}^{\tau,+} & \xrightarrow{det} & \hat{A}. \end{array} \tag{5.1.6}$$

Remark 5.1.4. As in the case of K3 surfaces (Remark 3.2.10), when τ has no fixed points, the irreducible components of $|C|^\tau$ are of the same dimension, and any one of them can be chosen to be $|C|^{\tau,+}$. Denoting the other one

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by $|C|^{\tau,-}$, we have two different relative compactified Prym varieties $\mathcal{P}_C^{\pm}$ associated to A, C, τ, as well as two different 0-Prym varieties of the same dimension with their Lagrangian fibrations $\mathcal{P}^{0,\pm} \to |C|^{\tau,\pm}$.

When τ has a fixed curve D, then only one of the components of $|C|^{\tau}$ is a base point free linear system, and the other component (when it exists) is a linear system with base locus D.

Also in this case the singular locus of $\mathcal{P}_C^0$ is contained in the η^0-invariant part of the singular locus of $\mathcal{K}_C$, so in the locus of η^0-invariant strictly semistable sheaves.

We remark that all the considerations about $\mathcal{P}_C$ that we presented in Chapter 3 are still valid. In particular, by Theorem 3.6.1, to have the regularity of η^0 we are forced to consider a singular space $\mathcal{K}_C$.

5.2 Singularities of $\mathcal{P}_C^0$

Lemma 5.2.1. *Let (A, τ) be an abelian surface with an antisymplectic involution. Let $C = C_1 \cup C_2$ be a curve of arithmetic genus g on A, which is a double cover of a curve C' of genus g', where C_i $(i = 1, 2)$ are smooth irreducible τ-invariant curves meeting transversely in $2\delta \geq 2$ points. Then $\bar{P}_C^0(C, \tau)$ has one irreducible component of dimension $g - g' - 1$, whose generic points parametrize stable sheaves, and the locus of strictly C-semistable sheaves is of dimension $g - g' - \delta - 1$, representing the $\mathcal{S}$-equivalence classes $[\mathcal{F}_1 \oplus \mathcal{F}_2]$ with* $\operatorname{supp}(\mathcal{F}_i) = C_i$ *and $Alb_1(\mathcal{F}_1) + Alb_2(\mathcal{F}_2) = 0$, where $Alb_i : J^{(-1)^{i+1}\delta}(C_i) \to A$.*

Proof. By (5.1.5) of Lemma 5.1.2, we have that $\bar{P}_C^0(C, \tau)$ has codimension one in $\bar{P}_C(C, \tau)$ on smooth curves. By degeneration this holds also on reducible curves. In Lemma 3.3.1 we described $\bar{P}_C(C, \tau)$ for a reducible curve of this type. On the component parametrizing stable sheaves, Alb gives a codimension one condition. On the stratum parametrizing strictly semistable sheaves, Alb splits as $Alb_1 + Alb_2$, where $Alb_i : J^{(-1)^{i+1}\delta}(C_i) \to A$. □

Lemma 5.2.2. *Let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \in \mathcal{K}_C$ be a polystable sheaf such that* $\operatorname{supp}(\mathcal{F}_i) =: C_i$ *are smooth irreducible τ-invariant curves of genus g_i meeting transversely at 2δ points. Then $(C_{[\mathcal{F}]}(\mathcal{K}_C), [\mathcal{F}])$ is locally analytically equivalent to $(\mathbb{C}^{2g_1+2g_2-4} \times \hat{Z}, 0)$, where $\hat{Z}$ is the affine cone over a hyperplane section of the Segre embedding $\sigma_{\delta-1,\delta-1} : \mathbb{P}^{\delta-1} \times \mathbb{P}^{\delta-1} \to \mathbb{P}^{\delta^2-1}$.*

Proof. For a stable sheaf $\mathcal{G}$, the natural decomposition

$$\operatorname{Ext}^1_S(\mathcal{G}, \mathcal{G}) = \operatorname{Ext}^1_C(\mathcal{G}, \mathcal{G}) \times H^0(\mathcal{N}_{C/A}) = T_0 \operatorname{Pic}^0(C) \times T_0 \operatorname{Pic}^0(C)^\vee \quad (5.2.1)$$

gives a natural way to describe the differential of the Albanese map:

$$Alb_* : \operatorname{Ext}^1_S(\mathcal{G}, \mathcal{G}) = T_0 \operatorname{Pic}^0(C) \times T_0 \operatorname{Pic}^0(C)^\vee \to T_0 A \times T_0 \hat{A}. \quad (5.2.2)$$

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Let $\mathcal{F} \in \mathcal{K}_C$ be as in the hypothesis. Using the same notation (2.3.2) as in Lemma 2.3.5, we can represent the differential of the Albanese map as a map

$$Alb_* : U_1 \times U_2 \times W \times W^* \to \mathbb{C}^4. \tag{5.2.3}$$

By (5.1.4), $\mathcal{J}_C$ splits into a direct product $A \times \hat{A} \times \mathcal{K}_C$ modulo an isogeny, so the restriction of Alb_* to $W \times W^*$ is zero. Thus setting $U := \ker(Alb|_{U_1 \times U_2})$, we have

$$k_2^{-1}|_{\ker(Alb_*)}(0)//G = (k_2^{-1}(0)//G) \cap \ker(Alb_*) = U \times (W \times W^*)//G. \tag{5.2.4}$$

We conclude by Lemma 2.3.5 and (5.2.4). □

Theorem 5.2.3. *Let (A, τ) be a generic abelian surface with an antisymplectic involution, C a smooth τ-invariant curve. Let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \in \mathcal{P}_C^0$ be a polystable sheaf such that* $\mathrm{supp}(\mathcal{F}_i) =: C_i$ $(i = 1, 2)$ *are smooth irreducible τ-invariant curves of genera g_i meeting transversely at 2δ points. Then $(\mathcal{P}_C^0, [\mathcal{F}])$ is locally analytically equivalent to $(\mathbb{C}^N \times (\mathbb{C}^{2\delta}/\pm 1), 0)$ with $N = 2(g_1 - g_1' + g_2 - g_2' - 2)$.*

Proof. It follows from the description of $(\mathcal{P}_C, [\mathcal{F}])$ in Theorem 3.3.2, which holds also for an abelian surface (by the same proof), and from the characterization of $(C_{[\mathcal{F}]}(\mathcal{K}_C), [\mathcal{F}])$ in Lemma 5.2.2. □

Corollary 5.2.4. *Let (A, τ) be a generic abelian surface with an antisymplectic involution, C a smooth τ-invariant curve. Suppose that $|C|^\tau$ contains a reducible curve $C_1 \cup C_2$, where C_i are smooth irreducible τ-invariant curves of genus g_i meeting transversely at $2\delta \geq 4$ points.*

Then $\mathcal{P}_C^0$ does not admit any symplectic desingularization.

Proof. The same proof as in Corollary 3.3.4, using Theorem 5.2.3. □

5.3 Abelian surfaces admitting an antisymplectic involution without fixed points

Fujiki classified the antisymplectic involutions of complex tori of dimension 2 in [19]. Essentially there are two cases:

- if $\mathrm{Fix}(\tau) = \emptyset$, then Y is a bielliptic surface;
- if $\mathrm{Fix}(\tau) \neq \emptyset$, then Y is a ruled surface over an elliptic curve.

It is possible to relate abelian surfaces with antisymplectic involutions to K3 surfaces with antisymplectic involutions. Indeed τ induces an involution

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on the Kummer surface, as -1 commutes with every involution. So we have the commutative diagram

$$\begin{array}{ccc} \tau \circlearrowright A & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \tau \circlearrowright A/(-1) & \longrightarrow & Y/(-1) \\ \uparrow & & \uparrow \\ \tau \circlearrowright S & \longrightarrow & Z \end{array} \tag{5.3.1}$$

But the Kummer surface has Picard number ≥ 17, so this is not a generic (S, τ), and hence we cannot use Nikulin's classification.

In the rest of the chapter we focus on antisymplectic involutions without fixed points. They give bielliptic surfaces, which are, by the Enriques classification of projective surfaces (see [6]), minimal surfaces with Kodaira dimension 0 and first Betti number 2. Bagnera and de Franchis classified them in [5] in the beginning of the last century.

Let A be an abelian surface with lattice Λ, and τ an antisymplectic involution on A without fixed points. Then τ is an affine map of $\mathbb{C}^2$,

$$\tau \begin{pmatrix} x \\ y \end{pmatrix} = \underline{M} \begin{pmatrix} x \\ y \end{pmatrix} + \underline{t}, \tag{5.3.2}$$

with

$$(\underline{M} + \underline{I})\underline{t} = 0 \bmod \Lambda \tag{5.3.3}$$

because τ is an involution, and

$$det(\underline{M}) = -1 \tag{5.3.4}$$

because τ is antisymplectic and $\tau(dx \wedge dy) = det(\underline{M}) dx \wedge dy$.

Choosing a basis of eigenvectors of $\underline{M}$, we have

$$\underline{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.3.5}$$

and changing the origin we can suppose $\underline{t} = \begin{pmatrix} t \\ 0 \end{pmatrix}$, thus by (5.3.3)

$$2\underline{t} = \begin{pmatrix} 2t \\ 0 \end{pmatrix} \in \Lambda. \tag{5.3.6}$$

The eigenspaces $\mathbb{C}^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbb{C}^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the corresponding lattices $\Lambda_\pm := \Lambda \cap \mathbb{C}^\pm$ give two elliptic curves

$$E^\pm := \mathbb{C}^\pm / \Lambda_\pm. \tag{5.3.7}$$

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By definition, we get the inclusions of lattices

$$2\Lambda \subset \Lambda_+ \oplus \Lambda_- \subset \Lambda, \tag{5.3.8}$$

which induce the isogenies

$$A \xrightarrow{(I+M)\times(I-M)} E^+ \times E^- \xrightarrow{\Sigma} A. \tag{5.3.9}$$

Clearly the composition of these two maps is $[\cdot 2]$, the multiplication by 2, which is 16:1. Hence the possible degrees of the isogenies are, up to symmetries, $(1,16),(2,8),(4,4)$. Thus we get the four possibilities

$$A = (E^+ \times E^-)/G, \qquad G = 1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, \tag{5.3.10}$$

where G is a group of translations.

Moreover τ induces an antisymplectic involution without fixed points on $E^+ \times E^-$, which by the previous considerations (5.3.5) and (5.3.6), is of type $T \times (-1)$, where T is a translation by a 2-torsion point. Since the translations by 2-torsion points commute with $T \times (-1)$, setting $X := (E^+ \times E^-)/\tau$ and $Y := A/\tau$, we get the commutative diagram

$$\begin{array}{ccc} G \circlearrowright E^+ \times E^- & \longrightarrow & A \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y. \end{array} \tag{5.3.11}$$

Let us study the first non-trivial case of (5.3.10), i.e. $G = \mathbb{Z}_2 = \langle T_+ \times T_- \rangle$, where $T_\pm$ is a translation of $E_\pm$ by a 2-torsion point. Of course the two translations cannot be trivial, otherwise A is again decomposable. Moreover $T_+ \neq T$, otherwise $\langle T_+ \times T_-, T \times (-1) \rangle = \langle 1 \times (-T_-), T \times (-1) \rangle$ and we would have the commutative diagram

$$\begin{array}{ccc} 1\times(-T_-) \circlearrowright E^+ \times E^- & \longrightarrow & A \\ \downarrow & & \downarrow \\ T\times 1_{\mathbb{P}^1} \circlearrowright E_+ \times \mathbb{P}^1 & \longrightarrow & E' \times \mathbb{P}^1 = Y, \end{array} \tag{5.3.12}$$

which is absurd because $Y := A/\tau$ is bielliptic.

Now we focus on the case $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle T_+^1 \times T_-^1, T_+^2 \times T_-^2 \rangle$, where $T_\pm^i$ is a translation by a 2-torsion point on $E_\pm$. Again, $T_\pm^i$ cannot be trivial, otherwise we are in the case $G = 1$ or $\mathbb{Z}_2$. For the same reason, also $T_\pm^1 \circ T_\pm^2$ is non-trivial. Hence one involution among $T_+^1, T_+^2, T_+^1 \circ T_+^2$ is equal to T. Assume $T = T_+^1$. So $1 \times (-T_-^1) \in \langle G, \tau \rangle$. But then we can factor the étale covering $E_+ \times E_- \to X$ through the quotient by $1 \times (-T_-^1)$, and this map has a branch locus, which is absurd.

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Finally, it remains the case $G = \mathbb{Z}_4 = \langle T_+ \times T_- \rangle$, where $T_+ \times T_-$ is a translation by a 4-torsion point. But since the composition of the two maps in (5.3.9) is $[\cdot 2]$, this case cannot occur.

Summing up, we have proven the following classical result:

Theorem 5.3.1. *Let A be a generic abelian surface with an antisymplectic involution τ without fixed points. Then there are only two possibilities:*

i) *$A = E_1 \times E_2$ and $\tau = T \times (-1)$, where E_1, E_2 are generic elliptic curves and T is a translation by a 2-torsion point of E_1;*

ii) *$A = E_1 \times E_2 / T_1 \times T_2$ and $\tau = T \times (-1)$, where E_1, E_2 are generic elliptic curves, T_i are translations by 2-torsion points on E_i, T is a translation by a 2-torsion point on E_1 such that $T \neq T_1$.*

Remark 5.3.2. By the classification of Bagnera and de Franchis [5], there are seven types of bielliptic surfaces, two generic (in the sense that E_1 and E_2 are generic) and five non-generic (in the sense that E_2 is not generic). The two described in Theorem 5.3.1 correspond to the two generic types.

5.4 Curves on bielliptic surfaces

From now on we will denote by A, $\bar{A}$ respectively the abelian surfaces from items *i)*, *ii)* of Theorem 5.3.1, and by Y, $\bar{Y}$ the corresponding bielliptic surfaces.

We first describe divisors and linear systems on A and $\bar{A}$.

Denoting by p_i the natural projection from A to E_i, clearly $\operatorname{Pic}(A) = p_1^* \operatorname{Pic}(E_1) \oplus p_2^* \operatorname{Pic}(E_2)$, so $\operatorname{Num}(A) = \langle E_1, E_2 \rangle$ and, endowed with the intersection form, it has a lattice structure of type U, the standard hyperbolic lattice of rank 2. We denote by $aE_1 + bE_2$ the numerical class of a line bundle on A of type $p_1^* \mathcal{L}_b \otimes p_2^* \mathcal{M}_a$, where $\mathcal{L}_b$ is a line bundle on E_1 of degree b and $\mathcal{M}_a$ a line bundle on E_2 of degree a.

If C has numerical class $aE_1 + bE_2$, by adjunction formula it has genus $g = ab + 1$, hence $|C| = \mathbb{P}^{ab-1}$. The induced map factors through the Segre embedding

$$\begin{array}{ccc} E_1 \times E_2 & \xrightarrow{\phi_{\mathcal{L}_b} \times \phi_{\mathcal{M}_a}} & \mathbb{P}^{b-1} \times \mathbb{P}^{a-1} \\ & \searrow^{\phi_{|C|}} & \downarrow \scriptstyle{Segre} \\ & & \mathbb{P}^{ab-1}. \end{array} \tag{5.4.1}$$

By their genericity, E_1 and E_2 are non-isogenous, so for a smooth ample curve we have $a, b > 1$.

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Let $\bar{E}_i := E_i/T_i$, and let $\bar{p}_i$ be the natural projection from $\bar{A}$ to E_i. To describe divisors and linear systems on $\bar{A}$, we can determine the $(T_1 \times T_2)$-invariant divisors on A.

Lemma 5.4.1. *Let C be a divisor of numerical class $aE_1 + bE_2$. Then $|C|_A^{T_1 \times T_2} \neq \emptyset$ if and only if a and b are even.*

Proof. For a line bundle $\mathcal{L} = p_1^*\mathcal{L}_b \otimes p_2^*\mathcal{M}_a$ on A and an automorphism $\alpha_1 \times \alpha_2$ on A, we have

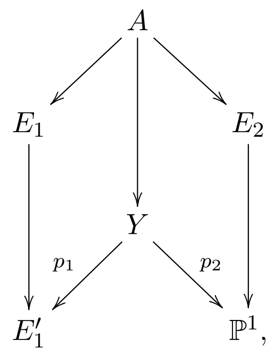
$$\alpha_*(\mathcal{L}) = p_1^*(\alpha_{1*}\mathcal{L}_b) \otimes p_2^*(\alpha_{1*}\mathcal{M}_a). \tag{5.4.2}$$

Since T_i has no fixed points, a T_i-invariant line bundle is a pullback of a line bundle on $\bar{E}_i$, hence it has even degree. □

A fiber of $\bar{p}_i$ is isomorphic to E_j with $i \neq j$. We denote by F_2 the fiber of $\bar{p}_1$ and by F_1 the fiber over $\bar{p}_2$. As $\mathrm{Num}(A) = \langle E_1, E_2 \rangle$, $\mathrm{Num}(\bar{A}) = \langle F_1, F_2 \rangle$. Using the 2-isogeny $\bar{A} \to A$ of (5.3.9), we determine the intersection form on $\bar{A}$: $\mathrm{Num}(\bar{A}) \cong U(2)$. A divisor $\bar{C}$ on $\bar{A}$ of numerical class $cF_1 + dF_2$ induces a divisor C on A of numerical class $2cE_1 + 2dE_2$, as described in Lemma 5.4.1. By Riemann-Hurwitz theorem, as C has genus $4cd + 1$, $\bar{C}$ has genus $\bar{g} = 2cd + 1$, so $|\bar{C}|_{\bar{A}} = \mathbb{P}^{2cd-1}$. By adjunction formula, we deduce again that $F_1 \cdot F_2 = 2$.

We characterize now divisors and linear systems on Y and $\bar{Y}$.

Setting $E_1' := E_1/T$ and $\bar{E}_1' := \bar{E}_1/T$, we get the following commutative diagrams



(5.4.3)

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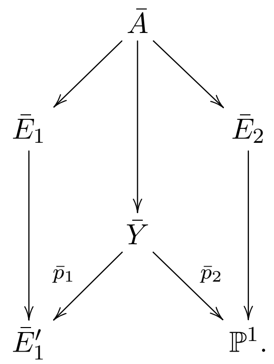

(5.4.4)

As for $\bar{A}$, we can easily relate the divisors on Y and $\bar{Y}$ to τ-invariant divisors on A and $\bar{A}$.

Lemma 5.4.2. *Let C (respectively $\bar{C}$) be a divisor on A (respectively $\bar{A}$) of numerical class $aE_1 + bE_2$ (respectively $aF_1 + bF_2$).*

Then $|C|^\tau \neq \emptyset$ (respectively $|\bar{C}|^\tau \neq \emptyset$) if and only if b is even.

Proof. The same proof as in Lemma 5.4.1, using the fact that T has no fixed points and (5.4.2). □

Remark 5.4.3. Denoting by H the group of involutions of A generated by $T_1 \times T_2$ and τ, Lemma 5.4.2 can be reformulated in the case of $\bar{A}$: if C is a curve on A of numerical class $aE_1 + bE_2$, then $|C|^H \neq \emptyset$ if and only if a is even and b is divisible by 4.

As T does not have any fixed point, a fiber of p_1 and of $\bar{p}_1$ is isomorphic respectively to E_2 and $\bar{E}_2$. A fiber of p_2 and of $\bar{p}_2$ over one of the four singular points is isomorphic respectively to twice E'_1 and twice $\bar{E}'_1$. Following Serrano [62], by the exponential sequence, we get, for $Z = Y, \bar{Y}$, the exact sequence

$$H^1(Z, \mathcal{O}^*) \to H^2(Z, \mathbb{Z}) \to H^2(Z, \mathcal{O}) = 0,$$

hence $\mathrm{Num}(Z)$ coincides with $H^2(Z, \mathbb{Z})$ modulo torsion. By Noether formula $\dim H^2(Z, \mathbb{Q}) = 2$, so $H^2(Y, \mathbb{Q}) = \langle E'_1, E_2 \rangle$ and $H^2(\bar{Y}, \mathbb{Q}) = \langle \bar{E}'_1, \bar{E}_2 \rangle$. Serrano proves in [62] that these are also bases of $\mathrm{Num}(Y)$ and $\mathrm{Num}(\bar{Y})$ respectively. Considering the intersection form, we get $\mathrm{Num}(Y) = \mathrm{Num}(\bar{Y}) = U$, the hyperbolic lattice of rank 2.

A divisor C' ($\bar{C}'$) on Y ($\bar{Y}$) of numerical class $aE'_1 + dE_2$ ($a\bar{E}'_1 + d\bar{E}_2$) induces a divisor C ($\bar{C}$) on A ($\bar{A}$) of numerical class $aE_1 + 2dE_2$ ($a\bar{E}_1 + 2d\bar{E}_2$), as described in Lemma 5.4.2. By Riemann-Hurwitz theorem, C' ($\bar{C}'$) has genus $g' = ad + 1$ ($\bar{g}' = ad + 1$). By Riemann-Roch theorem and Nakai-Moishezon criterion, C' ($\bar{C}'$) is ample if and only if $a, d > 0$ and moreover $h^0(C') = ad$ (respectively $h^0(\bar{C}') = ad$).

Using the description of Num and the fact that the canonical divisor on a bielliptic surface is numerically trivial, Reider criterion can be formulated as in Proposition 5 of [56]:

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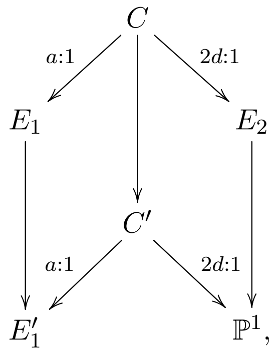
Theorem 5.4.4 (Reider criterion)**.** *Let C' ($\bar{C}'$) be a divisor on Y ($\bar{Y}$) of numerical class $aE_1' + dE_2$ ($a\bar{E}_1' + d\bar{E}_2$).*

Let $ad \geq 6$. Then C' (respectively $\bar{C}'$) is base point free if and only if $a, d \geq 2$.

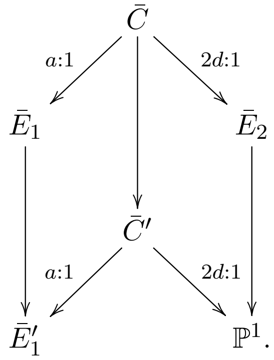
Let $ad \geq 10$. Then C' (respectively $\bar{C}'$) is very ample if and only if $a, d \geq 3$.

If we consider a τ-invariant curve C on A, (5.4.3) provides the diagram



(5.4.5)

If we consider a τ-invariant curve $\bar{C}$ on $\bar{A}$, (5.4.4) provides the diagram



(5.4.6)

5.5 Classification of the relative 0-Prym varieties from bielliptic surfaces

Lemma 5.5.1. *Let $a, d \geq 1$ and C_i (respectively $\bar{C}_i$) for $i = 1, 2$ smooth curves on A (respectively $\bar{A}$) of numerical class $aE_1 + 2dE_2$ (respectively $aF_1 + 2dF_2$). Then $\mathcal{J}_{C_1} = \mathcal{J}_{C_2}$ (respectively $\mathcal{J}_{\bar{C}_1} = \mathcal{J}_{\bar{C}_2}$).*

Moreover, if C_i are τ-invariant, then $\mathcal{P}_{C_1} = \mathcal{P}_{C_2}$ (respectively $\mathcal{P}_{\bar{C}_1} = \mathcal{P}_{\bar{C}_2}$) for an appropriate choice of $\{C_i\}^{\tau,+}$.

Proof. Under our hypothesis, $\{C_1\} = \{C_2\}$ and it is obvious that $\mathcal{J}_{C_1} = \mathcal{J}_{C_2}$. If, moreover, both C_1 and C_2 are τ-invariant, then $C_1 - C_2 \in \hat{A}^{\tau *}$. Choosing the plus-components of $\{C_i\}$ in such a way that $C_i \in \{C_i\}^{\tau,+}$, we have

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$\{C_1\}^{\tau,+} = \{C_2\}^{\tau,+}$, and then $\mathcal{P}_{C_1} = \mathcal{P}_{C_2}$ as well. The case of $\bar{C}_i \subset \bar{A}$ is similar. □

Thus the following definition makes sense.

Definition 5.5.2. *Let $d \geq 1$, $a > 1$. We denote by $\mathcal{J}_{a,d}$ (respectively $\mathcal{J}^{\sim}_{a,d}$) the relative compactified Jacobian associated to a smooth curve C (respectively $\bar{C}$) on A (respectively $\bar{A}$) of numerical class $aE_1 + dE_2$ (respectively $aF_1 + dF_2$), and by $\mathcal{K}_{a,d}$ (respectively $\mathcal{K}^{\sim}_{a,d}$) a fiber of the corresponding Albanese map. We denote by $\mathcal{P}_{a,d}$ (respectively $\mathcal{P}^{\sim}_{a,d}$) the relative Prym variety associated to a τ-invariant smooth curve C (respectively $\bar{C}$) on A (respectively $\bar{A}$) of numerical class $aE_1 + 2dE_2$ (respectively $aF_1 + 2dF_2$), and by $\mathcal{P}^0_{a,d}$ (respectively $\mathcal{P}^{\sim,0}_{a,d}$) its relative 0-Prym variety.*

We can reduce the study of the relative 0-Prym varieties to the study of curves on A via the following result.

Lemma 5.5.3. *Let $d \geq 1$, $a > 1$. Then*

$$\mathcal{P}^{\sim}_{a,d} = \mathrm{Fix}((T_1 \times T_2)^*) \subset \mathcal{P}_{2a,2d} \text{ and } \mathcal{P}^{\sim,0}_{a,d} = \mathrm{Fix}((T_1 \times T_2)^*) \subset \mathcal{P}^0_{2a,2d}, \tag{5.5.1}$$

or equivalently

$$\mathcal{P}^{\sim}_{a,d} = \mathrm{Fix}(H^*) \subset \mathcal{J}_{2a,4d} \text{ and } \mathcal{P}^{\sim,0}_{a,d} = \mathrm{Fix}(H^*) \subset \mathcal{K}_{2a,4d}, \tag{5.5.2}$$

where H is the group of involutions on A generated by $T_1 \times T_2$ and τ.

Proof. As the $(T_1 \times T_2)^*$-invariant sheaves on $\mathcal{J}_{a,2d}$ come from sheaves on $\mathcal{J}^{\sim}_{a,d}$ by pullback, $\mathcal{J}^{\sim}_{a,d} = \mathrm{Fix}((T_1 \times T_2)^*) \subset \mathcal{J}_{a,2d}$. The assertion follows form this identification and from the definition of relative Prym variety. □

As described in Lemma 5.1.2, the image of $P(C,\tau)$ (respectively $P(\bar{C},\tau)$) under the Albanese map is $\mathrm{Fix}^0_A(id \times (-1)) = E_1 \times 0$ (respectively $\mathrm{Fix}^0_{\bar{A}}(id \times (-1)) = \bar{E}_1 \times 0$).

If we consider a τ-invariant curve C (respectively $\bar{C}$) of numerical class $aE_1 + 2dE_2$ (respectively $a\bar{E}_1 + 2d\bar{E}_2$), $P^0(C,\tau)$ (respectively $P^0(\bar{C},\tau)$) has dimension $g - g' - 1 = 2ad + 1 - (ad + 1) - 1 = ad - 1$, as expected since $h^0(C') = ad$ ($h^0(\bar{C}') = ad$).

By Lemma 2.4.3, the primitivity of the Mukai vector $(0, C, 1 - g)$ is equivalent to the primitivity of the curve C (respectively $\bar{C}$) on A ($\bar{A}$) of numerical class $aE_1 + 2dE_2$ (respectively $aF_1 + 2dF_2$), which corresponds to $GCD(a, 2b) = 1$. But the primitivity of a curve on Y or $\bar{Y}$ with numerical class $aE'_1 + dE_2$ (respectively $a\bar{E}'_1 + d\bar{E}_2$) is equivalent to $GCD(a, d) = 1$. Indeed, if a curve on A, divisible by 2, belongs to the linear system of a curve lifted from Y or $\bar{Y}$, then it cannot be τ-invariant. Thus $\mathcal{K}^+$ has no singularities corresponding to sheaves supported on non-reduced curves if

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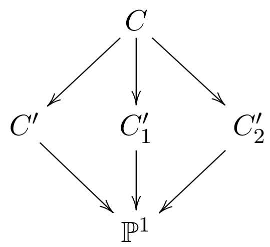
and only if $GCD(a, d) = 1$.

When $d = 1$, the curve on a bielliptic surface is primitive. By Lemma 5.1.2, supp : $\mathcal{P}^0_{a,1} \to \mathbb{P}^{a-1}$ (respectively supp : $\mathcal{P}^{\sim,0}_{a,1} \to \mathbb{P}^{a-1}$) is a Lagrangian fibration in principally polarized abelian varieties, as $C \cdot \mathrm{Fix}^0_A(L(\tau)) = (aE_1 + 2E_2) \cdot E_1 = 2$ (respectively $\bar{C} \cdot \mathrm{Fix}^0_{\bar{A}}(L(\tau)) = (a\bar{E}_1 + 2\bar{E}_2) \cdot \bar{E}_1 = 2$).

The following description of Prym varieties of double covers of hyperelliptic curves by Mumford [44] is crucial for the description of $\mathcal{P}^0_{a,1}$.

Lemma 5.5.4. *Let $C' \longrightarrow \mathbb{P}^1$ be a hyperelliptic curve with branch locus B, and let $\tau \circlearrowright C \longrightarrow C'$ be an étale double cover.*

Then there exist B_1, B_2 finite subsets of $\mathbb{P}^1$ such that $B_1 \cup B_2 = B$, $B_1 \cap B_2 = \emptyset$, $\deg B_i$ are even, and hyperelliptic curves C'_1, C'_2 with branch loci B_i, such that $C = \widetilde{C'_1 \times_{\mathbb{P}^1} C'_2}$, the normalization of the fiber product. Hence we have the following commutative diagram of double coverings



and, denoting by τ_i the involution on C associated to C'_i, we have $\tau = \tau_1 \circ \tau_2$. Moreover, $P(C, \tau) \cong J(C'_1) \times J(C'_2)$ and the natural polarization of $P(C, \tau)$ is given by $\Xi = J(C'_1) \times \Theta_2 + \Theta_1 \times J(C'_2)$.

Lemma 5.5.5. *Let C be a τ-invariant smooth curve on A numerically equivalent to $aE_1 + 2E_2$.*

Then $P(C, \tau) = J(C'') \times E_2$ with the product polarization, where $C'' := C/\tau_1$ and $\tau_1 = (-T) \times (-1)$. Moreover $P^0(C, \tau) = J(C'')$ and $\Xi = \Theta''$.

Proof. By the diagram (5.4.5), C' is hyperelliptic, hence we can apply Lemma 5.5.4.

We can assume that $\phi_{\mathcal{L}_2} : E_1 \to \mathbb{P}^1$ is the projection to the y-coordinates of $E_1 : \xi^2 = g_2(y_0^2, y_1^2)$ with $\deg g_2 = 2$, and $T : (\xi, y_0, y_1) \mapsto (-\xi, y_0, -y_1)$. By (5.4.1) the morphism induced by $|C|$ on A is

$$\begin{array}{ccccc} E_1 \times E_2 & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^{a-1} & \longrightarrow & \mathbb{P}^{2a-1} \\ (\xi, \underline{y}), (\underline{x}) & \mapsto & (\underline{y}), (\underline{x}) & \mapsto & (y_i x_j). \end{array} \tag{5.5.3}$$

The equation of C in A does not involve ξ because of its numerical class, and the hyperelliptic involution of E_1 induces on C the involution $\tau_2 = (-1) \times id$, corresponding to the double cover $C \to E_2$.

By Lemma 5.5.4, $\tau_1 = \tau \circ \tau_2 = (-T) \times (-1)$ and the assertion follows. □

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Remark 5.5.6. A similar result holds in the case of $\bar{A}$, since by diagram (5.4.6), $\bar{C}$ is hyperelliptic when it has numerical class $aF_1 + 2F_2$, so we can still apply Lemma 5.5.4.

Theorem 5.5.7. *For $a > 1$, $\mathcal{P}^0_{a,1}$ is birational to an irreducible symplectic manifold of $K3^{[a-1]}$-type.*

Proof. τ_1 induces an involution on A with 16 fixed points. The quotient $S := A/\tau_1$ is a K3 surface with 16 singular points, given by

$$\{(\pm\xi_{[0,1]}, 0, 1), (\pm\xi_{[1,0]}, 1, 0)\} \times E_2[2].$$

Blowing up in the 16 singular points we get a smooth K3 surface $\hat{S}$.

Thus we have the following commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\rho} & \hat{S} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ A & \xrightarrow{\rho'} & S \end{array} \tag{5.5.4}$$

where $Z \to A$ is the blowup of the surface at the 16 fixed points of the involution, and $Z \to \hat{S}$ is the double cover ramified along the 16 exceptional curves of τ_1.

Denote $\hat{C} := \delta'^{-1}C''$ and $\mathcal{J}_{\hat{C}} := \mathcal{J}^{\hat{C}}_{\hat{S}}(\hat{C})$. Let U be the open subset of smooth curves in $|C|^{\tau,+}$. It corresponds to an open subset of smooth curves in $|\hat{C}|_{\hat{S}}$. By Lemma 5.5.5, $\mathcal{P}^0_{a,1}$ and $\mathcal{J}_{\hat{C}}$ coincide fiberwise over U. Then the natural rational map $\delta_* \circ \rho^*$ factors as follows

$$\begin{array}{ccc} \mathcal{J}_{\hat{C}} & \dashrightarrow & \mathcal{P}^0_{a,1} \\ & {\scriptstyle\delta_*\circ\rho^*}\searrow & \downarrow \\ & & \mathcal{J}_C \end{array} \tag{5.5.5}$$

and $\mathcal{J}_{\hat{C}}$ and $\mathcal{P}^0_{a,1}$ are birational.

Moreover C primitive implies $\hat{C}$ primitive. Hence changing the polarization, we can obtain a smooth birational model of $\mathcal{J}_{\hat{C}}$ by Theorem 2.4.9, which is deformation equivalent to a Hilbert scheme of points on a K3 surface by Theorem 2.4.5.

□

Remark 5.5.8. The smallest dimensional example for A is given by $\mathcal{P}^0_{3,1}$, which is a 4-fold. In this case the reduced reducible members on A have 5 irreducible components, and the non-reduced ones have 3 or 4 components. Indeed $c = d = 1$ gives a linear system with only non-integral members.

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We now treat the case $d > 1$. By Lemma 5.1.2, the fibers of supp : $\mathcal{P}^0_{a,d} \to \mathbb{P}^{ad-1}$ (respectively supp : $\mathcal{P}^{\sim,0}_{a,d} \to \mathbb{P}^{ad-1}$) have polarization of type $(1, ..., 1, d)$.

Lemma 5.5.9. *Let C ($\bar{C}$) be a curve on A ($\bar{A}$) of numerical class aE_1+2dE_2 ($aF_1 + 2dF_2$), with $d > 1$, $a > 2$ and $(a-1)d \geq 6$. Then $|C|^{\tau,+}$ ($|\bar{C}|^{\tau,+}$) contains a curve with two smooth irreducible τ-invariant components meeting transversely in at least 2 points.*

Proof. Let C_1 ($\bar{C}_1$) be the fiber of p_2 ($\bar{p}_2$) over $0 \in E_2$ ($0 \in \bar{E}_2$). It is clearly τ-invariant. Let C_1' ($\bar{C}_1'$) be its image in Y ($\bar{Y}$). By Theorem 5.4.4, if $(a-1)d \geq 6$, then $|C' - C_1'|$ ($|\bar{C}' - \bar{C}_1'|$) is a base point free linear system. Consequently also $|C - C_1|^{\tau,+}$ ($|\bar{C} - \bar{C}_1|^{\tau,+}$) is base point free. Thus, by Bertini theorem, there exists a smooth connected curve C_2 ($\bar{C}_2$) of numerical class $(a-1)E_1 + dE_2$ ($(a-1)F_1 + dF_2$). Hence C_1 and C_2 ($\bar{C}_1$ and $\bar{C}_2$) meet transversely at $d \geq 2$ points, and their union is in $|C|^{\tau,+}$ ($|\bar{C}|^{\tau,+}$). □

Remark 5.5.10. The smallest dimensional example is given by $\mathcal{P}^0_{3,2}$ ($\mathcal{P}^{\sim,0}_{3,2}$), which has dimension 10.

Adapting Theorem 5.2.4 and Corollary 5.2.3 to this situation, we get:

Corollary 5.5.11. *Let $d > 1$ and $a > 2$ such that $(a-1)d \geq 6$. Then there exists a polystable sheaf $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ in $\mathcal{P}^0_{a,d}$ ($\mathcal{P}^{\sim,0}_{a,d}$) whose support is a curve with two smooth irreducible τ-invariant components meeting transversely in $2d$ points, and it is a singularity of analytic type $(\mathbb{C}^{2d(a-1)-2} \times (\mathbb{C}^{2d}/\pm 1), 0)$. Hence $\mathcal{P}^0_{a,d}$ ($\mathcal{P}^{\sim,0}_{a,d}$) does not admit any symplectic desingularization.*

Summing up, we have obtained the following result for relative 0-Prym varieties associated to curves on bielliptic surfaces.

Theorem 5.5.12. *Let E_1, E_2 be generic elliptic curves, $A := E_1 \times E_2$ ($\bar{A} := E_1 \times E_2 / T_1 \times T_2$), $\tau := T \times (-1)$. Let $Y := A/\tau$ ($\bar{Y} := \bar{A}/\tau$) be the corresponding bielliptic surface. Let $a, d \in \mathbb{Z}_+$. Then*

i) *$\mathcal{P}^0_{a,1}$ is birational to an irreducible symplectic manifold of $K3^{[a-1]}$-type;*

ii) *for $d > 1$ and $a > 2$ with $(a-1)d \geq 6$, $\mathcal{P}^0_{a,d}$ and $\mathcal{P}^{\sim,0}_{a,d}$ are singular symplectic varieties which do not admit any symplectic resolution.*

For a future work, it remains to analyze the cases

iii) $\mathcal{P}^{\sim,0}_{a,1}$;

iv) $d > 1$ and $a = 2$;

v) $d = 2$ and $a = 3$.

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